THE GEOMETRY OF LOOP SPACES II: CHARACTERISTIC CLASSES

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Abstract. In this follow-up paper to [15], we prove that $|\pi_1(\text{Diff}(M))| = \infty$ for the total space $M$ of circle bundles associated to high multiples of a Kähler class over integral Kähler surfaces. To detect nontrivial elements of $\pi_1(\text{Diff}(M))$, we develop a theory of Chern-Simons classes $CS_{2k-1}^{LM}(\nabla^0, \nabla^1) \in H^{2k-1}(LM^{2k-1}; \mathbb{R})$ on the loop space $LM$ of a Riemannian manifold. These classes use the Wodzicki residue of the connection and curvature forms of the $L^2$ connection $\nabla_0$ and the Sobolev parameter $s = 1$ connection $\nabla^1$ on $LM$, as these forms take values in pseudodifferential operators.

Dedicated to the memory of Prof. Shoshichi Kobayashi

1. Introduction

The loop space $LM$ of a Riemannian manifold $(M, g)$ admits a family of Riemannian metrics $g^s$ depending on $g$ and a Sobolev space parameter $s \geq 0$. In [15], we calculated the Levi-Civita connections associated to $g^s$. In this paper, we develop a theory of characteristic classes on the tangent bundle $TLM$ associated to these metrics. It turns out that the Pontrjagin classes are trivial, but the associated Chern-Simons classes are nontrivial in $H^{2k-1}(LM^{2k-1}; \mathbb{R})$ in general. As the main application, we use a specific Chern-Simons class to prove that $|\pi_1(\text{Diff}(M))| = \infty$ for the total space $M$ of circle bundles associated to high multiples of a Kähler class over integral Kähler surfaces. These circle bundles include several topologically distinct infinite families of 5-manifolds, including $S^2 \times S^3$ and some lens spaces.

To develop a Chern-Weil theory for these connections, we need invariant polynomials on the Lie algebra of the structure group of the Levi-Civita connections, which by [15] is the group of invertible zeroth order pseudodifferential operators ($\Psi$DOs). The naive choice is the standard polynomials $\text{Tr}(\Omega^k)$ of the curvature $\Omega = \Omega^s$, where $\text{Tr}$ is the operator trace. However, $\Omega^k$ is zeroth order and hence not trace class, and in any case the operator trace is impossible to compute in general. Instead, as in [17] we use the Wodzicki residue, the only trace on the full algebra of $\Psi$DOs. Following Chern-Simons [5] as much as possible, we build a theory of Wodzicki-Chern-Simons (WCS) classes, which gives classes in $H^{2k-1}(LM^{2k-1}; \mathbb{R})$ associated to the Wodzicki-Chern character of $TLM$.

There are several differences from finite dimensional Chern-Simons theory. The absence of a Narasimhan-Ramanan universal connection theorem means that we do not have a theory of differential characters [4]. Moreover, the use of the Wodzicki
residue allows to define WCS classes associated to the Chern character but not to the Chern classes. On the positive side, since we have a family of connections on $LM$, we can define the relative $\mathbb{R}$-valued, not just the absolute $\mathbb{R}/\mathbb{Z}$-valued, WCS classes associated to a Riemannian metric on $M$. Finally, if $\dim(M) = 3$, the WCS form vanishes, a result without a finite dimensional analogue.

In contrast to the operator trace, the Wodzicki residue is locally computable, so we can write explicit expressions for the WCS classes. In particular, we can see how the WCS classes depend on the Sobolev parameter $s$, and hence define “regularized” or $s$-independent WCS classes. As the main application, we compute the integral of $CS^W_5 \in H^5(LM, \mathbb{R})$ over a specific 5-cycle for $M$ as above. This leads to the results on $\pi_1(\text{Diff}(M))$.

For related results on characteristic classes on infinite rank bundles with a group of $\Psi$DOs as structure group, see [11, 17]. For the important case of loop groups, see [7].

This paper is organized as follows. In §2, we review finite dimensional Chern-Weil and Chern-Simons theory, and use the Wodzicki residue to define residue Chern character classes (RCC) and residue or Wodzicki-Chern-Simons (WCS) classes (Definition 2.1) on $LM$. We prove the necessary vanishing of the RCC classes for mapping spaces (and in particular for $LM$) in Proposition 2.2. In Theorem 2.6, we give the explicit local expression for the WCS class $CS^W_{2k-1}(g) \in H^{2k-1}(LM^{2k-1})$. We then define the regularized or $s$-independent WCS class. In Theorem 2.7, we give the vanishing result for the WCS class in dimension 3.

In particular, the WCS class which is the analogue of the classical dimension three Chern-Simons class vanishes on loop spaces of 3-manifolds, so we look for nontrivial examples on 5-manifolds. In §3, we consider the total spaces $\overline{M}_p$ of circle bundles over integral Kähler surfaces associated to integer multiples $p\omega$ of an integral Kähler class $\omega$. These 5-manifolds admit a Sasakian structure, which makes the computation of $CS^W_5$ tractable, if unpleasant. The $S^1$-action given by rotation in the fibers of $\overline{M}_p$ gives both a 5-cycle $[a]$ in $LM_p$ and an element of $\pi_1(\text{Diff}(\overline{M}_p))$. We compute that $\int_{[a]} CS^W_5 \neq 0$ for $|p| \gg 0$, which implies $|\pi_1(\text{Diff}(\overline{M}_p))| = \infty$ (Thm. 3.10). In the case of specific Kähler surfaces such as $T^4$, $\mathbb{C}P^2$, $S^2 \times S^2$, and K3 surfaces, we can give sharp estimates on $p$. Finally, we give an explicit computer computation to check that $|\pi_1(\text{Diff}(S^2 \times S^3))| = \infty$.

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Notation: $H^*$ always refers to de Rham cohomology for complex valued forms. By [1], $H^*(LM) \simeq H^*_\text{sing}(LM, \mathbb{C})$. Our convention for the curvature tensor is $R(\partial_j, \partial_k)\partial_h = R_{jkb}^a \partial_a$. 

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2. Chern-Simons Classes on Loop Spaces

We review Chern-Weil and Chern-Simons theory for finite dimensional vector bundles \cite{19} Ch. 4).

2.1. Chern-Weil and Chern-Simons Theory for Finite Dimensional Bundles. Let $G$ be a finite dimensional linear Lie group with Lie algebra $\mathfrak{g}$, and let $F \to M$ be a vector bundle over a manifold $M$ with structure group $G$. Set $\mathfrak{g}^k = \mathfrak{g}^\otimes k$ and let

$$I^k(G) = \{ P : \mathfrak{g}^k \to \mathbb{C} \mid P \text{ symmetric, multilinear, Ad-invariant} \}$$

be the degree $k$ Ad-invariant polynomials on $\mathfrak{g}$.

**Remark 2.1.** For $G = U(n)$, resp. $O(n)$, $I^k(G)$ is generated by the polarization of the Newton polynomials $P_k(A) = \text{Tr}(A^k)$, resp. $\text{Tr}(A^{2k})$, where $\text{Tr}$ is the usual trace on finite dimensional matrices.

For $\phi \in \Lambda^k(M, \mathfrak{g}^k)$, $P \in I^k(G)$, set $P(\phi) = P \circ \phi \in \Lambda^k(M)$.

**Theorem 2.1** (The Chern-Weil Homomorphism). Let $\nabla$ be a connection on $F \to M$ with curvature $\Omega \in \Lambda^2(M, \mathfrak{g})$. For $P \in I^k(G)$, $P(\Omega)$ is a closed $2k$-form on $M$, and so determines a de Rham cohomology class $[P(\Omega)] \in H^{2k}(M)$. The Chern-Weil map

$$\oplus_k I^k(G) \to H^*(M), \ P \mapsto [P(\Omega)]$$

is a well-defined algebra homomorphism, and in particular is independent of the choice of connection on $F$.

$[P(\Omega)]$ is the characteristic class of $P$. For example, the characteristic class associated to $\frac{1}{k!} \text{Tr}(A^k)$ is the $k^{th}$ component of the Chern character of $F$.

Part of the theorem’s content is that for any two connections on $F$, $P(\Omega_1) - P(\Omega_0) = dCS_P(\nabla_1, \nabla_0)$ for some odd form $CS_P(\nabla_1, \nabla_0)$. Explicitly, for $\nabla_i = d + \omega_i, i = 1, 2$, locally,

$$CS_P(\nabla_1, \nabla_0) = k \int_0^1 P(\omega_1 - \omega_0, \Omega_t, ..., \Omega_t) \, dt \tag{2.1}$$

where

$$\omega_t = t\omega_0 + (1-t)\omega_1, \ \Omega_t = d\omega_t + \omega_t \wedge \omega_t$$

\cite{19}. Note that $\omega_1 - \omega_0, \Omega_t$ are globally defined $\mathfrak{g}$-values forms on $M$. $CS_P(\nabla_1, \nabla_0)$ is the transgression or Chern-Simons form associated to $P$, relative to $\nabla_1, \nabla_0$.

**Remark 2.2.** We relate this construction to the absolute version of this construction on principal $G$-bundles $\tilde{F} \xrightarrow{\pi} M$ in \cite{5}. $\pi^*F \to F$ is trivial, so it has a flat connection $\theta_1$ on $\pi^*F$ with respect to a fixed trivialization. Let $\theta_1$ also denote the connection $\chi^*\theta_1$ on $F$, where $\chi$ is the global section of $\pi^*F$. For any other connection $\theta = \theta_0$ on $F$, $\theta_t = t\theta_0, \Omega_t = t\Omega_0 + (t^2 - t)\theta_0 \wedge \theta_0$. Assume an invariant polynomial $P$
takes values in $\mathbb{R}$. Then we obtain the formulas for the transgression form $TP(\theta)$ on $F$: for
\[ \phi_t = t\Omega_1 + \frac{1}{2}(t^2 - t)[\theta, \theta], \quad TP(\theta) = k \int_0^t P(\theta \wedge \phi_t^{k-1})dt, \quad (2.2) \]
d$TP(\theta) = P(\Omega_1) \in \Lambda^{2k}(F)$ [5]. $TP(\Omega_1)$ pushes down to an $\mathbb{R}/\mathbb{Z}$-class on $M$, the absolute Chern-Simons class.

2.2. Chern-Weil and Chern-Simons Theory for $\Psi DO^*_0$-Bundles. Fix $s > 0$. Let $LM$ be the space of loops from $S^1 \to M$ in a fixed Sobolev class $s' \gg s$. This restriction makes $LM$ into a paracompact Hilbert manifold, which is technically easier to work with than the Fréchet manifold of smooth loops.

The Riemannian metric $g^s$ on $LM$ associated to the Riemannian metric $g$ on $M$ is defined at a loop $\gamma \in LM$ by
\[ g^s(X, Y)_{\gamma} = \frac{1}{2\pi} \int_{S^1} g(X_{\gamma}(\theta), (1 + \Delta)^s Y_{\gamma}(\theta))d\theta, \]
where $\theta \mapsto X_{\gamma}(\theta), Y_{\gamma}(\theta) \in T_{\gamma}(\theta)M$ are elements of $T_sLM = \Gamma^{s-1}(g^s TM)$ (sections of Sobolev class $s - 1$), $\Delta = (\frac{D^s}{d\theta})^s (\frac{D^s}{d\theta})$ is the Laplacian built from covariant differentiation along $\gamma$, and $(1 + \Delta)^s$ is the associated pseudodifferential operator ($\Psi DO$). We call $g^s$ the Sobolev $s$-metric. The connection one-form and curvature two-form of the associated Levi-Civita connection $\nabla_s$ take values in the algebra $\Psi DO_{\leq 0}$ of nonpositive order $\Psi DO$s acting on sections of the trivial $\mathbb{R}^n$ bundle $\mathcal{R}^n$ over $S^1$, where $n = \dim M$. For $s = 0$, we have the usual $L^2$ metric and connection. See [13] §2 for details.

Even though the structure group of $TLM$ is the gauge group $\mathcal{G}$ of $\mathcal{R}^n$, the connection and curvature forms take values in an algebra larger than the Lie algebra $\text{Lie}(\mathcal{G})$. Thus we should extend the gauge group to the larger group $\Psi DO^*_0(\mathcal{R}^n)$ of zeroth order invertible $\Psi DO$s acting on sections of $\mathcal{R}^n$, since $\text{Lie}(\Psi DO^*_0) = \Psi DO^*_0$.

In general, let $\mathcal{E} \to \mathcal{M}$ be an infinite rank bundle over a paracompact Banach manifold $\mathcal{M}$, with the fiber of $\mathcal{E}$ modeled on a fixed Sobolev class of sections of a finite rank complex bundle $E \to N$, and with structure group $\Psi DO^*_0(E)$. For such $\Psi DO^*_0$-bundles, we can produce primary and secondary characteristic classes once we choose Ad-invariant polynomials of $\Psi DO_{\leq 0}$.

Since the adjoint action of $\Psi DO^*_0$ on $\Psi DO_{\leq 0}$ is by conjugation, any trace $T$ on $\Psi DO_{\leq 0}$ will produce invariant polynomials $A \mapsto T(A^k)$, just as for $u(n)$. These traces were classified in [13, 14], although there are slight variants in our special case $N = S^1$ [18]. Roughly speaking, the traces fall into two classes, the leading order symbol trace [17] and the Wodzicki residue. In this paper, we consider only the Wodzicki residue, and refer to [10, 11] for the leading order symbol trace.

