The Energy Distribution in a Static Spherically Symmetric Nonsingular Black Hole Space-Time

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Abstract

We calculate the energy distribution in a static spherically symmetric nonsingular black hole space-time by using the Tolman’s energy-momentum complex. All the calculations are performed in quasi-Cartesian coordinates. The energy distribution is positive everywhere and be equal to zero at origin. We get the same result as obtained by Y-Ching Yang by using the Einstein’s and Weinberg’s prescriptions.

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1 INTRODUCTION

The General Theory of Relativity is an excellent theory of space, time and gravitation and has been supported by experimental evidences. However, the subject of energy-momentum localization has been a problematic issue since the outset of this theory.

A large number of definitions of the gravitational energy have been given since now. Some of them are coordinate independent and other are coordinate-dependent. It is possible to evaluate the energy and momentum distribution
by using various energy-momentum complexes. There lies a dispute with
the importance of nontensorial energy-momentum complexes whose physi-
cal interpretation have been questioned by a number of physicists, including
Weyl, Pauli and Eddington. Also, there exist an opinion that the energy-
momentum complexes are not useful to get meaningful energy distribution
in a given geometry.

Several examples of particular space-times (the Kerr-Newman, the Einstein-
Rosen and the Bonnor-Vaidya) have been investigated and different energy-
momentum complexes are known to give the same energy distribution for a
given space-time [1]-[6]. Aguirregabiria, Chamorro and Virbhadra [7] showed
that several energy-momentum complexes coincide for any Kerr-Schild class
metric. Xulu obtained interesting results about the energy distribution of
a charged dilaton black hole [8] and about the energy associated with a
Schwarzschild black hole in a magnetic universe [9]. Also, recently, Xulu
[10] obtained the total energy of a model of universe based on the Bianchi I
type metric. Recently, Virbhadra [11] shows that different energy-momentum
complexes give the same and reasonable results for many space-times.

I. C. Yang [12] employing the Einstein’s and Weinberg’s energy-momentum
complexes obtained the energy distribution in the Dymnikova space-time that
is positive everywhere and be equal to zero at origin.

In this paper we compute the energy in a static spherically symmetric
nonsingular black hole space-time in the Tolman’s prescription [13] . We
obtain the same result as obtained by I. Ching-Yang [12] and also make a
discussion of the results. We use the geometrized units (\(G = 1, c = 1\)) and
follow the convention that the Latin indices run from 0 to 3.

2 THE ENERGY DISTRIBUTION IN THE
TOLMAN’S PRESCRIPTION

Dymnikova [14] obtained a static spherically symmetric nonsingular black
hole solution which is expressed by

\[
ds^2 = (1 - \frac{R_g(r)}{r})dt^2 - \frac{dr^2}{(1 - \frac{R_g(r)}{r})} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,
\]

where
\[ R_g(r) = r_g(1 - e^{-\left(\frac{r^3}{r_1}\right)}), \]  
\[ \text{and} \]
\[ r_1^3 = r_0^2 r_g. \]  
We have also
\[ r_0^2 = \frac{3}{8\pi\varepsilon_0}, \]  
and
\[ r_g = 2M. \]

This black hole solution is regular at \( r = 0 \) and everywhere else. The assumed form of the energy-momentum tensor is

\[ T^0_0 = \varepsilon_0 e^{-\left(\frac{r}{r_0}\right)}. \]

The structure of this solution is like a Schwarzschild solution whose singularity is replaced by the de Sitter core.

The Tolman’s energy-momentum complex [13] is given by

\[ \Upsilon^k_i = \frac{1}{8\pi} U^{k\ell}_{i\cdot\ell}, \]  
where \( \Upsilon^0_0 \) and \( \Upsilon^\alpha_\alpha \) are the energy and momentum components.

