Examples of Unbounded Homogeneous Domains in Complex Space ∗ †

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We construct several new examples of homogeneous domains in complex space that do not have bounded realisations. They are equivalent to tubes over affinely homogeneous domains in real space and have a real-analytic everywhere Levi non-degenerate non-umbilic boundary. Using the geometry of the boundary, we determine the full automorphism groups of the domains. We also discuss some interesting examples of tube domains with everywhere umbilic boundary (i.e., boundary equivalent to the corresponding quadric).

0 Introduction

The study of bounded holomorphically homogeneous domains in complex space goes back to É. Cartan [C] who determined all bounded symmetric domains in \( \mathbb{C}^n \) as well as all bounded homogeneous domains in \( \mathbb{C}^2 \) and \( \mathbb{C}^3 \). A fundamental theorem due to Vinberg, Gindikin, and Pyatetskii-Shapiro states that every bounded homogeneous domain is biholomorphically equivalent to a Siegel domain of the second kind (see [P-S]). Although this result does not immediately imply a complete classification of bounded homogeneous domains, it reduces the classification problem to that for domains of a very special form. Siegel domains are unbounded by definition (although they possess bounded realisations) and it is therefore natural to consider not necessarily bounded homogeneous domains. However, the classification problem in the unbounded case is extremely hard and is far from complete, despite the existence of a substantial theory of homogeneous manifolds and spaces (see, e.g., [A]). Therefore, any new examples of homogeneous domains that possess no bounded realisations are of interest. In this paper we give such examples.

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One way to attempt to produce examples of this sort is to consider tube domains, that is domains of the form $D_\Omega = \Omega + i\mathbb{R}^n$, where $\Omega$ is a domain in $\mathbb{R}^n \subset \mathbb{C}^n$. Such domains are clearly unbounded. Further, if $\Omega$ is affinely homogeneous, then $D_\Omega$ is homogeneous as a domain in $\mathbb{C}^n$ since every affine mapping of $\mathbb{R}^n$ can be lifted to an affine mapping of $\mathbb{C}^n$ and since $D_\Omega$ is invariant under imaginary translations. To construct $\Omega$ one can start with an affinely homogeneous hypersurface $\Gamma \subset \mathbb{R}^n$ and let $\Omega$ be a domain on one side of $\Gamma$. Such a domain $\Omega$ has a chance of being affinely homogeneous. More precisely, we might hope that the orbits of the affine symmetry group of $\Gamma$ be, in addition to $\Gamma$ itself, the domains to either side of it. Of course, in this case, the symmetry group must have dimension at least $n$ and so its action on $\Gamma$ must have isotropy. Examples of affinely homogeneous hypersurfaces $\Gamma \subset \mathbb{R}^n$ can be taken for instance from the explicit classifications of affinely homogeneous curves in $\mathbb{R}^2$ (see, e.g., [NS2]), surfaces in $\mathbb{R}^3$ [DKR], [EE1], or equiaffinely homogeneous hypersurfaces in $\mathbb{C}^3$ [NS1] and $\mathbb{C}^4$ [EE2]. A classification of affine homogeneous hypersurfaces with isotropy may be found in [EE3]. Of course, one must independently verify that the domain $D_\Omega$ so constructed is not biholomorphically equivalent to any bounded domain and is indeed homogeneous.

The paper is organised as follows. We construct our examples in Section 1. They are domains in $\mathbb{C}^4$ arising from domains on each side of certain affinely homogeneous hypersurfaces $\Gamma_\alpha \subset \mathbb{R}^4$, where $\alpha$ is a real parameter. We verify that the domains in question are indeed homogeneous and are not biholomorphically equivalent to any bounded domain. Further, for a homogeneous domain it is always desirable to know the full group of holomorphic automorphisms, and in Section 2 we determine the automorphism groups of the domains from Section 1. In order to do this, we write the domains in a non-tubular form and study the automorphism group of the boundary which is a real-analytic everywhere Levi non-degenerate non-umbilic hypersurface. In connection with this we recall (see e.g., [R]) that a bounded homogeneous domain with smooth boundary is biholomorphically equivalent to the unit ball. In the unbounded case there are many more domains with smooth boundary, and the examples that we construct in Section 1 are not equivalent to any “ball-like” domains that we discuss further in the paper.

Sometimes a non-trivially looking tube domain in $\mathbb{C}^n$, whose boundary is a tube hypersurface over a homogeneous hypersurface in $\mathbb{R}^n$, turns out to be holomorphically equivalent to a simple well-known domain. For example the
domains lying on either side of the quadric

\[ \text{Re } z_n = \langle z', z' \rangle, \tag{0.1} \]

where \( \langle z', z' \rangle \) is a Hermitian form in the space of the first \( n - 1 \) variables \( z' := (z_1, \ldots, z_{n-1}) \) are well-known to be homogeneous. Every such domain possesses a tubular realisation over an affinely homogeneous domain in \( \mathbb{R}^n \). Moreover, the convex side of the positive definite quadric admits a bounded realisation as the the unit ball. For \( \alpha = 1/12 \) the domains that we construct in Section 1 are, in fact, of this type. In Section 3 we discuss more examples of tube domains that are holomorphically equivalent to domains lying on one side of hypersurfaces (0.1) and announce a theorem that states that the number of such domains is quite substantial. Therefore, when considering homogeneous tube domains, one should carefully rule out domains arising in this manner from quadrics (0.1). This is often a non-trivial task. As an example, we mention a letter by D’Atri (reproduced in the preface to [G]) where he apparently found a new homogeneous domain, as one side of the tube over the Cayley hypersurface (3.1). In fact, as shown in Section 3, the domain that D’Atri considered is equivalent to a domain on one side of the quadric (0.1). In Section 3 more examples of this kind are given.

