SOME AUTOMORPHISM INVARIANCE PROPERTIES FOR MULTICONTRACTIONS

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Abstract. In the theory of row contractions on a Hilbert space, as initiated by Popescu, two important objects are the Poisson kernel and the characteristic function. We determine their behaviour with respect to the action of the group of unitarily implemented automorphisms of the algebra generated by creation operators on the Fock space. The case of noncommutative varieties, introduced recently by Popescu, is also discussed.

1. Introduction

Among the attempts to extend the dilation theory of contractions on a Hilbert space, as developed in [17], to multivariable operator theory, a most notable achievement is the theory of row contractions (which we will call below multicontractions), initiated by the work of Gelu Popescu [8, 9, 10]. It has been pursued in the last two decades by Popescu and others (see for instance, [1, 2, 6, 7, 11, 12]). Popescu’s theory is essentially noncommutative; later, starting with [3], interest has developed around the case of commuting multiopeators. This presents some specific features; on the other hand, many properties of the commuting case can be obtained from the noncommuting situation. The two recent papers of Popescu [13, 14] have pursued systematically the development of the commutative situation from the noncommutative one, putting it into the more general framework of constrained multioperators (see Section 6).

Two objects related to a multicontraction play a significant role in Popescu’s theory: the Poisson kernel and the characteristic function. The Poisson kernel is an important tool used by Popescu in order to prove the von Neumann inequality for row contractions; the characteristic function is essential in the model theory of completely noncoisometric multicontractions, while its commutative counterpart is related to Arveson’s curvature [3, 5, 15].

On the other hand, as the Sz.-Nagy–Foias theory of single contractions is related to classical function theory in the unit circle, an analogous role in Popescu’s theory is played by algebras generated by creation operators on the Fock space. There is a

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distinguished group of automorphisms of these algebras, that have been introduced
by Voiculescu in [19] (and discussed recently in [6]); they are the non commutative
analogues of the analytic automorphisms of the unit ball. These automorphisms
act on multicontractions, and it is interesting to see what is the effect of this
transformations on the Poisson kernel and on the characteristic function. The
purpose of this paper is to show, firstly, that these objects obey natural rules
of transformation, and secondly, that the rules of transformation also extend to
the case of constrained multicontractions. In particular, the relation between the
transformation rules for commuting and for noncommuting multicontractions is
clarified.

The first three sections following the introduction contain mostly preli-
minary material. The main results, Theorems 5.1 and 6.2, are proved in Sections 5
and 6. In connection to constrained objects, the last section discusses invariant ideals of
the noncommutative Toeplitz algebra.

The method of proof uses, in order to avoid complicated computations, the
machinery of Redheffer products. This machinery may seem unfamiliar, but it
provides a simple and short way to reach the main results. Up to a certain point,
using Redheffer products is equivalent to composing $J$-unitary operators, but there
is a slightly larger level of generality that happens to be important in our context.
For an illuminating discussion of these facts, see the first two sections of [20].

2. Preliminaries

2.1. The Fock space. A main object of study is formed by the Fock space and
the non-commutative Toeplitz algebras that act on it. We will follow mainly the
work of Popescu [8, 9, 11], as well as [6].

In the whole paper we will fix a positive integer $n$. We denote by $\mathbb{F}_n^+$ the free
semigroup with the $n$ generators $1, \ldots, n$ and unit $\emptyset$. An element $w = i_1 \cdots i_k \in \mathbb{F}_n^+$
is called a word in the letters $1, \ldots, n$, and its length is $|w| = k$.

If $A = (A_1, \ldots, A_n) \in \mathcal{L}(\mathcal{H})^n$ is a (not necessarily commuting) multioperator,
we denote $A_\emptyset = I_{\mathcal{H}}$ and, if, $w = i_1 \cdots i_k \in \mathbb{F}_n^+$, then $A_w = A_{i_1} \cdots A_{i_k} \in \mathcal{L}(\mathcal{H})$.

Consider an $n$-dimensional complex Hilbert space $\mathfrak{h}_n$, with basis vectors $e_1, \ldots, e_n$.
The full Fock space is then

$$
\mathfrak{F}_n = \bigoplus_{k \geq 0} \mathfrak{h}_n^\otimes k
$$

where $\mathfrak{h}_n^\otimes \emptyset = \mathbb{C}1$ and $\mathfrak{h}_n^\otimes k$ is the tensor product of $k$ copies of $\mathfrak{h}_n$. An orthonormal
basis of $\mathfrak{F}_n$ is given by $(e_w)_{w \in \mathbb{F}_n^+}$, where $e_\emptyset = 1 \in \mathfrak{h}_n^\otimes \emptyset$, while, if $w = i_1 \cdots i_k \in \mathbb{F}_n^+$,
then $e_w = e_{i_1} \otimes \cdots \otimes e_{i_k} \in \mathfrak{h}_n^\otimes k$. 

The left creation operators $L_i \in \mathcal{L}(\mathfrak{F}_n), \; i = 1, \ldots, n$, are defined by

$$L_i \xi = e_i \otimes \xi, \; \xi \in \mathfrak{F}_n.$$ 

The norm closed algebra generated by $L_1, \ldots, L_n$ is denoted by $\mathfrak{L}_n$, and the weakly closed algebra by $\mathfrak{L}_n^w$.

Similarly, we have right creation operators $R_i$ given by

$$R_i \xi = \xi \otimes e_i,$$

while the norm closed and weakly closed algebras they generate are denoted by $\mathfrak{R}_n$ and $\mathfrak{R}_n^w$ respectively. Each of the algebras $\mathfrak{L}_n$ and $\mathfrak{R}_n$ is the commutant of the other.

We can write any $f \in \mathfrak{L}_n$ as a formal series $f = \sum_w \hat{f}_w L_w$. For each $r < 1$ the series

$$(2.1) \quad f_r := \sum_w \hat{f}_w r^{|w|} L_w$$

converges uniformly, and thus $f_r \in \mathfrak{L}_n$; then $f = \text{SOT} - \lim_{r \to 1} f_r$. A similar statement is valid for $\mathfrak{R}_n$.

The flip operator is the involutive unitary $F \in \mathcal{L}(\mathfrak{F}_n)$ which acts on simple tensors by reversing the order of the components:

$$F(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = e_{i_n} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}.$$ 

We have then $R_i = FL_i F$.

