Measure valued solutions of the 2D Keller-Segel system.

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1 Introduction

In this paper we study the solutions of the following two-dimensional Keller-Segel system describing chemotaxis:

\[
\begin{align*}
\partial_t u - \Delta u + \nabla (u \nabla v) &= 0 \quad \text{in } \Omega, \quad \partial_{\nu} u|_{\partial \Omega} = 0 \\
-\Delta v &= u - \frac{1}{|\Omega|} \int_{\Omega} u \, dx \quad \text{in } \Omega, \quad \partial_{\nu} v|_{\partial \Omega} = 0 \\
u(x, 0) &= u_0(x) \quad \text{in } \Omega
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with boundary \( \partial \Omega \in C^4 \) and \( u_0 \) is a bounded, nonnegative function.

It is well known that the solutions of (1.1)-(1.3) blow-up in finite time, i.e.

\[
\lim_{t \to T} \int_{\Omega} u^p \, dx = \infty \quad \text{for all } p > 1
\]

for some \( T < \infty \) (cf. [5]).

The Keller-Segel system as well as the properties of the blow-up set has been extensively studied. An idea that was introduced in [6] to prove discreteness of the blow-up set is the symmetrization of the nonlinear term in (1.1) in the equation that describes the evolution of the mass of \( u \). The symmetrization idea has been used in a more general form in [8] to show that the solutions of a system analogous to (1.1)-(1.3) but with (1.2) replaced by

\[
-\Delta v + \gamma v = u \quad \text{in } \Omega, \quad \partial_{\nu} v|_{\partial \Omega} = 0
\]

blow-up in a finite set of points. The method used in [8] relies heavily in the symmetry properties of the operator \( u \nabla v \). Similar ideas to the ones in [8] can be applied to prove discreteness of the blow-up set for the solutions of (1.1)-(1.3).

Continuation beyond blow-up has been considered from several points of view. The usual approach used to extend the solutions of Keller-Segel systems beyond the blow-up time consists in regularizing some of the nonlinearities in the equations by means of a sequence of problems depending on a parameter \( \varepsilon > 0 \). The regularization is chosen in order to obtain a problem with global solutions in time and also to recover formally the original Keller-Segel system as the parameter \( \varepsilon \to 0 \). The papers [11], [12] study in detail one of these regularizations using matched asymptotics. In particular, it was obtained in those papers that formal limits of solutions of the system (1.1)-(1.3) with \( \Omega = \mathbb{R}^2 \) can be described by a set of Dirac measures whose positions and masses evolve according to a system of ODEs. A different regularization was considered in the papers [7], [2] where it was introduced a concept of weak solutions for systems analogous to (1.1)-(1.3) as the limit of regularized problems, different from the ones considered in [11], [12]. A key idea in [7], [2] is the use of a symmetrization procedure for the nonlinear term similar to the one in [8]. It was also seen in [2] that assuming that the measures were Dirac masses concentrated in smoothly moving curves, the masses and positions of the Dirac masses would evolve according to a system of ODEs, closely

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related to the one obtained in [11], but exhibiting some differences due to the different choice of regularization used.

In this paper we obtain generalized solutions for (1.1)-(1.3) in the sense of measures as the limit of two-different regularizations of it. We will show that the resulting limit measures, that are in some suitable sense global weak solutions of (1.1)-(1.3), depend in the regularization.

The plan of the paper is the following. In Section 2 we introduce two different regularizations of the Keller-Segel system. Section 3 contains some properties of the fundamental solution for the Laplace equation in bounded domains that will be used throughout the whole paper. In Section 4 we describe a key argument that allows to control the local change of mass in a given region. Section 5 contains an estimate for the solutions of the second regularization obtained using an entropy inequality. Local regularity estimates for the solutions of both regularized problems in the regions where the mass is small are obtained in Section 6. Section 7 describes how to obtain limit measure solutions for both limit problems, as well as the fact that such measures consist in a finite number of atoms plus a regular part. Section 8 describes the limit problems satisfied by such measures. Section 9 proves that both regularization yield different limit measures for masses above the critical value. Finally, Section 10 contains a formalism to describe the form of the nonlinear terms arising in the limit weak formulation using measured valued Young measures, since some fast oscillations could take place near the singularities.

2 Two regularizations of (1.1)-(1.3).

We will use in this paper two regularizations of the system (1.1)-(1.3). The first one is:

\[
\begin{align*}
\partial_t u - \Delta u + \nabla \left( f_\varepsilon(u) \nabla v \right) &= 0 \quad \text{in } \Omega, \quad \partial_v v |_{\partial \Omega} = 0 \\
-\Delta v &= f_\varepsilon(u) - \frac{1}{|\Omega|} \int_{\Omega} f_\varepsilon(u) \, dx \quad \text{in } \Omega, \quad \partial_v v |_{\partial \Omega} = 0
\end{align*}
\]  

(2.1)

where:

\[
f_\varepsilon(u) = \int_0^u \min \left\{ 1, \left( \frac{1}{\varepsilon} - s \right)_+ \right\} \, ds
\]  

(2.3)

The second regularization that we will use is:

\[
\begin{align*}
\partial_t u - \Delta \left( u + \varepsilon u^\frac{7}{6} \right) + \nabla (u \nabla v) &= 0 \quad \text{in } \Omega, \quad \partial_v v |_{\partial \Omega} = 0 \\
-\Delta v &= u - \frac{1}{|\Omega|} \int_{\Omega} u dx \quad \text{in } \Omega, \quad \partial_v v |_{\partial \Omega} = 0
\end{align*}
\]  

(2.4)

The choice of the exponent \( \frac{7}{6} \) in (2.4) is not essential. The arguments could be made in a similar manner for any number greater than one. However, some computations in Section 6 will become slightly simpler with the particular exponent \( \frac{7}{6} \).

We assume in both cases \( \varepsilon > 0 \). It would be possible to use in (2.1), (2.2) \( f_\varepsilon(u) = \frac{u}{1+u} \) or similar cutoff functions. A key feature of these regularizations is the symmetry of the nonlinear terms \( f_\varepsilon(u) \nabla v, u \nabla v \) respectively on the function \( u \). This restriction is needed, because the idea used in [8] to control the motion of the mass relies heavily on these symmetry properties.

It is trivially seen that the classical solution of the problems (2.1), (2.2) or (2.4), (2.5) with initial data \( u(x,0) = u_0(x) \) is globally defined in time for any \( \varepsilon > 0 \). In particular, \( ||u||_{L^\infty(\Omega)} \) is bounded in any interval \( 0 \leq t \leq T < \infty \), although the resulting estimate depends on \( \varepsilon \) and it can be expected to blow-up as \( \varepsilon \to 0^+ \). The choice of boundary conditions imply:

\[
\int_{\Omega} u(x,t) \, dx = \int_{\Omega} u_0(x) \, dx.
\]  

(2.6)
The rest of the paper is devoted to characterize the limit of the solutions of these two problems as $\varepsilon \to 0$.

3 A local approximation of the Green’s function for the Laplace operator with Neumann boundary conditions.

We will use the a detailed description of the Green’s function associated to the Laplace equation with Neumann boundary conditions near the boundary.

The following Lemma collects some basic geometrical results. The proof is elementary and it will be omitted.

**Lemma 1** Suppose that $\Omega \subset \mathbb{R}^2$ is an open set with $\partial \Omega \in C^4$. Let us denote as $d(y) = \text{dist}(y, \partial \Omega)$ the distance of $y$ to $\partial \Omega$. There exists $\sigma_0 > 0$ depending only on $\partial \Omega$ such that the function $d(y)$ is uniquely defined and it has two continuous derivatives in the set $\mathcal{U} = \{y \in \Omega : \text{dist}(y, \partial \Omega) \leq 2\sigma_0\}$. For any $D \in [0, 2\sigma_0]$ we define the curves $\Gamma_D = \{y \in \mathbb{R}^2; d(y) = D\}$. For any $y \in \Gamma_D$ the vector $\nu(y) = -\nabla d(y)$ is the normal unit vector to the curve at the point $y$ and $h(y) = \nabla \cdot (\nu(y))$ is the curvature of $\Gamma_D$ at $y$. Moreover, suppose that we denote as $t(y)$ the unit tangent vector to $\Gamma_D$ at $y$. Then:

$$\nabla \nu(y) = h(y) t(y) \otimes t(y).$$

(3.1)

The following general property of the Green’s function of the Laplace equation with Neumann boundary conditions will be useful:

**Lemma 2** Suppose that $G(y, x)$ is the unique solution of:

$$-\Delta_y G(y, x) = \delta_x(y) - \frac{1}{|\Omega|}, \quad y \in \Omega, \quad x \in \Omega$$

(3.2)

$$\frac{\partial G(y, x)}{\partial \nu_y} = 0, \quad y \in \partial \Omega$$

(3.3)

$$\int_{\Omega} G(y, x) \, dy = 0, \quad x \in \Omega$$

(3.4)

where $\nu_y$ denotes the outer normal to $\partial \Omega$ at $y \in \partial \Omega$. Suppose that $x, x_0 \in \Omega, \ x \neq x_0$. Then:

$$G(x, x_0) = G(x_0, x)$$

Proof. Multiplying (3.2)–(3.4) by $G(y, x_0)$, and integrating by parts we obtain:

$$\int_{\Omega} \nabla_y G(y, x_0) \nabla_y G(y, x) \, dy = G(x, x_0)$$

Exchanging the role of $x, x_0$ the result follows from the symmetry of the left-hand side. ■

3.1 Uniform regularity estimates near the boundary.

We now describe the above mentioned Green’s function. The main content of the next lemma is the uniform continuity of the remainder function $K(y, x)$.

**Lemma 3** Suppose that $G(y, x)$ is as in Lemma 2. Let $\sigma_0$ be as in Lemma 2 and let $Z \in C^\infty(\bar{\Omega})$ be a cutoff function satisfying with $Z(y) = 1$ for $d(y) \leq \sigma_0$, $Z(y) = 0$ for $d(y) \geq 2\sigma_0$.

Then, there exists $K \in C(\bar{\Omega} \times \bar{\Omega})$ with $\nabla_y K, \nabla_x K \in C(\bar{\Omega} \times \bar{\Omega})$ such that:

$$G(y, x) = -\frac{1}{2\pi} \left[\log |y - x| + Z(y) \log (|\tau(y) - x|)\right] + K(y, x)$$

(3.5)

where

$$\tau(y) = y + f(y) \nu(y), \quad f(y) = 2d(y) + h(y)(d(y))^2.$$
Proof. Existence and uniqueness of the function $G$ is standard (cf. [10]). Since the result is immediate for $d(y) \geq \sigma_0$ we restrict our analysis to the case $d(y) \leq \sigma_0$. We define a function $$G^*(y, x) = -\frac{1}{2\pi} \log \left( \frac{\tau(y) - x}{|x-y|} \right) .$$

Notice that $G^*$ satisfies:

$$\frac{\partial G^*}{\partial \nu_y} = 0, \ y \in \partial \Omega \quad (3.6)$$

as well as:

$$- \Delta_y G^*(y, x) = - \frac{1}{4\pi} \Delta_y \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) \text{ in } \Omega \quad (3.7)$$

In order to compute the right hand side of (3.7) we write:

$$\Delta_y \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) = \frac{1}{|\tau(y) - x|^2} \Delta_y \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) - \frac{1}{|\tau(y) - x|^4} \left( \nabla_y \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) \right)^2 . \quad (3.8)$$

Using Lemma [1] we obtain:

$$\nabla_y \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) = 2 \log \left( \frac{\tau(y) - x}{|x-y|} \right) \nabla_y f + 2h f \log \left( \frac{\tau(y) - x}{|x-y|} \right) t(y) \quad (3.9)$$

We will use also the equivalent formula:

$$\nabla_y \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) = 2 \left[ \log \left( \frac{\tau(y) - x}{|x-y|} \right) - 2 \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right] \nu(y) + \frac{1}{|\tau(y) - x|^2} \Delta_y f + 2h f \log \left( \frac{\tau(y) - x}{|x-y|} \right) t(y) . \quad (3.10)$$

Applying the operator $\nabla \cdot$ to (3.9), using that $d \leq C \log \left( \frac{\tau(y) - x}{|x-y|} \right)$, $f(d) \leq C_{d}$ where $C$ is a constant depending only on the regularity bounds of $\Omega$ and neglecting terms that are smaller than $|\tau(y) - x|^2$ we obtain:

$$\Delta_y \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) = 2 \nabla_y \cdot \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) + 2 \left[ \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right] \nabla_y f + 2h f \log \left( \frac{\tau(y) - x}{|x-y|} \right) t(y) + O \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) . \quad (3.11)$$

The first term on the right-hand side of (3.11) can be approximated, using the definition of $f$, as:

$$2 \nabla_y \cdot \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) = 2f h - 4dh + O \left( d^2 \right) = O \left( d^2 \right). \quad (3.12)$$

The second term on the right-hand side of (3.11) satisfies:

$$2 \left[ \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right] \Delta_y f = 2 \left[ \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right] \left[ -2 \nabla_y \cdot \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) + 2h \nabla_y \log \left( \frac{\tau(y) - x}{|x-y|} \right) t(y) \right] = O \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) . \quad (3.13)$$

The third term on the right-hand side of (3.11) can be written, using Lemma [1] as:

$$2 \left[ \nabla_y \left( \frac{\tau(y) - x}{|x-y|} \right) \right] \cdot \nabla_y f = -4 - 4dh + 2 \left[ 2 + 2hd \right] + O \left( d^2 \right) = 4 + 12dh + O \left( d^2 \right) . \quad (3.14)$$

We can estimate the fourth term on the right-hand side of (3.11), using the orthogonality of $t(y)$ and $\nu(y)$ as:

$$2 \left[ t(y) \cdot \left( \frac{\tau(y) - x}{|x-y|} \right) \right] \left( \nabla_y \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) = O \left( \log \left( \frac{\tau(y) - x}{|x-y|} \right) \right) . \quad (3.15)$$
Moreover, using (3.11) and the orthogonality of \( t(y) \) and \( \nu(y) \) we estimate the fifth term on the right-hand side of (3.11) as:

\[
2hf \left( \nabla_y \left[ t(y) \cdot (\tau(y) - x) \right] \cdot t(y) \right) = 2hf + O \left( |\tau(y) - x|^2 \right) = 4dh + O \left( |\tau(y) - x|^2 \right). \tag{3.16}
\]

Therefore, combining (3.11)-(3.16):

\[
\Delta_y \left( |\tau(y) - x|^2 \right) = 4 + 16dh + O \left( |\tau(y) - x|^2 \right). \tag{3.17}
\]

On the other hand we compute \( \left( \nabla_y \left( |\tau(y) - x|^2 \right) \right)^2 \). To this end we use (3.10). Using that \( [\nabla_y f + 2\nu(y)] = O(d) \) as well as the fact that \((I - 2\nu \otimes \nu)\) is an isometry we obtain, after some computations:

\[
\left( \nabla_y \left( |\tau(y) - x|^2 \right) \right)^2 = 4 |\tau(y) - x|^2 + 16dh |\tau(y) - x|^2 + O \left( |\tau(y) - x|^4 \right). \tag{3.18}
\]

Combining (3.8), (3.17), (3.18) we obtain:

\[
|\Delta_y (\log (|\tau(y) - x|))] \leq C \quad \text{in} \quad \Omega \tag{3.19}
\]

for some constant \( C \) depending only on \( \Omega \).

We now derive an estimate for \( \nabla_x (\Delta_y (\log (|\tau(y) - x|)) \) with respect to \( x \). To this end we differentiate (3.8):

\[
\nabla_x \left( \Delta_y \left( \log \left( |\tau(y) - x|^2 \right) \right) \right)
= 2 \frac{(\tau(y) - x)}{|\tau(y) - x|^2} \Delta_y \left( |\tau(y) - x|^2 \right) - \frac{2}{|\tau(y) - x|^2} \nabla_y (\tau(y) - x) - \\
- \frac{4 (\tau(y) - x)}{|\tau(y) - x|^6} \left( \nabla_y \left( |\tau(y) - x|^2 \right) \right)^2 + \frac{4}{|\tau(y) - x|^4} \nabla_y \left( |\tau(y) - x|^2 \right) \cdot \nabla_y (\tau(y) - x) \tag{3.20}
\]

for \( d(y) \leq \sigma_0 \). Notice that, using Lemma 4 and the definition of \( \tau(y) \) we obtain, after some computations:

\[
\Delta_y (\tau_i(y) - x_i) = \nabla_y \cdot (\nabla_y f \nu_i(y) + f \nabla_y \nu_i(y)) = O(d) \tag{3.21}
\]

On the other hand:

\[
\nabla_y (\tau_i(y) - x_i) = \nabla_y (\tau_i(y)) = e_i - [2 + 2dh] \nu(y) \nu_i(y) + fht_i(y) t(y) + O(d^2) \tag{3.22}
\]

where \( e_i \) is a unit vector in the direction of the \( y_i \) axis. Combining (3.10) and (3.22) it then follows that:

\[
\nabla_y \left( |\tau(y) - x|^2 \right) \cdot \nabla_y (\tau(y) - x) = [2 + 8dh] (\tau(y) - x) + O \left( |\tau(y) - x|^3 \right) \tag{3.23}
\]

It then follows from (3.17), (3.18), (3.20), (3.21), (3.22):

\[
\nabla_x (\Delta_y (\log (|\tau(y) - x|)) ) \]
\[
= 2 \frac{(\tau(y) - x)}{|\tau(y) - x|^4} \left[ 4 + 16dh \right] - \frac{4 (\tau(y) - x)}{|\tau(y) - x|^2} \left[ 4 |\tau(y) - x|^2 + 16dh |\tau(y) - x|^2 \right] + \\
+ \frac{4 (\tau(y) - x)}{|\tau(y) - x|^4} [2 + 8dh] + O \left( \frac{1}{|\tau(y) - x|^2} \right) \tag{3.24}
\]

\[
= O \left( \frac{1}{|\tau(y) - x|^4} \right)
\]
Combining (3.19), (3.21) we obtain:

\[ |\Delta_y (Z (y) \log (|\tau (y) - x|))| \leq C \]  \hspace{1cm} (3.25)

\[ |\nabla_x (\Delta_y (Z (y) \log (|\tau (y) - x|)))| \leq C \left[ \frac{Z (y)}{|\tau (y) - x| + 1} \right] \]  \hspace{1cm} (3.26)

uniformly on \((x, y) \in \Omega \times \Omega\).

We can now prove the continuity of the function \(K (x, y)\) defined in (3.26). Notice that (3.22)-(3.24), (3.25), (3.26) imply:

\[ -\Delta_y (K (y, x)) = R_1 (y, x) \quad , \quad |R_1 (y, x)| \leq C \]

\[ \frac{\partial K}{\partial y} (y, x) = 0 \quad , \quad y \in \partial \Omega \quad , \quad x \in \Omega \]

\[ \left| \int_{\Omega} K (y, x) \, dy \right| \leq C_1 \quad , \quad x \in \Omega \]  \hspace{1cm} (3.27)

with \(C_1\) independent on \(x\).

Therefore, multiplying (3.28) by \(\nabla_x K (y, x)\), integrating with respect to the \(y\) variable and integrating by parts and using Sobolev and Hölder inequalities we obtain:

\[ \int_{\Omega} |\nabla_x K (y, x)|^p \, dy + \int_{\Omega} |\nabla_y (\nabla_x K (y, x))|^2 \, dy \leq C \quad , \text{for any } p < \infty \]

uniformly on \(x \in \Omega\). On the other hand, multiplying (3.27) by \(K (y, x)\) and using a similar argument we obtain:

\[ \int_{\Omega} |K (y, x)|^p \, dy + \int_{\Omega} |\nabla_y (K (y, x))|^2 \, dy \leq C \quad , \text{for any } p < \infty \]

uniformly on \(x \in \Omega\). Classical regularity theory for (3.27) then shows that \(K (y, x)\), \(\nabla_y K (y, x)\) are uniformly bounded in \(\Omega \times \Omega\).

Since \(\frac{\tau (x)}{|\tau (x) - x|} \in L^q (\Omega)\) for any \(q < 2\), it follows from classical regularity theory that \(\nabla_x K (\cdot, x) \in W^{2, q} (\Omega)\) for any \(q < 2\) with uniform bounds on \(x \in \Omega\). Therefore, Sobolev embeddings yield \(\nabla_x K (\cdot, x) \in W^{1, \sigma} (\Omega)\) for any \(\sigma < \infty\) and then \(\nabla_x K (\cdot, x)\) is Hölder in the \(y\) variable, uniformly on \(x \in \Omega\). It remains to prove continuity in the \(x\) variable. To this end we use (3.28) as well as (3.2-3.4) to write the representation formula:

\[ \nabla_x K (y, x) = \int_{\Omega} G (y, z) R_2 (z, x) \, dz = \int_{\Omega} G^* (y, z) R_2 (z, x) \, dz + \int_{\Omega} K (y, z) R_2 (z, x) \, dz \]  \hspace{1cm} (3.29)

Using the inequalities:

\[ |G^* (y, z) R_2 (z, x)| \leq C \left[ \frac{Z (z)}{|\tau (z) - x| + 1} \right] | \log (|z - y|) | \]

\[ |K (y, z) R_2 (z, x)| \leq C \left[ \frac{Z (z)}{|\tau (z) - x| + 1} \right] \]

as well as the continuity of the function \(R_2 (x, z)\) with respect to the \(x\) variable for \(x \neq z\) it then follows that the right-hand side of (3.29) is continuous and the result follows.
3.2 A geometric representation formula for $\nabla_x G$.

We write the Green’s function $\nabla_x G(y, x)$ in a more convenient form. To this end we introduce the following notation to denote the closest point to $x$ at the boundary $\partial \Omega$.

\[
P_\theta (x) = x + d(x) \nu(x), \quad \text{dist}(x, \partial \Omega) \leq \sigma_0 \quad \tag{3.30}
\]

The next lemma provides a suitable approximation for $\nabla_x G$ near $\partial \Omega$.

**Lemma 4** Assume that $G$ is as in Lemma 3 Then we can write:

\[
\nabla_x G(y, x) = \frac{1}{2\pi} \frac{(x - y)}{|x - y|^2} - \frac{Z(y) P_\theta (x) - P_\theta (y) - [d(x) \nu(x) + d(y) \nu(y)]}{2\pi D} - \frac{Z(y) h(y)}{2\pi} G_t(Y(x, y), \lambda_1(x, y), \lambda_2(x, y)) + g_n(Y(x, y), \lambda_1(x, y), \lambda_2(x, y)) \nu(y) + W(x, y) \quad \tag{3.31}
\]

where the operator $P_\theta$ is defined in (3.30), $d(\cdot)$ is as in Lemma 1, $W(x, y)$ is continuous in $\bar{\Omega} \times \bar{\Omega}$, and the functions $G_t$, $g_n$, $Y$, $\lambda_1$, $\lambda_2$ are given as:

\[
G_t(Y, \lambda_1, \lambda_2) = -2(\lambda_1 + \lambda_2) \lambda_2^2 Y + (\lambda_1 - \lambda_2) |Y|^2 Y \quad \tag{3.32}
\]

\[
g_n(Y, \lambda_1, \lambda_2) = \left[ -\lambda_2^2 + 2\lambda_2^2 \left( \lambda_1 + \lambda_2 \right) + \left( \lambda _2^2 - \lambda_1 \right) |Y|^2 \right] \quad \tag{3.33}
\]

\[
D = |P_\theta(x) - P_\theta(y)|^2 + (d(x) + d(y))^2 \quad \tag{3.34}
\]

\[
Y(x, y) = \frac{P_\theta(x) - P_\theta(y)}{\sqrt{|P_\theta(x) - P_\theta(y)|^2 + (d(x) + d(y))^2}}
\]

\[
\lambda_1(x, y) = \frac{d(x)}{\sqrt{|P_\theta(x) - P_\theta(y)|^2 + (d(x) + d(y))^2}}
\]

\[
\lambda_2(x, y) = \frac{d(y)}{\sqrt{|P_\theta(x) - P_\theta(y)|^2 + (d(x) + d(y))^2}}.
\]

**Remark 5** The first two terms in (3.31) are homogeneous functions of order $-1$. The terms $G_t$, $g_n$ are homogeneous functions of order zero that in the limit $|x - \tau(y)| \to 0$ yield respectively a tangential component and a normal component to $\partial \Omega$.

**Proof.** Our goal is to approximate $\nabla_x G$ that is given as (cf. (3.35)):

\[
\nabla_x G(y, x) = -\frac{1}{2\pi} \frac{(x - y)}{|x - y|^2} - \frac{Z(y) x - \tau(y)}{2\pi |x - \tau(y)|^2} + \nabla_x K(y, x). \quad \tag{3.35}
\]

Using (3.30) we obtain:

\[
x - \tau(y) = P_\theta(x) - P_\theta(y) - [d(x) \nu(x) + d(y) \nu(y)] - h(y)(d(y))^2 \nu(y), \quad \tag{3.36}
\]

whence it follows, after some computations:

\[
|x - \tau(y)|^2 = |P_\theta(x) - P_\theta(y)|^2 + (d(x) + d(y))^2 - 2(P_\theta(x) - P_\theta(y)) \cdot (d(x) \nu(x) + d(y) \nu(y)) + 2h(y)(d(y))^2(d(x) + d(y)) + O((d(x))^4 + (d(y))^4 + |x - y|^4). \quad \tag{3.37}
\]
Let us define $\ell(x,y) = (P_0(x) - P_0(y)) \cdot t(y)$. Using also $\nu(y) = \nu(P_0(y))$, $t(y) = t(P_0(y))$ we obtain:

\[
(P_0(x) - P_0(y)) = \ell(x,y) t(y) - \frac{h(P_0(y)) (\ell(x,y))^2}{2} \nu(y) + O\left( (\ell(x,y))^4 \right),
\]

\[
d(y)(P_0(x) - P_0(y)) \cdot \nu(y) = -\frac{h(P_0(y)) (\ell(x,y))^2 d(y)}{2} + O\left( (\ell(x,y))^4 \right),
\]

\[
d(x)(P_0(x) - P_0(y)) \cdot \nu(x) = \frac{h(P_0(x)) (\ell(x,y))^2 d(x)}{2} + O\left( (\ell(x,y))^4 \right),
\]

\[
(\ell(x,y))^2 = |P_0(x) - P_0(y)|^2 + O\left( |x - y|^3 \right).
\]

Then:

\[
2 (P_0(x) - P_0(y)) \cdot (d(x) \nu(x) + d(y) \nu(y)) = -h(y)(d(y) - d(x))(\ell(x,y))^2 + O\left( |x - y|^4 + (d(x))^4 + (d(y))^4 \right). \tag{3.38}
\]

Plugging (3.38) into (3.37) and using Taylor’s expansion we obtain:

\[
x - \tau(y)\left|\frac{x - \tau(y)}{D}\right|^2 = \frac{P_0(x) - P_0(y) - [d(x) \nu(x) + d(y) \nu(y)]}{D} + \mathcal{G}_t(Y(x,y), \lambda_1(x,y), \lambda_2(x,y)) + g_n(Y(x,y), \lambda_1(x,y), \lambda_2(x,y)) \nu(y) + \tilde{W}(x,y), \tag{3.39}
\]

where $\mathcal{G}_t$, $g_n$ are as in (3.32), (3.38) and $\tilde{W}(x,y)$ is a continuous function in $\Omega \times \Omega$ satisfying:

\[
\tilde{W}(x,y) = O(d(x) + d(y) + |x - y|) .
\]

Combining (3.35), (3.39), (3.31) and Lemma 4 follows with $W(x,y) = -\frac{Z(y)}{4\pi} \tilde{W}(x,y) + \nabla_x K(x,y)$.

