Viscous dissipation in DG methods for turbulent incompressible flows

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Nowadays, (high-order) DG methods, or hybridised variants thereof, are widely used in the simulation of turbulent compressible flow problems. For turbulence simulations, and especially in the practically relevant situation of strong under-resolution, it is important to distinguish between the resolved physical dissipation rate and the contribution of numerical dissipation originating from the underlying method. In this note, a certain ambiguity related to such a decomposition for the viscous effects in a DG-discretised fluid flow problem, which is due to the discontinuity of the approximate solution, is addressed. A novel but rather natural decomposition into ‘physical’ and ‘numerical’ viscous dissipation is proposed for a class of DG methods. Based on a typical 3D benchmark problem for decaying turbulence, its meaningfulness is confirmed numerically. In order to justify the term ‘dissipation’, both the physical and the numerical contributions for the proposed additive decomposition are proved non-negative (possibly zero).

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1 Introduction

Let us consider an incompressible Navier–Stokes problem without acting outer forcing terms and periodic or no-stress boundary conditions. Given a physical domain $\Omega \subset \mathbb{R}^d$, the strong form of such a problem, equipped with a suitable initial condition $u_0: \Omega \to \mathbb{R}^d$, reads $\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0$, subject to $\nabla \cdot u = 0$. Here, $u: (0, T) \times \Omega \to \mathbb{R}^d$ indicates the velocity field, $p: (0, T) \times \Omega \to \mathbb{R}$ is the (zero-mean) kinematic pressure, and the underlying fluid is assumed to be Newtonian with kinematic viscosity $0 < \nu \ll 1$. We are especially interested in the situation where the corresponding Reynolds number is large enough such that a turbulent flow is expected and its approximation is performed in a strongly under-resolved setting.

For a finite element pair $V_h/Q_h$ for velocity/pressure, and assuming that the simulation is performed up to the time instance $T > 0$, a typical DG scheme (in primal form) for discretising the Navier–Stokes equations reads as follows:

$$\begin{align*}
\text{Find } (u_h, p_h): (0, T] \to V_h \times Q_h \text{ with } u_h(0) &= u_{0h} \text{ s.t., } (v_h, q_h) \in V_h \times Q_h, \\
(\partial_t u_h, v_h) + \nu a_h(u_h, v_h) + c_h(u_h; v_h, p_h) + j_h(u_h, v_h)_h = 0. 
\end{align*}$$

The form $a_h$ treats the viscosity effects, $c_h$ is the nonlinear convection term, $b_h$ connects pressure and incompressibility condition and $j_h$ is a possible additional stabilisation and/or turbulence model [4, 6]. In this note we focus on the viscous term $a_h$. Testing (1) symmetrically with $(v_h, q_h) = (u_h, p_h)$ leads to the discrete kinetic energy balance

$$-\partial_t K(u_h) = -\frac{1}{2} \frac{d}{dt} \|u_h\|_{L^2(\Omega)}^2 = \nu a_h(u_h, u_h) + c_h(u_h; u_h, u_h) + j_h(u_h, u_h).$$

The only physical dissipation present in the original Navier–Stokes model is due to viscosity. Therefore, in our opinion, every additional energy-dissipating (or even energy-producing) mechanism, which is frequently incorporated in $c_h$ and $j_h$, has to be characterised as an artificial (numerical) contribution.

To introduce a mathematically rigorous notion of viscous dissipation processes, let $a_h^{\text{phy}}$ denote the non-negative part in the discretisation of the viscous term that represents physical dissipation, which is supposed to fulfil $a_h^{\text{phy}}(u, u) = \|\nabla u\|_{L^2}^2$ for the exact solution $u$. We assume that the remainder of $a_h$ is a non-negative bilinear form $a_h^{\text{num}}$, which describes numerical dissipation in the discretisation of the viscous term. The decomposition is required to be consistent in the sense that $a_h^{\text{num}}(u_h, u_h)$ vanishes for $h/k \to 0$, with $u_h$ being a discrete solution converging to $u$ as $h/k \to 0$. Here, $h$ denotes the underlying mesh size and $k$ is the polynomial order of discrete velocities belonging to $V_h$.

Let us emphasise that the requirement that both parts of the decomposition be non-negative is a restriction and disallows some choices for $a_h^{\text{phy}}$ which may seem intuitive at first glance; cf. [5, Sec. 2]. Being able to identify the physical dissipation, the total numerical dissipation $\varepsilon_h^{\text{tot}}$ of the scheme can be defined as $\varepsilon_h^{\text{tot}} := -\partial_t K(u_h) - \nu a_h^{\text{phy}}(u_h, u_h)$. This $\varepsilon_h^{\text{tot}}$ then fulfils the reasonable and widely accepted expectation (for a meaningful discretisation) that it is non-negative; that is, $\varepsilon_h^{\text{tot}} \geq 0$.

