Anomalous Dimension of the Electrical Current in the Normal State of the Cuprates from the Fractional Aharonov-Bohm Effect

Kridsanaphong Limtragool and Philip W. Phillips*

Department of Physics and Institute for Condensed Matter Theory,
University of Illinois 1110 W. Green Street, Urbana, IL 61801, U.S.A.

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We show here that if the current in the normal state of the cuprates has an anomalous dimension, then the Aharonov-Bohm flux through a ring does not have the standard \( eBA/\hbar \) form, where \( A \) is the area, \( B \) is the external magnetic field, and \( e \) is the electric charge, but instead it is modified by a geometrical factor that depends directly on the anomalous dimension of the current. We calculate the Aharonov-Bohm flux in square and disk geometries. In both cases, the deviation from the standard result is striking and offers a fingerprint about what precisely is strange about the strange metal.

I. INTRODUCTION

Before Faraday discovered that moving charges induce magnetic fields \( (B) \), electric and magnetic fields were thought to be independent. A concise mathematical synthesis of the two requires an additional entity, the vector potential, \( A \), which in classical physics is experimentally undetectable. Aharonov and Bohm[1] showed, however, that in quantum mechanics, the principle of gauge invariance imbests the vector potential with physical content such that the wave function of a charged particle moving in a closed loop around a magnetic solenoid experiences a phase shift that is determined entirely by the line integral,

\[
\Delta \phi = \frac{e}{\hbar} \oint A \cdot d\ell, \tag{1}
\]

of the vector potential around a closed loop. Because \( \nabla \times A = B \) and Stokes’ theorem which allows us to convert a line integral to a surface one, the integral simplifies to \( eBA/\hbar \), where \( A \) is the cross sectional area of the magnetic solenoid, \( e \) is the electric charge, and \( \hbar \) is Planck’s constant divided by \( 2\pi \). The key physical surprise here is that charges outside the solenoid know about the magnetic field solely because of the spatial extent of the vector potential. The relationship between the vector potential and the magnetic and electric fields implies that all the equations of classical electromagnetism are invariant with respect to the transformation,

\[
A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \tag{2}
\]

where \( \partial_\mu = (\partial_t/c, \partial_x, \partial_y, \partial_z) \) and \( \Lambda \) is an arbitrary dimensionless function. Because \( \Lambda \) is dimensionless, this transformation fixes the dimension of \( A_\mu \) to be unity; that is, \( A_\mu \) has dimensions of inverse length. A further consequence of the invariance of electricity and magnetism to a choice in the gauge is that there has to be a corresponding conserved current whose dimension is set by the generators of the \( U(1) \) symmetry group. The resulting dimension of the conserved current in a relativistic theory is \( d \) where \( d \) is the spatial dimension. Clearly if \( |\Lambda| \neq 1 \), the underlying theory is not governed by the standard 1-form gauge-invariant principle of electricity and magnetism.

There are no known examples in nature of a conserved current in which the vector potential has a dimension other than unity. Perhaps possible exceptions to this rule could obtain in exotic materials such as the high-temperature copper-oxide superconductors. This problem remains unsolved because no knock-down experiment has revealed unambiguously the nature of the charge carriers in the normal state. What we know for sure is that the standard theory of metals and a single-parameter[2, 3] formulation of quantum criticality cannot simultaneously explain \( T \)-linear resistivity, power-law optical conductivity[4–6], breakdown of the Weidemann-Franz law[7], and the scaling of the Hall angle[8]. However, recent theoretical work[3, 9, 10] suggests that all of the transport properties of the normal state can be explained by positing a conserved current with an anomalous dimension.

Indeed, this is a striking prediction because a textbook problem[11–13] in quantum field theory is to prove that conserved quantities cannot acquire anomalous dimensions under renormalization. For a local theory away from the strict relativistic limit, the dimension of the current can change by two mechanisms: 1) reduction of the effective dimensionality, that is a violation of hyperscaling[14, 15] with exponent \( \theta \) or 2) space and time scale differently thereby requiring a dynamical exponent \( z > 1[19] \). The new scaling of the current is now \( d-\theta+z-1 \). Either of these can be modeled using holography with

* Guggenheim Fellow
II. FRACTIONAL GAUGE TRANSFORMATION

Indeed it is gauge invariance that makes the problem of anomalous dimensions for the gauge field highly problematic \textit{a priori}. Consider the transformation in Eq. (2) applied to the action

\[S = \int d^4x[F^2 + J_\mu A^\mu + \cdots].\]  

(3)

Since the field strength, $F$ is invariant under Eq. (2), the action transform

\[S \rightarrow S + \int d^4x J_\mu \partial^\mu \Lambda.\]  

(4)

Consequently, invariance under Eq. (2), upon integration by parts, results in the standard charge conservation equation

\[\partial^\mu J_\mu = 0.\]  

(5)

The natural question that arises is if an anomalous dimension is not compatible with Eq. (2), then what is the consequence for charge conservation? Indeed fractional formulations of electricity and magnetism do exist\cite{23–25} based on the gauge principle $a_\alpha A_\mu(x) \rightarrow a_\alpha A_\mu(x) + \partial^\mu \Lambda(x)$ which contain fractional derivatives (see Appendices A and B) of the phase $\Lambda(x)$, the power of which fixes the engineering dimension of $a_\alpha A_\mu$ to be $\alpha_\mu$. From the argument presented previously, such an implementation will affect the charge conservation equation. But an immediate problem with such constructions is that the gauge transformation is not rotationally invariant and hence this is not an acceptable theory.

