New and improved approximation algorithms for Steiner Tree Augmentation Problems

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Abstract

In the Steiner Tree Augmentation Problem (STAP), we are given a graph $G = (V, E)$, a set of terminals $R \subseteq V$, and a Steiner tree $T$ spanning $R$. The edges $L := E \setminus E(T)$ are called links and have non-negative costs. The goal is to augment $T$ by adding a minimum cost set of links, so that there are 2 edge-disjoint paths between each pair of vertices in $R$. This problem is a special case of the Survivable Network Design Problem which can be approximated to within a factor of 2 using iterative rounding [14].

We give the first polynomial time algorithm for STAP with approximation ratio better than 2. In particular we achieve a ratio of $(1 + \ln 2 + \varepsilon) \approx 1.69 + \varepsilon$. To do this, we use the Local Greedy approach of [22] for the Tree Augmentation Problem and generalize their main decomposition theorem from links (of size two) to hyper-links.

We also consider the Node-Weighted Steiner Tree Augmentation Problem (NW-STAP) in which the non-terminal nodes have non-negative costs. We seek a cheapest subset $S \subseteq V \setminus R$ so that $G[R \cup S]$ is 2-edge-connected. We provide a $O(\log^2(|R|))$-approximation algorithm for NW-STAP. To do this, we use a greedy algorithm leveraging the spider decomposition of optimal solutions.
1 Introduction

Network design problems are fundamental in combinatorial optimization and have motivated the development of broadly applicable algorithmic techniques, in addition to being of practical interest. The general theme of such problems is to satisfy a certain connectivity requirement in a graph while using the cheapest subset of edges. Many problems in network design are special cases of the Survivable Network Design Problem (SNDP) [14]. In SNDP, we are given a graph with non-negative costs on edges, and a connectivity requirement $r_{ij}$ for each pair of vertices $i, j \in V$. We want to find a cheapest subgraph of $G$ so that there are $r_{ij}$ pairwise edge-disjoint paths between all pairs of vertices $i$ and $j$.

SNDP can be approximated within a factor of 2 using Jain’s iterative rounding algorithm [14]. For most special cases of SNDP, this algorithm yields the best currently known approximation ratio. One notable exception is the Steiner tree problem for which the best-known approximation factor is $\ln 4 < 1.39$ [4]. Recently, algorithms for the Weighted Tree Augmentation Problem (WTAP) were designed with an approximation ratio below 2 [22, 23]. Interestingly, the initial improvement of the 2-approximation ratio for Steiner trees motivated the relative greedy heuristic of Zelikovsky [26], and the recent improvements for WTAP also employ the relative greedy heuristic [26].

In this paper, we expand the application of the relative greedy heuristic to break the best-known approximation factor of 2 for another problem in Network Design. In particular, we examine the “Steiner” variant of WTAP, in which we seek to cheaply augment a Steiner Tree on a set of terminals $R$ to a 2-edge-connected Steiner subgraph spanning $R$. Here the links may involve Steiner nodes, and we achieve an approximation factor of $(1 + \ln 2 + \epsilon)$-approximation, following the approach of Traub and Zenklusen in [22]. We also apply the relative greedy idea to the node-weighted version and obtain the first log-squared approximation, following the development of Klein and Ravi in [15].

1.1 Weighted Tree Augmentation

In the Weighted Tree Augmentation Problem (WTAP), we are given a tree $T = (V, E(T))$ and a set of additional edges $L$, called links, with non-negative costs $c_\ell$. We seek to find a cheapest subset $F \subseteq L$ such that $(V, E(T) \cup F)$ is 2-edge-connected [1]. In particular, this is the special case of SNDP where the graph contains a spanning tree of cost 0, and $r_{ij} = 2$ for all pairs of vertices $i$ and $j$.

Notice that the problem of augmenting a connected graph $G$ to a 2-edge-connected graph can be solved using WTAP, by contracting the 2-edge-connected components of $G$, yielding a tree to augment. In fact, the more general problem of augmenting a $k$-edge-connected graph to a $(k + 1)$-edge-connected graph – called the Connectivity Augmentation Problem – reduces to WTAP in an approximation preserving way when $k$ is odd [7].

There are several 2-approximation algorithms for WTAP. The first such result is due to Frederickson and Jája [10]. They also show that WTAP is NP-hard, even in the unweighted setting on trees of diameter 4. WTAP was shown to be APX-Hard by Kortsarz et. al. [16]. Other approaches that achieve a factor of 2 for this problem include the primal-dual method of [12] and Jain’s iterative rounding algorithm for general Survivable Network Design [14].

While there have been many results which improve upon the ratio of 2 in certain special cases [1, 6, 19], until recently, this was the best known approximation ratio for general, weighted TAP. In [22], Traub and Zenklusen, building on the ideas of Cohen and Nutov [6], use a greedy local search algorithm to achieve an approximation ratio of $(1 + \ln 2)$. They begin with a 2-approximate solution using only up-links (links going from a node to its ancestor), and iteratively improve on this solution using local moves. Each local improvement consists of adding a subset of links to the

\[^1\text{A graph is 2-edge-connected if there are two edge-disjoint paths between every pair of vertices.}\]
solution, and dropping any up-links that are rendered unnecessary for feasibility. The choice of the subset of links to add minimizes the ratio of their cost to the drop they effect and hence applies a relative greedy approach. The relative greedy method was first introduced by Zelikovsky [26] to improve upon the ratio of 2 for the Steiner Tree problem. It was then used in the context of WTAP by Cohen and Nutov [6], to achieve an approximation ratio of 1 + ln 2 for constant height trees.

Traub and Zenklusen [23] subsequently improved upon their previous algorithm, bringing the approximation ratio down to 1.5. The main idea in this improvement is to not only drop links from the initial up-link solution, but to also drop links that were added in previous iterations of the algorithm, leading to a more sophisticated potential-function based analysis.

1.2 Steiner Tree Augmentation Problem (STAP)

In this paper, we examine the “Steiner” variant of the classic weighted Tree Augmentation Problem. Here, we seek to cheaply augment a Steiner Tree on a set of terminals to a 2-edge-connected Steiner subgraph spanning these terminals. Importantly, the augmentation may use nodes that are not in the tree to be augmented.

Problem 1 (STAP). We are given as input a graph $G = (V, E)$, a set of terminals $R \subseteq V$, and a minimal Steiner Tree $T$ spanning $R$. The edges of $G$ which are not in $T$ are called links and are denoted by $L$. That is, $L := E(G) \setminus E(T)$. Note that $L$ may have endpoints in $V(T)$ or $V \setminus V(T)$. Finally, we have a cost function $c : L \rightarrow \mathbb{R}_{\geq 0}$.

The goal is to augment $T$ to be a 2-edge-connected Steiner subgraph spanning $R$. That is, we seek $S \subseteq V \setminus R$ and an $F \subseteq L$ of minimum cost such that the graph $(V(T) \cup S, E(T) \cup F)$ has two edge-disjoint paths between every pair of terminals. This is equivalent to requiring that $(V(T) \cup S, E(T) \cup F)$ is a 2-edge-connected graph. Thus, we assume in the remainder that $V(T) = R$.

Figure 1: An instance of STAP. The red edges are the edges of the given tree. The dashed edges are the links, and the blue links form a feasible solution.

STAP is a natural augmentation analogue of WTAP for augmenting Steiner minimal trees connecting only the terminals of interest in any application. It is a special case of the minimum-cost Steiner 2-edge-connected subgraph problem (S2ECSP) when the graph contains a Steiner tree of cost zero. S2ECSP was introduced by Monma et al. [17] in the context of the design of survivable telecommunication and logistics networks where it has been extensively studied [20, 5, 13]. Linear time algorithms are known in the special case of Halin [24] and series-parallel [25] graphs, a complete linear description of the dominant of the associated polytope is known for a class of graphs called perfectly Steiner 2-edge connected graphs, which generalize series-parallel graphs [2].
Notice that WTAP is the special case of STAP where $R = V$. However, STAP is also a special case of SNDP. Indeed take $r_{ij} = 2$ for $i, j \in R$, and 0 otherwise. By setting the costs of edges in $E(T)$ to be 0 we get an instance of SNDP which is equivalent to the STAP instance. Therefore, STAP can be approximated to within a factor of 2 using Jain’s algorithm [14]. In this paper, using a relative greedy algorithm, we are able to break the approximation barrier of 2 for STAP.

