Multifractal analysis in non-uniformly hyperbolic interval maps

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Abstract
In this paper, we establish a framework for the construction of Moran set driven by dynamics. Under this framework, we study the Hausdorff dimension of the generalized intrinsic level set with respect to the given ergodic measure in a class of non-uniformly hyperbolic interval maps with finitely many branches.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Let $T : \bigcup_{i=1}^{m} I_i \subset [0, 1] \to [0, 1]$ be a piecewise $C^1$ map, where $I_i$ is the closed interval for $1 \leq i \leq m$ such that $\text{int}(I_i) \cap \text{int}(I_j) = \emptyset$ for any distinct $i$ and $j$. Here, $\text{int}(I_i)$ means the interior of $I_i$. In this paper, we consider the following class of non-uniformly hyperbolic interval maps,

- $T|_{I_i} : I_i \to [0, 1]$ is a bijective and continuously differentiable for $1 \leq i \leq m$. There is a unique $x_i \in I_i$ such that $T(x_i) = x_i$ for each $i$.
- $|T'(x)| > 1$ for $x \not\in \{x_1, \ldots, x_m\}$. Here, we also allow that $|T'(x_i)| > 1$ for some $i \in \{1, 2, \ldots, m\}$.

\footnote{Author to whom any correspondence should be addressed. Recommended by Dr Mark Pollicott.}
If $T'(x_i) = 1$ or $T'(x_i) = -1$ for some $i$, we say that $x_i$ is a parabolic fixed point. The class of non-uniformly hyperbolic maps includes the important example of Manneville–Pomeau map \cite{20}, $T : [0, 1] \to [0, 1]$ defined by $Tx = x + x^{1+\beta} \pmod{1}$, where $0 < \beta < 1$, see figure 1.

We define the repeller $\Lambda$ of $T$ by

$$\Lambda := \left\{ x \in \bigcup_{i=1}^{m} I_i : T^n(x) \in [0, 1], \forall n \geq 0 \right\}.$$ 

We know that $(\Lambda, T)$ has a very natural Markov partition. Let $S_i$ be the inverse branch of $T|_{I_i} : I_i \to [0, 1]$ for $i = 1, \ldots, m$. Let $A = \{1, \ldots, m\}$, $\Sigma = A^\mathbb{N}$ and $\Sigma_n$ be the set of all $n$-blocks over $A$ for any $n \in \mathbb{N}$. Here we point out $\mathbb{N}$ is the set of positive integers in this paper. Let $\sigma : \Sigma \to \Sigma$ be the shift map,

$$\sigma((\omega_n)_{n \geq 0}) = (\omega_{n+1})_{n \geq 0}.$$ 

Define the semi-conjugacy map $\pi : \Sigma \to [0, 1]$ by

$$\pi(\omega) := \lim_{n \to \infty} S_{\omega_0} \circ S_{\omega_1} \circ \cdots \circ S_{\omega_{n-1}}([0, 1]).$$

It is easy to check that $\pi(\Sigma) = \Lambda$ and $\pi \circ \sigma(\omega) = T \circ \pi(\omega)$. We remark that $\pi$ is a bijective map from $\Sigma$ to $\Lambda$ except for countably many points.

Let $C(\Lambda, \mathbb{R}^d)$ be the set of continuous functions from $\Lambda$ to $\mathbb{R}^d$. For any $\phi \in C(\Lambda, \mathbb{R}^d)$, we denote the $n$th Birkhoff sum by $S_n \phi(x) = \sum_{j=0}^{n-1} \phi(T^j x)$ and $n$th Birkhoff average by $A_n \phi(x) = \frac{1}{n} S_n \phi(x)$ for any $x \in \Lambda$. For any $\alpha \in \mathbb{R}^d$, we define

$$A_n(\phi)(x) := \left\{ x \in \Lambda : \lim_{n \to \infty} A_n \phi(x) = \alpha \right\},$$

and more generally,

$$\tilde{A_n}(\phi)(x) := \left\{ x \in \Lambda : \lim_{k \to \infty} A_{n_k} \phi(x) = \alpha \text{ for some } \{n_k\}_{k=1}^{\infty} \text{ with } \lim_{k \to \infty} n_k = \infty \right\}.$$

Figure 1. Manneville–Pomeau map.
Since $\Lambda_{\phi}(\alpha)$ not only depends on $\alpha$, but also on the continuous function $\phi$, it is natural to introduce a set which is intrinsic in some sense.

We denote the set of all invariant measures by $M(\Lambda, T)$ and the set of all ergodic measures by $E(\Lambda, T)$. For any $\mu \in E(\Lambda, T)$, we define the set of all $\mu$-generic points by

$$\Lambda_\mu := \left\{ x \in \Lambda : (A_n) \delta_x = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \rightarrow \mu \right\},$$

where $\rightarrow$ stands for the convergence in the weak-* topology. Similarly, we define the generalized intrinsic level set

$$\tilde{\Lambda}_\mu := \left\{ x \in \Lambda : (A_{n_k}) \delta_x \rightarrow \mu \text{ for some } \{n_k\}_{k=1}^{\infty} \text{ with } \lim_{k \to \infty} n_k = \infty \right\}.$$

To some extent, in order to have a good understanding of the hyperbolicity in the non-uniform hyperbolic system, we need to handle the points which exhibit hyperbolicity along some subsequence of the iteration. Another issue is we also need to deal with the points without hyperbolicity. This is our motivation to introduce $\tilde{\Lambda}_\mu$ in this paper. We try to analyse the coexistence or competence of these two phenomena (mainly in the $C^1$ regularity condition) through a careful discussion of $\tilde{\Lambda}_\mu$.

The multifractal analysis for uniformly hyperbolic conformal dynamical system is well developed in past years, we refer the reader to [7, 8, 15–17] for entropy spectrum and Birkhoff spectrum of level set and dimension spectrum of Gibbs measure or weak Gibbs measure. The original method developed in [17] is based on the equilibrium state (Gibbs measure) in thermodynamic formalism method, which needs the further regularity conditions on $T$. Later, this method was further developed in [2, 7, 15, 16], and the regularity conditions have been completely removed in the uniformly hyperbolic case.

Recently, there has been a trend in understanding the multifractal analysis beyond the uniformly hyperbolic dynamical system. However, up to now, there is still not a complete picture in the direction of non-uniformly hyperbolic dynamical system. The topological entropy of these types of sets have been studied in [3, 18, 19, 21, 22, 24]. The dimension spectrum of Birkhoff ergodic limits in non-uniformly hyperbolic dynamic systems has been done in [1, 2, 9, 10, 12, 13]. In [11], Gelfert and Rams studied the dimension spectrum of $\Lambda_{\phi}$ for $\phi = \log |T'|$. For the general continuous function $\phi$, Johansson, Jordan, Oberg, and Pollicott established a formula of $\dim_H \Lambda_{\phi}$ in [13], where $\dim_H \Lambda_{\phi}$ is the Hausdorff dimension of the given set $\Lambda_{\phi}$.

For any ergodic measure $\mu$, we call $\mu$ is hyperbolic if the Lyapunov exponent of $\mu$

$$\lambda(\mu, T) := \int \log |T'| d\mu > 0,$$

otherwise $\mu$ is called to be parabolic. If $\mu$ is a parabolic measure, $\mu$ is only supported on some parabolic fixed point in the setting of this paper.

Given a compact and $T$-invariant set $K \subset \Lambda$, we say that $K$ is a hyperbolic set, if $|T'(z)| > 1$ for any $z \in K$. The hyperbolic dimension of $\Lambda$ is defined by

$$\dim_H^{hyp} \Lambda := \sup \{ \dim_H K : K \text{ is a hyperbolic set in } \Lambda \}.$$

One of the main goal in the study of non-uniformly hyperbolic dynamical system is to recover the sufficient hyperbolicity to dominate or balance the non-hyperbolic behaviour, we refer the reader to [4, 5]. Now, we will state our main result of the Hausdorff dimension of $\Lambda_\mu$ and $\tilde{\Lambda}_\mu$. 

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Firstly, we point out the topological entropy $h_{\text{top}}(\Lambda_\mu)$ and $h_{\text{top}}(\tilde{\Lambda}_\mu)$ can be deduced from [6, 19]. The following result is proved by Pfister and Sullivan in [19], and Fan et al [6] independently for the system with the (almost) specification property. It is known that this result holds in the non-uniformly hyperbolic interval maps we considered here.

**Theorem A ([6, 19]).** Let $m$ be an invariant measure in $\mathcal{M}(\Lambda, T)$. We have

$$h_{\text{top}}(\tilde{\Lambda}_m) = h_{\text{top}}(\Lambda_m) = h(m, T),$$

where $h(m, T)$ is the measure entropy of $m$. In particular,

$$h_{\text{top}}(\tilde{\Lambda}_m) = 0,$$

if $\lambda(m, T) = 0$.

