Introduction

A basic problem of inverse Galois theory is to find, for a given field $K$ and finite group $G$, an absolutely irreducible cover of the projective line $\mathbb{P}^1_K$ defined over $K$, whose Galois group equals $G$. The outstanding case is $K = \mathbb{Q}$ (viz., solving the inverse Galois theory problem over $\mathbb{Q}(t)$), but here we will be concerned with its cousin $K = \mathbb{F}_q(T)$, a global function field over a finite field $\mathbb{F}_q$ with $q$ elements. A famous theorem of Harbater-Pop ([11]) says that any $G$ can be realized over “large” fields (e.g., the separable closure of $K$), but the question remains over what smaller field such a cover can be defined (especially in connection with the so-called “arithmetic Abhyankar conjecture”, cf. [1], (9.2)).

In this direction, we have the following result of Nori ([10]): if $G = A(\mathbb{F}_{q^d})$ is the group of $\mathbb{F}_{q^d}$ points of a semi-simple simply connected algebraic group $A$ over $\mathbb{F}_q$, then $G$ is a Galois group over $K$ (ramified above at most one point) — this applies in particular to $SL(r, q)$, so there is also such a $PSL(r, q^d)$-cover. However, this gives only one such cover, not a “family” over $\mathbb{P}^1_k$ for $K$ a global field. In this paper, we link this problem with torsion of Drinfeld modules to show that there is at least an $(r - 1)$-dimensional family of $K$-regular unimodular covers (which can then be given explicitly by calculations with modular forms):

**Theorem.** Let $q$ be a power of an odd prime number $p$. For every pair of integers $(r, d)$, such that $r$ is even and $(r; q^d - 1) = 2$ there exists a $K$-regular Galois extension of the purely transcendental field $K(t_1, ..., t_{r-1})$ with Galois group the unimodular group $PSL(r, q^d)$.

To appreciate this result, let us describe two known related facts. The general linear group $GL(m, q^d)$ (and, a fortiori, its projectivized version) is known to
occur as a Galois group over \( K(t_1, \ldots, t_{r-1}) \), since it occurs as the image of the Galois representation \( \rho_{\phi,p,K} \) associated to the \( p \)-torsion of the “generic” rank-\( r \) Drinfeld module \( \phi \) over \( K \), where \( p \) is irreducible of degree \( d \) in \( F_q[T] \) (but note that these extensions are in general not regular, since they contain the \( p \)-th ray class field of \( K \), cf. infra); explicit proofs are due to Abhyankar and collaborators (cf. [2]). On the other hand, the unimodular group \( PSL(2,p) \) is known to be the Galois group of an extension of \( \mathbb{Q}(t) \) if 2, 3 or 7 is not a square modulo \( p \), by the work of K.-y. Shih ([12]). In modern language, he uses moduli of elliptic \( \mathbb{Q} \)-curves.

As a matter of fact, our proof of the above theorem will be a combination of the techniques used to prove these two results: we study a twisted version of moduli spaces of Drinfeld modules of rank \( r \) with an appropriate level structure. The main novelties are: (a) the fact that we use higher dimensional varieties – hence rationality questions related to moduli spaces become more difficult; (b) the fact that the Weil pairing is replaced by determinants in the category of \( t \)-motives; (c) the fact that there is no residue class condition in the final result.

Here is a more detailed outline: we start by looking at the moduli scheme \( Y_{0}^{r}(T) \) of Drinfeld modules over \( K \) of rank \( r \) “with level \( \Gamma_{0}(T) \)-structure” as classifying Drinfeld modules of rank \( r \) with a “full flag” of subgroups of its \( T \)-torsion. We give an explicit description of this variety. Let \( L = K(\sqrt{p}) \) for \( p \) prime in \( F_q[T] \) of degree \( d \). We define \( \tilde{Y} \) to be the quotient

\[
\tilde{Y} := (\langle w \rangle \times \text{Gal}(L/K)) \backslash (Y_{0}^{r}(T) \times \text{Spec } K \text{ Spec } L),
\]

where \( w \) is an Atkin-Lehner-style involution (whose action we describe explicitly) – this will turn out to make sense since \( r \) is even. We then show that \( \tilde{Y} \) has a rational function field by showing that a naive compactification is a Brauer-Severi variety with a \( K \)-rational point. If \((q^{\deg p - 1},r) = 2\), then to any \( K \)-rational point of \( \tilde{Y} \), there is associated a representation

\[
\rho : \text{Gal}(\bar{K}/K) \to PGL(r,F_q[T]/p),
\]

whose determinant is minus the quadratic character of \( \zeta T \) modulo \( p \) for a constant \( \zeta \in F_q^* \) that only depends on the rank \( r \). We then show that this character can take any value for a suitable choice of \( N \):

**Theorem** (continuation). There exists a computable constant \( \zeta \in F_q^* \) only depending on \( r \), such that if \( p \) is irreducible in \( F_q[T] \) of degree \( d \) with \( p(0)\zeta \) a non-square in \( F_q^* \), then the desired cover is the Galois closure of \( Y_{0}^{r}(p) \times_{Y_{r}(1)} \tilde{Y} \rightarrow \tilde{Y} \); furthermore, such \( p \) exist.

As this sketch of the proof indicates, the construction can be made explicit if one can find explicit equations for suitable moduli schemes. For example if \( r = 2 \), one has enough control over modular forms (and hence, function fields of modular curves), and one arrives at the following result.
Proposition. Let \( h \) be a modular form of weight \( q + 1 \) and character \( \det \) for \( GL(2, \mathbb{F}_q[T]) \) (which is then unique up to a constant). Let \( N \in \mathbb{F}_q[T] \) be a monic irreducible polynomial of degree two whose constant term is a square. There exists a polynomial \( P \in K[x, y] \) of the form
\[
P(x, y) = T^{q+1}y^2x^2 + a(y)x + y
\]
of bidegree \( (2, q^2 + 1) \) such that
\[
P(h(Tz)/h(z), \sqrt[p]{f_T(Nz)/f_T(z)}) = 0.
\]
Then the splitting field of the numerator of
\[
P(T^{\frac{q+1}{2}} \sqrt[8]{N + x}, \sqrt[8]{N + y}, \sqrt[8]{N - x}, \sqrt[8]{N - y})
\]
defines a \( K \)-regular extension of \( K(x) \) with Galois group \( \text{PSL}(2, q^2) \).