**Theorem 1.** For any $\varepsilon > 0$, there is a $(1 + \ln 2 + \varepsilon)$-approximation algorithm for STAP which runs in polynomial time.

The following $k$-edge-connected generalization of STAP is of interest: we are given a graph $G = (V, E)$ and a $k$-edge-connected subgraph $H = (R, E(H))$, with costs on the links $L := E \setminus E(H)$. The goal is to add nodes and a subset of links of minimum cost to $H$ so that there are $k+1$ pairwise edge-disjoint paths between each pair of vertices in $R$. This is the problem of augmenting a given $k$-edge-connected graph to be $(k+1)$-edge-connected, but where Steiner nodes may be included in the augmentation.

This problem is a special case of SNDP and hence admits a 2-approximation [14]. We note that this problem reduces to STAP when $k$ is odd. Indeed, the $k$-edge-connected graph $G$ can be replaced with a tree whose min-cuts correspond to the min-cuts of $G$ [7, 9]. Then we can apply our algorithm for STAP, and the solution will correspond to the desired augmentation of $G$. However, when $k$ is even, the resulting augmentation problem is a cactus augmentation problem arising from the cactus structure of the min-cuts separating $R$ [7, 9]. Even in the special case when the cactus is a single cycle, no better-than-2 approximation is known for the weighted case of the problem, though such algorithms are known in the unweighted case [11]. One of the open problems related to STAP is thus to find better approximations for the cactus augmentation problem or even the simpler cycle augmentation case.

Given Theorem 1 based on [22], it is natural to ask whether the improvement of [22] in [23] to an even better approximation ratio of 1.5 for WTAP can be extended to STAP to bring the approximation ratio down to 1.5. However, we have not been able to extend the corresponding charging arguments for WTAP to STAP, and this remains another interesting open direction.

### 1.3 Node Weighted STAP

**Problem 2** (NW-STAP). In node weighted STAP, we are given a graph $G = (V, E)$, a set of terminals $R \subseteq V$, and a Steiner Tree $T$ spanning $R$. The edges $L := E(G) \setminus E(T)$ are called links, and the nodes in $V \setminus R$ are called Steiner nodes. Each Steiner node has a non-negative cost $c_v$.

Our goal is to pick a minimum cost subset $S \subseteq V \setminus R$ so that the induced subgraph $G[R \cup S]$ is 2-edge-connected.

We may also allow costs on links $c_\ell$ for $\ell \in L$. However, by subdividing each link with a node of cost equal to the cost of the link, we may assume that the links have cost 0.

The node-weighted Steiner tree problem has an approximation ratio of $\Theta(\log |R|)$ due to Klein and Ravi [15]. The lower bound is due to an approximation-preserving reduction from the set-cover problem. Interestingly, for proving the upper bound, their algorithm is also a relative greedy heuristic. It uses “spiders” which are star homeomorphs, and merge terminal clusters that occur at the feet (leaves of the star). The algorithm proceeds by finding minimum cost-ratio spiders (minimizing the total node cost divided by the number of feet or terminal clusters connected), and adding them to the solution. Their key idea is to show how to decompose an optimal node-weighted Steiner tree into spiders, and thus argue that there is a spider whose ratio cost is at least as good as that of the optimal solution. Then, a set-cover based analysis is used to obtain a logarithmic guarantee.
We extend the relative greedy spider algorithm for NW-STAP.

**Theorem 2.** There is an \(O(\log^2 |R|)\)-approximation algorithm for the node weighted Steiner tree Augmentation Problem on a tree on \(|R|\) nodes which runs in polynomial time.

### 1.4 Organization of this Paper

In Section 2, we define notation and background. In Section 3, we outline our techniques for approximating STAP and NW-STAP. In Section 4, we give the improved approximation for STAP proving Theorem 1 and do the same for NW-STAP in Section 5 proving Theorem 2.

### 2 Preliminaries

We consider a solution to STAP \((S, F)\) where \(S \subseteq V\) and \(F \subseteq L\).

**Definition 1.** A full component of an STAP solution \((S, F)\), is a subtree of the solution where each leaf is a terminal (that is, a vertex of \(R\)), and each internal node is in \(V \setminus R\).

It is clear that any STAP solution can be uniquely decomposed into link-disjoint full components. We say that a full component “joins” the terminals that it contains.

**Definition 2.** Let \((S, F)\) be a solution to STAP. We say that a set \(A\) is joined by \((S, F)\) if there is a full component with leaves \(A\).

In fact, the feasibility of a solution is determined only by which sets of nodes it joins.

**Definition 3.** We say that a tree edge \(e \in E(T)\) is covered by a solution \((S, F)\) if \(e\) lies on the unique path in the tree \(P_{uv}\) between two nodes \(u\) and \(v\) which are joined by a full component of \((S, F)\).

**Lemma 1.** A solution \((S, F)\) is feasible for STAP iff all tree edges are covered.

**Proof.** Suppose all tree edges are covered, and consider any cut in the tree induced by an edge \(e \in E(T)\). The cut contains \(e\) and since \(e\) is covered, some link in \(F\) also crosses the cut. Thus, the augmentation is a 2-edge-connected graph.

Conversely, if some edge \(e \in E(T)\) is not covered, then the cut \(e\) induces in the tree contains no links, so this edge is a bridge, contradicting the feasibility of \((S, F)\).

In the WTAP problem, we must choose a set of links to cover all the edges of a given tree. In WTAP, each link joins exactly two tree vertices. The added difficulty in the case of STAP is that full components may join an arbitrary number of terminals. Thus, we introduce the Hyper-TAP problem as the natural generalization of WTAP to hyper-links, which join arbitrary subsets of tree vertices.

**Problem 3** (Hyper-TAP). In Hyper-TAP, we are given a tree \(T = (V, E)\), and a collection of hyper-links \(L \subseteq 2^V\), with non-negative costs \(c_\ell\) for \(\ell \in L\). The goal is to cover the edges of the given tree with the minimum cost subset of hyper-links.

Consider a hyper-link \(\ell = \{a_1, \ldots, a_k\}\). We say that the vertices \(a_1, \ldots, a_k\) are joined by \(\ell\). After fixing a root of \(T\), denote the least common ancestor of \(\{a_1, \ldots, a_k\}\) by \(\text{lca}(a_1, \ldots, a_k)\) and define \(\text{apex}(\ell) := \text{lca}(a_1, \ldots, a_k)\). Let \(P_{a,b}\) be the unique edge path in the tree from \(a\) to \(b\).
Let $T_\ell$ be the subtree of $T$ consisting of the union of all paths between vertices joined by $\ell$. Equivalently, $T_\ell := \bigcup_{a \in \ell} P_{a, \text{apex}(\ell)}$. We say that the link $\ell$ covers the edges in $T_\ell$.

Then, the Hyper-TAP problem is the following covering problem:

$$\min_{Z \subseteq \mathcal{L}} \left\{ \sum_{\ell \in Z} c(\ell) : \bigcup_{\ell \in Z} T_\ell = E \right\}.$$ 

Clearly, Hyper-TAP is an instance of Set Cover. However, they are in fact equivalent. Indeed, given any instance of Set Cover with ground set $E$ and subsets $S$, we can create an instance of Hyper-TAP in which $T$ is a star, and the edges of $T$ correspond to elements of $E$. Finally, we can create a hyper-link $\ell$ for each $S \in S$ covering exactly the edges corresponding to the elements covered by $S$.