## 4 Local mass change estimates.

In this Section we derive some crucial estimates for the local change of mass of $u$ for the solutions of (2.1), (2.2) or (2.4), (2.5). To this end we use the symmetrization argument as introduced in [8] and used also in [2], [7]. We will consider separately the cases of points $x_0$ that are at the interior of $\Omega$ and the points that are close to the boundary.

### 4.1 Interior estimates.

We will use an auxiliary test function $\varphi \in C^{1,1}(\mathbb{R}^+)$ defined as:

\[
\varphi(r) = 1 - \frac{r^2}{2}, \quad 0 \leq r \leq 1, \quad \varphi(r) = \frac{1}{2} - \log(r), \quad 1 \leq r \leq e^\frac{1}{4}
\]

\[
\varphi(r) = \frac{1}{e^\frac{1}{4}} \left( \frac{3e^{\frac{1}{4}}}{2} - r \right)^2, \quad e^{\frac{1}{4}} \leq r \leq \frac{3e^{\frac{1}{4}}}{2}, \quad \varphi(r) = 0, \quad r \geq \frac{3e^{\frac{1}{4}}}{2}
\] \tag{4.1}

Given $\rho > 0$ and $x_0 \in \Omega$, such that $d(x_0) \geq \frac{3e^{\frac{1}{4}}}{2}$ we define:

\[
\psi_\rho(x) = \varphi\left( \frac{|x - x_0|}{\rho} \right) \tag{4.2}
\]
Notice that (4.1) implies:
\[ \Delta \psi_\rho (x) = - \frac{2}{\rho^2} \text{ for } |x - x_0| < \rho , \quad \Delta \psi_\rho (x) = 0 \text{ for } \rho < |x - x_0| < e^{\frac{4}{5}} \rho , \] (4.3)
\[ \Delta \psi_\rho (x) \geq 0 \text{ for } e^{\frac{4}{5}} \rho < |x - x_0| \]

The letters \( \varphi, \psi \) will denote generic test functions that will change along the paper, but will be used consistently in each argument.

We have:

**Proposition 6** Suppose that \( u \) solves one of the problems (2.1), (2.2) or (2.4), (2.5). Let \( u \) fix \( \rho > 0 \) and let us assume that \( \text{dist} (x_0, \partial \Omega) \geq 2\rho \). Let \( \psi_\rho \) be as in (4.2). Then:
\[ \left| \partial_t \left( \int \Omega \psi_\rho u \right) \right| \leq \frac{\kappa}{\rho^2} \] (4.4)
if \( u \) solves (2.1), (2.2), and:
\[ \partial_t \left( \int \Omega \psi_\rho u \right) \geq - \frac{\kappa}{\rho^2} - \frac{2\varepsilon}{\rho^2} \int_{B_\rho(x_0)} u \frac{\partial \psi_\rho}{\partial y} dy \] (4.5)
if \( u \) solves (2.4), (2.5). The constant \( \kappa \) depends on \( \|u_0\|_{L^1(\Omega)} \), but it is independent on \( \varepsilon \) and \( \rho \).

**Proof.** Suppose that \( u \) solves (2.1), (2.2). Then, integrating by parts and using (2.1) we obtain:
\[ \partial_t \left( \int \Omega \psi_\rho u \right) - \int \Omega \Delta \psi_\rho (y) u \, dy - \int \Omega f_\varepsilon (u) \nabla \psi_\rho (y) \nabla v (y, t) \, dy = 0 \] (4.6)

We rewrite the fundamental solution \( G(x, y) \) using Lemma 3. It then follows that:
\[ G(y, x) = - \frac{1}{2\pi} \log (|x - y|) + G_0(y, x) \]
where, using that \( 3e^{\frac{4}{5}} < 2 \):
\[ |\nabla_y G_0(y, x)| \leq \frac{C}{\rho} \quad \text{, } |y - x_0| \leq \frac{3e^{\frac{4}{5}} \rho}{2} \] (4.7)

Then the following representation formula for \( \nabla v \) follows from (2.2) and Lemma 3
\[ \nabla_y v (y, t) = \frac{1}{2\pi} \int \Omega \frac{(x - y)}{|x - y|^2} f_\varepsilon (u(x, t)) \, dx + \int \Omega \nabla_y G_0(y, x) f_\varepsilon (u(x, t)) \, dx \]
and plugging this formula into (4.6) we obtain:
\[ \partial_t \left( \int_{B_\rho(x_0)} \psi_\rho u dy \right) - \int_{B_\rho(x_0)} \Delta \psi_\rho u dy - \frac{1}{2\pi} \int \Omega \int \Omega f_\varepsilon (u(x, t)) f_\varepsilon (u(y, t)) \frac{(x - y)}{|x - y|^2} \nabla \psi_\rho (y) \, dxdy - \int \Omega \int \Omega f_\varepsilon (u(x, t)) f_\varepsilon (u(y, t)) \nabla_y G_0(y, x) \nabla \psi_\rho (y) \, dxdy \]
\[ = 0 \]

The the third one in (4.8) can be estimated using the symmetrization argument introduced in [8]. Notice that:
\[ \left| \frac{(x - y)}{|x - y|^2} \cdot \left[ \nabla \psi_\rho (y) - \nabla \psi_\rho (x) \right] \right| \leq \frac{C}{\rho^2} \]
Then:
\[
\int_{\Omega} \int_{\Omega} f_{\varepsilon}(u(x, t)) f_{\varepsilon}(u(y, t)) \frac{(x - y)}{|x - y|^2} \nabla \psi_{\rho}(y) \, dx \, dy
\]
\[
= \frac{1}{2} \int_{\Omega} \int_{\Omega} f_{\varepsilon}(u(x, t)) f_{\varepsilon}(u(y, t)) \frac{(x - y)}{|x - y|^2} \cdot [\nabla \psi_{\rho}(y) - \nabla \psi_{\rho}(x)] \, dx \, dy \leq \frac{C}{\rho^2} . \quad (4.9)
\]

On the other hand, the linear term due to the laplacian can be estimated as:
\[
\left| \int_{B_{2\rho}(x_0)} \Delta \psi_{\rho} u dy \right| \leq \frac{C}{\rho^2} \quad (4.10)
\]
and (4.7) yields:
\[
\left| \int_{\Omega} \int_{\Omega} f_{\varepsilon}(u(x, t)) f_{\varepsilon}(u(y, t)) \nabla_y G_{\rho}(y, x) \nabla \psi_{\rho}(y) \, dx \, dy \right| \leq \frac{C}{\rho^2} . \quad (4.11)
\]

Combining (4.8)-(4.11) we obtain (4.4). If \( u \) solves (2.4), (2.5) a similar computation yields:
\[
\partial_t \left( \int_{\Omega} \psi_{\rho} u dy \right) - \int_{\Omega} \Delta \psi_{\rho} u dy - \varepsilon \int_{\Omega} \Delta \psi_{\rho} u^2 dy +
\]
\[
+ \frac{1}{2\pi} \int_{\Omega} \int_{\Omega} u(x, t) u(y, t) \frac{(x - y)}{|x - y|^2} \nabla \psi_{\rho}(y) \, dx \, dy -
\]
\[
- \int_{\Omega} \int_{\Omega} u(x, t) u(y, t) \nabla_y G_{\rho}(y, x) \nabla \psi_{\rho}(y) \, dx \, dy
\]
\[
= 0 . \quad (4.12)
\]

The last two terms on the left-hand side of (4.12) can be estimated as in the previous case. The main difference is in the nonlinear term in the laplacian that can be estimated using (cf. (2.6)):
\[
\int_{\Omega} \Delta \psi_{\rho} u^2 dy \geq - \frac{2}{\rho^2} \int_{B_{\rho}(x_0)} u^2 dy \quad (4.13)
\]
whence (4.5) and therefore Proposition 6 follows. \( \blacksquare \)

### 4.2 Boundary estimates.

We now derive the local mass growth estimate if the point \( x_0 \) is near the boundary. To this end we need to construct an auxiliary test function that will play a role analogous to the function \( \psi_{\rho} \) in Proposition 6. This will be made in the following lemma:

**Lemma 7** Let \( \sigma_0, \, \nu(x), \, d(x) \) be as in Lemma 7. There exists \( \rho_0 \leq \sigma_0 \) small enough, \( \Lambda > 1, \)
\( C > 0 \) depending only on \( \Omega \) such that, for any \( \rho \in (0, \rho_0) \) and any \( x_0 \in \Omega \) with \( \text{dist}(x_0, \partial \Omega) \leq 2\rho \) there exists a function \( \psi_{\rho} \in C^{1,1}(\Omega) \) with the following properties:

\[
\Delta \psi_{\rho} = - \frac{2}{\rho^2} \text{ in } B_{\rho}(x_0) \cap \Omega
\]
\[
\Delta \psi_{\rho} \geq 0 \text{ in } [B_{\Lambda \rho}(x_0) \setminus B_{\rho}(x_0)] \cap \Omega
\]
\[
\partial_{\nu} \psi_{\rho} = 0 \text{ in } B_{\Lambda \rho}(x_0) \cap \partial \Omega \quad , \quad \psi_{\rho} = 0 \text{ in } \Omega \setminus B_{\Lambda \rho}(x_0)
\]
\[
\psi_{\rho} \geq \frac{1}{2} \text{ in } B_{\rho}(x_0) \cap \Omega \quad , \quad 0 \leq \psi_{\rho} \leq 1 \text{ in } B_{\Lambda \rho}(x_0) \cap \Omega
\]
\[
\rho |\nabla \psi_{\rho}| + \rho^2 |\nabla^2 \psi_{\rho}| + \frac{\rho^2 |\nu(x) \cdot \nabla \psi_{\rho}(x)|}{d(x)} \leq C \text{ in } B_{\Lambda \rho}(x_0) \cap \Omega
\]
\textbf{Proof.} The main idea is that for $\rho_0$ sufficiently small the problem can be treated as a perturbation of the problem in the half-plane. We introduce a rescaled system of coordinates:

$$X = \frac{x - P_0(x_0)}{\rho}, \quad X_0 = \frac{x_0 - P_0(x_0)}{\rho}$$

where the operator $P_0(x_0)$ is defined as in (3.30). Notice that the assumption $\text{dist}(x_0, \partial \Omega) \leq 2\rho$ implies $|X_0| \leq 2$. Rotating the coordinate system we can assume that the normal vector $\nu(P_0(x_0))$ is $(0, -1)$. We construct $\Psi(X)$ in the half-plane solving the problem:

$$\Delta_X \Psi(X) = -1, \quad X \in B_1(X_0) \cap \{X = (X_1, X_2) : X_2 > 0\}$$

$$\partial_{\nu_0} \Psi(X) = 0, \quad X \in \partial [B_1(X_0) \cap \{X = (X_1, X_2) : X_2 > 0\}]$$

(4.14) \hspace{1cm} (4.15)

where $\nu_0 = (0, -1)$. This problem can be solved using the reflection method. We can obtain a family of solutions for it in the form:

$$\tilde{\Psi}(X) = A + \Psi(X), \quad \tilde{\Psi}(X) = -\frac{1}{2\pi} \int_{B_1(X_0) \cup B_1(X_0 + 2\nu_0)} \log(|X - Y|) \, dY$$

(4.16)

where $A$ is an arbitrary constant to be precised. Notice that $\tilde{\Psi}(X)$ is bounded in $|X| \leq 1$ and it satisfies:

$$\left| \tilde{\Psi}(X) + \frac{m}{2\pi} \log(|X|) \right| \leq \frac{C}{|X|^2} \quad \text{for} \quad |X| = \lambda_0$$

(4.17)

$$\left| \nabla_X \tilde{\Psi}(X) + \frac{m}{2\pi} \frac{X}{|X|^2} \right| \leq \frac{C}{|X|^3} \quad \text{for} \quad |X| = \lambda_0$$

(4.18)

with $\lambda_0$ sufficiently large and $C$ independent on $\lambda_0$ and where $m = |B_1(X_0) \cup B_1(X_0 + 2\nu_0)|$. Notice that $m$ is bounded above and below by constants independent on $X_0$. In the derivation of (4.17), (4.18) we have used the fact that $\int_{B_1(X_0) \cup B_1(X_0 + 2\nu_0)} Y \, dY = 0$. We then define:

$$\Phi(X) = \frac{m}{4\pi\lambda_0} (\lambda_0 + 1 - |X|)^2 \quad \text{for} \quad |X| \geq \lambda_0$$

and choose $A$ in (4.14) as:

$$A = \frac{m}{4\pi\lambda_0} + \frac{m}{2\pi} \log(\lambda_0)$$

It then follows from (4.17), (4.18) that:

$$\left| \tilde{\Psi}(X) - \Phi(X) \right| \leq \frac{C}{\lambda_0^2}, \quad \left| \nabla_X \tilde{\Psi}(X) - \nabla_X \Phi(X) \right| \leq \frac{C}{\lambda_0^3} \quad \text{for} \quad |X| = \lambda_0$$

$$\Delta_X \Phi(X) \geq \frac{m}{4\pi\lambda_0} \quad \text{for} \quad \lambda_0 \leq |X| \leq \lambda_0 + 1$$

$$\tilde{\Psi}(X) \geq 1 \quad \text{for} \quad |X| \leq 1$$

if $\lambda_0$ is sufficiently large. Let us consider an function $W \in C^2(\mathbb{R}^2 \setminus B_{\lambda_0}(0))$ and satisfying:

$$W(X) = \tilde{\Psi}(X) - \Phi(X), \quad \nabla_X W(X) = \nabla_X \tilde{\Psi}(X) - \nabla_X \Phi(X) \quad \text{for} \quad |X| = \lambda_0,$$

$$W(X) = 0 \quad \text{for} \quad |X| \geq \lambda_0 + 1, \quad |\Delta_X W(X)| \leq \frac{C}{\lambda_0^2}$$

Then, the function $\Psi \in C^{1,1}(\{X_2 \geq 0\})$ defined as:

$$\Psi(X) = \tilde{\Psi}(X) \quad \text{for} \quad |X| < \lambda_0$$

$$\Psi(X) = \Phi(X) + W(X) \quad \text{for} \quad |X| \geq \lambda_0$$
satisfies (4.14), (4.15) as well as:
\[
\Delta_X \Psi (X) \geq \frac{m}{8\pi \lambda_0} \text{ for } \lambda_0 \leq |X| < \lambda_0 + 1, \quad |X| \neq \lambda_0
\]
\[
\Psi (X) = 0 \text{ for } |X| \geq \lambda_0 + 1
\] (4.19)

if \( \lambda_0 \) is sufficiently large.

The function \( \Psi \) would provide a solution of the desired problem for planar \( \partial \Omega \). In order to take into account curvature effects we study the family of problems:
\[
\Delta_X \tilde{\Psi} = \Delta_X \Psi (X) \quad \text{in} \quad \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \cap B_{\lambda_0 + 2} (0)
\]
\[
\partial_{\nu} \tilde{\Psi} = 0 \quad \text{in} \quad \left[ \partial \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \right] \cap B_{\lambda_0 + 2} (0)
\]
\[
\tilde{\Psi} = 0 \quad \text{in} \quad \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \cap \partial B_{\lambda_0 + 2} (0)
\]

Classical continuous dependence results on the domain show that \( |\tilde{\Psi} - \Psi| \) can be made arbitrarily small for \( \rho \leq \rho_0 \) small. We now construct \( \tilde{W} \) satisfying:
\[
\tilde{W} = 0 \quad \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \cap \partial B_{\lambda_0 + 2} (0) \quad , \quad \partial_{\nu} \tilde{W} = 0 \quad \text{in} \quad \left[ \partial \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \right] \cap B_{\lambda_0 + 2} (0)
\]
and
\[
\partial_{\nu} \tilde{W} = \partial_{\nu} \tilde{\Psi} \quad \text{in} \quad \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \cap B_{\lambda_0 + 2} (0) \quad , \quad \tilde{W} = 0 \quad \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \cap B_\lambda (0)
\]
as well as \( |\Delta \tilde{W}| \) small in \( \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \cap B_{\lambda_0 + 2} (0) \), something that it is possible for \( \rho \leq \rho_0 \) small. Then the function \( \Psi_c = \tilde{\Psi} - \tilde{W} \) satisfies:
\[
\Delta_X \Psi_c = -1 \quad \text{in} \quad \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \cap B_1 (0) \quad , \quad \Delta_X \Psi_c \geq 0 \quad \text{in} \quad \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \setminus B_1 (0)
\]
\[
\partial_{\nu} \Psi_c = 0 \quad \text{in} \quad \left[ \partial \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right) \right]
\]
\[
\Psi_c \in C^{1,1} \left( \frac{\Omega - P_\theta (x_0)}{\rho} \right)
\]
\[
\Psi_c (X) = 0 \quad \text{for} \quad |X| \geq \lambda_0 + 2
\]

The function \( \psi_\rho (x) = \Psi_c \left( \frac{x - P_\theta (x_0)}{\rho} \right) \) then satisfies all the properties required in Lemma 7 for \( \rho \leq \rho_0 \). ■

**Proposition 8** Suppose that \( u \) solves one of the problems (2.1), (2.2) or (2.4), (2.5). Let us fix \( 0 < \rho < \rho_0 \) with \( \rho_0 \) as in Lemma 7 and let us assume that \( \text{dist} (x_0, \partial \Omega) \leq 4 \rho \). Then:
\[
|\partial_t \left( \int_\Omega \psi_\rho u \right) | \leq \frac{\kappa}{\rho^2} , \quad 0 < t < \infty
\] (4.20)

if \( u \) solves (2.1), (2.2), and:
\[
|\partial_t \left( \int_\Omega \psi_\rho u \right) | \geq -\frac{\kappa}{\rho^2} - \frac{2 \varepsilon}{\rho^3} \int_{B_\rho (x_0)} u^2 , \quad 0 < t < \infty
\] (4.21)

if \( u \) solves (2.4), (2.5). The constant \( \kappa \) depends only on \( \|u_0\|_{L^1 (\Omega)} \), but it is independent on \( \varepsilon \) and \( \rho \).
Proof. Arguing as in the derivation of (4.18), (4.12) and using Lemmas 2 and 3 we obtain:

\[
\begin{align*}
&\partial_t \left( \int_{\Omega} \psi_p u dx \right) - \int_{\Omega} \Delta \psi_p u dx + \\
&+ \frac{1}{2\pi} \int_{\Omega} \int_{\Omega} f_x \left( u(x,t) \right) f_x \left( u(y,t) \right) \frac{(x-y)}{|x-y|^2} \nabla \psi_p(x) dxdy - \\
&+ \frac{1}{2\pi} \int_{\Omega} \int_{\Omega} f_x \left( u(x,t) \right) f_x \left( u(y,t) \right) \frac{(x-\tau(y))}{|x-\tau(y)|^2} \nabla \psi_p(x) dxdy - \\
&- \int_{\Omega} \int_{\Omega} f_x \left( u(x,t) \right) f_x \left( u(y,t) \right) \nabla_x K(y,x) \nabla \psi_p(x) dxdy \\
&= 0
\end{align*}
\] (4.22)

where \( \psi_p \) is now chosen as in Lemma 7. Using this lemma we can estimate all the terms in (4.22) as in the proof of Proposition 6 except the fourth term in (4.22). We estimate first the contribution to this term of the region where \(|x-\tau(y)| \geq \rho\) using Lemma 7 and as well as the mass conservation property (2.6): Combining (4.23) with (4.25)-(4.28) we obtain:

\[
\left| \int_{\Omega \times \Omega \cap \{|x-\tau(y)| \geq \rho\}} f_x \left( u(x,t) \right) f_x \left( u(y,t) \right) \frac{(x-\tau(y))}{|x-\tau(y)|^2} \nabla \psi_p(x) dxdy \right| \leq \frac{C}{\rho^2}
\] (4.23)

In order to estimate the contribution of the region where \(|x-\tau(y)| \leq \rho\) we use the fact that for \(\rho_0\) sufficiently small

\[
\frac{1}{4} \left[ |x-\tau(x)| + |y-\tau(y)| \right] \leq d(x) + d(y) \leq 3|x-\tau(y)|.
\] (4.24)

Symmetrizing (4.23) we obtain:

\[
\begin{align*}
&f_x \left( u(x,t) \right) f_x \left( u(y,t) \right) \frac{(x-\tau(y))}{|x-\tau(y)|^2} \nabla \psi_p(x) \\
&= \frac{f_x \left( u(x,t) \right) f_x \left( u(y,t) \right)}{2|x-\tau(y)|^2} \left[ (x-\tau(y)) \nabla \psi_p(x) + (y-\tau(x)) \nabla \psi_p(y) \right]
\end{align*}
\] (4.25)

Notice that:

\[
\begin{align*}
&(x-\tau(y)) \nabla \psi_p(x) + (y-\tau(x)) \nabla \psi_p(y) \\
&= (x-\tau(x)) \nabla \psi_p(x) + (y-\tau(y)) \nabla \psi_p(y) + (\tau(x) - \tau(y)) \left[ \nabla \psi_p(x) - \nabla \psi_p(y) \right]
\end{align*}
\] (4.26)

Lemma 7 as well as the fact that \(|\tau(x) - \tau(y)| \leq 2|x-y| \leq 3|x-\tau(y)|\) yields:

\[
|\tau(x) - \tau(y)| \left| \nabla \psi_p(x) - \nabla \psi_p(y) \right| \leq \frac{C}{\rho^2} |x-\tau(y)|^2
\] (4.27)

On the other hand, using Lemma 7 we obtain:

\[
\left| (x-\tau(x)) \nabla \psi_p(x) \right| + \left| (y-\tau(y)) \nabla \psi_p(y) \right| \leq \frac{C}{\rho^2} \left( |x-\tau(x)| d(x) + |y-\tau(y)| d(y) \right)
\] (4.28)

Combining (4.23) with (4.25)-(4.28) we obtain:

\[
\left| \int_{\Omega \times \Omega} f_x \left( u(x,t) \right) f_x \left( u(y,t) \right) \frac{(x-\tau(y))}{|x-\tau(y)|^2} \nabla \psi_p(x) dxdy \right| \leq \frac{C}{\rho^2}
\]

This concludes the proof of (4.20). The proof of (4.21) is similar. \(\blacksquare\)
5 An entropy estimate.

Entropy estimates for the study of Keller-Segel models were introduced in [3] and they have been extensively used for the analysis of chemotaxis models. We will use the following estimate for the solutions of the second regularization considered above (2.4), (2.5).

**Lemma 9** Let us assume that \((u, v)\) solves (2.4), (2.5) with bounded initial data \(u(x, 0) = u_0(x)\) and \(\varepsilon > 0\). Then, for any \(\alpha > 0\) there exists \(C\) depending only on \(\alpha, u_0, \Omega\) such that:

\[
\varepsilon^{1+\alpha} \int_{\Omega} u^\frac{7}{6} dx \leq C, \quad 0 < t < \infty
\]

**Proof.** We use the following entropy formula that can be easily checked integrating by parts for the solutions of (2.4), (2.5):

\[
\partial_t \left( \int_{\Omega} \left[ u \left( \log (u) - 1 \right) + 6 \varepsilon u^\frac{7}{6} - \frac{|\nabla v|^2}{2} \right] dx \right) = - \int u \left[ \nabla \left( \log (u) + 7 \varepsilon u^\frac{1}{6} \right) - \nabla v \right]^2 dx \leq 0
\]

Then, since \(\int_{\Omega} u \log u \geq C\):

\[
\varepsilon \int_{\Omega} u^\frac{7}{6} dx \leq C + \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx
\]

where \(C\) depends only on \(u_0\) and \(\Omega\).

Classical regularity theory for the Poisson equation yields:

\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 dx \leq C \left( \int_{\Omega} u^p dx \right)^\frac{2}{p}
\]

for any \(p > 1\), with \(C\) depending only on \(p\) and \(\Omega\). Then:

\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 dx \leq C \left( \int_{\Omega} u^p dx \right)^\frac{2}{p} \leq C \left( \int_{\Omega} u^{\frac{p+q}{p-q}} dx \right)^\frac{2(q-1)}{pq} \left( \int_{\Omega} u^q dx \right)^\frac{p}{pq}
\]

for any \(q > 1\). Choosing \(1 < p < \frac{7}{6}\) and \(q = \frac{1}{1-6p}\) we obtain:

\[
\varepsilon \int_{\Omega} u^\frac{7}{6} dx \leq C \left( \int_{\Omega} u^\frac{7}{6} dx \right)^\frac{12(p-1)}{p} \left( \int_{\Omega} u_0 dx \right)^\frac{2(1-6p)}{p} \leq C \left( \int_{\Omega} u^\frac{7}{6} dx \right)^\frac{12(p-1)}{p}
\]

where \(p > 1\) can be chosen arbitrarily close to one. Young’s inequality then implies:

\[
\varepsilon^{1+\alpha} \int_{\Omega} u^\frac{7}{6} dx \leq C
\]

where \(C\) depends on \(\alpha, u_0\) and \(\Omega\). 

6 \(L^2\) estimates.

We now prove some estimates ensuring that the solutions of (2.1), (2.2) or (2.4), (2.5) are smooth in regions where the amount of mass of \(u\) is small.
6.1 Interior estimates: First regularization.

We consider first the regularization of Keller-Segel system in (2.1), (2.2).

**Proposition 10** Given \( M > 0, \kappa > 0 \) there exist \( m_0 > 0 \), independent of \( M, \kappa, \varepsilon \) and positive constants \( c_i, i = 1, 2 \) depending on \( M, \kappa \) but independent of \( \varepsilon \), such that for each \( 0 < \rho \leq 1 \) and any solution \((u, v)\) of

\[
\begin{align*}
\partial_t u - \Delta u + \nabla (f_\varepsilon(u) \nabla v) &= 0 \quad \text{in} \quad (x, t) \in B_{4\rho}(0) \times (\bar{t} - c_1\rho^2, \bar{t}) \quad (6.1) \\
-\Delta v &= f_\varepsilon(u) - h(t) \quad \text{in} \quad (x, t) \in B_{4\rho}(0) \times (\bar{t} - c_1\rho^2, \bar{t}) \quad (6.2)
\end{align*}
\]

satisfying:

\[
\begin{align*}
\partial_t \left( \int_{\mathbb{R}^2} \varphi \left( \frac{|x|}{2\rho} \right) u(x, t) \, dx \right) &\geq -\frac{\kappa}{\rho^2}, \quad t \in (\bar{t} - c_1\rho^2, \bar{t}) \quad (6.3) \\
\int_{B_{4\rho}(0)} u(x, \bar{t}) \, dx &\leq m_0 \\
\sup_{t \in (\bar{t} - c_1\rho^2, \bar{t})} \|v(\cdot, t)\|_{L^6(B_{4\rho}(0))} &\leq M \quad (6.4) \\
0 &\leq h(t) \leq M \quad (6.5)
\end{align*}
\]

with \( \varphi \) as in (4.1).

