A QUADRATIC REGRESSION PROBLEM FOR TWO-STATE ALGEBRAS WITH APPLICATION TO THE CENTRAL LIMIT THEOREM

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Abstract. We extend the free version \cite{Boz08} of the Laha-Lukacs theorem to probability spaces with two-states. We then use this result to generalize the noncommutative central limit theorem of Kargin \cite{Kar09} to the two-state setting.

1. Introduction

Both classical and free Meixner distributions first appeared in the theory of orthogonal polynomials in the works of Meixner \cite{Mei33}, Anshelevich \cite{Ans03}, and Saitoh and Yoshida \cite{Sai98a}. Morris \cite{Mor97} pointed out the relevance of classical Meixner distributions for the theory of exponential families in statistics; Diaconis, Khare and Saloff-Coste \cite{Dia04} gave an excellent overview of state of the art. Ismail and May \cite{Isma94} analyzed a mathematically equivalent problem from the point of view of approximation operators. A counterpart of (some aspects of) this theory for free Meixner distributions appear in an unpublished manuscript by Bryc and Ismail \cite{Bry06} and in \cite{Bry07}.

Laha and Lukacs \cite{Lah48} characterized all the (classical) Meixner distributions using a quadratic regression property and Bożejko and Bryc \cite{Boz08} proved the corresponding free version. Anshelevich \cite{Ans03} considered a Boolean version of this property showing that in the Boolean theory Laha-Lukacs property characterizes only the Bernoulli distributions.

According to Example 3 in \cite{Fraz07} and Proposition 3.1 of Franz \cite{Fraz05}, Boolean, monotone, and free independence are all special cases of the c-freeness for algebras with two states. Our primary goal in this paper is to extend \cite{Boz08} and \cite{Ans03} to the two-state setting under a weaker form of c-freeness, which we call $\langle \varphi|\psi \rangle$-freeness, and which shares with boolean and free independence a good description by cumulants.

As an application of our main result, we prove the central limit theorem under a certain type of “weak dependence” which includes the so called singleton condition, whose importance to central limit theorem was pointed out in Theorem 0 of Bożejko and Speicher \cite{Bosn02}; our assumptions are modeled on Kargin \cite{Kar09} who weakened freeness assumption in the free central limit theorem. Our result addresses a question of finding the “appropriate notions of independence or of weak dependence” for the quantum central limit theorem which was raised on page 11 of \cite{Bosn02} and describes the

\begin{itemize}
  \item [Date: Printed: January 27, 2009. File: noncom-clt-07-10.TEX.]
  \item [2000 Mathematics Subject Classification. Primary: 46L53; Secondary: 60E05, 05A18.]
  \item [Key words and phrases. generalized two-state freeness, generalized free Meixner distribution, Laha-Lukacs theorem, noncommutative quadratic regression.]
  \item [Research partially supported by the Taft Research Center, KBN Grant No 1 PO3A 01330, and NSF grant #DMS-0504198.]
\end{itemize}
limit law; if one is interested solely in convergence, it can be deduced from the general
theory of the quantum central limit theorem developed by Accardi, Hashimoto
and Obata [1]. Section 8.2 of Hora and Obata [20] discusses the role of singleton
condition and gives the central limit theorem under classical, free, boolean, and
monotone independence.

1.1. A two-state freeness condition. Let \( \mathcal{A} \) be a unital \( \ast \)-algebra with two states
\( \psi, \varphi : \mathcal{A} \to \mathbb{C} \). We assume that both states fulfill the usual assumptions of positivity
and normalization, and we assume tracial property \( \psi(ab) = \psi(ba) \) for \( \psi \), but not
for \( \varphi \).

A typical model of an algebra with two states is a group algebra of a group
\( G = \ast G_i \), a free product of groups \( G_i \). Here \( \varphi \) is the boolean product of the
individual states (which was also called ”regular free state”); the simplest example
is the free product of integers, \( G_i = \mathbb{Z} \), where \( G \) is a free group with arbitrary
number of generators, and \( \varphi \) is the Haagerup state, \( \Phi(x) = |x|^r \), where \( |x| \) is the
length of word \( x \in G \), \( -1 \leq r \leq 1 \), and state \( \psi \) is \( \delta(0) \). For details see Bozejko
[6, 7].

A self-adjoint element \( X \in \mathcal{A} \) with moments that fulfill appropriate growth con-
dition defines a pair \( \mu, \nu \) of probability measures on \((\mathbb{R}, \mathcal{B}) \) such that
\( \varphi(X^k) = \int_{\mathbb{R}} x^k \mu(dx) \) and \( \psi(X^k) = \int_{\mathbb{R}} x^k \nu(dx) \).

We will refer to measures \( \mu, \nu \) as the \( \varphi \)-law and the \( \psi \)-law of \( X \), respectively.

With each set of \( a_1, \ldots, a_n \in \mathcal{A} \) and a pair of states \( (\varphi, \psi) \) we associate the
cumulants \( R_k = R_{k, \varphi, \psi}, \ k = 1, 2, \ldots, \) which are the multilinear functions \( \mathcal{A}^k \to \mathbb{C} \)
defined by

\[
(1.1) \quad \varphi(a_1a_2 \ldots a_n) = \sum_{k=1}^{n} \sum_{s_1 < s_2 < \ldots < s_k \leq n} R_k(a_1, a_{s_2}, \ldots, a_{s_k}) \varphi(a_{s_k+1} \ldots a_n) \prod_{r=1}^{k-1} \psi \left( \prod_{j=s_r+1}^{s_{r+1}-1} a_j \right).
\]

We will use the notation
\[
(1.2) \quad r_n(a_1, \ldots, a_n) := R_{n, \psi, \psi}(a_1, \ldots, a_n).
\]

We remark that \( r_n \) are the free cumulants with respect to state \( \psi \), as defined by
Speicher [35, 36]; see also [32]. For more general theory of cumulants, see [29].

Fix \( a \in \mathcal{A} \), and consider the following formal power series

\[
(1.3) \quad R(z) = \sum_{n=1}^{\infty} R_n(a, \ldots, a) z^{n-1},
\]

\[
(1.4) \quad m(z) = \sum_{n=0}^{\infty} z^n \psi(a^n),
\]

\[
(1.5) \quad M(z) = \sum_{n=0}^{\infty} z^n \varphi(a^n).
\]

By Theorem 5.1 of [9], Eqtn. (1.1) is equivalent to the following relation

\[
(1.6) \quad M(z) (1 - z R(zm(z))) = 1.
\]
Definition 1.1. We say that subalgebras $A_1, A_2, \ldots$ are $(\varphi|\psi)$-free if for every choice of $a_1, \ldots, a_n \in \bigcup_j A_j$ we have
\[ R_n(a_1, \ldots, a_n) = 0 \] except if all $a_j$ come from the same algebra.

