Information Fusion Kalman Filter for Two-Sensor System with Time-Delayed Measurements

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Abstract

For the two-sensor linear discrete stochastic system with time-delayed measurements, an equivalent system without time-delayed measurements is obtained by introducing a new measurement process. Then a covariance intersection (CI) fusion steady-state Kalman filter is presented based on the modern time series analysis method. Compared with the optimal Kalman fusers weighted by matrices, diagonal matrices and scalars, this CI Kalman fuser avoids computing the cross-covariances among the local filtering errors. It is proved that its accuracy is higher than that of each local filter, and is lower than that of the Kalman fuser weighted by matrices. The geometric interpretations of these fusers’ accuracy relations are given based on the covariance ellipses. A Monte-Carlo simulation example for target tracking system verifies the correctness of the proposed accuracy relations, i.e. the actual accuracy of the CI Kalman fuser is close to that of the fuser weighted by matrices, and is higher than that of each local filter, so it has higher accuracy and good performances.

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Keywords: Multisensor information fusion; time-delayed measurements; covariance intersection; covariance ellipse; Kalman filter.

1. Introduction

Since the emergence of the Kalman filter, it is widely applied to solve the estimation problem of signal or state. The systems without time-delayed measurements are usually considered [1-3], however, in practical application, the systems with measurement delays are inevitable. Several basic methods are used

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to solve the state estimation problem in this situation: the augmented state method [4,5] requires large computations, especially when the dimension of the system is high or the measurement delays are very large. A re-organized innovation analysis method [6,7] requires to compute the multiple Kalman filters and multiple Riccati equations. A measurement transform approach presented in [8] can transform the system with measurement delays into an equivalent system without measurement delays. In the meanwhile, multisensor information fusion is widely applied to improve the accuracy of the filter. In order to compute the optimal weighted fusion Kalman filter, the computation of the local estimation error cross-covariances is required, which yields a large computational burden and computational complexity [8], the covariance intersection (CI) fusion method presented in [9] can solve the problem of fused estimator with unknown cross-covariances. This CI fusion algorithm is of consistency and robustness, because it can give an upper bound of actual filtering error variance, which is independent of unknown cross-covariances.

In this paper, using measurement transform approach, the optimal local Kalman filters are presented based on the modern time series analysis method. Then the CI fusion Kalman filter is presented, and the accuracy relation among the local Kalman filters, CI fusion Kalman filter and the fusion Kalman filters weighted by matrices, diagonal matrices and scalars is proved.

2. The local steady-state optimal Kalman filter

Consider two-sensor linear discrete time-invariant stochastic system with time-delayed measurements

\[ x(t+1) = \Phi x(t) + \Gamma w(t) \]
\[ z_i(t) = H_i x(t - \tau_i) + \xi_i(t), \quad i = 1, 2 \]

where \( t \) is the discrete time, \( \tau_i > 0 \) is the measurement delay of the \( i \)th sensor, \( x(t) \in \mathbb{R}^n, z_i(t) \in \mathbb{R}^m \) are the state and measurement, \( w(t) \in \mathbb{R}^r, \xi_i(t) \in \mathbb{R}^m \) are uncorrelated input and measurement white noises with zero mean and variances \( Q_w \) and \( Q_{\xi_i} \), respectively. \( \Phi, \Gamma \) and \( H_i \) are known constant matrices with proper dimensions, and \( (\Phi, H_i) \) is a completely observable pair with the observable index \( \beta_i \), \( (\Phi, \Gamma) \) is the completely controllable pair.

The aim is to find the local steady-state optimal Kalman filter \( \hat{x}_i(t | t) \), \( i = 1, 2 \), the optimal fusion Kalman filters \( \hat{x}_m(t | t) \) weighted by matrices, \( \hat{x}_d(t | t) \) weighted by diagonal matrices, \( \hat{x}_s(t | t) \) weighted by scalars, and the CI fusion Kalman filter \( \hat{x}_{CI}(t | t) \).

Introducing a new measurement \( y_i(t) \) and a new measurement noise \( v_i(t) \)

\[ y_i(t) = z_i(t + \tau_i), v_i(t) = \xi_i(t + \tau_i) \]

So the new measurement equation is obtained

\[ y_i(t) = H_i x(t) + v_i(t), \quad i = 1, 2 \]

where \( v_i(t) \) is white noise with zero mean and variance \( Q_v = Q_{\xi_i} \), and it is independent with \( w(t) \).

