A LIE INFINITY-ALGEBRA OF HAMILTONIAN FORMS IN
N-PLECTIC GEOMETRY

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Abstract. We propose a new definition of so-called Hamiltonian forms in n-plectic geometry and show that they have a non-trivial Lie ∞-algebra structure.

1. N-Plectic Manifold

Basically n-plectic is just another term for what was once called multisymplectic (See [14]). It generalizes the idea of symplectic geometry to manifolds with a distinguished closed and nondegenerate differential form of tensor degree higher than two.

A redefinition of the latter was necessary because multisymplectic now referees to a special kind of vector bundle, designed to have an n-plectic total space as well as a valid Darboux theorem.

Definition 1.1. For any $$n \in \mathbb{N}$$, an n-plectic manifold $$(M, \omega)$$ is a smooth manifold $$M$$, together with a differential form $$\omega \in \Omega^{n+1}M$$, such that $$\omega$$ is closed and the map

$$i_X \omega : T_mM \to \bigwedge^n T_m^*M : X \mapsto i_X \omega_m$$

is injective for all $$m \in M$$. In that case we call $$\omega$$ the n-plectic form of $$M$$.

At a first sight this might look quite similar to the definition of a symplectic manifold, but in fact it is a huge generalization. As it turns out for all cases of $$n$$ intermediate between symplectic forms ($$n = 1$$) and volume forms ($$n = \dim(M) - 1$$), there is no Darboux theorem without the assumption of additional structure [9] and according to this n-plectic geometry behave in a manner having very little in common with symplectic geometry where the Darboux theorem is such a central organizing fact [1].

Remark. In [2], Forger et al. looked in more detail on particular classes of n-plectic forms and derived conditions under which a generalized Darboux theorem can be expected. In particular a multisymplectic structure should be a fiber bundle of rank $$N$$ over an $$n$$-dimensional manifold equipped with a closed and nondegenerate $$(n + 1)$$-form $$\omega$$ defined on the total space $$P$$ which is $$(n - 1)$$-horizontal and admits an involutive and isotropic vector subbundle of the vertical bundle $$VP$$ of $$P$$ of codimension $$N$$ and dimension $$Nn + 1$$. Under these conditions a generalized Darboux’s theorem assures the existence of special local coordinates

$$x_{\mu}, \; q^j, \; p^j, \; p$$
of $P$, called Darboux coordinates, in which $\omega$ adopt the form

$$\omega = dq^j \land dp^m \land d^n x^\mu - dp \land d^n x.$$  \hfill (2)

In particular the total space of every multisymplectic fiber bundle over some $n$-dimensional base is an $n$-plectic manifold and in this sense 'n-plectic' is a generalization of multisymplectic.

Not much is said about morphisms and categories in $n$-plectic geometry so far. Even in symplectic geometry there is not a well accepted concept of an appropriate category yet. A naive but obvious first definition would be the following:

**Definition 1.2.** An $n$-plectic morphism is a smooth map $f : M \to N$, such that $(M, \omega_M)$ and $(N, \omega_N)$ are $n$-plectic manifolds and $f$ is subject to the condition

$$\omega_M = f^* \omega_N.$$  

If $f$ is a diffeomorphism in addition, such that the inverse is an $n$-plectic morphism, then $f$ is called an $n$-plectomorphism.

**2. The Lie $\infty$-Algebra of Hamiltonian Forms**

We define a Lie $\infty$-algebra on any $n$-plectic manifold, different from the one in [14]. After a short introduction, we suggest a new kind of Hamiltonian forms and exhibit their rich and non-trivial Lie $\infty$-algebra structure. We derive explicit expressions for the bilinear and the trilinear bracket and define the higher operators inductively.

Since we make extensive use use differential calculus, a short introduction is given in appendix (B). Moreover we have to deal a lot with graded vector spaces and sign factors and according to a better readable text we use our own sign symbols as defined in (B).

**2.1. Lie $\infty$-Algebras.** On the structure level Lie $\infty$-algebras generalize (differential graded) Lie-algebras to a setting where the Jacobi identity isn’t satisfied any more, but holds 'up to higher homotopies' only. For more on this topic and how these algebras are related to the homotopy theory of (co)chain complexes, see for example [12].

A Lie $\infty$-algebra can be defined in many different ways [12], but the one that works best for us is its 'graded symmetric, many bracket' version:

**Definition 2.1.** A Lie $\infty$-algebra $(V, (D_k)_{k \in \mathbb{N}})$ is a graded vector space $V$, together with a sequence $(D_k)_{k \in \mathbb{N}}$ of graded symmetric, $k$-multilinear morphisms $D_k : \times^k V \to V$, homogeneous of of degree $-1$, such that the strong homotopy Jacobi equation in dimension $n$

$$\sum_{i+j=n+1} \left( \sum_{s \in Sh(j,k-j)} e(s; v_1, \ldots, v_n) D_i(D_j(v_{s_1}, \ldots, v_{s_j}), v_{s_{j+1}}, \ldots, v_{s_n}) \right) = 0$$  \hfill (3)

is satisfied for any integer $n \in \mathbb{N}$ and any vectors $v_1, \ldots, v_n \in V$.  

In particular Lie $\infty$-algebras generalizes ordinary Lie algebras, if the grading is chosen right:

\footnote{$Sh(p,q)$ is the set of shuffle permutations. See (B)}
Example 1 (Lie Algebra). Every Lie algebra \((V, [\cdot, \cdot])\) is a Lie \(\infty\)-algebra if we consider \(V\) as concentrated in degree one and define \(D_k = 0\) for any \(k \neq 2\) as well as \(D_2(\cdot, \cdot) := [\cdot, \cdot]\).

It is beyond the scope of this paper to deal with morphisms or \(\infty\)-morphisms of Lie \(\infty\)-algebras. For more about that look for example at [12].

2.2. Hamiltonian Forms. We know from symplectic geometry that the non-degenerate symplectic 2-form \(\omega\) gives rise to a well defined pairing

\[
 i_X \omega = df
\]  

between functions \(f\) and vector fields \(X\) and that this is the origin of the Poisson bracket for smooth functions on a symplectic manifold.

In attempt to define something similar in a general \(n\)-plectic setting, we have to take the following into account:

For higher \(n\)-plectic forms the pairing is capable to exhibit a much richer inner structure, since it makes sense for differential forms and multivector fields in a range of tensor degrees. However the kernel of \(\omega\) is potentially non-trivial on multivector fields of degrees greater than one and consequently the association

\[
 \text{multivector fields} \leftrightarrow \text{differential forms}
\]

is not unique in either direction.

Moreover the associative product and the Jacobi identity of the Poisson bracket depend on properties that can’t be expected in a general \(n\)-plectic setting. If the Poisson structure has to be replaced by a more general Lie \(\infty\)-algebra, a combination of the strong homotopy Jacobi equation (3) and the fundamental pairing (4) leads to the equation

\[
 i_Y \omega = -\sum_{i+j=k+1} \left( \sum_{s \in Sh(j,k-j)} D_i \left( D_j \left( f_{s_1}, \ldots, f_{s_j}, f_{s_j+1}, \ldots, f_{s_k} \right) \right) \right).
\]

To ensure the existence of a multivector field solution \(Y\), a second fundamental pairing like

\[
 i_Y \omega = f
\]

is required to hold in addition to the first pairing in (4).

Fortunately as we will see in (2.3) the second pairing then also takes care of the ambiguity inherent in the first pairing due to a possible non-trivial kernel of \(\omega\). Consequently we propose both pairings [(4) and (5)] to have multivector field solutions for a given form.

Remark. Maybe strange at a first sight this is already true in symplectic geometry. In fact if \(\eta\) is the Poisson bivector field associated to the symplectic form \(\omega\), then the second equation has a solution for any function \(f\) since

\[
 i_{(f, \eta)} \omega = f.
\]

In conclusion, we propose the following new definition of what should be called Hamiltonian in \(n\)-plectic geometry:

Definition 2.2. A multivector field \(X\) on an \(n\)-plectic manifold \((M, \omega)\) is called semi-Hamiltonian if there is a differential form \(f \in \Omega M\) such that the first fundamental equation

\[
 i_X \omega = -df
\]
is satisfied. Conversely, a differential form \( f \) on \( M \) is called **semi-Hamiltonian** if there exists a multivector field such that (6) holds.

In addition a multivector field \( Y \) is called **Hamiltonian** if there exists a semi-Hamiltonian form \( f \) on \( M \) with

\[
i_Y \omega = -f.
\]

Conversely, a semi-Hamilton form \( f \) is called **Hamiltonian** if there is a multivector field \( X \) such that (7) holds.

The sign in equation (6) is necessary for consistency of our Lie \( \infty \)-algebra structure as we will see later on. The second sign in equation (5) is a convention we propose for fancy.

We write \( H(M) \) for the set of Hamiltonian forms on an \( n \)-plectic manifold \((M, \omega)\).

Since both fundamental equations are always satisfied by the zero form and any element of the kernel of \( \omega \) the set \( H(M) \) is not empty.

If a multivector field satisfies one of the fundamental equation for a given Hamiltonian form \( f \), we say that it is **associated** to \( f \). We exclusively use designators like \( X \) for multivector fields satisfying the first equation (i.e. semi-Hamiltonian multivector fields) and designators like \( Y \) for multivector fields satisfying the second equation (i.e. Hamiltonian multivector fields).

**Remark.** On 1-plectic (symplectic) manifolds, this just rephrases the common definition of Hamiltonian vector fields and functions.

On multisymplectic fiber bundles this is equivalent to the definition of so called Poisson forms.

**Example 2.** Let \((P \to M, \omega)\) be a multisymplectic fiber bundle and \( f \in \Omega(P) \) a semi-Hamiltonian form such that \( \ker(\omega) \subset \ker(f) \) (These forms are called Poisson forms in [4]). Then there exist a multivector field \( Y \) with \( i_Y \omega = -f \).

In symplectic geometry brackets are defined in terms of associated multivector fields, but in a general \( n \)-plectic setting this association is not necessarily well defined. To handle the inherent ambiguity, it is required that the kernel of \( \omega \) is part of the kernel of any Hamiltonian form (as first observed in [4]). We call this the **kernel property** and as the following proposition shows it is a consequence of the second pairing:

**Proposition 2.3.** Assume that \( f \in H(M) \) is a Hamiltonian form on an \( n \)-plectic manifold \((M, \omega)\). Then

\[
\ker(\omega) \subset \ker(f).
\]

If \( Z \) and \( Z' \) are semi-Hamiltonian (resp. Hamiltonian) multivector fields associated to \( f \), then their difference \( Z - Z' \) is an element of the kernel of \( \omega \) and the contractions \( i_Z g \) and \( i_{Z'} g \) are equal for any Hamiltonian form \( g \in H(M) \).