Recall that a classical $\Psi DO\ P$ acting on sections of $E \to N$ has an order $\alpha \in \mathbb{R}$ and a symbol expansion $\sigma^P(x, \xi) \sim \sum_{k=0}^{\infty} \sigma^P_{\alpha-k}(x, \xi)$, where $x \in N, \xi \in T^*_x N$, and $\sigma^P_{\alpha-k}(x, \xi)$ is homogeneous of degree $\alpha - k$ in $\xi$. For $(x, \xi)$ fixed, $\sigma^P(x, \xi), \sigma^P_{\alpha-k}(x, \xi) \in \Psi DO^*_0(N)$.
End(\mathcal{E}_\gamma). We note that real \( \Psi DOs \) on complexified bundles will have symbols which are real endomorphisms.

The Wodzicki residue of \( P \) is

\[
\text{res}^W(P) = \frac{1}{(2\pi i)^n} \int_{S^*N} \text{tr} \sigma_n(x, \xi) d\xi\ dx,
\]

where \( S^*N \) is the unit cosphere bundle over \( N \) with respect to a fixed Riemannian metric, and \( \dim(N) = n \). It is nontrivial that \( \text{res}^W \) is independent of coordinates and defines a trace: \( \text{res}^W[P, Q] = 0 \).

We will restrict attention to the generating invariant polynomials \( P_k(A_1, \ldots, A_k) \), the symmetrization of \( \frac{1}{k!} \text{Tr}(A_1 \cdots A_k) \), whose corresponding characteristic class \( \frac{1}{k!}[\text{Tr}(\Omega^k)] \) is the degree \( 2k \) component of the Chern character. We only consider \( \mathcal{E} = TLM \), which here denotes the complexified tangent bundle. Thus the corresponding Chern classes are really Pontrjagin classes. For \( TLM, E \longrightarrow N \) is the trivial complex bundle \( S^1 \times \mathbb{C}^n \longrightarrow S^1 \).

**Definition 2.1.**

(i) The \( k^{th} \) residue Chern character (RCC) form of a \( \Psi DO^{*}_0 \)-connection \( \nabla \) on \( TLM \) with curvature \( \Omega \) is

\[
\text{ch}^W_{[2k]}(\gamma) = \int_{S^*S^1} \text{tr} \sigma_{-1}(\Omega^k)\ d\xi\ dx. \tag{2.3}
\]

As above, for each \( \gamma \in LM, \sigma_{-1}(\Omega^k) \) is a \( 2k \)-form with values in endomorphisms of a trivial bundle over \( S^*S^1 \), so \( \text{ch}^W_{[2k]} \in \Lambda^{2k}(LM, \mathbb{C}) \). For convenience, we omit the usual \( \frac{1}{k!} \) factor from the RCC form, and the usual constant in the Wodzicki residue.

(ii) The \( k^{th} \) Wodzicki-Chern-Simons (WCS) form of two \( \Psi DO^{*}_0 \)-connections \( \nabla_0, \nabla_1 \) on \( TLM \) is

\[
CS^W_{2k-1}(\nabla_1, \nabla_0) = k \int_0^1 dt \int_{S^*S^1} \text{tr} \sigma_{-1}((\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1})\ d\xi\ dx \tag{2.4}
\]

\[
= k \int_0^1 \text{res}^W[(\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}]\ dt \in \Lambda^{2k-1}(LM, \mathbb{C}).
\]

(iii) The \( k^{th} \) Wodzicki-Chern-Simons form associated to a Riemannian metric \( g \) on \( M \), denoted \( CS^W_{2k-1}(g) \), is \( CS^W_{2k-1}(\nabla_1, \nabla_0) \in \Lambda^{2k-1}(LM, \mathbb{R}) \), where \( \nabla_0, \nabla_1 \) refer to the \( L^2 \) and Sobolev \( s = 1 \) Levi-Civita connections on \( LM \), respectively.

(iv) Let \( P = \sum a^K P_K \) be a polynomial in the \( U(n) \)-invariant polynomials \( P_k : A \mapsto \frac{1}{k!} \text{Tr}(A^k) \), with \( K = (k_1, \ldots, k_r) \) and \( P_K = P_{k_1} \cdots P_{k_r} \). Define the residue characteristic form by

\[
c^W_P(\Omega) = \sum a^K c^W_{k_1}(\Omega) \wedge \ldots \wedge c^W_{k_r}(\Omega).
\]

As in finite dimensions, \( c^W_P(\Omega) \) is a closed \( 2k \)-form, with de Rham cohomology class \( c_P(LM) \) independent of \( \nabla \), as

\[
c^W_P(\Omega_1) - c^W_P(\Omega_0) = dCS^W_P(\nabla_1, \nabla_0).
\]
Here $CS_P^W$ is defined as in (2.1), with $P$ replaced with $\text{res}^W$. In particular, in our notation

$$ch^W_{[2k]}(\Omega_1) - ch^W_{[2k]}(\Omega_0) = dCS^W_{2k-1}(\nabla_1, \nabla_0).$$

**Remark 2.3.** It is an interesting question to determine all $\Psi\text{DO}_0^*$-invariant polynomials on $\Psi\text{DO}_{\leq 0}$. As above, $U(n)$-invariant polynomials combine with the Wodzicki residue (or the other traces on $\Psi\text{DO}_{\leq 0}$) to give $\Psi\text{DO}_0^*$-polynomials, but there may be others.

We now prove that $TLM$ and more generally the tangent bundle to mapping spaces $\text{Maps}(N, M)$, with $N$ a closed manifold, have vanishing residue Chern classes. As above, we take a Sobolev topology on $\text{Maps}(N, M)$ for some large Sobolev parameter.

We denote the de Rham class of $c^W_P(\Omega)$ for a connection on $E$ by $c^W_P(E)$, and for the special case $E = T\text{Maps}(N, M) \otimes \mathbb{C}$ by $c^W_P(\text{Maps}(N, M))$.

**Proposition 2.2.** Let $N, M$ be closed manifolds, and let $\text{Maps}_f(N, M)$ denote the component of a fixed $f : N \rightarrow M$. Then the residue characteristic classes $c^W_P(\text{Maps}_f(N, M))$ vanish.

**Proof.** For $TLM$, the $L^2$ connection has curvature $\Omega$ which is a multiplication operator/pointwise endomorphism [15, Lemma 2.1]. Thus $\sigma^{-1}(\Omega)$ and hence $\sigma^{-1}(\Omega^i)$ are zero, so the residue characteristic forms $c_P(\Omega)$ vanish.

For $n \in N$ and $h : N \rightarrow M$, let $\text{ev}_n : \text{Maps}_f(N, M) \rightarrow M$ be $\text{ev}_n(h) = h(n)$. Then $D_XY(h)(n) \overset{\text{def}}{=} (\text{ev}_n^* \nabla^{LC,M})_X Y(h)(n)$ is the $L^2$ Levi-Civita connection on $\text{Maps}(N, M)$. As in [15, Lemma 2.1], the curvature of $D$ is a a multiplication operator. Details are left to the reader. □

In finite dimensions, characteristic classes are topological obstructions to the reduction of the structure group, and geometric obstructions to the existence of a flat connection. RCC classes for $\Psi\text{DO}_0^*$-bundles are also topological and geometric obstructions, but the geometric information is a little more refined due to the grading on the Lie algebra $\Psi\text{DO}_{\leq 0}$.

**Proposition 2.3.** Let $\mathcal{E} \rightarrow B$ be a $\Psi\text{DO}_0^*$-bundle, for $\Psi\text{DO}_0^*$ acting on $E \rightarrow N^n$. If $\mathcal{E}$ admits a reduction to the gauge group $\mathcal{G}(E)$, then $c^W_P(\mathcal{E}) = 0$ for all $P$. If $\mathcal{E}$ admits a $\Psi\text{DO}_0^*$-connection whose curvature has order $-k$, then $c_\ell(\mathcal{E}) = 0$ for $\ell \geq \lfloor n/k \rfloor$.

**Proof.** If the structure group of $\mathcal{E}$ reduces to the gauge group, there exists a connection one-form with values in $\text{Lie}(\mathcal{G}) = \text{End}(E)$, the Lie algebra of multiplication operators. Thus the Wodzicki residue of powers of the curvature vanishes, so the residue Chern character classes vanish. For the second statement, the order of $\Omega^\ell$ is less than $-n$ for $\ell \geq \lfloor n/k \rfloor$, so the Wodzicki residue vanishes in this range. □

However, we do not have examples of nontrivial RCC classes; cf. [11], where it is conjectured that these classes always vanish.
2.3. Local calculations. We now use local symbol calculations to compute WCS forms.

Notation and Conventions: For curvature conventions for $M$, we set
\[ \Omega^M(\partial_k, \partial_j)^a_b = R_{kjba} = R(\partial_k, \partial_j)^a_b, \quad R(\partial_k, \partial_j, \partial_b, \partial_a) = \langle R(\partial_k, \partial_j)\partial_b, \partial_a \rangle = R_{kjba}, \]
in agreement with [15]. Our convention for wedge product is $\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!} \text{Alt}(\omega \otimes \eta)$ for $\omega \in \Lambda^k, \eta \in \Lambda^\ell$.

For completeness, we restate some results from [15, Appendix A] about the symbols of the connections $\nabla_0, \nabla_1$ on $TLM$.

**Lemma 2.4.** Fix a metric $g$ on $M$ with connection form $\omega^M$ and curvature tensor $R = R^M$. Let $\omega^0, \Omega^0$ denote the connection and curvature forms for the $L^2$ ($s = 0$) metric on $LM$, let $\omega^1, \Omega^1$ denote the connection and curvature forms for the $s = 1$ metric, and let $\omega^M$ be the connection one-form for the Levi-Civita connection on $M$. Fix a loop $\gamma \in LM$. At $(\theta, \xi) \in T^*S^1$,

(i) $(\omega^M_X)_b = (\omega^M_X)_b^0$, where $\omega^M_X$ is computed at $\gamma(\theta)$. $\sigma_0(\Omega^0(X,Y))_b = R(X,Y)_b = R_{cb}aXY^d$. $\omega^0, \Omega^0$ are multiplication operators, so $\sigma_i(\omega^0) = \sigma_i(\Omega^0) = 0$ for $i < 0$.

(ii) $\sigma_0(\omega_X^1)_b = (\omega_X^M)_b^0$.

\[
\frac{1}{i|\xi|^{-2\xi}}\sigma_{-1}(\omega^1_X) = \frac{1}{2}(-2R(X, \dot{\gamma}) - R(\cdot, \dot{\gamma})X + R(X, \cdot)\dot{\gamma}).
\]

With this Lemma, the WCS forms become tractable.

**Proposition 2.5.** Let $\sigma$ be a permutation of $\{1, \ldots, 2k - 1\}$. Then
\[
CS_{2k-1}^W(g)(X_1, \ldots, X_{2k-1}) = k \int_{S^1} \text{sgn}(\sigma) \int_{S^1} \text{tr}[(\omega_{\sigma(1)} - \omega_0)(\Omega^M)^{k-1}(X_{\sigma(2)}\ldots X_{\sigma(2k-1)})].
\]

**Proof.** By Lemma 2.4(i), $\sigma_0((\omega_1 - \omega_0)_X) = 0$. Thus
\[
CS_{2k-1}^W(g) = k \int_0^1 \text{dt} \int_{S^1} \text{tr}[\sigma_{-1}(\omega_1 - \omega_0) \wedge (\sigma_0(\Omega^M))^{k-1}] d\xi dx.
\]

Moreover,
\[
\sigma_0(\Omega^i) = td(\sigma_0(\omega_0)) + (1 - t)d(\sigma_0(\omega_1)) + (t\sigma_0(\omega_0) + (1 - t)\sigma_0(\omega_1)) \wedge (t\sigma_0(\omega_0) + (1 - t)\sigma_0(\omega_1))
\]
\[
= dw^M + \omega^M \wedge \omega^M = \Omega^M.
\]

Therefore,
\[
CS_{2k-1}^W(g) = k \int_0^1 \text{dt} \int_{S^1} \text{tr}[\sigma_{-1}(\omega_1) \wedge (\Omega^M)^{k-1}] d\xi dx,
\]

(2.6)
since \(\sigma^{-1}(\omega_0) = 0\). We can drop the integral over \(t\). The integral over the \(\xi\) variable contributes a factor of 2: the integrand has a factor of \(|\xi|^{-2}\xi\), which equals \(\pm 1\) on the two components of \(S^*S^1\). Since the fiber of \(S^*S^1\) at a fixed \(\theta\) consists of two points with opposite orientation, the “integral” over each fiber is \(1 - (-1) = 2\). Thus

\[
CS_{2k-1}^W(g)(X_1, \ldots, X_{2k-1}) = \frac{2k}{2^{k-1}} \cdot \frac{1}{2} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr}([-2R(X_{\sigma(1)}, \gamma) - R(\gamma, \gamma)X_{\sigma(1)} + R(X_{\sigma(1)}, \gamma)\gamma]) \cdot (\Omega^M)^{k-1}(X_{\sigma(2)}, \ldots, X_{\sigma(2k-1)})]
\]

by Lemma 2.4(ii).

This produces odd classes in the de Rham cohomology of the loop space of an odd dimensional manifold.

