Local interferometric symmetries for Gaussian pure states

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We interpret optical Gaussian pure states as systems of entangled spins using the Schwinger representation of $SU(2)$. As these are continuous-variable systems, we are interested in the nullifiers for any given system in terms of Schwinger operators. Considering also a Schwinger representation of $U(2)$ and generalising this to $U(n)$ for $n$ modes, we show that these nullifiers correspond to passive interferometric symmetries of the system. We present necessary and sufficient conditions that the adjacency matrix for a given Gaussian pure state and its corresponding nullifiers must satisfy. For the class of Gaussian pure states with bipartite graphs, we provide a method to derive the corresponding nullifiers. We derive the class of Gaussian pure states on two modes that is left invariant under an arbitrary passive interferometric operation and provide examples of these for some simple interferometers. We derive nullifiers for optical Gaussian pure states on two, four, and $n$ modes. These nullifiers reveal information about the structure of the entangled Schwinger spins in the system, providing intuition to applying the Schwinger mapping to large, optical Gaussian states. These nullifiers also provide local, interferometric symmetries for the corresponding optical Gaussian systems. These symmetries could lend themselves to applications such as gate teleportation in continuous-variable systems, whereby the symmetry of the entangled resource enables specific gates to be teleported with minimal noise.

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I. INTRODUCTION

Quantum information science promises new and revolutionary technology [1]. The emergence of fields such as quantum communication [2], quantum cryptography [3], and quantum computation (QC) [4] has lead to significant improvements in the capabilities, efficiency, and power of the technologies associated with these information processing tasks. Quantum-optical systems are promising architectures for many of these technologies. Optical circuits are ideal for communication, and photons—which do not interact strongly with one another and their environment—exhibit quantum phenomena that can be easily examined and manipulated with simple optical elements.

Optical Gaussian states are states of light whose Wigner functions are Gaussian distributions over the real ($\hat{q}$) and imaginary ($\hat{p}$) parts of the complex quantised-mode (qumode) amplitude, where $[\hat{q}, \hat{p}] = i\hbar$ [5]. In the context of quantum information, such states have the desirable feature of being produced, manipulated, and measured with experimental ease using squeezing, linear optics, and homodyne detection [6].

Many important aspects of quantum-information technologies have already been realised in Gaussian systems, including quantum teleportation [7], entanglement swapping [8], quantum dense coding [9, 10], entanglement purification [11], measurement-based QC [12], cluster state preparation [13], quantum error correction [14], and quantum algorithms [15]. These capabilities, combined with the relative experimental ease with which Gaussian states are processed and measured, encourages a serious consideration of Gaussian states for quantum information tasks. Indeed, Gaussian—or continuous-variable (CV) [16]—cluster states can be generated efficiently in a highly scalable fashion and serve as resources for measurement-based QC [12, 13, 17–20].

However, the simplicity and ease of Gaussian states comes at a price. It has been shown that Gaussian noise cannot be corrected for using only Gaussian operations [21]. Furthermore, for CV cluster states, local Gaussian measurements are not sufficient to transport quantum information throughout the cluster [22]. These results highlight the need for specific, tailored implementations of quantum error correction and fault tolerance within Gaussian systems.

Gaussian states on $n$ qumodes are formally equivalent to $n$ harmonic oscillators, and thus a well-known mapping, known as the Schwinger representation [23], can be applied to such systems. This is a mapping from the state space of a pair of harmonic oscillators to the representation space of $SU(2)$ [24] and thus, to a single quantum spin. This mapping has been a beneficial tool when applied to many areas in quantum optics, including the $SU(2)$ symmetry of a beamsplitter [25, 26], demonstrating violations of Bell inequalities in macroscopic systems [27–31], the formulation of

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angular momentum coherent states [32], polarization squeezing and entanglement [33], and detecting entanglement of non-Gaussian states [34].

Considering the fruitful applications of applying the Schrödinger representation to systems of a few qumodes, it is of interest to investigate larger systems, such as many-qumode optical Gaussian states, within the Schwinger picture. In this context, pairs of qumodes within the system are known as Schwinger spins.

In a CV setting, the nullifiers for a given state are of interest—i.e., the operators that have the state as a zero eigenstate [35]. These operators play a similar role to what stabilizers do for qubit-based states [36]. Nullifiers can be used to uniquely define a state as well as to describe the evolution of a state [12]. We thus seek nullifiers that act on a system of Schwinger spins. We show in Section V that these nullifiers can be interpreted as Hamiltonians that generate passive unitary interferometric transformations on the systems they act on.

Thus, in this work we perform two complimentary analyses: First, we interpret Gaussian states on $n$ qumodes (where $n$ can be arbitrarily large) as systems of entangled Schwinger spins, which, as far as we are aware, has not been done to present. In this case, the nullifiers reveal information about the structure of such a spin system. The interpretation of CV systems using the Schwinger representation has also simultaneously been addressed by the authors in [37].

Second, the derived nullifiers are interpreted as passive interferometric symmetries for a given state. These are photon-number-conserving interferometric operations that leave the state invariant. Such symmetries can be of use in contexts such as minimal-noise gate teleportation (explained in Section IIIA), which exploits the symmetry of a teleportation resource. In the context of CV measurement-based QC [17], such symmetries can be exploited in protocols that aim to introduce minimal noise to the system. Generally speaking, symmetries also provide us with a deeper understanding about the structure of a given state. The interferometric symmetries explored in this work provide additional insight into optical Gaussian states which, as discussed above, have potential applications in a wide variety of contexts.

In Section II, we explain the Schrödinger representation of SU(2) [23] maps a pair of harmonic oscillators to a single spin whose spin quantum numbers are determined by the sum and difference of quanta in each of the oscillators. It is a bosonic realization of a Lie algebra and is a multiplicity-free direct sum of all the unitary irreducible representations of SU(2) [24]. In an optical setting, the harmonic oscillators are qumodes that occupy a Fock space—i.e., an infinite dimensional Hilbert space endowed with the orthonormal basis $\{|n\rangle\}_{n=0}$ and operators $\hat{a}, \hat{a}^\dagger$ satisfying $[\hat{a}, \hat{a}^\dagger] = 1$. The Schrödinger spin operators are

$$S_{ij}^x = \frac{1}{2}(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i),$$

$$S_{ij}^y = \frac{1}{2}(\hat{a}_i^\dagger \hat{a}_j - \hat{a}_j^\dagger \hat{a}_i),$$

$$S_{ij}^z = \frac{1}{2}(\hat{a}_i \hat{a}_j^\dagger - \hat{a}_j \hat{a}_i^\dagger),$$

(2.1)

where the $i,j$ subscripts denote qumodes $i$ and $j$, respectively.
The appropriate $\mathfrak{su}(2)$ Lie algebra relations are satisfied:

$$[\hat{S}^k, \hat{S}^l] = i \sum_m \epsilon_{klm} \hat{S}^m,$$  \hspace{1cm} (2.2)

where $\epsilon_{klm}$ is the Levi-Civita symbol. It is also important to introduce the $\hat{S}^0$ operator,

$$\hat{S}^0_{i,j} = \frac{1}{2}(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i),$$

\hspace{1cm} (2.3)

which is related to $\hat{S}^2 = (\hat{S}^x)^2 + (\hat{S}^y)^2 + (\hat{S}^z)^2$, the Casimir invariant of $\mathfrak{su}(2)$, as

$$\hat{S}^2 = \hat{S}^0(\hat{S}^0 + 1).$$

\hspace{1cm} (2.4)

Therefore, the $\hat{S}^0$ operator corresponds to the spin quantum number $s$,

$$\hat{S}^0 \mapsto s,$$

\hspace{1cm} (2.5)

while $\hat{S}^z$ corresponds to the spin quantum number $m_s$,

$$\hat{S}^z \mapsto m_s.$$  \hspace{1cm} (2.6)

We can therefore link the pair of qumodes, with basis states spanned by $|n_1\rangle|n_2\rangle$ for all $n_1, n_2 \in \mathbb{Z}$, to a Schwinger spin basis $|s, m_s\rangle$ (where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m = -s, -s+1, \ldots, s-1, s$). These basis states are eigenstates of the $\hat{S}^z$ and $\hat{S}^0$ operators. The Schwinger mapping is thus a map from the state space of two qumodes to a single spin:

$$|n_1\rangle \otimes |n_2\rangle \mapsto |s, m_s\rangle,$$

\hspace{1cm} (2.7)

with the following relationship between the number of quanta in each qumode and the spin values:

$$|s, m_s\rangle = \frac{1}{2}(n_1 + n_2)\frac{1}{2}(n_1 - n_2).$$

\hspace{1cm} (2.8)

Combining the Schwinger $\hat{S}^0$ operator with the three Schwinger spin operators, the Schwinger representation of $U(2)$ is defined by the generators:

$$\hat{S}^j := \frac{1}{2} \hat{a}^H \sigma_j \hat{a} \hspace{1cm} j = 0, x, y, z$$

\hspace{1cm} (2.9)

where $\sigma_j$ is the identity or a Pauli matrix, respectively, and $\hat{a} = (\hat{a}_1, \hat{a}_2)^T$ and $\hat{a}^H = (\hat{a})^T = (\hat{a}_1^T, \hat{a}_2^T)$ are vectors of operators.

### B. Local interferometric operations and the Schwinger $U(2)$ map

To describe local interferometric symmetries of optical Gaussian states, we first note that only Hamiltonians that are quadratic in the quadrature operators correspond to Gaussian unitary transformations—i.e., unitary operations that preserve the Gaussian form of the state.

There are two varieties of quadratic Hamiltonians: compact, which correspond to transformations that preserve the total photon number, and non-compact, which do not preserve the total photon number. In this work we are concerned with passive transformations—i.e., those that are generated by compact Hamiltonians. These can be employed with interferometry (beamsplitters and phase shifters). A set of operators that generate passive transformations on $n$ qumodes are known to correspond to the group $U(n)$, a subgroup of $Sp(2n, \mathbb{R})$ [39].

A general compact Hamiltonian for a system of $n$ qumodes has the following form:

$$\sum_{i,j=1}^{n} \Pi_{ij} \hat{a}_i^\dagger \hat{a}_j,$$

\hspace{1cm} (2.10)

where $\Pi_{ij} \in \mathbb{C}$ and $\Pi_{ij} = \Pi_{ji}^\dagger$ [40]. Considering a system of just two qumodes, the compact Hamiltonian in expression (2.10) can be written as a real linear combination of the Schwinger $U(2)$ operators of Equation (2.9). Thus, these operators generate the group of $U(2)$ transformations on the pair of qumodes.

