Radiation Reaction Friction: I. Resistive Material Medium

Martin Formanek, Andrew Steinmetz, and Johann Rafelski
Department of Physics, The University of Arizona, Tucson, AZ 85721-0081, USA
(Dated: April 14, 2020)

We explore a novel method of describing the radiation friction of particles traveling through a mechanically resistive medium. We introduce a modified metric in a form that assures that charged particle dynamics occurs subject to radiative energy loss described by the Larmor formula. We compare our description with the Landau-Lifshitz-like model for the radiation friction and show that the established model exhibits non-physical behavior. Our approach works as expected predicting in the presence of large mechanical friction an upper limit on radiative energy loss being equal to the energy loss due to the mechanical medium resistance.

I. INTRODUCTION

We study the motion of particles subject to a covariant mechanical friction force (MFF) caused by the presence of a material medium. In general, in the presence of any force, a charged particle emits radiation, a result obtained by Larmor considering properties of Maxwell’s equations. Emitted radiation complements the MFF as an induced radiation friction force (RFF). MFF will be used as an insightful model to learn how to accommodate the dynamics of radiation reaction force, i.e. radiation reaction (RR).

To the best of our knowledge, all prior covariant studies of RR employed the Lorentz force (LF) due to externally prescribed electromagnetic (EM) fields. Considered in the context of a Lorentz-Maxwell system of dynamical equations, it is well known that RR is an unsolved problem. The advantage of our approach is that we can focus on a better understanding of the effect of the radiation reaction on the mechanically accelerated particle, without need to reconcile the LF with Maxwell field dynamics.

We choose a MFF force which reduces to the familiar form of Newton’s friction force in the non-relativistic limit in the case of linear relativistic motion. In Sect. [1] we find that the relativistic generalization of Newtonian friction has a unique form used also in the study of Brownian motion [1]. For constant MFF we evaluate stopping distance, rapidity shift, and stopping power, which provide background for the later study of motion including RFF.

In this work we introduce, as a mathematical tool for making RR consistent with special relativity, a non-Minkowski space-time metric warped by particle acceleration and parameterized by the proper time of the particle. Since modification of the metric is not a field but is known only along the path of an accelerated particle, we prefer to speak of a warped metric rather than a curved metric. There have been earlier efforts to modify the space-time metric for accelerated particles spearheaded by Caianiello [2], whose work was driven by the postulate of a maximal proper acceleration. For a recent review see [3]. Our objective is different as we explore the physical environment of a resistive medium. Therefore, our approach can be seen as the ‘warping of the material medium’ due to accelerated motion.

To summarize the advantages of using the material medium:

1. The presence of a material medium provides a reference frame against which we measure particle motion. Hence the covariant form of MFF depends on the particle 4-velocity as well as the 4-velocity of the medium.

2. When a particle experiences energy loss due to RFF, this occurs in the model always at the expense of the well-defined relative motion with respect to the medium.

3. The specific form of the LF does not enter and thus the inconsistency between the EM field equations and the description of charged particle motion, see discussion on p. 745 in Jackson [4], is not introduced. We tacitly employ the Maxwell field equations when characterizing the magnitude of radiative energy loss for an accelerated charged particle as it is well known from Larmor’s work.

4. A metric warped by particle acceleration within a material medium can have the simple interpretation as being due to local material response to particle motion.

Our warped metric method avoids the introduction of higher order derivatives into the equations of motion, see discussion on p. 393 in Panofsky-Phillips [5]. The Lorentz-Abraham-Dirac (LAD) equation’s higher order derivative term was introduced in an ad-hoc manner to assure orthogonality of the equation of motion with respect to 4-velocity. This term leads to causality challenges and runaway solutions.

In view of these difficulties, Landau-Lifshitz [6] proposed an iterative scheme using dynamical equations to eliminate higher derivatives. We show in Sect. [11] that for a particle decelerated in the medium the Landau-Lifshitz-like model of radiation friction predicts non-physical behavior for particle motion. This alone demonstrates the
need to find another method to incorporate RR into in-medium particle dynamics.

In Sect. [V] we show that the ‘pure’ Larmor-RR term proportional to particle 4-velocity can be a natural consequence of a suitable warping of the metric tensor. To achieve this we allowed for the explicit dependence of the metric on the particle path and its acceleration. This naturally satisfies the requirement discussed by Langevin [7] that “being accelerated” marks the body in a distinct way in that the magnitude of time dilation depends on the history of acceleration. We note that Langevin’s remarks do not depend on the particle moving only in vacuum, they retain in full their meaning for motion in resistive material medium as well. We will return to the more difficult case of motion in vacuum in a follow-up work.

The present reformulation of RR contributes as well to a better understanding of LAD, which has been interpreted as the interaction of the charged particle with its own radiation field [8]. However, both classical and quantum particles do not move within their own Coulomb fields. With this in mind, we posit that such particles should not be allowed to move under the influence of their own radiation fields as well. It is a textbook exercise, see sect. 29.4 in Ref [9], to show that a charged accelerated particle in its instantaneous co-moving frame generates both the Coulomb field and the radiative field and there is no relative motion with an observer required to establish this. This is because, unlike velocity, acceleration has an absolute meaning, and only in the instantaneous co-moving frame does the acceleration 4-vector have the pure space-like format \(a^\mu = (0, \vec{a})\). Use of metric warping naturally prevents the particle from moving in its own radiation field.

In Sect. [V] we implement the numerical solution for the equations of motion and compare it to the motion without radiation friction. We show that in the limit of high rapidity and/or mechanical friction strength the radiation friction loss is at most matching the mechanical friction energy loss. We briefly consider application of such a model for high energy particle collisions.

II. FRICTION FORCE IN MEDIUM

In this section we describe motion of a particle under the influence of a covariant friction force in a resistive medium. The form of the force is such that it reduces to the Newtonian friction in the non-relativistic limit. We derive the expressions for stopping distance, rapidity shift and loss of energy and momentum. This force is present for both neutral and charged particles, and the energy loss manifests itself in general as heat dissipation.

