A SHARP THRESHOLD FOR COLLAPSE OF THE RANDOM TRIANGULAR GROUP

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Abstract. The random triangular group $\Gamma(n, p)$ is the group given by a random group presentation with $n$ generators in which every relator of length three is present independently with probability $p$. We show that in the evolution of $\Gamma(n, p)$ the property of collapsing to the trivial group admits a very sharp threshold.

1. Introduction

Let $P = \langle S | R \rangle$ denote a group presentation, where $S$ is the set of generators and $R$ is the set of relators. A group generated by a presentation $P$ is called a triangular group if $R$ consists of cyclically reduced words of length three over the alphabet $S \cup S^{-1}$, that is if $R$ consists of words of the form $abc$ such that $a \neq b^{-1}$, $b \neq c^{-1}$ and $c \neq a^{-1}$. Here we consider the following binomial model $\Gamma(n, p)$ of the random triangular group. $\Gamma(n, p)$ is a group given by a random triangular group presentation with $n$ generators such that each cyclically reduced word of length three over the alphabet $S \cup S^{-1}$ is present in $R$ independently with probability $p$.

We study the asymptotic properties of random triangular groups when the number of generators $n$ goes to infinity. Thus, for a group property $P$ and a function $p(n)$, we say that $\Gamma(n, p(n))$ has $P$ asymptotically almost surely (a.a.s.), if the probability that $\Gamma(n, p(n))$ has this property tends to 1 as $n \to \infty$.

Random triangular groups were introduced by Žuk [11]. In particular, he showed that for every constant $\epsilon > 0$, if $p \leq n^{-3/2-\epsilon}$, then a.a.s. $\Gamma(n, p)$ is an infinite, hyperbolic group, while for $p \geq n^{-3/2+\epsilon}$, a.a.s. $\Gamma(n, p)$ collapses to the trivial group. Antoniuk, Łuczak and Świątkowski [2] detected a one-side sharp threshold for collapsibility of $\Gamma(n, p)$ proving that there exists a constant $C > 0$ such that for $p \geq Cn^{-3/2}$ a.a.s. $\Gamma(n, p)$ collapses to the trivial group. In this paper...
we show that the bound for the property of $\Gamma(n, p)$ being infinite and hyperbolic can be also improved.

Our first result states that the group remains infinite and hyperbolic also for $p = n^{-3/2 + o(1)}$.

**Theorem 1.** Let $p \leq n^{-3/2 - (\log n)^{-1/3} \log \log n}$. Then a.a.s. $\Gamma(n, p)$ is infinite, torsion-free, and hyperbolic.

The proof of Theorem 1 follows closely the argument of Ollivier who in [8] showed that the hyperbolicity of the random triangular group admits a threshold behaviour. This result was initially stated by Gromov [5], however it seems that Ollivier was the first one who gave a complete proof of this statement. Ollivier considered Žuk’s model of the random group in which the set of relators $R$ in the random presentation is chosen uniformly at random among all the sets of size roughly $(2n - 1)^{3d}$, where the parameter $d$, called the density of the random triangular group, controls the size of the set of relators. As we have already mentioned if $d < 1/2$, then a.a.s. the random triangular group is infinite and hyperbolic, while for $d > 1/2$ a.a.s. it is trivial. Theorem 1 tells us what happens near the critical density $d = 1/2$. In particular, it states that a.a.s. random triangular group remains hyperbolic when we are approaching the critical density. Note however that in this case $\Gamma(n, p)$ is no longer ‘uniformly hyperbolic’ and the hyperbolicity constant depends on the number of generators.

Our second result shows that in $\Gamma(n, p)$ collapsibility has a ‘sharp’ threshold. Let us remark first that this property is monotone, i.e. if a group $\langle S | R \rangle$ is trivial then for any $R' \supseteq R$ the group $\langle S | R' \rangle$ is trivial as well. Hence, by a well known argument (see Bollobás and Thomason [3]), there exists a ‘coarse’ threshold function for collapsibility i.e. there exists a function $\theta(n)$ such that if $p(n)/\theta(n) \to 0$, then a.a.s. $\Gamma(n, p)$ is non-trivial, whereas for $p(n)/\theta(n) \to \infty$ a.a.s. $\Gamma(n, p)$ collapses. We show that in fact collapsibility has a sharp threshold, namely the following holds.

