Realisation of chiral symmetry in the domain model of QCD

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The domain model for the QCD vacuum has previously been developed and shown to exhibit confinement of quarks and strong correlation of the local chirality of quark modes and duality of the background domain-like gluon field. Quark fluctuations satisfy a chirality violating boundary conditions parametrized by a random chiral angle \( \alpha_j \) on the \( j \)-th domain. The free energy of an ensemble of \( N \to \infty \) domains depends on \( \{\alpha_j, j = 1 \ldots N\} \) through the logarithm of the quark determinant. Its parity odd part is given by the axial anomaly. The anomaly contribution to the free energy suppresses continuous axial \( U(1) \) degeneracy in the ground state, leaving only a residual axial \( Z(2) \) symmetry. This discrete symmetry and flavour \( SU(N_f)_L \times SU(N_f)_R \) chiral symmetry in turn are spontaneously broken with a quark condensate arising due to the asymmetry of the spectrum of Dirac operator. In order to illustrate the splitting between the \( \eta' \) from octet pseudoscalar mesons realised in the domain model, we estimate the masses of light pseudoscalar and vector mesons.

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I. INTRODUCTION

A mechanism which simultaneously provides for confinement of colour, spontaneously broken chiral symmetry and a resolution of the \( U_A(1) \) problem remains one of the open problems in QCD today. Partial solutions [1] based on specific semi-classical or topologically stable configurations can go someway to manifesting this triplet of phenomena but either founder on generating all three or in allowing for an effective model of the vacuum from which hadron spectroscopy can be derived. In any case, one expects that topological objects of various dimensions - point-like, string-like, and sheet-like - should contribute [2] complete with significant quantum fluctuations in a way that would be difficult to describe via an interacting microscopic model. In this paper, we continue the exploration of the “domain model” for the vacuum, originally proposed in [3], as a scenario for simultaneous appearance of all three phenomena: confinement, spontaneous chiral symmetry breaking via the appearance of a quark condensate and a continuous \( SU(N_f)_L \times SU(N_f)_R \) degeneracy of the vacuum for \( N_f \) massless quarks, but without a \( U_A(1) \) continuous degeneracy of ground states that would be indicative of an unwanted Goldstone boson. The purpose of studying a model of this type is to identify the typical features of relevant nonperturbative gluonic configurations. Such configurations would provide for as many gross features of nonperturbative QCD as possible. But this should preserve simultaneously the well-studied short distance regime and the model should be expressed in terms of quark-gluon degrees of freedom as well as in terms of colourless hadron bound states.

The model under consideration provides for confinement of both static (area law) and dynamical (propagators are entire functions of momentum) quarks [3]. It also displays specific chiral properties of quark eigenmodes. Namely, as will be discussed in more detail below, the spectrum of the Dirac operator is asymmetric with respect to \( \lambda \to -\lambda \) and zero quark modes are absent, but the local chirality of all nonzero modes at the centre of domains is correlated with the duality of the background field [4]. This has been observed on the lattice [5] and is usually considered as an indication of spontaneous breakdown of flavour chiral symmetry. The purpose of this article is to study the details of chiral symmetry realization in the domain model. The nonzero quark condensate and axial anomaly are generated as a result of spectral asymmetry and definite mean chirality of eigenmodes. We compute the quark condensate, study the degeneracies of the minima of the free energy of the domain ensemble with respect to chiral transformations and estimate the spectrum of pseudoscalar mesons.

The model is defined by a partition function describing an ensemble of hyperspherical domains, each characterised by a background covariantly constant self-dual or anti-self-dual gluon field of random orientation. Summing over all

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orientations and both self-dual and anti-self-dual fields guarantees Lorentz and CP invariance. Quarks are confined as demonstrated in the original work [3]. On the boundaries of each hypersphere, fermion fluctuations satisfy a chirality violating boundary condition

\[ i \gamma(x)e^{i\alpha\gamma_5}\psi(x) = \psi(x) \]  

which is 2\pi periodic in the chiral angle \(\alpha\). Here \(\eta_{\mu}\) is a unit radial vector at the boundary. Integrating over all such chiral angles guarantees chiral invariance of the ensemble. As a consequence of Eq. (1) the spectrum of eigenvalues \(\lambda\) of the Dirac operator in a single domain is asymmetric under \(\lambda \rightarrow -\lambda\). Such asymmetries have been studied in other contexts, for example by [6]. In the case of the domain model, the above boundary conditions are combined with the (anti-)self-dual gluon field which leads to a strong correlation between the local chirality of quark modes at the centres of domains with the duality of the background gluon field [4]. In this paper we study how these aspects contribute to quark condensate formation and the pattern of chiral symmetry breaking.

The vacua of the quantum problem associated with an ensemble of domains are the minima of the free energy determined from the partition function. The problem of the quark contributions to the free energy requires calculation of the determinant of the Dirac operator in the presence of chirality violating boundary conditions. For a choice of boundary condition with \(\alpha \rightarrow -i\theta - \pi/2\) this problem has been tackled in [7] without taking into account of the spectral asymmetry where the parity odd part of the logarithm of the determinant was identified as \(\ln \det(i\mathcal{D}) \sim 2q\theta\) with \(q\) the topological charge (not necessarily integer) of the underlying gluon field, namely the axial anomaly.

Our first goal in this paper is to address the analogous problem for the specific gluon field relevant to the domain model taking into account the asymmetry of the spectrum. For the parity odd part we obtain

\[ \ln \det(i\mathcal{D}) \sim 2iq(\alpha \mod \pi). \]  

This result is consistent with [7] up to a contribution coming from the asymmetry spectral function. However, we obtain an additional parity even part which also turns out to be \(\alpha\)-dependent. We consider this to be more an artifact of the incompleteness of our calculation than an established property of the determinant.

In the partition function all possible sets of chiral angles \(\{\alpha_1, \ldots, \alpha_N\}\) are summed ensuring the chiral invariance of the ensemble. Summation over all degrees of freedom besides chiral angles defines the free energy as a function of these chiral angles. In the limit \(N \rightarrow \infty\) the minima of the free energy density in \(\{\alpha_1, \ldots, \alpha_N\}\) determine the preferred chiral angles. More specifically, when self-dual and anti-self-dual configurations are summed, the anomaly Eq. (2) leads to a contribution to the free energy of the form \(-\ln \cos(2q\arctan(\tan \alpha))\) which vanishes when \(\alpha = 0, \pi\). The minima degenerate with respect to discrete \(Z_2\) chiral transformations. Each of these minima are characterised by a quark condensate of opposite sign, which arises due to the spectral asymmetry. An infinitesimally small quark mass removes the degeneracy between the two discrete minima, and a nonzero quark condensate is generated with value

\[ \langle \tilde{\psi}(x)\psi(x) \rangle = -(237.8\text{ MeV})^3 \]

with no additional modifications of the two model parameters after fixing in the gluonic sector of the theory. This gives a model with the chiral \(Z_2\) discrete subgroup of \(U_A(1)\) being spontaneously broken, and not the continuous \(U_A(1)\) itself. In the absence of the mass term the ensemble average of \(\tilde{\psi}\psi\) correctly vanishes. A similar argument based on minimisation of the free energy and thereby a relaxation of the effective \(\theta\)-parameter of QCD to zero is discussed in detail in [8] in the context of the strong CP-problem.

Moreover, the form of Eq.(2) means that the free energy does not depend on flavour nonsinglet chiral angles when more than one massless quark flavours are introduced. This allows for the correct degeneracy of vacua with respect to continuous \(SU(N_f)_L \times SU(N_f)_R\) chiral transformations. This vacuum structure implies the existence of Goldstone bosons in the flavour nonsinglet pseudoscalar channel but not in the singlet channel. To unveil more explicitly the singlet-octet splitting we analyse the structure of pseudoscalar correlation functions in the context of the domain model and estimate the masses of light pseudoscalar and vector mesons. The qualitative conclusion of this analysis is that the area law (confinement of static quarks) and the singlet-octet splitting in the model have the same origin: the finite range correlations of the background gluon field.

In the next section we briefly review the model, followed by a summary of the properties of the spectrum of the Dirac operator in the domain-like gluon field. We then discuss in detail the calculation of the logarithm of the quark determinant for one massless quark flavour, including the role of spectral asymmetry in domains in giving the anomaly for the parity odd part. This is followed by an analysis of the symmetries of the ground state of the domain ensemble and the computation of the condensate. In section V we generalise the result to \(N_f\) massless flavours in order to verify the spontaneous breaking of \(SU(N_f)_L \times SU(N_f)_R\) in the ensemble. The last section is devoted to calculation of meson masses. Details of calculations are relegated to the appendix.
II. THE DOMAIN MODEL

For motivation and a detailed description of the model we refer the reader to [3]. The essential definition of the model is given in terms of the following partition function for \( N \to \infty \) domains of radius \( R \)

\[
Z = N \lim_{V,N \to \infty} \prod_{i=1}^{N} \int d\sigma_i \int_{F^i_Q} \mathcal{D}\phi^{(i)}(x) \mathcal{D}\bar{\phi}^{(i)}(x) \int_{F^i_Q} \mathcal{D}Q^i \delta[D(\bar{\mathcal{E}}^{(i)})Q^{(i)}] \Delta_{FP}[\bar{\mathcal{E}}^{(i)},Q^{(i)}] e^{-S_{QCD}^{(i)}[Q^{(i)} + B^{(i)}(x),\bar{\phi}^{(i)},\phi^{(i)}]} \tag{3}
\]

where the functional spaces of integration \( F^i_Q \) and \( F^i_\psi \) are specified by the boundary conditions \((x-z_i)^2 = R^2\)

\[
\begin{align*}
\bar{n}_i Q^{(i)}(x) &= 0, \\
\bar{\psi}(i) e^{i\alpha_\gamma} \hat{\eta}_i(x) &= \psi^{(i)}(x), \\
\bar{\psi}(i) e^{i\alpha_\gamma} \hat{\eta}_i(x) &= -\bar{\psi}^{(i)}(x).
\end{align*}
\tag{4, 5, 6}
\]

Here \( \bar{n}_i = n_i^a t^a \) with the generators \( t^a \) of \( SU_c(3) \) in the adjoint representation and the \( \alpha_i \) are chiral angles associated with the boundary condition Eq.(5) with different values assigned to domains. We shall discuss this constraint in detail in later sections. The thermodynamic limit assumes \( V,N \to \infty \) but with the density \( v^{-1} = N/V \) taken fixed and finite. The partition function is formulated in a background field gauge with respect to the domain mean field, which is approximated inside and on the boundaries of the domains by a covariantly constant (anti-)self-dual gluon field with the field-strength tensor of the form

\[
B^a_{\mu\nu}(x) = \sum_{j=1}^{N} n_{(j)a} B^{(j)}_{\mu\nu} \theta(1 - (x - z_j)^2 / R^2), \\
B^{(j)}_{\mu\nu} B^{(j)}_{\mu\nu} = B^2 \delta_{\nu\rho}.
\]

Here \( z_j^\mu \) are the positions of the centres of domains in Euclidean space.

The measure of integration over parameters characterising domains is

\[
\int d\sigma_i \cdots = \frac{1}{48\pi^2} \int_{V} \frac{dz_i}{V} \int_{0}^{2\pi} d\alpha_i \int_{0}^{2\pi} d\varphi_i \int_{0}^{\pi} d\delta_i \int_{0}^{\pi} d\xi_i \int_{0}^{\pi} d\omega_i \sum_{l=0,1,2} \delta(\xi_l - (2l + 1)\pi/6) \int_{0}^{\pi} d\omega_i \sum_{k=0,1,2} \delta(\omega_i - \pi k) \ldots \tag{7}
\]

where \((\theta_i, \varphi_i)\) are the spherical angles of the chromomagnetic field, \( \omega_i \) is the angle between chromoelectric and chromomagnetic fields and \( \xi_i \) is an angle parametrising the colour orientation.

This partition function describes a statistical system of the domain-like structures, of density \( v^{-1} \) where the volume of a domain is \( v = \pi^2 R^4 / 2 \). Each domain is characterised by a set of internal parameters and whose internal dynamics are represented by fluctuation fields. Most of the symmetries of the QCD Lagrangian are respected, since the statistical ensemble is invariant under space-time and colour gauge transformations. For the same reason, if the quarks are massless then the chiral invariance is respected. The model involves only two free parameters: the mean field strength \( B \) and the mean domain radius \( R \). These dimensionful parameters break the scale invariance present originally in the QCD Lagrangian. In principle, they should be related to the trace anomaly of the energy-momentum tensor [9, 10] and, eventually, to the fundamental scale \( \Lambda_{QCD} \). Knowledge of the full quantum effective action of QCD would be required for establishing a relation of this kind.

A straightforward application of Eq. (3) to the vacuum expectation value of a product of \( n \) field strength tensors, each of the form

\[
B_{\mu\nu}^{a}(x) = \sum_{j=1}^{N} n_{(j)a} B^{(j)}_{\mu\nu} \theta(1 - (x - z_j)^2 / R^2),
\]

gives for the connected \( n \)-point correlation function

\[
\langle B_{\mu_1,\nu_1}^{a_1}(x_1) \cdots B_{\mu_n,\nu_n}^{a_n}(x_n) \rangle = \lim_{V,N \to \infty} \sum_{j} \int_{V} \frac{dz_j}{V} \int d\sigma_j n_{(j)a_1} \cdots n_{(j)a_n} B^{(j)}_{\mu_1,\nu_1} \cdots B^{(j)}_{\mu_n,\nu_n}
\]
\[
\times \theta(1 - (x_1 - z_j)^2 / R^2) \cdots \theta(1 - (x_n - z_j)^2 / R^2)
\]
\[
= B_{\mu_1,\nu_1}^{a_1,\ldots,a_n} \Xi_n(x_1, \ldots, x_n),
\]
where the tensor $t$ is given by the integral

$$
{t^{a_1 \ldots a_n}_{\mu_1 \nu_1 \ldots \mu_n \nu_n}} = \int d\sigma \, n^{(j)a_1 \ldots n^{(j)a_n} B^{(j)}_{\mu_1 \nu_1} \ldots B^{(j)}_{\mu_n \nu_n},}
$$

and can be calculated explicitly using the measure, Eq. (7). This tensor vanishes for odd $n$. In particular, the integral over spatial directions is defined by the generating formula

$$
\frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \, \sin \theta \, e^{J^{(j)}_{\mu\nu}} = \frac{\sin \sqrt{2B^2[J_{\mu\nu}J_{\mu\nu} \pm J_{\mu\nu}J_{\mu\nu}]} \sqrt{2B^2[J_{\mu\nu}J_{\mu\nu} \pm J_{\mu\nu}J_{\mu\nu}]}.
$$

The translation-invariant function

$$
\Xi_n(x_1, \ldots, x_n) = \frac{1}{v} \int d^4z \, \theta(1 - (x_1 - z)^2/R^2) \ldots \theta(1 - (x_n - z)^2/R^2)
$$

(8)
can be seen as the volume of the region of overlap of $n$ hyperspheres of radius $R$ and centres $(x_1, \ldots, x_n)$, normalised to the volume of a single hypersphere $v = \pi^2 R^4/2$.

