The non-anticoercive Hénon-Lane-Emden system

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Abstract

We use variational methods to study the existence of a principal eigenvalue for the non-anticoercive Hénon-Lane-Emden system on a bounded domain. Then we provide a detailed insight into the problem in the linear case.

1 Introduction

The Hénon-Lane-Emden system

\[
\begin{align*}
-\Delta u &= |x|^a |v|^{p-2} v \quad \text{in } \Omega \\
-\Delta v &= |x|^b |u|^{q-2} u \quad \text{in } \Omega \\
u &= 0 = v \quad \text{on } \partial \Omega
\end{align*}
\]

includes the second and fourth order Lane-Emden equations and the Hénon equation in astrophysics. Here Ω is a domain in \( \mathbb{R}^n \) containing the origin, and \( a, b, p, q \) are given, with \( a, b > -n, p, q > 1 \).

Most of the papers about problem (1) require \( n \geq 3 \) and deal with the so-called anticoercive case

\[
\frac{1}{p} + \frac{1}{q} < 1.
\]

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Since the celebrated paper [13] by P.L. Lions, where \( a = b = 0 \) and \( \Omega = \mathbb{R}^n \) are assumed, large efforts have been made in investigating (1) and related problems. We will not try to provide a complete list of references on this subject. We limit ourselves to cite [1, 2, 3, 4, 7, 9, 10, 11, 16, 20, 23, 25, 26, 27, 28] and references therein. Several challenging problems are still open. Some of them affect the critical hyperbola
\[
\frac{a + n}{p} + \frac{b + n}{q} = n - 2,
\]
that was firstly introduced by Mitidieri [15] in 1990, in the autonomous case \( a = b = 0 \) (see also [8, 21, 16]). Roughly speaking, to have existence of solutions one is lead to assume
\[
\frac{a + n}{p} + \frac{b + n}{q} > n - 2. \tag{2}
\]

The present paper deals with the case of a bounded and smooth domain \( \Omega \) in \( \mathbb{R}^n \), \( n \geq 1 \), and with the non-anticoercive case
\[
q = p' = \frac{p}{p - 1}.
\]
Due to the homogeneities involved, we are lead to study the eigenvalue problem
\[
\begin{cases}
-\Delta u = \lambda_1 |x|^a |v|^{p'-2}v & \text{in } \Omega \\
-\Delta v = \lambda_2 |x|^b |u|^{p'-2}u & \text{in } \Omega \\
u = 0 = v & \text{on } \partial \Omega.
\end{cases} \tag{3}
\]
We emphasize the fact that we include the lower dimensional cases \( n = 1, 2 \), that actually present some peculiarities.

As far as we know, only few references are available for (3). We mention the paper [18], where Montenegro uses degree theory to face problem (3) in a more general setting that includes non-self adjoint elliptic operators.

We adopt a variational approach that allows us to weaken the integrability assumptions on the coefficients from Montenegro’s \( L^n(\Omega) \) to \( L^1(\Omega) \). More precisely, we assume
\[
\begin{cases}
a, b > -n \tag{4a} \\
\frac{a}{p'} + \frac{b}{p} + 2 > 0. \tag{4b}
\end{cases}
\]
Notice that (4b) is automatically satisfied if \( n = 1, 2 \) and (4a) holds. Moreover, (4b) coincides with (2), as \( q = p' \) in our setting.

We look for finite energy solutions, accordingly with the next definition.
Definition 1. The pair \((u,v)\) is a finite-energy solution to (3) if
- \(u,v \in W^{2,1}(\Omega) \cap W^{1,1}_0(\Omega)\);
- \(u \in L^p(\Omega, |x|^b dx), v \in L^{p'}(\Omega, |x|^a dx)\), that is,
  \[
  \int_{\Omega} |x|^b |u|^p \, dx < \infty, \quad \int_{\Omega} |x|^a |v|^{p'} \, dx < \infty; \tag{5}
  \]
- \(u,v\) are weak solutions to the elliptic equations in (3). That is,
  \[
  \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \lambda_1 \int_{\Omega} |x|^a |v|^{p'-2} v \varphi \, dx, \quad \int_{\Omega} \nabla v \cdot \nabla \varphi \, dx = \lambda_2 \int_{\Omega} |x|^a |u|^{p-2} u \varphi \, dx
  \]
  for any test function \(\varphi \in C_\infty(\Omega)\).

Our approach is based on the equivalence\(^1\) between (3) and the fourth order eigenvalue problem
\[
\begin{align*}
\Delta \left( |x|^{-a(p-1)} |\Delta u|^{p-2} \Delta u \right) &= \mu |x|^b |u|^{p-2} u \quad \text{in } \Omega \\
u = \Delta u = 0 &= \text{on } \partial \Omega,
\end{align*}
\tag{6}
\]
where \(\mu, \lambda_1\) and \(\lambda_2\) satisfy
\[
|\lambda_1|^{p-1} \lambda_1 \lambda_2^{p'-1} \lambda_2 = \mu^{p'}, \tag{7}
\]
compare with Lemma 12. We refer to Section 3 for the proof of the next result.

