Unified Systems of FB-SPDEs/FB-SDEs with Jumps/Skew Reflections and Stochastic Differential Games

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Abstract

We study four systems and their interactions. First, we formulate a unified system of coupled forward and backward stochastic partial differential equations (FB-SPDEs) with Lévy jumps, which is vector-valued and whose drift, diffusion, and jump coefficients may involve partial differential operators. Under generalized local linear growth and Lipschitz conditions, the well-posedness concerning adapted strong solution to the FB-SPDEs is proved. Second, we consider a unified system of FB-SDEs, a special form of the FB-SPDEs, however, with skew reflections. Under generalized linear growth and Lipschitz conditions together with a general completely-S condition on reflection matrices, we prove the well-posedness of adapted weak solution to the FB-SDEs. In particular, if the spectral radii in certain sense for both reflection matrices are strictly less than the unity, a unique adapted strong solution will be concerned. Third, we formulate a stochastic differential game (SDG) problem with general number of players based on the FB-SDEs. By a solution to the FB-SPDEs, we determine a solution to the FB-SDEs under a given control rule and then obtain a Pareto optimal Nash equilibrium point to the non-zero-sum SDG problem. Fourth, we study the application of the FB-SPDEs in a queueing system and discuss how to use the queueing system to motivate the SDG problem.

Key words and phrases: Stochastic (Ordinary/Partial) Differential Equation, Lévy Jump, Skew Reflection, Skorohod Problem, Stochastic Differential Game, Nash Equilibrium, Queueing Network

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1 Introduction

We study four systems and their interactions: a unified system of coupled forward and backward stochastic partial differential equations (FB-SPDEs) with Lévy jumps; a unified system of FB-SDEs, a special form of the FB-SPDEs, however, with skew reflections; a stochastic differential game (SDG) problem with general number of players based on the FB-SDEs; and a queueing system with its associated reflecting diffusion approximation. More precisely, there are four interconnected and streamlined aims involved in our discussions.

The first aim is to study the adapted strong solution to the unified system of coupled FB-SPDEs with Lévy jumps,

\[
\begin{align*}
U(t,x) &= G(x) + \int_0^t \mathcal{L}(s^-, x, U) \, ds \\
&\quad + \int_0^t \mathcal{J}(s^-, x, U) \, dW(s) \\
&\quad + \int_0^t \int_{z>0} \mathcal{I}(s^-, x, U, z) \tilde{N}(\lambda ds, dz), \\
V(t,x) &= H(x) + \int_t^T \mathcal{L}(s^-, x, V) \, ds \\
&\quad + \int_t^T (\mathcal{J}(s^-, x, V) - \tilde{V}(s^-, x)) \, dW(s) \\
&\quad + \int_t^T \int_{z>0} (\tilde{I}(s^-, x, U, V, z) - \tilde{V}(s^-, x, z)) \tilde{N}(\lambda ds, dz).
\end{align*}
\]

The F-SPDE in (1.1) is with the given initial vector random field \( G \), while the B-SPDE in (1.1) has the known terminal vector random field \( H \). In (1.1), \( U \) and \( V \) are \( r \)-dimensional and \( q \)-dimensional vector random field processes respectively, \( W \) is a standard \( d \)-dimensional Brownian motion, and \( \tilde{N} \) is a \( h \)-dimensional centered Lévy jump process. Furthermore, the partial differential operators of \( r \)-dimensional vector \( \mathcal{L} \), \( r \times d \)-dimensional matrix \( \mathcal{J} \), and \( r \times h \)-dimensional matrix \( \mathcal{I} \) are functions of \( U, V, \tilde{V}, \tilde{\tilde{V}} \), and their partial derivatives of up to the \( k \)th order for \( k \in \{0, 1, 2, \ldots\} \),

\[
\begin{align*}
\mathcal{L}(s,x,U,V) &\equiv \mathcal{L}(s,x,U(s,x),\ldots,U^{(k)}(s,x)), \\
&\quad V(s,x), \ldots, V^{(k)}(s,x), \\
&\quad \tilde{V}(s,x), \ldots, \tilde{V}^{(k)}(s,x), \\
&\quad \tilde{\tilde{V}}(s,x,z), \ldots, \tilde{\tilde{V}}^{(k)}(s,x,z),
\end{align*}
\]

\[
\begin{align*}
\mathcal{J}(s,x,U,V) &\equiv \mathcal{J}(s,x,U(s,x),\ldots,U^{(k)}(s,x)), \\
&\quad V(s,x), \ldots, V^{(k)}(s,x), \\
&\quad \tilde{V}(s,x), \ldots, \tilde{V}^{(k)}(s,x), \\
&\quad \tilde{\tilde{V}}(s,x,z), \ldots, \tilde{\tilde{V}}^{(k)}(s,x,z),
\end{align*}
\]

\[
\begin{align*}
\mathcal{I}(s,x,U,V,z) &\equiv \mathcal{I}(s,x,U(s,x),\ldots,U^{(k)}(s,x)), \\
&\quad V(s,x), \ldots, V^{(k)}(s,x), \\
&\quad \tilde{V}(s,x), \ldots, \tilde{V}^{(k)}(s,x), \\
&\quad \tilde{\tilde{V}}(s,x,z), \ldots, \tilde{\tilde{V}}^{(k)}(s,x,z),
\end{align*}
\]

In addition, the partial differential operators of \( q \)-dimensional vector \( \tilde{\mathcal{L}} \), \( q \times d \)-dimensional matrix \( \tilde{\mathcal{J}} \), and \( q \times h \)-dimensional matrix \( \tilde{\mathcal{I}} \) are also functions of \( U, V, \tilde{V}, \tilde{\tilde{V}} \), and their partial
derivatives of up to the $k$th order for $k \in \{0, 1, 2, \ldots\}$,

\[
\tilde{L}(s,x,U,V) \equiv \tilde{L}(s,x,U(s,x),\ldots,U^{(k)}(s,x),
V(s,x),\ldots,V^{(k)}(s,x),
\tilde{V}(s,x),\ldots,\tilde{V}^{(k)}(s,x),
\tilde{V}(s,x,z),\ldots,\tilde{V}^{(k)}(s,x,z),\cdot),
\]

\[
\tilde{J}(s,x,V) \equiv \tilde{J}(s,x,U(s,x),\ldots,U^{(k)}(s,x),
V(s,x),\ldots,V^{(k)}(s,x),\cdot),
\]

\[
\tilde{I}(s,x,V,z) \equiv \tilde{I}(s,x,U(s,x),\ldots,U^{(k)}(s,x),
V(s,x),\ldots,V^{(k)}(s,x),z,\cdot).
\]

Under generalized local linear growth and Lipschitz conditions, we prove the existence and uniqueness of an adapted strong solution to the FB-SPDEs in a measurable functional topological space. The proof developed in this paper is extended from our earlier work summarized in Dai [12] (arxiv, 2011) for a unified B-SPDE. The solution to the unified system in (1.1) can be interpreted in a sample surface manner (see, e.g., $V(t,x)$ in Figure 1 for such an example).

Figure 1: Sample Surface Solution to the FB-SPDEs

The newly unified system of coupled FB-SPDEs in (1.1) covers many existing forward and/or backward SDEs/SPDEs as special cases, where the partial differential operators are taken to be special forms. For examples, specific strongly nonlinear F-SPDE and B-SPDE solely driven by Brownian motions can be respectively derived for the purpose of optimal-utility based portfolio choice (see, e.g, Musiela and Zariphopoulou [28]). Here, the strong nonlinearity is in the sense addressed by Lions and Souganidis [26] and Pardoux [32]. Furthermore, the stochastic Hamilton-Jacobi-Bellman (HJB) equations are also examples of our
unified system in (1.1), which are specific B-SPDEs (see, e.g., Øksendal et al. [31] and references therein). Note that the proof of the well-posedness concerning solution to the F-SPDE derived in Musiela and Zariphopoulou [28] and solution to the HJB equation derived in Øksendal et al. [31] is covered by the study in Dai [12] (arxiv, 2011) although the authors in both [28] and Øksendal et al. [31] claim it as an open problem. The proof of the well-posedness about solution to the F-SPDE derived in Musiela and Zariphopoulou [28] is covered by the even more unified discussion for the coupled FB-SPDEs in (1.1) of this paper. Actually, partial motivations to enhance the unified B-SPDE in Dai [12] (arxiv, 2011) to the current coupled FB-SPDEs in (1.1) are from optimal portfolio management in finance (see, e.g., Dai [11, 15]), and multi-channel (or multi-valued) image regularization such as color images in computer vision and network application (see, e.g., Caselles et al. [5]).

The second aim of the paper is to prove the well-posedness of adapted weak solution to the non-Markovian system of coupled FB-SDEs with Lévy jumps and skew reflections under a given control rule $u$,

\[
\begin{aligned}
&dX(t) = b(t, X(t), V(t), \hat{V}(t), \hat{V}(t, \cdot), u(t, X(t)))dt \\
&\quad + \sigma(t, X(t), V(t), \hat{V}(t), \hat{V}(t, \cdot), u(t, X(t)), z) dW(t) \\
&\quad + \int_{z>0} \eta(t, X(t), V(t), \hat{V}(t), \hat{V}(t, \cdot), u(t, X(t)), z) \tilde{N}(dt, dz) \\
&\quad + RdY(t),
\end{aligned}
\]

\[
\begin{aligned}
X(0) &= x,\\
dV(t) &= c(t, X(t), V(t), \hat{V}(t), \hat{V}(t, \cdot), u(t, X(t))) dt - \hat{V}(t) dW(t) \\
&\quad - \int_{z>0} \hat{V}(t, z) \tilde{N}(dt, dz) - SdF(t),
\end{aligned}
\]

\[
\begin{aligned}
V_0(t) &= \sum_{i=1}^n V_i(t), \\
V(T) &= H(X(T)).
\end{aligned}
\]

In (1.2), $X$ is a $p$-dimensional process governed by the F-SDE with skew reflection matrix $R$ and $V$ is a $q$-dimensional process governed by the B-SDE with skew reflection matrix $S$. Note that, comparing with the unified system in (1.1), the coefficients appeared in (1.2) do not contain any partial derivative operator but the FB-SDEs themselves involve skew reflections. The proof for the well-posedness of adapted weak solution to the FB-SDEs is based on two general conditions. The first one is a general completely-$S$ condition (see, e.g., Dai [9], Dai and Dai [7], and Figure 2 for an illustration). The second one is the generalized linear growth and Lipschitz conditions, where the conventional growth and Lipschitz constant is replaced by a possible unbounded but mean-squarely integrable adapted stochastic process (see, e.g., Dai [11, 15]). In particular, if the completely-$S$ condition becomes more strict, e.g., with additional requirements that the spectral radii in certain sense for both reflection matrices are strictly less than the unity, a unique adapted strong solution will be concerned.

Concerning coupled FB-SDEs, it motivates a hot research area (see, e.g., Øksendal et
However, to our best knowledge, the coupled system in (1.2) with double skew reflection matrices and the well-posedness study in terms of solution with Lévy jumps and under a general completely-$\mathcal{S}$ condition are new and are for the first time in this area.

The third aim of the paper involves two folds. On the one hand, we use the solution to the coupled FB-SPDEs in (1.1) to obtain an adapted solution to the system in (1.2); On the other hand, we use the obtained adapted solution to determine a Pareto optimal Nash equilibrium point to a non-zero-sum SDG problem in (1.3), which is newly formulated based on the FB-SDEs in (1.2). In this game, there are $q$-players and each player $l \in \{1, \ldots, q\}$ has his own value function $V^u_l$ subject to the system in (1.2) under an admissible control policy $u$. Every player $l$ chooses an optimal policy to maximize his own value function over an admissible policy set $\mathcal{C}$, i.e.,

$$\sup_{u \in \mathcal{C}} V^u_l(0) = V^{u^*_l}(0).$$

Furthermore, the value functions $\{V^u_l(0), l \in \{1, \ldots, q\}\}$ do not have to add up to a constant (e.g., zero), or in other words, the game is not necessarily a zero-sum one.

The contribution and literature review of the study associated with the game in (1.3) for the third aim can be summarized as follows. One of the important solution methods for SDE based optimal control is the dynamic programming. In general, this method is related to

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Figure 2: Skew and Inward Reflection Under Completely-$\mathcal{S}$ Condition
a special case of the unified system in (1.1) (or its earlier unified B-SPDE form in Dai [12] (arxiv, 2011)), e.g., the specific B-SPDE with $q = 1$ (called stochastic HJB equation) in Peng [34] with no jumps and Øksendal et al. [31] with jumps. Here, we extend the discussions in Peng [34] and Øksendal et al. [31] to a system of stochastic HJB equations with jumps corresponding to the case that $q > 1$. More importantly, this system provides an effective way to resolve a non-zero-sum SDG problem with jumps and general number of $q$ players, which subjects to a non-Markovian system of coupled FB-SDEs with Lévy jumps and skew reflection (see, e.g., Figure 3 for such a game platform (partially adapted from Dai [10])). By a solution to the FB-SPDEs in (1.1), we determine a solution to the FB-SDEs in (1.2).

![Figure 3: A 5-player game platform based on brain and satellite communication](image)

under a given control rule and then obtain a Pareto optimal Nash equilibrium point to the non-zero-sum SDG problem in (1.3). Note that, the concept and technique concerning the non-zero-sum SDG and Pareto optimality used in this paper is refined and generalized from Dai [13] and Karatzas and Li [24].

The fourth aim of the paper also involves two folds. On the one hand, we study the application of the unified system of FB-SPDEs presented by (1.1) in the system of queueing networks; On the other hand, we discuss how to use the queueing system and its associated reflecting diffusion approximation to motivate the SDG problem.

Queueing networks widely appear in many real-world applications such as those in service, cloud computing, and communication systems. They typically consist of arrival processes, service processes, and buffer storages with certain kind of service regime and network architecture. To be clear, we present such an example with $p$-job classes in Figure 4. The major performance measure for this system is the queue length process that is a generalized birth-death process or a reaction-diffusion process. More precisely, we use $Q(\cdot) = (Q_1(\cdot), ..., Q_p(\cdot))'$ to denote the $p$-dimensional process, where $Q_i(t)$ is the number of $i$th class jobs stored in
the $i$th buffer for each $i \in \{1, ..., p\}$ at time $t$. Both the arrival and service processes can be described by renewal processes (see, e.g., Dai [9], Dai and Dai [7]), renewal reward processes (see, e.g., Dai and Jiang [16]), or doubly stochastic renewal processes (see, e.g., Dai [13]). Comparing the widely studied Markovian queueing networks associated with Lévy processes (see, e.g., Dai [14] and Konstantopoulos et al. [25]), the physical renewal queueing networks are not of Markovian property. Thus, it is usually impossible to get a product-form solution concerning the distribution of $Q(\cdot)$ for such a renewal queueing network. However, under certain conditions (e.g., the arrival rates close to the associated service rates), one can show that the corresponding sequence of diffusion-scaled queue length processes converges in distribution to a $p$-dimensional reflecting Brownian motion (RBM) (see, e.g., Dai [9], Dai and Dai [7], Dai and Jiang [16]), or more generally, a reflecting diffusion with regime switching (RDRS) (see, e.g., Dai [13]). In other words, we have that

$$
\hat{Q}_r(\cdot) \equiv \frac{1}{r}Q(r^2 \cdot) \Rightarrow \hat{Q}(\cdot) \quad \text{along } r \in \{1, 2, \ldots\},
$$

where “$\Rightarrow$” means “converges in distribution” and $\hat{Q}(\cdot)$ is a RBM or a RDRS. Thus, it is possible for us to use the Fokker-Planck formula (a PDE or a SPDE) of $\hat{Q}(\cdot)$ to study the related queue performance and to employ $(\hat{Q}(\cdot))$ to construct a SDG.

The remainder of the paper is organized as follows. In Section 2 we introduce suitable functional topological space and state conditions required for our main theorems to guarantee the well-posedness of adapted strong solution to the unified system of coupled FB-SPDEs in
and Corollary 13.7 in Kallenberg [23]) for convenience, we take the constant
(2.1) $\lambda = (\lambda_1, ..., \lambda_l)$ in (2.1) and the non-zero-sum SDG problem in (1.3) by studying the unified system of coupled
FB-SDEs with skew reflections in (1.2). Related main theorems are also presented. Finally,
in Sections 4-5 we develop theory to prove our main theorems.

2 The Unified System of Coupled FB-SPDEs with Lévy Jumps

Let $(\Omega, F, P)$ be a fixed complete probability space on which a standard $d$-dimensional Brownian motion $W \equiv \{W(t), t \in [0,T]\}$ for a given $T \in [0, \infty)$ with $W(t) = (W_1(t), ..., W_d(t))'$ and a $h$-dimensional subordinator $L \equiv \{L(t), t \in [0,T]\}$ with $L(t) = (L_1(t), ..., L_h(t))'$ (see, e.g., Applebaum [1], Bertoin [4], and Sato [38]) are defined. Note that the prime appeared
in this paper is used to denote the corresponding transpose of a matrix or a vector. Furthermore, $W$, $L$, and their components are supposed to be independent of each other. For each
$\lambda = (\lambda_1, ..., \lambda_l, l > 0$, we let $L(\lambda s) = (L_1(\lambda s), ..., L_h(\lambda s))'$. Then, we denote a filtration by
$\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_t \equiv \sigma\{\mathcal{G}, W(s), L(\lambda s) : 0 \leq s \leq t\}$ for each $t \in [0,T]$, where $\mathcal{G}$ is $\sigma$-algebra independent of $W$ and $L$. In addition, let $I_A(\cdot)$ be the index function over the set $A$ and $\nu_i$ for each $i \in \{1, ..., h\}$ be a Lévy measure. Then, we use $N_i((0, t] \times A) \equiv \sum_{0<s \leq t} I_A(L(s) - L_i(s^-))$ to denote a Poisson random measure with a deterministic, time-homogeneous intensity measure $ds\nu_i(dz_i)$. Thus, each subordinator $L_i$ can be represented by (see, e.g., Theorem 13.4 and Corollary 13.7 in Kallenberg [23])

\begin{equation}
L_i(t) = a_it + \int_{(0,t]} \int_{z_i>0} z_iN_i(ds, dz_i), \ t \geq 0.
\end{equation}

For convenience, we take the constant $a_i$ to be zero.

In the subsequent two subsections, we first study the unified system in (1.1) over a closed space domain and then extend the discussion to an open space domain.