If $\mathcal{E}, \mathcal{E}_*$ are Hilbert spaces, then a linear operator $M : \mathfrak{F}_n \otimes \mathcal{E} \to \mathfrak{F}_n \otimes \mathcal{E}_*$ is called **multianalytic** if

$$M(L_i \otimes I_{\mathcal{E}}) = (L_i \otimes I_{\mathcal{E}_*})M \quad \forall i = 1, \ldots, n.$$ 

$M$ is then uniquely determined by the “coefficients” $m_w \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, defined by

$$\langle m_w k, k' \rangle = \langle M(1 \otimes k), e_w \otimes k_* \rangle, \quad k \in \mathcal{E}, k_* \in \mathcal{E}_*, w \in \mathbb{F}_n^+,$$

where $\tilde{w}$ is the reverse of $w$, i.e., $\tilde{w} = i_k \cdots i_1$ if $w = i_1 \cdots i_k$; we can then associate with $M$ the formal Fourier expansion

$$(2.2) \quad \hat{M}(R_1, \ldots, R_n) = \sum_{w \in \mathbb{F}_n^+} R_w \otimes m_w.$$
2.2. Redheffer products. Several computations that appear in the sequel can be gathered in a simple uniform framework if we use the formalism of Redheffer products. The basic reference is [15]; we will follow the exposition in [18].

Suppose

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad L_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \]

are bounded operators mapping \( \mathcal{X} \oplus \mathcal{U} \) (respectively \( \mathcal{X}_1 \oplus \mathcal{U}_1 \)) into \( \mathcal{Y} \oplus \mathcal{Z} \) (respectively \( \mathcal{Y}_1 \oplus \mathcal{Z}_1 \)); also \( \mathcal{U}_1 = \mathcal{Z} \) and \( \mathcal{X} = \mathcal{Y}_1 \). Under the assumption

\[ (*) \quad I - B_1 C \text{ is invertible} \]

it follows that \( I - CB_1 \) is also invertible, and we define the Redheffer product by

\[ M = L \circ L_1 = \begin{pmatrix} A(I - B_1 C)^{-1}A_1 & B + A(I - B_1 C)^{-1}B_1 D \\ C_1 + D_1 C(I - B_1 C)^{-1}A_1 & D_1(I - CB_1)^{-1}D \end{pmatrix}. \]

\( M \) is an operator from \( \mathcal{X}_1 \oplus \mathcal{U} \) to \( \mathcal{Y} \oplus \mathcal{Z}_1 \). It is useful to visualize the interlacing of spaces by input-output boxes, in a manner suggested by system theory:

\[ \begin{array}{ccc}
\mathcal{U} & \xrightarrow{L} & \mathcal{Y} \\
\mathcal{X} & \xrightarrow{L_1} & \mathcal{Z}_1 \\
\mathcal{Z} = \mathcal{U}_1 & \xrightarrow{L_1} & \mathcal{X}_1 \\
\mathcal{Y} = \mathcal{Y}_1 & \xrightarrow{L} & \mathcal{X} \\
\end{array} \]

We will write also \( \beta_L(A_1, B_1) \) and \( \alpha_L(B_1) \) for the entries in the first row of \( L \circ L_1 \) (as given by (2.3)).

The basic properties of the Redheffer product are gathered in the following proposition. In its statement it is tacitly assumed that condition (*) is satisfied, when necessary.

**Proposition 2.1.** (i) The identities matrices (on the corresponding spaces) act as unit elements also for the Redheffer products.

(ii) If \( L \) is invertible, \( L_1 = L^{-1} \), and one can form \( L \circ L_1 \), then \( L_1 \) is also the inverse of \( L \) with respect to the Redheffer product.

(iii) The Redheffer product is associative: if \( L, L_1, L_2 \) are given, and all Redheffer products in (2.4) can be formed, then

\[ (2.4) \quad L \circ (L_1 \circ L_2) = (L \circ L_1) \circ L_2. \]

(iv) \( L, L_1 \) contractions (isometries, coisometries, unitaries) imply \( L \circ L_1 \) contraction (isometry, coisometry, unitary respectively). In particular, if \( L \) and \( B_1 \) are contractions, then \( \alpha_L(B_1) \) is also a contraction.
A particular case that will be useful is $Z_1 = \{0\}$ (and thus $C_1 = D_1 = 0$).

In connection with Redheffer products, we need also a lemma concerning the structure of unitary $2 \times 2$ matrices. To state it, remember that if $E_1, E_2$ are two Hilbert spaces, and $C : E_1 \rightarrow E_2$ is a contraction, one defines the defect operator $D_C = (1_{E_1} - C^*C)^{1/2} \in \mathcal{L}(E_1)$ and the defect space $D_C = D_C E_1 \subset E_1$.

**Lemma 2.2.** A $2 \times 2$ operator matrix from $E_1 \oplus E_2$ to $E'_1 \oplus E'_2$ that has $A^*$ as its $(2,1)$ entry, while the $(1,1)$ entry has dense range, has the form

$$J = \begin{pmatrix} Z_* D_{A^*} & -Z_* A Z_* \\ A^* & D_A Z_* \end{pmatrix},$$

$Z_* : D_{A^*} \rightarrow E'_1$ and $Z : D_A \rightarrow E_2$ being unitary operators.

3. Automorphisms

The analytic automorphisms of the unit ball $\mathbb{B}^n$ act by composition on any Hilbert space of functions on $\mathbb{B}^n$. There exist corresponding unitarily implemented automorphisms on the non-commutative Toeplitz algebras on the Fock space.

3.1. The commutative case: automorphisms of the unit ball. There are two different descriptions of the automorphisms of the unit ball $\mathbb{B}^n$. In view of further extensions, we identify elements in $\mathbb{B}^n$ with row $1 \times n$ matrices. Naturally, the action of an $n \times n$ matrix on such an element will be done by multiplication on the right.

**First form.** Start with the group $U(1,n)$ of $(n+1) \times (n+1)$ matrices $X$ that are $J$-unitary, where $J = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$; that is, $X^* J X = J$. According to the decomposition $\mathbb{C}^{n+1} = \mathbb{C} \oplus \mathbb{C}^n$, one writes $X = \begin{pmatrix} x & y \\ z^t & X' \end{pmatrix}$; note that with these conventions $x$ is a scalar, while $y$ and $z$ are row matrices. Accordingly, there is a corresponding map $\phi_X : \mathbb{B}^n \rightarrow \mathbb{B}^n$, defined by

$$\phi_X(\lambda) = (x - \lambda z^t)^{-1}(\lambda X' - y).$$

Then the map $X \mapsto \phi_X$ is a group antihomomorphism from $U(1,n)$ to the group of automorphisms of $\mathbb{B}^n$ (the “anti” being due to our decision to see elements of $\mathbb{B}^n$ as row matrices and write the action of the group on the right); this antihomomorphism is onto, and its kernel is formed by scalar unitaries.

