The influence of difference in the surface properties on the axisymmetric vibrations of an oblate drop in an AC field

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Abstract. The forced oscillations of an incompressible fluid drop in the non-uniform electric field are considered. The external electric field acts as an external force that causes motion of the contact line. In order to describe this contact line motion the modified Hocking boundary condition is applied: the velocity of the contact line is proportional to the deviation of the contact angle and the rate of the fast relaxation processes, whose frequency is proportional to twice the frequency of the electric field. The equilibrium drop has the form of a cylinder bounded by axially parallel solid planes. These plates have different surface (wetting etc.) properties. The solution of the problem is represented as a Fourier series in eigenfunctions of the Laplace operator. The resulting system of heterogeneous equations for unknown amplitudes was solved numerically. The amplitude-frequency characteristics and the evolution of the drop shape are plotted for different values of the problem parameters.

1. Introduction
Control of inclusions (drops, bubbles, particles) in fluid systems is an important research topic. One of the promising methods is changing of wetting properties of a substrate by a liquid, for example, due to an external electric field (electrowetting-on-dielectric – EWOD) [1,2]. Now EWOD has found wide application in various fields, such as electronic display technology [3,4], variable-focus liquid lenses [5,6], digital (droplet) microfluidic devices for bioanalysis (lab-on-a-chip) [7,8], etc.

The Young–Lippmann equation is used in most articles for theoretical description of the contact angle $\theta$ at EWOD (fig. 1) [1,9]. Nevertheless, the experimental results show a considerable departure from the theoretical predictions of the Young–Lippmann equation: although a zero contact angle is obtained after reaching some critical voltage angle (complete wetting and the contact angle does not change), real experiments invariably yield a finite value of the angle [1,2,10]. The mechanism of the contact angle saturation has not yet been elucidated and is the object of discussion [1].

![Image of EWOD devices](image-url)

**Figure 1.** Typical schematic configurations of EWOD-devices. 1 – electrode, 2 – dielectric layer.
As an alternative, another effective boundary condition is proposed in [11] taking as a premise the Hocking equation [12] for a cylindrical drop:

$$\frac{\partial \zeta'}{\partial t} = \pm \Lambda' \left( \frac{\partial \zeta'}{\partial z} + A' \cos(2\omega' t') \right),$$  

(1)

where $\zeta'$ is the deviation of the drop interface from the equilibrium position, $z'$ is the axial coordinate, $\Lambda'$ is a phenomenological constant (the so-called wetting parameter or Hocking parameter), having the dimension of velocity, $A'$ is the effective AC amplitude, $\omega'$ is the AC frequency. We also investigated the effect of a non-uniform electric field [11,13] and heterogeneous substrates [13,14]. In the opinion of the authors of [15], a non-uniform field or heterogeneity of substrate can cause the observed effects. In the general case, the Hocking equation [12] is used for a theoretical description of the contact line dynamics under free and forced oscillations [16-22].

This paper is intended as an extension of work [11]. In contrast to all other works [11,13-14,19-22], we consider the case of different plates, i.e. the plates have different surface (wetting etc.) properties. In order to describe the motion of the contact line, the modified boundary condition (1) is used: the velocity of the contact line is proportional to the deviation of the contact angle and the rate of fast relaxation processes, whose frequency is proportional to twice the electric field frequency.

2. Problem formulation

The problem formulation largely coincides with that developed in articles [11,13,14]. An incompressible liquid drop of density $\rho_i^*$ is surrounded by another liquid of density $\rho_e^*$ (see figure 2). In what follows, the quantities with subscript $i$ refer to the drop, and those with subscript $e$ to the surrounding liquid. Both liquids are bounded by two parallel solid surfaces at a distance $h^*$ from one another. The equilibrium contact angle $\theta_0$ between the lateral surface of the drop and the solid surface is equal to $\pi/2$. The external non-uniform alternating electric field acts as an external force that causes the contact line motion.

Figure 2. Problem geometry (1 – electrode, 2 – dielectric layer).

Let the surface of the droplet be described by the equation $r^* = R_0^* + \zeta'(\alpha, z^*, t')$ according to the cylindrical coordinates $r^*$, $\alpha$, $z^*$. The azimuthal angle $\alpha$ is reckoned from the x-axis. Taking the length $R_0^*$, the height $h^*$, the density $\rho_e^* + \rho_i^*$, the time $\sigma^{-1/2} \sqrt{(\rho_e^* + \rho_i^*) R_0^*}$, the velocity $A' \sqrt{\sigma} \left( (\rho_e^* + \rho_i^*) R_0^* \right)^{-1/2}$, the pressure $A' \sigma \left( R_0^* \right)^{-2}$, and the deviation of the surface $A'$ as characteristic quantities, we pass to dimensionless variables and obtain the following linear problem:

$$p_j = -\rho_j \frac{\partial \varphi}{\partial t}, \quad \Delta \varphi = 0, \quad j = i, e,$$

