QUANTUM MAJORIZATION ON SEMIFINITE VON NEUMANN ALGEBRAS

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Abstract. We extend Gour et al’s characterization of quantum majorization via conditional min-entropy to the context of semifinite von Neumann algebras. Our method relies on a connection between conditional min-entropy and operator space projective tensor norm for injective von Neumann algebras. This approach also connects the tracial Hahn-Banach theorem of Helton, Klep and McCullough to noncommutative vector-valued $L_1$-space.

1. Introduction

Majorization is a fundamental tool introduced by Hardy, Littlewood, and Polya [Har29] that finds application in various fields [MOA79]. Among the different motivations for majorization, the core idea is a notion of “disorder”. For example, a probability distribution is majorized by another if it is less deviated from the uniform distribution. Recently, Gour, Jennings, Buscemi, Duan, and Marvian in [GJB+18] use the concept of “quantum majorization” to accommodate the ordering of states and processes in quantum mechanical systems.

Let $H$ be a finite dimensional Hilbert space and $B(H)$ be space of the bounded operators acting on $H$. A density operator $\rho \in B(H)$ (called a state on the quantum system $H$ in the quantum information theory literature) is positive and has trace 1. The process between quantum systems is modeled by completely positive trace preserving maps (also called quantum channels) which map density operators to density operators. For two bipartite density operators $\rho$ and $\sigma$ on the tensor product Hilbert space $H_A \otimes H_B$, $\sigma$ is said to be quantum majorized by $\rho$ if there exists a linear completely positive trace preserving (CPTP) map $\Phi : B(H_B) \to B(H_B)$ such that $\sigma = id \otimes \Phi(\rho)$. This concept has been studied in different contexts under various guises [Shm05, Che09, Bus12, BDS14, Jen16]. Intuitively, quantum majorization describes the disorder observed from the $B$ system. This can be witnessed from the data processing inequality of conditional entropy,

$$H(A|B)_\rho \leq H(A|B)_{id \otimes \Phi(\rho)} = H(A|B)_\sigma,$$

where $H(A|B) := H(\rho) - H(\tau \otimes id(\rho))$ and $H(\rho) = -\tau(\rho \log \rho)$ is the von Neumann entropy. The conditional entropy $H(A|B)_\rho$ describes the uncertainty of the bipartite density...
operator $\rho$ given its information on the $B$ system [HOW05]. The data processing inequality says such uncertainty is monotone non-decreasing under quantum majorization. As a converse to data processing inequality, Gour and his coauthors proved the following characterization of quantum majorization using conditional min-entropy $H_{\min}(A|B)$, defined as

$$H_{\min}(A|B)_\rho = -\log \inf \{ \tau(\omega) | \rho \leq \lambda I \otimes \omega \text{ for some positive } \omega \in B(H_B) \}. \quad (1.1)$$

Theorem ([GJB+18]). Let $H_A, H_B$ be finite dimensional Hilbert spaces. For two bipartite density operators $\rho$ and $\sigma$, $\sigma$ is quantum majorized by $\rho$ if and only if for all finite dimensional $H'_A$ and all CPTP maps $\Psi : B(H_A) \to B(H'_{A'})$,

$$H_{\min}(A'|B)_{\Psi \otimes \text{id}(\rho)} \leq H_{\min}(A'|B)_{\Psi \otimes \text{id}(\sigma)}. \quad (1.2)$$

$H_{\min}(A|B)$ is analogy of $H(A|B)$ as the Rényi $p$-version at $p = \infty$ [MLDS+13] and it connects to $H(A|B)$ by the quantum version of asymptotic equipartition property [TCR09]. The “only if” direction in the above theorem follows from the data processing inequality of $H_{\min}$, which is indeed self-evident from its definition (1.1). The other direction states that quantum majorization is actually determined by the data processing inequality of $H_{\min}$. In [GJB+18], the above theorem has been used to characterize quantum process under group symmetry and thermodynamic condition. It has further extensions from bipartite states to bipartite quantum channels [Gou19].

In this work, we revisit Gour et al’s theorem from a functional analytic perspective. Our starting point is the observation that the conditional min-entropy corresponds to the operator space tensor norm

$$H_{\min}(A|B)_\rho = -\log \| \rho \|_{S_1(H_B) \hat{\otimes} B(H_A)} \quad (1.3)$$

where $S_1(H_B)$ is the set of trace class operators on $H_B$ and $S_1(H_B) \hat{\otimes} B(H_A)$ is the operator space projective tensor product. This correspondence is based on an factorization expression for the norm of $S_1(H_B) \hat{\otimes} B(H_A)$ that Pisier used in [Pis98] to define noncommutative vector-valued $L_p$ space. On the other hand, it is known [EJ00, BP91] that the dual space of $S_1(H_B) \hat{\otimes} B(H_A)$ is the completely bounded maps $CB(B(H_A), B(H_B))$, where quantum channels correspond to unital completely positive maps by taking adjoints. From this perspective, $H_{\min}$ is the dual of CB norm with respect to quantum channels and Gour et al’s theorem is essentially a Hahn-Banach separation theorem. Using this approach, we prove the following characterization of quantum majorization using projective tensor norm which extend Gour et al’s results to the setting of tracial von Neumann algebra. We consider two semifinite von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ equipped with normal faithful semi-finite traces $\tau_\mathcal{M}$ (resp. $\tau_\mathcal{N}$). We denote $L_1(\mathcal{M})$ (resp. $L_1(\mathcal{N})$) as the space of 1-integrable operators with respect to $\tau_\mathcal{M}$ (resp. $\tau_\mathcal{N}$). Our main theorem is
Theorem 1.1 (c.f. Theorem 3.8). Let $\mathcal{M}$ and $\mathcal{N}$ be two semifinite von Neumann algebras. Suppose $\mathcal{M}$ is injective. Then for two density operators $\rho, \sigma \in L_1(\mathcal{M} \otimes \mathcal{N})$, there exists a CPTP map $\Phi : L_1(\mathcal{M}) \rightarrow L_1(\mathcal{M})$ such that $\Phi \otimes \text{id}(\rho) = \sigma$ if and only for any projection $e \in \mathcal{M}$ with $\tau_{\mathcal{M}}(e) < \infty$ and for any CPTP map $\Psi : L_1(\mathcal{N}) \rightarrow L_1(e_{\mathcal{M}e^\text{op}}) \cap e_{\mathcal{M}e^\text{op}}$,

$$\|\text{id} \otimes \Psi(\rho)\|_{L_1(\mathcal{M}) \otimes e_{\mathcal{M}e^\text{op}}} \geq \|\text{id} \otimes \Psi(\sigma)\|_{L_1(\mathcal{M}) \otimes e_{\mathcal{M}e^\text{op}}}$$

Here the $L_1(\mathcal{M}) \otimes \mathcal{N}$-norm gives the analogue of $H_{\text{min}}$ as in (1.3). We note that the assumption on injectivity is crucial in our argument. Indeed, we show that for semifinite von Neumann algebras, the conditional min-entropy $H_{\text{min}}$ coincides with the projective tensor norm $L_1(\mathcal{M}) \otimes \mathcal{N}$ if and only if $\mathcal{M}$ is injective. This can be viewed as a predual form of Haagerup’s characterization of injectivity via decomposability [Haa85]. Beyond injectivity, it is not clear whether the above equivalence holds and we do not know the information-theoretic meaning of the projective tensor norm.

The above theorem admits several variants. By taking $\mathcal{N} = l_\infty$, the commutative von Neumann algebra of bounded sequences, Theorem 1.1 concerns the quantum interpolation problem of converting an infinite family of density operators into another family of density operators using a CPTP map. On the other hand, the dual form of Theorem 1.1 provides a characterization for the factorization of CPTP maps (a problem known as channel factorization). A CPTP map $S$ is quantum majorized by $T$ if $S$ admits a factorization $S = \Phi \circ T$ for some CPTP map $\Phi$. Note that in finite dimensions, quantum majorization applies to CPTP maps via their Choi matrices. However, in infinite dimensions, the Choi matrix of a CPTP map is never trace class and our dual consideration is needed. Inspired by Jenova’s work [Jen16] on statistical deficiency for CPTP maps, we also consider the approximate case when the error $\text{id} \otimes \Phi(\rho) - \sigma$ is small but non-zero.

Our approach also has applications to the tracial Hahn-Banach theorem in [HKM14]. The tracial Hahn-Banach theorem is a dual form of Effros-Wrinkler’s separation theorem for matrix convex sets. We find that the duality behind the tracial Hahn-Banach theorem is the same duality as that between the operator space projective tensor product and completely bounded maps. Using an idea similar to the one in characterization of quantum majorization, we give a tracial Hahn-Banach theorem on $L_1(\mathcal{M}) \otimes E$ for a semifinite injective von Neumann algebra $\mathcal{M}$ and an arbitrary operator space $E$. If we replace $L_1(\mathcal{M})$ by an abstract operator space, our method gives some analogous results under the assumptions of 1-locally reflexivity and completely contractive approximation property.

Our paper is organized as follows. Section 2 reviews some basic operator space theory needed for the remainder of the paper. In Section 3, we first discuss the relation between $H_{\text{min}}$ and projective tensor norm and the connection to injectivity of von Neumann algebras. After that, we prove our main theorem and its variants with respect to channel factorization and the approximate case. In particular, all the results in this section apply to $B(H)$ with $H$ being infinite dimensional. As this is arguably the case of most interest in quantum information theory, we summarize the implications for $B(H)$ in Section 3.5.
Section 4 is devoted to the tracial Hahn-Banach theorem and the connection to noncommutative vector-valued $L_1$ space. Section 5 discusses the parallel results on projective tensor product of abstract operator spaces.

2. Operator Space Preliminaries

In this section we briefly recall some operator space basics that are needed in our discussion. We refer to the books [Pis03, EJ00] for more information on operator space theory. We denote by $B(H)$ the bounded operator on a complex Hilbert space $H$ and $M_n := M_n(\mathbb{C})$ the algebra of $n \times n$ complex-valued matrices. A (concrete) operator space $E$ is a closed subspace of some $B(H)$. We denote by $M_{n,m}(E)$ the set of $n \times m$ matrices with entries from $E$ and similar $M_{n,m}(E)$ for $n \times m$ rectangular matrix. The space $M_n(B(H))$ is naturally isomorphic to $B(H^n)$, where $H^n = \ell_2^n(H)$ is the Hilbert space direct sum of $n$ copies of $H$. For all $n \geq 1$, the inclusion $M_n(E) \subset M_n(B(H)) \cong B(H^n)$ induces a norm on the matrix level space $M_n(E)$ which we denote by $\| \cdot \|_{M_n(E)}$. The operator space structure of $E$ is given by the norm sequence $\| \cdot \|_{M_n(E)}$, $n \geq 1$.

Given a linear map $u : E \to F$ between two operator spaces $E$ and $F$, $u$ is completely bounded (or CB) if its completely bounded norm ($CB$-norm)

$$\| u \|_{cb} := \sup_{n \geq 1} \| \text{id}_n \otimes u : M_n(E) \to M_n(F) \|_{op}$$

is finite. Here $\text{id}_n$ is the identity map on $M_n$. We say $u$ is a complete isometry if for each $n$, $\text{id}_n \otimes u$ is an isometry. We denote by $CB(E, F)$ the Banach space of all completely bounded maps $E \to F$ equipped with the $CB$-norm. Moreover, $CB(E, F)$ is again an operator space with the operator space structure given by $M_n(CB(E, F)) = CB(E, M_n(F))$. In particular, the operator space dual is defined as

$$E^* = CB(E, \mathbb{C}).$$

Throughout the paper, we will use $\otimes$ for algebraic tensor product. Given two operator spaces $E \subset B(H_A)$ and $F \subset B(H_B)$, the operator space injective tensor product $E \otimes_{\min} F$ is defined by the (completely) isometric embedding

$$E \otimes_{\min} F \subset B(H_A \otimes_2 H_B) \tag{2.1}$$

where $H_A \otimes_2 H_B$ is the Hilbert space tensor product. Namely, $E \otimes_{\min} F$ is the norm completion of $E \otimes F$ for the inclusion $E \otimes F \subset B(H_A \otimes_2 H_B)$. Via injectivity, one has the (completely) isometric embedding

$$E^* \otimes_{\min} F \subset CB(E, F). \tag{2.2}$$

Another important tensor product for our work is the projective tensor product. We denote by $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm. The operator space projective tensor product $E \hat{\otimes} F$ is defined as the completion of $E \otimes F$ with respect to the following norm,

$$\| z \|_{E \hat{\otimes} F} = \inf \{ a \| H_S \| x \| M_n(E) \| y \| M_m(F) \| b \| H_S \}$$
where the infimum runs over all factorizations of rectangular matrices $a, b$ and $x = (x)_{i,j=1}^m \in M_n(E), y = (y)_{p,q=1}^m \in M_m(F)$ such that

$$z = \sum_{i,j=1}^l \sum_{p,q=1}^m a_{i,p} x_{i,j} \otimes y_{p,q} b_{j,q}. \quad (2.3)$$

For $z = (z)_{r,s=1}^n \in M_n(E \otimes F)$, we consider the following factorization

$$z_{rs} = \sum_{i,j=1}^l \sum_{p,q=1}^m a_{r,ip} x_{i,j} \otimes y_{p,q} b_{jq,s}, \quad (2.4)$$

where $a \in M_{n,m}, b \in M_{m,n}$ and $x \in M(E), y \in M_m(F)$. The operator space structure of $E \hat{\otimes} F$ is defined as

$$\| z \|_{M_n(E \otimes F)} = \inf \| a \|_{M_{n,m}} \| x \|_{M(E)} \| y \|_{M_m(F)} \| b \|_{M_{m,n}},$$

where the infimum runs over all factorizations in (2.4). An equivalent characterization is the following duality [EJ00, BP91]

$$(E \hat{\otimes} F)^* \cong CB(E, F^*). \quad (2.5)$$

For $x \in E, y \in F$ and $\Phi \in CB(E, F^*)$. The dual pairing is

$$\langle x \otimes y, \Phi \rangle = \langle \Phi(x), y \rangle_{(F^*, F)}. \quad (2.5)$$

Let us mention some basic examples related to our discussion. Let $K(H)$ denote the space of compact operators on $H$ and $S_1(H)$ the space of trace class operators. We have the operator space dual relations

$$S_1(H)^* = B(H), K(H)^* = S_1(H), \quad (2.6)$$

where both dual pairings are given by the trace

$$\langle b, a \rangle_{(B(H), S_1(H)^*)} = \tau(b^t a), \langle a, c \rangle_{(S_1(H), K(H)^*)} = \tau(a^t c)$$

where $a^t$ is the transpose of $a$ with respect to a (fixed) orthonormal basis. For two Hilbert spaces $H_A$ and $H_B$, by (2.1) and (2.2) we have the isometric embedding

$$B(H_A) \otimes_{\min} B(H_B) \subset B(H_A \otimes_2 H_B), B(H_A) \otimes_{\min} B(H_B) \subset CB(S_1(H_A), B(H_B)).$$

Indeed, one has the equality

$$B(H_A \otimes_2 H_B) \cong CB(S_1(H_A), B(H_B)). \quad (2.7)$$

Note that by (2.5) and (2.6),

$$CB(S_1(H_A), B(H_B)) = S_1(H_A) \hat{\otimes} S_1(H_B)^*, \quad B(H_A \otimes_2 H_B) = S_1(H_A \otimes_2 H_B)^*.$$
where the infimum is taken over all possible factorizations of \( x = (a \otimes 1_A)y(b \otimes 1_A) \) with \( a, b \in S_2(H_B) \) and \( 1_A \) denotes the identity operator on \( H_A \). For positive \( x \), it suffices to choose \( a = b^* \) and, by rescaling \( \| a \|_2 = 1 \), we obtain
\[
\| x \|_{S_1(H_B) \hat{\otimes} B(H_A)} = \inf \{ \| y \|_{B(H_B) \otimes_{min} B(H_A)} \mid x = (a \otimes 1)y(a^* \otimes 1) \text{ for some } \| a \|_{S_2(H_B)} = 1 \}
\]
\[
= \inf \{ \lambda \mid x \leq \lambda 1 \otimes \sigma \text{ for some density operator } \sigma \in S_1(H_B) \}.
\]
Therefore, this norm on \( S_1(H_B) \hat{\otimes} B(H_A) \) corresponds to the conditional min entropy \( H_{\min} \).
That is, for a bipartite density operator \( \rho \),
\[
H_{\min}(A|B)_\rho = -\log \| \rho \|_{S_1(H_B) \hat{\otimes} B(H_A)}.
\]
At the dual level, by (2.5) we have
\[
(S_1(H_B) \hat{\otimes} B(H_A))^* = CB(B(H_A), B(H_B)).
\]
Note that a CPTP map \( \Phi : S_1(H_B) \to S_1(H_A) \) is completely positive trace preserving, and hence
\[
\Phi \in CB(S_1(H_B), S_1(H_A)) \subset CB(B(H_A), B(H_B))
\]
where \( CB(S_1(H_B), S_1(H_A)) \subset CB(B(H_A), B(H_B)) \) as normal cb maps by taking adjoints.
Therefore, the \( S_1(H_B) \hat{\otimes} B(H_A) \) norm or equivalently \( H_{\min} \), is the dual of CB-norm with respect to quantum channels. This duality is implicitly used in Gour et al’s arguments in [GJB+18]. In quantum information literature, the CB-norm of \( CB(S_1(H_B), S_1(H_A)) \) is also called diamond norm. The diamond norm and its dual norm has been used by Jenčová in studying Le Cam’s deficiency for quantum channels [Jen16].