2.1 The System over Closed Space Domain

In this subsection, we let $D \in \mathbb{R}^p$ with a given $p \in \mathcal{N} = \{1, 2, \ldots\}$ be a closed and connected domain. Furthermore, we use $C^k(D, \mathbb{R}^l)$ for each $k \in \mathcal{N}$ and $l \in \{r, q\}$ to denote the Banach space of all functions $f$ having continuous derivatives up to the order $k$ with the uniform norm for each $f$ in this space,

\begin{equation}
\|f\|_{C^k(D, \mathbb{R}^l)} = \max_{c \in \{0, 1, ..., k\}} \max_{j \in \{1, ..., r(c)\}} \max_{x \in D} \sup \left| f^{(c)}_j(x) \right|.
\end{equation}

The $r(c)$ in (2.2) for each $c \in \{0, 1, ..., k\}$ is the total number of the partial derivatives of the order $c$

\begin{equation}
f^{(c)}_{i_1, i_2, ..., i_p}(x) = \frac{\partial^c f_i(x)}{\partial x_{i_1}^{i_1} \partial x_{i_2}^{i_2} \cdots \partial x_{i_p}^{i_p}}
\end{equation}
with \( i_l \in \{0, 1, \ldots, c\}, l \in \{1, \ldots, p\}, r \in \{1, \ldots, l\} \), and \( i_1 + \ldots + i_p = c \). Here, we remark that, whenever the partial derivative on the boundary \( \partial D \) is concerned, it is defined in a one-side manner. In addition, let

\[
(2.4) \quad f_{i_1,\ldots,i_p}^{(c)} = (f_{i_1,\ldots,i_p}^{(c)})_{q=i_1,\ldots,i_p},
\]

\[
(2.5) \quad f^{(c)}(x) = (f_{1}^{(c)}(x), \ldots, f_{r}^{(c)}(x)),
\]

where each \( j \in \{1, \ldots, r(c)\} \) corresponds to a \( p \)-tuple \((i_1, \ldots, i_p)\) and a \( r \in \{1, \ldots, l\} \). Then, we use \( C^{\infty}(D, R^l) \) to denote the Banach space

\[
(2.6) \quad C^{\infty}(D, R^l) = \left\{ f \in \bigcap_{c=0}^{\infty} C^{c}(D, R^l), \|f\|_{C^{\infty}(D,l)} < \infty \right\},
\]

where

\[
(2.7) \quad \|f\|_{C^{\infty}(D,q)}^2 = \sum_{c=0}^{\infty} \xi(c) \|f\|_{C^{c}(D,l)}^2
\]

for some discrete function \( \xi(c) \) in terms of \( c \in \{0, 1, 2, \ldots,\} \), which is fast decaying in \( c \). For convenience, we take \( \xi(c) = \frac{1}{((c^m)/(\eta(c)))^{c}} \) with

\[
\eta(c) = \left[ \max\{\|x_1| + \ldots + |x_p|, x \in D \} \right]^c,
\]

where the notation \( \| \| \) denotes the summation of the unity and the integer part of a real number.

Next, let \( L_{\mathcal{F}}^{2}([0,T], C^{\infty}(D; R^l)) \) denote the set of all \( R^l \)-valued (or called \( C^{\infty}(D; R^l) \)-valued) measurable random field processes \( Z(t, x) \) adapted to \( \{ \mathcal{F}_t, t \in [0, T] \} \) for each \( x \in D \), which are in \( C^{\infty}(D, R^l) \) for each fixed \( t \in [0, T] \), such that

\[
(2.8) \quad E \left[ \int_{0}^{T} \|Z(t)\|_{C^{\infty}(D,l)}^2 dt \right] < \infty.
\]

In particular, let \( L_{\mathcal{G}_l}^{2}(\Omega, C^{\infty}(D; R^l)) \) with \( l \in \{r, q\} \) denote the set of all \( R^l \)-valued and \( \mathcal{G}_l \)-measurable random variables \( \zeta(x) \) for each \( x \in D \), where \( \mathcal{G}_r = \mathcal{G} \) and \( \mathcal{G}_q = \mathcal{F}_T \), and \( \zeta(x) \in C^{\infty}(D, R^l) \) satisfies

\[
(2.9) \quad \|\zeta\|_{L_{\mathcal{G}_l}^{2}(\Omega, C^{\infty}(D, R^l))}^2 = E \left[ \|\zeta\|_{C^{\infty}(D, l)}^2 \right] < \infty.
\]

In addition, let \( L_{\mathcal{G}_l}^{2}([0, T] \times R^h, C^{\infty}(D, R^{l \times h})) \) be the set of all \( R^{l \times h} \)-valued predictable processes \( \tilde{V}(t, x, z) = (\tilde{V}_1(t, x, z_1), \ldots, \tilde{V}_h(t, x, z_h)) \) for each \( x \in D \) and \( z \in R^h \), satisfying

\[
(2.10) \quad E \left[ \sum_{i=1}^{h} \int_{0}^{T} \int_{z_i > 0} \|\tilde{V}_i(t, z_i)\|_{C^{\infty}(D, l)}^2 \nu_i(dz_i) dt \right] < \infty.
\]
Thus, we can define
\begin{equation}
Q^2_{\mathcal{F}}([0, T] \times D) \equiv L^2_{\mathcal{F}}([0, T], C^\infty(D, \mathbb{R}^r)) \\
\times L^2_{\mathcal{F}}([0, T], C^\infty(D, \mathbb{R}^q)) \\
\times L^2_{\mathcal{F}, p}([0, T], C^\infty(D, \mathbb{R}^{q \times d})) \\
\times L^p_{\mathcal{F}}([0, T] \times R^h_+, C^\infty(D, \mathbb{R}^{q \times h})).
\end{equation}

Finally, let
\begin{equation}
L^2_{\nu}(R^h_+, C^c(D, \mathbb{R}^{q \times h})) \equiv \left\{ \tilde{v} : R^h_+ \rightarrow C^c(D, \mathbb{R}^{q \times h}), \right. \\
\left. \sum_{i=1}^h \int_{z_i > 0} \|\tilde{v}_i(z_i)\|_{C^c(D, \mathbb{R})}^2 \nu_i(dz_i) < \infty \right\}
\end{equation}

that is endowed with the norm
\begin{equation}
\|\tilde{v}\|_{\nu, c}^2 \equiv \sum_{i=1}^h \int_{z_i > 0} \|\tilde{v}_i(z_i)\|_{C^c(D, \mathbb{R})}^2 \lambda_i \nu_i(dz_i)
\end{equation}

for any $\tilde{v} \in L^2_{\nu}(R^h_+, C^c(D, \mathbb{R}^{q \times h}))$ and $c \in \{0, 1, ..., \infty\}$.

In the sequel, we let $\|A\|$ be the largest absolute value of entries (or components) of the given matrix (or vector) $A$. Furthermore, for each $s \in [0, T]$ and $z \in R^h_+$, we let
\begin{equation}
\tilde{N}(\lambda ds, dz) = (\tilde{N}_1(\lambda_1 ds, dz_1), ..., \tilde{N}_h(\lambda_h ds, dz_h))',
\end{equation}

where
\begin{equation}
\tilde{N}_i(\lambda_i ds, dz_i) = N_i(\lambda_i ds, dz_i) - \lambda_i ds \nu_i(dz_i)
\end{equation}

for each $i \in \{1, ..., h\}$. Then, we impose some conditions to guarantee the unique existence of adapted strong solution to the unified system in [11].

First, for each partial differential operator $A \in \{\mathcal{L}, \mathcal{L}, \mathcal{J}, \mathcal{T}\}$ and every $c \in \{0, 1, 2, ..., \}$, we define
\begin{equation}
\Delta A^{(c)}(s, x, u^1, v^1, u^2, v^2) \equiv A^{(c)}(s, x, u^1, v^1) - A^{(c)}(s, x, u^2, v^2),
\end{equation}

and assume that, for any
\begin{equation}
(u^i, v^i, \tilde{u}^i, \tilde{v}^i) \in C^\infty(D, \mathbb{R}^r) \times C^\infty(D, \mathbb{R}^q) \times C^\infty(D, \mathbb{R}^{q \times d}) \times \tilde{L}^2_{\nu}(R^h_+, C^\infty(D, \mathbb{R}^{q \times h}))
\end{equation}

with $i \in \{1, 2\}$, the generalized local Lipschitz condition is true almost surely (a.s.)
\begin{equation}
\left\|\Delta A^{(c+i+o)}(s, x, u^1, v^1, u^2, v^2)\right\| \\
\leq K_{D,c} \left( \|u^1 - u^2\|_{C^{k+c}(D,x)} + \|v^1 - v^2\|_{C^{k+c}(D,q)} \\
+ \|\tilde{u}^1 - \tilde{u}^2\|_{C^{k+c}(D,qd)} + \|\tilde{v}^1 - \tilde{v}^2\|_{\nu,k+c} \right).
\end{equation}
Note that $K_{D,c}$ in (2.16) with each $c \in \{0,1,2,...\}$ is a nonnegative constant. It depends on the domain $D$ and the differential order $c$ and may be unbounded as $c \to \infty$ and $D \to \mathbb{R}^p$. $l \in \{0,1,2\}$ denotes the $l$th order of partial derivative of $\Delta \mathcal{L}^{(c)}(s,x,u,v)$ in time variable $t$. $o \in \{0,1,2\}$ denotes the $o$th order of partial derivative of $\Delta \mathcal{L}^{(c+l)}(s,x,u,v)$ in terms of a component of $u$ or $v$. Furthermore, the partial differential operators of $q \times d$-dimensional matrix $\tilde{\mathcal{J}} = (\tilde{J}_1, ..., \tilde{J}_d)$ and $q \times h$-dimensional matrix $\tilde{\mathcal{T}} = (\tilde{T}_1, ..., \tilde{T}_h)$ satisfy the conditions

\begin{align}
\left\| \Delta \tilde{\mathcal{J}}^{(c+l)}(s,x,u^1,u^2,v^2) \right\| \\
\leq K_{D,c} \left( \left\| u^1 - u^2 \right\|_{C^{k+c}(D,\mathbb{R}^h)} + \left\| v^1 - v^2 \right\|_{C^{k+c}(D,\mathbb{R}^h)} \right),
\end{align}

(2.17)

\begin{align}
\left\| \sum_{i=1}^{h} \frac{\partial}{\partial z_i} \tilde{\mathcal{I}}^{(c+l)}_i(s,x,u^1,u^2,v^2,v^2,\nu_i(z_i)) \right\| \\
\leq K_{D,c} \left( \left\| u^1 - u^2 \right\|_{C^{k+c}(D,\mathbb{R}^h)} + \left\| v^1 - v^2 \right\|_{C^{k+c}(D,\mathbb{R}^h)} \right),
\end{align}

(2.18)

Second, we suppose that, for each $A \in \{\mathcal{L}, \tilde{\mathcal{L}}, \tilde{\mathcal{J}}, \tilde{\mathcal{T}}\}$, every $c \in \{0,1,2,...,\}$, and any $(u,v,\bar{v},\bar{v}) \in C^\infty(D,R^r) \times C^\infty(D,R^q) \times C^\infty(D,R^{q \times d}) \times \tilde{L}^2_{D,c}(R^h_+, C^\infty(D,R^{q \times h}))$, the generalized local growth conditions hold

\begin{align}
\left\| A^{(c+l)}(s,x,u,v) \right\| \\
\leq K_{D,c} \left( \delta_{0,c} + \left\| u \right\|_{C^{k+c}(D,\mathbb{R}^h)} + \left\| v \right\|_{C^{k+c}(D,\mathbb{R}^h)} + \left\| \bar{v} \right\|_{C^{k+c}(D,\mathbb{R}^h)} \right),
\end{align}

(2.19)

and

\begin{align}
\left\| \tilde{\mathcal{J}}^{(c+l)}(s,x,u,v) \right\| & \leq K_{D,c} \left( \delta_{0,c} + \left\| u \right\|_{C^{k+c}(D,\mathbb{R}^h)} + \left\| v \right\|_{C^{k+c}(D,\mathbb{R}^h)} \right), \\
\left\| \sum_{i=1}^{h} \frac{\partial}{\partial z_i} \tilde{\mathcal{I}}^{(c+l)}_i(s,x,u,v,v,\nu_i(z_i)) \right\| & \leq K_{D,c} \left( \delta_{0,c} + \left\| u \right\|_{C^{k+c}(D,\mathbb{R}^h)} + \left\| v \right\|_{C^{k+c}(D,\mathbb{R}^h)} \right),
\end{align}

(2.20)

(2.21)

where $\delta_{0,c} = 1$ if $c = 0$ and $\delta_{0,c} = 0$ if $c > 0$.

**Theorem 2.1** Suppose that $(G,H) \in L^2_G(\Omega, C^\infty(D;\mathbb{R}^r)) \times L^2_{\mathcal{F}}(\Omega, C^\infty(D;\mathbb{R}^q))$ and conditions in [2.16]-[2.21] are true. Furthermore, assume that each $A \in \{\mathcal{L}, \tilde{\mathcal{L}}, \tilde{\mathcal{J}}, \tilde{\mathcal{T}}\}$ is $\{\mathcal{F}_t\}$-adapted for every fixed $x \in D$, $z \in R^h_+$, and any given

\begin{align}
(u,v,\bar{v},\bar{v}) & \in C^\infty(D,R^r) \times C^\infty(D,R^q) \times C^\infty(D,R^{q \times d}) \times \tilde{L}^2_{D,c}(R^h_+, C^\infty(D,R^{q \times h})),
\end{align}

with

\begin{align}
\mathcal{L}(\cdot,x,0) & \in L^2_{\mathcal{F}}([0,T], C^\infty(D,R^r)), \\
\tilde{\mathcal{J}}(\cdot,x,0) & \in L^2_{\mathcal{F}}([0,T], C^\infty(D,R^{r \times d})), \\
\tilde{\mathcal{T}}(\cdot,x,0,\cdot) & \in L^2_{\mathcal{F}}([0,T] \times R^h_+, C^\infty(D,R^{r \times h})),
\end{align}

(2.22)

(2.23)

(2.24)
\begin{align}
\bar{L}(\cdot, x, 0) & \in L^2_F([0, T], C^\infty(D, R^q)), \\
\bar{J}(\cdot, x, 0) & \in L^2_F([0, T], C^\infty(D, R^{q \times d})), \\
\bar{I}(\cdot, x, 0, \cdot) & \in L^2_F([0, T] \times R^h, C^\infty(D, R^{q \times h})).
\end{align}

Then, there exists a unique adapted strong solution to the system in (1.1), i.e.,
\begin{equation}
(U, V, \bar{V}, \tilde{V}) \in Q^2_F([0, T] \times D),
\end{equation}
and \((U, V)(\cdot, x)\) is càdlàg for each \(x \in D\).

The proof of Theorem 2.1 is provided in Section 4.

### 2.2 The System over Open Space Domain

In this subsection, we generalize the study in Subsection 2.1 to the case corresponding to an open (or partially open) space domain \(D\) (e.g., \(R^p\) or \(R^p_+\)). More exactly, assume that there exists a sequence of nondecreasing closed and connected sets \(\{D_n, n \in \{0, 1, \ldots\}\}\) such that
\begin{equation}
D = \bigcup_{n=0}^{\infty} D_n.
\end{equation}
Furthermore, let \(C^\infty(D, R^l)\) with \(l \in \{r, q\}\) be the Banach space endowed with the norm for each \(f\) in the space
\begin{equation}
\|f\|_{C^\infty(D, l)} \equiv \sum_{n=0}^{\infty} \xi(n + 1)\|f\|_{C^\infty(D_n, l)}.
\end{equation}
In addition, define \(Q^2_F([0, T] \times D)\) to be the corresponding space in (2.11) if the norm in (2.7) is replaced by the one in (2.30). Then, we have the following theorem.

**Theorem 2.2** Suppose that \((G, H) \in L^2_F(\Omega, C^\infty(D; R^r)) \times L^2_F(\Omega, C^\infty(D; R^q))\) and the system in (1.1) satisfies the conditions in (2.16)-(2.21) over \(D_n\) for each \(n \in \{0, 1, \ldots\}\) with associated (local) linear growth and Lipshitz constant \(K_{D_n, c}\). Furthermore, assume that each \(A \in \{\mathcal{L}, \bar{L}, \bar{J}, \tilde{J}, \mathcal{I}, \bar{I}\}\) is \(\mathcal{F}_t\)-adapted for every fixed \(x \in D, z \in R^h_+\), and any given
\begin{equation}
(u, v, \bar{v}, \tilde{v}) \in C^\infty(D, R^r) \times C^\infty(D, R^q) \times C^\infty(D, R^{q \times d}) \times \bar{L}^2_F(R^h_+, C^\infty(D, R^{q \times h}))
\end{equation}
with conditions in (2.22)-(2.27) being true. Then, the system in (1.1) has a unique adapted strong solution
\begin{equation}
(U, V, \bar{V}, \tilde{V}) \in Q^2_F([0, T] \times D),
\end{equation}
and \((U, V)(\cdot, x)\) is càdlàg for each \(x \in D\).

The proof of Theorem 2.2 is provided in Section 4.
3 Connections to Queues, FB-SEDs, and SDGs

3.1 Reflecting Diffusions for Queues

To be simple, we consider the case that the limit \( \hat{Q}(\cdot) \) in (1.4) is a RBM living in a state space \( D \) (e.g., a \( p \)-dimensional positive orthant or a \( p \)-dimensional rectangle). Furthermore, let \( D_i = \{ x \in \mathbb{R}^p, x \cdot n_i = b_i \} \) for \( i \in \{1, ..., b \} \) be the \( i \)-th boundary face of \( D \), where \( b_i = 0 \) for \( i \in \{1, ..., p \} \), \( b_i \) is some positive constant for \( i \in \{p + 1, ..., b \} \), and \( n_i \) is the inward unit normal vector on the boundary face \( D_i \). For convenience, we define \( N = (n_1, ..., n_b) \). Now, let \( \theta \) be a vector in \( \mathbb{R}^p \), \( \Gamma \) be a \( p \times p \) symmetric and positive definite matrix. Moreover, let \( R \) be a \( p \times b \) matrix with \( b \in \{p, 2p\} \), whose \( i \)-th column denoted by \( p \)-dimensional vector \( v_i \) is the reflection direction on \( D_i \). Then, we have the formal definition of a RBM (see, e.g., Dai [9]) as follows.

