Modeling Magnetohydrodynamic Equilibrium in Magnetars with Applications to Continuous Gravitational Wave Production

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ABSTRACT
Possessing the strongest magnetic fields in the Universe, magnetars mark an extremum of physical phenomena. The strength of their magnetic fields is sufficient to deform the shape of the stellar body, and when the rotational and magnetic axes are not aligned, these deformations lead to the production of gravitational waves (GWs) via a time-varying quadrupole moment. Such gravitational radiation differs from signals presently detectable by the Laser Interferometer Gravitational-Wave Observatory. These signals are continuous rather than the momentary ‘chirp’ waveforms produced by binary systems during the phases of inspiral, merger, and ringdown. We evaluate the sensitivity requirements of future iterations of GW detectors to continuous GW signals resulting from magnetars. Here, we construct a computational model for magnetar stellar structure with strong internal magnetic fields. We implement an \( n = 1 \) polytropic equation of state (EOS) and adopt a mixed poloidal and toroidal magnetic field model constrained by the choice of EOS. We utilize fiducial values for magnetar magnetic field strength and various stellar physical attributes. Via computational simulation, we measure the deformation of magnetar stellar structure to determine upper bounds on the strength of continuous GWs formed as a result of these deformations inducing non-axisymmetric rotation. We compute predictions of upper limit GW strain values for sources in the McGill Magnetar Catalog, an index of all detected magnetars.

Key words: stars: magnetars – gravitational waves – MHD

1 INTRODUCTION
Magnetars are an exceptional classification of pulsars, characterized by surface magnetic field strengths in excess of \( 10^{14} \) G and dipolar magnetic energies exceeding the star’s rotational energy (Thompson & Duncan 1995). Olausen & Kaspi (2014) provide a catalog of 23 confirmed and 6 candidate sources, and document considerable progress in magnetar detection via γ-ray burst events in recent years following the launch of the Swift and Fermi space telescopes. Given the rapid growth in confirmed magnetar sources, these stars present a wealth of opportunity for improving current understanding regarding the influence of strong magnetic fields in extreme stellar environments.

Chandrasekhar & Fermi (1953) first showed for an incompressible stellar model that a strong internal magnetic field will deform a star away from spherical symmetry. For deformations induced along a magnetic field axis which is misaligned with the stellar rotational axis, a time-varying gravitational quadrupole will result in the production of gravitational waves (GWs). Thus, magnetars are compelling candidates for the detection of GWs from deformed stellar sources.

Such GWs differ from former event detections, as unlike the ‘chirp’ waveform of binary inspiral mergers, GWs produced by a rapidly rotating stellar source are nearly constant-frequency, sinusoidal signals due to the source returning to the same spatial configuration in the span of a complete revolution about its rotational axis. Due to the consistent periodicity of these GW signals, they are referred to as ‘continuous’ GWs. Under extended survey, stellar spin down due to loss in rotational kinetic energy through magnetic braking or energy loss in the form of gravitational radiation will increase the rotational period and GWs emitted will drift to lower frequencies (Creighton & Anderson 2011). However, under shorter observation, continuous GWs appear as constant frequency sinusoidal waveforms.

Continuous GWs are expected to be detected following improvements in GW detector sensitivity, as their signals are often far fainter than GWs produced by binary inspiral
events. Evaluation of their signal strength, or wave strain, can be made by estimating the magnitude of stellar deformations responsible for producing such signals (Zimmermann & Szedenits 1979).

Recent work places upper limits on the GW strain of pulsar sources capable of producing GWs within the operating range of the Laser Interferometer Gravitational-Wave Observatory (LIGO) (Abbott et al. 2017). The authors compute the spin-down limit; the GW strain sensitivity produced by attributing the loss in rotational kinetic energy completely to gravitational radiation. For a rigidly rotating triaxial star, the frequency of gravitational waves produced by the source will be twice the rotational frequency. As magnetars are slowly rotating stars ($\tau \sim 5$–8 s) (Olausen & Kaspi 2014), GWs produced by these sources fall outside the sensitivity range of LIGO and corresponding wavestrain estimates were not addressed by Abbott et al. (2017).

The principal goal of this paper is to provide estimates for upper-limit calculations of the GW strain for all confirmed magnetar sources in the McGill Magnetar Catalog (Olausen & Kaspi 2014) by constructing a computational model for magnetar stellar structure and magnetic field configuration. We determine the degree of structural deformation introduced by a strong internal magnetic field as the stellar structure reaches magnetohydrodynamic (MHD) equilibrium. These results for stellar deformation subsequently inform wavestrain estimates.

