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VALUATION DOMAINS WITH A MAXIMAL IMMEDIATE EXTENSION OF FINITE RANK.

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Abstract. If $R$ is a valuation domain of maximal ideal $P$ with a maximal immediate extension of finite rank it is proven that there exists a finite sequence of prime ideals $P = L_0 \supset L_1 \supset \cdots \supset L_m \supset 0$ such that $R_{L_i}/L_{i+1}$ is almost maximal for each $j$, $0 \leq j \leq m - 1$ and $R_{L_m}$ is maximal if $L_m \neq 0$. Then we suppose that there is an integer $n \geq 1$ such that each torsion-free $R$-module of finite rank is a direct sum of modules of rank at most $n$. By adapting Lady’s methods, it is shown that $n \leq 3$ if $R$ is almost maximal, and the converse holds if $R$ has a maximal immediate extension of rank $\leq 2$.

Let $R$ be a valuation domain of maximal ideal $P$, $\hat{R}$ a maximal immediate extension of $R$, $\hat{R}$ the completion of $R$ in the $R$-topology, and $Q$, $\hat{Q}$, $\tilde{Q}$ their respective fields of quotients. If $L$ is a prime ideal of $R$, as in [5], we define the total defect at $L$, $d_R(L)$, the completion defect at $L$, $c_R(L)$, as the rank of the torsion-free $R/L$-module $\hat{R}$(R/L) and the rank of the torsion-free $R/L$-module $(\hat{R}/L)$, respectively. Recall that a local ring $R$ is Henselian if each indecomposable module-finite $R$-algebra is local and a valuation domain is strongly discrete if it has no non-zero idempotent prime ideal. The aim of this paper is to study valuation domains $R$ for which $d_R(0) < \infty$. The first example of such a valuation domain was given by Nagata [11]; it is a Henselian rank-one discrete valuation domain of characteristic $p > 0$ for which $d_R(0) = p$. By using a generalization of Nagata’s idea, Facchini and Zanardo gave other examples of characteristic $p > 0$, which are Henselian and strongly discrete. More precisely:

Example 0.1. [5, Example 6] For each prime integer $p$ and for each finite sequence of integers $\ell(0) = 1$, $\ell(1), \ldots, \ell(m)$ there exists a strongly discrete valuation domain $R$ with prime ideals $P = L_0 \supset L_1 \supset \cdots \supset L_m = 0$ such that $c_R(L_i) = p^{\ell(i)}$, $\forall i$, $1 \leq i \leq m$.

So, $d_R(0) = p^{\sum_{i=0}^{\infty} \ell(i)}$ by [5, Corollary 4].

Theorem 0.2. [5, Theorem 8] Let $\alpha$ be an ordinal number, $\ell : \alpha + 1 \to \mathbb{N} \cup \{\infty\}$ a mapping with $\ell(0) = 1$ and $p$ a prime integer. Then there exists a strongly discrete valuation domain $R$ and an antiisomorphism $\alpha + 1 \to \text{Spec}(R)$, $\lambda \mapsto L_\lambda$, such that $c_R(L_\lambda) = p^{\ell(\lambda)}$, $\forall \lambda \leq \alpha$.

So, if $\ell(\lambda) = 0$, $\forall \lambda \leq \alpha$, except for a finite subset, then $d_R(0) < \infty$ by [5, Corollary 4].

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In 1990, Vámos gave a complete characterization of non-Henselian valuation domains with a finite total defect, and examples of such rings. His results are summarized in the following:

**Theorem 0.3.** [12, Theorem 5] Let $R$ be a non-Henselian valuation domain and assume that $d_R(0) < \infty$. Then one of the following holds:

1. $d_R(0) = 2$. $R$ has characteristic zero, $\hat{Q}$ is algebraically closed and its cardinality $|\hat{Q}|^\omega = |\hat{Q}|$. Further, $Q$ is real-closed, the valuation on $Q$ has exactly two extensions to $\hat{Q}$ and $R$ is almost maximal.
2. There is a non-zero prime ideal $L$ of $R$ such that $R_L$ is a maximal valuation ring, and $R/P$ and its field of quotients satisfy $(rc)$.

As Vámos, if $R$ is a domain, we say that $fr(R) \leq n$ (respectively $fr^n(R) \leq n$) if every torsion-free module (respectively every submodule of a free module) of finite rank is a direct sum of modules of rank at most $n$. By [12, Theorem 3] $fr(R) \geq d_R(0)$ if $R$ is a valuation domain. So, the study of valuation domains $R$ for which $d_R(0) < \infty$ is motivated by the problem of the characterization of valuation domains $R$ for which $fr(R) < \infty$.

When $R$ is a valuation domain which is a $Q$-algebra or not Henselian, then $fr(R) < \infty$ if and only if $fr(R) = d_R(0) \leq 2$ by [12, Theorem 10]. Moreover, if $fr(R) = 2$, either $R$ is of type $(rc)$ and $fr^2(R) = 1$ or $R$ is of type $(y)$ and $fr^2(R) = 2$. When $R$ is a rank-one discrete valuation domain, then $fr(R) < \infty$ if and only if $fr(R) = d_R(0) \leq 3$ by [13, Theorem 8] and [1, Theorem 2.6].

In this paper we complete Vámos’s results. In Section 1, a description of valuation domains with a finite total defect is given by Theorem 1.7 and Proposition 1.8. In Section 2 we give some precisions on the structure of torsion-free $R$-modules of finite rank when $R$ satisfies a condition weaker than $d_R(0) < \infty$. In Section 3 we extend to every almost maximal valuation domain the methods used by Lady in [8] to study torsion-free modules over rank-one discrete valuation domains. If $R$ is an almost maximal valuation domain, we prove that $d_R(0) \leq 3$ if $fr(R) < \infty$ and that $fr(R) = d_R(0)$ if $d_R(0) \leq 2$.

For definitions and general facts about valuation rings and their modules we refer to the books by Fuchs and Salce [3] and [5].

1. **Maximal Immediate Extension of Finite Rank**

We recall some preliminary results needed to prove Theorem 1.7 which gives a description of valuation domains with a finite total defect.