Theorem 1. Let \(\Omega\) be a bounded and smooth domain in \(\mathbb{R}^n\). If (4) hold, then problem (3) has a positive principal eigenvalue \(\mu\). That is, for any pair of real numbers \((\lambda_1, \lambda_2)\) satisfying (4), problem (3) has a finite-energy solution \((u,v)\), such that \(u,v > 0\) in \(\Omega\).

The last part of the paper is focused on the linear case \(p = 2\), so that (4b) becomes
\[
a + b + 4 > 0. \tag{8}
\]
In Section 4 we prove that problem
\[
\begin{align*}
-\Delta u &= \lambda_1 |x|^a v \quad \text{in } \Omega \\
-\Delta v &= \lambda_2 |x|^b u \quad \text{in } \Omega \\
u = 0 &= v \quad \text{on } \partial \Omega
\end{align*}
\tag{9}
\]
\(^1\) already noticed for instance by Wang [29] and Calanchi-Ruf [5] in the anticoercive case.
has a unique and simple principal eigenvalue $\mu_1 > 0$, and a discrete spectrum 
$\{\mu_k\}_{k \in \mathbb{N}}$. More precisely, the following results hold.

**Theorem 2.** Let $\Omega$ be a bounded and smooth domain in $\mathbb{R}^n$. Let $a, b > -n$ and assume that (8) holds.

i) There exists an increasing, unbounded sequence of eigenvalues $\{\mu_k\}_{k \in \mathbb{N}}$ such that problem (9) has a nontrivial and finite-energy solution $(u, v)$ if and only if $\lambda_1 \lambda_2 = \mu_k$ for some integer $k \geq 1$.

ii) The first eigenvalue $\mu_1$ is the unique principal eigenvalue. In addition, $\mu_1$ is simple, that is, if $(\tilde{u}, \tilde{v})$ solves (9) and $\lambda_1 \lambda_1 = \mu_1$, then $\tilde{u} = \alpha u$ and $\tilde{v} = \beta v$ for some $\alpha, \beta \in \mathbb{R}$.

2 The variational approach

In this section we introduce and study certain second order weighted Sobolev spaces under Navier boundary conditions that are suitable for studying (6) via variational methods.

2.1 Functional setting and embedding results

To simplify notation we set $s = a(p - 1)$. Thus, from now on we assume that $s, b$ are given exponents such that

$$s > n - np, \quad b > -n$$

even if not explicitly stated. In addition, $\Omega \subset \mathbb{R}^n$ will always denote a bounded and smooth domain. We denote by $c$ any universal positive constant.

We introduce the function space

$$C^2_{N}(\Omega) := \{u \in C^2(\Omega) \mid u = 0 \mathrm{on} \partial \Omega\}.$$ 

Let $W^{2,p}_{N}(\Omega, |x|^{-s} dx)$ be the reflexive Banach space defined as the completion of the set

$$D_0 := \{u \in C^2_{N}(\Omega) \mid \Delta u \equiv 0 \mathrm{on a neighborhood of the origin}\},$$

with respect to the uniformly convex norm

$$\|u\|_s \equiv \|u\|_{p,s} := \left(\int_\Omega |x|^{-s} |\Delta u|^p \, dx\right)^{\frac{1}{p}}.$$
We begin to study the spaces $W^2_N(\Omega, |x|^{-s} \, dx)$ by pointing out few embedding results. Firstly, notice that the boundedness of the domain $\Omega$ implies

$$W^2_N(\Omega, |x|^{-s} \, dx) \hookrightarrow W^2_N(\Omega, |x|^{-s_0} \, dx) \quad \text{if } s_0 \leq s. \quad (10)$$

In order to simplify notation in the next lemma, we introduce the exponent

$$\hat{p}_s = p \quad \text{if } s \geq 0, \quad \hat{p}_s = \frac{np}{n - s} \quad \text{if } s < 0.$$

**Lemma 3.** Assume $s > n - np$. Then

$$W^2_N(\Omega, |x|^{-s} \, dx) \hookrightarrow W^{2, \tau}(\Omega) \cap W^{1,\tau}_0(\Omega) \quad \text{for any } \tau \in [1, \hat{p}_s).$$

**Proof.** Notice that $1 < \hat{p}_s < p$. For any $u \in D_0$ and $\tau \in [1, \hat{p}_s)$ we use elliptic regularity estimates, see for instance [12, Lemma 9.17], to get

$$\|u\|_{W^{2, \tau}(\Omega)} \leq c \left( \int_{\Omega} |\Delta u|^\tau \, dx \right)^{\frac{1}{\tau}} \left( \int_{\Omega} |x|^{-s} |\Delta u|^p \, dx \right)^{\frac{p-\tau}{p}}.$$

The last integral is finite as $s > n - np$, and the lemma readily is proved. $\square$

The next lemma will be used in the next section to rigorously prove the equivalence between the second order system (3) and the fourth order equation (6).

