UNDERLYING VARIETIES,
AND GROUP STRUCTURES

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To the memory of A. Białynicki-Birula

Abstract. We explore to what extent the underlying variety of a connected algebraic group or the underlying manifold of a real Lie group determines its group structure.

1. Introduction. The central theme of this paper is the question as to what extent the underlying variety of a connected algebraic group or the underlying manifold of a real Lie group determines its group structure. The author was naturally led to consideration of it by the positive answer he gave in [21] to the question of B. Kunyavsky [15] about the validity of the statement formulated below as Corollary of Theorem 1. This statement concerns the possibility to represent the underlying variety of a connected reductive algebraic group in the form of a product of underlying varieties of its derived subgroup and connected component of the center. It follows from it, in particular, that the underlying variety of any connected reductive non-semisimple algebraic group can be represented as a product of algebraic varieties of positive dimension.

Sections 2, 3 make up the content of [21], where the possibility of such representations is explored. For some of them, in Theorem 1 is proved their existence, and in Theorems 3–6, on the contrary, their non-existence.

Theorem 1 shows that there are non-isomorphic reductive groups whose underlying varieties are isomorphic. In Sections 4–10, we explore the problem, naturally arising in connection with this, of dependence of the group structure of a connected algebraic group on the geometric properties of its underlying variety. A striking illustration of this dependence is the classical theorem about the commutativity of a connected algebraic group whose underlying variety is complete. In an explicit or implicit form, this problem was considered in the classical papers of A. Weil [30], C. Chevalley [9], A. Borel [4], A. Grothendieck [12, p. 5-02, Cor.], M. Rosenlicht [27, Thm. 3], M. Lazar [16, Thm.].
In Theorems 8–16, it is proved that such group characteristics of a connected algebraic group as dimensions of its radical and unipotent radical, reductivity, semisimplicity, solvability, unipotency, toricity, the property of being a semi-abelian variety can be expressed in terms of the geometric properties and numerical invariants of its underlying variety.

Theorem 13 generalizes to the case of connected solvable affine algebraic groups M. Lazar’s theorem, which states that an algebraic group, whose underlying variety is isomorphic to an affine space, is unipotent.

Theorem 1, when applied to connected semisimple algebraic groups (in contrast to its application to reductive non-semisimple ones), does not give a way to construct non-isomorphic such groups with isomorphic underlying varieties. However, in fact, such groups do exist: in Sections 6, 7 we find a method for constructing them.

It is well known (see [7, §4, Exer. 18, p. 122]) that for \( n \geq 7 \), there exist infinite (parametric) families of pairwise non-isomorphic \( n \)-dimensional connected unipotent algebraic groups. Being isomorphic to the \( n \)-dimensional affine space \( \mathbb{A}^n \), their underlying varieties are isomorphic to each other. We show that the situation is different for connected reductive algebraic groups: in Theorem 20, is proven that for any such group \( R \), the number of all algebraic groups, considered up to isomorphism, whose underlying variety is isomorphic to the underlying variety of \( R \), is finite. Generally speaking, this number is greater than 1. We prove (Theorems 20, 22) that if the group \( R \) is either simply connected and semisimple, or simple, then it is equal to 1.

The proof of Theorem 20 relies on the general finiteness theorem for the number of connected reductive algebraic groups of a fixed rank, considered up to isomorphism (Theorem 24); it is proved in the appendix (Section 10). Theorem 24 is a fundamental fact of the theory of algebraic groups that is well known for semisimple groups (in which case it follows from the finiteness of their centers). However, in full generality (that is, for reductive, and not just semisimple groups), the author failed to find it in the literature.

The obtained results imply similar results for connected compact real Lie groups, in particular, the finiteness theorem for the number of such groups of any fixed dimension (Theorem 26). As in the case of Theorem 24, the author failed to find this fundamental fact of the theory of compact real Lie groups in the literature. The same applies to the finiteness theorem for the number of reduced root data of any fixed rank, which follows from the 24 theorem (Theorem 25).

The results of this paper were announced in [22], [23], [25], and [24].
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Conventions and notation.
We follow the point of view on algebraic groups adopted in [6], [14], [29], and use the following notation:
• $k$ is an algebraically closed field, over which all algebraic varieties considered below are defined.
• Groups are considered in multiplicative notation. The unit element of a group $G$ is denoted by $e$ (which group is meant will be clear from the context).
• For groups $G$ and $H$, the notation $G \simeq H$ means that they are isomorphic.
• $\mathcal{C}(G)$ is the center of a group $G$.
• $\mathcal{D}(G)$ is the derived group of a group $G$.
• $\langle g \rangle$ is the cyclic group generated by an element $g$.
• A torus means affine algebraic torus, and a homomorphism of algebraic groups means their algebraic homomorphism.
• $\mathcal{R}(G)$ and $\mathcal{R}_u(G)$ are, respectively, the radical and the unipotent radical of a connected affine algebraic group $G$.
• $G^o$ is the identity connected component of an algebraic group or a Lie group $G$.
• $\text{Lie}(G)$ is the Lie algebra of an algebraic group or a Lie group $G$.
• If $\alpha : G \to H$ is a homomorphism of algebraic groups, then
\[ d\alpha : \text{Lie}(G) \to \text{Lie}(H) \] (1)

is its differential at the unit element.
• $\mathbb{G}_a, \mathbb{G}_m$ are respectively, the one-dimensional additive and multiplicative algebraic groups.
• $\text{Hom}(G, H)$ and $\text{Aut}(G)$ are the groups of algebraic homomorphisms if $G$ and $H$ are algebraic groups. The character of such a group $G$ is an element of the group $\text{Hom}(G, \mathbb{G}_m)$.
• $\mathbb{A}_n$ is the $n$-dimensional coordinate affine space.
• $\mathbb{A}_n^m$ is the product of $n$ copies of the variety $\mathbb{A}_1 \setminus \{0\}$.
• Let $p := \text{char}(k)$ and $a \in \mathbb{Z}$. If $ap \neq 0$, then $a'$ denotes the quotient of dividing $a$ by the greatest power of $p$ that divides $a$. If $ap = 0$, then $a' := a$.

2. Reductive groups with isomorphic group varieties. In this section, we prove the existence of some representations of underlying varieties of affine algebraic groups in the form of products of algebraic varieties, and also the existence of connected non-isomorphic reductive
non-semisimple algebraic groups whose underlying varieties are isomorphic algebraic varieties.

Let $G$ be a connected reductive algebraic group. Then

$$D := D(G) \quad \text{and} \quad Z := C(G)^o$$

are respectively a connected semisimple algebraic group and a torus (see [6, Sect. 14.2, Prop. (2)]). The algebraic groups $D \times Z$ and $G$ are not always isomorphic; the latter is equivalent to the equality $D \cap Z = \{e\}$, that, in turn, is equivalent to the property that the isogeny of algebraic groups $D \times Z \to G$, $(d, z) \mapsto dz$, is their isomorphism.

**Theorem 1.** There is an injective homomorphism of algebraic groups $\iota: Z \hookrightarrow G$ such that the mapping

$$\varphi: D \times Z \to G, \quad (d, z) \mapsto d \cdot \iota(z),$$

is an isomorphism of algebraic varieties (but, in general, not a homomorphism of algebraic groups).

**Corollary.** The underlying varieties of (in general, non-isomorphic) algebraic groups $D \times Z$ and $G$ are isomorphic algebraic varieties.

**Remark 2.** The existence of $\iota$ in the proof of Theorem 1 is established by an explicit construction.

**Example** ([20, Thm. 8, Proof]). Let $G = \text{GL}_n$. Then $D = \text{SL}_n$, $Z = \{\text{diag}(t, \ldots, t) \mid t \in k^\times\}$, and one can take

$$\text{diag}(t, \ldots, t) \mapsto \text{diag}(t, 1, \ldots, 1)$$

as $\iota$. In this example, if $n \geq 2$, then $G$ and $D \times Z$ are non-isomorphic algebraic groups, because the center of $G$ is connected and that of $D \times Z$ is not.

**Proof of Theorem 1.** Let $T_D$ be a maximal torus of the group $D$, and let $T_G$ be a maximal torus of the group $G$ containing $T_D$. The torus $T_D$ is a direct factor of the torus $T_G$, i.e., in the latter, there is a torus $S$ such that the map $T_D \times S \to T_G$, $(t, s) \mapsto ts$, is an isomorphism of algebraic groups (see [6, 8.5, Cor.]). We shall show that the mapping

$$\psi: D \times S \to G, \quad (d, s) \mapsto ds,$$

is an isomorphism of algebraic varieties.

