ON NÉRON CLASS GROUPS OF ABELIAN VARIETIES

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A la memoria de Esbriel Avilés Cruz.*

Abstract. Let $F$ be a global field, let $S_{\infty}$ be the set of archimedean primes of $F$ and let $S$ be any nonempty finite set of primes of $F$ containing $S_{\infty}$. In this paper we study the Néron $S$-class group $C_{A,F,S}$ of an abelian variety $A$ defined over $F$. In the well-known analogy that exists between the Birch and Swinnerton-Dyer conjecture for $A$ over $F$ and the analytic class number formula for the field $F$ (in the number field case), the finite group $C_{A,F,S_{\infty}}$ (not the Tate-Shafarevich group of $A$) is a natural analog of the ideal class group of $F$.

1. Introduction

Let $F$ be a global field, let $S$ be a nonempty finite set of primes of $F$ containing all archimedean primes and let $O_{F,S}$ be the ring of $S$-integers of $F$. Further, we write $U = \text{Spec } O_{F,S}$. Note that $v \not\in S$ if, and only if, $v$ corresponds to a point of $U$.

A central problem in Number Theory is that of extending certain known results for tori to abelian varieties over $F$. Specifically, the analytic class number formula for a number field $F$ has long been regarded as a template for the Birch and Swinnerton-Dyer conjecture for an abelian variety $A$ over $F$. But the analogy between a theorem for the trivial torus $T = \mathbb{G}_{m,F}$ (as the analytic class number formula certainly is) and a conjecture about an arbitrary abelian variety is a distant one, and many researchers have come to view (incorrectly, we believe) the Tate-Shafarevich group $\text{III}^1(A)$ of $A$ as a natural analog of the ideal class group of $F$.

In this paper we introduce the Néron $S$-class group $C_{A,F,S}$ of $A$ over $F$ and establish a duality theorem for this group. The definition of $C_{A,F,S}$ is quite simple. For each prime $v \in U$, let $F_v$ be the completion of $F$ at $v$ and let $k(v)$ denote the corresponding residue field. There exists a canonical reduction map $A(F) \to \Phi_v(A)(k(v))$, where $\Phi_v(A)$ is the group scheme of

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connected components of the special fiber of the Néron model of $A_{F_v}$. Then

$$C_{A,F,S} = \text{Coker} \left[ A(F) \to \bigoplus_{v \in U} \Phi_v(A)(k(v)) \right],$$

where the points of $A(F)$ are mapped diagonally into $\bigoplus_{v \in U} \Phi_v(A)(k(v))$ via the preceding reduction maps. For reasons that are explained in Remark 3.3, we believe that this group is the correct analog of the ideal $S$-class group of $F$.

We will now state the main theorem of this paper. Let $B$ denote the dual (i.e., Picard) variety of $A$. For each $v \in U$, Grothendieck’s pairing

$$\Phi_v(A)(k(v)) \times \Phi_v(B)(k(v)) \to \mathbb{Q}/\mathbb{Z}$$

induces a nondegenerate pairing of finite groups

$$\Phi_v(A)(k(v)) \times H^1(k(v), \Phi_v(B)) \to \mathbb{Q}/\mathbb{Z}. $$

See [14], Theorem 4.8. Thus, since the Pontryagin dual of a direct sum is a direct product, there exists a nondegenerate pairing

$$(1.1) \quad \bigoplus_{v \in U} \Phi_v(A)(k(v)) \times \prod_{v \in U} H^1(k(v), \Phi_v(B)) \to \mathbb{Q}/\mathbb{Z}. $$

Note that, since the groups $\Phi_v(A)(k(v))$ are zero for all but finitely primes $v$, the above pairing is in fact a pairing of finite groups. Now, given a nondegenerate pairing of abelian groups $M \times N \to \mathbb{Q}/\mathbb{Z}$ and a subgroup $M' \subset M$, there exists an induced nondegenerate pairing $(M/M') \times (M')^\perp \to \mathbb{Q}/\mathbb{Z}$, where $(M')^\perp \subset N$ is the exact annihilator (i.e., orthogonal complement) of $M'$ in $N$. In the case at hand, a natural question is the following one: what is the exact annihilator, under (1.1), of the image of $A(F)$ in $\bigoplus_{v \in U} \Phi_v(A)(k(v))$? The main result of this paper is the determination of this annihilator under the assumption that $\Pi^1(A)$ is finite. More precisely, the following holds.

**Theorem 4.9.** Assume that $\Pi^1(A)$ is finite. Then (1.1) induces a nondegenerate pairing of finite groups

$$C_{A,F,S} \times C_{1,B,F,S} \to \mathbb{Q}/\mathbb{Z},$$

where $C_{1,B,F,S}$ is the group (3.7).

The subgroup $C_{1,B,F,S}$ of $\prod_{v \in U} H^1(k(v), \Phi_v(B))$ which appears in the statement of the theorem admits the following simple description. Consider the finite set of primes

$$P = \{ v \in U : \Phi_v(A)(k(v)) \neq 0 \}$$

and assume, to avoid trivialities, that $P \neq \emptyset$. Then the nonzero elements of $C_{1,B,F,S}$ are represented by principal homogeneous spaces for $B$ over $F$ which have a point defined over $F_v$ for every $v \notin P$ and a point defined over some unramified extension of $F_v$ for each $v \in P$, but no point defined over
Thus the theorem could be interpreted as saying that the Néron class group \( C_{A,F,S} \) of \( A \) controls via duality the existence (or lack of existence) of rational points on principal homogeneous spaces for \( B \) over \( F \) in various completions of \( F \).

Regarding the question of how the groups \( C_{A,F,S} \) and \( \mathbb{III}^1(A) \) are related, we show in Proposition 3.4 that there exists an exact sequence

\[
0 \to C_{A,F,S} \to D^1(U, \mathcal{O}_A) \to \mathbb{III}^1(A) \to 0,
\]

where \( D^1(U, \mathcal{O}_A) \) is the group \( [\mathbb{A}, \mathbb{A}] \). Hence the Tate-Shafarevich group of \( A \) is not only not the correct analogue of the ideal class group of \( F \) but, in a sense, is "perpendicular" to it.

Remark 1.1. The assumption \( S \neq \emptyset \) is only used in Section 3, where it guarantees the finiteness of the \( S \)-class group of an affine group scheme \( H \). However, when \( H = A \) is an abelian variety, this hypothesis is not needed and, consequently, our main theorem above remains valid when \( S = \emptyset \) in the function field case.

