On approximate solutions of the incompressible Euler and Navier-Stokes equations

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Abstract

We consider the incompressible Euler or Navier-Stokes (NS) equations on a torus $T^d$, in the functional setting of the Sobolev spaces $H^\infty_0(T^d)$ of divergence free, zero mean vector fields on $T^d$, for $n \in (d/2+1, +\infty)$. We present a general theory of approximate solutions for the Euler/NS Cauchy problem; this allows to infer a lower bound $T_c$ on the time of existence of the exact solution $u$ analyzing \textit{a posteriori} any approximate solution $u_a$, and also to construct a function $R_n$ such that $\|u(t) - u_a(t)\|_n \leq R_n(t)$ for all $t \in [0, T_c)$. Both $T_c$ and $R_n$ are determined solving suitable “control inequalities”, depending on the error of $u_a$; the fully quantitative implementation of this scheme depends on some previous estimates of ours on the Euler/NS quadratic nonlinearity [15] [16]. To keep in touch with the existing literature on the subject, our results are compared with a setting for approximate Euler/NS solutions proposed in [3]. As a first application of the present framework, we consider the Galerkin approximate solutions of the Euler/NS Cauchy problem, with a specific initial datum considered in [2]: in this case our methods allow, amongst else, to prove global existence for the NS Cauchy problem when the viscosity is above an explicitly given bound.

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1 Introduction

In recent years, there has been some activity about approximate solutions of the Euler and Navier-Stokes (NS) equations, viewed as tools to infer accurate a posteriori estimates on the exact solutions. We mention, in particular: the works by Chernyshenko et al. [3], Dashti and Robinson [4], Robinson and Sadowski [18], and our papers [12] [13] [14]. The present work seats within the same research area; here we consider the incompressible Euler/NS equations

$$\frac{\partial u}{\partial t} = -\mathfrak{L}(u \bullet \partial u) + \nu \Delta u + f,$$  \hspace{1cm} (1.1)

where: \( u = u(x,t) \) is the divergence free velocity field; the space variables \( x = (x_s)_{s=1,...,d} \) belong to the torus \( T^d \) (and yield the derivatives \( \partial_s := \partial/\partial x_s \)); \( \Delta := \sum_{s=1}^d \partial_{ss} \) is the Laplacian; \( (u \bullet \partial u)_r := \sum_{s=1}^d u_s \partial_s u_r \) \( (r = 1,...,d) \); \( \mathfrak{L} \) is the Leray projection onto the space of divergence free vector fields; \( \nu \) is the viscosity coefficient, so that \( \nu = 0 \) in the Euler case and \( \nu \in (0, +\infty) \) in the NS case; \( f = f(x,t) \) is the Leray projected density of external forces. The dimension \( d \) is arbitrary in the general setting of the paper, but we put \( d = 3 \) in a final application.

The functional setting that we consider for Eq. (1.1) relies on the Sobolev spaces

$$H^n_{\Sigma_0}(T^d) \equiv H^n_{\Sigma_0} := \{ v : T^d \to \mathbb{R}^d \mid \langle v \rangle = 0, \text{div} v = 0, \sqrt{-\Delta}^n v \in L^2(T^d) \} ,$$  \hspace{1cm} (1.2)

with \( \langle \rangle \) indicating the mean over \( T^d \); for any real \( n \), the above space is equipped with the inner product \( \langle v \mid w \rangle_n := \langle \sqrt{-\Delta}^n v \mid \sqrt{-\Delta}^n w \rangle_{L^2} \) and with the corresponding norm \( \| \|_n \). One of the main issues in this setting is the behavior of the bilinear map

$$\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet w)$$  \hspace{1cm} (1.3)

in the above mentioned Sobolev spaces. It is well known that there are positive constants \( K_{nd} \equiv K_n \) and \( G_{nd} \equiv G_n \) fulfilling the “basic inequality”

$$\| \mathcal{P}(v, w) \|_n \leq K_n \| v \|_n \| w \|_{n+1} \quad \text{for} \ n \in \left( \frac{d}{2}, +\infty \right), \ v \in H^n_{\Sigma_0}, \ w \in H^{n+1}_{\Sigma_0} ,$$  \hspace{1cm} (1.4)

and the so-called “Kato inequality”

$$\| \langle \mathcal{P}(v, w) \mid w \rangle \|_n \leq G_n \| v \|_n \| w \|_n^2 \quad \text{for} \ n \in \left( \frac{d}{2} + 1, +\infty \right), \ v \in H^n_{\Sigma_0}, \ w \in H^{n+1}_{\Sigma_0} ;$$  \hspace{1cm} (1.5)

fully quantitative upper and lower bounds on \( K_n \) and \( G_n \) were derived in our previous works [13] [16], for reasons related to the present setting and described more precisely in the sequel.

Independently of the problem to estimate \( K_n \) and \( G_n \), the above two inequalities play a major role in the very interesting paper [3] on approximate Euler/NS solutions.
and a posteriori estimates on exact solutions. To give an idea of the framework of our approach, we describe a result therein, using notations closer to our setting.

Consider the Euler/NS equation \( (1.1) \) with a specified initial condition \( u(x, 0) = u_0(x) \); let \( u_a : T^d \times [0, T_a) \rightarrow \mathbb{R}^d \) be an approximate solution of this Cauchy problem. Given \( n \in (d/2 + 1, +\infty) \) (and assuming suitable regularity for \( u_0, f, u_a \)), let \( u_a \) possess the differential error estimator \( \epsilon_n : [0, T_a) \rightarrow [0, +\infty) \), the datum error estimator \( \delta_n \in [0, +\infty) \) and the growth estimators \( D_n, D_{n+1} : [0, T_a) \rightarrow [0, +\infty) \); this means that, for \( t \in [0, T_a) \),

\[
\left\| \left( \frac{\partial u_a}{\partial t} + \varpi (u_a \cdot \partial u_a) - \nu \Delta u_a - f \right) (t) \right\|_n \leq \epsilon_n(t) , \tag{1.6}
\]

\[
\left\| u_a(0) - u_0 \right\|_n \leq \delta_n , \tag{1.7}
\]

\[
\left\| u_a(t) \right\|_n \leq D_n(t) , \quad \left\| u_a(t) \right\|_{n+1} \leq D_{n+1}(t) \tag{1.8}
\]

(with \( u_a(t) := u_a(\cdot, t) \), etc.). According to \([3]\), Eq. \((1.1)\) with datum \( u_0 \) has an exact (strong, \( H^n_{\text{loc}} \)-valued) solution \( u \) on a time interval \([0, T_b) \subset [0, T_a)\), if \( T_b \) (with the estimators for \( u_a \)) fulfills the inequality

\[
\delta_n + \int_0^{T_b} dt \epsilon_n(t) < \frac{1}{G_n T_b} e^{-\int_0^{T_b} dt \left( G_n D_n(t) + K_n D_{n+1}(t) \right)} . \tag{1.9}
\]

The present work aims to refine, to some extent, the approach of \([3]\) and to apply it to get fully quantitative estimates on the exact solution of the Euler/NS Cauchy problem on \( T^d \), with some specific initial datum. Our main result can be described as follows: assuming suitable regularity for \( u_0, f, u_a \), and intending \( \epsilon_n, D_n, D_{n+1} \) as above, suppose there is a function \( R_n \in C([0, T_c), [0, +\infty)) \), with \( T_c \in (0, T_a] \), fulfilling the control inequalities

\[
\frac{d^+ R_n}{dt} \geq -\nu R_n + (G_n D_n + K_n D_{n+1}) R_n + G_n R_n^2 + \epsilon_n \text{ on } [0, T_c), \quad R_n(0) \geq \delta_n \tag{1.10}
\]

(with \( d^+/dt \) the right upper Dini derivative, see Section \([2]\)). Then, the solution \( u \) of the Euler/NS equation \((1.1)\) with initial datum \( u_0 \) exists (in a classical sense) on the time interval \([0, T_c)\), and its distance from the approximate solution admits the bound

\[
\left\| u(t) - u_a(t) \right\|_n \leq R_n(t) \quad \text{for } t \in [0, T_c) . \tag{1.11}
\]

Some features distinguishing our approach from \([3]\) are the following ones.

(i) Differently from \((1.9)\), our control inequalities \((1.10)\) depend explicitly on \( \nu \) and thus could allow a more accurate analysis of the influence of viscosity on the regularity of the Euler/NS solutions.

(ii) Our approach promises better lower bounds on the time of existence of \( u \). For example, for \( \nu > 0 \) and under specific assumptions illustrated in the paper, the
inequalities (1.10) have solutions $\mathcal{R}_n$ with $T_c = +\infty$, implying the global nature of the NS solution $u$; on the contrary, if $\delta_n$ or $\epsilon_n$ are nonzero the inequality (1.9) cannot have a solution with very large $T_b$, since the right hand side is bounded by $1/(G_n T_b)$ and thus vanishes for $T_b \to +\infty$.

(iii) In [3] there is not an explicit bound on the distance between $u$ and $u_a$, such as (1.11) (however, our analysis yielding (1.11) is greatly indebted to [3] and, in a sense, it mainly refines and completes a chain of inequalities for $u - u_a$ appearing therein).

(iv) The constants $K_n$ and $G_n$ in the inequalities (1.4) (1.5) are not evaluated in [3]. On the contrary, here we have at hand our previous results [15] [16] on these constants; thus, in specific applications, we can implement the control inequalities (1.10) and their outcome (1.11) in a fully quantitative way.

As an example of our approach, in the final part of the paper we consider the Euler/NS equations on $\mathbb{T}^3$ with a specific initial datum $u_0$. Independently of the approach developed here, this datum has been already considered in an interesting paper by Behr, Nečas and Wu [2], where it is indicated as the origin of a possible blow-up for the Euler equations. However, in the cited work the blow-up is conjectured on the grounds of a merely “experimental” analysis of a finite number of terms in the power series $\sum_{i=0}^{+\infty} u_i(x) t^i$ solving formally the Euler Cauchy problem.

In the present work, dealing with the initial datum of [2] both for $\nu = 0$ and for $\nu > 0$, a different approach to the Cauchy problem is developed using the familiar Galerkin approximation (with a convenient set of Fourier modes), combined with our general setting for approximate solutions based on the control inequalities (1.10); in this case, the Sobolev order is $n = 3$, $u_a$ is the Galerkin solution and we use for it the required estimators, to be substituted in the control inequalities (1.10) (with the values for the constants $K_3$ and $G_3$ obtained in [15] [16]). We search for a solution $\mathcal{R}_3$ fulfilling Eqs. (1.10) as equalities (i.e., with $\leq$ replaced by $=$); this gives rise to an ordinary Cauchy problem for $\mathcal{R}_3$, which is solved very easily and reliably by numerical means. Admittedly, our computations are preliminary: they were performed using MATHEMATICA on a PC, with a fairly small set of 150 Fourier modes for the Galerkin approximation; we plan to develop the same approach with more powerful computational tools in a subsequent work.

In a few words, our results are as follows: in the case $\nu = 0$, the solution $\mathcal{R}_3$ of the control equations (1.10) exists on a finite time interval $[0, T_c)$ (after which it blows up); so, we can grant existence for the Euler Cauchy problem on the interval $[0, T_c)$, where we also have the estimate (1.11) on the $H^3_{\Sigma_0}$ distance between the exact solution $u$ and the Galerkin approximate solution $u_a$. (Unfortunately, $T_c$ is less than the blow-up time suggested in [2] for the Euler Cauchy problem, so we cannot disprove the conjecture of the cited paper; the situation could change using many more Galerkin modes, which is our aim for the future). For $0 < \nu \lesssim 8$, the situation is similar: $\mathcal{R}_3$ blows up in a finite time $T_c$, and we can grant existence for
the NS Cauchy problem only up to \( T_c \). On the contrary, for \( \nu \gtrsim 8 \), our approach grants global existence for the NS Cauchy problem (and a bound of the type (1.11) on the full interval \([0, +\infty)\)).

To conclude this Introduction, let us describe the organization of the paper. In Section 2 we present some preliminaries: these concern mainly the Sobolev spaces on \( T^d \), in view of their applications to the Euler/NS equations (1.1). In Section 3 we define formally the Euler/NS Cauchy problem, the general notion of approximate solution for this problem and the related error estimators. In Section 4 that contains the main theoretical results of the paper, we develop the general framework yielding the control inequalities (1.10), and prove the estimate (1.11) on the distance between the exact solution \( u \) of the Euler/NS Cauchy problem and an approximate solution \( u_a \) (here we also give more details on the connections of the present work with [3]). In Section 5 we present some analytical solutions of the control inequalities (1.10), under specific assumptions for their estimators and supposing, for simplicity, that the external forcing in (1.1) is zero; as anticipated, in certain cases our analytical solutions for the control inequalities are global, thus ensuring global existence for the Euler/NS Cauchy problem. In Section 6 we describe the general Galerkin method for (1.1); in particular, we give error and growth estimators for the Galerkin approximate solutions, to be used with our control inequalities (1.10). In Section 7 we consider the Galerkin method with the initial datum of [2], both for \( \nu = 0 \) an for \( \nu > 0 \). In Appendix A we review some comparison lemmas of the Čaplygin type about differential inequalities; these are employed in Section 4 in relation to the control inequalities. In Appendix B for completeness we report the proof of an essentially known statement on the Galerkin approximants for the Euler/NS Cauchy problem.

## 2 Preliminaries

**Dini derivatives.** Consider a function

\[
f : [0, T) \rightarrow \mathbb{R} , \quad t \mapsto f(t)
\]  

(with \( T \in (0, +\infty) \)). The right, lower and upper Dini derivatives of \( f \) at any point \( t_0 \in [0, T) \) are, respectively,

\[
\frac{d^+ f}{dt}(t_0) := \lim_{h \to 0^+} \inf \frac{f(t_0 + h) - f(t_0)}{h} \in \mathbb{R} ;
\]

\[
\frac{d^+ f}{dt}(t_0) := \lim_{h \to 0^+} \sup \frac{f(t_0 + h) - f(t_0)}{h} \in \mathbb{R} .
\]
Of course,
\[
\frac{d_+ f}{dt}(t_0) \leq \frac{d^+ f}{dt}(t_0);  \quad (2.4)
\]
furthermore, the opposite function \(-f : t \in [0,T) \mapsto -f(t)\) is such that
\[
\frac{d_+ (-f)}{dt}(t_0) = -\frac{d^+ f}{dt}(t_0). \quad (2.5)
\]
The left, lower and upper Dini derivatives \(\frac{d_-}{dt}, \frac{d_+}{dt}\) are defined similarly, with \(h \to 0^-\); however, left derivatives are not used in this paper. Of course, all Dini derivatives coincide with the usual derivative if this exists.

**Sobolev spaces of vector fields on the torus; Laplacian, Leray projection, and so on.** We work in any space dimension \(d \in \{2,3,\ldots\}\) (using \(r,s\) as indices in \(\{1,\ldots,d\}\)). For \(a,b\) in \(C^d\) we put \(a \cdot b := \sum_{r=1}^d a_r b_r\) and write \(\overline{a}\) for the complex conjugate \(d\)-tuple \((a_r); C^d\) carries the inner product \((a,b) \mapsto a \cdot b\) and the norm \(|a| := \sqrt{\overline{a} \cdot a}\). We often restrict the previous operations to \(R^d\).