We have (see [6, Sect. 14.2, Prop. (1),(3)]):

$$Z \subseteq T_G, \quad DZ = G, \quad |D \cap Z| < \infty. \quad (3)$$

Let $g \in G$. In view of (3)(b), there are the elements $d \in D$, $z \in Z$ such that $g = dz$, and in view of (3)(a) and the definition of $S$, there
are the elements $t \in T_D$, $s \in S$ such that $z = ts$. We have $dt \in D$ and $\psi(dt, s) = dts = g$. Therefore, the morphism $\psi$ is surjective.

Consider in $G$ a pair of mutually opposite Borel subgroups containing $T_G$. Their unipotent radicals $U$ and $U^-$ lie in the group $D$. Let $N_D(T_D)$ and $N_G(T_G)$ be the normalizers of the tori $T_D$ and $T_G$ in the groups $D$ and $G$, respectively. Then $N_D(T_D) \subseteq N_G(T_G)$ in view of (3)(b). The homomorphism $N_D/T_D \to N_G/T_G$ induced by this embedding is an isomorphism of groups (see [6, IV.13]), by which we identify them and denote by $W$. For every element $\sigma \in W$, we fix a representative $n_\sigma \in N_D(T_D)$. The group $U \cap n_\sigma U^- n_\sigma^{-1}$ does not depend on the choice of this representative because $T_D$ normalizes the group $U^-$; we denote it by $U'_\sigma$.

It follows from the Bruhat decomposition that for every element $g \in G$, there are uniquely defined elements $\sigma \in W$, $u \in U$, $u' \in U'_\sigma$, and $t_G \in T_G$ such that $g = u'n_\sigma ut_G$ (see [14, 28.4, Thm.]). In view of the definition of the torus $S$, there are uniquely defined elements $t_D \in T_D$ and $s \in S$ such that $t_G = t_D s$, and since $u', n_\sigma, u, t_D \in D$, the condition $g \in D$ is equivalent to the condition $s = e$. It follows from this and the definition of the morphism $\psi$ that the latter is injective.

Thus, $\psi$ is a bijective morphism. Therefore, to prove that it is an isomorphism of algebraic varieties, it remains to prove its separability (see [6, Sect. 18.2, Thm.]). We have $\text{Lie}(G) = \text{Lie}(D) + \text{Lie}(T_G)$ (see [6, Sect. 13.18, Thm.]) and $\text{Lie}(T_G) = \text{Lie}(T_D) + \text{Lie}(S)$ (in view of the definition of the torus $S$). Therefore,

$$\text{Lie}(G) = \text{Lie}(D) + \text{Lie}(S).$$

(4)

On the other hand, from (2) it is obvious that the restrictions of the morphism $\psi$ to the subgroups $D \times \{e\}$ and $\{e\} \times S$ in $D \times S$, are isomorphisms respectively with the subgroups $D$ and $S$ in $G$. Since $\text{Lie}(D \times S) = \text{Lie}(D \times \{e\}) + \text{Lie}(\{e\} \times S)$, from (4) it follows that the differential of the morphism $\psi$ at the point $(e, e)$ is surjective. Therefore (see [6, Sect. 17.3, Thm.]), the morphism $\psi$ is separable.

Since $\psi$ is an isomorphism, from (2) it follows that $\dim(G) = \dim(D) + \dim(S)$. On the other hand, from (3)(b),(c) it follows that $\dim(G) = \dim(D) + \dim(Z)$. Therefore, $Z$ and $S$ are equidimensional and hence isomorphic tori. Consequently, as $\iota$ one can take the composition of any tori isomorphism $Z \to S$ with the identity embedding $S \hookrightarrow G$. □

3. Properties of factors. In contrast to the previous section, this one, on the contrary, concerns the non-existence of some representations the underlying variety of an affine algebraic group as a product of algebraic varieties.
Theorem 3. An algebraic variety, on which there is a non-constant invertible regular function, cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.

Proof. If the statement of Theorem 3 were not true, then the existence of the non-constant invertible regular function specified in it would imply the existence of such a function on a connected semisimple algebraic group. Dividing this function by its value at the unit element, we would then get, according to [27, Thm. 3], a non-trivial character of this group, which contradicts the absence of non-trivial characters of any connected semisimple groups. □

Below, unless otherwise stated, we assume that $k = \mathbb{C}$. By the Lefschetz principle, Theorems 5, 6, 19, 20, 22 proved below are valid for any field $k$ of characteristics zero. Topological terms refer to classical topology, and homology and cohomology are singular.

Every complex reductive algebraic group $G$ has a compact real form, every two such forms are conjugate, and if $G$ is one of them, then the topological manifold $G$ is homeomorphic to the product of $G$ and a Euclidean space; see [19, Chap. 5, §2, Thms. 2, 8, 9]. Therefore, $G$ and $G$ have the same homology and cohomology. This is used below without further explanation.

Theorem 4. If a $d$-dimensional algebraic variety $X$ is a direct factor of the underlying variety of a connected reductive algebraic group, then

$$H_d(X,\mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad H_i(X,\mathbb{Z}) = 0 \quad \text{for} \quad i > d.$$

Proof. Suppose there is a connected reductive algebraic group $G$ and an algebraic variety $Y$ such that the underlying variety of the group $G$ is isomorphic to $X \times Y$. Let $n := \dim(G)$; then $\dim(Y) = n - d$. The algebraic varieties $X$ and $Y$ are irreducible non-singular and affine. Therefore (see [17, Thm. 7.1]),

$$H_i(X,\mathbb{Z}) = 0 \quad \text{for} \quad i > d, \quad H_j(Y,\mathbb{Z}) = 0 \quad \text{for} \quad j > n - d. \quad (5)$$

By the universal coefficient theorem, for every algebraic variety $V$ and every $i$, we have

$$H_i(V,\mathbb{Q}) \cong H_i(V,\mathbb{Z}) \otimes \mathbb{Q}, \quad (6)$$

and by the Künneth formula,

$$H_n(G,\mathbb{Q}) \cong H_n(X \times Y,\mathbb{Q}) \cong \bigoplus_{i+j=n} H_i(X,\mathbb{Q}) \otimes H_j(Y,\mathbb{Q}). \quad (7)$$

Therefore, from (5), (7) it follows that

$$H_n(G,\mathbb{Q}) \cong H_d(X,\mathbb{Q}) \otimes H_{n-d}(Y,\mathbb{Q}). \quad (8)$$
Consider a compact real form $G$ of the group $G$. Since $G$ is a closed connected orientable $n$-dimensional topological manifold, $H_n(G, \mathbb{Q}) \simeq \mathbb{Q}$. Hence, $H_n(G, \mathbb{Q}) \simeq \mathbb{Q}$. From this and (8) it follows that $H_d(X, \mathbb{Q}) \simeq \mathbb{Q}$. In turn, in view of (6), the latter implies $H_d(X, \mathbb{Z}) \simeq \mathbb{Z}$, because $H_d(X, \mathbb{Z})$ is a finitely generated (see [10, Sect. 1.3]) torsion-free abelian group (see [1, Thm. 1]). □

**Corollary.** A contractible algebraic variety (in particular, $\mathbb{A}^d$) of positive dimension cannot be a direct factor of the underlying variety of a connected reductive algebraic group.