**Second form.** A variant of (3.1) which uses a unitary instead of a $J$-unitary matrix is more natural in the context of Redheffer products. Namely, if $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
is a unitary \((n+1) \times (n+1)\) matrix, then we can consider the map (see (2.3))

\[
\alpha_Y(\lambda) = b + a\lambda(I - c\lambda)^{-1}d.
\]

The corresponding diagram is

\[
\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{L_Y} & \mathbb{C}^n \\
\mathbb{C} & \xleftarrow{\mathbb{C}} & \mathbb{C}
\end{array}
\]

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{L_1} & \{0\} \\
\mathbb{C} & \xleftarrow{\mathbb{C}} & \mathbb{C}
\end{array}
\]

where \(L_1 = \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} : \mathbb{C} \oplus \mathbb{C}^n \to \mathbb{C} \oplus \{0\}\).

By Proposition 2.1 (iv), for \(\lambda\) contractive, \(\alpha_Y(\lambda)\) is also contractive. Thus \(\alpha_Y\) is an analytic map from \(\mathcal{B}^n\) to \(\mathcal{B}^n\); it is even an automorphism, since again Proposition 2.1 (ii) implies \(\alpha_Y^{-1} = \alpha_Y^*\).

The passage from (3.1) to (3.2) is done by the formulas

\[
a = x^{-1}, \quad b = -x^{-1}y, \quad c = x^{-1}z^t, \quad d = X' - x^{-1}z^ty.
\]

These formulas can be inverted, provided \(a \neq 0\).

Working in the context of Redheffer products, (3.2) is more convenient; however, (3.1) is related to the automorphisms in Subsection 3.2. Also, while in (3.1) any \(J\)-unitary produces an automorphism, in (3.2) we must require \(a \neq 0\).

3.2. The noncommutative case: the Fock space. We shall introduce some facts and notations from [6]: in Section 4 therein the automorphisms of the algebra \(\mathfrak{L}_n\) are investigated. It is shown that all contractive automorphisms of \(\mathfrak{L}_n\) are actually unitarily implemented, and they are also automorphisms of the \(C^*\)-algebra \(\mathfrak{I}_n\).

A detailed description of these automorphisms can be obtained following [19].

As in Section 3, take \(X \in U(1, n)\), \(X = \begin{pmatrix} x & y \\ z & X' \end{pmatrix}\). Write also \(L[\zeta] = \sum_{i=1}^n \zeta_i L_i\) for \(\zeta \in \mathbb{C}^n\). Then there is an automorphism \(\Phi_X\) of \(\mathfrak{L}_n\) such that the restriction to the generators is given by

\[
\Phi_X(L[\zeta]) = (xI - L[z])^{-1}(L[X'\zeta] - (\zeta \cdot y')I).
\]

This automorphism is implemented by a unitary \(U_X \in \mathcal{L}(\mathfrak{F}_n)\), which satisfies

\[
U_X(Ae_\emptyset) = \Phi_X(A)(xI - L[z])^{-1}e_\emptyset
\]

for all \(A \in \mathfrak{L}_n\); this means that \(\Phi_X(A) = U_X AU_X^*\) for all \(A \in \mathfrak{L}_n\). The map \(X \mapsto \Phi_X\) from \(U(1, n)\) to the automorphisms of \(\mathfrak{L}_n\) has as image all unitarily implemented automorphisms (which actually coincide with all contractive automorphisms), and its kernel consists of the scalar matrices \(xI_{n+1}\), with \(x \in \mathbb{T}\).
To make the connection with (3.1) apply (3.4) for \( \zeta \) a basis vector; one obtains
\[
\Phi_X(L_i) = (xI - L[z])^{-1}\left(\sum_{j=1}^{n} X'_{ji}L_j - y_iI\right),
\]
while writing (3.1) on coordinates yields
\[
(\phi_X(\lambda))_i = (x - \sum_{j=1}^{n} z_j\lambda_j)^{-1}\left(\sum_{j=1}^{n} \lambda_jX'_{ji} - y_i\right).
\]
Consequently, (3.4) can be obtained by formally replacing \( \lambda \) in (3.1) with \( L_i \).

One can interpret also these automorphisms in terms of Redheffer products. Suppose that, as in (3.1), one defines the unitary matrix
\[
Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{L}(\mathbb{C} \oplus \mathbb{C}^n)
\]
by (3.3); denote by \( \iota : \mathbb{C} \rightarrow \mathfrak{F}_n \) the inclusion map that sends 1 to \( e_\emptyset \), and
\[
L_Y = \begin{pmatrix} aI_{\mathfrak{F}_n} & b \otimes I_{\mathfrak{F}_n} \\ c \otimes I_{\mathfrak{F}_n} & d \otimes I_{\mathfrak{F}_n} \end{pmatrix} : \mathfrak{F}_n \oplus (\mathfrak{F}_n \otimes \mathbb{C}^n) \rightarrow \mathfrak{F}_n \oplus (\mathfrak{F}_n \otimes \mathbb{C}^n)
\]
\[
L_1 = \begin{pmatrix} \iota & L \\ 0 & 0 \end{pmatrix} : \mathbb{C} \oplus (\mathfrak{F}_n \otimes \mathbb{C}^n) \rightarrow (\mathfrak{F}_n) \oplus \{0\}
\]
(we have implicitly used the fact that we can identify \( \mathfrak{F}_n \otimes \mathbb{C}^n \) with \( \mathfrak{F}_n^n \)).

Then it follows immediately from the discussion above that:
\[
(3.6) \quad \alpha_{L_Y}(L) = \Phi_X(L) = U_XLU_X^*.
\]
Moreover, from (3.5) and (3.4) we have
\[
U_X(e_\emptyset) = x^{-1}(I - x^{-1}\sum_{j=1}^{n} z_jL_j)^{-1}e_\emptyset = a(I - \sum_{j=1}^{n} c_jL_j)^{-1}e_\emptyset,
\]
whence
\[
\beta_{L_Y}(\iota, L) = U_X\iota.
\]
Let us also note that, if \( X \) and \( Y \) are related by (3.3), then
\[
(3.7) \quad \Phi_X^{-1}(L) = \Phi_X^{-1}(L) = \alpha_{L_Y^*}(L).
\]
Suppose now that we want to obtain similar relations with \( R \) instead of \( L \). We may immediately note that \( R_i = FL_iF \), which leads to
\[
\alpha_{L_Y}(R) = FU_XFRFU_X^*F.
\]
But we can actually say more. Since $\mathfrak{R}_n$ is the commutant of $\mathfrak{L}_n$, it follows that $U_X BU_X^* \in \mathfrak{R}_n$ for all $B \in \mathfrak{R}_n$; this can be made precise using the following lemma.

