Agent-based and continuous models of hopper bands for the Australian plague locust: How resource consumption mediates pulse formation and geometry

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Supporting Information

S3 Appendix Formulas for $N, c, R^+, R^−$

In S2 Appendix we show that there exists a traveling wave solution to the PDE model, Eq (9) in the main text. We now characterize this solution with explicit formulas that relate $N, c, R^+, \text{ and } R^−$. Given any two of these variables and model parameter values, the following equations determine the other two variables:

\[
\frac{I_1}{I_2} = \frac{c}{v - c}
\]

\[
N = \frac{c}{v - c} \ln \left(\frac{R^+}{R^-}\right)
\]

where

\[
I_1 = \int_{R^-}^{R^+} \frac{k_{sm}(R)}{R} dR, \quad \text{and} \quad I_2 = \int_{R^-}^{R^+} \frac{k_{ms}(R)}{R} dR.
\]

We prove that equivalent equations hold for the nondimensionalized PDE model in S2 Appendix, reproduced here

\[
R_t = -SR \\
S_t = -k_{sm}S + k_{ms}M \\
M_t = k_{sm}S - k_{ms}M - M_x.
\]

Theorem 1. Given a traveling wave solution to Eq (PDE). Then the quantities $N, c, R^+, R^-\text{ satisfy}$

\[
\frac{I_1}{I_2} = \frac{c}{1 - c}
\]

\[
N = \frac{c}{1 - c} \ln \left(\frac{R^+}{R^-}\right)
\]

where $I_1, I_2$ are given by Eq (3) with the nondimensional versions of $k_{sm}, k_{ms}$. 
Proof. A traveling wave solution satisfies the ODE in S2 Appendix reproduced below

\[ R_{\xi} = \frac{1 - c}{c} \rho R \]

\[ \rho_{\xi} = \left( \frac{k_{sm}}{c} - \frac{k_{ms}}{1 - c} \right) \rho. \]  

(ODE)

Since \( R_{\xi} > 0 \) in the first quadrant of the phase plane, we can write \( \rho \) as a function of \( R \) along any heteroclinic. Thus \( \frac{d\rho}{d\xi} = \frac{d\rho}{dR} \frac{dR}{d\xi} \) so integrating along a heteroclinic, we have

\[ \int_{R_{\xi}}^{R_{\xi}^+} \frac{\rho_{\xi}}{R_{\xi}} \, dR = \int_{R_{\xi}}^{R_{\xi}^+} \frac{d\rho}{dR} \, dR = \rho(R^+) - \rho(R^-) = 0. \]

We also have

\[ \int_{R_{\xi}}^{R_{\xi}^+} \frac{\rho_{\xi}}{R_{\xi}} \, dR = \int_{R_{\xi}}^{R_{\xi}^+} \frac{c}{1 - c} \frac{K(R)}{R} \, dR = \frac{1}{1 - c} I_1 - \frac{c}{(1 - c)^2} I_2 \]

where \( K(R) \) is given in Eq (??). Therefore we have proved Eq (4).

Dividing the equation for \( R_{\xi} \) in Eq (ODE) by \( R \) and integrating the left hand side, we have

\[ \int_{-\infty}^{\infty} \frac{R_{\xi}}{R} \, d\xi = \ln \left( \frac{R^+}{R^-} \right). \]

Meanwhile, the right hand side gives us

\[ \int_{-\infty}^{\infty} \frac{1 - c}{c} \rho \, d\xi = \frac{1 - c}{c} N, \]

proving Eq (5).

In fact, these equalities can be used to prove monotonicity of the mass-speed relation.

**Theorem 2.** Fix \( R^+ \). Then the speed \( c \) is a strictly increasing function of mass \( N \) and a strictly decreasing function of \( R^- \).

Proof. Let \( s = \frac{c}{1 - c} \). Then Eqs (4)-(5) become

\[ s = \frac{I_1}{I_2} \]

(6)

\[ s = \frac{N}{\ln \left( \frac{R^+}{R^-} \right)}. \]

(7)

First, we show \( \frac{ds}{dR^-} < 0 \). Taking the derivative of Eq (6), we get

\[ \frac{ds}{dR^-} = \frac{I_2 \frac{dI_1}{dR^-} - I_1 \frac{dI_2}{dR^-}}{I_2^2} = \frac{I_1 k_{ms}(R^-) - I_2 k_{sm}(R^-)}{R^- I_2^2}. \]

(8)

We will show the numerator, call it \( I \), is negative. Dividing \( I \) by \( k_{sm}(R^-) \cdot k_{ms}(R^-) \), we get

\[ \frac{I_1}{k_{sm}(R^-)} - \frac{I_2}{k_{ms}(R^-)} = \int_{R^-}^{R^+} \frac{1}{R} \left( k_{sm}(R) \frac{k_{ms}(R^-)}{k_{ms}(R^-)} - k_{ms}(R) \frac{k_{sm}(R^-)}{k_{sm}(R^-)} \right) \, dR. \]

(9)

The integrand \( \frac{k_{sm}(R)}{k_{ms}(R^-)} - \frac{k_{ms}(R)}{k_{sm}(R^-)} \) is less than zero for \( R^- < R < R^+ \). To see this, note that at \( R = R^- \) this integrand is 0. Also we know that \( k_{sm} \) is decreasing in \( R \) and \( k_{ms} \)
is increasing in \( R \), so the first term decreases and the second term (without the negative sign) increases. Then the integrand is indeed negative for all \( R > R^- \).

Second, we will show \( \frac{ds}{dN} > 0 \). Differentiating Eqs (6) and (7) respectively, we get

\[
\frac{dS}{dN} = \frac{\ln \left( \frac{R^+}{R^-} \right) + \frac{N}{R} \left( \frac{dR^-}{dN} \right)}{\ln \left( \frac{R^+}{R^-} \right)^2} \quad (10)
\]

and

\[
\frac{dS}{dN} = \frac{dR^-}{dN} \left[ \frac{I_1 k_{ms}(R^-) - I_2 k_{sm}(R^-)}{I_2^2 R^-} \right]. \quad (11)
\]

Setting these equal to each other, we obtain

\[
\frac{dR^-}{dN} = \frac{R^- \ln \left( \frac{R^+}{R^-} \right)}{\ln \left( \frac{R^+}{R^-} \right)^2 \left[ \frac{I_1 k_{ms}(R^-) - I_2 k_{sm}(R^-)}{I_2^2} \right] - N}. \quad (12)
\]

Substituting Eq (12) into either Eq (10) or Eq (11), we obtain

\[
\frac{ds}{dN} = \frac{\ln \left( \frac{R^+}{R^-} \right) \left[ I_1 k_{ms}(R^-) - I_2 k_{sm}(R^-) \right]}{\ln \left( \frac{R^+}{R^-} \right)^2 \left[ I_1 k_{ms}(R^-) - I_2 k_{sm}(R^-) \right] - N I_2^2} = \frac{\ln \left( \frac{R^+}{R^-} \right) \mathcal{I}}{\ln \left( \frac{R^+}{R^-} \right)^2 \mathcal{I} - N I_2^2}. \quad (13)
\]

We have already shown that \( \mathcal{I} < 0 \). Because \( R^+ > R^- \), the numerator is negative. The denominator is also negative, so have shown that \( \frac{ds}{dN} > 0 \). \( \square \)