Let $M$ be a non-zero module over a valuation domain $R$. As in [7, p.338] we set $M^2 = \{s \in R \mid sM \subseteq M\}$. Then $M^2$ is a prime ideal of $R$ and is called the **top prime ideal** associated with $M$.

**Proposition 1.1.** Let $A$ be a proper ideal of $R$ and let $L$ be a prime ideal such that $A^2 \subseteq L$ and $A$ is not isomorphic to $L$. Then $R/A$ is complete in its ideal topology if and only if $R_L/A$ is also complete in its ideal topology.

**Proof.** Let $(a_i + A_i)_{i \in I}$ be a family of cosets of $R_L$ such that $a_i \in A_j + A_j$ if $A_i \subseteq A_j$ and such that $A = \cap_{i \in I} A_i$. We may assume that $A_i \subseteq L$, $\forall i \in I$. So, $a_i + L = a_j + L$, $\forall i, j \in I$. Let $b \in a_i + L$, $\forall i \in I$. It follows that $a_i - b \in L$, $\forall i \in I$. If $R/A$ is complete in the $R/A$-topology, $\exists c \in R$ such that $c + b - a_i \in A_i$, $\forall i \in I$. Hence $R_L/A$ is complete in the $R_L/A$-topology too.
Conversely let \((a_i + A_j)_{i \in I}\) be a family of cosets of \(R\) such that \(a_i \in a_j + A_j\) if \(A_i \subseteq A_j\) and such that \(A = \cap_{i \in I} A_i\). We may assume that \(A \subseteq A_i \subseteq L\), \(\forall \ i \in I\). We put \(A'_i = (A_i)/L\). We know that \(A = \cap_{a \in A} La\). Consequently, if \(a \notin A\), there exists \(i \in I\) such that \(A_i \subseteq L \setminus a\), whence \(A' \subseteq L\). It follows that \(A = \cap_{i \in I} A'_i\). Similarly, \(a \in a_i + A'_i\) if \(A'_i \subseteq A'_j\). Then there exists \(c \in R/L\) such that \(c \in a_i + A'_i\), \(\forall i \in I\). Since \(A'_i \subseteq R\), \(\forall i \in I\), \(c \in R\). From \(A = \cap_{i \in I} A'_i\) and \(A \subseteq A_i\), \(\forall i \in I\) we deduce that \(\forall i \in I\), \(\exists j \in I\) such that \(A'_j \subseteq A_i\). We get that \(c \in a_i + A_i\) because \(c - a_j \in A'_j \subseteq A_i\) and \(a_j - a_i \in A_i\). So, \(R/A\) is complete in the \(R/A\)-topology.

**Proposition 1.2.** Exercise II.6.4] Let \(R\) be a valuation ring and let \(L\) be a non-zero prime ideal. Then \(R\) is (almost) maximal if and only if \(R/L\) is maximal and \(R_L\) is (almost) maximal.

**Proof.** If \(R\) is (almost) maximal, it is obvious that \(R/L\) is maximal and by Proposition 1.1 \(R_L\) is (almost) maximal. Conversely let \(A\) be a non-zero ideal and \(J = A^\perp R\). Suppose that either \(J \subseteq L\) or \(J = L\) and \(A\) is not isomorphic to \(L\). Since \(R_L\) is (almost) maximal it follows that \(R_L/A\) is complete in its ideal topology. From Proposition 1.3 we deduce that \(R/A\) is complete in its ideal topology. Now, suppose that \(L \subseteq J\). If \(A \subseteq L\) let \(i \in J \setminus L\). Thus \(A \subseteq t^{-1}A\). Let \(s \in t^{-1}A\), \(A\). Therefore \(L \subseteq \{d\} \subseteq s^{-1}A\). So, \(R/s^{-1}A\) is complete in its ideal topology because \(R/L\) is maximal, whence \(R/A\) is complete too. Finally if \(A \cong L\) the result is obvious.

**Proposition 1.3.** Let \((L_\lambda)_{\lambda \in \Lambda}\) be a non-empty family of prime ideals of \(R\) and let \(L = \cup \lambda \in \Lambda L_\lambda\). Then \(L\) is prime, \(R_L = \cap_{\lambda \in \Lambda} R_{L_\lambda}\) and \(R_L\) is maximal if and only if \(R_{L_\lambda}\) is maximal \(\forall \lambda \in \Lambda\).

**Proof.** It is obvious that \(L\) is prime. Let \(Q\) be the field of fractions of \(R\). If \(x \in Q \setminus R_L\) then \(x = \frac{1}{s}\) where \(s \in L\). Since \(L = \cup_{\lambda \in \Lambda} L_\lambda\), \(\exists \mu \in \Lambda\) such that \(s \in L_\mu\). We deduce that \(x \notin R_{L_\mu}\) and \(R_L = \cap_{\lambda \in \Lambda} R_{L_\lambda}\).

If \(R_L\) is maximal, we deduce that \(R_{L_\lambda}\) is maximal \(\forall \lambda \in \Lambda\) by Proposition 1.1. Conversely, by Proposition 4 \(R_L\) is linearly compact in the inverse limit topology. Since \(R_L\) is Hausdorff in this linear topology then every nonzero ideal is open and also closed. Hence \(R_L\) is linearly compact in the discrete topology.

Recall that a valuation domain \(R\) is Archimedean if its maximal \(P\) is the only non-zero prime ideal and an ideal \(A\) is Archimedean if \(A^\perp = P\).

**Proposition 1.4.** Corollary 9] Let \(R\) be an Archimedean valuation domain. If \(d_R(0) < \infty\), then \(R\) is almost maximal.

From Propositions 1.1, 1.2, 1.3 and 1.4 we deduce the following:

**Proposition 1.5.** Let \(R\) be a valuation domain such that \(d_R(0) < \infty\) and \(R/A\) is Hausdorff and complete in its ideal topology for each non-zero non-Archimedean ideal \(A\). Then \(R\) is almost maximal.