**Theorem 2.** Let $h(n, p)$ denotes the probability that $\Gamma(n, p)$ is trivial. There exists a function $c(n)$ such that for any $\epsilon > 0$,

$$\lim_{n \to \infty} h(n, (1 - \epsilon)c(n)n^{-3/2}) = 0 \quad \text{and} \quad \lim_{n \to \infty} h(n, (1 + \epsilon)c(n)n^{-3/2}) = 1.$$  

Our argument is based on a result of the second author [6] which states, roughly, that if a property does not have a sharp threshold it is ‘local’, i.e. its probability can be significantly changed by a local modification of the random structure (see Lemma 5 below). We show that it is not the case with the collapsibility i.e., adding to $R$ a few more specially selected relators affects the probability of collapsing less than a tiny increase of the probability $p$ which results in an increase of $R$ by a set of random relators.
Note that the above statement says nothing about the asymptotic behaviour of the function $c(n)$. From the result from [2] we have already mentioned it follows that $\limsup_{n \to \infty} c(n)$ is finite. We conjecture that, in fact, as $n \to \infty$ the function $c(n)$ converges to a positive constant.

2. Proof of Theorem [1]

Before we show Theorem [1] we need to introduce a number of somewhat technical definitions. Let $P = \langle S|R \rangle$ be a group presentation. A van Kampen diagram with respect to the presentation $P$ is a finite planar 2-cell complex $D$ satisfying the following conditions:

- $D$ is connected and simply connected,
- each edge $e$ is oriented and assigned a generator $s \in S$,
- each 2-cell $c$ is assigned a relator $r \in R$, the number of edges on the boundary of $c$ is equal to the length of the relator $r$,
- each 2-cell $c$ has a marked vertex on its boundary and an orientation at this vertex,
- the word read from the marked vertex of $c$ in the direction given by the orientation is the relator $r \in R$ assigned to $c$.

For a van Kampen diagram $D$ the size of the diagram, denoted by $|D|$, is the number of faces (2-cells) of $D$, $\partial D$ denotes the boundary of $D$ and $|\partial D|$ denotes the size of the boundary that is the number of edges in $\partial D$.

A van Kampen diagram is said to be reduced if there is no pair of adjacent faces which are assigned the same relator $r$, which have opposite orientations and such that the common edge corresponds to the same letter in the relator with respect to the starting point. A van Kampen diagram is said to be minimal if there is no other van Kampen diagram with smaller number of faces and having the same boundary word.

Let $\Gamma$ be the group given by a presentation $P = \langle S|R \rangle$. In order to verify whether $\Gamma$ is hyperbolic it is enough to consider minimal reduced van Kampen diagrams with respect to the presentation $P$ and to show that they fulfill a certain geometric condition. In particular, it is known that a group generated by a presentation $P = \langle S|R \rangle$ is hyperbolic if and only if there exists a constant $\delta > 0$ such that every minimal reduced van Kampen diagram $D$ with respect to the presentation $P$ satisfies the linear isoperimetric inequality $|D| \leq \delta |\partial D|$ (cf. [1]).

However, verifying that every reduced van Kampen diagram satisfies a certain isoperimetric inequality may turn out fairly hard since it requires showing that this inequality holds for all such diagrams and there are infinitely many of them. At this point, the so called local to global principle for hyperbolic geometry (or Cartan-Hadamard-Gromov-Papasoglu theorem) (cf. [9]) comes to an aid. This principle
states that it is enough to verify whether the isoperimetric inequality holds for a finite, but sufficiently large, family of van Kampen diagrams.

**Theorem 3** (Cartan-Hadamard-Gromov-Papasoglu). Let \( P = \langle S | R \rangle \) be a triangular group presentation. Assume that for some integer \( K > 0 \) every minimal reduced van Kampen diagram \( D \) w.r.t. \( P \) and of size \( K^2/2 \leq |D| \leq 240K^2 \) satisfies the inequality

\[
|D| \leq \frac{K}{200} |\partial D|.
\]

Then for every minimal reduced van Kampen diagram w.r.t. \( P \) the following isoperimetric inequality is true

\[
|D| \leq K^2 |\partial D|.
\]

Following Ollivier \[8\], in order to simplify the verification of the isoperimetric condition for van Kampen diagrams, we introduce a **decorated abstract van Kampen diagram** (davKd). The davKd is defined in a similar way as the van Kampen diagram except that in the davKd no generators are attached to edges and no relators are attached to faces, and instead each face in the diagram is given a label. If \( k \) is the number of distinct labels, without loss of generality we can assume that each face is labeled by a number between 1 and \( k \). We say that a given davKd is **fulfillable** with respect to the presentation \( P = \langle S | R \rangle \) if there exists an assignment of relators to faces and generators to edges such that any two faces with the same label are assigned the same relator and such that the diagram we obtain in this way is a valid van Kampen diagram with respect to the presentation \( P = \langle S | R \rangle \).

