One loop renormalization of soliton quantum mass corrections in (1+1)-dimensional scalar field theory models

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The bare one loop soliton quantum mass corrections can be expressed in two ways: as a sum over the zero-point energies of small oscillations around the classical configuration, or equivalently as the (Euclidean) effective action per unit time. In order to regularize the bare one loop quantum corrections (expressed as the sum over the zero-point energies) we subtract and add from it the tadpole graph that appear in the expansion of the effective action per unit time. The subtraction renders the one loop quantum corrections finite. Next, we use the renormalization prescription that the added tadpole graph cancels with adequate counterterms, obtaining in this way a finite unambiguous expression for the one loop soliton quantum mass corrections. When we apply the method to the solitons of the sine-Gordon and \( \phi^4 \) kink models we obtain results that agree with known results. Finally we apply the method to compute the soliton quantum mass corrections in the recently introduced \( \phi^2 \cos^2 \ln(\phi^2) \) model.

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A renewed interest in the computation of quantum energies around classical configurations has recently arose. See for example [1–7] and references therein. The methods used to approach the problem include the derivative expansion method [1], the scattering phase shift technique [2], the mode regularization method [4], the zeta-function regularization technique [5] and also the dimensional regularization method [7]. In this letter I will give a very simple derivation of the one loop renormalized soliton quantum mass correction in 1+1 dimensional scalar field theory models, using the scattering phase shift technique. The approach used here differ from those given in Ref. [2]. Consequently, it is obtained a formula, Eq. (19), that at first sight is different from the one obtained in Ref. [2].

Let us start with a Lagrangean density

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) ,
\]

where \( \mu = 0, 1 \). When \( U(\phi) \) has at least two degenerate trivial vacua the classical equation of motion admits static finite energy solutions \( \phi_c \). These solutions are called solitons [3]. After quantizing around one of these static solutions, it is showed that there is a quantum state corresponding to this static solution. This state is called the soliton quantum state [10]. The soliton quantum state behaves as a particle and in particular, the first contribution different from zero to their mass is given by the (classical) energy of the static solution. The one loop (bare) soliton quantum mass correction is given by

\[
\Delta M_{\text{bare}} = \frac{1}{2} \sum_n \omega_n - \frac{1}{2} \sum_k \omega_k^0 ,
\]

where \( \omega_n \) and \( \omega_k^0 \) are given respectively from the eigenvalue equations,

\[
\left[ -\frac{d^2}{dx^2} + U''(\phi_c(x)) \right] \psi_n(x) = \omega_n^2 \psi_n(x) \tag{3}
\]

and

\[
\left[ -\frac{d^2}{dx^2} + m^2 \right] \psi_k(x) = (\omega_k^0)^2 \psi_k(x) , \tag{4}
\]

where \( m^2 = U''(\phi_c(\pm \infty)) \) is the mass of the quantum fluctuations around the trivial vacua (we restrict ourselves to the case in which \( U''(\phi_c(\infty)) = U''(\phi_c(-\infty)) \)). From Eq. (3) it is easy to see that the soliton-free eigenfunctions are

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given by $\psi_k \propto e^{ikx}$ with $\omega_k^0 = \sqrt{k^2 + m^2}$. In general the eigenvalues of Eq. (3) fall in two sets, a finite discrete set, that we denote by $\omega_q$ and a continuum set, that we denote by $\omega^2(q)$ ranging from $m^2$ to $\infty$. The continuous eigenfunctions behave asymptotically as $\propto e^{iqx}$ with modes $\omega(q) = \sqrt{q^2 + m^2}$, i.e., equal to the soliton-free modes $\omega_k^0$. Naively we can expect that the sum in Eq. (2) will be reduced to a sum over the discrete modes $\omega_i$ only. But this is not the case, since there is a difference between the distributions of size over the continuum modes, we have different densities of states. We can divide the sum in Eq. (2) into a sum over the discrete modes and a sum over the continuum modes,

$$\Delta M_{\text{bare}} = \frac{1}{2} \sum_i \omega_i + \frac{1}{2} \sum_q \omega(q) - \frac{1}{2} \sum_k \omega_k^0. \tag{5}$$

