Steering heat engines: a truly quantum Maxwell demon

Konstantin Beyer,∗ Kimmo Luoma, † and Walter T. Strunz ‡
Institut für Theoretische Physik, Technische Universität Dresden, D-01062, Dresden, Germany
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We address the question of verifying the quantumness of thermal machines. A Szilárd engine is truly quantum if its work output cannot be described by a local hidden state (LHS) model, i.e. an objective local statistical ensemble. Quantumness in this scenario is revealed by a steering-type inequality which bounds the classically extractable work. A quantum Maxwell demon can violate that inequality by exploiting quantum correlations between the work medium and the thermal environment. While for a classical Szilárd engine an objective description of the medium always exists, any such description can be ruled out by a steering task in a truly quantum case.

Introduction — Experimental progress has led to unprecedented possibilities of preparation, control, and measurement of small quantum systems, where quantum and thermal fluctuations have to be considered on equal footing. In particular, fundamental concepts of thermodynamics have been revisited from a quantum point of view. This has led to a quantum interpretation of thermal states [1–4], the development of quantum fluctuation theorems [5–15] and the concepts of quantum heat engines [16–24]. One of the key points in these investigations is the question what is fundamentally quantum about these extensions. For instance, whether and how is a quantum heat engine qualitatively and quantitatively different from its classical counterpart? Is quantumness useful in thermodynamics? Influences of quantum features like coherence [25, 26], discord [27] and entanglement [28, 29] on the efficiency of quantum engines have been reported, which show that the answer can be positive under suitable conditions. However, other investigations show that quantumness can even be a hindrance for efficient thermal machines, which can be regarded as classical supremacy in such situations [30–32].

In this Letter we want to address quantumness of thermal machines from a different perspective. We consider a heat engine truly quantum if its work output cannot be explained by a local hidden state (LHS) model, i.e. by a local statistical model. Even though the issue of hidden classicality is fundamental to quantum information, it only rarely appears in the context of quantum thermodynamics [33]. In this Letter we give a verifiable criterion for the quantumness of thermodynamical systems, indicating the lack of a classical statistical description. Most remarkably, the classicality sets an upper bound on the extractable work for certain scenarios.

Quantum Szilárd engine — The prototypical example we want to study is a quantum modification of the Szilárd engine [34, 35]. The classical version consists of a single atom in a box which is in contact with a thermal bath. In equilibrium the atom is in a Gibbs state, a statistical mixture of different phase space points. For work extraction, the demon has knowledge about the microstate of the system.

So far, quantum versions of this heat engine have been investigated using different underlying systems [19, 21, 36–38]. In these examples the demon performs quantum measurements on the work medium, acquiring information about local properties of the heat engine only. Here, we want to exploit the fact that such a local thermal state may arise naturally from a global entangled state of the work medium and its environment, as supported, for instance, by the eigenstate thermalization hypothesis [1–4, 39]. In contrast to previous proposals, the demon obtains her information from measurements on the environment rather than the work medium [40, 41]. A truly quantum Szilárd engine can be revealed by deriving local work extraction bounds which cannot be violated by any local statistical ensemble description, that is a LHS model. These bounds do neither rely on the knowledge about the shared system-environment state, nor on any assumptions about the properties of the environment (semi-device-independent).

Work medium — Let us assume that the work medium is a finite quantum system $S$ with Hamiltonian $H_S$. Its Gibbs state reads

$$\rho^{\text{Gibbs}}_S = \sum_i \frac{e^{-\beta E_i}}{Z} |i\rangle\langle i| = \sum_i p_i |i\rangle\langle i|,$$

where $\beta = \frac{1}{k_B T}$, $E_i$ is the energy of the $i$th energy eigenstate $|i\rangle$ and $Z = \sum_i e^{-\beta E_i}$. Locally, the Gibbs state can be seen as a statistical mixture of the energy eigenstates but it can equally be decomposed into infinitely many other ensembles $D = \{p_k; \rho_k\}$ with $\sum_k p_k \rho_k = \rho^{\text{Gibbs}}_S$, $p_k \geq 0$ and $\sum_k p_k = 1$. Any such decomposition can be given by an extension to a bipartite state $\rho_{SE}$ of system and environment, with $\rho^{\text{Gibbs}}_S = \text{Tr}_E\{\rho_{SE}\}$, and a local POVM $\{M_k\}$ on $E$, such that $p_k = \text{Tr}\{\rho_{SE}(1 \otimes M_k)\rho_{SE}\}$ and $\rho_k = \text{Tr}_E\{1 \otimes M_k\rho_{SE}\}$.

To investigate the difference of a classical and a quantum Szilárd engine we introduce Alice and Bob. Alice is a demon who can prepare many copies of the global state $\rho_{SE}$. While she can perform measurements on $E$, she does not act directly on $S$ since this would in general disturb the local thermodynamical situation [42]. Bob has access only to the system $S$ and would like to extract work from its Gibbs state. As shown in [40] any non-
product joint state $\rho_{SE} \neq \rho_S \otimes \rho_E$ allows Bob to extract work from $S$ if Alice performs suitable measurements on $E$ and communicates classically with him.

