The relation between logarithmic corrections to the area law for black hole entropy, due to thermal fluctuations around an equilibrium canonical ensemble, and those originating from quantum spacetime fluctuations within a microcanonical framework, is explored for three and four dimensional asymptotically anti-de-Sitter black holes. For the BTZ black hole, the two logarithmic corrections are seen to precisely cancel each other, while for four dimensional adS-Schwarzschild black holes a partial cancellation is obtained. We discuss the possibility of extending the analysis to asymptotically flat black holes.

I. INTRODUCTION

Non-perturbative canonical Quantum General Relativity (QGR) [1] reveals an appealing picture [2] of microstates underlying the area law [3,4] for black hole entropy. One begins with a classical connection formulation of a black hole horizon where the usual event horizon (appropriate to a stationary situation) is replaced by an isolated horizon, defined entirely by boundary conditions locally, i.e., without reference to asymptotic null infinity \( I_\pm \). These boundary conditions lead uniquely to a Chern Simons theory ‘living’ on the horizon, with the bulk geometry (the pullback of the densitised triad to the two-sphere foliation of the horizon) playing the role of source currents for that theory. Quantization, using the spin network formalism of canonical QGR lead to a description of spacetime fluctuations on the horizon in terms of states of this Chern Simons theory. Links of the bulk spin network puncturing the horizon provide pointlike sources for this Chern Simons theory. Black hole entropy, defined in terms of the degeneracy of the boundary Chern Simons states, can be evaluated either by solving the theory directly [2], or by relating it to the dimensionality of the Hilbert space of the two dimensional boundary Wess-Zumino-Witten theory [5]. This dimensionality can be evaluated using standard techniques of two dimensional conformal field theory. In both cases, the area law ensues, with a specific choice of the Barbero-Immirzi parameter needed to obtain Hawking’s normalization of it. But the QGR formulation does more than simply reproduce a law that has been anticipated decades ago on the basis of semiclassical arguments; it provides an entire infinite series of quantum corrections to the area law, in decreasing powers of horizon area for macroscopic black holes with large horizons. Each term in the series has a fixed, finite coefficient, completely calculable in principle. The leading correction is logarithmic in the area, with the coefficient \(-3/2\) [6], [7].

From a thermodynamic perspective, the ‘log area’ corrections can be thought of as ‘finite size’ corrections to the thermodynamic limit of large areas. The isolated horizon boundary conditions require that nothing crosses the horizon, although radiation may exist arbitrarily close to it. The resulting constancy of the horizon area is therefore a restriction of crucial importance, implying that a microcanonical framework has been employed to calculate the entropy. \(^1\) This entropy is thus rightly called the microcanonical entropy. The horizon area never undergoes thermal fluctuations, because by fixing the area (energy) of the black hole horizon, such fluctuations have been eliminated. However, this does appear to be overly restrictive in physical terms; even the slightest amount of matter or radiation crossing the horizon will violate it. In other words, a canonical, rather than microcanonical, ensemble seems to be more suitable on physical grounds, since it allows thermal fluctuations of various quantities like energy (area).

The effect of such thermal fluctuations on equilibrium entropy has been investigated for asymptotically anti-de-Sitter black holes [9] using a canonical ensemble [10], and found to produce leading corrections logarithmic in the area (for large area) with calculable coefficients. These corrections were computed for adS-Schwarzschild black holes of arbitrary dimension including and beyond four, and also for the three dimensional BTZ black hole. Links were also established with an underlying conformal field theory structure.

The purpose of this paper is to clarify the precise relationship between the logarithmic corrections to the area law originating from quantum fluctuations of spacetime geometry, and those due to thermal fluctuations in a canonical ensemble of black holes where the area is no longer constrained to be a constant. This is done by deriving a relation,
using standard equilibrium statistical mechanics, between the canonical entropy (including fluctuations around thermal equilibrium), and the microcanonical entropy which equals the logarithm of the density of states. The distinct roles played by finite size corrections and thermal fluctuation corrections to black hole entropy are thus made explicit.

In Section II, major tenets of the canonical QGR computation of the microcanonical entropy are reviewed for general four dimensional non-rotating black holes. In Section III, the derivation of the relation between canonical and microcanonical entropy for general equilibrium statistical mechanical systems is presented. The effect of thermal fluctuations is included in the derivation. In subsection B of the same section, the relation is employed to compute the canonical entropy of asymptotically adS black holes in three and four dimensions, using results of earlier computation of the microcanonical entropy. The analysis thus far presented does not extend to the more commonly studied asymptotically flat black holes, which seem to have some kind of instability, variously described in the literature as ‘negative specific heat’, ‘exponentially exploding density of states’ and so on. In Section V we show how this problem manifests in a straightforward manner through the relation derived in Section III. We discuss how the problem may perhaps be obviated within canonical QGR.