We also mention that Jaerisch and Takahasi [14] studied the Hausdorff dimension of $\Lambda_\mu$ in the setting of non-uniformly hyperbolic interval maps even with infinitely many branches recently. We formulate their result in our setting as follows.

**Theorem B ([14]).** Let $m$ be an ergodic measure in $\mathcal{M}(\Lambda, T)$. We have

(a) $\dim_H \Lambda_m = \frac{h(m, T)}{\lambda(m, T)}$ if $\lambda(m, T) > 0$;

(b) $\dim_H \tilde{\Lambda}_m \geq \dim_H^\text{hyp} \Lambda > 0$ if $\lambda(m, T) = 0$. Moreover if $T$ is a $C^2$ map, we have

$$\dim_H \tilde{\Lambda}_m = \dim_H \Lambda.$$

To the best of our knowledge, the relation between $\dim_H \Lambda_m$ and $\dim_H \tilde{\Lambda}_m$ is still lacking even in the setting of the finitely many branches. We have the following result in this direction.

**Theorem 1.** Let $m$ be an ergodic measure in $\mathcal{M}(\Lambda, T)$. We have

(a) $\dim_H \tilde{\Lambda}_m = \frac{h(m, T)}{\lambda(m, T)}$ if $\lambda(m, T) > 0$;

(b) $\dim_H \tilde{\Lambda}_m \geq \dim_H \Lambda_m \geq \dim_H^\text{hyp} \Lambda > 0$ if $\lambda(m, T) = 0$. Moreover if $T$ is a $C^2$ map, we have

$$\dim_H \tilde{\Lambda}_m = \dim_H \Lambda_m = \dim_H \Lambda.$$

In [14], theorem B was proved in the framework of infinitely branches by a series of delicate approximation arguments. We point out that in our proof of theorem 1, we also recover a direct proof of theorem B by using our framework, which is inspired by [6, 13]. To the best of our knowledge, it seems open whether there exists an example in the $C^1$ regularity condition such that $\dim_H \tilde{\Lambda}_m \geq \dim_H \Lambda_m$. We also remark that the $C^2$ regularity condition in this paper is only used to ensure that

$$\dim_H^\text{hyp} \Lambda = \dim_H \Lambda.$$            

**Remark 1.** It follows from theorems A and 1 that if the measure $m$ is parabolic, we have

$$h_{\text{top}}(\tilde{\Lambda}_m) = h_{\text{top}}(\Lambda_m) = 0 \quad \text{and} \quad \dim_H \tilde{\Lambda}_m \geq \dim_H \Lambda_m \geq \dim_H^\text{hyp} \Lambda > 0.$$

One sees that if $m$ is a parabolic measure, there is a great difference in the size between topological entropy and Hausdorff dimension for both $\tilde{\Lambda}_m$ and $\Lambda_m$.

Indeed, the assumption of the ergodicity is not necessary in theorem 1. This can be removed by the careful approximating arguments and we do not address it in this paper.

Remark 1...
By the same technique in the proof of theorem 1, we can get the following result.

**Theorem 1'.** Let $\mathcal{C}$ be a compact and connected set in $\mathcal{M}(\Lambda, T)$ and

$$
\Lambda_\mathcal{C} := \{ x \in \Lambda : \text{Asym}(\{(A_n)_x, \delta_x\}_n) = \mathcal{C}\},
$$

where $\text{Asym}(\{(A_n)_x, \delta_x\}_n)$ is the set of all accumulating points of $\{(A_n)_x, \delta_x\}_n$ in $\mathcal{M}(\Lambda, T)$. Then we have

(a) $\dim_H \Lambda_\mathcal{C} = \inf \{ \frac{\lambda(m)}{\mu(m)} : m \in \mathcal{C}, \lambda(m, T) > 0 \}$ if there exists some invariant measure $\mu$ in $\mathcal{C}$ such that $\lambda(\mu, T) > 0$;

(b) $\dim_H \Lambda_\mathcal{C} = \dim_H \Lambda$ if $T$ is a $C^2$ map and

$$
\mathcal{C} \subset \{ m \in \mathcal{M}(\Lambda, T) : \lambda(m, T) = 0 \}.
$$

Here we do not pursue this generalization in this paper. Motivated by theorems 1 and B, we have the following corollary immediately.

**Corollary 1.** Let $m$ be an ergodic measure and $T$ is a $C^2$ map in the setting above. Then we have

$$
\dim_H \tilde{\Lambda}_m = \dim_H \Lambda_m.
$$

Our proof of corollary 1 can be modified to prove that $\dim_H \tilde{\Lambda}_m = \dim_H \Lambda_m$ holds for conformal repeller under the $C^1$ condition. We do not know whether this holds or not in the non-uniformly hyperbolic case under the $C^1$ condition. However, if $T$ has only one parabolic fixed point, we have the following result.

**Theorem 2.** Assume $T$ is a $C^1$ map with only one parabolic fixed point $p$ in $[0, 1)$. Then for any ergodic measure $m$ in $\mathcal{M}(\Lambda, T)$, we have

$$
\dim_H \tilde{\Lambda}_m = \dim_H \Lambda_m.
$$

We remark that if $T$ has more than one parabolic fixed point, we do not know how to control the persistent recurrence behaviours between (or among) the multiple parabolic fixed points in the $C^1$ regularity condition.

The rest of this paper is organized as follows. In section 2, some preliminaries and basic results are collected. In section 3, we will prove the upper bound of the first part in theorem 1. In section 4, we will make some effort to give a unified framework of geometric Moran construction in dimension 1, which may be of independent interest. We believe that this framework can be used to deal with many problems in multifractal analysis in non-uniformly hyperbolic dynamical systems. In section 5, under the framework established in section 4, we will give the proof of the lower bound for theorem 1, which also provides a direct proof of theorem B. Finally, theorem 2 will be proved in section 6.

## 2. Preliminaries

In this section, firstly, we will recall some basic results in the literature, then we define two sets of invariant measures which will be used throughout this paper.

For any $\omega \in \Sigma$, we define

$$
I_\omega(\omega) := S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_{n-1}}([0, 1]),
$$
and the diameter of $I_n(\omega)$ by
\[ D_n(\omega) := \max \{|x - y| : x, y \in I_n(\omega)\}. \]

Let $g(x) = \log|T'(x)|$ for any $x \in \Lambda$ and $G(\omega) = \log|T'(\pi(\omega))|$ for any $\omega \in \Sigma$. We know that $g(\pi(\omega)) = G(\omega)$ for $\omega \in \Sigma$. Given any $f \in C(\Lambda, \mathbb{R})$, we denote $F : \Sigma \to \mathbb{R}$ by
\[ F(\omega) = f(\pi(\omega)). \]

For any $n \in \mathbb{N}$, we define the $n$-variation of $f$ by
\[ \text{var}_n(f) := \sup \{|f(\pi(\omega)) - f(\pi(\tau))| : \omega_0 = \tau_0, \ldots, \omega_{n-1} = \tau_{n-1}, \omega, \tau \in \Sigma\}. \]

The following lemma shows the relation between the logarithm growth of $D_n(\omega)$ and the Birkhoff average of $G(\omega)$.

**Lemma 1** [13]. In the setting above, we have
\[ \lim_{n \to \infty} \sup_{\omega \in \Sigma} \left| \frac{\log D_n(\omega)}{n} - A_n G(\omega) \right| = 0. \]

In our following discussion, we also need the following fact.

**Lemma 2** [13]. Let $f$ be a continuous function on $\Lambda$. Then we have
\[ \lim_{n \to \infty} \text{var}_n(A_n f) = 0. \]

Since $\Lambda$ is a compact set, there exists a countable dense subset $\{f_n\}_{n=1}^\infty$ in $C(\Lambda, \mathbb{R})$ and we can assume $f_n \not\equiv 0$ for any $n \in \mathbb{N}$ without loss of generality. Let $\mathcal{M}(\Lambda)$ be the set of all probability measures. We introduce a metric $d$ on $\mathcal{M}(\Lambda)$ by
\[ d(\mu, \nu) := \sum_{n=1}^\infty \frac{\left| \int f_n \, d\mu - \int f_n \, d\nu \right|^2}{2^n \|f_n\|^2}. \]

And the topology induced by the metric $d$ on $\mathcal{M}(\Lambda)$ is compatible with the weak-* topology.