**Lemma 4.** If $s > n - np$, then $u \in W^2_N(\Omega, |x|^{-s} \, dx)$ if and only if

$$u \in W^{2,1} \cap W^{1,1}_0(\Omega) \quad \text{and} \quad -\Delta u \in L^p(\Omega, |x|^{-s} \, dx). \quad (11)$$

**Proof.** Clearly, any $u \in W^2_N(\Omega, |x|^{-s} \, dx)$ satisfies (11) by Lemma 3.

Conversely, fix $u$ satisfying (11). Assume in addition that $-\Delta u = 0$ almost everywhere on a ball $B_r$ about 0, so that $-\Delta u \in L^p(\Omega)$. Hence, $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ by elliptic regularity theory. Extend $u$ to a function $u$ in $W^{2,p}(\mathbb{R}^n)$ with compact support and take a sequence of mollifiers $\{\rho_k\}_{k \in \mathbb{N}}$. Since for $k$ large enough, $-\Delta(\rho_k * u) \equiv 0$ on $B_{r/2}$ and $\rho_k * u \to u$ in $W^{2,p}(\Omega)$, then $-\Delta(\rho_k * u) \to -\Delta u$ in $L^p(\Omega, |x|^{-s} \, dx)$. Let $u_k$ be the solution to

$$\begin{cases}
-\Delta u_k = -\Delta(\rho_k * u) & \text{in } \Omega \\
u_k = 0 & \text{on } \partial \Omega.
\end{cases}$$

It turns out that $u_k \in D_0 \cap W^{2,p}_N(\Omega)$, as $u_k$ is smooth up to the boundary of $\Omega$ by regularity theory, and $-\Delta u_k \equiv 0$ in $B_{r/2}$. In addition, $u_k \to u$ in
\[ W^{2,p}(\Omega) \] and \(-\Delta u_k \to -\Delta u\) in \(L^p(\Omega, |x|^{-s} dx)\), that is sufficient to conclude that \(u \in W^{2,p}_N(\Omega, |x|^{-s} dx)\).

For a general \(u\) satisfying (11) let \(u_k\) be the unique solution to
\[
\begin{aligned}
-\Delta u_k &= \chi_{\Omega_k}(-\Delta u) \text{ in } \Omega \\
 u_k &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]
where \(\Omega_k := \Omega \setminus B_{\varepsilon_k}\) and \(\varepsilon_k \to 0\). Then \(u_k \in W^{2,p}_N(\Omega, |x|^{-s} dx)\) by the first part of the proof. Clearly, the sequence \(\{u_k\}_{k \in \mathbb{N}}\) is bounded in \(W^{2,p}_N(\Omega, |x|^{-s} dx)\), and we can assume that \(u_k \to \bar{u}\) weakly in \(W^{2,p}_N(\Omega, |x|^{-s} dx)\). On the other hand, \(-\Delta u_k\) converges to \(-\Delta u\) in \(L^p(\Omega, |x|^{-s} dx)\) by Lebesgue’s theorem. Thus \(\bar{u} = u\), that is, \(u \in W^{2,p}_N(\Omega, |x|^{-s} dx)\).

The next corollary is an immediate consequence of Lemma 4.

**Corollary 5.** Assume \(s > n - np\). For any \(f \in L^p(\Omega, |x|^{-s} dx)\), the unique solution \(u\) to
\[
\begin{aligned}
-\Delta u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega
\end{aligned}
\]
belong to \(W^{2,p}_N(\Omega, |x|^{-s} dx)\).

Next we deal with embeddings in weighted \(L^p\) spaces.

**Lemma 6.** If \(s + b + 2p \geq 0\), then
\[
\Lambda(s, b) := \inf_{\substack{u \in W^{2,p}_N(\Omega, |x|^{-s} dx) \\
u \neq 0}} \frac{\int_{\Omega} |x|^{-s} |\Delta u|^p \, dx}{\int_{\Omega} |x|^b |u|^p \, dx} > 0.
\]

**Proof.** First of all, notice that \(L^p(\Omega, |x|^{b_0} dx) \hookrightarrow L^p(\Omega, |x|^{b} dx)\) if \(b_0 \leq b\), that together with (10) implies
\[
\Lambda(s, b) \geq c\Lambda(s_0, b_0) \quad \text{if } s_0 \leq s \text{ and } b_0 \leq b.
\]

We start with the lowest dimensions \(n = 1, 2\). Fix an exponent \(s_0 \leq s\), such that \(n - np < s_0 \leq b(p - 1)\). Then \(\Lambda(s, b) \geq c\Lambda(s_0, \frac{s_0}{p-1}) > 0\) by (12) and Lemma 15 in the Appendix.

Next, assume \(n \geq 3\). In addition, assume firstly that \(s < n - 2p\). By a Rellich-type inequality in [17], see also [19] Lemma 2.14, and using [19]
Lemma 2.9, one readily checks that there exists a positive and explicitly known constant \( c = c(n, p, s) \), such that

\[
c \int_{\Omega} |x|^{-s-2p} |u|^p \, dx \leq \int_{\Omega} |x|^{-s} |\Delta u|^p \, dx \quad \text{for any } u \in C_N^2(\overline{\Omega}),
\]

that is, \( c = \Lambda(s, -s - 2p) > 0 \). Thus \( \Lambda(s, b) \geq c\Lambda(s, -s - 2p) > 0 \) by (12).

