Nonextensive statistical mechanics, anomalous diffusion and central limit theorems

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We briefly review Boltzmann-Gibbs and nonextensive statistical mechanics as well as their connections with Fokker-Planck equations and with existing central limit theorems. We then provide some hints that might pave the road to the proof of a new central limit theorem, which would play a fundamental role in the foundations and ubiquity of nonextensive statistical mechanics. The basic novelty introduced within this conjectural theorem is the generalization of the hypothesis of independence of the $N$ random variables being summed. In addition to this, we also advance some nonlinear dynamical (possibly exact) relations which generalize the concepts of Lyapunov exponents, entropy production per unit time, and their interconnection as first proved by Pesin for chaotic systems.

I. INTRODUCTION

As well known, thermodynamics is the basic branch of physics which focuses on the generic connections between variables (temperature, pressure, volume, energy, entropy and many others) that play an important role in the description of the macroscopic world. Boltzmann and Gibbs provided a magnificent connection of thermodynamics with the microscopic world [1,2]. This connection, normally referred to as Boltzmann-Gibbs (BG) statistical mechanics (or simply statistical mechanics since it was basically the only one to be formulated along more than one century), turns out to be the appropriate one for ubiquitous systems in nature. It is based on the following axiomatic expression for the entropy:

$$S_{BG} \equiv -k \sum_{i=1}^{W} p_i \ln p_i ,$$  \hspace{1cm} (1)

with

$$\sum_{i=1}^{W} p_i = 1 ,$$  \hspace{1cm} (2)

where $p_i$ is the probability associated with the $i^{th}$ microscopic state of the system, and $k$ is Boltzmann constant. In the particular case of equiprobability, i.e., $p_i = 1/W$ ($\forall i$), Eq. (1) yields the celebrated Boltzmann principle (as referred to by Einstein himself [3]):

$$S_{BG} = k \ln W .$$  \hspace{1cm} (3)

From now on, and without loss of generality, we shall take $k$ equal to unity.

For continuous variables, the BG entropy is written as

$$S_{BG} \equiv -\int dx \, p(x) \ln p(x) \quad (x \in \mathbb{R}^d),$$  \hspace{1cm} (4)

with

$$\int dx \, p(x) = 1 .$$  \hspace{1cm} (5)

If $x$ happens to carry physical units, we write Eq. (4) as follows:

$$S_{BG} \equiv -\int dx \, p(x) \ln \left[ \left( \prod_{r=1}^{d} \sigma_r \right) p(x) \right] ,$$  \hspace{1cm} (6)

where $\sigma_r > 0$ carries the same units as the $r^{th}$ component of the $d$-dimensional variable $x$. Clearly, when $x$ carries no units (i.e., when $x \in \mathbb{R}^d$), we take $\sigma_r = 1$ ($\forall r$). In fact, everytime this is possible (and it is possible most of the time), we shall adapt the physical units in such a way that $\sigma_r = 1$ ($\forall r$) even when $x$ does have physical units. Consistently, unless otherwise specified, we shall use Eq. (4) for the general continuous case. If we consider the particular case $p(x) = \sum_{i=1}^{W} p_i \delta(x - x_i)$ (with $\sum_{i=1}^{W} p_i = 1$, $\{x_i\}$ being some set of values, and $\delta(z)$ being Dirac’s delta distribution), Eq. (4) recovers Eq. (1).

For quantum systems, the BG entropic form is written as

$$S_{BG} \equiv -\text{Tr} \rho \ln \rho ,$$  \hspace{1cm} (7)

with

$$\text{Tr} \rho = 1 ,$$  \hspace{1cm} (8)

$\rho$ being the density operator or matrix. When the $W \times W$ matrix $\rho$ is diagonalized, it shows the set $\{p_i\}$ in its diagonal. In what follows, depending on the context, we shall use either the discrete form (Eqs. (1) and (2)), or

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the continuous form (Eqs. (4) and (5)), or the matricial form (Eqs. (7) and (8)). In spite of its tremendous power and usefulness, the BG concepts and statistical mechanics appear to be not universally applicable. Indeed, there is a plethora of natural and artificial systems (see, for instance, [4] and references therein) for which they do not provide the adequate mathematical frame for handling physically relevant quantities. This fact started being explicitly recognized at least as early as in 1902 by Gibbs himself: see page 35 of [2], where he addresses anomalies related to systems such as gravitation. A formalism becomes therefore desirable which would address such anomalous systems. A vast class of them (although surely not all of them) appears to be adequately discussed within a generalization of the BG theory, frequently referred to as nonextensive statistical mechanics. This theory was first introduced in 1988 [5], and then refined in 1991 [6] and 1998 [7]. It is based on the following generalization of $S_{BG}$:

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \quad (q \in \mathbb{R}; \ S_1 = S_{BG}) . \quad (9)$$

Expressions (4), (6) and (7) are respectively generalized into

$$S_q = \frac{1 - \int dx \ [p(x)]^q}{q - 1} , \quad (10)$$

and

$$S_q = \frac{1 - Tr \rho^q}{q - 1} \quad (12)$$

For equiprobability (i.e., $p_i = 1/W, \forall i$), Eq. (9) yields

$$S_q = \ln_q W \ , \quad (13)$$

with the $q$-logarithm function defined as

$$\ln_q z \equiv \frac{z^{1-q} - 1}{1-q} \quad (z \in \mathbb{R}; \ z > 0; \ \ln_1 z = \ln z) . \quad (14)$$

Its inverse function, the $q$-exponential, is given by

$$e^z_q \equiv [1 + (1-q)z]^{1/(1-q)} \quad (e^1_q = e^z) \quad (15)$$

if the argument $1 + (1-q)z$ is positive, and equals zero otherwise.

In Section II we briefly review a few known results concerning the probability distributions which extremize the entropy, and are ultimately associated with macroscopically stationary states. In Section III we briefly review available results related to Fokker-Planck equations. In Section IV we review the standard and the Lévy-Gnedenko central limit theorems, and argue about the possible formulation of a new theorem that would generalize the standard limit theorem. Such a theorem would provide an important mathematical cornerstone for nonextensive statistical mechanics and its ubiquity in nature. Providing hints that could help formulating and proving the theorem constitutes the main reason of the present paper.

## II. STATISTICAL MECHANICS

Entropy is necessary to formulate statistical mechanics but it is not sufficient. Indeed, we must also introduce the concept of energy. The easiest (and more frequently used) way to do so is addressing the so called (by Gibbs) canonical ensemble. It corresponds to the ubiquitous physical situation in which the system of interest is in contact with a (large by definition) thermostat which, at equilibrium (or at the physically relevant stationary state, more generally speaking), imposes to the system its temperature. The system is typically described by a quantum Hamiltonian, and is characterized by the spectrum of energies $\{E_i\} (i = 1, 2, ..., W)$ defined as the eigenvalues associated with the Hamiltonian and its boundary conditions. The probability distribution at the relevant stationary state is the one which extremizes the entropy under the norm and the energy constraints.