**Work extraction scenarios** Bob wants to extract work from a certain decomposition $D = \{p_k; \rho_k\}$ of his local Gibbs state. We can describe this work extraction as a transfer of energy due to a suitable coupling between the work medium $S$ and a work storage system $W$ with Hamiltonians $H_S$ and $H_W$, respectively [5, 7, 43, 44]. The total Hamiltonian is given by $H = H_S \otimes 1 + 1 \otimes H_W$. Following the reasoning of [7] the coupling of $S$ and $W$ has to be a unitary transformation that conserves the total energy independently of the initial state of the work storage $\rho_W$. Further constraints ensuring that only work and no heat is transferred are presented in the Supplemental Material [45]. For each state $p_k$ in $D$ Bob can find a suitable unitary $U_k$ which transfers a non-negative amount of energy from $S$ to $W$. The average work extraction associated with this process is given by

$$\Delta W_k = \text{Tr}\{1 \otimes H_W(U_k \rho_k \otimes \rho_W U_k^\dagger - \rho_k \otimes \rho_W)\},$$

where $\rho_W$ is the initial state of the work storage. Due to energy conservation, the change of the inner energy in $S$ is $\Delta E_k = -\Delta W_k$ and $\Delta W_k \leq \text{Tr}\{H_S \rho_k\}$. The average work Bob can extract from $D$ by using the set of unitaries $U = \{U_k\}$ is then given by $\bar{W} = \sum_k p_k \Delta W_k$.

Special cases are pure state decompositions $D^{\text{pur}} = \{p_k; |\phi_k\rangle \langle \phi_k|\}$. If Bob knows the pure state $|\phi_k\rangle$ of his system, the largest possible amount of work can only be extracted if he performs a suitable local unitary operation $U_k$ on $S$ which acts as $U_k |\phi_k\rangle = |0\rangle$, where $|0\rangle$ is the ground state of the local Hamiltonian $H_S$ whose energy we set to $E_0 = 0$ [46]. As shown in [7], such a local unitary $U_k$ can indeed always be implemented by an energy conserving global coupling $U_k$ between $S$ and $W$ but requires the work storage to be initialized in a pure state $\rho_W$ which is a coherent superposition of energy eigenstates states of $H_W$. Thus, energy measurements on $W$ will in general yield probabilistic outcomes [47]. However, Bob is only interested in the average work output under the given $U_k$, which is, due to the energy conservation, given by the negative energy change in $S$, that is $\Delta W_k = -\Delta E_k = \langle \phi_k | H_S | \phi_k \rangle - \langle 0 | H_S | 0 \rangle = \langle \phi_k | H_S | \phi_k \rangle$. Accordingly, if Alice can provide the pure state decomposition $D^{\text{pur}}$, Bob can extract on average $\bar{W}^{\text{pur}} = \sum_k p_k \langle \phi_k | H_S | \phi_k \rangle = \langle H_S \rangle_{\rho_S^{\text{pur}}}$. This is, not surprisingly, the inner energy of the work medium. Thus, equivalently to the classical case, full knowledge about the state of the system allows for maximal work extraction. If Alice cannot announce correct pure states to Bob, the output will be $\bar{W} \leq \bar{W}^{\text{pur}}$. In fundamental contrast to the classical Szilárd scenario, a Gibbs state of a quantum system allows for infinitely many different ensembles of pure states.

Truly quantum features can be revealed when Bob wants to extract work from different decompositions $D_n$. For each decomposition he has a suitable set of unitaries $U_n = \{U_k^n\}$ as described above. He chooses randomly with probabilities $c_n$ one of the sets and asks Alice which unitary out of the particular set $U_n$ he should perform to extract the maximal amount of work. Depending on how well Alice can produce the desired decompositions, Bob will extract on average $\bar{W} \leq \sum_n c_n \bar{W}_n$.

The question now arises, under which conditions Bob can be sure that his Szilárd engine is truly quantum. He has no access to the global state $\rho_{SE}$ and, therefore, cannot check whether the state is quantum correlated. The only information he gets from Alice is which unitary $U_k^n$ he should use if he asks her for the decomposition $D_n$. Accordingly, Bob has to certify quantumness without any assumptions about the properties (for example the Hilbert space) of the environment $E$. Such a device-independent verification task is called quantum steering [48, 49]. Successful steering has important implications on the objectivity [50] of the local state in the system. In a classical scenario the system state is always objective, though unknown to Bob as long as the demon does not share her knowledge with him. In the quantum case, in general, it makes no sense to assign objective system states at all, as long as no observation of the environment is made. Particularly, a thermalized quantum system is not in one of its energy eigenstates and does not fluctuate between them while time is evolving, if these fluctuations are not given relative to measured states of the bath [51]. For a closer look on how steering can rule out objective quantum dynamics see [50, 52, 53]. In our Szilárd scenario we can use these ideas as follows: If a local objective statistical description of Bob’s system $S$ holds, it can be represented by a local hidden state (LHS) model $\mathcal{F} = \{p_k; \rho_k\}$ [49]. The hidden states $\rho_k$ are distributed randomly according to their probabilities $p_k$. Locally the Gibbs state in Bob’s system has to be recovered