**Proof.** The first part is an implication of the second fundamental pairing. Assume \( \xi \in \ker(\omega) \). Then there is a Hamiltonian multivector field \( Y \) with \( i_\xi f = i_\xi i_Y \omega = \pm i_Y i_\xi \omega = 0 \). For the second part compute \( 0 = f - f = i_Z \omega - i_{Z'} \omega = i_{(Z-Z')} \omega \) in case \( Z \) and \( Z' \) are Hamiltonian as well as \( 0 = df - df = i_{Z'} \omega - i_{Z} \omega = i_{(Z-Z')} \omega \) in

\[\text{This was shown in [5] only for the so called multiphase space of a vector bundle, but since the proof just requires the local form (2) of } \omega \text{ and the existence of the Euler vector field, it holds for any multisymplectic fiber bundle.}\]
the semi-Hamiltonian situation. Finally we get $i_\gamma g - i_{\gamma'} g = i_{(\gamma - \gamma')} g = 0$ from the kernel property of $g$. □

As we will see in the next sections, this forces our multilinear operators given in terms of associated multivector fields to be well defined. The reason is that the contraction of a Hamiltonian form along multivector fields which only differ in elements of the kernel of $\omega$ are equal.

Next we examine algebraic structures on Hamiltonian forms. Immediate from the linearity of the fundamental equations is the following proposition.

**Proposition 2.4.** Let $(M, \omega)$ be an $n$-plectic manifold. The set of Hamiltonian forms is a $\mathbb{N}_0$-graded vector space with respect to the tensor grading. In addition the tensor degree of every (non-trivial) Hamiltonian form $f \in H(M)$ is bounded by $0 \leq |f| \leq n + 1$.

**Proof.** Since both fundamental equations are linear in their arguments, $H(M)$ is a graded vector subspace of $\Omega M$.

The lower bound is obvious. The upper bound holds, since for a differential form $f$ of tensor degree $|f| > (n + 1)$ there can’t be a non-trivial solution to the second fundamental equation $i_Y \omega = -f$, since $\omega$ is of tensor degree $n + 1$ only. □

Hamiltonian forms of tensor degree $n + 1$ have to be closed, since there can’t be a non trivial solution to the first fundamental equation in that case. In contrast semi-Hamiltonian forms have no upper bound but have to be closed in tensor degrees greater than $n + 1$ for the same reason.

This is a rather restrictive property with far reaching implications on the algebraic structure of Hamiltonian forms.

**Corollary 2.5.** The set $H(M)$ of Hamiltonian forms is not a subalgebra of $\Omega(M)$.

**Proof.** As a consequence of the upper bound on non closed semi-Hamiltonian forms, the wedge product of semi-Hamiltonian forms is in general not a semi-Hamiltonian form. □

Nevertheless the wedge product still closes on Hamiltonian functions. This is an $n$-plectic generalization of the associative product for functions in symplectic geometry.

**Theorem 2.6.** Let $C^\infty(M)$ be the algebra of smooth functions on an $n$-plectic manifold $(M, \omega)$ and $H_0(M)$ the subset of Hamiltonian functions. Then $H_0(M)$ is a subalgebra of $C^\infty(M)$.

**Proof.** Since the zero function is Hamiltonian, $H_0(M)$ is not empty. By proposition (2.4) it is a vector subspace of $C^\infty(M)$ and it only remains to show that the product of $C^\infty(M)$ closes on Hamiltonian functions.

To see that let $f_1, f_2 \in H_0(M)$ be Hamiltonian functions, with associated semi-Hamiltonian multivector fields $X_1$ and $X_2$ and associated Hamiltonian multivector fields $Y_1$ and $Y_2$, respectively. A semi-Hamiltonian multivector field associated to the product $f_1 f_2$ is defined by $f_1 X_2 + f_2 X_1$ as the computation $i_{f_1 X_2 + f_2 X_1} \omega = -f_1 df_2 - f_2 df_1 = -d(f_1 f_2)$ shows and an associated Hamiltonian multivector field is given by $f_1 Y_2$ (or $f_2 Y_1$), since $i_{f_1 Y_2} \omega = -f_1 f_2$. □
An even more important consequence of the upper bound is the following corollary. It basically tells us that the set of Hamiltonian forms is not always the section space of a vector bundle over $M$.

**Corollary 2.7.** In general, the set $H(M)$ of Hamiltonian forms is not a $C^\infty(M)$-submodule of $\Omega(M)$.

**Proof.** To see that consider a Hamiltonian form $f$ of tensor degree $n + 1$. An associated Hamiltonian multivector field has to be of tensor degree zero (up to elements of the kernel of $\omega$) and hence is a function $\phi$ satisfying $\phi \omega = -f$. Since $f$ is closed we have

$$0 = -df = d(\phi \omega) = (d\phi) \wedge \omega + \phi (d\omega) = (d\phi) \wedge \omega$$

and it follows, that $d\phi$ is a zero divisor of $\omega$ with respect to the wedge product. Now suppose that there is a function $g \in C^\infty(M)$ such that $dg$ is not a zero divisor of $\omega$. Then $gf$ is not a Hamiltonian form since in general

$$dg \wedge f + gd f = dg \wedge (\phi \omega) = \phi (dg \wedge \omega) \neq 0.$$ 

Hence if there are functions $g \in C^\infty(M)$ such that their exterior derivative $dg$ is not a zero divisor of $\omega$, then $H(M)$ is not a $C^\infty(M)$-submodule of $\Omega(M)$. □

Fortunately it is enough to restrict to zero divisors of the $n$-pletic form to get an appropriate function ring, that qualifies in a module structure on Hamiltonian forms. We propose the following definition and proof the module structure later, since it is easier to use another grading, defined in the next section.

**Definition 2.8.** Let $(M, \omega)$ be an $n$-pletic manifold and $C^\infty(M)$ the algebra of smooth functions on $M$. We say that the set

$$C^\infty_\omega(M) := \{ f \in C^\infty(M) \mid df \wedge \omega = 0 \},$$

is the $n$-pletic function algebra of $M$. Moreover we call a function $f \in C^\infty_\omega(M)$ an $n$-pletic function.

Any constant function is $n$-pletic and so the $C^\infty_\omega(M)$ is not empty.

**Proposition 2.9.** $C^\infty_\omega(M)$ is a subalgebra of $C^\infty(M)$.

**Proof.** From the distributive laws we see that $C^\infty_\omega(M)$ is a vector subspace of $C^\infty(M)$. It only remains to show that the product of zero divisors is a zero divisor, but this is true, since the wedge product is graded commutative. □

**Example 3.** Let $M$ be an orientable manifold and $\omega$ the volume form on $M$. Then $C^\infty_\omega(M)$ equals $C^\infty(M)$, since $df \wedge \omega = 0$ for any differential form $f \in \Omega(M)$.

**Example 4.** Let $(P \to M, \omega)$ be an multisymplectic fiber bundle. Then $C^\infty_\omega(M)$ is the algebra of constant functions on $P$. This was shown in [5].

As the following theorem shows, the definition of (semi)-Hamiltonian forms is at least natural with respect to $n$-pletomorphisms.

**Theorem 2.10.** Assume that $(M, \omega_M)$ and $(N, \omega_N)$ are $n$-pletic manifolds and that $\phi : M \to N$ is an $n$-pletomorphism. The pullback $\phi^* f$ is a Hamiltonian form on $M$ for any Hamiltonian form $f \in H(N)$ on $N$.

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3Recall that functions are both: multivector fields as well as differential forms and the contraction is just multiplication.
Proof. Since $\phi$ is a diffeomorphism we can pull any multivector field $X$ on $N$ back to a multivector field $\phi^*X$ on $M$.

It remains to show that if $X$ is associated to $f$ then $\phi^*X$ is associated to $\phi^*f$, but that follows, since the exterior derivative is natural and the contraction has the natural property (B.2). In particular we have $i_X\omega_M = i_X\phi^*\omega_N = \phi^*i_X\omega_N = -\phi^*df = -d\phi^*f$ for the first pairing and a similar calculation for the second. \hfill $\Box$

2.3. The Differential. We define a differential for Hamiltonian forms and give another grading, slightly different from the usual tensor grading. The latter will simplify our calculations in the following sections and leads to a symmetric Lie $\infty$-algebra.

**Definition 2.11 (Symmetric Grading).** Let $(M, \omega)$ be an n-plectic manifold and $f \in H(M)$ a Hamiltonian form homogeneous of tensor degree $r$. The **symmetric degree** of $f$ is

$$\deg(f) := n - r.$$  \hfill (9)

To distinguish it from the usual tensor grading we use the symbol $| \cdot |$ exclusively for the tensor degree.

An immediate consequence of the fundamental equations is that the tensor degree of any Hamiltonian multivector field $Y$ associated to a homogeneous Hamiltonian form $f$ is given by

$$|Y| = \deg(f) + 1.$$ \hfill (10)

If $f$ is not closed, the tensor degree of an associated semi-Hamiltonian multivector field $X$ is

$$|X| = \deg(f).$$

**Proposition 2.12.** Let $(M, \omega)$ be an n-plectic manifold. The set of Hamiltonian forms is a $\mathbb{Z}$-graded vector space with respect to the symmetric grading. If $f \in H(M)$ is a Hamiltonian form its symmetric degree is bounded by

$$-1 \leq \deg(f) \leq n.$$  \hfill (2.4)

Proof. Follows from proposition (2.4). \hfill $\Box$

We choose this grading to simplify our calculations. If one wants Hamiltonian forms to be bounded by zero, the grading can be shifted. In that case we get a Lie $\infty$-algebra in its graded skew-symmetric incarnation. Unless otherwise stated, we assume that Hamiltonian forms are graded with respect to the symmetric grading.

Besides the vector space structure, Hamiltonian forms have an additional module structure related to the ring of $n$-plectic functions.

**Theorem 2.13.** Let $(M, \omega)$ be an n-plectic manifold and $C^\infty_\omega(M)$ the n-plectic function algebra. The set of Hamiltonian forms $H(M)$ is a $C^\infty_\omega(M)$-module.

If $f \in C^\infty_\omega(M)$ is an n-plectic function and $g$ a homogeneous Hamiltonian form with associated semi-Hamiltonian and Hamiltonian multivector field $X$ and $Y$, respectively, then an associated Hamiltonian multivector field of the scalar product $fg$ is given by

$$fY$$ \hfill (11)

and an associated semi-Hamiltonian multivector field of the scalar product $fg$ is given by

$$e(f)[f, Y] + fX.$$ \hfill (12)
Proof. The first part follows from the second, since we only have to show that the $C^\infty(M)$ scalar multiplication closes on Hamiltonian forms. This can be seen by the direct calculation

$$i_{Y\omega} f = -fg$$

in case of the associated Hamiltonian multivector field and

$$i_{-e(f)}[f,Y] + fX\omega = -e(f)i_{[f,Y]}\omega - fdg \quad = L_f i_Y \omega - i_Y L_f \omega - fdg 
\quad = -L_f g - i_Y d(f\omega) - fdg
\quad = -d(fg) + fdg - i_Y df \wedge \omega - fdg
\quad = -df \wedge g - fdg 
\quad = D_1(fg)$$

in case of the associated semi-Hamiltonian multivector field. □

Back on topic, the following proposition qualifies the exterior derivative as a valid differential on Hamilton forms.