Unless otherwise specified, we utilize the following fiducial values for stellar attributes: stellar mass, $M_\star = 1.4 \, M_\odot$; stellar radius, $R_\star = 10$ km; and central density, $\rho_c = 2.0 \times 10^{15} \, \text{g} \cdot \text{cm}^{-3}$.

2 STRUCTURAL MODEL

Prior authors (Owen (2005) and references therein) note that cumulative errors introduced by excluding relativistic gravity and rotational effects largely cancel; while relativistic effects result in a more compact model of stellar structure than the Newtonian framework, stellar rotation has an opposing effect. Thus, in constructing a stellar model, we adopt the Newtonian gravitational theory and neglect rotational effects.

2.1 Hydrostatic Equilibrium Conditions

Our choice of stellar model is constrained to configurations which are in static equilibrium. Thus, the construction of this stellar model requires a crucial balance between the force of gravity and stellar structure. The equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -G M_r \rho \frac{2}{r^2},$$

where $P$ and $\rho$ are the stellar pressure and density, respectively, and $M_r$ is the mass interior to the radius for $r < R_\star$, provides the basis for balancing the gravitational force with structural variation throughout the stellar interior.

The interior mass changes with radius, and thus introduces the following expression for mass conservation within the stellar medium:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho.$$

2.2 Polytropic Equations of State

The time-independent equations of hydrostatic equilibrium and mass conservation provide an initial description of stable Newtonian stars. In order to fully specify stellar structure, an equation of state (EOS) is required to relate pressure to a number of state variables describing stellar structure. We adopt a barytropic EOS, which defines the relationship between pressure and density as $P(\rho)$. While an EOS parameterized by numerous state variables such as $P(p, x, T, ...)$ is more physically representative of the interior of a neutron star, subsequent discussion will show that our particular choice of barytropic EOS allows analytic expressions for stellar structure and magnetic field expressions. We use a polytropic EOS of the form

$$P(\rho) = K \rho^n,$$

where $K$ is the polytropic constant and the real, positive constant $n$ is defined via the polytropic index $n$ as

$$n = \frac{n + 1}{n}.$$

Polytropic equations of state are often categorized by the compressibility of stellar matter, whereby a lowering of the constant $n$ corresponds to lower compression (Haensel et al. 2007). Thus, the structural composition of the stellar interior sets a constraint on representative equations of state.

Prior work has established neutron star structure as well approximated by the choice of polytropic EOS corresponding to $0 < n \lesssim 1$ (Cho & Lee (2010), Woosley (2014)). Under this consideration, we implement an $n = 1$ polytropic EOS in modeling neutron star structure. An expression for density as a function of radius can be determined via solutions to the Lane-Emden equation for a specified polytropic index, $n$.

The $n = 1$ polytropic possesses the following analytic solutions for $\rho(r)$ and $P(r)$ from which stellar structure can be fully determined:

$$\rho(r) = \rho_c \frac{\sin((n+1)\pi (r/R_\star))}{(n+1)\pi} \frac{R_\star}{r}$$

and

$$P(r) = K \rho_c^2 \left(\frac{r}{R_\star}\right)^2$$

where the polytropic constant $K = 4.25 \times 10^4 \, \text{cm}^5 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$ for a neutron star with radius $R_\star = 10$ km, mass $M_\star = 1.4 \, M_\odot$, and central density $\rho_c = 2.0 \times 10^{15} \, \text{g} \cdot \text{cm}^{-3}$.

2.3 Gravitational Potential Model

Hydrostatic equilibrium requires the balance of an inward gravitational force with the radial change in pressure. We determine solutions to the spherically symmetric form of Poisson’s equation for gravitational potential per unit mass,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi_k}{dr}\right) = 4\pi G \rho.$$

Solutions to $\Phi_k$ interior and exterior to the stellar surface are constrained by our determined expression for density given by Equation 5. Additionally, these solutions must satisfy the
following boundary conditions:

\[
\frac{d\Phi_g}{dr} = 0 \quad \text{at} \quad r = 0, \quad (8a)
\]

\[
\Phi_g \bigg|_{r = R^*_\text{inside}} = \Phi_g \bigg|_{r = R^*_\text{outside}}, \quad (8b)
\]

\[
\frac{d\Phi_g}{dr} \bigg|_{r = R^*_\text{inside}} = \frac{d\Phi_g}{dr} \bigg|_{r = R^*_\text{outside}}, \quad (8c)
\]