Before proceeding we would like to acknowledge that this work started while the second author was visiting the University of Adelaide.

## 1 The Examples

In accordance with the general scheme outlined above we consider a one-parameter family of affinely homogeneous hypersurfaces in \( \mathbb{R}^4 \) that occurs in the classifications in [EE2] and [EE3]:

\[ \Gamma_\alpha := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 = x_1 x_2 + x_3^2 + x_1^2 x_3 + \alpha x_1^4 \}, \alpha \in \mathbb{R}. \tag{1.1} \]

We are interested in the domains on each side of \( \Gamma_\alpha \):

\[ \Omega^\alpha_> := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 > x_1 x_2 + x_3^2 + x_1^2 x_3 + \alpha x_1^4 \}, \]

\[ \Omega^\alpha_< := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 < x_1 x_2 + x_3^2 + x_1^2 x_3 + \alpha x_1^4 \}. \]
One can verify that $\Omega^>_{\alpha}$ and $\Omega^<_{\alpha}$ are invariant under the following four subgroups of affine transformations of $\mathbb{R}^4$:

\[
\begin{align*}
\phi_q : & \\
& x_1 \mapsto qx_1, \\
& x_2 \mapsto q^3x_2, \\
& x_3 \mapsto q^2x_3, \\
& x_4 \mapsto q^4x_4,
\end{align*}
\]

with $q \in \mathbb{R}^*$,

\[
\psi_r : \\
x_1 \mapsto x_1 + r, \\
x_2 \mapsto -4\alpha(4\alpha - 1)r^2x_1 + x_2 + 2(4\alpha - 1)r x_3 - \frac{4}{3} \alpha(4\alpha - 1)r^3, \\
x_3 \mapsto -4\alpha r x_1 + x_3 - 2\alpha r^2, \\
x_4 \mapsto \frac{4}{3} \alpha(4\alpha - 1)r^3 x_1 + r x_2 + (4\alpha - 1)r^2 x_3 + x_4 - \frac{4}{3}\alpha(4\alpha - 1)r^4,
\]

with $r \in \mathbb{R}$,

\[
\mu_s : \\
x_1 \mapsto x_1, \\
x_2 \mapsto x_2 + s, \\
x_3 \mapsto x_3, \\
x_4 \mapsto sx_1 + x_4,
\]

with $s \in \mathbb{R}$,

\[
\nu_t : \\
x_1 \mapsto x_1, \\
x_2 \mapsto -tx_1 + x_2, \\
x_3 \mapsto x_3 + t, \\
x_4 \mapsto 2tx_3 + x_4 + t^2,
\]

with $t \in \mathbb{R}$.

We will now show that $\Omega^>_{\alpha}$ is affinely homogeneous. Take the point $\textbf{0} = (0,0,0,1) \in \Omega^>_{\alpha}$ and apply the mapping $F_{q,s,t,r} := \phi_q \circ \mu_s \circ \nu_t \circ \psi_r$ to it. The result is the point

\[
\begin{pmatrix}
qr, \\
q^3(-\frac{4}{3} \alpha(4\alpha - 1)r^3 - tr + s), \\
q^2(-2\alpha r^2 + t), \\
q^4(1 - \frac{4}{3} \alpha(4\alpha - 1)r^4 - 4\alpha tr^2 + t^2 + sr)
\end{pmatrix}.
\]
Let \((x_1^0, x_2^0, x_3^0, x_4^0)\) be any other point in \(\Omega_\alpha^>\). Then setting
\[
q = \left( x_4^0 - x_1^0 x_2^0 - (x_3^0)^2 - (x_1^0)^2 x_3^0 - \alpha (x_1^0)^4 \right)^{1/4},
\]
\[
r = \frac{x_1^0}{q},
\]
\[
s = \frac{1}{q^3} \left( x_2^0 + \frac{4}{3} \alpha (4 \alpha - 1) (x_1^0)^3 + x_1^0 x_3^0 + 2 \alpha (x_1^0)^3 \right),
\]
\[
t = \frac{1}{q^2} \left( x_3^0 + 2 \alpha (x_1^0)^2 \right),
\]
we obtain an affine automorphism \(F_{q,r,s,t,r}\) of \(\Omega_\alpha^>\) that maps \((0, 0, 0, 1)\) into \((x_1^0, x_2^0, x_3^0, x_4^0)\). This proves that \(\Omega_\alpha^>\) is affinely homogeneous. A similar
argument shows that \(\Omega_\alpha^<\) is affinely homogeneous as well.