**Lemma 3.1.** For all $X \in U(1, n)$, we have $U_X F = F U_X$.

**Proof.** We have to check the relation on simple tensors $e_w$, where $w = i_1 \cdots i_k$. Denote also $\bar{w} = i_k \cdots i_1$. According to (3.5), we have

$$U_X F(e_w) = U_X(e_{\bar{w}}) = U_X(L_{i_k} \cdots L_{i_1} e_\emptyset) = \Phi_X(L_{i_k} \cdots L_{i_1})(xI - L[z])^{-1} e_\emptyset = \Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\emptyset.$$

If we write $\frac{1}{x} z = (a_1, \ldots, a_n)$, we have

$$(xI - L[z])^{-1} = x \sum_{\text{all words } j = j_1 \cdots j_s} a_{j_1} L_{j_1} \cdots a_{j_s} L_{j_s},$$

where the sum is norm convergent.

To simplify notations, define the map $\tilde{F} : \mathfrak{L}_n \to \mathfrak{L}_n$ by the formula $\tilde{F}(L_v) = L_\emptyset$.

Then:

1. $\tilde{F}(AB) = \tilde{F}(B)\tilde{F}(A)$;
2. $\tilde{F}(L_v) = L_v$ if $v$ has length 0 or 1;
3. $\tilde{F}((xI - L[z])^{-1}) = (xI - L[z])^{-1}$;
4. $F L_v e_\emptyset = \tilde{F}(L_v)e_\emptyset$.

Therefore, applying (3.4), we have

$$\Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\emptyset = \Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\emptyset \cdots$$

$$(xI - L[z])^{-1} L[X' \zeta_{i_k}] - \langle \zeta_{i_k}, y \rangle I)(xI - L[z])^{-1} \cdots$$

$$= \Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\emptyset$$

$$= \tilde{F} \Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\emptyset$$

$$(xI - L[z])^{-1} L[X' \zeta_{i_1}] - \langle \zeta_{i_1}, y \rangle I)(xI - L[z])^{-1} \cdots$$

$$= \tilde{F} \Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\emptyset$$

$$= \Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\emptyset.$$

The lemma is proved.

As a consequence, $FU_X F = U_X$, and we have

$$\alpha_{L_i}(R) = U_XRU_X^*.$$
4. MULTICONTRACTIONS

Suppose $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^n$ is a multicontraction; that is,

$$\sum_{i=1}^{n} T_i T_i^* \leq 1.$$ 

This is the same as requiring the row operator $T = (T_1 \cdots T_n) : \mathcal{H}^n \to \mathcal{H}$ to be a contraction. (We will currently denote with the same letter $T$ the multioperator and the associated row contraction.) Accordingly, we have the operators $D_T = (1_{\mathcal{H}^n} - T^*T)^{1/2}$ and $D_T^* = (1_{\mathcal{H}} - TT^*)^{1/2}$, and the spaces $\mathcal{D}_T = D_T^* \mathcal{H}^n \subset \mathcal{H}^n$, $\mathcal{D}_T^* = D_T^* \mathcal{H} \subset \mathcal{H}$. If the row operator $T$ is a strict contraction, we will say that $T$ is a strict multicontraction.

There exists an $l_n$-functional calculus for a multicontraction $T$; it is the unique completely contractive homomorphism $\rho : l_n \to \mathcal{L}(\mathcal{H})$, such that $\rho(L_i) = T_i$. This homomorphism can be extended to $\mathfrak{L}_n$ in an important particular case. Namely, $T$ is called completely noncoisometric (c.n.c.) if there is no $h \in \mathcal{H}$, $h \neq 0$, such that

$$\sum_{|w|=k} \|T^*_w h\|^2 = \|h\|^2 \text{ for all } k \geq 0.$$ 

If $T$ is c.n.c., then $\rho$ can be extended to a completely contractive homomorphism defined on $\mathfrak{L}_n$, that we will denote with the same letter, $\rho : \mathfrak{L}_n \to \mathcal{L}(\mathcal{H})^{11}$. If $f \in \mathfrak{L}_n$, then $f_r \in l_n$, and we may apply $\rho$ to obtain

$$\rho(f_r) = \sum_w f_w r_{|w|} T_w$$

with the sum on the right converging absolutely. If $T$ is c.n.c., then we have also

$$\rho(f) = \text{SOT} - \lim_{r \to 1} \rho(f_r).$$

Similar results are valid for $\tau_n$ and $\mathfrak{R}_n$, the corresponding functional calculus being denoted by $\rho'$.

The next definition introduces two basic objects that appear in Popescu’s theory of multicontractions (see [12, 13]).

**Definition 4.1.** Suppose $T$ is a multicontraction. Then:

(a) The Poisson kernel $K_T$ is the operator $K_T : \mathcal{H} \to \mathfrak{F}_n \otimes D_T^*$ defined by

$$K_T h = \sum_w c_w \otimes D_T^* T^*_w h.$$ 

(b) The characteristic function $\Theta_T$ is the multianalytic operator

$$\Theta_T : \mathfrak{F}_n \otimes D_T \to \mathfrak{F}_n \otimes D_T^*.$$
having the formal Fourier representation

\[
\hat{\Theta}_T(R_1, \ldots, R_n) = -I_{\mathfrak{F}_n} \otimes T + \left( I_{\mathfrak{F}_n} \otimes D_T \cdot \right) \left( I_{\mathfrak{F}_n} \otimes H - \sum_{i=1}^n R_i \otimes T_i^* \right)^{-1} [R_1 \otimes I_H, \ldots, R_n \otimes I_H] (I_{\mathfrak{F}_n} \otimes D_T) | \mathfrak{F}_n \otimes D_T.
\]

(4.1)

The following proposition gathers several results from [13].

**Proposition 4.2.**

(i) The Poisson kernel and the characteristic function are contractions, and \( K_T K_T^* + \Theta_T \Theta_T^* = I_{\mathfrak{F}_n} \otimes D_T \).

(ii) If we define, for \( 0 < r \leq 1 \),

\[
K_T, r h = \sum_w r^{|w|} e_w \otimes D_T T_w^* h = (I_{\mathfrak{F}_n} \otimes D_T) \left( I_{\mathfrak{F}_n} \otimes H - \sum_{i=1}^n r R_i \otimes T_i^* \right)^{-1} (e_\emptyset \otimes h),
\]

then

\[
(4.2)
K_T = \text{SOT} - \lim_{r \to 1} K_T, r.
\]

(iii) If we replace \( R_i \) with \( r R_i \), \( 0 < r < 1 \), in (4.1), then the inverse in the right hand side exists, the equation can be used to define \( \Theta_T(r R) \), and

\[
(4.3)
\Theta_T = \text{SOT} - \lim_{r \to 1} \Theta_T(r R).
\]