### A. Boltzmann-Gibbs statistical mechanics

At thermal equilibrium we must optimize $S_{BG}$ (as given by Eq. (1)) with the norm constraint as given by Eq. (2), and the energy constraint given as follows:

$$\sum_{i=1}^{W} p_i E_i = U_{BG} , \quad (16)$$

where $U_{BG}$ is given and referred to as the internal energy. By following the Lagrange method, we define the quantity

$$\Phi_{BG} \equiv S_{BG} + \alpha \left( \sum_{i=1}^{W} p_i - 1 \right) - \beta \left( \sum_{i=1}^{W} p_i E_i - U_{BG} \right) , \quad (17)$$

where $\alpha$ and $\beta$ are the Lagrange parameters (their signs have been chosen following tradition). The extremizing condition $\delta \Phi_{BG} / \delta p_j = 0$ yields

$$p_j = e^{\alpha - 1 - \beta E_j} \quad (j = 1, 2, ..., W) . \quad (18)$$

The use of Eq. (2) allows the elimination of the parameter $\alpha$. We then obtain the celebrated Boltzmann-Gibbs weight

$$p_i = \frac{e^{-\beta E_i}}{Z_{BG}} \quad (i = 1, 2, ..., W) , \quad (19)$$
where $\beta$ is connected with the thermostat temperature $T$ through $\beta \equiv 1/T$, and the partition function is defined as follows:

$$Z_{BG}(\beta) \equiv \sum_{j=1}^{W} e^{-\beta E_j} . \quad (20)$$

It is precisely the present Eqs. (19) and (20) that are referred to as dogma by the mathematician Takens! [8].

Clearly, if we replace the probability distribution (19) into Eq. (16), we obtain the thermodynamically important relation between the inverse temperature $\beta$ and the internal energy $U_{BG}$.

As a subsidiary comment, whose relevance will become transparent later on, let us remark that the functional (17) is generalized into

$$\Phi_q \equiv S_q + \alpha \sum_{i=1}^{W} p_i q [E_i - U_q] - \beta \sum_{i=1}^{W} p_i^q (E_i - U_q) , \quad (25)$$

(we recall the cutoff of the $q$-exponential function for $q < 1$, i.e., the states for which $1 - \beta(E_i - U_0) < 0$ do not contribute).

As a subsidiary comment, let us remark that the weight (27) can be seen as the solution of the nonlinear ordinary differential equation

$$\frac{dy}{dx} = ay^q \quad (y(0) = 1) . \quad (29)$$

Indeed, the solution is given by

$$y = e^{ax} , \quad (30)$$

which reproduces Eq. (27) through the identification $(x, a, y) \equiv (E_i, -\beta, Z_{BG} p_i)$.

### B. Nonextensive statistical mechanics

We want now to optimize $S_q$ (as given by Eq. (9)) with the norm constraint still given by Eq. (2), and the energy constraint generalized as follows (see [21]):

$$\sum_{i=1}^{W} p_i E_i = U_q , \quad (23)$$

(referred to as $q$-expectation value or $q$-mean value) or, equivalently,

$$\sum_{i=1}^{W} p_i^q (E_i - U_q) = 0 . \quad (24)$$

The functional (17) is generalized into

$$\Phi_q \equiv S_q + \alpha \sum_{i=1}^{W} [p_i - 1] - \beta \sum_{i=1}^{W} p_i^q (E_i - U_q) , \quad (25)$$

and the extremizing condition $\delta \Phi_q / \delta p_j = 0$ yields

$$p_j = \left( \frac{q}{\alpha} \right)^{(1-q)/q} e^{-\beta (E_j - U_q)} \quad (j = 1, 2, ..., W) . \quad (26)$$

The use of Eq. (2) allows, as before, the elimination of the parameter $\alpha$, thus obtaining the generalized weight

$$p_i = \frac{e^{-\beta (E_i - U_q)}}{Z_q} \quad (i = 1, 2, ..., W) , \quad (27)$$

with

$$Z_q(\beta) \equiv \sum_{j=1}^{W} e^{\beta(E_j - U_q)} . \quad (28)$$

This probability distribution corresponds to a maximum (minimum) of $S_q$ for $q > 0$ ($q < 0$). For $q = 0$, the entropy is constant, namely $S_0 = W - 1$, and the distribution is given by $p_i = [1 - \beta(E_i - U_0)] / \sum_{j=1}^{W} [1 - \beta(E_j - U_0)]$ (we recall the cutoff of the $q$-exponential function for $q < 1$, i.e., the states for which $1 - \beta(E_i - U_0) < 0$ do not contribute).

We straightforwardly verify that constraint (31) is satisfied as well.

A more general case is to assume the following constraints:

$$\langle x \rangle \equiv \int_{-\infty}^{\infty} dx \; p(x) = C , \quad (35)$$
and
\[ \langle (x - \langle x \rangle)^2 \rangle = \int_{-\infty}^{\infty} dx \langle x - \langle x \rangle \rangle^2 p(x) = D. \]  
(36)
The extremizing distribution is then given by
\[ p(x) = \sqrt{\frac{\beta}{\pi}} e^{-\beta(x - \langle x \rangle)^2}, \]  
(37)
or, equivalently,
\[ p(x) = \frac{e^{-(x-C)^2/(2D)}}{\sqrt{2\pi D}}. \]  
(38)

D. The \(q\)-generalization of the Gaussian case

Let us assume that the constraints are now even more general, namely
\[ \langle x \rangle_q \equiv \frac{\int_{-\infty}^{\infty} dx \ x \ [p(x)]^q}{\int_{-\infty}^{\infty} dx \ [p(x)]^q} = C_q, \]  
(39)
and
\[ \langle (x - \langle x \rangle_q)^2 \rangle_q = \frac{\int_{-\infty}^{\infty} dx \ (x - \langle x \rangle_q)^2 [p(x)]^q}{\int_{-\infty}^{\infty} dx \ [p(x)]^q} = D_q \]  
(40)
where \(P(x) \equiv [p(x)]^q / \int dx [p(y)]^q\) is, in the literature, referred to as the escort distribution.

The associated distribution extremizing \(S_q\) is now given by
\[ p(x) = A_q \sqrt{\beta} e^{-\beta(x - \langle x \rangle_q)^2}, \]  
(41)
where we have \(A_q = \sqrt{(q-1)/\pi \Gamma(1/(q-1)) / \Gamma((3-q)/(2/(q-1)))}\) for \(q > 1\), and \(A_q = \sqrt{(1-q)/\pi \Gamma((5-3q)/(2(1-q))) / \Gamma((2-q)/(1-q))}\) for \(q < 1\), \(\Gamma(z)\) being the Riemann function.

The use of constraint (40) straightforwardly provides \(\beta = 1/(3-q)D_q\), which, replaced in Eq. (41), yields
\[ p(x) = \begin{cases} 
A_q \sqrt{x/\Gamma((3-q)/(2/(q-1)))} & (q > 1) \\
A_q \sqrt{(3-q)D_q^{-1}} \left[ 1 - \frac{(x-C_q)^2}{D_q} \right]^{1/(1-q)} & (q < 1) 
\end{cases} \]  
(42)
In the \(q < 1\) case, the support is compact, and the distribution vanishes outside the interval \(|x - C_q| \leq \sqrt{(3-q)D_q^{-1}(1-q)}\).

If \(\langle x \rangle_q = 0\), distributions (41) and (42) take respectively the simple forms
\[ p(x) = A_q \sqrt{\beta} e^{-\beta x^2}, \]  
(43)
and
\[ p(x) = \frac{A_q}{\sqrt{(3-q)D_q}} e^{-x^2/(3-q)D_q}, \]  
(44)
which appear frequently in various contexts.