It is obvious from this geometrical interpretation that $\Xi_n$ is a continuous function and vanishes if the distance between any two points $|x_i - x_j| \geq 2R$; correlations in the background field have finite range $2R$. The Fourier transform of $\Xi_n$ is then an entire analytical function and thus correlations do not have a particle interpretation. It should be stressed that the statistical ensemble of background fields is not Gaussian since all connected correlators are independent from each other and cannot be reduced to the two-point correlations.

Within this framework the gluon condensate to lowest order in fluctuations is $4B^2$, the absolute value of the topological charge per domain reads $q = B^2 R^4/16$ and the topological susceptibility turns out to be $\chi = B^3 R^4/128 \pi^2$. An area law is obtained for static quarks. Computation of the Wilson loop for a circular contour of a large radius $L \gg R$ gives a string tension $\sigma = BF(\pi BR^2)$ where $F$ is given for colour $SU(2)$ and $SU(3)$ in [3]. The area law emerges due to the finite range of background field correlators Eq. (48). On the other hand, the model cannot account for such a subtle feature as Casimir scaling: the adjoint Wilson loop naturally shows perimeter law, but trivially because of the abelian character of the domain mean field.

Estimations of the values of these quantities are known from lattice calculation or phenomenological approaches and can be used to fit $B$ and $R$. As described in [3] these parameters are fixed to be $\sqrt{B} = 947$ MeV, $R = (760 \text{MeV})^{-1} = 0.26 \text{fm}$ with the average absolute value of topological charge per domain turning out to be $q \approx 0.15$ and the density of domains $v^{-1} = 42 \text{fm}^{-3}$. The topological susceptibility is then $\chi = (197 \text{MeV})^4$, comparable to the Witten-Veneziano value [11]. This fixing of the parameters of the model remains unchanged in this investigation of the quark sector. The quark condensate at the origin of a domain where angular dependence drops out was estimated in paper [3] with a result of $-228 \text{MeV})^3$.

**III. DIRAC OPERATOR AND SPECTRUM**

The eigenvalue problem

$$
D\psi(x) = \lambda \psi(x),
$$

$$
i \gamma(x)e^{ix\gamma_5}\psi(x) = \psi(x), \quad x^2 = R^2
$$

was studied in [4]. Dirac matrices are in an anti-hermitean representation. For $\alpha$ assumed to be real a bi-orthogonal basis has to be constructed. Solutions can be labelled via the Casimirs and eigenvalues

$$
K_{1}^2 = K_{2}^2 \rightarrow k^2, \quad k = 0, 1, \ldots, \infty
$$

$$
K_{1,2} \rightarrow m_{1,2},
$$

$$
m_{1,2} = -k/2, -k/2 + 1, \ldots, k/2 - 1, k/2,
$$

corresponding to the angular momentum operators

$$
K_{1,2} = \frac{1}{2}(L \pm M)$$
with \( L \) the usual three-dimensional angular momentum operator and \( M \) the Euclidean version of the boost operator. The solutions for the self-dual background field are then
\[
\psi_{\kappa m_1}^- = i \not\! \! \! \! \partial \chi_{\kappa m_1}^- + \varphi_{\kappa m_1}^-,
\]
where \( \chi \) and \( \varphi \) must both have negative chirality in the self-dual field and \( \kappa \) is related to the polarisation of the field defined via the projector
\[
O_\kappa = N_+ \Sigma_\kappa + N_- \Sigma_{-\kappa}
\]
with
\[
N_{\pm} = \frac{1}{2} (1 \pm \hat{n}/|\hat{n}|), \quad \Sigma_{\pm} = \frac{1}{2} (1 \pm \Sigma_B/B)
\]
being respectively separate projectors for colour and spin polarizations. Significantly, the negative chirality for \( \chi \) and \( \varphi \) is the only choice for which the boundary condition Eq. (5) can be implemented for the self-dual background. The explicit form of the spinors \( \chi \) and \( \varphi \) can be found in [4], where it is demonstrated that the eigenspinor Eq. (9) has definite chirality at the centre of domain correlated with the duality of the gluon field. The boundary condition reduces to
\[
\chi = -e^{\mp i \alpha} \varphi, \quad \bar{\chi} = \bar{\varphi} e^{\mp i \alpha}, \quad x^2 = R^2,
\]
where upper (lower) signs correspond to \( \varphi \) and \( \chi \) with chirality \( \mp 1 \), which, using the solutions, amounts to equations for the two possible polarisations, for \( \Lambda^{++} \):
\[
e^{-i \alpha} M \left( k + 2 - \Lambda^2, k + 2, z_0 \right) - \frac{\sqrt{z_0}}{i \Lambda} \left[ M \left( k + 2 - \Lambda^2, k + 2, z_0 \right) - \frac{k + 2 - \Lambda^2}{k + 2} M \left( k + 3 - \Lambda^2, k + 3, z_0 \right) \right] = 0,
\]
and for \( \Lambda^{--} \):
\[
e^{-i \alpha} M \left( -\Lambda^2, k + 2, z_0 \right) + \frac{i \Lambda \sqrt{z_0}}{k + 2} M \left( 1 - \Lambda^2, k + 3, z_0 \right) = 0
\]
where \( z_0 = \hat{B} R^2 / 2 \) and \( \Lambda = \lambda / \sqrt{2B} \). For the present work Eqs. (12,13) are the starting point, from which we see by inspection that a discrete spectrum of complex eigenvalues emerges for which there is no symmetry of the form \( \lambda \to -\lambda \). For given chirality and polarisation and angular momentum \( k \), an infinite set of discrete \( \Lambda \) are obtained labelled by a “principal quantum number” \( n \).

### IV. Quark Determinant and Free Energy for a Single Domain

#### A. Massless case

We consider the one-loop contribution of the quarks to the free energy density \( F(B, R | \alpha) \) of a single (anti-)self-dual domain of volume \( v = \pi^2 R^4 / 2 \)
\[
\exp \left\{ - v F(B, R | \alpha) \right\} = \det_\alpha \left( \frac{i D}{i \partial} \right) = \prod_{\kappa, k, n, m_1} \frac{\lambda_{\kappa n}(B)}{\lambda_{\kappa n}(0)} = \exp \left\{ - \zeta'(s) \right\}_{s=0}.
\]
The normalization is chosen such that \( \lim_{B \to 0} F(B, R | \alpha) = 0 \). The free energy is then \( F = v^{-1} \zeta'(0) \).

In the zeta-regularised determinant an arbitrary scale \( \mu \) appears and it is convenient to work with scaled variables
\[
\beta = \sqrt{2B / \mu}, \quad \rho = \mu R, \quad \xi = \lambda(B) / \mu, \quad \xi_0 = \lambda(0) / \mu,
\]
and where the dimensionless quantity \( z = B R^2 / 2 = \beta^2 \rho^2 / 4 \) appears prominently. Moreover it is convenient to analytically continue \( \alpha \to -i \theta - \pi / 2 \) to guarantee a real spectrum of eigenvalues of the Dirac operator for all \( \theta \). We
shall regard the background domain field as being self-dual in the following. Then, Eqs.(12,13) can be recast into a form determining the rescaled eigenvalues $\xi$, namely

$$M(k + 2 - \xi^2/\beta^2, k + 2, z) + e^{\frac{\beta^2 \rho}{2\xi}} \left[ M(k + 2 - \xi^2/\beta^2, k + 2, z) - \frac{k + 2 - \xi^2/\beta^2}{k + 2} M(k + 3 - \xi^2/\beta^2, k + 3, z) \right] = 0 \quad (15)$$

for $\xi^+$ and

$$M(-\xi^2/\beta^2, k + 1, z) + e^{\frac{\rho}{2(k + 1)}} M(1 - \xi^2/\beta^2, k + 2, z) = 0 \quad (16)$$

for $\xi^-$. The zeta function $\zeta(s)$ breaks up into two parts [6], respectively symmetric (S) and antisymmetric (AS) with respect to $\xi \rightarrow -\xi$

$$\zeta(s) = \zeta_S(s) + \zeta_{AS}(s),$$

with

$$\zeta_S(s) = \frac{1}{2} \left( 1 + e^{\mp i\pi s} \right) \eta(s/2),$$

$$\zeta_{AS}(s) = \frac{1}{2} \left( 1 - e^{\mp i\pi s} \right) \eta(s). \quad (18)$$

The assumption behind these representations is that the spectrum of eigenvalues can be well ordered according to the magnitude of the eigenvalues, $|\xi|$, which is certainly the case with the solutions to Eqs.(15,16). However there is an ambiguity which can be fixed by specifying whether the smallest-in-magnitude eigenvalue is either positive or negative. This determines respectively the sign choice in Eqs.(17,18).

Thus the key quantities devolve into the zeta function for the squared Dirac operator and the asymmetry function respectively:

$$\zeta_{\rho^2}(s) = \sum_{k,n,\kappa} (k + 1) \left( \frac{1}{|\xi_{kn}(B)|^{2s}} - \frac{1}{|\xi_{kn}(0)|^{2s}} \right),$$

$$\eta(s) = \sum_{k,n,\kappa} (k + 1) \left( \frac{\text{sgn}(\xi_{kn}(B))}{|\xi_{kn}(B)|^s} - \frac{\text{sgn}(\xi_{kn}(0))}{|\xi_{kn}(0)|^s} \right). \quad (20)$$

It should be stressed that in the presence of bag-like boundary conditions $\zeta_S$ and $\zeta_{AS}$ do not have the meaning of parity conserving and parity violating terms since a parity transformation in terms of eigenvalues is given by $\xi(\vartheta) \rightarrow -\xi(-\vartheta)$ and both spectral functions contain parity conserving and violating terms. Thus the determinant for a given parameter $\vartheta$ is defined by

$$\zeta'(0) = \left( \frac{1}{2} \zeta_{\rho^2}'(0) \pm i \frac{\pi}{2} \eta_{\rho^2}'(0) \mp i \frac{\pi}{2} \eta(0) \right), \quad (21)$$

with the normalization chosen in Eq.(14) such that $\zeta'(0)$ vanishes as $B \rightarrow 0$.

Spectral sums over quantum labels $N$ can be computed using a representation of the sum as a contour integral of the logarithmic derivative of the function whose zeroes determine the spectrum,

$$\sum_N \frac{1}{\xi_N} = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi^s} \frac{d}{d\xi} \ln f(\xi), \quad (22)$$

see [12], where the zeroes of $f(\xi) = 0$ are $\xi = \xi_N$ and the contour is chosen such that all zeroes are enclosed. With real parameter $\vartheta$, the poles lie on the real axis, and there is no pole at the origin for any $\vartheta$. By deforming the contour and accounting for the vanishing of contributions to the integral at infinity, the expressions arising from Eq. (22) can be transformed into real integrals. The following representations for the two spectral functions are eventually obtained

$$\zeta_{\rho^2}(s) = \rho^2 s \sin(\pi s) \sum_{k=1}^{\infty} k^{1-2s} \int_0^\infty \frac{dt}{t^{2s}} \frac{d}{dt} \Psi(k, t | z, \vartheta),$$

$$\eta(s) = \rho^2 \cos(\pi s/2) \sum_{k=1}^{\infty} k^{-1-s} \int_0^\infty \frac{dt}{t^s} \frac{d}{dt} \Phi(k, t | z, \vartheta), \quad (23)$$
with $\Psi$ and $\Phi$ being the sum of contributions from the two polarisations, taking the form

$$\Psi(k,t|z,\vartheta) = \sum_{n=\pm\ast} \ln \left( \frac{A_n^2(k,t|z) + e^{2\vartheta} B_n^2(k,t|z)}{A^2(k,t) + e^{2\vartheta} B^2(k,t)} \right)$$

$$\Phi(k,t|z,\vartheta) = \sum_{n=\pm\ast} \ln \left( \frac{A_n(k,t|z) + ie^{\vartheta} B_n(k,t|z)}{A_n(k,t|z) - ie^{\vartheta} B_n(k,t|z)} \right)$$

and where

$$A_-(k,t|z) = M \left( \frac{k^2 t^2 \rho^2}{4z}, k + 1, z \right),$$

$$B_-(k,t|z) = \frac{kt\rho}{2(k+1)} M \left( \frac{k^2 t^2 \rho^2}{4z}, k + 2, z \right),$$

$$A_+(k,t|z) = M \left( \frac{k^2 t^2 \rho^2}{4z}, k + 1, -z \right),$$

$$B_+(k,t|z) = \frac{2z}{kt\rho} \left[ M \left( \frac{k^2 t^2 \rho^2}{4z}, k + 1, -z \right) - k + 1 + \frac{k^2 t^2 \rho^2}{4z} M \left( \frac{k^2 t^2 \rho^2}{4z}, k + 2, -z \right) \right],$$

$$A(k,t) = \frac{2^{k+1}}{k!} I_k(kt\rho),$$

$$B(k,t) = \frac{2^{k+1}}{k!} I_{k+1}(kt\rho).$$

The Bessel functions $I_k$ emerge from the limit $B \to 0$ with the normalization of the determinant as specified above. The next step is to expand the confluent hypergeometric functions in $1/k$, similar to the Debye expansion of Bessel functions, for example

$$M \left( \frac{k^2 t^2 \rho^2}{4z}, k + 1, z \right) = C(t\rho, k) \sum_{n=0}^{\infty} \frac{M_n(t\rho, z)}{k^n}.$$  \hspace{1cm} (25)

The form of the prefactors and the $M_n(x,z)$ functions are given in Appendix A for the various Kummer functions appearing in the $A_n$ and $B_n$.