The $\hat{S}^z$ and $\hat{S}^y$ generators correspond to two-mode mixing operators [40] that couple the qumodes together. Such operators can be modelled as different beamsplitter configurations [26]. The $\hat{S}^z$ generator—the relative-phase-shift operator—corresponds to an equal and opposite phase shift being applied to each qumode. The fourth generator, $\hat{S}^0$, imparts an equal phase shift on both qumodes. The unitary generated by this operator consists of two individual qumode phase-space rotation operators $\hat{R} = \exp(-i\theta \hat{a}_i^\dagger \hat{a}_i)$, where $i$ denotes qumode $i$. This operator acting on an eigenstate of the number operator simply results in an overall phase factor of $e^{-i\theta n}$. However, in general, such an operator has a nontrivial effect on a state. Considering the Heisenberg action of $\hat{R}(\theta)$ on the quadrature operators,

$$\hat{R}(\theta)^\dagger \hat{q} \hat{R}(\theta) = \hat{q} \cos(\theta) + \hat{p} \sin(\theta),$$

\hspace{1cm} (2.11)

$$\hat{R}(\theta)^\dagger \hat{p} \hat{R}(\theta) = \hat{p} \cos(\theta) - \hat{q} \sin(\theta),$$

\hspace{1cm} (2.12)

it is clear that this operator rotates the state in phase space, which is nontrivial if the state is anything but a number state.

Concerning some previous investigations of interferometry using group theory [25, 26, 41], it has been sufficient to only use a representation of $SU(2)$. This is because in those cases the effects of interest were either interference effects (that
only depend on the relative phase difference between qumodes and are thus not altered by the $S^0$ generator) and effects that can be measured with photodetectors (photodetectors are insensitive to the transformations generated by $S^0$). We are nonetheless interested in the full Schwinger $U(2)$ representation, which enables an arbitrary interferometric $U(2)$ transformation to be written in terms of the Schwinger $U(2)$ generators as

$$\hat{U}(\vec{\theta}) = \exp[-i\vec{\theta} \cdot \vec{\hat{S}}],$$

(2.13)

where $\vec{\hat{S}} = (\hat{S}^x, \hat{S}^y, \hat{S}^z, \hat{S}^0)$, $\vec{\theta} = (\theta_x, \theta_y, \theta_z, \theta_0)$, and $\theta_i \in \mathbb{R}$ for $i = x, y, z, 0$.

Generalising to an $n$-qumode system, the compact Hamiltonian of expression (2.10) can be written as a real linear combination of Schwinger $U(2)$ operators acting on all possible qumode pairings. Any $U(n)$ transformation generated by these operators can be decomposed into a sequence of $U(2)$ operations acting on two-qumode subspaces of the underlying $n$-qumode Hilbert space [42].

An arbitrary unitary passive interferometric operation on $n$ qumodes can thus be interpreted as a real linear combination of Schwinger $U(2)$ elements which in turn generate the total $U(n)$ transformation.

III. MOTIVATION

The Schwinger representation has been useful when applied to a range of topics in physics [43–45] and, more specifically, in quantum optics. Specific applications include demonstrating the $SU(2)$ symmetry of a beamsplitter [25, 26], whereby individual beamsplitter or phase-shift operations were written in terms of the Schwinger $SU(2)$ generators. It has been a useful tool in exploring tests of local realism in macroscopic systems, whereby four qumodes can be paired together into two entangled spins, providing a test bed for various Bell inequalities [27–31]. The Schwinger representation also naturally lends itself to analysing the polarization state of quantum light, as the Schwinger operators acting on a pair of horizontally and vertically polarized qumodes are the quantum Stokes operators [33].

In this work, we apply the Schwinger representation of $U(2)$ to various optical Gaussian states on an arbitrarily large number of qumodes. In some cases this can be interpreted in terms of the Schwinger representation of $SU(2)$, which enables interpretation of optical Gaussian states as entangled Schwinger spins. Considering both the wide variety of applications of optical Gaussian states and the usefulness of the Schwinger mapping, it is beneficial to strengthen the theoretical connection between the two. This research aims to enhance intuition about the Schwinger mapping and its potential uses concerning optical multi-qumode Gaussian states.

As we show in Section V, the Schwinger representation of $U(2)$ provides insight into optical Gaussian states. Nullifiers for optical Gaussian states that are written in terms of Schwinger $U(2)$ operators generate passive interferometric symmetries for those corresponding states. Throughout this work, we will derive various interferometric symmetries for optical Gaussian states. Some of these states are useful resources for quantum teleportation and measurement-based QC. The inherent symmetry of an entangled resource state used for teleportation is of interest as it enables gate teleportation of certain gates with minimal noise. Before we describe this process, we briefly explain quantum teleportation in the discrete-variable and continuous-variable cases, including gate teleportation.

Quantum teleportation in the discrete-variable (qubit) case involves the apparent transportation of an unknown qubit to a different physical location simply via a Bell measurement on that qubit and another qubit that is part of an EPR pair, along with classical communication [46]. In addition to simply teleporting the input qubit, it is possible to apply quantum gates to the state being teleported by applying a unitary gate to one of the EPR-pair qubits and then proceeding with the regular teleportation protocol [47]. A general gate teleportation scheme is depicted in Figure 1, where the gate $\hat{U}$ is applied to the second qubit of the EPR pair, which results in the original unknown state $|\psi\rangle$ being teleported, as well as having $\hat{U}$ applied to it—i.e., $|\psi'\rangle = \hat{U}|\psi\rangle$ [47].

The teleportation process just described is possible using any maximally entangled two-qubit pair, such as the Bell states. Each of these states have an inherent symmetry—i.e., a pair of operations

![FIG. 1. Circuit diagram for the gate teleportation of $\hat{U}$ onto the input state $|\psi\rangle$. Bob applies $\hat{U}$ to his qubit, followed by Alice performing a Bell basis measurement (B) on her two qubits. She then feeds two bits of classical information to Bob, that enables him to perform a corrective unitary ($\hat{C}$) on his qubit such that $|\psi'\rangle = U|\psi\rangle |47\rangle$.](image-url)
This state is invariant under $\hat{U}$ such that applying $\hat{U}$ to his qubit, Alice measures her Bell pair in a rotated basis, indicated by the dashed lines. She then feeds two bits of classical information to Bob, which enables him to perform a corrective unitary ($\hat{C}$) (that in general, depends on $\hat{U}$) on his qubit such that $|\psi'\rangle = \hat{U}|\psi\rangle$.

which, when performed together, leave the state invariant. For the Bell pair $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, this state is invariant under $\hat{U}^T \otimes \hat{U}^\dagger$:

$$\hat{U}^T \otimes \hat{U}^\dagger |\beta_{00}\rangle = |\beta_{00}\rangle,$$

(3.1)

where $\hat{U}$ is any unitary operator and $\hat{U}^T$ is the transpose of $\hat{U}$ in the computational basis. The symmetry relation of Equation (3.1) implies that

$$\left(\hat{U}^T \otimes I\right) |\beta_{00}\rangle = (I \otimes \hat{U}) |\beta_{00}\rangle.$$

(3.2)

This symmetry allows a certain freedom to the teleportation scheme, where, rather than applying $\hat{U}$ to the second of the EPR qubits, one can apply $\hat{U}^T$ to the first EPR qubit, or, as depicted in Figure 2, one can equivalently rotate the measurement basis and not perform any unitary directly. This equivalence enables unitary operations to be performed simply by measuring in a rotated basis, which could be much more practical and less noisy than performing $\hat{U}$ on one of the EPR qubits.

A direct analogue of the qubit-based teleportation scheme holds for the teleportation of a qumode, whereby the EPR pair is replaced with a two-mode squeezed (TMS) state [48, 49]. A TMS state is an entangled, bipartite Gaussian state of two qumodes that exhibits EPR-like correlations [50]. In this setting, Alice measures the joint quadratures on her two qumodes by first sending them through beamsplitters and then performing individual qumode measurements, acquiring two real-valued measurement outcomes. She then classically communicates them to Bob, who performs a phase-space-displacement operation $\hat{D}(q_0, p_0)$ on his qumode that depends on those outcomes. Continuous-variable gate teleportation is also possible [51], whereby a quantum optical gate $\hat{U}$ can be effected on the teleported qumode simply by Bob performing $\hat{U}$ on his half of the TMS state. This is depicted in Figure 3, where Alice and Bob each have a qumode from a TMS state and Bob performs $\hat{U}$ on his half. Alice then measures the joint quadratures on her two qumodes, and sends Bob two classical numbers $q_0$ and $p_0$. Bob then applies a displacement operation $\hat{D}(q_0, p_0)$, which depends on the classical numbers Alice sent as well as the unitary that he performed [51].

TMS states are only maximally entangled in the infinite squeezing limit, which is unphysical as this requires infinite energy. Thus, quantum teleportation of continuous-variable states is always restricted to using non-maximally entangled states, resulting in a degree of error or imperfect teleportation where the fidelity of the teleportation depends on the level of squeezing in the system.

A. Minimal-noise gate teleportation

As TMS states are not maximally entangled, any CV gate teleportation scheme never achieves perfect teleportation. Rather, there is always a Gaussian noise term that distorts the information passing through the teleportation channel in a way that depends specifically on which gate is being implemented. The order with which the gate is performed relative to the teleportation—i.e., whether the gate is performed before the state goes through the teleportation channel or afterwards—also has a non-trivial effect on the way the Gaussian noise propagates through the system. In this section, we show that an inherent symmetry of TMS states enables certain gates (those corresponding to the symmetries) to be applied to the input qumode with no additional noise, i.e., the noise introduced to the system is the same regardless of when the gate is performed. We call this a minimal-noise gate teleportation, as it enables a gate to be teleported onto a qumode but introduces the same amount of noise to the system as simply teleporting an identity gate.

We can exploit the inherent symmetry of TMS states to enable minimal-noise gate teleportation. A TMS state $|\text{TMS}\rangle$ is invariant under the following individual single-qumode rotations in phase space (which we prove in Section VII B 1):

$$\hat{R}_1(\theta) \otimes \hat{R}_2(-\theta) |\text{TMS}\rangle = |\text{TMS}\rangle,$$

(3.3)

where the rotation operator is $\hat{R}_i(\theta) = e^{-i\theta a_i^\dagger a_i}$, $0 \leq \theta < 2\pi$, and $a_i$ denotes the qumode $i$. This operation is generated by the Schwinger $\hat{S}^z$ operator and thus corresponds to a rotation about the z-axis in the Schwinger spin picture.

The symmetry of a TMS state under phase space rotations implies that an application of $\hat{R}(\theta)$ on one of the qumodes from the TMS state used
in the CV gate teleportation scheme (Figure 3) is equivalent to applying the same operator to the other qumode from the same pair. This implies that the phase-space-rotation gate can be implemented either after the teleportation (on Bob’s qumode) or before, simply via rotating Alice’s measurement basis. This equivalence is illustrated in Figure 4, where the unknown state to be teleported is |ψ⟩, the entangled resource state is a TMS state, and the phase-space-rotation operator \( R(\theta) \) is implemented on |ψ⟩.

Considering the usefulness of the symmetry of a TMS state, it is worthwhile to explore inherent symmetries in larger structures which are also key components of teleportation and measurement-based QC architectures. For entangled resource states on more than two qumodes, these symmetries could be used to develop generalised teleportation schemes that teleport multi-qumode gates. In Sections VII B, VIII B, and IX C, we derive inherent symmetries of multi-qumode entangled optical Gaussian states, including a resource for measurement-based QC.