II. MICROCANONICAL ENTROPY

A. Classical Aspects

This calculation [2] employs a classical Hamiltonian formulation of general relativity, in terms of a canonical pair consisting of the real $SU(2)$ connection $\gamma A'_{a} \equiv \Gamma_{a}^{i} + \gamma K_{a}^{i}$ and the rescaled densitized triad $\gamma^{-1} E_{a}^{i}$, with $E_{a}^{i} \equiv \det e_{a}^{i}$, where $e_{a}^{i}$ is the triad on a chosen spatial slice $\mathcal{M}$. Here, $\Gamma_{a}^{i} \equiv \frac{1}{2} q_{ab} e_{jk} \Gamma^{ijk}$, with $\Gamma^{ijk}$ being the pullback of the Levi-Civita spin connection to the spatial slice under consideration, and $q_{ab}$ is the 3-metric on the slice; the extrinsic curvature $K_{a}^{i} \equiv q_{ab} \Gamma^{abcd}$; $\gamma$, the Barbero-Immirzi parameter [11] is a real positive parameter. Four dimensional local Lorentz invariance has been partially gauge fixed to the ‘time gauge’ $e_{a}^{0} = -n_{a}$ where $n_{a}$ is the normal to the spatial slice. This choice leaves the residual gauge group to be $SU(2)$. It is convenient to introduce the quantity $\gamma \Sigma_{ab}^{ij} \equiv \gamma^{-1} e_{[a}^{i} e_{b]}^{j}$, in terms of which the symplectic two-form of general relativity can be expressed as

$$\Omega = \frac{1}{8\pi G} \int_{\mathcal{M}} Tr[\delta^{i} \Sigma \wedge \delta^{j} A' - \delta^{i} \Sigma' \wedge \delta^{j} A].$$

(1)

The expression (1) of course is subject to modification by boundary terms arising from the presence of boundaries of spacetime. The black hole horizon, assumed to have the topology $S^{2} \otimes R$, is intersected by $\mathcal{M}$ in a two-sphere which thus plays the role of an inner boundary.

Rather than using the notion of event horizon appropriate to stationary situations studied in earlier literature [12], we adopt here the concept of ‘isolated’ horizon [13]. This has the advantage of being characterized completely locally, without requiring a global timelike Killing vector field. The characterization, for non-rotating situations, involves a null surface $H$ with topology as assumed above, with preferred foliation by two-spheres and ruling by lines transverse to the spheres. $l^{a}$ and $n_{a}$ are null vector fields satisfying $l^{a} n_{a} = -1$ on the isolated horizon. $l^{a}$ is a tangent vector to the horizon, which is assumed to be geodesic, twist-free, divergenceless and most importantly, non-expanding. The Raychaudhuri equation is then used to prove that it is also free of shear. Similarly, the null normal one-form field $n_{a}$ is assumed to be shear- and twist-free, and have negative spherical expansion. Finally, while stationarity is not a part of the characterization of an isolated horizon, the vector direction field $l^{a}$ can be shown [13] to behave like a Killing vector field on the horizon, satisfying

$$l^{a} \nabla_{a} l^{b} = \kappa l^{b}.$$

(2)

Here, $\kappa$ is the acceleration of $l^{a}$ on the isolated horizon. Unlike standard surface gravity whose normalization is fixed by the requirement that the global timelike Killing vector generate time translations at spatial infinity, the normalization of $\kappa$ here varies with rescaling of $l^{a}$.

These features imply that while gravitational or other radiation may exist arbitrarily close to the horizon, nothing actually crosses the horizon, thereby emulating an ‘equilibrium’ situation. This, in turn, means that the area $A_{H}$ of the isolated horizon must be a constant. Lifting of this restriction leads to dynamical variants (the so-called ‘dynamical’ horizons) which have also been studied [14]; we shall however not consider these here.

The actual implementation of these properties of the isolated horizon require boundary conditions on the phase space variables on the 2-sphere foliate of the horizon. Recalling that the horizon is an inner boundary of spacetime, it is obvious that one needs to add boundary terms to the classical Einstein action, in order that the variational principle
can be used to derive equations of motion. It turns out [2], [13] that the ‘boundary action’ $S_H$ that one must add to the Einstein action (in the purely gravitational case)

$$S_E = -\frac{i}{8\pi G} \int_M Tr \Sigma \wedge F, \quad (3)$$

is an $SU(2)$ Chern Simons (CS) action

$$S_H = -\frac{i}{8\pi G} \frac{A_H}{4\pi} \int_H Tr [A \wedge dA + \frac{2}{3} A \wedge A \wedge A], \quad (4)$$

where, now, $A$ is the CS connection, and $F$ the corresponding curvature. The resultant modification to the symplectic structure (1) is given by the CS symplectic two-form

$$\Omega_H = -\frac{k}{2\pi} \oint_S Tr [\delta^\gamma A \wedge \delta^\gamma A^r], \quad (5)$$

where, $k \equiv A_H/8\pi \gamma G$. In writing the boundary action (4), we have suppressed other terms like the boundary term at infinity.