Denoting the linear space spanned by the stochastic variables \((z_i(t + N), z_i(t + N - 1), \cdots)\) as \( L(z_i(t + N), z_i(t + N - 1), \cdots) \), and the linear space spanned by the stochastic variables \((y_i(t + N - \tau_i), y_i(t + N - \tau_i - 1), \cdots)\) as \( L(y_i(t + N - \tau_i), y_i(t + N - \tau_i - 1), \cdots) \), and we have the relation as \( L(z_i(t + N), z_i(t + N - 1), \cdots) = L(y_i(t + N - \tau_i), y_i(t + N - \tau_i - 1), \cdots) \). Defining the linear minimum variance
predictor \( \hat{x}_i(t | t - \tau) \) of \( x(t) \) based on \( (y_i(t + N - \tau), y_j(t + N - \tau - 1), \cdots) \) and the linear minimum variance filter \( \hat{x}_j(t | t) \) of \( x(t) \) based on \( (z_i(t + N), z_j(t + N - 1), \cdots) \). And the relation between them is given by

\[
\hat{x}_i(t | t) = \hat{x}_i(t | t - \tau_i), \tau_i > 0
\]

Hence the relation of the errors \( \hat{x}_i(t | t) \) and \( \hat{x}_i(t | t - \tau) \) is \( \hat{x}_i(t | t) = \hat{x}_i(t | t - \tau) \). Furthermore, the steady-state filtering error variances \( P_{ii}^x = E[\hat{x}_i(t | t)\hat{x}_i(t | t)'] \) and the steady-state filtering error cross-covariances \( P_{ij}^x = E[\hat{x}_i(t | t)\hat{x}_j(t | t)'] \) have the relation as \( P_{ii}^x = P_i(-\tau_i, -\tau_i), P_{ij}^x = P_j(-\tau_i, -\tau_j), i, j = 1, 2, i \neq j \), where \( P_i(-\tau, -\tau) = E[\hat{x}_i(t | t - \tau)\hat{x}_i(t | t - \tau)] \).

Therefore, the problem of getting the local steady-state optimal Kalman filter \( \hat{x}_i(t | t) \) is converted to that of finding the local steady-state optimal Kalman predictor \( \hat{x}_i(t | t - \tau) \). From (1), (3) and (4), it yields that \( y_i(t) = H_i(I_n - q^{-1}\Phi)^{-1}q^{-1}w(t) + \xi_i(t + \tau_i) \), where \( q^{-1} \) is the backward shift operator. Introducing the left-coprime factorization \( H_i(I_n - q^{-1}\Phi)^{-1}q^{-1} = A_i(q^{-1})B_i(q^{-1}) \), where \( I_n \) denotes the \( n \times n \) unit matrix, \( A_i(q^{-1}) \) and \( B_i(q^{-1}) \) are polynomial matrices with form \( X_i(q^{-1}) = X_{i0} + X_{1i}q^{-1} + \cdots + X_{ni}q^{-n_i} \), \( X_{ni} \neq 0 \), \( X_{i0} = 0 \), \( H_i = 0 \), \( A_0 = I_n \), \( B_0 = 0 \). So the local ARMA innovation models are obtained by \( A_i(q^{-1})y_i(t) = D_i(q^{-1})\epsilon_i(t) \), where \( D_i(q^{-1}) = D_{i0} + D_{1i}q^{-1} + \cdots + D_{ni}q^{-n_i} \) is stable, \( D_{i0} = I_{n_i} \), \( \epsilon_i(t) \in \mathbb{R}^{n_i} \) is white noise with zero mean and variance matrix \( Q_{i} \). And \( D_i(q^{-1})\epsilon_i(t) = B_i(q^{-1})w(t) + A_i(q^{-1})v(t), D_i(q^{-1}) \) and \( Q_{i} \) are obtained by Gevers-Wouters iterative algorithm [10].