Thus, we cannot expect to achieve approximation algorithms for general Hyper-TAP better than those for general Set Cover. However, in proving Theorem 1, we exploit the structure of Hyper-TAP instances that come from instances of STAP to achieve improved approximations in this case.

As noted above, every STAP instance is equivalent to an instance of Hyper-TAP obtained as follows: for each subset of tree vertices $S \subseteq R$, find the cheapest full component joining $S$, and create a hyper-link $\ell_S$ joining $S$ with this cost. However, if we allow full components of unbounded cardinality this reduction cannot be carried out in polynomial time. In order to perform this reduction in polynomial time, we restrict the size of the full components we consider.

**Definition 4.** We say that a full component is $\gamma$-restricted if it joins at most $\gamma$ terminals. We say that a solution to STAP is $\gamma$-restricted if it uses only $\gamma$-restricted full components. Analogously, we say an instance of Hyper-TAP is $\gamma$-restricted if each hyper-link has size at most $\gamma$.

In Section 4.3, we show that up to a factor of $(1 + \varepsilon)$, it suffices to find the best $\gamma$-restricted solution to STAP for some constant $\gamma(\varepsilon)$. This allows us to reduce an instance of STAP to an instance of $\gamma$-restricted Hyper-TAP in polynomial time, while only losing a factor of $(1 + \varepsilon)$ in the approximation ratio.

## 3 Our Techniques

### 3.1 Edge Weighted STAP

Our algorithm for (edge-weighted) STAP is a local greedy algorithm which follows in the vein of the recent improved approximation algorithms for WTAP due to Traub and Zenklusen [22]. We briefly describe their methods here. The local greedy algorithm for WTAP begins with an initial 2-approximate feasible solution. It then makes local moves to improve the current solution by adding carefully chosen subsets of links and dropping links in the initial solution which are rendered unnecessary for feasibility.

An up-link is a link which joins two nodes having an ancestor-descendant relationship in the tree. The initial 2-approximate solution has a special structure: it only consists of up-links, and each tree edge is covered exactly once. In the WTAP setting, this structured 2-approximate solution can be found efficiently by replacing every link with two up-links to their least common ancestor and using dynamic programming to find the best up-links only solution [10].

For a given up-link solution $U$ and subset of links $C \subseteq L$, $\text{drop}_U(C)$ is the set of up-links in $U$ which can be removed from $U \cup C$ while preserving feasibility. In each iteration, the local greedy algorithm in [22] seeks to choose a subset of links $C$ minimizing the ratio between the cost of links
in $C$, and the cost of up-links in $\text{drop}_U(C)$. It then adds these links to the current solution and removes all the links in $\text{drop}_U(C)$.

However, the maximization cannot be done efficiently over all subsets of links. Thus, they define a notion of $k$-thin subsets of links. They show that one can compute the maximizer over all $k$-thin subsets in polynomial time using dynamic programming. Thus, these subsets of links are simple enough so that we can choose the best to add at each iteration.

Crucial for the analysis of the local greedy algorithm is the property that, as long as the current solution is expensive, there will always be an improving $k$-thin subset of links to add. This is the main decomposition theorem of Traub and Zenklusen [22].

In order to apply the local greedy algorithm to the STAP setting, we need to first find an initial structured 2-approximate up-link solution covering each tree edge exactly once. We show how to do this in Section 4.1 using Euler tours over the optimal solution components. In particular, we show the following.

**Lemma 2.** Given an instance of STAP, there is a polynomial time algorithm which returns a feasible up-link solution $U$, with $c(U) \leq 2OPT$ and where each edge $e \in E(T)$ is covered exactly once.

Next, we need to prove a decomposition result analogous to their theorem for TAP. We extend their decomposition theorem to arbitrary hyper-links in Section 4.4.

**Definition 5.** Let $Z$ be a collection of hyperlinks. We say that $Z$ is $k$-thin if for each $v \in V(T)$, we have $|\{\ell \in Z : v \in T_\ell\}| \leq k$.

Let $P_u$ be the edges covered by up-link $u$. Given a set of up-links $U$ and a collection of hyper-links $Z \subseteq \mathcal{L}$, let

$$\text{drop}_U(Z) = \{u \in U : P_u \subseteq \bigcup_{\ell \in Z} T_\ell\}.$$  

**Theorem 3 (Decomposition Theorem).** Given an instance of Hyper-TAP $(T, \mathcal{L})$, suppose $U$ is an up-link solution such that $P_u$ are pairwise edge-disjoint for $u \in U$. Suppose $F^* \subseteq \mathcal{L}$ is any solution. Then there exists a partition $Z$ of $F$ into parts so that for any $\epsilon > 0$:

- For each $Z \in Z$, $Z$ is $k$-thin, for $k = \lceil 1/\epsilon \rceil$.
- There exists $O \subseteq U$ with $c(O) \leq \epsilon$, such that for all $u \in U \setminus O$, there is some $Z \in Z$ with $u \in \text{drop}_U(Z)$. That is, $U \setminus O \subseteq \bigcup_{Z \in Z} \text{drop}_U(Z)$.

Following the method of Traub and Zenklusen [22], these two results are enough to prove that the local greedy algorithm achieves an approximation ratio of $1 + \ln 2$. However, in order to attain a polynomial runtime, we cannot afford to search over arbitrary size hyper-links. This is where we use the notion of $\gamma$-restricted hyperlinks. We extend the result of Borchers and Du for Steiner Trees [3], which shows a bounded ratio between optimal $k$-restricted Steiner trees and optimal unrestricted Steiner trees, to the case of STAP. This allows us to efficiently approximately reduce an instance of STAP to an instance of Hyper-TAP, where each hyper-link has size bounded by a constant.

**Lemma 3 (Bounding loss through restriction).** Given an instance of STAP, let $OPT$ be the optimal value and $OPT_\gamma$ be the optimal value over all $\gamma$-restricted solutions. Then for all $\epsilon > 0$, there exists $\gamma(\epsilon) = 2\lceil \frac{1}{\epsilon} \rceil$ such that

$$\frac{OPT_\gamma}{OPT} \leq 1 + \epsilon.$$
This ensures that we can generate Hyper-TAP instances with only constant sized hyper-links, which allows us to find the greedy local move used in the local search algorithm in polynomial time using dynamic programming.

### 3.2 Node Weighted STAP

In our algorithm for NW-STAP, we will use the notion of minimum-ratio spiders introduced by Klein and Ravi [15].

**Definition 6.** A spider is a tree with at least 3 vertices and at most vertex of degree greater than 2. The degree 1 vertices are called feet and the unique vertex of degree greater than 2 is called the head. The paths from the head of the spider to its feet are called legs.

A spider decomposition of a tree $T$ is a collection of node-disjoint spiders whose feet are exactly the leaves of the tree. Given the tree $T = (R, E(T))$ to be augmented and a spider $(S, F)$, recall that the spider covers the subtree of the tree $T$ that is induced by the set of nodes in $S \cap R$. The following theorem shows that starting with $O(\log |R|)$ copies of an optimal augmentation, they can be decomposed into a collection of spiders which together cover the edges of $T$. In other words, one can decompose $O(\log |R|)$ copies of the optimal solution into a collection of spiders that collectively covers the edges of $T$.

**Theorem 4** (Naor et. al. [18]). *Given a tree $T$ with $k$ leaves, there exists a sequence of at most $O(\log k)$ spider decompositions whose union covers the edges of the tree.*

Note that there is a key difference between using minimum ratio spiders to build a minimum node-weighted Steiner tree as in [15] and using them for augmenting a Steiner tree. The coverage of a spider in the former case is simply the number of terminal connected components it merges, while in the latter, it is the number of edges of the Steiner tree to be augmented that it covers. Thus, in order to be able to efficiently compute the minimum ratio spider-like subgraph to add to the solution at each step, we will need find a way to account for coverage of the subtree induced by the points of attachment of the spider. For this, we use the observation that the coverage function of the subtree induced by a subset of nodes in a tree is submodular. To account for the cost of the spider, we relax the spider to the notion of a pseudo-spider, where we pay for each leg of the spider separately even if the legs are not node-disjoint.

