RETURN PROBABILITY AND SELF-SIMILARITY OF THE RIESZ WALK

RYOTA HANAOKA∗ NORIO KONNO†

Abstract

The quantum walk is a counterpart of the random walk. The 2-state quantum walk in one dimension can be determined by a measure on the unit circle in the complex plane. As for the singular continuous measure, results on the corresponding quantum walk are limited. In this situation, we focus on a quantum walk, called the Riesz walk, given by the Riesz measure which is one of the famous singular continuous measures. The present paper is devoted to the return probability of the Riesz walk. Furthermore, we present some conjectures on the self-similarity of the walk.

keywords: Quantum walks, Singular continuous measure, Return probability, Self-similarity

1 Introduction

Quantum walk (QW) was introduced as a quantum version of random walk (RW) and has been widely studied since around 2000 [1, 2, 11, 12]. Some properties that appear in QWs but not in RWs are linear spreading and localization. One of the approaches to the study on QWs is the CGMV method introduced by Cantero, Grünbaum, Moral, and Velázquez [4]. By using this method, we can associate a QW with a measure on the unit circle in the complex plane. This method has explained the characteristics of QWs, such as recursion and localization conditions [3, 7]. However, the results for the case of singular continuous measures on the unit circle in the complex plane are not much obtained. As one example, Grünbaum and Velázquez [6] introduced a QW constructed by the Riesz measure in 2012. We call this QW the Riesz walk.

In this paper, we compute the return probability of the Riesz walk. For the Riesz walk on a half line starting from the origin, we calculate the measure of the origin for any time. Therefore, we obtain a specific behavior corresponding to the singular continuous measure. Furthermore, numerical simulations suggest some interesting conjectures of the evolution of the Riesz walk.

The rest of this paper is organized as follows. Section 2 gives the definition of the Riesz walk. In Section 3 we present our main result (Theorem 3.1) related to the return probability at the origin. Section 4 shows some conjectures on the self-similarity of the Riesz walk suggested by numerical calculations. Section 5 summarizes this paper.

∗Department of Applied Mathematics, Graduate School of Engineering Science, Yokohama National University, Tokiwadai, Hodogaya, Yokohama, 240-8501, Japan
†Department of Applied Mathematics, Graduate School of Engineering, Yokohama National University, Tokiwadai, Hodogaya, Yokohama, 240-8501, Japan
The Riesz measure on the unit circle

2.1 CGMV method

The QW on $\mathbb{Z}_\geq$, considered in this paper, can be determined by a measure $\mu$ on the unit circle $\partial \mathbb{D}$ in the complex plane $\mathbb{C}$, where $\mathbb{Z}_\geq = \{0, 1, 2, \ldots\}$, $\partial \mathbb{D}$ is the set of complex numbers. Given a measure $\mu$ on $\partial \mathbb{D}$, the Carathéodory and Schur functions are defined as follows.

$$F(z) = \int \frac{t + z}{t - z} d\mu(z), \quad f(z) = \frac{1}{z} \frac{F(z) - 1}{F(z) + 1}.$$  

Then, the Verblunsky parameters $\alpha_k$ are given by the following algorithm.

$$f_{k+1}(z) = \frac{f_k(z) - f_k(0)}{z(1 - f_k(0)f_k(z))}, \quad \alpha_k = f_k(0) \quad (k \geq 0),$$

where $f_0(z) = f(z)$. From the Verblunsky parameters, the corresponding QW on $\mathbb{Z}_\geq$ is determined by

$$U^{(s)} = \begin{bmatrix} R_0 & Q_1 & O & O & O & \cdots \\ P_0 & R_1 & Q_2 & O & O & \cdots \\ O & P_1 & R_2 & Q_3 & O & \cdots \\ O & O & P_2 & R_3 & Q_4 & \cdots \\ O & O & O & P_3 & R_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix},$$

$$R_0 = \begin{bmatrix} \alpha_0 & \rho_0 \\ \bar{\rho}_0 & -\alpha_0 \end{bmatrix}, \quad P_x = \begin{bmatrix} \rho_{2x}\rho_{2x+1} & -\alpha_{2x}\rho_{2x+1} \\ 0 & 0 \end{bmatrix},$$

