Tree Level Recursion Relations 
In General Relativity

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Recently, tree-level recursion relations for scattering amplitudes of gluons in Yang-Mills theory have been derived. In this note we propose a generalization of the recursion relations to tree-level scattering amplitudes of gravitons. We use the relations to derive new simple formulae for all amplitudes up to six gravitons. In particular, we present an explicit formula for the six graviton non-MHV amplitude. We prove the relations for MHV and next-to-MHV \( n \)-graviton amplitudes and for all eight-graviton amplitudes.
1. Introduction

Lately, there has been a lot renewed progress in understanding the tree-level and one-loop gluon scattering amplitudes in Yang-Mills theory. Among other things, a new set of recursion relations for computing tree-level amplitudes of gluons have been recently introduced in [1]. A proof of the recursion relations was given in [2]. A straightforward application of these recursion relations gives new and simple forms for many amplitudes of gluons. Many of these have been obtained recently using somewhat related methods [3,4,5]. Application of recursion relations to amplitudes with fermions also lead to very compact formulas [6,7].

It has been known that tree level graviton amplitudes have remarkable simplicity that cannot be expected from textbook recipes for computing them. The tree level $n$ graviton amplitudes vanish if more than $n - 2$ gravitons have the same helicity. The maximally helicity violating (MHV) amplitudes are thus, as in Yang-Mills case, those with $n - 2$ gravitons of one helicity and two of the opposite helicity. These have been computed by Berends, Giele, and Kuijf (BGK) [8] from the Kawai, Lewellen and Tye (KLT) relations [9]. The four particle case was first computed by DeWitt [10].

This raises the question whether there are analogous recursion relations for amplitudes of gravitons which would explain some of the simplicity of the tree-level graviton amplitudes. The possibility of such recursion relations has been recently raised in [11].

In this note, we propose tree-level recursion relations for amplitudes of gravitons. The recursion relations can be schematically written as follows

$$A_n = \sum_{I, h} A_I^h \frac{1}{P_I^2} A_J^{-h}. \quad (1.1)$$

Here $A$ denotes a tree level graviton amplitude. In writing a recursion relation for $n$ graviton amplitude $A_n$, one marks two gravitons and sums over products of subamplitudes with external gravitons partitioned into sets $I \cup J = (1, 2, \ldots, n)$, among the two subamplitudes so that $i \in I$ and $j \in J$. $P_I$ is the sum of the momenta of gravitons in the set $I$ and $h$ is the helicity of the internal graviton. The momenta of the internal and the marked gravitons are shifted so that they are on-shell.

We use the recursion relations to derive new compact formulas for all amplitudes up to six gravitons. In particular, we give the first published result for the six graviton non-MHV amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. 

1
We attempt to prove the recursion relations along the lines of [2]. The first part of the proof that rests on basic facts about tree-level diagrams, such as the fact that their singularities come only from the poles of the internal propagators can be easily adapted to the gravity case. To have a complete proof of the recursion relations, it is necessary to prove that certain auxiliary function $A(z)$ constructed from the scattering amplitude vanishes as $z \to \infty$.

We are able to prove this fact from the KLT relations for all amplitudes up to eight gravitons. For amplitudes with nine or more gravitons, the KLT relations suggest that the function $A(z)$ does not vanish at infinity unless there is an unexpected cancellation between different terms in the KLT relations.

While we are not able to prove that $A(z)$ vanishes at infinity for a general $n$ graviton scattering amplitude, we show that $A(z)$ does vanish at infinity for MHV amplitudes with arbitrary number of gravitons from the BGK formula. Hence, the recursion relations are valid for all MHV amplitudes contrary to the expectation from KLT relations.

Finally, we introduce an auxiliary set of recursion relations for NMHV amplitudes which are easier to prove but give more complicated results for the amplitudes. This auxiliary recursion relation is then used to prove the vanishing of $A(z)$ for any NMHV amplitudes.

This raises the hope, that the recursion relations hold for other scattering amplitudes of gravitons as well.

2. Recursion Relations

Just like gauge theory scattering amplitudes, the graviton scattering amplitudes are efficiently written in terms of spinor-helicity formalism. In a nutshell, a on-shell momentum of a massless particle can be written as a bispinor, $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$. Here, $\lambda$ and $\tilde{\lambda}$ are commuting spinors with invariant products $\langle \lambda, \lambda' \rangle = \epsilon_{ab} \lambda^a \lambda'^b$ and $\{ \tilde{\lambda}, \tilde{\lambda}' \} = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{\dot{a}} \tilde{\lambda}'^{\dot{b}}$. Notice that $2p \cdot p' = \langle \lambda, \lambda' \rangle [\tilde{\lambda}, \tilde{\lambda}']$. We will also find it convenient to introduce the symbol $\langle \lambda | p | \tilde{\lambda}' \rangle = \lambda^a p_{a\dot{a}} \tilde{\lambda}'^{\dot{a}}$. The polarization tensors of the gravitons can be expressed in terms of gluon polarization vectors

$$
\epsilon_{a\dot{a}}^{\pm, \pm} = \epsilon_{a\dot{a}}^{\pm} \epsilon_{bb}^{\pm}, \quad \epsilon_{a\dot{a}, bb}^{\pm, \pm} = \epsilon_{a\dot{a}}^{\pm} \epsilon_{bb}^{\pm}.
$$
The polarization vectors of positive and negative helicity gluons are respectively

\[ \epsilon_{a\dot{a}}^- = \frac{\lambda_a\tilde{\mu}_{\dot{a}}}{[\lambda, \tilde{\mu}]} \quad \epsilon_{a\dot{a}}^+ = \frac{\mu_a\tilde{\lambda}_{\dot{a}}}{(\mu, \lambda)}, \tag{2.2} \]

where \( \mu \) and \( \tilde{\mu} \) are fixed reference spinors.

