Asymptotic convergence rates for coordinate descent in polyhedral sets

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Abstract We consider a family of parallel methods for constrained optimization based on projected gradient descents along individual coordinate directions. In the case of polyhedral feasible sets, local convergence towards a regular solution occurs unconstrained in a reduced space, allowing for the computation of tight asymptotic convergence rates by sensitivity analysis, this even when global convergence rates are unavailable or too conservative. We derive linear asymptotic rates of convergence in polyhedra for variants of the coordinate descent approach, including cyclic, synchronous, and random modes of implementation. Our results find application in stochastic optimization, and with recently proposed optimization algorithms based on Taylor approximations of the Newton step.

1 Introduction

The interest for the coordinate descent methods lies in their simplicity of implementation and flexibility [12]. Yet their performances in terms of speed of convergence are generally modest compared to their centralized counterparts and still subject to active research. In this work we derive the asymptotic convergence rates of parallel implementations of the gradient projection algorithm [3] in the context of the constrained minimization of a strictly convex, continuously differentiable function over a polyhedral feasible set—this class of problems is met for instance in bound-constrained optimization or in dual optimization. Our developments rely on the property of projected gradient methods to asymptotically behave, when applied in a polyhedral feasible set specified by a collection of affine inequality constraints, like unconstrained gradient descents on the surface of the polyhedron, provided that the gradient of the cost function at the point of convergence be a negative combination of the
normal vectors of the active constraints. This property facilitates the derivation of rates of convergence in the form of matrices playing roles analogous to those of system matrices in linear state space models, thus reducing the question of the convergence to the spectral analysis of matrices.

Outline — Section 2 formulates the gradient projection algorithm and identifies certain properties enjoyed by the method in polyhedral sets. From the initial algorithm we derive parallelized implementations operating gradient descents along coordinate directions, each of them characterized by the way these operations are organized (synchronously, cyclically, randomly, etc.). In Section 3 we compute asymptotic convergence rates for the parallel algorithms under the hypothesis of twice continuous differentiability at the point of convergence. Our developments are then reconsidered for non-twice differentiable cost functions and from the perspective of stochastic optimization settings.

Notation — In this paper vectors are column vectors and denoted by $x = (x_1, ..., x_n)$, where $x_1, ..., x_n$ are the coordinates of $x$. Subscripts are reserved for vector coordinates. The transpose of a vector $x \in \mathbb{R}^p$ is denoted by $x^\top$ and its Euclidean norm by $\|x\|_2$; for any $M \in \mathbb{R}^{p \times p}$ symmetric, positive definite, we define the scaled norm $\|x\|_M := (x^\top M x)^{\frac{1}{2}}$. Let $S$ be a finite set, $\{a^k\}$ a sequence in $S$, and $a \in S$. We write $a^k \rightarrow a$ if there is a $k$ such that $a^k = a$ for $k > \bar{k}$. Similarly, for $A \subset S$ and a sequence $\{A^k\}$ of subsets of $S$, we write $A^k \rightarrow A$ if there is a $k$ such that $A^k \equiv A$ for $k > \bar{k}$.

2 The gradient projection algorithm

2.1 Formulation

Consider a closed convex subset $X$ of a real vector space $\mathbb{R}^m$ and a function $f \in F(m)$, where, for any space $\mathbb{R}^p$, $F(p)$ denotes the set of the functions $\mathbb{R}^p \rightarrow \mathbb{R}$ strictly convex, continuously differentiable with gradient $\nabla f$ Lipschitz continuous. Lipschitz continuity of $\nabla f$ can be understood as the existence of a symmetric, positive definite matrix $L \in \mathbb{R}^{p \times p}$ satisfying $[\nabla f(x) - \nabla f(y)]'(x - y) \leq \|x - y\|^2_L$ for any $x, y \in \mathbb{R}^p$. It follows from this condition that

$$\nabla f(x)'(y - x) \geq f(y) - f(x) - \frac{1}{2}\|x - y\|^2_L, \quad \forall x, y \in \mathbb{R}^p.$$  (1)

Let $\lambda$ and $\bar{\lambda}$ be two positive scalar constants such that $0 < \lambda \leq \bar{\lambda} < \infty$. For any real space $\mathbb{R}^p$, we let $\mathcal{T}(p)$ define the set of the symmetric, positive definite scaling matrices in $\mathbb{R}^{p \times p}$ with eigenvalues bounded by $\lambda$ and $\bar{\lambda}$, i.e. $\mathcal{T}(p) = \{T \in \mathbb{R}^{p \times p} : \lambda I \leq T \leq \bar{\lambda} I\}$. We consider the following algorithm.

Algorithm 1 (Scaled gradient projection $\mathcal{G}^{(T,X)}$). Consider a closed, convex set $X \subset \mathbb{R}^p$, a function $f \in F(p)$, a scaling mapping $T : F(p) \times X \mapsto \mathbb{R}^p$.
Proposition 3.7 in Section 3.3] that \( \bar{x} \) from

\[
x^{k+1} = G^{(T,X)}(f,x^k), \quad k = 0, 1, 2, ..., \tag{2}
\]

with \( G^{(T,X)} \) defined for \( x \in X \) by \( G^{(T,X)}(f,x) := \bar{x}(a) \), where

\[
\bar{x}(a) \in \arg\min_{y \in X} \nabla f(x)'(y - x) + \frac{1}{2} \|y - x\|^2_{aT(f,x)}^{-1}, \quad \forall a > 0, \tag{3}
\]

and \( a \) is an appropriate step size bounded above \( 0 \).

Any point \( x \in X \) such that \( G^{(T,X)}(f,x) = x \) is called stationary. Since by assumption \( f \) is convex, the stationary points coincide with the solutions of the minimization of \( f \). The first-order optimality condition of a point \( x \in X \) is therefore given by

\[
\nabla f(x)'(y - x) \geq 0, \quad \forall y \in X. \tag{4}
\]

If \( f \) is strictly convex, (4) holds for at most one point and there is at most one solution. Notice that condition (4) reduces, for the subproblem (3) and any step size \( a > 0 \), to

\[
[\nabla f(x) + [aT(f,x)]^{-1}(\bar{x}(a) - x)']'(y - \bar{x}(a)) \geq 0, \quad \forall y \in \mathbb{R}^p. \tag{5}
\]

The notion of ‘gradient projection’ in Algorithm 1 can be explained by the observation that \( \bar{x}(a) \) in (3) coincides with the scaled projection on \( X \),

\[
\bar{x}(a) \in \arg\min_{y \in X} \|y - z\|^2_{T(f,x)}^{-1}, \tag{6}
\]

of the vector \( z = x - aT(f,x)\nabla f(x) \) obtained by scaled gradient descent from \( x \). It follows from the convexity of \( X \) and from the projection theorem [1] Proposition 3.7 in Section 3.3] that \( \bar{x}(a) \) is uniquely defined in (6) and (3).

Global convergence of (2) is commonly guaranteed by using an approximate line search rule of the type Armijo [4], which consists of setting \( \hat{a} = \hat{a}(f,x^k) \) where, for \( x \in X \), \( \hat{a}(f,x) \) is defined as the largest \( a \in \{\beta^m\}_{m=0}^{\infty} \) satisfying

\[
f(x) - f(\bar{x}(a)) \geq \sigma \|\bar{x}(a) - x\|_{\hat{a}T(f,x)}^{-1}. \tag{7}
\]

From [4] we know that the step-sizes computed by (7) are restricted to a set \([a, 1]\), where \( a > 0 \) is a function of the Lipschitz constant of \( \nabla f \). When the algorithm is appropriately designed, (7) becomes asymptotically trivial. This is illustrated by the next result, shown in the Appendix, where \( L \) denotes the Lipschitz constant in the sense of [1].

**Proposition 1 (Line search efficiency).** Suppose that Algorithm 1 is implemented with the step-size selection rule (7) and generates a sequence \( \{x^k\} \) converging to a stationary point \( x^* \). If

\[
2(1 - \sigma)T(f,x)^{-1} \succeq L, \quad \forall x \in X, \tag{8}
\]

then \( \hat{a}(f,x^k) = 1 \) for all \( k \). If \( T(f,\cdot) \) is continuous, \( f \) is twice continuously differentiable in a neighborhood of \( x^* \), and

\[
2(1 - \sigma)T(f,x^*)^{-1} \succeq \nabla^2 f(x^*), \tag{9}
\]

then \( \hat{a}(f,x^k) \rightarrow 1. \)
2.2 Descent in polyhedral sets

Throughout the paper we consider the following problem.

**Problem 1.** Solve

\[
\min_{x \in X} f(x)
\]

where \( f \in F(m) \), \( \nabla f \) satisfies (7) with Lipschitz constant \( L \), and \( X \) is the nonempty polyhedron \( X := \{ x \in \mathbb{R}^m \mid a'_1 x \leq b_1, a'_2 x \leq b_2, \ldots, a'_p x \leq b_p \} \), with \( a_1, \ldots, a_p \in \mathbb{R}^m \) and \( b_1, \ldots, b_p \in \mathbb{R} \).

