Finite-Time Cluster Synchronization of Delayed Fractional-Order Fully Complex-Valued Community Networks

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This work was supported in part by the National Natural Science Foundation of China under Grant 61305076.

ABSTRACT This paper focuses on the finite-time (FNT) cluster synchronization issues for a class of delayed fractional-order fully complex-valued community networks (FFCVCNs). A new mathematical expression of the complex networks is developed with internal delay, non-delayed and delayed couplings, complex-valued state variables, system function, coupling strengths, inner coupling matrices and outer coupling matrices. Instead of transforming the complex-valued (CV) networks into two independent real-valued (RV) systems, the delay-dependent controllers are designed based on the quadratic norm and a novel norm composed of the absolute-valued norm to realize the cluster synchronization for the proposed complex networks in FNT, respectively. In addition, the upper bounds of the settling time (ST) when the system could reach finite-time cluster synchronization are estimated. The obtained results are less conservative than some of the existing studies due to the characteristics of fully fractional-order complex-valued (FOCV) dynamical networks. The feasibility and effectiveness of the main results are demonstrated by simulation examples.

INDEX TERMS Finite-time synchronization, cluster synchronization, fully complex-valued community networks, delay.

I. INTRODUCTION

Cluster synchronization is a universal synchronization phenomenon in many fields, such as traffic control, engineering control, brain science, communication engineering, ecology, etc. In the past decades, numerous investigations have been conducted into the cluster synchronization issues of dynamical networks due to their feasibility in explaining some natural phenomena and potential applications in biomathematics, secure communication, signal processing, and mechanical engineering. In cluster synchronization of dynamical networks, the nodes can be divided into several clusters and only nodes belonging to the same cluster are synchronized to a corresponding state, while nodes in different clusters have different synchronization behaviors [1], [2], [3], [4]. The FNT synchronization has the advantages of a faster convergence rate, better robustness against uncertainties, and disturbance rejection properties [5]. Therefore, various studies on FNT cluster synchronization of dynamical networks have been conducted [3], [6], [7], [8], [9], [10], [11].

However, in the existing studies about FNT cluster synchronization of dynamical networks, an overwhelming majority of the models are assumed to be integer-order and RV. Compared with integer-order RV dynamical networks, fractional-order and CV networks have wider application value [12], [13], [14], [15]. The literature shows that the FOCV dynamical networks are more realistic and superior because they allow describing the characteristics of memory and heredity of various materials and dynamical processes [16], [17], [18], [19]. Moreover, the transmitted signals in the FOCV dynamical networks acquire stronger versatility and anti-attack ability since the dynamic systems in complex space can evolve in various directions [20]. Several noticeable studies on FNT synchronization of FOCV dynamical networks have been conducted. For example, the synchronization problems for FOCV neural networks have been investigated in [15], [21], and [22] by transforming the CV network into two independent RV systems (namely,
The separation method brings the dimension of two relatively independent RV systems twice that of the original CV system, which significantly increases the complexity of theoretical analysis and mathematical derivation. Therefore, it is necessary to explore new and more suitable methods to deal with the synchronization problem of CV complex dynamic networks. To avoid the complexity and trivialness of theoretical analysis and mathematical derivation caused by the classic separation method, the synchronization problems for FO dynamical networks and neural networks have been explored in [23] and [24], respectively, by introducing some novel norms based on the signum function of complex numbers. However, there is less discussion about the FNT cluster synchronization of FOCV complex networks, which has gradually become one of the most important research directions. The FNT cluster synchronization problem for FOVC networks with nonlinear coupling has been studied in [3] through the control strategy related to complex symbolic functions.

However, the mathematical models in the above studies do not consider the time delay term. Time delay is a very common phenomenon in real complex dynamical networks for describing the effects of the finite speeds and spreading as well as traffic congestion for nodes’ behaviors, respectively [25], [26]. Time delay in complex dynamical networks can make the nodes’ dynamical behaviors more complicated, and may destabilize the stable equilibria and admit periodic oscillation, bifurcation and even chaos [28], [29]. Coupling delay can result in the networks crash. In [3], [23], [24], and [30], the authors constructed complex networks without delay, single time delay \( x_i(t - \tau) \), or single coupling delay \( Gx_i(t - \tau) \), which differs from complex systems in the actual world. In [31], although the authors discussed the complex networks with internal delays and coupling delays, they investigated them in the integer-order and RV domain. In contrast, fractional-order (FO) complex networks with fully CV have more extensive application value. To the best of the authors’ knowledge, there are no studies considering the FNT cluster synchronization of FFOCVNs with internal delay and coupling delay.

Motivated by the above discussion, this paper focuses on the FNT cluster synchronization for a class of FFOCVNs with internal delay as well as non-delayed and delayed couplings. Instead of transforming the CV networks into two independent real-valued systems, some delay-dependent controllers are designed based on different norms of complex numbers and several sufficient criteria are acquired to ensure the finite-time cluster synchronization of the proposed complex dynamical networks.

The main contribution of this paper can be summarized as follows: 1) Most of the previous research on FNT cluster synchronization are investigated by general complex dynamical networks, as the first attempt, internal delay, coupling delay, FO calculus as well as CV state variables, system function, coupling strengths, inner coupling matrices and outer coupling matrices are taken into account to study the FNT cluster synchronization of delayed FFCVCNs. 2) Lyapunov functions/functionals based on the quadratic norm and a novel norm composed of the absolute-valued norm are constructed to reduce the computational complexity, trivialness and limitations. 3) Two more comprehensive delay-dependent controllers are designed to realize the FNT cluster synchronization, less conservative results are acquired compared with the existing literature [23], [32], [33], [34]. 4) Compared with the separation method, we analyze the FNT cluster synchronization problem of delayed FFCVCNs by introducing the CV signum function and using the non separation method, which effectively avoids the complexity of theoretical analysis and calculation analysis.

The following is the structure of the paper. In Section II, the problem statement and preliminaries are given. In Section III, two different controllers are proposed to ensure FNT cluster synchronization for the addressed delayed FFCVCNs. Numerical simulations are presented in Section IV to illustrate the validity and practicality of the proposed theoretical solutions. Section V concludes this paper.

**Notations.** In this paper, \( \mathbb{R} \) represents the real field, \( \mathbb{R}^+ \) denotes the positive real field and \( \mathbb{R}^n \) symbolizes \( n \)-dimensional real space. \( \mathbb{C} \) represents the complex field, for any \( v = p + iq \in \mathbb{C}, \bar{v} = p - iq \) denotes the conjugate of \( v, |v|_1 = |p| + |q|, |v|_2 = \sqrt{v^*v} \), where \( i \) meets \( i^2 = -1 \), and \( p, q \in \mathbb{R} \) are the real and imaginary parts of \( v \), respectively, this is, \( Re(v) = p \) and \( Im(v) = q \). \( \mathbb{C}^n \) symbolizes \( n \)-dimensional complex field, for any \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{C}^n, v = Re(v) + iIm(v), v^H \) denotes its conjugate transposition, \( ||v||_1 = ||Re(v)||_1 + i||Im(v)||_1 \), \( ||v||_2 = \sqrt{v^Hv} \). \( \mathbb{C}^{m \times n} \) denotes the set of all \( m \times n \)-dimensional complex matrices. \( I_N \) denotes the \( n \)-dimensional column vector with each element equal to 1. \( E_N \) represents the \( n \)-dimensional diagonal identity matrix. The notation \( C^m([t_0, +\infty), \mathbb{C}) \) denotes the family of all continuous \( n \)-differential functions from \([t_0, +\infty)\) into \( \mathbb{C} \).