What the charge conservation equation lays plain is that any operator, $\hat{Y}$, which commutes with the total differential can be used to redefine the current operator and hence will change its dimension without affecting the linear nature of Eq. (5). However, a key restriction on the operator $\hat{Y}$ is that it cannot change the order of the form of either the current or the dual current ($\kappa J$). If such an operator exists, it would also offer a loophole around the general argument advanced by Gross\cite{11} that it is the commutator of the charge density with any $U(1)$ field, $\phi(x)$,

\[\delta(x_0 - y_0)[J_0(x), \phi(y)] = \delta \phi(y) \delta(D)(x - y),\]  

(6)

that fixes the scaling dimension of the conserved current. Here $\delta \phi(y)$ is the change in the field $\phi$ to linear order upon acting with the $U(1)$ transformation and $J_0$ is the charge density. Letting $J_0 \rightarrow \hat{Y} J_0$ we see that the current no longer has dimension $D$ but rather $D - D_Y$ where $D_Y = [\hat{Y}]$. 

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1 In the cuprates, the power law in the ac conductivity\cite{4–6} appears in the mid-infrared and hence does not persist down to zero frequency. For the multi-band construction with a mass-dependent relaxation time\cite{9, 10} to match this feature, the summation over mass needs to have a cutoff\cite{9}. In this case, there exists an onset energy scale $\tau_0^{-1}$ for the power law to appear.

\[ S = \int d^4x[F^2 + J_\mu A^\mu + \cdots]. \]  

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Elsewhere[27] we have shown how to construct \( \hat{Y} \) explicitly for dilaton actions of the form,

\[
S = \int d^{d+1}x \sqrt{-\hat{g}}Z(\phi)F^2 + \cdots ,
\]

used by holographic models[17, 18] to yield either anomalous dimensions for the gauge field or hyperscaling violation exponents. Here, \( Z(\phi) \sim e^{\gamma \phi} \) is a dilaton field and \( F \) the field strength. The equations of motion for the Maxwell part of the action are

\[
\nabla^\mu (Z(\phi)F_{\mu\nu}) = 0 ,
\]

where \( \nabla^\mu \) is the covariant divergence. A typical solution[17, 18] for the dilaton field is \( \phi \sim \ln kr \). Consequently the equations of motion are equivalent to

\[
\nabla^\mu (g^\alpha F_{\mu\nu}) = 0 .
\]

In the language of differential forms, this equation becomes

\[
d (g^\alpha \star dA) = 0
\]

which clearly illustrates that for any slice perpendicular to the radial direction, the standard \( U(1) \) gauge transformation applies. To determine what happens at the boundary, we note that these equations are reminiscent of those studied by Caffarelli and Silvestre[26] (CS) for the case of a scalar field,

\[
\nabla \cdot (g^\alpha \nabla u),
\]

which is just a recasting of the elliptic differential equation

\[
u(x, y = 0) = f(x) \quad \Delta_x u + \frac{a}{y} u_y + u_{yy} = 0 .
\]

What they were interested in is what form does this differential equation acquire at the boundary, \( y \to 0 \). What Caffarelli/Silvestre showed is that any equation of this kind satisfies

\[
\lim_{y \to 0} (g^\alpha u_y) = C_d,\gamma (-\Delta)^\gamma f(x). \quad (14)
\]

where \( \gamma = (1 - a)/2 \).

The exact same result holds for the gauge field as it is just a 1-form generalization of the CS extension theorem. In a separate paper, we have generalized[27] the CS extension theorem to p-forms. The result is as expected. The p-form generalization of the CS extension theorem yields the boundary action of the form,

\[
S = \frac{1}{2} \int A_i (-\nabla)^{2\gamma} A^i ,
\]

whose propagator in Lorentz gauge, \( \partial^\gamma_i A^i = 0 \), is

\[
D^{ij} (k) = \left< A^i (k) A^j (-k) \right> = \frac{-i\eta^{ij}}{(k^2)^{\gamma}} ,
\]

Clearly at the boundary \( [A_i] \neq 1 \). The corresponding field strength is the 2-form,

\[
F = d_\gamma A = d(-\Delta)^{\gamma/2} A,
\]

with gauge-invariant condition,

\[
A \to A + d_\gamma \Lambda,
\]

with

\[
d_\gamma \equiv (-\Delta)^{\gamma/2} d ,
\]

which preserves the 1-form nature of the gauge-field with dimension \( [A_\mu] = \gamma_\mu \), rather than unity. Note \( [d, (-\Delta)^{\gamma/2}] = 0 \). Consequently, we identify \( \hat{Y} = (-\Delta)^{\gamma/2} \) which is a completely rotationally invariant operator. In general, the total differential commutes with any power of the Laplacian operator and hence the conservation equation is uniquely specified up to \( (-\Delta)^{\gamma} \). This added ambiguity in the formulation of electricity and magnetism does not seem to have been noticed until now.

What the p-form generalization[27] of the CS extension theorem lays plain in the context of holographic models that yield an anomalous dimension for the gauge field is that the anomalous dimension enters the boundary theory (see Eq. (15)) as a result of the rotationally invariant entity,

\[
\partial^\gamma_\mu = (-\Delta)^{\gamma/4} \partial_\mu ,
\]

which we take to be our operational definition of the fractional derivative. As expected, the action in terms of the electromagnetic field strength defined by Eq. (17)

\[
S = \int \frac{1}{4} F_{ij} F^{ij}
\]

is identical to Eq. (15). Simply integrate by parts and pick a gauge \( \partial^\gamma_\mu A^i = 0 \) and the action reduces to the action, Eq. (15), which results from the CS extension theorem. Consequently, the boundary actions of the holographic models that generate anomalous dimensions or hyperscaling violation exponents all contain fractional Laplacians and hence transform under the non-local gauge transformation, Eq. (18).