Before proceeding with more detail, let us give an overview of the subsequent steps. The expansions Eq. (25) are first inserted in $\zeta(s)$ and $\eta(s)$. Then the integrals over $t$ can be evaluated term by term in the series in $1/k$. The order of summation over $k$ and $n$ are then exchanged. This is the most subtle step, which we discuss below. But it means that now the sums over $k$ can be read off in terms of the Riemann zeta function. The resulting $\zeta_{p2}(s)$ then has the structure

$$\zeta_{p2}(s) = sp^2 \frac{\sin(\pi s)}{i\pi} \sum_{n=0}^{\infty} \zeta_n(2s + n - 1) f(z^2|n) + \delta \zeta_{p2}(s),$$  \hspace{1cm} (26)

where the term $\delta \zeta$ denotes those potentially present contributions coming from interchange of the order of summations over $n$ and $k$. A similar structure appears for the asymmetry spectral function as well. Then we are interested in the decomposition of the resulting expressions around $s = 0$. In this limit the first term in Eq.(26) provides only a contribution from the $n = 2$ term and can be calculated with relative ease since only the lowest coefficients $M_1$ and $M_2$ in Eq. (25) contribute. However, as occurs in even simple problems [13], the second term in Eq.(26) is much more difficult to compute. To achieve this one needs to know coefficients $f(z^2|n)$ as a function of continuous variable $n$. Below we will elucidate on the contribution from the first term alone, bearing in mind the necessity of a complete analysis.

Now in more detail, using the expansions of the type Eq.(25) as given in Appendix A, we arrive at the following expansions of $\Psi$ and $\Phi$ in $1/k$:

$$\Psi(k,t|z,\vartheta) = z^2 \left[ \frac{\Psi_1(y)}{k} + \frac{\Psi_2(y|\vartheta)}{k^2} \right] + O(1/k^3),$$

$$\Phi(k,t|z,\vartheta) = z^2 \frac{\Phi_2(y|\vartheta)}{k^2} + O(1/k^3),$$
with the coefficient functions

\[ \Psi_1(y) = -\frac{2}{3} \frac{y(y - 1)(3y + 1)}{(y + 1)^2}, \]
\[ \Psi_2(y|\vartheta) = \frac{y^2(y - 1)}{(y^2 - 1)^2 e^{2\vartheta}} \times \left[ (y^2 - 1)(2y^3 + 6y^2 + 7y + 1) - 2(y + 1)(2y^4 + 5y^3 + 5y^2 - y + 1)e^{2\vartheta} + (y - 1)(2y^4 + 6y^3 + 5y^2 - 4y - 1)e^{4\vartheta} \right], \]
\[ \Phi_2(y|\vartheta) = -2ie^{\vartheta}y^2 \sqrt{1 - y^2} \left(1 + 2s\ln \rho\right) \left[-\frac{1}{4} - \ln 2 + \ln \left(1 + e^{2\vartheta}\right)\right] + O(s^2), \]

with \( y = \frac{1}{\sqrt{1 + t^2}} \). We thus get for \( s \ll 1 \)

\[ \zeta_{\rho^2}(s) = z^2(1 + 2s\ln \rho)s \left[ -\frac{1}{2}I_1 + \left(\frac{1}{2s} + \gamma\right)I_2 \right] \]
\[ I_1 = \int_0^\infty \frac{dt}{t^{2s}} d\Psi_1(y) = -s + O(s^2) \]
\[ I_2 = \int_0^\infty \frac{dt}{t^{2s}} d\Psi_2(y|\vartheta) = 2s \left[ -\frac{1}{4} - \ln 2 + \ln \left(1 + e^{2\vartheta}\right)\right] + O(s^2) \]

and

\[ \eta(s) = -\frac{z^2}{i\pi}(1 + s\ln \rho)\frac{1}{s}J_2 \]
\[ J_2 = 2ie^{\vartheta}\int_0^\infty \frac{dt}{t^{2s}} d\Phi_2(y|\vartheta) = -s\frac{i}{2} \left[ \pi + 2\text{Arctan}(\sinh(\vartheta)) \right] + O(s^2) \]

upon performing the \( t - \) (or \( y - \)) integrations and where we have used that

\[ \zeta_R(0) = -\frac{1}{2}, \quad \zeta_R(1 + 2s) = \frac{1}{2s} + \gamma + O(s). \]

The final results for the first term in Eq. (26) and its analogue in the asymmetry function \( \eta(s) \) are then summarised in the following equations:

\[ \zeta_{\rho^2}(0) = 0, \]
\[ \zeta_{\rho^2}(0) = z^2 \left[ -\frac{1}{4} - \ln 2 + \ln \left(1 + e^{2\vartheta}\right)\right] \]
\[ \eta(0) = \frac{z^2}{2} + \frac{z^2}{2\pi}\text{Arctan}(\sinh(\vartheta)) \]

and hence

\[ \zeta'(0) = \frac{z^2}{2} \left[ -\frac{1}{4} - \ln 2 + \pi \right] \]
\[ + \ln \left(1 + e^{2\vartheta}\right) \mp i\text{Arctan}(\sinh(\vartheta))\] \hspace{1cm} (27)
\[ + \delta\zeta'(0), \]

Now we can straightforwardly continue \( \vartheta \rightarrow i\alpha + i\pi/2 \). The final result for the free energy \( F \) is complex with the imaginary part of the form

\[ \Im F = \pm 2q\text{Arctan}(\tan(\alpha)) \] \hspace{1cm} (28)

where \( q \) is the absolute value of topological charge in a domain. This charge is not integer here in general but the anomalous term is \( \pi n \) periodic in \( \alpha \). This is the Abelian anomaly as observed within the context of bag-like boundary conditions by [7]. Its appearance here is in the spirit of the derivation by Fujikawa [14], where the phase appears as an extra contribution under a chiral transformation on the fermionic measure of integration.
However the analytic continuation of Eq.(27) also exposes an $\alpha$ dependent real part,

$$F(\alpha) = \frac{z^2}{2} \left[ -\frac{1}{4} + \ln(1 \pm \cos(\alpha)) \right]$$

where the signs are according to the prescription in Eqs.(17,18). There is no reason to expect that anything other than the anomaly should appear in the logarithm of the determinant. Thus the likelihood is that this additional real part contribution should be cancelled by the contributions $\delta \zeta$ in Eq.(26), which remains to be verified.

Below we assume that the anomaly Eq. (28) provides the entire result for the $\alpha$-dependent part of the free energy of massless fermions in a domain, which leads to intriguing consequences.

**B. Massive fermions**

Inclusion of an infinitesimally small fermion mass leads to a modification of the free energy by a term which is linear in mass to leading order, namely [6]

$$F = F_{m=0} + \frac{m}{\mu} \eta(1).$$

(29)

Using again the representation Eq.(23) but now in the vicinity of $s = 1$ one can find the following representation for the summed contributions of both polarisations in the self-dual domain to the asymmetry function

$$\eta(s) = i\mu R e^{i\pi s/2} \cos(\pi s/2) \sum_{k=1}^{\infty} k^{1-s} \frac{k}{k+1} \left[ 1 + M(1, k + 2, z) - \frac{z}{k+2} M(1, k + 3, -z) \right].$$

(30)

We next evaluate the asymptotic behaviour in $k$. A singular term as $s \rightarrow 1$ can be extracted, which turns out to be independent of the field (that is, $B$) and is canceled by the normalization. The final expression for the free energy density for a self-dual domain, including the contribution linear in mass, is

$$F^{(sd)} = 2i q \arctan(\tan(\alpha)) - e^{i\alpha} m \langle \bar{\psi} \psi \rangle,$$

where we have used the suggestive notation

$$\langle \bar{\psi} \psi \rangle = \frac{1}{\pi^2 R^2} \sum_{k=1, z=1, z_1, z_2}^{\infty} \frac{k}{k+1} \left[ M(1, k + 2, z) - \frac{z}{k+2} M(1, k + 3, z) - 1 \right]$$

(31)

coming from $\eta(1)$ with the sum over $z$ corresponding to a colour trace.

Let us summarise the results of this section as follows. The $\alpha$ dependent part of the free energy of a self-dual domain for massive quarks is complex with the following real and imaginary parts:

$$F = \Re F + i \Im F$$

$$\Re F = -m \cos \alpha \langle \bar{\psi} \psi \rangle$$

$$\Im F = 2q \arctan(\tan(\alpha)) - m \sin \alpha \langle \bar{\psi} \psi \rangle.$$  

(32)

(33)

The free energy of an anti-self-dual domain is obtained via complex conjugation.

**V. ENSEMBLE FREE ENERGY AND CHIRAL SYMMETRIES.**

**A. One flavour case: quark condensate and $U_A(1)$**

Under the assumption that only the anomalous term depends on the chiral angle, the part of the free energy density $\mathcal{F}$ relevant for the present consideration of an ensemble of $N \rightarrow \infty$ domains with both self-dual and anti-self-dual
configurations takes the form

\[ e^{-v N \mathcal{F}} = \mathcal{N} \prod_{j}^{N} \int_{0}^{2\pi} d\alpha_j \frac{1}{2} \left[ e^{i v \mathcal{F}(\alpha_j)} + e^{-i v \mathcal{F}(\alpha_j)} \right] \]

\[ = \mathcal{N} \prod_{j}^{N} \int_{0}^{2\pi} d\alpha_j e^{i v \ln(\cos(\mathcal{F}(\alpha_j)))} \]

\[ = \mathcal{N} \exp \left( N \max_{\alpha} \ln(\cos(\mathcal{F}(\alpha))) \right). \]

The maxima (minima of the free energy density) are achieved at \( \alpha_1 = \cdots = \alpha_N = \pi n. \)

In the absence of a quark mass, only the anomaly contribution in the imaginary part, \( \Im \mathcal{F}, \) of the free energy of a single domain appears under the logarithm of the cosine and defines the minima of the free energy density, \( \ln(\cos(v \mathcal{F}(\alpha))) = 0. \) Thus for massless quarks there is no continuous \( U_A(1) \) symmetry in the ground states, rather a discrete \( \mathbb{Z}_2 \) chiral symmetry. The anomaly plays a peculiar role here: selecting out those chiral angles which minimise the free energy so that the full \( U_A(1) \) group is no longer reflected in the vacuum degeneracy. It should be stressed here that this residual discrete degeneracy is sufficient to ensure a zero value for the quark condensate in the absence of a mass term or some other external chirality violating sources.

Now switching on the quark mass, we see this discrete symmetry spontaneously broken, and one of the two vacua selected in the infinite volume limit according to the sign of the mass. In this case (for these conventions of boundary condition and mass term), the minimum at \( \alpha = 0 \) is selected. The quark condensate can be now extracted from the free energy via

\[ \langle \bar{\psi}(x) \psi(x) \rangle = - \lim_{m \to 0} \lim_{N \to \infty} (v N)^{-1} \frac{d}{dm} e^{-v N \mathcal{F}(m)}. \]

Taking the thermodynamic limit \( N \to \infty \) and then \( m \to 0 \) gives a nonzero condensate

\[ \langle \bar{\psi}(x) \psi(x) \rangle = - \langle \bar{\psi} \psi \rangle >. \]

(34)

According to Eq. (31) the condensate is equal to

\[ \langle \bar{\psi}(x) \psi(x) \rangle = -(237.8 \text{ MeV})^3 \]

for the values of field strength \( B \) and domain radius \( R \) fixed earlier by consideration of the pure gluonic characteristics of the vacuum – string tension, topological susceptibility and gluon condensate. A nonzero condensate is generated without a continuous degeneracy of the ground states of the system.

Dependence of the condensate on the domain radius \( R \) is illustrated in Fig.1. As expected the condensate diverges with \( R \to \infty, \) since in this limit the number of low-lying strongly chiral modes is growing. We will return to the discussion of this feature again in the section on the meson spectrum.

B. Multiflavour case: spontaneous breakdown of \( SU_L(N_f) \times SU_R(N_f) \)

The question remains whether any continuous directions in the space of vacua are to be expected when the full flavour chiral symmetry is brought into play. For this we must generalise the analysis. We consider \( N_f \) massless quark flavours. Firstly, we observe that the fermion boundary condition in Eq.(5) explicitly breaks all chiral symmetries, flavour singlet and non-singlet (see also [7]). Thus the procedure we have used here of integrating over all \( \alpha \) does not suffice to restore the full chiral symmetry of the massless QCD action. Rather, the boundary condition must be generalised to include flavour non-singlet angles,

\[ \alpha \to \alpha + \beta^a T^a, \]

with \( T^a \) the \( N_f^2 - 1 \) generators of \( SU(N_f). \) Then integration over \( N_f^2 \) angles \( \alpha, \beta^a \in [0, 2\pi] \) must be performed for a fully chiral symmetric ensemble. The spectrum of the Dirac problem now proceeds quite analogously, except that the boundary condition mixes flavour components, thus an additional projection into flavour sectors is required in order to extract the eigenvalue equation analogous to Eq.(11).

