SHORT COMMUNICATION

An edge-based interface-tracking method for multiphase flows

Leonardo Chirco1   | Stéphane Zaleski1,2

1Sorbonne Université and CNRS, Institut Jean Le Rond d’Alembert, Paris, France
2Institut Universitaire de France, Paris, France

Correspondence
Stéphane Zaleski, Sorbonne Université and CNRS, Institut Jean Le Rond d’Alembert, Place Jussieu 4, 75252 Paris Cedex 05, France.
Email: stephane.zaleski@sorbonne-universite.fr

Funding information
European Research Council

Abstract
We propose a novel class of edge-based interface-tracking (EBIT) methods in the field of multiphase flows for advecting the interface. The position of the interface is tracked by marker points located on the edges of the underlying grid, making the method flexible with respect to the choice of spatial discretization and suitable for parallel computation. In this article we present a simple EBIT method based on two-dimensional Cartesian grids and on a linear interface representation.

KEYWORDS
front-tracking, interface tracking, level-set, two-phase flows

1 INTRODUCTION

Many methods for following an interface or front exist, the simplest and most popular being the front-tracking, the level-set and the volume-of-fluid method.1 In this article we first consider a new class of methods, which could be called edge-based interface-tracking (EBIT) methods. In these methods, the basic information about the front position is known or “tracked” by the position of marker points, which makes the method a kind of front-tracking. However, the additional requirement is that the markers are located on the edges of the underlying grid. When the connecting interface lines between the marker points are linear, the method bears an obvious similarity with the volume-of-fluid method of Piecewise Linear Interface Calculation type (PLIC-VOF). Finally, since the position of the markers gives an explicit information about the distance of the vertices of the underlying grid to the interface, it is a kind of distance information as in the level-set method, where the implicit definition of the interface is given by a function as close as possible to the signed distance function. In particular, a linear interface has the same representation using EBIT and level-set methods.

Several prior works have attempted a combination of pairs of the three main methods and may result in methods similar to this one, such as the combination of markers and VOF2 or the combination of level-set and front-tracking.3 However, the EBIT method adds the simplifying requirement that only the position of the markers on the grid lines or grid edges needs to be known. This is true both in 2D and 3D and whatever the grid type, structured, unstructured, or hierarchical/quadtree, see Figure 1. The use of iso-faces to perform the advection of interfaces on general meshes consisting of arbitrary polyhedral cell is the core of the isoAdvector algorithm as well, see Reference 4. Perhaps the most important advantage of EBIT methods is that they allow for almost automatic parallelization. In fact, since the marker points are constrained to move along the grid edges, their re-distribution among processes follows naturally that of the grid cells. Another potential advantage is that as the grid is adapted, refined, or unrefined the front is adapted consistently. Finally, since information about the connectivity of the marker points does not need to be stored (it can be...
reconstructed and is thus known implicitly) the addition or removal of points or grid cells is easier than in traditional front-tracking. In this article we focus on a special case of EBIT methods, the Semushin method, in which the underlying grid is a 2D square grid, the intersections are at most two per square edge of the grid and the interpolation between the marker points is linear. This is clearly a “bare bones” version of the EBIT method and is inspired by Semushin’s preprint and by personal communications received from its author. This article is organized as follows. The method is described is Section 2 and then in Section 3.2 the numerical results are presented. Finally, the conclusions are given in the last section.

2 | THE SEMUSHIN METHOD

In Semushin’s method for tracking the interface, the reference phase is enclosed by a set of marker points placed on the grid lines. The advection of the interface is done by moving these points along the grid lines. Thanks to this constraint, the $n$-dimensional advection algorithm can be split into a succession of $n$ times the one-dimensional scheme, one for each direction.

The equation of motion for the interface point is

$$\frac{dx}{dt} = u,$$  \hspace{1cm} \text{(1)}

that can be integrated as

$$x = x_0 + \int_{t_0}^{t} u(x(t'), t')dt',$$ \hspace{1cm} \text{(2)}

where the initial position $x_0$ is known. For the sake of simplicity, in this work we use a first-order explicit Euler method such that $x = x_0 + u_0 \Delta t = x_0 + \Delta x$.