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2. Preliminaries

Let \( F \) be a global field, i.e. \( F \) is a finite extension of \( \mathbb{Q} \) (the number field case) or is finitely generated and of transcendence degree 1 over a finite field of constants (the function field case). We fix a separable algebraic closure \( \overline{F} \) of \( F \). Further, \( S \) is a nonempty finite set of primes of \( F \) containing the archimedean primes in the number field case, \( \mathcal{O}_{F,S} \) is the corresponding ring of \( S \)-integers of \( F \) and \( U = \text{Spec} \mathcal{O}_{F,S} \). For every \( v \in U \), \( \mathcal{O}_v \) will denote the completion of the local ring of \( U \) at \( v \), \( F_v \) will denote its field of fractions and \( k(v) \) is the corresponding (finite) residue field. Further, for each prime \( v \) of \( F \), we fix a prime \( \mathfrak{p} \) of \( \overline{F} \) lying above \( v \), let \( \overline{F}_v = \text{Gal}(\overline{F}_v/F_v) \). The inertia subgroup of \( G_{F_v} \) is the group \( I_{\mathfrak{p}} = \text{Gal}(\overline{F}_v/F_v^{nr}) \), where \( F_v^{nr} \) is the maximal unramified extension of \( F_v \) lying inside \( \overline{F}_v \). We will write \( i_v \) for the canonical closed immersion \( \text{Spec} k(v) \to U \).

For any abelian group \( M \) and positive integer \( n \), we will write \( M_n \) for the \( n \)-torsion subgroup of \( M \) and \( M/n \) for the quotient \( M/nM \). Further, \( M(1) = \bigcup_{r \geq 1} M_{r^n} \) is the \( n \)-primary torsion component of \( M \) and \( M_{\text{div}} = \bigcap nM \) is the subgroup of \( M \) of infinitely divisible elements. For simplicity, we will write \( M/\text{div} \) for \( M/M_{\text{div}} \). Further, we define \( M^D = \text{Hom}(M, \mathbb{Q}/\mathbb{Z}) \).

A pairing of abelian groups \( A \times B \to \mathbb{Q}/\mathbb{Z} \) is called nondegenerate on the right (resp. left) if the induced homomorphism \( B \to A^D \) (resp. \( A \to B^D \)) is injective. It is called nondegenerate if it is nondegenerate both on
the right and on the left. The pairing is said to be perfect if the induced homomorphisms \( B \to A^D \) and \( A \to B^D \) are isomorphisms. Note that a pairing of finite abelian groups is nondegenerate if and only if it is perfect.

For the convenience of the reader, we now recall the well-known 4-lemmas from Homological Algebra.

**Lemma 2.1.** Let

\[
\begin{array}{cccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & W \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & W'
\end{array}
\]

be an exact commutative diagram of abelian groups and group homomorphisms. If the maps \( \alpha \) and \( \gamma \) are epimorphisms and \( \delta \) is a monomorphism, then the map \( \beta \) is an epimorphism. Dually, if

\[
\begin{array}{cccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & W \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & W'
\end{array}
\]

is an exact commutative diagram in which \( \alpha \) is an epimorphism and \( \beta \) and \( \delta \) are monomorphisms, then \( \gamma \) is a monomorphism. \( \Box \)

**Lemma 2.2.** Let \( n \) be a positive integer and let

\[
\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_3} & & \downarrow{f_4} & & \downarrow{f_5} \\
B_5 & \longrightarrow & B_4 & \longrightarrow & B_3 & \longrightarrow & B_2 & \longrightarrow & B_1 \\
\end{array}
\]

be exact sequences of abelian groups. Assume that there exist pairings

\[
\varphi_i : A_i \times B_i \to \mathbb{Q}/\mathbb{Z} \quad (1 \leq i \leq 5)
\]

such that the following conditions hold.

(i) The map \( A_1/n \to [(B_1)_n]^D \) induced by \( \varphi_1 \) is surjective.

(ii) The maps \( A_i/n \to [(B_i)_n]^D \) and \( B_i/n \to [(A_i)_n]^D \) induced by \( \varphi_i \) are injective for \( i = 2 \) and \( i = 4 \).

(iii) The map \( (A_5)_n \to [B_5/n]^D \) induced by \( \varphi_5 \) is injective.

Then the maps \( A_3/n \to [(B_3)_n]^D \) and \( B_3/n \to [(A_3)_n]^D \) induced by \( \varphi_3 \) are injective.

**Proof.** Hypothesis (i) implies that the canonical map \( d : \text{Im} f_1/n \to [(\text{Im} g_1)_n]^D \) is surjective. Now Lemma 2.1 applied to the exact commutative diagram

\[
\begin{array}{cccccc}
\text{Im} f_1/n & \longrightarrow & A_2/n & \longrightarrow & \text{Im} f_2/n & \longrightarrow & \mathbb{Q} \\
\downarrow{d} & & \downarrow{(ii)} & & \circlearrowleft{b} & & \downarrow{} \\
[(\text{Im} g_1)_n]^D & \longrightarrow & [(B_2)_n]^D & \longrightarrow & [(\text{Im} g_2)_n]^D & \longrightarrow & 0
\end{array}
\]
(whose top and bottom rows are induced by the short exact sequences $0 \to \text{Im} f_1 \to A_2 \to \text{Im} f_2 \to 0$ and $0 \to \text{Im} g_2 \to B_2 \to \text{Im} g_1 \to 0$, respectively) shows that the map labeled $b$ above is injective. On the other hand, applying Lemma 2.1 to the diagrams
\[
\begin{array}{cccccc}
0 & \to & (\text{Im } f_3)_n & \to & (A_4)_n & \to & (\text{Im } f_4)_n \\
\downarrow & & \downarrow a & & \downarrow (ii) & & \downarrow (iii) \\
0 & \to & (\text{Im } g_3/n)^D & \to & (B_4/n)^D & \to & (\text{Im } g_4/n)^D
\end{array}
\]
and
\[
\begin{array}{cccccc}
(A_4)_n & \to & (\text{Im } f_4)_n & \to & \text{Im } f_3/n & \to & A_4/n \\
\downarrow & & \downarrow \text{c} & & \downarrow (ii) & & \downarrow (iii) \\
[(B_4)_n]^D & \to & (\text{Im } g_4/n)^D & \to & [(\text{Im } g_3)_n]^D & \to & [(B_4)_n]^D
\end{array}
\]
shows that the maps labeled $a$ and $c$ above are surjective and injective, respectively. Further, applying Lemma 2.1 to the diagram
\[
\begin{array}{cccccc}
(\text{Im } f_3)_n & \to & \text{Im } f_2/n & \to & A_3/n & \to & \text{Im } f_3/n \\
\downarrow a & & \downarrow b & & \downarrow & & \downarrow c \\
(\text{Im } g_3/n)^D & \to & [(\text{Im } g_2)_n]^D & \to & [(B_3)_n]^D & \to & [(\text{Im } g_3)_n]^D
\end{array}
\]
we obtain the injectivity of $A_3/n \to [(B_3)_n]^D$. To check the injectivity of $B_3/n \to [(A_3)_n]^D$, i.e., the surjectivity of $(A_3)_n \to [B_3/n]^D$, one considers the diagrams
\[
\begin{array}{cccccc}
(A_2)_n & \to & (\text{Im } f_2)_n & \to & \text{Im } f_1/n & \to & A_2/n \\
\downarrow (ii) & & \downarrow & & \downarrow d & & \downarrow (ii) \\
(B_2/n)^D & \to & (\text{Im } g_2/n)^D & \to & [(\text{Im } g_1)_n]^D & \to & [(B_2)_n]^D
\end{array}
\]
and
\[
\begin{array}{cccccc}
(\text{Im } f_2)_n & \to & (A_3)_n & \to & (\text{Im } f_3)_n & \to & \text{Im } f_2/n \\
\downarrow & & \downarrow a & & \downarrow & & \downarrow b \\
(\text{Im } g_2/n)^D & \to & (B_3/n)^D & \to & (\text{Im } g_3/n)^D & \to & [(\text{Im } g_2)_n]^D.
\end{array}
\]

