S-Matrices of $\phi_{1,2}$ perturbed unitary minimal models: IRF-Formulation and Bootstrap-Program

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Abstract

We analyze the algebraic structure of $\phi_{1,2}$ perturbed minimal models relating them to graph-state models with an underlying Birman-Wenzl-Murakami algebra. Using this approach one can clarify some physical properties and reformulate the bootstrap equations. These are used to calculate the $S$-matrix elements of higher kinks, and to determine the breather spectrum of the $\phi_{1,2}$ perturbations of the unitary minimal models $\mathcal{M}_{r,r+1}$. 
1 Introduction

It has been proven by A. Zamolodchikov that certain deformations of minimal models of conformal field theory (CFT) are described by integrable massive field theories \[1\]. The corresponding Hamiltonian can be written as

\[ H_p = H_{CFT} + \lambda \int \phi(x) \, dx \quad . \tag{1.1} \]

Integrability of the perturbed theory is achieved only for few specific operators \( \phi \) of the space of states of the CFT, in general only for the primary fields \( \Phi_{1,2} \), \( \Phi_{2,1} \) and \( \Phi_{1,3} \) of the Kac-table \([1]\).

The on-shell behaviour of massive quantum field theories is described by the \( S \)-matrix. For integrable massive quantum field theories the \( S \)-matrix is factorized, \textit{i.e.} \( n \)-particle scattering amplitudes can be decomposed into 2-particle ones. There is a large variety of methods in order to compute the \( S \) matrix (see \textit{e.g.} [2]), but in many cases the latter is just conjectured on the basis of physical features and symmetries of the model under consideration.

We will discuss mainly \( \phi_{1,2} \)-perturbed minimal models. The scattering theories corresponding to the hamiltonian

\[ H = H_{CFT} + \lambda \int \Phi_{1,2}(x) \, dx \quad , \tag{1.2} \]

have first been discussed by Smirnov [3]. He wrote down the \( S \)-matrix of the fundamental particle, which in the IRF (Interaction around Face) representation takes the form

\[ S(\beta) = \left( \sinh \frac{\pi}{\xi} (\beta - i\pi) \sinh \frac{\pi}{\xi} \left( \beta - \frac{2\pi i}{3} \right) \right)^{-1} \]

\[ \times \exp \left( -2i \int_{0}^{\infty} \frac{dx}{x} \sin \beta x \sinh \frac{\pi x}{2} \cosh \left( \frac{\pi}{6} - \frac{\xi}{2} \right) x \right) \times R_{bd}^{ac} \quad . \tag{1.3} \]

For our purpose we write the \( R \)-matrix as

\[ R_{bd}^{ac} = (-1) \left[ \begin{array}{ccc} 1 & b & a \\ d & c \end{array} \right] \sin \frac{i\pi \beta}{\xi} \cos \left( \frac{i\pi \beta}{\xi} + (\alpha + 3)\gamma \right) + \delta_{ac} \cos 3\gamma \sin 2\gamma \] \quad , \tag{1.4}
which is equivalent to the expression originally given by Smirnov. The parameter $\gamma = \frac{\xi}{\pi}$ corresponds to the model $M_{r,s}$ and determines the reduction of the quantum group, that is neighboring states $a_k, a_{k+1}$ are restricted by the IRF rules

$$|a_k - 1| \leq a_{k+1} \leq \min(a_k + 1, r - 3 - a_k).$$

The parameter $\xi = \frac{2}{3} \frac{\pi \gamma}{2\pi - \gamma}$ relates the rapidity to the spectral parameter of the quantum group [3].

The spectrum of these theories is rather complicated. This, because the $S$-matrix also contains poles which generate kinks of higher mass. These in turn can form further scalar bound-states or even higher kinks. This mechanism explains the difficulty of finding the spectrum for arbitrary coupling constant $\gamma$.

In section 2 we analyze the algebraic structure of $S$-matrices for $\phi_{1,2}$-perturbed minimal models, cast them into a graph-state formulation. In section 3 we confront this construction with the $\phi_{1,2}$ scattering theories and draw some physical consequences for the ultraviolet limit and the bootstrap equations. In section 4 we apply the bootstrap, find the $S$-matrices for the higher kink, analyze the pole-structure and conjecture the full $S$-matrix of all unitary minimal models $M_n$ for $n > 4$ perturbed by $\phi_{1,2}$. Our conclusions are presented in section 5.

2 Algebraic structure of $\phi_{12}$-perturbed $S$-matrices

The approach of Smirnov was to use the vector-representation of the $R$-matrix of $A_2^{(2)}$ and then to perform a change to the IRF-representation. This approach has the disadvantage that it passes through the vector-representation of $A_2^{(2)}$, which gives an inconsistent field theoretic model, since the corresponding hamiltonian is not hermitian. In order to construct a consistent field theory, one needs to restrict the Hilbert space, i.e. one has to go into the IRF-representation.

