WEYL-PEDERSEN CALCULUS FOR SOME SEMIDIRECT PRODUCTS OF NILPOTENT LIE GROUPS

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Abstract. For certain nilpotent real Lie groups constructed as semidirect products, algebras of invariant differential operators on some coadjoint orbits are used in the study of boundedness properties of the Weyl-Pedersen calculus of their corresponding unitary irreducible representations. Our main result is applicable to all unitary irreducible representations of arbitrary 3-step nilpotent Lie groups.

1. Introduction

Weyl quantization for representations of Lie groups of various types has been an area of quite active research (see for instance [Ca07], [Ca13] and the references therein). In this connection, the aim of the present paper is to study $L^2$-boundedness properties of a certain operator calculus for unitary representations of nilpotent Lie groups of lower nilpotence step, including all 3-step nilpotent Lie groups. More specifically, the deep work of N.V. Pedersen ([Pe94]) shows that for each unitary irreducible representation of a nilpotent Lie group one can set up a Weyl correspondence between tempered distributions on any coadjoint orbit and unbounded linear operators in the representation space of any unitary irreducible representation associated with that coadjoint orbit. This correspondence has remarkable properties and in the special case of the Heisenberg group it recovers the pseudo-differential Weyl calculus which has been widely used in the theory of PDEs.

The construction of the Weyl-Pedersen calculus is to some extent noncanonical, in that it depends on the choice of a Jordan-Hölder basis in the nilpotent Lie algebra under consideration. Nevertheless, the construction is fully canonical in the case of the irreducible representations that are square integrable modulo the center, that is, for the representations associated to the flat coadjoint orbits, and one can then even prove that it is covariant with respect to the coadjoint action (see [BB11]). For the representations of this type, we also recently characterized the maximal space of smooth functions on the coadjoint orbit that is invariant under the coadjoint action and gives rise to bounded linear operators via the Weyl-Pedersen calculus (see [BB13a]). The characterization involves growth properties of the functions expressed in terms of the differential operators on the coadjoint orbit which are invariant under the coadjoint action. It is then natural to ask about the coadjoint
orbits which are not necessarily flat. In the present paper we will take a first step in that direction, as we will now explain.

For the purposes of the irreducible representation theory, one may restrict the attention to the groups with 1-dimensional center and moreover it is reasonable to begin by considering the groups of a lower nilpotence step. Note that the 2-step nilpotent Lie algebras having 1-dimensional center are precisely the Heisenberg algebras, and the coadjoint orbits of the Heisenberg groups are always flat. So the first nontrivial case to consider is the one mentioned in the following problem:

**Problem 1.1.** Investigate the $L^2$-boundedness properties of the Weyl-Pedersen calculus on coadjoint orbits of 3-step nilpotent Lie groups.

We will address that problem in Corollary 4.3 below, by relying on some results announced in [BBP13]. Specifically, we will prove that the aforementioned result from [BB13a] carries over to coadjoint orbits of any 3-step nilpotent Lie group, thereby establishing a version of the Calderón-Vaillancourt theorem for the Weyl-Pedersen calculus, in which the role of the partial derivatives is played by invariant differential operators on the coadjoint orbits of 3-step nilpotent Lie groups.

**General notation.** Throughout this paper, Lie groups are finite-dimensional and are denoted by upper case Roman letters, and their corresponding Lie algebras are denoted by the corresponding lower case Gothic letters. For any real Lie algebra $g$ we denote by $U(g)$ the complexification of the universal associative enveloping algebra of $g$ (see for instance [CG90]), by $\text{Aut} g$ the automorphism group of $g$, and by $\text{Der} g$ the Lie algebra of all derivations of $g$. For any set $A$, the notation $B \subset A$ means that $B$ is a subset of $A$ and $B \neq A$.

2. **Weyl-Pedersen calculus**

In this preliminary section we recall the Weyl-Pedersen calculus set forth in [Pe94]. More details on this calculus, additional references, and connections with other operator calculi can be found for instance in [BBP13].

Let $g$ be any nilpotent Lie algebra of dimension $m \geq 1$ with its corresponding nilpotent Lie group $G = (g, \cdot)$ whose multiplication is defined by the Baker-Campbell-Hausdorff formula. Select any Jordan-Hölder basis $\{X_1, \ldots, X_m\}$ in $g$. So for $j = 1, \ldots, m$ if we define $g_j := \text{span} \{X_1, \ldots, X_j\}$ then $[g, g_j] \subseteq g_{j-1}$, where $g_0 := \{0\}$. Let $\pi : G \to B(\mathcal{H})$ be a unitary representation associated with a coadjoint orbit $O \subseteq g^*$ via Kirillov’s correspondence. Pick $\xi_0 \in O$ with its corresponding coadjoint isotropy algebra $g_{\xi_0} := \{X \in g \mid [X, g] \subseteq \text{Ker} \xi_0\}$, and define the set of jump indices

$$e := \{j \mid X_j \not\in g_{j-1} + g_{\xi_0}\}.$$

Denoting $g_e := \text{span} \{X_j \mid j \in e\}$, one has $g = g_{\xi_0} \oplus g_e$, and the mapping $O \to g_e^*$, $\xi \mapsto \xi|_{g_e}$, is a diffeomorphism. Hence one can define an orbital Fourier transform $\mathcal{S}'(O) \to \mathcal{S}'(g_e^*)$, $a \mapsto \hat{a}$ which is a linear topological isomorphism and for every $a \in \mathcal{S}(O)$ one has

$$\forall X \in g_e^*, \quad \hat{a}(X) = \int_O e^{-i\langle \xi, X \rangle} a(\xi) d\xi.$$

Here we have the Lebesgue measure $dx$ on $g_e$ corresponding to the basis $\{X_j \mid j \in e\}$ and $d\xi$ is the Borel measure on $O$ for which the above diffeomorphism $O \to g_e^*$ is a
measure preserving mapping and the Fourier transform \( L^2(\mathcal{O}) \to L^2(g_e) \) is unitary. The inverse of this orbital Fourier transform is denoted by \( a \mapsto \hat{a} \).