$$Q_x = \begin{bmatrix} 0 & 0 \\ \rho_{2x-1}\rho_{2x} & \rho_{2x-1}\rho_{2x+1} \end{bmatrix},$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$ and $\bar{\alpha}_k$ is the complex conjugate of $\alpha_k$. The amplitude of location $x \in \mathbb{Z}_\geq$ at time $t \in \mathbb{Z}_\geq$ is denoted by $\Psi_t(x) = [\Psi^{L}_t(x), \Psi^{R}_t(x)]^T$, where $T$ is the transposed operator. The evolution is defined by the equation $\Psi_{t+1} = U^{(s)} \Psi_t$, where $\Psi_t = [\Psi_t(0), \Psi_t(1), \Psi_t(2), \ldots]^T$. Then the probability of location $x$ at time $t$ is given by $\mu_t(x) = \|\Psi_t(x)\|^2 = |\Psi^{L}_t(x)|^2 + |\Psi^{R}_t(x)|^2$.

2.2 Riesz walk

The Riesz measure on the unit circle $\partial \mathbb{D}$ in $\mathbb{C}$ is defined by

$$d\mu(z) = \prod_{k=1}^{\infty} \frac{1 + \cos(4^k \theta)}{2\pi} \frac{d\theta}{2\pi} = \prod_{k=1}^{\infty} \left(1 + \frac{z^{4^k} + z^{-4^k}}{2}\right) \frac{dz}{2\pi iz} = \sum_{j=-\infty}^{\infty} \pi_j z^j \frac{dz}{2\pi iz},$$

where $z = e^{i\theta}$ with $\theta \in [0, 2\pi)$ and $\mu_j$ is the $j$-th moment of $\mu$. The $\mu_j$ can be written as follows (see [4]).

$$\mu_j = \begin{cases} 1, & j = 1, \\
1/2^p, & j = \pm 4k_1 \pm 4k_2 \pm \cdots \pm 4k_p, \\
0, & \text{otherwise,}
\end{cases} \quad (1)$$
where \( k_1 > k_2 > \cdots > k_p \geq 1 \). The Carathéodory and Schur functions of the Riesz measure are computed in the following fashion.

\[
F(z) = 1 + 2 \sum_{n=1}^{\infty} \frac{\mu_n}{z^n} = 1 + z^4 + \frac{z^{12}}{2} + z^{16} + \frac{z^{20}}{4} + \frac{z^{44}}{2} + \frac{z^{48}}{4} + \frac{z^{52}}{2} + \frac{z^{60}}{4} + \cdots,
\]

\[
f(z) = \frac{1}{z} F(z) - 1 = \frac{z^3}{2} - \frac{z^7}{4} + \frac{3z^{11}}{8} + \frac{3z^{15}}{16} - \cdots.
\]

Here we introduce \( G(z) \) and \( g(z) \) satisfying the following relations respectively, \( F(z) = G(z^4) \) and \( f(z) = z^3 g(z^4) \). Thus we have

\[
G(z) = 1 + z + \frac{z^3}{2} + z^4 + \cdots, \quad g(z) = \frac{z}{2} - \frac{z^2}{4} + \frac{3z^3}{8} + \frac{3z^4}{16} - \cdots. \tag{2}
\]

Then, the non-zero Verblunsky parameters \( \xi_k \) can be obtained by using the following algorithm.

\[
g_{k+1}(z) = \frac{1}{z} \frac{g_k(z) - g_k(0)}{1 - \frac{g_k(0) g_k(z)}{z}}, \quad \xi_k = g_k(0) \quad (k \geq 1), \tag{3}
\]

where \( g_1(z) = g(z) \). By the relationship between \( f(z) \) and \( g(z) \), we see that \( \xi_k = \alpha_{4k-1} \) \( (k \geq 1) \). Note that \( \alpha_n = 0 \) for \( n \neq 4k - 1 \). We call the QW defined by the Riesz measure the Riesz walk here.

### 3 Return probability

In this section, we calculate the return probability of the Riesz walk starting from the origin. First, we present our main result.

**Theorem 3.1** For the Riesz walk on \( \mathbb{Z}_+ \) with an initial state \( \Psi_0 = [\alpha, \beta]^T \delta_0 (|\alpha|^2 + |\beta|^2 = 1) \) at the origin, the return probability of the origin at time \( t \) is given by

\[
\mu_t(0) = \begin{cases} 
|\alpha|^2, & t = 4^{k_1} \pm 4^{k_2} \pm \cdots \pm 4^{k_p} - 1, \\
|\beta|^2, & t = 4^{k_1} \pm 4^{k_2} \pm \cdots \pm 4^{k_p}, \\
1, & t = 0, \\
0, & \text{otherwise},
\end{cases}
\]

where \( k_1 > k_2 > \cdots > k_p \geq 1 \) and \( \delta_m(x) = 1 (x = m), = 0 (x \neq m) \).