$$\rho_S^{\text{Gibbs}} = \sum_\zeta p_\zeta \rho_\zeta.$$

Thus, among all the copies of his local state, a fraction $p_\zeta$ will be in state $\rho_\zeta$. Bob does not know which state he has for a particular copy but he can assume that, if the LHS model holds, the best knowledge Alice can possibly have about his system is the particular hidden state for each of his copies. Therefore, any decomposition Alice can provide has to be either the LHS ensemble $\mathcal{F}$ itself or a coarse graining of the latter [49]. However, this does not mean that the real state $\rho_{SE}$ shared by Alice and Bob has to be separable. It only means that Bob could explain his statistics also by a state without quantum correlations. A truly quantum Szilárd engine can therefore be defined by the condition $\bar{W} > \bar{W}_cl$, that is, Bob’s average work output is larger than what could be obtained from a state which can be described by a LHS ensemble $\mathcal{F}$. Clearly, the work output of a single
decomposition $D$ can always be explained by a classically correlated state because we can always identify $D = F$. Bob needs at least two different sets of unitaries $U_n$.

We should note that the observables on Bob’s side needed to perform a steering task are represented by the work extraction. In order to determine the average energy transferred to the work storage he has to measure $W$ in its energy basis. According to Naimark’s dilation theorem, this measurement, together with a unitary $U_x$, defines a POVM on $S$. The set of POVMs which can be implemented by the described work extraction scenario is strictly smaller than the set of all local POVMs on $S$. For example, the only implementable projective measurement is the one diagonal in the energy eigenbasis of $H_S$. It is an open question whether the work extraction POVMs can demonstrate steering for any steerable state $\rho_{SE}$ which respects the local Gibbs state.

Whether the work output on Bob’s side can also be provided by a classical demon is in general not trivial to answer. As in a standard steering scenario a suitable inequality has to be derived which depends on the properties of the work medium $S$ and the work extracting unitaries $\{U_x\}$. It is crucial for quantum steering that the inequality does not depend on the part $E$ which is inaccessible for Bob.

**Qubit work medium** — To illustrate the concept we consider a qubit work medium $S$ with local Hamiltonian $H_S = |1\rangle\langle 1|$. Its thermalized Gibbs state is given by

$$\rho_{\text{Gibbs}}^S = \frac{1 + \eta}{2\beta} |1\rangle\langle 1| + \frac{1 - \eta}{2\beta} |0\rangle\langle 0|, \quad (4)$$

with $\eta = \frac{e^{-\beta} - 1}{e^{-\beta} + 1} < 0$ and $\beta = \frac{1}{k_B T}$. As stated above, Bob needs at least two different sets of work extracting unitaries to verify a quantum Szilard engine. Let us assume that he would like to extract work from two dichotomic pure state decompositions $D_1$ and $D_2$. The first one is a decomposition into energy eigenstates $\{|0\rangle, |1\rangle\}$, the second one is given by the two Bloch vectors $\vec{r}_\pm = \left(\frac{\pm \sqrt{1 - \eta^2}}{1 - \eta^2}, 0, \eta\right)$. The local unitaries for $D_1$ are $U_{\uparrow}^1 = \sigma_x$ for the state $|1\rangle$ and $U_{\downarrow}^1 = 1$ for the state $|0\rangle$. For $D_2$ the suitable unitaries $U_{\pm}^2$ are rotations around the $y$-axis about an angle $\alpha = \pm \arctan\left(\frac{\eta}{\sqrt{1 - \eta^2}}\right)$ (see Fig. 1).

Accordingly, Bob needs two different kinds of work extraction devices. We represent them by red and blue cells which both have two buttons to trigger the different work extraction units and measure the work storage $W$ in the energy basis (Fig. 1). In each cell Bob can place one qubit. The red cells can perform $U_{\uparrow}^1$ and $U_{\downarrow}^1$, the blue cells apply either $U_{+}^2$ or $U_{-}^2$. In the Supplemental Material [45] we construct an explicit model, how the energy conserving unitaries can be realized by using only two qubit interactions.

Let us first assume that Alice prepares the global state $\rho_{SE}^{D_1} = \frac{1 + \eta}{2\beta} |1\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{1 - \eta}{2\beta} |0\rangle\langle 0| \otimes |0\rangle\langle 0|$, compatible with the local Gibbs state. Bob places his qubit into a red cell and asks Alice which button he should press. Alice measures $E$ in the $\sigma_z$-basis and tells Bob to press the button 1 if the outcome is 1 and button 0 if the outcome is 0. The cell will apply either $U_{\uparrow}^1$ or $U_{\downarrow}^1$. On average — Bob has many red cells which he wants to charge — he will extract $W_x^1 = \frac{1 + \eta}{2\beta}$ because Alice tells him to press button 1 with probability $p_1^1 = \frac{1 + \eta}{2}$.

If Bob wants to charge his blue cells, Alice could help him by preparing the state $\rho_{SE}^{D_2} = \rho_+ \otimes |1\rangle\langle 1| + \rho_- \otimes |0\rangle\langle 0|$, where $\rho_\pm$ are the density matrices corresponding to the Bloch vectors $\vec{r}_\pm$. Locally, the Gibbs state is again recovered. Depending on her outcome, Alice tells Bob to press either the button which applies $U_{\uparrow}^2$ or $U_{\downarrow}^2$ (see Fig. 1). On average Bob can again extract $W_x^2 = \frac{1 + \eta}{2\beta} = W_x^1$. Thus, the decomposition $D_1$ into energy eigenstates is by no means better for the work extraction than decomposition $D_2$. Both $\rho_{SE}^{D_1}$ and $\rho_{SE}^{D_2}$ are separable and describe situations where Alice exploits only classical correlations. We call such a demon a classical one because the same result can be obtained from a local statistical model for $\rho_{SE}$ without any reference to a global quantum state $\rho_{SE}$.