3. Quantum majorization on von Neumann algebras

3.1. \( H_{\min} \) and injectivity of von Neumann algebras. We first discuss the connection between the conditional min entropy \( H_{\min} \) and the projective tensor product in the setting of tracial von Neumann algebras. Throughout this paper, we assume that \( (\mathcal{M}, \tau_\mathcal{M}) \) and \( (\mathcal{N}, \tau_\mathcal{N}) \) are semifinite von Neumann algebras with normal faithful semifinite traces \( \tau_\mathcal{M} \) (resp. \( \tau_\mathcal{N} \)). We introduce the notation
\[
\mathcal{M}_0 := \cup_{\epsilon} e\mathcal{M}e,
\]
where the union runs over all projections with \( \tau_\mathcal{M}(e) < \infty \) which forms a lattice. For \( 1 \leq p < \infty \), the space \( L_p(\mathcal{M}) \) is the completion of \( \mathcal{M}_0 \) with respect to the \( L_p \)-norm
\[
\| a \|_{L_p(\mathcal{M})} = \tau_\mathcal{M}(|a|^p)^{1/p}, a \in \mathcal{M}_0.
\]
We will often use the shorthand notation \( \| : \|_p \) for the \( p \)-norm and \( \| : \|_\infty \) for the operator norm in \( \mathcal{M} \). Let \( \mathcal{M}^{op} = \{ a^{op} \mid a \in \mathcal{M} \} \) be the opposite algebra equipped with reversed multiplication \( a^{op} \cdot b^{op} = (ba)^{op} \) and trace \( \tau_\mathcal{M}^{op}(a^{op}) = \tau_\mathcal{M}(a) \). The predual of \( \mathcal{M} \) can be identified with \( \mathcal{M}_* = L_1(\mathcal{M}^{op}) \), via the pairing \( \langle a^{op}, b \rangle = \tau_\mathcal{M}(ab) \) for \( a \in L_1(\mathcal{M}) \) and \( b \in \mathcal{M} \). Recall that the von Neumann algebra tensor product \( \mathcal{M} \hat{\otimes} \mathcal{N} \) is the weak*-closure of \( \mathcal{M} \otimes_{min} \mathcal{N} \). The Effros-Ruan isomorphism [EJ00] gives a complete isometry
\[
\mathcal{N} \hat{\otimes} \mathcal{M} \cong CB(\mathcal{N}_*, \mathcal{M}) \cong CB(L_1(\mathcal{N}^{op}), \mathcal{M}).
\]
This isomorphism is order preserving. Indeed, a positive operator \( x \in \mathcal{N} \otimes \mathcal{M} \) corresponds to a completely positive map \( T_x \in CB(L_1(\mathcal{N}^{op}), \mathcal{M}) \). As for the pre dual of (3.1), we have

\[
L_1(\mathcal{M}) \otimes L_1(\mathcal{N}) = L_1(\mathcal{M} \overline{\otimes} \mathcal{N}) = (\mathcal{M}^{op} \overline{\otimes} \mathcal{N}^{op})_*.
\]

The conditional min entropy \( H_{\min} \) is related to the vector-valued \( L_1 \)-space introduced in [Pis98]. We will use the shorthand notation that for \( a, b \in \mathcal{M}, y \in \mathcal{M} \overline{\otimes} \mathcal{N} \),

\[
a \cdot y \cdot b := (a \otimes 1_{\mathcal{N}}) y (b \otimes 1_{\mathcal{N}}).
\]

We define the \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \) norm for \( x \in \mathcal{M}_0 \otimes \mathcal{N} \) as follows,

\[
\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = \inf \{ \| a \|_{L_2(\mathcal{M})} \| y \|_{\mathcal{M} \overline{\otimes} \mathcal{N}} \| b \|_{L_2(\mathcal{M})} \mid x = a \cdot y \cdot b, a, b \in \mathcal{M}_0, y \in \mathcal{M} \otimes \mathcal{N} \},
\]

where the infimum is over all factorizations \( x = a \cdot y \cdot b \). Then \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \) is defined as the completion of \( \mathcal{M}_0 \otimes \mathcal{N} \) under the above norm. The triangle inequality for this norm is verified in [Pis98, Lemma 3.5]. We will also use the shorthand notation

\[
\mathcal{M} \overline{\otimes} \mathcal{N}_0 = \cup_q \mathcal{M} \overline{\otimes} q \mathcal{N} \subset \mathcal{M} \overline{\otimes} \mathcal{N},
\]

where the union runs over all projections \( q \in \mathcal{N} \) with \( \tau_\mathcal{N}(q) < \infty \). For \( x \in \mathcal{M} \overline{\otimes} \mathcal{N}_0 \), we define the \( L_\infty(\mathcal{M}, L_1(\mathcal{N})) \) norm as

\[
\| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = \sup \{ \| a \cdot x \cdot b \|_{L_1(\mathcal{M} \overline{\otimes} \mathcal{N})} \mid \| a \|_{L_2(\mathcal{M})} = \| b \|_{L_2(\mathcal{M})} = 1 \}.
\]

This norm clearly satisfies the triangle inequality. The space \( L_\infty(\mathcal{M}, L_1(\mathcal{N})) \) is defined as the norm completion of \( \mathcal{M} \overline{\otimes} \mathcal{N}_0 \). Both spaces contain the corresponding algebraic tensor

\[
L_1(\mathcal{M}) \otimes \mathcal{N} \subset L_1(\mathcal{M}, L_\infty(\mathcal{N})), \mathcal{M} \otimes L_1(\mathcal{N}) \subset L_\infty(\mathcal{M}, L_1(\mathcal{N})).
\]

Indeed, for \( a \otimes b \) with \( a \in L_1(\mathcal{M}) \) and \( b \in \mathcal{N} \), let \( e_n \) be the spectral projection of \( |a| \) for the interval \([1/n, n]\). Then \( e_n a e_n \otimes b \) converges in \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \) and the limit can be identified with \( a \otimes b \). It is clear from the definitions that

i) a complete contraction \( T : L_\infty(\mathcal{N}_1) \to L_\infty(\mathcal{N}_2) \) extends to a contraction

\[
\text{id}_\mathcal{M} \otimes T : L_1(\mathcal{M}, L_\infty(\mathcal{N}_1)) \to L_1(\mathcal{M}, L_\infty(\mathcal{N}_2)).
\]

ii) a complete contraction \( S : L_1(\mathcal{N}_1) \to L_1(\mathcal{N}_2) \) extends to a contraction

\[
\text{id}_\mathcal{M} \otimes S : L_\infty(\mathcal{M}, L_1(\mathcal{N}_1)) \to L_\infty(\mathcal{M}, L_1(\mathcal{N}_2)).
\]

For the trivial case \( \mathcal{N} = \mathbb{C} \), we have \( L_1(\mathcal{M}, \mathbb{C}) = L_1(\mathcal{M}) \) and \( L_\infty(\mathcal{M}, \mathbb{C}) = L_\infty(\mathcal{M}) \). In general, \( L_\infty(\mathcal{M}, L_1(\mathcal{N})) \) is a subspace of \( \left( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \right)^* \). Indeed,

\[
\| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = \sup \{ \| a \cdot x \cdot b \|_1 \mid \| a \|_2 = \| b \|_2 = 1, a, b \in \mathcal{M}_0 \}
= \sup \{ \| \tau(y(a \cdot x \cdot b)) \|_2 \mid \| a \|_2 = \| b \|_2 = 1, a, b \in \mathcal{M}_0, \| y \|_{\mathcal{M} \overline{\otimes} \mathcal{N}} = 1 \}
= \sup \{ \| \tau((b \cdot y \cdot a) x) \|_2 \mid \| a \|_2 = \| b \|_2 = 1, a, b \in \mathcal{M}_0, \| y \|_{\mathcal{M} \overline{\otimes} \mathcal{N}} = 1 \}
= \sup \{ \| \tau(z x) \|_2 \| \tau(z) \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = 1, z \in \mathcal{M}_0 \otimes \mathcal{N} \}.
\]

Here and in the following we will use \( \tau := \tau_\mathcal{M} \otimes \tau_\mathcal{N} \) for the product trace.
Lemma 3.1. i) For any self-adjoint \( x \in \mathcal{M}_0 \otimes \mathcal{N} \),
\[
\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = \inf \{ \| a \|_{L_2(\mathcal{M})} \| y \|_{\mathcal{M} \overline{\otimes} \mathcal{N}} \| a^* \|_{L_2(\mathcal{M})} \mid x = a \cdot y \cdot a^*, a \in \mathcal{M}_0, y \text{ self-adjoint} \}.
\]

ii) For any positive \( x \in \mathcal{M} \overline{\otimes} \mathcal{N}_0 \),
\[
\| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = \sup \{ \tau(a \cdot x \cdot a^*) \mid \| a \|_{L_2(\mathcal{M})} = 1 \}.
\]

Proof. For ii), Hölder’s inequality gives,
\[
\| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = \sup_{\| a \|_2 = 1} \| (a \otimes 1) x (b \otimes 1) \|_1 \leq \sup_{\| a \|_2 = 1} \| (a \otimes 1) x \|_2 \sup_{\| b \|_2 = 1} \| x \frac{1}{2} (b \otimes 1) \|_2 = \sup_{\| a \|_2 = 1} \| (a \otimes 1) x (a^* \otimes 1) \|_1 = \sup_{a} \tau(a \cdot x \cdot a^*).
\]

For i), choose \( x = (a \otimes 1) y (b \otimes 1) \) such that \( a, b \in e\mathcal{M}e \) and
\[
\| a \|_{L_2(\mathcal{M})} = \| b \|_{L_2(\mathcal{M})} = 1, \| y \|_{\mathcal{M} \overline{\otimes} \mathcal{N}} < \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} + \epsilon.
\]

Take \( d = (aa^* + b^*b + \delta e)^{\frac{1}{2}} \). Then \( d > 0 \) is invertible in \( e\mathcal{M}e \) and \( \| d \|_2 = (2 + \delta \tau(e))^{\frac{1}{2}} \).

Note that \( x = x^* \) implies that
\[
\begin{align*}
x &= \frac{1}{2} \left( a \cdot y \cdot b + b^* \cdot y^* \cdot a \right) \\
&= \frac{1}{2} d \cdot \left( d^{-1} a \cdot y \cdot bd^{-1} + d^{-1} b^* \cdot y^* \cdot ad^{-1} \right) \cdot d \\
&= d \cdot \tilde{y} \cdot d,
\end{align*}
\]

where
\[
\tilde{y} = \frac{1}{2} \left( d^{-1} a \cdot y \cdot bd^{-1} + d^{-1} b^* \cdot y^* \cdot ad^{-1} \right) = \frac{1}{2} \left[ \begin{array}{cc} d^{-1} a & d^{-1} b^* \\ 0 & 0 \end{array} \right] \cdot \left[ \begin{array}{c} 0 \\ y^* \\ 0 \end{array} \right] \cdot \left[ \begin{array}{c} a^* d^{-1} \\ bd^{-1} \end{array} \right].
\]

Since
\[
\left\| \begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix} \right\|_{M_2(\mathcal{M})} = \| y \|_{\mathcal{M}}, \quad \left\| \begin{bmatrix} d^{-1} a & d^{-1} b^* \\ 0 & 0 \end{bmatrix} \right\|_{M_{1,2}(\mathcal{M})} = \| d^{-1} (aa^* + b^*b) d^{-1} \|_{\mathcal{M}} \leq 1
\]
and similarly \( \left\| \begin{bmatrix} a^* d^{-1} \\ bd^{-1} \end{bmatrix} \right\|_{M_{2,1}(\mathcal{M})} \leq 1 \), it follows that
\[
\| \tilde{y} \|_{\infty} \leq \frac{1}{2} \left\| \begin{bmatrix} d^{-1} a & d^{-1} b^* \end{bmatrix} \right\| \left\| \begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} a^* d^{-1} \\ bd^{-1} \end{bmatrix} \right\| \leq \frac{1}{2} \| y \|_{\infty}.
\]

Thus we have \( x = d \cdot \tilde{y} \cdot d \) with
\[
\| d \|_2^2 \leq 2 + \delta \tau(e), \quad \| \tilde{y} \|_{\infty} \leq \frac{1}{2} \| y \|_{\infty}.
\]

Choosing \( \delta \) small enough yields the assertion. \( \blacksquare \)
Lemma 3.2. Let $H_{\min}$ be self-adjoint. Define

$$\lambda(x) = \inf \{ \lambda \mid x \leq \lambda \sigma \otimes 1 \text{ for some density operator } \sigma \in L_1(\mathcal{M}) \}.$$ 

Then

i) $\lambda(x) \leq \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))}$,

ii) $\lambda(x) = \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))}$ if $x$ is positive.

Proof. We first discuss the case $x \in \mathcal{M}_0 \otimes \mathcal{N}$. Suppose $x = (a \otimes 1)y(a^* \otimes 1)$ for some self-adjoint $y \in \mathcal{M} \otimes \mathcal{N}$ and $\| a \|_2 = 1$ with $a \in \mathcal{M}_0$. Then $x \leq \| y \|_\infty a a^* \otimes 1$, where $aa^* \in \mathcal{M}_0$. Then by Lemma 3.1, we have

$$\lambda(x) \leq \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))}$$

for $x \in \mathcal{M}_0 \otimes \mathcal{N}$. Note that $\lambda(x_1 + x_2) \leq \lambda(x_1) + \lambda(x_2)$ and hence

$$|\lambda(x_1) - \lambda(x_2)| \leq \lambda(x_1 - x_2)$$

For general $x$ and $\epsilon > 0$, we can find a self-adjoint sequence $x_n \in \mathcal{M}_0 \otimes \mathcal{N}$ such that $x = \sum_{n=1}^\infty x_n$ converges absolutely and

$$\sum_n \| x_n \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} + \epsilon.$$ 

Then $\lambda(x) \leq \sum_n \lambda(x_n) \leq \sum_n \| x_n \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} + \epsilon$. Since $\epsilon$ is arbitrary, this proves i).

To prove ii), first let $x \in e\mathcal{M}e \otimes \mathcal{N}$ be positive. If $x \leq \lambda \sigma \otimes 1$ for some density operator $\sigma \in \mathcal{M}_0$, we can choose $\tilde{\sigma} = \sigma + \delta e$ invertible in $e\mathcal{M}e$ with $\tau_\mathcal{M}(\tilde{\sigma}) \leq 1 + \epsilon$. Then, we have

$$0 \leq y = \tilde{\sigma}^{-\frac{1}{2}} \cdot x \cdot \tilde{\sigma}^{-\frac{1}{2}} \leq \lambda 1, \ x = \tilde{\sigma}^{\frac{1}{2}} \cdot y \cdot \tilde{\sigma}^{\frac{1}{2}}.$$ 

Hence, we obtain

$$\| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \inf \{ \lambda \mid x \leq \lambda \sigma \otimes 1, \ \sigma \in \mathcal{M}_0 \text{ density operator} \} \quad (3.2)$$

Then it suffices to show that $\lambda(x)$ equals the right hand side. Suppose $x \leq \lambda \sigma \otimes 1$ for some density operator $\sigma \in L_1(\mathcal{M})$. Without losing generality, we can assume that $\sigma$ is invertible on $e\mathcal{M}e$. By definition, for any positive $y \in \mathcal{M} \otimes \mathcal{N}_0$, 

$$\lambda \tau((\sigma \otimes 1)y) \geq \tau(xy).$$
This implies \( \| \sigma^{-\frac{1}{2}} x \sigma^{-\frac{1}{2}} \| \leq \lambda + \epsilon \). We modify \( \sigma \) to a density operator \( \tilde{\sigma} \in \mathcal{M} \) such that \( \tilde{\sigma} = \sigma e_{(0,k)} + ke_{(k,\infty)} \) where \( e_{(0,k)} \) is the spectral projection of \( \sigma \) for the interval \([0,k]\). Note that for any \( z \geq 0 \),
\[
(min\{z,k\})^{-1} - z^{-1} = (z - min\{z,k\})/z(min\{z,k\}) = \begin{cases} 0, & \text{if } z \leq k \\ \frac{z-k}{zk}, & \text{if } z > k. \end{cases}
\]
Then by functional calculus, \( \| \tilde{\sigma}^{-\frac{1}{2}} - \sigma^{-\frac{1}{2}} \|_\infty \leq \frac{1}{k} \). Therefore,
\[
\| \tilde{\sigma}^{-\frac{1}{2}} x \tilde{\sigma}^{-\frac{1}{2}} - x \| = \| x^{\frac{1}{2}} (\tilde{\sigma}^{-1} - \sigma^{-1}) x^{\frac{1}{2}} \| + \| (\tilde{\sigma}^{-\frac{1}{2}} - \sigma^{-\frac{1}{2}}) x^{\frac{1}{2}} \| = \lambda + \epsilon + \frac{1}{k} \| x \|_\infty.
\]
By choosing \( k \) large enough, we have
\[
x \leq (\lambda + 2\epsilon) \tilde{\sigma} \otimes 1.
\]
where \( \| \tilde{\sigma} \|_\infty \leq k \) hence belongs to \( \mathcal{M}_0 \). This proves ii) for positive \( x \in \mathcal{M}_0 \otimes \mathcal{N} \). For a general positive element \( x \in L_1(\mathcal{M}, L_\infty(\mathcal{N})) \), let \( x_n \) be a sequence of positive operators in \( \mathcal{M}_0 \otimes \mathcal{N} \) such that \( \| x_n - x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \to 0 \) Then by i), we know
\[
\lambda(x) = \lim_n \lambda(x_n) = \lim_n \| x_n \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))},
\]
which completes the proof. 

The next lemma shows that \( \lambda(\rho) \) is attained by the duality
\[
L_\infty(\mathcal{M}, L_1(\mathcal{N})) \subset \left( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \right)^*.
\]

**Lemma 3.3.** Let \( \rho \in L_1(\mathcal{M}, L_\infty(\mathcal{N})) \) be self-adjoint. Then
\[
\lambda(\rho) = \sup \{ \tau(\rho x) \mid x \in \mathcal{M} \bar{\otimes} \mathcal{N}_0, x \geq 0, \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = 1 \}.
\]
In particular, if \( \rho \in L_1(\mathcal{M}, L_\infty(\mathcal{N})) \) is positive, then
\[
\| \rho \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} = \sup \{ \tau(\rho x) \mid x \in \mathcal{M} \bar{\otimes} \mathcal{N}_0, x \geq 0, \| x \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))} = 1 \}.
\]

**Proof.** By Lemma 3.2, it suffices to consider \( \rho \in \mathcal{M}_0 \otimes \mathcal{N} \). Let \( \rho \in e\mathcal{M}e \otimes \mathcal{N} \) for some \( \tau_\mathcal{M}(e) < \infty \). We can assume \( \mathcal{M} \) is finite by restricting to \( e\mathcal{M}e \). Let us first consider the case that \( \mathcal{N} \) is finite. We use a standard Grothendieck-Pietsch separation argument. Let \( \lambda \) be a positive number such that \( \lambda < \lambda(\rho) \). We know from (3.2) that for any density operator \( \sigma \in \mathcal{M}_0 \), \( \lambda(1 \otimes \sigma) - \rho \) is not positive and hence has nontrivial negative part. Then there exists a positive \( x \in L_\infty(\mathcal{M} \bar{\otimes} \mathcal{N}) \) such that \( \| x \|_\infty = 1 \) and
\[
\tau(\rho x) - \lambda \tau(\sigma \otimes 1 x) > 0.
\]
Consider the weak* compact subset
\[
B = \{ x \in \mathcal{M} \bar{\otimes} \mathcal{N} \mid \| x \|_\infty = 1, x \geq 0 \}.
\]
For each positive operator \( \sigma \in \mathcal{M}_0 \) with \( \tau_\mathcal{M}(\sigma) \leq 1 \), we define the function \( f_\sigma : B \to \mathbb{R} \) as follows (we suppress the dependence on \( \rho \) since \( \rho \) is fixed)
\[
f_\sigma(x) = \tau(\rho x) - \lambda \tau(\sigma \otimes 1 x), x \in B.
\]
These \( f_\sigma \) are continuous with respect to weak\(^*\) topology on \( B \) because \( \mathcal{N} \) is finite and both \( \sigma \otimes 1 \) and \( \rho \) are in \( L_1(\mathcal{M} \otimes \mathcal{N}) \). Denote \( C(B, \mathbb{R}) \) as the space \( w^*\)-continuous real function on \( B \). We define two subsets

\[
\mathcal{F} = \{ f_\sigma \in C(B, \mathbb{R}) \mid \sigma \in \mathcal{M}_0, \sigma \geq 0, \tau_\mathcal{M}(\sigma) \leq 1 \} \\
\mathcal{F}_- = \{ f \in C(B, \mathbb{R}) \mid \sup f < 0 \}.
\]

Both \( \mathcal{F} \) and \( \mathcal{F}_- \) are convex sets and \( \mathcal{F}_- \) is open. Moreover, \( \mathcal{F} \) and \( \mathcal{F}_- \) are disjoint because for each \( f_\sigma \in \mathcal{F} \), \( \sup_x f_\sigma(x) > 0 \). Then by the Hahn-Banach Theorem, there exists a norm-one linear function \( \psi : C(B, \mathbb{R}) \to \mathbb{R} \) such that for any \( f_- \in \mathcal{F}_- \) and \( f_\sigma \in \mathcal{F} \), there exists a real number \( r \) such that

\[
\phi(f_-) < r \leq \phi(f_\sigma).
\]

Because \( \mathcal{F}_- \) is a cone, \( r \geq 0 \). Similarly, \( r \leq 0 \) because for any \( 0 < \delta < 1 \), \( \delta \mathcal{F} \subset \mathcal{F} \). Then \( r = 0 \) and \( \phi \) is a positive linear functional because \( \phi(f_-) < 0 \) for any \( f_- \in \mathcal{F}_- \). By the Riesz Representation Theorem, \( \phi \) is given by a Borel probability measure \( \mu \) on \( B \). Namely,

\[
\phi(f) = \int_B f(x) \mu(x).
\]

Denote \( x_0 = \int_B x d\mu(x) \). We have for any positive operator \( \sigma \in \mathcal{M}_0 \) with \( \tau_\mathcal{M}(\sigma) \leq 1 \), that

\[
\phi(f_\sigma) = \int_B f(x) d\mu(x) = \int_B \tau(\rho x) - \lambda \tau((\sigma \otimes 1)x) d\mu(x) = \tau(\rho x_0) - \lambda \tau((\sigma \otimes 1)x_0) \geq 0.
\]

By Lemma 3.1,

\[
\tau(\rho x_0) \geq \lambda \sup_\sigma \{ \tau((\sigma \otimes 1)x_0) \mid \sigma \in \mathcal{M}_0, \tau_\mathcal{M}(\sigma_0) \leq 1, \sigma \geq 0 \} = \lambda \| x_0 \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))}.
\]

Normalizing \( x_0 = \| x_0 \|_{L_\infty(\mathcal{M}, L_1(\mathcal{N}))}^{-1} x_0 \), we have \( \tau(\rho \tilde{x}_0) \geq \lambda \). This proves the case for finite \( \mathcal{N} \). For semifinite \( \mathcal{N} \), we define for each projection \( p \in \mathcal{N} \) with \( \tau_\mathcal{N}(p) < \infty \),

\[
\lambda_p = \inf \{ \lambda \mid (1 \otimes p)\rho(1 \otimes p) \leq \lambda \sigma \otimes p \text{ for some density operator } \sigma \in \mathcal{M}_0 \}.
\]

For two projections \( p_1 \leq p_2 \), we have \( \lambda_{p_1} \leq \lambda_{p_2} \). Thus \( \lambda_p \) is monotone non-decreasing over \( p \) for the natural ordering. Based on the finite case, it suffices to show that \( \lim_p \lambda_p \geq \lambda(\rho) \).