It is easy to see that the variational principle for the full action is valid, provided we have, on the two-sphere foliation of $H$, the restriction,

$$\frac{k}{2\pi} F_{ab}^i + \Sigma_{ab}^i = 0. \quad (6)$$

Eq. (6) has the physical interpretation of Gauss law for the CS theory, with the two-form $\Sigma$ playing the role of source current. We shall see shortly that this has crucial implications for the quantum version of the theory.

**B. Quantum Aspects: General**

The classical configuration space consists of the space of smooth, real $SU(2)$ Lie Algebra-valued connections modulo gauge transformations [15]. Alternatively, the space can be described in terms of three dimensional oriented, piecewise analytic networks or graphs embedded in the spatial slice $M$ [15]. Consider a particular graph $C$ with $n$ links (or edges) $e_1, \ldots, e_n$; consider also the pullback of the connection $A$ to $C$. Consider the holonomies defined as

$$h_C(e_i) \equiv \mathcal{P} \exp \oint_{e_i \in C} \gamma A_C, \quad i = 1, 2, \ldots, n, \quad (7)$$

where $\gamma A_C$ represents the restriction of the connection to the graph $C$; these span the configuration space $\mathcal{A}_C$ of connections on the graph $C$. This space consists of $[SU(2)]^n$ group elements obtained as $n$-fold compositions of $SU(2)$ group elements characterised by the spin $j_i$ of the edge $e_i$ for $i = 1, 2, \ldots, n$. The edges of $C$ terminate at vertices $v_1, \ldots, v_n$ which, in their turn, are characterised by group elements $g(v_1), \ldots, g(v_m)$, which together constitute a set of $[SU(2)]^m$ group elements for a given graph $C$. The union of spaces $\mathcal{A}_C$ for all networks is then an equally good description of the classical configuration space.

The transition to the *quantum* configuration space is made, first by enhancing the space of connections to include connections $\gamma A$ which are not smooth but distributional, and then considering the space $\mathcal{H}_C$ of square-integrable functions $\Psi_C[\gamma A]$ of connections. For the integration measure, one uses $n$-copies of the $SU(2)$-invariant Haar measure. For a given network $C$, the wave function $\Psi_C[\gamma A]$ can be expressed in terms of a smooth function $\psi$ of the holonomies $h_C(e_1), \ldots, h_C(e_n)$ of distributional connections,

$$\Psi_C[\gamma A] = \psi(h_C(e_1), \ldots, h_C(e_n)). \quad (8)$$

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2Strictly speaking, the boundary conditions considered by [2] involve a partial gauge fixing whereby the only independent connection on the horizon is actually an internal radial $U(1)$ projection of the $SU(2)$ CS connection. However, we ignore this subtlety at this point and continue to work with the $SU(2)$ CS theory. The modification to the final answer, had we chosen not to ignore this subtlety, will be discussed later.
The inner product of these wave functions can be defined as
\[ \langle \Psi_1C, \Psi_2C \rangle = \int d\mu \bar{\Psi}_1C \Psi_2C. \] (9)

Basic dynamical variables include the holonomy operator \( \hat{h}_C(e) \) and the operator version of the canonically conjugate \( \gamma \)-rescaled densitized triad \( \hat{E}_a^i \). The holonomy operator acts diagonally on the wave functions,
\[ [\hat{h}_C(e) \Psi_C][\gamma \mathcal{A}] = \hat{h}_C(e) \Psi_C[\gamma \mathcal{A}]. \] (10)

The canonical conjugate densitized triad operators \( \hat{E}_a^i \) act as derivatives on \( \Psi_C[\gamma \mathcal{A}] \):
\[ \gamma \hat{E}_a^i \Psi[\gamma \mathcal{A}] = \frac{\gamma l_p^2}{i} \delta A_i^a \Psi[\gamma \mathcal{A}]. \] (11)

One defines the kinematical Hilbert space \( \mathcal{H} \) as the union of the spaces of wave functions \( \Psi_C \) for all networks.\(^3\)

Particularly convenient bases for the wave functions are the spin network bases. Typically, the spin network (spinet) states can be schematically exhibited as
\[ \psi_C(\{h_C\}; \{v\}) = \sum \prod_{v \in C} I_v \prod_i D_i^{..}, \] (12)
where, \( D^i \) is the \( SU(2) \) representation matrix corresponding to the \( i \)th edge of the network \( C \), carrying spin \( j_i \), and \( I_v \) is the invariant \( SU(2) \) tensor inserted at the vertex \( v \). If one considers all possible spin networks, the set of spinet states corresponding to these is dense in the kinematical Hilbert space \( \mathcal{H} \). Spinet states diagonalize the densitized triad (momentum) operators and hence operators corresponding to geometrical observables like area, volume, etc. constructed out of the the triad operators. The spectra of these observables turn out to be discrete; e.g., for the area operator corresponding to the area of a two dimensional spacelike physical surface \( s \) (like the intersection of a spatial slice with a black hole horizon), one considers the spins \( j_1, j_2, \ldots, j_p \) on \( s \) at the \( p \) punctures made by the \( p \) edges of the spinet assumed to intersect the surface. The area operator is defined as \[ \hat{A}_s \equiv \left\{ \sqrt{n_a n_b \hat{E}_a^i \hat{E}_b^i} \right\}_\text{reg}, \] (13)
where, \( n_a \) is the normal to the surface, and \( \text{reg} \) indicates that the operator expression within the braces is suitably regularized. The eigenspectrum turns out to be \[ a_s(p; \{j_i\}) = 8\pi g l_p^2 \sum_{i=1}^{p} \sqrt{j_i(j_i + 1)}. \] (14)