A. Covariant equation of motion

The covariant equation of motion and the friction force are given by

\[
\dot{u}^\mu = \frac{1}{m} F^\mu, \quad F^\mu = r P^\mu_{\ \nu} \eta^\nu,
\]

where \(r\) is the strength of the friction and \(P^\mu_{\ \nu}\) is the projector on the orthogonal direction to 4-velocity \(u^\mu\) of the particle

\[
P^\mu_{\ \nu} = \delta^\mu_{\ \nu} - \frac{u^\mu u_\nu}{c^2}.
\]

This ensures that our friction force is automatically orthogonal to the 4-velocity. Finally, we denote the 4-velocity of the medium as \(\eta^\nu\). For general choice of the 4-velocities

\[
\eta^\mu = (\gamma_M c, \gamma_M \mathbf{v}_M), \quad u^\mu = (\gamma c, \gamma \mathbf{v}),
\]

\[
\eta \cdot u = \gamma_M \gamma c^2 (1 - \beta_M \cdot \beta),
\]

the zeroth and spatial components of the equation of the motion Eq. (1) read

\[
\gamma \frac{d\gamma}{dt} = \frac{r}{mc} \gamma_M (1 - \gamma^2 (1 - \beta_M \cdot \beta)),
\]

\[
\gamma \frac{d\beta}{dt} = \frac{r}{mc} \gamma_M (\beta_M - \gamma^2 (1 - \beta_M \cdot \beta) \beta).
\]

The energy balance, given by Eq. (5), is overall negative when

\[
\beta_M \cdot \beta < \beta^2,
\]

which means the particle loses energy due to friction. In the opposite case the medium is moving faster than the particle and the particle is being accelerated to match the velocity of the medium. When \(\beta_M = \beta\) the particle reaches an equilibrium state of rest with respect to the medium and the friction force completely disappears.

Let’s explore further the behavior of the friction force in the rest frame of the medium when \(\beta_M = 0, \gamma_M = 1\) or in 4-vector notation

\[
\eta^\mu = (c, 0, 0, 0), \quad \eta \cdot u = \gamma c^2.
\]

In this case the components of Eq. (1) are

\[
\gamma \frac{d\gamma}{dt} = \frac{r}{mc} (1 - \gamma^2),
\]

\[
\frac{d}{dt} (\gamma \beta) = -\frac{r}{mc} \gamma \beta.
\]

We can always orient our coordinate system so that the initial velocity of the particle coincides with one of the coordinate axes. Consider that the particle enters the medium in the \(x\)-direction. The perpendicular velocity then remains zero for the duration of the particle’s travel
and the motion is entirely one-dimensional. In terms of rapidity \( y \) satisfying
\[
\gamma = \cosh y, \quad \gamma \beta = \sinh y, \quad \beta = \tanh y, \quad (11)
\]
we can re-write both equations Eq. (9) and Eq. (10) as
\[
\frac{dy(t)}{dt} = -\frac{r}{mc} \tanh y(t), \quad (12)
\]
which has a solution for initial rapidity \( y(0) = y_0 \)
\[
y(t) = \text{Arcsinh} \left( \sinh(y_0) \exp \left( -\frac{r}{mc} t \right) \right). \quad (13)
\]
Note that if the velocity of the particle with respect to the medium is small, \( \beta \ll 1 \), the equation of motion Eq. (10) becomes
\[
m \frac{dv}{dt} = -\frac{r}{c} v, \quad (14)
\]
which is a familiar Newtonian friction force linearly proportional to velocity. In order to account for friction with more complicated behavior than linear dependence on velocity we need to replace the constant \( r \) with a function of relative velocity \( r(\eta \cdot u) \), which is manifestly a Lorentz scalar. In such case the solution Eq. (13) would have to be replaced by a numerical solution of Eq. (12) with a specific function \( r(y) \). The friction strength \( r \) is also in general a function of the medium density.

### B. Distance traveled and rapidity shift

Rewriting the solution \( y(t) \) in Eq. (13) as
\[
\sinh y(t) = \sinh y_0 \exp \left( -\frac{r}{mc} t \right), \quad (15)
\]
and using \( \sinh y = \gamma \beta \), we can find solution for \( \beta(t) \) as
\[
\beta(t) = \frac{\gamma_0 \beta_0}{\sqrt{\exp \left( \frac{2r}{mc} t \right) + \gamma_0^2 \beta_0^2}}. \quad (16)
\]
The distance traveled in the rest frame of the medium is given by the integral
\[
x(t) = x_0 + \int_0^t \beta(t') c dt',
\]
\[
= x_0 + \int_0^t \frac{\gamma_0 \beta_0}{\sqrt{\exp \left( \frac{2r}{mc} t' \right) + \gamma_0^2 \beta_0^2}} c dt', \quad (17)
\]
which can be evaluated as
\[
x(t) = x_0 + \frac{mc^2}{r} \left( \text{Arctanh}(\beta_0) - \text{Arctanh}(\beta(t)) \right)
\]
\[
= x_0 + \frac{mc^2}{r} (y_0 - y(t)). \quad (18)
\]
The total distance traveled \( D \) until the particle comes to a stop \( y(t_s) = 0 \) at time \( t_s \) is simply
\[
D = (x(t) - x_0) \big|_{y(t_s)=0} = \frac{mc^2}{r} y_0. \quad (19)
\]
By inverting Eq. (18) we can obtain \( y \) as a function of \( x \)
\[
y(x) = y_0 - \frac{r}{mc^2} (x - x_0). \quad (20)
\]
The rapidity shift per change in distance \( dy/dx \) is therefore a constant
\[
\frac{dy}{dx} = -\frac{r}{mc^2}. \quad (21)
\]
This is the mechanical rapidity shift caused by the medium’s resistance. In the case of charged particle motion we also need to account for the additional radiation rapidity shift effect. The description of this contribution is the main focus of Sect. III and Sect. IV.