Then, the following inequality holds:

\[
\sup_{s \in [\bar{t} - c_1\rho^2, \bar{t}]} \int_{B_{4\rho}(0)} u^2(x, s) \, dx \leq \frac{C_2}{\rho^4}.
\]

An essential ingredient in the proof of Proposition 10 is the following lemma that has been obtained before in slightly different forms, but that we prove here by the reader’s convenience.

**Lemma 11** For any \( \delta > 0 \) there exists \( C > 0 \) independent on \( \delta \) such that for any \( u \in W^{1,2}_\text{loc}(\mathbb{R}^2) \), any compactly supported function \( \eta \in C^\infty(\mathbb{R}^2) \) there holds:

\[
\int u^3 \eta^6 \, dx \leq \frac{9(1 + \delta)}{16\pi} \left[ \int |\nabla u|^2 \eta^6 \, dx \right]^{\frac{1}{2}} \left[ \int_{\text{supp}(\eta)} u \, dx \right] + \frac{C}{\delta^5} \|\nabla \eta\|_{L^\infty}^6 \left( \int_{\text{supp}(\eta)} u \, dx \right)^3 \left( \int_{\text{supp}(\eta)} \, dx \right)
\]

**Proof.** We apply the following classical Sobolev estimate in the critical case

\[
\int w^2 \, dx \leq \frac{1}{4\pi} \left( \int |\nabla w| \, dx \right)^2, \quad w \in W^{1,1}(\mathbb{R}^2)
\]

to the particular function \( w = \frac{u^2}{4} \eta^3 \). Then:

\[
\int u^3 \eta^6 \, dx \leq \frac{9(1 + \frac{\delta}{2})}{16\pi} \left[ \int |\nabla u|^2 \eta^6 \, dx \right]^{\frac{1}{2}} \left[ \int_{\text{supp}(\eta)} u \, dx \right] + \frac{C}{\delta} \left[ \int u^2 \eta^2 |\nabla \eta| \, dx \right]^2
\]

Applying Hölder and Young’s inequalities:

\[
\left[ \int u^2 \eta^2 |\nabla \eta| \, dx \right]^2 \leq \delta^2 \int u^3 \eta^6 \, dx + \frac{C}{\delta^4} \|\nabla \eta\|_{L^\infty}^6 \left( \int_{\text{supp}(\eta)} u \, dx \right)^3 \left( \int_{\text{supp}(\eta)} \, dx \right)
\]

and the result follows. \( \blacksquare \)
Proof of Proposition 10. Let \( \eta = \eta(x) \) be a cutoff function satisfying \( \eta(x) = 1 \) for \( |x| \leq \rho \), \( \eta(x) = 0 \) for \( |x| \geq 2\rho \), \( \eta \in C^\infty \), decreasing on \( |x| \) and satisfying \( \rho |\nabla \eta| + \rho^2 |\nabla^2 \eta| \leq C \). Let us denote \( t_0 = t - 2c_1\rho^2 \), where \( c_1 \) will be precised later. Multiplying (6.1) by the test function \( u\eta^6 (t-t_0)^\beta \) with \( \beta \geq 2 \) we obtain, after integrating by parts:

\[
\partial_t \left( \int \frac{u^2}{2} \eta^6 (t-t_0)^\beta \, dx \right) = -\int |\nabla u|^2 \eta^6 (t-t_0)^\beta \, dx + \frac{\beta}{2} \int u^2 \eta^6 (t-t_0)^{\beta-1} \, dx - 6 \int u \nabla u \eta^5 \nabla \eta (t-t_0)^\beta \, dx + \int f_\varepsilon (u) \nabla u \eta^6 (t-t_0)^\beta \, dx + 6 \int u f_\varepsilon (u) \nabla v \eta^5 \nabla \eta (t-t_0)^\beta \, dx
\]

We integrate by parts again to bring the eliminate the derivatives of \( u \) in the fourth term on the right. Then, if we define \( F_\varepsilon (u) = \int_0^\varepsilon f_\varepsilon (s) \, ds \) and use (6.2) we obtain:

\[
\partial_t \left( \int \frac{u^2}{2} \eta^6 (t-t_0)^\beta \, dx \right) = -\int |\nabla u|^2 \eta^6 (t-t_0)^\beta \, dx + \frac{\beta}{2} \int u^2 \eta^6 (t-t_0)^{\beta-1} \, dx - 6 \int u \nabla u \eta^5 \nabla \eta (t-t_0)^\beta \, dx + \int F_\varepsilon (u) \eta^6 (t-t_0)^\beta \, dx + 6 \int [u f_\varepsilon (u) - F_\varepsilon (u)] \eta^5 \nabla v \eta (t-t_0)^\beta \, dx
\]

Eliminating the derivatives of \( v \) in the last integral we arrive at:

\[
\partial_t \left( \int \frac{u^2}{2} \eta^6 (t-t_0)^\beta \, dx \right) = -\int |\nabla u|^2 \eta^6 (t-t_0)^\beta \, dx + \frac{\beta}{2} \int u^2 \eta^6 (t-t_0)^{\beta-1} \, dx - 6 \int u \nabla u \eta^5 \nabla \eta (t-t_0)^\beta \, dx + \int F_\varepsilon (u) f_\varepsilon (u) \eta^6 (t-t_0)^\beta \, dx - 6 \int \nabla [u f_\varepsilon (u) - F_\varepsilon (u)] \eta^5 v \nabla \eta (t-t_0)^\beta \, dx - 30 \int [u f_\varepsilon (u) - F_\varepsilon (u)] \eta^4 v (\nabla \eta)^2 (t-t_0)^\beta \, dx - \int F_\varepsilon (u) h(t) \eta^6 (t-t_0)^\beta \, dx - 6 \int [u f_\varepsilon (u) - F_\varepsilon (u)] \eta^5 v \Delta \eta (t-t_0)^\beta \, dx
\]

The last two terms can be estimated easily:

\[
\left\| \int [u f_\varepsilon (u) - F_\varepsilon (u)] \eta^4 v (\nabla \eta)^2 (t-t_0)^\beta \, dx \right\| \leq \frac{C}{\rho^2} \left( \int u^3 \eta^6 (t-t_0)^\beta \, dx \right)^\frac{1}{2} \left( \int v^3 (t-t_0)^\beta \, dx \right)^\frac{1}{2} \leq \frac{C}{\rho^2} \left( \int u^3 \eta^6 (t-t_0)^\beta \, dx \right)^\frac{1}{2} \left( \int v^3 (t-t_0)^\beta \, dx \right)^\frac{1}{2} \leq \delta \int u^3 \eta^6 (t-t_0)^\beta \, dx + \frac{C}{\rho^2} \int_{\text{supp} (\eta)} v^3 (t-t_0)^\beta \, dx
\]

\[
\left\| \int [u f_\varepsilon (u) - F_\varepsilon (u)] \eta^5 v \Delta \eta (t-t_0)^\beta \, dx \right\| \leq \frac{C}{\rho^2} \left( \int u^3 \eta^6 (t-t_0)^\beta \, dx \right)^\frac{1}{2} \left( \int v^3 \eta^3 (t-t_0)^\beta \, dx \right)^\frac{1}{2} \leq \delta \int u^3 \eta^6 (t-t_0)^\beta \, dx + \frac{C}{\rho^2} \int v^3 \eta^3 (t-t_0)^\beta \, dx
\]
On the other hand, using that $|f'_{\varepsilon}| \leq 1$, $|F'_{\varepsilon}| = |f_{\varepsilon}| \leq |u|$ we obtain:

$$\left| \int u\nabla u \eta^5 \nabla (t-t_0)^{\beta} \, dx \right| + \left| \int \nabla [u f_{\varepsilon}(u) - F_{\varepsilon}(u)] \eta^5 \nabla (t-t_0)^{\beta} \, dx \right| \leq \frac{C}{\rho} \left( \int u^3 \eta^6 (t-t_0)^{\beta} \, dx \right)^{\frac{1}{2}} \left( \int |\nabla u|^2 \eta^6 (t-t_0)^{\beta} \, dx \right)^{\frac{1}{2}} \left( \int (1+|v|)^6 (t-t_0)^{\beta} \, dx \right)^{\frac{1}{2}}$$

(6.10)

$$\leq \delta \int u^3 \eta^6 (t-t_0)^{\beta} \, dx + \delta \int |\nabla u|^2 \eta^6 (t-t_0)^{\beta} \, dx + \frac{C}{\rho^2 \delta^2} \int (1+|v|)^6 (t-t_0)^{\beta} \, dx$$

We also have, using $|F'_{\varepsilon}(u)| \leq \frac{\eta^2}{2}$ and Hölder’s inequality

$$\left| \int u^2 \eta^6 (t-t_0)^{\beta-1} \, dx \right| + \left| \int F_{\varepsilon}(u) h(t) \eta^6 (t-t_0)^{\beta} \, dx \right| \leq \delta \int u^3 \eta^6 (t-t_0)^{\beta} \, dx + \frac{C (t-t_0)^\beta}{\delta} \int u \eta^6 \, dx$$

(6.11)

where $C > 0$ depends only on $M$.

The most delicate term is the fourth one on the right-hand side of (6.7). This term can be estimated using Lemma 11 as:

$$\left| \int F_{\varepsilon}(u) f_{\varepsilon}(u) \eta^6 (t-t_0)^{\beta} \, dx \right| \leq \frac{9 (1+\delta)}{32\pi} \left[ \int |\nabla u|^2 \eta^6 (t-t_0)^{\beta} \, dx \right] \left[ \int_{\text{supp}(u)} u \, dx \right] + \frac{C (t-t_0)^\beta}{\delta^5} \left( \int_{\text{supp}(\eta)} u \, dx \right)^3 \left( \int_{\text{supp}(\eta)} \, dx \right)$$

(6.12)

Combining (6.7)-(6.12) we obtain:

$$\partial_t \left( \int \frac{u^2}{2} \eta^6 (t-t_0)^{\beta} \, dx \right)$$

(6.13)

$$\leq - (1-\delta) \int |\nabla u|^2 \eta^6 (t-t_0)^{\beta} \, dx + \frac{9 (1+\delta)}{32\pi} \left[ \int |\nabla u|^2 \eta^6 (t-t_0)^{\beta} \, dx \right] \left[ \int_{\text{supp}(u)} u \, dx \right] + \frac{C (t-t_0)^\beta}{\delta^5 \rho^2} \left( \int_{\text{supp}(\eta)} u \, dx \right)^3 \left( \int_{\text{supp}(\eta)} \, dx \right) + \frac{C}{\rho^2 \delta^2} \int_{\text{supp}(\eta)} \eta^6 (t-t_0)^{\beta} \, dx + \frac{C (t-t_0)^{\beta-2}}{\delta} \int_{\text{supp}(\eta)} \, dx$$

where we have estimated all the terms $\delta \int u^3 \eta^6 (t-t_0)^{\beta} \, dx$ on the right-hand side of (6.8)-(6.11) using Lemma 11. The values of $\delta$ and $C$ have been then changed.

Let us write $m_0 = \frac{8\pi}{27}$. Using assumptions (6.3), (6.4) as well as the definition of $\varphi$ it follows that, if $c_1$ is chosen sufficiently small (although independent on $\rho$), we have:

$$\int_{B_{2\rho}(0)} u(x,t) \, dx \leq 3m_0 \text{ for } t \in (\bar{t} - 2c_1 \rho^2, \bar{t})$$

Then, if $\delta$ is small enough, it follows from (6.5), (6.13) that:

$$\partial_t \left( \int \frac{u^2}{2} \eta^6 (t-t_0)^{\beta} \, dx \right) \leq \frac{C (t-t_0)^\beta}{\delta^5 \rho^2} m_0^3 \rho^2 + \frac{C (t-t_0)^\beta}{\rho^2 \delta^2} M^3 \rho + \frac{C (t-t_0)^{\beta-2}}{\delta} \left[ \rho^2 + M^6 \right]$$

$$+ \frac{C m_0 (t-t_0)^{\beta-2}}{\delta}$$
and assuming that \( \rho_0 \) is small enough and \( M \) is of order one, without loss of generality, we obtain:

\[
p_t \left( \int \frac{u^2}{\rho^6} (t - t_0)^\beta \, dx \right) \leq \frac{K (t - t_0)^\beta}{\rho^6 \delta^5} + \frac{C_{m_0} (t - t_0)^{\beta - 2}}{\delta}
\]

where \( K \) depends on \( M \). Integrating this formula, with \( \beta = 2 \) in the interval \( t \in (t_0, \bar{t}) \), and using that \( t_0 = \bar{t} - 2c_1 \rho^2 \) it follows that:

\[
\int (u(x, t))^2 \eta^6 \, dx \leq K \left[ \frac{\bar{t} - t_0}{\rho^5 \delta^5} + \frac{1}{\delta (t - t_0)} \right] \leq \frac{K}{\delta^5 \rho^4} \quad s \in \left[ \bar{t} - c_1 \rho^2, \bar{t} \right]
\]

and since \( \delta \) is of order one (although small) the result follows just changing \( \frac{\rho^2}{\delta^2} \) by \( c_1 \). 

6.2 Interior estimates: Second regularization.

We now derive interior estimates for the regularization in (2.4), (2.5).

**Proposition 12** Given \( M > 1, \kappa > 0 \) there exist \( m_0 > 0 \) independent of \( M, \kappa, \varepsilon \), and positive constants \( c_i, i = 1, 2, \varepsilon_0 > 0, \rho_0 > 0 \) depending on \( M, \kappa \) but independent of \( \varepsilon \), such that for each \( 0 < \rho \leq \rho_0, 0 < \varepsilon \leq \varepsilon_0 \) and any solution \((u, v)\) of

\[
\begin{align*}
\partial_t u - \Delta \left( u + \varepsilon u^{\frac{2}{3}} \right) + \nabla (u \nabla v) &= 0 \quad \text{in} \quad (x, t) \in B_{4\rho}(0) \times \left( \bar{t} - c_1 \rho^2, \bar{t} \right) \quad (6.14) \\
\Delta v &= u - h(t) \quad \text{in} \quad (x, t) \in B_{4\rho}(0) \times \left( \bar{t} - c_1 \rho^2, \bar{t} \right) \quad (6.15)
\end{align*}
\]

satisfying:

\[
\begin{align*}
\partial_t \left( \int_{\Omega} \varphi \left( \frac{|x|}{2\rho} \right) u(x, t) \, dx \right) &\geq -\frac{\kappa}{\rho^2} - \frac{2\varepsilon}{\rho^2} \int_{B_{\rho}(0)} u^{\frac{2}{3}}(x, t) \, dx \quad t \in \left( \bar{t} - c_1 \rho^2, \bar{t} \right) \quad (6.16) \\
\sup_{t \in \left( \bar{t} - 2c_1 \rho^2, \bar{t} \right)} \left[ \frac{1}{\rho} \int_{B_{\rho}(0)} u^{\frac{2}{3}}(x, t) \, dx \right] &\leq M \quad (6.18) \\
\sup_{t \in \left( \bar{t} - c_1 \rho^2, \bar{t} \right)} \| v(\cdot, t) \|_{L^6(B_{4\rho}(0))} &\leq M \quad (6.19) \\
0 &\leq h(t) \leq M \quad (6.20)
\end{align*}
\]

with \( \varphi \) as in (4.7).

Then, the following inequality holds:

\[
\sup_{s \in \left( \bar{t} - c_1 \rho^2, \bar{t} \right)} \int_{B_{\rho}(0)} u^{\frac{2}{3}}(x, s) \, dx \leq \frac{c_2}{\rho^6} \quad (6.21)
\]

**Proof.** Let us assume that \( \eta \) is the same cutoff function used in the proof of Proposition 10.

Arguing as there we obtain the inequality (6.13) for any \( t_0 \in \left[ \bar{t} - 2c_1 \rho^2, \bar{t} \right] \). We can estimate the
terms in \((6.13)\) containing \(v\) as in the proof of Proposition \(10\) This gives:

\[
\partial_t \left( \int \frac{u^2}{2} \eta^\beta (t - t_0) dx \right) \leq \left[ -1 + \delta + \frac{9 (1 + \delta)}{32 \pi} \int_{\text{supp}(\eta)} \| \nabla u \|^2 \eta^\beta (t - t_0) dx + K (t - t_0)^\beta \right]
\]

\[
+ \frac{C (t - t_0)^\beta}{\rho^6 \delta^5} + \frac{C (t - t_0)^\beta - 2}{\delta} \int_{\text{supp}(\eta)} u dx - \frac{7 \varepsilon}{6} \int \eta^6 (t - t_0)^\beta u^\frac{7}{6} (\nabla u)^2 dx \]  \quad \quad (6.22)
\]

with \(K\) depending only on \(M\). The last two terms are due to the regularizing term \(\varepsilon \Delta \left( u^\frac{7}{6} \right) \). The term \(-\frac{7 \varepsilon}{6} \int \eta^6 (t - t_0)^\beta u^\frac{7}{6} (\nabla u)^2 dx\) is nonpositive and therefore it can be estimated above by zero. It remains to estimate the additional term \(-6\varepsilon \int \eta^5 (t - t_0)^\beta u \nabla \eta \nabla \left( u^\frac{7}{6} \right) dx\). To this end we use the estimate:

\[
\left| \int \eta^5 (t - t_0)^\beta u \nabla \eta \nabla \left( u^\frac{7}{6} \right) dx \right| \leq \frac{C (t - t_0)^\beta}{\rho} \left[ \int_{\text{supp}(\eta)} u dx \right]^\frac{1}{2} \left[ \int \eta^6 u^3 dx \right] \frac{1}{2} \left[ \int \eta^6 |\nabla u|^2 dx \right]^\frac{1}{2}
\]

Using now Lemma \(11\) we obtain:

\[
\left| \int \eta^5 (t - t_0)^\beta u \nabla \eta \nabla \left( u^\frac{7}{6} \right) dx \right| \leq \delta \left| \int |\nabla u|^2 \eta^6 (t - t_0)^\beta dx \right| + \frac{C (t - t_0)^\beta}{\rho^6 \delta^5} \left[ 1 + \left[ \int_{\text{supp}(\eta)} u dx \right]^3 \right]
\]

Using this estimate in \((6.22)\) we arrive at:

\[
\partial_t \left( \int \frac{u^2}{2} \eta^\beta (t - t_0) dx \right) \leq \left[ -1 + 2\delta + \frac{9 (1 + \delta)}{16 \pi} \int_{\text{supp}(\eta)} \| \nabla u \|^2 \eta^\beta (t - t_0) dx \right]
\]

\[
+ \frac{K (t - t_0)^\beta}{\rho^6 \delta^5} + \frac{C (t - t_0)^\beta}{\delta^5 \rho^5} \left( \int_{\text{supp}(\eta)} u dx \right)^3 + \frac{C (t - t_0)^\beta - 2}{\delta} \int_{\text{supp}(\eta)} u dx \]  \quad \quad (6.23)
\]

with \(K\) depending only on \(M\) and \(C\) just a numerical constant.

For any \(t\), let us define \(t^*\) as:

\[
t^* = \inf \left\{ t \in \left[ \bar{t} - 2c_1 \rho^2, \bar{t} \right] : \sup_{s \in [t, \bar{t}]} \int_{B_{2\rho}(0)} u(x, s) dx \leq 3m_0 \right\}
\]

with \(m_0 = \frac{4\pi}{\bar{t}}\).

Our goal is to show that \(t^* = \bar{t} - 2c_1 \rho^2\) if \(c_1\) is sufficiently small (with \(c_1\) independent on \(\varepsilon, \rho\)).

First we notice that the assumptions of Proposition \(12\) imply \(t^* \leq \min \left\{ \bar{t} - \frac{m_0^2 \sqrt{\varepsilon}}{2(\kappa \sqrt{\varepsilon} + 2M)}, \bar{t} - 2c_1 \rho^2 \right\} \).

Indeed, this is an easy consequence of the fact that the definition of \(\varphi\) combined with \((6.10)-(6.18)\) imply:

\[
\int_{B_{2\rho}(0)} u(x, t) dx \leq 2m_0 + 2 \left( \kappa + \frac{2M}{\sqrt{\varepsilon}} \right) \frac{(\bar{t} - t)}{\rho^2}
\]

(6.24)
and the left hand side of (6.24) is smaller than $3m_0$ for any $t$ satisfying $(\bar{t} - t) \leq \frac{m_0 \rho^5}{2(\kappa + \frac{\kappa}{\rho^2})}$.

Let us suppose that $t^* > \bar{t} - c_1 \rho^2$. Then, (6.23) implies that, if $\delta = \frac{1}{7}$, for $t_0 = t^*$ and $t \in [t^*, \bar{t}]$:

$$\partial_t \left( \int \frac{u^2}{2} \eta^\beta (t - t^*)^\beta \ d\xi \right) \leq K \left[ \frac{(t - t^*)^\beta}{\rho^\beta} + (t - t^*)^{\beta - 2} \right]$$

where $K$ whose value can change, depends only on $M$. Assuming that $\beta = 2$, and integrating in $[t^*, \bar{t}]$ we obtain:

$$\int_{B_{\rho}(0)} u^2(x,t) \ dx \leq K \left[ \frac{(t - t^*)^\beta}{\rho^\beta} + \frac{1}{(t - t^*)^\beta} \right], \ t \in [t^*, \bar{t}] \quad (6.25)$$

This estimate yields:

$$\int_{B_{\rho}(0)} u^\beta(x,t) \ dx \leq \left[ \int_{B_{\rho}(0)} u^2(x,t) \ dx \right]^{\frac{\beta}{2}} \left[ \int_{B_{\rho}(0)} u(x,t) \ dx \right]^{\frac{\beta}{2}} \leq K \left[ \frac{(t - t^*)^\beta}{\rho^\beta} + \frac{1}{(t - t^*)^\beta} \right]$$

with $K$ depending only on $M$. Using (6.16):

$$\partial_t \left( \int_\Omega \varphi \left( \frac{|x|}{2\rho} \right) u(x,t) \ dx \right) \geq -\frac{\kappa}{\rho^2} - \frac{K\varepsilon}{\rho^2} \left[ \frac{(t - t^*)^{\frac{\beta}{2}}}{\rho^\frac{\beta}{2}} + \frac{1}{(t - t^*)^{\frac{\beta}{2}}} \right]$$

Integrating this formula between $t$ and $\bar{t}$ and using that $t \geq t^*$ we obtain:

$$\frac{1}{2} \int_{B_{2\rho}(0)} u(x,t) \ dx \leq m_0 + \frac{K\varepsilon}{\rho^2} (\bar{t} - t^*) + \frac{K\varepsilon}{\rho^2} (\bar{t} - t^*)^{\frac{\beta}{2}} + \frac{K\varepsilon}{\rho^2} (\bar{t} - t^*)^{\frac{\beta}{2}}$$

Suppose first that $\varepsilon \leq \rho^{\frac{\beta}{2}}$. Then, since $(\bar{t} - t^*) \leq \rho^2$ we obtain:

$$\frac{1}{2} \int_{B_{2\rho}(0)} u(x,t) \ dx \leq m_0 + 2\kappa c_1 + c_1^\frac{\beta}{2} + c_1^\frac{\beta}{2} \leq m_0 + 2\kappa c_1 + K \left[ c_1^{\frac{\beta}{2}} + c_1^{\frac{\beta}{2}} \right]$$

Then:

$$\int_{B_{2\rho}(0)} u(x,t) \ dx \leq 2m_0 + 4\kappa c_1 + 2K \left[ c_1^{\frac{\beta}{2}} + c_1^{\frac{\beta}{2}} \right], \ t \in [t^*, \bar{t}]$$

Choosing $c_1$ small enough we obtain $4\kappa c_1 + 2K \left[ c_1^{\frac{\beta}{2}} + c_1^{\frac{\beta}{2}} \right] < m_0$. This contradicts the definition of $t^*$ unless $t^* = \bar{t} - 2c_1 \rho^2$. Using (6.23) with $t = \bar{t}$ it then follows that:

$$\sup_{s \in [t^* - c_1 \rho^2, \bar{t}]} \int_{B_{\rho}(0)} u^2(x,s) \ dx \leq \frac{c_2}{\rho^4} \quad \text{if} \quad \varepsilon \leq \rho^{\frac{\beta}{2}} \quad (6.26)$$

Suppose now that $\varepsilon \geq \rho^{\frac{\beta}{2}}$. Then (6.16) implies:

$$\sup_{s \in [t^* - c_1 \rho^2, \bar{t}]} \int_{B_{\rho}(0)} u^\beta(x,s) \ dx \leq \frac{c_2}{\varepsilon^\frac{\beta}{2}} \leq \frac{c_2}{\rho} \quad \text{if} \quad \varepsilon \geq \rho^{\frac{\beta}{2}} \quad (6.27)$$

\[\Box\]

### 6.3 Boundary estimates.

Estimates analogous to Propositions (10) and (12) can be obtained near the boundary points. The proof is similar, with the only difference of using test functions $\varphi$ and $\eta$ with homogeneous boundary conditions at $\partial\Omega$. We formulate the results by completeness, but the details of the proofs will be omitted.

In the case of the first regularization we have:
Proposition 13 Let \( \Lambda \) be as in Lemma 4. Given \( M > 0, \kappa > 0 \) there exist \( m_0 > 0 \), independent of \( M, \kappa, \varepsilon \) and positive constants \( c_i, i = 1, 2 \) depending on \( M, \kappa \) but independent of \( \varepsilon \), such that for each \( 0 < \rho \leq 1 \), any \( x_0 \in \Omega \) with \( d(x_0) \leq 4\rho \) and any solution \((u,v)\) of

\[
\partial_t u - \Delta u + \nabla (f_\varepsilon(u) \nabla v) = 0 \quad \text{in} \quad (x,t) \in [B_{\Lambda \rho}(x_0) \cap \Omega] \times (\bar{t} - c_1 \rho^2, \bar{t})
\]

\[-\Delta v = f_\varepsilon(u) - h(t) \quad \text{in} \quad (x,t) \in [B_{\Lambda \rho}(x_0) \cap \Omega] \times (\bar{t} - c_1 \rho^2, \bar{t})
\]
satisfying:

\[
\partial_t \left( \int_{\Omega} \psi_\rho(x) u(x,t) \, dx \right) \geq -\frac{\kappa}{\rho^2}, \quad t \in (\bar{t} - c_1 \rho^2, \bar{t})
\]

\[
\int_{[B_{\Lambda \rho}(x_0) \cap \Omega]} u(x,t) \, dx \leq m_0
\]

\[
\sup_{t \in (\bar{t} - c_1 \rho^2, \bar{t})} \|v(\cdot,t)\|_{L^6([B_{\Lambda \rho}(x_0) \cap \Omega])} \leq M
\]

\[
0 \leq h(t) \leq M
\]

with \( \psi_\rho \) as in Lemma 4.