IV. THE NULLIFIER FORMALISM FOR GAUSSIAN PURE STATES

A. Continuous-variable cluster states

The nullifier formalism is well suited for analysis of CV cluster states [12, 17, 35]. A CV cluster state is a Gaussian analogue of the qubit cluster state [52] and serves as a resource for CV measurement-based QC [53].

In a CV setting, the unit of information for computation is a qumode, with the computational and conjugate basis being that of position \( \hat{q} \) and momentum \( \hat{p} \), respectively [16]. The space of a single qumode is an infinite-dimensional Hilbert space, spanned by orthogonal states \( |s⟩_q \) (\( s \in \mathbb{R} \)) with orthogonality condition \( \langle r|s⟩_q = \delta(r-s) \), where \( \delta \) is the Dirac delta function. Position and momentum eigenstates satisfy the following eigenvalue relations

\[
\hat{q}|s⟩_q = s|s⟩_q, \quad \hat{p}|s⟩_p = s|s⟩_p .
\]

Qubit cluster states can be completely specified by the generators of their stabilizer group—i.e., the group of operators \( G \) such that \( G|ψ⟩ = |ψ⟩ \). Similarly, CV cluster states can be completely specified through the stabilizer formalism [12, 54]. The analogue of the Pauli group for a single qumode is the Heisenberg-Weyl group [55], which consists of the phase space-displacement operators \( X(s) = e^{-is\hat{p}} \) and \( Z(\theta) = e^{i\theta\hat{q}} \). As this is a Lie group, we can define a CV cluster state by the Lie algebra that generates its stabilizers. For a system of \( n \) qumodes, such an algebra is spanned by \( n \) independent operators \( N_i \), for \( i = 1, ..., n \), which are called nullifiers, as they have the system as a zero eigenstate.

A canonical CV cluster state is defined as a collection of zero-momentum eigenstates \( |0⟩_p \) on a square lattice which have had a controlled-Z gate applied between adjacent states [17]. However, momentum or position eigenstates are not normalizable and thus are not physical. They can be approximated by squeezed states of light, which are quantum states of light whose quadratures are correlated such that a small degree of noise can be present in one quadrature, at the cost of increased noise in the other. The single-qumode squeezing operator is \( S(\xi) = e^{\frac{i}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^2)} \), where \( \xi = re^{i\theta} \) is the squeezing parameter, \( 0 \leq r < \infty \) and \( 0 \leq \theta < 2\pi \). Momentum eigenstates are attained in the limit of infinite squeezing:

\[
\lim_{\xi \to \infty} S(\xi)|0⟩ = |0⟩_p .
\]

Approximate CV cluster states thus consist of a collection of squeezed qumodes that have undergone an entangling operation.

B. Mathematical formalism for Gaussian pure states

Any Gaussian pure state (including approximate CV cluster states) can be defined uniquely (up to phase-space displacements) by an undirected graph with complex-valued edge weights, given by the matrix \( \mathbf{Z} = \mathbf{V} + i\mathbf{U} \), where \( \mathbf{U} = \mathbf{U}^T > 0 \) and \( \mathbf{V} = \mathbf{V}^T \) [35]. Every such \( \mathbf{Z} \) also defines a unique Gaussian pure state. The position-space wavefunction for a Gaussian pure state on \( n \)
interfering a parametric down-conversion or, equivalently, by TMS states can be generated by non-degenerate partite entangled Gaussian state on two qumodes. The uniqueness of $K$ is related to $Z$ by:

$$\psi_Z(q) = \frac{\text{det}(U)_1^{1/4}}{\pi^{n/4}} \exp \left(\frac{i}{2} q^T Z q \right). \quad (4.3)$$

The nullifiers for a Gaussian pure state are also captured within this formalism. For an $n$-qumode Gaussian pure state $|\psi_Z\rangle$, $n$ independent nullifiers can be written succinctly as

$$\langle \hat{p} - Z \hat{q} | \psi_Z \rangle = 0, \quad (4.4)$$

where $\hat{p} = (\hat{p}_1, ..., \hat{p}_n)^T$ and $\hat{q} = (\hat{q}_1, ..., \hat{q}_n)^T$ are column vectors of quadrature operators.

We now introduce the complex, symmetric matrix $K$ that is related to $Z$ by [35]

$$K = (I + iZ)(I - iZ)^{-1} \quad (4.5)$$

and thus also uniquely specifies the state. The symmetry of $K$ is an immediate consequence of the symmetry of $Z$. The requirements on $U$ and $V$ mentioned above induce the following restriction on $K$:

$$||K|| < 1, \quad (4.6)$$

where $||K||$ is the spectral norm (i.e., the largest singular value) of $K$. For proof of this, see Appendix B.

The uniqueness of $K$ enables a useful graphical representation of the Gaussian pure state that it defines, whereby the graphical depiction of the state is simple, yet precise, as it represents the wavefunction for the corresponding state [35]. We illustrate this graphical formalism with an example using a TMS state and its corresponding $K$ matrix. As mentioned previously, a TMS state is a bipartite entangled Gaussian state on two qumodes. TMS states can be generated by non-degenerate parametric down-conversion or, equivalently, by interfering a $\hat{q}$-squeezed state and a $\hat{p}$-squeezed state on a 50:50 beamsplitter. The $K$ matrix for a TMS state is [35]

$$K = \begin{pmatrix} 0 & \tanh(\alpha) \\ \tanh(\alpha) & 0 \end{pmatrix}, \quad (4.7)$$

where $\alpha > 0$ is an overall squeezing parameter. Figure 5 displays the graphical representation of $K$, whereby qumodes are depicted by nodes and the edge weight corresponds to the off-diagonal terms in $K$.

**Definition 1.** Any Gaussian pure state can be uniquely specified by the complex, symmetric matrix $K$, which we define to be the adjacency matrix of the state.

The nullifier relation of Equation (4.4) can be written in terms of $K$ and the qumode annihilation operators. First, we multiply Equation (4.4) by $i$ on both sides,

$$(i\hat{p} - iZ\hat{q})|\psi_Z\rangle = 0. \quad (4.8)$$

We then use the relation between quadrature operators and annihilation operators, $\hat{a} = \frac{1}{\sqrt{2}} (\hat{q} + i\hat{p})$, to rewrite this expression as

$$(\hat{a} - \hat{a}^\dagger) - iZ(\hat{a} + \hat{a}^\dagger)|\psi_Z\rangle = 0 \quad (4.9)$$

$$(I - iZ)\hat{a} - (I + iZ)\hat{a}^\dagger|\psi_Z\rangle = 0. \quad (4.10)$$

Multiplying both sides of this equation by $(I - iZ)^{-1}$ results in the expression

$$(\hat{a} - K\hat{a}^\dagger)|\psi_Z\rangle = 0. \quad (4.11)$$
For $K$ defined through Equation (4.5), the state defined by $K$ is the same as the state defined by $Z$ ($|\phi_K\rangle = |\psi_Z\rangle$) and we can write the nullifier relation for a Gaussian pure state on $n$ qumodes in terms of $K$:

$$\langle \hat{a} - K\hat{a}^\dagger |\phi_K\rangle = 0.$$ (4.12)

This is written in terms of the column vectors of annihilation and creation operators $\hat{a} = (\hat{a}_1, ..., \hat{a}_n)^T$ and $\hat{a}^\dagger = (\hat{a}_1^\dagger, ..., \hat{a}_n^\dagger)^T$.

Now that we have provided the reader with the relevant graphical formalism for Gaussian pure states as well as the nullifier relations in terms of the adjacency matrix $K$, we proceed to define Schwinger nullifiers for Gaussian pure states and what these entail. We then present the main result of this paper, which is a mathematical connection between the adjacency matrix for a Gaussian pure state and the corresponding Schwinger nullifiers.

V. SCHWINGER NULLIFIERS FOR A GAUSSIAN PURE STATE

Definition 2. Let $|\phi_K\rangle$ be a Gaussian pure state on $n$ qumodes. A Schwinger nullifier for $|\phi_K\rangle$ is a real linear combination of Schwinger $U(2)$ operators that is a nullifier for $|\phi_K\rangle$.

Lemma 3. Any real linear combination of Schwinger $U(2)$ operators acting on $n$ qumodes can be written concisely as

$$\hat{a}^H \hat{M} \hat{a},$$ (5.1)

where $\hat{M}$ is an $n \times n$ Hermitian matrix, the column vector $\hat{a} = (\hat{a}_1, ..., \hat{a}_n)^T$, and the row vector $\hat{a}^H = (\hat{a}_1^\dagger, ..., \hat{a}_n^\dagger)^T$. Furthermore, any expression of this form corresponds to some real linear combination of Schwinger $U(2)$ operators acting on $n$ qumodes.

Proof. As discussed in Section II B, for a system of $n$ qumodes, real linear combinations of Schwinger $U(2)$ operators correspond to the compact Hamiltonians for the system. The most general compact Hamiltonian for a system of $n$ qumodes has the following form [40]:

$$\sum_{i,j=1}^{n} M_{ij} \hat{a}_i^\dagger \hat{a}_j,$$ (5.2)

where $M_{ij}$ is a complex number, and $M_{ij} = M_{ji}^*$. Using the row and column vectors of creation/annihilation operators defined above, we can rewrite expression (5.2) as

$$\sum_{i,j=1}^{n} M_{ij} \hat{a}_i^\dagger \hat{a}_j = \hat{a}^H \hat{M} \hat{a},$$ (5.3)

where $\hat{M}$ is a Hermitian matrix with entries $M_{ij}$.

To prove the converse, consider a pair $(r, s)$ of the $n$ qumodes. Let $\sigma^{(r,s)}_j$ be the $n \times n$ Hermitian matrix whose entries are all zero except for $(r, r), (s, s), (r, s), (s, r)$, and $(s, s)$, which contain the corresponding entries from the Pauli matrix $\sigma_j$. Then, a Schwinger $U(2)$ operator acting on that pair can be written concisely as

$$\hat{S}^{(r,s)}_{j} := \frac{1}{2} \hat{a}^H \sigma^{(r,s)}_j \hat{a}, \quad j = 0, x, y, z.$$ (5.4)

The collection of all $\sigma^{(r,s)}_j$ for all index pairs $(r, s)$ and all $j = 0, x, y, z$ form an overcomplete basis for all $n \times n$ Hermitian operators. As such, we can write

$$\hat{M} = \sum_{(r,s)} \sum_{j} c^{(r,s)}_j \sigma^{(r,s)}_j$$ (5.5)

for some real constants $c^{(r,s)}_j$. When $\hat{M}$ is sandwiched between the vectors of creation and annihilation operators, we have

$$\hat{a}^H \hat{M} \hat{a} = \sum_{(r,s)} \sum_{j} 2^{i_0} \hat{S}^{(r,s)}_{r,s}$$ (5.6)

which is explicitly a real linear combination of Schwinger $U(2)$ operators. □

Theorem 4. Let $|\phi_K\rangle$ be a Gaussian pure state on $n$ qumodes. Schwinger nullifiers for this state generate passive interferometric symmetries that apply to the state—i.e., passive interferometric operations that leave the state invariant.