Write \( \lambda_1 = \lim_p \lambda_p \). Given \( \epsilon > 0 \), for \( p \) large enough there exists a density operator \( \sigma_p \in \mathcal{M}_0 \) such that

\[
(1 \otimes p)\rho(1 \otimes p) \leq (\lambda_1 + \epsilon)\sigma_p \otimes p.
\]

Let \( \psi_p : \mathcal{M} \otimes \mathcal{N} \to \mathbb{C} \) be the positive normal linear functional \( \psi_p(x) = \tau((1 \otimes p)\rho(1 \otimes p)x) \) and let \( \psi \) be the normal weight \( \psi(x) = \tau(\rho x) \). Let \( \xi_p : \mathcal{M} \to \mathbb{C} \) be the normal state \( \xi_p(\sigma_p y) = \tau_\mathcal{M}(\sigma_p y) \). Let \( \xi \) be a weak*-limit point of \( \xi_p \) in \( \mathcal{M}^* \). Then \( \xi \) is a state on \( \mathcal{M} \) and it decomposes into a normal part and a singular part \( \xi = \xi_n + \xi_s \). For any positive \( x \in (\mathcal{M} \otimes \mathcal{N}_0)_+ \)

\[
\psi_p(x) \leq (\lambda_1 + \epsilon) \lim_p \xi_p \otimes \tau_\mathcal{N}(x) = (\lambda_1 + \epsilon) \xi \otimes \tau_\mathcal{N}(x).
\]
By the normality of $\psi_p$, we have $\psi_p \leq (\lambda_1 + \epsilon)\xi \otimes \tau_N$ as normal states on $M \bar{\otimes} pN_p$. For any positive $x \in (M \otimes N_0)_+$, there exists a $p_x$ such that for $p \geq p_x$, $\psi(x) = \psi_p(x)$ and hence

$$\psi(x) = \psi_p(x) \leq (\lambda_1 + \epsilon)\xi_n \otimes \tau_N(x).$$

By normality, $\psi \leq (\lambda_1 + \epsilon)\xi_n \otimes \tau_N$ as weights. Since $\xi_n$ is a sub-state (that is, a positive linear functional with norm $\leq 1$), $\xi_n(y) = \tau_M(y\sigma)$ for some positive $\sigma \in L_1(M)$ with $\tau_M(\sigma) \leq 1$. Then we have

$$\rho \leq (\lambda_1 + \epsilon)\sigma \otimes 1.$$

Since $\epsilon$ is arbitrary, we complete the proof. 

This next lemma is an analogue of the Choi matrix.

**Lemma 3.4.** There is a contraction

$$L_\infty(M, L_1(N)) \to CB(L_1(M^{op}), L_1(N)),$$

$$x \mapsto T_x \in CB(L_1(M), L_1(N)), \quad T_x(\rho^{op}) = \tau_M \otimes id_N((\rho \otimes 1)x).$$

Moreover,

i) for any positive $x$, $\|x\|_{L_\infty(M,L_1(N))} = \|T_x\|_{cb}$.

ii) $T_x$ is completely positive if and only if $x$ is positive.

iii) $T_x$ is trace preserving if and only if $\text{id} \otimes \tau_N(x) = 1_M$.

iv) for $S \in CB(L_1(N), L_1(N))$, $S \circ T_x = T_{\text{id} \otimes S(x)}$.

v) for any finite rank $T : L_1(M^{op}) \to L_1(N)$, $T = T_x$ for some $x \in M \otimes L_1(N)$.

**Proof.** By a density argument, it suffices to discuss $x \in M \bar{\otimes} pN_p$ with $\tau_N(p) < \infty$. Given $\rho \in L_1(M)$, $(\rho \otimes 1_N)x = (\rho \otimes p)x \in L_1(M \bar{\otimes} pN_p)$ hence the map $T_x(\rho) = \tau_M \otimes id_N((\rho \otimes 1_N)x) \in L_1(N)$ is well defined. For $\|\rho^{op}\|_{L_1(M^{op})} = 1$, we have $\rho = ba$ for some $a, b \in M \bar{\otimes} pN_p$ such that $\|a\|_2 = \|b\|_2 = 1$. Note that $\tau_M \otimes id_N((ba \otimes 1_N)x) = \tau_M \otimes id_N((a \otimes 1)x(b \otimes 1))$. Then

$$\|T_x(\rho^{op})\|_{L_1(N)} \leq \|a \cdot x \cdot b\|_{L_1(L_\infty(M,L_1(N)))} \leq \|x\|_{L_\infty(M,L_1(N))}.$$
Using bracket notation,
\[ \phi = \langle h \rangle \langle h \rangle = \sum_{i=1}^{\infty} |i \rangle \langle i | \]
where \( \{ |i \rangle \} \) is the standard basis in \( l_2^n \). We have
\[ \| \omega_k \cdot \phi \cdot \sigma_l \|_1 = \| \omega_k \otimes 1 \langle h \rangle \|_{l_2} \left\| \sigma_l^* \otimes 1 \langle h \rangle \right\|_{l_2} = \| \omega_k \|_2 \| \sigma_l \|_2 . \]
Here \( \| \cdot \|_{l_2} \) is the vector norm and
\[ \| \omega_k \otimes 1 \langle h \rangle \|_{l_2}^2 = \langle k \rangle \omega_k^* \omega_k \otimes 1 \langle h \rangle = \tau (\omega_k^* \omega_k) = \| \omega_k \|_2 . \]
Therefore,
\[ \| \sum_{k,l} (\omega_k \cdot \phi \cdot \sigma_l) \otimes (a_k \cdot x \cdot b_l) \|_1 \leq \sum_{k,l} \| \omega_k \cdot \phi \cdot \sigma_l \|_1 \| a_k \cdot x \cdot b_l \|_1 \]
\[ \leq \sum_{k,l} \| \omega_k \|_2 \| \sigma_l \|_2 \| x \|_{L_\infty(M,L_1(N))} \leq \| x \|_{L_\infty(M,L_1(N)))} \]
By \( \| \text{id}_{M_n} \otimes T_x(ab) \|_1 \leq \sum_{k,l} (\omega_k \cdot \phi \cdot \sigma_l) \otimes (a_k \cdot x \cdot b_l) \|_1 \), this implies
\[ \| \text{id}_{M_n} \otimes T_x : L_1(M_n(M)^{op}) \rightarrow L_1(M_n(N)) \|_1 \leq \| x \|_{L_\infty(M,L_1(N)))} . \]
Then by \( L_1(M_n(M)^{op}, \tau \otimes \tau_M) \cong S_1^n(L_1(M)^{op}) \) and [Pis98, Lemma 1.2], we obtain
\[ \| T_x : L_1(M)^{op} \rightarrow L_1(N) \|_{cb} = \sup_n \| \text{id}_{n} \otimes T_x : S_1^n(L_1(M)^{op}) \rightarrow S_1^n(L_1(N)) \| \]
\[ \leq \| \phi \otimes x \|_{L_\infty(M_n(M),L_1(M_n(N)))} = \| x \|_{L_\infty(M,L_1(N)))} . \]
Now suppose \( x \) is positive. For a density operator \( \rho \in L_1(M)^{op} \),
\[ T_x(\rho^{op}) = \tau_M \otimes \text{id}_N((\rho \otimes 1)x) = \tau_M \otimes \text{id}_N(\rho^{1/2} \cdot x \cdot \rho^{1/2}) \geq 0 \]
Applying the same argument for \( \phi \otimes x \), we know \( T_x \) is completely positive. Then taking the supremum over all density operators \( \rho \),
\[ \sup_{\rho} \| T_x(\rho^{op}) \|_1 = \sup_{\rho} \tau_M \otimes \tau_N(\rho^{1/2} \cdot x \cdot \rho^{1/2}) = \| x \|_{L_\infty(M,L_1(N)))} . \]
Thus for positive \( x \), we find \( \| T_x \|_{cb} = \| x \|_{L_\infty(M,L_1(N)))} = \| T_x \|_{cb} \), which proves i).
For ii), we note that the “if” statement follows by the construction of \( T_x \). To prove the “only if” statement, we conversely suppose \( x \) is not positive and we show that \( T_x \) is not completely positive. There exists a vector \( h = \sum_{j=1}^{\infty} a_j \otimes b_j \in L_2(M) \otimes_2 L_2(N) \) such that \( a_j \in M_0, b_j \in N_0, \langle h, x h \rangle \not\geq 0 \) (that is, the inner product is either not real or is negative). This means
\[ \langle h, x h \rangle = \sum_{i,j=1}^{n} \tau_M \otimes \tau_N((a_i^* \otimes b_i^*)x(a_j \otimes b_j)) \]
\[ = \tau_N\left( \sum_{i,j=1}^{n} b_i^* \tau_M \otimes \text{id}_N((a_i^* \otimes 1)x(a_j \otimes 1))b_j \right) \not\geq 0 . \]
Thus, \( (\tau_M \otimes \text{id}_N)((a_i^* \otimes 1)x(a_j \otimes 1)) \sum_{i,j=1}^{n} \) is not positive in \( S_1^n(L_1(N)) \). Note that \( \omega^{\text{op}} = \sum_{i,j=1}^{n} e_{ij} \otimes (a_i^*)^{\text{op}}(a_j)^{\text{op}} = \sum_{i,j=1}^{n} e_{ij} \otimes (a_j^*a_i^*)^{\text{op}} \) is positive in \( S_1^n(L_1(M^{\text{op}})) \). Then \( T_x \) is not completely positive because

\[
id_n \otimes T_x(\omega) = \sum_{i,j=1}^{n} e_{ij} \otimes \left( \tau_M \otimes \text{id}_N((a_ja_i^*) \otimes 1)x(\alpha) \right)
= \sum_{i,j} e_{ij} \otimes \left( \tau_M \otimes \text{id}_N((a_i^*a_j)(a_j \otimes 1)) \right) \not\geq 0.
\]

This proves ii). For any \( \rho \in L_1(M) \), by Fubini’s theorem,

\[
\tau_N(T_x(\rho^{\text{op}})) = \tau_N(\tau_M \otimes \text{id}_N((\rho \otimes 1)x)) = \tau_M(\rho \text{id}_M \otimes \tau_N(x)).
\]

Thus \( T_x \) is trace preserving if and only \( \text{id}_M \otimes \tau_N(x) = 1 \). This verifies iii). For iv), let \( S \in CB(L_1(N), L_1(N)) \). For \( \rho \in L_1(M) \),

\[
S \circ T_x(\rho^{\text{op}}) = S(\tau_M \otimes \text{id}_N((\rho \otimes 1)x)) = \tau_M(\rho \text{id}_M \otimes \tau_N(x)) = T_{\text{id} \otimes S(x)}(\rho^{\text{op}}).
\]

Finally, for v), let \( T \) be a finite rank map from \( L_1(M^{\text{op}}) \) to \( L_1(N) \). Then there exists finite \( y_j \in M \) and \( z_j \in L_1(N) \) such that \( T(\rho^{\text{op}}) = \sum_{j=1}^{n} \tau_M(\rho y_j)z_j \). Then \( T = T_x \) for \( x = \sum_{j=1}^{n} y_j \otimes z_j \) which belongs to \( M \otimes L_1(N) \). That completes the proof.

The above lemma gives a contraction

\[
L_\infty(M, L_1(N)) \to CB(L_1(M^{\text{op}}), L_1(N)) \subset CB(N^{\text{op}}, M).
\]

Note that \( CB(N^{\text{op}}, M)_s = L_1(M^{\text{op}}) \otimes L_\infty(N^{\text{op}}) \). The pairings for an algebraic tensor \( \rho^{\text{op}} = \sum_{j=1}^{n} y_j^{\text{op}} \otimes z_j^{\text{op}} \subset L_1(M^{\text{op}}) \otimes N^{\text{op}} \) to \( L_\infty(M, L_1(N)) \) and to \( CB(N^{\text{op}}, M) \) coincide,

\[
\langle x, \rho^{\text{op}} \rangle_{L_\infty(M, L_1(N)), L_1(M^{\text{op}}), L_\infty(N^{\text{op}})} = \tau_M \otimes \tau_N(\sum_{j=1}^{n} y_j \otimes z_j) = \tau_N(\sum_{j} z_j \tau_M \otimes \text{id}(y_j \otimes 1)) = \tau_N(\sum_{j} z_j T_{\text{x}}(y_j)) = \tau_N(\sum_{j} T_{\text{x}}^*(z_j)y_j) = \langle T_x, \rho^{\text{op}} \rangle_{CB(N^{\text{op}}, M), L_1(M^{\text{op}}), L_\infty(N^{\text{op}})}.
\]

Then, for an algebraic tensor \( x = \sum_{j=1}^{n} y_j \otimes z_j \in L_1(M) \otimes N \), we have

\[
\| x \|_{L_1(M, L_\infty(N))} \leq \| x \|_{L_1(M) \otimes N}.
\]

It was proved in [Pis98, Theorem 3.4] that for hyperfinite \( M \) (i.e. \( M = (\bigcup_\alpha \mathcal{M}_\alpha)^{w^*} \)), where the union is of an increasing net of finite-dimensional subalgebras \( \mathcal{M}_\alpha \), we have the
isometric isomorphism
\[ L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M}) \hat{\otimes} \mathcal{N}. \]  
(3.3)

We shall show that this isomorphism is characterized by the injectivity of \( \mathcal{M} \). Recall that a von Neumann algebra \( \mathcal{M} \) is injective if there exists an embedding \( \mathcal{M} \subset B(H) \) and a completely positive projection \( P : B(H) \to \mathcal{M} \) with \( \| P \| = 1 \). An equivalent condition is the weak* completely positive approximation property (weak*-CPAP). A von Neumann algebra \( \mathcal{M} \) has weak*-CPAP if there exists a net of normal finite rank completely positive maps \( \Phi_\alpha \) such that for any \( x \in \mathcal{M} \), \( \Phi_\alpha(x) \to x \) in the weak* topology. In general, hyperfinite implies injective. The converse (say, when \( \mathcal{M} \subset B(H) \) on a separable Hilbert space \( H \)) is a celebrated result of Connes [Con76]. We refer to [Pis03] for more information about these properties.

The next theorem is a dual form of Haagerup’s characterization of injectivity by decomposability [Haa85]. It suggests that the conditional min entropy connects to the projective tensor norm if and only if \( \mathcal{M} \) is injective.

**Theorem 3.5.** Let \( \mathcal{M}, \mathcal{N} \) be semi-finite von Neumann algebras. Suppose \( \mathcal{N} \) is infinite dimensional. The following are equivalent

i) \( \mathcal{M} \) is injective.

ii) \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M}) \hat{\otimes} \mathcal{N} \) isomorphically

iii) \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M}) \hat{\otimes} \mathcal{N} \) isometrically

In particular, \( L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op})) \cong L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op} \) if and only if \( \mathcal{M} \) is injective.

**Proof.** We first prove i) \( \Rightarrow \) iii). Suppose \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \neq L_1(\mathcal{M}) \hat{\otimes} \mathcal{N} \) isometrically. Because both spaces are norm completions of the algebraic tensor \( L_1(\mathcal{M}) \otimes \mathcal{N} \), there exists \( \rho = \sum_{j=1}^n y_j \otimes z_j \) such that
\[ \| \rho \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} < 1 = \| \rho \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{N}}. \]

Then by the duality \( (L_1(\mathcal{M}) \hat{\otimes} \mathcal{N})^* = CB(\mathcal{N}, \mathcal{M}^{op}) \), there exists a CB map \( S \in CB(\mathcal{N}, \mathcal{M}^{op}) \) with \( \| S \|_{cb} = 1 \) such that
\[ 1 = \langle S, \rho \rangle = \langle \text{id}, \text{id}_\mathcal{M} \otimes S(\rho) \rangle. \]

Here we have
\[ \| \text{id}_\mathcal{M} \otimes S(\rho) \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} \leq \| S \|_{cb} \| \rho \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))} < 1. \]

If \( \mathcal{M} \) is injective, then there exists a net of finite-rank, normal, unital, completely positive maps \( \Phi_\alpha \) approximating the identity map \( \text{id}_\mathcal{M} \) in the point-weak* topology. By Lemma 3.4, \( \Phi_\alpha = T_{x_\alpha} \) for some \( x_\alpha \in \mathcal{M}^{op} \otimes L_1(\mathcal{N}^{op}) \) with
\[ \| x_\alpha \|_{L_\infty(\mathcal{M}^{op}, L_1(\mathcal{N}^{op}))} = \| \Phi_\alpha \|_{cb} = 1. \]

This leads to a contraction:
\[ 1 = \langle \text{id}, \text{id}_\mathcal{M} \otimes S(\rho) \rangle = \lim \langle T_{x_\alpha}, \text{id}_\mathcal{M} \otimes S(\rho) \rangle \]
\[ = \lim \langle x_\alpha, \text{id}_\mathcal{M} \otimes S(\rho) \rangle \]
\[ \leq \lim \| x_\alpha \|_{L_\infty(\mathcal{M}^{op}, L_1(\mathcal{N}^{op}))} \| \text{id}_\mathcal{M} \otimes S(\rho) \|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}))}. \]
algebra. Because Neumann algebras and $H$ such that $M \R$ embedding

Suppose $L_1(\mathcal{M}, L_\infty(\mathcal{N})) \subset L_1(\mathcal{M}, L_\infty(\mathcal{N}))$.

Thus we have the isometric imbeddings $L_1(\mathcal{M}, L_\infty(\mathcal{N})) \subset L_1(\mathcal{M}, L_\infty(\mathcal{N}))$.

Suppose $L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M}) \hat{\otimes} \mathcal{N}$ isometrically. We have for each $i$, $L_1(\mathcal{M}_i, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M}_i) \hat{\otimes} \mathcal{N}$ isometrically. It suffices to show that this implies $\mathcal{M}_i$ is injective.