**Lemma 3.** Let $m$ be an invariant measure in $\mathcal{M}(\Lambda, T)$ and
\[ \Lambda_{m,k} := \left\{ x \in \Lambda : \lim_{n \to \infty} A_n f_i(x) = \int f_i \, dm, \quad \text{for } 1 \leq i \leq k \right\}. \]

Then we have
\[ \Lambda_m = \bigcap_{k=1}^\infty \Lambda_{m,k}. \]

The proof of lemma 3 is simple, we will omit the proof here. The existence of parabolic fixed points has important impact on the size of $m$-generic set. Assume that $T$ has $l + 1$ different parabolic fixed points $\{p_i, 0 \leq i \leq l\}$ and we denote
\[ \mathcal{M}^p(\Lambda, T) := \left\{ \sum_{i=0}^l \lambda_i \delta_{p_i} : \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=0}^l \lambda_i = 1 \right\}. \]

It is obvious that $\mathcal{M}^p(\Lambda, T)$ is an $l$-dimensional simplex in $\mathcal{M}(\Lambda, T)$ and
\[ \mathcal{M}^p(\Lambda, T) = \{ \mu \in \mathcal{M}(\Lambda, T) : \lambda(\mu, T) = 0 \}. \]
For any invariant measure $\mu$ in $\mathcal{M}(\Lambda, T)$, we also define the following set

$$\mathcal{M}_{\mu,k} := \left\{ \nu \in \mathcal{M}(\Lambda, T) : \int f_i \, d\nu = \int f_i \, d\mu \text{ for } 1 \leq i \leq k \right\}.$$ 

Then we have the following important observation.

**Lemma 4.** Let $\mu$ be an invariant measure in $\mathcal{M}(\Lambda, T)$ such that $\lambda(\mu, T) > 0$. Then we have $\mathcal{M}_{\mu,k} \cap \mathcal{M}'(\Lambda, T) = \emptyset$, for $k$ sufficiently large.

**Proof of lemma 4.** We assume that there exists a sequence of invariant measures $\{\mu_i\}_{i=1}^{\infty}$ such that $\mu_i \in \mathcal{M}_{\mu,k} \cap \mathcal{M}'(\Lambda, T)$.

It follows that

$$\int f_j \, d\mu_i = \int f_j \, d\mu,$$

for any $j \leq k$, which implies that

$$d(\mu_i, \mu) \leq \frac{1}{2k-1}.$$

Then we have

$$\lim_{i \to \infty} \mu_i = \mu.$$

Recalling that $\lambda(\mu_i, T) = 0$ for any $i \in \mathbb{N}$, we get $\lambda(\mu, T) = 0$, which contradicts the fact that $\lambda(\mu, T) > 0$. \hfill $\square$

### 3. Proof for the upper bound of theorem 1

Recall that $g(x) = \log|T'(x)|$ for any $x \in \Lambda$. For any $\delta > 0$, $\epsilon \in (0, \delta)$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ and $n \in \mathbb{N}$, let $G_\delta(\alpha, \delta; n, \epsilon)$ be the set of all cylinders in $\Sigma_n$ such that for any $[\omega]_n \in G_\delta(\alpha, \delta; n, \epsilon)$ there exists $x \in I_n(\omega)$ satisfying the following properties:

(a) $\left| \frac{1}{n} \sum_{i=1}^{n} \phi_i(x) - \alpha_i \right| < \epsilon$ for $1 \leq i \leq k$;

(b) $A_\delta g(x) \geq \delta - \epsilon$.

We define two pressure like functions $g_\delta(\alpha, \delta; s, n, \epsilon)$ and $g^*_\delta(\alpha, \delta; s, n, \epsilon)$ associated to $G_\delta(\alpha, \delta; n, \epsilon)$ as follows:

$$g_\delta(\alpha, \delta; s, n, \epsilon) := \sum_{[\omega]_n \in G_\delta(\alpha, \delta; n, \epsilon)} (\text{diam}(I_{s}(\omega)))^s,$$

and

$$g^*_\delta(\alpha, \delta; s, n, \epsilon) := \sum_{[\omega]_n \in G_\delta(\alpha, \delta; n, \epsilon)} \sup_{x \in I_{s}(\omega)} \exp(-sS_n g(x)).$$
where \( \text{diam}(I_n(\omega)) \) is the diameter of \( I_n(\omega) \) and \( s \in [0, \infty) \). Next, let
\[
\mathbb{F}(\alpha, \delta; s) := \lim lim_{\epsilon \to 0} \sup_{n \to \infty} \frac{1}{n} \log g_\epsilon(\alpha, \delta; s, n, \epsilon),
\]
and
\[
g_\epsilon(\alpha, \delta; s) := \lim lim_{\epsilon \to 0} \inf_{n \to \infty} \frac{1}{n} \log g_\epsilon(\alpha, \delta; s, n, \epsilon).
\]
Similarly, we define \( g_\epsilon^*(\alpha, \delta; s) \) and \( \mathbb{F}_\epsilon(\alpha, \delta; s) \). Then we have the following result.

**Lemma 5.** In the setting above, we have
\[
g_\epsilon(\alpha, \delta; s) = \mathbb{F}_\epsilon(\alpha, \delta; s) = g_\epsilon^*(\alpha, \delta; s) = \mathbb{F}_\epsilon^*(\alpha, \delta; s),
\]
for any \( s \geq 0 \).

**Proof.** The proof can be essentially deduced from [7, proposition 4.3]. \( \square \)

We denote the common limit above by \( g_\epsilon(\alpha, \delta; s) \). Recalling that there exists a dense set \( \{f_n\}_{n=1}^\infty \in C(\Lambda, \mathbb{R}) \) and \( f_n \neq 0 \) for any \( n \in \mathbb{N} \). Denote \( F_k : \Lambda \to \mathbb{R}^k \) by
\[
F_k(x) := (f_1(x), \ldots, f_k(x)).
\]
And for any \( \mu \in \mathcal{M}(\Lambda, T) \), we define
\[
\int F_k d\mu := \left( \int f_1 d\mu, \ldots, \int f_k d\mu \right).
\]
For any \( \alpha = (\alpha_1, \ldots, \alpha_k) \), we introduce the norm \( \| \cdot \|_\infty \) on \( \mathbb{R}^k \) by
\[
\| \alpha \|_\infty := \sup_{1 \leq i \leq k} |\alpha_i|.
\]

**Proposition 1.** Let \( m \) be an invariant measure in \( \mathcal{M}(\Lambda, T) \) such that \( \lambda(m, T) > 0 \) and \( \alpha = \int F_k d\mu \in \mathbb{R}^k \). Then, for any \( \delta \in (0, \lambda(m, T)) \) and \( \tau \in (0, \delta) \), we have
\[
g_\epsilon(\alpha, \delta; s) \leq \sup \{ h(\mu, T) - s \lambda(\mu, T) : \mu \in X(\alpha, \delta, k, \tau) \},
\]
where
\[
X(\alpha, \delta, k, \tau) := \left\{ \mu \in \mathcal{M}(\Lambda, T) : \left\| \int F_k d\mu - \alpha \right\|_\infty \leq \tau, \lambda(\mu, T) \geq \delta - \tau \right\}.
\]

**Proof.** Let \( a = g_\epsilon(\alpha, \delta; s) \). We have
\[
a = \lim lim_{\epsilon \to 0} \sup_{n \to \infty} \frac{1}{n} \log \left( \sum_{|\omega| \in G_k(\alpha, \delta, \tau, 1) \cap I_n(\omega)} \sup_{x \in I_n(\omega)} \exp(-sS_n g(x)) \right).
\]
We first introduce a probability measure \( \widetilde{\mu}_n \) on \( \Sigma_n \). For any \([\omega]_n \in G_k(\alpha, \delta; n, \epsilon)\), we define
\[
\widetilde{\mu}_n(\omega) := \frac{\sum_{|\omega| \in G_k(\alpha, \delta, \tau, 1) \cap I_n(\omega)} \sup_{x \in I_n(\omega)} \exp(-sS_n g(x))}{\sum_{|\omega| \in G_k(\alpha, \delta, \tau, 1) \cap I_n(\omega)} \sup_{x \in I_n(\omega)} \exp(-sS_n g(x))},
\]
Now, we construct an $n$-Bernoulli measure $(\tilde{\mu}_n)\circ$ on $\Sigma$, which is just the products of countably many copies of $(\Sigma_n, \tilde{\mu}_n)$. Here, we identify the probability spaces $((\Sigma_n)\circ, (\tilde{\mu}_n)\circ)$ with $(\Sigma, (\tilde{\mu}_n)\circ)$ by the following isomorphism $\Theta : ((\Sigma_n)\circ, (\tilde{\mu}_n)\circ) \rightarrow (\Sigma, (\tilde{\mu}_n)\circ)$

\[
\Theta(\theta_1, \theta_2, \ldots, \theta_k, \ldots) = (\theta_1, \theta_2 \ldots \theta_k \ldots),
\]

for any $(\theta_1, \ldots, \theta_k, \ldots) \in ((\Sigma_n)\circ, (\tilde{\mu}_n)\circ)$. It is easy to see that $(\tilde{\mu}_n)\circ$ is ergodic measure on $(\Sigma, \sigma^n)$. We refer the interested readers to [13] for a very good introduction of the $n$-Bernoulli measure. It follows that

\[
h((\tilde{\mu}_n)\circ, \sigma^n) = H(\tilde{\mu}_n) = s \int \inf_{x \in I_n(\omega)} S_n g(x) \, d\mu_n(\omega) + I_n,
\]

where

\[
I_n := \log \left( \sum_{[\omega]_n \in G_k(\alpha, \delta, x, \tau) \in I_n(\omega)} \sup_{x \in I_n(\omega)} \exp(-sS_n g(x)) \right).
\]