The proposition is entirely algorithmical since one can determine the polynomial \( P \) by looking at series expansions of modular forms.

Example. The polynomial in \( \mathbb{F}_3((T))[(x, y)] \) given by
\[
T^2x^2y^9 - T^2Nxy^9 - N^2(x^2 - N)(Ty^8 - Ny^6 - N^2y^4 - TN^3y^2) + T^2N^5xy - T^2N^6
\]
with \( N = T^2 + 1 \) defines a regular extension of \( \mathbb{F}_3(T)(x) \) with Galois group \( \text{PSL}(2, 9) \).

Remark. The branching of the cover in the proposition is above the two cusps of \( \tilde{Y} \), whereas Nori’s covers are unramified over \( \mathbb{A}^1 \).

1. Drinfeld modular schemes

In this section we review the main definitions and properties from the theory of moduli spaces of Drinfeld modules, cf. [13]. Let us note from the start that we need only naive étale level structures, as we don’t bother about bad fibers of the moduli spaces we will consider.

1.1 Notations. Let \( \mathbb{F}_q \) be a finite field with \( q \) elements (\( q \) a power of an odd prime), \( A = \mathbb{F}_q[T] \) the polynomial ring over \( \mathbb{F}_q \), \( K = \mathbb{F}_q(T) \) the field of rational functions, \( K_\infty = \mathbb{F}_q((T^{-1})) \) its completion for the valuation \( |a| = q^\text{deg}(a) \) for \( a \in A \) and \( C \) the completion of an algebraic closure of \( K_\infty \).

1.2 Drinfeld modules. For any field \( L \) containing \( A \), a Drinfeld module \( \phi \) over \( L \) is a non-trivial representation \( \phi \) of \( A \) in the endomorphism ring of the additive group scheme \( \mathbb{G}_{a/L} \) that induces the identity on the tangent space at zero. Said otherwise, it is a ring morphism
\[
\phi : A \to L \{ \tau \} : a \to \phi_a
\]
from $A$ to the twisted polynomial ring $L\{\tau\}$, where $\tau$ is the $q$-Frobenius element, i.e., satisfies $\tau x = x^{q^r}$ for all $x \in L$. Furthermore, $\phi$ satisfies $\phi_a(0) = a$ and $\phi(A) \not\subseteq L$. It turns out that $\deg_r(\phi_a) = \deg(a) \cdot r$ for some constant integer $r$, which is called the rank of $\phi$. A morphism between two Drinfeld modules $\phi$ and $\psi$ over $L$ is an element $u \in \text{End}(G_{a/L})$ such that $u\phi = \psi u$.

A Drinfeld module $\phi$ induces a new $A$-module structure on $G_{a/L}$ given by $a \cdot x = \phi_a(x)$ for $a \in A, x \in L$, so it makes sense to speak of the $n$-torsion $\phi[n]$ (as a subgroup scheme of $G_{a/L}$) of $\phi$ for any $n \in A$. It turns out that $\phi[n] \cong (n^{-1}A/A)^r$. The $n$-torsion comes naturally with an action of the Galois group of the separable closure $\bar{K}$ of $K$, so that we get an (equicharacteristic) Galois representation

$$\rho_{\phi,n,K} : G_K \to \text{Aut} \phi[n] = GL(r, A/n),$$

where $G_K$ is the absolute (separable) Galois group of $K$.

1.3 Moduli spaces with étale level structures. It makes sense to speak of a Drinfeld module over an $A$-scheme $S$: it consists of a pair consisting of a line bundle $\mathcal{L}$ on $S$ and a map $\phi$ from $A$ to the endomorphisms of $\mathcal{L}$ as an $S$-group scheme such that the differential of $\phi(a)$ equals $a$ for all $a \in A$ and such that for any field $k$ with a morphism $\text{Spec}(k) \to S$, the base change of $\phi$ to $k$ is an ordinary Drinfeld module. Let $n \in A$. An étale level $n$ structure on $(\mathcal{L}, \phi)$ is an isomorphism of group schemes

$$\lambda : (n^{-1}A/A)^r \times_S T \to \phi[n] \times_S T$$

(for some finite étale base change $T \to S$) commuting with the $A$-action. The functor $\text{Sch}/A \to \text{Sets}$ that associates to an $A$-scheme $S$ the set of $S$-Drinfeld modules of rank $r$ with étale level $n$-structure up to isomorphism is representable by a scheme $M^r(n)$ of finite type over $A$ (here, isomorphism is given locally by an isomorphism of Drinfeld modules compatible with level structures). Seen over $C$, $M^r(n)$ is the disjoint union of irreducible components parametrized by $(A/n)^*/F_q^*$, and these irreducible components are isomorphic over $C$ to the rigid analytic space $\Gamma(n)/\Omega^r$, where $\Omega^r$ is Drinfeld’s $r$-dimensional rigid analytic space and $\Gamma(n)$ is the principal congruence group defined by $\Gamma(n) = \{ \gamma \in GL(r, A) : \gamma = 1 \mod n \}$.