**Proposition 2.14.** Let $(M, \omega)$ be an $n$-plectic manifold. With respect to the symmetric grading, the exterior derivative $d$ is a differential on $H(M)$.

**Proof.** Since the exterior derivative is a codifferential on differential forms with respect to the tensor grading it only remains to show that it is homogeneous of symmetric degree $-1$ and closes on Hamiltonian forms.

The former can be seen from $\text{deg}(df) = n - (|f| + 1) = (n - |f|) - 1 = \text{deg}(f) - 1$. For the latter, assume $f \in H(M)$ with associated semi-Hamiltonian multivector field $X$. Then $df$ is closed and an associated semi-Hamiltonian multivector is given by any element of the kernel of $\omega$, while an associated Hamiltonian multivector field is given by $X$. □

Regarding our proposed Lie $\infty$-algebra on Hamiltonian forms we will use the negative exterior derivative as the differential for consistency reasons.

**Definition 2.15.** Let $(M, \omega)$ be an $n$-plectic manifold and $H(M)$ the $\mathbb{Z}$-graded vector space of Hamiltonian forms on $M$. The $n$-plectic differential

$$D_1 : H(M) \rightarrow H(M)$$

of $H(M)$ is defined for any Hamiltonian form $f \in H(M)$ by the negative exterior derivative

$$D_1 f := -df .$$

In terms of the $n$-plectic differential, the first fundamental equation then just reads as $i_X \omega = D_1 f$.

2.4. **The Bilinear Operator.** We generalize the usual Poisson bracket of functions on a symplectic manifold (see for example [1]) to Hamiltonian forms in a general $n$-plectic framework. Inspired by the Poisson bracket in [4], our bilinear operator is just a (graded symmetric) adoption of the former to a differential graded setting where no Hamiltonian primitive of the $n$-plectic form is available.
Definition 2.16. Let \((M,\omega)\) be an \(n\)-plectic manifold and \(H(M)\) the \(\mathbb{Z}\)-graded vector space of Hamiltonian forms on \(M\). The strong homotopy Lie 2-bracket

\[
D_2 : H(M) \times H(M) \to H(M)
\]

is defined for any homogeneous \(f_1, f_2 \in H(M)\) and associated semi-Hamiltonian multivector fields \(X_1\) and \(X_2\)

\[
D_2(f_1, f_2) := e(f_1)L_{X_1}f_2 + e(f_1, f_2)e(f_2)L_{X_2}f_1
\]

and is then extended to \(H(M)\) by linearity.

Also similar operators \([\cdot,\cdot],[\cdot,\cdot]\) are usually called Poisson bracket, this is misleading in our context, since there is no known product to define a Poisson \((\infty)\)-algebra on \(H(M)\) in general. In addition we propose the 'strong homotopy' modifier since theorem (2.20) shows that the Jacobi identity does not vanish but holds 'up to higher homotopies' as we will see in the next section.

On the technical level, a first thing to show is, that the bracket is independent of the particular chosen associated semi-Hamiltonian multivector fields. This is guaranteed by proposition (2.3).

Theorem 2.17. For any two \(f_1, f_2 \in H(M)\), the image \(D_2(f_1, f_2)\) is a well defined Hamiltonian form. If \(Y_1\) resp. \(Y_2\) are associated Hamiltonian multivector fields and \(X_1\) resp. \(X_2\) are associated semi-Hamiltonian multivector fields, then an associated Hamiltonian multivector field \(Y_{D_2}(f_1, f_2)\) is given by

\[
[Y_2, X_1] + e(f_1, f_2)[Y_1, X_2]
\]

and an associated semi-Hamiltonian multivector field \(X_{D_2}(f_1, f_2)\) by

\[
-2e(f_1)[X_2, X_1].
\]

Proof. To see that \(D_2\) is well defined, suppose \(\xi\) is a multivector fields from the kernel of \(\omega\). Then \(L_\xi f = d\xi f - (-1)^{|\xi|}i_\xi df = 0\) since \(f\) as well as \(df\) has the kernel property and we get \(L_{X+\xi} f = L_X f\) for any Hamiltonian form \(f\) and semi-Hamiltonian multivector field \(X\).

By prop. (2.3) the difference of multivector fields associated to the same Hamiltonian form is an element of the kernel of \(\omega\) and consequently the image \(D_2(f_1, f_2)\) does not depend on the particular chosen associated semi-Hamiltonian multivector field.

To see that \([Y_2, X_1] + e(f_1, f_2)[Y_1, X_2]\) is an associated Hamiltonian multivector field, compute

\[
i_{Y_{D_2}(f_1, f_2)}\omega = i_{[Y_2, X_1]}\omega + e(f_1, f_2)i_{[Y_1, X_2]}\omega
\]

\[
= -e(f_1, f_2)e(f_2)i_{[X_1, Y_2]}\omega - e(f_1)i_{[X_2, Y_1]}\omega
\]

\[
= -e(f_1, f_2)e(f_2)(-e(f_1, f_2)e(f_2)e(f_2)L_{X_1}i_{Y_2}\omega - i_{Y_2}L_{X_1}\omega)
\]

\[
- e(f_1)(-e(f_1, f_2)e(f_1)e(f_2)L_{X_2}i_{Y_1}\omega - i_{Y_1}L_{X_2}\omega)
\]

\[
= e(f_1)L_{X_1}i_{Y_2}\omega + e(f_1, f_2)e(f_2)L_{X_2}i_{Y_1}\omega
\]

\[
= -e(f_1)L_{X_1}f_2 - e(f_1, f_2)e(f_2)L_{X_2}f_1
\]

\[
= -D_2(f_1, f_2)
\]
and to see that $-2e(f_1)[X_2, X_1]$ is an associated semi-Hamilton multivector field use $L_X\omega = 0$ (for a semi-Hamiltonian multivector field) and compute

$$-2e(f_1)i_{[X_2, X_1]}\omega = -e(f_1)i_{[X_2, X_1]}\omega - e(f_2)e(f_2)i_{[X_1, X_2]}\omega$$
$$= -e(f_1, f_2)L_{X_2}i_{X_1}\omega + e(f_1)i_{X_1}L_{X_2}\omega$$
$$= -L_{X_1}i_{X_2}\omega + e(f_1, f_2)e(f_2)i_{X_2}L_{X_1}\omega$$
$$= -e(f_1, f_2)e(f_2)L_{X_2}D_1f_1 - e(f_1)e(f_1)L_{X_1}D_1f_2$$
$$= e(f_1, f_2)e(f_2)e(f_2)L_{X_2}dL_1f_1 + e(f_1)e(f_1)L_{X_1}df_2$$
$$= -e(f_1, f_2)e(f_2)dL_{X_2}f_1 - e(f_1)dL_{X_1}f_2$$
$$= e(f_1, f_2)e(f_2)D_1L_{X_2}f_1 + e(f_1)D_1L_{X_1}f_2$$
$$= D_1D_2(f_1, f_2).$$

\[ \square \]

In what follows we will sometimes refer to $X_{D_2}$ as a semi-Hamiltonian multivector field, associated to $D_2$, without stating the arguments explicit.

**Corollary 2.18.** The $\mathbb{N}_0$-graded vector space of semi-Hamiltonian multivector fields is a subalgebra of the Schouten algebra.

**Proof.** Semi-Hamiltonian multivector fields associated to $(n + 1)$-forms are trivial, so the grading is valid. Since the first fundamental equation is linear it only remains to show that the Schouten bracket closes on semi-Hamiltonian multivector fields, but this is guaranteed by the previous theorem. \( \square \)

**Remark.** The associated multivector field $Y_{D_2}(f_1, f_2)$ is graded symmetric in its arguments and from the super-symmetry of the Schouten bracket we get

$$-2e(f_1)[X_2, X_1] = -2e(f_1, f_2)e(f_2)[X_1, X_2],$$

so that $X_{D_2}(f_1, f_2)$ is graded symmetric too. If at least one of the arguments is a closed form, $X_{D_2}(f_1, f_2)$ vanishes. Consequently closed forms are a two-sided ideal in the non-associative algebra $(H(M), D_2)$.

The next theorem shows that the strong homotopy Lie 2-bracket qualifies as the bilinear operator in a Lie $\infty$-algebra, in the sense that it has the right symmetry and interacts with the differential as required.

**Theorem 2.19.** The bilinear operator $D_2$ is graded symmetric and homogeneous of degree $-1$ with respect to the symmetric grading. Moreover the strong homotopy Jacobi equation in dimension two

$$D_1D_2(f_1, f_2) + D_2(D_1f_1, f_2) + e(f_1, f_2)D_2(D_1f_2, f_1) = 0$$

is satisfied for any Hamiltonian forms $f_1, f_2 \in H(M)$.

**Proof.** Assume $X_1, X_2 \in \mathfrak{X}M$ are semi-Hamiltonian multivector fields, associated to $f_1$ and $f_2$, respectively.

Bilinearity is a straightforward implication of the definition, since a multivector field associated to any linear combination $\lambda_1f_1 + \lambda_2f_2$ is given by the linear...
combination $\lambda_1X_1 + \lambda_2X_2$. To see graded symmetry, compute

$$D_2(f_1, f_2) = e(f_1)L_{X_2}f_2 + e(f_1, f_2)e(f_2)L_{X_1}f_1$$

$$= e(f_1, f_2)(e(f_1, f_2)e(f_1)L_{X_2}f_2 + e(f_2)L_{X_2}f_1)$$

$$= e(f_1, f_2)(e(f_2)L_{X_2}f_2 + e(f_2, f_1)e(f_1)L_{X_2}f_2)$$

$$= e(f_1, f_2)D_2(f_2, f_1).$$

If $f_i$ is homogeneous of symmetric degree $\text{deg}(f_i)$ the Lie derivative along $X_i$ is homogeneous of degree $\text{deg}(f_i) - 1$. Consequently $\text{deg}(D_2(f_1, f_2)) = \text{deg}(f_1) + \text{deg}(f_2) - 1$ and $D_2$ is homogeneous of symmetric degree $-1$.

Finally, compute the strong homotopy Jacobi equation in dimension two:

$$D_1D_2(f_1, f_2) = -e(f_1)dL_{X_1}f_2 - e(f_1, f_2)e(f_2)dL_{X_2}f_1$$

$$= e(f_1)e(f_1)X_{L_{X_2}f_2} + e(f_1, f_2)e(f_2)L_{X_2}df_1$$

$$= e(f_1)e(f_1)L_{X_1}(-df_2) - e(D_1f_1, f_2)e(f_2)L_{X_2}(-df_1)$$

$$= -D_2(D_1f_1, f_2) - e(f_1)D_2(f_1, f_2)$$

$$= -D_2(D_1f_1, f_2) - e(f_1, df_2)e(f_1)D_2(D_1f_2, f_1)$$

$$= -D_2(D_1f_1, f_2) - e(f_1, f_2)D_2(D_1f_2, f_1).$$

In case $\omega$ is symplectic, (non-closed) Hamiltonian forms are just functions and definition (2.16) rephrases the usual Poisson bracket from symplectic geometry to a differential graded setting. However a big difference is, that in general the Jacobi identity does not vanish any more.