**Definition 3.1** A semimartingale RBM associated with the data \((S, \theta, \Gamma, R)\) that has initial distribution \( \pi \) is a continuous, \( \{F_t\} \)-adapted, \( p \)-dimensional process \( Z \) defined on some filtered probability space \((\Omega, \mathcal{F}, \{F_t\}, \mathbb{P})\) such that under \( \mathbb{P} \),

\[
X(t) = Z(t) + RY(t) \quad \text{for all } t \geq 0,
\]

where

1. \( X \) has continuous paths in \( S \), \( \mathbb{P} \)-a.s.,
2. under \( \mathbb{P} \), \( Z \) is a \( p \)-dimensional Brownian motion with drift vector \( \theta \) and covariance matrix \( \Gamma \) such that \( \{Z(t) - \theta t, F_t, t \geq 0\} \) is a martingale and \( P Z^{-1}(0) = \pi \),
3. \( Y \) is a \( \{F_t\} \)-adapted, \( b \)-dimensional process such that \( \mathbb{P} \)-a.s., for each \( i \in \{1, ..., b\} \), the \( i \)-th component \( Y_i \) of \( Y \) satisfies
   
   (a) \( Y_i(0) = 0 \),
   
   (b) \( Y_i \) is continuous and non-decreasing,
   
   (c) \( Y_i \) can increase only when \( Z \) is on the face \( D_i \), i.e.,

\[
\int_0^t I_{D_i}(X(s))dY_i(s) = Y_i(t) \quad \text{for all } t \geq 0.
\]

From the physical viewpoint of queueing system (see, e.g., Dai [9, 13]) and the discussion in Reiman and Williams [37], the pushing process \( Y \) in Definition 3.1 can be assumed to a.s. satisfy

\[
Y_i(t) = \int_0^t I_{D_i}(X(s))ds.
\]

Furthermore, to guarantee the existence of a RBM, we need to impose the completely-\( S \) condition on the reflection matrix \( R \).
Definition 3.2 A $p \times p$ square matrix $R$ is called completely-S if and only if there is $x > 0$ such that $Rx > 0$ for each principal sub-matrix $\tilde{R}$ of $R$, where vector inequalities are to be interpreted componentwise. Furthermore, a $p \times b$ matrix $R$ is called completely-S if and only if each $p \times p$ square sub-matrix of $N'R$ is completely-S.

Note that the completely-S condition on the reflection matrix guarantees that the RBM is of inward reflection on each boundary and corner of the orthant or the rectangle (see, e.g., Figure 2 and Dai [9]). Furthermore, the reflection appeared here is called skew reflection that is a generalization of the conventional mirror (or called symmetry) reflection.

Now, assume that $H(x)$ is the stationary distribution that we expect for the RBM $X$. For example, in reality, it is the given distribution of the long-run average queue lengths among different users or job classes. Theoretically, it can be computed by a method (e.g., the finite element method designed and implemented in Dai et al. [9, 39]). Then, we can use a B-PDE or a B-SPDE (a special form of the system in (1.1)) to get the transition function at each time point to reach the targeted or limiting stationary distribution $H(x)$ for the RBM $X$ for a given initial distribution (e.g., $X(0) = 0$ a.s. in many situations). Hence, the corresponding performance measures of the physical queueing system can be estimated. More precisely, we have the following theorem and related remark.

Theorem 3.1 Suppose that the reflection matrix satisfies the completely-S condition. Then, the transition function of the RBM $X$ over $[0,T]$ is determined by

$$V(t, x) = H(x) + \int_t^T \mathcal{L}(s, x, V)ds,$$

where $V$ is a 1-dimensional function. Furthermore, $\mathcal{L}$ is the following form of partial differential operator

$$\mathcal{L}(t, x, V, \cdot) = (K(t, x, V, \cdot), D_1(t, x, V, \cdot), ..., D_b(t, x, V, \cdot)),$$

$$K(t, x, V, \cdot) = \sum_{i,j=1}^p \Gamma_{ij} \frac{\partial^2 V(t, x)}{\partial x_i \partial x_j} + \theta \cdot \nabla V(t, x) + \sum_{i=1}^b D_i(t, x, V, \cdot),$$

$$D_i(t, x, V, \cdot) = (v_i \cdot \nabla V(t, x))I_{D_i}(x)$$

for $x \in D_i$ with $i \in \{1, ..., b\}$,

where $\nabla V$ is the gradient vector of $V$ in $x$ and $I_{F_i}$ is the indicator function over the set $F_i$.

Proof. It follows from the completely-S condition that the RBM $X$ is a strong Markov process (see, e.g., Dai and Williams [3]). Then, by applying the Itô’s formula (see, e.g., Protter [36]) and Fokker-Planck’s formula (or called Kolmogorov’s forward/backward equations, see, e.g., Øksendal [29]), we know that the claim stated in the theorem is true. □

Remark 3.1 Owing to the uncertainty error of measurement, $H(x)$ could be random. Furthermore, the coefficients in (3.4) may also be random, e.g., for the case that the limit $\hat{Q}(\cdot)$ is a RDRS. Thus, a B-SPDE can be introduced. Furthermore, the indicator function $I_{F_i}(x)$ can be approximated by a sufficient smooth function in order to apply Theorem 2.1 to the equation in (3.2), which is reasonable from the viewpoint of numerical computation.
3.2 The System of Coupled FB-SDEs with Skew Reflections

In this subsection, we suppose that $S$ in (1.2) is a $q \times b$ matrix for a known $b \in \{q, 2q\}$, $V$ tales value in a region $D$ with boundary face $D_i = \{v \in R^q, v \cdot \bar{n}_i = b_i\}$ for $i \in \{1, ..., b\}$, where $\bar{n}_i$ is the inward unit normal vector on the boundary face $D_i$. For convenience, we define $\bar{N} = (\bar{n}_1, ..., \bar{n}_b)$. In finance, the given constant $b_i$ is called early exercise reward. Furthermore, $F(\cdot)$ in (1.2) is a nondecreasing predictable process with $F(0) = 0$ and satisfies

$$f(u) = \int_0^T I_{\bar{D}_i}(V_i(t))dF_i(t)$$

for each $i \in \{1, ..., b\}$. Similarly, $R$ is assumed to be a $p \times b$ reflection matrix and the process $Y(\cdot)$ in (1.2) satisfies the property (3) in Definition 3.1. Finally, the given functions in (1.2)

$$\begin{align*}
b(u) &\equiv b(t, x, v, \bar{v}, u, \omega) : [0, T] \times R^p \times R^{q \times d} \times R^{q \times h} \times U \times \Omega \to R^p, \\
\sigma(u) &\equiv \sigma(t, x, v, \bar{v}, u, \omega) : [0, T] \times R^p \times R^{q \times d} \times R^{q \times h} \times U \times \Omega \to R^p, \\
\gamma(u) &\equiv \gamma(t, x, v, \bar{v}, u, z, \omega) : [0, T] \times R^p \times R^{q \times d} \times R^{q \times h} \times U \times R^h \times \Omega \to R^p, \\
c(u) &\equiv c(t, x, v, \bar{v}, u, \omega) : [0, T] \times R^p \times R^{q \times d} \times R^{q \times h} \times U \times \Omega \to R^{p+1}
\end{align*}$$

are $\mathcal{F}_t$-predictable, satisfying

$$\begin{align*}
\|f(u)\| &\leq L(t, \omega) \left(1 + \|x\| + \|v\| + \|\bar{v}\| + \|\bar{v}\|_\nu\right), \\
\|f^2(u) - f^1(u)\| &\leq L(t, \omega) \left(\|x^2 - x^1\| + \|v^2 - v^1\| + \|\bar{v}^2 - \bar{v}^1\| + \|\bar{v}^2 - \bar{v}^1\|_\nu\right)
\end{align*}$$

for any $f, f^1, f^2 \in \{b, \sigma, \gamma, c\}$. Furthermore, $L$ is assumed to be a known non-negative stochastic process that is $\mathcal{F}_t$-adapted and mean-square integrable, i.e.,

$$E \left[\int_0^T L^2(t)dt\right] < \infty.$$

**Theorem 3.2** Under conditions (3.7)-(3.9), the following two claims are true:

1. If $S$ and $R$ satisfy the completely-$S$ condition, there exists a unique adapted weak solution to the system in (1.2) when at least one of the forward and backward SDEs has reflection boundary;

2. Furthermore, if each $q \times q$ sub-principal matrix of $\bar{N}^tS$ and each $p \times p$ sub-principal matrix of $N^tR$ are invertible or if both of the SDEs have no reflection boundaries, there is a unique adapted strong solution to the system in (1.2).

Due to the length, the proof of Theorem 3.2 is postponed to Section 5.
3.3 Non-Zero-Sum SDGs

Let $u(\cdot)$ be a $B$-valued ($B \subset R^b$) and $\{\mathcal{F}_t\}$-adapted control process, whose $l$th component $u_l(\cdot)$ for each $l \in \{1, ..., q\}$ is the $l$th player’s control policy. Furthermore, suppose that the utility function for each player $l \in \{1, ..., q\}$ is given by

$$
\begin{align*}
(3.10) & \quad \begin{cases} 
c_l(u) & \equiv c_l(t, X(t), V(t, X(t)), \bar{V}(t, X(t)), \bar{V}(t, X(t)), u(t, X(t))), \\
c_0(u) & \equiv \sum_{l=1}^q c_l(u). 
\end{cases} 
\end{align*}
$$

Note that the 4-tuple $(X, \bar{V}, \bar{V}, F)$ in (3.10) is part of a solution $(X, Y, V, \bar{V}, \bar{V}, F)$ to the non-Markovian system of coupled FB-SDEs with Lévy jumps and skew reflections in (1.2).

Next, we consider a specific case of the FB-SPDE in (1.1), which corresponds to the special forms of partial differential operators $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{I}$. More precisely, for each $l \in \{0, 1, ..., q\}$, we define

$$
(3.11) \quad \hat{L}_l(t, x, U, V, u) 
\equiv \sum_{i,j=1}^p (\sigma \sigma')_{ij}(t, x, u) \frac{\partial^2 V_l(t, x)}{\partial x_i \partial x_j} + \sum_{i=1}^p b_i(t, x, u) + \sum_{j=1}^b v_{ij} \gamma_{ij}(t, x) \frac{\partial V_l(t, x)}{\partial x_i} 
+ \sum_{j=1}^d \sum_{i=1}^p \sigma_i(t, x, u) \frac{\partial \bar{V}_l(t, x)}{\partial x_i} - c_l(t, x, V(t, x), \bar{V}(t, x), \bar{V}(t, x), u(t, x)) + \sum_{k=1}^q s_k \beta_k(t, x) 
- \sum_{j=1}^h \int_{z_j > 0} \left( V_l(t, x + \eta_j(t, x, z_j)) - V_l(t, x) - \sum_{i=1}^p \frac{\partial V_l(t, x)}{\partial x_i} \eta_{ij}(t, x, z_j) \right) \nu_j(dz_j) 
- \sum_{j=1}^h \int_{z_j > 0} \left( \bar{V}_{ij}(t, x + \eta_j(t, z_j), z_j) - \bar{V}_{ij}(t, x, z_j) \right) \nu_j(dz_j),
\end{align*}
$$

where $\eta_j$ is the $j$th column of $\eta$, $\gamma_{ij}(t, x)$ for $j \in \{1, ..., b\}$ and $\beta_k(t, x)$ for $k \in \{1, ..., q\}$ are some functions in $t$ and $x$. In addition, we define

$$
\begin{align*}
(3.12) & \quad \mathcal{J}(t, x, U, V) = 0, \\
(3.13) & \quad \mathcal{I}(t, x, U, V, z) = 0, \\
(3.14) & \quad V(T, x) = H(x).
\end{align*}
$$

Then, we have the following definitions.

**Definition 3.3** $C$ is called the admissible set of adapted control processes if $\{\mathcal{L}_l(t, x, U, V, u), l \in \{0, 1, ..., q\}\}$ together with $\{\mathcal{L}, \mathcal{J}, \mathcal{I}\}$ satisfy the conditions as stated in Theorem 2.7 (or Theorem 2.2).

**Definition 3.4** By a non-zero-sum SDG to the system in (1.2), we mean that each player $l \in \{1, ..., q\}$ chooses an optimal policy to maximize his own value function expressed in (1.3). Furthermore, the value functions $\{V_l^u(0), l \in \{1, ..., q\}\}$ do not have to add up to a constant (e.g., zero), or in other words, the SDG is not necessarily a zero-sum one.
Definition 3.5 $u^*()$ is called a Pareto optimal Nash equilibrium point if, the point is also an optimal point to the sum of all the $q$ players’ value functions; no player will profit by unilaterally changing his own policy when all the other players’ policies keep the same. Mathematically,

\begin{equation}
V_0^{u^*}(0) \geq V_0^u(0), \quad V_i^{u^*}(0) \geq V_i^{u_{-l}}(0)
\end{equation}

for each $l \in \{0, 1, \ldots, q\}$ and any given admissible control policy $u$, where

$$u^*_{-l} = (u^*_1, \ldots, u^*_{l-1}, u_l, u^*_{l+1}, \ldots, u^*_q).$$

Definition 3.6 \{\bar{\mathcal{L}}_i(t, x, U, V, u), l \in \{0, 1, \ldots, q\}\} together with \{\mathcal{L}, \mathcal{J}, \mathcal{I}\} are called satisfying the comparison principle in terms of $u$ if, for any two $u^i \in \mathcal{C}$ with $i \in \{1, 2\}$ and any two $\mathcal{F}_T$-measurable $H^1$ with associated two solutions ($U^i, V^i, \bar{V}^i(t, x)$, respectively, of (1.2)) such that

$$\bar{\mathcal{L}}_i(t, x, U^1, V^1, u^1) \leq \bar{\mathcal{L}}_i(t, x, U^2, V^2, u^2), \quad H^1(x) \leq H^2(x)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^p$, we have

$$V^1(t, x) \leq V^2(t, x).$$

**Theorem 3.3** Let $(U(t, x), V(t, x), \bar{V}(t, x), \bar{V}(t, x, \cdot))$ be the unique adapted strong solution to the $(r, q + 1)$-dimensional FB-SPDEs in (1.2), which corresponds to specific \{\mathcal{L}, \mathcal{J}, \mathcal{I}\} in (3.11)-(3.13), terminal condition in (3.14), and a control process $u \in \mathcal{C}$. If $S$ and $R$ satisfy the completely-$S$ condition, the following claims in Part I and Part II are true.

**Part I:**

1. There exists a unique adapted weak solution $((X(t), Y(t)), (V(t), \bar{V}(t), \bar{V}(t, z), F(t)))$ to the system in (1.2) when at least one of the SDEs has reflection boundary, where

\begin{align}
(3.16) \quad & V_t(t) = V_1(t, X(t)), \\
(3.17) \quad & \bar{V}_{ij}(t) = -\left(\bar{V}_{ij}(t, X(t)) + \sum_{i=1}^{p} \sigma_i(t, x, u) \frac{\partial \bar{V}_{ij}(t, x)}{\partial x_i}\right), \\
\end{align}

\begin{align}
(3.18) \quad & \bar{V}_{ij}(t, z) = -\left(V_1(t, X(t) + \eta_j(t, z)) - V_1(t, X(t))\right) \\
& \quad - \left(\bar{V}_{ij}(t, X(t) + \eta_j(t, z_j), z_j) - \bar{V}_{ij}(t, X(t), z_j)\right)
\end{align}

for $l \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, h\};$

2. There is a unique adapted strong solution to the system in (1.2) when each $q \times q$ sub-principal matrix of $N'S$ and each $p \times p$ sub-principal matrix of $N'R$ are invertible or when both of the SDEs have no reflection boundaries.
Part II: Furthermore, if \( \{ \hat{L}_l(t, x, U, V, u), l \in \{0, 1, \ldots, q\} \} \) together with \( \{ L, J, I \} \) for suitably chosen \( \gamma(t, x) \) and \( \beta(t, x) \) satisfy the comparison principle in terms of \( u \), the following two claims are true:

1. There is a Pareto optimal Nash equilibrium point \( u^*(t, X(t)) \) to the non-zero-sum SDG problem in (1.3) when both of the SDEs in (1.2) have no reflection boundaries and if \( \gamma(t, x) = \beta(t, x) \equiv 0 \);

2. There is an approximated Pareto optimal Nash equilibrium point \( u^*(t, X(t)) \) to the non-zero-sum SDG problem in (1.3) when at least one of the SDEs in (1.2) has reflection boundary and if \( \gamma(t, x), \beta(t, x) \) are taken to be infinitely smooth approximated functions of \( \frac{dF}{dt}(t, x) \) and \( \frac{dY}{dt}(t, x) \) in \( x \).

Example 3.1 (Queueing based game problem) From the information system displayed in Figure 3 and Figure 4 (presenting a parallel-server queueing system with \( q = p \)), we can give an explanation about the decision process for such a game problem. In this game, each player (or called user in Dai [13]) relates to a control process and/or fair control process can be determined by the utility functions of all players, queueing control policy at each time point by the central information administrative. Then, an optimal and/or fair control process can be determined by the utility functions of all players, queueing process, and the available resource constraint in a cooperative way (see, e.g., Jones [22]).

4 Proofs of Theorem 2.1 and Theorem 2.2

We justify the two theorems by first proving three lemmas in the following subsection.

4.1 The Lemmas

Lemma 4.1 Assume that the conditions stated in Theorem 2.1 hold and take a quadruplet for each fixed \( x \in D \),

\[
\begin{align*}
(U^1(\cdot, x), V^1(\cdot, x), \bar{V}^1(\cdot, x), \tilde{V}^1(\cdot, x, z)) \in Q^2_x([0, T] \times D).
\end{align*}
\]

Then, there exists another quadruplet \((U^2(\cdot, x), V^2(\cdot, x), \bar{V}^2(\cdot, x), \tilde{V}^2(\cdot, x, z))\) such that

\[
\begin{align*}
U^2(t, x) &= G(x) + \int_t^T L(s^-, x, U^1, V^1)ds \\
&\quad + \int_t^T J(s^-, x, U^1, V^1)dW(s) \\
&\quad + \int_t^T \int_{z>0} \bar{I}(s^-, x, U^1, V^1, z)\tilde{N}(\lambda ds, dz), \\
V^2(t, x) &= H(x) + \int_t^T \hat{L}(s^-, x, U^1, V^1)ds \\
&\quad + \int_t^T \int_{z>0} \bar{J}(s^-, x, U^1, V^1) - \bar{V}^2(s^-, x)dW(s) \\
&\quad + \int_t^T \int_{z>0} \left( \tilde{I}(s^-, x, U^1, V^1, z) - \tilde{V}^2(s^-, x, z) \right) \tilde{N}(\lambda ds, dz),
\end{align*}
\]
where \((U^2, V^2)\) is a \(\{\mathcal{F}_t\}\)-adapted càdlàg process and \((\tilde{V}^2, \tilde{V}^2)\) is the corresponding predictable process. Furthermore, for each \(x \in D\),

\[
(4.3) \quad E \left[ \int_0^T \|U^2(t,x)\|^2 dt \right] < \infty,
\]

\[
(4.4) \quad E \left[ \int_0^T \|V^2(t,x)\|^2 dt \right] < \infty,
\]

\[
(4.5) \quad E \left[ \int_0^T \|\tilde{V}^2(t,x)\|^2 dt \right] < \infty,
\]

\[
(4.6) \quad E \left[ \sum_{i=1}^h \int_0^T \int_{z_i > 0} \|\tilde{V}^2_i(t,x,z_i)\|^2 \nu_i(dz_i) dt \right] < \infty.
\]

**Proof.** For each fixed \(x \in D\) and a quadruplet as stated in (4.1), it follows from conditions (2.16)-(2.27) that

\[
(4.7) \quad \mathcal{L}(\cdot, x, U^1, V^1) \in L^2_F([0,T], C^\infty(D, \mathbb{R}^d)),
\]

\[
(4.8) \quad \mathcal{J}(\cdot, x, U^1, V^1) \in L^2_F([0,T], C^\infty(D, \mathbb{R}^{q \times d})),
\]

\[
(4.9) \quad \mathcal{I}(\cdot, x, U^1, V^1, \cdot) \in L^2_F([0,T] \times \mathbb{R}^h, C^\infty(D, \mathbb{R}^{q \times d})),
\]

\[
(4.10) \quad \mathcal{L}(\cdot, x, U^1, V^1) \in L^2_F([0,T], C^\infty(D, \mathbb{R}^d)),
\]

\[
(4.11) \quad \mathcal{J}(\cdot, x, U^1, V^1) \in L^2_F([0,T], C^\infty(D, R^{q \times d})),
\]

\[
(4.12) \quad \mathcal{I}(\cdot, x, U^1, V^1, \cdot) \in L^2_F([0,T] \times \mathbb{R}^h, C^\infty(D, R^{q \times d})).
\]

By considering \(\mathcal{L}, \mathcal{J}, \) and \(\mathcal{I}\) in (4.7)-(4.9) as new starting \(\mathcal{L}(\cdot, x, 0), \mathcal{J}(\cdot, x, 0),\) and \(\mathcal{I}(\cdot, x, 0, \cdot),\) we can define \(U^2\) by the forward iteration in (4.12). Furthermore, \(U^2\) is a \(\{\mathcal{F}_t\}\)-adapted càdlàg process that is square-integrable for each \(x \in D\) in the sense of (4.3).