Then, the following inequality holds:

\[
\sup_{s \in [t - c_1 \rho^2, t]} \int_{B_{\rho}(x_0)} u^2(x,s) \, dx \leq \frac{c_2}{\rho^2}.
\]

In the case of the second regularization we have:

Proposition 14 Let \( \Lambda \) be as in Lemma 4. Given \( M > 1, \kappa > 0 \) there exist \( m_0 > 0 \) independent of \( M, \kappa, \varepsilon \) and positive constants \( c_i, i = 1, 2, \varepsilon_0 > 0, \rho_0 > 0 \) depending on \( M, \kappa \) but independent of \( \varepsilon \), such that for each \( 0 < \rho \leq \rho_0, 0 < \varepsilon \leq \varepsilon_0 \), any \( x_0 \in \Omega \) with \( d(x_0) \leq 4\rho \) and any solution \((u,v)\) of

\[
\partial_t u - \Delta \left( u + \varepsilon u^2 \right) + \nabla (u \nabla v) = 0 \quad \text{in} \quad (x,t) \in [B_{\Lambda \rho}(x_0) \cap \Omega] \times (\bar{t} - c_1 \rho^2, \bar{t})
\]

\[-\Delta v = u - h(t) \quad \text{in} \quad (x,t) \in [B_{\Lambda \rho}(x_0) \cap \Omega] \times (\bar{t} - c_1 \rho^2, \bar{t})
\]
satisfying:

\[
\partial_t \left( \int_{\Omega} \psi_\rho(x) u(x,t) \, dx \right) \geq -\frac{\kappa}{\rho^2} - \frac{2\varepsilon}{\rho^2} \int_{B_{\rho}(x_0)} u^2(x,t) \, dx, \quad t \in (\bar{t} - c_1 \rho^2, \bar{t})
\]

\[
\int_{[B_{\Lambda \rho}(x_0) \cap \Omega]} u(x,t) \, dx \leq m_0
\]

\[
\sup_{t \in (\bar{t} - 2c_1 \rho^2, \bar{t})} \left[ \varepsilon \frac{2}{\rho^2} \int_{B_{\rho}(x_0)} u^2(x,t) \, dx \right] \leq M
\]

\[
\sup_{t \in (\bar{t} - c_1 \rho^2, \bar{t})} \|v(\cdot,t)\|_{L^6([B_{\Lambda \rho}(x_0) \cap \Omega])} \leq M
\]

\[
0 \leq h(t) \leq M
\]

with \( \psi_\rho \) as in Lemma 4.

Then, the following inequality holds:

\[
\sup_{s \in [t - c_1 \rho^2, t]} \int_{B_{\rho}(x_0)} u^2(x,s) \, dx \leq \frac{c_3}{\rho^4}
\] (6.28)
7 The limit $\varepsilon \to 0$: Measured valued solutions.

The regularity results above allow to obtain convergence of the solutions of the problems (2.1), (2.2) and (2.4), (2.5) to some measures $\mu$ whose "singular set" is a finite set of points for each time $t$.

We will denote in the following as $M^+(\Omega \times \mathbb{R}^+)$ the space of positive Radon measures in $\Omega \times \mathbb{R}^+$. We will also denote as $N$ the number:

$$N = \left[ \frac{4 \int_{\Omega} u_0 dx}{m_0} \right]$$

(7.1)

where $[x]$ denotes the integer part of $x \geq 0$ and $m_0$ is as in Proposition 10.

We begin considering the limit of the solutions of (2.1), (2.2):

**Proposition 15** Suppose that $u^\varepsilon$ is the solution of (2.1), (2.2) with initial value $u^\varepsilon (x, 0) = u_0 (x), x \in \Omega$ and $\varepsilon > 0$. Then, there exist Radon measures $\mu, \mu^- \in M^+(\Omega \times \mathbb{R}^+)$ and a subsequence $\{\varepsilon \kappa\}_{k=1}^\infty$ such that:

$$u^\varepsilon_k \rightharpoonup \mu, \quad f_{\varepsilon_k} (u^\varepsilon_k) \rightharpoonup \mu^- \quad \text{as} \quad k \to \infty, \quad \mu^- \leq \mu$$

in the $*-\text{weak}$ topology. Moreover, the measures $\mu, \mu^-$ can be written as the product:

$$d\mu = d\mu_t dt, \quad d\mu^- = d\mu^-_t dt$$

with

$$\mu_t (\Omega) = \int_{\Omega} u_0 (x) \, dx \quad \mu^-_t \leq \mu_t. \quad (7.2)$$

We define the singular set of $\mu$, and denote it as $S$, as the set of points $(x_0, t_0) \in \bar{\Omega} \times [0, \infty)$ where:

$$\lim_{\delta \to 0} \inf \left[ \frac{\mu (B_\rho (x_0) \times [t_0, t_0 + \delta]))}{\delta} \right] = \lim_{\delta \to 0} \inf \left[ \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mu_t (B_\rho (x_0)) \, dt \right] \geq \frac{m_0}{2} \quad \text{for any } \rho > 0 \quad (7.3)$$

with $m_0$ as in (6.4). Then we can write:

$$\mu = \bar{\mu} + u, \quad \mu^- = \bar{\mu}^- + u \quad (7.4)$$

where $\bar{\mu}, \bar{\mu}^- \in M^+(\Omega \times \mathbb{R}^+)$ are supported in the set $S$ and $u \in C^\infty (\Omega \times \mathbb{R}^+ \setminus S) \cap L^1 (\Omega \times [0, T])$ for any $T < \infty$. Moreover, for a.e. $t_0 \in [0, \infty)$ the set $S_{t_0} \equiv S \cap \{(x, t) : t = t_0\}$ contains at most $N$ points and the measures $\mu_t, \mu^-_t$ can be represented as:

$$\mu_t = \sum_{x_j(t) \in S_t} \alpha_j (t) \delta_{x_j(t)} + u (\cdot, t) \, dx, \quad \text{a.e.} \quad t \in [0, \infty) \quad (7.5)$$

$$\mu^-_t = \sum_{x_j(t) \in S_t} \beta^-_j (t) \delta_{x_j(t)} + u (\cdot, t) \, dx, \quad \text{a.e.} \quad t \in [0, \infty) \quad (7.6)$$

with $\alpha_j (t) > 0, \alpha_j (t) \geq \beta^-_j (t), u (\cdot, t) \in L^1 (\bar{\Omega}), \int_{\Omega} u (x, t) \, dx \leq \int_{\Omega} u_0 (x) \, dx$.

**Proof.** We will just make the arguments for points at the interior of $\Omega$, since in the case of boundary points the arguments are similar. We define the family of measures $\mu^\varepsilon \in M^+(\Omega \times \mathbb{R}^+)$ by means of:

$$\mu^\varepsilon (B) = \int_B u^\varepsilon (x, t) \, dx \, dt$$
for any Borel set $B \subset \bar{\Omega} \times [0, \infty)$. Taking a subsequence we have $\mu^k \to \mu$ as $k \to \infty$. Using the mass conservation property for (2.4), (2.5) we then have:

$$\left| \int_{\Omega \times A} \varphi d\mu \right| \leq \|u_0\|_{L^1(\Omega)} \int_A \|\varphi \cdot \cdot\cdot \|_{L^\infty(\Omega)} dt . \quad (7.7)$$

For any $\varphi \in C_0(\bar{\Omega} \times \mathbb{R}^+)$ we define a signed measure $\omega_\varphi \in M(\mathbb{R}^+)$ by means of:

$$\omega_\varphi (A) = \int_A \int_{\Omega} \varphi(x,t) \, d\mu . \quad (7.8)$$

Notice that (7.7) implies:

$$|\omega_\varphi (A)| \leq \|u_0\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega \times \mathbb{R}^+)} \int_A dt . \quad (7.9)$$

Similar estimates hold for $\omega_\varphi^+, \omega_\varphi^-$. Therefore the measure $|\omega_\varphi| = \omega_\varphi^+ + \omega_\varphi^-$ is absolutely continuous with respect to the Lebesgue measure in $[0, \infty)$. It then follows from the Radon-Nikodym theorem that:

$$\omega_\varphi (A) = \int_A g_\varphi dt \quad (7.10)$$

for some $g_\varphi \in L^1(\mathbb{R}^+)$. Moreover, due to (7.9) we have

$$\|g_\varphi\|_{L^\infty(\mathbb{R}^+)} \leq \|u_0\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega \times \mathbb{R}^+)} \quad (7.11)$$

Notice that:

$$g_{\alpha_1 \varphi_1 + \alpha_2 \varphi_2} = \alpha_1 g_{\varphi_1} + \alpha_2 g_{\varphi_2} \quad (7.12)$$

for any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in C_0(\bar{\Omega} \times \mathbb{R}^+)$. We now remark that for any $\varepsilon > 0$, $t_0 \in [0, \infty)$ and any smooth cutoff function $\eta$ such that $\eta(t) = 1$, $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\eta(t) = 0$, $|t - t_0| \geq 2\varepsilon$ and any $\psi \in C(\bar{\Omega})$ we have:

$$g_{\eta \psi}(t) = g_{\varphi_T}(t) = \int_{\Omega} \psi d\mu , \quad t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap \mathcal{U} \quad (7.13)$$

due to (7.3), (7.10). We will denote $L_t [\psi] = \lim_{T \to \infty} L_{t,T} [\psi]$. This defines a family of continuous linear functionals $L_t$ for a.e. $t \in [0, \infty)$. Therefore, Riesz-Markov Theorem implies that there exists a family of signed measures $\mu_t \in M(\bar{\Omega})$ defined a.e. $t \in [0, \infty)$ such that:

$$L_t [\psi] = \int_{\Omega} \psi d\mu , \quad \psi \in C(\bar{\Omega}) , \text{ a.e. } t \in [0, \infty)$$

with $|\mu_t| \leq \|u_0\|_{L^1(\Omega)}$, a.e. $t \in [0, \infty)$. We now remark that for any $\varepsilon > 0$, $t_0 \in [0, \infty)$ and any smooth cutoff function $\eta$ such that $\eta(t) = 1$, $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\eta(t) = 0$, $|t - t_0| \geq 2\varepsilon$ and any $\psi \in C(\bar{\Omega})$ we have:

$$g_{\eta \psi}(t) = g_{\varphi_T}(t) = \int_{\Omega} \psi d\mu , \quad t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap \mathcal{U} \quad (7.13)$$

for $T$ sufficiently large. This is just a consequence of (7.8), (7.10). A density argument then yields:

$$\int_{[0,\infty)} \int_{\Omega} \varphi(x,t) \, d\mu = \int_{[0,\infty)} \int_{\Omega} \varphi(x,t) \, d\mu dt , \quad \varphi \in C_0(\bar{\Omega} \times \mathbb{R}^+)$$
or shortly \(d\mu = d\mu_t dt\).

Moreover, assuming that \(\varphi = \varphi (t)\) and using

\[
\int_{[0,\infty)} \varphi (t) \int_\Omega u_0 (x) \, dxdt = \int_{\Omega \times [0,\infty)} \varphi (t) u^\varepsilon_k (x,t) \, dxdtd
\]

\[
= \int_{\Omega \times [0,\infty)} \varphi (t) \, d\mu^\varepsilon_k \to \int_{[0,\infty)} \mu_t (\Omega) \, \varphi (t) \, dt
\]

it follows that \(\mu_t (\Omega) = \int_\Omega u_0 (x) \, dx\), a.e. \(t \in [0,\infty)\).

We now define the singular set by means of (7.3) and decompose \(\mu\) as in (7.4) with \(\nu = \mu \chi_S\), with \(\chi_S\) denoting the characteristic function of \(S\). We now show that \(u = \mu - \nu\) is a smooth function. To this end, notice that by definition of \(S\) we have, for any \((x_0, t_0) \in [\Omega \setminus S] \times [0, \infty)\) there exists \(\rho = \rho (x_0, t_0)\) such that:

\[
\lim \inf_{\delta \to 0} \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mu_t (B_{4\rho} (x_0)) \, dt < \frac{m_0}{2}
\]

Then, there exists a sequence \(\delta_n \to 0\) such that:

\[
\frac{1}{\delta_n} \int_{t_0}^{t_0 + \delta_n} \mu_t (B_{4\rho} (x_0)) \, dt \leq \frac{3m_0}{4}
\]

and the weak convergence \(u^\varepsilon_k \rightharpoonup \mu\) yields:

\[
\frac{1}{\delta_n} \int_{t_0}^{t_0 + \delta_n} \int_{B_{4\rho} (x_0)} u^\varepsilon_k (x,t) \, dxdtd \leq m_0
\]

for \(k\) sufficiently large. Therefore, there exists a sequence \(t_{n,k} \in [t_0, t_0 + \delta_n]\) such that:

\[
\int_{B_{4\rho} (x_0)} u^\varepsilon_k (x,t_{n,k}) \, dxdt \leq m_0
\]

(7.13)

for \(k\) sufficiently large. We can now apply Proposition 10 to the functions \(u^\varepsilon_k\) for large \(k\). Indeed, notice that (6.3) holds due to (4.4) in Proposition 6 and, on the other hand, \(\|u^\varepsilon\|_{L^p (\Omega)} \leq C \int u_0\) due to classical regularity theory for the Poisson equation, whence (6.5) also holds. Finally (7.13) implies (6.4). Proposition 10 then yields:

\[
\sup_{s \in [t_0, t_0 + \delta_n]} \int_{B_{4\rho} (x_0)} (u^\varepsilon_k)^2 (x,t) \, dxdt \leq \frac{c_2}{\rho^4}
\]

and since \(\delta_n\) can be assumed to be arbitrarily close to 0 we have:

\[
\sup_{s \in [t_0 - \frac{\rho^2}{4}, t_0]} \int_{B_{4\rho} (x_0)} (u^\varepsilon_k)^2 (x,t) \, dxdt \leq \frac{c_2}{\rho^4}
\]

Notice that the order of the limits is, first we fix \(\rho\), then we choose \(n\) to have \(\delta_n \leq \frac{c_2 \rho^2}{4}\), and then \(k\) large.

Classical regularity results for parabolic equations (cf. [11]) then imply that

\(u^\varepsilon_k \in C^\infty (B_{4\rho} (x_0) \times [t_0 - \frac{\rho^2}{4}, t_0])\) as well as uniform estimates for the derivatives of \(u^\varepsilon_k\) in the same set. Then \(u \in C^\infty ([\Omega \setminus S] \times \mathbb{R}^+)\). Moreover, using the estimate \(\int_{\Omega \times [0,T]} u \leq T \int_\Omega u_0\) for any \(T > 0\), and the positivity of \(u\), we obtain \(u \in L^1 (\Omega \times \mathbb{R}^+)\) for any \(T < \infty\).

We now prove that for a.e. \(t_0 \in [0,\infty)\) the set \(S_{t_0}\) contains at most \(N\) points. Suppose that \(S_{t_0}\) contains at least \((N+1)\) points \(x_1, ..., x_{N+1}\). Let us choose \(\rho < \frac{1}{2} \min \{|x_i - x_j|\}\). Due to the definition of the singular set there exists \(\delta\) (depending on \(\rho\)) such that:

\[
\frac{1}{\delta} \mu (B_{\rho} (x_i) \times [t_0, t_0 + \delta]) = \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mu_t (B_{\rho} (x_i)) \, dt \geq \frac{m_0}{4}
\]

(7.14)
for $i = 1, ..., (N + 1)$.

Since the balls $\{B_\rho (x_i)\}_{i=1}^{N+1}$ are disjoint, we have:

$$
\frac{1}{\delta} \sum_{i=1}^{N+1} \mu(B_\rho (x_i) \times [t_0, t_0 + \delta]) = \frac{1}{\delta} \mu \left( \bigcup_{i=1}^{N+1} B_\rho (x_i) \times [t_0, t_0 + \delta] \right) \\
\leq \frac{1}{\delta} \mu (\Omega \times [t_0, t_0 + \delta]) = \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mu_t(\Omega) \, dt = \int_\Omega u_0 \, dx
$$
due to (7.2).

Using (7.1), (7.14) it then follows that:

$$
\frac{m_0 (N + 1)}{4} \leq \int_\Omega u_0 \, dx < \frac{m_0 (N + 1)}{4}
$$

that yields a contradiction. Then $S_{t_0}$ contains at most $N$ points for a.e. $t_0 \in [0, \infty)$.

Since a measure concentrated in a finite set of points is a sum of Dirac masses it then follows that:

$$
\mu_t \chi_{S_{t_0}} = \sum_{x_j(t) \in S_t} \alpha_j(t) \delta_{x_j(t)} , \text{ a.e. } t \in [0, \infty)
$$

for at most $N$ points $\{x_j(t)\}$ and positive numbers $\{\alpha_j(t)\}$. On the other hand, if $(x_0, t_0)$ is not in the singular set, we can represent $\mu$ in $B_\mathbb{R}^2(x_0) \times [t_0 - \frac{\varepsilon_0^2}{4}, t_0]$ as a smooth function. Then:

$$
\mu_t(1 - \chi_{S_{t_0}}) = u(\cdot, t) \, dx
$$

and:

$$
\mu_t = \sum_{x_j(t) \in S_t} \alpha_j(t) \delta_{x_j(t)} + u(\cdot, t) \, dx , \text{ a.e. } t \in [0, \infty)
$$

This gives (7.3) and concludes the proof of Proposition 15.

The convergence of a subsequence $\{f_{\varepsilon_k}(u_{\varepsilon_k})\}$ to $\mu^-$ as well as the properties of this measure can be obtained exactly as for the measure $\mu$. The property $\mu^- \leq \mu$ is just a consequence of the inequalities $f_{\varepsilon}(u_{\varepsilon}) \leq u_{\varepsilon}$. The fact that $\mu^-$ can be represented by means of $u$ at the regular points is a consequence of the fact that $u_{\varepsilon}$ is bounded at the regular set, and therefore $f_{\varepsilon}(u_{\varepsilon}) \to u$, there.

We now prove a result similar to Proposition 15 for the regularization (2.4), (2.5).

**Proposition 16** Suppose that $u_{\varepsilon}$ is the solution of (2.4), (2.5) with initial value $u_{\varepsilon}(x, 0) = u_0(x)$, $x \in \Omega$ and $\varepsilon > 0$. Then, there exist a Radon measure $\mu \in M^+(\Omega \times \mathbb{R}^+)$ and a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ such that:

$$
u_{\varepsilon_k} \rightharpoonup \mu , \ u_{\varepsilon_k} + \varepsilon_k(\nabla u_{\varepsilon_k})^2 \rightharpoonup \mu^+ \text{ as } k \to \infty , \mu \leq \mu^+
$$
in the $*-\text{weak}$ topology. Moreover, the measures $\mu$, $\mu^+$ can be written as the product:

$$
d\mu = d\mu^- \, dt , \ d\mu^+ = d\mu^+ \, dt , \mu_t \leq \mu_t^+
$$

where the family of Radon measures $\{\mu_t\}_{t \geq 0}$ satisfy a.e. $t \in [0, \infty)$:

$$
\mu_t(\Omega) = \int_{\Omega} u_0(x) \, dx
$$

We define the singular set of $\mu$, and denote it as $S$, as the set of points $(x_0, t_0) \in \bar{\Omega} \times [0, \infty)$ where:

$$
\lim_{\delta \to 0} \inf_{\delta} \left[ \frac{\mu(B_\rho (x_0) \times [t_0, t_0 + \delta])}{\delta} \right] = \lim_{\delta \to 0} \inf_{\delta} \left[ \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mu_t(B_\rho (x_0)) \, dt \right] \geq \frac{m_0}{2} \text{ for any } \rho > 0
$$
with $m_0$ as in (6.4). Then, we can write:

$$\mu = \bar{\mu} + u \quad , \quad \mu^+ = \bar{\mu}^+ + u$$

(7.15)

where $\bar{\mu}, \bar{\mu}^+ \in M^+(\Omega \times \mathbb{R}^+)$ are supported in the set $S$ and $u \in C^\infty(\Omega \times \mathbb{R}^+ \setminus S) \cap L^1(\Omega \times [0,T])$ for any $T < \infty$. Moreover, for a.e. $t_0 \in [0,\infty)$ the set $S_{t_0} \equiv S \cap \{(x,t) : t = t_0\}$ contains at most $N$ points and the measure $\mu_t$ can be represented as:

$$\mu_t = \sum_{x_j(t) \in S_t} \alpha_j(t) \delta_{x_j(t)} + u(\cdot, t) \, dx, \quad \text{a.e. } t \in [0,\infty)$$

(7.16)

$$\mu^+_{t_0} = \sum_{x_j(t) \in S_t} \beta_j^+(t) \delta_{x_j(t)} + u(\cdot, t) \, dx, \quad \text{a.e. } t \in [0,\infty)$$

(7.17)

with $\beta_j^+(t) \geq \alpha_j(t) > 0$, $u(\cdot, t) \in L^1(\Omega)$, $\int_{\Omega} u(x, t) \, dx \leq \int_{\Omega} u_0(x) \, dx$.

**Proof.** Arguing as in the proof of Proposition 15 it follows that $d\mu = d\mu_t dt$ as well as $\mu_t(\Omega) = \int_{\Omega} u_0(x) \, dx$. We have also that for $(x_0, t_0) \in [\Omega \times [0,\infty)) \setminus S$ there exists $\rho = \rho(x_0, t_0) > 0$ such that:

$$\lim_{\delta \to 0} \inf \left[ \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mu_t(B_{4\rho}(x_0)) \, dt \right] \leq \frac{m_0}{2}$$

(7.18)

Arguing as in the proof of Proposition 15 we can apply Proposition 10 to the functions $u^{\varepsilon}$ for large $k$. Indeed, notice that (6.10) holds due to (1.5) in Proposition 6 and, on the other hand, $\|v^\varepsilon\|_{L^6(\Omega)} \leq C \int \rho(u) dt$ due to classical regularity theory for the Poisson equation, whence (6.19) also holds. The entropy estimate (6.1) implies (6.18). Finally (7.18) implies the existence of a subsequence $t_{n,k} \downarrow t_0$ such that $\int_{B_{4\rho}(x_0)} u^\varepsilon(x, t_{n,k}) \, dx \leq m_0$ (cf. (6.17)). Proposition 12 as well as the fact that $t_{n,k} \downarrow t_0$ then yields:

$$\sup_{s \in [t_0 - c_1 \rho^2, t_0]} \int_{B_{\rho}(x)} (u^\varepsilon)^2(y, s) \, dy \leq \frac{c_2}{\rho^4}.$$  

Classical regularity results for parabolic equations then imply that $u^\varepsilon \in C^\infty(\Omega \times [0,\infty)) \setminus S)$ (cf. 9). The uniform estimate $\int_{\Omega} u \leq \int_{\Omega} u_0$ and the positivity of $u$ imply that $u \in L^1(\Omega \times \mathbb{R}^+)$. We can now prove as in Proposition 15 that for a.e. $t_0 \in [0,\infty)$ the set $S_{t_0}$ contains at most $N$ points. The measures $\mu_t$ have then the structure (7.10). This concludes the proof of Proposition 10.

Finally we prove the convergence properties of $u^\varepsilon + \varepsilon (u^\varepsilon)^2$. To this end we need to obtain an estimate for $\int \int (u^\varepsilon)^2 \, dx dt$. This can be obtained as follows. Suppose that $(x_0, t_0) \in \Omega \times [0, T]$, with $0 < T < \infty$. Either $(x_0, t_0) \in S$ or $(x_0, t_0) \in \Omega \setminus S$. In the second case, there exists $\rho = \rho(x_0, t_0)$ such that $u^\varepsilon$ is bounded by some $C = C(x_0, t_0, \rho)$ but independent on $\varepsilon$ for $\varepsilon \leq \varepsilon_0(x_0, t_0, \rho)$ in $B_{2\rho}(x_0) \times [t_0 - 2c\rho^2, t_0]$ (cf. Proposition 12). Then, $u$ is smooth in $B_{\rho(x_0, t_0)}(x_0) \times [t_0 - c\rho^2, t_0]$ and we have:

$$\int_{t_0}^{t_0 + \delta} \int_{B_{\rho(x_0, t_0)}(x_0)} \left[ u^\varepsilon(x, \delta) + \varepsilon (u^\varepsilon(x, t))^2 \right] \, dx dt \leq C(x_0, t_0, \rho)$$

(7.19)

with $C$ independent on $\varepsilon$.

Suppose that, on the contrary, $(x_0, t_0) \in S$. Since the number of points in $S_{t_0}$ is finite, we can choose $\rho > 0$, such that $D = \left( \left[ B_{2\rho}(x_0) \setminus B_{\rho}(x_0) \right] \times \{ t_0 \} \right) \cap S = \emptyset$. It then follows that for any $(\bar{x}, t_0) \in D$ there exists $\delta = \delta(\bar{x}, t_0)$ such that $u^\varepsilon$ is uniformly bounded in $B_{\delta(\bar{x}, t_0)}(\bar{x}) \times [t_0 - 2c\delta^2, t_0]$.

Since $D$ is compact we can cover it with a finite number of sets of the form $B_{\delta(\bar{x}, t_0)}(\bar{x}) \times [t_0 - 2c\delta^2, t_0]$.

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Therefore, there exists $\delta_0 = \delta_0 (x_0, t_0, \rho) > 0$ such that $u^\varepsilon$ is uniformly bounded in $\mathcal{D} \times [t_0 - \delta_0, t_0]$ for $\varepsilon \leq \varepsilon_0 (x_0, t_0, \rho)$. We now consider a test function $\psi_\rho$ such that $\Delta \psi_\rho = -\frac{1}{\rho^2}$ for $|x - x_0| \leq \rho$ and $|\Delta \psi_\rho| \leq C_\rho$ for $\rho \leq |x - x_0| \leq 2\rho$, and $\psi_\rho = 0$ for $|x - x_0| \geq 2\rho$.

Using $\psi_\rho$ as test function we obtain the following estimate:

$$\left| \partial_t \left( \int_{B_{2\rho} (x_0)} u^\varepsilon (x, t) \psi_\rho (x) \, dx \right) + \frac{1}{\rho^2} \int_{B_\rho (x_0)} \left[ u^\varepsilon (x, t) + \varepsilon (u^\varepsilon (x, t))^\sharp \right] \, dx \right| \leq C_\rho \, , \, t \in [t_0 - \delta_0, t_0]$$

where the error term on the right is due to the contribution of $\int_\mathcal{D} \Delta \psi_\rho (x) [u^\varepsilon (x, t)] \, dx$, as well as the nonlinear terms that can be estimated using the symmetrization argument as in the Proof of Proposition 12. Notice that we use the smoothness of $u^\varepsilon$ in $\mathcal{D} \times [t_0 - \delta_0, t_0]$. Therefore, integrating (7.20) in $[t_0 - \delta_0, t_0]$:

$$\int_{t_0 - \delta_0}^{t_0} \int_{B_\rho (x_0)} \left[ u^\varepsilon (x, t) + \varepsilon (u^\varepsilon (x, t))^\sharp \right] \, dx \, dt \leq C_\rho$$

(7.21)

Therefore, (7.19), (7.21) imply the existence for any $(x_0, t_0) \in \Omega \times [0, T]$ of a cylinder $B_\rho (x_0) \times [t_0 - \delta_0, t_0]$ for which (7.21) holds. Since $\Omega$ is compact, we can find a finite covering of it by means of some of these cylinders. Then:

$$\int_0^T \int_\Omega \left[ u^\varepsilon (x, t) + \varepsilon (u^\varepsilon (x, t))^\sharp \right] \, dx \, dt \leq C$$

with $C > 0$ independent on $\varepsilon$, assuming that $\varepsilon \leq \varepsilon_0$.