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} + b^2 \frac{\partial^2}{\partial z^2},$$  

(2)
\[ r = 1: [\varphi] = 0, \quad \zeta = \varphi, \quad [p] = \zeta + \zeta_{\text{ext}} + b^2 \zeta_{\text{int}}, \quad (3) \]
\[ z = \pm 0.5: \quad \varphi = 0, \quad (4) \]
\[ r = 1, \quad z = \pm 0.5: \quad \zeta = \pm \lambda_{\text{h}}(\zeta_z + a f(\alpha) \cos(2\omega t)), \quad (5) \]

where \( p \) is the fluid pressure, \( \varphi \) is the velocity potential, \( f(\alpha) \) is the function of the non-uniform electric field, the square brackets denote the jump in the quantity at the interface between the external liquid and the drop, \( \lambda_{\text{h}} \) and \( \lambda_{\text{b}} \) are the Hocking parameter of the “top” \((z = 0.5)\) and “bottom” \((z = -0.5)\) substrate, respectively. The subscripts \((t, r, \alpha \) or \( z)\) at the unknown functions denote differentiation with respect to the corresponding variables. The boundary-value problem \((2)-(5)\) involves six parameters: the aspect ratio, the dimensionless density, the wetting parameter, the AC frequency and amplitude

\[ b = R_0 h, \quad \rho_i = \rho_i(\rho_i^* + \rho_i^*), \quad \rho_c = \rho_c(\rho_c^* + \rho_c^*), \quad \lambda = \lambda^* \sigma^{-1/2} b(\rho_i^* + \rho_i^*)R_0, \]
\[ \omega = \omega \sigma^{-1/2} R_0, \quad a = 0.5 \sigma \epsilon^3 R_0, \]

where \( C = \epsilon_0 \epsilon_d d^{-1} \) is capacitance per unit area, \( \epsilon_0 \) and \( \epsilon_d \) are the vacuum and the dielectric layer permittivity, respectively, \( d \) is the thickness of the dielectric film.

3. Forced oscillations

The function \( f(\alpha) \) is expanded in the Fourier series in terms of the eigenfunctions of the Laplace operator. Let us consider a particular case of the non-uniform electric field \( f(\alpha) = \sin(k_0 \cos(\alpha)) \), where \( k_0 \) are the wavenumber. Note, that we used this function \( f(\alpha) \) in [11]. The solutions for the velocity potential and the surface deviation are written as

\[ \varphi(r, \alpha, z, t) = \text{Re} \left( i2\omega \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{mk} R_{mk}^i(r) \sin((2k + 1)\pi z) + a_{mk} R_{mk}^j(r) \cos(2\pi k z)) \cos(2m\alpha) e^{i2\omega t} \right), \]
\[ \varphi(r, \alpha, z, t) = \text{Re} \left( i2\omega \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (b_{mk} R_{mk}^c(r) \sin((2k + 1)\pi z) + b_{mk} R_{mk}^d(r) \cos(2\pi k z)) \cos(2m\alpha) e^{i2\omega t} \right), \]
\[ \zeta(z, t) = \text{Re} \left( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (c_{mk} \sin((2k + 1)\pi z) + c_{mk} \cos(2\pi k z)) \cos(2m\alpha) e^{i2\omega t} \right), \]

where \( I_\alpha \) and \( K_\alpha \) are the modified Bessel functions of the m-th order. Substituting solutions \((6)-(8)\) into \((2)-(5)\), we obtain the expressions for the unknown amplitudes \( a_{mk}, b_{mk}, c_{mk}, d_{j\alpha}, d_{m\alpha} \). These expressions are equivalent to the similar solutions obtained in [11,13,14] for \( \lambda_{\text{h}} = \lambda_{\text{b}} = \lambda \).

For convenience, as a maximum deviation of the drop surface from the equilibrium position, we prescribe on the “upper” plate \( z = 0.5 - \zeta_u = \max(\zeta(0,0.5,0)) \), on the “bottom” plate \( z = -0.5 - \zeta_u = \max(\zeta(0,-0.5,0)) \), in the center of the layer \( z = 0 - \zeta_0 = \max(\zeta(0,0,0)) \) and a “quarter” position \( z = 0.25 - \zeta_q = \max(\zeta(0,0.25,0)) \); the values of the internal contact angle \( \gamma \) on the “upper” plate are \( \gamma_u \), at the “bottom” plate \( - \gamma_u \); the deviation from the equilibrium contact angle on the “upper” plate is \( \delta_u = \max(\gamma_u - 0.5\pi) \) and on the “bottom” plate \( \delta_u = \max(\gamma_u - 0.5\pi) \).