**Theorem 2.6.** (i) Let \(\dim(M) = 2k - 1\). Then \(ch_{2k}^W(\Omega) \equiv 0\) for any \(\Psi\)DO\(_0\)-connection \(\nabla\) on TLM. Thus the \(k\)th Wodzicki-Chern-Simons form \(CS_{2k-1}^W(\nabla_1, \nabla_0)\) is closed and defines a class \([CS_{2k-1}^W(\nabla_1, \nabla_0)] \in H^{2k-1}(LM)\). In particular, we can define \([CS_{2k-1}^W(g)] \in H^{2k-1}(LM)\) for a Riemannian metric \(g\) on \(M\).

(ii) For dim(\(M\)) = \(2k - 1\), \(CS_{2k-1}^W(g)\) simplifies to

\[
CS_{2k-1}^W(g)(X_1, \ldots, X_{2k-1}) = \frac{k}{2^{k-2}} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr}((R(X_{\sigma(1)}, \gamma))\gamma)(\Omega^M)^{k-1}(X_{\sigma(2)}, \ldots, X_{\sigma(2k-1)})]
\]

**Proof.** (i) Let \(\Omega\) be the curvature of \(\nabla\). \(ch_{2k}^W(\Omega)(X_1, \ldots, X_{2k})(\gamma)\) is an integral of the pointwise expression which is a 2\(k\)-form on \(M\), and hence vanishes.

(ii) Denote the three terms on the right hand side of the second line of (2.5) by (I), (II), (III), resp. Since

\[
\text{tr}[R(X_1, \gamma) \cdot (\Omega^M)^{k-1}(X_2, \ldots, X_{2k-1})] = [i_\gamma \text{tr}(\Omega^k)](X_1, \ldots, X_{2k-1})
\]

(2.10)

(I) vanishes on a \((2k - 1)\)-manifold. By the Bianchi identity, (II) equals

\[
[R(\gamma, X_{\sigma(1)}) \cdot + R(X_{\sigma(1)}, \gamma)\gamma](\Omega^M)^{k-1}(X_{\sigma(2)}, \ldots, X_{\sigma(2k-1)}).
\]

The first term is of type (I), so its contribution vanishes, and the second term equals (III). Thus the right hand side of (2.5) simplifies to 2(III).

**Remark 2.4.** (i) The argument in Thm. 2.6(ii) fails for the general invariant polynomials in Defn. 2.1(iv). For expressions like \(c_{k_1}^W(\Omega) \wedge \ldots \wedge c_{k_r}(\Omega)(X_1, \ldots, X_{2k})\), \(r > 1\), are not the integral of a 2\(k\)-form around a loop, but are products of integrals of lower degree forms around this loop. In particular, we do not construct residue Chern
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(ii) There are several variants to the construction of relative WCS classes.

(a) If we define the transgression form \( T_{c_k}(\nabla) \) with the Wodzicki residue replacing the trace in (2.2), it is easy to check that \( T_{c_k}(\nabla) \) involves \( \sigma_{-1}(\Omega) \). For \( \nabla \) the \( L^2 \) connection, this WCS class vanishes. For \( \nabla \) the Levi-Civita connection on \( LM \) for the Sobolev \( s \)-metric, \( s > 1/2 \), \( \sigma_{-1}(\Omega) \) involves the covariant derivative of the curvature of \( M \) (cf. \([15, \text{Lemma A.2}]\) for \( s = 1 \)) Thus the relative WCS class is easier for computations than the absolute class \( T_{c_k}(\nabla) \).

(b) If we define \( CS^W_k(g) \) using the Levi-Civita connection for the Sobolev \( s \)-metric instead of the \( s = 1 \) metric, the WCS class is simply multiplied by the artificial parameter \( s \) by \([15, \text{Lemma A.3}]\). Therefore setting \( s = 1 \) is not only computationally convenient, it regularizes the WCS, in that it extracts the \( s \)-independent information. This justifies the following definition:

**Definition 2.2.** The *regularized\( k \)-th WCS class* associated to a Riemannian metric \( g \) on \( M \) is \( CS^{W, reg}_k(g) = CS^W_k(\nabla^1, \nabla^0) \), where \( \nabla^1 \) is the Levi-Civita connection for the Sobolev \( s = 1 \) metric, and \( \nabla^0 \) is the \( L^2 \) Levi-Civita connection.

We conclude this section with a vanishing result that has no finite dimensional analogue.

**Proposition 2.7.** The WCS form \( CS^W_3(g) \) vanishes.

**Proof.** Fix a loop \( \gamma \) and a parameter \( \theta \), and let \( \{ e_r \} \) be an orthonormal frame at \( \gamma(\theta) \). Since the integrand at \( \theta \) vanishes if \( \dot{\gamma}(\theta) = 0 \), we can assume that \( \dot{\gamma}(\theta) \) is a multiple of \( e_1 \). For simplicity, we assume \( \dot{\gamma}(\theta) = e_1 \).

We may assume that \( X_r = e_r \) in (2.9). A typical term in (2.9) is

\[
I_\sigma = \text{sgn}(\sigma) R(e_{\sigma(1)}, e_{r_1}, e_{r_2}, e_{r_3}) R(e_{\sigma(2)}, e_{\sigma(3)}, e_{r_2}, e_{r_1}) R(e_{\sigma(4)}, e_{\sigma(5)}, e_{r_3}, e_{r_2}) \\
\quad \quad \quad \quad \quad \quad \cdots R(e_{\sigma(2k-2)}, e_{\sigma(2k-1)}, e_{r_k}, e_{r_{k-1}})
\]

\[
= \text{sgn}(\sigma) R_{\sigma(1)r_1r_2r_3} R_{\sigma(2)r_2r_1r_3} R_{\sigma(4)r_3r_2r_1} \cdots R_{\sigma(2k-2)r_kr_{k-1}}
\]

For \( k = 2 \), we have

\[
\frac{1}{2} CS^W_3(g)(e_1, e_2, e_3) = \sum_\sigma I_\sigma = \sum_\sigma \text{sgn}(\sigma) R_{\sigma(1)s1t} R_{\sigma(2)s3t} \\
= R_{1s1t} R_{23ts} - R_{2s1t} R_{13ts} - R_{3s1t} R_{21ts} \\
- R_{1s1t} R_{32ts} + R_{2s1t} R_{31ts} + R_{3s1t} R_{12ts}
\]

\[
= 2[R_{1s1t} R_{23ts} + R_{2s1t} R_{31ts} + R_{3s1t} R_{12ts}]
\]

\[
= 2[(a) + (b) + (c)].
\]
Then

\[(a) \quad R_{1213}R_{2332} + R_{1312}R_{2323} = R_{1213}R_{2332} - R_{1213}R_{2323} = 0,\]
\[(b) \quad R_{2112}R_{3121} + R_{2113}R_{3131} + R_{2312}R_{3123} = (\alpha_1) + (\beta_1) + (\gamma_1),\]
\[(c) \quad R_{3112}R_{1221} + R_{3113}R_{1231} + R_{3213}R_{1232} = (\alpha_2) + (\beta_2) + (\gamma_2).\]

Since

\[(\alpha_1) + (\alpha_2) = R_{2112}R_{3121} + R_{3112}R_{1221} = R_{2112}R_{3121} - R_{2112}R_{3121} = 0,\]
\[(\beta_1) + (\beta_2) = R_{3113}R_{3131} + R_{3113}R_{1231} = R_{3113}R_{3131} - R_{3113}R_{3131} = 0,\]
\[(\gamma_1) + (\gamma_2) = R_{3212}R_{3212} + R_{3213}R_{1232} = R_{3212}R_{3212} - R_{3212}R_{3212} = 0,\]

we obtain \(CS^W_3(g) = 0\).

\[\square\]

**Remark 2.5.** Theorem [2.7] highlights the difference between finite dimensional CS classes on \(M\) and WCS classes on \(LM\). Let \(\dim(M) = 3\). The only invariant monomials of degree two involving the ordinary trace are \(\text{tr}(A_1A_2)\) and \(\text{tr}(A_1)\text{tr}(A_2)\) (corresponding to \(ch_2\) and \(c_1^2\), respectively).

\(\text{tr}(A_1A_2)\) gives rise to the classical Chern-Simons invariant for \(M\). The Chern-Simons class associated to \(\text{tr}(A_1)\text{tr}(A_2)\) is trivial: this form involves \(\text{tr}(\omega_1 - \omega_0)\text{tr}(\Omega_t)\), which vanishes since \(\omega_1 - \omega_0, \Omega_t\) take values in skew-symmetric endomorphisms.

On \(LM\), we know that the WCS class \(CS^W_3\) associated to \(\text{tr}(A_1A_2)\) vanishes. The WCS form associated to \(\text{tr}(A_1)\text{tr}(A_2)\) involves \(\text{tr}\sigma_{-1}(\omega_1 - \omega_0) = \text{tr}\sigma_{-1}(\omega_1)\) and \(\text{tr}\sigma_{-1}(\Omega_t)\). Both \(\omega_1\) and \(\Omega_t\) take values in skew-symmetric \(\Psi\)DOs, but it does not follow that these lower order terms in their symbol expansions are skew-symmetric. In fact, a calculation using [15, Lemma A.1] shows that \(\sigma_{-1}(\omega_1)\) is not skew-symmetric. Thus the WCS form associated to \(\text{tr}(A_1)\text{tr}(A_2)\) may be nonzero. However, by Remark 2.3(i), this WCS form may not be closed.

3. **WCS Classes and Diffeomorphism Groups of Sasakian 5-Manifolds**

In this section we produce several infinite families of nonhomeomorphic 5-manifolds \(\overline{M}\) with \(\pi_1(\text{Diff}(\overline{M})) = \pi_1(\text{Diff}(\overline{M}), \text{Id})\) infinite. These manifolds are the total space of circle bundles over integral Kähler surfaces. To give some context, although there are many results about diffeomorphism groups of manifolds of dimension less than four and a few results in dimension four, the best results in higher dimensions work only in the stable range. In particular, \(\pi_1(\text{Diff}(S^5))\) is not well understood; we will see that this is a surprisingly difficult case from our viewpoint.

The fundamental observation is that a circle action \(a : S^1 \times \overline{M} \rightarrow \overline{M}\) on a closed, oriented manifold \(\overline{M}\) can be thought of both as a map \(a^P : S^1 \rightarrow \text{Diff}(\overline{M})\) and as a map \(a_L : M \rightarrow \text{Maps}(S^1, \overline{M}) = LM\). The first interpretation gives an element \([a^P] \in \pi_1(\text{Diff}(\overline{M}))\), and the second gives an element \([a^L] = a^L_t[\overline{M}] \in H_*((LM))\). As we will prove, the nontriviality of \([a^P]\) is guaranteed if \(\int_{[a^L]} CS^W_5 \neq 0\).
The 5-manifolds we consider come with circle actions given by rotating the fibers. The calculation of \( \int_{[a_L]} C_{5}^{W} \) is feasible only for special metrics. Fortunately, circle bundles over Kähler manifolds come with Sasakian structures, and in particular with specific metrics related to the Kähler form. For these metrics, we can estimate the integral and often show it is nonzero. The main result (Thm. 3.10) is that for an integral Kähler surface (integral and often show it is nonzero. The main result (Thm. 3.10) is that for an integral Kähler surface \( (M, \omega) \), \( \pi_1(\text{Diff}(\overline{M}_k)) \) is infinite for \( k \in \mathbb{Z}, |k| \gg 0 \), where \( \overline{M}_k \) is the circle bundle associated to \( k\omega \).

In §3.1 we discuss the relationship between \( \pi_1(\text{Diff}(\overline{M})) \) and \( L\overline{M} \). The main result is proven in §3.2, and examples are given in §3.3.

**Notation:** In this section, \( \pi_1(\text{Diff}(N)) = \pi_1(\text{Diff}(N), \text{Id}) \) for a manifold \( N \).

### 3.1. Circle actions.

Recall that \( H^*(LM) \) denotes de Rham cohomology of complex valued forms. In particular, integration of closed forms over homology cycles gives a pairing of \( H^*(LM) \) and \( H_*(LM, \mathbb{C}) \).

For a closed, oriented manifold \( M \), let \( a_0, a_1 : S^1 \times M \to M \) be two smooth actions.

**Definition 3.1.** (i) \( a_0 \) and \( a_1 \) are smoothly homotopic if there exists a smooth map

\[
F : [0, 1] \times S^1 \times M \to M, \quad F(0, \theta, m) = a_0(\theta, m), \quad F(1, \theta, m) = a_1(\theta, m).
\]

(ii) \( a_0 \) and \( a_1 \) are smoothly homotopic through actions if \( F(t, \cdot, \cdot) : S^1 \times M \to M \)

is an action for all \( t \).

An action \( a \) can be rewritten in two equivalent ways.

- \( a \) determines (and is determined by) \( a^D : S^1 \to \text{Diff}(M) \) given by \( a^D(\theta)(m) = a(\theta, m) \). Since \( a^D(0) = \text{Id} \), we get a class \( [a^D] \in \pi_1(\text{Diff}(M), \text{Id}) \). Here \( \text{Diff}(M) \) is a Banach manifold as an open subset of the Banach manifold of Maps(\( M \to M \) of some fixed high Sobolev class.