In order to write the continuous sum in the above equation in an integral form, we first enclose the system in a box of size $L$, apply periodic boundary conditions and finally take the limit $L \to \infty$. For simplicity we start with the simplest case in which $U''[\phi_c(x)]$ is a reflectionless potential (this is the case for example, in the sine-Gordon and $\phi^4$ kink models), then we generalize the result for other potentials. For reflectionless potentials the asymptotic behavior of $\psi_q(x)$ is given by,

$$\psi_q(x) = \begin{cases} e^{iqx} & x \to -\infty \\ e^{iqx+\delta(q)} & x \to \infty \end{cases}, \tag{6}$$

where $\delta(q)$ is the so called phase shift. Placing the system into a box of size $L$ and imposing periodic boundary conditions for the free eigenfunctions $\psi_k \propto e^{ikx}$, we have $e^{-ikL/2} = e^{ikL/2}$, from which we obtain $k_n = \frac{2\pi n}{L}, n = \pm 1, \pm 2$. In this case the free density of states is $1/(k_n+1 - k_n) = L/(2\pi)$. On the other hand from Eq. (3), imposing periodic boundary conditions we get,

$$q_n = 2\pi n - \frac{\delta(q_n)}{L} = k_n - \frac{\delta(k_n)}{L} + O(L^{-2}), \tag{7}$$

where when passing to the second line we have solved iteratively in terms of $k_n$. In Eq. (6) we disregarded terms of order higher than $O(L^{-1})$, since in the limit $L \to \infty$ their contributions to $\Delta M_{\text{bare}}$ vanish. Replacing Eq. (6) in $\sqrt{q_n^2 + m^2}$ and expanding up to $O(L^{-2})$, we obtain

$$\sqrt{q_n^2 + m^2} = \sqrt{k_n^2 + m^2} - \frac{k_n\delta(k_n)}{L\sqrt{k_n^2 + m^2}} + O(L^{-2}). \tag{8}$$

Replacing Eq. (6) in Eq. (5), using the free density of states $\frac{1}{2\pi}$, going to the continuum limit $L \to \infty$ and integrating by parts we obtain,

$$\Delta M_{\text{bare}} = \frac{1}{2} \sum_i \omega_i - \frac{\omega_k \delta(k)}{4\pi} \bigg|_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \omega(k) \frac{d}{dk} \delta(k). \tag{9}$$

Note the appearance of a surface integration term. This term is not considered in other treatments, see for example [2], [4] and [8]. Eq. (6) can be generalized to the case of potentials $U''[\phi_c(x)]$ that are not necessarily reflectionless. The expression is the same, we have only to give a generalized expression for the phase shift in terms of the S-matrix associated to the one dimensional scattering problem given by the continuous solutions of Eq. (3). In terms of the reflection and transmission coefficient amplitudes, $R$ and $T$, the S-matrix reads [11],

$$S(k) = \begin{pmatrix} T(k) & -R^*(k)T(k)/T^*(k) \\ R(k) & T(k) \end{pmatrix}, \tag{10}$$

and the phase shift is given by [11]

$$\delta(k) = \frac{1}{2i} \ln \det S(k) = \frac{1}{2i} \ln \left[ \frac{T(k)}{T^*(k)} \right], \tag{11}$$

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where in passing to the second equality we have used the known property \(|R|^2 + |T|^2 = 1\). Since the \(S\)-matrix is unitary it is sure that \(\delta(k)\) as given by Eq. (11) is a real function of \(k\). In the case of reflectionless potentials, since we have \(T = e^{i\delta(k)}\), Eq. (11) is trivially satisfied.

The bare quantum mass corrections as given by Eq. (9), is logarithmically divergent, since the phase shift behaves as \(1/k\) for large \(k\). Then, we have to renormalize such expression. In order to do this we write Eq. (2) in other equivalent form \([3] [12]\)

\[
\Delta M_{\text{bare}} = \frac{1}{2} \sum_i \omega_i \delta(k_i) \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{k^2 + m^2}}
\]

The above expression is obtained with functional methods as the Euclidean effective action per unit time evaluated at the static soliton configuration. Eq. (12) can be expanded in terms of Feynman graphs,

\[
\Delta M_{\text{bare}} = \quad + \quad + \quad + \ldots ,
\]

where the background field \(V(x) = U''[\phi_c(x)] - m^2\). From the expansion in Feynman graphs we note that the only divergent term is the tadpole graph,

\[
\langle V \rangle = \int_{-\infty}^{\infty} dx V(x) .
\]

As expected the tadpole graph is logarithmically divergent. Since Eqs. (9) and (13) are equivalent, in order to regularize the quantum mass correction as given by Eq. (9), we subtract and add to it the tadpole graph given by Eq. (14) obtaining

\[
\Delta M_{\text{bare}} = \left\{ \frac{1}{2} \sum_i \omega_i \delta(k_i) \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{k^2 + m^2}} \right\} + \langle V \rangle .
\]