3. Class groups and Tate-Shafarevich groups

In the remainder of the paper we will need to consider flat cohomology groups $H^i_{\text{flat}}(U, \mathcal{F}) = H^i(U_{\text{fl}}, \mathcal{F})$, where $U_{\text{fl}}$ is the category of $U$-schemes locally of finite type endowed with the flat topology. If $\mathcal{F}$ is represented by a smooth, quasi-projective and commutative $U$-group scheme, then $H^i_{\text{flat}}(U, \mathcal{F}) = \square$
$H^i_{fet}(U, \mathcal{F})$ (see [13], Theorem III.3.9, p.114). On the other hand, if $Y$ is any scheme, $Y_0$ will denote the set of closed points of $Y$. Further, if $V$ is a scheme, $Y$ is a $V$-scheme and $\text{Spec} \ A$ is an affine $V$-scheme, $Y(A)$ will denote $\text{Hom}_V(\text{Spec} \ A, Y)$.

Let $V$ be a nonempty open subscheme of $U$. The ring of $V$-integral adeles of $U$ is by definition

$$\mathbb{A}_V(U) = \prod_{v \in U \setminus V} F_v \times \prod_{v \in V_0} \mathcal{O}_v.$$ 

Since $V = \text{Spec} \left( \bigcap_{v \in V_0} \mathcal{O}_v \right)$ and there exists a canonical map $\bigcap_{v \in V_0} \mathcal{O}_v \rightarrow \mathbb{A}_V(U)$, $\text{Spec} \mathbb{A}_V(U)$ is canonically an affine $V$-scheme. Now let $H$ be a smooth algebraic group over $F$ and let $\mathcal{M}$ be a quasi-projective $U$-model of $H$ of finite type. We will write $\mathcal{M}_v$ for $\mathcal{M} \times_U V$. The projections $\mathbb{A}_V(U) \rightarrow F_v$ (for $v \in U \setminus V$) and $\mathbb{A}_V(U) \rightarrow \mathcal{O}_v$ (for $v \in V_0$) induce a bijection

$$\mathcal{M}_v(\mathbb{A}_V(U)) \rightarrow \prod_{v \in U \setminus V} H(F_v) \times \prod_{v \in V_0} \mathcal{M}_v(\mathcal{O}_v),$$

where $\mathcal{M}_v = \mathcal{M} \times_U \text{Spec} \mathcal{O}_v$. See [3], Theorem 3.5. We now define a partial ordering on the family of nonempty open subschemes of $U$ by setting $V \leq V'$ if $V' \subset V$. Then, for every pair $V, V'$ of such schemes such that $V \leq V'$, there exists a canonical injection $\mathbb{A}_V(U) \hookrightarrow \mathbb{A}_V(V')$, namely the product of the identity map on $\prod_{v \in U \setminus V} F_v$ and the canonical injection

$$\prod_{v \in V_0} \mathcal{O}_v \hookrightarrow \prod_{v \in V \setminus V'} F_v \times \prod_{v \in V_0'} \mathcal{O}_v.$$ 

We will view $\mathbb{A}_U(V)$ as a subring of $\mathbb{A}_U(V')$ through the above map. The ring of adeles of $U$ is by definition

$$\mathbb{A}_U = \lim_{V \hookrightarrow U} \mathbb{A}_U(V).$$

Clearly, for any $V$ as above, we may regard $\mathbb{A}_U(V)$ as a subring of $\mathbb{A}_U$ and $\mathcal{M}_v(\mathbb{A}_U(V))$ as a subgroup of $\mathcal{M}((\mathbb{A}_U(U)$). Then, by [3], p.5, the natural map

$$\lim_{V \hookrightarrow U} \mathcal{M}_v(\mathbb{A}_U(V)) \rightarrow \mathcal{M}(\mathbb{A}_U)$$

is a bijection. We conclude that there exists a canonical bijection

$$(3.1) \quad \mathcal{M}(\mathbb{A}_U) = \lim_{V \hookrightarrow U} \left( \prod_{v \in U \setminus V} H(F_v) \times \prod_{v \in V_0} \mathcal{M}_v(\mathcal{O}_v) \right).$$

Remark 3.1. For each non-archimedean prime $v$ of $F$, $H(F_v)$ has a natural locally compact Hausdorff topology containing $\mathcal{M}_v(\mathcal{O}_v)$ as a compact open subgroup. Thus, $\lim_{V \hookrightarrow U} \big\{ \prod_{v \in U \setminus V} H(F_v) \times \prod_{v \in V_0} \mathcal{M}_v(\mathcal{O}_v) \}$ has a natural locally compact Hausdorff topology [12], 6.16(c), p.57. This topology can then be transferred to $\mathcal{M}(\mathbb{A}_U)$ via (3.1) so that (3.1) is a homeomorphism. See [3], Theorem 3.5.
It is shown in [3], Theorem 4.3, that $H(F)$ injects into $\mathcal{M}(\mathbb{A}_U)$. Further, as noted above, $\mathcal{M}(\mathbb{A}_U(U)) \subset \mathcal{M}(\mathbb{A}_U)$. We define the class set $C(\mathcal{M})$ of $\mathcal{M}$ as the double coset space

$$C(\mathcal{M}) = \mathcal{M}(\mathbb{A}_U(U)) \setminus \mathcal{M}(\mathbb{A}_U) / H(F).$$

Now, although the arguments of [17], Chapter I, §2 (which are reproduced in [7], §3) are valid in principle only when $\mathcal{M}$ is affine, they admit a straightforward generalization to arbitrary $\mathcal{M}$ as above. In particular, the pointed set $C(\mathcal{M})$ admits the following Nisnevich-cohomological interpretation (see [7], Theorem 3.5):

$$C(\mathcal{M}) = \text{H}^1_{\text{Nis}}(U, \mathcal{M}).$$

Assume now that, in addition to being smooth, $H$ is commutative, connected and admits a Néron model $\mathcal{H}$ over $U$. Thus $\mathcal{H}$ is a smooth and separated $U$-group scheme (in particular, it is locally of finite type) and represents the sheaf $j_*(\mathcal{O}_H)$ on the small smooth site over $U$. Its identity component $\mathcal{H}^0$ is a smooth $U$-model of $H$ of finite type and quasi-projective. See [10], VI B, Proposition 3.9, p.344, and [1], Theorem 6.4.1, p.153. We call the corresponding class group $C(\mathcal{H}^0)$ the Néron $S$-class group of $H$ and denote it by $C_{H,F,S}$. Thus

$$C_{H,F,S} = \mathcal{H}^0(\mathbb{A}_U) / H(F) \mathcal{H}^0(\mathbb{A}_U(U)).$$

The group $C_{H,F,S}$ is known to be finite if $H$ is affine (see, e.g., [2], §1.3). It is also finite if $H = A$ is an abelian variety (this is immediate from Theorem 3.2 below since $\bigoplus_{v \in U} \Phi_v(A)(k(v))$ is finite).