We consider the torus \(T^d\), i.e., the product of \(d\) copies of \(T := R/(2\pi \mathbb{Z})\); a point of \(T^d\) is generically written as \(x = (x_s)_{s=1,\ldots,d}\). In the sequel we often refer to the space \(D(T^d) \equiv D\) (2.6)
of the real distributions on \(T^d\), and to the space
\[
\mathbb{D}'(T^d) \equiv \mathbb{D}' := \{v = (v_r)_{r=1,\ldots,d} \mid v_r \in D\text{ for all }r\}. \quad (2.7)
\]
Elements of \(\mathbb{D}'\) can be interpreted as “generalized functions \(T^d \to R^{d_0}\); in the sequel, we call them (distributional) vector fields on \(T^d\). \(D\) and \(\mathbb{D}'\) will be equipped with their weak topologies. For more details on distributions (and on the function spaces mentioned in the sequel) we refer, e.g., to [14].

Using distributional derivatives, we can give a meaning to several differential operators acting on vector fields, e.g. the Laplacian \(\Delta : \mathbb{D}' \to \mathbb{D}'\) and the divergence \(\text{div} : \mathbb{D}' \to D\). Any real distribution \(w \in D\) has a mean \(\langle w \rangle := \langle w, 1/(2\pi)^d \rangle \in R\) (the right hand side in this definition indicates the action of \(w\) on the constant test function \(1/(2\pi)^d\)). Passing to a vector field \(v \in \mathbb{D}'\), we can define componentwise the mean \(\langle v \rangle \in R^d\).

Each vector field \(v \in \mathbb{D}'\) has a unique (weakly convergent) Fourier series expansion
\[
v = \sum_{k \in Z^d} v_k e_k, \quad v_k \in C^d, \quad e_k(x) := \frac{1}{(2\pi)^{d/2}} e^{ik \cdot x} \text{ for } x \in T^d; \quad (2.8)
\]
of course, the \(r\)-th component of \(v_k\) is \(\langle v_r, e_{-k} \rangle\) (i.e., it equals the action of \(v_r\) on the test function \(e_{-k}\)). Due to the reality of \(v\), the Fourier coefficients have the property \(\overline{v_k} = v_{-k}\) for all \(k \in Z^d\), one has \(\langle v \rangle = v_0/(2\pi)^{d/2}\).
In the sequel we often refer to the space of zero mean vector fields, of the divergence free (or solenoidal) vector fields and to their intersection; these are, respectively,
\[ \mathcal{D}_0' := \{ v \in \mathcal{D}' \mid \langle v \rangle = 0 \}, \quad \mathcal{D}_\Sigma' := \{ v \in \mathcal{D}' \mid \text{div} v = 0 \}, \quad \mathcal{D}_{\Sigma 0} := \mathcal{D}_\Sigma' \cap \mathcal{D}_0'. \] (2.9)
Elements of \( \mathcal{D}_0' \) and \( \mathcal{D}_\Sigma' \) are characterized, respectively, by the conditions \( v_0 = 0 \) and \( k \cdot v_k = 0 \) for all \( k \); handling with the Fourier components of vector fields in \( \mathcal{D}_0' \), it is convenient to put
\[ Z_0^d := Z^d \setminus \{ 0 \}. \] (2.10)
Of course, for each \( v \in \mathcal{D}' \) one has \( (\Delta v)_k = -|k|^2 v_k \); this suggests to define, for any \( n \in \mathbb{R} \),
\[ \sqrt{-\Delta}^n : \mathcal{D}_0' \to \mathcal{D}_0', \quad v \mapsto \sqrt{-\Delta}^n v \text{ such that } (\sqrt{-\Delta}^n v)_k = |k|^n v_k \text{ for } k \in \mathbb{Z}_0^d. \] (2.11)
We denote with \( \mathcal{L}^2(T^d) \) the space of square integrable vector fields \( v : T^d \to \mathbb{R}^d \); this is a real Hilbert space with the inner product \( \langle v|w \rangle_{L^2} := \int_{T^d} dx \langle v(x)\bullet w(x) \rangle = \sum_{k \in \mathbb{Z}^d} \sqrt{v_k} \sqrt{w_k} \) and the norm \( \|v\|_{L^2} := \sqrt{\langle v|v \rangle_{L^2}}. \) For any \( n \in \mathbb{R} \), let us consider the Sobolev space
\[ H^n_0(T^d) \equiv H_0^n := \{ v \in \mathcal{D}_0' \mid \sqrt{-\Delta}^n v \in \mathcal{L}^2(T^d) \} = \{ v \in \mathcal{D}' \mid v_0 = 0, \sum_{k \in \mathbb{Z}_0^d} |k|^{2n}|v_k|^2 < +\infty \} ; \] (2.12)
this is a real Hilbert space with the inner product and the norm
\[ \langle v|w \rangle_n := \langle \sqrt{-\Delta}^n v|\sqrt{-\Delta}^n w \rangle_{L^2} = \sum_{k \in \mathbb{Z}_0^d} |k|^{2n}v_k \sqrt{w_k} , \] (2.13)
\[ \|v\|_n := \sqrt{\langle v|v \rangle_n} = \sqrt{\sum_{k \in \mathbb{Z}_0^d} |k|^{2n}|v_k|^2} . \] (2.14)
Clearly, \( n \leq n' \) implies \( H_0^{n'} \subset H_0^n \) and \( \| \cdot \|_n \leq \| \cdot \|_{n'} \). In this paper, we mainly fix the attention on the divergence free Sobolev space
\[ H_{\Sigma 0}^n(T^d) \equiv H_{\Sigma 0}^n := \mathcal{D}_\Sigma' \cap \mathcal{D}_0' = \{ v \in H_0^n \mid k \cdot v_k = 0 \} \] (2.15)
\( (n \in \mathbb{R}) \); this is a closed subspace of \( H_0^n \), and thus a real Hilbert space with the restriction of \( \langle \cdot | \cdot \rangle_n \).
For \( m \in \{ 0, 1, 2, \ldots, +\infty \} \) let us consider the space \( \mathcal{C}^m(T^d,\mathbb{R}^d) \equiv \mathcal{C}^m \) of vector fields \( v : T^d \to \mathbb{R}^d \) of class \( C^m \) in the ordinary sense; if \( m < +\infty \), equip this with the norm
\[ \|v\|_{C^m} := \max_{\ell = 0, \ldots, m} \max_{r, s_1, \ldots, s_\ell = 1, \ldots, d} \max_{x \in T^d} |\partial_{s_1 \ldots s_\ell} u_r(x)| . \] (2.16)
There is a well known Sobolev imbedding
\[ \mathbb{H}_0^n \subset \mathbb{C}^m, \quad \| \mathbb{C}^m \|_{n} \leq S_{mn} \| \mathbb{H}_0^n \|_{n} \quad \text{for } m \in \{0, 1, 2, \ldots \}, \quad n \in \mathbb{R}, \quad n > \frac{d}{2} + m, \quad (2.17) \]

involving suitable positive constants \( S_{mnd} \equiv S_{mn} \).

For arbitrary \( n \in \mathbb{R} \), we have
\[ \Delta \mathbb{H}_0^{n+2} = \mathbb{H}_0^n, \quad \Delta \mathbb{H}_0^{n+2} = \mathbb{H}_0^n, \quad (2.18) \]
and \( \Delta \) is continuous between the above spaces; furthermore,
\[ \langle \Delta v | v \rangle_n = -\|v\|_{n+1}^2 \leq -\|v\|_n^2 \quad \text{for each } v \in \mathbb{H}_0^{n+2} \quad (2.19) \]
(all the above statements are made evident by the Fourier representations).

By definition, the Leray projection is the map
\[ \mathfrak{L} : \mathcal{D'} \to \mathcal{D'_{\Sigma}}, \quad v \mapsto \mathfrak{L} v \text{ such that } (\mathfrak{L} v)_k = \mathfrak{L}_k v_k \text{ for } k \in \mathbb{Z}^d; \quad (2.20) \]
here \( \mathfrak{L}_k \) is the orthogonal projection of \( \mathbb{C}^d \) onto \( k^\perp = \{ a \in \mathbb{C}^d \mid k \cdot a = 0 \} \) (given explicitly by \( \mathfrak{L}_k c = c - (k \cdot c)k/|k|^2 \) and \( \mathfrak{L}_0 c = c \), for \( k \in \mathbb{Z}^d \) and \( c \in \mathbb{C}^d \)). For each real \( n \), we have a continuous linear map
\[ \mathfrak{L} | \mathbb{H}_0^n : \mathbb{H}_0^n \to \mathbb{H}_0^{n_{\Sigma_0}} \quad (2.21) \]
(which, in fact, is the orthogonal projection of \( (\mathbb{H}_0^n, \langle \cdot | \cdot \rangle_n) \) onto the subspace \( \mathbb{H}_0^{n_{\Sigma_0}} \)).

**The fundamental bilinear map in the Euler/NS equations.** In this subsection we assume \( n \in (d/2, +\infty) \). With this condition, we have a continuous bilinear map \( \mathbb{H}_0^{n_{\Sigma_0}} \times \mathbb{H}_0^{n_{\Sigma_0}+1} \to \mathbb{H}_0^n \), \( (v, w) \mapsto v \cdot \partial w \), where \( v \cdot \partial w \) is the vector field on \( \mathbb{T}^d \) with components \( (v \cdot \partial w)_r := \sum_{s=1}^d v_s \partial_s w_r \). By composition with the Leray projection \( \mathfrak{L} \), we obtain a continuous bilinear map
\[ \mathcal{P} : \mathbb{H}_{\Sigma_0}^{n_{\Sigma_0}} \times \mathbb{H}_{\Sigma_0}^{n_{\Sigma_0}+1} \to \mathbb{H}_{\Sigma_0}^n, \quad (v, w) \mapsto \mathcal{P}(v, w) := -\mathfrak{L}(v \cdot \partial w). \quad (2.22) \]
This appears in the Euler/NS equations, and is referred in the sequel as the fundamental bilinear map for such equations. As is known, for all \( v \in \mathbb{H}_{\Sigma_0}^n \) and \( w \in \mathbb{H}_{\Sigma_0}^{n_{\Sigma_0}+1} \) of Fourier components \( v_k \) and \( w_k \), \( \mathcal{P}(v, w) \) has Fourier components \( \mathcal{P}(v, w)_0 = 0 \) and
\[ \mathcal{P}(v, w)_k = -\frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_0^d} v_h \cdot (k - h) \mathfrak{L}_k w_{k-h} \quad (2.23) \]
for all \( k \in \mathbb{Z}_0^d \), where \( \mathfrak{L}_k \) is the already mentioned projection of \( \mathbb{C}^d \) onto \( k^\perp \).

The continuity of \( \mathcal{P} \) is equivalent to the existence of a constant \( K_{nd} \equiv K_{n} \in (0, +\infty) \) such that
\[ \| \mathcal{P}(v, w) \|_n \leq K_n \| v \|_n \| w \|_{n+1} \quad \text{for } v \in \mathbb{H}_{\Sigma_0}^n, \quad w \in \mathbb{H}_{\Sigma_0}^{n+1}; \quad (2.24) \]
we refer to this as the basic inequality about $\mathcal{P}$. With the stronger assumption
$n \in \left( \frac{d}{2} + 1, +\infty \right)$, it is known that there is a constant
$G_n \equiv G_n \in (0, +\infty)$ such that
\[ |\langle \mathcal{P}(v, w)|w \rangle_n| \leq G_n \|v\|_n \|w\|_n^2 \quad \text{for } v \in H^{n}_{\Sigma_0}, \ w \in H^{n+1}_{\Sigma_0}; \tag{2.25} \]
we call this the Kato inequality, since it originates from Kato’s seminal paper [6] (for completeness, we mention that $\langle \mathcal{P}(v, w)|w \rangle_n = -\langle v \cdot \partial w|w \rangle_n$ for $v, w$ as above).

In our previous works [15] [16], we derived upper and lower bounds for the sharp constants in the above inequalities; throughout this paper, $K_n$ and $G_n$ are any two constants fulfilling Eqs. (2.24) (2.25).

3 The Cauchy problem for the Euler/NS equations: exact and approximate solutions

From here to the end of Section 6 we fix any space dimension $d \in \{2, 3, \ldots \}$, we consider the Sobolev spaces of vector fields on $T^d$, and we choose a real number $n$ such that
\[ n \in \left( \frac{d}{2} + 1, +\infty \right). \tag{3.1} \]

Euler and NS equations: the Cauchy problem. Let us choose a “viscosity coefficient”
\[ \nu \in [0, +\infty), \tag{3.2} \]
a “forcing”
\[ f \in C([0, +\infty), H^n_{\Sigma_0}) \tag{3.3} \]
and an initial datum
\[ u_0 \in H^{n+2}_{\Sigma_0}. \tag{3.4} \]

3.1 Definition. The Cauchy problem for the (incompressible) fluid with viscosity $\nu$, initial datum $u_0$ and forcing $f$ is the following:

\[ \text{Find } u \in C([0, T), H^{n+2}_{\Sigma_0}) \cap C^1([0, T), H^n_{\Sigma_0}) \text{ such that} \]
\[ \frac{du}{dt} = \nu \Delta u + \mathcal{P}(u, u) + f, \quad u(0) = u_0 \]
(with $T \in (0, +\infty]$, depending on $u$). As usually, we speak of the “Euler Cauchy problem” if $\nu = 0$, and of the “NS Cauchy problem” if $\nu > 0$. \[ \Diamond \]
3.2 Remark. (i) The map \( u \mapsto \mathcal{P}(u, u) \) sends continuously \( \mathbb{H}^{n+1}_{\Sigma_0} \) into \( \mathbb{H}^n_{\Sigma_0} \), while \( \Delta \) sends continuously \( \mathbb{H}^{n+2}_{\Sigma_0} \) into \( \mathbb{H}^n_{\Sigma_0} \); this explains the appearing of \( \mathbb{H}^{n+2}_{\Sigma_0} \) in the previous definition, at least in the NS case. In the Euler case \( \nu = 0 \), \( \Delta \) is absent from (3.5); so, in the previous definition and in the subsequent theoretical developments one could systematically replace \( \mathbb{H}^{n+2}_{\Sigma_0} \) with \( \mathbb{H}^{n+1}_{\Sigma_0} \); this is not done just to avoid tedious distinctions between the Euler and the NS case.

(ii) Consider any function \( u \in C([0, T), \mathbb{H}^{n+2}_{\Sigma_0}) \). By the Sobolev imbedding (2.17), one has
\[
    u \in C([0, T), \mathbb{C}^m) \quad \text{for any } m \in \{0, 1, 2, \ldots \} \text{ such that } n + 2 > \frac{d}{2} + m
\]
(in particular, for \( m = 0, 1, 2, 3 \)). (3.6)

The following results are well known, and reported for completeness.

3.3 Proposition. For any \( \nu, f \) and \( u_0 \) as above, (i)-(iii) hold.

(i) The Cauchy problem (3.5) has a unique maximal (i.e., nonextendable) solution, hereafter denote with \( u \), with a suitable domain \([0, T) \) \((0 < T \leq +\infty)\). All the other solutions of (3.5) are restrictions of \( u \).

(ii) \( u \) has the property (3.6). Furthermore, if \( \nu > 0 \) and the forcing \((x, t) \mapsto f(t)(x) \equiv f(x,t)\) is \( C^\infty \) from \( T^d \times (0, T) \) to \( \mathbb{R}^d \), the function \((x, t) \mapsto u(t)(x) \equiv u(x,t)\) is \( C^\infty \) as well from \( T^d \times (0, T) \) to \( \mathbb{R}^d \).