**When is the demon really quantum?** — In the remainder we will consider the case where Bob would like to charge both the red and the blue cells. He has $N$ blue cells, $M$ red cells and $N + M$ thermal qubits. The ratio between red and blue cells is given by $c = \frac{N}{M}$. First, he distributes the qubits over the cells which fixes the decomposition he needs to extract maximal work with any given cell. Subsequently, he announces the color of each single cell to Alice and asks her which button he should press. In the end he can read off the extracted work from each work meter and average over them to see how efficient the procedure has been. If Alice’s knowledge for

![FIG. 1. Work extraction. Bob has blue and red work extraction cells with locally thermal qubits. The red cells can extract work from each work meter and average over them to see how efficient the extraction has been. ](image-url)
each single qubit is described by $\rho_{D1}^SE$ or $\rho_{D2}^SE$, his average work output in the limit $N \to \infty$ can never reach the optimal $W_{opt} = \frac{1+\eta}{2}$ (we assume that $c > 0$ is kept constant). The blue and the red cells are not compatible with the same statistical mixture of states. On the other hand the entangled state of the form

$$|\Psi\rangle_{SE} = \sqrt{\frac{1+\eta}{2}} |1\rangle_S \otimes |1\rangle_E + \sqrt{\frac{1-\eta}{2}} |0\rangle_S \otimes |0\rangle_E.$$  

would do the job. Alice could measure either $\sigma_z$ or $\sigma_x$ depending on the color of the cell for which Bob would like to know which button he should press. If Alice is indeed a demon who can prepare Bob’s thermal state to be the partial trace of a pure entangled state, he can extract the optimal average $W_{opt}$.

We will now calculate which average work Bob can maximally obtain if a LHS model holds. The best Alice can do is if she knows the state $\rho_x$ for each cell is to tell Bob which button he should press in order to obtain the maximal work output. Accordingly, for the red cells Alice would tell him to press the button triggering $U^*_x$ whenever $z_x = \text{Tr}[\sigma_x \rho_x] > 0$ for the state in this cell. The average work output for the red cells will then be $W_x = \frac{1}{2} \sum_z \rho_x (|z_x\rangle \langle z_x|)$. The work output for the blue cells Alice announces for $U^*_z$ if $x_z = \text{Tr}[\sigma_x \rho_x] > 0$ and the button for $U^*_z$ if $x_z \leq 0$. On average the blue cells will then reach $W_x = \frac{1}{2} (\eta + \eta^2 + \sum_z \rho_z (x_z) \sqrt{1-\eta^2}) [45]$. The work average over all cells which can be expected for a LHS model is, thus, given by $W_{cl} = \frac{W_x + W_z}{1+c}$. For the choice $c = \frac{1}{\sqrt{1-\eta^2}}$ we can bound the average work for any LHS model by [45]

$$W_{cl} \leq \frac{\eta \left( \sqrt{1 - \eta^2} + \eta + 1 \right) + \sqrt{2 - 2\eta^2}}{2 \left( \sqrt{1 - \eta^2} + 1 \right)}.$$  

If Bob extracts an average work beyond the classical limit $W_{cl}$ he can be satisfied that Alice is indeed a quantum demon who exploits non-classical correlations. Any local statistical model has to be discarded in this case.

If Alice can, for example, instead prepare the entangled state (5), the work value which can be reached is $W_{quan} = W_{opt} = \frac{1+\eta}{2}$. Thus, Alice can violate the inequality for

$$\eta > -\sqrt{2(\sqrt{2} - 1)} \approx -0.91,$$  

that is, for temperatures $k_B T > 0.33$. For the case of infinite temperature ($\eta \to 0$) the steering inequality simplifies to $W_{cl} \leq \frac{1}{2\sqrt{q^2}}$, while $W_{opt} = \frac{1}{2}$.

For an intermediate regime we can consider the following mixture

$$\rho_{SE} = q \rho_{quan} + (1-q) \rho_{cl},$$  

where $\rho_{quan} = |\Psi\rangle_S |\Psi\rangle_{SE}$ as in Eq. (5) and $\rho_{cl}$ is the classically correlated state $\rho_{cl} = \frac{1+\eta}{2} |1\rangle_S \otimes |1\rangle_E + \frac{1-\eta}{2} |0\rangle_S \otimes |0\rangle_E$. The parameter $q$ tunes between the fully quantum case ($q = 1$) and the scenario which represents a classical Szilárd demon ($q = 0$). The extractable work in the blue cells is now $W_x = \frac{1}{2} (q + \eta + \eta^2 - q\eta^2)$ [45]. Fig. 2 shows the relation between the non-classicality of the demon and the two parameters $\eta$ and $q$. For parameters above the red line Alice can demonstrate that she is a quantum Szilárd demon.