### C. Energy and momentum loss

We now consider the stopping power in terms of the change of energy \( E = mc^2 \gamma \) per unit of distance
\[
\frac{dy}{dx} = \frac{d \text{Arccosh}(\gamma)}{dx} = \frac{1}{\sqrt{\gamma^2 - 1}} \frac{d\gamma}{dx} = \frac{1}{mc \sinh y} \frac{dE}{dx}. \quad (22)
\]
Therefore by substituting Eq. (22) into Eq. (21), we obtain
\[
\frac{dE}{dx} = -r \sinh y(x). \quad (23)
\]
Clearly, when rapidity reaches zero in the rest frame of the medium, the particle energy stops changing as expected. Another way to write this expression is uses
\[
\sinh y = \gamma \beta = \frac{p}{mc}, \quad (24)
\]
where \( p = m \gamma v \) is particle’s momentum. Then
\[
\frac{dE}{dx} = -\frac{r}{mc} p, \quad (25)
\]
and conversely using the relativistic expression for energy
\[
E = \sqrt{m^2 c^4 + p^2 c^2}
\]
\[
\frac{dp}{dx} = -\frac{r}{mc^2} E. \quad (26)
\]
Finally, energy and momentum can be expressed as
\[
E = mc^2 \gamma = mc^2 \cosh y, \quad (27)
\]
\[
p = mc \gamma \beta = mc \sinh y, \quad (28)
\]
which we can evaluate either as a function of laboratory time or position, using the solutions for rapidity \( y(t) \) Eq. (13) and \( y(x) \) Eq. (20), respectively.
III. RADIATION FRICTION

This section introduces: a) the ‘standard model’ of RFF, the Lorentz-Abraham-Dirac (LAD) equations of motion for description of RR; and b) Landau-Lifshitz reduction of LAD differential order. We apply this procedure to our problem of the charged particle moving in a resistive medium. We present Landau-Lifshitz (LLL) equations of motion for both Newtonian friction and friction with strength generally dependent on \( \gamma \cdot u \) and discuss their behavior. We show that the LLL model leads to non-physical behavior for the particle motion.

A. LAD radiation friction in vacuum

The unresolved question of the consistent description of accelerated charged particle motion including its radiation and radiation friction is now well over a century old. Indeed, the power radiated by such particle was first described by Larmor at the end of the 19th century. In a covariant form

\[ P = m \tau_0 \dot{u}^2, \]

where the characteristic time \( \tau_0 \) reads

\[ \tau_0 = \frac{2}{3 \cdot 4 \pi \varepsilon_0 me^3} = \frac{2 \alpha \hbar}{3 mc^2} \approx 6.26 \times 10^{-24} \text{ s}, \]

(30)

for an electron, where \( \alpha \approx 1/137.036 \). If we add a corresponding momentum change to Eq. (1) the equation of motion reads

\[ \dot{u}^\mu = \frac{1}{m} F^\mu + \tau_0 \ddot{u}^2 \dot{u}^\mu \frac{c^2}{c^2}. \]

(31)

We see that this expression does not preserve \( u^2 = c^2 \) because \( \dot{u} \cdot u \neq 0 \). Further work by Abraham [11], Dirac [12], and Lorentz [13] resulted in the formulation of the LAD equation for accelerated charged particle motion

\[ \dot{u}^\mu = \frac{1}{m} F^\mu + \tau_0 \left( \ddot{u}^\mu + \dot{u}^2 \frac{u^\mu}{c^2} \right), \]

(32)

where the term proportional to the second derivative of 4-velocity, the so-called Schott term, is added ad-hoc to ensure \( u^2 = \text{const} \). This term contains unwelcome second order proper-time derivative of the 4-velocity and is plagued by issues with initial conditions, causality, run-away and pre-accelerated solutions [16–18]. This field of study remains very active till this day, with recent publications exploring particles of finite extent and their point limit [18] and methods improving the Dirac’s derivation [8].

Currently there are two main approaches aiming to resolve the issue of the LAD formulation. The first is to impose appropriate boundary and asymptotic conditions on the solution so that the non-physical solutions are discarded [20]. The second is the Landau-Lifshitz (LL) model [6], or related order reducing schemes, which approximate the LAD equation by iterating the acceleration due to external force and expanding into powers of the small parameter \( \tau_0 \). This approach reduces the order of the equation of motion and thus resolves the known issues of the LAD formulation at the expense of truncating the full radiation reaction.

Spohn [21] showed that LAD restricted onto a physical manifold produces the LL series, so both approaches lead in certain environment to the same dynamics. However, this is not the case in the study of electron stopping by a truncated light plane wave, see Ref. [19]. This occurs because a traveling light wave front creates a quasi-material edge for an incoming particle. This resembles, but is not exactly the same, the case of the material medium we look at next.

B. Landau-Lifshitz-like RR in medium

1. Constant material friction

For our system in the zeroth order in \( \tau_0 \), the acceleration is given by the external force

\[ \dot{u}^\mu_{(0)} = \frac{r}{mc} P^\mu, \eta \cdot \eta'. \]

(33)

By substituting this expression to the radiation friction term in Eq. (32) we obtain for the second derivative of 4-velocity

\[ \dot{u}^\mu_{(0)} = \frac{r}{mc} P^\mu, \eta \cdot \eta' = \frac{r}{mc} \left( \frac{\dot{u}^\mu_{(0)}(\eta \cdot u)}{c^2} - \frac{u^\mu_{(0)}(\dot{u} \cdot \eta)}{c^2} \right) \]

\[ = -\frac{r^2}{m^2 c^4}(P^\mu, \eta \cdot u) + u^\mu(\eta \cdot P \cdot \eta') \]

(34)

and for the square of acceleration in the zeroth order in \( \tau_0 \)

\[ \ddot{u}^2_{(0)} = \frac{r^2}{m^2 c^2}(\eta \cdot P \cdot \eta) = \frac{r^2}{m^2 c^2}(\eta \cdot P \cdot \eta), \]

(35)

because of the property of the projector \( P^2 = P \). We see that the Larmor term cancels with one of the two terms arising from the Schott term and the final Landau-Lifshitz-like (LLL) equation of motion reads

\[ \dot{u}^\mu_{(1)} = \frac{r}{mc} P^\mu, \eta \cdot \eta' - \tau_0 \frac{r^2}{m^2 c^4}(\eta \cdot u) P^\mu, \eta \cdot \eta'. \]

(36)

The zeroth component of this equation in the rest frame of the medium is

\[ \gamma \frac{d \gamma}{dt} = \frac{r}{mc} (1 - \gamma^2) - \tau_0 \frac{r^2}{m^2 c^2} \gamma (1 - \gamma^2), \]

(37)

and in terms of rapidity Eq. (11) we have

\[ \frac{dy}{dt} = -\frac{r}{mc} \tanh y + \tau_0 \frac{r^2}{m^2 c^2} \sinh y. \]

(38)
In the second term of Eq. (38) we see a reversal in the sign. The effect of radiation friction is then, up to the first power in $\tau_0$, to increase the energy of the particle experiencing deceleration in a medium. Moreover, the rapidity has to satisfy
\[ y < \text{arccosh} \left( \frac{mc}{\tau_0^2} \right), \] (39)otherwise the radiation friction overpowers the mechanical friction in the medium and we get a runaway solution which is clearly an unacceptable behavior. As the Lorentz force did not appear in this derivation, the incompatibility we discovered must originate in the LAD extension of Larmor radiation-reaction.