The affine constraint functions in Problem 1 can be rewritten as \( c_j(x) \leq 0 \), where \( c_j(x) := a'_j x - b_j \) and \( \nabla c_j(x) = a_j \) for \( x \in \mathbb{R}^m \) \((j = 1, \ldots, p)\). A constraint \( c_j(x) \leq 0 \) is said to be inactive at a point \( x \in \mathbb{R}^m \) if \( c_j(x) < 0 \), and active if \( c_j(x) = 0 \), in which case we write \( j \in \mathcal{A}(x) \), where \( \mathcal{A}(x) \subset \{1, \ldots, p\} \) denotes the index set of the active constraints at \( x \). If a constraint qualification holds for Problem 1 (e.g. Slater’s condition [5]), then the first-order optimality condition (4) for a point \( x \) holds at strict complementarity. By Proposition 2 (Identification of the active constraints), assume that \( \mathcal{A}(x) = \mathcal{A}(x^*) \) holds for Problem 1 and consider Algorithm 1. Any sequence \( \{x^k\} \) generated by Algorithm 1 and converging towards \( x^* \) is such that \( \mathcal{A}(x^k) \to \mathcal{A}(x^*) \).

The proof is given in the Appendix. A consequence of Proposition 2 is that, under strict stationarity at a point of convergence \( x^* \), local convergence occurs in a subspace \( \{ x \in \mathbb{R}^m \mid a'_j x = b_j, j \in \mathcal{A}(x^*) \} \) with dimension \( \tilde{m} \leq m \), called the reduced space at \( x^* \), and orthogonal to the normal vectors of all the active constraints at \( x^* \). By \( E(x^*) \) we denote any matrix whose columns form an orthonormal basis of the reduced space at \( x^* \). For any \( x \in X \) such that \( \mathcal{A}(x) = \mathcal{A}(x^*) \), there is a unique vector \( \tilde{x} \in \mathbb{R}^{\tilde{m}} \) satisfying \( x = x^* + E(x^*) \tilde{x} \). The following result states that the gradient projections reduce to mere gradient descents in the vicinity of \( x^* \) and derives asymptotics for \( G^{(T,X)} \). We denote by \( \tilde{I} \) the identity matrix in \( \mathbb{R}^{\tilde{m} \times \tilde{m}} \).

**Proposition 3 (Descent in the reduced space).** Let \( x^* \) be a solution of Problem 1 where strict complementarity holds, and consider Algorithm 1. Any vectors \( x, y \in X \) such that \( y = G^{(T,X)}(f,x) \) with step size \( \alpha \), and \( \tilde{x}, \tilde{y} \in \mathbb{R}^{\tilde{m}} \) such that \( x = x^* + E(x^*) \tilde{x} \) and \( y = x^* + E(x^*) \tilde{y} \), satisfy

\[
\tilde{y} = \tilde{x} - \alpha \tilde{T}(f,x) \tilde{\nabla} f(x),
\]
where \( \bar{f} := E(x^*)'\bar{\nabla} f \) and \( \bar{T}(f, x) := [E(x^*)'T(f, x)^{-1}E(x^*)]^{-1} \).

Further, if \( f \) is twice continuously differentiable and \( T(f, \cdot) \) is continuous at \( x^* \), then
\[
\tilde{y} = \left[ \tilde{I} - \hat{a}\bar{T}(f, x)\bar{\nabla}^2 f(x) \right] \tilde{x} + \rho(f, \tilde{x}),
\]
where \( \bar{\nabla}^2 f(x^*) := E(x^*)'\nabla^2 f(x^*)E(x^*) \) and \( \rho(f, \tilde{x}) = o(\|\tilde{x}\|) \). If \( f \) is smooth and \( T(f, \cdot) \) is continuously differentiable at \( x^* \), then the remainder rewrites as \( \rho(f, \tilde{x}) = g(f, \tilde{x})(\tilde{x} - y)(\tilde{x} - y)' \), where \( g(f, \tilde{x}) \) is a function of the derivatives at \( \tilde{x} \) of \( \bar{\nabla}^2 f \) and \( \bar{T}(f, \cdot) \) uniformly bounded in a neighborhood of 0.

**Proof.** Since the proposition is trivial when \( \tilde{m} = 0 \), we suppose that \( \tilde{m} > 0 \). Let \( X^* := \{x \in X | A(x) = A(x^*)\} \), and \( \tilde{X} := \{E(x^*)'(x - x^*) | x \in X^*\} \), which is an open subset of \( \mathbb{R}^{\tilde{m}} \) by continuity of the constraint functions. Moreover, the function \( h(\tilde{x}) := x^* + E(x^*)\tilde{x} \) is a bijection between \( \tilde{X} \) and \( X^* \), i.e. \( X^* \equiv \{h(\tilde{y}) | \tilde{y} \in \tilde{X}\} \). It follows the assumptions on \( x \) and \( y \) that
\[
\tilde{y} = \begin{array}{c}
\underset{\tilde{y} \in \tilde{X}}{\arg\min} \\
\left\{ \bar{f}(y)'(\xi - \tilde{x}) + \frac{1}{2}\|\xi - \tilde{x}\|^2(\bar{a}\bar{T}(f, x))^{-1} \right\}
\end{array}
\]
\[
\tilde{x} - \hat{a}\bar{T}(f, x)\bar{\nabla} f(x)
\]
where (13) follows from the fact that \( \tilde{X} \) is an open set containing \( \tilde{y} \), thus \( \tilde{y} \) is the projection on \( \tilde{X} \) of only one point: itself.

The remaining statements follow directly from (12) and Taylor’s theorem, by linear approximation at \( x^* \) of the displacement \( d(\tilde{x}) := \bar{G}(T, X^*\tilde{x}, x) - x \).

### 2.3 Parallel analysis and coordinate descent

This section considers parallel implementations of Algorithm 1 for Problem 1, where assumption is made that \( X \) is a Cartesian product set.

**Assumption 1 (Parallel analysis).** The feasible set of Problem 1 is given by \( X = X_1 \times \ldots \times X_n \), where each \( X_i \) is a polyhedron in \( \mathbb{R}^{m_i} \), and \( m_1 + \ldots + m_n = m \). The Lipschitz continuity of \( \nabla f \) is considered coordinate-wise and (3) holds for \( L = \text{diag}(L_1, \ldots, L_n) \).

Assumption 1 implicitly defines a set \( N = \{1, \ldots, n\} \) of coordinate directions with respective dimensions \( m_1, \ldots, m_n \), suggesting parallel optimization by coordinate descent.

#### 2.3.1 Coordinate descent

In this study the optimization of \( f \in F(m) \) at a point \( x \in X \) along a particular coordinate direction \( i \in N \) is symbolized by the function \( f_{i:x} \in F(m_i) \) obtained from \( f(x) \) by fixing the other coordinates, i.e.
\[
f_{i:x}(y) := f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n), \quad \forall y \in \mathbb{R}^{m_i}, \ i \in N.
\]
Using (10), we consider in each direction $i$ a scaling mapping $T_i : F(m_i) \times X_i \mapsto T(m_i)$ and define the associated coordinate gradient projection mapping
\[
G_i(f, x) := G_i^{(T, \pi)}(f, x), \quad \forall x \in X, \; i \in N.
\]  
(17)

Based on (17), we formulate a synchronous coordinate descent algorithm, modeled on the Jacobi method, as $x^{k+1} = J(f, x^k)$, where
\[
J(f, x) := \left(G_1, G_2, \ldots, G_n\right)(f, x) \quad \forall x \in X.
\]  
(18)

Notice in (15) that the coordinate descents $G_1, \ldots, G_n$ are applied simultaneously along the $n$ directions. Global convergence of $J$ is conditioned by its quality to guarantee sufficient descent along the considered function at each step, which, in the general case, requires not only synchronism for the applications of $G_1, \ldots, G_n$, but also consensus at the global level on the choice of scaling matrices and step-sizes in each direction [7]. An alternative is to process the coordinate descents sequentially, using the directional mappings
\[
\hat{G}_i(f, x) := (x_1, \ldots, x_{i-1}, G_i(f, x), x_{i+1}, \ldots, x_n), \quad \forall x \in X, \; i \in N.
\]  
(19)

A cyclic coordinate descent algorithm can then be designed by applying the coordinate descent mappings in a predefined order as in the Gauss-Seidel method, i.e. $x^{k+1} = S(f, x^k)$, where
\[
S(f, x) := (\hat{G}_n \circ \ldots \circ \hat{G}_2 \circ \hat{G}_1)(f, x), \quad \forall x \in X.
\]  
(20)

and $\circ$ denotes the composition operator, defined for any $i, j \in N$ by $(\hat{G}_i \circ \hat{G}_j)(f, x) := \hat{G}_i(f, \hat{G}_j(f, x))$. The global convergence of $S$ is guaranteed by approximate line search in each coordinate direction, i.e. using, for $i \in N$ and at every $x \in X$, the step sizes $\hat{a}(f, x, x_i)$ specified by (7). One shows that convergence is conserved when the mappings $G_i$ are applied in random order provided that each coordinate direction is visited an infinite number of times [7]. A random coordinate descent algorithm is given by $x^{k+1} = R^k(f, x^k)$, where
\[
R^k(f, x) := \hat{G}_{\phi^k}(f, x), \quad \forall x \in X, \; k = 0, 1, 2, \ldots,
\]  
(21)

and $\{\phi_i\}_{i=0}^\infty$ is a sequence of coordinate directions randomly selected in $N$, so that each $\phi^k$ is a realization of a discrete random variable defined on a probability space $(N, 2^N, \pi)$, with $\pi = (\pi_1, \ldots, \pi_n) \in (0, 1)^n$. More sophisticated parallel implementations of $G$ involving (block-) coordinate selection routines (e.g. Gauss-Southwell methods [5]) can also be devised.