We can interpret the Poisson kernel and the characteristic function by means of the Redheffer product (see Section 2.2). Remember that \( \iota : \mathbb{C} \to \mathfrak{F}_n \) is the embedding \( z \mapsto ze_\emptyset \). Take \( r < 1 \), and define then

\[
(4.4)
L_T = \begin{pmatrix} I_{\mathfrak{F}_n} \otimes D_T & -I_{\mathfrak{F}_n} \otimes T \\ I_{\mathfrak{F}_n} \otimes T^* & I_{\mathfrak{F}_n} \otimes D_T \end{pmatrix} : (\mathfrak{F}_n \otimes H) \oplus (\mathfrak{F}_n \otimes D_T) \to (\mathfrak{F}_n \otimes D_T) \oplus (\mathfrak{F}_n \otimes H^n)
\]

\[
(4.5)
L_r = \begin{pmatrix} \iota \otimes I_H & r R \otimes I_H \\ 0 & 0 \end{pmatrix} : \mathcal{H} \oplus (\mathfrak{F}_n \otimes H^n) \to (\mathfrak{F}_n \otimes H) \oplus \{0\}
\]

\[
\begin{array}{ccc}
\mathfrak{F}_n \otimes D_T & \mathfrak{F}_n \otimes H^n & \mathfrak{F}_n \otimes H \\
\mathfrak{F}_n \otimes D_T^* & \mathfrak{F}_n \otimes H & {\{0\}} \rightarrow \mathcal{H}
\end{array}
\]

Then from (4.4) it follows that

\[
(4.6)
L_T \circ L_r = \begin{pmatrix} K_{T, r} & \Theta_T(r R) \\ 0 & 0 \end{pmatrix} : \mathcal{H} \oplus (\mathfrak{F}_n \otimes D_T) \to (\mathfrak{F}_n \otimes D_T) \oplus \{0\}.
\]
Otherwise stated,
\begin{align}
\Theta_T(rR) = \alpha_{L_T}(rR \otimes I_{\mathcal{H}}), \quad K_{T,r} = \beta_{L_T}(r \otimes I_{\mathcal{H}}, rR \otimes I_{\mathcal{H}}).
\end{align}

5. Multicontractions and Automorphisms

By using the functional calculus \( \rho \) (see Section 4), we can extend the action of automorphisms \( \Phi_X \) to a multicontraction \( T \). This is done by defining
\[
\Phi_X(T) = \rho(\Phi_X(L)),
\]
and it follows from (3.6) that we have then also
\[
\Phi_X(T) = \alpha_{L_Y}(T),
\]
where, as usually, \( Y \) is connected to \( X \) by formulas (3.3). Since the functional calculus \( \rho \) is completely contractive, \( \Phi_X(T) \) is also a multicontraction. According to (3.7), we have also
\begin{align}
(5.1) \quad & \Phi^{-1}_X(T) = \alpha_{L_Y^*}(T).
\end{align}

The main result of this section is given by the next theorem.

**Theorem 5.1.** For each \( X \) there exist unitary operators \( Z : \mathcal{D}_{\Phi_X^{-1}(T)} \to \mathcal{D}_T \) and \( Z^* : \mathcal{D}_{\Phi_X^{-1}(T)^*} \to \mathcal{D}_{T^*} \), such that:
\begin{enumerate}
\item \( \Theta_{\Phi_X^{-1}(T)} = (U_X \otimes Z^*) \Theta_T(U_X \otimes Z) \).
\item \( K_{\Phi_X^{-1}(T)} = (U_X \otimes Z^*) K_T \).
\end{enumerate}

**Proof.** Let us define \( L_T \) and \( L_r \) by formulas (4.4) and (4.5) respectively, and \( L_X \) by
\[
L_X = \begin{pmatrix}
a \otimes I_{\mathcal{H}_n \otimes \mathcal{H}} & b \otimes I_{\mathcal{H}_n \otimes \mathcal{H}} \\
c \otimes I_{\mathcal{H}_n \otimes \mathcal{H}} & d \otimes I_{\mathcal{H}_n \otimes \mathcal{H}}
\end{pmatrix} : (\mathcal{F}_n \otimes \mathcal{H}) \oplus (\mathcal{F}_n \otimes \mathcal{H}^n) \to (\mathcal{F}_n \otimes \mathcal{H}) \oplus (\mathcal{F}_n \otimes \mathcal{H}^n),
\]
where \( a, b, c, d \) are related to \( X \) by formulas (3.3).

We want to apply the associativity of the Redheffer product, as stated in Proposition 2.1 (iii):
\[
(5.2) \quad (L_T \circ L_X) \circ L_r = L_T \circ (L_X \circ L_r).
\]
Thus we have computed the right hand side of (5.2). and we will use Lemma 2.2 to obtain its other entries. 

First, we have 

\[
LX \circ Lr = \begin{pmatrix}
(U_X t) \otimes I_H & (U_X r RU_X^* \otimes I_H) \\
0 & 0
\end{pmatrix}.
\]

It follows that 

\[
\alpha_{Lr}((U_X RU_X^*) \otimes I_H) = \left( I_{\tilde{F}_n} \otimes T + (I_{\tilde{F}_n} \otimes D_{T'}) \right) \left( I_{\tilde{F}_n} \otimes - \sum_{i=1}^{n} (U_X r_i RU_X^*) \otimes T_i^* \right)^{-1} \left[ U_X r RU_X^* \otimes I_H, \ldots, U_X r RU_X^* \otimes I_H \right] (I_{\tilde{F}_n} \otimes D_T) |_{\tilde{F}_n} \otimes D_T
\]

\[
= (U_X \otimes I_{D_{T'}}) \Theta_T(rR)(U_X^* \otimes I_{D_{T'}})
\]

and 

\[
\beta_{Lr}((U_X t) \otimes I_H, (U_X r RU_X^*) \otimes I_H)
\]

\[
= (I_{\tilde{F}_n} \otimes D_{T'}) \left( I_{\tilde{F}_n} \otimes - \sum_{i=1}^{n} (U_X r_i RU_X^*) \otimes T_i^* \right)^{-1} (U_X t) \otimes I_H
\]

\[
= (U_X \otimes I_{D_{T'}}) K_{T,r}.
\]

Thus

\[
(5.3) \quad L_T \circ (L_X \circ L_r) = \begin{pmatrix}
(U_X \otimes I_{D_{T'}}) K_{T,r} & (U_X \otimes I_{D_{T'}}) \Theta_T(rR)(U_X^* \otimes I_{D_{T'}}) \\
0 & 0
\end{pmatrix},
\]

and we have thus computed the right hand side of (5.2).

As for the left hand side, let us first remark that, computing \( L_T \circ L_X \) according to (2.3), we obtain as (2, 1) entry \( I_{\tilde{F}_n} \otimes (\alpha_{L_Y^*}, (T))^* \). To avoid messy computations, we will use Lemma 2.2 to obtain its other entries.