Proof. Let $\hat{N}$ be a Schwinger nullifier for $|\phi_K\rangle$. Therefore, it is a real linear combination of Schwinger $U(2)$ operators. Consider the unitary operator $\hat{U}(\theta) = e^{-i\theta \hat{N}}$, where $\theta \in \mathbb{R}$. This unitary is generated by Schwinger $U(2)$ elements and therefore is a passive interferometric operation on two qumodes. As $\hat{N}$ is a nullifier for $|\phi_K\rangle$, $\hat{U}(\theta)$ is a stabilizer for $|\phi_K\rangle$—i.e., $\hat{U}(\theta)|\phi_K\rangle = |\phi_K\rangle$—which is an operator that leaves the state invariant. □

Theorem 5. Let $|\phi_K\rangle$ be a Gaussian pure state on $n$ qumodes. Let $\hat{M}$ be an $n \times n$ Hermitian matrix. $\hat{a}^H \hat{M} \hat{a}$ is a Schwinger nullifier for $|\phi_K\rangle$ if and only if $\hat{M} = - (\hat{M} K)^T$.

Proof. Starting with the known nullifier relation for a Gaussian pure state,

$$\langle \hat{a} - K\hat{a}^\dagger |\phi_K\rangle = 0,$$ (5.7)

we note that multiplying this expression on the left by $\hat{a}^H \hat{M}$ results in the following (which is also a nullifier relation):

$$\langle \hat{a}^H \hat{M} - \hat{a}^H \hat{M} K \hat{a}^\dagger |\phi_K\rangle = 0.$$ (5.8)
By Lemma 3, the first term in parentheses is a real linear combination of Schwinger $U(2)$ operators. The second term contains $\hat{a}_i^\dagger \hat{a}_j$ and $\hat{a}_i \hat{a}_j$, terms which correspond to active transformations and are not linear combinations of Schwinger $U(2)$ operators. Thus, in order for expression (5.8) to be a Schwinger nullifier, the second term must vanish.

Assume $MK = -(MK)^T$. We now show that the second term in (5.8) thus vanishes. We use the result that for any anti-symmetric, complex matrix $A$, $x^T A x = 0$ for any vector $x$, where $x^T$ denotes the transpose of $x$. Therefore, $\hat{a}^H MK \hat{a}^\dagger = 0$. Although we are dealing with vectors of operators in this case, we can still use this result as all the operators commute.

Conversely, assume $MK \neq -(MK)^T$. The second term in (5.8) never equals zero. To see this, decompose $MK$ into its symmetric and anti-symmetric components:

$$MK = \frac{1}{2}[(MK + (MK)^T) + (MK - (MK)^T)].$$

(5.9)

The second term in (5.8) can then be written as:

$$\frac{1}{2} \hat{a}^H [MK + (MK)^T] \hat{a}^\dagger + \frac{1}{2} \hat{a}^H [MK - (MK)^T] \hat{a}^\dagger.$$  

(5.10)

The second term in this expression vanishes (according to the same anti-symmetric argument used previously), resulting in

$$\frac{1}{2} \hat{a}^H [MK + (MK)^T] \hat{a}^\dagger.$$  

(5.11)

This expression will never vanish. To see this, let $MK + (MK)^T = A$ and note

$$\hat{a}^H A \hat{a}^\dagger = \sum_{i,j} \hat{a}_i^\dagger \hat{a}_j A_{ij}.$$  

(5.12)

There are two cases in which this expression could potentially vanish. The first is if $A = 0$, which is not the case as we are assuming $MK \neq -(MK)^T$. The second case would be if $\forall i,j$:

$$A_{ij} \hat{a}_i^\dagger \hat{a}_j + A_{ji} \hat{a}_j^\dagger \hat{a}_i = 0,$$

(5.13)

but since $A_{ij} = A_{ji}$, by assumption there exists some $i,j$ such that $A_{ij} \neq 0$, implying

$$A_{ij} (\hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i) \neq 0.$$  

(5.14)

□

**Theorem 6.** Let $|\phi_K\rangle$ be a Gaussian pure state on $n$ qumodes with bipartite adjacency matrix $K$. Schwinger nullifiers can be derived for this state, simply using the singular value decomposition of $K$ and an additional diagonal matrix.

**Proof.** Starting with bipartite $K$:

$$K = \begin{pmatrix} 0 & K_0 \\ K_0^T & 0 \end{pmatrix},$$

(5.15)

we decompose $K_0$ into its singular value decomposition:

$$K = \begin{pmatrix} 0 & U \Sigma V^H \\ V^* \Sigma U^T & 0 \end{pmatrix},$$

(5.16)

where $U$ and $V$ are unitary $n \times n$ matrices, $\Sigma$ is an $n \times n$ diagonal matrix with non-negative values, and $V^H$ denotes the conjugate transpose of $V$. We can then write $K$ as:

$$K = (U \oplus V^*)(\sigma_x \otimes \Sigma)(U^T \oplus V^H),$$

(5.17)

where $\sigma_x$ is the Pauli-$x$ matrix. If we define $M$ to be of the form

$$M = (U \oplus V^*)(\sigma_x \otimes D)(U^H \oplus V^T),$$

(5.18)

where $D$ is any symmetric matrix that commutes with $\Sigma$ and $\sigma_x$ is the Pauli-$z$ matrix, the equation $MK = -(MK)^T$ holds. If the equation $MK = -(MK)^T$ holds, $\hat{a}^H \hat{M} \hat{a}$ is a Schwinger nullifier for $|\phi_K\rangle$. Therefore, this method can be used to derive Schwinger nullifiers for any Gaussian pure state with bipartite $K$. □

**VI. H-GRAPH STATES**

There exists a particular class of Gaussian pure states whose adjacency matrix $K$ depends on a real, symmetric matrix $G$:

$$K = \tanh(\alpha G),$$

(6.1)

where $\alpha > 0$ is an overall squeezing parameter. These states have been previously discussed in terms of the $Z$ matrix formalism [35], where $Z = i e^{-2\kappa G}$. For proof of Equation (6.1) in light of this formalism, see Appendix A. Such states can be generated via parametric down conversion in an optical parametric oscillator [56], whereby the multi-qumode squeezing interaction is defined by the Hamiltonian

$$\mathcal{H} = i \hbar \sum_{m,n} G_{mn}(\hat{a}_m^\dagger \hat{a}_n^\dagger - \hat{a}_m \hat{a}_n),$$

(6.2)

where $\kappa$ is a global coupling strength, and $G$ is a matrix that specifies the multi-qumode interactions. Such states are called $H$-graph states [18, 19], as they are defined in terms of a matrix (or graph) $G$ that determines the multimode-squeezing Hamiltonian. H-graph states can also be generated via different optical methods. For
instance, they can be generated by sending pairs of TMS states through a series of beamsplitters [38].

The class of H-graph states with self-inverse and bipartite matrices are particularly useful for measurement-based QC as it has been shown that these states are equivalent to CV cluster states up to certain phase shifts [18]. H-graph states of this form that are on a square-lattice topology are universal resources for measurement-based QC [18]. Due to the scalable, compact nature [57] with which these states are prepared, H-graph states with bipartite and self-inverse adjacency matrices are of particular interest in continuous-variable implementations of QC [19, 38, 56]. H-graph states with self-inverse and can lead to theoretical insight of continuous-variable QC and quantum simulation of topologically ordered condensed-matter systems [58].

In the next section, we apply the method of Theorem 6 to H-graph states with bipartite and self-inverse G and show that Schwinger nullifiers can be easily derived in this case.

A. Method to derive Schwinger nullifiers for H-graph states with bipartite and self-inverse adjacency matrices

When G is self-inverse, K can be written as

$$K = \tanh(\alpha)G.$$  \hspace{1cm} (6.3)

For proof of this, see Appendix A. When G is also bipartite,

$$K = \tanh(\alpha) \begin{pmatrix} 0 & G_0^* \\ G_0 & 0 \end{pmatrix},$$  \hspace{1cm} (6.4)

which allows a straightforward derivation of M (as defined in Theorem 6) as:

$$M = (U \oplus V^*)(\sigma_z \otimes D)(U^H \oplus V^T).$$  \hspace{1cm} (6.5)

Here, U, V and Σ are the singular value decomposition matrices of G_0, and D is any symmetric matrix that commutes with Σ.

This method can be used to derive local Schwinger nullifiers for any H-graph state with bipartite, self-inverse G. One example of such a state is the dual-rail quantum wire, which is a useful resource for CV measurement-based QC and is discussed in Section IX.

VII. TWO-QUMODE GAUSSIAN PURE STATES

Considering a Gaussian pure state on two qumodes, Theorem 5 enables derivation of the class of two-qumode Gaussian pure states that is nullified by an arbitrary Schwinger U(2)—i.e., $\alpha \hat{S}^x + \beta \hat{S}^y + \gamma \hat{S}^z + \delta \hat{S}^0$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. By Theorem 4, this derivation enables one to calculate what two-qumode Gaussian pure states are left invariant by any passive interferometric operation.

A. Two-qumode Gaussian pure states nullified by an arbitrary Schwinger U(2) element

Theorem 7. Let $\hat{U} = \exp[-i(\alpha \hat{S}^0 + \beta \hat{S}^x + \gamma \hat{S}^y + \delta \hat{S}^z)]$ be a two-qumode interferometric operation, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The class of Gaussian pure states on two qumodes that are left invariant by $\hat{U}$ have corresponding K matrices that satisfy the following constraints:

$$\begin{align*}
(\alpha + \delta)k_{11} + (\beta - i\gamma)k_{12} &= 0, \\
(\beta + i\gamma)k_{12} + (\alpha - \delta)k_{22} &= 0, \\
2\alpha k_{12} + (\beta - i\gamma)k_{22} + (\beta + i\gamma)k_{11} &= 0.
\end{align*}$$  \hspace{1cm} (7.3)

Proof. The arbitrary Schwinger U(2) element $\alpha \hat{S}^0 + \beta \hat{S}^x + \gamma \hat{S}^y + \delta \hat{S}^z$ (where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$) can be specified by a $2 \times 2$ Hermitian matrix M where

$$M = \sum_{j=0}^{3} m_j \sigma_j,$$  \hspace{1cm} (7.4)

and $m_j \in \mathbb{R}$. To see this, recall that each Schwinger operator can be written as

$$\hat{S}^k := \frac{1}{2} \hat{a}^H \sigma_k \hat{a}, \hspace{1cm} k = 0, x, y, z,$$  \hspace{1cm} (7.5)

where $\hat{a} = (\hat{a}_1, \hat{a}_2)^T$ and $\hat{a}^H = (\hat{a}^\dagger)^T = (\hat{a}_1^\dagger, \hat{a}_2^\dagger)$. Thus, setting $m_j$ to $\alpha, \beta, \gamma$ and $\delta$ for $j = 0, ..., 3$, respectively, corresponds to $\alpha \hat{S}^0 + \beta \hat{S}^x + \gamma \hat{S}^y + \delta \hat{S}^z$. An arbitrary Gaussian pure state on two qumodes can be specified by a complex, symmetric adjacency matrix K:

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix},$$  \hspace{1cm} (7.6)

where $k_{11}, k_{22}, k_{12} \in \mathbb{C}$. To derive the form of K such that the state specified by K is nullified by the operator specified by M, we require that $MK = -(MK)^T$. This results in Equations (7.1) through to (7.3). By Theorem 5, if $MK = -(MK)^T$, the operator corresponding to M is a nullifier for the state specified by K. $\hat{U}$ is therefore a stabilizer for the state specified by K. \hspace{1cm} $\square$

For any desired two-qumode interferometric operation, this result enables direct derivation of the...
class of Gaussian pure states that is left invariant by this operation. In the next section we apply this theorem to a few examples of simple two-qumode passive interferometric operations.