With the notation above, the \textit{Weyl-Pedersen calculus} associated to the unitary irreducible representation \( \pi \) is

\[
\text{Op}_\pi : \mathcal{S}(\mathcal{O}) \to \mathcal{B}(\mathcal{H}), \quad \text{Op}_\pi(a) = \int_{g_e} \hat{a}(x) \pi(x) dx.
\]

The space of smooth vectors \( \mathcal{H}_\infty := \{ v \in \mathcal{H} \mid \pi(\cdot)v \in C^\infty(g,\mathcal{H}) \} \) is dense in \( \mathcal{H} \) and has the natural topology of a nuclear Fréchet space with the space of the antilinear functionals denoted by \( \mathcal{H}_{-\infty} := \mathcal{H}_\infty^* \) (with the strong dual topology). One can show that the Weyl-Pedersen calculus extends to a linear bijective mapping

\[
\text{Op}_\pi : \mathcal{S}'(\mathcal{O}) \to \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty}), \quad (\text{Op}_\pi(a) v \mid w) = (\hat{a}, (\pi(\cdot)v \mid w))
\]

for \( a \in \mathcal{S}'(\mathcal{O}), v, w \in \mathcal{H}_\infty \), where in the left-hand side \( (\cdot \mid \cdot) \) denotes the extension of the scalar product of \( \mathcal{H} \) to the sesquilinear duality pairing between \( \mathcal{H}_\infty \) and \( \mathcal{H}_{-\infty} \).

It turned out in [BBH13] and [BBP13] that the invariant differential operators on coadjoint orbits are a quite effective tool in the study of boundedness properties of the Weyl-Pedersen calculus. Specifically, one defines \( \text{Diff}(\mathcal{O}) \) as the space of all linear differential operators \( D \) on \( \mathcal{O} \) that are \textit{invariant} to the coadjoint action, in the sense that

\[
(\forall x \in g)(\forall a \in C^\infty(\mathcal{O})) \quad D(a \circ \text{Ad}^*_G(x) | \mathcal{O}) = (Da) \circ \text{Ad}^*_G(x) | \mathcal{O}.
\]

By using these differential operators one can introduce the Fréchet space of symbols

\[
\mathcal{C}^\infty_b(\mathcal{O}) = \{ a \in C^\infty(\mathcal{O}) \mid Da \in L^\infty(\mathcal{O}) \text{ for all } D \in \text{Diff}(\mathcal{O}) \},
\]

with the topology given by the seminorms \( \{ a \mapsto \| Da \|_{L^\infty(\mathcal{O})} \}_{D \in \text{Diff}(\mathcal{O})} \). This space of symbols will be needed in our main result (Theorem 4.2 below).

### 3. Some properties of coadjoint orbits

In this section we study some simple properties of coadjoint orbits (Propositions 3.1 and 3.2) that we were not able to find in the literature in this degree of generality, so we include full details of their proofs. These facts will be needed in the proof of Theorem 4.2.

**Proposition 3.1.** Let \( G \) be a Lie group with the Lie algebra \( g \) and assume that we have a closed normal subgroup \( G_0 \trianglelefteq G \) whose Lie algebra is denoted by \( g_0 \). Pick \( \xi \in g^* \) with its coadjoint orbit \( \mathcal{O}_\xi^G := \text{Ad}^*_G(\xi) \) and the \( G \)-coadjoint isotropy group

\[
G_\xi := \{ g \in G \mid \text{Ad}^*_G(g)\xi = \xi \}.
\]

Set \( \xi_0 := \xi|_{g_0} \in g^*_0 \) and consider its coadjoint orbit \( \mathcal{O}_{\xi_0}^{G_0} := \text{Ad}_{G_0}(\xi_0) \) and the \( G_0 \)-coadjoint isotropy group

\[
(G_0)_{\xi_0} := \{ g \in G_0 \mid \text{Ad}^*_{G_0}(g)\xi_0 = \xi_0 \}.
\]

Define \( \rho : g^* \to g_0^*, \eta \mapsto \eta|_{g_0} \). Then the following assertions are equivalent:

1. The restriction mapping \( \rho \) gives a well-defined bijection \( \mathcal{O}_\xi^G \to \mathcal{O}_{\xi_0}^{G_0} \).
2. We have:
   (a) \( (G_0)_{\xi_0} = G_0 \cap G_\xi \);
   (b) \( G = G_0 \cdot G_\xi \).
3. The restriction mapping \( \rho \) gives a well-defined diffeomorphism \( \mathcal{O}_\xi^G \to \mathcal{O}_{\xi_0}^{G_0} \).
Proof. It is clear that \[ (3) \Rightarrow (1) \], so it suffices to prove \[ (1) \Rightarrow (2) \Rightarrow (3) \].

“\[ (1) \Rightarrow (2) \]” For proving \[ (2\alpha) \], note that we always have \((G_0)_{\xi_0} \supseteq G_0 \cap G_\xi\). For the converse inclusion let \(g \in (G_0)_{\xi_0}\) arbitrary, hence \(g \in G_0\) and \(\xi_0 \circ \text{Ad}_{G_0}(g) = \xi_0\). Since \(G_0 \subseteq G\), it follows that \(g_0 \leq g\). Then we have

\[
\xi_0 = \xi_0 \circ \text{Ad}_{G_0}(g) = (\xi|_{g_0}) \circ (\text{Ad}_G(g)|_{g_0}) = (\xi \circ \text{Ad}_G(g))|_{g_0}
\]

hence we get \((\xi \circ \text{Ad}_G(g))|_{g_0} = \xi|_{g_0}\). Since the restriction mapping \(\rho|_{\mathcal{O}_\xi} \) is injective, it then follows that \(\xi \circ \text{Ad}_G(g) = \xi\), that is, \(g \in G_\xi\). Therefore \(g \in G_0 \cap G_\xi\).