Our goal is to construct the IRF-amplitudes directly. We want to emphasize, that we will not derive new models with respect to those of Smirnov, but our construction will allow us to understand better the algebraic structure of the scattering amplitudes. We will need this formulation to find new physical results in section 3.
2.1 Temperly Lieb Algebra

The construction is based on models which intrinsically present the restriction of the Hilbert space: the so-called graph-state models [4, 5, 6]. These graphs are usually picturized fusion algebras of some Wess-Zumino-Witten (WZW) model. For our purpose, describing perturbations of conformal field theory, we use graphs based on the fusion-rules of $SU(2)$ WZW-models, which read as

$$\phi_{j_1} \times \phi_{j_2} = \sum_{j_3 = |j_1 - j_2|}^{\min(j_1 + j_2, k - j_1 - j_2)} \phi_{j_3}. \quad (2.6)$$

For the fundamental representation (spin $j = \frac{1}{2}$) they coincide with the graph of the $A_{k+1}$ Dynkin-Diagrams, where $k$ denotes the level of the underlying Kac-Moody algebra,

$$0 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2 \quad \frac{5}{2} \quad \ldots \quad \frac{k}{2} \quad \ldots \quad (2.7)$$

In order to construct $\phi_{1,2}$-perturbed models, we are interested in the spin $j = 1$ representation. The corresponding graphs are

$$k = 3 \quad \quad k = 4 \quad \quad k = 5 \quad \ldots \quad (2.8)$$

where we indicated also the corresponding level of the Kac-Moody algebra. Given a graph, one can find a representation of the Temperly-Lieb algebra (TLA),

$$E_i E_j = E_j E_i \quad \text{for} \quad |i - j| \geq 2, \quad \quad (2.9)$$

$$E_i E_{i \pm 1} E_i = E_i, \quad E_i^2 = \sigma(j)^{\frac{1}{2}} E_i, \quad \quad (2.10)$$

by diagonalizing the incidence matrix of the diagram. The generators $E_i$ operate in an $n$-particle space. For graph-state models this action can be visualized as

$$\ldots \quad l - 2 \quad l - 1 \quad \begin{array}{c} \nu \end{array} \quad l + 1 \quad l + 2 \quad \ldots \sim E_{l-1,l+1}^{l,l'}, \quad (2.11)$$
and the generators are matrices in the indices $l$ and $l'$. It is amazing how similar the results are for the two different families of graphs (2.7) and (2.8). Let us define the parameter $\lambda \equiv \frac{\pi}{k+2}$. Then the largest Eigenvalue, corresponding to the Perron-Frobenius Eigenvector\(^\dagger\) for the case (2.7) is $\sigma(\frac{1}{2}) = 2\cos \lambda$ whereas for the case (2.8) it is $\sigma(1) = 1 + 2\cos 2\lambda$. This becomes more similar introducing a notion borrowed from the quantum-group language. Let us define $q = e^{i\lambda}$ and the quantum-symbol $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. Then we find that $\sigma(\frac{1}{2}) = [\frac{1}{2}]$ and $\sigma(1) = [1]$.

Also the eigenvectors have the same structure. They are $\psi(a) = [2a + 1]$, where the numbers $a$ take the values of the labels on the nodes of the corresponding diagram, that is half-integers for (2.7) and integers for (2.8) respectively. These indices $a$ are restricted in both cases by the bound $a \leq \frac{k}{2}$. Finally, the generators (2.11) are constructed out of the eigenvectors $[4]$ as

$$E_{bd}^{ac} = \frac{[2a + 1]^{\frac{1}{2}}[2c + 1]^{\frac{1}{2}}}{[2b + 1]^{\frac{1}{2}}[2d + 1]^{\frac{1}{2}}} \delta_{bd} .$$

### 2.2 Braid Group

In order to introduce the spectral parameter one needs to go to a braid-group representation, that is elements $b_i$ satisfying

$$b_ib_j = b_jb_i \text{, for } |i - j| \geq 2 ,$$

$$b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1} \text{ .}$$

The usual approach, which leads to $\phi_{1,3}$ perturbed models, defines the braid-group generators by the linear transformation

$$b_i = 1 - e^{i\lambda} E_i \text{ .}$$

In that way one obtains a Hecke algebra, since the linear transformation supplies a quadratic relation for the Braid group generators,

$$(b_i - 1)(b_i + e^{2i\lambda}) = 0 \text{ .}$$

\(^\dagger\) The other eigenvalues lead in general to imaginary Boltzmann weights, for an example see \[7\]
For the spin-1 algebra the natural choice \[4\] is the Birman-Wenzl-Murakami (BWM) algebra \[8\], which is given by the relations

\[ g_i g_j = g_j g_i , \text{ for } |i - j| \geq 2 , \]
\[ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} ; \]
\[ e_i e_j = e_j e_i , \text{ for } |i - j| \geq 2 , \]
\[ e_i e_{i\pm 1} e_i = e_i , \quad e_i^2 = (m^{-1}(l + l^{-1}) - 1)e_i ; \quad (2.16) \]
\[ g_i + g_i^{-1} = m(1 + e_i) , \quad g_i^2 = m(g_i + l^{-1}e_i) - 1 , \]
\[ g_{i+1} g_i e_{i\pm 1} = e_i g_i g_{i+1} = e_i e_{i\pm 1} , \]
\[ g_{i\pm 1} e_i g_{i\pm 1} = g_i^{-1} e_i g_i^{-1} , \quad e_{i\pm 1} e_i e_{i\pm 1} = g_i^{-1} e_{i\pm 1} , \]
\[ e_{i\pm 1} e_i g_{i\pm 1} = e_{i\pm 1} g_{i\pm 1} , \quad g_{i\pm 1} e_i = e_{i\pm 1} g_{i\pm 1} = l^{-1} e_i , \quad e_i g_{i\pm 1} e_i = l e_i \]

with \( e_i = -E_i \) and \( g_i = -ib_i \). The parameters appearing in the algebra are \( m = -i(q^2 - q^{-2}) \) and \( l = iq^4 \). This algebra implies a third order relation for the braid group generators \[4\], which in our notation reads as

\[ (b_i - q^{-2})(b_i + q^2)(b_i + q^{-4}) = 0 \quad . \quad (2.17) \]

Not all of the relations in \((2.16)\) are independent \[9\], but in order to clearly see the relation of braid group and Temperley-Lieb algebra we listed them anyway.

