Thin-shell wormholes: Linearization stability

Eric Poisson and Matt Visser

Physics Department, Washington University, St. Louis, Missouri 63130-4899, USA
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The class of spherically-symmetric thin-shell wormholes provides a particularly elegant collection of exemplars for the study of traversable Lorentzian wormholes. In the present paper we consider linearized (spherically symmetric) perturbations around some assumed static solution of the Einstein field equations. This permits us to relate stability issues to the (linearized) equation of state of the exotic matter which is located at the wormhole throat.

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I. INTRODUCTION

The class of thin-shell wormholes provides a particularly elegant collection of exemplars for the study of traversable Lorentzian wormholes. Within the class of spherically-symmetric thin-shell wormholes, the dynamics of the wormhole is completely specified (up to overall integration constants such as the total mass of the wormhole system) once an equation of state is specified for the “exotic matter” which is located at the wormhole throat.

The stability of such wormholes has previously been considered for certain specially chosen equations of state. This type of analysis addresses the question of stability in the sense of proving bounded motion for the wormhole throat; no static solution to the wormhole equations need exist.

In the present paper we analyse the stability of spherically-symmetric thin-shell wormholes by considering linearized radial perturbations around some assumed static solution of the Einstein field equations. This permits us to rephrase stability issues in terms of constraints on the equation of state of the “exotic matter”. Our analysis follows closely that of Ref. [1].

The two types of stability analyses are complementary and permit the extraction of somewhat different information regarding the wormhole systems.

II. SCHWARZSCHILD SURGERY

To construct the wormholes of interest, apply the by now standard cut-and-paste construction: Take two copies Schwarzschild spacetime, and remove from each manifold the four-dimensional regions described by

$$\Omega_{1,2} \equiv \{ r_{1,2} \leq a \mid a > 2M \}. \quad (1)$$

where $a$ is a constant. The resulting manifolds have boundaries given by the timelike hypersurfaces $\partial \Omega_{1,2} \equiv \{ r_{1,2} = a \mid a > 2M \}$. Now identify these two timelike hypersurfaces (i.e., $\partial \Omega_1 \equiv \partial \Omega_2$). The resulting manifold $\mathcal{M}$ is geodesically complete and possesses two asymptotically flat regions connected by a traversable Lorentzian wormhole. The throat of the wormhole is at $\partial \Omega$.

Because $\mathcal{M}$ is piecewise Schwarzschild, the Einstein tensor is zero everywhere except at the throat, where it is formally singular: the Einstein tensor is a Dirac distribution on the manifold. Using the field equations, the surface stress-energy tensor can be calculated in terms of the jump in the second fundamental form across $\partial \Omega$. This prescription, known as the “thin-shell formalism”, is standard. We now review it briefly.

Adopt Gaussian normal coordinates at the throat. Let $\eta$ denote proper distance away from the throat (in the normal direction), with $\eta$ positive in $\Omega_1$ and negative in $\Omega_2$. The second fundamental forms are then

$$K^i_j \pm = \frac{1}{2} g^{ik} \frac{\partial g_{kj}}{\partial \eta} \bigg|_{\eta = \pm 0}; \quad (3)$$

they are functions over the surface $\partial \Omega$. The Ricci tensor at the throat is easily calculated in terms of the discontinuity in the second fundamental forms. Define $[K_{ij}] = K_{ij}^+ - K_{ij}^-$. Then

$$R^\mu_\nu = \left( \begin{array}{cc} [K_{ij}] & 0 \\ 0 & [K] \end{array} \right) \delta(\eta), \quad (4)$$

where $[K]$ denotes the trace of $[K_{ij}]$. This, together with the Einstein field equations, implies that the stress-energy tensor is localized at the throat:

$$T^{\mu\nu} = S^{\mu\nu} \delta(\eta), \quad (5)$$

with

$$S_j^i = -\frac{\epsilon^i}{8\pi G} \left( [K_{ij}] - \delta_{ij} [K] \right). \quad (6)$$