Using separation of variables and substitution to solve Equation 7 for \(\Phi_g(r)\), we determine the following expressions for the gravitational potential:

\[
\Phi^\text{core}_g = 4G\rho_c \left( -\frac{R^2}{r} - \frac{M}{4R^*_\text{poly} \rho_c} \right), \quad \text{for} \quad r = 0 \quad (9a)
\]

\[
\Phi^\text{inside}_g(r) = 4G\rho_c \left( -\frac{R^2}{r} \sin(\pi r/R_*) \right) + \frac{M}{4R^*_\text{poly} \rho_c}, \quad \text{for} \quad 0 < r < R_* \quad (9b)
\]

\[
\Phi^\text{outside}_g(r) = -\frac{GM}{r}, \quad \text{for} \quad r > R_* \quad (9c)
\]

As the stellar model evolves and the magnetic field induces morphological changes in the stellar structure, the model’s gravitational potential remains static; the potential adheres to the form determined here as assigned via initial conditions of the simulation. This approximation is referred to as the Cowling approximation, and provides considerable accuracy under direct comparison between static and dynamic potentials for modeling stellar structure which, under evolution, become slightly perturbed from initial conditions (Yoshida & Kojima (1997), Yoshida (2013)).

2.4 Structural Validation

We test the condition of hydrostatic equilibrium for our stellar model by generating a 3-D spherical computational simulation of an \(n = 1\) polytrope with unit radius \(R_*\) = 1 in dimensionless code units and accompanying gravitational potential in the form of Equation 9. The computational simulation is composed of finite-volume voxels, each assigned structural parameter values interpolated via the analytic form of Equations 5, 6, and 9.

Throughout the simulation, we monitor the evolution of the internal energy, defined as

\[
W = \int_V \rho \cdot \frac{dV}{d\Phi} \quad (10)
\]

where the pressure \(P\) is integrated over the stellar volume. Changes to the internal energy give indication of the degree to which the virial theorem is satisfied assuming a static gravitational potential and constant potential energy. We find initial perturbations present in the stellar pressure and density due to the relaxation of interpolated values in the discretized computational domain. These perturbations manifest as minute variations in the internal energy, and Figure 1 shows the evolution of the normalized internal energy. The amplitude of these changes to the internal energy are quite small and do not appear to impact large-scale stellar structure as the simulation is allowed to evolve. We conclude that our simulation is at hydrostatic equilibrium and trivially validates the structural expressions for an \(n = 1\) polytrope.

### 3 MAGNETIC FIELD MODEL

3.1 Hydromagnetic Conditions

We follow the works of Haskell et al. (2008), Roxburgh (1966), and references therein, which determine magnetic field solutions for spherically symmetric stars, treating the magnetic energy as a perturbation of the total stellar energy. In turn, the equation of hydrostatic equilibrium is modified to include a magnetic term due to the Lorentz force as

\[
\nabla P \rho + \nabla \Phi_g = \frac{(\nabla \times B) \times B}{4\pi \rho} = \frac{L}{4\pi \rho} \quad (11)
\]

where \(P\) is the pressure, \(\rho\) is the density, \(\Phi_g\) is the gravitational potential, \(B\) is the vector-valued magnetic field, and \(L\) is the Lorentz force. We refer to Expression 11 as the equation for hydromagnetic equilibrium.

Roxburgh (1966) shows that magnetic field configurations satisfying Equation 11 must meet an additional constraint, determined by taking the curl of the equation for hydromagnetic equilibrium. Since the curl of a gradient is zero, i.e. \(\nabla \times \nabla \Phi = 0\) for \(A\) any scalar-valued function, the left-hand side of Equation 11 vanishes under such operation, and we arrive at the following constraint:

\[
\nabla \times \frac{B \times (\nabla \times B)}{\rho} = 0 \quad (12)
\]

where \(\rho\) is a barotropic EOS of the form \(\rho(P)\). Because the density \(\rho\) is present in this constraint, Expression 12 must be solved alongside Equation 11. Thus, the choice of barotropic EOS constrains allowable magnetic field models, motivating our reasoning for choosing an EOS and determining structural expressions in the preceding section. Subsequent discussion will specialize to magnetic field solutions which adhere to the EOS determined for the \(n = 1\) polytrope in Equations 5 and 6.

Additionally, we ensure that our magnetic field model adheres to Maxwell’s equations, where particular significance is given to preserving the divergence condition,

\[
\nabla \cdot B = 0 \quad (13)
\]

for the purposes of accurate magnetic field evolution in MHD simulations.

3.2 Mixed Field Equations

The choice of magnetic field configuration requires careful consideration of dynamically stable models which preserve
field geometry under evolution and unallowable configurations which rapidly evolve and alter the stellar field structure. G. Flowers & A. Ruderman (1977) discuss the instability of pure-poloidal stellar magnetic fields with uniform, unclosed field lines in the stellar interior. Pure toroidal configurations are also unstable, as instabilities form along the magnetic axis (Tayler 1973). Mixed magnetic fields, including both poloidal and toroidal components, offer promising stable configurations (G. Flowers & A. Ruderman (1977), Braithwaite & Spruit (2006)).