For proving \( (2\beta) \), let \(g \in G\) arbitrary. We have \(\rho(\mathcal{O}_\xi^G) = \mathcal{O}_{\xi_0}^{G_0}\), hence there exists \(g_0 \in G_0\) such that \((\xi \circ \text{Ad}_G(g^{-1}))|_{g_0} = \xi_0 \circ \text{Ad}_{G_0}(g_0^{-1})\). Therefore \((\xi \circ \text{Ad}_G(g^{-1}))|_{g_0} = (\xi \circ \text{Ad}_G(g_0^{-1}))|_{g_0}\) and then by the hypothesis that \(\rho|_{\mathcal{O}_\xi} \) is also injective we get \(\xi \circ \text{Ad}_G(g_0^{-1}) = \xi \circ \text{Ad}_G(g_0^{-1})\). This implies \(\xi \circ \text{Ad}_G(g_0^{-1}) = \xi\), hence \(g_0^{-1}g \in G_\xi\), that is, \(g \in G_0G_\xi\). Thus \(G \subseteq G_0\), and the converse inclusion is obvious.

“\[ (2) \Rightarrow (3) \]” If both \( (2\alpha) \) and \( (2\beta) \) hold true, then there exists the commutative diagram

\[
\begin{array}{c}
\mathcal{O}_\xi^G \\
\downarrow^\rho|_{\mathcal{O}_\xi^G} \\
(\mathcal{O}_{\xi_0}^{G_0})/\mathcal{O}_{\xi_0} \end{array}
\begin{array}{c}
\mathcal{O}_{\xi_0}^{G_0} \\
\downarrow_{\rho|_{\mathcal{O}_\xi}^{G_0}} \\
(\mathcal{O}_{\xi_0}^{G_0}) \end{array}
\begin{array}{c}
G_0/G_\xi \\
\downarrow^\beta \\
G_0/(G_0 \cap G_\xi) \\
\downarrow^{\text{id}} \\
G_0/(G_0)_{\xi_0}
\end{array}
\]

where the vertical arrows are the diffeomorphisms defined by the coadjoint actions of the groups \(G (= G_0G_\xi)\) and \(G_0\), respectively, and whose bottom arrow \(\beta\) is given by \(g(G_0 \cap G_\xi) \mapsto gG_\xi\). It is straightforward to check that \(\beta\) is a bijective immersion, hence either it is a diffeomorphism, or its differential fails to be surjective at every point. The latter variant is impossible because of Sard’s theorem, hence \(\beta\) is a diffeomorphism. (Alternatively, since \(G_0 \subseteq G\), it follows by the second isomorphism theorem for groups that we have a natural isomorphism of Lie groups \(G/G_0 \cong G_\xi/(G_0 \cap G_\xi)\), and this implies \(\dim G - \dim G_\xi = \dim G_\xi - \dim(G_0 \cap G_\xi)\). Then \(\beta\) is a bijective immersion between manifolds with equal dimensions, hence \(\beta\) is a diffeomorphism.)

Note that the inclusion map \(\rho(\mathcal{O}_\xi^G) \hookrightarrow \mathcal{O}_{\xi_0}^{G_0}\) in the above diagram is actually bijective since for every \(g \in G = G_0G_\xi\) there exists \(g_0 \in G_0\) such that \(g \in g_0G_\xi\), hence \(\text{Ad}_G(g)\xi = \text{Ad}_G(g_0)\xi\), and then

\[
\rho(\text{Ad}_G(g)\xi) = \rho(\text{Ad}_G(g_0)\xi) = (\text{Ad}_G^*(g_0)\xi)|_{g_0} = \text{Ad}_G^*(g_0)\xi_0 \in \mathcal{O}_{\xi_0}^{G_0}.
\]

We thus get the well-defined onto map \(\rho|_{\mathcal{O}_\xi^G} : \mathcal{O}_\xi^G \to \mathcal{O}_{\xi_0}^{G_0}\). Since in the above commutative diagram the vertical arrows and the bottom ones are diffeomorphisms, it follows that \(\rho|_{\mathcal{O}_\xi^G} : \mathcal{O}_\xi^G \to \mathcal{O}_{\xi_0}^{G_0}\) is also a diffeomorphism, and this concludes the proof.

Some implications from the statement of the following simple result can be obtained by the disintegration of restrictions of irreducible representations, but we give here an alternative, direct proof.

**Proposition 3.2.** Let \(G = (\mathfrak{g}, \cdot)\) be a nilpotent Lie group with an irreducible representation \(\pi : G \to \mathcal{B}(\mathcal{H})\) associated with the coadjoint orbit \(\mathcal{O} \subset \mathfrak{g}^*\). If we consider an ideal \(\mathfrak{h} \subseteq \mathfrak{g}\) and the corresponding normal subgroup \(H = (\mathfrak{h}, \cdot)\) of \(G\), then the following assertions are equivalent:
(1) The restricted representation $\pi|_H: H \to \mathcal{B}(\mathcal{H})$ is irreducible.
(2) The mapping $g^* \mapsto h^*, \xi \mapsto \xi|_h$ gives a bijection of $\mathcal{O}$ onto a coadjoint orbit of $H$, which will be denoted $\mathcal{O}|_h$.
(3) For some/any $\xi_0 \in \mathcal{O}$ we have $\mathfrak{g} = \mathfrak{g}_{\xi_0} + \mathfrak{h}$.
(4) For some/any $\xi_0 \in \mathcal{O}$ we have $\dim(h/(\mathfrak{g}_{\xi_0} \cap \mathfrak{h})) = \dim(\mathfrak{g}/\mathfrak{g}_{\xi_0})$.
(5) For some/any Jordan-Hölder sequence $\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ with $\mathfrak{g}_k = \mathfrak{h}$ for $k = \dim \mathfrak{h}$, the corresponding set of jump indices of $\mathcal{O}$ is contained in $\{1, \ldots, k\}$.

If this is the case, then the irreducible representation $\pi|_H$ is associated with the coadjoint orbit $\mathcal{O}|_h$ of $H$.

Proof. First note that (3) $\iff$ (4) since there exists the canonical linear isomorphism $(\mathfrak{g}_{\xi_0} + \mathfrak{h})/\mathfrak{g}_{\xi_0} \simeq \mathfrak{h}/(\mathfrak{g}_{\xi_0} \cap \mathfrak{h})$, while (3) $\iff$ (5) by the definition of the jump indices.

