METRIC ENTROPY AND $n$-WIDTHS OF FUNCTION SPACES ON DATA DEFINED MANIFOLDS AND QUASI-METRIC MEASURE SPACES

MARTIN EHLER AND FRANK FILBIR

ABSTRACT. We first determine the asymptotics of the Kolmogorov metric entropy and $n$-widths of Sobolev spaces on some classes of data defined manifolds and quasi-metric measure spaces. Secondly, we develop constructive algorithms to represent those functions within a prescribed accuracy. The constructions can be based on either spectral information or scattered samples of the target function. Our algorithmic scheme is asymptotically optimal in the sense of $n$-widths and asymptotically optimal up to a logarithmic factor with respect to the metric entropy.

1. INTRODUCTION

In classical computational mathematics, it is customary to represent a function by using finitely many parameters, e.g., the coefficients of some truncated series expansion. Representing a function in terms of binary bits rather than a sequence of real numbers is the problem of quantization. A well known method for image compression is to consider the discrete cosine transform of sub-blocks of an image, and manipulate these by using a so-called quantization mask. The integers thus obtained are represented as a bit string to which coding techniques can be applied to achieve the final compressed image. Similarly, in wireless communication, one needs to transform an analogue signal into a stream of bits, from which the original signal should be recovered at the receiving end with a minimal distortion. The theory of bit representation of functions pre-dates these modern requirements and was already studied by Kolmogorov. The notion of metric entropy in the sense of Kolmogorov gives a measurement of the minimal number of bits needed to represent an arbitrary function from a compact subset of a function space. Babenko, Kolmogorov, Tikhomirov, Vitushkin, and Yerokhin have given many estimates on the metric entropy for several compact subsets of the standard function spaces, cf. [15, 31].

Constructive algorithms were derived in [18] to represent functions in suitably defined Besov spaces on the sphere using asymptotically the same number of bits as the metric entropy of these classes, except for a logarithmic factor. A generalization was obtained for compact smooth Riemannian manifolds $\mathbb{X}$ and global Sobolev spaces $W^s(L_p(\mathbb{X}))$ for $p = \infty$ in [5]. Related measures of complexity are $n$-widths [14] and were studied for some classical function spaces in [24, 25]. Both concepts, metric entropy and $n$-widths, are important complexity measures for the analysis of functions on high-dimensional datasets occurring in biology, medicine, and related areas. Many computational schemes are categorized into the field of manifold learning, where functions need to be learned from finitely many training data that are assumed to lie on some (unknown) manifold [3, 9, 22, 27, 28, 29]. While much of the recent research in this direction focuses on the understanding of data geometry, approximation theory methods were introduced in [6, 7, 17, 19, 20] to obtain certain wavelet-like frame representations of functions on such data defined spaces.

The purpose of the present paper is to generalize results on metric entropy and $n$-widths to the context of functions on data defined quasi-metric measure spaces and covering the entire range $1 \leq p \leq \infty$. Indeed, we determine the asymptotics of the metric entropy and $n$-widths for global Sobolev spaces. We will use wavelet frame expansions to obtain a representation of functions in Sobolev spaces, and this scheme is asymptotically optimal with respect to the $n$-widths and asymptotically optimal up to a logarithmic factor in the sense of the metric entropy. Our results can be extended easily to the case of Besov spaces, but we restrict ourselves to Sobolev spaces both for clarity and because no new ideas are involved in the extension to the case of Besov spaces. In addition to obtaining theoretical bounds on the metric entropy and $n$-widths, our results have the following notable features:

- The computational scheme is based on a linear approximation operator to asymptotically match the optimal bounds in the sense of $n$-widths.
- We give explicit schemes for converting the target function into a near minimal number of bits by combining the linear approximation operator with linear quantization, and we derive a reconstruction scheme from such bits to a prescribed accuracy.
- Our constructions can deal with both, spectral information as well as finitely many training data consisting of function evaluations at scattered data points.

In addition, we determine the asymptotics of the metric entropy of local Sobolev spaces. A local bit representation scheme is derived for certain alternative Sobolev spaces that express local smoothness. These spaces resemble the local Sobolev space used for the metric entropy but may not coincide exactly. Hence, we cannot claim that those local representations are near optimal although we believe that this is true and a rigorous treatment of the local case is part of our ongoing research.

The outline of this paper is as follows: In Section 2 we introduce the setting and define metric entropy and n-widths. The asymptotics of the metric entropy of global and local Sobolev spaces is determined in Section 3. In Section 4, we introduce our approximation schemes for global Sobolev spaces based on wavelet expansions and compute the asymptotics of the n-widths of global Sobolev spaces. In Section 5 we verify that linear quantization of the approximation scheme leads to optimal bit representations up to a logarithmic factor for the global Sobolev space. Local versions of Sobolev spaces are considered in Section 6. For the readers convenience, Appendix A contains a list and brief discussion of the technical assumptions used for the main results of the present paper.

2. Sobolev spaces and their metric entropy and n-widths

2.1. Diffusion measure space. To fix the setting and introduce some technical assumptions used throughout the paper, let us recall that the classical heat kernel in $\mathbb{R}^d$ is the fundamental solution to the heat equation and given by

$$G_t^{\mathbb{R}^d}(x, y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x-y\|^2}{4t}\right).$$

Note that the term $(2\pi t)^{d/2}$ is the volume of a ball of radius $\sqrt{t}$. The Laplacian on the unit sphere $S^{d-1}$ induces the spherical heat kernel, which has the expansion

$$G_t^{S^{d-1}}(x, y) = \sum_{k=0}^{\infty} \exp(-\lambda_k t) \varphi_k(x) \varphi(y),$$

where $\{\varphi_k\}_{k=0}^{\infty}$ are the eigenfunctions and $\{\lambda_k\}_{k=0}^{\infty}$ the eigenvalues of the Laplacian (i.e., the Laplace-Beltrami operator) on the sphere. Those eigenfunctions are called the spherical harmonics and form an orthonormal basis for $L^2(S^{d-1})$. More generally suppose that $\mathcal{X}$ is a compact Riemannian manifold without boundary. The spectral decomposition of the Laplace-Beltrami operator yields a sequence of nondecreasing eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$ and smooth eigenfunctions $\{\varphi_k\}_{k=0}^{\infty}$ that form an orthonormal basis for $L^2(\mathcal{X})$, so that the corresponding heat kernel can formally be written as in (2).

The technical assumptions that we shall introduce are indeed guided by (2). Let $(\mathcal{X}, \rho)$ be a quasi-metric space endowed with a Borel probability measure $\mu$. The system $\{\varphi_k\}_{k=0}^{\infty} \subset L^2(\mathcal{X}, \mu)$ is supposed to be an orthonormal basis of continuous functions with $\varphi_0 \equiv 1$ and our results also involve a sequence of nondecreasing real numbers $\{\lambda_k\}_{k=0}^{\infty}$ such that $\lambda_0 = 0$ and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. Let $N$ be a positive integer and we shall restrict us to $N = 2^n$, where $n$ is some nonnegative integer. The space of diffusion polynomials up to degree $N$ is $\Pi_N := \text{span}\{\varphi_k : \lambda_k \leq N\}$. Moreover, we imitate (2) and make use of the generalized heat kernel

$$G_t(x, y) = \sum_{k=0}^{\infty} \exp(-\lambda_k^2 t) \varphi_k(x) \varphi(y), \quad t > 0.$$
Definition 2.1 ([2]). Under the above notation, a quasi-metric space \( X \) is called a diffusion measure space if each of the following properties is satisfied:

(i) For each \( x \in X \) and \( t > 0 \), the closed ball \( B_t(x) \) of radius \( t \) at \( x \) is compact, and there is \( \alpha > 0 \) such that
\[
\mu(B_t(x)) \lesssim t^\alpha, \quad x \in X, \ t > 0.
\]

(ii) There is \( c > 0 \) such that
\[
|G_t(x,y)| \lesssim t^{-\alpha/2} \exp \left( -c \frac{d(x,y)^2}{t} \right), \quad x, y \in X, \ 0 < t \leq 1.
\]

(iii) We have \( t^{-\alpha/2} \lesssim G_t(x,x), \quad x \in X, \ 0 < t < 1 \).

This definition relates \( G_t(x,y) \) to the Euclidean heat kernel [1]. In other words, the generalized heat kernel is supposed to describe some “(artificial) heat propagation” on the diffusion measure space \( X \) that resembles the one in Euclidean space when \( d \) is meant with respect to the measure \( \mu \).

From here on, we suppose that \( X \) is a diffusion measure space throughout the present paper. It is also noteworthy that the conditions of a diffusion measure space imply that \( \mu(B_t(x)) \approx t^\alpha \), for all \( 0 < t < 1 \), cf. [6]. Thus, the volume of a ball behaves locally as in \( \mathbb{R}^\alpha \). The conditions of a diffusion measure space imply the following estimate on the Christoffel function,
\[
\sum_{\lambda_k \leq N} |\varphi_k(x)|^2 \approx N^\alpha, \quad x \in X, \ N > 0,
\]
see [2] [6] [7] for a discussion and references. By integrating over \( X \), we obtain that \( \dim(\Pi_N) \approx N^{\alpha} \), which would follow from Weyl’s law in case of a compact Riemannian manifold of dimension \( \alpha \).