For any parallel implementation of Algorithm [4] designed with coordinate scaling $\{T_i\}_{i \in N}$, we let $T(f, x) := \text{diag}(T_1(f_{1, x}, x_1), \ldots, T_n(f_{n, x}, x_n))$ for $x \in X$. Proposition [4] extends to coordinate descent, and the step sizes computed by (7) in each direction reduce to 1 if (5) holds [7]. If the scaling mappings are continuous and $f$ is twice continuously differentiable, the Hessian $\nabla^2 f = (\nabla^2 f_{ij})$ may be regarded as a block matrix and decomposed into $\nabla^2 f = \nabla^2 f - \nabla^2 f - \nabla^2 f$, where $\nabla^2 f := \text{diag}(\nabla^2 f_{11}, \ldots, \nabla^2 f_{nn})$ is block diagonal and $\nabla^2 f$ strictly lower triangular. The step sizes of the parallel algorithms then reduce to 1 in the vicinity of $x^*$ if $2(1 - \sigma)T_i(f, x^*)^{-1} > \nabla^2 f(x^*)$ holds for $i \in N$, i.e. if
\[
2(1 - \sigma)T(f, x^*)^{-1} > \nabla^2 f(x^*).
\]  
(22)
2.3.2 Asymptotics

By extension of Proposition 2 (7), one shows the existence of a reduced space where every sequence generated by a parallel implementation of Algorithm 1 and converging to the solution $x^*$ accumulates under strict complementarity for $f$ at $x^*$ or, equivalently, strict complementarity for $f_i(x^*)$ at $x^*_i$ in every direction $i \in N$. The reduced space at $x^* = \{x^*_1, ..., x^*_n\}$ is a Cartesian product and the matrix $E(x^*)$, introduced in Section 2.2, takes the block-diagonal form $E(x^*) = \text{diag}(E_1(x^*_1), ..., E_n(x^*_n))$, where the columns of $E_i(x^*_i)$ form an orthonormal basis of the reduced space at $x^*_i$ in the coordinate direction $i$.

Under Assumption 1, $\nabla f$ rewrites as the n-dimensional composite composite vector $\nabla f = (\nabla_1 f, ..., \nabla_n f)$. For $x \in X$, we define $\nabla_i f(x) := E_i(x^*_i)\nabla f(x)$ ($i \in N$). Similarly, setting $T(f, x) := [E(x^*)^Tf(x)^{-1}E(x^*)]^{-1}$ yields the block-diagonal form $T(f, x) = \text{diag}(T_1(f_{1:x}, x_1), ..., T_n(f_{n:x}, x_n))$, where the diagonal elements are given by $T_i(f_{i:x}, x_i) := [E_i(x^*_i)^Tf_i(f_{i:x}, x_i)^{-1}E_i(x^*_i)]^{-1}$.

In the reduced space at $x^*$, a gradient projection $y = \mathcal{G}_i(f, x)$ with step size $\bar{a}$ in direction $i \in N$ at a point $x \in X$ near $x^*$ reduces, by translation of Proposition 3 into the coordinate descent framework (7), to $x = x^* + \bar{E}(x^*)\bar{y}$ and $y = x^*_i + E_i(x^*_i)\bar{y}$ for some $\bar{x} \in \mathbb{R}^m$ and $\bar{y} \in \mathbb{R}^{m_i}$ satisfying

$$\bar{y} = \bar{x}_i - \bar{T}(f_{i:x}, x_i)\nabla_i f(x), \quad \forall i \in N.$$ (23)

If $f$ is twice continuously differentiable and $T(f, \cdot)$ is continuous at $x^*$, then (23) asymptotically reduces to

$$\bar{y} = [\bar{I}_i - (0, ..., 0, \bar{\alpha}_i, 0, ..., 0)^T\bar{T}(f, x^*)\nabla^2 f(x^*)]\bar{x} + o(\|\bar{x}\|), \quad \forall i \in N,$$ (24)

where $\nabla^2 f(x) := E(x^*)^T\nabla^2 f(x)E(x^*)$ and $\bar{I}_i$ denotes the identity matrix in $\mathbb{R}^{m_i \times m_i}$. Similarly, we write $\nabla^2 f = \nabla^2 f - \nabla^2 f - \nabla^2 f'$, where $\nabla^2 f(x) := E(x^*)^T\nabla^2 f(x)E(x^*)$, and $\nabla^2 f(x) := E(x^*)^T\nabla^2 f(x)E(x^*)$.

3 Asymptotic convergence rates

The developments of this section rely on the following assumption.

**Assumption 2.** Problem 7 has a unique solution $x^*$ where strict complementarity holds and in the vicinity of which $f$ is twice continuously differentiable.

**Terminology.** We derive asymptotic rates of convergence for the algorithms of Section 2.3 by first-order sensitivity analysis around $x^*$. Our aim is to find a matrix $H$ which satisfies an equation of the type $h(x^{k+1}) = H h(x^k) + o(h(x^k))$ for some continuous function $h$ and any sequence $\{x^k\}$ generated by the considered algorithm. If this equation holds for $H \leq 1$, we say that $\{h(x^k)\}$ converges towards $h(x^*)$ with asymptotic rate $H$. Convergence is called sublinear if $H = 1$, and linear if $H < 1$. If $H$ satisfies the inequality $h(x^{k+1}) \leq H h(x^k) + o(h(x^k))$ for any generated sequence and the algorithm may produce a sequence for which the latter inequality holds with equality sign, then we speak of convergence with limiting asymptotic rate $H$. 

Proposition 4 extends to polyhedral sets and arbitrary scaling a result derived in [9] for a cyclic coordinate descent algorithm used in the non-negative orthant with coordinate-wise Newton scaling. The proof provided in this paper is arguably simpler and the requirements less restrictive. The spectral radius of any matrix \( M \in \mathbb{R}^{p \times p} \), defined as the supremum among the absolute values of the eigenvalues of \( M \), is denoted by \( \rho(M) \).

**Proposition 4 (Asymptotic convergence rate of \( S \)).** Let Assumptions 2 and 3 hold for Problem 7. Consider the cyclic algorithm \( x^{k+1} = S(f, x^k) \) implemented with the step-size selection rule 7 and with scaling \( \{ T_i(f_{i=1}, x_i) \}_{i \in \mathbb{N}} \) continuous at \( x^* \), and satisfying condition (22) for the step sizes. Any sequence \( \{ x^k \} \) generated by the algorithm converges towards \( x^* \) with asymptotic rate \( E(x^*) \tilde{S}(f, x^*) \), where

\[
\tilde{S}(f, x^*) = \left[ \tilde{T}(f, x^*)^{-1} - \nabla^2 f(x^*) \right]^{-1} \left[ \tilde{T}(f, x^*)^{-1} - \nabla^2 f(x^*) + \nabla^2 f(x^*) \right],
\]

with \( \rho(\tilde{S}(f, x^*)) \leq 1 \), while \( |f(x^k) - f(x^*)| \) vanishes with limiting asymptotic rate \( \rho(\tilde{S}(f, x^*))^2 \). If \( \nabla^2 f(x^*) \) is positive definite, then \( \rho(\tilde{S}(f, x^*)) < 1 \).

**Proof.** It follows from the assumptions, Proposition 1 and (23), that one can find a \( k < \infty \) such that, for any \( k \geq \tilde{k} \), we have \( x^k = x^* + E(x^*)\tilde{x}^k \) and \( x^{k+1} = x^* + E(x^*)\tilde{x}^{k+1} \) for some \( \tilde{x}^k, \tilde{x}^{k+1} \in \mathbb{R}^m \), with

\[
\tilde{x}^{k+1} = \tilde{G}_n(f, x^*) \tilde{G}_{n-1}(f, x^*) \cdots \tilde{G}_1(f, x^*) \tilde{x}^k + o(\|\tilde{x}^k\|),
\]

where

\[
\tilde{G}_i(f, x^*) := I - \text{diag}(0, \ldots, 0, \tilde{I}, 0, \ldots, 0)\tilde{T}(f, x^*)\nabla^2 f(x^*), \quad \forall i \in \mathbb{N},
\]

embodies the effect of a gradient projection along coordinate direction \( i \). Applying Lemma 2 from the Appendix yields \( \tilde{G}_n(f, x^*) \cdots \tilde{G}_1(f, x^*) = \tilde{S}(f, x^*) \).