For \( N_f = 2 \) the boundary condition can be chosen as

\[ i \not{n}(x) e^{i(\alpha + \beta^a T^a) \gamma_5} \bar{\psi}(x) = \psi(x) \]

(36)
where flavours now mix on the boundary. We must now solve the Dirac spectrum in the presence of this mixing. The spinors $\chi$ and $\varphi$ used in our solutions become now isospin doublets. We need to project the boundary condition onto separate equations for eigenvalues.

Things proceed much as before in the spin sector with our previous decompositions and projectors. The boundary condition will devolve to the structure

$$\chi[\lambda] = e^{i(\alpha + |\vec{\beta}|/2)} \varphi[\lambda]$$

where the $\gamma_5$ in the exponent is eliminated via the projection into chirality eigenspinors. The equation still mixes the flavour components and cannot be solved for an eigenvalue $\lambda$.

We introduce projectors

$$P_\pm(\beta) = \frac{1 \pm 2}{2}$$

which have the property

$$P_\pm(\beta)e^{-i\vec{\beta} \cdot \vec{\sigma}/2} = e^{\pm i|\vec{\beta}|/2} P_\pm(\beta).$$

Note here the appearance of the magnitude of the triplet $\vec{\beta}$ in the exponent. Projecting the spinors thereby we obtain separate equations for eigenvalues $\lambda[\beta, \pm]$ where we suppress all other quantum numbers previously dealt with,

$$\chi[\lambda[\beta, \pm]] = e^{i(\alpha \pm |\vec{\beta}|/2)} \varphi[\lambda[\beta, \pm]]$$

It is convenient to define two new angles,

$$\alpha_\pm = \alpha \pm |\vec{\beta}|/2$$

and in terms of these angles we can calculate the quark determinant separately for each flavour projection as before. The free energy for each flavour is now a function of four constrained angles

$$F_\pm = F_\pm(\alpha, \beta_+, \beta_0, \beta_-)$$

where we use this suggestive notation instead of $\beta_1, \beta_2, \beta_3$, and where $\pm$ denotes the isospin projection for the two quark flavours. For a given self-dual domain the total free energy will be a sum

$$F_T(\alpha, \vec{\beta}) = F(\alpha_+) + F(\alpha_-)$$
where $F$ is the result from the one-flavour case. Under the assumption that the anomaly provides the entire chiral angular dependence of the determinant, the nonsinglet angles $\beta^a$ drop out due to the form $2q \arctan(\tan(\alpha)) = 2q(\alpha + 2n\pi)$. This expresses the known result that the anomaly only depends on the flavour singlet directions or is Abelian. Thus for an ensemble of domains the free energy is identical to that for one massless flavour, namely it depends only on the Abelian angle $\alpha$. Thus for $N_f = 2$, the $U_A(1)$ direction remains fixed by energy minimisation while the $SU(2)_L \times SU(2)_R$ directions represent degeneracies in the space of ground states in the thermodynamic limit.

For $N_f = 3$ the boundary condition will now involve the flavour mixing matrix $e^{i\beta^a \lambda^a / 2}$ in terms of the Gell-Mann matrices. It is now harder to explicitly diagonalise and project into flavour sectors. Nonetheless the form of the result is clear. The Cartan subalgebra consists of the diagonal generators $\lambda^3$ and $\lambda^8$.

Thus the result of a diagonalisation of the argument of the exponential will have the form,

$$e^{i\beta^a \lambda^a / 2} \rightarrow \text{diag}\left(e^{b^3 + \frac{1}{\sqrt{3}} b^8}, e^{-b^3 + \frac{1}{\sqrt{3}} b^8}, e^{-\frac{2}{\sqrt{3}} b^8}\right)$$

So projection of the combined $U(1) \times SU(3)$ chiral boundary condition will lead to three sets of equations with the eigenvalues depending on the combinations of angles

$$B_1 = \alpha + b^3 + \frac{1}{\sqrt{3}} b^8,$$
$$B_2 = \alpha - b^3 + \frac{1}{\sqrt{3}} b^8,$$
$$B_3 = \alpha - \frac{2}{\sqrt{3}} b^8$$

where evidently

$$\sum_i B_i = 3\alpha$$

reflecting ultimately the tracelessness of the generators. The free energy for a given domain will be the sum of three terms

$$F_T = \sum_{i=1}^3 F(B_i)$$

with $F$ being the same expression evaluated for the one-flavour case. Once again, the assumption of the anomaly means cancellation of the $\beta^a$ dependent functions $b^A$ leaving only the $\alpha$ dependence in the free energy. Once again energy minimisation constraints the $U_A(1)$ direction, leaving the $SU(3)_L \times SU(3)_R$ directions unconstrained. Thus for $N_f = 3$ one expects eight, not nine, continuous directions in the space of vacua.

The argument for arbitrary $N_f$ is now evident, the key ingredients being the tracelessness of $SU(N_f)$ generators and the specific form of the anomaly for the quark determinant.

VI. ESTIMATION OF MESON MASSES.

The consequences of the above realisation of chiral symmetries should be seen in the meson spectrum providing the splitting between pseudoscalar and vector mesons and between the pseudoscalar octet and $\eta'$. The source for the difference between the masses of the octet states and $\eta'$ can be recognised in the drastic difference between correlators of the flavour octet $J_P^a(x)$ and singlet $J_P(x)$ pseudoscalar quark currents as they appear in the domain model,

$$\langle J_P^a(x) J_P^b(y) \rangle = \frac{\langle \langle J_P^a(x) J_P^b(y) \rangle \rangle}{\langle \langle J_P(x) J_P(y) \rangle \rangle}$$
$$\langle J_P(x) J_P(y) \rangle = \frac{\langle \langle J_P(x) J_P(y) \rangle \rangle - \langle \langle J_P(x) \rangle \rangle \langle \langle J_P(y) \rangle \rangle}{\langle \langle J_P(x) J_P(y) \rangle \rangle}.
(37)$$
Here double brackets denote integration over quantum fluctuation fields and the overline means integration over all configurations in the domain ensemble. The second term in the RHS of the flavour singlet correlator contains two quark loops and is subleading in $1/N_c$ compared with the first one-loop contribution. The second term is entirely determined by the correlation function of the background gluon field $B$ in the ensemble (8) and is proportional to the quark condensate squared. The analogous two-loop term in the flavour nonsinglet correlator is equal to zero due to the trace over flavour indices. It should be added that the pseudoscalar condensate, $\langle \bar{\psi} \gamma_5 \psi \rangle$, naturally vanishes since parity is not broken in the ensemble of domains. Thus massless modes can be expected in the nonsinglet channel, but not in the flavour singlet due to the additional term in the correlator. This general structure of correlators is exactly the same as in the instanton liquid model [15] and manifests the mechanism for eta-prime mass generation proposed by Witten in [11] and appreciated in chiral perturbation theory by [16]. The correspondence between domain model correlators to original QCD diagrams is illustrated in Fig. 2. The grey background denotes averaging over the domain ensemble and is intended to represent the nonperturbative intermediate range part of the gluon exchange in the original QCD diagrams (a’) and (b’). Diagrams (a) and (b) correspond to the first and second terms in the RHS of Eq. (37). Averaging in diagram (b) relates to both quark loops, which is the domain model representation of the exchange between quark loops in (b’) by infinitely many gluons in the original QCD representation. It should be noted that diagrams (a) and (b) in Fig. 2 are the lowest order contributions in the fluctuation gluon fields $Q$ which are treated as perturbations of the background. Higher orders include exchange by gluonic fluctuations in the presence of the domain mean field.

![Diagram](image)

**FIG. 2**: Diagrammatic representation of two types of contributions to the flavour singlet correlator: (a) and (b) as in the domain model; (a’) and (b’) as in full QCD. The grey background in (a,b) represents the nonperturbative part of the gluon exchange indicated in diagrams (a’,b’).

Direct calculation of meson masses within the domain model is not available yet. However we can estimate the effect of contributions subleading in $1/N_c$ in the flavour singlet pseudoscalar correlator within a calculational scheme which is quite close to the domain model in this respect. This is the main purpose of this section. Simultaneously we will schematically expose the form of the effective action for collective colourless meson-like modes as is expected to emerge in the domain model and estimate the value of the typical dimensionless parameter $BR^2$ but now from the meson spectrum. The model which will be used for this purpose is based on the bosonisation of a one gluon exchange interaction between quark currents and in which both quark and gluon propagators are exact solutions in the presence of a background (anti-)self-dual homogeneous gluon field as in the papers [17–19], but here with additional correlations of the type (b) in Fig. 2. In [17–19], as well as the formulation of the model based on a homogeneous gluon field, applications to the calculation of the spectrum of light mesons, their orbital excitations, heavy-light mesons, heavy quarkonia, decay constants and form-factors are given. In the presence of a homogeneous (anti-)self-dual background field the quark condensate emerges due to chiral zero
modes, but, since there is a continuum of such modes in the homogeneous field, “an overkill” occurs [20, 21]:

\[ m \langle \bar{\psi} \psi \rangle \propto -B^2, \quad m \to 0, \]

and the limit \( m \to 0 \) cannot be defined properly. However, due to the same zero modes the momentum representation correlators of scalar, pseudoscalar, vector and axial vector currents have the following limits for \( m \to 0 \)

\[
\tilde{\Pi}_{P/S} \to \mp \frac{B^2}{m^2} F_1(p^2/B),
\]

\[
\tilde{\Pi}_{V/A} \to \mp BF_2(p^2/B).
\]

(38)

Such a qualitatively different behaviour of correlators becomes manifest already at \( m^2/B \sim O(1) \) and leads to a strong splitting of the masses of the corresponding mesons ensuring light pions and heavy \( \rho \)-mesons in particular [18] (see also [15]). The fitted values of quark masses in the model correspond to the constituent masses.

The definition of the chiral limit in that model thus cannot be given in terms of quark masses. However, as we discuss below it is nonetheless possible to define a regime when the calculated pion mass vanishes. The curious side of this approach to the description of chiral properties of the light mesons is that the zero-mode mechanism of condensate formation alone is exploited without any use made of a Nambu-Jona-Lasinio or, more generally, Dyson-Schwinger equation type mechanism [22]. The inclusion in this model of the additional correlations (b,b') of Fig. 2 for the flavour singlet channel enables this approach to also work phenomenologically for \( \eta - \eta' \) masses with the chiral limit so-defined.

In the domain model studied in the first part of this paper the quark condensate appears also due to specific chiral properties of (now nonzero) Dirac modes in gluon (domain-like) background fields and not due to a four-fermion interaction. The chiral properties of these low-lying modes resemble the properties of zero modes in the homogeneous-field model. The condensate diverges as \( R \to \infty \) as shown in Fig. 1 much like the condensate in the homogeneous field model diverges as \( 1/m \). We expect that this dependence on \( R \) in the domain model should have a similar effect on meson correlators and masses as the \( 1/m \) singularity does in the homogeneous field case. The characteristic scale of higher modes is defined by the field strength \( B \) rather than the domain radius for \( BR^2 \gg 1 \). Thus one would expect that the heuristic consideration below should be more consistent with the domain model picture if \( BR^2 \gg 1 \).

The difference is that in the domain model flavour chiral symmetry is broken spontaneously, which is not the case in the homogeneous field model. So we should expect a massless pion for massless quarks: the current mass and the chiral limit are well-defined, as has been discussed in previous sections. However a detailed description of the emergence of Goldstone modes is a question of detailed study of correlators and the bound state problem in the domain model which is still beyond our efforts.

To conclude these preliminary comments, the direct purpose of the following calculations is to demonstrate the \((\pi, K) - \eta - \eta'\) splitting due to correlations Fig.2b. An auxiliary purpose, justified by the similarity of chiral properties of zero modes in the homogeneous model and nonzero eigenmodes in the domain model, is a demonstration that the pseudoscalar singlet-octet mass splitting occurs simultaneously with a correct description of pseudoscalar-vector meson splitting.

FIG. 3: Diagrammatic representation of the three-point vertex function \( \Gamma^{(3)}_{Q_1Q_2Q_3} \). Diagram (a) corresponds to the first term in the RHS of Eq. (44), (b) and (c) to the second and third terms respectively. Solid lines are quark propagators \( S \) in the background field, vertices correspond to nonlocal quark-meson vertices \( V_{Q_i} \) and the dashed lines represent meson fields \( \Phi_{Q_i} \).
A. Effective action for colourless composite fields.

We shall estimate meson masses within the model described by the following partition function

$$Z = N \lim_{V \to \infty} \int D\Phi_Q \exp \left\{ -B \frac{h_Q^2}{2 \gamma^2 C_Q} \int dx \Phi_Q^2(x) - \sum_k \frac{1}{k} W_k[\Phi] \right\}, \quad (39)$$

$$1 = \frac{g^2 C_Q}{B} \Gamma_Q^{(2)}(-M_Q^2|B), \quad (40)$$

$$h_Q^{-2} = \frac{d}{dp^2} \Gamma_Q^{(2)}(p^2)|_{p^2 = -M_Q^2}. \quad (41)$$

The effective action in Eq. (39) is expressed in terms of colourless composite meson fields $\Phi_Q(x)$ with the mass $M_Q$ defined by Eq. (40), where the condensed index $Q$ denotes isotopic and space-time indices as well as all possible mesonic quantum numbers (iso-spin, spin-parity in the ground state, total momentum, radial quantum number), and $k$-point nonlocal vertices $\Gamma_Q^{(k)}$.

$$W_k[\Phi] = \sum_{Q_1,\ldots,Q_k} h_{Q_1} \cdots h_{Q_k} \int dx_1 \cdots \int dx_k \Phi_{Q_1}(x_1) \cdots \Phi_{Q_k}(x_k) \Gamma_Q^{(k)}_{Q_1\ldots Q_k}(x_1,\ldots,x_k|B), \quad (42)$$

$$\Gamma_Q^{(1)} = \frac{G_Q^{(1)}}{h_Q}, \quad (43)$$

$$\Gamma_Q^{(2)}_{Q_1 Q_2} = \frac{G_Q^{(2)}_{Q_1 Q_2}(x_1,x_2) - \Xi_2(x_1-x_2)G_Q^{(1)}_{Q_1}G_Q^{(1)}_{Q_2}}{2\Xi_2(x_1-x_2)G_Q^{(1)}_{Q_1}G_Q^{(1)}_{Q_2}}, \quad (44)$$

$$\Gamma_Q^{(3)}_{Q_1 Q_2 Q_3} = \frac{G_Q^{(3)}_{Q_1 Q_2 Q_3}(x_1,x_2,x_3) - 3\Xi_2(x_1-x_2)G_Q^{(2)}_{Q_1 Q_2}(x_1,x_2)G_Q^{(1)}_{Q_3}(x_3) + \frac{1}{2}\Xi_3(x_1,x_2,x_3)G_Q^{(1)}_{Q_1}(x_1)G_Q^{(1)}_{Q_2}(x_2)G_Q^{(1)}_{Q_3}(x_3)}{6\Xi_2(x_1-x_2)G_Q^{(1)}_{Q_1}G_Q^{(1)}_{Q_2}G_Q^{(1)}_{Q_3}}, \quad (45)$$

and analogous expressions for the higher vertices. Defining the meson-quark coupling constants $h_Q$ by Eq. (41) provides for the correct residue of the meson propagators at the poles and is known as a compositeness condition [23].