Remark 2.2. It was pointed out in [2] that all technical assumptions are satisfied when \( X \subset \mathbb{R}^d \) is an \( \alpha \)-dimensional compact, connected, Riemannian manifold without boundary, with non-negative Ricci curvature, geodesic distance \( p \), and \( \mu \) being the Riemannian volume measure on \( X \) normalized with \( \mu(X) = 1 \), \( \{\varphi_k\}_{k=0}^\infty \) are the eigenfunctions of the Laplace-Beltrami operator on \( X \), and \( \{-\lambda_k\}_{k=0}^\infty \) are the corresponding eigenvalues arranged in nonincreasing order, see also [12]. For further discussions, we refer to [6] [7] [19].

Given an arbitrary normed space \( X \) and a subset \( Y \subset X \), we define, for \( f \in X \),
\[
E(f, Y, X) := \inf_{g \in Y} \|f - g\|_X.
\]

Definition 2.3. For a nontrivial ball \( B \subset X \) and \( 1 \leq p \leq \infty \), the Sobolev space of order \( s > 0 \) is
\[
W^s(L_p(B)) = \{ f \in L_p(B) : \|f\|_{W^s(L_p(B))} < \infty \},
\]
where the Sobolev norm is given by
\[
\|f\|_{W^s(L_p(B))} := \|f\|_{L_p(B)} + \sup_{N \geq 1} N^s E(f, \Pi_N, L_p(B)).
\]
The ball of radius \( r > 0 \) in \( W^s(L_p(B)) \) is denoted by
\[
\overline{W}_r^s(L_p(B)) := \{ f \in L_p(B) : \|f\|_{W^s(L_p(B))} \leq r \}.
\]

In the above definition functions in \( \Pi_N \) are simply identified with their restrictions to \( B \) and \( L_p(B) \) is meant with respect to the measure \( \mu \) restricted to \( B \).

2.2. Kolmogorov metric entropy and \( n \)-widths. Metric entropy as studied in [16] refers to the minimal number of bits needed to represent a function \( f \) up to precision \( \varepsilon \). This number determines the maximal compression when loss of information is bounded by \( \varepsilon \). For a more stringent mathematical exposition, let \( Y \) be a compact subset of a metric space \( (X, \rho) \). Given \( \varepsilon > 0 \), let \( N_\varepsilon(Y) \) be the \( \varepsilon \)-covering number of \( Y \) in \( X \), i.e., the minimal number of balls of radius \( \varepsilon \) that cover \( Y \). Suppose that \( g_1, \ldots , g_{N_\varepsilon(Y)} \) be a list of centers of these balls. Given any \( f \in Y \), there is \( g_j \) such that \( \rho(f, g_j) \leq \varepsilon \). We may then represent \( f \) using the binary representation of \( j \), and use \( g_j \) as the reconstruction of \( f \) based on this representation. Any binary enumeration of these centers takes \( \log_2(N_\varepsilon(Y)) \) many bits, which somewhat measures the complexity of \( Y \):
Definition 2.4. Let \( Y \) be a compact subset of a metric space \((X, \rho)\) and, for \( \varepsilon > 0 \), let \( N_\varepsilon(Y) \) be the \( \varepsilon \)-covering number of \( Y \) in \( X \). Then
\[
H_\varepsilon(Y, X) := \log_2(N_\varepsilon(Y))
\]
is called the metric entropy of \( Y \) in \( X \).

Thus, the metric entropy is the minimal number of bits necessary to represent any \( f \) with precision \( \varepsilon \). Let us also introduce some alternative notions of complexity:

Definition 2.5 (25). Let \( Y \) be a subset of a linear normed space \((X, \| \cdot \|)\) and let \( n \geq 1 \) be an integer.

(i) The Kolmogorov \( n \)-width of \( Y \) in \( X \) is
\[
\mathcal{K}_n(Y, X) := \inf \sup_{L_n} \inf_{x \in L_n} \| x - y \|,
\]
where the infimum is taken over all \( n \)-dimensional linear subspaces \( L_n \) in \( X \).

(ii) The linear \( n \)-width of \( Y \) in \( X \) is
\[
\mathcal{L}_n(Y, X) := \inf \sup_{F_n} \| x - F_n(x) \|,
\]
where the infimum is taken over all bounded linear operators \( F_n \) on \( X \) whose range is of dimension at most \( n \).

(iii) The Gelfand \( n \)-width of \( Y \) in \( X \) is
\[
\mathcal{G}_n(Y, X) := \inf \sup_{L_n} \| x \|,
\]
where the infimum is taken over all closed subspaces \( L_n \) of \( X \) of codimension at most \( n \).

(iv) The Bernstein \( n \)-width of \( Y \) in \( X \) is
\[
\mathcal{B}_n(Y, X) := \sup_{X_{n+1}} \sup_{\lambda} \{ \lambda : \lambda X_{n+1} \subset Y \}
\]
where the supremum is taken over all subspaces \( X_{n+1} \) of \( X \) of dimension at least \( n + 1 \) and \( X_{n+1} \) denotes the unit ball in \( X_{n+1} \).

All 4 types of widths measure the complexity of \( Y \) with respect to \( X \). It should be mentioned that especially the Gelfand \( n \)-width is an important factor in optimal recovery problems, cf. \([4, 5, 21, 23, 30]\).

In the subsequent sections we shall compute the metric entropy \((8)\) and the \( n \)-widths of the Sobolev ball \( W^s_r(L_p(\mathbb{X})) \) of radius \( r \) given by \((7)\) in the ambient space \( L_p(\mathbb{X}) \).

3. The metric entropy of global and local Sobolev spaces

We shall determine the asymptotics of the metric entropy of global and local Sobolev spaces, so let \( B \subset \mathbb{X} \) be some nontrivial ball, which is allowed to coincide with \( \mathbb{X} \). Since \( W^s(L_p(B)) \) is not finite-dimensional, \( W^s_r(L_p(B)) \) is not compact in the Sobolev space. Here, we consider \( W^s(L_p(B)) \) as a subspace of \( L_p(B) \), in which it is compact, see \([5]\) for \( B = \mathbb{X} \) and the case \( B \subset \mathbb{X} \) can be proven analogously. The following result extends findings in \([18]\) from the sphere to balls in diffusion measure spaces:

**Theorem 3.1.** If \( s > 0 \) is fixed, \( B \) is a nontrivial ball in \( \mathbb{X} \), and \( 0 < \varepsilon \leq r \), then
\[
H_\varepsilon(W^s_r(L_p(B)), L_p(B)) \asymp (r/\varepsilon)^{\alpha/s}
\]
holds, where the generic constants neither depend on \( \varepsilon \) nor on \( r \), and \( \alpha \) is the constant in Definition \([2, 1]\).

It is obvious that increased precision requires more bits, and smoother functions can be represented with fewer bits. The exact growth condition \((9)\) reflects these thoughts in a quantitative fashion.

If we can verify that the system \( \{ \varphi_k|B \} \) is linearly independent, then \([3]\) allows us to follow the lines in \([5]\) to derive Theorem \((8, 1)\).

**Proposition 3.2.** If \( B \subset \mathbb{X} \) is a nontrivial ball, then the functions \( \{ \varphi_k|B \}_{k=0}^\infty \) are linearly independent.

The proof of Proposition \((5, 2)\) requires some machinery of localization:
**Definition 3.3.** We call an infinitely often differentiable and non-increasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}$ a low-pass filter if $H(t) = 1$ for $t \leq 1/2$ and $H(t) = 0$ for $t \geq 1$.

A standard example of a low-pass filter is

$$H(x) = \begin{cases} 1, & x \leq 1/2, \\ \exp\frac{(x-\frac{1}{2})^2 (2x^2 - 2x - 1)}{x^2 (x-1)^2}, & 1/2 \leq x \leq 1, \\ 0, & 1 \leq x. \end{cases}$$

For notational convenience, we define the kernel

$$K_N(x, y) := \sum_{k=0}^{\infty} H\left(\frac{\lambda_k}{2N}\right) \varphi_k^*(x) \varphi_k(y).$$

According to [6, 17, 19], any low-pass filter induces a localization result, namely, for fixed $S > \alpha$ and all $x \neq y$ with $N = 1, 2, \ldots$,

$$|K_N(x, y)| \lesssim \frac{N^{\alpha-S}}{\rho(x, y)^S}.$$

Alternatively, we also have for fixed $S > \alpha$ and all $x, y$ with $N = 1, 2, \ldots$,

$$|K_N(x, y)| \lesssim \frac{N^\alpha}{\max(1, (NS\rho(x, y)^S))}.$$

We find in [6] Inequality (3.12) that

$$\sup_{y \in \mathcal{Y}} \int_{\mathcal{X}} |K_N(x, y)| d\mu(x) \lesssim 1$$

holds. Now, we can take care of the proposition:

**Proof of Proposition 3.2.** Let $f = \sum_{\lambda_k < N} c_k \varphi_k$ and assume $f|_B = 0$. We shall check that $f$ vanishes on the entire quasi-metric space, so that the linear independence on $\mathcal{X}$ implies that $c_k = 0$, for all $\lambda_k < N$. For all $M > 2N$, $0 < r \leq 1$, and $x \in \mathcal{X} \setminus B$, we have

$$f(x) = \sum_{\lambda_k < N} \langle f, \varphi_k \rangle_{L_2(\mathcal{X})} \varphi_k(x)$$

$$= \sum_{k=0}^{\infty} \frac{\lambda_k}{M} \int_{\mathcal{X}} f(y) \varphi_k^*(x) \varphi_k(y) d\mu(y)$$

$$= \int_{\mathcal{X} \setminus B_r(x)} f(y) K_M(x, y) d\mu(y) + \int_{B_r(x)} f(y) K_M(x, y) d\mu(y).$$

In order to derive $f(x) = 0$, we shall estimate the two above terms separately. First, we can apply the localization property [13] with $S = \alpha + 2$ and obtain

$$\left| \int_{\mathcal{X} \setminus B_r(x)} f(y) K_M(x, y) d\mu(y) \right| \lesssim \|f\|_{L_1(\mathcal{X})} M^{-2r-S},$$

which tends to zero when $r = M^{-1/S}$ and $M$ tends to infinity. We can bound the second term by applying the Hölder inequality, the estimate [13], and the volume decay of the ball $B_r(x)$.

$$\left| \int_{B_r(x)} f(y) K_M(x, y) d\mu(y) \right| \leq \int_{B_r(x)} \left| K_M(x, y) \right| d\mu(y) \|f\|_{L_1(\mathcal{X})}$$

$$\lesssim \mu(B_r(x)) \|f\|_{L_1(\mathcal{X})} \lesssim r^{\alpha}\|f\|_{L_1(\mathcal{X})}.$$
Since the system \(\{\varphi_k|_{\mathcal{B}}\}\) is indeed linearly independent and \([4]\) holds, we can follow the approach in \([3]\) with \(B\) in place of \(X\), which proves Theorem \([5,1]\) that serves as a benchmark for function representation on diffusion measure spaces. The remaining part of the present work is dedicated to develop a scheme that matches the optimality bound at least up to a logarithmic factor.

4. Approximating functions globally from scattered data and \(n\)-widths

This section is dedicated to introduce our approximation scheme, to discuss its capabilities to characterize certain smoothness spaces, and to determine the asymptotics of the \(n\)-widths.

4.1. Quadrature measures. This section is dedicated to some technical details needed to develop our approximation scheme. We first aim to replace the integral over diffusion polynomials with a finite sum or at least with an integral over a “simpler” measure.

**Definition 4.1.** We say that the **strong product assumption** holds if there is a constant \(a > 0\) such that 
\[
f \cdot g \in \Pi_{aN} \quad \text{for all } f, g \in \Pi_N.
\]

It should be mentioned that similar product assumptions were used in \([10]\) and that we suppose that the strong product assumption holds throughout the remaining part of the present paper. Note that the strong product assumption holds in compact smooth Riemannian manifolds and the function system are eigenfunctions of the Laplace-Beltrami operator, cf. \([2]\) Theorem \(A.1\). It is also noteworthy that the theory of localized summation kernels was adapted to a so-called weak product assumption introduced in \([19]\), but we shall restrict us to its strong counterpart to avoid some technical issues and to not lose focus in the presentation.

**Definition 4.2.** A signed Borel measure \(\nu\) on \(X\) is called a **quadrature measure** of order \(N\) if
\[
\int_X f(x) d\mu(x) = \int_X f(x) d\nu(x), \quad \text{for all } f \in \Pi_{aN}.
\]

**Definition 4.3.** For fixed \(1 \leq p \leq \infty\), a signed Borel measure \(\nu\) on \(X\) is called a **Marcinkiewicz-Zygmund measure** of order \(N\) if the \(L_p\)-norm \(\|f\|_{L_p(X)}\) of \(f\) with respect to \(|\nu|\) satisfies
\[
\|f\|_{|\nu|,L_p(X)} \asymp \|f\|_{L_p(X)}, \quad \text{for all } f \in \Pi_{aN},
\]
and \(|\nu|\) denotes the total variation measure of \(\nu\). A signed Borel measure is called a **Marcinkiewicz-Zygmund quadrature measure** of order \(N\) if it is both, a quadrature and a Marcinkiewicz-Zygmund measure of order \(N\).

**Definition 4.4.** For fixed \(1 \leq p \leq \infty\), a family \((\nu_N)_{N=1}^\infty\) of Marcinkiewicz-Zygmund (quadrature) measures of order \(N\), respectively, is called **uniform** if the generic constants in \([11]\) can be chosen independently of \(N\).

The existence of uniform families of Marcinkiewicz-Zygmund quadrature measures are proven for fairly general smooth Riemannian manifolds in \([9,7]\), where a construction procedure is outlined. Also, note that there are families of uniform Marcinkiewicz-Zygmund quadrature measures with finite support.

The Definitions \([8]\) and \([7]\) imply that \(W^s(L_p(B))\) and \(W^s(L_p(B))\) are contained in the \(L_p(B)\)-closure of the diffusion polynomials. In the proof of Theorem \([3,1]\) we have already used this closure and make it a formal definition here:

**Definition 4.5.** Let \(X_p(X)\) denote the \(L_p(X)\)-closure of the diffusion polynomials \(\bigcup_{N \geq 1} \Pi_N\).

This section is supposed to provide a complementary perspective on our approximation scheme. Here, we derive the reconstruction formulas from a wavelet perspective by summarizing the approach in \([20]\), which was later applied in \([13]\) to build wavelet frames on graphs.

4.2. Tight wavelet frames. In a series of papers \([3,7,17,19,20]\) the theory of diffusion wavelets and localized summation kernels was developed, which shall be the basis of our computational scheme matching the metric entropy. Before we construct wavelets on the diffusion measure space \(X\) though, we review the standard setting of wavelets on the real line. Given a function \(\Phi \in L_2(\mathbb{R})\) (sometimes called the mother
wavelet), we define \( \Phi_{0,y}(x) = \Phi(x - y) \), for \( y \in \mathbb{Z} \). We call a function \( \Psi \in L_2(\mathbb{R}) \) an orthonormal wavelet (or father wavelet) if the collection
\[
\{ \Phi_{0,y} : y \in \mathbb{Z} \} \cup \{ \Psi_{j,y} : y \in \mathbb{Z}, \ j = 1, 2, \ldots \}
\]
is an orthonormal basis for \( L_2(\mathbb{R}) \), where \( \Psi_{j,y}(x) = 2^{j/2}\Psi(2^j x - y) \). The parameter \( j = 0, 1, 2, \ldots \) refers to the scaling, and we therefore speak of a multiscale system. Clearly, \( \Phi \) and \( \Psi \) are closely tied to each other, and the definitions of \( \Phi_{0,y} \) and \( \Psi_{j,y} \) are equivalent to
\[
\widehat{\Phi}_{0,y}(\omega) = \widehat{\Phi}(\omega) e^{-2\pi i \omega y}, \quad \text{and} \quad \widehat{\Psi}_{j,y}(\omega) = 2^{-j/2} \widehat{\Psi}(2^{-j} \omega) e^{-2\pi i \omega 2^{-j} y},
\]
respectively, where the Fourier transform of \( f \) with weights \( w_{\mu} \) combined with the quadrature property of \( \Psi \).

Next, we suppose that the strong product assumption holds. In [26, Theorem 3], the relation (18) was then
\[
|\langle f, e_\omega \rangle| = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx.
\]
Here, we have applied the notation \( e_\omega(x) = e^{2\pi i \omega x} \), so that \( \{ e_\omega \}_{\omega \in \mathbb{R}} \) are the eigenfunctions of the Laplace operator on \( \mathbb{R} \).

To derive wavelets on the diffusion measure space \( X \), we shall use [15] as a guiding scheme. We replace \( \mathbb{R} \) with \( X \) and use the notation of Section 27. In analogy to (10), the Fourier transform is defined by
\[
\hat{f}(k) = \langle f, \varphi_k \rangle = \int_X f(x) \varphi_k^*(x) d\mu(x).
\]
If \( X \) is a smooth Riemannian manifold, then the functions \( \{ \varphi_k \}_{k=0}^\infty \) can be the eigenfunctions of the Laplace-Beltrami operator on \( X \), but it is not a requirement, and we only suppose that \( X \) is a diffusion measure space. Let \( h, g : \mathbb{R}_+ \rightarrow \mathbb{R} \) be continuous functions with \( h(x), g(x) \rightarrow 0 \), for \( x \rightarrow \infty \) sufficiently fast, such that
\[
\hat{\Phi}_{0,y}(k) := h(\lambda_k) \varphi_k^*(y), \quad \hat{\Psi}_{j,y}(k) := g(2^{-j} \lambda_k) \varphi_k^*(y),
\]
is square-summable in \( k \), so that \( \Phi_{0,y} \) and \( \Psi_{j,y} \) are well-defined. Note that (17) is the analogue of (15) in the sense that \( h \) and \( g \) play the roles of \( \hat{\Phi} \) and \( \hat{\Psi} \), respectively, \( \lambda_k \) replaces the frequency \( \omega \), and \( j \) is still the scaling parameter. To match the translation in (15) given by \( 2^{-j}y \), we will choose \( y \) in (17) depending on the scaling parameter \( j \), which needs some preparation.

