Integrability and the Kerr-(A)dS black hole in five dimensions

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In this note we prove that the Hamilton-Jacobi equation for a particle in the five dimensional Kerr-(A)dS black hole is separable, for arbitrary rotation parameters. As a result we find an irreducible Killing tensor. We also consider the Klein-Gordon equation in this background and show that this is also separable. Finally we comment on extensions and implications of these results.

INTRODUCTION

It is a remarkable fact that the geodesic equation of the four dimensional Kerr black hole is integrable, in the sense of Liouville [1]. The result is at first unexpected. The black hole has two Killing vectors which, together with the geodesic constraint, give three constants of motion in the four dimensional configuration space of a particle moving in this background. Nevertheless one can show there exists an extra conserved quantity rendering the system integrable. This ‘hidden symmetry’ is due to another constant of motion on the phase space of the particle and furthermore may be thought of as resulting from non trivial properties of the space-time. These results were generalised to include the case of a rotating black hole with non-zero cosmological constant [2].

In this paper we will consider the integrability properties of two important partial differential equations defined on the space-time consisting of the five dimensional Kerr-(A)dS black hole with two arbitrary angular momentum parameters. Namely, we will consider the Hamilton-Jacobi (HJ) equation and the Klein-Gordon (KG) equation. The first is of relevance in classical mechanics, and its separability implies that geodesic motion on the space-time is integrable. The KG equation is of course relevant when considering quantum theory on the space-time. The separability of these equations is closely related to geometric properties of the manifold, in particular the existence of second rank Killing tensors. This is a symmetric tensor $K_{\mu\nu}$ which satisfies $\nabla_\mu K_{\nu\rho} = 0$. Note that unlike Killing vectors, Killing tensors do not give rise to Noether charges of Lagrangian theories built on the space-time with the corresponding metric. They are symmetries which can only be seen in phase space.

Our work generalises previous work as follows. Firstly, the case where the angular momenta are equal can be read off from the results of [3], in which these two equations were shown to be separable in all odd dimensional Kerr-(A)dS spacetimes when all rotation parameters are equal. The Killing tensor found however turned out to be reducible in the sense it can be written as a linear combination of direct products of Killing vectors. In this degenerate case, the enhancement of symmetry of the space-time is enough to ensure that the HJ equation separates, thus rendering this result less surprising.

Secondly, a non trivial Killing tensor was shown to exist in the case with only one non vanishing parameter in all dimensions [2]. It is clearly a remarkable property of five dimensions that separability can in fact be proven in the general case. However, it is not wholly unexpected as the five dimensional vacuum Myers-Perry black hole has been shown to possess a Killing tensor for arbitrary angular momentum parameters [3, 4].

The five dimensional Kerr-(A)dS metric was first constructed in [5] and subsequently generalised to all dimensions in the work of [6]. The metric is most compactly written in Kerr-Schild coordinates. However, for our purposes Boyer-Lindquist coordinates are more suitable due to the absence of any off-diagonal components involving $dr$. Explicitly the metric is:

$$
\begin{align*}
\text{ds}^2 &= -W (1 - \lambda r^2) \, dt^2 + \frac{\rho^2 \, dr^2}{V - 2M} + \frac{\rho^2}{\Delta_\theta} \, d\theta^2 \\
&+ \frac{2M}{\rho^2} \left( dr - \sum_{i=1}^{2} \frac{\mu_i}{1 + \lambda a_i^2} \, d\varphi_i \right)^2 \\
&+ \sum_{i=1}^{2} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \, (d\varphi_i - \lambda a_i \, dt)^2,
\end{align*}
$$

(1)

where we have the definitions:

$$
\begin{align*}
\rho^2 &= r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \\
\Delta_\theta &= 1 + \lambda a^2 \cos^2 \theta + \lambda b^2 \sin^2 \theta, \\
V &= r^{-2} (1 - \lambda a^2) (r^2 + a^2) (r^2 + b^2), \\
W &= \sum_{i=1}^{2} \frac{\mu_i^2}{1 + \lambda a_i^2} = \frac{\Delta_\theta}{\Xi_a \Xi_b}
\end{align*}
$$

(2)

(3)

(4)