We will now consider the corresponding tube domains \(D_{\Omega_\alpha^>}, D_{\Omega_\alpha^<} \subset \mathbb{C}^4\). Since \(\Omega_\alpha^>\) and \(\Omega_\alpha^<\) are affinely homogeneous, \(D_{\Omega_\alpha^>}\) and \(D_{\Omega_\alpha^<}\) are holomorphically homogeneous. We may write these domains in a different form. Suppose firstly that \(\alpha \neq 1/12\). Then the mapping
\[
z_1 \mapsto \left| \frac{3}{2} \left( \alpha - \frac{1}{12} \right) \right|^{1/4} z_1,
\]
\[
z_2 \mapsto z_2 + z_1 z_3 + \alpha z_1^3,
\]
\[
z_3 \mapsto \left| \frac{3}{2} \left( \alpha - \frac{1}{12} \right) \right|^{1/4},
\]
\[
z_3 \mapsto \sqrt{2} \left( z_3 + \frac{z_1^2}{4} \right),
\]
\[
z_4 \mapsto 4z_4 - 2z_1 z_2 - 2z_3^2 - z_1 z_3 - \frac{\alpha}{2} z_1^4
\]
transforms \(D_{\Omega_\alpha^>}\) and \(D_{\Omega_\alpha^<}\) into
\[
D_{\Omega_\alpha^>}^> := \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \text{Re} \ z_4 > z_1 \overline{z_2} + z_2 \overline{z_1} + |z_3|^2 + |z_1|^4 \right\} \quad (1.2)
\]
and
\[
D_{\Omega_\alpha^<}^< := \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \text{Re} \ z_4 < z_1 \overline{z_2} + z_2 \overline{z_1} + |z_3|^2 + |z_1|^4 \right\} \quad (1.3)
\]
respectively, if $\alpha > 1/12$, and into
\[ D^> := \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \Re z_4 > z_1z_2 + z_2z_1 + |z_3|^2 - |z_1|^4 \right\} \] (1.4)
and
\[ D^<_\alpha := \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \Re z_4 < z_1z_2 + z_2z_1 + |z_3|^2 - |z_1|^4 \right\} \] (1.5)
respectively, if $\alpha < 1/12$. If, however, $\alpha = 1/12$, then the mapping
\[ z_1 \mapsto \frac{1}{\sqrt{2}} \left( z_1 + z_2 + z_1z_3 + \frac{1}{12}z_1^3 \right), \]
\[ z_2 \mapsto \sqrt{2} \left( z_3 + \frac{z_3^2}{4} \right), \]
\[ z_3 \mapsto \frac{1}{\sqrt{2}} \left( z_1 - z_2 - z_1z_3 - \frac{1}{12}z_1^3 \right), \]
\[ z_4 \mapsto 4z_4 - 2z_1z_2 - 2z_2^2 - z_1^2 z_3 - \frac{1}{24}z_1^4 \]
transforms $D^>_{\Omega_1}$ and $D^<_\alpha$ into
\[ D^>_{\Omega_1} := \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \Re z_4 > |z_1|^2 + |z_2|^2 - |z_3|^2 \right\} \] (1.7)
and
\[ D^<_\alpha := \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \Re z_4 < |z_1|^2 + |z_2|^2 - |z_3|^2 \right\} \] (1.8)
respectively.

We discuss the domains $D^>_{\Omega_1}$ and $D^<_\alpha$ in greater generality in Section 3 and for the moment concentrate on the domains $D^>_{\pm}, D^<_\pm$. We note that none of these domains is biholomorphically equivalent to a bounded domain. In fact, all these domains are not Kobayashi-hyperbolic (we remark here that it is shown in [N] that any connected homogeneous Kobayashi-hyperbolic manifold is biholomorphically equivalent to a bounded domain in complex space). Indeed, domains $D^>_{\pm}, D^<_\pm$ contain the affine complex line $\{ z_1 = 0, z_3 = 0, z_4 = 1 \}$ and the domains $D^<_\pm, D^<_\pm$ the affine complex line $\{ z_1 = 0, z_3 = 0, z_4 = -1 \}$. Hence, we have proved the following

**THEOREM 1.1** The domains $D^>_{\pm}, D^<_\pm \subset \mathbb{C}^4$ defined by (1.2)–(1.5) are holomorphically homogeneous and not biholomorphically equivalent to any bounded domain.
2 The Automorphism Groups of $D^>_{\pm}$ and $D^<_{\pm}$

In this section we determine the groups Aut ($D^>_{\pm}$) and Aut ($D^<_{\pm}$) of holomorphic automorphisms of $D^>_{\pm}$ and $D^<_{\pm}$ respectively. Let $P_{\pm}$ be the following subgroup of holomorphic transformations of $\mathbb{C}^4$:

\begin{align*}
t_1 &\mapsto qe^{i\phi}t_1 + \rho, \\
t_2 &\mapsto (\mp 2|\rho|^2qe^{i\phi} + q^2b)t_1 + q^3e^{i\phi}\bar{t}_1 + qdtd_3 + 2\overline{\rho}q^2e^{2i\phi}\bar{t}_1 + \sigma, \\
t_3 &\mapsto -de^{i(\phi + \psi)}t_1 + q^2e^{i\psi}\bar{t}_3 + \tau, \\
t_4 &\mapsto (2\sigma qe^{i\phi} + 2\overline{\rho}q^2b - 2\overline{\rho}de^{i(\phi + \psi)})t_1 + 2\overline{\rho}q^3e^{i\phi}\bar{t}_1 + + (2\overline{\rho}d + 2\overline{\sigma}q^2e^{i\psi})t_3 + q^4t_4 + 2\overline{\rho}q^2e^{2i\phi}\bar{t}_1^2 + \rho\sigma + \sigma\overline{\rho} + |
\end{align*}

where $q > 0$, $\phi, \psi, u \in \mathbb{R}$, $\rho, \sigma, \tau, b, d \in \mathbb{C}$, $\text{Re}(e^{i\phi}d) \leq 0$ and

\begin{equation}
|d|^2 = -2q^3\text{Re}(e^{i\phi}d). 
\end{equation}

It can be checked directly that $P_\pm$ preserves $D^>_{\pm}$ and $D^<_{\pm}$, so $P_\pm \subset \text{Aut}(D^>_{\pm})$ and $P_\pm \subset \text{Aut}(D^<_{\pm})$.