- \( a \) determines (and is determined by) \( a^L : M \to LM \) given by \( a^L(m)(\theta) = a(\theta, m) \). This determines a class \( [a^L] = a^L_+ [M] \in H_n(LM, \mathbb{Z}) \) with \( n = \text{dim}(M) \).

We give a series of elementary lemmas comparing these maps.

**Lemma 3.1.** (i) \( a_0 \) is smoothly homotopic to \( a_1 \) through actions implies \( [a_0^D] = [a_1^D] \in \pi_1(\text{Diff}(M)) \).

(ii) \( [a_0^D] = [a_1^D] \in \pi_1(\text{Diff}(M)) \) implies \( a_0 \) is smoothly homotopic to \( a_1 \).

**Proof.** (i) Given \( F \) as above, set \( G : [0, 1] \times S^1 \to \text{Diff}(M) \) by \( G(t, \theta)(m) = F(t, \theta, m) \). We have \( G(0, \theta)(m) = a_0(\theta, m) = a^D(\theta)(m), \quad G(1, \theta)(m) = a_1(\theta, m) = a^D_1(\theta)(m) \).

\( G(t, \theta) \in \text{Diff}(M) \), as

\[
G(t, -\theta)(G(t, \theta)(m)) = F(t, -\theta, F(t, \theta, m)) = F(t, 0, m) = m,
\]
since $F(t, \cdot, \cdot)$ is an action. Since $F$ is smooth, $G$ is a continuous (in fact smooth) map of $\text{Diff}(M)$. Thus $a_0^D, a_1^D$ are homotopic as elements of $\text{Maps}(S^1, \text{Diff}(M))$, so $[a_0^D] = [a_1^D]$.

(ii) Let $G : [0, 1] \times S^1 \to \text{Diff}(M)$ be a continuous homotopy from $a_0^D(\theta) = G(0, \theta)$ to $a_1^D(\theta) = G(1, \theta)$ with $G(t, 1) = \text{Id}$ for all $t$. It is possible to approximate $G$ arbitrarily well by a smooth map, also called $G$, since $[0, 1] \times S^1$ is compact. Set $F : [0, 1] \times S^1 \times M \to M$ by $F(t, \theta, m) = G(t, \theta)(m)$. $F$ is smooth. Then $F(0, \theta, m) = G(0, \theta)(m) = a_0^D(\theta)(m) = a_0(\theta, m)$, and $F(1, \theta, m) = a_1(\theta, m)$. Thus $a_0$ and $a_1$ are smoothly homotopic.

**Lemma 3.2.** $a_0$ is smoothly homotopic to $a_1$ iff $a_0^L, a_1^L : M \to LM$ are smoothly homotopic.

**Proof.** Let $F$ be the homotopy from $a_0$ to $a_1$. Set $H : [0, 1] \times M \to LM$ by $H(t, m)(\theta) = F(t, \theta, m)$. Then $H(0, m)(\theta) = F(0, \theta, m) = a_0(\theta, m) = a_0^L(m)(\theta)$, $H(1, m)(\theta) = a_1^L(m)(\theta)$, so $H$ is a homotopy from $a_0^L$ to $a_1^L$. It is easy to check that $H$ is smooth.

Conversely, if $H : [0, 1] \times M \to LM$ is a smooth homotopy from $a_0^L$ to $a_1^L$, set $F(t, \theta, m) = H(t, m)(\theta)$. $\square$

**Corollary 3.3.** If $a_0$ is smoothly homotopic to $a_1$, then $[a_0^L] = [a_1^L] \in H_n(LM, \mathbb{Z})$.

**Proof.** By the last Lemma, $a_0^L$ and $a_1^L$ are homotopic. Thus $[a_0^L] = a_0^L_*[M] = a_1^L_*[M] = [a_1^L]$. $\square$

This yields a technique to use WCS classes to distinguish actions and to investigate $\pi_1(\text{Diff}(M))$. From now on, “homotopic” means “smoothly homotopic.”

**Proposition 3.4.** Let $\dim(M) = 2k - 1$. Let $a_0, a_1 : S^1 \times M \to M$ be actions.

(i) If $\int_{[a_0^L]} CS^W_{2k-1} \neq \int_{[a_1^L]} CS^W_{2k-1}$, then $a_0$ and $a_1$ are not homotopic through actions, and $[a_0^D] \neq [a_1^D] \in \pi_1(\text{Diff}(M))$.

(ii) If there exists an action $a$ with $\int_{[a]} CS^W_{2k-1} \neq 0$, then $\pi_1(\text{Diff}(M))$ is infinite.

**Proof.** (i) If $[a_0^L] = [a_1^L] \in H_n(LM, \mathbb{C})$, then $a_0^L = a_1^L + \partial \alpha$ for some $2k$-chain $\alpha$. Consider $CS^W_{2k-1}$ as an element of singular cohomology. For $\langle \ , \rangle$ the pairing of cochains and chains, we have

$$
\int_{[a_0^D]} CS^W_{2k-1} = \langle CS^W_{2k-1}, a_0^L \rangle = \langle CS^W_{2k-1}, a_1^L + \partial \alpha \rangle
$$

$$
= \int_{[a_1^L]} CS^W_{2k-1} + \langle \delta CS^W_{2k-1}, \alpha \rangle = \int_{[a_1^L]} CS^W_{2k-1} + \int_{[a_1^L]} dCS^W_{2k-1}
$$

$$
= \int_{[a_1^L]} CS^W_{2k-1}.
$$

Thus $[a_0^L] \neq [a_1^L]$, so $a_0$ is not homotopic to $a_1$ by Cor. 3.3. By Lem. 3.2 (ii), $[a_0^D] \neq [a_1^D]$.  

(ii) Let $a_n$ be the $n$th iterate of $a$, i.e. $a_n(\theta, m) = a(n\theta, m)$. We claim that 
$\int_{[a_k]} CS_{2k-1}^W = n \int_{[a_1]} CS_{2k-1}^W$. By (2.5), every term in $CS_{2k-1}^W$ is of the form 
$\int S^1 \dot{\gamma}(\theta) f(\theta)$, where $f$ is a periodic function on the circle. Each loop $\gamma \in a_1^\mathbb{L}(M)$ corresponds to the loop $\gamma(n\cdot) \in a_n^\mathbb{L}(M)$. Therefore the term $\int S^1 \dot{\gamma}(\theta) f(\theta)$ is replaced by 
$\int_{S^1} \frac{d}{d\theta} \gamma(n\theta) f(n\theta) d\theta = n \int_0^{2\pi} \dot{\gamma}(\theta) f(\theta) d\theta$.

Thus $\int_{[a_k]} CS_{2k-1}^W = n \int_{[a_1]} CS_{2k-1}^W$. By (i), the $[a_n^\mathbb{L}] \in \pi_1(\text{Diff}(M))$ are all distinct.

\[\square\]

**Remark 3.1.** If two actions are homotopic through actions, the $S^1$-index of an equivariant differential operator of the two actions is the same. (Here equivariance means for each action $a_t, t \in [0, 1]$.) In contrast to Proposition 3.4(ii), the $S^1$-index of an equivariant operator cannot distinguish actions on odd dimensional manifolds, as the $S^1$-index vanishes. This can be seen from the local version of the $S^1$-index theorem [2, Thm. 6.16]. For the normal bundle to the fixed point set is always even dimensional, so the fixed point set consists of odd dimensional submanifolds. The integrand in the fixed point submanifold contribution to the $S^1$-index is the constant term in the short time asymptotics of the appropriate heat kernel. In odd dimensions, this constant term is zero.

In [10], we interpret the $S^1$-index theorem as the integral of an equivariant characteristic class over $[a^\mathbb{L}]$.

### 3.2. Sasakian structures over Kähler surfaces and diffeomorphism groups.

Let $(M^4, g, J, \omega)$ be a compact integral Kähler surface, i.e. $J$ is the complex structure, $g$ is the Kähler metric, and with Kähler form $\omega \in H^2(M, \mathbb{Z})$. Recall that $\omega(X, Y) = g(JX, Y) = -g(X, JY)$. Recall that $M$ is integral iff it is projective algebraic.

Fix $p \in \mathbb{Z}$. As in geometric quantization, we can construct a $S^1$-bundle $L_p \to M$ with connection $\eta$ with curvature $d\eta = p\omega$. Let $\overline{M}_p$ be the total space of $L_p$. Our goal is to show that $\pi_1(\text{Diff}(\overline{M}_p))$ is infinite for $|p| \gg 0$.

$\overline{M}_p$ has a Sasakian structure, as we now sketch; see [3, §4.5], [16, Lemma 1] for details. The horizontal space of the connection is $\mathcal{H} = \text{Ker}(\overline{\eta})$. We choose a normalized vertical vector $\overline{\xi}$ satisfying

$\overline{\eta}(\overline{\xi}) = 1$.

Note that $d\overline{\eta}(\overline{\xi}, \cdot) = 0$, since the curvature form is horizontal. In the language of contact geometry, $\xi$ is the characteristic vector field of the fibration $\overline{M}_p \to M$. Define a metric $\overline{g}$ on $\overline{M}_p$ by

$\overline{g}(\overline{X}, \overline{Y}) = g(\pi_*\overline{X}, \pi_*\overline{Y}) + \overline{\eta}(\overline{X})\overline{\eta}(\overline{Y})$. 


Then $\xi \perp \mathcal{H}, \varphi(\xi, \xi) = 1, \varphi(\xi, X) = \eta(X)$. Moreover, $\xi$ is a Killing vector field, since

$$
\mathcal{L}_\xi g = \mathcal{L}_\xi \pi^*g + \mathcal{L}_\xi \eta \otimes \eta + \eta \otimes \mathcal{L}_\xi \eta = 0 + [d\eta(\xi, \cdot) + d(\eta(\xi))] \otimes \eta + \eta \otimes [d\eta(\xi, \cdot) + d(\eta(\xi))]
$$

$$
= 0.
$$

Thus the flow lines of $\xi$ are geodesics, making $\pi: \overline{M}_p \to M$ a Riemannian submersion with totally geodesic fiber.

Let $\Phi$ be the $(1, 1)$-tensor on $\overline{M}_p$ defined by

$$
\Phi(X_{\overline{p}}) = (J[\pi_*(\overline{X})]_{\pi(p)})^L,
$$

with $X^L$ the horizontal lift of $X \in T_{\pi(p)}M$. It is easy to check that

$$
\Phi(\xi) = 0, \Phi(X^L) = (JX)^L, \Phi^2 = -I + \eta \otimes \xi.
$$

Let $\overline{X} = \overline{X}^H + \overline{X}^V \in T\overline{M}_p$ be the decomposition of $\overline{X}$ into horizontal and vertical components for the Levi-Civita connection $\nabla$ associated to $\overline{\eta}$. Define $A: TM \to TM, H: TM \otimes TM \to \mathbb{R}$ by $AX = A(X) = \pi_*(\nabla_{\overline{X}^L} \xi), (\nabla_{\overline{X}^L} Y^V)^V = H(X, Y)\xi$.

Using $(\nabla_{\overline{X}^L} Y^V)^L = (\nabla_X Y)^L$ [16] Lem. 1 and $\varphi(\nabla_{\overline{X}^L} \xi, \xi) = 0 = 2X^L(\varphi(\xi, \xi))$, we get

$$
\nabla_{\overline{X}^L} Y^L = (\nabla_X Y)^L + H(X, Y)\xi, \nabla_{\overline{X}^L} \xi = (AX)^L.
$$

Also,

$$
\varphi(Y^L, \xi) = 0 \Rightarrow \varphi(\nabla_{\overline{X}^L} Y^L, \xi) + \varphi(\nabla_{\overline{X}^L} \xi, Y^L) = 0 \Rightarrow H(X, Y) = -g(AX, Y).
$$

**Lemma 3.5.** $AX = pJX$. Equivalently, $H(X, Y) = -pg(JX, Y)$.

**Proof.** We compute

$$
(\nabla_{\overline{X}} \eta)(Y) = \nabla_{\overline{X}}(\eta(Y)) - \eta(\nabla_{\overline{X}} Y) = \nabla_{\overline{X}}(\varphi(\xi, Y)) - \eta(\nabla_{\overline{X}} Y) = \varphi(\nabla_{\overline{X}^L} \xi, Y) + \varphi(\xi, \nabla_{\overline{X}} Y) - \eta(\nabla_{\overline{X}} Y) = \varphi(\nabla_{\overline{X}^L} \xi, Y).
$$

Thus

$$
d\eta(X, Y) = \frac{1}{2}[\nabla_{\overline{X}}(\eta(Y)) - (\nabla_{\overline{Y}} \eta)(X)]
$$

$$
= \frac{1}{2}[\varphi(\nabla_{\overline{X}^L} \xi, Y) - \varphi(\nabla_{\overline{X}^L} \xi, X)] = \varphi(\nabla_{\overline{X}^L} \xi, Y),
$$

by [16] Lemma 2. Thus

$$
pg(JX, Y) = d\eta(X^L, Y^L) = \varphi(\nabla_{\overline{X}^L} \xi, Y^L) = \varphi((AX)^L, Y^L) = g(AX, Y),
$$

so

$$
AX = pJX, \ H(X, Y) = -pg(JX, Y).
$$

$\square$
In summary, we have
\[ \nabla_{X^L} Y^L = (\nabla_X Y)^L - pg(JX, Y)\Xi, \quad \nabla_{X^L} \xi = \nabla_{\xi^L} X^L = p(JX)^L, \quad \nabla_{\xi^L} \xi = 0, \] (3.1)
which implies
\[ \nabla_{X^L} \xi = p\Phi \Xi \] (3.2)
Moreover,
\[ g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \]
\[ (\nabla_{X^L} \Phi)(Y) = -p(\nabla_{X^L} \xi - \eta(Y)^L), \]
which can be checked case by case for \( X, Y \) vertical or horizontal. By definition, the data \((M_p, \Phi, \xi, \eta, g)\) satisfying (3.2) and the second equation in (3.3) defines a Sasakian structure [3, §6.3].