(iii) For any \( t \in [0, T) \) consider the vorticity matrix \( \omega(t) \), of elements \( \omega_{rs}(t) := \partial_r u_s(t) - \partial_s u_r(t) \) \((r, s = 1, \ldots, d)\); let \( \|\omega(t)\|_{C^0} := \max_{r, s = 1, \ldots, d, T^d} |\omega_{rs}(t)| \). If \( T < +\infty \), one has
\[
    \limsup_{t \to T^-} \|\omega(t)\|_{C^0} = +\infty ;
\]
this implies
\[
    \limsup_{t \to T^-} \|u(t)\|_n = +\infty .
\]

Proof. (i) See [7].

(ii) The property (3.6) holds because \( u \in C([0, T), \mathbb{H}^{n+2}_{\Sigma_0}) \). For \( \nu > 0 \) and \( f \) of class \( C^\infty \) on \( T^d \times (0, T) \), the same smoothness property is granted for \( u \) by Theorem 6.1 of [5] (with the domain \( D \) considered therein replaced by \( T^d \)).

(iii) Let \( T < +\infty \). Eq. (3.7) is the celebrated Beale-Kato-Majda blow-up criterion [1] (see also [8]). To prove (3.8) we note that, for each \( t \in [0, T) \),
\[
    \|\omega(t)\|_{C^0} \leq \text{const.} \cdot \|u(t)\|_{C^1} \leq \text{const.} \cdot \|u(t)\|_n ;
\]
here the first inequality follows from the definition of \( \omega(t) \) in terms of the derivatives \( \partial_r u_s(t) \), and the second one from the Sobolev imbedding (2.17) (with \( m = 1 \)). The relations (3.7) and (3.9) immediately give Eq. (3.8). □
Approximate solutions. Our treatment uses systematically the present terminology.

3.4 Definition. An approximate solution of the problem (3.3) is any map $u_a \in C([0, T_a), \mathbb{H}^{n+2}_{\Sigma_0}) \cap C^1([0, T_a), \mathbb{H}_\Sigma^{n})$ (with $T_a \in (0, +\infty]$). Given such a function, we stipulate
(i) The differential error of $u_a$ is
$$e(u_a) := \frac{du_a}{dt} - \nu \Delta u_a - P(u_a, u_a) - f \in C([0, T_a), \mathbb{H}^n_{\Sigma_0}) ;$$
(ii) Let $m \in \mathbb{R}, m \leq n$. A differential error estimator of order $m$ for $u_a$ is a function
$$\epsilon_m \in C([0, T_a), [0, +\infty)) \text{ such that } \|e(u_a)(t)\|_m \leq \epsilon_m(t) \text{ for } t \in [0, T_a).$$
Let $m \in \mathbb{R}, m \leq n+2$. A datum error estimator of order $m$ for $u_a$ is a real number
$$\delta_m \in [0, +\infty) \text{ such that } \|u_a(0) - u_0\|_m \leq \delta_m ;$$
a growth estimator of order $m$ for $u_a$ is a function
$$D_m \in C([0, T_a), [0, +\infty)) \text{ such that } \|u_a(t)\|_m \leq D_m(t) \text{ for } t \in [0, T_a).$$
In particular the function $\epsilon_m(t) := \|e(u_a)(t)\|_m$, the number $\delta_m := \|u_a(0) - u_0\|_m$ and the function $D_m(t) := \|u_a(t)\|_m$ will be called the tautological estimators of order $m$ for the differential error, the datum error and the growth of $u_a$.

4 Main theorems about approximate solutions

Assumptions and notations. Throughout this section we fix a viscosity coefficient, a forcing and an initial datum as in Eqs. (3.2)–(3.4) (recalling the condition $n > d/2 + 1$). We consider for the Cauchy problem (3.5) an approximate solution
$$u_a \in C([0, T_a), \mathbb{H}^{n+2}_{\Sigma_0}) \cap C^1([0, T_a), \mathbb{H}_\Sigma^{n}) ;$$
in the sequel $\epsilon_n \in C([0, T_a), [0, +\infty)$ and $\delta_n \in [0, +\infty)$ are differential error and datum error estimators of order $n$ for $u_a$, while $D_n, D_{n+1} \in C([0, T_a), [0, +\infty))$ are growth estimators of orders $n, n+1$ (see Definition 3.4). Finally, we denote with
$$u \in C([0, T), \mathbb{H}^{n+2}_{\Sigma_0}) \cap C^1([0, T), \mathbb{H}_\Sigma^{n})$$
the maximal solution of (3.5) (typically unknown, as well as $T$).
Some lemmas. For the sake of brevity, we put

\[ T_w := \min(T_a, T), \quad w := u - u_a \in C([0, T_w), \mathbb{H}_\Sigma^{n+2}) \cap C^1([0, T_w), \mathbb{H}_\Sigma^n). \quad (4.3) \]

Furthermore, we introduce the function

\[ W_n : [0, T_w) \to [0, +\infty), \quad W_n(t) := \|w(t)\|_n; \quad (4.4) \]

this is clearly continuous, and \( C^1 \) in a neighborhood of any instant \( t_0 \in [0, T_w) \) such that \( w(t_0) \neq 0 \). In the sequel we often consider the right, upper Dini derivative \( d^+ W_n/dt \) (see Eq. (2.3)), that is just the ordinary derivative at any \( t_0 \) with \( w(t_0) \neq 0 \).

The forthcoming two lemmas review and partially refine some results of [3] (after adaptation to our slightly different setting; for example, Dini derivatives are not even mentioned in [3]).

4.1 Lemma. One has

\[ \frac{dw}{dt} = \nu \Delta w + P(u_a, w) + P(w, u_a) + P(w, w) - e(u_a) \quad \text{on} \quad [0, T_w), \quad (4.5) \]

\[ w(0) = u_0 - u_a(0). \quad (4.6) \]

Proof. By definition of the differential error \( e(u_a) \), we have

\[ \frac{du_a}{dt} = \nu \Delta u_a + P(u, u_a) + f + e(u_a); \quad (4.7) \]

of course, \( du/dt = \nu \Delta u + P(u, u) + f \) whence, writing \( u = u_a + w, \)

\[ \frac{du_a}{dt} + \frac{dw}{dt} = \nu \Delta (u_a + w) + P(u_a + w, u_a + w) + f \]

\[ = \nu \Delta u_a + \nu \Delta w + P(u_a, u_a) + P(u_a, w) + P(w, u_a) + P(w, w) + f. \quad (4.8) \]

Subtracting Eq. (4.7) from (4.8) we obtain Eq. (4.5); Eq. (4.6) is obvious. \( \square \)

4.2 Lemma. Consider the above mentioned estimators \( \epsilon_n, \delta_n, D_n, D_{n+1} \), and the function \( W_n \) defined in Eq. (4.4). Then

\[ \frac{d^+ W_n}{dt} \leq -\nu W_n + (G_n D_n + K_n D_{n+1}) W_n + G_n W_n^2 + \epsilon_n \quad \text{everywhere on} \quad [0, T_w), \quad (4.9) \]

\[ W_n(0) \leq \delta_n. \quad (4.10) \]
Proof. We proceed in three steps.

Step 1. Verification of Eq. (4.14) at a time $t_0$ such that $w(t_0) \neq 0$. In this case, $w(t) \neq 0$ for all $t$ in some interval $I \ni t_0$; $W_n$ is nonzero and $C^1$ on $I$. In the same interval, we have:

$$\frac{d^+ W_n}{dt} = \frac{dW_n}{dt} = \frac{1}{2W_n} \frac{dW_n^2}{dt} = \frac{1}{2W_n} \frac{d}{dt} \langle w \rangle_n = \frac{1}{W_n} \langle \frac{dw}{dt} \rangle_n \quad (4.11)$$

$$= \frac{1}{W_n} \left( \nu \langle \Delta w \rangle_n + \langle \mathcal{P}(u_a, w) \rangle_n + \langle \mathcal{P}(w, u_a) \rangle_n + \langle \mathcal{P}(w, w) \rangle_n - \langle e(u_a) \rangle_n \right).$$

On the other hand, we have the following inequalities:

$$\langle \Delta w \rangle_n \leq -\|w\|_n^2 = -W_n^2; \quad (4.12)$$

$$\langle \mathcal{P}(u_a, w) \rangle_n \leq |\langle \mathcal{P}(u_a, w) \rangle_n| \leq G_n \|u_a\|_n \|w\|_n^2 \leq G_n D_n W_n^2; \quad (4.13)$$

$$\langle \mathcal{P}(w, u_a) \rangle_n \leq |\langle \mathcal{P}(w, u_a) \rangle_n| \leq \|\mathcal{P}(w, u_a)\|_n \|w\|_n \leq K_n \|u_a\|_{n+1} \|w\|_n^2 \leq K_n D_{n+1} W_n^2; \quad (4.14)$$

$$\langle \mathcal{P}(w, w) \rangle_n \leq |\langle \mathcal{P}(w, w) \rangle_n| \leq G_n \|w\|_n^3 = G_n W_n^3; \quad (4.15)$$

$$- \langle e(u_a) \rangle_n \leq |\langle e(u_a) \rangle_n| \leq \|e(u_a)\|_n \|w\|_n \leq \epsilon_n W_n; \quad (4.16)$$

inserting the relations (4.12)–(4.16) into Eq. (4.11), we obtain the inequality (4.9) at all times $t \in I$ and, in particular, at time $t_0$.

Step 2. Verification of Eq. (4.15) at a time $t_0$ such that $w(t_0) = 0$. At this instant we have $W_n(t_0) = 0$, and the relation (4.9) to be proved becomes

$$\frac{d^+ W_n}{dt}(t_0) \leq \epsilon_n(t_0). \quad (4.17)$$

To go on we recall that, given a $C^1$ function $z : [0, T_z) \to E$ with values in a Banach space $E$ equipped with the norm $\| \|$, we have $d^+ \|z\|/dt \leq \|dz/dt\|$ at all times, including instants when $z$ vanishes (see, e.g., [17] and references therein); if this result is applied to the function $w$ with norm $\|w\|_n = W_n$, we get

$$\frac{d^+ W_n}{dt}(t_0) \leq \| \frac{dw}{dt}(t_0) \|_n. \quad (4.18)$$

Now, we evaluate $(dw/dt)(t_0)$ via Eq. (4.5), taking into account that $w(t_0) = 0$; this gives

$$\| \frac{dw}{dt}(t_0) \|_n = \| e(u_a)(t_0) \|_n \leq \epsilon_n(t_0), \quad (4.19)$$

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and Eqs. (4.18) (4.19) yield the thesis (4.17).

Step 3. Verification of Eq. (4.10). In fact, the definitions of $W_n, w_n \text{ and } \delta_n$ give

$$W_n(0) = \| u(0) - u_a(0) \| \leq \delta_n.$$  \hfill (4.20)

Now, the proof is concluded. \hfill \Box

To go on, we need a comparison lemma of the Čaplygin type. This is a variant of some known results \[9\] \[11\]: for more details, see Appendix A.

4.3 **Lemma.** Let $T \in (0, +\infty]$; consider a function $\psi \in C(\mathbb{R} \times [0, T), \mathbb{R})$, $(s, t) \mapsto \psi(s, t)$ possessing partial derivative $\partial \psi / \partial s \in C(\mathbb{R} \times [0, T), \mathbb{R})$; furthermore, let $s_0 \in \mathbb{R}$. Suppose that $W, R \in C([0, T_c), \mathbb{R})$ are functions such that

$$d^+ W(t) \leq \psi(W(t), t) \quad \text{for all } t \in [0, T), \ W(0) \leq s_0,$$  \hfill (4.21)

$$d^+ R(t) \geq \psi(R(t), t) \quad \text{for all } t \in [0, T), \ R(0) \geq s_0.$$  \hfill (4.22)

Then

$$W(t) \leq R(t) \quad \text{for all } t \in [0, T).$$  \hfill (4.23)

\hfill \Diamond

Hereafter, interesting conclusions will arise from the combination of the above comparison result with Lemma 4.2.

**The control inequalities and the main theorem.** We begin with a definition, which is followed by the main result of the section.

4.4 **Definition.** Consider a function $R_n \in C([0, T_c), [0, +\infty))$, with $T_c \in (0, T_a]$. This function is said to fulfill the control inequalities (with respect to the estimators $\varepsilon_n, \delta_n, D_n, D_{n+1}$) if

$$d^+ R_n(t) \geq -\nu R_n + (G_n D_n + K_n D_{n+1}) R_n + G_n R_n^2 + \varepsilon_n \quad \text{everywhere on } [0, T_c],$$  \hfill (4.24)

$$R_n(0) \geq \delta_n.$$  \hfill (4.25)

4.5 **Proposition.** Suppose there is a function $R_n \in C([0, T_c), [0, +\infty))$ fulfilling the control inequalities, and consider the maximal solution $u$ of the Euler/NS Cauchy problem (3.5). Then, the existence time $T$ of $u$ is such that

$$T \geq T_c;$$  \hfill (4.26)

furthermore,

$$\| u(t) - u_a(t) \|_n \leq R_n(t) \quad \text{for } t \in [0, T_c).$$  \hfill (4.27)
Proof. Let us employ the following notation, already used in the previous subsections: \( T_w := \min(T_a, T) \), \( w := u - u_a \) and \( \mathcal{W}_n : t \mapsto \|w(t)\|_n \) (see Eqs. (4.3) (4.4); \( w \) and \( \mathcal{W}_n \) have domain \([0, T_w]\)). We further define \( \mathcal{I} := \min(T_c, T) \) (and note that \( T_c \leq T_a \) implies \( \mathcal{I} \leq T_w \)). For the moment we have not yet proved (4.26), so we do not know whether \( \mathcal{I} \) equals \( T_c \), or not. We proceed in three steps.

Step 1. One has

\[
\|u(t) - u_a(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, \mathcal{I}].
\]  

Consider the functions \( \mathcal{W}_n \) and \( \mathcal{R}_n \) on \([0, \mathcal{I}]\). Due to Lemma 4.2 and to the control inequalities, these fulfill the relations

\[
\frac{d^+ \mathcal{W}_n}{dt}(t) \leq \frac{d^+ \mathcal{R}_n}{dt}(t) \leq \psi(\mathcal{W}_n(t), t) \text{ for all } t \in [0, \mathcal{I}], \quad \mathcal{W}_n(0) \leq \delta_n;
\]

\[
\frac{d^+ \mathcal{R}_n}{dt}(t) \geq \psi(\mathcal{R}_n(t), t) \text{ for all } t \in [0, \mathcal{I}], \quad \mathcal{R}_n(0) \geq \delta_n,
\]

where \( \psi : [0, \mathcal{I}] \times \mathbb{R} \to \mathbb{R}, \psi(s, t) := -\nu s + (G_n \mathcal{D}_n(t) + K_n \mathcal{D}_{n+1}(t))s + G_n s^2 + \epsilon_n(t). \)

Therefore, by Lemma 4.3 we have \( \mathcal{W}_n(t) \leq \mathcal{R}_n(t) \) for \( t \in [0, \mathcal{I}] \); this is just the relation (4.28).

Step 2. One has the relation (4.26) \( T \geq T_c \). To prove this, we assume \( T < T_c \) and try to infer a contradiction. To this purpose we note that, according to (3.8),

\[
\limsup_{t \to T^-} \|u(t)\|_n = +\infty.
\]

On the other hand, recalling Eq. (4.28) we can write \( \|u(t)\|_n \leq \|u_a(t)\|_n + \|u(t) - u_a(t)\|_n \leq \|u_a(t)\|_n + \mathcal{R}_n(t) \) for \( t \in [0, T] \); from here and from the continuity of \( \|u_a(\cdot)\|_n, \mathcal{R}_n \) on \([0, T_c) \supset [0, T] \) we get

\[
\sup_{t \in [0, T]} \|u(t)\|_n \leq \max_{t \in [0, T]} (\|u_a(t)\|_n + \mathcal{R}_n(t)) < +\infty.
\]  

The results (3.8) and (4.29) are contradictory.