Below we prove the following theorem.

**THEOREM 2.1** $\text{Aut}(D^>_{\pm}) = \text{Aut}(D^<_{\pm}) = P_\pm$.

To prove Theorem 2.1 we deal with the automorphism group of $M_\pm := \partial D^>_{\pm} = \partial D^<_{\pm}$. Denote by Aut$(M_\pm)$ the group of CR-automorphisms of $M_\pm$ equipped with the topology of uniform convergence of the derivatives of all orders of the component functions on compact subsets of $M_\pm$. The Levi form of $M_\pm$ at every point is non-degenerate, and therefore Aut$(M_\pm)$ is a Lie group in this topology [T]. We need the following theorem that implies Theorem 2.1.

**THEOREM 2.2** $\text{Aut}(M_\pm) = P_\pm$. 

We first show how Theorem 2.1 follows from Theorem 2.2. Since the Levi form of $M_\pm$ at every point has eigenvalues of opposite signs, every element $f$ of $\text{Aut}(D_+^\pm)$ or $\text{Aut}(D_-^\pm)$ extends past $M_\pm$ to a biholomorphic map between neighbourhoods of $D_+^\pm$ and $D_-^\pm$ respectively thus giving rise to an element \( g \in \text{Aut}(M_\pm) \). Clearly, $f$ is uniquely determined by $g$. Since $\text{Aut}(M_\pm) = P_\pm$ by Theorem 2.2, we obtain $\text{Aut}(D_+^\pm) = \text{Aut}(D_-^\pm) = P_\pm$ as required.

**Proof of Theorem 2.2:** First we note that $P_\pm$ considered as a subgroup of $\text{Aut}(M_\pm)$ is closed and is therefore a Lie subgroup of $\text{Aut}(M_\pm)$. Clearly, the topology induced on $P_\pm$ by $\text{Aut}(M_\pm)$ coincides with that induced on $P_\pm$ by its parameters as in formula (2.1), and hence the dimension of $P_\pm$ as a subgroup of $\text{Aut}(M_\pm)$ is equal to 13. Therefore $\dim \text{Aut}(M_\pm) \geq 13$.

We will now show that $\dim \text{Aut}(M_\pm) \leq 13$. Since $M_\pm$ is homogeneous, it is sufficient to prove that $\dim I_0(M_\pm) \leq 6$, where $I_0(M_\pm) \subset \text{Aut}(M_\pm)$ is the isotropy subgroup of the point $0 \in M_\pm$. More generally, we prove the following proposition.

**Proposition 2.3** Let $S \subset \mathbb{C}^4$ be a real-analytic hypersurface passing through the origin and assume that the Levi form of $S$ is everywhere non-degenerate and has signature $(2,1)$. Suppose further that $S$ is non-umbilic at 0. Let $I_0(S)$ denote the Lie group of all CR-automorphisms of $S$ preserving 0. Then $\dim I_0(S) \leq 6$.

**Proof of Proposition 2.3:** We will use the Chern-Moser normal form [CM] for $S$. Namely, we will say that $S$ in coordinates $z = (z_1, z_2, z_3), w = u + iv$ near 0 is written in the Chern-Moser normal form if $S$ near 0 is given by an equation

$$u = \langle z, z \rangle + \sum_{k,l \geq 2} F_k(z, \overline{z}, v),$$

(2.3)

where $\langle z, z \rangle$ is a non-degenerate Hermitian form of signature $(2,1)$ and $F_k(z, \overline{z}, v)$ are polynomials of degree $k$ in $z$ and $\overline{z}$ in $v$ whose coefficients are analytic functions of $v$ such that the following conditions hold

$$\text{tr} F_k \equiv 0,$$

$$\text{tr}^2 F_k \equiv 0,$$

$$\text{tr}^3 F_k \equiv 0.$$
with the operator $\text{tr}$ defined as follows. Let $\langle z, z \rangle = \sum_{\alpha, \beta=1}^{4} h^{\alpha \beta} z_\alpha z_\beta$ and let $(g_{\alpha \beta})$ be the matrix inverse to $(h^{\alpha \beta})$. Then we have

$$\text{tr} := \sum_{\alpha, \beta=1}^{4} g_{\alpha \beta} \frac{\partial^2}{\partial z_\alpha \partial z_\beta}.$$ 