In the above equation, the first part is finite and the full one loop divergence is contained in the added tadpole graph only. Now we use the simplest renormalization prescription that the added tadpole graph cancels exactly with adequate counterterms (that are present in the classical bare mass), obtaining in this way unambiguously the renormalized soliton quantum mass correction at one loop order. This renormalization prescription is equivalent to using a normal ordering prescription for the field operators, that in 1+1 dimensions render finite any scalar field theoretical model \([13]\). Then, we can identify the finite renormalized one loop soliton quantum mass correction, that we denote by \(\Delta M\), with the first part of Eq. (16),

\[
\Delta M = \frac{1}{2} \sum_i \omega_i \delta(k_i) \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{k^2 + m^2}} ,
\]

where we used Eq. (14) and it is understood that now the parameters that appear in Eq. (17) are the physical ones. In order to compute the surface term in Eq. (17) we only need to know the behavior of the phase shift for large \(k\) and this is given by the Born approximation. The only term that contributes to the surface term is the first Born approximation, given by \([14]\)

\[
\delta^1(k) = -\frac{\langle V \rangle}{2k} .
\]

Replacing Eq. (18) in Eq. (17) we obtain,
\[ \Delta M = \frac{1}{2} \sum_i \omega_i + \frac{\langle V \rangle}{4\pi} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \omega(k) \frac{d\delta(k)}{dk} - \frac{\langle V \rangle}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{k^2 + m^2}}. \]  
(19)

In order to check that our formula given by the above equation is correct, we use it to compute the one loop soliton quantum mass correction in the sine-Gordon and \( \phi^4 \) kink models.

**The sine-Gordon Model**

For this model the density potential is given by

\[ U(\phi) = \frac{m^4}{\alpha^2} \left[ 1 - \cos \left( \frac{\alpha}{m} \phi \right) \right] \]  
(20)

and \( V(x) \) is given by

\[ V(x) = -\frac{2m^2}{\cosh^2(mx)} \]  
(21)

The potential \( U''[\phi_c(x)] = V(x) + m^2 \) admits only one discrete eigenvalue, \( \omega_0^2 = 0 \). The phase shift \( \delta(k) = 2 \tan^{-1}(m/k) \) and from Eq. (15) we obtain \( \langle V \rangle = -4m \). Substituting these values in Eq. (19) we obtain,

\[ \Delta M = -m - \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{2m}{\sqrt{k^2 + m^2}} + m \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{k^2 + m^2}} \]
\[ = -\frac{m}{\pi}. \]  
(22)

**The \( \phi^4 \) kink model**

In this case the density potential is given by

\[ U(\phi) = \frac{m^4}{32\alpha^2} \left[ 1 - \frac{4\alpha^2}{m^2 \phi^2} \right]^2, \]  
(23)

and \( V(x) \) is given by

\[ V(x) = -\frac{3m^2}{2\cosh^2(mx/2)}. \]  
(24)

In this case the potential \( U[\phi_c(x)] \) admits two discrete eigenvalues, the zero mode eigenvalue \( \omega_0 = 0 \) and \( \omega_1 = \frac{m\sqrt{3}}{2} \). The phase shift in this case is given by \( \delta(k) = -2 \tan^{-1}[3mk/(m^2 - 2k^2)] \). From Eq. (15) we obtain \( \langle V \rangle = -6m \). Replacing these values in Eq. (19) we obtain,

\[ \Delta M = \frac{m\sqrt{3}}{4} - \frac{3m}{2\pi} - \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{6m(2k^2 + m^2)}{(4k^2 + m^2)\sqrt{k^2 + m^2}} + \frac{3m}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{k^2 + m^2}} \]
\[ = \frac{m\sqrt{3}}{4} - \frac{3m}{2\pi} - \frac{3}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{m^3}{(4k^2 + m^2)\sqrt{k^2 + m^2}} \]
\[ = -m \left( \frac{3}{2\pi} - \frac{1}{4\sqrt{3}} \right). \]  
(25)

Note that Eqs. (22) and (25) are the same obtained previously, with other methods. These results were obtained in Ref. [18]. Then we can conclude that our formula, Eq. (19), is correct. As we have mentioned previously, in the sine-Gordon and \( \phi^4 \) kink models, the potential \( V(x) \) (or \( U''[\phi_c(x)] \)) is reflectionless. This property makes the calculation easy. Next we use Eq. (19) to compute the soliton quantum mass corrections in a model where \( V(x) \) is not reflectionless.