Now, consider \(\tilde{\mathcal{L}}, \tilde{\mathcal{J}}, \) and \(\tilde{\mathcal{I}}\) in (4.10)-(4.12) as new starting \(\tilde{\mathcal{L}}(\cdot, x, 0), \tilde{\mathcal{J}}(\cdot, x, 0),\) and \(\tilde{\mathcal{I}}(\cdot, x, 0, \cdot),\) it follows from the Martingale representation theorem (see, e.g., Theorem 5.3.5 in page 266 of Applebaum [11]) that there are unique predictable processes \(\tilde{V}^2(\cdot, x)\) and \(\tilde{V}^2(\cdot, x, z)\) such that

\[
(4.13) \quad \tilde{V}^2(t,x) \equiv E \left[ H(x) + \int_0^T \tilde{\mathcal{L}}(s^-, x, U^1, V^1) ds 
\right. \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \left. \int_0^T \tilde{\mathcal{J}}(s^-, x, U^1, V^1) dW(s) 
\right. \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \left. \int_0^T \int_{z > 0} \tilde{\mathcal{I}}(s^-, x, U^1, V^1, z) \tilde{N}(\lambda ds, dz) \bigg| \mathcal{F}_t \right]
\]

\[
= \tilde{V}^2(0,x) + \int_0^t \tilde{V}^2(s^-, x) dW(s) \\
\quad + \int_0^t \int_{z > 0} \tilde{V}^2(s^-, x, z) \tilde{N}(\lambda ds, dz).
\]
Furthermore, $\hat{V}^2$ and $\tilde{V}^2$ are square-integrable for each $x \in D$ in the sense of (4.5)-(4.6), and
\begin{align*}
(4.14) \quad \hat{V}^2(0, x) &= \hat{V}^2(T, x) - \int_0^T \hat{V}^2(s, x) dW(s) \\
&\quad - \int_0^T \int_{z>0} \hat{V}^2(s-, x, z) \tilde{N}(\lambda ds, dz) \\
&= H(x) + \int_0^T \hat{L}(s, x, U^1, V^1) ds \\
&\quad + \int_0^T (\hat{J}(s, x, U^1, V^1) - \hat{V}^2(s, x)) dW(s) \\
&\quad + \int_0^T \int_{z>0} (\hat{I}(s-, x, U^1, V^1, z) - \hat{V}^2(s-, x, z)) \tilde{N}(\lambda ds, dz).
\end{align*}

Owing to the corollary in page 8 of Protter [36], $\hat{V}^2(\cdot, x)$ can be taken as a càdlàg process. Now, define a process $V^2$ given by
\begin{align*}
(4.15) \quad V^2(t, x) &= E \left[ H(x) + \int_t^T \hat{L}(s, x, U^1, V^1) ds \\
&\quad + \int_t^T \hat{J}(s, x, U^1, V^1) dW(s) \\
&\quad + \int_t^T \int_{z>0} (\hat{I}(s-, x, U^1, V^1, z) - \hat{V}^2(s-, x, z)) \tilde{N}(\lambda ds, dz) \bigg| F_t \right].
\end{align*}

Thus, it follows from (2.17)-(2.20) and simple calculation that $V^2(\cdot, x)$ is square-integrable in the sense of (4.4). In addition, by (4.13)-(4.16), we know that
\begin{align*}
(4.17) \quad V^2(t, x) &= \hat{V}^2(t, x) - \int_0^t \hat{L}(s, x, U^1, V^1) ds \\
&\quad - \int_0^t \hat{J}(s, x, U^1, V^1) dW(s) \\
&\quad - \int_0^t \int_{z>0} (\hat{I}(s-, x, U^1, V^1, z) - \hat{V}^2(s-, x, z)) \tilde{N}(\lambda ds, dz),
\end{align*}

which implies that $V^2(\cdot, x)$ is a càdlàg process.

Hence, for a given quadruplet in (4.1), it follows from (4.13)-(4.14) and (4.17) that the associated quadruplet $(U^2(\cdot, x), V^2(\cdot, x), \hat{V}^2(\cdot, x), \tilde{V}^2(\cdot, x, z))$ satisfies the equation (4.2) as stated in the lemma. Furthermore, we know that
\begin{align*}
(4.18) \quad V^2(t, x) &\equiv V^2(0, x) - \int_0^t \hat{L}(s, x, U^1, V^1) ds \\
&\quad - \int_0^t (\hat{J}(s, x, U^1, V^1) - \hat{V}^2(s, x)) dW(s) \\
&\quad - \int_0^t \int_{z>0} (\hat{I}(s-, x, U^1, V^1, z) - \hat{V}^2(s-, x, z)) \tilde{N}(\lambda ds, dz).
\end{align*}

Thus, we complete the proof of Lemma 4.1. □
Lemma 4.2 Under the conditions of Theorem 2.1, consider a quadruplet as in (4.2) for each fixed \( x \in D \) and define \((U(t, x), V(t, x), \bar{V}(t, x), \tilde{V}(t, x))\) by (4.2). Then, \((U^{(c)}(\cdot, x), V^{(l)}(\cdot, x), \bar{V}^{(l)}(\cdot, x), \tilde{V}^{(l)}(\cdot, x))\) for each \( c \in \{0, 1, \ldots\} \) exists a.s. and satisfies

\[
\left\{
\begin{array}{l}
U^{(c)}_{i_1 \ldots i_p}(t, x) = G^{(c)}_{i_1 \ldots i_p}(x) + \int_0^t E^{(c)}_{i_1 \ldots i_p}(s, x, U^1, V^1) \, ds \\
V^{(l)}_{i_1 \ldots i_p}(t, x) = H^{(l)}_{i_1 \ldots i_p}(x) + \int_0^T \mathcal{L}^{(l)}_{i_1 \ldots i_p}(s, x, U^1, V^1) \, ds \\
\end{array}
\right.
\]

(4.19)

where \( i_1 + \ldots + i_p = c \) and \( i_l \in \{0, 1, \ldots, c\} \) with \( l \in \{1, \ldots, p\} \). Furthermore, \((U^{(c)}_{i_1 \ldots i_p}, V^{(c)}_{i_1 \ldots i_p})\) for each \( c \in \{0, 1, \ldots\} \) is a \( \mathcal{F}_t \)-adapted càdlàg process and \((\bar{V}^{(c)}_{i_1 \ldots i_p}, \tilde{V}^{(c)}_{i_1 \ldots i_p})\) is the associated predictable processes. All of them are square-integrable in the senses of (4.4)-(4.6).

**Proof.** Without loss of generality, we only consider the point \( x \in D \), which is an interior one of \( D \). Otherwise, we can use the corresponding derivative in a one-side manner to replace the one in the following proof.

First, we show that the claim in the lemma is true for \( c = 1 \). To do so, for each given \( t \in [0, T], x \in D, z \in \mathbb{R}_+^h \), and \((U^1(t, x), V^1(t, x), \bar{V}^1(t, x), \tilde{V}^1(t, x), \bar{V}^1(t, x), \tilde{V}^1(t, x))\) as in the lemma, let

\[
(U^{(1)}_{i_1}(t, x), V^{(1)}_{i_1}(t, x), \bar{V}^{(1)}_{i_1}(t, x), \bar{V}^{(1)}_{i_1}(t, x))
\]

be defined by (4.2) but each \( K \in \{L, \mathcal{J}, \tilde{L}, \mathcal{J}, \tilde{L}\} \) is replaced by its first-order partial derivative

\[
\mathcal{K}^{(1)}_{i_1} \in \{\mathcal{L}^{(1)}_{i_1}, \mathcal{J}^{(1)}_{i_1}, \mathcal{L}^{(1)}_{i_1}, \mathcal{J}^{(1)}_{i_1}, \mathcal{J}^{(1)}_{i_1}, \tilde{F}^{(1)}_{i_1}\}
\]

with respect to \( x_l \) for \( l \in \{1, \ldots, p\} \) if \( i_l = 1 \). Then, we can show that the quadruplet defined in (4.20) for each \( l \) is the required first-order partial derivative of \((U, V, \bar{V}, \tilde{V})\) in (4.2) for the given \((U^1, V^1, \bar{V}^1, \tilde{V}^1)\).

In fact, for each \( f \in \{U, V, \bar{V}, \tilde{V}, U^1, V^1, \bar{V}^1, \tilde{V}^1\} \), sufficiently small positive constant \( \delta, i_l = 1, \) and \( l \in \{1, \ldots, p\} \), define

\[
f_{i_1, \delta}(t, x) \equiv f(t, x + \delta e_l),
\]

where \( e_l \) is the unit vector whose \( l \)th component is one and others are zero. Furthermore, let

\[
\Delta f^{(1)}_{i_1, \delta}(t, x) = \frac{f_{i_1, \delta}(t, x) - f(t, x)}{\delta} - f^{(1)}_{i_1}(t, x).
\]

In addition, let

\[
\Delta \mathcal{K}^{(1)}_{i_1, \delta}(s, x, U^1, V^1) = \frac{1}{\delta} \left( \mathcal{K}(s, x + \delta e_l, U^1(s, x + \delta e_l), V^1(s, x + \delta e_l)) - \mathcal{K}(s, x, U^1(s, x), V^1(s, x)) \right)
\]

(4.23)
for each \( K \in \{ \mathcal{L}, \mathcal{J}, \mathcal{I}, \mathcal{L}, \mathcal{J}, \mathcal{I} \} \).

Now, let \( \text{Tr}(A) \) denote the trace of the matrix \( A' A \) for a given matrix \( A \) and let \( (\text{Tr}(A))_j \) be the \( j \)th term in the summation of the trace. Furthermore, for each fixed \( t \in [0, T] \), \( \sigma > 0 \), and \( \gamma > 0 \), define

\[
Z_\sigma(t, x) = \zeta(\Delta U_{i,\delta}^{(1)}(t, x) + \Delta V_{i,\delta}^{(1)}(t, x)) = \left( \text{Tr} \left( \Delta U_{i,\delta}^{(1)}(t, x) \right) + \text{Tr} \left( \Delta V_{i,\delta}^{(1)}(t, x) \right) \right) e^{2\gamma t}.
\]

Then, it follows from \( \text{[18]} \) and the Itô's formula (see, e.g., Theorem 1.14 and Theorem 1.16 in pages 6-9 of Øksendal and Sulem \( \text{[30]} \) that

\[
Z_\sigma(t, x) + \int_t^T \text{Tr} \left( \Delta \bar{J}_{i,\delta}^{(1)}(s, x, U^1, V^1) - \Delta \bar{V}_{i,\delta}^{(1)}(s, x) \right) e^{2\gamma s} ds
+ \sum_{j=1}^h \int_t^T \int_{z_j > 0} \left( \text{Tr} \left( \Delta \bar{J}_{i,\delta}^{(1)}(s, x, U^1, V^1) - \Delta \bar{V}_{i,\delta}^{(1)}(s, x, z) \right) \right) e^{2\gamma s} N_j(\lambda_j ds, dz_j)
= 2 \int_0^t \left( -\gamma \text{Tr} \left( \Delta U_{i,\delta}^{(1)}(s, x) \right) + \left( \Delta U_{i,\delta}^{(1)}(s, x) \right)' \left( \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right) \right) e^{2\gamma s} ds
+ 2 \int_0^t \left( -\gamma \text{Tr} \left( \Delta V_{i,\delta}^{(1)}(s, x) \right) + \left( \Delta V_{i,\delta}^{(1)}(s, x) \right)' \left( \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right) \right) e^{2\gamma s} ds
- M_\delta(t, x)
\leq \left( -2\gamma + \frac{1}{\hat{\gamma}} \right) \left( \int_0^t \text{Tr} \left( \Delta U_{i,\delta}^{(1)}(s, x) \right) e^{2\gamma s} ds + \int_t^T \text{Tr} \left( \Delta V_{i,\delta}^{(1)}(s, x) \right) e^{2\gamma s} ds \right)
+ \hat{\gamma} \left( \int_0^t \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds + \int_t^T \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds \right)
- M_\delta(t, x)
= \hat{\gamma} \left( \int_0^t \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds + \int_t^T \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds \right)
- M_\delta(t, x)
\]

if, in the last equality, we take

\[
\hat{\gamma} = \frac{1}{2\gamma} > 0.
\]

Note that \( M_\delta(t, x) \) in \( \text{[125]} \) is a martingale of the form,

\[
M_\delta(t, x) = -2 \sum_{j=1}^d \int_0^t \left( \Delta U_{i,\delta}^{(1)}(s, x) \right)' \left( \Delta J_{j,\delta}^{(1)}(s, x, U^1, V^1) \right) e^{2\gamma s} dW_j(s)
-2 \sum_{j=1}^h \int_0^t \int_{z_j > 0} \left( \Delta U_{i,\delta}^{(1)}(s, x) \right)' \left( \Delta J_{j,\delta}^{(1)}(s, x, U^1, V^1, z_j) \right) e^{2\gamma s} d\bar{N}_j(\lambda_j ds, dz_j)
\]

\[
22
\]
\begin{align*}
+2 \sum_{j=1}^{d} \int_{t}^{T} \left( \Delta V_{i,j}(s^{-}, x) \right)' \left( \Delta(\bar{J})_{i,j}(s^{-}, x, U^{1}, V^{1}) - \Delta(\bar{V})_{i,j}(s^{-}, x) \right) e^{2\gamma_s} dW_j(s) \\
+2 \sum_{j=1}^{h} \int_{t}^{T} \int_{z_j > 0} \left( \Delta V_{i,j}(s^{-}, x) \right)' \left( \Delta(\bar{J})_{i,j}(s^{-}, x, U^{1}, V^{1}, z_j) - \Delta(\bar{V})_{i,j}(s^{-}, x, z_j) \right) e^{2\gamma_s} \bar{N}_j(\lambda_j ds, dz_j).
\end{align*}

Next, for each fixed \( t \in [0, T] \), \( x \in D \), and \( \sigma > 0 \), consider the random variable set \( \{Z_{\delta}(t, x), \delta \in [0, \sigma]\} \). It follows from Lemma 1.3 in pages 6-7 of Peskir and Shiryaev \[35\] that there is a countable subset \( C = \{\delta_1, \delta_2, \ldots\} \subset [0, \sigma] \) such that

\begin{equation}
(4.28) \quad \text{esssup}_{\delta \in [0, \sigma]} Z_{\delta}(t, x) = \sup_{\delta \in C} Z_{\delta}(t, x), \quad \text{a.s.}
\end{equation}

Furthermore, take

\begin{equation}
(4.29) \quad \begin{cases}
\tilde{Z}_{\delta_1}(t, x) = Z_{\delta_1}(t, x), \\
\tilde{Z}_{\delta_{n+1}}(t, x) = Z_{\delta_n}(t, x) \lor Z_{\delta_n+1}(t, x) \quad \text{for } n \in \{1, 2, \ldots\},
\end{cases}
\end{equation}

where \( \alpha \lor \beta = \max\{\alpha, \beta\} \) for any two real numbers \( \alpha \) and \( \beta \). Obviously,

\begin{equation}
(4.30) \quad \begin{cases}
Z_{\delta}(t, x) \leq \tilde{Z}_{\delta}(t, x) \quad \text{for each } \delta \in C \\
\tilde{Z}_{\delta_1}(t, x) \leq \tilde{Z}_{\delta_2}(t, x) \quad \text{for any } \delta_1, \delta_2 \in C \text{ satisfying } \delta_1 \leq \delta_2.
\end{cases}
\end{equation}

The second inequality in (4.30) implies that the set \( \{\tilde{Z}_{\delta}(t, x), \delta \in C\} \) is upwards directed. Hence, for each \( t \in [0, T] \), \( x \in D \), \( \sigma > 0 \), and the associated sequence of \( \{\delta_n, n = 1, 2, \ldots\} \), it follows from (4.28) that

\begin{equation}
(4.31) \quad E \left[ \text{esssup}_{0 \leq \delta \leq \sigma} Z_{\delta}(t, x) \right] \leq E \left[ \text{esssup}_{\delta \in C} \tilde{Z}_{\delta}(t, x) \right] = \lim_{n \to \infty} E \left[ \tilde{Z}_{\delta_n}(t, x) \right] = \lim_{n \to \infty} E \left[ \max_{\delta \in \{\delta_1, \ldots, \delta_n\}} Z_{\delta}(t, x) \right].
\end{equation}

In addition, for each fixed \( n \in \{2, 3, \ldots\} \), let

\begin{equation}
(4.32) \quad \bar{M}_{\delta_n}(t, x) = M_{\delta_n}(t, x)I_{\{Z_{\delta_n} \geq \tilde{Z}_{\delta_{n-1}}\}} + M_{\delta_{n-1}}(t, x)I_{\{Z_{\delta_n} < \tilde{Z}_{\delta_{n-1}}\}}.
\end{equation}

Then, we know that \( \bar{M}_{\delta_n}(t, x) \) is still a \( \{\mathcal{F}_t\} \)-martingale. Thus, by the induction method in terms of \( n \in \{1, 2, \ldots\} \) and (4.25), we know that

\begin{equation}
(4.33) \quad E \left[ \max_{\delta \in \{\delta_1, \ldots, \delta_n\}} Z_{\delta}(t, x) \right] \leq \hat{\gamma} \lim_{n \to \infty} E \left[ \int_{0}^{t} \max_{\delta \in \{\delta_1, \ldots, \delta_n\}} \left\| \Delta L_{i,\delta}(s, x, U^1, V^1) \right\|^2 e^{2\gamma_s} ds \right]
+ \int_{t}^{T} \max_{\delta \in \{\delta_1, \ldots, \delta_n\}} \left\| \Delta \bar{L}_{i,\delta}(s, x, U^1, V^1) \right\|^2 e^{2\gamma_s} ds
\end{equation}

23
where “esssup” denotes the essential supremum. Owing to the mean-value theorem and the
condition in (2.16), we know that
\[
\|\Delta L_{i,\delta}^{(1)}(t, x, U^1, V^1)\| \leq \gamma E \left[ \int_0^t \text{esssup}_{0 \leq \delta \leq \sigma} \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds \right] + \int_t^T \text{esssup}_{0 \leq \delta \leq \sigma} \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds,
\]

Therefore, it follows from (4.33) and the Lebesgue’s dominated convergence theorem that
\[
\lim_{\sigma \to 0} E \left[ \text{esssup}_{0 \leq \delta \leq \sigma} Z_{\delta}(t, x) \right] \leq \gamma E \left[ \int_0^t \text{esssup}_{0 \leq \delta \leq \sigma} \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds \right] + \int_t^T \text{esssup}_{0 \leq \delta \leq \sigma} \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds.
\]
Hence, by (4.34) and the Fatou’s lemma, we know that, for any sequence \( \sigma_n \) satisfying \( \sigma_n \to 0 \) along \( n \in \mathcal{N} \), there is a subsequence \( \mathcal{N}' \subset \mathcal{N} \) such that
\[
\text{esssup}_{0 \leq \delta \leq \sigma_n} Z_{\delta}(t, x) \to 0 \quad \text{along} \quad n \in \mathcal{N}' \quad \text{a.s.}
\]
The convergence in (4.35) implies that the first-order derivatives of \( U \) and \( V \) in terms of \( x_t \) for each \( l \in \{1, ..., p\} \) exists. More exactly, they equal \( U_{i_l}^{(1)}(t, x) \) and \( V_{i_l}^{(1)}(t, x) \) a.s. respectively for each \( t \in [0, T] \) and \( x \in D \). Furthermore, they are \( \{\mathcal{F}_t\}\)-adapted.