Step 3. Conclusion of the proof. Eq. (4.26) proved in Step 2 implies \( \mathcal{I} = T_c \); now, the inequality (4.28) of Step 1 coincides with the thesis (4.27). \( \square \)

Of course, the control inequalities (4.24) (4.25) are fulfilled by a function \( \mathcal{R}_n \in C^1([0, T_c), [0, +\infty)) \) such that \( T_c \in (0, T_a] \), and

\[
\frac{d\mathcal{R}_n}{dt} = -\nu \mathcal{R}_n + (G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1})\mathcal{R}_n + G_n \mathcal{R}_n^2 + \epsilon_n \text{ everywhere on } [0, T_c),
\]  

\[
\mathcal{R}_n(0) = \delta_n.
\]  

(4.31)
4.6 Definition. Eqs. (4.30) (4.31) are referred in the sequel as the control equations, or the control Cauchy problem for $\mathcal{R}_n$.

5 Analytic solutions of the control inequalities (with no external forcing)

Throughout this section we consider again the Euler/NS Cauchy problem (3.5), for a given viscosity $\nu \in [0, +\infty)$, external forcing $f$ and initial datum $u_0$ as in (3.2)–(3.4). After choosing an approximate solution $u_a$ for (3.5) and determining the corresponding estimators, one is faced with the problem of solving the control inequalities (4.24) (4.25), or the control equations (4.30) (4.31), for the unknown real function $t \mapsto \mathcal{R}_n(t)$. In many applications, such as the one of the next section on the Galerkin approximate solutions, a numerical treatment of the control equations is recommended (and the result is generally reliable: (4.30) (4.31) is a typically nonstiff Cauchy problem). However, an analytic approach to the control equations/inequalities has its own interest, both for theoretical reasons and for building simple user-ready criteria.

In this section we propose an analytic approach for special forms of the approximate solution and/or its estimators. For simplicity, throughout the section we assume zero external forcing:

$$ f(t) := 0 \quad \text{for } t \in [0, +\infty) ; $$

(5.1)

however, many results presented in the section could be extended to the case of nonzero $f$, with suitable assumptions on this function.

We begin by considering the approximate solution $u_a := 0$; this choice seems to be very trivial but, in spite of this, it can be used to obtain nontrivial estimates on the time of existence of the exact solution of the Euler/NS Cauchy problem. In the NS case $\nu > 0$, these estimates include a condition for global existence under a fully quantitative norm bound on the initial datum.

The second case considered in the section is much more general: the approximate solution is unspecified, even though a certain form is assumed for its estimators.

Results from the zero approximate solution. Let us choose for (3.5) the approximate solution

$$ u_a : [0, +\infty) \to \mathbb{H}_\Sigma^{n+2} , \quad u_a(t) := 0 \quad \text{for all } t . $$

(5.2)
5.1 Lemma. (i) The differential and datum errors of the zero approximate solution are

\[ e(u_a)(t) = 0 \quad \text{for all } t \in [0, +\infty), \quad u_a(0) - u_0 = -u_0; \]  

consequently, this approximate solution has the differential and datum estimators

\[ \epsilon_n(t) := 0, \quad \delta_n := \|u_0\|_n, \]  

and the growth estimators

\[ D_n(t) := D_{n+1}(t) := 0 \quad \text{for } t \in [0, +\infty). \]  

The control equations (4.30) (4.31) with the above estimators take the form

\[ \frac{dR_n}{dt} = -\nu R_n + G_n R_n^2, \quad R_n(0) = \|u_0\|_n. \]  

(ii) The Cauchy problem (5.6) has the solution \( R_n \in C^1([0, T_c), [0, +\infty)) \), determined as follows:

\[ T_c := \begin{cases} +\infty & \text{if } \nu > 0, \|u_0\|_n \leq \nu/G_n, \\ -\frac{1}{\nu} \log \left( 1 - \frac{\nu}{G_n \|u_0\|_n} \right) & \text{if } \nu > 0, \|u_0\|_n > \nu/G_n, \\ \frac{1}{G_n \|u_0\|_n} & \text{if } \nu = 0 \end{cases} \]  

(intending \( 1/(G_n \|u_0\|_n) := +\infty \) if \( u_0 = 0 \);

\[ R_n(t) := \frac{\|u_0\|_n e^{-\nu t}}{1 - G_n \|u_0\|_n e^{-\nu t}} \quad \text{for } t \in [0, T_c), \]  

\[ e_{\nu}(t) := \begin{cases} \frac{1 - e^{-\nu t}}{t} & \text{if } \nu > 0, \\ \frac{1 - e^{-\nu t}}{\nu} & \text{if } \nu = 0 \end{cases} \]  

(note that \( t = \lim_{\nu \to 0^+} \frac{1 - e^{-\nu t}}{\nu} \)).

Proof. (i) Obvious.

(ii) The Cauchy problem (5.6) is solved by the quadrature formula

\[ \int_{\|u_0\|_n}^{R_n(t)} \frac{dr}{G_n r^2 - \nu r} = t; \]  

\[ \int_{\|u_0\|_n}^{R_n(t)} \frac{dr}{G_n r^2 - \nu r} = t; \]  

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the integral in the left hand side equals \((1/\nu) \log \frac{G_n - \nu / R_n(t)}{G_n - \nu / \|u_0\|_n}\) if \(\nu > 0\), and \((1/G_n)(1/\|u_0\|_n - 1/R_n(t))\) if \(\nu = 0\); some elementary manipulations yield for the solution \(R_n\) the expression \((5.7)-(5.9)\).

From the previous lemma and from the main theorem on approximate solutions (Proposition 4.5), here applied with \(u_a = 0\) and \(R_n\) as in \((5.8)\), we obtain the following result.

5.2 Proposition. Consider the Cauchy problem \((3.5)\) for the Euler/NS equations with zero external forcing, and any datum \(u_0 \in H^{n+2}_{\Sigma_0}\); let \(u \in C([0,T), H^{n+2}_{\Sigma_0}) \cap C^1([0,T), H^n_{\Sigma_0})\) be the maximal solution. Define \(T_c\) and \(e_\nu\) as in Eqs. \((5.7)-(5.9)\); then

\[ T \geq T_c, \quad \|u(t)\|_n \leq \frac{\|u_0\|_n e^{-\nu t}}{1 - G_n \|u_0\|_n e_\nu(t)} \quad \text{for } t \in [0,T_c). \quad (5.11) \]

In particular, due to \((5.7)\),

\[ T = T_c = +\infty \quad \text{if } \|u_0\|_n \leq \frac{\nu}{G_n}; \quad (5.12) \]

in this case \(u\) is global and, if \(\nu > 0\), it decays exponentially.

Other sufficient conditions for global existence. Consider again the Cauchy problem \((3.5)\) with zero external forcing and any datum \(u_0 \in H^{n+2}_{\Sigma_0}\); let \(u \in C([0,T), H^{n+2}_{\Sigma_0}) \cap C^1([0,T), H^n_{\Sigma_0})\) be the maximal solution. An obvious implication of Proposition 5.2 is the following.

5.3 Corollary. Assume

\[ \|u(t_1)\|_n \leq \frac{\nu}{G_n} \quad \text{for some } t_1 \in [0,T). \quad (5.13) \]

Then:

\[ T = +\infty, \quad \|u(t)\|_n \leq \frac{\|u(t_1)\|_n e^{-\nu(t-t_1)}}{1 - G_n \|u(t_1)\|_n e_\nu(t-t_1)} \quad \text{for } t \in [t_1, +\infty), \quad (5.14) \]

with \(e_\nu\) as in Eq. \((5.9)\).

Proof. The function \(u \upharpoonright [t_1, T)\) is the maximal solution of the Cauchy problem with datum \(u(t_1)\) specified at time \(t_1\), rather than at time 0. By Eqs. \((5.11)-(5.12)\), with an obvious shift in time, we obtain the thesis \((5.14)\). \(\square\)

A consequence of the previous result, of more practical use, is the following.
5.4 Corollary. Let \( u_a \in C([0, T_a), \mathbb{H}^{n+2}_0) \cap C^1([0, T_a), \mathbb{H}^n_0) \) be any approximate solution of (5.3), with estimators \( \epsilon_n, \delta_n, D_n, D_{n+1} \); assume that the corresponding control inequalities (4.24) (4.25) have a solution \( R_n \in C^1([0, T_c), [0, +\infty)) \), with \( T_c \in (0, T_a) \). Finally, assume that

\[
(D_n + R_n)(t_1) \leq \frac{\nu}{G_n} \quad \text{for some } t_1 \in [0, T_c).
\]

Then:

\[
T = +\infty, \quad \|u(t)\|_n \leq \frac{(D_n + R_n)(t_1)e^{-\nu(t-t_1)}}{1 - G_n(D_n + R_n)(t_1)e^{\nu(t-t_1)}} \quad \text{for } t \in [t_1, +\infty). \tag{5.16}
\]

Proof. By Proposition 4.5 we have \( \|u(t) - u_a(t)\|_n \leq R_n(t) \) for all \( t \in [0, T_c) \) and, in particular, for \( t = t_1 \); we further write \( \|u(t_1)\|_n \leq \|u_a(t_1)\|_n + \|u(t_1) - u_a(t_1)\|_n \), which implies

\[
\|u(t_1)\|_n \leq (D_n + R_n)(t_1). \tag{5.17}
\]

Eq. (5.17) and the assumption (5.15) gives the inequality

\[
\|u(t_1)\|_n \leq \frac{\nu}{G_n},
\]

which has the form (5.13). By the previous corollary, we have Eq. (5.14) and this result, combined with (5.17), gives the thesis (5.16).

A general result, under conditions of exponential decay (\( \nu > 0 \)) or boundedness (\( \nu = 0 \)) for the approximate solution. In this paragraph we exhibit a solution of the control equations (4.30) (4.31), holding for any approximate solution whose estimators have certain features. Such features are described in the forthcoming Eq. (5.18); these indicate that the norms of orders \( n \), \( n + 1 \) of \( u_a \) behave like \( e^{-\nu t} \), while the \( n \)-th norm of the differential error behaves like \( e^{-2\nu t} \). (In the NS case these are conditions of exponential decay, while in the Euler case they simply indicate the boundedness of \( u_a \) and its error.)

The assumption that \( u_a \) behaves like \( e^{-\nu t} \) corresponds to a rather typical behavior of the approximate solutions under zero external forcing; for example, this behavior occurs in the case of the Galerkin approximate solutions discussed in the next section (see Lemma 6.4 and the subsequent Remark). The differential error of \( u_a \) typically behaves like \( e^{-2\nu t} \) when \( u_a \) is bounded by \( e^{-\nu t} \) and the differential error depends only on the quadratic function \( P(u_a, u_a) \); again, this situation occurs in the example of the Galerkin approximate solutions with no forcing (see Lemma 6.8).

Other approximation methods suggested for the NS equations (\( \nu > 0 \)) yield approximate solutions with a behavior as in (5.18); as an example, this happens if one assumes no external forcing and takes for \( u_a \) the truncation to any order of the power series solution introduced by Sinai [19].

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5.5 Lemma. For any \( \nu \in [0, +\infty) \), consider for (3.5) an approximate solution \( u_n \in C([0, +\infty), \mathbb{H}^{\nu+2}) \cap C^1([0, +\infty), \mathbb{H}^{\nu}_2) \). Assume this to have growth and differential error estimators of the forms

\[
D_n(t) := D_n e^{-\nu t}, \quad D_{n+1}(t) := D_{n+1} e^{-\nu t}, \quad E_n(t) := E_n e^{-2\nu t},
\]

(5.18)

with \( D_n, D_{n+1} \in (0, +\infty), \ E_n \in [0, +\infty) \); furthermore, assume this to have any datum error estimator \( \delta_n \in [0, +\infty) \). From the above constants, let us define

\[
F_n := \frac{1}{2}(G_n D_n + K_n D_{n+1}) ;
\]

(5.19)

furthermore, assume the “error bound”

\[
G_n E_n < F_n^2
\]

(5.20)

and define

\[
W_n^\pm := F_n \pm \sqrt{F_n^2 - G_n E_n}.
\]

(5.21)

Finally, consider the function \( t \in [0, +\infty) \mapsto e_\nu(t) \) defined by (5.9), and put

\[
\eta_n : [0, +\infty) \to [1, +\infty) , \quad \eta_n(t) := e^{(W_n^+ - W_n^-)e_\nu(t)}.
\]

(5.22)

Then, (i) (ii) hold.

(i) The control equations (4.30) (4.31) with the above estimators take the form

\[
\frac{d\mathcal{R}_n}{dt} = -\nu \mathcal{R}_n + 2F_n e^{-\nu t} \mathcal{R}_n + G_n \mathcal{R}_n^2 + E_n e^{-2\nu t} , \quad \mathcal{R}_n(0) = \delta_n.
\]

(5.23)

(ii) The Cauchy problem (5.23) has the solution \( \mathcal{R}_n \in C^1([0, T_c), [0, +\infty)) \), determined as follows:

\[
T_c := \left\{ \begin{array}{ll}
+\infty & \text{if } \nu > 0, \ W_n^+/G_n \delta_n \geq e^{(W_n^+ - W_n^-)/\nu} \\
-\frac{1}{\nu} \log \left( 1 - \frac{\nu}{W_n^+/G_n \delta_n} \log \frac{W_n^+ + G_n \delta_n}{W_n^- + G_n \delta_n} \right) & \text{if } \nu > 0, \ W_n^+/G_n \delta_n < e^{(W_n^+ - W_n^-)/\nu} \\
\log \frac{1}{W_n^+ - W_n^-} & \text{if } \nu = 0;
\end{array} \right.
\]

(5.24)

\[
\mathcal{R}_n(t) := \frac{1}{G_n} \frac{W_n^+ (W_n^- + G_n \delta_n) \eta_n(t) - W_n^- (W_n^+ + G_n \delta_n) \eta_n(t)}{(W_n^+ + G_n \delta_n) - (W_n^- + G_n \delta_n) \eta_n(t)} e^{-\nu t} \text{ for } t \in [0, T_c).
\]

(5.25)

Proof. (i) Obvious.

(ii) We write the unknown solution \( \mathcal{R}_n \) of (5.23) as

\[
\mathcal{R}_n(t) = Z_n(t) e^{-\nu t}
\]

(5.26)
where \( \mathcal{Z}_n \in C^1([0, T_c), [0, +\infty)) \) (and \( T_c \)) are to be found. Eq. (5.23) is equivalent to the Cauchy problem

\[
\frac{d\mathcal{Z}_n}{dt} = (E_n + 2F_n \mathcal{Z}_n + G_n \mathcal{Z}_n^2) e^{-\nu t}, \quad \mathcal{Z}_n(0) = \delta_n ,
\]

(5.27)

with \( F_n \) as in (5.19); this is solved by the quadrature formula

\[
\int_{\delta_n}^{\mathcal{Z}_n(t)} \frac{dz}{E_n + 2F_n z + G_n z^2} = \int_0^t ds e^{-\nu s} .
\]

(5.28)

On the other hand

\[
\int_0^t ds e^{-\nu s} = e^{-\nu t} ;
\]

(5.29)

furthermore \( E_n + 2F_n z + G_n z^2 = (1/G_n)(W_n^+ + G_n z)(W_n^- + G_n z) \), which implies

\[
\int_{\delta_n}^{\mathcal{Z}_n(t)} \frac{dz}{E_n + 2F_n z + G_n z^2} = -\frac{1}{W_n^+ - W_n^-} \left[ \log \frac{W_n^+ + G_n z}{W_n^- + G_n z} \right]_{z=\mathcal{Z}_n(t)} \]

(5.30)

Inserting Eqs. (5.29) (5.30) into Eq. (5.28) we easily obtain an explicit expression for \( \mathcal{Z}_n(t) \) which implies Eq. (5.25) for \( \mathcal{R}_n(t) = \mathcal{Z}_n(t)e^{-\nu t} \). The interval \([0, T_c)\) where \( \mathcal{Z}_n \) is well defined is also easily determined, by the same manipulation employed to make explicit \( \mathcal{Z}_n(t) \).

\[ \square \]

From the previous lemma and from the main theorem on approximate solutions (Proposition 4.3), we obtain the following result.