Consider a tree level graviton scattering amplitude \( A(1, 2, \ldots, n) \). The amplitude is invariant under any permutations of the gravitons because there is no color ordering.

To write down the recursion relations, we single out two gravitons. Without loss of generality, we call these gravitons \( i \) and \( j \). Define the shifted momenta \( p_i(z) \) and \( p_j(z) \), where \( z \) is a complex parameter, to be

\[ p_i(z) = \lambda_i(\tilde{\lambda}_i + z\tilde{\lambda}_j) \quad p_j(z) = (\lambda_j - z\lambda_i)\tilde{\lambda}_j. \tag{2.3} \]

Note that \( p_i(z) \) and \( p_j(z) \) are on-shell for all \( z \) and that \( p_i(z) + p_j(z) = p_i + p_j \). Hence, the following function

\[ A(z) = A(p_1, \ldots, p_i(z), \ldots, p_j(z), \ldots, n) \tag{2.4} \]

is a physical on-shell scattering amplitude for all values of \( z \).

Consider the partitions of the gravitons \( (1, 2, \ldots, i, \ldots, j, \ldots, n) = I \cup J \) into two groups such that \( i \in I \) and \( j \in J \). Then the recursion relation for a tree-level graviton amplitude is

\[ A(z) = \sum_{I,J} \sum_h A_L(I, -P_I^h(z_I), z_I) \frac{1}{P^2_I(z)} A_R(J, P_I^{-h}(z_I), z_I), \tag{2.5} \]

where

\[ P_I(z) = \sum_{k \in I, k \neq i} p_k + p_i(z) \tag{2.6} \]

\[ z_I = \frac{P^2_I}{\langle i|P_I|j \rangle}. \]

The sum in (2.5) is over the partitions of gravitons and over the helicities of the intermediate gravitons. The physical amplitude is obtained by taking \( z \) in equation (2.5) to be zero

\[ A(1, 2, \ldots, n) = A(0). \tag{2.7} \]

We will give evidence below that the recursion relation is valid for gravitons \( i \) and \( j \) of helicity \((+, +), (−, −)\) and \((−, +)\) respectively.
Fig. 1: This is a schematic representation of the recursion relations (2.5). The thick lines represent the reference gravitons. The sum here is over all partitions of the gravitons into two groups with at least two gravitons on each subamplitude and over the two choices of the helicity of the internal graviton.

3. Explicit Examples

In this section, we compute all tree-level amplitudes up to six gravitons to illustrate the use of the recursion relations (2.5).

Consider first the four-graviton MHV amplitude \(A(1^-, 2^-, 3^+, 4^+).\) The amplitude is invariant under arbitrary permutations of external gravitons so the order of gravitons does not matter. Hence, this is the only independent four graviton amplitude. In contrast, in gauge theory, there are two independent amplitudes \(A_{YM}(1^-, 2^-, 3^+, 4^+)\) and \(A_{YM}(1^-, 3^+, 2^-, 4^+)\) because the Yang-Mills scattering amplitudes are color ordered.

Fig. 2: Two configurations contributing to the four graviton amplitude \(A(1^-, 2^-, 3^+, 4^+).\) Notice that the diagrams are related by the interchange \(2 \leftrightarrow 3.\)

We single out gravitons \(1^-\) and \(4^+.\) Then, there are two possible configurations contributing to the recursion relations (2.3), see fig. 2. We refer to the configuration from fig. 2(a) as \((2, 1^|4, 3)\) and from fig. 2(b) as \((3, 1^|4, 3).\) To evaluate the diagrams we use the known form of three graviton scattering amplitudes

\[
A(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle^6}{\langle 23 \rangle^2 \langle 31 \rangle^2}, \quad A(1^+, 2^+, 3^-) = \frac{[12]^6}{[23]^2 [31]^2}. \tag{3.1}
\]
The sum of the two contributions from fig. 2 is
\[
A(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12 \rangle^5 \langle 34 \rangle^2}{\langle 12 \rangle \langle 23 \rangle^2 \langle 14 \rangle^2} + \frac{\langle 12 \rangle^8 \langle 24 \rangle^2}{\langle 13 \rangle \langle 13 \rangle \langle 23 \rangle^2 \langle 14 \rangle^2}. \tag{3.2}
\]
A short calculation shows that (3.2) equals to the known result \[8\] obtained from KLT relations
\[
A(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12 \rangle^8 \langle 12 \rangle}{\langle 12 \rangle \langle 13 \rangle \langle 14 \rangle \langle 23 \rangle \langle 24 \rangle \langle 34 \rangle^2}. \tag{3.3}
\]
We picked the reference gravitons to have opposite helicity because this leads to most compact expressions for graviton scattering amplitudes. We could have chosen reference gravitons of the same helicity, ie. 1$^-$ and 2$. This leads to a longer expression because there are more diagrams contributing to the scattering amplitude. In the rest of the paper, we will always choose reference gravitons of opposite helicity. The actual choice of reference gravitons does not matter, because the amplitude is invariant under permutations that preserve the sets of positive and negative helicity gravitons. All choices lead to the same answer up to relabelling of the gravitons.

The next amplitude to consider is the five graviton MHV amplitude $A(1^-, 2^-, 3^+, 4^+, 5^+)$. Just as in the four graviton example, this is the only independent five graviton amplitude. All other five graviton amplitudes are related to it by permutation and/or conjugation symmetry.