Now, we prove the claim for \( \tilde{V} \). In fact, it follows from the proof as displayed in (4.31)-(4.33) that
\[
\lim_{\sigma \to 0} E \left[ \int_t^T \text{esssup}_{0 \leq \delta \leq \sigma} \text{Tr} \left( \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) - \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right) e^{2\gamma s} ds \right] \leq \gamma E \left[ \int_0^t \text{esssup}_{0 \leq \delta \leq \sigma} \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds \right] + \int_t^T \text{esssup}_{0 \leq \delta \leq \sigma} \left\| \Delta L_{i,\delta}^{(1)}(s, x, U^1, V^1) \right\|^2 e^{2\gamma s} ds.
\]
Thus, by (4.35) and (4.36), we know that
\[
\lim_{\delta \to 0} \Delta \tilde{V}_{i_l}(t, x) = \lim_{\delta \to 0} \Delta \tilde{V}_{i_l}(t, x, U) = 0 \quad \text{a.s.}
\]
Hence, the first-order derivative of $\bar{V}$ in $t_i$ for each $l \in \{1, ..., p\}$ exists and equals $\bar{V}^{(1)}_{t_i}(t, x)$ a.s. for every $t \in [0, T]$ and $x \in D$. Furthermore, it is a $\{\mathcal{F}_t\}$-predictable process. Similarly, we can get the conclusion for $\bar{V}^{(1)}_{t_i}(t, x, z)$ associated with each $l, t, x,$ and $z$.

Second, we suppose that $(U^{(c-1)}(t,x), V^{(c-1)}(t,x), \bar{V}^{(c-1)}(t,x), \bar{V}^{(c-1)}(t,x,z))$ corresponding to a given $(U^1(t,x), V^1(t,x), \bar{V}^1(t,x), \bar{V}^1(t,x,z)) \in \mathcal{Q}_2^2([0,T] \times D)$ exists for any given $c \in \{1,2,\ldots\}$. Then, we can show that

\begin{equation}
(U^{(c)}(t,x), V^{(c)}(t,x), \bar{V}^{(c)}(t,x), \bar{V}^{(c)}(t,x,z))
\end{equation}

exists for the given $c \in \{1, 2, \ldots\}$.

In fact, consider any fixed nonnegative integer numbers $i_1, \ldots, i_p$ satisfying $i_1 + \ldots + i_p = c - 1$ for the given $c \in \{1, 2, \ldots\}$. Take $f \in \{U, V, \bar{V}, \bar{V}\}$, $l \in \{1, \ldots, p\}$, and sufficiently small $\delta > 0$. Then, let

\begin{equation}
f^{(c-1)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x) \equiv f^{(c-1)}_{i_1, \ldots, i_p}(t,x + \delta e_l)
\end{equation}

correspond to the $(c - 1)$th-order partial derivative $K^{(c-1)}_{i_1, \ldots, i_p}(s, x + \delta e_l, U^1(s, x + \delta e_l), V^1(s, x + \delta e_l))$ of $K \in \{\mathcal{L}, \mathcal{J}, \mathcal{I}, \bar{L}, \bar{J}, \bar{I}\}$ via (4.2). Similarly, let

\begin{equation}
(U^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x), V^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x), \bar{V}^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x), \bar{V}^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x,z))
\end{equation}

be defined by (4.2), where $K \in \{\mathcal{L}, \mathcal{J}, \mathcal{I}, \bar{L}, \bar{J}, \bar{I}\}$ are replaced by their $c$th-order partial derivatives $K^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}$ corresponding to a given $t, x, U^1(t,x), V^1(t,x), \bar{V}^1(t,x), \bar{V}^1(t,x,z)$. Furthermore, let

\begin{equation}
\Delta f^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x) = \frac{f^{(c-1)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x) - f^{(c-1)}_{i_1, \ldots, i_p}(t,x)}{\delta} - f^{(c-1)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x)
\end{equation}

for each $f \in \{U, V, \bar{V}, \bar{V}, U^1, V^1, \bar{V}^1, \bar{V}^1\}$. Then, define

\begin{equation}
\Delta K^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x, U^1, V^1) \equiv \frac{1}{\delta} \left( K^{(c-1)}_{i_1, \ldots, i_p}(t,x + \delta e_l, U^1(t,x + \delta e_l) - V^1(t,x + \delta e_l), \cdot) - K^{(c-1)}_{i_1, \ldots, i_p}(s,x,U^1(s,x), V^1(s,x)) \right)
\end{equation}

for each $K \in \{\mathcal{L}, \mathcal{J}, \mathcal{I}, \bar{L}, \bar{J}, \bar{I}\}$. Thus, by the Itô’s formula and repeating the procedure as used in the first step, we know that

\begin{equation}
(U^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x), V^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x), \bar{V}^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x), \bar{V}^{(c)}_{i_1, \ldots, (i_l+1) \ldots i_p}(t,x,z))
\end{equation}

exist for the given $c \in \{1, 2, \ldots\}$ and all $l \in \{1, \ldots, p\}$. Therefore, the claim in (4.37) is true.

Third, it follows from the induction method with respect to $c \in \{1, 2, \ldots\}$ that the claims stated in the lemma are true. Hence, we finish the proof of Lemma 4.2. □
To state and prove the next lemma, let $D^2_\gamma([0, T], C^\infty(D, R^l))$ with $l \in \{r, q\}$ be the set of $R^l$-valued $\{\mathcal{F}_t\}$-adapted and square integrable càdlàg processes as in (2.8). Furthermore, for any given number sequence $\gamma = \{\gamma_c, c = 0, 1, 2, \ldots\}$ with $\gamma_c \in R$, define $\mathcal{M}^D_\gamma[0, T]$ to be the following Banach space (see, e.g., the related explanation in Yong and Zhou [41], and Situ [40])

$$\mathcal{M}^D_\gamma[0, T] \equiv D^2_\gamma([0, T], C^\infty(D, R^l))$$

$$\times D^2_\gamma([0, T], C^\infty(D, R^q))$$

$$\times L^2_{2,p}([0, T], C^\infty(D, R^{q \times d}))$$

$$\times L^2_{2,p}([0, T] \times R^h_+, C^\infty(D, R^{q \times h})),$$

which is endowed with the norm

$$\| (U, V, \tilde{V}, \tilde{V}) \|^2_{\mathcal{M}^D_\gamma} = \sum_{c=0}^{\infty} \xi(c) \| (U, V, \tilde{V}) \|^2_{\mathcal{M}^D_\gamma},$$

for any given $(U, V, \tilde{V}, \tilde{V}) \in \mathcal{M}^D_\gamma[0, T]$, and

$$\| (U, V, \tilde{V}, \tilde{V}) \|^2_{\mathcal{M}^D_\gamma} = E \left[ \sup_{0 \leq t \leq T} \| U(t) \|^2_{C^\infty(D, q)} e^{2\gamma_c t} \right]$$

$$+ E \left[ \sup_{0 \leq t \leq T} \| V(t) \|^2_{C^\infty(D, q)} e^{2\gamma_c t} \right]$$

$$+ E \left[ \int_0^T \| \tilde{V}(t) \|^2_{C^\infty(D, q)} e^{2\gamma_c t} dt \right]$$

$$+ E \left[ \int_0^T \| \tilde{V}(t) \|^2_{\nu_{\gamma}} e^{2\gamma_c t} dt \right].$$

Then, we have the following lemma.

**Lemma 4.3** Under the conditions of Theorem 2.1, all the claims in the theorem are true.

**PROOF.** By (4.2), we can define the following map 

$$\Xi : (U^1(\cdot, x), V^1(\cdot, x), \tilde{V}^1(\cdot, x), \tilde{V}^1(\cdot, x, z)) \rightarrow (U(\cdot, x), V(\cdot, x), \tilde{V}(\cdot, x), \tilde{V}(\cdot, x, z)).$$

Then, we show that $\Xi$ forms a contraction mapping in $\mathcal{M}^D_\gamma[0, T]$. In fact, consider 

$$(U^i(\cdot, x), V^i(\cdot, x), \tilde{V}^i(\cdot, x), \tilde{V}^i(\cdot, x, z)) \in \mathcal{M}^D_\gamma[0, T]$$

for each $i \in \{1, 2, \ldots\}$, satisfying

$$(U^{i+1}(\cdot, x), V^{i+1}(\cdot, x), \tilde{V}^{i+1}(\cdot, x), \tilde{V}^{i+1}(\cdot, x, z))$$

$$= \Xi(U^i(\cdot, x), V^i(\cdot, x), \tilde{V}^i(\cdot, x), \tilde{V}^i(\cdot, x, z)).$$
Furthermore, define
\[ \Delta f^i = f^{i+1} - f^i \quad \text{with} \quad f \in \{U, V, \bar{V}, \bar{V} \} \]
and take
\[ \zeta(\Delta U^i(t, x) + \Delta V^i(t, x)) = (\text{Tr} (\Delta U^i(t, x)) + \text{Tr} (\Delta V^i(t, x))) e^{2\gamma t}. \]
Thus, it follows from (2.16) and the similar argument as used in proving (4.25) that, for a \( \gamma_0 > 0 \) and each \( i \in \{2, 3, \ldots\} \),
\[ (4.44) \quad \zeta(\Delta U^i(t, x) + \Delta V^i(t, x)) \]
\[ + \int_t^T \text{Tr} (\Delta J(s, x, U^i, U^{i-1}, V^i, V^{i-1}) - \Delta \bar{V}^i(s, x)) e^{2\gamma s} ds \]
\[ + \sum_{j=1}^h \int_t^T \int_{z_j>0} \left( \text{Tr} \left( \Delta \bar{J}(s^-, x, U^i, U^{i-1}, V^i, V^{i-1}, z^-, z^+), \bar{V}^i(s^-, x, z^+) \right) \right)_j e^{2\gamma s} N_j(\lambda_j ds, dz_j) \]
\[ \leq \hat{\gamma}_0 \left( \int_0^t \|\Delta \mathcal{L}(s, x, U^i, U^{i-1}, V^i, V^{i-1})\|^2 e^{2\gamma s} ds \right. \]
\[ + \int_t^T \|\Delta \mathcal{L}(s, x, U^i, U^{i-1}, V^i, V^{i-1})\|^2 e^{2\gamma s} ds \right) - M^i(t, x) \]
\[ \leq \hat{\gamma}_0 K_{a,0} N^{i-1}(t) - M^i(t, x), \]
where \( K_{a,0} \) is some nonnegative constant depending only on \( K_{D,0} \). For the last inequality in (4.45), we have taken
\[ (4.46) \quad \hat{\gamma}_0 = \frac{1}{2\gamma_0} > 0. \]
Furthermore, \( N^{i-1}(t) \) appeared in (4.45) is given by
\[ (4.47) \quad N^{i-1}(t) \]
\[ = \int_0^t \|\Delta U^{i-1}(s)\|_{C^k(D,x)}^2 e^{2\gamma s} ds \]
\[ + \int_t^T \left( \|\Delta V^{i-1}(s)\|_{C^k(D,q)}^2 + \|\Delta \bar{V}^{i-1}(s)\|_{C^k(D,qd)}^2 + \|\Delta \bar{J}^{i-1}(s)\|_{\nu,k}^2 \right) e^{2\gamma s} ds. \]
In addition, \( M^i(t, x) \) in (4.45) is a martingale of the form,
\[ (4.48) \quad M^i(t, x) = \]
\[ -2 \sum_{j=1}^h \int_0^t \left( \Delta U^i(s^-, x) \right)' \Delta J_j(s^-, x, U^i, U^{i-1}, V^i, V^{i-1}, 1) e^{2\gamma s} dW_j(s) \]
\[ -2 \sum_{j=1}^h \int_0^t \int_{z_j>0} \left( \Delta U^i(s^-, x) \right)' \Delta J_j(s^-, x, U^i, U^{i-1}, V^i, V^{i-1}, z_j) e^{2\gamma s} \tilde{N}_j(\lambda_j ds, dz_j) \]
\[ \qquad - \int_t^T \int_{z_j>0} \left( \Delta \bar{U}^i(s^-, x) \right)' \Delta \bar{J}_j(s^-, x, U^i, U^{i-1}, V^i, V^{i-1}, z_j, z_j) e^{2\gamma s} \tilde{N}_j(\lambda_j ds, dz_j) \]
Then, it follows from (4.45)-(4.48) and the martingale properties related to the Itô’s stochastic integral that

\[
2 \leq \sum_{j=1}^{d} \int_{t}^{T} (\Delta V_j(s^{-}, x))' + \int_{t}^{T} \nu_{j} \left( (\Delta V_j(s^{-}, x))' \right) \\text{d}W_j(s)
\]

\[
+ 2 \sum_{j=1}^{d} \int_{t}^{T} \int_{z_j > 0} \left( (\Delta V_j(s^{-}, x))' \right) \\text{d}W_j(s)
\]

\[
+ \frac{h}{2} \sum_{j=1}^{d} \int_{t}^{T} \int_{z_j > 0} \left( (\Delta V_j(s^{-}, x))' \right) \\text{d}W_j(s)
\]

\[
\left( \Delta \tilde{V}_j(s^{-}, x, U_i, V_i, V_i^{-1}, z_j) - (\Delta \tilde{V}_j(s^{-}, x, z_j)) \right) e^{2\gamma_0 s} \tilde{N}_j(\lambda_j \text{d}s, dz_j).
\]

Then, it follows from (4.45)-(4.48) and the martingale properties related to the Itô’s stochastic integral that

(4.49) \quad E \left[ \left( \| \Delta U^i(t, x) \|^{2} + \| \Delta V^i(t, x) \|^{2} \right) e^{2\gamma_0 t} \right] \quad e^{2\gamma_0 t}

\quad + \int_{t}^{T} \text{Tr} \left( \Delta \tilde{J}(s, x, U^i, V_i, V_i^{-1}, V_i^{-1}) - \Delta \tilde{V}(s, x) \right) e^{2\gamma_0 s} \\text{d}s

\quad + \sum_{j=1}^{d} \int_{t}^{T} \int_{z_j > 0} \left( \text{Tr} \left( \Delta \tilde{I}(s, x, U^i, V_i, V_i^{-1}, V_i^{-1}, z) - \Delta \tilde{V}(s^{-}, x, z) \right) \right)_{j}

\quad \leq \hat{\gamma}_0 (T + 1) K_{a,0} \left\| (\Delta U_i^{-1}, V_i^{-1}, \Delta \tilde{V}_i^{-1}, \Delta \tilde{V}_i^{-1}) \right\|_{M^{t,0,4}}^{2} .

Next, it follows from (4.48) that

(4.50) \quad E \left[ \sup_{0 \leq t \leq T} \left| M^i(t, x) \right| \right]

\quad \leq 2 \sum_{j=1}^{d} E \left[ \sup_{0 \leq t \leq T} \left| \int_{0}^{t} (\Delta U^i(s^{-}, x))' \Delta J_j(s^{-}, x, U^i, U_i^{-1}) e^{2\gamma_0 s} \\text{d}W_j(s) \right| \right]

\quad + 2 \sum_{j=1}^{d} E \left[ \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \int_{z_j > 0} (\Delta U^i(s^{-}, x))' \Delta I_j(s^{-}, x, U^i, U_i^{-1}, z_j) e^{2\gamma_0 s} \tilde{N}(\lambda_j \text{d}s, dz_j) \right| \right]

\quad + 4 \sum_{j=1}^{d} E \left[ \sup_{0 \leq t \leq T} \left| \int_{0}^{t} (\Delta V^i(s^{-}, x))' \right. \left. \left( \Delta \tilde{J}_j(s^{-}, x, U^i, V_i, V_i^{-1}, V_i^{-1}) - (\Delta \tilde{V}_j(s^{-}, x)) \right) e^{2\gamma_0 s} \\text{d}W_j(s) \right| \right]

\quad + 4 \sum_{j=1}^{d} E \left[ \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \int_{z_j > 0} (\Delta V^i(s^{-}, x))' \left( \Delta \tilde{I}_j(s^{-}, x, U^i, V_i, V_i^{-1}, V_i^{-1}, z_j) - (\Delta \tilde{V}_j(s^{-}, x, z_j)) \right) e^{2\gamma_0 s} \tilde{N}(\lambda_j \text{d}s, dz_j) \right| \right] .

