EFFECTIVENESS OF STRONG OPENNESS PROPERTY IN $L^p$

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Abstract. In this article, we obtain an effectiveness result of strong openness property in $L^p$ with some applications.

1. Introduction

The concept of multiplier ideal sheaf helps to translate $L^2$ estimates into algebraic conditions, which plays an important role and was widely discussed in several complex variables, complex geometry and algebraic geometry (see e.g. 29 [24, 23, 20, 2, 6, 7, 27, 28, 3]).

Recall the definition of the multiplier ideal sheaf $I(\varphi)$ (see [2, 3]): a germ of holomorphic function $(f, z) \in I(\varphi)_z$ if and only if $|f|^2 e^{-\varphi}$ is integrable near $z$, where $\varphi$ is a plurisubharmonic function on a complex manifold $X$.

The strong openness property for multiplier ideal sheaves i.e. $I(\varphi) = I_+(\varphi) := \cup_{q>1} I(q\varphi)$, was conjectured by Demailly (see [2, 3]) and proved by Guan-Zhou [16] (see also [18, 23], 2-dimensional case was proved by Jonsson-Mustaţă [19]).

There is an important special case of the strong openness property, which was called the openness property i.e. if $I(\varphi) = O$, then $I(\varphi) = I_+(\varphi)$. The openness property was conjectured by Demailly-Kollár [6] and proved by Berndtsson [1] (2-dimensional case was proved by Favre-Jonsson [9]).

We would like to recall that Berndtsson’s proof of the openness property established the effectiveness result of the openness property [1]. Stimulated by the effectiveness in Berndtsson’s proof of the openness property and continuing Guan-Zhou’s proof of the strong openness property, Guan-Zhou [17] established the effectiveness result of the strong openness property.

Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing the origin $o$. Let $F$ be a holomorphic function on $D$ and let $\varphi$ be a negative plurisubharmonic function on $D$. Recall a notation in [17] that

$$K_{\varphi,F}(o) := \frac{1}{\inf\{\int_D |F|^2 : (\tilde{F} - F, o) \in I_+(2\theta_q(\varphi)\varphi)_o & \tilde{F} \in \mathcal{O}(D)\}^\dagger},$$

where $\theta_q(\varphi) := \sup\{c \geq 0 : |F|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } o\}$ is the jumping number (see [20]). Let

$$\theta(q) := \left(\frac{1}{(q-1)(2q-1)}\right)^\dagger,$$

where $q \in (1, +\infty)$.

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Theorem 1.1. [17] Let $C_1$ and $C_2$ be two positive constants. We consider the set of the pairs $(F, \varphi)$ satisfying

1. $\int_{D} |F|^2 e^{-\varphi} \leq C_1$;
2. $(K_{\varphi, F})^{-1}(a) \geq C_2$.

Then for any $q > 1$ satisfying

$$\theta(q) > \frac{C_1}{C_2},$$

we have

$$(F, o) \in \mathcal{I}(q \varphi).$$

When $F \equiv 1$, Theorem 1.1 implies Berndtsson’s effectiveness result of the openness property [1], see also [17].

By considering the minimal $L^2$ integrals related to a multiplier ideal sheaf $\mathcal{I}(\varphi)$ on the sublevel sets of the weight $\varphi$, Guan [13] obtained a sharp version of Theorem 1.1, i.e.

$$\theta(q) = \frac{q}{q-1},$$

and presented a concavity property of the minimal $L^2$ integrals. After that, Guan [14] (see also [15]) generalized the above concavity property.

In [10], Fornæss established the following strong openness property in $L^p$:

Let $F$ be a holomorphic function on a domain $D \subset \mathbb{C}^n$ containing the origin $o$, $\varphi$ a plurisubharmonic function on $D$ and $p \in (0, +\infty)$. If $|F|^p e^{-q\varphi}$ is $L^1$ on a neighborhood of $o$, then there exists $q > 1$ such that $|F|^p e^{-q\varphi}$ is $L^1$ on a neighborhood of $o$.

In this article, we obtain an effectiveness result of the strong openness property in $L^p$ by using the general concavity in [13].

