Relation between the Weyl group orbits of fundamental weights for multiply-laced finite dimensional simple Lie algebras and d-complete posets

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Abstract

It is known that there exists an order isomorphism between the Weyl group orbit through a minuscule weight of a simply-laced finite-dimensional simple Lie algebra and the set of all order filters in a self-dual connected d-complete poset. In this paper, we try to extend this fact to the case of multiply-laced finite-dimensional simple Lie algebras by using the “folding” technique with respect to a Dynkin diagram automorphism.

1 Introduction.

A d-complete poset, introduced by Robert A. Proctor ([P1, P2]), is a finite poset that satisfies some local conditions described in terms of double-tailed diamonds (see Section 2). A d-complete poset is one of the extensions of Young diagrams or shifted Young diagrams; in fact, a d-complete poset has an extension of some Young diagram’s properties such as the hook length property ([P3]) and the jeu de taquin property ([P4]). So, it is expected that d-complete posets will be used in combinatorial representation theory as well as Young diagrams and shifted Young diagrams are used.

Now, we recall the fundamental relation between d-complete posets and finite-dimensional simple Lie algebras (see Section 5). Let \( g \) be a simply-laced finite-dimensional simple Lie algebra, with \( W = \langle s_i \mid i \in I \rangle \) the Weyl group. Let \( \lambda \) be a dominant integral weight of \( g \), and set \( W_\lambda := \{ w \in W \mid w\lambda = \lambda \} \). We define an order \( \leq_s \) (resp., \( \leq_w \)) on \( W\lambda \) which corresponds to the Bruhat order (resp., weak Bruhat order) under the canonical map \( W\lambda \cong W/W_\lambda \subset W \). If \( \lambda \) is minuscule (in this case, \( \leq_s \) is identical to \( \leq_w \)), then there exists a connected self-dual d-complete poset \((P_\lambda, \leq)\) such that \((W\lambda, \leq_s) = (W\lambda, \leq_w)\) and \((\mathcal{F}(P_\lambda), \subseteq)\)

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are isomorphic as posets ([P1, Section 14]), where \( \mathcal{F}(P_{\lambda}) \) is the set of order filters of \( P_{\lambda} \). Furthermore, using a unique map \( \kappa : P_{\lambda} \to I \) called coloring, we construct an \( I \)-colored \( d \)-complete poset \( (P_{\lambda}, \leq, \kappa, I) \). Then, there exist a unique order isomorphism \( f : W\lambda \to \mathcal{F}(P_{\lambda}) \) satisfying the condition that \( \mu \to s_i \mu \) is a cover relation in \( W\lambda \) if and only if \( f(s_i \mu) \setminus f(\mu) \) consists of one element \( x \) with \( \kappa(x) = i \) ([P2, Proposition 9.1]). There are some important applications of these results. For example, the problem counting the \( \lambda \)-minuscule elements is reduced to the combinatorial problem counting the “standard tableaux” of the corresponding \( d \)-complete posets ([S, Theorem 3.5]). Also, constructing the “colored hook formula” for the corresponding from the reflection on Cartan subalgebra’s dual space \( h^* \) to the action “removing hook” on \( d \)-complete poset ([N]).

In this paper, we study the relation between the Weyl group orbit through a dominant integral weight and the set of order filters in a \( d \)-complete poset in the case that \( g \) is multiply-laced. To do this, we use the “folding” technique (see Section 3). Assume that \( g \) is of type \( A_n, D_n, E_6 \). Let \( \sigma \) be a non-trivial automorphism of the Dynkin diagram of \( g \); note that \( \sigma \) canonically induces a Lie algebra automorphism of \( g \) and a linear automorphism of \( h^* \). Then the fixed point subalgebra \( g(0) := \{ x \in g \mid \sigma(x) = x \} \) is isomorphic to a multiply-laced finite-dimensional simple Lie algebra with \( h(0) := \{ h \in h \mid \sigma(h) = h \} \) the Cartan subalgebra. Let \( J \) be the set of \( \sigma \)-orbits in \( I \), and let \( \tilde{W} = (\tilde{s}_p \mid p \in J) \subset GL(h(0)^*) \) be the Weyl group of \( g(0) \). Then, \( \tilde{W} := \{ w \in W \mid \sigma w \sigma^{-1} = w \} \) is group isomorphic to \( W \). Let \( \text{res} : h^* \to h(0)^* \) be the restriction map. The map \( \text{res}|_{\tilde{W}_{\lambda}} \) gives a bijection \( \tilde{W}_{\lambda} \) onto \( \tilde{W} \text{res}(\lambda) \) for a dominant integral weight \( \lambda \) of \( g \). Now, let \( \lambda \) be a minuscule dominant integral weight of \( g \). We define \( \tilde{f} : \tilde{W} \text{res}(\lambda) \to \mathcal{F}(P_{\lambda}) \) by \( \tilde{f} \circ \text{res} = f \), and set \( \tilde{\mathcal{F}}(P_{\lambda}) := \text{Im}(\tilde{f}) \subset \mathcal{F}(P_{\lambda}) \).

**Theorem 1.1** (= Theorem 7.2; main theorem).

1. The poset \((\tilde{W} \text{res}(\lambda), \leq_w)\) is isomorphic to the poset \((\tilde{\mathcal{F}}(P_{\lambda}), \trianglelefteq)\), where \( \trianglelefteq \) is a partial order on \( \mathcal{F}(P_{\lambda}) \) defined in terms of an involution \( \tilde{S}_p(p \in J) \) on \( \mathcal{F}(P_{\lambda}) \).

2. The poset \((\tilde{W} \text{res}(\lambda), \leq_s)\) is isomorphic to the poset \((\tilde{\mathcal{F}}(P_{\lambda}), \subseteq)\).

In addition, in the case that \( g \) is of type \( A_n \), we give an explicit description of \( \tilde{\mathcal{F}}(P_{\lambda}) \) (see Theorem 9.4).

This paper is organized as follows. In Sections 2 and 3, we explain a (colored) \( d \)-complete poset and introduce an involution \( S_c \) on \( \mathcal{F}(P) \) for a \( d \)-complete poset \( P \) and a color \( c \). In Section 4, we fix our notation for finite-dimensional simple Lie algebras and explain the orders \( \leq_s, \leq_w \) on \( W\lambda \). In Section 5, we explain the fundamental relation between \( d \)-complete posets and finite-dimensional simple Lie algebras. In Section 6, we review the “folding” technique for a simply-laced finite-dimensional simple Lie algebra. In Section 7, we introduce “\( J \)-colored” \( d \)-complete posets by using the folding technique. In Section 8, we prove Theorem
above. In Section 9, we give an explicit description of $\tilde{F}(P_\lambda)$ in the case that $g$ is of type $A_n$.

Acknowledgements. The author would like to thank Professor Daisuke Sagaki, who is his supervisor, for his helpful advice.

2 d-complete posets.

Let $(P, \leq)$ be a poset. When $x$ is covered by $y$ in $P$, we write $x \to y$. For $x, y \in P$, we set $[x, y] := \{ z \in P \mid x \leq z \leq y \}$, which we call an interval. A subset $F$ is called an order filter if every element in $P$ greater than an element in $F$ is always contained in $F$. Let $\mathcal{F}(P)$ be the set of all order filters in $P$. Let $(P, \leq)^*$ denote the order dual set of $(P, \leq)$. If $(P, \leq)$ is isomorphic, as a poset, to $(P, \leq)^*$, then $(P, \leq)$ is said to be self-dual. If the Hasse diagram of $P$ is connected, then the $P$ is said to be connected.

Definition 2.1 ([P1, Section 2]). For $k \geq 3$, we define a poset $d_k(1)$ by the following conditions (1) and (2) (see also Figure 1):

1. $d_k(1)$ consists of $2k - 2$ elements $w_k, w_{k-1}, \ldots, w_3, x, y, z_3, \ldots, z_{k-1}, z_k$.

2. The partial order on $d_k(1)$ is as follows:

$$w_k < w_{k-1} < \cdots < w_3, \quad w_3 < x < z_3, \quad w_3 < y < z_3,$$

$$x \not\leq y, \quad x \not\geq y, \quad z_3 < \cdots < z_{k-1} < z_k.$$

We call $d_k(1)$ the double-tailed diamond. Also, we define $d_k^-(1) := d_k(1) \setminus \{z_k\}$ for $k \geq 3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.9\textwidth]{figure1.png}
\caption{Double-tailed diamonds.}
\end{figure}

Definition 2.2 ([P1, Section 2]). Let $P$ be a poset, and $x, y \in P$. For $k \geq 3$ (resp., $k \geq 4$), if the interval $[x, y]$ is isomorphic to $d_k(1)$ (resp., $d_k^-(1)$), then we say that $[x, y]$ is a $d_k$-interval (resp., $d_k^-$-interval). If $w, x, y \in P$ satisfy $w \to x$ and $w \to y$, then we say that $\{w, x, y\}$ is a $d_k^-$-interval.
Definition 2.3 ([P1, Section 3]). Let $P$ be a poset. Let $k \geq 4$ (resp., $k = 3$), and let $I = [x, y]$ (resp., $I = \{w, x, y\}$) be a $d_k^-$-interval in $P$. If $I \cup \{z\}$ is not a $d_k$-interval for any $z \in P$, then the $d_k^-$-interval $I$ is called an incomplete $d_k^-$-interval. If there is another $d_k^-$-interval $I' = [x', y']$ (resp., $I' = \{w', x', y'\}$) such that $I \setminus \{\min I\} = I' \setminus \{\min I'\}$ and $\min I \neq \min I'$, then the $d_k^-$-interval $I$ is called an overlapping $d_k^-$-interval.

Definition 2.4 ([P1, Section 3]). A finite poset $P$ is called a $d$-complete poset if $P$ satisfies the following conditions (D1)-(D3):

(D1) There is no incomplete $d_k^-$-interval in $P$ for any $k \geq 3$.

(D2) If $I$ is a $d_k$-interval in $P$ for some $k \geq 3$, then there is no element that is not included in $I$ and is covered by $\max I$.

(D3) There is no overlapping $d_k^-$-interval in $P$ for any $k \geq 3$.

![Figure 2: Connected, self-dual d-complete posets.](image)

Definition 2.5 ([P1, Section 4]). Let $P$ be a $d$-complete poset. We define the top tree $T_P$ of $P$ to be the subset of $P$ consisting of all elements $x \in P$ satisfying the condition that

(T) $\#\{z \in P \mid y \rightarrow z\} \leq 1$ for every $y \in P$ such that $x \leq y$.

Proposition 2.6 ([P1, Sections 3 and 14], [P2, Proposition 8.6]). Let $P$ be a $d$-complete poset.

(1) If $P$ is connected, then $P$ has a unique maximum element.

(2) For each $w \in P \setminus T_P$, there are unique $z \in P$ and $k \geq 3$ such that $[w, z]$ is a $d_k$-interval.