Lemma 1[1]. For the two-sensor system (1) and (4), the local steady-state Kalman predictor \( \hat{x}_i(t + 1 | t) \) is given by

\[
\hat{x}_i(t + 1 | t) = \Psi_{pi}\hat{x}_i(t | t - 1) + K_{pi}y_i(t), i = 1, 2
\]

\[
\Psi_{pi} = \Phi - K_{pi}H_i, K_{pi} = \begin{bmatrix} H_i \\ H_i\Phi \\ \vdots \\ H_i\Phi^{i-1} \\ \vdots \\ H_i\Phi^{n_i-1} \\ M_{i,1} \\ \vdots \\ M_{i,1} \\ M_{i,2} \\ \vdots \\ M_{i,2} \\ \vdots \\ M_{i,n_i} \\ \vdots \\ M_{i,n_i} \end{bmatrix}
\]

where \( \Psi_{pi} \) is a stable matrix, \( K_{pi} \) is the predictor gain, the pseudo-inverse of matrix \( X \) is defined as \( X^+ = (X'X)^{-1}X^T \). \( M_i \) can recursively be computed as \( M_{ji} = -A_{ji}M_{i,j-1} - \cdots - A_{ji}M_{i,n_i} + D_{ji} \), with \( M_{ji} = 0 \) \((j < 0)\), \( M_{i0} = I_{n_i} \). The prediction error variances \( \Sigma_i \) and the prediction error cross-covariances \( \Sigma_{ij} \) satisfy the Lyapunov equations

\[
\Sigma_i = \Psi_{pi}\Sigma_{pi}' + \Gamma Q_i\Gamma^T + K_{pi}Q_iK_{pi}' \quad \Sigma_{ij} = \Psi_{pi}\Sigma_{pj}' + \Gamma Q_i\Gamma^T, i, j = 1, 2, i \neq j
\]

The \(-k_i \) steps steady-state Kalman predictor \( \hat{x}_i(t | t + k_i) \) is given by

\[
\hat{x}_i(t | t + k_i) = \Phi^{-k_i}\hat{x}_i(t + k_i + 1 | t + k_i), k_i \leq 2
\]

without loss of generality, taking \( \tau_i < \tau_j \), \( k_i = -\tau_i, k_j = -\tau_j \), the local steady-state Kalman predicting error variances \( P_i(k, k_i) = E[\hat{x}_i(t | t + k)\hat{x}_i(t | t + k)] \) and the local steady-state Kalman predicting error cross-covariances \( P_{ij}(k, k_i) = E[\hat{x}_i(t | t + k)\hat{x}_j(t | t + k)] \) are obtained by

\[
P_i(k, k_i) = \Phi^{-k_i}\Sigma_i\Phi^{i+k_i-1}\Gamma + \sum_{l=0}^{i+k_i-2}\Phi^{i+k_i-l}\Gamma Q_i\Gamma^T\Phi^{l}, k_i \leq 2
\]
\[ P_{\theta}(k_i,k_j) = \Phi^{k_j,k_i} \Psi_{\theta}^{k_j,k_i} \sum_{\alpha} \Phi^{k_j,k_i} \Gamma_{\theta}^{k_j,k_i} \Phi^{\alpha} + \sum_{\tau=0}^{k_j-2} \Phi^{k_j,k_i} \Omega_{\theta}^{k_j,k_i} \Gamma_{\theta}^{\tau} \Phi^{\tau}, k_j \leq k_i \leq 2 \] \hspace{1cm} (11)

3. The fused steady-state optimal Kalman filter

Defining \( P_i = P_i(k_i,k_j), P_\theta = P_\theta(k_i,k_j) \) for convenience, \( i, j = 1, 2, i \neq j \), when they are known, three weighting fusers are as follows:

3.1. Kalman filter weighted by matrices

For the two-sensor system (1) and (2), the optimal information fusion Kalman filter weighted by matrices [1] is given by \( \hat{x}_i(t | t) = \sum_{i=1}^{2} \Theta_i \hat{x}_i(t | t) = \sum_{i=1}^{2} \Theta_i \hat{x}_i(t | t - \tau_i) \), where the weighting matrix is given by \([\Omega_1, \Omega_2] = (e^\tau P_i^{-1} e^{-\tau})^{-1} e^\tau P_i^{-1} e^{-\tau}, e^\tau = [I_s, I_s] \), \( P = (P_i)_{2m \times 2n} \), the fusion error variance \( P_\theta \) weighted by matrices is obtained by \( P_\theta = (e^\tau P_i^{-1} e^{-\tau})^{-1} \).