**Proof.** Let \(L, L'\) be prime ideals such that \(L' \subseteq L\). Since \((R_L)\) is a summand of \((R)\) we have \(d_{R_L}(0) \leq d_R(0)\). On the other hand, by tensoring a pure-composition series of \((R_L)\) with \(R_L/L'\) we get a pure-composition series of \((R_L/L')\). So, \(d_{R_L}(L') \leq d_R(0)\).
If \( R \) is Archimedean the result follows from Proposition 4. Suppose that \( R \) is not Archimedean, let \( J \) be a non-zero ideal and let \((L_\Lambda)_{\Lambda \in \Lambda}\) be the family of prime ideals properly containing \( J \) and properly contained in \( P \). If \( \Lambda = \emptyset \) we get that \( R \) is almost maximal by applying Propositions 1.4 and 1.2. Else, let \( L' = \bigcup_{\Lambda \in \Lambda} L_\Lambda \). By Proposition 1.4, \( R_{L'}/J \) is maximal for each non-zero prime \( J \). If \( L' \neq P \) then \( R/L' \) is maximal by Proposition 1.4 and it follows that \( R/J \) is maximal by Proposition 1.2. If the intersection \( K \) of all non-zero primes is zero then \( R \) is almost maximal. If \( K \neq 0 \) then \( R_K \) is Archimedean. We conclude by using Propositions 1.4 and 1.2.

Given a ring \( R \), an \( R \)-module \( M \) and \( x \in M \), the content ideal \( c(x) \) of \( x \) in \( M \), is the intersection of all ideals \( A \) for which \( x \in AM \). We say that \( M \) is a content module if \( x \in c(x)M \), \( \forall x \in M \).

**Lemma 1.6.** Let \( U \) be a torsion-free module such that \( U = PU \). Then:

1. \( \forall x \in U, \; x \neq 0, \; x \notin c(x)U \);
2. let \( 0 \neq x, y \in U \) and \( t \in R \) such that \( x = ty \). Then \( c(y) = t^{-1}c(x) \);
3. if \( U \) is uniserial then, for each \( x \in U, \; x \neq 0, \; c(x)^{\sharp} = U^{\sharp} \).

**Proof.** (1). If \( x \in c(x)U \), there exist \( a \in R \) and \( z \in U \) such that \( x = az \) and \( c(x) = Ra \). But, since \( z \in PU \), we get a contradiction.

(2). Let \( 0 \neq x, y \in U \) such that \( x = ty \). If \( s \notin c(y) \) then \( x = tsz \) for some \( z \in U \) and \( st \notin c(x) \). So, \( st \notin t^{-1}c(x) \). Conversely, if \( s \notin t^{-1}c(x) \) then \( st \notin c(x) \). We have \( x = stz \) for some \( z \in U \). We get that \( y = sz \). So, \( s \notin c(y) \).

(3). We put \( A = c(x) \) and \( L = A^\sharp \). Let \( s \notin L \) and \( y \in U \) such that \( x = ty \) for some \( t \in R \). Then \( c(y) = t^{-1}A \) and \( t \notin A \). So, \( t^{-1}A \subseteq L \). Consequently \( y \in sU \). Let \( s \in L \). If \( s \notin A \) then \( x \notin sU \). If \( s \in L \setminus A \) let \( t \in s^{-1}A \setminus A \). There exists \( y \in U \) such that \( x = ty \). Since \( c(y) = t^{-1}A \) and \( s \in t^{-1}A \) we deduce that \( y \notin sU \).

This lemma and the previous proposition allow us to show the following theorem.

**Theorem 1.7.** Let \( R \) be a valuation domain such that \( d_R(0) < \infty \). Then there exists a finite family of prime ideals \( P = L_0 \supset L_1 \supset \cdots \supset L_{m-1} \supset L_m \supset 0 \) such that \( R_{L_k}/L_{k+1} \) is almost maximal, \( \forall k, \; 0 \leq k \leq m-1 \) and \( R_{L_m} \) is maximal if \( L_m \neq 0 \) (or equivalently, for each proper ideal \( A \nsubseteq L_k \), \( \forall k, \; 0 \leq k \leq m, \; R/A \) is Hausdorff and complete in its ideal topology). Moreover, \( d_R(0) = \prod_{k=1}^m c_R(L_k) \).

**Proof.** Let \( n = d_R(0) \). Then \( \widehat{R} \) has a pure-composition series

\[
0 = G_0 \subset R = G_1 \subset \cdots \subset G_{n-1} \subset G_n = \widehat{R}
\]

such that, \( \forall k, \; 1 \leq k \leq n, \; U_k = G_k/G_{k-1} \) is a uniserial torsion-free module. The family \( (L_0, \ldots, L_m) \) is defined in the following way: \( \forall j, \; 0 \leq j \leq m, \) there exists \( k, \; 1 \leq k \leq n \) such that \( L_j = U_k^\sharp \).

Now, let \( A \) be a proper ideal such that \( R/A \) is Hausdorff and non-complete in its ideal topology. By Lemma V.6.1] there exists \( x \in \widehat{R} \setminus R \) such that \( A = c(x + R) \) (Clearly \( c(x + R) = B(x) \), the breadth ideal of \( x \)). Let \( U \) be a pure uniserial submodule of \( \widehat{R}/R \) containing \( x + R \) and let \( M \) be the inverse image of \( U \) by the natural map \( \widehat{R} \to \widehat{R}/R \). From the pure-composition series of \( M \) with factors \( R \) and \( U \), and a pure-composition series of \( \widehat{R}/M \) we get a pure-composition series...
of \( \tilde{R} \). Since each pure composition series has isomorphic uniserial factors by \( \tilde{R} \) Theorem XV.1.7], it follows that \( U \cong U_k \) for some \( k, \ 2 \leq k \leq n \). So, by Lemma \[ 4 \]
\[ A^j = U^j = U^j_k \]

We apply Proposition \[ 1,3 \] and deduce that \( R_{L_k}/L_{k+1} \) is almost maximal \( \forall k, \ 0 \leq k \leq m - 1 \) and \( R_{L_m} \) is maximal if \( L_m \neq 0 \).

To prove the last assertion we apply \[ 3 \] Lemma 2] (The conclusion of this lemma holds if \( R_L/L' \) is almost maximal, where \( L \) and \( L' \) are prime ideals, \( L' \subset L \)). \( \square \)

The following completes the previous theorem.