Spinet basis states correspond to networks without any ‘hanging’ edge, so that they transform as gauge singlets under the gauge group \( SU(2) \). Furthermore, invariance under spatial diffeomorphisms is implemented by the stipulation that the length of any edge of any graph is without physical significance.

### C. Quantum Aspects: entropy calculation

As discussed in subsection A, the sphere \( S_M \) formed by the intersection of the isolated horizon and a spatial slice \( M \) can be thought of as an inner boundary of \( M \). The dynamics of the isolated horizon is described by an \( SU(2) \) Chern Simons theory with the bulk gravitational degrees of freedom playing the role of source current. This picture can be implemented at the quantum level in a straightforward manner. Because of the isolation implied by the boundary conditions, the kinematical Hilbert space \( \mathcal{H} \) can be decomposed as

\(^3\)Unfortunately, \( \mathcal{H} \) is not the physical Hilbert space of the theory; that space is the algebraic dual of \( \mathcal{H} \) with no natural scalar product defined on it. However, for the purpose of calculation of the microcanonical entropy, it will turn out to be adequate to use \( \mathcal{H} \).
\[ \mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S , \]  
where, \( \mathcal{H}_V (\mathcal{H}_S) \) corresponds to quantum states with support on the spatial slice \( M \) (on the inner boundary, i.e., these are the Chern Simons states). The boundary conditions also imply the Chern Simons Gauss law, eq. (6); the quantum operator version of this equation may be expressed as

\[ \frac{k}{2\pi} \mathbf{1} \otimes \hat{F}_{ab}^i + \hat{\Sigma}_{ab} \otimes \mathbf{1} = 0 \] on \( S_H \).

Now, the bulk spinet states diagonalize the operator \( \hat{\Sigma} \) with distributional eigenvalues,

\[ \hat{\Sigma}(\vec{x}) |\psi\rangle_V \otimes |\psi\rangle_S = \gamma l_p^2 \sum_{i=1}^p \lambda(j_i) \delta^{(2)}(\vec{x}, \vec{x}_i) |\psi\rangle_V \otimes |\psi\rangle_S . \]  

Eq. (16) then requires that the boundary Chern Simons states also diagonalize the Chern Simons curvature operator \( \hat{F} \). In other words, edges of the bulk spin network punctures the horizon foliate \( S_H \), endowing the \( i \)th puncture with a deficit angle \( \theta_i \equiv \theta(j_i) \) for \( i = 1, 2, \ldots, p \), such that

\[ \sum_{i=1}^p \theta(j_i) = 4\pi . \]  

The curvature on \( S_H \) is thus vanishingly small everywhere else except at the location of the punctures. This manner of building up the curvature of the two-sphere \( S_H \) out of a large but finite number of deficit angles requires that the number of such angles must be as large as possible. This is achieved for the smallest possible value of all spins \( j_i \), namely \( j_i = 1/2 \) for all \( i \). We shall come back to this point later.

The calculation of the entropy now proceeds by treating the isolated horizon as a microcanonical ensemble with fixed area. Recalling the semiclassical relationship between horizon area and mass of the isolated horizon, this is equivalent to considering a standard equilibrium microcanonical ensemble where the (average) energy of the ensemble does not fluctuate thermally. The number of configurations of such a system is equal to the exponential of the microcanonical entropy \( S_{MC} \). Likewise, in this case, the number of boundary Chern Simons states \( dim \mathcal{H}_S \) with pointlike sources, as depicted in eq. (16) (keeping (17) in view) yields \( exp S_{MC} \). This number has been calculated for all four dimensional non-rotating isolated horizons \([2],[5]\) of large macroscopic fixed horizon area \( A_H \gg l_p^2 \). In ref. \([5]\), the computation makes use of the well-known relation between the dimensionality of the boundary Chern Simons Hilbert space and the number of conformal blocks of the corresponding two dimensional \( SU(2)_k \) Wess-Zumino-Witten model that ‘lives’ on the punctured two-sphere \( S_H \). This number is given by

\[ dim \mathcal{H}_S = \sum_p \prod_{i=1}^p \sum_{j_i} \mathcal{N}(p, \{ j_i \}) , \]  

subject to the constraint that the area eigenvalues are fixed (to within a factor of the Planck area) to the constant macroscopic area \( A_H \),

\[ A_H = 8\pi \gamma l_p^2 \sum_{i=1}^p \sqrt{j_i(j_i+1)} , \]  

where,

\[ \mathcal{N}(p, \{ j_i \}) = \sum_{m_1=-j_1}^{j_1} \cdots \sum_{m_p=-j_p}^{j_p} \left[ \delta^{(p)}(m_1,0) - \frac{1}{2} \delta^{(p-1)}(m_1,1) - \frac{1}{2} \delta^{(p-1)}(m_1,-1) \right] . \]  