\section*{5.6 Propositon.}

Consider the Euler/NS Cauchy problem (3.5), and an approximate solution \( u_a \in C([0, +\infty), \mathbb{H}_{50}^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_{50}^n) \); assume this to have growth and differential error estimators fulfilling all conditions in Lemma 5.5, and define \( T_c, \mathcal{R}_n \) via Eqs. (5.24) (5.25) of the same lemma. Now, consider the maximal solution \( u \in C([0, T), \mathbb{H}_{50}^{n+2}) \cap C^1([0, T), \mathbb{H}_{50}^n) \) of (3.5); then

\[
T \geq T_c , \quad \|u(t) - u_a(t)\| \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c) .
\]

(5.31)

(In particular, \( u \) is global under the conditions giving \( T_c = +\infty \) in Eq. (5.24).)

\section*{5.7 Remark.}

One could easily derive a variant of Lemma 5.5 (and Proposition 5.6) dealing with the limit case \( G_n E_n = F_n^2 \), where \( W_n^- = W_n^+ \). The expressions for \( T_c \) and \( \mathcal{R}_n \) in this case coincide with the ones derivable from Eqs. (5.24) (5.25) in the limit \( W_n^- \to W_n^+ \). The results in Lemma 5.1 (and Proposition 5.2) could also be derived from Lemma 5.5 (and Proposition 5.6), in the limit case \( E_n = 0 \) and \( D_n, D_{n+1} \to 0. \)
6 The Galerkin approximate solutions for the Euler/NS equations, and their errors

Throughout this section, we consider a set $G$ with the following features:

\[ G \subset \mathbb{Z}_0^d, \quad G \text{ finite}, \quad k \in G \Leftrightarrow -k \in G. \quad (6.1) \]

Hereafter we write $\langle e_k \rangle_{k \in G}$ for the linear subspace of $\mathbb{D}'_0$ made of the sums $\sum_{k \in G} v_k e_k$ (which is, in fact, contained in $\mathbb{C}^\infty$). The forthcoming definitions follow closely the setting proposed in our previous work [13] for the Galerkin method.

Galerkin subspaces and projections. We define them as follows.

6.1 Definition. The Galerkin subspace and projection corresponding to $G$ are, respectively:

\[ H^G_{\Sigma_0} := \mathbb{D}'_{\Sigma_0} \cap \langle e_k \rangle_{k \in G} = \{ \sum_{k \in G} v_k e_k \mid v_k \in \mathbb{C}^d, \overline{v}_k = v_{-k}, k \bullet v_k = 0 \text{ for all } k \}; \quad (6.2) \]

\[ P^G : \mathbb{D}'_{\Sigma_0} \to H^G_{\Sigma_0}, \quad v = \sum_{k \in \mathbb{Z}_0^d} v_k e_k \mapsto P^G v := \sum_{k \in G} v_k e_k. \quad (6.3) \]

It is clear that

\[ H^G_{\Sigma_0} \subset \mathbb{C}^\infty \cap \mathbb{D}'_{\Sigma_0}, \quad \Delta(H^G_{\Sigma_0}) = H^G_{\Sigma_0}, \quad H^G_{\Sigma_0} \subset H^m_{\Sigma_0}, \quad P^G(H^m_{\Sigma_0}) = H^G_{\Sigma_0} \text{ for all } m \in \mathbb{R}. \quad (6.4) \]

The following result is useful in the sequel.

6.2 Lemma. Let $m, p \in \mathbb{R}$, $m \leq p$ and $v \in H^p_{\Sigma_0}$. Then,

\[ \|(1 - P^G)v\|_m \leq \frac{\|v\|_p}{|G|^{p-m}}, \quad |G| := \min_{k \in \mathbb{Z}_0^d \setminus G} |k|. \quad (6.5) \]

Proof. We have $(1 - P^G)v = \sum_{k \in \mathbb{Z}_0^d \setminus G} v_k e_k$; thus,

\[ \|(1 - P^G)v\|^2_m = \sum_{k \in \mathbb{Z}_0^d \setminus G} |k|^{2m} |v_k|^2 = \sum_{k \in \mathbb{Z}_0^d \setminus G} \frac{|k|^{2p}}{|k|^{2p-2m}} |v_k|^2 \]

\[ \leq \left( \sup_{k \in \mathbb{Z}_0^d \setminus G} \frac{1}{|k|^{2p-2m}} \right) \sum_{k \in \mathbb{Z}_0^d \setminus G} |k|^{2p} |v_k|^2 \leq \frac{1}{|G|^{2p-2m} \|v\|^2_p}. \]

Galerkin approximate solutions. Let us be given $\nu \in [0, +\infty)$, $f \in C([0, +\infty), \mathbb{D}'_{\Sigma_0})$ and $u_0 \in \mathbb{D}'_{\Sigma_0}$.
6.3 Definition. The Galerkin approximate solution of the Euler/NS equations corresponding to the datum $u_0$, to the forcing $f$ and to the set of modes $G$ is the maximal (i.e., nonextendable) solution $u_{f,u_0,G} \equiv u_G$ of the following Cauchy problem, in the finite dimensional space $\mathbb{H}_{\Sigma_0}^G$:

$$\text{Find } u_G \in C^1([0,T_G), \mathbb{H}_{\Sigma_0}^G) \text{ such that}$$

$$\frac{du_G}{dt} = \nu \Delta u_G + \mathcal{P}^G \mathcal{P}(u_G,u_G) + \mathcal{P}^G f , \quad u_G(0) = \mathcal{P}^G u_0 .$$

Eq. (6.7) describes the Cauchy problem for an ODE in the finite-dimensional vector space $\mathbb{H}_{\Sigma_0}^G$; this relies on the continuous function $\mathbb{H}_{\Sigma_0}^G \times [0, +\infty) \to \mathbb{H}_{\Sigma_0}^G$, $(v,t) \mapsto \nu \Delta v + \mathcal{P}^G \mathcal{P}(v,v) + \mathcal{P}^G f(t)$, which is smooth with respect to the variable $v$. So, the standard theory of ODEs in finite dimensional spaces grants existence and uniqueness for the solution of (6.7).

The following facts are known (see, e.g., [20]); a proof suitable for the present setting is given, for completeness, in Appendix B.

6.4 Lemma. (i) For any $\nu \in [0, +\infty)$, $f \in C([0, +\infty), \mathbb{D}^\prime_{\Sigma_0})$ and $u_0 \in \mathbb{D}^\prime_{\Sigma_0}$, the maximal solution of problem (6.7) is in fact global:

$$T_G = +\infty .$$

(ii) If the forcing $f$ is identically zero, one has

$$\|u_G(t)\|_{L^2} \leq \begin{cases} \|\mathcal{P}^G u_0\|_{L^2} & \text{if } \nu = 0, \quad t \in [0, +\infty), \\ \|\mathcal{P}^G u_0\|_{L^2} e^{-\nu t} & \text{if } \nu > 0, \quad t \in [0, +\infty). \end{cases}$$

(iii) Let $\nu \in [0, +\infty)$. For any $f \in C([0, +\infty), \mathbb{D}^\prime_{\Sigma_0})$ one has

$$\|u_G(t)\|_{L^2} \leq \left( \|\mathcal{P}^G u_0\|_{L^2} + \int_0^t ds e^{\nu s} \|\mathcal{P}^G f(s)\|_{L^2} \right) e^{-\nu t} \quad \text{for } t \in [0, +\infty) .$$

In particular, let

$$J := \int_0^{+\infty} ds e^{\nu s} \|\mathcal{P}^G f(s)\|_{L^2} < +\infty ;$$

then,

$$\|u_G(t)\|_{L^2} \leq \left( \|\mathcal{P}^G u_0\|_{L^2} + J \right) e^{-\nu t} \quad \text{for } t \in [0, +\infty) .$$

6.5 Remark. Let $\nu \in [0, +\infty)$. If $f = 0$ or, more generally, if $J < +\infty$, the previous lemma gives a bound of the type $\|u_G(t)\|_{L^2} \leq \text{const. } e^{-\nu t}$. This fact,
with the equivalence of all norms on the finite dimensional space $\mathbb{H}^G_{\Sigma_0}$, implies the following: for any real $m$,

$$\|u_G(t)\|_m \leq U_m e^{-\nu t} \quad \text{for } t \in [0, +\infty),$$

with a suitable constant $U_m \in [0, +\infty)$ (depending on the initial datum and on the forcing).

**6.6 Definition.** (i) For all $k \in \mathbb{Z}^d_0$, $f_k \in C([0, +\infty), \mathbb{C}^d)$ and $u_{0k} \in \mathbb{C}^d$ are the Fourier components of the forcing and the initial datum: $f(t) = \sum_{k \in \mathbb{Z}^d_0} f_k(t)e_k$, $u_0 = \sum_{k \in \mathbb{Z}^d_0} u_{0k}e_k$.

(ii) For $k \in G$, $\gamma_{Gk} \equiv \gamma_k \in C^1([0, +\infty), \mathbb{C}^d)$ are the Fourier components of $u_G$:

$$u_G(t) = \sum_{k \in G} \gamma_k(t)e_k.$$  \hspace{1cm} (6.14)

(Note the relations $f_k = f_{-k}$, $k \cdot f_k = 0$, and the analogous relations for $u_{0k}$, $\gamma_k$.) ♦

Let us review a well known fact.

**6.7 Proposition.** Under the correspondence $u_G \mapsto (\gamma_k)$, Eq. (6.7) for $u_G$ is equivalent to the following problem: find a family of functions $\gamma_k \in C^1([0, +\infty), \mathbb{C}^d)$ ($k \in G$) such that

$$\frac{d\gamma_k}{dt} = -\nu|k|^2\gamma_k - \frac{i}{(2\pi)^d/2} \sum_{h \in G} [\gamma_h \cdot (k - h)] \mathfrak{L}_k \gamma_{k-h} + f_k, \quad \gamma_k(0) = u_{0k} \hspace{1cm} (6.15)$$

(intending $\gamma_{k-h}(t) := 0$ if $k-h \notin G$; recall that $\mathfrak{L}_k$ is the orthogonal projection of $\mathbb{C}^d$ onto $k^\perp$). A family of functions $\gamma_k$ ($k \in G$) fulfilling Eqs. (6.15) automatically possesses the properties $\overline{\gamma_{-k}} = \gamma_{-k}$ and $k \cdot \gamma_k = 0$.

**Proof.** Clearly, Eq. (6.7) is equivalent to

$$\left(\frac{du_G}{dt}\right)_k = \nu(\Delta u_G)_k + \mathfrak{P}(u_G, u_G)_k + f_k \quad \text{for } k \in G;$$

making explicit the above Fourier components via Eq. (2.23), we obtain Eqs. (6.15). Let us show that Eqs. (6.15) imply $\overline{\gamma_{-k}} = \gamma_{-k}$ and $k \cdot \gamma_k = 0$. The first implication is proved noting that the functions $(\overline{\gamma_k})_{k \in G}$ and $(\gamma_{-k})_{k \in G}$ are solutions of the same Cauchy problem. The second implication follows noting that, for each $k \in G$, $(d/dt)(k \cdot \gamma_k) = -\nu|k|^2(k \cdot \gamma_k)$ and $(k \cdot \gamma_k)(0) = 0$; the solution of this Cauchy problem is identically zero. ☐

The Galerkin solutions in the general framework for approximate solutions. From now on we stick to the framework of the previous sections, i.e.: we
consider the Sobolev spaces of orders $n, n+1, n+2$, for a fixed $n \in (d/2 + 1, +\infty)$; as in (3.2)–(3.4), we choose a viscosity $\nu \in [0, +\infty)$, a forcing $f \in C([0, +\infty), \mathbb{H}^n_{\Sigma_0})$ and an initial datum $u_0 \in \mathbb{H}^{n+2}_{\Sigma_0}$; we consider the Euler/NS Cauchy problem (3.5). Having fixed a set of modes $G$ as in (6.1), we regard the corresponding Galerkin solution $u_G$ as an approximate solution of the Euler/NS Cauchy problem (3.5); our aim is to apply the general theory of the previous sections with $u_a = u_G$.

The desired application requires to give growth and error estimators for $u_G$. Of course, we have the tautological growth estimators

$$D_m(t) := \|u_G(t)\|_m = \sqrt{\sum_{k \in G} |k|^{2m} |\gamma_k(t)|^2} ,$$

(6.16)

to be used with $m = n$ or $m = n+1$; these are employed systematically in the sequel. Let us pass to the errors of $u_G$ and their estimators.

6.8 Lemma. (i) The Galerkin solution $u_G$ has the datum error

$$u_G(0) - u_0 = -(1 - \mathcal{P}^G)u_0 = - \sum_{k \in \mathbb{Z}^d \setminus G} u_{0k} e_k$$

(6.17)

and its tautological estimator

$$\delta_n := \|u_G(0) - u_0\|_n = \sqrt{\sum_{k \in \mathbb{Z}^d \setminus G} |k|^{2n} |u_{0k}|^2} .$$

(6.18)

There is a rougher estimator

$$\|u_G(0) - u_0\|_n \leq \delta'_{np} , \quad \delta'_{np} := \frac{\|u_0\|_p}{|G|^p-n} ,$$

(6.19)

where $p$ is any real number such that $p \geq n$, $u_0 \in \mathbb{H}^p_{\Sigma_0}$.

(ii) The differential error of $u_G$ is

$$e(u_G) = -(1 - \mathcal{P}^G)\mathcal{P}(u_G, u_G) - (1 - \mathcal{P}^G) f ;$$

(6.20)

the Fourier representation of the above summands is

$$(1 - \mathcal{P}^G)f = \sum_{k \in \mathbb{Z}^d \setminus G} f_k e_k ;$$

(6.21)

$$\mathcal{P}(u_G, u_G) = \sum_{k \in G} p_k e_k ;$$

(6.22)
\[ dG := (G + G) \setminus (G \cup \{0\}) , \quad p_k := -\frac{i}{(2\pi)^{d/2}} \sum_{h \in G} \gamma_h \cdot (k - h) \mathbf{e}_{h} \gamma_{h}^{-} \cdot \mathbf{e}_{h} \]

(In the above: \( G + G := \{ p + q \mid p, q \in G \}; \setminus \) is the usual set-theoretical difference; again, we intend \( \gamma_{h}^{-} := 0 \) if \( k - h \not\in G \).)