One can again find generators \( b_i \) satisfying \((2.16)\) which are of the same form as those satisfying \((2.15)\) for the corresponding spin, if we write them in quantum group language.

For that it is necessary to introduce the 6\( j \)-symbols \[10\],

\[
\left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\}_q = \sqrt{[2e + 1][2f + 1]} \quad (a \neq c) \quad \Delta(abc) \Delta(acf) \Delta(acd) \Delta(dbc) \sum_z (-1)^z [z + 1]! \times (2.18)
\]

\[
([z + a - b - e][z - a - c - f][z - b - d - f][z - d - c - e]^{-1} \times [a + b + c + d - z][a + d + e + f - z][b + c + e + f - z]^{-1},
\]

wherein we use the conventions that \([0]! = 1\) and the sum runs only over \( z \) such that no factor \([x]\) is less than zero. Further,

\[
\Delta(abc) = \left( \frac{[-a + b + c]![a - b + c]![a + b - c]!}{[a + b + c + 1]!} \right)^{\frac{1}{2}} .
\]
With this definition the braid group generators can be written as

$$
b_{bd}^{ac} = q^{(c_d-c_a-c_a+c_b)}(-1)^{(b + d - a - c)} \times \left\{ \begin{array}{c} j \\ b \\ a \\ j \\ d \\ c \end{array} \right\}_q \times (-1)^{(a-c)\frac{1}{2}} . \tag{2.19}\$$

These generators satisfy a further property: the crossing symmetry,

$$
b_{bd}^{ac} = \left( b_{ac}^{bd} \right)^{-1} \left( \frac{[2a + 1][2c + 1]}{[2b + 1][2d + 1]} \right)^{\frac{1}{2}} . \tag{2.20}\$$

### 2.3 Introducing the Spectral Parameter

In [11] it is shown that given a representation of the braid group which factors either through the Hecke algebra or through the BWM algebra, one can introduce a spectral parameter with a mechanism called *universal baxterization*. In the Hecke algebra case one finds

$$
R_i(x) = q^{-1} e^x b_i - q e^{-x} b_i^{-1} , \tag{2.21}\$$

whereas for the BWM case the spectral parameter depending solution is written as

$$
R_i(x) = (x^{-1} - 1) k g_i + m(k + k^{-1}) + (x - 1) k^{-1} g_i^{-1} , \tag{2.22}\$$

with $k = q^3$. Using the BWM-algebra (2.16) and the relation (2.20) one can show, that the $R$-matrix (2.22) satisfies crossing

$$
R_{bd}^{ac}(x) = \left( \frac{[2a + 1][2c + 1]}{[2b + 1][2d + 1]} \right)^{\frac{1}{2}} R_{ac}^{bd}(-x^{-1}q^6) , \tag{2.23}\$$

and the completeness relation

$$
\sum_e R_{bd}^{ac}(x) R_{bd}^{ce}(-x) = \delta_{ac} \times (x^{-1}q^3 + xq^{-3})(x^{-1}q^2 - xq^{-2})(xq^3 + x^{-1}q^{-3})(xq^{-2} - x^{-1}q^2) . \tag{2.24}\$$

Similar results are well known for the $R$-matrix based on the Hecke algebra (2.21) [4].

As a last note we mention the so-called *symmetry-breaking transformations* [4], which leave untouched the Yang-Baxter equation and the completeness-relation, but can change the parameters appearing in the crossing-relation. The transformations we will need are

$$
R_{bd}^{ac}(e^u) \rightarrow \tilde{R}_{bd}^{ac}(e^u) = \alpha_{abcd}(u) \beta_{abcd} \times R_{bd}^{ac}(e^u) , \tag{2.25}\$$
\[ \alpha_{abcd}(u) = e^{[−p(a)+p(b)+p(c)+p(d)]u}, \]  
\[ \beta_{abcd} = \frac{p'(a)}{p'(c)}. \]  

Herein \( p(.) \) and \( p'(.) \) are arbitrary functions. The transformation (2.26) can be used to eliminate the parameters appearing in the crossing relation \([12, 16]\). The second transformation is of particular importance for relating the above \( R \)-matrix to the Izergin-Korepin \( R \)-matrix used by Smirnov. In order to transform (2.22) to become equal to (1.3), we need to perform a gauge-transformation of the form \( R_{acbd}(x) \rightarrow (−1)^{a−c}R_{acbd}(x) \). This is of physical significance because it changes the signs of some amplitudes. Since the signs of the residues in a unitary theory are fixed, this simple gauge-transformation cannot be obscured. Anyhow, note that with this gauge transformation also the underlying braid-group and TLA undergo the same transformation. To simplify the discussion, we consider from now on the \( R \)-matrix (2.22), inserting the factor \((−1)^{a−c}\), leaving though the form with non-trivial crossing factors.

### 3 Application to Scattering Theories

Now we want to apply this mathematical construction to the problem of scattering theories describing deformations of conformal field theories. From now on, we will concentrate our discussion mainly onto the \( R \)-matrix built on the BWM-algebra, since this is the one describing \( φ_{1,2} \) perturbed models, which we are mainly interested in. Analogous results hold for the \( R \)-matrix constructed from the Hecke algebra describing \( φ_{1,3} \) perturbed models. These theories have been analyzed in \([12, 13]\).