It can be seen that these distributions are analytic extensions (to real values of \(q\)) of the Student’s \(t\)-distribution and the \(r\)-distribution, for \(q > 1\) and \(q < 1\) respectively. Consistently, there is an asymptotic long power-law tail for \(q > 1\), and a compact support for \(q < 1\). There is an upper bound for \(q\), namely \(q = 3\), imposed by the norm constraint (5). More precisely, the admissible values of \(q\) are \(q < 3\). It deserves to be mentioned that also constraint (40) has an upper bound in order to be finite, which happens to be precisely the same, i.e., \(q = 3\).

III. DIFFUSION AND FOKKER-PLANCK EQUATIONS

Normal diffusion (i.e., the one which satisfies \(\langle x^2 \rangle \propto t\), typical of Brownian motion) is characterized by the heat equation (the simplest form of the Fokker-Planck equation)
\[ \frac{\partial p(x,t)}{\partial t} = D \frac{\partial^2 p(x,t)}{\partial x^2} \]  
(45)
By assuming, at \(t = 0\), the paradigmatic form
\[ p(x,0) = \delta(x), \]  
(46)
we obtain the following exact solution:
\[ p(x,t) = \frac{e^{-x^2/2Dt}}{\sqrt{2\pi Dt}} \]  
(47)
There is of course an infinity of possible generalizations of Eq. (45), which is linear and defined through integer derivatives. We address here two important such generalizations, both of them associated with anomalous diffusion, i.e., violating the relation \(\langle x^2 \rangle \propto t\). The first one remains linear but replaces the second derivative of the right term by a fractional derivative; it yields \(\langle x^2 \rangle \rightarrow \propto (\forall t > 0)\). The second one is nonlinear but preserves the integer derivatives; it yields \(\langle x^2 \rangle \propto t^\alpha \) \((\alpha \neq 1)\).

The first one is as follows:
\[ \frac{\partial p(x,t)}{\partial t} = D \frac{\partial^\gamma p(x,t)}{\partial x^\gamma} \]  
(48)
The exact solution associated with the \(t = 0\) condition (46) is given by
\[ p(x,t) = \frac{1}{(Dt)^{1/\gamma}} L_\gamma(x/(Dt)^{1/\gamma}) \]  
(49)
where \(L_\gamma(z)\) is the Lévy distribution of index \(\gamma\). The \(\gamma \rightarrow 2\) limit obviously corresponds to the Gaussian solution (47). The \(\gamma = 1\) particular case corresponds to the Cauchy-Lorentz distribution
\[ p(x,t) = L_1(x/(Dt)) = \frac{Dt}{\pi (Dt)^2 + x^2} \]  
(50)
For all other values of $\gamma$ different from 1 and 2, the Lévy distribution has no direct analytic expression, and is expressed only through its Fourier transform.

The second generalization of Eq. (45) is as follows:

$$\frac{\partial p(x, t)}{\partial t} = D \frac{\partial^2 [p(x, t)]^\nu}{\partial x^2} \quad (\nu \in \mathbb{R}).$$  \hspace{1cm} (51)

The exact solution associated with the $t = 0$ condition (46) is given by

$$p(x, t) = \frac{A_q}{\sqrt{(3 - q) (Dt)^{1/(3 - q)}}} e^{-x^2 / (2(Dt)^{2/(3 - q)})} \quad (t \geq 0)$$  \hspace{1cm} (52)

with

$$q = 2 - \nu < 3,$$  \hspace{1cm} (53)

with $D > 0$ if $\nu > 0$, and $D < 0$ if $\nu < 0$. If we rewrite the diffusion coefficient as $D = \bar{D}/\nu$, Eq. (51) can be rewritten as follows:

$$\frac{\partial p(x, t)}{\partial t} = \bar{D} \frac{\partial^2 [p(x, t)]^\nu}{\partial x^2} \quad (\nu \in \mathbb{R}).$$  \hspace{1cm} (54)

which, in the limit $\nu \to 0$, becomes

$$\frac{\partial p(x, t)}{\partial t} = \bar{D} \frac{\partial^2 [\ln p(x, t)]}{\partial x^2} \quad (\nu \in \mathbb{R}).$$  \hspace{1cm} (55)

whose solution is known to be the Cauchy-Lorentz distribution.

If we start from any $t = 0$ distribution different from (46), the solution $p(x, t)$ does not coincide with Eq. (52), but, nevertheless, it asymptotically approaches Eq. (52) for $t \to \infty$. In other words, the solution (52) constitutes an attractor in the space of the distributions, i.e., it is a solution which is robust. For all $q < 3$, $x$ scales with $t^{1/(3-q)}$. Consequently, in all cases for which the second moment of $p(x, t)$ is finite (i.e., for $q < 5/3$) we obtain $\langle x^2 \rangle \propto t^{2/(3-q)}$. What happens for $0 \leq 5/3 < 3$ is that the prefactor of $t^{2/(3-q)}$ diverges. In this case, it is convenient to focus on the width of the distribution, or, equivalently, on $\langle x^2 \rangle_q$.

To close these remarks about Fokker-Planck equations, let us write down a quite general one, namely

$$\frac{\partial^4 p(x, t)}{\partial t^4} = D \frac{\partial^2 [p(x, t)]^{2-q}}{\partial |x|^\gamma} \quad (\delta, \gamma, q) \in \mathbb{R}^3),$$  \hspace{1cm} (56)

where we have used, for convenience, $q$ rather than $\nu$ (related through Eq. (53)). The regions we address are $0 < \delta \leq 1$, $0 < \gamma \leq 2$ and $q < 3$. Not surprising, the solutions for arbitrary $(\delta, \gamma, q)$ are not known. But, nevertheless, we depict the $\delta = 1$ regions in Fig. 1. This will clarify the particular cases that are related to the central limit theorems we are interested in.

**IV. CENTRAL LIMIT THEOREMS**

In one of its simplest versions, the standard central limit theorem may be formulated as follows. Let $\{x_l\}$

\begin{equation}
Z \equiv \sum_{l=1}^{N} x_l \quad (l = 1, 2, ..., N) \text{ be a set of independent random variables, each of them satisfying the same symmetric (with regard to } x = 0) \text{ distribution } p(x). \text{ Let also } p(x) \text{ be such that } \langle x^2 \rangle = \int \limits_{-\infty}^{\infty} dx x^2 p(x) \text{ is finite. Let us define now the sum variable } \langle x^2 \rangle_N = \sum_{l=1}^{N} x_l \text{.} \end{equation}

The question to which the central limit theorem answers is what is the probability distribution of the random variable $Z$ when $N \to \infty$? This answer happens to be very simple, namely a Gaussian (in the properly rescaled variables). If we call $p(Z, N)$ this distribution (with $p(Z, 1) = p(x)$), to exhibit the Gaussian we must use as abscissa $Z/\sqrt{N}$, and as ordinate $\sqrt{N} p(Z, N)$. Then, we gradually see emerging the Gaussian as $N \to \infty$. And, what specific Gaussian? The one whose second moment precisely coincides with $\langle x^2 \rangle$.