For each prime $v \in U$, let $\Phi_v(H) = i_v^*(\mathcal{H} / \mathcal{H}^0)$ be the $k(v)$-sheaf of connected components of $\mathcal{H}$ at $v$. It is representable by an étale $k(v)$-group scheme of finite type, and there exists a canonical exact sequence of étale sheaves on $U$

$$0 \to \mathcal{H}^0 \to \mathcal{H} \to \bigoplus_{v \in U} (i_v)_* \Phi_v(H) \to 0. \tag{3.2}$$

For each $v \in U$, the natural map $\mathcal{H}_v \to (i_v)_* \Phi_v(H)$ induces a map $\vartheta_v = \vartheta_{H,v}: H(F_v) \to \Phi_v(H)(k(v))$. Let

$$\vartheta_{H,S}: H(F) \to \bigoplus_{v \in U} \Phi_v(H)(k(v)) \tag{3.3}$$

be the map induced by the composition

$$H(F) \to \prod_{v \in U} H(F_v) \xrightarrow{\prod \vartheta_v} \prod_{v \in U} \Phi_v(H)(k(v)),$$

where the first map is the natural “diagonal homomorphism”.

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Note, however, that if topological considerations are relevant, then this generalization is nontrivial. See [3], §3, and Remark 3.1 above.

In [1], Chapter 10, this is called a Néron lift-model.

That the image of $\vartheta_{H,S}$ is contained in $\bigoplus_{v \in U} \Phi_v(H)(k(v))$ follows from [20], Lemma 10.1.1, p.157.
Theorem 3.2. Let $H$ be a smooth, connected and commutative algebraic group over $F$. Assume that $H$ admits a Néron model $\mathcal{H}$ over $U$. Then (3.2) induces an exact sequence

$$0 \to \mathcal{H}^\circ(U) \to H(F) \xrightarrow{\partial_{H,S}} \bigoplus_{v \in U} \Phi_v(H)(k(v)) \to C_{H,F,S} \to 0,$$

where $\partial_{H,S}$ is the map (3.3).

Proof. Taking étale cohomology of (3.2) and using the identification $C_{H,F,S} = H^1_{\text{Nis}}(U, \mathcal{H}^\circ)$, we are immediately reduced to checking that $H^1_{\text{Nis}}(U, \mathcal{H}^\circ)$ is canonically isomorphic to the kernel of the natural map $H^1_{\text{ét}}(U, \mathcal{H}^\circ) \to H^1_{\text{ét}}(U, \mathcal{H})$.

The Cartan-Leray spectral sequence shows that the maps $H^1_{\text{ét}}(U, \mathcal{H}^\circ) \to H^1_{\text{ét}}(U, \mathcal{H})$ and $H^1_{\text{ét}}(U, \mathcal{H}^\circ) \to H^1_{\text{ét}}(F, H)$ have the same kernel (see [8], proof of Lemma 3.1). On the other hand, Ye.Nisnevich has shown [18], Example 1.44, p.286, that the canonical sequence

$$0 \to H^1_{\text{Nis}}(U, \mathcal{H}^\circ) \to H^1_{\text{ét}}(U, \mathcal{H}^\circ) \to H^1(F, H)$$

is exact provided the local maps $H^1_{\text{ét}}(\mathcal{O}^h_v, \mathcal{H}^\circ) \to H^1_{\text{ét}}(F^h_v, \mathcal{H}^\circ)$ are injective for each $v \in U_0$, where $\mathcal{O}^h_v$ denotes the henselization of the local ring of $U$ at $v$ and $F^h_v$ is its field of fractions. Since $H^1_{\text{ét}}(\mathcal{O}^h_v, \mathcal{H}^\circ)$ is in fact zero [15], Remark III.3.11, p.116, and Lang’s Theorem [13], the proof is now complete. \qed

Remark 3.3. The literature records the following assertions: “If $A$ is an abelian variety over a number field $F$, then $A(F)$ is the natural analog of the group of units of $F$ and the Tate-Shafarevich group of $A$ is the natural analog of the ideal class group of $F$”. We believe that these assertions are incorrect. Indeed, the exact sequence of the theorem for $H = \mathbb{G}_{m,F}$ is the familiar exact sequence

$$(3.4) \quad 1 \to \mathcal{O}^*_F \to F^* \to \bigoplus_{v \in U} \mathbb{Z} \to C_{F,S} \to 0,$$

where $\mathcal{H}^\circ(U) = \mathbb{G}_{m,U}(U) = \mathcal{O}^*_F$, is the group of $S$-units of $F$, $\bigoplus_{v \in U} \mathbb{Z}$ is (isomorphic to) the group $\mathcal{O}_{F,S}$ of fractional $S$-ideals of $F$ and $C_{F,S}$ is the ideal $S$-class group of $F$. See [8], Remark 2.1. Thus (3.4) and the exact sequence of the theorem when $H = A$ is an abelian variety with Néron model $\mathcal{A}$ show that $\mathcal{A}^\circ(U), \bigoplus_{v \in U} \Phi_v(A)(k(v))$ and $C_{A,F,S}$ are natural analogs of $\mathcal{O}^*_F, \mathcal{O}_{F,S}$ and $C_{F,S}$, respectively.

Let $H$ be as in the statement of the theorem. For each $v \notin S$, set

$$H^1_{\text{nr}}(F_v, H) = \text{Ker} \left[ H^1(F_v, H) \to H^1(F^\text{nr}_v, H) \right],$$

where the map involved is the restriction map in Galois cohomology. The inflation map in Galois cohomology induces an isomorphism

$$H^1(G_{F_v}/\mathcal{P}, H(F^\text{nr}_v)) \simeq H^1_{\text{nr}}(F_v, H).$$
On the other hand, by a straightforward generalization of [16], proof of Proposition I.3.8, p.57, the reduction map $H(F_{nr}v) \to \Phi_v(H)(k(v))$ induces an isomorphism $H^1(G_{F_v}/I_v, H(F_{nr}v)) \simeq H^1(k(v), \Phi_v(H))$. Thus there exists a canonical isomorphism

$H^1_{nr}(F_v, H) \cong H^1(k(v), \Phi_v(H))$.

We will henceforth identify $H^1_{nr}(F_v, H)$ and $H^1(k(v), \Phi_v(H))$ via the above map.