2. Variable material friction in LLL approach

The derivation above assumes that the radiation friction $r$ is constant. If we evaluate LLL model for non-constant $r$ as a function of $\eta \cdot u$, then the covariant equation of motion up to first order in $\tau_0$ is
\[ \ddot{u}_\mu + \frac{P_{\nu} \eta^\nu}{mc} \eta^\mu + \tau_0 \frac{r}{m^2 c^2} \left( \frac{\eta \cdot P}{\eta^\nu} \frac{d\eta}{d(\eta \cdot u)} \eta^\nu - \frac{\eta^\nu}{\eta^\nu} \right) \eta^\mu + \frac{d\eta}{d\gamma} \eta^\mu, \] (40)and the zeroth component in the rest frame of the medium is
\[ \gamma \frac{d\gamma}{dt} = \frac{r}{mc} (1-\gamma^2) + \tau_0 \frac{r}{m^2 c^2} \left( -r + \frac{dr}{d\gamma} (1-\gamma^2) \right) (1-\gamma^2). \] (41)Such LLL friction term has a chance of having negative contribution to energy if
\[ \frac{dr}{d\gamma} < -\frac{r \gamma}{\gamma^2 - 1}. \] (42)This cannot happen in the non-relativistic limit, because $dr/d\gamma$ would have to go to minus infinity. The terms exactly cancel when
\[ r \propto \exp \left( -\sqrt{\gamma^2 - 1} \right) \propto \exp \left( -\gamma \beta \right) \propto \exp \left( -\frac{p}{mc} \right). \] (43)

In such a case there is no radiation friction according to the LLL approach, which we consider an unphysical outcome. As the word friction implies, an energy loss occurs for particle motion and one must not allow a particle to slip through a mechanically resistive medium unimpeded. Even if the format of radiation friction required in Eq. (43) is unusual, this is what naive evaluation predicts. We conclude that the conventional LAD radiation reaction combined with LLL reduction of the order of differentiation produces an unacceptable description of radiation friction in a material medium, even when the friction strength is an arbitrary function of relative velocity.

IV. METRIC WARping

Here we propose an alternative radiation friction model. We show that formally we can introduce radiation friction through space-time metric warping while keeping the form of the covariant Larmor formula. This allows us to formulate the dynamics without higher order derivatives and with self-consistent formula for the magnitude of acceleration. As the driving force we take the covariant mechanical friction force. We establish equations of motion for such a system in the warped-metric model and evaluate stopping power.

A. General considerations

We start with the equation of motion with only the Larmor term present
\[ \dot{u}^\mu = \frac{1}{m} F^\mu + \tau_0 \dot{u}^2 u^\mu, \] (44)and instead of adding a second order derivative Schott term we assume that the space-time metric is not flat anymore. This will allow us to preserve
\[ u^2 = g_{\mu\nu} u^\mu u^\nu = c^2 \] (45)while assuming that 4-force remains orthogonal to 4-velocity. We can multiply Eq. (44) by $g_{\mu\nu} u^\nu$ to obtain
\[ \dot{u} \cdot u = \tau_0 \dot{u}^2. \] (46)
If we use this identity in Eq. (44) we derive an expression which is explicitly orthogonal to $u^\mu$
\[ \dot{u}^\mu - (\dot{u} \cdot u) \frac{u^\mu}{c^2} = \frac{1}{m} F^\mu, \] (47)suggesting that the covariant friction is proportional to the product of 4-velocity and 4-acceleration $\dot{u} \cdot u$, which is in flat space-time zero. Upon differentiating the condition $u^2 = c^2$ Eq. (45) with respect to proper time we obtain
\[ \dot{u} \cdot u = -\frac{1}{2} \frac{d g_{\mu\nu} u^\mu u^\nu}{d\tau}, \] (48)which is a condition for components of the metric. Eq. (44) provides a consistent expression for the magnitude of acceleration $\dot{u}^2$. The square of this expression reads
\[ \dot{u}^2 = \frac{1}{m^2} F^2 + \frac{\tau_0}{c^2} \dot{u}^4, \] (49)which is a quadratic equation for $\dot{u}^2$ with solutions
\[ \dot{u}^2 = \frac{c^2}{2\tau_0} \left( 1 \pm \sqrt{1 - 4 \tau_0^2 \frac{F^2}{c^2 m^2}} \right). \] (50)
We take the minus sign as the physical solution, because it reduces in the limit \( t \rightarrow 0 \) to the usual expression \( \hat{u}^2 = F^2/m^2 \). This expression can be further simplified to

\[
\hat{u}^2 = \frac{2F^2/m^2}{1 + \sqrt{1 - 4\xi^2F^2/m^2}}.
\]  

(51)

It is worth noting that as \( F^2 \rightarrow 0 \) then also \( \hat{u}^2 \rightarrow 0 \) and conversely as \( F^2 \rightarrow -\infty \) the growth of \( \hat{u}^2 \) is damped.

**B. Specific metric model**

In order to avoid mixing the spatial and time components we will assume that the metric is diagonal and in our 1D situation we choose a parametrization

\[
g_{\mu\nu} = \text{diag}(f_0^2, -f^2, -1, -1).
\]  

(52)

In the following we will suppress the two trivial degrees of freedom. The proper time of the particle is by definition

\[
d\tau = \frac{1}{c} \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^n}{dt}} dt = \sqrt{f_0^2 - f^2 \left( \frac{dx}{dt} \right)^2} \; dt,
\]  

(53)

where we took the position 4-vector as \( x^\mu = (ct, x) \). We can then perform a coordinate transformation from \( dt \) and \( dx \) to measurable quantities

\[
dt_{\text{lab}} = f_0 dt, \quad dx_{\text{lab}} = f dx,
\]  

(54)

which simplifies the increment of proper time to

\[
d\tau = \sqrt{1 - \frac{1}{c^2} \left( \frac{dx_{\text{lab}}}{dt_{\text{lab}}} \right)^2} \; dt_{\text{lab}}.
\]  

