AMENABLE ACTIONS, INVARIANT MEANS AND BOUNDED COHOMOLOGY

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ABSTRACT. We show that topological amenability of an action of a countable discrete group on a compact space is equivalent to the existence of an invariant mean for the action. We prove also that this is equivalent to vanishing of bounded cohomology for a class of Banach G-modules associated to the action, as well as to vanishing of a specific cohomology class. In the case when the compact space is a point our result reduces to a classic theorem of B.E. Johnson characterising amenability of groups. In the case when the compact space is the Stone-Čech compactification of the group we obtain a cohomological characterisation of exactness for the group, answering a question of Higson.

1. INTRODUCTION

An invariant mean on a countable discrete group G is a positive linear functional on $\ell^\infty(G)$ which is normalised by the requirement that it pairs with the constant function 1 to give 1, and which is fixed by the natural action of G on the space $\ell^\infty(G)^*$. A group is said to be amenable if it admits an invariant mean. The notion of an amenable action of a group on a topological space, studied by Anantharaman-Delaroche and Renault [1], generalises the concept of amenability, and arises naturally in many areas of mathematics. For example, a group acts amenably on a point if and only if it is amenable, while every hyperbolic group acts amenably on its Gromov boundary.

In this paper we introduce the notion of an invariant mean for a topological action and prove that the existence of such a mean characterises amenability of the action. Moreover, we use the existence of the mean to prove vanishing of bounded cohomology of G with coefficients in a suitable class of Banach G modules, and conversely we prove that vanishing of these cohomology groups characterises amenability of the action. This generalises the results of Johnson [6] on bounded cohomology for amenable groups.

Another generalisation of amenability, this time for metric spaces, was given by Yu [10] with the definition of property A. Higson and Roe [7] proved a remarkable result that unifies the two approaches: A finitely generated discrete group G (regarded as a metric space) has Yu’s property A if and only if the action of G on its Stone-Čech compactification $\beta G$ is topologically amenable, and this is true if and only if G acts amenably on any compact space. Ozawa proved [9] that such groups are exact, and indeed property A and exactness are equivalent for countable discrete groups equipped with a proper left-invariant metric.

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To generalise the concept of invariant mean to the context of a topological action, we introduce a Banach $G$-module $W_0(G, X)$ which is an analogue of $\ell^1(G)$, encoding both the group and the space on which it acts. Taking the dual and double dual of this space we obtain analogues of $\ell^\infty(G)$ and $\ell^\infty(G)^*$. A mean for the action is an element $\mu \in W_0(G, X)^*$ satisfying the normalisation condition $\mu(\pi) = 1$, where the element $\pi$ is a summation operator, corresponding to the pairing of $\ell^1(G)$ with the constant function 1 in $\ell^\infty(G)$. A mean $\mu$ is said to be invariant if $\mu(g \cdot \varphi) = \mu(\varphi)$ for every $\varphi \in W_0(G, X)$, (Definition 13).

With these notions in place we give the following very natural characterisation of amenable actions.

**Theorem A.** Let $G$ be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space $X$. The action is amenable if and only if there exists an invariant mean for the action.

We then turn to the question of a cohomological characterisation of amenable actions. Given an action of a countable discrete group $G$ on a compact space $X$ by homeomorphisms we introduce a submodule $N_0(G, X)$ of $W_0(G, X)$ associated to the action and which is analogous to the submodule $\ell^1_0(G)$ of $\ell^1(G)$ consisting of all functions of sum 0. Indeed when $X$ is a point these modules coincide. We also define a cohomology class $[J]$, called the Johnson class of the action, which lives in the first bounded cohomology group of $G$ with coefficients in the module $N_0(G, X)^*$. We have the following theorem.

**Theorem B.** Let $G$ be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space $X$. Then the following are equivalent

1. The action of $G$ on $X$ is topologically amenable.
2. The class $[J] \in H^1_b(G, N_0(G, X)^*)$ is trivial.
3. $H^p_b(G, E^*) = 0$ for $p \geq 1$ and every $\ell^1$-geometric $G$-$C(X)$ module $E$.

The definition of $\ell^1$-geometric $G$-$C(X)$ module is given in Section ???. When $X$ is a point our theorem reduces to Johnson’s celebrated characterisation of amenability [6]. As a corollary we also obtain a cohomological characterisation of exactness for discrete groups, which answers a question of Higson, and which follows from our main result when $X$ is the Stone-Čech compactification $\beta G$ of the group $G$. In this case, $C(\beta G)$ can be identified with $\ell^\infty(G)$, and we obtain the following.

**Corollary.** Let $G$ be a countable discrete group. Then the following are equivalent.

1. The group $G$ is exact;
2. The Johnson class $[J] \in H^1_b(G, N_0(G, \beta G)^*)$ is trivial;
3. $H^p_b(G, E^*) = 0$ for $p \geq 1$ and every $\ell^1$-geometric $G$-$\ell^\infty(G)$-module $E$. 
This paper builds on the cohomological characterisation of property A developed in [3] and on the study of cohomological properties of exactness in [5].

2. Geometric Banach modules

Let $C(X)$ denote the space of real-valued continuous functions on $X$. For a function $f : G \to C(X)$ we shall denote by $f_g$ the continuous function on $X$ obtained by evaluating $f$ at $g \in G$. We define the sup $-\ell^1$ norm of $f$ to be

$$\|f\|_{\infty,1} = \sup_{x \in X} \sum_{g \in G} |f_g(x)|,$$

and denote by $V$ the Banach space of all functions on $G$ with values in $C(X)$ that have finite norm. We introduce a Banach $G$-module associated to the action.

**Definition 1.** Let $W_{00}(G, X)$ be the subspace of $V$ consisting of all functions $f : G \to C(X)$ which have finite support and such that for some $c \in \mathbb{R}$, depending on $f$, $\sum_{g \in G} f_g = c1_X$, where $1_X$ denotes the constant function 1 on $X$. The closure of this space in the sup $-\ell^1$-norm will be denoted $W_0(G, X)$.

Let $\pi : W_{00}(G, X) \to \mathbb{R}$ be defined by $\sum_{g \in G} f_g = \pi(f)1_X$. The map $\pi$ is continuous with respect to the sup $-\ell^1$ norm and so extends to the closure $W_0(G, X)$; we denote its kernel by $N_0(G, X)$.

In the case of $X = \beta G$ and $C(\beta G) = \ell^\infty(G)$ the space $W_0(G, \beta G)$ was introduced in [5]. For every $g \in G$ we define the function $\delta_g \in W_{00}(G, X)$ by $\delta_g(h) = 1_X$ when $g = h$, and zero otherwise.

The $G$-action on $X$ gives an isometric action of $G$ on $C(X)$ in the usual way: for $g \in G$ and $f \in C(X)$, we have $(gf)(x) = f(g^{-1}x)$. The group $G$ also acts isometrically on the space $V$ in a natural way: for $g, h \in G, f \in V, x \in X$, we have $(gf)(x) = f_{g^{-1}h}(g^{-1}x) = (g \cdot f_{g^{-1}h})(x)$.

Since the summation map $\pi$ is $G$-equivariant (we assume that the action of $G$ on $\mathbb{R}$ is trivial) the action of $G$ restricts to $W_{00}(G, X)$ and so by continuity it restricts to $W_0(G, X)$. We obtain a short exact sequence of $G$-vector spaces:

$$0 \to N_0(G, X) \to W_0(G, X) \xrightarrow{\pi} \mathbb{R} \to 0.$$

**Definition 2.** Let $E$ be a Banach space. We say that $E$ is a $C(X)$-module if it is equipped with a contractive unital representation of the Banach algebra $C(X)$.

If $X$ is a $G$-space then a $C(X)$-module $E$ is said to be a $G$-$C(X)$-module if the group $G$ acts on $E$ by isometries and the representation of $C(X)$ is $G$-equivariant.

Note that the fact that we will only ever consider unital representations of $C(X)$ means that there is no confusion between multiplying by a scalar or by the corresponding constant function. For instance, for $f \in W_0(G, X)$ multiplication by $\pi(f)$ agrees with multiplication by $\pi(f)1_X$. 
Example 3. The space $V$ is a $G$-$C(X)$-module. Indeed, for every $f \in V$ and $t \in C(X)$ we define $tf \in V$ by $(tf)_{g}(x) = t(x)f_{g}(x)$, for all $g \in G$. This action is well-defined as $\|tf\|_{\infty,1} \leq \|t\|_{\infty}\|f\|_{\infty,1}$; this also implies that the representation of $C(X)$ on $V$ is contractive. As remarked above, the group $G$ acts isometrically on $V$. The representation of $C(X)$ is clearly unital and also equivariant, since for every $g \in G$, $f \in V$ and $t \in C(X)$

$$(g(tf))_{h}(x) = (tf)_{g^{-1}h}(g^{-1}x) = t(g^{-1}x)f_{g^{-1}h}(g^{-1}x) = (g \cdot t)(x)(gf)_{h}(x)$$

Thus we have $g(tf) = (g \cdot t)(gf)$.

The equivariance of the summation map $\pi$ implies that both $W_{0}(G, X)$ and $N_{0}(G, X)$ are $G$-invariant subspaces of $V$. Note however, that $W_{0}(G, X)$ is not invariant under the action of $C(X)$ defined above, as for $f \in W_{0}(G, X)$ and $t \in C(X)$ we have

$$\sum_{g \in G} (tf)_{g}(x) = \sum_{g \in G} t(x)f_{g}(x) = t(x)\sum_{g \in G} f_{g}(x) = ct(x).$$

However, the same calculation shows that the subspace $N_{00}(G, X)$ is invariant under the action of $C(X)$, and so is a $G$-$C(X)$-module, and hence so is its closure $\overline{N_{00}(G, X)}$.

Let $E$ be a $G$-$C(X)$-module, let $E^{*}$ be the Banach dual of $E$ and let $\langle -, - \rangle$ be the pairing between the two spaces. The induced actions of $G$ and $C(X)$ on $E^{*}$ are defined as follows. For $\alpha \in E^{*}$, $g \in G$, $f \in C(X)$, and $v \in E$ we let

$$\langle g\alpha, v \rangle = \langle \alpha, g^{-1}v \rangle, \quad \langle f\alpha, v \rangle = \langle \alpha, fv \rangle.$$

Note that the action of $C(X)$ is well-defined since $C(X)$ is commutative. It is easy to check the following.