Let $J_0(S)$ denote the group of all local CR-automorphisms of $S$ defined near the origin and preserving it. It follows from [E] that near 0 one can choose holomorphic coordinates in which $S$ is given in the Chern-Moser normal form and every element of $J_0(S)$ is written as

$$z \mapsto \lambda U z, \quad w \mapsto \lambda^2 w,$$

(2.4)

where $U$ is a matrix such that $\langle U z, U z \rangle = \langle z, z \rangle$ and $\lambda > 0$. Without loss of generality we assume that $\langle z, z \rangle = |z_1|^2 + |z_2|^2 - |z_3|^2$ and hence the matrix $U$ satisfies

$$U^t H U = H,$$

(2.5)

where

$$H := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

The group $G$ of all matrices given by condition (2.5) is isomorphic to the usual group $U(2, 1)$ of pseudo-unitary matrices $V$ satisfying

$$V H V^* = H$$

by means of the mapping

$$V = (U^t)^{-1}.$$ 

It follows from [B], [L] that, for every element of $J_0(S)$, $\lambda$ is uniquely determined by the corresponding matrix $U$ by means of an algebraic relation. This implies that the subgroup $G_0 \subset G$ of matrices $U$ arising from automorphisms in $J_0(S)$ is a closed subgroup of $G$. In addition, the mapping $U \mapsto \lambda$ is a Lie group homomorphism from $G_0$ into $\mathbb{R}^*$. 

We will now plug an automorphism of the form (2.4) into equation (2.3). We obtain:

$$\frac{1}{\lambda^2} \sum_{k,l \geq 2} F_{k,l}(\lambda U z, \lambda^2 v) = \sum_{k,l \geq 2} F_{k,l}(z, v).$$

(2.6)
Since $S$ is not umbilic at $0$, $F_{2\Gamma}(z,\overline{z},0) \neq 0$. Extracting from (2.6) terms independent of $v$ of degree 2 in each of $z$ and $\overline{z}$ we obtain:

$$F_{2\Gamma}(Uz,\overline{Uz},0) = \frac{1}{\lambda^2} F_{2\Gamma}(z,\overline{z},0). \quad (2.7)$$

Hence a necessary condition for a matrix $U \in G$ to belong to $G_0$ is the preservation of the term $F_{2\Gamma}(z,\overline{z},0)$ up to a scalar multiple as in (2.7). We will show that condition (2.7) implies that $\dim G_0 \leq 6$.

Suppose first that $\dim G_0 = 9$, i.e., $G_0 = G$. Since there does not exist a non-trivial homomorphism from $G$ into $\mathbb{R}^*$, we have $\lambda = 1$ for all $U$. Then (2.7) gives that $F_{2\Gamma}(z,\overline{z},0)$ is a function of $\langle z, z \rangle$, i.e. $F_{2\Gamma}(z,\overline{z},0) = c\langle z, z \rangle^2$, for some $c \in \mathbb{R}^*$. This is impossible since then $\text{tr} F_{2\Gamma}(z,\overline{z},v) = 8c\langle z, z \rangle \not\equiv 0$.

To deal with the cases $\dim G_0 = 7, 8$ we need the following lemma.

**Lemma 2.4**

(i) The only closed subgroup of $U(2,1)$ is codimension 1 is $SU(2,1)$.

(ii) There does not exist a closed subgroup of $U(2,1)$ of codimension 2.

**Proof of Lemma 2.4:** It suffices to prove this on the level of Lie algebras, which we shall denote by $u(2,1)$ and $su(2,1)$, respectively. Certainly, $su(2,1)$, has codimension 1 in $u(2,1)$ and any other subalgebra of codimension 1 or 2 would intersect $su(2,1)$ in a subalgebra of codimension 1 or 2. Therefore, it suffices to show that $su(2,1)$ has no subalgebras of codimension 1 or 2, equivalently of dimension 7 or 6. To do this, we shall show that the complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ has no complex subalgebras of dimension 7 and classify those of dimension 6. The result concerning $su(2,1)$ will follow if we can show that these 6-dimensional complex subalgebras have no real form in $su(2,1)$.

Recall that the Killing form

$$\langle X, Y \rangle = \text{trace} \ (XY), \quad \text{for} \ X, Y \in \mathfrak{sl}(3, \mathbb{C}),$$

is a non-degenerate symmetric form. Therefore, if $\mathfrak{s} \subset \mathfrak{sl}(3, \mathbb{C})$ is a subalgebra, the linear subspace $\mathfrak{s}^\perp$ with respect to the Killing form, will have dimension equal to the codimension of $\mathfrak{s}$. Furthermore, invariance of the Killing form with respect to the adjoint representation implies that, if $P \in \mathfrak{s}^\perp$, then the linear mapping

$$[P, \cdot] : \mathfrak{s} \rightarrow \mathfrak{sl}(3, \mathbb{C}) \quad (2.8)$$
has range contained in $s^\perp$. Now, if $\dim s^\perp \leq 2$, then the dimension of the kernel of $[P, \mathord{\cdot}]$ must be at least 4 and this will prove to be very restrictive. In particular, since $s \subset P^\perp$, we conclude that

$$[P, \mathord{\cdot}]: P^\perp \rightarrow \mathfrak{sl}(3, \mathbb{C}) \text{ has kernel of dimension } \geq 4. \quad (2.9)$$

Notice that this constraint depends only on the element $P \in \mathfrak{sl}(3, \mathbb{C})$ and so we may test it for any particular $P$. Furthermore, we may assume without loss of generality that $P$ is in Jordan canonical form in which case a simple computation shows that there is only one $P$ satisfying (2.9), namely

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

This immediately rules out subalgebras of codimension 1 since $P^\perp$ is not a subalgebra.