**II. MODELING AND PRELIMINARY**
Throughout this work, \( \alpha \)-order Caputo derivative is selected.

**Definition 1 ([35], [37]):** For an integrable function \( f(t) : [t_0, +\infty) \rightarrow \mathbb{C} \), its \( \alpha \)-order fractional integral is defined as

\[
_{t_0}I^\alpha_t f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s)ds, \quad t \geq t_0, 
\]

where \( \Gamma(\cdot) \) is the Gamma function and \( \alpha > 0 \).

**Definition 2 ([35], [37]):** For \( f \in C^m([t_0, +\infty), \mathbb{C}), \) its \( \alpha \)-order Caputo derivative is defined as

\[
_{t_0}D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^{(m)}(s)}{(t-s)^{\alpha}} ds, \quad t \geq t_0, 
\]

where \( m \) is a positive integer such that \( \alpha \in (m-1, m) \). Especially, if \( \alpha \in (0, 1) \), then

\[
_{t_0}D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{f^{(1)}(s)}{(t-s)^{\alpha}} ds. 
\]
Definition 3 ([23]): For any real numbers $v, w > 0$, the integral
\[
\int_0^1 x^{v-1}(1-x)^{w-1} \, dx
\]
is called the Beta-function denoted by $\Gamma(v, w)$. It is obvious that
\[
\Gamma(v, w) = \frac{\Gamma(v)\Gamma(w)}{\Gamma(v + w)}.
\]

Definition 4: For any $v \in \mathbb{C}$, $[v] = \text{sign}(\text{Re}(v)) + i\text{sign}(\text{Im}(v))$ is the signum function of $v$. For any $v = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$, $[v] = (\text{sign}(\text{Re}(v_1)) + i\text{sign}(\text{Im}(v_1)), \ldots, \text{sign}(\text{Re}(v_n)) + i\text{sign}(\text{Im}(v_n))^T$ is the signum function vector of $v$.

The delayed FFCVCNs with $r$ communities and $N$ identical dynamical nodes can be described as:

\[
C_{l_0}D_l^\alpha x_i(t) = f_{\mu_i}(x_i(t), x_i(t - \tau_1)) + c_1 \sum_{k=1}^r a_{ij}G_1x_j(t) + c_2 \sum_{k=1}^r b_{ij}G_2x_j(t - \tau_2),
\]
where $i = 1, 2, \ldots, N, k = 1, 2, \ldots, r; C_k$ represents the set of all nodes belonging to the $k$th community; $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{C}^n$; define the function $\mu : \{1, 2, \ldots, N\} \rightarrow \{1, 2, \ldots, r\}$, if node $j \in C_k$, then one has $\mu_j = k$; $f_{\mu_i} : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a continuous nonlinear vector function; $c_i \in \mathbb{C}$ ($l = 1, 2$ is the coupling strength, $G_l = \text{diag}(\delta_1^{(l)}, \delta_2^{(l)}, \ldots, \delta_n^{(l)}) \in \mathbb{C}^{n \times n}$ ($l = 1, 2$) is the inner matrix linking the coupled nodes, $\tau_1$ is the internal delay occurring inside the dynamical node, $\tau_2$ represents the coupling delay, $A = (a_{ij})_{N \times N}$ and $B = (b_{ij})_{N \times N}$ denote the topological structure of the complex networks for non-delayed and delayed configurations, respectively. If nodes $i$ and nodes $j$ ($i \neq j$) have a link, then $a_{ij}, b_{ij} \neq 0 \in \mathbb{C}$, else $a_{ij} = b_{ij} = 0$ ($i \neq j$). The diagonal elements of $A$ and $B$ are defined as

\[
a_{ii} = -\sum_{j=1, j \neq i}^N a_{ij}, \quad b_{ii} = -\sum_{j=1, j \neq i}^N b_{ij}, \quad i = 1, 2, \ldots, N.
\]

The nodes can be separated into $r$ nonempty communities, namely,
\[
\{1, 2, \ldots, N\} = C_1 \cup C_2 \cup \cdots \cup C_r,
\]
where $C_1 = \{1, \ldots, m_1\}, C_2 = \{m_1 + 1, \ldots, m_2\}, \cdots, C_r = \{m_{r-1} + 1, \ldots, N\}$. In turn, matrices $A$ and $B$ can be rewritten as

\[
A = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1r} \\
A_{21} & A_{22} & \cdots & A_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1} & A_{r2} & \cdots & A_{rr}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1r} \\
B_{21} & B_{22} & \cdots & B_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
B_{r1} & B_{r2} & \cdots & B_{rr}
\end{bmatrix}.
\]

Remark 1: The coupling configuration matrices $A$ and delayed one $B$ do not need to be identical, symmetric or irreducible, that is, the corresponding graphs can be directed, weakly connected and even have no rooted spanning directed tree. Moreover, the internal delay as well as the non-delayed and delayed couplings are considered in this paper. The state variables, system function, coupling strengths, inner coupling matrices and outer coupling matrices in this model are all defined as CV. Therefore, the complex dynamical network considered in this paper is more general and realistic.

Suppose $s_{\mu_i}(t) \in \mathbb{C}^n$ is the state vector of an isolated node satisfying
\[
C_{l_0}D_l^\alpha s_{\mu_i}(t) = f_{\mu_i}(s_{\mu_i}, s_{\mu_i}(t - \tau)), \quad i = 1, 2, \ldots, N,
\]
with $\lim_{t \to +\infty} \|s_{\mu_i}(t) - s_{\mu_j}(t)\| 
\neq 0$ ($\mu_i \neq \mu_j$). Here, $s_{\mu_i}(t)$ maybe an equilibrium point, a periodic orbit, or a chaotic orbit.

To achieve the FNT cluster synchronization, the following definition, assumptions and lemmas are necessary.

Definition 6: Define the error variables as $e_i(t) = x_i(t) - s_{\mu_i}(t)$ with $i = 1, 2, \ldots, N$. The community network (1) is said to realize cluster synchronization in FNT under controllers $u_i(t)$ ($i = 1, 2, \ldots, N$), if there exists a ST $T(e(t_0))$ that is dependent on the initial synchronization error, such that $\lim_{t \to T(e(t_0))} \|e_i(t)\| = 0$ and $\|e_i(t)\| \equiv 0$ for $t > T(e(t_0))$ for all $i = 1, 2, \ldots, N$, in which $e(t_0) = (e_1(t_0), e_2(t_0), \ldots, e_N(t_0))^T$.

Assumption 1: Each block matrices $A_{kl}$ and $B_{kl}$ are zero-row-sum-matrices, $k, l = 1, 2, \ldots, r$.