It is instructive to compute the current-current correlator for the action with \( F = d_\gamma A \). Consider the action with a coupling to matter field (through the current \( J^i \))

\[
S = \int \frac{1}{4} F_{ij} F^{ij} + J^i A_i .
\]
In momentum space, the generators, \( L^a = (\partial^a / \partial z)^a \), are governed by fractional calculus to the strange metal is new, numerous physical processes abound such as anomalous diffusion or Levy flights\[31\] which have been described using fractional equations of motion. We advocate here that the anomalies in the strange metal are tailor-made for fractional calculus.

III. FRACTIONAL AHARONOV-BOHM EFFECT

To derive the new result, we introduce a gauge connection into the Schrödinger equation. Let us define the covariant derivative \( D_i = \partial_i - i \xi a_i \) with the associated gauge connection \[24\]

\[
a_i \equiv [\partial_i, J^a_i \alpha A_i] = \partial_i J^a_i \alpha A_i
\]
where $I^{\alpha}$ is the fractional integral (see Appendices A and B). The fundamental theorem of fractional calculus[32] states that $I^{\alpha}\partial^{\alpha}\Lambda = \Lambda$. As a consequence, $a_{\mu} \to a_{\mu} + \partial_{\mu}\Lambda$ and our physical theory is gauge invariant although $a_{\mu}$ is directly related to the fractional gauge field. Choosing $A_{0} = 0$, we reduce the Schrödinger equation to

$$\left(-\frac{\hbar^{2}}{2m}(\partial_{t} - i\frac{e}{\hbar}a)_{\mu}^{2} + V\right)\psi = i\hbar\partial_{t}\psi. \quad (32)$$

To derive the AB phase, let us consider a particle confined on the $x,y$ plane with a fractional magnetic field applied along the $z$ axis. Assume a particle can move from point $r_{i}$ to $r_{f}$ along path $\gamma_{1}$ (with wave function $\psi_{1}$) and along path $\gamma_{2}$ (with wave function $\psi_{2}$). The total wave function at the point $r_{f}$ at zero fractional magnetic field ($a_{i} = 0$) is $\psi = \psi_{1} + \psi_{2}$. When the fractional magnetic field is turned on, the total wave function at $r_{f}$ changes to

$$\psi(r_{f}, t) = e^{i\frac{\pi}{2}a_{(r)} \cdot d\gamma_{1}}\psi_{1}(r_{f}, t) + e^{i\frac{\pi}{2}a_{(r)} \cdot d\gamma_{2}}\psi_{2}(r_{f}, t)$$

$$= C\left(\psi_{1}(r_{f}, t) + e^{i\frac{\pi}{2}a_{(r)} \cdot d\gamma_{2}}\psi_{2}(r_{f}, t)\right). \quad (33)$$

Here $C$ is an over all phase factor $= e^{i\frac{\pi}{2}a_{(r)} \cdot d\gamma_{2}}$. The phase difference between the two paths due to the gauge field is

$$\Delta\phi = \frac{e}{\hbar} \oint a(r) \cdot dl. \quad (34)$$

In the strange metal, we posit that the current carrying degrees of freedom which emerge in the infrared couple to the fractional electromagnetic fields. By definition, the propagating degrees of freedom are weakly interacting thereby warranting the Schrödinger propagator approach we have adopted here.

We consider two different geometries in which an external magnetic field, $B$, pierces the sample and vanishes outside the shaded region in Figs. (1) and (2). We postulate that the $B$-field interacts with the material in such a way that the $B$-field acquires an anomalous dimension and hence becomes fractional, $B\alpha$.[2] The charged particles in the sample now directly couple to the fractional vector potential $B\alpha A$ instead of coupling to the external field $A$. Hence, we can use Eq. (34) to calculate the AB phase shift that these particles experience.

We work with five different definitions of fractional calculi (see Appendix A). We show below only the result of the Feller calculus (for Fig. 1) and the rotationally invariant definition (for Fig. 2) because these definitions are odd under parity and thus the fractional gauge field formulated with these definitions will resemble the regular gauge field. The results for other definitions can be found in Appendices D and E. For the rectangle geometry in Fig. (1), the AB phase for the Feller calculus when $a,b,c,d \gg \ell$ is

$$\Delta\phi_{R} = \frac{e}{\hbar} B\ell^{2}\left(\frac{(a^{\alpha-1} + b^{\alpha-1})(c^{\alpha-1} + d^{\alpha-1})}{4\Gamma^{2}(\alpha)\sin^{2}\frac{\pi\alpha}{2}}\right). \quad (35)$$

The phase picks up a geometric factor that is directly determined by the anomalous dimension $\alpha$ of the vector potential. The limiting value is $eB\ell^{2}/\hbar$ as $\alpha \to 1$. The convention that we have used is that the anomalous dimension is carried by the $\alpha B$-field not the charge such that $|\alpha B| = 2\alpha$. As a result $\Delta\phi$ is dimensionless.

FIG. 1: Rectangle geometry that confines particle motion. The fractional magnetic field is confined to the red region of size $\ell$ in the figure.

The more experimentally tractable setup is most likely the disk in Fig. (2). The AB phase shift for the rotationally invariant definition is:

2 Depending on how $B$ and $B\alpha$ are related in the material, there is a possibility of having finite magnetic monopoles in the system. However, it turns out that when $\alpha > 0$ magnetic monopoles does not exist for the field configurations we consider in Figs. (1) and (2). We discuss this issue in Appendix C.
\[ \Delta \phi_D = \frac{e}{\hbar} \pi r^2 \alpha BR^{2\alpha-2} \left( \frac{2^{2-2\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)} 2F_1(1-\alpha, 2-\alpha, 2; \frac{r^2}{R^2}) \right). \] (36)

Here \( 2F_1(a, b; c; z) \) is a hypergeometric function and the terms in the parenthesis reduce to unity in the limit \( \alpha \to 1 \).