Consider the sequence of function values \( \{ f(x^k) \} \). It follows from Proposition 2 and (10) that, for \( k \) large enough, we have \( \nabla f(x^*)' (x^k - x^*) = 0 \). Setting \( \tilde{x}^k := \nabla^2 f(x^*)' \tilde{x}^k \) and \( \tilde{S}(f, x^*) := \nabla^2 f(x^*)' \tilde{S}(f, x^*) \nabla^2 f(x^*)' \), the Taylor theorem yields

\[
f(x^{k+1}) - f(x^*)
\]
\[
= \frac{1}{2}(x^{k+1} - x^*)' \nabla^2 f(x^*)(x^{k+1} - x^*) + o(\|x^{k+1} - x^*\|^2) \tag{28}
\]
\[
= \frac{1}{2}(\tilde{x}^{k+1})' \nabla^2 f(x^*)\tilde{x}^{k+1} + o(\|\tilde{x}^{k+1}\|^2) \tag{29}
\]
\[
\leq \frac{1}{2}\|\tilde{S}(f, x^*)\tilde{x}^k\|^2 + o(\|\tilde{x}^k\|^2) \tag{30}
\]
\[
\leq \frac{1}{2}\rho(\tilde{S}(f, x^*))\|\tilde{x}^k\|^2 + o(\|\tilde{x}^k\|^2) \tag{31}
\]
\[
= \frac{1}{2}\rho(\tilde{S}(f, x^*))\|x^k - x^*\|^2 + o(\|x^k - x^*\|^2). \tag{32}
\]

We now characterize \( \rho(\tilde{S}(f, x^*)) \). First assume now that \( \nabla^2 f(x^*) \) is positive definite. Observe that \( \tilde{S}(f, x^*) = (\tilde{D} - \tilde{E})^{-1} \tilde{E} \), where \( \tilde{D} = 2\tilde{T}(f, x^*)^{-1} - \tilde{T}(f, x^*)^{-1} \)}
\[ f \ \text{equivalently,} \ \nabla^2 f(x^*) \text{ is positive definite and} \ (\hat{D} - E) \text{ is nonsingular, the Ostrowski-Reich theorem} \ [10, \ \text{Theorem 3.12]} \text{ states that} \ \rho(\hat{S}(f, x^*)) < 1 \text{ if} \ \hat{D} > 0, \ \text{i.e. if} \]

\[ 2\tilde{T}(f, x^*)^{-1} \succ \nabla^2 f(x^*). \tag{34} \]

Because (22) implies (34), we infer that \( \rho(\hat{S}(f, x^*)) < 1 \), and the algorithm converges linearly.

If, however, \( \nabla^2 f(x^*) \) is only positive semi-definite, then by computing under (22) the asymptotic rate for the function \( f(x) + \frac{1}{2} \|x\|^2 \) with Hessian \( \nabla^2 f + \epsilon \) and letting \( \epsilon \to 0 \), we find \( \rho(\hat{S}(f, x^*)) \leq 1 \) by continuity of the eigenvalues of \( \hat{S}(f, x^*) \) with respect to \( \nabla^2 f(x^*) \), which completes the proof.

**Remark 1.** Since condition (7) is the conjunction of \( n \) conditions verifiable along individual directions, i.e. \( 2(1 - \sigma)T_i(f_{i,x}, x^*)^{-1} \succ \nabla^2 f(x^*) \) for \( i = 1, \ldots, N \), the cyclic algorithm \( \mathcal{S} \) is an attractive candidate for distributed optimization.

**Remark 2 (Coordinate-wise Newton scaling).** When Newton scaling is used in each direction, i.e. \( T_i(f_{i,x}, x) = \nabla^2 f(x)^{-1} \) for \( i = 1, \ldots, n \) or, equivalently, \( T(f, x) = \nabla^2 f(x)^{-1} \), (9) holds at the point of convergence, and the asymptotic convergence rate (24) reduces to \( \hat{S}(f, x^*) = [\nabla^2 f(x^*) - \nabla^2 f(x^*)]^{-1} \nabla^2 f(x^*)' \).

**Remark 3.** In the case when condition (9) is not met and \( \hat{a}(f, \cdot) \) is discontinuous at \( x^* \), then \( \mathcal{S} \) then proves to converge locally like a stable discrete-time switching system defined by a rate set \( \{\hat{S}^{(\psi)}(f, x^*)\}_{\psi \in \Psi} \) with \( \rho(\hat{S}^{(\psi)}(f, x^*)) \leq 1 \) (or \( \rho(\hat{S}^{(\psi)}(f, x^*)) < 1 \) if \( \nabla^2 f(x^*) \) is positive definite) for all \( \psi \in \Psi \), reducing for large \( k \) to \( x^{k+1} = \hat{S}^{(\psi(k))}(f, x^*) x^k + o(\|x^k\|) \), where \( \psi(\cdot) \) is a switching function.

### 3.1 Synchronous implementations

When implemented with identical step sizes in all coordinate directions, the synchronous algorithm \( \mathcal{J} \) proves to be equivalent to \( \mathcal{G} \) endowed with block-diagonal scaling. We directly infer from (13) the asymptotic convergence rate of \( \mathcal{J} \), by setting \( n \equiv 1 \) in Proposition (20).

**Proposition 5 (Asymptotic convergence rate of \( \mathcal{J} \)).** Let Assumptions (4) and (9) hold for Problem (7). Consider the synchronous algorithm \( x^{k+1} = \hat{S}(f, x^*) \) implemented with step size 1 in every coordinate direction and with scaling \( \{T_i(f_{i,x}, x_i)\}_{i \in N} \) continuous at \( x^* \). If \( \{x^k\} \) in a sequence generated by the algorithm converging towards \( x^* \), then \( \{x^k\} \) converges with asymptotic rate

\[ \bar{J}(f, x^*) = \bar{I} - \bar{T}(f, x^*) \nabla^2 f(x^*). \tag{35} \]
Remark 4. In contrast with Remark 1, global and linear convergence of $J$ may be difficult to assess by inspection along individual directions. In the particular case when $m_i = 1$ for $i \in N$ and $\rho(\tilde{J}(f,x^*))$ is satisfied, then $\rho(\tilde{J}(f,x^*))$ is jointly characterized by Proposition 4 and the Stein-Rosenberg theorem [10, Theorem 3.8], which claim that either $\rho(\tilde{S}(f,x^*)) = \rho(\tilde{J}(f,x^*)) = 1$, or $\rho(\tilde{S}(f,x^*)) < \rho(\tilde{J}(f,x^*)) < 1$ when $\nabla^2 f(x^*)$ is positive definite, in which case any convergent sequence $\{x^k\}$ generated by the algorithm $J$ converges linearly. Notice that convergence is then asymptotically faster for the cyclic implementation $S$ than for the synchronous implementation $J$.

Synchronous algorithms based on approximations of the Newton method — A particular approach explored e.g. in [11, 12, 13] is to find a compromise between the computational and organizational attractiveness of the coordinate descent methods, which set $T(f,x) \equiv \nabla^2 f(x)^{-1}$ for $x \in X$ under strong convexity of $f$ and converge linearly, and the quadratic convergence of the centralized Newton method, for which $T(f,x) \equiv \nabla^2 f(x)^{-1}$. In these studies $\nabla^2 f(x)$ is assumed to be sparse and such that the quantity $Q(f,x) := \nabla^2 f(x)^{1/2} [\nabla^2 f(x) + \nabla^2 f(x)'] \nabla^2 f(x)^{-1/2}$ can be computed in a distributed manner, while the inverse of the Hessian of $f$ rewrites as the series

$$\nabla^2 f(x)^{-1} = \nabla^2 f(x)^{-1/2} \sum_{i=0}^{\infty} Q(f,x)^i \nabla^2 f(x)^{-1/2}$$

(36)

provided that $\rho(Q(f,x)) < 1$, which holds under a strict diagonal dominance condition for $\nabla^2 f(x)^{-1/2} \nabla^2 f(x) \nabla^2 f(x)^{-1/2}$ in virtue of the Gershgorin circle theorem [10]. The approach suggested by (36) is to generate vector sequences such that $x^{k+1} = Z[q](f,x^k)$, where $Z[q] \equiv g(T,X)$ with scaling strategy

$$T(f,x) = \nabla^2 f(x)^{-1/2} \sum_{i=0}^{q} Q(f,x)^i \nabla^2 f(x)^{-1/2},$$

(37)

and $q$ is a parameter symbolizing the implementability vs. rapidity trade-off, and directly proportional to the computational complexity of the algorithm. By setting (37) in (35), we obtain for $\{x^k\}$ the asymptotic convergence rate $E(x^*) Z[q](f,x^*)$, where

$$Z[q](f,x^*) = T(f,x^*) E(x^*) T(f,x^*)^{-1} Z[q](f,x^*) E(x^*)$$

(38)

and $Z[q](f,x^*) := \nabla^2 f(x)^{-1/2} Q(f,x^*)^{q+1} \nabla^2 f(x)^{1/2}$ is the asymptotic convergence rate for the unconstrained problem (i.e. $X \equiv \mathbb{R}^m$). It can be seen that $\rho(Z[q](f,x^*))$ vanishes with growing $q$. When $q = 0$, (38) reduces to the rate of $J$ with coordinate-wise Newton scaling.