We should note that the steering inequality (6) is not ideal for the detection of non-classical correlations in the state $|\Psi\rangle_{SE}$. It is well known that any pure entangled state is steerable [48], thus, $|\Psi\rangle_{SE}$ is steerable for any $-1 < \eta < 1$. More generally, any non-zero temperature Gibbs state is mixed and therefore has an entangled and steerable purification. However, we have restricted Bob’s observables to a special class of operations, namely the work extraction. It is therefore not surprising that the given inequality cannot detect every steerable state. We could improve the bound by adding additional work extraction options on Bob’s side, but this does not add anything conceptually new to the framework. Furthermore, motivated by the concept of a Szilárd engine, the inequality is based on the assumption that Bob’s reduced state is indeed a Gibbs state. This property can of course be locally verified by Bob. It has to be emphasized that the construction of the steering inequality only depends on the device-dependent part of the steering task, such as the Hilbert space of Bob’s system $S$ and the work extracting operations he uses. There are no assumptions made about the structure of the environment or the operations Alice performs.

Conclusions In this Letter we have shown how the concept of quantum steering can be applied to quantum thermodynamics in order to verify quantumness. The violation of a steering inequality is connected to the macroscopic average work. The use of a quantum steering task for the verification of quantumness is motivated by the asymmetric setting in quantum heat engines. The work system under control is taken to be the device-dependent
part in the scenario, whereas the environment is treated device-independently.

Our concept is of particular interest for the investigation of bath-induced fluctuations in quantum thermodynamics. A violation of the steering inequality rules out any possible objective (though statistical) description of fluctuations in the system. Notably, the assumption that a system fluctuates between its energy eigenstates is not valid if genuine quantum correlations are taken into account. Statements about the fluctuations in the system can only be made with respect to the observed fluctuations of the environment which will depend on how the environment is measured.

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* konstantin.beyer@tu-dresden.de
† kimmo.luoma@tu-dresden.de
‡ walter.strunz@tu-dresden.de

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Of course, objective states can be assigned by direct measurements on the system, but this case is not considered in our setting. Furthermore, we should note that ensemble representations of mixed states can be computationally useful in order to calculate averaged quantities more efficiently.

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WORK EXTRACTION THROUGH COUPLING TO A WORK STORAGE

The work extraction framework we use has been studied thoroughly in [1]. We will summarize here some details of the underlying concept which are necessary for our scenario.

The system $S$ and a work storage $W$ with local Hamiltonians $H_S$ and $H_W$ are coupled with the aim to transfer work between the two. The work storage is in general assumed to be a continuous degree of freedom with $H_W = \int x|\psi(x)\rangle\langle x|\,dx$, with an orthonormal basis $\{|x\rangle, \forall x \in \mathbb{R}\}$.

The coupling is established by a joint unitary operation $U_{SW}$. In order to avoid unaccounted energy contributions due to the coupling, the unitary has to be energy conserving, meaning it commutes with the global Hamiltonian

$$[U_{SW}, H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_W] = 0. \quad (1)$$

Since the Hamiltonians are purely local the change of energy in $W$ due to the coupling $U_{SW}$ has to be exactly compensated by the change of energy in $S$

$$\Delta W_W = -\Delta E_S. \quad (2)$$

Furthermore, the whole process should be invariant under translations of the work storage system, that is $[U_{SW}, \Delta W] = 0$, where $\Delta W$ is defined by $[H_W, \Delta W] = i$. This condition is necessary in order to avoid that the work storage can be used to lower the entropy of the system which would allow for an energy transfer in form of heat instead of work [1, 2].

As shown in the Appendix of [1] it is always possible to implement a global unitary $U_{SW}$ which satisfies the conditions above and which acts locally on $S$ as a unitary $U_S$:

$$\text{Tr}_W\{U_{SW}(\rho_S \otimes \rho_W)U_{SW}^\dagger\} = U_S \rho_S U_S^\dagger. \quad (3)$$

Bob achieves the largest work extraction for a given pure state $|\psi_S\rangle$ if he transforms it to the ground state $|0\rangle_S$ of his local Hamiltonian (he could not lower the energy of his system any further). If he implements the unitary which maps $U_S|\psi_S\rangle = |0\rangle_S$ in the manner described above, he can be sure that the energy growth in the work storage is maximized and given by

$$\Delta W_W = \langle \psi_S | H_S | \psi_S \rangle - \langle 0 | H_S | 0 \rangle.$$

It might be counter intuitive that the global unitary reduces to a local one although $U_{SW} \neq U_S \otimes U_W$ (if $U_{SW} = U_S \otimes U_W$ the unitary could in general not be energy conserving for every input state). As shown in [1] this behavior can in general only be obtained if the work storage $W$ is initialized in an eigenstate of the translation operator $\Delta W$. Thus, the state $\rho_W = |\psi_W\rangle\langle \psi_W|$ is pure, and $\langle x|\psi_W \rangle \neq 0, \forall x$.

Even though this general approach is mathematically elegant, it is questionable how physical such an initial state of the work storage is. In particular the measurement of the energy change in the work storage (that’s the quantity Bob is interested in) could be hard to implement because the measured energies in the work storage must fluctuate strongly as shown in [3] and therefore the averaging would require a large amount of measured data. We therefore construct for the qubit example an explicit collision model which is able to approximate the work extraction to an arbitrary precision by discrete two-qubit interactions. Such an approach could in principle be implemented as a quantum circuit with current quantum technologies.