Lemma 4. If $E$ is a $G$-$C(X)$-module, then so is $E^{*}$.

We will now introduce a geometric condition on Banach modules which will play the role of an orthogonality condition. To motivate the definition that follows, let us note that if $f_{1}$ and $f_{2}$ are functions with disjoint supports on a space $X$ then (assuming that the relevant norms are finite) the sup-norm satisfies the identity $\|f_{1} + f_{2}\|_{\infty} = \sup(\|f_{1}\|_{\infty}, \|f_{2}\|_{\infty})$, while for the $\ell^{1}$-norm we have $\|f_{1} + f_{2}\|_{\ell^{1}} = \|f_{1}\|_{\ell^{1}} + \|f_{2}\|_{\ell^{1}}$.

Definition 5. Let $E$ be a Banach space and a $C(X)$-module. We say that $v_{1}$ and $v_{2}$ in $E$ are disjointly supported if there exist $f_{1}, f_{2} \in C(X)$ with disjoint supports such that $f_{1}v_{1} = v_{1}$ and $f_{2}v_{2} = v_{2}$.

We say that the module $E$ is \textit{$\ell^{\infty}$-geometric} if, whenever $v_{1}$ and $v_{2}$ have disjoint supports, $\|v_{1} + v_{2}\| = \sup(\|v_{1}\|, \|v_{2}\|)$.

We say that the module $E$ is \textit{$\ell^{1}$-geometric} if for every two disjointly supported $v_{1}$ and $v_{2}$ in $E$ $\|v_{1} + v_{2}\| = \|v_{1}\| + \|v_{2}\|$.

If $v_{1}$ and $v_{2}$ are disjointly supported elements of $E$ and $f_{1}$ and $f_{2}$ are as in the definition, then $f_{1}v_{2} = f_{1}f_{2}v_{2} = 0$, and similarly $f_{2}v_{1} = 0$. 


Note also that the functions \( f_1 \) and \( f_2 \) can be chosen to be of norm one in the supremum norm on \( C(X) \). To see this, note that Tietze’s extension theorem allows one to construct continuous functions \( f_1' \), \( f_2' \) on \( X \) which are of norm one, have disjoint supports and such that \( f_1' \) takes the value \( 1 \) on \( \text{Supp } f_1 \). Then \( f_1' \phi_1 = (f_1' f_1) \phi_1 = f_1 \phi_1 = \phi_1 \). Now replace \( f_1 \) with \( f_1' \).

Finally, if \( f_1, f_2 \in C(X) \) have disjoint supports then, again by Tietze’s extension theorem, \( f_1 v_1 \) and \( f_2 v_2 \) are disjointly supported for all \( v_1, v_2 \in \mathcal{E} \).

**Lemma 6.** If \( \mathcal{E} \) is an \( \ell^1 \)-geometric module then \( \mathcal{E}^* \) is \( \ell^\infty \)-geometric.

If \( \mathcal{E} \) is an \( \ell^\infty \)-geometric module then \( \mathcal{E}^* \) is \( \ell^1 \)-geometric.

**Proof.** Let us assume that \( \phi_1, \phi_2 \in \mathcal{E}^* \) are disjointly supported and let \( f_1, f_2 \in C(X) \) be as in Definition 5 chosen to be of norm 1.

If \( \mathcal{E} \) is \( \ell^1 \)-geometric, then for every vector \( v \in \mathcal{E} \), \( \|f_1 v\| + \|f_2 v\| = \|f_1 + f_2\| v \leq \|v\| \). Furthermore,

\[
\|\phi_1 + \phi_2\| = \sup_{\|v\|=1} |\langle \phi_1 + \phi_2, v \rangle| = \sup_{\|v\|=1} |\langle f_1 \phi_1, v \rangle + \langle f_2 \phi_2, v \rangle|
\]

\[
= \sup_{\|v\|=1} |\langle \phi_1, f_1 v \rangle + \langle \phi_2, f_2 v \rangle|
\]

\[
\leq \sup_{\|v\|=1} (\|\phi_1\| \|f_1 v\| + \|\phi_2\| \|f_2 v\|)
\]

\[
\leq \sup (\|\phi_1\|, \|\phi_2\|) \sup_{\|v\|=1} (\|f_1 v\| + \|f_2 v\|)
\]

\[
\leq \sup (\|\phi_1\|, \|\phi_2\|)
\]

Since \( f_1 \phi_2 = 0 \) we have that \( \|\phi_1\| = \|f_1 (\phi_1 + \phi_2)\| \leq \|f_1\| \|\phi_1 + \phi_2\| = \|\phi_1 + \phi_2\| \).

Similarly, we have \( \|\phi_2\| \leq \|\phi_1 + \phi_2\| \), and the two estimates together ensure that \( \|\phi_1 + \phi_2\| = \sup (\|\phi_1\|, \|\phi_2\|) \) as required.

For the second statement, let us assume that \( \mathcal{E} \) is \( \ell^\infty \)-geometric and that \( \phi_1, \phi_2 \in \mathcal{E}^* \) are disjointly supported. Then

\[
\|\phi_1\| + \|\phi_2\| = \sup_{\|v_1\|, \|v_2\|=1} \langle \phi_1, v_1 \rangle + \langle \phi_2, v_2 \rangle
\]

\[
= \sup_{\|v_1\|, \|v_2\|=1} \langle \phi_1, f_1 v_1 \rangle + \langle \phi_2, f_2 v_2 \rangle
\]

\[
= \sup_{\|v_1\|, \|v_2\|=1} \langle \phi_1 + \phi_2, f_1 v_1 + f_2 v_2 \rangle
\]

\[
\leq \sup_{\|v_1\|, \|v_2\|=1} (\|\phi_1 + \phi_2\| \|f_1 v_1 + f_2 v_2\|)
\]

\[
\leq \|\phi_1 + \phi_2\| \leq \|\phi_1\| + \|\phi_2\|.
\]

where the last inequality is just the triangle inequality, so the inequalities are equalities throughout and \( \|\phi_1\| + \|\phi_2\| = \|\phi_1 + \phi_2\| \) as required.  \(\square\)
We have already established that \( N_0(G,X) \) is a \( G\)-\( C(X) \)-module. Let \( \phi^1 \) and \( \phi^2 \) be disjointly supported elements of \( N_0(G,X) \); this means that there exist disjointly supported functions \( f_1 \) and \( f_2 \) in \( C(X) \) such that \( \phi^i = f_i \phi^i \) for \( i = 1, 2 \). Then

\[
\|\phi^1 + \phi^2\|_{\infty,1} = \|f_1\phi^1 + f_2\phi^2\| = \sup_{x \in X} \sum_{g \in G} |f_1(x)\phi^1_g(x) + f_2(x)\phi^2_g(x)|
\]

We note that the two terms on the right are disjointly supported functions on \( X \) and so

\[
\|\phi^1 + \phi^2\|_{\infty,1} = \sup_{x \in X} \left( \sum_{g \in G} |f_1(x)\phi^1_g(x)| + \sum_{g \in G} |f_2(x)\phi^2_g(x)| \right) = \sup(\|\phi^1\|_{\infty,1}, \|\phi^2\|_{\infty,1}).
\]

Thus we obtain

**Lemma 7.** The module \( N_0(G,X) \) is \( \ell^\infty \)-geometric. Hence the dual \( N_0(G,X)^* \) is \( \ell^1 \)-geometric and the double dual \( N_0(G,X)^{**} \) is \( \ell^\infty \)-geometric.

We now assume that \( E \) is an \( \ell^1 \)-geometric \( C(X) \)-module, so that its dual \( E^* \) is \( \ell^\infty \)-geometric.

**Lemma 8.** Let \( f_1, f_2 \in C(X) \) be non-negative functions such that \( f_1 + f_2 \leq 1_X \). Then for every \( \phi_1, \phi_2 \in E^* \)

\[
\|f_1\phi_1 + f_2\phi_2\| \leq \sup(\|\phi_1\|, \|\phi_2\|).
\]

**Proof.** Let \( M \in \mathbb{N} \) and \( \varepsilon = 1/M \). For \( i = 1, 2 \) define \( f_{i,0} = \min(f_i, \varepsilon), f_{i,1} = \min(f_i - f_{i,0}, \varepsilon) \), \( f_{i,2} = \min(f_i - f_{i,0} - f_{i,1}, \varepsilon) \), and so on, to \( f_{i,M-1} \).

Then \( f_{i,j}(x) = 0 \) iff \( f_i(x) \leq j\varepsilon \), so \( f_{i,j} > 0 \) iff \( f_i(x) > j\varepsilon \) which implies that \( \text{Supp} f_{i,j} \subseteq f_{i,j}^{-1}([j\varepsilon, \infty)) \). So for \( j \geq 2 \), \( \text{Supp}(f_{i,j}) \subseteq f_j^{-1}([j\varepsilon, \infty)) \) and \( \text{Supp} f_{2,M+1-j} \subseteq f_2^{-1}(([M+1-j]\varepsilon, \infty)) \).

If \( x \in \text{Supp}(f_{1,j}) \cap \text{Supp}(f_{2,M+1-j}) \) then \( 1 \geq f_1(x) + f_2(x) \geq j\varepsilon + (M+1-j)\varepsilon = 1+\varepsilon \), so the two supports \( \text{Supp}(f_{1,j}) \), \( \text{Supp}(f_{2,M+1-j}) \) are disjoint.

We have that

\[
f_1 = f_{1,0} + f_{1,1} + \sum_{j=2}^{M-1} f_{1,j}
\]

\[
f_2 = f_{2,0} + f_{2,1} + \sum_{j=2}^{M-1} f_{2,M+1-j}.
\]
So using the fact that \( \| f_{1,j} \phi_1 + f_{2,M+1-j} \phi_2 \| \leq \sup(\| f_{1,j} \phi_1 \|, \| f_{2,M+1-j} \phi_2 \|) \leq \varepsilon \sup_i \| \phi_i \| \) we have the following estimate:

\[
\| f_1 \phi_1 + f_2 \phi_2 \| \leq \|(f_{1,0} + f_{1,1}) \phi_1\| + \|(f_{2,0} + f_{2,1}) \phi_2\| + \sum_{j=2}^{M} \| f_{1,j} \phi_1 + f_{2,M+1-j} \phi_2 \|
\]

\[
\leq 4\varepsilon \sup_i \| \phi_i \| + \sum_{j=2}^{M-1} \varepsilon \sup_i \| \phi_i \|
\]

\[
= (4\varepsilon + (M - 2)\varepsilon) \sup_i \| \phi_i \|
\]

\[
= (1 + 2\varepsilon) \sup_i \| \phi_i \|.
\]

Lemma 9. Let \( f_1, \ldots, f_N \in \mathcal{C}(X) \), \( f_i \geq 0 \), \( \sum_{i=1}^{N} f_i \leq 1_X \), \( \phi_1, \ldots, \phi_N \in \mathcal{E}^* \).