(3) A connected self-dual $d$-complete poset is isomorphic, as a poset, to one of those in Figure 2.
Example 2.7. (1) For \( m, n \geq 1 \), we set \( Y_{m,n} := \{(i,j) \mid i, j \in \mathbb{Z}, \ 1 \leq i \leq m, \ 1 \leq j \leq n\} \); we identify \( Y_{m,n} \) with a Young diagram of rectangular shape as the left diagram in Figure 3. We define a partial order on \( Y_{m,n} \) as follows. If \( i_1 \geq i_2 \) and \( j_1 \geq j_2 \), then \((i_1,j_1) \leq (i_2,j_2)\). Then the poset \((Y_{m,n}, \leq)\) is a d-complete poset of Shape class in Figure 2. The top tree \( T_{Y_{m,n}} \) of \( Y_{m,n} \) is identical to the set of those cells in the first row or in the first column; see the right diagram in Figure 3.

\[
\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & (1,4) \\
(2,1) & (2,2) & (2,3) & (2,4)
\end{array}
\quad
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet
\end{array}
\]

Figure 3: The Young diagram corresponding to \( Y_{2,4} \) and its top tree.

(2) For \( n \geq 1 \), we set \( SY_n = \{(i,j) \mid i, j \in \mathbb{Z}, \ 1 \leq i \leq n, i \leq j \leq n\} \); we identify \( SY_n \) with a shifted Young diagram of “triangular shape” as the left diagram in Figure 4. We define a partial order on \( SY_n \) as that on \( Y_{m,n} \). Then the poset \((SY_n, \leq)\) is a d-complete poset of Shifted Shape class in Figure 2. The top tree \( T_{SY_n} \) of \( SY_n \) is identical to the set of those cells in the first row or in the second column; see the right diagram in Figure 4.

\[
\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & (1,4) \\
(2,2) & (2,3) & (2,4) & (2,5)
\end{array}
\quad
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet
\end{array}
\quad
\begin{array}{cccc}
(3,3) & (3,4) & (3,5) \\
(4,4) & (4,5) \\
(5,5)
\end{array}
\]

Figure 4: The shifted Young diagram corresponding to \( SY_5 \) and its top tree.

In what follows, we use Young diagrams and shifted Young diagrams for d-complete posets of Shape and Shifted Shape classes. For a given subset \( X \) in these d-complete posets \( P \), we indicate an element in \( X \) (resp., in \( P \setminus X \)) by a white cell (resp., gray cell). For example, the left diagram in Figure 5 indicates the subset \{\((1,1), (1,2), (1,3), (2,1)\)\} of \( Y_{2,4} \), which is in fact an order filter of \( Y_{2,4} \). The right diagram in Figure 5 indicates the subset \{\((1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (3,3)\)\} of \( SY_5 \), which is in fact an order filter of \( SY_5 \).
3 Colored d-complete posets and involutions on $\mathcal{F}(P)$.

Let $(P, \leq)$ be a poset, and let $C$ be a set. We call a map $\kappa : P \to C$ a coloring of $P$ with $C$ the set of colors the quadruple $(P, \leq, \kappa, C)$ a colored poset.

**Proposition 3.1** ([P2, Proposition 8.6]). Let $(P, \leq)$ be a d-complete poset, and let $C$ be a set such that $\#C = \#T_P$. There exists a coloring $\kappa : P \to C$ of $P$ satisfying the following conditions (a) and (b):

(a) The restriction of $\kappa : P \to C$ to the top tree $T_P$ is a bijection from $T_P$ onto $C$. Namely, each element of $T_P$ has a different color from each other.

(b) If $[w, z]$ is a $d_k$-interval for some $k \geq 3$, then $\kappa(w) = \kappa(z)$.

Moreover, this coloring of $P$ with $C$ the set of colors is unique, up to the coloring of the top tree $T_P$ in (a). In this case, we call the quadruple $(P, \leq, \kappa, C)$ a colored d-complete poset.

**Proposition 3.2** ([P2, Section 3]). Let $(P, \leq, \kappa, C)$ be a colored d-complete poset.

1. Let $x, y \in P$. If there is the covering relation between $x$ and $y$, or if $x$ and $y$ are incomparable, then $\kappa(x) \neq \kappa(y)$, that is, $x$ and $y$ have distinct colors.
(2) Let $I$ be an interval of $P$. If $I$ is a totally order set, then $\kappa(x) \neq \kappa(y)$ for all elements $x, y \in I$ with $x \neq y$, that is, each element in $I$ has a distinct color from each other.

(3) For each $c \in C$, the subset $\kappa^{-1}\{c\}$ consisting of elements in $P$ having the color $c$ is a totally order set.

**Definition 3.3.** Let $(P, \leq, \kappa, C)$ be a finite colored poset. For each $c \in C$, we define maps $A_c, R_c, S_c : \mathcal{F}(P) \to \mathcal{F}(P)$ as follows. For each $F \in \mathcal{F}(P)$,

$$A_c(F) := \bigcup_{F' \in \mathcal{F}(P)} \setminus F' \subseteq \kappa^{-1}\{c\},$$

$$R_c(F) := \bigcap_{F' \in \mathcal{F}(P)} F \setminus F' \subseteq \kappa^{-1}\{c\},$$

$$S_c(F) := \begin{cases} (A_c(F) \setminus F) \cup R_c(F) & \text{if } (A_c(F) \setminus F) \cup R_c(F) \in \mathcal{F}(P), \\ F & \text{otherwise.} \end{cases}$$

**Remark 3.4.** It is obvious by the definition that $A_c(F) \supseteq F \supseteq R_c(F)$. If $F$ satisfies $R_c(F) = F$ (resp., $A_c(F) = F$), then $S_c(F) = A_c(F)$ (resp., $S_c(F) = R_c(F)$). Also, it can be easily verified that $A_c(F) \supseteq S_c(F) \supseteq R_c(F)$.

**Example 3.5.** Let $P = Y_{2,4} = \begin{array}{cccc} & & & \\
& & & \\
& & & \\
& & & \\
\end{array}$, and define a coloring $\kappa : P \to \{1, 2, 3\}$ for $P$ by $\begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array}$. Let $F = \begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array}$; notice that $F$ is an order filter of $P$. Then, $A_2(F), R_2(F), S_2(F)$ are as follows:

$$A_2\left(\begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array}\right) = \begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array},$$

$$R_2\left(\begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array}\right) = \begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array},$$

$$S_2\left(\begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array}\right) = \begin{array}{cccc} 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 2 \\
\end{array}.$$
(1) \( A_c(S_c(F)) = A_c(F) \).

(2) \( R_c(S_c(F)) = R_c(F) \).

(3) \( S_c(S_c(F)) = F \). Namely, the map \( S_c : \mathcal{F}(P) \to \mathcal{F}(P) \) is an involution on \( \mathcal{F}(P) \).

**Proof.** By the definition of \( S_c(F) \), it suffices to consider the case that \( (A_c(F) \setminus F) \cup R_c(F) \) is an order filter of \( P \).

(1) Since all elements of \( A_c(F) \setminus R_c(F) \) have the color \( c \) and since \( S_c(F) \supseteq R_c(F) \), all elements of \( A_c(F) \setminus S_c(F) \) also have the color \( c \). Hence, \( A_c(F) \in \{ F' \in \mathcal{F}(P) \mid F' \setminus S_c(F) \subseteq \kappa^{-1}(\{c\}) \} \), and hence \( A_c(S_c(F)) \supseteq A_c(F) \) by the definition of \( A_c \). This inclusion relation also implies that all elements in \( A_c(S_c(F)) \setminus A_c(F) \) have the color \( c \). By the definition of \( A_c \), all elements in \( A_c(F) \setminus F \) have the color \( c \). Hence, \( A_c(S_c(F)) \in \{ F' \in \mathcal{F}(P) \mid F' \setminus F \subseteq \kappa^{-1}(\{c\}) \} \). By the definition of \( A_c \), we obtain \( A_c(S_c(F)) \subseteq A_c(F) \). Therefore, \( A_c(S_c(F)) = A_c(F) \).

(2) Similar to Part (1).

(3) We compute

\[
\begin{align*}
(A_c(S_c(F)) \setminus S_c(F)) \cup R_c(S_c(F)) &= (A_c(F) \setminus ((A_c(F) \setminus F) \cup R_c(F))) \cup R_c(F) \\
&= ((A_c(F) \setminus (A_c(F) \setminus F)) \cap (A_c(F) \setminus R_c(F))) \cup R_c(F) \\
&= (F \cup R_c(F)) \cap ((A_c(F) \setminus R_c(F)) \cup R_c(F)) \\
&= F \cap A_c(F) \\
&= F
\end{align*}
\]

Therefore, \( (A_c(S_c(F)) \setminus S_c(F)) \cup R_c(S_c(F)) \) is an order filter of \( P \), and \( S_c(S_c(F)) = F \).

\[ \square \]

**Definition 3.7.** Let \( (P, \leq, \kappa, C) \) be a colored poset. We define an order \( \sqsubseteq \) on \( \mathcal{F}(P) \) as follows. For \( F, F' \in \mathcal{F}(P) \), \( F \sqsubseteq F' \) if there exists a sequence of order filters \( F = F_0, F_1, \ldots, F_{n-1}, F_n = F' \) such that for all \( i \in \{0, 1, \ldots, n - 1\} \), there exist \( c_i \in C \) such that \( S_{c_i}(F_i) = F_{i+1} \supset F_i \).

**Lemma 3.8.** Let \( (P, \leq, \kappa, C) \) be a colored d-complete poset. For an order filter \( F \) of \( P \) and a color \( c \in C \), the symmetric difference of \( F \) and \( S_c(F) \) has at most one element.
Proof. Suppose, for a contradiction, that the symmetric difference of $F$ and $S_c(F)$ has more than one element. Let $x, y$ be the elements of the symmetric difference, with $x \neq y$. Because both $x$ and $y$ have the color $c$, it follows from Proposition 3.2(3) that either $x < y$ or $x > y$ holds; we may assume that $x < y$. Because both $F$ and $S_c(F)$ are order filters, we deduce that either $x, y \in F \setminus S_c(F)$ or $x, y \in S_c(F) \setminus F$ holds. Assume that $x, y \in F \setminus S_c(F)$. Since $x < y$, there exists an element $z \in P$ such that $x \rightarrow z$ and $z \leq y$. Because $x \in F$, and $F$ is an order filter, we see that $z \in F$. Similarly, because $y \notin S_c(F)$, and $S_c(F)$ is an order filter, we see that $z \notin S_c(F)$. Thus we get $z \in F \setminus S_c(F)$; in particular, $z$ has the color $c$. However, this contradicts Proposition 3.2(1); recall that $x \rightarrow z$, and $x$ has the color $c$. A proof for the case that $x, y \in S_c(F) \setminus F$ is similar. □

Remark 3.9. Let $(P, \leq, \kappa, C)$ be a colored d-complete poset. By Lemma 3.8 it is clear that for $F, F' \in \mathcal{F}(P)$, $F \subseteq F'$ if and only if $F \subseteq F'$. In particular, $(\mathcal{F}(P), \leq)$ and $(\mathcal{F}(P), \subseteq)$ are order isomorphic.