Writing the surface stress-energy tensor in terms of the surface energy density $\sigma$, and surface pressure $p$, one has...
Now specializing to spherical symmetry and adopting units such that \( G = c = 1 \), the thin-shell equations become

\[
\sigma = -\frac{1}{4\pi} [K^\theta_\theta], \quad p = +\frac{1}{8\pi} ([K^\tau_\tau + [K^\theta_\theta]).
\] (8)

To analyse the dynamics of the wormhole, we permit the radius of the throat to become a function of time, \( a \rightarrow a(t) \). A simple computation then yields

\[
[K^\theta_\theta] = \frac{2}{a}\sqrt{1 - 2M/a + \dot{a}^2},
\] (9)

\[
[K^\tau_\tau] = \frac{\dot{a} + M/a^2}{\sqrt{1 - 2M/a + \dot{a}^2}}.
\] (10)

Note that \( \dot{a} \) denotes \( da/d\tau \), where the parameter \( \tau \) measures proper time along the wormhole throat.

The Einstein field equations reduce to

\[
\sigma = -\frac{1}{2\pi a} \sqrt{1 - 2M/a + \dot{a}^2};
\] (11)

\[
p = +\frac{1}{4\pi a} \frac{1}{\sqrt{1 - 2M/a + \dot{a}^2}}.
\] (12)

It is easy to check that these imply energy conservation:

\[
\frac{d}{d\tau} \sigma A + p \frac{d}{d\tau} A = 0,
\] (13)

where \( A = 4\pi a^2 \). In this equation, the first term corresponds to a change in the throat’s internal energy, while the second term corresponds to the work done by the throat’s internal forces.

### III. Stability Analysis

The Einstein equations obtained in the previous section may be recast as

\[
\dot{a}^2 - 2M/a - (2\pi \sigma a)^2 = -1;
\] (14)

\[
\dot{\sigma} = -2(\sigma + p) \frac{\dot{a}}{a}.
\] (15)

If we choose a particular equation of state, in the form \( p = p(\sigma) \), then we can formally integrate the conservation equation and obtain

\[
\ln(a) = -\frac{1}{2} \int \frac{d\sigma}{\sigma + p(\sigma)}.
\] (16)

This relationship may then be formally inverted to yield \( \sigma \) as a function of the wormhole radius: \( \sigma = \sigma(a) \). Once this is done, the first Einstein equation can be written in the form

\[
\dot{a}^2 = -V(a); \quad V(a) = 1 - 2M/a - [2\pi \sigma(a) a]^2.
\] (17)

This single dynamical equation completely determines the motion of the wormhole throat.

One may now choose to investigate particular equations of state, as is done for instance in \ref{2,5}. Alternatively, one might attempt a more general analysis. We have found that a particularly simple, though still instructive, choice is to consider linearized fluctuations around an assumed static solution characterized by the constants \( a_0, \sigma_0 \), and \( p_0 \). Note that these constants are (by assumption) inter-related:

\[
\sigma_0 = -\frac{1}{2\pi a_0} \sqrt{1 - 2M/a_0};
\] (18)

\[
p_0 = +\frac{1}{4\pi a_0} \frac{1}{\sqrt{1 - 2M/a_0}}.
\] (19)

We now insert these relations into the dynamical equation, expanding to second order in \( a - a_0 \). Generically we would have

\[
V(a) = V(a_0) + V'(a_0)[a - a_0] + \frac{1}{2} V''(a_0)[a - a_0]^2 + O([a - a_0]^3),
\] (20)

where a prime denotes \( d/da \). However, because we are linearizing around a static solution at \( a = a_0 \), we know that \( V(a_0) = 0 \), and \( V'(a_0) = 0 \). To leading order, therefore, \( V(a) = \frac{1}{2} V''(a_0)[a - a_0]^2 \).