Finally, if \( s \geq n - 2p \), we fix a parameter \( s_0 \) such that

\[
\max \{ n - np, -2p - b \} < s_0 < n - 2p \leq s,
\]

that is possible as \( b > -n \) and \( n \geq 3 \). Then (12) and (13) with \( s \) replaced by \( s_0 \) give \( \Lambda(s, b) \geq c\Lambda(s_0, -s_0 - 2p) > 0 \), and the lemma is proved.

**Remark 7.** If \( \Omega \) contains the origin and \( s + b + 2p < 0 \) then \( \Lambda(s, b) = 0 \). Indeed, fix a nontrivial \( \psi \in C_c^\infty(B_1 \setminus \{0\}) \). For \( k \) large enough the function \( \psi_k(x) = \psi(kx) \) has compact support in \( \Omega \setminus \{0\} \). Thus

\[
\Lambda(s, b) \leq \frac{\int_{\Omega} |x|^{-s} |\Delta \psi_k|^p \, dx}{\int_{\Omega} |x|^b |\psi_k|^p \, dx} = Ck^{s+2p+b} = o(1) \quad \text{as } k \to \infty.
\]

**Remark 8.** If \( n - np < s < n - 2p \), then \( C_N^2(\Omega) \subset W_N^{2,p}(\Omega, |x|^{-s} \, dx) \) and the space

\[
C_N^2(\Omega \setminus \{0\}) := \{ u \in C_N^2(\Omega) \mid u \equiv 0 \text{ on a neighborhood of the origin} \}
\]

is dense in \( W_N^{2,p}(\Omega, |x|^{-s} \, dx) \), see Lemma 2.14 in [19].

**Remark 9.** By Lemma 4, the set \( D_0 \) is dense in the standard Sobolev space \( W_N^{2,p}(\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \). The smaller set \( C_N^2(\Omega \setminus \{0\}) \) is dense in \( W_N^{2,p}(\Omega) \) if and only if \( n > 2p \), compare with Remark 8.

The next compactness result is a crucial point for studying the eigenvalue problem (6).

**Lemma 10.** If \( s + b + 2p > 0 \) then \( W_N^{2,p}(\Omega, |x|^{-s} \, dx) \) is compactly embedded into \( L^p(\Omega, |x|^b \, dx) \).

**Proof.** It suffices to show that any sequence \( \{u_k\}_{k \in \mathbb{N}} \) that converges weakly to the null function in \( W_N^{2,p}(\Omega, |x|^{-s} \, dx) \) actually converges in \( L^p(\Omega, |x|^b \, dx) \). Fix such a sequence, and take \( \varepsilon > 0 \) small. Since clearly \( \{u_k\}_{k \in \mathbb{N}} \) is bounded.
in $W^{2,p}(\Omega \setminus B_\varepsilon)$, then $|x|^b|u_k|^p \to 0$ in $L^1(\Omega \setminus B_\varepsilon)$ by Rellich theorem. Therefore, for any $b_0 \in (-n, b)$ we have that
\[
\int_\Omega |x|^b|u_k|^p \, dx = \int_{B_\varepsilon} |x|^b|u_k|^p \, dx + o(1) \leq \varepsilon^{b-b_0} \int_\Omega |x|^{b_0}|u_k|^p \, dx + o(1).
\]
Now, if $b_0$ is close enough to $b$, then $s+b_0+2p > 0$. Hence
\[
\int_\Omega |x|^b|u_k|^p \, dx \leq c\varepsilon^{b-b_0} + o(1)
\]
by Lemma 6. The conclusion follows, as $\varepsilon > 0$ was arbitrarily chosen. 

3 Two equivalent problems

In this section we furnish a rigorous proof of the equivalence between the eigenvalue problems (3) and (6). We start with a preliminary result.

Lemma 11. Assume that (4) hold. For any $f \in L^p(\Omega, |x|^b \, dx)$, the problem

\[
\begin{cases}
-\Delta u = |x|^a |v|^{p'-2} v \\
-\Delta v = |x|^b |f|^{p-2} f \\
u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx), \ v \in W^{2,p'}_N(\Omega, |x|^{-b(p'-1)} \, dx)
\end{cases}
\]  

admits a unique solution.

Proof. First of all, notice that $a(p-1) > n - np$, $b(p'-1) > n - np'$. Thus the results in Subsection 2.1 apply to the spaces $W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx)$ and $W^{2,p'}_N(\Omega, |x|^{-b(p'-1)} \, dx)$.