**Theorem 2.20** (Jacobi Identity). The graded Jacobi identity does not vanish. Instead for any three Hamiltonian forms $f_1$, $f_2$, $f_3 \in H(M)$ the equation

$$\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3})$$

$$= -\frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)e(f_{s_2})L_{X_{D_2(f_{s_1}, f_{s_2})}}f_{s_3}$$

is satisfied.

**Proof.** Let $X_1$, $X_2$, $X_3$ be associated semi-Hamiltonian multivector fields, respectively. We apply the definition of $D_2$ to rewrite the left side into

$$-\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)e(f_{s_1})e(f_{s_2})L_{X_{D_2(f_{s_1}, f_{s_2})}}f_{s_3}$$

$$+ \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)e(f_{s_1}, f_{s_3})e(f_{s_2})L_{X_{s_2}}$$

$$\cdot (e(f_{s_1})L_{X_{s_1}}f_{s_2} + e(f_{s_1}, f_{s_2})e(f_{s_2})L_{X_{s_2}}f_{s_1}) =$$

$$-\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)e(f_{s_1})e(f_{s_2})L_{X_{D_2(f_{s_1}, f_{s_2})}}f_{s_3}$$

$$+ \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)e(f_{s_2})L_{X_{s_1}}L_{X_{s_2}}f_{s_3}$$

$$+ \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)e(f_{s_1})e(f_{s_1}, f_{s_2})L_{X_{s_2}}L_{X_{s_2}}L_{X_{s_3}}f_{s_3} =$$
multivector fields and is a closed Hamiltonian form. If
expression
Proof.

\[
- \sum_{s \in Sh(2,1)} c_2 e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{X_{D_2}}(f_{s_1}, f_{s_2}) f_{s_3}
- \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_2}) \\
\cdot (-e(f_{s_1}, f_{s_2}) e(f_{s_2}) L_{X_2} L_{X_{s_1}} f_{s_3} - L_{X_{s_1}} L_{X_{s_2}} f_{s_3}).
\]

Using (B.2) the second shuffle sum can be rewritten in terms of the Schouten bracket
to get the expression

\[
- \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{X_{D_2}}(f_{s_1}, f_{s_2}) f_{s_3}
- \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_2}) L_{[X_{s_2}, X_{s_1}]} f_{s_3}
\]

and expanding this

\[
- \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{X_{D_2}}(f_{s_1}, f_{s_2}) f_{s_3}
+ \frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{-2 e(f_{s_1}) [X_{s_2}, X_{s_1}]} f_{s_3}
\]

we apply the definition of the associated semi-Hamiltonian multivector field and collect the shuffle terms to arrive at

\[
- \frac{1}{2} \sum_{s \in Sh(2,1)} c_2 e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{X_{D_2}}(f_{s_1}, f_{s_2}) f_{s_3}. \
\]

Remark. The factor 'two' in \(-2e(f_1)[X_2, X_1]\) is the only reason for the graded
Jacobi expression to not vanish. In addition the Jacobi equation as given above, is
still valid for semi-Hamiltonian forms.

The next proposition uses the vanishing Jacobi identity of the Schouten bracket
to show, that the Hamiltonian Jacobi expression is a closed form.

**Theorem 2.21.** For any three Hamiltonian forms \(f_1, f_2, f_3 \in H(M)\), the Jacobi
expression

\[
\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3})
\]

is a closed Hamiltonian form. If \(X_1, X_2, X_3\) are associated semi-Hamiltonian
multivector fields and \(Y_1, Y_2, Y_3\) are associated Hamiltonian multivector field,
respectively, then an associated Hamiltonian multivector field is given by

\[
\frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_2}) [[X_{s_2}, X_{s_1}], Y_{s_1}] \\
\]

(18)

**Proof.** To see that (18) is an associated Hamiltonian multivector field compute

\[
\frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_2}) i_{[[X_{s_2}, X_{s_1}], Y_{s_1}]} \omega
\]

\[
= -\frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_2}) e(f_{s_1}, f_{s_2}) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) L_{[X_{s_2}, X_{s_1}]} i_{Y_{s_1}} \omega
\]

\[
= \frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{-2 e(f_{s_1}) [X_{s_2}, X_{s_1}]} i_{Y_{s_1}} \omega
\]

\[
= \frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{X_{D_2}}(f_{s_1}, f_{s_2}) i_{Y_{s_1}} \omega
\]

\[
= \frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) L_{X_{D_2}}(f_{s_1}, f_{s_2}) f_{s_3}
\]

\[
= - \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3}).
\]
The closedness follows from the Jacobi identity of the Schouten bracket for multivector fields. Since \([X_i, X_j]\) is a semi-Hamiltonian multivector field, \(L_{[X_i, X_j]} \omega = 0\) and we have

\[
0 = \sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) e(f_{s1}) e(f_{s2}) e(f_{s3}) \omega_{[X_{s3}, X_{s2}, X_{s1}]} \omega
= \sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) e(f_{s1}) e(f_{s2}) e(f_{s3}) \omega_{[X_{s3}, X_{s2}, X_{s1}]} i_{X_{s1}} \omega
= \sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) e(f_{s3}) L_{[X_{s2}, X_{s1}]} i_{X_{s1}} \omega
= -\frac{1}{2} \sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) e(f_{s1}) e(f_{s2}) e(f_{s3}) L_{-2e(f_{s1}) [X_{s2}, X_{s1}]} df_{s3}
= -\frac{1}{2} e(f_1) e(f_2) e(f_3) \sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) L_{X_{D_{2s}}} (f_{s1}, f_{s2}) df_{s3}
= -\frac{1}{2} e(f_1) e(f_2) e(f_3) \sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) e(f_{s1}) e(f_{s2}) dL_{X_{D_{2s}}} (f_{s1}, f_{s2}) f_{s3}
= e(f_1) e(f_2) e(f_3) d \left( \sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) D_2 (f_{s1}, f_{s2}, f_{s3}) \right).
\]

\(\square\)

The existence of the associated multivector field (2.21) is tied to the assumption that for any Hamiltonian form \(f\) there is a multivector field \(Y\) satisfying \(i_Y \omega = -f\). If we drop that assumption and work with semi-Hamiltonian forms instead, the equation

\[
i_Y \omega = -\sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) D_2 (f_{s1}, f_{s2}, f_{s3})
\]  

must not have a solution any more, as the following simple counterexample shows:

**Example 5.** Let \(M\) be the linear manifold \(\mathbb{R}^6\) with global coordinates \(x_1, \ldots, x_6\) and \(\omega\) the differential 4-form

\[
dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^6 + dx^2 \wedge dx^5 \wedge dx^6.
\]

\((\mathbb{R}^6, \omega)\) is a 3-plectic manifold, since \(\omega\) is closed and the contraction \(i_{X^j} \omega\) is the zero form, only if any coordinate \(X^j\) is the zero function. Define

\[
f_1 := (x_4 - x_1^2 x_3) \, dx^5 \wedge dx^6 \quad \text{and} \quad f_2 := (x_3 + x_2^2 x_4) \, dx^5 \wedge dx^6.
\]

These forms are Hamiltonian, since associated Hamiltonian multivector fields are given (for example) by

\[
Y_1 := (x_1^2 x_3 - x_4) \, \partial_3 \wedge \partial_1, \quad Y_2 := -(x_1^2 x_4 + x_3) \, \partial_4 \wedge \partial_2
\]

and associated semi-Hamiltonian multivector fields by

\[
X_1 := x_1^2 \partial_1 - \partial_2 - 2x_1 x_3 \partial_3, \quad X_2 := -\partial_1 - x_2^2 \partial_2 + 2x_2 x_4 \partial_4.
\]

In this case the Schouten bracket reduces to the usual Lie bracket and is given by

\[
[X_2, X_1] = 2x_1 \partial_1 + 2x_2 \partial_2 - 2x_3 \partial_3 - 2x_4 \partial_4.
\]

Next define the form

\[
f_3 := dx^1 \wedge dx^2.
\]

Any closed form is semi-Hamiltonian and so is \(f_3\). In contrast \(f_3\) is not Hamiltonian, because there can’t be a multivector field satisfying \(i_Y \omega = -f_3\).

To compute the Jacobi expression \(\sum_{s \in \text{Sh}(2,1)} e(s; f_1, f_2, f_3) D_2 (f_{s1}, f_{s2}, f_{s3})\) we use the fact that equation (2.20) is still valid for semi-Hamiltonian forms. Since \(f_3\) is closed we have to compute \(L_{[X_2, X_1]} f_3\) only, but this is \(4 dx^1 \wedge dx^2\). It follows that the Jacobi expression is not a Hamiltonian form and that equation (19) has no solution.
This justifies our proposed definition of Hamiltonian forms. If we want a trilinear operator $D_3$, related to our sh-Lie 2-bracket by the strong homotopy Jacobi equation in dimension three a solution to the equation

$$i_{X_{D_3}}(f_1, f_2, f_3) = - \sum_{s \in \text{Sh}(2,1)} \epsilon(s; f_1, f_2, f_3) D_2(D_2(f_1, f_2), f_3) + \sum_{s \in \text{Sh}(1,2)} \epsilon(s; f_1, f_2, f_3) D_3(D_1 f_1, f_2, f_3)$$

is required and since the contraction operator is linear, this needs a solution of equation (19).

2.5. The Trilinear Operator. The strong homotopy Lie 2-bracket does not satisfy the graded Jacobi identity and it is the subject of this section to define a trilinear operator $D_3$ such that the strong homotopy Jacobi equation (3) in dimension three is satisfied instead.

**Definition 2.22.** Let $(M, \omega)$ be an $n$-plectic manifold and $H(M)$ the $\mathbb{Z}$-graded vector space of Hamiltonian forms on $M$. The **strong homotopy Lie 3-bracket**

$$D_3 : H(M) \times H(M) \times H(M) \to H(M)$$

is defined for any homogeneous $f_1, f_2, f_3 \in H(M)$ and semi-Hamiltonian multivector fields $X_{D_2}(\cdot, \cdot)$ associated to the sh-Lie 2-bracket, by

$$D_3(f_1, f_2, f_3) := -\frac{1}{2} \sum_{s \in \text{Sh}(2,1)} \epsilon(s; f_1, f_2, f_3) e(f_3) i_{X_{D_2}(f_1, f_2)} f_3$$

and is then extended to $H(M)$ by linearity.