A davKd is said to be **reduced** if it satisfies the same conditions as the reduced van Kampen diagram. It is called **minimal** if it is fulfillable and there exists an assignment of generators and relators such that in this assignment we get a minimal van Kampen diagram.

Our aim is to show that for a function \( f = f(n) = \log \log n / \log 1/3 n \) and \( p = n^{-3/2-f} \), a.a.s. all minimal reduced davKd’s with respect to the random presentation in \( \Gamma(n, p) \) satisfy the isoperimetric inequality with a constant \( \delta = \delta(n) = (200/f)^2 \), i.e. we show that the following statement holds.

**Lemma 4.** If \( p = p(n) = n^{-3/2-(\log n)^{-1/3} \log \log n} \), then a.a.s. for each minimal reduced davKd \( \mathcal{D} \) with respect to the random presentation \( \Gamma(n, p) \) we have

\[
|\mathcal{D}| \leq \left( \frac{200 \log^{1/3} n}{\log \log n} \right)^2 |\partial \mathcal{D}|.
\]

**Proof.** Let \( f = f(n) = \log \log n / \log 1/3 n \). From Theorem \[8\] it is enough to show that for each given davKd \( \mathcal{D} \) of size at most \( |\mathcal{D}| \leq 240(200/f)^2 \) one of the following two possibilities holds:

(i) \( \mathcal{D} \) satisfies the isoperimetric inequality with the constant \( 1/f \);
(ii) the probability that \( \mathcal{D} \) is fulfillable by \( \Gamma(n, p) \) is bounded by 
\( n^{-f/2} \), which in turn would imply that a.a.s. no such \( \mathcal{D} \) is fulfillable in \( \Gamma(n, p) \).

Let \( \mathcal{D} \) be a davKd with \( m = |\mathcal{D}| \) faces having \( k \) distinct labels and with \( l_1 \) internal edges and \( l_2 = |\partial \mathcal{D}| \) boundary edges. Let \( m_i \) be the number of faces labeled with \( i \). Without loss of generality we may assume that \( m_1 \geq m_2 \geq \ldots \geq m_k \). We want to count the probability that \( \mathcal{D} \) is fulfillable with respect to the random presentation given by \( \Gamma(n, p) \). If each face is assigned a different label, i.e. each cell of \( \mathcal{D} \) correspond to a different relator, this probability is bounded above by 
\( (2n - 1)^{l_1 + l_2} p^m \). This is in fact a rather easy case and showing that for all diagrams with different labels and fulfillable in \( \Gamma(n, p) \) an isometric inequality holds with a constant \( 1/f \) is rather straightforward. The main challenge is to deal with diagrams where some of the labels may appear more than once. On one hand, this reduces the number of distinct relators used to fulfill the diagram. On the other hand, this also imposes some restrictions on the generators used in this assignment. To control the influence of these two factor we follow Ollivier and introduce an auxiliary graph \( G = G(\mathcal{D}) \) which captures all the constraints resulting from the structure of the davKd.

We construct \( G \) in \( k \) steps building in each step graphs \( G_1, G_2, \ldots, G_k \), with \( G_k = G \). Let \( r_1, \ldots, r_k \) denote the relators we want to assign to the faces of \( \mathcal{D} \) labeled \( 1, \ldots, k \) respectively. The vertices of each \( G_i \) represent the elements of \( r_1, \ldots, r_i \) and the edges of \( G_i \) represent the constraints given by the davKd.

Each \( G_i \) has \( 3i \) vertices arranged in \( i \) parts of \( 3 \) vertices. Call vertices of \( G_i \) corresponding to faces labeled \( j \) the \( j \)-th part of \( G_i \). Recall that in davKd each face has a marked vertex and an orientation, which gives us an ordering of the edges belonging to this face. Label the edges 1, 2, 3 accordingly.