Let $\mu_n\circ = \pi_* (\tilde{\mu}_n)\circ$ and $\nu_n = (\Lambda_n)\circ \mu_n\circ$. We have $\nu_n \in \mathcal{M}(\Lambda, T)$ and

\[
h(\nu_n, T) - s \int \log |T'(x)| \, d\nu_n(x) \geq -\frac{1}{n} \sup_{[\omega]_n} \sup_{x \in I_n(\omega)} |S_n g(x) - S_n g(y)| + \frac{I_n}{n}.
\]

Also, it is not difficult to prove that for $n$ sufficiently large, we have

\[
\int g \, d\nu_n = \int A_n g \, d\mu_n\circ \geq \delta - 2\epsilon,
\]

and

\[
\int F_k \, d\nu_n = \alpha \leq 2\epsilon.
\]

Then for any $\epsilon < \frac{\delta}{2}$, we have $\nu_n \in X(\alpha, \delta, k, \tau)$ and

\[
\sup \{ h(\mu, T) - s\lambda(\mu, T) : \mu \in X(\alpha, \delta, k, \tau) \}
\geq \limsup_{n \to \infty} \left( h(\nu_n, T) - s \int \log |T'(x)| \, d\nu_n(x) \right)
\geq \limsup_{n \to \infty} \frac{1}{n} \log \left( \sum_{[\omega]_n \in G_k(\alpha, \delta, x, \tau) \in I_n(\omega)} \sup_{x \in I_n(\omega)} \exp(-sS_n g(x)) \right).
\]

Taking $\epsilon \to 0$, we complete the proof of proposition 1. \qed

Now we are going to prove the upper bound

\[
\dim_{\mathcal{H}}(X_m) \leq \frac{h(m, T)}{\lambda(m, T)}.
\]
for the first part of theorem 1. Here, we need some ideas of thermodynamic formalism method. We will make some delicate modification of the arguments in [7]. Some key estimates there break down in our cases, so we need to be very careful to deal with the hyperbolicity.

**Proof.** Let
\[ h_k(\alpha, \delta, s, \tau) := \sup \{ h(\mu, T) - s\lambda(\mu, T) : \mu \in X(\alpha, \delta, k, \tau) \} . \]

It is not difficult to prove that the map
\[ s \mapsto h_k(\alpha, \delta, s, \tau), \]
is strictly decreasing in \([0, +\infty)\). Moreover,
\[ h_k(\alpha, \delta, 0, \tau) \geq 0 \quad \text{and} \quad \lim_{s \to +\infty} h_k(\alpha, \delta, s, \tau) = -\infty. \]

Thus, there exists unique \( s^* = s^*(\alpha, \delta, k, \tau) \in [0, \infty) \) such that
\[ h_k(\alpha, \delta, s^*, \tau) = 0. \]
Indeed, we have
\[ s^* = \sup \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \mu \in X(\alpha, \delta, k, \tau) \right\} . \]

Since the map \( s \mapsto h_k(\alpha, \frac{1}{j}, s, \tau) \) is decreasing and has a zero at \( s = s^* \), by proposition 1, we have
\[ t = g_k \left( \alpha, \frac{1}{j}, s^* + \epsilon \right) < 0, \]
for any \( \epsilon > 0 \). Thus, there exists \( L_0 = L_0(j) \geq j + 1 \), for any \( l \geq L_0 \), we have
\[ \limsup_{n \to \infty} \frac{1}{n} \log g_k \left( \alpha, \frac{1}{j}, s^* + \epsilon, n, \frac{1}{T} \right) < -\frac{t}{4}. \]

It follows that there exists \( U_0 = U_0(l) \in \mathbb{N} \) such that for any \( u \geq U_0 \), we have
\[ \sum_{[\omega] \in G_k(\alpha, \frac{1}{j}, u, \frac{1}{T})} (\text{diam}_{I_u(\omega)})^{s^* + \epsilon} \leq \exp \left( -\frac{t}{12} \right). \]

Recalling that
\[ \tilde{\Lambda}_m = \left\{ x \in \Lambda : (A_n)_{n=\infty} \xrightarrow{\mathcal{P}} m \quad \text{for some} \ \{ n_k \}_{k=1}^{\infty} \ \text{with} \ \lim_{k \to \infty} n_k = \infty \right\}, \]
it is straightforward to see that
\[ \tilde{\Lambda}_m \subset \bigcup_{j=1}^{\infty} \bigcup_{l=L_0(j)}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{u=n}^{\infty} \prod_{k} G_k \left( \alpha, \frac{1}{j}, u, \frac{1}{T} \right) \]
\[ \subset \bigcup_{j=1}^{\infty} \bigcup_{l=L_0(j)}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{u \geq U_0}^{\infty} \prod_{k} G_k \left( \alpha, \frac{1}{j}, u, \frac{1}{T} \right). \]
Let
\[ E_{j,k} = \bigcup_{u \geq U_0} \mathbb{P} G_k \left( \alpha, \frac{1}{j}; u, \frac{1}{T} \right). \]

Thus, we proved that
\[ \mathcal{H}^{\tau_\epsilon, \mu}(E_{j,k}) \leq \sum_{u \geq U_0} \sum_{\omega_i \in \Gamma_k} (\text{diam}(J_\mu(\omega)))^{\tau_\epsilon} \leq \sum_{u \geq U_0} \exp \left( -u \frac{t}{2} \right) < +\infty. \]

Note that \( \epsilon > 0 \) is arbitrary. It follows that \( \dim_H E_{j,k} \leq s_\ast \). And this implies that \( \dim_H \tilde{\Lambda}_m \leq s_\ast \), that is
\[ \dim_H \tilde{\Lambda}_m \leq \sup \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \mu \in X(\alpha, \delta, k, \tau) \right\} \leq \sup \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \mu \in X(\alpha, k, \tau) \right\}, \tag{1} \]

where
\[ X(\alpha, k, \tau) := \left\{ \mu \in M(\Lambda, T) : \left\| \int F_k \, d\mu - \alpha \right\|_\infty \leq \tau, \lambda(\mu, T) > 0 \right\}. \]

Noting that \( \lambda(m, T) > 0 \), hence there exists \( k_0 \in \mathbb{N} \) such that
\[ M_{m,k} \cap M^p(\Lambda, T) = \emptyset, \]
for any \( k \geq k_0 \) by lemma 4.

Since both \( M_{m,k} \) and \( M^p(\Lambda, T) \) are compact, we know that there exists \( \gamma > 0 \) such that
\[ d(\nu_1, \nu_2) = \sum_{n=1}^{\infty} \frac{1}{2^n \| \rho_n \|} \left| \int f_n \, d\nu_1 - \int f_n \, d\nu_2 \right| \geq \gamma, \tag{2} \]
for any \( \nu_1 \in M_{m,k}, \nu_2 \in M^p(\Lambda, T) \). Fix \( k_\ast \in \mathbb{N} \) such that \( k_\ast \geq k_0 \) and \( \frac{1}{n+1} < \gamma \).

We claim that for any \( k \geq k_\ast \), there exists \( \tau_0 = \tau_0(k) \) such that for any \( \tau \in (0, \tau_0) \) we have
\[ \tilde{X}(\alpha, k, \tau) = X(\alpha, k, \tau), \tag{3} \]
where
\[ \tilde{X}(\alpha, k, \tau) := \left\{ \mu \in M(\Lambda, T) : \left\| \int F_k \, d\mu - \alpha \right\|_\infty \leq \tau \right\}. \]

Otherwise, we know that \( X(\alpha, k, \tau) \subseteq \tilde{X}(\alpha, k, \tau). \) Thus, there exists a sequence \( \{ \tau_n \}_{n=1}^{\infty} \) with \( \lim_{n \to \infty} \tau_n = 0 \) and invariant measures \( \{ \mu_n \} \) in \( M(\Lambda, T) \) with \( \lambda(\mu_n, T) = 0 \) such that
\[ \left\| \int F_k \, d\mu_n - \int F_k \, d\mu \right\|_\infty = \left\| \int F_k \, d\mu_n - \alpha \right\|_\infty \leq \tau_n. \]
We could assume that \( \lim_{n \to \infty} \mu_n = \nu \) for some \( \nu \in \mathcal{M}(\Lambda, T) \) up to a sub-sequence. It follows that
\[
\lambda(\nu, T) = \lim_{n \to \infty} \lambda(\mu_n, T) = 0 \quad \text{and} \quad \int F_k d\nu - \int F_k dm = 0.
\]