Then \( \| \sum_i f_i \phi_i \| \leq \sup_{1,\ldots,N} \| \phi_i \| \).

Proof. We proceed by induction. Assume that the statement is true for some \( N \). Then let \( f_0, f_1, \ldots, f_N \in \mathcal{C}(X) \), \( f_i \geq 0 \), \( \sum_{i=1}^{N} f_i \leq 1_X \), and let \( \phi_0, \phi_1, \ldots, \phi_N \in \mathcal{E}^* \).

Let \( f'_i = f_0 + f_1 \) and leave the other functions unchanged. For \( \delta > 0 \) let

\[
\phi'_{i,\delta} = \frac{1}{f'_0 + f'_1 + \delta} (f_0 \phi_0 + f_1 \phi_1).
\]

Since we clearly have

\[
\frac{f_0}{f'_0 + f'_1 + \delta} + \frac{f_1}{f'_0 + f'_1 + \delta} \leq 1_X
\]

by the previous lemma we have that \( \| \phi'_{i,\delta} \| \leq \sup \{ \| \phi_0 \|, \| \phi_1 \| \} \), and so by induction

\[
\| f'_i \phi'_{i,\delta} + f_2 \phi_2 + \cdots + f_N \phi_N \| \leq \sup \{ \| \phi'_1 \|, \| \phi_2 \|, \ldots, \| \phi_N \| \} \leq \sup_{i=0,\ldots,N} \| \phi_i \|.
\]

Consider now

\[
f'_i \phi'_{i,\delta} = \frac{(f_0 + f_1)}{f'_0 + f'_1 + \delta} (f_0 \phi_0 + f_1 \phi_1) = \frac{(f_0 + f_1) f_0}{f'_0 + f'_1 + \delta} \phi_0 + \frac{(f_0 + f_1) f_1}{f'_0 + f'_1 + \delta} \phi_1.
\]

We note that for \( i = 0, 1 \)

\[
f_i - \frac{(f_0 + f_1) f_i}{f'_0 + f'_1 + \delta} = \frac{\delta f_i}{f'_0 + f'_1 + \delta} \leq \delta
\]

and so \( \frac{(f_0 + f_1) f_i}{f'_0 + f'_1 + \delta} \) converges to \( f_i \) uniformly on \( X \), as \( \delta \to 0 \), which implies that \( f'_i \phi'_{i,\delta} \) converges to \( f_0 \phi_0 + f_1 \phi_1 \) in norm, and the lemma follows. \( \square \)
Lemma 10. If $f_1, \ldots, f_N \in C(X)$ (we do not assume that $f_i \geq 0$) are such that $\sum_{i=1}^N |f_i| \leq 1_X$ and $\phi_1, \ldots, \phi_N \in E^*$ then

$$\|\sum_{i=1}^N f_i \phi_i\| \leq 2 \sup_{i=1,\ldots,N} \|\phi_i\|.$$ 

Proof. If $f_i = f_i^+ - f_i^-$, then $|f_i| = f_i^+ + f_i^-$ and $\sum f_i^+ + \sum f_i^- \leq 1$.

Then by the previous lemma $\|\sum_{i=1}^N f_i^+ \phi_i\| \leq \sup_{i=1,\ldots,N} \|\phi_i\|$ so

$$\|\sum_{i=1}^N f_i^+ \phi_i - \sum_{i=1}^N f_i^- \phi_i\| \leq 2 \sup_{i=1,\ldots,N} \|\phi_i\|.$$ 

3. AMENABLE ACTIONS AND INVARIANT MEANS

In this section we will recall the definition of a topologically amenable action and characterise it in terms of the existence of a certain averaging operator. For our purposes the following definition, adapted from [4, Definition 4.3.1] is convenient.

Definition 11. The action of $G$ on $X$ is amenable if and only if there exists a sequence of elements $f^n \in W_00(G, X)$ such that

1. $f^n_g \geq 0$ in $C(X)$ for every $n \in \mathbb{N}$ and $g \in G$,
2. $\pi(f^n) = 1$ for every $n$,
3. for each $g \in G$ we have $\|f^n - gf^n\|_V \to 0$.

Note that when $X$ is a point the above conditions reduce to the definition of amenability of $G$. On the other hand, if $X = \beta G$, the Stone-Čech compactification of $G$ then amenability of the natural action of $G$ on $X$ is equivalent to Yu’s property A by a result of Higson and Roe [7].

Remark 12. In the above definition we may omit condition 1 at no cost, since given a sequence of functions satisfying conditions 2 and 3 we can make them positive by replacing each $f^n_g(x)$ by

$$\frac{|f^n_g(x)|}{\sum_{h \in G} |f^n_h(x)|}.$$

Conditions 1 and 2 are now clear, while condition 3 follows from standard estimates (see e.g. [5, Lemma 4.9]).
Definition 13. Let $G$ be a countable group acting on a compact space $X$ by homeomorphisms. A mean for the action is an element $\mu \in W_0(G, X)^{**}$ such that $\mu(\pi) = 1$. A mean $\mu$ is said to be invariant if $\mu(g\varphi) = \mu(\varphi)$ for every $\varphi \in W_0(G, X)^*$. 

We now state our first main result.

Theorem A. Let $G$ be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space $X$. The action is amenable if and only if there exists an invariant mean for the action.

Proof. Let $G$ act amenably on $X$ and consider the sequence $f^n$ provided by Definition 11. Each $f^n$ satisfies $\|f^n\| = 1$. We now view the functions $f^n$ as elements of the double dual $W_0(G, X)^{**}$. By the weak-* compactness of the unit ball there is a convergent subnet $f^\lambda$, and we define $\mu$ to be its weak-* limit. It is then easy to verify that $\mu$ is a mean. Since
\[
\left| \langle f^\lambda - gf^\lambda, \varphi \rangle \right| \leq \|f^\lambda - gf^\lambda\| \|\varphi\|
\]
and the right hand side tends to 0, we obtain $\mu(g\varphi) = \mu(\varphi)$.

Conversely, by Goldstine’s theorem, (see, e.g., [8, Theorem 2.6.26]) as $\mu \in W_0(G, X)^{**}$, $\mu$ is the weak-* limit of a bounded net of elements $f^\lambda \in W_0(G, X)$. We note that we can choose $f^\lambda$ in such a way that $\pi(f^\lambda) = 1$. Indeed, given $f^\lambda$ with $\pi(f^\lambda) = c_\lambda \to \mu(\pi) = 1$ we replace each $f^\lambda$ by
\[
f^\lambda + (1 - c_\lambda)\delta_e.
\]
Since $(1 - c_\lambda)\delta_e \to 0$ in norm in $W_0(G, X)$, $\mu$ is the weak-* limit of the net $f^\lambda + (1 - c_\lambda)\delta_e$ as required.

Since $\mu$ is invariant, we have that for every $g \in G$, $gf^\lambda \to g\mu = \mu$, so that $gf^\lambda - f^\lambda \to 0$ in the weak-* topology. However, for every $g \in G$, $gf^\lambda - f^\lambda \in W_0(G, X)$, and so the convergence is in fact in the weak topology on $W_0(G, X)$.

For every $\lambda$, we regard the family $(gf^\lambda - f^\lambda)_{g \in G}$ as an element of the product $\prod_{g \in G} W_0(G, X)$, noting that this sequence converges to 0 in the Tychonoff weak topology.

Now $\prod_{g \in G} W_0(G, X)$ is a Fréchet space in the Tychonoff norm topology, so by Mazur’s theorem there exists a sequence $f^n$ of convex combinations of $f^\lambda$ such that $(gf^n - f^n)_{g \in G}$ converges to zero in the Fréchet topology. Thus there exists a sequence $f^n$ of elements of $W_0(G, X)$ such that for every $g \in G$, $\|gf^n - f^n\| \to 0$ in $W_0(G, X)$.

The result then follows from Remark 12.

4. Equivariant Means on Geometric Modules

Given an invariant mean $\mu \in W_0(G, X)^{**}$ for the action of $G$ on $X$ and an $\ell^1$-geometric G-$C(X)$ module $E$, we define a G-equivariant averaging operator $\mu_E : \ell^\infty(G, E^*) \to E^*$ which we will also refer to as an equivariant mean for the action.
To do so, following an idea from [3], we introduce a linear functional \( \sigma_{\tau,v} \) on \( W_{00}(G, X) \). Given a Banach space \( \mathcal{E} \) define \( \ell^\infty(\mathcal{E}) \) to be the space of functions \( f : G \to \mathcal{E} \) such that \( \sup_{g \in G} \|f(g)\|_\mathcal{E} < \infty \). If \( G \) acts on \( \mathcal{E} \) then the action of the group \( G \) on the space \( \ell^\infty(\mathcal{E}) \) is defined in an analogous way to the action of \( G \) on \( V \), using the induced action of \( G \) on \( \mathcal{E} \):

\[
(g\tau)_h = g(\tau^{-1}h),
\]

for \( \tau \in \ell^\infty(G, \mathcal{E}) \) and \( g \in G \).

Let us assume that \( \mathcal{E} \) is an \( \ell^1 \)-geometric \( G \)-C(X) module, and let \( \tau \in \ell^\infty(G, \mathcal{E}^*) \). Choose a vector \( v \in \mathcal{E} \) and define a linear functional \( \sigma_{\tau,v} : W_{00}(G, X) \to \mathbb{R} \) by

\[
\sigma_{\tau,v}(f) = \left\langle \sum_{h \in G} f_h \tau_h, v \right\rangle
\]

for every \( f \in W_{00}(G, X) \). If we now use Lemma 10 together with the support condition required of elements of \( W_{00}(G, X) \) then we have the estimate

\[
|\sigma_{\tau,v}(f)| \leq \left\| \sum_{h} f_h \tau_h \right\|_v \leq 2\|f\|_1 \|\tau\|_v \|v\|.
\]

This estimate completes the proof of the following.