Noting that in \( L_T \) and \( L_X \) all spaces have \( \tilde{F}_n \) as a tensor factor, and all operators have \( I_{\tilde{F}_n} \) as a factor, we shall write (a slight abuse of notation) \( L_T = I_{\tilde{F}_n} \otimes L'_T \), \( L_X = I_{\tilde{F}_n} \otimes L'_X \). Since both \( L'_T \) and \( L'_X \) are unitary operators, the same is true of \( L'_T \circ L'_X \). Its (2, 1) entry is 

\[
c + dT^*(I - bT^*)^{-1}a = (c^* + a^*T(I - b^*T)^{-1}d^*)^* = (\alpha_{L_Y^*}, (T))^*,
\]

while its (1, 1) entry is \( D_{T'}(I - bT^*)^{-1}a \). This last operator has obviously dense range from \( H \) to \( D_{T'} \) (remember that \( a \neq 0 \)), and we may therefore apply Lemma 2.2. Consequently, the operators \( Z_* : D_{\alpha_{L_Y^*}, (T)}^* \rightarrow D_{T'} \) and \( Z : D_{\alpha_{L_Y^*}, (T)} \rightarrow D_{T} \), defined by 

\[
Z_* D_{\alpha_{L_Y^*}, (T)}^* = D_{T'}(I - bT^*)^{-1}a
\]

\[
Z D_{\alpha_{L_Y^*}, (T)} = D_{T}(I - b^*T)d^*
\]

and

\[
(5.3) \quad L_T \circ (L_X \circ L_r) = \begin{pmatrix}
(U_X \otimes I_{D_{T'}}) K_{T,r} & (U_X \otimes I_{D_{T'}}) \Theta_T(rR)(U_X^* \otimes I_{D_{T'}}) \\
0 & 0
\end{pmatrix},
\]
are unitary, and

\[
L_T' \circ L_X' = \begin{pmatrix}
Z_s D_{\alpha_{L_Y} (T)} & -Z_s \alpha_{L_Y} (T) Z^* \\
\alpha_{L_Y} (T)^* & D_{\alpha_{L_Y} (T)} Z^*
\end{pmatrix}.
\]

Therefore

\[
L_T \circ L_X = \begin{pmatrix}
I_{\delta_n} \otimes (Z_s D_{\alpha_{L_Y} (T)} ) & -I_{\delta_n} \otimes (Z_s \alpha_{L_Y} (T) Z^*) \\
I_{\delta_n} \otimes \alpha_{L_Y} (T)^* & I_{\delta_n} \otimes (D_{\alpha_{L_Y} (T)} Z^*)
\end{pmatrix} = L'' \circ L_{\alpha_{L_Y} (T)},
\]

where \( L'' = \left( \begin{smallmatrix} I_{\delta_n} \otimes Z^* \\ 0 \\ 0 \\ I_{\delta_n} \otimes Z^* \end{smallmatrix} \right) \). Thus

\[
(L_T \circ L_X) \circ L_r = (L'' \circ L_{\alpha_{L_Y} (T)}) \circ L_r = L'' \circ (L_{\alpha_{L_Y} (T)} \circ L_r)
\]

\[
= L'' \circ \begin{pmatrix}
K_{\alpha_{L_Y} (T)} & \Theta_{\alpha_{L_Y} (T)} (r R) \\
0 & 0
\end{pmatrix}
\]

\[(5.4) \]

\[
= \left( \begin{smallmatrix} (I_{\delta_n} \otimes Z_s) K_{\alpha_{L_Y} (T)} & (I_{\delta_n} \otimes Z_s) \Theta_{\alpha_{L_Y} (T)} (r R) (I_{\delta_n} \otimes Z^*) \\ 0 & 0 \\ 0 & 0 \\ (I_{\delta_n} \otimes Z_s) K_{\alpha_{L_Y} (T)} \\ (I_{\delta_n} \otimes Z_s) \Theta_{\alpha_{L_Y} (T)} (r R) (I_{\delta_n} \otimes Z^*)
\end{smallmatrix} \right)
\]

(we have used (5.1) for the last equality. We compare now (5.3) with (5.4), and make \( r \to 1 \). The resulting limits exist by (4.2) and (4.3), and we obtain the assertion of the theorem. \( \square \)

6. Constrained row contractions

We introduce now some definitions from \[13, 14\], where the notion of constrained objects appears. Let \( J \) be a WOT-closed two-sided ideal in \( L_n, J \neq L_n \). We define two subspaces of \( F_n \) by

\[
M_J = J F_n, \quad N_J = F_n \ominus M_J.
\]

Then \( M_J \) and \( F M_J \) are invariant to \( L \) and to \( R \), while \( N_J \) and \( F N_J \) are invariant to \( L^* \) and \( R^* \).

The constrained left and right creation operators belong to \( \mathcal{L}(N_J) \) and are given by

\[
L^3_I = P_{N_J} L_i | N_J, \quad R^3_I = P_{N_J} R_i | N_J
\]

An operator \( M \in \mathcal{L}(N_J \otimes E, N_J \otimes E) \) is called multianalytic if

\[
M(L^3_I \otimes I_E) = (L^3_I \otimes I_E) M.
\]

We want to define constrained row contractions by using the functional calculus with respect to elements of the ideal. A problem appears, since for a general multicontraction the functional calculus is only defined for elements in \( I_n \); it can
be extended to $\mathcal{L}$ only for completely noncoisometric contractions. Thus, if $T$ is a general multicontraction, and $j \subset I_n$ is a two-sided norm closed ideal, we say that $T$ is $j$-constrained if $f(T) = 0$ for all $f \in j$. If $T$ is c.n.c., and $\mathfrak{J} \subset \mathfrak{L}$, we say that $T$ is $\mathfrak{J}$-constrained if $f(T) = 0$ for all $f \in \mathfrak{J}$. If $\mathfrak{J}$ is the wot-closure of $j$, and $T$ is c.n.c., then it is $j$-constrained iff it is $\mathfrak{J}$-constrained.

The next result connects the constraints with the automorphisms.

**Proposition 6.1.** If $T$ is $j$-constrained, then $\Phi^{-1}_X(T)$ is $\Phi_X(j)$-constrained (and similarly for $\mathfrak{J}$-constraints, in case $T$ is c.n.c.).

**Proof.** If we denote $T' = \Phi^{-1}_X(T)$, and $\rho : I \to \mathcal{L}(\mathcal{H})$ is, as above, the functional calculus for $T'$, then

$$T'_i = \rho((\Phi^{-1}_X(L))_i) = \rho(U_X^* L_i U_X) = \rho(\Phi^{-1}_X(L_i)).$$

Since the functional calculus for $T'$ is the unique homomorphism algebra that maps $L_i$ into $T'_i$, it must be $\rho \circ \Phi^{-1}_X$, and therefore

$$f(T') = \rho(\Phi^{-1}_X(f)) = (\Phi^{-1}_X(f))(T).$$

Thus $f(T') = 0$ is equivalent to $(\Phi^{-1}_X(f))(T) = 0$, whence the statement of the proposition follows. 