(55)

If we define the true physical velocity of the particle as

\[
v = \frac{dx_{\text{lab}}}{dt_{\text{lab}}} = \frac{f}{f_0} \frac{dx}{dt},
\]  

(56)

we can write the gamma factor in usual form

\[
\gamma = \frac{dt_{\text{lab}}}{d\tau} = \sqrt{1 - v^2/c^2}.
\]  

(57)

Note that the transformation Eq. (54) is a transformation to flat space coordinates, as can be seen by evaluating

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2 dt_{\text{lab}}^2 - dx_{\text{lab}}^2.
\]  

(58)

The coordinate 4-velocity \( u^\mu \) that enters our equation of motion is given by

\[
u^\mu = \frac{dx^\mu}{d\tau} = \gamma \left( \frac{c}{f_0}, \frac{v}{f} \right) = \gamma \left( c, \frac{v}{f} \right).
\]  

(59)

This expression satisfies \( u^2 = c^2 \) Eq. (45) as expected and energy of the particle is given by the usual expression

\[E = \gamma mc^2.
\]  

(60)

For the 4-velocity of the medium we can write analogically

\[
\eta^\mu = \gamma_M \left( \frac{c}{f_0}, \frac{v_M}{f} \right),
\]  

(61)

preserving \( \eta^2 = c^2 \). Therefore the RHS of the equation of motion Eq. (47) in the rest frame of the medium reads

\[
\frac{1}{m} F^\mu = \frac{r}{mc} P^\mu v \eta^\nu = \frac{r}{mc} \left( \eta^\mu - \frac{u^\mu}{c^2} (\eta \cdot u) \right)
\]  

\[
= \frac{r}{mc} \left( \frac{c}{f_0} (1 - \gamma^2), -\frac{\gamma^2}{f} v \right),
\]  

(62)

which is equivalent to rescaling the zeroth component of the force by \( f_0 \) and spatial component by \( f \). Note that \( r \) can be in general a function of \( \eta \cdot u \) to account for more complicated mechanical friction than Newtonian friction, but this fact does not modify our derivation and holds true throughout the current section (Sec. [V]). Finally, we can evaluate the dot product \( \hat{u} \cdot u \) using Eq. (48)

\[
\hat{u} \cdot u = -\frac{1}{2} \frac{df_0^2}{d\tau} \gamma^2 c^2 + \frac{1}{2} \frac{df^2}{d\tau} \gamma^2 v^2 = -A \gamma^2 c^2 + B \gamma^2 v^2,
\]  

(63)

where we denoted

\[
A \equiv \frac{d}{d\tau} \ln f_0, \quad B \equiv \frac{d}{d\tau} \ln f.
\]  

(64)

Now we are prepared to establish the equations of motion for the particle with this choice of metric.

**C. Radiation energy loss in metric model**

If we substitute the 4-velocity Eq. (59), the force Eq. (62), and the dot product Eq. (63) to the equation of motion Eq. (47) we obtain for the zeroth component

\[
\frac{d}{d\tau} \left( \gamma \frac{c}{f_0} \right) + (A \gamma^2 c^2 - B \gamma^2 v^2) \gamma \frac{f_0}{mc} = \frac{r}{mc f_0} (1 - \gamma^2). \quad (65)
\]

The first term can be further expanded

\[
\frac{d}{d\tau} \left( \gamma \frac{c}{f_0} \right) = \gamma \frac{d\gamma}{f_0 dt_{\text{lab}}} - \frac{1}{f_0^2} \frac{df_0}{dt_{\text{lab}}} \gamma \frac{d\gamma}{f_0 dt_{\text{lab}}} - \gamma A. \quad (66)
\]

Finally, by substituting back to Eq. (65) and multiplying by \( f_0/\gamma \) we have an expression for the change in gamma factor

\[
\frac{d\gamma}{dt_{\text{lab}}} = \frac{r}{mc \gamma} (1 - \gamma^2) + A (1 - \gamma^2) + B \gamma^2 \beta^2. \quad (67)
\]

Similarly for the spatial component of Eq. (47)

\[
\frac{d}{d\tau} \left( \frac{\gamma v}{f} \right) + (A \gamma^2 c^2 - B \gamma^2 v^2) \frac{\gamma v}{fc^2} = -\frac{\gamma^3 v^2}{mf}. \quad (68)
\]
For the first term in Eq. (68) we evaluate the derivative
\[
\frac{d}{dt} \left( \gamma v \right) = \gamma \frac{d\gamma}{dt} \frac{v}{f} - \frac{1}{f^2} \frac{df}{dt} \gamma v + \frac{\gamma^2}{f} \frac{dv}{dt}.
\]
If we use Eq. (67) for change in \(\gamma\) and substitute back to Eq. (68) then after several cancellations we obtain
\[
\frac{\gamma^2}{f} \frac{dv}{dt} = -\frac{r}{mc} \frac{v}{f} - (A - B) \frac{\gamma}{f}, \tag{70}
\]
which if we multiply by \(f/\gamma^2 c\) becomes
\[
\frac{d\beta}{dt} = -\frac{r}{mc} \frac{\beta}{c^2} - (A - B) \frac{\gamma}{c^2}. \tag{71}
\]

We can use identity \(1 + \gamma^2 \beta^2 = \gamma^2\) to further simplify the zeroth part Eq. (67) and write the components of the covariant equation of motion in a final form
\[
\frac{d\gamma}{dt} = \frac{r}{mc} (1 - \gamma^2) + (A - B) (1 - \gamma^2), \tag{72}
\]
\[
\frac{d\beta}{dt} = -\frac{r}{mc} \frac{\beta}{c^2} - (A - B) \frac{\gamma}{c^2}, \tag{73}
\]
which are mutually equivalent. Although the metric \(g_{\mu\nu}\) is specified by two unknowns, \(A\) and \(B\), the dynamics of the particle motion depends only on their difference
\[
A - B = \frac{d}{dt} \ln \frac{f_0}{f}, \tag{74}
\]
which means that any arbitrary factor re-scaling the whole metric does not change the motion. Therefore we can set either \(A\) or \(B\) to zero and evaluate the other without any loss of generality.