We now assume $\mathcal{M} = \mathcal{M}_i$ finite. Let $l^n_\infty$ be the $n$-dimensional commutative $C^*$-algebra. Because $\mathcal{N}$ is infinite dimensional, for any $n$ there exists completely positive and contractive maps (see [Haa85, Lemma 2.7])

$$ Q : l^n_\infty \to \mathcal{N}, \quad R : \mathcal{N} \to l^n_\infty $$

such that $R \circ Q = \text{id}_{l^n_\infty}$. Both $\text{id}_\mathcal{M} \otimes R$ and $\text{id}_\mathcal{M} \otimes Q$ extend to complete contractions

$$ L_1(\mathcal{M}) \hat{\otimes} l^n_\infty \xrightarrow{\text{id}_\mathcal{M} \otimes R} L_1(\mathcal{M}) \hat{\otimes} \mathcal{N} \xrightarrow{\text{id}_\mathcal{M} \otimes Q} L_1(\mathcal{M}) \hat{\otimes} l^n_\infty, $$

$$ L_1(\mathcal{M}, l^n_\infty) \xrightarrow{\text{id}_\mathcal{M} \otimes R} L_1(\mathcal{M}, L_\infty(\mathcal{N})) \xrightarrow{\text{id}_\mathcal{M} \otimes Q} L_1(\mathcal{M}, l^n_\infty). $$

Thus we have the isometric imbeddings

$$ L_1(\mathcal{M}, l^n_\infty) \subset L_1(\mathcal{M}, L_\infty(\mathcal{N})) \quad \text{and} \quad L_1(\mathcal{M}) \hat{\otimes} l^n_\infty \subset L_1(\mathcal{M}) \hat{\otimes} \mathcal{N}. $$

Suppose $L_1(\mathcal{M}, L_\infty(\mathcal{N})) \cong L_1(\mathcal{M}) \hat{\otimes} \mathcal{N}$ isomorphically. Then we have $L_1(\mathcal{M}, l^n_\infty) \cong L_1(\mathcal{M}) \hat{\otimes} l^n_\infty$ for each $n$, and moreover a uniform constant $c$ such that for all $n$,

$$ c \| \rho \|_{L_1(\mathcal{M}) \hat{\otimes} l^n_\infty} \leq \| \rho \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{N}} \leq \| \rho \|_{L_1(\mathcal{M}, l^n_\infty)} \leq \| \rho \|_{L_1(\mathcal{M}) \hat{\otimes} l^n_\infty}. $$

At the dual level, for each $T : l^n_\infty \to \mathcal{M}^\text{op}$,

$$ \| T \|_{cb} = \| T_x \|_{cb} \leq \| x \|_{L_\infty(\mathcal{M}^\text{op}, l^n_\infty)} \leq c^{-1} \| T_x \|_{cb}. \quad (3.4) $$

Here $T = T_x$ as in Lemma 3.4, for $x = \sum_{j=1}^n T(e_j) \otimes e_j \in \mathcal{M}^\text{op} \otimes l^n_1$ with $e_j \in l^n_1$ being the dual standard basis of $l^n_\infty$. We shall suppress the “op” notation since it is equivalent to consider $\mathcal{M}$ and $\mathcal{M}^\text{op}$ here. For any $n$ unitaries $u_j$ and a central projection $q$ in $\mathcal{M}$, we consider $x_u = q \sum_{j=1}^n u_j \otimes e_j$. We have

$$ \| x_u \|_{L_\infty(\mathcal{M}, l^n_1)} = \text{sup} \{ \| \sum_{j=1}^n a u_j b \otimes e_j \|_{L_1(\mathcal{M}, l^n_\infty)} \mid \| a \|_{L_2(\mathcal{M})} = \| b \|_{L_2(\mathcal{M})} = 1 \} $$

$$ = \text{sup} \{ \sum_{j} \| q a u_j b \|_{L_1(\mathcal{M})} \mid \| a \|_{L_2(\mathcal{M})} = \| b \|_{L_2(\mathcal{M})} = 1 \} $$

$$ \geq \sum_{j} T_\mathcal{M}(q)^{-1} \| q u_j \|_{L_1(\mathcal{M})}. $$
\[= \sum_{j=1}^{n} 1 = n.\]

Here we have chosen \( a = b = \tau_\mathcal{M}(q)^{-1/2}q \). Then by (3.4), we have
\[
\|T_u\|_{cb} \geq c \|x_u\|_{L_\infty(\mathcal{M},\mathcal{T})} = cn, \quad T_u : l_\infty^n \to \mathcal{M}, \quad T_u((c_j)_j) = q \sum_{j=1}^{n} c_j u_j.
\]

Then it follows from [Haa85, Lemma 2.3 & Lemma 2.5] that \( \mathcal{M} \) is injective. Since iii) \( \Rightarrow \) ii) is trivial, this completes the proof. \( \blacksquare \)

3.2. Quantum Majorization. We now discuss quantum majorization for semifinite von Neumann algebras. We will focus on the case where \( \mathcal{M} \) is injective, because by Theorem 3.5, beyond injectivity we lose the duality between \( \text{H}_{\text{min}} \) entropy and CPTP maps.

We say \( T : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \) is completely positive trace preserving (resp. trace non-increasing) if its adjoint \( T^* : \mathcal{M}^{\text{op}} \to \mathcal{M}^{\text{op}} \) is normal completely positive and unital (resp. sub-unital). We will use the abbreviation CPTP for completely positive trace preserving, CPTNI for completely positive trace non-increasing and UCP for unital completely positive. We start with a consequence of Lemma 3.3 and Theorem 3.5.

**Proposition 3.6.** Let \( \mathcal{M} \) be injective.

i) For a self-adjoint \( x \in L_1(\mathcal{M}) \otimes \mathcal{N} \),
\[
\lambda(x) = \sup\{ \langle \Phi, x \rangle | \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{\text{op}}) \text{ CPTNI} \}
\]

ii) Define the real part of \( x \in L_1(\mathcal{M}) \otimes \mathcal{N} \) as \( \text{Re} x = (x + x^*)/2 \). Then
\[
\lambda(\text{Re} x) = \sup\{ \text{Re} \langle \Phi, x \rangle | \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{\text{op}}) \text{ CPTNI} \}
\]

iii) For positive \( \rho \),
\[
\|\rho\|_{L_1(\mathcal{M}) \otimes \mathcal{N}} = \sup\{ \langle \Phi, \rho \rangle | \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{\text{op}}) \text{ CPTNI} \}
\]

\[= \sup\{ \langle \Phi, \rho \rangle | \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{\text{op}}) \text{ CPTP} \}.
\]

**Proof.** We first show that \( \langle \Phi, y \rangle \geq 0 \) for a positive \( y \in L_1(\mathcal{M}) \otimes \mathcal{N} \) and CP \( T : \mathcal{N} \to \mathcal{M}^{\text{op}} \). By density argument, it suffices to consider \( \rho \in \mathcal{M}_0 \otimes \mathcal{N} \). Suppose \( y = (\sum_{j=1}^{n} a_j \otimes b_j^*)(\sum_{j=1}^{n} a_j \otimes b_j) \) for some \( a_j \in \text{e} \mathcal{M} \text{e} \) and \( b_j \in \mathcal{N} \). Then \( \text{id}_n \otimes T(\sum_{i,j=1}^{n} e_{ij} \otimes b_i^* b_j) = \sum_{i,j=1}^{n} e_{ij} \otimes T(b_i^* b_j) \) is positive in \( M_n(\mathcal{M}^{\text{op}}) \). Therefore,
\[
\langle T, y \rangle = \tau_{\mathcal{M}}(\sum_{i,j=1}^{n} (a_i^* a_j)^{\text{op}} T(b_i^* b_j)) = \sum_{i,j=1}^{n} \tau_{\mathcal{M}^{\text{op}}}(a_i^{\text{op}}(a_i^{\text{op}})^* T(b_i^* b_j)a_j^{\text{op}}) = \sum_{i,j=1}^{n} (a_i^{\text{op}}|T(b_i^* b_j)|a_j^{\text{op}}) \geq 0,
\]

where \( |a_j^{\text{op}}| \in L_2(\mathcal{M}^{\text{op}}, \tau_\mathcal{M}) \) is the vector of \( a_j^{\text{op}} \) in the GNS representation. Thus, \( \langle T, y \rangle \geq 0 \) for CP \( T : \mathcal{N} \to \mathcal{M}^{\text{op}} \) and also CP \( T : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{\text{op}}) \) as normal maps. Then if \( x \leq \lambda 1 \otimes \sigma \) and \( \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}^{\text{op}}) \) CPTNI, we have
\[
\langle \Phi, x \rangle \leq \lambda \langle \Phi, 1 \otimes \sigma \rangle = \lambda \tau_{\mathcal{N}}(\Phi(\sigma)) \leq \lambda,
\]

which implies \( \langle \Phi, \rho \rangle \leq \lambda(\rho) \). On the other hand, by Theorem 3.5 and Lemma 3.3,
\[
\lambda(x) = \sup\{ \tau(xy) | y \in \mathcal{M} \otimes \mathcal{N}_0, y \geq 0, \|y\|_{L_1(\mathcal{M}, L_\infty(\mathcal{N}^{\text{op}}))} = 1 \}
\]
that for any $x$ where the density operator $\sigma$ double adjoint map $\Phi$ where $\Phi$ $\otimes$ a normal part and a singular part. Let $\Phi : L_1(M) \to L_1(N)$ CPTNI. Then it is clear that $\Phi$ of adjoints $\Phi \otimes \beta$-compactness, there exists a sub-net $\Phi = \Phi_{x}$ such that $\Phi_{x}$: $L_1(M) \otimes L_1(N)$ $\to$ $L_1(M)$. Taking $x = 1_M \otimes 1_N$, this implies $\tau(\sigma) = \lim_{\alpha} \tau(\Phi_{x} \otimes \beta(\rho)) = 1$. Note that the 

\[ CB(L_1(M), L_1(M)) \subset CB(M^{\text{op}}, M^{\text{op}}) = (L_1(M) \otimes M^{\text{op}})^{*}. \]

By weak$^*$-compactness, there exists a sub-net $\Phi_{\beta}$ such that their corresponding subnet of adjoints $\Phi_{\beta}^{\dagger} : M^{\text{op}} \to M^{\text{op}}$ converges to some $\Phi_{\beta}^{\dagger} : M^{\text{op}} \to M^{\text{op}}$ in the point-weak$^*$ topology. That is, for $x \in L_1(M), y^{\text{op}} \in M^{\text{op}}$, we have

\[ \lim_{\beta} \tau_{M}(x\Phi_{\beta}^{\dagger}(y^{\text{op}})) = \tau_{M}(x\Phi_{\beta}^{\dagger}(y^{\text{op}})) \].

Then it is clear that $\Phi^{\dagger}$ is UCP. Note that $(M^{\text{op}})^{*} = L_1(M) \oplus L_1(M)^{\perp}$ decomposes into a normal part and a singular part. Let $\Phi : L_1(M) \to (M^{\text{op}})^{*}$ be the restriction of the double adjoint map $\Phi^{\dagger} : (M^{\text{op}})^{*} \to (M^{\text{op}})^{*}$. Then $\Phi_{\beta} \otimes \beta(\rho) \to \Phi \otimes \beta(\rho)$ in the sense that for any $x \in M \otimes N$

\[ \tau(x\Phi_{\beta} \otimes \beta(\rho)) = \tau(\Phi_{\beta}^{\dagger} \otimes \beta(\rho)) \to \Phi \otimes (\beta(\rho)(x) \),

where $\Phi \otimes \beta(\rho) \in (M^{\text{op}})^{*} \otimes L_1(N)$. Then by (3.5), for any $x \in M \otimes N$,

\[ \Phi \otimes (\beta(\rho)(x) = \tau(\beta \rho) := \beta(\rho) \]

where the density operator $\sigma$ is viewed as a normal state. Decompose the map $\Phi = \Phi_{n} + \Phi_{s}$ where $\Phi_{n} \in CB(L_1(M), L_1(M))$ is the normal part and $\Phi_{s} \in CB(L_1(M), L_1(M)^{\perp})$ is the singular map. Then for any $x \in M \otimes N$,

\[ (\sigma - \Phi_{n} \otimes \beta(\rho))(x) = \Phi_{s} \otimes \beta(\rho)(x) \]
where $\sigma - \Phi_n \otimes \text{id}(\rho) \in L_1(\mathcal{M}) \otimes L_1(\mathcal{N})$ and $\Phi_s \otimes \text{id}(\rho) \in (\mathcal{M}^{\text{op}})^* \otimes L_1(\mathcal{N})$. Let $\omega_1, \omega_2 : \mathcal{M} \to \mathbb{C}$ be the linear functionals defined by

$$\omega_1(y) := (\sigma - \Phi_n \otimes \text{id}(\rho))(y \otimes 1), \quad \omega_2(y) := \Phi_s \otimes \text{id}(\rho)(y \otimes 1), \quad y \in \mathcal{M}.$$ 

Then $\omega_1$ is normal and $\omega_2$ is singular. By (3.6), $\omega_1 = \omega_2$ which implies $\omega_1 = \omega_2 = 0$. Therefore,

$$\Phi_s \otimes \text{id}(\rho)(1 \otimes 1) = \omega_2(1) = 0.$$ 

Hence $\Phi_s \otimes \text{id}(\rho) = 0$. We have $\sigma = \Phi_n \otimes \text{id}(\rho)$ for $\Phi_n : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ CPTNI. Define $\Phi_0(x) = \tau(\Phi_n(x) - x)\omega$ for any density operator $\omega \in L_1(\mathcal{M})$. Then $\Phi = \Phi_n + \Phi_0$ is a CPTP map and $\Phi \otimes \text{id}(\rho) = \sigma$. This completes the proof.

We say a CPTP map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ is entanglement-breaking if $\Phi(\rho) = \sum_{j=1}^n \tau(x_j \rho)\omega_j$ for some set of $x_j$, $j = 1, 2, \ldots$, satisfying $\sum_{j=1}^n x_j = 1$ and $x_j \geq 0$ (such a set $\{x_j\}$ is called a measurement in quantum mechanics) and density operators $\omega_j$. Such a CPTP map is a quantum channel that admits a factorization through $\mathbb{I}_\infty$, which is the state space of a classical system. We now prove our main theorem with respect to quantum majorization for injective semifinite von Neumann algebra.

**Theorem 3.8.** Let $\mathcal{M}$ and $\mathcal{N}$ be two semifinite von Neumann algebras and let $\mathcal{M}$ be injective. Let $\rho, \sigma$ be two density operators in $L_1(\overline{\mathcal{M} \otimes \mathcal{N}})$. The following are equivalent:

i) there exists a CPTP map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ such that $\Phi \otimes \text{id}(\rho) = \sigma$

ii) for any CP and CB $\Psi : L_1(\mathcal{N}) \to \mathcal{M}^{\text{op}},$

$$\|\text{id} \otimes \Psi(\rho)\|_{L_1(\mathcal{M}) \otimes (e\mathcal{M} e)^{\text{op}}} \geq \|\text{id} \otimes \Psi(\sigma)\|_{L_1(\mathcal{M}) \otimes (e\mathcal{M} e)^{\text{op}}}$$

iii) for any projection $e \in \mathcal{M}$ with $\tau_\mathcal{M}(e) < \infty$ and for any entanglement-breaking CPTP map $\Psi : L_1(\mathcal{N}) \to L_1(e\mathcal{M} e^{\text{op}}) \cap e\mathcal{M} e^{\text{op}},$

$$\|\text{id} \otimes \Psi(\rho)\|_{L_1(\mathcal{M}) \otimes e\mathcal{M} e^{\text{op}}} \geq \|\text{id} \otimes \Psi(\sigma)\|_{L_1(\mathcal{M}) \otimes e\mathcal{M} e^{\text{op}}}$$

**Proof.** The direction i) $\Rightarrow$ ii) and iii) follows from the factorization $\text{id} \otimes \Psi(\sigma) = \Phi \otimes \text{id}\left(\text{id} \otimes \Psi(\rho)\right)$ and

$$\|\Phi \otimes \text{id} : L_1(\mathcal{M}) \otimes \mathcal{M}^{\text{op}} \to L_1(\mathcal{M}) \otimes \mathcal{M}^{\text{op}}\| \leq \|\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})\|_{cb} = 1.$$ 

Let $C(\rho)$ be the convex set from Lemma 3.7

$$C(\rho) = \{\Phi \otimes \text{id}(\rho) | \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}), \text{CPTP}\}$$ 

for some bipartite density operator $\rho$. Suppose by way of contradiction that $\sigma \notin C$. Because $C(\rho)$ is closed with respect to the weak topology induced by $\mathcal{M}^{\text{op}} \otimes_{\text{min}} \mathcal{N}^{\text{op}}$, by the Hahn-Banach theorem there exists $x_1 \in \mathcal{M} \otimes_{\text{min}} \mathcal{N}$ such that

$$\text{Re} \; \tau(\sigma x_1) > \text{Re} \;\sup_{\Phi} \tau(\Phi \otimes \text{id}(\rho)x_1).$$

We can replace $x_1$ with a finite tensor $x_2 = \sum_j a_j \otimes b_j \in \mathcal{M} \otimes \mathcal{N}$ such that $\|x_1 - x_2\| < \epsilon$ is small enough and