Assumption 2: For the vector-valued function $f_k : \mathbb{C}^n \to \mathbb{C}^n$ ($k = 1, 2, \ldots, r$), there exist positive constants $\eta_k^{(p)}$ and $\zeta_k^{(p)}$ ($p = 1, 2$) such that $f_k$ satisfies
\[
\|f_k(x(t), x(t - \tau_1)) - f_k(s(t), y(t - \tau_1))\|_p \leq \eta_k^{(p)} \|x(t) - s(t)\|_p + \zeta_k^{(p)} \|x(t - \tau_1) - s(t - \tau_1)\|_p
\]
for any $x(t), s(t) \in \mathbb{C}^n$ and $t \geq 0$.

Lemma 1 ([23], [38]): Assume that $\lambda(t) \in \mathbb{C}^n$ is differentiable, then for $t \geq t_0$ and $\alpha \in (0, 1)$, the following is obtained:
\[
C_{l_0}D_l^\alpha \lambda(t) \leq \lambda(t)C_{l_0}D_l^\alpha \lambda(t) + (C_{l_0}D_l^\alpha \lambda(t))\lambda(t).
\]

Lemma 2 ([23], [33]): For any $u \in \mathbb{C}$, $y(t) \in \mathbb{C}^n$, the following statements are true for $\alpha \in (0, 1)$.

1. $u + \bar{u} = 2\text{Re}(u) \leq |u|_2 \leq |u|_1$.
2. $\mu^{H(t)}[A(t)] + [A(t)]^H \mu \leq 2\|A(t)\|_1$.
3. $A^H(t)[A(t)] + [A(t)]^H A(t) = 2\|A(t)\|_1 \geq 2\|A(t)\|_2$.
4. $C_{l_0}D_l^\alpha (A^H(t)[A(t)] + [A(t)]^H A(t)) \leq [A(t)]^H C_{l_0}D_l^\alpha A(t) + [A(t)]^H C_{l_0}D_l^\alpha [A(t)]$. 

\[
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\]
Lemma 3 ([23]): Suppose there exist positive constants \( \gamma \in (0, \alpha) \) and \( \lambda \) such that
\[
\mathcal{C}D_t^q V(t) \leq -\lambda V(t), \quad V(t) \in \mathbb{R}^+, \tag{4}
\]
then
\[
limit_{t \to T^*} V(t) = 0 \quad \text{and} \quad V(t) \equiv 0 \quad \text{for all} \quad t \geq T^*, \tag{5}
\]
in which
\[
T^* = t_0 + \left( \frac{\alpha}{\lambda} V_{0}^{\frac{1}{1-\gamma}} (t_0) B(\alpha, 1 - \gamma) \right)^{1/\alpha}.
\]

Lemma 4 ([39]): Assume \( \lambda_i \geq 0 \) for \( i = 1, 2, \ldots, n \), \( p > 1, 0 < q < 1 \), then
\[
\frac{\sum_{i=1}^{n} \lambda_i q}{n} \geq \left( \frac{\sum_{i=1}^{n} \lambda_i}{n} \right)^q, \quad \frac{\sum_{i=1}^{n} \lambda_i^p}{n} \geq \left( \frac{\sum_{i=1}^{n} \lambda_i}{n} \right)^p.
\]

According to Assumption 1, it is easy to see that
\[
\sum_{j \in \mathcal{C}_k} a_j G_1(s_k) = 0 \quad \text{and} \quad \sum_{j \in \mathcal{C}_k} b_j G_2(s_k - \tau_2) = 0 \quad \text{for all} \quad i = 1, 2, \ldots, N.
\]
Therefore, the error dynamical system can be derived as
\[
\mathcal{C}D_t^q V_i(t) = f_{\mu_i}(e_i(t), e_i(t - \tau_1)) + c_2 \sum_{k=1}^{r} \sum_{j \in \mathcal{C}_k} b_j G_2 e_j(t - \tau_2) + u_i(t), \tag{6}
\]
where \( i = 1, 2, \ldots, N \), \( k = 1, 2, \ldots, r \), \( \mu_i, \mu_j \), \( e_i(t - \tau_1) \) and \( e_j(t - \tau_2) \) are some suitable controllers that should be designed later.

III. MAIN RESULTS

The FNT cluster synchronization problems for a class of delayed FFVCVCNs are explored by designing delay-dependent controllers. The following are the key results.

Theorem 1: Suppose Assumptions 1 and 2 hold, the control law is designed as
\[
u_i(t) = -d_i e_i(t) - \beta [e_i(t)]^\gamma - \frac{1}{2} \sum_{p=1}^{2} e_i^{(p)}(t - \tau_p), \tag{7}
\]
and \( d_i, \beta, \gamma \in (1, \gamma) > 0 \), \( 0 < \gamma < 2\alpha - 1 \), \( i = 1, 2, \ldots, N \) and satisfy
\[
\begin{align*}
2\eta^{(2)}_{\tau_1} + c_1 \delta^{(1)}_1 A + c_1 \delta^{(1)}_1 A^H - 2D & \leq 0, \\
\zeta^{(2)} - \Pi^{(1)} & \leq 0, \\
c_2 \delta^{(2)}_2 B - \Pi^{(2)} & \leq 0, \\
c_2 \delta^{(2)}_2 B^H - \Pi^{(2)} & \leq 0,
\end{align*}
\]
for all \( i = 1, 2, \ldots, n \), in which
\[
\zeta^{(2)} = \text{diag}(\zeta^{(2)}_1, \ldots, \zeta^{(2)}_1, \zeta^{(2)}_2, \ldots, \zeta^{(2)}_m, \ldots, \zeta^{(2)}_N),
\]
\[
\Pi^{(p)} = \text{diag}(\epsilon^{(p)}_1, \epsilon^{(p)}_2, \ldots, \epsilon^{(p)}_N), \quad (p = 1, 2).
\]
Then the community network (1) is said to achieve cluster synchronization in FNT under controller (7), and the ST can be estimated by
\[
T \leq T_1 = t_0 + \left( \frac{\alpha}{\lambda} V_{1}(t_0) \frac{1}{\gamma} (1 - \gamma) \right)^{\gamma}, \tag{9}
\]
where \( V_1(t_0) = \frac{1}{2} \sum_{i=1}^{N} e_i^H(t_0)e_i(t_0), \lambda = 2^{(1+\gamma)/2}, \gamma = (1 + \gamma)/2 \).

Proof: Define the Lyapunov function
\[
V_1(e_i(t)) = \frac{1}{2} \sum_{i=1}^{N} e_i^H(t)e_i(t). \tag{10}
\]

According to Lemma 1,
\[
\mathcal{C}D_t^q V_1(e_i(t)) \leq \frac{1}{2} \sum_{i=1}^{N} (e_i^H(t)g_{\mu_i}(e_i(t), e_i(t - \tau_1)) + e_i^H(t)c_1 a_i G_1 e_i(t) + e_i^H(t)c_1 \bar{a}_i G_1^H e_i(t)) + \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in \mathcal{C}_k} b_i G_2 e_j(t - \tau_2) + u_i(t) + \beta \frac{N}{2} \sum_{i=1}^{N} [e_i(t)]^\gamma e_i^H(t) + e_i^H(t)[e_i(t)]^\gamma e_i(t)^\gamma,
\]
where \( i = 1, 2, \ldots, N \), \( k = 1, 2, \ldots, r \), \( g_{\mu_i}, g_{\mu_j} \), and \( e_i(t - \tau_1) \) are some suitable controllers that should be designed later.