3.2. Kalman filter weighted by diagonal matrices

The optimal information fusion Kalman filter weighted by diagonal matrices [2,11] is given by \( \hat{x}_d(t | t) = \sum_{i=1}^{2} \Theta_d \hat{x}_d(t | t) = \sum_{i=1}^{2} \Theta_d \hat{x}_d(t | t - \tau_i) \), where \( \Theta_d = \text{diag}(\alpha_1, \cdots, \alpha_n), i = 1, 2 \). The optimal weighting vectors are given by \( [\alpha_1, \alpha_2] = (e^\tau (P_i^\alpha)^{-1} e^{-\tau})^{-1} e^\tau (P_i^\alpha)^{-1}, s = 1, \cdots, n \), where \( e^\tau = [1,1] \), \( P_i^\alpha = (P_i^\alpha)_{2 \times 2} \), \( P_i^\alpha \) is the \((s,s)\) diagonal element of \( P_i \), the fusion error variance \( P_d \) weighted by diagonal matrices is obtained by \( P_d = \sum_{i=1}^{2} \sum_{i=1}^{2} \Theta_d P_i \).

3.3. Kalman filter weighted by scalars

The optimal information fusion Kalman filter weighted by scalars [3] is given by \( \hat{x}_s(t | t) = \sum_{i=1}^{2} \omega_i \hat{x}_s(t | t) = \sum_{i=1}^{2} \omega_i \hat{x}_s(t | t - \tau_i) \), where the optimal weights is given by \( [\omega_1, \omega_2] = (e^\tau P_i^{-1} e^{-\tau})^{-1} e^\tau P_i^{-1}, e^\tau = [1,1], P_\theta = (tr P_i)_{2 \times 2} \), the notation \( \text{tr} \) means the trace of a matrix. The fusion error variance \( P_s \) weighted by scalars is obtained by \( P_s = \sum_{i=1}^{2} \sum_{i=1}^{2} \omega_i P_s \).

3.4. CI fusion Kalman filter

When \( P_i, P_\theta \) are unknown, for the two-sensor system (1) and (2), the CI fusion Kalman filter [9] is given by \( \hat{x}_c(t | t) = P_c \omega P_i^{-1} \hat{x}_c(t | t) + (1 - \omega) P_i^{-1} \hat{x}_i(t | t) \), where the CI fusion error variance matrix \( P_c \) is given by \( P_c = [\omega P_i^{-1} + (1 - \omega) P_i^{-1}]^{-1} \), the weighted coefficient \( \omega \in [0,1] \) and it minimizes the performance index, \( J = \min_{\omega} tr P_c = \min_{\omega} tr \{[\omega P_i^{-1} + (1 - \omega) P_i^{-1}]^{-1} \} \). The optimal weighting coefficient \( \omega \) can be solved by applying the gold section method or the Fabonacci method. Then according to the definition
\[ \ddot{x}(t|t) = x(t) - \ddot{x}(t|t), \] it yields that the filtering error \[ \dddot{x}(t|t) = P^{-1}_c\{\omega P^{-1}_c \dddot{x}(t|t) + (1-\omega)P^{-1}_c \dddot{x}(t|t) \}. \] The actual filtering error variance \[ \dddot{P}_c = E[\dddot{x}(t|t)\dddot{x}(t|t)^T] \] is also easily obtained by

\[
\dddot{P}_c = P^{-1}_c\{\omega^2 P^{-1}_c + \omega(1-\omega)P^{-1}_c P^{-1}_d + \omega(1-\omega)P^{-1}_d P^{-1}_c + (1-\omega)^2 P^{-1}_d P^{-1}_d \} P^{-1}_c.
\]

**Theorem 1.** For the two-sensor system (1) and (2), the accuracy relations of local and fused filtering error variance matrices are

\[
P_m \leq P_d \leq P_s \leq P_j, \quad i = 1,2 \tag{12}
\]

\[
trP_m \leq trP_d \leq trP_s \leq trP_j, \quad i = 1,2 \tag{13}
\]

