On finitely nondegenerate closed homogeneous CR manifolds

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Abstract
A complex flag manifold \( F = G / Q \) decomposes into finitely many real orbits under the action of a real form \( G^0 \) of \( G \). Their embedding into \( F \) defines on them CR manifold structures. We characterize and list all the closed real orbits which are finitely nondegenerate.

Keywords Equivalence problem · CR · Levi nondegenerate · Real orbits · Simple model

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Introduction

In order to solve the biholomorphic equivalence problem for smooth complex domains, one needs to study biholomorphic invariants of their boundaries. These are closed smooth hypersurfaces and are the codimension one prototypes of the more general objects that we call now Cauchy–Riemann (or CR) manifolds.

Within specific classes of CR manifolds, one aims at determining complete systems of geometric invariants characterizing equivalence. An important step in this context is investigating transformation groups preserving these invariants.

In this paper, we will consider CR manifolds which are orbits of real forms in complex flag manifolds (cf., e.g., [25]).

Let $G$ be a complex semisimple Lie group, $Q$ a parabolic subgroup and $G^\sigma$ a real form of $G$. The complex flag manifold $F = G/Q$ decomposes into finitely many $G^\sigma$-orbits; among them there exists exactly one of minimal dimension, which is closed and hence compact (see [3] for details on its topology). Their embedding into $F$ yields natural CR manifold structures.

An important characteristic of a CR manifold is the degeneracy/nondegeneracy of its Levi form. This is a vector valued sesquilinear form which, roughly speaking, measures the non-integrability of its CR distribution. In several instances, this form has a nontrivial kernel and, by considering its higher-order iterations, we are lead to weaker notions of nondegeneracy (see, e.g., [10, 23]). This leads to a notion of Levi order, expressed by a positive integer $\kappa$ (degeneracy corresponding to $\kappa = +\infty$). In [15], we found the upper bound $\kappa = 3$ for the finitely nondegenerate orbits of real forms in complex flag manifolds, which drops to 2 when restricting to the closed ones (or, more generally, to those of minimal type). In this paper, we characterize and list all the closed real orbits which are finitely nondegenerate.

Other interesting classes of homogeneous CR manifolds were considered by many Authors, as those of hypersurface type (see, e.g., [9, 19, 23, 24]), or those maximally symmetric simple with associated symbol of length 2 (see, e.g., [12]), or those which are reductive compact (see, e.g., [5]).

A fundamental tool, while studying homogeneous CR manifolds, is employing the notion of CR algebra $(g^\sigma, q_\phi)$ introduced in [18] (see Sect. 2). In this note, we compute the Levi order of minimal orbits, by using the description of their CR algebras given by cross-marked Satake diagrams, improving and completing results of [10].

Closed orbits in complex flag manifolds are in a one-to-one correspondence with cross-marked Satake diagrams $(S, \phi)$, where connected Satake diagrams $S$’s classify simple real Lie algebras (see, e.g., [7]) and different subsets of simple roots $\phi$’s encode properties of the minimal parabolic CR algebras.

Split real forms $g^\sigma$ yield totally real CR algebras $(g^\sigma, q_\phi)$ for all choices of $\phi \subseteq B$, while all compact forms (for all $\phi \subseteq B$) and real forms of types A II and D II, with the $\phi$’s listed in Table 1, yield totally complex CR algebras $(g^\sigma, q_\phi)$ which are associated with the corresponding complex flag manifolds. We will characterize the $\phi$’s yielding fundamental CR algebras and for which $(g^\sigma, q_\phi)$ is finitely nondegenerate.

The paper is structured as follows. In Sect. 1, we summarize basic notions of CR geometry and, for later use, we provide Lemma 1.2, which states that a CR manifold admitting a complex CR fibration cannot be finitely nondegenerate. In Sect. 2, we focus on homogeneous CR manifolds; the notions of the previous section are rewritten in terms of CR algebras:
Table 1 Totally complex CR algebras \((g^\sigma, q_\phi)\) with simple \(g\)

| Type | \(g^\sigma\) | \(S\) | \(\phi\) |
|------|--------------|--------|---------|
| A II | \(sl(\mathbb{H}, p), \ell = 2p - 1\) | \(\alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_\ell\) | \(\phi = \{\alpha_1\}, \{\alpha_\ell\}\) |
| D II | \(so(1, 2\ell - 1)\) | \(\alpha_1 \rightarrow \cdots \rightarrow \alpha_\ell\) | \(\phi = \{\alpha_{\ell-1}\}, \{\alpha_\ell\}\) |

these objects are pairs \((g^\sigma, q_\phi)\), consisting of a real Lie algebra \(g^\sigma\), whose elements are infinitesimal CR automorphism of the manifold, and a complex Lie subalgebra \(q_\phi\) of its complexification \(g^\sigma \cong \mathbb{C} \otimes g^\sigma\), which encodes its \(G^\sigma\)-invariant CR structure. In Sect. 3, we describe the CR algebras of orbits of real forms in complex flag manifolds, concentrating upon the unique closed one (cf., e.g., [1]). In Sect. 4, we characterize those which are finitely nondegenerates (cf. [10, Sec.5]), and, adapting a more general result [15, Th.2.11], we show that 2 is an upper bound for the length of the chain of kernels of iterated Levi’s forms. In Sect. 5, we describe canonical CR fibrations in terms of the subsets of simple roots \(\phi\) employed to define the parabolic CR algebra \((g^\sigma, q_\phi)\). In Sect. 6, by using the results of the previous sections, we give a complete classification of all closed simple homogeneous CR manifolds which are 1- and 2-nondegnerate. The final classification is summarized in Tables 2 and 3.

1 Preliminaries on CR manifolds

A CR manifold of type \((n, d)\) is a pair \((M, T^{0,1}M)\), consisting of a smooth manifold \(M\) of real dimension \(2n+d\), and a rank \(n\) smooth complex subbundle \(T^{0,1}M\) of its complexified tangent bundle \(T^CM \cong \mathbb{C} \otimes TM\), satisfying

1. \(T^{0,1}M \cap T^{0,1}M = \{0\}\);
2. \([\Gamma^\infty(M, T^{0,1}M), \Gamma^\infty(M, T^{0,1}M)] \subseteq \Gamma^\infty(M, T^{0,1}M)\).

The integers \(n, d\) are its CR dimension and CR codimension, respectively, and (2) is the formal integrability condition.

Notation 1.1 We use the following notation:

- \(T^{1,0}M \cong T^{0,1}M\);
- \(H^CM \cong T^{1,0}M \oplus T^{0,1}M\);
- \(HM \cong H^CM \cap TM\).

The rank \(2n\) real subbundle \(HM\) of \(TM\) is the real contact distribution underlying the CR structure of \(M\). Condition (1) defines a smooth complex structure \(J_M\) on the fibers of \(HM\) by

\[T^{0,1}M = \{X + i J_M X \mid X \in HM\}.\]  (1.1)

The map \(J_M\) squares to \(-\text{Id}_{HM}\) and is called the partial complex structure of \(M\). An equivalent definition of the CR structure can be given by assigning first an even dimensional distribution \(\mathcal{H}\) and then a smooth partial complex structure \(J_M\) on \(\mathcal{H}\) in such a way that the complex distribution (1.1) satisfies the integrability condition (2).
Table 2  List of all finitely nondegenerate CR algebras $(g^\sigma, q_\phi)$ of closed orbits with classical simple $g$

| Name | $g^\sigma$ | $\mathcal{S}$ | Levi order 1 | Levi order 2 |
|------|-------------|---------------|--------------|--------------|
| AII  | $\mathfrak{sl}(\mathfrak{h}, p)$, $\ell=2p-1$ | $\bullet \circ \circ \circ \circ \bullet$ | $B_\bullet \ni \phi$, $|\phi| > 1$. | $B_\bullet \ni \phi \neq \{\alpha_1\}, \{\alpha_\ell\}$, $|\phi| = 1$. |
| AIIIa| $\mathfrak{su}(p, q)$, $p+q=\ell+1$, $2\leq p \leq \ell$ | | | |
| AIIIb| $\mathfrak{su}(p, p)$, $2 \leq p = \frac{(\ell+1)}{2}$ | | | |
| AIV  | $\mathfrak{su}(1, \ell)$ | | | |
| BI   | $\mathfrak{so}(p, 2\ell+1-p)$, $2 \leq p \leq \ell$ | | | |
| BII  | $\mathfrak{so}(1, 2\ell)$ | | | |
| CIIa | $\mathfrak{sp}(p, \ell-p)$, $2p < \ell$ | | | |
| CIIb | $\mathfrak{sp}(p, p)$, $2p = \ell$ | | | |
| Name | $g^\sigma$ | $S$ | Levi order 1 | Levi order 2 |
|------|------------|------|--------------|--------------|
| DIA  | $\mathfrak{so}(p, 2\ell - p)$ | \(2 \leq p \leq \ell - 2\) | $\phi = \{\alpha_{p+1}\}$. | $\phi = \{\alpha_i\}, \; p+1 < i \leq \ell$; $\phi = \{\alpha_{\ell-1}, \alpha_\ell\}$. |
| DIB  | $\mathfrak{so}(\ell-1, \ell+1)$ | $\alpha_1$ | $\phi = \{\alpha_{\ell-1}\}$; $\phi = \{\alpha_\ell\}$. | $\emptyset$ |
| DII  | $\mathfrak{so}(1, 2\ell - 1)$ | | $\phi = \{\alpha_2\}$. | $\phi = \{\alpha_i\}, \; 2 < i \leq \ell$; $\phi = \{\alpha_{\ell-1}, \alpha_\ell\}$. |
| DIIA | $\mathfrak{su}_{2p}^+(\mathbb{H}), \ell = 2p$ | | $\phi = \{\alpha_{2i-1}, \ldots, \alpha_{2i_k-1}\}, \; 1 \leq i_1 < \ldots < i_k \leq p, \; k > 1$. | $\phi = \{\alpha_{2i-1}\}, \; 1 \leq i \leq p$. |
| DIIIB | $\mathfrak{su}_{2p+1}^+(\mathbb{H}), \ell = 2p+1$ | | $\phi \cap \mathcal{B}_0 = \emptyset, \{\alpha_{\ell-1}\}, \{\alpha_\ell\}$ | $\phi \cap \mathcal{B}_0 = \emptyset, \{\alpha_{2i-1}\}, \; 1 \leq i \leq p$. |
Table 3 List of all finitely nondegenerate CR algebras \((g^σ, q_{q_0})\) of closed orbits with exceptional simple \(g\)

| Name | \(\mathcal{S}\) | Levi order 1 | Levi order 2 |
|------|----------------|--------------|--------------|
| EII  | \(\alpha_1\) \(\alpha_3\) \(\alpha_5\) \(\alpha_6\) | \(\phi = \{\alpha_1\}, \{\alpha_3\}, \{\alpha_5\}, \{\alpha_6\}\), \(\{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}\) | \(\emptyset\) |
| EIII | \(\alpha_1\) \(\alpha_2\) \(\alpha_3\) \(\alpha_4\) \(\alpha_5\) \(\alpha_6\) | \(\phi = \{\alpha_1\}, \{\alpha_2\}, \{\alpha_1, \alpha_3\}, \{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_2, \alpha_3\}, \{\alpha_2, \alpha_4\}, \{\alpha_2, \alpha_5\}, \{\alpha_3, \alpha_4\}, \{\alpha_3, \alpha_5\}, \{\alpha_4, \alpha_5\}\) | \(\emptyset\) |
| EIV  | \(\alpha_1\) \(\alpha_2\) \(\alpha_3\) \(\alpha_4\) \(\alpha_5\) \(\alpha_6\) | \(\phi \subseteq \mathcal{R}_*, |\phi| \leq 2, \\phi \neq \{\alpha_3\}, \{\alpha_3, \alpha_4\}\) | \(\phi = \{\alpha_3\}\) |
| EVI  | \(\alpha_1\) \(\alpha_2\) \(\alpha_3\) \(\alpha_4\) \(\alpha_5\) \(\alpha_6\) | \(\phi \subseteq \mathcal{R}_*, |\phi| = 2, 3\) | \(\phi = \{\alpha_1\}, \{\alpha_3\}, \{\alpha_5\}\) |
| EVII | \(\alpha_1\) \(\alpha_2\) \(\alpha_3\) \(\alpha_4\) \(\alpha_5\) \(\alpha_6\) | \(\phi \subseteq \mathcal{R}_*, |\phi| \leq 2, \\phi \neq \{\alpha_4\}, \{\alpha_4, \alpha_5\}\) | \(\phi = \{\alpha_4\}\) |
| EIX  | \(\alpha_1\) \(\alpha_2\) \(\alpha_3\) \(\alpha_4\) \(\alpha_5\) \(\alpha_6\) | \(\phi \subseteq \mathcal{R}_*, |\phi| \leq 2, \\phi \neq \{\alpha_5\}, \{\alpha_5, \alpha_6\}\) | \(\phi = \{\alpha_5\}\) |
| FII  | \(\alpha_1\) \(\alpha_2\) \(\alpha_3\) \(\alpha_4\) \(\alpha_5\) \(\alpha_6\) | \(\phi = \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}\) | \(\emptyset\) |

**Definition 1.1** A CR manifold \((M, HM, J_M)\) is *(locally) embeddable* if admits a (local) CR embedding into a complex manifold \((M, J)\), i.e., \(M \hookrightarrow M\) is a smooth (local) embedding and \(H_xM = T_xM \cap JT_xM, J_M = J|_{HM}\).

Every real-analytic CR manifold \((M, HM, J_M)\) admits a CR embedding into a complex manifold \((M, J)\) (see [6]).

**Notation 1.2** We denote by \(\mathcal{H}\) (resp. \(T, T^C, T^C, T^{0,1}, T^{1,0}\)) the sheaf of germs of smooth sections of \(HM\) (resp. \(TM, T^CM, H^CM, T^{0,1}M, T^{1,0}M\)).

**Definition 1.2** A CR manifold \(M\) is called *fundamental* at its point \(x\) if \(\mathcal{H}_x\) generates under Lie brackets the Lie algebra \(T_x\).

The *Levi form* of \((M, T^{0,1}M)\) at the point \(x \in M\) is the Hermitian symmetric map

\[
\mathcal{L}_x : T^{0,1}_xM \otimes T^{1,0}_xM \to T^C_xM/H^C_xM
\]

(1.2)
defined by
\[ L_x(Z, \bar{W}) = \frac{1}{2t} \pi([Z, \bar{W}]_x), \quad \forall Z, W \in T_x^0 M, \tag{1.3} \]
where \( \pi \) is the canonical projection \( T_x^0 M \rightarrow T_x^C M/H_x^C M \), and \( Z, W \in T_x^0 M \) are smooth germs with \( Z_x = Z, W_x = W \).

The Levi kernel at \( x \in M \) is the null space of the Levi form
\[ F_x M = \{ Z \in T_x^0 M \mid L_x(Z, \bar{W}) = 0, \ \forall W \in T_x^0 M \}. \]

When the Levi kernel is not trivial, one needs to consider higher order Levi forms. A formulation analogous to (1.2), (1.3) would involve jet bundles. We may simplify the argument by considering germs of smooth sections.