#### 3.1 Construction of the Scattering Amplitudes

In order to identify the corresponding scattering theory one needs to relate the spectral parameter \( x \) to the rapidity variable. This causes that a whole series of scattering theories get related to the same \( R \)-matrix. Let us explain this mechanism for the \( R \)-matrix (2.22).
Since we have in mind the scattering theories proposed by Smirnov, we make an ansatz, \( x = e^{\frac{2\pi\beta}{\xi}} \). Crossing symmetry in scattering theories means \( S(\beta)_{bd}^{ac} = S(i\pi - \beta)_{ac}^{bd} \). For the \( R \)-matrix we have the relation (2.23), which includes also the factors arising from a symmetry-breaking transformation. In order to achieve crossing symmetry the transformation \( \beta \to i\pi - \beta \) must correspond to \( x \to -x^{-1}q^6 \). This implies the constraint

\[
\frac{2\pi^2i}{\xi} = i\pi + \frac{6i\pi}{r} + 2n\pi i ,
\]

from which we find the relation for the parameter \( \xi = \frac{2\pi r}{\pm 6 + 2nr - 3r} \), with \( n \in \mathbb{Z} \) and with \( r \equiv k + 2 \). But in order to implement the symmetry of the diagram (2.8) dynamically we need a bound state at the pole \( \beta = \frac{2\pi}{3} \). But this requires that at this point the \( R \)-matrix has to degenerate into a 3-dimensional projector. In the appendix we have collected some information on the projectors and the necessary 6\( j \)-symbols. Therefore we need that

\[
e^{\frac{2\pi i}{\xi}} \bigg|_{\beta = \frac{2\pi}{3}} = q^4.
\]

This condition eliminates part of the possible values for the parameter \( \xi \), leaving \( \xi = \frac{2\pi r}{{\pm 6 + 6nr - 3r}} \).

This condition is equivalent to an idea raised by Zamolodchikov [14], constructing a factorizable scattering theory for the tricritical Ising model perturbed by the subleading magnetization (\( M_{4,5} + \phi_{2,1} \)). He required that one of the amplitudes needs to become zero at the pole at \( \frac{2\pi}{3} \). The amplitude corresponds to \( R_{10}^{11} \), since a kink interpolating the vacuum 0 to the vacuum 0 does not exist (see the graph (2.8), which has no tadpole at the node 0). This condition is automatically fulfilled if the amplitudes degenerates into a three-dimensional projector at this point.

To show this we need the formulation of the previous paragraph of the \( R \)-matrix in terms of 6\( j \)-symbols. Since this \( R \)-matrix is an affinization of a quantum group in the shadow-world representation [10], we can also express the projectors as 6\( j \)-symbols, that is

\[
P_{ac}^{bd} = \left\{ \begin{array}{ccc}
1 & 1 & j \\
b & d & a
\end{array} \right\} \left\{ \begin{array}{ccc}
1 & 1 & j \\
b & d & c
\end{array} \right\} .
\]

The exact relation for the 3d-projector is

\[
R_{bd}(x = q^4, q) = [2][4]P_{bd}^{ac;1} .
\]
From the expressions given in the appendix, we easily can calculate the residues at the pole. The general amplitude needed in order to verify the Zamolodchikov condition is

\[ R_{ll}^{l+1,l+1}(q^4,q) = (q^2 - q^{-2})[2][2l][2l + 3][2l + 2][2l + 1]. \]

This becomes zero for \( l = 0. \)

As a last ingredient for a physical scattering theory one needs unitarity, that is \( S(\beta)S^*(\beta) = 1. \) Since the elements of the \( R \)-matrix are real, \( R \) satisfies also real analyticity, i.e. \( S^*(\beta) = S(-\beta). \) Additionally we have the completeness property \( (2.24), \) and therefore the \( R \)-matrix multiplied by a scalar factor \( S_0, \) which eliminates the terms on the right hand side of \( (2.24) \) is unitary. But this factor has been determined by Smirnov, and is the prefactor in \( (1.3) \) with the corresponding parameter \( \xi. \)

Confronting the resulting theories with those of Smirnov \( (1.3), \) we find that all of the perturbed conformal scattering theories \( M_{r,mr\pm 1}\phi_{1,2} \) correspond to the \( R \)-matrix \( (2.22). \) Here \( m \) takes the values \( m = 1, 2, \ldots. \) Of course the ‘formal’ theory \( M_{r,r-1}\phi_{1,2} \) is the physical one \( M_{r-1,r}\phi_{2,1}. \) For all of these theories the scattering matrix of the fundamental particle is unitary, that is \( SS^* = 1. \) Since through the bootstrap this property is preserved also for other particles, all of these models are supposed to be consistent scattering theories.

We want to insert a comment here. We found that one \( R \)-matrix corresponds to many different scattering theories, according to how one relates the rapidity variable to the spectral parameter. Up to now, there was the believe that there is a unique way to find a physical scattering theory given an \( R \)-matrix, using the principle of “minimality”. This principle was commonly used in order to eliminate ambiguities deriving from the fact that the factor \( S_0 \) can not be derived uniquely, but has always an ambiguity of so-called CDD-factors. Minimality said, that the physical scattering theory corresponding to a given \( R \)-matrix is that one, which introduces the minimal number of poles and zeros in the physical strip. We see now, that this is no fundamental principle. We find, that the theories belonging to one \( R \)-matrix depend on how the spectral parameter is related to the rapidity variable, and the \( S \)-matrix of the fundamental particle differ from each other.

\[ ^2 \text{This is related to the fact, that we used the highest eigenvalue in diagonalizing the incidence matrix of the diagrams \((2.8), \) whose corresponding eigenvector is the Frobenius-Perron eigenvector.} \]
by CDD-factors. These factors of course usually introduce further poles in the physical strip, and therefore generate a completely different physical scattering theory. This fact was explicitly discussed for scattering theories of perturbed minimal models $\mathcal{M}_{5,n}$ in \[7\].