**Definition 7.** Given a node-weighted graph $G = (V, E)$, a pseudo-spider of $G$ with head $h \in V$ and feet $P \subseteq V$ is the (not necessarily disjoint) union of node-weighted shortest paths from the head to each foot $f \in P$.

Note that if there exists a spider with head $h$ which joins feet $P$, then there is a pseudo-spider with head $h$ joining $P$ of at most the same cost. Hence, Theorem 4 implies that there is a solution consisting of pseudo-spiders with cost at most $O(\log |R|)OPT$.

Now, to find the minimum ratio pseudo-spider, we can fix a budget on the cost of the pseudo-spider and use monotone, submodular maximization under a knapsack constraint [21] (where the items are the different legs whose costs are paid separately in the knapsack) to implement the greedy step.

Our algorithm will proceed by finding, in each step, a pseudo-spider which (approximately) minimizes the ratio $\frac{\text{cost of the pseudo-spider}}{\text{number of new tree edges of } T \text{ covered}}$. We continue greedily adding pseudo-spiders until all tree edges have been covered. As discussed, the greedy step can be implemented with a constant factor approximation in polynomial time by using an algorithm for monotone submodular maximization subject to a knapsack constraint. Coupled with the analysis of the greedy algorithm as in [15], this gives us a proof of Theorem 2. Details are in Section 5.
4 Edge Weighted STAP

We prove Theorem 1 in this section using the outline in Section 3.1. In Section 4.1 we show how to achieve a 2-approximation algorithm for STAP which returns a structured solution involving up-links with disjoint coverage. In Section 4.2 we define the algorithm for STAP. In Section 4.3 we show that we can restrict our search for solutions to STAP to γ-restricted ones while only losing a factor of ε in the approximation ratio. In Section 4.4 we prove the decomposition theorem and finally in Section 4.5 we describe the dynamic program which allows us to implement the algorithm in polynomial time.

4.1 A Structured 2-Approximation for STAP

Consider an instance of STAP and let \((S^*, F^*)\) be an optimal augmentation. We describe an approximation algorithm for STAP which yields a feasible solution \((S, F)\) of cost at most \(2c(F^*)\). This approximation ratio is already achievable by using Jain’s algorithm for SNDP. However, we show that we can find an approximate solution \((S, F)\) with nice structural properties. In particular, we will have \(S \subseteq V(T)\), \(F\) consisting of only up-links, and the coverage of each \(\ell \in F\) being pairwise disjoint.

We will perform a metric completion step on the instance without loss of generality. For every pair of nodes \(u\) and \(v\) in \(R\), consider the shortest path from \(u\) to \(v\) in the graph \((V, L)\). We add a link \((u, v)\) with cost equal to the shortest path length. This does not change the problem since any solution which uses one of the added links may instead use the shortest path instead, paying the same cost.

Next, we perform a shadow completion. Let \(P_{uv}\) be the path in \(T\) between terminals \(u\) and \(v\). If there exists a link \(\ell = (u, v)\) of cost \(c\), we will also add links \(\ell' = (u', v')\) of cost \(c\) where \(u', v' \in P_{uv}\). Again, this can be done without loss of generality since any solution which uses an added link may be converted to a solution to the original instance of the same cost.

It is easy to see that both of these preprocessing steps can be done in polynomial time. For the following we will consider the given tree \(T\) to be rooted at an arbitrary node \(r\).

Lemma 4. Given any feasible solution \((S^*, F^*)\) to STAP, there exists a solution \((S, F)\) with \(c(F) \leq 2c(F^*)\) such that \(S \subseteq R\). Furthermore, \(F \subseteq L\) involves only up-links.

Proof. Denote the full components of \((S^*, F^*)\) by \((A_1^*, H_1^*), \ldots, (A_p^*, H_p^*)\). For each full component \((A_i^*, H_i^*)\), we will take an Eulerian tour which traverses each link in \(H_i^*\) exactly twice. This yields an ordering of the terminals in \(S^* \cap R\), say \(r_1, \ldots, r_k\). Now, let \(v_i := \text{lca}(r_i, r_{i+1})\) for \(i = 1, \ldots, n\), where \(r_{n+1} := r_1\).

We will consider the set of links \(F := \{(r_1, v_1), (r_2, v_2), \ldots, (r_k, v_k)\}\). It is clear that \(F\) contains only up-links between nodes in \(R\). We will show that \(c(F) \leq 2c(F^*)\).

Note that the total cost of \(F' := \{(r_1, r_2), (r_2, r_3), \ldots, (r_k, r_1)\}\) is at most \(2c(F^*)\) because of the metric completion step, and the fact that the Eulerian tour traverses each link in \(F^*\) exactly twice. Furthermore, the link \((r_i, v_i)\) is a shadow of \((r_i, r_{i+1})\), so it exists and has at most the cost. Thus, \(c(F) \leq c(F') \leq 2c(F^*)\).

We will now show that \(F\) is a feasible solution. We want to show that \(G' = (R \cup S, E(T) \cup F)\) is a 2-edge-connected graph. It suffices to show that \(G'\) is connected after any edge \(e \in E(T)\) is deleted.

Thus, we fix some \(e \in E(T)\) and consider the cut \((W, \bar{W})\) it induces on the tree \(T\). Notice that since \(F'\) is a feasible solution and \(e\) must be covered, there must be terminals \(r_i\) and \(r_j\) in the Euler
tour with \( r_i \in W \) and \( r_j \in \bar{W} \). We assume \( i < j \). In fact, since the tour is a cycle and therefore returns to its starting node, we also have a pair of vertices \( r_{i'} \in \bar{W} \) and \( r_{j'} \in W \) with \( i' < j' \).

We claim that either link \((r_i, v_i)\) or link \((r_{i'}, v_{i'})\) covers \( e \). Indeed, it is impossible for \( v_i \in W \) and \( v_{i'} \in \bar{W} \) since if the root \( r \in W \), then both \( v_i, v_{i'} \in W \), and if \( r \in \bar{W} \), then both \( v_i, v_{i'} \in \bar{W} \).

Thus, \( F \) is a feasible solution with the desired properties.

Finally, we will need the following standard result, showing that we can shorten the up-link solution by replacing certain links with their shadows, so that each tree edge is covered exactly once.

**Lemma 5.** Given an up-link solution \( U \), we can in polynomial time find an up-link solution \( U' \) with \( c(U') \leq c(U) \) and with \( |\{\ell \in U' : e \in P_\ell\}| = 1 \) for all \( e \in E(T) \).

By Lemma 4 if the optimal solution to a STAP instance has cost \( OPT \), then there is an up-link solution of cost at most \( 2OPT \). Since the optimal up-link solution can be easily computed in polynomial time using dynamic programming (see e.g. [10]), we obtain Lemma 2.

### 4.2 Local Greedy for STAP

We now give the algorithm for STAP which, given any \( \varepsilon > 0 \), computes a solution \( F \) with cost at most \( (1 + \ln 2 + \varepsilon)OPT \) and runs in polynomial time. See Algorithm 1.

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**Algorithm 1** Local greedy algorithm for STAP

**Input:** A shadow-complete, metric-complete instance of STAP with graph \( G = (V, E) \), tree \( T = (R, E(T)) \), links \( L = E \setminus E(T) \), and \( c : L \to \mathbb{R} \). Also an \( \varepsilon > 0 \).

**Output:** A solution \( F \subseteq L \) with \( c(F) \leq (1 + \ln(2) + \varepsilon)OPT \).

1. Compute a 2-approximate up-link solution \( U \) such that each edge \( e \in E(T) \) is covered exactly once (Lemma 2).

2. Let \( \varepsilon' := \frac{\varepsilon/2}{1+\ln(2)+\varepsilon/2} \) and \( \gamma := 2^{[1/\varepsilon']} \).

3. For each \( S \subseteq R \) where \( |S| \leq \gamma \), compute the cheapest full component joining \( S \) and denote the cost by \( c_S \).