We begin our construction with an empty graph \( G_1 \) which has three vertices. If in the davKd we have a pair of adjacent faces each labeled with 1, then we put an edge in \( G_1 \) corresponding to the labels of this edge coming from the adjacent faces.

Suppose now that we have already constructed the graph \( G_i \). We want to build the graph \( G_{i+1} \). First, we add three new vertices to \( G_i \). Then, if in davKd a face labeled \( i + 1 \) is adjacent to a face labeled \( j \in \{1, \ldots, i+1\} \) by an edge \( e \), we put an edge in \( G_{i+1} \) between a vertex in the \( (i + 1) \)-th part and a vertex in the \( j \)-th part, each corresponding to a suitable label of the edge \( e \) coming from the two adjacent faces.

Now, the number of connected components in the graph \( G \) is the number of the degrees of freedom we have while choosing generators used to fulfill the diagram \( \mathcal{D} \). Indeed, if two vertices are adjacent in \( G \), then the corresponding edges in \( \mathcal{D} \) must bear the same generator. Therefore, if we denote the number of connected components in the
graph $G$ by $C$, then we obtain the estimate
\[
\operatorname{Pr}(D \text{ is fulfillable}) \leq n^C p^k.
\]
A similar argument works for the graphs $G_i$, which correspond to a partial assignment, namely we assign relators to faces bearing labels $1, \ldots, i$ and we assign generators to edges belonging to these faces. Let $C_i$ denote the number of connected components in $G_i$. Then
\[
\operatorname{Pr}(D \text{ is fulfillable}) \leq n^{C_i} p^i = n^{C_i - i(3/2 + \ell)},
\]
therefore putting
\[
d_i = C_i - i(3/2 + \ell)
\]
we get the estimate
\[
\operatorname{Pr}(D \text{ is fulfillable}) \leq n^{\min d_i}.
\]
Thus, if for some $i$ we have $d_i < -\ell/2$, then
\[
\operatorname{Pr}(D \text{ is fulfillable}) \leq n^{-\ell/2}.
\]
On the other hand, we claim that in the case of $\min d_i \geq -\ell/2$, the diagram $D$ satisfies the isoperimetric inequality with the constant $1/\ell$. Indeed, following Ollivier [8] we get that
\[
|\partial D| \geq 3|D|(1 - 2d) + 2 \sum_{i=1}^{k} d_i (m_i - m_{i+1}),
\]
where the parameter $d$ is the density of the random triangular group, which in our notation is equal to $1/2 - \ell/3$. Thus
\[
|\partial D| \geq 2f|D| + 2 \sum_{i=1}^{k} d_i (m_i - m_{i+1}).
\]
Next, observe that $m_i - m_{i+1} \geq 0$ for every $i$ and $\sum m_i = |D|$. Hence, if $\min d_i \geq -\ell/2$, then
\[
|\partial D| \geq 2f|D| - f \sum_{i=1}^{k} (m_i - m_{i+1}) \geq f|D|,
\]
and we get the desired isoperimetric inequality
(1) \[ |D| \leq \frac{1}{f}|\partial D|. \]

To complete our argument we use the local to global principle. In our case the constant $K$ from Theorem [3] is equal to $200/f$. We need to show that the probability that there exists a diagram of size at most $240(200/f)^2$ violating the isoperimetric inequality (1) tends to 0. If this is the case, then the random presentation in the $\Gamma(n, p)$ model a.a.s. meets the assumptions of the local to global principle, hence a.a.s. the group $\Gamma(n, p)$ is hyperbolic.
First, we need to count the number of all possible davKd’s with precisely $m$ faces. To do this we take the number of all possible triangulations of a polygon which consist of exactly $m$ triangles, and then for each triangle we choose the orientation in 2 ways, the starting point in 3 ways and the label of this face in $m$ ways.