Since $|x|^b |f|^{p-2} f \in L^{p'}(\Omega, |x|^{-b(p'-1)} \, dx)$, then Corollary 5 guarantees (14b) has a unique solution $v \in W^{2,p'}_N(\Omega, |x|^{-b(p'-1)} \, dx)$. The embedding Lemma 6 gives that $|x|^a |v|^{p'-2} v \in L^p(\Omega, |x|^{-a(p-1)} \, dx)$. Thus there exists a unique solution $u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx)$ to (14a), thanks again to Corollary 5.

We are ready to prove the claimed equivalence result.

Lemma 12. Assume that (4) hold. Let $\mu, \lambda_1, \lambda_2 \in \mathbb{R}$ satisfying (7). Then the following sentences are equivalent.

i) $u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx)$ is a weak solution to (6).
\textbf{ii)} The pair \(u, v := -|x|^{-a(p-1)}|\Delta u|^{p-2} \Delta u\) solves
\begin{align}
-\Delta u &= |x|^a |v|^{p'-2} v \\
-\Delta v &= \mu |x|^b |u|^{p-2} u, \\
u &\in W^{2,p}_N(\Omega, |x|^{-a(p-1)}) dx, \quad v \in W^{2,p'}_N(\Omega, |x|^{-b(p'-1)}) dx.
\end{align}

\textbf{iii)} The pair \(u, v := -|x|^{-a(p-1)}|\Delta u|^{p-2} \Delta u\) is a finite-energy solution to (3), in the sense of Definition 7.

\textbf{Proof.} If \(u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}) dx\), then \(u \in L^p(\Omega, |x|^b dx)\) by Lemma 6. Thus we can apply Lemma \ref{lem:existence} to find a unique pair \(u_0, v_0\) such that
\begin{align}
-\Delta u_0 &= |x|^a |v_0|^{p'-2} v_0 \\
-\Delta v_0 &= \mu |x|^b |u_0|^{p-2} u_0, \\
u_0 &\in W^{2,p}_N(\Omega, |x|^{-a(p-1)}) dx, \quad v_0 \in W^{2,p'}_N(\Omega, |x|^{-b(p'-1)}) dx.
\end{align}

Notice that \(v_0 = |x|^{-a(p-1)}|\Delta u_0|^{p-2}(-\Delta u_0)\) almost everywhere in \(\Omega\). Therefore, if \(u\) solves (5), then for any \(\varphi \in D_0\) it holds that
\begin{align}
\int_\Omega |x|^{-a(p-1)}|\Delta u|^{p-2} \Delta u \Delta \varphi \, dx &= \mu \int_\Omega |x|^b |u|^{p-2} u \varphi \, dx \\
&= \int_\Omega (-\Delta v_0) \varphi \, dx = \int_\Omega v_0(-\Delta \varphi) \, dx \\
&= \int_\Omega |x|^{-a(p-1)}|\Delta u_0|^{p-2} \Delta u_0 \Delta \varphi \, dx,
\end{align}
that readily gives that \(u = u_0\), since \(u, u_0 \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}) dx\) and \(D_0\) is dense in \(W^{2,p}_N(\Omega, |x|^{-a(p-1)}) dx\). Hence, also \(v = v_0\), the pair \(u, v\) solves (15), and the first implication is proved.

The equivalence between \textbf{ii)} and \textbf{iii)} is immediate, thanks to Lemma 8 and Corollary 5. It remains to show that \textbf{ii)} implies i). If \((u, v)\) solves (15), then for every \(\varphi \in D_0\) it holds that
\begin{align}
\mu \int_\Omega |x|^b |u|^{p-2} u \varphi \, dx &= \int_\Omega v(-\Delta \varphi) \, dx \\
&= \int_\Omega |x|^{-a(p-1)}|\Delta u|^{p-2} \Delta u \Delta \varphi \, dx,
\end{align}
that is, \(u\) solves (6). \hfill \blacksquare
Lemma 12 shows that finite energy solutions to (3) are the stationary points of the functional

\[ u \mapsto \int_{\Omega} |x|^{-a(p-1)} |\Delta u|^p \, dx \]

on the constraint

\[ M = \left\{ u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx) \mid \int_{\Omega} |x|^b |u|^p \, dx = 1 \right\}. \]

If (4) hold, then $M$ is compact in the weak $W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx)$ topology, by Lemma 10. Thus the infimum

\[ \mu := \Lambda(a(p-1), b) = \inf_{\substack{u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx) \setminus \{0\}}} \frac{\int_{\Omega} |x|^{-a(p-1)} |\Delta u|^p \, dx}{\int_{\Omega} |x|^b |u|^p \, dx} \]  

is positive and attained. The next lemma deals with minimizers for $\mu$.

**Lemma 13.** Assume that (4) hold. If $u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx)$ achieves $\mu$, then, up to a change of sign, $u$ is positive and superharmonic.