It is important to note that $(\varphi|\psi)$-freeness is weaker than $c$-freeness, as explained before Lemma 1.1. Thus we could have used the term weak $c$-freeness instead of $(\varphi|\psi)$-freeness.

When the algebras are $(\psi|\psi)$-free, we will abbreviate this to $\psi$-free. From Ref. [35] it follows that $\psi$-freeness coincides with the usual concept of freeness as introduced by Voiculescu [37].

We will say that $X, Y$ are $(\varphi|\psi)$-free if the unital algebras $C(X)$ and $C(Y)$ are $(\varphi|\psi)$-free.

A related concept is the following.

Definition 1.2 (See Refs. [10] and [9]). We say that subalgebras $A_1, A_2, \ldots$ are $c$-free if for every choice of $i_1 \neq i_2 \neq \cdots \neq i_n$ and every choice of $a_j \in A_j$ such that $\psi(a_j) = 0$ (thus $a_j \neq 1$) we have
\[ \varphi(a_{i_1} \ldots a_{i_n}) = \prod_{k=1}^{n} \varphi(a_{i_k}). \] (1.7)

1.2. Properties of $(\varphi|\psi)$-freeness. If $A_1, A_2$ are $(\varphi|\psi)$-free then for $a \in A_1, b \in A_2$
\[ \varphi(ab) = \varphi(a) \varphi(b). \] (1.8)

For $a_1, a_2 \in A_1, b \in A_2$ we have
\[ \varphi(a_1 a_2 b) = \varphi(b) \varphi(a_1 a_2) - \varphi(b) \varphi(a_1) \varphi(a_2) + \varphi(b) \varphi(a_1) \varphi(a_2). \] (1.9)

For $a_1, a_2 \in A_1, b_1, b_2 \in A_2$ we have
\[ \varphi(a_1 a_2 b_1 b_2) = \varphi(a_1 a_2) \varphi(b_1) \varphi(b_2) - \varphi(a_1) \varphi(a_2) \varphi(b_1) \varphi(b_2) + \varphi(a_1) \varphi(a_2) \varphi(b_1) \varphi(b_2) - \varphi(a_1) \varphi(a_2) \varphi(b_1) \varphi(b_2). \] (1.10)

Formulas (1.8), (1.9) and are identical to formulas under $c$-freeness as given in Lemma 2.1 of Ref. [9]. Together with formula (1.10) they imply that for a pair of $(\varphi|\psi)$-free algebras, (1.7) holds for $n \leq 4$. One can check that if $a, b$ are $(\varphi|\psi)$-free and $\psi(a) = \psi(b) = 0$ but $\psi(abab) \neq 0$ then $\varphi(abab) \neq \varphi(a)^2 \varphi(b)^2$; thus the concepts of $c$-freeness and of $(\varphi|\psi)$-freeness are not equivalent. Nevertheless they coincide for $\psi$-free algebras as noted in the following.

Lemma 1.1 (page 368 of Ref. [9]). Suppose $A_1, A_2, \ldots$ are $\psi$-free. Then the algebras $A_1, A_2, \ldots$ are $(\varphi|\psi)$-free if and only if they are $c$-free.

(3) It would be interesting to characterize $(\varphi|\psi)$-freeness without the freeness assumption on $\psi$.)

We will also rely on the following fact.

Lemma 1.2 (Ref [9]). Given a noncommutative random variable $X$ in a two-state probability space, there exist a two-state algebra (which one can take as the algebra of noncommutative polynomials $C(X, Y)$ in two variables) and two non-commutative random variables $X, Y$ which are $\psi$-free, $(\varphi|\psi)$-free, and both have the same $\varphi$-law and $\psi$-law as $X$. 

Definitions and Lemmas
Proof. Theorem 1 of Ref. [10], see also Theorem 2.2 of Ref. [9], shows how to extend both states to the free product of the original algebra so that the resulting algebras are c-free and \( \psi \)-free. By Lemma 1.1 they are thus \((\varphi|\psi)\)-free.

2. A \((\varphi|\psi)\)-free QUADRATIC REGRESSION PROBLEM

In this section we prove a two-state version of Theorem 3.2 of Ref. [8]. The statement is fairly technical, but we found it useful for our proof of the central limit theorem (Theorem 4.1 below).

**Theorem 2.1.** Suppose \( X, Y \) are self-adjoint \((\varphi|\psi)\)-free and

\[
\varphi(X^n) = \varphi(Y^n), \; \psi(X^n) = \psi(Y^n)
\]

for all \( n \). Furthermore, assume that \( \varphi(X) = 0, \; \varphi(X^2) = 1 \). (This can always be achieved by a shift and dilation, as long as \( \varphi(X^2) \neq 0 \).)

Let \( S = X + Y \) and suppose that there are \( a, c \in \mathbb{R} \) and \( b > -2 \) such that

\[
\varphi \left( \left( X - Y \right)^2 S^n \right) = c \varphi \left( \left( I + 2a S + b S^2 \right) S^n \right), \; n = 0, 1, 2, \ldots.
\]

Then the \( \varphi \)-moment generating functions \( M_S(z) := \sum_{k=0}^{\infty} z^k \varphi(S^k) \) and \( m_S(z) := \sum_{k=0}^{\infty} z^k \psi(S^k) \), which are defined as formal power series, are related as follows

\[
M_S(z) = \frac{2 + b - (2az + b)m_S(z)}{2 + b - (4z^2 + 2az + b)m_S(z)}.
\]

**Remark 2.1.** We will apply (2.3) to the case when \( m_S(z) \) converges for small enough \(|z|\), in the form as written. In general, the right hand side of (2.3) needs to be interpreted correctly. Recall that the composition \( p(q(z)) \) of two power series \( p, q \) is well defined if \( q(z) \) has no constant term. Note that the formal power series \(-b + (4z^2 + 2az + b)m_S(z)\) has no constant term, so it can be composed with the formal power series \( \sum_{n=0}^{\infty} \frac{1}{2^n} z^n \), which is a formal power expansion of the function \( \frac{1}{2-z} \). It is therefore natural to denote such a composition by

\[
\frac{1}{2 - (-b + (4z^2 + 2az + b)m_S(z))}.
\]

The right hand side of (2.2) is then interpreted as the product of this power series with the formal power series \( 2 + b - (2az + b)m_S(z) \).