3.2 Random implementations

We consider the asymptotic convergence of the random algorithm $\{R^k\}$ given in (21) and used with probabilities $q^k \sim \pi = (\pi_1, \ldots, \pi_n)$ for the coordinate directions. In this context we formulate a strong convexity assumption.
Assumption 3. The function $f$ is strongly convex so that there exist a symmetric, positive definite block matrix $U = (U_{i,j})$ with $U_{i,j} \in \mathbb{R}^{m_i \times m_j}$ satisfying $\|\nabla f(x) - \nabla f(y)\|_1(x - y) \geq \|x - y\|_2^2$ for every $x, y \in X$.

In polyhedral feasible sets, the local convergence of a generated sequence $\{x^k\}$ converging to a solution $x^*$ where strict complementarity holds occurs in the reduced space at $x^*$. In that case we find from (23) and for $k \geq 1$ large, $x^k = x^* + E(x^*)\tilde{x}_k$ and $x^{k+1} = x^* + E(x^*)\tilde{x}_{k+1}$, with $\tilde{x}_k, \tilde{x}_{k+1} \in \mathbb{R}^n$ and

$$\tilde{x}_{k+1} = \tilde{G}_o(x^*)\tilde{x}_k + o(\|\tilde{x}_k\|).$$

The expectation of (39) in $\phi_k$ gives $E[\tilde{x}_{k+1}|x^k, \theta^k] = \tilde{R}(f, x^*)\tilde{x}_k + o(\|\tilde{x}_k\|)$, where $\tilde{R}(f, x^*) := \bar{I} - \text{diag}(\pi_1 \bar{T}_1, ..., \pi_n \bar{T}_n)\tilde{\nabla}^2 f(x)$ and $\theta^k$ symbolizes the event that $\{x^k\}_{k \geq 1}$ is confined in the reduced space at $x^*$. In order to derive asymptotic convergence rates for $\{R_k\}$, we need to find out what happens when $\theta^k$ is false, ideally making sure that $[1 - \Pr(\theta^k)] E[\|h(x^{k+1}) - h(x^k)\|_2^2, -\theta^k] = o(h(x^k))$ for some residual $h(\cdot)$. This information can be partially inferred from the following lemma, which extends to arbitrary distributions a convergence result derived in [14] Theorem 5 for the algorithm known as UCDM, which is a version of $\{R_k\}$ using fixed scaling in each direction and equal probabilities $\pi_i = \frac{1}{n}$ for all directions $i \in N$.

Lemma 1 (Convergence of $\{R_k\}$). Assume that Problem 1 has a unique solution $x^*$ and that the feasible set is the Cartesian product $X = X_1 \times ... \times X_n$. Consider the cyclic algorithm $x^{k+1} = R_k(f, x^k)$ implemented with $\phi^k \sim \pi = (\pi_1, ..., \pi_n)$ at every step $k$, the step-size selection rule (7) where $\sigma \leq \frac{1}{2}$, and fixed scaling $T(f, x) \equiv \bar{T} = \text{diag}(\bar{T}_1, ..., \bar{T}_n)$ with $\bar{T}_i \leq U_i^{-1}$ for $i \in N$. Define

$$\Psi : x \in X \mapsto \Psi(x) := \frac{n \pi}{2} \|x - x^*\|_2^2 + f(x) - f(x^*) \in \mathbb{R}_{>0},$$

where $\pi := \min \{\pi_1, ..., \pi_n\}$ and $V := [n\text{diag}(\pi_1 \bar{T}_1, ..., \pi_n \bar{T}_n)]^{-1}$. For any sequence $\{x^k\}$ generated by the algorithm, we have

$$E[f(x^k) - f(x^*)] \leq \frac{1}{1 + \pi \Psi(x^0)}, \quad k = 0, 1, 2, ...$$

If, in addition, $f$ is strongly convex as in Assumption 3 then

$$E[\Psi(x^{k+1}) | x^k] \leq \frac{1 - \frac{2}{u} \pi}{\frac{u}{u + n \pi}} \Psi(x^k), \quad k = 0, 1, 2, ...,$$

where the constant $u > 0$ satisfies $uV \leq U$.

The proof is similar to that of [14] Theorem 5 and reported in the Appendix. We are now able to characterize the convergence of the algorithm in polyhedra.

Proposition 6 (Asymptotic convergence of $\{R_k\}$). Let Assumptions 7 and 3 hold for Problem 1. Consider the cyclic algorithm $x^{k+1} = R_k(f, x^k)$ implemented with $\phi^k \sim \pi = (\pi_1, ..., \pi_n)$ at all $k$, the step-size selection rule (7) where $\sigma \leq \frac{1}{2}$, and fixed scaling $T(f, x) \equiv \bar{T} = \text{diag}(\bar{T}_1, ..., \bar{T}_n)$ with $\bar{T}_i \leq U_i^{-1}$ for $i \in N$. The function $f$ is strongly convex so that there exist a symmetric, positive definite block matrix $U = (U_{i,j})$ with $U_{i,j} \in \mathbb{R}^{m_i \times m_j}$ satisfying $\|\nabla f(x) - \nabla f(y)\|_1(x - y) \geq \|x - y\|_2^2$ for every $x, y \in X$.
\( L_i^{-1} \) for \( i \in N \). For any sequence \( \{x^k\} \) generated by the algorithm, \( E[f(x^k)] \) converges towards \( f(x^*) \) with limiting asymptotic rate

\[
\tilde{R}_t(f, x^*) := \rho(\sum_{i=1}^{n} \pi_i \hat{G}_i(f, x^*)^\top \nabla^2 f(x^*) \hat{G}_i(f, x^*) \nabla^2 f(x^*)^{-1}) \leq 1.
\] (43)

If \( \nabla^2 f(x^*) \) is positive definite, then \( \tilde{R}_t(f, x^*) < 1 \).

Moreover, if \( f \) is strongly convex as in Assumption 3, then \( E[\Psi(x^k)] \) vanishes at least linearly with limiting asymptotic rate

\[
\tilde{R}_t^\theta(f, x^*) := \max \left\{ \tilde{R}_t(f, x^*), \rho(\sum_{i=1}^{n} \pi_i \hat{G}_i(f, x^*)^\top \tilde{V} \hat{G}_i(f, x^*) \tilde{V}^{-1}) \right\} < 1,
\] (44)

where \( \tilde{V} = E(x^*)'VE(x^*) \), and \( \Psi \) and \( V \) are defined as in (46).

**Proof.** Let \( \{x^k\} \) be a sequence generated by the algorithm. Proposition 2 claims that one can find a \( \delta > 0 \) such that \( A(x^t) = \mathcal{A}(x^*) \) for \( t \geq k + 1 \) when \( \|x^k - x^*\| < \delta \). If \( f^\delta := \max \{f(x) \mid \|x - x^*\| < \delta, x \in X\} \) and \( \theta^k \) is defined as above as the event that \( A(x^t) = \mathcal{A}(x^*) \) for \( t \geq k \), it follows from Lemma 4 that

\[
\Pr(\theta^{k+1}) \geq \Pr(\|x^k - x^*\| < \delta) \geq \Pr(f(x^k) < f^\delta) \geq 1 - \frac{\Psi(x^\delta)}{(1 + \pi k)(f^\delta - f(x^*))}.
\] (45)

Hence \( \Pr(\theta^{k+1}) \to 1 \). From Proposition 1 we also know that the step sizes are equal to 1, and from (23) that, when \( \theta^k \) is true, then \( x^k = x^* + E(x^*) \hat{x}^k \) and \( x^{k+1} = x^* + E(x^*) \hat{x}^{k+1} \) for some \( \hat{x}^k, \hat{x}^{k+1} \in \mathbb{R}^{n} \) satisfying (39).

Consider the sequence of function values \( \{f(x^k)\} \) and a step \( k \). If \( \theta^k \) is true, we have \( \nabla f(x^*)' (x^k - x^*) = 0 \). By setting \( \hat{x}^k := \nabla^2 f(x^*) \hat{x}^k \) and \( \hat{G}_i(f, x^*) := \nabla^2 f(x^*) \hat{x}^k \hat{G}_i(f, x^*) \nabla^2 f(x^*)^{-1} \hat{x} \) for \( i \in N \), and proceeding as in (25)-(33), we find

\[
f(x^{k+1}) - f(x^*) = \frac{1}{2} (\hat{x}^k)' \hat{G}_{\phi^k}(f, x^*)' \hat{G}_{\phi^k}(f, x^*) \hat{x}^k + o(\|\hat{x}^k\|^2).
\] (46)

Thus,

\[
E[f(x^{k+1}) - f(x^*) \mid x^k, \theta^k] \leq \frac{1}{2} (\hat{x}^k)' \sum_{i=1}^{n} \pi_i \hat{G}_i(f, x^*)' \hat{G}_i(f, x^*) \hat{x}^k + o(\|\hat{x}^k\|^2)
\]