This theorem is closely related to normal diffusion, hence to Eq. (45) and to its solution (47), where $t$ plays the role of $N$.

Let us now address the other central limit theorem which is known, namely the Lévy-Gnedenko one. In its simplest version, it can be formulated as follows. As before, the set $\{x_l\}$ $l = 1, 2, ..., N$ is constituted of independent random variables, each of them satisfying

![Central Limit Theorems Diagram](image)
Then \( \gamma \) has been recently introduced \cite{16}. It is called \( Z \downarrow N \) when \( \gamma \). Various converging paths are nevertheless available. Let us be more specific: the typical case is that for which \( p(x) \) decays as \( 1/|x|^\mu \) with \( 1 < \mu < 3 \). Then \( \gamma = \mu - 1 \). And what specific Lévy distribution \( L_\gamma \)? The one which has precisely the same coefficient of the asymptotically dominant term. In other words, let us consider the finite value \( \lim_{|x| \to \infty} |p(x)|/|x|^\mu \). The particular Lévy distribution which is asymptotically approached when \( |x| \to \infty \) is the one which has the same value for \( \lim_{|x| \to \infty} |L_\gamma(x)|/|x|^\mu \). To see, on a graphic representation, the gradual emergence of the Lévy distribution while \( N \) increases, the abcissa must be \( Z/N^{1/\gamma} \), and the ordinate must be \( N^{1/\gamma} p(Z, N) \).

This theorem is closely related to the anomalous diffusion characterized by Eq. (48) and by its solution (49), where, as before, \( t \) plays the role of \( N \).

Let us finally address the conjectural theorem that we are focusing in this paper. It is of course the one to be associated with the nonlinear Eq. (51) and its solution (52). What hypothesis is to be violated in the two preceding and well studied theorems? The hypothesis of independence! Indeed, we believe that the variables \( \{x_i\} \) are to be assumed somehow collectively correlated in such a persistent manner that the correlation does not disappear even in the \( N \to \infty \) limit. For the standard central limit theorem, the quantity which is preserved is the second moment \( \langle x^2 \rangle \equiv \int_{-\infty}^{\infty} dx^2 p(x) \). For the Lévy-Gnedenko theorem, the quantity which is preserved is the coefficient \( \lim_{|x| \to \infty} |p(x)|/|x|^\mu \). For this conjectural theorem, some quantity is expected to be preserved. Could it be \( \langle x^2 \rangle_q \equiv \int_{-\infty}^{\infty} dx x^2 |p(x)|^q \)?

It should be transparently clear at this point that we have no definitive arguments for proving this conjectural theorem. Various converging paths are nevertheless available that might inspire a (professional or amateur) mathematician the way to prove it. Galileo used to say that knowing a result is not neglectable in order to prove it! It is our best hope that his saying does apply in the present case! So, what are these converging paths? Although naturally intertwined, let us expose them along six different lines.

1. The \( q \)-product hint

The following generalization of the product operation has been recently introduced \cite{16}. It is called \( q \)-product and is defined through

\[
X \otimes_q Y = [X^{1-q} + Y^{1-q} - 1]^{1/(1-q)} \quad (q \in \mathbb{R})
\]

We shall address the case where \( X \geq 1 \) and \( Y \geq 1 \). This product has the following properties:

(i) \( X \otimes Y = XY \);

(ii) \( \ln_q(X \otimes Y) = \ln_q X + \ln_q Y \) (whereas \( \ln_q(XY) = \ln_q X + \ln_q Y + (1-q)\ln_q Y \));

(iii) \( 1/(X \otimes q Y) = (1/X) \otimes_{2-q} (1/Y) \);

(iv) \( X \otimes_q (Y \otimes_q Z) = (X \otimes_q Y) \otimes_q Z = X \otimes_q Y \otimes_q Z = (X^{1-q} + Y^{1-q} + Z^{1-q} - 2)^{1/(1-q)} \);

(v) \( X \otimes_q 1 = X \);

(vi) \( X \otimes_q Y = Y \otimes_q X \);

(vii) For fixed \((X, Y)\), \( X \otimes_q Y \) monotonically increases with \( q \).

Property (ii) is particularly important since it is directly relevant to the extensivity of \( S_q \) that will be addressed soon.

Notice that property (iii) involves, like the solution of Eq. (51) with Eq. (53), a \( q \leftrightarrow (2-q) \) transformation.

We may apply this product to the number \( \mathcal{W}_{A_1+A_2+...+A_N} \) of allowed states in a composed system whose subsystems \( A_1, A_2, ..., A_N \) have respectively \( W_{A_1}, W_{A_2}, ..., W_{A_N} \) possible states (by allowed we mean that their probability is essentially nonzero). If \( q = 1 \) we have the total number \( \mathcal{W}_{A_1+A_2+...+A_N} = \prod_{l=1}^{N} W_{A_l} \) of states that are not only possible a priori, but even generically allowed. But, if \( q \neq 1 \), say \( q < 1 \), we expect correlations to inhibit (even forbidden occasionally) some of the states, i.e., to be associated with a probability close (in some sense) to zero. In this case, the effective total number \( \mathcal{W}_{A_1+A_2+...+A_N} \) of allowed states is expected to be smaller that \( \mathcal{W}_{A_1+A_2+...+A_N} \). We expect to have basically

\[
\mathcal{W}_{A_1+A_2+...+A_N}^\text{eff} \sim \mathcal{W}_{A_1} \otimes_{q} \mathcal{W}_{A_2} \otimes_{q} \cdots \otimes_{q} \mathcal{W}_{A_N} = \left[ \left( \sum_{l=1}^{N} W_{A_l}^{1-q} \right)^{1/(1-q)} - (N-1) \right]^{1/(1-q)} < W_{A_1+A_2+...+A_N} = \prod_{l=1}^{N} W_{A_l} \text{.}
\]

In particular, for \( q = 0 \), we have

\[
\mathcal{W}_{A_1+A_2+...+A_N}^\text{eff} = \left( \sum_{l=1}^{N} W_{A_l} \right) - N + 1 \text{.}
\]

If the \( N \) subsystems are all equal, we have that

\[
\ln_q \mathcal{W}_{A_1+A_2+...+A_N}^\text{eff} (N) \sim N \ln_q W(1),
\]

the changement of notation clearly being \( \mathcal{W}_{A_1+A_2+...+A_N}^\text{eff} = \mathcal{W}_{A_1+A_2+...+A_N} \) and \( W(1) \equiv W_{A_1} \). Consequently, for \( q = 1 \), we have

\[
\mathcal{W}_{A_1+A_2+...+A_N}^\text{eff} (N) = W(N) = |W(1)|^N,
\]

whereas, for \( q < 1 \), we have

\[
\mathcal{W}_{A_1+A_2+...+A_N}^\text{eff} (N) \sim \left\{ N \left[ |W(1)|^{1-q} - 1 \right] + 1 \right\}^{1/(1-q)} < W(N) = |W(1)|^N \text{.}
\]
We therefore see that, for \( q = 1 \), the number \( W^{\text{eff}}(N) \) of nonzero-probability states grows exponentially with \( N \), whereas, for \( q < 1 \), \( W^{\text{eff}}(N) \) grows like a power-law with \( N \), more precisely like \( N^{1/(1-q)} \). For \( q = 0 \), it grows linearly with \( N \), more precisely \( W^{\text{eff}}(N) = N[W(1) - 1] + 1 \).