**Lemma 14.** Let \( \mathcal{E} \) be an \( \ell^1 \)-geometric \( G \)-C(X) module. For every \( \tau \in \ell^\infty(G, \mathcal{E}^*) \) and every \( v \in \mathcal{E} \) the linear functional \( \sigma_{\tau,v} \) on \( W_{00}(G, X) \) is continuous and so it extends to a continuous linear functional on \( W_0(G, X) \).

**Lemma 15.** The map \( \ell^\infty(G, \mathcal{E}^*) \times \mathcal{E} \to W_0(G, X)^* \) defined by \( (\tau, v) \mapsto \sigma_{\tau,v} \) is \( G \)-equivariant.

**Proof.**

\[
\sigma_{g\tau,gv}(f) = \left\langle \sum_{h} f_h (g^{-1} \tau_h), g^* v \right\rangle = \left\langle g \sum_{h} (g^{-1} \cdot f_h) \tau_g^{-1}h, g^* v \right\rangle = \left\langle \sum_{h} (g^{-1} \cdot f_h) \tau_g^{-1}h, v \right\rangle = \sigma_{\tau,v}(g^{-1}f) = (g\sigma_{\tau,v})(f).
\]

**Definition 16.** Let \( \mathcal{E} \) be an \( \ell^1 \)-geometric \( G \)-C(X) module, and let \( \mu \in W_0(G, X)^{**} \) be an invariant mean for the action. We define \( \mu_\mathcal{E} : \ell^\infty(\mathcal{E}) \to \mathcal{E}^* \) by

\[
\langle \mu_\mathcal{E} (\tau), v \rangle = \langle \mu, \sigma_{\tau,v} \rangle,
\]

for every \( \tau \in \ell^\infty(G, \mathcal{E}^*) \), and \( v \in \mathcal{E} \).

**Lemma 17.** Let \( \mathcal{E} \) be an \( \ell^1 \)-geometric \( G \)-C(X) module, and let \( \mu \in W_0(G, X)^{**} \) be an invariant mean for the action.

1. The map \( \mu_\mathcal{E} \) defined above is \( G \)-equivariant.
If \( \tau \in \ell^\infty(G, E^*) \) is constant then \( \mu_E(\tau) = \tau e \).

**Proof.**

\[
\langle \mu_E(g\tau), v \rangle = \mu(\sigma_{g\tau,v}) = \mu(g \cdot \sigma_{\tau,g^{-1}v}) = \mu(\sigma_{\tau,g^{-1}v})
\]

\[
= \langle \mu_E(\tau), g^{-1}v \rangle = \langle g \cdot (\mu_E(\tau)), v \rangle.
\]

If \( \tau \) is constant then

\[
\sigma_{\tau,v}(f) = \left\langle \sum_h f_h \tau_h, v \right\rangle = \left\langle \left( \sum_h f_h \right) \tau, v \right\rangle
\]

\[
= \langle (\pi(f)1_X)\tau, v \rangle = \langle \pi(f)\tau, v \rangle = \langle \tau, v \rangle \pi(f).
\]

So \( \sigma_{\tau,v} = (\tau, v)\pi \) and

\[
\langle \mu_E(\tau), v \rangle = \mu(\sigma_{\tau,v}) = \mu((\tau, v)\pi) = \langle \tau, v \rangle,
\]

hence \( \mu_E(\tau) = \tau e \). □

### 5. AMENABLE ACTIONS AND BOUNDED COHOMOLOGY

Let \( \mathcal{E} \) be a Banach space equipped with an isometric action by \( G \). Then we consider a cochain complex \( C^b_m(G, \mathcal{E}^*) \) which in degree \( m \) consists of \( G \)-equivariant bounded cochains \( \phi : G^{m+1} \to \mathcal{E}^* \) with values in the Banach dual \( \mathcal{E}^* \) of \( \mathcal{E} \) which is equipped with the natural differential \( d \) as in the homogeneous bar resolution. Bounded cohomology with coefficients in \( \mathcal{E}^* \) will be denoted by \( H_*^{b}(G, \mathcal{E}^*) \).

**Definition 18.** Let \( G \) be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space \( X \). The function

\[
J(g_0, g_1) = \delta_{g_1} - \delta_{g_0}
\]

is a bounded cochain of degree 1 with values in \( N_{00}(G, X) \), and in fact it is a bounded cocycle and so represents a class in \( H_*^{b}(G, N_{00}(G, X)^{**}) \), where we regard \( N_{00}(G, X) \) as a subspace of \( N_{0}(G, X)^{**} \). We call \( [J] \) the Johnson class of the action.

**Theorem B.** Let \( G \) be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space \( X \). Then the following are equivalent

1. The action of \( G \) on \( X \) is topologically amenable.
2. The class \([J] \in H_*^{b}(G, N_{0}(G, X)^{**})\) is trivial.
3. \( H_*^{b}(G, \mathcal{E}^*) = 0 \) for \( p \geq 1 \) and every \( \ell^1 \)-geometric \( G \)-C(X) module \( \mathcal{E} \).

**Proof.** We first show that (1) is equivalent to (2). The short exact sequence of \( G \)-modules

\[
0 \to N_{0}(G, X) \to W_{0}(G, X) \xrightarrow{\pi} \mathbb{R} \to 0
\]
leads, by taking double duals, to the short exact sequence
\[ 0 \to N_0(G, X)^{**} \to W_0(G, X)^{**} \to \mathbb{R} \to 0 \]
which in turn gives rise to a long exact sequence in bounded cohomology
\[ H^0_b(G, N_0(G, X)^{**}) \to H^1_b(G, W_0(G, X)^{**}) \to H^1_b(G, \mathbb{R}) \to H^1_b(G, N_0(G, X)^{**}) \to \ldots \]
The Johnson class \([J]\) is the image of the class \([I]\) \(\in \mathbb{H}^1_b(G, \mathbb{R})\) under the connecting homomorphism \(d : \mathbb{H}^0_b(G, \mathbb{R}) \to \mathbb{H}^1_b(G, N_0(G, X)^{**})\), and so \([J]\) = 0 if and only if \(d[I] = 0\). By exactness of the cohomology sequence, this is equivalent to \([I]\) \(\in \text{Im } \pi^*\), where \(\pi^* : \mathbb{H}^0_b(G, W_0(G, X)^{**}) \to \mathbb{H}^0_b(G, \mathbb{R})\) is the map on cohomology induced by the summation map \(\pi\). Since \(\mathbb{H}^0_b(G, W_0(G, X)^{**}) = (W_0(G, X)^{**})^G\) and \(\mathbb{H}^0_b(G, \mathbb{R}) = \mathbb{R}\) we have that \([J]\) = 0 if and only if there exists an element \(\mu \in W_0(G, X)^{**}\) such that \(\mu = g\mu\) and \(\mu(\pi) = 1\). Thus \(\mu\) is an invariant mean for the action and the equivalence with amenability of the action follows from Theorem A.

We turn to the implication (1) implies (3). Since \(G\) acts amenably on \(X\) there is, by Theorem A, an invariant mean \(\mu\) associated with the action. For every \(h \in G\) and for every equivariant bounded cochain \(\phi\) we define \(s_h\phi : G^p \to \mathcal{E}^*\) by \(s_h\phi(g_0, \ldots, g_{p-1}) = \phi(g_hg_0, \ldots, g_{p-1})\); we note that for fixed \(h\), \(s_h\phi\) is not equivariant in general. However, the map \(s_h\) does satisfy the identity \(ds_h + s_hd = 1\) for every \(h \in G\), and we will now construct an equivariant contracting homotopy, adapting an averaging procedure introduced in [3].

For \(\phi \in C^p_b(G, \mathcal{E}^*)\) let \(\tilde{\phi} : G^p \to \ell^\infty(G, \mathcal{E}^*)\) be defined by \(\tilde{\phi}(g)(h) = s_h\phi(g)\), for \(g = (g_0, \ldots, g_{p-1})\).

Note that for every \(k, h \in G\),
\[
\tilde{\phi}(k_{g_0}, \ldots, k_{g_{p-1}})(h) = \phi(h, k_{g_0}, \ldots, k_{g_{p-1}}) = k(\phi(k^{-1}h, g_0, \ldots, g_{p-1}))
\]
\[
= k(\tilde{\phi}(g_0, \ldots, g_{p-1})(k^{-1}h))
\]
\[
= (k(\tilde{\phi}(g_0, \ldots, g_{p-1}))(h)
\]
so \(\tilde{\phi}(kg) = k(\tilde{\phi}(g))\).

We can now define a map \(s : C^p(G, \mathcal{E}^*) \to C^{p-1}(G, \mathcal{E}^*)\):
\[ s\phi(g) = \mu_{\mathcal{E}}(\tilde{\phi}(g)), \]
where \(\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \to \mathcal{E}^*\) is the map defined in Lemma [17] using the invariant mean \(\mu\). Note that \(\|\mu_{\mathcal{E}}\| \leq 2\|\mu\|\), and \(\|\tilde{\phi}(g)\| \leq \sup\{\|\phi(k)\| : k \in G^{p+1}\}\). Hence \(s\phi\) is bounded.

For every cochain \(\phi\), \(k(s\phi) = s(k\phi) = s\phi\) since \(\tilde{\phi}\) and \(\mu_{\mathcal{E}}\) are equivariant.

The map \(s\phi\) provides a contracting homotopy for the complex \(C^*_b(G, \mathcal{E}^*)\) which can be seen as follows. As \(\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \to \mathcal{E}^*\) is a linear operator it follows that for a given \(\phi \in C^p_b(G, \mathcal{E}^*)\), and a \(p + 1\)-tuple of arguments \(k = (k_0, \ldots, k_p)\), \(ds\phi\) is obtained by applying the mean \(\mu_{\mathcal{E}}\) to the map \(g \mapsto ds\phi(k)\), while \(sd\phi\) is obtained by applying \(\mu_{\mathcal{E}}\) to the function \(g \mapsto s_d\phi(k)\). Thus
\[
(s + ds)\phi(k) = \mu_{\mathcal{E}}(g \mapsto (ds\phi + s_d\phi)(k)).
\]
Given that $ds_g + s_g d = 1$ for every $g \in G$, for every $g \in G^{p+1}$ the function $g \mapsto (ds_g + s_g d)\phi(k) = \phi(k) \in \mathcal{E}$ is constant, and so by Lemma 17

$$(sd + ds)\phi(k) = (ds_e + s_e d)\phi(k) = \phi(k).$$

Thus $sd + ds = 1$, as required.