(47)

\[
\leq \tilde{R}_t^\theta(f, x^*)|f(x^{k+1}) - f(x^*)| + o(f(x^k) - f(x^*))
\] (48)

where \( \tilde{R}_t^\theta(f, x^*) \) is given by (43). If, however, \( \theta^k \) is false, then \( |f(x^{k+1}) - f(x^*)| \leq |f(x^k) - f(x^*)| \) by (4). All in all, we have

\[
E[f(x^{k+1}) - f(x^*) \mid x^k] \leq \tilde{R}_t^\theta(f, x^*)|f(x^{k+1}) - f(x^*)| + v^k + o(f(x^k) - f(x^*)),
\] (49)

where \( v^k = [1 - \Pr(\theta^k)][1 - \tilde{R}_t^\theta(f, x^*)]|f(x^{k+1}) - f(x^*)| = o(f(x^k) - f(x^*)) \), and the rate \( \tilde{R}_t^\theta(f, x^*) \) is tight.
Let Assumption 3 hold. Similarly, one can write \( \|x^{k+1} - x^*\|_V^2 = \|x^{k+1}\|_V^2 \) when \( \theta_k \) is true. Using (40), and (43), one finds

\[
E \left[ \Psi(x^{k+1}) \right] \leq \hat{R}[\Psi](f, x^*) \Psi(x^k) + \epsilon_k + o(\Psi(x^k)),
\]

where \( \epsilon_k = [1 - Pr(\theta_k)][1 - 2\pi u(y + n\pi)^{-1} - \hat{R}[\Psi](f, x^*)] \Psi(x^k) = o(\Psi(x^k)) \), and \( \hat{R}[\Psi](f, x^*) \) is given by (44) and tight. It follows from Lemma 4 and 12 that \( \hat{R}[\Psi](f, x^*) < 1 \), and thus \( \hat{R}[f](f, x^*) < 1 \). Otherwise there would exist a vector \( y^0 = x^* + E(x^*) \epsilon \in X \) such that \( \hat{R}[\Psi](f, x^*) \geq 1 \) and (50) holds with equality sign, which contradicts (12) if we take \( \epsilon \) small enough.

Assume now that \( f \) is not necessarily strongly convex, yet \( \nabla^2 f(x^*) \) is positive definite. Because \( \hat{R}[f](f, x^*) \) depends only on local properties of \( f \), applying the same algorithm within \( X \) to a strongly convex function \( g \) with the same derivative and Hessian as \( f \) in a neighborhood of \( x^* \) will see \( E[\|g(x^{k+1}) - g(x^*)]\] converge with asymptotic rate \( \hat{R}[f](f, x^*) < 1 \). When \( \nabla^2 f(x^*) \) is positive semidefinite, we find \( \hat{R}[f](f, x^*) \leq 1 \) by considering the function \( f + \frac{\epsilon}{2}\|x\|^2 \) and using the same continuity arguments as in the proof of Proposition 3.

Remark 5. Let us compare the rates given in (12) and (14). For this purpose we place ourselves in the conditions which optimize the precision of (12) by supposing (i) that the bound \( \gamma \) is tight, in the sense that \( yV \leq U \) is satisfied with equality sign (thus implying that \( U \) is block diagonal), (ii) that \( \nabla^2 f(x^*) \) is defined and equal to \( U \) so that the strong convexity constant \( \gamma \) is itself tight and we have the constraint \( y \leq n\pi \) imposed by \( U \leq L^{-1} \leq U^{-1} \), and (iii) that \( A(x^*) = 0 \). After computations, we find \( \hat{R}[\Psi](f, x^*) = 1 - \frac{n\pi}{\pi} \). The ratio with the rate (14) gives

\[
\frac{1 - \hat{R}[\Psi](f, x^*)}{1 - (1 - 2\pi u)} = 1 + \frac{u(n\pi - u)}{2(n\pi)^2} \geq 1.
\]

It can be inferred from (51) that the rate (14) is (for this problem) generally conservative, and equal to the asymptotic rate \( \hat{R}[\Psi](f, x^*) \) if \( y = n\pi \) holds, i.e., when we use a scaled version \( T_i = \frac{n\pi}{\pi} \nabla^2 f(x^*)^{-1} \) \((i \in N)\) of the asymptotic expression of coordinate-wise Newton scaling approach previously discussed in Remark 2—in that case the convergence rate reduces to 1 - \( \pi \).

### 3.3 Non-twice differentiable cost functions

In the previous sections we have assumed that \( \nabla^2 f \) existed at the point of convergence \( x^* \). Suppose instead that \( \nabla^2 f(x^*) \) is not defined but that \( f \) satisfies a strong convexity condition at least locally in a neighborhood \( X^* \) of \( x^* \), i.e. there is a symmetric, positive definite matrix \( U \) such that \( [\nabla f(x) - \nabla f(y)](x - y) \geq \|x - y\|^2_U \) holds for \( x, y \in X^* \).

Under Assumption 1, consider any algorithm based on \( \{G_i\}_{i=1}^n \), such as those introduced in Section 2.3, and generate a sequence \( \{x^k\} \) with step-size...
The bounded sequences \( \{x^k\} \) asymptotically efficient in the sense of Proposition 1. Assume that strict complementarity holds at \( x^* \), so that convergence occurs in the reduced space at \( x^* \) and, for large \( k \), we have \( x^k = x^* + E(x^*) \tilde{x}^k \) with \( \tilde{x}^k \in \mathbb{R}^m \). In view of Remark 3 and by considering directional derivatives of \( \nabla f \) in the developments that lead to (24), we find

\[
\hat{G}_i(f, x^k) - x^* = G_i^k(x^k - x^*) + o(\|x^k - x^*\|)
\]  

(52)

where \( G_i^k := E(x^*)[I - \text{diag}(0, ..., 0, I_i, 0, ..., 0)\hat{G}_i(f, x^*)E(x^*)M^kE(x^*)] \) for some matrix \( M^k \in \Sigma \), where \( \Sigma \) denotes the set of all the symmetric matrices \( M \) satisfying \( U \preceq M \preceq L \). Suppose now that, for any strongly convex function \( g \) which realizes its minimum on \( X \) at \( x^* \) and satisfies Assumption 2, the algorithm produces sequences \( y^k \) linearly convergent towards \( x^* \) with respect to some residual \( h(y^k) \) and with rate \( H(\nabla^2 g(x^*)) \), i.e.

\[
h(y^{k+1}) \leq H(\nabla^2 g(x^*))h(y^k) + o(h(y^k)),
\]  

(53)

where \( \rho(H(M)) < 1 \) for any \( M \in \Sigma \), and \( H(\cdot) \) is a continuous mapping. It follows from the compactness of \( \Sigma \) that we can find a matrix \( \tilde{H} \in \Sigma \) such that \( \rho(\tilde{H}) = \max_{M \in \mathcal{S}} \{\rho(H(M))\} < 1 \) and

\[
h(x^{k+1}) \leq \tilde{H}h(x^k) + o(h(x^k)).
\]  

(54)

3.4 Stochastic optimization based on gradient projections

Some stochastic optimization problems are concerned with the minimization of a cost function unknown in closed form that can only be estimated through measurement or simulation. Assume in Problem 1 that \( f \) is unknown, while a sequence \( \{f^k\} \) of estimates in \( F(m) \) is available for \( f \) with common Lipschitz constant for every \( \nabla f^k \), and that \( \{f^k\} \) converges almost surely towards \( f \) in the sense that \( \sup_{x \in C} |f^k(x) - f(x)| + \sup_{x \in C} ||\nabla f^k(x) - \nabla f(x)||| \) vanishes almost surely for any compact set \( C \subset X \). An approach for solving this problem consists of sequentially applying an iterative optimization algorithm \( \mathcal{M} \) along the sequence of function estimates, i.e.

\[
x^{k+1} = \mathcal{M}(f^k, x^k), \quad k = 0, 1, 2, \ldots
\]  

(55)

The bounded sequences \( \{x^k\} \) generated by (55) are known to converge almost surely towards a nonempty solution set provided that \( \mathcal{M} \) is closed and a descent algorithm with respect to \( f \) and the set of solutions [13 Theorem 2.1]. Possible choices for \( \mathcal{M} \) include the gradient projection algorithm \( \mathcal{G}^{(T, X)} \) and (under Assumption 4) the parallel implementations of Section 2.3 whose convergence in stochastic settings is specifically addressed in [16].