2. The hint of the extensivity of \( S_q \)

We have recently argued (see references therein) that special correlations may exist between the \( N \) subsystems \( A_1, A_2, ..., A_N \) of a composed system such that one (and only one) value of the index \( q \) exists which ensures the additivity of \( S_q \). The trivial case of course is that of \( S_{BG} \). We consider the case of independency, i.e., the joint probabilities given by

\[
p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} = \prod_{i=1}^{N} p_{i_i}^{A_i}, \forall(i_1, i_2, ..., i_N).
\]

We straightforwardly verify that

\[
S_{BG}(A_1 + A_2 + ... + A_N) = \sum_{i=1}^{N} S_{BG}(A_i) \tag{65}
\]

where

\[
S_{BG}(A_1 + A_2 + ... + A_N) \equiv \sum_{i_1=1}^{W_{A_1}} \sum_{i_2=1}^{W_{A_2}} ... \sum_{i_N=1}^{W_{A_N}} \ln p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} \tag{66}
\]

and

\[
S_{BG}(A_i) \equiv - \sum_{i=1}^{W_{A_i}} p_{i_i}^{A_i} \ln p_{i_i}^{A_i}, \forall i. \tag{67}
\]

We can also verify

\[
\ln[1+(1-q)S_q(A_1+A_2+...+A_N)] = \sum_{i=1}^{N} \ln[1+(1-q)S_q(A_i)]. \tag{68}
\]

If the subsystems are all equal, we have

\[
S_{BG}(N) = NS_{BG}(1), \tag{69}
\]

the notation being self-explanatory.

If we consider \( N = 2 \) in Eq. (68) we obtain

\[
S_q(A_1 + A_2) = S_q(A_1) + S_q(A_2) + (1-q)S_q(A_1)S_q(A_2). \tag{70}
\]

Therefore, \( S_{BG} \) is said to be extensive (or additive). Consistently, \( S_q \) is, unless \( q = 1 \), nonextensive (or nonadditive). It is in fact from this property that the statistical mechanics we are talking about has been named nonextensive. As we shall exhibit in what follows, this early denomination might (unfortunately) be somewhat misleading. Indeed, \( S_q \) is, for \( q \neq 1 \), nonextensive if the subsystems are (explicitly or tacitly) assumed independent. But, if they are specially correlated, \( S_q \) can in fact be extensive for some special value of \( q \neq 1 \).

Consider now the following set of joint probabilities (clearly corresponding to nonindependent subsystems):

\[
p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} = \left( \sum_{i=1}^{N} p_{i_i}^{A_i} \right) - N + 1, \tag{71}
\]

and zero otherwise. Consequently, we have only

\[
W^{\text{eff}}_{A_1+A_2+...+A_N} = \left( \sum_{i=1}^{N} W_{A_i} \right) - N + 1 \text{ joint probabilities}
\]

which are generically nonzero, consistently with Eq. (60). We straightforwardly verify

\[
S_0(A_1 + A_2 + ... + A_N) = \sum_{i=1}^{N} S_0(A_i), \tag{73}
\]

where

\[
S_0(A_1 + A_2 + ... + A_N) = W^{\text{eff}}_{A_1+A_2+...+A_N} - 1, \tag{74}
\]

and

\[
S_0(A_i) = W_{A_i} - 1 \text{ (} \forall i \text{)}. \tag{75}
\]

Therefore, for the particular correlations involved in Eqs. (71) and (72), it is for \( q = 0 \) (and not for \( q = 1 \)) that \( S_q \) is additive.

Unfortunately, we do not know the general explicit form of the set of joint probabilities for an arbitrary number \( N \) of (not necessarily equal) subsystems, corresponding to \( q \neq 0, 1 \). We do know however, for \( 0 \leq q \leq 1 \), some special cases, such as the \( N = 2, 3 \) generic binary subsystems. They are presented and analyzed in [18]. We also know, for binary equal subsystems, exact recurrence relations that enable the calculation of the entire set of \( 2^N \) joint probabilities associated with \( N \) subsystems given the entire set of \( 2^{N-1} \) joint probabilities associated with \( N - 1 \) subsystems. This calculation is rather lengthy and will be the object of a separate paper.

Now we present here (in a form which is slightly different, and possibly more transparent, than the one used in [18]) the case of \( N = 2 \) equal binary subsystems. Consider the set of joint probabilities indicated in Table 1. Let us impose the additivity of the entropy, i.e., \( S_q(2) = 2S_q(1) \), where

\[
S_q(2) = \frac{1 - [p^2 + \kappa]q - 2[p(1-p) - \kappa]q - [(1-p)^2 + \kappa]q}{q - 1}, \tag{76}
\]

and

\[
S_q(1) = \frac{1 - p^q - (1-p)^q}{q - 1}. \tag{77}
\]
The relation between the correlation $\kappa$, $p$ and $q$ is then given by
\begin{equation}
2[p^q - (1-p)^q] - [p^2 + \kappa]^q - 2[p(1-p) - \kappa]^q - [(1-p)^2 + \kappa]^q = 1.
\end{equation}
(With the notation $f_q(p) = p^q + \kappa$, this relation coincides with the one indicated in Eq. (78).) In Fig. 2, typical $\kappa$ versus $p$ curves are presented. The lower curve corresponds to $p^2 + \kappa = 0$ and to $(1-p)^2 + \kappa = 0$. The upper curve corresponds to $p(1-p) - \kappa = 0$. The lower curve undoubtedly corresponds to $q = 0$. The upper curve could in principle correspond to both cases $q \to -\infty$ and $q \to \infty$. Indeed, for $\kappa = p(1-p)$, we have that $p^3 + \kappa = p$ and $(1-p)^2 + \kappa = 1 - p$, hence $S_q(2) = S_q(1)$. Since we must also satisfy $S_q(2) = 2S_q(1)$, only two possibilities emerge a priori, and these are $S_q(1) = 0$ (which corresponds to $q \to \infty$), and $S_q(1) \to \infty$ (which corresponds to $q \to -\infty$). Monotonicity with regard to $q$ suggests that the upper curve should correspond to $q \to \infty$. But on the other hand, having, among the four joint probabilities that are a priori possible in the present $N = 2$ system, two zeros (which definitively is what corresponds to the upper curve) appears as more restrictive than having one zero (which definitively is what corresponds to the lower curve). The generic four nonzero-probability case corresponds to $q = 1$, and the generic three nonzero-probability case corresponds to $q = 0$. Consequently, it appears as reasonable that the generic two nonzero-probability case would correspond to $q \to -\infty$. This kind of paradoxal situation appears to need further discussion to be clarified. The situation would probably be clear-cut if we had — but we do not! —, for arbitrary values of $N$ arbitrary subsystems, the general answer to be associated with arbitrary $q$. This point thus remains open.

| $A \setminus B$ | 1   | 2   |
|------------------|-----|-----|
| 1                | $p^2 + \kappa$ | $p(1-p) - \kappa$ |
| 2                | $p(1-p) - \kappa$ | $(1-p)^2 + \kappa$ | $1-p$ |

TABLE I: Joint probabilities for two binary subsystems $A$ and $B$. The marginal probabilities are indicated as well. The correlation $\kappa$ and the probability $p$ are such that all terms $p^2 + \kappa$, $p(1-p) - \kappa$, and $(1-p)^2 + \kappa$ remain within the interval $[0, 1]$. The case of independency corresponds to $\kappa = 0$.