Instead of the area constraint, one may now recall eq. (18) which also is a constraint on the spins and number of punctures. Using this result in the area formula (20) yields the maximal number of punctures

\[ p_0 = \frac{A_H}{4\pi \sqrt{3} \gamma l_p^2} . \]  

The corresponding number of Chern Simons states for this assignment of spins is given via (21) by
Now, the (microcanonical) entropy of the isolated horizon is given by

\[ S_{IH} \equiv \log \dim \mathcal{H}_S , \] (24)
as remarked earlier. For isolated horizons with large macroscopic area, the largest contribution to the rhs of eq.(19) is given by the contribution of the single term of the multiple sum, corresponding to \( j_i = 1/2 \forall i \) and \( p = p_0 \). This contribution dominates all others in the multiple sum, so that, one has, using eq.(23), the microcanonical entropy formula [6]-[8]

\[ S_{IH} = S_{MC} = S_{BH} + \Delta_Q , \] (25)

where,

\[ S_{BH} \equiv A_H / 4l_p^2, \] (26)
is the Bekenstein-Hawking Area Law (BHAL), and we have set the Barbero-Immirzi parameter \( \gamma = \log 2/\pi \sqrt{3} \) [2] in order to reproduce the BHAL with the correct normalization. \( \Delta_Q \), given by

\[ \Delta_Q = - \frac{3}{2} \log S_{BH} + \text{const.} + O(S_{BH}^{-1}) , \] (27)

constitutes an infinite series (in decreasing powers of \( S_{BH} \)) of corrections to the BHAL due to quantum fluctuations of spacetime, and can be thought of as ‘finite size’ corrections. One important aspect of the formula (25) is that the coefficient of each correction term is finite and unambiguously calculable, after \( \gamma \) has been fixed as mentioned.

### III. CANONICAL ENTROPY

#### A. Thermal fluctuations in a canonical ensemble

In this subsection we present a derivation of the canonical entropy of a standard equilibrium canonical ensemble, when small thermal fluctuations around equilibrium are taken into account. The derivation is a variant of the version given in [9] and also in textbooks. We begin with the formula for the canonical partition function of a classical system in equilibrium

\[ Z_C(\beta) = \int_0^\infty dE \exp -\beta E \rho(E) , \] (28)

where, \( \rho(E) \) is the density of states. In what follows, we shall employ the identification \( \rho(E) \equiv \exp S_{MC}(E) \), where, \( S_{MC}(E) \) is the microcanonical entropy of an isolated subsystem whose energy is held fixed at \( E \). The integral in eq.(28) can be performed in general by the saddle point approximation, provided the microcanonical entropy \( S_{MC}(E) \) can be Taylor-expanded around the average equilibrium energy \( E_0 \),

\[ S_{MC}(E) = S_{MC}(E_0) + (E - E_0) S'_{MC}(E_0) + \frac{1}{2} (E - E_0)^2 S''_{MC}(E_0) + \ldots , \] (29)

and higher order terms in powers of the energy fluctuation represented by the \( \ldots \) in this expansion can be neglected in comparison to terms of second order. The integration is then performed by requiring that the inverse temperature \( \beta \) is determined in terms of the average energy \( E_0 \),

\[ \beta = S'_{MC}(E_0) . \] (30)

The resulting Gaussian integration leads to the result

\[ Z_C(\beta) = e^{[\beta E_0 + S_{MC}(E_0)]} \left[ \frac{2\pi}{-S''_{MC}(E_0)} \right]^{1/2} . \] (31)

Using the standard formula from equilibrium statistical mechanics,
\[ S_C = \beta E_0 + \log Z_C \]  
(32)

It is easy to deduce that the canonical entropy is given in terms of the microcanonical entropy by

\[ S_C(E_0) = S_{MC}(E_0) + \Delta F, \]  
(33)

where,

\[ \Delta F = \frac{1}{2} \log[-S'_{MC}(E_0)], \]  
(34)

is the leading correction to the canonical entropy due to thermal fluctuations in the energy. Using the definition of the specific heat of the system

\[ C = -\beta^2 \left( \frac{\partial E_0}{\partial \beta} \right), \]  
(35)

this correction may be reexpressed as

\[ \Delta F = \frac{1}{2} \log \left( \frac{C}{\beta^2} \right). \]  
(36)

Clearly formulae (34)-(36) make sense provided \(-S''_{MC} < 0\), i.e., the specific heat is positive, as is true in general for equilibrium thermodynamic systems.