With consideration to stable field configurations, we adopt a mixed magnetic field model derived by Haskell et al. (2008) for an $n = 1$ polytropic EOS. Haskell et al. (2008) find that for the $n = 1$ polytrope and for eigenvalues $\lambda$, solutions to the mixed magnetic field configuration take the form

$$B = \left\{ \frac{2A \cos \theta}{r^2}, \frac{-A' \sin \theta}{r}, \frac{\pi A \sin \theta}{rR_\star} \right\},$$

where $A$ is

$$A = \frac{B_k R_\star^2}{(\lambda^2 - 1)^2 y} \left[ \frac{2\pi}{\lambda y} \cos(\lambda y) - \sin(\lambda y) \right] \frac{\pi A \cos(\lambda y) - \sin(\lambda y)}{\pi \lambda} + \left[ (1 - \lambda^2) y^2 - 2 \right] \sin(y) + 2y \cos(y) \right] .$$

The dimensionless radial parameter $y$ has been implemented to simplify Equation 15, where

$$y = \frac{\pi r}{R_\star}.$$ 

The constant $B_k$ sets the strength of the magnetic field. The field strength at the stellar surface, which we label $B_\star$, imposes a constraint on the value of $B_k$, as we wish for the value of $B_k$ to adhere to magnetar surface field strengths of order $10^{15}$ G. By computing the average field strength for voxels along the surface of the stellar medium in the computational domain, we experimentally determine that a value of $B_k = 8 \times 10^{16}$ G results in an average surface field strength $B_\star \sim 1.59 \times 10^{15}$ G, with a maximum equal to $B_\star^{\text{max}} \sim 2.02 \times 10^{15}$ G in the equatorial plane of the star. This assignment is consistent with the notion that internal magnetic field strengths can range up to a few orders of magnitude higher than surface fields (Haensel et al. 2007).

3.3 Magnetic Field Model Stability Validation

The stability of a magnetic field configuration is dependent on its evolution under Alfvén time scales, which define the period necessary for tension-induced Alfvén waves to propagate throughout the magnetic field. These waves determine the geometric evolution of the field configuration, and thus provide a strong basis for studying the stability of stellar magnetic fields (Goedbloed & Poedts 2004). For a homogeneous plasma with uniform density $\rho_0$ and magnetic field strength $B_0$, the velocity of an Alfvén wave is

$$v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}.$$ 

As density and magnetic field strength vary in our model, we determine a volume averaged value for the Alfvén velocity, $\bar{v_A}$, where $\bar{v_A} \approx 2.736 \times 10^9$ cm$^2$s$^{-1}$. The Alfvén crossing time for wave propagation is then

$$t_A = \frac{d}{v_A}.$$ 

where $d$ is the wavelength of the Alfvén wave, which is approximated in the stellar interior by the radius $R_\star \approx 10$ km (Suzuki & Nagataki 2005). We compute the volume-averaged Alfvén crossing time for our model to be $t_A \approx 0.4$ ms, in agreement with prior evaluation of the Alfvén crossing time for interior magnetic fields in highly magnetic neutron stars (Suzuki & Nagataki 2005).

In assessing the stability of our model's magnetic field configuration, the computed Alfvén crossing time indicates that robust analysis of the field's stability may be conducted by analyzing the field configuration after several Alfvén crossings. We conduct stability analysis of the magnetic field configuration by comparing the geometry of the initial field configuration to the evolved state after 100 Alfvén crossings.

We use streamlines to label the geometry of the field. Streamlines represent the trajectories of fluid elements in the presence of an axisymmetric stellar magnetic field, and evolution of their form provides immediate awareness of changes to the magnetic field structure. We plot streamlines for both the poloidal component field in Figures 2 and 3 and for the toroidal component field in Figures 4 and 5. After 100 Alfvén crossings, consistent arrangement of magnetic field streamlines indicate little change in the structure of the field, and we conclude that the field configuration is well preserved. Our findings provide evidence that the chosen magnetic field expression (Equations 14 and 15) correspond to a stable configuration.
gravitational radiation will be produced with wave strain

$$h_0 = \frac{4\pi^2G I_0 f_{SW}^2}{c^4} \epsilon,$$  \hfill (19)$$

where we assume that the rotational axis, taken to be the z-axis, is optimally pointed towards an observer on Earth, and where the ellipticity, $\epsilon$, is a measure of stellar deformation and is defined as

$$\epsilon = \frac{I_{zz} - I_{xx}}{I_0}.$$  \hfill (20)$$

If the stellar ellipticity is negative, whereby $I_{zz} < I_{xx}$, the star is considered prolate. Conversely, a positive stellar ellipticity, such that $I_{zz} > I_{xx}$, corresponds to an oblate star. These stellar ellipticity scenarios are illustrated in Figure 6.