We should remark that \( \mu_t(0) = 1/4 \) for \( t = 4^k (k = 1, 2, \ldots) \). We call “localization occurs” if there exists \( x \in \mathbb{Z}_+ \) such that \( \limsup_{t \to \infty} \mu_t(x) > 0 \). Therefore, localization occurs for the Riesz walk, since \( \limsup_{t \to \infty} \mu_t(0) > 0 \).

In order to prove Theorem 3.1 we explain the following properties (1) and (2) on the evolution of the Riesz walk. After that, we consider an initial state \( \Psi_0 = [1, 0]^T \). Moreover, we extend the initial state to general form \( \Psi_0 = [\alpha, \beta]^T \) at the end of the proof of Theorem 3.1.
Based on the above properties, we introduce another QW. Let $\tilde{\Psi}_n(x)$ be the amplitude of the QW at location $x$ at time $n$. The relationship between the introduced QW $\tilde{\Psi}_n(x)$ and the Riesz walk $\Psi_n(x)$ is as follows.

$$
\tilde{\Psi}_n(x) = \begin{bmatrix}
\tilde{\Psi}_n^L(x) \\
\tilde{\Psi}_n^R(x)
\end{bmatrix} = \begin{bmatrix}
\Psi_{2n+1}^L(2x+1) \\
\Psi_{2n+1}^R(2x+1)
\end{bmatrix},
$$

where

$$
C_x = \begin{bmatrix}
\xi_x & \rho_x \\
\rho_x & -\xi_x
\end{bmatrix}.
$$

Here $\xi_x$ is a non-zero Verbinsky parameter and $\rho_x = 1 - \xi_x^2$. At the origin, we add $\Psi_1(1)$ to the entire system and set $\xi_0 = -1$ and $\rho_0 = 0$. Since $\xi_0 = -1$, the relationship $\Psi_{2n}^L(0) = \Psi_{2n-1}^R(0)$ holds at the origin.

Based on the above properties, we introduce another QW. Let $\tilde{\Psi}_n(x)$ be the amplitude of the QW at location $x$ at time $n$. The relationship between the introduced QW $\tilde{\Psi}_n(x)$ and the Riesz walk $\Psi_n(x)$ is as follows.

$$
\tilde{\Psi}_n(x) = \begin{bmatrix}
\tilde{\Psi}_n^L(x) \\
\tilde{\Psi}_n^R(x)
\end{bmatrix} = \begin{bmatrix}
\Psi_{2n+1}^L(2x+1) \\
\Psi_{2n+1}^R(2x+1)
\end{bmatrix},
$$

where $\tilde{C}$ is a coin operator, $\tilde{S}$ is a shift operator, and the evolution of the entire system $\tilde{U}$ is defined by $\tilde{U} = \tilde{S}\tilde{C}$. Note that the initial state of the introduced QW is the state at time 1 of the Riesz walk, that is, $\tilde{\Psi}_0 = [1, 0]^{T}\delta_1$.

We want to calculate the return probability of the Riesz walk. From $\Psi_{2n+1}^L(1) = \Psi_{2n}^L(0) = \Psi_{2n-1}^R(1)$ by the above mentioned properties, we need to calculate $\Psi_{2n+1}^L(1) = \tilde{\Psi}_n^L(1)$. To do so, we get the following generating function of $\tilde{\Psi}_n^L(1)$.

**Lemma 3.2** The generating function of $\tilde{\Psi}_1(x)$ is defined as $\tilde{\Psi}_1^M(z) = \sum_{i=0}^{\infty} \tilde{\Psi}_1^M(x)z^i (M \in \{L, R\})$, then $\tilde{\Psi}_1^L(z)$ is given by

$$
\tilde{\Psi}_1^L(z) = \frac{1 + \xi_1 \tilde{f}_1^{(+)}(z)}{(1 - \xi_1 z^2) + (\xi_1 - z^2)\tilde{f}_1^{(+)}(z)},
$$
where \( \hat{f}_{k}^{(+)} \) is the following continued fraction.