It follows from Assumption 1 and Lemma 2 that
\[
\frac{1}{2} \sum_{i=1}^{N} (e_i^H(t)g_{\mu_i}(e_i(t), e_i(t - \tau_1)) + g_{\mu_i}(e_i(t), e_i(t - \tau_1))e_i(t)) \leq \frac{1}{2} \sum_{i=1}^{N} [e_i^H(t)]^2 \times [g_{\mu_i}(e_i(t), e_i(t - \tau_1))]^2 + \frac{1}{2} \sum_{i=1}^{N} (e_i^H(t)[e_i(t)]^\gamma e_i(t)) + e_i^H(t)[e_i(t)]^\gamma e_i(t)
\]
\[
\leq \sum_{i=1}^{N} \sum_{l=1}^{n} \eta^{(2)}_{\tau_1} e_i^H(t) e_i(t) + \frac{1}{2} \sum_{i=1}^{N} \sum_{l=1}^{n} \epsilon^{(2)}_{\tau_1} (e_i(t)[e_i^H(t)]^\gamma e_i^H(t) + e_i^H(t)[e_i(t)]^\gamma e_i^H(t)) + \frac{1}{2} \sum_{i=1}^{N} \sum_{l=1}^{n} \epsilon^{(2)}_{\tau_1} (e_i(t)[e_i^H(t)]^\gamma e_i^H(t) + e_i^H(t)[e_i(t)]^\gamma e_i^H(t))
\]
\[
= \sum_{j=1}^{n} e_i^H(t) \eta^{(2)}_{\tau_1} e_i(t).
\]
where
\[ \eta^{(2)} = \text{diag}(\eta^{(2)}_1, \ldots, \eta^{(2)}_N, \eta^{(2)}_1, \ldots, \eta^{(2)}_N) \]

By utilizing Lemma 2 again, it follows that
\[ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{r} \sum_{k \in C_k} \left( e_i^H(t) c_1 a_j G_1 e_j(t) + e_j^H(t) c_1 a_j G_1^H e_i(t) \right) \]
\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{r} \sum_{k \in C_k} \left( e_i^H(t) c_1 a_j \delta^{(1)}_l e_j(t) + e_j^H(t) c_1 a_j \delta^{(1)}_l e_i(t) \right) \]
\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{r} \left( e_i^H(t) (c_1 \delta^{(1)}_1 A + \overline{c}^{(1)}_1 A^H) e_j(t) \right) , \]
and
\[ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{r} \sum_{k \in C_k} \left( e_i^H(t) c_2 b_j G_2 e_j(t) - \tau_2 \right) \]
\[ + e_j^H(t) (c_2 \delta^{(2)}_l B + \Pi^{(2)} e_i(t) - \tau_2) \]
\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{r} \left( e_i^H(t) c_2 b_j \overline{\delta^{(2)}_l} e_j(t) + e_j^H(t) c_2 b_j \delta^{(2)}_l e_i(t) \right) \]
\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{r} \left( e_i^H(t) (c_2 \delta^{(2)}_l B) e_j(t) \right) + e_j^H(t) (c_2 \delta^{(2)}_l B) e_i(t) \right) . \]

Furthermore, according to Lemmas 2 and 4, the following can be obtained:
\[ \frac{1}{2} \sum_{i=1}^{N} \| e_i(t) \|_2^2 < 0 \]
\[ \leq -\beta \sum_{i=1}^{N} \| e_i(t) \|_2^{1+\gamma} \]
\[ \leq -\beta \left( \sum_{i=1}^{N} \| e_i(t) \|_2^2 \right)^{1+\gamma} . \]

Substituting (12)-(15) into (11) gives
\[ \sum_{i=1}^{N} \left( e_i^H(t) \right) e_i(t) + \left( e_i^H(t) \right) \sum_{i=1}^{N} \left( e_i^H(t) \right) \]
\[ + \frac{1}{2} \sum_{i=1}^{N} \left( e_i^H(t) (c_2 \delta^{(2)}_l B - \Pi^{(2)} e_i(t) - \tau_2) \right) \]
\[ + e_j^H(t) (c_2 \delta^{(2)}_l B) e_i(t) \right) , \]
in which \( D = \text{diag}(d_1, d_2, \ldots, d_N) \).

According to (8),
\[ \sum_{i=1}^{N} \| e_i(t) \|_2^{1+\gamma} \leq \beta \left( \sum_{i=1}^{N} \| e_i(t) \|_2^2 \right)^{1+\gamma} . \]

Based upon Lemma 3, the delayed FFCCVCNs (1) under controller (7) can achieve cluster synchronization within \( T_1 \). The proof is complete. \( \Box \)

**Remark 2:** In Theorem 1, the FNT cluster synchronization problem for a class of FFCCVCNs with linearly non-delayed and delayed couplings is deliberated based on the quadratic norm. The limitation conditions \( 0 < \gamma < 2 - 1 \) and \( 0 < \alpha < 1 \) in Theorem 1 result in \( \frac{1}{2} < \alpha < 1 \), which has certain limitations and is difficult to apply in many practices. Therefore, the results for FNT cluster synchronization are provided in terms of a novel norm composed of the absolute values of each part.

**Theorem 2:** Suppose Assumptions 1 and 2 hold, the control law is designed as
\[ u_i(t) = -d_i e_i(t) - \beta \| e_i(t) \|_2^\gamma \]
\[ - \frac{1}{2} \sum_{p=1}^{N} \left( e_p^H(t) e_i(t) \right) \]
\[ + \frac{1}{2} \sum_{i=1}^{N} \| e_i(t) \|_2^{1+\gamma} \]
\[ \leq -\beta \left( \sum_{i=1}^{N} \| e_i(t) \|_2^2 \right)^{1+\gamma} . \]

and \( d_i, \beta, \varepsilon_i^{(1)}, \varepsilon_i^{(2)} > 0 \), \( 0 < \gamma < \alpha \), \( i = 1, 2, \ldots, N \) and satisfy
\[ \begin{cases} \eta^{(1)} + \Xi^{(l)} - D \leq 0 \\ \zeta^{(1)} - \Pi^{(1)} \leq 0 \\ \Omega^{(l)} - \Pi^{(2)} \leq 0, \end{cases} \]
for all \( l = 1, 2, \ldots, n \), in which
\[ \zeta^{(1)} = \text{diag}(s_1^{(1)}, s_2^{(1)}, \ldots, s_r^{(1)}, s_1^{(1)}, s_2^{(1)}, \ldots, s_r^{(1)}) \]
\[ \Pi^{(p)} = \text{diag}(s_1^{(p)}, s_2^{(p)}, \ldots, s_N^{(p)}, (p = 1, 2), \Omega^{(l)} = (\omega_{ij}^{(l)})_{N \times N} \]
and
\[ \omega_{ij} = |\text{Re}(c_2 b_j \delta^{(2)}_l)| + |\text{Im}(c_2 b_j \delta^{(2)}_l)|. \]