We then have:

$$u^{\varepsilon_k} + \varepsilon_k (u^{\varepsilon_k})^\sharp \rightharpoonup \mu^+ \text{ as } k \to \infty$$

The fact that the singular set of $\mu^+$ and $\mu$ are the same is a consequence of the fact that for every regular point of $\mu$ we have $\varepsilon_k (u^{\varepsilon_k})^\sharp \rightarrow 0$ in a neighbourhood of the regular point, as it can be seen from the estimate (6.21).

**Remark 17** We have denoted the limit of the sequence $u^\varepsilon$ as $\mu$ for both regularizations. Notice, however, that there is not any reason to expect both limits to be the same measure.

### 7.1 A continuity result for the singular set.

**Lemma 18** Suppose that the measure $\mu_t$ and the set $S_t$ are as in Proposition 13 or Proposition 16. Let $T > 0$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_0 \in [0, T]$, $S_t \subset S_{t_0} + B_\varepsilon (0)$ if $t \in [t_0 - \delta, t_0]$. Moreover, suppose that for $(x_0, t_0) \in S$ we have $\int_{B_R (x_0) \setminus \{x_0\}} d\mu_0 \leq \frac{m_0}{|t_0|^{2/3}}$, with $m_0$ as in Propositions 10, 13 and $R > 0$ fixed. Then, there exists $c > 0$, $L > 0$ depending only on $\int_{\Omega} d\mu_0 (x)$ and $\Omega$ such that $S_t \cap B_R (x_0) \subset B_{R \sqrt{|t - t_0|}} (x_0)$ for $t \in [t_0 - cR^2, t_0]$.

**Proof.** It is just a consequence of Propositions 10, 12, 13, 14. ■

We include now some auxiliary results that will be required later.

### 7.2 Mass continuity. Characterization of the limit of some quadratic terms.

A basic characteristic of the regularizations of the Keller-Segel system in (2.1) - (2.2), (2.4) - (2.5) is the fact that the mass in each neighbourhood changes in a continuous way. More precisely, we have the following result:
Lemma 19  Suppose that $0 < T < \infty$. Let $u^\varepsilon$ be the solution of (7.1)\textendash(7.2) or (7.1)\textendash(7.3). Given $(x_0, t_0) \in \Omega \times [0, T]$ there exists a cutoff function $\psi(x; x_0, t_0)$ and $\rho = \rho(x_0, t_0)$, $\tau = \tau(x_0, t_0)$ independent on $\varepsilon$ satisfying:

$$\left| \partial_t \left( \int_{B_\rho(x_0)} u^\varepsilon(y, t) \psi(y; x_0, t_0) \, dy \right) \right| \leq C(x_0, t_0)$$

(7.22)

for $t \in [t_0 - \tau, t_0]$ with $C(x_0, t_0)$ independent on $\varepsilon$.

Proof. We just consider interior points, since boundary estimates can be obtained similarly using Lemma 7. If $(x_0, t_0) \in (\Omega \times [0, T]) \setminus S$ there exists $\rho = \rho(x_0, t_0) > 0$, $\tau = \tau(x_0, t_0)$ such that $B_\rho(x_0) \times [t_0 - \tau, t_0] \in (\Omega \times [0, T]) \setminus S$ (cf. Propositions 10, 12). We take a cutoff function $\psi(y)$ such that $\psi(y) = 1$ for $|y - x_0| \leq \rho$ and $\psi(y) = 0$ for $|y - x_0| \geq 2\rho$. Then, the estimates for the linear terms $\Delta(u^\varepsilon)$, $\Delta(u^\varepsilon + \varepsilon u^\varepsilon)$ are immediate and the nonlinear terms can be estimated using the symmetric argument in the proof of Propositions 10, 12. If $(x_0, t_0) \in S$ we have $[B_{2\rho}(x_0) \setminus B_\rho(x_0)] \times [t_0 - \tau, t_0] \cap S = \emptyset$ if $\rho > 0$ and $\tau > 0$ are sufficiently small (see Lemma 18). The result then follows choosing $\psi(y)$ with $\nabla \psi \neq 0$ only in $[B_{2\rho}(x_0) \setminus B_\rho(x_0)]$.

Lemma [18] allows to obtain the weak limit of some quadratic terms.

Lemma 20  Suppose that $u^\varepsilon$ is as in Lemma 19 and that, for suitable subsequences $u^{\varepsilon_k} \rightharpoonup \mu$, with $\mu$ as in Proposition 15 or Proposition 16. Let $\varphi \in C(\bar{\Omega} \times \bar{\Omega} \times \mathbb{R}^+)$.

Then:

$$\int \int \int u^{\varepsilon_k}(x, t) u^{\varepsilon_k}(y, t) \varphi(x, y, t) \, dx \, dy \, dt \rightarrow \int \int \int \varphi(x, y, t) \, d\mu_t(x) \, d\mu_t(y) \, dt$$

(7.23)

as $k \to \infty$.

Proof. The compactness of $\bar{\Omega} \times [0, T]$ implies the existence, for each $\delta_0 > 0$ of functions $\{\psi_{\ell, m}(y)\}$ satisfying $\sum_{\ell} \psi_{\ell, m}(y) = 1$ for $y \in \bar{\Omega}$, and such that:

$$\left| \partial_t \left( \int u^\varepsilon(y, t) \psi_{\ell, m}(y) \, dy \right) \right| \leq C, \quad t \in [m\delta_0, (m + 1)\delta_0], \quad m\delta_0 < T$$

(7.24)

whence:

$$\left| \int u^\varepsilon(y, t) \psi_{\ell, m}(y) \, dy - \int u^\varepsilon(y, t_m) \psi_{\ell, m}(y) \, dy \right| \leq C |t - t_m|, \quad t \in [m\delta_0, (m + 1)\delta_0]$$

(7.25)

Combining Propositions 15 and 16 and (7.24) we obtain the existence of $t_m \in [m\delta_0, (m + 1)\delta_0]$ for any $m = 0, 1, \ldots$ such that:

$$\int u^\varepsilon(y, t_m) \psi_{\ell, m}(y) \, dy \rightarrow \int \psi_{\ell, m}(y) \, d\mu_m$$

(7.26)

We rewrite the left-hand side of (7.23) as:

$$\int \int \int u^\varepsilon(x, t) u^\varepsilon(y, t) \varphi(x, y, t) \, dx \, dy \, dt = \lim_{\sigma_0 \to 0} \int \int \int_{m\delta_0}^{((m+1)\delta_0) \land T} dt \sum_{\ell} \int dx u^\varepsilon(x, t) \int u^\varepsilon(y, t) \varphi(x, y, t) \, dy$$

Using the continuity of $\varphi$ and (7.25) we then obtain for any $\sigma_0 > 0$:

$$\left| \int \int \int u^\varepsilon(x, t) u^\varepsilon(y, t) \varphi(x, y, t) \, dx \, dy \, dt \right| \rightarrow \int_{0}^{T} dt \int d\mu(x) \int d\mu(y) \varphi(x, y, t) \, dy \right| \leq C(\sigma_0 + \delta_0)$$

with $C$ independent on $\sigma_0, \delta_0$ whence the result follows.
8 Formulation of the limit problem.

We now pass to the limit in the problems \((2.1)-(2.2), (2.4)-(2.5)\) to derive the problems satisfied by the pairs of measures \((\mu, \mu^-), (\mu, \mu^+)\) respectively. As a matter of fact, in order to write the weak equations satisfied by the measures \(\mu\) we will need to introduce some auxiliary measures \(\hat{\mu}, \hat{\mu}\) defined in a space larger than \(\bar{\Omega} \times [0, \infty)\) at the singular points. We begin with the first regularization \((2.1)-(2.2)\).

8.1 First regularization.

We begin rewriting the weak formulation of the regularized problem \((2.1)-(2.2)\) in a more convenient form:

**Lemma 21** Suppose that \(u^\varepsilon\) solves \((2.1)-(2.2)\). Then, for any test function \(\psi \in C^\infty (\Omega \times \mathbb{R}^+)\) satisfying

\[
\frac{\partial \psi}{\partial \nu_x} (x,t) = 0, \quad x \in \partial \Omega
\]  

we have:

\[
L_1 + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 = 0,
\]  

where:

\[
L_1 = -\int \psi (x,0) u_0 (x) dx - \int \int \psi \Delta \psi dxdt - \int \int u^\varepsilon \Delta \psi dxdt \tag{8.3}
\]

\[
Q_1 = \frac{1}{4\pi} \int \int \frac{[x - y \cdot (\nabla \psi (x,t) - \nabla \psi (y,t))]}{|x - y|^2} d\omega^-_x (x,t) \tag{8.4}
\]

\[
Q_2 = \frac{1}{4\pi} \int \int [\nabla \psi (x,t) Z (y) - \nabla \psi (y,t) Z (x)] \left( \frac{P_\partial (x) - P_\partial (y)}{D} \right) d\omega^-_x (x,t) \tag{8.5}
\]

\[
Q_3 = -\frac{1}{4\pi} \int \int [\nabla \psi (x,t) Z (y) + \nabla \psi (y,t) Z (x)] \left[ \frac{d (x) \nu (x) + d (y) \nu (y)}{D} \right] d\omega^-_x (x,t) \tag{8.6}
\]

\[
Q_4 = \int \int \nabla \psi (x,t) \frac{Z (y) h (y)}{2\pi}.
\]  

\[
\cdot \left| G_t (Y (x,y), \lambda_1 (x,y), \lambda_2 (x,y)) + g_t (Y (x,y), \lambda_1 (x,y), \lambda_2 (x,y), \nu (y) \right| d\omega^-_x (x,t) \tag{8.7}
\]

\[
Q_5 = -\int \int \nabla \psi (x,t) W (x,y) d\omega^-_x (x,y,t) \tag{8.8}
\]

and:

\[
d\omega^-_x (x,y,t) = f_\varepsilon (u (x,t)) f_\varepsilon (u (y,t)) dxdydt
\]

**Proof.** Multiplying the regularized equations by a test function \(\psi (x,t)\) compactly supported in \(\bar{\Omega} \times [0,\infty)\), solving the Poisson equation using the corresponding Green’s function and integrating in \(\bar{\Omega} \times [0,\infty)\) we obtain:

\[
0 = -\int \psi (x,0) u_0 (x) dx - \int \int \psi \Delta \psi dxdt - \int \int u^\varepsilon \Delta \psi dxdt - \int \int f_\varepsilon (u (x,t)) f_\varepsilon (u (y,t)) \nabla \psi (x) \nabla G (y, x) dxdydt
\]

where \(G\) is the Green’s function for the Poisson equation described in Lemma 2.

Using Lemma 3 it then follows that:

\[
L_1 + \frac{1}{2\pi} \int \int \nabla \psi (x,t) \frac{Z (y)}{|x - y|^2} d\omega^-_x (x,y,t) + \frac{1}{2\pi} \int \int \nabla \psi (x,t) \frac{P_\partial (x) - P_\partial (y) - \frac{d (x) \nu (x) + d (y) \nu (y)}{D}}{D} d\omega^-_x (x,y,t) (8.9)
\]

\[
+ \int \int \nabla \psi (x,t) Z (y) d\omega^-_x (x,y,t) + Q_4 + Q_5 = 0
\]

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Symmetrizing the second term in (8.9) we can rewrite it as $Q_1$. On the other hand, symmetrizing the third term in (8.9) it becomes $Q_2 + Q_3$. ■

We now proceed to identify the limit of the quadratic terms $Q_k$, $k = 1, \ldots, 5$ in (8.4)-(8.8). The sequence $\{\omega^- (x, y, t)\}$ has good properties to pass to the limit in $M^+ (\Omega \times \Omega \times \mathbb{R}^+)$, however these measures are multiplied by functions like $\frac{|(x-y) \cdot (\nabla \psi (x, t) - \nabla \psi (y, t))|}{|x-y|^2}$ that are bounded but not continuous near the diagonal $\{x = y\}$. To deal with such a terms will require to introduce measures defined in larger sets than $\Omega \times \Omega \times [0, \infty)$.

### 8.1.1 Limit of the nonlinear terms: The term $Q_1$.

**Lemma 22** There exist measures $\omega^\pm \in M^+ (\Omega \times \Omega \times \mathbb{R}^+)$, $\mu^- \in M^+ (\Omega \times S^1 \times \mathbb{R}^+)$ satisfying $\omega^- (\{x = y\} \times [0, \infty)) = 0$, $\text{supp} (\tilde{\mu}^-) \subset S \times \mathbb{R}^1$ such that for any test function as in Lemma 22 we have:

$$Q_1 \to \int_{[\Omega \times S^1] \times [0, \infty)} \frac{(\nu \cdot \nabla^2 \psi (x, t) \cdot \nu)}{4\pi} d\tilde{\mu}^- + \frac{1}{4\pi} \int_{[\Omega \times \Omega \times [0, \infty)] \cap \{x \neq y\}} \frac{|(x-y) \cdot (\nabla \psi (x, t) - \nabla \psi (y, t))|}{|x-y|^2} d\omega_i^- dt$$

(8.10)

for some subsequence $\varepsilon_k \to 0$, $k \to \infty$.

Moreover, we have:

$$d\mu^- = d\tilde{\mu}_i^- dt , \quad d\omega^- = d\omega_i^- dt$$

(8.11)

and:

$$\int_{S^1} d\tilde{\mu}_i^- (\cdot, \nu) = \sum_{x_i(t) \in S_i} \gamma_i^- (t) \delta_{x_i(t)} \quad , \quad \gamma_i^- (t) \geq (\beta_i^- (t))^2 , \quad \text{a.e. } t \in [0, \infty)$$

(8.12)

$$d\omega^- (x, y) = \sum_{\{x_i(t) \neq x_j(t)\}} \lambda_{i,j}^- (t) \delta_{x_i(t)} (x) \delta_{x_j(t)} (y) +$$

$$+ \sum_{x_i(t) \in S_i} \beta_i^- (t) \left[ \delta_{x_i(t)} (x) u (y, t) dy + \delta_{x_i(t)} (y) u (x, t) dx \right] + u (x, t) u (y, t) dxdy$$

(8.13)

for some $\lambda_{i,j}^- (t) \geq 0$ defined for $i \neq j$.

**Proof.** Let $\eta \in C^\infty (\mathbb{R}^+)$ be a cutoff function satisfying:

$$\eta (s) = 1 , \quad |s| \leq \frac{1}{2} , \quad \eta (s) = 0 , \quad |s| \geq 1 , \quad 0 \leq \eta \leq 1$$

We then write:

$$Q_1 = \int \int \int H_1 (x, y, t) \eta \left( \frac{|x-y|}{\delta} \right) d\omega^- (x, y, t) + \int \int \int H_1 (x, y, t) \left[ 1 - \eta \left( \frac{|x-y|}{\delta} \right) \right] d\omega_i^- (x, y, t)$$

$$\equiv I_1 + I_2$$

where:

$$H_1 (x, y, t) = \frac{1}{4\pi} \left[ \frac{|(x-y) \cdot (\nabla \psi (x, t) - \nabla \psi (y, t))|}{|x-y|^2} \right]$$

(8.14)

Given $\varphi \in C^\infty (\Omega \times S^1 \times \mathbb{R}^+)$ we define:

$$\tilde{\mu}_{\delta,x}^- (\varphi) = \int \int \varphi \left( x, \frac{x-y}{|x-y|}, t \right) \eta \left( \frac{|x-y|}{\delta} \right) d\omega^- (x, y, t)$$
The family of nonnegative measures $\hat{\mu}_{\delta,\varepsilon}(\chi_{[0,T]}\varphi)$ is compact in $M^+ (\Omega \times S^1 \times [0,T])$ for each $T < \infty$ with the weak topology, since $\hat{\mu}_{\delta,\varepsilon}(\chi_{[0,T]} \cdot 1) \leq T (\int u_0 dx)^2$. Taking a subsequence if needed we can define $\hat{\mu}_{\delta,T}(\varphi) = \lim_{\varepsilon \to \infty} \hat{\mu}_{\delta,\varepsilon}(\chi_{[0,T]}\varphi)$ where the limit is taken in the weak topology. We can now take the limit $\delta \to 0$ for suitable subsequences. Then:

$$\hat{\mu}_{\delta,T} \to \hat{\mu}_T$$ \hspace{1cm} (8.15)

Moreover, we can write $d\hat{\mu}_T = d\hat{\mu}_T^{-}$ arguing as in the Proof of Proposition [13].

We can now compute the limit of the term $f_1$ using the measures $\hat{\mu}_T$. To this end, we approximate the test function $\chi_{[\delta,\varepsilon]}(x, y, \cdot)$ by a test function having the form $\varphi(x, \frac{x-y}{|x-y|}, t)$. Indeed, Taylor’s Theorem yields:

$$\frac{(x-y) \cdot (\nabla \varphi(x,y,t) - \nabla \varphi(y,t))}{|x-y|^2} = \frac{(x-y) \cdot \nabla^2 \varphi(x,y,t) \cdot (x-y)}{|x-y|^2} + O(\delta)$$

whence:

$$I_1 = \hat{\mu}_{\delta,\varepsilon}(\chi_{[0,T]}\varphi) + O(\delta), \quad \varphi(x,\nu,t) = \frac{\mu \cdot \nabla^2 \varphi(x,y,t)}{4\pi} , \ x \in \Omega , \nu \in S^1$$

Therefore, the limit $\varepsilon \to 0$, $\delta \to 0$, for suitable subsequences, yields:

$$I_1 \to \int \int_{\Omega \times S^1} \left( \frac{\mu \cdot \nabla^2 \varphi(x,y,t) \cdot \nu}{4\pi} \right) d\hat{\mu}_T^{-}(x,\nu) dt$$ \hspace{1cm} (8.16)

It only remains to compute the limit of $I_2$ as $\varepsilon \to 0$, $\delta \to 0$. Notice that the family $\{\omega_{\varepsilon^{-}} (x, y, t)\}$ is weakly compact in $M^+ \left( (\Omega \times \Omega) \cap \{|x-y| \geq \delta \} \times [0,T] \right)$. Therefore there exists $\omega_{\varepsilon^{-}}\delta$, such that $\omega_{\varepsilon^{-}}\delta \biggl[1 - \eta \left(\frac{|x-y|}{\delta}\right)\biggr] \to \omega_{\delta,T}$. Taking then the limit $\delta \to 0$, we finally obtain:

$$I_2 \to \frac{1}{4\pi} \int_{\Omega \times \Omega \times [0,\infty]} \frac{[(x-y) \cdot (\nabla \varphi(x,y,t) - \nabla \varphi(y,t))]}{|x-y|^2} d\omega_t^{-} dt$$ \hspace{1cm} (8.17)

Combining (8.16), (8.17) we obtain (8.10). The representation of $\omega^{-}$ given in (8.11) follows as in Proposition [13].

To derive (8.12) we compute the measure $\hat{\mu}_T^{-}$ acting over test functions independent on $\nu$. We then need to consider the limit as $\varepsilon \to 0$, $\delta \to 0$, (for suitable subsequences) of integrals with the form:

$$\int \int f_\varepsilon(u^\varepsilon(x,t)) f_\varepsilon(u^\varepsilon(y,t)) \varphi(x,t) \eta \left(\frac{|x-y|}{\delta}\right) dxdydt$$ \hspace{1cm} (8.18)

Writing:

$$f_\varepsilon(u^\varepsilon(x,t)) = \mu^{-}(x,t) + [f_\varepsilon(u^\varepsilon(x,t)) - \mu^{-}(x,t)]$$

we obtain:

$$f_\varepsilon(u^\varepsilon(x,t)) f_\varepsilon(u^\varepsilon(y,t))$$

$$= \mu^{-}(x,t) \mu^{-}(y,t) + \mu^{-}(x,t) [f_\varepsilon(u^\varepsilon(y,t)) - \mu^{-}(y,t)] +$$

$$+ \left[f_\varepsilon(u^\varepsilon(x,t)) - \mu^{-}(x,t)\right] \mu^{-}(y,t) +$$

$$+ \left[f_\varepsilon(u^\varepsilon(x,t)) - \mu^{-}(x,t)\right] \left[f_\varepsilon(u^\varepsilon(y,t)) - \mu^{-}(y,t)\right]$$ \hspace{1cm} (8.19)

Plugging (8.19) into (8.18) it follows that the limit of the second and third term approach zero as $\varepsilon \to 0$. On the other hand, in order to estimate the contribution of the last term in (8.19) we remark that, estimating $f_\varepsilon(u^\varepsilon)$ by $u^\varepsilon$, and using that outside a ball of radius $\rho$ of the singular
set $u^\varepsilon$ converges uniformly to $u$, we can estimate the contribution outside the singular set by a $L^1$ function and the resulting integral contribution approaches zero as $\delta \to 0$, since the measure of the considered set approaches zero. Therefore, the integration in (8.18) is restricted to $S + B_\rho(0) \times \{0\}$ with $\rho$ very small. In such a set we can assume that $\eta \left( \frac{|x-y|}{\delta} \right)$ is constant, and $\varphi(x,t)$ can be approximated by the values at the singular set, therefore, by functions depending only on time. It then follows that the last term in (8.19) gives a contribution with the form:

$$
\int \sum_{x_j(t) \in S_i} \varphi(x_j(t), t) \left( \int f_\varepsilon(u^\varepsilon(x,t)) - \mu^-(x,t) \right) dt \geq 0
$$

except for a small error. $\varphi(x,t)$ in (8.18) can be approximated as $\varepsilon \to 0$ by the sum of the values at the singular set.

It then follows that:

$$
\int_{S^1} d\tilde{\mu}_t(\cdot, \nu) \geq \left( \mu_{\text{sing}}(\cdot) \right)^2
$$

whence using the fact that $f_\varepsilon(u) \leq u$ as well as Corollary 23 (8.12) follows. The representation formula (8.13) is a consequence of the fact that the measures \( \omega^- (x, y, t) \left[ 1 - \eta \left( \frac{|x-y|}{\delta} \right) \right] \) are supported in the region $\{ |x-y| \geq \delta \}$, as well as (7.6). If we consider points $(x_0, y_0)$ at the singular set we can obtain smoothness of the solutions in an neighbourhood, and obtain strong convergence of $f(u^\varepsilon(x,t)) f (u^\varepsilon(y,t))$. This gives the term $u(x,t) u(y,t)$ in (8.13). If, say $x_0 \in S_i$ and $y_0$ is a regular point, we obtain strong convergence of the function $f(u^\varepsilon(y,t))$ in a neighbourhood and, taking the product of weak convergence with strong convergence to obtain the terms $\sum_{x_i(t) \in S_i} \beta_i^- (t) \left[ \delta_{x_i(t)}(x) u(y,t) + \delta_{x_i(t)}(y) u(x,t) \right] dx$ in (8.13). Finally, in a neighbourhood of the points $(x,y) = (x_i(t), x_j(t)) \in S_i \times S_i$ with $i \neq j$ we can only prove the existence of a singular set with weights $\lambda_{i,j}(t) \geq 0$. Unfortunately it is not possible to ensure that $\lambda_{i,j}(t) = \beta_i^-(t) \beta_j^- (t)$ without a careful study of the possible fast oscillations in time of the functions $f_\varepsilon(u)$. A Young measure formalism that allows to describe the possible effect of oscillations in short time scales is given in Section 10.

A consequence of Lemmas 19 and 22 is the following:

**Corollary 23** Suppose that $\mu$, $\omega^-$ are as in Proposition 16. Then:

$$
d\omega^- (x,y) \leq d\mu_t(x) d\mu_t(y)
$$

(8.20)

### 8.1.2 Limit of the nonlinear terms: The term $Q_2$.

In order to characterize the limit of the term $Q_2$ we need to define a manifold that will play the role of $\Omega \times S^1$ for the points at the boundary. We define also some auxiliary sets.

**Definition 24** For any $y \in \partial \Omega$ we define the following manifold with boundary:

$$
\mathcal{M}_2[y] = \left\{ (Y, \lambda_1, \lambda_2) : Y \in TM_y(\partial \Omega), \lambda_1 \geq 0, \lambda_2 \geq 0, |Y|^2 + (\lambda_1 + \lambda_2)^2 = 1 \right\}
$$

Notice that $\mathcal{M}_2[y]$ is isomorphic to the intersection of a two-dimensional cylinder with the quadrant $\{ \lambda_1 \geq 0, \lambda_2 \geq 0 \}$.

**Definition 25** We will denote as $\mathcal{M}_2$ the set $\{(y, \sigma) : y \in \partial \Omega, \sigma \in \mathcal{M}_2[y]\}$ endowed naturally with a structure of three-dimensional manifold with boundary.
**Definition 26** We will denote as $M_2^{(\varepsilon_0)}$ the set $\{(y, \sigma) : y \in \Omega, \text{dist}(y, \partial \Omega) \leq \varepsilon_0, \ \sigma \in M_2[y_0]\}$, where $y_0$ is the closest point to $y$ in $\partial \Omega$ and $\varepsilon_0 > 0$ is fixed sufficiently small.

**Lemma 27** Let $Q_2$ as in (8.21). There exists a measure $\hat{\mu}_{b,\varepsilon}^\infty \in M^+(M_2 \times \mathbb{R}^+)$, $d\hat{\mu}_{b,\varepsilon}^\infty = d\hat{\mu}_{b,\varepsilon}dt$ such that, for suitable subsequences $\varepsilon_k \rightarrow 0$, $\delta_k \rightarrow 0$, $k \rightarrow \infty$:

$$Q_2 \rightarrow \hat{\mu}_{b,\varepsilon}^\infty (\varphi_1) +$$

$$+ \frac{1}{4\pi} \int \int \int_{[\Omega \times \Omega \times [0, \infty)]} \left[ \nabla \psi(x, t) Z(y) - \nabla \psi(y, t) Z(x) \right] \left( \frac{P_0(x) - P_0(y)}{D} \right) d\omega_\delta^\infty(x, y, t) dt \quad (8.21)$$

with $\omega_\delta^\infty(x, y)$ as in Lemma 22 and with:

$$\varphi_1(y, y, \lambda_1, \lambda_2, t) = [Y + (\lambda_2 - \lambda_1) \nu(y)] \nabla^2 \psi(y, t) Y, \quad y \in \partial \Omega, \quad (Y, \lambda_1, \lambda_2) \in M_2[y], \quad t \in [0, \infty) \quad (8.22)$$

Moreover:

$$\int_{\mathcal{M}_2[y]} d\hat{\mu}_{b,\varepsilon}^\infty(\cdot, \sigma) = \sum_{x_i(t) \in S_i} \gamma_{b,1}(t) \delta_{x_i(t)}(\cdot), \quad \gamma_{b,1}(t) \geq (\beta_i^\infty(t))^2, \quad a.e. \ t \in [0, \infty) \quad (8.23)$$

**Proof.** We use the same argument as in the proof of Lemma 22. Using the same cutoff function $\eta$ we write:

$$\int \int \int H_2(x, y, t) d\omega_\delta^\infty(x, y, t)$$

$$= \int \int \int H_2(x, y, t) \eta \left( \frac{|x - \tau(y)|}{\delta} \right) d\omega_\delta^\infty(x, y, t) +$$

$$+ \int \int \int H_2(x, y, t) \left[ 1 - \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \right] d\omega_\delta^\infty(x, y, t)$$

$$\equiv I_1 + I_2$$

with:

$$H_2(x, y, t) = \frac{1}{4\pi} \left[ \nabla \psi(x, t) Z(y) - \nabla \psi(y, t) Z(x) \right] \left( \frac{P_0(x) - P_0(y)}{D} \right) \quad (8.24)$$

In order to define the measure $\hat{\mu}_{b}^\infty \in M^+(M_2 \times \mathbb{R}^+)$ we argue as follows. Given a test function $\varphi \in C^\infty(M_2 \times \mathbb{R}^+)$ we extend it to $M_2^{(\varepsilon_0)}$ for small $\varepsilon_0$ as $\varphi(y, \sigma) = \varphi(y_0, \sigma)$ with $\sigma \in M_2[y_0]$ and we define the auxiliary linear functional for $\delta > 0$ sufficiently small:

$$\hat{\mu}_{b,\delta,\varepsilon}^\infty(\varphi) = \int \int \int \varphi \left( \frac{x + y}{2}, \frac{(P_0(x) - P_0(y)) \cdot t(y)}{\kappa \sqrt{D}} \right) t(y) \cdot \frac{d(x)}{\sqrt{D}} \frac{d(y)}{\sqrt{D}} \cdot t(t) \eta \left( \frac{|x - \tau(y)|}{\delta} \right) d\omega_\delta^\infty(x, y, t) \quad (8.25)$$

where $\kappa = \frac{(P_0(x) - P_0(y)) \cdot t(y)}{(P_0(x) - P_0(y)) \cdot t(y)}$. Notice that for any $x, y$ satisfying $|x - \tau(y)| \leq \delta$, we have also $\text{dist}(x, \partial \Omega) \leq \delta$, $\text{dist}(y, \partial \Omega) \leq 2\delta$ for $\delta$ sufficiently small. In particular $Z(y) = 1$. The sequence of measures $\hat{\mu}_{b,\delta,\varepsilon}^\infty$ is weakly compact in $C(M_2 \times [0, T])^*$ for any $0 < T < \infty$. Then, taking suitable subsequences:

$$\hat{\mu}_{b,\delta,\varepsilon}^\infty \rightarrow \hat{\mu}_{b}^\infty$$

where $\hat{\mu}_{b}^\infty \in M^+(M_2 \times \mathbb{R}^+)$.