\[
\hat{f}_{x}^{(+)}(z) = \frac{z^2 \left( \hat{f}_{x+1}^{(+)}(z) + \xi_{x+1} \right)}{1 + \xi_{x+1} \hat{f}_{x+1}^{(+)}(z)} = \frac{z^2}{\xi_{x+1}} \left( 1 - \frac{\rho_{x+1}^2}{1 + \xi_{x+1} \hat{f}_{x+1}^{(+)}(z)} \right).
\]

**Proof of Lemma 3.2** From Lemma 3.1 in [10], we consider \( a_x = d_x = \rho_x, b_x = -c_x = \xi_x \), and

\[
\begin{align*}
\bar{P}_x = \begin{bmatrix} 0 & b_x \\ b_x & a_x \end{bmatrix}, & \quad \bar{Q}_x = \begin{bmatrix} d_x & c_x \\ 0 & 0 \end{bmatrix}, & \quad \bar{R}_x = \begin{bmatrix} 0 & 0 \\ d_x & c_x \end{bmatrix}, & \quad \bar{S}_x = \begin{bmatrix} b_x & c_x \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]

Define \( F^{(+)}(x,n) \) (resp. \( F^{(-)}(x,n) \)) as the weight of all passages starting from location \( x \) and returning to \( x \) for the first time at \( n \) moving only in \( \{y \in \mathbb{Z} : y \geq x\} \) (resp. \( \{y \in \mathbb{Z} : y \leq x\} \)). The generating function \( \hat{F}_{x}^{(+)} = \sum_{n=2}^{\infty} F^{(+)}(x,n)z^n \) can be expressed as \( \hat{F}_{x}^{(+)} = \hat{f}_{x}^{(+)}(z)R_x \), where \( \hat{f}_{x}^{(+)}(z) \) is a complex number. Furthermore, \( \Xi^{(+)}(x,n) \) (resp. \( \Xi^{(-)}(x,n) \)) is defined as the weight of all passages starting from location \( x \) and returning to \( x \) at time \( n \) moving only in \( \{y \in \mathbb{Z} : y \geq x\} \) (resp. \( \{y \in \mathbb{Z} : y \leq x\} \)). The generating function \( \hat{\Xi}_{x}^{(+)}(z) = \sum_{n=0}^{\infty} \Xi^{(+)}(x,n)z^n \) can be expressed as \( \hat{\Xi}_{x}^{(+)}(z) = I + \hat{f}_{x}^{(+)}(z)\hat{\Xi}_{x}^{(+)}(z) \), therefore \( \hat{\Xi}_{x}^{(+)}(z) = \left(I - \hat{F}_{x}^{(+)}(z)\right)^{-1}. \) In addition, \( \hat{F}_{x}^{(+)}(z) = z\bar{P}_x + \hat{f}_{x}^{(+)}(z)\bar{Q}_x \) also holds, so we have

\[
\hat{f}_{x}^{(+)}(z) = \frac{z^2 \left( \rho_{x+1} + \Delta_{x+1}\hat{f}_{x+1}^{(+)}(z) \right)}{1 - \xi_{x+1}\hat{f}_{x+1}^{(+)}(z)}, \quad \Delta_x = a_x d_x - b_x c_x.
\]

In a similar way, we get

\[
\hat{F}_{x}^{(-)}(z) = z\bar{Q}_x - \hat{\Xi}_{x-1}^{(-)}z\bar{P}_x, \quad \hat{f}_{x}^{(-)}(z) = \frac{z^2 \left( c_{x-1} + \Delta_{x-1}\hat{f}_{x-1}^{(-)}(z) \right)}{1 - b_{x-1}\hat{f}_{x-1}^{(-)}(z)}.
\]

Next, \( \Xi_y(x,n) \) is defined as the weight of all passages starting from location \( y \) arriving at \( x \) at time \( n \) and \( \hat{\Xi}_{x,y}(z) = \sum_{n=0}^{\infty} \Xi_y(x,n)z^n \) is defined as generating function. From \( \hat{\Xi}_{1,1}(z) = I + \left( \hat{F}_{1}^{(+)}(z) + \hat{F}_{1}^{(-)}(z) \right)\hat{\Xi}_{1,1}(z) \), we obtain

\[
\hat{\Xi}_{1,1}(z) = \frac{1}{\gamma_1(z)} \begin{bmatrix} 1 - c_1\hat{f}_{1}^{(+)}(z) \\ d_1\hat{f}_{1}^{(-)}(z) \end{bmatrix},
\]

where \( \gamma_1(z) = \text{det} \left( \hat{F}_{1}^{(+)}(z) + \hat{F}_{1}^{(-)}(z) \right) = 1 - b_1\hat{f}_{1}^{(-)}(z) - c_1\hat{f}_{1}^{(+)}(z) - \Delta_1\hat{f}_{1}^{(+)}(z)\hat{f}_{1}^{(-)}(z). \)

