Differential Geometry of the $q$-Quaternions

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Abstract

Differential calculus on the quantum quaternionic group $\text{GL}(1, H_q)$ is introduced.
1. Introduction

Differential geometry in the theory of (quantum) Lie groups plays an important role in the mathematical modelling of physics theories. In the classical differential geometry one has a choice between two dual and equivalent descriptions: one can either work with points on a manifold $M$ or with the algebra $C(M)$ of smooth functions on $M$. The idea that the algebra $C(M)$ need not be commutative gives rise to the noncommutative geometry. Such a space is called a quantum space. This is analogy with the quantization of the commutative algebra of functions on phase space that yields the noncommutative operator algebra of quantum mechanics.

A class of noncommutative Hopf algebras have been found in the discussions of integrable systems. These Hopf algebras are $q$-deformed function algebras of classical groups and this structure is called quantum group [1]. The quantum group can also be regarded as a generalization of the notion of a group [2]. Noncommutative geometry [3] is one of the most attractive mathematical concepts in physics and has started to play an important role in different fields of mathematical physics for the last few years. The basic structure giving a direction to the noncommutative geometry is a differential calculus [4] on an associative algebra.

Quantum quaternionic algebra and its Hopf algebra structure is important in physics. The importance of differential geometry in the quantum quaternionic algebra should not be underestimated. Quantum quaternion is an example of a quantum space, and to investigate its differential geometry may be interesting. This is considered in the present work.
2. Review of Hopf algebra $H_q$

Elementary properties of quantum quaternionic group $\text{GL}(1, H_q) = H_q$ are described in Refs. 5 and 6. We state briefly the properties we are going to need in this work.

2.1 The algebra of functions on $H_q$

The quantum quaternionic algebra $H_q$ is defined as a pair $(A, M)$ equipped with $*$ structure, where $A$ is an algebra and $M$ is an $A$-module, and they have the following properties:

1. $A$ is an unital associative algebra generated by generators $a_k$ ($k = 0, 1, 2, 3$) with the commutation relations

   \[
   \begin{align*}
   a_0a_1 &= a_1a_0 - \frac{i}{2}(q - q^{-1})(a_2^2 + a_3^2), \\
   a_0a_2 &= \frac{q + q^{-1}}{2}a_2a_0 + \frac{i}{2}(q - q^{-1})a_2a_1, \\
   a_0a_3 &= \frac{q + q^{-1}}{2}a_3a_0 + \frac{i}{2}(q - q^{-1})a_3a_1, \\
   a_1a_2 &= \frac{q + q^{-1}}{2}a_2a_1 - \frac{i}{2}(q - q^{-1})a_2a_0, \\
   a_1a_3 &= \frac{q + q^{-1}}{2}a_3a_1 - \frac{i}{2}(q - q^{-1})a_3a_0, \\
   a_2a_3 &= a_3a_2,
   \end{align*}
   \]

   where $i^2 = -1$ and $q$ is a nonzero real number.

2. The $*$ antiinvolution in $A$ is defined by

   \[
   \begin{align*}
   a_j^* &= a_j, & j &= 0, 1 \\
   a_2^* &= \frac{1}{2} \left[ (q + q^{-1})a_2 - i(q - q^{-1})a_3 \right], \\
   a_3^* &= \frac{i}{2} \left[ (q - q^{-1})a_2 - i(q + q^{-1})a_3 \right].
   \end{align*}
   \]

   Note that

   \[(a_k^*)^* = a_k, \quad k = 0, 1, 2, 3.\]
(3) $M$ is an $A$-module generated by the quaternionic units $e_k$ with the relations

$$e_k e_l = -\delta_{kl} e_0 + \epsilon_{klm} e_m$$

and

$$a_k e_l = e_l a_k,$$  

where $\delta_{kl}$ denotes the Kronecker delta and

$$\epsilon_{klm} = \frac{1}{2} (k - l)(l - m)(m - k).$$

The quaternionic conjugation (*-antiinvolution) in $M$ is defined by

$$e^*_k = 2\delta_{0,k} e_0 - e_k, \quad k = 0, 1, 2, 3.$$  

(4) Assume that any quaternion $h$ has a representation

$$h = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$$

in terms of the generators of $A$. $h$ will be called the $q$-quaternion, and in this case we shall say that $q$-quaternion $h$ belongs to $H_q$.