1.4 Rank one Drinfeld modules and explicit class field theory. For $r = 1$, the scheme $M^1(n)$ is the spectrum of the integral closure of $A[n^{-1}]$ in the $n$-th ray class field $K_{+}(n)$ of $K$ (i.e., $K_{+}(n)$ is the fixed field of the maximal abelian extension of $K$ that is totally split at $\infty$ by the $1$-units modulo $n$). The natural action of $(A/n)^*/F_q^*$ on $M^1(n)$ is given by class field theory. This means the following: let $K(n)$ be the field extension of $K$ given by adjoining the $n$-torsion of any rank one Drinfeld module $\psi$ over $K$ to $K$; then $K_{+}(n)$ is its maximal “real” subfield (this does not depend on the choice of $\psi$). Then there is an explicit
isomorphism
\[(A/n)^* \to \text{Gal}(K(n)/K)\]
\[a \mapsto [\alpha \in \psi[n] \mapsto \psi_a(\alpha)]\]
such that \(\psi_a(\alpha) = \alpha^{q^{\deg(a)}} \mod n\) (i.e., \(a \mapsto \text{Frob}_a\)).

As before, the scheme \(M^1(n)\) splits over \(K_+(n)\) in a number of absolutely irreducible copies of \(\text{Spec} K_+(n)\) parametrized by \((A/n)^*/\mathbb{F}_q^*\). Note that if we denote by \(Y^r(n)\) a fixed irreducible component of \(M^r(n)\) over \(K_+(n)\), \(Y^r(n)\) is defined over \(K_+(n)\), absolutely irreducible, and isomorphic to \(\Gamma(n)\backslash \Omega^r\) over \(C\) (this also follows from the theory of modular forms, in particular, calculating the field of definition of the Fourier coefficients of Eisenstein series of weight one).

We prove a lemma about rank one Drinfeld modules which will be used later on. Let \(C\) be the Carlitz module given by \(C_T = T + \tau\).

1.5 Lemma. For any rank one module \(\psi\), its \(n\)-torsion \(\psi[n]\) is isomorphic as \(G_K\)-module to \(C[n]\), where \(C\) is the Carlitz module.

Proof. There exists an isomorphism \(v \in \hat{K}\) with \(v\psi v^{-1} = C\), and this maps a \(n\)-torsion point \(x \in \psi[n]\) to \(vt \in C[n]\). Now the \(G_K\)-action factors through \(K(n)\), and the action of \(a \in (A/n)^* = \text{Gal}(K(n)/K)\) is given by \(t \mapsto \psi_a(t)\) and \(t \mapsto C_a(t)\) respectively. From this, \(G_K\)-equivariance follows immediately. \(\square\)

1.6 \(T\)-motives (Anderson, [3]). Fix an \(A\)-Drinfeld module \(\phi\). The skew \(L\)-algebra \(L\{\tau\}\) becomes a \(B = L \otimes_{\mathbb{F}_q} A\)-algebra, denoted \(M_\phi\), if we let
\[\kappa \otimes a \cdot m = \kappa \cdot m \circ \phi_a \text{ and } m(\kappa \otimes a) = \kappa^a \cdot m \circ \phi_a,\]
for every \(m \in M_\phi\), \(a \in A\) and \(\kappa \in L\). We note that \(B\) is a polynomial ring in one variable over \(L : B = L[Y]\) where \(Y = 1 \otimes T\); and \(M_\phi\) is a free \(B\)-module of rank \(r\) with \(\{1, \tau, \tau^2, \ldots, \tau^{r-1}\}\) as a basis. We call \(M_\phi\) the \(T\)-motive associated to \(\phi\). The category of \(T\)-motives admits tensor products, so it makes sense to speak of the highest \((r\text{-fold})\) exterior power (=determinant) \(\wedge^r M_\phi\) of \(M_\phi\) as a \(B\)-module. It is a free rank one \(B\)-module with as basis \(\{1 \wedge \tau \wedge \cdots \wedge \tau^{r-1}\}\); so it is again a \(T\)-motive, induced by a rank one Drinfeld module which we denote by \(\wedge \phi\). Actually, one can calculate that \(\wedge \phi_T = T + (-1)^{r-1} a \sigma\) if \(\phi\) has rank \(r\) and \(a\) is the leading term of \(\phi_T\). (cf. [7]). The determinant is compatible with principal level structures in the sense that \((\wedge \phi)[n] \cong \wedge (\phi[n])\) as \(A[G_K]\)-modules. It implies in particular the following

1.7 Lemma. Let \(\rho_{\phi,n,K} : G_K \to GL(r, A/n)\) be the Galois representation associated to the \(n\)-torsion of a rank \(r\) Drinfeld module \(\phi\). Then \(\det \rho_{\phi,n,K} : G_K \to (A/n)^*\) is the Galois representation associated to the \(n\)-torsion of the rank one Drinfeld module \(\wedge \phi\); in particular, the action is given by class field theory. \(\square\)

2. The scheme \(Y_0^r(n)\) and its twist
We will now introduce the higher dimensional moduli space $Y^r_0(n)$ and describe an Atkin-Lehner style involution that acts on it.

2.1 The scheme $Y^r_0(n)$. We consider the moduli functor over Sch/$A$ associated to the moduli problem of pairs $(\phi, F)$ over an $A$-scheme $S$, where $\phi$ is a rank $r$ Drinfeld module over $S$ and $F$ is a full flag of $S$-subgroup schemes of the $n$-torsion of $\phi$, i.e.,

$$F = (\phi[n] = F_r \supset F_{r-1} \supset \ldots \supset F_1 \supset F_0 = 0)$$

with $F_i/F_{i-1} \times_S T \cong A^{-1}/A \times_S T, \forall i,$

where $T \to S$ is a finite étale base change. Isomorphism is given by an isomorphism of Drinfeld modules that induces isomorphisms on the different $F_i$ for all $i$. This functor is coarsely represented by a quasi-projective scheme $M^r_0(n)$ of finite type over $A$, since it is actually the quotient of $M^r(n)$ by the action of the upper triangular matrices in $GL(2, A/n)$. Since $(A/n)^r$ (as diagonal matrices) belongs to this subgroup, the components of $M^r(n)(C)$ are acted upon transitively, so that $M^r_0(n)$ is absolutely irreducible. We will denote it by $Y^r_0(n)$.