28
By the Burkholder-Davis-Gundy’s inequality (see, e.g., Theorem 4.8 in page 193 of Protter [36]), the right-hand side of the inequality in (4.50) is bounded by

\begin{align}
(4.51) \quad K_{b,0} \left[ \sum_{j=1}^{d} E \left[ \left( \int_{0}^{T} \| \Delta U^{i}(s, x) \|^{2} \right) \right]^{\frac{1}{2}} \right. \\
+ \sum_{j=1}^{h} E \left[ \left( \int_{0}^{T} \int_{z_{j}>0} \| \Delta U^{i}(s, x) \|^{2} \right) \right]^{\frac{1}{2}} \\
+ \sum_{j=1}^{d} E \left[ \left( \int_{0}^{T} \| \Delta V^{i}(s, x) \|^{2} \right) \right]^{\frac{1}{2}} \\
+ \sum_{j=1}^{h} E \left[ \left( \int_{0}^{T} \int_{z_{j}>0} \| \Delta V^{i}(s, x) \|^{2} \right) \right]^{\frac{1}{2}} \\
\left. \left[ \sum_{j=1}^{d} \left( \int_{0}^{T} \| \Delta \mathcal{J}^{i}(s, x, V^{i}, U^{i-1}, V^{i-1}) \|^{2} \right) \right]^{\frac{1}{2}} \right]
\end{align}

where \( K_{b,0} \) is some nonnegative constant depending only on \( K_{D,0} \) and \( T \). Furthermore, it follows from the direct observation that the quantity in (4.51) is bounded by

\begin{align}
(4.52) \quad K_{b,0} \left[ \left( \sup_{0 \leq t \leq T} \| \Delta U^{i}(t, x) \|^{2} e^{2\gamma_{0} t} \right) \right]^{\frac{1}{2}} \\
\left. \left[ \sum_{j=1}^{d} \left( \int_{0}^{T} \| \Delta \mathcal{J}^{i}(s, x, U^{i}, V^{i}, U^{i-1}, V^{i-1}) \|^{2} e^{2\gamma_{0} s} ds \right) \right]^{\frac{1}{2}} \right]
\end{align}
Furthermore, it follows from (4.45) and (2.17) that, for

\[-(\Delta \tilde{V}_j(s, x, z_j))^2 e^{2\gamma_0 s} \lambda_j \nu_j (dz_j) ds \right) \right] \right) .

In addition, by the direct computation, we know that the quantity in (4.52) is dominated by

\[
\frac{1}{2} E \left[ \sup_{0 \leq t \leq T} \| \Delta U^i (t, x) \|^2 e^{2\gamma_0 t} \right] + dK^2_{b,0} \left[ \int_0^T \text{Tr} \left( \Delta \tilde{\mathcal{J}}(s, x, U^i, \bar{V}^i, V^{i-1}, V^{i-1}) e^{2\gamma_0 s} ds \right) \right] + K^2_{b,0} \left[ \int_0^T \int_{x > 0} \text{Tr} \left( \Delta \tilde{\mathcal{T}}(s^-, x, U^i, \bar{V}^i, V^{i-1}, z) ) e^{2\gamma_0 s} \lambda \nu (dz) ds \right) \right] + \frac{1}{2} E \left[ \sup_{0 \leq t \leq T} \| \Delta V^i (t, x) \|^2 e^{2\gamma_0 t} \right] + dK^2_{b,0} \left[ \int_0^T \int_{x > 0} \text{Tr} \left( \Delta \tilde{\mathcal{T}}(s, x, U^i, \bar{V}^i, V^{i-1}, V^{i-1}, z) - \Delta \tilde{V}(s) \right) e^{2\gamma_0 s} \lambda \nu (dz) ds \right] .
\]

Due to (4.49), the quantity in (4.53) is bounded by

\[
\frac{1}{2} \left( E \left[ \sup_{0 \leq t \leq T} \| \Delta U^i (t, x) \|^2 e^{2\gamma_0 t} \right] + E \left[ \sup_{0 \leq t \leq T} \| \Delta V^i (t, x) \|^2 e^{2\gamma_0 t} \right] \right) + \gamma_0 (T + 1)dK_{a,0} \left( \gamma_0 \right) \sup_{0 \leq t \leq T} \| (\Delta U^{i-1}, \Delta V^{i-1}, \Delta \tilde{V}^{i-1}) \|_{\mathcal{M}^D_{\gamma_0, k}}^2 ,
\]

where \( K_{a,0} \) is some nonnegative constant depending only on \( T, d, \) and \( K_{D,0} \). Thus, it follows from (2.16) and (4.45)-(4.54) that

\[
\frac{1}{2} E \left[ \sup_{0 \leq t \leq T} \| \Delta U^i (t, x) \|^2 C^0 (q) e^{2\gamma_0 t} \right] + E \left[ \sup_{0 \leq t \leq T} \| \Delta V^i (t) \|^2 C^0 (q) e^{2\gamma_0 t} \right] \leq 2 \left( 1 + dK^2_{b,0} \right) \gamma_0 (T + 1) \left( \| (\Delta U^{i-1}, \Delta V^{i-1}, \Delta \tilde{V}^{i-1}) \|_{\mathcal{M}^D_{\gamma_0, k}}^2 .
\]

Furthermore, it follows from (4.45) and (2.17) that, for \( i \in \{3, 4, \ldots \} ,

\[
\frac{1}{2} E \left[ \int_0^T \text{Tr} \left( \Delta \tilde{V}^i (s, x) \right) e^{2\gamma_0 s} ds \right] \leq 2E \left[ \int_0^T \text{Tr} \left( \Delta \tilde{\mathcal{J}}(s, x, U^i, \bar{V}^i, V^{i-1}, V^{i-1}) - \Delta \tilde{V}^i (s, x) \right) e^{2\gamma_0 s} ds \right] + 2E \left[ \int_0^T \text{Tr} \left( \Delta \tilde{\mathcal{J}}(s, x, U^i, V^{i-1}, V^{i-1}) \right) e^{2\gamma_0 s} ds \right] \leq 2\gamma_0 K_{C,0} \left( \| (\Delta U^{i-1}, \Delta V^{i-1}, \Delta \tilde{V}^{i-1}) \|_{\mathcal{M}^D_{\gamma_0, k}}^2 .
\]

30
where $K_{C,0}$ is some nonnegative constant depending only on $K_{D,0}$ and $T$. Similarly, it follows from (2.18) that

$$
E \left[ \sum_{j=1}^{h} \int_{t}^{T} \int_{z > 0} \left( \text{Tr} \left( \Delta \tilde{V}^i(s^-, x, z) \right) \right) e^{2\gamma t} \lambda_j ds \nu_j (dz_j) \right]
$$

(4.57)

$$
\leq 2E \left[ \sum_{j=1}^{h} \int_{t}^{T} \int_{z > 0} \left( \text{Tr} \left( \Delta \tilde{T}(s, x, U^i, V^i, U^{i-1}, V^{i-1}, z) - \Delta \tilde{V}^i(s^-, x, z) \right) \right) e^{2\gamma t} \lambda_j ds \nu_j (dz_j) \right]
$$

$$
+ 2E \left[ \sum_{j=1}^{h} \int_{t}^{T} \int_{z > 0} \left( \text{Tr} \left( \Delta \tilde{T}(s, x, U^i, V^i, U^{i-1}, V^{i-1}, z) \right) \right) e^{2\gamma t} \lambda_j ds \nu_j (dz_j) \right]
$$

$$
\leq 2\gamma_0 K_{C,0} \left( \|(\Delta U^{i-1}, \Delta V^{i-1}, \Delta \tilde{V}^{i-1}, \Delta \tilde{V}^{i-1})\|_{\mathcal{M}_{K_0}^{D,k}} \right)
$$

(4.57)

$$
+ \left( \|(\Delta U^{i-2}, \Delta V^{i-2}, \Delta \tilde{V}^{i-2}, \Delta \tilde{V}^{i-2})\|_{\mathcal{M}_{K_0}^{D,k}} \right).
$$

Thus, by (4.45), (4.55)-(4.57), and the fact that all functions and norms used in this paper are continuous in terms of $x$, we have

$$
\|(\Delta U^i, \Delta V^i, \Delta \tilde{V}^i, \Delta \tilde{V}^i)\|_{\mathcal{M}_{K_0}^{D,0}}^2
$$

(4.58)

$$
\leq \gamma_0 K_{d,0} \left( \|(\Delta U^{i-1}, \Delta V^{i-1}, \Delta \tilde{V}^{i-1}, \Delta \tilde{V}^{i-1})\|_{\mathcal{M}_{K_0}^{D,k}} \right)^2
$$

$$
+ \left( \|(\Delta U^{i-2}, \Delta V^{i-2}, \Delta \tilde{V}^{i-2}, \Delta \tilde{V}^{i-2})\|_{\mathcal{M}_{K_0}^{D,k}} \right)^2.
$$

(4.58)

where $K_{d,0}$ is some nonnegative constant depending only on $K_{D,0}$ and $T$.}

Now, by Lemma 4.2 and the similar construction as in (4.44), for each $c \in \{1, 2, \ldots\}$, we can define

$$
\zeta(\Delta U^{c,i}(t, x) + \Delta V^{c,i}(t, x)) \equiv \left( \text{Tr} \left( \Delta U^{c,i}(t, x) \right) + \text{Tr} \left( \Delta V^{c,i}(t, x) \right) \right) e^{2\gamma t},
$$

(4.59)

where

$$
\Delta U^{c,i}(t, x) = (\Delta U^{(0),i}(t, x), \Delta U^{(1),i}(t, x), \ldots, \Delta U^{(c),i}(t, x))^t,
$$

$$
\Delta V^{c,i}(t, x) = (\Delta V^{(0),i}(t, x), \Delta V^{(1),i}(t, x), \ldots, \Delta V^{(c),i}(t, x))^t.
$$
Then, it follows from the Itô’s formula and the similar discussion for (4.58) that

\[
\left\| (U^i, V^i, \tilde{V}^i) \right\|_{\mathcal{M}_{g,c,k+c}^D}^2
\]

\[
\leq \hat{g}_c K_{g,c,k+c} \left( \left\| (U^{i-1}, V^{i-1}, \tilde{V}^{i-1}) \right\|_{\mathcal{M}_{g,c,k+c}^D}^2
+ \left\| (U^{i-2}, V^{i-2}, \tilde{V}^{i-2}) \right\|_{\mathcal{M}_{g,c,k+c}^D}^2 \right)
\]

\[
\leq \frac{(c + 1)^{10}(c + 2)^{10}...((c + k)^{10})(\eta(c + 1)\eta(c + 2)...\eta(c + k))}{\delta}
\left( \left\| (U^{i-1}, V^{i-1}, \tilde{V}^{i-1}) \right\|_{\mathcal{M}_{g,c,k+c}^D}^2
+ \left\| (U^{i-2}, V^{i-2}, \tilde{V}^{i-2}) \right\|_{\mathcal{M}_{g,c,k+c}^D}^2 \right),
\]

where, for the last inequality of (4.60), we have taken the number sequence \( \gamma \) such that \( \gamma_0 < \gamma_1 < ... \) and

\[
\hat{g}_c K_{d,c}((c + 1)^{10}(c + 2)^{10}...((c + k)^{10})(\eta(c + 1)\eta(c + 2)...\eta(c + k)) \leq \delta
\]

for some \( \delta > 0 \) such that \( 2\sqrt{e^k\delta} \) is sufficiently small. Hence, we have

\[
\left\| (U^i, V^i, \tilde{V}^i) \right\|_{\mathcal{M}_{c}^D}^2
\]

\[
\leq e^k \delta \left( \left\| (U^{i-1}, V^{i-1}, \tilde{V}^{i-1}) \right\|_{\mathcal{M}_{c}^D}^2
+ \left\| (U^{i-2}, V^{i-2}, \tilde{V}^{i-2}) \right\|_{\mathcal{M}_{c}^D}^2 \right).\]

Since \( (a^2 + b^2)^{1/2} \leq a + b \) for \( a, b \geq 0 \), we have

\[
\left\| (U^i, V^i, \tilde{V}^i) \right\|_{\mathcal{M}_{c}^D}
\]

\[
\leq \sqrt{e^k} \delta \left( \left\| (U^{i-1}, V^{i-1}, \tilde{V}^{i-1}) \right\|_{\mathcal{M}_{c}^D}
+ \left\| (U^{i-2}, V^{i-2}, \tilde{V}^{i-2}) \right\|_{\mathcal{M}_{c}^D} \right).\]

Therefore, by (4.62), we know that

\[
\sum_{i=3}^{\infty} \left\| (U^i, V^i, \tilde{V}^i) \right\|_{\mathcal{M}_{c}^D}
\leq \frac{\sqrt{e^k} \delta}{1 - 2\sqrt{e^k} \delta} \left( 2 \left\| (U^2, V^2, \tilde{V}^2) \right\|_{\mathcal{M}_{c}^D}
+ \left\| (U^1, V^1, \tilde{V}^1) \right\|_{\mathcal{M}_{c}^D} \right).
\]

\[
< \infty.
\]
Thus, from (4.63), we see that \((U^i, V^i, \tilde{V}^i, \tilde{\tilde{V}}^i)\) with \(i \in \{1, 2, \ldots\}\) forms a Cauchy sequence in \(\mathcal{M}_0^D[0, T]\), which implies that there is some \((U, V, \tilde{V}, \tilde{\tilde{V}})\) such that

\[(U^i, V^i, \tilde{V}^i, \tilde{\tilde{V}}^i) \to (U, V, \tilde{V}, \tilde{\tilde{V}}) \text{ as } i \to \infty \text{ in } \mathcal{M}_0^D[0, T].\]

Finally, by (4.64) and the similar procedure as used for Theorem 5.2.1 in pages 68-71 of Øksendal [29], we can complete the proof of Lemma 4.3. \(\square\)

4.2 Proof of Theorem 2.1

By combining Lemmas 4.1, 4.3 we can reach a proof for Theorem 2.1. \(\square\)

4.3 Proof of Theorem 2.2

This proof is along the line of the one for Lemma 4.3. More precisely, for any given number sequence \(\gamma = \{\gamma_{D_c}, c = 0, 1, 2, \ldots\}\) with \(\gamma_{D_c} \in R\), replace the norm for the Banach space \(\mathcal{M}_0^D[0, T]\) defined in (4.41) by the one

\[(4.65) \quad \| (U, V, \tilde{V}, \tilde{\tilde{V}}) \|_{\mathcal{M}^D_{D_c}}^2 = \sum_{c=0}^{\infty} \xi(c) \| (U, V, \tilde{V}, \tilde{\tilde{V}}) \|_{\mathcal{M}^D_{D_c} \times c}^2,
\]

for any given \((U, V, \tilde{V}, \tilde{\tilde{V}})\) in this space, where

\[
\| (U, V, \tilde{V}, \tilde{\tilde{V}}) \|_{\mathcal{M}^D_{D_c}}^2 = E \left[ \sup_{0 \leq t \leq T} \|U(t)\|_{C^0(D_c, \mathbb{R})}^2 e^{2\gamma_{D_c} t} \right] \\
+ E \left[ \sup_{0 \leq t \leq T} \|V(t)\|_{C^0(D_c, \mathbb{R})}^2 e^{2\gamma_{D_c} t} \right] \\
+ E \left[ \int_0^T \|\tilde{V}(t)\|_{C^0(D_c, \mathbb{R})}^2 e^{2\gamma_{D_c} t} dt \right] \\
+ E \left[ \int_0^T \|\tilde{\tilde{V}}(t)\|_{\psi_c}^2 e^{2\gamma_{D_c} t} dt \right].
\]

Then, it follows from the similar argument used for (4.61) in the proof of Lemma 4.3 that

\[ (U^1(\cdot, x), V^1(\cdot, x), \tilde{V}^1(\cdot, x), \tilde{\tilde{V}}^1(\cdot, x, z)) \in \bar{Q}_T^2([0, T] \times D) \]

with \((U^0, V^0, \tilde{V}^0, \tilde{\tilde{V}}^0) = (0, 0, 0, 0)\), where \((U^1, V^1, \tilde{V}^1, \tilde{\tilde{V}}^1)\) is defined through (4.2) in Lemma 4.1. Furthermore, over each \(D_c\) with \(c \in \{0, 1, \ldots\}\), we have that

\[(4.66) \quad \| (\Delta U^i, \Delta V^i, \Delta \tilde{V}^i, \Delta \tilde{\tilde{V}}^i) \|_{\mathcal{M}^D}^2 \leq e^k \delta \left( \| (\Delta U^{i-1}, \Delta V^{i-1}, \Delta \tilde{V}^{i-1}, \Delta \tilde{\tilde{V}}^{i-1}) \|_{\mathcal{M}^D}^2 \right) + \| (\Delta U^{i-2}, \Delta V^{i-2}, \Delta \tilde{V}^{i-2}, \Delta \tilde{\tilde{V}}^{i-2}) \|_{\mathcal{M}^D}^2,
\]

33
where \(\delta\) is a constant that can be determined by suitably choosing a number sequence \(\gamma\) such that \(\gamma_{D_0} < \gamma_{D_1} < \ldots\) and \(0 < \sqrt{e^k \delta} \left(1 - 2 \sqrt{e^k \delta}\right) < 1\) (note that \(\gamma_{D_c}\) may depend on both \(D_c\) and \(c\) for each \(c \in \{0, 1, \ldots\}\)). Thus, it follows from (4.66) that the remaining justification for Theorem 2.2 can be conducted along the line of proof for Theorem 2.1. \(\Box\)

5 Proofs of Theorem 3.2 and Theorem 3.3

To provide the proofs for Theorem 3.2 and Theorem 3.3, we first recall the Skorohod problem and study some related properties.

5.1 The Skorohod Problem

Let \(D([0, T], R^b)\) with \(b \in \{p, 2p\}\) be the space of all functions \(z : [0, T] \rightarrow R^b\) that are right-continuous with left limits and are endowed with Skorohod topology (see, e.g., Billingsley [3], Jacod and Shiryaev [21]). Then, we can introduce the Skorohod problem as follows.

**Definition 5.1 (The Skorohod problem).** Given \(z \in D([0, T], R^p)\) with \(z(0) \in D\), a \((D, R)\)-regulation of \(z\) over \([0, T]\) is a pair \((x, y) \in D([0, T], D) \times D([0, T], R^b_+)\) such that

\[
x(t) = z(t) + R_{y(t)}\]

for all \(t \in [0, T]\),

where, for each \(i \in \{1, \ldots, b\}\),

1. \(y_i(0) = 0\),

2. \(y_i\) is nondecreasing,

3. \(y_i\) can increase only at a time \(t \in [0, T]\) with \(x(t) \in F_i\).

Furthermore, we define the modulus of continuity with respect to a function \(z(\cdot) \in D([0, T], R^b)\) and a real number \(\delta > 0\) by

\[
w(z, \delta, T) \equiv \inf_{T} \max_{t_i} \text{Osc} \left( z, [t_{i-1}, t_i] \right),
\]

where the infimum takes over the finite sets \(\{t_l\}\) of points satisfying \(0 = t_0 < t_1 < \ldots < t_m = T\) and \(t_l - t_{l-1} > \delta\) for \(l = 1, \ldots, m\), and

\[
\text{Osc}(z, [t_{l-1}, t_l]) = \sup_{t_1 \leq s \leq t_2} \| z(t) - z(s) \|.
\]

Then, we have the following lemma.