The amplitude has contribution from three diagrams $(1, 4, 2\hat{3}, 5), (1, 5, 2\hat{3}, 4), (4, 5, 2\hat{3}, 1)$. These contributions give the following three terms
\[
A(1^-, 2^-, 3^+, 4^+, 5^+) = \frac{\langle 12 \rangle^7}{\langle 14 \rangle \langle 15 \rangle \langle 23 \rangle \langle 24 \rangle^2 \langle 45 \rangle} \left( \frac{\langle 14 \rangle \langle 35 \rangle}{\langle 24 \rangle \langle 35 \rangle} - \frac{\langle 15 \rangle \langle 34 \rangle}{\langle 25 \rangle \langle 34 \rangle} - \frac{\langle 12 \rangle \langle 13 \rangle \langle 45 \rangle}{\langle 13 \rangle \langle 24 \rangle \langle 25 \rangle} \right). \tag{3.4}
\]
This expression agrees with the BGK result \[8\]
\[
A(1^-, 2^-, 3^+, 4^+, 5^+) = \langle 12 \rangle^7 \left( \frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle - \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}{\langle 13 \rangle \langle 14 \rangle \langle 15 \rangle \langle 23 \rangle \langle 24 \rangle \langle 25 \rangle \langle 34 \rangle \langle 35 \rangle \langle 45 \rangle} \right). \tag{3.5}
\]
At six gravitons, there are two independent scattering amplitudes, the MHV amplitude $A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ and the first non-MHV amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. The MHV amplitude $A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ has contribution from four configurations, $(4, 3\hat{2}, 1, 5, 6), (5, 3\hat{2}, 1, 4, 6), (6, 3\hat{2}, 1, 4, 5)$ and $(1, 3\hat{2}, 4, 5, 6)$. Notice that the first three diagrams are related by interchange of 4, 5, 6 gravitons, so there are only two diagrams to compute.
The first configuration \((4, \hat{3}, 2, 1, 5, 6)\) evaluates to
\[
D_1 = (12)^7[34]\frac{(2\{3 + 4\}5\{4\}2 + 3\{1\}\{51\}) - (12)p_{234}^2(45)\{51\}}{(14)\{15\}\{16\}\{23\}\{25\}\{26\}\{34\}\{45\}\{46\}\{56\}} \tag{3.6}
\]

The last configuration \((1, \hat{3}, 2, 4, 5, 6)\) gives
\[
D_2 = (12)^8[13]\frac{(14)\{45\}\{52\}p_{123}^2 - (45)\{2\}1 + 3\{4\}\{1\}2 + 3\{5\}}{(13)\{14\}\{15\}\{16\}\{23\}\{24\}\{25\}\{26\}\{45\}\{46\}\{56\}} \tag{3.7}
\]

Adding all four contributions, we get
\[
A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+) = D_1 + D_1(4 \leftrightarrow 5) + D_1(4 \leftrightarrow 6) + D_2. \tag{3.8}
\]

(3.8) agrees with the known result for the six graviton MHV amplitude.

The non-MHV amplitude \(A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)\) has contribution from six classes of diagrams \(D_1 = (2, \hat{3}, 4, 5, 6, 1) + (1 \leftrightarrow 2), D_2 = (1, 6, \hat{3}, 4, 2, 5) + (1 \leftrightarrow 2) + (5 \leftrightarrow 6) + (1 \leftrightarrow 2, 5 \leftrightarrow 6), D_3 = (2, 5, 6, \hat{3}, 4, 1) + (1 \leftrightarrow 2), D_4 = D_{1\text{flip}}^f, D_5 = D_{3\text{flip}}^f\) and \(D_6 = (5, 6, \hat{3}, 4, 1, 2)\). The 'conjugate flip' \(D_{\text{flip}}^f\) exchanges the spinor products \(\langle \rangle \leftrightarrow [\rangle and the labels \(i \leftrightarrow 7 - i\).

The first class of diagrams \(D_1 : (2, \hat{3}, 4, 5, 6, 1) + (1 \leftrightarrow 2)\) evaluates to
\[
D_1 = \frac{(23)\{1\}2 + 3\{4\}7((12 + 3\{4\}5\{3\} + 4\{2\}51) + [12]45\{51\}p_{234}^2)}{(15)\{16\}\{23\}\{34\}\{56\}\{234\}\{1\}3 + 4\{2\}5\{3\} + 4\{2\}5\{2\} + 3\{4\}6\{3\} + 4\{2\}6\{2\} + 3\{4\}} \tag{3.9}
\]

The second group, \(D_2 : (1, 6, \hat{3}, 4, 2, 5) + \text{permutations}, \) gives
\[
D_2 = -\frac{(13)^7\{25\}\{45\}7\{16\}}{(16)\{24\}\{25\}\{36\}\{245\}\{1\}2 + 5\{4\}\{6\}2 + 5\{4\}\{3\}1 + 6\{5\}\{3\}1 + 6\{2\}} \tag{3.10}
\]

The third class \(D_3 : (2, 5, 6, \hat{3}, 4, 1) + (1 \leftrightarrow 2)\) is
\[
D_3 = \frac{(13)^8\{14\}\{56\}7((23)\{56\}\{62\}\{1\}3 + 4\{5\} + (35)\{56\}\{62\}\{1\}3 + 4\{2\})}{(14)\{25\}\{26\}\{34\}2\{1\}3 + 4\{2\}\{1\}3 + 4\{5\}\{1\}3 + 4\{6\}\{3\}1 + 4\{2\}\{3\}1 + 4\{5\}\{3\}1 + 4\{6\}} + (1 \leftrightarrow 2). \tag{3.11}
\]