(iii) The above terms in \( e(u_G) \) have norms

\[
\| (1 - \mathcal{P}^G) f \|_n = \sqrt{\sum_{k \in Z_d \setminus G} |k|^{2n} |f_k|^2} , \tag{6.23}
\]

\[
\| (1 - \mathcal{P}^G) \mathcal{P}(u_G, u_G) \|_n = \sqrt{\sum_{k \in dG} |k|^{2n} |p_k|^2} ; \tag{6.24}
\]

these admit the bounds

\[
\| (1 - \mathcal{P}^G) f \|_n \leq \frac{|f|_p}{|G|^{p-n}} , \tag{6.25}
\]

\[
\| (1 - \mathcal{P}^G) \mathcal{P}(u_G, u_G) \|_n \leq \frac{K_q}{|G|^{q-n}} u_G \|_q u_G \|_{q+1} , \tag{6.26}
\]

where: \( p \) is any real number such that \( p \geq n \) and \( f \in C([0, +\infty), \mathbb{H}^p_{\Sigma 0}) \); \( q \) is any real number such that \( q \geq n \); \( K_q \in (0, +\infty) \) is such that \( \| \mathcal{P}(v, w) \|_q \leq K_q \| v \|_q \| w \|_{q+1} \)

for all \( v \in \mathbb{H}^q_{\Sigma 0}, w \in \mathbb{H}^{q+1}_{\Sigma 0} \) (of course \( \| u_G \|_q = \sqrt{\sum_{k \in G} |k|^{2q} |\gamma_k|^2} \), and similarly for \( \| u_G \|_{q+1} \)). Thus, a differential error estimator \( \epsilon_n \) of order \( n \) for \( u_G \) is obtained setting

\[
\epsilon_n := (\text{r.h.s. of (6.24) or (6.26)}) + (\text{r.h.s. of (6.23) or (6.25)}) . \tag{6.27}
\]

**Proof.** (i) Eqs. (6.17) (6.18) are obvious. Eq. (6.19) follows writing \( \| u_G(0) - u_0 \|_n = \| (1 - \mathcal{P}^G) u_0 \|_n \) and using the inequality (6.5).

(ii) We have

\[
e(u_G) = \frac{du_G}{dt} - \nu \Delta u_G - \mathcal{P}(u_G, u_G) - f = \mathcal{P}^G \mathcal{P}(u_G, u_G) + \mathcal{P}^G f - \mathcal{P}(u_G, u_G) - f ,
\]

where the first equality is just the definition of the differential error, and the second follows from (6.7); so, Eq. (6.20) is proved.

The Fourier representation (6.21) for \( (1 - \mathcal{P}^G) f \) follows immediately from the definition of \( \mathcal{P}^G \). To derive the representation (6.22) for \( (1 - \mathcal{P}^G) \mathcal{P}(u_G, u_G) \), let us consider the Fourier components of \( u_G \) that we indicate in any case with \( \gamma_k \), intending \( \gamma_k := 0 \) for \( k \in Z_d \setminus G \). Eq. (2.23) gives

\[
\mathcal{P}(u_G, u_G) = \sum_{k \in Z_d} p_k \mathbf{e}_k ,
\]
where, for any $k$, $p_k$ is defined following Eq. (6.22). On the other hand: for each $k \in \mathbb{Z}^d_0$, $p_k$ is a sum over $h \in G$ containing terms of the form $\gamma_{k-h}$; if $k \notin G + G$, for all $h \in G$ we have $k-h \notin G$ (since $k-h \in G$ would imply $k = (k-h) + h \in G + G$); $k-h \notin G$ implies $\gamma_{k-h} = 0$. Due to the above considerations we have $p_k = 0$ for $k \notin G + G$, whence
\[
P(u_G, u_G) = \sum_{k \in (G+G) \setminus \{0\}} p_k e_k .
\]
(6.28)

Application of $1 - \Psi^G$ removes from the above sum the terms with $k \in G$; thus
\[
(1 - \Psi^G)P(u_G, u_G) = \sum_{k \in (G+G) \setminus (G \cup \{0\})} p_k e_k ,
\]
in agreement with Eq. (6.22).

(iii) Eqs. (6.23) (6.24) are straightforward consequences of the Fourier representations (6.21) (6.22). Eq. (6.25) follows immediately from (6.5). To infer Eq. (6.26), we note that (6.5) and the inequality involving $K_q$ imply
\[
\|(1 - \Psi^G)P(u_G, u_G)\|_n \leq \frac{K_q}{|G|^{q-n}} \|P(u_G, u_G)\|_q \leq \frac{K_q}{|G|^{q-n}} \|u_G\|_q \|u_G\|_{q+1} .
\]
Finally, Eq. (6.20) implies
\[
\|e(u_G)\| \leq \|(1 - \Psi^G)P(u_G, u_G)\|_n + \|(1 - \Psi^G)f\|_n ;
\]
binding the two summands in the right hand side via Eqs. (6.24) or (6.26), (6.23) or (6.25) we obtain the estimator $\epsilon_n$ in (6.27).

6.9 Remarks. (i) The exact expression (6.24) for $\|(1 - \Psi^G)P(u_G, u_G)\|_n$ involves a sum over the finite set $dG$; for each $k \in dG$, the term $p_k$ in the above sum is itself a finite sum, that can be computed explicitly from the Fourier coefficients $\gamma_k = \gamma_k(t)$ of the Galerkin approximate solution (assuming these ones to be known, say, from the numerical integration of the system (6.15)). However, when $G$ is large the set $dG$ is typically very large, and this can make too expensive the computation of the sum over $dG$. In these cases one can use for $\|(1 - \Psi^G)P(u_G, u_G)\|_n$ the bound (6.26); this requires computation of $\|u_G\|_q = \sqrt{\sum_{k \in G} |k|^2 \gamma_k|^2}$ and of $\|u_G\|_{q+1}$, both of them less expensive since these sums are over $G$, rather than $dG$.
(ii) We are aware that, if the Galerkin equations (6.15) are solved numerically by some standard method for ODEs, one does not obtain the exact solutions $\gamma_k(t)$ ($k \in G$) but, rather, some approximants whose distance from the $\gamma_k$’s should be estimated on the grounds of the employed numerical scheme. In the application presented in the next section, relying on a relatively small set $G$ of modes, the intrinsic error in the numerical integration of (6.15) has been regarded as negligible; the situation would be different using a much larger set of modes.

\[\diamond\]
7 An application of the previous framework for the Galerkin method

A preliminary. In this section we frequently report the results of computations performed with MATHEMATICA. A formula like \( r = a.bcd e... \) must be intended as follows: computation of the real number \( r \) via MATHEMATICA produces as an output \( a.bcd e... \), followed by other digits not reported for brevity.

Setting up the problem. Throughout this section, we work in space dimension

\[ d = 3 , \]

with any viscosity \( \nu \in [0, +\infty) \). We consider the Euler/NS equations on \( T^3 \) with no external forcing, in the Sobolev framework of order \( n = 3 \). So, the Cauchy problem \((3.5)\) takes the form

\[ \text{Find } u \in C([0,T), \mathbb{H}^5_{\Sigma 0}) \cap C^1([0,T), \mathbb{H}^3_{\Sigma 0}) \text{ such that} \]

\[ \frac{du}{dt} = \nu \Delta u + \mathcal{P}(u,u) , \quad u(0) = u_0 . \]

The initial datum \( u_0 \) in \( \mathbb{H}^5_{\Sigma 0} \) (in fact, in \( \mathbb{H}^m_{\Sigma 0} \) for any real \( m \)) is chosen of this form:

\[ u_0 = \sum_{k=\pm a, \pm b, \pm c} u_{0k} e_k , \]

\( a := (1, 1, 0) , \quad b := (1, 0, 1) , \quad c := (0, 1, 1) ; \)
\( u_{0,\pm a} := (2\pi)^{3/2}(1, -1, 0) , \quad u_{0,\pm b} := (2\pi)^{3/2}(1, 0, -1) , \quad u_{0,\pm c} := (2\pi)^{3/2}(0, 1, -1) . \)

For \( \nu = 0 \), this initial datum has been considered by Behr, Nečas and Wu [2] as the origin of a possible blow-up for the Euler equations. More precisely, the authors of the cited work try a solution \( u \) of the Euler equations in the form of a power series \( u(t) = \sum_{i=0}^{+\infty} u_i t^i \), where the zero order term corresponds to the initial datum, and \( u_1, u_2, \ldots : T^3 \to \mathbb{R}^3 \) are determined recursively. By a means of \( C++ \) program, the authors compute the terms \( u_i \) for \( i = 1, \ldots, 35 \), allowing to construct the partial sums \( u_N(t) := \sum_{i=0}^{N} u_i t^i \) up to \( N = 35 \). A merely “experimental” analysis of these partial sums and of their \( \mathbb{H}^3_{\Sigma 0} \) norms brings the authors to conjecture that the exact solution \( u(t) \) of the Euler Cauchy problem blows up in \( \mathbb{H}^3_{\Sigma 0} \) for \( t \to T_* \), for some \( T_* \in (0.32, 0.35) \).

For subsequent use, we mention that Eq. \((7.3)\) implies

\[ \|u_0\|_m = \sqrt{3\pi^{3/2}2^m + 5} \]

for any real \( m \). This formula gives, for example,

\[ \|u_0\|_1 = 77.15... , \quad \|u_0\|_2 = 109.1... , \]

\[ 27 \]
\[ \|u_0\|_3 = 154.3..., \quad \|u_0\|_4 = 218.2..., \quad \|u_0\|_5 = 308.6... . \]

**Introducing our approach.** In this section we propose a different approach to the Euler Cauchy problem of \([2]\), that we apply as well to the NS case; so, we consider the problem \((7.2)\) \((7.3)\), for arbitrary \(\nu \in [0, +\infty)\).

We refer to the general setting developed in the present paper, using the spaces \(H^1_{\Sigma_0}, H^{2+1}_{\Sigma_0}, H^{3+2}_{\Sigma_0}\) with
\[ n = 3 . \tag{7.6} \]

This requires, amongst else, the numerical values of two constants \(K_3, G_3\) fulfilling for \(n = 3\) the “basic inequality” \((2.24)\) and the “Kato inequality” \((2.25)\). Due to the computations in \([15]\) \([16]\), these can be taken as follows:
\[ K_3 = 0.323 , \quad G_3 = 0.438 . \tag{7.7} \]

To illustrate our setting, we start with a very elementary result.

**A simple sufficient condition for global existence (and exponential decay).**

According to Proposition 5.2, the solution \(u\) of the NS Cauchy problem \((7.2)\) \((7.3)\) is global and exponentially decaying for \(t \to +\infty\), if
\[ \nu \geq G_3\|u_0\|_3 = 67.58... . \tag{7.8} \]

(in the last passage, we have used the numerical values in \((7.5)\) \((7.7)\) for \(G_3\) and \(\|u_0\|_3\) \((3)\). As shown hereafter, a more refined application of our setting for approximate solutions significantly improves the above condition: in fact, in the next pages, combining this setting with the Galerkin method we will be able to infer global existence for \(\nu \geq 8\).

**Going on in our approach: Galerkin approximants.** The idea developed in the sequel, both in the Euler case (\(\nu = 0\)) and in the NS case (\(\nu > 0\)), is the following: to compute numerically the Galerkin approximate solution \(u_G\) for a suitable set of modes \(G\); to construct for it error and growth estimators, in the Sobolev norms of orders 3 or 4; to solve numerically the control equations \((4.30)\) \((4.31)\) for an unknown function \(R_3 : \left[0,T_c\right) \to [0, +\infty)\). After finding the solution of this control problem, we can grant on theoretical grounds that the solution \(u\) of the Euler/NS Cauchy problem \((7.2)\) exists at least up to time \(T_c\), and that \(\|u(t) - u_G(t)\|_3 \leq R_3(t)\) for \(t \in \left[0,T_c\right)\).

\(^2\)For completeness, we mention a criterion for NS global existence in \(H^1_{\Sigma_0}(T^3)\), discussed in \([14]\): this can be written as \(\nu \geq \|u_0\|_1/0.407\), where \(u_0\) is an arbitrary initial datum in \(H^1_{\Sigma_0}\). With the datum \(u_0\) in \((7.3)\) (and the value of \(\|u_0\|_1\) in \((7.5)\)), we conclude that \([14]\) ensures global existence in \(H^1_{\Sigma_0}\) for \(\nu \geq 189.5...\). By a known regularity theorem about NS equations \([10]\), this result of global existence in \(H^1_{\Sigma_0}\) also implies global existence in \(H^3_{\Sigma_0}\); however, the condition \(\nu \geq 189.5...\) arising from the \(H^1\) setting of \([14]\) is manifestly weaker than the condition \((7.8)\).
Our computation has been performed using Mathematica on a PC, with the relatively small set of 150 modes

\[ G := S \cup -S ; \quad -S := \{ -k \mid k \in S \} ; \quad (7.9) \]

\[ S := \{(0, 0, 2), (0, 1, -3), (0, 1, 1), (0, 1, 3), (0, 2, 0), (0, 2, 2), (0, 3, -1), (0, 3, 1), (0, 3, 3), (1, -3, -2), (1, -3, 0), (1, -3, 2), (1, -2, -3), (1, -2, -1), (1, -2, 1), (1, -2, 3), (1, -1, -2), (1, -1, 0), (1, 0, -3), (1, 0, 1), (1, 0, 3), (1, 1, -2), (1, 1, 0), (1, 1, 2), (1, 2, -3), (1, 2, -1), (1, 2, 1), (1, 2, 3), (1, 3, -2), (1, 3, 0), (1, 3, 2), (2, -3, -3), (2, -3, -1), (2, -3, 1), (2, -2, 2), (2, -2, 0), (2, -1, -3), (2, -1, -1), (2, -1, 1), (2, -1, 3), (2, 0, 0), (2, 0, 2), (2, 1, -3), (2, 1, -1), (2, 1, 1), (2, 1, 3), (2, 2, -2), (2, 2, 0), (2, 3, -3), (2, 3, -1), (2, 3, 1), (2, 3, 3), (3, -3, -2), (3, -3, 2), (3, -2, -3), (3, -2, -1), (3, -2, 1), (3, -2, 3), (3, -1, -2), (3, -1, 0), (3, -1, 2), (3, 0, -1), (3, 0, 1), (3, 0, 3), (3, 1, -2), (3, 1, 0), (3, 1, 2), (3, 2, -3), (3, 2, -1), (3, 2, 1), (3, 2, 3), (3, 3, -2), (3, 3, 0), (3, 3, 2) \} .

The results we present here are somehow provisional; we plan to attack the problem by more powerful numerical tools in a future work, using for \( G \) a larger set of modes.

**A sketch of the operations to be done.** The list of such operations is the following:

(i) First of all, one chooses a value \( \nu \in [0, +\infty) \) for the viscosity, and a finite time interval \([0, T_1]\) for the numerical computation of the Galerkin solution.

(ii) The Galerkin solution \( u_G \) for the set of modes \( G \) in (7.9) and for the initial datum \( u_0 \) is found numerically on the chosen time interval \([0, T_1]\). More precisely, one solves numerically the system of differential equations (6.15) for the Fourier components \( (\gamma_k)_{k \in G} \) of \( u_G \), with the initial conditions \( u_{0k} \) corresponding to Eq. (7.3). We recall (see Lemma 6.3) that the general theory of the Galerkin solutions for zero external forcing ensures the global existence of \( u_G \) (and its exponential decay, if \( \nu > 0 \)); thus, from a theoretical viewpoint there is no obstruction to the computation of \( u_G \) on any finite interval \([0, T_1]\).

(iii) We apply to \( u_G \) on \([0, T_1]\) the framework of the present paper for the approximate Euler/NS solutions using (we repeat it) the spaces \( \mathbb{H}^n_{\Sigma_0}, \mathbb{H}^{n+1}_{\Sigma_0} \) and \( \mathbb{H}^{n+2}_{\Sigma_0} \) with \( n = 3 \).