4 Finite-dimensional simple Lie algebras.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, with $A = (a_{ij})_{i,j \in I}$ the Cartan matrix. Denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathfrak{h}$ the set of simple coroots, $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ the set of simple roots, $\Delta_+ \subset \mathfrak{h}^*$ the set of positive roots, $\Delta_- \subset \mathfrak{h}^*$ the set of negative roots, $\Lambda_i \in \mathfrak{h}^*(i \in I)$ the fundamental weights, and $e_i, f_i \in \mathfrak{g}(i \in I)$ the Chevalley generators. Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$, where $s_i$ is the simple reflection in $\alpha_i$ for $i \in I$. For $\beta \in \Delta_+$, $\beta^\vee \in \mathfrak{h}$ denotes the dual root of $\beta$, and $s_\beta \in W$ denotes the reflection in $\beta$; recall that if $\beta = w(\beta')$ for $\beta' \in \Delta_+$ and $w \in W$, then $s_\beta = s_{w(\beta')} = ws_\beta w^{-1}$.

Definition 4.1. Let $\lambda$ be a dominant integral weight of $\mathfrak{g}$. We define the order $\preceq_s$ on the Weyl group orbit $W\lambda$ through $\lambda$ as follows. For $\mu, \mu' \in W\lambda$, $\mu \preceq_s \mu'$ if there exists a finite sequence $\mu = \mu_0, \mu_1, \ldots, \mu_{k-1}, \mu_k = \mu'$ of elements in $W\lambda$ and a finite sequence $\beta_0, \ldots, \beta_{k-1}$ of elements in $\Delta_+$ such that $s_{\beta_i}(\mu_i) = \mu_{i+1}$ and $\mu_i(\beta_i^\vee) > 0$ for each $i \in \{0, 1, \ldots, k - 1\}$.

Lemma 4.2. Let $\mu$ be an integral weight of $\mathfrak{g}$, and $\beta \in \Delta_+$. For $w \in W$, if $\mu <_s s_\beta(\mu)$ and $w(\beta) \in \Delta_+$, then $w(\mu) <_s ws_\beta(\mu)$.

Proof. Since $s_{w(\beta)}(w\mu) = ws_\beta w^{-1}(w\mu) = ws_\beta(\mu)$, and since $w(\mu) \neq ws_\beta(\mu)$, either $w(\mu) <_s ws_\beta(\mu)$ or $w(\mu) >_s ws_\beta(\mu)$ holds. By the definition of $\preceq_s$, there exists $n \in \mathbb{Z}_{>0}$ such that $s_\beta(\mu) = \mu - n\beta$. Thus, $ws_\beta(\mu) = w(\mu - n\beta) = w(\mu) - nw(\beta)$. Because $w(\beta) \in \Delta_+$, we obtain $w(\mu) <_s ws_\beta(\mu)$, as desired. □

Proposition 4.3 ([L] Lemma 4.1). Let $\mu_1, \mu_2 \in W\lambda$, and $i \in I$. 

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(1) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \geq 0$ and $\mu_2(h_i) \leq 0$, then $\mu_1 \leq_s s_i(\mu_2)$.

(2) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \geq 0$ and $\mu_2(h_i) \leq 0$, then $s_i(\mu_1) \leq_s \mu_2$.

(3) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \leq 0$ and $\mu_2(h_i) \leq 0$, then $s_i(\mu_1) \leq_s s_i(\mu_2)$.

(4) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \geq 0$ and $\mu_2(h_i) \geq 0$, then $s_i(\mu_1) \leq_s s_i(\mu_2)$.

**Definition 4.4.** Let $\lambda$ be a dominant integral weight of $\mathfrak{g}$. We define the order $\leq_w$ on $W\lambda$ as follows. For $\mu, \mu' \in W\lambda$, $\mu \leq_w \mu'$ if there exists a finite sequence $\mu = \mu_0, \mu_1, \ldots, \mu_k = \mu'$ of elements in $W\lambda$ and a finite sequence $j_0, \ldots, j_{k-1}$ of elements in $I$ such that $s_{j_i}(\mu_i) = \mu_{i+1}$ and $\mu_i(h_{j_j}) > 0$ for each $i \in \{0, 1, \ldots, k-1\}$.

**Remark 4.5** (see, e.g., [G, Section 4.3] and [B, Section 2.4]). Let $\lambda$ be a dominant integral weight, and $W_\lambda := \{w \in W \mid w\lambda = \lambda\}$ the stabilizer of $\lambda$; we have the canonical bijection $W/W_\lambda \to W\lambda, wW_\lambda \mapsto w\lambda$. It is known that $W_\lambda$ is the subgroup of $W$ generated by $s_i$ for $i \in I$ such that $\lambda(h_i) = 0$, and each coset in $W/W_\lambda$ has a unique element whose length is minimal among the element in the coset; we regard $W/W_\lambda$ as a subset of $W$ by taking the minimal-length coset representative from each coset in $W/W_\lambda$. The poset $W/W_\lambda$ in the restriction of the Bruhat order (resp., the weak Bruhat order) on $W$ is order isomorphic to $(W\lambda, \leq_s)$ (resp., $(W\lambda, \leq_w)$) under the canonical map $W/W_\lambda \to W\lambda$ above.

## 5 Order isomorphism between $W\lambda$ and $\mathcal{F}(P)$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$.

**Definition 5.1.** Let $\lambda$ be a dominant integral weight of $\mathfrak{g}$. We call $\lambda$ a minuscule weight if $\lambda$ satisfies $(w\lambda)(h_i) \in \{-1, 0, 1\}$ for all $w \in W$ and $i \in I$.

Table 1 below is the list of minuscule weights of simply-laced finite-dimensional simple Lie algebras; the vertices of the Dynkin diagram are numbered as Figure 7.

| $\mathfrak{g}$ | minuscule weight $\lambda$ |
|---------------|--------------------------|
| $A_n$         | $\Lambda_1, \ldots, \Lambda_n$ |
| $D_n$         | $\Lambda_1, \Lambda_{n-1}, \Lambda_n$ |
| $E_6$         | $\Lambda_1, \Lambda_5$ |
| $E_7$         | $\Lambda_6$ |
| $E_8$         | none |

Table 1: Minuscule weights; simply-laced case.
Remark 5.2 ([G] Lemma 11.1.18) and Remark 4.5. Assume that $\lambda$ is minuscule. For $\mu, \mu' \in W\lambda$, $\mu \leq_s \mu'$ if and only if $\mu \leq_w \mu'$. Therefore, $(W\lambda, \leq_s)$ and $(W\lambda, \leq_w)$ are order isomorphic.

Proposition 5.3 ([P1] Section 14). Assume that $\mathfrak{g}$ is simply-laced. Let $\lambda$ be a minuscule weight of $\mathfrak{g}$. There exists a connected, self-dual d-complete poset $P_\lambda$ such that $(W\lambda, \leq_s)$ and $(\mathcal{F}(P_\lambda), \subseteq)$ are isomorphic, as posets (see also Table 2).

![Diagram of Dynkin diagrams](image.png)

Figure 7: Simply-laced Dynkin diagrams.

### Table 2: The d-complete posets $P_\lambda$ corresponding to minuscule weights $\lambda$.

| $\mathfrak{g}$ | minuscule weight $\lambda$ | corresponding d-complete poset $P_\lambda$ |
|----------------|---------------------------|-----------------------------------------|
| $A_n$          | $\Lambda_i (1 \leq i \leq n)$ | $Y_{i,n-i+1}$ (Shape)                  |
| $D_n$          | $\Lambda_1$               | $SY_{n-1}$ (Shifted Shape)             |
| $E_6$          | $\Lambda_1, \Lambda_5$   | $d_n(1)$ (Inset)                       |
| $E_7$          | $\Lambda_6$               | Bat                                     |

Keep the setting in Proposition 5.3 with $\lambda = \Lambda_i$ for some $i \in I$ such that $\Lambda_i$ is minuscule. We know from [P2] Proposition 8.6 that the graph obtained from the Hasse diagram of the top tree $T_{P_\lambda}$ of $P_\lambda$ by replacing each allow by an edge is identical to the Dynkin diagram of $\mathfrak{g}$; in particular, $\#I = \#T_{P_\lambda}$. By Proposition 3.1 we can obtain the colored poset $(P_\lambda, \leq, \kappa, I)$ such that $\kappa|_{T_{P_\lambda}} : T_{P_\lambda} \to I$ is the
graph isomorphism and the maximum element of $P$ (notice that it is contained in $T_p$) is sent to the $i$ under the map $\kappa$. We call $(P, \leq, \kappa, I)$ the $I$-colored $d$-complete poset for the minuscule weight $\lambda$.

**Proposition 5.4** ([P2, Proposition 9.1]). Keep the notation and setting in Proposition 5.3. Let $(P, \leq, \kappa, I)$ be the $I$-colored $d$-complete poset. There exists a unique order isomorphism $f : (W\lambda, \leq_s) \to (F(P), \subseteq)$ such that for $\mu \in W\lambda$ and $i \in I$, there exists the cover relation $\mu \to s_i\mu$ in $W\lambda$ if and only if $f(s_i(\mu)) \setminus f(\mu)$ consists of one element having the color $i$.

**Example 5.5.** Let $g$ be of type $A_5$, and $\lambda = \Lambda_2$; in this case, the corresponding (connected, self-dual) $d$-complete poset $P_{A_2}$ is $Y_{2,4}$. Let $(P_{A_2}, \leq, \kappa, I)$ be the $I$-colored $d$-complete poset, with the coloring $\kappa$ as in Figure 8. The Hasse diagrams of $(W\Lambda_2, \leq_s)$ and $(F(P_{A_2}), \subseteq)$ are given in Figure 8 below:

![Hasse diagrams](image)

Figure 8: $(W\Lambda_2, \leq_s)$ and $(F(P_{A_2}), \subseteq)$ of type $A_5$

The next corollary follows from Remark 3.9 and Proposition 5.4.

**Corollary 5.6.** Assume that $g$ is simply-laced. Let $\lambda$ be a minuscule weight of $g$, and let $(P, \leq)$ be the $d$-complete poset such that $(F(P), \subseteq)$ is isomorphic to $(W\lambda, \leq_s)$ (see Proposition 5.3). Let $(P, \leq, \kappa, I)$ be the $I$-colored $d$-complete
For $\mu \in W\lambda$ and $i \in I$,

$$f(s_i(\mu)) = S_i(f(\mu)).$$

For $F \in \mathcal{F}(P\lambda)$ and $i \in I$, we define $c_i(F) := \#\{x \in F \mid \kappa(x) = i\}$. Because $\lambda$ is minuscule, we see that if there exists the cover relation $\mu \rightarrow s_i(\mu)$ in $W\lambda$, then $\mu(h_i) = 1$ and $s_i(\mu) = \mu - \alpha_i$. Hence we have the next corollary.