Now, for any prime $v$ of $F$, there exists a canonical “localization” map $H^1(F, H) \to H^1(F_v, H)$. We let

$$\lambda_S: H^1(F, H) \to \prod_{v \in \mathcal{U}} H^1(F_v, H)$$

be the induced map and set $\Pi^1_S(H) = \text{Ker} \lambda_S$. Further, we recall the Tate-Shafarevich group of $H$:

$$\Pi^1(H) = \text{Ker} \left[ H^1(F, H) \to \prod_{v} H^1(F_v, H) \right].$$

Now define

$$\Pi^1_{S \setminus P}(H) = \text{Ker} \left[ H^1(F, H) \to \prod_{v \in S} H^1(F_v, H) \right]$$

and let

$$\lambda'_S: \Pi^1_{S \setminus P}(H) \to \prod_{v \in \mathcal{U}} H^1(F_v, H)$$

be the restriction of (3.6) to $\Pi^1_{S \setminus P}(H)$. Set

$$C^1_{H,F,S} = \left( \prod_{v \notin S} H^1_{nr}(F_v, H) \right) \cap \text{Im} \lambda'_S \subset \prod_{v \notin S} H^1(F_v, H).$$

The elements of $C^1_{H,F,S}$ can be described as follows. Consider the set of primes

$$P = \{ v \in \mathcal{U} : H^1(k(v), \Phi_v(H)) \neq 0 \}$$

and assume, to avoid trivialities, that $P \neq \emptyset$. Then the nonzero elements of $C^1_{H,F,S}$ are represented by principal homogeneous spaces for $H$ over $F$ which have a point defined over $F_v$ for every $v \notin P$ and a point defined over some unramified extension of $F_v$ for each $v \in P$, but no point defined over $F_v$ itself for some $v \notin P$.

In order to explain how the groups $C^1_{H,F,S}, C^1_{H,F,S}$ and $\Pi^1(H)$ are related, we need another definition. For any prime $v$ of $F$ and any sheaf $\mathcal{F}$ on $U_{\overline{\mathbb{F}}}$, let $H^1_{fl}(F_v, \mathcal{F}) = H^1_{fl}(\text{Spec } F_v, \mathcal{F}_v)$, where $\mathcal{F}_v$ is the sheaf on $(\text{Spec } F_v)_{\overline{\mathbb{F}}}$ obtained by pulling back $\mathcal{F}$ relative to the composite morphism

$$\text{Spec } F_v \to \text{Spec } F \to U.$$
If \( v \) is archimedean, \( H^i_{fl}(\text{Spec} \, F_v, \mathcal{F}_v) \) will denote the \( i \)-th reduced (Tate) cohomology group of \( \mathcal{F}_v \). Clearly, the preceding morphism induces a map 
\[ H^i_{fl}(U, \mathcal{F}) \rightarrow H^i_{fl}(F_v, \mathcal{F}). \]

Set

\[
D^i(U, \mathcal{F}) = \text{Ker} \left[ H^i_{fl}(U, \mathcal{F}) \rightarrow \prod_{v \in S} H^i_{fl}(F_v, \mathcal{F}) \right].
\]  

Recall that \( H^i_{fl}(U, \mathcal{F}) = H^i_{\acute{e}t}(U, \mathcal{F}) \) if \( \mathcal{F} \) is represented by a smooth, quasi-projective and commutative \( U \)-group scheme.

**Proposition 3.4.** Let \( H \) be a smooth, connected and commutative algebraic group over \( F \). Assume that \( H \) admits a Néron model \( \mathcal{H} \) over \( U \). Then there exist canonical exact sequences

\[
0 \rightarrow C_{H,F,S} \rightarrow D^1(U, \mathcal{H}^0) \rightarrow \Pi^1(H) \rightarrow 0
\]

and

\[
0 \rightarrow \Pi^1(H) \rightarrow D^1(U, \mathcal{H}) \rightarrow C_{H,F,S}^1 \rightarrow 0.
\]

**Proof.** The Cartan-Leray spectral sequence \cite{15}, Theorem 1.18(a), p.89, and \cite{16}, Remark I.3.10, p.58, yield an exact sequence

\[
0 \rightarrow H^1_{\acute{e}t}(U, \mathcal{H}) \rightarrow H^1(F, H) \rightarrow \prod_{v \in U} H^1(F^\text{nr}_v, H).
\]

See \cite{16}, proof of Lemma II.5.5, p.247. Now, applying the kernel-cokernel exact sequence \cite{16}, Proposition I.0.24, p.19) to the pair of maps

\[
H^1(F, H) \rightarrow \prod_{v \in U} H^1(F_v, H) \rightarrow \prod_{v \in U} H^1(F^\text{nr}_v, H),
\]

we obtain an exact sequence

\[
0 \rightarrow \Pi^1_S(H) \rightarrow H^1(U, \mathcal{H}) \rightarrow \prod_{v \in U} H^1_{\text{nr}}(F_v, H) \rightarrow \text{Coker} \lambda_S,
\]

where \( \lambda_S \) is the localization map \cite{3.6}. Thus, there exists an exact sequence

\[
0 \rightarrow \Pi^1_S(H) \rightarrow H^1(U, \mathcal{H}) \rightarrow \left( \prod_{v \in U} H^1_{\text{nr}}(F_v, H) \right) \cap \text{Im} \lambda_S \rightarrow 0.
\]

On the other hand, via the identification \cite{3.5}, the exact sequence \cite{3.2} induces an exact sequence

\[
0 \rightarrow C_{H,F,S} \rightarrow H^1(U, \mathcal{H}^0) \rightarrow H^1(U, \mathcal{H}) \rightarrow \prod_{v \notin S} H^1_{\text{nr}}(F_v, H).
\]

Thus \cite{3.9} yields an exact sequence

\[
0 \rightarrow C_{H,F,S} \rightarrow H^1(U, \mathcal{H}^0) \rightarrow \Pi^1_S(H) \rightarrow 0.
\]
The proposition now follows from (3.7) and the exact commutative diagrams

\[
\begin{array}{cccc}
\prod_{v \in S} H^1(F_v, H) & \longrightarrow & H^1(U, \mathcal{F}) & \longrightarrow & \left( \prod_{v \in S} H^1_{nr}(F_v, H) \right) \cap \text{Im } \lambda_S \\
\downarrow & & \downarrow & & \\
\prod_{v \in S} H^1(F_v, H) & \longrightarrow & \prod_{v \in S} H^1(F_v, H)
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & \longrightarrow & C_{H,F,S} & \longrightarrow & H^1(U, \mathcal{F}^\circ) & \longrightarrow & \prod_{v \in S} H^1(F_v, H) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\prod_{v \in S} H^1(F_v, H) & \longrightarrow & \prod_{v \in S} H^1(F_v, H),
\end{array}
\]

whose top rows are (3.9) and (3.10), respectively.

\[\square\]

4. Proof of the main theorem

Let \( A \) be an abelian variety defined over \( F \) and let \( B \) be the abelian variety dual to \( A \). The Néron model of \( B \) over \( U \) will be denoted by \( \mathcal{B} \).