**Theorem 5.** An algebraic curve cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.

**Proof.** Suppose an algebraic curve $X$ is a direct factor of the underlying variety of a connected semisimple algebraic group $G$. Then $X$ is irreducible, non-singular, affine, and there is a surjective morphism $\pi: G \to X$. Due to rationality of the underlying variety of $G$ (see [6, Sect. 14.14]), the existence of $\pi$ implies unirationality, and hence, by Luroth’s theorem, rationality of $X$. Therefore, $X$ is isomorphic to an open subset $U$ of $\mathbb{A}^1$. The case $U = \mathbb{A}^1$ is impossible due to Theorem 4. If $U \neq \mathbb{A}^1$, then there is a non-constant invertible regular function on $X$, which is impossible in view of Theorem 3. □

**Theorem 6.** An algebraic surface cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.

**Proof.** Suppose there are a connected semisimple algebraic group $G$ and the algebraic varieties $X$ and $Y$ (necessarily irreducible and smooth) such that $X$ is a surface, and the product $X \times Y$ is isomorphic to the underlying variety of the group $G$. We keep the notation of the proof of Theorem 4. Since the group $G$ is semisimple, the group $G$ is semisimple too. Hence, $H^1(G, \mathbb{Q}) = H^2(G, \mathbb{Q}) = 0$ (see [18, §9, Thm. 4, Cor. 1]). Insofar as the $\mathbb{Q}$-vector spaces $H^1(G, \mathbb{Q})$ and $H_1(G, \mathbb{Q})$ are dual to each other, this gives

$$H_1(G, \mathbb{Q}) = H_2(G, \mathbb{Q}) = 0. \quad (9)$$

Since the group $G$ is connected, the topological manifolds $X$ and $Y$ are connected too. Therefore,

$$H_0(X, \mathbb{Q}) = H_0(Y, \mathbb{Q}) = \mathbb{Q}. \quad (10)$$

From (7), (9), and (10) it follows that $H_2(X, \mathbb{Q}) = 0$. In view of (6), this contradics Theorem 4, which completes the proof. □

**Remark 7.** Theorem 5 can be proved in the same way as Theorem 6. Namely, in the proof of the latter one only needs to consider
$X$ being a curve, and then the arguments used in it lead to the equality $H_1(X, \mathbb{Q}) = 0$, which contradicts Theorem 4. The other proof is given in the hope that it raises the chances of carrying over Theorem 5 to positive characteristic.

4. Group properties determined by properties of underlying variety. Theorem 1 naturally leads to the question as to what extent the underlying variety of an algebraic group determines its group structure.

Explicitly or implicitly, this question has long been considered in the literature.

For example, M. Lazar proved in [16] that if the underlying variety of an algebraic group is isomorphic to an affine space, then this group is unipotent (for a short proof, see Remark 14 below).

By Chevalley’s theorem, every connected algebraic group $G$ contains the largest connected affine normal subgroup $G_{\text{aff}}$, and the group $G/G_{\text{aff}}$ is an abelian variety. M. Rosenlicht in [27] considered $G$ such that $G_{\text{aff}}$ is a torus; this property is equivalent to the absence of connected one-dimensional unipotent subgroups in $G$. In modern terminology (see [8, Sect. 5.4]), such groups are called semi-abelian varieties (M. Rosenlicht called them toroidal). Next Theorem 8 gives a criterion that the group $G$ is a semi-abelian variety in terms of geometric properties of its underlying variety (the proof does not use the restriction on the characteristic of the field $k$; constraint (b) in Theorem 8 is weaker than the constraint made in [8, Prop. 5.4.5]):

**Theorem 8** (semi-abelianess criterion). *The following properties of a connected algebraic group $G$ are equivalent:

(sa$_1$) $G$ is toroidal;

(sa$_2$) $G$ does not contain subvarieties isomorphic to $\mathbb{A}^1$.

**Proof.** Let $\pi: G \to G/G_{\text{aff}}$ be the natural epimorphism, and let $X$ be a subvariety of $G$ isomorphic to $\mathbb{A}^1$. Shifting it by an appropriate element of the group $G$, we can assume that the unit element $e$ of the group $G$ lies in $X$. Since the variety $X$ is isomorphic to the underlying variety of the group $\mathbb{G}_a$, we can endow the variety $X$ with a structure of an algebraic group isomorphic to group $\mathbb{G}_a$ with the unit element $e$. Then $\pi|_X: X \to G/G_{\text{aff}}$ is a homomorphism of algebraic groups in view of [27, Thm. 3]. Since $X$ is an affine and $G/G_{\text{aff}}$ is a complete algebraic variety, this yields $X \subseteq G_{\text{aff}}$. Therefore, the matter comes down to proving the equivalence of the following properties:

(sa$_1'$) $G_{\text{aff}}$ is a torus;

(sa$_2'$) $G_{\text{aff}}$ does not contain subvarieties isomorphic to $\mathbb{A}^1$. 
(sa′_1) ⇒ (sa′_2): Let the subvariety \( X \) of the torus \( G_{\text{aff}} \) be isomorphic to \( \mathbb{A}^1 \). The algebra of regular functions on \( G_{\text{aff}} \) is generated by invertible functions. This means that this is also the case for the algebra of regular functions on \( X \). This contradicts the fact that there are no non-constant invertible regular functions on \( \mathbb{A}^1 \).

(sa′_2) ⇒ (sa′_1): If (sa′_2) holds, then \( G_{\text{aff}} \) is reductive, since the variety \( \mathcal{R}_u(G_{\text{aff}}) \) is isomorphic to \( \mathbb{A}^d \) (see [12, p. 5-02, Cor.]), which for \( d > 0 \) contains affine lines. In addition, \( \mathcal{D}(G_{\text{aff}}) = \{e\} \), because root subgroups in a semisimple group are isomorphic to \( \mathbb{G}_a \), whose underlying variety is isomorphic to \( \mathbb{A}^1 \). Hence, \( G_{\text{aff}} \) is a torus. \( \square \)

**Corollary.** The following properties of a connected algebraic group \( G \) are equivalent:

(a) in the underlying variety of the group \( G \), there are no subvarieties isomorphic to \( \mathbb{A}^1 \);

(b) in the group \( G \), there are no algebraic subgroups isomorphic to \( \mathbb{G}_a \).

**Proof.** According to [27, Prop.], property (b) is equivalent to \( G \) being semi-abelian variety. Therefore, the claim follows from Theorem 8. \( \square \)

Below is listed a series of group properties of connected affine algebraic groups determined by the properties of their underlying varieties. In the formulations of the corresponding statements, the following numerical invariants of underlying varieties are used.

Let \( X \) be an irreducible algebraic variety. The multiplicative group \( k[X]^\times \) of invertible regular functions on \( X \) contains the subgroup of non-zero constants \( k^\times \), and the quotient \( k[X]^\times /k^\times \) is a free abelian group of a finite rank (see [27, Thm. 1]). Let us denote

\[
\text{units}(X) := \text{rank}(k[X]^\times /k^\times). \tag{11}
\]

According to [27, Thms. 2, 3], this invariant has the following properties:

(i) If \( X \) and \( Y \) are irreducible algebraic varieties, then

\[
\text{units}(X \times Y) = \text{units}(X) + \text{units}(Y). \tag{12}
\]

(ii) If \( G \) is a connected algebraic group, then

\[
\text{units}(G) = \text{rank}(\text{Hom}(G, \mathbb{G}_m)). \tag{13}
\]

**Lemma 9.** Let \( G \) be a connected algebraic group. Then

(i) \( \text{units}(G) \leq \text{dim}(G) \);

(ii) the equality in (i) is equivalent to the property that \( G \) is a torus.

**Proof.** By [26, Cor. 5 of Thm. 16], the kernel of every character of the group \( G \) contains the smallest normal algebraic subgroup \( D \) of \( G \) such
that the group $G/D$ is affine. In view of this and (13), in what follows we can (and will) assume that $G$ is affine. Similarly, since $R_u(G)$ lies in the kernel of every character of $G$, we can (and will) assume that $G$ is reductive. Let $T$ be a maximal torus of $G$. Since the set $\bigcup_{g \in G} gTg^{-1}$ is dense in $G$ (see [6, Sect. 12.1, Thm.(a), (b) and Sect. 13.17, Cor. 2(c)]), the restriction of characters of $G$ to $T$ is a group embedding $\text{Hom}(G, \mathbb{G}_m) \hookrightarrow \text{Hom}(T, \mathbb{G}_m)$; whence, in view of (13) and [6, Sect. 8.5, Prop.], we get $\text{units}(G) \leq \text{units}(T) = \dim(T) \leq \dim(G)$.