Hence, we get
\[
d(\nu, m) = \sum_{n=k+1}^{\infty} \frac{1}{2^{n-1}} \left| \int f_n d\nu - \int f_n dm \right| \\
\leq \sum_{n=k+1}^{\infty} \frac{1}{2^{n-1}} \leq \frac{1}{2^{k-1}} < \gamma.
\]

It is contradictory to the fact that \( d(\nu, m) \geq \gamma \) due to \( \nu \in \mathcal{M}^0(\Lambda, T) \) and \( m \in \mathcal{M}_{m,k} \) by (2). Then by (1) together with (3), for any \( k \geq k_* \), \( \tau \in (0, \tau_0) \), we have
\[
\dim_{\mu} \tilde{\Lambda}_m \leq \sup \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \int F_k d\mu = \int F_k dm \right\}.
\]

Since \( \tau \) is arbitrary, it follows that
\[
\dim_{\mu} \tilde{\Lambda}_m \leq \sup \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \int F_k d\mu = \int F_k dm \right\},
\]
for any \( k \geq k_* \). It is not difficult to check the fact that
\[
\bigcap_{k=k_*}^{\infty} \left\{ \mu \in \mathcal{M}(\Lambda, T) : \int F_k d\mu = \int F_k dm \right\} = \{m\}.
\]
Thus we have
\[
\dim_{\mu} \tilde{\Lambda}_m \leq \frac{h(m, T)}{\lambda(m, T)}. \quad \square
\]

As the ending of this section, we give a more generalized result on the intrinsic level set of an invariant measure, which will be used to prove theorem 2 in section 6. For any invariant measure \( m \) in \( \mathcal{M}(\Lambda, T) \) and \( R > 0 \), we define
\[
B_d(m, R) := \{ \mu \in \mathcal{M}(\Lambda, T) : d(\mu, m) \leq R \quad \text{and} \quad \lambda(\mu, T) > 0 \}.
\]

We can prove the following result by almost the same method presented above with minor changes. So we omit its proof here.

**Proposition 2.** Let \( m \) be an invariant measure in \( \mathcal{M}(\Lambda, T) \) such that \( \lambda(m, T) > 0 \) and \( B_d(m, R) \) as above. Furthermore let
\[
\mathcal{A}_r := \{ x \in \Lambda : \text{Asym}\{\{A_n, \delta_x\}_{n=1}^{\infty} \cap B_d(m, r) \neq \emptyset \} \},
\]
for any \( r \in (0, R) \). Then we have
\[
\dim_{\mu} \mathcal{A}_r \leq \sup \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \mu \in B_d(m, r) \right\}.
\]
4. Moran constructions

Since the proof of the lower bound of theorem 1 relies heavily on the constructions of Moran sets, which can be seen as the generalized Cantor sets with product structures.

We first establish a framework of abstract construction of the Moran set, which may be of independent interest. Indeed this kind of techniques have been greatly used in a certain amount of literature, for example see [1, 6, 7]. However, it seems that there is no unified framework.

4.1. Basic settings in Moran construction

Let

\[ I := \{ I_{n,j} : 1 \leq j \leq m_n, n \in \mathbb{N} \}, \]

be a sequence of the intervals in \([0, 1]\) such that

(a) \( \text{int} I_{n,j} \cap \text{int} I_{n,j} = \emptyset \) for any \( n \in \mathbb{N} \) and \( 1 \leq i \neq j \leq m_n; \)

(b) \( I_{n,j} \) contains at least one element \( I_{n+1,j} \) for any \( n \in \mathbb{N} \), and \( 1 \leq i \leq m_n; \)

(c) For each \( n \in \mathbb{N} \), \( I_{n+1,j} \) is contained in one of the elements in \( \{ I_{n,i} : 1 \leq i \leq m_n \} \) for \( 1 \leq j \leq m_{n+1}. \)

Let \( r_n = \min_{1 \leq i \leq m_n} \text{diam}(I_{n,i}) \) and \( R_n = \max_{1 \leq i \leq m_n} \text{diam}(I_{n,i}). \) We assume that \( \lim_{n \to \infty} R_n = 0. \) We define the Moran set \( Y \) associated to \( I \) by

\[ Y := \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{m_n} I_{n,j}. \]

For \( 1 \leq i \leq m_n, I_{n,i} \) is said to be the \( i \)th fundamental interval in the \( n \)th level of \( Y. \)

The following observation is crucial in the estimates of the lower bound of the Hausdorff dimension of the Moran set.

Lemma 6. Assume that the fundamental intervals satisfy the following conditions

\[ \lim_{n \to \infty} \frac{\log R_n}{\log r_n} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{\log r_{n+1}}{\log r_n} = 1. \]

Let \( \eta \) be a measure supported on \( Y \) which satisfies the balanced property

\[ \lim_{n \to \infty} \frac{\min_{1 \leq i \leq m_n} \log \eta(I_{n,i})}{\max_{1 \leq i \leq m_n} \log \eta(I_{n,i})} = 1. \]

Then we have

\[ \lim_{r \to 0} \frac{\log \eta(B(x, r))}{\log r} = \lim_{n \to \infty} \frac{\log \eta(I_{n+1,k})}{\log R_n}, \]

\[ \lim_{r \to 0} \frac{\log \eta(B(x, r))}{\log r} = \lim_{n \to \infty} \frac{\log \eta(I_{n,k})}{\log r_n}, \]

for any \( x \in Y. \)

Proof. For any \( x \in Y, r > 0 \), there exists a nested sequence of intervals \( \{ I_{k,x}(r) \}_{k=1}^{\infty} \) such that \( x \in I_{k,x}(r) \) for any \( k \in \mathbb{N}. \) And there always exits a unique integer \( n \) such that \( R_{n+1} \leq r < R_n. \)
We assume that there exists \( N = N(x, n, r) \) intervals in \( \{I_{n,j}\}_{j=1}^{m_n} \) which intersect with \( B(x, r) \). It follows that
\[
Nr_n \leq \sum_{B(x,j) \cap I_{n,j} \neq \emptyset} |I_{n,j}| \leq 4R_n.
\]
Thus, we get \( N \leq \frac{4R_n}{r_n} \). Combining the fact that \( B(x, r) \) contains at least one element in \( \{I_{n+1,j} : 1 \leq j \leq m_{n+1}\} \), we have
\[
\frac{\log \eta(B(x, r))}{\log r} \geq \frac{\min_{1 \leq j \leq m_{n+1}} \log \eta(I_{n+1,j})}{\log R_n}.
\]
and
\[
\frac{\log \eta(B(x, r))}{\log r} \geq \frac{\log R_n - \log r_n}{\log R_{n+1}} + \frac{\max_{1 \leq j \leq m_n} \log \eta(I_{n,j})}{\log R_{n+1}}.
\]
By (4) and (5), we proved the desired conclusion. \( \square \)

It seems that the conditions in lemma 6 are too strong and awkward at the first glance. Indeed they are very useful especially in estimating the lower bound of the Hausdorff dimension in the Moran constructions.

4.2. Moran construction driven by dynamical system

In many applications, the fundamental intervals in the different levels in Moran construction was obtained by dynamical system. In this section, we try to provide a framework for Moran construction driven by dynamical system.

Let \( \mathcal{F}_n := \{I_{n,j} : 1 \leq j \leq s_n\} \) be a family of \( s_n \) disjoint closed intervals in \([0, 1]\) and \( \mu_n \) be a probability measure on \( \mathcal{F}_n \) for any \( n \in \mathbb{N} \). We assume that there exists a sequence \( \{I_{n,j}\}_{n=1}^{\infty} \subset \mathbb{N} \) such that \( \mathcal{F}_n(\hat{I}_{n,j}) \supset \bigcup_{k=1}^{m_{n+1}} I_{n+1,k} \) for \( 1 \leq j \leq s_n \) and \( n \in \mathbb{N} \).

Let \( S_k := \mathcal{F}_k \) for any \( k \in \mathbb{N} \). Thus, for each \( k \in \mathbb{N} \), we know that \( S_k^{-1}(\mathcal{F}_{k+1}) \subset \mathcal{F}_k \). Let
\[
\mathcal{I}_k := S_1^{-1} \circ \cdots \circ S_k^{-1}(\mathcal{F}_{k+1}),
\]
for any \( k \in \mathbb{N} \). It is easy to see that \( \{\mathcal{I}_k\}_{k=1}^{\infty} \) is a sequence of nested compact sets. Indeed, \( \mathcal{I}_k \) is the union of all fundamental intervals of level \( k \) and \( \mathcal{I} = \bigcup_{k=1}^{\infty} \mathcal{I}_k \) is the set of all fundamental intervals of different levels.