For each point \( x \in M \) and any germ \( Z \in T_x^0 M \) with \( Z_x \neq 0 \), we can take the infimum \( \kappa(x, Z) \) of the set of positive integers \( k \) for which there are germs \( Z_1, \ldots, Z_k \in T_x^0 M \) such that
\[ [Z_1, [Z_2, \ldots, [Z_k, \tilde{Z}]]_x] \notin H_x^C M. \tag{1.4} \]

The nondegeneracy of the higher-order Levi form is measured by
\[ \kappa_x = \sup \{ \kappa(x, Z) \mid Z \in T_x^0 M, \ Z_p \neq 0 \}. \tag{1.5} \]

To investigate the degeneracy/nondegeneracy of the Levi form, Freeman in [11, Thm.3.1] defined a nested sequence of sheaves of germs of smooth complex-valued vector fields on \( M \)
\[ \mathcal{F}^{(0)} \supseteq \mathcal{F}^{(1)} \supseteq \cdots \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{F}^{(k+1)} \supseteq \cdots \tag{1.6} \]
by setting
\[ \left\{ \begin{array}{l} \mathcal{F}^{(0)} = T_x^0 M, \\ \mathcal{F}^{(k)} = \bigcup_{x \in M} \left\{ Z \in \mathcal{F}^{(k-1)} \mid [Z, T_x^{1,0}] \subseteq \mathcal{F}^{(k-1)} \oplus T_x^{1,0} \right\} \end{array} \right., \ \text{for} \ k \geq 1. \tag{1.7} \]

The \( k \)-th order Levi form may be defined by the map
\[ L_x^{(k)} : \mathcal{F}^{(k-1)}_p \otimes T_x^{0,1} \rightarrow \left( \mathcal{F}^{(k-1)}_x \oplus T_x^{1,0} \right) / \left( \mathcal{F}^{(k-1)}_x \oplus T_x^{1,0} \right), \tag{1.8} \]
with
\[ L_x^{(k)}(Z, \bar{W}) = \frac{1}{2t} \pi^{(k)}([Z, \bar{W}]), \tag{1.9} \]
where \( \pi^{(k)} \) is the canonical projection
\[ \pi^{(k)} : \mathcal{F}^{(k-2)}_x \oplus T_x^{1,0} \rightarrow \left( \mathcal{F}^{(k-2)}_x \oplus T_x^{1,0} \right) / \left( \mathcal{F}^{(k-1)}_x \oplus T_x^{1,0} \right), \tag{1.10} \]
and \( Z, W \) are smooth germs in \( \mathcal{F}^{(k-1)}_x \) and \( T_x^{0,1} \), respectively.

**Definition 1.3** A CR manifold \( M \) is \( \kappa \)-nondegenerate at \( p \) iff there is a positive integer \( \kappa \) such that
\[ \mathcal{F}^{(k-1)}_p \supsetneq \mathcal{F}^{(k)}_p = \{ 0 \}. \tag{1.11} \]

In this case, \( M \) is called finitely nondegenerate, otherwise holomorphically degenerate. The integer \( \kappa \) of (1.11) is called its Levi order.

\( ^1 \kappa(x, Z) = +\infty \) if there is not such an integer \( k \).
The standard notion of Levi nondegeneracy in the literature is equivalent to 1-nondegeneracy, or Levi order 1, in definition (1.3).

**Example 1.1** ([19]) The tube over the future light cone

\[ M \doteq \{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid (\text{Re} z_1)^2 + (\text{Re} z_2)^2 - (\text{Re} z_3)^2 = 0, \text{Re} z_3 > 0 \} \subset \mathbb{C}^3 \]

is a hypersurface in \( \mathbb{C}^3 \) and hence has a CR structure of type (2, 1), being a smooth hypersurface immersed in \( \mathbb{C}^3 \), which is 2-nondegenerate.

If \( M \) and \( M' \) are CR manifolds, a CR map \( f : M \rightarrow M' \) is a \( \mathcal{C}^\infty \)-smooth map such that

\[ f_C^*(T_{1,0}^0 x M) \subseteq T_{f(x)}^1 M' \text{ for all } x \in M. \]

A smooth fibration \( \pi : M \rightarrow M' \) is said to be CR if \( \pi_C^*(T_{1,0}^0 x M) = T_{f(x)}^1 M' \text{ for all } x \in M. \)

We recall the following:

**Lemma 1.2** (Prop.4.1 [2]) Let \( M \) and \( M' \) be CR manifolds. Assume that \( M' \) is locally embeddable and that there exists a CR fibration

\[ \pi : M \rightarrow M' \]

with complex fibers of positive dimension. Then, \( M \) is holomorphically degenerate. \( \square \)

### 2 Homogeneous CR manifolds and CR algebras

Let \( G^\sigma \) be a real Lie group of CR diffeomorphisms acting transitively on a CR manifold \( M \).

Fix a point \( x \) in \( M \) and let

\[ \pi : G^\sigma \ni g \rightarrow g \cdot x \in M \]

be the natural projection. Its differential at \( p \) maps the Lie algebra \( g^\sigma \) of \( G^\sigma \) onto the tangent space to \( M \) at \( x \). By the formal integrability condition, the pullback of \( T_{\pi(x)}^0 M \) by the complexification \( \hat{\pi}_* : g \rightarrow T_{\pi(x)}^C M \) of \( \pi_* : g^\sigma \rightarrow T_x M \) is a complex Lie subalgebra \( q \) of \( g \doteq g^\sigma \otimes \mathbb{C} : \)

\[ q \doteq \hat{\pi}_*(T_{\pi(x)}^0 M). \tag{2.1} \]

We will denote by \( \sigma \) the anti-\( \mathbb{C} \)-linear involution of \( g \) fixing \( g^\sigma \).

A different choice of the base point \( x \) would yield another \( \text{Ad}(G^\sigma) \)-conjugated complex Lie subalgebra.

Vice versa, the assignment of a complex Lie subalgebra \( q \) of \( g \) yields a \( G^\sigma \)-equivariant structure of CR manifold on a (locally) \( G^\sigma \)-homogeneous space \( M \), by requiring that \( T_{\pi(x)}^0 M \) is generated by the pushforward of \( q \) (see [18] for more details).

**Definition 2.1** A CR algebra is a pair \((g^\sigma, q)\) consisting of a real Lie algebra \( g^\sigma \) and a complex Lie subalgebra \( q \) of its complexification \( g = \mathbb{C} \otimes_R g^\sigma \), such that the quotient \( g^\sigma / (g^\sigma \cap q) \) is a finite dimensional real vector space.

The real Lie algebra \( g^\sigma \) encodes the transitive group of CR diffeomorphisms and the complex Lie subalgebra \( q \) the CR structure. The CR dimension \( n \) and CR codimension \( d \) of \( M \) are expressed in terms of its associated CR algebra \((g^\sigma, q)\) by

\[
\begin{align*}
n &= \dim_{\mathbb{C}} q - \dim_{\mathbb{C}} (q \cap \sigma(q)), \\
d &= \dim_{\mathbb{C}} g - \dim_{\mathbb{C}} (q + \sigma(q)).
\end{align*}
\tag{2.2}
\]

A CR algebra \((g^\sigma, q)\) is called
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- **totally real**: if \( q = q \cap \sigma(q) \), i.e., the CR dimension \( n \) is 0;
- **totally complex**: if \( q + \sigma(q) = g \), i.e., the CR codimension \( d \) is 0.

We call the intersection \( q \cap g^{\sigma} \) its **isotropy subalgebra**, and the subspace \( H = (q + \sigma(q)) \cap g^{\sigma} \) its **holomorphic tangent space**. Moreover, a CR algebra \((g^{\sigma}, q)\) is said to be effective if no nontrivial ideal of \( g^{\sigma} \) is contained in \( g^{\sigma} \cap q \).

The strong nonintegrability condition for the CR distribution of Definition 1.2 translates to the following notion for CR algebras.

**Definition 2.2** A CR algebra \((g^{\sigma}, q)\) is called **fundamental** if \( q + \sigma(q) \) generates \( g \) as a Lie algebra.

**Remark 2.1** The hypothesis of being fundamental is a key element for the Tanaka’s theory of prolongations (see [16]), necessary to compute the maximal CR automorphism group of the CR manifold \( M \). Moreover, if we consider a CR algebra \((g^{\sigma}, q)\) which is not fundamental, i.e., \( q + q \) generates a subalgebra \( g' \subset g \), then we can always construct a fibration with totally real base \((g^{\sigma}, g')\) and fundamental fiber \((g' \cap g^{\sigma}, q),\) named the **fundamental reduction**, e.g., [1]. In fact by Frobenius theorem, we know that the distribution associated with \( q' \) gives rise to a foliation of \( M \) with fundamental CR homogeneous leaves with the same CR dimension of \( M \).

For a finitely nondegenerate (locally) homogeneous CR manifold \( M \), the Levi order \( k \) can be computed by using its associated CR algebra \((g^{\sigma}, q)\). Noticing that \( q \cap \sigma(q) \) is the complexification of its isotropy subalgebra \( g^{\sigma} \cap q \), the \( 1 \)-nondegeneracy of the Levi form can be stated by

\[
\forall Z \in q \setminus \sigma(q), \quad \exists Z' \in q \quad \text{such that} \quad [Z, \sigma(Z')] \notin q + \sigma(q),
\]

and this is equivalent to

\[
q^{(1)} \doteq \{ Z \in q \mid [Z, \sigma(q)] \subseteq q + \sigma(q) \} = q \cap \sigma(q).
\]

Similar to (1.4), in the homogeneous case one can seek whether, for any given \( Z \in q \setminus (q \cap \sigma(q)) \) one can find a positive integer \( k \) and \( Z_1, \ldots, Z_k \in q \) such that

\[
[Z_1, \ldots, Z_k, Z] \notin q + \sigma(q).
\]

To this aim, in analogy to the Freenman’s sequence (1.6) of Sect. 1, it is convenient to consider the descending chain (see, e.g., [10, 11, 14])

\[
q^{(0)} \supseteq q^{(1)} \supseteq \cdots \supseteq q^{(k-1)} \supseteq q^{(k)} \supseteq q^{(k+1)} \supseteq \cdots,
\]

with

\[
\begin{cases}
q^{(0)} = q, \\
q^{(k)} = \{ Z \in q^{(k-1)} \mid [Z, \sigma(q)] \subseteq q^{(k-1)} + \sigma(q) \}, \quad \text{for } k \geq 1.
\end{cases}
\]

Note that \( q \cap \sigma(q) \subseteq q^{(k)} \) for all integers \( k \geq 0 \). If \( q/(q \cap \sigma(q)) \) is finite dimensional, then there is a smallest nonnegative integer \( k \) such that \( q^{(k')} = q^{(k)} \) for all \( k' \geq k \).

**Definition 2.3** A CR algebra \((g^{\sigma}, q)\) is said to be **\( k \)-nondegenerate** iff

\[
q^{(k-1)} \supseteq q^{(k)} = q \cap \sigma(q).
\]

**Proposition 2.2** ([15]) The terms \( q^{(k)} \) of (2.4) are Lie subalgebras of \( q \).
Remark 2.3 ([15], [16]) We point out that the notion of finite nondegeneracy defined in [18] by the requirement that for any complex Lie subalgebra \( g' \) of \( g \),
\[
q \subseteq q' \subseteq q + \sigma(q) \implies q' = q,
\]
(2.6)
is equivalent to that of Definition 2.3.

3 Structure of the closed orbit

A real Lie algebra \( g^\sigma \) is a real form of a complex Lie algebra \( g \) if
\[
g \simeq \mathbb{C} \otimes_{\mathbb{R}} g^\sigma.
\]
The real forms of \( g \) are the fixed points loci
\[
g^\sigma = \{ X \in g \mid \sigma(X) = X \}
\]
of its anti-\( \mathbb{C} \)-linear involutions \( \sigma \).

A complex flag manifold \( F \) is a smooth compact algebraic variety that can be described as the quotient of a complex semisimple Lie group \( G \) by a parabolic subgroup \( Q \). In [25, Theorem 2.6], J. A. Wolf shows the action of any real form \( G^\sigma \) of \( G \) partitions \( F \) into finitely many orbits. With the partial complex structures induced by \( F \), these orbits make a nice class of homogeneous CR manifolds that were studied by many authors (see, e.g., [1, 10, 15, 17]). Among these orbits, only one is closed; it is also compact, connected and of minimal dimension [25, Theorem 3.3].

Being connected and simply connected, a complex flag manifold \( F = G/Q \) is completely described by the Lie pair \((g, q)\) consisting of the complex Lie algebras of \( G \) and \( Q \) and vice versa to any Lie pair \((g, q)\) of a complex semisimple Lie algebra and a parabolic subalgebra \( q \) corresponds a unique flag manifold \( F \). Therefore, the classification of complex flag manifolds reduces to that of parabolic subalgebras of semisimple complex Lie algebras.

Parabolic subalgebras \( q \) of \( g \) are classified, modulo automorphisms, by a finite set of parameters as follows. Having fixed a Cartan subalgebra \( h \) of \( q \), a Weyl chamber \( C \) of the root system \( R \) of \((g, h)\), the classes of parabolic subalgebras of \( g \) are in a one to one correspondence with the subsets \( \phi \) of the basis \( B(C) \) of simple positive roots for the lexicographic order of \( C \) (see, e.g., [8, Ch.VIII,§3.4]). In the following, we will drop, for simplicity, the explicit reference to \( C \), writing, e.g., \( B \) instead of \( B(C) \).

Each root \( \beta \) in \( R \) can be written in a unique way as a nontrivial linear combination
\[
\beta = \sum_{\alpha \in B} n_{\beta, \alpha} \alpha,
\]
(3.2)
with integral coefficients \( n_{\beta, \alpha} \) which are either all \( \geq 0 \), or all \( \leq 0 \). Set
\[
\text{supp}(\beta) = \{ \alpha \in B \mid n_{\beta, \alpha} \neq 0 \}.
\]
(3.3)
The parabolic subset and the parabolic subalgebra associated with \( \phi \subseteq B \) are
\[
Q_\phi = \{ \beta \in R \mid n_{\beta, \alpha} \geq 0, \forall \alpha \in \phi \} \subseteq R,
\]
(3.4)
\[
q_\phi = h \oplus \sum_{\beta \in Q_\phi} g_\beta, \quad \text{with} \quad g_\beta = \{ Z \in g \mid [H, Z] = \beta(H)Z \}.
\]
(3.5)
The fact that $Q_\phi$ is parabolic means that
\[(Q_\phi + Q_\phi) \cap \mathcal{R} \subseteq Q_\phi \quad \text{and} \quad Q_\phi \cup (-Q_\phi) = \mathcal{R}.
\]

**Notation 3.1** We denote by $\xi_\phi$ the $\mathbb{R}$-linear functional on the $\mathbb{R}$-linear span $E$ of $\mathcal{R}$ which satisfies
\[
\xi_\phi(\alpha) = \begin{cases} 1, & \text{if } \alpha \in \phi, \\ 0, & \text{if } \alpha \in \mathcal{B} \setminus \phi. \end{cases} \tag{3.6}
\]
Then,
\[Q_\phi = \{ \beta \in \mathcal{R} \mid \xi_\phi(\beta) \geq 0 \} \tag{3.7}\]
and we get the partitions
\[Q_\phi = Q^{r}_\phi \cup Q^n_\phi, \quad \mathcal{R} = Q^{r}_\phi \cup Q^n_\phi \cup Q^{-n}_\phi, \tag{3.8}\]
where
- $\mathcal{Q}^{r}_\phi = \{ \beta \in Q_\phi \mid -\beta \notin Q_\phi \} = \{ \beta \in \mathcal{R} \mid \xi_\phi(\beta) = 0 \}$,
- $\mathcal{Q}^n_\phi = \{ \beta \in Q_\phi \mid -\beta \notin Q_\phi \} = \{ \beta \in \mathcal{R} \mid \xi_\phi(\beta) > 0 \}$,
- $\mathcal{Q}^{-n}_\phi = \{ \beta \in \mathcal{R} \mid -\beta \in Q^n_\phi \} = \{ \beta \in \mathcal{R} \mid \xi_\phi(\beta) < 0 \}$.

We recall (see, e.g., [8, Ch.VIII,§3]):
- $\mathcal{Q}^{r}_\phi = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{Q}^{r}_\phi} g_\alpha$ is a reductive complex Lie algebra;
- $\mathcal{Q}^n_\phi = \sum_{\alpha \in \mathcal{Q}^n_\phi} g_\alpha$ is the nilradical of $q_\phi$;
- $\mathcal{Q}^{-n}_\phi = \mathcal{Q}^n_\phi \oplus \mathcal{Q}^r_\phi$ is a Levi-Chevalley decomposition of $q_\phi$;
- $\mathcal{Q}^{-n}_\phi = \sum_{\alpha \in \mathcal{Q}^{-n}_\phi} g_\alpha$ is a nilpotent Lie algebra;

Given a Cartan subalgebra $\mathfrak{h}^\sigma$ of a semisimple real Lie algebra $\mathfrak{g}^\sigma$, we can find a compatible Cartan decomposition $\mathfrak{g}^\sigma = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}^\sigma$, such that $\mathfrak{h}^\sigma = (\mathfrak{h}^\sigma \cap \mathfrak{k}) \oplus (\mathfrak{h}^\sigma \cap \mathfrak{p})$. The two summands are its toroidal and vector parts. Among the equivalence classes of Cartan subalgebras of $\mathfrak{g}^\sigma$, there are only one with maximal toroidal part and only one with maximal vector part.