Following [16], we now compute the curvature \( R \) of \( g \) in terms of the curvature \( R \) of \( M \), although our computations are simpler since the fiber of \( M_p \) is one-dimensional.

**Proposition 3.6.**
\[ g(R(X^L, Y^L)Z^L, W^L) = \langle R(X, Y)Z, W \rangle + p^2[\langle JY, Z \rangle \langle JX, W \rangle + \langle JX, Z \rangle \langle JY, W \rangle + 2\langle JX, Y \rangle \langle JZ, W \rangle], \]
\[ g(R(X^L, Y^L)Z, \xi) = 0, \]
\[ g(R(\xi, X^L)Y^L, \xi) = p^2\langle X, Y \rangle. \]

**Proof.** Denoting \( g(X, Y) \) by \( \langle X, Y \rangle \), we have
\[ \nabla_{X^L}(\nabla_{Y^L} Z^L) = \nabla_{X^L}(\langle \nabla_Z Y \rangle^L - p\langle JY, Z \rangle \xi) \]
\[ = ([\nabla_X \nabla_Y Z]^L - p\langle JX, \nabla_Y Z \rangle \xi - pX^L\langle JY, Z \rangle \xi - p\langle JY, Z \rangle \nabla_{X^L} \xi]
\[ = (\nabla_X \nabla_Y Z)^L - p\langle JX, \nabla_Y Z \rangle \xi - p\langle J\nabla_X Y, Z \rangle + \langle JY, \nabla_X Z \rangle \xi \]
\[ -p^2\langle JY, Z \rangle \langle JX \rangle^L. \]

Also,
\[ [X^L, Y^L] = \nabla_{X^L} Y^L - \nabla_{Y^L} X^L = (\nabla_X Y)^L - p\langle JX, Y \rangle \xi - (\nabla_Y X)^L + p\langle JY, X \rangle \xi \]
\[ = [X, Y]^L - 2p\langle JX, Y \rangle \xi, \]
so
\[ \nabla_{[X^L, Y^L]} Z^L = \nabla_{[X, Y]^L} Z^L - 2p\langle JX, Y \rangle \nabla_{\xi^L} Z^L \]
\[ = (\nabla_{X, Y} Z)^L - p\langle J[X, Y], Z \rangle \xi - 2p^2\langle JX, Y \rangle \langle JZ \rangle^L. \]
Thus
\[
\mathcal{R}(X^L, Y^L)Z^L = \nabla_{X^L} \nabla_{Y^L} Z^L - \nabla_{Y^L} \nabla_{X^L} Z^L - \nabla_{[X^L, Y^L]} Z^L
\]
\[
= (\nabla_X \nabla_Y Z)^L - p^2(\langle JY, Z \rangle (JX)^L - (\nabla_Y \nabla_X Z)^L + p^2(\langle JX, Z \rangle (JY)^L
\]
\[
- (\nabla_{[X,Y]} Z)^L + 2p^2(\langle JX, Y \rangle (JZ)^L
\]
\[
- p(\langle JX, \nabla_Y Z \rangle + (\langle J \nabla_X Y, Z \rangle + (\langle JY, \nabla_X Z \rangle - (\langle JX, \nabla_Y Z \rangle - (\langle J[X,Y], Z \rangle)] \xi
\]
\[
= (R(X,Y)Z)^L + p^2[-(\langle JY, Z \rangle (JX)^L + (\langle JX, Z \rangle (JY)^L + 2\langle JX, Y \rangle (JZ)^L].
\]
This implies the first two statements of the Proposition.

For the last statement, we have
\[
\nabla_{X^L} (\nabla_{\xi^L} Y^L) = \nabla_{X^L} (\nabla_{\xi^L} \xi) = -p\nabla_{X^L} JY^L = -p[(\langle J \nabla_X Y \rangle)^L + p(\langle JX, Y \rangle \xi],
\]
\[
\nabla_{\xi^L} \nabla_{X^L} Y^L = 0,
\]
\[
\nabla_{\xi^L} \nabla_{X^L} Y^L = \nabla_{\xi^L} ((\nabla_X Y)^L + p(\langle JX, Y \rangle \xi) = \nabla_{\xi^L} (\nabla_X Y)^L
\]
\[
= \nabla_{(\nabla_X Y)^L} \xi = -p(\langle J \nabla_X Y \rangle)^L,
\]
using \(\nabla J = J \nabla, \xi \langle JX, Y \rangle = 0, \nabla \xi = 0\). Thus
\[
\mathcal{R}(\xi, X^L)Y^L = p^2(\langle X, Y \rangle \xi).
\]

\[\square\]

We now compute \(CS_5^W\) on \(\overline{M}_p\). Set \(\hat{\xi} = \xi\) and let \(\{e_i\}\) be an orthonormal frame of \(T\overline{M}_p\) with \(\xi = e_1\). Since \(\xi_i\) is perpendicular to \(e_i\) for \(i \neq 1\), we can write \(e_i = e_i^L\) for an orthonormal frame \(\{e_i^L\}_{i=2}^5\) of \(TM\).

Let \(a : S^1 \times \overline{M}_p \longrightarrow \overline{M}_p\) be given by the rotation action in the \(S^1\) fibers. As in §3.1, we want to compute
\[
\int_{[a^L]} CS_5^W = \int_{\overline{M}_p} CS_5^W = \int_{\overline{M}_p} a^* CS_5^W. \tag{3.4}
\]

By (2.9)
\[
a^* CS_5^W(e_1, \ldots, e_5)
\]
\[
= \frac{3}{5} \sum_{\sigma} \text{sgn}(\sigma) \overline{R}(a_* \overline{e}_{\sigma(1)}, a_* \overline{e}_t, a_* \overline{\xi}, a_* \overline{e}_n) \overline{R}(a_* \overline{e}_{\sigma(2)}, a_* \overline{e}_{\sigma(3)}, a_* \overline{e}_r, a_* \overline{e}_t) \tag{3.5}
\]
\[
= \overline{R}(a_* \overline{e}_{\sigma(4)}, a_* \overline{e}_{\sigma(5)}, a_* \overline{e}_n, a_* \overline{e}_r).
\]

Because the action is via isometries, we have e.g. \(\overline{R}(a_* \overline{e}_{\sigma(1)}, a_* \overline{e}_t, a_* \overline{\xi}, a_* \overline{e}_n) = \overline{R}(\overline{e}_{\sigma(1)}, \overline{e}_t, \overline{\xi}, \overline{e}_n)\), and since \(CS_5^W\) involves the average of each loop of these curvatures, we will omit \(a_*\) in what follows. In particular, we will write \(\int_{[a^L]} CS_5^W\) just as \(\int_{\overline{M}_p} CS_5^W\).

Choose a local orthonormal frame of \(M\) of the form \(\{e_2, Je_2, e_3, Je_3\}\).
Lemma 3.7. The Chern-Simons form \((3.3)\) equals
\[
a^*CS^W_5(\xi, e_2, Je_2, e_3, Je_3)
= \frac{3p^2}{5} \left\{ 32\pi^2 p_1(\Omega)(e_2, Je_2, e_3, Je_3) + 32p^2 \left[ 3R(e_2, Je_2, e_3, Je_3) - R(e_2, e_3, e_2, e_3)
- R(e_2, Je_3, e_2, Je_3) + R(e_2, Je_2, e_2, Je_2) + R(e_3, Je_3, e_3, Je_3) \right] \right\}
\tag{3.6}
\]
where \(p_1(\Omega)\) is the first Pontryagin form of \((M, g)\).

The proof is in the Appendix.

Set
\[|R|_\infty = \max \{|R(i, j, k, \ell)| : i, j, k, \ell \in \{2, 3, 4, 5\}\}.\]
Here we use the previous notation for the orthonormal frame \(\{e_2, e_3, e_4, e_5\}\).

Proposition 3.8. \(\int_{M_p} CS^W_5 > 0\) if
\[p^2 \left( 32\pi^2 p_1(\Omega)(e_2, Je_2, e_3, Je_3) - 224p^2 |R|_\infty + 192p^4 \right) > 0\]
pointwise on \(M\). Moreover, \(\int_{M_p} CS^W_5 > 0\) if
\[p^2 \left( 96\pi^2 \sigma(M) - 224p^2 |R|_\infty \vol(M) + 192p^4 \cdot \vol(M) \right) > 0.\]
\(\text{(3.7)}\)
Here \(\sigma(M)\) is the signature of \(M\).

Proof. The first statement follows from \((3.6)\), noting that \(\int_{M_p} CS^W_5 = \int_{M_p} a^*CS^W_5\) in our notation. For the second statement, we use \(\vol(M_p) = \lambda \vol(M)\), where the circle fiber has constant length \(\lambda\) [3 p. 37]. Also, \(p_1(\Omega) = p_1(\Omega)(e_2, Je_2, e_3, Je_3)e_3^* \wedge (Je_2)^* \wedge e_3^* \wedge (Je_3)^*\) in the obvious notation, so \(\int_{M_p} p_1(\Omega)(e_2, Je_2, e_3, Je_3)\,d\vol_M = \int_{M} p_1(\Omega) = 3\sigma(M)\). This gives
\[p^2 \left( 96\pi^2 \lambda \sigma(M) - 224p^2 |R|_\infty \lambda \vol(M) + 192p^4 \cdot \lambda \vol(M) \right) > 0,
from which the second statement follows.

Corollary 3.9. The loop of diffeomorphisms of \(\overline{M}_p\) given by rotation in the circle fiber gives an element of infinite order in \(\pi_1(\Diff(\overline{M}_p))\) provided
\[|R|_\infty < \frac{6}{\pi^2} + \frac{3\pi^2 \sigma(M)}{\vol(M)p^2}, \text{ if } p \neq 0.\]

Proof. This follows from Prop. \(3.4(ii)\) and \(3.7\).

Theorem 3.10. Let \((M^4, J, g, \omega)\) be a compact integral Kähler surface, and let \(\overline{M}_p\) be the circle bundle associated to \(p[\omega] \in H^2(M, \Z)\) for \(p \in \Z\). Then the loop of diffeomorphisms of \(\overline{M}_p\) given by rotation in the circle fiber gives an element of infinite order in \(\pi_1(\Diff(\overline{M}_p))\) if \(|p| \gg 0\).

Proof. The estimate \((3.7)\) holds for \(|p| \gg 0\). Prop. \(3.4(ii)\) then gives the result.
Remark 3.2. (i) The Theorem does not apply to the simplest case $p = 0$, where $\overline{M}_0 = M \times S^1$. In this case, the projection $p : \overline{M}_0 \rightarrow S^1$ induces $p_* : \pi_1(\text{Diff}(\overline{M}_0)) \rightarrow \pi_1(S^1) = \mathbb{Z}$. The fiber rotation maps to a generator of $\pi_1(S^1)$, so we immediately conclude $|\pi_1(\text{Diff}(\overline{M}_0))| = \infty$.

(ii) It is not the case that we can scale the metric on $M$ so that the estimate in Cor. 3.9 holds for any fixed $p \neq 0$. For if we scale the metric $g$ on $M$ to $\eta g$, $\eta \in \mathbb{Z}^+$, then $|R|_\infty$ scales to $\eta^{-1}|R|_\infty$, $\text{vol}(M)$ scales to $\eta^2\text{vol}(M)$, and $\overline{M}_p$ changes to $\overline{M}_{\eta p}$. Thus for fixed $p$, the estimate in Cor. 3.9 holds for $\eta \gg 0$, which is just a restatement of Thm. 3.10.

(iii) By Prop. 3.4(i), these fiber rotations give examples of actions which are not smoothly homotopic to the trivial action.

Corollary 3.11. Under the hypothesis in Thm. 3.10, $\pi_1(\text{Isom}(\overline{M}_p))$ is infinite.

Proof. The loop $\beta$ of fiber rotations consists of isometries. Under the inclusion $i : \text{Isom}(\overline{M}_p) \hookrightarrow \text{Diff}(\overline{M}_p)$, if $\beta \in \pi_1(\text{Isom}(\overline{M}_p))$ has finite order, then $i_\ast \beta \in \pi_1(\text{Diff}(\overline{M}_p))$ has finite order. This contradicts the hypothesis. \qed

3.3. Examples. For specific Kähler surfaces we can precisely determine for which $p$ we have $|\pi_1(\text{Diff}(\overline{M}_p))| = \infty$.

Example 3.12. $M = T^4$. For the flat metric, $|R|_\infty = 0$ and $p_1(\Omega) = 0$. For the standard Kähler class, by (3.4), (3.6), we have $\int_{\overline{M}_p} CS^W_5 > 0$ for $p \neq 0$. By Prop. 3.4(ii), $|\pi_1(\text{Diff}(\overline{M}_p))| = \infty$ for $p \neq 0$. As noted above, this also holds trivially for $p = 0$.