Now, we describe the simple one-dimensional advection algorithm used, see Figure 2. We recall that we study a two-dimensional problem, admit at most two interface intersections (and then markers) per face (edge in 2D) of the grid, and that the interpolation between the marker points is linear. The extension to three-dimensional problems or unstructured grids is straightforward, see Figure 1. The points placed on the grid lines aligned with the velocity are called aligned points, while the remaining ones are unaligned. Starting from the initial configuration (Figure 2A), the new position of the aligned points (Figure 2B) is directly obtained by integrating (2). To place the unaligned points (in this example on the vertical grid lines), we first advect them using the same Equation (2) obtaining the fictitious gray points in Figure 2C. Finally, the new position of the unaligned points (in red in Figure 2C) is obtained by connecting with a segment either one blue and one gray point or two consecutive gray points and by finding the intersection with the grid lines. The position of the points and of the interface after the advection along the $x$-direction is shown in Figure 2D.

![Figure 1](https://wileyonlinelibrary.com)
Figure 2  The steps of the one-dimensional advection scheme of Semushin’s method. (A) Initial markers position for the dashed circle; (B) advection of the (blue) points aligned with the velocity; (C) fictitious advection of the unaligned (gray) points and (red) intersections; (D) final position of the markers and interface [Colour figure can be viewed at wileyonlinelibrary.com]

3 | RESULTS

We define the surface error $E_{area}(t)$ between the total area of the reference phase at the initial time $t_0$ and time $t$ as

$$E_{area} = \frac{|A(t) - A(t_0)|}{A(t_0)}.$$  

(3)

We define the shape error, in a $L^\infty$ norm, as the maximum distance between any marker point $x_i$ on the interface and the corresponding closest point on the analytical shape as

$$E_{shape} = \max \{|\text{dist}(x_i)|\}.$$  

(4)

We recall that for a circle centered in $(x_c, y_c)$ and radius $R$, we have $\text{dist}(x_i) = \sqrt{(x_i - x_c)^2 + (y_i - y_c)^2} - R$. The order of convergence of the method is computed by comparing the errors on successively refined grids as

$$\text{order} = \log_2(E(h)/E(h/2)),$$  

(5)

where $E(h)$ is the norm of the error on the grid with spacing $h$, with respect to the exact solution. We perform three well-known tests to evaluate the accuracy of interface advecting methods.7

3.1 | Translation with uniform velocity

In the first test a circular shape of radius $r = 0.15$ and center $(0.25, 0.75)$ is placed inside the unit box. The box is meshed with $N_x \times N_y$ square cells of size $h = 1/N_x$, where $N_x = 64, 128, 256, 512$. A uniform and constant velocity field $(u, v)$ with $u = -v$ is imposed in the box, so that the reference phase is advected along the diagonal of the box. After one time unit, the velocity field is reversed and the circular fluid body should return to its initial position with no distortion, allowing error measurement with (3) and (4). For this test, we employ two constant $CFL$ numbers $CFL = u\Delta t/h$, where $\Delta t$ is the time step. For example, if $CFL = 1$, the circle is displaced of exactly one grid spacing per time step, while if $CFL < 1$, the circle advances only by a fraction of the grid spacing.

In Figure 3, the position of the reference phase is shown after two full diagonal translations (solid line) and one (dashed line). When using the coarser grids, the circular shape is shrunk radially. In Table 1 we report the surface error $E_{area}$, the shape error $E_{shape}$, and order of convergence for two complete translations along the main diagonal, at different resolutions and $CFL$ numbers. In purely kinematic tests, smaller errors are obtained using $CFL = 1$, since fewer substeps of the algorithm are necessary to obtain a given displacement. However, since the intended use of EBIT methods is advecting the interface in multiphase flows where the $CFL$ has to be limited for stability reasons, the accumulation of errors will affect the performance.
FIGURE 3 Final circular shape (solid line) and after half diagonal translation (dashed line). (A) $CFL = 0.125$; (B) $CFL = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Surface error $E_{\text{area}}$, shape error $E_{\text{shape}}$, and order of convergence for two complete translations along the main diagonal, at different resolutions and CFL numbers.