To compute the various derivatives, it is useful to rewrite the conservation equation as

\[
[\sigma(a) a]' = -(\sigma + 2p).
\] (21)

Differentiating once more,

\[
[\sigma(a) a]'' = -(\sigma' + 2p')
\]

\[
= -\sigma' \left( 1 + 2 \frac{\partial p}{\partial \sigma} \right)
\]

\[
= 2 \left( 1 + 2 \frac{\partial p}{\partial \sigma} \right) \frac{\sigma + p}{a}.
\] (22)

We now define a parameter \( \beta \) by the relation

\[
\beta^2(\sigma) \equiv \frac{\partial p}{\partial \sigma} \bigg|_{\sigma}.
\] (23)

The physical interpretation of \( \beta \) is a matter of some subtlety which we shall subsequently discuss. For now, we simply consider \( \beta \) to be a useful parameter related to the equation of state. Using this definition,

\[
[\sigma(a) a]'' = 2 \left( 1 + 2\beta^2 \right) \frac{\sigma + p}{a}.
\] (24)

Collecting the preceding results, we obtain

\[
V'(a) = \frac{2M}{a^2} + 8\pi^2 \sigma a \left( \sigma + 2p \right),
\] (25)
and
\[ V''(a) = -\frac{4M}{a^3} - 8\pi^2 \left[ (\sigma + 2p)^2 + 2\sigma (1 + 2\beta^2) (\sigma + p) \right]. \] (26)

When evaluated at the static solution \( a = a_0 \) these equations yield the expected results \( V(a_0) = 0 \), and \( V'(a_0) = 0 \). Furthermore,
\[ V''(a_0) = -2a_0^2 \left[ \frac{2M}{a_0} + \frac{M^2/a_0^2}{1 - 2M/a_0} + (1 + 2\beta_0^2) \left( 1 - \frac{3M}{a_0} \right) \right]. \] (27)

The equation of motion for the wormhole throat is, at this order of approximation,
\[ \ddot{a}^2 = -\frac{1}{2} V''(a_0) [a - a_0]^2 + O([a - a_0]^3). \] (28)

So for \( V''(a_0) > 0 \) the wormhole is stable, while for \( V''(a_0) < 0 \) perturbations can grow (at least until the nonlinear regime is reached). Thus the wormhole is stable if and only if:
\[ \frac{2M}{a_0} + \frac{M^2/a_0^2}{1 - 2M/a_0} + (1 + 2\beta_0^2) \left( 1 - \frac{3M}{a_0} \right) < 0. \] (29)

We shall now study this equation in detail.

If one treats \( a_0 \) and \( M \) as specified quantities, stability may be rephrased as a restriction on the parameter \( \beta_0 \):
\[ \beta_0^2 (1 - \frac{3M}{a_0}) < -\frac{1 - 3M/a_0 + 3(M/a_0)^2}{2(1 - 2M/a_0)}. \] (30)

The RHS of this inequality is always negative, while the LHS flips sign at \( a_0 = 3M \). One deduces
\[ \beta_0^2 < -\frac{1 - 3M/a_0 + 3(M/a_0)^2}{2(1 - 2M/a_0)(1 - 3M/a_0)}; \quad a_0 > 3M. \] (31)

\[ \beta_0^2 > -\frac{1 - 3M/a_0 + 3(M/a_0)^2}{2(1 - 2M/a_0)(1 - 3M/a_0)}; \quad a_0 < 3M. \] (32)

The region of stability, in the \( (\beta_0^2) \)-(\( a_0/M \)) plane, is depicted in Fig. 1. Note that as we have formulated the problem, it is meaningless to ask what happens for \( a_0 < 2M \).

On the other hand, if one treats \( \beta_0 \) as an externally specified quantity, stability may be rephrased as a restriction on the the allowable radius \( (a_0) \) of the assumed static solution. The boundary of the region of stability is given by the curve \( V''(a_0/M; \beta_0) = 0 \). This gives a quadratic equation for \( a_0/M \),
\[ 3(1 + 4\beta_0^2) \left( \frac{M}{a_0} \right)^2 - (3 + 10\beta_0^2) \left( \frac{M}{a_0} \right) + 1 + 2\beta_0^2 = 0. \] (33)

with roots
\[ a_0^\pm = \frac{6(1 + 4\beta_0^2)M}{3 + 10\beta_0^2 \mp \sqrt{4\beta_0^2 - 12\beta_0^4 - 3}}. \] (34)

The discriminant is real only for \( \beta_0^2 \geq \frac{1}{2} + \sqrt{3} \approx 3.23205 \), and for \( \beta_0^2 \leq \frac{3}{2} - \sqrt{3} \approx -0.23205 \). After a bit of fiddling to make sure that one is in the physically relevant region, the regions of stability are given by
\[ I: \quad \beta_0^2 \geq \frac{3}{2} + \sqrt{3}, \quad a_0^- < a_0 < a_0^+; \] \[ II: \quad \beta_0^2 \leq -1/2, \quad a_0 > a_0^- . \] (35) (36)

For \(-\frac{1}{2} < \beta_0^2 < \frac{3}{2} + \sqrt{3} \), the wormhole is unstable for all values of \( a_0 \). Of course, this is just a reformulation of the result previously discussed, as plotted in Fig. 1.