The fourth and fifth group are related by conjugate flip to the third and first group respectively. The last group to evaluate consists of a single diagram \(D_6 : (5, 6, \hat{3}, 4, 1, 2)\)
\[
D_6 = \frac{(12)\{56\}\{3\}1 + 2\{4\}8}{[21]4\{24\}\{35\}\{36\}\{56\}\{245\}\{1\}1 + 2\{4\}\{6\}1 + 2\{4\}\{3\}5 + 6\{1\}\{3\}5 + 6\{2\}}. \tag{3.12}
\]
Adding the pieces together, the six graviton non-MHV amplitude reads

\[ A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = D_1 + D_1^{\text{flip}} + D_2 + D_3 + D_3^{\text{flip}} + D_6. \]  (3.13)

4. Derivation of the Recursion Relations

The derivation of the tree-level recursion relations (2.5) goes, with few modifications, along the same lines as the derivation of the tree-level recursion relations for scattering amplitudes of gluons [2], so we will be brief.

We start with the scattering amplitude \( A(z) \) defined at shifted momenta, see (2.4) and (2.3). \( A(z) \) is a rational function of \( z \) because the \( z \) dependence enters the scattering amplitude only via the shifts \( \tilde{\lambda}_i \to \tilde{\lambda}_i + z\lambda_j \) and \( \lambda_j \to \lambda_j - z\lambda_i \) and because the original tree-level scattering amplitude is a rational function of the spinors.

Actually, for generic momenta, \( A(z) \) has only single poles in \( z \). These come from the singularities of the propagators in Feynman diagrams. To see this, recall that for tree level amplitudes, the momentum through a propagator is always a sum of momenta of external particles \( P_\mathcal{I} = p_{i_1} + p_{i_2} + \ldots + p_{i_l} \), where \( \mathcal{I} \) is a group of not necessarily adjacent gravitons. At nonzero \( z \), the momentum becomes \( P_\mathcal{I}(z) = p_{i_1}(z) + p_{i_2}(z) + \ldots + p_{i_l}(z) \). Here, \( p_k(z) \) is independent of \( z \) for \( k \neq i, j \) and \( p_i(z) + p_j(z) \) is independent of \( z \). Hence, \( P_\mathcal{I}(z) \) is independent of \( z \) if both \( i \) and \( j \) are in \( \mathcal{I} \) or if neither of them is in \( \mathcal{I} \). In the remaining case, one of \( i \) and \( j \) is in the group \( \mathcal{I} \) and the other is not. Without loss of generality, we take \( i \in \mathcal{I} \). Then \( P_\mathcal{I}(z) = P_\mathcal{I} + z\lambda_i \tilde{\lambda}_j \) and \( P_\mathcal{I}^2(z) = P_\mathcal{I}^2 - z\langle i | P_\mathcal{I} | j \rangle \). Clearly, the propagator \( 1/P_\mathcal{I}(z)^2 \) has a simple pole for

\[ z_\mathcal{I} = \frac{P_\mathcal{I}^2}{\langle i | P_\mathcal{I} | j \rangle}. \]  (4.1)

For generic momenta, \( P_\mathcal{I} \)’s are distinct for distinct groups \( \mathcal{I} \), hence the \( z_\mathcal{I} \)’s are distinct. So all singularities of \( A(z) \) are simple poles.

To continue the argument, we need to assume that \( A(z) \) vanishes as \( z \to \infty \). In the next section we will argue that the tree level graviton amplitudes obey this criterium. A rational function \( A(z) \) that has only simple poles and vanishes at infinity can be expressed as

\[ A(z) = \sum_\mathcal{I} \frac{\text{Res} A(z_\mathcal{I})}{z - z_\mathcal{I}}, \]  (4.2)
where $\text{Res} A(z_I)$ are the residues of $A(z)$ at the simple poles $z_I$. The physical scattering amplitude is simply $A(0)$

$$A = - \sum_I \frac{\text{Res} A(z_I)}{z_I}. \quad (4.3)$$

It follows from the above discussion that the sum is over $I$ such that $i$ is in $I$ while $j$ is not.

The residue $\text{Res} A(z_I)$ has contribution from Feynman diagrams which contain the propagator $1/P^2_I$. The propagator divides the tree diagram into “left” part containing gravitons in $I$ and “right” part containing gravitons in $J = (1, 2, \ldots, n) - I$. For $z \to z_I$, the propagator with momentum $P_I$ goes on-shell and the left and right part of the diagram approach tree-level diagrams for on-shell amplitudes. The contribution of these diagrams to the pole is

$$\sum_h A^h_L(z_I) \frac{1}{P^2_I(z)} A^{-h}_R(z_I), \quad (4.4)$$

where the sum is over the helicity $h = \pm$ of the intermediate graviton. This gives the recursion relation (2.5).

5. Large $z$ Behavior of Gravity Amplitudes

To complete the proof, it remains to show that the amplitude $A(z)$ goes to zero as $z$ approaches infinity. We were able to obtain only partial results in this direction, which we now discuss.

5.1. Vanishing of the MHV Amplitudes

Let us firstly consider the large $z$ behavior of the $n$ graviton MHV amplitude

$$A(1^-, 2^-, 3^+, \ldots, n) = \langle 12 \rangle^8 \left\{ \frac{[23]\langle n| P_{2,3}|4\rangle \langle n| P_{2,4}|5\rangle \ldots \langle n| P_{2,n-2}|n-1\rangle}{\langle 12\rangle\langle 23\rangle \ldots \langle n-2, n-1\rangle \langle n-1, 1\rangle \langle 1n\rangle^2 \langle 2n\rangle \langle 3n\rangle \ldots \langle n-1, n\rangle} \right\} + \text{permutations of (3, 4, \ldots, n-1)} \right\}, \quad (5.1)$$

where $P_{i,j} = \sum_{k=i}^j p_k$. The formula is valid for $n \geq 5$. It follows from supersymmetric Ward identities that the expression in the bracket is totally symmetric, although this is not manifest.