**Remark 2.2.** Our assumptions on \( \varphi \) do not allow us to use conditional expectations. However, it is still natural to ask which properties of conditional expectations would have implied assumptions of Theorem 2.1. To this end, we denote by \( \varphi(\cdot|S) \) the conditional expectation onto the commutative algebra generated by \( S \).

From equality of the laws (2.1) and \((\varphi|\psi)\)-freeness, one can deduce that

\[
\varphi(XS^n) = \frac{1}{2} \varphi \left( S^{n+1} \right), \; n = 0, 1, 2, \ldots.
\]

(See (2.7) below.) When the conditional expectation exists, this property follows from \( \varphi(X|S) = \frac{1}{2} S \). We can then derive (2.22) from the quadratic variance property

\[
\varphi(X^2|S) - (\varphi(X|S))^2 = c \left( I + \frac{a}{2} S + \frac{b}{4} S^2 \right).
\]
2.1. **Proof of Theorem 2.1** We first remark that \( c = (2 + b)^{-1} \). This follows from (2.2) with \( n = 0 \) since \( \varphi(X + Y)^2 = 2 \pm \varphi(XY) \pm \varphi(YX) = 2 \).

By definition, \( R_n(S, \ldots, S) = R_n(X, \ldots, X) + R_n(Y, \ldots, Y) \). From (2.1) we see that \( R_n(X, \ldots, X) = R_n(Y, \ldots, Y) \). Thus

\[ R_n(X - Y, S, \ldots, S) = R_n(X, S, \ldots, S) - R_n(Y, S, \ldots, S) \]

\[ = R_n(X, \ldots, X) - R_n(Y, \ldots, Y) = 0 \]

for all \( n \). By (1.1) this implies

\[ \varphi((X - Y)S^n) = 0. \]

Similarly, using multilinearity of \( R \),

\[ R_n(X - Y, X - Y, S, \ldots, S) \]

\[ = R_n(X, X - Y, S, \ldots, S) - R_n(Y, X - Y, S, \ldots, S) \]

\[ = R_n(X, \ldots, X) + R_n(Y, \ldots, Y) = R_n(S, \ldots, S) \]

for all \( n \geq 2 \). Formula (1.1) therefore implies that

\[ \varphi((X - Y)^2 \mathbb{S}^n) \]

\[ = \sum_{k=2}^{n+2} \sum_{1 \leq b_1 < b_2 < \cdots < b_k \leq n+2} R_k(X - Y, X - Y, S, \ldots, S) \varphi(S^{n-b_k-1}) \prod_{r=1}^{k-1} \psi(S^{b_{r+1} - b_r - 1}) \]

\[ + \sum_{k=1}^{n+2} \sum_{1 \leq b_1 < b_2 < \cdots < b_k \leq n+2} R_k(X - Y, S, \ldots, S) \varphi(S^{n-b_k-1}) \prod_{r=1}^{k-1} \psi(S^{b_{r+1} - b_r - 1}). \]

By (2.6), the second sum vanishes. Using (2.8) we get

\[ \varphi((X - Y)^2 \mathbb{S}^n) \]

\[ = \sum_{k=2}^{n+2} \sum_{1 \leq b_1 < b_2 < \cdots < b_k \leq n+2} R_k(S, S, \ldots, S) \varphi(S^{n-b_k-1}) \prod_{r=1}^{k-1} \psi(S^{b_{r+1} - b_r - 1}). \]

Comparing this with the decomposition for \( \varphi(S^{n+2}) \) we see that

\[ \varphi((X - Y)^2 \mathbb{S}^n) = \varphi(S^{n+2}) \]

\[ - \sum_{k=2}^{n+2} \sum_{1 \leq b_1 < b_2 < \cdots < b_k \leq n+2} R_k(S, S, \ldots, S) \varphi(S^{n-b_k-1}) \prod_{r=1}^{k-1} \psi(S^{b_{r+1} - b_r - 1}). \]

We now rewrite the last sum based on the value of \( m = b_2 - b_1 \), compare Ref. [8]. We have

\[ \varphi((X - Y)^2 \mathbb{S}^n) = \varphi(S^{n+2}) \]

\[ - \sum_{m=1}^{n} \psi(S^m) \sum_{k=2}^{n+2} \sum_{1 \leq b_1 < b_2 < \cdots < b_k \leq n+2} R_k(S, S, \ldots, S) \varphi(S^{n-b_k-1}) \prod_{r=1}^{k-1} \psi(S^{b_{r+1} - b_r - 1}). \]
Thus from (2.2) we get

\[ \psi\text{-law of } b \text{ probability measures which is determined by the following procedure. Consider the Cauchy transforms are from Theorem 5.2 in Ref. [9] is especially convenient for explicit calculations.} \]

compactly supported probability measures is a pair \((\mu, \nu)\) from Refs. [9, 12, 13, 26, 27]. The generalized convolution is a binary operation on the \((\mu, \nu)\) pairs of compactly supported probability measures \((\mu, \nu) \odot \nu\), which was introduced by Bożejko and Speicher [10] and studied in Refs. [9, 12, 13, 26, 27]. The generalized convolution is a binary operation on the pairs of compactly supported probability measures \((\mu, \nu)\).

A routine argument now relates the formal power series:

\[ M_\psi(z) = 1 + z^2 \sum_{n=0}^{\infty} z^n \varphi(S^n) \]

\[ = 1 + \frac{z^2}{2 + b} \sum_{n=0}^{\infty} \sum_{j=0}^{n} z^j \psi(S^j) z^{n-j} \left( 4 \varphi(S^{n-j}) + 2a \varphi(S^{n-j+1}) + b \varphi(S^{n-j+2}) \right) \]

\[ = 1 + \frac{z^2}{2 + b} \sum_{j=0}^{\infty} \psi(S^j) \sum_{n=j}^{\infty} z^{n-j} \left( 4 \varphi(S^{n-j}) + 2a \varphi(S^{n-j+1}) + b \varphi(S^{n-j+2}) \right) \]

\[ = 1 + \frac{m_\psi(z)}{2 + b} \left( 4z^2 M_\psi(z) + 2az(M_\psi(z) - 1) + b(M_\psi(z) - 1) \right). \]