Analyzing the allowed parameters $\xi$ we find that the theory with the minimal number of poles and zero’s is that one, corresponding to a deformed unitary conformal theory. But note that also the $S$-matrices of the fundamental particles of $\mathcal{M}_{r-1,r} + \phi_{2,1}$ and the theory $\mathcal{M}_{r,r+1}$ correspond to the same $R$-matrix $R(x)$.

### 3.2 Ultraviolet Limit

As a next point, let us discuss the ultraviolet limit. For $\beta \to \infty$ the $S$-matrix becomes again proportional to the braid-group generators (2.19), but with the gauge-transformation, that is

$$b_{bc}^{ad} = S_0(\infty) q^{(c_d-c_a-c_b)}(-1)^{(b+d-a-c)} \times \left\{ \begin{array}{ccc} j & b & a \\ d & c \\ j & c & a \end{array} \right\}_q .$$

This expression is valid also for $\phi_{1,3}$ perturbations, which correspond to the spin $j = \frac{1}{2}$.

One notices that these expressions are proportional to the braiding matrices of conformal blocks of the WZW-models [15], as one expects.

Now we use the algebraic structure. Let us view these braid-group generators as matrices in the indices $a$ and $c$. Now since they satisfy (for spin $j = 1$) a third order relation (2.17), there can only be 3 independent eigenvalues. The same fact holds also for the corresponding $R$-matrices, whose non-diagonal components are given by the braid group generators. Diagonalizing those one finds that the eigenvalues correspond to the amplitudes $S_{00}^{11}(\beta)$, $S_{01}^{11}(\beta)$ and $S_{02}^{11}(\beta)$ which define three independent phase-shifts. We study now their asymptotic behaviour

$$\lim_{\beta \to \infty} S_{00}^{11}(\beta) = e^{2i\pi \Delta_{3,1}} ,$$
$$\lim_{\beta \to \infty} S_{01}^{11}(\beta) = e^{i\pi \Delta_{3,1}} ,$$
$$\lim_{\beta \to \infty} S_{02}^{11}(\beta) = e^{i\pi (2\Delta_{3,1} - \Delta_{5,1})} ,$$

where $\Delta_{3,1}$ and $\Delta_{5,1}$ are the anomalous dimensions of the corresponding fields of the original conformal field theory. These are exactly the dimensions appearing in the operator
product expansion (OPE) of $\Psi \equiv \Phi_{3,1}$ of the original minimal model $M_{r,mr \pm 1}$:

$$
\Psi(z)\Psi(0) = \frac{1}{z^{2\Delta_{3,1}}} 1 + \frac{C_{\Psi,\Psi,\Psi}}{z^{\Delta_{3,1}}} \Psi(0) + \frac{C_{\Psi,\Psi,\Phi_{5,1}}}{z^{2\Delta_{3,1} - \Delta_{5,1}}} \Phi_{5,1}(0) + \ldots \quad (3.33)
$$

Of course if one considers the series of theories $M_{r-1,r} + \phi_{2,1}$ one finds that the corresponding field is $\phi_{1,3}$ instead of $\phi_{3,1}$. This correspondence for $\phi_{2,1}$ perturbed unitary theories has been found in [16].

Similar one can analyze the asymptotic phase-shifts of the $\phi_{1,3}$-perturbed models. They satisfy a second order relation (2.15) and therefore the braid group generators as well as the $R$-matrix (2.21) have only two eigenvalues. They correspond to the amplitudes $S_{00}(\beta)$ and $S_{01}(\beta)$. It is not surprising that their asymptotic phase-shifts determine the dimensions of the OPE of the field $\phi_{2,1}$,

$$
S_{00}(\infty) = e^{2i\pi\Delta_{2,1}}, \quad S_{01}(\beta) = e^{i\pi(2\Delta_{2,1} - \Delta_{3,1})} \quad . \quad (3.34)
$$

### 3.3 Bootstrap Equations

The last formal application involves the bootstrap-equations. For degenerate particles in the IRF description they were developed in [17]. The equations are

$$
S_{ab}^{bd}f_{abc} = \sum_g f_{egd}S_{ag}^{eb}(\theta + i\bar{u}) S_{db}^{cg}(\theta - i\bar{u}) \quad . \quad (3.35)
$$

The relation of the constants $f$ with the scattering matrix [18] is

$$
\text{Res}_{\theta = iu} S_{bd}^{ac} = i f_{bad} f_{bcd} \quad , \quad (3.36)
$$

where $u$ is the corresponding $S$-matrix pole. It is useful to exploit the quantum-group symmetry in order to reformulate the above equations, since the above definition of the constants $f$ leads to a system of quadratic equations to solve and therefore leaves an ambiguity of a sign.

Since the $S$-matrix for $\phi_{12}$ perturbed minimal models is proportional to the $A_2^{(2)}$ quantum group $R$-matrix, one can also rewrite the bootstrap-equations in terms of the pentagon-identity. This determines the constants $f$ as $6j$-symbols. Or more explicitly,

$$
f_{abc} = \left\{ \begin{array}{ccc} 1 & 1 & j \\ a & c & b \end{array} \right\} \quad , \quad (3.37)
$$
where the spin $j$ corresponds to the projector, into which the $S$-matrix degenerates at the pole. This correspondence can also be seen from the form of the projectors (3.29).