D. Stopping power and limiting cases

We assume that the more general equation of motion Eq. (47) is in the form of Eq. (44) where the friction term is given by the Larmor formula. Combining expressions for \(\dot{u} \cdot u\) in Eq. (46) and Eq. (48) we can equate
\[
\tau_0 \dot{u}^2 = -\frac{1}{2} \frac{dg_{\mu\nu}}{d\tau} u^\mu u^\nu, \tag{75}
\]
where the self-consistent magnitude of acceleration \(\dot{u}^2\) was evaluated in Eq. (51) and the right hand side is given in our metric as Eq. (63)
\[
\frac{2 \tau_0 F^2/m^2}{1 + \sqrt{1 - \frac{4 \tau_0^2 2 F^2}{m^2}}} = -A \gamma^2 c^2 + B \gamma^2 v^2. \tag{76}
\]
The square of the external force can be computed using equation Eq. (62)
\[
\frac{1}{m^2} F^2 = \frac{1}{m^2} g_{\mu\nu} F^\mu F^\nu = \frac{r^2}{m^2} (1 - \gamma^2). \tag{77}
\]
Notice that this expression does not depend on the metric and is equal to the square of the external force in the flat space-time. As discussed above we can set \(A = 0\) and evaluate \(B\)
\[
B = \frac{-2 \tau_0 r^2}{m^2 c^2 + m^2 c^2 \gamma^2 \beta^2}, \tag{78}
\]
where we used the identity \(1 - \gamma^2 = -\gamma^2 \beta^2\). Using this solution in the equations of motion Eq. (72) and Eq. (73) yields
\[
\frac{d\gamma}{dt} = \frac{r}{mc} (1 - \gamma^2) + \frac{2 \tau_0 r^2}{m^2 c^2 + m^2 c^2 \gamma^2 \beta^2} (1 - \gamma^2), \tag{79}
\]
\[
\frac{d\beta}{dt} = -\frac{r}{mc} \frac{\beta}{c^2} - \frac{2 \tau_0 r^2}{m^2 c^2 + m^2 c^2 \gamma^2 \beta^2} \frac{\beta}{c^2}, \tag{80}
\]
From Eq. (79) we can calculate the radiation energy loss in powers of \(\tau_0\)
\[
\frac{dE}{dt}_{RF} = \frac{mc^2}{dt} \frac{d\gamma}{dt}|_{RF} = \frac{r^2}{m} (1 - \gamma^2) + O(\tau_0^3), \tag{81}
\]
which matches the covariant Larmor energy loss formula Eq. (29) with 4-acceleration given purely by the external force.

In terms of stopping power \(dE/dx\) we can convert Eq. (79) to
\[
\frac{dE}{dt}_{lab} = \left. \frac{dE}{dt}_{lab} \right|_{RF} - \frac{2 \tau_0 \left( \frac{dE}{dt}_{lab} \right|_{M})}{1 + \sqrt{1 + 4 \tau_0^2 2 F^2/m^2}} \coth y, \tag{82}
\]
where the \(dE/dx_{lab}|_M\) is the stopping power caused by the medium friction given by Eq. (23). The two important limiting cases are determined by critical mechanical stopping power
\[
\frac{dE}{dt}_{lab|_{crit}} = \frac{mc^2}{c\tau_0} \approx \frac{3 (mc^2)^2}{2 \pi ahc} \approx 0.27 \text{ MeV/fm}, \tag{83}
\]
where the value given is for an electron. This value scales accordingly for strongly interacting quarks considering mass and coupling constant dependence: for a strange quark \(M_s \approx 100\text{ MeV}\) and strong interaction coupling constant \(\alpha_s = 0.6\) the corresponding value is \(120\text{ MeV/fm}\). This corresponds to a situation when particle losses due to the friction in the medium equivalent of its rest mass energy while traveling distance \(c\tau_0\).

If mechanical stopping power is much higher than the critical stopping power the radiation friction part of the stopping power is approximately
\[
\left. \frac{dE}{dt}_{lab} \right|_{RF} \approx \left. \frac{dE}{dt}_{lab} \right|_{M} \coth y. \tag{84}
\]
In the opposite case, when mechanical stopping power is much less than the critical stopping power,

\[ \frac{dE}{dx_{\text{lab}}}|_{\text{RF}} \approx \frac{dE}{dx_{\text{lab}}}|_{\text{M}} \left( \frac{dE}{dx_{\text{lab}}}|_{\text{crit}} \right) \coth y. \]  

(85)

Note that if \( y \to 0 \) then the stopping power in medium also goes to zero as \( \sinh y \), so the expression is well behaved.

V. MOTION EXAMPLES

With the dynamics developed in the previous section (Sect. IV) we can evaluate the motion of the radiating charged particles and compare to the LLL model (Sect. III) and to the motion without any radiation friction (Sect. II). This section presents numerical solutions for the particle motion in each of the three situations with the underlying Newtonian mechanical friction force. We show that our model unlike the LLL model increases, as expected, the energy loss due to radiation friction. This additional energy loss can at most match the mechanical friction loss in the medium in the limit of high rapidity or friction strength. Finally, we discuss possible experimental applications of our model for high energy particle collisions.

A. Solution of dynamical equations

Let us define a unitless friction strength

\[ \tilde{r} = \frac{r\tau_0}{mc} = \frac{r}{dE/dx_{\text{lab}}|_{\text{crit}}}, \]  

(86)

then we can write the equation of motion Eq. (80) as

\[ \frac{d\beta}{dt_{\text{lab}}} = -\tilde{r} \frac{\beta}{\gamma^2} - \frac{2\tilde{r}^2}{1 + 4\gamma^2} \frac{\beta}{\gamma}. \]  

(87)

Switching to rapidity Eq. (11) we obtain

\[ \frac{dy}{dt_{\text{lab}}/\tau_0} = -\tilde{r} \tanh y + 2\tilde{r}^2 \frac{\sinh y}{1 + 4\tilde{r}^2 \sinh^2 y}. \]  

(88)

These expressions are suitable for numerical analysis.

Let us consider motion with initial rapidity \( y_0 = 7 \) and \( \tilde{r} = 10^{-3} \) to demonstrate the character of the solution. Figure 2 shows rapidity as a function of time for both our model and the LLL model as propagated by the RK4 integration scheme. Note that the condition Eq. (39) is satisfied to prevent runaway solutions in the LLL model. The dashed line indicates the analytical solution without any radiation friction Eq. (13). We see that in the LLL model, the particle decelerates more slowly than without radiation friction and in our model faster. The behavior of our model should match our intuition as the ‘correct’ physical behavior where both sources of friction impede the particle’s motion.