The vertices $\Gamma_Q^{(k)}$ are expressed via quark loops $G_Q^{(n)}$ with $n$ quark-meson vertices

$$G_Q^{(k)}_{Q_1\ldots Q_k}(x_1,\ldots,x_k) = \int d\sigma_j \text{Tr} V_{Q_1}(x_1|B^{(j)})S(x_1,x_2|B^{(j)}) \cdots V_{Q_k}(x_k|B^{(j)})S(x_k,x_1|B^{(j)})$$

$$G_Q^{(4)}_{Q_1\ldots Q_4}(x_1,\ldots,x_4)G_Q^{(k)}_{Q_1\ldots Q_k}(x_{i+1},\ldots,x_k) = \int d\sigma_j$$

$$\times \text{Tr} \left\{ V_{Q_1}(x_1|B^{(j)})S(x_1,x_2|B^{(j)}) \cdots V_{Q_k}(x_k|B^{(j)})S(x_k,x_1|B^{(j)}) \right\}$$

$$\times \text{Tr} \left\{ V_{Q_{i+1}}(x_{i+1}|B^{(j)})S(x_{i+1},x_{i+2}|B^{(j)}) \cdots V_{Q_k}(x_k|B^{(j)})S(x_k,x_{i+1}|B^{(j)}) \right\}, \quad (46)$$

where bar denotes integration over all configurations of the background field with measure $d\sigma_j$. The quark propagator

$$S(x,y) = \exp \left( -\frac{i}{2} x_{\mu} \tilde{B}_{\mu\nu} y_{\nu} \right) H(x-y),$$

$$\tilde{B}(p) = \frac{1}{2\beta\Lambda^2} \int_0^1 ds e^{-p^2/2\alpha \lambda^2} \left( \frac{1-s}{1+s} \right) \frac{m^2/4\alpha \lambda^2}{1-s}$$

$$\times \left[ p_\alpha \gamma_\alpha \pm is\gamma_5 \gamma_\alpha f_{\alpha\beta} p_\beta + m \left( P_\pm + P_+ \frac{1+s^2}{1-s^2} - \frac{i}{2} \gamma_\alpha f_{\alpha\beta} \gamma_\beta \frac{s}{1-s^2} \right) \right].$$
is the exact Dirac propagator in the presence of the (anti-)self-dual homogeneous field

$$\hat{B}_\mu(x) = \frac{1}{2} \hat{n}B_{\mu\nu}x_\nu, \quad \hat{B}_{\mu\nu}\hat{B}_{\rho\sigma} = 4v^2\Lambda^4\delta_{\nu\rho},$$

$$f_{\alpha\beta} = \frac{\hat{n}}{v\Lambda^2}B_{\mu\nu}, \quad v = \text{diag}(1/6, 1/6, 1/3),$$

where we introduced a scale $\Lambda$ related to the field strength $B$ as

$$\Lambda^2 = \frac{\sqrt{3}}{2}B.$$

For arbitrary orbital momentum $l$ and radial quantum number $n$ the vertex function (more precisely – vertex differential operator) is factorised into a radial part

$$F_{nl}(s) = \int_0^1 dt t^{l+n}e^{st}, \quad s = \frac{s^2}{\Lambda^2},$$

$$\hat{\nabla}_{f'} = \xi_f \hat{\nabla} - \xi_{f'} \hat{\nabla}, \quad \xi_f = m_f/(m_f + m_{f'}),$$

$$\hat{\nabla}_\mu = \partial_\mu + iB_\mu, \quad \hat{\nabla}_\mu = \partial_\mu - iB_\mu,$$

and angular part. In our particular calculation below we will not deal with the excited states. For the explicit form of the angular part of the vertex the reader is referred to [18], where technical details of the derivation of meson-quark vertices $V_{Q_1}(x_k|B^{(ij)})$ and discussion of approximations and assumptions behind the derivation of the effective action for composite fields can be found. A simplified scalar field model allowing exact implementation of the method has been considered, and a variational procedure for approximate solution of realistic problems has been formulated in [24].

The relation of this method to the Bethe-Salpeter equation and the nonrelativistic limit are analysed in [24, 25].

Thus the $n$-point vertex $\Gamma^{(n)}$ includes contributions of two types – the usual one-loop contributions (averaged over the background field) as diagram (a) in Fig. 3 and products of two or more such one-loop terms simultaneously averaged over the background field and multiplied by the corresponding correlators $\Xi$ of the background field, for example as in diagrams (b) and (c) of Fig. 3 for the three-point vertex. In the absence of a background field these additional terms would be just disconnected diagrams and would not appear in the effective action. For the purely constant field $B$ they would correspond to infinite length correlations breaking the cluster decomposition property in the effective action ($\Xi_n \equiv 1$). The idea of domains is implemented in the above expressions by means of finite length correlations of the background field which ensures cluster decomposition.

The $n$-point correlation functions $\Xi_n$ of the background field are defined in Eq. (8). In particular the two-point correlator which will be used below can be written in explicit form

$$\Xi_2(x - y) = \frac{N}{V} \int d\theta(x - z)\theta(y - z) = \frac{2}{3\pi} \phi\left(\frac{(x - y)^2}{4R^2}\right),$$

$$\phi(\rho^2) = \left[\frac{3\pi}{2} - 3\arcsin(\rho) - 3\rho\sqrt{1 - \rho^2} - 2\rho(1 - \rho^2)^{3/2}\right].$$

Geometrically, the correlation function $\Xi_2(x - y)$ is equal to the volume of overlap between spherically symmetric four-dimensional regions determined by the characteristic function $\theta$ and central points $x$, $y$, normalized to the volume of a single such region.

Now all the elements of the effective action Eq. (39) are fixed: nonlocal meson-quark vertices $V_{Q_1}(x_k|B^{(ij)})$ and quark propagators $S(x_k, x_j|B^{(ij)})$ and background field correlators are given in explicit analytical form.

It should be noted here that meson-quark vertices are determined by the gluon propagator in the presence of the background field. In momentum representation both quark propagator, meson-quark vertices and correlators of the background field are entire functions in the complex momentum plane, which means that quarks and gluons are absent in the model as asymptotic states, which is treated as colour confinement. In the momentum representation the vertices $\Gamma^{(n)}$ have no imaginary parts and the probability of a meson to decay into a quark-antiquark pair is zero. As is shown in [17, 18] within the model formulated above and clarified in a simplified exactly solvable model in [24] entire propagators and vertices lead to a Regge spectrum of excited bound states represented by composite fields $\Phi_Q$.

The only free parameters of the model are the same as in the domain model discussed in the first part of this paper: the background field strength $B$, mean domain radius $R$, strong coupling constant $g$ and quark masses $m_f$. The factor...
$C_{\phi}$ in front of the gauge coupling constant $g$ in Eqs.(39) and (40) is known explicitly

$$C_{Jnl} = C_J \frac{l + 1}{2^n l! (l + n)!}, \quad C_{S/P} = \frac{1}{9}, \quad C_{S/P} = \frac{1}{18}.$$  

The only place where the gauge coupling constant $g$ enters this scheme is the equation for meson masses Eq. (40); the rest of the effective action contains the meson-quark coupling constants calculated by means of Eq. (41).

In the next subsection we present results for the masses of pseudoscalar and vector nonets coming straightforwardly from this formalism with a special emphasis on the $\eta$ and $\eta'$.

B. The masses and decay constants of $\eta$ and $\eta'$.

With this representation, the calculation of masses of light mesons and weak decay constants, as performed in [18], is modified only with respect to the $\eta$ and $\eta'$ masses. Equations for these masses contain terms with additional correlation $\Xi_2$, and this splits them from the remaining light pseudoscalar mesons.

As follows from Eqs. (40), the second term of (42) and (46), the simplest quark loops relevant to the calculation of masses are the constants

$$G^{(1)}_{aP} = -\text{Tr} i \gamma_5 F_{00} (x|B^{(j)}) S(x, x|B^{(j)}),$$  \hspace{1cm} (49)$$
corresponding to the quark loops in Fig.2b. Here $a$ is the flavour index ($a=0, \ldots, 8$). In the iso-singlet case ($a=0$) this quantity is nonzero. If the $s$-quark mass is not degenerate with the masses of $u$- and $d$-quarks then $G^{(1)}_{8P}$ is also nonzero. For $a=1, \ldots, 7$ the constants are identically equal to zero.

Calculation of $G^{(1)}_{aP}$ is quite simple. The result of the action of the vertex operator $F_{00} (x|B)$ onto the quark propagator can be found as follows (in our case $\xi = 1/2$ in Eq. (47)):

$$F_{00} (x|B) S(x, x|B) = \int_0^1 dt \exp \left\{ \frac{t}{4\Lambda^2} \left( \nabla_x - \nabla_y \right)^2 \right\} S(y, x|B)|_{x=y}$$

$$= \int_0^1 dt \int \frac{d^4p}{(2\pi)^4} \tilde{H}(p) e^{-tp^2/\Lambda^2}. \hspace{1cm} (50)$$

Furthermore,

$$\text{Tr} i \gamma_5 \tilde{H}(p) = \mp \frac{2im}{v\Lambda^2} \int_0^1 ds e^{-sp^2/2v\Lambda^2} \left( \frac{1-s}{1+s} \right)^{m^2/4v\Lambda^2} s^2 \frac{1-s^2}{1-s^2}. \hspace{1cm} (51)$$

Here the sign “$\mp$” relates to self- and anti-self-dual configurations of the vacuum field.

Substitution of Eqs. (51) and (50) in Eq. (49) and integration over the loop momentum $p$ and the parameter $t$ leads to the following integral representation

$$G^{(1)}_{aP} = \pm i \frac{\Lambda^3}{2\pi^2} \sum_f \lambda_j^a R_f, \quad R_f = \sum_v \frac{m_f}{\Lambda} \int_0^1 ds s \left( \frac{1-s}{1+s} \right)^{m^2/4v\Lambda^2}, \hspace{1cm} (52)$$

where summation over $f$ and $v$ corresponds to traces in the flavour and colour spaces. As indicated in Eq. (47), $v$ is a diagonal matrix relating to the direction of the vacuum field in colour space.

Eq. (52) results in the following momentum representation of the two-point correlation function, Eq. (42)

$$\Lambda^{-2} \Pi^{(2)}_{aP,bP}(p^2) = \Pi_{ab}(p^2) = \Pi_{ab}(p^2) - \delta \Pi_{ab}(p^2),$$
where
\[ \delta \Pi_{ab}(p^2) = -\frac{8}{3\pi^2}(AR)^4 T^{ab} F(p^2), \quad T^{ab} = \sum_{f'f} \chi^a_{f'} \chi^b_f R_f R_{f'}, \]
\[ F(p^2) = \int_0^1 dt \sqrt{1-t^2} \int_0^1 ds s \cos \left( \sqrt{4p^2R^2}t^2s \right) \left[ \frac{3\pi}{2} - 3 \arcsin \sqrt{s - (5 - 2s)\sqrt{s(1-s)}} \right]. \]
\[ T^{00} = \frac{1}{3}(2R_u + R_s)^2, \quad T^{ss} = \frac{2}{3}(R_u - R_s)^2, \quad T^{0s} = \frac{\sqrt{2}}{3}(2R_u + R_s)(R_u - R_s), \]
\[ \Pi_{00} = \frac{1}{3}(2\Pi_u + \Pi_s), \quad \Pi_{ss} = \frac{1}{3}(\Pi_u + 2\Pi_s), \quad \Pi_{0s} = \frac{\sqrt{2}}{3}(\Pi_u - \Pi_s). \]  

The function \( \Pi_f(p^2) \) has been calculated in Ref. [18],
\[ \Pi_f(p^2) = -\frac{1}{4\pi^2} \text{Tr}_v \int_0^1 dt_1 \int_0^1 dt_2 \int_0^1 ds_1 \int_0^1 ds_2 \left( \frac{1 - s_1}{1 + s_1} \right)^{m^2_{\text{eff}}/(4\pi^2)} \left( \frac{1 - s_2}{1 + s_2} \right)^{m^2_{\text{eff}}/(4\pi^2)} \]
\[ \times \left[ -\frac{p^2}{\Lambda^2} F_1(t_1, t_2, s_1, s_2) + m^2_{\text{eff}} \frac{F_2(s_1, s_2)}{\Lambda^2 (1 - s_1^2)(1 - s_2^2)} \right] \exp \left\{ -\frac{p^2}{2e\Lambda^2} \varphi(t_1, t_2, s_1, s_2) \right\}, \]
\[ \varphi = \varphi_1(t_1, t_2, s_1, s_2), \quad \varphi_1 = \frac{1}{2} \epsilon(t_1 + t_2)[s_1 + s_2] + s_1 s_2, \]
\[ \varphi_2 = 2 \epsilon(t_1 + t_2)(1 + s_1 s_2) + (1 + 4\epsilon^2 t_1 t_2)(s_1 + s_2), \]
\[ F_1 = (1 + s_1 s_2)[(s_1 + \epsilon(t_1 + t_2))(s_2 + \epsilon(t_1 + t_2)) + \epsilon^2(t_1 - t_2)^2 s_1 s_2], \]
\[ F_2 = (1 + s_1 s_2)^2, \quad F_3 = 2(1 + s_1 s_2). \]