To simplify notation, let \( N_j := 2^{j+1} \) and suppose that \( \{ \mu_j \}_{j=0}^\infty \) is a family of quadrature measures of order \( (N_j)^{\infty}_{j=0} \), respectively. Note that \( \Psi_{j,y} \in \Pi_{N_j} \) holds. In addition, suppose that \( h : \mathbb{R}_+ \rightarrow \mathbb{R} \) is a nonincreasing function with support in \([0, 1]\) and \( h(t) = 1 \) on \([0, c_0]\), where \( 0 < c_0 < 1 \). If we define \( g(t) := \sqrt{h^2(t/2) - h^2(t)} \), then
\[
|h(\lambda_k)|^2 + \sum_{j=1}^\infty |g(2^{-j} \lambda_k)|^2 = 1, \quad \text{for all } k = 0, 1, \ldots
\]
Next, we suppose that the strong product assumption holds. In [26, Theorem 3], the relation (18) was then combined with the quadrature property of \( \{ \mu_j \}_{j=0}^\infty \) to derive, for all \( f \in L_2(X) \),
\[
f = \sum_{j=0}^\infty \int_X \langle f, \Psi_{j,y} \rangle \Psi_{j,y} d\mu_j(y), \quad \|f\|_{L_2} = \sum_{j=0}^\infty \int_X |\langle f, \Psi_{j,y} \rangle|^2 d\mu_j(y),
\]
where we have applied \( \Psi_{0,y} := \Phi_{0,y} \). Thus, if the support of \( \mu_j \) is compact and given by \( \{y_{j,1}, \ldots, y_{j,n_j}\} \) with weights \( w_{j,i} := \mu_j(\{y_{j,i}\}) \), then the system
\[
\{ \sqrt{w_{j,i}} \Psi_{j, y_{j,i}} : i = 1, \ldots, n_j, \ j = 0, 1, \ldots \}
\]
is a tight frame for \( L_2(X) \), which refers to the following concept: Given a countable index set \( I \), a collection \( \{f_i : i \in I\} \) in a Hilbert space \( \mathcal{H} \) is called a frame in \( \mathcal{H} \) if there exist two constants \( A, B > 0 \) such that
\[
A \|f\|_{\mathcal{H}}^2 \leq \|\langle (f, f_i)_{i \in I}\rangle_{L_2(I)}\|_{\mathcal{H}}^2 \leq B \|f\|_{\mathcal{H}}^2, \quad \text{for all } f \in \mathcal{H}.
\]
If we can choose \( A = B \) in (20), then \( \{f_i : i \in I\} \) is called a tight frame, and we obtain the reconstruction formula
\[
f = \frac{1}{A} \sum_{i \in I} \langle f, f_i \rangle f_i,
\]
for all $f \in \mathcal{H}$. Thus, although being a much more flexible concept, tight frames provide many features of orthogonal bases. This comes in handy when the construction of an orthogonal basis is cumbersome or even impossible in some situations.

4.3. Summation kernels and wavelet frames. This section is dedicated to write the wavelet expansion in a more compact form by means of a summation kernel.

For some signed Borel measure $\nu$ on $\mathbb{X}$ and $f \in L_1(\mathbb{X},|\nu|)$, we can define, for $N = 2^n$, $n = 0, 1, 2, \ldots$,

$$
\sigma_N(f, \nu) := \sum_{k=0}^\infty H\left(\frac{\lambda_k}{N}\right) \int_\mathbb{X} f(y) \varphi_k^*(y) \varphi_k d\nu(y) = \int_\mathbb{X} f(y) K_N(\cdot, y) d\nu(y).
$$

We shall verify that the approximation $\sigma_N(f, \mu)$ can be thought of as the wavelet expansion in $\mathbb{P}$, where the scales $j$ are bounded by $n$ with $N = 2^{n+1}$. Let $h := \sqrt{H}$, $N = 2^{n+1}$ and $j = 0, \ldots, n$. As before, if we choose $g(t) = \sqrt{h^2\left(\frac{t}{2}\right) - h^2(t)}$ implying, for all $k = 0, 1, \ldots$,

$$
|h(\lambda_k)|^2 + \sum_{j=0}^n |g(2^{-j} \lambda_k)|^2 = H\left(\frac{\lambda_k}{N}\right).
$$

By using the latter and the strong product assumption, a straightforward calculation yields

$$
\sigma_N(f, \mu) = \sum_{k=0}^\infty H\left(\frac{\lambda_k}{N}\right) \hat{f}(k) \varphi_k = \sum_{j=0}^n \int_\mathbb{X} \langle f, \psi_{j,y} \rangle \psi_{j,y} d\mu_j(y).
$$

Thus, $\sigma_N(f, \mu)$ is a wavelet expansion up to the scale $n$.

For a fully discrete scheme, we still need to approximate the inner product $\langle f, \psi_{j,y} \rangle$ using the quadrature measure $\mu_n$, i.e.,

$$
\langle f, \psi_{j,y} \rangle \approx \int_\mathbb{X} f(x) \Psi_{j,y}(x) d\mu_n(x).
$$

By applying $\mu_j$ to approximate $\hat{f}(k)$, we turn into our approximation scheme

$$
\sigma_N(f, \mu_n) := \sum_{k=0}^\infty H\left(\frac{\lambda_k}{N}\right) \varphi_k \int_\mathbb{X} f(y) \varphi_k^*(y) d\mu_n(y)
$$

$$
= \sum_{j=0}^n \int_\mathbb{X} \int_\mathbb{X} f(x) \Psi_{j,y}(x) d\mu_n(x) \Psi_{j,y} d\mu_j(y).
$$

Thus, we make use of $\mu_n$ and $\mu_j$ to derive $\sigma_N(f, \mu_n)$.

4.4. Characterization of global Sobolev spaces and their $n$-widths. We have seen that $\sigma_N$ can be derived from a wavelet expansion. In this section, we shall verify that it can be used to characterize membership in Sobolev spaces.

Fix $1 \leq p \leq \infty$ and suppose now that $(\mu_N)_{N=1}^\infty$ is a uniform family of Marcinkiewicz-Zygmund quadrature measures of order $N$, respectively. According to \cite{6} Inequality (3.13), we have

$$
\|\sigma_N(f, \mu_N)\|_{L_p} \lesssim \|f\|_{\|\mu_N\|, L_p},
$$

as long as $f \in L_p(\mathbb{X},|\mu_N|)$, and the generic constant can be chosen independently of $f$ and $N$. Later, we shall need that also $\|f\|_{\|\mu_N\|, L_p} \lesssim \|f\|_{L_p(\mathbb{X})}$ holds, for all $f \in X_p(\mathbb{X})$ (not just $f \in \Pi_N$), which is obvious for $p = \infty$, because $X_\infty(\mathbb{X})$ consists of continuous functions. For $1 \leq p < \infty$, we have not yet found any explicit example except for the measure $\mu$ itself. Therefore, we shall simply restrict us to $\mu_N = \mu$, $N = 1, 2, 4, \ldots$ in this case.

We can also estimate

$$
\sup_{x \in \mathbb{X}} \int_\mathbb{X} |K_N(x, y)| d\mu_N(y) \lesssim 1,
$$

8
Remark 4.7. Note that (28) is the quadrature version of (13). The transition from one scale to the next is defined by
\[
\tau_n(f, \mu_N) := \sigma_N(f, \mu_N) - \sigma_{N/2}(f, \mu_N).
\]
Next, we shall characterize global Sobolev smoothness using \(\sigma_N\) and \(\tau_N\), see [6, 17, 19, 20].

**Theorem 4.6.** Suppose that \(1 \leq p \leq \infty\) and assume that \((\mu_N)_{N=1}^\infty\) is a uniform family of Marcinkiewicz-Zygmund quadrature measures of order \(N\), respectively, if \(p = \infty\). For \(1 \leq p < \infty\) we choose \(\mu_N = \mu\), \(N = 1, 2, 4, \ldots\). Assume further that \(H\) is a low-pass filter. Then, for all \(f \in W^s(L_p(\mathbb{X}))\), we have
\[
\|f - \sigma_N(f, \mu_N)\|_{L_p(\mathbb{X})} \lesssim N^{-s}\|f\|_{W^s(L_p(\mathbb{X}))}, \quad \|\tau_N(f, \mu_N)\|_{L_p(\mathbb{X})} \lesssim N^{-s}\|f\|_{W^s(L_p(\mathbb{X}))},
\]
where the generic constants do not depend on \(N\) or \(f\). On the other hand, if, for \(f \in L_p(\mathbb{X})\), there are generic constants not depending on \(N\) such that
\[
\|f - \sigma_N(f, \mu_N)\|_{L_p(\mathbb{X})} \lesssim N^{-s}, \quad \text{or} \quad \|\tau_N(f, \mu_N)\|_{L_p(\mathbb{X})} \lesssim N^{-s},
\]
then \(f \in W^s(L_p(\mathbb{X}))\).