The constraint on mapping (2.8) now pins down $s$ as two possibilities, namely matrices of the form either

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \text{ or } \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{pmatrix}.$$ 

Both of these are, indeed, subalgebras and have geometric interpretations. The first is the stabiliser up to scale of the first standard basis vector in the defining representation. If it were the complexification of a subalgebra $\mathfrak{s}_0 \subset \mathfrak{su}(2, 1) \subset \mathfrak{sl}(3, \mathbb{C})$ then $\mathfrak{s}_0$ would stabilise up to scale some vector $v \in \mathbb{C}$. Up to conjugation and scale, there are only three possibilities for $v$ according to $\langle v, v \rangle > 0$, $\langle v, v \rangle < 0$, or $\langle v, v \rangle = 0$. Taking $v$ in some convenient normal form it is easy to check that the corresponding stabiliser has dimension only 4, 4, or 5 respectively. The second possibility for $\mathfrak{s} \subset \mathfrak{sl}(3, \mathbb{C})$ is similarly eliminated thanks to a geometric interpretation in the dual representation.

The lemma is proved. \qed
Remark 2.5 This proof of Lemma 2.4 extends to higher dimensions, where it yields the subalgebras of \(\mathfrak{sl}(n, \mathbb{C})\) of maximal dimension. They are parabolic and their real forms may be determined following the Satake classification. More generally, the maximal subalgebras of the complex classical Lie algebras were determined by Dynkin [D]. The real case was considered by Komrakov [K] and, in principle, the lemma follows from his classification. However, no proofs are given in [K]. One may construct another proof by considering how potential subalgebras of \(\mathfrak{su}(2,1)\) intersect \(\mathfrak{u}(2)\), whereupon Lemma 2.1 of [IK], dealing with large compact subgroups of \(GL(n, \mathbb{C})\), may be used to eliminate the various possibilities. Finally, it was pointed out to us by Vladimir Ezhov that a careful investigation of the proof of his linearisation result in [E], shows that, for a hypersurface of the form

\[
\text{Re } z_4 = z_1 \overline{z_2} + z_2 \overline{z_1} + |z_3|^2 \pm |z_1|^4
\]

as we have, any local CR-automorphism near the origin is already linear. By this means one can avoid Lemma 2.4 if one so chooses.

We now finish the proof of Proposition 2.3. Suppose that \(\dim G_0 = 8\). Then by Lemma 2.4, \(G_0\) is the subgroup of \(G\) given by the condition \(\det U = 1\). In this case we again get that \(F_{2\overline{2}}(z, \overline{z}, 0)\) is a function of \(\langle z, z \rangle\) and obtain a contradiction as before. Further, Lemma 2.4 gives that \(\dim G_0 \neq 7\), and hence \(\dim G_0 \leq 6\).

Finally, since the mapping \(f \mapsto U\) is an injective Lie group homomorphism from \(I_0(S)\) into \(G_0\), we obtain that \(\dim I_0(S) \leq 6\), and the proposition is proved. \(\square\)

Remark 2.6 Instead of the non-umbilicity of \(S\) at 0 it is sufficient to assume in Proposition 2.3 that at least one of \(F_{2\overline{2}}, F_{2\overline{3}}\) and \(F_{3\overline{3}}\) is not identically zero.

We will now continue with the proof of Theorem 2.2. Proposition 2.3 gives that \(\dim \text{Aut}(M_{\pm}) \leq 13\) and hence in fact \(\dim \text{Aut}(M_{\pm}) = 13\). Since \(P_\pm\) is connected in \(\text{Aut}(M_{\pm})\), it coincides with the connected component of the identity of \(\text{Aut}(M_{\pm})\). To finish the proof of Theorem 2.2 we need the following proposition.

Proposition 2.7 The group \(\text{Aut}(M_{\pm})\) is connected.
Proof of Proposition 2.7: Since $M_{\pm}$ is homogeneous and connected, it is sufficient to prove that $I_0(M_{\pm})$ is connected. As in the proof of Proposition 2.3, let $J_0(M_{\pm})$ denote the group of all local CR-automorphisms of $M_{\pm}$ defined near the origin and preserving it. Let $f \in J_0(M_{\pm})$. Since $f$ extends holomorphically to a neighbourhood of the origin, we can write $f$ in the form:

$$
\begin{align*}
    z^* &= f_1(z, w), \\
    w^* &= f_2(z, w),
\end{align*}
$$

where $f_1, f_2$ are holomorphic in a neighbourhood of the origin in $\mathbb{C}^4$. Let

$$
U_f := \frac{1}{\sqrt{\frac{\partial f_1}{\partial z}(0)\frac{\partial f_2}{\partial w}(0)}}\frac{\partial}{\partial z},
$$

(note that $\frac{\partial f_2}{\partial w}(0)$ is necessarily positive). Clearly, $U_f$ is a matrix satisfying

$$
\langle U_f z, U_f z \rangle = \langle z, z \rangle, \quad (2.10)
$$

where $\langle z, z \rangle := z_1\overline{z_2} + z_2\overline{z_1} + |z_3|^2$. It follows from [B], [L] that $f$ is uniquely determined by the corresponding $U_f$ and that the collection of all matrices $U_f$ arising in this way from elements of $J_0(M_{\pm})$ form a closed subgroup $G_0$ in the group $G$ of all matrices satisfying (2.10) (of course, $G$ is isomorphic to $U(2, 1)$).