We can then pass to the limit in $I_1$ as follows. Using Taylor’s we can write:

$$I_1 = \int \int \int \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \cdot \left[ [(P_0(x) - P_0(y)) \cdot t(y)] t(y) + (d(y) - d(x)) \nu(y) \right] \frac{\nabla^2 \psi(y, t)}{4\pi} \left( \frac{P_0(x) - P_0(y)}{D} \right) d\omega_\delta^\infty(x, y, t)$$

$$+ O(\delta)$$

$^4$See if it is possible to put $T = \infty$ directly.
Then:

\[ I_1 = \hat{\mu}_{b, \delta, \varepsilon} (\varphi_1) + O (\delta) \]

Then, taking suitable subsequences:

\[ I_1 \to \hat{\mu}_{b} (\varphi_1), \; \varepsilon_k \to 0, \; \delta_k \to 0 \; \text{as} \; k \to \infty \]

On the other hand, taking the limit \( \varepsilon \to 0 \) and \( \delta \to 0 \), also for suitable subsequences:

\[ I_2 \to \frac{1}{4\pi} \int \int \int_{[\Omega \times [0, \infty] \cap \{x \neq y\}}} |\nabla \psi (x, t) Z (y) - \nabla \psi (y, t) Z (x)| \left( \frac{P_b (x) - P_b (y)}{D} \right) d\omega^- (x, y) dt \]

where the measure \( \omega^- \) is as in (8.17).

We can compute the action of the measure \( \hat{\mu}_{b}^- \) over test functions \( \psi \in C (\mathcal{M}_2 \times \mathbb{R}^+) \) depending only on \( (x, t) \in \partial \Omega \times [0, \infty) \). To this end we need to consider the limit of:

\[ \int \int \varphi \left( \frac{x + y}{2}, t \right) \eta \left( \frac{|x - \tau (y)|}{\delta} \right) d\omega^- (x, y, t) \]

that converge in the limit \( \varepsilon_k \to 0, \delta_k \to 0 \) as \( k \to \infty \) to:

\[ \int_0^\infty \int_\partial \Omega \int_{\mathcal{M}_2 [x]} \varphi (x, t) d\hat{\mu}_{b, \varepsilon}^- (x, \sigma) dt \]

Arguing as in the Proof of Lemma 22 we obtain (8.28). \( \blacksquare \)

8.1.3 Limit of the nonlinear terms: The term \( Q_3 \).

We now compute the limit of \( Q_3 \).

**Lemma 28** Suppose that \( \psi \in C^\infty (\bar{\Omega} \times \mathbb{R}^+) \) satisfies \( \text{(8.1)} \). Then, taking suitable subsequences \( \varepsilon_k \to 0, \; \delta_k \to 0 \; \text{as} \; k \to \infty \) we have:

\[ Q_3 = \hat{\mu}_{b}^- (\varphi_2 + \varphi_3) - \frac{1}{4\pi} \int \int \int_{[\bar{\Omega} \times [0, \infty] \cap \{x \neq y\}}} \frac{[d (x) \nu (x) + d (y) \nu (y)]}{D} \]

\[ \cdot \left( Z (y) |\nabla \psi (x, t) - Z (x) \nabla \psi (P_b (x), t) | + Z (x) |\nabla \psi (y, t) - Z (y) \nabla \psi (P_b (y), t) | \right) d\omega^- (x, y) dt - \]

\[ - \frac{1}{4\pi} \int \int \int_{[\bar{\Omega} \times [0, \infty] \cap \{x \neq y\}}} \frac{Z (x) Z (y) |\nabla \psi (P_b (x), t) - \nabla \psi (P_b (y), t) |}{D} \]

\[ \cdot \left[ d (y) \nu (y) - d (x) \nu (x) \right] d\omega^- (x, y) dt \]

where \( Q_3 \) is as in (8.10), the measure \( \hat{\mu}_{b}^- \) as in Lemma 27, \( \omega^- (x, y) \) is as in Lemma 22 and the test functions \( \varphi_2, \varphi_3 \) are given by:

\[ \varphi_2 (y, Y, \lambda_1, \lambda_2) = \frac{1}{4\pi} \left[ \nu (y) \cdot \nabla^2 \psi (y, t) \cdot \nu (y) \right] (\lambda_1 + \lambda_2)^2 \]  

\[ \varphi_3 (y, Y, \lambda_1, \lambda_2) = \frac{1}{4\pi} \left( \lambda_1 - \lambda_2 \right) \left[ Y \cdot \nabla^2 \psi (y, t) \cdot \nu (y) \right] \]  

with \( (Y, \lambda_1, \lambda_2) \in \mathcal{M}_2 [y] \).

**Proof.** In order to rewrite \( Q_3 \) we use the identity:

\[ |\nabla \psi (x, t) Z (y) + \nabla \psi (y, t) Z (x) | [d (x) \nu (x) + d (y) \nu (y)] \]

\[ = |\nabla \psi (x, t) - Z (x) \nabla \psi (P_b (x), t) | Z (y) + |\nabla \psi (P_b (x), t) Z (x) Z (y) | [d (x) \nu (x) + d (y) \nu (y)] \]

\[ + |\nabla \psi (y, t) - Z (y) \nabla \psi (P_b (y), t) | Z (x) + |\nabla \psi (P_b (y), t) Z (x) Z (y) | [d (x) \nu (x) + d (y) \nu (y)] \]
Using \( \nu(P_0(x)) = \nu(x) \) as well as (8.1), we obtain, after symmetrizing:

\[
\begin{align*}
\nabla \psi(x,t) Z(y) + \nabla \psi(y,t) Z(x) \left[ d(x) \nu(x) + d(y) \nu(y) \right] \\
= [Z(y) [\nabla \psi(x,t) - Z(x) \nabla \psi(P_0(x),t)] + Z(x) [\nabla \psi(y,t) - Z(y) \nabla \psi(P_0(y),t)]] \cdot [d(x) \nu(x) + d(y) \nu(y)] + \\
+ Z(x) Z(y) [\nabla \psi(P_0(x),t) - \nabla \psi(P_0(y),t)] \cdot [d(y) \nu(y) - d(x) \nu(x)] 
\end{align*}
\]

Therefore, we can write \( Q_3 \) as:

\[ Q_3 = Q_{3,1} + Q_{3,2} \]

with:

\[ Q_{3,1} = - \int \int \int H_{3,1}(x,y,t) \, d\omega^-_\varepsilon(x,y,t) \quad , \quad Q_{3,2} = - \int \int \int H_{3,2}(x,y,t) \, d\omega^-_\varepsilon(x,y,t) \]

\[
H_{3,1}(x,y,t) = \frac{[d(x) \nu(x) + d(y) \nu(y)]}{4\pi D} \cdot [Z(y) [\nabla \psi(x,t) - Z(x) \nabla \psi(P_0(x),t)] + \\
+ Z(x) Z(y) [\nabla \psi(P_0(x),t) - \nabla \psi(P_0(y),t)] \cdot [d(y) \nu(y) - d(x) \nu(x)]
\]

(8.29)

\[
H_{3,2}(x,y,t) = \frac{Z(x) Z(y) [\nabla \psi(P_0(x),t) - \nabla \psi(P_0(y),t)] \cdot [d(y) \nu(y) - d(x) \nu(x)]}{4\pi D} \quad (8.30)
\]

In order to compute the contribution near the diagonal \( \{ x = y \} \) and away from it we split \( Q_{3,1}, Q_{3,2} \) as in the Proof of Lemmas 22, 27. More precisely:

\[ Q_{3,1} = - \int \int \int H_{3,1}(x,y,t) \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \, d\omega^-_\varepsilon(x,y,t) \\
- \int \int \int H_{3,1}(x,y,t) \left[ 1 - \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \right] \, d\omega^-_\varepsilon(x,y,t) \\
= I_{1,1} + I_{1,2} \]

\[ Q_{3,2} = - \int \int \int H_{3,2}(x,y,t) \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \, d\omega^-_\varepsilon(x,y,t) \\
- \int \int \int H_{3,2}(x,y,t) \left[ 1 - \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \right] \, d\omega^-_\varepsilon(x,y,t) \\
= I_{2,1} + I_{2,2} \]

Using Taylor’s expansion as well as the definition of \( P_0(x), P_0(y) \):

\[
I_{1,1} = \frac{1}{4\pi} \int \int \int \frac{[d(x) + d(y)]^2}{D} \left[ \nu(y) \cdot \nabla^2 \psi(y,t) \cdot \nu(y) \right] \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \, d\omega^-_\varepsilon(x,y,t) + O(\delta)
\]

\[
I_{2,1} = \frac{1}{4\pi} \int \int \int \left[ (P_0(x) - P_0(y)) \cdot \nabla^2 \psi(y,t) \cdot \nu(y) \right] \frac{[d(x) + d(y)]}{D} \eta \left( \frac{|x - \tau(y)|}{\delta} \right) \, d\omega^-_\varepsilon(x,y,t)
\]

Taking the limit \( \varepsilon_k \to 0, \delta_k \to 0 \) for suitable subsequences we then obtain:

\[ I_{1,1} + I_{2,1} \to \tilde{\mu}_b(\varphi_2 + \varphi_3) \]

On the other hand, using the boundedness of the integrands we can pass to the limit as \( \varepsilon_k \to 0, \delta_k \to 0 \) in \( I_{1,2}, I_{2,2} \) to obtain:

\[
I_{1,2} \to - \frac{1}{4\pi} \int \int \int \frac{[d(x) \nu(x) + d(y) \nu(y)]}{D} \cdot [Z(y) [\nabla \psi(x,t) - Z(x) \nabla \psi(P_0(x),t)] + Z(x) [\nabla \psi(y,t) - Z(y) \nabla \psi(P_0(y),t)]] \, d\omega^-_\varepsilon(x,y,t) \, dt
\]

\[
I_{2,2} \to - \frac{1}{4\pi} \int \int \int \frac{Z(x) Z(y)}{D} [\nabla \psi(P_0(x),t) - \nabla \psi(P_0(y),t)] \cdot [d(y) \nu(y) - d(x) \nu(x)] \, d\omega^-_\varepsilon(x,y,t) \, dt
\]

This concludes the proof of the lemma.
8.1.4 Limit of the nonlinear terms: The terms $Q_4$ and $Q_5$.

We finally precise the limit of the terms $Q_4$ and $Q_5$.

**Lemma 29** Taking suitable subsequences $\varepsilon_k \to 0$, $\delta_k \to 0$ as $k \to \infty$ we have:

$$Q_4 \to \bar{\mu}_b \left( \varphi_4 \right) + \int \int \int_{\Omega \times \Omega \times [0,\infty)} \nabla \psi (x,t) \frac{Z(y) h(y)}{2\pi} dt$$

\[ \cdot \left[ G_t \left( Y(x,y), \lambda_1(x,y), \lambda_2(x,y) \right) + g_n \left( Y(x,y), \lambda_1(x,y), \lambda_2(x,y) \right) \nu(y) \right] d\omega_t^{-} (x,y) dt \]

where $Q_4$, $Q_5$ are as in (8.7), (8.8) respectively, $\varphi_4$ is:

$$\varphi_4 (y,Y,\lambda_1,\lambda_2) = \frac{h(y)}{2\pi} \nabla \psi (y,t) \cdot \left[ G_t \left( Y, \lambda_1, \lambda_2 \right) + g_n \left( Y, \lambda_1, \lambda_2 \right) \nu(y) \right]$$

and $G_t$, $g_n$ are as in (3.32), (3.33). The measures $\omega_t^{-} (x,y)$, $\bar{\mu}_b$, are respectively as in Lemmas 24, 27.

**Proof.** The proof is essentially similar to the one in Lemmas 24, 27. We split $Q_4$ using the cutoff function $\eta$. Then:

$$Q_4 = \int \int \int [\cdot] \eta \left( \frac{|x-\tau(y)|}{\delta} \right) d\omega_t^{-} (x,y,t) +$$

$$+ \int \int \int [\cdot] \left[ 1 - \eta \left( \frac{|x-\tau(y)|}{\delta} \right) \right] d\omega_t^{-} (x,y,t) \equiv Q_{4,1} + Q_{4,2}$$

Using (8.25) we have:

$$Q_{4,1} = \bar{\mu}_b \left( \varphi_4 \right) + O(\delta)$$

with $\varphi_4$ as in (8.33). Taking the limit $\varepsilon_k \to 0$, $\delta_k \to 0$ we then obtain (8.34). On the other hand, we can take directly the limit in $Q_5$ due to the continuity of $W(x,y)$, whence (8.32) follows. \[\square\]

**Remark 30** It is important to notice that the domain of integration in (8.32) includes the diagonal $\{x = y\}$ differently from the other formulas where the measures $\{\omega_t^{-}\}$ appear (cf. (8.14), (8.21), (8.20), (8.31)).

8.1.5 Weak formulation limit equation for the first regularization.

We can now collect the previous results as follows.

**Theorem 31** Let $u^\varepsilon, v^\varepsilon$ be a solution of (2.1), (2.2). Suppose that the measures $\mu$, $\bar{\mu}_b$, $\omega_b$, $\bar{\mu}_b$ are defined as in Proposition 17 and Lemmas 24, 27 respectively. Then, for any test function $\psi \in C^\infty$ satisfying (8.1) we have:

$$- \int \psi (x,0) u_0 (x) dx - \int \int \psi_t d\mu_t dt - \int \int \Delta \psi d\mu_t dt +$$

$$+ \frac{1}{4\pi} \int \int \int_{\Omega \times S^1} \nu \cdot \nabla^2 \psi (x,t) \cdot \nu \cdot d\mu_t^{-} (x,\nu) dt + \bar{\mu}_b (\varphi) -$$

$$- \int \int \int_{\Omega \times \Omega \times (0,\infty)} \nabla \psi (x,t) W(x,y) d\omega_t^{-} (x,y) dt -$$

$$- \int \int \int_{\Omega \times \Omega \times (0,\infty)} \nabla \cdot G(x,y) \cdot \nabla \psi (x,t) d\omega_t^{-} (x,y) dt =$$

$$= 0 \quad \text{(8.34)}$$
with the test function $\varphi$ given by:
\[
\varphi(y, \lambda_1, \lambda_2, t) = \frac{Y \cdot \nabla^2 \psi(y, t) \cdot Y}{4\pi} + \frac{[\nu(y) \cdot \nabla^2 \psi(y, t) \cdot \nu(y)]}{4\pi} (\lambda_1 + \lambda_2)^2 + \frac{h(y)}{2\pi} \nabla \psi(y, t) \cdot [G_t(Y, \lambda_1, \lambda_2) + g_n(Y, \lambda_1, \lambda_2) \nu(y)]
\]
(8.35)

Moreover, the measures $\hat{\mu}^{-}_t$, $\hat{\mu}^{-}_{b,t}$ satisfy (8.12), (8.23).

**Proof.** The result just follows taking the limit in (8.2). The limit of the term $L$ is immediate. The limit of the terms $Q_1, \ldots, Q_5$ can be obtained using Lemmas 22-29. We have $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ with the functions $\varphi_j$, $j = 1, \ldots, 4$ as in (8.22), (8.27), (8.28), (8.33). We then obtain:

\[
\begin{align*}
&- \int \psi(x, 0) u_0(x) dx - \int \int \psi \, d\mu_t dt - \int \int \Delta \psi \, d\mu_t dt + \\
&+ \int \int_{\Omega \times S^1} \left( \frac{\nu \cdot \nabla^2 \psi(x, t) \cdot \nu}{4\pi} \right) d\hat{\mu}^{-}_t(x, \nu) dt + \hat{\mu}^{-}_{b}(\varphi) + \\
&\quad + \int \int_{\Omega \times [0, \infty]} \int H_1(x, y, t) d\omega^{-}_t(x, y) dt + \\
&\quad + \int \int \int_{\Omega \times [0, \infty]} H_2(x, y, t) d\omega^{-}_t(x, y) dt - \\
&\quad - \int \int \int_{\Omega \times [0, \infty]} \nabla \psi(x, t) W(x, y) d\omega^{-}_t(x, y) dt \\
&\quad - \int \int \int_{\Omega \times [0, \infty]} H_{3,1}(x, y, t) d\omega^{-}_t(x, y) dt - \\
&\quad - \int \int \int_{\Omega \times [0, \infty]} H_{3,2}(x, y, t) d\omega^{-}_t(x, y) dt + \\
&\quad + \int \int \int_{\Omega \times [0, \infty]} \nabla \psi(x, t) \frac{Z(y)}{2\pi} h(y) d\mu_t dt \\
&\quad \cdot [G_t(Y(x, y), \lambda_1(x, y), \lambda_2(x, y)) + g_n(Y(x, y), \lambda_1(x, y), \lambda_2(x, y)) \nu(y)] d\omega^{-}_t(x, y) dt \\
&= 0
\end{align*}
\]

(8.36)

where the functions $H_1, H_2, H_{3,1}, H_{3,2}$ are as in (8.14), (8.24), (8.29), (8.30) respectively.

Formula (8.36) is particularly convenient in order to check the convergence of the different terms arising in the integrals, because the measures $\omega^{-}_t(x, y)$ are integrated against continuous functions in the region of integration. However, it can be written in a more convenient form reversing the computations in Lemma 28 in order to rewrite $H_{3,1}, H_{3,2}$, using (8.1) and Lemma 4 □

**Remark 32** It is important to take into account the presence in (8.34) of the integral term containing $W$ and integrated in $\{x = y\}$. This term gives a nonzero contribution in the singular points of the measure $\omega^{-}_t(x, y)$.

### 8.2 Second regularization

Arguing in a completely similar manner with the second regularization (2.4), (2.5) we can obtain the following result:

**Theorem 33** Let $\nu^-, \nu^+$ be a solution of (2.4), (2.5). Suppose that $\mu$, $\mu^+$ are as in Proposition 16. There exists measures $\hat{\mu} \in M^+ (\Omega \times S^1 \times \mathbb{R}^+) \cup M^+ (\Omega \times \mathbb{R}^+)$, $\hat{\mu}_0 \in M^+ (M \times \mathbb{R}^+)$.
defined as in Lemmas 22, 27 with the functions \( f_c (u^e) \) replaced by \( u^e \). For any test function \( \psi \in C^\infty \) satisfying (8.1) this family of measures satisfy:

\[
- \int \psi (x, 0) u_0 (x) \, dx - \int \int \psi (x, t) \, d\mu_t - \int \int \Delta \psi \, d\mu_t^+ \, dt + \frac{1}{4\pi} \int \int_{\Omega \times S^1} (\nu \cdot \nabla^2 \psi (x, t) \cdot \nu) \, d\hat{\mu}_t (x, \nu) \, dt + \hat{\mu}_b (\varphi) - \int \int \int_{[\Omega \times [0, \infty)] \cap \{x = y\}} \nabla \psi (x, t) \, W (x, y) \, d\omega_t (x, y) \, dt - \int \int \int_{[\Omega \times [0, \infty)] \cap \{x \neq y\}} (\nabla_x G (x, y) \cdot \nabla \psi (x, t)) \, d\omega_t (x, y) \, dt
\]

\[
= 0 \quad (8.37)
\]

with the test function \( \varphi \) as in Section 8. Moreover, we have:

\[
\int_{S^1} d\hat{\mu_t} (\cdot, \nu) = \sum_{x_i (t) \in S_t} \gamma_i (t) \delta_{x_i (t)} (\cdot), \quad \gamma_i (t) = (\alpha_i (t))^2, \quad \text{a.e. } t \in [0, \infty) \quad (8.38)
\]

\[
d\omega_t (x, y) = \sum_{\{x_i (t) \neq x_j (t)\}} \alpha_i (t) \alpha_j (t) \delta_{x_i (t)} (x) \delta_{x_j (t)} (y) + \sum_{x_i (t) \in S_t} \alpha_i (t) \left[ \delta_{x_i (t)} (x) u (y, t) \, dy + \delta_{x_i (t)} (y) u (x, t) \, dx \right] + u (x, t) u (y, t) \, dx \, dy
\]

\[
\int_{\mathcal{M}_2 (\cdot)} d\hat{\mu}_{b,t} (\cdot, \sigma) = \sum_{x_i (t) \in S_t} \gamma_{b,i} (t) \delta_{x_i (t)} (\cdot), \quad \gamma_{b,i} (t) = (\alpha_i (t))^2, \quad \text{a.e. } t \in [0, \infty)
\]

9 On the connection between the measures \( \mu, \mu^- \) and \( \mu^+ \).

Separation Lemma. Proof of the different evolutions for the two regularizations.

The main result in this Section is the following:

**Theorem 34** The measure \( \hat{\mu} \) in Proposition 14 (cf. (7.4)) is related with the measures \( \hat{\mu}^- \), \( \hat{\mu}^- \) defined in Lemma 28 by means of:

\[
\frac{1}{8\pi} \int_{S^1} d\hat{\mu}_t (\cdot, \nu) + \frac{1}{8\pi} \int_{\mathcal{M}_2} d\hat{\mu}_t (\cdot, \nu) = d\hat{\mu}_t (\cdot) \quad \text{a.e. } t \in [0, \infty)
\]

The measure \( \hat{\mu}^+ \) in Proposition 16 (cf. (7.15)) is related with the measures \( \hat{\mu}, \hat{\mu}_b \) in Theorem 5 by means of:

\[
\frac{1}{8\pi} \int_{S^1} d\hat{\mu}_t (\cdot, \nu) + \frac{1}{8\pi} \int_{\mathcal{M}_2} d\hat{\mu}_t (\cdot, \nu) = d\hat{\mu}_t^+ (\cdot) \quad \text{a.e. } t \in [0, \infty)
\]

A relevant consequence of Theorem 34 is the following.
for some constant $\eta$ regularization we use (7.17), (8.38) and Theorem 34 to obtain:

$$\frac{\pi}{\eta}$$

whence

Then $\alpha_i(0) \geq \beta_i(0) \leq \beta_i(0)$ and the result follows. In the case of the second regularization we use (8.17), (8.38) and Theorem 34 to obtain:

$$\alpha_i(t)^2 = \gamma_i(t) = 8\pi \beta_i^+(t)$$

whence the conclusion follows in a similar way.

**Remark 36** Notice that a consequence of Corollary 35 is that the weak formulations (8.34), (8.38) cannot define the same evolution as soon as one of the Dirac masses at any singular point becomes larger than $8\pi$. Notice that intuitively, Corollary 35 states that the effect of the cutoff $\pi$ in the case of the first regularization, or the term $\epsilon u^2$ in the case of the second regularization becomes visible as soon as the masses become larger than $8\pi$.

We will give the details of the proof of Theorem 34 for the measures $\hat{\mu}^-$, $\hat{\mu}$ for the points placed at the interior of $\Omega$, since the points at the boundary $\partial\Omega$ or the case of the measures $\mu$, $\hat{\mu}$ can be studied with similar arguments.

The starting point in the proof of Theorem 34 will be the following inequality that measures the rate of change of the mass in the neighborhood of a singular point in terms of the values of the measures $\hat{\mu}^-$, $\hat{\mu}$ near such a point.

**Lemma 37** Suppose that the measures $\hat{\mu}^-$, $\hat{\mu}$ solve (8.34) for any $\psi \in C^\infty(\Omega \times \mathbb{R}^+)$ satisfying (8.7). Let $x_0 \in \Omega$, $2\rho < R$, $B_{2R}(x_0) \subset \Omega$, $0 \leq T_1 \leq T_2 < \infty$. Suppose that $\eta_R(x) = \eta\left(\frac{|x-x_0|}{R}\right)$ with $\eta \in C^\infty(\mathbb{R})$, $\eta(r) = 1$ if $0 \leq r \leq 1$, $\eta'(r) \leq 0$, $\eta(r) = 0$ if $r \geq 2$. We define also the test function:

$$\varphi_\rho(x) = \begin{cases} 
1 - \frac{|x-x_0|^2}{2\rho^2}, & |x-x_0| < \rho \\
\frac{|x-x_0|^2-2\rho^2}{2\rho^2}, & |x-x_0| > \rho
\end{cases}$$

(9.1)

where $(s)_- = s$ if $s \leq 0$ and $(s)_- = 0$ if $s > 0$. We define also:

$$U_\rho(t; x_0) = \int_{B_\rho(x_0) \times S^1} d\hat{\mu}_t^-(x, \nu)$$

(9.2)

Then, the following inequalities hold:

$$\left| \int \varphi_\rho(x) d\mu_{T_2} - \int \varphi_\rho(x) d\mu_{T_1} - \frac{1}{\rho^2} \int_{T_1}^{T_2} \left[ \frac{U_\rho(t; x_0)}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t \right] dt \right| \leq C \rho^2 \int_{T_1}^{T_2} \int_{\Omega \cap B_{2R}(x_0)} \eta_R(x) d\mu_t \int d\mu_t \leq + C \rho^2 \int_{T_1}^{T_2} \int \left[ \frac{\eta_R(x)}{R} + 1 \right] d\mu_t \int d\mu_t$$

(9.3)
Proof. We will give the details of the proof for the points placed at the interior of \( \Omega \), since the boundary points can be treated similarly. Suppose that \( \psi \) is any test function supported in a ball \( B_{2\rho} (x_0) \) with \( 2\rho < R \). Using Lemma 3 and symmetrizing we can rewrite the last term on the right-hand side of (8.34) as:

\[
\int \int \int_{[\Omega \times [0, \infty)] \cap \{x \neq y\}} \left[ \nabla_x G (x, y) \cdot \nabla \psi (x, t) \right] d\omega^-_1 (x, y) dt + \int \int_{|x-y| > 0} H_1 (x, y, t) d\omega^-_1 (x, y) dt \leq C \| \nabla_x \psi \|_{L^\infty (\Omega \times \mathbb{R}^+)} \int \int d\omega^-_1 (x, y)
\]

(9.4)

where \( H_1 (x, y, t) \) as in (8.14). Using the test function \( \eta_R (x) \) we can write:

\[
\int \int_{|x-y| > 0} H_1 (x, y, t) d\omega^-_1 (x, y) dt = \int \int_{|x-y| > 0} H_1 (x, y, t) \eta_R (x) \eta_R (y) d\omega^-_1 (x, y) dt + \int \int_{|x-y| > 0} H_1 (x, y, t) (1 - \eta_R (x)) \eta_R (y) d\omega^-_1 (x, y) dt + \int \int_{|x-y| > 0} H_1 (x, y, t) \eta_R (x) (1 - \eta_R (y)) d\omega^-_1 (x, y) dt
\]

(9.5)

where we use that a term containing the product \( (1 - \eta_R (x)) (1 - \eta_R (y)) \) vanishes due to the choice of the supports of \( \eta_R, \psi \).

