Disorder Averaging and Finite Size Scaling

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We propose a new picture of the renormalization group (RG) approach in the presence of disorder, which considers the RG trajectories of each random sample (realization) separately instead of the usual renormalization of the averaged free energy. The main consequence of the theory is that the average over randomness has to be taken after finding the critical point of each realization. To demonstrate these concepts, we study the finite-size scaling properties of the two-dimensional random-bond Ising model. We find that most of the previously observed finite-size corrections are due to the sample-to-sample fluctuation of the critical temperature and scaling is more adequate in terms of the new scaling variables.

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Finite size scaling (FSS) is a very powerful tool of theoretical physics: it allows us to extract some properties of the infinite system near a phase transition by studying finite, numerically accessible samples. It is therefore of major interest to have a clear theoretical background behind the bold extrapolation from finite to infinite sizes. Though the basic concepts can be summarized in a few lines, the theory of FSS is far from trivial even for clean systems. Randomness brings in additional complexity, and a deeper understanding of FSS in disordered systems is still lacking. The main difference with the clean case is that somehow we have to average over the different random samples. There is an on-going discussion of whether the way the disorder average is taken influences the FSS results or not, and if it does, what is the “correct” average? The importance of the details of averaging is demonstrated most spectacularly by the so-called Chayes et al theorem, which claims that a certain finite-size correlation length exponent cannot be smaller than 2/d, d being the dimension of the disorder, for any phase transition driven by quenched randomness. It turns out that the proof of this quite general statement relies entirely on the specific manner the disorder was generated: a slight change in the ensemble of the random samples gives a different final result. Further studies along these lines showed that in some cases even the numerically measured quantities do depend on the set of the disorder realizations, though there are claims that they shouldn’t.

In this Letter we propose to understand the role of the disorder based on the scaling of a single realization instead of renormalizing the averaged free energy. We argue that, eventhough the difference between the two approaches is expected to vanish for infinite systems and short-range interactions, it might be crucial for finite samples and/or long-range forces. From a practical point of view, our main result is that disorder averaging should be done after finding the critical point of each sample independently. We demonstrate how this works in practice by performing an extensive numerical study of the two-dimensional random-bond Ising model.

First, recall some basic ideas of FSS in clean systems. Close to a continuous phase transition the correlation length, \( \xi \), diverges as \( \xi(T) \sim T - T_c \) with \( T \) the correlation length critical exponent and \( \tau = |T - T_c| \) the distance from the critical temperature \( T_c \) of the infinite system. For a finite system, the size \( L \) itself is measured in units of \( \xi \), i.e. a physical quantity \( Q \) depends on \( L \) only through the ratio \( L/\xi \), i.e.

\[
Q(T, L) = L^y \psi(L^{1/\nu} \tau),
\]

where \( y \) describes the \( L \)-dependence at criticality.

The surprise in Eq. (1) is that it contains the infinite system’s correlation length (or critical temperature \( T_c \)), eventhough in a finite system the actual characteristic length \( \xi_L(T) \) is typically different from \( \xi(T) \). Indeed, while in the high-temperature phase \( \xi_L \sim \xi \), there is a temperature \( T_c(L) \) where the correlation length reaches the system size, i.e. \( \xi_L \sim L \). Below this temperature the whole sample becomes correlated and \( \xi_L \) is defined by subtracting this overall correlation. We call the temperature \( T_c(L) \), the critical temperature at size \( L \). In terms of RG flows, the trajectories bend towards high temperatures for \( T > T_c(L) \), towards zero for \( T < T_c(L) \), and they “stick around” a fixed point for \( T = T_c(L) \).

Admittedly, \( T_c(L) \) is not a very well defined quantity, but the peak in a susceptibility or specific heat may give it a sensible meaning. Still, both the RG picture and the behaviour of \( \xi_L \) suggest that the scaling variable of the problem is \( \tau_L = T - T_c(L) \) instead of \( \tau \), leading to the FSS formula

\[
Q(T, L) = L^y f(L^{1/\nu} \tau_L).
\]

For clean systems the connection between Equations (1) and (2) is delivered by the scaling of \( T_c(L) \):

\[
T_c(L) = L^{\nu_y} T_c(L^{1/\nu_x}),
\]

where \( y \) and \( x \) are the FSS exponents of the critical temperature and correlation length, respectively.
The constant $C$ in this equation is not universal, it depends e.g. on the boundary conditions. But once the details are fixed, $C$ is constant for large $L$’s. Substituting Eq. (3) into Eq. (2) gives us the usual form of FSS (Eq. (4)).