Then \( \mathcal{Y} := \bigcap_{n=1}^{\infty} \mathcal{I}_n \) is the Moran set determined by \( \mathcal{I} \). For each interval \( J \) in \( \mathcal{I}_n \), there exits \( t_k = t_k(J) \leq s_k \) for any \( 1 \leq k \leq n \), such that
\[
J := \bigcap_{k=1}^{n} S_1^{-1} \circ \cdots \circ S_k^{-1}(\hat{I}_{k+1,j_{k+1}}).
\]
We can define a family of product type measures \( \{\eta_n\}_{n=1}^{\infty} \) on \( \mathcal{I}_n \) such that
\[
\eta_n(J) := \prod_{k=1}^{n} \mu_k(\hat{I}_{k,j_{k}}).
\]
Moreover, the family of measures \( \eta_n \) are compatible to each other in the sense that
\[
\eta_n(J) = \sum_{J \in \mathcal{I}_n, J \supset J} \eta_m(J),
\]
(6)
for any \( m \geq n \). The equality (6) follows from the following observation,

\[
\sum_{\tilde{J} \in I_{m,n} \cup J} \eta_{n}(\tilde{J}) = \sum_{l_n=1}^{s_m} \cdots \sum_{l_{n+1}=1}^{s_{n+1}} \left( \prod_{k=1}^{n} \mu_k(\tilde{I}_{k,l_n}) \right) \left( \prod_{k=1}^{n+1} \mu_{n+1}(I_{n+1,k,l_{n+1}}) \cdots \mu_m(I_{m,n}) \right) \\
= \left( \sum_{l_n=1}^{s_m} \mu_l(I_{m,n}) \right) \cdots \left( \sum_{l_{n+1}=1}^{s_{n+1}} \mu_{n+1}(I_{n+1,k,l_{n+1}}) \right) \prod_{k=1}^{n} \mu_k(\tilde{I}_{k,l_n}) \\
= \prod_{k=1}^{n} \mu_k(\tilde{I}_{k,l_n}) = \eta_{n}(J).
\]

It is standard to get a measure \( \eta \) on the Moran set \( \mathcal{Y} \) by just taking the weak-* limit of \( \eta_{n} \). It follows that

\[
\eta(J) = \prod_{k=1}^{n} \mu_k(\tilde{I}_{k,l_n}),
\]

for any \( J \in \mathcal{I}_n \).

For any \( x \in \mathcal{Y} \) and \( k \in \mathbb{N} \), there exists \( J_k = J_k(x) \) in \( \mathcal{I}_k \) and \( \tilde{I}_{k,l_n}(x) = \tilde{I}_{k,l_n}(x) \) in \( \mathcal{F}_k \) such that

\[
x \in \bigcap_{k=1}^{\infty} J_k \quad \text{and} \quad T^{n_k}(x) \in \tilde{I}_{k+1,l_{n+1}},
\]

where \( n_k = l_1 + l_2 + \cdots + l_k \).

We can transfer lemma 6 to the following result, which will be very helpful in the construction of Moran set.

**Lemma 7.** Let \( \eta \) satisfies the balanced property defined in lemma 6. If the following asymptotic additive property

\[
\lim_{n \to \infty} \frac{\log \text{diam}(I_n(x))}{\sum_{i=0}^{n} \log \text{diam}(I_{i,l}(x))} = 1,
\]

and the tempered growth property

\[
\lim_{n \to \infty} \frac{\log \text{diam}(I_{n+1,l_{n+1}}(x))}{\sum_{i=0}^{n+1} \log \text{diam}(I_{i,l}(x))} = 1,
\]

hold, we have

\[
\limsup_{r \to 0} \frac{\log \eta(B(x,r))}{\log r} = \limsup_{i \to \infty} \frac{\log \mu_i(I_{i,l}(x))}{\log \text{diam}(I_{i,l}(x))},
\]

and

\[
\liminf_{r \to 0} \frac{\log \eta(B(x,r))}{\log r} = \liminf_{i \to \infty} \frac{\log \mu_i(I_{i,l}(x))}{\log \text{diam}(I_{i,l}(x))},
\]

for any \( x \in \mathcal{Y} \).

The asymptotic additive property and tempered growth property are crucial in the construction of Moran set and Moran measure which is a Bernoulli-like measure on the Moran set. By lemma 7, we only need to gluing a sequence of subsystems carefully in the construction.
4.3. Preparation of geometric Moran construction

In order to prove the lower bound of theorem 1, we need to introduce two Moran sets, while the constructions in the both of the Moran sets share some similar steps in the initial constructions.

Since it is more convenient to explain the construction in the symbolic space \( \Sigma \) rather than in \( \Lambda \), here, we slightly abuse the notation. We identify the invariant measures \( M(\Lambda, T) \) with its lift on the symbolic space \( \Sigma \) for simplicity. We also use the same conventions to identify \( h(\mu, \sigma) \) and \( \lambda(\mu, \sigma) \) with \( h(\mu, T) \) and \( \lambda(\mu, T) \) in the following constructions. We first try to deal with the common steps in the two constructions of a hyperbolic measure \( \mu \). In the first case, we will set \( \mu = m \) in section 5.1, where \( m \) is the given hyperbolic measure in theorem 1.

In the second case, \( \mu \) is an arbitrary given hyperbolic measure in section 5.2. It is very convenient to transfer most of our discussion to the symbolic space. In the rest of the paper, let \( \{ \epsilon_i \}_{i=1}^{\infty} \) be a decreasing sequence such that \( \lim_{i \to \infty} \epsilon_i = 0 \). Recall that \( \{ f_j \}_{j=1}^{\infty} \) and \( g = \log |T'| \) are uniformly continuous on \( \Lambda \), there exists \( k_i \in \mathbb{N} \) such that

\[
\begin{align*}
\var_{f_j} A_n f_j &< \epsilon_i, \\
\var_{A_n g} &< \epsilon_i, \\
|\frac{1}{n} \log D_n(\omega) - A_n g(\pi(\omega))| &< \epsilon_i \quad \text{for any } \omega \in \Sigma,
\end{align*}
\]

for any \( n \geq k_i \). Noting that \( \mu \) is an ergodic measure, we have for \( \mu \) a.e. \( \omega \),

\[
\begin{align*}
A_n f_j(\pi(\omega)) &\to \int f_j \circ \pi \, d\mu \quad \text{for } 1 \leq j \leq i, \\
A_n g(\pi(\omega)) &\to \lambda(\mu, \sigma), \\
-\frac{1}{n} \log \mu[\omega|n] &\to h(\mu, \sigma)
\end{align*}
\]

by Birkhoff’s ergodic theorem and Shannon–Mcmillan–Breiman’s theorem. For any \( \delta > 0 \), there exists a compact set \( \Omega'(i) \subset \Sigma \) such that

\[
\mu(\Omega'(i)) > 1 - \delta,
\]

and (7) holds uniformly on \( \Omega'(i) \) by Egorov’s theorem. Then there exists \( m_i \geq k_i \) such that for any \( n \geq m_i \) and any \( \omega \in \Omega'(i) \), we have

\[
\begin{align*}
\sup_{1 \leq j \leq i} |A_n f_j(\pi(\omega)) - \int f_j \, d\mu| &< \epsilon_i, \\
|A_n g(\pi(\omega)) - \lambda(\mu, \sigma)| &< \epsilon_i, \\
|\frac{1}{n} \log \mu[\omega|n] - h(\mu, \sigma)| &< \epsilon_i.
\end{align*}
\]

In this way, we get a good block of length at least \( m_i \) with nice statistical behaviour. A very naive idea is to product the blocks we have got in each step to construct the Moran set and introduce a product measure on itself. While, this construction is not really good since it is possible that

\[
\lim_{i \to \infty} \frac{m_{i+1}}{m_i} = \infty,
\]

and this makes it difficult to study the statistical behaviours of the points in the Moran set. To overcome this difficulty, we introduce a new sequence with very slow growth and the corresponding sequence of blocks still have well controlled statistical behaviours.
We can assume that $m_{i+1} \geq 2m_i$. The key point in the construction of Moran set is to construct a suitable sequence $\{l_{i,j}: 0 \leq j \leq p_i, i \in \mathbb{N}\}$ such that

(a) $p_i = m_{i+1} - m_i - 1$;
(b) $l_{i,j} = m_i + j$.

It is a simple observation that $l_{i,p_i} = m_i + 1$ and

$$\lim_{i \to \infty} \sup_{0 \leq j < p_i} \frac{l_{i,j+1}}{l_{i,j}} = 1. \quad (9)$$

5. Proof for the lower bound

In this section, we will use the framework developed in section 4 to prove the lower bound for theorem 1. For the convenience of the reader, we will separate the proof into two cases.

5.1. The construction of the first Moran set

Let $m$ be a hyperbolic measure. Let

$$\Sigma(i,j) = \{\omega_1\omega_2 \ldots \omega_{l_{i,j}} | \omega = (\omega_n)_{n=1}^{\infty} \in \Omega'(i)\},$$

and

$$\Omega(i,j) = \{\omega \in \Sigma: \omega_1\omega_2 \ldots \omega_{l_{i,j}} \in \Sigma(i,j)\}.$$

Now, fix $\mu = m$. By the construction discussed in section 4.3, we have

$$\mu(\Omega(i,j)) \geq \mu(\Omega'(i)) \geq 1 - \delta,$$

for $1 \leq j \leq p_i$.