The conjugation of a $q$–quaternion $h$ is introduced as

$$h^* = e_0 a_0^* - e_1 a_1^* - e_2 a_2^* - e_3 a_3^*.$$  

Hence, we can introduce the $q$-norm of $h$ as

$$N_q(h) = hh^* = a_0^2 + a_1^2 + \frac{1}{2} (q + q^{-1})(a_2^2 + a_3^2).$$

Note that $N_q(h)$ belongs to the center of $H_q$.

2. 2 The Hopf Algebra Structure of $H_q$

The action of comultiplication $\Delta$ on the generators $a_k$ of $A$ can be introduced as

$$\Delta(a_0) = a_0 \otimes a_0 - (a_1 \otimes a_1 + a_2 \otimes a_2 + a_3 \otimes a_3),$$

$$\Delta(a_1) = a_0 \otimes a_1 + a_1 \otimes a_0 + a_2 \otimes a_3 - a_3 \otimes a_2.$$  

(9)
\[ \Delta(a_2) = a_0 \otimes a_2 + a_2 \otimes a_0 + a_3 \otimes a_1 - a_1 \otimes a_3, \]
\[ \Delta(a_3) = a_0 \otimes a_3 + a_3 \otimes a_0 + a_1 \otimes a_2 - a_2 \otimes a_1. \]

Note that
\[ \Delta(e_0) = e_0 \otimes e_0. \] (10)

It also easy to show that
\[ \Delta(N_q(h)) = N_q(h) \otimes N_q(h). \] (11)

The action of counit \( \varepsilon \) on the generators \( a_k \) of \( A \) is given by
\[ \varepsilon(a_k) = \delta_{0,k}e_0, \quad k = 0, 1, 2, 3 \] (12)
\[ \varepsilon(e_0) = e_0. \]

The action of antipode \( S \) on the generators \( a_k \) of \( A \) is introduced as
\[ S(a_k) = N_q^{-1}(h)(2\delta_{0,k}a_0 - a_k^*) \] (13)
for \( k = 0, 1, 2, 3. \)

3. Differential calculus on \( H_q \)

In this section, we shall build up the differential calculus on the quantum quaternionic algebra \( H_q \). The differential calculus on \( H_q \) involves functions on the algebra, differentials and differential forms.

3.1 Classical case

Let’s begin with differential calculus on the classical quaternionic group \( \text{GL}(1, H) \). In a classical Lie group \( G \), one-order differential calculus is a linear map
\[ d : C^\infty(G) \longrightarrow \Gamma, \]
where \( C^\infty(G) \) is a \( \mathcal{C} \)-algebra consisting of all smooth functions on \( G \) and \( \Gamma \) is a \( C^\infty(G) \)-bimodule consisting of all differential forms. This linear map satisfies
(i) the nilpotency
\[ d^2 = 0, \]  
(14)

(ii) for all \( f, g \in C^\infty(G) \),
\[ d(fg) = (df)g + (-1)^\hat{f}f(dg) \]  
(15)

where \( f \) and \( g \) are functions of the generators and \( \hat{f} \) is the corresponding grading of \( f \).

According to the ideas of noncommutative geometry [3], the differential calculus can be defined on a more general noncommutative algebra.

Let \( \mathcal{A}_1 \) be a Hopf algebra with unit generated by the generators \( a_k \) (\( k = 0, 1, 2, 3 \)). We denote differentials of \( a_k \) by \( da_k \). Then one can construct the one-form \( \Omega \), where
\[ \Omega = dh h^*. \]  
(16)

Explicitly,
\[ w_0 = da_0a_0 + da_1a_1 + da_2a_2 + da_3a_3, \]  
(17)

etc. It can be easily checked that these one-forms construct a four dimensional Grassmann algebra. The anti-commutation relations of the one-forms allow us to construct the algebra of the generators. To obtain the Lie algebra of the algebra generators we first write the one-forms as
\[ da_0 = w_0a_0 - w_1a_1 - w_2a_2 - w_3a_3, \]  
(18)

etc. The differential \( d \) can then the expressed in the form
\[ d = 2(w_0\nabla_0 - w_1\nabla_1 + w_2\nabla_2 - w_3\nabla_3), \]  
(19)

where \( \nabla_k \) (\( k = 0, 1, 2, 3 \)) are the Lie algebra generators. We wish to obtain the commutation relations of these generators. Let \( f \) be an arbitrary function of the generators of \( \mathcal{A}_1 \). Then, using the nilpotency of the exterior differential \( d \) we obtain
\[ (-1)^kdw_k \nabla_kf = (-1)^{k+j}w_k(w_j\nabla_j)\nabla_kf, \quad k, j = 0, 1, 2, 3 \]  
(20)