3. The \(\varphi\text{-law of } X\)

In this section we are interested in one explicit case when Theorem 2.1 allows us to determine the \(\varphi\text{-law of } X\) from the \(\psi\text{-law of } X\). This case arises when \(X, Y\) are \(\psi\)-free and \((\varphi, \psi)\)-free with compactly supported laws. Then the \(\varphi\text{-law and the } \psi\text{-law of } X + Y\) are determined uniquely from the laws of \(X, Y\) by the generalized convolution \(\odot\) which was introduced by Bożejko and Speicher [10] and studied in Refs. [9, 12, 13, 26, 27]. The generalized convolution is a binary operation on the pairs of compactly supported probability measures \((\mu, \nu)\). The analytic approach from Theorem 5.2 in Ref. [9] is especially convenient for explicit calculations. According to this result, the generalized convolution \((\mu_1, \nu_1) \odot (\mu_2, \nu_2)\) of pairs of compactly supported probability measures is a pair \((\mu, \nu)\) of compactly supported probability measures which is determined by the following procedure. Consider the Cauchy transforms

\[ G_j(z) = \int \frac{1}{z - x} \mu_j(dx) \quad g_j(z) = \int \frac{1}{z - x} \nu_j(dx), \quad j = 1, 2. \]
Let \( k_j(z) \) be the inverse function of \( g_j(z) \) in a neighborhood of \( \infty \), and define
\[
(3.1) \quad r_j(z) = k_j(z) - 1/z.
\]
On the second component the \( c \)-convolution acts as the free convolution \([37]\), \( \nu = \nu_1 \boxplus \nu_2 \). Recall that the free convolution \( \nu \) of measures \( \nu_1 \) and \( \nu_2 \) is the unique probability measure with the Cauchy transform \( g(z) \) which solves the equation
\[
g(z) = \frac{1}{z - r_1(g(z)) - r_2(g(z))}.
\]

To define the action of the generalized convolution on the first component, let
\[
R_j(z) = k_j(z) - 1/G_j(k_j(z)).
\]
Thus
\[
(3.2) \quad G_j(z) = \frac{1}{z - R_j(g_j(z)).
\]
The first component of the generalized convolution is defined as the unique probability measure \( \mu \) with the Cauchy transform
\[
G(z) = \frac{1}{z - R_1(g(z)) - R_2(g(z))}.
\]
We write
\[
(\mu, \nu) = (\mu_1, \nu_1) \boxast (\mu_2, \nu_2).
\]
We remark that
\[
r(z) = \sum_{k=1}^{\infty} r_k z^{k-1}, \quad R(z) = \sum_{k=1}^{\infty} R_k z^{k-1}
\]
are the generating functions for the \( \psi \)-free and \((\varphi|\psi)\)-free cumulants respectively, see \([13]\). We also note that the above relations can be interpreted as combinatorial relations between \( \psi \)-moments and \( \varphi \)-moments; the assumption of compact support allows us to determine the laws uniquely from moments.

### 3.1. The case of “constant conditional variance”.

**Proposition 3.1.** Suppose \( X, Y \) are \( \psi \)-free with the same compactly supported \( \psi \)-law \( \nu \), and are \((\varphi|\psi)\)-free with the same \( \varphi \)-law. If \((2.2)\) holds with \( a = b = 0 \), then the \( \varphi \)-law of \( X \) is compactly supported and uniquely determined by \( \nu \).

**Proof.** The \( \psi \)-law of \( S \) is the free convolution \( \nu \boxplus \nu \), so it is compactly supported. Therefore \( m_S(z) \) is given by a series that converges for small enough \( |z| \). Then \((2.3)\) reduces to
\[
M_S(z) = \frac{1}{1 - 2z^2 m_S(z)},
\]
and \( M_S(z) \) is also given by a convergent series. In particular, the \( \varphi \)-law of \( S \) is compactly supported. So for \( \exists z > 0 \), the Cauchy transform is
\[
(3.3) \quad G_S(z) = \frac{1}{z} M_S(1/z) = \frac{1}{z - 2g_S(z)}.
\]
Thus \( R_k(S, \ldots, S) = 0 \) for all \( k \) except for \( R_2(S, S) = 2 \). This shows that \( R_k(X, \ldots, X) = 0 \) for all \( k \) except for \( R_2(X, X) = 1 \). Thus \( R_X(z) = z \) and \((1.6)\) gives
\[
(3.4) \quad M_X(z) = \frac{1}{1 - z^2 m_X(z)}.
\]
This implies that ϕ-law of X has compact support, and its Cauchy transform is uniquely determined by

\[ G_X(z) = \frac{1}{z - g_X(z)}. \]  

In particular, suppose \( \nu \) is the semicircle law with mean zero and variance \( \sigma^2 \), so that \( g_X(z) = \frac{z}{\sqrt{z^2 - 4\sigma^2}} \). Proposition 3.1 then shows that the ϕ-law of X has Cauchy-Stieltjes transform

\[ G_X(z) = \frac{(\sigma^2 - \frac{1}{2}) z - \frac{1}{2} \sqrt{z^2 - 4\sigma^2}}{1 + (\sigma^2 - 1) z^2}. \]

This law plays the role of the “Gaussian limit” in Ref. [9].

3.2. The case of “linear conditional variance”. Suppose (2.2) holds with \( b = 0 \). Then (2.3) reduces to

\[ M_S(z) = \frac{1 - azm_S(z)}{1 - (2z + a)zm_S(z)}. \]

So again the Φ-law of S is compactly supported, if the ψ-law is, and the Cauchy transform is

\[ G_S(z) = \frac{1 - ag_S(z)}{z - (2 + az)g_S(z)} = \frac{1}{z - R_S(g_S(z))} \]

with

\[ R_S(u) = \frac{2u}{1 - au}. \]

This shows that \( R_X(z) = \frac{1}{1 - az} \) and

\[ G_X(z) = \frac{1 - ag_X(z)}{z - (1 + az)g_X(z)}. \]

In particular, suppose that the ψ-law of X is Marchenko-Pastur with parameter \( \lambda > 0 \), so that

\[ g_X(z) = \frac{z + (1 - \lambda) - \sqrt{(z - 1 - \lambda)^2 - 4\lambda}}{2z}. \]

If \( a = 1 \), then the ϕ-law of X is compactly supported, with Cauchy transform

\[ G_X(z) = \frac{1 + \lambda - z(1 - 2\lambda) - \sqrt{(z - 1 - \lambda)^2 - 4\lambda}}{2(1 + z(1 + \lambda) - z^2(1 - \lambda))}. \]

Related laws appear in Eqtn. (17) of Ref. [17] and on page 380 in Ref. [9].