**Proposition 1.8.** Let \( R \) be a valuation domain such that \( d_R(0) < \infty \), let \( (U_k)_{1 \leq k \leq n} \) be the family of uniserial factors of all pure-composition series of \( \tilde{R} \) and let \( (L_j)_{0 \leq j \leq n} \) be the family of prime ideals defined in Theorem \[ 1,3 \]. Then:

1. \( \forall k, \ 1 \leq k \leq n, \ U_k \cong R_{U_k} \)
2. \( \tilde{R} \) has a pure-composition series

\[ 0 = F_0 \subset R = F_1 \subset \cdots \subset F_{m-1} \subset F_m = \tilde{R} \]

where \( F_j+1/F_j \) is a free \( R_{L_j} \)-module of finite rank, \( \forall j, \ 0 \leq j \leq m - 1 \).

**Proof.** (1). Let \( A \) be an ideal such that \( \exists j, \ 0 \leq j \leq m, \ A^j = L_j \) and \( A \not\cong L_j \). In the sequel we put \( L_{m+1} = 0 \) if \( L_m \neq 0 \).

First, for each uniserial torsion-free module \( U \), we will show that each family \( (x_r + rU)_{r \in R \setminus A} \) has a non-empty intersection if \( x_r \in x_r + tU, \forall r, t \in R \setminus A, \ r \in tR \).

As in the proof of Proposition \[ 1,2 \] we may assume that \( L_{j+1} \subset A \). Since \( R_{L_j}/L_{j+1} \) is almost maximal and \( A \) is an ideal of \( R_{L_j} \) the family \( (x_r + rU_{L_j})_{r \in R \setminus A} \) has a non-empty intersection. If \( r \in L_j \setminus A \), we have \( r^{-1}A \subset L_j \). So, if \( t \in L_j \setminus r^{-1}A \) then \( rt \not\in A \) and \( rtU_{L_j} \subseteq rU \). It follows that we can do as in the proof of Proposition \[ 1,3 \] to show that the family \( (x_r + rU)_{r \in R \setminus A} \) has a non-empty intersection.

Let \( 0 = G_0 \subset R = G_1 \subset \cdots \subset G_{n-1} \subset G_n = \tilde{R} \) be a pure-composition series of \( \tilde{R} \) whose factors are the \( U_k \), \( 1 \leq k \leq n \). By induction on \( k \) and by using the pure-exact sequence \( 0 \to G_{k-1} \to G_k \to U_k \to 0 \), we get that each family \( (x_r + rG_k)_{r \in R \setminus A} \) for which \( x_r \in x_r + tG_k, \forall r, t \in R \setminus A, \ r \in tR \), has a non-empty intersection.

Let \( k, \ 2 \leq k \leq n, \) be an integer, let \( 0 \neq x \in G_k \setminus G_{k-1} \) and let \( A = c(x + G_{k-1}) \). Then \( A^j = U^j_k = L_j \) for some \( j, \ 1 \leq j \leq m \). We shall prove that \( A \cong L_j \). For each \( r \in R \setminus A, \ x = gr + r y_r \) for some \( y_r \in G_{k-1} \) and \( y_r \in G_k \). Let \( r, t \in R \setminus A \) such that \( r \in tR \). Then we get that \( y_r \in y_r + tG_{k-1} \cap G_{k-1} = y_r + tG_{k-1} \) since \( G_{k-1} \) is a pure submodule. If \( A \not\cong L_j \) the family \( (y_r + rG_{k-1})_{r \in R \setminus A} \) has a non-empty intersection. Let \( y_r \in y_r + rG_{k-1}, \forall r \in R \setminus A \). Then \( (x - y) \in rG_k, \forall r \in R \setminus A \). Since \( G_k \) is a pure-essential extension of a free module, \( G_k \) is a content module by \[ 4 \] Proposition 23]. It follows that \( (x - y) \in AG_k \). So \( x + G_{k-1} \in AU_k \). But, since \( k \geq 2, \ U_k = PU_k \) because \( \tilde{R}/PR \cong R/PR \). So, \( x + G_{k-1} \not\subseteq AU_k \). From this contradiction we get that \( A = L_j \) for some \( 0 \neq s \in R \). If \( sL_j \neq L_j \) then \( x + G_{k-1} = sy + G_{k-1} \) for some \( y \in G_k \) because \( s \notin A \). If follows that \( c(y + G_{k-1}) = L_j \). We put \( y' = y + G_{k-1} \). Then, for each \( z \in U_k \setminus R_{L_j} \) there exists \( t \in R \setminus L_j \) such that \( y' = tz \). We get that \( U_k = L_{L_j}y' \).

2. Let \( M = \tilde{R}/R \). Then \( L_1 = M^2 \). From above we get that \( M/L_1M \neq 0 \). By \[ 4 \] Proposition 21] \( M \) contains a pure free \( R_{L_1} \)-submodule \( N \) such that \( N/L_1N \cong \tilde{R} \). \( \square \)
$M/L_1M$. It follows that $(M/N)^1 = L_2$. We set $E_2$ the inverse image of $N$ by the natural map $\bar{R} \to M$. We complete the proof by induction on $j$. \hfill \square

2. Torsion-free modules of finite rank.

In this section we give some precisions on the structure of torsion-free $R$-modules of finite rank when $R$ satisfies a condition weaker than $d_R(0) < \infty$. The following lemmas are needed.

**Lemma 2.1.** Let $R$ be a valuation ring (possibly with zerodivisors), let $U$ be a uniserial module and let $L$ be a prime ideal such that $L \subset U\!^2$. Then $U_L$ is a cyclic $R_L$-module.

**Proof.** Let $s \in U\!^2 \setminus L$ and let $x \in U \setminus sU$. Let $y \in U \setminus Rx$. There exists $t \in P$ such that $x = ty$. Then $t \notin Rs$, whence $t \notin L$. It follows that $U_L = R_Lx$. \hfill \square

**Lemma 2.2.** Let $R$ be a valuation ring (possibly with zerodivisors), let $U$ and $V$ be uniserial modules such that $V^2 \subset U^2$. Assume that $U_L$ is a faithful $R_L$-module, where $L = V^2$. Then $\text{Ext}^1_R(U, V) = 0$.