$$\text{Re} \; \tau(\sigma x_2) > \sup_{\Phi} \text{Re} \; \tau(\text{id} \otimes \Phi(\rho)x_2).$$
Take $x_3 = (x_2 + x_2^*)/2$ be the real part of $x_2$:
\[
x_3 = \frac{1}{2}(x_2 + x_2^*) = \frac{1}{2} \sum_j (a_j \otimes b_j + a_j^* \otimes b_j^* )
\]
\[
= \frac{1}{4} \left( \sum_j (a_j + a_j^*) \otimes (b_j + b_j^*) + \sum_j i(a_j - a_j^*) \otimes (-i)(b_j - b_j^*) \right),
\]
which is a finite sum of tensor products of self-adjoint elements. Since $\sigma$ and $\Phi \otimes \text{id}(\rho)$ are positive,
\[
\tau(\sigma x_3) = \text{Re} \tau(\sigma x_2) > \sup_{\Phi} \text{Re} \tau(\text{id} \otimes \Phi(\rho)x_2) = \sup_{\Phi} \tau(\text{id} \otimes \Phi(\rho)x_3).
\]
For each $j$,
\[
a_j \otimes b_j + \|a_j\| \|b_j\| 1 \otimes 1 = \frac{1}{2} \left( (a_j + \|a_j\| 1) \otimes (b_j + \|b_j\| 1) + (\|a_j\| (1-a_j)) \otimes (\|b_j\| (1-b_j)) \right).
\]
is a sum of tensor products of positive elements. Take $K = \sum_j \|a_j\| \|b_j\|$. Then
\[
x_4 = x_3 + K1 \otimes 1 \in B(H_A) \otimes B(H_B)
\]
is a sum of tensor products of positive elements. Since $\tau(\text{id} \otimes \Phi(\rho)) = \tau(\sigma) = 1$, we have
\[
\tau(\sigma x_4) = \tau(\sigma x_3) + K > \sup_{\Phi} \tau(\text{id} \otimes \Phi(\rho)x_3) + K > \sup_{\Phi} \tau(\text{id} \otimes \Phi(\rho)x_4).
\]
The opposite element $x_4^{op} \in \mathcal{M}^{op} \hat{\otimes} \mathcal{N}^{op}$ corresponds to a CP map $T \in CB(L_1(\mathcal{N}), \mathcal{M}^{op})$. Note that $\text{id} \otimes T(\sigma) \in L_1(\mathcal{M})$ and $\text{id} \otimes T(\rho) \in L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}$. We have by Proposition 3.6
\[
\tau(x_4 \sigma) = \langle T, \sigma \rangle = \langle \text{id}_M, \text{id} \otimes T(\sigma) \rangle \leq \|\text{id} \otimes T(\sigma)\|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}},
\]
and
\[
\sup_{\Phi \text{ CPTP}} \tau(x_4 \Phi \otimes \text{id}(\rho)) = \sup_{\Phi} \langle T, \Phi \otimes \text{id}(\rho) \rangle = \sup_{\Phi} \langle \Phi, \text{id} \otimes T(\rho) \rangle
\]
\[
= \|\text{id} \otimes T(\rho)\|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}}.
\]
Here the bracket is the pairing for $(L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op})^* \cong CB(\mathcal{M}^{op}, \mathcal{M}^{op})$ and $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ is a normal map in $CB(\mathcal{M}^{op}, \mathcal{M}^{op})$. Then the inequality (3.8) implies that
\[
\|\text{id} \otimes T(\sigma)\|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} > \|\text{id} \otimes T(\rho)\|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}}
\]
which violates ii). This proves the direction ii) $\Rightarrow$ i). For the direction iii) $\Rightarrow$ i), we shall further modify $T$ to get a CPTP map. There exists a projection $e \in \mathcal{M}$ such that $\tau_\mathcal{M}(e) < \infty$ and $\|e \otimes 1\sigma(e \otimes 1) - \sigma\|_i < \epsilon$. Then for small enough $\epsilon$ we have
\[
\tau(\sigma(e \otimes 1)x_4(e \otimes 1)) - \epsilon > \sup_{\Phi} \tau(\text{id} \otimes \Phi(\rho)x_4).
\]
Take $x_5 := (e \otimes 1)x_4(e \otimes 1) = \sum_{i=1}^n c_i \otimes d_j \in e\mathcal{M}e \otimes \mathcal{N}$ as a finite sum of tensor product of positive operators. Then $x_5^{op} \in e\mathcal{M}e^{op} \otimes \mathcal{N}^{op}$ corresponds to the CP map $T_1 : L_1(\mathcal{N}) \to e\mathcal{M}e^{op}$ given by
\[
T_1(\omega) = \sum_{j=1}^n \tau_{\mathcal{N}}(d_j \omega)c_j.
\]
By (3.9), we have
\[ \tau(\sigma x_5) > \sup_{\Phi \text{ CPTP}} \tau(\id \otimes \Phi(\rho)x_4) = \sup_{\Phi \text{ CPTP}} \langle \Phi, \id \otimes T(\rho) \rangle = \| \id \otimes T(\rho) \|_{L_1(M) \hat{\otimes} M^{op}}. \]
Take the map \( T_1(\cdot) = eT(\cdot)e \). Because the map \( y \mapsto ey \) is a complete contraction from \( M^{op} \) to \( eM^{op} \), we have
\[ \| \id \otimes T(\rho) \|_{L_1(M) \hat{\otimes} M^{op}} \geq \| (1 \otimes e) \id \otimes T(\rho)(1 \otimes e) \|_{L_1(M) \hat{\otimes} M^{op}} = \| \id \otimes T_1(\rho) \|_{L_1(M) \hat{\otimes} eM^{op}}. \]
On the other hand,
\[ \tau(\sigma x_5) = \langle \id_M, \id \otimes T_1(\sigma) \rangle \leq \sup_{\Phi \text{ CPTP}} \langle \Phi, \id \otimes T_1(\sigma) \rangle = \| \id \otimes T_1(\sigma) \|_{L_1(M) \hat{\otimes} eM^{op}}. \]
Thus \( T_1 : L_1(M) \rightarrow eM^{op} \) is a CP and CB map and
\[ \| \id \otimes T_1(\sigma) \|_{L_1(M) \hat{\otimes} eM^{op}} > \| \id \otimes T_1(\rho) \|_{L_1(M) \hat{\otimes} eM^{op}}. \]  
Note that \( eM^{op} \subset L_1(eM^{op}) \) because \( \tau_M(e) < \infty \). Since \( T_1 \) is CP and finite rank, we have
\[ \| T_1 : L_1(N) \rightarrow L_1(eM^{op}) \|_{cb} = \| T_1 : L_1(N) \rightarrow L_1(eM^{op}) \| < \infty. \]
Then \( T_2 = \| T_1 : L_1(N) \rightarrow L_1(eM^{op}) \|^{-1} T_1 \) is CPTP and satisfies the inequality (3.10). Finally, we modify \( T_2 \) to be trace preserving.

Denote by \( \rho_M = \id \otimes \tau_N(\rho) \) and \( \rho_N = \tau_M \otimes \id(\rho) \) the reduced density operator of \( \rho \) and similarly for \( \sigma \). For the case \( \rho_N = \sigma_N \), we define \( T_3 = T_2 + T_0 \) where \( T_2(x) = (\tau(x) - \tau(\Psi(x))) \frac{e}{\tau_M(e)} \). Then \( T_3 : L_1(M) \rightarrow L_1(eM^{op}) \) is CPTP. We have
\[ \id \otimes T_3(\rho) = \id \otimes T_2(\rho) + \frac{\lambda_1}{\tau_M(e)} \rho_M \otimes e, \]
\[ \id \otimes T_3(\sigma) = \id \otimes T_2(\sigma) + \frac{\lambda_2}{\tau_M(e)} \sigma_M \otimes e, \]
where \( \lambda_1 = \tau(\rho_N) - \tau(T_2(\rho_N)) \) is equal to \( \lambda_2 = \tau(\sigma_N) - \tau(T_2(\sigma_N)) \). Note that for any density operator \( \omega \in L_1(M) \) and \( \lambda > 0 \)
\[ \id \otimes T_3(\rho) = \id \otimes T_2(\rho) + \frac{\lambda_1}{\tau_M(e)} \rho_1 \otimes e \leq \lambda \omega \otimes e \iff \id \otimes T_2(\rho) \leq (\lambda - \frac{\lambda_1}{\tau_M(e)}) \rho_1 \otimes e. \]
Therefore we have
\[ \| \id \otimes T_3(\rho) \|_{L_1(M) \hat{\otimes} eM^{op}} = \| \id \otimes T_2(\rho) \|_{L_1(M) \hat{\otimes} eM^{op}} + \frac{\lambda_1}{\tau_M(e)} \]
\[ < \| \id \otimes T_2(\sigma) \|_{L_1(M) \hat{\otimes} eM^{op}} + \frac{\lambda_1}{\tau_M(e)} = \| \id \otimes T_3(\sigma) \|_{L_1(M) \hat{\otimes} eM^{op}}. \]
Thus \( T_3 \) is a CPTP map that violates the condition iii). For the case \( \rho_N \neq \sigma_N \), we choose \( q_1 \in N \) to be projection onto the support of \( (\sigma_N - \rho_N)_+ \) and \( q_2 = 1 - q_1 \). We define the CPTP map \( T_4 : L_1(N) \rightarrow M^{op} \) as
\[ T_4(x) = \tau_N(qx) \frac{e_0}{\tau_M(e_0)} + \tau_N(q_2x) \frac{e}{\tau_M(e)}. \]
where \( e_0 < e \) is a projection (by choosing \( e \) large enough, we can always assume such \( e_0 \) exists). Denote \( \sigma_{M,j} = \text{id} \otimes \tau_N((1 \otimes q_j)\sigma) \) and \( \rho_{M,j} = \text{id} \otimes \tau_N((1 \otimes q_j)\sigma) \) with \( j = 1, 2 \). Note that

\[
\tau_M(\sigma_{M,1}) + \tau_M(\sigma_{M,2}) = \tau_M(\sigma_M) = 1 \quad \text{and} \quad \tau_M(\rho_{M,1}) + \tau_M(\rho_{M,2}) = \tau_M(\rho_M) = 1 ,
\]

and

\[
\tau_M(\sigma_{M,1}) - \tau_M(\rho_{M,1}) = \tau((1 \otimes q_j)(\sigma - \rho)) = \tau_N((\sigma_N - \rho_N)q_1) > 0.
\]

Therefore,

\[
\|\text{id} \otimes T_4(\sigma)\|_{L_1(M) \hat{\otimes} E_{Meop}} = \|\sigma_{M,1} \otimes \frac{e_0}{\tau_M(e_0)} + \sigma_{M,2} \otimes \frac{e}{\tau_M(e)}\|_{L_1(M) \hat{\otimes} E_{Meop}} = \frac{\tau_M(\sigma_{M,1})}{\tau_M(e_0)} + \frac{\tau_M(\sigma_{M,2})}{\tau_M(e)} > \frac{\tau_M(\rho_{M,1})}{\tau_M(e_0)} + \frac{\tau_M(\rho_{M,2})}{\tau_M(e)} = \|\text{id} \otimes T_4(\rho)\|_{L_1(M) \hat{\otimes} E_{Meop}}.
\]

Note that both \( T_3 \) and \( T_4 \) are entanglement-breaking. Then in both case, we reach a contradiction to condition iii). This proves iii) \( \Rightarrow \) i).

\[\Box\]

**Remark 3.9.** In the above work, the only result that uses the injectivity of \( \mathcal{M} \) is Proposition 3.6. In fact, Theorem 3.8 holds for any von Neumann algebras \( \mathcal{M} \) for which the conclusions of Proposition 3.6 hold (in particular, if the identify map satisfies item (iii)), even for Type III cases. Proposition 3.6 uses the equivalence between \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \) and \( L_1(\mathcal{M}) \hat{\otimes} \mathcal{N} \) for semifinite injective \( \mathcal{M} \). It is possible to extend the definition of \( L_1(\mathcal{M}, L_\infty(\mathcal{N})) \) and its connection to \( L_1(\mathcal{M}) \hat{\otimes} \mathcal{N} \) for non-tracial \( \mathcal{M} \). (See [JX07] for the case of \( L_1(\mathcal{M}, l_\infty) \) for general \( \mathcal{M} \).) Nevertheless, beyond the injective case the projective norm \( L_1(\mathcal{M}) \hat{\otimes} \mathcal{N} \) loses its connection to the conditional \( H_{min} \) entropy by Theorem 3.5.

We shall now discuss the special case of \( \mathcal{N} = l_\infty \). Let \( \{\rho_i\} \) and \( \{\sigma_i\} \) be two families of density operators in \( L_1(\mathcal{M}) \). Consider the bipartite density operator \( \rho, \sigma \in L_1(\mathcal{M}) \hat{\otimes} l_1 \cong l_1(L_1(\mathcal{M})) \) given by

\[
\rho = (\lambda_i \rho_i)_{i}, \quad \sigma = (\lambda_i \sigma_i)_{i},
\]

where \( \lambda_i > 0, \sum_{i=1}^{\infty} \lambda_i = 1 \) is a probability distribution. Then there exists a CPTP map such that \( \sigma = \Phi \otimes \text{id}_{l_1}(\rho) \) if and only if there exists a CPTP map \( \Phi \) such that \( \sigma_i = \Phi(\rho_i) \) for each \( i \). The latter statement, called the quantum interpolation problem in [HKM14], concerns the convertibility from one family of density operators to another using a quantum process (CPTP map). For finite families of finite dimensional density operators, it was shown in [HKM14] that the quantum interpolation problem is solvable by semi-definite programming (SDP). The \( H_{min} \) characterization of quantum interpolation problem was used in [GJB+18] as a key lemma to prove the bipartite matrix case and has applications in the study of quantum thermal processes. A similar theorem for finite families of self-adjoint operators is obtained in [HKM14, Theorem 7.6]. We will discuss the connection in Section 4. The following theorem is an extension in two ways: it addresses infinite sequences and density operators on von Neumann algebras.
Theorem 3.10. Let \( \mathcal{M} \) be an injective semi-finite von Neumann algebra. Let \( \{\rho_i\}_{i \in \mathbb{N}} \) and \( \{\sigma_i\}_{i \in \mathbb{N}} \) be two countable families of density operators in \( L_1(\mathcal{M}) \). TFAE

i) there exists a CPTP map such that \( \Phi(\rho_i) = \sigma_i \) for all \( i \in \mathbb{N} \)

ii) for any finitely supported probability distribution \( (\lambda_i)_{i \in \mathbb{N}} \) and any set of density operators \( \{\omega_i\} \in L_1(\mathcal{M}^{op}) \cap \mathcal{M}^{op} \)

\[
\| \sum_i \lambda_i \rho_i \otimes \omega_i \|_{L_1(\mathcal{M}) \mathcal{M}^{op}} \leq \| \sum_i \lambda_i \sigma_i \otimes \omega_i \|_{L_1(\mathcal{M}) \mathcal{M}^{op}}.
\]

Proof. Choose a probability distribution \( (\mu_i)_{i \in \mathbb{N}} \) such that \( \mu_i > 0 \) for each \( i \in \mathbb{N} \). Let \( \rho = (\mu_i \rho_i) \) and \( \sigma = (\mu_i \sigma_i) \) be density operators in \( L_1(\mathcal{M}) \mathcal{M} \cong L_1(L_1(\mathcal{M})) \). Then \( i) \Rightarrow ii) \) again follows from the factorization \( \Phi \otimes \text{id}(\rho) = \sigma \) and

\[
\| \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \|_{cb} \leq 1.
\]

Assume that such \( \Phi \) does not exists. Then by Theorem 3.8 there exists a CPTP map \( \Psi : l_1^\infty \to L_1(\mathcal{M}^{op}) \cap \mathcal{M}^{op} \) such that

\[
\| \text{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M}) \mathcal{M}^{op}} > \| \text{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M}) \mathcal{M}^{op}}.
\]

Note that the map \( \Psi \) constructed is also CB from \( l_1^\infty \) to \( \mathcal{M}^{op} \). We can choose \( N \) such that \( \sum_{i > N} \mu_i < \epsilon \). Write \( \rho_N = (\rho_i)_{i \leq N} \oplus 0 \) and \( \sigma_N = (\sigma_i)_{i \leq N} \oplus 0 \) as the corresponding truncated sequences. Then

\[
\| \text{id} \otimes \Psi(\sigma) - \text{id} \otimes \Psi(\sigma_N) \|_{L_1(\mathcal{M}) \mathcal{M}^{op}} \leq \| \sigma - \sigma_N \|_{L_1(\mathcal{M}) \mathcal{M} \cap l_1} \leq \sum_{i > N} \mu_i < \epsilon.
\]

For large enough \( N \), we have

\[
\| \text{id} \otimes \Psi(\sigma_N) \|_{L_1(\mathcal{M}) \mathcal{M}^{op}} \geq \| \text{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M}) \mathcal{M}^{op}} - \epsilon > \| \text{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M}) \mathcal{M}^{op}} \geq \| \text{id} \otimes \Psi(\rho_N) \|_{L_1(\mathcal{M}) \mathcal{M}^{op}}.
\]

Write \( \omega_i = \Psi(e_i) \) where \( e_i \) is the standard basis of \( l_1 \). We have

\[
\text{id} \otimes \Psi(\sigma_N) = \sum_{i \leq N} \mu_i \sigma_i \otimes \omega_i, \text{id} \otimes \Psi(\rho) = \sum_{i \leq N} \mu_i \rho_i \otimes \omega_i.
\]

Renormalizing the coefficient \( \lambda_i = \mu_i (\sum_{i=1}^N \mu_i)^{-1} \), we have a violation of \( ii) \). This completes the proof.

Note that the condition \( ii) \) above only concerns finite subsets of \( \{\rho_i\} \) and \( \{\sigma_i\} \). This leads to the following “compactness” result. It says that to ask whether there is a CPTP map that sends an infinite family of density operators to another infinite family of density operators, it suffices to check the convertibility for every finite subfamily of the two infinite families.

Corollary 3.11. Let \( \{\rho_i\}_{i \in \mathbb{N}} \) and \( \{\sigma_i\}_{i \in \mathbb{N}} \) be two infinite families of density operators in \( L_1(\mathcal{M}) \). There exists a CPTP map \( \Phi \) such that \( \Phi(\rho_i) = \sigma_i \) for all \( i \in \mathbb{N} \) if and only if for any finite subset \( I \subset \mathbb{N} \), there exists a CPTP map \( \Phi_I(\rho_i) = \sigma_i \) for all \( i \in I \).
3.3. Channel factorization. The dual picture of quantum majorization is channel factorization: given two CPTP maps $T, S$, determine if there exists a third CPTP $\Phi$ such that $\Phi \circ T = S$. Such a factorization relation for two CPTP maps has many implications in quantum information theory. In particular, the channel $T$ has larger capacity than $S$ for various communication tasks. For a finite dimensional CPTP map $\Phi : M_n \to M_m$, its Choi matrix is

$$\chi_{\Phi} = \sum_{i,j=1}^{n} e_{ij} \otimes \Phi(e_{ij})$$

where $e_{ij}$ are the matrix unit in $M_n$. As noted in [GJB+18], for two CPTP maps $S, T : M_n \to M_m$, there exists a CPTP $\Phi$ such that $\Phi \circ T = S$ if and only if there exists a CPTP $\Phi$ such that $\text{id} \otimes \Phi(\chi_T) = \chi_S$. So in finite dimensions channel factorization corresponds to quantum majorization of Choi matrices. However, in the infinite dimensional case, such a correspondence fails because the Choi matrix of a CPTP map is never a density operator (since its trace is unbounded). We shall use again the duality $C(B(L_1(\mathcal{M}), L_1(\mathcal{M})) \subset (L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op})^*$ to give a characterization of channel factorization in infinite dimensions and von Neumann algebras. We start with a lemma.

**Lemma 3.12.** Let $T : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ be a CPTP map. Define the set of CPTP maps

$$C_{\text{post}}(T) = \{ \Phi \circ T \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \text{ CPTP} \},$$

$$C_{\text{pre}}(T) = \{ T \circ \Phi \mid \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \text{ CPTP} \}.$$

Then both $C_{\text{post}}(T)$ and $C_{\text{pre}}(T)$ are weak*-closed in $(L_1(\mathcal{N}) \hat{\otimes} \mathcal{M}^{op})^* = C(B(\mathcal{M}^{op}, \mathcal{N}^{op})$. Namely, both set are closed in the point-weak topology.

**Proof.** We first argue for $C_{\text{post}}(T)$. Let $(\Phi_\alpha)$ be a net such that $\Phi_\alpha \circ T \to S$ in the weak*-topology. That is, for any $x \in L_1(\mathcal{N}), y \in \mathcal{M}$

$$\lim_{\alpha} \tau_\mathcal{M}(y \Phi_\alpha \circ T(x)) = \tau_\mathcal{M}(y S(x)) \quad (3.11)$$

Let $(\Phi_\beta)$ be a sub-net such that $\Phi_\beta \to \Phi$ for some $\Phi : L_1(\mathcal{N}) \to (\mathcal{M}^{op})^*$ in the weak*-topology $C(B(L_1(\mathcal{N}), (\mathcal{M}^{op})^*) \cong (L_1(\mathcal{N}) \hat{\otimes} \mathcal{M}^{op})^*$. Note that

$$C(B(L_1(\mathcal{N}), (\mathcal{M}^{op})^*) \cong C(B(\mathcal{M}^{op}, \mathcal{N}^{op})$$

by taking adjoint. The map $\Phi^\dagger$ is UCP because for any positive $x \in L_1(\mathcal{N})$

$$\tau_\mathcal{M}(x \Phi^\dagger(1)) = \lim_{\alpha} \tau_\mathcal{M}(x \Phi^\dagger_\alpha(1)) = \lim_{\alpha} \tau_\mathcal{M}(\Phi_\alpha(x)) = \tau_\mathcal{N}(x)$$

We have $\Phi_\beta \circ T \to \Phi \circ T$ because for any $x \in L_1(\mathcal{N}), y \in \mathcal{M}$

$$\lim_{\alpha} \tau_\mathcal{M}(y \Phi_\beta \circ T(x)) = \lim_{\alpha} \tau_\mathcal{M}(\Phi^\dagger_\beta(y) T(x)) = \tau_\mathcal{M}(\Phi^\dagger(y) T(x)) = \Phi \circ T(x)(y^{op}) \quad .$$

where $\Phi \circ T(x) \in (\mathcal{M}^{op})^*$. Then by (3.11), $\Phi \circ T(x) = S(x) \in L_1(\mathcal{M})$. This implies $\Phi_n \circ T = S$ for $\Phi_n : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ being the normal part of $\Phi$. Since $\Phi^\dagger_n$ is normal CP and sub-unital, $\Phi_n$ is CPTNI. Define $\Phi_0(\rho) = \tau_\mathcal{M}(\Phi_n(\rho) - \rho)\sigma$ where $\sigma$ is some density operator. Then $\tilde{\Phi} = \Phi_n + \Phi_0$ is CPTP. Moreover, $\Phi_0 \circ T = 0$ because both $\tilde{\Phi} \circ T$ and $\Phi_n \circ T = S$ are CPTP. Thus, we obtain $\tilde{\Phi} \circ T = \Phi_n \circ T = S$. 