Collecting these results together, we have proved that (1) implies (3).

The fact that (3) implies (2), follows from the fact that $N_0(G, X)^*$ is an $\ell^1$-geometric $G\cdot C(X)$-module, proved in Lemma 7.

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AMENABLE ACTIONS, INVARIANT MEANS AND BOUNDED COHOMOLOGY

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ABSTRACT. We show that topological amenability of an action of a countable discrete group on a compact space is equivalent to the existence of an invariant mean for the action. We prove also that this is equivalent to vanishing of bounded cohomology for a class of Banach G-modules associated to the action, as well as to vanishing of a specific cohomology class. In the case when the compact space is a point our result reduces to a classic theorem of B.E. Johnson characterising amenability of groups. In the case when the compact space is the Stone-Čech compactification of the group we obtain a cohomological characterisation of exactness for the group, answering a question of Higson.

1. INTRODUCTION

An invariant mean on a countable discrete group $G$ is a positive linear functional on $\ell^\infty(G)$ which is normalised by the requirement that it pairs with the constant function $1$ to give $1$, and which is fixed by the natural action of $G$ on the space $\ell^\infty(G)^*$. A group is said to be amenable if it admits an invariant mean. The notion of an amenable action of a group on a topological space, studied by Anantharaman-Delaroche and Renault [1], generalises the concept of amenability, and arises naturally in many areas of mathematics. For example, a group acts amenably on a point if and only if it is amenable, while every hyperbolic group acts amenable on its Gromov boundary.

In this paper we introduce the notion of an invariant mean for a topological action and prove that the existence of such a mean characterises amenability of the action. Moreover, we use the existence of the mean to prove vanishing of bounded cohomology of $G$ with coefficients in a suitable class of Banach $G$ modules, and conversely we prove that vanishing of these cohomology groups characterises amenability of the action. This generalises the results of Johnson [6] on bounded cohomology for amenable groups.

Another generalisation of amenability, this time for metric spaces, was given by Yu [10] with the definition of property A. Higson and Roe [7] proved a remarkable result that unifies the two approaches: A finitely generated discrete group $G$ (regarded as a metric space) has Yu’s property A if and only if the action of $G$ on its Stone-Čech compactification $\beta G$ is topologically amenable, and this is true if and only if $G$ acts amenably on any compact space. Ozawa proved [9] that such groups are exact, and indeed property A and exactness are equivalent for countable discrete groups equipped with a proper left-invariant metric.

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To generalise the concept of invariant mean to the context of a topological action, we introduce a Banach $G$-module $W_0(G, X)$ which is an analogue of $\ell^1(G)$, encoding both the group and the space on which it acts. Taking the dual and double dual of this space we obtain analogues of $\ell^\infty(G)$ and $\ell^\infty(G)^*$. A mean for the action is an element $\mu \in W_0(G, X)^*$ satisfying the normalisation condition $\mu(\pi) = 1$, where the element $\pi$ is a summation operator, corresponding to the pairing of $\ell^1(G)$ with the constant function 1 in $\ell^\infty(G)$. A mean $\mu$ is said to be invariant if $\mu(g \cdot \varphi) = \mu(\varphi)$ for every $\varphi \in W_0(G, X)^*$, (Definition 13).

With these notions in place we give the following very natural characterisation of amenable actions.

**Theorem A.** Let $G$ be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space $X$. The action is amenable if and only if there exists an invariant mean for the action.

We then turn to the question of a cohomological characterisation of amenable actions. Given an action of a countable discrete group $G$ on a compact space $X$ by homeomorphisms we introduce a submodule $N_0(G, X)$ of $W_0(G, X)$ associated to the action and which is analogous to the submodule $\ell^1_0(G)$ of $\ell^1(G)$ consisting of all functions of sum 0. Indeed when $X$ is a point these modules coincide. We also define a cohomology class $[J]$, called the Johnson class of the action, which lives in the first bounded cohomology group of $G$ with coefficients in the module $N_0(G, X)^*$. We have the following theorem.

**Theorem B.** Let $G$ be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space $X$. Then the following are equivalent

1. The action of $G$ on $X$ is topologically amenable.
2. The class $[J] \in H^1_b(G, N_0(G, X)^*)$ is trivial.
3. $H^p_b(G, E^*) = 0$ for $p \geq 1$ and every $\ell^1$-geometric $G$-$C(X)$ module $E$.

The definition of $\ell^1$-geometric $G$-$C(X)$ module is given in Section ???. When $X$ is a point our theorem reduces to Johnson’s celebrated characterisation of amenability [6]. As a corollary we also obtain a cohomological characterisation of exactness for discrete groups, which answers a question of Higson, and which follows from our main result when $X$ is the Stone-Čech compactification $\beta G$ of the group $G$. In this case, $C(\beta G)$ can be identified with $\ell^\infty(G)$, and we obtain the following.

**Corollary.** Let $G$ be a countable discrete group. Then the following are equivalent.

1. The group $G$ is exact;
2. The Johnson class $[J] \in H^1_b(G, N_0(G, \beta G)^*)$ is trivial;
3. $H^p_b(G, E^*) = 0$ for $p \geq 1$ and every $\ell^1$-geometric $G$-$\ell^\infty(G)$-module $E$. 


This paper builds on the cohomological characterisation of property A developed in [3] and on the study of cohomological properties of exactness in [5].

2. Geometric Banach modules

Let \( C(X) \) denote the space of real-valued continuous functions on \( X \). For a function \( f : G \to C(X) \) we shall denote by \( f_g \) the continuous function on \( X \) obtained by evaluating \( f \) at \( g \in G \). We define the sup \(-\ell^1\) norm of \( f \) to be
\[
\|f\|_{\infty,1} = \sup_{x \in X} \sum_{g \in G} |f_g(x)|,
\]
and denote by \( V \) the Banach space of all functions on \( G \) with values in \( C(X) \) that have finite norm. We introduce a Banach \( G \)-module associated to the action.

**Definition 1.** Let \( W_{00}(G,X) \) be the subspace of \( V \) consisting of all functions \( f : G \to C(X) \) which have finite support and such that for some \( c \in \mathbb{R} \), depending on \( f \), \( \sum_{g \in G} f_g = c1_X \), where \( 1_X \) denotes the constant function \( 1 \) on \( X \). The closure of this space in the sup \(-\ell^1\)-norm will be denoted \( W_0(G,X) \).

Let \( \pi : W_00(G,X) \to \mathbb{R} \) be defined by \( \sum_{g \in G} f_g = \pi(f)1_X \). The map \( \pi \) is continuous with respect to the sup \(-\ell^1\) norm and so extends to the closure \( W_0(G,X) \); we denote its kernel by \( N_0(G,X) \).

In the case of \( X = \beta G \) and \( C(\beta G) = \ell^\infty(G) \) the space \( W_0(\beta G) \) was introduced in [5]. For every \( g \in G \) we define the function \( \delta_g \in W_00(G,X) \) by \( \delta_g(h) = 1_X \) when \( g = h \), and zero otherwise.

The \( G \)-action on \( X \) gives an isometric action of \( G \) on \( C(X) \) in the usual way: for \( g \in G \) and \( f \in C(X) \), we have \((g \cdot f)(x) = f(g^{-1}x)\). The group \( G \) also acts isometrically on the space \( V \) in a natural way: for \( g,h \in G \), \( f \in V \), \( x \in X \), we have \((gf)(h)(x) = f_{g^{-1}h}(g^{-1}x) = (g \cdot f_{g^{-1}h})(x)\).

Since the summation map \( \pi \) is \( G \)-equivariant (we assume that the action of \( G \) on \( \mathbb{R} \) is trivial) the action of \( G \) restricts to \( W_00(G,X) \) and so by continuity it restricts to \( W_0(G,X) \). We obtain a short exact sequence of \( G \)-vector spaces:
\[
0 \to N_0(G,X) \to W_0(G,X) \xrightarrow{\pi} \mathbb{R} \to 0.
\]

**Definition 2.** Let \( E \) be a Banach space. We say that \( E \) is a \( C(X) \)-module if it is equipped with a contractive unital representation of the Banach algebra \( C(X) \).

If \( X \) is a \( G \)-space then a \( C(X) \)-module \( E \) is said to be a \( G \)-\( C(X) \)-module if the group \( G \) acts on \( E \) by isometries and the representation of \( C(X) \) is \( G \)-equivariant.

Note that the fact that we will only ever consider unital representations of \( C(X) \) means that there is no confusion between multiplying by a scalar or by the corresponding constant function. For instance, for \( f \in W_0(G,X) \) multiplication by \( \pi(f) \) agrees with multiplication by \( \pi(f)1_X \).
Example 3. The space $V$ is a $G$-$C(X)$-module. Indeed, for every $f \in V$ and $t \in C(X)$ we define $tf \in V$ by $(tf)_g(x) = t(x)f_g(x)$, for all $g \in G$. This action is well-defined as $\|tf\|_{\infty,1} \leq \|t\|_{\infty}\|f\|_{\infty,1}$; this also implies that the representation of $C(X)$ on $V$ is contractive. As remarked above, the group $G$ acts isometrically on $V$. The representation of $C(X)$ is clearly unital and also equivariant, since for every $g \in G$, $f \in V$ and $t \in C(X)$

$$(g(tf))_h(x) = (tf)g^{-1}h(g^{-1}x) = t(g^{-1}x)f_g^{-1}h(x) = (g \cdot t)(x)(gf)_h(x)$$

Thus we have $g(tf) = (g \cdot t)(gf)$.

The equivariance of the summation map $\pi$ implies that both $W_0(G,X)$ and $N_0(G,X)$ are $G$-invariant subspaces of $V$. Note however, that $W_0(G,X)$ is not invariant under the action of $C(X)$ defined above, as for $f \in W_0(G,X)$ and $t \in C(X)$ we have

$$\sum_{g \in G} (tf)_g(x) = \sum_{g \in G} t(x)f_g(x) = t(x)\sum_{g \in G} f_g(x) = ct(x).$$

However, the same calculation shows that the subspace $N_{00}(G,X)$ is invariant under the action of $C(X)$, and so is a $G$-$C(X)$-module, and hence so is its closure $N_0(G,X)$.