1.1. Main result. Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing the origin $o$. Let $F$ be a holomorphic function on $D$, and let $\varphi$ be a negative plurisubharmonic function on $D$. Let $p \in (0, +\infty)$, and denote that $F_1 = F[\varphi/2]$, $\varphi_1 = \frac{2}{2(p-1)} \log |F|$, where $[m] := \min\{n \in \mathbb{Z} : n \geq m\}$. Then $\int_{D} |F|^2 e^{-\varphi} = \int_{D} |F_1|^2 e^{-\varphi_1}.

Take $c_{a,p}(\varphi) := \sup\{c \geq 0 : |F|^p e^{-c\varphi} \text{ is } L^1 \text{ on a neighborhood of } o\}$. Especially, when $p = 2$, $c_{a,p}(\varphi)$ degenerates to the jumping number $c^2_{\varphi}(\varphi)$. If $c^2_{\varphi}(\varphi) < +\infty$, we generalize $K_{\varphi, F}$ as follows:

$$K_{\varphi, F, a}(o) := \frac{1}{\inf\{\int_{D} \tilde{F}^2 e^{-\varphi_{1}-(1-a)\varphi} : \tilde{F} \in \mathcal{O}(D)\}}.$$ 

where $a \in (0, +\infty)$. Especially, when $p = 2$ and $a = 1$, $K_{\varphi, F, a}(o)$ degenerates to $K_{\varphi, F}$. Note that $K_{\varphi, F, a}(o) < +\infty$ (see Appendix).

We obtain an effectiveness result of the strong openness property in $L^p$.

Theorem 1.2. Let $C_1$ and $C_2$ be two positive constants. If there exists $a > 0$, such that

1. $\int_{D} |F|^p e^{-\varphi} \leq C_1$;
2. $(K_{\varphi, F, a})^{-1}(a) \geq C_2$.

Then for any $q > 1$ satisfying

$$\theta_a(q) > \frac{C_1}{C_2},$$

where $\theta_a(q)$ is the jumping number of $a$.
we have $|F|^pe^{-q\varphi}$ is $L^1$ on a neighborhood of $o$, where $\theta_s(q) = \frac{a+q-1}{q-1}$.

Let $D = \Delta$, $F = z$ and $\varphi = \frac{a+2}{q} \log |z|$, then $e^{\varphi}_{\partial D}(\varphi) = \frac{2}{q}$. By some calculations, we have $\int_D |F|^p e^{-\varphi} = \frac{2^{q+1}}{(q-1)(p+2)}$ and $K^{(p)}_{\varphi,F,\alpha}(\theta) = \frac{(q+\alpha-1)(p+2)}{2^{q+1}}$, then $K^{(p)}_{\varphi,F,\alpha}(\theta) \int_D |F|^p e^{-\varphi} = \frac{a+q-1}{q-1}$, which implies the sharpness of Theorem \ref{thm: openness_prop}.

When $F \equiv 1$, $K_{D,(1-a)\varphi}(\theta) \geq K^{(p)}_{\varphi,F,\alpha}(\theta)$, where $K_{D,(1-a)\varphi}$ is the Bergman kernel with weight $e^{-(1-a)\varphi}$ on $D$. Note that $K_{D,(1-a)\varphi}(\theta) < +\infty$ (see Appendix).

Theorem \ref{thm: openness_prop} implies the following effectiveness result of the openness property.

**Corollary 1.3.** Let $C_1$ and $C_2$ be two positive constants. If there exists $a > 0$, such that

1. $\int_D e^{-\varphi} \leq C_1$;
2. $(K_{D,(1-a)\varphi})^{-1}(\theta) \geq C_2$.

Then for any $q > 1$ satisfying

\begin{equation}
\theta_s(q) > C_1 \frac{C_2}{C_1}.
\end{equation}

we have $e^{-q\varphi}$ is $L^1$ on a neighborhood of $o$, where $\theta_s(q) = \frac{a+q-1}{q-1}$.

When $D = \Delta$ and $\varphi = \frac{2}{q} \log |z|$. By some calculations, we have $\int_D e^{-\varphi} = \frac{2\pi}{q-1}$ and $K_{D,(1-a)\varphi}(\theta) = \frac{a+q-1}{q-1}$, then $K_{D,(1-a)\varphi}(\theta) \int_D e^{-\varphi} = \frac{a+q-1}{q-1}$, which implies the sharpness of Corollary \ref{cor: openness_prop}.