**Proof.** Let $M$ be an extension of $V$ by $U$. By lemma 2.1 $U_L$ is a free cyclic $R_L$-module. Since $V$ is a module over $R_L$, it follows that $V$ is a summand of $M_L$. We deduce that $V$ is a summand of $M$ too. \hfill \square

**Lemma 2.3.** Let $R$ be a valuation domain for which there exists a prime ideal $L \neq P$ such that $R/L$ is almost maximal. Then $\text{Ext}^1_R(U, V) = 0$ for each pair of ideals $U$ and $V$ such that $L \subset U\!^2 \cap V\!^2$.

**Proof.** Let $M$ be an extension of $V$ by $U$. It is easy to check that $U/LU$ and $V/LV$ are non-zero and non divisible $R/L$-modules. Since $R/L$ is almost maximal $M/LM \cong U/LU \oplus V/LV$ by \[3\] Proposition VI.5.4. If $L \neq 0$, it follows that there exist two submodules $H_1$ and $H_2$ of $M$, containing $LM$, such that $H_1/LM \cong U/LU$ and $H_2/LM \cong V/LV$. For $i = 1, 2$ let $x_i \in H_i \setminus LM$ and let $A_i$ be the submodule of $H_i$ such that $A_i/Rx_i$ is the torsion submodule of $H_i/Rx_i$. Then $A_i + LM/LM$ is a non-zero pure submodule of $H_i/LM$ which is of rank one over $R/L$. It follows that $H_i = A_i + LM$. By Lemma 2.1 $LM \cong V_L \oplus U_L$. We deduce that $LM \cong LM_L$ is a direct sum of uniserial modules. Since $A_i \cap LM$ is a non-zero pure submodule of $LM$ there exists a submodule $C_i$ of $LM$ such that $LM = (A_i \cap LM) \oplus C_i$ by \[6\] Theorem XII.2.2. It is easy to check that $H_i = A_i \oplus C_i$. From $M = H_1 + H_2$ and $LM = H_1 \cap H_2$ we deduce that the following sequence is pure exact:

$$0 \to LM \to H_1 \oplus H_2 \to M \to 0,$$

where the homomorphism from $LM$ is given by $x \mapsto (x, -x)$, $x \in LM$, and the one onto $M$ by $(x, y) \mapsto x + y$ if $x \in H_1$, $y \in H_2$. Since $H_1 \oplus H_2$ is a direct sum of uniserial modules, so is $M$ by \[6\] Theorem XII.2.2. Consequently $M \cong V \oplus U$. \hfill \square

**Proposition 2.4.** Let $R$ be a valuation domain. Let $G$ be a torsion-free $R$-module of finite rank. Then $G$ has a pure-composition series with uniserial factors $(U_k)_{1 \leq k \leq n}$ such that $U_k^2 \subset U_{k+1}^2$, $\forall k$, $1 \leq k \leq n - 1$.

**Proof.** $G$ has a pure-composition series

$$0 = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$$
Proposition 2.5. Let $\square$ be a valuation domain. Assume that there exists a finite family of prime ideals $P = L_0 \supset L_1 \supset \cdots \supset L_m = 0$ such that $R_{L_j}/L_{j+1}$ is almost maximal $\forall k$, $0 \leq k \leq m - 1$. Let $G$ be a torsion-free $R$-module of finite rank. Then $G$ has a pure-composition series

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m \subseteq G_{m+1} = G$$

where $G_{j+1}/G_j$ is a finite direct sum of ideals of $R_{L_j}$, $\forall j$, $0 \leq j \leq m$.

Proof. By Proposition 2.4 $G$ has a pure-composition series

$$0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that, $\forall k$, $1 \leq k \leq n$, $U_k = H_k/H_{k-1}$ is a uniserial torsion-free module and $U_k^r \supseteq U_{k+1}^r$, $\forall k$, $1 \leq k \leq n - 1$. Now, for each $j$, $1 \leq j \leq m$, let $k_j$ be the greatest index such that $L_j \subseteq U_{k_j}^r$. We put $G_j = H_{k_j}$. Then $G_{j+1}/G_j$ is an $R_{L_j}$-module which is a direct sum of ideals by Lemma 2.3. □

3. Valuation domains $R$ with $fr(R) < \infty$.

First we extend to every almost maximal valuation domain the methods used by Lady in [8] to study torsion-free modules over rank-one discrete valuation domains. So, except in Theorem 3.4, we assume that $R$ is an almost maximal valuation ring. We put $K = Q/\overline{R}$. For each $R$-module $M$, $d(M)$ is the divisible submodule of $M$ which is the union of all divisible submodules and $M$ is said to be reduced if $d(M) = 0$. We denote by $\overline{M}$ the pure-injective hull of $M$ (see [5, chapter XIII]). If $U$ is a uniserial module then $\overline{U} \cong \overline{\overline{U}}$ because $\overline{R}$ is almost maximal. Let $G$ be a torsion-free module of finite rank $r$. By Proposition 2.3 $G$ contains a submodule $B$ which is a direct sum of ideals and such that $G/B$ is a $Q$-vector space. We put $\text{corank } G = \text{rank } G/B$. Now, it is easy to prove the following.

Proposition 3.1. Let $G$ be a torsion-free $R$-module of rank $r$ and corank $c$. Then:

1. $G$ contains a pure direct sum $B$ of ideals, of rank $r - c$, such that $G/B$ is a $Q$-vector space of dimension $c$.
2. $G$ contains a pure direct sum $B'$ of ideals, of rank $r$ such that $G/B'$ is isomorphic to a quotient of $K^c$.