Calculation of the continuous GW strain, $h_0$, depends on the degree to which the distribution of mass is spherically non-uniform about the rotational axis, i.e., when $I_{zz} \neq I_{xx}$. The presence of a strong internal magnetic field in magnetars modifies the equilibrium configuration of the stellar structure, perturbing the density profile, $\rho$, through quadrupolar ($\ell = 2$) deformations. Perturbation of the density expression changes the ellipticity via modification of the principal moments of inertia: $I_{xx}$, $I_{yy}$, and $I_{zz}$. 

**Figure 3.** Streamlines for the poloidal component field after 100 Alfvén crossings.

**Figure 4.** Streamlines for the initial configuration of the toroidal component field. The colormap corresponds to the strength of the magnetic field along computed streamlines.

**Figure 5.** Streamlines for the toroidal component field after 100 Alfvén crossings.

**Figure 6.** Three spheroids representing stars of varying ellipticity. From left to right, a prolate spheroid with $\epsilon < 0$, a uniform sphere with $\epsilon = 0$, and an oblate spheroid with $\epsilon > 0$. 

$I_{zz}$ and $I_{xx}$ are principal moments of inertia, determined via the inertia tensor,

$$I_{jk} = \int_V \rho(r)(r^2\delta_{jk} - x_j x_k) dV.$$  \hfill (21)$$
4.1 Modeling Deformations in the Computational Domain

We use the astrophysical fluid dynamics code PLUTO (Mignone et al. 2007) to specify and simulate our computational stellar model. Flux computation is made via the Hartman, Lax, Van Leer (hllc) solver. The data visualization platform VisIt (Childs et al. 2012) is used to analyze simulation data, including evaluation of the moment of inertia tensor in a specified computational domain. We determine \( I_{zz} \) and \( I_{xx} \) by evaluating the moment of inertia tensor over domain voxels for which \( r < R_* \) at simulation time steps of 100 ms.

As the inertia tensor (Equation 21) is a volume integral over the stellar interior, we anticipate numerical limitations on the accuracy of computed values for \( I_{zz} \) and \( I_{xx} \), as each tensor component must be computed over a discretized domain of finite three-dimensional voxels.

For the initial configuration of the stellar model at \( t = 0 \) s, the inertia tensor is expressed as

\[
I_{jk} = \int_V \rho(r, t = 0) \left( r^2 \delta_{jk} - x_j x_k \right) dV.
\]

(22)

where \( \rho(r, t = 0) \) takes the form of Equation 5, such that the analytic evaluation of Equation 22 for all principal moments of inertia gives

\[
I_0 = I_{xx} = I_{yy} = I_{zz} = \frac{8(\pi^2 - 6)R_*^3 \rho_c}{3\pi^3}.
\]

(23)

The initial error in numerically computing \( I_{zz} \) and \( I_{xx} \) is determined via the difference between each numerically computed tensor component at simulation time \( t = 0 \) and the analytic result of expression 23. We plot the absolute value of the initial error for \( I_{xx}(t = 0) \) and \( I_{zz}(t = 0) \) as a function of voxel resolution in Figures 7 and 8, respectively. The radial resolution of the spherical mesh is kept constant over domain voxels for which \( r < R_* \).

Although higher angular resolution allows greater precision in both \( I_{xx} \) and \( I_{zz} \), such improvements come at the cost of greater wall time, or the elapsed real time necessary to complete a computational modelling run through a specified simulation duration. Thus, consideration is given to balancing the trade-off between resolution and compute time, and an angular resolution of \( n_{\theta,\phi} = 30 \) is selected for simulations.

Crucial to the evaluation of the ellipticity, \( \epsilon \), is the difference \( I_{zz} - I_{xx} \), which we refer to as \( \Delta I \). Because the finite-difference integration scheme for these inertia tensor components over the spherical mesh provide slightly different values for \( I_{zz}(t = 0) \) and \( I_{xx}(t = 0) \), the value

\[
\delta I_{xx} = |I_{zz}(t = 0) - I_{xx}(t = 0)| \neq 0
\]

(25)

is of significance, representing a systematic error in our evaluation of \( I_{xx} \) and \( I_{zz} \). Therefore, we represent numerical evaluations for these tensor components as

\[
I_{xx} = I_{xx} \pm \delta I_{xx}
\]

(26)

and

\[
I_{zz} = I_{zz} \pm \delta I_{zz}.
\]

(27)

4.2 Deformation Results

In order to determine whether our simulated deformation results are in accordance with expectation, we simulate the instance of stellar hydrostatic equilibrium by removing the magnetic field model. For the instance of hydrostatic equilibrium, the null hypothesis is that \( I_{xx} \) and \( I_{zz} \) do not change from their initial configuration, such that \( I_{xx} = I_{zz} \) and \( \epsilon = 0 \).