The terms in the curly brackets are completely symmetric so they contribute the same power of $z$ independently of $i$ and $j$. To find the contribution of the terms in the brackets,
we pick a convenient value \((i, j) = (1, n)\) for the reference gravitons. Recall that \(\tilde{\lambda}_i(z)\) and \(\lambda_j(z)\) are linear in \(z\) while \(\lambda_i\) and \(\tilde{\lambda}_j\) do not depend on \(z\). It follows that the numerator of each term in the brackets (5.1) goes like \(z^{n-4}\) and the denominator gives a factor of \(z^{n-2}\). Hence, the terms in the brackets give a factor of \(1/z^2\). This factor is the same for all choices of reference momenta by the complete symmetry of the terms in the brackets. For the helicity configurations \((h_i, h_j) = (-, +), (+, +)\) and \((-,-)\), the factor \(\langle 12 \rangle^8\) does not contribute, so the amplitude vanishes at infinity as \(A_{MHV} \sim 1/z^2\).

A recent paper \cite{12} relates the MHV amplitudes to current correlators on curves in twistor space. This raises the possibility of a twistor string description of perturbative \(\mathcal{N} = 8\) supergravity \cite{13}. In the gauge theory case, the twistor string leads to an MHV diagrams construction for the tree level scattering amplitudes \cite{14}. One computes the tree-level amplitudes from tree-level Feynman diagrams in which the vertices are MHV amplitudes, continued off-shell in a suitable manner, and the propagators are ordinary Feynman propagators.

The vanishing of the gluon scattering amplitude \(A(z)\) at infinity follows very easily from the vanishing of the MHV diagrams via the MHV diagrams construction \cite{2}. We would like to speculate, that it might be possible to prove the vanishing of graviton scattering amplitude \(A(z)\) along the same lines using the hypothetical MHV diagrams construction.

### 5.2. Analysis of the Feynman Diagrams

In this section we study the large \(z\) behavior of Feynman diagrams contributing to \(A(z)\) following \cite{2}.

Recall that any Feynman diagram contributing to \(A(z)\) is linear in the polarization tensors \(\epsilon_{a\dot{a},b\dot{b}}\) of the external gravitons. The polarization tensors of all but the \(i^{th}\) and \(j^{th}\) graviton are independent of \(z\). To find the \(z\) dependence of the polarization tensors of the reference gravitons, recall that \(\tilde{\lambda}_i(z), \lambda_j(z)\) are linear in \(z\) and \(\lambda_i, \tilde{\lambda}_j\) do not depend on \(z\). It follows from (2.1) and (2.2) that the polarization tensors of the reference gravitons give a factor of \(z^{\pm 2}\) depending on their helicities. Hence, the polarization tensors can suppress \(A(z)\) by at most a factor of \(z^4\).

The remaining pieces in Feynman diagrams are constructed from vertices and propagators that connect them. Perturbative gravity has infinite number of vertices coming from the expansion of the Einstein-Hilbert Lagrangian

\[
\mathcal{L} = -\sqrt{-g} R \tag{5.2}
\]
around the flat vacuum $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The graviton vertices have two powers of momenta coming from the two derivatives in the Ricci scalar.

The $z$ dependence in a tree level diagram ”flows” along a unique path of Feynman propagators from the $i^{th}$ to the $j^{th}$ graviton. In a path composed of $k$ propagators, there are $k + 1$ vertices. Each propagator contributes a factor of $1/z$ and each vertex contributes a factor of $z^2$. Altogether, the propagators and vertices give a factor of $z^{k+2}$.

The product of polarization tensors vanishes at best as $1/z^4$, so the contribution of individual Feynman diagrams to $A(z)$ seems to grow at infinity as $z^{k-2}$, where $k$ is the number of propagators from the $i^{th}$ to the $j^{th}$ graviton. Clearly, in a generic Feynman diagram, this number grows with the number of external gravitons. So this analysis suggests that $A(z)$ grows at infinity with a power of $z$ that grows as we increase the number of external gravitons.

This is in contrast to the above analysis of MHV amplitude that vanishes at infinity as $1/z^2$. The vanishing of $A(z)$ at infinity depends on unexpected cancellation between Feynman diagrams.

### 5.3. KLT Relations and the Vanishing of Gluon Amplitudes

A different line of attack is to express the graviton scattering amplitudes via the KLT relations in terms of the gluon scattering amplitudes. One then infers behavior of $A(z)$ at infinity from the known behavior of the gauge theory amplitudes. The KLT relations have been used in past to show that $\mathcal{N} = 8$ supergravity amplitudes have better than expected ultraviolet behavior so we expect that the KLT relations give us a better bound on $A(z)$ than the analysis of Feynman diagrams. Indeed, we will use them to prove the vanishing of $A(z)$ at infinity up to six gravitons and up to eight gravitons in the appendix.