Then, inserting \( a_0 = d_0 = 0, \quad b_0 = -c_0 = -1 \) into Eq. (13), we get \( \hat{f}_{1}^{(-)}(z) = z^2 \) and \( \gamma_1(z) = (1 - \xi_1 z^2) + (\xi_1 z^2)\hat{f}_{1}^{(+)}(z). \) Finally, from \( \hat{\Psi}_1(z) = \hat{\Xi}_{1,1}(z)[1,0]^{T} \), we obtain \( \hat{\Psi}_1(z) = \left( 1 + \xi_1\hat{f}_{1}^{(+)}(z)/\gamma_1(z). \right) \)

By Lemma 3.2 and the properties of the Riesz walk, we see that

\[
\hat{\Psi}_1(z^2) = \sum_{t=0}^{\infty} \hat{\Psi}_t^{L}(1)z^{2t} = \sum_{t=0}^{\infty} \hat{\Psi}_{2t+1}^{L}(1)z^{2t} = \sum_{t=0}^{\infty} \hat{\Psi}_{2t}^{0}(0)z^{2t} = \hat{\Psi}_0^{L}(z).
\]
Then we have the following generating function of $\Psi^L_n(0)$.

**Proposition 3.3** For the Riesz walk on $\mathbb{Z}_+$ with an initial state $\Psi_0 = [1, 0]^T \delta_0$, the generating function of amplitude at the origin $\hat{\Psi}^L_0(z) = \sum_{n=0}^{\infty} \Psi^L_n(0) z^n$ is given by

$$\hat{\Psi}^L_0(z) = \frac{1 + \xi_1 \hat{f}^+(z^2)}{(1 - \xi_1 z^4) + (\xi_1 - z^4) \hat{f}^+(z^2)},$$

where $\hat{f}^+(k)$ is the following continued function.

$$\hat{f}^+(z^2) = \frac{z^4 (\hat{f}^+(z^2) + \xi_{x+1})}{1 + \xi_{x+1} \hat{f}^+(z^2)}.$$

From now on, we will prove Theorem 3.1.

**Proof of Theorem 3.1** First, the generating function of the Riesz walk is given by

$$\hat{\Psi}^L_0(z) = \frac{1}{1 - \xi_1 z^4 + (\xi_1 - z^4) \hat{f}^+(z^2)} = \frac{1}{1 - z^4 \xi_1 \hat{f}^+(z^2)} = \frac{1}{1 - h^+(z^2)},$$

where $h^+(n)$ is defined as

$$h^+(z) = \frac{z^2 (\xi_n + \hat{f}^+(z^2))}{1 + \xi_n \hat{f}^+(z^2)} \quad (n \geq 1).$$

Noting that

$$\hat{f}^+(z^2) = \frac{z^4 (\xi_{n+1} + \hat{f}^+(z^2))}{1 + \xi_{n+1} \hat{f}^+(z^2)} = h^+(z^2) \quad (n \geq 1),$$

$h_n(z)$ is the continued functions.

$$h^+(z^2) = \frac{z^4 (\xi_n + h^+(n+1)(z^2))}{1 + \xi_n h^+(n+1)(z^2)} \quad (n \geq 1).$$

On the other hand, it follows from Eqs. 2 and 3 that $G(z) = (1 + zg(z)) / (1 - zg(z))$ and $g_k(z) = (zg_{k+1}(z) + \xi_k) / (1 + \xi_k zg_{k+1}(z)) (k \geq 1)$. Thus,

$$\frac{1}{2} (G(z^4) + 1) = \frac{1}{2} \left( \frac{1 + z^4 g_1(z^4)}{1 - z^4 g_1(z^4)} + 1 \right) = \frac{1}{1 - z^4 g_1(z^4)} = \frac{1}{1 - \bar{g}(z^4)},$$

where $\bar{g}(z)$ is defined as

$$\bar{g}_n(z) = zg_n(z) \quad (n \geq 1).$$
Then, \( \tilde{g}_{n+1}(z) \) is the following continued function.