We now focus on the case $n = T$, which is the only one that will be important for us.

2.2 Lemma. $Y^r_0(T)$ is an open reduced subscheme of $\mathbb{P}^{r-1}$, in particular, $Y^r_0(T) \times L$ for any $K$-field $L$ has a rational function field.

Proof. We can restrict the functor to Sch/$K$. The scheme we have in mind in the theorem (which we call $M$) is the complement of the variety cut out by the equation $\prod a_i = 0$ in $\mathbb{P}^{r-1}$, where $a_i$ are coordinates on projective space. First of all, we have to check that there is a bijection between the $C$-points of $Y^r_0(T)$ and that scheme. This follows from the following considerations: if $(\phi, F)$ is a point of $Y^r_0(T)(C)$, then we can write

$$\phi_T = (a_1\tau + T)(a_2\tau + 1) \ldots (a_r\tau + 1)$$

for $a_i \in C^*$ such that $F_i = \ker(a_{r-i+1}\tau + \ldots + a_r\tau + 1)$. We map this point to $(a_1 : \ldots : a_r)$ in $\mathbb{P}^{r-1}$. The map is well-defined. Indeed, if $u \in C^*$ is an isomorphism $(\phi, F) \cong (\phi', F')$ (corresponding to $a'_i$), then

$$\phi'_T = u\phi_T u^{-1} = (u^{1-q}a_r\tau + T) \ldots (u^{1-q}a_1\tau + 1).$$

Then $u(F_1) = F'_1$, so that $u^{q-1}a_1 = a_1$. Next, we have $u(F_2) = F'_2$, since $C\{\tau\}$ has a right division algorithm (Ore, cf. [5]), this implies $u^{q-1}a_2 = a'_2$, etc. In the end, the vectors $(a_i)$ and $(a'_i)$ are identical up to a scalar factor $u^{q-1}$.

Secondly, if $(a_i)$ is a point of $\mathbb{P}^{r-1}$ such that $\prod a_i \neq 0$, then it clearly corresponds to a point of $Y^r_0(T)$. This is because knowing $\phi_T$ fixes the Drinfeld module (so there are no algebraic relations between the coordinates $a_i$ – this is wrong for higher level). This construction gives a section of the natural transformation
between our functor and Mor$_M$, so if we are given a natural transformation $\Psi'$ between our functor and Mor$_{M'}$ for another scheme $M'$, then we can define a map from Mor$_M$ to Mor$_{M'}$ by postcomposing this section with $\Psi'$. This shows that $M$ represents $Y^r_0(T)$. □

2.3 Remark. Picking a basis for $\phi[T]$ over $C$ allows one to represent the moduli space $Y^r_0(n) \times C$ as the quotient of Drinfelds $(r-1)$-dimensional symmetric space $\Omega^r$ by the arithmetic subgroup of $GL(r, A)$ given by matrices congruent to an upper triangular matrix modulo $n$, but we will not need this description.

2.4 The involution $w$. Assume that $r = 2s$ is even. We define the Atkin-Lehner involution $w = w_T$ on $Y^r_0(T)$ to be the map that maps the point

$$(\phi_T = (a_1 \tau + T)(a_2 \tau + 1) \ldots (a_r \tau + 1), F)$$

to its image under the isogeny

$$u = (a_{s+1} \tau + 1) \ldots (a_r \tau + 1).$$

One computes immediately that

$$u \phi_T u^{-1} = (a_{s+1} \tau + 1) \ldots (a_r \tau + 1)(a_1 \tau + T) \ldots (a_s \tau + 1)$$

$$= (a_{s+1} \tau + 1) \cdot T \cdot T^{-1} \ldots (a_r \tau + 1) \cdot T \cdot T^{-1}(a_1 \tau + T) \ldots (a_s \tau + 1)$$

$$= (a_{s+1} T^q \tau + T) \ldots (a_r T^{q-1} \tau + 1)(T^{-1} a_1 \tau + 1) \ldots (a_s \tau + 1),$$

so that

2.5 Lemma. With respect to the natural embedding $Y^r_0(T) \to \mathbf{P}^{r-1} : (\phi, F) \mapsto (a_i)$ of the previous lemma, the involution $w$ maps $(a_i)$ to

$$(T^q a_{s+1} : T^{q-1} a_{s+2} : T^{q-1} a_{s+3} : \ldots : T^{q-1} a_r : T^{-1} a_1 : a_2 : a_3 : \ldots : a_s).$$

In this way, one sees directly that $w$ is an involution and that it induces an automorphism of $Y^r_0(T)$.

2.6 The twisted moduli space $\tilde{Y}$. We now assume that $L/K$ is a quadratic field with non-trivial $K$-automorphism $\sigma$, and we look at the twisted moduli space

$$\tilde{Y} := (w \times \sigma) \backslash (Y^r_0(T) \times_{\text{Spec } K} \text{Spec } L),$$

i.e., the quotient on which $w$ acts like $\sigma$.

2.7 Lemma. $\tilde{Y}$ is a rational variety over $K$ of dimension $r - 1$.

Proof. The action of $w$ extends to $\mathbf{P}^{r-1}$ in the obvious way described by the previous lemma. We then set

$$P := (w \times \sigma) \backslash (\mathbf{P}^{r-1}_K \times_{\text{Spec } K} \text{Spec } L).$$
Since $Y_0^r(T) \times K$ is dense open in $P_K^{r-1}$, $\tilde{Y} \times K$ is dense open in $P$, an in particular, they have the same function field.