**Remark 4.7.** Let us point out again that we suppose \(\mu_N = \mu\), \(N = 1, 2, 4, \ldots\) for \(1 \leq p < \infty\). In this case, the term \(\sigma_N(f, \mu_N)\) contains spectral information \(\hat{f}(k)\) since \(\sigma_N(f, \mu) = \sum_{k=0}^\infty \hat{H}(\frac{N}{N_k}) \hat{f}(k)\phi_k\). If \(p = \infty\) and \(\mu_N\) has finite support, then we have an approximation scheme that uses finitely many training data consisting of function evaluations at scattered data points.

We have already determined the asymptotics of the Kolmogorov metric entropy. Here, we shall determine the \(n\)-widths for the global Sobolev space.

**Theorem 4.8.** The \(n\)-widths of \(\Pi^s_p(L_p(\mathbb{X}))\) in \(L_p(\mathbb{X})\) satisfy
\[
r_n^{-s/\alpha} \asymp \mathcal{K}_n \asymp \mathcal{L}_n \asymp \mathcal{G}_n \asymp \mathcal{B}_n.
\]

**Proof.** According to [25, Theorem 1.1], we have the following relationships among the four widths:
\[
\mathcal{L}_n \asymp \mathcal{K}_n, \quad \mathcal{G}_n \asymp \mathcal{B}_n.
\]
The operator \(\sigma_N\) is bounded on \(L_p(\mathbb{X})\) and \(\sigma_N(f)\) is an element in \(\Pi_N\). Since \(\dim(\Pi_N) \asymp N^\alpha\), we have \(n^{1/\alpha} \asymp N\). Theorem 4.6 yields \(r_n^{-s/\alpha} \asymp \mathcal{L}_n\).

To verify the appropriate lower bound on \(\mathcal{B}_n\), we take the subspace \(\Pi_{N+1}\) and aim to derive a generic constant \(c > 0\) such that \(cN^{-s}\Pi_{N+1} \subset \Pi_{N}^s(L_p(\mathbb{X}))\), where \(\Pi_{N+1} = \{f \in \Pi_{N+1} : \|f\|_{L_p(\mathbb{X})} \leq 1\}\). For \(f \in \Pi_{N+1}\), let \(g := rN^{-s}f\). We obtain \(\|g\|_{L_p(\mathbb{X})} \leq rN^{-s}\) and, for \(M > N\), we derive \(E(g, \Pi_M, L_p(\mathbb{X})) = 0\). The choice \(M \leq N\) yields
\[
E(g, \Pi_M, L_p(\mathbb{X})) = rN^{-s}\|f - \sigma_M(f)\|_{L_p(\mathbb{X})} \lesssim rN^{-s},
\]
because \(\|f\|_{L_p(\mathbb{X})} \leq 1\) and \(\|\sigma_M(f)\|_{L_p(\mathbb{X})} \lesssim \|f\|_{L_p(\mathbb{X})}\). Thus, \(\sup_{M \geq 1} M^sE(g, \Pi_M, L_p(\mathbb{X})) \lesssim r\) holds. We have verified that there is a generic constant \(c\) such that \(cN^{-s}\mathcal{K}_{N+1} \subset \Pi_{N+1}^s(L_p(\mathbb{X}))\). Therefore, we obtain that \(\mathcal{B}_n \gtrsim rN^{-s/\alpha}\) holds.

It should be mentioned that upper bounds on the Kolmogorov and linear \(n\)-widths for compact Riemannian manifolds were already derived in [11], where also the exact asymptotics were obtained for compact homogeneous manifolds. The Kolmogorov \(n\)-width for Besov spaces on the sphere was studied in [4].

5. Bit representation in global Sobolev spaces

This section is dedicated to verify that linear quantization of the approximation scheme \(\sigma_N(f, \mu_N)\) enables bit representations matching the optimality bounds derived in Theorem 3.1 up to a logarithmic factor. First, we recall the formula (21),
\[
\sigma_N(f, \mu_N) = \sum_{k=0}^{\infty} H\left(\frac{\lambda_k}{N}\right) \int_{\mathbb{X}} f(y)\phi_k^*(y) \phi_k d\mu_N(y),
\]
where \((\mu_N)_{N=1}^\infty\) is a uniform family of Marcinkiewicz-Zygmund quadrature measures of order \(N\), respectively, if \(p = \infty\). Again, if \(1 \leq p < \infty\), then we choose \(\mu_N = \mu\), \(N = 1, 2, 4, \ldots\).
Since $H(t) = 1$, for $t \in [0, 1/2]$ and $H(t) = 0$, for $t > 1$, we observe that $H(\frac{t}{2})H(\frac{t}{2^n}) = H(\frac{t}{2^n})$. If $(\nu_N)_{N=1}^{\infty}$ is a family of quadrature measures of order $N$, respectively, then a straight-forward calculation using the strong product assumption yields

$$\sigma_N(f, \mu_N) = \int_X \sigma_N(f, \mu_N, y) \sum_{k=0}^{\infty} H(\frac{\lambda_k}{2^n}) \phi_k^*(y) d\nu_N(y) \phi_k,$$

The representation (31) involves the quadrature measure $\nu_N$ and the Marcinkiewicz-Zygmund quadrature measure $\mu_N$. To design the final approximation scheme, we fix some $S > 1$ and apply the quantization

$$I_N(f, \mu_N, y) = [N^S \sigma_N(f, \mu_N, y)],$$

and define the actual approximation by

$$\sigma^*_N(f, \mu_N, \nu_N) := N^{-S} \int_X I_N(f, \mu_N, y) \sum_{k=0}^{\infty} H(\frac{\lambda_k}{2^n}) \phi_k^*(y) d\nu_N(y) \phi_k.$$

In other words, we replace $\sigma_N(f, \mu_N, y)$ in (31) with a number on the grid $\frac{1}{N}Z$.

We have the following result for the ball $\mathcal{F}(L_p(\mathbb{X}))$ of radius $r$ of the global Sobolev space given by (30). It extends results in [5] from compact Riemannian manifolds and $p = \infty$ to diffusion measure spaces and to the entire range $1 \leq p \leq \infty$:

**Theorem 5.1.** Suppose that $(\mu_N)_{N=1}^{\infty}$ is a uniform family of Marcinkiewicz-Zygmund quadrature measures of order $N$, respectively, if $p = \infty$. For $1 \leq p < \infty$ we choose $\mu_N = \mu$, $N = 1, 2, 4, \ldots$. Assume further that $H$ is a low-pass filter. We also suppose that $(\nu_N)_{N=1}^{\infty}$ are Marcinkiewicz-Zygmund quadrature measures with $\# \text{supp} (\nu_N) \lesssim N^\alpha$. For fixed $s > 0$ and $S > \max(1, s)$, we apply the discretizations (32) and (33). Then there is a constant $c > 0$ such that, for all $f \in \mathcal{F}(L_p(\mathbb{X}))$,

$$||f - \sigma^*_N(f, \mu_N, \nu_N)||_{L_p(\mathbb{X})} \leq crN^{-s}$$

holds. For $crN^{-s} = \varepsilon \leq 1$ and $\varepsilon \leq r$, the number of bits needed to represent all integers $\{I_N(f, \mu_N, y) : y \in \text{supp}(\nu_N)\}$ does not exceed a positive constant (independent of $\varepsilon$ and $r$) times

$$(r/\varepsilon)^{\alpha/s}(1 + \log_2(r/\varepsilon)).$$

**Remark 5.2.** To avoid some technicalities in the presentation, we stated the Theorems 3.1 and 6.7 using Sobolev smoothness. Straight-forward modifications would cover the full scale of Besov spaces as for the spherical setting in [18].

Before we take care of the proof, we discuss a fairly general example, in which the above assumptions are satisfied. It was pointed out in [2] that if the diffusion measure space $\mathbb{X}$ is a compact Riemannian manifold without boundary and with nonnegative Ricci curvature, $\rho$ the geodesic distance, and $\mu$ the normalized Riemannian volume measure, then the technical assumptions in Theorem 3.1 can indeed be satisfied: the smooth cut-off property is satisfied, and [7] Theorem A.1 implies the strong product assumption. Moreover, the results in [6] Theorem 3.1 imply that there is a uniform family of finitely supported Marcinkiewicz-Zygmund quadrature measures $(\mu_N)_{N=1}^{\infty}$ of order $N$, respectively, such that $\# \text{supp}(\mu_N) \asymp N^\alpha$, where $\alpha$ as in Definition 2.1. Thus, we have the following result:

**Corollary 5.3.** Suppose that the diffusion measure space $\mathbb{X}$ is a compact Riemannian manifold without boundary and with nonnegative Ricci curvature, $\rho$ the geodesic distance, and $\mu$ the normalized Riemannian volume measure. Let $H$ be a low-pass filter. Then there are two uniform families $(\mu_N)_{N=1}^{\infty}$ and $(\nu_N)_{N=1}^{\infty}$ of finitely supported Marcinkiewicz-Zygmund quadrature measures of order $N$, respectively, such that the following holds: For fixed $s > 0$ and $S > \max(1, s)$, the discretizations (32) and (33) yield that there is a constant $c > 0$ such that, for all $f \in \mathcal{F}(L_p(\mathbb{X}))$,

$$||f - \sigma^*_N(f, \mu_N, \nu_N)||_{L_{\infty}(\mathbb{X})} \leq crN^{-s}$$

holds. For $crN^{-s} = \varepsilon \leq 1$ and $\varepsilon \leq r$, the number of bits needed to represent all integers $\{I_N(f, \mu_N, y) : y \in \text{supp}(\nu_N)\}$ does not exceed a positive constant (independent of $\varepsilon$ and $r$) times

$$(r/\varepsilon)^{\alpha/s}(1 + \log_2(r/\varepsilon)).$$
Proof of Theorem 5.1. The triangle inequality yields
\[ \|f - \sigma_N^s(f, \mu_N)\|_{L_p(X)} \lesssim \|f - \sigma_N(f, \mu_N)\|_{L_p(X)} + \|\sigma_N(f, \mu_N) - \sigma_N^s(f, \mu_N, \nu)\|_{L_p(X)}. \]