\[
\tilde{g}_n(z) = z^4 \times \frac{z^4 g_{n+1}(z^4) + \xi_n}{1 + \xi_n z^4 g_{n+1}(z^4)} = \frac{z^4 (\xi_n + \tilde{g}_{n+1}(z^4))}{1 + \xi_n g_{n+1}(z^4)} \quad (n \geq 1). \tag{7}
\]

Combining Eqs. (5) and (6) with Eq. (7), we have

\[
\hat{\Psi}_0^L(z) = \frac{1}{2} (G(z^4) + 1) = \frac{1}{2} (F(z) + 1) = \frac{1}{2} \left\{ 1 + 2 \sum_{n=1}^\infty \bar{\mu}_n z^n \right\} = \sum_{n=0}^\infty \bar{\mu}_n z^n. \tag{8}
\]

Therefore, Eq. (1) implies that the amplitude at the origin at time \( n \) is

\[
\Psi_n^L(0) = \begin{cases} 1, & n = 1, \\ 1/2^p, & n = \pm 4^{k_1} \pm 4^{k_2} \pm \cdots \pm 4^{k_p}, \quad k_1 > k_2 > \cdots > k_p \geq 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Furthermore, the properties of the evolution give

\[
\Psi_n^R(0) = \Psi_{n+1}^L(0) = \begin{cases} 1/2^p, & n = \pm 4^{k_1} \pm 4^{k_2} \pm \cdots \pm 4^{k_p} - 1, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( k_1 > k_2 > \cdots > k_p \geq 1 \). Therefore, we see that

\[
\mu_n(0) = \begin{cases} |\Psi_n^L(0)|^2, & (n = \text{even}), \\ |\Psi_n^R(0)|^2, & (n = \text{odd}), \end{cases}
\]

so we have the return probability of the Riesz walk with initial state \( \Psi_0 = [\alpha, 0]^T \delta_0 \). In the end, considering the Riesz walk with an initial state \( \Psi_0 = [0, 1]^T \delta_0 \), the amplitude at time 1 is

\( \Psi_1 = [1, 0]^T \delta_0 \). Therefore, from the parity of the amplitude with initial states both \( [\alpha, 0]^T \delta_0 \) and \( [0, 0]^T \delta_0 \), we have the desired conclusion. \( \square \)

In particular, for an initial state \( \Psi_0 = [1, 0]^T \), we have

**Corollary 3.4** For the Riesz walk on \( \mathbb{Z}_2 \) with an initial state of \( \Psi_0 = [1, 0]^T \delta_0 \), the return probability of the origin at time \( t \) is as follows.

\[
\mu_t(0) = \begin{cases} \frac{1}{2}, & t = 4^{k_1} \pm 4^{k_2} \pm \cdots \pm 4^{k_p} - \delta, \quad k_1 > k_2 > \cdots > k_p \geq 1, \quad \delta \in \{0, 1\}, \\ 1, & t = 0, \\ 0, & \text{otherwise}. \end{cases}
\]

Let \( s(k) = \sum_{t=1}^k 4^t \). In Corollary 3.3, the return probability from time \( 4^{k+1} - s(k-1) \) to \( 4^{k+1} + s(k-1) \) is equal to the return probability from time \( 4^k - s(k-1) \) to \( 4^k + s(k-1) \). Furthermore, quadratic of the return probability from time \( 4^{k+1} \pm 4^k - s(k-1) \) to \( 4^{k+1} \pm 4^k + s(k-1) \) is equal to the return probability from time \( 4^k - s(k-1) \) to \( 4^k + s(k-1) \). Therefore, the return probability has self-similar sets between \( 4^k - s(k-1) \) and \( 4^k + s(k-1) \) for each \( k \) (see Fig. 1).
4 Conjectures on the Riesz walk

In Section 3, we calculated the probability at the origin only. This section is devoted to conjectures on the evolution of the Riesz walk with initial state $\Psi_0 = [1, 0]$ based on numerical simulations.