4. Central limit theorem for non-identical summands

The central limit theorem and the Poisson convergence theorem for sums of (ϕ|ψ)-free random variables that are also ψ-free appear in Theorems 4.3 and 4.4 of Ref. [9]. Recently Kargin [23] observed that in the free case one can dispense with the assumption of identical laws and at the same time relax the freeness assumption. A similar result in classical probability is due to Komlos [24] who assumes a much weaker version of singleton condition (4.1) and has an inequality in his condition (6) that substitutes for (4.3). Komlos’ conditions were motivated by (classical) central limit theorem for the so called multiplicative systems. We
also note that in classical probability Jakubowski and Kwapień [22] discovered a beautiful connection between multiplicative systems and independent sequences. No counterpart of this result is known in noncommutative setting; compare also non-commutative $p$-orthogonality and Remark 2.4 of Pisier [33], and work of Köstler and Speicher [25] on noncommutative versions of de Finetti’s theorem.

In this section we use Theorem 2.1 to deduce a two-state version of Kargin’s result. The convergence of moments can also be obtained as a corollary of Theorem 3 in Accardi Hashimoto and Obata [2], see also Theorem 3.3 in [1], Theorem 0 of [11], and Section 8.2 in [20]. This theorem says that under the singleton condition (4.1), in order to complete the proof of CLT, it suffices to control ergodic averages of totally entangled pair partitions. The disentanglement can be achieved from various conditions that include statistical conditions, such as the free case or the generalized freeness given by conditions (4.2) and (4.3). This approach, as well as classical CLT in Ref [24], suggests that one should seek a weaker version of (4.3) that perhaps would be stated as an inequality. On the other hand, our proof from Theorem 2.1 gives directly the formula for the Cauchy-Stieltjes transform of the limit law which would require additional work if the techniques from [1] were applied.

We also note that Wang [38] uses analytical methods to study limit theorems for additive $c$-convolution with measures of unbounded support. It is not obvious how Kargin’s condition A should be generalized to this setting. In fact, a generalization of Theorem 2.1 to unbounded random variables would be interesting even in the free case studied in [8].

**Definition 4.1.** We will say that a sequence of random variables $X_1, X_2, \ldots$ satisfies Kargin’s Condition A with respect to $(\varphi|\psi)$, if:

(i) For every $k \notin \{j_1, \ldots, j_r\}$ the following singleton conditions hold:

\[
\varphi(X_k X_{j_1} \ldots X_{j_r}) = \varphi(X_{j_1} X_k X_{j_2} \ldots X_{j_r}) = \ldots = \varphi(X_{j_1} \ldots X_{j_r} X_k) = 0.
\]

(ii) For every $k \notin \{j_1, \ldots, j_r\}$, and $0 \leq p \leq r$,

\[
\psi(X_k X_{j_1} \ldots X_{j_p} X_{j_{p+1}} \ldots X_{j_r}) = \varphi(X_k^2) \psi(X_{j_1} \ldots X_{j_p}) \psi(X_{j_{p+1}} \ldots X_{j_r}).
\]

We remark that conditions (4.1) and (4.3) are automatically satisfied if $X_1, X_2, \ldots$ are $\varphi$-centered and $(\varphi|\psi)$-free; clearly, condition (4.2) holds true if $X_1, X_2, \ldots$ are $\psi$-centered and $\psi$-free but of course it is weaker and can hold also for classical (commutative) independent random variables.

**Theorem 4.1.** Suppose that

(i) $X_1, X_2, \ldots$ satisfies Kargin’s Condition A with respect to $(\varphi|\psi)$;

(ii) All joint moments of order $k$ are uniformly bounded

\[
\sup_{j_1, \ldots, j_k \geq 1} |\varphi(X_{j_1} \ldots X_{j_k})| \leq C_k < \infty \text{ for } k = 1, 2, \ldots.
\]
(iii) Sequences $s_j^n := \psi(X_j^n)$ and $S_j^n := \varphi(X_j^n)$ satisfy
\begin{equation}
(4.5) \quad (s_1^n + \cdots + s_k^n)/n \to s \text{ and } (S_1^n + \cdots + S_k^n)/n \to S.
\end{equation}
(iv) $0 < s, S < \infty$.
(v) The $\psi$-moments of
\[
\frac{1}{s_1^n + \cdots + s_k^n} \sum_{j=1}^n X_j
\]
converge to the corresponding moments of a compactly supported probability measure $\nu$.

Then the $\varphi$-moments of
\[
\frac{1}{S_1^n + \cdots + S_k^n} \sum_{j=1}^n X_j
\]
converge to the moments of the unique compactly supported law $\mu$ with Cauchy transform $\psi$,
where \( g_\mu(z) = \int \frac{s}{s^2 - z^2} \nu(dx) \).

Combining Theorem 4.1 with Ref. [23] and formula (3.6) we get the following generalization of Theorem 4.3 in Ref. [9].

**Corollary 4.2.** Suppose that

(i) $X_1, X_2, \ldots$ satisfies Kargin’s Condition A with respect to $(\psi|\psi)$ and with respect to $(\psi|\psi)$.
(ii) All moments are uniformly bounded: \[4.4\] holds true, and $\sup_n |\psi(X_n^k)| < \infty$ for $k = 1, 2, \ldots$.
(iii) Sequence $s_j^n := \psi(X_j^n) = s_j^n$ and $S_j^n := \varphi(X_j^n)$ satisfy \[4.5\] with $0 < s, S < \infty$.

Then the $\varphi$-law of
\[
\frac{1}{S_1^n + \cdots + S_k^n} \sum_{j=1}^n X_j
\]
converges to the law $\mu$ with the Cauchy-Stieltjes transform \[4.6\] and $\sigma = s/S$.

Our proof of the central limit theorem is based on reduction to Laha-Lukacs theorem which in classical probability was introduced in Section 7.3.1 of Bryc [14].

### 4.1. Proof of Theorem 4.1

By Ref. [10] without loss of generality we may assume that we have a two-state probability space with two copies of the original sequence: $(X_k)$ and $(Y_k)$ each of them separately having the same $\psi$-moment and $\varphi$-moments as the original sequence, but such that the algebras $\mathcal{A}_X$ and $\mathcal{A}_Y$ generated by $(X_k)$ and by $(Y_k)$, respectively, are $\psi$-free and $(\varphi|\psi)$-free.