Then the delayed FFCCVCNs (1) is said to achieve cluster synchronization in FNT under controller (18), and the ST can be estimated as
\[ T \leq T_2 = t_0 + \left( \frac{\alpha V(t_0) \eta^{(1)} - \gamma}{\beta} \Gamma(\alpha, 1 - \gamma) \right)^{1/\alpha} , \]
where \( V(t_0) = \sum_{i=1}^{N} \| e_i(t_0) \|_1 \).
Proof: Define the Lyapunov function
\[
V_2(t) = \frac{1}{2} \sum_{i=1}^{N} \left( e_i^H(t) [e_i(t)] + [e_i(t)]^H e_i(t) \right).
\] (21)

According to Lemma 2, the following can be obtained,
\[
\begin{align*}
\dot{C}_{\infty} D_{\infty} V_2(t) & \leq \frac{1}{2} \sum_{i=1}^{N} \{ (e_i(t))^H g_{ii}(e_i(t), e_i(t - \tau_1)) \\
& \quad + \delta_{ii}(e_i(t), e_i(t - \tau_1)) [e_i(t)] \} \\
& \quad + \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in C_k} \{ (e_i(t))^H c_{1} a_{ij} G_1 e_j(t) \\
& \quad + \delta_{ij}(e_i(t), e_i(t - \tau_1)) [e_i(t)] \} \\
& \quad + \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in C_k} \{ (e_i(t))^H c_{2} b_{ij} G_2 e_j(t - \tau_2) \\
& \quad + \delta_{ij}(e_i(t), e_i(t - \tau_1)) [e_i(t)] \} \\
& \quad + \frac{1}{2} \sum_{i=1}^{N} d_i [(e_i(t))^H e_i(t) + e_i^H(t) [e_i(t)]] \\
& \quad - \beta \sum_{i=1}^{N} (e_i(t))^H [e_i(t)]^\top e_i(t) \\
& \quad - \frac{1}{2} \sum_{i=1}^{N} \sum_{r=1}^{N} e_i^r ((e_i(t) - \tau_r))^H e_i(t - \tau_r) \\
& \quad + e_i^H(t) - \tau_1) [e_i(t) - \tau_1)]). \tag{22}
\end{align*}
\]

From Assumption 1 and Lemma 2, the following can be obtained,
\[
\begin{align*}
\frac{1}{2} \sum_{i=1}^{N} \{ & (e_i(t))^H g_{ii}(e_i(t), e_i(t - \tau_1)) \\
& + \delta_{ii}(e_i(t), e_i(t - \tau_1)) [e_i(t)] \} \\
= & \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in C_k} \{ (e_i(t))^H g_{ij}(e_j(t), e_i(t - \tau_1)) \\
& + \delta_{ij}(e_i(t), e_i(t - \tau_1)) [e_i(t)] \} \\
= & \sum_{i=1}^{N} \sum_{l=1}^{N} \left( \text{sign}(\text{Re}(e_i(t))) \text{Re}(g_{ii}(e_i(t), e_i(t - \tau_1))) \\
& + \text{sign}(\text{Im}(e_i(t))) \text{Im}(g_{ii}(e_i(t), e_i(t - \tau_1))) \right) \\
\leq & \sum_{i=1}^{N} \|g_{ii}(e_i(t), e_i(t - \tau_1))\|_1 \\
\leq & \sum_{i=1}^{N} \eta_{ii}^{(1)} \|e_i(t)\|_1 + \sum_{i=1}^{N} \zeta_{ii}^{(1)} \|e_i(t - \tau_1)\|_1 \\
= & \sum_{i=1}^{N} I_{N}^{\top} \eta_{i}^{(1)} e_i(t) + \sum_{i=1}^{N} I_{N}^{\top} \zeta_{i}^{(1)} e_i(t - \tau_1), \tag{23}
\end{align*}
\]

where
\[
\begin{align*}
e_i(t) & = ([e_1(t)]^1, [e_2(t)]^1, \ldots, [e_N(t)]^1)^T, e_i(t - \tau_1) \\
& = ([e_1(t) - \tau_1], [e_2(t) - \tau_1], \ldots, [e_N(t) - \tau_1])^T
\end{align*}
\]

and
\[
\eta^{(1)} = \text{diag}(\eta_{11}^{(1)}, \ldots, \eta_{N1}^{(1)}, \ldots, \eta_{N1}^{(1)}, \ldots, \eta_{N1}^{(1)}, \ldots, \eta_{N1}^{(1)}, \ldots, \eta_{N1}^{(1)}).
\]

In addition, it follows from Lemma 2 that
\[
\begin{align*}
\frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in C_k} \{ & (e_i(t))^H c_1 a_{ij} G_1 e_j(t) \\
& + e_i^H(t) c_{1} a_{ij} [e_i(t)] \} \\
= & \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in C_k} \{ (e_i(t))^H c_1 a_{ij} G_1^H [e_i(t)] \\
& + e_i^H(t) c_{1} a_{ij} [e_i(t)] \} \\
\leq & \sum_{i=1}^{N} \sum_{l=1}^{N} \{ \text{Re}(c_1 a_{ij} \delta_i^{(1)}) + \|\text{Im}(c_1 a_{ij} \delta_i^{(1)})\| \} |e_j(t)|_1 \\
& + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \sum_{l=1}^{N} |c_1 a_{ij} \delta_i^{(1)}| |e_j(t)|_1 \\
= & \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ij}^{(1)} |e_j(t)|_1 \\
= & \sum_{i=1}^{N} I_{N}^{\top} \Xi^{(1)} e_i(t), \tag{24}
\end{align*}
\]

and
\[
\begin{align*}
\frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in C_k} \{ & (e_i(t))^H c_2 b_{ij} G_2 e_j(t - \tau_2) \\
& + e_i^H(t - \tau_2) c_{2} b_{ij} G_2^H [e_i(t)] \} \\
= & \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{r} \sum_{j \in C_k} \{ (e_i(t))^H c_2 b_{ij} G_2^H [e_i(t)] \\
& + e_i^H(t - \tau_2) c_{2} b_{ij} \delta_j^{(2)} [e_j(t)] \} \\
\leq & \sum_{i=1}^{N} \sum_{l=1}^{N} \{ \|\text{Re}(c_2 b_{ij} \delta_i^{(2)})\| + \|\text{Im}(c_2 b_{ij} \delta_i^{(2)})\| \} |e_j(t) - \tau_2)|_1 \\
& + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \sum_{l=1}^{N} \omega_{ij}^{(l)} |e_j(t) - \tau_2)|_1 \\
= & \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ij}^{(l)} |e_j(t) - \tau_2)|_1 \\
= & \sum_{i=1}^{N} I_{N}^{\top} \Xi^{(l)} e_i(t - \tau_2), \tag{25}
\end{align*}
\]

in which \( \Xi^{(l)} = \text{diag}(\lambda_{ij}^{(l)})_{N \times N} \) and
\[
\lambda_{ij}^{(l)} = \begin{cases} 
\text{Re}(c_1 a_{ij} \delta_i^{(1)}) + |\text{Im}(c_1 a_{ij} \delta_i^{(1)})|, & i = j, \\
|c_1 a_{ij} \delta_i^{(1)}|, & i \neq j.
\end{cases}
\]
On the other hand, according to Lemma 2,

\[
-\beta \sum_{i=1}^{N} e_i(t) H e_i(t) \sum_{i=1}^{N} \|e_i(t)\|_1^2 \\
= -\beta \sum_{i=1}^{N} \|e_i(t)\|_1 \|e_i(t)\|_1^2 \\
\leq -\beta \sum_{i=1}^{N} \|e_i(t)\|_1^2 \\
\leq -\beta \left( \sum_{i=1}^{N} \|e_i(t)\|_1 \right)^2. \tag{26}
\]