4. Create an instance of \( \gamma \)-restricted Hyper-TAP on tree \( T = (R, E(T)) \) with hyper-links \( \mathcal{L} = \{\ell_S : \emptyset \subseteq R, |S| \leq \gamma \} \). Set the cost of hyper-link \( \ell_S \) to be \( c_S \).

5. Initialize \( F := \emptyset \)

6. Let \( k := \lceil 4/\varepsilon \rceil \)

7. While \( U \neq \emptyset \):

   - Compute the \( k \)-thin subset of hyper-links \( Z \subseteq 2^\mathcal{L} \) minimizing \( \frac{c(Z)}{c(\text{drop}_U(Z))} \).
   - Let \( F := F \cup Z \) and let \( U := U \setminus \text{drop}_U(Z) \).

8. Return A STAP solution with full components corresponding to the hyper-links in \( F \).
We assume we are given a shadow-complete, metric-complete instance of STAP with graph \( G = (V, E) \), tree \( T = (R, E(T)) \), links \( L := E \setminus E(T) \) with costs \( c : L \to \mathbb{R} \).

The algorithm uses Lemma \( 2 \) to compute an up-link solution \( U \) that has cost at most \( 2\text{OPT} \) and can be chosen to have \( P_u \) disjoint for each \( u \in U \).

By Lemma \( 3 \) we may restrict our attention to finding a \( \gamma \)-restricted solution to the STAP instance, for \( \gamma = 2^{1/\varepsilon} \). Our algorithm now constructs an equivalent instance of \( \gamma \)-restricted Hyper-TAP. It enumerates over all subsets \( S \subseteq R \) of at most \( \gamma \) terminals, and computes the cheapest full component joining \( S \). Notice that there are at most \( n^\gamma \) such sets. Furthermore, for each subset \( S \), we can compute the cheapest full component joining \( S \) in polynomial time by solving an instance of Steiner tree with constantly many terminals using the following result.

**Lemma 6** (Dreyfus, Wagner \([8]\)). *There is an algorithm for Steiner Tree which returns the optimal Steiner Tree and runs in time \( O(n^3 \cdot 3^p) \), where \( p \) is the number of terminals.*

For the remainder of the procedure the algorithm works with this Hyper-TAP instance and the previously computed up-link solution \( U \).

We iteratively improve the current solution by finding the best \( k \)-thin subset of hyper-links to add. In particular, we find the \( k \)-thin subset of hyper-links \( Z \) which minimizes the ratio \( c(Z) / c(\text{drop}_U(Z)) \). This can be done in polynomial time via dynamic programming, see Lemma \( 11 \).

Finally, since at least one up-link is dropped in each iteration of the while-loop, this algorithm runs in polynomial time overall.

We now turn to analyzing the quality of the solution returned. This relies on the Decomposition Theorem \( 3 \) for Hyper-TAP which we prove in Section \( 4.4 \). With this Decomposition Theorem we can immediately conclude that the local greedy procedure computes a \( (1 + \ln 2 + \varepsilon) \)-approximation of the optimal Hyper-TAP solution by leveraging the results of previous work (See \([6, 22]\) Theorem 6). This proves our main Theorem 1.

**Proof of Theorem 7.** We prove that for every \( \varepsilon > 0 \), Algorithm 7.1 returns a solution \( F \) of cost at most \( (1 + \ln 2 + \varepsilon)\text{OPT} \).

As usual \( \text{OPT} \) denotes the cost of the optimal solution to the original STAP problem. Let \( \text{OPT}' \) denote the cost of the optimal \( \gamma \)-restricted STAP solution. Notice that by taking \( \varepsilon' = \frac{\varepsilon/2}{1 + \ln 2 + \varepsilon/2} \), and \( \gamma = 2^{1/\varepsilon'} \), we have that \( \text{OPT}' \leq (1 + \varepsilon')\text{OPT} \) by Lemma \( 3 \).

By the Decomposition Theorem \( 3 \) the local greedy procedure returns a solution \( F \) with cost at most \( (1 + \ln 2 + \varepsilon/2)\text{OPT}' \).

Hence our overall cost is at most

\[
c(F) \leq (1 + \ln 2 + \varepsilon/2)(1 + \frac{\varepsilon/2}{1 + \ln 2 + \varepsilon/2})\text{OPT} = (1 + \ln 2 + \varepsilon)\text{OPT}.
\]

\( \square \)

### 4.3 Effects of \( \gamma \)-restriction

In this section, we prove that, for any \( \varepsilon > 0 \), there is a large enough \( \gamma \) so that the cost of the optimal \( \gamma \)-restricted solution to STAP costs at most \( (1 + \varepsilon) \) times the optimal cost of an unrestricted STAP solution. This is an extension of the result of Borchers and Du \([3]\) for Steiner trees.

Recall that a full component of an STAP solution \( (S, F) \), is a subtree of the solution where each leaf is a terminal (that is, a vertex in \( R \)), and each internal node is in \( V \setminus R \). Also recall that an STAP solution is \( \gamma \)-restricted if each each of its full components joins at most \( \gamma \) terminals.
Notice that finding a minimum cost 2-restricted STAP solution is simply the Weighted Tree Augmentation Problem. We show that, for $\gamma$ large enough, the optimal $\gamma$-restricted STAP solution is close (in cost) to the optimal unrestricted solution.

First, we recall the following result for Steiner trees. Given a Steiner tree solution, a full component of this solution is a subtree whose leaves are terminals and whose non-leaves are non-terminals. A $\gamma$-restricted Steiner Tree solution is a solution whose full components all have at most $\gamma$ terminals.

**Theorem 5** (Borchers and Du [3]). Let $\varepsilon > 0$. Fix any instance of Steiner tree and let $T^*$ be the optimal Steiner Tree and $T_\gamma$ be the optimal $\gamma$-restricted Steiner Tree. Then for $\gamma = 2^{\lceil \frac{1}{\varepsilon} \rceil}$, we have

$$\frac{c(T_\gamma)}{c(T^*)} \leq 1 + \varepsilon.$$

Now, we prove an analogous result for STAP. For an instance of STAP, let $(S^*, F^*)$ be the optimal solution and $(S_\gamma, F_\gamma)$ be the optimal $k$-restricted solution.

**Proof of Lemma** Let $\varepsilon > 0$. Then we show that for $\gamma = 2^{\lceil \frac{1}{\varepsilon} \rceil}$, we have

$$\frac{c(F_\gamma)}{c(F^*)} \leq 1 + \varepsilon.$$

Let $(S^*, F^*)$ be the optimal STAP solution. Each of its full components $(A^*_i, H^*_i)$ is a tree joining some set of terminals $R_i \subseteq R$.

Since $(S^*, F^*)$ is optimal, and $(A^*_i, H^*_i)$ is a full component, it must be the cheapest way to join the nodes in $R_i$. That is, $(A^*_i, H^*_i)$ is an optimal solution to the Steiner Tree instance on the graph $(R_i \cup (V \setminus R), L)$, with terminals $R_i$.

By Theorem 5 above, there is a $\gamma$-restricted Steiner Tree solution of cost at most $(1 + \varepsilon)c(H^*_i)$. Applying this to each full component of $(S^*, F^*)$ yields a solution of cost at most $(1 + \varepsilon)c(F^*)$ which only has full components joining at most $\gamma$ terminals.

### 4.4 The Decomposition Theorem

In this section, we prove our main decomposition theorem (Theorem 3). In order to prove this result, we will follow the methods of [22], and extend them from WTAP to Hyper-TAP. At a high level, the argument is as follows. We will build a directed graph $D$ whose vertices correspond to the hyper-links $\ell \in F^*$. For each up-link $u \in U$, we will choose a minimal set of hyper-links $F_u$ such that $u \in \text{drop}_U(F_u)$. Each set $F_u$ will correspond to a directed path $A_u$ in $D$.