**IV. CONCLUSION**

We have shown here that the presence of an anomalous dimension leads to a significant deviation from the standard AB phase. Appearing in the AB phase is a geometric factor in which the size of the sample is raised to a power involving the anomalous dimension. This extra sample-size dependence reflects the non-locality of the current. The correction is sizeable as it involves a ratio of the sample size to the region where the flux is threaded. As a result, we have provided an experimental diagnostic that is independent of any scaling ansatz. One possible way to detect this AB phase is to perform a current interference experiment on a strange metal ring with a magnetic field at the center. This is the same geometry as Fig. 2. We predict that the periodicity of a magnetoresistance is directly proportional to the fractional AB phase as opposed to the standard AB phase. One can then extract the anomalous dimension by varying the ring’s radius. This setup is based on the experiment in which the standard AB phase was observed in a metallic ring[33]. Of course the success or failure of the experiment will be determined by how well phase coherence can be maintained along the excursion around the solenoid. Nonetheless, the clarity of our theoretical diagnostic should provide sufficient impetus for experiments along these lines to be performed which should serve to definitively settle that what is strange about the strange metal is that the current possesses an anomalous dimension.

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**Appendix A: Fractional Calculus in Fourier-Space Representation**

Vector potentials with anomalous dimensions require fractional calculus. It can be defined by extending the standard integer derivatives and integrals to those involving fractional powers

\[ \{ I^n_x, \frac{\partial^n}{\partial x^n} \} \rightarrow \{ I^n_x, \frac{\partial^n}{\partial x^n} \}. \] (A1)

Here \( I^n_x \) is defined as a repeated integral \( n \) times over \( x \). We focus on five definitions of fractional calculi: left and right Liouville, Feller, Riesz[24, 32, 34–37], and the rotationally invariant definition. The rotationally invariant definition, \( \partial^n_i \equiv (\nabla^2)^{\frac{n}{2}} \partial_i \), is based on the fractional Laplacian and thus needs to be defined in dimensions greater or equal to two. These definitions of fractional calculi can be formulated in real or
Fourier space. For our purposes, it is most useful to implement the Fourier-space formulation,

\[
\partial_x^\alpha f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(\alpha, k) \tilde{f}(k) \tag{A2}
\]

\[
I_x^\alpha f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F^{-1}(\alpha, k) \tilde{f}(k), \tag{A3}
\]

where \(F(\alpha, k) = (ik)^\alpha\) for left Liouville, \(F(\alpha, k) = (-ik)^\alpha\) for right Liouville, \(F(\alpha, k) = \text{sgn}(k)|k|^\alpha\) for Feller, and \(F(\alpha, k) = |k|^\alpha\) for Riesz. For the rotationally invariant definition of fractional calculus, one has the kernel for the fractional derivative/integral on the \(x_i\) coordinate \(F_i(\alpha, k) = |k|^{\alpha-1} ik_i\) with \(k\) being a \(d\)-dimensional momentum vector. Here \(\partial_x^\alpha\) and \(I_x^\alpha\) denote the fractional derivative and integral. The convention of the branch cut we use is \(-\pi < \theta \leq \pi\). Left and right Liouville are spatially asymmetric because, in real space, the operations involve an integration on the left and on the right of \(x\), respectively (Eqs. (B1) - (B4)). Feller calculus is odd under parity, and thus it resembles an odd-integer-order calculus. On the other hand, since Riesz calculus is even under parity, its behavior is similar to an even-integer-order calculus. The rotationally invariant definition is rotational invariant and odd under parity. In terms of formal mathematical operations, the methods outlined have restrictions regarding the range of validity of \(\alpha\). For both left and right Liouville calculi, one needs \(0 < \alpha < 1\). For the Feller and the Riesz calculi, one needs \(0 < \alpha < 2\). Nonetheless, the results can be analytically continued outside this range.

The important property of these definition is that when \(\alpha > 0\) the fractional derivative of a constant is zero. Let \(f(x) = C\) where \(C\) is a constant. The Fourier component of \(f(x)\) is \(\tilde{f}(k) = 2\pi C \delta(k)\). Consequently,

\[
\partial_x^\alpha f(x) = CF(\alpha, 0) = 0. \tag{A4}
\]

For other definitions such as the left Riemann derivative (Eq. (B5) with \(a = 0\)) and the right Riemann derivative (Eq. (B7) with \(b = 0\)), \(\partial_x^\alpha f(x)\) can be nonzero.

**Appendix B: Fractional Calculus in Coordinate-Space Representation**

The fractional calculi in the previous section are formulated in the Fourier-space representations. Alternatively, they can be defined in coordinate space [24, 32, 34–37]. Let \(a\) and \(b\) be real numbers. We define the following notations for the fractional derivative and the corresponding integral for \(x > a\):

\[
D_a^\alpha(x) f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x dx'(x-x')^{n-\alpha-1} f(x') \tag{B1}
\]

\[
I_a^\alpha(x) f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x dx'(x-x')^{\alpha-1} f(x'). \tag{B2}
\]

When \(x < b\), we define

\[
D_b^\alpha(x) f(x) = \frac{1}{\Gamma(n-\alpha)} \left(- \frac{d}{dx}\right)^n \int_x^b dx'(x'-x)^{n-\alpha-1} f(x') \tag{B3}
\]

\[
I_b^\alpha(x) f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b dx'(x'-x)^{\alpha-1} f(x') \tag{B4}
\]

where \(n = [\alpha] + 1\) and \([\alpha]\) denotes the integer part of \(\alpha\).