$P$ is by definition a Brauer-Severi variety (cf. [8]) with splitting field $L$. Indeed, over $L$, it is isomorphic to $w\backslash P_L^{r-1}$, which in its turn is isomorphic to $P_L^{r-1}$ (since $\langle w \rangle$ is an abelian group of order coprime to $p$ acting linearly on projective space, one can assume that it acts by characters (of order $\leq 2$) on suitable variables of the homogeneous coordinate ring $R = L[x_1, \ldots, x_r]$ of $P_L^{r-1}$, so that $R^{(w)} = L[x_1, \ldots, x_l, x_{l+1}^2, \ldots, x_{r}^2]$ for $l$ the dimension of the invariant subspace – which, by the way, equals $s$).

Hence to prove that $P$ is $K$-isomorphic to $P^{r-1}$, it suffices to find a point in $P(K)$ (Châtelet, cf. [8], 4.8). For example, the point

$$(T^{\frac{2^{2k+1}}{2}} \cdot a_{s+1} : 0 : \ldots : 0 : a_{s+1} : 0 : \ldots : 0)$$

is a $K$-rational fixed point of $w$ on $P^{r-1}$ and hence descends to a $K$-rational point of $P$. This proves that $\tilde{Y}$ also has function field $K(t_1, \ldots, t_{r-1})$. \hfill \Box

### 3. A Galois representation associated to rational points of $\tilde{Y}$.

#### 3.1 Rational points of $\tilde{Y}$. A $K$-rational point of $\tilde{Y}$ is by definition a pair $(\phi, F)$ defined over $L$ such that

$$(\phi^\sigma, F^\sigma) = (u\phi u^{-1}, u(F)).$$

In particular, $\phi$ is a Drinfeld module over $L$ which admits a $T$-isogeny (i.e., with kernel a subspace of $\phi[T](L)$) of rank $s$ to its Galois conjugate $\phi^\sigma$. Note furthermore that $w(\phi[T]) = (\ker u)^\sigma$, since both are equal to $\ker(\hat{u})$ for $\hat{u}$ the isogeny dual to $u$ (i.e., $\hat{u}u = \phi_T$).

#### 3.2 Choice of the field $L$. Fix a prime $p$ of degree $d > 1$ in $A$. From now on, we set $L = K(\sqrt{p})$. We claim that $L$ is a subfield of $K(p)$. Indeed, $K(p)$ is the field generated by the $p$-torsion of the Carlitz module, i.e., the roots of $C_p$ if $C_T = TX + X^q$. We calculate the discriminant $D$ of $C_p$ as the resultant of $C_p$ and its $X$-derivative $C_p = p$, to find $D = p^{q^d}$. Hence $L = K(\sqrt{D})$, which clearly is a subfield of $K(p)$.

#### 3.3 The associated Galois representation. Given such a pair $(\phi, F)/L$ corresponding to a rational point of $\tilde{Y}$, we construct a Galois representation

$$\rho : G_K \to PGL(r, A/p)$$

as follows. Recall that $G_L$ has index 2 in $G_K$, the quotient being generated by $\sigma$ (the generator of of $Gal(L/K)$). If $\tau \in G_L$, then we set $\rho(\tau) = \rho_{\phi,p,L}(\tau)$ modulo scalars in $GL(r, A/p)$. On the other hand, the isogeny $u$ induces an isomorphism $\phi[p] \cong \phi[p]^{\sigma}$ since $T$ and $p$ are coprime (the dual does the same in the other direction), so it makes sense to set $\rho(\sigma)(x) = \hat{u}(x^\sigma)$ as an automorphism of $\phi[p]$, i.e., an element of $GL(r, A/p)$, which we then again consider modulo scalars.
3.4 Lemma. If $\text{Aut}(\phi) = \mathbb{F}_q^*$, this $\rho$ is a well-defined group representation of $G_K$.

Proof. We have to check that $\rho(\tau_1 \tau_2) = \rho(\tau_1)\rho(\tau_2)$ for all $\tau_1, \tau_2 \in G_K$. It is clear if both $\tau_i$ belong to $G_L$.

Now assume $\tau_1 \notin G_L, \tau_2 \in G_L$, so $\tau_1 \tau_2 \notin G_L$. Then if $x \in \phi[p]$, $x^{\rho(\tau_1 \tau_2)}$ is by definition $\hat{u}(x^{\tau_1 \tau_2})$, whereas

$$x^{\rho(\tau_1)\rho(\tau_2)} = (\hat{u}(x^{\tau_1}))^{\tau_2} = \hat{u}(x^{\tau_1 \tau_2}).$$

The case $\tau_1 \in G_L, \tau_2 \notin G_L$ is similar.

Finally, if $\tau_1 \notin G_L, \tau_2 \notin G_L$, then $\tau_1 \tau_2 \in G_L$, so on the one hand, $x^{\rho(\tau_1 \tau_2)} = x^{\tau_1 \tau_2}$, and on the other hand,

$$x^{\rho(\tau_1)\rho(\tau_2)} = \hat{u}(\hat{u}(x^{\tau_1}))^{\tau_2} = \hat{u}\hat{u}^{\sigma}(x^{\tau_1 \tau_2}).$$

We claim that $\hat{u}^\sigma = \lambda u$ for some $\lambda \in \mathbb{F}_q^*$. Taking this claim for granted, we conclude (using $\hat{u}u = \phi_T$) that

$$x^{\rho(\tau_1)\rho(\tau_2)} = \lambda T x^{\tau_1 \tau_2}$$

where the multiplication with $\lambda T$ is that of a scalar matrix modulo $p$ via the canonical identification $\phi[p] = (A/p)^*$; hence after dividing out scalars, we indeed get $x^{\tau_1 \tau_2}$.