For each \( v \in U \), the étale \( k(v) \)-sheaves \( \Phi_v(A) \) and \( \Phi_v(B) \) will be identified with the \( G_{k(v)} \)-modules \( \Phi_v = \Phi_v(A)(k(v)) \) and \( \Phi'_v = \Phi_v(B)(k(v)) \), respectively. Let \( \Gamma_v \) and \( \Gamma'_v \) be subsheaves of \( \Phi_v(A) \) and \( \Phi_v(B) \). These will be regarded as \( G_{k(v)} \)-submodules of \( \Phi_v \) and \( \Phi'_v \), respectively. We will assume throughout that

\( \Gamma'_v \) is the exact annihilator of \( \Gamma_v \) under Grothendieck’s pairing

\[ \Phi_v \times \Phi'_v \to \mathbb{Q}/\mathbb{Z}. \]

Since the above pairing is known to be nondegenerate in the context of this paper (see, e.g., [14], Theorem 4.8), \( \Gamma'_v = \Phi'_v \) and \( \Gamma_v = 0 \) is a valid choice. We will make this choice in the proof of Corollary 4.8 below, and Corollary 4.8 is used in the proof of Theorem 4.9 (the main theorem). It is in this way that the nondegeneracy of Grothendieck’s pairing intervenes in the derivation of our main result.

We will need the cohomology groups with compact support \( H^*_c(U_{fl}, \mathcal{F}) \) defined in [16], pp.270-271. For any sheaf \( \mathcal{F} \) on \( U_{fl} \), there exists an exact sequence

\[ (4.1) \]

\[ \ldots \to H^*_c(U_{fl}, \mathcal{F}) \to H^1_{fl}(U, \mathcal{F}) \to \prod_{v \in S} H^1_{fl}(F_v, \mathcal{F}) \to H^1_{fl}(U_{fl}, \mathcal{F}) \to \ldots. \]

See [16], Remark III.0.6(b), p.274.
Let \( \Gamma = \bigoplus_{v \in U} (i_v)_* \Gamma_v \) and \( \Gamma' = \bigoplus_{v \in U} (i_v)_* \Gamma'_v \), and define \( \mathcal{A} \Gamma \) and \( \mathcal{B} \Gamma' \) by the exactness of the sequences

\[
0 \to \mathcal{A} \to \mathcal{A} \Gamma \to \Gamma \to 0
\]

and

\[
0 \to \mathcal{B} \to \mathcal{B} \Gamma' \to \Gamma' \to 0.
\]

By [9], Theorem VIII.7.1(b), the canonical Poincaré biextension of \((A, B)\) by \(\mathbb{G}_{m,F}\) extends uniquely to a biextension of \((\mathcal{A} \Gamma, \mathcal{B} \Gamma')\) by \(\mathbb{G}_{m,U}\). This biextension induces a map

\[
\mathcal{A} \Gamma \otimes^L \mathcal{B} \Gamma' \to \mathbb{G}_{m,U}[1]
\]

in the derived category of the category of smooth sheaves on \(U\), and this map induces pairings

\[
\langle -, - \rangle : H^1_c(U_{\text{fl}}, \mathcal{A} \Gamma) \times H^1_{\text{fl}}(U, \mathcal{B} \Gamma') \to \mathbb{Q}/\mathbb{Z}
\]

and

\[
H^1_{\text{fl}}(U, \mathcal{A} \Gamma) \times H^1_{\text{fl}}(U, \mathcal{B} \Gamma') \to \mathbb{Q}/\mathbb{Z}.
\]

See [16], comments preceding Theorem III.0.16, and [4].

Now recall the group \((3.8)\):

\[
D^i(U, \mathcal{F}) = \text{Ker} \left[ H^i_{\text{fl}}(U, \mathcal{F}) \to \prod_{v \in S} H^i_{\text{fl}}(F_v, \mathcal{F}) \right]
\]

\[
= \text{Im} \left[ H^i_c(U_{\text{fl}}, \mathcal{F}) \to H^i_{\text{fl}}(U, \mathcal{F}) \right].
\]

There exists a canonical exact commutative diagram

\[
0 \to D^1(U, \mathcal{B} \Gamma') \to H^1_{\text{fl}}(U, \mathcal{B} \Gamma') \to \prod_{v \in S} H^1_{\text{fl}}(F_v, B) \to H^1_c(U_{\text{fl}}, \mathcal{A} \Gamma)^D \oplus \bigoplus_{v \in S} H^0_{\text{fl}}(F_v, A)^D,
\]

where the middle vertical map is induced by \((4.3)\) and the right-hand vertical map is induced by the local pairings [16], I.3.4, I.3.7 and III.7.8. The horizontal maps come from \((4.1)\) (for appropriate choices of \(i\) and \(\mathcal{F}\)) and the definition of \(D^1(U, \mathcal{B} \Gamma')\). It follows that there exists a well-defined pairing

\[
\{ -, - \} : D^1(U, \mathcal{A} \Gamma) \times D^1(U, \mathcal{B} \Gamma') \to \mathbb{Q}/\mathbb{Z}
\]
given by \( \{a, a'\} = \langle \tilde{a}, a' \rangle \), where \( a' \in D^1(U, \mathcal{B}^\Gamma) \subset H^1_{\text{fl}}(U, \mathcal{B}^\Gamma) \) and \( \tilde{a} \) is a preimage of \( a \) in \( H^1_{\text{fl}}(U, \mathcal{A}_\Gamma) \). We will show (Theorem 4.7) that, if \( \Xi^1(A) \) and \( \Xi^1(B) \) are finite, then (4.5) is a perfect pairing of finite groups.

**Lemma 4.1.** Let \( V \) be a nonempty open subscheme of \( U \) such that \( \mathcal{A}|_V \) is an abelian scheme. Let \( n \) be any positive integer. Then the maps

\[
H^1_c(V_{\text{fl}}, \mathcal{B})/n \to (H^1_{\text{fl}}(V, \mathcal{A})_n)^D
\]

and

\[
H^1_{\text{fl}}(V, \mathcal{A})/n \to (H^1_c(V_{\text{fl}}, \mathcal{B})_n)^D,
\]

induced by (4.4) over \( V \), are injective.

**Proof.** The lemma is immediate from the commutativity of the diagrams

\[
\begin{array}{ccc}
H^1_c(V_{\text{fl}}, \mathcal{B})/n & \overset{\cong}{\longrightarrow} & H^2_c(V_{\text{fl}}, \mathcal{B})_n \\
\downarrow & & \downarrow \\
(H^1_{\text{fl}}(V, \mathcal{A})_n)^D & \overset{\sim}{\longrightarrow} & H^1_{\text{fl}}(V, \mathcal{A})_n^D
\end{array}
\]

and

\[
\begin{array}{ccc}
H^1_{\text{fl}}(V, \mathcal{A})/n & \overset{\cong}{\longrightarrow} & H^2_{\text{fl}}(V, \mathcal{A})_n \\
\downarrow & & \downarrow \\
(H^1_{\text{fl}}(V_{\text{fl}}, \mathcal{B})_n)^D & \overset{\sim}{\longrightarrow} & H^1_{\text{fl}}(V_{\text{fl}}, \mathcal{B})_n^D
\end{array}
\]

where the right-hand vertical maps are isomorphisms by [16], corollary III.3.2, p.313, and theorem III.8.2, p.361.

