Iterations and groups of formal transformations

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Abstract
In this paper, we consider the problem of formal iteration. We construct an area preserving mapping which does not have any square root. This leads to a counterexample to Moser’s existence theorem for an interpolation problem. We give examples of formal transformation groups such that the iteration problem has a solution for every element of the groups.

Keywords: iteration, formal transformations, functional equations.

1 Introduction.
Iterated functions are objects of study in computer science, fractals, dynamical systems and renormalization group physics [1, 2, 3, 4]. Here we will consider continuous iterations of mappings. Let \( K \) denote either the set of real numbers or the set of complex numbers. Suppose we are given a local diffeomorphism \( u \) of a neighborhood of the origin \( 0 \in K^n \) onto another and leaves \( 0 \) fixed. The problem of continuous iteration consists in finding a one-parameter family of mappings (a flow) \( f(t, x) = f^t(x) \) such that

\[
f^t \circ f^s = f^{t+s}, \quad f^1 = u, \quad f^0(x) = x \quad \forall t, s \in \mathbb{R}.
\]

The iteration problem was investigated by Koenigs, Lewis, Baker, Chen, Sternberg and others. Bibliographical references can be found in [4, 5, 6].

Every smooth flow \( f^t \) is defined by a system of ordinary differential equations

\[
y' = X(y)
\]

with initial condition \( y(0) = x \). Thus the iteration problem is equivalent the following question. Given a a local diffeomorphism \( u \), does there exist a system of ordinary differential equations such that \( y(1) = u \)? If the answer to this question is affirmative then we say that the map \( u \) is embedded in the flow \( f^t \).

The problem is of great interest in the study of the exponential mapping of infinite-dimensional Lie algebras of vector fields [5, 7, 8]. Let \( \exp(tX) \) denote an one-parameter group generated by a vector field \( X \), then the map \( \exp : X \mapsto \exp(X) \) is called the exponential map or time-one map. Let \( G \) be a group of smooth (or formal) maps, and we are given the mapping \( u \in G \). The question which arises is this: under what conditions is there a vector field \( X \) such that \( u = \exp(X) \)? If such a vector field \( X \) exists, then it is called the logarithm of \( u \). We will also say that the formal transformation \( u \) possesses a logarithm.

Let us denote by \( GS_n(K) \) the group of formal power series transformations [9]. Lewis [10] proved that if a transformation \( u \in GS_n(K) \) satisfies so-called pseudo-incommensurable condition, then the iteration problem has a formal power series solution. This Lewis result has been repeatedly proved by different authors [5, 6, 9].
In this paper we discuss the iteration problem for some subgroups of the group $G_S_n(K)$. It turns out that there are mappings $u$ to which the problem does not even have formal solution, namely, we give an example of a polynomial mapping $u : \mathbb{R}^2 \to \mathbb{R}^2$ preserving the area such that there does not exist a 2-tuple $g = (g_1, g_2)$ of formal power series $g_1, g_2 \in \mathbb{R}[[x, y]]$ with $g \circ g = u$. This is a counterexample to Moser’s statement [3] about the existence of a solution to the iteration problem for area-saving mappings. We present sufficient conditions for the existence of a solution of the iteration problem. These conditions allow to indicate some groups of formal transformations such that any element of a group possesses a logarithm and the corresponding iteration problem has a formal solution.

2 Examples and condition for the existence of solutions

We begin with the case of a linear mapping

$$u(x) = Ux, \quad x \in \mathbb{K}^n,$$

where $U$ is an invertible matrix. In this case, a solution of the iteration problem has the form

$$u^t(x) = U^t x = e^{t \ln(U)} x$$

whenever the matrix $\ln(U)$ is correctly defined. When $\mathbb{K} = \mathbb{C}$ the matrix $\ln(U)$ exists but in general it is not unique. If $\mathbb{K} = \mathbb{R}$ and $U$ is positive definite then $\ln(U)$ is a real matrix. Some details of the linear case can be found in [10]. Sometimes a nonlinear problem (1) can be reduced to a linear one. This is true if an analytical map $u$ is conjugate to a linear map. Some of the most known results in this direction are Poincaré and Siegel-Sternberg theorems [11, 12, 13].

We now consider the groups of formal transformations. Let $\mathbb{K}[[x]]$ denote the ring of formal power series in indeterminate $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$. The ring has a maximal ideal $\mathfrak{M}_1$ and an ideal $\mathfrak{M}_2$ consisting of series without constant and linear terms. Denote by $\mathfrak{M}_i^n$ ($i = 1, 2$) the $n$-ary Cartesian product of $\mathfrak{M}_i$. Obviously $\mathfrak{M}_i^n$ is a monoid under substitution of series. We denote by $G_S_n(\mathbb{K})$ the set of all invertible elements of $\mathfrak{M}_1^n$. We shall call elements of $G_S_n(\mathbb{K})$ formal transformations. It is clear that $G_S_n(\mathbb{K})$ is a group. As usual, the general linear group of degree $n$ over $\mathbb{K}$ is denoted by $GL_n(\mathbb{K})$.

Example 1. Let us consider the group $GS_1(\mathbb{C})$ and a polynomial map

$$u = e^{i \pi / 3} z + z^7.$$

It is easy to see that there is no a formal power series

$$g = c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \ldots$$

such that

$$g \circ g = u. \quad (2)$$
Actually, comparing coefficients of $z$ in (2), we have

$$c_1^2 = e^{i\pi/3}.$$ 

Then comparing coefficients of $z^2, \ldots, z^6$ yields $c_2 = \cdots = c_6 = 0$. Finally, comparing coefficients of $z^7$, we obtain

$$c_1c_7(c_6^1 + 1) = 1.$$ 

This is a contradiction, because $c_6^1 + 1 = 0$. This example shows that there is no one-parameter group passing through the polynomial $e^{i\pi/3}z + z^7$. If such a group $f^t$ exists, then $f^{1/2} \circ f^{1/2} = u$. But it is not possible as we just proved. This example shows that polynomial map $u = e^{i\pi/3}z + z^7 \in GS_1(\mathbb{C})$ does not possess a logarithm.