COLLISION MODEL APPROACH TO WORK EXTRACTION

The work extraction is implemented by performing a local unitary $U_S$ on the system. The change of the inner energy is then equal to the work performed (or absorbed) by the system. However, this energy change has to be compensated by an energy supply or storage. Instead of a single work storage quantum system with a single continuous degree of freedom, as shown in the section above, we formulate the work extraction using a qubit collision model approach which reproduces the local unitary on the system and preserves the energy on average [4–6]. Thus, the mean value of extracted work is equal to the change of the inner energy in the work medium of the engine as required. Our approach can be approximated to arbitrary precision by a finite number of two-qubit interactions.

The local unitaries $U_S$ implemented in the work storage cells of the main text are rotations around the $y$-axis:

$$U_S = e^{-i\frac{\phi}{2}\sigma_y}. \quad (4)$$

In a first step we construct this unitary from a collision model. The system interacts with a sequence of ancilla qubits. The interaction between the system and a single
ancilla is given by
\[ U_{S,A} = e^{-i\Delta \phi (\sigma_y \otimes \sigma_x + \sigma_x \otimes \sigma_y)}. \] (5)
Each ancilla is initialized in the \( y \)-eigenstate \(|+y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\). The map for the state of the system \( S \) after a single collision can be written as:
\[ \rho_S' = \text{Tr}_A\{U_{S,A}(\rho_S \otimes \rho_A)U_{S,A}^\dagger\} \]
(6)
where
\[ \rho_S' = \rho_S - \frac{i\Delta \phi}{2}[\sigma_y, \rho_S]. \] (7)
We assume each collision to be weak and expand the interaction to first order in \( \Delta \phi \), hence we get
\[ \rho_S' = \rho_S - \frac{i}{2} \Delta \phi \sigma_y. \]
In the limit of infinitesimal steps \( \Delta \phi \to d\phi \) we therefore obtain the desired local unitary evolution \( U_S \).

**The ancillas as a stochastic work storage** — The state of a single ancilla after its collision with the system reads to first order in \( \Delta \phi \):
\[ \rho_A' = \text{Tr}_S\{U_{S,A}(\rho_S \otimes \rho_A)U_{S,A}^\dagger\} \]
(8)
where \( a \) is the \( x \)-component of the state \( \rho_S \). Thus, the \( z \)-component of the ancilla after the interaction contains information about the \( x \)-component of the system before the collision.

The Hamiltonian of each ancilla is given by \( H_A = \sigma_z \). Accordingly, the change of the average energy of a single ancilla due to its collision with the work medium is given by
\[ \Delta E = \text{Tr}\{H_A(\rho_A' - \rho_A)\} = \Delta \phi a. \] (9)
As we have derived above, the system state evolves under the unitary \( U_S \) during the collisions. Thus, in a continuous limit (\( \Delta \phi \to d\phi \)) the component \( a \) becomes a function of \( \phi \). 

**WORK EXTRACTION BOUND FOR A LHS ENSEMBLE - INFINITE TEMPERATURE**

Before we derive the steering inequality for a finite temperature we consider the infinite temperature case. In this limit (\( \eta \to 0 \)) the local Gibbs state is given by the fully mixed state \( \rho_S^{\text{Gibbs}} = \frac{1}{2} I \). We will calculate the average work which Bob can expect to extract with his red cells from a LHS ensemble \( F = \{\rho_{\xi}; p_{\xi}\} \), where \( p_{\xi} \geq 0 \) and \( \sum_{\xi} p_{\xi} = 1 \). Bob assumes that Alice might know the current state \( \rho_{\xi} \) of his system because of classical correlations with the environment. If Alice indeed knows the current \( \rho_{\xi} \), the best she can do is to announce \( U_{\xi} \) whenever \( z_{\xi} = \text{Tr}\{\sigma_z \rho_{\xi}\} > 0 \). These are the states which give a positive work output. For \( z_{\xi} = \text{Tr}\{\sigma_z \rho_{\xi}\} \leq 0 \) Alice announces \( U_{\xi} = I \) and Bob will leave his system unchanged.

For suitably many runs the work output averaged over the ensemble \( \{\rho_{\xi}; p_{\xi}\} \) reads
\[ \overline{W} = \sum_{\xi, z_{\xi} > 0} p_{\xi}\left(\text{Tr}(H_S \rho_{\xi}) - \text{Tr}(H_S U_{\xi} \rho_{\xi} U_{\xi}^\dagger)\right) \]
(10)
where the fourth line comes from
\[ \sum_{\xi} p_{\xi} z_{\xi} = \sum_{\xi, z_{\xi} > 0} p_{\xi} z_{\xi} + \sum_{\xi, z_{\xi} \leq 0} p_{\xi}(-z_{\xi}) = 0 = \text{Tr}\{\sigma_z \rho_S\}. \] (11)
We have to note that Bob does not know the ensemble \( \{\rho_{\xi}; p_{\xi}\} \).