(iv) Some important characters in our approach are the norms

\[ D_n(t) := \| u_G(t) \|_n = \sqrt{\sum_{k \in G} |k|^{2n} |\gamma_k(t)|^2} \quad (n = 3, 4), \quad (7.10) \]

which can be obtained from the numerical values \( \gamma_k(t) \) of the Fourier components.

Further, we need a datum error estimator \( \delta_0 \) and a differential error estimator \( \epsilon_3 \). On the other hand the datum error is zero in this case, since the initial condition \( u_0 \)
is in the subspace $\mathbb{H}^G_{\gamma_0}$ spanned by the chosen set (7.9) of modes; thus, we can take
$$\delta_3 = 0.$$ (7.11)

As for the differential error estimator, we take the precise expression coming from Eqs. (6.24) (6.27) (taking into account that the forcing $f$ is zero in this case); these yield the expression
$$\epsilon_3(t) = \sqrt{\sum_{k \in dG} |k| |p_k(t)|^2}$$ (7.12)
where $dG := (G+G) \setminus (G \cup \{0\})$ and $p_k(t) := -i (2\pi)^{-d/2} \sum_{h \in G} \gamma_h(t) \cdot (k-h) \gamma_{k-h}(t)$, as in Eq. (6.22). The computation of $\epsilon_3(t)$ following Eq. (7.12) is rather expensive: the set $dG$ consists of 929 modes and, for each one of them, one must perform the nontrivial computation of $p_k(t)$. (For these reasons, a computation with a set of modes much larger than the $G$ in (7.9) would suggest to replace this $\epsilon_3$ with a rougher estimator, coming from Eqs. (6.26) (6.27)).

(v) Now we consider the control equations (4.30) (4.31), taking the form
$$\frac{dR_3}{dt} = -\nu R_3 + (G_3 D_3 + K_3 D_4) R_3 + G_3 R_3^2 + \epsilon_3 ,$$ (7.13)
$$R_3(0) = 0 ,$$ (7.14)
with $K_3, G_3$ as in Eq. (7.7) and $D_3(t), D_4(t), \epsilon_3(t)$ as in Eqs. (7.10) (7.12). The unknown is a function $R_3 \in C^1([0,T_c), [0, +\infty))$, with $0 < T_c \leq T_I$; this is determined numerically (with a package for self-adaptive integration, allowing to detect a possible blow-up of $R_3$; in this case, $T_c$ is the blow-up time).

Once we have $R_3$, the general theory allows to state the following. (a) The maximal solution $u$ of the Euler/NS Cauchy problem (7.2) (7.3) has a domain containing $[0, T_c)$, and
$$\|u(t) - u_G(t)\|_3 \leq R_3(t) \quad \text{for} \quad t \in [0, T_c)$$ (7.15)
(see Proposition 4.3).
(b) Suppose $\nu > 0$, and
$$(D_3 + R_3)(t_1) \leq \frac{\nu}{G_3} \quad \text{for some} \quad t_1 \in [0, T_c) .$$ (7.16)
Then, the solution $u$ of the NS Cauchy problem (7.2) is global, and
$$\|u(t)\|_3 \leq \frac{(D_3 + R_3)(t_1) e^{-\nu(t-t_1)}}{1 - G_3(D_3 + R_3)(t_1) e_\nu(t-t_1)} \quad \text{for} \quad t \in [t_1, +\infty) ,$$ (7.17)
with $e_\nu$ as in Eq. (5.9) (see Corollary 5.4).
Case $\nu = 0$. First of all, the Galerkin equations (6.15) have been solved for the set of modes (7.9) on a time interval of length $T_I = 2$. (This required, approximately, 15 seconds of CPU time on our machine.) In Figures 1a, 1b and 1c we report, as examples, the graphs of $|\gamma_k(t)|$ for $t \in [0, 2)$, in the cases $k = (1, 1, 0)$, $k = (0, 0, 2)$ and $k = (0, 1, -3)$, respectively. (Of course, we could consider many alternative choices, such as plotting the norms $|\gamma_k(t)|$ for other modes, or the real parts of the components $\gamma_k^r(t)$ ($r = 1, 2, 3$), or the imaginary parts of the same components.)

As a supplementary information, we mention that the graphs of $|\gamma_k(t)|$ are identical in the three cases $k = (1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$.

After computing numerically all the $\gamma_k(t)$ (for $k \in G$), one obtains from Eqs. (7.10) and (7.12) the norms $D_n(t) := \|u_G(t)\|_n$ ($n = 3, 4$) and the differential error estimator $\epsilon_3(t)$ (to give some more detail, we mention that the functions $D_3(t)$, $D_4(t)$ and $\epsilon_3(t)$ have been computed for a set of sample values of $t \in [0, 2)$, and then interpolated by means of the MATHEMATICA algorithms. The computation of $\epsilon_3(t)$ at each sample value $t$ is rather expensive, for it involves a sum on the large set $dG$; the CPU time is about 15 seconds for each $t$ and, for this reason, we have used only 30 sample values).

The final step is the numerical solution of the Cauchy problem (4.30) (4.31) for the unknown function $\mathcal{R}_3(t)$ (a task performed almost instantaneously by our PC). The MATHEMATICA self-adaptive routines for ODEs indicate a divergence of $\mathcal{R}_3(t)$ for $t \to T_\ast$, with $T_\ast = 0.06666...$. Figure 1f gives the graph of $\mathcal{R}_3(t)$ for $t \in [0, T_\ast)$. The conclusion of these computations is the following: the solution $u$ of the Euler Cauchy problem (7.2) (7.3) has a domain containing $[0, T_\ast)$, and its distance from $u_G$ is bounded by $\mathcal{R}_3(t)$ on this interval.

Some remarks on the $\nu = 0$ case. We have already mentioned that the blow-up of $u(t)$ is conjectured in [2] for $t \to T_\ast \in (0.32, 0.35)$. The basis of this conjecture is an experimental analysis of the first 35 terms in the power series (in time) solving formally the Cauchy problem; in principle, this analysis does not even prove existence of $u(t)$ at any specified time $t < T_\ast$. Our lower bound $T_\ast = 0.06666...$ for the interval of existence of $u$ is about 1/5 of the blow-up time $T_\ast$ suggested by [2], but it relies on an analytic theory for approximate solutions, their errors, etc., summarized by Proposition 4.5; in this sense, it is theoretically grounded.

Perhaps, our method could give a sensibly larger lower bound $T_\ast$ on the time of existence, when implemented with a set $G$ of Galerkin modes much larger than (7.9). Alternatively, one could apply our theoretical framework using as an approximate solution the partial sum $u_N(t) := \sum_{i=1}^N u_it^i$, with $N = 35$ as in [2] or with a larger $N$. Both tasks require much more expensive numerical computations (to be done
with more appropriate hardware and software): we plan to do this elsewhere. One cannot exclude that an attack with more powerful devices could finally result into a theoretically grounded lower bound \( T_c \) larger than the suspected blow-up time \( T_c \) of [2]; however, at present this is just a hope.

**Cases \( \nu = 3 \) and \( \nu = 7 \).** We use again the Galerkin solution \( u_G \) with \( G \) as in (7.9). Due to the positivity of \( \nu \), all components \( \gamma_k(t) \) of the Galerkin solution decay exponentially for \( t \to +\infty \) (recall Lemma 6.4); the same happens of the norms \( \mathcal{D}_3(t) \), \( \mathcal{D}_4(t) \) and of the differential error estimator \( \epsilon_3(t) \), which are essential objects for our purposes.

The system (6.15) for the Galerkin components \( \gamma_k(t) \) has been solved numerically on a time interval of length \( T_t = 1 \) (which required a CPU time of about 15 seconds for \( \nu = 3 \), and 25 seconds for \( \nu = 7 \)). Subsequently, \( \mathcal{D}_3(t) \), \( \mathcal{D}_4(t) \) and \( \epsilon_3(t) \) have been computed from the components \( \gamma_k(t) \) and Eqs. (7.10) (7.12) (indeed, some interpolation has been done as in the case \( \nu = 0 \): as in that case, for the computation of \( \epsilon_3(t) \) we have used only 30 sample values of \( t \) in \([0, 1]\), with a CPU time of about 15 seconds for each one).

For both the above values of \( \nu \), the final step has been the (very fast) numerical solution of the Cauchy problem (1.30) (1.31) for the unknown function \( R_3(t) \). According to the MATHEMATICA routines for ODEs, \( R_3(t) \) diverges for \( t \to T_c \), where \( T_c = 0.09025 \ldots \) for \( \nu = 3 \), and \( T_c = 0.2386 \ldots \) for \( \nu = 7 \). We repeat that these results grant existence on a domain \([0, T_c]\) for the solution \( u \) of the NS Cauchy problem (7.2) (7.3), and the bound (7.15) on this interval.

As examples, in Figures 2a-2f we have reported some details on computations for \( \nu = 7 \). More precisely, Figures 2a, 2b and 2c are the graphs of \( |\gamma_k(t)| \) for \( t \in [0, 1] \), in the cases \( k = (1, 1, 0) \), \( k = (0, 0, 2) \) and \( k = (0, 1, -3) \), respectively. (In fact, the graphs of \( |\gamma_k(t)| \) for \( k = (1, 0, 1) \) and \( k = (0, 1, 1) \) are identical to the graph of the case \( k = (1, 1, 0) \).) Figures 2d and 2e are the graphs of \( \mathcal{D}_3(t) \) and \( \epsilon_3(t) \), for \( t \in [0, 1] \). Figure 2f gives the graph of \( R_3(t) \) that, as anticipated, diverges for \( t \to T_c = 0.2386 \ldots \).

**Case \( \nu = 8 \).** Again, we have used the Galerkin solution \( u_G \) with \( G \) as in (7.9). All the components \( \gamma_k(t) \), the norms \( \mathcal{D}_3(t) \), \( \mathcal{D}_4(t) \) and the differential error estimator \( \epsilon_3(t) \) decay exponentially for \( t \to +\infty \). The system (6.15) for the Galerkin components \( \gamma_k(t) \) has been solved numerically on a time interval of length \( T_t = 1 \) (which required a CPU time of about 25 seconds). The forthcoming Figures 3a,3b,3c are the graphs of \( |\gamma_k(t)| \) for \( t \in [0, 1] \), \( k = (1, 1, 0) \), \( k = (0, 0, 2) \) and \( k = (0, 1, -3) \), respectively. Subsequently, the norms \( \mathcal{D}_4(t) \) (\( n = 3, 4 \)) and the error \( \epsilon_3(t) \) have been computed from the components \( \gamma_k(t) \) and Eqs. (7.10) (7.12) (making some interpolations, as in the previous cases). Figures 2d and 2e are the graphs of \( \mathcal{D}_3(t) \) and \( \epsilon_3(t) \), for \( t \in [0, 1] \).

The final step has been the (very fast) numerical solution of the Cauchy problem
for the unknown function $R_3(t)$. Differently from all the previous cases, the numerical solution $R_3(t)$ determined by MATHEMATICA is defined on the whole interval $[0, 1)$; its graph is reported in Figure 3f which suggests, via some extrapolation, that $R_3(t)$ should be defined for all $t \in [0, +\infty)$, with $R_3(t) \to 0^+$ for $t \to +\infty$; of course, this would imply global existence for the solution $u$ of the NS Cauchy problem (7.2) (7.3).

However, global existence of $u$ can even be inferred without extrapolating the behavior of $R_3$ outside the interval $[0, 1)$. In fact, global existence is granted if the condition (7.16) $(D_3 + R_3)(t_1) \leq \nu/G_3$ holds at any instant $t_1 > 0$; in the present case $\nu/G_3 = 18.26\ldots$, and the numerical computation performed in the interval $[0, 1)$ shows that (7.16) is satisfied for any $t_1 \in [0.1567\ldots, 1)$. In conclusion, we can take for granted that we have global existence for the solution $u$ of the NS Cauchy problem. Of course, $\|u(t) - u_4(t)\|_3$ is bounded by the numerically computed function $R_3(t)$, for $t \in [0, 1)$. After choosing a $t_1 \in [0.1567, +\infty)$, we obtain as well a bound of the form (5.14) for $\|u(t)\|_3$, which also implies $\|u(t)\|_3$ to vanish exponentially for $t \to +\infty$.

For example, let us choose $t_1 = 0.9$. We have $D_3(0.9) = 8.580\ldots \times 10^{-5}$, $R_3(0.9) = 0.06100\ldots$; with the already known value $G_3 = 0.438$, Eq. (5.14) gives

$$\|u(t)\|_3 \leq \frac{0.0614 e^{-8(t-0.9)}}{1 + 0.00335 e^{-8(t-0.9)}} \leq 0.0614 e^{-8(t-0.9)} \quad \text{for } t \in [0.9, +\infty) . \quad (7.18)$$

(Here the first inequality follows directly from (5.14), recalling that $e_8(t) = (1 - e^{-8t})/8$; the second inequality is obvious.)

**Summary of the previous results, and final comments.** Our method to treat the Galerkin approximant $u_G$, with $G$ as in (7.9), has given the following results about the Euler/NS Cauchy problem (7.2) (7.3).

a) $\nu = 0, 3, 7$: we can grant existence of the solution $u$ of (7.2) (7.3) on an interval containing $[0, T_c)$, with $T_c = 0.06666, 0.09025, 0.2386$, respectively, for these three choices of $\nu$. We have $\|u(t) - u_G(t)\|_3 \leq R_3(t)$ for $t \in [0, T_c)$, where $R_3$ is computed numerically solving (4.30) (4.31) (for $\nu = 0$ and $\nu = 7$, the graph of $R_3$ is reported in Figures 1f and 2f).

b) $\nu = 8$: we can grant global existence for the solution $u$ of (7.2) (7.3). For $t \in [0, 1)$ we have a bound $\|u(t) - u_G(t)\|_3 \leq R_3(t)$, with $R_3$ obtained again from the numerical solution of (4.30) (4.31); the graph of this function is reported in Figure 3f. For $t \in [0.9, +\infty)$ we have a bound of the form (7.18) on $\|u(t)\|_3$, decaying exponentially for large $t$.

By extrapolation, we are led to conjecture that results similar to (a) should be obtained for all $\nu \in [0, \nu_{cr})$, while results similar to (b) should be obtained for all $\nu \in [\nu_{cr}, +\infty)$, for some $\nu_{cr} \in (7, 8)$. (From a qualitative viewpoint, this is just the behavior described by Lemma 5.5 on the analytical solution of the control inequality.)
However, here we are applying the control equalities with the tautological growth and error estimators, given directly by the numerical solution of the Galerkin equations; these are more precise than the simple analytical estimators considered in the cited lemma.)
Figure 1a. $\nu = 0$. Graph of $|\gamma_k(t)|$ when $k = (1, 1, 0)$, for $t \in [0, 2)$.

Figure 1b. $\nu = 0$. Graph of $|\gamma_k(t)|$ when $k = (0, 0, 2)$, for $t \in [0, 2)$.

Figure 1c. $\nu = 0$. Graph of $|\gamma_k(t)|$ when $k = (0, 1, -3)$, for $t \in [0, 2)$.

Figure 1d. $\nu = 0$. Graph of $D_3(t) = \|u_G(t)\|_3$, for $t \in [0, 2)$. One has $D_3(0) = 154.3..., D_3(0.02) = 156.4..., D_3(0.04) = 162.7..., D_3(0.06) = 172.7...$.