1.2. **Applications: more precise versions of some known results.** In this section, using Theorem \ref{thm: openness_prop} and Corollary \ref{cor: openness_prop} we give more precise versions of some known effectiveness results of the strong openness property in $L^2$ and the openness property.

When $p = 2$, Theorem \ref{thm: openness_prop} is the following effectiveness result of the strong openness property in $L^2$.

**Corollary 1.4.** Let $C_1$ and $C_2$ be two positive constants. If there exists $a > 0$, such that

1. $\int_D |F|^2 e^{-\varphi} \leq C_1$;
2. $(K^{(2)}_{\varphi,F,\alpha})^{-1}(\theta) \geq C_2$.

Then for any $q > 1$ satisfying

\begin{equation}
\theta_s(q) > C_1 \frac{C_2}{C_1},
\end{equation}

we have $|F|^2 e^{-q\varphi}$ is $L^1$ on a neighborhood of $o$, where $\theta_s(q) = \frac{a+q-1}{q-1}$.

Note that $K^{(2)}_{\varphi,F,\alpha} = K_{\varphi,F}$. When $a = 1$, Corollary \ref{cor: openness_prop} is the sharp version of Theorem \ref{thm: openness_prop} in \ref{thm: openness_prop}.

Let $C_1$ and $C_2$ be two positive constants. We consider the set of the pairs $(F, \varphi)$ satisfying: $\int_D |F|^2 e^{-\varphi} \leq C_1$ and $(K_{\varphi,F})^{-1}(\theta) \geq C_2$. Then for any $q > 1$ satisfying

\begin{equation}
\theta(q) > C_1 \frac{C_2}{C_1},
\end{equation}

we have $|F|^2 e^{-q\varphi}$ is $L^1$ on a neighborhood of $o$, where $\theta(q) = \frac{a}{q-1}$. 

It is clear that \( \frac{q}{q-1} > \left( \frac{1}{(q-1)(2q-1)} \right)^{\frac{1}{q}} \) for any \( q > 1 \), then Corollary 1.4 implies Theorem 1.1. The following remark shows that Corollary 1.4 is more precise than the sharp version of Theorem 1.1 in [13].

**Remark 1.5.** Let \( D \) be \( \Delta^2 \in \mathbb{C}^2 \). Let \( F = z_1 + z_2 \) and \( \varphi = \log |z_1| \), then \( c_o^F(\varphi) = 1 \). By some calculations, we have \( \int_D |F|^2 e^{-\varphi} = \frac{5}{3} \pi^2 \), and \( K_{\varphi,F,a}(o) = \frac{2n+1}{\pi^2} \). Then inequality (1.1) implies \( q < \frac{5}{3} \), and inequality (1.3) implies \( q < \frac{10}{7} \).

When \( F \equiv 1 \), the sharp version of Theorem 1.1 in [13] implies the following result (13):

Let \( C_1 \) and \( C_2 \) be two positive constants. We consider the set of \( \varphi \) satisfying: \( \int_D e^{-\varphi} \leq C_1 \) and \( (K_D)^{-1}(o) \geq C_2 \). Then for any \( q > 1 \) satisfying

\[
\theta(q) > \frac{C_1}{C_2},
\]

we have \( e^{-q\varphi} \) is \( L^1 \) on a neighborhood of \( o \), where \( \theta(q) = \frac{q}{q-1} \) and \( K_D \) is the Bergman kernel on \( D \).

The above result is the case \( a = 1 \) of Corollary 1.3.

It is known that Guan-Zhou’s effectiveness result of strong openness property (17) implies Berndtsson’s effectiveness result of the openness property (11), see also [17]. Note that \( \frac{q}{q-1} > \left( \frac{1}{(q-1)(2q-1)} \right)^{\frac{1}{q}} \) for any \( q > 1 \), then Corollary 1.3 is a general version of Berndtsson’s effectiveness result of the openness property.

The following remark shows that Corollary 1.3 is more precise than the above result (13) with condition (1.4).