For $C_{pre}(T)$, let $Ψ_α$ be a net such that $T \circ Φ_α → S$ in the weak-star-topology. Let $Ψ_β$ be a sub-net of $Ψ_α$ such that $Ψ_β → Ψ$ for some $Φ ∈ CB(L_1(抵抗力), (抵抗力^*)^*)$. For any $x ∈ L_1(抵抗力)$ and $y ∈ M,$

$$\lim_β τ(yT \circ Ψ_β(x)) = \lim_β τ(T^t(y)Ψ_β(x)) = Ψ(x)(T^t(y)) = T^t \circ Ψ(x)(y)$$

This means $T \circ Ψ_β → T^t \circ Ψ$ in the weak-star-topo of $CB(L_1(抵抗力), (抵抗力^*)^*)$. Let $Ψ_n$ be the normal part of $Ψ$. Since $T^t|_{L_1(M)} = T$, we have

$$S = T^t \circ Ψ_n = T^t|_{L_1(M)} \circ Ψ_n = T \circ Ψ_n.$$  

The argument to modify $Φ_n$ to be CPTP is similar. ■

We say a bipartite density operator $ρ ∈ L_1(抵抗力 \otimes抵抗力)$ is separable if $ρ$ can be written as $ρ = \sum_{j=1}^{∞} λ_j ω_j \otimes σ_j$, for some $λ_j ≥ 0, \sum_{j=1}^{∞} λ_j = 1$ and $ω_j ∈ L_1(抵抗力), σ_j ∈ L_1(抵抗力)$ are density operators.

**Theorem 3.13.** Assume that $抵抗力$ is injective. Let $T, S : L_1(抵抗力) → L_1(抵抗力)$ be two CPTP maps. TFAE

i) there exists a CPTP $Φ : L_1(抵抗力) → L_1(抵抗力)$ such that $Φ \circ T = S$

ii) for any projection $e ∈抵抗力$ with $τ(e) < ∞$ and any separable density operator $ρ ∈ L_1(抵抗力) \otimes e抵抗力^op$,

$$\|T \otimes id(ρ)\|_{L_1(M) \otimes抵抗力^op} ≥ \|S \otimes id(ρ)\|_{L_1(M) \otimes抵抗力^op}$$

**Proof.** i)⇒ ii) follows from $(Φ \circ T) \otimes id(ρ) = S \otimes id(ρ).$ For ii) ⇒ i), we again argue by contradiction. Suppose $S \notin C_{post}(T) = \{Φ \circ T | Φ$ CPTP$\}$. Then by Lemma 3.12, there exists $x_1 ∈ L_1(抵抗力) \otimes抵抗力^op$ such that

$$Re\langle S, x_1 \rangle > Re \sup_Φ \langle Φ \circ T, x_1 \rangle.$$  

We can replace $x_1$ by a finite tensor sum $x_2 = \sum_{j=1}^{n} a_j \otimes b_j$ with $\|x_1 - x_2\|_{L_1(抵抗力) \otimes抵抗力^op}$ small enough. Moreover, following the same argument in (3.7), $a_j ∈ L_1(抵抗力)$ and $b_j ∈抵抗力^op$ can be self-adjoint. Note that for any $ω ∈ L_1(抵抗力)$,

$$\langle S, ω \otimes 1 \rangle = τ抵抗力(S(ω)) = τ抵抗力(ω) = τ抵抗力(Φ \circ T(ω)) = \langle Φ \circ T, ω \otimes 1 \rangle$$

because $S$ and $Φ \circ T$ are trace preserving. Then we can replace $x_2$ by

$$x_3 = \sum_j a_j \otimes b_j + \|b_j\| (|a_j| \otimes 1) = \sum_j (a_j)_+ \otimes (\|b_j\| 1 + b_j) + (a_j)_- \otimes (\|b_j\| 1 - b_j)$$

which is a finite tensor of positive elements. Let $e ∈抵抗力$ be a projection with finite trace such that

$$\left| \sum_j τ抵抗力(t_j^op S(a_j)) - \sum_j τ抵抗力(eb_j^op eS(a_j)) \right| < ε.$$  

Take $x_4 = (1 ⊗ e)x_3(1 ⊗ e)$. We have for small $ε$

$$\langle S, x_4 \rangle > \langle S, x_3 \rangle - ε > \sup_Φ \langle Φ \circ T, x_3 \rangle.$$  

(3.12)
Now we reinterpret the duality pairing

\[ \langle S, x_4 \rangle = \langle \text{id}, S \otimes \text{id}(x_4) \rangle \leq \sup_{\Phi \text{ CPTP}} \langle \Phi, S \otimes \text{id}(x_4) \rangle = \| S \otimes \text{id}(x_4) \|_{L_1(\mathcal{M}) \hat{\otimes} e\mathcal{M}^{op}} \]

\[ \sup_{\Phi \text{ CPTP}} \langle \Phi \circ T, x_3 \rangle = \sup_{\Phi \text{ CPTP}} \langle \Phi, T \otimes \text{id}(x_3) \rangle = \| T \otimes \text{id}(x_3) \|_{L_1(\mathcal{M}) \hat{\otimes} e\mathcal{M}^{op}} \]

\[ \geq \| T \otimes \text{id}(x_4) \|_{L_1(\mathcal{M}) \hat{\otimes} e\mathcal{M}^{op}} \]

Thus we have a violation of ii),

\[ \| S \otimes \text{id}(x_4) \|_{L_1(\mathcal{M}) \hat{\otimes} e\mathcal{M}^{op}} > \| T \otimes \text{id}(x_4) \|_{L_1(\mathcal{M}) \hat{\otimes} e\mathcal{M}^{op}} . \]

Here \( x_4 \in L_1(\mathcal{M}) \hat{\otimes} e\mathcal{M}^{op} \) is a finite tensor of positive element with finite trace. Replacing \( x_4 \) by its normalization, we get a separable density operator. This completes the proof. ■

The above theorem gives the characterization for “post”-factorization. Similarly, we consider the “pre”-factorization, which is equivalent to the “post”-factorization of normal UCP maps.

**Theorem 3.14.** Assume that \( \mathcal{M} \) is injective. Let \( T, S : L_1(\mathcal{M}) \to L_1(\mathcal{N}) \) be two CPTP maps. TFAE

i) there exists a CPTP \( \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \) such that \( T \circ \Phi = S \),

ii) for any positive \( x \in \mathcal{N}^{op} \otimes \mathcal{M} \),

\[ \| T^{\dagger} \otimes \text{id}(x) \|_{\mathcal{M}^{op} \hat{\otimes} \mathcal{M}^{op}} \leq \| S^{\dagger} \otimes \text{id}(x) \|_{\mathcal{M}^{op} \hat{\otimes} \mathcal{M}^{op}} . \]

**Proof.** By taking the adjoint, \( \Phi^{\dagger} \circ T^{\dagger} = S^{\dagger} \) as normal UCP maps. Then i) \( \Rightarrow \) ii) follows from

\[ \| S^{\dagger} \otimes \text{id}(x) \|_{\infty} = \| \Phi^{\dagger} \circ T^{\dagger} \otimes \text{id}(x) \|_{\infty} \leq \| T^{\dagger} \otimes \text{id}(x) \|_{\infty} . \]

For ii) \( \Rightarrow \) i), suppose \( S \notin C_{pre}(T) := \{ T \circ \Phi \mid \Phi \text{ CPTP} \} \). By the same argument as for Theorem 3.13, there exists a finite tensor \( x_2 = \sum_j a_j \otimes b_j \in L_1(\mathcal{M}) \hat{\otimes} \mathcal{N}^{op} \) with \( a_j, b_j \) positive such that

\[ \langle S, x_2 \rangle > \sup_{\Phi \text{ CPTP}} \langle T \circ \Phi, x_2 \rangle . \]

Then we choose a finite trace projection \( \epsilon \in \mathcal{M} \) such that \( ea_j \epsilon \in \mathcal{M} \) are bounded and for \( x_3 = (\epsilon \otimes 1)x_2(\epsilon \otimes 1) = \sum_j ea_j \epsilon b_j \),

\[ \langle S, x_3 \rangle > \langle S, x_2 \rangle - \epsilon > \sup_{\Phi \text{ CPTP}} \langle T \circ \Phi, x_2 \rangle \]  \hspace{1cm} (3.13)

Reinterpret the pairings

\[ \langle S, x_3 \rangle = \langle \text{id}, \text{id} \otimes S^{\dagger}(x_3) \rangle \leq \sup_{\Phi \text{ CPTP}} \langle \Phi, \text{id} \otimes S^{\dagger}(x_3) \rangle = \| \text{id} \otimes S^{\dagger}(x_3) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} \]

and

\[ \langle T \circ \Phi, x_2 \rangle = \sup_{\Phi \text{ CPTP}} \langle \Phi, \text{id} \otimes T^{\dagger}(x_2) \rangle = \| \text{id} \otimes T^{\dagger}(x_2) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} \]

This implies

\[ \| \text{id} \otimes S^{\dagger}(x_3) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} > \| \text{id} \otimes T^{\dagger}(x_2) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} \]

Because \( \text{id} \otimes T^{\dagger}(x_2) \) is a positive operator in \( e\mathcal{M} \otimes \mathcal{M}^{op} \), we have

\[ \| \text{id} \otimes T^{\dagger}(x_2) \|_{L_1(e\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} = \inf_{\sigma} \| (\sigma^{-\frac{1}{2}} \otimes 1) \text{id} \otimes T^{\dagger}(x_2)(\sigma^{-\frac{1}{2}} \otimes 1) \|_{\infty} . \]
where the infimum is over all invertible density operators $\sigma \in \epsilon Me$. It suffices to consider invertible $\sigma$ with $\|\sigma^{-1}\|_{eMe} < \infty$ because we can always replace $\sigma$ by an invertible density operator $\tilde{\sigma} = (\sigma + \delta e)$. Thus we choose an invertible density operator $\sigma \in eMe$ such that
\[
\| (\sigma^{-\frac{1}{2}} \otimes 1) \text{id} \otimes T^\dagger(x_3)(\sigma^{-\frac{1}{2}} \otimes 1) \|_{eMe\overline{M}^\text{op}} < \| \text{id} \otimes T^\dagger(x_3) \|_{L_1(eMe)\otimes M^\text{op}} + \epsilon
\]
\[
< \| \text{id} \otimes S^\dagger(x_3) \|_{L_1(eMe)\otimes M^\text{op}}
\]
\[
\leq \| (\sigma^{-\frac{1}{2}} \otimes 1) \text{id} \otimes S^\dagger(x_3)(\sigma^{-\frac{1}{2}} \otimes 1) \|_{eMe\overline{M}^\text{op}}.
\]
Then $x_4 = (\sigma^{-\frac{1}{2}} \otimes 1)x_3(\sigma^{-\frac{1}{2}} \otimes 1)$ is positive in $\mathcal{M} \otimes N^\text{op}$, and we have
\[
\| \text{id} \otimes T^\dagger(x_4) \|_{\mathcal{M}\overline{M}^\text{op}} < \| \text{id} \otimes S^\dagger(x_4) \|_{\mathcal{M}\overline{M}^\text{op}}.
\]
which is a violation to condition ii). This proves ii) $\Rightarrow$ i). \qed

3.4. **Approximate case.** In [Jen16], Jenčová gives a characterization for the approximate post-channel factorization in finite dimensions that
\[
\inf_{\Phi \in \text{CPTP}} \| S - \Phi \circ T \|_{cb} < \delta
\]
is small but nonzero. Inspired by Jenčová’s work, we consider the approximate case of quantum majorization. The following lemma is an analogue of [Jen16, Proposition 1].

**Lemma 3.15.** i) Let $\rho, \sigma$ be two density operators in $L_1(\mathcal{M})$. Then
\[
\frac{1}{2} \| \rho - \sigma \|_1 = \sup \{ \tau(x(\rho - \sigma)) \mid x \geq 0, \| x \|_\infty \leq 1 \}.
\]

ii) Let $\mathcal{M}$ be injective. Let $T, S : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ be two CPTP maps. Then
\[
\frac{1}{2} \| T - S \|_{cb} = \sup \{ \langle T - S, \rho \rangle \mid \rho \geq 0, \| \rho \|_{L_1(\mathcal{M}, L\infty(\mathcal{M}^\text{op}))} \leq 1 \}.
\]

**Proof.** For i), note that
\[
x = x^*, \| x \| \leq 1 \iff x + 1 \geq 0, \| x + 1 \| \leq 2.
\]
Since $\tau(\rho - \sigma) = 0$,
\[
\| \rho - \sigma \|_1 = \sup \{ \text{Re} \tau(x(\rho - \sigma)) \mid \| x \|_\infty \leq 1 \}
\]
\[
= \sup \{ \tau(x(\rho - \sigma)) \mid \| x \|_\infty \leq 1, x \text{ self-adjoint} \}
\]
\[
= \sup \{ \tau((x + 1)(\rho - \sigma)) \mid \| x \|_\infty \leq 1, x \text{ self-adjoint} \}
\]
\[
= \sup \{ \tau(y(\rho - \sigma)) \mid \| y \|_\infty \leq 2, y \geq 0 \}
\]
\[
= 2 \sup \{ \tau(y(\rho - \sigma)) \mid \| y \|_\infty \leq 1, y \geq 0 \}.
\]
For ii), let $x$ be self-adjoint and satisfy $\| x \|_{L_1(\mathcal{M}, L\infty(\mathcal{M}^\text{op}))} < 1$. There exists a density operator $\sigma \in L_1(\mathcal{M})$ such that $-\sigma \otimes 1 \leq x \leq \sigma \otimes 1$. Then
\[
x + \sigma \otimes 1 \geq 0, \| x + \sigma \otimes 1 \|_{L_1(\mathcal{M}, L\infty(\mathcal{M}^\text{op}))} \leq 2.
\]
Conversely, let $y \geq 0, \| y \|_{L_1(\mathcal{M}, L\infty(\mathcal{M}^\text{op}))} < 2$. There there exists a density operator $\sigma \in L_1(\mathcal{M})$ such that $0 \leq y \leq 2\sigma \otimes 1$. Then
\[
-\sigma \otimes 1 \leq y - \sigma \otimes 1 \leq \sigma \otimes 1, \| y - \sigma \otimes 1 \|_{L_1(\mathcal{M}, L\infty(\mathcal{M}^\text{op}))} \leq 1.
\]
Since $\mathcal{M}$ is injective, we have $L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op})) \cong L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}$. Then using the fact that $\langle T - S, \sigma \otimes 1 \rangle = \tau(T(\sigma)) - \tau(S(\sigma)) = 0$, we have

$$
\| T - S \|_{cb} = \sup \{ \text{Re} \langle T - S, x \rangle \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 1 \} \\
= \sup \{ \langle T - S, x \rangle \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 1, x = x^* \} \\
= \sup \{ \langle T - S, x \rangle \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 2, x \geq 0 \} \\
= 2 \sup \{ \langle T - S, x \rangle \| x \|_{L_1(\mathcal{M}, L_\infty(\mathcal{M}^{op}))} < 1, x \geq 0 \} .
$$

**Theorem 3.16.** Let $\mathcal{M}, \mathcal{N}$ be semi-finite von Neumann algebras and $\mathcal{M}$ be injective and $\tau_\mathcal{M}(1) = +\infty$. Suppose $\rho$ and $\sigma$ are two density operators in $L_1(\mathcal{M} \hat{\otimes} \mathcal{N})$ such that $\tau_\mathcal{M} \otimes \text{id}(\rho) = \tau_\mathcal{M} \otimes \text{id}(\sigma)$. TFAE

i) $\inf_{\Phi \text{ CPTP}} \| \sigma - \Phi \otimes \text{id}(\rho) \|_1 \leq \delta$.

ii) for any CPTP map $\Psi : L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M}^{op}) \cap \mathcal{M}^{op}$, we have

$$
\| \text{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} \leq \| \text{id} \otimes \Psi(\rho) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} + \frac{\delta}{2} \| \Psi : L_1(\mathcal{N}) \rightarrow \mathcal{M}^{op} \|_{cb}
$$

**Proof.** For a CPTP $\Psi$, we can choose $R : L_1(\mathcal{M}) \rightarrow L_1(\mathcal{M})$ CPTNI such that such that

$$
\langle R, \text{id} \otimes \Psi(\sigma) \rangle \geq \| \text{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} - \epsilon .
$$

Then

$$
\| \text{id} \otimes \Psi(\sigma) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} \leq \epsilon + \langle R, \text{id} \otimes \Psi(\rho) \rangle + \langle R, \text{id} \otimes \Psi(\sigma) - \Phi \otimes \Psi(\rho) \rangle \\
\leq \epsilon + \langle R \circ \Phi, \text{id} \otimes \Psi(\rho) \rangle + \langle \Psi^\dagger \circ R, \sigma - \Phi \otimes \text{id}(\rho) \rangle \\
\leq \epsilon + \| \text{id} \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} + \frac{1}{2} \| \Psi^\dagger \circ R \|_{cb} \| \sigma - \Phi \otimes \text{id}(\rho) \|_1 \\
\leq \epsilon + \| \text{id} \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \hat{\otimes} \mathcal{M}^{op}} + \frac{1}{2} \| \Psi^\dagger \|_{cb} \| \sigma - \Phi \otimes \text{id}(\rho) \|_1
$$

where in the second last inequality we used Lemma 3.15 i). Then i) $\Rightarrow$ ii) follows from taking the infimum over all CPTP $\Phi$ and $\epsilon \rightarrow 0$. Conversely, suppose $\inf_{\Phi \text{ CPTP}} \| \sigma - \Phi \otimes \text{id}(\rho) \|_1 > \delta$.