Under this representation, the $\psi$-distribution of
\[
\frac{1}{s_1^n + \cdots + s_k^n} \sum_{j=1}^n (X_j + Y_j)
\]
converges to $\nu \boxplus \mu$. Our goal is to show that the $\varphi$-distribution of
\[
\frac{1}{S_1^n + \cdots + S_k^n} \sum_{j=1}^n (X_j + Y_j)
\]
has the unique limit determined by the law with Cauchy-Stieltjes transform \[3.3\]. To do so, denote
\[
U_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n X_j, \quad V_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n Y_j, \quad S_n = U_n + V_n.
\]

Denote
\[
Z_j^{(\varepsilon)} = X_j^{\varepsilon} Y_{j+1}^{1 - \varepsilon}, \quad \varepsilon = 0, 1.
\]

Since the variables do not commute, we adopt a special convention for the product notation convention which relies on the order of the index set:
\[
\varphi \left( \prod_{s=1}^P Z_{j(s)}^{(\varepsilon(s))} \right) := \varphi \left( Z_{j(1)}^{(\varepsilon(1))} Z_{j(2)}^{(\varepsilon(2))} \cdots Z_{j(P)}^{(\varepsilon(P))} \right).
\]

**Lemma 4.3.** In the above setting, if $\{X_j\}$ satisfies Kargin’s Condition A, then $\{X_1, Y_1, X_2, Y_2, \ldots\}$ satisfies Kargin’s Condition A.

**Proof.** We first note the following.
Claim 1. Singleton properties \((4.1)\), \((4.2)\) for \(\{X_j\}\) are equivalent to the following: for \(k \not\in \{j_1, \ldots, j_p\}\) with \(p = 0, 1, 2, \ldots\), we have

\[
(4.6) \quad R_{p+1}(X_k, X_{j_1}, \ldots, X_{j_p}) = R_{p+1}(X_{j_1}, X_k, X_{j_2}, \ldots, X_{j_p}) = \ldots = R_{p+1}(X_{j_1}, \ldots, X_{j_p}, X_k) = 0,
\]

and

\[
(4.7) \quad r_{p+1}(X_k, X_{j_1}, \ldots, X_{j_p}) = 0.
\]

Proof. Clearly, \((4.7)\) implies \((4.2)\) by \((1.1)\) applied to \(\varphi = \psi\). Conversely, suppose that \(r_{p+1}(X_k, X_{j_1}, \ldots, X_{j_p}) \neq 0\) for some \(p \geq 0\), and take the smallest \(p\). Since for \(F = \{f_1, f_2, \ldots\} \subset \{j_1, \ldots, j_p\}\),

\[
\psi(X_k \prod_{f \in F} X_f) = 0,
\]

the only non-zero terms in \((1.1)\) must come from cumulants that have \(X_k\) as their argument. Thus, with \(\Pi_F\) denoting the appropriate products of moments,

\[
0 = \psi(X_k, X_{j_1} \ldots X_{j_p}) = \sum_{F} r_{|F|+1}(X_k, X_{f_1}, X_{f_2}, \ldots) \Pi_F
\]

\[
= r_{p+1}(X_k, X_{j_1}, \ldots, X_{j_p}) + \text{lower order terms}.
\]

Since by assumption all lower order cumulants vanish, we see that \(r_{p+1}(X_k, X_{j_1}, \ldots, X_{j_p})\) in fact must be zero. \(\square\)

Claim 2. Suppose \(\{X_j\}\) satisfies singleton properties \((4.1)\) and \((4.2)\). Then \((4.3)\)

\[
(4.8) \quad R_{r+2}(X_k, X_{j_1}, \ldots, X_{j_r}, X_k, X_{j_{p+1}} \ldots X_{j_r}) = 0.
\]

Proof. Suppose \((4.6)\) and \((4.8)\) hold. Then in \((1.1)\), \(X_k\) must appear twice in the argument of \(R\). Thus

\[
\varphi(X_k, X_{j_1} \ldots X_{j_r}, X_k, X_{j_{p+1}} \ldots X_{j_r})
\]

\[
= R_2(X_k, X_k) \psi(X_{j_1} \ldots X_{j_r}) \varphi(X_{j_{p+1}} \ldots X_{j_r}) + \text{sum involving higher cumulants}
\]

\[
= \varphi(X_k)^2 \psi(X_{j_1} \ldots X_{j_r}) \varphi(X_{j_{p+1}} \ldots X_{j_r}) + 0.
\]

Conversely, suppose that \(R_{r+2}(X_k, X_{j_1}, \ldots, X_{j_r}, X_k, X_{j_{p+1}} \ldots X_{j_r}) \neq 0\), for some \(r \geq 1\), and take the smallest such \(r\). By \((4.6)\), expansion \((1.1)\) has no singleton appearances of \(X_k\). Thus

\[
\varphi(X_k, X_{j_1} \ldots X_{j_r}, X_k, X_{j_{p+1}} \ldots X_{j_r}) = R_{r+2}(X_k, X_{j_1}, \ldots, X_{j_r}, X_k, X_{j_{p+1}} \ldots X_{j_r})
\]

\[
+ \sum_{\alpha=0}^{r-1} \sum_{\# F = \alpha} R_{\alpha+2}(X_k, X_{f_1}, \ldots, X_{f_\alpha}, X_k, X_{f_{\alpha+1}}, \ldots)
\]

\[
= R_{r+2}(X_k, X_{j_1}, \ldots, X_{j_r}, X_k, X_{j_{p+1}} \ldots X_{j_r})
\]

\[
+ R_2(X_k, X_k) \psi(X_{j_1} \ldots X_{j_r}) \varphi(X_{j_{p+1}} \ldots X_{j_r}).
\]

Thus \(R_{r+2}(X_k, X_{j_1}, \ldots, X_{j_r}, X_k, X_{j_{p+1}} \ldots X_{j_r}) = 0.\) \(\square\)
We will show that \( \{ Z_j^{\varepsilon(j)} \} \) satisfies Kargin’s Condition A for any choice of indices \((j, \varepsilon(j)) \in \mathbb{N} \times \{0, 1\} \). Since the assumptions are symmetric with respect to \( \{ X_j \} \) and \( \{ Y_j \} \), it is enough to analyze the case when the distinguished element is \( X_k = Z_k^{(1)} \).

