Asymptotically free lattice gauge theory in five dimensions

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(Dated: November 19, 2014)

A lattice formulation of Lifshitz-type gauge theories is presented. While the Lorentz-invariant Yang-Mills theory is not renormalizable in five dimensions, non-Abelian Lifshitz-type gauge theories are renormalizable and asymptotically free. We construct a lattice gauge action and numerically examine the continuum limit and the bulk phase structure.

I. INTRODUCTION

Since olden times, the Lifshitz-type anisotropic field theory [1, 2] has been considered in various condensed matter systems. In recent years, the Hořava-Lifshitz-type gravity [3] has received much interest. Its analogues in non-gravitational quantum field theories have also been discussed intensively [4–46], in the context of physics beyond the Standard Model; see [47] for a review. Such theories may prove useful as a UV completion of extra-dimensional models considered e.g., in the gauge-Higgs unification scenario. Anisotropic non-Abelian gauge theories are expected to arise as an effective theory at quantum critical points in certain condensed matter systems, see [48–50] and references therein. Cold atomic gases may also provide a venue for non-Abelian gauge theories [51–53].

In this work we propose a lattice formulation of an anisotropic non-Abelian gauge theory put forward by Hořava [6]. The Euclidean action of this Hořava-Lifshitz-type gauge theory in \((1 + D)\) dimensions reads

\[
S = \frac{1}{2} \int dx_0 d^D x \left[ \frac{1}{e^2} \text{Tr}(E_i E_i) + \frac{1}{g^2} \text{Tr}(D^a_i F_{ik})(D^a_{jk} F_{jk}) \right],
\]

where the indices \(i, j, k\) run from 1 to \(D\), and

\[ E_i = F_{0i}, \]

\[ F_{ij} = -i[D_i, D_j] = \partial_i A_j - \partial_j A_i + i[A_i, A_j] \]

\[ D_i = \partial_i + iA_i \]

\[ D^a_i F = \partial_i F + i[A_i, F]. \]

The gauge field \(A_i \equiv A^a_i T^a\) takes values in the Lie algebra of a non-Abelian compact Lie group. For the second term in (1) to be nonzero, \(D \geq 2\) is required. There are two couplings, \(e^2\) and \(g^2\). In a weighted power counting with the dimensions of fields \([A_0] = 2\) and \([A_1] = 1\), we find \([e^2] = [g^2] = 4 - D\). In the critical dimension \((d \equiv 1 + D = 1 + 4)\), the theory is asymptotically free [6]. This has been shown by using the fact that the action (1) satisfies the detailed balance condition; it ensures that the theory in \(d = 5\) is renormalizable with no need for other terms of the same dimension, like \(\text{Tr}(F_{ij} F_{jk} F_{ki})\) and \(\text{Tr} \{ (D^a_i F_{ij})(D^a_{jk} F_{jk}) \}\). It also implies, surprisingly, that the theory (1) inherits quantum mechanical properties from the Lorentz-invariant Yang-Mills theory in \(d = 4\). As a result, the beta functions in both theories coincide [6].

While renormalizability in the continuum requires \(d \leq 5\), we will shortly see that the theory can be discretized on a lattice in any \(d \geq 3\) dimensions, thus opening a way toward a non-perturbative study of Hořava-Lifshitz-type gauge theories. With a soft deformation term, the theory restores effective Lorentz invariance in the infrared [6], hence the theory may be considered as a UV completion of the non-renormalizable Yang-Mills theory in five dimensions [54–63].

This paper is structured as follows. In Section II we present a lattice action for the Hořava-Lifshitz-type gauge theory and discuss its continuum limit. In Section III the setup of our lattice simulation is outlined and the first numerical results of this theory for the SU(3) gauge group are presented. Section IV is devoted to summary and conclusions. Some technical details on the classical continuum limit are presented in appendix A. Lattice actions for more general terms in the continuum are discussed in appendix B.