For $x \in \mathcal{N} \hat{\otimes} \mathcal{M}$,

$$
\tau(x(\sigma - \Phi \otimes \text{id}(\rho))) = \langle T, \sigma - \text{id} \otimes \Phi(\rho) \rangle ,
$$

where $T$ is the map corresponding to $x^{op}$ via the Effros-Ruan isomorphism

$$
CB(L_1(\mathcal{N}), \mathcal{M}^{op}) \cong \mathcal{N}^{op} \hat{\otimes} \mathcal{M}^{op}.
$$

Because this pairing is linear for both $T$ and $\Phi$, we have by min-max theorem,

$$
\delta < \inf_{\Phi \text{ CPTP}} \| \sigma - \Phi \otimes \text{id}(\rho) \|_1 \\
= 2 \inf_{\Phi \text{ CPTP}} \sup_{T \in \text{CP}, \|T\|_{cb} \leq 1} \langle T, \sigma - \Phi \otimes \text{id}(\rho) \rangle \\
= 2 \sup_{T \in \text{CP}, \|T\|_{cb} \leq 1} \inf_{\Phi \text{ CPTP}} \langle T, \sigma - \Phi \otimes \text{id}(\rho) \rangle \\
= 2 \sup_{T \in \text{CP}, \|T\|_{cb} \leq 1} \langle T, \sigma \rangle - \sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \text{id}(\rho) \rangle .
$$
Rescaling the above inequality, there exist CP and CB $T : L_1(\mathcal{N}) \to \mathcal{M}^{op}$ such that

$$\langle T, \sigma \rangle - \sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \text{id}(\rho) \rangle > \frac{\delta}{2} \| T : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} .$$

For a projection $e \in \mathcal{M}$, denote the map $T_e(\cdot) = eT(\cdot)e$. There exists $e$ with $\tau_{\mathcal{N}}(e) < \infty$ such that $|\langle T, (e \otimes 1)\sigma(e \otimes 1) - \sigma \rangle|$ is small and

$$\langle T_e, \sigma \rangle = \langle T, (e \otimes 1)\sigma(e \otimes 1) \rangle > \sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \text{id}(\rho) \rangle + \frac{\delta}{2} \| T : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb} .$$

$$= \| \text{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} + \frac{\delta}{2} \| T : L_1(\mathcal{N}) \to \mathcal{M}^{op} \|_{cb}$$

$$\geq \| \text{id} \otimes T_e(\rho) \|_{L_1(\mathcal{M}) \otimes e\mathcal{M}^{op}} + \frac{\delta}{2} \| T_e : L_1(\mathcal{N}) \to e\mathcal{M}^{op} \|_{cb} .$$

Here we use the fact that

$$\sup_{\Phi \text{ CPTP}} \langle T, \Phi \otimes \text{id}(\rho) \rangle = \sup_{\Phi \text{ CPTP}} \langle \Phi, \text{id} \otimes T(\rho) \rangle = \| \text{id} \otimes T(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}}$$

and the projection from $\mathcal{M}$ to $e\mathcal{M}e$ is a complete contraction. Also, we have

$$\langle T_e, \sigma \rangle = \langle \text{id}, \text{id} \otimes T_e(\sigma) \rangle \leq \| \text{id} \otimes T_e(\sigma) \|_{L_1(\mathcal{M}) \otimes e\mathcal{M}^{op}} .$$

Therefore, we have a violation of ii) for $T_e : L_1(\mathcal{N}) \to e\mathcal{M}^{op}$ is CP and CB,

$$\| \text{id} \otimes T_e(\sigma) \|_{L_1(\mathcal{M}) \otimes e\mathcal{M}^{op}} > \| \text{id} \otimes T_e(\rho) \|_{L_1(\mathcal{M}) \otimes e\mathcal{M}^{op}} + \frac{\delta}{2} \| T_e : L_1(\mathcal{N}) \to e\mathcal{M}^{op} \|_{cb} (3.14)$$

By linearity, we can assume $T_e$ is CPTNI. Denote $\rho_{\mathcal{N}} = \tau_{\mathcal{N}} \otimes \text{id}(\rho)$ and $\sigma_{\mathcal{N}} = \tau_{\mathcal{N}} \otimes \text{id}(\sigma)$. Because $\rho_{\mathcal{N}} = \sigma_{\mathcal{N}}$, we follow the argument in Theorem 3.16 to replace $T_e$ by

$$\tilde{T} = T_e + T_0 \quad T_0(x) = \frac{\tau_{\mathcal{M}}(x - T_e(x))}{\tau_{\mathcal{M}}(e)} .$$

Note that $\| T_0 : L_1(\mathcal{N}) \to e\mathcal{M}^{op} \|_{cb} = \frac{1}{\tau_{\mathcal{M}}(e)}$. Then we can always choose $\tau_{\mathcal{M}}(e)$ large enough such that $\| \tilde{T} \|_{cb} - \| T_e \|_{cb}$ is small and (3.14) is satisfied for $\tilde{T}$.

**Remark 3.17.** If, in addition, $\inf\{\tau_{\mathcal{M}}(e_0) | e_0 \text{ nonzero projection}\} = 0$, we do not need the assumption $\rho_{\mathcal{N}} = \sigma_{\mathcal{N}}$ in Theorem 3.16. In the case of $\rho_{\mathcal{N}} \neq \sigma_{\mathcal{N}}$, by the corresponding discussion in Theorem 3.8, we have a CPTP map $T_1$ such that

$$\| \text{id} \otimes T_1(\sigma) \|_{L_1(\mathcal{M}) \otimes e\mathcal{M}^{op}} - \| \text{id} \otimes T_1(\rho) \|_{L_1(\mathcal{M}) \otimes e\mathcal{M}^{op}} > \left( \frac{1}{\tau_{\mathcal{M}}(e_0)} - \frac{1}{\tau_{\mathcal{M}}(e)} \right) \tau_{\mathcal{N}}((\rho_{\mathcal{N}} - \sigma_{\mathcal{N}})_-)$$

where $e_0 \leq e$ is a sub-projection. This difference can be arbitrarily large if $\inf_{e_0 \neq 0} \tau_{\mathcal{M}}(e_0) = 0$.

This following is a generalization of [Jen16, Theorem 1].

**Theorem 3.18.** Let $\mathcal{M}, \mathcal{N}$ be semi-finite von Neumann algebras and $\mathcal{M}$ be injective. Let $S, T : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ be two CPTP maps. TFAE

i) $\inf_{\Phi \text{ CPTP}} \| S - \Phi \circ T \|_{cb} \leq \delta$

ii) $\| S - T \|_{cb} \leq \delta$

iii) $\| S - T \|_{cb} \leq \delta$

iv) $\| S - T \|_{cb} \leq \delta$
ii) for any density operator \( \rho \in L_1(\mathcal{N} \otimes \mathcal{M}^{op}) \), we have

\[
\| S \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} \leq \| T \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} + \frac{\delta}{2} \| \rho \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}}.
\]

**Proof.** Let \( \rho \in L_1(\mathcal{N} \otimes \mathcal{M}^{op}) \) be a density operator. For any \( \epsilon > 0 \), we can choose \( R : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \) CPTP such that

\[
\langle R, S \otimes \text{id}(\rho) \rangle \geq \| S \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} - \epsilon.
\]

Then

\[
\| S \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} \leq \epsilon + \langle R, S \otimes \text{id}(\rho) \rangle
\]

\[
\leq \epsilon + \langle R, \Phi \circ T \otimes \text{id}(\rho) \rangle + \langle R, (S - T) \otimes \text{id}(\rho) \rangle
\]

\[
\leq \epsilon + \| \Phi \circ T \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} + (S - \Phi \circ T, \text{id} \otimes R^\dagger(\rho)) \]

\[
\leq \epsilon + \| \Phi \circ T \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} + \frac{1}{2} \| S - \Phi \circ T \|_{cb} \| \text{id} \otimes R^\dagger(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}}
\]

\[
\leq \epsilon + \| \Phi \circ T \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} + \frac{1}{2} \| S - \Phi \circ T \|_{cb} \| \rho \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}}
\]

where in the second last inequality we used Lemma 3.15 ii). Then i) \( \Rightarrow \) ii) follows from taking the infimum over all CPTP \( \Phi \) and \( \epsilon \to 0 \). For ii) \( \Rightarrow \) i), suppose \( \inf_{\Phi \text{CPTP}} \| S - \Phi \circ T \|_{cb} > \delta \). Let us use the shorthand notation \( \| \cdot \|_{1,\infty} = \| \cdot \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} \). Using the min-max theorem,

\[
\delta < \inf_{\Phi \text{CPTP}} \| S - \Phi \circ T \|_{cb}
\]

\[
= 2 \inf_{\Phi \text{CPTP}} \sup_{\rho \geq 0, \| \rho \|_{1,\infty} \leq 1} \langle S - \Phi \circ T, \rho \rangle
\]

\[
= 2 \sup_{\rho \geq 0, \| \rho \|_{1,\infty} \leq 1} \inf_{\Phi \text{CPTP}} \langle S - \Phi \circ T, \rho \rangle
\]

\[
= 2 \sup_{\rho \geq 0, \| \rho \|_{1,\infty} \leq 1} \langle \text{id}, S \otimes \text{id}(\rho) \rangle - \sup_{\Phi \text{CPTP}} \langle \Phi, T \otimes \text{id}(\rho) \rangle
\]

\[
\leq 2 \sup_{\rho \geq 0, \| \rho \|_{1,\infty} \leq 1} \sup_{\Phi \text{CPTP}} \langle \Phi, S \otimes \text{id}(\rho) \rangle - \sup_{\Phi \text{CPTP}} \langle \Phi, T \otimes \text{id}(\rho) \rangle
\]

\[
= 2 \sup_{\rho \geq 0, \| \rho \|_{1,\infty} \leq 1} \| S \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}} - \| T \otimes \text{id}(\rho) \|_{L_1(\mathcal{M}) \otimes \mathcal{M}^{op}}.
\]

Thus there exists a positive \( \rho \in L_1(\mathcal{M}) \otimes \mathcal{M}^{op} \) violating the inequality in ii). One can then replace \( \rho \) by a bipartite density operator \( \tilde{\rho} \in L_1(\mathcal{M} \otimes \mathcal{M}^{op}) \) as in Theorem 3.13. 

**Remark 3.19.** In Theorem 3.16 & 3.18, we cannot reduce condition ii) to entanglement-breaking CPTP maps and respectively separable density operator as in the case for \( \delta = 0 \). This is because Lemma 3.15 fails when we restrict the pairing to entanglement-breaking or separable elements.

### 3.5 Results in \( B(H) \) setting.

The results of the previous subsections subsume the case of \( B(H) \) where \( H \) is infinite dimensional. However, since this is the case most relevant to quantum information theory, we briefly restate some of our results for \( B(H) \) in terms of the conditional min entropy \( H_{\min} \). \( H_{\min}(A|B) \) is the sandwiched Rényi \( p \)-version of
$H(A|B)$ at $p = \infty$ and the smooth version of $H_{\min}(A|B)$ connects to $H(A|B)$ by quantum asymptotic equipartition property [TCR09]. While the operational meaning of $H(A|B)$ is in i.i.d. asymptotic regime, $H_{\min}(A|B)$ has many applications in the one shot setting ([Tom15] and reference therein). The following theorem summarizes the results on quantum majorization, state convertibility and channel factorization.

**Theorem 3.20.** Let $H_A, H_B$ be two infinite dimensional Hilbert spaces. The following statements hold.

1) For two bipartite density operators $\rho^{AB}, \sigma^{AB} \in S_1(H_A \otimes_2 H_B)$, there exists a quantum channel $\Phi : S_1(H_B) \to S_1(H_B)$ such that $\id_A \otimes \Phi(\rho) = \sigma$ if and only if for any entanglement-breaking channel $\Psi : S_1(H_A) \to S_1(H_A)$

$$H_{\min}(A|B)_{\Psi \circ \id(\rho)} \leq H_{\min}(A|B)_{\Psi \circ \id(\sigma)}.$$ 

2) For two families of density operators $\{\rho_i\}_{i \in \mathbb{N}}$ and $\{\sigma_i\}_{i \in \mathbb{N}}$ in $B(H_B)$, there exists a quantum channel such that $\Phi(\rho_i) = \sigma_i$ for all $i \in \mathbb{N}$ if and only if for any finitely supported probability distribution $\lambda_i$ on $\mathbb{N}$ and any set of density operators $\{\omega_i\} \subset B(H_A)$

$$H_{\min}(A|B)_{(\sum_i \lambda_i \omega_i \otimes \rho_i)} \leq H_{\min}(A|B)_{(\sum_i \lambda_i \omega_i \otimes \sigma_i)}.$$ 

3) For two quantum channels $T,S : S_1(H_B) \to S_1(H_B)$, there exists a quantum channel $\Phi$ such that $\Phi \circ T = S$ if and only if for any separable density operator $\rho \in S_1(H_A \otimes_2 H_B)$,

$$H_{\min}(A|B)_{\id \otimes T(\rho)} \leq H_{\min}(A|B)_{\id \otimes S(\rho)}.$$ 

The above theorem make senses even when $H_{\min}$ equals $-\infty$. We know by Theorem 3.10 and 3.13 that it suffices to consider all finite dimensional $H_A$ in the equivalence ii) and iii). Similarly, for the equivalence it suffices to consider channels $\Psi : S_1(H_A) \to S_1(H_A')$ into a finite dimensional $H_A'$. In these situation, $H_{\min}$ will always take finite values. In general, $H_{\min}(A|B)$ can be $-\infty$, where the inequalities in above theorem are trivially satisfied.

4. **Tracial convex sets in Vector-valued noncommutative $L_1$-space**

In this section, we discuss the analogue of quantum majorization in vector-valued noncommutative $L_1$-space and the connection to the tracial Hahn-Banach Theorem. Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra equipped with a normal faithful semifinite trace $\tau$. Let $E$ be a operator space. The $E$-valued noncommutative $L_1$-space was introduced by Pisier in [Pis98]. For $x \in \mathcal{M}_0 \otimes E$ in the algebraic tensor, we define the $L_1(\mathcal{M}, E)$ norm as follows,

$$\|x\|_{L_1(\mathcal{M}, E)} = \inf\{\|a\|_{L_2(\mathcal{M})}\|b\|_{L_2(\mathcal{M})}\|y\|_{\mathcal{M} \otimes \min E} \mid x = a \cdot y \cdot b\},$$

(4.1)

where the infimum runs over all factorizations $x = a \cdot y \cdot b := (a \otimes 1_E)y(b \otimes 1_E)$ with $a, b \in \mathcal{M}_0$ and $y \in \mathcal{M} \otimes E$. The space $L_1(\mathcal{M}, E)$ is defined as the norm completion of $\mathcal{M}_0 \otimes E$. The $L_1(\mathcal{M}, L_\infty(\mathcal{N}))$ space we discussed in the previous section is the special
case of $E$ being a von Neumann algebra $\mathcal{N}$. Recall that a von Neumann algebra $\mathcal{M}$ is hyperfinite if $\mathcal{M} = \bigcup \mathcal{M}_\alpha$ is the $w^*$-closure of the union of an increasing net of finite dimensional von Neumann algebras $\mathcal{M}_\alpha$. It was proved in [Pis98, Theorem 3.4] that for hyperfinite $\mathcal{M}$,

$$L_1(\mathcal{M}, E) \cong L_1(\mathcal{M}) \hat{\otimes} E \quad (4.2)$$

isometrically. Namely, for hyperfinite $\mathcal{M}$, the vector-valued noncommutative $L_1$ space is identified with projective tensor product. Following that, we introduce the following definition of a tracial set in $L_1(\mathcal{M}) \hat{\otimes} E$.

**Definition 4.1.** A subset $V \subset L_1(\mathcal{M}) \hat{\otimes} E$ is called a contractively tracial set if for any CPTNI map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$, $\Phi \otimes \text{id}_E(V) \subset V$.

The matrix level tracial sets are discussed in [HKM14, Section 6.2] as the dual concept of matrix convex set. We refer to their definition as matrix tracial set.

**Definition 4.2.** A matrix contractively tracial set $(V_n)_n$ is a sequence of subsets $V_n \subset M_n(\mathcal{E})$ such that for any CPTNI map $\Phi : M_n \to M_m$, $\Phi \otimes \text{id}(V_n) \subset V_m$.

This definition was considered in [HKM14] for finite dimensional $E$. Indeed, for $\dim E = m$, each element in $V_n \subset M_n(\mathcal{E}) \cong M_n^m$ can be identified with a finite sequence $(x_j) \in (M_n)^m$. We discuss the relations of these two definitions in the following proposition.

**Proposition 4.3.** Let $H$ be an separable Hilbert space and $(e_n)_n$ be a sequence of projections such that $\dim(e_nH) = n$ and $e_n \to 1$ weakly. Identify $M_n \cong S_1(eH_n)$ as subspace of $S_1(H)$.

i) Given a contractively tracial set $V \subset S_1(H) \hat{\otimes} E$, then the set

$$V[n] = e_n \cdot V \cdot e_n$$

forms a matrix contractively tracial set such that $\bigcup_n V[n] = \overline{\bigcup_n V[n]} \subset S_1(H) \hat{\otimes} E$.

ii) Given a matrix contractively tracial set $(V_n)_n \subset M_n(\mathcal{E})$, then the set

$$V = \left( \bigcup_n V_n \right) \|\cdot\| \subset S_1(H) \hat{\otimes} E$$

is a closed contractively tracial set such that $V[n] = V[n] \subset V$.

**Proof.** i) Let $e \in B(H)$ be a projection. Because the map $\rho \to e\rho e$ is CPTNI on $S_1(H)$, $x \in V$ implies that $e \cdot x \cdot e \in V$. Then for any $\Phi : M_n \to M_m$ CPTNI, $\Phi \otimes \text{id}(e_n \cdot x \cdot e_n) \in V[m] \subset V$. Thus $(V[n])_n$ is a matrix contractively tracial set. Moreover, for any $\epsilon > 0$ and $x \in S_1(H) \hat{\otimes} E$, $\lim_n \| e_n \cdot x \cdot e_n - x \|_{S_1(H) \hat{\otimes} E} \to 0$. Then $\overline{\bigcup_n V[n]} \subset S_1(H) \hat{\otimes} E$ and the other inclusion follows from $V[n] \subset V$.

ii) Let $x \in V_n$. For $\Phi : S_1(H) \to S_1(H)$ CPTNI and $\rho \in V_n$, we find that

$$e_m \cdot \Phi \otimes \text{id}(\rho) \cdot e_m \in V_m$$

because $x \to e_m \Phi(x)e_m$ can be viewed as a CPTNI map from $M_n$ to $M_m$. Let $x_k \in V_{n(k)}$ be a sequence such that $x_k \to x$ in $S_1(H) \hat{\otimes} E$. Then $\Phi \otimes \text{id}(x_m) \to \Phi \otimes \text{id}(x)$, which implies
Φ ⊗ id(x) ∈ V. This verifies that V is contractively tracial. In particular, the fact that 
\( e_n \cdot x_k \cdot e_n \) converges to \( e_n \cdot x \cdot e_n \) implies that \( V[n] \subset V_n \). □

The above proposition shows that Definition 4.1 and Definition 4.2 are closely related for the case \( \mathcal{M} = B(H) \). In particular, they coincide for closed sets. It is easy to see that the convex hull of a contractively tracial set is again contractively tracial. In general, contractively tracial sets are not necessary convex.

The next theorem is the tracial Hahn-Banach separation theorem for convex contractively tracial sets. For matrix contractively tracial sets with \( \dim E < \infty \), this was obtained in [HKM14, Theorem 7.6]. Using projective tensor product, we can now consider semi-finite injective \( \mathcal{M} \) and a general operator space \( E \).