The masses \( \eta \) and \( \eta' \) can be calculated by means of Eqs. (40), which take the form
\[ 1 + \frac{\eta^2}{9} \Pi_{\eta}(-M^2_{\eta}) = 0, \quad 1 + \frac{\eta'^2}{9} \Pi_{\eta'}(-M^2_{\eta'}) = 0, \]
\[ \Pi_{\eta} = \frac{1}{2} \left[ \Pi_{00} + \Pi_{ss} - \sqrt{(\Pi_{00} - \Pi_{ss})^2 + 4\Pi_{0s}^2} \right], \]
\[ \Pi_{\eta'} = \frac{1}{2} \left[ \Pi_{00} + \Pi_{ss} + \sqrt{(\Pi_{00} - \Pi_{ss})^2 + 4\Pi_{0s}^2} \right]. \]

where \( \Pi = \Pi - \delta \Pi \) includes both diagrams in Fig.2. These equations can be written more transparently if we introduce the mixing angle
\[ \eta = \eta_8 \cos \theta + \eta_0 \sin \theta, \]
\[ \eta' = \eta_0 \cos \theta - \eta_8 \sin \theta, \]  

(54)
where the angle \( \theta \) is a function of momentum,
\[ \tan 2\theta(p^2) = 2\Pi_{0s}(p^2)/\left( \tilde{\Pi}_{ss}(p^2) - \tilde{\Pi}_{00}(p^2) \right). \]  

(55)
The polarization functions of \( \eta \) and \( \eta' \) and their derivatives with respect to \( p^2 \) take the form
\[ \Pi_{\eta} = \tilde{\Pi}_{ss} \cos^2 \theta + \tilde{\Pi}_{00} \sin^2 \theta + \tilde{\Pi}_{0s} \sin 2\theta, \]
\[ \Pi_{\eta'} = \tilde{\Pi}_{00} \cos^2 \theta + \tilde{\Pi}_{ss} \sin^2 \theta - \tilde{\Pi}_{0s} \sin 2\theta, \]
\[ \Pi'_{\eta} = \tilde{\Pi}_{ss} \cos^2 \theta + \tilde{\Pi}_{00} \sin^2 \theta + \tilde{\Pi}_{0s} \sin 2\theta, \]
\[ \Pi'_{\eta'} = \tilde{\Pi}_{00} \cos^2 \theta + \tilde{\Pi}_{ss} \sin^2 \theta - \tilde{\Pi}_{0s} \sin 2\theta, \]  

(56)
where for computing derivatives Eq. (55) has been used.
With the values of the parameters

\[ m_u = m_d = 177.85 \text{ MeV}, \quad m_s = 400.98 \text{ MeV}, \quad \sqrt{\Lambda} = 469.52 \text{ MeV}, \quad g = 8.94, \]

fitted from the masses of \( \pi, \rho, K \) and \( K^* \) mesons such that

\[
\begin{align*}
M_\pi &= 140 \text{ MeV}, \\
M_K &= 496 \text{ MeV}, \\
f_\pi &= 129.9 \text{ MeV}, \\
f_K &= 150.8 \text{ MeV}, \\
M_\rho &= M_\omega = 770 \text{ MeV}, \\
M_{K^*} &= 890 \text{ MeV}, \\
M_\phi &= 1035 \text{ MeV},
\end{align*}
\]

we find that

\[ M_\eta = 640 \text{ MeV}, \quad M_{\eta'} = 950 \text{ MeV}, \]

\[ h_\eta = 4.72, \quad h_{\eta'} = 2.55 \]

for

\[ \sqrt{\mathcal{B} R} = 1.56. \]

The parameter \( R \) was fitted to \( \eta' \) mass. The domain model value for this dimensionless parameter quoted in the first part of the paper \( \sqrt{\mathcal{B} R} = 1.24 \) is close to Eq. (59). Thus we see that all features of the spectrum of light vector and pseudoscalar mesons usually associated with chiral symmetries are correctly reproduced by the model quantitatively. The origin of the splitting between the pseudoscalar and vector mesons was discussed in detail in [18]. The new feature is the shift in the masses of \( \eta' \) and \( \eta \) with respect to the other pseudoscalar mesons and their mutual splitting, which occurs here due to the additional contribution of Eq. (53) to their polarization functions. The momentum dependent mixing angle takes different values at the scale of \( \eta \) and \( \eta' \) masses

\[ \theta_\eta = \theta(-M_\eta^2) = -19.8^\circ, \quad \theta_{\eta'} = \theta(-M_{\eta'}^2) = 46.1^\circ. \]

It is notable that \( \theta_\eta \) coincides with the value of the mixing angle in the naive quark model, while \( \theta_{\eta'} \) is completely different from \( \theta_\eta \) both in sign and magnitude.

We quote also the result for the weak decay constants of \( \eta \) and \( \eta' \):

\[
\begin{align*}
f_\eta^0 &= 4.13 \text{ MeV}, \\
f_\eta^8 &= 165.7 \text{ MeV}, \\
f_{\eta'}^0 &= 154.5 \text{ MeV}, \\
f_{\eta'}^8 &= 288.6 \text{ MeV}, \\
f_{\eta'}^6 &= 23.67 \text{ MeV}, \\
f_{\eta'}^0 &= 183.3 \text{ MeV}.
\end{align*}
\]

These constants are defined by the matrix elements

\[
\langle 0| J_{5\mu}^a(0)|\phi(p)\rangle = ip_\mu f_\phi^a/\sqrt{2},
\]

\[
J_{5\mu}^a = \bar{q}\gamma_\mu\gamma_5 t^a q, \quad t^0 = \lambda^0/2, \quad t^8 = \lambda^8/2,
\]

and \( f_\phi^a \) corresponds to the mixings:

\[
\begin{align*}
f_{\eta}^0 &= f_\eta^8 \cos \theta + f_\eta^0 \sin \theta, \\
f_{\eta}^8 &= f_\eta^8 \cos \theta - f_\eta^0 \sin \theta.
\end{align*}
\]

Here

\[
\begin{align*}
f_{\eta}^0 &= h_\eta [f_s^8 \cos \theta + f_s^0 \sin \theta], \\
f_{\eta}^8 &= h_\eta [f_s^0 \cos \theta - f_s^8 \sin \theta], \\
f_{\eta'}^0 &= \frac{1}{3}(2f_u + f_s), \\
f_{\eta'}^8 &= \frac{1}{3}(f_u + 2f_s), \\
f_s^0 &= \frac{\sqrt{2}}{3}(f_u - f_s).
\end{align*}
\]

The function \( f_f(-M^2) \) was calculated in [18],

\[
f_f = \frac{m_f}{4\pi^2} \sum_v \int_0^1 \int_0^1 ds_1 ds_2 (1 + s_1 s_2) \left[ \frac{1 - s_1}{s_1 + s_2 + 2vt(1 + s_1 s_2)} \right]^3 \left[ \frac{(1 - s_1)(1 - s_2)}{(1 + s_1)(1 + s_2)} \right]^{m_f^2/4\nu\lambda^2} \times \left[ 2vt + \frac{s_1}{1 - s_1^2} + \frac{s_2}{1 - s_2^2} \right] \exp \left( \frac{M^2}{2\nu\lambda^2} \Psi(t, s_1, s_2) \right),
\]

\[
\Psi = \frac{s_1 s_2 + vt(s_1 + s_2)/2}{s_1 + s_2 + 2vt(1 + s_1 s_2)}.
\]
As we have already mentioned, the quark masses here should be considered as constituent quark masses. The massless limit is ill-defined due to the contribution of zero modes to the quark propagator. However, as is discussed in [18], a peculiar feature is that all the necessary shifts and splittings in the meson spectrum occur explicitly due to the strong (and, for pseudoscalar, scalar, vector and axial vector states, very different) dependence of meson masses on the quark masses, which is driven entirely by the zero modes. In order to visualize this picture in a quantitative manner, including now also the $\eta$ and $\eta'$, let us consider a regime in the model which can be called “a chiral limit”.

C. Massless pseudoscalar octet.

This limit can be defined in terms of composite meson fields as the condition that the $\pi$ and $K$ mesons become massless particles. Within the above-formulated meson theory this requirement is fulfilled if constituent quark masses satisfy the relation

$$m_s = m_d = m_u = m^*$$

and the mass $m^*$ is defined from the equation for pion mass with $m_\pi = 0$ substituted, that is

$$1 + \frac{g^2}{9\Lambda^2}\Pi_\pi(0) = 0.$$ 

(61)

Since in this limit flavour $SU(3)$ becomes exact symmetry the kaons are also massless. Furthermore, as follows from Eqs. (53)

$$\Pi_{08} = T^{08} = T^{88} \equiv 0,$$

which means that the $\eta$ meson is degenerate with the $\pi$ and $K$. The mixing angle between $\eta_0$ and $\eta_8$ is equal to zero and the $\eta'$ meson corresponds to a pure flavour singlet state. As seen from Eqs. (53),

$$\delta\Pi_{00} \propto T^{00}F(p^2) \neq 0$$

and the mass of the singlet state is split from the flavour octet states. Vector mesons are massive in this regime. Numerically one finds

$$m^* = 168.83 \text{ MeV},$$

$$M_\eta = M_\pi = M_K = 0, \ M_{\eta'} = 897 \text{ MeV},$$

$$h_\eta = h_\pi = h_K = 4.28, \ h_{\eta'} = .63,$$

$$f^8_\eta = f_K = f_\pi = 127.76 \text{ MeV}, \ f^0_\eta = 281.87 \text{ MeV},$$

$$f^0_\eta = f^8_\eta = 0.$$ 

and

$$M_\rho = M_{K^*} = M_{\omega} = 762 \text{ MeV}, \ M_\phi = 950 \text{ MeV}.$$ 

This result displays a typical chiral limit picture with both flavour $SU_R(3) \times SU_L(3)$ and $U_A(1)$ symmetries correctly implemented de facto: while the masses of octet states are drastically reduced from physical values to zero their decay constants are subject only to minor change. Simultaneously we see that the $\eta'$ does not look like a ”Goldstone” particle: both its mass and decay constant $f^0_{\eta'}$ are practically unchanged. The vector states are subject to minor change also. For completeness we mention that the scalar and axial vector particles as ground state mesons are absent in the spectrum in this model, but they appear in the hyperfine structure of orbital excitations of vector mesons with quantitatively correct masses [18], which stays unchanged also in the chiral limit described here.

In this picture $m^*$ looks like the condensate part of the constituent quark mass. This impression can be enhanced by estimating the current quark masses and their ratio,

$$\mu_u = m_u - m^* = 9.02 \text{ MeV}, \ \mu_s = m_s - m^* = 232.15 \text{ MeV},$$

$$\mu_s/\mu_u = 25.73.$$ 

The ratio, which is the only meaningful quantity, practically coincides with the generally accepted value $\mu_s/\mu_u = 25.$
We have shown that the correlation between the chirality of low-lying Dirac modes and the duality of the domain-like background field indeed drives the spontaneous breakdown of flavour chiral symmetry, as was hinted at in [4], and as indicated in lattice calculations [5]. We have also shown that the mechanism generating the $\eta - \eta'$ mass splitting is the same as that causing area law confinement in the domain model, namely the finite range correlations induced by the domain mean field. In more detail, we have extracted the parity-odd part of the logarithm of the quark determinant and seen that the axial anomaly is recovered. The chirality properties of the Dirac modes in domains generate the anomaly in this context. We see however that in this formulation with domains with bag-like boundary conditions both the symmetric zeta and the asymmetric eta functions are necessary to obtain this result. We have then explored the consequences of the anomaly for the realisation of chiral symmetry in the domain model. The contribution of the anomaly to the free energy of an ensemble of domains leads to a spontaneous breaking of $Z(2)$ symmetry rather than a $U_A(1)$ for one massless fermion flavour. Quark condensation occurs because of the spectral asymmetry coming from the bag-like boundary conditions. In the multflavour case, the Abelian nature of the axial anomaly guarantees that the discrete $Z(2)$ symmetry remains spontaneously broken in addition to the correct continuous non-singlet $SU(N_f)_L \times SU(N_f)_R$ symmetries. One would thus naively expect only $N_f^2 - 1$ Goldstone bosons based on the number of continuous degenerate directions in the space of ground states of an ensemble of domains.

On this basis we expect massless pions, kaons and eta mesons in the chiral limit. The domain model manifests correlations Fig.2b in the singlet channel which generates the splitting in $\eta - \eta'$ masses via the Witten mechanism. The contribution, Fig.2b, is entirely driven by the correlators of the background field Eq. (48). Simultaneously the background field correlators are also entirely responsible for the area law [3]. Thus the origin of both mechanisms is identical. In the $N_c \rightarrow \infty$ limit the $\eta'$ is massless since the contribution of Fig.2b is $1/N_c$ suppressed with respect to Fig.2a.

We have seen that the calculational scheme does work well phenomenologically, but the dimensionless parameter turns out to be $BR^2 \approx O(1)$ such that for a self-consistent consideration of meson physics one needs to use the propagators of the domain background field: neglecting boundary conditions inside quark loops is not consistent with $BR^2 \approx O(1)$. There is no clear separation of two scales as has already been observed in the consideration of static parameters of the vacuum – quark and gluon condensates, the string constant and the topological susceptibility within the domain model.