**Proof.** It has been proved that \( P_m \leq P_d \leq P_s \leq P_j \) [1] and \( \dddot{P}_c \leq \dddot{P}_j \) [12]. The unbiasedness of local filter \( \dddot{x}(t|t) \) can yield that the fused filters \( \dddot{x}_m(t|t) \), \( \dddot{x}_d(t|t) \), \( \dddot{x}_s(t|t) \), and \( \dddot{x}_C(t|t) \) are also unbiased. Reference [13] has proved that the error variance matrix of the linear minimum variance unbiased fused filter weighted by matrices is less than or equal to that of any other linear unbiased filters. CI fused filter \( \dddot{x}_C(t|t) \) can be considered as one kind of linear unbiased predictor weighted by matrices, so (12) holds. And in the CI fusion algorithm, if taking \( \omega = 0 \), we have \( J = trP_j \), and if taking \( \omega = 1 \), we have \( J = trP_j \). Hence the optimal weighting coefficient \( \omega \in [0,1] \) yields \( trP_m \leq trP_j, i = 1,2 \). Then applying (12), it is obvious that (13) holds.

**4. Monte-Carlo simulation example**

Consider the two-sensor target tracking system with time-delayed measurements (1) and (2), in simulation we take \( \Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.5T^2 \\ T \end{bmatrix}, T = 0.5 \), \( H_1 = [1 \ 0], H_2 = I_2, \tau_1 = 2, \tau_2 = 3, Q_1 = 2, Q_2 = 3, Q_{11} = \begin{bmatrix} 4 & 0 \\ 0 & 0.05 \end{bmatrix}, \) the aim is to obtain the local steady-state optimal Kalman filter \( \dddot{x}(t|t) \), the optimal fusion Kalman filters \( \dddot{x}_m(t|t) \), \( \dddot{x}_d(t|t) \), \( \dddot{x}_s(t|t) \), and the CI fusion Kalman filter \( \dddot{x}_C(t|t) \).

In order to give a powerful geometric interpretation with respect to accuracy relations of local and fused filters, the covariance ellipse for a covariance matrix \( P \) is defined as the locus of point \( \{x : x^TP^{-1}x = c\} \) where \( c = 1 \) will be assumed without loss of generality. It was proved [9] that \( P_m \leq P_d \) is equivalent to that the ellipse for \( P_m \) is enclosed in the ellipse for \( P_d \). The simulation results are shown in Fig1. The accuracy relation has the following geometric interpretations: the ellipse for \( P_m \) is enclosed in the ellipses for \( P_d, P_s \), and \( P_j \), the ellipse for \( P_d \) is enclosed in the ellipse for \( P_s \), and the ellipse for \( P_s \) encloses the intersection of the ellipses for \( P_m \) and \( P_d \).

In order to verify the theoretical results for accuracy relation, N=50 Monte-Carlo runs for \( t = 1,\cdots,300 \), are performed. The mean square errors (MSE) at time \( t \) for local and fused Kalman filters is defined as sampled average for \( trP_j = trE[(\dddot{x}(t|t) - x(t))(\dddot{x}(t|t) - x(t))^T] \), i.e.

\[
MSE_j(t) = \frac{1}{N} \sum_{i=1}^{N} [(\dddot{x}^{(i)}_j(t|t) - x^{(i)}(t))^T (\dddot{x}^{(i)}_j(t|t) - x^{(i)}(t))] , \quad j = 1,2,m,d,s,CI \],
\[
MSE_j(t) \rightarrow trP_j, j = 1,2,m,d,s,CI.
\]
The simulation results are shown in Table 1, Fig 1 and Fig 2. From Table 1, it is obvious that the accuracy relation (12) and (13) hold. From Fig 2, it is easy to find that the $MSE_j(t)$ values of the local and fused Kalman filters fluctuate around the corresponding theoretical values $trP_j$, so $MSE_m(t) \leq MSE_{CI}(t) \leq MSE_d(t) \leq MSE_s(t) \leq MSE_c(t) \leq MSE(t)$. Also, we see that the ellipse of $P_{CI}$ is close to the ellipse of $P_m$, and $trP_{CI}$ is also close to $trP_m$, so the CI fusion algorithm has good performance.

![Fig.1. The accuracy comparison of $P_j$, $j=1,2,m,d,s,CI$ and $P_{CI}$ based on covariance ellipses](image1)

![Fig.2. Comparison of MSE curves for local and fused Kalman filters](image2)
Table 1. The accuracy comparison of local and fused Kalman filters

| trP_1 | trP_2 | trP_3 | trP_4 | trP_5 | trP_6 |
|-------|-------|-------|-------|-------|-------|
| 3.5422 | 4.0137 | 4.5553 | 4.5899 | 5.9547 | 6.5582 | 7.1766 |

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