Since $M_{\pm}$ is non-umbilic at 0, it follows from the proof of Proposition 2.3 that $\dim G_0 \leq 6$. Let $G'_0$ be the subgroup of $G_0$ that consists of matrices $U_f$ for $f \in I_0(M_{\pm})$. We will prove the connectedness of $I_0(M_{\pm})$ by showing that $G'_0$ is connected. It follows from (2.1) that $G'_0$ contains matrices of the following form:

$$
\begin{pmatrix}
    \frac{1}{q}e^{i\phi} & 0 & 0 \\
    b & qe^{i\phi} & \frac{d}{q} \\
    -\frac{d}{q^2}e^{i(\phi+\psi)} & 0 & e^{i\psi}
\end{pmatrix}, \quad (2.11)
$$

where $q > 0, \phi, \psi \in \mathbb{R}, b, d \in \mathbb{C}, \Re(e^{ib\phi}) \leq 0$ and condition (2.2) holds. The group $G''_0$ of matrices (2.11) is a closed connected subgroup of $G$ of dimension 6. Hence $G'_0$ is the connected component of the identity of $G'_0$.

We now need the following lemma.
Lemma 2.8 There does not exist a disconnected (not necessarily closed) subgroup of $G$ whose connected component of the identity coincides with $G''_0$.

Proof of Lemma 2.8: Let $H \subset G$ be a group whose connected component of the identity is $G''_0$. Then any other connected component of $H$ is of the form $gG''_0$ for some $g \in G$, so we have $H = \cup_\alpha g_\alpha G''_0$. For every pair of indices $\alpha, \beta$ there exists an index $\gamma$ such that

$$g_\alpha G''_0 g_\beta G''_0 = g_\gamma G''_0,$$

or

$$g_\alpha G''_0 g_\beta = g_\gamma G''_0.
$$

Choosing $g_\alpha G''_0 = G''_0$ we obtain

$$G''_0 g_\beta = g_\gamma G''_0.	ag{2.12}$$

We now apply both sides of (2.12) to the vector $v := (0, 1, 0)$. For every $g \in G''_0$ we have $gv = \lambda(g)v$ with $\lambda(g) \in \mathbb{C}^*$, and therefore for every $g \in G''_0$ we obtain

$$g(g_\beta v) = \lambda(g)(g_\gamma v),$$

or, denoting $w_1 := g_\alpha v$ and $w_2 := g_\gamma v$,

$$gw_1 = \lambda(g)w_2,$$

i.e., the whole group $G''_0$ maps the vector $w_1$ into the complex line generated by $w_2$. It is then easy to see that such a vector $w_1$ has to be proportional to $v$.

Hence we obtain that $g_\beta$ preserves $v$ up to a multiple. This implies that $g_\beta$ has the form (2.11), i.e., $g_\beta \in G''_0$ for all $\beta$, and $H = G''_0$.

The lemma is proved. □

We can now finish the proof of Proposition 2.7. Indeed, Lemma 2.8 implies that $G'_0 = G''_0$ is connected. Since the mapping $f \mapsto U_f$ is an injective Lie group homomorphism from $I_0(M_\pm)$ onto $G'_0$ and since the image of the connected component of the identity of $I_0(M_\pm)$ under this mapping is $G'_0$, we obtain that $I_0(M_\pm)$ is connected. □

Further, since $\text{Aut}(M_\pm)$ is connected by Proposition 2.7, we obtain that $\text{Aut}(M_\pm) = P_\pm$, and Theorem 2.2 is proved. □
3 Domains with Boundary Equivalent to the Quadric

We will now turn to domains $D_0^>$ and $D_0^<$ defined in (1.7) and (1.8). In fact, they belong to the following well-known class of domains in $\mathbb{C}^n$. Let $(z_1, \ldots, z_n, z_{n+1})$ be coordinates in $\mathbb{C}^{n+1}$, and $x_j = \text{Re} z_j$, $j = 1, \ldots, n+1$. Set $z := (z_1, \ldots, z_n)$ and $x := (x_1, \ldots, x_n)$. Consider a non-degenerate Hermitian form $H_{p,n}(z, \overline{z})$ on $\mathbb{C}^n$, where $p$ is the number of positive eigenvalues of $H_{p,n}$ and suppose that $n \leq 2p$. Without loss of generality we assume that $H_{p,n}$ is given in the diagonal form

$$H_{p,n}(z, \overline{z}) = \sum_{j=1}^p |z_j|^2 - \sum_{j=p+1}^n |z_j|^2.$$

We now set

$$D_{H_{p,n}}^> := \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : x_{n+1} > H_{p,n}(z, \overline{z})\},$$
$$D_{H_{p,n}}^< := \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : x_{n+1} < H_{p,n}(z, \overline{z})\}.$$

It is easy to check that the mappings

$$z \mapsto az + b,$$
$$z_{n+1} \mapsto 2a H_{p,n}(z, \overline{b}) + a^2 z_{n+1} + H_{p,n}(b, \overline{b}) + ic,$$

with $a \in \mathbb{R}^*$, $b \in \mathbb{C}^n$, $c \in \mathbb{R}$, act transitively on each of $D_{H_{p,n}}^>$ and $D_{H_{p,n}}^<$ so these domains are homogeneous. The domain $D_{H_{p,n}}^<$ is never Kobayashi-hyperbolic as it contains the complex line $\{z_2 = \ldots = z_n = 0, z_{n+1} = -1\}$. The domain $D_{H_{p,n}}^>$ is not Kobayashi hyperbolic for $p < n$ as it contains the complex line $\{z_1 = \ldots = z_{n-1} = 0, z_{n+1} = 1\}$. The domain $D_{H_{p,n}}^>$ is biholomorphically equivalent to the unit ball.