**Corollary 5.7.** For $\mu \in W\lambda$ and $F = f(\mu)$,

$$\mu = \sum_{i \in I}((\#(S_i(F)) - \#(F))\Lambda_i = \lambda - \sum_{i \in I}c_i(F)\alpha_i.$$ 

For $F \in \mathcal{F}(P\lambda)$, we define

$$g(F) := \sum_{i \in I}((\#(S_i(F)) - \#(F))\Lambda_i = \lambda - \sum_{i \in I}c_i(F)\alpha_i.$$ 

By Corollary 5.7, $g : (\mathcal{F}(P\lambda), \subseteq) \sim (W\lambda, \leq_s)$ is the inverse of $f$.

We will use the following proposition later.

**Proposition 5.8** ([P2, Proposition 8.6]). Keep the notation and setting in Proposition 5.3. Let $(P\lambda, \leq, \kappa, I)$ be the $I$-colored d-complete poset. If there exists the covering relation between $x, y \in P\lambda$, then the color $\kappa(x)$ of $x$ is adjacent to the color $\kappa(y)$ of $y$ in the Dynkin diagram of $g$.

## 6 Folding of a Lie algebra.

We review the “folding” of a simply-laced finite-dimensional simple Lie algebra; for the details, see [K, Sections 7.9 and 7.10] and [C, Section 9.5] in example.

Let $g$ be the finite-dimensional simple Lie algebra of type $A_n, D_n$ or $E_6$; we use the notation in Section 4. Let $\sigma$ be a non-trivial graph automorphism of the Dynkin diagram of $g$. Denote by $\langle \sigma \rangle$ the cyclic group generated by $\sigma$ (in the group of permutations on $I$), and $J$ the set of $\langle \sigma \rangle$-orbits on $I$. We say that $p \in J$ satisfies the **orthogonality condition** if $a_{ij} = a_{ji} = 0$ for all $i, j \in p$ with $i \neq j$; notice that $p \in J$ does not satisfy the orthogonality condition if and only if $g$ is of type $A_{2n}$ and $p = \{n, n + 1\}$. It is known that the graph automorphism $\sigma$ induces a (unique) Lie algebra automorphism of $g$ such that $\sigma(e_i) = e_{\sigma(i)}, \sigma(f_i) = f_{\sigma(i)}, \sigma(h_i) = h_{\sigma(i)}$ for $i \in I$; we set $g(0) := \{x \in g \mid \sigma(x) = x\}$. For each $p \in J$, we define $H_p, E_p, F_p \in g(0)$ as follows:
If $p$ satisfies the orthogonality condition, then

$$H_p := \sum_{i \in p} h_i, \quad E_p := \sum_{i \in p} e_i, \quad F_p := \sum_{i \in p} f_i.$$ 

(2) If $p$ does not satisfy the orthogonality condition, then

$$H_p := 2 \sum_{i \in p} h_i, \quad E_p := \sum_{i \in p} e_i, \quad F_p := 2 \sum_{i \in p} f_i.$$ 

**Proposition 6.1** (see, e.g., [K, Sections 7.9 and 7.10]). The fixed point subalgebra $\mathfrak{g}(0)$ is generated by $\{H_p, E_p, F_p\}_{p \in J}$, and is isomorphic to a multiply-laced finite-dimensional simple Lie algebra; see Figure 9 and Table 3. 

Let $\mathfrak{h}(0)$ be the subspace of $\mathfrak{h}$ spanned by $\{H_p\}_{p \in J}$, which is a Cartan subalgebra of $\mathfrak{g}(0)$. Denote by $\text{res} : \mathfrak{h}^* \to \mathfrak{h}(0)^*, \mu \mapsto \mu|_{\mathfrak{h}(0)}$, the restriction map, and set $\beta_p := \text{res}(\alpha_i) \in \mathfrak{h}(0)^*$ for $p \in J$, where $i$ is an arbitrary element in the $\langle \sigma \rangle$-orbit $p$. 

![Figure 9: The Dynkin diagram of $\mathfrak{g}$, its (non-trivial) graph automorphism $\sigma : I \to I$, and the Dynkin diagram of the fixed point subalgebra $\mathfrak{g}(0)$.](image-url)
Table 3: \(g\), \(\sigma\), and \(g(0)\). The vertices of the Dynkin diagram of \(g(0)\) are “num-
bered” as Figure 9.

| type of \(g\) | \(A_{2n}\) | \(A_{2n-1}\) | \(D_{n+1}\) | \(E_6\) | \(D_4\) |
|---------------|-------------|-------------|------------|------|-----|
| order of \(\sigma\) | 2 | 2 | 2 | 2 | 3 |
| type of \(g(0)\) | \(B_n\) | \(C_n\) | \(B_n\) | \(F_4\) | \(G_2\) |

note that \(\beta_p\) is independent of the choice of \(i \in p\). The set of simple coroots and the set of simple roots of \(g(0)\) are given by \(\{H_p\}_{p \in J}\) and \(\{\beta_p\}_{p \in J}\), respectively. Denote by \(\Delta_+ \subset \mathfrak{h}(0)^*\) the set of positive roots of \(g(0)\), and \(\Delta_- \subset \mathfrak{h}(0)^*\) the set of negative roots of \(g(0)\). For \(p \in J\), we define \(\hat{s}_p(\nu) := \nu - \nu(H_p)\beta_p\) for \(\nu \in \mathfrak{h}(0)^*\). Then, \(\tilde{W} := \langle \hat{s}_p \mid p \in J \rangle\) is the Weyl group of \(g(0)\).

For each \(p \in J\), we define \(\hat{s}_p \in W\) as follows:

1. If \(p\) satisfies the orthogonality condition, then
   \[
   \hat{s}_p := \prod_{k \in p} s_k.
   \]
2. If \(p\) does not satisfy the orthogonality condition, that is, if \(g\) is of type \(A_{2n}\) and \(p = \{n, n+1\}\) (see also page 13), then
   \[
   \hat{s}_p := s_n s_{n+1} s_n = s_{n+1} s_n s_{n+1}.
   \]

**Lemma 6.2.** For \(p \in J\), \(\hat{s}_p(\text{res}(\mu)) = \text{res}(\hat{s}_p(\mu))\) for all \(\mu \in \mathfrak{h}^*\).

**Proof.** If \(p\) satisfies the orthogonality condition, then we compute

\[
\text{res}(\hat{s}_p(\mu)) = \text{res}\left(\mu - \sum_{i \in p} \mu(h_i)\alpha_i\right) = \text{res}(\mu) - \text{res}(\mu)(H_p)\beta_p = \hat{s}_p(\text{res}(\mu)).
\]

If \(p\) does not satisfy the orthogonality condition, then we compute

\[
\text{res}(\hat{s}_p(\mu)) = \text{res}(\mu - \mu(h_n + h_{n+1})(\alpha_n + \alpha_{n+1})) = \text{res}(\mu) - \text{res}(\mu)(H_p)\beta_p = \hat{s}_p(\text{res}(\mu)).
\]

Since \(\sigma\) acts on \(\mathfrak{h} = \bigoplus_{i \in I} \mathbb{C}h_i\), \(\sigma\) naturally acts also on \(\mathfrak{h}^*\) by \((\sigma(\mu))(h) = \mu(\sigma^{-1}(h))\) for \(\mu \in \mathfrak{h}^*\) and \(h \in \mathfrak{h}\); we see that \(\sigma(\Lambda_i) = \Lambda_{\sigma(i)}\), \(\sigma(\alpha_i) = \alpha_{\sigma(i)}\) for \(i \in I\). Notice that \(\sigma s_i \sigma^{-1} = s_{\sigma(i)}\) for \(i \in I\) in \(GL(\mathfrak{h}^*)\). Hence, \(\sigma W \sigma^{-1} \subseteq W\).
Proposition 6.3 ([C Proposition 9.17]). Set $\hat{W} := \{ w \in W \mid \sigma w \sigma^{-1} = w \}$. There is a group isomorphism from $\hat{W}$ onto $\hat{W}$ such that $\hat{s}_p \mapsto \hat{s}_p$. Therefore $\hat{W}$ is the subgroup of $W$ generated by $\{ \hat{s}_p \}_{p \in J}$.

Remark 6.4. Because $\tilde{W}$ and $\hat{W}$ are generated by $\{ \hat{s}_p \}_{p \in J}$ and $\{ \hat{s}_p \}_{p \in J}$, we see by Lemma 6.2 that $\text{res}(\hat{W} \lambda) = \hat{W} \text{res}(\lambda)$ for every (dominant) integral weight $\lambda$.

Let $\tilde{\Lambda}_p \in \mathfrak{h}(0)^*(p \in J)$ be the fundamental weights of $\mathfrak{g}(0)$. We can easily show the following lemma.

Lemma 6.5. Let $p \in J$, and $i \in p$.

(1) If $p$ satisfies the orthogonality condition, then $\text{res}(\Lambda_i) = \tilde{\Lambda}_p$.

(2) If $p$ does not satisfy the orthogonality condition, then $\text{res}(\Lambda_i) = 2\tilde{\Lambda}_p$.

Lemma 6.6. Let $\lambda$ be a dominant integral weight of $\mathfrak{g}$, and let $\mu_1, \mu_2 \in \hat{W} \lambda$. If $\text{res}(\mu_1) = \text{res}(\mu_2)$, then $\mu_1 = \mu_2$. Therefore the map $\text{res}|_{\hat{W} \lambda} : \hat{W} \lambda \to \hat{W} \text{res}(\lambda)$ is bijective (see Remark 6.4).

Proof. For each $i = 1, 2$, let $\hat{w}_i \in \hat{W}$ be such that $\mu_i = \hat{w}_i \lambda$, and let $\tilde{w}_i \in \tilde{W}$ be such that $\text{res} \circ \hat{w}_i = \tilde{w}_i \circ \text{res}$ (see Lemma 6.2). We have $\hat{w}_1 \text{res}(\lambda) = \text{res}(\hat{w}_1 \lambda) = \text{res}(\mu_1) = \text{res}(\mu_2) = \text{res}(\hat{w}_2 \lambda) = \hat{w}_2 \text{res}(\lambda)$. Since $\text{res}(\lambda)$ is a dominant integral weight for $\mathfrak{g}(0)$ by Lemma 6.5 it follows that $\hat{w}_1^{-1} \hat{w}_2 \in \langle \tilde{s}_p \mid (\text{res}(\lambda))(H_p) = 0 \rangle$, and hence $\hat{w}_1^{-1} \hat{w}_2 \in \langle \tilde{s}_p \mid (\text{res}(\lambda))(H_p) = 0 \rangle$. Observe that $(\text{res}(\lambda))(H_p) = 0$ if and only if $\lambda(h_i) = 0$ for all $i \in p$. Thus we obtain $\hat{w}_1^{-1} \hat{w}_2(\lambda) = \lambda$, and hence $\mu_1 = \hat{w}_1 \lambda = \hat{w}_2 \lambda = \mu_2$, as desired.