Suppose \((1, k) \notin \{(\varepsilon(1), j_1), (\varepsilon(2), j_2), \ldots, (\varepsilon(p), j_p)\}\). Then
\[
R_{p+1}(X_k, Z_{j_1}^{\varepsilon(1)}, \ldots, Z_{j_p}^{\varepsilon(p)}) = 0.
\]
Indeed, this holds true by \( \psi \)-freeness of \( A_X, A_Y \) if one of the \( \varepsilon(i) = 0 \). On the other hand, if all \( \varepsilon(i) = 1 \), then this holds true by \((4.0)\). Similarly, \((\varphi|\psi)\)-freeness of \( A_X, A_Y \) implies that
\[
R_{p+1}(X_k, Z_{j_1}^{\varepsilon(1)}, \ldots, Z_{j_p}^{\varepsilon(p)}) = 0
\]
either because some of the \( \varepsilon(j) = 0 \), or by \((4.0)\). Thus \((4.1)\) and \((4.2)\) hold for \( \{ Z_j^{\varepsilon(j)} \} \) \( \{ Z_j^{\varepsilon(j)} \} \) by Claim [1].

Similarly, if \( r \geq 1 \),
\[
R_{r+2}(X_k, Z_{j_1}^{\varepsilon(1)}, \ldots, Z_{j_p}^{\varepsilon(p)}, X_k, Z_{j_{p+1}}^{\varepsilon(p+1)}, \ldots, Z_{j_{r}}^{\varepsilon(r)}) = 0
\]
either because some of \( \varepsilon(i) = 0 \) and \( A_X, A_Y \) are \((\varphi|\psi)\)-free, or by \((4.8)\). Therefore \((4.3)\) holds for \( \{ Z_j^{\varepsilon(j)} \} \) by Claim [2].

**Lemma 4.4.** For fixed \( j, k, m \geq 0 \),
\[
\sup_n |\varphi \left( \bigcup_n^{k} \bigcup_n^{\infty}(U_n + V_n)^m \right) | < \infty.
\]

**Proof.** Expanding the product, by Lemma \((4.3)\) we see that
\[
(4.9) \quad \varphi \left( \bigcup_n^{k} \bigcup_n^{\infty}(U_n + V_n)^m \right)
= n^{-(j+k+m)/2} \sum_{J: \{1, \ldots, j+k+m\} \rightarrow \{1, \ldots, n\}} \sum_{\varepsilon \in \mathcal{E}} \varphi \left( \prod_{s=1}^{j+k+m} \varepsilon(\varepsilon(s)) \right)
= n^{-(j+k+m)/2} \sum_{J \in \mathcal{J}_{\geq 2}} \sum_{\varepsilon \in \mathcal{E}} \varphi \left( \prod_{s=1}^{j+k+m} \varepsilon^{(s)} \right),
\]
where
\[
\mathcal{J}_{\geq 2} = \{ J : \#J^{-1}(s) \neq 1 \text{ for all } 1 \leq s \leq n \}
\]
is the set of mappings \( J : \{1, \ldots, j+k+m\} \rightarrow \{1, \ldots, n\} \) that take no singleton values, and
\[
\mathcal{E} = \{ \varepsilon \in 2^{\{1, \ldots, j+k+m\}} : \varepsilon(1) = \cdots = \varepsilon(j) = 1, \varepsilon(j+1) = \cdots = \varepsilon(j+k) = 0 \}.
\]

The cardinality of the first set can be bounded above by \( \#\mathcal{J}_{\geq 2} \leq n^{(j+k+m)/2} \), and \( \#\mathcal{E} = 2^m \), so by \((4.3)\),
\[
\left| \sum_{J \in \mathcal{J}_{\geq 2}} \sum_{\varepsilon \in \mathcal{E}} \varphi \left( \prod_{s=1}^{j+k+m} \varepsilon^{(s)} \right) \right| \leq C_{j+k+m} 2^m n^{(j+k+m)/2}.
\]
\[\square\]
Let 
\[ J_2 = \{ J : \#J^{-1}(s) = 0, 2 \text{ for all } 1 \leq s \leq n \} \]
be the subset of \( J_{\geq 2} \) that consists of all mappings \( J : \{1, \ldots, j+k+m\} \to \{1, \ldots, n\} \) that are two-to-one valued. (Clearly \( J_2 = \emptyset \) when \( j+k+m \) is odd.)

**Lemma 4.5.** For \( j, k, m \geq 0 \),
\[
(4.10) \limsup_{n \to \infty} \left| \varphi(U_n V_n^k (U_n + V_n)^m) - n^{-\frac{j+k+m}{2}} \sum_{J \in J_2 \setminus J_2} \varphi\left( \prod_{s=1}^{j+k+m} Z_{J(s)}^{\ell(s)} \right) \right| = 0,
\]

**Proof.** If there is a value \( s \in \{1 \ldots n\} \) that is taken by \( J \) at three or more different points, then there are at most \( j+k+m-1 \) points on which \( J \) is two-to-one. Therefore,
\[
\# (J_{\geq 2} \setminus J_2) \leq \left( \frac{j+k+m}{3} \right) n^{\frac{j+k+m-1}{2}},
\]
and by (4.4), the result follows from (4.9),
\[
\sum_{J \in J_{\geq 2} \setminus J_2} \sum_{\ell} \left| \varphi\left( \prod_{s=1}^{j+k+m} Z_{J(s)}^{\ell(s)} \right) \right| \leq \left( \frac{j+k+m}{3} \right) C_{j+k+m} 2^{m} n^{\frac{j+k+m-1}{2}}.
\]

We remark that since \( J_2 = \emptyset \) for odd \( j+k+m \), Lemma 4.5 implies that
\[
\limsup_{n \to \infty} |\varphi(U_n V_n)^m)| = 0 \text{ for odd } m.
\]

The next lemma is the main tool in identifying the limit via Theorem 2.1.

**Lemma 4.6.** For \( m \geq 1 \),
\[
(4.11) \limsup_{n \to \infty} \left| \varphi(U_n (U_n - V_n)^2 S_n^m) - 2 \varphi(S_n^m) \sum_{j=1}^{n} S_j^2 / n \right| = 0.
\]

**Proof.** Since \((x - y)^2 = x(x - y) + y(y - x)\), and the joint moments of \((U_n, V_n)\) are symmetric in \((U_n, V_n)\), it is enough to show that
\[
\limsup_{n \to \infty} \left| \varphi(U_n (U_n - V_n) S_n^m) - \varphi(S_n^m) \sum_{j=1}^{n} S_j^2 / n \right| = 0.
\]