II. LATTICE FORMULATION

In the following, for convenience, we call the isotropic \(D\) dimensions “space” and the other one dimension “time” although it is not necessarily so. The spatial lattice spacing is denoted by \(a\) and the temporal lattice spacing by \(b\). The mass dimensions are \([a] = -1\) and \([b] = -2\) according to the standard weighted power counting for Lifshitz-type theories [47]. Unit vectors in \(x^d\) direction will be denoted as \(\hat{\mu}\) for \(\mu = 0, 1, \ldots, D\).

The temporal and spatial link variables are defined as

\[ U_0(x) = \text{P} \exp \left( i \int_x^{x + b \hat{0}} \text{d} y A_0(y) \right) \simeq \exp(ibA_0(x)) \quad \text{and} \quad U_i(x) = \text{P} \exp \left( i \int_x^{x + a \hat{i}} \text{d} y A_i(y) \right) \simeq \exp(iA_i(x)) \]

respectively.
We define the lattice Hořava-Lifshitz gauge theory as

$$Z = \int DU \exp(-S_{\text{lat}})$$

with

$$S_{\text{lat}} = \frac{1}{\epsilon_{\text{lat}}} \sum_x \sum_{i=1}^{D} \text{Re} \text{Tr} \left\{ 1 - P_{0i}(x) \right\}$$

$$+ \frac{1}{g_{\text{lat}}^{2}} \sum_x \sum_{j=1}^{D} \text{Re} \text{Tr} \left\{ 1 - \prod_{i \neq j} T_{ij}(x) \right\},$$

where 1 denotes the unit matrix. The temporal component of $S_{\text{lat}}$ includes a $1 \times 1$ plaquette $P_{0\mu}(x)$, which is well known in the lattice Yang-Mills theory, while the spatial component of $S_{\text{lat}}$ includes a $2 \times 1$ twisted loop $T_{\mu\nu}(x)$, which is shown in Fig. 1. Such a rectangular loop has been considered for improved lattice actions [64]. We remark that the ordering of $T$'s in the product $\prod T_{\mu\nu}(x)$ is inessential, because as we shall shortly see, only subleading terms irrelevant in the continuum limit are affected by this ordering. Note also that gauge invariance is maintained, since all the twisted loops begin and end at the same point $x$.

We can check the naive continuum limit of this lattice action using the Baker-Campbell-Hausdorff (BCH) formula. The temporal plaquette may be evaluated as

$$P_{0i}(x) = \exp \left( i a b F_{0i}(x) + \mathcal{O}(a^2 b, a b^2) \right).$$

Hence

$$\sum_{x=1}^{D} \text{Re} \text{Tr} \left\{ 1 - P_{0i}(x) \right\} = \frac{a^2 b^2}{2} \sum_{x=1}^{D} \text{Tr} \left\{ F_{0i}(x) \right\} + \mathcal{O}(a^3 b^2, a^2 b^3).$$

Next, the twisted loop is given (cf. appendix A) by

$$T_{ij}(x) = \exp \left( i a^3 D_{ij}^a F_{ij}(x) + \mathcal{O}(a^4) \right).$$

Then

$$\sum_{x=1}^{D} \sum_{j=1}^{D} \text{Re} \text{Tr} \left\{ 1 - \prod_{i=1}^{D} T_{ij}(x) \right\}$$

$$= \sum_{x=1}^{D} \sum_{j=1}^{D} \text{Re} \text{Tr} \left\{ 1 - \exp \left( i a^3 \sum_{i=1}^{D} D_{ij}^a F_{ij}(x) + \mathcal{O}(a^4) \right) \right\}$$

$$= \frac{a^6}{2} \sum_{x=1}^{D} \sum_{j=1}^{D} \text{Tr} \left\{ \left( \sum_{i=1}^{D} D_{ij}^a F_{ij}(x) \right)^2 \right\} + \mathcal{O}(a^7).$$

Collecting Eqs. (6) and (8),

$$S_{\text{lat}} \to \frac{1}{\epsilon_{\text{lat}}} \int dx_0 d^D x \left[ \frac{1}{\epsilon_{\text{lat}}} \frac{b}{a^{D-2}} \sum_{i=1}^{D} \text{Tr} \left\{ F_{0i}(x)^2 \right\} \right.$$

$$+ \frac{1}{g_{\text{lat}}^{6-D}} \frac{a^{6-D}}{b} \sum_{j=1}^{D} \text{Tr} \left\{ \left( \sum_{i=1}^{D} D_{ij}^a F_{ij}(x) \right)^2 \right\} \left. \right\}$$

as $a, b \to 0$. This reproduces the continuum action (1).