It follows from Lemma 2 that

\[
-\frac{1}{2} \sum_{i=1}^{N} d_i \left[ (e_i(t))^H e_i(t) + e_i(t) e_i(t)^H \right] \\
= -\sum_{i=1}^{N} I_N^T D e_i(t), \tag{27}
\]

and

\[
-\frac{1}{2} \sum_{p=1}^{2} \sum_{i=1}^{N} \varepsilon_i^{(p)} \left[ (e_i(t-\tau_p))^H e_i(t-\tau_p) \\
+ e_i(t-\tau_p) e_i(t-\tau_p) \right] \\
= -\sum_{p=1}^{2} \sum_{i=1}^{N} I_N^T \Pi^{(p)} e_i(t-\tau_p). \tag{28}
\]

in which \(D = \text{diag}(d_1, d_2, \cdots, d_N)\).

Substituting (23)-(28) into (22) gives

\[
C_{t_0} I_{q(t)} V_2(t) \\
\leq \sum_{i=1}^{N} I_N^T [\varepsilon_i^{(1)} + \Xi^{(i)} - D] e_i(t) \\
+ \sum_{i=1}^{N} I_N^T (\varepsilon_i^{(1)} - \Pi^{(1)}) e_i(t-\tau_1) \\
+ \sum_{i=1}^{N} I_N^T (\Omega^{(i)} - \Pi^{(2)}) e_i(t-\tau_2) - \beta \left( \sum_{i=1}^{N} \|e_i(t)\|_1 \right)^2. \tag{29}
\]

Choose appropriate \(d_i (i = 1, 2, \cdots, N)\) such that \(\eta^{(1)} + \Xi^{(i)} - D \leq 0\) for all \(i = 1, 2, \cdots, n\), according to (19):

\[
C_{t_0} I_{q(t)} V_2(t) \leq -\beta \left( \sum_{i=1}^{N} \|e_i(t)\|_1 \right)^2. \tag{30}
\]

Similarly, based upon Lemma 3, the delayed FFCVCNs (1) under controller (18) is said to achieve cluster synchronization within \(T_2\). The proof is complete. \(\square\)

Remark 3: Selecting \(d_i, \beta, \varepsilon_i^{(1)}, \varepsilon_i^{(2)}\), \(i = 1, 2, \cdots, N\) parameters, can choose freely within the range of satisfying conditions (8) and condition (19). However, to increase the complexity and authenticity of the system, the next section will select parameters that can make the system show a chaotic state.

Remark 4: Two delay-dependent control strategies are proposed in this paper. Each part of the controllers (7) and (18) has a unique contribution to the FNT cluster synchronization of delayed FFCVCNs. The controllers (7) and (18) include the time delays \(\tau_p\) and the terms \(\frac{1}{2} \sum_{p=1}^{2} \varepsilon_i^{(p)} e_i(t-\tau_p)\) or \(-\frac{1}{2} \sum_{p=1}^{2} e_i^{(p)} e_i(t)|H(t-\tau_p)| e_i(t-\tau_p)|\), which means larger control inputs. Moreover, the terms \(-d_i e_i(t), -\beta|e_i(t)||e_i(t)|^4\) and \(-\beta|e_i(t)||e_i(t)|^4\) play a key part in achieving FNT cluster synchronization. In addition, designing the delay-independent controllers that do not include the time-delay term but can achieve better synchronization control effect and obtain smaller ST will be the focus of the authors’ future research.

Remark 5: In [15], [21], [22], and [36], the original complex valued network is decomposed into two real valued subnetworks, and controllers are designed for the two real valued subnetworks respectively. In this paper, based on the introduced complex-valued signum function, the power-law control strategy of the community network is designed directly in the complex domain, and the Lyapunov function based on the complex-valued variable is constructed to realize the FNT cluster synchronization of the complex community network. The complexity of the calculation process is reduced, and the universality and universality of the theoretical results are improved.

Remark 6: Many studies on FNT synchronization have been reported for CV dynamical networks [23], [32], [33], [34]. However, these studies have the following limitations. First, the coupling strength, inner coupling matrix and outer coupling matrix of the mathematical models considered in [23] and [34] are RV although the state variables and the system function are CV. The issues of integrating fully CV coupling strengths and couplings into the research of synchronization for complex networks are more realistic and require more complicated analysis. Second, a class of integer-order fully CV dynamical networks has been investigated in [32] and [33] without considering the FO calculus. Third, all the above-mentioned studies have not considered time delay or cluster synchronization. Therefore, it is reasonable to state that the proposed technique and results for the FNT synchronization of complex networks are non-trivial and novel when compared with the literature [23], [33], [34]. Moreover, the results in this paper represent the enhancement and generalization of the already established results.

Remark 7: The FNT cluster synchronization for delayed FFCVCNs is studied by designing two delay-dependent controllers in this paper. The community networks contain internal delay, non-delayed and delayed couplings, which represent a more general situation and have a wider range of
applications. It can be applied to complex networks analysis with cluster effect, such as traffic control, engineering control, brain science, communication engineering, ecology, etc.

Remark 8: FNT synchronization denotes that synchronization may be realized in a certain FNT, in which the ST is dependent on the initial states. However, not all of the initial states of the complex networks are available and accessible. Thus, one future research direction will be the fixed-time cluster synchronization of delayed FFCVCNs.

IV. NUMERICAL SIMULATIONS

A class of delayed FFCVCNs of 9 dynamical nodes with 3 communities $C_1 = \{1, 2, 3\}$, $C_2 = \{4, 5, 6\}$, $C_3 = \{7, 8, 9\}$ is considered. Fig. 1 shows the topology of the proposed directed community network.

The following FFCVCNs are considered.

$$\frac{d}{dt}C_i x_i(t) = f_{\mu_i}(x_{\mu_i}(t), x_i(t - \tau_1)) + c_1 \sum_{k=1}^{r} \sum_{j \in C_k} a_{ij} G_1 x_j(t) + c_2 \sum_{k=1}^{r} \sum_{j \in C_k} b_{ij} G_2 x_j(t - \tau_2) + u_i(t),$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T$, $i = 1, 2, \ldots, 9$.