Now for the proof of the claim, we observe that $u^\sigma$ and $\hat{u}$ are two $T$-isogenies from $\phi^\sigma$ to $\phi$ defined over $L$. We have remarked in 3.1 that the two isogenies have the same kernel, so using the right division algorithm in $L\{\tau\}$, we see that they differ by a scalar element in $\lambda \in L^*$. On the other hand, we compute (using that $\sigma$ has order two and that $u$ is an isogeny) for any $a \in A$:

$$\lambda \phi_a^\sigma u = \lambda u \phi_a = \hat{u}^\sigma \phi_a = (\hat{u} \phi_a^\sigma)^\sigma = (\phi_a \hat{u})^\sigma = \phi_a^\sigma \hat{u}^\sigma = \phi_a^\sigma \lambda u.$$

Canceling $u$, this says that $\lambda$ is an endomorphism of $\phi^\sigma$, which together with the fact that it is a scalar implies $\lambda \in \text{Aut}(\phi^\sigma)$. Finally, our assumption that $\text{Aut}(\phi) = \mathbb{F}_q^*$ gives the claim.

As a preparation for the statement of the next lemma, note that if $(r; q^d - 1) = 2$, then $(A/p)*/(A/p)^r = (A/p)^*/(A/p)^{\star 2} \cong \mathbb{Z}/2 \cong \{\pm 1\}$.

3.5 Lemma. Assume that $\text{Aut}(\phi) = \mathbb{F}_q^*$, that $r$ is even and $(r; q^d - 1) = 2$. Then $\det(\rho(L)) = 1$ as an element of $(A/p)^*/(A/p)^r \cong \{\pm 1\}$, and

$$\det \rho(\sigma) = -\left[\frac{\zeta T}{p}\right],$$

where $[\zeta]_p$ denotes the quadratic character modulo $p$ and $\zeta \in \mathbb{F}_q^*$ is a constant depending only on $r$. 

9
Proof. We compute the action of Galois on the determinants of the occurring Drinfeld modules (\(\wedge \phi\) and \(\wedge \phi^\sigma\)) via 1.7. For an element \(\tau \in G_L\), \(\det \rho(\tau)\) (acting on the Carlitz module, which we can assume by lemma 1.5) equals \(\det \rho_{\phi, \eta, L}(\tau)\), which is a square in \(\text{Gal}(K(p)/K) = (A/p)^*\) since \(L\) is the subfield of \(K(p)\) of degree two over \(K\). So we only have to compute \(\det \rho(\sigma)\).

Using lemma 1.5, \(\sigma : \wedge \phi[p] \to \wedge \phi^\sigma[p]\) is equivalent to the action of \(\sigma\) on \(\mathcal{C}[p]\), which is given by class field theory as an element of order two in \((A/p)^*\) (since \(L\) lies as a quadratic subfield in \(K(p)\)).

For the action of \(\hat{u} : \wedge \phi^\sigma \to \wedge \phi\), we rather look at the induced (\(B\)-algebra) homomorphism between associated \(T\)-motives \(\hat{u} : M_{\wedge \phi^\sigma} \to M_{\wedge \phi}\). On both sides, we pick the basis \(e = 1 \wedge \ldots \tau^{r-1}\). Since the modules are one-dimensional, we know that \(\hat{u}\) acts like multiplication with an element \(1 \otimes a\) of \(B\). To determine this element, we observe that \(u\hat{u}(= \phi_T)\) acts like \(1 \otimes T\) on \(M_{\phi}\); hence it acts like \(1 \otimes T^s\) on \(\wedge \phi\). Clearly, \(\deg_T a = \deg_T u = s\), so \(a = \zeta T^s\) for some \(\zeta \in F_q^*\).

Taking these two calculations together, we see that \(\det \rho\) acts like multiplication with the product of a non-square (coming from \(\sigma\)) and \(\zeta T^s\). Since \(s\) is odd we get that modulo squares, the determinant equals \(\left[\frac{\zeta^2}{p}\right]\).

From this description, it is not yet clear that \(\zeta\) does not depend on the particular “point” \((u : \phi \to \phi^\sigma)\) of \(\hat{Y}\). This, however, can be seen as follows. Suppose that we calculate formally, starting with a Drinfeld module

\[
\phi_T = (a_1 \tau + T)(a_2 \tau + 1) \ldots (a_r \tau + 1)
\]

over the function field \(L' = L(a_i)\) of \(\hat{Y}\). The isogeny \(u\) is (on the level of motives) a \(B\)-module homomorphism between \(M_{\phi}\) and \(M_{\phi^\sigma}\), and if we express \(u\) as a matrix w.r.t. the basis \(\{1, \ldots, \tau^{r-1}\}\) on both sides, it has entries in \(B' := L' \otimes A\). Hence the same holds for its determinant, which we already know is of the form \(l \otimes T^s\) for some \(l \in L'\). We have expressed \(\det(u)\) w.r.t. the bases \(1 \wedge \ldots \wedge \tau^{r-1}\) of \(M_{\wedge \phi}\) and \(M_{\wedge \phi^\sigma}\). Now we can make both of these one-dimensional \(B'\)-modules into the Carlitz module by an isomorphism, which here is of the form \(i \otimes 1 \in B'\) as a \(B'\)-module homomorphism. Now we see that \(\det(u)\) w.r.t. these bases is given by \(li \otimes T^s\), which we know from before equals \(\zeta \otimes T^s\). Identifying both results, we see that \(\zeta\) depends as a rational function on the coordinates \(a_i\) of the variety \(\hat{Y}\). This means that the finite stratification of \(\hat{Y}\) induced by what \(\zeta\) occurs in the corresponding Galois representation is actually by subvarieties. Since there are only finitely many, at least one of them is equidimensional with \(\hat{Y}\), but \(\hat{Y}\) is absolutely irreducible, so it equals this subvariety. It follows that \(\zeta\) is constant on all of \(\hat{Y}\) (namely, what it is on that subvariety).

\[\square\]