Example 3.13. $M = \mathbb{CP}^2$. We give this case to show sharp results for $p$ and to check the constants in Prop. 3.8. The total space $\mathbb{CP}^2_1$ of the hyperplane bundle over $\mathbb{CP}^2$ is $S^5$. As above, the rotational action of the fiber is via isometries and gives an element of $\pi_1(\text{Isom}(S^5)) = \pi_1(SO(6)) \simeq \mathbb{Z}_2$, with generator given by this rotational action. Thus this element is of order at most two in $\pi_1(\text{Diff}(S^5))$. The line bundle $L_1$ associated to $\mathbb{CP}^2_1$ is diffeomorphic to the dual line bundle $L_1^* \simeq L_{-1}$, the line bundle associated to $\mathbb{CP}^2_{-1}$, by the map $v \mapsto \langle , v \rangle$ for a hermitian metric on $\mathbb{CP}^2_1$. Thus $\mathbb{CP}^2_1 \simeq \mathbb{CP}^2_{-1}$, and the rotational action also has order at most two for $p = -1$. (In general, $\mathbb{CP}^2_p$ is diffeomorphic to $\mathbb{CP}^2_{-p}$.)

Thus we must have $\int_{\mathbb{CP}^2_p} CS^W_5 = 0$, which we verify by computing (3.6) explicitly. Using the formula for the curvature tensor of $\mathbb{CP}^2$ in [9] Vol. II, p. 166\footnote{With $c = 4$ in their notation, and noting that $R(X, Y, Z, W)$ in [9] is the negative of our $R(X, Y, Z, W)$.} a direct computation gives $b_2 = 0, b_3 = b_4 = -192p^2$ in (A.12) in the Appendix. Since
1 = σ(\mathbb{C}P^2) = \frac{1}{3} \int_{\mathbb{C}P^2} p_1, \text{ this gives}

\int_{\mathbb{CP}^2 = S^5} C_{S^5}^W = \frac{3p^2}{5} \left( 64\pi^3 \int_{\mathbb{C}P^2} p_1(\Omega) - 384p^2 \text{vol}(S^5) + 192p^4 \text{vol}(S^5) \right)

= \frac{3p^2}{5} (192\pi^3 - 384p^2\pi^3 + 192p^4\pi^3)

= \frac{586p^2\pi^3}{5}(p^2 - 1)^2\),

which vanishes iff \(p = \pm 1\). Note that the WCS form vanishes pointwise at \(p = \pm 1\).

In particular, for any Kähler form \(\omega = \omega^{FS} + \partial \bar{\partial} \log f\) in the same cohomology class as the Fubini-Study form \(\omega^{FS}\), the curvature estimate in Cor. 3.9 fails at \(p = 1\). This yields a lower bound for \(|R|_\infty\). Since \(\text{vol}(\mathbb{C}P^2) = \frac{\pi^2}{2}\) independent of \(\omega\), we have

\(|R|_\infty \geq \frac{12}{7}\)

for any such \(\omega\).

**Proposition 3.14.** (i) For \(p = \pm 1\), the element of \(\pi_1(\text{Diff}(\mathbb{C}P^2_p))\) given by rotation in the fiber of \(\mathbb{C}P^2_p \to \mathbb{C}P^2\) has order at most two. For \(p \neq \pm 1\), this element has infinite order.

(ii) For \(|p| \neq |\ell|\), \(\mathbb{C}P^2_p\) is not diffeomorphic to \(\mathbb{C}P^2_\ell\).

(iii) \(\mathbb{C}P^2_p\) is diffeomorphic to the lens space \(L_p = S^5/\mathbb{Z}_p\), where \(z \sim e^{2\pi i/p}z\) for \(z \in S^5\).

**Proof.** Part (i) follows from Prop. 3.4(ii), (3.4), and (3.8).

(iii) implies (ii), but for later purposes we prove (ii) directly. For (ii), the Euler class of the fibration \(S^1 \to \mathbb{C}P^2_p \to \mathbb{C}P^2\) is \(p\) times the generator \([\omega = \omega^{FS}]\) of \(H_2(\mathbb{C}P^2, \mathbb{Z})\). As in [8, Appendix A], one piece of the Gysin sequence of this fibration is

\[ 0 \to H^3(M_p) \to \mathbb{Z} \xrightarrow{[\omega]} \mathbb{Z} \to H^4(M_p) \to 0, \]

which implies \(H^4(\mathbb{C}P^2_p, \mathbb{Z}) \simeq \mathbb{Z}/|p|\mathbb{Z}\) for all \(p\). This gives (ii).

For (iii), let \(\phi_i : S^5|_{U_i} = L_1|_{U_i} \to U_i \times S^1\) be local trivializations of \(L_1\). Set \(e_p : S^1 \to S^1, z \mapsto z^p\). The \(\mathbb{Z}_p\) action preserves the fibers of \(L_1\), so the maps \(\phi_i^p = e_p \circ \phi_i : \mathcal{L}_p|_{U_i} \to U_i \times S^1\), are local trivializations of \(\mathcal{L}_p\) as a circle bundle over \(\mathbb{C}P^2\).

The first Chern class \(c_1(L_1)\) is a generator of \(H^2(\mathbb{C}P^2)\), and has the Čech representative \((2\pi i)^{-1}(\ln \phi_{ij} + \ln \phi_{jk} + \ln \phi_{kl})\), where \(\phi_{ij} = \phi_i \circ \phi_j^{-1}\). Thus \(c_1(\mathcal{L}_p) = pc_1(L_1) = c_1(\mathbb{C}P^2_p)\). Therefore \(\mathcal{L}_p \simeq \mathbb{C}P^2_p\) as line bundles, from which it follows that they are diffeomorphic.

\[ \square \]
Example 3.15. $M = \mathbb{CP}^1 \times \mathbb{CP}^1$. As for $\mathbb{CP}^2$, we can compute explicitly the Chern-Simons class. Let $\omega_1, \omega_2$ be the standard Kähler form on each $\mathbb{CP}^1$ of sectional curvature 1. Let $R$ be the curvature of the Kähler metric for the Kähler form $\omega = a\omega_1 + b\omega_2$ for $a, b > 0$. For $\{e_2, Je_2\}$ and $\{e_3, Je_3\}$ orthonormal frames for the first and second $\mathbb{CP}^1$, we have

\[
R(e_2, Je_2, e_3, Je_3) = R(e_2, e_3, e_2) = R(e_2, Je_3, e_2, Je_3) = 0,
\]
\[
R(e_2, Je_2, e_2, Je_2) = -a^{-1}, \quad R(e_3, Je_3, e_3, Je_3) = -b^{-1}.
\]

For $a, b \in \mathbb{Z}^+$, $\omega = a\omega_1 + b\omega_2$ is an integral Kähler form. Let $M_{p(a,b)}$ be the total space of the line bundle associated to $p\omega$. Since $\sigma(M) = 0$, the integral of $\frac{3p^2}{5} \int_{M_{p(a,b)}} (-32p^2(-a^{-1} - b^{-1}) + 192p^4) d\text{vol},$

which is positive for $p \neq 0$.

This produces a new infinite family of five manifolds with $\pi_1(\text{Diff}(M_{p(a,b)}))$ infinite.

Proposition 3.16. (i) Let $M = \mathbb{CP}^1 \times \mathbb{CP}^1 = S^2 \times S^2$. For $a, b \in \mathbb{Z}^+, p \in \mathbb{Z}$, the element of $\pi_1(\text{Diff}(M_{p(a,b)}))$ given by rotation in the fiber of $M_{p(a,b)} \longrightarrow M$ has infinite order.

(ii) For $\gcd(a, b) \neq \gcd(c, d)$, $M(a,b)$ is not diffeomorphic to $M(c,d)$. Furthermore, $M(a,b)$ is not diffeomorphic to $\mathbb{CP}^2_p$ for any $p$.

Proof. (i) is proven as in the previous examples.

(ii) Part of the Gysin sequence with $\mathbb{Z}$ coefficients is

\[
\begin{align*}
H^3(M) \longrightarrow H^3(M_{a,b}) & \xrightarrow{\beta} H^2(M) \xrightarrow{\alpha} H^4(M) \longrightarrow H^4(M_{a,b}) \longrightarrow H^3(M),
\end{align*}
\]

where $\alpha = \cup(a[\omega_1] + b[\omega_2])$. This reduces to

\[
\begin{align*}
0 \longrightarrow H^3(M_{a,b}) & \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow H^4(M_{a,b}) \longrightarrow 0.
\end{align*}
\]

Since $\alpha(x, y) = ax + by$, the image of $\alpha$ is $\gcd(a, b)\mathbb{Z}$. Thus $H^4(M(a,b)) \simeq \mathbb{Z}_{\gcd(a,b)}$, which gives the first statement in (ii).

Since $\text{Im}(\beta) \simeq \text{Ker}(\alpha) \simeq \{(bx, -ax) : x \in \mathbb{Z}\} \simeq \mathbb{Z}$, we have $H^3(M_{a,b}) \simeq \mathbb{Z}$. The corresponding part of the Gysin sequence for $\mathbb{CP}^2_p$ yields $H^3(\mathbb{CP}^2_p) \simeq 0$, which gives the second statement in (ii).

Example 3.17. Let $M$ be a compact Kähler surface with a Ricci flat metric, i.e. a compact K3 surface. We orient $M$ with the opposite of the orientation coming from the complex structure. Let $R : \Lambda^2(M) \longrightarrow \Lambda^2(M)$ be the curvature operator. According to the decomposition $\Lambda^2(M) = \Lambda^2_+ \oplus \Lambda^2_-$ into $\pm 1$ eigenspaces of the star operator, we have
\[
R = \begin{pmatrix}
\Lambda^2_+ & \Lambda^2 \\
R_{++} & R_{+-} \\
R_{-+} & R_{--} \\
\end{pmatrix} \Lambda^2_+ \\
\Lambda^2
\]

\text{Tr}(R_{++}) + \text{Tr}(R_{--}) = 4r = 0, \text{ where } r \text{ is the scalar curvature. } R_{+-} \text{ and } R_{-+} \text{ are determined by the traceless Ricci tensor, and so vanish.}

Let \((e_2, Je_2, e_3, Je_3)\) be a local orthonormal frame for \(TM\) with \(*e_2 = -Je_2 \wedge e_3 \wedge Je_3, \text{ etc.} \); the minus sign reflects the change in orientation. Then a basis of \(\Lambda^2\) is

\[\{f_1^+ = e_2 \wedge Je_2 - e_3 \wedge Je_3, f_2^+ = e_2 \wedge e_3 + Je_2 \wedge Je_3, f_3^+ = e_2 \wedge Je_3 - Je_2 \wedge e_3\},\]

and a basis of \(\Lambda^2_+\) is

\[\{f_1^- = e_2 \wedge Je_2 + e_3 \wedge Je_3, f_2^- = e_2 \wedge e_3 - Je_2 \wedge Je_3, f_3^- = e_2 \wedge Je_3 + Je_2 \wedge e_3\}.
\]

Since \(R_{++}, R_{--}\) are symmetric, there are bases \(\{\omega_1^+, \omega_2^+, \omega_3^+\}\) of \(\Lambda^2_+\) and \(\{\omega_1^-, \omega_2^-, \omega_3^-\}\) of \(\Lambda^2\) for which \(R_{++}, R_{--}\) diagonalize:

\[
R_{++} = \lambda_1^+ \omega_1^+ \otimes \omega_1^+ + \lambda_2^+ \omega_2^+ \otimes \omega_2^+ + \lambda_3^+ \omega_3^+ \otimes \omega_3^+ \\
R_{--} = \lambda_1^- \omega_1^- \otimes \omega_1^- + \lambda_2^- \omega_2^- \otimes \omega_2^- + \lambda_3^- \omega_3^- \otimes \omega_3^-,
\]

for \(\omega_1^\pm = (\omega_i^\pm)^*\). Thus

\[
\sum_{i=1}^{3} \lambda_1^+ + \sum_{i=1}^{3} \lambda_1^- = 0. \quad (3.9)
\]

We claim that the curvature terms in \((3.6)\) add to zero. By the Bianchi identity, the first curvature term is

\[
3R(e_2, Je_2, e_3, Je_3) = -3R(e_2, e_3, Je_3, Je_2) - 3R(e_2, Je_3, Je_2, e_3) \\
= 3R(e_2, e_3, e_2, e_3) + 3R(e_2, Je_3, e_2, e_3).
\]

Thus the curvature terms in \((3.6)\) become

\[
2R(e_2, e_3, e_2, e_3) + 2R(e_2, Je_3, e_2, Je_3) + R(e_2, Je_2, e_2, Je_2) + R(e_2, Je_3, e_3, Je_3). \quad (3.10)
\]