Nevertheless, the calculations presented in the final section resemble two features of the domain model we have addressed in the first part of the article. The first feature is quite obvious: in both cases the additional contributions to the correlators (polarization functions) for $\eta'$ and, if $m_u \neq m_s$ also for $\eta$, are crucial for their splitting from the other pseudoscalar mesons, thus resolving this aspect of the $U_A(1)$ problem. Certainly, the successful (but not self-consistent) quantitative description provided by the purely homogeneous background field need not be equally successful in the domain model, and verification of this is one of our first priorities. The second feature is not quite so obvious. The splitting between pseudoscalar and vector meson masses in the case of the purely homogeneous field is determined by the singular behaviour of the quark condensate for $m \rightarrow 0$ (as mentioned, the condensate diverges in the massless limit due to a continuum of zero modes [20, 21]). This singular behaviour is not present in the domain model – zero modes do not exist at all and the limit $m \rightarrow 0$ is regular. However a quark condensate is generated in the domain model by the asymmetry in the spectrum of the Dirac operator, and this condensate diverges for $R \rightarrow \infty$ as follows from Eqs. (31) and (34) and as shown in Fig. 1, as discussed above. This divergence is expected to play the same role for correlators of the domain model as $m \rightarrow 0$ in the homogeneous field, thus generating strong pseudoscalar-vector splitting. The value of quark condensate given in Eq. (35) corresponds to $R/R_0 = 1$ in Fig. 1. But, unlike the model based on a purely homogeneous field, the domain model manifests spontaneous breaking of flavour chiral symmetry and hence has the potential to reproduce a genuine picture of the chiral limit. This is only mimicked in the homogeneous field considerations above, though in a surprisingly successful manner.

The zeta function calculation nonetheless remains incomplete, where the contributions that should eliminate the chiral angle dependent parity even part of the quark determinant’s logarithm need to be calculated. The observation of lack of separation of scales, mentioned above, also means that a more careful study of collective modes within the domain model per se is necessary to put the phenomenological results above on a sound footing. In the same context, explicit correlation functions in the domain model would enable a study of the (anomalous) Ward identities and issues [26] related to the realisation of Goldstone’s theorem (or otherwise) in the $U_A(1)$ sector.

There are two issues naturally related to this work and requiring at least some preliminary comment within the constraints of the calculations realised thus far.

At first superficial glance, the mechanism of quark condensate generation through the asymmetry in the spectrum of the Dirac operator looks to be different to the approach of Banks and Casher [27] where the spectrum is symmetric and the condensate arises due to a finite density of chiral zero modes. However, these seemingly different formulations in fact have more in common. In particular, in the thermodynamic limit the system is characterised by a finite
density of low-lying (but non-zero) modes which are strongly chiral. Even if for a given fixed domain the spectrum is asymmetric, as indicated by the spectral asymmetry function $\eta$, on average all positive and negative non-zero eigenvalues appear in the ensemble in a symmetric way, and neither left nor right chirality modes prevail in the ensemble. This can be seen in the symmetric distribution of the local chirality parameter given in our previous work [4]. The chiral condensate arises due to the existence of a finite density of strongly chiral eigenmodes both in the Banks-Casher formulation and in the model under consideration. The difference is that, in the domain model the corresponding eigenvalues are nonzero in the thermodynamic limit and none of the fermionic modes are purely chiral, rather they can be characterised by their average chirality with a definite sign correlated with the duality of the mean gluonic field in the domain: the lower the Dirac operator eigenmode the closer this average chirality to $\pm 1$. Thus, we would take the liberty to say that the domain model gives a ”smeared” realisation of the Banks-Casher scenario.

Another interesting question inspired by the chiral boundary condition used in the model is the manifestation or otherwise of features seen in chiral bag models of the nucleon. How much is in common here beyond the similar boundary condition? There is no simple answer to this question yet. We recall that the domain model is formulated in Euclidean four-dimensional space, colourless hadrons (if any) are not associated with domains themselves but are anticipated to arise as collective excitations of quantum fluctuations of quark (and possibly gluon) fields in the domain ensemble. The description of colourless hadrons requires analytical continuation to physical Minkowski space. Quark and gluon fluctuations in this approach are localised both in space and time: no asymptotic particle-like states can be associated with them. The model requires substantial use of methods of nonlocal quantum field theory. Various issues related to quantisation, unitarity, causality, Froissart-type bounds at high energies and interpretation of nonlocal fields as fluctuations localised in space and time can be found in [23, 24, 28–30]. The nonlocality appears here primarily due to the presence of strong background gluon fields which eliminate the pole in the momentum space quark propagator [17, 18, 31] rendering it an entire analytical function. The nature of coloured fluctuations as localised in space and time prohibits a straightforward resolving in the domain model of the manifestation of such interesting features as baryon number fractionalisation in chiral bags [32]. This phenomenon appears due to asymmetry (and a corresponding $\eta$ invariant) in the spectrum of the Dirac Hamiltonian, rather than the Euclidean Dirac operator, after a rearrangement of energy levels in the solitonic background chiral bag. There is no automatic one-to-one correspondence between phenomena in the domain model and those of the chiral bag model for the nucleon. In particular, the answer to the question of baryon number is intimately connected to the analytical properties of propagators (for example see [33]) and thus to the mode of dynamical confinement and the realisation of hadrons as propagating excitations in the domain ensemble, the subject of further work.

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APPENDIX A: EXPANSION OF KUMMER FUNCTIONS

1. Asymptotic form of Kummer function $M(k^2 x^2/(4z), k+1, z)$ for $k \gg 1, z, x$ fixed.

We use the representation [34]

$$M(a/4z, k+1, z) = \frac{k!}{\Gamma(a/4z)} \int_0^\infty dt t^{a/4z-1} e^{-t} I_k(2\sqrt{zt})$$

and change the integration variable $2\sqrt{zt} = s$ giving

$$M(a/4z, k+1, z) = \frac{k!}{\Gamma(a/4z)} \int_0^\infty ds s^{a/2z-k-1} e^{-s^2/4z} I_k(s).$$

The asymptotic behaviour of

$$M_k = \int_0^\infty ds s^{a/2z-k-1} e^{-s^2/4z} I_k(s)$$
at $a \propto k^2$, $k \gg 1$ can be found by the saddle point method. Denoting $\alpha = a/2z$, $\zeta = 2z$ and $f(s) = [1 - (k + 1)/\alpha] \ln(s) - s^2/2\alpha \zeta$, we arrive at

$$M_k = \int_0^{\infty} ds e^{af(s)} I_k(s)$$

with $f(s)$ having a maximum at

$$s_0 = \sqrt{\alpha(k - 1)}.$$

Expanding the exponent about the saddle point and using the addition theorem

$$\left( \frac{z_1 - z_2 e^{-i\phi}}{z_1 - z_2 e^{i\phi}} \right)^\nu I_\nu(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n I_{\nu+n}(z_2) e^{in\phi},$$

$$\omega = \sqrt{z_1^2 + z_2^2 - 2z_1z_2 \cos(\phi)}$$

with $\phi = 0$ and $\pi$ one obtains the following representation

$$M(a/z, k + 1, z) = \sqrt{2\pi} \frac{k! z^{-a/4z} k^{k+1-a/2z}}{\Gamma(a/4z)} \left[ a - 2z(k + 1) \right]^{a/4z-(k+1)/2} e^{-a/4z+(k+1)/2} e^{-z/4}$$

$$\times \sum_{n=-\infty}^{\infty} I_{2n+k}(s_0) I_n(z/4) \left[ 1 + W_1(x, z, n)/k + W_2(x, z, n)/k^2 \right]$$

$$+ \sum_{n=-\infty}^{\infty} I_{2n+k+1}(s_0) I_n(z/4) \frac{W_3(x, z)}{k},$$

(A1)

$$W_1 = \frac{z(z - 2n + 3)}{6x^2},$$

$$W_2 = \frac{z}{72x^4} \left[ (27z - 8zn + z^3 + 15z^2 + 4zn^2) \sqrt{1 + x^2} + 8znx^2 - 12x^2 + 15z^2 - 9zx^2 + 2z^2nx^2 + 8n^3x^2 + z^3x^2 + 4z^2x^2 - 56zn + 27z - 48n^2x^2 + 46nx^2 + 26znx^2 + 4zn^2 + z^3 \right]$$

$$W_3 = \frac{z(z + 3)}{6x},$$

$$s_0 = \sqrt{a - 2z(k + 1)} = k^2x^2, \quad x = tp,$$

and calculation of the asymptotic form of $M$ is reduced to the asymptotic decomposition of $I_{2n+k}(s_0)$ and $I_{2n+k+1}(s_0)$ for $k \gg n$.

Decomposition of the factors in front of the sum in Eq. (A2) at $k \gg 1$ gives

$$\frac{k! z^{-a/4z} k^{k+1-a/2z}}{\Gamma(a/4z)} \left[ a - 2z(k + 1) \right]^{a/4z-(k+1)/2} e^{-a/4z+(k+1)/2} e^{-z/4} \sqrt{2\pi}$$

$$\frac{k! 2^k}{(k \pi)^{k/2+1/2}} \left[ 1 + \frac{z(3x^2 + z)}{3x^4} + \frac{z(3x^6 + 27x^4z + 12x^2z^2 + z^3)}{18x^8 k^2} + O(1/k^3) \right]$$

The representation

$$I_\nu(s_0) = \frac{1}{\sqrt{\pi}} \frac{s_0^\nu}{2^{\nu+1/2} \Gamma(\nu + 1/2)} \int_{-1}^{1} dt e^{-s_0 t^2} (1 - t^2)^{\nu-1/2}$$

is now suitable for determining an asymptotic decomposition of the $I_{2n+k}(s_0)$ again by means of the saddle point approximation. With

$$g_1(x, n, z) = \frac{8z^2 + 4xz(2 + 4n + z) + x^4(-1 + 16n^2 + 8z)}{8x^4 \sqrt{1 + x^2}},$$

$$g_2(x, n, z) = \frac{[8z^2 + 4xz(2 + 4n + z) + x^4(-1 + 16n^2 + 8z)]^2}{128x^8(1 + x^2)}$$

$$- \frac{1}{24(1 + x^2)^{3/2} x^6} \left[ 32z^3 + 48x^2z^2(1 + n + z) + x^6[-1 + 6n - 32n^3 - 3z + 48n(1 + n)z + 24z^2] \right. \right.$$  

$$+ \left. 12x^4[z(6 + z) + n(4 + 6z)] \right].$$
and

\[ h_1(x, n, z) = \frac{2(x^2 - 2)\sqrt{1 + x^2} - 5}{24(1 + x^2)^{3/2}} \]

\[ h_2(x, n, z) = \frac{1}{1152(1 + x^2)^3} \left[ -8 - 68x^2 - 720z + 480n - 960x^2n \right] \sqrt{1 + x^2} + 17 - 144x^2 + 276x^4 + 52x^6 + 384n - 576x^2n - 1152x^4n - 192x^6n - 576z x^2 - 576z \]

we obtain the following full expression for the asymptotics of the Kummer function up to \( O(1/k^2) \)

\[
M(a/4z, k + 1, z) = \frac{k!2^k}{(k\pi)^k} \frac{1}{\sqrt{2\pi k}(1 + x^2)^{1/4}} \left( k \sqrt{x^2 + 1} + k \ln \left( \frac{x}{\sqrt{x^2 + 1}} + 1 \right) \right) \times \exp \left\{ \frac{z}{4} + \frac{z}{2x^2} - \frac{z}{x^2 + 1} \right\} \left( 1 + \frac{z(3x^2 + z)}{3x^4 k} + \frac{z(3x^6 + 27x^4 z + 12x^2 z^2 + z^3)}{18x^8 k^2} + O(1/k^3) \right) \times \sum_{n=-\infty}^{\infty} I_n(z/4) \left( \frac{x}{1 + \sqrt{x^2 + 1}} \right)^{2n} \left( 1 + \frac{2z + 1 - 4n}{4(1 + x^2)^{k}} + \frac{16n^2(3 - 2x^2) + 8n(2x^2 - 3 - 10z) + 16z x^2 + 36z - 2x^2 + 20z^2 + 3}{32(1 + x^2)^2 k^2} \right) \times \left( 1 + \frac{1}{24} - \frac{2n^2}{k} \right) \left( 1 + \frac{1}{1152} - \frac{n^2}{12} - \frac{n^3}{12} + \frac{4n^4}{3} + \frac{2n^4}{3} \right) \frac{1}{k^2} \right) \times \left( 1 + \frac{16n^2 - 1}{8k} \right) \left( 1 + \frac{768n^4 - 512n^3 - 96n^2 + 96n - 13}{384k^2} \right) \times \left( 1 + \frac{g_1(x, n, z)}{k} + \frac{g_2(x, n, z)}{k^2} \right) \left( 1 + \frac{h_1(x, n, z)}{k} + \frac{h_2(x, n, z)}{k^2} \right) \times \left( 1 + \frac{W_1(x, n, z)}{k} + \frac{W_2(x, n, z)}{k^2} \right) \times \left( 1 + \frac{g_1(x, n + 1/2, z)}{k} \right) \left( 1 + \frac{h_1(x, n + 1/2, z)}{k} + O(1/k^3) \right) \right\}

The series over \( n \) can now be resummed in terms of the generating function of Bessel functions and its derivatives,

\[
\sum_{n=-\infty}^{\infty} v^n I_n(u) = \exp \left\{ \frac{u}{2} \left( v + \frac{1}{v} \right) \right\},
\]

\[
\sum_{n=-\infty}^{\infty} n^l v^n I_n(u) = \left( \frac{d}{du} \right)^l \exp \left\{ \frac{u}{2} \left( v + \frac{1}{v} \right) \right\}
\]

where

\[
u = z/4, \quad v = \left( \frac{x}{1 + \sqrt{x^2 + 1}} \right)^2, \quad l = 0, 1, 2, 3, 4.
\]