**Notation 3.2** Fix a Cartan subalgebra $\mathfrak{h}^\sigma$ of $\mathfrak{g}^\sigma$. Its complexification $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The conjugation $\sigma$ induces a symmetry on $\mathcal{R} = \mathcal{R}(\mathfrak{g}, \mathfrak{h})$ that we will still denote by the same symbol $\sigma$.

A root $\alpha \in \mathcal{R}$ is called:
- real if $\sigma(\alpha) = \alpha$;
- imaginary if $\sigma(\alpha) = -\alpha$;
- complex if $\sigma(\alpha) \neq \pm \alpha$.

We denote by $\mathcal{R}_\bullet$ the set of imaginary roots in $\mathcal{R}$.

We say that $C$ is an $S$-chamber if the $\sigma$-conjugate of any positive complex root stays positive:
\[\sigma(\alpha) \in \mathcal{R}^+, \quad \forall \alpha \in \mathcal{R}^+ \setminus \mathcal{R}_\bullet. \tag{3.9}\]
We can always find Weyl chambers with this property (see e.g. [2, Proposition 6.1]) and, on the basis $B$ of simple positive roots of an $S$-chamber an involution $\varepsilon : B \to B$ is defined, which keeps fixed the elements of $B^\sigma = B \cap R^\sigma$ and such that, for nonnegative $n_{\alpha, \beta} \in \mathbb{Z}$,
\[
\begin{cases}
\sigma(\alpha) = -\alpha, & \forall \alpha \in B^\sigma. \\
\sigma(\alpha) = \varepsilon(\alpha) + \sum_{\beta \in B^\sigma} n_{\alpha, \beta} \beta, & \forall \alpha \in B \setminus B^\sigma.
\end{cases}
\quad (3.10)
\]

**Definition 3.1** A Satake diagram $S$ is obtained from a Dynkin diagram, with the simple roots of an $S$-chamber, by painting black the roots in $B^\sigma \bullet$ and joining by an arch the pairs of distinct complex roots $\alpha, \beta$ with $\varepsilon(\alpha) = \beta$.

The Satake diagrams associated with an $h^\sigma$ with maximal vector part classify (modulo equivalence) the real forms of the complex Lie algebra described by the underlying Dynkin diagram (see e.g. [7]).

Fix a subset $\phi$ of $B$ and consider the diagram $(S, \phi)$ obtained from $S$ by adding a cross-mark on each node of $S$ corresponding to a root in $\phi$.

**Example 3.1** Fix the real form $g^\sigma = \text{su}(1, 3)$, then the CR algebra $(g^\sigma, q_\phi)$, where $\phi = \{\alpha_2\} \subseteq \{\alpha_1, \alpha_2, \alpha_3\} = B$, is described by the cross-marked Satake diagram

\[
\alpha_1 = e_1 - e_2 \quad \alpha_2 = e_2 - e_3 \quad \alpha_3 = e_3 - e_4
\]

where the cross-mark indicates the root in $\phi$.

This is the CR algebra of the minimal orbit $M$ of $\text{SU}(1, 3)$, consisting of the totally isotropic two planes of $\mathbb{C}^4$ for a hermitian symmetric form of signature $(1, 3)$.

Here $g \simeq \mathfrak{sl}_4(\mathbb{C})$, $R = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 4\}$, with a basis of simple roots $B = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$ for an orthonormal basis $e_1, e_2, e_3, e_4$ of $\mathbb{R}^4$.

Then, the linear functional $\xi_\phi$ on $R$ in (3.6) is given by
\[
\xi_\phi(e_i) = \begin{cases} 1, & i = 1, 2, \\ 0, & i = 3, 4 \end{cases}
\]

and the conjugation $\sigma$ induced by $\text{su}(1, 3)$ is
\[
\sigma(e_i) = -e_4, \quad \sigma(e_i) = -e_i, \quad i = 2, 3.
\]

Summarizing, to a parabolic subalgebra $q_\phi$ we associate a cross-marked Dynkin diagram, consisting of the Dynkin diagram of $g$ with marks on the nodes corresponding to roots in $\phi$. The correspondence between isomorphism classes of parabolic subalgebras and cross-marked Dynkin diagrams is bijective.

An effective parabolic minimal CR algebra $(g^\sigma, q)$ can always be described by a cross-marked Satake diagram: in fact, we can find a Cartan subalgebra $h^\sigma$ of $g^\sigma$ with maximal vector part and an $S$-chamber $C$ such that, for a subset $\phi$ of $B$, $q = q_\phi$. We have indeed

**Theorem 3.2** ([2, Th.6.3]) Having fixed a Cartan subalgebra $h^\sigma$ of $g^\sigma$ with maximal vector part, there is a one to one correspondence
\[
(S, \phi) \leftrightarrow (g^\sigma, q_\phi)
\quad (3.11)
\]

\[\text{Springer}\]
between cross-marked Satake diagrams (modulo automorphism of cross-marked Satake diagrams) and minimal effective parabolic CR algebras (modulo CR algebra isomorphisms).

Remark 3.3 Having fixed an $h^\sigma$ with maximal vector part and $\phi \subset B$ for a base of simple positive roots of a Weyl chamber $C$ satisfying (3.9) (see [2, Prop.6.1, 6.2]) on $R$, the symmetry $\vartheta(\alpha) = -\sigma(\alpha)$, corresponding to a Cartan involution on $g^\sigma$, leaves invariant $Q_n^{-\sigma} \cap \sigma(Q_n^\sigma)$: this set is therefore the union of its fixed points, which are in $R^\sigma_\bullet$, and of pairs $(\alpha, \sigma(\alpha))$ of symmetric distinct roots.

If $(g^\sigma, q_\phi)$ is the effective parabolic CR algebra of a minimal orbit, then by (3.9) these set of roots $Q_n^{\phi}$ associated with $q_\phi$ satisfies $Q_n^{\phi} \cap \sigma(Q_n^{\phi}) \subseteq R^\sigma_\bullet$, or, equivalently $Q_n^{\phi} \cap \sigma(Q_n^{\phi}) \subseteq R^\sigma_\bullet$.

Remark 3.4 ([1, Prop.6.2]) In the following, we will further restrict our consideration to the case of a simple $g^\sigma$. In fact, if $(g^\sigma, q_\phi)$ is an effective parabolic CR algebra and $g^\sigma = g_1^\sigma \oplus \cdots \oplus g_\ell^\sigma$ the decomposition of $g^\sigma$ into the direct sum of its simple ideals, then the following holds:

(1) $q = q_1 \oplus \cdots \oplus q_\ell$, where $q_j = q \cap g_j$;
(2) for each $j = 1, \ldots, \ell$, the pair $(g_j^\sigma, q_j)$ is an effective fundamental parabolic CR algebra;
(3) $(g^\sigma, q_\phi)$ is fundamental iff $(g_j^\sigma, q_j)$ is fundamental for all $j \in \{1, \ldots, \ell\}$;
(4) $(g^\sigma, q_\phi)$ is finitely nondegenerate iff $(g_j^\sigma, q_j)$ is finitely nondegenerate for all $j \in \{1, \ldots, \ell\}$.

Theorem 3.5 ([2, Th.10.2]) A simple effective parabolic minimal CR algebra $(g^\sigma, q_\phi)$ with associated cross-marked Satake diagram $(S, \phi)$ is totally complex if and only if

$$Q_n^{\phi} \cap \sigma(Q_n^{\phi}) = \emptyset,$$

i.e., if and only if one of the following holds:

(1) $g^\sigma$ is the compact real form of $g$;
(2) $g^\sigma$ is of the complex type (i.e., its complexification $g$ is semisimple, but nonsimple) and all cross-marked nodes belong to the same connected component of $S$;
(3) $(S, \phi)$ is one of the cases of the following table:

Proof Condition (3.13) is a consequence of the fact that $R$ is a disjoint union

$$R = (Q_\phi \cup \sigma(Q_\phi)) \cup (Q_n^{\phi} \cap \sigma(Q_n^{\phi})).$$

An open minimal orbit $M$, being also closed, coincides with the flag manifold $F$. This is always the case when $g^\sigma$ is compact (see, e.g., [20, 26]. Cases (2) and (3) follow easily from (3.13) (see also [26, Corollary 1.7]).

4 A bound for the k-nondegeneracy of the closed orbit

Finite nondegeneracy of the closed orbits of real forms was characterized in [2, Th. 11.5] in terms of their description by cross-marked Satake diagrams. Using Prop.2.2, we can...
discuss the order of finite nondegeneracy in terms of the chain (2.4) for \((g^\sigma, q_\phi)\) and can be investigated by using the combinatorics of the root system. Let us set
\[
Q^{(k)}_\phi = \{\alpha \in \mathcal{R} \mid g_\alpha \subseteq q^{(k)}_\phi\}, \quad \text{so that} \quad q^{(k)}_\phi = \mathfrak{h} \oplus \sum_{\alpha \in Q^{(k)}_\phi} g_\alpha. \tag{4.1}
\]
They can be defined recursively by
\[
\begin{cases}
Q^{(0)}_\phi &= Q_\phi, \\
Q^{(1)}_\phi &= \{\alpha \in Q_\phi \mid (\alpha + \sigma(Q_\phi)) \cap \mathcal{R} \subseteq Q_\phi \cup \sigma(Q_\phi)\}, \\
Q^{(k)}_\phi &= \{\alpha \in Q^{(k-1)}_\phi \mid (\alpha + \sigma(Q_\phi)) \cap \mathcal{R} \subseteq Q^{(k-1)}_\phi \cup \sigma(Q_\phi)\}, & \text{for} \ k > 1.
\end{cases} \tag{4.2}
\]
This yields a characterization of finite nondegeneracy in terms of roots:

**Proposition 4.1** A necessary and sufficient condition for \((g^\sigma, q_\phi)\) being finitely nondegenerate of order \(k\) is that \(Q^{(k)}_\phi \supseteq Q^{(k-1)}_\phi \supseteq \mathcal{Q}_\phi = Q_\phi \cap \sigma(Q_\phi)\). \(\Box\)

By Proposition 4.1, we obtain:

**Remark 4.2** ([2, Lemma 12.1]) A necessary and sufficient condition for \((g^\sigma, q_\phi)\) being finitely nondegenerate is that for all \(\beta \in Q_\phi \cap \sigma(Q^{-n}_\phi)\), one can find \(k \in \mathbb{Z}_+\), and \(\alpha_1, \ldots, \alpha_k \in \sigma(Q_\phi)\) such that
\[
\begin{cases}
\gamma_h = \beta + \sum_{i=1}^h \alpha_i \in \mathcal{R}, & \forall 1 \leq h \leq k, \\
\gamma_k \in Q^{-n}_\phi \cap \sigma(Q^{-n}_\phi). 
\end{cases} \tag{4.3}
\]

**Notation 4.1** Denote by \(k^\sigma_\phi(\beta)\) the smallest positive integer for which (4.3) holds true. Set \(k^\sigma_\phi(\beta) = +\infty\) if there is no \(k\) for which (4.3) is satisfied.

**Proposition 4.3** ([15, Lemma 2.6]) Let \(\beta \in Q_\phi \cap \sigma(Q^{-n}_\phi)\) and assume that \(k = k^\sigma_\phi(\beta) < +\infty\). Then, any sequence \(\alpha_1, \ldots, \alpha_k\) satisfying (4.3) has the properties:
- (i) \(\alpha_i \in \sigma(Q_\phi) \cap Q^{-n}_\phi\) for all \(1 \leq i \leq k\);
- (ii) \(\beta + \sum_{i=1}^h \alpha_i \in Q_\phi \cap \sigma(Q^{-n}_\phi)\) for all \(h < k\);
- (iii) (4.3) is satisfied by all permutations of \(\alpha_1, \ldots, \alpha_k\);
- (iv) \(\alpha_i + \alpha_j \notin \mathcal{R}\) for all \(1 \leq i < j \leq k\). \(\Box\)

**Example 4.4** We continue with Example 3.1. We obtain
\[
Q^{-n}_\phi \cap \sigma(Q^{-n}_\phi) = \{\alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4\},
\]
\[
\sigma(Q_\phi) \cap Q^{-n}_\phi = \{\alpha_1 + \alpha_2 = e_1 - e_3, \alpha_2 = e_2 - e_3, \alpha_2 + \alpha_3 = e_2 - e_4\},
\]
\[
Q_\phi \cap \sigma(Q^{-n}_\phi) = \{\alpha_3 = e_3 - e_4, -\alpha_2 = e_3 - e_2, \alpha_1 = e_1 - e_2\}.
\]
We obtain that \((g^\sigma, q_\phi)\) is fundamental and finitely nondegenerate of order 2, since \(k^\sigma_\phi(e_3 - e_2) \geq 2\) and equals 2 because
\[
e_1 - e_4 = (e_3 - e_2) + (e_1 - e_3) + (e_2 - e_4).
\]

**Remark 4.5** Since \(\xi_\phi(\alpha) \leq -1\) for all \(\alpha \in Q^{-n}_\phi\), if \(\beta \in Q_\phi \cap \sigma(Q^{-n}_\phi)\) and \(k^\sigma_\phi(\beta) < +\infty\), then
\[
k^\sigma_\phi(\beta) \leq 1 + \xi_\phi(\beta). \tag{4.4}
\]
Corollary 4.6 If $\alpha \in \mathcal{Q}_1^r \cap \sigma(\mathcal{Q}^{-n}_1)$, then $\kappa^\sigma_1(\alpha) = 1$ or $\kappa^\sigma_1(\alpha) = +\infty$.

We obtain also a useful criterion of finite nondegeneracy (cf. [4, Thm.6.4])

Proposition 4.7 The parabolic CR algebra $(\mathfrak{g}^\sigma, q_{\phi})$ is finitely nondegenerate if and only if

\[ \forall \beta \in \mathcal{Q}_1 \cap \sigma(\mathcal{Q}^{-n}_1) \exists \alpha \in \sigma(\mathcal{Q}_1) \cap \mathcal{Q}^{-n}_1 \text{ such that } \beta + \alpha \in \sigma(\mathcal{Q}^{-n}_1). \quad (4.5) \]

We rehearse the result contained in [15, Cor.2.19], in a form which is suitable for our purposes.

Theorem 4.8 The closed orbit is either holomorphically degenerate or is finitely nondegenerate with Levi order $\kappa$ less or equal than two.

Proof Let $(\mathfrak{g}^\sigma, q_{\phi})$ be a parabolic CR algebra of the closed orbit, $\phi$ being the set of crossed roots in a cross-marked Satake diagram. Since all roots $\beta$ in $\mathcal{Q}^{-n}_1$ are positive, by (3.9), if $\beta \in \mathcal{Q}^{-n}_1 \setminus \mathcal{R}_\bullet$, then its conjugate $\sigma(\beta)$ is positive and hence has $\xi_{\phi}(\sigma(\beta)) \geq 0$, i.e.,

\[ \xi_{\phi}(\beta) \geq 0, \ \forall \beta \in \sigma(\mathcal{Q}^{-n}_1) \setminus \mathcal{R}^\sigma_\bullet. \quad (4.6) \]

Let $\beta \in \mathcal{Q}_1 \cap \sigma(\mathcal{Q}^{-n}_1)$. If $\beta \notin \mathcal{R}^\sigma_\bullet$, then $\xi_{\phi}(\beta) = 0$ by (4.6), and hence, by Cor. 4.6, $\kappa^\sigma_{\phi}(\beta)$ is either 1 or $+\infty$.