B. Examples of interferometric symmetries on two-qumode Gaussian pure states

In this section, we use the results of Theorem 7 to determine what classes of Gaussian pure states on two qumodes are nullified by each of the three Schwinger spin operators (operators of Equation (2.1)). Equivalently, we determine what classes of Gaussian pure states on two qumodes are left invariant by the interferometers that are generated by these operators. In the Schwinger spin picture, these interferometers can be interpreted as rotations of the spin about a given axis.

1. The relative-phase-shift operator

Consider the two-qumode interferometric operation

$$\hat{U}(\theta) = \exp[-i\theta\hat{S}^z],$$

(7.7)

which, as described in Section (II B), imparts an equal and opposite phase shift to the qumodes. To derive the class of Gaussian pure states on two qumodes that is left invariant under $\hat{U}(\theta)$, we use Theorem 7 and set $\alpha = \beta = \gamma = 0$ and $\delta = \theta = 1$. This results in a $K$ matrix satisfying

$$k_{11} = k_{22} = 0,$$

(7.8)

with no constraint on $k_{12}$. A general $K$ matrix that satisfies this condition can be written as

$$K = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix},$$

(7.9)

where $x \in \mathbb{C}$ and, as discussed in Section IV B, $|x| < 1$. Thus, the class of Gaussian pure states on two qumodes that is left invariant under $\exp[-i\theta\hat{S}^z]$ has adjacency matrices of the form of (7.9). The most general expression for such a matrix is

$$K = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & |x| \\ |x| & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

(7.10)

As $|x| < 1$, we can write $K$ as

$$K = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & \tanh(\alpha) \\ \tanh(\alpha) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

(7.11)

where $\alpha \geq 0$. The form of this $K$ suggests that the class of two-qumode Gaussian pure states that are left invariant under $\exp[-i\theta\hat{S}^z]$ is the class of TMS states that have had a phase shift of $\theta$ applied to qumode 2, where $\theta \in \mathbb{R}$. To see this, note that the $K$ matrix transforms as follows under phase shifts [35]:

$$K' = e^{-i\theta}Ke^{-i\theta},$$

(7.12)

where $\theta$ is a diagonal matrix whereby $\theta_{ii}$ specifies the desired phase shift on qumode $i$. Setting $\theta = 0$ in (7.11) provides us with one example of a state that falls within this class. This is a TMS state, whose $K$ matrix was introduced in Section IV B. Another state within this class is a two-qumode CV cluster state, which corresponds to setting $\theta = -\frac{\pi}{2}$ in (7.11) [35]. The relative-phase-shift operator can be decomposed into two single-qumode rotation operators:

$$\exp[-i\theta\hat{S}^z] = \exp\left(i\theta(-\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2)\right)$$

(7.13)

$$= \hat{R}_1(\theta)\hat{R}_2(-\theta),$$

(7.14)

where the subscripts 1 and 2 denote qumodes 1 and 2, respectively. The symmetry relation of a TMS state, which we discussed in Section III A, is therefore confirmed. A corollary of this symmetry is that a TMS state is nullified by the Schwinger $\hat{S}^z$ operator, acting on both qumodes in either order. This result has been shown previously, in the context of determining eigenstates of the various Stokes operators [33]. We have re-derived this result by using Theorem 7 and also making use of the graphical calculus formalism [35], which describes Gaussian pure states by $K$ (and, equivalently, $Z$).

2. Beamsplitters

The operators $\exp[-i\theta\hat{S}^x]$ and $\exp[-i\theta\hat{S}^y]$, where $\theta \in \mathbb{R}$, correspond to different beamsplitter interactions. A two-port lossless beamsplitter can be modelled by a $2 \times 2$ unitary matrix $B$, that transforms the annihilation operators of each qumode:

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = B_{11} \hat{a}_1 + B_{12} \hat{a}_2,$$

(7.15)

The $\exp[-i\theta\hat{S}^x]$ operator corresponds to the following beamsplitter [25]:

$$B = \begin{pmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix},$$

(7.16)

which imparts a phase shift of $\frac{\theta}{2}$ between the reflected components of each qumode [26]. Using Theorem 7 with $\alpha = \gamma = \delta = 0$ and $\beta = \theta = 1$, the $K$ matrix for the two-qumode state that is left
invariant by this beamsplitter has entries satisfying
\[ k_{12} = 0, \quad (7.17) \]
\[ k_{11} = -k_{22}, \quad (7.18) \]
and can therefore be written as
\[ K = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \quad (7.19) \]
Thus, the class of Gaussian pure states on two qumodes that is left invariant by the beamsplitter operation \( \exp[-i\theta S^x] \) has corresponding adjacency matrices given by (7.19), where \( x \in \mathbb{C} \) and \(|x| < 1\). Putting \( K \) into the form of (7.12),
\[ \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad (7.20) \]
reveals that this class consists of all identical two-qumode states that have had a phase shift of \(-\frac{\pi}{2}\) applied to the second qumode. One such example is a pair of qumodes that are identically squeezed along \( \hat{p} \) and \( \hat{q} \).

The \( \exp[-i\theta S^y] \) operator corresponds to the beamsplitter \([25]\)
\[ B = \begin{pmatrix} \cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \end{pmatrix}, \quad (7.21) \]
which doesn’t impart any phase shifts on the input and output qumodes. We use Theorem 7 with \( \alpha = \beta = \delta = 0 \) and \( \gamma = \theta = 1 \), giving a resultant \( K \) matrix of the form:
\[ K = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}. \quad (7.22) \]
Such a state is a product state of two identical qumodes. Thus, the class of two-qumode states consisting of two independent copies of the same pure Gaussian state is left invariant under beamsplitters of the form \( \exp[-i\theta S^y] \), for all \( \theta \in \mathbb{R} \).

3. Combinations of beamsplitters

Consider a rotation of the Schwinger spin along some rotated axis, described by the linear combination of the beamsplitter generators just discussed, \( \cos \theta S^x + \sin \theta S^y \). This would correspond to a beamsplitter that imparts a phase shift of \( \theta \) on the reflected qumodes. We use Theorem 7 to explore this, with \( \alpha = \delta = 0 \), \( \beta = \cos \theta \), \( \gamma = \sin \theta \). These constraints result in the following \( K \):
\[ K = \begin{pmatrix} x & 0 \\ 0 & -e^{i\theta}x \end{pmatrix}, \quad (7.23) \]
which can be put into the following form:
\[ K = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}. \quad (7.24) \]
As discussed previously, the second matrix in this \( K \) corresponds to the class of two qumodes that are out of phase by \( \frac{\pi}{2} \). Thus, the form of \( K \) reveals that the class of two-qumode Gaussian states that are left invariant under a beamsplitter that imparts a phase shift of \( \theta \) on the reflected qumodes are the class of two qumodes that are initially out of phase by \( \frac{\pi}{2} \) and have had a further phase shift of \(-\theta \) applied to the second qumode.

VIII. FOUR-QUMODE STATES

A. Entangled spins

We now discuss four-qumode systems which, when interpreted in the Schwinger representation of \( SU(2) \), can be interpreted as a pair of entangled spins. The matrix representation theory of \( SU(2) \) is a finite-dimensional representation, whereby the spin quantum number \( s \) (\( s=0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \)) labels which representation the system is in (for example, \( s=\frac{1}{2} \) corresponds to a spin-\( \frac{1}{2} \) system). The Schwinger representation is a multiplicity-free, direct sum of all the unitary irreducible representations of \( SU(2) \) \([24]\). A specific representation is established by measuring the total spin operator \( S^0 \) (discussed in Section II A), which projects to a specific spin-\( s \) representation.

Consider a pair of TMS states, depicted according to the graphical formalism in Figure 6a, where qumodes are labelled 1—4. Pairing qumodes 1 and 3 into a spin (labelled spin \( A \)) and qumodes 2 and 4 into a spin (labelled spin \( B \)) results in a pair of entangled Schwinger spins. This is illustrated in Figure 6b. If the squeezing parameter, \( r \), of each TMS state is equal, this state takes the following form in the number basis:
\[ |\psi\rangle = \sum_{n_1, n_2=0}^{\infty} \frac{(\tanh^2 r)^{n_1+n_2}}{(\cosh r)^2} |n_1\rangle|n_1\rangle \otimes |n_2\rangle|n_2\rangle. \quad (8.1) \]
Converting to the Schwinger spin basis, this state can be written \([31]\)
\[ |\psi\rangle = \sum_{s \in \mathbb{N}/2} \frac{1}{\cosh r}^{2s} \sum_{m=-s}^{s} |s, m\rangle_A |s, m\rangle_B. \quad (8.2) \]
This is an entangled state on the Schwinger spins labelled \( A \) and \( B \) (each containing qumodes 1, 3 and 2, 4 respectively).
nullifiers only hold when the squeezing parameter is expected, as TMS states have identical photon total photon numbers in each spin are equal. This is indicated as acting on a single spin where we have introduced the notation where this, note that for a specific down to a finite-dimensional representation—the summation over $s$ in Equation (8.2) disappears, resulting in an overall constant factor that depends on $s$, leaving a state of the form $\sum_{i=0}^{2s} |i\rangle |i\rangle$ [27, 31, 60].

By phase shifting qumode 3 by $\pi$, the transformed spin state is nullified by

\[
\hat{S}_A^x + \hat{S}_B^x, \quad \text{(8.6)}
\]
\[
\hat{S}_A^y - \hat{S}_B^y, \quad \text{(8.7)}
\]
\[
\hat{S}_A^z - \hat{S}_B^z, \quad \text{(8.8)}
\]

where now both Y and Z are correlated and X is anti-correlated. This transformed state can be seen as the Schwinger spin analogue of the Bell pair $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. The transformed pair of TMS states is graphically represented in Figure 7, where we note that the edge weight of the second TMS state has changed sign from positive to negative and thus, the squeezing parameters are no longer identical.

The other two Bell basis states can be generated by performing rotations about different axes on spins $A$ and $B$. First, by performing a rotation of $180^\circ$ about the $y$-axis on spin $A$, we generate a pair of spins whose nullifiers are correlated and anti-correlated analogously to the Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. By performing a further $180^\circ$ rotation about the $z$-axis on spin-$B$ to the pair just described, we generate the Schwinger analogue of the Bell pair $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$. These are also illustrated in Figure 8.