This completes the proof. $\square$

In what follows, we use the following notation:

$$mh(X) := \max\{d \in \mathbb{Z}_{\geq 0} \mid H_d(X, \mathbb{Q}) \neq 0\}. \quad (14)$$

If $X$ is a non-singular affine algebraic variety, then, according to [17, Thm. 7.1],

$$mh(X) \leq \dim(X).$$

**Theorem 10.** If $G$ is a connected affine algebraic group, then

$$\dim(\mathcal{R}_u(G)) = \dim(G) - mh(G), \quad (15)$$

$$\dim(\mathcal{R}(G)) = \dim(G) - mh(G) + \text{units}(G). \quad (16)$$

**Proof.** By [12, p. 5-02, Cor.], the underlying variety of the group $\mathcal{R}_u(G)$ is isomorphic to an affine space. Therefore, the underlying varieties of the groups $G$ and $R := G/\mathcal{R}_u(G)$, considered as topological manifolds, are homotopy equivalent. Therefore, $H_i(G, \mathbb{Q}) \simeq H_i(R, \mathbb{Q})$ for every $i$, and hence

$$mh(G) = mh(R). \quad (17)$$

Since the group $R$ is reductive, it follows from (6) and Theorem 4 that

$$mh(R) = \dim(R). \quad (18)$$

In view of $\dim(R) = \dim(G) - \dim(\mathcal{R}_u(G))$, equalities (17) and (18) imply (15).

The group $\mathcal{R}(G)$ is a semi-direct product of its maximal torus $T$ and the group $\mathcal{R}_u(G)$ (see [6, Sect. 10.6, Thm.]), so

$$\dim(\mathcal{R}(G)) = \dim(T) + \dim(\mathcal{R}_u(G)). \quad (19)$$

Let $\pi : G \to G/\mathcal{R}_u(G)$ be the canonical projection. Then (see [6, Sect. 11.21])

$$\pi(T) = \pi(\mathcal{R}(G)) = \mathcal{C}(G/\mathcal{R}_u(G))^c. \quad (20)$$
Since the group $G/\mathcal{R}_u(G)$ is reductive, it follows from (13) and (20) that

$$\text{units}(G) = \text{rank}(\text{Hom}(G, \mathbb{G}_m))$$

$$= \text{rank}(\text{Hom}(G/\mathcal{R}_u(G), \mathbb{G}_m))$$

$$= \dim(\mathcal{O}(G/\mathcal{R}_u(G)))$$

$$= \dim(\pi(T)) = \dim(T).$$

(21)

Now equality (16) follows from (15), (19), and (21). \hfill \Box

Since reductivity (respectively, semisimplicity) of a connected affine algebraic group is equivalent to the triviality of its unipotent radical (respectively, radical), Theorem 10 gives the following criteria for reductivity and semisimplicity in terms of the geometric properties of the underlying variety:

**Theorem 11** (reductivity criterion). The following properties of a connected affine algebraic group $G$ are equivalent:

- $(\text{red}_1)$ $G$ is reductive;
- $(\text{red}_2)$ $\dim(G) = \text{mh}(G)$.

If these properties hold, then $\dim(\mathcal{C}(G)) = \text{units}(G)$.

**Proof.** The first claim follows from (15), and the second from (20) and (21). \hfill \Box

**Theorem 12** (semisimplicity criterion). The following properties of a connected affine algebraic group $G$ are equivalent:

- $(\text{ss}_1)$ $G$ is semisimple;
- $(\text{ss}_2)$ $\dim(G) = \text{mh}(G) - \text{units}(G)$;
- $(\text{ss}_3)$ $\dim(G) = \text{mh}(G)$ and $\text{units}(G) = 0$.

**Proof.** $(\text{ss}_1) \iff (\text{ss}_2)$ follows from (16). In view of reductivity of semisimple groups and finiteness of their centers, from Theorem 11 it follows that $(\text{ss}_1) \Rightarrow (\text{ss}_3)$. Clearly, $(\text{ss}_3) \Rightarrow (\text{ss}_2)$. \hfill \Box

The following Theorem 13 generalizes M. Lazar’s theorem [16] to the case of connected solvable affine algebraic groups and shows that solvability of a connected affine algebraic group also can be characterized in terms of the geometric properties of its underlying variety.

**Theorem 13** (solvability criterion). The following properties of a connected affine algebraic group $S$ are equivalent:

- $(\text{sol}_1)$ $S$ is solvable;
- $(\text{sol}_2)$ $\text{mh}(S) = \text{units}(S)$;
there are nonnegative integers $t$ and $r$ such that the underlying variety of the group $S$ is isomorphic to $\mathbb{A}^t_\ast \times \mathbb{A}^r$, and in this case, necessarily $t = \text{units}(S)$.

If these properties hold, then the dimension of maximal tori of the group $S$ is equal to $\text{units}(S)$, and the following equality holds

$$\dim(\mathcal{R}_u(S)) = \dim(S) - \text{units}(S).$$

(22)

Proof. $(\text{sol}_1) \Leftrightarrow (\text{sol}_2)$: Let $G := S/\mathcal{R}_u(S)$; it is a connected reductive algebraic group. We shall use the same notation as in the proof of Theorem 1. Solvability of the group $S$ is equivalent to the equality $G = Z$, whence, in view of connectedness of the groups $G$ and $Z$, it follows that $S$ is solvable $\iff \dim(G) = \dim(Z)$. 

(23)

Given (15) and (17), we have

$$\dim(G) = \text{mh}(S).$$

(24)

The elements of the group $\text{Hom}(S, \mathbb{G}_m)$ (respectively, $\text{Hom}(G, \mathbb{G}_m)$) are trivial on the group $\mathcal{R}_u(S)$ (respectively, $D$). It follows this and (3)(b) that

$$\text{Hom}(S, \mathbb{G}_m) \simeq \text{Hom}(G, \mathbb{G}_m),$$

$$\text{Hom}(G, \mathbb{G}_m) \simeq \text{Hom}(Z/(Z \cap D), \mathbb{G}_m).$$

(25)

From (13), (25), and (3)(c) we obtain

$$\text{units}(S) = \text{rank}(\text{Hom}(Z/(Z \cap D), \mathbb{G}_m))$$

$$= \dim(Z/(Z \cap D)) = \dim(Z).$$

(26)

Matching (23), (24), and (26) completes the proof the equivalence $(\text{sol}_1) \Leftrightarrow (\text{sol}_2)$.

$(\text{sol}_1) \Rightarrow (\text{sol}_3)$: This is proven in [12, p. 5-02, Cor.] for the field $k$ of any characteristic.

$(\text{sol}_3) \Rightarrow (\text{sol}_2)$: Let $(\text{sol}_3)$ holds. It follows from (12), (13), and the evident equality $\text{units}(\mathbb{A}^r) = 0$ that

$$\text{units}(\mathbb{A}^t_\ast \times \mathbb{A}^r) = t.$$ 

(27)

On the other hand, since the topological manifold $\mathbb{A}^r$ is contractible, and $\mathbb{A}^t_\ast$ is homotopically equivalent to the product $t$ circles, we have

$$H_j(\mathbb{A}^t_\ast \times \mathbb{A}^r, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & j = t, \\ 0, & j > t; \end{cases}$$

we conclude from this and (14) that

$$\text{mh}(\mathbb{A}^t_\ast \times \mathbb{A}^r) = t.$$ (28)
Comparing (27) with (28) completes the proof of the implication \((\text{sol}_3) \Rightarrow (\text{sol}_2)\), and with it the proof of the first claim of the theorem.

Let the properties specified in the first statement of the theorem be satisfied. Then it follows from the property \((\text{sol}_1)\) and [6, Sect. 10.6, Thm. (4)] that the dimension of maximal tori in \(S\) is equal to \(\dim(S/\mathcal{R}_u(S)) = \dim(G)\), what is equal to units\((S)\) in view of (23) and (26). Equality (22) follows from equalities (15) and \((\text{sol}_2)\). This proves the second claim of the theorem.

The group \(S\) is unipotent (respectively, is a torus) if and only if it is solvable (i.e., by \((c)\), its underlying variety is isomorphic to \(A^r \times A^r\)), and by Theorem 10, the equality \(\text{mh}(S) = 0\) (respectively, \(\text{mh}(S) = \dim(S)\)) holds. Now the last claim of the theorem follows from (28).

**Remark 14.** Here is a short proof of M. Lazard’s theorem [16], suitable for the field \(k\) of any characteristic.

**Proof of M. Lazard’s theorem** [16]. Let the underlying variety of the group \(G\) be isomorphic to \(A^r\). If \(G\) is not unipotent, then \(G\) contains a non-identity semisimple element, and therefore, also a non-identity torus (see [6, Thms. 4.4(1), 11.10]). The action of this torus on \(G\) by left translations has no fixed points. This contradicts the fact that every algebraic torus action on \(A^r\) has a fixed point, see [2, Thm. 1].