If $\kappa^\sigma_{\phi}(\beta)$ is an integer $k>1$, then $\beta \in \mathcal{R}^\sigma_\bullet$. Let $(\alpha_1, \ldots, \alpha_k)$ be a sequence satisfying (4.3) and thus the conditions in Prop.4.3. Since $\mathcal{Q}^{-n}_1 \cap \sigma(\mathcal{Q}^{-n}_1) \cap \mathcal{R}_\bullet = \emptyset$, there is at least one root $\alpha_i$ which does not belong to $\mathcal{R}^\sigma_\bullet$. Again by (iii) of Proposition 4.3, we can assume it is $\alpha_1$. Then, $\beta + \alpha_1$ belongs to $(\mathcal{Q}_1 \cap \sigma(\mathcal{Q}^{-n}_1)) \setminus \mathcal{R}^\sigma_\bullet$, and therefore, by the first part of the proof, $\kappa^\sigma_{\phi}(\beta + \alpha_1) = 1$. This implies that $k=2$. The proof is complete.

Remark 4.9 Theorem 4.8 is a special case of a more general result proved in [15] which state that all finitely nondegenerate parabolic CR algebras have Levi order less or equal 3. A special class, generalizing minimal orbits, and called of minimal type are proved there to have also $\kappa \leq 2$.

Proposition 4.10 Let $(\mathfrak{g}^\sigma, q_{\phi})$ be a parabolic CR algebra, corresponding to a minimal orbit. Then,

\[ \mathcal{Q}_1 \cap \sigma(\mathcal{Q}^{-n}_1) = (\mathcal{Q}^n_1 \cap \mathcal{R}^\sigma_\bullet) \cup (\mathcal{Q}^{n-1}_1 \cap \sigma(\mathcal{Q}^{-n}_1)). \quad (4.7) \]

If $(\mathfrak{g}^\sigma, q)$ is finitely nondegenerate, then its order of nondegeneracy is

\begin{itemize}
  \item 1. if $\mathcal{Q}^n_1 \cap \mathcal{R}^\sigma_\bullet = \emptyset$;
  \item $\sup \{ \kappa^\sigma_{\phi}(\beta) \mid \beta \in \mathcal{Q}^n_1 \cap \mathcal{R}_\bullet \}$, if $\mathcal{Q}^n_1 \cap \mathcal{R}^\sigma_\bullet \neq \emptyset$
\end{itemize}

Proof We assume, as we can, that $(\mathfrak{g}^\sigma, q_{\phi})$ is described by a cross-marked Satake diagram. In particular, complex positive roots have a positive $\sigma$-conjugate. This implies that all complex roots that do not belong to $\sigma(\mathcal{Q}_1)$ are negative, and hence, if they are in $\mathcal{Q}_1$, they belong to $\mathcal{Q}^n_1$. On the other hand, imaginary roots not belonging to $\sigma(\mathcal{Q}_1)$ belong to $\mathcal{Q}^{n-1}_1$.

The statement about the order of nondegeneracy follows from Corollary 4.6.
5 $G^\sigma$-equivariant CR fibrations

In this section, we describe the canonical $G^\sigma$-equivariant CR fibrations of the closed orbits and their characterization in terms of associated cross-marked Satake diagrams. Together with Lemma 1.2, this will give us effective criteria of finite nondegeneracy.

**Remark 5.1** Let $(g^\sigma, q)$ be a CR algebra, $a^\sigma$ an ideal of $g^\sigma$ and $a$ its complexification. Set

\[
\begin{align*}
&f = (q + a) \cap (\sigma(q) + a), \\
&f^\sigma = f \cap g^\sigma, \\
&q' = q \cap (\sigma(q) + a).
\end{align*}
\]

(5.1)

Since

\[q + (q + a) \cap (\sigma(q) + a) = q + a,
\]

the following fibration is CR:

\[
0 \longrightarrow (f^\sigma, q') \longrightarrow (g^\sigma, q) \longrightarrow (g^\sigma, q + a) \longrightarrow 0.
\]

(5.2)

If $\psi \subseteq \phi \subseteq B$, then $q_{\phi} \subseteq q_{\psi}$ and we obtain a $g^\sigma$-equivariant fibration

\[
0 \longrightarrow (f^\sigma_{\psi}, q_{\phi, \psi}) \longrightarrow (g^\sigma, q_{\phi}) \longrightarrow (g^\sigma, q_{\psi}) \longrightarrow 0
\]

(5.3)

where the typical fiber $(f^\sigma_{\psi}, q_{\phi, \psi})$ is described by

\[
\begin{align*}
f^\sigma_{\psi} &= q_{\phi} \cap \sigma(q_{\psi}), \\
f^\sigma_{\phi} &= q_{\phi} \cap g^\sigma, \\
q_{\phi, \psi} &= q_{\phi} \cap \sigma(q_{\psi}).
\end{align*}
\]

(5.4)

We recall that (5.3) is a CR fibration iff

\[q_{\psi} = q_{\phi} \cap \sigma(q_{\psi}) + q_{\phi}.
\]

(5.5)

In terms of roots, this equality can be rewritten by

\[
Q_{\psi} = (Q_{\psi} \cap \sigma(Q_{\phi})) \cup Q_{\phi}
\]

iff

\[
\begin{align*}
\mathcal{R} \setminus Q^{-n}_{\psi} &= (\mathcal{R} \setminus Q^{-n}_{\phi}) \cap \sigma(\mathcal{R} \setminus Q^{-n}_{\psi}) \cup (\mathcal{R} \setminus Q^{-n}_{\phi}) \\
\mathcal{R} \setminus Q^{-n}_{\phi} &= \mathcal{R} \setminus (Q^{-n}_{\psi} \cup \sigma(Q^{-n}_{\psi})) \cup (\mathcal{R} \setminus Q^{-n}_{\phi}) \\
Q^{-n}_{\psi} &= (Q^{-n}_{\phi} \cup \sigma(Q^{-n}_{\phi})) \cap Q^{-n}_{\phi} \\
Q^{-n}_{\phi} &= (Q^{-n}_{\psi} \cup \sigma(Q^{-n}_{\psi})) \cap Q^{-n}_{\phi} \\
Q^{-n}_{\psi} \cap \sigma(Q^{-n}_{\psi}) &\subseteq Q^{-n}_{\psi}.
\end{align*}
\]

We summarize the above discussion by the following proposition.

**Proposition 5.2** Let $C$ be an adapted Weyl chamber for the parabolic CR algebra $(g^\sigma, q_{\phi})$, described by the subset $\phi$ of the basis $B = B(C)$ of simple positive roots of $\mathcal{R}^+(C)$. If there is a $g^\sigma$-equivariant CR algebra homomorphism of $(g^\sigma, q_{\phi})$ onto a CR algebra $(g^\sigma, q')$, then this is also parabolic and $q' = q_{\psi}$ for a subset $\psi$ of $B$ with $\psi \subseteq \phi \subseteq B$. The corresponding fibration (5.3) is CR if and only if

\[Q^{-n}_{\phi} \cap \sigma(Q^{-n}_{\psi}) \subseteq Q^{-n}_{\psi}.
\]

(5.6)
In general, \( g_{\Phi}^\sigma \) and \( g_{\Psi} \) may not be semisimple. However, both \( f_{\Psi} \) and \( q_{\Phi, \Psi} \) are regular subalgebras of \((g, h)\). With

\[
\mathcal{F}_\Psi = Q'_\Psi \cap \sigma(Q'_\Psi), \quad \mathcal{F}_\Psi = (Q_\Psi \cap \sigma(Q_\Psi^n)) \cup (Q_\Psi^n \cap \sigma(Q_\Psi)), \quad \mathcal{F}_\Psi = \mathcal{F}_\Psi \cup \mathcal{F}_\Psi^n, \quad (5.7)
\]

we get a Levi-Chevalley decomposition of \( f_{\Psi} \):

\[
f_{\Psi} = f_{\Psi}^r + f_{\Psi}^n, \quad \begin{cases} f_{\Psi}^r = h \oplus \sum_{\alpha \in \mathcal{F}_\Psi} g_{\alpha}^r \quad \text{(reductive Levi factor)}, \\ f_{\Psi}^n = \sum_{\alpha \in \mathcal{F}_\Psi} g_{\alpha}^n \quad \text{(nilradical)}. \end{cases} \quad (5.8)
\]

Analogously, with

\[
Q_{\Phi, \Psi} = Q_\Phi \cap \sigma(Q_\Psi), \quad Q'_{\Phi, \Psi} = Q'_\Phi \cap \sigma(Q'_\Psi), \quad Q''_{\Phi, \Psi} = (Q''_\Phi \cap \sigma(Q_\Psi)) \cup (Q_\Phi \cap \sigma(Q_\Psi^n)),
\]

we get a Levi–Chevalley decomposition of \( q_{\Phi, \Psi} \):

\[
q_{\Phi, \Psi} = q_{\Phi, \Psi}^r + q_{\Phi, \Psi}^n, \quad \begin{cases} q_{\Phi, \Psi}^r = h \oplus \sum_{\alpha \in Q_{\Phi, \Psi}^r} g_{\alpha}^r \quad \text{(reductive Levi factor)}, \\ q_{\Phi, \Psi}^n = \sum_{\alpha \in Q_{\Phi, \Psi}^n} g_{\alpha}^n \quad \text{(nilradical)}. \end{cases} \quad (5.9)
\]

**Lemma 5.3** Assume that \( C \) has property (3.9) for \( \sigma \). Then

\[
\mathcal{F}_\Psi^n \subseteq \mathcal{R}^+(C) \setminus \mathcal{R}^\sigma \subseteq Q_\Phi \implies f_{\Psi}^r \subseteq q_{\Phi}.
\]

**Proof** Let us check that \( Q_{\Psi}^n \cap \sigma(Q_\Psi) \cap \mathcal{R}^\sigma = \emptyset \). Assume that \( \alpha \in Q_{\Psi}^n \cap \mathcal{R}^\sigma \). Then, \(-\alpha \in \sigma(Q_{\Psi}^-)\) shows that \( \alpha \in \sigma(Q_{\Psi}^{-n}) \), i.e., that \( \alpha \notin \sigma(Q_\Psi) \). Since \( Q_{\Psi}^n \) is contained in \( \mathcal{R}^+(C) \) and we assumed that \( C \) satisfies (3.9), we obtain that also \( \sigma(Q_{\Psi}^r \cap \sigma(Q_\Psi)) = Q_\Psi \cap \sigma(Q_{\Psi}^n) \subseteq \mathcal{R}^+(C) \). The proof is complete. \( \square \)

**Lemma 5.4** The set \( \mathcal{F}_\Phi \) is a subsystem of roots of \( \mathcal{R} \),

\[
B_\Phi = \{ \alpha \in B \mid \alpha, \sigma(\alpha) \notin Q_{\Psi}^n \}
\]

a basis of simple positive roots of \( \mathcal{F}_\Phi \) and \( Q_{\Phi, \Psi} \cap \mathcal{F}_\Phi \) a parabolic subset of \( \mathcal{F}_\Phi \). \( \square \)

**Proposition 5.5** Let \( \Phi \subseteq \Phi \subseteq B \) and assume that \( (g^\sigma, q_{\Phi}) \rightarrow (g^\sigma, q_{\Psi}) \) is a CR fibration. A necessary and sufficient condition for the typical fiber being totally complex is that

\[
q_{\Phi} + \sigma(q_{\Phi}) = q_{\Psi} + \sigma(q_{\Psi}).
\]

**Proof** Indeed, we have

\[
q_{\Phi, \Psi} + \sigma(q_{\Phi, \Psi}) = f_{\Psi} \iff q_{\Psi} \cap \sigma(q_{\Phi}) + q_{\Phi} \cap \sigma(q_{\Psi}) = q_{\Psi} \cap \sigma(q_{\Psi}).
\]

By the assumption that the fibration is CR, we get

\[
q_{\Psi} = q_{\Phi} + q_{\Psi} \cap \sigma(q_{\Phi}) = q_{\Phi} + q_{\Psi} \cap \sigma(q_{\Phi}) + q_{\Phi} \cap \sigma(q_{\Psi}) \subseteq q_{\Psi} + \sigma(q_{\Psi})
\]

and this implies that the right-hand side is contained in the left hand side of (5.13). Since the opposite inclusion is obvious, the proof is complete. \( \square \)
Remark 5.6 Condition (5.13) is equivalent to any of

\[ q_{\phi} \subseteq q_{\psi} \subseteq q_{\phi} + \sigma(q_{\phi}), \]  
\[ q_{\phi} + q_{\psi} \cap \sigma(q_{\phi}) = q_{\psi}, \]  
\[ q_{\phi} \cap \sigma(q_{\psi}) \subseteq q_{\psi}. \]  

In terms of roots, we can translate (5.13) into

\[ (f^\sigma, q_{\phi}, q_{\psi}) \text{ is totally complex } \iff Q_{\psi} \cup \sigma(Q_{\psi}) = Q_{\phi} \cup \sigma(Q_{\phi}). \]  

We have the following criterion on its cross-marked Satake diagram to establish whether a minimal orbit is fundamental (see Def. 1.2):

Proposition 5.7 ([2, Th. 9.1]) The effective parabolic CR algebra \((g^\sigma, q_{\phi})\) of a closed orbit is fundamental if and only if its corresponding cross-marked Satake diagram \((\mathcal{S}, \phi)\) has the property

\[ \alpha \in \phi \setminus R^\sigma_\phi \Rightarrow \epsilon(\alpha) \notin \phi. \]  

□

Notation 5.1 Given a Satake diagram \(\mathcal{S}\) and a subset of simple roots \(\phi \subset B\), we will use the notation:

- Let \(\alpha\) be any positive root. We denote by \(\phi^\circ(\alpha)\) the connected component of \(\text{supp}(\alpha)\) in \((B \setminus \phi) \cup \text{supp}(\alpha)\).
- The exterior boundary \(\delta_e \eta\) of a given a subset \(\eta\) of \(B\) is the set

\[ \delta_e \eta = \{ \alpha \in R \mid \exists \beta \in \eta \text{ s.t. } \alpha + \beta \in R \}. \]

Proposition 5.8 ([2, Th. 7.10]) Let \(\psi \subseteq \phi \subseteq B\). A necessary and sufficient condition for (5.3) to be a \(g^\sigma\)-equivariant CR fibration is that for each \(\alpha \in \phi \setminus \psi\) either one of the following conditions holds:

1. \(\psi^\circ(\alpha) \subset R^\sigma\phi\);
2. \(\psi^\circ(\alpha) + \delta_e \phi^\circ(\alpha) \cap R^\sigma\phi = \emptyset\), and \(\delta_e(\psi^\circ(\alpha) \setminus R^\sigma\phi) = \emptyset\).

□

Remark 5.9 Let \((g^\sigma, q_{\phi})\) be the effective parabolic CR algebra of a closed orbit. By Prop. 5.5, \((g^\sigma, q_{\phi})\) is finitely degenerate if and only if we can find a \(\psi \subsetneq \phi\), satisfying conditions (1) and (2) of Proposition 5.8, such that

\[ q_{\psi} \subset q_{\phi} + \sigma(q_{\phi}). \]

Theorem 5.10 ([2, Th. 11.5]) Let \((g^\sigma, q_{\phi})\) be a simple fundamental effective parabolic minimal CR algebra and assume that it is not totally complex. Let \(\Pi \subseteq \phi\) be the set of simple roots \(\alpha \in \phi\) that satisfy either one of:

1. \(\phi^\circ(\alpha) \subset R^\sigma\phi\);
2. \((\phi^\circ(\alpha) \cup \delta_e \phi^\circ(\alpha)) \cap R^\sigma\phi = \emptyset\) and \(\epsilon(\phi^\circ(\alpha) \setminus R^\sigma\phi) = \emptyset\).

Then, \((g^\sigma, q_{\phi})\) is finitely nondegenerate if and only if \(\Pi = \emptyset\).

In general, if we set \(\psi = \phi \setminus \Pi\), then (5.3) is a \(g^\sigma\)-equivariant CR fibration with fundamental finitely nondegenerate base and totally complex typical fiber. □
Theorem 5.11 Let \((g^\sigma, q_\phi)\) be a finitely nondegenerate fundamental minimal effective parabolic CR algebra, with \(g^\sigma\) simple.