Since Theorem 4.3 implies \( \|f - \sigma_N(f, \mu_N)\|_{L_p(X)} \lesssim N^{-s}r\|f\|_{W^s(L_p(X))}, \) we only need to take care of the term on the far most right. The quantization (32) immediately yields
\[ |\sigma_N(f, \mu_N, y) - N^{-s}I_N(f, \mu_N, y)| \leq N^{-s}, \quad \text{for all } y \in \text{supp}(\nu_N), \]
so that (31) and (28) imply
\[ \|\sigma_N(f, \mu_N) - \sigma_N^s(f, \mu_N)\|_{L_p(X)} = \| \int_X (\sigma_N(f, y) - N^{-s}I_N(f, \mu_N, y))K_N(. , y) d\nu_N(y) \|_{L_p(X)} \lesssim N^{-s} \leq N^{-s}. \]
Hence, we have derived (34).

To tackle (35), we observe that the localization property (12) yields \( \|g\|_{L_\infty} \lesssim N^\alpha \|g\|_{L_1}, \) for all \( g \in \Pi_N, \) see also [19, Lemma 5.5] for more general Nikolskii inequalities. We apply (36) and then use \( \sigma_N(f, \mu_N) \in \Pi_N \) with \( L_p \to L_1, \) which yields
\[ |I_N(f, \mu_N, y)| \lesssim N^\alpha \|\sigma_N(f, \mu_N)\|_{L_\infty(X)} \lesssim N^{\alpha+s} \|\sigma_N(f, \mu_N)\|_{L_p(X)}. \]

According to [19] Theorem 5.1, \( \|\sigma_N(f, \mu_N)\|_{L_p(X)} \lesssim \|f\|_{L_p(X)} \) holds. Since \( f \) is contained in the ball of radius \( r, \) so that \( \|f\|_{L_p(X)} \leq r, \) we see that
\[ |I_N(f, \mu_N, y)| \lesssim \epsilon N^{\alpha+s}. \]
Thus, the number of bits needed to represent each single \( I_N(f, \mu_N, y) \) at most \( \log_2(c_1 \epsilon N^{\alpha+s}), \) where \( c_1 \geq 1 \) is a positive constant. Note that we can assume that \( c_1 \epsilon N^{\alpha+s} \geq 1 \) because otherwise \( I_N(f, \mu_N, y) \) would be zero. Since \( \#\text{supp}(\nu_N) \lesssim N^\alpha, \) we have \( \#\{I_N(f, \mu_N, y) : y \in \text{supp}(\nu_N)\} \lesssim N^\alpha. \) Therefore, the total number of bits needed to represent all numbers \( \{I_N(f, \mu_N, y) : y \in \text{supp}(\nu_N)\} \) is at most \( c_2 N^{\alpha} \log_2(c_1 \epsilon N^{\alpha+s}), \) where \( c_2 \) is a positive constant. By using \( \epsilon N^{-s} = \epsilon \leq r \) and \( \epsilon \leq 1, \) we derive that the number of necessary bits does not exceed
\[ c_2 \epsilon^{\alpha/s}(r/\epsilon)^{\alpha/s} \log_2(c_1 \epsilon N^{\alpha+s}) \lesssim (r/\epsilon)^{\alpha/s} \log_2(c_1 \epsilon^{(S+\alpha)/s} N^{\alpha+s}/\epsilon) \lesssim (r/\epsilon)^{\alpha/s} \log_2((c_1 \epsilon)^{S+\alpha}/\epsilon) \lesssim (r/\epsilon)^{\alpha/s}(1 + \log_2(r/\epsilon)), \]
which concludes the proof.

The Theorem 5.1 and Corollary 5.3 yield that our bit representation scheme is optimal with respect to the metric entropy as derived in Theorem 5.1 at least up to a logarithmic factor.

6. Bit representation of locally smooth functions

Our metric entropy result in Theorem 5.1 covers \( W^s(L_p(B)), \) where \( B \) is some ball in \( X, \) and in the previous section we derived a bit-representation scheme for the global Sobolev space, i.e., \( B = X. \) It turns out that the case \( B \subseteq X \) is more involved because we do not have results that characterize \( W^s(L_p(B)) \) by means of \( \sigma_N \) and \( \tau_N. \) In fact, \( \sigma_N \) and \( \tau_N \) require functions to be defined globally so that one would be forced to deal with boundary effects. On the other hand, \( B \) itself may not be a diffusion measure space satisfying all required assumptions. Thus, we try to circumvent such difficulties by defining another Sobolev space that "resembles" \( W^s(L_p(B)) \) and for which we can construct a bit representation scheme.

6.1. Characterization of local smoothness by local approximation rates. Before we can discuss local smoothness, few technical details need to be introduced and we make use of \( C^\infty(X) := \bigcap_{s > 0} W^s(L_\infty(X)). \)

**Definition 6.1.** We say that \( X \) satisfies the smooth cut-off property if for any \( s > 0 \) and any two concentric balls \( B', B \) with \( B' \subseteq B \) there is \( \phi \in C^\infty(X) \) such that \( \phi = 1 \) on \( B' \) and \( \phi \) vanishes outside of \( B. \)

Note that any smooth manifold satisfies the smooth cut-off property.

**Definition 6.2.** Given \( x \in X, \) the local Sobolev space in \( x \) is denoted by \( W^s(L_p(X), x) \) and defined as the collection of \( f \in X_p(X) \) such that there is an open ball \( B \) containing \( x \) with \( f \phi \in W^s(L_p(X)), \) for all \( \phi \in C^\infty(X) \) with support in \( B. \)
It turns out that the approximation rate of \( \sigma_N(f, \mu_N) \) characterizes the Sobolev smoothness of \( f \), see [6] [17] [19] [20]:

**Theorem 6.3.** Let \( X \) satisfy the smooth cut-off property. For \( p = \infty \), suppose that \( (\mu_N)_{N=1}^{\infty} \) is a uniform family of Marcinkiewicz-Zygmund quadrature measures of order \( N \), respectively. For \( 1 \leq p < \infty \), we choose \( \mu_N = \mu, \ N = 1, 2, 4, \ldots \) If \( H \) is a low-pass filter, then the following points are equivalent:

(i) \( f \in W^s(L_p(X), x) \),

(ii) there is a ball \( B \) centered at \( x \) such that

\[
\|\tau_N(f, \mu_N)\|_{L_p(B)} \lesssim N^{-s},
\]

(iii) there is a ball \( B \) centered at \( x \) such that

\[
\|f - \sigma_N(f, \mu_N)\|_{L_p(B)} \lesssim N^{-s}.
\]

Note that the generic constants in (37) and (38) may depend on \( f \). Nonetheless, the above theorem characterizes local Sobolev spaces by means of approximation rates and decay properties of \( \sigma_N \) and \( \tau_N \), respectively. The local Sobolev space \( W^s(L_p(X), x) \), for \( x \in X \), is not endowed with any norm. In view of Theorem 6.3, we fix some ball \( B \) and introduce a new Sobolev space:

**Definition 6.4.** Let \( B \) be a nontrivial ball in \( X \). For \( p = \infty \), suppose that \( (\mu_N)_{N=1}^{\infty} \) is a uniform family of Marcinkiewicz-Zygmund quadrature measures of order \( N \), respectively. For \( 1 \leq p < \infty \), we choose \( \mu_N = \mu, \ N = 1, 2, 4, \ldots \) If \( H \) is a low-pass filter, then we define the local Sobolev space in \( B \) by

\[
W^s(L_p(X), B) := \{ f \in X_p(X) : \|f\|_{W^s(L_p(X), B)} < \infty \}
\]

endowed with its norm

\[
\|f\|_{W^s(L_p(X), B)} := \|f\|_{L_p(X)} + \sup_{N \geq 1} N^s \|\tau_N(f, \mu_N)\|_{L_p(B)},
\]

where \( \tau_N \) is given by [20].

Note that if \( p = \infty \), then the space \( W^s(L_p, B) \) implicitly depends on the uniform family \( (\mu_N)_{N=1}^{\infty} \) of Marcinkiewicz-Zygmund quadrature measures of order \( N \), respectively. As opposed to \( W^s(L_p(B)) \) defined in [6], the space \( W^s(L_p(X), B) \) consists of functions defined globally that inherit Sobolev smoothness locally. By definition, we have

\[
\|\tau_N(f, \mu_N)\|_{L_p(B)} \lesssim N^{-s} \|f\|_{W^s(L_p(X), B)}, \quad \|f - \sigma_N(f, \mu_N)\|_{L_p(B)} \lesssim N^{-s} \|f\|_{W^s(L_p(X), B)},
\]

where the generic constants can be chosen independently of \( f \in W^s(L_p(X), B) \) (indeed, the constants can be 1). Since \( \sigma_N(f, \mu_N) \) is a diffusion polynomial, we observe that

\[
W^s(L_p(X), B)|_B \hookrightarrow W^s(L_p(B)).
\]

However, we cannot claim that the reverse embedding also holds.