In [22], the authors show that based on the choices for the minimal sets $F_u$, the dependency graph satisfies the following key properties, which allow for the selection $O \subseteq U$ and a partition of $F^*$ into the desired $k$-thin components.

1. The dependency graph is a branching.

2. Let $(Z, A)$ be a connected component of the dependency graph. If the arc set of every directed path in $(Z, A)$ has non-empty intersection with $A_u$ for at most $k$ up-links $u \in U$, then $Z$ is $(k + 1)$-thin.

In [22], Theorem 5, shows how one can use these two properties to select $O \subseteq U$ and the partition of $F^*$ into the desired $k$-thin components, proving the Decomposition Theorem.
We argue that the same properties hold in the hyper-link setting. The crucial property that allows us to do this is that if \( u \) is an up-link and \( \ell \) is a hyper-link, then the intersection of \( P_u \) and \( T_\ell \) is a subpath of \( P_u \).

We now describe how to construct for each \( u \in U \) a minimal set \( F_u \) such that \( u \in \text{drop}_{T_\ell}(F_u) \). Suppose \( u = (t, b) \) where \( t \) is an ancestor of \( b \). We define \( v_u \) to be the lowest ancestor of \( t \), i.e., the ancestor farthest away from the root \( r \), such that \( P_u \) is covered by hyper-links in

\[
B_{v_u} := \{ \ell \in F^* : \text{apex}(\ell) \text{ is a descendant of } v_u \}
\]

i.e., \( P_u \subseteq \bigcup_{\ell \in B_{v_u}} P_\ell \). Then we choose \( F_u \subseteq B_{v_u} \) minimal such that \( P_u \subseteq \bigcup_{\ell \in F_u} T_\ell \).

There is a natural ordering on the hyper-links in \( F_u \). First, we make some observations about how each hyper-link interacts with an up-link. Let \( P_{u,\ell} := P_u \setminus \bigcup_{\ell' \in F_u \setminus \{\ell\}} T_{\ell'} \).

**Lemma 7.** For any up-link \( u \), let \( \ell \in F_u \). Then \( P_{u,\ell} \) is non-empty and the edge set of a path.

For \( \ell_1, \ell_2 \in F_u \), we define \( \ell_1 \prec \ell_2 \) if and only if the edges in \( P_{u,\ell_1} \) appear before the edges of \( P_{u,\ell_2} \) on the \( t - b \) path in \( T \).

The arcs of the dependency graph are determined by the orderings on \( F_u \) for \( u \in U \). For every up-link \( u \in U \), let \( \ell_1 \prec \cdots \prec \ell_q \) be the links in \( F_u \). Then

\[
A_u := \{ (\ell_1, \ell_2), \ldots, (\ell_{q-1}, \ell_q) \}
\]

The arcs of the dependency graph consist of the union of \( A_u \) over all \( u \in U \). We will now show that the dependency graph has the two key properties which were introduced in [22], which allow the selection \( O \subseteq U \) and a partition of \( F^* \) into the desired \( k \)-thin components.

First, we consider property (1). This property simply follows from the minimality of \( F_u \) for each \( u \in U \) and has been shown in [6].

**Lemma 8** (Cohen and Nutov [6]). The dependency graph \( D \) is a node-disjoint collection of arborescences.

To prove property (2), we need the following lemma, which relies on the particular choice of minimal \( F_u \). Fix any connected component of the dependency graph \((Z, A)\).

**Lemma 9.** Let \( \ell_1, \ell_2 \in Z \) with \( V_{\ell_1} \cap V_{\ell_2} \neq \emptyset \). Then \( \ell_1 \) and \( \ell_2 \) have an ancestry relationship in the arborescence \((Z, A)\), i.e., either \( \ell_1 \) is an ancestor of \( \ell_2 \) or \( \ell_2 \) is an ancestor of \( \ell_1 \).

Lemma 9 has been shown in the context of WTAP in [22]. The proof extends verbatim to the case of Hyper-TAP, so we don’t rewrite the proof here. Just as in [22], Lemma 9 along with the property that \( F_u \) is 2-thin for each \( u \in U \) (which follows from the minimality of \( F_u \)) imply property (2).

**Lemma 10.** Let \( k \) be a positive integer, and \((Z, A)\) be a connected component of the dependency graph. If every directed path in \((Z, A)\) intersects at most \( k \) sets \( A_u \), then \( Z \) is \((k + 1)\)-thin.

Thus, the dependency graph satisfies the 2 properties described earlier in the section. We can use the identical technique to [22] to select a set of up-links \( O \subseteq U \) with \( c(O) \leq \varepsilon \). We will delete the arc sets \( A_u \) for \( u \in O \) from the dependency graph, and the connected components of the remaining digraph will each be \((k + 1)\)-thin. We reproduce the proof below.

**Theorem 6.** Given an instance of Hyper-TAP \((T, \mathcal{L})\), suppose \( U \) is an up-link solution such that the sets \( P_u \) are pairwise edge-disjoint for \( u \in U \). Suppose \( F^* \subseteq \mathcal{L} \) is any solution. Then there exists a partition \( Z \) of \( F \) into parts so that for any \( \varepsilon > 0 \):
• For each \( Z \in \mathcal{Z} \), \( Z \) is \( k \)-thin, for \( k = \lceil 1/\varepsilon \rceil \).

• There exists \( O \subseteq U \) with \( c(O) \leq \varepsilon \), such that for all \( u \in U \setminus O \), there is some \( Z \in \mathcal{Z} \) with \( u \in \text{drop}_U(C) \). That is, \( U \setminus O \subseteq \bigcup_{Z \in \mathcal{Z}} \text{drop}_U(Z) \).

Proof. Let \( k := \lceil \frac{1}{\varepsilon} \rceil \). We will construct an arc labeling for each connected component \((Z, A)\) of the dependency graph. The arcs in the same set \( A_u \) will receive the same label.

For each directed path \((F_u, A_u)\) which begins at the root of the arborescence \((Z, A)\), we set the labels of the arcs in this path to be 0. For a directed path \((F_u, A_u)\) which does not begin at the root, let \( \ell \) be its starting node and suppose the arc entering \( \ell \) has label \( j \in \mathbb{Z}_{\geq 0} \). We set the labels of arcs in \( A_u \) to be \( j + 1 \). Since \((Z, A)\) is an arborescence, this labeling is well-defined.

For \( i \in \{0, \ldots, k-1\} \), let \( O_i \subseteq U \) be the set of up-links in \( U \) for which the arcs in \( A_u \) have a label \( j \) with \( j \equiv i \) (mod \( k \)). Since \( O_0, O_1, \ldots, O_{k-1} \) is a partition of \( U \), the average cost of the sets \( O_i \) is \( c(U)/k \). Hence, the cheapest set \( O_i \) has cost at most \( c(U) \leq c(U)/k \), and we set \( O := O_i \).

Based on the choice of \( O \), we obtain a partition of \( F \) by removing from the dependency graph all arcs in \( A_u \) where \( u \in O \). Then, the links in the connected components of the resulting directed graph form the partition \( Z \) of \( F \). By the choice of labeling, every directed path in the dependency graph after deleting these arcs intersects at most \( k-1 \) distinct sets \( A_u \). Therefore, by Lemma 10, each part \( Z \in \mathcal{Z} \) is \( k \)-thin as desired.

\[ \square \]

4.5 Dynamic Programming to find the best \( k \)-thin component

In this section, we prove that we can find the \( k \)-thin subset of hyper-links \( Z \subseteq \mathcal{L} \) minimizing \( \frac{c(Z)}{c(\text{drop}_U(Z))} \) in polynomial time using dynamic programming. A similar result was needed in [22]. However, in general Hyper-TAP, there may be exponentially many hyper-links, so we cannot enumerate over all \( \binom{\mathcal{L}}{k} \) sets efficiently. Thus, we again make use of the results in Section 4.3. In our algorithm, we work with an instance of \( \gamma \)-restricted Hyper-TAP for some constant \( \gamma \). Therefore, there are at most \( O(n^\gamma) \) hyper-links overall, which will be necessary for the efficiency of the dynamic program.