We define the concatenation of $\Sigma(i,0), \Sigma(i,1), \ldots, \Sigma(i,p_i)$ as follows:

$$\prod_{j=0}^{p_i} \Sigma(i,j) := \{\omega_1\omega_2 \ldots \omega_{l_{i,0}}\omega_1\omega_2 \ldots \omega_{l_{i,j}} \ldots, \omega_1\omega_2 \ldots \omega_{l_{i,p_i}}\}.$$

Similarly, we define the geometric Moran construction in the following,

$$M := \Pi_{i=1}^{\infty} \prod_{j=0}^{p_i} \Sigma(i,j).$$

(See figure 2 for the illustration of the construction of the Moran set.)

We relabel the sequences $\{l_{i,j}: 0 \leq j \leq p_i, i \in \mathbb{N}\}$ and $\{\Sigma(i,j): 0 \leq j \leq p_i, i \in \mathbb{N}\}$ by $\{l_{i,j}^*\}_{i=1}^{\infty}$ and $\{\Sigma^*(i)\}_{i=1}^{\infty}$ for convenience. Correspondingly, we also use the notations $\{\Omega^*_i\}$ and $\{\epsilon^*_i\}_{i=1}^{\infty}$. It follows from (9) that

$$\lim_{i \to \infty} \frac{l_{i,j+1}}{l_{i,j}^*} = 1.$$

By the Stolz’s theorem, we have

$$\lim_{n \to \infty} \frac{l_1^* + l_2^* + \ldots + l_{n+1}^*}{l_1^* + l_2^* + \ldots + l_n^*} = 1. \quad (10)$$
Lemma 8. For any $i \in \mathbb{N}$ and $\omega \in M$, we have
\[
\lim_{{n \to \infty}} A_n f_i(\pi(\omega)) = \int f_i \, d\mu.
\]

Proof of lemma 8. We denote $n_q = \sum_{j=1}^{q} l_j^*$ for any $q \in \mathbb{N}$. By (10), we have
\[
\lim_{{q \to \infty}} \frac{n_{q+1}}{n_q} = 1.
\]
It is easy to see that
\[
\lim_{{q \to \infty}} A_{n_q} f_i(\pi(\omega)) = \int f_i \, d\mu,
\]
for each $i \in \mathbb{N}$. For $n_q \leq n < n_{q+1}$, we have
\[
\frac{1}{n} \left| S_n f_i(\pi(\omega)) - n \int f_i \, d\mu \right|
\leq \frac{1}{n} \left| S_{n_q} f_i(\pi(\omega)) - n_q \int f_i \, d\mu \right|
+ \frac{1}{n} \left| S_{n-n_q} f_i(\pi(\omega) - n_q) \int f_i \, d\mu \right|
\leq \frac{1}{n} \left| S_{n_q} f_i(\pi(\omega)) - n_q \int f_i \, d\mu \right|
+ \frac{2(n-n_q)}{n} \| f_i \|
\leq \frac{1}{n_q} \left| S_{n_q} f_i(\pi(\omega)) - n_q \int f_i \, d\mu \right|
+ \frac{2(n_{q+1}-n_q)}{n_q} \| f_i \|.
\]
Taking $n$ goes to infinity, we complete the proof of lemma 8. \qed

By lemma 8, it is easy to check that $\pi(M) \subset \Lambda_\mu$. Now, we will construct a probability measure $\eta$ on $M$, and we call it Moran measure. For each $w \in \Sigma^*(i)$, we define
\[
\rho^i_w := \frac{\mu([w])}{\mu(\Sigma^*(i))}.
\]
It is seen that $\sum_{w \in \Sigma^*(i)} \rho^j_w = 1$. Let $C_n := \{ [w] : w \in \prod_{i=1}^n \Sigma^*(i) \}$. For each $w = w_1 \ldots w_n \in C_n$, we define

$$\nu([w]) := \prod_{j=1}^n \rho^j_{w_j}.$$ 

We still denote the extension of $\nu$ to Borel $\sigma$-algebra $\sigma(C_n : n \geq 1)$ of $M$ by $\nu$. Let $\eta = \pi_* \nu$ and $\{I_{n,j} : 1 \leq j \leq k_n \} = \pi(\prod_{i=1}^n \Sigma^*(i))$.

We will split the remaining part of the proof into two steps. In the first step, we are going to prove the balanced property for $\eta$. For each $I_{n,j}$, there exists $\omega = \omega_1 \omega_2 \ldots \omega_n \in C_n$ such that $\pi(\omega) = I_{n,j}$. By the definition of $\eta$, we know that $\eta(I_{n,j}) = \prod_{i=1}^n \rho^j_{\omega_i}$.

It follows from (8), we have

$$-\sum_{i=1}^n L_i^j (h(\mu, \sigma) + \epsilon_i) \leq \log \eta(I_{n,j}) \leq -n \log(1 - \delta) - \sum_{i=1}^n L_i^j (h(\mu, \sigma) - \epsilon_i).$$

(11)

Noting that both sides of the above inequality are independent of the index $j$, we obtain

$$\min_{1 \leq j \leq k_n} \frac{\log \eta(I_{n,j})}{\max_{1 \leq j \leq k_n} \log \eta(I_{n,j})} \geq \frac{n \log(1 - \delta) + \sum_{i=1}^n L_i^j (h(\mu, \sigma) - \epsilon_i)}{\sum_{i=1}^n L_i^j (h(\mu, \sigma) + \epsilon_i)}$$

and

$$\min_{1 \leq j \leq k_n} \frac{\log \eta(I_{n,j})}{\max_{1 \leq j \leq k_n} \log \eta(I_{n,j})} \leq \frac{\sum_{i=1}^n L_i^j (h(\mu, \sigma) + \epsilon_i)}{n \log(1 - \delta) - \sum_{i=1}^n L_i^j (h(\mu, \sigma) - \epsilon_i)}.$$ 

This proves the balanced property for $\eta$ as $n$ goes to infinity.

In the second step, we will show that the fundamental intervals $\{I_{n,k} : 1 \leq k \leq k_n, n \in \mathbb{N} \}$ satisfy the tempered growth property. By lemma 1 we can get

$$\log \text{diam} I_{n,k} = \log D_{\omega^{j=1} L_j^*} = -\left( S_{\omega^{j=1} L_j^*} g(\pi(\omega)) - \sum_{j=1}^n L_j^* \epsilon_n \right)$$

$$= -\sum_{j=1}^n S_{L_j^*} g(\pi(\sigma^{j=1} L_j^* \omega)) + \sum_{j=1}^n L_j^* \epsilon_n^*$$

$$\leq -\sum_{j=1}^n L_j^* \lambda(\mu, \sigma) + \sum_{j=1}^n L_j^* (\epsilon_n^* + \epsilon_j^*).$$

Thus, we have

$$\log R_n \leq -\sum_{j=1}^n L_j^* \lambda(\mu, \sigma) + \sum_{j=1}^n L_j^* (\epsilon_n^* + \epsilon_j^*).$$

(12)
By the Stolz’s theorem, it is not difficult to see that
\[ \lim_{n \to \infty} \frac{\log R_n}{\log r_n} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{\log r_{n+1}}{\log r_n} = 1. \]
It follows from (11), (12) and (13) that
\[ \lim_{n \to \infty} \max_j \left| \frac{\log \eta(I_{n,j})}{\log \text{diam}(I_{n,j})} - \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} \right| = 0. \]
By lemma 6, we have
\[ \lim_{r \to 0} \frac{\log \eta(B(x, r))}{\log r} = \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)}, \]
for any \( x \in \pi(M) \). Recall that \( \mu = m \), we obtain
\[ \dim_H \Lambda_m \geq \dim_H \pi(M) \geq \frac{h(m, \sigma)}{\lambda(m, \sigma)}. \]
Noting that \( \dim_H \Lambda_m \leq \dim_H \Lambda_m \) and \( \dim_H \Lambda_m \leq \frac{h(m, \sigma)}{\lambda(m, \sigma)} \), we have
\[ \dim_H \Lambda_m = \dim_H \Lambda_m = \frac{h(m, \sigma)}{\lambda(m, \sigma)}. \]
This also recover first part of theorem B in our setting.