3.6 Example. One can make the considerations in the lemma very explicit if one fixes \(r\). For example, if \(r = 2\), write

\[
\phi_T = (a_1 \tau + T)(a_2 \tau + 1) = a_1 a_2^* \tau^2 + (T a_2 + a_1) \tau + T
\]
and \( u = a_2 \tau + 1 \). W.r.t. the basis \( \{1, \tau\} \), \( u \) is the matrix

\[
u = \begin{pmatrix} 1 & a_1^{-1}(1 \otimes T - T) \\ a_2 & -T a_1^{-1} a_2 \end{pmatrix},
\]

so the determinant is \( \det(u) = -a_1^{-1} a_2 \otimes T \). One calculates that

\[
\wedge \phi_T = T - a_1 a_2^3 \tau \quad \text{and} \quad \wedge \phi_T^\sigma = \wedge u \phi_T u^{-1} = T - a_2 a_1^3 \tau,
\]

so that

the isomorphism \( C \cong \wedge \phi_T \) is multiplication by \( (a_2 \otimes \sqrt{-a_1 a_2})^{-1} \otimes 1 \)

the isomorphism \( \wedge \phi_T^\sigma \cong C \) is multiplication by \( a_1 \otimes \sqrt{-a_1 a_2} \otimes 1 \).

Finally, we get that as an automorphism of the Carlitz module, \( \det(u) = -1 \otimes T \) (so here, \( \zeta \) is not a square in \( F_q^* \)).

4. Proof of the main theorem.

The moduli functor, and hence also the moduli space \( M^r(p) \), comes with a natural action of \( GL(r, A/p) \) and the quotient of this action is \( M^r(1) \). So \( \rho_{\phi,p,K(M^r(1))} \) is surjective for \( F \) the function field of \( M^r(1) \) and \( \phi \) the “universal” Drinfeld module over \( M^r(1) \). Since \( \tilde{Y} \to M^r(1) \) is finite, the same holds if we replace \( K(M^r(1)) \) by \( K(\tilde{Y}) \).

If we replace \( K \) by \( K(\tilde{Y}) \) and \( L \) by \( L(\tilde{Y}) \), the construction of \( \S 3 \) provides us with a representation

\[
\rho : G_{K(\tilde{Y})} \to PGL(r, q^d)
\]
whose image we will now determine. The arguments in lemma 3.5 show that \( \det \rho = 1 \) if the generic \( \phi \) does not have non-trivial automorphisms and if we can choose \( p \) such that \( \left[\frac{T}{p}\right] = -1 \). That the first is true is obvious, since \( \text{Aut}(T + a_1 \tau + \ldots + a_r \tau^r) \neq F_q^* \) implies the non-generic condition that \( a_1 = 0 \). That we can accomplish the second condition comes out of the following lemma:

4.1 Lemma. For any \( d > 1 \), there exists an irreducible polynomial \( p \) of degree \( d \) in \( A \) such that \( \left[\frac{T}{p}\right] \) assumes a given value (\( \pm 1 \)).

Proof. By quadratic reciprocity in function fields, it suffices to show that \( \left[\frac{T}{p}\right] \) can attain a given value. But this depends on the fact whether or not the constant term of \( p \) is a square in \( F_q \). Hence it certainly suffices to show that for any given \( \xi \in F_q^* \), there exists an irreducible polynomial \( p \) of degree \( d \) over \( K \) whose constant term equals \( \xi \); for a proof of this, see Hansen-Mullen ([6], p. 642).

Since \( \rho_{\phi,p,K(\tilde{Y})} \) is surjective, we get that

\[
\rho : G_{K(\tilde{Y})} \to PSL(r, q^d)
\]
is surjective, and by lemma 2.7, we find that $K(\tilde{Y}) = K(t_1, \ldots, t_{r-1})$.

Finally, the fixed field of the separable closure of $K(\tilde{Y})$ under the kernel of $\rho$ gives a $PSL(r, q^d)$-extension of $K(\tilde{Y})$, which is regular, since the non-regular subextension $K_+(p)$ of $K(M^r(p))$ corresponds to the (inverse image of) the center of $GL(r, q^d)$, which intersects $\ker(\rho)$ trivially (since the determinant of $\rho$ is trivial). This finishes the proof of the theorem. □

5. Proof of the proposition — an explicit example.

5.1 Setup. Let $N$ be irreducible in $A$ of degree 2. We will use a method similar to the one in [12], §5 to make the construction explicit — in more complicated cases the method of [9] should be applied. We will assume here that the reader is familiar with the basic notions from the theory of Drinfeld modular forms, as explained for example in [4]. We make the convention that all of our modular forms and functions are normalized so the leading term of their expansion at the infinite cusp is $= 1$. The curves $X_0(T)$ and $X_0(N)$ are rational, and good Hauptmoduln for them are given by

$$f_T(z) := \frac{h(Tz)}{h(z)}$$

and

$$f_N(z) := \sqrt[{{q+1}}]{\frac{h(Nz)}{h(z)} \frac{h(Tz)}{h(Nz)}}$$

respectively, where $h$ is a Poincaré series of weight $q + 1$ for $GL(2, A)$. The curve $X_0(NT)$ is of genus $q$, but its quotient $X_+(NT)$ by the Atkin-Lehner involution $w_{NT}$ is rational. Note that for $a \in A$, $w_a$ acts like $w_a(z) = -\frac{1}{az}$ on the upper half plane. Also note that for the uniformizer $s(z) := t(z)^q = 1$ of $X(1)$ at the infinite cusp, we have

$$t(az) = \frac{t^{[a]}}{C_a(t(z)^{-1})t^{[a]}}$$

where $C$ is the Carlitz module. In particular, the order of $f(az)$ at $\infty$ is $[a]$ times the order of $f$ at $\infty$.