A triangulation of a polygon with $m$ triangles has at most $m + 2$ vertices. Thus, the number of such triangulations is bounded from above by the number of distinct triangulations $t(N)$ of a 2-dimensional sphere with $N$ vertices, where $N \leq m + 3$, which in turn we bound from above by $a^m$ for some absolute constant $a > 0$ (see Tutte [10]). Hence, the total number of davKd’s with exactly $m$ faces can be bounded by $a^m \cdot 6^m \cdot m^m$. Therefore, the probability that a davKd of size at most $240(200/f)^2$ violates the isoperimetric inequality (1) is at most

$$\sum_{m \leq 240(200/f)^2} (6am)^m n^{-f/2} \leq \left(\frac{b}{f^2}\right)^{b/f^2} n^{-f/2},$$

for some constant $b$. It is easy to verify that the right hand side of this inequality tends to 0 as $n \to \infty$ provided $f = f(n) = (\log n)^{-1/3} \log \log n$. Hence, by Theorem 2 a.a.s. for every diagram $D$ with $|D| \leq 240(200/f)^2$ fulfillable in $\Gamma(n, p)$ the isoperimetric inequality holds with a constant $1/f$ and so the assertion follows from Theorem 3.

Proof of Theorem 4. Observe first that a.a.s. $\Gamma(n, p)$ is aspherical, i.e. there exists no reduced spherical van Kampen diagram with respect to the random presentation $\Gamma(n, p)$. Indeed, such a spherical reduced van Kampen diagram has zero boundary, which violates the isoperimetric inequality proved in Lemma 4. Since $\Gamma(n, p)$ is aspherical, it is torsion-free (see, for instance, Brown [11], p. 187). Consequently, a.a.s. $\Gamma(n, p)$ is an infinite, hyperbolic group.

3. Proof of Theorem 2

As mentioned in the introduction, the tool we use in order to prove the sharpness of the threshold, as expressed in Theorem 2, is a result from Friedgut [6]. In [6] the author gives a general necessary condition for a property to have a coarse threshold, namely that it can be well approximated by a local property. Although the main theorem in that paper refers to graphs, the proof extends to hypergraph-like settings where the number of isomorphism types of bounded size, is bounded. This includes random hypergraphs, random SAT Boolean formulae, and also the model of random groups that we are addressing in the current paper. A different, but very similar tool that can be used here is Bourgain’s theorem that appears in the appendix of [6], which has a weaker conclusion, but does not assume the symmetry of the property in question, such as we have in our current problem. To make things simpler we will use the ”working-mathematicians-version”
of these theorems, as described in Friedgut [7]. We present below the lemma we will use, stated in terms of the problem at hand, but first let us introduce some notation. For each value of $n$ we denote by $S$ the set $S_n$ of generators, and assume that $S_n \subset S_{n+1}$, so that any fixed relator is meaningful for all sufficiently large values of $n$. Next, let $\Gamma(n, p)$ be given by a presentation $P = \langle S | R \rangle$ where $R$ is random, and let $R^*$ be a set of relators. We use the notation 

$$h(n, p | R^*) := \Pr[\langle S | R \cup R^* \rangle \text{ is trivial}].$$

We will use this notation both for $R^* = R_{\text{fixed}} = \{r_1, \ldots, r_k\}$, a fixed set of cyclically reduced relators of length three, and for $R^* = R_{\epsilon p}$, a random set of relators chosen from $S_n$ with probability $\epsilon p$ (in which case the probability is over both the choice of $R$ and of $R^*$). The following is an adaptation of theorems 2.2, 2.3, and 2.4 from [7] to the current setting.

**Lemma 5.** Assume Theorem 2 does not hold, i.e. there exists a function $p = p(n)$, and constants $0 < \alpha, \epsilon < 1$, such that there exist infinitely many values of $n$ for which it holds that

$$\alpha < h(n, p) < h(n, (1 + \epsilon)p) < 1 - \alpha.$$

Then there exists a fixed (finite, independent of $n$) set $R_{\text{fixed}} = \{r_1, \ldots, r_k\}$ of cyclically reduced relators of length three, and a constant $\delta > 0$ such that for all such $n$

1. $h(n, p | R_{\text{fixed}}) > h(n, p) + 2\delta$
2. $h(n, p | R_{\epsilon p}) < h(n, p) + \delta$

We will now see how this lemma, together with the information given by Theorem 1 regarding the location of the threshold, implies Theorem 2.