**The model \( \phi^2 \cos^2 \ln(\phi^2) \)**

In Ref. [15] it has been introduced the model with density potential given by

\[ U(\phi) = \frac{1}{2} m^2 B^2 \phi^2 \cos^2 \left[ \frac{1}{2B} \ln \left( \frac{\phi^2}{3m^2} \right) \right]. \]  
(26)

This model has infinitely degenerate trivial vacua at the points...
\[ \phi_n = \pm \frac{3m^2}{\alpha} \exp \left( \frac{2n + 1}{2} \pi B \right), \quad n = 0, \pm 1, \pm 2, \ldots \]  

The static soliton and anti-soliton solutions are given by

\[ \phi_c(x) = \pm \frac{3m^2}{\alpha} \exp \left[ n \pi B \pm B \tan^{-1} \sinh(mx) \right], \quad n = 0, \pm 1, \pm 2, \ldots \]  

Using the above equation in the formula for \( U''(\phi) \) obtained from Eq. (27) we get

\[ U''[\phi_c(x)] = m^2 \left[ 1 + \frac{(B^2 - 2)}{\cosh^2(mx)} \pm 3B \frac{\tanh(mx)}{\cosh(mx)} \right], \]  

from which we obtain for \( V(x) \),

\[ V(x) = m^2 \frac{(B^2 - 2) \pm 3B \sinh(mx)}{\cosh^2(mx)}. \]  

The potential given by Eq. (29) is exactly solvable. It has only one discrete eigenvalue, the zero mode solution \( \omega_0 = 0 \). Also the reflection and transmission coefficient amplitudes can be obtained exactly. For the transmission coefficient amplitude the result obtained is

\[ T(k) = \frac{\Gamma(-1 - ik/m)\Gamma(2 - ik/m)\Gamma(1/2 \mp iB - ik/m)\Gamma(1/2 \pm iB - ik/m)}{\Gamma(-ik/m)\Gamma(1 - ik/m)\Gamma^2(1/2 - ik/m)}, \]  

from which the phase shift \( \delta(k) \) can be obtained using Eq. (11). Replacing Eq. (30) in Eq. (15) we get \( \langle V \rangle = 2m(B^2 - 2) \). Then using the preceding formulas in Eq. (19), integrating by parts and after some algebraic manipulations we obtain,

\[ \frac{\Delta M}{m} = -\frac{1}{\pi} - \frac{1}{2} \int_{-\infty}^{\infty} \frac{dq}{2\pi \sqrt{\alpha^2 + 1}} \left\{ \frac{1}{2i} \ln \left[ \frac{\Gamma(1/2 - iB - iq)\Gamma(1/2 + iB - iq)\Gamma^2(1/2 + iq)\Gamma^2(1/2 - iq)}{\Gamma(1/2 + iB + iq)\Gamma(1/2 - iB + iq)\Gamma^2(1/2 - iq)} \right] + \frac{B^2}{q} \right\}, \]  

where we have defined \( q = k/m \). The integral in Eq. (32) cannot be performed exactly, but for \( B = 0 \) it vanishes and in this case we obtain, \( \Delta M = -m/\pi \), equal to the soliton quantum mass correction in the sine-Gordon Model. This is understood since for \( B = 0 \) the potential \( V(x) \) given by Eq. (30) is equal to those given by Eq. (27). For other values of \( B \) we can perform numerically the integration in Eq. (32). The result obtained is showed in Fig. 1. From Fig. 1 we see that the soliton quantum mass correction is negative and decrease for increasing \( B \). Although we do not compute exactly \( \Delta M \) in this case, we call attention to the fact that all the steps have been done analytically and that only the final integration is done numerically. Then our computation, although not exact is precise. Frequently the sine-Gordon and \( \phi^4 \) kink models are used to test approximate or numerical methods developed to compute quantum corrections around static classical configurations, see for example [10]. Here we present another model that can be used to this end. We believe that this model is more adequate to test approximate or numerical methods since the sine-Gordon and \( \phi^4 \) kink models are very special: their quantum fluctuations are described by reflectionless potentials.

I would like to remark that Eq. (13) is equivalent to the one presented in Ref. [2], in which the authors have worked the case in which \( V(x) \) is symmetric in \( x \). The interested reader can prove this equivalence using the one-dimensional Levinson theorem. Although the final formula is equivalent the derivation is different. The starting point in Ref. [2] is Eq. (8) without the surface term. Then the authors subtract the first Born approximation to the phase shift as given by Eq. (18) in order to render finite the quantum mass correction. But, since the phase shift is divergent for \( k = 0 \), they use (before doing the subtraction) the one dimensional Levinson theorem [1]. By using the Levinson theorem they cure the infrared divergence of the first Born approximation to the phase shift and also the surface term (not considered at the start) emerges. Finally they use the same renormalization prescription that the added tadpole graph cancels, obtaining in this way a formula that is equivalent to Eq. (13). Also in Ref. [3], the surface term in Eq. (12) is not considered at the start but, it emerges after using the called mode number cutoff. Finally I would like to call attention about the fact that if one consider in Ref. [3] as starting point Eq. (12) with the surface term, the energy-momentum cutoff regularization will give the correct answer for the soliton quantum mass corrections. I believe that the origin of part of the controversy raised in the literature about how to compute correctly the soliton quantum mass corrections has been to use (incorrectly) as starting point Eq. (12) without the surface term.
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