It is interesting that thermal fluctuations produce a positive correction to the canonical entropy. This is in contrast to the case of the black hole microcanonical entropy where QGR effects tend to reduce the semiclassical value by restricting the set of degenerate states to those that are singlets under the residual gauge group SU(2).

### B. Canonical entropy of anti de Sitter black holes

The corrections to the canonical entropy due to thermal fluctuations can be calculated in principle for all isolated horizons which includes all stationary black holes. We shall first deal with the case of adS black holes where the calculation makes sense for a certain range of parameters of the black hole solution. Computation of such corrections has been performed in ref. [9]. Here, we recount the computation in a slightly different form, and compare the result with the corrections to the BHAL due to quantum spacetime fluctuations.

1. **BTZ**

The non-rotating BTZ metric is given by [18]

\[ ds^2 = - \left( \frac{r^2}{\ell^2} - 8G_3M \right) dt^2 + \left( \frac{r^2}{\ell^2} - 8G_3M \right)^{-1} dr^2 + r^2 d\phi^2, \]  
(37)

where, \( \ell^2 = -1/\Lambda^2 \) and \( \Lambda \) is the cosmological constant. The BH entropy is

\[ S_{BH} = \frac{\pi r_H^2}{2G_3}, \]  
(38)

where, the horizon radius \( r_H = \sqrt{8G_3M} \). Quantum spacetime fluctuations produce corrections to the microcanonical BHAL, given for \( r_H \gg \ell \) by [19]

\[ \Delta Q = - \frac{3}{2} \log S_{BH}. \]  
(39)

Using (39), and identifying the mass \( M \) of the black hole with the equilibrium energy \( E_0 \), the microcanonical entropy \( S_{MC} \) has the properties

\[ S''_{MC}(M) < 0 \text{ for } r_H > \ell \]
\[ S''_{MC}(M) > 0 \text{ for } r_H < \ell. \]
Alternatively, the specific heat of the BTZ black hole is positive, so long as \( r_H \geq \ell \). The system can therefore be thought of as being in equilibrium for parameters in this range. It follows that the calculation of \( \Delta_F \) yields a sensible result in this range,

\[
\Delta_F = \frac{3}{2} \log S_{BH} = - \Delta_Q .
\]

The import of this for the canonical entropy is rather intriguing, using eq.(33)

\[
S_C = S_{BH} .
\]

The quantum corrections to the BHAL in this case are cancelled by corrections due to thermal fluctuations of the area (mass) of the black hole horizon. We do not know the complete significance of this result yet.\(^4\)

2. 4 dimensional \( \text{adS} \) Schwarzschild

Such black holes have the metric

\[
ds^2 = -V(r) \, dt^2 + V(r)^{-1} \, dr^2 + r^2 \, d\Omega^2 ,
\]

where,

\[
V(r) = 1 - \frac{2GM}{r} + \frac{r^2}{\ell^2} ,
\]

with \( \ell^2 \equiv -3/\Lambda \). The horizon area \( A_H = 4\pi r_H^2 \), where the Schwarzschild radius obeys the cubic \( V(r_H) = 0 \). It is easy to see that the cubic yields the mass-area relation

\[
M = \frac{1}{2G} \left( \frac{A_H}{4\pi} \right)^{1/2} \left( 1 + \frac{A_H}{4\pi \ell^2} \right) .
\]

It is clear from eq.(44) that

\[
S_{MC}''(M) < 0 \text{ for } A_H > \frac{4}{3} \pi \ell^2 ,
\]

so that, once again the specific heat is positive in this range. The thermal fluctuation contribution for this parameter range is

\[
\Delta_F = \log \left( \frac{A_H}{\ell^2} \right) .
\]

The net effect on the canonical entropy is a partial cancellation of the effects due to quantum spacetime fluctuations and thermal fluctuations,

\[
S_C = S_{BH} - \frac{1}{2} \log S_{BH} .
\]

Note that the thermal and quantum fluctuation effects compete with each other in both cases considered above, with the net result that the canonical entropy is still superadditive

\[
S_C(A_1 + A_2) \geq S_C(A_1) + S_C(A_2) .
\]

The point \( r_H \sim \ell \) in parameter space signifies the breakdown of thermal equilibrium; this point has been identified with the so-called Hawking-Page phase transition [10] from the black hole phase to a phase which has been called an ‘\( \text{adS} \) gas’. In this latter phase, the black hole is supposed to have ‘evaporated away’, leaving behind a gas of massless particles in an asymptotically \( \text{adS} \) spacetime.