The left Riemann-Liouville fractional calculus corresponds to

\[
D_{LRL}^\alpha = D_a^\alpha(\alpha) \tag{B5}
\]

\[
I_{LRL}^\alpha = I_a^\alpha(\alpha), \tag{B6}
\]
while the right Riemann-Liouville fractional calculus is
\begin{align}
    D_R^a &= D_x^b(a) \\
    I_R^a &= I_x^b(a).
\end{align}
\(B7\)
\(B8\)

The Liouville fractional calculi is the special case of the Riemann-Liouville calculi with \(a = -\infty\) and \(b = \infty\). The Feller fractional calculus corresponds to
\begin{align}
    D_F^\alpha &= \frac{1}{2} \sin \frac{\pi \alpha}{2} (D_\infty^x(\alpha) - D_x^\infty(\alpha)) \\
    I_F^\alpha &= \frac{1}{2} \sin \frac{\pi \alpha}{2} (I_\infty^x(\alpha) - I_x^\infty(\alpha)).
\end{align}
\(B9\)
\(B10\)

and the Riesz fractional calculus corresponds to
\begin{align}
    D_{RZ}^\alpha &= \frac{1}{2} \cos \frac{\pi \alpha}{2} (D_\infty^x(\alpha) + D_x^\infty(\alpha)) \\
    I_{RZ}^\alpha &= \frac{1}{2} \cos \frac{\pi \alpha}{2} (I_\infty^x(\alpha) + I_x^\infty(\alpha)).
\end{align}
\(B11\)
\(B12\)

The Fourier-space formulations can be shown to be the same as the coordinate space representation. We explicitly show this for the case of the left Liouville calculus. We start by rewriting Eq. (A3) in the case of left Liouville to
\begin{align}
    I_{LL}^\alpha f(x) &= \int_{-\infty}^{\infty} dx' K(x - x') f(x') \tag{B13}
\end{align}
where the kernel \(K(x - x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} (ik)^{-\alpha}\) and the subscript LL denotes left Liouville. This integral can be evaluated to be
\begin{align}
    K(x - x') &= \Theta(x - x') \frac{(x - x')^{\alpha - 1}}{\Gamma(\alpha)} \tag{B14}
\end{align}
when \(0 < \alpha < 1\). Thus, the left Liouville integral in coordinate space is
\begin{align}
    I_{LL}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} dx' (x - x')^{\alpha - 1} f(x') = I_{-\infty}^x(\alpha). \tag{B15}
\end{align}
Similarly, we rewrite the left Liouville derivative from Eq. (A2) to
\begin{align}
    \partial_{LL}^\alpha f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} (ik)^\alpha \tilde{f}(k) \\
    &= \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} (ik)^{-(n-\alpha)} \tilde{f}(k) \\
    &= \frac{d^n}{dx^n} I_{LL}^{n-\alpha} f(x) \\
    &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{x} dx' (x - x')^{n-\alpha - 1} f(x') \\
    &= D_{-\infty}^x(\alpha) f(x) \tag{B16}
\end{align}
where \(n = [\alpha] + 1\). The equivalences between the Fourier-space and the coordinate-space formulations of the right Liouville, Feller, and Riesz can be shown in similar manner.
Appendix C: Absence of Magnetic Monopoles in Constant Fractional Magnetic Field

Let us consider the possibility of having magnetic monopoles (or magnetic charges) in a fractional electromagnetic system.\(^3\) Let \(\alpha B\) denote the fractional magnetic field and \(B\) denotes the actual magnetic field. One can define the magnetic charge density \(\rho_m\) as a fractional divergence of the fractional magnetic field,

\[
\rho_m = \nabla^\alpha \cdot \alpha B. \tag{C1}
\]

The question whether \(\rho_m\) equals zero depends on how one associates \(\alpha B\) with \(B\) and on the definition of the fractional derivative we consider. We focus on the four definitions discussed in Appendix A. If we assume \(\alpha B \propto B\), then \(\nabla \cdot \alpha B = 0\). However, this does not necessarily imply that \(\nabla^\alpha \cdot \alpha B = 0\). So it is possible to have a nonzero \(\rho_m\).

It turns out that for the field configurations in Figs. (1) and (2) \(\rho_m\) vanishes when \(\alpha > 0\). From Eqs. (D1) and (E1), we have

\[
\alpha B(x, y, z) = \alpha B_z \hat{z}, \tag{C2}
\]

with \(\alpha B_z = \alpha B \Theta(\ell^2/4 - x^2) \Theta(\ell^2/4 - y^2)\) for the rectangle geometry and \(\alpha B_z = \alpha B \Theta(r - \sqrt{x^2 + y^2})\) for the disk geometry. We can directly compute \(\rho_m\) by taking the fractional divergence. We find that

\[
\rho_m = \nabla^\alpha \cdot \alpha B = \partial_z^\alpha \alpha B_z = 0, \tag{C3}
\]

with \(\alpha > 0\) and we have used Eq. (A4) since \(B_z\) does not depend on \(z\). Consequently, for the system considered here magnetic monopoles do not exist.

Appendix D: Fractional Aharonov-Bohm Effect in Rectangular geometry

The expression for \(\alpha B\) from Fig. (1) is

\[
\alpha B(x, y) = \alpha B \Theta(\ell^2/4 - x^2) \Theta(\ell^2/4 - y^2) \hat{z}. \tag{D1}
\]

The Fourier transform of \(\alpha B(x, y)\) is

\[
\alpha B(k) = \alpha B_z(k) \hat{z} = 4(\alpha B) \frac{\sin \frac{k_x \ell}{\ell} \sin \frac{k_y \ell}{\ell}}{k_x k_y} \hat{z}. \tag{D2}
\]

Below we directly use the Fourier-space formulations to evaluate fractional derivatives and integrals.