The dynamic behavior of isolated nodes in the three communities can be described as

$$\frac{d}{dt}C_i x_{\mu_i}(t) = f_{\mu_i}(s_{\mu_i}(t), s_{\mu_i}(t - \tau_1)), \mu_i = 1, 2, 3.$$

In the formula,

$$f_1(s_1(t), s_1(t - \tau_1)) = D_1 s_1(t) + g_{11}(s_1(t)) + g_{12}(s_1(t - \tau_1)),$$

$$f_2(s_2(t), s_2(t - \tau_1)) = D_2 s_2(t) + g_{21}(s_2(t)) + g_{22}(s_2(t - \tau_1)),$$

$$f_3(s_3(t), s_3(t - \tau_1)) = D_3 s_3(t) + g_{31}(s_3(t)) + g_{32}(s_3(t - \tau_1)),$$

where

$$s_1(t) = (s_{11}(t), s_{12}(t), s_{13}(t))^T \in \mathbb{C}^3,$$

$$g_{11}(s) = (0, -s_{11s_{13}}, \frac{s_{11}s_{12}}{2}, \frac{s_{11}s_{12}}{2})^T,$$

$$g_{12}(s) = (0, (0.1 + 0.01i)\cos(s_{12}^R) + \sin(s_{12}^I)i, 0)^T,$$

$$g_{21}(s) = (0, -s_{21}s_{23}, \frac{s_{21}s_{22}}{2}, \frac{s_{21}s_{22}}{2})^T,$$

$$g_{22}(s) = (0, 0, (1 - i)\cos(s_{22}^R) + \sin(s_{22}^I)i)^T,$$

$$g_{31}(s) = (0, -s_{31}s_{33}, \frac{s_{31}s_{32}}{2}, \frac{s_{31}s_{32}}{2} + is_{31}s_{33})^T,$$

$$g_{32}(s) = (0, 0.01 + 0.01i)(\cos(s_{32}^R) + \sin(s_{32}^I)i, 0)^T,$$

$$D_1 = \begin{bmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -2 & 2 & 0 \\ 60 + 0.02i & -1 - 0.06i & 0 \\ 0 & 0 & -0.8 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} -21 & 21 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -6 \end{bmatrix}.$$

In the following numerical simulation, the initial conditions of the system (31) are selected as

$$x_{i1}(\tau) = -5 + 0.9i + (-3.8 + 0.8i)i,$$

$$x_{i2}(\tau) = 3.5 - 0.7i + (-3.1 + 0.6i)i,$$

$$x_{i3}(\tau) = 0.9 - 0.4i + (1.9 - 0.9i)i,$$

where $i = 1, 2, \ldots, 9$, $\tau \in [-0.5, 0]$. Under the initial values $s_1(\tau) = (-0.35 - 0.2i, -0.2 + 0.3i, -1.3 - 0.3i)^T$, $s_2(\tau) = (-1 - 1.5i, 0.8 + 1.3i, 2 - 0.6i)^T$, $s_3(\tau) = (1 + 2.5i, -2 + 1.6i, -2.1 - 0.6i)^T$, $\tau \in [-0.5, 0]$. The phase trajectories of the models (33)-(35) are shown in Figs. 2-7, respectively, when $\alpha = 0.97$.

Next, two different controllers are designed to realize the FNT cluster synchronization of system (31).

A. EXAMPLE 1

First, considering the FNT cluster synchronization of the system (31) with the controller (7). Choose $d_1 = 81,$
$d_2 = 82, d_3 = 81, d_4 = 71.7, d_5 = 63.4, d_6 = 63.4, d_7 = 76.7, d_8 = 76.7, d_9 = 76.7, \varepsilon_{1}^{(1)} = \varepsilon_{2}^{(1)} = \varepsilon_{3}^{(1)} = 0.1005, \varepsilon_{4}^{(1)} = \varepsilon_{5}^{(1)} = \varepsilon_{6}^{(1)} = 1.4142, \varepsilon_{7}^{(1)} = \varepsilon_{8}^{(1)} = \varepsilon_{9}^{(1)} = 0.0141, \varepsilon_{1}^{(2)} = 2.5, \varepsilon_{2}^{(2)} = 4.6, \varepsilon_{3}^{(2)} = 2.5, \varepsilon_{4}^{(2)} = 1.8, \varepsilon_{5}^{(2)} = 3.6, \varepsilon_{6}^{(2)} = 3.6, \varepsilon_{7}^{(2)} = 2.7, \varepsilon_{8}^{(2)} = 2.7, \varepsilon_{9}^{(2)} = 2.7, \alpha = 0.97,$

\[ c_1 = 10 - 3i, c_2 = 1 - i, \tau_1 = 0.5, \tau_2 = 0.1, \]

\[ G_1 = \begin{pmatrix} 1 + 2i & 0 & 0 \\ 0 & 1.5 + 1.6i & 0 \\ 0 & 0 & 1.5 + 1.6i \end{pmatrix}, \]

\[ G_2 = \begin{pmatrix} 2 + 1i & 0 & 0 \\ 0 & 1.3 + 1.2i & 0 \\ 0 & 0 & 1.8 + 1.2i \end{pmatrix}, \]

\[ A = \begin{pmatrix} -1 + li & 1 + li & 0 & -1 - li & 0 & 1 + li & 0 & 0 & 0 \\ 1 + li & -2 - 2i & 1 + li & 0 & 0 & 0 & -1 - li & 1 + li & 0 \\ 0 & 1 + li & -1 - li & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 - li & 1 + li & 0 & -1 - li & 1 + li & 0 \\ -1 - li & 1 + li & 0 & 1 + li & -2 - 2i & 1 + li & 0 & -1 - li & 0 \\ 0 & 0 & 0 & 1 + li & 1 + li & -2 - 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 - li & 0 & 1 + li & -1 - li & 1 + li & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 + li & -1 - li & 1 - li \end{pmatrix}, \]

\[ B = 0.5A. \]
ST can be estimated as $T_1 \leq 1.6116$. Figs. 8-10 show the trajectories of synchronization errors of the internal nodes of the 3 communities under the controller (7). It can be seen from Figs. 8-10 that the synchronization errors of the internal nodes of the same community converges to zero.

The trajectories of state variables $x_{i1}, x_{i2}$ and $x_{i3}$ ($i = 1, 2, \ldots, 9$) of the system (31) under controller (7) are shown in Figs. 11-13 respectively.

The dynamic behaviors of nodes in the same community are consistent, while the dynamic behaviors of nodes from different communities are different.