Now we argue that in the presence of randomness, fixing the disorder distribution and the boundary conditions is not enough to keep the value of $C$ in Eq. (3) constant. Due to the randomness, $C$ will fluctuate from sample to sample and under renormalization. Consequently, $T_c(L)$ of a given disorder realization will fluctuate as well, preventing the use of the infinite system’s $T_c$ for all samples and sizes, like in Eq. (4). At the same time $\tau_L$ remains a good scaling variable and, after an appropriate averaging, Eq. (3) holds.

The basic observation in support of the above is that the RG trajectories for disordered systems are not smooth, but rather look like a random walk. Each time that the RG trajectories for disordered systems are not smooth, but rather look like a random walk. Each time that the RG trajectories for disordered systems, we find that the critical surface will be random and contain some randomness. Accordingly, the renormalized temperature will pick up a random part, too. Of course, this fluctuation of the RG trajectory will scale as a negative power of $L$, in the gaussian case as $L^{-d/2}$, and disappear if $L \to \infty$. But in the case of FSS, we are comparing temperatures as close as $\sim L^{-1/\nu}$, so for a $\nu$ close to $2/d$ the random walk of the RG trajectories becomes important. Since $T_c(L)$ itself changes under renormalization, we find that the critical surface will be random and different for each disorder realization.

Now let’s take a random sample of size $L$ and consider the RG trajectory starting at a temperature $T$ close to the sample’s $T_c(L)$. After a renormalization step we get the renormalized values $L'$, $T'$, and $T_c'(L')$. According to the above arguments both $T'$ and $T_c'(L')$ have a random part. But both $T'$ and $T_c'$ are temperatures, and they are close to each other, so it is natural to suppose that their fluctuating part will be almost the same, i.e. they are correlated. The main consequence of this correlation is that $T'(L') - T_c'(L')$ will be a smooth function of $L'$, scaling with the exponent $1/\nu$, while $T'(L') - T_c$ will show the large fluctuations of the random walk (see Fig. 1).

The standard (grand canonical) average uses $T_c$ only, and completely neglects the correlations. Such an approach is justified as long as the fluctuations of the RG trajectories are much smaller than their distance from $T_c$. In the case of FSS, however, they might be of the same order and the correlations become important: one has to use $\tau_L = T - T_c(L)$ to extract the critical exponents. If $T$ is at some distance (but not too far) from $T_c(L)$, so the sample contains many correlated regions, the system is almost self averaging. But around $T_c(L)$ the remaining randomness in other quantities does not necessarily scale to zero and, in order to use Eq. (3), we have to get rid of this extra noise by averaging. The “correlated average” requires them to find the critical temperature of a given sample, and average over realizations with the same $\tau_L$. In practice, this means “shifting” and superposing the curves of $Q(T)$ measured on different random samples of the same size.

We now test the above theoretical concepts on the two-dimensional random-bond Ising model. We simulated $L \times L$ systems ($L = 32, \ldots, 128$) with periodic boundary conditions using the Wolff single-cluster algorithm to overcome critical slowing down. Disorder was generated from a bimodal distribution: bonds had two values, $J_1$ and $J_2$ (all positive) with equal probabilities. The strength of randomness was tuned by changing the ratio $r = J_1/J_2$ ($r = 0.25$, $0.5$). The exact critical temperature $\beta_c = 1/k_B T_c$ of the model is known as a function of $r$ through $\sinh(2\beta_c J_2) \sinh(2\beta_c r J_2) = 1$. For each measurement, we used up to $10^4$ Monte Carlo (MC) steps each comprising 10 cluster updatings and we used $10^4$ steps for equilibration. To avoid inaccuracies due to
RG approach predicts that small disorder is marginally expected scaling of the very large system. In many cases, the different parameters are identified only on the disorder strength order for all studied system sizes, depending on the different samples’ susceptibilities demonstrates that 

\[ \tau_c(L) = T - T_c(L) \]

is indeed the good scaling variable. Fig. 3 also shows that, as expected, disorder fluctuations are pronounced only at, or around \( T_c(L) \). Our data indicate that the relative fluctuations of the peak heights, are in the same order for all studied system sizes, depending only on the disorder strength \( r \).

In terms of averaging over disorder, Figures 2 and 3 correspond to the grand canonical and correlated averages, respectively: the latter achieves a spectacular noise reduction, but it is still to see, which one reproduces the expected scaling of the very large system.