The last two terms on the right-hand side of (9.5) can be bounded by:

\[
\frac{C}{R - 2\rho} \| \nabla_x \psi \|_{L^\infty (\Omega \times \mathbb{R}^+)} \int \int \eta_R (x) d\omega^-_1 (x, y)
\]

Then (9.5) becomes:

\[
\int \int_{|x-y| > 0} H_1 (x, y, t) d\omega^-_1 (x, y) dt = \int \int_{|x-y| > 0} H_1 (x, y, t) \eta_R (x) \eta_R (y) d\omega^-_1 (x, y) dt + O \left( \frac{\| \nabla \psi \|_{L^\infty (\Omega \times \mathbb{R}^+)}}{R - 2\rho} \int \int \eta_R (x) d\omega^-_1 (x, y) \right)
\]

(9.6)

Combining (9.4), (9.6), and using the fact that \( \psi = 0 \) at \( \partial \Omega \) we can rewrite (8.34) as:

\[
- \left( \int \psi (x, 0) u_0 (x) dx \right) - \int \int \psi \mu_t (x) dt - \int \int \Delta \psi d\mu_t (x) = \int \int_{\Omega \times S^1} \left( \frac{\nu \cdot \nabla^2 \psi (x, t) \cdot \nu}{4\pi} \right) d\mu^-_1 (x, \nu) dt + \frac{1}{4\pi} \int \int_{|x-y| > 0} \frac{[-(x-y) \cdot (\nabla \psi (x, t) - \nabla \psi (y, t))] \eta_R (x) \eta_R (y) d\omega^-_1 (x, y) dt}{|x-y|^2} + O \left( \int \int \left[ \frac{\eta_R (x)}{R - 2\rho} \| \nabla_x \psi \|_{L^\infty (\Omega \times \mathbb{R}^+)} + \| \nabla_x \psi \|_{L^\infty (\Omega \times \mathbb{R}^+)} \right] d\omega^-_1 (x, y) dt \right)
\]

(9.7)

Let us consider a function \( H_\varepsilon (t; T_1, T_2) \), \( H_\varepsilon \in C^\infty \), \( 0 \leq H_\varepsilon \leq 1 \) such that \( H_\varepsilon \to \chi_{[T_1, T_2]} \) as \( \varepsilon \to 0 \) in \( L^1 (\mathbb{R}^+) \), \( H_\varepsilon \to \delta_{T_1} - \delta_{T_2} \) in \( (C (\mathbb{R}^+))^* \), where \( \chi_{[T_1, T_2]} \) is the characteristic function of
the interval \([T_1, T_2]\) and \(\delta_T\) is a Dirac mass at \(t = T\). Replacing \(\psi(x, t)\) by \(\psi(x, t) H_\varepsilon(t; T_1, T_2)\) and taking the limit \(\varepsilon \to 0\) we obtain for a.e. \(T_1, T_2 > 0\), and then for all \(T_1, T_2 > 0\) due to the absolute continuity of the integration on \(t\)

\[
- \left( \int \psi(x, T_1) \, d\mu_{T_1}(x) \right) + \int \psi(x, T_2) \, d\mu_{T_2}(x) - \int_{T_1}^{T_2} \int \frac{\partial \psi}{\partial t} \, d\mu_t(x) \, dt - \\
\int_{T_1}^{T_2} \int \Delta \psi \, d\mu_t(x) \, dt + \int_{T_1}^{T_2} \int_{\Omega \times S^1} \left( \frac{\nu \cdot \nabla^2 \psi(x, t) \cdot \nu}{4\pi} \right) \, d\hat{\mu}_t(x, \nu) \, dt + \\
\int_{T_1}^{T_2} \int_{|x-y|>0} H_1(x, y, t) \, \eta_R(x) \, \eta_R(y) \, d\omega^-_t(x, y) \, dt + \\
O \left( \int_{T_1}^{T_2} \|\nabla \psi\|_{L^\infty(\Omega \times R^+)} \int \int \left[ \frac{\eta_R(x)}{R - 2\rho} + 1 \right] \, d\omega^-_t(x, y) \, dt \right) = 0
\]

(9.8)

We now use in (9.8) the test function \(\psi(x, t) = \varphi_\rho(x)\) (cf. (9.1)). Notice that \(\varphi_\rho \notin C^\infty\), but since \(\varphi_\rho \in C^1,1\) it can be used by means of a density argument. Then:

\[
- \left( \int \varphi_\rho(x) \, d\mu_{T_1}(x) \right) + \int \varphi_\rho(x) \, d\mu_{T_2}(x) - \int_{T_1}^{T_2} \int \Delta \varphi_\rho \, d\mu_t(x) \, dt + \\
\int_{T_1}^{T_2} \int_{\Omega \times S^1} \left( \frac{\nu \cdot \nabla^2 \varphi_\rho(x) \cdot \nu}{4\pi} \right) \, d\hat{\mu}_t(x, \nu) \, dt + \\
\frac{1}{4\pi} \int_{T_1}^{T_2} \int_{|x-y|>0} \frac{|(x-y) \cdot (\nabla \varphi_\rho(x) - \nabla \varphi_\rho(y))|}{|x-y|^2} \, \eta_R(x) \, \eta_R(y) \, d\omega^-_t(x, y) \, dt + \\
\int_{T_1}^{T_2} O \left( \frac{1}{\rho} \int \int \left[ \frac{\eta_R(x)}{R - 2\rho} + 1 \right] \, d\omega^-_t(x, y) \right) \, dt = 0
\]

(9.9)

We now estimate the term containing the measure \(\hat{\mu}^-\). Notice that:

\[
\int_{T_1}^{T_2} \int_{\Omega \times S^1} \left( \frac{\nu \cdot \nabla^2 \varphi_\rho(x) \cdot \nu}{4\pi} \right) \, d\hat{\mu}_t(x, \nu) \, dt \\
= - \frac{1}{4\pi \rho^2} \int_{T_1}^{T_2} \int_{B_{\rho}(x_0) \times S^1} \, d\hat{\mu}_t(x, \nu) \, dt + \\
\int_{T_1}^{T_2} \int_{B_{\rho}(x_0) \setminus B_{\rho}(x_0)} \left( \frac{\nu \cdot \nabla^2 \varphi_\rho(x) \cdot \nu}{4\pi} \right) \, d\hat{\mu}_t(x, \nu) \, dt
\]

whence:

\[
\int_{T_1}^{T_2} \int_{\Omega \times S^1} \left( \frac{\nu \cdot \nabla^2 \varphi_\rho(x) \cdot \nu}{4\pi} \right) \, d\hat{\mu}_t(x, \nu) \, dt + \frac{1}{4\pi \rho^2} \int_{T_1}^{T_2} \int_{B_{\rho}(x_0) \times S^1} \, d\hat{\mu}_t(x, \nu) \, dt \\
\leq \frac{C}{\rho^2} \int_{T_1}^{T_2} \int_{B_{\rho}(x_0) \setminus B_{\rho}(x_0)} \, d\mu_t(x) \, dt
\]

(9.10)
Using the fact that $|\nabla \varphi_\rho (x) - \nabla \varphi_\rho (y)| \leq \frac{c}{r} |x - y|$ for $|x - x_0| \leq \rho, \ |y - x_0| \geq \rho$ as well as (9.9) and (9.10) it then follows that:

\[
\int \varphi_\rho (x) \, d\mu_{T_2} (x) - \int \varphi_\rho (x) \, d\mu_{T_1} (x) = \frac{2}{\rho^2} \int_{T_1} \int_{B_{2\rho}(x_0) \setminus B_\rho(x_0)} \frac{|x - x_0| - \rho}{|x - x_0|} \, d\mu_t (x) \, dt + \\
+ \frac{1}{\rho^2} \int_{T_1} \int_{B_\rho(x_0)} \left[ \frac{U_\rho(t; x_0)}{4\pi} - 2 \int_{B_\rho(x_0)} \, d\mu_t (x) \right] \, dt + \\
+ \int_{T_1} \left( \frac{1}{\rho^2} \int_{\Omega \setminus B_\rho(x_0)} \int_{\Omega} \eta_R (x) \eta_R (y) \, d\omega^- (x, y) + \frac{1}{\rho^2} \int_{B_{2\rho}(x_0) \setminus B_\rho(x_0)} \, d\mu_t (x) \right) \, dt + \\
+ \int_{T_1} \left( \frac{1}{\rho} \int_{\Omega \setminus B_\rho(x_0)} \int_{\Omega} \left[ \frac{\eta_R (x)}{R - 2\rho} + 1 \right] \, d\omega^- (x, y) \right) \, dt + \\
+ \int_{T_1} \int_{B_\rho(x_0) \setminus B_{2\rho}(x_0) \cap \{ x \neq y \}} \eta_R (x) \eta_R (y) \, d\omega^- (x, y) \, dt.
\]

(9.11)

Using (8.20) as well as the inequality $\frac{|x - x_0| - \rho}{|x - x_0|} \leq \eta_R (x)$ for $|x - x_0| \leq 2\rho$ we obtain:

\[
\int \varphi_\rho (x) \, d\mu_{T_2} (x) - \int \varphi_\rho (x) \, d\mu_{T_1} (x) \leq \frac{1}{\rho^2} \int_{T_1} \int_{B_\rho(x_0)} \frac{U_\rho(t; x_0)}{4\pi} \, d\mu_t (x) \, dt + \frac{C}{\rho^2} \int_{T_1} \int_{\Omega \setminus B_\rho(x_0)} \eta_R (x) \, d\mu_t (x) \, dt + \\
+ \frac{C}{\rho} \int_{T_1} \int_{\Omega \setminus B_\rho(x_0)} \left[ \frac{\eta_R (x)}{R} + 1 \right] \, d\mu_t (x) \, d\mu_t (y) \, dt + \\
+ \frac{1}{4\pi \rho^2} \int_{T_1} \int_{B_{2\rho}(x_0) \setminus B_\rho(x_0) \cap \{ x \neq y \}} \eta_R (x) \eta_R (y) \, d\omega^- (x, y) \, dt.
\]

We define some auxiliary sets that will be used in the following.

**Definition 38** For any $\sigma, \delta_1, \delta_2 > 0$ we define:

\[
I_{\delta_1, \delta_2} = \left\{ t \in [0, \infty) : \forall \ x \in S_t, \ B_{2\delta_1} (x) \cap S_t = \{ x \} , \ \forall \ Y \in \Omega, \ \int_{B_{2\delta_1}(Y) \setminus S_t} \, d\mu_t (x) < \delta_2 \right\}
\]

\[
I_{\delta_1, \delta_2}^+ (X) = \left\{ t \in [0, \infty) : \frac{U_{\delta_1}(t; X)}{4\pi} > 2 \int_{B_{\delta_1}(X)} \, d\mu_t (x) + \delta_2 \right\}
\]

\[
I_{\delta_1, \delta_2}^- (X) = \left\{ t \in [0, \infty) : \frac{U_{\delta_1}(t; X)}{4\pi} < 2 \int_{B_{\delta_1}(X)} \, d\mu_t (x) - \delta_2 \right\}
\]

where $U_{\delta_1}(t; X)$ is defined as in (9.3) and $X \in \Omega$.

In the proof of the following result, it will be convenient to make more explicit the dependence on the singular point of the Dirac masses $\alpha_j, \gamma_j$ in (7.3), (8.12). We write:

\[
\mu_t = \sum_{x_j(t) \in S_t} \alpha (t; x_j) \delta_{x_j(t)} + u (\cdot , t) \, dx , \ u (\cdot , t) \in L^1 (\Omega)
\]

(9.12)

\[
\int_{S_t} \, d\tilde{\mu}_t (\cdot , \nu) = \sum_{x_j(t) \in S_t} \gamma^- (t; x_j) \delta_{x_j(t)}
\]

(9.13)
Lemma 39 There exists $\delta_0 > 0$ small depending only on $\int_{\Omega} d\mu_0 (x)$ such that for any $\delta_1$, $\delta_2 \in (0, \delta_0)$ we have
\[
\left| I_{\delta_1, \delta_2} \cap I_{\delta_1, \delta_2}^+ (X) \right| = 0 \tag{9.14}
\]
\[
\left| I_{\delta_1, \delta_2} \cap I_{\delta_1, \delta_2}^- (X) \right| = 0 \tag{9.15}
\]
for any $X \in \Omega$.

Proof. We will prove (9.14) since the proof of (9.15) is similar. We argue by contradiction. Suppose that for some $X \in \Omega$, $\left| I_{\delta_1, \delta_2} \cap I_{\delta_1, \delta_2}^+ (X) \right| > 0$.

A well known result \[\text{[4]}\] states that for any measurable set $A \subset \mathbb{R}$:
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left| \{ t - t_0 < \varepsilon \} \cap A \right| = 1 \quad \text{a.e.} \quad t_0 \in A \tag{9.16}
\]

Let $A = I_{\delta_1, \delta_2} \cap I_{\delta_1, \delta_2}^+ (X)$ and fix $n$ integer such that $n\delta_2 \in (1, 2)$. For every $t_0 \in A$ such that (9.16) holds, there exists a sequence $\{ \varepsilon_\ell > 0 \}$, $\varepsilon_\ell \to 0$ such that
\[
\left| \{ t - t_0 < \varepsilon_\ell \} \cap A \right| \leq \frac{\delta_2}{K} \varepsilon_\ell \tag{9.17}
\]
where $K > 0$ is a fixed numerical constant independent on $\delta_1, \delta_2$ that will be precised later.

Suppose that $K > 8$. Then, since $\frac{\delta_2 \varepsilon_\ell}{K} \leq \frac{2\varepsilon_\ell}{K} \leq \frac{\varepsilon_\ell}{2(n+1)}$, we can obtain, for each $\ell$, $n$ times $t_\ell^i \in A$, $i = 1, \ldots, n$ such that:
\[
t_0 = t_\ell^1 < \ldots < t_\ell^n = t_0 + \varepsilon_\ell
\]
\[
\frac{\varepsilon_\ell}{2n} \leq \left( t_\ell^{i+1} - t_\ell^i \right) \leq \frac{\varepsilon_\ell}{n}, \quad i = 1, \ldots, (n-1) \tag{9.18}
\]

We now prove that for $t \in I_{\delta_1, \delta_2} \cap I_{\delta_1, \delta_2}^+$ there exists a singular point $Y \in S_t \cap B_{\delta_1} (X)$. Indeed, notice that for $t \in A = I_{\delta_1, \delta_2} \cap I_{\delta_1, \delta_2}^+$ we have:
\[
U_{\delta_1} (t; X) > \delta_2
\]
On the other hand by (8.12):
\[
U_{\delta_1} (t; X) = \int_{B_{\delta_1} (X) \times S^1} d\hat{\mu}_t (x, \nu) \leq \left( \int_{B_{\delta_1} (X)} d\mu_t (x) \right)^2
\]

Suppose that $S_t \cap B_{\delta_1} (X) = \emptyset$. Then, since $t \in A \subset I_{\delta_1, \delta_2}$ it follows from the definition of $I_{\delta_1, \delta_2}$ that:
\[
\int_{B_{\delta_1} (X)} d\mu_t (x) \leq \delta_2^2
\]

Therefore:
\[
\delta_2 < U_{\delta_1} (t; X) \leq \delta_2^4
\]
and this gives a contradiction for $\delta_2$ sufficiently small. Then for $t \in A$ we have $S_t \cap B_{\delta_1} (X) \neq \emptyset$. Moreover $S_t \cap B_{\delta_1} (X) = \{ Y \}$ for some $Y \in \Omega$ and $[B_{2\delta_1} (Y) \setminus \{ Y \}] \cap S_t = \emptyset$.

Given the sequence of times $\{ t_\ell^i : i = 1, \ldots, n \}$, let us denote as $Y_\ell^i$ the corresponding singular points $Y_\ell^i \in S_t \cap B_{\delta_1} (X)$.

It now follows using Lemma [18] that
\[
\left| Y_\ell^i - Y_\ell^{i+1} \right| \leq L \sqrt{|t_\ell^i - t_\ell^{i+1}|} \leq L \sqrt{\frac{\varepsilon_\ell}{n}}, \quad 1 = 1, \ldots, (n-1)
\]
where $L > 0$ is a constant depending only on $\int_{\Omega} d\mu_0 (x)$ and $\Omega$. 

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We then apply \((9.3)\) with \(T_1 = t^\ell_i, T_2 = t^\ell_{i+1}, x_0 = \frac{y^\ell_i + y^\ell_{i+1}}{2}\). We will assume also that:

\[
\rho = 4L\sqrt{\delta_2 \varepsilon_\ell}, \quad R = D\rho
\]

(9.19)

where the constant \(D\) will be chosen depending only on \(\int_{\Omega} d\mu_0\). Notice that \(t \in A\) implies \(t \in \bar{I}_{\delta_1, \delta_2} \cap I^\ell_{\delta_1, \delta_2}(X)\) for the above mentioned value of \(\delta_1\). We only assume for the moment that:

\[
D \geq 8
\]

(9.20)

Then:

\[
\int \varphi_{\rho}(x) \, d\mu_{t^\ell_{i+1}}(x) - \int \varphi_{\rho}(x) \, d\mu_{t^\ell_{i}}(x)
\]

\[
\geq \frac{1}{\rho^2} \int_{[t^\ell_{i}, t^\ell_{i+1}] \cap A} \left[ \frac{U_\rho(t; Y^\ell_i)}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t(x) \right] dt + \frac{1}{\rho^2} \int_{[t^\ell_{i}, t^\ell_{i+1}] \cap A} \left[ \frac{U_\rho(t; Y^\ell_{i+1})}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t(x) \right] dt -
\]

\[
-C \int_{t^\ell_{i}}^{t^\ell_{i+1}} \int_{\Omega \setminus B_\rho(x_0)} \eta(t, x_0) \, d\mu_t(x) \, dt -
\]

\[
-C \int_{t^\ell_{i}}^{t^\ell_{i+1}} \int \left[ \frac{\eta(t, x_0)}{R} + 1 \right] \, d\mu_t(x) \, dt - \frac{1}{4\pi \rho^2} \int_{t^\ell_{i}}^{t^\ell_{i+1}} \int_{B_\rho(x_0) \times B_\rho(x_0) \cap \{x \neq y\}} d\mu_t(x) \, d\mu_t(y) \, dt
\]

Notice that, due to our choice of \(x_0\) and the definition of \(\varphi_{\rho}\), we have that

\[
|Y^\ell_{i+1} - x_0| \leq \rho, \quad |Y^\ell_i - x_0| \leq \rho
\]

\[
\varphi_{\rho}(Y^\ell_i) = \varphi_{\rho}(Y^\ell_{i+1}) \geq \frac{1}{2}
\]

(It is important to take into account that \(x_0\) depends on \(i\)). Then:

\[
\alpha \left( Y^\ell_{i+1}; t^\ell_{i+1} \right) - \alpha \left( Y^\ell_i; t^\ell_i \right) + \int_{\Omega \setminus \{Y^\ell_{i+1}\}} \varphi_{\rho}(x; Y^\ell_{i}) \, d\mu_{t^\ell_{i+1}}(x) - \int_{\Omega \setminus \{Y^\ell_{i}\}} \varphi_{\rho}(x; Y^\ell_{i}) \, d\mu_{t^\ell_{i}}(x)
\]

\[
\geq \frac{1}{\rho^2} \int_{[t^\ell_{i}, t^\ell_{i+1}] \cap A} \left[ \frac{U_\rho(t; Y^\ell_i)}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t(x) \right] dt + \frac{1}{\rho^2} \int_{[t^\ell_{i}, t^\ell_{i+1}] \cap A} \left[ \frac{U_\rho(t; Y^\ell_{i+1})}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t(x) \right] dt -
\]

\[
-C \int_{t^\ell_{i}}^{t^\ell_{i+1}} \int_{\Omega \setminus B_\rho(x_0)} \eta(t, x_0) \, d\mu_t(x) \, dt -
\]

\[
-C \int_{t^\ell_{i}}^{t^\ell_{i+1}} \int \left[ \frac{\eta(t, x_0)}{R} + 1 \right] \, d\mu_t(x) \, dt - \frac{1}{4\pi \rho^2} \int_{t^\ell_{i}}^{t^\ell_{i+1}} \int_{B_\rho(x_0) \times B_\rho(x_0) \cap \{x \neq y\}} d\mu_t(x) \, d\mu_t(y) \, dt
\]

where we use the fact that \(t^\ell_i, t^\ell_{i+1} \in I_{\delta_1, \delta_2}\) and we write explicitly the dependence on the center \(x_0\) for \(\varphi_{\rho} = \varphi_{\rho}(x; x_0)\). Notice that we use also the fact that \(U_\rho(t; Y^\ell_i) = U_\rho(t; x_0)\) for \(t \in [t^\ell_i, t^\ell_{i+1}] \cap A\).
Using the global boundedness of \( \int d\mu_t(x) \) as well as (9.18) as the fact that \( R \leq 1 \) and the definition of \( \eta_R \) we obtain:

\[
\begin{align*}
\alpha \left( Y_{t_{i+1}}^t, t_{i+1}^t \right) - \alpha \left( Y_{t_i}^{\ell}, t_i^{\ell} \right) + \int_{\Omega \setminus \{Y_i^t\}} \varphi_\rho \left( x; Y_i^t \right) d\mu_{t_{i+1}}(x) - \int_{\Omega \setminus \{Y_i^t\}} \varphi_\rho \left( x; Y_i^t \right) d\mu_{t_i}(x) \\
\geq \frac{1}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \left[ \frac{U_\rho(t; x_0)}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t(x) \right] dt - \frac{C}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} dt \\
- \frac{C}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \int_{B_\rho(x_0) \setminus B_\rho(x_0)} d\mu_t(x) dt - \frac{C}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \int_{B_\rho(x_0) \setminus B_\rho(x_0)} d\mu_t(x) dt \\
- \frac{1}{4\pi \rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \int_{B_\rho(x_0) \setminus B_\rho(x_0) \cap \{x \neq y\}} d\mu_t(x) d\mu_t(y) dt
\end{align*}
\]

and using again (9.17) as well as the boundedness of \( \int d\mu_t(x) = \int d\mu_0(x) \) we obtain:

\[
\begin{align*}
\alpha \left( Y_{t_{i+1}}^t, t_{i+1}^t \right) - \alpha \left( Y_{t_i}^{\ell}, t_i^{\ell} \right) + \int_{\Omega \setminus \{Y_i^t\}} \varphi_\rho \left( x; Y_i^t \right) d\mu_{t_{i+1}}(x) - \int_{\Omega \setminus \{Y_i^t\}} \varphi_\rho \left( x; Y_i^t \right) d\mu_{t_i}(x) \\
\geq \frac{1}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \left[ \frac{U_\rho(t; x_0)}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t(x) \right] dt - \frac{C}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} dt \\
- \frac{C}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \int_{B_\rho(x_0) \setminus B_\rho(x_0)} d\mu_t(x) dt \\
- \frac{C}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \int_{B_\rho(x_0) \setminus B_\rho(x_0) \cap \{x \neq y\}} d\mu_t(x) d\mu_t(y) dt
\end{align*}
\]

Then since \( B_{2R}(x_0) \setminus B_\rho(x_0) \subset B_{2\delta_1}(Y) \setminus \{Y\} \) due to (9.20) we have:

\[
\int_{B_{2R}(x_0) \setminus B_\rho(x_0)} d\mu_t(x) \leq \int_{B_\rho(Y) \setminus S_\epsilon} d\mu_t(x) < \delta_2^2
\]
due to the definition of \( I_{\delta_1, \delta_2} \). Then:

\[
\frac{C}{\rho^2} \int_{[t_i^t, t_{i+1}^t] \cap A} \int_{B_\rho(x_0) \setminus B_\rho(x_0)} d\mu_t(x) dt \leq \frac{C \delta_2^2}{\rho^2} \epsilon_\ell
\]

Moreover, since \( t \in A \subset I_{\delta_1, \delta_2} \) and \( X = x_0 \) we have

\[
\frac{U_\rho(t; x_0)}{4\pi} - 2 \int_{B_\rho(x_0)} d\mu_t(x) > \delta_2
\]

We finally estimate the last term in (9.21). Since \( t \in A \subset I_{\delta_1, \delta_2} \) there is only one singular point \( Y \in B_\rho(x_0) \). Then:

\[
\int_{B_\rho(x_0) \setminus B_\rho(x_0) \cap \{x \neq y\}} d\mu_t(x) d\mu_t(y) \leq 2 \left[ \int_{B_\rho(x_0)} d\mu_t(x) \right] \cdot \left[ \int_{B_\rho(x_0) \setminus \{Y\}} d\mu_t(x) \right] \leq C \int_{B_\rho(Y) \setminus \{Y\}} d\mu_t(x)
\]

and using the definition of \( I_{\delta_1, \delta_2} \) we obtain:

\[
\int_{B_\rho(x_0) \setminus B_\rho(x_0) \cap \{x \neq y\}} d\mu_t(x) d\mu_t(y) \leq C \delta_2^2
\]
we obtain:

\[ \alpha (Y_{i+1}^\ell; t_{i+1}^\ell) - \alpha (Y_i^\ell; t_i^\ell) + \int_{\Omega \setminus \{Y_{i+1}^\ell\}} \varphi_\rho (x; Y_i^\ell) \, d\mu_{t+1} (x) - \int_{\Omega \setminus \{Y_i^\ell\}} \varphi_\rho (x; Y_i^\ell) \, d\mu_t (x) \]

\[ \geq \frac{\delta_2}{\rho^2} \int_{[t_i, t_{i+1}) \cap A} dt - \frac{C}{\rho^2} \int_{[t_i, t_{i+1}) \setminus A} dt - \frac{C \delta_2^2 \varepsilon_t}{\rho^2} \]

Adding for all \( i = 1, ..., (n-1) \):

\[ \alpha (Y_{n}^\ell; t_0 + \varepsilon) - \alpha (Y_1^\ell; t_0) + \sum_{i=1}^{n-1} \left[ \int_{\Omega \setminus \{Y_{i+1}^\ell\}} \varphi_\rho (x; Y_i^\ell) \, d\mu_{t+1} (x) - \int_{\Omega \setminus \{Y_i^\ell\}} \varphi_\rho (x; Y_i^\ell) \, d\mu_t (x) \right] \]

\[ \geq \frac{\delta_2}{\rho^2} \int_{[t_0, t_0 + \varepsilon]) \cap A} dt - \frac{C}{\rho^2} \int_{[t_0, t_0 + \varepsilon]) \setminus A} dt - \frac{C \delta_2^2 \varepsilon_t}{\rho^2} \]

We now choose \( \delta_2 \) such that \( C \delta_2 \leq \frac{1}{10} \), \( \frac{C}{\delta_2} = \frac{1}{10} \), and \( K \) such that \( \frac{C}{K} \leq \frac{1}{10} \). Notice that \( R = D \rho = 4L \sqrt{2 \varepsilon_t} = \frac{40L \sqrt{2 \varepsilon_t}}{\delta_2} \rightarrow 0 \) as \( \varepsilon_t \rightarrow \infty \). Finally we choose \( D \) satisfying \( 9.20 \) and using the choice of \( \rho \) in \( 9.19 \) we obtain:

\[ \alpha (Y_n^\ell; t_0 + \varepsilon) - \alpha (Y_1^\ell; t_0) \geq \frac{1}{2 (4L)^2} \sum_{i=1}^{n-1} \left[ \int_{\Omega \setminus \{Y_{i+1}^\ell\}} \varphi_\rho (x; Y_i^\ell) \, d\mu_{t+1} (x) - \int_{\Omega \setminus \{Y_i^\ell\}} \varphi_\rho (x; Y_i^\ell) \, d\mu_t (x) \right] \]

We estimate the sum as:

\[ \leq \sum_{i=1}^{n-1} \delta_2^2 \leq \delta_2^2 n \leq 2 \delta_2 \]

Then:

\[ \alpha (Y_n^\ell; t_0 + \varepsilon) - \alpha (Y_1^\ell; t_0) \geq \frac{1}{32L^2} - 2 \delta_2 \geq \frac{1}{64L^2} = \theta > 0 \quad (9.25) \]

where \( \theta \) depends only on \( \int_{\Omega} \, d\mu_0 (x) \) and \( \Omega \).