**Proof.** Let $v = |x|^{-a(p-1)}|\Delta u|^{p-2}(-\Delta u)$, so that the pair $(u, v)$ solves (15).

Use Corollary 5 to introduce $u_0$ via

\[ \begin{cases} -\Delta u_0 = |x|^a |v|^{p'-1} \\ u_0 \in W^{2,p}_N(\Omega, |x|^{-a(p-1)} \, dx) \end{cases} \]

In particular, $u_0$ is superharmonic and positive on $\Omega$. Next, put

\[ g = |x|^a |v|^{p'-2} v. \]

Thus $u$ and $u_0$ solve, for some $\tau \in [1, p)$,

\[ \begin{cases} -\Delta u = g \\ u \in W^{2,\tau} \cap W^{1,\tau}_0(\Omega), \\ -\Delta u_0 = |g| \\ u_0 \in W^{2,\tau} \cap W^{1,\tau}_0(\Omega). \end{cases} \]

Since $-\Delta (u_0 \pm u) \geq 0$ and $u_0 \pm u = 0$ on the boundary of $\Omega$, then $u_0 \pm u \geq 0$, that is, $u_0 \geq |u|$. On the other hand, $|\Delta u_0| = |g| = |\Delta u|$. Therefore

\[ \mu \leq \frac{\int_{\Omega} |x|^{-a(p-1)} |\Delta u_0|^p \, dx}{\int_{\Omega} |x|^b |u_0|^p \, dx} \leq \frac{\int_{\Omega} |x|^{-a(p-1)} |\Delta u|^p \, dx}{\int_{\Omega} |x|^b |u|^p \, dx} = \mu, \]

that is, $u_0$ attains $\mu$ and $u_0 = |u|$. Since $u_0$ is positive in $\Omega$, then $u$ and $-\Delta u$ have constant sign, as desired. \qed
Proof of Theorem 1. Immediate, thanks to Lemmata 10 and 13 and the equivalence given by Lemma 12.

We conclude the section pointing out a symmetry result about the infimum in (16). It is convenient to use the notation $\mu(a,b,p)$ to emphasize the dependence of $\mu$ on the parameters.

Proposition 14. If (4) hold, then $\mu(b,a,p')^p = \mu(a,b,p)^p$.

Proof. Let $u$ be an extremal for $\mu(a,b,p)$. By Lemma 12 the pair $u,v = -|x|^{-a(p-1)}|\Delta u|^{p-2}\Delta u$ solves (15) with $\mu = \mu(a,b,p)$ and hence

$$\int_{\Omega} |x|^{-b(p'-1)}|\Delta v|^{p'} dx = \mu(a,b,p)^{p'} \int_{\Omega} |x|^b |u|^p dx$$

$$= \mu(a,b,p)^{p'-1} \int_{\Omega} |x|^{-a(p-1)}|\Delta u|^p dx = \mu(a,b,p)^{p'-1} \int_{\Omega} |x|^a |v|^p' dx.$$ 

Thus $\mu(b,a,p') \leq \mu(a,b,p)^{p'-1}$, or equivalently $\mu(b,a,p')^p \leq \mu(a,b,p)^{p'}$.

Exchanging the roles of $u$ and $v$, we get the opposite inequality. □

4 Proof of Theorem 2

Since $p = 2$, then equation (6) reduces to

$$\Delta (|x|^{-a} \Delta u) = \mu |x|^b u.$$  

(17)

Proof of i). We denote by $X_a$ the Hilbert space $W^{2,2}_N(\Omega, |x|^{-a}dx)$, endowed with norm $\| \cdot \|_a$ and scalar product $(\cdot | \cdot)_a$.

We formally introduce the “solution operator” to (17) under Navier boundary conditions. More precisely, we define the linear operator

$$T: L^2(\Omega, |x|^b dx) \to X_a, \quad (Tf|w)_a = \int_{\Omega} |x|^b f w dx \quad \text{for every } w \in X_a.$$ 

Then $T$ is continuous, positive and self-adjoint. Let $j: X_a \to L^2(\Omega, |x|^b dx)$ be the embedding in Lemma 6. Then the operator

$$T \equiv j \circ T \circ T: L^2(\Omega, |x|^b dx) \to L^2(\Omega, |x|^b dx)$$
is compact. Thus the point spectrum $\sigma_p(T)$ of $T$ is a sequence $\{\nu_k\}_{k \in \mathbb{N}}$ of positive numbers converging to 0, and

$$\frac{1}{\nu_k} = \min \left\{ \frac{\int_{\Omega} |x|^{-a} |\Delta u|^2 \, dx}{\int_{\Omega} |x|^b u^2 \, dx} : u \in \Lambda^+_i, 1 \leq i \leq k-1 \right\},$$

where $\Lambda_i := \Lambda(\nu_i)$ is the eigenspace relative to the eigenvalue $\nu_i$.

By the results in the previous sections, $u$ is an eigenfunction for $T$ if and only if the couple $(u, - |x|^{-a} \Delta u)$ is the only solution to (14). This concludes the proof.

**Proof of ii).** We will use the theory of abstract positive operators on Banach lattices, for which we refer to the monograph [24]. Recall that $L^2(\Omega, |x|^b \, dx)$ has a natural Banach lattice structure induced by the cone $P_+ \subset \text{nonnegative}$ functions. We will show that $T$ is positive and irreducible. Then, the conclusion will follows thanks to an adaptation of Theorem V.5.2 in [24], that guarantees the following facts hold:

(a) The spectral radius $r(T) \in \mathbb{R}_+$ is an eigenvalue.