1. Left Liouville Fractional Calculus

We solve \(\alpha A(k)\) from

\[
\alpha B(k) = (ik)^\alpha \times \alpha A(k) \tag{D3}
\]

where \((ik)^\alpha = \{(ik_x)^\alpha, (ik_y)^\alpha, 0\}\). A choice of \(\alpha A(k)\) that satisfies Eq. (D3) is

\[
\alpha A(k) = \frac{\alpha B_z(k)}{(ik_x)^{2\alpha} + (ik_y)^{2\alpha}} \{-(ik_y)^\alpha, (ik_x)^\alpha, 0\}. \tag{D4}
\]

Next, using Eq. (7) of the main text, we obtain \(\alpha(k)\) as

\[
\alpha(k) = \frac{\alpha B_z(k)}{(ik_x)^{2\alpha} + (ik_y)^{2\alpha}} \{-\alpha(ik_y)^\alpha, (ik_x)^\alpha (ik_y)^{1-\alpha}, 0\}. \tag{D5}
\]

\(^3\) We mean here the system in which its gauge transformation is defined according to Eq. (18).
It is easiest to work with \( b(\mathbf{k}) = (\mathbf{i} \mathbf{k}) \times \mathbf{a}(\mathbf{k}) \). We obtain
\[
b(\mathbf{k}) = a B z(\mathbf{k})(ik_x)^{1-\alpha}(ik_y)^{1-\alpha} \hat{z},
\]
which in position space becomes
\[
b_z(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} b_z(\mathbf{k})
\]
\[
= 4(a B) \ell^{2\alpha-2} f_1(\frac{x}{\ell}) f_1(\frac{y}{\ell}),
\]
where
\[
f_1(s) = \int_{-\infty}^{\infty} \frac{dz}{2\pi i} i(z)^{-\alpha} \sin z e^{i z s}
\]
\[
= \frac{1}{2\Gamma(\alpha)} \left( \Theta(s + \frac{1}{2})(s + \frac{1}{2})^{\alpha-1} - \Theta(s - \frac{1}{2})(s - \frac{1}{2})^{\alpha-1} \right).
\]
Consequently, we obtain
\[
b_z(x, y) = \frac{a B}{\Gamma^2(\alpha)} \left( \Theta(x + \frac{\ell}{2})(x + \frac{\ell}{2})^{\alpha-1} - \Theta(x - \frac{\ell}{2})(x - \frac{\ell}{2})^{\alpha-1} \right)
\]
\[
\times \left( \Theta(y + \frac{\ell}{2})(y + \frac{\ell}{2})^{\alpha-1} - \Theta(y - \frac{\ell}{2})(y - \frac{\ell}{2})^{\alpha-1} \right).
\]
The phase difference is
\[
\Delta \phi = \frac{e}{\hbar} \int_{-a}^{b} dx \int_{-c}^{d} dy b_z(x, y)
\]
\[
= \frac{e}{\hbar a^{2} \Gamma^2(\alpha)} B b^\alpha \ell^\alpha \left( (1 + \frac{\ell}{2b})^\alpha - (1 - \frac{\ell}{2b})^\alpha \right) \left( (1 + \frac{\ell}{2c})^\alpha - (1 - \frac{\ell}{2c})^\alpha \right).
\]
In the limit \( b \gg \ell \) and \( d \gg \ell \),
\[
\Delta \phi \approx \frac{e}{\hbar} a B \ell^2 \left( \frac{b^{\alpha-1} d^{\alpha-1}}{\Gamma^2(\alpha)} \right).
\]
The AB phase from the left Liouville calculus is not symmetric. It involves only the length \( b \) and \( d \), but not \( a \) and \( c \). This result can be understood from the fact that the left Liouville calculus is spatially asymmetric.

2. Right Liouville Fractional Calculus

The resulting \( b_z(\mathbf{k}) \) is the same as Eq. (D7) but the function \( f_1(s) \) is replaced with
\[
f_2(s) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} i(-z)^{-\alpha} \sin z e^{i z s}
\]
\[
= \frac{1}{2\Gamma(\alpha)} \left( \Theta(-s - \frac{1}{2})(-s - \frac{1}{2})^{\alpha-1} - \Theta(-s + \frac{1}{2})(-s + \frac{1}{2})^{\alpha-1} \right).
\]
Performing the area integral, we find that
\[
\Delta \phi = \frac{e}{\hbar a^{2} \Gamma^2(\alpha)} B a^\alpha e^\alpha \left( (1 + \frac{\ell}{2a})^\alpha - (1 - \frac{\ell}{2a})^\alpha \right) \left( (1 + \frac{\ell}{2c})^\alpha - (1 - \frac{\ell}{2c})^\alpha \right).
\]
In the limit of $a \gg \ell$ and $c \gg \ell$,

$$\Delta \phi \approx \frac{e}{\hbar} B \ell^2 \left( \frac{a^{\alpha-1}c^{\alpha-1}}{\Gamma^2(\alpha)} \right). \quad (D14)$$

As in the case of the Left Liouville calculus, the phase is not symmetric, because the right Liouville calculus is also spatially asymmetric.

### 3. Feller Fractional Calculus

The resulting $b_z(k)$ is the same as Eq. (D7) but the function $f_1(s)$ is replaced with

$$f_3(s) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} \text{sgn}(z) |z|^{-\alpha} \sin \frac{z}{2} e^{izs}$$

$$= -\frac{1}{4\Gamma(\alpha) \sin \frac{\pi \alpha}{2}} \left( \Theta(s + \frac{1}{2})(s + \frac{1}{2})^{\alpha-1} - \Theta(-s - \frac{1}{2})(-s - \frac{1}{2})^{\alpha-1} \right) - \Theta(s - \frac{1}{2})(s - \frac{1}{2})^{\alpha-1} + \Theta(-s + \frac{1}{2})(-s + \frac{1}{2})^{\alpha-1}. \quad (D15)$$