**Theorem 4.4.** Let \( \mathcal{M} \) be an injective semifinite von Neumann algebra. Let \( V \) be a closed convex contractively tracial set in \( L_1(\mathcal{M}) \otimes E \) and \( x \in L_1(\mathcal{M}) \otimes E \). Then \( x \not\in V \) if and only if there exists a CB map \( T : E \to \mathcal{M}^{\text{op}} \) such that for each \( y \in V \), there exists a density operator \( \omega_y \in L_1(\mathcal{M}) \) depending on \( y \) such that

\[
\Re \id \otimes T(y) \leq \omega_y \otimes 1
\]

and for any density operator \( \omega \),

\[
\Re \id \otimes T(x) \not\leq \omega \otimes 1
\]

**Proof.** The “if” direction is trivial. For the other direction, suppose \( \sigma \not\in V \). Using the duality \( L_1(\mathcal{M}) \otimes E^* = CB(E, \mathcal{M}^{\text{op}}) \), there exists a CB map \( T : E \to \mathcal{M}^{\text{op}} \)

\[
\Re \langle T, x \rangle > \sup_{\rho \in V} \Re \langle T, y \rangle.
\]

Reinterpreting the dual pairing and using the Proposition 3.6,

\[
\Re \langle T, x \rangle = \Re \langle \id_{\mathcal{M}}, \id \otimes T(x) \rangle \leq \sup_{\Phi \text{ CPTNI}} \Re \langle \Phi, \id \otimes T(x) \rangle
\]

\[
= \inf \{ \tau(\omega) | \Re \id \otimes T(x) \leq \omega \otimes 1, \omega \geq 0 \}.
\]

On the other hand, because \( V \) is contractively tracial,

\[
\sup_{\rho \in V} \Re \langle T, y \rangle \geq \sup_{\rho \in V, \Phi \text{ CPTNI}} \Re \langle T, \Phi \otimes \id(y) \rangle
\]

\[
= \sup_{\rho \in V} \inf \{ \tau(x) | \Re T \otimes \id(y) \leq x, x \geq 0 \}
\]

Take \( \lambda \) such that \( \Re \langle T, x \rangle > \lambda > \sup_{\rho \in V} \Re \langle T, y \rangle \). Then for the map \( \tilde{T} = \frac{1}{\lambda} T \),

\[
\sup \inf \{ \tau(\omega) | \Re \tilde{T} \otimes \id(y) \leq \omega \otimes 1, \omega \geq 0 \} < 1 < \inf \{ \tau(\omega) | \Re \tilde{T} \otimes \id(x) \leq \omega \otimes 1, \omega \geq 0 \}
\]

which completes the proof. □

Using the similar idea, we obtain a variant of Effros-Winkler’s separation theorem [EW97]. Recall a CP map \( \Phi \) is sub-unital if \( \Phi(1) \leq 1 \).
Theorem 4.5. Let $E$ be a operator space. Let $V \subset M_n(E)$ be a closed convex set such that $\Phi \otimes \text{id}(V) \subset V$ for any CP sub-unital $\Phi : M_n \to M_n$. Then $x \notin V$ if and only if there exists a map $T : E \to M_n$ such that for each $y \in V$, there exists a density operator $\omega_y \in M_n$ depending on $y$ such that
\[
\Re \text{id} \otimes T(y) \leq 1 \otimes \omega_y ,
\]
and for any density operator $\omega$,
\[
\Re \text{id} \otimes T(x) \not\leq 1 \otimes \omega .
\]

Proof. Suppose $x \notin V$. Because $M_n$ is finite dimensional, we have $M_n(E)^* = S_1^n \hat{\otimes} E^*$. Then there exists an element $T \in E^* \hat{\otimes} S_1^n$ such that
\[
\Re \langle T, x \rangle > \sup_{y \in V} \Re \langle T, y \rangle .
\]
We identify $T \in E^* \hat{\otimes} S_1^n$ with a map $T : E \to S_1^n$. Then the pairing on the left hand side of (4.3) can be rewritten as
\[
\Re \langle T, x \rangle = \Re \langle \text{id}_{M_n}, \text{id} \otimes T(x) \rangle \leq \inf \{ \tau(\omega) | \Re \text{id} \otimes T(x) \leq 1 \otimes \omega, \omega \geq 0 \}.
\]
Here the second pairing is between $CB(M_n, M_n) = (M_n \hat{\otimes} S_1^n)^*$. For the right hand side of (4.3),
\[
\sup_{y \in V} \Re \langle T, y \rangle = \sup_{y \in V} \sup_{\Phi \text{ CP sub-unital}} \Re \langle T, \Phi \otimes \text{id}(y) \rangle = \sup_{y \in V} \sup_{\Phi} \Re \langle \Phi, \text{id} \otimes T(x) \rangle
\]
\[
\leq \sup_{y \in V} \inf \{ \tau(\omega) | \Re \text{id} \otimes T(y) \leq 1 \otimes \omega, \omega \geq 0 \} .
\]
Then the assertion follows from the inequality (4.3).

Recall that a contractively matrix convex set is a sequence $(V_n) \subset M_n(E)$ such that i) for any CP sub-unital $\Phi : M_m \to M_n$, $\Phi \otimes \text{id}(V_m) \subset V_n$; and ii) for any $a \in V_m, b \in V_n$, $a \oplus b \in V_{n+m}$. Effros-Winkler's theorem stated for matrix convex set admits a stronger separation: there exists a density operator $\omega$ uniform for all $y$ such that $\Re \text{id} \otimes T(y) \leq 1 \otimes \omega$. A similar lemma for tracial sets was given in [HKM14, Lemma 7.4]. The above Theorem 4.5 leads to a weaker separation because we consider convex sets closed under CP sub-unital maps without assumption ii).

5. Norm separations on projective tensor product

In this section, we discuss the analogue of quantum majorization on projective tensor product. Recall that a operator space $G$ is 1-locally reflexive if for any finite dimensional operator space $G$, we have the complete isometry
\[
CB(E, G^{**}) \cong CB(E, G)^{**}.
\]
It is clear from the definition that $G = G^{**}$ is reflexive implies that $G$ is 1-locally reflexive. It was proved by Effros, Junge, and Ruan [EJR00] that the predual of von Neumann algebras are 1-locally reflexive. Another property needed in our discussion is completely contractive approximation property (CCAP). A operator space $E$ is CCAP if there exists a
Let $\Phi_\alpha : E \to E$ such that for any $x$, $\Phi_\alpha(x) \to x$ in norm. In the setting of operator spaces, this is an analog of $w^*$-CPAP (or injectivity).

The following lemma shows that these two properties combined give the desired norm attaining property similar to Proposition 3.6. Throughout this section, we write $CB$ for completely bounded and $CC$ for completely contractive.

**Lemma 5.1.** Let $E$ be CCAP. Then $CB(E,G) \subset CB(E,G^{**})$ is $w^*$-dense in the sense of $CB(E,G^{**}) = (E \hat{\otimes} G^*)^*$. If in additional $G$ is 1-locally reflexive, then

$$\|\rho\|_{\hat{E} \hat{\otimes} \hat{G}} = \sup \{ Re \langle \Psi, \rho \rangle | \Psi : E \to G \text{ CC} \}.$$ 

**Proof.** Let $\Phi_\alpha : E \to E$ be a net of CC maps such that $\Phi_\alpha(x) \to x$ in norm for any $x \in E$. For $\rho \in E \hat{\otimes} G^*$ with $\|\rho\|_{E \hat{\otimes} G} = 1$, we can choose a finite tensor sum $\rho_0 = \sum_{j=1}^n x_j \otimes y_j$ such that $\|\rho - \rho_0\|_{E \hat{\otimes} G} \leq \epsilon$. Then for $T : E \to G^{**}$ with $\|T\|_{cb} = 1$, there exists an $\alpha$ such that

$$|\langle T \circ \Phi_\alpha - T, \rho \rangle| \leq |\langle T \circ \Phi_\alpha - T, \rho - \rho_0 \rangle| + |\langle T \circ \Phi_\alpha - T, \rho_0 \rangle| \leq 2\epsilon + \epsilon.$$ 

Let $E_\alpha$ be the range of $\Phi_\alpha$ as a finite dimensional subspace of $E$ and $T|_{E_\alpha} \in CB(E_\alpha, G^{**})$ be the restriction of $T$ to $E_\alpha$. There exists $T_\alpha \in CB(E_\alpha, G)$ such that

$$|\langle T_\alpha - T, \Phi_\alpha \otimes \text{id}(\rho_0) \rangle| = |\langle (T_\alpha - T) \circ \Phi_\alpha, \rho_0 \rangle| \leq \epsilon.$$ 

Therefore $T_\alpha \circ \Phi_\alpha : E \to G$ is CB and

$$|\langle T_\alpha \circ \Phi_\alpha - T, \rho \rangle| \leq |\langle T_\alpha \circ \Phi_\alpha - T, \rho \rangle| + |\langle (T_\alpha - T) \circ \Phi_\alpha, \rho - \rho_0 \rangle| + |\langle (T_\alpha - T) \circ \Phi_\alpha, \rho_0 \rangle| \leq 3\epsilon + 2\epsilon + \epsilon = 6\epsilon$$

which proves the $w^*$-density of $CB(E,G) \subset CB(E,G^{**})$. If $G$ is 1-locally reflexive, $T_\alpha$ and $T_\alpha \circ \Phi_\alpha$ can be CC because of the isometry $CB(E_\alpha, G^{**}) \cong CB(E_\alpha, G)^{**}$. 

The following theorem is the analog of quantum majorization and channel factorization in the abstract operator space setting.

**Theorem 5.2.** Let $E, F, G$ be operator spaces. Suppose one of the following condition holds:

a) $G$ is reflexive;

b) $G$ is 1-locally reflective and $F$ is CCAP

Then the following two statements hold:

i) For $\rho \in E \hat{\otimes} F$ and $\sigma \in E \hat{\otimes} G$, there exists a sequence of CC maps $u_n : F \to G$ such that $\text{id} \otimes u_n(\rho) \to \sigma$ in the norm of $E \hat{\otimes} G$ if and only if for any CB map $v : E \to G^*$,

$$\|v \otimes \text{id}(\rho)\|_{G^* \hat{\otimes} F} \geq \|v \otimes \text{id}(\sigma)\|_{G^* \hat{\otimes} G}.$$ 

ii) For $T \in CB(E, F)$ and $S \in CB(E, G)$, there exists a net of CC $u_\alpha : F \to G$ such that $u_\alpha \circ T \to S$ in the point-weak topology if and only if for any $x \in E \hat{\otimes} G^*$,

$$\|T \otimes \text{id}(x)\|_{F \hat{\otimes} G} \geq \|S \otimes \text{id}(x)\|_{G \hat{\otimes} G^*}.$$
Proof. i) The “only if” direction is easy. For the if part, consider the norm-closed convex set

\[ C(\rho) = \{ \text{id} \otimes u(\rho) | u : F \to G, CC \} \subset E \hat{\otimes} G. \]

If \( \sigma \notin C \), there exists a \( v \in CB(E, G^*) = (E \hat{\otimes} G)^* \) such that

\[ \text{Re} \left\langle v, \sigma \right\rangle > \sup_{u} \text{Re} \left\langle v, \text{id} \otimes u(\rho) \right\rangle. \]

Let \( \iota_G : G \to G^{**} \) be the embedding. Note that

\[ \text{Re} \left\langle v, \sigma \right\rangle = \text{Re} \left( \iota_G, v \otimes \text{id}(\sigma) \right) \leq \| v \otimes \text{id}(\sigma) \|_{G^* \hat{\otimes} G}, \]

\[ \sup_{u} \text{Re} \left\langle v, \text{id} \otimes u(\rho) \right\rangle = \sup_{u} \text{Re} \left\langle u, v \otimes \text{id}(\rho) \right\rangle = \| v \otimes \text{id}(\rho) \|_{G^* \hat{\otimes} F} \]

where the last equality follows Lemma 5.1.

ii) Suppose \( u_\alpha \) is a net of CC maps such that \( u_\alpha \circ T \to S \) in the point-wise topology. Then for any \( R \in CB(G^*, G^*) = (G \hat{\otimes} G^*)^* \) and \( x \in E \hat{\otimes} G^* \)

\[ \lim_{\alpha} \langle R, u_\alpha \circ T \otimes \text{id}(x) \rangle = \lim_{\alpha} \langle u_\alpha \circ T, \text{id} \otimes R(x) \rangle = \langle S, \text{id} \otimes R(x) \rangle = \langle R, S \otimes \text{id}(x) \rangle \]

which implies \( \| T \otimes \text{id}(x) \|_{F \hat{\otimes} G^*} \geq \| S \otimes \text{id}(x) \|_{G \hat{\otimes} G^*} \). For the converse, consider the \( w^* \)-closure of convex set

\[ C(T) = \overline{\{ u \circ T | u : F \to G, CC \}}^w \subset CB(E, G^{**}) = (E \hat{\otimes} G^*)^*. \]

If \( S \notin C \), there exists a \( \rho \in E \hat{\otimes} G^* \) such that

\[ \text{Re} \left\langle S, \rho \right\rangle > \sup_{u} \text{Re} \left\langle u \circ T, \rho \right\rangle. \]

By a density argument, we can further assume \( \rho \in E \otimes G^* \) in the algebraic tensor product. Note that

\[ \text{Re} \left\langle S, \rho \right\rangle = \text{Re} \left( \rho, S \otimes \text{id}(\sigma) \right) \leq \| S \otimes \text{id}(\sigma) \|_{G \otimes G^*}, \]

\[ \sup_{u} \text{Re} \left\langle u \circ T, \rho \right\rangle = \sup_{u} \text{Re} \left\langle u, T \otimes \text{id}(\rho) \right\rangle = \| \rho \|_{F \hat{\otimes} G^*} \]

where again the last equality uses Lemma 5.1. \( \blacksquare \)

The following proposition discusses the case when the limits in above theorem can be replaced by equality.

**Proposition 5.3.** Let \( E, F, G \) be operator spaces. Let \( T \in CB(E, F) \) and \( \rho \in E \hat{\otimes} F \). Suppose \( G = (G_*)^* \) is a dual space. Then \( \{ u \circ T | u : F \to G, CC \} \) is \( w^* \)-closed in \( CB(E, G) \). If, in addition, \( E \) is CCAP or \( G \) is reflexive, \( \{ \text{id} \otimes u(\rho) | u : F \to G, CC \} \) is norm-closed in \( E \hat{\otimes} G \).

**Proof.** To prove the first statement, let \( u_k : F \to G \) be a sequence of CC maps such that \( \lim_k u_k \circ T = S \) in the \( w^* \)-topology of \( CB(E, G) = (E \hat{\otimes} G_*)^* \). Because \( CB(F, G) = (F \hat{\otimes} G_*)^* \), we choose \( u \) as \( w^* \)-limit of \( (u_k) \) such that the subsequence \( u_{k_i} \to u \). Then \( u_{k_i} \circ T \to u \circ T \) in the point \( w^* \)-topology hence \( S = T \). For the second statement, we assume \( E \) is CCAP or \( G \) is reflexive. Let \( u_k : F \to G \) be a sequence of CC such that
id ⊗ u_k(ρ) → σ in the norm of $E \hat{\otimes} G$. Choose a subsequence $u_k \to u$ in the $w^*$-topology for some CC $u$. For any $T \in CB(E, G^*)$, 

$$\lim_i \langle T, id \otimes u_k(\rho) \rangle = \lim_i \langle u_k, T \otimes id(\rho) \rangle = \langle T, u \otimes id(\rho) \rangle = \langle T, u \otimes id(\rho) \rangle.$$ 

Thus $id \otimes u_k(\rho) \to id \otimes u(\rho)$ in $E \hat{\otimes} G$ with the topology induced by $CB(E, G^*) \subset CB(E, G^*)$. Note that by Lemma 5.1, this topology is separating. Hence we have $\sigma = \lim_i id \otimes u_k(\rho) = id \otimes u(\rho)$.

Theorem 5.2 also holds for Banach space tensor products. We can replace the operator space concepts with their Banach space counterparts: replace “operator spaces” by “Banach spaces”, “CB (resp. CC)” by “bounded (resp. contractive)” and “CCAP” by “contractive approximation property (or 1-AP)”. Moreover, all Banach spaces have 1-local reflexivity. We refer to the book [LT96] for definitions of the above mentioned Banach space concepts. Here we state the result analogous to Theorem 5.2. Let $\otimes_\pi$ denote the Banach space projective tensor product and $B(E, F)$ be the set of bounded maps from Banach space $E$ to $F$.

**Theorem 5.4.** Let $E, F, G$ be Banach spaces. Suppose one of the following conditions holds:

a) $G$ is reflexive;

b) $F$ is nuclear.

Then the following two statements hold:

i) for $\rho \in E \otimes_\pi F$ and $\sigma \in E \otimes_\pi G$, there exists a sequence of contraction map $u_n : F \to G$ such that $id \otimes u_n(\rho) \to \sigma$ in the norm of $E \otimes_\pi G$ if and only if for any bounded map $v : E \to G^*$, 

$$\| v \otimes id(\rho) \|_{G^* \otimes_\pi F} \geq \| v \otimes id(\sigma) \|_{G^* \otimes_\pi G}.$$ 

ii) for $T \in B(E, F)$ and $S \in B(E, G)$, there exists a net of contraction $u_\alpha : F \to G$ such that $u_\alpha \circ T \to S$ in the point-weak topology if and only if for any $x \in E \otimes_\pi G^*$, 

$$\| T \otimes id(x) \|_{F \otimes_\pi G^*} \geq \| S \otimes id(x) \|_{G \otimes_\pi G^*}.$$ 

The proof is identical to Theorem 5.2 and the details are left to the reader.

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References

[BDS14] Francesco Buscemi, Nilanjana Datta, and Sergii Strelchuk, *Game-theoretic characterization of antidegradable channels*, Journal of Mathematical Physics 55 (2014), no. 9, 092202. ↑1

[BP91] David P Blecher and Vern I Paulsen, *Tensor products of operator spaces*, Journal of Functional Analysis 99 (1991), no. 2, 262–292. ↑2, 5

[Bus12] Francesco Buscemi, *Comparison of quantum statistical models: equivalent conditions for sufficiency*, Communications in Mathematical Physics 310 (2012), no. 3, 625–647. ↑

[Che09] Anthony Chefly, *The quantum blackwell theorem and minimum error state discrimination*, arXiv preprint arXiv:0907.0866 (2009). ↑1

[Con76] Alain Connes, *Classification of injective factors cases II1, II∞, IIIλ, λ ≠ 1*, Annals of Mathematics (1976), 73–115. ↑15

[EJ00] Edward G Effros and Ruan Zhong Jin, *Operator spaces*, Clarendon Press, 2000. ↑2, 4, 5

[EJR00] Edward G Effros, Marius Junge, and Zhong-Jin Ruan, *Integral mappings and the principle of local reflexivity for noncommutative L1-spaces*, Annals of Mathematics 151 (2000), no. 1, 59–92. ↑34

[EW97] Edward G Effros and Soren Winkler, *Matrix convexity: operator analogues of the bipolar and Hahn–Banach theorems*, Journal of functional analysis 144 (1997), no. 1, 117–152. ↑33

[GJB+18] Gilad Gour, David Jennings, Francesco Buscemi, Runyao Duan, and Iman Marvian, *Quantum majorization and a complete set of entropic conditions for quantum thermodynamics*, Nature communications 9 (2018), no. 1, 5352. ↑1, 2, 6, 22, 24

[Gou19] Gilad Gour, *Comparison of quantum channels by superchannels*, IEEE Transactions on Information Theory (2019). ↑2

[Haa85] Uffe Haagerup, *Injectivity and decomposition of completely bounded maps*, Operator algebras and their connections with topology and ergodic theory, 1985, pp. 170–222. ↑3, 15, 16, 17

[Har29] Godfrey H Hardy, *Some simple inequalities satisfied by convex functions*, Messenger Math. 58 (1929), 145–152. ↑1

[HKM14] J William Helton, Igor Klep, and Scott McCullough, *The tracial Hahn-Banach theorem, polar duals, matrix convex sets, and projections of free spectrahedra*, arXiv preprint arXiv:1407.8198 (2014). ↑3, 22, 32, 33, 34

[HOW05] Michal Horodecki, Jonathan Oppenheim, and Andreas Winter, *Partial quantum information*, Nature 436 (2005), no. 7051, 673. ↑2

[Jen16] Anna Jenčová, *Comparison of quantum channels and statistical experiments*, 2016 IEEE international symposium on information theory (ISIT), 2016, pp. 2249–2253. ↑1, 3, 6, 27, 29

[JX07] Marius Junge and Quanhua Xu, *Noncommutative maximal ergodic theorems*, Journal of the American Mathematical Society 20 (2007), no. 2, 385–439. ↑22

[LT96] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces I and II, classics in mathematics*, Springer-Verlag, 1996. ↑37

[MLDS+13] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel, *On quantum rényi entropies: A new generalization and some properties*, Journal of Mathematical Physics 54 (2013), no. 12, 122203. ↑2

[MOA79] Albert W Marshall, Ingram Olkin, and Barry C Arnold, *Inequalities: theory of majorization and its applications*, Vol. 143, Springer, 1979. ↑1

[Pis03] Gilles Pisier, *Introduction to operator space theory*, Vol. 294, Cambridge University Press, 2003. ↑4, 15

[Pis98] _, *Non-commutative vector valued Lp-spaces and completely p-summing maps*, Astérisque (1998). ↑2, 5, 7, 13, 14, 31, 32

[Shm05] Eran Shmaya, *Comparison of information structures and completely positive maps*, Journal of Physics A: Mathematical and General 38 (2005), no. 44, 9717. ↑1
[Tak79] Masamichi Takesaki, *Theory of operator algebras. I*, Springer-Verlag, 1979. ↑16

[TCR09] Marco Tomamichel, Roger Colbeck, and Renato Renner, *A fully quantum asymptotic equipartition property*, IEEE Transactions on information theory 55 (2009), no. 12, 5840–5847. ↑2, 31

[Tom15] Marco Tomamichel, *Quantum information processing with finite resources: Mathematical foundations*, Vol. 5, Springer, 2015. ↑31

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