Again the kernel property (2.3) guarantees that this definition does not depend on the particular chosen associated semi-Hamiltonian multivector fields. Moreover it qualifies as the trilinear operator in a Lie $\infty$-algebra structure:

**Theorem 2.23.** The operator $D_3$ is well defined, graded symmetric and homogeneous of degree $-1$. For any three Hamiltonian forms $f_1, f_2, f_3 \in H(M)$ the strong homotopy Jacobi equation in dimension four

$$D_1 D_3(f_1, f_2, f_3) + \sum_{s \in \text{Sh}(1,2)} \epsilon(s; f_1, f_2, f_3) D_3(D_1 f_1, f_2, f_3) + \sum_{s \in \text{Sh}(2,1)} \epsilon(s; f_1, f_2, f_3) D_2(D_2(f_1, f_2), f_3) = 0$$

is satisfied and for associated Hamiltonian multivector fields $Y_1, Y_2, Y_3$ an associated Hamiltonian multivector field $Y_{D_3}(f_1, f_2, f_3)$ is given by

$$\frac{1}{2} \sum_{s \in \text{Sh}(2,1)} \epsilon(s; f_1, f_2, f_3) e(f_3) e(f_2) Y_3 \wedge X_{D_2}(f_1, f_2),$$

while for associated semi-Hamiltonian multivector fields $X_1, X_2, X_3$ an associated semi-Hamiltonian multivector field $X_{D_3}(f_1, f_2, f_3)$ is given by

$$\sum_{s \in \text{Sh}(2,1)} \epsilon(s; f_1, f_2, f_3) e(f_3) ([X_{s_3}, X_{s_2}], Y_{s_1}) + X_{s_3} \wedge X_{D_2}(f_1, f_2)).$$
Proof. To see that the definition does not depend on the particular chosen associated Hamiltonian multivector fields, we use (2.3) and proceed as in (2.17).

Graded symmetry follows from the graded symmetry of the semi-Hamiltonian multivector fields \( X_{D_2}(f_1, f_2) \).

The contraction \( i_X \) is graded linear of (symmetric) degree \(|X|\) for any homogeneous multivector field \( X \) and since \(|X_{D_2}(f_1, f_2)| = \text{deg}(f_1) + \text{deg}(f_2) - 1\) for any \( i, j \in \mathbb{N}_3 \), the sh-Lie 3-bracket is homogeneous of degree \(-1\).

To see the strong homotopy Jacobi equation in dimension three, we apply the definition of the differential and of \( D_3 \) to rewrite the left side

\[
D_1 D_3(f_1, f_2, f_3) = -d D_3(f_1, f_2, f_3) = \frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) d i_{X_{D_2}(f_{s_1}, f_{s_2})} f_{s_3}.
\]

Regarding (2.20) and (3.2) we use \( e(D_2(f_1, f_2)) = -e(f_1) e(f_2) \) and insert appropriate correction terms

\[
\frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_1}) e(f_{s_2}) \left( d i_{X_{D_2}(f_{s_1}, f_{s_2})} f_{s_3} + e(f_{s_1}) e(f_{s_2}) i_{X_{D_2}(f_{s_1}, f_{s_2})} d f_{s_3} \right)
\]

\[-\frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) i_{X_{D_2}(f_{s_1}, f_{s_2})} df_{s_3}.
\]

According to (2.20) the first shuffle sum is just the negative Jacobi expression and rewriting the latter we get

\[-\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3})
\]

\[-\frac{1}{2} \sum_{s \in Sh(2,1)} e(s; f_{s_1}, f_{s_2}, df_{s_3}) e(f_{s_2}) i_{X_{D_2}(f_{s_1}, f_{s_2})} df_{s_3}.
\]

which we can rewrite into

\[-\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3})
\]

\[-\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_3(f_{s_1}, f_{s_2}, D_1 f_{s_3}).
\]

To see that \( Y_{D_3}(f_1, f_2, f_3) \) is a Hamiltonian multivector field associated to the image \( D_3(f_1, f_2, f_3) \), just apply the contraction of \( \omega \) along \( Y_{D_3} \) using (3.1).

To see that \( X_{D_3}(f_1, f_2, f_3) \) is a semi-Hamiltonian multivector field associated to \( D_3(f_1, f_2, f_3) \), we write \( Y := \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) e(f_{s_2}) [[X_{s_1}, X_{s_2}], Y_{s_1}] \) and use (18) to compute:

\[
i_{X_{D_3}(f_1, f_2, f_3)} \omega = i_Y \omega + \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) i_{X_{s_1} \wedge X_{D_2}(f_{s_1}, f_{s_2})} \omega
\]

\[= -\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3}) + \sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) i_{X_{s_1} \wedge X_{D_2}(f_{s_1}, f_{s_2})} \omega
\]

\[= -\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3}) + \sum_{s \in Sh(1,2)} e(s; f_1, f_2, f_3) i_{X_{D_2}(f_{s_1}, f_{s_2})} df_{s_3}
\]

\[= -\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3) D_2(D_2(f_{s_1}, f_{s_2}), f_{s_3})
\]
\[ + \sum_{s \in Sh(1,2)} e(s; f_1, f_2, f_3)D_3(df_{s_1}, f_{s_2}, f_{s_3}) = D_1D_3(f_1, f_2, f_3). \]

\[ \square \]

**Remark.** At this point we should stress again, that if there is no Hamiltonian multivector field \( Y \) satisfying

\[ i_Y \omega = -\sum_{s \in Sh(2,1)} e(s; f_1, f_2, f_3)D_3(D_2(f_{s_1}, f_{s_2}), f_{s_3}) \]

then the previous proof shows, that the image \( D_3(f_1, f_2, f_3) \) is not semi-Hamiltonian. Regarding example (5) this justifies our definition of Hamiltonian forms as differential forms satisfying both fundamental equations.

### 2.6. The general multilinear Operator

We give an inductive definition of \( k \)-linear operators, based at the sh-Lie 3-bracket from the previous section. These operators satisfy strong homotopy Jacobi equations in any dimension and define a Lie \( \infty \)-algebra structure on the set of Hamiltonian forms.

**Definition 2.24.** Let \( (M, \omega) \) be an \( n \)-plectic manifold and \( H(M) \) the \( \mathbb{Z} \)-graded vector space of Hamiltonian forms on \( M \). The **strong homotopy Lie \( k \)-bracket**

\[ D_k : H(M) \times \cdots \times H(M) \to H(M), \quad (24) \]

is defined inductively for any \( k > 3 \), homogeneous \( f_1, \ldots, f_k \in H(M) \) and semi-Hamiltonian multivector fields \( X_{D_{k-1}}(\cdot, \cdots, \cdot) \) associated to the strong homotopy Lie \((k-1)\)-bracket \( D_{k-1} \) by

\[ D_k(f_1, \ldots, f_k) := -\sum_{s \in Sh(k-1,1)} e(s; f_1, \ldots, f_k)e(f_{s_1}) \cdots e(f_{s_k-1})i_{X_{D_{k-1}}(f_{s_1}, \ldots, f_{s_k-1})}f_{s_k} \]

and is then extended to \( H(M) \) by linearity.

The induction base is the sh-Lie 3-bracket. If we refere to the sh-Lie \( k \)-bracket for any \( k \in \mathbb{N} \), then the differential \( D_1 \) is meant to be the sh-Lie 1-bracket.

The following theorem is the central statement in this work and basically says, that the sequence of sh-Lie \( k \)-brackets defines a Lie \( \infty \)-algebra on the vector space of Hamiltonian forms.

**Theorem 2.25.** The operator \( D_k \) is well defined, graded symmetric and homogeneous of degree \(-1\) for any \( k \in \mathbb{N} \) and the strong homotopy Jacobi equation

\[ \sum_{i+j=n+1} \left( \sum_{s \in Sh(j,i-1)} e(s; f_1, \ldots, f_n)D_1(D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_{j+1}}, \ldots, f_{s_n}) \right) = 0 \]

is satisfied for any Hamiltonian forms \( f_1, \ldots, f_n \in H(M) \) and in any dimension \( n \in \mathbb{N} \).

If \( Y_1, \ldots, Y_k \) are associated Hamiltonian multivector fields, an associated Hamiltonian multivector field \( Y_{D_k}(f_1, \ldots, f_k) \) is given by

\[ \sum_{s \in Sh(k-1,1)} e(s; f_1, \ldots, f_k)e(f_{s_1}) \cdots e(f_{s_k-1})Y_{s_k} \wedge X_{D_{k-1}}(f_{s_1}, \ldots, f_{s_k-1}) \quad (25) \]

If \( X_1, \ldots, X_k \) are associated semi-Hamiltonian multivector field, an associated semi-Hamiltonian multivector field \( X_{D_k}(f_1, \ldots, f_k) \) is given by

\[ \sum_{s \in Sh(k-1,1)} e(s; f_1, \ldots, f_k)X_{s_k} \wedge X_{D_{k-1}}(f_{s_1}, \ldots, f_{s_k-1}) - Y_{f_k}(f_{s_1}, \ldots, f_{s_k}) \quad (26) \]
where the $Y_k(f_1, \ldots, f_{k+1})$ is defined by the equation

$$i Y_k(f_1, \ldots, f_{k+1}) \omega = - \sum_{i+j=k+1} (s \in S_{h(k,j)}) e(s) D_i(D_j(f_1, \ldots, f_s, f_{s+1}, \ldots, f_{k+1}))$$

**Proof.** (By induction) For $k \leq 3$ this was shown in the previous sections. For the induction step assume that all statements of the theorem are true for some $k \in \mathbb{N}$. We proof that they are true for $(k + 1)$:

First of all let’s see that the definition does not depend on the particular chosen associated semi-Hamiltonian multivector fields. This follows from proposition (2.3). Since the difference of multivector fields associated to the same Hamilton form differ only in elements of the kernel of $\omega$ we can write $i X_{D_k}(f_1, \ldots, f_k) f_{k+1} = i X_{D_k}(f_1, \ldots, f_k) + \xi f_{k+1} = i X_{D_k}(f_1, \ldots, f_k) f_{k+1}$ because each $f_i$ has the kernel property.

To see that $D_{k+1}$ is graded symmetric we use the assumed symmetry of any associated semi-Hamiltonian multivector field $X_{D_k}(f_1, \ldots, f_k)$ (up to elements of the kernel of $\omega$) and rewrite the expression $D_{k+1}(f_1, \ldots, f_{k+1})$ in terms of the symmetric group like

$$\frac{1}{k!} \sum_{s \in S_{k+1}} e(s; f_1, \ldots, f_k) e(D_k(f_1, \ldots, f_s)) i X_{D_k}(f_1, \ldots, f_s) f_{s+1},$$

which is graded symmetric.

To see that the operator is homogeneous of degree $-1$, assume that every argument is homogeneous. Then the degree $deg(i X_{D_k}(f_1, \ldots, f_k) f_{k+1}) = \sum deg(f_j) - 1$ follows from the assumption that $D_k(f_1, \ldots, f_k)$ is homogeneous of degree $-1$.

The proof of the strong homotopy Jacobi equation is a very long calculations. According to a better readable text, we put it into appendix (A).

To see that (25) is a Hamiltonian multivector field associated to $D_k(f_1, \ldots, f_k)$ just compute $i Y_{D_k}(f_1, \ldots, f_k) \omega$.