**Lemma 5.1** Suppose that the reflection matrix \(R\) in Definition satisfies the completely-\(S\) condition. Then, any \((D, R)\)-regulation \((x, y)\) of \(z \in D([0, T], R^p)\) with \(z(0) \in D\) satisfies the oscillation inequality over \([t_1, t_2]\) with \(t_1, t_2 \in [0, T]\)

\[
\text{Osc}(x, [t_1, t_2]) \leq \kappa \text{Osc}(z, [t_1, t_2]),
\]

\[
\text{Osc}(y, [t_1, t_2]) \leq \kappa \text{Osc}(z, [t_1, t_2]),
\]
where $\kappa$ is some nonnegative constant depending only on the inward normal vector $N$ and the reflection matrix $R$.

**Proof.** For each $t \in [t_1, t_2]$, define

\begin{align}
\Delta z(t) &\equiv z(t) - z(t^-), \\
\Delta x(t) &\equiv x(t) - x(t^-), \\
\Delta y(t) &\equiv y(t) - y(t^-).
\end{align}

Since the reflection matrix $R$ satisfies the completely-\$ condition, it is easy to check that the linear complementarity problem (LCP)

\begin{align}
\Delta x(t) &= \Delta z(t) + R\Delta y(t), \\
\Delta x(t) &\in D, \\
\Delta y(t) &\geq 0, \\
\Delta x_i(t)\Delta y_i(t) &= 0 \text{ for } i = 1, \ldots, p, \\
(b_i - \Delta x_i(t))\Delta y_i(t) &= 0 \text{ for } i = p + 1, \ldots, b,
\end{align}

is completely solvable (see also Theorem 2.1 in Mandelbaum [27] for the related discussion). Furthermore, we can conclude that

\begin{equation}
\Delta y(t) \leq C \Delta z(t)
\end{equation}

for some nonnegative constant $C$ depending only on the inward normal vector $N$ and the reflection matrix $R$. Then, the rest of the proof is the direct conclusion of the one for Theorem 3.1 in Dai [9] or the one for Theorem 4.2 in Dai and Dai [7]. □

**Lemma 5.2** Assume that $(x^n, y^n) \to (x, y)$ along $n \in \{1, 2, \ldots\}$ in $D([0, T], R^p) \times D([0, T], R^b)$ and $y^n(\cdot)$ is of bounded variation for each $n \in \{1, 2, \ldots\}$. Furthermore, suppose that

\begin{equation}
\int_0^t f(x^n(s))dy^n(s) = 0
\end{equation}

for all $n \in \{1, 2, \ldots\}$ and each $t \in [0, T]$, where $f \in C^b([0, T], R^b)$ is a $b$-dimensional bounded vector function. Then, for each $t \in [0, T]$, we have that

\begin{equation}
\int_0^t f(x(s))dy(s) = 0.
\end{equation}

**Proof.** It follows from the definition in pages 123-124 of Billingsley [3] or Theorem 1.14 in page 328 of Jacod and Shiryaev [21] that there is a sequence $\{\gamma_n, n \in \{1, 2, \ldots\}\}$ of continuous and strictly increasing functions mapping from $[0, T] \to [0, T]$ with $\gamma_n(0) = 0$ and $\gamma_n(T) = T$ such that

\begin{align}
\sup_{t \in [0, T]} |\gamma_n(t) - t| &\to 0, \\
\sup_{t \in [0, T]} |(x^n, y^n)(\gamma_n(t)) - (x, y)(t)| &\to 0.
\end{align}
Then, by the uniform convergence in (5.11)-(5.12) and the condition in (5.9), we know that
\[
\int_0^t f(x(s))dy(s) = \lim_{n \to \infty} \int_0^t f(x^n(\gamma_n(s)))dy^n(\gamma_n(s)) \\
= \lim_{n \to \infty} \int_0^{\gamma_n^{-1}(t)} f(x^n(u))dy^n(u) \\
= 0,
\]
where \(\gamma_n^{-1}(\cdot)\) is the inverse function of \(\gamma_n(\cdot)\) for each \(n \in \{1, 2, \ldots\}\). Hence, we complete the proof of Lemma 5.2. □

5.2 Proof of Theorem 3.2

We divide the proof of the theorem into four parts: Part A (Existence, Uniqueness), Part B, Part C, and Part D, which correspond to different boundary reflection conditions.

**Part A (Existence).** We consider the case that \(L(t, \omega)\) appeared in (3.7)-(3.8) is a constant and both of the forward and the backward SDEs have reflection boundaries. In this case, we need to prove the claim that there is an adapted weak solution \(((X, Y), (V, \bar{V}, \tilde{V}, F))\) to the system in (1.2).

In fact, for a positive integer \(b\), let \(D^2_{\mathcal{F}}([0, T], \mathbb{R}^b)\) be the space of \(\mathbb{R}^b\)-valued and \(\{\mathcal{F}_t\}\)-adapted processes with sample paths in \(D([0, T], \mathbb{R}^b)\). Furthermore, each \(Y \in D^2_{\mathcal{F}}([0, T], \mathbb{R}^b)\) is square-integrable in the sense that
\[
E \left[ \int_0^T \|Y(t)\|^2 dt \right] < \infty. \tag{5.13}
\]

In addition, we use \(D^2_{\mathcal{F}, p}([0, T], \mathbb{R}^b)\) to denote the corresponding predictable space. Then, for a given \(n \in \{1, 2, \ldots\}\) and a 4-tuple
\[
(X^n, V^n, \bar{V}^n, \tilde{V}^n) \in D^2_{\mathcal{F}}([0, T], \mathbb{R}^b) \times D^2_{\mathcal{F}}([0, T], \mathbb{R}^q) \times D^2_{\mathcal{F}, p}([0, T], \mathbb{R}^{q \times d}) \times D^2_{\mathcal{F}, p}([0, T] \times R^h_+, \mathbb{R}^{q \times h}) \tag{5.14}
\]
with \(X^n(0) = X(0) \in D\) and \(V^n(T) = V(T) \in \bar{D}\), we have the following observation.

By the study concerning the continuous dynamic complementarity problem (DCP) in Bernard and El Kharrouri [2] (see also the related discussions in Mandelbaum [27], Reiman and Williams [37]), Theorem 2.1 (and its proof) in the current paper, there is a 6-tuple
\[
((X^{n+1}, Y^{n+1}), (V^{n+1}, \bar{V}^{n+1}, \tilde{V}^{n+1}, F^{n+1})) \in D^2_{\mathcal{F}}([0, T], \mathbb{R}^b) \times D^2_{\mathcal{F}}([0, T], \mathbb{R}^b) \times D^2_{\mathcal{F}, p}([0, T], \mathbb{R}^{q \times d}) \times D^2_{\mathcal{F}, p}([0, T] \times R^h_+, \mathbb{R}^{q \times h}) \times D^2_{\mathcal{F}, p}([0, T] \times R^h_+, \mathbb{R}^{q \times h}) \times D^2_{\mathcal{F}, p}([0, T], \mathbb{R}^{q \times \bar{b}})
\]
for each \( n \in \{1, 2, \ldots \} \), satisfying the properties along each sample path:

\[
X^{n+1}(t) = Z^n(t) + RY^{n+1}(t) \in D,
\]

with

\[
\begin{align*}
Z^n(t) &= X(0) + Z^n_1(t) + Z^n_2(t), \\
Z^n_1(t) &= \int_0^t b(s, X^n(s), V^n(s), \bar{V}^n(s), u(s, X^n(s))) ds, \\
Z^n_2(t) &= \int_0^t \sigma(s, X^n(s), V^n(s), \bar{V}^n(s), u(t, X^n(s))), z) dW(s) \\
&\quad + \int_0^t \int_{z>0} \eta(s, X^n(s), V^n(s), \bar{V}^n(s), u(s, X^n(s)), z) \tilde{N}(ds, dz);
\end{align*}
\]

and

\[
V^{n+1}(t) = U^n(t) - S(F^{n+1}(T) - F^{n+1}(t)) \in \bar{D},
\]

with

\[
\begin{align*}
U^n(t) &= V(T) + U^n_1(t) - U^n_2(t), \\
U^n_1(t) &= \int_t^T c(s, X^n(s), V^n(s), \bar{V}^n(s), u(s, X^n(s))) ds, \\
U^n_2(t) &= \int_t^T \bar{V}^{n+1}(s) dW(s) + \int_t^T \int_{z>0} \bar{V}^{n+1}(s, z) \tilde{N}(ds, dz).
\end{align*}
\]

Furthermore, \((X^{n+1}, Y^{n+1})\) satisfies the property (3) in Definition 3.1. In other words, \(Y^{n+1}\) is a \(b\)-dimensional \(\mathcal{F}_t\)-adapted process such that the \(i\)th component \(Y^{n+1}_i\) of \(Y^{n+1}\) for each \(i \in \{1, \ldots, b\}\) \(\mathbb{P}\)-a.s. has the properties that \(Y^{n+1}_i(0) = 0\), \(Y^{n+1}_i\) is non-decreasing, and \(Y^{n+1}_i\) can increase only when \(X^{n+1}\) is on the boundary face \(D_i\), i.e.,

\[
\int_0^t I_{D_i}(X^{n+1}(s)) dY^{n+1}_i(s) = Y^{n+1}_i(t) \quad \text{for all } t \geq 0.
\]

Similarly, \((V^{n+1}, F^{n+1})\) also satisfies the property (3) in Definition 3.1. More precisely, \(F^{n+1}\) is a \(q\)-dimensional \(\mathcal{F}_t\)-adapted process such that the \(i\)th component \(F^{n+1}_i\) of \(F^{n+1}\) for each \(i \in \{1, \ldots, \tilde{b}\}\) \(\mathbb{P}\)-a.s. has the properties that \(F^{n+1}_i(0) = 0\), \(F^{n+1}_i\) is non-decreasing, and \(F^{n+1}_i\) can increase only when \(V^{n+1}\) is on the boundary face \(\bar{D}_i\), i.e.,

\[
\int_0^t I_{D_i}(V^{n+1}(s)) dF^{n+1}_i(s) = F^{n+1}_i(t) \quad \text{for all } t \geq 0.
\]

Next, we prove that the following sequence of stochastic processes along \( n \in \{1, 2, \ldots \} \),

\[
\Xi^n = ((X^{n+1}, Y^{n+1}), (V^{n+1}, \bar{V}^{n+1}, \tilde{V}^{n+1}, F^{n+1})), \quad (X^1, V^1, \bar{V}^1, \tilde{V}^1) = 0,
\]

37
is relatively compact in the Skorohod topology over the space

\[
\mathcal{P}[0,T] \equiv D_2^2([0,T], R^p) \times D_2^2([0,T], R^b) \\
\times D_2^2([0,T], R^q) \times D_2^2([0,T], R^{q \times d}) \\
\times D_2^2([0,T], R^h) \times D_2^2([0,T], R^{q \times h}).
\]

Along the line of Dai [9, 13], Dai and Dai [7] and by Corollary 7.4 in page 129 of Ethier and Kurtz [18], it suffices to prove the following two conditions to be true: First, for each \( \epsilon > 0 \) and rational \( t > 0 \), there is a constant \( C(\epsilon, t) \) such that

\[
\lim_{n \to \infty} \inf P \left\{ \| \mathcal{X}_n \|^2 \leq C(\epsilon, t) \right\} \geq 1 - \epsilon;
\]

Second, for each \( \epsilon > 0 \) and \( T > 0 \), there is a constant \( \delta > 0 \) such that

\[
\lim_{n \to \infty} \sup P \left\{ w(\mathcal{X}_n, \delta, T) \geq \epsilon \right\} \leq \epsilon.
\]

To prove the two conditions stated in (5.21) and (5.22), we first define the norm along each sample path for each \( f \in \{ X^n, \bar{Z}^n, U^n, (V^n, \bar{V}^n, \tilde{V}^n) \} \) and each \( a, b \in [0, T] \). Then, we introduce the space endowed with the norm

\[
\left\| (X, V, \bar{V}, \tilde{V}) \right\|_{Q, [0, T]}^2 \\
\equiv E \left[ \sup_{t \in [0,T]} (\|X(t)\|^2 + \|V(t)\|^2) e^{2\gamma t} \right] + E \left[ \int_0^T \|\bar{V}(t)\|^2 e^{2\gamma t} dt \right] \\
+ E \left[ \int_0^T \|\tilde{V}(t, \cdot)\|_\nu^2 e^{2\gamma t} dt \right]
\]

for each \( (X, V, \bar{V}, \tilde{V}) \in Q, [0, T] \). Thus, by Lemma 5.1 there is a positive constant \( C_1 \) such that

\[
\left\| (X^{n+1}, Y^{n+1})(t) \right\| \leq \left\| (X^{n+1}, Y^{n+1})(0) \right\| + \kappa \text{Osc}(Z^n, [0, T]) \leq C_1 \left( \|X(0)\| + \|Z^n\|_{[0,T]} \right).
\]
and

\[
\begin{align*}
&\|(V^{n+1}, \tilde{V}^{n+1}, \tilde{\nu}^{n+1}(\cdot), F^{n+1})(t)\| \\
&\leq \|(V^{n+1}, \tilde{V}^{n+1}, \tilde{\nu}^{n+1}(\cdot), F^{n+1})(T)\| + \kappa \text{Osc}(U^n, [0, T]) \\
&\leq C_1 \left( \|V(T)\| + \|U^n\|_{[0,T]} \right).
\end{align*}
\]

Thus, for each \(n \in \{1, 2, \ldots\}\), the given linear growth constant \(L \geq 0\) in (3.7), and any constant \(K > LT\), it follows from the Markov’s inequality that

\[
P\left\{\left\|(Z_1^n)(t)\right\| \geq K \right\} \leq \frac{LT}{K - LT} E \left[ \left\|(X^n, V^n, \tilde{V}^{n}(\cdot))\right\|_{[0,T]} \right].
\]

Furthermore, by Lemma 4.2.8 in page 201 of Applebaum [1] (or related theorem in page 20 of Gihman and Skorohod [19]) and the linear growth condition, we know that

\[
P\left\{\left\|(Z_2^n)(t)\right\| \geq K \right\} \leq \frac{\bar{K}}{K^2} + \frac{L^2T}{K - L^2T^2} E \left[ \left\|(X^n, V^n, \tilde{V}^{n}(\cdot))\right\|^2_{[0,T]} \right]
\]

for all nonnegative constant \(\bar{K} > L^2T\). In addition, similar to the illustration of (5.27), we have that

\[
P\left\{\left\|(U_1^n)(t)\right\| \geq K \right\} \leq \frac{1}{K - LT} E \left[ \left\|(X^n, V^n, \tilde{V}^{n}(\cdot))\right\|_{[0,T]} \right]
\]

Next, by the similar demonstration for (5.31), it follows from the linear growth condition and the proof of Proposition 5.3 in Dai [15] that

\[
P\left\{\left\|(U_2^n)(t)\right\| \geq K \right\} \leq \frac{\bar{K}}{K^2} + \frac{K_1T}{(K - L^2T)^2} + \frac{\bar{K}_2}{(K - L^2T)^2} E \left[ \left\|(X^n, V^n, \tilde{V}^{n}(\cdot))\right\|^2_{Q,[0,T]} \right]
\]

for some nonnegative constants \(K_1\) and \(K_2\). Therefore, for each given \(\epsilon > 0\), it follows from (5.27)-(5.30), suitably chosen constants \(K\) and \(\bar{K}\), and the initial condition in (5.19) that there is a nonnegative constant \(C\) such that

\[
\inf_n P\left\{\|\Xi^n(t)\| \leq C, \ 0 \leq t \leq T \right\} \\
\geq \inf_n \min \left\{ P\left\{\|X^{n+1}, Y^{n+1}(t)\| \leq C, \ 0 \leq t \leq T \right\}, P\left\{\|V^{n+1}, \tilde{V}^{n+1}, F^{n+1}(t)\| \leq C, \ 0 \leq t \leq T \right\} \right\} \\
\geq 1 - \epsilon.
\]

Thus, the condition in (5.21) is satisfied by the sequence of \(\{\Xi^n\}\).
Similarly, for any $t \in [0, T]$, it follows from the proof of Proposition 18 for a BSDE with jumps in Dai [11] and Lemma [5.1] that

\begin{equation}
(5.32) \quad \left\| \langle U^n, \bar{V}^n, \bar{V}^n \rangle \right\|_{Q_t\gamma}^2 \\
\leq 2L^2(T-t) + K_\gamma \left\| (V^{n-1}, \bar{V}^{n-1}, \bar{V}^{n-1}) \right\|_{Q_t\gamma}^2 \\
\leq K_\gamma \left( 2L^2(T-t) + e^{2\gamma T} E \left[ \left\| V^{n-1} \right\|_{[t,T]}^2 \right] + \left\| (U^{n-1}, \bar{V}^{n-1}, \bar{V}^{n-1}) \right\|_{Q_t\gamma}^2 \right) \\
\leq K_\gamma \left( 2L^2(T-t) + 2\kappa^2 e^{2\gamma T} \left\| V(T) \right\|_{[t,T]}^2 \right) \\
+ K_\gamma (2\kappa^2 e^{2\gamma T} + 1) \left\| (U^{n-1}, \bar{V}^{n-1}, \bar{V}^{n-1}) \right\|_{Q_t\gamma}^2
\end{equation}

where $K_\gamma < 1$ depending only on $L, T, d$, and $h$ for some suitable chosen $\gamma > 0$. Thus, by Lemma [5.1] the Itô’s isometry formula, and (5.33), we have that

\begin{equation}
(5.33) \quad E \left[ \left\| V^n \right\|_{[t,T]}^2 \right] \\
\leq 2E \left[ \left\| V(T) \right\|_{[t,T]}^2 \right] + 2\kappa^2 E \left[ \text{Osc}(U^{n-1}, [t, T])^2 \right] \\
\leq 2E \left[ \left\| V(T) \right\|_{[t,T]}^2 \right] + 12\kappa^2 L^2(T-t)^2 \\
+ 12\kappa^2 L^2(T-t) \left( \int_t^T E \left[ \left\| X^{n-1} \right\|_{[0,s]}^2 \right] ds + E \left[ \left\| V^{n-1} \right\|_{[t,T]}^2 \right] \right) \\
+ 12\kappa^2 L^2(T-t) \left\| (U^{n-1}, \bar{V}^{n-1}, \bar{V}^{n-1}) \right\|_{Q_t\gamma}^2 \\
+ 2\kappa^2 \left\| (U^n, \bar{V}^n, \bar{V}^n) \right\|_{Q_t\gamma}^2
\end{equation}