Notice that $\sigma$ preserves $\Delta$ and $\Delta_+, \Delta_-$.

Lemma 6.7. Let $\lambda$ be a dominant integral weight of $\mathfrak{g}$.

(1) For each $\mu \in \hat{W} \lambda$ and $p \in J$, either $\mu(h_i) \geq 0$ for all $i \in p$ or $\mu(h_i) \leq 0$ for all $i \in p$.

(2) For each $\mu \in \hat{W} \lambda$ and $p \in J$, if $\mu(h_i) > 0$ (resp., $\mu(h_i) < 0$) for some $i \in p$, then $\mu <_w \hat{s}_p(\mu)$ (resp., $\mu >_w \hat{s}_p(\mu)$).

Proof. (1) Let $w \in \hat{W}$ be such that $\mu = w \lambda$. Because $\mu(h_i) = (w \lambda)(h_i) = \lambda(w^{-1}h_i) = \lambda((w^{-1} \alpha_i)^\vee)$, and because $\lambda$ is a dominant integral weight, it suffices to show that either $w^{-1} \alpha_i \in \Delta_+$ for all $i \in p$ or $w^{-1} \alpha_i \in \Delta_-$ for all $i \in p$. If $w^{-1} \alpha_i \in \Delta_+$ (resp., $w^{-1} \alpha_i \in \Delta_-$) for some $i \in p$, then $w^{-1} \sigma \alpha_i = w^{-1} \sigma \alpha_i = \sigma w^{-1} \alpha_i \in \Delta_+$ (resp., $\in \Delta_-$). Since $p$ is a $\langle \sigma \rangle$-orbit, the assertion above follows.

(2) We give a proof only for the case that $\mu(h_i) > 0$ for some $i \in p$, and $\#p = 2$; the proofs for the other cases are similar. Since $\mu(h_i) > 0$, it follows
that $\mu < w s_i(\mu)$. If $p = \{i, j\}$, then we see by part (1) that $\mu(h_j) \geq 0$. Assume that $p$ satisfies the orthogonality condition. Then,

$$s_j s_i(\mu) = s_j(\mu - \mu(h_i)\alpha_i) = s_j(\mu) - \mu(h_i)s_j(\alpha_i)$$

$$= \mu - \mu(h_j)\alpha_j - \mu(h_i)\alpha_i = s_i(\mu) - \mu(h_j)\alpha_j \geq_w s_i(\mu).$$

Thus we obtain $\mu <_w s_i(\mu) \leq_w s_j s_i(\mu) = \hat{s}_p(\mu)$, as desired. Assume that $p$ does not satisfy the orthogonality condition. Then,

$$s_j s_i(\mu) = s_j(\mu - \mu(h_i)\alpha_i) = s_j(\mu) - \mu(h_i)s_j(\alpha_i)$$

$$= \mu - \mu(h_j)\alpha_j - \mu(h_i)\alpha_i = s_i(\mu) - \mu(h_i)\alpha_j - \mu(h_j)\alpha_j$$

$$\geq_w \mu - \mu(h_i)\alpha_j - \mu(h_i + h_j)\alpha_j = s_j s_i(\mu).$$

Thus we obtain $\mu <_w s_i(\mu) <_w s_j s_i(\mu) \leq_w s_i s_j s_i(\mu) = \hat{s}_p(\mu)$, as desired. □

**Definition 6.8.** We set $Q_+ := \sum_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i$. For $\nu = \sum_{i \in I} m_i \alpha_i \in Q_+$, we define the height $ht(\nu)$ of $\nu$ by $ht(\nu) := \sum_{i \in I} m_i$. Similarly, we set $\tilde{Q}_+ := \sum_{p \in J} \mathbb{Z}_{\geq 0} \beta_p$. For $\xi = \sum_{p \in J} n_p \beta_p \in \tilde{Q}_+$, we define the height $ht(\xi)$ of $\xi$ by $ht(\xi) := \sum_{p \in J} n_p$.

**Lemma 6.9.** Let $\lambda$ be a dominant integral weight, and $\mu_1, \mu_2 \in \hat{W}\lambda$. Then, $\mu_1 \leq_s \mu_2$ if and only if $res(\mu_1) \leq_s res(\mu_2)$.

**Proof.** First, we show the “if” part. We see that $res(\lambda) - res(\mu_2) \in \tilde{Q}_+$ since $res(\lambda)$ is dominant and $res(\mu_2) \in \hat{W}res(\lambda)$. We show the assertion by induction on $\tilde{h} := ht(res(\lambda) - res(\mu_2))$. If $\tilde{h} = 0$, then $res(\mu_2) = res(\lambda)$. Because $res(\lambda) - res(\mu_1) \in \tilde{Q}_+$, and because $res(\mu_1) - res(\lambda) = res(\mu_1) - res(\mu_2) \in \tilde{Q}_+$ by the definition of $\leq_s$ on $\hat{W}res(\lambda)$, we get $res(\mu_1) = res(\lambda)$. Now, for $i = 1, 2$, we see that $\lambda - \mu_i \in Q_+$. Since $res(\lambda - \mu_i) = res(\lambda) - res(\mu_i) = res(\lambda) - res(\lambda) = 0$, we deduce that $\lambda = \mu_i$. Thus, we obtain $\mu_1 = \lambda \leq_s \lambda = \mu_2$.

Assume that $\tilde{h} > 0$. In this case, there exists $p \in J$ such that $res(\mu_2)(H_p) < 0$, because $res(\lambda)$ is a unique dominant integral weight in $\hat{W}res(\lambda)$; note that $ht(res(\lambda) - \hat{s}_p res(\mu_2)) < \tilde{h}$. Here, we give a proof only for the case that $p = \{i, j\}$ with $i \neq j$, and $p$ satisfies the orthogonality condition; the proofs for the other cases are similar. If $res(\mu_1)(H_p) \geq 0$, then we get $res(\mu_1) \leq s \hat{s}_p res(\mu_2)$ by Proposition 4.3(1). By the induction hypothesis, it follows that $\mu_1 \leq s \hat{s}_p(\mu_2)$. Because $\mu_2(h_i + h_j) = res(\mu_2)(H_p) < 0$, we see by Lemma 6.7 that $\hat{s}_p(\mu_2) = s_j s_i(\mu_2) \leq s_i(\mu_2) \leq s_\mu_2$. Thus we obtain $\mu_1 \leq \mu_2$. If $res(\mu_1)(H_p) \leq 0$, then we get $\hat{s}_p res(\mu_1) \leq s \hat{s}_p res(\mu_2)$ by Proposition 4.3(3). By the induction hypothesis,
it follows that $\hat{s}_p(\mu_1) \leq_s \hat{s}_p(\mu_2)$. Similarly to the case above, we deduce that $\hat{s}_p(\mu_k)(h_i) \leq 0$ and $s_i\hat{s}_p(\mu_k)(h_j) \leq 0$ for $k = 1, 2$. By Proposition 4.3 (4), we obtain $s_i\hat{s}_p(\mu_1) \leq_s s_i\hat{s}_p(\mu_2)$, and then $\mu_1 = s_i s_i\hat{s}_p(\mu_1) \leq_s s_i s_i\hat{s}_p(\mu_2) = \mu_2$, as desired.

Next, we show the “only if” part by the induction on $h := \text{ht}(\lambda - \mu_2)$. If $h = 0$, then we see by the same argument as above that $\mu_1 = \mu_2 = \lambda$. Hence, $\text{res}(\mu_1) \leq_s \text{res}(\mu_2)$. Assume that $h > 0$. Then there exists $i \in I$ such that $\mu_2(h_i) < 0$. Let $p \in J$ be such that $i \in p$. Here, we give a proof only for the case that $p = \{i, j\}$ with $i \neq j$, and $p$ satisfies the orthogonality condition; the proofs for the other cases are similar. By Lemma 6.7, $\mu_2(h_j) \leq 0$ and $\hat{s}_p(\mu_2) = s_j s_i(\mu_2) \leq s_j s_i(\mu_2) < \mu_2$; note that $\text{ht}(\lambda - \hat{s}_p(\mu_2)) < h$. Assume that $\mu_1(h_i) \geq 0$. It follows from Proposition 4.3 (1) that $\mu_1 \leq_s s_i(\mu_2)$. Also, we see by Lemma 6.7 (1) that $\mu_1(h_j) \geq 0$. By Proposition 4.3 (1), we get $\mu_1 \leq_s s_j s_i(\mu_2) = \hat{s}_p(\mu_2)$. By the induction hypothesis, it follows that $\text{res}(\mu_1) \leq_s \hat{s}_p\text{res}(\mu_2)$. Because $\text{res}(\mu_2)(H_p) = \mu_2(h_i + h_j) < 0$, we have $\hat{s}_p\text{res}(\mu_2) \leq_s \text{res}(\mu_2)$, and hence $\text{res}(\mu_1) \leq_s \text{res}(\mu_2)$. Assume that $\mu_1(h_i) \leq 0$. It follows from Proposition 4.3 (3) that $s_i(\mu_1) \leq s_i(\mu_2)$. Also, we see by Lemma 6.7 (2) that $(s_i(\mu_1))(h_j) \leq 0$. By Proposition 4.3 (3), we get $\hat{s}_p(\mu_1) = s_j s_i(\mu_1) \leq s_j s_i(\mu_2) = \hat{s}_p(\mu_2)$. By the induction hypothesis, it follows that $\hat{s}_p\text{res}(\mu_1) \leq_s \hat{s}_p\text{res}(\mu_2)$. Because $\text{res}(\mu_1)(H_p) \leq 0$ and $\text{res}(\mu_2)(H_p) \leq 0$, we obtain $\text{res}(\mu_1) \leq_s \text{res}(\mu_2)$ by Proposition 4.3 (4), as desired.

7 $J$-colored d-complete poset.