(5)

and $\Xi_a = 1 + \lambda a^2$, $\Xi_b = 1 + \lambda b^2$, $a_1 = a$, $a_2 = b$, $\mu_1 = \sin \theta$ and $\mu_2 = \cos \theta$. This metric satisfies $R_{\mu\nu} = 4 \lambda \eta_{\mu\nu}$.
and has the isometry group $\mathbb{R} \times U(1) \times U(1)$. It will be convenient to write down the inverse of the metric at this stage as we will need it in several instances in the later sections. Unfortunately it is rather unsightly:

\[ g^{rr} = g^{\varphi r} = 0 , \]
\[ g^{rr} = \frac{V - 2M}{\rho^2} , \]
\[ g^{r\varphi} = \frac{Q - \frac{4M^2}{\rho^2(1 - \lambda r^2)^2(V - 2M)}}{\rho^2(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)} , \]
\[ g^{\varphi \varphi} = \frac{\lambda a_i Q - \frac{4M^2 a_i(1 + \lambda a_i^2)}{\rho^2(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)}}{2M - \frac{2M}{\rho^2(1 - \lambda r^2)^2(r^2 + a_i^2)} , \]
\[ g^{\varphi j} = \frac{(1 + \lambda a_i^2)(\varphi^j_{ij} + \lambda^2 a_i a_j Q + Q^{ij})}{4M^2 a_i a_j(1 + \lambda a_i^2)(1 + \lambda a_j^2)} + \frac{2M}{\rho^2(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)} \]
\[ g^{\theta \theta} = \frac{\Delta \theta}{\rho^2} \] (6)

where $Q$ and $Q^{ij}$ are defined to be

\[ Q = \frac{1}{W(1 - \lambda r^2)} - \frac{2M}{\rho^2(1 - \lambda r^2)^2} \]
\[ Q^{ij} = -\frac{8M^2 \lambda a_i a_j(1 + \lambda a_i^2)(1 + \lambda a_j^2)}{\rho^2(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)} + \frac{2M a_i a_j}{\rho^2(1 - \lambda r^2)^2(r^2 + a_i^2)(r^2 + a_j^2)} \]
\[ -\frac{2M \lambda a_i a_j}{\rho^2(1 - \lambda r^2)^2} \left( \frac{1}{r^2 + a_i^2} + \frac{1}{r^2 + a_j^2} \right) + \frac{4M^2 a_i a_j(1 + \lambda a_i^2)(1 + \lambda a_j^2)}{\rho^2(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)} . \] (8)

Finally one more useful quantity is the determinant of the metric:

\[ \sqrt{-g} = \frac{r \rho^2 \sin \theta \cos \theta}{\Xi_a \Xi_b} . \] (9)

We should note that to compute these quantities it is easiest to use the Kerr-Schild form of the metric and then perform a coordinate transformation.

**THE HAMILTON-JACOBI EQUATION**

The Hamilton-Jacobi equation for the problem at hand is

\[ \frac{\partial S}{\partial l} + \frac{1}{2} g^{\mu \nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0 \] (10)

where $S$ is Hamilton’s principal function. Recall it is a type II generating function for a canonical transformation $(x^\mu, p_\nu) \to (X^\mu, P_\nu)$ which implies $p_\mu = \partial S/\partial x^\mu$ and $X^\mu = \partial S/\partial P_\mu$. Due to the presence of the Killing vectors we know that:

\[ S = \frac{1}{2} m^2 l - E r + \sum_{i=1}^{2} L_i \phi_i + F(r, \theta) \] (11)

and we will show that the problem is completely separable so $F(r, \theta) = S_r(r) + S_\theta(\theta)$. The proof is as follows. It is apparent that the inverse metric is largely composed of terms which are of the form $f(r)/\rho^2$. Thus one is led to multiplying the HJ equation by $\rho^2$ in order to achieve separability. The only non-trivial terms are the first one in the function $Q$ and the first term in $g^{\varphi \varphi}$. However, simple algebra shows that

\[ \frac{\rho^2}{W(1 - \lambda r^2)} = -\frac{\Xi_a \Xi_b}{\lambda \Delta \theta} + \frac{\Xi_a \Xi_b}{\lambda(1 - \lambda r^2)} \] (12)

which takes care of the first term in $Q$. The identity

\[ \sum_{i=1}^{2} \frac{L_i^2 \Xi_i}{\mu_i^2 (r^2 + a_i^2)} = \frac{\sum_{i=1}^{2} L_i^2 \Xi_i + \Xi_a \Xi_b (a^2 - b^2)}{r^2 + b^2} \] (13)

takes care of the first term in $g^{\varphi \varphi}$. Thus, as promised, multiplying the HJ equation by $\rho^2$ renders it separable. The $\theta$ dependent part of the separated HJ equation reads:

\[ m^2 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) + \frac{\Xi_a \Xi_b}{\lambda \Delta \theta} (E - \lambda a_i L_i)^2 + \Delta \theta \left( \frac{\partial S_\theta}{\partial \theta} \right)^2 = K . \] (14)