We will prove by induction on $\dim(\mathfrak{g}/\mathfrak{h})$ that (1) $\iff$ (2) $\iff$ (3). This follows at once by [CG90, Th. 2.5.1(b)] if $\dim(\mathfrak{g}/\mathfrak{h}) = 1$, since in this case we have $\mathfrak{g}_{\xi_0} \not\subseteq \mathfrak{h}$ if and only if $\mathfrak{g} = \mathfrak{g}_{\xi_0} + \mathfrak{h}$. Now assume $\dim(\mathfrak{g}/\mathfrak{h}) \geq 2$. We will prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1). Since $\mathfrak{h} \subseteq \mathfrak{g}$, there exists a Jordan-Hölder sequence as in Assertion (5). If (1) holds true, then also the representation $\pi|_{G_{m-1}}: G_{m-1} \to \mathcal{B}(\mathcal{H})$ is irreducible and its restriction to $H$ is $\pi|_H$. Therefore, by using [CG90, Th. 2.5.1(b)] we see that the restriction mapping $\mathfrak{g}^* \to \mathfrak{g}_{m-1}^*$ gives a bijection from $\mathcal{O}$ onto a coadjoint $G_{m-1}$-orbit $\mathcal{O}|_{\mathfrak{g}_{m-1}}$ of $G_{m-1}$. Then by the induction hypothesis it follows that the restriction mapping $\mathfrak{g}_{m-1}^* \to \mathfrak{h}^*$ gives a bijection from $\mathcal{O}|_{\mathfrak{g}_{m-1}}$ onto a coadjoint $H$-orbit $\mathcal{O}^1|_{\mathfrak{g}_{m-1}}|_h$. Therefore Assertion (2) holds true. Moreover, if we know that Assertion (2) holds true, then Proposition [8.1.2a] shows that we have $\mathcal{O}|_h \cong H/(H \cap G_{\xi_0})$, and then

$$\dim(\mathfrak{g}/\mathfrak{g}_{\xi_0}) = \dim \mathcal{O} = \dim(\mathcal{O}|_h) = \dim(\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{g}_{\xi_0}))$$

hence we get Assertion (1), and we saw at the very beginning of the proof that (1) $\iff$ (5). Now assume that Assertion (5) holds true. Then we have

$$\mathfrak{h} \subseteq \mathfrak{g}_{m-1} \subset \mathfrak{g} = \mathfrak{h} + \mathfrak{g}_{\xi_0}$$

hence $\mathfrak{g}_{\xi_0} \not\subseteq \mathfrak{g}_{m-1}$. Now, using [CG90, Th. 2.5.1] again along with the notation $G_{m-1} = (\mathfrak{g}_{m-1}, \cdot)$, it follows that the representation $\pi|_{G_{m-1}}: G_{m-1} \to \mathcal{B}(\mathcal{H})$ is irreducible. Furthermore, since $\mathfrak{h} \subset \mathfrak{g}_{m-1} \subset \mathfrak{h} + \mathfrak{g}_{\xi_0}$, it follows that

$$\mathfrak{g}_{m-1} = \mathfrak{h} + (\mathfrak{g}_{m-1} \cap \mathfrak{g}_{\xi_0})$$

If we denote $\xi_{0,m-1} := \xi|_{\mathfrak{g}_{m-1}}$ then it is clear that $\mathfrak{g}_{m-1} \cap \mathfrak{g}_{\xi_0}$ is contained in the $G_{m-1}$-coadjoint isotropy algebra

$$(\mathfrak{g}_{m-1})_{\xi_{0,m-1}} = \{x \in \mathfrak{g}_{m-1} \mid \langle \xi, [x, \mathfrak{g}_{m-1}] \rangle = \{0\}\}$$

hence by the above equality we get $\mathfrak{g}_{m-1} = \mathfrak{h} + (\mathfrak{g}_{m-1})_{\xi_{0,m-1}}$. Since $\dim(\mathfrak{g}_{m-1}/\mathfrak{h}) < \dim(\mathfrak{g}/\mathfrak{h})$, we can apply the induction hypothesis for the irreducible representation $\pi|_{G_{m-1}}: G_{m-1} \to \mathcal{B}(\mathcal{H})$. It thus follows that $(\pi|_{G_{m-1}})|_H$ is an irreducible representation. Since $(\pi|_{G_{m-1}})|_H = \pi|_H$, we see that Assertion (1) holds true, and this completes the induction step. \qed
4. Weyl-Pedersen calculus for some semidirect products

We established in [BB13a] a quite natural version of the Calderón-Vaillancourt theorem for the Weyl-Pedersen calculus of the unitary irreducible representations of nilpotent Lie groups which are square integrable modulo the center of the group. In this section we use the above Proposition 3.2 along with some results of [BBP13] in order to prove that the same statement holds true for some unitary irreducible representations of more general nilpotent Lie groups, irrespective of whether these representations are square integrable or not.

For the statement of Theorem 4.2 we need the following remark.

Remark 4.1. Let \((g_0, \omega)\) be any symplectic nilpotent Lie algebra. This means that \(g_0\) is a nilpotent Lie algebra and \(\omega: g_0 \times g_0 \to \mathbb{R}\) is a skew-symmetric non-degenerate bilinear functional satisfying the 2-cocycle condition

\[
(\forall x, y, z \in g_0) \quad \omega([x, y, z]) + \omega([y, z, x]) + \omega([z, x, y]) = 0.
\]

Denote by \(g := \mathbb{R} + \omega g_0\) the corresponding 1-dimensional central extension, that is, \(g = \mathbb{R} \times g_0\) as a vector space and the Lie bracket of \(g\) is defined by

\[
(\forall t, s \in \mathbb{R})(\forall x, y \in g_0) \quad [(t, x), (s, y)]_g := (\omega(x, y), [x, y]_{g_0}).
\]