### B. Four-qumode interferometric symmetries

A result of the nullifiers of (8.3), (8.4) and (8.5) is that a pair of TMS states (with the same squeezing parameter) are left invariant by the following four-qumode interferometers:

\[
\hat{U}_x(\theta) = e^{-i\theta(S_{1,3}^x - S_{2,4}^x)}, \quad \text{(8.9)}
\]
\[
\hat{U}_y(\theta) = e^{-i\theta(S_{1,3}^y + S_{2,4}^y)}, \quad \text{(8.10)}
\]
\[
\hat{U}_z(\theta) = e^{-i\theta(S_{1,3}^z - S_{2,4}^z)}, \quad \text{(8.11)}
\]
FIG. 8. Graphical depiction of the adjacency matrices for two distinct pairs of entangled Schwinger spins. (a) Starting with the pair of spins from Figure 7, a 180° degree rotation about the y-axis on spin A results in the Schwinger analogue of the Bell pair \( \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle) \), depicted here. (b) Starting with the state depicted in (a), a further 180° degree rotation about the z-axis on spin B results in the Schwinger analogue of the Bell pair \( \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) \). Note that the edge weights are ± tanh(\( \alpha \)) (blue and yellow, respectively).

where we now denote the precise qumodes that the Schwinger operators act on and \( \theta \in \mathbb{R} \).

The unitary operators of Equations (8.9), (8.10) and (8.11) correspond to two distinct two-qumode interferometers that can be applied in any order to the system. The \( \hat{U}_z(\theta) \) operator of Equation (8.11) corresponds to an equal and opposite phase shift being applied to qumodes 1 and 3 and an equal and opposite phase shift being applied to qumodes 2 and 4. This is a direct consequence of the single TMS-state symmetry discussed in Section VII B 1. Equations (8.9) and (8.10) reveal specific beamsplitters that leave a pair of TMS states (with identical squeezing parameters) invariant. The \( \hat{U}_y(\theta) \) operator corresponds to a beamsplitter applied successively (in any order) to the qumodes (1, 3) and (2, 4) that imparts a phase shift of \( \pm \hat{a}_n \) to the reflected qumodes. The \( \hat{U}_y(\theta) \) operator acts on the same qumode pairings but corresponds to a beamsplitter that doesn’t impart any phase shifts on the qumodes.

IX. N-QUMODE ENTANGLED GAUSSIAN PURE STATES

In this section, we interpret a Gaussian state that has more structure than those explored so far. The dual-rail quantum wire is a Gaussian cluster state with one-dimensional topology. Such a state can be generated through an OPO [18] or, equivalently, by sending a collection of TMS states through beamsplitter operations [38]. Suitably weaving multiple dual-rail quantum wires together results in a structure that is equivalent to a 2D cluster state and is thus a universal resource for measurement-based QC with Gaussian states.

We depict a dual-rail quantum wire on periodic boundaries in Figure 9, which shows the precise graphical representation for the state, as well as additional labels that clarify how we re-interpret this state as entangled Schwinger spins. We interpret a dual-rail quantum wire on 2n qumodes as a collection of n entangled Schwinger spins (labelled along the horizontal axis in Figure 9), with qumodes paired vertically (labelled a and b along the vertical axis). A single qumode can thus be labelled according to which spin it is part of as well as where it lies on the vertical axis. For example, the first Schwinger spin in Figure 9 contains qumodes 1a and 1b.

The dual-rail quantum wire is an H-graph state with self-inverse and bipartite \( \mathbf{G} \) and thus, the method outlined in Section VI A can be applied to derive local Schwinger nullifiers for the state. In this section, we introduce another method that can be applied to such states, which is more intuitive. This method involves reinterpreting the dual-rail quantum wire (or any H-graph state with self-inverse and bipartite \( \mathbf{G} \)) in a different basis.

A. Local Schwinger nullifiers for the dual-rail quantum wire

It is possible to represent the dual-rail quantum wire in a different conceptual way, which is illustrated in Figure 10. This is an equivalent description of the state, whereby the labelling of the qumodes has changed from \((a, b)\) to \((\pm)\). To do this, the following change of basis (and thus a re-definition of qumode operators) is required:

\[
\hat{a}_{n\pm} = \frac{1}{\sqrt{2}} (\hat{a}_{na} \pm \hat{a}_{nb}) .
\]  

In effect, this change of basis allows the beamsplitter interaction to be absorbed into the qumode re-definition and the system to be interpreted as a collection of TMS states in the \( \pm \) basis even after the beamsplitter operation has been done [20, 61]. We will call this new basis the distributed-qumodes [18].
basis and the original basis the physical-qumodes basis from now on.

We now show how to derive local Schwinger nullifiers via this change of basis. Recall that $\hat{S}^z$ nullifies a single TMS state (described in Section VII B 1). This operator acting on any of the individual TMS states in the new distributed-qumodes basis is a nullifier for the state and, upon performing the basis change of (9.1), is equivalent to a sum of Schwinger spin operators acting on neighbouring spins in the physical-qumodes basis.

One nullifier for the dual-rail quantum wire in the distributed-modes picture is $\hat{S}^z_{2,-1}$, which can be rewritten in the physical-modes basis as:

$$\hat{S}^z_{2,+1} = \frac{1}{2} (\hat{a}^\dagger_2 \hat{a}_2 + \hat{a}^\dagger_{-1} \hat{a}_{-1})$$

$$= \frac{1}{2} (\hat{S}^0_{2a,2b} - \hat{S}^x_{2a,2b} - \hat{S}^0_{3a,3b} - \hat{S}^x_{3a,3b}).$$

(9.2)

For a dual-rail quantum wire (on periodic boundaries) on $n$ Schwinger spins, this technique applied to every TMS state in the system results in the derivation of $n$ independent Schwinger nullifiers:

$$\hat{N}_i = \hat{S}^0_{ia,ib} - \hat{S}^x_{ia,ib} - \hat{S}^0_{i+1a,i+1b} - \hat{S}^x_{i+1a,i+1b},$$

(9.3)

$i \in \mathbb{Z}_n$

where the qumodes are labelled $a$ and $b$ and paired vertically into Schwinger spins that are labelled by $i$, and addition is modulo $n$. These nullifiers are interpreted in terms of what they reveal about the Schwinger spin structure of the dual-rail quantum wire in Section IX B and also analysed in terms of the interferometric symmetries they generate in Section IX C.

We illustrate the local nullifier from (9.2) in Figure 11, where the qumodes (labelled by $a$ and $b$) are paired vertically into Schwinger spins, labelled 1–5. The nullifier is illustrated with coloured ellipses, where the green ellipse represents the operator $\hat{S}^0_{2a,2b} - \hat{S}^x_{2a,2b}$ and the red ellipse represents the operator $-\hat{S}^0_{3a,3b} - \hat{S}^x_{3a,3b}$.

The technique described in Section VIA, which derives $\mathcal{M}$ for any $H$-graph state with bipartite, self-inverse $\mathcal{G}$ could also have been used to derive the nullifiers in this section. However, the technique of switching bases lends itself to a more natural and straightforward derivation in this case.

B. Entangled Schwinger spins on the dual-rail quantum wire

The nullifiers of Equation (9.3) reveal information about the underlying Schwinger spin structure of the dual-rail quantum wire on periodic boundaries. For instance, the nullifier from Figure 11 reveals that for any pair of Schwinger spins $i$ and $j$ along the dual-rail quantum wire, the values obtained by measuring $\hat{S}^0_{ia,ib} - \hat{S}^0_{ja,jb}$ determine the total spin along the $x$-axis for the pair of spins. The $\hat{S}^0_{ia,ib}$ operator corresponds to the total photon number in the Schwinger spin $i$, and thus, the spin along the $x$-axis at any given two spins on the ring of entangled Schwinger spins is directly determined by the photon-number difference between the spins.

It is worthwhile to note that these nullifiers do not solely correspond to rotations of the corresponding Schwinger spins, as these nullifiers include the $\hat{S}^0$ operator, which is not a generator of the Schwinger representation of $SU(2)$. It is however, a generator of the Schwinger representation of $U(2)$, as discussed in Section II A.

C. Local interferometric symmetries on the dual-rail quantum wire

The nullifiers of Equation (9.3) can also be interpreted as local interferometers that act on four-qumode subspaces of the dual-rail quantum wire and leave it invariant. In this section, we define these interferometers and also present some examples of larger symmetries that act on six-qumode subspaces. As CV measurement-based QC using the dual-rail quantum wire is basically concatenated gate teleportation [20, 61], these symmetries
FIG. 12. A local four-qumode subspace of the dual-rail quantum wire (on periodic boundary conditions) with qumodes labelled 1–4. The coloured edges in this graph correspond to weights of $\pm \frac{1}{2} \tanh(\alpha)$ in the corresponding adjacency matrix, where $\alpha > 0$ is an overall squeezing parameter.

reveal an additional structure that might be exploited via rotated measurement bases to perform computation.

1. Local interferometric symmetries on four qumodes

Consider a local portion of the dual-rail quantum wire consisting of four qumodes (depicted in Figure 12). Using the nullifiers derived in the previous section, such a portion is left invariant under the following interferometric operation:

$$\exp[-i\theta(\hat{S}_{1,2}^{0} - \hat{S}_{1,2}^{x} - \hat{S}_{3,4}^{0} - \hat{S}_{3,4}^{x})], \quad (9.4)$$

for all $\theta \in \mathbb{R}$. This operator corresponds to two distinct operators, acting independently on two-qumode subspaces of the wire:

$$\exp[-i\theta(\hat{S}_{1,2}^{0})] \exp[i\theta(\hat{S}_{3,4}^{0} + \hat{S}_{3,4}^{x})]. \quad (9.5)$$

As mentioned in Section II A, the $\hat{S}^{x}$ and $\hat{S}^{0}$ operators commute. Thus, these interferometers can be further decomposed and correspond to a sequence of single-qumode phase shifts and beamsplitters:

$$\hat{R}_{1}(\theta)\hat{R}_{3}(-\theta)\hat{R}_{2}(\theta)\hat{R}_{4}(-\theta)e^{-i\theta \hat{S}_{1,2}^{x}}e^{-i\theta \hat{S}_{3,4}^{x}}. \quad (9.6)$$

for all $\theta \in \mathbb{R}$. Another way of stating this symmetry is that, for a given dual-rail quantum wire $|\psi_{\text{QW}}\rangle$, performing these separate operators is equivalent:

$$e^{-i\theta(\hat{S}_{1,2}^{0} - \hat{S}_{1,2}^{x})}|\psi_{\text{QW}}\rangle = e^{-i\theta(\hat{S}_{3,4}^{0} + \hat{S}_{3,4}^{x})}|\psi_{\text{QW}}\rangle. \quad (9.7)$$

The symmetry of Equation (9.7) could be useful when considering the dual-rail quantum wire as a resource for measurement-based QC [18, 38]. In this setting, subsequent measurements in rotated bases are performed on pairs of vertically paired qumodes along the wire. Measuring in a rotated basis corresponding to the operation $e^{-i\theta(\hat{S}^{0} - \hat{S}^{x})}$ automatically effects the operator $e^{-i\theta(\hat{S}^{0} + \hat{S}^{x})}$ on adjacent qumodes.