**Remark 1.6.** Let \( D \) be \( \Delta^2 \in \mathbb{C}^2 \), and let \( \varphi = \log |z_1| + \log |z_2| \). By some calculations, we have \( \int_D e^{-\varphi} = 4\pi^2 \), and \( K_{D,(1-o)\varphi}(o) = \frac{(a+1)^2}{4\pi^2} \). Then inequality (1.1) implies \( q < \frac{3}{2} \), and inequality (1.4) implies \( q < \frac{4}{3} \).

## 2. Proof of Theorem 1.2

Firstly, we recall two lemmas which will be used in the proof of Proposition 2.3.

**Lemma 2.1.** [13] Let \( F \) be a holomorphic function on pseudoconvex domain \( D \). Let \( \psi \) be a negative plurisubharmonic function on \( D \), and let \( \varphi' \) be a plurisubharmonic function on \( D \). Assume that \( \int_D |F|^2 e^{-\varphi'} < +\infty \). Then

\[
\int_D |F|^2 e^{-\varphi'} = \int_{-\infty}^{+\infty} \left( \int_{\{\psi < -t\}} |F|^2 e^{-\varphi' + \psi} \right) e^t dt.
\]

**Proof.** It is clear that the lemma directly follows from the basic formula (11)

\[
\int_X f d\mu = \int_{0}^{+\infty} \mu(\{x \in X : f(x) > l\}) dl
\]

for nonnegative measurable function \( f : X \to \mathbb{R}^n \) with \( X = D \), \( f = e^{-\psi} \), and \( d\mu = |F|^2 e^{-\varphi' + \psi} dV_{2n} \), where \( dV_{2n} \) is the Lebesgue measure in \( \mathbb{C}^n \).
Next, we will prove the equality (2.2).

\[
\int_X f \, d\mu = \int_X \left( \int_0^{f(x)} \theta_f(l, x) \, dl \right) \, d\mu
\]
\[
= \int_X \left( \int_0^{+\infty} \theta_f(l, x) \, dl \right) \, d\mu
\]
\[
= \int_0^{+\infty} \left( \int_X \theta_f(l, x) \, d\mu \right) \, dl
\]
\[
= \int_0^{+\infty} \mu(\{x \in X : f(x) > l\}) \, dl,
\]

where \( \theta_f(l, x) = \begin{cases} 1 & \text{if } l < f(x) \\ 0 & \text{if } l \geq f(x) \end{cases} \) is a function defined on \( \mathbb{R} \times X \). Then the equality (2.2) has thus been completed. \( \square \)

We recall some notations and definitions in [14]. Let \( \tilde{\psi} < 0 \) be a plurisubharmonic function on \( D \) satisfying \( \tilde{\psi}(o) = -\infty \), and let \( \tilde{\varphi} \) be a Lebesgue measurable function on \( D \) such that \( \tilde{\varphi} + \tilde{\psi} \) is a plurisubharmonic function on \( D \). We call a positive smooth function \( c \) on \((0, +\infty) \) in class \( \mathcal{P}_0 \) if the following three statements hold:

1. \( \int_0^{+\infty} c(t) e^{-t} \, dt < +\infty \);
2. \( c(t) e^{-t} \) is decreasing with respect to \( t \);
3. for any compact subset \( K \subset D \), \( e^{-\tilde{\varphi}} c(-\tilde{\psi}) \) has a positive lower bound on \( K \).

Define a function \( G(t; c) : [0, +\infty) \rightarrow [0, +\infty] \) (\( G(t) \) for short without misunderstanding) by

\[
\inf \left\{ \int_{\{\psi < -t\}} |\tilde{\varphi}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) : (\tilde{\varphi} - F, o) = I(\tilde{\varphi} + \tilde{\psi})_o \& \tilde{\varphi} \in \mathcal{O}(\{\psi < -t\}) \right\},
\]

where \( c \in \mathcal{P}_0 \).

We will use the following concavity of \( G(t) \) in the proof of Proposition 2.3. Let \( h(t) = \int_t^{+\infty} c(t_1) e^{-t_1} \, dt_1 \).

Lemma 2.2. ([14], see also [15]) Let \( c \in \mathcal{P}_0 \). If \( G(0) < +\infty \), then \( G(h^{-1}(r)) \) is concave with respect to \( r \in (0, \int_0^{+\infty} c(t) e^{-t} \, dt] \).