If \(\alpha \in Q_\phi \cap \sigma(Q'^{-n}_{\phi})\) has \(k^\sigma_\phi(\alpha) = 2\), then \(\alpha \in Q'^{n}_{\phi} \cap R^\sigma_\bullet\) and \(\phi \subseteq \text{supp}(\alpha)\).

Proof Since \(C\) is an S-chamber, \(Q_\phi \cap \sigma(Q'^{-n}_{\phi})\) is the disjoint union

\[ Q_\phi \cap \sigma(Q'^{-n}_{\phi}) = \left( Q'_\phi \cap \sigma(Q'^{-n}_{\phi}) \right) \cup \left( Q'^{n}_{\phi} \cap R^\sigma_\bullet \right), \]

the first set in the right-hand side being contained in \(R^-(C)\), the second in \(R^+(C)\).

Fix any \(\alpha \in Q_\phi \cap \sigma(Q'^{-n}_{\phi})\). We want to prove that, if either \(\alpha \in Q'_\phi \cap \sigma(Q'^{-n}_{\phi})\) or \(\alpha \in Q'^{n}_{\phi} \cap R^\sigma_\bullet\) and \(\phi \not\subseteq \text{supp}(\alpha)\), then \(k^\sigma_\phi(\alpha) = 1\).

By Corollary 4.6, we get \(k^\sigma_\phi(\alpha) = 1\) when \(\alpha \in Q'_\phi \cap \sigma(Q'^{-n}_{\phi})\).

Suppose \(\alpha \in Q'^{n}_{\phi} \cap R^\sigma_\bullet\) and let \(\psi\) be the connected component of \(\text{supp}(\alpha)\) in \(B^\sigma_\bullet\).

Assume that \(\psi \setminus \text{supp}(\alpha)\) contains an element \(\alpha'\) of \(\phi\). By Theorem 5.10, \(\phi^\sigma(\alpha') \subseteq B^\sigma_\bullet\).

Thus we can find a simple path \((\alpha_0, \ldots, \alpha_p)\) in \(\phi^\sigma(\alpha) \setminus \text{supp}(\alpha)\) such that \(\alpha_0 \notin B^\sigma_\bullet\), \(\alpha_i \in \psi\) for \(0 < i \leq p\), joining \(\alpha_0\) to \(\text{supp}(\alpha)\), i.e., satisfying

\[
\begin{aligned}
(\alpha_i | \alpha_j) &= 0, & & \text{for } |i - j| > 1, \\
(\alpha_{i-1} | \alpha_i) &< 0, & & \text{for } 1 \leq i \leq p, \\
(\alpha_p | \alpha) &< 0.
\end{aligned}
\]

(\(*)\)

Let \(\gamma = -\sum_{i=0}^{p} \alpha_i\) and \(\beta = \sigma(\gamma)\). Then, \(\beta\) is a negative root in \(\sigma(Q'_\phi) \cap Q'^{-n}_{\phi}\), because \(\alpha'\), belonging to the support of \(\sigma(\alpha_0)\), also belongs to the support of \(\beta\). The inequality

\[(\alpha | \beta) = (\sigma(\alpha) | \gamma) = - (\alpha, \gamma) = (\alpha | \alpha_p) < 0\]

\[(***)\]

implies that \(\alpha + \beta\) is a negative root, containing \(\alpha'\) in its support, while \(\sigma(\alpha + \beta) = -\alpha + \gamma\) contains \(\alpha\) in its support. Hence \(\alpha + \beta \in Q'^{-n}_{\phi} \cap \sigma(Q'^{-n}_{\phi})\) and this shows that \(k^\sigma_\phi(\alpha) = 1\).

If \(\phi \setminus \psi \subseteq \text{supp}(\alpha)\), but \(\phi \not\subseteq \text{supp}(\alpha)\), then we can find a simple path \((\alpha_0, \ldots, \alpha_p)\) in \(B\), satisfying (\(*\)), with \(\alpha_0 \in \phi\), \(\alpha_i \notin \phi\) for \(1 \leq i \leq p\).

If we can find such a sequence with \(\alpha_0 \in B^\sigma_\bullet\), then \(p \geq 1\) and \(\gamma = -\sum_{i=1}^{p} \alpha_i\) is a not imaginary negative root in \(Q'_\phi\). Set \(\hat{\phi} = \sigma(\gamma)\).

Since \(\{\alpha_0\} \cup \text{supp}(\alpha) \subseteq \text{supp}(\beta)\), we get \(\beta \in \sigma(Q'_\phi) \cap Q'^{-n}_{\phi}\). Also (\**) holds true, so that \(\alpha + \beta \in Q'^{-n}_{\phi} \cap \sigma(Q'^{-n}_{\phi})\) and hence \(k^\sigma_\phi(\alpha) = 1\).

Otherwise, we consider the set \(\hat{\phi} = \phi \cup \varepsilon(\phi \setminus B^\sigma_\bullet)\). Let \(\hat{\psi}\) be the connected component of \(\psi\) in \((\hat{B} \setminus \hat{\phi}) \cup \psi\). This set is \(\varepsilon\)-invariant and hence also its exterior boundary is \(\varepsilon\)-invariant. In particular, it contains an element \(\alpha_0\) of \(\phi \setminus B^\sigma_\bullet\) and its symmetrical \(\varepsilon(\alpha_0)\), which, by Proposition 5.7, does not belong to \(\phi\). We take simple paths \((\alpha_0, \ldots, \alpha_p)\) and \((\alpha_{p+1}, \ldots, \alpha_q)\), with \(\alpha_i \in \hat{B} \setminus \phi\) for \(i \neq 0, q\), joining \(\text{supp}(\alpha)\) to \(\alpha_0\) and to \(\varepsilon(\alpha_0)\), respectively. This means that (\*) holds true, and moreover,

\[
\begin{aligned}
(\alpha_i | \alpha_j) &= 0, & & \text{for } 1 \leq i < 1 + 2 \leq j \leq q, \\
(\alpha_p | \alpha_{p+1}) &= 0, \\
(\alpha_{p+1} | \alpha) &< 0.
\end{aligned}
\]

Then, \(\gamma = -\sum_{i=p+1}^{q} \alpha_i \in Q'_\phi\) and \(\beta = \sigma(\gamma) \in \sigma(Q'_\phi) \cap Q'^{-n}_{\phi}\), since \(\beta\) is a negative root whose support contains \(\alpha_0\). Since (\***) holds true, \(\alpha + \beta \in Q'^{-n}_{\phi} \cap \sigma(Q'^{-n}_{\phi})\), proving that \(k^\sigma_\phi(\alpha) = 1\). The proof is complete. \(\square\)
6 Finitely nondegenerate closed orbits

Putting together the results of the previous sections, we obtain a complete classification of finitely Levi-nondegenerate closed orbits of real forms in complex flag manifolds and may compute their order of Levi nondegeneracy. Since, by Remark 3.4, they are Cartesian products of orbits of simple real Lie groups, we can restrain to the case of a simple $G^\sigma$. We first consider the case where $G$ is also a simple Lie group.

**Theorem 6.1** Let $(g^\sigma, q_\phi)$ be the CR algebra of a closed orbit and assume that $g^\sigma$ is simple and of the real type.\(^2\) If $(g^\sigma, q_\phi)$ is finitely nondegenerate, then $g^\sigma$ in not compact and the subset $\phi$ is one of those listed in Tables 2 and 3, where also its Levi order is indicated.

**Proof** By Theorem 3.2, closed orbits in complex flag manifolds are in a one-to-one correspondence with cross-marked Satake diagrams $(S, \phi)$. Connected Satake diagrams classify simple real Lie algebras of the real type (see, e.g., [7]) and properties of the minimal parabolic CR algebras defined by different $\phi$’s can be discussed by utilizing the results of the previous section.

For the classification of simple real forms of the real type we use the notation of [13, Table VI, pp.532-534].

Real types A I, C I, E I, E V, E VIII, F I, G and the split B I, D I, being split real forms, yield totally real $(g^\sigma, q_\phi)$ for all choices of $\phi \subseteq \mathcal{B}$.

Types A II and D II, with the $\phi$’s listed Table 1, and all compact forms, for all $\phi \subseteq \mathcal{B}$, yield totally complex CR algebras $(g^\sigma, q_\phi)$ which are associated with the corresponding complex flag manifold.

By using Proposition 5.7, we can recognize the $\phi$’s yielding fundamental CR algebras and by Theorem 5.10 the $\phi$’s for which $(g^\sigma, q_\phi)$ is finitely nondegenerate.

By Corollary 4.6, we know that a finitely nondegenerate minimal parabolic $(g^\sigma, q_\phi)$ has Levi order less or equal 2 and by Theorem 5.11 that, when $g^\sigma$ is simple and of the real type, only those with $\phi$ contained in the support of an imaginary root may have Levi order 2.

Hence we will restrain the computation of this invariant to these cases.

\[ \begin{array}{c} \text{g}^\sigma \text{ OF TYPE AII} \quad \text{We have} \quad g^\sigma \cong sl_p(\mathbb{H}), \text{for some integer} \quad p \geq 2. \\
\end{array} \]

The $\phi$’s for which $(g^\sigma, q_\phi)$ is fundamental are those with $\phi \subseteq B^\sigma_{\phi}$. Those with $\phi$ equal to $\{\alpha_1\}$ of $\{\alpha_{2q-1}\}$ are totally complex. The other have Levi order 1 if $|\phi| \geq 2$ and if $\phi = \{\alpha_{2q-1}\}$ with $2 \leq q \leq p - 1$.

Indeed, in the last case, $\xi_\phi (\alpha_{2q-1}) = 1 \geq \sup_{\alpha \in \mathcal{R}} |\xi(\alpha)|$ and hence no root of the form $\alpha_{2q-1} + \alpha$, with $\alpha \in \mathcal{R}$, may belong to $Q_{\Phi}^\sigma$.

\[ \begin{array}{c} \text{g}^\sigma \text{OF TYPE AIII, AIV} \quad \text{We have} \quad g^\sigma \cong su(p, q) \text{ for integers} \quad 1 \leq p \leq q. \\
\end{array} \]

We have, for an orthonormal basis $e_1, \ldots, e_{p+q}$ of $\mathbb{R}^{p+q}$,

\[ \begin{align*}
\mathcal{R} &= \{ \pm (e_i - e_j) \mid 1 \leq i < j \leq p+q \}, \\
\mathcal{B} &= \{ \alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq p+q-1 \}, \\
\mathcal{S} &= \{ \sigma(e_i) = -e_{p+q+1-i}, \quad 1 \leq i \leq p < q \leq p+q, \\
\sigma(e_i) &= -e_i, \quad p < i \leq q. \\
\mathcal{E} &= \{ \varepsilon(\alpha_i) = \alpha_{p+q-i}, \quad 1 \leq i < j \leq p \lor q < i < p+q, \\
\varepsilon(\alpha_i) &= \alpha_i, \quad p < i \leq q. \\
\end{align*} \]

The CR algebra $(g^\sigma, q_\phi)$ is fundamental iff $\varepsilon(\alpha_i) \notin \phi$ for all $\alpha_i \in \phi \setminus B^\sigma_{\phi}$ and finitely nondegenerate iff $\phi \setminus B^\sigma_{\phi}$ either contains at most one element, or, for a sequence $1 \leq i_1 < \cdots < i_h \leq p$,

\[ \text{if} \quad \varepsilon(\alpha_i) \notin \phi \quad \text{for all} \quad i \in \{i_1, \ldots, i_h\}. \]

\(^2\) This means that its complexification $g$ is simple.
equals either one of
\[ \{ \alpha_{i2j-1} \mid 1 \leq (2j-1) \leq h \} \cup \{ \varepsilon(\alpha_{i2j}) \mid 1 \leq 2j \leq h \}, \]
\[ \{ \varepsilon(\alpha_{i2j-1}) \mid 1 \leq (2j-1) \leq h \} \cup \{ \alpha_{i2j} \mid 1 \leq 2j \leq h \}, \]
and
\[
\begin{align*}
| \phi \cap B_\sigma^0 | & \leq 2, \quad \text{if } \phi \cap \{ \alpha_p, \alpha_{q+1} \} = \emptyset, \\
| \phi \cap B_\sigma^0 | & \leq 1, \quad \text{if } \phi \cap \{ \alpha_p, \alpha_{q+1} \} \neq \emptyset.
\end{align*}
\]

The CR algebra \((g^\sigma, q_\phi)\) has Levi order 2 if and only if, moreover, \(\phi \subseteq B_\sigma^0\).

Indeed, if \(\emptyset \neq \phi \subseteq B_\sigma^0\) and \(\alpha\) is the largest positive imaginary root, then \(\alpha \in Q_\phi \cap \sigma(Q_\phi^{-n})\) and \(\varepsilon(\alpha) \geq \sup_{\alpha \in R} |\varepsilon(\alpha)|\).

\[ g^\sigma \text{ OF TYPE BI, BII} \]
We have \(g^\sigma \simeq so(p, 2\ell+1-p)\), for \(1 \leq p \leq \ell\).

When \(p = \ell\), we get the split real form and all minimal parabolic CR algebras \((g^\sigma, q_\phi)\) are totally real.

If \(1 \leq p < \ell\), \((g^\sigma, q_\phi)\) is fundamental iff \(\phi \subseteq B_\sigma^0\) and finitely nondegenerate if, moreover, \(|\phi| = 1\).

Let us describe the root system, the basis of positive simple roots and the symmetry \(\sigma\) by
\[
\begin{align*}
\mathcal{R} &= \{ \pm e_i \mid 1 \leq i \leq \ell \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq \ell \}, \\
\mathcal{B} &= \{ \alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq \ell - 1 \} \cup \{ \alpha_\ell = e_\ell \}, \\
\sigma(e_i) &= e_i, \quad 1 \leq i \leq p, \\
\sigma(e_i) &= -e_i, \quad p+1 \leq i \leq \ell.
\end{align*}
\]
If \(\phi = \{ \alpha_{p+1} \}\), then
\[
\begin{align*}
Q^n_{\alpha_{p+1}} \cap R_\sigma^0 &= \{ e_{p+1} \} \cup \{ e_{p+1} \pm e_i \mid p+2 \leq i \leq \ell \}, \\
\beta &= -e_1 - e_{p+1} = \sigma(e_{p+1} - e_1) \in \sigma(\mathcal{Q}_\phi^n) \cap \mathcal{Q}_\phi^{-n}, \\
e_{p+1} + \beta &= -e_1 \in Q_{\alpha_{p+1}}^n \cap \sigma(\mathcal{Q}_{\alpha_{p+1}}^n), \\
e_{p+1} + \beta &= -e_1 \pm \varepsilon_i \in Q_{\alpha_{p+1}}^{-n} \cap \sigma(\mathcal{Q}_{\alpha_{p+1}}^{-n}), \quad \forall p+2 \leq i \leq \ell,
\end{align*}
\]
shows that \((g^\sigma, q_{\alpha_{p+1}})\) has Levi order 1.

If \(\phi = \{ \alpha_q \}\) with \(p+2 \leq q \leq \ell\), then \(e_{p+1} + e_{p+2} \in Q_\phi^n \cap \mathcal{R}_\sigma^0\) and
\[
2 = \varepsilon(\alpha_q)(e_{p+1} + e_{p+2}) \geq \sup_{\beta \in R} |\varepsilon(\alpha_q)(\beta)|
\]
implies that \((g^\sigma, q_{\alpha_q})\) has Levi order 2.

\[ g^\sigma \text{ OF TYPE CII} \]
We have \(g^\sigma \simeq sp_{p,2\ell-p}\) for some integer \(1 \leq p \leq \ell\). With
\[
\begin{align*}
\mathcal{R} &= \{ \pm 2e_i \mid 1 \leq i \leq \ell \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq \ell \}, \\
\mathcal{B} &= \{ \alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq \ell - 1 \} \cup \{ \alpha_\ell = 2e_\ell \}, \\
\sigma(e_{2i-1}) &= e_{2i}, \quad \sigma(e_{2i}) = e_{2i-1}, \quad 1 \leq i \leq p, \\
\sigma(e_i) &= -e_i, \quad 2p+1 \leq i \leq \ell,
\end{align*}
\]
we have
\[ B_\sigma^0 = \{ \alpha_{2h-1} \mid 1 \leq h \leq p \} \cup \{ \alpha_h \mid 2p + 1 \leq h \leq \ell \} \]
Fundamental \((g^\sigma, q_\phi)\) have \(\phi \subseteq B_\sigma^0\); those which are finitely nondegenerate have \(|\{ \alpha_h \in \phi \mid 2p < h \leq \ell \}| \leq 1\). Hence, when \(|\phi| \geq 2\) and \((g^\sigma, q_\phi)\) is finitely nondegenerate, each root of \(\phi\) belongs to a different connected component of \(\mathcal{R}_\sigma\). By Theorem 5.10 this implies that \((g^\sigma, q_\phi)\) has Levi order 1.
We are left to compute Levi orders when $|\phi| = 1$.