Stopping power can be evaluated by diving the whole expression Eq. (88) with \( \beta = \tanh y \) because \( dx_{\text{lab}} = \)
FIG. 3. Distance traveled by the particle as a function of time for initial rapidity $y_0 = 7$ and friction strength $\tilde{r} = 10^{-3}$. We show results with and without radiation friction and for the LLL model (dashed). Dotted line marks the stopping distance without radiation friction.

$\beta c dt_{\text{lab}}$

$$\frac{dy}{dx_{\text{lab}}/c t_0} = -\tilde{r} \left( 1 + \frac{2\tilde{r} \cosh y}{1 + \sqrt{1 + 4\tilde{r}^2 \sinh^2 y}} \right) \equiv \tilde{r}(1 + \Delta), \quad (90)$$

where distance is measured in units of $c t_0$ and $\Delta$ is the relative change from the case without any radiation friction. Fig. 2 shows possible values for $\Delta$ for selected unitless friction strengths $\tilde{r}$ and range of rapidities $y$. We see that the radiation friction at most doubles the stopping power $dy/dx_{\text{lab}}$.

With our choice of parameters the stopping distance $D$ is in the case without radiation friction given by Eq. (19), which in unitless quantities reads

$$\frac{D}{c t_0} = \frac{y_0}{\tilde{r}} = 7 \times 10^3. \quad (91)$$

As can be seen from the trajectories in Fig. 3 with radiation friction present, the particle in our model stops in a significantly shorter distance and for the LLL model it travels further. Figure 2 demonstrates that initially the stopping power is doubled for high rapidities and as particle slows down the radiation friction contributes less and less.

B. Experimental verification

From the expression for the critical stopping power Eq. (53) we see that a very high energy loss is needed to reach significant radiation friction. However, the typical stopping power of proton beams in a material medium is on the order of $1 - 100 \text{ MeV/cm}$ and this applies also to many other particles. This value is many orders of magnitude too small to induce sizable radiation energy production. This value is of course dependent on the density of the medium and varies with the energy of the particle. But even at the high end of the range, such stopping power is only a $10^{-11}$ fraction of the critical stopping power.

We conclude that in normal materials with atomic structure, RR induced by the material stopping power is negligible, hence RR induced by MFF is negligible too. However, microscopically the particle motion is also experiencing Coulomb scattering off atomic nuclei of the medium with high accelerations, which at each scattering event contribute to the bremsstrahlung [23]. In this work we do not consider these microscopic processes as we do not want to deal with EM forces presently.

The RR effect in medium we have presented could be of interest to following experimental environments:

1. Ultra-relativistic cosmic particles entering our atmosphere. Such particles can have energies on the order of $10^{20} \text{ eV}$ and while slowing down induce a shower of particles interacting directly with the nuclei of atoms making the atmosphere - where the nuclear density creates appropriate stopping power to enhance the energy found in EM radiation in comparison to hadronic energy.

2. CERN-LHC head-on particle collisions. According to Ref. [24] the ratio of electromagnetic to hadronic energy in the proton proton collisions at LHC is about 0.4 for central collision (pseudorapidity $\eta = 0$). In heavy ion nuclear CERN-LHC collisions any RR effect could be of more interest considering the much longer ‘nuclear’ distances available to slow
down charged partons, and in general the much greater energy accumulation near to $\eta = 0$ as compared to $pp$ collisions.

3. The EM processes we considered appear in analogous situations in strong interaction environments. The method proposed here should be generalized in order to improve the understanding of energy and parton stopping for ultra-relativistic $pp$ and nuclear heavy ion collisions.

Regarding the $pp$ value 0.4 above: This is just what is expected even if there must be considerable stopping power, since all particles produced in the central rapidity region originate in initially rapidly moving incoming charged parton constituents of colliding protons.

To see how 0.4 arises, consider that all particles produced are $\pi^+, \pi^-, \pi^0$. The charge pions in general decay into charged particles and thus manifest themselves in hadronic energy from $\pi^0$ to be 1/2 of hadronic energy from $\pi^+, \pi^-$. The presence of particles other than pions (strange particle component, baryons and antibaryons etc) reduce this value, thus experimentally the observed 0.4 is qualitatively what we expect. Both the theoretical analysis and the experiment can be refined allowing the determination, in a quantitative manner, the contribution of RR to this energy ratio. Any effect would be amplified considering the case of heavy ion collisions.

VI. FUTURE WORK AND CONCLUSIONS

In the forthcoming work we hope to investigate other, less material environments. A study of EM interaction has, as we have mentioned, the challenge of reconciling the form of the Lorentz force with that of Maxwell’s equations. Therefore a training non-material problem in study of RR is an exploration of charged particle dynamics in presence of an external scalar force field. In this case the field dynamics providing the Larmor RR term is decoupled from the force form, which is non-material and thus reaching beyond the case considered in this work. We plan to return to this example within a short delay.

A further training ground is the study of RR for particles under influence of an externally prescribed constant electromagnetic field. There is good reason to take a second look at the constant field case: The LAD or LL format of RR may not correctly describe the physical reality of a constant EM field. For example, a well known prediction of the LL model is that a particle linearly accelerated by an electric field (so-called hyperbolic motion) does not feel any radiation friction and yet produces radiation. This is due to the contribution of the Schott term in the equation of motion, which in this case exactly balances out the Larmor term, a situation that is subject to ongoing discussion.

Even though a metric RR in this case requires establishment of consistency between the Maxwell field equations and the equation governing the particle motion we are optimistic that our metric approach may succeed. What encourages us to pursue constant fields is that for an observer, for whom the energy-momentum tensor $T^\mu_\nu$ of the constant external electromagnetic field is diagonal, we obtain the metric
g_{\mu\nu}(\tau) = \eta_{\mu\nu} \exp \left( \frac{2}{m^2} T^{\alpha\nu} \right),

which exactly reproduces the Landau-Lifshitz format of the equations of motion in the first order in $\tau_0$.

Another particularly interesting RR study involves the motion of particles in plane wave fields. The well studied case of electron interaction with a light wave edge is here of particular interest. This case was explored using the LL approach, and critical acceleration effects were demonstrated. Therefore a self-consistent metric formulation of RR could lead to directly verifiable experimental outcomes using present day experimental facilities.