It is worth stressing a logical consequence of what has been said up to now. Unless $q = 1$, we cannot simultaneously have equal probabilities for the set of joint probabilities of the allowed states and for the associated marginal probabilities. For example, for the $q = 0$ case of two equal binary subsystems, equal probabilities for the joint set means $p_{11}^{A+B} = p_{12}^{A+B} = p_{21}^{A+B} = 1/3$ and $p_{22}^{A+B} = 0$, hence $p = 2/3$ (which differs from $1 - p = 1/3$), whereas equal probabilities for the marginal sets means $p = 1/2$, hence $p_{11}^{A+B} = 0 \neq p_{12}^{A+B} = p_{21}^{A+B} = 1/2$. The $q = 1$ case simultaneously allows for $p_{11}^{A+B} = p_{12}^{A+B} = p_{21}^{A+B} = p_{22}^{A+B} = 1/4$ and $p = 1 - p = 1/2$. It is clear that, if such behavior (i.e., generic impossibility of equal probabilities for both joint and marginal ones simultaneously) persists up to the thermodynamic limit $N \to \infty$, it will have heavy consequences for the macroscopic statistics of the system.

Let us recall at this stage that, depending on whether the subsystems that we are composing are or are not independent, it is $S_{BG}$ or a different entropy which is additive. If they are correlated in the special form that has been illustrated above, and that persists up to the thermodynamic limit $N \to \infty$, it is $S_q$ with a specific value of $q \neq 1$ which becomes extensive. The corresponding statistical mechanics should consistently be based on $S_q$ (or on a directly related one, such as say $S_{2-q}$) and not anymore on $S_{BG}$. We may summarize this scenario through the following statement: Unless the composition law of the subsystems is specified, the question whether an entropy (or some similar quantity) is or is not extensive has no sense. Allow us a quick digression. The situation is totally analogous to the quick or slow motion of a body. Ancient greeks considered motion to be an absolute property. It was not until Galileo that it was clearly perceived that motion has no sense unless the referential is specified.
3. The hint of the q-generalization of the Pascal triangle

Very recently, Suyari and Tsukuda \cite{21} used the concept of q-product to consistently generalize various results that are widely known in the context of BG statistical mechanics and its mathematical structure. The generalization of the n! (n-factorial) operation, as well as of the binomial (and even multinomial) coefficients was performed. As a corollary, the Pascal triangle itself was q-generalized as well. It is known that the coefficients of the n-th line of the Pascal triangle yield (after appropriate centralization and rescaling) the Gaussian distribution in the $n \rightarrow \infty$ limit. This is well known to be a simple consequence of the standard central limit theorem as applied to independent binary variables. Suyari claims that, in the $n \rightarrow \infty$ limit of the q-generalized Pascal triangle, what is obtained is precisely the q-Gaussian distributions!

4. The hint of the scale-free networks

The mathematical study of random networks (or random graphs) started many decades ago. But quite recently, it has acquired great interest due to the fact that such structures appear to be ubiquitous in physical, social, internet and other complex phenomena (see \cite{21, 22} and references therein). A central quantity of such structures is the so-called connectivity or degree distribution, defined as the probability distribution of the number of links that are connected to the same site (or node); being more explicit, what one counts is the percentage of nodes that have a given number of links. For the important class of networks that are referred to as scale-free ones, this distribution is systematically found to be precisely a q-exponential. Many examples do exist in the literature. Three recent illustrations can be found in \cite{21, 22, 25}.

These scale-invariant structures exhibit hubs and sub-hubs, and so on. If embedded into a d-dimensional space, they tend to have zero Lebesgue measure. They appear to be like fractals, which implies that most of the locations of the d-dimensional space are forbidden (like the flights of any air company are expected to start and end only at the airports where that company operates, and not in any place of the territory!).

5. The nonlinear dynamical hint

We review at this point a few nonlinear dynamical results which, although not directly related to the possible third central limit theorem we are seeking, provide what we consider important connections within the mathematical structure which sustains nonextensive statistical mechanics.

We focus on classical nonlinear dynamical systems, either conservative (e.g., Hamiltonian systems) or dissipative. The basic ideas are quite general, but their presentation becomes easier if we illustrate them on a simple, one-dimensional, system. Let us address unimodal one-dimensional maps, such as for example the z-logistic map defined through

$$x_{t+1} = 1 - a|x_t|^z$$

$$(t = 0, 1, 2, ... \ ; \ z \geq 1; \ 0 \leq a \leq 2; \ -1 \leq x_t \leq 1)$$ \hspace{2em} (79)

The sensitivity $\xi$ to the initial conditions is defined as follows:

$$\xi(t) \equiv \lim_{\Delta x(0) \rightarrow 0} \frac{\Delta x(t)}{\Delta x(0)},$$ \hspace{2em} (80)

where $\Delta x(t)$ is the discrepancy at time $t$ of two values of $x_0$ initially separated by $\Delta x(0)$. The typical behavior of $\xi$ is given by

$$\xi(t) = e^{\lambda_1 t},$$ \hspace{2em} (81)

where $\lambda_1$ is referred to as the Lyapunov exponent (the subindex $1$ will become clear in a few lines). The following ordinary differential equation is satisfied: $d\xi / dt = \lambda_1 \xi$.

For most values of the control parameter $a$ (see Eq. (79)), $\lambda_1$ is nonzero. When it is positive, we shall say that there is strong chaos. When it is negative, we have regular orbits as attractors (e.g., fixed points, cycles-2, cycles-3, cycles-4, etc.). But there is an infinity of values of $a$ for which $\lambda_1$ vanishes. Examples are (i) all the values of $a$ for which there is a doubling-period bifurcation from one cycle to its double; (ii) all the values of $a$ for which there is a tangent bifurcation; (iii) all the values of $a$ for which there is an edge to chaos. All these cases are interesting, but by far the richest one is case (iii), referred to as weak chaos. Indeed, it is there that we expect the appearance of complexity (in physics, biology, economics, linguistics, and elsewhere), since it is the frontier between considerable order (regular orbits) and considerable disorder (strong chaos). So, the following question arises: what is the behavior of $\xi(t)$ when $\lambda_1$ vanishes? The typical behavior is as follows:

$$\xi(t) = e^{\lambda_2 t},$$ \hspace{2em} (82)

where we have $q > 1$ (with $\lambda_q > 0$) for the above cases (i) and (ii), and $q < 1$ (with $\lambda_q > 0$) for the above case (iii). The following ordinary differential equation is satisfied: $d\xi / dt = \lambda_q \xi^q$. The behavior (82) was conjectured in 1997 \cite{20} and was proved in 2003 \cite{21}. For example, for $z = 2$, the first edge of chaos appears at $a = a_e \approx 1.4011$. Its associated values are $q_{sen} = 0.2445...$ and $\lambda_{qsen} = 1/(1 - q_{sen}) = 1.3236...$, where the subscript $sen$ stands for sensitivity.