4 Bootstrap for the unitary series

The unitary minimal series perturbed by the operator $\phi_{12}$ was analyzed by Smirnov[3]. He established the spectrum of all theories except $M_{5,6}$ and wrote down the $S$ matrix of the fundamental kink as well as that one of the fundamental breather. We apply now the bootstrap-equations in the IRF formulation in order to write down the complete $S$-matrix of kinks and their bound-states, the breathers. As a byproduct we also find that the theory $M_{5,6}$ has two kinks and four breathers.

The calculation of the $S$-matrices is tedious, but straightforward. For that we give only the results. Let us use the abbreviations

$$ (x)^\pm = \frac{\Gamma(\frac{2k\pi}{\xi} + x + \frac{i\beta}{\xi})}{\Gamma(\frac{2k\pi}{\xi} + x - \frac{i\beta}{\xi})}, \quad (4.38) $$

and

$$ \langle x \rangle = \frac{\tanh(\frac{\pi}{2} + i\pi x)\tanh(\frac{\pi}{2} - i\pi x)}{\tanh(\frac{\pi}{2} + i\pi x)}. \quad (4.39) $$

Then the $S$-matrices of the kinks are

$$ S_{K_1,K_1}(\beta) = \left(\sinh\frac{\pi}{\xi}(\beta - i\pi)\sinh\frac{\pi}{\xi}(\beta - \frac{2\pi i}{3})\right)^{-1} \times $$

$$ \prod_{k=0}^{\infty} \left(\frac{\pi}{\xi}\right)^{-2}\pi^{1}(\frac{1}{2} + \frac{\pi}{\xi})^{\frac{4\pi}{3\xi}}(1 + \frac{2\pi}{3\xi})^{-1}(1 + \frac{5\pi}{3\xi}) \times R^{ac}_{bd} \quad (4.40) $$

$$ S_{K_1,K_2}(\beta) = \left(\cosh\frac{\pi}{\xi}(\beta - i\pi)\cosh\frac{\pi}{\xi}(\beta - \frac{2\pi i}{3})\right)^{-1} \times $$

$$ \prod_{k=0}^{\infty} \left(\frac{1}{2} + \frac{\pi}{\xi}\right)^{-\frac{5\pi}{3\xi}}(\frac{1}{2} + \frac{2\pi}{3\xi})^{\frac{4\pi}{3\xi}}(1 + \frac{4\pi}{3\xi})^{\frac{2\pi}{3\xi}}(1 + \frac{2\pi}{3\xi})^{-1}(1 + \frac{5\pi}{3\xi}) \times $$

$$ \tilde{R}^{ac}_{bd} \times \langle \frac{2i\pi}{3} - \frac{\xi}{2} \rangle \langle \frac{2\pi}{3} + \frac{\xi}{2} \rangle \quad (4.41) $$

where $\tilde{R}$ is the $R$-matrix with a spectral parameter shifted by a phase-factor of $\frac{\pi}{2}$. Finally,

$$ S_{K_2,K_2}(\beta) = S_{K_1,K_1}(\beta)\langle \frac{2i\pi}{3} - \xi \rangle \langle \frac{2\pi}{3} \rangle^{2}\langle \xi \rangle \quad (4.42) $$
The analytic structure is exhibited in figure 1 and 2. In both cases we showed only the direct channel poles, the crossed ones being in a one to one correspondence. The double poles in the kink-kink $S$-matrices can all be explained in terms of elementary scattering processes [19]. They are exhibited in figure 3.

The $S$-matrix elements involving the breathers are the following:

\[
S_{K_1,B_1}(\beta) = \langle \frac{\pi}{2} - \frac{\xi}{2} \rangle_{K_1} \langle -\frac{\xi}{2} \rangle_{K_2}
\]

\[
S_{K_1,B_2}(\beta) = \langle \frac{2\pi}{3} \rangle^{2} \langle \frac{2\pi}{3} - \xi \rangle_{K_2} \langle \xi \rangle
\]

\[
S_{K_2,B_1}(\beta) = \langle \frac{\pi}{2} \rangle \langle \frac{\pi}{6} \rangle_{K_1} \langle \frac{\pi}{6} + \xi \rangle \langle -\frac{\pi}{6} + \xi \rangle
\]

\[
S_{K_2,B_2}(\beta) = \langle \frac{\pi}{3} + \frac{\xi}{2} \rangle^{3} \langle \pi - \frac{\xi}{2} \rangle \langle \frac{\pi}{3} - \frac{\xi}{2} \rangle_{K_1} \langle \pi - \frac{3\xi}{2} \rangle \langle -\frac{\pi}{3} + \frac{3\xi}{2} \rangle
\]

\[
S_{B_1,B_1}(\beta) = \langle \xi \rangle \langle \frac{2\pi}{3} \rangle_{B_1} \langle -\frac{\pi}{3} + \xi \rangle_{B_2}
\]

\[
S_{B_1,B_2}(\beta) = \langle \frac{\pi}{2} - \frac{\xi}{2} \rangle^{2} \langle -\frac{\pi}{6} + \frac{3\xi}{2} \rangle \langle -\frac{\pi}{2} + \frac{3\xi}{2} \rangle \langle \frac{\pi}{2} + \frac{\xi}{2} \rangle \langle -\pi + \frac{3\xi}{2} \rangle_{B_1}
\]

\[
S_{B_2,B_2}(\beta) = \langle \frac{2\pi}{3} \rangle^{3} \langle \xi \rangle^{3} \langle -\frac{\pi}{3} + \xi \rangle^{2} \langle \frac{\pi}{3} + \xi \rangle \langle -\frac{\pi}{3} + 2\xi \rangle \langle -\frac{2\pi}{3} + 2\xi \rangle
\]

The lower indices indicate the bound state corresponding to that pole.