The next two theorems show that unipotency and toricity of a connected affine algebraic group can also be characterized in terms of the introduced numerical invariants of its underlying variety.

**Theorem 15** (unipotency criterion). *The following properties of a connected affine algebraic group \(G\) are equivalent:*

\[
\begin{align*}
(\text{u}_1) & \quad \text{\(G\) is unipotent;} \\
(\text{u}_2) & \quad \text{\(\text{mh}(G) = \text{units}(G) = 0\);} \\
(\text{u}_3) & \quad \text{the underlying variety of the group \(G\) is isomorphic to } A^{\dim(U)}.
\end{align*}
\]

**Proof.** In view of solvability of unipotent group, the equivalence \((\text{u}_1) \Leftrightarrow (\text{u}_2)\) (respectively, \((\text{u}_1) \Leftrightarrow (\text{u}_3)\)) follows from the equivalence \((\text{sol}_1) \Leftrightarrow (\text{sol}_2)\) (respectively, \((\text{sol}_1) \Leftrightarrow (\text{sol}_3)\)) and formula (22) in Theorem 13.

In the following theorem, the characteristic of the field \(k\) can be arbitrary.

**Theorem 16** (toricity criterion). *The following properties of a connected affine algebraic group \(G\) are equivalent:*

\[
\begin{align*}
(\text{t}_1) & \quad \text{\(G\) is a torus};
\end{align*}
\]
(t_2) \( \dim(G) = \text{units}(G) \);
(t_3) the underlying variety of the group \( G \) is isomorphic to \( \mathbb{A}_*^{\dim(G)} \).

Proof. Lemma 9 gives \( (t_1) \Leftrightarrow (t_2) \). The implication \( (t_3) \Rightarrow (t_2) \) follows from (12) and \( \text{units}(\mathbb{A}_1^1) = 1 \), and \( (t_1) \Rightarrow (t_3) \) is evident. \( \square \)

5. Different group structures on the same variety. As is known (see [7, §4, Exer. 18, p.122]), there are infinitely many pairwise non-isomorphic connected unipotent algebraic groups of any fixed dimension \( \geq 7 \); their underlying varieties, however, all are isomorphic to each other (see Theorem 13). On the other hand, there are types of connected algebraic groups for which underlying varieties define group structure unambiguously. The following Theorem 17 shows that such types include semi-abelian varieties (the proof is not uses restrictions on the characteristic of the field \( k \)). Below we will find other types of algebraic groups that have the indicated uniqueness property, see Theorems 20(b) and 22.

**Theorem 17.** Let \( G_1 \) and \( G_2 \) be algebraic groups, one of which is a semi-abelian variety. The following properties are equivalent:
(a) the underlying varieties of the groups \( G_1 \) and \( G_2 \) are isomorphic;
(b) the algebraic groups \( G_1 \) and \( G_2 \) are isomorphic.

Proof. Let \( G_1 \) be a semi-abelian variety. Then by [27, Thm. 3], the composition of an isomorphism of the underlying varieties \( G_2 \to G_1 \) with a suitable left translation of \( G_1 \) is an isomorphism of algebraic groups, which proves (a)\( \Rightarrow \) (b). \( \square \)

**Corollary.** Isomorphism of algebraic groups, among which there is either a torus or an abelian variety, is equivalent to isomorphism of their underlying varieties.

From this, in particular, one obtains the fact, discovered by A. Weil, that isomorphism of abelian varieties is equivalent to isomorphism of their underlying varieties (see [30]).

**Remark 18.** Semi-abelian varieties are commutative. In preprint [11] published after preprint [22] of the first version of this paper, it is proved that, for \( \text{char}(k) = 0 \), isomorphism of the underlying varieties of two connected commutative algebraic groups implies isomorphism of these algebraic groups. The existence of Witt groups shows that in this statement the condition \( \text{char}(k) = 0 \) cannot be dropped.

We now investigate the problem of determinability of group structure by the properties of underlying variety for reductive algebraic groups.
Theorem 19. Let $G_1$ and $G_2$ be connected affine algebraic groups, and let $R_i$ be a maximal reductive algebraic subgroup of $G_i$, $i = 1, 2$. If the underlying varieties of the groups $G_1$ and $G_2$ are isomorphic, then the Lie algebras of the connected algebraic groups $R_1$ and $R_2$ are isomorphic.

Proof. From $\text{char}(k) = 0$ it follows that the group $G_i$ is a semidirect product of the groups $R_i$ and $\mathcal{R}_u(G_i)$ (see [6, 11.22]). Hence the group $R_i$ is connected (because $G_i$ is connected), and the underlying manifolds of the groups $G_i$ and $R_i$ are homotopy equivalent topological manifolds (see the proof of Theorem 10).

Consider a compact form $R_i$ of the reductive algebraic group $R_i$. The underlying manifolds of the groups $R_i$ and $R_i$ are homotopy equivalent.

Suppose that the underlying varieties of the groups $G_1$ and $G_2$ are isomorphic algebraic varieties, and therefore, homeomorphic topological manifolds. Then the underlying manifolds of the groups $R_1$ and $R_2$ are homotopy equivalent topological manifolds. In view of [28, Satz], this implies that the real Lie algebras $\text{Lie}(R_1)$ and $\text{Lie}(R_2)$ are isomorphic. Now the claim of the theorem follows from the fact that the real Lie algebra $\text{Lie}(R_i)$ is a real form of the complex Lie algebra $\text{Lie}(R_i)$. □

Theorem 20. Let $R$ be a connected reductive algebraic group.

(i) If $G$ is an algebraic group such that the underlying varieties of $G$ and $R$ are isomorphic, then

(a) $G$ is connected and reductive, and the Lie algebras $\text{Lie}(R)$ and $\text{Lie}(G)$ are isomorphic;

(b) in the case of a semisimple simply connected group $R$, the algebraic groups $R$ and $G$ are isomorphic.

(ii) The number of all algebraic groups, considered up to isomorphism, whose underlying varieties are isomorphic to that of $R$, is finite.

Proof. (i)(a) It follows from connectedness of the group $R$ and the condition on the group $G$ that the group $G$ is connected. In view of Theorem 19 and reductivity of the group $R$, the Lie algebra of a maximal reductive subgroup in $G$ is isomorphic to $\text{Lie}(R)$. In particular, the dimension of this subgroup is equal to $\dim(R)$. Since $\dim(R) = \dim(G)$, this subgroup coincides with $G$.

(i)(b) From the condition on the group $G$ and simply connectedness of the underlying manifold of the group $R$ it follows that the underlying manifold of the group $G$ is simply connected. In view of (a), the Lie
algebras $\text{Lie}(R)$ and $\text{Lie}(G)$ are isomorphic. Consequently, the algebraic groups $R$ and $G$ are isomorphic (see [19, Chap. 1, §3, 3°, Chap. 3, §3, 4°]).

Statement (ii) follows from (i)(a) and finiteness of the numbers of all, considered up to isomorphism, connected reductive algebraic groups of a fixed dimension. This finiteness theorem, which the author failed to find in the literature, is proved below in appendix (Section , Theorem 24, and Remark 28).

\[ \square \]

**Remark 21.** Using the proof of Theorem 24 given in Section , one can obtain an upper bound for the number specified in Theorem 20(ii) (see also Remark 27 below).

Theorem 1 provides examples of non-isomorphic connected reductive algebraic groups whose underlying varieties are isomorphic (according to Theorem 20(a), the Lie algebras of these groups are isomorphic). In the case of connected semisimple algebraic groups (that is, when $Z = \{e\}$), Theorem 1 degenerates into a trivial statement that does not provide such examples. However, non-isomorphic connected semisimple algebraic groups whose group varieties are isomorphic do exist. Below is described a method that allows one to construct them. It is suitable for a field $k$ of any characteristic.

**6. Construction of non-isomorphic semisimple algebraic groups with isomorphic underlying varieties.** Fix a positive integer $n$ and an abstract group $H$. Consider the group

$$ G := H^n := H \times \cdots \times H \quad (n \text{ factors}). $$

We have $\mathcal{C}(G) = \mathcal{C}(H)^n$.

Let $F_n$ be a free group of rank $n$ with a free system of generators $x_1, \ldots, x_n$. For any elements $g = (h_1, \ldots, h_n) \in G$, where $h_j \in H$, and $w \in F_n$, denote by $w(g)$ the element of $H$, which is the image of the element $w$ under the homomorphism $F_n \to H$, mapping $x_j$ to $h_j$ for every $j$.