Consider such an algorithm \( \mathcal{M} \), and suppose that strict complementarity holds at \( x^* \) (Assumption 2) and that each function \( f^k \) has a unique minimizer \( y^k \) on \( X \) where \( \nabla^2 f^k \) is defined, continuous and positive definite at \( y^k \). By Lipschitz continuity of \( f \), the sequence \( \{y^k\} \) is such that \( \mathcal{A}(y^k) \rightarrow \mathcal{A}(x^*) \),
i.e. \( y^k = x^* + E(x^*)\hat{\gamma}^k \) for \( \hat{\gamma}^k \in \mathbb{R}^m \), and, say, \( k > \hat{k} \), with strict complementarity holding at \( y^k \) for \( f^k \). Assume that the considered algorithm \( \mathcal{M} \) produces, when applied to any \( f^k \) with \( k > \hat{k} \), sequences in \( X \) converging towards \( y^k \) in the subspace at \( x^* \) with rate \( M(f^k, y^k) < 1 \). Any bounded sequence \( \{ x^k \} \) generated by (55) will then be such that \( x^k = x^* + E(x^*)\tilde{x}^k \) and \( x^{k+1} = x^* + E(x^*)\tilde{y}^{k+1} \) for \( k \) large enough and for some \( \tilde{x}^k, \tilde{y}^{k+1} \in \mathbb{R}^m \) satisfying

\[
\tilde{x}^{k+1} - \tilde{y}^k = \bar{M}(f^k, y^k)(\tilde{x}^k - \tilde{y}^k) + \rho(f^k, \tilde{x}^k),
\]

where \( \rho(f^k, \tilde{x}^k) = o(\| \tilde{x} - y^k \|) \) for \( k \geq \hat{k} \). Further, if \( f \) and all \( f^k \) are smooth, the scale \( T(f, \cdot) \) of \( \mathcal{M} \) is continuously differentiable at \( x^* \), and, almost surely, \( \nabla^2 f \) and its derivatives converge uniformly on a neighborhood of \( x^* \) towards \( \nabla^2 f \) and its derivatives, respectively, then \( \rho(f^k, \tilde{x}^k) \equiv \rho(f^k, \tilde{x}^k)(\tilde{x}^k - \tilde{y}^k)(\tilde{x}^k - \tilde{y}^k)' \), where \( \rho(f^k, \tilde{x}^k) \) is a function of derivatives at \( x^* \) of \( \nabla^2 f \) and \( T(f, \cdot) \) and uniformly bounded for all \( k \geq \hat{k} \) on a neighborhood of \( x^* \) in accordance with Proposition 3. Then, (56) rewrites (with probability one) as

\[
x^{k+1} - x^* = A^k(x^k - x^*) + B^k(y^k - x^*) + o(\| x^k - x^* \|),
\]

where \( A^k = E(x^*)(\bar{M}f^k, y^k)E(x^*)' \) and \( B^k = E(x^*)(\bar{I} - \bar{M}(f^k, y^k))E(x^*)' \).

The asymptotics of \( \{ y^k - x^* \} \) ensue from the nature of the function sequence \( \{ f^k \} \). In many problems, \( f \) is an expectation function of the type

\[
f(x) = \int_{\Omega} \tilde{y}(x, \omega)P(d\omega), \quad \forall x \in \mathbb{R}^m,
\]

where \( \omega \) is a random parameter defined on a probability space \((\Omega, \mathcal{F}, P)\), and \( \tilde{y}(\cdot, \omega) \) serves as a random measurement of \( f \), modeling for instance the optimal value of the second-stage problem of a two-stage stochastic program [17]. Based on (55) and the simulation of a sequence of samples \( \{ \{ \omega_i \} \}_{i=0}^{q(k)-1} \) of independent realizations of \( \omega \), with \( q(k) \to \infty \) as \( k \to \infty \), it is common to consider the sample average approximation (SAA)

\[
f^k(x) = \frac{1}{q(k)} \sum_{i=0}^{q(k)-1} \tilde{y}(x, \omega_i^k), \quad k = 0, 1, 2, \ldots,
\]

which converges almost surely and uniformly towards \( g \) on any compact set \( C \subset X \) under certain continuity and integrability conditions for \( \tilde{y} \) [13]. The sequence \( \{ y^k \} \) is then known as the (SAA) estimator, and it follows from the central limit theorem that (59) is asymptotically normal, i.e.

\[
q(k)^{-\frac{1}{2}}[f^k(x) - f(x)] \overset{d}{\to} \nu(x), \quad \forall x \in X,
\]

where \( \overset{d}{\to} \) denotes convergence in distribution and \( \nu(x) \) is a centered normal random variable with variance \( \sigma^2(x) = \text{Var}[\tilde{y}(x, \omega)] \). Since the hypotheses of [17] Theorem 5.8 are then satisfied at \( x^* \), the first order asymptotics of the SAA estimator \( y^k \) can be inferred from the second order Taylor series expansion of \( f \) at \( x^* \) and the Delta theorem, and we find

\[
q(k)^{-\frac{1}{2}}[y^k - x^*] \overset{d}{\to} -E(x^*)\nabla^2 f(x^*)^{-1}E(x^*)'\nabla \nu(x^*),
\]
where \( \tilde{\nabla}^2 f(x) := E(x^*)\nabla^2 f(x)E(x^*) \).

We see from (67) and (61) that the convergence of the sequence \( \{x^k\} \) is then asymptotically analogous to that of a discrete-time random dynamical system characterized by (i) the affine mapping sequence \( \{A^k\} \), which converges almost surely towards the asymptotic convergence rate \( A^\infty = E(x^*)M(f, x^*)E(x^*)' \) of the (typically linearly convergent) algorithm \( \mathcal{M} \), and (ii) a random noise process with variance vanishing sublinearly like \( O(q(k)^{-1}) \), thus hindering the whole optimization process and dictating its actual asymptotic performance.

**Remark 6.** The impact of variance of the SAA estimator can be lessened using variance reduction [19,20,17] or scenario reduction techniques [21]. Reducing the computational charge due to sample averaging is possible for instance by controlling the sample generation process [22], or by synchronizing—possibly in parallel—the application of the descent algorithm (55) with the increasing precision of \( \{f^k\} \) [10].

**Appendix: proofs and auxiliary results**

**Proof of Proposition 1.** Consider any \( x \in X \) and the gradient projection \( G(T,X)(f, x) \) with a tentative step size \( a \in (0, 1] \). We have

\[
\begin{align*}
    f(x) - f(\bar{x}(a)) & \geq -\nabla f(x)'(\bar{x}(a) - x) - \frac{1}{2}\|\bar{x}(a) - x\|_2^2 \\
    & \geq (\bar{x}(a) - x)'M(\bar{x}(a) - x)
\end{align*}
\]

with \( M := [aT(f, x)]^{-1} - \frac{1}{2} \), and by (3) the condition (4) is satisfied for \( a = 1 \).

Suppose now that \( T(f, x) \) is continuous and \( f \) is twice continuously differentiable in a neighborhood \( X^* \) of \( x^* \). Taylor’s theorem yields

\[
f(x) - f(y) = -\nabla f(x)'(y - x) - \frac{1}{2}\|y - x\|_2^2 T(x, y) + o(\|y - x\|_2^2), \; \forall x, y \in X^*. \tag{64}
\]

Consider the sequence \( \{x^k\} \) converging to \( x^* \) and the sequence \( \{y^k\} \) such that \( y^k = G(T,X)(f, x^k) \) with step size 1 for all \( k \). Since \( G(T,X)(f, x^*) = x^* \) for any step size by stationarity of \( x^* \), we find that \( y^k \rightarrow x^* \) by continuity of \( G(T,X) \). Thus, for \( k \) large enough, \( x^k, y^k \in X^* \), and it follows from (3) and (5) that

\[
f(x^k) - f(y^k) \geq \|y^k - x^k\|_Q^2 + o(\|y^k - x^k\|_2^2) \tag{65}
\]

with \( Q := T(f, x^k)^{-1} - \frac{1}{2} \tilde{\nabla}^2 f(x^k) \). By (3) and continuity arguments, (6) is satisfied at \( x^k \) for large \( k \) if \( a = 1 \), i.e. \( x^k+1 = y^k \). Hence \( \hat{a}(f, x^k) \rightarrow 1 \).

**Proof of Proposition 2.** By strict complementarity at \( x^* \) we know that (11) is satisfied with coefficients \( \{a_i\}_{i \in \mathcal{A}(x^*)} \) all positive. For any \( \delta > 0 \), denote by \( \mathcal{A}(x^*) := \{x \in X : \|x - x^*\| \leq \delta\} \) a neighborhood of \( x^* \) in \( X \). We first show that one can find a \( \delta > 0 \) such that \( \mathcal{A}(x^*) \subseteq \mathcal{A}(x^*) \) for any \( x \in \mathcal{A}(x^*) \). Otherwise there would be \( j \in \{1, ..., p\} \setminus \mathcal{A}(x^*) \) and a sequence \( \{y^k\} \) in \( X \) converging towards \( x^* \), such that \( c_j(y^k) = 0 \) for all \( k \). By continuity of \( c_j \), we would find \( c_j(x^*) = 0 \) and thus \( c_j \in \mathcal{A}(x^*) \), which is a contradiction. Since the proposition becomes trivial if \( \mathcal{A}(x^*) = \emptyset \), we suppose in the rest of the proof that \( \mathcal{A}(x^*) \neq \emptyset \), and thus \( \|\nabla f(x^*)\| \neq 0 \) by strict complementarity at \( x^* \).
Consider a point \(x \in X\) where \(A(x) = B\) with \(B \subset A(x^*)\) and \(B \neq A(x^*)\). The affine constraints can be rewritten as \(c_j(x) = \nabla c_j(x^*)(x - x^*)\) for all \(x \in X\) and \(j \in A(x^*)\). We find
\[
\sum_{j \in A(x^*)} \alpha_j c_j(x) = \left[ \sum_{j \in A(x^*)} \alpha_j \nabla c_j(x^*) \right] (x - x^*) \quad \text{for all } x \in X.
\] (66)