Let us now address another very important quantity, namely the entropy production $K$ per unit time. It is basically defined as follows. Take the admissible phase space (the interval $[-1, 1]$ for the variable $x$ in the map (79), for instance), and make a partition of it in $W$ little parts, identified through $i = 1, 2, ..., W$. Then consider
$N_0$ initial values of $x_0$ within one of the $W$ intervals. As a function of time, the points within the initial window will possibly spread around, in such a way that we have $N_i(t)$ points in the $i$-th interval ($\sum_{i=1}^W N_i(t) = N_0$). We define next the set of probabilities $p_i \equiv N_i(t)/N_0$ ($\forall i$), and finally we calculate the entropy $S_q(t)$ through Eq. (9) for any chosen value of $q$. By construction we have $S_q(0) = 0$. We then define the entropy production per unit time as follows

$$K_q \equiv \lim_{t \to \infty} \lim_{W \to \infty} \frac{S_q(t)}{t}.$$  

The rate $K_1$ coincides, in most cases, with the so-called Kolmogorov-Sinai entropy rate. The rate $K_q$ represents its $q$-generalization. Consistently with the Pesin theorem, we expect, whenever $\lambda_1 \geq 0$,

$$K_1 = \lambda_1.$$  

This is indeed verified (see, for instance, 28). For example, for $(z,a) = (2,2)$ in (79), we obtain $K_1 = \ln 2$. What happens however at the edge of chaos (say at $a_c(z)$)? Can we state something more informative than just $K_1 = \lambda_1 = 0$? Yes, we can. It can be numerically shown 29 and analytically proved 30 that

$$K_{q_{\text{sen}}} = \lambda_{q_{\text{sen}}}.$$  

This interesting relation (which refers in fact to upper bounds: see details in 31) basically generalizes the celebrated Pesin theorem, and was also conjectured in 1997 29. To be more specific, what happens is that there is a value of $q$ (and only one), noted $q_{\text{sen}}$, such that $K_q = 0$ for $q > q_{\text{sen}}$, and $K_q \to \infty$ for $q < q_{\text{sen}}$. And for $q = q_{\text{sen}}$ we obtain a finite value $K_{q_{\text{sen}}}$ which precisely coincides with $\lambda_{q_{\text{sen}}}$. When $\lambda_1 > 0$, $q_{\text{sen}} = 1$ (strong chaos); when $\lambda_1 = 0$ and the system is at the edge of chaos, $q_{\text{sen}} < 1$ and generically $\lambda_{q_{\text{sen}}} > 0$ (weak chaos). For example, for the edge of chaos of the universality class to which the map (79) belongs, $q_{\text{sen}}$ increases from $-\infty$ to slightly below unity when $\mu$ increases from 1 to $\infty$.

What happens when our nonlinear dynamical system has more than one, say $\mu$ nonnegative Lyapunov exponents $\{\lambda_{(m)}\}$ ($m = 1, 2, ..., \mu$)? From the Pesin theorem, we certainly expect

$$K_1 = \sum_{m=1}^{\mu} \lambda_{(m)}.$$  

What happens then if the system is at an edge of chaos, where $\lambda_{(m)} = 0, \forall m$? We expect 31 the following (conjectural) behavior for the Lebesgue measure $\xi$ associated with the dynamically expanding directions:

$$\xi \simeq \prod_{m=1}^{\mu} \xi_{(m)} = \prod_{m=1}^{\mu} \exp_{q_{\text{sen}}}^{(m)} (\lambda_{(m)}^{\text{(m)}} t),$$  

where the $(\geq)$ sign might become just (=) under some simplifying hypothesis (like orthogonality of the directions along which the expansions associated with the $\lambda_{(m)}^{(m)}$'s occur).

Since we essentially expect, for $t \to \infty$,

$$S_{q_e} \sim \ln_{q_e} \xi,$$  

(the subscript $e$ stands for entropy), we possibly have the following relation:

$$S_{q_e} \sim \ln_{q_e} \left[ \prod_{m=1}^{\mu} \exp_{q_{\text{sen}}}^{(m)} (\lambda_{(m)}^{(m)} q_{\text{sen}}^{-1} t) \right].$$  

Using the definition (83) we obtain the following interesting relation:

$$K_{q_e} = \lim_{t \to \infty} \ln_{q_e} \left[ \prod_{m=1}^{\mu} \exp_{q_{\text{sen}}}^{(m)} (\lambda_{(m)}^{(m)} q_{\text{sen}}^{-1} t) \right].$$  

Two important cases must be distinguished, namely strong and weak chaos. Strong chaos corresponds to $\lambda_{(m)} > 0, \forall m$. In this case, we have $q_e = q_{\text{sen}} = 1, \forall m$, hence Eq. (90) straightforwardly recovers relation (86), consistently with the Pesin theorem. Weak chaos corresponds to $\lambda_{(m)} = 0, \forall m$. In this case, by focusing on the $t \to \infty$ asymptotic region, we have

$$K_{q_e} = \prod_{m=1}^{\mu} \left[ 1 - (1 - q_{\text{sen}}) \lambda_{(m)}^{(m)} q_{\text{sen}}^{-1} t \right]^{-1} \times $$

$$\lim_{t \to \infty} \frac{\ln_{q_e} \left[ \prod_{m=1}^{\mu} \exp_{q_{\text{sen}}}^{(m)} (\lambda_{(m)}^{(m)} q_{\text{sen}}^{-1} t) \right]}{t},$$  

consequently (since $q_e$ must be chosen so that $K_{q_e}$ is finite)

$$1 - q_{\text{sen}} = \sum_{m=1}^{\mu} \frac{1}{1 - \lambda_{(m)}^{(m)} q_{\text{sen}}^{-1}},$$  

and

$$K_{q_e} = \prod_{m=1}^{\mu} \left[ 1 - (1 - q_{\text{sen}}) \lambda_{(m)}^{(m)} q_{\text{sen}}^{-1} t \right]^{-1} \times \frac{1}{1 - q_e}.$$  

Eq. (93) can be rewritten in a more symmetric form, namely

$$[(1 - q_{\text{sen}})^{-1}]^{-1} = \prod_{m=1}^{\mu} \left[ 1 - (1 - q_{\text{sen}}) \lambda_{(m)}^{(m)} q_{\text{sen}}^{-1} t \right]^{-1} \times \frac{1}{1 - \lambda_{(m)}^{(m)} q_{\text{sen}}^{-1}}.$$  

If $\mu = 1$, Eq. (92) recovers $q_e = q_{\text{sen}}$, and Eq. (94) recovers Eq. (85).

Another interesting particular case is when $q_{\text{sen}}^{(m)} = q_{\text{sen}} (\forall m)$, and $\lambda_{(m)}^{(m)} = \lambda_{q_{\text{sen}}} (\forall m)$. From Eqs. (92) and (94) it follows then

$$1 - q_e = \frac{1 - q_{\text{sen}}}{\mu}.$$  


and
\[ K_{q_e} = \mu \lambda_{q_{en}}. \tag{96} \]

An example which verifies Eq. (95) can be found in [31].