where summation over repeated indices is understood. Using one-forms one easily obtain the two-forms
\[ dw_0 = 0, \quad dw_1 = 2w_2w_3, \]
\[
dw_2 = 2w_3 w_1, \quad dw_3 = -2w_2 w_1, \quad (21)
\]
since
\[
d\Omega = -dh \, dh^* = \Omega^2. \quad (22)
\]
We now find the following commutation relations for the Lie algebra
\[
[\nabla_1, \nabla_0] = [\nabla_2, \nabla_0] = [\nabla_3, \nabla_0] = 0,
\]
\[
[\nabla_1, \nabla_2] = -2\nabla_3, \quad [\nabla_2, \nabla_3] = -2\nabla_1, \quad [\nabla_3, \nabla_1] = -2\nabla_2. \quad (23)
\]

3.2 Quantum case

A differential algebra on \(H_q\) is an associative algebra \(\Gamma\) equipped with an operator \(d\). Also the algebra \(\Gamma\) has to be generated by \(\mathcal{A} \cup d\mathcal{A}\).

Firstly, to obtain the relations between the generators of \(\mathcal{A}\) and their differentials.

We shall use the method of Ref. 7. Using the consistency of a differential calculus, as the final result one has the following commutation relations
\[
a_0 \, dx_+ = \frac{q^2 + 1}{2} dx_+ a_0 + i\frac{q^2 - 1}{2} dx_+ a_1,
\]
\[
a_1 \, dx_+ = \frac{q^2 + 1}{2} dx_+ a_1 - i\frac{q^2 - 1}{2} dx_+ a_0,
\]
\[
a_0 \, dx_- = \frac{q^2 + 1}{2} dx_- a_0 - i\frac{q^2 - 1}{2} dx_- a_1 + \frac{(q-q^{-1})^2}{2} dx_+ x_- -(q-q^{-1})(da_2 a_2 + da_3 a_3),
\]
\[
a_1 \, dx_- = \frac{q^2 + 1}{2} dx_- a_1 + i\frac{q^2 - 1}{2} dx_- a_0 - i\frac{(q-q^{-1})^2}{2} dx_+ x_- + i(q-q^{-1})(da_2 a_2 + da_3 a_3),
\]
\[
a_0 \, da_2 = qda_2 a_0 + \frac{q^2 - 1}{2} (da_0 + ida_1)a_2,
\]
\[
a_0 \, da_3 = qda_3 a_0 + \frac{q^2 - 1}{2} (da_0 + ida_1)a_3,
\]
\[
a_1 \, da_2 = qda_2 a_1 - \frac{q^2 - 1}{2} (da_0 + ida_1)a_2,
\]
\[
a_1 \, da_3 = qda_3 a_1 - \frac{q^2 - 1}{2} (da_0 + ida_1)a_3,
\]
\[
a_2 \, da_0 = qda_0 a_2 + \frac{q^2 - 1}{2} da_2 (da_0 - ia_1),
\]
\[
a_2 \, da_1 = qda_1 a_0 - \frac{q^2 - 1}{2} da_2 (a_0 - ia_1), \quad (24)
\]
\[ a_2 \, da_2 = \frac{q^2 + 1}{2} da_2 a_2 - \frac{q^2 - 1}{2} da_3 a_3 - \frac{q - q^{-1}}{2} dx_+ x_-, \]
\[ a_2 \, da_3 = \frac{q^2 + 1}{2} da_3 a_2 + \frac{q^2 - 1}{2} da_2 a_3, \]
\[ a_3 \, da_0 = q da_0 a_3 + \frac{q^2 - 1}{2} da_3 (a_0 - i a_1), \]
\[ a_3 \, da_1 = q da_1 a_3 - i \frac{q^2 - 1}{2} da_3 (a_0 - i a_1), \]
\[ a_3 \, da_2 = \frac{q^2 + 1}{2} da_2 a_3 + \frac{q^2 - 1}{2} da_3 a_2, \]
\[ a_3 \, da_3 = \frac{q^2 + 1}{2} da_3 a_3 - \frac{q^2 - 1}{2} da_2 a_2 - \frac{q - q^{-1}}{2} dx_+ x-, \]

where
\[ dx_\pm = da_0 \pm ida_1. \]