Moreover, the presence of the factor

\[
\exp \left\{ \frac{z}{4} + \frac{z}{2x^2} - \frac{z}{x^2 + 1} \right\}
\]

allows further simplification

\[
\exp \left\{ \frac{z}{4} + \frac{z}{2x^2} - \frac{z}{x^2 + 1} \right\} \sum_{n=-\infty}^{\infty} \left( \frac{x}{1 + \sqrt{x^2 + 1}} \right)^{2n} I_n(z/4) = \exp \left\{ \frac{z}{2} \sqrt{x^2 + 1} - 1 \right\} \frac{1}{2 \sqrt{x^2 + 1}}.
\]
The remaining formulae required can be obtained by differentiation. The final result is then

\[
M(a/4z, k + 1, z) = \frac{k!2^k}{(k\pi)^k} \frac{1}{\sqrt{2\pi k}} \frac{1}{(1 + x^2)^{1/4}} \exp \left\{ k\sqrt{x^2 + 1} + k \ln \left( \frac{x}{1 + \sqrt{x^2 + 1}} \right) \right\} \\
\times \exp \left\{ \frac{z}{2} \sqrt{x^2 + 1} - 1 \right\} \left[ M_1(x, z) + \frac{M_2(x, z)}{k} + O(1/k^3) \right]
\]

where

\[
M_1(x, z) = -\frac{y(5y^4 + (-12z + 10)y^3 + (12z^2 + 2 - 24z)y^2 + (-8z^2 + 12z - 6)y + 24z - 4z^2 - 3)}{24(y + 1)^2}
\]

and

\[
M_2(x, z) = \frac{y^2}{1152(y + 1)^3} \left[ 385y^8 + (-840z + 1540)y^7 + (1848 + 840z^2 - 3360z)y^6 + (-308 + 3040z^2 - 4584z - 480z^3)y^5 \right. \\
+ (144z^4 - 2306 - 1152z^3 + 3304z^2 - 1200z)y^4 + (-1408z^2 - 192z^4 + 3336z - 1524 + 768z^3)y^3 \\
+ (24 - 32z^4 + 1344z^3 + 4128z - 4456z^2)y^2 + (-1632z^2 + 324 + 2088z + 64z^4 - 288z^3)y \\
+ 16z^4 + 432z - 192z^3 + 312z^2 + 81 \right]
\]

with

\[
y = \frac{1}{\sqrt{x^2 + 1}}
\]

For the particular case \(z = 0\) we get

\[
M_1(x, 0) = -\frac{y(5y^2 - 3)}{24}
\]

and

\[
M_2(x, 0) = \frac{y^2}{1152} \left[ 385y^4 - 462y^2 + 81 \right]
\]

which reproduces the known asymptotic formulae [35]. This, as well as cancellation of powers of \(y\) in the denominators of \(M_i\), gives a quite reliable criterion for correctness of the obtained expression.

2. Asymptotic form of the Kummer function \(M(k^2x^2/4z + 1, k + 2, z)\) for \(k \gg 1, z, x\) fixed.

Computations analogous to those of the previous section lead to the result

\[
M(k + 1 + a/4z, k + 2, z) = \frac{k!2^k}{(k\pi)^k} \frac{1}{\sqrt{2\pi k}} \frac{1}{(1 + x^2)^{1/4}} \exp \left\{ k\sqrt{x^2 + 1} + k \ln \left( \frac{x}{1 + \sqrt{x^2 + 1}} \right) \right\} \\
\times \exp \left\{ \frac{z}{2} \sqrt{x^2 + 1} - 1 \right\} \left[ M_0(x) + \frac{M_1(x, z)}{k} + \frac{M_2(x, z)}{k^2} + O(1/k^3) \right]
\]

where

\[
M_0(x) = \frac{2y}{1 + y},
\]

\[
M_1(x, z) = -\frac{y(5y^3 + (-12z + 22)y^2 + (-48z + 38 + 12z^2)y^3 + (-36z + 6 - 8z^2)y^2 + (-39 - 4z^2)y - 24)}{12(y + 1)^3}
\]
3. Asymptotic form of Kummer function $M(-k^2x^2/4z, k+1, -z)$ for $k \gg 1$, $z$, $x$ fixed.

Use of previous calculations can be maximised by writing the function to be decomposed as

$$M(-a/4z, k+1, -z) = e^{-z}M(a/4z + k + 1, k + 1, z)$$

$$= \frac{k!2^k}{(4z)^{k+1}} \frac{1}{\Gamma(a/4z)} \frac{1}{\Gamma(a/4z + k + 1)} e^{-z} \int_0^\infty ds s^{a/2z + k + 1} e^{-s^2/4z} I_k(s).$$

The final result is

$$M(-a/4z, k+1, -z) = e^{-z}M(a/4z + k + 1, k + 1, z) = \frac{k!2^k}{(4z)^{k+1}} \frac{1}{\Gamma(a/4z)} \frac{1}{\Gamma(a/4z + k + 1)} e^{-z} \int_0^\infty ds s^{a/2z + k + 1} e^{-s^2/4z} I_k(s).$$

where

$$M_1(x, z) = -\frac{y(5y^4 + (12z + 10)y^3 + (12z^2 + 2 + 24z)y^2 + (-8z^2 - 12z - 6)y - 24z - 4z^2 - 3)}{24(y + 1)^2}$$

and

$$M_2(x, z) = \frac{y^2}{1152(y + 1)^3} \left[385y^8 + (840z + 1540)y^7 + (840z^2 + 1848 + 3360z)y^6 + (-308 + 3040z^2 + 480z^3 + 4584z)y^5ight.$$

$$+ (1152z^3 - 2306 + 1200z + 144z^4 + 3304z^2)y^4 + (-1408z^2 - 192z^4 - 3336z - 1524 - 768z^3)y^3$$

$$+ (24 - 4456z^2 - 1344z^3 - 32z^4 - 4128z)y^2 + (-2088z + 324 + 288z^3 + 64z^4 - 1632z^2)y$$

$$+ 16z^4 + 192z^3 + 81 + 312z^2 - 432z]\right]$$

with $y = 1/\sqrt{x^2 + 1}$.

4. Asymptotic form of Kummer function $M(-k^2x^2/(4z), k+2, -z)$ for $k \gg 1$, $z$ fixed.

The relation of $M(-k^2x^2/(4z), k+2, -z)$ to $M(-k^2x^2/(4z), k+1, -z)$ is analogous to that of $M(k^2x^2/(4z) + 1, k + 2, z)$ to $M(k^2x^2/(4z), k + 1, z)$, thus the calculation is similar to that in Sec.A 3. The result is

$$M(-a/4z, k+2, -z) = e^{-z}M(k + 2 + a/4z, k + 2, z)$$

$$= \frac{k!2^k}{(4z)^{k+1}} \frac{1}{\Gamma(a/4z)} \frac{1}{\Gamma(a/4z + k + 2)} e^{-z} \int_0^\infty ds s^{a/2z + k + 2} e^{-s^2/4z} I_k(s).$$

where

$$M_0(x) = \frac{2y}{1+y}.$$
\[ M_1(x, z) = -\frac{y(5y^5 + (12z + 22)y^4 + (48z + 12z^2 + 38)y^3 + (6 - 8z^2 - 12z)y^2 + (-4z^2 - 48z - 39)y - 24)}{12(y + 1)^3} \]

and

\[ M_2(x, z) = \frac{y^2}{576(y + 1)^5} \left[ 385y^9 + (2380 + 840z)y^8 + (5040z + 6048 + 840z^2)y^7 + \right. \]
\[ + (4480z^2 + 7636 + 12264z + 480z^3)y^6 + \]
\[ + (3478 + 1728z^2 + 5008z^2 + 144z^3 + 13152z)y^5 + (-2376z - 1792z^2 - 1152z^3 - 192z^4 - 3660)y^4 + \]
\[ + (-19056z - 32z^4 - 11560z^2 - 2112z^3 - 7440)y^3 + (-2880z^2 + 672z^3 - 13032z - 5892 + 64z^4)y^2 + \]
\[ + (384z^3 - 2439 + 864z + 2712z^2 + 16z^4)y - 432 + 192z^2 + 2304z \]

with

\[ y = \frac{1}{\sqrt{x^2 + 1}} \]

[1] For example, vortices. Some references are: P. de Forcrand, M. D’Elia, Phys.Rev.Lett. 82, 4582 (1999); J. Greensite, S. Olejnik, hep-lat/0302018; H. Reinhardt, Nucl.Phys. B628, 133 (2002); M. Engelhardt, Nucl.Phys. B638, 81 (2002); P. de Forcrand, L. v. Smekal, Phys.Rev. D66, 011504 (2002).

[2] P. van Baal, At the frontier of particle physics, vol. 2, 760-821, ed. by M. Shifman, World Scientific, 2001 (hep-ph/0008206).

[3] A.C. Kalloniatis, S. N. Nedelko, Phys. Rev. D 64 (2001) 114025.

[4] A.C. Kalloniatis, S.N. Nedelko, Phys. Rev. D 66 (2002) 074020.

[5] R.G. Edwards, U.M. Heuer, Phys.Rev. D65, 014505 (2002); T. De Grand, A. Hasenfratz, Phys.Rev. D65 014503 (2002); I. Hip et al, Phys. Rev. D65, 014506 (2002); T. Blum et al, Phys.Rev. D65, 014504 (2002); C. Gattringer et al, Nucl. Phys. B618, 205 (2001); C. Gattringer et al, Nucl. Phys. B617, 101 (2001).

[6] S. Deser, L. Griguolo, D. Seminara, Phys. Rev. Lett. 79 (1998) 1976.

[7] A. Wipf and S. Duerr, Nucl. Phys. B 443 (1995) 201.

[8] L. Bergamin, P. Minkowski, hep-th/0003097 (2000); M. Leibundgut, P. Minkowski, Nucl. Phys. B531 (1998) 95.

[9] N.K. Nielsen, Nucl. Phys. B29 (1977) 212.

[10] N.K. Nielsen, Nucl. Phys. B29 (1977) 212.

[11] P. Minkowski, Nucl. Phys. B177 (1981) 203.

[12] E. Witten, Nucl. Phys. B 156 (1979) 269; G. Veneziano, Nucl. Phys. B 159 (1979) 213.

[13] M. Bordag, E. Elizalde, K. Kirsten, J. Math. Phys. 37 (1996) 895; K. Kirsten, Spectral functions in mathematics and physics, Chapman&Hall/CRC Press, Boca Raton, FL, 2001; for recent application see C.G. Beneventano, E.M. Santangelo, A. Wipf, hep-th/0205199, (2002).

[14] E. Elizalde, Ten Physical Applications of Spectral Zeta Functions, Lecture Notes in Physics, m35, Springer-Verlag, Berlin, Heidelberg, New York, 1995.

[15] K. Fujikawa, Phys. Rev. D 21 (1980) 2848; Erratum-ibid. D 22 (1980) 1499.

[16] Th. Shafer, E. Shuryak, Rev. Mod. Phys. 70 (1998) 323.

[17] See for example, S.R. Sharpe, Phys. Rev. D 41 (1990) 3233; C.W. Bernard, M.F. Golterman, Phys. Rev. D 46 (1992) 853. Some more recent applications of this include, R.D. Young, et al. Phys. Rev. D 66 (2002) 094507; S.J. Dong et al., hep-lat/0304005.

[18] G.V. Efimov, S.N. Nedelko, Phys. Rev. D 51, 174 (1995).

[19] Ja.V. Burdanov, G.V. Efimov, S.N. Nedelko, S.A. Solomin, Phys. Rev. D 54, 4483 (1996).

[20] Ja.V. Burdanov, G.V. Efimov, S.N. Nedelko, S.A. Solomin, hep-ph/9806478 (1998).

[21] M.S. Dubovikov, A.V. Smilga, Nucl.Phys. B185 (1981) 109.

[22] G.V. Efimov, S.N. Nedelko, Eur.Phys.J.C1 (1998) 343.

[23] The difference between the NJL/DSE and instanton-like (zero mode) mechanisms has been discussed in recent work, P. Faccoli, T. A. De Grand, hep-ph/0304219 (2003), where a specific combination of scalar and pseudoscalar correlators has been used to test the character of chiral symmetry realisation in lattice QCD.

[24] G.V. Efimov, M.A. Ivanov, "The Quark Confinement of Hadrons", IOP, Bristol (1993); A. Salam, Nuovo Cim. 25 (1965) 224; S Weinberg, Phys. Rev. 130 (1963) 776; K. Hayashi et al, Fortsch. der Phys. 15 (1967) 625.

[25] G.V. Efimov, G. Ganbold, Phys.Rev.D65 (2002) 054012.

[26] G.V. Efimov, hep-ph/0304194 (2003).

[27] See both R.J. Crewther, “Chiral Properties of Quantum Chromodynamics”, in NATO Adv.Study Inst.Ser.B Phys.55:59,1980 and G. ’t Hooft, Phys. Rept. 142 (1986) 357 for a thorough list of references on this issue.

[28] T. Banks, A. Casher, Nucl. Phys. B169 (1980) 103.
[29] Fainberg V. Ya., and Soloviev M. A. Theor. Math. Phys. 93 (1992) 1438; Teor. Mat. Fiz. 93:514-528, 1992; *ibid* Annals Phys. 113 (1978) 421; *ibid* Commun. Math. Phys. 57 (1977) 149.

[30] R. Alkofer, and L. von Smekal, Phys. Rep. 353 (2001) 281.

[31] H. Leutwyler, Phys. Lett. B96 (1980) 154.

[32] A.J. Niemi, G.W. Semenoff, Phys. Rept. 135 (1986) 99.

[33] K.W. Mak, Phys. Rev. D 49 (1994) 6939; A.N. Sisakian, O. Yu. Shevchenko, S.B. Solganik, Nucl. Phys. B518 (1998) 455.

[34] W. Magnus, *et al*, "Formulas and Theorems for the Special Functions of Mathematical Physics", Springer-Verlag, Berlin Heidelberg New York (1966).

[35] See Eq.(9.7.7) for expansion of Bessel functions in M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions", Dover Publications, New York (1968).