For completeness we outline the lattice discretization of other possible terms in the action in appendix B.

Matching with the continuum action (1) yields

$$\frac{1}{\epsilon^2} = \frac{1}{\epsilon_{\text{lat}}} \frac{b}{a^{D-2}} \quad \text{and} \quad \frac{1}{g^2} = \frac{1}{g_{\text{lat}}} \frac{a^{6-D}}{b}.$$ (10)

The two terms in Eq. (4) are of the same order only if we take the limit $a, b \to 0$ with $b/a^2 \sim \mathcal{O}(1)$. Plugging this scaling into Eq. (10), we find $\epsilon^2 \sim a^{4-D}$ and $g^2 \sim g_{\text{lat}}^{2a^{4-D}}$. Now, let us consider the continuum limit in each dimension:

- $D = 2 (d = 2 + 1): \epsilon_{\text{lat}}^{2}, g_{\text{lat}}^{2} \propto a^{2} \Rightarrow \epsilon_{\text{lat}}, g_{\text{lat}} \to 0$
  with $\epsilon_{\text{lat}}/g_{\text{lat}} \sim \mathcal{O}(1)$.

- $D = 3 (d = 3 + 1): \epsilon_{\text{lat}}^{2}, g_{\text{lat}}^{2} \propto a^{2} \Rightarrow \epsilon_{\text{lat}}, g_{\text{lat}} \to 0$
  with $\epsilon_{\text{lat}}/g_{\text{lat}} \sim \mathcal{O}(1)$.

- $D = 4 (d = 4 + 1): \epsilon_{\text{lat}}^{2}, g_{\text{lat}}^{2} \propto a^{0} \Rightarrow$ It is unclear how to take the continuum limit at tree level.

This means that the continuum limit for $D = 2$ and $3$ ($d = 3$ and $4$) is reached trivially by sending $\epsilon_{\text{lat}}$ and $g_{\text{lat}}$ to $0$. However, $D = 4$ ($d = 5$) is the critical dimension where there is no scaling of the couplings at tree level. In $D = 4$, the one-loop $\beta$ functions [6] are given by

$$\frac{d}{d \log \mu} \epsilon_{\text{lat}}(\mu) = -\frac{3}{2} C_2 e^2 g + \cdots$$

$$\frac{d}{d \log \mu} g_{\text{lat}}(\mu) = -\frac{35}{6} C_2 e g^2 + \cdots,$$ (11a)

or, with $g_{\text{YM}} \equiv \sqrt{e g}$ and $\lambda \equiv g/e$,

$$\frac{d}{d \log \mu} g_{\text{YM}}(\mu) = -\frac{11}{3} C_2 g_{\text{YM}}^3 + \mathcal{O}(g_{\text{YM}}^4)$$

$$\frac{d}{d \log \mu} \lambda(\mu) = -\frac{13}{3} C_2 g_{\text{YM}}^4 \lambda + \mathcal{O}(g_{\text{YM}}^4 \lambda),$$ (12a)

where $C_2 \equiv N/(4\pi^2)$ for the gauge group SU$(N)$. The theory is asymptotically free and therefore the continuum
limit is achieved by sending both $g_{\text{YM}}$ and $\lambda$ to 0. Solving Eqs. (12a) and (12b) simultaneously, we find

$$\lambda(\mu) \propto (g_{\text{YM}}(\mu))^{13/11}, \quad \text{i.e.,} \quad g \propto e^{35/9}. \quad (13)$$

This scaling defines lines of constant physics in the weak-coupling region on the $(e, g)$ plane. The renormalization group flow of $e$ and $g$ is displayed in Fig. 2. (Since $C_2$ only enters the $\beta$ functions (11) as a multiplicative factor, the flow pattern is the same for all $N \geq 2$.) Integrating Eq. (12a), we encounter an infrared energy scale which survives the continuum limit:

$$\Lambda = \frac{1}{a} \exp \left( - \frac{24\pi^2}{11} \frac{1}{Ng_{\text{YM}}(\frac{1}{a})} \right). \quad (14)$$

This is the phenomenon called dimensional transmutation.