To see that (26) is a semi-Hamiltonian multivector field associated to $D_k(f_1, \ldots, f_k)$ we compute

$$i X_{D_k}(f_1, \ldots, f_k) \omega$$

$$= \sum_{s \in S_{h(k-1,1)}} e(s; f_1, \ldots, f_k) i X_{s_{k+1}} X_{D_{k+1}}(f_1, \ldots, f_{k+1}) \omega - i Y_{s_{k+1}}(f_1, \ldots, f_{k+1}) \omega$$

$$= \sum_{s \in S_{h(k-1,1)}} e(s; f_1, \ldots, f_k) i X_{D_{k+1}}(f_1, \ldots, f_{k+1}) df_{s_{k+1}} - \sum_{i+j=k+1} (s \in S_{h(k,j)}) e(s; f_1, \ldots, f_k) D_i(D_j(f_1, \ldots, f_s, f_{s+1}, \ldots, f_{k+1}))$$

$$= - \sum_{i+j=k+1} (s \in S_{h(k,j)}) e(s; f_1, \ldots, f_k) D_i(D_j(f_1, \ldots, f_s, f_{s+1}, \ldots, f_{k+1}))$$

And since the strong homotopy Jacobi equation is satisfied in dimension $k$ the last sum equals $D_1 D_k(f_1, \ldots, f_k)$ and $X_{D_k}(f_1, \ldots, f_k)$ is a solution to the first fundamental equation. \hfill \Box

Since the set of Hamiltonian forms is an $\mathbb{Z}$-graded vector space an immediate consequence is

**Corollary 2.26** (The Lie $\infty$-Algebra of Hamiltonian Forms). Let $(M, \omega)$ be an $n$-plectic manifold and $H(M)$ the $\mathbb{Z}$-graded vector space of Hamiltonian forms. The sequence $(D_k)_{k \in \mathbb{N}}$ of sh-Lie $k$-brackets defines a Lie $\infty$-algebra on $H(M)$. 
3. Conclusion and Outlook

We defined an Lie $\infty$-algebra on Hamiltonian forms on any $n$-plectic manifold $M$. However since differential forms are moreover sections of a vector bundle, the question arises, whether or not Hamiltonian forms are vector bundle sections and hence have a Lie $\infty$-algebroid structure in addition.

As seen in (2.7) the answer is not trivial, since Hamiltonian forms are at least not a $C^\infty(M)$-module, but a $C^\infty_\omega(M)$-module instead.

Appendix A. Proof of the sh-Jacobi Equation in 2.25

We proof the strong homotopy Jacobi equation in dimension $k$ under the assumption that it is satisfied in dimension $(k-1)$. According to a better readable text we start with some auxiliary calculations, necessary to keep the main part as simple as possible.

The computation is different for $k = 4$ and will be treated after the general situation. We assume $k > 4$, that $f_1, \ldots, f_k \in H(M)$ are homogeneous Hamiltonian forms with associated semi-Hamiltonian multivector fields $X_1, \ldots, X_k$ and that graded symmetric associated semi-Hamiltonian multivector fields of $D_j$ are given by (20) for any $j \in [k]$.

The first step is to rewrite the strong homotopy Jacobi equation, according to the definition of the sh-Lie $k$-bracket. Since the sh-Lie 3-bracket differs from the higher brackets by the factor $\frac{1}{2}$ only, we define $c_k := 1$ for any $k \neq 3$ and $c_3 := \frac{1}{2}$, to handle them equally. Nevertheless the sh-Lie 2-bracket is different and we consider the situation $i > 2$ first:

Equation 1.

$$\sum_{s \in \text{Sh}(j,i-1)} e(s; f_1, \ldots, f_k) D_s(D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_j+1}, \ldots, f_{s_k}) =$$

$$= c_i \sum_{s \in \text{Sh}(j,i-2,1)} e(s; f_1, \ldots, f_k) e(f_1) e(f_{s_k}) \cdot e(f_{s_k-1}) i_{X_{D_{s_k-1}(f_1, \ldots, f_{s_k-1})}} f_{s_k}$$

$$- c_i \sum_{s \in \text{Sh}(i-1,j)} e(s; f_1, \ldots, f_k) i_{X_{D_{i-1}(f_1, \ldots, f_{i-1})}} D_j(f_1, \ldots, f_{i-1})$$

for any $i, j \in \mathbb{N}$ with $i > 2$ and $i + j = k + 1$.

Proof. We can split the definition of the sh-Lie $k$-bracket into two parts. A summation over shuffles that fix the first element and a remaining term, where the first element is shuffled to the last position.

In particular let $Sh(1, i, j)$ be the set of permutations $(1, \mu_1, \ldots, \mu_i, \nu_1, \ldots, \nu_j)$, subject to the conditions $\mu_1 < \ldots < \mu_i$ and $\nu_1 < \ldots < \nu_j$. Then

$$D_i(f_1, \ldots, f_i)$$

$$= -c_i \sum_{s \in Sh(1, i, i-1)} e(s; f_1, \ldots, f_i) e(f_2) \cdots e(f_{s_i}) i_{X_{D_{s_i}(f_1, f_2, \ldots, f_{s_i-1})}} f_{s_i}$$

$$- c_i e(i, 1, \ldots, i-1; f_1, \ldots, f_i) e(f_2) \cdots e(f_i) i_{X_{D_{i-1}(f_2, \ldots, f_i)}} f_1$$

If we substitute the operator $D_j$ for the first argument of $D_i$, we can rewrite the left side of equation (1) into

$$- c_i \sum_{s \in Sh(1, j, i-1)} e(s; f_1, \ldots, f_k) \left[ \sum_{t \in Sh(1, j, i-2,1)} e(t; D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_j+1}, \ldots, f_{s_k}) \right]$$

$$\cdot e(D_j(f_{s_1}, \ldots, f_{s_j})) e(f_{t_{s_1+j})} \cdots e(f_{t_{s_k-1}})$$

$$\cdot i_{X_{D_{i-1}(f_{s_1}, \ldots, f_{s_j}), f_{t_{s_1+j}}, \ldots, f_{t_{s_k-1}}}} f_{t_{s_k}}$$
For any shuffle $s$, the expression is exactly one shuffle $i$ over all possible combinations of $s$.

Equation 3.

To simplify this, apply $e(D_j(f_s, f_{s+1}, \ldots, f_k)) = -e(f_s) \cdot e(f_{s+1}) \cdots e(f_k)$ and the bijective map

$$\text{Sh}(j, i - 1) \times \text{Sh}(i, i - 2, 1) \rightarrow \text{Sh}(j, i - 2, 1)$$

$$((\mu_1, \ldots, \mu_j, \nu_1, \ldots, \nu_{i-1}), (1, \lambda_1, \ldots, \lambda_{i-2}, \kappa)) \mapsto (id_{S_j} \times (\lambda_1, \ldots, \lambda_{i-2}, \kappa)) \circ (\mu_1, \ldots, \mu_j, \nu_1, \ldots, \nu_{i-1}),$$

on the first part, expand $e((2, \ldots, i, 1); D_j(f_s, f_{s+1}, \ldots, f_k))$ according to

$$e((2, \ldots, i, 1); D_j(f_s, f_{s+1}, \ldots, f_k)) =$$

$$e(D_j(f_s, f_{s+1}, \ldots, f_k)) \cdot e(D_j(f_{s+1}, \ldots, f_k)) \cdots e(f_k) =$$

$$e(f_s) \cdot e(f_{s+1}) \cdots e(f_{s+k-1}) \cdot e(f_k)$$

and use the bijective map

$$\text{Sh}(j, i - 1) \rightarrow \text{Sh}(i - 1, j); (\mu_1, \ldots, \mu_j, \nu_1, \ldots, \nu_{i-1}) \mapsto (\nu_1, \ldots, \nu_{i-1}, \mu_1, \ldots, \mu_j)$$

on the second part to rewrite the substituted expression into the right side of equation (1).

The case $i \leq 2$ will be treated later. First we applying the definition of $D_j$ to expend equation (1) further. Again, since $D_2$ is different, we additionally assume $j > 2$ and get

**Equation 2.**

$$\sum_{s \in \text{Sh}(j, i-1)} e(s; f_1, \ldots, f_k) D_i(D_j(f_s, f_{s+1}, \ldots, f_k)) =$$

$$c_i \sum_{s \in \text{Sh}(j, i-1)} e(s; f_1, \ldots, f_k) e(f_s) \cdot e(f_{s+1}) \cdots e(f_{s+k-1}) \cdot$$

$$\cdot X_{D_j-1}(f_{s+k-1}, f_{s+k})$$

for any $i, j > 2$ and $i + j = k + 1$.

This is obvious from equation (1) and we skip the calculation. We can exploit the symmetry of the wedge product in the previous expression when we sum over all possible combinations of $i$ and $j$. To be more precise

**Equation 3.**

$$\sum_{i+j=k+1} c_i c_j \sum_{s \in \text{Sh}(i-j-1, i-j-1)} e(s; f_1, \ldots, f_k)$$

$$\cdot e(f_s) \cdot e(f_{s+1}) \cdots e(f_{s+k-1}) \cdot X_{D_j-1}(f_{s+k-1})$$

$$= 0$$

**Proof.** For any shuffle $s := (\mu_1, \ldots, \mu_i, \nu_1, \ldots, \nu_{j-1}, \delta) \in \text{Sh}(i-1, j-1, 1)$ there is exactly one shuffle $s^* := (\nu_1, \ldots, \nu_{j-1}, \mu_1, \ldots, \mu_{i-1}, \delta) \in \text{Sh}(j-1, i-1, 1)$ and
the identity $s^{**} = s$ holds. From the graded symmetry of the wedge product we get
\[
e(s; f_1, \ldots, f_k) e(f_{i_1}) \cdots e(f_{i_{r_1}}) i_{X_{D_{j-1}}} (f_{j_1}, \ldots, f_{j_{r_1}}) f^\delta\]
\[
= e(s; f_1, \ldots, f_k) e(f_{i_1}) \cdots e(f_{i_{r_1}}) i_X (D_{i_1-1} (f_{j_1}, \ldots, f_{j_{r_1}}), D_{j_1-1} (f_{j_1}, \ldots, f_{j_{r_1}}))
\]
\[
- e(s^*; f_1, \ldots, f_k) e(f_{\mu_1}) \cdots e(f_{\mu_{r_1}}) i_X (f_{j_1}, \ldots, f_{j_{r_1}}) f^\delta\]
and consequently each terms indexed by $s$ cancel with the unique term indexed by $s^*$ in (3).
\]

Now we look at the situation where either $i = 2$ or $j = 2$. As it turns out it is advantageous to join them in a single equation. Since $k > 4$ and $i + j = k + 1$ we get

**Equation 4.**
\[
\sum_{s \in Sh(k-1, 1)} e(s; f_1, \ldots, f_k) D_2 (D_{k-1} (f_{s_1}, \ldots, f_{s_{k-1}}), f_{s_k})
+
\sum_{s \in Sh(2, k-2)} e(s; f_1, \ldots, f_k) D_{k-1} (D_2 (f_{s_1}, f_{s_2}), f_{s_3}, \ldots, f_{s_k})
=\]
\[
- \sum_{s \in Sh(k-1, 1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}
+
\sum_{s \in Sh(2, k-3)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}
\]