The phase difference is

$$\Delta \phi = \frac{e(\alpha B)}{4\hbar \alpha^2 \Gamma(\alpha) \cos^2 \frac{\pi \alpha}{2}} \left( a^{\alpha}[(1 + \frac{\ell}{2a})^{\alpha} - (1 - \frac{\ell}{2a})^{\alpha}] + b^{\alpha}[(1 + \frac{\ell}{2b})^{\alpha} - (1 - \frac{\ell}{2b})^{\alpha}] \right) \times \left( c^{\alpha}[(1 + \frac{\ell}{2c})^{\alpha} - (1 - \frac{\ell}{2c})^{\alpha}] + d^{\alpha}[(1 + \frac{\ell}{2d})^{\alpha} - (1 - \frac{\ell}{2d})^{\alpha}] \right), \quad (D16)$$

which in the limit of $a, b, c, d \gg \ell$, reduces to

$$\Delta \phi \approx \frac{e(\alpha B)\ell^2}{\hbar} \left( \frac{(a^{\alpha-1} + b^{\alpha-1})(c^{\alpha-1} + d^{\alpha-1})}{4\Gamma^2(\alpha) \cos^2 \frac{\pi \alpha}{2}} \right). \quad (D17)$$

### 4. Riesz Fractal Calculus

The resulting $b_z(k)$ is the same as Eq. (D7) but the function $f_1(s)$ is replaced with

$$f_4(s) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} ||z||^{-\alpha} \sin \frac{z}{2} e^{izs}$$

$$= -\frac{1}{4\Gamma(\alpha) \cos \frac{\pi \alpha}{2}} \left( \Theta(s + \frac{1}{2})(s + \frac{1}{2})^{\alpha-1} + \Theta(-s - \frac{1}{2})(-s - \frac{1}{2})^{\alpha-1} \right) - \Theta(s - \frac{1}{2})(s - \frac{1}{2})^{\alpha-1} + \Theta(-s + \frac{1}{2})(-s + \frac{1}{2})^{\alpha-1}. \quad (D18)$$

The phase difference is

$$\Delta \phi = \frac{e}{4\hbar \alpha^2 \Gamma(\alpha) \cos^2 \frac{\pi \alpha}{2}} \left( a^{\alpha}[(1 + \frac{\ell}{2a})^{\alpha} - (1 - \frac{\ell}{2a})^{\alpha}] - b^{\alpha}[(1 + \frac{\ell}{2b})^{\alpha} - (1 - \frac{\ell}{2b})^{\alpha}] \right) \times \left( c^{\alpha}[(1 + \frac{\ell}{2c})^{\alpha} - (1 - \frac{\ell}{2c})^{\alpha}] - d^{\alpha}[(1 + \frac{\ell}{2d})^{\alpha} - (1 - \frac{\ell}{2d})^{\alpha}] \right). \quad (D19)$$

In the limit of $a, b, c, d \gg \ell$,

$$\Delta \phi \approx \frac{e}{\hbar} B \ell^2 \left( \frac{(a^{\alpha-1} - b^{\alpha-1})(c^{\alpha-1} - d^{\alpha-1})}{4\Gamma^2(\alpha) \cos^2 \frac{\pi \alpha}{2}} \right). \quad (D20)$$

The limiting value of the phase is not $eB\ell^2/\hbar$ as $\alpha \to 1$. We can understand this result from the fact that the Riesz calculus has an even parity, so one cannot expect it to have the same behavior as the first order derivative.
Appendix E: Fractional Aharonov-Bohm Effect of Disk Geometry

We consider now the disk geometry shown in Fig. (2). The fractional magnetic field is given by

\[ a B = a B_z(\rho) \hat{z} = a B \Theta(r - \rho) \hat{z}. \]  

(E1)

In Fourier space,

\[ a B_z(k) = a B \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-ik \rho \Theta(r - \rho)}. \]  

(E2)

We now change to polar coordinates, \( \rho = \rho \cos \phi \hat{x} + \rho \sin \phi \hat{y} \) and \( k = k \cos \xi \hat{x} + k \sin \xi \hat{y} \). The result is

\[ a B_z(k) = 2 \pi r k a B J_1(kr) \]  

(E3)

1. Left Liouville Fractional Calculus

We perform the same calculation as in the rectangle case to obtain

\[ b_z(k) = a B_z(k)(ik_x)^{1-\alpha}(ik_y)^{1-\alpha} = 2 \pi r a B k^{1-2\alpha} J_1(kr)(i \cos \xi)^{1-\alpha}(i \sin \xi)^{1-\alpha}. \]  

(E4)

In position space,

\[ b_z(\rho, \theta) = \frac{1}{4 \pi^2} \int_0^{2 \pi} d\phi \int_0^\infty d\rho e^{ik \rho \cos(\theta - \xi)} 2 \pi r a B J_1(kr)(i \cos \xi)^{1-\alpha}(i \sin \xi)^{1-\alpha}. \]  

(E5)

The phase difference is the area integral of \( b_z \) over the disk of radius \( R \) in Fig. 2,

\[ \Delta \phi = \frac{e}{\hbar} \int_0^R d\rho \int_0^{2 \pi} d\theta \rho b_z(\rho, \theta) = \frac{e r a B}{2 \pi \hbar} \int_0^\infty dk \int_0^{2 \pi} d\xi \int_0^R d\rho e^{ik \rho \cos(\theta - \xi)} J_1(kr)(i \cos \xi)^{1-\alpha}(i \sin \xi)^{1-\alpha}. \]  

(E6)

The \( \theta \) integration yields

\[ \int_0^{2 \pi} d\theta e^{ik \rho \cos(\theta - \xi)} = 2 \pi J_0(k \rho). \]  

(E7)