**Proof of Theorem 2.** Assume, by way of contradiction, that as in the hypothesis of the Lemma 5 Theorem 2 does not hold. Let $R_{\text{fixed}}$ be the guaranteed set of relators, and let $Z := \{z_1, z_2, \ldots, z_\ell\}$ be the set of all generators involved in $R_{\text{fixed}}$ and all of their inverses. Let $R_{\text{strong}}$ be the following relation: $z_1 = z_2 = \ldots = z_\ell = e$. Clearly

$$h(n, p | R_{\text{strong}}) \geq h(n, p | R_{\text{fixed}}).$$

Next, consider the graph $G = (V, E)$ defined as follows. $V = (S \cup S^{-1}) \setminus Z$, and $E$ consists of all pairs $xy$ such that there is a relator in $R$ which involves $x, y$ and an element of $Z$ (implying $x = y^{-1}$, since all elements in $Z$ are set by $R_{\text{strong}}$ to be equal to $e$). The probability that a given pair $xy$ forms an edge is less than $6\ell p$, and these events are independent, so $G = G(2n - \ell, q)$ with $q = O(n^{-3/2})$. Elementary first moment estimates imply that a.a.s. $G$ has fewer than $n^{0.6}$ non-trivial components each of them consisting of at most two edges.
Let $R'$ be the set of relators in $R$ that are disjoint from $Z$. (Note that a.a.s. there are no relators in $R$ containing two elements from $Z$.) Slightly abusing the notation we will also use $E$ to denote the set of relations $\{x = y^{-1} : \{x, y\} \in E\}$. We have

\[
(2) \quad h(n, p|R_{\text{strong}}) = \Pr[\langle S \setminus Z | R' \cup E \rangle \text{ is trivial}] + o(n^{-0.4}),
\]

where the $o(n^{-0.4})$ accounts for the case where there exists in $R$ a relator involving two elements of $Z$. Note that both $E$ and $R'$ are random.

Now let us consider the effect of $R_{\text{np}}$. First let us choose arbitrarily a set $M$, $|M| = m = \lfloor n^{1.9} \rfloor$, of pairs of generators $\{a, b\}$, $a, b \in S \cup S^{-1}$. Define a graph $G' = (V', E')$ with $V' = (S \cup S^{-1}) \setminus Z$ and $E'$ consisting of all pairs $xy$ such that $R_{\text{np}}$ involves two elements of the form $abx$ and $aby^{-1}$, where $\{a, b\} \in M$. Note that the existence of such two relators clearly implies that $x = y^{-1}$. Let $X$ denote the number of paths of length two in $G' = (V', E')$. It is easy to see that for the expectation of $X$ we have

\[
\mathbb{E}X \geq 0.5n^3n^2\mathbb{E}(\epsilon p)^4 = 0.5n^3n^{3.8}n^{4(-3/2+o(1))} = 0.5n^{0.8-o(1)} \geq 4n^{0.75}.
\]

It is also easy to check that the standard deviation of $X$ is also of the order $O(n^3m^2p^4)$, so from Chebyshev’s inequality we infer that a.a.s. the number of such paths is larger than $3n^{0.75}$. On the other hand, let $Y$ be the number of pairs of paths which share at least one vertex. The expectation of $Y$ is dominated by the number of pairs which share one edge and is bounded from above by

\[
\mathbb{E}Y \leq n^4m^3\mathbb{E}(\epsilon p)^6 = n^4n^{5.7}n^{6(-3/2+o(1))} = n^{0.7+o(1)}.
\]

Thus, from Markov’s inequality, a.a.s. the number of such pairs is smaller than $n^{0.75}$. Consequently, a.a.s. $G' = (V', E')$ contains at least $n^{0.75}$ disjoint paths of length two.

Now

\[
(3) \quad h(n, p|R_{\text{np}}) = \Pr[\langle S \setminus Z | R' \cup E' \rangle \text{ is trivial}] - o(n^{-0.4}),
\]

where the term $o(n^{-0.4})$ accounts for the fact that even if the group generated by $S \setminus Z$ collapses there are the generators in $Z$ to account for. However, if all generators in $S \setminus Z$ are set to be equal to the identity it suffices that for each element $z \in Z$ there will be in $R$ a relator involving $z$ and two elements of $S \setminus Z$. The probability of this event is at least as large as $1 - o(n^{-0.4})$ as $n$ tends to infinity.

Since we have shown that $G'$ contains at least $n^{0.1}$ disjoint subgraphs isomorphic to the graph spanned by the edges of $G$, it is clear that the equations (2) and (3) contradict the items 1 and 2 in the conclusion of Lemma 5. Hence the hypothesis of the lemma cannot hold, and the property in question must have a sharp threshold. \qed
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