It can be observed from the estimation expression (9) of $T_1$, that the FO of the system $\alpha$ and the FO power law $\beta$ significantly impact the estimation of the ST. Fig. 14 describes the relationship between the estimated ST $T_1$, the FO of the system $\alpha$ and the control parameter $\gamma$ under the conditions of $0 < \gamma < 2\alpha - 1$ and $\beta = 4$. It can be seen from Fig. 14 that the synchronization error converges to the equilibrium point.
faster with the decrease of $\alpha$. That is, when $0.5 < \alpha < 1$, $T_1$ will increase with the increase of $\alpha$. Let $E_1(t), E_2(t)$ and $E_3(t)$ express as the synchronization errors of the three cluster communities in the community networks, respectively, that is

$$
\begin{align*}
E_1(t) &= \sqrt{\sum_{\mu=1}^{\mu_{\max}} ||x_{\mu}(t) - s_1(t)||^2}, \\
E_2(t) &= \sqrt{\sum_{\mu=2}^{\mu_{\max}} ||x_{\mu}(t) - s_2(t)||^2}, \\
E_3(t) &= \sqrt{\sum_{\mu=3}^{\mu_{\max}} ||x_{\mu}(t) - s_3(t)||^2}.
\end{align*}
$$

Under the same initial conditions of the system (31), choose $\gamma = 0.6$, $\alpha = 0.97$. Under the action of the controller (7), Figs. 15-17 show the trajectories of synchronization errors of 3 communities under different FO power laws $\beta$ of the system (31). It can be seen from the figures that the synchronization error converges to the equilibrium point faster with the increase of $\beta$. That is, $T_1$ will decrease with the increase of $\beta$.

**B. EXAMPLE 2**

Next, consider the FNT cluster synchronization of the system (31) with the controller (18). Choose $d_1 = d_2 = d_3 = 148, d_4 = d_5 = d_6 = 159, d_7 = d_8 = d_9 = 124, \varepsilon_1^{(1)} = \varepsilon_2^{(1)} = \varepsilon_3^{(1)} = 0.11, \varepsilon_4^{(1)} = \varepsilon_5^{(1)} = \varepsilon_6^{(1)} = 1.45, \varepsilon_7^{(1)} = \varepsilon_8^{(1)} = \varepsilon_9^{(1)} = 0.02, \varepsilon_1^{(2)} = 3, \varepsilon_2^{(2)} = 6, \varepsilon_3^{(2)} = 3, \varepsilon_4^{(2)} = 2.5, \varepsilon_5^{(2)} = 5, \varepsilon_6^{(2)} = 5, \varepsilon_7^{(2)} = 3.8, \varepsilon_8^{(2)} = 3.8, \varepsilon_9^{(2)} = 3.8, \alpha = 0.97, \gamma = 0.7, \beta = 4$. Afterwards, the conditions in
Theorem 2 (18) are met. According to Theorem 2, the system (31) can achieve FNT cluster synchronization under the controller (18) and the ST can be estimated as $T_2 \leq 2.8191$. Figs. 18-20 show the trajectories of the synchronization errors of the internal nodes of the 3 communities under the controller (18). It can be seen from Figs. 18-20 that the synchronization errors of the internal nodes of the same community converges to zero. The trajectories of state variables $x_{i1}, x_{i2}$ and $x_{i3}$ ($i = 1, 2, \ldots 9$) of the system (31) under controller (18) are shown in Figs. 21, 22 and 23.

Similarly, it can be seen from the estimation expression (20) of $T_2$ that the FO of system $\alpha$ and the FO power law $\beta$ significantly impact the estimation of the ST. Under the conditions of $0 < \gamma < \alpha$ and $\beta = 4$, Fig. 24 illustrates the relationship between the estimated ST $T_2$, the FO of the system $\alpha$ and the control parameter $\gamma$. It can be seen from Fig. 24 that the synchronization error converges to the equilibrium point faster with the decrease of $\alpha$. That is, when $0 < \alpha < 1$, $T_2$ will increase with the increase of $\alpha$.

Under the same initial conditions of the system (31), choose $\gamma = 0.6$ and $\alpha = 0.97$. Under the action of the
controller (18). Figs. 25-27 show the trajectories of the synchronization errors of 3 communities under different values of the FO power law β of the system (31). It can be observed from the figures that the synchronization error converges to the equilibrium point faster with the increase in β. That is, \( T_2 \) will increase with the decrease in β.

Table 1 compares the settling times of the system (31) under the controllers (7) and (18) with different groups of different parameters. It can be seen from Table 1 that the ST for FNT cluster synchronization of the system (31) will decrease with the increase in β or decrease in α under both controllers.

**Remark 9:** To increase the complexity and authenticity of the system, the parameters that can make the system show a chaotic state are selected in Examples 1 and 2. In addition, the selection of FO should meet the limiting condition in Theorem 1 and Theorem 2 (\( \frac{1}{2} < \alpha < 1 \) in Theorem 1, \( 0 < \alpha < 1 \) in Theorem 2). To illustrate the effectiveness of the controllers when α and β are in other range, different groups of parameters are set. Table 1 shows the settling times of the system (31) under the controllers (7) and (18) with different parameters.

**Remark 10:** From inequalities (9) and (20), as well as Table 1, it can be seen that the FO of the system α and FO power law β play key roles in the upper limit of the ST of FNT cluster synchronization. Particularly, \( T_1 \) and \( T_2 \) will decrease with the increase of β or the decrease of α, meaning that the FNT cluster effect is better when α is small or β is large. Therefore, in practical application, appropriate
parameters can be selected after weighing the control cost and synchronization effect according to the actual situation.

Remark 11: The ST is related to the initial value of the system. Under the initial value conditions given in this paper, controller (7) performs better than controller (18). Furthermore, the comparison of the restrictions of the two controllers on FO shows that the restriction of the controller (18) is more relaxed than the controller (7). Therefore, different controllers can be selected according to different conditions and constraints to achieve the effect of FNT cluster synchronization.

Remark 12: It can be observed from Figs. 8-13 and 18-23 that the actual ST is smaller than the estimated ST. The reason is that the inequality technique in Lemmas 1 to 4 is used for the FNT synchronization criteria of the system (31). In this process, the estimated ST will be amplified, which means the estimated ST may be larger than the actual ST and there is a certain error. How to improve Lemma 3 and estimate ST more accurately will be the focus of the authors’ future research.

V. CONCLUSION

This paper explores the FNT cluster synchronization problems for a class of FFCVCNs with internal delay as well as non-delayed and delayed couplings. Instead of transforming the CV networks into two independent RV systems, two delay-dependent controllers are designed based on the quadratic norm and a novel norm composed of the absolute-valued norm and some useful synchronization criteria are obtained to guarantee the proposed complex networks to be cluster synchronized in FNT. The derived estimations of the ST reveal that the FO of the system α and the FO power law β play important roles in the upper bound of the ST for FNT cluster synchronization. Finally, the simulations indicate the practicability, effectiveness and flexibility of the acquired theoretical findings. It is worth mentioning that the obtained new criteria offer advantages over certain previous criteria due to being less cautious, in addition to extending some current results. Note that discontinuous behaviors of dynamical systems can be found everywhere such as impacting machines, dry friction [40], [41]. In this paper, FNT cluster synchronization is achieved by applying continuous control to each node. However, in reality, the complex networks are usually composed of numerous nodes, and it is not easy to control each node at the same time [42]. Moreover, continuous control of the system may increase costs [43]. Hence, investigating the synchronization of FFCVCNs with internal delay as well as non-delayed and delayed couplings via periodically intermittent pinning control is essential and interesting. It will be one of the future research directions for the authors.

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