Let $E$ be a $G$-$C(X)$-module, let $E^*$ be the Banach dual of $E$ and let $\langle -, - \rangle$ be the pairing between the two spaces. The induced actions of $G$ and $C(X)$ on $E^*$ are defined as follows. For $\alpha \in E^*$, $g \in G$, $f \in C(X)$, and $v \in E$ we let

$$\langle g\alpha, v \rangle = \langle \alpha, g^{-1}v \rangle, \quad \langle f\alpha, v \rangle = \langle \alpha, fv \rangle.$$ 

Note that the action of $C(X)$ is well-defined since $C(X)$ is commutative. It is easy to check the following.

Lemma 4. If $E$ is a $G$-$C(X)$-module, then so is $E^*$.

We will now introduce a geometric condition on Banach modules which will play the role of an orthogonality condition. To motivate the definition that follows, let us note that if $f_1$ and $f_2$ are functions with disjoint supports on a space $X$ then (assuming that the relevant norms are finite) the sup-norm satisfies the identity $\|f_1 + f_2\|_{\infty} = \sup(\|f_1\|_{\infty},\|f_2\|_{\infty})$, while for the $\ell^1$-norm we have $\|f_1 + f_2\|_{\ell^1} = \|f_1\|_{\ell^1} + \|f_2\|_{\ell^1}$.

Definition 5. Let $E$ be a Banach space and a $C(X)$-module. We say that $\nu_1$ and $\nu_2$ in $E$ are disjointly supported if there exist $f_1, f_2 \in C(X)$ with disjoint supports such that $f_1\nu_1 = \nu_1$ and $f_2\nu_2 = \nu_2$.

We say that the module $E$ is $\ell^\infty$-geometric if, whenever $\nu_1$ and $\nu_2$ have disjoint supports, $\|\nu_1 + \nu_2\| = \sup(\|\nu_1\|,\|\nu_2\|)$.

We say that the module $E$ is $\ell^1$-geometric if for every two disjointly supported $\nu_1$ and $\nu_2$ in $E$ $\|\nu_1 + \nu_2\| = \|\nu_1\| + \|\nu_2\|$.

If $\nu_1$ and $\nu_2$ are disjointly supported elements of $E$ and $f_1$ and $f_2$ are as in the definition, then $f_1\nu_2 = f_1f_2\nu_2 = 0$, and similarly $f_2\nu_1 = 0$. 


Note also that the functions \( f_1 \) and \( f_2 \) can be chosen to be of norm one in the supremum norm on \( C(X) \). To see this, note that Tietze’s extension theorem allows one to construct continuous functions \( f_1', f_2' \) on \( X \) which are of norm one, have disjoint supports and such that \( f_1' \) takes the value 1 on \( \text{Supp} \ f_1 \). Then \( f_1' \phi_i = \langle f_1', f_i \rangle \phi_i = f_i \phi_i = \phi_i \). Now replace \( f_1 \) with \( f_1' \).

Finally, if \( f_1, f_2 \in C(X) \) have disjoint supports then, again by Tietze’s extension theorem, \( f_1 v_1 \) and \( f_2 v_2 \) are disjointly supported for all \( v_1, v_2 \in E \).

**Lemma 6.** If \( E \) is an \( \ell^1 \)-geometric module then \( E^* \) is \( \ell^\infty \)-geometric.

If \( E \) is an \( \ell^\infty \)-geometric module then \( E^* \) is \( \ell^1 \)-geometric.

**Proof.** Let us assume that \( \phi_1, \phi_2 \in E^* \) are disjointly supported and let \( f_1, f_2 \in C(X) \) be as in Definition 5 chosen to be of norm 1.

If \( E \) is \( \ell^1 \)-geometric, then for every vector \( v \in E \), \( \| f_1 v \| + \| f_2 v \| = \|(f_1 + f_2) v\| \leq \|v\| \). Furthermore,

\[
\| \phi_1 + \phi_2 \| = \sup_{\|v\|=1} |\langle \phi_1 + \phi_2, v \rangle| = \sup_{\|v\|=1} |\langle f_1 \phi_1, v \rangle + \langle f_2 \phi_2, v \rangle|
\]

\[
= \sup_{\|v\|=1} |\langle \phi_1, f_1 v \rangle + \langle \phi_2, f_2 v \rangle|
\]

\[
\leq \sup_{\|v\|=1} (\| \phi_1 \| \| f_1 v \| + \| \phi_2 \| \| f_2 v \|)
\]

\[
\leq \sup (\| \phi_1 \|, \| \phi_2 \|) \sup_{\|v\|=1} (\| f_1 v \| + \| f_2 v \|)
\]

\[
\leq \sup (\| \phi_1 \|, \| \phi_2 \|)
\]

Since \( f_1 \phi_2 = 0 \) we have that

\[
\| \phi_1 \| = \| f_1 (\phi_1 + \phi_2) \| \leq \| f_1 \| \| \phi_1 + \phi_2 \| = \| \phi_1 + \phi_2 \|.
\]

Similarly, we have \( \| \phi_2 \| \leq \| \phi_1 + \phi_2 \| \), and the two estimates together ensure that \( \| \phi_1 + \phi_2 \| = \sup (\| \phi_1 \|, \| \phi_2 \|) \) as required.

For the second statement, let us assume that \( E \) is \( \ell^\infty \)-geometric and that \( \phi_1, \phi_2 \in E^* \) are disjointly supported. Then

\[
\| \phi_1 \| + \| \phi_2 \| = \sup_{\|v_1\|,\|v_2\|=1} \langle \phi_1, v_1 \rangle + \langle \phi_2, v_2 \rangle
\]

\[
= \sup_{\|v_1\|,\|v_2\|=1} \langle \phi_1, f_1 v_1 \rangle + \langle \phi_2, f_2 v_2 \rangle
\]

\[
= \sup_{\|v_1\|,\|v_2\|=1} \langle \phi_1 + \phi_2, f_1 v_1 + f_2 v_2 \rangle
\]

\[
\leq \sup_{\|v_1\|,\|v_2\|=1} \| \phi_1 + \phi_2 \| \| f_1 v_1 + f_2 v_2 \|
\]

\[
\leq \| \phi_1 + \phi_2 \| \leq \| \phi_1 \| + \| \phi_2 \|.
\]

where the last inequality is just the triangle inequality, so the inequalities are equalities throughout and \( \| \phi_1 \| + \| \phi_2 \| = \| \phi_1 + \phi_2 \| \) as required. \( \square \)
We have already established that $N_0(G, X)$ is a $G$-$C(X)$-module. Let $\phi^1$ and $\phi^2$ be disjointly supported elements of $N_0(G, X)$; this means that there exist disjointly supported functions $f_1$ and $f_2$ in $C(X)$ such that $\phi^i = f_i\phi^i$ for $i = 1, 2$. Then

$$\|\phi^1 + \phi^2\|_{\infty,1} = \|f_1\phi^1 + f_2\phi^2\| = \sup_{x \in X} \sum_{g \in G} |f_1(x)\phi^1_g(x) + f_2(x)\phi^2_g(x)|$$

We note that the two terms on the right are disjointly supported functions on $X$ and so

$$\|\phi^1 + \phi^2\|_{\infty,1} = \sup_{x \in X} \left( \sum_{g \in G} |f_1(x)\phi^1_g(x)| + \sum_{g \in G} |f_2(x)\phi^2_g(x)| \right) = \sup(\|\phi^1\|_{\infty,1}, \|\phi^2\|_{\infty,1}).$$

Thus we obtain

**Lemma 7.** The module $N_0(G, X)$ is $\ell^\infty$-geometric. Hence the dual $N_0(G, X)^*$ is $\ell^1$-geometric and the double dual $N_0(G, X)^{**}$ is $\ell^\infty$-geometric.

We now assume that $E$ is an $\ell^1$-geometric $C(X)$-module, so that its dual $E^*$ is $\ell^\infty$-geometric.

**Lemma 8.** Let $f_1, f_2 \in C(X)$ be non-negative functions such that $f_1 + f_2 \leq 1_X$. Then for every $\phi_1, \phi_2 \in E^*$

$$\|f_1\phi_1 + f_2\phi_2\| \leq \sup(\|\phi_1\|, \|\phi_2\|).$$

**Proof.** Let $M \in \mathbb{N}$ and $\varepsilon = 1/M$. For $i = 1, 2$ define $f_{i,0} = \min(f_i, \varepsilon)$, $f_{i,1} = \min(f_i - f_{i,0}, \varepsilon)$, $f_{i,2} = \min(f_i - f_{i,0} - f_{i,1}, \varepsilon)$, and so on, to $f_{i,M-1}$.

Then $f_{i,j}(x) = 0$ iff $f_i(x) \leq j\varepsilon$, so $f_{i,j} > 0$ iff $f_i(x) > j\varepsilon$ which implies that $\text{Supp} f_{i,j} \subseteq f_i^{-1}([j\varepsilon, \infty))$. So for $j \geq 2$, $\text{Supp}(f_{i,j}) \subseteq f_i^{-1}([j\varepsilon, \infty))$ and $\text{Supp} f_{2,M+1-j} \subseteq f_{2,M}^{-1}([1/M + 1 - j, \infty))$.

If $x \in \text{Supp}(f_{1,j}) \cap \text{Supp}(f_{2,M+1-j})$ then $1 \geq f_1(x) + f_2(x) \geq j\varepsilon + (M + 1 - j)\varepsilon = 1 + \varepsilon$, so the two supports $\text{Supp}(f_{1,j}), \text{Supp}(f_{2,M+1-j})$ are disjoint.