Applying the exterior differential \( d \) on the relations (24) and using the nilpotency of \( d \) one obtains
\[ da_0 \, da_1 = -da_1 da_0, \]
\[ (da_0)^2 = 0 = (da_1)^2, \]
\[ da_0 \, da_2 = -\frac{q + q^{-1}}{2} da_2 da_0 + i \frac{q - q^{-1}}{2} da_2 da_1, \]
\[ da_0 \, da_3 = -\frac{q + q^{-1}}{2} da_3 da_0 + i \frac{q - q^{-1}}{2} da_3 da_1, \]
\[ da_1 \, da_2 = -\frac{q + q^{-1}}{2} da_2 da_1 - i \frac{q - q^{-1}}{2} da_2 da_0, \]
\[ da_1 \, da_3 = -\frac{q + q^{-1}}{2} da_3 da_1 - i \frac{q - q^{-1}}{2} da_3 da_0, \]
\[ da_2 \, da_3 = -da_3 da_2, \]
\[ (da_2)^2 = i (q - q^{-1}) da_1 da_0 = (da_3)^2. \]

There is an interesting case which gives rise to the second kind of quaternionic variables, Grassmann quaternion is defined by
\[ \psi = \psi_0 e_0 + \psi_1 e_1 + \psi_2 e_2 + \psi_3 e_3, \]
where components $\psi_k (k = 0, 1, 2, 3)$ are Grassmann variables. Essentially the relations (25) are the relations between the components $\psi_k$, in $q$-deformation. More details will be given in Appendix.

To complete the differential calculus, we need the Cartan-Maurer one-forms. In analogy with the one-forms on a Lie group in classical differential geometry, one can construct the one-form $\Omega$ where

$$\Omega = dh \, h^* = w_0 e_0 + w_1 e_1 + w_2 e_2 + w_3 e_3.$$ 

So we can write the one-forms as follows

$$w_0 = \alpha_0 a_0 + \alpha_1 a_1 + \frac{q + q^{-1}}{2} (\alpha_2 a_2 + \alpha_3 a_3) + \sqrt{i} \frac{q - q^{-1}}{2} (\alpha_3 a_2 - \alpha_2 a_3),$$

$$w_1 = -\alpha_0 a_1 + \alpha_1 a_0 - \frac{q + q^{-1}}{2} (\alpha_2 a_3 - \alpha_3 a_2) - \sqrt{i} \frac{q - q^{-1}}{2} (\alpha_2 a_2 + \alpha_3 a_3),$$

$$w_2 = \alpha_2 a_0 - \alpha_3 a_1 + \frac{q + q^{-1}}{2} (\alpha_1 a_3 - \alpha_0 a_2) + \sqrt{i} \frac{q - q^{-1}}{2} (\alpha_0 a_3 + \alpha_1 a_2),$$

$$w_3 = \alpha_3 a_1 + \alpha_3 a_0 - \frac{q + q^{-1}}{2} (\alpha_0 a_3 + \alpha_1 a_2) + \sqrt{i} \frac{q - q^{-1}}{2} (\alpha_1 a_3 - \alpha_0 a_2),$$

where $\alpha_0 = da_0$, etc.

We wish to find the commutation relations of the generators of $\mathcal{A}$ with those of the components of $\Omega$ which may be computed directly, as follows:

$$a_0 w_+ = \frac{q^2 + 1}{2} w_+ a_0 + \sqrt{i} \frac{q^2 - 1}{2} w_+ a_1,$$

$$a_1 w_+ = \frac{q^2 + 1}{2} w_+ a_1 - \sqrt{i} \frac{q^2 - 1}{2} w_+ a_0,$$

$$a_0 w_2 = qw_2 a_0 + \frac{q - q^{-1}}{2} w_+ a_2,$$

$$a_1 w_2 = qw_2 a_1 - \frac{q - q^{-1}}{2} w_+ a_3,$$

$$a_0 w_3 = qw_3 a_0 + \frac{q - q^{-1}}{2} w_+ a_3,$$

$$a_1 w_3 = qw_3 a_1 + \frac{q - q^{-1}}{2} w_+ a_2,$$

$$a_0 w_- = \frac{q^2 + 1}{2} w_- a_0 - \sqrt{i} \frac{q^2 - 1}{2} w_- a_1 + \frac{(q - q^{-1})^2}{2} w_+ (a_0 + ia_1) + (1 - q^2)(w_2 a_2 + w_3 a_3),$$