**Proof.** We transform the first shuffle sum according to the definition of the sh-Lie 2-bracket into
\[
\sum_{s \in Sh(k-1, 1)} e(s; f_1, \ldots, f_k) D_2 (D_{k-1} (f_{s_1}, \ldots, f_{s_{k-1}}), f_{s_k})
=\]
\[
- \sum_{s \in Sh(k-1, 1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}
+
\sum_{s \in Sh(2, k-1)} e(s; f_1, \ldots, f_k) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) D_{k-1} (f_{s_2}, \ldots, f_{s_k})
\]

use the map $Sh(k-1, 1) \to Sh(1, k-1); (\mu_1, \ldots, \mu_{k-1}, \nu) \mapsto (\nu, \mu_1, \ldots, \mu_{k-1})$ and the identity $e(D_{k-1} (f_{s_1}, \ldots, f_{s_{k-1}}), f_{s_k}) e(f_{s_k}) = e(f_{s_1}, f_{s_k}) \cdots e(f_{s_{k-1}}, f_{s_k})$ to canonicalize the expression
\[
- \sum_{s \in Sh(k-1, 1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}
+
\sum_{s \in Sh(1, k-1)} e(s; f_1, \ldots, f_k) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) D_{k-1} (f_{s_2}, \ldots, f_{s_k})
\]

Then apply the definition of $D_{k-1}$, taking $(k-1) \cdot |Sh(1, k-1)| = |Sh(1, k-2, 1)|$ into account and rewrite the second shuffle sum using equation (1) to combine both
\[
- \sum_{s \in Sh(k-1, 1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}
- \sum_{s \in Sh(1, k-2, 1)} e(s; f_1, \ldots, f_k) e(f_{s_2}) \cdots e(f_{s_{k-1}}) L_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}
+
\sum_{s \in Sh(2, k-3, 1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}
- \sum_{s \in Sh(k-2, 1, 1)} e(s; f_1, \ldots, f_k) e(f_{s_{k-1}}) i_X (f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}.$
Using \( \text{Sh}(1, k - 2, 1) \to \text{Sh}(k - 2, 1, 1) \); \( (\lambda, \mu_1, \ldots, \mu_{k-2}, \nu) \mapsto (\mu_1, \ldots, \mu_{k-2}, \lambda, \nu) \)
we reorder the first shuffle sum and join it with the last shuffle sum, applying the
second equation in (3.2) to get:
\[
\sum_{s \in \text{Sh}(k-2,1,1)} e(s; f_1, \ldots, f_k) e(f_{s_{k-2}}) \cdot \iota_{X_{s_{k-1}}}, X_{D_{k-2}}(f_{s_1}, \ldots, f_{s_{k-2}}) f_{s_k} \\
- \sum_{s \in \text{Sh}(k-1,1)} e(s; f_1, \ldots, f_k) \sum_{s_i \in \text{Sh}(k-1)} e(s_{s_{k-3}}, f_{s_{k-2}}) \cdot \iota_{X_{s_{k-2}}}(f_{s_1}, \ldots, f_{s_{k-2}}) f_{s_k} \\
+ \sum_{s \in \text{Sh}(k-3,1)} e(s; f_1, \ldots, f_k) \cdot \iota_{X_{s_{k-2}}}(f_{s_1}, \ldots, f_{s_{k-2}}) f_{s_k}.
\]

Finally, the right side of equation (2) follows from the definition of the associated
semi-Hamiltonian multivector field \( X_{D_2}(X_n, X_m) \), for appropriate \( n, m \in \mathbb{N} \).

Terms where either \( i = 1 \) or \( j = 1 \) needs no precalculation. Nevertheless we need
the sh-Jacobi equation in its semi-Hamiltonian multivector field incarnation:

**Proposition A.1.** Suppose that the strong homotopy Jacobi equation is satisfied in dimension \( (k - 1) \). Then there is a multivector field \( \xi \in \ker(\omega) \) with
\[
\sum_{i + j = k} \sum_{s \in \text{Sh}(j, i - 1)} e(s; f_1, \ldots, f_k) X_{D_1}(D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_{j+1}}, \ldots, f_{s_{k-1}}) \\
- \sum_{s \in \text{Sh}(k, 1-2)} e(s; f_1, \ldots, f_k) X_{D_{k-1}}(D_1 f_{s_1}, \ldots, f_{s_{k-1}}) + \xi
\]

**Proof.** Apply the n-plectic differential \( D_1 \) to the strong homotopy Jacobi equation
in dimension \( (k - 1) \). Since \( D_1 D_1 = 0 \) we get
\[
\sum_{i + j = k} \sum_{s \in \text{Sh}(j, i - 1)} e(s; f_1, \ldots, f_k) D_1 D_1(D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_{j+1}}, \ldots, f_{s_{k-1}}) \\
- \sum_{s \in \text{Sh}(1, k-2)} e(s; f_1, \ldots, f_k) D_1 D_{k-1}(D_1 f_{s_1}, \ldots, f_{s_{k-1}})
\]

and using the fundamental pairing this transforms into
\[
\sum_{i + j = k} \sum_{s \in \text{Sh}(j, i - 1)} e(s; f_1, \ldots, f_k) \iota_{X_{D_1}}(D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_{j+1}}, \ldots, f_{s_{k-1}}) \omega \\
- \sum_{s \in \text{Sh}(1, k-2)} e(s; f_1, \ldots, f_k) \iota_{X_{D_{k-1}}}(D_1 f_{s_1}, \ldots, f_{s_{k-1}}) \omega
\]

By proposition (2.3), the multivector fields on both sides of the equation only differ
by an element of the kernel of \( \omega \).

Now we have all we need to calculate the strong homotopy Jacobi equation
in dimension \( k \). Again, since the definition of the differential \( D_1 \) and the sh-Lie
2-bracket is different from the general situation, we separate appropriate terms
according to:
\[
\sum_{i + j = k+1} \left( \sum_{s \in \text{Sh}(j, i - 1)} e(s; f_1, \ldots, f_k) D_1(D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_{j+1}}, \ldots, f_{s_k}) \right) \\
= D_1 D_0(f_1, \ldots, f_k) + \sum_{s \in \text{Sh}(1, k-1)} e(s; f_1, \ldots, f_k) D_1 D_1(f_{s_1}, f_{s_2}, \ldots, f_{s_k}) \\
+ \sum_{s \in \text{Sh}(k, 1-2)} e(s; f_1, \ldots, f_k) D_2(D_{k-1}(f_{s_1}, \ldots, f_{s_{k-1}}), f_{s_k}) \\
+ \sum_{s \in \text{Sh}(2, k-3)} e(s; f_1, \ldots, f_k) D_{k-1}(D_1 f_{s_1}, f_{s_2}, f_{s_3}, \ldots, f_{s_k}) \\
+ \sum_{i + j = k+1} \left( \sum_{s \in \text{Sh}(j, i - 1)} e(s; f_1, \ldots, f_k) D_1(D_j(f_{s_1}, \ldots, f_{s_j}), f_{s_{j+1}}, \ldots, f_{s_k}) \right).
\]
Applying the definition of $D_1$ and $D_k$ and using the previously calculated expressions accordingly we rewrite this into

$$- \sum_{s \in S(k-1,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) D_k X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}}) f_{s_k}$$

$$+ \sum_{s \in Sh(k-2,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_{X_{D_k^{-1}}(D_1 f_{s_1}, \ldots, f_{s_{k-1}})} D_k f_{s_k}$$

$$- \sum_{s \in Sh(k-1,1)} e(s; f_1, \ldots, f_k) i_{X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}})} D_1 f_{s_k}$$

$$- \sum_{s \in Sh(k-1,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_{X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}})} f_{s_k}$$

$$+ \frac{1}{2} \sum_{s \in Sh(k-2,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_{X_{D_2^{-1}}(D_1 f_{s_1}, \ldots, f_{s_{k-2}})} D_2 f_{s_k}$$

$$+ \sum_{s \in Sh(k-2,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_{X_{D_2^{-1}}(D_1 f_{s_1}, f_{s_2}, \ldots, f_{s_{k-1}})} f_{s_k}$$

where we already used the vanishing of equation (3). In the next step we collect terms and substitute $l := i - 1$ to get

$$\sum_{s \in S(k-1,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}})$$

$$\left( d_{i_{X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}})}} f_{s_k} + e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_{X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}})} df_{s_k} \right)$$

$$+ \sum_{s \in Sh(k-2,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_{X_{D_k^{-1}}(D_1 f_{s_1}, \ldots, f_{s_{k-1}})} D_k f_{s_k}$$

$$- \sum_{s \in Sh(k-1,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_{X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}})} f_{s_k}$$

$$+ \sum_{l+j=k+1} \sum_{s \in Sh(j, i-1, 1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}})$$

$$i_{X_{D_1}(D_1 f_{s_1}, \ldots, f_{s_j})} i_{X_{D_2}(D_2 f_{s_1}, f_{s_2}, \ldots, f_{s_{k-1}})} f_{s_k}$$

Now apply the definition of the Lie derivative to the first shuffle sum and the induction assumption together with proposition (A.1) to the last term. Since each argument $f_j$ has the kernel property we get

$$\sum_{s \in S(k-1,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_{X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}})} f_{s_k}$$

$$+ \sum_{s \in Sh(k-2,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_{X_{D_k^{-1}}(D_1 f_{s_1}, \ldots, f_{s_{k-1}})} f_{s_k}$$

$$- \sum_{s \in Sh(k-1,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) L_{X_{D_k^{-1}}(f_{s_1}, \ldots, f_{s_{k-1}})} f_{s_k}$$

$$- \sum_{s \in Sh(k-2,1)} e(s; f_1, \ldots, f_k) e(f_{s_1}) \cdots e(f_{s_{k-1}}) i_{X_{D_k^{-1}}(D_1 f_{s_1}, \ldots, f_{s_{k-1}})} f_{s_k}$$

and consequently the strong homotopy Jacobi equation vanishes in dimension $k$ for $k \neq 4$.