Bob assumes that Alice’s knowledge about her current state does not depend on whether he has placed the qubit in a red or a blue cell. Therefore, he takes the ensemble \( \{\rho_{\xi}; p_{\xi}\} \) to be the same also for the work extraction in the blue cells. Of course, Alice again would announce the unitary which leads to larger work extraction. Thus, she tells him to perform \( U_{\xi} \) whenever \( z_{\xi} = \text{Tr}\{\sigma_z \rho_{\xi}\} > 0 \) and \( U_{\xi} \) otherwise. For the work extraction in the blue
cells we then find:

\[
W_x = \sum_{\xi, x_0 > 0} p_{\xi} \left( \text{Tr} \{ H_S \rho_\xi \} - \text{Tr} \{ H_S \rho_\xi \rho_\xi^{+1} \} \right) \\
+ \sum_{\xi, x_0 \leq 0} p_{\xi} \left( \text{Tr} \{ H_S \rho_\xi \} - \text{Tr} \{ H_S \rho_\xi \rho_\xi^{+1} \} \right) \\
= \sum_{\xi} p_{\xi} \eta \left( |x_0| - x_0 \right) \\
= \sum_{\xi} p_{\xi} \frac{|x_0|}{2},
\]

(14)

where the last equality comes from \( \sum_{\xi} p_{\xi} z_\xi = 0 \).

**FINITE TEMPERATURE CASE**

In order to respect the \( \langle \sigma_z \rangle \) expectation value of the Gibbs state we have:

\[
\eta = \text{Tr} \{ \sigma_z \rho_S^{\text{Gibbs}} \} = \sum_{\xi, z_\xi > 0} p_{\xi} z_\xi \\
= \sum_{\xi, z_\xi > 0} p_{\xi} z_\xi + \sum_{\xi, z_\xi \leq 0} p_{\xi} z_\xi.
\]

(15)

Thus, for the \( z_\xi > 0 \) we get:

\[
\frac{1}{2} \sum_{\xi, z_\xi > 0} p_{\xi} z_\xi = \frac{1}{2} \eta - \frac{1}{2} \sum_{\xi, z_\xi \leq 0} p_{\xi} z_\xi \\
= \frac{1}{2} \eta + \frac{1}{2} \sum_{\xi, z_\xi \leq 0} p_{\xi} |z_\xi|.
\]

(16)

(17)

and we find for the red cells:

\[
W_x = \sum_{\xi, z_\xi \leq 0} p_{\xi} z_\xi \\
= \frac{1}{2} \sum_{\xi, z_\xi > 0} p_{\xi} z_\xi + \frac{1}{2} \sum_{\xi, z_\xi \leq 0} p_{\xi} |z_\xi| + \frac{\eta}{2} \\
= \frac{1}{2} \sum_{\xi} p_{\xi} (|z_\xi| + \eta) = \frac{1}{2} \left( \eta + \sum_{\xi} p_{\xi} |z_\xi| \right).
\]

(18)

The unitaries for the blue cells are given by:

\[
U_\xi^+ = \left( \begin{array}{cc}
\frac{-\sqrt{1-\eta}}{\sqrt{\eta + 1}} & \frac{\sqrt{\eta + 1}}{\sqrt{1-\eta}} \\
\frac{\sqrt{\eta + 1}}{\sqrt{1-\eta}} & \frac{-\sqrt{1-\eta}}{\sqrt{\eta + 1}}
\end{array} \right), \quad U_\xi^- = \left( \begin{array}{cc}
\frac{-\sqrt{\eta + 1}}{\sqrt{1-\eta}} & \frac{\sqrt{1-\eta}}{\sqrt{\eta + 1}} \\
\frac{\sqrt{1-\eta}}{\sqrt{\eta + 1}} & \frac{-\sqrt{\eta + 1}}{\sqrt{1-\eta}}
\end{array} \right).
\]

(19)

Then, the expected work output is

\[
W_x = \sum_{\xi, x_0 > 0} p_{\xi} \left( \text{Tr} \{ H_S \rho_\xi \} - \text{Tr} \{ H_S \rho_\xi \rho_\xi^{+1} \} \right) \\
+ \sum_{\xi, x_0 \leq 0} p_{\xi} \left( \text{Tr} \{ H_S \rho_\xi \} - \text{Tr} \{ H_S \rho_\xi \rho_\xi^{+1} \} \right) \\
= \frac{1}{2} \sum_{\xi} p_{\xi} (z_\xi (1 + \eta) + |x_0| \sqrt{1 - \eta^2}) \\
= \frac{1}{2} \left( \eta + \eta^2 + \sum_{\xi} p_{\xi} |x_0| \sqrt{1 - \eta^2} \right).
\]

(20)

To obtain the steering inequality in the finite temperature case we calculate the expected classical work for the ratio \( c = \frac{1}{\sqrt{1-\eta^2}} \) of red and blue cells:

\[
W_{cl} = W_x + c W_x \\
= \frac{1}{2} \eta + c(\eta + \eta^2) + \sum_{\xi} p_{\xi} (|z_\xi| + |x_0|) \\
\leq \frac{1}{2} \eta + c(\eta + \eta^2) + \sqrt{2} \\
\leq \frac{1}{2} \eta + \sqrt{2 - 2\eta^2} \\
= \eta \left( \sqrt{1 - \eta^2} + \eta + 1 \right) + \sqrt{2 - 2\eta^2} \\
= \frac{\eta \left( \sqrt{1 - \eta^2} + \eta + 1 \right) + \sqrt{2 - 2\eta^2}}{2 \left( \sqrt{1 - \eta^2} + 1 \right)}.
\]