The evolution over simulation time of \( I_{xx} \) and \( I_{zz} \) for...
Figure 9. Quadratic regression of elapsed real (wall) time to solution \((t = 1 \text{ s})\) against angular resolution \((n_{\theta, \phi} = \pi/9)\). Higher resolution causes increasingly prohibitive compute time.

The instance of hydrostatic equilibrium is displayed in Figure 10. Both inertia components are assigned an error margin as expressed in equations 26 and 27, represented by the lighter shaded regions surrounding each curve.

Because the error margins for both \(I_{xx}\) and \(I_{zz}\) overlap for the duration of the simulation, we strictly cannot distinguish a non-zero value for the ellipticity, \(\epsilon\). Thus, we verify the trivial null hypothesis for hydrostatic equilibrium. As an aside, we note that for Figures 9 and 11, the moment of inertia is given in \(\text{g}\cdot\text{cm}^{-2}\) because the choice of normalized radius, \(R_\star = 1.0\), leaves expressions for the moment of inertia such as the analytic result of Equation 23 with dimensions of density.

For the non-trivial instance in which \(I_{xx}\) and \(I_{zz}\) evolve such that their error margins do not overlap, we can determine experimental measurements for ellipticity. We calculate error bounds for the ellipticity, \(\epsilon\), as expressed in

\[
\epsilon = \frac{\Delta I}{I_0} \pm \frac{2\delta I_{xx}}{I_0}.
\]

We reintroduce the magnetic field model and graph the evolution of \(I_{xx}\) and \(I_{zz}\) through a simulation time of \(t = 5.0\ \text{s}\) in Figure 11, where the magnetic field is assigned the magnitude \(B_k = 1 \times 10^{17}\ \text{G}\) in accordance with a surface field strength of order \(10^{15}\ \text{G}\). In distinct difference to the instance of hydrostatic equilibrium in Figure 10, \(I_{xx}\) and \(I_{zz}\) become distinguishable such that measurements of ellipticity can be performed.

The computed ellipticity for the results of Figure 11 is plotted in Figure 12. The value of the ellipticity becomes increasingly negative, whereby the star becomes steadily more prolate under evolution. By simulation time \(t = 5.0\ \text{s}\), we find the ellipticity to be \(\epsilon = (7.907 \pm 0.408) \times 10^{-2}\).

4.3 Effect of Angular Resolution on Ellipticity Measurement

To determine whether greater mesh discretization modifies the calculation of stellar ellipticity, we test the effect of increasing angular resolution of our simulation.

Angular resolution is increased from \(n_{\theta, \phi} = 30\) to \(n_{\theta, \phi} = 50\) and ellipticity evolution is determined as in previous analysis. We find that for \(t \lesssim 3\ \text{s}\), our results agree for both higher and regular resolution ellipticity measurements within error margins set by the ellipticity error margin expressed in Equation 29. We plot our results in Figure 13.

For \(t \gtrsim 3\ \text{s}\), ellipticity measurement for higher angular resolution is plotted in Figure 14.
Figure 12. Ellipticity as measured for each instance of the evolution of $I_{xx}$ and $I_{zz}$ as depicted in Figure 11 in the presence of a magnetic field.

Figure 13. Comparison of ellipticity as calculated in Figure 12 (red) for angular resolution $n_{\theta,\phi} = 30$ and results for $n_{\theta,\phi} = 50$ (blue).

Figure 14. The first time derivative for $I_{xx}$ and $I_{zz}$ in an extended simulation through simulation time $t = 14$ s. Dramatic evolution from the initial configuration the stellar medium is constrained as the star approaches MHD equilibrium.

Figure 15. The first time derivative for $I_{xx}$ and $I_{zz}$ in an extended simulation through simulation time $t = 14$ s. Dramatic evolution from the initial configuration the stellar medium is constrained as the star approaches MHD equilibrium.

4.4 Extended Simulation Results

The continual evolution of $I_{xx}$ and $I_{zz}$ through simulation time $t = 5$ s motivates us to run an extended simulation through time $t = 14$ s. We analyze the first and second derivative of $I_{xx}$ and $I_{zz}$ to determine whether extended simulation time indicates that the evolution of these principal moments of inertia are constrained as the star approaches MHD equilibrium. In Figures 14 and 15, we plot the first and second time derivative for $I_{xx}$ and $I_{zz}$, where we find strong evidence of a decaying envelope which constrains the evolution of each moment of inertia.