The model $M_{5,6}$ exhibits two more breathers, even though no more kinks are generated. This can be seen from fig 1 and 2. Since $\xi \leq \frac{\pi}{2}$ new breathers are created. But since $\xi \geq \frac{\pi}{3}$ no new kink poles come into the physical strip. The third breather with the mass

\[
M_3 = 4m \sin \frac{5}{21} \pi \sin \frac{3}{7} \pi
\]

is a bound state of $K_1$ and $K_2$ at the pole $u_{K_1,K_2} = \frac{2}{7} \pi$. The heaviest breather is a bound state of $K_2$ and $K_2$ at rapidity $u_{K_2,K_2} = \frac{1}{21} \pi$ and has mass

\[
M_4 = 4m \cos \frac{2\pi}{21} \cos \frac{\pi}{42}
\]

In [20] the truncation method was performed for this model, and the all but the heaviest particle were found. This is not surprising since for heavy particles the finiteness of the basis in the truncated Hilbert space causes rather big systematical errors. The remaining
breather part of the $S$-matrix of this model is

$$S_{13} = \langle \frac{31}{14} \rangle^2 \langle \frac{3}{4} \rangle^4 \langle \frac{13}{17} \rangle^2 \langle \frac{11}{12} \rangle^2 \langle \frac{19}{12} \rangle^2 , \quad S_{14} = \langle \frac{9}{7} \rangle^3 \langle \frac{3}{4} \rangle^2 \langle \frac{10}{21} \rangle \langle \frac{4}{21} \rangle \langle \frac{5}{21} \rangle ,$$

$$S_{23} = \langle \frac{5}{7} \rangle^1 \langle \frac{5}{17} \rangle^2 \langle \frac{13}{32} \rangle^2 \langle \frac{1}{2} \rangle^2 \langle \frac{3}{11} \rangle \langle \frac{19}{32} \rangle , \quad S_{24} = \langle \frac{10}{21} \rangle^2 \langle \frac{2}{57} \rangle^2 \langle \frac{8}{21} \rangle \langle \frac{2}{57} \rangle \langle \frac{3}{7} \rangle \langle \frac{1}{7} \rangle \langle \frac{10}{21} \rangle ,$$

$$S_{33} = \langle \frac{2}{3} \rangle^3 \langle \frac{1}{2} \rangle^2 \langle \frac{10}{21} \rangle \langle \frac{2}{3} \rangle \langle \frac{8}{21} \rangle \langle \frac{4}{21} \rangle ,$$

$$S_{34} = \langle \frac{10}{14} \rangle^1 \langle \frac{17}{14} \rangle^4 \langle \frac{3}{14} \rangle^2 \langle \frac{10}{32} \rangle^3 \langle \frac{11}{32} \rangle^3 \langle \frac{5}{14} \rangle \langle \frac{5}{42} \rangle \langle \frac{5}{21} \rangle ,$$

$$S_{44} = \langle \frac{2}{3} \rangle^5 \langle \frac{2}{7} \rangle^2 \langle \frac{1}{7} \rangle^2 \langle \frac{10}{21} \rangle^5 \langle \frac{8}{21} \rangle^3 \langle \frac{4}{21} \rangle^3 \langle \frac{3}{7} \rangle \langle \frac{1}{21} \rangle \langle \frac{5}{21} \rangle .$$



Herein the indices of the $S$-matrix elements correspond to breathers. This is the complete breather-part of this $S$-matrix.

A final confirmation of these $S$-matrices is expected from the thermodynamic Bethe ansatz. It involves higher level Bethe ansatz techniques, and gets rather complicated since there the spectrum consists of *two* degenerate particles.

## 5 Conclusions

We have analyzed the IRF structure which lies under $\phi_{1,2}$-perturbed conformal field theories. They can be built as graph-state models, but not in the usual Hecke-algebra sense but on a BWM-algebra. The advantage of this construction is that it avoids using the vector-representation, which leads to non-unitary scattering matrices [3].

The disadvantage of this approach lies in the fact, that there is still one step which essentially requires guess-work. There is no well-defined mechanism in order to get the braid group generators fulfilling the BWM algebra given the TLA. This is of course, because the TLA-generators are quadratic functions of the braid group generators. Since they do not have inverses the resolution of this quadratic relation is highly non-trivial. If one can succeed in this point, and find the constraints on the TLA generators, such
that they give rise to a braid group satisfying the BWM-algebra, one will have a means to define general BWM-graph-state models. Work on this problem is in progress.

We have used this construction in order to compare the corresponding $R$-matrix with the scattering matrices described by Smirnov. We found that a whole series of unitary scattering matrices corresponds to one Yang-Baxter geometry. The difference between the corresponding models is the relation of the spectral parameter of the $R$-matrix to the rapidity variable, and the scalar prefactors, which differ by so-called CDD factors from each other.

The BWM geometry plays a fundamental role in the ultraviolet limit. One finds that the asymptotic phase-shifts are in relation to the dimensions appearing in the operator product expansion of certain fields of the underlying CFT.

Having an explicit expression of the residues in form of $6j$ symbols, we have rewritten the bootstrap-equations in a form, which is easier to apply. We then used that to calculate the $S$-matrix of the higher kink, which appears in the unitary series $M_{r,r+1} + \phi_{1,2}$. A non-trivial degeneracy structure persists for the models $r \geq 5$. Using the principle that breathers are supposed to be bound states of kinks, we find the whole $S$-matrices involving kinks and breathers of these theories. For $r = 5$ the theory has 4 breathers and for $r \geq 6$ only 2. The $S$-matrix elements among kinks exhibit double poles which can all be described by elementary scattering processes of the lightest kink and the lightest breather.