By Lemma 4.5, once we expand the sums in \( \varphi(U_n^2 S_n^m - V_n S_n^m - S_n^m \sum_{j=1}^{n} S_j^2 / n) \), the only contributing terms come from the sum over the two-to-one functions \( J : \{1 \ldots m + 2\} \to \{1 \ldots n\} \). Therefore, it is enough to show that before taking the limit, we have the following identity:
\[
(4.12) \frac{1}{n} \sum_{J \in J_2} \sum_{\ell} \left( \varphi\left( X_{J(1)} X_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\ell(s)} \right) - \varphi\left( X_{J(1)} X_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\ell(s)} \right) \right) = 0.
\]
Let \( \mathcal{J}_1 \subset \mathcal{J}_2 \) denote the set of two-to-one functions with \( J(1) = J(2) \). Expanding the products we see that for \( J \in \mathcal{J}_2 \), each term in (4.12) can be written as
\[
\varphi \left( X_{J(1)} X_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) - \varphi \left( X_{J(1)} Y_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) - S_{J(1)}^2 \varphi \left( \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right).
\]

Since \( Y_{J(2)} \) is a singleton, by Lemma 4.3, \( \varphi \left( X_{J(1)} Y_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) = 0 \). The same lemma gives
\[
\varphi \left( X_{J(1)} X_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) = \varphi \left( X_{J(1)} Y_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) = S_{J(1)}^2 \varphi \left( \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right).
\]

Thus
\[
\sum_{J \in \mathcal{J}_2} \sum_{\varepsilon} \left( \varphi \left( X_{J(1)} X_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) - \varphi \left( X_{J(1)} Y_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) \right) = \varphi \left( S_n^m \right) \sum_{j=1}^{n} S_j^2.
\]

To end the proof, we need to show that the sum over \( J \in \mathcal{J}_2 \setminus \mathcal{J}_1 \) is zero. In fact, we observe that for each \( J \in \mathcal{J}_2 \setminus \mathcal{J}_1 \),
\[
\sum_{\varepsilon} \left( \varphi \left( X_{J(1)} X_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) - \varphi \left( X_{J(1)} Y_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) \right) = 0.
\]

To see this, denote by \( r > 2 \) the unique index with \( J(1) = J(r) \). Given \( \varepsilon \in 2^{(3, \ldots, m+2)} \), let
\[
\varepsilon'(s) = \begin{cases} 
1 - \varepsilon(s) & \text{if } s < r, \\
\varepsilon(s) & \text{if } s \geq r.
\end{cases}
\]

Clearly, the mapping \( \varepsilon \mapsto \varepsilon' \) is a bijection of \( \mathcal{E} \). Therefore, (4.14) follows from
\[
\sum_{\varepsilon} \left( \varphi \left( X_{J(1)} X_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) - \varphi \left( X_{J(1)} Y_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right) \right) = \sum_{\varepsilon} \varphi \left( X_{J(1)} Y_{J(2)} \prod_{s=3}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right).
\]

The latter holds true because by Lemma 4.3, for a fixed \( \varepsilon \), the left hand side of (4.15) is
\[
\varphi \left( X_{J(1)}^2 \right) \psi \left( X_{J(2)} \prod_{s=3}^{r-1} Z_{J(s)}^{\varepsilon(s)} \right) \varphi \left( \prod_{s=r+1}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right),
\]
while the right hand side of (4.15) is
\[
\varphi \left( X_{J(1)}^2 \right) \psi \left( Y_{J(2)} \prod_{s=3}^{r-1} Z_{J(s)}^{\varepsilon'(s)} \right) \varphi \left( \prod_{s=r+1}^{m+2} Z_{J(s)}^{\varepsilon(s)} \right).
\]

The two expressions are equal, because the joint (mixed) \( \psi \)-moments of \( X_1, X_2, \ldots, Y_1, Y_2, \ldots \) by construction do not change when we swap the roles of the sequences \( \{X_j\} \) and \( \{Y_j\} \). Of course, such a transformation converts \( \psi \left( X_{J(2)} \prod_{s=3}^{r-1} Z_{J(s)}^{\varepsilon(s)} \right) \) into \( \psi \left( Y_{J(2)} \prod_{s=3}^{r-1} Z_{J(s)}^{\varepsilon'(s)} \right) \).

\( \square \)
Proof of Theorem 4.1. Since convergence of moments is a metric convergence, we use the standard lemma: to show convergence it suffices to show that every subsequence has a subsequence that converges to the same limit.

The joint $\psi$-moments of $U_n, V_n, S_n$ converge, as the $\psi$-moments of $U_n$ converge by assumption and (4.5), and $U_n, V_n$ are $\psi$-free so their joint $\psi$-moments are uniquely determined from the moments of $U_n$ alone.

By Lemma 4.4, from any subsequence $U_{n_k}$ by diagonal method we can extract a further sub-subsequence such that the joint $\phi$-moments of $U_n, V_n, S_n$ converge along that sub-subsequence. Taken together, the limits of these $\psi$-moments and $\phi$-moments define a pair of states on $C\langle U, V \rangle$, which we will denote again by $\psi$ and $\phi$.

Since $U_n, V_n$ are $\psi$-free and ($\phi | \psi$)-free under the limit state $U, V$ are also $\psi$-free and ($\phi | \psi$)-free. From Lemma 4.6, we see that the pair $X := U/S, Y := V/S$ satisfies the assumptions of Theorem 2.1 with $a = b = 0$. By Proposition 3.1 this determines the $\phi$-law of $U$ uniquely. Therefore, the original sequence $\{U_n\}$ converges in $\phi$-moments to $U$, and the $\phi$-law of $\frac{1}{\sqrt{S_1^2 + \ldots + S_n^2}} \sum_{j=1}^n X_j = \sqrt{\frac{n}{S_1^2 + \ldots + S_n^2}} U_n$ converges in $\phi$-moments to $U/S$. Since the $\phi$-law of $U/S$ is the dilations by $s/S$ of measure $\nu$, we get formula (3.5).

□

Proof of Corollary 4.2. By Ref. [23], or by repeating the proof of Theorem 4.1 in the special case when $\varphi = \psi$ with Ref. [8] used instead of Theorem 2.1, we know that the $\psi$-moments of $\frac{1}{\sqrt{S_1^2 + \ldots + S_n^2}} \sum_{j=1}^n X_j$ converge to the semicircle law of variance $\sigma^2 = s^2/S^2$.

Since the semicircle law has compact support, we can use Theorem 4.1. the limiting distribution is then given by (3.6).

□

Acknowledgements. This research was partially supported by the Taft Research Center, KBN Grant No 1 PO3A 01330, and NSF grant #DMS-0504198. The second named author thanks Magda Peligrad for bringing Ref. [23] to his attention. The paper benefited from comments by Luigi Accardi.

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