1) Assume that $\phi = \{\alpha_{2q-1}\}$, with $1 \leq q \leq p$.

Then, $Q^\sigma_{\phi} \cap R^\sigma_\bullet = \{\alpha_{2q-1}\}$ and $\beta = -2e_{q-1} = \sigma(-e_{2q}) \in \sigma(Q^\sigma_{\{\alpha_{2q-1}\}} \cap Q^{-n}_{\{\alpha_{2q-1}\}})$ and $\alpha_{2q-1} + \beta = -2e_{q-1} - e_{2q} \in Q^{-n}_{\{\alpha_{2q-1}\}} \cap \sigma(Q^{-n}_{\{\alpha_{2q-1}\}})$ shows that also in this case the Levi order is 1.

2) Assume that $\phi = \{\alpha_q\}$ with $2p+1 \leq q \leq \ell$. Then, $2e_q \in Q^n_{\{\alpha_q\}} \cap R^\sigma_\bullet$ and

$$2 = \xi_{\{\alpha_q\}}(2e_q) \geq \sup_{\beta \in R} |\xi_{\{\alpha_q\}}(\beta)|$$

shows that the Levi order of $(g^\sigma, q_{\{\alpha_q\}})$ is 2.

**g^\sigma OF TYPE DI, DII** We have $g^\sigma \simeq so_{p,2\ell-p}$, with $1 \leq p \leq \ell$.

For $p = \ell$ we obtain the split real form and $(g^\sigma, q_{\phi})$ is totally real for every $\phi \subseteq B$. For $p = \ell - 1$ we obtain the quasi-split real form. There are no imaginary roots and $(g^\sigma, q_{\phi})$ is fundamental if and only if $\phi$ equals either $\{\alpha_{\ell-1}\}$ or $\{\alpha_{\ell}\}$. In both cases, it is also finitely nondegenerate and of Levi order 1.

If $1 \leq p \leq \ell - 2$, then $(g^\sigma, q_{\phi})$ is fundamental iff $\phi \subseteq B^\sigma_\bullet = \{\alpha_{p+1}, \ldots, \alpha_{\ell}\}$ and finitely nondegenerate in the following cases

1. $p \geq 2$ and $|\phi| = 1$ or $\phi = \{\alpha_{\ell-1}, \alpha_{\ell}\}$;
2. $p = 1$ and either $\phi = \{\alpha_q\}$ with $2 \leq q \leq \ell - 2$, or $\phi = \{\alpha_{\ell-1}, \alpha_{\ell}\}$.

To discuss different cases, we recall that, for an orthonormal basis $e_1, \ldots, e_\ell$ of $\mathbb{R}^\ell$, we can set

$$\begin{align*}
\mathcal{R} &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq \ell\}, \\
\mathcal{B} &= \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq \ell - 1\} \cup \{\alpha_{\ell} = e_{\ell-1} + e_{\ell}\},
\end{align*}$$

while $\sigma$ is the symmetry

$$\sigma(e_i) = \begin{cases} 
 e_i, & 1 \leq i \leq p, \\
 -e_i, & p < i \leq \ell.
\end{cases}$$

We consider first the cases where $p \geq 1$ and $\phi = \phi_q = \{\alpha_q\}$ with $p < q \leq \ell - 2$ or $\phi = \phi_{\ell-1} = \{\alpha_{\ell-1}, e_\ell\}$. Then, $Q_{\phi_q} = \{\alpha \in \mathcal{R} \mid \xi_q(\alpha) \geq 0\}$ with

$$\xi_q(e_i) = \begin{cases} 
 1, & 1 \leq i \leq q, \\
 0, & q < i \leq \ell.
\end{cases}$$

We have

$$\begin{align*}
Q^n_{\phi_{p+1}} &= \{e_{p+1} \pm e_i \mid p + 2 \leq i \leq \ell\}, \\
\sigma(e_{p+1} - e_1) &= -e_{p+1} - e_1 \in \sigma(Q^\sigma_{\phi_q}) \cap Q^{-n}_\phi, \\
(e_{p+1} \pm e_i) + (-e_{p+1} - e_1) &= -e_1 \pm e_i \in Q^{-n}_\phi \cap \sigma(Q^{-n}_\phi),
\end{align*}$$

and hence, $(g^\sigma, q_{\{\alpha_{p+1}\}})$ has Levi order 1.

If $p+1 < q \leq \ell - 1$

$$\alpha = \alpha_{p+1} + \alpha_{\ell-1} + \alpha_{\ell} + 2 \sum_{i=p+2}^{\ell-2} \alpha_i \in Q^n_{\phi_q} \cap R^\sigma_\bullet$$

and $\xi_{\phi_q}(\alpha) = \sup_{\beta \in R} |\xi_{\phi_q}(\beta)|$ shows that $(g^\sigma, q_{\phi_q})$ has Levi order 2.
By the symmetry of the Satake diagram, the cases of \( \varphi \) equal to \( \{ \alpha_{\ell-1} \} \) or \( \{ \alpha_{\ell} \} \) have the same Levi order. Take \( \varphi = \{ \alpha_{\ell} \} \). Then,

\[
\xi_{\{ \alpha_{\ell} \}}(e_i) = \frac{1}{2}, \quad \forall 1 \leq i \leq \ell
\]

and \( 1 = \xi_{\{ \alpha_{\ell} \}} \geq \sup_{\beta \in \mathcal{R}} |\xi_{\{ \alpha_{\ell} \}}(\beta)| \) shows that also in this case the Levi order is 2.

\[\text{g}^\varphi \text{ OF TYPE DIIIa}\]

We have \( g^\varphi \simeq \mathfrak{su}_{2p}^*(\mathbb{H}) \), for an integer \( p \geq 2 \). In all these cases it is finitely nondegenerate and has Levi order 1 when \( |\varphi| \geq 2 \).

The conjugation on the root system is

\[
\begin{cases}
\sigma(e_{2i-1}) = e_{2i}, & 1 \leq i \leq p, \\
\sigma(e_{2i}) = e_{2i-1}, & 1 \leq i \leq p.
\end{cases}
\]

The CR algebra \( (g^\varphi, q_\varphi) \) is fundamental if and only if \( \Phi \subseteq \mathcal{B}_\varphi = \{ \alpha_{2i-1} \mid 1 \leq i \leq p \} \) and in all these cases is also finitely nondegenerate. By Theorem 5.10 it has Levi order 1 when \( |\varphi| \geq 2 \).

Let us consider the case where \( \varphi = \{ \alpha_{2q-1} \} \) for some \( 1 \leq q \leq p \).

We distinguish two cases.

1) If \( \varphi = \{ \alpha_{2q-1} \} \) with \( 1 \leq q < p \), then the roots in \( \sigma(Q_\varphi) \cap Q_\varphi^- \) which are not orthogonal to \( \alpha_{2q-1} \) are either of the form \( -e_{2q-1} \pm e_h \), with \( h > 2q \), or of the form \( e_{2q} - e_h \) with \( h \leq 2q - 2 \). We have

\[
\begin{cases}
\alpha_{2q-1} + (-e_{2q-1} \pm e_h) = e_{2q} \pm e_h \notin Q_\varphi^- \cap \sigma(Q_\varphi^-) & \text{for } h > 2q, \\
\alpha_{2q-1} + (e_{2q} - e_h) = e_{2q-1} - e_h \notin Q_\varphi^- \cap \sigma(Q_\varphi^-) & \text{for } 1 \leq h \leq 2q - 2.
\end{cases}
\]

This shows that \( k_\varphi^\varphi(\alpha_{2q-1}) \neq 1 \) and therefore \( (g^\varphi, q_\varphi) \) has Levi order 2.

2) Consider finally the case \( \varphi = \{ \alpha_{2p-1} \} \). The roots in \( \sigma(Q_\varphi) \cap Q_\varphi^- \) which can be added to \( \alpha_{2p-1} \) are of the form \( e_{2p} - e_h \) with \( h \leq 2p - 2 \). Since

\[
\alpha_{2p-1} + (e_{2p} - e_h) = e_{2p-1} - e_h \notin Q_\varphi^- \cap \sigma(Q_\varphi^-) & \text{for } 1 \leq h \leq 2p - 2,
\]

we have \( k_\varphi^\varphi(\alpha_{2p-1}) \neq 1 \) and therefore \( (g^\varphi, q_\varphi) \) has Levi order 2.

\[\text{g}^\varphi \text{ OF TYPE DIIIb}\]

We have \( g^\varphi \simeq \mathfrak{su}_{2p+1}^*(\mathbb{H}) \), for an integer \( p \geq 2 \). The conjugation on the root system is

\[
\begin{cases}
\sigma(e_{2i-1}) = e_{2i}, & 1 \leq i \leq p, \\
\sigma(e_{2i}) = e_{2i-1}, & 1 \leq i \leq p, \\
\sigma(e_{2p+1}) = -e_{2p+1}.
\end{cases}
\]

The minimal parabolic fundamental CR algebras \( (g^\varphi, q_\varphi) \) have

\[
\varphi \subseteq \{ \alpha_{2i-1} \mid 1 \leq i \leq p \} \cup \{ \alpha_{2p}, \alpha_{2p+1} \}, \quad \text{with } |\varphi \cap \{ \alpha_{2p}, \alpha_{2p+1} \}| \leq 1
\]

and are all finitely nondegenerate.

By repeating the previous arguments, it turns out that \( (g^\varphi, q_\varphi) \) is

- 1-nondegenerate when \( \varphi \) contains \( \alpha_{2p} \) or \( \alpha_{2p+1} \), or \( |\varphi| \geq 2 \);
- 2-nondegenerate when \( \varphi = \{ \alpha_{2q-1} \} \) for \( 1 \leq q \leq p \).

To treat in the following real forms of the exceptional Lie algebras of type E, we shall use suitable root system, in which we will use elements of the following form. After fixing an
orthonormal basis $e_1, \ldots, e_8$ of $\mathbb{R}^8$, we set, for every permutation $(i_1, \ldots, i_8)$ of $\{1, \ldots, 8\}$,

$$
\begin{align*}
\zeta_0 &= -\frac{1}{2} (e_1 + \cdots + e_8), \\
\zeta_i &= \frac{1}{2} (e_{i_1} - e_{i_2} - \cdots - e_{i_8}), \\
\zeta_{i_1, \ldots, i_h} &= \frac{1}{2} (e_{i_1} + \cdots + e_{i_h} - e_{i_{h+1}} - \cdots - e_{i_8}), \quad 2 \leq h \leq 6, \\
\zeta_{i_1, \ldots, i_7} &= \frac{1}{2} (e_{i_1} + \cdots + e_{i_7} - e_{i_8}).
\end{align*}
$$

(6.1)

**$g^\sigma$ of Type EIII** This is a quasi-split real form, which, in the representation

$$
\begin{align*}
\mathcal{R} &= \{ \pm(e_i - e_j)|1 \leq i < j \leq 6\} \cup \{ \pm(e_7 - e_8)\} \cup \{ \pm\zeta_i, j, k, \ell|1 \leq i < j < k < \ell \leq 6\}, \\
\mathcal{B} &= \{ \alpha_i = e_i - e_{i+1} | 1 \leq i \leq 5\} \cup \{ \alpha_6 = \zeta_4, 5, 6, 7\}.
\end{align*}
$$

(6.2)

is described by the symmetry

$$
\varepsilon(e_i) = \begin{cases} 
-e_{7-i}, & 1 \leq i \leq 6, \\
-e_{9-i}, & i = 7, 8.
\end{cases}
$$

The $\phi$’s for which $(g^\sigma, q_{\phi})$ is fundamental are those with $\phi \cap \varepsilon(\phi) = \emptyset$. Those for which it is also finitely nondegenerate are those with $|\phi| = 1$ and $\phi \subset \{ \alpha_1, \alpha_2, \alpha_4, \alpha_5 \}$ and also $\{ \alpha_1, \alpha_4 \}, \{ \alpha_2, \alpha_5 \}$. In all these cases the Levi order is 1, since $\mathcal{R}$ does not contain imaginary roots.

**$g^\sigma$ of Type EIII** This is a root system of type E, which, in the setting (6.2), is defined by the symmetry

$$
\begin{align*}
\sigma(e_1) &= -e_6, \quad \sigma(e_6) = -e_1, \\
\sigma(e_7) &= -e_8, \quad \sigma(e_8) = -e_7, \\
\sigma(e_i) &= -e_i, \quad 2 \leq i \leq 5.
\end{align*}
$$

The $\phi$’s for which $(g^\sigma, q_{\phi})$ is fundamental are those containing neither $\alpha_6$ nor the couple $\{ \alpha_1, \alpha_5 \}$ and they are all finitely nondegenerate, with the exceptions:

$$
\{ \alpha_1, \alpha_2, \alpha_3 \}, \{ \alpha_3, \alpha_4, \alpha_5 \}, \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \quad \text{and} \quad \{ \alpha_2, \alpha_3, \alpha_4, \alpha_5 \}.
$$

Those containing either $\alpha_1$ or $\alpha_5$ are 1-nondegenerate, because $\phi$ contains a nonimaginary root.

Let us consider the cases where $\phi \subseteq \mathcal{B}_\sigma^\phi$.

If either $\phi = \{ \alpha_2, \alpha_3, \alpha_4 \} = \mathcal{B}_\sigma^\phi$, or $\phi = \{ \alpha_2, \alpha_4 \}$, by Theorem 5.10 the single root in $Q^-_\phi \cap R^-_\phi$ which may have Levi order 2 is $\alpha = \alpha_2 + \alpha_3 + \alpha_4 = e_2 - e_5$.

Since $(e_6 - e_2) = \sigma(e_2 - e_1) \in \sigma(Q^-_\phi) \cap Q^-_\phi$ and

$$(e_2 - e_5) + (e_6 - e_2) = e_6 - e_5 \in Q^-_\phi \cap \sigma(Q^-_\phi),$$

we obtain $k^\sigma_\phi(\alpha) = 1$, showing that $(g^\sigma, q_{\{\alpha_2, \alpha_3, \alpha_4\}})$ and $(g^\sigma, q_{\{\alpha_2, \alpha_4\}})$ have Levi order 1.

The cases with $\phi \subseteq \mathcal{B}_\sigma^\phi$ and $|\phi| = 2$ are 1-nondegenerate: the argument is the one used in the discussion of AIII.

Let $\phi = \{ \alpha_2 \}$. Then, $Q^\phi_\phi = \{ \alpha \in \mathcal{R} | \xi(\alpha) \geq 0 \}$ with

$$
\begin{align*}
\xi_\phi(e_1) &= 1, \quad \xi_\phi(e_2) = 1, \quad \xi_\phi(e_3) = 0, \quad \xi_\phi(e_4) = 0, \\
\xi_\phi(e_5) &= 0, \quad \xi_\phi(e_6) = 0, \quad \xi_\phi(e_7) = 2, \quad \xi_\phi(e_8) = 0.
\end{align*}
$$
We have \( Q^\alpha_\phi \cap R^\sigma_\phi = \{ e_2 - e_i \mid 3 \leq i \leq 5 \} \) and
\[
\begin{align*}
(e_2 - e_3) + \zeta_{2,4,5,8} &= \zeta_{2,4,5,8} \in Q^\alpha_\phi \cap \sigma(Q^{-\alpha}_\phi), \text{ with } \zeta_{2,4,5,8} \in \sigma(Q\phi) \cap Q^{-\alpha}_\phi, \\
(e_2 - e_4) + \zeta_{2,3,5,8} &= \zeta_{2,3,5,8} \in Q^\alpha_\phi \cap \sigma(Q^{-\alpha}_\phi), \text{ with } \zeta_{2,3,5,8} \in \sigma(Q\phi) \cap Q^{-\alpha}_\phi, \\
(e_2 - e_5) + \zeta_{2,3,4,8} &= \zeta_{2,3,4,8} \in Q^\alpha_\phi \cap \sigma(Q^{-\alpha}_\phi), \text{ with } \zeta_{2,3,4,8} \in \sigma(Q\phi) \cap Q^{-\alpha}_\phi,
\end{align*}
\]
shows that \((g^\sigma, q_{[\alpha_3]})\) has Levi order 1. Symmetrically, also \((g^\sigma, q_{[\alpha_4]})\) has Levi order 1.