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\[
a_{1}w_{-} = \frac{q^{2} + 1}{2}w_{-a_{1}} + i \frac{q^{2} - 1}{2}w_{-a_{0}} - i (q - q^{-1})^{2}w_{+}(a_{0} + ia_{1}) - (1 - q^{2})(w_{2}a_{3} - w_{3}a_{2}),
\]
\[
a_{2}w_{-} = \frac{q^{2} + 1}{2}w_{-a_{2}} - i \frac{q^{2} - 1}{2}w_{-a_{3}} + \frac{(q - q^{-1})^{2}}{2}w_{+}(a_{2} + ia_{3}) + (q^{2} - 1)(w_{2}a_{0} + w_{3}a_{1}),
\]
\[
a_{3}w_{-} = \frac{q^{2} + 1}{2}w_{-a_{3}} + i \frac{q^{2} - 1}{2}w_{-a_{2}} - i \frac{(q - q^{-1})^{2}}{2}w_{+}(a_{2} + ia_{3}) + (1 - q^{2})(w_{2}a_{1} - w_{3}a_{0}),
\]
where
\[
w_{\pm} = w_{0} \pm iw_{1}.
\]

We now obtain the commutation relations of the Cartan-Maurer forms
\[
w_{0}^{2} = i \frac{(q - q^{-1})^{2}}{2}w_{3}w_{2}, \quad w_{1}^{2} = i \frac{q^{2} - q^{-2}}{2}w_{3}w_{2},
\]
\[
w_{0}w_{1} = -w_{1}w_{0} + (q^{-2} - 1)w_{2}w_{3},
\]
\[
w_{0}w_{2} = -w_{2}w_{0} + \frac{q^{2} - q^{-2}}{2}w_{3}w_{1} + i \frac{(q - q^{-1})^{2}}{2}w_{2}w_{1},
\]
\[
w_{0}w_{3} = -w_{3}w_{0} + \frac{q^{2} - q^{-2}}{2}w_{2}w_{1} + i \frac{(q - q^{-1})^{2}}{2}w_{3}w_{1},
\]
\[
w_{1}w_{2} = -\frac{q^{2} + q^{-2}}{2}w_{2}w_{1} + i \frac{q^{2} - q^{-2}}{2}w_{3}w_{1}, \quad (28)
\]
\[
w_{1}w_{3} = -\frac{q^{2} + q^{-2}}{2}w_{3}w_{1} + i \frac{q^{2} - q^{-2}}{2}w_{2}w_{1},
\]
\[
w_{2}w_{3} = -w_{3}w_{2}, \quad w_{2}^{2} = 0 = w_{3}^{2}.
\]

The conjugation of one-forms is defined by
\[
w_{0}^{*} = q^{-2}w_{0} + i(q^{-2} - 1)w_{1}, \quad w_{1}^{*} = w_{1},
\]
\[
w_{2}^{*} = w_{2}, \quad w_{3}^{*} = w_{3}. \quad (29)
\]

Note that, we have
\[
\Omega + \overline{\Omega} = (1 - q^{-2})(w_{0} + iw_{1})e_{0}. \quad (30)
\]

To obtain the quantum algebra of the algebra generators we first write the Cartan-Maurer forms as
\[
d a_{0} = w_{0}a_{0} - (w_{1}a_{1} + w_{2}a_{2} + w_{3}a_{3}),
\]
\[
d a_{1} = w_{0}a_{1} + w_{1}a_{0} + w_{2}a_{3} - w_{3}a_{2},
\]
\[
d a_{2} = w_{0}a_{2} - w_{2}a_{0} + w_{3}a_{1},
\]
\[
d a_{3} = w_{0}a_{3} - w_{3}a_{0} - w_{2}a_{1},
\]
\[
\Omega = \frac{1}{2} \sum_{i=0}^{3} a_{i}w_{i}. \quad (31)
\]

The anticommutation relations are given by
\[
[a_{i}, a_{j}] = 0, \quad (i, j = 0, 1, 2, 3), \quad (32)
\]
\[
[a_{i}, \Omega] = 0, \quad (i = 0, 1, 2, 3). \quad (33)
\]
\[ da_2 = w_0 a_2 - w_1 a_3 + w_2 a_0 + w_3 a_1, \quad (31) \]
\[ da_3 = w_0 a_3 + w_1 a_2 - w_2 a_1 + w_3 a_0. \]

Note that
\[ d^* = q^2 d. \quad (32) \]