The above formulation is straightforwardly applicable to Abelian gauge theories as well. The geometrical structure of the lattice action is the same. The SU($N$) link variable is replaced by the the U(1) link variable. However, the resultant compact U(1) gauge theory is not asymptotically free in $D = 4$ ($d = 5$).

### III. NUMERICAL SIMULATION

We apply the above formulation to the lattice Monte Carlo simulation. The simulation can be done with the standard algorithms in the lattice Yang-Mills theory. In this work, we performed a simulation of the lattice Hořava-Lifshitz theory for the case of SU($N = 3$) gauge group.

First we examine the bulk phase structure on the $(e, g)$ plane. We calculated the action density $s \equiv \langle S_{\text{lat}} \rangle/N_{\text{lat}}$ for various values of the lattice couplings defined as

$$\beta_e \equiv \frac{2N}{e_{\text{lat}}} \quad \text{and} \quad \beta_g \equiv \frac{2N}{g_{\text{lat}}}. \quad (15)$$

The lattice size is $N_{\text{lat}} = 6^5$. In Fig. 3, we show the simulation results for isotropic couplings $\beta \equiv \beta_e = \beta_g$. For comparison, we also show simulation results of the isotropic Yang-Mills theory in five dimensions. As already known, there is a jump at $\beta = 4-5$ in the five-dimensional lattice Yang-Mills theory [63]. This jump indicates a bulk first-order phase transition. This bulk phase transition is a lattice artifact. Its existence makes it impossible to take the continuum limit of the lattice Yang-Mills theory in five dimensions. On the other hand, there is no phase transition in the Hořava-Lifshitz theory. As shown in Fig. 4, there is no discontinuity in the region $1 \leq \beta_e \leq 9$ and $1 \leq \beta_g \leq 9$. Thus, we can smoothly take the continuum limit of the lattice Hořava-Lifshitz theory.

Next we study a rectangular Wilson loop $W_{0i}$ of size $t \times x$ lying in the $(x_0, x_i)$ plane. This temporal Wilson loop may be interpreted as the infinite mass limit of a quark-antiquark system and then it gives the color singlet
of this theory on a lattice for the SU(3) gauge group. Numerical results suggest that the continuum limit can be taken smoothly, in contrast to the ordinary Yang-Mills theory in five dimensions which is beset with a bulk phase transition. Using the present framework one can study various nonperturbative aspects of the Hořava-Lifshitz-type gauge theories by means of numerical lattice simulations. For example, it is straightforward to compactify a temporal or spatial direction and study possible center symmetry breaking. Of course one can perform simulations for other gauge groups and in other spacetime dimensions. Lattice simulations may also be performed with additional terms in the action, such as $\text{Tr}(F_{ij}^2)$, $\text{Tr}(F_{ij}F_{jk}F_{ki})$, and $\{D_{ij}^{ad}F_{jk}\}^2$, as discussed in appendix B. The interplay of these terms is an interesting subject. A more ambitious generalization is to include fermions coupled to the gauge field and study spontaneous chiral symmetry breaking. These issues are left for future works.

ACKNOWLEDGMENTS

TK was supported by the RIKEN iTHES Project and JSPS KAKENHI Grants Number 25887014. The numerical simulations were performed by using the RIKEN Integrated Cluster of Clusters (RICC) facility.

Appendix A: Classical continuum limit

It has been known from \cite{64, Eq. (16)} that a spatial plaquette in the naive continuum limit $a \to 0$ becomes

\[ P_{ij}(x) \equiv U_i(x)U_j(x + a\hat{a})U_i(x + a\hat{a})^\dagger U_j(x) = \exp \left( ia^2 F_{ij}(x) + \frac{i}{2} a^3 (D_i^{ad} + D_j^{ad})F_{ij}(x) + O(a^4) \right). \]

Because a twisted $2 \times 1$ Wilson loop is a product of two neighboring spatial plaquettes, we get

\[ T_{ij}(x) = \exp \left( ia^2 F_{ij}(x) + \frac{i}{2} a^3 (D_i^{ad} + D_j^{ad})F_{ij}(x) + O(a^4) \right) \times \exp \left( -ia^2 F_{ij}(x) - \frac{i}{2} a^3 (-D_i^{ad} + D_j^{ad})F_{ij}(x) + O(a^4) \right) = \exp \left( ia^3 D_i^{ad}F_{ij}(x) + O(a^4) \right), \]

which proves (7).