The main contribution to a change in the drop dynamics is made by the Hocking parameter \( \lambda \) [16-22]. Figure 3 shows the oscillation amplitude of the drop surface and the deviation of the contact angle.
as a function of the frequency of the uniform electric field for several values of the Hocking parameters $\lambda_u$ and $\lambda_b$. The amplitudes of the surface oscillations and the contact angle reach maximum values in a linear resonance. It is also seen from the graphs that the values of the resonant frequencies decrease with an increase of $\lambda_u$ or $\lambda_b$. Despite weak dissipation at small values of the parameter $\lambda$, the amplitude of surface oscillations is finite (fig. 3a) and the amplitude of oscillations of the contact line is small (fig. 3c). The contact angle varies in a wide range (fig. 3e, f). It is important to note that if both parameters $\lambda_u$ and $\lambda_b$ are small but not equal to each other, the amplitude of surface oscillations is always finite. However, if these parameters are small but equal, the amplitude of oscillations increases indefinitely.

**Figure 3.** The accuracy of representation of the drop surface (a-d) and the contact angle (d-f) vs the frequency $\omega$

($a = 10$, $b = 1$, $\rho_i = 0.7$, $\lambda_u = 1$, $f(\alpha) = 1$), $\lambda_b = 0.1$ – solid line, $\lambda_b = 1$ – dashed, $\lambda_b = 10$ – dotted.

**Figure 4.** Evolution of the drop surface shape (a), the shape of the contact line (b,c) and the contact angles (d). $T = \frac{2\pi}{\omega}$ is the oscillation period.

($b = 1$, $\rho_i = 0.7$, $\alpha = 10$, $\lambda_u = 1$, $\lambda_b = 0.1$, $\omega = 1$, $\varepsilon = 0.1$, $f(\alpha) = 1$),

(a-c) $t = 0$ – solid line, $t = 0.125T$ – dashed, $t = 0.25T$ – dotted, $t = 0.375T$ – dash-dotted.
For clarity, fig. 3 shows the case of equality of the Hocking parameters $\lambda_u = \lambda_b = 1$. In this situation, the external force excites only odd spatial modes, so that there is no deviation of the drop surface in the center of the layer (fig. 3 a). Recall that for finite values of the parameter $\lambda$, a dissipation is maximum during the movement of the contact line, and therefore, the plots do not display pronounced resonance peaks (fig. 3 b, c). The motion of a drop does not depend on the value of $\lambda$ at certain frequencies $\omega$: the contact line is like a fixed contact line for any values of $\lambda$ (fig. 3 b, c). The values of such “anti-resonant” frequencies are determined from the solution (6)–(8).

Figure 4 shows the profile of the lateral surface (fig. 4 a) and the contact line (fig. 4 b, c) and changes in the internal contact angle (fig. 4 d) at different moments of the oscillation period. The shape of the drop surface depends on the frequency of the electric field. For example, in fig. 4, most of the vibration energy at a given frequency $\omega = 1$ (see fig. 3 a) is transmitted to the lowest spatial mode.

**Figure 5.** Deviation of the contact angle as a function of the square root of the amplitude ($b = 1, \rho_i = 0.7, \alpha = 10, \lambda_u = 1, f(\alpha) = 1$), (a) $\omega = 1$, (b) $\omega = 2.5$, (c) $\omega = 5$, (d) $\omega = 10$,

- $\gamma_u$: $\lambda_b = 0.1$ – solid line, $\lambda_b = 1$ – dotted, $\lambda_b = 10$ – dash-2-dotted;
- $\gamma_b$: $\lambda_b = 0.1$ – dashed line, $\lambda_b = 1$ – dash-dotted, $\lambda_b = 10$ – 2-dash-2-dotted.

**Figure 6.** The accuracy of representation of the drop surface (a-d) and the contact angle (d-f) vs frequency $\omega$ ($\alpha = 5, b = 1, \rho_i = 0.7, \lambda_u = 1, k_a = 1$), $\lambda_b = 0.1$ – solid line, $\lambda_b = 1$ – dashed, $\lambda_b = 10$ – dotted.
The deviation of the contact angle as a function of the square root of the amplitude $\alpha$ (i.e. proportional to AC potential $V$) is given in fig. 5 for different values of the Hocking parameter $\lambda$ and AC frequency $\omega$. The responses obtained are in qualitative agreement with the experimental data. The maximum deviation of the contact angle tends to $\pi/2$, i.e. $\theta \to 0$ or $\theta \to \pi$ whereas in experiments the contact angle is finite.

The non-uniform electric field $f(\alpha) = \sin(k_x \cos(\alpha))$ excites axisymmetric and azimuthal modes. Consequently the dynamics of the drop differs significantly from its behavior in the uniform field. Similar dependences are shown in fig. 6-8. The amplitude of the lateral surface oscillations has identical local maximum unlike the situation with the uniform electric field. Additional resonance amplitude peaks are associated with the excitation of azimuth modes. A quadrupole mode has a significant effect (fig. 7b): the drop is compressed along field inhomogeneity.