Let us consider \( \phi = \{ \alpha_3 \} \). Then \( Q_\phi = \{ \alpha \in R \mid \xi(\alpha) \geq 0 \} \) with
\[
\begin{align*}
\xi_\phi(e_1) &= 1, \quad \xi_\phi(e_2) = 1, \quad \xi_\phi(e_3) = 1, \quad \xi_\phi(e_4) = 0, \\
\xi_\phi(e_5) &= 0, \quad \xi_\phi(e_6) = 0, \quad \xi(e_7) = 3, \quad \xi(e_8) = 0.
\end{align*}
\]
We have \( Q^\alpha_\phi \cap R^\sigma_\phi = \{ e_2 - e_i \mid 4 \leq i \leq 5 \} \cup \{ e_3 - e_i \mid 4 \leq i \leq 5 \} \) and
\[
\begin{align*}
(e_2 - e_4) + \zeta_{1,4,5,8} &= \zeta_{1,2,5,8} \in Q^\alpha_\phi \cap \sigma(Q^{-\alpha}_\phi), \text{ with } \zeta_{1,4,5,8} \in \sigma(Q\phi) \cap Q^{-\alpha}_\phi, \\
(e_2 - e_5) + \zeta_{1,4,5,8} &= \zeta_{1,4,5,8} \in Q^\alpha_\phi \cap \sigma(Q^{-\alpha}_\phi), \text{ with } \zeta_{1,4,5,8} \in \sigma(Q\phi) \cap Q^{-\alpha}_\phi, \\
(e_3 - e_4) + \zeta_{1,4,5,8} &= \zeta_{1,3,5,8} \in Q^\alpha_\phi \cap \sigma(Q^{-\alpha}_\phi), \text{ with } \zeta_{1,4,5,8} \in \sigma(Q\phi) \cap Q^{-\alpha}_\phi, \\
(e_3 - e_5) + \zeta_{1,4,5,8} &= \zeta_{1,4,5,8} \in Q^\alpha_\phi \cap \sigma(Q^{-\alpha}_\phi), \text{ with } \zeta_{1,4,5,8} \in \sigma(Q\phi) \cap Q^{-\alpha}_\phi,
\end{align*}
\]
shows that also \((g^\sigma, q_{[\alpha_3]})\) has Levi order 1.

To discuss the real form EIV, we found convenient to describe the root system and a basis of simple roots by (the upper sign is for the root in \( R^+ \))
\[
\begin{align*}
R &= \{ \pm(e_i \pm e_j), \pm \zeta_{i,j} \mid 1 \leq i < j \leq 5 \} \cup \{ \pm \zeta_\phi \} \cup \{ \mp \zeta_i, 6, 7, 8 \mid 1 \leq i \leq 5 \}, \\
B &= \{ \alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq 4 \} \cup \{ \alpha_5 = e_4 + e_5 \} \cup \{ \alpha_6 = \zeta_\phi \}.
\end{align*}
\]
(6.3)

The symmetry yielding the form EIV is defined by
\[
\begin{align*}
\sigma(e_i) &= e_i, \quad i = 1, 6, 7, 8, \\
\sigma(e_i) &= -e_i, \quad 1 = 2, 3, 4, 5.
\end{align*}
\]
The corresponding Satake diagram is

The fundamental \((g^\sigma, q_\phi)\)'s have \( \phi \subseteq B^\sigma_\phi \) and they are finitely nondegenerate iff \( |\phi| \leq 2 \) and \( \phi \notin \{ \alpha_3, \alpha_4 \} \).

By the argument used in case DII, \((g^\sigma, q_\phi)\) has Levi order 1 when \( \phi \) equals either \( \{ \alpha_2 \} \) or \( \{ \alpha_5 \} \).

Let \( \phi = \{ \alpha_3 \} \). We have
\[
\xi_{[\alpha_3]}(e_i) = \begin{cases} 
1, & i = 1, 2, 3, \\
0, & i = 4, 5, \\
-1, & i = 6, 7, 8.
\end{cases}
\]
The root $\beta = (e_2 + e_3)$ belongs to $Q_\phi^n \cap \mathcal{R}_\sigma$ and $\bar{\xi}_{\xi(e_3)}(\beta) = 0$. Since only the roots $\zeta_{4,6,7,8}$ and $\xi_{5,6,7,8}$ have a $\xi(e_3)$ less than $(-2)$, and they both belong to $Q^n \cap \sigma(Q_\phi^n)$, we obtain $k_{\xi(e_3)}(\beta) = 0$. Hence $(q^\sigma, q_{\xi(e_3)})$ has Levi order 2.

Let $\phi = \{\alpha_4\}$. We have

$$\bar{\xi}_{\xi(e_4)}(e_i) = \begin{cases} \frac{1}{2}, & i = 1, 2, 3, 4 \\ -\frac{1}{2}, & i = 5, 6, 7, 8 \end{cases}$$

and $Q_\phi^n \cap \mathcal{R}_\sigma = \{e_i \in e_5 \mid 2 \leq i \leq 4\} \cup \{e_i + e_j \mid 2 \leq i < j \leq 4\}$. Taking into account that $(-e_i - e_j)$, for $2 \leq i < j \leq 4$, and $\zeta_{4,6,7,8} = \sigma(-\zeta_{\phi})$ belong to $\sigma(Q_\phi^n) \cap Q_\phi^n$, the equalities

$$\begin{cases} (e_i - e_5) + (e_1 - e_i) = (e_1 - e_5) \in Q_\phi^n \cap \sigma(Q_\phi^n), & 2 \leq i \leq 4, \\ (e_i + e_j) + \zeta_{1,6,7,8} = \zeta_{1,6,7,8} \in Q_\phi^n \cap \sigma(Q_\phi^n), & 2 \leq i < j \leq 4, \end{cases}$$

show that $(g^\sigma, q_{\xi(e_4)})$ has Levi order 1.

Next we consider the subsets $\phi$ with $|\phi| = 2$.

Let $\phi = \phi_{2,3} = \{\alpha_2, \alpha_3\}$. We have $Q_\phi = \{\alpha \in \mathcal{R} \mid \bar{\xi}_{\phi_{2,3}}(\alpha) \geq 0\}$ with

$$\bar{\xi}_{\phi_{2,3}} = \begin{cases} 2, & i = 1, 2, \\ 1, & i = 3, \\ 0, & i = 4, 5, \\ -\frac{5}{3}, & i = 6, 7, 8. \end{cases}$$

Then

$$Q_\phi^n \cap \mathcal{R}_\sigma = \{e_i - e_j \mid i = 2, 3, \text{ or } i < j \leq 5\} \cup \{e_i + e_j \mid 2 \leq i \leq j, \text{ or } i < j \leq 5\}.$$

We note that $(-e_1 - e_2) = \sigma(e_2 - e_1)$, $-\zeta_{2,3} = \sigma(\zeta_{4,5})$ and $\zeta_{1,6,7,8} = \sigma(-\zeta_{\phi})$ belong to $\sigma(Q_\phi^n) \cap Q_\phi^n$. Therefore,

$$\begin{cases} (e_2 - e_1) + (-e_1 - e_2) = e_1 - e_i \in \mathcal{R}_{\phi_{2,3}} \cap \sigma(Q_\phi^n)_{\phi_{2,3}}, & i = 3, 4, 5, \\ (e_1 - e_2) + (-e_2 - e_1) = \zeta_{1,2,3} \in \mathcal{R}_{\phi_{2,3}} \cap \sigma(Q_\phi^n)_{\phi_{2,3}}, & i = 4, 5, \\ (e_2 + e_1) + \zeta_{1,6,7,8} = \zeta_{1,6,7,8} \in \mathcal{R}_{\phi_{2,3}} \cap \sigma(Q_\phi^n)_{\phi_{2,3}}, & i = 3, 4, 5, \\ (e_1 + e_2) + \zeta_{1,6,7,8} = \zeta_{1,6,7,8} \in \mathcal{R}_{\phi_{2,3}} \cap \sigma(Q_\phi^n)_{\phi_{2,3}}, & i = 4, 5, \end{cases}$$

shows that $(g^\sigma, q_{\phi_{2,3}})$ has Levi order 1. By the symmetry or the Satake diagram E IV, also $(g^\sigma, q_{\phi_{3,5}})$, with $\phi_{3,5} = \{\alpha_3, \alpha_5\}$ has Levi order 1.

Let now $\phi = \phi_{2,4} = \{\alpha_2, \alpha_4\}$. In this case we take

$$\bar{\xi}_{\phi_{2,4}} = \begin{cases} \frac{3}{7}, & i = 1, 2, \\ \frac{1}{3}, & i = 3, 4, \\ -\frac{1}{7}, & i = 5, \\ -\frac{7}{6}, & i = 6, 7, 8. \end{cases}$$

By Theorem 5.10, the elements of $Q_\phi \cap \sigma(Q_\phi^n)$ which can have Levi order 2 are roots of $Q_\phi^n \cap \mathcal{R}_\sigma$, whose support contains $\{\alpha_2, \alpha_4\}$. Thus, we need to check the Levi order of the three roots $e_2 + e_3$, $e_2 + e_4$, $e_2 - e_5$. We have $\sigma(e_2 - e_1) = (-e_1 - e_2) \in \sigma(Q_\phi^n) \cap Q_\phi^n$ and
then
\[
\begin{align*}
(e_2 + e_3) + (-e_1 - e_2) &= e_3 - e_1 \in Q_\phi^- \cap \sigma(Q_\phi^-), \\
(e_2 + e_4) + (-e_1 - e_2) &= e_4 - e_1 \in Q_\phi^- \cap \sigma(Q_\phi^-), \\
(e_2 - e_5) + (-e_1 - e_2) &= -e_1 - e_5 \in Q_\phi^- \cap \sigma(Q_\phi^-),
\end{align*}
\]
shows that \((g^\sigma, q_{i(2,\alpha_4)})\) has Levi order 1. By the symmetry of the Satake diagram EVI, also \((g^\sigma, q_{i(2,\alpha_4)})\), with \(\phi_{4,5} = \{\alpha_4, \alpha_5\}\) has Levi order 1.

Let now \(\phi = \phi_{2,5} = \{\alpha_2, \alpha_5\}\). We have
\[
\xi_{\phi_{2,5}}(e_i) = \begin{cases} 
2, & i = 1, 2, \\
1, & i = 3, 4, 5, \\
-3, & i = 6, 7, 8,
\end{cases}
\]
By Theorem 5.10, the elements of \(Q_\phi \cap \sigma(Q_\phi^-)\) which can have Levi order 2 are roots of \(Q_\phi^- \cap R^\sigma\) whose support contains \(\{\alpha_2, \alpha_5\}\). Thus, we need to check the Levi order of the three roots \(e_2 + e_3, e_2 + e_4, e_2 + e_5\). We have \(\sigma(e_2 - e_1) = (-e_1 - e_2) \in \sigma(Q_\phi^-) \cap Q_\phi^-\) and then
\[
\begin{align*}
(e_2 + e_3) + (-e_1 - e_2) &= e_3 - e_1 \in Q_\phi^- \cap \sigma(Q_\phi^-), \\
(e_2 + e_4) + (-e_1 - e_2) &= e_4 - e_1 \in Q_\phi^- \cap \sigma(Q_\phi^-), \\
(e_2 + e_5) + (-e_1 - e_2) &= e_5 - e_1 \in Q_\phi^- \cap \sigma(Q_\phi^-),
\end{align*}
\]
shows that \((g^\sigma, q_{i(2,\alpha_5)})\) has Levi order 1.

**g^\sigma OF TYPE EVI** This is a real form of E7. Let us take the root system of type E7 and the basis of simple roots consisting of (the upper sign is for the root in \(R^+\))
\[
\begin{align*}
\mathcal{R} &= \{\pm(e_i \pm e_j), \pm \zeta_{i,j}, \mp \zeta_{i,j}, 1 \leq i < j \leq 6 \} \cup \{\pm(e_7 + e_8), \pm \zeta_{0}, \mp \zeta_{7,8}\} \\
\mathcal{B} &= \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq 5\} \cup \{\alpha_6 = e_5 + e_6\} \cup \{\alpha_7 = \zeta_{0}\}.
\end{align*}
\]
(6.4)
The symmetry yielding the form EVI is defined by
\[
\begin{align*}
\sigma(e_{2i-1}) &= e_{2i}, & 1 \leq i \leq 4, \\
\sigma(e_{2i}) &= e_{2i-1}, & 1 \leq i \leq 4.
\end{align*}
\]
The corresponding Satake diagram is

The \(\phi\)’s yielding fundamental minimal parabolic CR algebras \((g^\sigma, q_{\phi_i})\) are those with \(\phi \subseteq B^\sigma = \{\alpha_1, \alpha_3, \alpha_5\}\) and they are all finitely nondegenerate. By the discussion of the case AII we know that those with \(|\phi| = 2, 3\) are 1-nondegenerate. Let us discuss the cases where \(|\phi| = 1\).
If \( \phi = \phi_1 = \{\alpha_1\} \), we have \( \mathcal{Q}_{\phi_1} = \{\alpha \in \mathcal{R} \mid \xi_1(\alpha) \geq 0\} \) with

\[
\xi_1(e_i) = \begin{cases} 1, & i = 1, \\ 0, & 2 \leq i \leq 6, \\ -\frac{1}{2}, & i = 7, 8. \end{cases}
\]

Since \( |\xi_1(\alpha)| \leq 1 \) for all roots \( \alpha \) of \( \mathcal{R} \) and \( \xi_1(\alpha_1) = 1 \), we get \( k_{\phi_1}^\sigma(\alpha_1) = 2 \) and hence \((\mathfrak{g}^\sigma, q_{\phi_1})\) has Levi order 2.

If \( \phi = \phi_3 = \{\alpha_3\} \), we have \( \mathcal{Q}_{\phi_3} = \{\alpha \in \mathcal{R} \mid \xi_3(\alpha) \geq 0\} \) with

\[
\xi_3(e_i) = \begin{cases} 1, & i = 1, 2, 3 \\ 0, & i = 4, 5, 6 \\ -\frac{3}{2}, & i = 7, 8. \end{cases}
\]

By the discussion of DIIIA, we know that there is no root \( \beta \) of the form \( \pm e_i \pm e_j \) such that \( \alpha_3 + \beta \in Q_{\phi_3}^- \cap \sigma(Q_{\phi_3}^-) \). The negative \( \xi \)-roots in \( Q_{\phi_3}^- \) are \(-\xi_i(\alpha)\) and \(-\xi_i(\alpha)\) with \( 4 \leq i < j \leq 6 \).

We have \( \sigma(-\xi_0) = (-\xi_0), \sigma(-\xi_4) = (-\xi_3, 6), \sigma(-\xi_4, 5) = (-\xi_3, 5), \sigma(-\xi_5, 6) = (-\xi_5, 6) \).

Thus, the \( \xi \)-roots in \( \sigma(Q_{\phi_3}^-) \cap Q_{\phi_3}^- \) are \(-\xi_3, 5 \) and \(-\xi_3, 6 \) and both \( \alpha_3 + (-\xi_3, 5) = (-\xi_4, 5) \) and \( \alpha_3 + (-\xi_3, 6) = (-\xi_4, 6) \) are in \( Q_{\phi_3}^- \). This shows that \( k_{\phi_3}^\sigma(\alpha_3) = 2 \), and hence, \((\mathfrak{g}^\sigma, q_{\phi_3})\) has Levi order 2.