To compute these terms, we note that e.g. \(R(e_2, Je_2, e_3, Je_3) = -\langle R(e_2 \wedge Je_2), e_3 \wedge Je_3\rangle\). Write \(f_j^+ = b_j^+ \omega_1^+, f_j^- = d_j^- \omega_i^-\). After normalizing \(\{\omega_i^\pm\}, \{f_j^\pm\}\), we may assume that \((b_j^\pm), (d_j^-)\) are orthonormal. Then

\[
R(e_2, e_3, e_2, e_3) = -\langle R(e_2 \wedge e_3), e_2 \wedge e_3\rangle \\
= -\left< R_{++} \left( \frac{f_2^+ + f_2^-}{2} \right), \frac{f_2^+ + f_2^-}{2} \right> - \left< R_{--} \left( \frac{f_2^+ + f_2^-}{2} \right), \frac{f_2^+ + f_2^-}{2} \right> \\
= -\left< R_{++} f_2^+, f_2^- \right> - \frac{1}{4} \left< R_{--} f_2^-, f_2^- \right> \\
= -\frac{1}{4} \left[ \lambda_1^+(b_2^+)^2 + \lambda_2^+(b_2^-)^2 + \lambda_3^+(b_3^-)^2 + \lambda_1^-(d_2^+)^2 + \lambda_2^-(d_2^-)^2 + \lambda_3^-(d_3^-)^2 \right].
\]
Similarly computing the other terms in (3.10), we get
\[ 2R(e_2, e_3, e_2, e_3) + 2R(e_2, Je_3, e_2, Je_3) + R(e_2, Je_2, e_2, Je_2) + R(e_3, Je_3, e_3, Je_3) \]
\[ \quad = -\frac{1}{2} (\lambda_1^+ (b_2^2) + \lambda_2^+ (b_3^2) + \lambda_3^+ (b_1^2) + \lambda_1^- (d_2^2) + \lambda_2^- (d_3^2) + \lambda_3^- (d_1^2)) \]
\[ \quad = -\frac{1}{2} (\lambda_1^+ (b_2^2) + \lambda_2^+ (b_3^2) + \lambda_3^+ (b_1^2) + \lambda_1^- (d_2^2) + \lambda_2^- (d_3^2) + \lambda_3^- (d_1^2)) \]
\[ \quad = -\frac{1}{2} (\lambda_1^+ + \lambda_2^+ + \lambda_3^+ + \lambda_1^- + \lambda_2^- + \lambda_3^-) \]
\[ \quad = 0. \]

Since \( p_1(\Omega) = \frac{1}{4\pi^2} (|W_+|^2 - |W_-|^2) = \frac{1}{4\pi^2} |W_+|^2 \geq 0 \), \( \sigma(M) \geq 0 \). (In fact, \( \sigma(M) = 16 \) in this orientation.) Thus \( \int_{\overline{M}_p} CS^W \geq 192p^6 > 0 \) for \( p \neq 0 \).

**Remark 3.3.** By [12], the vanishing of scalar curvature on a Kähler surface implies the metric is self-dual (after a change of orientation): for the decomposition of the Weyl tensor \( W = W_+ + W_- \), we have \( W_- = 0 \). Therefore all the pieces in the decomposition of \( R_- \) into projected traceless Ricci, \( W_- \) and \( r \) vanish, so \( R_- = 0 \). Thus the calculation above can be somewhat shortened.

This result provides another infinite family of examples. To introduce the notation, recall that \( H^2(M; \mathbb{Z}) \cong \mathbb{Z}^{22} \). Fix an integral Kähler class \([\omega] = [\omega_1, \ldots, \omega_{22}] \) in the obvious notation and take \( a_1, \ldots, a_{22} \in \mathbb{Z}^+ \setminus \{0\} \). For \( p \in \mathbb{Z} \), let \( \overline{M}_{p;i} \) be the total space of the line bundle associated to \( p \sum_{i=1}^{22} a_i \omega_i \).

**Proposition 3.18.** Let \( M \) be a compact projective algebraic K3 surface.

(i) The element of \( \pi_1(\text{Diff}(\overline{M}_{p;i})) \) given by rotation in the fiber of \( \overline{M}_{p;i} \rightarrow M \) has infinite order.

(ii) For \( p = 1 \), there are infinitely many nondiffeomorphic \( \overline{M}_{p;i} \). None of these examples are diffeomorphic to \( \mathbb{CP}^2 \) or to the manifolds \( \overline{M}_{(a,b)} \) in Proposition 3.16.

**Proof.** (i) This follows from \( \int_{\overline{M}_p} CS^W > 0 \) as before.

(ii) As before, \( H^4(\overline{M}_{p;i}) \cong \mathbb{Z}_{\gcd(a_1, \ldots, a_{22})} \), so infinitely many of the \( \overline{M}_{p;i} \) are nondiffeomorphic. Similarly, \( H^4(\overline{M}_{p;i}) \cong \mathbb{Z}_{\gcd(a_1, \ldots, a_{22})} \), which easily implies that \( H^3(\overline{M}_{p;i}, \mathbb{R}) \cong \mathbb{R}^{21} \). This gives the last statement in (ii).

**Example 3.19.** \( S^2 \times S^3 \). We now apply a variant of these methods to a Sasaki-Einstein metric on \( S^2 \times S^3 \) constructed in [8] to prove the following:

**Proposition 3.20.** \( \pi_1(\text{Diff}(S^2 \times S^3)) \) is infinite.
Proof. For \( a \in (0, 1] \), the metric
\[
g = \frac{1-y}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y) q(y)} dy^2 + \frac{q(y)}{9} [d\psi^2 - \cos \theta d\phi^2]
\]
\[
+ w(y) \left[ d\alpha + \frac{a-2y+y^2}{6(a-y^2)} [d\psi - \cos \theta d\phi] \right]^2,
\]
with
\[
w(y) = \frac{2(a-y^2)}{1-y}, q(y) = \frac{a-3y^2+2y^3}{a-y^2}, a \in \mathbb{R},
\]
is a family of Sasaki-Einstein metrics on a 5-ball in the variables \((\phi, \theta, \psi, y, \alpha)\) \([8]\). For \( p, q \) relatively prime, \( q < p \), and satisfying \( 4p^2 - 3q^2 = n^2 \) for some integer \( n \), and for \( a = a(p, q) \in \{0, 1\} \), the metric extends to a 5-manifold \( Y^{p,q} \) which has the coordinate ball as a dense subset. In this case, \((\phi, \theta, \psi, y)\) are spherical coordinates on \( S^2 \times S^2 \) with a nonstandard metric, and \( \alpha \) is the fiber coordinate of an \( S^1 \)-fibration \( S^2 \times S^2 \rightarrow S^2 \times S^2 \). \( Y^{p,q} \) is diffeomorphic to \( S^2 \times S^2 \), and has first Chern class which integrates over the two \( S^2 \) factors to \( p+q \) and \( p \) \([8, \S 2]\). The coordinate ranges are \( \phi \in (0, 2\pi), \theta \in (0, \pi), \psi \in (0, 2\pi), \alpha \in (0, 2\pi \ell) \), where \( \ell = \ell(p, q) \), and \( y \in (y_1, y_2) \), with the \( y_i \) two smaller roots of \( a - 3y^2 + 2y^3 = 0 \). \( p \) and \( q \) determine \( a, \ell, y_1, y_2 \) explicitly \([8, (3.1), (3.4), (3.5), (3.6)]\).

For these choices of \( p, q \), we get an \( S^1 \)-action \( a \) on \( Y^{p,q} \) by rotation in the \( \alpha \)-fiber. We claim that for e.g. \((p, q) = (7, 3)\),
\[
\int_{[a^L]} CS^W_5(g) \neq 0.
\]
By Proposition \([8, \S 4](ii)\), this implies \( \pi_1(\text{Diff}(S^2 \times S^3)) \) is infinite.

Set \( M = S^2 \times S^3 \). Since \( a^L : M \rightarrow LM \) has degree one on its image,
\[
\int_{[a^L]} CS^W_5(g) = \int_M a^{L,*} CS^W_5(g).
\]
(13.13)

For \( m \in M \),
\[
a^{L,*} CS^W_5(g)_m = f(m) d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha
\]
for some \( f \in C^\infty(M) \). We determine \( f(m) \) by explicitly computing \( a^L_\phi(\partial_\phi), ..., a^L_\alpha(\partial_\alpha) \), (e.g. \( a^L_\phi(\partial_\phi)(a^L(m))(t) = \partial_\phi|_{a(m,t)} \)), and noting
\[
f(m) = f(m) d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha(\partial_\phi, \partial_\theta, \partial_y, \partial_\psi, \partial_\alpha)
\]
\[
= a^{L,*} CS^W_5(g)_m(\partial_\phi, ..., \partial_\alpha)
\]
\[
= CS^W_5(g) a^{L,*}_{(m)}(a^L_\phi(\partial_\phi), ..., a^L_\alpha(\partial_\alpha)).
\]
(14.14)

Since \( CS^W_5(g) \) is explicitly computable from \((2.23)\), we can compute \( f(m) \) from \((3.14)\). Then \( \int_{[a^L]} CS^W_5(g) = \int_M f(m) d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha \) can be computed as an ordinary integral in the dense coordinate space.
Via this method, in the Mathematica file ComputationsChernSimonsS2xS3.pdf at http://math.bu.edu/people/sr/, \( \int_{[a \ell]} C_{W}^{S_{5}}(g) \) is computed as a function of \((p, q)\).

For example, for \((p, q) = (7, 3)\),
\[
\int_{[a \ell]} C_{W}^{S_{5}}(g) = -\frac{3}{5} \cdot \frac{1849\pi^{4}}{22050} = -\frac{1849\pi^{4}}{37750}.
\]

This formula is exact; the rationality up to \(\pi^{4}\) follows from \(4p^{2} - 3q^{2}\) being a perfect square, as then the various integrals computed in (3.13) with respect to our coordinates are rational functions evaluated at rational endpoints. In particular, (3.12) holds.

**Remark 3.4.** For \(a = 1\), the metric extends to the closure of the coordinate chart, but the total space is \(S^{5}\) with the standard metric. For \(n \gg 0\), \(\pi_{1}(\text{Diff}(S^{n}))\) is torsion \([6]\). If this holds below the stable range, specifically for \(n = 5\), then by Proposition 3.4(ii), \(\int_{[a \ell]} C_{W}^{S_{5}} = 0\) for any circle action on \(S^{5}\). In the formulas in the Mathematica file, \(\int_{[a \ell]} C_{W}^{S_{5}}\) is proportional to \((-1 + a)^{2}\), which vanishes at \(a = 1\). This gives some plausibility to the conjecture that \(\pi_{1}(\text{Diff}(S^{5}))\) is torsion.

**Appendix A. Proof of Lemma 3.7**

Recall that we choose a local orthonormal frame of \(\overline{M}_{p}\) of the form \(\bar{e}_{1} = \xi, \bar{e}_{i} = e_{L_{i}}, i = 2, 3, 4, 5\), where \(\{e_{i}\}\) is a local orthonormal frame of \(M\) and \(e_{L_{i}}\) are horizontal lifts. Eventually we will refine the frame of \(M\) to be of the form \(\{e_{2}, Je_{2}, e_{3}, Je_{3}\}\).

We wish to simplify (3.5):

\[
a^{\ast} C_{W}^{S_{5}, \gamma}(e_{1}, \ldots, e_{5}) \]
\[
= \frac{3}{5} \sum_{\sigma} \text{sgn}(\sigma) R(a_{s} e_{\sigma(1)}, a_{s} e_{t}, a_{s} \xi, a_{s} n) R(a_{s} e_{\sigma(2)}, a_{s} e_{\sigma(3)}, a_{s} e_{r}, a_{s} e_{t}) R(a_{s} e_{\sigma(4)}, a_{s} e_{\sigma(5)}, a_{s} n, a_{s} e_{r}) \quad (A.1)
\]

We divide the permutations into five cases \(A_{i}\) where \(\sigma(i) = 1\), i.e., \(\overline{e}_{\sigma(i)} = \overline{\xi}\). In formulas like (A.1), we refer to the curvature terms on the right hand side as the first, second or third term.

We claim that the \(A_{2}\) case contributes zero to the integrand in (A.1). Ignoring the factor \(\frac{3}{5}\), we have

\[
A_{2} = \sum_{\sigma, \sigma(2) = 1} \text{sgn}(\sigma) R(e_{\sigma(1)}, e_{\xi}, e_{n}) R(\xi, e_{\sigma(3)}, e_{r}, e_{t}) R(\overline{e}_{\sigma(4)}, \overline{e}_{\sigma(5)}, \overline{e}_{n}, e_{r}) \quad (A.2)
\]

By Prop. 3.6 in the second term exactly one of \(e_{r}, e_{t}\) must be \(\overline{\xi}\) for a nonzero contribution. If \(e_{r} = \overline{\xi}\), then the first term must have \(e_{n} = \overline{\xi}\). The third term is then
zero. If \( \tau_\ell = \xi \), then \( \tau_r = e^L_\ell \) and the first term forces \( \tau_n = e^L_n \). By the last equation in Prop. 3.6 we have

\[
A_2 = \sum_{\sigma(2)=1} \text{sgn}(\sigma) \overline{R}(e^L_{\sigma(1)}, \xi, \xi, e^L_n) \overline{R}(\xi, e^L_{\sigma(3)}, e^L_k, \xi) \overline{R}(e^L_{\sigma(4)}, e^L_{\sigma(5)}, e^L_n, e^L_k)
\]

\[
= p^2 \sum_{\sigma(2)=1} \text{sgn}(\sigma) \delta_{\sigma(1),n} \delta_{\sigma(3),k} \overline{R}(e^L_{\sigma(4)}, e^L_{\sigma(5)}, e^L_n, e^L_k)
\]

\[
= p^2 \sum_{\sigma(2)=1} \text{sgn}(\sigma) \overline{R}(e^L_{\sigma(4)}, e^L_{\sigma(5)}, e^L_{\sigma(1)}, e^L_{\sigma(3)})
\]

\[
= p^2 \sum_{\sigma(2)=1} \sum_{i=2}^4 \sum_{\sigma(4)=i} \text{sgn}(\sigma) \overline{R}(e^L_{\sigma(4)}, e^L_{\sigma(5)}, e^L_{\sigma(1)}, e^L_{\sigma(3)})
\]

\[
= 0,
\]

since by the Bianchi identity, for each \( i \) we have

\[
\sum_{\sigma(2)=1} \sum_{\sigma(4)=i} \text{sgn}(\sigma) \overline{R}(e^L_{\sigma(4)}, e^L_{\sigma(5)}, e^L_{\sigma(1)}, e^L_{\sigma(3)}) = 0.
\]

For \( A_3 \), again exactly one of \( \tau_r, \tau_\ell \) must be \( \xi \). If \( \tau_r = \xi \), then in the third term we must have \( \tau_n = \xi \), which forces this term to vanish. If \( \tau_\ell = \xi \), as above we use Bianchi to get zero. For \( A_4 \), either \( \tau_n \) or \( \tau_r \) equals \( \xi \). However, when \( \tau_n = \xi \) the first term is zero by Prop. 3.6, and when \( \tau_r = \xi \) the third term vanishes for the same reason. Thus \( A_4 = 0 \). The exact same argument shows that \( A_5 = 0 \).