We can now derive a contradiction as follows. Let us denote as \( A \) the set of density points of \( A \). More precisely:

\[ A = \left\{ t \in A : \lim_{\varepsilon \to 0} \frac{|A \cap [t, t + \varepsilon]|}{\varepsilon} = 1 \right\} \]
We now use that (cf. [4]) \( |A \setminus A| = 0 \). Therefore \( \lim_{\varepsilon \to 0} |A \cap (t + \varepsilon)| = \lim_{\varepsilon \to 0} |A \cap (t + \varepsilon)| \) and all the points of \( A \) are density points. By assumption \( |A| = |A| > 0 \). We have proved in (9.25) the following.

For any \( t_0 \in A \) there exists \( \varepsilon (t_0) \) such that, for any \( \varepsilon \leq \varepsilon (t_0) \) we have:

\[
\alpha (Y (t_0 + \varepsilon); t_0 + \varepsilon) - \alpha (Y (t_0); t_0) \geq \theta
\]  

(9.26)

where \( Y (t) \) is the unique point in \( S_1 \cap B_{\delta_1} (X) \) that exists for any \( t \in A = I_{\delta_1, \delta_2} \cap I_{\delta_1, \delta_2}^+ (X) \).

Moreover:

\[
|A \cap [t_0, t_0 + \varepsilon]| \geq \left( 1 - \frac{\delta_2}{K} \right) \varepsilon
\]  

(9.27)

for any \( \varepsilon \leq \varepsilon (t_0) \).

We now argue iteratively. Due to (9.27) we can find \( t_1 \in A \cap (t_0, t_0 + \varepsilon (t_0)) \), and there exists \( \varepsilon (t_1) \leq (t_0 + \varepsilon (t_0) - t_1) \) such that for any \( \varepsilon \leq \varepsilon (t_1) \) we have:

\[
\alpha (Y (t_1 + \varepsilon); t_1 + \varepsilon) - \alpha (Y (t_1); t_1) \geq \theta
\]  

(9.28)

\[
|A \cap [t_1, t_1 + \varepsilon]| \geq \left( 1 - \frac{\delta_2}{K} \right) \varepsilon
\]  

(9.29)

Taking \( \varepsilon = t_1 - t_0 \) in (9.26) and using also (9.28) we obtain:

\[
\alpha (Y (t_1 + \varepsilon); t_1 + \varepsilon) - \alpha (Y (t_0); t_0) \geq 2 \theta
\]  

for any \( \varepsilon \leq \varepsilon (t_1) \).

Iterating the argument, something that it is possible due to (9.29) we obtain the existence of sequences \( \{t_n\} \subset A, \{\varepsilon (t_n)\} \subset [0, \infty) \) such that:

\[
\alpha (Y (t_n + \varepsilon); t_n + \varepsilon) - \alpha (Y (t_0); t_0) \geq (n + 1) \theta
\]

for any \( \varepsilon \leq \varepsilon (t_n) \). Since \( \alpha (Y; t) \) is bounded for the total mass \( \int_{\Omega} d\mu_0 (x) \) this gives a contradiction.

Using (9.14), (9.15) we can now conclude the Proof of Theorem 34.

**Proof of Theorem 34.** Notice that for any \( \delta_2 > 0 \) fixed the sets \( I_{\delta_1, \delta_2} \) are an decreasing sequence of sets in the sense that:

\[
0 < \tilde{\delta}_1 < \delta_1 \implies I_{\delta_1, \delta_2} \subset I_{\tilde{\delta}_1, \delta_2}
\]

Moreover, for any \( \delta_2 > 0 \) fixed we have:

\[
\left| (0, \infty) \setminus \bigcup_{\delta_1 > 0} I_{\delta_1, \delta_2} \right| = 0
\]  

(9.30)

Let us write:

\[
Z = (0, \infty) \setminus \bigcup_{\delta_1 > 0} I_{\delta_1, \delta_2}
\]

Then:

\[
[0, \infty) = Z \cup \bigcup_{\delta_1 > 0} I_{\delta_1, \delta_2} = Z \cup \bigcup_{n=1}^{\infty} I_{\delta_n, \delta_2}
\]  

(9.31)

\[
|Z| = 0
\]

Let us consider a countable set \( \mathcal{F} \) dense in \( \Omega \). We have:

\[
\Omega = \bigcup_{X \in \mathcal{F}} B_{\delta_1} (X)
\]  

(9.32)
We define
\[ \mathcal{U}_{\delta_2} = \{ t \in [0, \infty) : \exists Y \in S_t, \gamma^- (Y; t) - 8\pi \alpha (Y; t) \geq 8\pi \delta_2 \} \] (9.33)

Using (9.31) we obtain:
\[ \mathcal{U}_{\delta_2} = [\mathcal{Z} \cap \mathcal{U}_{\delta_2}] \cup \bigcup_{n=1}^{\infty} \left( I_{n, \delta_2}^+ \cap \mathcal{U}_{\delta_2} \right) \] (9.34)

Suppose that \( t \in \left[ I_{n, \delta_2}^{+} \cap \mathcal{U}_{\delta_2} \right] \). Then, there exists \( Y \in S_t \) such that \( \gamma^- (Y, t) - 8\pi \alpha (Y, t) \geq 8\pi \delta_2 \) and, due to (9.32) there exists \( \tilde{X} \in \mathcal{F} \) such that \( Y \in B_1 \left( \tilde{X} \right) \). Then:
\[ \int_{B_1 \left( \tilde{X} \right) \times S^1} d\hat{\mu}_t (x, \nu) - 8\pi \int_{B_1 \left( \tilde{X} \right)} d\mu_t \geq 8\pi \delta_2 - 8\pi \delta_2^2 \geq 4\pi \delta_2 \]

Therefore \( t \in I_{n, \delta_2}^{+} \left( \tilde{X} \right) \subset \bigcup_{X \in \mathcal{F}} I_{n, \delta_2}^{+} (X) \). Then:
\[ \left[ I_{n, \delta_2}^{+} \cap \mathcal{U}_{\delta_2} \right] \subset \bigcup_{X \in \mathcal{F}} \left[ I_{n, \delta_2}^{+} \cap I_{n, \delta_2}^{+} (X) \right] \]

It then follows from (9.34) that:
\[ \mathcal{U}_{\delta_2} \subset [\mathcal{Z} \cap \mathcal{U}_{\delta_2}] \cup \bigcup_{n=1}^{\infty} \bigcup_{X \in \mathcal{F}} \left[ I_{n, \delta_2}^{+} \cap I_{n, \delta_2}^{+} (X) \right] \]

Then:
\[ |\mathcal{U}_{\delta_2}| \leq |\mathcal{Z} \cap \mathcal{U}_{\delta_2}| + \sum_{n=1}^{\infty} \sum_{X \in \mathcal{F}} \left| I_{n, \delta_2}^{+} \cap I_{n, \delta_2}^{+} (X) \right| = 0 + \sum_{n=1}^{\infty} \sum_{X \in \mathcal{F}} 0 = 0 \]

whence:
\[ |\mathcal{U}_{\delta_2}| = 0 \]

for any \( \delta_2 > 0 \) sufficiently small. Due to the definition of \( \mathcal{U}_{\delta_2} \) in (9.33) it follows that:
\[ \forall Y \in S_t, \quad \frac{\gamma^- (Y, t)}{8\pi} - \alpha (Y, t) \leq \delta_2 \quad \text{a.e. } t \in [0, \infty) \]

Then, since \( \delta_2 \) can be made arbitrarily small it follows that:
\[ \forall Y \in S_t, \quad \frac{\gamma^- (Y, t)}{8\pi} - \alpha (Y, t) \leq 0 \quad \text{a.e. } t \in [0, \infty) \]

A similar argument taking as starting point (9.15) yields:
\[ \forall Y \in S_t, \quad \frac{\gamma^- (Y, t)}{8\pi} - \alpha (Y, t) \geq 0 \quad \text{a.e. } t \in [0, \infty) \]

whence:
\[ \forall Y \in S_t, \quad \frac{\gamma^- (Y, t)}{8\pi} = \alpha (Y, t) \quad \text{a.e. } t \in [0, \infty) \]

or, equivalently:
\[ \frac{1}{8\pi} \int_{S^1} d\hat{\mu}_t (\cdot, \nu) = d\mu^\text{sing}_t (\cdot) \]

The previous argument yields the contribution of the interior singular points. A similar argument could be used to compute the contribution of the boundary terms. The main idea needed is now sketched. Taking a test function that is quadratic near a singular boundary point (with Neumann
boundary conditions), and using the test function $\varphi$ in (8.35) we obtain formally that the main
contribution is due to the terms containing $\nabla^2 \psi$ that give, assuming that curvature effects in the
test function are higher order terms (as well as the curvature term in $\varphi$ that seems to give also a
negligible contribution):

$$
\psi \approx \frac{|x - x_0|^2}{2 \rho^2}, \quad \Delta \psi \approx \frac{2}{\rho^2}
Y \cdot \nabla^2 \psi (y, t) \cdot Y
\frac{\nu (y) \cdot \nabla^2 \psi (y, t) \cdot \nu (y)}{4\pi}
(\lambda_1 + \lambda_2)^2
\approx - \frac{1}{4\pi \rho^2} \left[ Y^2 + (\lambda_1 + \lambda_2)^2 \right] = - \frac{1}{4\pi \rho^2}
$$

This gives the boundary contributions. ■

10 Characterizing oscillations in the microscopic scale: Measure valued Young measures.

10.1 Generalities.

A possible feature of the solutions obtained in this paper that we do not rule out in this paper is the possibility of having oscillations at a microscopic time scale of order $\varepsilon^2$ for functions like $f_\varepsilon (u^\varepsilon)$ in the case of the first regularization or $u^\varepsilon + \varepsilon (u^\varepsilon)^\frac{3}{2}$. This is the reason because we obtained in the weak limits of $f_\varepsilon (u^\varepsilon (x, t)) f_\varepsilon (u^\varepsilon (y, t)) dxdy$ measures defined in $\bar{\Omega} \times \bar{\Omega} \times [0, \infty)$ having the form $d\omega^\varepsilon (x, y)$. It is not clear if this limit measure can be decomposed as $d\mu^- (x) d\mu^- (y)$. In this section we introduce some general formalism that allows to characterize the limits of this nonlinear expressions even if such oscillations take place. We will also prove that the limit objects, that will be denoted as Measure valued Young measures, can be characterized by means of a standard set of Young measures that depend only on the oscillations of the masses near the singular points.

Let us assume that $\{\mu_1, \mu_2, \ldots, \mu_L\}$ is a set of measures in $M_+ (\bar{\Omega} \times \mathbb{R}^+) \times \mathbb{R}^+$ satisfying:

$$
d\mu_k (x, t) = d\mu_{k,t} (x) dt, \quad k = 1, \ldots, L
$$

(10.1)

$$
\int_{\bar{\Omega}} d\mu_{k,t} (x) \leq A, \quad k = 1, \ldots, L, \quad a.e \ t \in [0, \infty)
$$

(10.2)

for some $A > 0$.

We will assume also that these measures can be approximated in the weak topology by means of sequences $\{d\mu_{k,t} (x) dt\}$ as $\varepsilon \to 0^+$, where $d\mu_{k,t} (x) = U_{\varepsilon} (x, t) dx$, and $U_{\varepsilon} \in C^\infty (\bar{\Omega} \times \mathbb{R}^+)$. It will be always understood that convergence takes place for suitable subsequences.

Specific examples would be the sequences $\{f_\varepsilon (u^\varepsilon) dx dt\}$, $\{u^\varepsilon + \varepsilon (u^\varepsilon)^\frac{3}{2} dx dt\}$, that converge respectively to $d\mu^- dt$, $d\mu^+ dt$. We could also consider sequences like $\{u^\varepsilon dx dt\}$ that converge to $d\mu dt$, but since in this case oscillations do not take place, the formalism presented below would be trivial and the resulting measure valued Young measures would be suitable Dirac masses.

Given measures $\{\mu_k\}_{k=1}^L$ satisfying (10.1) and (10.2), test functions $\varphi_{k,j} \in C (\bar{\Omega})$, $k = 1, \ldots, L$, $j = 1, \ldots, M_k$, $M_k \geq 1$ as well as $T \in (0, \infty)$ we define the following functional:

$$
L_{\{\varphi_{k,j}\}} : C_0 (\mathbb{R}^M \times [0, T]) \to \mathbb{R}, \quad M = \sum_{j=1}^L M_j
$$

$$
\Phi \to L_{\{\varphi_{k,j}\}} [\Phi], \quad \Phi \in C \left( [\bar{-B, B}]^M \times [0, T] \right)
$$
where:

\[
L_{(\varphi_{k,j})}^\varepsilon \Phi = \int_0^T \Phi \left( \int_{\Omega} \varphi_{1,1} d\mu_{\varepsilon}^1, \ldots, \int_{\Omega} \varphi_{1,M_1} d\mu_{\varepsilon}^1, \int_{\Omega} \varphi_{2,1} d\mu_{\varepsilon}^2, \ldots, \int_{\Omega} \varphi_{2,M_2} d\mu_{\varepsilon}^2, \ldots, \int_{\Omega} \varphi_{L,M_L} d\mu_{\varepsilon}^L, t \right) dt
\]

Due to (10.1), (10.2) it follows that \( |\int_{\Omega} \varphi_{k,j} d\mu_{\varepsilon}^k| \leq B, k = 1, \ldots, L, j = 1, \ldots, M_k \) depending only on \( A \), \( \|\varphi_{k,j}\|_{L^\infty(\Omega)} \). Therefore the functional defined in (10.3) can be considered as a functional in \( C([-B,B]^M \times [0,T]) \) and it satisfies:

\[
|L_{(\varphi_{k,j})}^\varepsilon \Phi| \leq T \|\Phi\|_{C([-B,B]^M \times [0,T])}
\]

Therefore, there exists a Radon measure \( d\lambda_{(\varphi_{k,j})}^\varepsilon, T \) such that:

\[
L_{(\varphi_{k,j})}^\varepsilon \Phi = \int_{[-B,B]^M \times [0,T]} \Phi (\xi, t) d\lambda_{(\varphi_{k,j})}^\varepsilon, T (\xi, t)
\]

where:

\[
\xi = (\xi_{1,1}, \ldots, \xi_{1,M_1}, \xi_{2,1}, \ldots, \xi_{2,M_2}, \ldots, \xi_{L,M_L})
\]

The compactness of \([-B,B]^M \times [0,T] \) implies that for suitable subsequences:

\[
d\lambda_{(\varphi_{k,j})}^\varepsilon, T \rightarrow d\lambda_{(\varphi_{k,j})}^\varepsilon, T
\]

Therefore (for subsequences):

\[
L_{(\varphi_{k,j})}^\varepsilon \Phi \rightarrow \int_{[-B,B]^M \times [0,T]} \Phi (\xi, t) d\lambda_{(\varphi_{k,j})}^\varepsilon, T (\xi, t)
\]

Extending the measure \( d\lambda_{(\varphi_{k,j})}^\varepsilon, T \) by zero for \( |\xi_{k,j}| > B \) we can ensure the existence of a family of measures \( d\lambda_{(\varphi_{k,j})}^\varepsilon, T \) such that:

\[
L_{(\varphi_{k,j})}^\varepsilon \Phi \rightarrow \int_{\mathbb{R}^M \times [0,T]} \Phi (\xi, t) d\lambda_{(\varphi_{k,j})}^\varepsilon, T (\xi, t)
\]

(10.4)

The family of measures \( \{\lambda_{(\varphi_{k,j})}^\varepsilon, T\} \) will be denoted as measure valued Young measures. Notice that they allow to compute weak limits for the weak limit of any finite sequence of "macroscopic" magnitudes obtained using sequences of measures \( d\mu_{\varepsilon}^k \).

Our goal is to reduce the computation of the measures \( d\lambda_{(\varphi_{k,j})}^\varepsilon, T \) in the case of the limits of sequences \( \{f_\varepsilon (u^\varepsilon) dxdt\} \), \( \left\{u^\varepsilon + \varepsilon (u^\varepsilon)^2 \right\} dxdt \) to the Young measures associated to the sequences that yield the masses near the singular set \( S_k \).

### 10.2 On the family of Young measures describing the aggregation of \( f_\varepsilon (u^\varepsilon) \), \( u^\varepsilon + \varepsilon (u^\varepsilon)^2 \).

We now consider the first regularization, and we define a family of Young measures labelled by the spatial and time positions and describing the weak limits of the weak limits associated to \( \{f_\varepsilon (u^\varepsilon)\} \), \( \left\{u^\varepsilon + \varepsilon (u^\varepsilon)^2 \right\} \). We will include also the dependence on the sequence \( \{u^\varepsilon\} \) that will yield a trivial Dirac mass contribution, but we will include it by completeness.

In order to obtain a general result describing the possible limits of nonlinear functionals associated to the sequences \( \{f_\varepsilon (u^\varepsilon) dxdt\} \) we will need detailed information on the behaviour of these sequences near the singular set.
The singular set varies in a continuous manner due to Lemma 18. We consider the set of continuous functions on $S$, $C(S)$. Since $S$ is a closed set in $\Omega \times [0, \infty)$, given $\psi \in C(S)$ it is possible to find a continuous extension of $\psi$ to $\Omega \times [0, \infty)$. We will denote as $\psi \in C(\Omega \times \mathbb{R}^+)\times [0, \infty)$ a generic continuous extension of $\psi$. Our goal is to define a family of measures that will describe the structure of oscillations of the sequence $\psi$. Then:

$$\{\tilde{\psi}_m, \tilde{\varphi}_m\}_{m=1}^N$$

for suitable subsequences. Then:

$$M^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \int [\phi] \leq T ||\phi||_{C([0,B]^{2N} \times [0,T])} \tag{10.6}$$

Therefore, there exist Radon measures $\nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \in M^+ \left([0,B]^{2N} \times [0,T]\right)$ such that:

$$M^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \int [\phi] = \int _{\Omega} \Phi \phi \nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \int [\phi] \leq T ||\phi||_{C([0,B]^{2N} \times [0,T])} \tag{10.6}$$

The weak compactness of the sequence $\{\nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \}_{m=1}^N$ implies the existence of measures $\left\{\nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \right\}_{m=1}^N$ such that:

$$\nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \rightarrow \nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \quad \text{as} \quad \varepsilon \rightarrow 0$$

$$\nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \rightarrow \nu^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \quad \text{as} \quad \sigma \rightarrow 0$$

for suitable subsequences. Then:

$$M^\varepsilon,\sigma_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \int [\phi] \rightarrow \int _{\Omega} \Phi \phi \nu_{\tilde{\psi}_m, \tilde{\varphi}_m}^N \int [\phi] \tag{10.7}$$

We now remark that the measures $\nu_{\tilde{\psi}_m, \tilde{\varphi}_m}^N$ depend only on the values of the functions $\{\tilde{\psi}_m, \tilde{\varphi}_m\}_{m=1}^N$ at the singular set. Indeed, for any extension of the functions $\{\psi_m, \varphi_m\}_{m=1}^N$ to $S + B_\sigma (0)$ we can choose $\sigma$ small such that the functions $\{\tilde{\psi}_m, \tilde{\varphi}_m\}_{m=1}^N$ differ from the values of $\{\psi_m, \varphi_m\}_{m=1}^N$ in the closest point in $S$ by an arbitrarily small amount. On the other hand we have the estimates:

$$\int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\psi}_m f \sigma (u^\sigma) dx \leq C \int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\varphi}_m u^\sigma dx$$

$$\int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\psi}_m u^\sigma dx \leq C \int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\varphi}_m u^\sigma dx$$

$$\int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\psi}_m f \sigma (u^\sigma) dx \leq C \int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\varphi}_m u^\sigma dx$$

$$\int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\psi}_m u^\sigma dx \leq C \int _{\Omega \setminus \{S + B_{\sigma_0} (0)\}} \tilde{\varphi}_m u^\sigma dx$$
and the right hand side of these formulas can be made arbitrarily small, uniformly on \( \varepsilon \), if \( \sigma \) is small for a.e. \( t \). It then follows that taking the limit in the order indicated above we obtain:

\[
\nu_{\{\psi_m, \varphi_m\}_{m=1}^N}^{N} \tag{10.8}
\]

The measures \( \{\nu_{\{\psi_m, \varphi_m\}_{m=1}^N}\} \) that can be extended to describe the possible oscillations of the sequences \( \{f_\varepsilon(u^\varepsilon)\}, \{u^\varepsilon\} \) near the singular set. Since nontrivial oscillations can take place only at the singular set, they are sufficient to describe the measures \( \{\lambda_{\{\varphi_k,j\},T}\} \) described above. Moreover, since the functions \( \int_\Omega \varphi_m u^\varepsilon dx \) change continuously in the macroscopic scale, they do not contribute to the oscillations. Therefore, there exist measures \( \nu_{\{\psi_m\}_{m=1}^N}^{N} \in (C(R^N_+ \times R_+))^N \) such that:

\[
d\nu_{\{\psi_m, \varphi_m\}_{m=1}^N} (\xi_1, ..., \xi_N, \eta_1, ..., \eta_N, t) = d\nu_{\{\psi_m\}_{m=1}^N} (\xi_1, ..., \xi_N, t) \prod_{m=1}^N \delta_{f_\varepsilon} \varphi_m d\mu_t (\eta_m) \tag{10.9}
\]

Therefore, the only non trivial measures that need to be computed are \( \nu_{\{\psi_m\}_{m=1}^N}^{N} \). Intuitively these measures describe the statistical distribution of the oscillations as well as their correlations at different points of the singular set.

Notice that, as mentioned above, it is possible to compute the measures \( \{\lambda_{\{\varphi_k,j\},T}\} \) in terms of the simplest family of measures \( \nu_{\{\psi_m\}_{m=1}^N}^{N} \). More precisely, suppose that we define a family of measures \( \{\lambda_{\{\varphi_k,j\},T}\} \) using as measures \( \{\mu_m\} \) the sequences \( \{f_\varepsilon(u^\varepsilon)\}, \{u^\varepsilon\} \). Suppose that we define measures \( \{\lambda_{\{\varphi_k,j\},T}\} \) by means of:

\[
L_{\{\varphi_{k,j}\},T}[\Phi] = \int_0^T \Phi \left( \int \varphi_{1,1} f_\varepsilon (u^\varepsilon) dx, ..., \int \varphi_{1,M_1} f_\varepsilon (u^\varepsilon) dx, \int \varphi_{2,1} u^\varepsilon dx, ..., \int \varphi_{2,M_2} u^\varepsilon dx, t \right) dt
\]

and limits:

\[
L_{\{\varphi_{k,j}\},T}[\Phi] \to L_{\{\varphi_{k,j}\},T}[\Phi]
\]

as \( \varepsilon \to 0 \).

Given a function \( \Phi \in C^{M_1+M_2+1}(R^{M_1} \times R^{M_2} \times R_+) \) and real numbers \( \{\alpha_{1,j}\}_{j=1}^{M_1} \) we define:

\[
T_{\{\alpha_{1,j}\}_{j=1}^{M_1}}(\Phi) (\sigma_{1,1}, ..., \sigma_{1,M_1}, \sigma_{2,1}, ..., \sigma_{2,M_2}, t) = \Phi (\sigma_{1,1} + \alpha_{1,1}, ..., \sigma_{1,M_1} + \alpha_{1,M_1}, \sigma_{2,1}, ..., \sigma_{2,M_2}, t)
\]

Using the convergence properties of the sequences \( \{u^\varepsilon\}, \{f_\varepsilon(u^\varepsilon)\} \) outside the singular set, as well as the definition of the measures \( \nu_{\{\psi_m\}_{m=1}^N}^{N} \) we obtain the following representation formula:

\[
\int_0^T \Phi (\sigma_{1,1}, ..., \sigma_{1,M_1}, \sigma_{2,1}, ..., \sigma_{2,M_2}, t) d\lambda_{\{\varphi_{k,j}\},T} (\sigma_{1,1}, ..., \sigma_{1,M_1}, \sigma_{2,1}, ..., \sigma_{2,M_2}, t)
\]

\[
= \int_0^T T_{\{f_\varepsilon, \varphi_1,j u dx\}_{j=1}^{M_1}} (\Phi) \cdot \left( \sigma_{1,1}, ..., \sigma_{1,M_1}, \int \varphi_{2,1} d\mu_t, ..., \int \varphi_{2,M_2} d\mu_t, t \right) d\nu_{\{\psi_m\}_{m=1}^N}^{N} (\sigma_{1,1}, ..., \sigma_{1,M_1}, t)
\]

This formula provides the desired representation formula for nonlinear functions of limits of sequences \( \{f_\varepsilon(u^\varepsilon)\}, \{u^\varepsilon + \varepsilon (u^\varepsilon)^p\}, \{u^\varepsilon\} \) in terms of the measures \( \nu_{\{\psi_m\}_{m=1}^N}^{N} \).
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