**Lemma 4.2.** Let \( V \) and \( n \) be as in the previous lemma. Then there exists a canonical exact sequence

\[
0 \to (\Xi^1(A)_{\text{div}} \cap \Xi^1(A))^D \to H^2_c(V_{\text{fl}}, \mathcal{B})_n \to (H^0_{\text{fl}}(V, \mathcal{A})_n)^D \to 0.
\]

In particular, if \( \Xi^1(A)(n) \) is finite, then the canonical map \( H^2_c(V_{\text{fl}}, \mathcal{B})_n \to (H^0_{\text{fl}}(V, \mathcal{A})_n)^D \) is an isomorphism.

**Proof.** There exist exact commutative diagrams

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1_c(V_{\text{fl}}, \mathcal{B})/n & \longrightarrow & H^2_c(V_{\text{fl}}, \mathcal{B})_n & \longrightarrow & H^3_c(V_{\text{fl}}, \mathcal{B})_n & \longrightarrow & 0 \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (H^1_{\text{fl}}(V, \mathcal{A})_n)^D & \longrightarrow & H^1_{\text{fl}}(V, \mathcal{A})_n^D & \longrightarrow & (H^0_{\text{fl}}(V, \mathcal{A})_n)^D & \longrightarrow & 0
\end{array}
\]

Recall that \( \Xi^1(A) \) is finite if, and only if, \( \Xi^1(B) \) is finite [16], Remark I.6.14(c), p.102.
and
\[
\prod_{v \not\in V} H^0_{\mathfrak{f}}(F_v, B)/n \rightarrow H^1_c(V_{\mathfrak{f}}, \mathcal{B})/n \rightarrow D^1(V, \mathcal{B})/n \rightarrow 0
\]
\[
\bigoplus_{v \not\in V} (H^1_{\mathfrak{f}}(F_v, A)/n)^D \rightarrow (H^1_{\mathfrak{f}}(V, \mathcal{A})/n)^D \rightarrow (D^1(V, \mathcal{A})_n)^D \rightarrow 0
\]
(see [16], p.245). Further, there exist canonical isomorphisms \(D^1(V, \mathcal{A}) \simeq \mathbb{III}^1(A)\) and \(D^1(V, \mathcal{B}) \simeq \mathbb{III}^1(B)\) [16], Lemma II.5.5 p.247. The lemma now follows from the above diagrams and the existence of a perfect pairing \(\mathbb{III}^1(A)/\text{div} \times \mathbb{III}^1(B)/\text{div} \rightarrow \mathbb{Q}/\mathbb{Z}\) [11], Theorem 4.8, and [8], Theorem 6.6.

\[\square\]

**Proposition 4.3.** Let \(n\) be any integer such that \(\mathbb{III}^1(A)(n)\) is finite. Then the maps
\[
H^1_c(U_{\mathfrak{f}}, \mathcal{B}^{\Gamma'})/n \rightarrow (H^1_{\mathfrak{f}}(U, \mathcal{A}^{\Gamma})_n)^D
\]
and
\[
H^1_{\mathfrak{f}}(U, \mathcal{A}^{\Gamma})/n \rightarrow (H^1_{\mathfrak{f}}(U_{\mathfrak{f}}, \mathcal{B}^{\Gamma'})_n)^D,
\]
induced by (4.3), are injective.

**Proof.** We wish to apply Lemma 2.2. Let \(V\) be a nonempty open subscheme of \(U\) such that \(\mathcal{A}^{\Gamma}|_{V} = \mathcal{A}|_{V}\) and \(\mathcal{B}^{\Gamma}|_{V} = \mathcal{B}|_{V}\) are abelian schemes. There exist canonical exact sequences of abelian groups
\[
\bigoplus_{v \not\in U \setminus V} H^0_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{B}^{\Gamma'}) \rightarrow H^1_c(V_{\mathfrak{f}}, \mathcal{B}) \rightarrow H^1_c(U_{\mathfrak{f}}, \mathcal{B}^{\Gamma'}) \rightarrow \bigoplus_{v \in U \setminus V} H^1_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{B}^{\Gamma'}) \rightarrow H^2_c(V_{\mathfrak{f}}, \mathcal{B})
\]
and
\[
H^0_{\mathfrak{f}}(V, \mathcal{A}) \rightarrow \bigoplus_{v \not\in U \setminus V} H^1_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{A}^{\Gamma'}) \rightarrow H^1(U_{\mathfrak{f}}, \mathcal{A}^{\Gamma}) \rightarrow \bigoplus_{v \in U \setminus V} H^1_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{A}^{\Gamma}) \rightarrow H^2_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{A}^{\Gamma}).
\]

See [16], Proposition III.0.4(c), p.272, and Remark III.0.6(b), p.275. Further, for each \(v \in U\), there exist pairings \(\varphi_{1,v}: H^0_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{B}^{\Gamma'}) \times H^1_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{A}^{\Gamma}) \rightarrow \mathbb{Q}/\mathbb{Z}\) and \(\varphi_{4,v}: H^1_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{A}^{\Gamma'}) / n \rightarrow [H^2_{\mathfrak{f}}(\mathcal{O}_v, \mathcal{A}^{\Gamma})]_n^D\), and the second one is a perfect pairing of finite groups. See [16], Theorem III.2.7, p.307, and Theorem III.7.13, p.358. On the other hand, by the previous lemma, the map \(H^2_c(V_{\mathfrak{f}}, \mathcal{B}) \rightarrow (H^0_V(V, \mathcal{A}))^D\) induced by the pairing \(H^2_c(V_{\mathfrak{f}}, \mathcal{B}) \times H^1_c(V, \mathcal{A}) \rightarrow \mathbb{Q}/\mathbb{Z}\) is an isomorphism. Finally, Lemma 4.1 shows that all the conditions needed to apply Lemma 2.2 hold, and the proposition follows.

\[\square\]

Now let \(\ell\) be any prime number and let \(r\) be a positive integer. Set
\[
\text{Sel}(U, \mathcal{B}^{\Gamma'})_{\ell} = \ker \left[ H^1_{\mathfrak{f}}(U, \mathcal{B}^{\Gamma'}) \rightarrow \prod_{v \in S} H^1_{\mathfrak{f}}(F_v, B) \right],
\]
where the map involved is the composite
\[ H_1^\fl(U, B'_\ell) \to H_1^\fl(U, B'_\ell)_{\ell'} \to \prod_{v \in S} H_1^\fl(F_v, B) \]

The kernel-cokernel exact sequence \[16\], Proposition I.0.24, applied to the above pair of maps yields an exact sequence
\[ 0 \to H_0^\fl(U, B'_\ell) \to \text{Sel}(U, B'_\ell)_{\ell'} \to D_1(U, B'_\ell)_{\ell'} \to 0. \]

Now, by \[9\], VIII.2.2.5, the biextension \((A_{\Gamma}, B'_\ell; \mathbb{G}_m, U)\) induces a map
\[ A_{\Gamma} \times B'_\ell \to \mathbb{G}_m, U. \]

The above map induces a (possibly degenerate) pairing
\[ H_2^c(U_\ell, A_{\Gamma}^\ell) \times H_1^\fl(U, B'_\ell)_{\ell'} \to \mathbb{Q}/\mathbb{Z}, \]
and we let
\[ \eta: H_2^2(U_\ell, A_{\Gamma}^\ell) \to H_1^1(U, B'_\ell^\ell) D \]
be the map induced by this pairing.