Every element $\sigma \in \text{End}(F_n)$ determines the map

$$ \hat{\sigma} : G \to G, \quad g \mapsto (\sigma(x_1)(g), \ldots, \sigma(x_n)(g)). \tag{29} $$

It is not hard to see that

$$ \hat{\sigma} \circ \hat{\tau} = \hat{\tau} \circ \hat{\sigma} \quad \sigma, \tau \in \text{End}(F_n), $$

$$ \hat{\text{id}}_{F_n} = \text{id}_G. \tag{30} $$

It follows from (29) and the definition of $w(g)$ that

(i) $\hat{\sigma}(S^n) \subseteq S^n \quad S \in H$;

(ii) $\hat{\sigma}(gz) = \hat{\sigma}(g)\hat{\sigma}(z) \quad g \in G, z \in \mathcal{C}(G)$. 
In particular, the restriction of the map $\hat{\sigma}$ to the group $C(G)$ is its endomorphism.

From (30) it follows that if $\sigma \in \text{Aut}(F_n)$, then $\hat{\sigma}$ is a bijection (but, in general, not an automorphism of the group $G$). Moreover, if $H$ is an algebraic group (respectively, a Lie group), then $\hat{\sigma}$ is an automorphism of the algebraic variety (respectively, a diffeomorphism of the differentiable manifold) that is $G$.

Consider now an element $\sigma \in \text{Aut}(F_n)$ and a subgroup of $C$ in $C(G)$. Then from (ii) it follows $C$-equivariance of the bijection $\hat{\sigma}: G \to G$ if we assume that every element $c \in C$ acts on the left copy of $G$ as the translation (multiplication) by $c$, and on the right one as the translation by $\hat{\sigma}(c)$. The quotient for the first action is the group $G/C$, and for the second is the group $G/\hat{\sigma}(C)$. Hence $\hat{\sigma}$ induces a bijection $G/C \to G/\hat{\sigma}(C)$. Moreover, if $H$ is an algebraic group (respectively, real Lie group), then this bijection is an isomorphism of algebraic varieties (respectively, a diffeomorphism of differentiable manifolds); see [6, 6.1]. Thus, $G/C$ and $G/\hat{\sigma}(C)$ are isomorphic algebraic varieties (respectively, diffeomorphic differentiable manifolds). But, in general, $G/C$ and $G/\hat{\sigma}(C)$ are not isomorphic as algebraic groups (respectively, as Lie groups).

Indeed, take for $H$ a simply connected semisimple algebraic group (respectively, a real compact real Lie group). Then $G$ is also a simply connected algebraic group (respectively, a real compact Lie group), so the group $C(G)$ is finite. Consider the natural epimorphisms $\pi: G \to G/C$ and $\pi_{\hat{\sigma}(C)}: G \to G/\hat{\sigma}(C)$. Since the group $C$ is finite, the differentials

$$d_{\pi_C}: \text{Lie}(G) \to \text{Lie}(G/C) \quad d_{\pi_{\hat{\sigma}(C)}}: \text{Lie}(G) \to \text{Lie}(G/\hat{\sigma}(C))$$

are the Lie algebra isomorphisms. Suppose there is an isomorphism of algebraic groups (respectively, real Lie groups) $\alpha: G/C \to G/\hat{\sigma}(C)$. Then

$$\left(d_{\pi_{\hat{\sigma}(C)}}\right)^{-1} \circ d_{\alpha} \circ d_{\pi_C}: \text{Lie}(G) \to \text{Lie}(G)$$

is the Lie algebra automorphism of $\text{Lie}(G)$. Since $G$ is simply connected, it has the form $d\tilde{\alpha}$ for some automorphism $\tilde{\alpha} \in \text{Aut}(G)$ (see [19, Thm. 6, p. 30]). It follows from the construction that the diagram

$$\begin{array}{ccc}
G & \overset{\tilde{\alpha}}{\longrightarrow} & G \\
\pi_C \downarrow & & \downarrow \pi_{\hat{\sigma}(C)} \\
G/C & \overset{\alpha}{\longrightarrow} & G/\hat{\sigma}(C)
\end{array}$$

is commutative, which, in turn, implies that $\tilde{\alpha}(C) = \hat{\sigma}(C)$. 


Thus, the algebraic groups (respectively, real Lie groups) \( G/C \) and \( G/\hat{\sigma}(C) \) are isomorphic if and only if the subgroups \( C \) and \( \hat{\sigma}(C) \) of the group \( G \) lie in the same orbit of the natural action of the group \( \text{Aut}(G) \) on the set of all subgroups of the group \( \mathcal{G}(G) \). This action is reduced to the action of the group \( \text{Out}(G) \) (isomorphic to the automorphism group of the Dynkin diagram of the group \( G \); see [19, Chap. 4, §4, no. 1]) because the group \( \text{Int}(G) \) acts on \( \mathcal{G}(G) \) trivially. It is not difficult to find \( H, \sigma, \) and \( C \) such that the groups \( C \) and \( \hat{\sigma}(C) \) do not lie in the same \( \text{Out}(G) \)-orbit. Here is a concrete example.

7. Example. Consider a simply connected simple algebraic (respectively, real compact Lie) group \( H \) with a nontrivial center. Take \( n = 2 \), so that

\[
G = H \times H. \tag{31}
\]

Let the element \( \sigma \in \text{End}(F_2) \) be defined by the equalities

\[
\sigma(x_1) = x_1, \quad \sigma(x_2) = x_1 x_2^{-1}; \tag{32}
\]

clearly, \( x_1, x_1 x_2^{-1} \) is a free system of generators of the group \( F_2 \), so \( \sigma \in \text{Aut}(F_2) \). Let \( S \) be a non-trivial subgroup of the group \( \mathcal{G}(H) \). Take

\[
C := \{(s, s) \mid s \in S\}. \tag{33}
\]

Then from (29), (32), (33) it follows that

\[
\hat{\sigma}(C) = \{(s, e) \mid s \in S\}. \tag{34}
\]

In view of simplicity of the group \( H \), the elements of \( \text{Out}(G) \) carry out permutations of the factors on the right-hand side of the equality (31). This, (33), and (34) imply that \( C \) and \( \hat{\sigma}(C) \) do not lie in the same \( \text{Out}(G) \)-orbit. Therefore,

\[
G/C = (H \times H) / C \quad \text{and} \quad G/\hat{\sigma}(C) = (H/S) \times H
\]

are non-isomorphic connected semisimple algebraic groups (respectively, real compact Lie groups), whose underlying varieties (respectively, manifolds) are isomorphic (respectively, diffeomorphic).

For example, let \( H = \text{SL}_d, d \geq 2 \), and \( S = \langle z \rangle \), where \( z = \text{diag}(\varepsilon, \ldots, \varepsilon) \in H, \varepsilon \in k \) is a primitive \( d \)-th root of 1. In this case, we obtain non-isomorphic algebraic groups

\[
G/C = (\text{SL}_d \times \text{SL}_d) / \langle (z, z) \rangle, \quad G/\hat{\sigma}(C) = \text{PSL}_d \times \text{SL}_d,
\]

whose underlying varieties are isomorphic. Note that if \( d = 2 \), then \( G = \text{Spin}_4, G/C = \text{SO}_4 \).

For \( H = \text{SU}_d \) and the same group \( S \), we obtain that

\[
G/C = K_1 := (\text{SU}_d \times \text{SU}_d) / C, \quad G/\hat{\sigma}(C) = K_2 := \text{PU}_d \times \text{SU}_d
\]
are non-isomorphic connected semisimple compact real Lie groups whose underlying manifolds are diffeomorphic. For $d = p^r$ with prime $p$, this is proved in [3, p. 331], where non-isomorphness of the groups $K_1$ and $K_2$ is deduced from the non-isomorphness of their Pontryagin rings $H_*(K_1, \mathbb{Z}/p\mathbb{Z})$ and $H_*(K_2, \mathbb{Z}/p\mathbb{Z})$ (discribing these rings is a non-trivial problem). Note that if $d = 2$, then

$$K_1 = \text{SO}_4, \quad K_2 = \text{SO}_3 \times \text{SU}_2.$$ (35)

That the underlying manifolds (35) are diffeomorphic was known long ago: in [13, Chap. 3, §3.D], a diffemorphism between them is constructed by means of quaternions.