The $r$ dependent part is rather more complicated:

\[ (V - 2M) \left( \frac{\partial S_r}{\partial r} \right)^2 + \bar{V}(r; E, L_i, m) = -K , \] (15)

where we have defined the “effective” potential

\[ \bar{V}(r; E, L_i, m) = m^2 r^2 \]
\[ -(E - \lambda a_i L_i)^2 \left( \frac{\Xi_a \Xi_b}{\lambda(1 - \lambda r^2)} + \frac{2M}{1 - \lambda r^2} \right) \]
\[ + \rho^2 Q^{ij} L_i L_j - E^2 \frac{8M^2 E L_i a_i(1 + \lambda a_i^2)}{(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)} + 4M E \frac{L_i a_i}{1 - \lambda r^2(r^2 + a_i^2)} \]
\[ + 4M \frac{L_i a_i(1 + \lambda a_i^2)(1 + \lambda a_j^2)}{(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)} \]
\[ \frac{L_i^2 \Xi_a (b^2 - a^2)}{r^2 + a^2} + \frac{L_i^2 \Xi_b (a^2 - b^2)}{r^2 + b^2} . \] (16)
and $K$ is the separation constant. Thus we have reduced the problem of solving for Hamilton's principal function $S$ to quadratures. Note that as $r \to \infty$ the potential $V \sim m^2 r^2$ and $V \sim -\lambda r^4$; upon inspection of \[12\] this shows that when $\lambda < 0$ (AdS) only bound orbits are possible.

We see that there exists an extra constant of motion $K$, as a consequence of the separability of the HJ equation. This is due to the presence of a Killing tensor which may be read off most easily from (14), using $K = K^{\mu \nu} p_{\mu} p_{\nu}$ and $g^{\mu \nu} p_{\mu} p_{\nu} = -m^2$ to give:

$$K^{\mu \nu} = -g^{\mu \nu} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) + \frac{\Xi_a \Xi_b}{\lambda \Delta_\theta} (\delta_{i}^{\mu} \delta_{i}^{\nu} - 2 \alpha_i \delta_{i}^{(\mu} \delta_{i}^{\nu)}) + \sum_{i=1}^{2} \frac{\Xi_{\mu} \Xi_{\nu}}{\mu_i} \delta_{i}^{\mu} \delta_{i}^{\nu} + \Delta \theta \delta_{\mu} \delta_{\nu}. \quad (17)$$

One may be concerned by the fact that the geodesic motion in the Kerr-(A)dS spacetime which can be read off most easily from (14), using $K = K^{\mu \nu} p_{\mu} p_{\nu}$ and $g^{\mu \nu} p_{\mu} p_{\nu} = -m^2$ to give:

Finally we should remark that the phase space functions $H, K, p_{\varphi i}, p_{r}$ are in involution under the Poisson bracket, thus proving Liouville integrability.

### THE KLEIN-GORDON EQUATION

Now we investigate the separability of the KG equation

$$\frac{1}{\sqrt{-g}} g^{\mu \nu} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu} \Phi) = m^2 \Phi, \quad (20)$$

describing a massive spinless field in the Kerr-(A)dS background. Once again separability relies crucially on the fact that the functions $\rho^2 \delta_{\mu}^{\nu}$ are separable. Using the expressions (19) and (20) one may express the KG equation as

$$\frac{1}{r} \partial_r (r(2M) \partial_r \Phi) + \frac{\partial_{\theta} (\Delta \theta \sin \theta \cos \theta \partial_{\theta} \Phi)}{\sin \theta \cos \theta} + \rho^2 g^{\tau \tau} \partial_{\tau}^{2} \Phi + 2 \rho^2 g^{r \tau} \partial_r \partial_{\tau} \Phi + \rho^2 g^{r \varphi} \partial_r \partial_{\varphi} \Phi = m^2 \rho^2 \Phi. \quad (21)$$

The obvious separation of variable ansatz gives $\Phi = e^{-i \omega r} e^{i m \varphi} R(r) \Theta(\theta)$ where $m_1, m_2 \in \mathbb{Z}$. We are then left with non-trivial equations for the functions $R(r)$ and $\Theta(\theta)$:

$$\frac{d}{dr} \left( \Delta_\theta \sin \theta \cos \theta \frac{dR}{dr} \right) \Theta + \frac{\partial_{\theta} (\Delta_\theta \sin \theta \cos \theta \frac{d\Theta}{d\theta})}{\sin \theta \cos \theta} = - \left( \frac{m_1^2 \Xi_a}{\sin^2 \theta} + \frac{m_2^2 \Xi_b}{\cos^2 \theta} - \frac{\Xi_a \Xi_b}{\lambda \Delta_\theta} (\omega - \lambda a_i a_j) \right)^2 - m^2 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) = k, \quad (22)$$

where $k$ is the separation constant. Now let us briefly analyse this equation. Firstly, we change variables to $z = \sin^2 \theta$ which gives:

$$\frac{d^2 \Theta}{dz^2} + \left( \frac{1}{z} + \frac{1}{z-1} - \frac{1}{1-z} - \frac{1}{d} \right) \frac{d \Theta}{dz} - \left( \frac{1}{4 \Delta z (1-z)} \left( \frac{m_1^2 \Xi_a}{z} + \frac{m_2^2 \Xi_b}{1-z} \right) + \frac{\Xi_a \Xi_b (\omega - \lambda a_i a_j)^2}{4 \lambda \Delta_\theta z (1-z)} \right) \Theta = 0, \quad (23)$$

where we have $k' = k - \frac{m^2}{\lambda}$, $d = -\frac{\Xi_a}{\lambda (b^2 - a^2)}$ and $\Delta_\theta = \Xi_a + \lambda (b^2 - a^2) z$. We immediately recognise this as a second order Fuchsian equation. It is easily verified that it has four regular singular points located at $z = 0, 1, d, \infty$. Therefore by a transformation of the form $\Theta(z) = z^A (z - 1)^B (z - d)^C y(z)$, one may show that $y(z)$ satisfies Heun’s equation for suitable choice of $A, B, C$. A welcome simplification occurs in the degenerate case $a = b$. Then it is easy to see that the resulting equation for $\Theta(z)$ only has three regular singular points.
located at \( z = 0, 1, \infty \). Thus the solutions are immediately expressible in terms of hypergeometric functions. The only solution which is regular at \( \theta = 0 \) is:

\[
\Theta(z) = Nz^{\alpha_1}(1-z)^{\beta_1}F_1(\alpha, \beta, \gamma; z),
\]

where \( (\alpha_1, -\alpha_1), (\beta_1, -\beta_1), (\gamma_1, 1-\gamma_1) \) are the indices of the equation for \( \Theta \) at \( 0, 1, \infty \) respectively. The indices at infinity are solutions to the quadratic equation:

\[
x^2 - x + \frac{(\omega - \lambda a_i m_i)^2}{4\lambda} + \frac{m^2}{4\lambda} + \frac{k'}{4E_n} = 0.
\]

Requiring that the solution is also regular at \( \theta = \pi/2 \) implies that \( \beta = -n \) where \( n \in \mathbb{N} \), which can be thought of as a quantization condition of the separation constant \( k \). This implies one can write the solution in terms of Jacobi polynomials:

\[
\Theta(z) = N^\prime z^{\alpha_1}(1-z)^{\beta_1}P_n^{(2\alpha_1, \alpha-\gamma-n)}(1-2z).
\]

For completeness we give the radial equation (valid for general rotation parameters) which may be expressed in the form:

\[
-\frac{1}{r} \frac{d}{dr} \left( r(V - 2M) \frac{d}{dr} R(r) \right) + \bar{V}(\rho; \varpi, m_1, m) R(r) = k R(r).
\]

This is also a second order Fuchsian equation. We will not concern ourselves with its analysis here.

**COMMENTS**

An interesting question is whether these integrability results we have found generalise to \( D > 5 \) for general rotation parameters. As remarked earlier for \( D > 5 \) the best result to date is when all rotation parameters are set equal, or when only one is non-zero. The situation for the asymptotically flat black holes of Myers and Perry is a little better; a slightly stronger result has been obtained where integrability has been proved when the rotation parameters \( a_i \) take on at most two different values \( \mathbb{R} \) or \( S^1 \). This gives rise to an irreducible Killing tensor. The difficulty in tackling the higher dimensional cases rests on the fact that the metric contains cross terms of the form \( d\mu_i d\mu_j \). A consequence of separability of the \( D > 5 \) black holes would be the presence of more Killing tensors. For suppose the HJ equation was separable for the general Myers-Perry black hole. Then we know that the number of Killing vectors is \( 1 + [(D - 1)/2] \). Together with the geodesic constraint this implies that to achieve integrability, which requires \( D \) constants of motion, one would need \( D - 2 - [(D - 1)/2] \) Killing tensors. Thus for \( D > 5 \) the number of Killing tensors would be greater than one.

Another generalisation of \([10]\) would be to investigate the existence of Killing tensors in the vacuum black ring \([11]\) which has the same isometry group as the five dimensional Myers-Perry black hole. However the coordinates in which it is usually written do not appear to be adapted to separability.

Finally, we note that for \( \lambda < 0 \) the black hole considered here is asymptotically AdS. These black holes are known to be dual to a thermal conformal field theory (CFT) on \( S^3 \) \([12]\). It is well known that isometries in the bulk theory give rise to conserved charges on the dual CFT living on the boundary. It would be interesting to see whether the existence of Killing tensors may correspond to some analogous quantity in the dual theory.

We would like to thank Gary Gibbons for useful comments and reading through the manuscript. HKK would like to thank St. John’s College, Cambridge, for financial support.