An element of $Q \otimes_R \text{Hom}_R(G, H)$ is called a quasi-homomorphism from $G$ to $H$, where $G$ and $H$ are $R$-modules. Let $C_{ab}$ be the category having weakly polserial $R$-modules (i.e modules with composition series whose factors are uniserial) as objects and quasi-homomorphisms as morphisms and let $C$ be full subcategory of $C_{ab}$ having torsion-free $R$-modules of finite rank as objects. Then $C_{ab}$ is abelian by [4, Lemma XII.1.1]. If $G$ and $H$ are torsion-free of finite rank, then the quasi-homomorphisms from $G$ to $H$ can be identified with the $Q$-linear maps $\phi : Q \otimes_R G \to Q \otimes_R H$ such that $r\phi(G) \subseteq H$ for some $0 \neq r \in R$. We say that $G$ and
$H$ are quasi-isomorphic if they are isomorphic objects of $C$. A torsion-free module of finite rank is said to be strongly indecomposable if it is an indecomposable object of $C$.

**Lemma 3.2.** Let $\phi : Q \otimes_R G \to Q \otimes_R H$ be a $Q$-linear map. Then $\phi$ is a quasi-homomorphism if and only if $(\hat{R} \otimes_R \phi)(d(\hat{R} \otimes_R G)) \subseteq d(\hat{R} \otimes_R H)$.

**Proof.** Assume that $\phi$ is a quasi-homomorphism. There exists $0 \neq r \in R$ such that $r\phi(G) \subseteq H$. It successively follows that $r(\hat{R} \otimes_R \phi)(\hat{R} \otimes_R G) \subseteq \hat{R} \otimes_R H$ and $(\hat{R} \otimes_R \phi)(d(\hat{R} \otimes_R G)) \subseteq d(\hat{R} \otimes_R H)$.

Conversely, let $B$ be a finite direct sum of ideals which satisfies that $G/B$ is a $Q$-vector space. There exists a free submodule $F$ of $Q \otimes_R G$ such that $B \subseteq F$. So, $\exists \neq r \in R$ such that $r\phi(B) \subseteq r\phi(F) \subseteq H$. Since $\hat{R} \otimes_R G = (\hat{R} \otimes_R B) \oplus (d(\hat{R} \otimes_R G))$, it follows that $r(\hat{R} \otimes_R \phi)(\hat{R} \otimes_R G) \subseteq (\hat{R} \otimes_R H)$. We deduce that $r\phi(G) \subseteq (\hat{R} \otimes_R H) \cap (Q \otimes_R H) = H$. □

**Proposition 3.3.** Let $G$ be a torsion-free $R$-module of rank $r$ and corank $c$.

1. If $G$ has no summand isomorphic to an ideal, then $\text{End}(G)$ can be embedded in the ring of $c \times c$ matrices over $\hat{Q}$. In particular if $c = 1$, $\text{End}(G)$ is a commutative integral domain.

2. If $G$ is reduced, then $\text{End}(G)$ can be embedded in the ring of $(r-c) \times (r-c)$ matrices over $\hat{R}$. In particular if $c = r - 1$, $\text{End}(G)$ is a commutative integral domain.

**Proof.** See the proof of [Reference Theorem 3.1]. □

In the sequel we assume that $n = c_R(0) < \infty$. So, there are $n - 1$ units $\pi_2, \ldots, \pi_n$ in $\hat{R} \setminus R$ such that $1, \pi_2, \ldots, \pi_n$ is a basis of $\hat{Q}$ over $Q$. By [Reference Theorem XV.6.3] there exists an indecomposable torsion-free $R$-module $\hat{E}$ with rank $n$ and corank 1. We can define $E$ in the following way: if $(e_k)_{2 \leq k \leq n}$ is the canonical basis of $\hat{R}^{n-1}$, if $e_1 = \sum_{k=2}^{n} \pi_k e_k$ and $V$ is the $Q$-vector subspace of $\hat{Q}^{n-1}$ generated by $(e_k)_{1 \leq k \leq n}$, then $E = V \cap \hat{R}^{n-1}$. Then a basis element for $d(\hat{R} \otimes E)$ can be written $u_1 + \pi_2 u_2 + \cdots + \pi_n u_n$, where $u_1, \ldots, u_n \in E$. Since $E$ is indecomposable it follows that $u_1, \ldots, u_n$ is a basis for $Q \otimes E \cong V$.

**Theorem 3.4.** Let $G$ be a torsion-free $R$-module of rank $r$ and corank $c$. Then the following assertions hold:

1. The reduced quotient of $G$ is isomorphic to a pure submodule of $\hat{B}$ where $B$ is a direct sum of $(r-c)$ ideals.

2. $G$ is the direct sum of ideals of $R$ with a quasi-homomorphic image of $E^c$.

**Proof.** (1) can be shown as the implication (1) $\Rightarrow$ (2) of [Reference Theorem 4.1] and (2) as the implication (1) $\Rightarrow$ (3) of [Reference Theorem 4.1]. □

**Corollary 3.5.** Let $G$ be a torsion-free $R$-module of rank $r$ and corank $c$. Then:

1. If $G$ has no summand isomorphic to an ideal, then $r \leq nc$.

2. If $G$ is reduced, then $nc \leq (n-1)r$.

**Proof.** This corollary is a consequence of Theorem 3.4 and can be shown as [Reference Corollary 4.2]. □
Theorem 3.6. Let $R$ be a valuation domain such that $d_R(0) = 2$. Then $fr(R) = 2$. Moreover $fr^a(R) = 1$ if $c_R(0) = 2$ and $fr^a(R) = 2$ if $c_R(0) = 1$.

Proof. First suppose that $c_R(0) = 2$. So, $R$ is almost maximal and $fr^a(R) = 1$. Let $G$ be an indecomposable torsion-free module with rank $r$ and corank $c$ which is not isomorphic to $Q$ and to an ideal. Then $G$ is reduced and has no summand isomorphic to an ideal of $R$. From Corollary 3.5 we deduce that $r = 2c$. By Theorem 3.4 $G$ is isomorphic to a pure submodule of $B$ where $B$ is a direct sum of $c$ ideals. Since rank $B = 2c$ it follows that $G \cong B$. So, $c = 1$ and $G \cong \hat{A}$ for a non-zero ideal $A$.

If $c_R(0) = 1$ let $L$ be the non-zero prime ideal such that $c_R(L) = d_{R/L}(0) = 2$. Then $fr(R/L) = 2$. Since $R_L$ is maximal it follows that $fr(R) = fr^a(R) = 2$ by Lemma 9 and Lemma 4.