8. The case of connected simple algebraic groups. The following theorem shows that the phenomenon under exploration is not possible for simple groups.

**Theorem 22.** Let $G_1$ and $G_2$ be algebraic groups, one of which is connected and simple. The following properties are equivalent:

(a) the underlying varieties of the groups $G_1$ and $G_2$ are isomorphic;

(b) algebraic groups $G_1$ and $G_2$ are isomorphic.

**Proof.** Let the group $G_1$ be connected and simple.

Suppose (a) holds. Let $\tilde{G}_1$ be a simply connected algebraic group with the Lie algebra isomorphic to $\text{Lie}(G_1)$. Then $G_1$ is isomorphic to $\tilde{G}_1/Z_1$ for some subgroup $Z_1$ of $\mathcal{C}(\tilde{G}_1)$. From Theorem 20 it follows that the group $G_2$ is isomorphic to $\tilde{G}_1/Z_2$ for some subgroup $Z_2$ of $\mathcal{C}(\tilde{G}_1)$. As explained above, statement (b) is equivalent to the property that the subgroups $Z_1$ and $Z_2$ lie in the same orbit of the natural action of the group $\text{Out}(\tilde{G}_1)$ (isomorphic to the automorphism group of the Dynkin diagram of the group $\tilde{G}_1$) on the set of all subgroups of the group $\mathcal{C}(\tilde{G}_1)$. We shall show that the subgroups $Z_1$ and $Z_2$ indeed lie in the same $\text{Out}(\tilde{G}_1)$-orbit.

Since the fundamental groups of topological manifolds $G_1$ and $G_2$ are isomorphic to, respectively, $Z_1$ and $Z_2$, it follows from (a) that that the finite groups $Z_1$ and $Z_2$ are isomorphic. Let $d$ be their order.

The structure of the group $\mathcal{C}(\tilde{G}_1)$ is known (see [19, Table 3, pp. 297–298]). Namely, if the type of the simple group $\tilde{G}_1$ is different from

$$D_\ell \text{ with even } \ell \geq 4,$$ (36)

then $\mathcal{C}(\tilde{G}_1)$ is a cyclic group. In the case of the group $\tilde{G}_1$ of type (36), the group $\mathcal{C}(\tilde{G}_1)$ is isomorphic to the Klein four-group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since there is at most one subgroup of a given finite order in any cyclic
group, we get that if the type of $\tilde{G}_1$ is different from (36), then $Z_1 = Z_2$, so in this case, the subgroups $Z_1$ and $Z_2$ lie in the same $\text{Out}(G_1)$-orbit.

Now, let $\tilde{G}_1$ be of type (36). This means that $\tilde{G}_1 = \text{Spin}_{4m}$ for some integer $m \geq 2$. Since $|\mathcal{C}(\tilde{G}_1)| = 4$, only the cases $d = 1, 2, 4$ are possible. It is clear that $Z_1 = Z_2$ for $d = 1$ and 4, so in these cases, as above, the subgroups $Z_1$ and $Z_2$ lie in the same $\text{Out}(G_1)$-orbit. Therefore, it remains to consider only the case $d = 2$.

There are exactly three subgroups of order 2 in $\mathcal{C}(\tilde{G}_1)$. The natural action on $\mathcal{C}(\tilde{G}_1)$ of the group $\text{Out}(\text{Spin}_{4m})$ (isomorphic to the automorphism group of the Dynkin diagram of the group $\tilde{G}_1$) can be easily described explicitly using the information specified in [19, Table 3, p. 297–298]. This description shows that the number of $\text{Out}(\text{Spin}_{4m})$-orbits on the set of these subgroups equals 1 for $m = 2$ and equals 2 for $m > 2$. Thus, for $m = 2$, the groups $G_1$ and $G_2$ are isomorphic and it remains for us to consider the case $m > 2$.

The quotient group of the group $\text{Spin}_{4m}$ by a subgroup of order 2 in $\mathcal{C}(\tilde{G}_1)$, which is not fixed (respectively, fixed) with respect to the group $\text{Out}(\text{Spin}_{4m})$, is the half-spin group $\text{SSpin}_{4m}$ (respectively, the orthogonal group $\text{SO}_{4m}$). Let $\text{SSpin}_{4m}$ and $\text{SO}_{4m}$ be the compact real forms of the groups $\text{SSpin}_{4m}$ and $\text{SO}_{4m}$, respectively. If the underlying varieties of the groups $\text{SSpin}_{4m}$ and $\text{SO}_{4m}$ were isomorphic, then the underlying manifolds of the groups $\text{SSpin}_{4m}$ and $\text{SO}_{4m}$ were homotopy equivalent. But according to [3, Thm. 9.1], for $m > 2$, they are not homotopy equivalent because $H^*(\text{SSpin}_{4m}, \mathbb{Z}/2\mathbb{Z})$ and $H^*(\text{SO}_{4m}, \mathbb{Z}/2\mathbb{Z})$ for $m > 2$ are not isomorphic as algebras over the Steenrod algebra. Hence the underlying varieties of the groups $\text{SSpin}_{4m}$ and $\text{SO}_{4m}$ for $m > 2$ are not isomorphic. This completes the proof of implication (a)$\Rightarrow$(b). Implication (b)$\Rightarrow$(a) is obvious.

Considerations used in the proof of Theorem 22 yield a proof of the following Theorem 23, which was published in [3] without proof.

**Theorem 23 ([3, Thm. 9.3])**. The underlying manifolds of two connected real compact simple Lie groups are homotopy equivalent if and only if these Lie groups are isomorphic.

---

1 In the notation of [19, Table 3, p. 297–298], for $m = 2$, each permutation of the vectors $h_1, h_3, h_4$ with fixed $h_2$ is realized by some automorphism of the Dynkin diagram (identified with the corresponding outer automorphism), and for $m > 2$, the only non-trivial automorphism of the Dynkin diagram swaps $h_{2m}$ and $h_{2m-1}$ and leaves all the rest $h_i$'s fixed.

2 Note that $H^*(\text{SSpin}_{n}, \mathbb{Z}/2\mathbb{Z})$ and $H^*(\text{SO}_{n}, \mathbb{Z}/2\mathbb{Z})$ are isomorphic as algebras over $\mathbb{Z}/2\mathbb{Z}$ if (and only if) $n$ is a power of 2, see [3, p. 330].
Proof. It repeats the proof of Theorem 22 if in it one assumes that $G_1$ and $G_2$ are connected real compact simple Lie groups, whose underlying manifolds are homotopy equivalent, and replaces $\text{Spin}_{4m}$, $\text{SSpin}_{4m}$, and $\text{SO}_{4m}$, respectively, with $\text{Spin}_{4m}$, $\text{SSpin}_{4m}$, and $\text{SO}_{4m}$. □

9. Questions.

1. Previous considerations naturally lead to the question of finding a classification of pairs of non-isomorphic connected reductive algebraic groups, whose underlying varieties are isomorphic. Is it possible to obtain it?

2. The same for connected real compact Lie groups, whose underlying manifolds are homotopy equivalent.

3. It seems plausible that, using, in the spirit of [5], étale cohomology in place of singular homology and cohomology, it is possible to prove Theorems 5 and 6 and implication $(c) \Rightarrow (a)$ of Theorem 13 in the case of positive characteristic of the field $k$. Are Theorems 19, 20, 22 true for such $k$?

4. The author is not aware of examples of connected simple algebraic groups whose underlying variety is a product of algebraic varieties of positive dimension. Do they exist (N. L. Gordeev’s question)?

5. The problems considered in this paper are obviously reformulated taking into account rationality questions, i.e., definability of varieties over an algebraically non-closed field $\ell$. How are the results of this paper modified in this context? Some of them, for example, Theorem 17, do not change: the above proof of this theorem, with the added remark that the specified left shift is performed by a rational over $\ell$ element, is transferred to the context of definability over $\ell$ and shows that two algebraic groups defined over $\ell$, one of which is a semi-abelian variety, are isomorphic over $\ell$ if and only if their underlying varieties are isomorphic over $\ell$. In particular, for tori or abelian varieties defined over $\ell$, their isomorphism over $\ell$ is equivalent to isomorphism of their underlying varieties over $\ell$.