(b) The eigenspace $\Lambda(r(T))$ has dimension one, and is spanned by a (unique, normalized) quasi-interior point of $P_+$.

(c) $r(T)$ is the unique eigenvalue of $T$ with a positive eigenvector.

To check that $T$ is irreducible we first recall that the only closed ideals in $L^2(\Omega, |x|^b \, dx)$ are the ones of the form

$$I_A = \left\{ f \in L^2(\Omega, |x|^b \, dx) \middle| f = 0 \text{ on } A \right\},$$

where $A$ is a measurable set, see for instance [24, p. 157]. Therefore, we have to show that if $A$ satisfies

$$0 < \int_{\Omega} |x|^b \chi_A \, dx < \int_{\Omega} |x|^b \, dx,$$

then $I_A$ is not fixed by $T$.

Let $f \in I_A$ be a nonnegative fixed function. Then the problem

$$\begin{cases}
-\Delta v = |x|^b f \\
v \in X_b
\end{cases}$$
admits a solution by Corollary 5 and \( v \in W^{2,\tau} \cap W^{1,\tau}_0(\Omega) \) for some \( \tau > 1 \). The minimum principles imply that \( v \) is strictly positive in \( \Omega \). For the same reason, the problem
\[
\begin{cases}
-\Delta u = |x|^a v \\
u \in X_a,
\end{cases}
\]
defines a function \( u \) that is strictly positive in \( \Omega \). Hence \( u \equiv Tf \notin I_A \), and this proves the irreducibility property. The same argument proves also the positivity property. \( \square \)

Appendix. An inequality in lower dimensions

We sketch here the proof of some second order integral estimates in low dimensions by using, in essence, the Rellich-type identity in [17]. Details and further applications of the underlying ideas will be given in [6].

**Lemma 15.** Assume \( n = 1 \) or \( 2 \) and let \( \Omega \) be an open interval or a bounded domain in \( \mathbb{R}^2 \) of class \( C^2 \). If \( s > n - np \), then there exists a constant \( c > 0 \) such that
\[
c \int_{\Omega} \frac{|x|^{s-1}}{|x|^p} |u|^p \, dx \leq \int_{\Omega} |\Delta u|^p \, dx
\]
for any \( u \in C^2_N(\Omega) \) such that \( \Delta u = 0 \) in a neighborhood of 0.

**Proof.** We can assume that \( \Omega \) is contained in the unit ball about the origin. Put
\[
a = \frac{s}{p-1},
\]
and notice that \( a > -n \). We argue in a heuristic way. A more rigorous proof requires a suitable approximation of the weight \( |x|^{a+2} \) by smooth functions, see also [6].

Fix \( u \in C^2_N(\Omega) \) such that \( \Delta u = 0 \) in a neighborhood of 0. For \( p \geq 2 \) one clearly has
\[
(p - 1) \int_{\Omega} |\nabla u|^2 |u|^{p-2} \, dx = \int_{\Omega} (-\Delta u) |u|^{p-2} u \, dx.
\]
For general \( p > 1 \), one can check that \( |\nabla u|^2 |u|^{p-2} \in L^1(\Omega) \) and
\[
(p - 1) \int_{\Omega} |\nabla u|^2 |u|^{p-2} \, dx \leq \int_{\Omega} |\Delta u| |u|^{p-1} \, dx. \tag{18}
\]
Next, we are allowed to use integration by parts again and Hölder inequality to estimate

\[(a + 2)(a + n) \int_{\Omega} |x|^a |u|^p \, dx = - \int_{\Omega} (\Delta |x|^{a+2}) |u|^p \, dx\]

\[= p \int_{\Omega} (\nabla |x|^{a+2} \cdot \nabla u) |u|^{p-2} u \, dx \leq p(a + 2) \int_{\Omega} |x|^{a+1} |\nabla u||u|^{p-1} \, dx\]

\[\leq p(a + 2) \left( \int_{\Omega} |\nabla u|^2 |u|^{p-2} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |x|^{2a+2} |u|^p \, dx \right)^{\frac{1}{2}}.\]

Therefore, from \(a + 2 \geq a + n > 0\) we infer

\[\left(\frac{a + n}{p}\right)^2 \int_{\Omega} |x|^a |u|^p \, dx \leq \int_{\Omega} |\nabla u|^2 |u|^{p-2} \, dx \leq \frac{1}{p - 1} \int_{\Omega} |u|^{p-1} |\Delta u| \, dx\]

by (18). It remains to use Hölder inequality to estimate

\[c \int_{\Omega} |x|^a |u|^p \, dx \leq \left( \int_{\Omega} |x|^a |u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |x|^{-a(p-1)} |\Delta u|^p \, dx \right)^{\frac{1}{p}}\]

where \(c = c(a, n, p) > 0\). Thus

\[c \int_{\Omega} |x|^a |u|^p \, dx \leq \int_{\Omega} |x|^{-a(p-1)} |\Delta u|^p \, dx,\]

as desired. \(\square\)

**References**

[1] M. F. Bidaut-Veron and H. Giacomini, A new dynamical approach of Emden-Fowler equations and systems, *Adv. Differential Equations* **15** (2010), no. 11-12, 1033–1082.