Next, we prove two propositions. Let \( I \) be an ideal in \( \mathcal{O}_o \), and take

\[
C_{F,I,\Phi_1,\Phi_2}(D) := \inf \left\{ \int_D |\tilde{\varphi}|^2 e^{-\Phi_1 + \Phi_2} : (\tilde{\varphi} - F, o) \in I \& \tilde{\varphi} \in \mathcal{O}(D) \right\}.
\]

where \( \Phi_1 \neq -\infty \) and \( \Phi_2 \neq -\infty \) are plurisubharmonic functions on \( D \). \( C_{F,I,\Phi_1,\Phi_2}(D) = 0 \) if and only if \( (F, o) \in I \) (see Appendix).

Proposition 2.3. Let \( F \) be a holomorphic function on pseudoconvex domain \( D \). Let \( \psi \) be a negative plurisubharmonic function on \( D \) satisfying \( \psi(o) = -\infty \), and let \( \varphi' \) be a plurisubharmonic function on \( D \). Then for any \( q' > 1 \), we have

\[
\int_D |F|^2 e^{-\varphi'} \geq \frac{q'}{q' - 1} C_{F,I,(\varphi'+(q'-1)\psi)_o,\varphi'-\psi}(D).
\]

Proof. It suffices to consider the situation that \( \int_D |F|^2 e^{-\varphi'} < +\infty \). For any \( l \in \mathbb{N}^+ \), we can find an increasing smooth function \( c_l(t) \) on \((0, +\infty)\), such that \( c_l(t) = 1 \) when
Let $F$ be a holomorphic function on pseudoconvex domain $D \subset \mathbb{C}^n$, which contains the origin $o \in \mathbb{C}^n$. Let $\psi$ be a negative plurisubharmonic function on $D$ satisfying $\psi(o) = -\infty$, and let $\varphi'$ be a plurisubharmonic function on $D$.

Define $c^F_o(\varphi'; \psi) := \sup\{c \geq 0 : |F|^2 e^{-\varphi'-(2c-1)\psi} \text{ is } L^1 \text{ on a neighborhood of } o\}$. Especially, when $\psi = \varphi'$, $c^F_o(\varphi'; \psi)$ degenerates to the jumping number $c^F_o(\varphi')$ (see [20]).

**Proposition 2.4.** Assume that $\int_D |F|^2 e^{-\varphi'} < +\infty$, and $c^F_o(\varphi'; \psi) < +\infty$. Then for any $q' > 1$ satisfying

\[
\frac{q'}{q' - 1} > \frac{\int_D |F|^2 e^{-\varphi'}}{C_{F, I_+ (\varphi'+(2c^F_o(\varphi'; \psi)-1)\psi)_{o, \varphi'-\psi}(D)}},
\]

we have $|F|^2 e^{-\varphi'-(q'-1)\psi} \text{ is } L^1 \text{ on a neighborhood of } o$.

**Proof.** We give the proof of Proposition 2.4 by using Proposition 2.3.

If $q' > 2c^F_o(\varphi'; \psi)$, then

\[
C_{F, I_+ (\varphi'+(q'-1)\psi)_{o, \varphi'-\psi}(D)} \geq C_{F, I_+ (\varphi'+(2c^F_o(\varphi'; \psi)-1)\psi)_{o, \varphi'-\psi}(D)}.
\]

Combining inequality (2.3), we have

\[
\int_D |F|^2 e^{-\varphi'} \geq \frac{q'}{q' - 1} C_{F, I_+ (\varphi'+(2c^F_o(\varphi'; \psi)-1)\psi)_{o, \varphi'-\psi}(D)}.
\]
Let \( q' \to 2c_o^F(\varphi'; \psi) \), we have inequality (2.7) also holds when \( q' = 2c_o^F(\varphi'; \psi) \). Then we obtain that, if
\[
\int_D |F|^2 e^{-\varphi'} < \frac{q'}{q' - 1} C_F, I_+ (\varphi' + (2c_o^F(\varphi'; \psi) - 1)\psi)\varphi' - \psi (D),
\]
then \( q' < 2c_o^F(\varphi'; \psi) \), \( |F|^2 e^{-\varphi'} - (q' - 1)\psi \) is \( L^1 \) on a neighborhood of \( o \).