Now, if $T$ is a $j$-constrained contraction, and $\mathfrak{J}$ is the wot-closure of $j$, we define as in [13]:

(a) the constrained Poisson kernel $K_{\mathfrak{J},T} : \mathcal{H} \to \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T$ by

$$K_{\mathfrak{J},T} = P_{\mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T} K_T;$$

(b) the constrained characteristic function $\Theta_{\mathfrak{J},T} : \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T \to \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T$ by

$$\Theta_{\mathfrak{J},T} = P_{\mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T} \Theta_T | \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T.$$ 

We can then give the following consequence of Theorem 5.1 for $j$-constrained multicontractions.

**Theorem 6.2.** Suppose $T$ is $j$-constrained, and denote $j' = \Phi_X(j)$, $\mathfrak{J}' = \Phi_X(\mathfrak{J})$, $T' = \Phi^{-1}_X(T)$. Then $\Theta_{j',T'} = (U_X \otimes Z_*') \Theta_{\mathfrak{J},T} (U_X^* \otimes Z)$ and $K_{\mathfrak{J}',T'} = (U_X \otimes Z_*') K_{\mathfrak{J},T}.$

**Proof.** By Theorem 5.1, we have

$$\Theta_{T'} = (U_X \otimes Z_*') \Theta_T (U_X^* \otimes Z),$$

where $Z : \mathcal{D}_{T'} \to \mathcal{D}_T$ and $Z_* : \mathcal{D}_{T'}^* \to \mathcal{D}_T^*$ are unitary operators.

We have

$$P_{N'_j} = U_X P_{N_j} U_X^*, \quad P_{D_{T'}} = Z^* P_{D_T} Z, \quad P_{D_{T'}^*} = Z_*^* P_{D_{T^*}} Z_*$$

for
whence
\[ P_{N_j} \otimes D_T = (U_X \otimes Z^*) P_{N_j} \otimes D_T (U_X^* \otimes Z). \]
Therefore
\[ \Theta_{T', \gamma} = P_{N_j} \otimes D_{T'}, \Theta_{T'} P_{N_j} \otimes D_T, \]
\[ = (U_X \otimes Z^*) P_{N_j} \otimes D_T (U_X^* \otimes Z) \]
\[ = (U_X \otimes Z^*) \Theta_T (U_X^* \otimes Z) (U_X \otimes Z^*) P_{N_j} \otimes D_T (U_X^* \otimes Z) \]
\[ = (U_X \otimes Z^*) P_{N_j} \otimes D_T \Theta_T P_{N_j} \otimes D_T (U_X^* \otimes Z) \]
\[ = (U_X \otimes Z^*) \Theta_{T, \gamma} (U_X^* \otimes Z). \]

The computations for the Poisson kernel are similar. \( \square \)

If \( j \) is the commutator ideal \( \mathfrak{c} = [l_n, l_n] \subset l_n \) (and correspondingly \( \mathfrak{c} = [\mathcal{L}_n, \mathcal{L}_n] \subset \mathcal{L}_n \)), then the constraint becomes just commutativity. Then \( \Phi_X(\mathfrak{c}) = \mathfrak{c}, \Phi_X(\mathfrak{c}) = \mathfrak{c} \) for all \( X \), which translates in the fact that applying the automorphism \( \Phi_X \) to a commuting multicontraction produces also a commuting multicontraction. Theorem \( 6.2 \) yields then the transformation rule of the commutative characteristic function with respect to automorphisms of the ball, as shown in \( [4] \) (see Theorem 6.3 therein).

7. Invariant ideals

In connection to Theorem \( 6.2 \), it is interesting to discuss bilateral ideals \( \mathfrak{J} \) of \( \mathcal{L}_n \) which are invariant with respect to all automorphisms \( \Phi_X \). They have the property that, if \( T \) is a \( \mathfrak{J} \)-constrained multicontraction, \( \alpha_X(T) \) is then also a \( \mathfrak{J} \)-constrained multicontraction.

We have already encountered the commutator ideal \( \mathfrak{c} \). Other examples of invariant ideals are given by the iterated commutators \( \mathfrak{c}^k \), defined by \( \mathfrak{c}^{k+1} = [\mathcal{L}_n, \mathfrak{c}^k] \). These form a decreasing sequence contained in \( \mathfrak{c} \). We will prove below that there are no invariant ideals larger than \( \mathfrak{c} \). But we need for this some more preparatory results.

First, it is shown in \( [9, 10] \) that any \( \mathcal{L}_n \)-invariant subspace of \( \mathcal{F}_n \) is of the form \( \Theta(\mathfrak{F}_n \otimes \mathcal{E}) \), for \( \mathcal{E} \) a Hilbert space and \( \Theta : \mathfrak{F}_n \otimes \mathcal{E} \to \mathcal{F}_n \) a multianalytic operator that is also an isometry (such a \( \Theta \) is called \textit{inner}). This multianalytic operator is essentially uniquely determined by the subspace: if \( \Theta' : \mathfrak{F}_n \otimes \mathcal{E}' \to \mathcal{F}_n \) satisfies \( \Theta'(\mathfrak{F}_n \otimes \mathcal{E}') = \Theta(\mathfrak{F}_n \otimes \mathcal{E}) \), then there exists a unitary \( V : \mathcal{E}' \to \mathcal{E} \) such that \( \Theta' = \Theta(I_{\mathfrak{F}_n} \otimes V) \). Based on these results, one proves in \( [9] \) that the map \( \mathfrak{J} \mapsto \mathcal{M}_J = \overline{J_{\mathfrak{c}^0}(= \overline{\mathfrak{F}_n})} \) is a one to one map from the set of all bilateral ideals in \( \mathcal{L}_n \) onto the set of subspaces in \( \mathfrak{F}_n \) invariant both to \( \mathcal{L}_n \) and to \( \mathcal{R}_n \). 


Finally, in \[1, 7\] one identifies the eigenvectors of \(L_n^\ast\). Namely, for any \(\lambda \in \mathbb{B}^n\), one defines

\[
\nu_\lambda = (1 - \|\lambda\|^2)^{1/2}(I - L[\hat{\lambda}])^{-1}e_\emptyset.
\]

Then \(L_n^\ast \nu_\lambda = \bar{\lambda}_i \nu_\lambda\), whence \(\langle Lw \nu_\lambda, \nu_\lambda \rangle = \lambda w\) for any \(w \in \mathbb{F}_n^\ast\). Note that \(\nu_\lambda\) are also eigenvectors of \(R_n^\ast\) (corresponding to the same eigenvalues). The space spanned by all \(\nu_\lambda\) (\(\lambda \in \mathbb{B}^n\)) is \(M_\perp C_n^\ast\). This last space is the symmetric Fock space, which we will denote by \(\mathfrak{F}_n^s\), and the map \(e_w \mapsto \lambda w\) identifies it with a space of functions on \(\mathbb{B}^n\). If \(A \in \mathfrak{R}_n\), then the projection of \(A1\) onto \(\mathfrak{F}_n^s\) is identified with the function \(\langle A \nu_\lambda, \nu_\lambda \rangle\).