\(^4\)We should mention that this result ensues only if one takes recourse to the classical relation between the horizon area and the mass. The validity of that relation in the domain in which the QGR calculation has been performed, is not obvious at this point.
IV. 4 DIMENSIONAL ASYMPTOTICALLY FLAT BLACK HOLES

One may want to repeat the computation of the thermal fluctuation correction to the canonical entropy for black holes which are asymptotically flat, i.e., the Kerr-Newman family of solutions of the Einstein-Maxwell equations without a cosmological constant. However, this is stymied by a pathology which can be seen most easily from the case of the Schwarzschild black hole: the classical relation between horizon area $A_H$ and mass $M$ is the well known $A_H = 16\pi GM^2$. If one uses this relation in the formula for the microcanonical entropy (25), it is obvious that $S^\text{MC}(M) > 0$ for all nonvanishing positive values of the mass. The canonical entropy, as a consequence, acquires an imaginary part, signifying a thermodynamic instability. Alternatively, the specific heat turns out to be negative. Thus the standard approach of including the effect of thermal fluctuations around equilibrium fails in this case, as the canonical description is no longer adequate. This failure persists with the inclusion of electric charge and angular momentum, so long as one has a well-defined stationary event horizon and stays away from extremality. It is therefore a generic conundrum vis-a-vis the use of the canonical ensemble in such cases. The only way asymptotically flat black holes can be described in terms of equilibrium ensembles is by restricting the energy available to them to a constant, i.e., by using a microcanonical ensemble. Unphysical as it may be, there does not appear to be any alternative, if we continue to use the classical relation between black hole mass and horizon area.

The origin of the malaise can be traced [10] to the extraordinarily large degeneracy of asymptotically flat black holes, delineated in the density of states growing as $\rho(M) \sim \exp M^2$, notwithstanding the power law suppression due to quantum spacetime fluctuations. Defining the classical canonical partition function as

$$Z_C(\beta) = \int_0^\infty dM \exp -\beta M \rho(M),$$

it is obvious that the integral can never converge for large $M$. Contrast this to the case of the adS Schwarzschild black hole for horizon areas $A_H > \frac{4}{3}\pi\ell^2$: the density of states grows only as $\rho(M) \sim \exp M^{2/3}$ for large masses, allowing thereby the Boltzmann factor to tame the integral in (49).

As we had remarked earlier (footnote 4), the validity of the classical relation between mass and horizon area, and identifying the mass with the internal energy, are of course not guaranteed in the domain of QGR. We start with the canonical partition function in the quantum case

$$Z_C(\beta) = \text{Tr} \exp -\beta \hat{H}.$$  

Recall that in classical general relativity in the Hamiltonian formulation, the bulk Hamiltonian is a first class constraint, so that the entire Hamiltonian consists of the boundary contribution $H_S$ on the constraint surface. In the quantum domain, the Hamiltonian operator can be written as

$$\hat{H} = \hat{H}_V + \hat{H}_S,$$

with the subscripts $V$ and $S$ signifying bulk and boundary terms respectively. The Hamiltonian constraint is then implemented by requiring

$$\hat{H}_V |\psi\rangle_V = 0$$

for every physical state $|\psi\rangle_V$ in the bulk Hilbert space. This relation implies that the partition function may be written as

$$Z_C = \sum_{V,S} s \langle \chi | \exp -\beta \hat{H}_S |\chi\rangle_S.$$  

Thus, the relevance of the bulk physics seems rather limited due to the constraint (52). The partition function thus reduces to

$$Z_C(\beta) = \text{dim } \mathcal{H}_V Z_S(\beta),$$

where $Z_S$ is the ‘boundary’ partition function given by

$$Z_S(\beta) = \text{Tr}_S \exp -\beta \hat{H}_S .$$

Since we are considering situations where, in addition to the boundary at asymptopia, there is also an inner boundary at the black hole horizon, we may infer that quantum fluctuations of this boundary lead to black hole thermodynamics.
The factorization in eq. (54) manifests in the canonical entropy as the appearance of an additive constant proportional to \( \text{dim} \mathcal{H}_V \). Since thermodynamic entropy is defined only up to an additive constant, perhaps one may argue that the bulk states do not play any role in black hole thermodynamics. It is not clear yet if this can be thought of as the origin of the holographic hypothesis [20].

The next step is to evaluate the boundary partition function \( Z_S \). For this, one needs to express the boundary Hamiltonian \( \hat{H}_S \) as a function of other observables pertaining to the boundary, notably the area operator. Since the horizon states \(|\chi\rangle_S\) are an eigenbasis for the area operator, the boundary partition function can be written as

\[
Z_S(\beta) = \sum_p \sum_{j_1} \cdots \sum_{j_p} \left( \exp -\beta \mathcal{E}(a_S(p; \{ j_k \})) \right) N(p; \{ j_k \}),
\]

where, \( a_S \) is given by eq. (14), and \( N(p; \{ j_k \}) \) by eq. (21), and the function \( \mathcal{E}(a_S) \) is to be determined, after appropriate regularization, in QGR. It is not difficult to show that lifting the classical relation between black hole mass and area to the quantum level naively, for instance for the ordinary Schwarzschild black hole, does not lead to a convergent result. The sum over \( p \) in eq. (58) invariably diverges for large \( p \), once again because of the exponentially large contribution from the degeneracy factor \( \mathcal{N} \) in the partition function. In the Appendix, we sketch a derivation of this result for the case when all spins are identical. This means that even within QGR the thermodynamic instability discerned in the classical analysis persists, if we naively use classical relations at the operator level. We mention in passing that there is a possible lower bound on the area eigenvalue spectrum. It is clear that a physical 2-surface \( S \) must have at least two punctures with spins \( j_1 = j_2 = 1/2 \) to correspond to an SU(2) singlet state. This leads to an area eigenvalue

\[
a_{S_{\text{min}}}^0 = 8l_P^2 \log 2, \text{ with } \gamma = \log 2/\pi \sqrt{3}.
\]