We remark that such examples have been known for a long time (see, for example [4, 9]).

**Example 2.** Let $SS_n(K)$ denote the set \{f \in GS_n(K) : det(Df) = 1\}, where $Df$ is the Jacobian matrix of $f$, i.e. $SS_n(K)$ is a group of volume preserving formal transformations. Consider an area preserving polynomial mapping $v \in SS_2(K)$ given by

$$\tilde{x}_1 = x_1 + x_2^{m+1}, \quad \tilde{x}_2 = x_2$$

and the rotation matrix

$$M = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},$$

where $\alpha = 2\pi/m$ and $m \geq 2$ is an even number. Thus $u = Mv$ is an area preserving mapping.

It is convenient to use the complex variables $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. Then the mapping $u$ has the form

$$u(z, \bar{z}) = e^{2i\pi/m} \left( z + \left( \frac{z - \bar{z}}{2i} \right)^{m+1} \right). \quad (3)$$

Let us show that there does not exist a formal series

$$g(z, \bar{z}) = c_{10}z + c_{01}\bar{z} + c_{20}z^2 + c_{11}z\bar{z} + c_{02}\bar{z}^2 + \ldots$$

satisfying the condition (2). We assume that such series exists and try to find his coefficients.

Collect all terms belonging to $z, \bar{z}$ in (3). Then we have two equations

$$e^{2i\pi/m} = c_{10}^2 + |c_{01}|^2, \quad (4)$$

$$c_{01}(c_{10} + \bar{c}_{10}) = 0.$$ 

It follows that

$$c_{01} = 0, \quad c_{10} = \pm \exp(i\pi/m).$$

Then comparing coefficients of $z^k\bar{z}^l$ ($1 < k + l < m + 1$) yields equation

$$c_{kl}(c_{10} + c_{10}^k\bar{c}_{10}) = 0.$$
Obviously, the following inequality holds
\[ c_{10} + c_{10}^{k l} \neq 0 \]
whenever \( 1 < k + l < m + 1 \). Thus we have \( c_{kl} = 0 \).

Finally, we collect all terms belonging to \( z^{m+1} \) and obtain equalities
\[
\frac{\exp(2i\pi/m)}{(2i)^{m+1}} = c_{(m+1)0}c_{10}(1 + c_{10}^m) = 0,
\]
since \( c_{10} = \pm \exp(i\pi/m) \) and \( m \) is an even number. This contradiction proves our assertion.

This example implies that Moser’s theorem [3] on the solvability of the iteration problem in the class of formal series is not true even for polynomial mappings. Moreover, it is impossible to find the square root of a area preserving mapping in the general case. This example shows that the polynomial map [3] does not possess a logarithm. We shall see that the above examples are related to resonances.

Let \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) be characteristic values of a matrix \( U \in GL_n(\mathbb{C}) \). We recall that an identity of the form
\[ \lambda^s = \lambda_1^{m_1} \cdots \lambda_n^{m_n} \quad m_i \in \mathbb{N}, \quad \sum_{i=1}^{n} m_i > 1 \] (5)
is called the resonance (induced by \( U \)). We say that the resonance (5) is not obstructive if
\[ \lambda^t = \lambda_1^{tm_1} \cdots \lambda_n^{tm_n} \quad \forall t \in \mathbb{R}. \] (6)
It is easy to see that we have resonances of the form
\[ \lambda = \lambda^{m+1}, \]
in Examples 1 and 2 above. These resonances are obstructive since
\[ \lambda^{\frac{1}{2}} \neq \lambda^{\frac{m+1}{2}}. \]

Using the theory of normal forms we proved the following statement in [14].

Lemma. Let \( u = Ux + g \in GS_n(\mathbb{C}) \) be a formal transformation with \( U \in GL_n(\mathbb{C}) \) and \( g \in M_2^n \). If any resonance induced by the matrix \( U \) is not obstructive then \( u \) possesses a logarithm.

Now we show that the conditions (5), (6) are equivalent to Lewis’s ones. Indeed, it follows from (5) that
\[ \exp(\log \lambda) = \exp(m_1 \log \lambda_1 + \ldots + m_n \log \lambda_n). \]
The last equality is equivalent to
\[ \log(\lambda) - \sum_{j=1}^{n} m_j \log(\lambda_j) \in 2\pi i \mathbb{Z}. \] (7)
Similarly, the condition (6) yields
\[ t \log(\lambda_s) - \sum_{j=1}^{n} m_j \log(\lambda_j) \in 2\pi i \mathbb{Z} \quad \forall t \in \mathbb{R}. \]

It follows that
\[ \log(\lambda_s) = \sum_{j=1}^{n} m_j \log(\lambda_j). \quad (8) \]

Conversely, it is easy to see that the equality (8) gives (6) and (7) implies (5).

We recall that Lewis’s condition means that any relation (7) implies the equality (8) (see [9, 10]).

One can apply Lemma to obtain subgroups \( G \) of \( GS_n(\mathbb{K}) \) such that any \( u \in G \) possesses a logarithm. For example, consider subgroup \( B_l \) which consists of formal transformations
\[ u = Ux + g, \quad g \in \mathfrak{M}_{2}^n, \]
where \( U \) is a lower triangular matrix with real positive eigenvalues.

**Corollary.** Any formal transformation \( u \in B_l \) possesses a logarithm.

The analogous result holds for subgroup of formal transformations \( B^u \) with upper triangular matrices.

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