We denote by \(G_0\) and \(G\) the Lie groups obtained from \(g_0\) and \(g\) by using the multiplication given by the Baker-Campbell-Hausdorff formula. We define

\[
\text{Aut}(g_0, \omega) := \{\alpha \in \text{Aut} g_0 \mid (\forall x, y \in g_0) \quad \omega(\alpha(x), \alpha(y)) = \omega(x, y)\}
\]

and we note that this has a natural embedding as a closed subgroup

\[
\text{Aut}(g_0, \omega) \hookrightarrow \text{Aut} g = \text{Aut} G
\]

since each \(\alpha \in \text{Aut}(g_0, \omega)\) can be extended to an automorphism \(\alpha \in \text{Aut} g\) with \(\alpha(t, 0) = (t, 0)\) for all \(t \in \mathbb{R}\). A unipotent automorphism group of \(g_0\) is a closed subgroup \(S \subseteq \text{Aut} g_0\) with the property that for every \(\alpha \in S\) there exists an integer \(m \geq 1\) for which \((\alpha - \text{id}_{g_0})^m = 0\) on \(g_0\).

For any closed subgroup \(S \subseteq \text{Aut}(g_0, \omega)\), its Lie algebra is

\[
s := \{D \in \text{Der}(g_0, \omega) \mid (\forall t \in \mathbb{R}) \exp(tD) \in S\} \hookrightarrow \text{Der} g
\]

where

\[
\text{Der}(g_0, \omega) := \{D \in \text{Der} g_0 \mid (\forall x, y \in g_0) \quad \omega(Dx, y) + \omega(x, Dy) = 0\}.
\]

Each \(D \in \text{Der}(g_0, \omega)\) can be extended to a derivation \(D \in \text{Der} g\) with \(D(t, 0) = 0\) for all \(t \in \mathbb{R}\). That is, if we denote by \(\mathfrak{z} := \mathbb{R} \times \{0\}\) the center of \(g\) (= \(\mathbb{R} \times g_0\)), then we obtain a canonical isomorphism of Lie algebras

\[
\text{Der}(g_0, \omega) \simeq \{D \in \text{Der} g \mid \mathfrak{z} \subseteq \text{Ker} D, \ \text{Ran} D \subseteq \{0\} \times g_0\}.
\]

Theorem 4.2. Let \((g_0, \omega)\) be any symplectic nilpotent Lie algebra. Define the nilpotent Lie algebra \(g := \mathbb{R} + \omega g_0\) and let \(G\) be its corresponding connected, simply connected nilpotent Lie group. Pick any connected, simply connected unipotent automorphism group \(S \subseteq \text{Aut}(g_0, \omega)\) and also define the corresponding semidirect product \(\widetilde{G} := S \rtimes G\).

Then \(\widetilde{G}\) is a connected, simply connected, nilpotent Lie group with 1-dimensional center. Moreover, if \(\tilde{\pi}: \widetilde{G} \to \mathcal{B}(\mathcal{H})\) is any unitary irreducible representation associated with some coadjoint orbit \(\tilde{O}\) for which there exists \(\tilde{\xi} \in \tilde{O}\) with \(\tilde{\xi}|_{\mathfrak{z}} \neq 0, \tilde{\xi}|_{g_0} = 0\),
and \( \tilde{\xi}|_{[s,a]} = 0 \), then for all \( a \in C^\infty_b(\tilde{O}) \) one has

\[
(\forall D \in \text{Diff}(\tilde{O})) \quad \text{Op}(Da) \in \mathcal{B}(\mathcal{H}).
\]

In addition, the Weyl-Pedersen calculus defines a continuous linear map

\[
\text{Op}: C^\infty_b(\tilde{O}) \rightarrow \mathcal{B}(\mathcal{H}).
\]

**Proof.** Recall that we denote the center of \( \mathfrak{g} \) by \( \mathfrak{z} \), and denote the center of \( \tilde{\mathfrak{g}} \) by \( \tilde{\mathfrak{z}} \). The bracket in \( \mathfrak{g} \) is given by

\[
[(D_1, x_1), (D_2, x_2)] := ([D_1, D_2], D_1(x_2) - D_2(x_1) + [x_1, x_2])
\]

for all \( D_1, D_2 \in \mathfrak{s} \) and \( x_1, x_2 \in \mathfrak{g} \). It follows by the above equality along with (1.2) that \( \{0\} \times \tilde{\mathfrak{z}} \subseteq \tilde{\mathfrak{z}} \). Conversely, let \( (D_1, x_1) \in \tilde{\mathfrak{z}} \). Then by the above equality for \( x_2 = 0 \) we see that \( D_1 \) belongs to the center of \( \mathfrak{s} \) and \( x_1 \in \bigcap_{D \in \mathfrak{s}} \text{Ker} D \). Moreover, by that equality with \( D_2 = 0 \) we obtain

\[
(\forall x_2 \in \mathfrak{g}) \quad D_1(x_2) + [x_1, x_2] = 0.
\]

By writing \( x_j = (t_j, x_{j0}) \in \mathbb{R} \times \mathfrak{g}_0 = \mathfrak{g} \) for \( j = 1, 2 \), one obtains by the above equation along with (1.1) and (1.2) that

\[
0 = D_1(x_2) + [x_1, x_2] = (\omega(x_{10}, x_{20}), D_1(x_{20}) + [x_{10}, x_{20}]_{\mathfrak{g}_0})
\]

for all \( x_{20} \in \mathfrak{g}_0 \), hence \( x_{10} = 0 \) (since \( \omega \) is nondegenerate) and then \( D_1 = 0 \). Consequently \( \tilde{\mathfrak{z}} = \{0\} \times \mathfrak{z} \). Therefore we will henceforth write \( \tilde{\mathfrak{z}} = \mathfrak{z} \), and \( \text{dim} \mathfrak{z} = 1 \).

Since \( \tilde{O} \) is a coadjoint orbit and \( \xi \in \tilde{O} \), it follows that \( \langle \tilde{O}, \tilde{\mathfrak{z}} \rangle \neq \{0\} \), so \( \langle \tilde{O}, x_0 \rangle = \{1\} \) for some \( x_0 \in \mathfrak{z} \).