Figure 1e. $\nu = 0$. Graph of $\varepsilon_3(t)$, for $t \in [0, 2)$. One has $\varepsilon_3(0.02) = 9.027..., \varepsilon_3(0.04) = 36.17..., \varepsilon_3(0.06) = 81.65...$.

Figure 1f. $\nu = 0$. Graph of $R_3(t)$; this function diverges for $t \to T_c = 0.06666...$. One has $R_3(0) = 0$, $R_3(0.02) = 0.1439..., R_3(0.04) = 4.685..., R_3(0.06) = 182.3...$. 

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Figure 2a. $\nu = 7$. Graph of $|\gamma_k(t)|$ when $k = (1,1,0)$, for $t \in [0,1)$.

Figure 2b. $\nu = 7$. Graph of $|\gamma_k(t)|$ when $k = (0,0,2)$, for $t \in [0,1)$.

Figure 2c. $\nu = 7$. Graph of $|\gamma_k(t)|$ when $k = (0,1,-3)$, for $t \in [0,1)$.

Figure 2d. $\nu = 7$. Graph of $D_3(t)$, for $t \in [0,1)$. One has $D_3(0) = 154.3...$, $D_3(0.025) = 109.5...$, $D_3(0.07) = 58.40...$, $D_3(0.15) = 18.89...$, $D_3(0.23) = 6.153...$.

Figure 2e. $\nu = 7$. Graph of $\epsilon_3(t)$, for $t \in [0,1)$. One has $\epsilon_3(0) = 0$, $\epsilon_3(0.025) = 2.024...$, $\epsilon_3(0.07) = 0.5788...$, $\epsilon_3(0.15) = 0.01089...$, $\epsilon_3(0.23) = 1.338... \times 10^{-4}$.

Figure 2f. $\nu = 7$. Graph of $R_3(t)$; this function diverges for $t \to T_c = 0.2386...$. One has $R_3(0) = 0$, $R_3(0.07) = 2.096...$, $R_3(0.15) = 20.90...$, $R_3(0.23) = 265.0...$. 

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The condition (7.16) for global existence of the NS Cauchy problem, i.e., $(D_3 + R_3)(t_1) \leq \nu/G_3$, is satisfied for any $t_1 \in [0, 1)$. One has $D_3(0) = 154.3\ldots$, $D_3(0.2) = 6.280\ldots$, $D_3(0.4) = 0.2559\ldots$, $D_3(0.6) = 0.01043\ldots$, $D_3(0.9) = 8.580\ldots \times 10^{-5}\ldots$.
A Some comparison lemmas of the Čaplygin type

Let \( \mathcal{I} \in (0, +\infty) \); suppose we are given

\[ \psi \in C(\mathbb{R} \times [0, \mathcal{I}), \mathbb{R}) \text{,} \quad (s, t) \mapsto \psi(s, t) \text{ such that} \quad \frac{\partial \psi}{\partial s} \in C(\mathbb{R} \times [0, \mathcal{I}), \mathbb{R}) \; ; \quad (A.1) \]

\[ s_0 \in \mathbb{R} \; . \quad (A.2) \]

Under the above assumptions, the Cauchy problem

\[ \frac{dS}{dt}(t) = \psi(S(t), t) \; , \quad S(0) = s_0 \; . \quad (A.3) \]

has a unique maximal (i.e., nonextendable) solution \( S \in C^1([0, T_S), \mathbb{R}) \). (Of course \( T_S \in (0, \mathcal{I}) \); later on we give conditions under which \( T_S = \mathcal{I} \).)

The following is a known result, of the Čaplygin type: see [9] or [11].

A.1 Lemma. Suppose there is a function \( W \in C([0, \mathcal{I}), \mathbb{R}) \) such that

\[ \frac{d^+ W}{dt}(t) \leq \psi(W(t), t) \text{ for } t \in [0, \mathcal{I}) \; , \quad W(0) \leq s_0 \; . \quad (A.4) \]

(with \( d^+/dt \) the right lower Dini derivative, see Eq. (2.2)). Then

\[ W(t) \leq S(t) \text{ for } t \in [0, T_S) \; . \quad (A.5) \]

A straightforward variant of the previous result is the following.

A.2 Lemma. Suppose there is a function \( R \in C([0, \mathcal{I}), \mathbb{R}) \) such that

\[ \frac{d^+ R}{dt}(t) \geq \psi(R(t), t) \text{ for } t \in [0, \mathcal{I}) \; , \quad R(0) \geq s_0 \; , \quad (A.6) \]

(with \( d^+/dt \) the right upper Dini derivative, see Eq. (2.3)). Then

\[ R(t) \geq S(t) \text{ for } t \in [0, T_S) \; . \quad (A.7) \]

Proof. We consider the function \(-R\), recalling that \( d^+ (-R)/dt = -d^+ R/dt \); from \( (A.6) \) we infer

\[ \frac{d^+ (-R)}{dt}(t) \leq -\psi(R(t), t) \text{ for } t \in [0, \mathcal{I}) \; , \quad -R(0) \leq -s_0 \; . \quad (A.8) \]

On the other hand, the function \( -S \in C^1([0, T_S), \mathbb{R}) \) is the maximal solution of the Cauchy problem

\[ \frac{d(-S)}{dt}(t) = -\psi(S(t), t) \; , \quad -S(0) = -s_0 \; . \quad (A.9) \]
Therefore, Lemma A.1 with \( \mathcal{W} \) replaced by \(-\mathcal{R}\) (and other obvious substitutions) gives
\[
-\mathcal{R}(t) \leq -\mathcal{S}(t) \quad \text{for } t \in [0, T_S),
\]
yielding the thesis (A.7). \(\square\)

**A.3 Lemma.** Suppose there are functions \( \mathcal{W}, \mathcal{R} \in C([0, T], \mathcal{R}) \) fulfilling Eqs. (A.4), (A.6) respectively. Then
\[
T_S = T, \quad \mathcal{W}(t) \leq \mathcal{S}(t) \leq \mathcal{R}(t) \quad \text{for } t \in [0, T). \tag{A.10}
\]

**Proof.** From Lemmas A.1 and A.2 we infer
\[
\mathcal{W}(t) \leq \mathcal{S}(t) \leq \mathcal{R}(t) \quad \text{for } t \in [0, T_S); \tag{A.11}
\]
we claim that this inequality implies
\[
T_S = T. \tag{A.12}
\]
In fact, whenever \( T_S \) is finite, the nonextendability of \( \mathcal{S} \) beyond this time implies that \( \mathcal{S} \) is unbounded in any left neighborhood of \( T_S^- \); on the other hand, if it were \( T_S < T \), Eq. (A.11) would ensure the boundedness of \( \mathcal{S} \) on \( [0, T_S) \).

Now, the combination of (A.11) and (A.12) gives the thesis (A.10). \(\square\)

If we forget the solution \( \mathcal{S} \) of the Cauchy problem (A.3), the inequality (A.10) becomes \( \mathcal{W}(t) \leq \mathcal{R}(t) \) for \( t \in [0, T); \) this is just the statement in Eq. (4.23) of Lemma 4.3 which is thus recognized as a weaker version of Lemma A.3.

**B Appendix. Proof of Lemma 6.4 on the Galerkin approximants**

Let \( \nu \in [0, +\infty), f \in C([0, +\infty), \mathbb{D}'_{\Sigma^0}) \) and \( u_0 \in \mathbb{D}'_{\Sigma^0} \). In the finite dimensional space \( H^G_{\Sigma^0} \), we consider the maximal solution \( u_G \) of the Cauchy problem (6.7), here reproduced for the reader’s convenience:
\[
\frac{du_G}{dt} = \nu \Delta u_G + \mathcal{P}^G_P(u_G, u_G) + \mathcal{P}^G f, \quad u_G(0) = \mathcal{P}^G u_0.
\]
For the moment, the maximal time of existence \( T_G \) is not known; one of our aims is to show that \( T_G = +\infty \). Let us proceed to prove this claim and all the other statements of Lemma 6.4; our arguments are divided in several steps.
Step 1 ("energy balance equation"). Everywhere on \([0, T_G)\), one has
\[
\frac{d}{dt} \|u_G\|_{L^2}^2 = 2\nu \langle \Delta u_G | u_G \rangle_{L^2} + 2 \langle \mathcal{P}^G f | u_G \rangle_{L^2} .
\] (B.1)

In fact,
\[
\frac{d}{dt} \|u_G\|_{L^2}^2 = 2\langle \frac{du_G}{dt} | u_G \rangle_{L^2}
= 2\nu \langle \Delta u_G | u_G \rangle_{L^2} + 2 \langle \mathcal{P}^G u_G | u_G \rangle_{L^2} + 2 \langle \mathcal{P}^G f | u_G \rangle_{L^2} .
\] (B.2)

On the other hand,
\[
\langle \mathcal{P}^G u_G | u_G \rangle_{L^2} = \langle \mathcal{P} u_G | u_G \rangle_{L^2} = 0 ;
\] (B.3)

in the first passage above, the projection \( \mathcal{P} \) on \( \mathbb{H}_G^G \) is omitted since we are taking the inner product with \( u_G = u_G(t) \in \mathbb{H}_G^G \); the second passage reflects the well known identity \( \langle \mathcal{P}(v, w) | w \rangle_{L^2} = 0 \), holding for all zero mean, divergence free vector fields \( v, w \) on \( \mathbb{T}^d \) sufficiently regular to give meaning to the indicated inner product (see, e.g., [16], Lemma 2.3). Eqs. (B.2), (B.3) yield the thesis (B.1).

Step 2. Everywhere on \([0, T_G)\), one has
\[
\frac{d}{dt} \|u_G\|_{L^2}^2 \leq -2\nu \|u_G\|_{L^2}^2 + 2 \|\mathcal{P}^G f\|_{L^2} \|u_G\|_{L^2} .
\] (B.4)

This follows from the energy equation (B.1), combined with the inequalities
\[
\langle \Delta u_G | u_G \rangle_{L^2} \leq -\|u_G\|_{L^2}^2 , \quad \langle \mathcal{P}^G f | u_G \rangle_{L^2} \leq \|\mathcal{P}^G f\|_{L^2} \|u_G\|_{L^2} .
\] (B.5)

The first of these relations follows using (2.19), with \( n = 0 \); the second one is an application of the Cauchy-Schwarz inequality for \( \langle \ | \ \rangle_{L^2} \).

Step 3. Everywhere on \([0, T_G)\), one has
\[
\frac{d^+}{dt} \|u_G\|_{L^2} \leq -\nu \|u_G\|_{L^2} + \|\mathcal{P}^G f\|_{L^2} .
\] (B.6)

with \( d^+/dt \) the upper Dini derivative. Let us first prove Eq. (B.6) at a time \( t_0 \in [0, T_G) \) such that \( u_G(t_0) \neq 0 \). In this case, \( \|u_G(t)\|_{L^2} \neq 0 \) for all \( t \) in an interval \( I \) containing \( t_0 \); in this interval the function \( t \mapsto \|u_G\|_{L^2} \) is \( C^1 \), and we have
\[
\frac{d^+}{dt} \|u_G\|_{L^2} = \frac{d}{dt} \|u_G\|_{L^2} = \frac{1}{2\|u_G\|_{L^2}^2} \frac{d}{dt} \|u_G\|_{L^2}^2 \leq -\nu \|u_G\|_{L^2} + \|\mathcal{P}^G f\|_{L^2} ,
\]

the last passage following from (B.4). Now, consider an instant \( t_0 \) such that \( u_G(t_0) = 0 \). Due to a general result already mentioned (see the comments before Eq. (4.18)), we can write
\[
\left. \frac{d^+}{dt} \right|_{t_0} \|u_G\|_{L^2} \leq \left. \frac{du_G}{dt} \right|_{t_0} \|u_G\|_{L^2} ;
\]
on the other hand, Eq. (6.7) and the assumption \( u_G(t_0) = 0 \) give \( (du_G/dt)(t_0) = \mathcal{P}^G f(t_0) \), so

\[
\left. \frac{d^+}{dt} \right|_{t_0} \| u_G \|_{L^2} \leq \| \mathcal{P}^G f(t_0) \|_{L^2} ;
\]

this is just the thesis (B.6), at the instant under consideration.

Step 4. One has

\[
\| u_G(t) \|_{L^2} \leq \left( \| \mathcal{P}^G u_0 \|_{L^2} + \int_0^t ds \, e^{\nu s} \| \mathcal{P}^G f(s) \|_{L^2} \right) e^{-\nu t} \quad \text{for} \ t \in [0, T_G) . \tag{B.7}
\]

Let us consider the continuous function \( t \in [0, T_G) \mapsto \| u_G(t) \|_{L^2} ; \) due to Step 3 and to \( u_G(0) = \mathcal{P}^G u_0 \), we have

\[
\frac{d^+}{dt} \| u_G(t) \|_{L^2} \leq \psi(\| u_G(t) \|_{L^2}, t) , \quad \| u_G(t) \|_{L^2} = \| \mathcal{P}^G u_0 \|_{L^2} \tag{B.8}
\]

where

\[
\psi : \mathbb{R} \times [0, +\infty) \to \mathbb{R} , \quad \psi(s, t) := -\nu s + \| \mathcal{P}^G f(t) \|_{L^2} . \tag{B.9}
\]

On the other hand, the Cauchy problem

\[
\frac{dS}{dt}(t) = \psi(S(t), t) , \quad S(0) = \| \mathcal{P}^G u_0 \|_{L^2} \tag{B.10}
\]

has the global solution

\[
S(t) := \left( \| \mathcal{P}^G u_0 \|_{L^2} + \int_0^t ds \, e^{\nu s} \| \mathcal{P}^G f(s) \|_{L^2} \right) e^{-\nu t} \quad \text{for} \ t \in [0, +\infty) . \tag{B.11}
\]

So, by the comparison results reviewed in the previous Appendix (see, in particular, Lemma A.2), we have

\[
\| u_G(t) \|_{L^2} \leq S(t) \quad \text{for} \ t \in [0, T_G) ;
\]

this is just the thesis (B.7).

Step 5. One has \( T_G = +\infty \) (which justifies statement (6.8) in the lemma under proof). If it were \( T_G < +\infty \), we would have \( \limsup_{t \to T_G} \| u_G(t) \|_{L^2} = +\infty ; \) this would be contradicted by Eq. (B.7), implying \( \sup_{t \in [0, T_G]} \| u_G(t) \|_{L^2} \leq \| \mathcal{P}^G u_0 \|_{L^2} + \int_0^{T_G} ds \, e^{\nu s} \| \mathcal{P}^G f(s) \|_{L^2} < +\infty . \)

Step 6. Proof of statement (6.9) in the lemma, under the assumption of zero forcing.

We use the results of Steps 1 and 4, putting therein \( T_G = +\infty . \) For \( f = 0 \) and \( \nu = 0 , \) the energy balance equation (B.1) takes the form \( (d/dt) \| u_G \|_{L^2}^2 = 0 ; \) thus,

\[
\| u_G(t) \|_{L^2} = \| u_G(0) \|_{L^2} = \| \mathcal{P}^G u_0 \|_{L^2} .
\]
for all $t \in [0, +\infty)$. For $f = 0$ and $\nu > 0$, Eq. (B.7) gives
\[
\|u_G(t)\|_{L^2} \leq \|P_G u_0\|_{L^2} e^{-\nu t},
\]
again for $t \in [0, +\infty)$. The above two equations correspond to the content of (6.9).

**Step 7. Proof of statements (6.10)-(6.12) in the lemma.** Statement (6.10) is just Eq. (B.7), with $T_G = +\infty$. Having proved (6.10), statements (6.11) (6.12) are obvious.

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