If \( \phi = \phi_5 = \{\alpha_5\} \), we have \( \mathcal{Q}_{\phi_5} = \{\alpha \in \mathcal{R} \mid \xi_5(\alpha) \geq 0\} \) with

\[
\xi_5(e_i) = \begin{cases} \frac{1}{2}, & i = 1, 2, 3, 4, 5 \\ -\frac{1}{2}, & i = 6, \\ -1, & i = 7, 8. \end{cases}
\]

By the discussion of DIIIA, we know that there is no root \( \beta \) of the form \( \pm e_i \pm e_j \) with \( \alpha_3 + \beta \in Q_{\phi_3}^- \cap \sigma(Q_{\phi_3}^-) \). The negative \( \xi \)-roots in \( Q_{\phi_3}^- \) are \(-\xi_i(\alpha)\) and \(-\xi_i(\alpha)\) with \( 1 \leq i \leq 5 \).

We have \( \sigma(-\xi_0) = (-\xi_0), \sigma(-\xi_2) = (-\xi_1, 6) = (-\xi_2, 5), \sigma(-\xi_3) = (-\xi_2, 1, 5) \) for \( i = 1, 2 \) and \( \sigma(-\xi_5, 6) = (-\xi_5, 6) \). Thus, the \( \xi \)-roots in \( \sigma(Q_{\phi_3}^-) \cap Q_{\phi_3}^- \) are \(-\xi_i(\alpha)\) for \( i = 1, 2, 3, 4 \).

We have \( \alpha_1 + (-\xi_1, 5) = (-\xi_1, 6) \in Q_{\phi_5}^- \) for all \( i = 1, 2, 3, 4 \) and hence \( k_{\phi_5}^\sigma(\alpha_5) = 2 \), showing that \((\mathfrak{g}^\sigma, q_{\phi_5})\) has Levi order 2.

\[ \mathfrak{g}^\sigma \text{ of type EVII} \]

This is a real form of E7. We keep the notation introduced above for EVI.

The symmetry yielding EVII is defined by

\[ \sigma(e_i) = \begin{cases} e_i, & i = 1, 2, 7, 8, \\ -e_i, & i = 3, 4, 5, 6. \end{cases} \]

The corresponding Satake diagram is

\[
\begin{array}{c}
\bullet \alpha_1 \quad \bullet \alpha_2 \quad \bullet \alpha_3 \quad \bullet \alpha_4 \quad \bullet \alpha_5 \\
\alpha_6 \quad \bullet \alpha_7
\end{array}
\]

The fundamental minimal parabolic \((\mathfrak{g}^\sigma, q_{\phi})\) correspond to \( \phi \subseteq \mathcal{B}_r^\sigma = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6\} \).
and are finitely nondegenerate if $|\phi|\leq 2$ and $\phi \neq \{\alpha_4, \alpha_5\}$. By the discussion of E IV, we know that in the cases listed above $(g^\sigma, q_\phi)$ has Levi order 1 for $\phi \neq \{\alpha_4\}$.

Let $\phi = \phi_4 = \{\alpha_4\}$. Then $Q_{\phi_4} = \{\alpha \in \mathcal{R} | \xi(\alpha) \geq 0\}$ with

$$\xi(e_i) = \begin{cases} 1, & i = 1, 2, 3, 4, \\ 0, & i = 5, 6, \\ -2, & i = 7, 8. \end{cases}$$

We have

$$Q_{\phi_4}^\sigma \cap \mathcal{R}_{\phi_4}^\sigma = \{e_3 + e_4, e_3 \pm e_5, e_3 \pm e_6, e_4 \pm e_5, e_4 \pm e_6\}.$$ 

Since $\xi(e_3 + e_4) = 2$, there is no root $\beta$ of the form $\pm e_i \pm e_j$ in $\mathcal{R}$ for which $(e_3 + e_4) + \beta \in Q_{\phi_4}^\sigma$. The $\zeta$-roots in $Q_{\phi_4}^\sigma$ are $\pm \zeta_6$ and $\pm \zeta_5, 6$. We have $\sigma(-\zeta_6) = \zeta_{1, 2, 7, 8}$ and $\sigma(-\zeta_5, 6) = -\zeta_{3, 4}$. Since

$$(e_3 + e_4) + \zeta_{1, 2, 7, 8} = -\zeta_5, 6 \in Q_{\phi_4}^\sigma \quad \text{and} \quad (e_3 + e_4) + (-\zeta_{3, 4}) = -\zeta_6 \in Q_{\phi_4}^\sigma,$$

we obtain that $k_{\phi_4}^\sigma(e_3 + e_4) = 2$ and therefore $(g^\sigma, q_{\phi_4})$ has Levi order 2.

**$g^\sigma$ of Type E IX** This is a real form of $E_8$. Let us take the root system of type $E_8$ and the basis of simple roots consisting of

$$\mathcal{R} = \{\pm \zeta_6\} \cup \{\pm (e_i \pm e_j), \pm \zeta_{i, j}, |1 \leq i \neq j \leq 8\} \cup \{\zeta_{i, j, h, \ell} | 1 \leq i < j < h < \ell \leq 8\}$$

$$\mathcal{B} = \{\alpha_i = e_i - e_{i+1} | 1 \leq i \leq 6\} \cup \{\alpha_7 = e_6 + e_7\} \cup \{\alpha_8 = \zeta_6\}.$$ 

The symmetry yielding E IX is defined by

$$\sigma(e_i) = \begin{cases} e_i, & i = 1, 2, 3, 8, \\ -e_i, & i = 4, 5, 6, 7. \end{cases}$$

The corresponding Satake diagram is

```
\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6
```

```
\alpha_7
```

```
\alpha_8
```

The fundamental minimal parabolic $(g^\sigma, q_{\phi})$ correspond to

$$\phi \subseteq \mathcal{B}_{\phi}^\sigma = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7\}$$

and are finitely nondegenerate if $|\phi|\leq 2$ and $\phi \neq \{\alpha_5, \alpha_6\}$. By the discussion of E IV, we know that in the cases listed above $(g^\sigma, q_\phi)$ has Levi order 1 for $\phi \neq \{\alpha_5\}$, while the same argument employed in that case shows that $(g^\sigma, q_{\{\alpha_5\}})$ has Levi order 2.

**$g^\sigma$ of Type F II** This is a real form of a root system of type F. We take the root system and basis of simple positive roots defined by

$$\mathcal{R} = \{\pm e_i | 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq 4\} \cup \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\},$$

$$\mathcal{B} = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3, \alpha_4 = \frac{1}{2}(e_4 - e_1 - e_2 - e_3)\}.$$
and the symmetry defining F II is

\[ \sigma(e_i) = \begin{cases} 
- e_i, & i = 1, 2, 3, \\
- e_4, & i = 4.
\end{cases} \]

The corresponding Satake diagram is

\[ \sigma \]

The fundamental CR algebras \( (g^\sigma, q_{\phi}) \) are those with \( \phi \subseteq B^\sigma = \{ \alpha_1, \alpha_2, \alpha_3 \} \), which are finitely nondegenerate iff \( |\phi| = 1 \).

If \( \phi = \phi_1 = \{ \alpha_1 \} \), then \( Q_{\phi_1} = \{ \alpha \in R \mid \xi_1(\alpha) \geq 0 \} \), with

\[ \xi_1(e_i) = \begin{cases} 
1, & i = 1, 4, \\
0, & i = 2, 3.
\end{cases} \]

We have \( Q_{\phi_1}^n \cap R^\sigma_0 = \{ e_1 \} \cup \{ e_1 \pm e_i \mid i = 2, 3 \} \).

Since \( (-e_1 - e_4) = \sigma(e_1 - e_4) \in \sigma(Q_{\phi_1}^r) \cap Q_{\phi_1}^{-n} \),

\[ e_1 + (-e_1 - e_4) = -e_4 \in Q_{\phi_1}^{-n} \cap \sigma(Q_{\phi_1}^{-n}), \]

\[ e_1 \pm e_i + (-e_1 - e_4) = \pm e_i - e_4 \in Q_{\phi_1}^{-n} \cap \sigma(Q_{\phi_1}^{-n}), \]

for \( i = 2, 3 \),

shows that \( (g^\sigma, q_{\phi_1}) \) has Levi order 1.

If \( \phi = \phi_2 = \{ \alpha_2 \} \), then \( Q_{\phi_2} = \{ \alpha \in R \mid \xi_2(\alpha) \geq 0 \} \), with

\[ \xi_1(e_i) = \begin{cases} 
1, & i = 1, 2, \\
0, & i = 3, \\
2, & i = 4.
\end{cases} \]

We have \( Q_{\phi_2}^n \cap R^\sigma_0 = \{ e_1, e_2, e_1 \pm e_3, e_2 \pm e_3 \} \).

Since \( \frac{1}{2} (\pm e_3 - e_1 - e_2 - e_4) = \sigma(\frac{1}{2} (e_1 + e_2 + e_3 - e_4) \in \sigma(Q_{\phi_2}^r) \cap Q_{\phi_2}^{-n} \) and

\[ e_1 + \frac{1}{2} (e_3 - e_1 - e_2 - e_4) = \frac{1}{2} (e_3 + e_1 - e_2 - e_4) \in Q_{\phi_2}^{-n} \cap \sigma(Q_{\phi_2}^{-n}), \]

\[ e_2 + \frac{1}{2} (e_3 - e_1 - e_2 - e_4) = \frac{1}{2} (e_3 - e_1 - e_2 - e_4) \in Q_{\phi_2}^{-n} \cap \sigma(Q_{\phi_2}^{-n}), \]

\[ (e_1 \pm e_3) + \frac{1}{2} (\mp e_3 - e_1 - e_2 - e_4) = \frac{1}{2} (\pm e_3 - e_1 - e_2 - 4) \in Q_{\phi_2}^{-n} \cap \sigma(Q_{\phi_2}^{-n}), \]

\[ (e_2 \pm e_3) + \frac{1}{2} (\mp e_3 - e_1 - e_2 - e_4) = \frac{1}{2} (\pm e_3 - e_1 + e_2 - e_4) \in Q_{\phi_2}^{-n} \cap \sigma(Q_{\phi_2}^{-n}), \]

\( (g^\sigma, q_{\phi_2}) \) has Levi order 1.

If \( \phi = \phi_3 = \{ \alpha_3 \} \), then \( Q_{\phi_3} = \{ \alpha \in R \mid \xi_3(\alpha) \geq 0 \} \), with

\[ \xi_1(e_i) = \begin{cases} 
1, & i = 1, 2, 3, \\
3, & i = 4.
\end{cases} \]

We have \( Q_{\phi_3}^n \cap R^\sigma_0 = \{ e_1, e_2, e_3 \} \cup \{ e_1 + e_j \mid 1 \leq i < j \leq 3 \} \).
Since \((-\frac{1}{2}(e_1+e_2+e_3+e_4)) = \sigma(\frac{1}{2}((e_1+e_2+e_3+e_4))) \in \sigma(Q^r_{\phi_3}) \cap Q^{-n}_{\phi_3}\) and
\[
\begin{align*}
  e_1 + \frac{1}{2}(-e_1-e_2-e_3-e_4) &= \frac{1}{2}(e_1-e_2-e_3-e_4) \in Q^{-n}_{\phi_3} \cap \sigma(Q^{-n}_{\phi_3}), \\
  e_2 + \frac{1}{2}(-e_1-e_2-e_3-e_4) &= \frac{1}{2}(-e_1+e_2-e_3-e_4) \in Q^{-n}_{\phi_3} \cap \sigma(Q^{-n}_{\phi_3}), \\
  e_3 + \frac{1}{2}(-e_1-e_2-e_3-e_4) &= \frac{1}{2}(-e_1-e_2+e_3-e_4) \in Q^{-n}_{\phi_3} \cap \sigma(Q^{-n}_{\phi_3}), \\
  (e_i+e_j)+\frac{1}{2}(-e_1-e_2-e_3-e_4) &= \frac{1}{2}(e_i+e_j-e_l-e_4) \in Q^{-n}_{\phi_3} \cap \sigma(Q^{-n}_{\phi_3})
\end{align*}
\]
if \(i, j, l \in \{1, 2, 3\}\).

\[\square\]

**Further remarks: the complex type cases**

In Tables 2, 3 we considered simple real Lie algebras \(g^\sigma\) having a simple complexification \(g\) (real type). Our approach generalizes these approach to complex type simple real algebras, i.e., simple complex Lie algebras considered as real Lie algebras by restriction of the field of scalars. The complexification \(g\) of a \(g^\sigma\) of the complex type is the direct sum of two copies of \(g^\sigma\). Having fixed any real form of the underlying complex Lie algebra \(g_C^\sigma\) of \(g^\sigma\), an involution \(\sigma\) defining the real form \(g^\sigma\) can be defined by using the corresponding conjugation of \(g_C^\sigma\) and setting
\[
g \simeq g_C^\sigma \oplus g_C^\sigma \ni (Z, W) \longrightarrow (\bar{W}, \bar{Z}) \in g_C^\sigma \oplus g_C^\sigma \simeq g. \tag{6.6}\]

Since the root system of \(g\) does not contain imaginary roots, by the criteria of Lemma 5.8 the parabolic CR algebras \((g^\sigma, q_{\phi_3})\) with \(g^\sigma\) of the complex type are either holomorphically degenerate or 1-nondegenerate.

A criterion of finitely nondegeneracy was given in [2, Thm. 11.5] in terms of cross-marked Satake diagrams. Those corresponding to minimal orbits consist of two copies of the Dynkin diagram of \(g_C^\sigma\), with homologous simple roots joined by a curved arrow. We denote by \(e\) the symmetry exchanging homologous roots of \(B\).

As above, the minimal parabolic CR algebras have the form \((g^\sigma, q_{\phi_3})\) for a subset \(\phi\) of the set \(B\) of positive simple roots.

If \(\alpha \in \phi\), we denote by \(\phi^\circ(\alpha)\) the connected component of \(\alpha\) in \((B\setminus \Phi)\cup[\alpha]\).

**Proposition 6.2** Let \((g^\sigma, q_{\phi_3})\) be a minimal parabolic CR algebra, with a simple \(g^\sigma\) of the complex type. The following are equivalent

1. \((g^\sigma, q_{\phi_3})\) is finitely nondegenerate;
2. \((g^\sigma, q_{\phi_3})\) is Levi 1-nondegenerate;
3. \(\forall \alpha \in \phi\), we have \(e(\phi^\circ(\alpha)) \cap \phi \neq \emptyset\). \[\square\]

**Example 6.3** The CR algebra \((g^\sigma, q_{\phi_3})\) described by the cross-marked Satake diagram

with \(g^\sigma \simeq sl_7(\mathbb{C})\), is fundamental and 1-nondegenerate by criteria of (2) Lemma 5.8.
Table 4  Elementary conjugation diagrams

| Type | S | σ |
|------|---|---|
| $A_1$ | $\sigma(\alpha_1) = \alpha_1$ | |
| $A_1 \times A_1$ | $\sigma(\alpha_1) = \alpha_2$ | |
| $A_3$ | $\sigma(\alpha_2) = \alpha_1 + \alpha_2 + \alpha_3$ | |
| $A_\ell$ | $\sigma(\alpha_1) = \alpha_2 + \cdots + \alpha_\ell$ | |
| $B_\ell$ | $\sigma(\alpha_1) = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_\ell)$ | |
| $C_\ell$ | $\sigma(\alpha_2) = \alpha_1 + \alpha_2 + 2(\alpha_3 + \cdots + \alpha_{\ell-1}) + \alpha_\ell$ | |
| $D_\ell$ | $\sigma(\alpha_1) = \alpha_1 + 2(\alpha_3 + \cdots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell$ | |
| $F_4$ | $\sigma(\alpha_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$ | |

Appendix A. Elementary conjugation diagrams

We conclude presenting a table, taken from explicit computations contained in [7, pp.18–21], that summarize all roots conjugation rules in all different subgraph types.

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