Thus Proposition 2.4 holds. \( \square \)

Proof of Theorem 1.2: \( c_o^F(\varphi) < +\infty \) shows that \( \varphi(o) = -\infty \). We use Proposition 2.4 to prove Theorem 1.2. Replace \( F, \psi \) by \( F_1, a\varphi \) and \( \varphi_1 + \varphi \), respectively, where \( a > 0 \). It is clear that \( I_+(\varphi_1 + \varphi + (2c_o^{F_1}(\varphi_1 + \varphi; a\varphi) - 1)a\varphi)_o = I_+(\varphi_1 + 2c_o^{F_1}(\varphi)\varphi)_o \).

Let \( q = (q' - 1)a + 1 \), then it follows from Proposition 2.4 that Theorem 1.2 holds. \( \square \)

3. Appendix

In this section, we prove that \( C_{F,I,\Phi_1 - \Phi_2}(D) = 0 \) if and only if \((F, o) \in I\).

As \( C_{F_1, I_+ (\varphi_1 + 2c_o^{F_1}(\varphi)\varphi)_o}(D) = \frac{1}{K_{\varphi, F,a}^{(p)}(o)} \) and \( F_1 \not\in I_+ (\varphi_1 + 2c_o^{F_1}(\varphi)\varphi)_o \), therefore we get \( K_{\varphi, F,a}^{(p)}(o) < +\infty \). Similarly, we have \( K_{D,(1-a)\varphi}(o) < +\infty \).

It is clear that \((F, o) \in I \Rightarrow C_{F_1, I, \Phi_1 - \Phi_2}(D) = 0 \). Thus, it suffices to prove that \( C_{F_1, I, \Phi_1 - \Phi_2}(D) = 0 \Rightarrow (F, o) \in I \).

By definition of \( C_{F_1, I, \Phi_1 - \Phi_2}(D) \), there exist holomorphic functions \( \{F_j\}_{j \in \mathbb{N}^+} \) on \( D \) such that \( \lim_{j \to +\infty} \int_D |F_j|^2 e^{-\Phi_1 + \Phi_2} = 0 \) and \((F_j - F, o) \in I\). Fixed open subsets \( D' \) and \( D'' \) of \( D \) satisfying \( D'' \subset D' \subset D \). As \( \Phi_1 \) is plurisubharmonic on \( D \), there exists a constant \( C_1 > 0 \) such that \( \int_{D''} |F_j|^2 e^{\Phi_2} \leq C_1 \int_D |F_j|^2 e^{-\Phi_1 + \Phi_2} \) for any \( j \in \mathbb{N}^+ \), therefore

\[
\lim_{j \to +\infty} \int_{D''} |F_j|^2 e^{\Phi_2} = 0.
\]

Since \( \Phi_2 \) is plurisubharmonic on \( D \) and \( \Phi_2 \neq -\infty \), there exists \( s_0 > 0 \) such that \( \int_{D''} e^{-s\Phi_2} < +\infty \) for any \( s \in [0, s_0) \).

Take a positive number \( r \) satisfying \( \frac{1}{r} \in (0, s_0) \), then \( \int_{D''} e^{-\frac{1}{r}\Phi_2} < +\infty \). Following from \( |F_j|^2 e^{\Phi_2} \) is plurisubharmonic on \( D \), we obtain that
\[
|F_j(z)|^{2r} \leq C_2 \int_{D''} |F_j|^2 e^{\Phi_2} \leq C_2 \int_{D''} |F_j|^2 e^{\Phi_2} (\int_{D''} e^{-\frac{1}{r}\Phi_2})^{1-r}
\]
holds for any \( z \in D'' \) and \( j \in \mathbb{N}^+ \), where \( C_2 \) is a constant independent of \( j \) and \( z \).

Then there exists a constant \( C_3 \) such that
\[
|F_j(z)|^2 \leq C_3 \int_{D''} |F_j|^2 e^{\Phi_2}
\]
holds for any \( z \in D'' \) and \( j \in \mathbb{N}^+ \). Combining equality (3.1) and inequality (3.2), we have \( \{F_j\}_{j \in \mathbb{N}^+} \) is uniformly convergent to 0 on any compact subset of \( D \). Hence, \( I \) is closed under local uniform convergence (see [12]) and \((F_j - F, o) \in I \) imply that \((F, o) \in I \).
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