The \( A_1 \) term is

\[
A_1 = \sum_{\sigma(1)=1} \text{sgn}(\sigma) \overline{R}(\xi, \tau_\ell, \xi, \tau_n) \overline{R}(\tau_{\sigma(2)}, \tau_{\sigma(3)}, \tau_r, \tau_\ell) \overline{R}(\tau_{\sigma(4)}, \tau_{\sigma(5)}, \tau_n, \tau_r).
\]
It is easy to check that if any of $\xi, \xi_n, \xi_r$ equals $\xi$, then the contribution to $A_1$ is zero.

By Prop. 3.6 (and denoting $(Je_x)^L$ by $Je_x^L$), we have

$$A_1 = \sum_{\sigma(1)=1} \text{sgn}(\sigma) R(\xi, e_{\sigma(1)}^L, \xi_n, e_{\sigma(1)}^L, e_{\sigma(2)}^L, e_{\sigma(3)}^L, e_{\sigma(4)}^L)$$

$$= -p^2 \sum_{\sigma(1)=1} \text{sgn}(\sigma) \delta_{\xi_n} R(e_{\sigma(2)}^L, e_{\sigma(3)}^L, e_{\sigma(4)}^L)$$

$$= p^2 \sum_{\sigma(1)=1} \text{sgn}(\sigma) R(e_{\sigma(2)}^L, e_{\sigma(3)}^L, e_{\sigma(4)}^L)$$

$$= p^2 \sum_{\sigma(1)=1} \text{sgn}(\sigma) R(e_{\sigma(2)}^L, e_{\sigma(3)}^L, e_{\sigma(4)}^L) [R(e_{\sigma(2)}^L, e_{\sigma(3)}^L, e_{\sigma(4)}^L) - p^2 \langle Je_{\sigma(2)}, Je_{\sigma(3)}, Je_{\sigma(4)} \rangle$$

$$+ p^2 \langle Je_{\sigma(2)}, Je_{\sigma(3)}, Je_{\sigma(4)} \rangle + 2p^2 \langle Je_{\sigma(2)}, Je_{\sigma(3)}, Je_{\sigma(4)} \rangle]$$

$$= p^2 \sum_{\sigma(1)=1} \text{sgn}(\sigma) R(e_{\sigma(2)}^L, e_{\sigma(3)}^L, e_{\sigma(4)}^L)$$

$$+ p^2 \langle Je_{\sigma(2)}, Je_{\sigma(3)}, Je_{\sigma(4)} \rangle = 0.$$
Thus
\[
A_{1c} = R(e_{\sigma(2)}, e_{\sigma(3)}, e_r, Je_r) \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle + 2p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle + 8p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle
\]
\[
= R(e_{\sigma(2)}, e_{\sigma(3)}, e_r, Je_r) \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle + 10p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle.
\]
Finally,
\[
A_{1a} = \overline{R}(e_{\sigma(2)}^L, e_{\sigma(3)}^L, e_r^L, e_\ell^L) R(e_{\sigma(4)}, e_{\sigma(5)}, e_r, e_\ell)
\]
\[
= [R(e_{\sigma(2)}, e_{\sigma(3)}, e_r, e_\ell) - p^2 \langle Je_{\sigma(2)}, e_r \rangle \langle Je_{\sigma(3)}, e_\ell \rangle + p^2 \langle Je_{\sigma(2)}, e_r \rangle \langle Je_{\sigma(3)}, e_\ell \rangle] R(e_{\sigma(4)}, e_{\sigma(5)}, e_r, e_\ell)
\]
\[
= R(e_{\sigma(2)}, e_{\sigma(3)}, e_r, e_\ell) R(e_{\sigma(4)}, e_{\sigma(5)}, e_r, e_\ell) - p^2 R(e_{\sigma(4)}, e_{\sigma(5)}, Je_{\sigma(3)}, Je_{\sigma(2)})
\]
\[
\quad + p^2 R(e_{\sigma(4)}, e_{\sigma(5)}, Je_{\sigma(2)}, Je_{\sigma(3)}) + 2p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle - 2p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle
\]
\[
= R(e_{\sigma(2)}, e_{\sigma(3)}, e_r, e_\ell) R(e_{\sigma(4)}, e_{\sigma(5)}, e_r, e_\ell) + 2p^2 R(e_{\sigma(4)}, e_{\sigma(5)}, Je_{\sigma(2)}, Je_{\sigma(3)})
\]
\[
\quad + 2p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle R(e_{\sigma(4)}, e_{\sigma(5)}, e_r, Je_r).
\]
Plugging $A_{1a}, A_{1b}, A_{1c}$ into $A_1$, we get
\[
A_1 = p^2 \sum_{\sigma(1)=1} \text{sgn}(\sigma) \left\{ \left[ R(e_{\sigma(2)}, e_{\sigma(3)}, e_r, e_\ell) R(e_{\sigma(4)}, e_{\sigma(5)}, e_r, e_\ell) + 2p^2 R(e_{\sigma(4)}, e_{\sigma(5)}, Je_{\sigma(2)}, Je_{\sigma(3)}) \right. \right.
\]
\[
\quad + 2p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle - 4p^4 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle \left[ R(e_{\sigma(4)}, e_{\sigma(5)}, Je_{\sigma(2)}, Je_{\sigma(3)}) + 2p^2 R(e_{\sigma(4)}, e_{\sigma(5)}, Je_{\sigma(2)}, Je_{\sigma(3)}) \right. \left. \left. \right. \right. \right.
\]
\[
\left. \left. \right. + 2p^2 \langle Je_{\sigma(2)}, e_{\sigma(3)} \rangle \langle Je_{\sigma(4)}, e_{\sigma(5)} \rangle \right\}.
\]
Denote the five terms in the last expression by $b_1, \ldots, b_5$. Note that
\[
b_1 = -4 \text{tr}(\Omega \wedge \Omega)(e_2, Je_2, e_3, Je_3) = 32\pi^2 p_1(\Omega)(e_2, Je_2, e_3, Je_3), \tag{A.3}
\]
where $p_1(\Omega)$ is the first Pontrjagin form. (As noted before, since the orbits of $\overline{\xi}$ have constant length $\lambda$, we have
\[
\int_{\mathcal{M}_p} b_1 dvol_{\mathcal{M}_p} = \frac{32\pi^2}{\lambda} \int_{\mathcal{M}} p_1(\Omega) = 96\pi^2 \lambda \sigma(M), \tag{A.4}
\]
where $\sigma(M)$ is the signature of $M$.)
Consider $b_2 = \sum_{\sigma(1)=1} \text{sgn}(\sigma) R(e_{\sigma(2)}, e_{\sigma(3)}, Je_{\sigma(4)}, Je_{\sigma(5)})$. As in the statement of Lemma \[3.7], choose an orthonormal frame $\{e_2, Je_2, e_3, Je_3\}$. The 24 permutations of $\{2, 3, 4, 5\}$ break into three groups of eight permutations, such that
\(\text{sgn}(\sigma)R(e_{\sigma(2)}, e_{\sigma(3)}, J e_{\sigma(4)}, J e_{\sigma(5)})\) is constant on each group. For example, the permutations

\[
\text{id}, (23), (23)(45), (45), (24)(35), (2534), (2435), (25)(34) \tag{A.5}
\]

have, for \(\sigma = \text{id}\),

\[
\text{sgn}(\sigma)R(e_{\sigma(2)}, e_{\sigma(3)}, J e_{\sigma(4)}, J e_{\sigma(5)}) = R(e_2, J e_2, J e_3, J^2 e_3) = R(e_2, J e_2, J e_3, -e_3) = R(e_2, J e_2, e_3, J e_3),
\]

and have, for \(\sigma = (25)(34)\),

\[
\text{sgn}(\sigma)R(e_{\sigma(2)}, e_{\sigma(3)}, J e_{\sigma(4)}, J e_{\sigma(5)}) = R(J e_3, e_3, J^2 e_2, J e_2) = -R(e_2, J e_2, J e_3, e_3) = R(e_2, J e_2, e_3, J e_3).
\]

In fact, all eight permutations in this group give

\[
\text{sgn}(\sigma)R(e_{\sigma(2)}, e_{\sigma(3)}, J e_{\sigma(4)}, J e_{\sigma(5)}) = R(e_2, J e_2, e_3, J e_3). \tag{A.6}
\]

The eight permutations consisting of the product of (34) with the eight permutations in (A.5) (with (34) concatenated on the right) give

\[
\text{sgn}(\sigma)R(e_{\sigma(2)}, e_{\sigma(3)}, J e_{\sigma(4)}, J e_{\sigma(5)}) = -R(e_2, e_3, e_2, e_3), \tag{A.7}
\]

and the eight permutations consisting of the product of (354) with the eight permutations in (A.5) give

\[
\text{sgn}(\sigma)R(e_{\sigma(2)}, e_{\sigma(3)}, J e_{\sigma(4)}, J e_{\sigma(5)}) = -R(e_2, J e_2, e_3, J e_3). \tag{A.8}
\]

Combining (A.6), (A.7), (A.8) gives

\[
b_2 = 32p^2[R(e_2, J e_2, e_3, J e_3) - R(e_2, e_3, e_2, e_3) - R(e_2, J e_2, e_2, J e_3)]. \tag{A.9}
\]

For \(b_3\), up to a factor the permutations \(\text{id}, (23), (45), (23)(45)\) contribute \(R(e_3, J e_3, e_\ell, J e_\ell)\), the permutations \((25)(34), (24)(35), (2534), (2435)\) contribute \(R(e_2, J e_2, e_\ell, J e_\ell)\), and all other permutations contribute zero. Thus

\[
b_3 = 8p^2[R(e_2, J e_2, e_\ell, J e_\ell) + R(e_3, J e_3, e_\ell, J e_\ell)].
\]

Letting \(e_\ell\) run through \(\{e_2, J e_2, e_3, J e_3\}\) gives

\[
b_4 = 16p^2[R(e_2, J e_2, e_2, J e_2) + 2R(e_2, J e_2, e_3, J e_3) + R(e_3, J e_3, e_3, J e_3)]. \tag{A.10}
\]

Using the same grouping of permutations as for \(b_3\), we get the same contribution as \(b_3\), but with the roles of \(e_2\) and \(e_3\) switched:

\[
b_4 = 8p^2[R(e_3, J e_3, e_\ell, J e_\ell) + R(e_2, J e_2, e_\ell, J e_\ell)].
\]

This is the same as \(b_3\), so

\[
b_4 = b_3 = 16p^2[R(e_2, J e_2, e_2, J e_2) + 2R(e_2, J e_2, e_3, J e_3) + R(e_3, J e_3, e_3, J e_3)]. \tag{A.11}
\]
Combining (A.9), (A.10), (A.11) gives
\[ b_2 + b_3 + b_4 = 32p^2 [3R(e_2, J e_2, e_3, J e_3) - R(e_2, e_3, e_2, e_3) - R(e_2, J e_3, J e_3) + R(e_2, J e_2, e_2, J e_2) + R(e_3, J e_3, e_3, J e_3)] \] (A.12)

Finally, up to a factor \( b_5 \) contributes one for each permutation in (A.5) and zero otherwise, so
\[ b_5 = 192p^4. \] (A.13)

**Summary:** The Chern-Simons form (A.1) reduces to calculating the term \( A_1 = b_1 + \ldots + b_5 \). By (A.3), (A.12), (A.13), this becomes
\[
a^*CS_{W_5}^W(\xi, e_2, J e_2, e_3, J e_3) = \frac{3p^2}{3} \left\{ 32\pi p_1(\Omega)(e_2, J e_2, e_3, J e_3) + 32p^2 [3R(e_2, J e_2, e_3, J e_3) - R(e_2, e_3, e_2, e_3) - R(e_2, J e_3, J e_3) + R(e_2, J e_2, e_2, J e_2) + R(e_3, J e_3, e_3, J e_3)] + 192p^4 \right\}
\]

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