Equation 20 relates the ellipticity to the principal moments of inertia $I_{xx}$ and $I_{zz}$. Taking a derivative of the equation for ellipticity with respect to time, we trivially find that

$$\frac{\partial \epsilon}{\partial t} \propto \frac{\partial I_{zz}}{\partial t} - \frac{\partial I_{xx}}{\partial t}. \quad (30)$$

As the time derivative of stellar ellipticity is proportional to the difference between the time derivatives of the principal moments of inertia, a stellar medium which approaches MHD equilibrium (whereby $\frac{\partial}{\partial t}(I_{xx})$, $\frac{\partial}{\partial t}(I_{zz})$, and higher order derivatives approach zero) will also approach constant ellipticity.

Our results for the evolution of the first and second time derivatives for $I_{xx}$ and $I_{zz}$ indicate that the timescale for perturbation of the stellar structure to be strongly damped by MHD forces is of order 10 s.

We compute the stellar ellipticity for our extended simulation and plot our results in Figure 16. We find that over the course of our simulation, the maximum magnitude of the ellipticity is $\epsilon(t = 5.0) \approx 7.1 \times 10^{-2}$.

4.5 Upper-Limit Estimate for Magnetar Gravitational Wavestrain

With the adoption of a canonical value for the unperturbed stellar moment of inertia, $I_0$, the gravitational wave strain (Equation 19) is dependent on three stellar parameters: ellipticity, rotational period, and distance to the source.
In determining accepted values for average magnetar rotational period, $\bar{\tau}$, and distance, $\bar{d}$, we utilize data for stellar sources in the McGill Magnetar Catalog (Olausen & Kaspi 2014), where $\bar{\tau} = 6.65$ s and $\bar{d} = 11.43$ kpc, indicating that these are slowly rotating, galactic sources. Following the work of Lasky (Lasky 2015) in expressing wave strain as

$$|\epsilon| \approx 7.1 \times 10^{-2} \text{ and average values for rotational period and distance via the McGill Magnetar Catalog, we find that}$$

$$|\epsilon| \approx 7.1 \times 10^{-2} \left( \frac{10^{-6}}{10 \text{ ms}} \right)^{-2} \left( \frac{11.43 \text{ kpc}}{1 \text{ kpc}} \right)^{-1}$$

$$\approx 5.9 \times 10^{-28}. \quad (32)$$

### 4.6 Strain Estimates for the McGill Magnetar Catalog

The ellipticity results presented in this paper are determined for a magnetar with surface field strength $B_S \approx 2.0 \times 10^{15}$ G. Based on our choice of surface field strength, our results for magnetar ellipticity represent an upper limit for known sources, as average surface dipolar magnetic field strength for sources in the McGill Magnetar Catalog is $\sim 3.65 \times 10^{14}$ G and the maximum detected field strength for an individual source is $2.0 \times 10^{15}$ G.

We compute upper limits of gravitational wave strain estimates for magnetars in the McGill Magnetar Catalog by utilizing Equation 31 where we set ellipticity to our determined value, $7.1 \times 10^{-2}$, and vary rotational period, $\tau$, and distance, $d$, in accordance with each source. We plot our results in Figure 17.

We compare our results qualitatively against prior gravitational wave strain predictions computed for pulsar sources. Lasky (2015) compute wave strain estimates for known pulsars in the ATNF catalog, and find conventional pulsars with mixed magnetic field configurations and field strengths $|B| < 10^{14}$ G to have strain sensitivities in the range of $\sim 10^{-34}$ to $10^{-31}$ Hz$^{-1/2}$. We anticipate that higher magnetic field strengths will correspond to greater deformation and increased strain sensitivity magnitudes, and our results for magnetars support this reasoning while lying reasonably within the range of past predictions.

Additionally, our computed wave strain estimates are listed in Table 1 with the exception of catalog source MG J1833-0831 due to the lack of data for the source’s stellar distance.

### 5 CONCLUSIONS

We implement a computational model for the stellar structure and magnetic field configuration of a magnetar to evaluate the structural changes the star undergoes as magnetic and hydrodynamic forces approach stable equilibrium. These structural changes are manifest in the principal moments of inertia which allow measurement of the stellar ellipticity. Via measurement of the principal moments of inertia in extended simulation of the stellar system, we find that the star approaches MHD equilibrium on the order of $10$ s, providing evidence that longer term structural evolution will be highly damped by magnetic and structural forces. Because stellar ellipticity is derived from measurement of the principal moments of inertia via Equation 20, damped structural evolution will limit future large-scale changes in the stellar ellipticity.
Based on these findings, we compute upper-limit estimates for the gravitational wave strain for sources in the McGill Magnetar Catalog. In comparing our computed upper limits against prior gravitational wave strain prediction for pulsar sources (Lasky 2015), we find that our results are in accordance with prediction; magnetars, possessing the strongest magnetic field strengths, are deformed more than conventional pulsars by their respective fields, thus resulting in higher wave strain estimates.