We have that

$$f_1 = f_{1,0} + f_{1,1} + \sum_{j=2}^{M-1} f_{1,j},$$

$$f_2 = f_{2,0} + f_{2,1} + \sum_{j=2}^{M-1} f_{2,M+1-j}.$$
So using the fact that \( \| f_{1,j} \phi_1 + f_{2,M+1-j} \phi_2 \| \leq \sup \{ \| f_{1,j} \phi_1 \|, \| f_{2,M+1-j} \phi_2 \| \} \leq \varepsilon \sup_i \| \phi_i \| \) we have the following estimate:

\[
\| f_1 \phi_1 + f_2 \phi_2 \| \leq \| (f_{1,0} + f_{1,1}) \phi_1 \| + \| (f_{2,0} + f_{2,1}) \phi_2 \| + \sum_{j=2}^M \| f_{1,j} \phi_1 + f_{2,M+1-j} \phi_2 \|
\]

\[
\leq 4\varepsilon \sup_j \| \phi_i \| + \sum_{j=2}^{M-1} \varepsilon \sup_i \| \phi_i \|
\]

\[
= (4\varepsilon + (M - 2)\varepsilon) \sup_i \| \phi_i \|
\]

\[
= (1 + 2\varepsilon) \sup_i \| \phi_i \|.
\]

Lemma 9. Let \( f_1, \ldots, f_N \in C(X), f_i \geq 0, \sum_{i=1}^N f_i \leq 1_X, \phi_1, \ldots, \phi_N \in \mathcal{E}^* \).

Then \( \| \sum_i f_i \phi_i \| \leq \sup_{1, \ldots, N} \| \phi_i \| \).

Proof. We proceed by induction. Assume that the statement is true for some \( N \). Then let \( f_0, f_1, \ldots, f_N \in C(X), f_i \geq 0, \sum_{i=1}^N f_i \leq 1_X, \) and let \( \phi_0, \phi_1, \ldots, \phi_N \in \mathcal{E}^* \).

Let \( f'_1 = f_0 + f_1 \) and leave the other functions unchanged. For \( \delta > 0 \) let

\[
\phi'_{1,\delta} = \frac{1}{f_0 + f_1 + \delta} (f_0 \phi_0 + f_1 \phi_1).
\]

Since we clearly have

\[
\frac{f_0}{f_0 + f_1 + \delta} + \frac{f_1}{f_0 + f_1 + \delta} \leq 1_X
\]

by the previous lemma we have that \( \| \phi'_{1,\delta} \| \leq \sup \{ \| \phi_0 \|, \| \phi_1 \| \} \), and so by induction

\[
\| f'_1 \phi'_{1,\delta} + f_2 \phi_2 + \cdots + f_N \phi_N \| \leq \sup \{ \| \phi'_{1,\delta} \|, \| \phi_2 \|, \ldots, \| \phi_N \| \} \leq \sup_{i=0, \ldots, N} \| \phi_i \|.
\]

Consider now

\[
f'_1 \phi'_{1,\delta} = \frac{(f_0 + f_1)}{f_0 + f_1 + \delta} (f_0 \phi_0 + f_1 \phi_1) = \frac{(f_0 + f_1)f_0}{f_0 + f_1 + \delta} \phi_0 + \frac{(f_0 + f_1)f_1}{f_0 + f_1 + \delta} \phi_1.
\]

We note that for \( i = 0, 1 \)

\[
f_i - \frac{(f_0 + f_1)f_i}{f_0 + f_1 + \delta} = \frac{\delta f_i}{f_0 + f_1 + \delta} \leq \delta
\]

and so \( \frac{(f_0 + f_1)f_i}{f_0 + f_1 + \delta} \) converges to \( f_i \) uniformly on \( X \), as \( \delta \to 0 \), which implies that \( f'_1 \phi'_{1,\delta} \) converges to \( f_0 \phi_0 + f_1 \phi_1 \) in norm, and the lemma follows. \( \square \)
Lemma 10. If $f_1, \ldots, f_N \in C(X)$ (we do not assume that $f_i \geq 0$) are such that $\sum_{i=1}^{N} |f_i| \leq 1_X$ and $\phi_1, \ldots, \phi_N \in \mathcal{E}^*$ then

$$\left\| \sum_{i=1}^{N} f_i \phi_i \right\| \leq 2 \sup_{i=1,\ldots,N} \| \phi_i \|.$$ 

Proof. If $f_i = f_i^+ - f_i^-$, then $|f_i| = f_i^+ + f_i^-$ and $\sum f_i^+ + \sum f_i^- \leq 1$. Then by the previous lemma $\left\| \sum_{i=1}^{N} f_i^+ \phi_i \right\| \leq \sup_{i=1,\ldots,N} \| \phi_i \|$ so

$$\left\| \sum_{i=1}^{N} f_i^+ \phi_i - \sum f_i^- \phi_i \right\| \leq 2 \sup_{i=1,\ldots,N} \| \phi_i \|.$$

3. AMENABLE ACTIONS AND INVARIANT MEANS

In this section we will recall the definition of a topologically amenable action and characterise it in terms of the existence of a certain averaging operator. For our purposes the following definition, adapted from [4, Definition 4.3.1] is convenient.

Definition 11. The action of $G$ on $X$ is amenable if and only if there exists a sequence of elements $f^n \in W_{00}(G, X)$ such that

1. $f^n_g \geq 0$ in $C(X)$ for every $n \in \mathbb{N}$ and $g \in G$,
2. $\pi(f^n) = 1$ for every $n$,
3. for each $g \in G$ we have $\|f^n - gf^n\|_V \to 0$.

Note that when $X$ is a point the above conditions reduce to the definition of amenability of $G$. On the other hand, if $X = \beta G$, the Stone-Čech compactification of $G$ then amenability of the natural action of $G$ on $X$ is equivalent to Yu’s property A by a result of Higson and Roe [7].

Remark 12. In the above definition we may omit condition 1 at no cost, since given a sequence of functions satisfying conditions 2 and 3 we can make them positive by replacing each $f^n_g(x)$ by

$$\frac{|f^n_g(x)|}{\sum_{h \in G} |f^n_h(x)|}.$$

Conditions 1 and 2 are now clear, while condition 3 follows from standard estimates (see e.g. [5, Lemma 4.9]).

The first definition of amenability of a group $G$ given by von Neumann was in terms of the existence of an invariant mean on the group. The following definition gives a version of an invariant mean for an amenable action on a compact space.
Definition 13. Let G be a countable group acting on a compact space X by homeomorphisms. A mean for the action is an element \( \mu \in W_0(G, X)^{**} \) such that \( \mu(\pi) = 1 \). A mean \( \mu \) is said to be invariant if \( \mu(g\varphi) = \mu(\varphi) \) for every \( \varphi \in W_0(G, X)^* \).

We now state our first main result.

Theorem A. Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X. The action is amenable if and only if there exists an invariant mean for the action.

Proof. Let G act amenably on X and consider the sequence \( f^n \) provided by Definition 11. Each \( f^n \) satisfies \( \|f^n\| = 1 \). We now view the functions \( f^n \) as elements of the double dual \( W_0(G, X)^{**} \). By the weak-* compactness of the unit ball there is a convergent subnet \( f^\lambda \), and we define \( \mu \) to be its weak-* limit. It is then easy to verify that \( \mu \) is a mean. Since

\[
|\langle f^\lambda - gf^\lambda, \varphi \rangle| \leq \|f^\lambda - gf^\lambda\|_V \|\varphi\|
\]

and the right hand side tends to 0, we obtain \( \mu(\varphi) = \mu(g\varphi) \).

Conversely, by Goldstine’s theorem, (see, e.g., [8, Theorem 2.6.26]) as \( \mu \in W_0(G, X)^{**} \), \( \mu \) is the weak-* limit of a bounded net of elements \( f^\lambda \in W_0(G, X) \). We note that we can choose \( f^\lambda \) in such a way that \( \pi(f^\lambda) = 1 \). Indeed, given \( f^\lambda \) with \( \pi(f^\lambda) = c_\lambda \rightarrow \mu(\pi) = 1 \) we replace each \( f^\lambda \) by

\[
f^\lambda + (1 - c_\lambda)\delta_e.
\]

Since \( (1 - c_\lambda)\delta_e \rightarrow 0 \) in norm in \( W_0(G, X) \), \( \mu \) is the weak-* limit of the net \( f^\lambda + (1 - c_\lambda)\delta_e \) as required.

Since \( \mu \) is invariant, we have that for every \( g \in G \), \( gf^\lambda \rightarrow g\mu = \mu \), so that \( gf^\lambda - f^\lambda \rightarrow 0 \) in the weak-* topology. However, for every \( g \in G \), \( gf^\lambda - f^\lambda \in W_0(G, X) \), and so the convergence is in fact in the weak topology on \( W_0(G, X) \).

For every \( \lambda \), we regard the family \( (gf^\lambda - f^\lambda)_{g \in G} \) as an element of the product \( \prod_{g \in G} W_0(G, X) \), noting that this sequence converges to 0 in the Tychonoff weak topology.

Now \( \prod_{g \in G} W_0(G, X) \) is a Fréchet space in the Tychonoff norm topology, so by Mazur’s theorem there exists a sequence \( f^n \) of convex combinations of \( f^\lambda \) such that \( (gf^n - f^n)_{g \in G} \) converges to zero in the Fréchet topology. Thus there exists a sequence \( f^n \) of elements of \( W_0(G, X) \) such that for every \( g \in G \), \( \|gf^n - f^n\| \rightarrow 0 \) in \( W_0(G, X) \).

The result then follows from Remark 12. \( \square \)

4. Equivariant Means on Geometric Modules

Given an invariant mean \( \mu \in W_0(G, X)^{**} \) for the action of G on X and an \( \ell^1 \)-geometric G-C(X) module \( E \), we define a G-equivariant averaging operator \( \mu_E : \ell^\infty(G, \mathcal{E}^*) \rightarrow \mathcal{E}^* \) which we will also refer to as an equivariant mean for the action.
To do so, following an idea from [3], we introduce a linear functional $\sigma_{\tau,v}$ on $W_0^0(G, X)$. Given a Banach space $E$ define $\ell^\infty(G, E)$ to be the space of functions $f : G \to E$ such that $\sup_{g \in G} \|f(g)\|_E < \infty$. If $G$ acts on $E$ then the action of the group $G$ on the space $\ell^\infty(G, E)$ is defined in an analogous way to the action of $G$ on $V$, using the induced action of $G$ on $E$:

$$(g\tau)_h = g(\tau_{g^{-1}h}),$$

for $\tau \in \ell^\infty(G, E)$ and $g \in G$.