Assume that \(\nabla f(x^*)\) is a linear combination of elements of \(\{\nabla c_j(x^*)\}_j \subset B\). We have \(c_j(x) = 0\) for \(j \in B\) and the expression in (66) is equal to 0. Since \(c_j(x) < 0\) for \(j \in A(x^*) \setminus B\), we find \(\sum_{j \in A(x^*)} \alpha_j c_j(x) = \sum_{j \in A(x^*) \setminus B} \alpha_j c_j(x) < 0\), a contradiction. Hence \(\nabla f(x^*)\) cannot be expressed as a linear combination of elements of \(\{\nabla c_j(x^*)\}_j \subset B\) and there exists a \(\Delta > 0\) independent of \(x\) such that
\[
\left\| \nabla f(x^*) + \sum_{j \in B} \alpha_j \nabla c_j(x^*) \right\| > \Delta, \quad \forall \{\alpha_j\}_j \in B.
\] (67)

For \(\delta > 0\), consider the function \(\theta(\delta) := \max(\delta, \max_{x \in X^*} \|G^{(T,X)}(f, x) - x\|)\) with any bounded scaling strategy \(T\) and step-size policy in \([\gamma, 1]\) \((\gamma > 0)\). Since \(G^{(T,X)}(f, x^*) = x^*\), we find by Lipschitz continuity of \(\nabla f\) and other continuity arguments that \(\theta(\delta) \downarrow 0\) whenever \(\delta \downarrow 0\). It follows that for any \(\rho > 0\), one can find a \(\delta > 0\) such that \(x \in X^*(\delta)\) yields both \(A(x) \subset A(x^*)\) and \(\theta(\delta) < \rho\). By Lipschitz continuity of \(\nabla f\), we also have \(\|\nabla f(x) - \nabla f(x^*)\| \leq l\|x - x^*\| < l\rho\) for any \(x \in X\), where \(l\) denotes the Lipschitz constant. Suppose now that \(A(x) = B\) for some \(x \in X^*(\delta)\) and set \(y = G^{(T,X)}(f, x)\) with step size \(a \in [\gamma, 1]\). From (66) and (67), we infer the existence of nonnegative coefficients \(\{\tilde{\alpha}_j\}_j \subset B\) satisfying
\[
\nabla f(x) + [aT(f, x)]^{-1}(y - x) = -\sum_{j \in B} \tilde{\alpha}_j \nabla c_j(y).
\] (68)

Then,
\[
\left\| \nabla f(x^*) + \sum_{j \in B} \tilde{\alpha}_j \nabla c(x^*) \right\| \leq \left\| \nabla f(x) - \nabla f(x^*) \right\| + \left\| [aT(f, x)]^{-1}(y - x) \right\|
\leq l\|a\lambda\|^{-1}\rho,
\]
which contradicts (67) if, initially, \(\rho < \Delta/[l + 1/(a\lambda)]\). Hence \(A(y) \neq B\), which proves the first statement considering that the number of constraints \(p\) is finite. The second statement is then immediate.

**Proof of Lemma 4** We already know from Proposition 1 that the step sizes chosen by (2) are equal to 1. The rest of the proof—herein provided for completeness and comparison—follows the lines of that of [11, Theorem 5] with the difference that we reason with the norm \(\|\cdot\|_V\). We have
\[
\left\| x^{k+1} - x^* \right\|_V^2 = \left\| x^k - x^* \right\|_V^2 + 2\langle x^k - x^* \rangle V(x^{k+1} - x^k) + \left\| x^{k+1} - x^k \right\|_V^2.
\]
\[
\leq \left\| x^k - x^* \right\|_V^2 + \frac{2}{n\pi \phi_k} \sum_{j \in B} \nabla f(x^k)(x^k - x^{k+1}) - \left\| x^{k+1} - x^k \right\|_V^2.
\]
\[
\leq \frac{2}{n\pi \phi_k} \sum_{j \in B} \nabla f(x^k)(x^k - x^{k+1}) - \left\| x^{k+1} - x^k \right\|_V^2.
\]
\[
\leq \frac{n\pi \phi_k}{n\pi \phi_k} \sum_{j \in B} \nabla f(x^k)(x^k - x^{k+1}) + \left\| f(x^k) - f(x^{k+1}) \right\|.
\]
which yields, by expectation in \(\phi^k\) and rearrangement of the terms,
\[
E \left[ \Psi^{(k+1)} \mid x^k \right] \leq \Psi(x^k) - \pi \nabla f(x^k)(x^k - x^*).
\] (69)
Since $\nabla f(x^k)'(x^k - x^*) \geq f(x^k) - f(x^*)$ by convexity of $f$, we find by computing successive conditional expectations,

$$E \left[ \Psi(x^{k+1}) \right] \leq \Psi(x^k) - \sum_{t=0}^{k} E \left[ f(x^t) - f(x^*) \right]$$

(70)

which shows (41).

When Assumption 3 holds, we proceed as in (1) and find,

$$\nabla f(x^k)'(x^k - x^*) \leq f(x^k) - f(x^*) - \frac{\beta}{2} \| x^k - x^* \|^2_2 \leq -\beta \| x^k - x^* \|^2_2.$$  

(72)

Substituting the two inequalities (72) into (69) with relative weights $\mu = 2\beta(y + n\pi)^{-1} \in (0, 1]$ and $1 - \mu$ yields (12).

**Lemma 2.** Let $H = (H_{ij})$ be a symmetric block matrix of $\mathbb{R}^{p \times p}$ such that $H = D - L - L'$, where $D = \text{diag}(D_1, ..., D_p)$ is block diagonal and $L$ is strictly lower triangular. If $T = \text{diag}(T_1, ..., T_p)$ is a symmetric, positive definite, block diagonal matrix of $\mathbb{R}^{p \times p}$ and $G_i := I_p - \text{diag}(0, ..., 0, I_i, 0, ..., 0)TH$ for $i = 1, ..., p$, then $G_pG_{p-1} \cdots G_1 = (T^{-1} - L)^{-1}(T^{-1} - D + L')$.

**Proof.** Since the result is trivial for $p = 1$ we suppose that $p \geq 2$. We proceed by induction on $p$. Let $M_1 := G_1$. For $2 \leq i \leq p$, define $M_i := G_iG_{i-1} \cdots G_1$ and decompose $T$, $D$, $L$ and the $p \times p$ identity matrix $I_p$ into

$$T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_i \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_i \end{pmatrix}, \quad L = \begin{pmatrix} I_i & 0 & 0 \\ 0 & I_i & 0 \\ 0 & 0 & I_i \end{pmatrix}, \quad I_p = \begin{pmatrix} I_i & 0 & 0 \\ 0 & I_i & 0 \\ 0 & 0 & I_i \end{pmatrix}.$$  

(73)

We can write

$$G_i = \begin{pmatrix} I_i & 0 & 0 \\ T_iI_i & I_i - T_iD_i & T_i \lambda_i' \end{pmatrix}, \quad 2 \leq i \leq p.$$  

(74)

For some $i \geq 2$, notice that $T_i$ is nonsingular, as well as $T_i^{-1} - L_i$, and suppose that

$$M_{i-1} = \begin{pmatrix} (T_{i-1}^{-1} - L_i^{-1})(T_{i-1}^{-1} - D_i + L_i') & (T_{i-1}^{-1} - L_i^{-1})(0, I_i)' \\ T_iI_i & I_i - T_iD_i \end{pmatrix}$$

(75)

$$= \begin{pmatrix} Z_i(T_{i-1}^{-1} - D_i + L_i') & Z_i' \lambda_i' \\ 0 & I_i \end{pmatrix},$$

(76)

where $Z_i := (T_{i-1}^{-1} - L_i^{-1})^{-1}$. By block matrix inversion of $T_{i+1}^{-1} - L_{i+1}$, we have

$$\bar{T}_{i+1}^{-1} - \bar{L}_{i+1} = \begin{pmatrix} \bar{Z}_i & 0 \\ T_iI_i & T_i \end{pmatrix}, \quad \bar{T}_{i+1}^{-1} - \bar{D}_{i+1} + \bar{L}_{i+1} = \begin{pmatrix} T_{i-1}^{-1} - D_i + L_i' & 0 \\ T_iI_i & I_i \end{pmatrix}. $$

(77)

It follows from (73), (76) and $M_i = G_iM_{i-1}$ that

$$M_i = \begin{pmatrix} Z_i(T_{i-1}^{-1} - D_i + L_i') & Z_i' \lambda_i' \\ T_iI_iZ_i(T_{i-1}^{-1} - D_i + L_i') & T_iI_iZ_i' \lambda_i' + I_i - T_iD_i & T_iI_iZ_i' \lambda_i' + T_i \lambda_i' \end{pmatrix},$$

(78)

$$= \begin{pmatrix} (T_{i-1}^{-1} - L_{i+1})^{-1}(T_{i+1}^{-1} - D_{i+1} + L_{i+1}') & (T_{i+1}^{-1} - L_{i+1})^{-1}(I_i + l_{i+1})' \\ 0 & I_i \end{pmatrix},$$

(79)

where we have used $(l_{i+1}, l_{i+1}) = (l_i, \lambda_i)$. Since (75) holds for $i = 1$, we find by induction

$$G_p \cdots G_1 = M_p = (T^{-1} - L)^{-1}(T^{-1} - D + L').$$

(79)
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