We have

\[ U^i_{k\ell} = \sqrt{-g}(-g^{pk}V^l_{ip} + \frac{1}{2}g^k_{im}V^m_{pl}), \]  
with

\[ V^i_{jk} = -\Gamma^i_{jk} + \frac{1}{2}g^i_{jk} - \frac{1}{2}g^k_{jm} \Gamma^m_{jk}. \]

The energy-momentum complex \( \Upsilon^k_i \) also satisfies the local conservation laws

\[ \frac{\partial \Upsilon^k_i}{\partial x^k} = 0. \]
The Tolman’s energy-momentum complex gives the correct result if the calculations are carried out in quasi-Cartesian coordinates.

We transform the line element (1) to quasi-Cartesian coordinates $t, x, y, z$ according to

\[
x = r \sin \theta \cos \varphi, \\
y = r \sin \theta \sin \varphi, \\
z = r \cos \theta,
\]

and

\[
r = (x^2 + y^2 + z^2)^{\frac{1}{2}}.
\]

The line element (1) becomes

\[
ds^2 = (1 - \frac{R_g(r)}{r})dt^2 - (dx^2 + dy^2 + dz^2) - \frac{R_g(r)}{r^2(r - R_g(r))}(xdx + ydy + zdz)^2.
\]

The only required components of $U_{ikl}$ in the calculation of the energy are the following

\[
U^{01} = \frac{xR_g(r)}{r^3}, \\
U^{02} = \frac{yR_g(r)}{r^3}, \\
U^{03} = \frac{zR_g(r)}{r^3}.
\]

The components of $U_{ikl}$ are calculated with the program Maple GR Tensor II Release 1.50.

The energy and momentum in the Tolman’s prescription are given by

\[
P_i = \iiint \Upsilon_0^i dx^1 dx^2 dx^3.
\]

Using the Gauss’s theorem we obtain

\[
P_i = \frac{1}{8\pi} \iint U_{i}^{0\alpha} n_\alpha dS
\]
where \( n_\alpha = (x/r, y/r, z/r) \) are the components of a normal vector over an infinitesimal surface element \( dS = r^2 \sin \theta d\theta d\phi \).

Using (14) and applying the Gauss’s theorem (16) we evaluate the integral over the surface of a sphere with radius \( r \)

\[
E(r) = \frac{1}{8\pi} \int \int \frac{R_\phi(r)}{r^2} r^2 \sin \theta d\theta d\phi. \tag{17}
\]

We find that the energy within a sphere with radius \( r \) is given

\[
E(r) = \frac{r^2}{2} \left(1 - e^{-\left(\frac{r}{r_1}\right)^3}\right). \tag{18}
\]

As \( r \to 0 \), \( E(r) \to 0 \) and as \( r \to \infty \), \( E(r) \to M \). Thus the total energy is given by the parameter \( M \) which is the same as the ADM mass for this metric. Note that \( E(r) > 0 \) for all \( r: 0 \leq r < \infty \).

3 DISCUSSION

The subject of the localization of energy continues to be an interesting one. Bondi [15] sustained that a nonlocalizable form of energy is not admissible in relativity. Other authors consider that because the energy-momentum complexes are not tensorial objects and give results which are coordinate dependent they are not adequate for describing the gravitational field.

Due to this reason, this subject was not taken seriously for a long time and was re-opened by the results obtained by Virbhadra, Rosen, Chamorro and Aguirregabiria. Some interesting results which have been found recently [1]-[6], [7]-[12] lend support to the idea that the several energy-momentum complexes can give the same and acceptable result for a given space-time. Also, in his recent paper Virbhadra [11] emphasized that though the energy-momentum complexes are non-tensors under general coordinate transformations, the local conservation laws with them hold in all coordinate systems. Chang, Nester and Chen [16] showed that there exist a direct relationship between energy-momentum complexes and quasilocal expressions because every energy-momentum complexes is associated with a legitimate Hamiltonian boundary term.

Our result also lends support to the idea that the energy-momentum complexes can give the same result for a given space-time.
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