The situation $k = 4$ is just a long but straightforward computation. To see that the expression

$$D_1 D_4 (f_1, \ldots, f_4) + \sum_{s \in Sh(1,3)} e(s; f_1, \ldots, f_4) D_k (D_1 f_{s_1}, f_{s_2}, f_{s_3}, f_{s_4})$$

$$+ \sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) D_2 (D_3 f_{s_1}, f_{s_2}, f_{s_3}, f_{s_4})$$

$$+ \sum_{s \in Sh(2,2)} e(s; f_1, \ldots, f_4) D_3 (D_2 f_{s_1}, f_{s_2}, f_{s_3}, f_{s_4})$$
Applying the explicit expressions to arrive at

\[
\begin{align*}
\sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) & \left( \delta X_{D_3} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4} \right) \\
- \sum_{s \in Sh(1,2,1)} e(s; f_1, \ldots, f_4) e(f_1) e(f_{s_2}) e(f_{s_3}) & \left( i_{X_{D_3}} (f_1, f_{s_2}, f_{s_3}) f_{s_4} \right) \\
- \sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) i_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) & \left( D_1 f_{s_4} \right) \\
- \sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) & \left( L_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4} \right) \\
+ \sum_{s \in Sh(1,3)} e(s; f_1, \ldots, f_4) L_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) & \left( D_3 f_{s_4} \right) \\
+ \frac{1}{2} \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) & \left( i_{X_{D_3}} (D_3 (f_1, f_{s_2}, f_{s_3}) f_{s_4} \right) \\
- \frac{1}{2} \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) & \left( i_{X_{D_3}} (f_{s_1}, f_{s_2}) D_2 (f_{s_3}, f_{s_4}) \right)
\end{align*}
\]

from splitting the shuffle sum into parts that fixes the first argument and a single term, where the first argument is suffixed to the last position. Then rewrite like

\[
\sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3})
\cdot \left( \delta X_{D_3} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4} + e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) i_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4} \right)
+ \sum_{s \in Sh(1,2,1)} e(s; f_1, \ldots, f_4) e(f_1) e(f_{s_2}) e(f_{s_3}) i_{X_{D_3}} (f_1, f_{s_2}, f_{s_3}) f_{s_4}
- \sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) L_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4}
- \frac{1}{2} \sum_{s \in Sh(1,2,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) L_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4}
+ \frac{1}{2} \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) i_{X_{D_3}} (D_3 (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4})
- \frac{1}{2} \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) i_{X_{D_3}} (f_{s_1}, f_{s_2}) D_2 (f_{s_3}, f_{s_4})
\]

and collect terms using (1.2) to arrive at

\[
\begin{align*}
\sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) & \left( L_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4} \right) \\
+ \sum_{s \in Sh(1,2,1)} e(s; f_1, \ldots, f_4) e(f_1) e(f_{s_2}) e(f_{s_3}) & \left( i_{X_{D_3}} (f_1, f_{s_2}, f_{s_3}) f_{s_4} \right) \\
- \sum_{s \in Sh(3,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) & \left( L_{X_{D_3}} (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4} \right) \\
\frac{1}{2} \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) & \left( -e(f_{s_1}, f_{s_2}, f_{s_3}) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) L_{X_{D_3}} i_{X_{D_3}} (f_{s_1}, f_{s_2}) f_{s_4} \right) \\
+ \frac{1}{2} \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) & \left( i_{X_{D_3}} (D_3 (f_{s_1}, f_{s_2}, f_{s_3}) f_{s_4}) \right) \\
- \frac{1}{2} \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) e(f_{s_3}) & \left( i_{X_{D_3}} (f_{s_1}, f_{s_2}) D_2 (f_{s_3}, f_{s_4}) \right)
\end{align*}
\]

Applying the explicit expressions (23) and (2.17) for the multivector fields \( X_{D_3} \) and \( X_{D_2} \), using \( Y_{D_3} f_j = X_j \), we transform this into
\[
\sum_{s \in Sh(1,2,3)} e(s; f_1, \ldots, f_3) e(f_{s_1}) e(f_{s_2}) i_{[x_{s_2}, x_{s_3}], x_{s_1}] + x_{s_2} \wedge x_{s_3} (D_{s_1} f_{s_1}, f_{s_2}) f_{s_3} \\
- 3 \sum_{s \in Sh(2,1,1)} e(s; f_1, \ldots, f_4) e(f_{s_1}) e(f_{s_2}) i_{[x_{s_2}, x_{s_3}], x_{s_1}] f_{s_4}
\]

The former sum over shuffles vanishes due to the Jacobi identity of the Schouten bracket and since the multivector field \( X_{D_2} \) vanishes if an argument is a closed form. The latter expression vanishes due to the Jacobi identity of the Schouten bracket. This completes the proof of the strong homotopy Jacobi equation.

Appendix B. Calculus of Differential Forms and Multivector Fields

B.1. Shuffle Permutation. Let \( S_k \) be the symmetric group, i.e. the group of all bijective maps of the ordinal \([k]\).

**Definition B.1** (Shuffle Permutation). For any \( p, q \in \mathbb{N} \) a \((p, q)\)-shuffle is a permutation \((\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q) \in S_{p+q}\) subject to the condition \( \mu_1 < \ldots < \mu_p \) and \( \nu_1 < \ldots < \nu_q \). We write \( Sh(p, q) \) for the set of all \((p, q)\)-shuffles.

For more on shuffles, see for example at [15].

B.2. Graded Vector Spaces. We recall the most basic facts about graded vector spaces.

A \( \mathbb{Z} \)-graded \( \mathbb{K} \)-vector space \( V \) is the direct sum \( \oplus_{n \in \mathbb{Z}} V_n \) of \( \mathbb{K} \)-vector spaces \( V_n \). The elements of \( V_n \) are said to be **homogeneous** of degree \( n \). Obviously every vector has a decomposition into homogeneous elements. When the degree of a vector \( \nu \in V \) is well defined, i.e. when the vector is homogeneous we denote it by \( \deg(v) \) (or by \(|\nu|\) if we have are dealing with more than one grading). In what follows we assume all vector space to be defined over \( \mathbb{R} \) and consequently we just write vector space instead of \( \mathbb{R} \)-vector space.

A morphism \( f : V \to W \) of graded vector spaces is a sequence of linear maps \( f_n : V_n \to W_{n+r} \) for all \( n \in \mathbb{Z} \). The integer \( r \) is called the degree of \( f \) and is as well denoted by \( \deg(f) \) (or \(|f|\)).

A \( k \)-linear morphism \( f : V_1 \times \ldots \times V_k \to W \) of graded vector spaces is a sequence of \( k \)-linear maps \( f_{n_1, \ldots, n_k} : (V_1)_{n_1} \times \ldots \times (V_k)_{n_k} \to W_{\sum n_i+r} \) for all \( n_i \in \mathbb{Z} \).

The (graded) tensor product \( V \otimes W \) of two graded \( \mathbb{K} \)-vector spaces \( V \) and \( W \) is given by

\[
(V \otimes W)_n := \oplus_{i+j=n} (V_i \otimes W_j)
\]

and the twisting morphism by \( \tau : V \otimes W \to W \otimes V \) on homogeneous elements \( v \otimes w \in V \otimes W \) by

\[
\tau(v \otimes w) := (-1)^{\deg(v) \deg(w)} w \otimes v
\]

and then extended to \( V \otimes W \) by linearity.

According to a better readable text we define \( e(v) := (-1)^{\deg(v)} \) as well as \( e(v, w) := (-1)^{\deg(v) \deg(w)} \). For any permutation \( s \in S_k \) and any homogeneous vectors \( v_1, \ldots, v_k \in V \) we define the **Koszul sign** \( e(s; v_1, \ldots, v_k) \in \{-1, +1\} \) by

\[
e(v_1 \otimes \ldots \otimes v_k) = e(s; v_1, \ldots, v_k) v_{s_1} \otimes \ldots \otimes v_{s_k},
\]

(27)

**Remark.** In an actual computation the Koszul sign can be determined by the following rules: If a permutation \( s \in S_k \) interchanges \( j \) and \( j+1 \), then \( e(s; v_1, \ldots, v_k) = (-1)^{\deg(v_j) \deg(v_{j+1})} \). If \( t \in S_k \) is another permutation, then \( e(ts; v_1, \ldots, v_k) = e(t; v_{s_1}, \ldots, v_{s_k}) e(s; v_1, \ldots, v_k) \).
A $k$-linear morphism $f : \mathbb{K}^k V \to W$ is called \textbf{graded symmetric} if

$$f(v_1, \ldots, v_k) = e(s; v_1, \ldots, v_k)f(v_{a(1)}, \ldots, v_{a(k)})$$

for all $s \in S_k$.

\textbf{B.3. Calculus on Multivector Fields.} For a comprehensive definition of multivector fields and the Schouten bracket see for example [13] or [10].

Let $M$ be a smooth manifold. A multivector field $X$ of tensor degree $r$ is a section of the $r$-th exterior power $\wedge^r TM$ of the tangent bundle. We write $\mathfrak{X}(M)$ for the set of all multivector fields.

The \textbf{Schouten bracket} is a graded antisymmetric, natural $\mathbb{R}$-bilinear operator (in the sense of [10])

$$[,] : \mathfrak{X}M \times \mathfrak{X}M \to \mathfrak{X}M,$$ (28)

homogeneous of tensor degree $-1$, that satisfies the graded Leibniz rule

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{|X||Y|} Y \wedge [X, Z],$$ (29)

as well as the graded Jacobi identity

$$\sum_{s \in S_h(2,1)} e(s; X_1, X_2, X_3)[[X_1, X_2], X_3] = 0.$$ (30)

Moreover it coincides with the standard Lie bracket on vector fields.

If $\alpha \in \Omega(M)$ is a differential form and $X \in \mathfrak{X}M$ a multivector field of tensor degree $r$, the \textbf{contraction} $i_X \alpha$ of $\alpha$ along $X$, is defined for decomposable multivector fields $X_1 \wedge \ldots \wedge X_r$ by repeated contraction

$$i_{X_1 \wedge \ldots \wedge X_r} \alpha = i_{X_r} \cdots i_{X_1} \alpha$$ (31)

and is then extended to arbitrary multivector fields $X$ by linearity. The \textbf{Lie derivative} $L_X \alpha$ along $X$ is defined in analogy to the Cartan formula for vector fields, as the graded commutator of the exterior derivative $d$ and the contraction operator $i_X$ according to:

$$L_X \alpha = d(i_X \alpha) - (-1)^r i_X d\alpha.$$ (32)

\textbf{Proposition B.2.} For multivector fields $X$ of tensor degree $r$ and $Y$ of tensor degree $s$, the equations

\begin{align*}
d L_X \alpha &= (-1)^{r-1} L_X d\alpha \\
i_{[X,Y]} \alpha &= (-1)^{(r-1)s} L_X i_Y \alpha - i_Y L_X \alpha \\
L_{[X,Y]} \alpha &= (-1)^{(r-1)(s-1)} L_X L_Y \alpha - L_Y L_X \alpha \\
L_{X \wedge Y} \alpha &= (-1)^s i_Y L_X \alpha + L_Y i_X \alpha
\end{align*} (33)

are satisfied for any $\alpha \in \Omega(M)$.

\textbf{Proof.} See for example [4]. \qed

\textbf{Definition B.3.} Let $f : M \to N$ be a smooth map. Two $r$-multivector fields $X \in \mathfrak{X}M$ and $Y \in \mathfrak{X}N$ are called \textbf{$f$-related}, if

$$\wedge^r Tf \circ X = Y \circ f.$$

If $f : M \to N$ is a diffeomorphism and $f^* Y$ is the pullback of a multivector field $Y \in \mathfrak{X}N$, then $f^* Y$ and $Y$ are $f$-related.

\textbf{Proposition B.4.} If $f : M \to N$ is a smooth map and $X \in \mathfrak{X}M$ and $Y \in \mathfrak{X}N$ are $f$-related multivector fields, then $i_X \circ f^* = f^* \circ i_Y$. 

**Definition B.5 (Kernel).** Let $\alpha$ be differential form on a manifold $M$. The kernel of $\alpha$ is the set

$$\ker(\alpha) := \{ X \in \mathfrak{X}(M) \mid i_X \alpha = 0_{C^\infty(M)} \} .$$

(34)

of all multivector fields $X$ on $M$, such that the contraction of $\alpha$ along $X$ vanishes.
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