Now denote \( \xi := \tilde{\xi}|_{\mathfrak{g}} \in \mathfrak{g}^* \) and \( \mathcal{O} \) be the coadjoint \( G \)-orbit of \( \xi \). Recalling the notation introduced in Proposition 3.2, we now prove that

\[
\tilde{O}|_{\mathfrak{g}} = \mathcal{O}.
\]

(4.3)

In fact, one has the direct sums of vector spaces \( \tilde{\mathfrak{g}} = \mathfrak{s} \oplus \mathfrak{g} \) and \( \mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_0 \) with the Lie brackets in \( \tilde{\mathfrak{g}} \) satisfying

\[
[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}, \quad [\mathfrak{s}, \mathfrak{z}] = \{0\}, \quad \text{and} \quad [\mathfrak{s}, \mathfrak{g}_0] \subseteq \mathfrak{g}_0,
\]

hence the hypothesis \( \mathfrak{g}_0 + [\mathfrak{s}, \mathfrak{g}] \subseteq \text{Ker} \tilde{\xi} \) implies \( \langle \tilde{\xi}, [\mathfrak{s}, \tilde{\mathfrak{g}}] \rangle = \{0\} \), that is, \( \mathfrak{s} \subseteq \tilde{\mathfrak{g}} \). It then follows that \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_z \oplus \mathfrak{g} \) and now (1.3) follows by the implication \( 3 \Rightarrow 2 \) in Proposition 3.2.

Since \( \tilde{O}|_{\mathfrak{g}} = \mathcal{O} \), it follows by the implication \( 2 \Rightarrow 1 \) in Proposition 3.2 that \( \tilde{\pi}|_G: G \rightarrow B(\mathcal{H}) \) is an unitary irreducible representation associated with \( \mathcal{O} \). If we introduce the \( G \)-equivariant diffeomorphism \( \rho_G: \tilde{O} \rightarrow \mathcal{O}, \xi \mapsto \xi|_{\mathfrak{g}} \), then by [BBP13] (3.2) we have

\[
(\forall a \in \mathcal{S}'(\mathcal{O})) \quad \text{Op}_a(a) = \text{Op}_a(a \circ \rho_G^{-1})
\]

and one has an injective homomorphism of associative algebras

\[
\rho_G^*: \text{Diff}(\tilde{O}) \rightarrow \text{Diff}(\mathcal{O}).
\]

Just as in [BBP13] Rem. 3.3 we have \( D(a \circ \rho_G) = (\rho_G^*(D)a) \circ \rho_G \) for all \( a \in C^\infty(\mathcal{O}) \) and \( D \in \text{Diff}(\tilde{O}) \), hence one has an injective map \( C^\infty(\mathcal{O}) \rightarrow C^\infty_b(\mathcal{O}), a \mapsto a \circ \rho_G^{-1} \).

Finally, since \( G \) is a nilpotent Lie group whose coadjoint orbit \( \mathcal{O} \) is flat, by [BH13a] Th. 1.1 for \( a \in C^\infty(\mathcal{O}) \) we have

\[
a \in C^\infty_b(\mathcal{O}) \Longrightarrow (\forall D \in \text{Diff}(\mathcal{O})) \quad \text{Op}_a(Da) \in \mathcal{B}(\mathcal{H}).
\]
and
\[ \text{Op}_z : C^\infty_b(\mathcal{O}) \to \mathcal{B}(\mathcal{H}) \]
is continuous. The above remarks then show that these assertions hold true for \( \text{Op}_z \), and this concludes the proof. \( \square \)

Now we show that Theorem 1.2 is applicable for every unitary irreducible representation of any connected, simply connected, 3-step nilpotent Lie group with 1-dimensional center.

**Corollary 4.3.** Let \( \tilde{G} \) be any connected, simply connected, 3-step nilpotent Lie group with its Lie algebra \( \tilde{\mathfrak{g}} \). If \( \pi : \tilde{G} \to \mathcal{B}(\mathcal{H}) \) is any unitary irreducible representation associated with some coadjoint orbit \( \mathcal{O} \subseteq \tilde{\mathfrak{g}}^* \), then for all \( a \in C^\infty_b(\mathcal{O}) \) one has
\[ (\forall D \in \text{Diff}(\mathcal{O})) \quad \text{Op}(Da) \in \mathcal{B}(\mathcal{H}). \]
In addition, the Weyl-Pedersen calculus defines a continuous linear map
\[ \text{Op} : C^\infty_b(\mathcal{O}) \to \mathcal{B}(\mathcal{H}). \]