From Corollary 3.4, the probability at the origin at time $4^n$ is $1/4$. On the other hand, the probabilities except the origin cannot be calculated. Therefore, we first show the numerical results for the probability distribution at time $4^n$. Let $\nu_t(x) = \mu_t(x - 1) + \mu_t(x)$. The probability distribution at time 4 (Fig. 2 (a)) is $\nu_4(4) = 3/4$. The probability distribution at time 16 (Fig. 2 (b)) is $\nu_{16}(4^2 - 4) = \nu_{16}(4^2) = 3/8$. Furthermore, the probability distribution at time 64 (Fig. 2 (c)) is $\nu_{64}(4^3 - 4^2) = 0$. Note that $3/16 = 0.1875$, which means that the probabilities for each location are close to $3/16$. Similarly, the probability distribution at time 256 (Fig. 2 (d)) is that $\nu_{256}(x)$ is close to $3/32$ for $x \in \{4^4 - 4^2k_1 - 4^2k_2 - 4k_3 | k_1, k_2, k_3 \in \{0, 1\}\}$. From these results, we have the following conjecture on the probability distribution at time $4^n$.

**Conjecture 4.1** The probability distribution of the Riesz walk on $\mathbb{Z}_+$ at time $4^n$ with initial state of $\Psi_0 = [1, 0]^T \delta_0$ is in the following. There exists $\varepsilon_{x,n} \in \mathbb{R}$ with $|\varepsilon_{x,n}| < 0.03$ such that

$$\mu_{4^n}(x) + \mu_{4^n}(x - 1) =\begin{cases} \frac{1}{3} \times \frac{1}{2^{n-1}} \times (1 \pm \varepsilon_{x,n}) , & x = 0, \\ 0 , & x \in K_n, \\ \text{otherwise}, & \end{cases}$$

where $\mathbb{R}$ is the set of real numbers and

$$K_n = \{4^n - (4^{n-1}k_1 + 4^{n-2}k_2 + \cdots + 4^1k_{n-1}) | k_1, k_2, \ldots, k_{n-1} \in \{0, 1\}\}.$$

We should remark that numerical simulations suggest $|\varepsilon_{x,n}| < 0.03$. The probabilities at time $4^n$ on $K_n$ are not equal to $3/(4^n-1)$. 

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Figure 1: Probability at the origin from time 0 to 1365.
Figure 2: Probability distributions of the Riesz walk with initial state $\Psi_0 = [1, 0]^T$ for (a) $t = 4$, (b) $t = 16$, (c) $t = 64$, and (d) $t = 256$. 
Next, we consider the evolution by comparing the probability distributions at different times. Comparing the distribution of time 16 and 64 (Fig. 2 (b), (c)) in more detail, we find that \[ \nu_{64}(4^3 - 4^2 - 4) + \nu_{64}(4^3 - 4^2) = \nu_{16}(4^2 - 4) = 3/8 \] and \[ \nu_{64}(4^3 - 4) + \nu_{64}(4^3) = \nu_{16}(4^2) = 3/8. \] Comparing the distribution of time 64 and 256 (Fig. 2 (c), (d)), we find that \[ \nu_{256}(x - 4) + \nu_{256}(x) = \nu_{64}(x/4) \] for \( x \in \{4^4 - 4^3 k_1 - 4^2 k_2 | k_1, k_2 \in \{0, 1\}\}. \) In fact, this relationship holds at time not just \( 4^n. \)

Figure 3: Probability distribution profile in the plane of position \( x \) vs time \( t \) until (a) \( t = 4, \) (b) \( t = 16, \) (c) \( t = 64, \) and (d) \( t = 256. \)

The time and space spread of probability distribution until 4 (Fig. 3 (a)) reproduces the time and space spread of probability distribution until 16 (Fig. 3 (b)). Similarly, considering the time and space spreads of probability distribution until times 64 and 256 (Fig. 3 (c), (d)), the time and space spread of the probability distribution is self-similarity until each quadruple time. From this self-similarity, the above relationship at time not just \( 4^n \) are confirmed by numerical calculation, and the following conjecture about the probability distributions between any even time and its quadruple time is obtained.

**Conjecture 4.2** Let \( \mathbb{Z}_\succ = \{1, 2, 3, \ldots\}. \) For each \( n \in \mathbb{Z}_\succ, \) we put \( T_0(n) = \{0\}, \) and \( T_k(n) = \{(k - 1)/n, k/n\} \) \((1 \leq k \leq n). \) Then, for the measure of the Riesz walk with initial state of
\[ \Psi_0 = [1,0]^T \delta_0, \]
\[
P\left( \frac{X_{2t}}{2t} \in T_k(t) \right) = P\left( \frac{X_{8t}}{8t} \in T_k(t) \right),
\]
for any \( t \in \mathbb{Z} > 0. \)