**Theorem 7.1.** If \(\mathfrak{J} \supset \mathfrak{C}\) is a bilateral ideal in \(\mathfrak{L}_n\), and \(\Phi_X(\mathfrak{J}) = \mathfrak{J}\) for all \(X\), then either \(\mathfrak{J} = \mathfrak{C}\) or \(\mathfrak{J} = \mathfrak{L}_n\).

**Proof.** Let \(\mathcal{M} = \mathfrak{J} \mathfrak{S}_n\) be the invariant subspace determined by \(\mathfrak{J}\); then \(\Phi_X(\mathfrak{J}) = \mathfrak{J}\) implies \(U_X \mathcal{M} = \mathcal{M}\). Suppose \(\mathcal{M} = \Theta(\mathfrak{S}_n \otimes \mathfrak{G})\); define \(\Gamma : \mathbb{B}^n \otimes \mathcal{E} \to \mathbb{C}\) by the formula

\[
\Gamma(\lambda, h) = \langle \Theta(\nu_\lambda \otimes h), \nu_\lambda \rangle.
\]

If \(\Theta = \sum_w R_w \otimes m_w\), then

\[
\Gamma(\lambda, h) = \lim_{r \to 1} \sum_w (\rho_t^w)(R_w \otimes m_w)(\nu_\lambda \otimes h), \nu_\lambda
\]

\[
= \lim_{r \to 1} \sum_w (R_w \nu_\lambda, \nu_\lambda) \langle m_w h, e_\emptyset \rangle = \lim_{r \to 1} \sum_w \lambda^w \langle m_w h, e_\emptyset \rangle.
\]

For \(r < 1\) the series on the right is uniformly convergent and thus defines an analytic function \(\lambda \in \mathbb{B}^n\). It follows then that \(\Gamma(\lambda, h)\) is analytic in \(\lambda\). It is obviously linear in \(h\); so we may consider \(\lambda \mapsto \Gamma(\lambda, \cdot)\) as an analytic map \(\tilde{\Gamma}\) from \(\mathbb{B}^n\) into \(\mathcal{E}\) (actually, in the dual of \(\mathcal{E}\), which can be identified with \(\mathcal{E}\)).

On the other hand, since \(U_X\) implements an automorphism of \(\mathfrak{R}_n\), one checks easily that \(\Theta_X = U_X \Theta(U_X^\ast \otimes I_\mathcal{E})\) is also an multianalytic inner operator. The invariance of \(\mathcal{M}\) with respect to \(U_X\) implies that \(\mathcal{M} = \Theta_X(\mathfrak{S}_n \otimes \mathcal{E})\). The essential uniqueness of this representation implies then that for any \(X \in U(1, n)\) there exists \(V_X \in \mathfrak{L}(\mathcal{E})\) such that

\[
U_X \Theta(U_X^\ast \otimes I_\mathcal{E}) = \Theta(I_{\mathfrak{S}_n} \otimes V_X).
\]

Let us take now \(X\) such that \(x > 0\) (remember that the mappings \(X \mapsto \Phi_X\) and \(X \mapsto \phi_X\) have as kernel the constant unitaries). From (5.5) and (7.1) it follows then that

\[
U_X^\ast e_\emptyset = U_X^\ast \nu_0 = \nu_{\phi_X(0)}.
\]
Therefore
\[
\Gamma(\phi_X(0), h) = \langle \Theta(\nu_{\phi_X(0)} \otimes h), \nu_{\phi_X(0)} \rangle \\
= \langle U_X \Theta(U_X^* \otimes I_\mathcal{E})(e_\emptyset \otimes h), e_\emptyset \rangle \\
= \langle \Theta(I_X \otimes V_X)(e_\emptyset \otimes h), e_\emptyset \rangle = \Gamma(0, V_X(h)).
\]
This last relation can be rewritten as \( \tilde{\Gamma} \circ \phi_X(0) = V_X^* \tilde{\Gamma}(0) \). Since \( V_X \) is unitary, we obtain that \( \| \tilde{\Gamma}(\phi_X(0)) \| = \| \tilde{\Gamma}(0) \| \). The image of \( \{X \in U(1, n) : x > 0 \} \) under the mapping \( X \mapsto \phi_X(0) \) is the whole \( \mathbb{B}^n \); therefore \( \tilde{\Gamma} \) is an analytic function on \( \mathbb{B}^n \) with values in the Hilbert space \( \mathcal{E} \), of constant norm, which must be actually constant.

For any \( h \in \mathcal{E} \) we can define an element \( \Theta_h \in \mathfrak{R} \) by the formula \( \Theta_h \xi = \Theta(\xi \otimes h) \).

We have then, by (7.2) and the remarks before the statement of the theorem,
\[
\Gamma(\lambda, h) = \langle \Theta_h \nu_\lambda, \nu_\lambda \rangle = (P_{\mathfrak{R}_n}(\Theta_h e_\emptyset))(\lambda).
\]

Two cases present now. If \( \tilde{\Gamma} \) is identically 0, then \( (P_{\mathfrak{R}_n}(\Theta_h e_\emptyset))(\lambda) = 0 \) for all \( h \in \mathcal{E} \), and thus the image of \( \Theta \) is included in \( \mathfrak{C} \). It follows that \( \mathfrak{J} \subset \mathfrak{C} \); and then the assumption implies \( \mathfrak{J} = \mathfrak{C} \).

In the opposite case, take \( h \in \mathcal{E} \) such that \( \Gamma(\lambda, h) \) is a nonnull constant. Then \( (P_{\mathfrak{R}_n}(\Theta_h e_\emptyset))(\lambda) \) is a nonnull multiple of \( e_\emptyset \). Thus \( \mathcal{M} \) contains a vector of the form \( ae_\emptyset + \xi_0 \), with \( a \neq 0 \) and \( \xi_0 \in \mathcal{M}_\mathcal{E} \). But the assumption \( \mathfrak{C} \subset \mathfrak{J} \) implies \( \mathcal{M}_\mathcal{E} \subset \mathcal{M} \); therefore \( e_\emptyset \in \mathcal{M} \). Since \( \mathcal{M} \) is invariant, it follows that \( \mathcal{M} = \mathfrak{R}_n \), whence \( \mathfrak{J} = \mathfrak{L} \). \( \square \)

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