**Proof.** By standard arguments, one may assume that the center of \( \tilde{G} \) is 1-dimensional. Then \( \tilde{\mathfrak{g}} \) is a 3-step nilpotent Lie algebra whose center \( \mathfrak{z} \) is 1-dimensional. It follows by [BB13b, Th. 5.1] that there exists a decomposition
\[ \tilde{\mathfrak{g}} = \mathfrak{h} \times ((\mathfrak{z} + \mathfrak{c} + \mathfrak{V}) \times (\mathfrak{h}_1 + \mathfrak{s})) \]
some subalgebras \( \mathfrak{h}, \mathfrak{c}, \mathfrak{h}_1, \mathfrak{s} \), and a linear subspace \( \mathfrak{V} \) of \( \tilde{\mathfrak{g}} \). Here we use the operations of reduced direct product \( \times \) and reduced semidirect product \( \rtimes \) introduced in [BB13b] Def. 2.10] on Lie algebras with 1-dimensional centers, and the following conditions are satisfied:
\begin{itemize}
    \item \( \tilde{\mathfrak{g}} = \mathfrak{z} + \mathfrak{c} + \mathfrak{V} + \mathfrak{h}_1 + \mathfrak{s} \);
    \item \( \mathfrak{h} \) and \( \mathfrak{h}_1 \) are Heisenberg algebras that contain \( \mathfrak{z} \), and \( [\mathfrak{h}, \tilde{\mathfrak{g}}] \subseteq \mathfrak{z} \);
    \item \( \mathfrak{h} \cap (\mathfrak{z} + \mathfrak{c} + \mathfrak{V} + \mathfrak{h}_1 + \mathfrak{s}) = \mathfrak{z} \); \( (\mathfrak{z} + \mathfrak{c} + \mathfrak{V}) \cap (\mathfrak{h}_1 + \mathfrak{s}) = \mathfrak{z} \);
    \item \( \mathfrak{c} \) is an abelian subalgebra, \( [\mathfrak{V}, \mathcal{V}] \subseteq \mathfrak{c} \), and \( [\mathfrak{c}, \mathfrak{c}] : \mathfrak{V} \times \mathfrak{c} \to \mathfrak{z} \) is a nondegenerate bilinear map (hence in particular \( \dim \mathfrak{V} = \dim \mathfrak{c} \), since \( \mathfrak{z} \cong \mathbb{R} \));
    \item \( [\mathfrak{h}_1 + \mathfrak{s}, \mathfrak{c}] = \{0\} \) and \( [\mathfrak{h}_1 + \mathfrak{c}, \mathfrak{V}] \subseteq \mathfrak{c} \);
    \item \( [\mathfrak{h}_1, \mathfrak{c}] = \{0\} \).
\end{itemize}
If we define
\[ \mathfrak{g} := \mathfrak{h} \times ((\mathfrak{z} + \mathfrak{c} + \mathfrak{V}) \rtimes \mathfrak{h}_1) \]
then we obtain \( \tilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{s}, [\mathfrak{s}, \mathfrak{g}] \subseteq \mathfrak{g} \). Moreover it follows by [BB13b, Th. 5.2(1)] that the connected simply connected Lie group associated with \( (\mathfrak{z} + \mathfrak{c} + \mathfrak{V}) \rtimes \mathfrak{h}_1 \) has flat generic coadjoint orbits. It is easily checked that the operation of reduced direct product preserves the class of nilpotent Lie algebras with 1-dimensional center and generic flat coadjoint orbits, \( \tilde{\mathfrak{g}} \) hence also the connected simply connected Lie group

\[ 1 \text{Let } \mathfrak{t} \text{ be any nilpotent Lie algebra whose center } \mathfrak{z} \text{ is 1-dimensional, and } \mathfrak{t}_1 \text{ and } \mathfrak{t}_2 \text{ be subalgebras of } \mathfrak{t} \text{ with generic flat coadjoint orbits and 1-dimensional centers, such that } \mathfrak{t} = \mathfrak{t}_1 \times \mathfrak{t}_2, \text{ that is, } \mathfrak{t}_1 = \mathfrak{t}_1 + \mathfrak{z} = \mathfrak{t}_1 \cap \mathfrak{t}_2, \text{ and } [\mathfrak{t}_1, \mathfrak{t}_2] = \{0\}. \text{ For } j = 1, 2 \text{ pick any linear subspace } \mathfrak{t}_j \subseteq \mathfrak{t}_j \text{ with } \mathfrak{z} + \mathfrak{t}_j = \mathfrak{t}_j. \text{ We claim that for every } \xi \in \mathfrak{t}^* \text{ with } \text{Ker } \xi = \mathfrak{t}_1^0 + \mathfrak{t}_2^0 \text{ one has } \mathfrak{t}_j = \mathfrak{z}, \text{ or, equivalently, for every } X \in \mathfrak{t}_j \text{ there exists } Y \in \mathfrak{t} \text{ with } [X, Y] \notin \text{Ker } \xi. \text{ To check that condition, let } X \in \mathfrak{t} = \mathfrak{z} + \mathfrak{t}_1^0 + \mathfrak{t}_2^0 \text{ with } X \notin \mathfrak{z}, \text{ hence } X \in \mathfrak{X}_0^0 + \mathfrak{X}_1^0 + \mathfrak{X}_2^0, \text{ where } \mathfrak{X}_0^0 \in \mathfrak{t}_0^0 \text{ for } j = 1, 2 \text{ with } \mathfrak{X}_1^0 + \mathfrak{X}_2^0 \neq 0. \text{ If } j \in \{1, 2\} \text{ and } X_j \neq 0, \text{ use the hypothesis that the generic coadjoint orbits of } \mathfrak{t}_j \text{ are flat and } \mathfrak{z}, \text{ does not vanish identically on the center } \mathfrak{z} \text{ of } \mathfrak{t}_j, \text{ to find } Y_j \in \mathfrak{t}_j \text{ with } \langle \xi, [X_j^0, Y_j^0] \rangle = 1. \text{ If } X_j^0 = 0 \text{ then let } Y_j^0 := 0. \text{ By the hypothesis } [\mathfrak{t}_1, \mathfrak{t}_2] = \{0\} \text{ we obtain } \langle \xi, [X, Y_1 + Y_2] \rangle \geq 1, \text{ hence } [X, Y_1 + Y_2] \notin \text{Ker } \xi. \]
$G$ associated to $\mathfrak{g}$ has generic flat coadjoint orbits.

$$\mathfrak{g}_0 := (\mathfrak{h}/3) \times ((\mathfrak{c} + \mathcal{V}) \times (\mathfrak{h}_1/3))$$

$S$ as the connected subgroup of $\text{Aut} \mathfrak{g}_0$ defined by integrating the representation of $s$ on $\mathfrak{g}_0$ coming from the commutation relation $[s, \mathfrak{g}] \subseteq \mathfrak{g}$, then we see that all the hypotheses of Theorem 4.2 are satisfied.