It should be mentioned that \( \sigma_N(f, \mu_N) \) in (39) approximates \( f \) locally but its definition needs global knowledge of \( f \) or at least on sup\(\mu_N\) if \( p = \infty \). To enable the design of an approximation scheme that involves local information on \( f \) exclusively, we define one more Sobolev space by using some cut-off function:

**Definition 6.5.** For some fixed \( \phi \in C^\infty(X) \), define

\[
W^s(L_p(X), \phi) := \{ f \in X_p(X) : f\phi \in W^s(L_p(X)) \}
\]

endowed with the norm \( \|f\|_{W^s(L_p(X), \phi)} := \|f\|_{L_p(X)} + \sup_{N \geq 1} N^s E(f\phi, \Pi_N, L_p(X)) \).

To study local approximation, choose two concentric balls \( B', B \) with \( B' \subset B \). If \( X \) satisfies the smooth cut-off property, then we can fix some \( \phi \in C^\infty(X) \) that is one on \( B' \) and zero outside of \( B \). The Definition [19] yields that \( f \in W^s(L_p(X), \phi) \) implies

\[
\|f\phi - \sigma_N(f\phi, \mu_N)\|_{L_p(X)} \lesssim N^{-s} \|f\phi\|_{W^s(L_p(X), \phi)}, \quad \|f - \sigma_N(f\phi, \mu_N)\|_{L_p(B')} \lesssim N^{-s} \|f\phi\|_{W^s(L_p(X), \phi)}.
\]
Moreover, the localization property of \( K_N \) yields \( W^s(L_p(\mathbb{X}), \phi) \hookrightarrow W^s(L_p(\mathbb{X}), B) \). In the subsequent section, we shall consider balls of radius \( r \) for both spaces,

\[
W_r^p(\mathbb{X}, B) := \{ f \in X_p(\mathbb{X}) : \| f \|_{W^s(L_p(\mathbb{X}), B)} \leq r \},
\]

\[
W_r^p(\mathbb{X}, \phi) := \{ f \in X_p(\mathbb{X}) : \| f \|_{W^s(L_p(\mathbb{X}), \phi)} \leq r \},
\]

and aim to develop approximation schemes requiring only few bits.

**Remark 6.6.** Intuitively, both spaces \( W^s(L_p(\mathbb{X}), B') \) and \( W^s(L_p(\mathbb{X}), \phi) \) almost coincide with the proper local Sobolev space \( W^s(L_p(B')) \), for which we have computed the metric entropy.

**6.2. Local bit-representation of Sobolev functions.** To design an approximation scheme for the spaces \( W_r^p(\mathbb{X}, B) \) and \( W_r^p(\mathbb{X}, \phi) \), let \( B \) be a ball in \( \mathbb{X} \) and let \( B' \subset B \) be another ball concentric with \( B \) and of radius strictly less. It will turn out that the following scheme enables us to approximate \( f \) on \( B' \). For some fixed \( S > 1 \), we apply the quantization \( I_N(f, \mu_N, y) \) as in (42) and define the local approximation in a different fashion by

\[
\sigma_N^r(f, \mu_N, \nu_N, B) := N^{-S} \int_B I_N(f, \mu_N, y) \sum_{k=0}^{\infty} H\left(\frac{\lambda_k}{2N}\right) \varphi_k(y) d\nu_N(y) \varphi_k,
\]

We have the following result for the ball \( W_r^p(\mathbb{X}, B) \) of radius \( r \) of the localized Sobolev space given by (11):

**Theorem 6.7.** Suppose that \( \mathbb{X} \) satisfies the smooth cut-off property and that \( (\mu_N)_{N=1}^{\infty} \) is a uniform family of Marcinkiewicz-Zygmund quadrature measures of order \( N \), respectively, if \( p = \infty \). For \( 1 \leq p < \infty \) we choose \( \mu_N = \mu \), \( N = 1, 2, 4, \ldots \). Assume further that \( H \) is a low-pass filter. Let \( B, B' \) be two concentric balls, so that \( B' \subset B \). We also suppose that \( (\nu_N)_{N=1}^{\infty} \) are Marcinkiewicz-Zygmund quadrature measures with \( \# \text{supp}(\nu_N) \lesssim N^a \). For fixed \( s > 0 \) and \( S > \max(1, s) \), we apply the discretizations (32) and (13). Then there is a constant \( c > 0 \) such that, for all \( f \in W_r^p(\mathbb{X}, B) \),

\[
\| f - \sigma_N^r(f, \mu_N, \nu_N, B) \|_{L_p(B')} \leq crN^{-s}
\]

holds. For \( crN^{-s} = \varepsilon \leq 1 \) and \( \varepsilon \leq r \), the number of bits needed to represent all integers \( \{ I_N(f, \mu_N, y) : y \in \text{supp}(\nu_N) \cap B \} \) does not exceed a positive constant (independent of \( \varepsilon \) and \( r \)) times

\[
(r/\varepsilon)^{a/s}(1 + \log_2(r/\varepsilon)).
\]

**Proof of Theorem 6.7.** For \( f \in W_r^p(\mathbb{X}, B) \), we use the localization property (11) and the embedding \( L_p \hookrightarrow L_1 \) in the compact case to derive, for \( x \in B' \),

\[
\int_{\mathbb{X} \setminus B} |\sigma_N(f, \mu_N, y)K_N(x, y)| d\nu_N(y) \lesssim N^{-S} \| \sigma_N(f, \mu_N) \|_{\nu_N, L_p} \lesssim N^{-S} \| \sigma_N(f, \mu_N) \|_{\mu_N, L_p}.
\]

The latter estimate holds because both \( (\nu_N)_{N=1}^{\infty} \) and \( (\mu_N)_{N=1}^{\infty} \) are uniform families of Marcinkiewicz-Zygmund measures. The quantization (32) immediately yields

\[
|\sigma_N(f, \mu_N, y) - N^{-S} I_N(f, \mu_N, y)| \leq N^{-S}, \quad \text{for all} \ y \in \text{supp}(\nu_N).
\]

By using (17), (31), and (28), we derive

\[
\| \sigma_N(f, \mu_N) - \sigma_N^r(f, \mu_N, B) \|_{L_p(B')} = \| \sigma_N(f, \mu_N) - \int_B N^{-S} I_N(f, \mu_N, y)K_N(\cdot, y) d\nu_N(y) \|_{L_p(B')} \lesssim \| \sigma_N(f, \mu_N) - \int_B \sigma_N(f, y)K_N(\cdot, y) d\nu_N(y) \|_{L_p(B')} + N^{-S}.
\]

Next, we make use of (31) and (45) to obtain

\[
\| \sigma_N(f, \mu_N) - \sigma_N^r(f, \mu_N, B) \|_{L_p(B')} \lesssim \int_{\mathbb{X} \setminus B} \sigma_N(f, \mu_N, y)K_N(\cdot, y) d\nu_N(y) \|_{L_p(B')} + N^{-S} \lesssim N^{-S} \| f \|_{\mu_N, L_p} + N^{-S} \lesssim rN^{-S}.
\]
where the very last inequality is due to $f \in W_p^r(L_p, B)$. Here, it is important that we assume $\mu_N = \mu$, for $1 \leq p < \infty$, so that $\|f\|_{L_p(\mu_N)} \lesssim \|f\|_{L_p(\mu)} \leq r$ holds for the entire range $1 \leq p \leq \infty$. The triangle inequality with (49) and the above estimate yield

$$
\|f - \sigma_N(f, \mu_N, B)\|_{L_p(B')} \lesssim \|f - \sigma_N(f, \mu_N)\|_{L_p(B')} + \|\sigma_N(f, \mu_N) - \sigma_N(f, \mu_N, B)\|_{L_p(B')} \lesssim rN^{-s} + rN^{-s} \lesssim rN^{-s},
$$

which verifies (50).