[2] D. Bonheure, E. Moreira dos Santos and M. Ramos, Ground state and non-ground state solutions of some strongly coupled elliptic systems, *Trans. Amer. Math. Soc.* **364** (2012), no. 1, 447–491.

[3] D. Bonheure, E. Moreira dos Santos and M. Ramos, Symmetry and symmetry breaking for ground state solutions of some strongly coupled elliptic systems, *J. Funct. Anal.* **264** (2013), no. 1, 62–96.

[4] J. Busca and R. Manásevich, A Liouville-type theorem for Lane-Emden systems, *Indiana Univ. Math. J.* **51** (2002), no. 1, 37–51.
[5] M. Calanchi and B. Ruf, Radial and non radial solutions for Hardy-Hénon type elliptic systems, *Calc. Var. Partial Differential Equations* **38** (2010), no. 1-2, 111–133.

[6] A. Carioli, Rellich type inequalities related to degenerate elliptic differential operators, in progress.

[7] G. Caristi, L. D’Ambrosio and E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, *Milan J. Math.* **76** (2008), 27–67.

[8] Ph. Clément, D. G. de Figueiredo and E. Mitidieri, Positive solutions of semilinear elliptic systems, *Comm. Partial Differential Equations* **17** (1992), no. 5-6, 923–940.

[9] D. G. de Figueiredo, I. Peral and J. D. Rossi, The critical hyperbola for a Hamiltonian elliptic system with weights, *Ann. Mat. Pura Appl. (4)* **187** (2008), no. 3, 531–545.

[10] M. Fazly, Liouville type theorems for stable solutions of certain elliptic systems, *Adv. Nonlinear Stud.* **12** (2012), no. 1, 1–17.

[11] M. Fazly and N. Ghoussoub, On the Hénon-Lane-Emden conjecture, *Discrete Contin. Dyn. Syst.* **34** (2014), no. 6, 2513–2533.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second edition, Grundlehren der Mathematischen Wissenschaften, 224, Springer, Berlin, 1983.

[13] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, *Rev. Mat. Iberoamericana* **1** (1985), no. 1, 145–201.

[14] E. Mitidieri, A Rellich identity and applications, *Rapporti interni Università di Udine* **25** (1990), 1-35.

[15] E. Mitidieri, A Rellich type identity and applications, *Comm. Partial Differential Equations* **18** (1993), no. 1-2, 125–151.

[16] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Differential Integral Equations* **9** (1996), no. 3, 465–479.

[17] E. Mitidieri, A simple approach to Hardy inequalities, *Mat. Zametki* **67** (2000), no. 4, 563–572; translation in *Math. Notes* **67** (2000), no. 3-4, 479–486.
[18] M. Montenegro, The construction of principal spectral curves for Lane-Emden systems and applications, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **29** (2000), no. 1, 193–229.

[19] R. Musina, Optimal Rellich-Sobolev constants and their extremals, *Differential Integral Equations* **27** (2014), no. 5-6, 579–600.

[20] R. Musina and K. Sreenadh, Radially symmetric solutions to the Hénon–Lane–Emden system on the critical hyperbola, *Commun. Contemp. Math.* **16** (2014), no. 3, 1350030 (16 pages).

[21] L. A. Peletier and R. C. A. M. Van der Vorst, Existence and nonexistence of positive solutions of nonlinear elliptic systems and the biharmonic equation, *Differential Integral Equations* **5** (1992), no. 4, 747–767.

[22] Q. H. Phan, Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems, *Adv. Differential Equations* **17** (2012), no. 7-8, 605–634.

[23] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, Part I: Elliptic systems, *Duke Math. J.* **139** (2007), 555–579.

[24] H. H. Schaefer, *Banach lattices and positive operators*, Springer, New York, 1974.

[25] J. Serrin and H. Zou, Non-existence of positive solutions of semilinear elliptic systems, in *A tribute to Ilya Bakelman (College Station, TX, 1993)*, 55–68, *Discourses Math. Appl.*, 3 Texas A & M Univ., College Station, TX.

[26] J. Serrin and H. Zou, Non-existence of positive solutions of Hénon-Lane-Emden systems, *Differential Integral Equations* **9** (1996), 635–653.

[27] J. Serrin and H. Zou, Existence of positive solutions of the Hénon-Lane-Emden system, *Atti Semin. Mat. Fis. Univ. Modena* **46** (1998), 369–380.

[28] P. Souplet, The proof of the Lane-Emden conjecture in four space dimensions, *Adv. Math.* **221** (2009), no. 5, 1409–1427.

[29] X. J. Wang, Sharp constant in a Sobolev inequality, *Nonlinear Anal.* **20** (1993), no. 3, 261–268.