FIG. 13. The generators of the local interferometer from (9.8) illustrated on a four-qumode subspace of the dual-rail quantum wire (on periodic boundary conditions), with qumodes labelled 1–4. The $-\hat{S}^{x}$ operator is indicated by the blue ellipses and $\hat{S}^{x}$ by the red ellipses. The coloured edges in this graph correspond to weights of $\pm \frac{1}{2} \tanh(\alpha)$ in the corresponding adjacency matrix, where $\alpha > 0$ is an overall squeezing parameter.

2. Overlapping Schwinger-spin structure of the dual-rail quantum wire

By rewriting the interferometer given in (9.4) in terms of the relative-phase shift and two-mode mixing operators (generated by $\hat{S}^{z}$ and $\hat{S}^{x}$), we can rewrite (9.4) as

$$\exp[-i\theta(\hat{S}_{3,1}^{z} + \hat{S}_{4,2}^{z} - \hat{S}_{1,2}^{z} - \hat{S}_{3,4}^{z})]. \quad (9.8)$$

for all $\theta \in \mathbb{R}$. The generators of this transformation are illustrated in Figure 13, which indicates $-\hat{S}^{z}$ with blue ellipses and $\hat{S}^{z}$ with purple ellipses. This local interferometer reveals an interesting Schwinger-spin structure of the dual-rail quantum wire, one that involves multiple-qumode pairings. To see this, note that the spins that $\hat{S}^{z}$ acts on constitute one spin structure (one where the qumodes are paired horizontally), and the spins that $\hat{S}^{z}$ acts on constitute a different spin structure (one where the qumodes are paired vertically). As mentioned in Section II A, the $\hat{S}^{z}$ and $\hat{S}^{x}$ operators generally do not commute. However, as (9.8) is equivalent to (9.4), the applications of $\hat{S}^{z}$ and $\hat{S}^{x}$ on the different spin structures can be performed in either order.

3. Local interferometric symmetries on six qumodes

Using the basis-change technique described in Section IX A, and noting that sums and differences of nullifiers are also nullifiers, a vast array of nullifiers can be derived for a given state. Applying the basis-change technique whereby we use the nullifiers for a pair of TMS states (presented in Section VIII), the following is a valid Schwinger nullifier that acts on a six-qumode portion of the dual-rail quantum wire:

$$-2\hat{S}_{3,4}^{z} + \hat{S}_{1,6}^{z} + \hat{S}_{1,5}^{z} - \hat{S}_{2,6}^{z} - \hat{S}_{2,5}^{z}, \quad (9.9)$$

The coloured edges in this graph correspond to weights of $\pm \frac{1}{2} \tanh(\alpha)$ in the corresponding adjacency matrix, where $\alpha > 0$ is an overall squeezing parameter.
for $\theta \in \mathbb{R}$. We illustrate this nullifier in Figure 14, where the different types of interferometers are indicated with different colours linking the qumodes that are being acted on. This corresponds to a six-qumode interferometer that can be decomposed into two distinct interferometers acting on the six-qumode subspace:

$$
\exp(2i\theta \hat{S}_{3,4}^{y}) \exp(-i\theta [\hat{S}_{1,6}^{x} + \hat{S}_{1,5}^{x} - \hat{S}_{2,6}^{x} - \hat{S}_{2,5}^{x}]).
$$  \hspace{1cm} (9.10)

The first of these operators corresponds to applying an equal and opposite phase shift of $2\theta$ on qumodes 3 and 4. The second interferometer seems to generate a more complex, four-qumode operation but nonetheless can be decomposed into a series of distinct $U(2)$ interferometric operations [42]. To see this, note that the operator $\hat{S}_{1,6}^{x} + \hat{S}_{1,5}^{x} - \hat{S}_{2,6}^{x} - \hat{S}_{2,5}^{x}$ can be written as

$$
(\hat{a}_{1}^{\dagger} \quad \hat{a}_{2}^{\dagger} \quad \hat{a}_{3}^{\dagger} \quad \hat{a}_{6}^{\dagger}) \begin{pmatrix} 0 & U & 0 \\ UT & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_{1} \\ \hat{a}_{2} \\ \hat{a}_{3} \\ \hat{a}_{6} \end{pmatrix},
$$  \hspace{1cm} (9.11)

where $U = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. This expression can be put into a different but equivalent form:

$$
(\hat{a}_{1}^{\dagger} \hat{a}_{2}^{\dagger} \hat{a}_{3}^{\dagger} \hat{a}_{6}^{\dagger}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_{1}^{\dagger} \\ \hat{a}_{2}^{\dagger} \\ \hat{a}_{3}^{\dagger} \\ \hat{a}_{6}^{\dagger} \end{pmatrix},
$$  \hspace{1cm} (9.12)

where the new annihilation operators for each qumode have undergone the following transformation:

$$
\begin{pmatrix} \hat{a}_{1}^{\dagger} \\ \hat{a}_{2}^{\dagger} \\ \hat{a}_{3}^{\dagger} \\ \hat{a}_{6}^{\dagger} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{a}_{1} \\ \hat{a}_{2} \\ \hat{a}_{3} \\ \hat{a}_{6} \end{pmatrix}.
$$  \hspace{1cm} (9.13)

The transformation of Equation (9.13) corresponds to a 50:50 beamsplitter interaction between qumodes 2 and 1, as well as between qumodes 6 and 5. Therefore, the second interferometer of expression (9.10) can be decomposed into the following distinct interferometers:

$$
\exp \left( \frac{i\pi}{2} \hat{S}_{1,2}^{y} \right) \exp \left( \frac{i\pi}{2} \hat{S}_{5,6}^{y} \right) \exp(-2i\theta \hat{S}_{2,5}^{x}).
$$  \hspace{1cm} (9.14)

These interferometers correspond to first performing a 50:50 beamsplitter interaction on qumodes 2 and 1, as well as 6 and 5. Following this, a beamsplitter interaction that imparts a phase shift of $\frac{\pi}{2}$ on the reflected qumodes is performed on qumodes 2 and 5. Finally, the original 50:50 beamsplitters are reversed.

Another exotic local interferometric symmetry is given by:

$$
\exp(-i\theta [\hat{S}_{1,3}^{y} + \hat{S}_{1,4}^{y} + \hat{S}_{2,3}^{y} + \hat{S}_{2,4}^{y} + \hat{S}_{4,5}^{y} + \hat{S}_{4,6}^{y} + \hat{S}_{5,4}^{y} + \hat{S}_{6,5}^{y}]),
$$  \hspace{1cm} (9.15)

which is illustrated in Figure 15, whereby the different coloured links indicate whether the $\hat{S}^{y}$ operator is acting from left to right (blue) or right to left (red). This interferometer can be decomposed into the following pair of beamsplitter interactions:

$$
\exp(-i\theta [\hat{S}_{1,3}^{y} + \hat{S}_{1,4}^{y} + \hat{S}_{2,3}^{y} + \hat{S}_{2,4}^{y}])
$$  \hspace{1cm} exp(-i\theta [\hat{S}_{3,5}^{y} + \hat{S}_{3,6}^{y} + \hat{S}_{4,5}^{y} + \hat{S}_{4,6}^{y}]),
$$  \hspace{1cm} (9.16)

which can be performed in either order. Each of these interferometers can be further decomposed into:

$$
\begin{pmatrix} \exp(-i\theta [\hat{S}_{1,3}^{y} + \hat{S}_{1,4}^{y}]) \exp(-i\theta [\hat{S}_{2,4}^{y}]) \\ \exp(-i\theta [\hat{S}_{3,5}^{y} + \hat{S}_{3,6}^{y}]) \exp(-i\theta [\hat{S}_{4,6}^{y}]) \end{pmatrix},
$$  \hspace{1cm} (9.17)

where the operators within the square brackets commute with one another and thus can be done in either order. This collection of commuting beamsplitter interactions corresponds to a 50:50 beamsplitter when $\theta = \frac{\pi}{2}$. It should be noted that although the operators within the square brackets commute with one another and that each entire square bracket commutes with the other, the individual beamsplitters within each one cannot necessarily be performed in any order.

We have shown that the local Schwinger nullifiers for the dual-rail quantum wire (on periodic boundary conditions) from Equation (9.3) can be reinterpreted as local interferometers that leave the state invariant. Some of these interferometers are single applications of phase shifters or beamsplitters, while others have more complicated decompositions, consisting of sequences of various
nullifiers that is illustrated in Figure 16 reveals about any segment of the dual-rail quantum wire. nullifiers of expression (9.19). Lnullifiers have been summed, resulting in the global

\[ N \] along the wire until all of the

fiers to the summed expression, this chain grows

directions along the wire, giving it a chain-like na-

tion (9.3) with

\[ \frac{a}{b} \]

tion are only examples from using such a tech-

nique.

D. Chain-like Schwinger spin nullifiers for the dual-rail quantum wire

Any linear combination of the nullifiers given in Equation (9.3) is also a Schwinger nullifier. In this section, we show that by taking positive combinations of some of these local nullifiers, chain-like nullifiers emerge. These provide information about a segment of the ring of entangled Schwinger spins on the dual-rail quantum wire, where the length of the segment depends on the amount of nullifiers summed. We illustrate this by considering a dual-rail quantum wire on eight Schwinger spins, which has corresponding nullifiers given by Equation (9.3) with \( n = 8 \). Summing nullifiers \( \hat{N}_1 \) to \( \hat{N}_3 \) results in a nullifier of the form:

\[
\hat{S}_{x,1a}^0 - \hat{S}_{x,1b}^r - 2\hat{S}_{x,2a}^r - \hat{S}_{x,3a}^r - \hat{S}_{x,4a}^0 - \hat{S}_{x,4b}^r \tag{9.18}
\]

which we illustrate graphically in Figure 16. As can be seen from Figure 16, this resultant nullifier resembles one of the independent nullifiers illustrated in Figure 11 but is slightly extended in both directions along the wire, giving it a chain-like nature. It can be shown that by adding more nullifiers to the summed expression, this chain grows along the wire until all of the \( n \) independent nullifiers have been summed, resulting in the global nullifier of expression (9.19).

These chain-like nullifiers reveal information about any segment of the dual-rail quantum wire. The nullifier that is illustrated in Figure 16 reveals

D. Chain-like Schwinger spin nullifiers for the dual-rail quantum wire

E. Global Schwinger spin nullifiers for the dual-rail quantum wire

We just discussed how, by taking linear combinations of some of the local nullifiers presented in Equation (9.3), we can derive new nullifiers that act on segments of the dual-rail quantum wire. As we will show in this section, taking linear combinations of all the independent nullifiers for a given system results in global nullifiers—i.e., nullifiers that act on every Schwinger spin in the system simultaneously. In terms of systems of entangled Schwinger spins, these global nullifiers reveal global properties of the system, such as what the overall spin is along a certain direction. As will become clear in this section, the differing global nullifiers also illustrate different spin structures—i.e., different potential qumode pairings. When interpreted as interferometric symmetries, these global nullifiers enable us to deduce global interferometric symmetries for the given system.