Lemma 3.7. Every proper subobject of $E$ in $\mathcal{C}$ is a direct sum of ideals.

Proof. Let $G$ be a proper object of $E$ in $\mathcal{C}$ and let $H$ be the pure submodule of $E$ such that $H/G$ is the torsion submodule of $E/G$. Since $E$ is indecomposable, $E$ has no summand isomorphic to a direct sum of ideals. So, corank $E/H = 1$ and corank $H = 0$. As corank $H \geq$ corank $G$ we get that $G$ is a direct sum of ideals.

Proposition 3.8. $E$ is an indecomposable projective object of $\mathcal{C}$.

Proof. Let $\phi : H \to E$ be a quasi-epimorphism where $H$ is a torsion-free module of finite rank. Suppose that $H = F \oplus G$ where $F$ is a direct sum of ideals. By Lemma 5.5, $\phi(G)$ is quasi-isomorphic to $E$. So, we may assume that $H$ has no summand isomorphic to an ideal. By Theorem 5.4 there is a quasi-epimorphism $\psi : E^c \to H$ where $c = \text{corank } H$. It is sufficient to see that $\phi \circ \psi$ is a split epimorphism in $\mathcal{C}$. But by Proposition 3.3(1), $Q \otimes \text{End}(E)$ is a subfield of $\hat{Q}$, so every quasi-homomorphism $E \to E$ is either a quasi-isomorphism or trivial and the splitting follows immediately.

In the sequel, $Q \otimes_R \text{Hom}_R(R \oplus E, M)$ is denoted by $\hat{M}$ for each $R$-module $M$ and the ring $Q \otimes_R \text{End}_R(R \oplus E)$ by $\Lambda$.

Theorem 3.9. The functor $Q \otimes_R \text{Hom}_R(R \oplus E, \cdot)$ is an exact fully faithful functor from $\mathcal{C}$ into mod-$\Lambda$, the category of finitely generated right $\Lambda$-modules.

Proof. By Theorem 5.4 and Proposition 5.5, $R \oplus E$ is a progenerator of $\mathcal{C}$. For each finite rank torsion-free $R$-module $H$, the natural map $Q \otimes_R \text{Hom}_R(R \oplus E, H) \to \text{Hom}_\Lambda(\hat{R} \oplus \hat{E}, \hat{H})$ is an isomorphism because $\Lambda = \hat{R} \oplus \hat{E}$. Thus $Q \otimes_R \text{Hom}_R(F, H) \to \text{Hom}_\Lambda(\hat{F}, \hat{H})$ is an isomorphism if $F$ is a summand of a finite direct sum of modules isomorphic to $R \oplus E$. Let $G$ be a finite rank torsion-free $R$-module. We may assume that $G$ has no summand isomorphic to an ideal of $R$. By Proposition 5.5 and Lemma 5.7 there is an exact sequence $0 \to R^{nc-r} \to E^c \to G \to 0$ in $\mathcal{C}$. Since both functors are left exact, we get that $Q \otimes_R \text{Hom}_R(G, H) \cong \text{Hom}_\Lambda(\hat{G}, \hat{H})$.

Lemma 3.10. If $M$ is a right $\Lambda$-module and $M \subseteq \hat{G}$ for some finite rank torsion-free $R$-module $G$, then $M \cong \hat{H}$ for some torsion-free $R$-module $H$.

Proof. See the proof of [8, Lemma 5.2].
Proposition 3.11. The ring $\Lambda$ is a hereditary Artinian $Q$-algebra such that $(\text{rad } \Lambda)^2 = 0$. There are two simple right $\Lambda$-modules, $\tilde{R}$ which is projective and $\tilde{K}$ which is injective.

Proof. See the proof of [8, Proposition 5.3]. $\square$

Proposition 3.12. $\tilde{Q}$ is an injective hull for $\tilde{R}$ and $\tilde{E}$ is a projective cover for $\tilde{K}$.

Proof. See the proof of [8, Proposition 5.4]. $\square$

Theorem 3.13. The image of $C$ under the functor $Q \otimes_R \text{Hom}_R(R \oplus E, \cdot)$ is the full subcategory of $\text{mod} - \Lambda$ consisting of modules with no summand isomorphic to $\tilde{K}$.

Proof. See the proof of [8, Theorem 5.5]. $\square$

Let $M$ be a finitely generated (i.e., finite length) right $\Lambda$-module. We define rank $M$ to be the number of factors in a composition series for $M$ isomorphic to $\tilde{R}$ and corank $M$ to be the number of composition factors isomorphic to $\tilde{K}$.

Proposition 3.14. The foncteur $Q \otimes_R \text{Hom}_R(R \oplus E, \cdot)$ preserves rank and corank.

Proof. See the proof of [8, Proposition 5.6]. $\square$

We now consider the functors $D = \text{Hom}_R(\cdot, Q)$ and $\text{Tr} = \text{Ext}_\Lambda(\cdot, \Lambda)$ which take right $\Lambda$-modules to left $\Lambda$-modules and conversely. It is well known that $D$ is an exact contravariant length preserving functor taking projectives to injectives and conversely, and that $D^2$ is the identity for finitely generated $\Lambda$-modules. Since $\Lambda$ is hereditary, $\text{Tr}$ is right exact and $\text{Tr}^2 M \cong M$ if $M$ has no projective summand, $\text{Tr} M = 0$ if $M$ is projective. We consider the Coxeter functors $C^+ = D\text{Tr}$ and $C^- = \text{Tr}D$. Thus $C^+ : \text{mod} - \Lambda \rightarrow \text{mod} - \Lambda$ is left exact and $C^- : \text{mod} - \Lambda \rightarrow \text{mod} - \Lambda$ is right exact. If $M$ has no projective (respectively injective) summand, it is easy to check that $M$ is indecomposable if and only if $C^+ M$ (respectively $C^- M$) is indecomposable.

Proposition 3.15. Let $M$ be a right $\Lambda$-module with rank $r$ and corank $c$.