The KLT relations of string theory relate the closed string amplitudes to the open string amplitudes. They arise from representing each closed string vertex operator $C$ as the product of two open string vertex operators $O$

$$C(z_i, \bar{z}_i) = O(z_i) \overline{O}(\bar{z}_i)$$

(5.3) and deforming the closed string integration contour into two sets of open string integration contours.
In the infinite tension limit, the KLT relations relate the gravity amplitudes to the gauge theory amplitudes \[8\]. The tree level KLT relations up to six gravitons are

\[
\begin{align*}
A(1, 2, 3) &= A(1, 2, 3)^2 \\
A(1, 2, 3, 4) &= s_{12}A(1, 2, 3, 4)A(1, 2, 4, 3) \\
A(1, 2, 3, 4, 5) &= s_{12}s_{34}A(1, 2, 3, 4, 5)A(2, 1, 4, 3, 5) + s_{13}s_{24}A(1, 3, 2, 4, 5)A(3, 1, 4, 2, 5) \\
A(1, 2, 3, 4, 5, 6) &= s_{12}s_{45}A(1, 2, 3, 4, 5, 6)\left\{s_{35}A(2, 1, 5, 3, 4, 6) + (s_{34} + s_{35})A(2, 1, 5, 4, 3, 6)\right\} \\
&\quad + \text{permutations of (234)},
\end{align*}
\]

where \(s_{ij} = (p_i + p_j)^2\). \(A(1, 2, \ldots, n)\) is the \(n\) graviton scattering amplitude and \(A(i_1, i_2, \ldots, i_n)\) is the color ordered gauge theory amplitude. Each graviton state on the left is the product of two gauge theory states on the right. The decomposition of the graviton states comes from the infinite tension limit of the decomposition of the closed string vertex operators \[5.3\]. It is reflected in the factorization of the graviton polarization tensor \[\text{(2.1)}\] in terms of the gluon polarization vectors. The KLT relations for any number of gravitons are written down in Appendix A of \[19\] and schematically in the appendix.

The KLT relations express an \(n\) graviton scattering amplitude as a sum of products of two gluon scattering amplitudes and \(n - 3\) \(s_{ij}\) invariants. The gluon scattering amplitudes vanish at infinity as \(1/z\) or faster \[2\]. Hence, KLT relations imply the vanishing at infinity of the graviton amplitudes as long as the products of \(s_{ij}\)’s in \[5.4\] grow at most linearly with \(z\).

For \(n \leq 6\) gravitons, a quick glance at \[5.4\] shows that this is the case. We rename the gravitons so that the reference gravitons are 1 and \(n\). The products of \(s_{ij}\)’s in \[5.4\] are independent of \(p_n\) and linear in \(p_1\). Hence they give one power of \(z\) because \(p_1(z)\) and \(p_n(z)\) are linear in \(z\) and \(p_k\) for \(k \neq 1, n\) is independent of \(z\). It follows that \(A(z)\) vanishes as \(1/z\) or faster as \(z \to \infty\) for less than seven gravitons.

For seven or more gravitons, an analysis of the general KLT relations shows that on the right hand side of KLT relations, there are always some products of \(n - 3\) \(s_{ij}\)’s that have more than one power of the reference momenta. The corresponding terms in the KLT relations are not expected to vanish at infinity. Hence, the function \(A(z)\) does not vanish at infinity unless there is an unexpected cancellation between different terms in the KLT relations.

In the appendix we present a more careful study of KLT relations that reveals that \(A(z)\) vanishes for \(n \leq 8\).
5.4. Proof of Vanishing of $A(z)$ for NMHV Amplitudes

NMHV amplitudes are those with three negative helicity gravitons and any number of plus helicity gravitons, $A(p_1^-, p_2^-, p_3^-, p_4^+, \ldots, p_n^+)$.

Consider the following function of $z$, $A_a(p_1^-(z), p_2^-, p_3^-, p_4^+(z), \ldots, p_n^+(z))$, where

$$p_1(z) = \lambda_1 \left( \tilde{\lambda}_1 + z \sum_{i=4}^{n} \tilde{\lambda}_i \right), \quad p_k(z) = (\lambda_k - z\lambda_1)\tilde{\lambda}_k \quad (5.5)$$

for $k = 4, \ldots, n$. The subscript $a$ in $A_a(z)$ stands for auxiliary. The idea is to derive a new set of recursion relations for $A_a(z)$ which we use later on to prove that $A(z)$ vanishes at infinity.

In order to get the auxiliary recursion relations we start by proving from Feynman diagrams that $A_a(z)$ vanishes as $z \to \infty$. Note that $(n - 2)$ polarization tensors depend on $z$ and with the choice made in (5.3) all of them vanish as $1/z^2$. The most dangerous Feynman diagram is the one with the largest number of vertices. Such a diagram must only have cubic vertices. For $n$ gravitons there are $n - 2$ vertices. Each vertex contributes a factor of $z^2$. Altogether, the polarization tensors contribute a factor of $1/z^{2(n-2)}$ and the vertices contribute a factor of $z^{2(n-2)}$ which gives a constant for large $z$. Now we have to consider propagators. Each propagator that depends on $z$ goes like $1/z$. Therefore, all we need is that in every diagram at least one propagator depends on $z$. From (5.3) it is easy to see that the only propagator that does not depend on $z$ is $1/(p_2^2 + p_3^2)$. A diagram with only this propagator has exactly two vertices and therefore our proof is complete for $n > 4$.

The shift in (5.5) can be thought of as iterating the shift introduced in [2]. Now we can follow the same steps as in section 4 to derive recursion relations based on the pole structure of $A_a(z)$.

We find

$$A_a(z) = \sum_{\mathcal{I}} \sum_h A_\mathcal{I}(z_\mathcal{I}, P_\mathcal{I}^h(z_\mathcal{I})) \frac{1}{P_\mathcal{I}^2(z)} A_{\mathcal{J}}(z_\mathcal{I}, -P_\mathcal{I}^h(z_\mathcal{I})). \quad (5.6)$$

where the sum is over all possible sets of two or more gravitons $\mathcal{I} \neq \{2, 3\}$, such that the graviton 1 is not in $\mathcal{I}$. Here, $\mathcal{J}$ is the complement of $\mathcal{I}$.

\[\text{This iteration procedure was used recently in [20] to find recursion relations for all plus one-loop amplitudes of gluons.}\]
The main advantage of choosing the same negative helicity graviton in (5.5) to pair up with all plus helicity gravitons is that \( P^2_I(z) \) is a linear function of \( z \). Therefore, the location of the poles \( z_I \) is easily computed to be of the form

\[
z_I = \frac{P^2_I}{\sum_j \langle 1 | P_I[j] \rangle}
\]

(5.7)

where the sum in \( j \) runs over all gravitons in \( I \) that depend on \( z \).