Recall that we seek to find the minimizer \( \rho^* \) of \( \frac{c(Z)}{c(\text{drop}_U(Z))} \) over all \( k \)-thin subsets \( Z \subseteq \mathcal{L} \). Using binary search, we can reduce this problem to deciding whether a given \( \rho \) is greater or less than \( \rho^* \in [0, 1] \).

For a given \( \rho \) and \( Z \subseteq \mathcal{L} \), define

\[
\text{slack}_\rho(Z) := \rho \cdot c(\text{drop}_U(Z)) - c(Z).
\]

Notice that the question of whether \( \frac{c(Z)}{c(\text{drop}_U(Z))} \leq \rho \) is equivalent to whether \( \text{slack}_\rho(Z) \geq 0 \).

Lemma 11. The maximizer among all \( k \)-thin hyper-links

\[
\max_{Z \subseteq \mathcal{L}} \{\text{slack}_\rho(Z) : Z \text{ is } k \text{-thin}\}
\]

can be found efficiently by using dynamic programming.

Proof. The proof is an extension from [22] where we can do this for links of size 2. We discuss the crux of the previous proof and show how the proof can be extended for hyper-links.

We denote by \( D_v \subseteq V \) the set of all descendants of \( v \) in \( G \), \( X[D_v] \subseteq X \) the set of hyper-links in \( X \subseteq L \) with all endpoints in \( D_v \), \( \delta_X(D_v) \subseteq X \) the set of links with at least one endpoint in \( D_v \) and at least one not in \( D_v \).

The dynamic program maintains a triple \( \{v, Y, x\} \):
• \( v \in V \) represents the subtree \( D_v \) we are considering.

• A set of hyper-links \( Y \subseteq \delta_L(D_v) \) with \(|Y| \leq k \). These are the hyper-links that do not interact solely with \( D_v \), but nevertheless effect the choices in the subproblem rooted at \( v \). However, since we are seeking a \( k \)-thin set of hyper-links, and each member of \( \delta_L(D_v) \) goes through \( v \), we have \(|Y| \leq k \).

• \( x \in \{+,-\} \): Note that since the sets \( P_u \) are disjoint for \( u \in U \), there is at most one up-link in \( \delta_U(D_v) \). If \( x = + \), the \( k \)-thin set is required to cover the edges of \( P_u \) that are under \( v \). If \( x = - \), there is no requirement.

We create a table \( \mathcal{T} \) with an entry for each such triple. The dimensions of this table are \( \mathcal{T} \subseteq V \times 2^\mathcal{L} \times \{+,-\} \), and since \(|\mathcal{L}| \leq O(n^\gamma)\), this table has polynomial size for any constant \( k \).

We will proceed to fill this table from the leaves up to the root of the tree, and use previously computed entries to ensure that we can fill each entry in polynomial time.

Let

\[
\text{slack}_\rho(Z,Y,v) := \rho \cdot c(\text{Drop}_{U[D_v]}(Z \cup Y)) - c(Z).
\]

If \( x = - \) then

\[
\mathcal{T}[v,Y,x] := \max_{Z: Z \subseteq \mathcal{L}[D_v], Z \cup Y \text{ is } k\text{-thin}} \text{slack}_\rho(Z,Y,v),
\]

and if \( x = + \), then \( \mathcal{T}[v,Y,x] \) is the solution to the same optimization problem, with the additional constraint that \( Z \cup Y \) must cover the edges of the unique up-link going through \( v \), if it exists. Let \( Z(v,Y,x) \) be the associated maximizer. Notice that the the answer to our problem is the table entry \( \mathcal{T}[r,\emptyset,-] \).

Fix an entry \( \mathcal{T}[v,Y,x] \) and let the children of \( v \) be \( v_1, v_2, \ldots, v_m \). We partition the problem into computing the \( Z[v_i,Y_i,x] \) for some choices of \( Y_i \) and \( x \). We enumerate to find the correct \( Y_i \) for each \( D[v_i] \). We use the following rule to partition \( Z \cup Y \):

• \( Z_i := Z \cap \mathcal{L}[D_{v_i}] \), which is the set of hyper-links that are contained fully in some \( D_{v_i} \)

• \( \mathcal{Y} := Y \cup \{ \ell \in Z : v \in V_\ell \} \), which is the set of hyper-links with least one endpoint in some \( D_{v_i} \) and at least one endpoint outside of that \( D_{v_i} \).

Note that the hyper-links in the set \( Z \cup Y \) should be \( k \)-thin because we are maximizing over all \( k \)-thin hyper-links. Since all hyper-links in \( \mathcal{Y} \) covers vertex \( v \), so \(|\mathcal{Y}| \leq k \). We consider the following set \( \mathcal{Y} \) which is the set of all feasible \( \mathcal{Y} \):

• \(|\mathcal{Y}| \leq k \);

• \( \mathcal{Y} \cup \delta_L(D_v) = Y \);

• Any hyperlink \( \ell \in \mathcal{Y} \) covers vertex \( v \);

• If \( x \) equals + and if the link \( u \in \delta_U(D_v) \) interacts with some subtree \( D_{v_i} \), i.e at least one endpoint of \( u \) is in some \( D_{v_i} \), then we have \( \mathcal{Y} \cap \delta_L(D_{v_i}) \neq \emptyset \).

Since \(|\mathcal{Y}| \leq k \), we can bound the size of \( \mathcal{Y} \). For \( \gamma \)-restricted hyper-links, the choice of one hyper-link is \( \sum_{j=1}^k \binom{|Y|}{j} \leq \gamma|V|^\gamma \) for constant parameter \( \gamma \). The size of \( \mathcal{Y} \) should satisfy \(|\mathcal{Y}| \leq \binom{|V|^{\gamma}}{k} \leq \gamma^k|V|^\gamma k \) for \( k \)-thin set, which is polynomially tractable. Thus, we can enumerate among all \( \mathcal{Y} \) that satisfy the above four conditions and we obtain all information we need before breaking our dynamic program into sub-problems for \( D_{v_i} \).
Let’s fix some set $\overline{Y} \in \mathcal{Y}$, then we have

$$\text{slack}_\rho(Z_{\overline{Y}}, Y, v) = \sum_{i=1}^{m} \text{slack}_\rho(Z_i, Y \cap \delta_L(D_{v_i}, v_i)) + \rho \cdot \sum_{u_i \in \text{Drop}_U(Z_i \cup \overline{Y})} c(u_i) - c(\overline{Y}/Y).$$

To compute $Z_i$, we need to determine whether $(v_i, Y_i, +)$ is feasible. There are three cases:

- $\delta_U(D_{v_i}) = \emptyset$: then $(v_i, Y_i, +)$ is infeasible due to previous definition, we only need to compute $Z(v_i, Y_i, -)$;
- The up-link $u_i$ only interacts with $v$ in vertex set $V/D_{v_i}$: then we need to compare if we want to drop $u_i$ or not, i.e. if $\text{slack}_\rho(Z_i^+, Y_i, v_i) + \rho \cdot w(u_i) \geq \text{slack}_\rho(Z_i^-, Y_i, v_i)$, then we will choose $Z(v_i, Y_i, +)$, otherwise we choose $Z(v_i, Y_i, -)$.
- The up-link $u_i$ interacts with $V/D_{v_i}$, then we choose same sign for $Z(v_i, Y_i, x)$ as $(v, Y, x)$.

Thus, we can enumerate over all choices for $\overline{Y}$ and choices of $x$ for each child $v_i$, and pick the best of these cases. Thus, we can compute $Z[v, Y, x]$ in polynomial time by relying on solutions to sub-problems on the children of $v$. By proceeding from the leaves to the root, we can compute the value of $T[r, \emptyset, -]$ and the associated maximizer as desired.