While we do not know at this point the actual physical significance of this minimal area, it is amusing to speculate that black hole radiation culminates in a remnant of this size.

V. CONCLUSIONS

The exact cancellation of the logarithmic corrections to the BHAL for the BTZ black hole canonical entropy, due to quantum spacetime fluctuations and thermal fluctuations, may hold a deeper significance which warrants further analysis. In ref. [19], the microcanonical entropy is calculated from the exact Euclidian partition function of the SU(2) \( \times \) SU(2) Chern Simons theory which describes the BTZ black hole. Perhaps this calculation can be extended to a finite temperature canonical quantum treatment of the problem, to include the effect of thermal fluctuations. Such an analysis is necessary to allay suspicions about using (semi)classical relations between the mass and the area of the black hole. Likewise, for four dimensional black holes, the precise relation admitted within QGR between the boundary Hamiltonian and the area operator needs to be ascertained. Presumably, this relation will not qualitatively change the results for adS black holes, although it might lead to a better understanding of the Hawking-Page phase transition. More importantly, this issue needs to be addressed in order to determine if the thermodynamic instability found for generic asymptotically flat black holes is an artifact of a semiclassical approach.

It is conceivable that inclusion of charge and/or angular momentum for adS black holes in dimensions \( \geq 4 \) will present no conceptual subtleties, so long as the (outer) horizon area exceeds in magnitude the inverse cosmological constant. However, to the best of our knowledge, the formulation of a higher dimensional (i.e., \( > 4 \)) QGR has not been completed; this will have to be done before comparison of quantum and thermal fluctuation effects can be made in higher dimensions.

The thermodynamic instability discerned for asymptotically flat black holes also appears to emerge for de Sitter Schwarzschild black holes [9]. Since current observations appear to point enticingly to an asymptotically de Sitter universe, this instability must be better understood. In its present incarnation, it would imply that massive black holes will continue to get heavier without limit. On the other end of the scale, the instability can be interpreted in terms of disappearance of primordial black holes due to Hawking radiation, except for the possible existence of Planck scale remnants. It should be possible to estimate the density of such remnants and check with existing bounds from cosmological data.

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VI. APPENDIX

We begin with the formula (58) for the boundary partition function
\[ Z_S(\beta) = \sum_p \sum_{j_1} \cdots \sum_{j_p} [\exp -\beta \mathcal{E}(a_S(p; \{j_k\})] N(p; \{j_k\}) . \]  

(58)

For the ordinary Schwarzschild black hole, classically the black hole mass \( M = (A_H/16\pi G)^{1/2} \); the quantum generalization of this can be taken to be
\[ \mathcal{E}(a_S(p; \{j_i\})) = \left( \frac{a_S(p; \{j_i\})}{16\pi G} \right)^{1/2} \]  

(59)

Now, for simplicity, let all spins \( j_i = J \forall i = 1, 2, \ldots, n \); the multiple sum over the spins then reduce to a single one over all possible half-integral values of \( J \)
\[ Z_S(\beta) = \sum_p \sum_J [\exp -\beta (a_S(p; J)/16\pi G)^{1/2}] N(p; J) , \]  

(60)

where,
\[ a_S(p; J) = 8\pi \gamma l^2 p \sqrt{J(J+1)} , \]  

(61)

and
\[ N(p; J) \simeq \frac{(2J+1)^p}{p^{f(J)}} \text{ for large } p . \]  

(62)

The function \( f(J) \) is at most a polynomial for large \( J \).

The object is to argue that the summation over \( p \) in eq.(60) diverges for large \( p \gg 1 \). This is best demonstrated by replacing the double sum in (60) with double integrals over \( p \) and \( J \) and using (61) and (62),
\[ Z_S(\beta) \sim \int_{p_0}^{\infty} dp \int_{j_0}^{\infty} \frac{1}{p^{f(J)}} \exp \left[ -K p^{1/2}(J+1)^{1/4} + p \log(2J+1) \right] , \]  

(63)

where, the dimensionless constant \( K \sim \gamma \beta lP \). It is obvious from eq.(63) that for large \( p \), the \( O(p) \) term in the exponent dominates the integral, irrespective of how large \( J \) is, and returns to the same divergent behaviour as in the classical case.

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