6. The results of this paper concern the uniqueness of the structure of a connected reductive algebraic group (respectively, real compact Lie group) on an algebraic variety (respectively, differentiable compact manifold) that admits at least one such structure. Is it possible to give a criterion for the existence of at least one such structure on an algebraic variety (respectively, differentiable compact manifold) in terms of its geometric characteristics?

10. Appendix: finiteness theorems for connected reductive algebraic groups and compact Lie groups. In this section, the characteristic of $k$ can be arbitrary.
Theorem 24. The number of all, considered up to isomorphism, connected reductive algebraic groups of any fixed rank is finite.

Proof. For every connected reductive group $G$, there is a torus $Z$ and a simply connected semisimple algebraic group $S$ such that the group $G$ is the quotient group of the group $Z \times S$ by a finite central subgroup. Indeed, let $S$ be the universal covering group of the connected semisimple group $\mathcal{D}(G)$, let $\pi: S \to \mathcal{D}(G)$ be the natural projection, and let $Z = \mathcal{C}(G)^0$. Then the map $Z \times S \to G$, $(z,s) \mapsto z \cdot \pi(s)$ is an epimorphism with a finite kernel, i.e., the natural projection to the quotient group by a finite central subgroup.

Being simply connected, the group $S$ is, up to isomorphism, uniquely determined by the type of its root system. Insofar as the set of types of root systems of any fixed rank is finite, tori of the same dimension are isomorphic, and $\mathcal{C}(S)$ is a finite group, the problem comes down to proving that, although for $\dim(Z) > 0$ there are infinitely many finite subgroups $F$ in $\mathcal{C}(Z \times S)$, the set of all, up to isomorphism, groups of the form $(Z \times S)/F$ is finite. Note that for every element $\sigma \in \text{Aut}(Z \times S)$, the groups $(Z \times S)/F$ and $(Z \times S)/\sigma(F)$ are isomorphic.

Proving this, we put

$$n := \dim(Z) > 0,$$

and let $\varepsilon_1, \ldots, \varepsilon_n$ be a basis of the group $\text{Hom}(Z, \mathbb{G}_m) \simeq \mathbb{Z}^n$.

For every positive integer $r$, denote by $\mathcal{D}_{r \times n}$ the set of all matrices $(m_{ij}) \in \text{Mat}_{r \times n}(\mathbb{Z})$ such that

(a) $m_{ij} = 0$ for $i \neq j$;

(b) $m_{ii}$ divides $m_{i+1,i+1}$;

(c) $m_{ii} = m'_{ii}$ (see the notation in Section 1).

Consider a matrix $M = (m_{ij}) \in \text{Mat}_{r \times n}(\mathbb{Z})$. Then

$$Z_M := \bigcap_{i=1}^r \ker(\varepsilon_{1}^{m_{11}} \cdots \varepsilon_{n}^{m_{nn}})$$

is an algebraic $(n - \text{rk}(M))$-dimensional subgroup of the group $Z$. Every algebraic subgroup of the group $Z$ is obtained in this way. If the matrix $M = (m_{ij})$ shares properties (a) and (b), then $Z_M = Z_{M'}$, where $M' := (m'_{ij})$, because $\ker(\varepsilon_{1}^{m_{11}}) = \ker(\varepsilon_{1}^{m'_{11}})$. If $r = n$, and the matrix $M$ is non-degenerate and shares properties (a), (b), (c), then $Z_M$ is a finite abelian group with the invariant factors $|m_{11}|, \ldots, |m_{nn}|$.

The elementary transformations of rows of the matrix $M$ do not change the group $Z_M$. If $\tau_1, \ldots, \tau_n$ is another basis of the group $\text{Hom}(Z, \mathbb{G}_m)$, then $\tau_i = \varepsilon_{1}^{c_{i1}} \cdots \varepsilon_{n}^{c_{in}}$, where $C = (c_{ij}) \in \text{GL}_n(\mathbb{Z})$. The automorphism of the group $\text{Hom}(Z, \mathbb{G}_m)$, which maps $\varepsilon_i$ to $\tau_i$ for every
i, has the form $\sigma_C$, where $\sigma_C$ is an automorphism of the group $Z$. The mapping $\text{GL}_n(Z) \rightarrow \text{Aut}(Z)$, $C \mapsto \sigma_C$, is a group isomorphism and the following equality holds:

$$Z_{MC} = \sigma_C(Z_M) \tag{38}$$

Since the elementary transformations of the columns of the matrix $M$ are realized by multiplying the matrix $M$ on the right by the corresponding matrices from $\text{GL}_r(Z)$, and by means of the elementary transformations of rows and columns the matrix $M$ can be transformed into its Smith diagonal normal form, (38) implies the existence of an automorphism $\nu \in \text{Aut}(Z)$ and a matrix $D \in \mathcal{D}_{n \times n}$ such that $\nu(Z_M) = Z_D$.

Now consider a finite subgroup $F$ in $\mathcal{C}(Z \times S) = Z \times \mathcal{C}(S)$ and the canonical projections

$$Z \xrightarrow{\pi_Z} F \xrightarrow{\pi_S} \mathcal{C}(S).$$

The groups $(Z \times S)/F$ and $\left( (Z \times S)/(F \cap Z) \right)/(F/(F \cap Z))$ are isomorphic. Being an $n$-dimensional torus, the group $Z/(F \cap Z)$ is isomorphic to the torus $Z$. Therefore, the groups $(Z \times S)/(F \cap Z)$ and $Z \times S$ are isomorphic. Hence, without changing, up to isomorphism, the group $(Z \times S)/F$, we can (and will) assume that $F \cap Z = \{e\}$. Then $\ker(\pi_S) = \{e\}$, and therefore, $\pi_S$ is an isomorphism between $F$ and the subgroup $\pi_S(F)$ in the group $\mathcal{C}(S)$. Let $\alpha : \pi_S(F) \rightarrow \pi_Z(F)$ be an epimorphism that is the composition of the isomorphism inverse to $\pi_S$ with $\pi_Z$. Then

$$F = \{\alpha(g) \cdot g \mid g \in \pi_S(F)\}.$$

The subgroup $\pi_Z(F) = \alpha(\pi_S(F))$ in $Z$ is finite and therefore has the form $Z_M$ for some non-degenerate matrix $M \in \text{Mat}_{n \times n}(Z)$. According to the above, there is an element $\nu \in \text{Aut}(Z)$ such that $\nu(\pi_Z(F)) = Z_D$, where $D$ is a non-degenerate matrix from $\mathcal{D}_{n \times n}$; we denote by the same letter the extension of $\nu$ up to an element of $\text{Aut}(Z \times S)$, which is the identity on $S$. Replacing the group $F$ by the group $\nu(F)$ shows that, without changing, up to isomorphism, the group $(Z \times S)/F$, we can (and will) assume that $\pi_Z(F) = Z_D$.

Thus, if $\mathcal{F}$ is the set all subgroups in $Z \times \mathcal{C}(S)$ of the form

$$\{\gamma(h) \cdot h \mid h \in H\},$$

where $H$ runs through all subgroups of $\mathcal{C}(S)$, and $\gamma$ through all epimorphisms $H \rightarrow Z_D$ with a non-degenerate matrix $D \in \mathcal{D}_{n \times n}$, then $F \in \mathcal{F}$. Since the group $\mathcal{C}(S)$ is finite, and the order of the group $Z_D$ is $|\det(D)|$, the set $\mathcal{F}$ is finite. This completes the proof. \[\square\]

**Theorem 25.** The number of all, considered up to isomorphism, root data of any fixed rank is finite.
Proof. This follows from Theorem 24 because connected reductive algebraic groups are classified by their root data, see [29, Thms. 9.6.2, 10.1.1]. □

Theorem 26. The number of all, considered up to isomorphism, connected real compact Lie groups of any fixed rank is finite.

Proof. This follows from Theorem 24 in view of the correspondence between connected reductive algebraic groups and connected real compact Lie groups, given by passing to a real compact form, see [19, Thm. 5.2.12]. □

Remark 27. Using this proof of Theorem 24, one can obtain an upper bound for the numbers specified in it and in Theorems 25 and 26.

Remark 28. Since the rank does not exceed the dimension of the group, Theorem 24 (respectively, Theorem 26) shows that the number of all, considered up to an isomorphism, connected reductive algebraic groups (respectively, connected real compact Lie groups) of any fixed dimension is finite.

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