5.2 Lemma. A Hauptmodul for $X_+(NT)$ is given by

$$f(z) := \sqrt[{{q^2-1}}]{\frac{h(NTz)h(z)}{h(Tz)h(Nz)}}$$

Proof. Observe that the quantity underneath the root sign is clearly $w_{NT}$-invariant, and has divisor on $X_+(NT)$ supported at the cusps, of which there are exactly two. Hence we can extract a $d$-th root of this function in the function field of $X_+(NT)$ for $d$ precisely equal to its order at, say, the cusp at infinity. Now since $h = t(1 + O(s))$, the expression under the root is $s^{q^2-1} +$ higher terms, whence the result. □

5.3 Lemma. $w_T f = f^{-1}$.
Proof. The divisor of $w_T f$ is the image of the divisor of $f$ under $w_T$. As $w_T$ interchanges the two cusps of $X_+(NT)$, we get that $\text{div}(w_T f) = -\text{div}(f)$. Hence $w_T f$ differs from $f^{-1}$ by a constant, which is seen to be $= 1$ by expanding. □

5.4 Function fields of $\tilde{Y}$’s. The function field of $X_0(NT)$ is generated by $f$ and $f_T$. As $w_T f_T = \frac{1}{T^{q+1}f_T}$, the function field of $\tilde{Y}(T)$ is generated by $x$ and that of $\tilde{Y}(NT)$ by $x$ and $y$, where

$$x = \sqrt{N} \frac{T^{\frac{q+1}{2}} f_T - 1}{T^{\frac{q+1}{2}} f_T + 1} \quad \text{and} \quad y = \sqrt{N} \frac{f - 1}{f + 1}.$$ 

Note that $X(\text{sth})$ is the Galois closure of $X_0(\text{sth})$, so our theoretical result implies that for suitable $N$, the splitting field of the minimal polynomial of $y$ over $\bar{K}(x)$ is a $K$-regular Galois extension with Galois group $PSL(2,q^2)$. Since $\zeta$ is a non-square for $r = 2$ (cf. 3.6), we should choose $[\frac{T}{T}] = 1$, i.e., the constant term of $N$ should be a square modulo $T$ (by reciprocity).

To find an explicit equation, it suffices to find a relation between $f$ and $f_T$, and solve it for $x$ and $y$. Now one calculates that

$$w_{NT} f_T = \frac{1}{T^{q+1} f_T f^{q^2-1}},$$

so

$$T^{q+1} f_T f^{q^2} + \frac{f}{f_T}$$

is invariant under $w_{NT}$ and hence belongs to $K(f)$. We see in particular by looking at the order of poles and zeros that if is in fact a polynomial in $f$ of degree $q^2 + 1$. This proves the proposition. □

5.5 Explicit equations. To find this polynomial, it suffices to compare series expansions, and for this one can use the fact that

$$h = t(-\frac{1}{U_1} + s^{q-1} \frac{1}{U_1} - \frac{1}{U_1^2}) - s^{q-1} + O(s^{q^2}),$$

where $U_1 = 1 - s^{q-1} + (T^q - T) s^q$ (cf. Gekeler, [4], (10.4)). The example from the introduction was computed setting $q = 3, N = T^2 + 1$ in gp- pari and Maple.

Acknowledgments. The first author is honorary fellow of the Fund for Scientific Research - Flanders (FWO-Vlaanderen). The second author is partially supported by IKY (Greece). Part of this work was done at the MPIM. We thank Gert-Jan van der Heiden for useful comments.

References

[1] S.S. Abhyankar, Mathieu group coverings and linear group coverings, in: Recent developments in the inverse Galois problem (Seattle, WA, 1993), 293–319, Contemp. Math., 186, Amer. Math. Soc., Providence, RI, 1995.
[2] S.S. Abhyankar and P.H. Keskar, *Descent principle in modular Galois theory*, Proc. Indian Acad. Sci. Math. Sci. **111** (2001), no. 2, 139–149.

[3] G.W. Anderson, *t-Motives*, Duke Math. J. **53** (1986), 457–502.

[4] E.-U. Gekeler, *On the coefficients of Drinfeld modular forms*, Invent. Math. **93** (1988), no. 3, 667–700.

[5] D. Goss, *Basic structures of function field arithmetic*, Ergebn. Math. Grenzg. (3) vol. 35, Springer-Verlag, Berlin, 1996.

[6] T. Hansen and G.L. Mullen, *Primitive polynomials over finite fields*, Math. Comp. **59** (1992), 639–643.

[7] G.-J. van der Heiden, *Weil-pairing for Drinfeld modules of rank 2*, preprint, Universiteit Groningen, 2001.

[8] J. Jahnel, *The Brauer-Severi variety associated with a central simple algebra: a survey*, Lin. Alg. Grps. and Related Structures (electronic) **52** (2000), 1–60.

[9] G. Malle, *Polynome mit Galoisgruppen* \(\text{PGL}_2(p)\) und \(\text{PSL}_2(p)\) über \(\mathbb{Q}(t)\), Comm. Algebra **21** (1993), no. 2, 511–526.

[10] M.V. Nori, *Unramified coverings of the affine line in positive characteristic*, in: Algebraic Geometry and its applications. Collection of papers from S. Abhyankar’s birthday conference (West Lafayette, 1990) (C.L. Bajaj, ed.), 209-212, Springer-Verlag, 1994.

[11] F. Pop, *Embedding problems over large fields*, Ann. of Math. (2) **144** (1996), no. 1, 1–34.

[12] K.-y. Shih, *p-division points on certain elliptic curves*, Compos. Math. **36** (1978), 113–129.

[13] S. Vladut, *Kronecker’s Jugendtraum and modular functions*, Studies in the Development of Modern Mathematics, Vol. 2, Gordon and Breach, New York, 1991.

Department of Mathematics, Utrecht University, P.O. Box 80010, 3508 TA Utrecht, The Netherlands (gc)

Askoutsi 20, TK 74100 Crete, Greece (mt)

e-mail: cornelissen@math.uu.nl, tripolit@csd.uoc.gr

---

1All correspondence should be sent to this author at this address