We note that Conjecture 4.2 means
\[
\mu_{2t}(0) = \mu_{8t}(0), \quad \mu_{2t}(2x - 1) + \mu_{2t}(2x) = \sum_{y=8x-7}^{8x} \mu_{8t}(y) \quad (t, x \geq 1).
\]

Finally, we describe a limit theorem of the Riesz walk. The well-known weak limit theorem of the one-dimensional two-state QW was given by \([8, 9]\). The limit density is an inverse-bell shape. On the other hand, the corresponding limit theorem for QW defined by singular continuous measure is not known. If Conjecture 4.1 holds, the following limit theorem of the Riesz walk might be obtained. Put \( \delta_{[a,b]}(x) = 1, (x \in [a,b]), = 0, (x \notin [a,b]). \)

**Conjecture 4.3** The limit theorem of the Riesz walk on \( \mathbb{Z} \geq 0 \) with initial state \( \Psi_0 = [1,0]^T \delta_0 \) in the limit \( n \to \infty \) along time \( 4^n \) is given by
\[
\lim_{n \to \infty} \frac{X_{4^n}}{4^n} = Z,
\]
where \( Z \) has the following measure:
\[
\frac{1}{4} \delta_0 + "a self-similar set" \times \delta_{[2/3,1]}.
\]

From Theorem 4.3 we see that the return probability at the origin at time \( 4^n \) is 1/4. From now on, we explain "a self-similar set" in Conjecture 4.3. First, we consider the Cantor set on closed interval \([0,1]\), which is the well-known self-similar set \([5]\). The Cantor set is constructed by repeatedly removing the middle third of intervals. The Cantor set \( C \) is defined by \( \cap_{n=0}^\infty C_n \), where \( C_n \) consists of \( 2^n \) disjoint closed intervals \( C^i_n = [a^i_n, b^i_n] (i = 1, 2, \ldots, 2^n) \). The Cantor set can be expressed in another way. Let \( R_n \) be a set of right-hand points of intervals \( C^i_n \in C_n \), and we have
\[
R_n = \{ b^i_n | i = 1, 2, \ldots, 2^n \} = \{ 1 - 2 \left( 3^{-1} k_1 + 3^{-2} k_2 + \cdots + 3^{-n} k_n \right) | k_1, k_2, \ldots, k_n \in \{0,1\} \}.
\]
Then, the Cantor set is \( C = \lim_{n \to \infty} R_n \). Second, as in the case of the Cantor set \( C \), we consider the set \( D \) that consists of each interval repeatedly divided into four sections and removing the middle two. Similarly, defining a set \( M_n \) of right-hand points of interval, we have
\[
M_n = \{ 1 - 3 \left( 4^{-1} k_1 + 4^{-2} k_2 + \cdots + 4^{-n} k_n \right) | k_1, k_2, \ldots, k_n \in \{0,1\} \}.
\]
Then, \( D = \lim_{n \to \infty} M_n \). Third, when the space is divided by time in Conjecture 4.4, location \( K_n \) where the measures are positive is as follows.
\[
\tilde{K}_n = \{ 1 - \left( 4^{-1} k_1 + 4^{-2} k_2 + \cdots + 4^{-(n-1)} k_{n-1} \right) | k_1, k_2, \ldots, k_{n-1} \in \{0,1\} \}.
\]
Now, we have \( (1/3)M_{n-1} + 2/3 = \tilde{K}_n \). Therefore, \( \tilde{K}_n \) can be considered as a mapping of \( M_{n-1} \) to an interval \([2/3, 1]\), and \( \lim_{n \to \infty} \tilde{K}_n \) is the set of \( D \) mapped to interval \([2/3, 1]\). Thus, in the limit along \( 4^n \) of the Riesz walk, the measure lies on “a self-similar set” \( (1/3)D + 2/3 (= \lim_{n \to \infty} \tilde{K}_n) \) in Conjecture 4.3.

5 Conclusion

Non-trivial rigorous results on QWs defined by singular continuous measures are not known. Therefore, this paper focused on the Riesz walk determined by the Riesz measure, which is one of the well-known singular continuous measures. The return probability of the Riesz walk starting from the origin on \( \mathbb{Z}_\geq \) was calculated. In addition, some conjectures on self-similar properties of the Riesz walk were presented by using numerical simulations. As a future work, it would be fascinating to give proofs of own conjectures and compute the return probability of the Riesz walk starting from any location.

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