Setting \( z \) to zero in (5.6) gives us a new representation of the original amplitude, i.e.,

\[
A(p_1^-, p_2^-, p_3^+, p_4^+, \ldots, p_n^+) = \sum_I \sum_h A_L(z_I, P^h_I(z_I)) \frac{1}{P^2_I} A_R(z_I, -P^{-h}_I(z_I)).
\]

(5.8)

This is a new set of recursion relations for NMHV amplitudes. However, the expressions obtained from (5.8) are naturally more complicated than the ones obtained from the one introduced in section 2. Instead of computing amplitudes with (5.8), the idea is to use it to prove that \( A(z) \) of section 2 vanishes for large \( z \).

Consider \( A(z) \) constructed from (5.8) by defining

\[
p_1(z) = \lambda_1(\tilde{\lambda}_1 + z\tilde{\lambda}_4), \quad p_4(z) = (\lambda_4 - z\lambda_1)\tilde{\lambda}_4.
\]

(5.9)

There are two different kind of terms in (5.8). One class consists of those where \( p_1 \) and \( p_4 \) are on the same side. This implies that neither \( P_I \) nor \( z_I \) depends on \( z \). Therefore, the \( z \) dependence is confined into one of the amplitudes, say \( A_L \). But this is an amplitude with less gravitons and by induction we assume that it vanishes for large \( z \).

The second class of terms is more subtle. Since \( p_1(z) \) and \( p_4(z) \) are on different sides, both \( P_I \) and \( z_I \) become functions of \( z \).

It turns out that \( z_I \) is a linear function of \( z \). More explicitly,\footnote{Had we chosen a different negative helicity graviton in (5.9), we would have found that \( z_I \) becomes a rational function of \( z \).}

\[
z_I = \frac{P^2_I + z \langle 1 | P_I[4] \rangle}{\sum_j \langle 1 | P_I[j] \rangle}.
\]

(5.10)

Recall that \( P_I \) denotes \( P_I(0) \).

Now we are left with \( A_I \) and \( A_J \) in (5.8), one with \( n_1 + 1 \) and the other with \( n_2 + 1 \) gravitons. Note that \( n = n_1 + n_2 \). In a Feynman diagram expansion of each of them we can single out the most dangerous diagrams and multiply them to get the most dangerous
terms in (5.8). Each diagram contributes a factor of $z^{2(n_i-1)}$ from the vertices. Therefore we find $z^{2(n-2)}$. From the polarization tensors we find $z^{-2(n-2)}$, this comes from the $z$ dependence of $z_T$. The polarization tensors for the internal gluons give a factor of

$$\sum_h \epsilon^h_{\mu\nu} \epsilon^{h\lambda\rho} = d_{\mu\rho} d_{\nu\lambda} + d_{\mu\lambda} d_{\nu\rho} - d_{\mu\nu} d_{\rho\lambda},$$

where

$$d_{\mu\nu} = \eta_{\mu\nu} - \frac{k_{\mu} n_{\nu} + k_{\nu} n_{\mu}}{k \cdot n}. \quad (5.12)$$

Here, $k = P_T(z)$ is the momentum of the internal propagator and $n_{a\dot{a}} = \mu_a \tilde{\mu}_{\dot{a}}$ is an auxiliary vector used in the definition (2.2). $n_{a\dot{a}}$ is taken no collinear with $k$. For large $z$, the tensor $d_{\mu\nu}$ does not depend on $z$ so the polarization tensors of the internal gravitons do not contribute a factor of $z$.

Finally, the propagator in (5.8) is $1/P^2_T(z)$, which vanishes as $1/z$. Therefore, the most dangerous term in $A(z)$ vanishes as $1/z$.

For large $z$ the $z$-dependence of $P_T(z_T)$ in (5.8) can be made trivial, for the product $A_T A_J$ must be homogeneous of degree zero under scalings of $P_T(z_T)$.

This completes the proof of the recursion relations of section 2 for next-to-MHV amplitudes of gravitons.

While we are not able to prove that $A(z)$ vanishes at infinity for general amplitudes with more than eight gravitons, we showed above that $A(z)$ vanishes at infinity for MHV and NMHV amplitudes with arbitrary number of gravitons. Hence, the recursion relations are valid for all MHV and NMHV amplitudes contrary to the expectations from KLT relations. This raises the hope, that the recursion relations are valid for other scattering amplitudes of gravitons as well. In particular, one might expect that by considering more general auxiliary recursion relations one could prove that $A(z)$ vanishes at infinity for general gravity amplitudes.

Note:

After this work was completed, [21] appeared which has some overlap with our results.

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Appendix A. Proof of Vanishing of $A(z)$ for $z \to \infty$ up to Eight Gravitons.

In this appendix we provide further evidence for validity of the recursion relations (2.3). We show that the recursion relations hold for any graviton amplitude up to eight gravitons. Recall that we need to prove that the auxiliary function $A(z)$ (2.4) vanishes at infinity. We demonstrate this using the KLT relations.

The basic fact we will use is that the function $A(z)$ for a gluon scattering amplitude goes like $1/z^2$ at infinity for non-adjacent marked gluons. Hence, picking the marked gluons so that they are non-adjacent in all terms in KLT relations, the product of two Yang-Mills amplitudes in each term goes like $1/z^4$.