Especially, the frequently used characterisation $a_h^{\text{phy}}(u_h, u_h) = \|\nabla u_h\|_{L^2}^2$ [3] can be misleading in DG methods as will be shown below. However, let us mention that the difference between different notions of physical dissipation in DG methods is only relevant in the under-resolved case. We will demonstrate that a lifting technique can be used to define a suitable/natural decomposition of the total viscous dissipation into a physical and a numerical contribution. In doing so, we restrict ourselves to the symmetric interior penalty (SIP) method as a very frequently used DG method.

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2 A natural decomposition of viscous dissipation for DG methods

The SIP bilinear form is given by [4]
\[ a_h(u_h, v_h) := \int_{\Omega} \nabla_h u_h : \nabla_h v_h \, dx - \sum_{F \in F_h} \frac{\lambda}{h_F} \left[ \nabla_h u_h \right]_{n_F} \cdot \left[ v_h \right]_{n_F} + \left[ u_h \right]_{n_F} \cdot \left[ \nabla_h v_h \right]_{n_F} - \frac{\lambda}{h_F} \int_{F} \left[ u_h \right]_{n_F} \cdot \left[ v_h \right]_{n_F} \, ds, \tag{3} \]

where \( \lambda > 0 \) is a sufficiently large (due to a discrete inverse inequality) penalty parameter.

We can interpret the DG formulation (3) in a mixed setting, cf. [1], which gives a natural definition of a discrete diffusive flux (scaled with \( \nu^{-1} \)) \( \sigma_h = \sigma_h(u_h) \), which is defined element-wise for all \( \tau_h \in \nabla_h V_h|_K \) by the following operation on any \( K \in T_h \), cf. [1, eqn. (1.2)]:
\[ \int_{K} \sigma_h : \tau_h \, dx = -\int_{\partial K} \widehat{u}_h \cdot (\nu \cdot \nabla_h \tau_h) \, ds + \int_{\partial K} (\widehat{u}_h - u_h) \cdot (\tau_h n_K) \, ds. \tag{4} \]

The second equality is due to integration by parts and \( \widehat{u}_h \) denotes a ‘numerical trace’ which characterises different DG methods, see [1, Table 3.1]. In the following, for simplicity, we only want to consider the SIP method where \( \widehat{u}_h = \{ u_h \} \).

For SIP, \( \widehat{u}_h - u_h = \{ u_h \} - u_h = -1/2 \|[ u_h ]| \) and thus we define the lifting operator \( L : V_h|_{\partial K} \rightarrow \nabla_h V_h|_K \) by
\[ \int_{\partial K} (\widehat{u}_h - u_h) \cdot (\tau_h n_K) \, ds = -\int_{\partial K} \|[ u_h ]\| \cdot \frac{1}{2} (\tau_h n_K) \, ds = -\int_{K} L([u_h]) : \tau_h \, dx, \quad \forall K \in T_h. \tag{5} \]

With the notion (5) of the lifting operator \( L \), (4) can finally be used to obtain the characterisation \( \sigma_h(u_h) = \nabla_h u_h - L([u_h]) \).

Using this definition of \( \sigma_h \), one can rewrite the symmetrically tested bilinear form \( a_h \) from (3) as follows:
\[ a_h(u_h, v_h) = \int_{\Omega} \| \sigma_h \|^2 \, dx + \sum_{F \in F_h} \frac{\lambda}{h_F} \int_{F} \|[ u_h ]\|^2 \, ds - \int_{\Omega} \| L([u_h]) \|^2 \, dx = a_{\text{phy}}(u_h, v_h) + a_{\text{num}}(u_h, v_h). \tag{6} \]

We notice that the usual assumption on the parameter \( \lambda \) guarantees that both parts \( a_{\text{phy}}(u_h, v_h) \) and \( a_{\text{num}}(u_h, v_h) \) are non-negative for any discrete function \( u_h \in V_h \); a detailed explanation for this statement can be found in [5, App. B]. Further, note that \( a_{\text{num}}(u_h, v_h) \to 0 \) as \( h/k \to 0 \).

As shown in detail in [5, App. A] the bilinear form \( a_{\text{phy}} \) in (6) corresponds to the DG method by Bassi and Rebay [2]. It can be seen as a central flux approximation to diffusion/viscosity (to the corresponding first order system). Moreover, note that for SIP a piecewise constant function will not induce physical dissipation if exclusively the broken gradient is used for the definition of \( a_{\text{phy}} \). In contrast, using the definition (6) of \( a_{\text{phy}} \) proposed here, also piecewise constant functions induce physical dissipation.

In Fig. 1 the difference between two different interpretations of physical and numerical dissipation is demonstrated for a highly under-resolved homogeneous decaying turbulence example. We observe that the broken gradient is not a good metric for characterising physical dissipation in that example as it – misleadingly – suggests that the numerical method contributes with negative numerical dissipation. Using \( \sigma_h \) as in (4), on the other hand, leads to a physically meaningful interpretation.

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