Appendix B: More general lattice action

Besides $\text{Tr}\{(D_i^{ad}F_{ik})(D_j^{ad}F_{jk})\}$, there are many other terms that could have been added to the action (1). In this appendix we discuss how to discretize them on a lattice.
Firstly, the term $\text{Tr} \left( F_{ij} F_{jk} F_{ki} \right)$ can be realized on a lattice as follows. Let us consider

$$\text{Tr} \left\{ (I - P_{ij}(x)) (I - P_{jk}(x)) (I - P_{ki}(x)) \right\}. \quad (B1)$$

This expression is manifestly gauge invariant. By plugging in (A1) for each $P$ and expanding in powers of $a$ we get

$$\text{Tr} \left\{ (I - P_{ij}(x)) (I - P_{jk}(x)) (I - P_{ki}(x)) \right\} = a^6 \text{Tr} \left\{ F_{ij}(x) F_{jk}(x) F_{ki}(x) \right\} + \mathcal{O}(a^7), \quad (B2)$$

which is the desired term.

The second term of our interest is $\text{Tr} \left\{ D^a_{ij} F_{ij} (x) D^a_{jk} F_{jk}(x) \right\}$. The case with $k = i$ or $k = j$ follows from $T_{ij}(x)$ as given in (7), so it is enough to assume here that $i, j$ and $k$ are distinct from each other, which requires $D \geq 3$.

Let us start from a Wilson loop $W_{ijk}(x)$ shown in Fig. 6:

$$W_{ijk}(x) \equiv P_{ij}(x) U_k(x) P_{ij}(x + a k) U_k(x)^\dagger \equiv \exp \left\{ a^2 \mathcal{P}_1 \phi a^2 \mathcal{P}_1(x) + \mathcal{O}(a^2) \right\},$$

where from (A1)

$$\mathcal{P}_1 \equiv i F_{ij}(x) + \frac{i}{2} a D^a_{ij} D^a_{ij}(x) + \mathcal{O}(a^2),$$

$$\mathcal{P}_2 \equiv i F_{ij}(x + a k) + \frac{i}{2} a D^a_{ij} + D^a_{ij}(x + a k) + \mathcal{O}(a^2).$$

Using the BCH formula,

$$W_{ijk}(x) = \exp \left( a^2 \mathcal{P}_1 - \mathcal{P}_2 - i a^3 [A_k(x), \mathcal{P}_2] + \mathcal{O}(a^4) \right) = \exp \left( - i a^3 D^a_{k} F_{ij}(x) + \mathcal{O}(a^4) \right),$$

so that

$$\text{Re} \text{Tr} \left\{ I - W_{ijk}(x) \right\} = \frac{1}{2} a^6 \text{Tr} \left\{ D^a_{ij} F_{ij}(x) D^a_{jk} F_{jk}(x) \right\} + \mathcal{O}(a^7). \quad (B4)$$

However, it has been known from [66, Eq. (2.10)] that $\text{Tr} \left\{ D^a_{ij} F_{ij} \left( D^a_{jk} F_{jk} \right) \right\}, \text{Tr} \left\{ F_{ij}(x) F_{jk}(x) F_{ki}(x) \right\}$ and $\text{Tr} \left\{ D^a_{ij} F_{ij} \left( D^a_{jk} F_{jk} \right) \right\}$ are linearly dependent, up to a total derivative. Thus it is sufficient to keep only two of them in the action.

The lattice actions for other possible terms like $\varepsilon_{jklm} \text{Tr} \left\{ D^a_{ij} F_{jk} (x) D^a_{j} F_{lm}(x) \right\}$ (for $D = 4$) can be worked out along similar lines.

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