Remark 4.4. In connection with other situations where Theorem 4.2 is applicable, we note the following. If $(\mathfrak{g}_0, \omega)$ is any symplectic nilpotent Lie algebra, it is easily checked that if a closed subgroup $S \subseteq \text{Aut} (\mathfrak{g}_0, \omega)$ is connected, then $S$ is unipotent if and only if $s$ consists of nilpotent derivations of $\mathfrak{g}_0$. This remark leads directly to two types of examples of such situations:

1. If $\mathfrak{g}_0$ is a characteristically nilpotent Lie algebra, then $\text{Aut} \mathfrak{g}_0$ is unipotent, and so are all its connected closed subgroups. Characteristically nilpotent Lie algebras with symplectic structures were studied in [Bu06].
2. Let $\mathfrak{p}$ be any polarization of $(\mathfrak{g}_0, \omega)$, hence $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}_0$ with $\omega|_{\mathfrak{p} \times \mathfrak{p}} = 0$ and $\dim \mathfrak{g}_0 = 2 \dim \mathfrak{p}$. If we define

$$S_\mathfrak{p} := \{ \alpha \in \text{Aut} (\mathfrak{g}_0, \omega) \mid (\forall x \in \mathfrak{p}) \, \alpha(x) = x \}$$

then $S_\mathfrak{p}$ is a unipotent automorphism group of $\mathfrak{g}_0$ and for every $\alpha \in S_\mathfrak{p}$ we have $(\alpha - \text{id}_{\mathfrak{g}_0})^2 = 0$. This follows from the corresponding statement for an abelian algebra $\mathfrak{g}_0 = \mathbb{R}^{2n}$, and in that special case $S_\mathfrak{p}$ is just the group $S_n$ from [Ra85, Sect. 3].

Problem 4.5. It would be interesting to know whether some version of Beals’ commutator criterion for recognizing pseudo-differential operators can be established for the Weyl-Pedersen calculus associated with coadjoint orbits that are not flat. Regarding the Weyl-Pedersen calculus on flat coadjoint orbits, we recall from [BH13a, Th. 1.1] that Beals’ criterion holds true under the form of the converse of the implication (4.5) from the proof of Theorem 4.2 above.

5. Specific examples

We now discuss some situations considered in the earlier literature, in particular providing an uncountable family of pairwise nonisomorphic examples of 3-step nilpotent Lie algebras with 1-dimensional centers. (Recall that there exist only countably many isomorphism classes of 2-step nilpotent Lie algebras with 1-dimensional centers, since these are precisely the Heisenberg algebras, hence there exists precisely one isomorphism class for every odd dimension, and no isomorphism classes for the even dimensions.)

Example 5.1 ([La05 Ex. 3.5], [La06 Ex. 5.2]). For all $s, t \in \mathbb{R} \setminus \{0\}$ let $\mathfrak{g}_0(s, t)$ be the 6-dimensional 2-step nilpotent Lie algebra defined by the commutation relations

$$[X_6, X_5] = sX_3, \ [X_6, X_4] = (s + t)X_2, \ [X_5, X_4] = tX_1.$$  

It follows by [La05 Ex. 3.5] that

$$\{ \mathfrak{g}_0(s, t) \mid s^2 + st + t^2 = 1, \ 0 < t \leq 1/\sqrt{3} \}$$

is a family of isomorphic Lie algebras that are however pairwise non-isomorphic as symplectic Lie algebras with the common symplectic structure $\omega$ given by the
algebra gω and the center of g is 1-dimensional, specifically 3 = R X1. Therefore the Lie algebra g0 = g/3 = span \{X2, X3, X4, X5\} is defined by the commutation relation

\[ [X5, X4] = X2 \]

and the center of g0 is spanned by \{X2, X3\}. The skew-symmetric bilinear functional \( \omega: g0 \times g0 \to \mathbb{R} \) is defined by \( \omega(X2, X5) = \omega(-X3, X4) = -1 \) and \( \omega(Xi, Xj) = 0 \) if \( 2 \leq i < j \leq 5 \) and \( (i, j) \not\in \{(2, 5), (3, 4)\} \), and this corresponds to the matrix

\[
J_\omega = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

while g is an ideal of \( \bar{g} \) and \( D = (ad_{\bar{g}}X0)|_g: g \to g \) is a derivation that vanishes on \( \bar{3} \), hence it induces a derivation of \( g/\bar{3} = g0 \) that is given by the matrix

\[
D = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
The above matrices satisfy the equation $J_\omega D + D^\top J_\omega = 0$, which is equivalent to $\omega(Dx, y) + \omega(x, Dy) = 0$ for all $x, y \in g_0$, and further to $\exp(tD) \in \text{Aut}(g_0, \omega)$ for all $t \in \mathbb{R}$. Thus the group $S := \{\exp(tD) \mid t \in \mathbb{R}\}$ satisfies the hypothesis of Theorem 4.2 and we have $\tilde{G} = S \ltimes G$, where $\tilde{G}$ and $G$ are the connected simply connected Lie groups that correspond to the Lie algebras $\tilde{g}$ and $g$, respectively.

Now let $\xi \in \tilde{g}^*$ with $\langle \tilde{\xi}, X_1 \rangle =: a \neq 0$, $\langle \tilde{\xi}, X_j \rangle = 0$ for $j = 2, 3, 4, 5$ (that is, $g_0 \subseteq \text{Ker}\tilde{\xi}$), and $\langle \tilde{\xi}, X_6 \rangle =: b$, and denote by $\tilde{O}$ the coadjoint $G$-orbit of $\tilde{\xi}$. It follows by [Pe89] page 562 that $\tilde{O}$ is given by the equation

$$y_6 = \frac{1}{6}(6a^2b + 6ay_2y_4 - 3ay_2^2 + 2y_2^3)$$

where $(y_1, y_2, y_3, y_4, y_5, y_6)$ are the Cartesian coordinates in $\tilde{g}^*$ with respect to the dual basis of $\langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle$.

As indicated also on [Pe89] page 562, an irreducible representation associated with $\tilde{O}$ is $\pi: \tilde{G} \to B(L^2(\mathbb{R}^2))$ given by

$$d\pi(X_1) = ia$$
$$d\pi(X_2) = it_1$$
$$d\pi(X_3) = it_2$$
$$d\pi(X_4) = a \frac{\partial}{\partial t_2}$$
$$d\pi(X_5) = \frac{1}{a}(-a^2 \frac{\partial}{\partial t_1} - it_1 t_2)$$
$$d\pi(X_6) = \frac{1}{6a^2}(6a^2t_1 \frac{\partial}{\partial t_2} + 6iat_2 - 3iat_2^2 + 2t_2^3)$$

where $(t_1, t_2)$ are the Cartesian coordinates in $\mathbb{R}^2$.

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