Consequently, we reduce the phase difference to

\[ \Delta \phi = \frac{e r a B}{\hbar} \int_0^\infty dk \int_0^{2 \pi} d\xi \int_0^R d\rho \rho k^{2-2\alpha} e^{ik \rho \cos(\theta - \xi)} J_1(kr)(i \cos \xi)^{1-\alpha}(i \sin \xi)^{1-\alpha}. \]  

(E8)

The integral over \( \rho \) can be done analytically,

\[ \int_0^R d\rho J_0(k \rho) = \frac{R}{k} J_1(kR). \]  

(E9)
Hence, the phase difference becomes

\[ \Delta \phi = \frac{e_r \alpha B R}{h} \int_0^\infty d\kappa k^{1-2\alpha} J_1(kr) J_1(kR) \int_0^{2\pi} d\xi (i \cos \xi)^{1-\alpha} (i \sin \xi)^{1-\alpha}. \]  

(E10)

The two integrals can be evaluated as

\[ \int_0^\infty d\kappa k^{1-2\alpha} J_1(kr) J_1(kR) = \frac{2^{1-2\alpha} r R^{2\alpha-3} \Gamma(2-\alpha)}{\Gamma(\alpha)} \binom{1}{2-\alpha} \sqrt{\pi} \Gamma(1 - \frac{\alpha}{2}) \]

and

\[ \int_0^{2\pi} d\xi (i \cos \xi)^{1-\alpha} (i \sin \xi)^{1-\alpha} = \frac{2^\alpha \sin^2 \frac{\pi \alpha}{2} \sqrt{\pi} \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2})}. \]  

(E11)

Here \( \binom{a}{b} \) is a hypergeometric function. Finally, the phase difference is

\[ \Delta \phi = \frac{e}{h} \pi r^2 \alpha B R^{2\alpha-2} \left( \frac{2^{1-\alpha} \Gamma(2-\alpha) \Gamma(1 - \frac{\alpha}{2})}{\sqrt{\pi} \Gamma(\alpha) \Gamma(\frac{3}{2} - \frac{\alpha}{2})} \sin^2 \frac{\pi \alpha}{2} \right)^2 \int_0^{2\pi} d\xi (i \cos \xi)^{1-\alpha} (i \sin \xi)^{1-\alpha}. \]  

(E12)

The terms in the parenthesis reduce to 1 in the limit \( \alpha \to 1 \).

2. Right Liouville Fractional Calculus

The phase difference from this fractional calculus is the same as the phase in Eq. (E13) because one can show that

\[ b_z(k) = 2\pi r^2 \alpha B k^{1-2\alpha} J_1(kr) (-i \cos \xi)^{1-\alpha} (-i \sin \xi)^{1-\alpha} \]  

(E14)

and the integral

\[ \int_0^{2\pi} d\xi (-i \cos \xi)^{1-\alpha} (-i \sin \xi)^{1-\alpha} = \frac{2^\alpha \sin^2 \frac{\pi \alpha}{2} \sqrt{\pi} \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2})}. \]  

(E15)

3. Feller Fractional Calculus

For this definition, one can show that

\[ b_z(k) = 2\pi r^2 \alpha B k^{1-2\alpha} J_1(kr) |\cos \xi|^{1-\alpha} |\sin \xi|^{1-\alpha}. \]  

(E16)

The only difference from the right Liouville calculus is the integration over \( \xi \). One finds

\[ \int_0^{2\pi} d\xi |\cos \xi|^{1-\alpha} |\sin \xi|^{1-\alpha} = \frac{2^\alpha \sqrt{\pi} \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2})}. \]  

(E17)

And hence the phase difference is

\[ \Delta \phi = \frac{e}{h} \pi r^2 \alpha B R^{2\alpha-2} \left( \frac{2^{1-\alpha} \Gamma(2-\alpha) \Gamma(1 - \frac{\alpha}{2})}{\sqrt{\pi} \Gamma(\alpha) \Gamma(\frac{3}{2} - \frac{\alpha}{2})} \right)^2 \int_0^{2\pi} d\xi (i \cos \xi)^{1-\alpha} (i \sin \xi)^{1-\alpha} - (E18) \]
4. Riesz Fractional Calculus

For this definition, one can show that
\[ b_\alpha(k) = -2\pi r_\alpha B k^{1-2\alpha} J_1(kr) \cos \xi \cos |\xi|^{\alpha} \sin \xi \sin |\xi|^{-\alpha}. \]  
(E19)
The integral over \( \xi \) vanishes because \( \cos \xi \cos |\xi|^{\alpha} \sin \xi \sin |\xi|^{-\alpha} \) is an odd function. As a result
\[ \Delta \phi = 0. \]  
(E20)
This result is not surprising, because from Eq. (D20), the AB phase from the Riesz calculus when \( a = b \) and \( c = d \) is zero.

5. Rotationally Invariance Definition

The fractional Laplacian in the definition, \( \partial_\alpha = (-\nabla^2)^{\alpha/2} \partial_\xi \), is to be interpreted as a two-dimensional operator. Hence, in the kernel \( F_\epsilon(\alpha, k) = |k|^{\alpha-1} k_\alpha \), one has \( k^2 = k^2_x + k^2_y \). The calculation is proceeded in the same manner as what we have done for other definitions. One can show that
\[ b_\alpha(k) = k^{1-2\alpha} 2\pi r_\alpha B J_1(kr). \]  
(E21)
Unlike other definitions, there is no dependence on \( \xi \) because this definition is rotationally invariance. The phase shift is
\[ \Delta \phi = \frac{e^{2\pi r^2_\alpha B R^{2\alpha-2}}}{\Gamma(\alpha)} \left( \frac{2^{2-2\alpha} \Gamma(2-\alpha)}{\Gamma(\alpha)} \right) \frac{1}{2^{1-2\alpha}} \left( 1 - \frac{2}{B} \right). \]  
(E22)

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