A formidable open problem is to apply the bootstrap to the $S$-matrices of non-unitary minimal models perturbed by the operator $\phi_{1,2}$. In that case we have seen that the $S$-matrix is unitary for the models $M_{r,mr\pm 1}$. These models exhibit a much more complicated bound-state structure of kinks and breathers.

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Appendix

Here we collected some necessary information on the projectors. The \( R \)-matrix \((2.22)\) can be written in terms of projectors as

\[
R(x) = (q^2 x^{-\frac{1}{2}} - x^\frac{3}{2} q^{-2})(q^{-3} x^{-\frac{1}{2}} + q^3 x^\frac{1}{2})P_0 + (q^2 x^\frac{1}{2} - x^{-\frac{1}{2}} q^{-2})(q^{-3} x^\frac{1}{2} + q^3 x^{-\frac{1}{2}})P_1
+ (q^2 x^{-\frac{1}{2}} - x^\frac{3}{2} q^{-2})(q^{-3} x^\frac{1}{2} + q^3 x^{-\frac{1}{2}})P_2.
\] (A.1)

We look for points where \( R(x) \sim P_i \), and therefore the other terms must vanish. We find:

\[
P_0 \text{ vanishes if: } x = q^4 \text{ or } x = iq^{-6},
\]

\[
P_1 \text{ vanishes if: } x = q^{-4} \text{ or } x = iq^6,
\]

\[
P_2 \text{ vanishes if: } x = q^4 \text{ or } x = iq^6.
\] (A.2)

This means that \( R \sim P_0 \) at \( x = iq^6 \) and \( R \sim P_1 \) at \( x = q^4 \). Note that \( R \) never becomes proportional to \( P_2 \), and therefore we can not form bound states of spin 2 in an hypothetical \( S \)-matrix based on the \( R \)-matrix \((2.22)\).

We give now the 6\( j \) symbols which are necessary to carry out the bootstrap. Note that they correspond to the fusion coefficient, graphically displayed as

\[
\begin{array}{c}
\begin{array}{c}
\downarrow j_1 \\
\downarrow j_2 \\
\downarrow j_3
\end{array}
\begin{array}{c}
\downarrow j_1 2 \\
\downarrow j 23
\end{array}

\begin{array}{c}
\downarrow j_3 2 \\
\downarrow j 23
\end{array}
\end{array}
\rightarrow \left\{ \begin{array}{ccc}
\begin{array}{c}
\downarrow j_3 \\
\downarrow j_2 \\
\downarrow j_3 2 \\
\downarrow j 23
\end{array}
\begin{array}{c}
\downarrow j_1 \\
\downarrow j
\end{array}
\end{array} \right\}_q.
\] (A.3)

The only non-zero fusion coefficients for spin 1 are

\[
\begin{align*}
\left\{ \begin{array}{ccc}
1 & 1 & 1 \\
l & l & l
\end{array} \right\}_q &= f_{l,l,l} = (q^{2l+1} + q^{-2l-1}) \left( \frac{[2]}{[4][2l][2l+2]} \right)^{\frac{1}{2}}, \\
\left\{ \begin{array}{ccc}
1 & 1 & 1 \\
l & l-1 & l
\end{array} \right\}_q &= f_{l,l-1,l} = \left( \frac{[2][2l+2]}{[4][2l]} \right)^{\frac{1}{2}}, \\
\left\{ \begin{array}{ccc}
1 & 1 & 1 \\
l & l+1 & l
\end{array} \right\}_q &= f_{l,l+1,l} = -\left( \frac{[2][2l]}{[4][2l+2]} \right)^{\frac{1}{2}}, \\
\left\{ \begin{array}{ccc}
1 & 1 & 1 \\
l & l & l-1
\end{array} \right\}_q &= f_{l,l-1,l} = -\left( \frac{[2l-1][2l+2]}{[2l+3][4][2l]} \right)^{\frac{1}{2}}.
\end{align*}
\]
\[
\begin{array}{c}
\begin{align*}
\left\{ \begin{array}{ccc}
1 & 1 & 1 \\
\ell & \ell & \ell + 1 \\
\end{array} \right\}_q &= f_{\ell,\ell+1,\ell} = \left( \frac{[2\ell + 3][2\ell][2\ell]}{[2\ell + 1][2][2\ell + 2]} \right)^{\frac{1}{2}}.
\end{align*}
\end{array}
\]

The 6\textit{j}-symbols for spin 0, that is the fusion coefficients for the breathers, are

\[
\begin{align*}
\left\{ \begin{array}{ccc}
1 & 1 & 0 \\
\ell & \ell & \ell \\
\end{array} \right\}_q &= -\left( \frac{1}{[3]} \right)^{\frac{1}{2}}, \\
\left\{ \begin{array}{ccc}
1 & 1 & 0 \\
\ell & \ell & \ell + 1 \\
\end{array} \right\}_q &= \left( \frac{[2\ell + 3]}{[3][2\ell + 1]} \right)^{\frac{1}{2}}, \\
\left\{ \begin{array}{ccc}
1 & 1 & 0 \\
\ell & \ell + 1 & \ell \\
\end{array} \right\}_q &= \left( \frac{[2\ell - 1]}{[3][2\ell + 1]} \right)^{\frac{1}{2}}.
\end{align*}
\]
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Figure 1: Pole structure of the $S$-matrix $S_{K_1,K_2}$

Figure 2: Pole structure of the $S$-matrix $S_{K_2,K_2}$
Figure 3: Multi scattering processes responsible for higher order poles in the $S$-matrices $S_{K_1,K_2}$ and $S_{K_2,K_2}$. 