Thus the domains $D_{\Omega_1}^>$ and $D_{\Omega_1}^<$ serve as a warning that sometimes non-trivially looking homogeneous tube domains can in fact be biholomorphically equivalent to a well-known domain (in our example $D_0^>$ and $D_0^<$).

It is interesting to remark that $D_{H_{p,n}}^>$ and $D_{H_{p,n}}^<$ can always be realised as tube domains. The mapping

$$z \mapsto \sqrt{2} z,$$
$$z_{n+1} \mapsto z_{n+1} + H_{p,n}(z, z)$$
transforms $D_{H, n, p}$ and $D_{H, n, p}$ into the tube domains
\[ \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : x_{n+1} > H_{p, n}(x, x)\} \]
and
\[ \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : x_{n+1} < H_{p, n}(x, x)\} \]
respectively. The bases of these domains are affinely homogeneous in $\mathbb{R}^{n+1}$.

In [DY], [I1], [IM] all tube hypersurfaces locally equivalent to the quadric
\[ Q_{H, n, p} := \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : x_{n+1} = H_{p, n}(z, z)\} \]
were determined for $n - p \leq 2$. Among such hypersurfaces those given by polynomial graphs are always globally equivalent to $Q_{H, n, p}$, and their bases are affinely homogeneous in $\mathbb{R}^{n+1}$. Moreover, the domains on each side of such a polynomial graph are affinely homogeneous.

For $p = n$ the only polynomial tube hypersurface (up to affine equivalence) is
\[ \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : x_{n+1} = H_{n, n}(x, x)\}. \]

As $n - p$ grows, higher order polynomial hypersurfaces appear. The tube over the hypersurface $\Gamma_{1/12}$ defined in (1.1) is an example for $n = 3, p = 2$, and mapping (1.6) establishes equivalence between this tube and $Q_{H, 2, 3}$ (see also [IM]). Another non-trivial example occurs for $n = 2, p = 1$. Consider the hypersurface in $\mathbb{R}^3$ given by the equation:
\[ x_3 = x_1 x_2 + x_1^3. \] (3.1)

It is called the Cayley surface (see [NS2] for a discussion of its properties). The Cayley surface is affinely homogeneous, and the domains on each side of it are affinely homogeneous as well. The tube over the Cayley surface appears in the classification in [IM], where its equivalence to $Q_{H, 1, 2}$ was explicitly established. Namely, the mapping
\[ z_1 \mapsto \frac{1}{\sqrt{2}} \left( z_1 + z_2 + \frac{3}{2} z_1^2 \right), \]
\[ z_2 \mapsto \frac{1}{\sqrt{2}} \left( z_1 - z_2 - \frac{3}{2} z_1^2 \right), \]
\[ z_3 \mapsto 4 z_3 - 2 z_1 z_2 - z_1^3 \]
transforms the tube over the Cayley surface into

\[ \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \text{Re } z_3 = |z_1|^2 - |z_2|^2 \}. \]

For \( n - p = 2 \) very complicated polynomial hypersurfaces equivalent to \( Q_{H_p,n} \) occur. For example, if \( n = 7, p = 5 \) we have the following one-parameter family of pairwise affinely non-equivalent hypersurfaces [I1]:

\[
x_8 = x_1^2 + x_2^2 + x_3^2 + x_4x_5 + x_6x_7 + \frac{2\sqrt{2}(1 + \sigma)x_1x_4x_6 + 2\sqrt{3}\sigma x_2x_6 + \frac{1 + \sigma}{\sqrt{3}\sigma}x_2x_4^2 + \sqrt{-\sigma^2 + 34\sigma - \frac{1}{3\sigma}}x_3^2}{(x_2^2 + x_6^2)(x_4^2 + \sigma x_6^2)}, \quad \sigma \in [1, 17 + 12\sqrt{2}).
\]

Every such a hypersurface and the domains on each side of it are affinely homogeneous, and it is possible to write explicitly a polynomial biholomorphism that maps the tubes over these hypersurfaces onto \( Q_{H_5,7} \). In general, the following holds.

**THEOREM 3.1** For every \( p \leq n \) and every \( 2 \leq s \leq 2(n - p) + 2 \) there exists a polynomial \( P(x_1, \ldots, x_n) \) of degree \( s \) such that:

(i) the graph of \( P \) is an affinely homogeneous hypersurface in \( \mathbb{R}^{n+1} \);

(ii) the tube hypersurface over the graph of \( P \) is equivalent to \( Q_{H_p,n} \) by means of a polynomial mapping.

Moreover, if \( n \geq 7 \) and \( n - p \geq 2 \), then for some \( 2 \leq s_0 \leq 2(n - p) + 2 \) there exists a family \( \{ P_\sigma \} \) of polynomials of degree \( s_0 \) depending on a continuous parameter, that possess properties (i) and (ii), such that the graphs of \( P_\sigma \) are pairwise affinely non-equivalent.

We do not prove Theorem 3.1 here; it follows from the techniques developed in [I2]. Theorem 3.1 shows that one has to exercise caution when constructing homogeneous domains from tubes over affinely homogeneous hypersurfaces in real space since a substantial number of such domains are either \( D_{H_p,n}^> \), or \( D_{H_p,n}^< \) in disguise.
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