1. If $M$ has no projective summand, then corank $C^+ M = (n - 1)c - r$ and rank $C^+ M = nc - r$.
2. If $M$ has no injective summand, then rank $C^- M = (n - 1)r - nc$ and corank $C^- M = r - c$.

Proof. See the proof of [8, Proposition 5.7]. $\square$

Theorem 3.16. The following assertions hold:

1. If $n = 3$, then, up to quasi-homomorphism, the strongly indecomposable torsion-free $R$-modules are $R$, $Q$, $E$, $\tilde{R}$ and an $R$-module with rank 2 and corank 1 (corresponding to $C^+ \tilde{Q} = C^- \tilde{R}$).
2. If $n \geq 4$, there are strongly indecomposable torsion-free $R$-modules with arbitrarily large rank.

Proof. We show (1) and (2) as Lady in the proof of [8, Theorem 5.11] by using Proposition 3.15 and [10, Theorem 2]. $\square$
4. Some other results and open questions.

Let $G$ be a finite rank torsion-free module over an almost maximal valuation domain $R$. A splitting field for $G$ is a subfield $Q'$ of $\hat{Q}$ containing $Q$ such that $(Q' \cap \hat{R}) \otimes_R G$ is a completely decomposable $(Q' \cap \hat{R})$-module (i.e. a direct sum of rank one modules). If $Q'$ is a splitting field for $G$, $G$ is called $Q'$-decomposable. By [9, Theorem 7], each finite rank torsion-free module $G$ has a unique minimal splitting field $Q'$ such that $[Q' : Q] < \infty$. So, Lady’s results on splitting fields of torsion-free modules of finite rank over rank one discrete valuation domains can be extended to almost maximal valuation domains by replacing $\hat{Q}$ by $Q'$ in the previous section and by taking $C$ to be the category whose objects are the finite rank torsion-free $Q'$-decomposable modules.

Now, $R$ is a valuation domain which is not necessarily almost maximal. We say that an $R$-module $G$ is strongly flat if it is an extension of a free module by a divisible torsion-free module (see [2]). By [6, Lemma V.1.1 and Proposition V.1.2] $\hat{Q}$ is a splitting field for each finite rank strongly flat module. So, each finite rank strongly flat module $G$ has a unique minimal splitting field $Q' \subseteq \hat{Q}$ and $[Q' : Q] < \infty$. We also can extended Lady’s results. In particular:

**Theorem 4.1.** Let $R$ be a valuation domain. We consider the following conditions:

1. $\exists l \geq 1$ such that each finite rank strongly flat module is a direct sum of modules of rank at most $l$;
2. $c_R(0) \leq 3$.

Then (1) $\Rightarrow$ (2). If $c_R(0) \leq 2$ then (1) holds and $l = c_R(0)$.

If $c_R(0) = 3$, it is possible that the proof of [1] Theorem 2.6 should be generalized.

**Proof.** (1) $\Rightarrow$ (2). We show that $c_R(0) \leq \infty$ in the same way that the condition (a) of [12, Theorem 3] is proven. Then, as above, we use Lady’s methods to get $c_R(0) \leq 3$.

If $c_R(0) = 2$ we do as in the proof of Theorem 3.6. $\square$

**Some open questions:**

1. Does henselian valuation domains with finite total defect, which are not strongly discrete, exist?
2. For a valuation domain $R$, does the condition $d_R(0) = 3$ imply $fr(R) = 3$ or $< \infty$? It is possible that the proof of [1] Theorem 2.6 should be generalized.
3. For a valuation domain $R$ with finite total defect, does the condition $c_R(L_i) = p$, $\forall i$, $1 \leq i \leq m$, where $p = 2$ or 3 and $(L_i)_{1 \leq i \leq m}$ is the family of prime ideals defined in Theorem 3.7, imply $fr(R) < \infty$ for some $m > 1$? (If $m > 1$, by [12, Theorem 10], $R$ is Henselian and $p$ is the characteristic of its residue field.)

**References**

[1] D. Arnold and M. Dugas. Indecomposable modules over Nagata valuation domains. *Proc. Amer. Math. Soc.*, 122(3):689–696, (1993).

[2] S. Bazzoni and L. Salce. On strongly flat modules over integral domains. *Rocky Mountain J. Math.*, 34(2):417–439, (2004).
[3] F. Couchot. Local rings of bounded module type are almost maximal valuation rings. *Comm. Algebra*, 33(8):2851–2855, (2005).
[4] F. Couchot. Flat modules over valuation rings. *J. Pure Appl. Algebra*, 211:235–247, (2007).
[5] A. Facchini and P. Zanardo. Discrete valuation domains and ranks of their maximal extensions. *Rend. Sem. Mat. Univ. Padova*, 75:143–156, (1986).
[6] L. Fuchs and L. Salce. *Modules over valuation domains*, volume 97 of *Lecture Notes in Pure and Appl. Math.* Marcel Dekker, New York, (1986).
[7] L. Fuchs and L. Salce. *Modules over Non-Noetherian Domains*. Number 84 in Mathematical Surveys and Monographs. American Mathematical Society, Providence, (2001).
[8] E.L. Lady. Splitting fields for torsion-free modules over discrete valuation rings, i. *J. Algebra*, 49:261–275, (1977).
[9] A. Menini. The minimal splitting field for a finite rank torsion free module over an almost maximal valuation domain. *Comm. Algebra*, 11(16):1803–1815, (1983).
[10] W. Muller. On Artin rings of finite representation type. In *Proceedings of the International Conference on Representations of Algebras*, volume 488 of *Lect. Notes in Mathematics*, pages 236–243. Springer-Verlag, (1974).
[11] M. Nagata. *Local rings*. Intersciences Publishers, New York and London, (1962).
[12] P. Vámos. Decomposition problems for modules over valuation domains. *J. London Math. Soc.*, 41:10–26, (1990).
[13] P. Zanardo. Kurosch invariants for torsion-free modules over Nagata valuation domains. *J. Pure Appl. Algebra*, 82:195–209, (1992).
[14] D. Zelinsky. Linearly compact modules and rings. *Amer. J. Math.*, 75:79–90, (1953).

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