Let $\mathfrak{g}$ be a simply-laced finite-dimensional Lie algebra, and let $\sigma$ be a non-trivial graph automorphism of the Dynkin diagram of $\mathfrak{g}$ (see Figure 9). Let $\lambda$ be a minuscule weight of $\mathfrak{g}$. Recall from Proposition 5.3 that there exists a connected self-dual d-complete poset $(P_\lambda, \leq)$ such that $(W \lambda, \leq_s)$ and $(\mathcal{F}(P_\lambda), \subseteq)$ are isomorphic. Let $(P_\lambda, \leq, \kappa, I)$ be the $I$-colored d-complete poset (see the comment after Proposition 5.3). By Proposition 5.4 and Corollary 5.6, there exists a unique order isomorphism $f : (W \lambda, \leq_s) \cong (\mathcal{F}(P_\lambda), \subseteq)$ such that $f(s_i(\mu)) = S_i(f(\mu))$ for all $\mu \in W \lambda$ and $i \in I$. Because the map $\text{res}|_{\hat{W}\lambda} : \hat{W}\lambda \to \hat{W}\text{res}(\lambda)$ is bijective (see Lemma 6.6), we can define a map $\hat{f} : \hat{W}\text{res}(\lambda) \to \mathcal{F}(P_\lambda)$ by the following commutative diagram (7.1):

$$
\begin{array}{ccc}
W \lambda & \xrightarrow{f} & \mathcal{F}(P_\lambda), \subseteq \\
\subseteq & \xrightarrow{f|_{\hat{W}\lambda}} & (\mathcal{F}(P_\lambda), \subseteq) \\
\hat{W}\lambda & \xrightarrow{\text{res}} & \hat{W}\text{res}(\lambda) \\
\end{array}
$$

(7.1)
We define $\tilde{\mathcal{F}}(P_\lambda) := \text{Im}(\tilde{f}) \subseteq \mathcal{F}(P_\lambda)$ (see also (8.2) below).

**Definition 7.1.** Keep the setting above. We define a map $\tilde{\kappa} : P_\lambda \to J$ to be the composition of $\kappa : P_\lambda \to I$ and the canonical projection $I \to J$. We call the colored poset $(P_\lambda, \leq, \tilde{\kappa}, J)$ the $J$-colored d-complete poset corresponding to $g(0)$ and $\text{res}(\lambda)$.

For $F \in \mathcal{F}(P_\lambda)$ and $p \in J$, we define $\tilde{c}_p(F) := \#\{x \in F \mid \tilde{\kappa}(x) = p\}$. By Corollary 5.7, it follows that for $\mu \in \hat{W}_\lambda$ and $F = \tilde{f}(\mu)$,

$$\text{res}(\mu) = \text{res}(\lambda) - \sum_{p \in J} \left( \sum_{i \in p} c_i(F) \right) \beta_p = \text{res}(\lambda) - \sum_{p \in J} \tilde{c}_p(F) \beta_p.$$ 

We define $\tilde{g} : \tilde{\mathcal{F}}(P_\lambda) \to \hat{W}\text{res}(\lambda)$ by

$$\tilde{g}(F) := \text{res}(\lambda) - \sum_{p \in J} \tilde{c}_p(F) \beta_p$$

for $F \in \tilde{\mathcal{F}}(P_\lambda)$. It can be easily checked that $\tilde{g}$ is the inverse of $\tilde{f}$.

Denote by $\tilde{A}_p, \tilde{R}_p, \tilde{S}_p : \mathcal{F}(P_\lambda) \to \mathcal{F}(P_\lambda)$ ($p \in J$) the maps in Definition 3.3 for the J-colored d-complete poset $(P_\lambda, \leq, \tilde{\kappa}, J)$. Also, we define the order $\preceq$ on $\mathcal{F}(P_\lambda)$ in exactly the same way as Definition 3.7. Namely, for $F, F' \in \mathcal{F}(P_\lambda)$, $F \preceq F'$ if there exists a sequence of order filters $F = F_0, F_1, \ldots, F_{n-1}, F_n = F'$ in $\mathcal{F}(P_\lambda)$ such that for each $i \in \{0, 1, \ldots, n-1\}$, there exists $p_i \in J$ such that $\tilde{S}_{p_i}(F_i) = F_{i+1} \supset F_i$.

**Theorem 7.2** (main result). Keep the notation and setting above.

1. The poset $(\hat{W}\text{res}(\lambda), \leq_w)$ is isomorphic to the poset $(\tilde{\mathcal{F}}(P_\lambda), \preceq)$ under the map $\tilde{f} : \hat{W}\text{res}(\lambda) \to \tilde{\mathcal{F}}(P_\lambda)$.

2. The poset $(\hat{W}\text{res}(\lambda), \leq_s)$ is isomorphic to the poset $(\tilde{\mathcal{F}}(P_\lambda), \subseteq)$ under the map $\tilde{f} : \hat{W}\text{res}(\lambda) \to \tilde{\mathcal{F}}(P_\lambda)$.

**Example 7.3.** Let $g$ be of type $A_5$, and $\lambda = \Lambda_2$. Recall from Example 5.5 that the corresponding (connected, self-dual) d-complete poset $P_{\Lambda_2}$ is $Y_{2,4}$, and the $I$-colored d-complete poset $(P_{\Lambda_2}, \leq, \kappa, I)$ is the left diagram in Figure 3. In this case, $g(0)$ is of type $C_3$, and $\text{res}(\Lambda_2) = \tilde{\Lambda}_2'$. The J-colored d-complete poset $(P_{\Lambda_2}, \leq, \tilde{\kappa}, J)$ is below. The Hasse diagrams of $(\hat{W}\tilde{\Lambda}_2', \leq_w)$ and $(\tilde{\mathcal{F}}(P_{\Lambda_2}), \preceq)$ are given in Figure 10.
8 Proof of Theorem 7.2

Keep the notation and setting in the previous section.

Definition 8.1. For \( p \in J \), we define \( \hat{S}_p : \mathcal{F}(P_\lambda) \to \mathcal{F}(P_\lambda) \) as follows:

1. If \( p \) satisfies the orthogonality condition, then
   \[
   \hat{S}_p := \prod_{k \in p} S_k;
   \]

   we see by Lemma 5.6 that \( \hat{S}_p \) does not depend on the order of the product of \( S_k \)'s.

2. If \( p \) does not satisfy the orthogonality condition, that is, if \( g \) is of type \( A_{2n} \) and \( p = \{n, n+1\} \) (see page 13), then
   \[
   \hat{S}_p := S_nS_{n+1}S_n = S_{n+1}S_nS_{n+1};
   \]

   the second equality follows from Lemma 5.6 together with \( s_ns_{n+1}s_n = s_{n+1}s_ns_{n+1} \).

We need the following fact to prove Lemma 8.3 below.

Proposition 8.2 (Page 23). Let \( P \) be an arbitrary poset, and let \( F \in \mathcal{F}(P) \) be an order filter of \( P \).

1. For \( x \in F \), \( x \) is a minimal element of \( F \) if and only if \( F \setminus \{x\} \) is an order filter.

2. For \( x \in F \), \( x \) is a maximal element of \( P \setminus F \) if and only if \( F \cup \{x\} \) is an order filter.
Lemma 8.3. Let \( \mu \in W \lambda \), and set \( F := f(\mu) \in \mathcal{F}(P_\lambda) \). It holds that
\[
\tilde{S}_p(F) = \check{S}_p(F) \text{ for all } p \in J. 
\]

Proof. First, we assume that \( p \in J \) satisfies the orthogonality condition. The case that \( \#p = 1 \) is easy. Assume that \( \#p = 2 \) (the proof for the case that \( \#p = 3 \) is similar). Let we write \( p \) as: \( p = \{i, j\} \), with \( i, j \in I, i \neq j \). We deduce by Lemma 3.8 that for each \( k \in p = \{i, j\} \), \( S_k(F) \) satisfies one of the following:

(i) \( S_k(F) = A_k(F) = F \cup \{x_k\} \) for some \( x_k \in P_\lambda \setminus F \); in this case, \( R_k(F) = F \).

(ii) \( S_k(F) = R_k(F) = F \setminus \{x_k\} \) for some \( x_k \in F \); in this case, \( A_k(F) = F \).

(iii) \( S_k(F) = A_k(F) = R_k(F) = F \).

Here, we give a proof only for the case that both \( S_i(F) \) and \( S_j(F) \) satisfy (i); the proofs for the other cases are similar. In this case, there exist \( x_i, x_j \in P_\lambda \setminus F \) such that \( S_i(F) = F \cup \{x_i\} \) and \( S_j(F) = F \cup \{x_j\} \); note that \( \kappa(x_i) = i \) and \( \kappa(x_j) = j \). By Definition 8.1, we have \( S_j(F \cup \{x_i\}) = S_jS_i(F) = S_iS_j(F) = S_i(F \cup \{x_j\}) \). Since \( \kappa(x_i) = i \) and \( i \neq j \), we see from the definition of \( S_j \) that when we apply \( S_j \) to \( F \cup \{x_i\} \), \( x_i \) is not removed. Hence, \( x_i \in S_j(F \cup \{x_i\}) = S_jS_i(F) \). Similarly,
\[ x_j \in S_j S_i(F). \] Since the symmetric difference of \( S_j S_i(F) \) and \( F \) has at most two element by Lemma 3.3, we see that \( S_j S_i(F) = F \cup \{ x_i \} \cup \{ x_j \} \). Therefore, it suffices to show that \( \tilde{S}_p(F) = F \cup \{ x_i \} \cup \{ x_j \} \).

Suppose, for a contradiction, that \( F \supseteq \tilde{R}_p(F) \). Let \( y \) be a minimal element of \( F \setminus \tilde{R}_p(F) \). Because \( \tilde{R}_p(F) \) is an order filter by the definition of \( \tilde{R}_p \), we deduce that \( y \) is a minimal element of \( F \). Hence, by Proposition 8.2 (1), \( F \setminus \{ y \} \) is an order filter of \( P_\lambda \). Note that \( \tilde{\kappa}(y) = p \), and recall that \( \tilde{\kappa}(y) = p \) if and only if \( \kappa(y) = i \) or \( \kappa(y) = j \). Assume that \( \kappa(y) = i \). Since \( F \setminus \{ y \} \) is an order filter of \( P_\lambda \) satisfying \( F \setminus (F \setminus \{ y \}) = \{ y \} \), \( \kappa^{-1}(\{ i \}) \), we see by the definition of \( R_i \) that \( R_i(F) \neq F \). Similarly, if \( \kappa(y) = j \), then \( R_j(F) \neq F \). Thus we conclude that \( R_i(F) \neq F \) or \( R_j(F) \neq F \). However, this contradicts the assumption that both \( S_i(F) \) and \( S_j(F) \) satisfy (i). Therefore, \( \tilde{R}_p(F) = F \), and hence \( \tilde{S}_p(F) = \tilde{A}_p(F) \).

Since \( F \cup \{ x_i \} \cup \{ x_j \} = S_j S_i(F) \) is an order filter of \( P_\lambda \), we see by the definition of \( \tilde{A}_p(F) \) that \( F \cup \{ x_i \} \cup \{ x_j \} \subseteq \tilde{A}_p(F) = \tilde{S}_p(F) \).