**Figure 7.** Evolution of the drop surface shape (a), the shape of the “upper” contact line (b) and the contact angles (c,d). $T = 2\pi \omega^{-1}$ is the oscillation period.

$$(b = 1, \rho = 0.7, \alpha = 10, \lambda = 0.1, \omega = 1, \epsilon = 0.1, f(\alpha) = 1),$$

(a-c) $t = 0$ – solid line, $t = 0.125T$ – dashed, $t = 0.25T$ – dotted, $t = 0.375T$ – dash-dotted.

**Figure 8.** Deviation of the contact angle as a function of the square root of the amplitude

$$(b = 1, \rho = 0.7, \alpha = 10, \lambda = 0.1, f(\alpha) = 1),$$

(a) $\omega = 1$, (b) $\omega = 2.5$, (c) $\omega = 5$, (d) $\omega = 10$.

$\gamma_a$: $\lambda = 0.1$ – solid line, $\lambda = 1$ – dotted, $\lambda = 10$ – dash-2-dotted;

$\gamma_b$: $\lambda = 0.1$ – dashed line, $\lambda = 1$ – dash-dotted, $\lambda = 10$ – 2 dash-2-dotted.

4. Conclusions

In this paper we studied the behavior of the cylindrical drop between solid plates under the action of the non-uniform electric field $\sin(k_x \alpha)$ taking into account the dynamics of the contact angle. The solid plates had different surfaces.

The study made allowed us to estimate the deviation of frequency and surface characteristics as a function of the Hocking parameter, frequency and amplitude of the external non-uniform electric field
and geometric parameters of the system. It was found that a non-uniform field can excite azimuthal modes as was observed, for example, in experiments [15]. The maximum deviation of the contact angle tends to $\pi/2$, i.e. $\theta \to 0$ or $\theta \to \pi$ whereas in experiments the contact angle is always finite. However, the estimates of dimensionless values indicate that the dimensionless amplitude $\sqrt{a}$ is achieved in experiments $\sqrt{a} \leq 10$. At this amplitude the contact angle is finite in a large range of the problem parameters. This allows us to expect good agreement between the theoretical predictions and experimental data in the case of a more detailed comparison. The main problem is to measure the value of the Hocking parameter $\lambda$ at this stage.

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5. References
[1] Mugele F and Baret J-C 2005 J. Phys.: Condens. Matter. 17
[2] Chen L and Bonaccurso E 2014 Adv. Colloid Interface Sci. 210
[3] Hayes R A and Feenstra B J 2003 Nature 425 383–5
[4] Roques-Carriès T, Hayes R A, Feenstra B J and Schlangen L J M 2004 J. Appl. Phys 95 4389–96
[5] Kuiper S and Hendriks B H W 2004 Appl. Phys. Lett. 85, 1128–30
[6] Li C and Jiang H 2014 Micromachines 5 432-41
[7] Hua Z, Rouse J L, Eckhardt A E, Srinivasan V, Pamula V K, Schell W A, Benton J L, Mitchell T G and Pollack M G 2010 Anal Chem 82 2310-16
[8] Li J, Wang Y, Chen H and Wan J 2014 Lab Chip 14 4334-37
[9] Berge B 1993 Comptes Rendus Acad. Sci. II 317
[10] Chevalliot S, Kuiper S and Heikenfeld J 2012 J. of Adhesion Sci. Tech. 26 1909–30
[11] Alabuzhev A A and Kashina M A 2016 J. Phys.: Conf. Ser. 681 012042
[12] Hocking L M 1987 J. Fluid Mech. 179 253-66
[13] Kashina M A and Alabuzhev A A 2018 Microgravity Sci. Technol. 30 (1–2) 11–17
[14] Kashina M A and Alabuzhev A A 2018 J. Phys.: Conf. Ser. 955 012016
[15] Mampallil D, Eral H B, Staicu A, Mugele F and van den Ende D 2013 Phys. Rev. E 88 053015
[16] Shklyaev S and Straube A V 2008 Phys. Fluids 20 052102
[17] Fayzrakhmanova I S and Straube A V 2009 Phys. Fluids 21 072104
[18] Fayzrakhmanova I S, Straube A V and Shklyaev S 2011 Phys. Fluids 23 102105
[19] Borkar A and Tsamopoulos J 1991 Phys. Fluids A 3 2866–74
[20] Perlin M, Schultz W W and Liu Z 2004 Wave Motion 40 41–56
[21] Alabuzhev A A 2016 Appl. Mech. Tech. Phys. 53 9–19
[22] Alabuzhev A A 2018 Microgravity Sci. Technol. 30 (1–2) 25–32