Without randomness \( \nu_{\text{pure}} = 1 \), and a perturbative RG approach predicts that small disorder is marginally

\[ \frac{d\Delta}{dx} = -8\Delta^2 + \mathcal{O}(\Delta^3), \]

where \( \Delta \) is proportional to the square dispersion of the random bonds, and \( x \propto \ln(L^{-1}) \). According to Eq. (5), the disorder scales to zero, but only logarithmically with \( L \), so we have to take it into account in the RG equations of other quantities, like the reduced temperature \( \tau \),

\[ \frac{d\tau}{dx} = (1 - 4\Delta)\tau + \ldots. \]

This equation predicts an effective exponent \( \nu_{\text{eff}} \sim 1 + 4\Delta \), which approaches \( \nu_{\text{pure}} = 1 \) very slowly. Since the randomness does not couple to the magnetization in first order, one expects that the susceptibility exponent \( \gamma/\nu \) remains unchanged. Even though these results were obtained by using replicas and grand-canonical disorder average, for a short-range-interaction model and very large system sizes we still expect them to be correct.

The detailed form of the above scaling corrections is still under debate even today [11,12]. Here we wish to concentrate only on their qualitative nature: disorder introduces corrections to \( \nu \) (the width of the susceptibility peak), but not to \( \gamma/\nu \) (the height of the peak at criticality). As we will see, this expectation is satisfied only with the correlated average.

The major difficulty of the correlated average is to find \( T_c(L) \) of a given disorder realization, and “shift” the different samples’ curves as in Fig. 3. Trying to identify the peak of the susceptibility for each realization is one possibility [4], but both thermal and random fluctuations are biggest at this point. Instead, we used the entire susceptibility curves and minimized the “distance” between them. We verified that the final results do not depend on the details of this procedure, and the average critical point \( T_c(L) \) scales to the exact \( T_c \) when \( L \to \infty \).

The exact values are \( T_c = 1.641018 \) for \( r = 0.5 \) and \( T_c = 1.239078 \) for \( r = 0.25 \). Our extrapolated \( L \to \infty \) numerical results are \( T_c = 1.640(1) \) and \( T_c = 1.239(1) \) respectively.

In the case of small disorder, \( r = 0.5 \), we found corrections to scaling in the case of grand canonical average both for \( \nu \) and \( \gamma/\nu \), though both of these corrections are relatively small. This violates what is expected for \( \gamma/\nu \). On the other hand, for the available sizes, the correlated average gives an almost perfect scaling plot with the pure exponents, as shown in Figure 3 (inset). No corrections were visible here.

The differences between the two disorder averages are even more pronounced for stronger disorder, \( r = 0.25 \). Clearly, for the grand canonical average (Fig. 4) not only the widths but also the heights of the peaks show sizable corrections to scaling. Note that there are no corrections for the heights (scaled by \( \gamma/\nu \) when the data...
have been evaluated with correlated average (Fig. 5). For this disorder strength the corrections in $\nu$ already appear, and an effective thermal exponent $1/\nu_{eff} \sim 0.92$ gives a good description of the data within this range of sizes (see the inset of Fig. 5). We emphasize that only the results of the correlated average reproduce our expectations for the infinite system scaling.

In addition to the susceptibility, other singular quantities, like the specific heat, show critical behaviour. A question of consistency arises: Do the same temperature shifts calculated from the susceptibilities of different samples give the best collapse of the specific heat curves? Indeed, the answer is yes, as can be seen in Fig. 6. This supports our theory that $\tau_L = T - T_c(L)$ of a given sample is a good scaling variable for any critical quantity.

We have proposed a new picture of the RG in random systems, which leads to a recently introduced way of disorder averaging for FSS, the so-called “correlated average” [2]. We studied in detail the FSS properties of the $d = 2$ disordered Ising model, and found that only the correlated average reproduces the expected behaviour of the susceptibility, in addition to spectacular noise reduction in averaged quantities. A detailed account of our simulations’ results will be published elsewhere.

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FIG. 4. Grand canonical average, $r = 0.25$ ($\nu = 1$, $\gamma = 1.75$).

FIG. 5. Correlated average, $r = 0.25$ ($\nu = 1$ and $\gamma = 1.75$, inset: $1/\nu \sim 0.92$ and $\gamma/\nu = 1.75$).

FIG. 6. Collapse of the specific-heat curves using the same shifts of temperature as in Figure 3 ($r = 0.5$, $L = 64$).