Suppose, for a contradiction, that \( \tilde{S}_p(F) \supseteq F \cup \{ x_i \} \cup \{ x_j \} \). Since \( F \cup \{ x_i \}, F \cup \{ x_j \} \in \mathcal{F}(P_\lambda) \), it follows from Proposition 8.2 (1) that \( x_i \) and \( x_j \) are minimal elements of \( F \cup \{ x_i \} \cup \{ x_j \} \). Let \( z \) be a maximal element of \( \tilde{S}_p(F) \setminus (F \cup \{ x_i \} \cup \{ x_j \}) \); note that \( \tilde{\kappa}(z) = p \), which implies that \( \kappa(z) \in p = \{ i, j \} \). If \( z \) and \( x_i \) are comparable, then \( z \rightarrow x_i \) because \( F \cup \{ x_i \} \cup \{ x_j \} \) is an order filter, and \( x_i \) is a minimal element of \( F \cup \{ x_i \} \cup \{ x_j \} \) as seen above. By Proposition 5.8 \( \kappa(z) \in \{ i, j \} \) and \( \kappa(x_i) = i \) are adjacent in the Dynkin diagram of \( g \). However, this contradicts that \( p \) satisfies the orthogonality condition. Thus, \( z \) and \( x_i \) are incomparable. Similarly, we can show that \( z \) and \( x_j \) are incomparable. Thus, \( z \) is a maximal element of \( \tilde{S}_p(F) \setminus F \). Since \( \tilde{S}_p(F) \) is an order filter of \( P_\lambda \), we see that \( z \) is a maximal element of \( P_\lambda \setminus F \). Hence, by Proposition 8.2 (2), \( F \cup \{ z \} \) is an order filter of \( P_\lambda \). Since \( \kappa(z) \in p = \{ i, j \} \), we see by the definitions of \( A_i \) and \( A_j \) that \( z \) is contained in either \( A_i(F) \) or \( A_j(F) \). However, this contradicts the assumption that both \( S_i(F) \) and \( S_j(F) \) satisfy (i). Therefore, we obtain \( F \cup \{ x_i \} \cup \{ x_j \} = \tilde{S}_p(F) \), as desired.

Next, we assume that \( p \) does not satisfy the orthogonality condition, that is, \( g \) is of type \( A_{2n} \) and \( p = \{ n, n + 1 \} \). Let \( \lambda = \Lambda_\gamma \). In this case, the corresponding d-complete poset \( P_\lambda \) is \( Y_{\gamma, 2n-i+1} \) (see Example 2.7 (1)), and its \( I \)-coloring \( \kappa : P_\lambda \rightarrow I \) is given as follows (see also Figure 6):
In this proof, the cells having the color $n$ or $n + 1$ are important; if $1 \leq i \leq n$ (resp., $n + 1 \leq i \leq 2n$), then $\kappa^{-1}(\{n\}) = \{(1, n - i + 1), (2, n - i + 2), \ldots, (i, n)\}$ and $\kappa^{-1}(\{n + 1\}) = \{(1, n - i + 2), (2, n - i + 3), \ldots, (i, n + 1)\}$ (resp., $\kappa^{-1}(\{n\}) = \{(i - n + 1, 1), (i - n + 2, 2), \ldots, (n + 1, 2n - i + 1)\}$ and $\kappa^{-1}(\{n + 1\}) = \{(i - n, 1), (i - n + 1, 2), \ldots, (n, 2n - i + 1)\}$). Notice that the subset $\kappa^{-1}(\{n, n + 1\}) \subset P_\lambda$ is a totally order set. Similarly to the case that $p$ satisfies the orthogonality condition, each of $S_n(F)$ and $S_{n+1}(F)$ satisfies one of (i),(ii),(iii). Suppose, for a contradiction, that both $S_n(F)$ and $S_{n+1}(F)$ satisfy (i). Then, there exist $x_n, x_{n+1}$ such that both $x_n$ and $x_{n+1}$ are maximal elements of $P_\lambda \setminus F$, and $\kappa(x_n) = n, \kappa(x_{n+1}) = n + 1$. However, this contradict the fact that $\kappa^{-1}(\{n, n + 1\})$ is a totally order set. Therefore, the case that both $S_n(F)$ and $S_{n+1}(F)$ satisfy (i) does not happen. Similarly, we deduce that the case that both $S_n(F)$ and $S_{n+1}(F)$ satisfy (ii) does not happen. So, it suffices to consider the other 7 cases.

Now, we give a proof only for the case that $S_n(F)$ satisfies (i), and $S_{n+1}(F)$ satisfies (iii); the proofs for the other cases are similar. Then, under the description mentioned at the end of Section 2, $F$ has a “block” of the following form:

Here, each element corresponding to the right-gray cell (with the color $n + 3$ or $n - 2$) is not necessarily an element of $F$. Then, $\hat{S}_p(F)$ and $\tilde{S}_p(F)$ are as follows:
\[
\begin{align*}
\hat{S}_p & = S_n S_{n+1} S_n \\
& = S_n S_{n+1} \\
& = S_n \\
& = \ldots \\
& = \hat{S}_p \\
& = \ldots \\
\end{align*}
\]
Thus we obtain $\tilde{S}_p(F) = \hat{S}_p(F)$, as desired. \hfill \Box

**Lemma 8.4.** For $\mu \in \hat{W}\lambda$ and $p \in J$,

$$\tilde{f}(\tilde{s}_p(\text{res}(\mu))) = \hat{S}_p(\tilde{f}(\text{res}(\mu))).$$

In particular,

$$\tilde{F}(P_\lambda) = \{\tilde{S}_{p_n} \cdots \tilde{S}_{p_2}\tilde{S}_{p_1}(f(\lambda)) \mid n \geq 0, p_k \in J(1 \leq k \leq n)\}. \quad (8.2)$$

**Proof.** We compute that

$$\tilde{f}(\tilde{s}_p(\text{res}(\mu))) = \tilde{f}(\text{res}(\hat{s}_p(\mu))) \quad \text{(by Lemma 6.2)}$$

$$= f(\hat{s}_p(\mu)) \quad \text{(by the definition of } \tilde{f})$$

$$= \hat{S}_p(f(\mu)) \quad \text{(by Corollary 5.6)}$$

$$= \hat{S}_p(\tilde{f}(\text{res}(\mu))) \quad \text{(by the definition of } \tilde{f})$$

$$= \hat{S}_p(\tilde{f}(\text{res}(\mu))) \quad \text{(by Lemma 8.3).}$$

\hfill \Box

**Proof of Theorem 7.2.** (1) By the definitions of $\leq_w$ and $\preceq$, it suffices to show that for $\mu \in \hat{W}\lambda$ and $p \in J$, $\text{res}(\mu) <_w \tilde{s}_p(\text{res}(\mu))$ if and only if $\tilde{f}(\text{res}(\mu)) \preceq \hat{S}_p(\tilde{f}(\text{res}(\mu)))$.

First, we assume that $\text{res}(\mu) <_w \tilde{s}_p(\text{res}(\mu)) = \text{res}(\hat{s}_p(\mu))$. Because $\text{res}(\mu)(H_p) > 0$, there exists $i \in p$ such that $\mu(h_i) > 0$. Then we deduce by Lemma 6.7(2) that $\mu <_w \hat{s}_p(\mu)$ in $(W\lambda, \leq_w)$. By the definition of $<_w$ and $<_s$, we have $\mu <_s \hat{s}_p(\mu)$ in $(W\lambda, \leq_s)$. So we compute

$$f(\mu) \subseteq f(\hat{s}_p(\mu)) \quad \text{(by Proposition 5.3)}$$

$$= \hat{S}_p(f(\mu)) \quad \text{(by Corollary 5.6)}$$

$$= \hat{S}_p(\tilde{f}(\text{res}(\mu))) \quad \text{(by the definition of } \tilde{f} \text{ and Lemma 8.3).}$$

Therefore, we obtain $\tilde{f}(\text{res}(\mu)) \preceq \hat{S}_p(\tilde{f}(\text{res}(\mu)))$, as desired.

Next, we assume that $f(\text{res}(\mu)) \preceq \hat{S}_p(f(\text{res}(\mu)))$. Then we have $\tilde{f}(\text{res}(\mu)) \preceq \hat{S}_p(\tilde{f}(\text{res}(\mu)))$. Since $\tilde{f}(\text{res}(\mu)) = f(\mu)$ and $\hat{S}_p(\tilde{f}(\text{res}(\mu))) = f(\hat{s}_p(\mu))$ as seen above, we get $f(\mu) \subseteq f(\hat{s}_p(\mu))$. Hence, by Proposition 5.3, $\mu <_s \hat{s}_p(\mu)$ in $(W\lambda, \leq_s)$. Write $\hat{s}_p(\mu)$ as: $\hat{s}_p(\mu) = \mu - \sum_{i \in I} m_i \alpha_i$; since $\mu <_s \hat{s}_p(\mu)$, we see that $m_i \geq 0$ for all $i \in I$, and $m := \sum_{i \in p} m_i > 0$. Because $\hat{s}_p(\text{res}(\mu)) = \text{res}(\mu) - m\beta_p$, we obtain $\text{res}(\mu) <_w \hat{s}_p(\text{res}(\mu))$, as desire.
(2) For $\mu_1, \mu_2 \in \hat{W}_\lambda$, we deduce

$$\text{res}(\mu_1) <_s \text{res}(\mu_2) \iff \mu_1 <_s \mu_2 \quad \text{(by Lemma 6.9)}$$

$$\iff f(\mu_1) \subset f(\mu_2) \quad \text{(by Proposition 5.3)}$$

$$\iff \tilde{f}(\text{res}(\mu_1)) \subset \tilde{f}(\text{res}(\mu_2)) \quad \text{(by the definition of } \tilde{f}).$$

\[\square\]

9 Explicit description of $\tilde{F}(P_\lambda)$.

Keep the notation and setting in Section 7. We give an explicit description of $\tilde{F}(P_\lambda)$ in the case that $g$ is of type $A_n$; in fact, our description, Theorem 9.4 below, and its proof are essentially restatements of [M, Theorem 1.1 and its proof]; however, we give a proof (in terms of our notation) for the convenience of the readers.

Assume that $g$ is of type $A_n$, and $\lambda = \Lambda_m$ with $1 \leq m \leq (n + 1)/2$. We regard $P_\lambda$ as a rectangular Young diagram $Y_{m,n-m+1}$ (see Example 2.7). Note that $\kappa((i, j)) = j - i + m$ and $\tilde{\kappa}((i, j)) = (\min\{j - i + m, i - j + n - m + 1\})'$.

For $i, j, p \in \mathbb{Z}$, we set $[i, j] := \{k \in \mathbb{Z} \mid i \leq k \leq j\}$ and $\binom{[i, j]}{p} := \{I \subseteq [i, j] \mid \#I = p\}$.

**Definition 9.1.** Let $Y \in \mathcal{F}(Y_{m,n-m+1})$. For $1 \leq l \leq m$, we set $k_l := \#\{j \mid (l, j) \in Y\}$, and $k(Y) := (k_1, \ldots, k_m)$. Also, we set $I(Y) := \{k_i + m + 1 - i \mid i \in [1, m]\}$, $\tilde{I}(Y) := \{n + 2 - i \mid i \in I(Y)\}$. Observe that the map $I : \mathcal{F}(Y_{m,n-m+1}) \to \binom{[1, n+1]}{m}, Y \mapsto I(Y)$, is a bijection.