5.2. The construction of second Moran set

In order to prove the second part of theorem 1, we need to make some modifications of the Moran set \( M \) constructed above. Let \( \mu \) be arbitrary hyperbolic measure. Since \( m \) is a parabolic ergodic measure, we assume that \( m \) supports on \( \pi(1^\infty) \) without loss of generality. There exists a sequence \( \{k_i\}_{i=1}^\infty \) such that
\[ \lim_{i \to \infty} k_i = \infty, \quad \lim_{i \to \infty} \frac{k_{i+1}}{k_i} = 1, \quad \lim_{i \to \infty} k_i \epsilon_i = 0. \]
Let \( k_{i,j} = k_i \) for \( 1 \leq j \leq p_i \). Denote
\[ \Sigma(i, j) = \{ \omega_1 \omega_2 \ldots \omega_{i,j} 1_{k_i \epsilon_i,j} | \omega \in \Omega'(i) \}, \]
and
\[ \Omega(i, j) = \{ w : \omega_1 \omega_2 \ldots \omega_{i,j} 1_{k_i \epsilon_i,j} \in \Sigma(i, j) \}. \]
Next, we still use the framework discussed in section 4.3. By the construction in section 4.3, we have
\[ \mu(\Omega(i, j)) \geq \mu(\Omega'(i)) \geq 1 - \delta. \]
We define a new Moran set in the following,

\[ \hat{M} := \prod_{i=1}^{\infty} \prod_{j=0}^{p_i} \Sigma(i, j), \]

and relabel the sequences \( \{b_i : 0 \leq j \leq p_i, i \in \mathbb{N}\} \) and \( \{\Sigma(i, j) : 0 \leq j \leq p_i, i \in \mathbb{N}\} \) by \( \{l_i^{**}\} \) \( \infty \) and \( \{\Sigma^{**}(i)\} \) \( \infty \). Analogously, we also take the relabelling sequences \( \{\Omega^{**}(i)\} \) \( i=1 \), \( \{K_i^{**}\} \) \( i=1 \) and \( \{epsilon_i^{**}\} \) \( i=1 \). For each \( \tilde{w} = w^1 \cdots w_n \in \Sigma^{**}(i) \), we define a probability measure on \( \Sigma^{**}(i) \) by

\[ \rho_{\tilde{w}}(\mu) := \frac{\mu([w1^{k_i^{**}}])}{\mu(\Omega^{**}(i))}. \]

Denote \( C_n := \{[w] : w \in \prod_{i=1}^{n} \Sigma^{**}(i) \} \). For each \( w = w_1 \ldots w_n \in C_n \), we define

\[ \tilde{\nu}([w]) := \prod_{i=1}^{n} \rho_{w_i}. \]

This measure can be uniquely extended to \( \hat{M} \) and we still denote it by \( \tilde{\nu} \). Let \( \{\tilde{L}_{n,i} : 1 \leq j \leq k_n\} = \pi(\prod_{i=1}^{n} \Sigma^{**}(i)) \) and denote \( \tilde{\eta} = \pi_{T} \tilde{\nu}. \) We can get the following estimates,

\[ -\sum_{i=1}^{n} l_i^{**}(h(\mu, \sigma) + \epsilon_i^{**}) \leq \log(\tilde{\eta}(\tilde{L}_{n,k})) \leq -\sum_{i=1}^{n} l_i^{**}(h(\mu, \sigma) - \epsilon_i^{**}), \]

\[ \log \text{diam}(\tilde{L}_{n,k}) \leq -\sum_{j=1}^{n} l_j^{**}(1 + k_j^{**}) \epsilon_j^{**}, \]

\[ \log \text{diam}(\tilde{L}_{n,k}) \geq -\sum_{j=1}^{n} l_j^{**}(1 + k_j^{**}) \epsilon_j^{**}, \]

for any \( 1 \leq k \leq k_n. \) It is almost the same to prove that \( \pi(M) \subset \Lambda_m \) and \( \dim_{H} \pi(M) \geq \frac{h_{\mu(T)}(\Lambda)}{h_{\mu(T)}}. \)

Noting that \( \mu \) is arbitrary, we have

\[ \dim_{H} \Lambda_m \geq \sup_{\mu} \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \lambda(\mu, T) > 0, \mu \text{ is ergodic} \right\}. \]

We need the following lemma about the hyperbolic dimension of \( \Lambda. \)

**Lemma 9 ([23])**. In the setting above, we have

\[ \dim_{H}^{\text{hyp}} \Lambda = \sup_{\mu} \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \lambda(\mu, T) > 0, \mu \text{ is ergodic} \right\}. \]

By lemma 9, we obtain \( \dim_{H} \Lambda_m \geq \dim_{H}^{\text{hyp}} \Lambda. \) Additionally, if \( T \) is a \( C^2 \) map, it is proved in [10] that

\[ \dim_{H} \Lambda = \sup_{\mu} \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \lambda(\mu, T) > 0, \mu \text{ is ergodic} \right\}. \]

Hence, we get

\[ \dim_{H} \Lambda_m = \dim_{H} \Lambda. \]
if \( T \) is a \( C^2 \) map. This also recovers the second part of theorem B in our setting. Noting that \( \dim_H \tilde{\Lambda}_m \geq \dim_H \Lambda \) we complete the second part of theorem 1.

6. Proof of theorem 2

In this section, we will prove theorem 2. Recall that \( g(x) = \log|T'(x)| \) for any \( x \in \Lambda \). By theorems 1 and B, we only need to prove that \( \dim_H \Lambda_{\delta_P} = \dim_H \tilde{\Lambda}_{\delta_P} \) with the assumption

\[
\dim_{\text{hyp}} \Lambda < \dim_H \Lambda. \tag{14}
\]

Let

\[
\Lambda_+ := \{ x \in \Lambda : \liminf_{n \to \infty} A_n g(x) > 0 \},
\]

and

\[
\Lambda^+ := \{ x \in \Lambda : \limsup_{n \to \infty} A_n g(x) > 0 \}.
\]

It is proved in [13] that

\[
\dim_H \Lambda = \dim_{\text{hyp}} \Lambda_+. \tag{15}
\]

By (15) and (14), we know that

\[
\dim_H \Lambda = \dim_H \Lambda. \Lambda_+.
\]

We write \( \Lambda^+ = \bigcup_{k=1}^{\infty} \Lambda_+^k \), where \( \Lambda_+^k = \{ x \in \Lambda : \limsup_{n \to \infty} A_n g(x) > \frac{1}{k} \} \) for any \( k \in \mathbb{N} \). For any \( n \in \mathbb{N} \), we denote

\[
C_n := \left\{ \mu \in \mathcal{M}(\Lambda, T) : d(\mu, \delta_P) \geq \frac{1}{n} \right\}.
\]

Since \( T \) has only one fixed point, we know that \( \lambda(\mu, T) > 0 \) for any \( \mu \in C_n \). Noting that \( \mathcal{C}_n \) is compact, then for any positive real number \( \rho_n < \frac{1}{n} \), there exists finitely many invariant measures \( \{ \mu_i \}_{i=1}^{l_n} \subset C_n \) for some \( l_n \in \mathbb{N} \) such that \( C_n \subset \bigcup_{i=1}^{l_n} \mathcal{B}(d(\mu_i, \rho_n)) \) and \( \delta_P \notin \bigcup_{i=1}^{l_n} \mathcal{B}(d(\mu_i, \rho_n)) \), where \( \mathcal{B}(d(\mu_i, \rho_n)) \) is the interior of the closed ball \( B(d(\mu_i, \rho_n)) = \{ \mu : d(\mu, \mu_i) \leq \rho_n \} \).

Let

\[
B_n := \{ x \in \Lambda : \text{Asym}(\{ A_m \delta_i \}_{m=1}^{\infty}) \cap C_n \neq \emptyset \},
\]

and

\[
B_n' := \{ x \in \Lambda : \text{Asym}(\{ A_m \delta_i \}_{m=1}^{\infty}) \cap B(d(\mu_i, \rho_n) \neq \emptyset \},
\]

for \( 1 \leq i \leq l_n \). It is not difficult to check that

\[
\Lambda^+ = \bigcup_{n=1}^{\infty} B_n = \Lambda \setminus \Lambda_{\delta_p}.
\]
By proposition 2, we have
\[ \dim_H B_i \leq \sup \left\{ \frac{h(\mu, T)}{\lambda(\mu, T)} : \mu \in B_i(\mu_i, \rho_n) \right\} \leq \dim_H^{\text{hyp}} \Lambda. \]

It follows that
\[ \dim_H B_i \leq \max_{1 \leq i \leq n} \dim_H B_i \leq \dim_H^{\text{hyp}} \Lambda. \]

Now, we get
\[ \dim_H \Lambda^* \leq \dim_H^{\text{hyp}} \Lambda. \]

Together with (15), we have
\[ \dim_H \Lambda^* = \dim_H^{\text{hyp}} \Lambda. \]

This implies that \( \dim_H \Lambda_\delta^* = \dim_H \Lambda. \) Since
\[ \dim_H \Lambda_\delta^* \geq \dim_H \Lambda_\delta, \]
we get
\[ \dim_H \Lambda_\delta^* = \dim_H \Lambda_\delta. \]

Thus, we finish the proof of theorem 2.

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