1. Global properties of entangled Schwinger spins on a ring

The Schwinger representation of \( SU(2) \) applied to the dual-rail quantum wire on periodic boundaries maps the state to a ring of entangled Schwinger spins. For such a system on a total of \( n \) Schwinger spins, by taking the sum of all nullifiers
that are paired horizontally far. This new spin structure consists of qumodes than the one that has been discussed so far. We have presented two global interferometers that act on the entire dual-rail quantum wire invariant under applying an equal and opposite phase shift to all of the horizontally paired qumodes in the system, whereby pairings do not overlap.

We have presented two global interferometers that act on the entire dual-rail quantum wire (on periodic boundary conditions) and leave it invariant. Such global symmetries could enable treatment of the dual-rail quantum wire as an entangled resource for a generalised, multi-qumode gate teleportation scheme. For example, consider a dual-rail quantum wire with eight qumodes, $|\psi_{QW}\rangle$.

Such a system has the following global symmetry relation:

$$e^{-i\theta \hat{S}^z_{x0}} e^{-i\theta \hat{S}^z_{x1}} |\psi_{QW}\rangle = e^{i\theta \hat{S}^z_{x1}} e^{i\theta \hat{S}^z_{x0}} |\psi_{QW}\rangle ,$$

(9.23)

where the spin operators act on vertically paired qumodes $i$ and $j$. This symmetry suggests that

These are global interferometers that act on every qumode of the system. The first of these is

$$\hat{U}(\theta)_{x} = \exp(-i\theta \hat{S}^x_{a1,1} + \ldots + \hat{S}^x_{b1,b1}) .$$

(9.21)

This operation can be decomposed into a series of individual beamsplitters that act successively on each pair of vertically paired qumodes along the wire. Thus, the dual-rail quantum wire on periodic boundary conditions is invariant under sending each vertical pair of qumodes through a beamsplitter with a phase shift of $\frac{\pi}{2}$ on the reflected qumodes.

If $n$ is even, the relation (9.20) implies that the following $2n$-qumode interferometric operation leaves the dual-rail quantum wire invariant:

$$\hat{U}(\theta)_{z} = \exp(-i\theta \hat{S}^z_{a1,2} + \hat{S}^z_{b1,2} + \ldots + \hat{S}^z_{(n-1)a,n} + \hat{S}^z_{(n-1)b,n}) .$$

(9.22)

This interferometer is an application of the relative-phase-shift operator $e^{-i\theta \hat{S}^z_{x}}$ on horizontally paired qumodes, performed in any order. This implies that a dual-rail quantum wire on $n$ Schwinger spins (where $n$ is even and on periodic boundary conditions) is invariant under applying an equal and opposite phase shift to all of the horizontally paired qumodes in the system, whereby pairings do not overlap.

FIG. 17. Schwinger spin structure for the dual-rail quantum wire on periodic boundary conditions, where the qumodes (labelled by $a$ and $b$) are paired together vertically into Schwinger spins, labelled 1—4. The blue ellipses are an illustration of the global nullifier for this state—i.e., a $\hat{S}^x$ operator acting on every Schwinger spin of the system.

FIG. 18. Alternate Schwinger spin structure for the dual-rail quantum wire on periodic boundary conditions (where the number of Schwinger spins is even). Note that the qumodes are now paired horizontally (indicated by the $a$ and $b$ labelling along the horizontal axis) and are paired together into Schwinger spins that are labelled 1—4. A global nullifier for this state is illustrated by the purple ellipses, which denote the $\hat{S}^x$ operator acting on all of the Schwinger spins in the system.

2. Global interferometric symmetries on the dual-rail quantum wire

For a dual-rail quantum wire on $n$ Schwinger spins, the global nullifiers defined in the previous section correspond to $2n$-qumode interferometric operations that leave the state invariant.

from the set given in Equation (9.3) we arrive at a global nullifier:

$$\sum_{i=1}^{n} \hat{S}^x_{a_{i}, b_{i}} ,$$

(9.19)

where $i$ denotes the Schwinger spin labelled $i$ along the horizontal direction and the $a$ and $b$ labels indicate the vertically paired qumodes. This is illustrated graphically in Figure 17, which indicates the $\hat{S}^x$ operator with blue ellipses. From this nullifier, we can conclude that the ring of entangled Schwinger spins (that is of variable length) always has a total spin along the $x$-axis equal to zero.

If $n$ is even, taking a sum of the differences of pairs of nullifiers from Equation (9.3) results in another global nullifier that acts on a different spin structure than the one that has been discussed so far. This new spin structure consists of qumodes that are paired horizontally into spins. The following nullifier applies to the structure:

$$\sum_{i=1}^{\frac{n}{2}} (\hat{S}^x_{(2i-1)a_{i}, 2a_{i}} + \hat{S}^x_{(2i-1)b_{i}, 2b_{i}}) ,$$

(9.20)

where $i$ denotes the Schwinger spin labelled $i$, which consists of horizontally paired qumodes. This is illustrated in Figure 18, where the purple ellipses correspond to the $\hat{S}^x$ nullifiers acting the corresponding qumodes. Thus, a dual-rail quantum wire with an even number of Schwinger spins (made out of horizontally paired qumodes rather than vertically) is a ring of entangled Schwinger spins in which the total spin along the $z$-axis is equal to zero at all times.

The following global symmetry relation:

$$e^{-i\theta \hat{S}^z_{x0} e^{-i\theta \hat{S}^z_{x1}} |\psi_{QW}\rangle = e^{i\theta \hat{S}^z_{x1}} e^{i\theta \hat{S}^z_{x0}} |\psi_{QW}\rangle ,$$

(9.23)
the dual-rail quantum wire could be used for a minimal-noise gate teleportation whereby the gate to be teleported is the four-qumode operator $e^{-\theta \hat{S}_{i,j}} e^{-\theta \hat{S}_{k,l}}$ for some vertically paired qumodes $(i, j)$ and $(k, l)$.

X. CONCLUSION

We have introduced the concept of entangled Schwinger spins on multi-qumode entangled Gaussian pure states and explored these systems using the Schwinger nullifier formalism. We explored a particular optical Gaussian state—the dual-rail quantum wire—in much detail in Section IX. In doing so, we have explored the properties of this state when considered within the Schwinger framework, adding some intuition to the interpretation of large-scale optical Gaussian states as entangled Schwinger spins. Specifically, we presented two distinct methods that enable derivation of Schwinger nullifiers for the dual-rail quantum wire. Furthermore, we extrapolated other nullifiers that are consequences of these, adding semi-local and global nullifiers to our understanding and thus, deepening our understanding of the spin structure of the system.

A direct result of the Schwinger nullifiers presented in this work are the existence of interferometric symmetries that apply to the Gaussian states in question. This is a consequence of these nullifiers being generators of passive interferometric transformations (involving beamsplitters and phase shifters). Thus, the nullifiers presented throughout this work serve as indications of the underlying structure of the systems they apply to. We present local, interferometric symmetries for a range of optical Gaussian states, including the dual-rail quantum wire. These symmetries could be harnessed to enable generalised gate-teleportation schemes whereby multi-qumode gates are applied in the minimal-noise setting described in Section III A.

We presented a method that enables derivation of the class of Gaussian pure states on two qumodes that is left invariant by an arbitrary unitary passive interferometric operation. In Section VII B, this method was highlighted with some examples of well-known, simple two-qumode interferometers (single beamsplitters and phase shifters).

We have discovered a relationship between the Schwinger nullifiers for any Gaussian pure state and the adjacency matrix for that state. This is presented in Theorem 5, which provides necessary and sufficient conditions that any Schwinger operators and adjacency matrix must satisfy in order for the operators to be Schwinger nullifiers for the state specified by the adjacency matrix.

The dual-rail quantum wire is a component of a more complex, two-dimensional structure, which is a universal resource for measurement-based QC [18, 38]. Extensions of the methods presented in Sections VIA and IX A will enable the further exploration of this structure and potentially other useful optical H-graph states. One such example is the recently proposed hypercubic H-graph state, which has potential applications in simulating the states of topologically ordered condensed-matter systems and in quantum error correction [58].

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Appendix A: Proof of relationship between $K$ and $G$ and further simplification when $G$ is self-inverse

Noting the relationship between $Z$ and the H-graph $G$,

$$Z = i e^{-2\alpha G}$$

and recalling the relationship between $Z$ and $K$,

$$K = (I + iZ)(I - iZ)^{-1},$$

we can rewrite $K$ as

$$K = (I + iZ)(I - iZ)^{-1}$$

$$= (I - e^{-2\alpha G})(I + e^{-2\alpha G})^{-1}. (A4)$$

Noting that $G$ is diagonalizable and that

$$(1 - e^{-2\alpha})(1 + e^{-2\alpha})^{-1} = \tanh(x), (A5)$$

we can write $K$ as

$$K = \tanh(\alpha G). (A6)$$

Expanding this out in a Taylor series,

$$\tanh(\alpha G) = \alpha G - \frac{\alpha G^3}{3} + 2\frac{\alpha G^5}{15} - 17\frac{\alpha G^7}{315} + ..., (A7)$$

which converges for $|\alpha| < \frac{\pi}{2}$. If $G$ is self-inverse, this simplifies to

$$\tanh(\alpha G) = \left[ \alpha - \frac{(\alpha^2)}{3} + 2\frac{(\alpha^5)}{15} - 17\frac{(\alpha^7)}{315} + ... \right] G$$

$$= \tanh(\alpha G). (A8)$$

Even though the series expansion for $\tanh$ only holds when $|\alpha| < \frac{\pi}{2}$, since we are able to re-sum the series analytically, we can use analytic continuation to extend this result to all $\alpha \in \mathbb{R}$. 20
Appendix B: Derivation of relations for K

The following relations between $Z$ and $K$ will be useful:

$$\frac{1}{2}(I - iZ) = (I + K)^{-1}, \quad (B1)$$

$$\frac{1}{2}(I + iZ^*) = (I + K^*)^{-1}, \quad (B2)$$

$$\frac{1}{2}(I + iZ) = (I + K^{-1})^{-1}, \quad (B3)$$

$$\frac{1}{2}(I - iZ^*) = (I + K^{*-1})^{-1}. \quad (B4)$$

To prove that $||K|| < 1$, first we write $U$ (defined in Section IV B) in terms of $K$ as follows:

$$U = \frac{1}{2i}(Z - Z^*) \quad (B5)$$

$$= \frac{1}{2}(I - iZ) + \frac{1}{2}(I + iZ^*) - I \quad (B6)$$

$$= (I + K)^{-1} + (I + K^*)^{-1} - I$$

$$= (I + K)^{-1}(I - KK^*)(I + K^*)^{-1}. \quad (B7)$$

Since $K = K^T$, and $(I + K)$ is invertible, the condition that $U > 0$ implies that $I - KK^* > 0$. This means that the eigenvalues of $KK^*$ are all less than 1. Since these eigenvalues are the squares of the singular values of $K$, the spectral norm condition follows.

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