| $N_x$ | CFL   | $E_{\text{area}}$ | $E_{\text{shape}}$ | Order |
|-------|-------|-------------------|--------------------|-------|
| 64    | 1.0   | 2.89e−2           | 2.78e−2            | 2.17  |
|       | 0.125 | 6.89e−1           | 7.01e−2            | 1.01  |
| 128   | 1.0   | 6.42e−3           | 1.23e−2            | 1.33  |
|       | 0.125 | 3.43e−1           | 3.12e−2            | 1.00  |
| 256   | 1.0   | 2.56e−3           | 6.33e−3            | 0.93  |
|       | 0.125 | 1.72e−1           | 1.55e−2            | 1.00  |
| 512   | 1.0   | 1.34e−3           | 3.57e−3            |       |
|       | 0.125 | 8.57e−2           | 7.87e−3            |       |

3.2 Single vortex rotation

The single vortex or “vortex-in-a-box” problem has been designed to test the ability of interface tracking methods when the reference phase is highly stretched, see Reference 8. A circular shape of radius $r = 0.15$ and center $(0.5, 0.75)$ is placed inside the unit box. The divergence-free velocity $\mathbf{u} = (u, v)$ is obtained from the following stream function $\psi = \pi^{-1}\sin^2(\pi x)\sin^2(\pi y)\cos(\pi t/T)$, as $u_x = \partial \psi / \partial y$ and $u_y = -\partial \psi / \partial x$. On the sides of the box, homogeneous Dirichlet boundary conditions are imposed. The cosinusoidal time-dependence slows down and reverses the flow, so that the maximum deformation occurs at $t = T/2$ and at time $T$ the reference phase returns to its initial position with no distortion, allowing again to measure the error with (3) and (4), see Reference 9. For this test we use a constant time step $\Delta t = 0.0005$. The position of the reference phase at $t = 1$, corresponding to its maximum deformation, and at $t = T = 2$ back to the initial position is shown in Figure 4. By refining the grid, the main fluid becomes thinner and more elongated at $t = 1$, while tends to the reference initial shape at $t = 2$. In Table 2 we report the surface error $E_{\text{area}}$, the shape error $E_{\text{shape}}$, and order of convergence at different grid resolutions.
FIGURE 4  The interface at maximum deformation at $t = 1.0$ (dotted line) and back to the initial position at $t = 2.0$ (solid line) for the single vortex field test with $T = 2.0$. [Colour figure can be viewed at wileyonlinelibrary.com]

### TABLE 2  Surface error $E_{\text{area}}$, shape error $E_{\text{shape}}$, and order of convergence for the single vortex test with $T = 2.0$, at different resolutions.

| $N_x$ | $E_{\text{area}}$ | $E_{\text{shape}}$ | Order |
|------|-----------------|-----------------|-------|
| 64   | 3.77e-1         | 6.43e-2         | 1.14  |
| 128  | 1.71e-1         | 2.95e-2         | 1.09  |
| 256  | 8.04e-2         | 1.45e-2         | 1.10  |
| 512  | 3.76e-2         | 7.42e-3         |       |

### TABLE 3  Surface error $E_{\text{area}}$ and order of convergence for the Zalesak’s disk rotation test with $T = 1.0$, at different resolutions.

| $N_x$ | $E_{\text{area}}$ | Order |
|------|-----------------|-------|
| 64   | 7.56e-1         | 0.87  |
| 128  | 4.12e-1         | 1.27  |
| 256  | 1.71e-1         | 1.36  |
| 512  | 6.69e-2         |       |