Here, we utilize the Newtonian formulation of hydrostatic equilibrium and mass conservation, which lead to analytic structural expressions and a static gravitational potential. Our results provide a firm starting point for subsequent determination of magnetar wave strain upper limits, and considerable opportunity exists to extend beyond the scope of this work, including considerations for relativistic effects, dynamic gravitational potentials that evolve with the structure of the star, and adoption of a more physically representative EOS.

While these results provide valuable indication of the instrument sensitivity required to measure continuous GWs from magnetars, the operational frequency range of current GW detectors falls outside the range of frequencies produced by relatively slowly rotating magnetars. We anticipate future advancements in GW detector design to improve sensitivity to frequencies produced by magnetars, which are sure to bring about significant advancement in the scientific body of knowledge on pulsars and highly magnetic stars.

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Table 1. McGill catalog sources, associated attributes, and wavestrain estimates. To differentiate catalog data from our wavestrain findings, we place our estimates in a shaded column. We adopt the naming scheme assigned to magnetar sources by the catalog authors, including the prefix ‘MG’ followed by the source right ascension and declination in J2000 epoch.

| MG Name          | Distance (kpc) | Period (s) | $f_{gw}$ (Hz) | $B$ ($10^{14}$ G) | GW strain |
|------------------|----------------|------------|---------------|------------------|-----------|
| MG J0100-7211    | 62.4(1.6)      | 8.020392(9)| 0.24938       | 3.9              | 1.05×10⁻²⁸ |
| MG J0146+6145    | 3.6(4)         | 8.68832877(2)| 0.2302       | 1.3              | 1.55×10⁻²⁷ |
| MG J0418+5372    | ~2             | 9.07838882(5)| 0.22031     | 0.061            | 2.55×10⁻²⁷ |
| MG J0501+4516    | ~2             | 5.76209653(3)| 0.3471       | 1.9              | 6.32×10⁻²⁷ |
| MG J0526-6604    | 53.6(1.2)      | 8.0544(2)  | 0.24832       | 5.6              | 1.21×10⁻²⁸ |
| MG J1050-5953    | 9.0(1.7)       | 6.4578754(25)| 0.3097      | 3.9              | 1.12×10⁻²⁷ |
| MG J1550-5418    | 4.5(5)         | 2.0721255(1)| 0.96525      | 3.2              | 2.17×10⁻²⁶ |
| MG J1622-4950    | ~9             | 4.3261(1)   | 0.46231       | 2.7              | 2.49×10⁻²⁷ |
| MG J1635-4735    | 11.0(3)        | 2.594578(6)| 0.77086       | 2.2              | 5.67×10⁻²⁷ |
| MG J1647-4552    | 3.9(7)         | 10.610644(17)| 0.18849     | <0.66            | 9.57×10⁻²⁸ |
| MG J1708-4008    | 3.8(5)         | 11.003027(1)| 0.18177      | 4.6              | 9.13×10⁻²⁸ |
| MG J1714-3810    | ~13.2          | 3.825352(4)| 0.52283       | 5                | 2.17×10⁻²⁷ |
| MG J1745-2900    | ~8.5           | 3.763553(2)| 0.53141       | 1.6              | 3.49×10⁻²⁷ |
| MG J1808-2024    | 8.7⁺⁻⁴⁻¹⁻⁸     | 7.547728(17)| 0.26498      | 20               | 8.47×10⁻²⁸ |
| MG J1809-1943    | 3.5⁺⁻³⁻⁸⁻⁴⁻⁸   | 5.5403537(2)| 0.36099      | 2.1              | 3.91×10⁻²⁷ |
| MG J1822-1604    | 1.6(3)         | 8.43771958(6)| 0.23703     | 0.51             | 3.69×10⁻²⁷ |
| MG J1833-0831    | ...            | 7.5654084(4)| 0.26436      | 1.6              | ...       |
| MG J1834-0845    | 4.2(3)         | 2.4823018(1)| 0.8057       | 1.4              | 1.62×10⁻²⁶ |
| MG J1841-0456    | 8.5⁺⁻³⁻¹⁻⁰⁻⁰   | 11.782898(1)| 0.16974      | 6.9              | 3.56×10⁻²⁸ |
| MG J1907+0919    | 12.5(1.7)      | 5.19987(7)  | 0.38463       | 7                | 1.24×10⁻²⁷ |
| MG J2301+5852    | 3.2(2)         | 6.978948446(4)| 0.28658    | 0.59             | 2.69×10⁻²⁷ |