5 Node Weighted STAP

In this section, we prove Theorem 2. The formal description of our algorithm is in Algorithm 2. Recall that $\text{cov}(A)$ is the set of tree edges covered by joining the nodes of $A \subseteq R$.

**Algorithm 2** Greedy Pseudo-Spiders Algorithm for Node-Weighted STAP

**Input:** An instance of NW-STAP with graph $G = (V, E)$, tree $T = (R, E(T))$, links $L = E \setminus E(T)$, and $c: V \setminus R \to \mathbb{R}_{\geq 0}$.

**Output:** A solution $(S, F)$ with $c(S) \leq O(\log^2(|R|)OPT$.

1. Initialize $S := \emptyset$ and $F := \emptyset$.
2. Initialize $U := E(T)$.
3. while $U \neq \emptyset$:
   - Find the pseudo-spider $(S', F')$ approximately minimizing the ratio $\frac{c(S')}{|U \setminus \text{cov}(S' \cap R)|}$ (Lemma 14).
   - Add this pseudo-spider to our solution: $S := S \cup S'$ and $F := F \cup F'$.
   - Contract the covered tree edges: $U := U \setminus \text{cov}(S', F')$.
4. Return the feasible solution $(S, F)$. 

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5.1 Analysis

Recall that in each iteration of the algorithm, we obtain a pseudo-spider $(S', F')$ which approximately minimizes the ratio $\frac{c(S')}{|U \cap \text{cov}(S' \cup A)|}$, where $A$ is the set of nodes covered by the pseudo-spider. We begin by showing how one can do this in polynomial time. The main idea is to enumerate over all choices for the head $h \in V \setminus R$ of the pseudo-spider, and as well over one choice of a foot $v \in R$. There are at most $|V|$ choices for each, hence we may assume that our algorithm knows the correct choice of head $h$ and one correct foot $v$.

We will find the other feet using an algorithm for submodular maximization subject to a knapsack constraint. Since a pseudo-spider is determined by the choice of its head and feet, this allows us to find the minimum-ratio pseudo-spider after several node-weighted shortest path computations.

**Definition 8.** Let $f$ be a set function $f : 2^{|V|} \to \mathbb{R}$. We say that $f$ is **monotone** if $f(A) \leq f(B)$ for all $A \subseteq B \subseteq [n]$. We say that $f$ is **submodular** if $f(A + x) - f(A) \geq f(B + x) - f(B)$ for all $A \subseteq B$ and $x \in [n] \setminus B$.

**Lemma 12.** Fix some vertex $v \in R$. Let the function $\text{cov}_v : 2^{R \setminus v} \to \mathbb{Z}$ be defined by

$$\text{cov}_v(P) := |\text{cov}(P \cup v)|.$$

Then $f(P) := |\text{cov}_v(P)|$ is monotone and submodular.

**Proof.** Suppose that $A \subseteq B \subseteq R \setminus v$. Clearly, $f(A) \leq f(B)$, so $f$ is monotone.

To show that $f$ is submodular, we fix $x \in (R \setminus v) \setminus B$. We want to show that $f(A + x) - f(A) \geq f(B + x) - f(B)$. This amounts to showing that $|\text{cov}(A + v + x)| - |\text{cov}(A + v)| \geq |\text{cov}(B + v + x)| - |\text{cov}(B + v)|$. For ease of notation, let $A' := A + v$ and $B' := B + v$. In fact, we will show the stronger claim that $|\text{cov}(A' + x) \setminus \text{cov}(A')| \geq |\text{cov}(B' + x) \setminus \text{cov}(B')|$. Note that $A'$ and $B'$ are non-empty. Thus, let $\emptyset \neq T_{A'} \subseteq R$ and $\emptyset \neq T_{B'} \subseteq R$ be the subtrees of $T$ which are covered by the nodes in $A'$ and $B'$ respectively. Notice that $|\text{cov}(A' + x) \setminus \text{cov}(A')| \geq |\text{cov}(B' + x) \setminus \text{cov}(B')|$ as desired.

**Lemma 12** allows us to exploit the following algorithm for submodular maximization, due to Srividenko [21].

**Lemma 13 (Srividenko [21]).** There is a polynomial time $(1 - \frac{1}{e})$-approximation algorithm to maximize a monotone submodular function subject to a knapsack constraint.

**Lemma 14.** Fix a head $h \in V \setminus R$ and one foot $p \in R$. There is an $(\frac{e - 1}{2e})$-approximation algorithm to find the minimum-ratio pseudo-spider with head $h$ and containing foot $p$.

**Proof.** First we assume that we know the cost of the minimum-ratio pseudo-spider, i.e. the sum of the node-weighted shortest paths from head to all feet. For a fixed cost, the problem of finding the minimum-ratio pseudo-spider is equivalent to finding the maximum number of edges of $T$ covered by a set of feet subject to a knapsack constraint, where each potential foot $x$ has cost equal to the length of the shortest node-weighted path from $h$ to $x$ (not including the cost of $h$). Using Lemma 13 we can find a $(1 - \frac{1}{e})$ approximate set of feet for this problem. This means that we can find a spider covering least $(1 - \frac{1}{e})$ of the edges covered in the optimal pseudo-spider under the fixed head and foot.

To fix the issue that we don’t know the cost of the minimum-ratio pseudo-spider in advance, we can use a doubling search: we guess the cost to be $1, 2, 4, \ldots, B$, where $B$ is the sum of total
cost on Steiner nodes. Suppose the true cost of the minimum-ratio pseudo-spider is $C$ that satisfies $2^m < C < 2^{m+1}$. Consider the iteration when we guess the cost to be $2^{m+1}$. The number of edges covered by the pseudo-spider in this iteration is at least $(1 - \frac{1}{e})$ times the number of edges covered in the optimum ratio pseudo-spider and the cost of the pseudo-spider that the algorithm finds is at most twice that of the optimum ratio pseudo-spider. Thus the approximation factor of this pseudo-spider of the optimum ratio is $\left(\frac{e}{2e} - \frac{1}{2e}\right)$.

For the running time, there are $\log B$ rounds in total, which is bounded by the complexity of reading the input of the cost value. In each round, the running time is polynomial due to Lemma 14.

Suppose $(S, E)$ is an instance of Set Cover with where $S \subseteq 2^E$ and each set $S \in S$ has some non-negative cost $c_S$. It is well known that the greedy algorithm, which chooses in each iteration a set $P \in S$ with minimum ratio of “cost per coverage” achieves an approximation ratio of $O(\log |E|)$. That is, we choose in each iteration a set $P$ satisfying

$$P = \arg \min_{S \in S} \left\{ \frac{c(S)}{\text{number of uncovered elements covered by } S} \right\}.$$ 

Furthermore, if in each step $P$ is chosen to be an $\alpha$-approximate minimizer, then the approximation ratio is bounded above by $O(\alpha \log |E|)$.

**Proof of Theorem 2.** We show that Algorithm 2 is an $O\left(\log^2 |R|\right)$-approximation for node-weighted STAP, and runs in polynomial time.

Let $OPT$ denote the cost of the optimal augmentation. By Theorem 4, there is a solution consisting only of pseudo-spiders with cost at most $O(\log(|R|))OPT$.

In each iteration, Algorithm 2 selects a pseudo-spider which has a cost-ratio at most $\frac{e-1}{2e}$ times that of the minimum cost-ratio pseudo spider. Thus by the standard analysis of the greedy algorithm for Set Cover, after all edges of the tree are covered, the total cost $c(S)$ of our algorithm is at most $O(|E(T)|) \times \log |E(T)|$ times the cost of the optimal pseudo spider solution. Thus, overall, our algorithm returns a feasible solution $(S, F)$ with cost

$$c(S) \leq O(\log |E|) OPT \leq O(\log^2(|R|))OPT.$$ 

Lemma 14 shows that we can implement each iteration of the while loop in polynomial time. Since in each iteration some tree edge is covered, the number of iterations is bounded by $|E(T)|$. Thus the algorithm runs in polynomial time overall.

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