Let us assume that $E$ is an $\ell^1$-geometric $G$-$C(X)$ module, and let $\tau \in \ell^\infty(G, E^*)$. Choose a vector $v \in E$ and define a linear functional $\sigma_{\tau,v} : W_0^0(G, X) \to \mathbb{R}$ by

$$\sigma_{\tau,v}(f) = \left \langle \sum_{h \in G} f_h \tau_h, v \right \rangle$$

for every $f \in W_0^0(G, X)$. If we now use Lemma 10 together with the support condition required of elements of $W_0^0(G, X)$ then we have the estimate

$$|\sigma_{\tau,v}(f)| \leq \left \| \sum_{h} f_h \tau_h \right \|_V \leq 2\|f\|\|\tau\|\|v\|.$$

This estimate completes the proof of the following.

**Lemma 14.** Let $E$ be an $\ell^1$-geometric $G$-$C(X)$ module. For every $\tau \in \ell^\infty(G, E^*)$ and every $v \in E$ the linear functional $\sigma_{\tau,v}$ on $W_0^0(G, X)$ is continuous and so it extends to a continuous linear functional on $W_0(G, X)$.

**Lemma 15.** The map $\ell^\infty(G, E^*) \times E \to W_0(G, X)^*$ defined by $(\tau, v) \mapsto \sigma_{\tau,v}$ is $G$-equivariant.

**Proof.**

$$\sigma_{g\tau,gv}(f) = \left \langle \sum_{h} f_h g(\tau_{g^{-1}h}), gv \right \rangle = \left \langle g \sum_{h} (g^{-1}f_h)\tau_{g^{-1}h}, gv \right \rangle$$

$$= \left \langle \sum_{h} (g^{-1}f_h)\tau_{g^{-1}h}, v \right \rangle = \left \langle \sum_{h} (g^{-1}f)g^{-1}\tau_{g^{-1}h}, v \right \rangle$$

$$= \sigma_{\tau,v}(g^{-1}f) = (g\sigma_{\tau,v})(f).$$

**Definition 16.** Let $E$ be an $\ell^1$-geometric $G$-$C(X)$ module, and let $\mu \in W_0(G, X)^{**}$ be an invariant mean for the action. We define $\mu_E : \ell^\infty(G, E^*) \to E^*$ by

$$\langle \mu_E(\tau), v \rangle = \langle \mu, \sigma_{\tau,v} \rangle,$$

for every $\tau \in \ell^\infty(G, E^*)$, and $v \in E$.

**Lemma 17.** Let $E$ be an $\ell^1$-geometric $G$-$C(X)$ module, and let $\mu \in W_0(G, X)^{**}$ be an invariant mean for the action.

1. The map $\mu_E$ defined above is $G$-equivariant.
If $\tau \in \ell_\infty(G, E^*)$ is constant then $\mu_E(\tau) = \tau e$.

**Proof.**

$$\langle \mu_E(g\tau), v \rangle = \mu(\sigma_{g\tau}, v) = \mu(g \cdot \sigma_{\tau, g^{-1}v}) = \mu(\sigma_{\tau, g^{-1}v}) = \langle \mu_E(\tau), g^{-1}v \rangle = \langle g \cdot (\mu_E(\tau)), v \rangle.$$ 

If $\tau$ is constant then

$$\sigma_{\tau, v}(f) = \left\langle \sum_h f_h \tau_h, v \right\rangle = \left\langle \left( \sum_h f_h \right) \tau_e, v \right\rangle = \langle \tau_e, v \rangle \pi(f).$$

So $\sigma_{\tau, v} = (\tau_e, v)\pi$ and

$$\langle \mu_E(\tau), v \rangle = \mu(\sigma_{\tau, v}) = \mu((\tau_e, v)\pi) = \langle \tau_e, v \rangle,$$

hence $\mu_E(\tau) = \tau e$. □

5. **Amenable actions and bounded cohomology**

Let $E$ be a Banach space equipped with an isometric action by $G$. Then we consider a cochain complex $C^b_m(G, E^*)$ which in degree $m$ consists of $G$-equivariant bounded cochains $\phi: G^{m+1} \to E^*$ with values in the Banach dual $E^*$ of $E$ which is equipped with the natural differential $d$ as in the homogeneous bar resolution. Bounded cohomology with coefficients in $E^*$ will be denoted by $H^*_b(G, E^*)$.

**Definition 18.** Let $G$ be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space $X$. The function

$$J(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$$

is a bounded cochain of degree 1 with values in $N_{00}(G, X)$, and in fact it is a bounded cocycle and so represents a class in $H^1_b(G, N_{00}(G, X)^{**})$, where we regard $N_{00}(G, X)$ as a subspace of $N_0(G, X)^{**}$. We call $[J]$ the Johnson class of the action.

**Theorem B.** Let $G$ be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space $X$. Then the following are equivalent

1. The action of $G$ on $X$ is topologically amenable.
2. The class $[J] \in H^1_b(G, N_0(G, X)^{**})$ is trivial.
3. $H^p_b(G, E^*) = 0$ for $p \geq 1$ and every $\ell^1$-geometric $G$-$C(X)$ module $E$.

**Proof.** We first show that (1) is equivalent to (2). The short exact sequence of $G$-modules

$$0 \to N_0(G, X) \to W_0(G, X) \xrightarrow{\pi} \mathbb{R} \to 0$$
leads, by taking double duals, to the short exact sequence
\[ 0 \to N_0(G,X)^{**} \to W_0(G,X)^{**} \to \mathbb{R} \to 0 \]
which in turn gives rise to a long exact sequence in bounded cohomology
\[ H_b^0(G,N_0(G,X)^{**}) \to H_b^1(G,W_0(G,X)^{**}) \to H_b^2(G,\mathbb{R}) \to H_b^1(G,N_0(G,X)^{**}) \to \ldots \]

The Johnson class \([J]\) is the image of the class \([1]\) \in \(H_b^1(G,\mathbb{R})\) under the connecting homomorphism \(d : H_b^0(G,\mathbb{R}) \to H^1(G,N_0(G,X)^{**})\), and so \([J]\) = 0 if and only if \(d[1] = 0\). By exactness of the cohomology sequence, this is equivalent to \([1]\) \in \(\text{Im } \pi^{**}\), where \(\pi^{**} : H_b^0(G,W_0(G,X)^{**}) \to H_b^0(G,\mathbb{R})\) is the map on cohomology induced by the summation map \(\pi\). Since \(H_b^0(G,W_0(G,X)^{**}) = (W_0(G,X)^{**}G\) and \(H_b^0(G,\mathbb{R}) = \mathbb{R}\) we have that \([J]\) = 0 if and only if there exists an element \(\mu \in W_0(G,X)^{**}\) such that \(\mu = g\mu\) and \(\mu(\pi) = 1\). Thus \(\mu\) is an invariant mean for the action and the equivalence with amenability of the action follows from Theorem A.

We turn to the implication (1) implies (3). Since \(G\) acts amenably on \(X\) there is, by Theorem A, an invariant mean \(\mu\) associated with the action. For every \(h \in G\) and for every equivariant bounded cochain \(\phi\) we define \(s_h \phi : G^p \to \ell^\infty(G,\mathcal{E}^*)\) by \(s_h \phi(g_0,\ldots,g_p-1) = \phi(g,g_0,\ldots,g_p-1)\); we note that for fixed \(h\), \(s_h \phi\) is not equivariant in general. However, the map \(s_h\) does satisfy the identity \(ds_h + s_h d = 1\) for every \(h \in G\), and we will now construct an equivariant contracting homotopy, adapting an averaging procedure introduced in [3].

For \(\phi \in C_b^p(G,\mathcal{E}^*)\) let \(\widehat{\phi} : G^p \to \ell^\infty(G,\mathcal{E}^*)\) be defined by \(\widehat{\phi}(g)(h) = s_h \phi(g)\), for \(g = (g_0,\ldots,g_p-1)\).

Note that for every \(k,h \in G\),
\[
\widehat{\phi}(kg_0,\ldots,kg_{p-1})(h) = \phi(h,kg_0,\ldots,kg_{p-1}) = k(\phi(k^{-1}h,g_0,\ldots,g_{p-1})) \\
= k(\phi(g_0,\ldots,g_{p-1})(k^{-1}h)) \\
= (k(\phi(g_0,\ldots,g_{p-1})))'(h)
\]
so \(\widehat{\phi}(kg) = k(\widehat{\phi}(g))\).

We can now define a map \(s : C^p(G,\mathcal{E}^*) \to C^{p-1}(G,\mathcal{E}^*)\):
\[
s\phi(g) = \mu_\ell(\widehat{\phi}(g)),
\]
where \(\mu_\ell : \ell^\infty(G,\mathcal{E}^*) \to \mathcal{E}^*\) is the map defined in Lemma [17] using the invariant mean \(\mu\). Note that \(\|\mu_\ell\| \leq 2\|\mu\|\), and \(\|\widehat{\phi}(g)\| \leq \sup\{\|\phi(k)\| : k \in G^{p+1}\}\). Hence \(s\phi\) is bounded.

For every cochain \(\phi\), \(k(s\phi) = s(k\phi) = s\phi\) since \(\widehat{\phi}\) and \(\mu_\ell\) are equivariant.

The map \(s\) provides a contracting homotopy for the complex \(C_b^*(G,\mathcal{E}^*)\) which can be seen as follows. As \(\mu_\ell : \ell^\infty(G,\mathcal{E}^*) \to \mathcal{E}^*\) is a linear operator it follows that for a given \(\phi \in C_b^p(G,\mathcal{E}^*)\), and a \(p+1\)-tuple of arguments \(k = (k_0,\ldots,k_p)\), \(ds\phi\) is obtained by applying the mean \(\mu_\ell\) to the map \(g \mapsto ds_g\phi(k)\), while \(sd\phi\) is obtained by applying \(\mu_\ell\) to the function \(g \mapsto s_g d\phi(k)\). Thus
\[
(sd + ds)\phi(k) = \mu_\ell(g \mapsto (ds_g + s_g d)\phi(k)).
\]
Given that \( d_s g + s_g d = 1 \) for every \( g \in G \), for every \( g \in G^{p+1} \) the function \( g \mapsto (d_s g + s_g d) \phi(k) = \phi(k) \in E^* \) is constant, and so by Lemma 17

\[
(s + d)(k) = (d + s)(k) = \phi(k).
\]

Thus \( s + d = 1 \), as required.

Collecting these results together, we have proved that (1) implies (3).

The fact that (3) implies (2), follows from the fact that \( N_0(G, X)^* \) is an \( \ell^1 \)-geometric \( G \)-\( C(X) \)-module, proved in Lemma 7.

\[\square\]

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