Similarly, for any $t \in [0, T]$, we have that

\begin{equation}
(5.34) \quad E \left[ \left\| X^n \right\|_{[0,t]}^2 \right] \leq 2E \left[ \left\| X(0) \right\|_{[0,t]}^2 \right] + 2\kappa^2 E \left[ \text{Osc}(Z^{n-1}, [0, t])^2 \right] \\
\leq 2E \left[ \left\| X(0) \right\|_{[0,t]}^2 \right] + 6\kappa^2 L^2 t^2 \\
+ 6\kappa^2 L^2 t \left( \int_0^t E \left[ \left\| X^{n-1} \right\|_{[0,s]}^2 \right] ds + E \left[ \left\| V^{n-1} \right\|_{[0,T]}^2 \right] \right) \\
+ 6\kappa^2 L^2 t \left\| (U^{n-1}, \bar{V}^{n-1}, \bar{V}^{n-1}) \right\|_{Q_t[0,T]}^2
\end{equation}
\[ \begin{align*}
&\leq 2E \left[ \|X(0)\|^2 \right] + 12\kappa^2 L^2 t^2 + 6\kappa^2 L^2 t E \left[ \|V^2(T)\| \right] \\
&\quad + 6\kappa^2 L^2 t \int_0^t E \left[ \|X^{n-1}\|^2_{[0,s]} \right] ds \\
&\quad + 6\kappa^2 L^2 t \left( 1 + 2\kappa^2 \right) \left\| (U^{n-1}, \tilde{V}^{n-1}) \right\|_{Q_\gamma[0,T]}^2 .
\end{align*} \]

Therefore, for any \( \epsilon > 0 \) and a constant \( \delta > 0 \), consider a finite set \( \{t_i\} \) of points satisfying 
\[ 0 = t_0 < t_1 < \ldots < t_m = T \text{ and } t_{i-1} - t_i = \delta < \epsilon/L \text{ with } l \in \{1,\ldots,m\} \].
It follows from (5.19), (5.21), and the similar explanation for (5.27) that
\[ \begin{align*}
P \{ w(Z^n_{1}, \delta, T) \geq \epsilon \} &
\leq \frac{3L^2 \delta}{(\epsilon - L\delta)^2} \left( E \left[ \|X^n\|^2_{[0,T]} \right] + E \left[ \|V^n\|^2_{[0,T]} \right] + E \left[ \| (U^n, \tilde{V}^n) \|_{Q_\gamma[0,T]} \right] \right) \\
&\leq \frac{3L^2 \delta}{(\epsilon - L\delta)^2} \left( A_0 + \sum_{k=1}^n A_{1}^{k+1} T^{k+1} (k+1)! \left( 1 + K_\gamma^k \right) + A_2 \sum_{k=1}^n K_\gamma^k \right) ,
\end{align*} \]

where \( A_0, A_1, \) and \( A_2 \) are some constants depending only on \( L, T, d, \) and \( h \). Furthermore, by Lemma 4.2.8 in page 201 of Applebaum [[11] (or related theorem in page 20 of Gihman and Skorohod [19])] and the linear growth condition, we know that
\[ \begin{align*}
P \{ w(Z^n_{2}, \delta, T) \geq \epsilon \} &
\leq \frac{\bar{\epsilon}}{\epsilon^2} + \frac{3L^2}{\epsilon - 3L^2 \delta} \left( \delta E \left[ \|X^n\|^2_{T} \right] + \delta E \left[ \|V^n\|^2_{T} \right] + E \left[ \| (U^n, \tilde{V}^n) \|_{Q_\gamma[0,T]}^2 \right] \right) \\
&\leq \frac{\bar{\epsilon}}{\epsilon^2} + \frac{3L^2}{\epsilon - 3L^2 \delta} \left( \delta \left( A_0 + \sum_{k=1}^n B_{1}^{k+1} T^{k+1} (k+1)! \left( 1 + K_\gamma^k \right) + A_2 \sum_{k=1}^n K_\gamma^k \right) + A_3 \sum_{k=1}^n K_\gamma^k \right) ,
\end{align*} \]

for all nonnegative constant \( \bar{\epsilon} > 3L^2\delta \), where \( A_3 \) is some constant depending only on \( L, T, \) \( d, \) and \( h \).

Similarly, there are some constants \( B_0, B_1, B_2, \) and \( B_3 \) depending only on \( L, T, d, \) and \( h \) such that
\[ \begin{align*}
P \{ w(U^n_{1}, \delta, T) \geq \epsilon \} &
\leq \frac{3L^2 \delta}{(\epsilon - L\delta)^2} \left( B_0 + \sum_{k=1}^n B_{1}^{k+1} T^{k+1} (k+1)! \left( 1 + K_\gamma^k \right) + B_2 \sum_{k=1}^n K_\gamma^k \right) ,
\end{align*} \]

and
\[ \begin{align*}
P \{ w(Z^n_{2}, \delta, T) \geq \epsilon \} &
\leq \frac{\bar{\epsilon}}{\epsilon^2} + \frac{3L^2}{\epsilon - 3L^2 \delta} \left( \delta \left( B_0 + \sum_{k=1}^n B_{1}^{k+1} T^{k+1} (k+1)! \left( 1 + K_\gamma^k \right) + B_2 \sum_{k=1}^n K_\gamma^k \right) + B_3 \sum_{k=1}^n K_\gamma^k \right) .
\end{align*} \]

Hence, for each given \( \epsilon > 0 \), it follows from (5.35)-(5.38) and suitably chosen constants \( \bar{\epsilon}, \delta, \) and \( \gamma \) that
\[ \begin{align*}
\limsup_{n \to \infty} P \{ w(\Xi^n), \delta, T) \geq \epsilon \} &\leq \epsilon .
\end{align*} \]
Thus, the condition in (5.22) is true for the sequence of \( \{\Xi^n\} \). Hence, by (5.31), (5.39), and Corollary 7.4 in page 129 of Ethier and Kurtz \([13]\), this sequence is relatively compact. Therefore, there is a subsequence of \( \{\Xi^n\} \) that converges weakly to \( \Xi \equiv ((X, Z, Y), (V, \bar{V}, \bar{F}, F)) \) over the space \( \mathcal{P}[0, T] \). For convenience, we suppose that the subsequence is the sequence itself, i.e.,

(5.40) \[ \Xi^n \Rightarrow \Xi. \]

Then, by the Skorohod representation theorem (see, e.g., Theorem 1.8 in page 102 of Ethier and Kurtz\([13]\)), we can assume that the convergence in (5.40) is a.s. in the Skorohod topology. Thus, by the claim (a) in Theorem 1.14 (or the claim (a) in Proposition 2.1) of Jacod and Shiryayev \([21]\) and the facts that \( Y^{n+1}(0) = 0 \) and \( Y^{n+1} \) is nondecreasing, we can conclude that \( Y(0) = 0 \) and \( Y \) is nondecreasing. Furthermore, by Lemma 5.2 and (5.17)

\[ \int_0^t I_{D_i}(X(s))dY_i(s) = Y_i(t) \] for all \( t \geq 0, i \in \{1, ..., b\} \).

Similarly, we know that \( F(0) = 0, F \) is non-decreasing, and

\[ \int_0^t I_{D_i}(V(s))dF_i(s) = F_i(t) \] for all \( t \geq 0, i \in \{1, ..., \bar{b}\} \).

Therefore, by the Lipschitz condition in (3.3), we know that \( ((X, Y), (V, \bar{V}, \bar{F}, F)) \) satisfies the FB-SDEs in (1.2) a.s. Thus, by the Skorohod representation theorem again, it is a weak solution to the FB-SDEs in (1.2).

**Part A (Uniqueness).** Assume that \( ((X, Y), (V, \bar{V}, \bar{F}, F)) \) is a weak solution to the FB-SDEs in (1.2). To prove its uniqueness, we introduce some additional notations. Let \( D_0 = D, \bar{D}_0 = \bar{D} \), and define

(5.43) \[ D_K \equiv \cap_{i \in K} D_i, \quad \bar{D}_{\bar{K}} \equiv \cap_{i \in \bar{K}} \bar{D}_i \]

for each \( \emptyset \neq K \subset \{1, ..., b\} \) and each \( \emptyset \neq \bar{K} \subset \{1, ..., \bar{b}\} \). In the sequel, we call a set \( K \in \{1, ..., b\} \) “maximal” if \( K \neq \emptyset, D_K \neq \emptyset \), and \( D_K \neq \bar{D}_{\bar{K}} \) for any \( \bar{K} \supset K \) such that \( \bar{K} \neq K \). Similarly, we can define the maximal set corresponding to a set \( \bar{K} \in \{1, ..., \bar{b}\} \).

Furthermore, let \( d(x, D_K) \) and \( d(\bar{x}, \bar{D}_{\bar{K}}) \) respectively denote the Euclidean distance between \( x \) and \( D_K \) for a point \( x \in D \) and the Euclidean distance between a point \( \bar{x} \in \bar{D} \) and \( \bar{D}_{\bar{K}} \).

Then, it follows from Lemma 3.2 in Dai \([9]\) or Lemma B.1 in Dai and Williams \([8]\) that there exist two constants \( C \geq 1 \) and \( \bar{C} \geq 1 \) such that

(5.44) \[ d(x, D_K) \leq C \sum_{i \in K} (n_i \cdot x - b_i), \quad d(\bar{x}, \bar{D}_{\bar{K}}) \leq \bar{C} \sum_{i \in \bar{K}} (\bar{n}_i \cdot \bar{x} - \bar{b}_i) \]

Now, for each \( \epsilon \geq 0, K \in \{1, ..., b\} \), and \( \bar{K} \in \{1, ..., \bar{b}\} \) (including the empty set), we let

(5.45) \[ D_K^\epsilon \equiv \{ x \in R^q : 0 \leq n_i \cdot x - b_i \leq C_\epsilon \quad \text{for all} \quad i \in K, \]

\[ n_i \cdot x - b_i > \epsilon \quad \text{for all} \quad i \in \{1, ..., b\} \setminus K \}, \]

(5.46) \[ \bar{D}_{\bar{K}}^\epsilon \equiv \{ \bar{x} \in R^q : 0 \leq \bar{n}_i \cdot \bar{x} - \bar{b}_i \leq \bar{C}_\epsilon \quad \text{for all} \quad i \in \bar{K}, \]

\[ \bar{n}_i \cdot \bar{x} - \bar{b}_i > \epsilon \quad \text{for all} \quad i \in \{1, ..., \bar{b}\} \setminus \bar{K} \} \]
where \( C_\varepsilon = C p \varepsilon \) and \( \tilde{C}_\varepsilon = \tilde{C} q \varepsilon \). Thus, by Lemmas 4.1-4.2 in Dai and Williams [8], we know that

\[
D = \bigcup_{K \in \mathcal{G}} D'_K, \quad \tilde{D} = \bigcup_{\tilde{K} \in \tilde{\mathcal{G}}} \tilde{D}'_{\tilde{K}},
\]

where, \( \mathcal{G} \) is the collection of subsets of \( \{1, \ldots, b\} \) consisting of all maximal sets in \( \{1, \ldots, b\} \) and \( \tilde{\mathcal{G}} \) is defined in the same way in terms of subsets of \( \{1, \ldots, \tilde{b}\} \). For convenience, we order the sets in \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \). Then, we can define a sequence of 3-dimensional points \( \{(r_n, \tilde{r}_n, \tau_n), n \in \{1, 2, \ldots\}\} \) with \( \tau_0 = 0 \) by induction.

In fact, since \( ((X, Y), (V, \tilde{V}, \tilde{\tilde{V}}, F)) \) is a weak solution to the FB-SDEs in (1.2), both \( X(0) \) and \( V(0) \) are defined. Thus, if \( (r_1, \tilde{r}_1) \) is the first \( K \times \tilde{K} \in \mathcal{G} \times \tilde{\mathcal{G}} \) such that \((x, \tilde{x}) \in D_{r_1} \times D_{\tilde{r}_1}\), we let

\[
\tau_1 = \inf \left\{ t \geq 0 : (X(t), V(t)) \notin D'_{r_1} \times \tilde{D}'_{\tilde{r}_1} \right\}.
\]

Furthermore, if \( (r_n, \tilde{r}_n, \tau_n) \) has been defined on \( \{\tau_n < \infty\} \), we let \( (r_{n+1}, \tilde{r}_{n+1}) \) be the first \( K \times \tilde{K} \in \mathcal{G} \times \tilde{\mathcal{G}} \) such that \((X(\tau_n), V(\tau_n)) \in D'_{r_n} \times \tilde{D}'_{\tilde{r}_n} \). Then, we can define

\[
\tau_{n+1} = \inf \left\{ t \geq \tau_n : (X(t), V(t)) \notin D'_{r_{n+1}} \times \tilde{D}'_{\tilde{r}_{n+1}} \right\}.
\]

On \( \{\tau_n = +\infty\} \), we define \( r_{n+1} = r_n, \tilde{r}_{n+1} = \tilde{r}_n, \) and \( \tau_{n+1} = \tau_n \). Due to the right-continuity of the sample paths of solution \((X, V)\) by the related property of Lévy process driven stochastic integral (see, e.g., Theorem 4.2.12 in page 204 of Applebaum [1]), \( \{\tau_n\} \) is a nondecreasing sequence of \( \mathcal{F}_t \}-stopping times, satisfying \( \tau_n \to \infty \) a.s. as \( n \to \infty \).

Hence, it suffices to prove the weak uniqueness of \( ((X, Y), (V, \tilde{V}, \tilde{\tilde{V}}, F)) (\cdot \wedge \tau_n) \) for each \( n \). Note that both \( D'_{r_n} \) and \( \tilde{D}'_{\tilde{r}_n} \) for each \( n \) are subsets of cones. Thus, without loss of generality, we assume that both \( D \) and \( \tilde{D} \) are cones. Therefore, we can prove the weak uniqueness by induction in terms of the numbers of boundary faces of \( D \) and \( \tilde{D} \).

In fact, for the case that \( b = \tilde{b} = 1 \), it follows from the uniqueness of the Skorohod mapping given by Lemma 3.1 in Dai [9] or Lemma 4.5 in Dai and Dai [7] that the weak uniqueness is true. Now, we suppose that the weak uniqueness is true for the case that \( b + \tilde{b} = m \geq 2 \) with \( b \geq 1 \) and \( \tilde{b} \geq 1 \). Then, we can prove the case for \( b + \tilde{b} = m + 1 \). In this case, we need to consider two folds indexed by two pairs of \((b + 1, \tilde{b})\) and \((b, \tilde{b} + 1)\). Both of the folds can be proved by the similar discussion for Theorem 5.4 in Dai and Williams [8]. Therefore, we finish the proof of weak uniqueness.

**Part B.** We consider the case that \( L(t, \omega) \) appeared in (3.7)-(3.8) is a constant and the spectral radii of \( S \) and each \( p \times p \) sub-principal matrix of \( N'R \) are strictly less than one. In this case, we need to prove that there is a unique strong adapted solution \( ((X, Y), (V, \tilde{V}, \tilde{\tilde{V}}, F)) \) to the system of in (1.2).

In fact, it follows from the discussions in Reiman and Harrison [20], Dai [13], Lemma 7.1 and Theorem 7.2 in pages 164-165 of Chen and Yao [6] that there exist two Lipschitz
continuous mappings \( \Phi \) and \( \Psi \) such that
\[
(5.50) \quad (X^{n+1}, Y^{n+1}) = \Phi(Z^n)
\]
\[
(5.51) \quad (V^{n+1}, F^{n+1}) = \Psi(U^n)
\]
for each \( n \in \{1, 2, \ldots\} \). Then, it follows from (5.50)-(5.51), the related estimates in Part A, and the conventional Picard’s iterative method, we can reach a proof for the claim in Part B.

**Part C.** We consider the case that \( L(t, \omega) \) appeared in (3.7)-(3.8) is a constant and both of the SDEs have no reflection boundaries. In this case, we need to prove that there is a unique strong adapted solution \((X, Y, (V, \tilde{V}, \check{V}, F))\) to the system of in (1.2). In fact, by the related estimates in Part A, this case can be proved by directly generalizing the conventional Picard’s iterative method. Actually, this case is a special one of Theorem 2.1 or Theorem 2.2.

**Part D.** We consider the case that \( L(t, \omega) \) appeared in (3.7)-(3.8) is a general adapted and mean-squarely integrable stochastic process. The proofs corresponding to the cases stated in Part A, Part B, and Part C can be accomplished along the lines of proofs for Lemma 4.1 in Dai [11] associated with a forward SDE under random environment and Proposition 18 in Dai [15] for a backward SDE under random environment. The key in the proofs is to introduce the following sequence of \( \{F_t\} \)-stopping times, i.e.,
\[
(5.52) \quad \tau_n \equiv \inf\{t > 0, \|L(t)\| > n\} \quad \text{for each} \quad n \in \{1, 2, \ldots\}.
\]
By the condition in (3.9), \( \tau_n \) is nondecreasing and a.s. tends to infinity as \( n \to \infty \).

Finally, by summarizing the cases presented in Part A to Part D, we finish the proof of Theorem 3.2. \( \square \)

### 5.3 Proof of Theorem 3.3

**Proof of Part I**

For a control process \( u^* \in \mathcal{C} \), it follows from Theorem 2.1 that the \((r, q+1)\)-dimensional FB-SPDEs in (1.1) with the partial differential operators \( \{\bar{L}, \bar{J}, \bar{I}\} \) given by (3.11)-(3.13) and terminal condition in (3.14) indeed admits a well-posed 4-tuple solution \((U(t, x), V(t, x), \bar{V}(t, x), \check{V}(t, x, \cdot))\). Thus, substituting
\[
(V(t), \bar{V}(t), \check{V}(t, \cdot)) \equiv (V(t, X(t)), \bar{V}(t, X(t)), \check{V}(t, X(t), \cdot))
\]
into the system in (1.2), it follows from Theorem 3.2 that the claims in Part I are true.

**Proof of Part II**

The proof of Part II(1) is the direct extension of the single-dimensional case (i.e., \( p = q = 1 \)) for the related optimal control problem in Øksendal et al. [34].

The proof of Part II(2) can be done as follows. For each \( u \in \mathcal{C} \) and \( \gamma(t, x) = \beta(t, x) \equiv 0 \), it follows from Part I that the regulator processes \( F(t) \) and \( Y(t) \) exist. Since they are
nondecreasing with respect to time variable $t$, the derivatives $\frac{dF}{dt}(t, x)$ and $\frac{dY}{dt}(t, x)$ exist a.e. in terms of time variable $t$ along each sample path a.s. Furthermore, if each $q \times q$ sub-principal matrix of $\bar{N}'S$ and each $p \times p$ sub-principal matrix of $N'R$ are invertible, these derivatives are uniquely determined owing to the Skorohod mapping. Nevertheless, if only the general completely-$S$ condition is imposed, these derivatives are weakly unique in a probability distribution sense. In addition, it follows from Proposition 7.1 in Ethier and Kurtz [18] that these derivatives can be approximated by polynomials in terms of variable $x$ for each given $t$, which are denoted by $\gamma(t, x)$ and $\beta(t, x)$. Then, the proof for the claim in Part II(2) follows from the one for the claim in Part II(1).

Finally, owing to the proofs for Part I and Part II, we reach a proof for Theorem 3.3. □

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