Hence, $A(z)$ vanishes at infinity as long as the products of $s_{ij}$'s in the KLT relations do not contribute more than a factor of $z^3$. An inspection of the KLT relations will show that this holds at least up to eight gravitons which will complete the proof.

Let us begin by showing that the gluon amplitudes go as $1/z^2$ as $z \to \infty$ for non-adjacent marked gluons with helicities $(h_i, h_j) = (+, +), (-, -), (-, +)$. The argument uses MHV rules and is a simple generalization of the argument given in [2], which showed that the amplitudes vanish as $1/z$. We assume that $h_j = +$. For $h_j = -$ one makes the same argument using the opposite helicity MHV rules.

Firstly, consider the $n$ gluon MHV amplitude

$$A(r^-, s^-) = \frac{\langle r, s \rangle^4}{\prod_{k=1}^{n} \langle k, k + 1 \rangle}.$$  \hfill (A.1)

Recall that $\lambda_j(z) = \lambda_j - z\lambda_i$ is linear in $z$ and $\lambda_i(z) = \lambda_i$ is independent of $z$. For $h_j = +$, $\lambda_j$ does not occur in the numerator. In the denominator it appears in the two factors $\langle \lambda_{j-1}, \lambda_j \rangle$ and $\langle \lambda_j, \lambda_{j+1} \rangle$, both of which are linear in $z$ for $i$ not adjacent to $j$. Hence for $h_i = +$ and $|i - j| > 1$, the MHV amplitude goes like $1/z^2$ at infinity.

For general amplitudes, we use MHV diagram constructions. In this construction, the amplitudes are built from Feynman vertices which are suitable off-shell continuations of the MHV amplitudes. The vertices are connected with ordinary scalar propagators.
The Feynman vertices are the MHV amplitudes (A.1), where we take \( \lambda^a = P^{a\dot{a}} \eta_{\dot{a}} \) for an off-shell momentum \( P \). Here \( \eta \) is an arbitrary positive helicity spinor. The physical amplitude, which is a sum of MHV diagrams, is independent of the choice of \( \eta \) [14].

The internal momentum \( P \) can depend on \( z \) only through a shift by the null vector \( z\lambda_j \tilde{\lambda}_j \). Taking \( \eta = \tilde{\lambda}_j \), \( \lambda^a = P^{a\dot{a}} \tilde{\lambda}_{j\dot{a}} \) becomes independent of \( z \). Hence, the internal lines do not introduce additional \( z \) dependence into the MHV vertices. The MHV vertices give altogether a factor of \( \frac{1}{z^2} \) from the two powers of \( \lambda_j(z) \) in the denominator of one of the vertices. The propagators \( \frac{1}{k^2} \) are either independent of \( z \) or contribute a factor of \( \frac{1}{z} \). So, a general gluon amplitude goes like \( \frac{1}{z^2} \) at infinity for non-adjacent marked gluons.

The KLT relations [19] for \( n \) gravitons are

\[
A(1, 2, \ldots, n) = \left( A(1, 2, \ldots, n) \sum_{\text{perm}} f(1, i_1, \ldots, i_j) \overline{f}(n - 1, l_1, \ldots, l_{j'}) \right. \\
\left. \times A(i_1, \ldots, i_j, 1, n - 1, l_1, \ldots, l_{j'}, n) \right) + \mathcal{P}(2, \ldots, n - 2),
\]

where \( j = \lfloor n/2 \rfloor - 1, j' = \lfloor (n - 1)/2 \rfloor - 1 \) and the permutations are \( (i_1, \ldots, i_j) \in \mathcal{P}(2, \ldots, \lfloor n/2 \rfloor) \) and \( (l_1, \ldots, l_{j'}) \in \mathcal{P}(\lfloor n/2 \rfloor + 1, \ldots, n - 2) \). The exact form of the functions \( f \) and \( \overline{f} \) does not concern us here. The only property we need is that \( f \) and \( \overline{f} \) are homogeneous polynomials of degree \( j \) and \( j' \) in the Lorentz invariants \( p_m \cdot p_n \) with \( m, n \in (1, i_1, \ldots, i_j) \) or \( m, n \in (l_1, \ldots, l_{j'}, n - 1) \) respectively.

Consider \( A(z) \) with marked gravitons \( n \) and \( k \) where \( k \) is any label from the set \( (2, \ldots, n - 2) \). In the KLT relations (A.2), the gluon amplitude \( A(1, 2, \ldots, n) \) contributes a factor of \( 1/z^2 \) since \( k \) and \( n \) are non-adjacent. For \( k \in (i_1, \ldots, i_j) \) the second gluon amplitude gives a factor of \( 1/z^2 \) and \( f \) gives at most a factor of \( z^j, j = \lfloor n/2 \rfloor - 1 \). Hence, the terms with \( k \in (i_1, \ldots, i_j) \) are bounded at infinity by \( z^\alpha \) where \( \alpha = \lfloor n/2 \rfloor - 5 \). For \( k \in (l_1, \ldots, l_{j'}) \) the second gluon amplitude gives a factor of \( 1/z \) because \( k \) and \( n \) might be adjacent. \( \overline{f} \) contributes \( z^{j'}, j' = \lfloor (n - 1)/2 \rfloor - 1 \) so the graviton amplitude is bounded by \( z^{\alpha'}, \alpha' = \lfloor (n - 1)/2 \rfloor - 4 \). The exponents \( \alpha, \alpha' \) are negative for \( n \leq 8 \), which completes the proof of the recursion relations up to eight gravitons.
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