Another possible extension concerns the development of a variational principle for RR force. Unlike LAD or LL models where a variational principle was never established, our metric deformation formulation has a better chance of arising from a specific covariant action since it does not contain higher order derivative terms in the equation of motion.

To conclude: We have shown that it is possible to describe mechanically decelerated particle energy loss due to radiation friction in terms of metric modification. Our approach resolves contradictions: For example, the Landau-Lifshitz-like procedure predicts that particles would gain energy, presumably due to interaction with its own radiation field. The modified metric approach does not introduce such an interaction and the total energy loss remains consistent with the Larmor radiation energy loss formula.

We have shown that when solved consistently, radiative fields are given by particle acceleration due to both external force and radiation friction. The prediction of such a self-consistent calculation is that the radiation friction at most doubles the ’mechanical’ energy loss, see Figure 2. This result is intuitively and theoretically satisfactory and it can have some interesting experimental consequences awaiting study in ultra relativistic collisions of cosmic particles and collider experiments.

ACKNOWLEDGMENTS

Martin Formanek and Johann Rafelski would like to express their gratitude to Dr. Tamás Biró and Dr. Péter Lévai for their hospitality during the summer 2019 at the Wigner Research Centre for Physics, Budapest and 2019 Balaton Workshop when part of this work was conducted. Johann Rafelski was a Fulbright Fellow during this period.
[1] J. Dunkel and P. Hänggi, “Theory of relativistic Brownian motion: The \((1+1)\)-dimensional case,” Phys. Rev. E 71, 016124 (2005)
[2] E. R. Caianiello, A. Feoli, M. Gasperini and G. Scarpetta, “Quantum Corrections to the Space-time Metric From Geometric Phase Space Quantization,” Int. J. Theor. Phys. 29, 131 (1990) doi:10.1007/BF00671323
[3] R. G. Torrom and P. Nicolini, “Theories with maximal acceleration,” Int. J. Mod. Phys. A 33, no. 22, 1830019 (2018) doi:10.1142/S0217751X18300193 [arXiv:1805.07126 [gr-qc]]
[4] J. D. Jackson, Classical Electrodynamics, Third Edition, Hoboken, NJ: John Wiley & Sons, Inc., p. 745 (1999)
[5] W. J. H. Panofsky and M. Phillips, Classical Electricity and Magnetism, Second Edition Reading, MA: Addison-Wesley (1962)
[6] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Second Edition, London, England: Pergamon (1962)
[7] P. Langevin, “L’Évolution de l’espace et du temps,” Scientia 10 pp 3154 (1911) English translation at Wikisources [https://en.wikisource.org/wiki/Translation:The_Evolution_of_Space_and_Time]
[8] C. Bild, H. Ruhl and D. A. Deckert, “Radiation reaction in classical electrodynamics,” Phys. Rev. D 99, no. 9, 096001 (2019) doi:10.1103/PhysRevD.99.096001 [arXiv:1812.09282 [physics.class-ph]]
[9] J. Rafelski, Relativity Matters, Berlin, Germany: Springer, p. 446 (2017)
[10] J. Larmor, “On the theory of the magnetic influence on spectra; and on the radiation from moving ions,” The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 44(271), 503 (1897)
[11] M. Abraham, “Classical theory of radiating electrons,” Ann. Physik 10, 105 (1903)
[12] P. A. M. Dirac, “Classical theory of radiating electrons,” Proc. Roy. Soc. Lond. A 167, 148 (1938) doi:10.1098/rspa.1938.0124
[13] H. A. Lorentz, The Theory of Electrons, New York, USA: Dover publications (1952)
[14] F. Rohrlich, Classical charged particles, Singapore: World Scientific Publishing Company (2007; First edition 1965, Second edition 1990)
[15] A. O. Barut, Electrodynamics and classical theory of fields and particles, New York, USA: Dover (1980)
[16] H. Spohn, Dynamics of Charged Particles and Their Radiation Field, Cambridge, England: Cambridge University Press (2004)
[17] E. Poisson, “Introduction to the Lorentz-Dirac equation,” arXiv preprint gr-qc/9912045 (1999)
[18] S. E. Gralla, A. I. Harte and R. M. Wald, “A Rigorous Derivation of Electromagnetic Self-force,” Phys. Rev. D 80, 024031 (2009) doi:10.1103/PhysRevD.80.024031 [arXiv:0905.2391 [gr-qc]]
[19] Y. Hadad, L. Labun, J. Rafelski, N. Elkina, C. Klier and H. Ruhl, “Effects of Radiation-Reaction in Relativistic Laser Acceleration,” Phys. Rev. D 82, 096012 (2010) doi:10.1103/PhysRevD.82.096012 [arXiv:1005.3980 [hep-ph]]
[20] G. N. Plass, “Classical electrodynamics equations of motion with radiative reaction,” Rev. of Mod. Phys., 33(1), 37 (1961)
[21] H. Spohn, “The critical manifold of the Lorentz-Dirac equation,” Europhys. Lett., 50(3), 287 (2000)
[22] M. J. Berger, J. S. Coursey, and M. A. Zucker, “ESTAR, PSTAR, and ASTAR: computer programs for calculating stopping-power and range tables for electrons, protons, and helium ions (version 1.21)” (1999) URL: http://physics.nist.gov/Star
[23] S. M. Seltzer and M. J. Berger, “Bremsstrahlung spectra from electron interactions with screened atomic nuclei and orbital electrons,” Nuclear Instruments and Methods in Physics Research Section B: Beam Interactions with Materials and Atoms, 12(1), 95 (1985)
[24] S. Baur, H. Dembinski, T. Pierog, R. Ulrich, and K. Werner, “The ratio of electromagnetic to hadronic energy in high energy hadron collisions as a probe for collective effects, and implications for the muon production in cosmic ray air showers,” arXiv preprint arXiv:1902.09265 (2019)
[25] E. Eriksen and . Grn, “Electrodynamics of Hyperbolically Accelerated Charges: II. Does a Charged Particle with Hyperbolic Motion Radiate?” Annals of Physics, 286(2), 343 (2000)
[26] E. Eriksen and . Grn, “Electrodynamics of hyperbolically accelerated charges: IV. energy-momentum conservation of radiating charged particles” Annals of Physics, 297(2), 243 (2002)