### 3.3  Zalesak’s disk rotation

In this test a notched circle of radius $r = 0.15$ and center $(0.5, 0.75)$ is placed inside the unit box. The notched width is 0.05 and the length is 0.25. Imposing the constant velocity field $(u, v) = (2\pi(0.5 - y), 2\pi(x - 0.5))$ the disk performs a full rotation around the box center and returns to the initial position at $T = 1.0$. At the lowest resolution the notch disappears, while increasing the resolution the notch is maintained with smoothed corners (Figure 5). Interestingly, our method recovers final shapes that are symmetrical with respect to the notch vertical axis, which is not always observed in literature especially at low resolution, see References 10 and 11. In Table 3 we report the surface error $E_{\text{area}}$ and the order of convergence at different grid resolutions. The method exhibits a first-order convergence rate upon grid refinement.
4 | CONCLUSIONS

In this article, we have studied a new interface-tracking method, where the interface is tracked by marker points located on the edges of the underlying grid. We have implemented the two-dimensional version of the method using linear interface reconstruction. We have used three well-known benchmark tests to validate the numerical method, recovering a first-order convergence rate of the surface error, lower than the one obtained with other methods, such as VOF, level-set, or isoAdvector.\textsuperscript{4,10,11} In future works we aim to use the EBIT method for multiphase simulations, developing models for topology changes and surface tension and extending the method to three dimensions.

ACKNOWLEDGMENTS

Stéphane Zaleski recalls meeting Sergei Semushin in March 1995 and learning about his method. He thanks him for the explanation of the method and the gift of the preprint.\textsuperscript{6} The authors benefited from the ERC Grant TRUFLOW.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Leonardo Chirco © https://orcid.org/0000-0002-2157-6573
Stéphane Zaleski © https://orcid.org/0000-0003-2004-9090

REFERENCES

1. Tryggvason G, Scardovelli R, Zaleski S. Direct Numerical Simulations of Gas–Liquid Multiphase Flows. Cambridge University Press; 2011.
2. Aulisa E, Manservisi S, Scardovelli R. A surface marker algorithm coupled to an area-preserving marker redistribution method for three-dimensional interface tracking. J Comput Phys. 2004;197(2):555-584.
3. Shin S, Juric D. A hybrid interface method for three-dimensional multiphase flows based on front tracking and level set techniques. Int J Numer Methods Fluids. 2009;60(7):753-778.
4. Roenby J, Bredmose H, Jasak H. A computational method for sharp interface advection. Royal Soc Open Sci. 2016;3(11):160405.
5. Glimm J, Grove JW, Li XL, Tan DC. Robust computational algorithms for dynamic interface tracking in three dimensions. SIAM J Sci Comput. 2000;21(6):2240-2256.
6. Semushin S. Flow Calculation with a Contact Surface on a Rectangular Lattice. Keldysh Institute of Applied Mathematics, USSR Academy of Sciences; 1988. Preprint Nr 134, trans Zenodo. doi:10.5281/zenodo.7025161.
7. Aulisa E, Manservisi S, Scardovelli R, Zaleski S. A geometrical area–Preserving Volume-of–Fluid advection method. J Comput Phys. 2003;192(1):355-364.
8. Enright D, Fedkiw R, Ferziger J, Mitchell I. A hybrid particle level set method for improved interface capturing. J Comput Phys. 2002;183(1):83-116.
9. Leveque RJ. High-resolution conservative algorithms for advection in incompressible flow. *SIAM J Numer Anal.* 1996;33(2):627-665.
10. Henri F, Coquerelle M, Lubin P. Geometrical level set reinitialization using closest point method and kink detection for thin filaments, topology changes and two-phase flows. *J Comput Phys.* 2022;448:110704.
11. Boniou V, Schmitt T, Vié A. Comparison of interface capturing methods for the simulation of two-phase flow in a unified low-Mach framework. *Int J Multiphase Flow.* 2022;103957.

**How to cite this article:** Chirco L, Zaleski S. An edge-based interface-tracking method for multiphase flows. *Int J Numer Meth Fluids.* 2023;95(3):491-497. doi: 10.1002/fld.5144