**Lemma 9.2.** Let $Y \in \mathcal{F}(Y_{m,n-m+1})$, and $k \in [1, n]$. Then,

1. $k \in I(Y)$ and $k + 1 \notin I(Y)$ if and only if $S_k(Y) \supset Y$;
2. $k \notin I(Y)$ and $k + 1 \in I(Y)$ if and only if $S_k(Y) \subset Y$;
3. $k, k + 1 \in I(Y)$ or $k, k + 1 \notin I(Y)$ if and only if $S_k(Y) = Y$.

**Proof.** Notice that for $Y \in \mathcal{F}(Y_{m,n-m+1})$ and $k \in [1, n]$, $S_k(Y)$ satisfies one of the following (see Lemma 3.8 and Corollary 5.6):

1. $S_k(Y) = A_k(Y) = Y \cup \{(i, j)\}$ for some $(i, j) \in Y_{m,n-m+1} \setminus Y$.
2. $S_k(Y) = R_k(Y) = Y \setminus \{(i, j)\}$ for some $(i, j) \in Y$.
3. $S_k(Y) = A_k(Y) = R_k(Y) = Y$.
(1) First, we show the “only if” part. Because $k \in \mathcal{I}(Y)$, there exists $i \in [1, m]$ such that $k = k_i + m + 1 - i$. Then we see that $(i, k_i) = (i, k - m - 1 + i) \in Y$ or $k_i = 0$. In both cases, we get $(i, k - m + 1) \notin Y$. Note that $\kappa(i, k - m + i) = k$. If $i = 1$, then we get $Y \cup \{(i, k - m + i)\} \in \mathcal{F}(Y_{m,n-m+1})$ and $S_k(Y) = Y \cup \{(i, k - m + i)\} \supseteq Y$. If $i > 1$, then $k_{i-1} + m + 1 - (i - 1) > k + 1$ by $k + 1 \notin \mathcal{I}(Y)$. Hence we get $Y \cup \{(i, k - m + i)\} \in \mathcal{F}(Y_{m,n-m+1})$ and $S_k(Y) = Y \cup \{(i, k - m + i)\} \supseteq Y$.

Next, we show the “if” part. Because $S_k(Y) \supseteq Y$, there exists $(i, j) \in S_k(Y)$ such that $S_k(Y) = Y \cup \{(i, j)\}$ and $\kappa(i, j) = j - i + m = k$. Then we see that $(i, j - 1) \in Y$ or $j - 1 = 0$. In both cases, we get $k_i = j - 1 = i - m + k - 1$. Hence, $k = k_i + m + 1 - i \in \mathcal{I}(Y)$. If $i = 1$, then $\max(\mathcal{I}(Y)) = k$, and hence $k + 1 \notin \mathcal{I}(Y)$. If $i > 1$, then $(i - 1, j) \in Y$ and $k_{i-1} \geq j$. Thus we obtain $k_{i-1} + m + 1 - (i - 1) \geq j + m + 1 - i + 1 = k + 2$, which implies that $k + 1 \notin \mathcal{I}(Y)$.

(2) Similar to part (1).

(3) Since $S_k(Y)$ satisfies one of (i)-(iii), the assertion is obvious from parts (1) and (2).

**Remark 9.3.** By Lemma 9.2, if $k \in \mathcal{I}(Y)$ and $k + 1 \notin \mathcal{I}(Y)$, then $k \notin \mathcal{I}(S_k(Y))$ and $k + 1 \in \mathcal{I}(S_k(Y))$. Moreover, either $k' \in \mathcal{I}(Y), k' \in \mathcal{I}(S_k(Y))$ or $k' \notin \mathcal{I}(Y), k' \notin \mathcal{I}(S_k(Y))$ for $k' \in [1, n + 1]$ with $k' \neq k, k + 1$.

For $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0}$ such that $1 \leq m \leq (n+1)/2$, we set $\mathcal{S}\mathcal{S}(Y_{m,n-m+1}) := \{Y \in \mathcal{F}(Y_{m,n-m+1}) \mid \mathcal{I}(Y) \cap \overline{\mathcal{I}(Y)} = \emptyset\}$.

**Theorem 9.4 (cf. [M Theorem 1.1]).** It holds that $\tilde{\mathcal{F}}(Y_{m,n-m+1}) = \mathcal{S}\mathcal{S}(Y_{m,n-m+1})$.

**Proof.** We will show that $Y \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$ if and only if $Y \in \mathcal{S}\mathcal{S}(Y_{m,n-m+1})$ by induction on $\#Y$. If $\#Y = 0$, then $Y = \emptyset$. It is obvious that $\emptyset \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$.

Also, because $I(\emptyset) = \{1, 2, \ldots, m\}$ and $\overline{I(\emptyset)} = \{n + 1, n, \ldots, n + 2 - m\}$, with $m < n + 2 - m$, it follows that $\mathcal{I}(\emptyset) \cap \overline{\mathcal{I}(\emptyset)} = \emptyset$, and hence $\emptyset \in \mathcal{S}\mathcal{S}(Y_{m,n-m+1})$.

Assume that $\#Y > 0$. First, we will show the “only if” part. Because $Y \neq \emptyset$, there exists $p \in J$ such that $\tilde{S}_p(Y) \subset Y$. Since $Y \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$, we have $\tilde{S}_p(Y) \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$. By the induction hypothesis, it follows that $\tilde{S}_p(Y) \in \mathcal{S}\mathcal{S}(Y_{m,n-m+1})$. Here, we give a proof only for the case that $\#p = 2$; the proof for the case that $\#p = 1$ is similar (and simpler). Assume that $p$ satisfies the orthogonality condition. We write $p$ as: $p = \{i, n + 1 - i\}$ with $i \neq n + 1 - i$.

By Lemma 6.7, $\tilde{S}_p(Y)$ satisfies one of the following:

(i) $\tilde{S}_p(Y) \subset S_i \tilde{S}_p(Y), \tilde{S}_p(Y) = S_{n+1-i} \tilde{S}_p(Y)$.

(ii) $\tilde{S}_p(Y) = S_i \tilde{S}_p(Y), \tilde{S}_p(Y) \subset S_{n+1-i} \tilde{S}_p(Y)$.

(iii) $\tilde{S}_p(Y) \subset S_i \tilde{S}_p(Y), \tilde{S}_p(Y) \subset S_{n+1-i} \tilde{S}_p(Y)$.

We see by Lemma 9.2 that (i) (resp., (ii), (iii)) holds if and only if the following (i)' (resp., (ii)', (iii)') holds:
Moreover, it can be easily checked that (i)' (resp., (ii)', (iii)') holds if and only if
\[ p \notin I(\tilde{S}_p(Y)), i + 1 \notin I(\tilde{S}_p(Y)), n + 1 - i \notin I(\tilde{S}_p(Y)), n + 2 - i \notin I(\tilde{S}_p(Y)). \]

By Remark 9.3, we obtain
\[ Y = 2 + 1 \in I \] for any cases. Assume that \( p \) does not satisfy the orthogonality condition; in this case, \( n \) is even, and \( p = \{i, i + 1\} \) with \( i = n/2 \). By Lemmas 6.7 and 9.2, \( \tilde{S}_p(Y) \) satisfies \( i \in I(\tilde{S}_p(Y)), i + 1 \notin I(\tilde{S}_p(Y)), \) and \( i + 2 \notin I(\tilde{S}_p(Y)) \). Also, \( Y \) satisfies \( i \notin I(Y), i + 1 \notin I(Y), \) and \( i + 2 \in I(Y) \). Thus we obtain \( Y \in SS(Y_{m,n-m+1}) \), as desired.

Next, we will show the “if” part. Because \( Y \neq \emptyset \), there exists \( k \in [1, n] \) such that \( k \notin I(Y) \) and \( k + 1 \in I(Y) \); we set \( p := \{k, n + 1 - k\} \in J \). Let \( Y' \in \mathcal{F}(Y_{m,n-m+1}) \) be such that \( I(Y') = I(Y) \cup \{k\} \setminus \{k + 1\} \); note that \#\( Y' \) = \#\( Y \) - 1.

Assume that \( n + 2 - k \notin I(Y') \). By Remark 9.3, we have \( Y' \in SS(Y_{m,n-m+1}) \). By the induction hypothesis, it follows that \( Y'' \in \mathcal{F}(Y_{m,n-m+1}) \). Notice that
\[ n/2 + 1 \notin I(Y), \] because \( n + 2 - (n/2 + 1) = n/2 + 1 \). Because \( k + 1 \in I(Y) \), we have \( n + 1 - k \neq k + 1 \). Also, because \( k \in I(Y') \) and \( n + 2 - k \notin I(Y') \), we have \( k \neq n + 2 - k \), and hence \( n + 1 - k \neq k - 1 \). Thus, \( p \) satisfies the orthogonality condition. If \#\( p = 1 \), then \( p = \{k\} \), and \( \tilde{S}_p(Y') = S_k(Y') = Y \) by Lemma 9.2. If \#\( p = 2 \), then \( k \neq n + 1 - k \) and \( \{k, k + 1\} \cap \{n + 1 - k, n + 2 - k\} = \emptyset \), which implies that \( n + 2 - k \notin I(Y) \) by Remark 9.3, and \( n + 1 - k \notin I(Y) \) by Lemmas 6.7 and 9.2. Hence we have \( \tilde{S}_p(Y') = S_{n+1-k}S_k(Y') = S_{n+1-k}(Y) = Y \). In both cases, we obtain \( Y \in \mathcal{F}(Y_{m,n-m+1}) \). Assume that \( n + 2 - k \in I(Y') \). Let \( Y'' \in \mathcal{F}(Y_{m,n-m+1}) \) be such that \( I(Y'') = I(Y') \cup \{n + 1 - k\} \setminus \{n + 2 - k\} \); note that \#\( Y'' \) = \#\( Y' \) - 1.

Because \( n + 2 - (n + 1 - k) = k + 1 \notin I(Y) \), we have \( Y'' \in SS(Y_{m,n-m+1}) \). By the induction hypothesis, it follows that \( Y'' \in \mathcal{F}(Y_{m,n-m+1}) \). Because \#\( Y'' \) = \#\( Y \) - 2, we have \#\( p = 2 \). We see by Lemmas 6.7 and 9.2 that if \( p \) satisfies the orthogonality condition, then \( \tilde{S}_p(Y'') = S_kS_{n+1-k}(Y'') = S_k(Y') = Y \). If \( p \) does not satisfy the orthogonality condition, then \( n + 1 - k = k + 1 \). Thus we obtain \( k - 1 \in I(Y'') \), \( k, k + 1 \notin I(Y'') \), and hence \( \tilde{S}_p(Y'') = S_{k-1}S_kS_{k-1}(Y'') = S_{k-1}Y = Y \). In both cases, we obtain \( Y \in \mathcal{F}(Y_{m,n-m+1}) \), as desired.

\[ \square \]

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