One more particular case which is interesting is when the expansion is linear in time for all directions, i.e., \( q^{(m)}_{en} = 0 (\forall m) \). We then have, from Eqs. (92) and (94),
\[ q_e = 1 - \frac{1}{\mu}, \tag{97} \]

and
\[ K_{q_e} = \mu \left( \prod_{m=1}^{\mu} \lambda_{q_{en}}^{(m)} \right)^{1/\mu}. \tag{98} \]

Relation (97) has already emerged in the literature through various forms. A first example concerns \( \mu = 1 \), hence we expect \( q_e = 0 \). This is precisely what can be verified \[32\] for the Casati-Prosen triangle map \[33\], which is a two-dimensional, conservative (hence \( \mu = 1 \)), mixing, ergodic one, with linear instability. In addition to \( q_e = 0 \), it has been verified that \( K_e = \lambda_0 \), in accordance with Eq. (98). A second example is the \( \mu \)-dimensional lattice Lotka-Volterra, for which it has precisely been verified \[34, 35\], the result (97). A third example is a specific \( d \)-dimensional Boltzmann lattice model \[36\]. Its Hamiltonian-like behavior leads to \( \mu = d/2 \), which, from Eq. (97), implies \( q_e = 1 - 2/d \). It is precisely this result that the authors \[36\] have obtained by imposing Galilean invariance to the dynamical equations of their model.

It is clear that the interesting (and possibly exact) relations (92) and (94) remain to be proved. At the present stage they constitute but conjectures.

6. The empirical hint

Last but not least, let us present a very pragmatic, epistemological-like, reason. There are, in the literature, already so many natural and artificial systems (and their number constantly increases) whose central quantities are well fitted by \( q \)-exponentials and \( q \)-Gaussians, that one is compelled to believe that only a limit theorem could explain such an ubiquity.

V. CONCLUSIONS

In this final section, let us summarize the scenario within which we are working. It appears that the entropic form \( S(N; \{ p_i \}) \) to be used for constructing a statistical mechanics that is naturally compatible with usual macroscopic thermodynamics should be \textit{generically} extensive, i.e., such that
\[ 0 \leq \lim_{N \to \infty} \frac{S(N; \{ p_i \})}{N} < \infty, \tag{99} \]

the equality occurring only for the case of certainty or (thermodynamically) close to it (the typical example is a system in thermal equilibrium at \textit{zero temperature}). Within this (axiomatic) viewpoint, we can distinguish the following cases:

(i) The \( N \) subsystems (typically physical elements such as particles free to move translationally and/or rotationally, localized spins, and similar entities) are \textit{probabilistically independent} (the paradigmatic case is that of ideal gases; their total energy is strictly proportional to \( N \), i.e., additive, hence \textit{extensive}). Then we must use \( S_{BG} \) since \( S_{BG}(N; \{ p_i \}) \propto N (\forall \{ p_i \}) \).

(ii) The \( N \) subsystems are \textit{locally correlated} (the paradigmatic case is that of \textit{short-range-interacting} many-body Hamiltonian systems; their total energy is asymptotically proportional to \( N \), i.e., it is \textit{extensive} once again). Then, once again, we must use \( S_{BG} \) since it satisfies Eq. (99), \( \forall \{ p_i \} \).

(iii) The \( N \) subsystems are \textit{globally correlated} in the special manner addressed in this paper (the paradigmatic case appears to be that of \textit{long-range-interacting} many-body Hamiltonian systems; their total energy asymptotically increases with \( N \) faster than linearly, i.e., it is \textit{nonextensive}). Then, we must use \( S_q \) for that unique value of \( q \) which guarantees Eq. (99), \( \forall \{ p_i \} \).

(iv) The \( N \) subsystems are \textit{globally correlated} in a manner which is more complex (or just as complex but in a different manner) than the one addressed in this paper. Then we must use an entropy which is \textit{not} included in the family \( S_q \) for any value of \( q \). Such an entropy would have to be either more general than \( S_q \) (see for instance \[37\]), or just of a completely different type (see for instance \[38, 40, 41\]).

The cases (i) and (ii), which we may call \textit{simple} in the sense of \textit{plectics} \[42\], belong to the world within which the concepts used in \textit{BG} statistical mechanics have, since more than one century, been profusely shown to be the appropriate ones.

The cases (iii) and (iv), which we may call \textit{complex} in the sense of \textit{plectics} \[42\], belong to the world within which the concepts used in nonextensive statistical mechanics (as well as in its possible generalizations or alternatives) have been shown and keep being shown (since more than one decade, by now) to be the appropriate ones.

The cause for a system to be \textit{simple} or \textit{complex} in the above sense lies basically (see \[3, 43\]) on its microscopic dynamics in the \textit{full} space of microscopic possibilities (Gibbs’ \( \Gamma \)-space for many-body Hamiltonian systems). It is believed to be so because it is this dynamics which is expected to determine the possible persistent correlations that would be responsible for the geometrical structure within which the system tends to “live”, given its initial conditions.

If its elementary nonlinear dynamics is controlled by \textit{strong} sensitivity to the initial conditions (i.e., at least one \textit{positive} Lyapunov exponent if the system is a classical one), we expect the system to be \textit{simple}. For virtually any initial condition, the system quickly visits the
neighborhood of virtually all possible states; it does so in such a way that the probability of any small part of the (macroscopically) admissible full space asymptotically becomes proportional to its size (Lebesgue measure). Its main time-dependent functions (typically) exponentially depend on time, the number of allowed stated exponentially increases with $N$, and, if the system is Hamiltonian, its stationary state (called thermal equilibrium) is characterized by an energy distribution given by the celebrated $BG$ weight. Summarizing, its paradigmatic ordinary differential equation is $dy/dx \propto y$.

If the elementary nonlinear dynamics of the system is nonregular and controlled by weak sensitivity to the initial conditions (i.e., no positive Lyapunov exponents if the system is a classical one), we expect the system to be complex. For given initial conditions, the system essentially visits a network of states (a scale-free network for many if not all $q$-systems) whose typical Lebesgue measure is zero. The particular network depends from the initial conditions and is highly inhomogeneous (like, as mentioned before, the network of airports on which a specific air company operates), but its geometry (both topology and metrics) is basically the same for virtually all initial conditions. To recover a homogeneous occupancy of the full space we are obliged to make averages over all the initial conditions (whereas no such thing is necessary for simple systems). The main time-dependent functions of the system typically depend on time slower than exponentially (typically like power-laws, more precisely like $q$-exponentials for most if not all the $q$-systems), the number of allowed states increases with $N$ like a power-law, and, if the system is Hamiltonian, its stationary or quasi-stationary (metastable) state (out of thermal equilibrium) is expected to be characterized by an energy distribution given by a $q$-exponential weight, which asymptotically approaches a power-law for high energies. Summarizing, its paradigmatic ordinary differential equation is $dy/dx \propto y^q$.

The standard central limit theorem plays a crucial role for the simple systems. We expect a similar theorem to exist for the complex systems of the $q$-class. The proof of such a theorem would be priceless.

In addition to the above, we have presented here more two conjectures (Eqs. (92) and (94)) concerning the entropy production per unit time for a nonlinear dynamical system at the edge of chaos, having $\mu$ vanishing Lyapunov exponents. The entropy $S_q$ which increases linearly with time (when the system is exploring its phase space) appears to be that whose entropic index is $q_e$ as given by Eq. (92). The associated entropy production $K_{q_e}$ per unit time appears to be as given by Eq. (94). The proof of these two connected conjectures (including, naturally, the precise conditions for their validity) also remains as a mathematically open problem.

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