**IV. CONCLUSION**

Under normal circumstances one might (naively) try to interpret \( \beta_0 \) as the speed of sound in the exotic matter located at the wormhole throat. Furthermore, under normal circumstances one would require \( \beta_0 \) to lie in the interval \( \beta_0 \in [0, 1] \). (One would normally deduce \( \beta_0 > 0 \) from the assumed stability of matter, and argue that \( \beta_0 \leq 1 \) based on the requirement that the speed of sound should not exceed the speed of light.) If we restrict the speed of sound to lie in this standard range, then a glance at the figure shows that spherically-symmetric thin-shell wormholes are always unstable to linearized radial perturbations. Naively therefore, insisting on stability seems possible only at the cost of requiring somewhat perverse restrictions on the speed of sound.

A certain level of caution in this interpretation is in order: To start with, we are already dealing with “exotic matter,” in the sense that the surface energy density \( \sigma_0 \) is negative. Thus, naive arguments depending on the stability of matter and the positivity of energy should be taken with a grain of salt. In particular, the usual proof that \( \beta_0 > 0 \) is here completely side stepped. Our analysis indeed shows that once the effect of the wormhole’s gravitational field is included, stability implies that large traversable wormhole \( (a_0 > 3M) \) are stable only for \( \beta_0^2 < 0 \! \! \! . \)

There are several known examples of such exotic \( \beta_0 \) < 0 behavior. In the test field limit, the Casimir vacuum between parallel plates is known to be of the form \( T^{\mu\nu} \propto \text{diag}(−1, 1, 1, −3) \). By integrating over the region between the plates, the three-dimensional surface stress-energy takes the form \( T^{ij} \propto \text{diag}(−1, 1, 1) \). In this case \( \beta_2^2 = \partial \sigma/\partial p = -1 \). A similar argument shows that \( \beta_2^2 = -1 \) for false vacuum, for which \( T^{\mu\nu} = \Lambda g^{\mu\nu} \), where \( \Lambda \) is a constant.)

Furthermore, the interpretation of \( \beta_0 \) as the speed of sound is itself problematic insofar as one does not have a detailed microphysical model for the exotic matter. This
is simply not available within the confines of the present analysis. Although $\beta_0$ has the dimensions of a speed, and although one might expect $\beta_0$ to be of the same order of magnitude as the speed of sound, there is no guarantee that $\beta_0$ actually is the speed of sound. A detailed microphysical model for the exotic matter would be necessary in order to settle such issues with any certainty. We therefore conclude that wormhole configurations with $|\beta_0^2| > 1$ should not be ruled out a priori.

We have studied the linearization stability of a thin, spherically symmetric, traversable wormhole against radial perturbations. We have explicitly presented the region of stability in terms of the mass of the wormhole $M$, the radius of the wormhole throat $a_0$, and a parameter $\beta_0$ related to the equation of state. Although the region of stability lies in a somewhat unexpected region of the $(\beta_0^2) - (a_0/M)$ plane, stable wormholes might nevertheless be physically acceptable. To decide for sure would require a detailed microphysical model for the exotic matter; this lies outside the scope of the present paper.

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* Electronic mail: poisson@wuphys.wustl.edu
† Electronic mail: visser@kiwi.wustl.edu
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FIG. 1. Regions of stability: Traversable wormholes in the indicated regions of the $\beta_0^2$ versus $a_0/M$ plane are stable against radial perturbations. The higher dashed line is the curve $\beta_0^2 = \frac{3}{2} + \sqrt{3}$; the lower dashed line is the curve $\beta_0^2 = -\frac{1}{2}$. 