For the remaining part, we can follow the lines of the proof of Theorem 5.1. \hfill \Box

Note that Theorem 6.7 stills requires global knowledge of $f$ because we need to build $\sigma_N(f, \mu_N)$. Due to the localization property of the kernel $K_N$, we only need to feed in local information by using a cut-off function:

**Theorem 6.8.** Under the same assumption as in Theorem 6.7, let $\phi \in C^\infty(\mathbb{X})$ be one on $B'$ and zero outside of $B$. Then there is a constant $c > 0$ such that, for all $f \in W_p^r(L_p, \phi)$,

$$
(48) \quad \|f - \sigma_N(f, \mu_N, \nu_N, B)\|_{L_p(B')} \leq crN^{-s}, \quad \|f - \sigma_N(f, \mu_N, \nu_N, \mathbb{X})\|_{L_p(\mathbb{X})} \leq crN^{-s}
$$

hold. For $crN^{-s} = \varepsilon \leq r$, the number of bits needed to represent all integers $\{I_N(f, \mu_N, y) : y \in \text{supp}(\nu_N)\}$ does not exceed a positive constant (independent of $\varepsilon$ and $r$) times

$$
(49) \quad (r/\varepsilon)^{\alpha/s}(1 + \log_2(r/\varepsilon)).
$$

**Proof.** Since $W^r(L_p, \phi) \hookrightarrow W^s(L_p, B)$, we can simply replace $f$ with $f \phi$ in Theorem 6.7 to derive (51) and the left-hand side of (48). It only remains to check the right-hand side of (48). By using (14) and (28), we obtain

$$
\|\sigma_N(f, \mu_N) - \sigma_N(f, \mu_N, \mathbb{X})\|_{L_p(\mathbb{X})} \lesssim N^{-s},
$$

so that we can derive

$$
\|f - \sigma_N(f, \mu_N, \mathbb{X})\|_{L_p(\mathbb{X})} \lesssim \|f - \sigma_N(f, \mu_N)\|_{L_p(\mathbb{X})} + \|\sigma_N(f, \mu_N) - \sigma_N(f, \mu_N, \mathbb{X})\|_{L_p(\mathbb{X})} \lesssim rN^{-s} + N^{-s} \lesssim rN^{-s}. \hfill \Box
$$

The estimate (49) can be derived by following the lines in the proof of Theorem 5.1.

**Appendix A. Summary of the technical assumptions**

The asymptotic bounds on the metric entropy in Theorem 3.1 hold for any diffusion measure space $\mathbb{X}$ as introduced in Definition 2.1. Our computational scheme that can achieve this bound up to a logarithmic factor needs few additional technical assumptions that are distributed within the present paper. Here, we list all the required assumptions for the sake of completeness:

(I) $\mathbb{X}$ is a diffusion measure space (Definition 2.1),

(ii) the strong product assumption holds (Definition 4.1),

(iii) there exists a uniform family $(\nu_N)_{N=1}^\infty$ of Marcinkiewicz-Zygmund quadrature measures of order $N$, respectively, satisfying $\# \text{supp}(\nu_N) \lesssim N^\alpha$ (Definitions 4.2, 4.3, 4.4),

(iv) the smooth cut-off property holds (Definition 6.1).

Condition (I) is the general framework, and the additional conditions are more technical details that may still hold in many practical situations. Note that (14) is only needed in Section 3 and it is well-known that this holds for smooth manifolds. All of the above conditions hold for compact homogeneous manifolds, e.g., the sphere and the Grassmann manifold, if the function system $\{\varphi_k\}_{k=0}^\infty$ are eigenfunctions of the Laplace operator, cf. [10]. Moreover, it was pointed out in [2] that the conditions are also satisfied for smooth compact Riemannian manifolds without boundary and with nonnegative Ricci curvature. Uniform families of Marcinkiewicz-Zygmund quadrature measures $(\mu_N)_{N=1}^\infty$ were then constructed in [4], such that $\# \text{supp}(\mu_N) \asymp N^\alpha$. We expect that the technical assumptions may hold in a much wider class of spaces and this is indeed part of our ongoing research.
Acknowledgment

M.E. has been funded by the Vienna Science and Technology Fund (WWTF) through project VRG12-009. The research of F.F. was partially funded by Deutsche Forschungsgemeinschaft grant FI 883/3-1. The authors also thank H. M. Mhaskar for many fruitful discussions.

References

[1] G. Brown, Kolmogorov width of classes of smooth functions on the sphere $S^{d-1}$, J. Complexity 18 (2002), 1001–1023.
[2] C. K. Chui and H. N. Mhaskar, Smooth function extension based on high dimensional unstructured data, to appear in Mathematics of Computation (2013).
[3] R. R. Coifman, S. Lafon, A. B. Lee, M. Maggioni, B. Nadler, F. J. Warner, and S. W. Zucker, Geometric diffusions as a tool for harmonic analysis and structure definition of data. part i: Diffusion maps, Proc. Nat. Acad. Sci. 102 (2005), 7426–7431.
[4] D. Dung and T. Ullrich, $n$-widths and $\varepsilon$-dimensions for high-dimensional approximations, to appear in Found. Comput. Math. (2013).
[5] M. Ehler and F. Filbir, $\varepsilon$-coverings of sobolev spaces on data-defined manifolds, preprint (2013).
[6] F. Filbir and H. N. Mhaskar, A quadrature formula for diffusion polynomials corresponding to a generalized heat kernel, J. Fourier Anal. Appl. 16 (2010), no. 5, 629–657.
[7] ———, Marcinkiewicz–Zygmund measures on manifolds, Journal of Complexity 27 (2011), no. 6, 568–596.
[8] S. Foucart, A. Pajor, H. Rauhut, and T. Ullrich, The gelfand widths of $l_p$-balls for $0 < p \leq 1$, J. Complexity 26 (2010), 629–640.
[9] M. Gavish, B. Nadler, and R. R. Coifman, Multiscale wavelets on trees, graphs and high dimensional data: Theory and applications to semi supervised learning, ICML, 2010, pp. 367–374.
[10] D. Geller and I. Z. Pesenson, Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds, J. Geom. Anal. 21 (2011), no. 2, 334–371.
[11] ———, Kolmogorov and linear widths of balls in Sobolev spaces on compact manifolds, arXiv:1104.1632v2 (2012).
[12] A. Grigor’yan, Estimates of heat kernels on riemannian manifolds, Spectral Theory and Geometry. ICMS Instructional Conference, Edinburgh, 1998 (B. Davies and Yu. Safarov, eds.), London Math. Soc. Lecture Notes, vol. 273, Cambridge Univ. Press, 1999, pp. 140–225.
[13] D. K. Hammond, P. Vandergheynst, and R. Gribonval, Wavelets on graphs via spectral graph theory, Appl. Comput. Harmon. Anal. 30 (2011), 129–150.
[14] A. N. Kolmogorov, über die beste annährung von funktionen einer gegebenen funktionenklasse, Ann. Math. 37 (1936), 107–111.
[15] A. N. Kolmogorov and V. M. Tihomirov, $\varepsilon$-entropy and $\varepsilon$-capacity of sets in function spaces, Uspehi Mat. Nauk., vol. 14 (1959), 3-86, English transl. in Amer. Math. Soc. Transl. 17, 277-364. MR0112032 (22:2890).
[16] G. G. Lorentz, Metric entropy and approximation, Bull. Amer. Math. Soc. 72 (1966), no. 6, 903–937.
[17] M. Maggioni and H. N. Mhaskar, Diffusion polynomial frames on metric measure spaces, Appl. Comput. Harmon. Anal. 24 (2008), no. 3, 329–353.
[18] H. N. Mhaskar, On the representation of functions on the sphere using finitely many bits, Appl. Comput. Harmon. Anal. 18 (2005), no. 3, 215–233.
[19] ———, Eigenets for function approximation, Appl. Comput. Harmon. Anal. 29 (2010), 63–87.
[20] ———, A generalized diffusion frame for parsimonious representation of functions on data defined manifolds, Neural Networks 24 (2011), 345–359.
[21] C. A. Micchelli and T. J. Rivlin, Optimal estimation in approximation theory, ch. A survey of optimal recovery, pp. 1–54, Plenum Press, 1977.
[22] B. Nadler and M. Galun, Fundamental limitations of spectral clustering, Neural Information Processing systems 19 (2007), 1017–1024.
[23] E. Nowak, Optimal recovery and $n$-widths for convex classes of functions, J. Approx. Theory 80 (1995), no. 3, 390–408.
[24] A. Pinkus, $n$-widths in approximation theory, Springer-Verlag, New York, 1985.
[25] ———, $n$-widths of Sobolev spaces in $L^p$, Constr. Approx. 1 (1985), 15–62.
[26] J. Prestin and H. N. Mhaskar, Polynomial frames: a fast tour, Approximation Theory XI, Gatlinburg, 2004 (Brentwood) (C. K. Chui, M. Neamtu, and L. Schumaker, eds.), Nashboro Press, 2005, pp. 287–318.
[27] S. T. Roweis and L. K. Saul, Nonlinear dimensionality reduction by locally linear embedding, Science 290 (2000), no. 5500, 2323–2326.
[28] B. Schölkopf and A. Smola, Learning with kernels, MIT Press, Cambridge, MA, 2002.
[29] J. B. Tenenbaum, V. de Silva, and J. C. Langford, A global geometric framework for nonlinear dimensionality reduction, Science 290 (2000), no. 5500, 2319–23.
[30] J. F. Traub and H. Wozniakowski, A general theory of optimal algorithms, Academic Press, New York, 1980.
[31] A. G. Vitushkin, Theory of the transmission and processing of information, Pergamon Press, 1961.
M. Ehler University of Vienna, Department of Mathematics, Oskar-Morgenstern-Platz 1 A-1090 Vienna
E-mail address: martin.ehler@univie.ac.at

F. Filbir Helmholtz Zentrum München, Institute of Computational Biology, Ingolstädter Landstrasse 1, D-85764 Neuherberg
E-mail address: filbir@helmholtz-muenchen.de