(21)

**CONNECTION TO ADDITIVE CONVEX CRITERIA**

Our steering inequality is an adaption of the well studied additive convex criteria [7]. Each work extraction unitary defines an observable whose expectation values for a certain state \( \rho_\xi \) are given by:

\[
\langle W_x^- \rangle = z_\xi \\
\langle W_x^0 \rangle = 0 \\
\langle W_x^+ \rangle = \frac{1}{2} (z_\xi (1 + \eta) + x_\xi \sqrt{1 - \eta^2}) \\
\langle W_x^- \rangle = \frac{1}{2} (z_\xi (1 + \eta) - x_\xi \sqrt{1 - \eta^2}).
\]

(22)

In contrast to the standard derivations of convex criteria we want to include the fact that the ensemble of states has to respect the Gibbs state. Accordingly, instead of bounding a convex function of the observables (22) for each single \( \rho_\xi \) we directly look at the average over the whole ensemble under the assumption that Alice wants to maximize the work output (that’s the best she can
do):
\[ W_{\text{cl}} = \frac{1}{1 + c} \left( \sum_{\xi, z \xi > 0} p_{\xi} \langle W_{z}^{1} \rangle + \sum_{\xi, z \xi \leq 0} p_{\xi} \langle W_{z}^{0} \rangle + \frac{c}{2} \left( \sum_{\xi, x \xi > 0} p_{\xi} \langle W_{x}^{+} \rangle + \sum_{\xi, x \xi \leq 0} p_{\xi} \langle W_{x}^{-} \rangle \right) \right) \] (23)

For \( c = \frac{1}{\sqrt{1 - \eta^2}} \) this value \( W_{\text{cl}} \) can be bounded as shown in the previous section.

**EXTRACTABLE WORK UNDER IMPERFECT CORRELATIONS**

We consider correlations between system and environment which are described by the state:
\[ \rho_{SE} = q \rho_{qu} + (1 - q) \rho_{cl}, \] (24)
as described in the main text. For the red cells Alice can help Bob to still extract on average \( W_{z} = \frac{1}{2} q \). For the blue cells we now have \( W_{x} = \frac{1}{2} (q + \eta + \eta^2 - q\eta^2) \). Weighted with \( c \), therefore, the average amount of work will be:
\[ W_{\text{qu}} = \frac{(\eta + 1) \left( \sqrt{1 - \eta^2} + \eta - \eta q + q \right)}{2 \left( \sqrt{1 - \eta^2} + 1 \right)} \] (25)

**QUANTUM SZILÁRD ENGINE SIMULATION**

We will now construct an exemplary model, how the quantum correlations Alice exploits can emerge from a thermalization process. A suitable framework known from open quantum system theory is based on collision models [4–6]. The dynamics – the relaxation to a thermal state of Bob’s system – is given by a sequence of short interactions between the system and subenvironments. Initially each subenvironment is in the two-qubit state
\[ |\Psi\rangle_{EE} = \sqrt{\frac{1 + \eta}{2}} |11\rangle + \sqrt{\frac{1 - \eta}{2}} |00\rangle. \] (26)
The part \( E \) interacts with the system by a joint unitary \( Q_{SE} = \exp[-i\sqrt{\Delta t} \text{SWAP}] \), where \( \gamma \) is a positive coupling constant, \( \Delta t \) is a short time interval and SWAP is the two-qubit swap-gate. The reduced dynamics for Bob’s system is given by the one-collision map
\[ \rho'_{S} = \text{Tr}_{EE'} \{ Q (\rho_{S} \otimes |\Psi\rangle \langle \Psi|_{EE'}) Q^\dagger \}. \] (27)

A time-continuous limit may be obtained by expanding \( Q \) to first order in \( \Delta t \) and taking the limit \( \Delta t \to dt \) [8]. The steady state in Bob’s system is indeed the Gibbs state (21). Instead of tracing out the environment and discarding the information, Alice couples the part \( E' \) to another ancilla qubit, called the control qubit \( C \). The interaction is taken to be the same as the one between \( S \) and \( E \), that is \( Q_{SE'C} = Q_{SE} \). The full unitary transformation in the collision model is then given by
\[ T_{SEEE'C} = Q_{SE} \otimes 1_{EE'C} + 1_{SE} \otimes Q_{EE'C}, \] (28)
and the one-collision map for the joint state \( \rho_{SC} \) reads
\[ \rho'_{SC} = \text{Tr}_{EE'} \{ T (\rho_{SC} \otimes |\Psi\rangle \langle \Psi|_{EE'}) T^\dagger \}. \] (29)
The steady state of this dynamics is the state \( |\Psi\rangle_{SC} = |\Psi\rangle_{SE} \) which Alice can use to violate the steering inequality.

\[ \text{konstantin.beyer@tu-dresden.de} \]
\[ \text{kimmo.luoma@tu-dresden.de} \]
\[ \text{walter.strunz@tu-dresden.de} \]

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