Indeed, for example,
\[
(d_0)^* = a_0^* w_0^* - a_1^* w_1^* - a_2^* w_2^* - a_3^* w_3^*
= q^{-2} a_0 (w_0 + i w_1) - i (a_0 - i a_1) w_1 - \frac{1}{2} a_2 [(q + q^{-1}) w_2 - i (q - q^{-1}) w_3]
+ \frac{1}{2} a_3 [(q - q^{-1}) w_2 + i (q + q^{-1}) w_3]
= q^2 (w_0 a_0 - w_1 a_1 - w_2 a_2 - w_3 a_3)
\]

so that
\[
d^* a_0^* = q^2 (w_0 a_0 - w_1 a_1 - w_2 a_2 - w_3 a_3) = q^2 da_0 = q^2 da_0^*
\]

implies that
\[
d^* - q^2 d = 0.
\]

Using the nilpotency of the differential \( d \), we can write the two-forms as
\[
d w_0 = i (q^{-2} - 1) w_2 w_3, \quad d w_1 = (q^{-2} + 1) w_2 w_3,
\]
\[
d w_2 = i (q^{-2} - 1) w_2 w_1 + (q^{-2} + 1) w_3 w_1, \quad (33)
\]
\[
d w_3 = i (q^{-2} - 1) w_3 w_1 - (q^{-2} + 1) w_2 w_1.
\]

Using the Cartan-Maurer equations with together (27) we find the following commutation relations for the quantum algebra:
\[
\nabla_0 \nabla_1 = \nabla_1 \nabla_0, \quad \nabla_0 \nabla_2 = \nabla_2 \nabla_0, \quad \nabla_0 \nabla_3 = \nabla_3 \nabla_0,
\]
\[
\nabla_1 \nabla_2 = \frac{q^2 + q^{-2}}{2} \nabla_2 \nabla_1 - (q^{-2} + 1) \nabla_3 + i (q^{-2} - 1) \nabla_2 - \frac{i (q - q^{-1})^2}{2} \nabla_0 \nabla_1
- \frac{q^2 - q^{-2}}{2} \nabla_3 (\nabla_0 + i \nabla_1),
\]

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\[ \nabla_1 \nabla_3 = \frac{q^2 + q^{-2}}{2} \nabla_3 \nabla_1 + (q^{-2} + 1) \nabla_2 - i(q^{-2} - 1) \nabla_3 - \frac{i(q - q^{-1})^2}{2} \nabla_0 \nabla_1 \\
+ \frac{q^2 - q^{-2}}{2} \nabla_2 (\nabla_0 + i \nabla_1), \]

\[ \nabla_3 \nabla_2 = \nabla_2 \nabla_3 + (q^{-2} + 1) \nabla_1 - i(q^{-2} - 1) \nabla_0 + i \frac{q^2 - q^{-2}}{2} \nabla_0^2 \\
+ \frac{(q - q^{-1})^2}{2} \nabla_1^2 + (1 - q^{-2}) \nabla_0 \nabla_1. \]

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Appendix

\( q \)-Deformation of Grassmann quaternionic algebra

Quantum Grassmann quaternionic algebra may be introduced using the idea of the quantum matrix theory [2,8].

Quantum Grassmann quaternionic algebra \( \hat{H}_q \) is defined as a pair \((\hat{A}, \mathcal{M})\), where \( \hat{A} \) is an algebra and has the following properties:

\[
\psi_0^2 = 0 = \psi_1^2, \quad \psi_0 \psi_1 + \psi_1 \psi_0 = 0,
\]
\[
\psi_0 \psi_2 = -\frac{q + q^{-1}}{2} \psi_2 \psi_0 + i \frac{q - q^{-1}}{2} \psi_2 \psi_1,
\]
\[
\psi_0 \psi_3 = -\frac{q + q^{-1}}{2} \psi_3 \psi_0 + i \frac{q - q^{-1}}{2} \psi_3 \psi_1,
\]
\[
\psi_1 \psi_2 = -\frac{q + q^{-1}}{2} \psi_2 \psi_1 - i \frac{q - q^{-1}}{2} \psi_2 \psi_0,
\]
\[
\psi_1 \psi_3 = -\frac{q + q^{-1}}{2} \psi_3 \psi_1 - \frac{i(q - q^{-1})}{2} \psi_3 \psi_0,
\]
\[
\psi_2^2 = \frac{i(q - q^{-1})}{2} \psi_1 \psi_0 = \psi_3^2, \quad \psi_2 \psi_3 + \psi_3 \psi_2 = 0.
\]
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