On the other hand, the exact sequences of flat sheaves
\[ 0 \to A_{\Gamma}^\ell \to A_{\Gamma}^\ell \\ B'_\ell^\ell \to B'_\ell^\ell \to 0 \]
induce maps
\[ \partial_c: H_1^1(U, A_{\Gamma}^\ell) \to H_2^2(U_\ell, A_{\Gamma}^\ell), \]
\[ \vartheta: H_1^1(U, B'_\ell^\ell) \to H_1^1(U, B'_\ell^\ell)_{\ell'} \]
such that the following holds. If
\[ [-, -]: H_1^1(U, A_{\Gamma}^\ell) D \times H_1^1(U, B'_\ell^\ell) \to \mathbb{Q}/\mathbb{Z} \]
is the evaluation pairing and \( \langle - , - \rangle \) is the pairing \[4.3\], then
\[ \eta \partial_c(\zeta), \xi \rangle = \langle \zeta, \vartheta(\xi) \rangle \]
for every \( \zeta \in H_1^1(U_\ell, A_{\Gamma}^\ell) \) and \( \xi \in H_1^1(U, B'_\ell^\ell) \), where \( \eta, \partial_c \) and \( \vartheta \) are the maps \[4.7\], \[4.8\] and \[4.9\]. Now consider
\[ \delta': \prod_{v \in S} H_1^1(F_v, A_{\ell'}) \to H_2^2(U_\ell, A_{\ell'}^\ell) \]
(see \[4.11\]) and set
\[ \delta = \eta \circ \delta': \prod_{v \in S} H_1^1(F_v, A_{\ell'}) \to H_1^1(U, B'_\ell^\ell)^D. \]
By [16, Corollary I.2.3, p.34, and Theorem III.6.10, p.344], there exists a perfect “cup-product” pairing defined by

\[(c_v), (c'_v) = \sum_{v \in S} \text{inv}_v(c_v \cup c'_v),\]

where \(\text{inv}_v : \text{Br}(F_v) \to \mathbb{Q}/\mathbb{Z}\) is the usual invariant map of local class field theory. Now the dual of (4.13) is a map

\[\delta : H^1_{fl}(U, B_{\Gamma'}^e) \to \bigoplus_{v \in S} H^1_{fl}(F_v, B_{\ell^r})\]

and the following holds: if \(c \in \prod_{v \in S} H^1_{fl}(F_v, A_{\ell^r})\) and \(x \in H^1_{fl}(U, B_{\Gamma'}^e)\), then

\[(c, \delta(x)) = [\delta(c), x],\]

where \([-,-]\) is the evaluation pairing (4.10). Next, let

\[\varrho : \bigoplus_{v \in S} H^0_{fl}(F_v, A) \to \prod_{v \in S} H^1_{fl}(F_v, A_{\ell^r})\]

be the composite

\[\bigoplus_{v \in S} H^0_{fl}(F_v, A) \to \bigoplus_{v \in S} H^0_{fl}(F_v, A)/\ell^r \cong \prod_{v \in S} H^0_{fl}(F_v, A)/\ell^r \to \prod_{v \in S} H^1_{fl}(F_v, A_{\ell^r})\]

(recall that \(S\) is finite).

**Lemma 4.4.** Let \(c \in \prod_{v \in S} H^1_{fl}(F_v, A_{\ell^r})\). Then \(\langle c, \delta^D(x) \rangle = 0\) for every \(x \in \text{Sel}(U, B_{\Gamma'}^e) \subset H^1_{fl}(U, B_{\Gamma'}^e)\) if, and only if, \(c = c_1 + c_2\), with \(c_1 \in \text{Im} \varrho\) and \(c_2 \in \ker \delta\).

**Proof.** The proof is similar to the proof of [8, lemma 5.5], using (4.14) and the commutative diagram

\[
\begin{array}{ccc}
\prod_{v \in S} H^1_{fl}(F_v, A_{\ell^r}) & \xrightarrow{\varrho} & \bigoplus_{v \in S} H^0_{fl}(F_v, A) \\
\delta & & \\
\bigoplus_{v \in S} H^0_{fl}(F_v, A) & \xrightarrow{\delta} & H^1_{fl}(U, B_{\Gamma'}^e)^D \\
& & \downarrow \text{Sel}(U, B_{\Gamma'}^e)^D \\
& & (\text{Sel}(U, B_{\Gamma'}^e)^e)^D,
\end{array}
\]

where \(\delta\) and \(\varrho\) are the maps (4.13) and (4.15), respectively (the exactness of the bottom row of this diagram follows from the definition of \(\text{Sel}(U, B_{\Gamma'}^e)^e\) and the local duality theorems for abelian varieties).  

\[\text{Recall that, in general, direct products and direct sums are in natural duality. Note, however, that } \prod_{v \in S} H^1_{fl}(F_v, A_{\ell^r}) \text{ and } \bigoplus_{v \in S} H^1_{fl}(F_v, A_{\ell^r}) \text{ are canonically isomorphic since } S \text{ is finite.}\]
Lemma 4.5. Assume that $\text{III}^1(B)(\ell)$ is finite and let $a \in D^1(U, \mathcal{A}^\Gamma)$. If $a \in \ell^r \text{H}^1_{\text{fl}}(U, \mathcal{A}^\Gamma)$ and $\{a, a'\} = 0$ for every $a' \in D^1(U, \mathcal{B}^\Gamma)'_\ell$, where $\{-, -\}$ is the pairing (4.5), then $a \in \ell^r D^1(U, \mathcal{A}^\Gamma)$.

Proof. Consider the exact commutative diagram

$$
\begin{array}{ccc}
H^1_{\text{fl}}(U, \mathcal{A}^\Gamma) & \longrightarrow & H^1_{\text{fl}}(U, \mathcal{A}^\Gamma) \\
\bigoplus_{v \in S} H^0_{\text{fl}}(F_v, A) & \longrightarrow & H^1_{\text{fl}}(U, \mathcal{A}^\Gamma) \\
\prod_{v \in S} H^0_{\text{fl}}(F_v, A^\ell') & \delta' & H^2_{\text{fl}}(U, \mathcal{A}^\Gamma) \\
\end{array}
$$

where the middle row comes from (4.1) and the maps $\varphi$, $\delta'$ and $\partial_c$ are given by (4.13), (4.12) and (4.8), respectively. Since $a \in D^1(U, \mathcal{A}^\Gamma) = \text{Im} \psi$, there exists $\overline{a} \in H^1_{\text{fl}}(U, \mathcal{A}^\Gamma)$ such that $\psi(\overline{a}) = a$. Now, by hypothesis, $0 = \partial(a) = \partial(\psi(\overline{a}))$, whence $\partial_c(\overline{a}) = \delta'(c)$ for some $c \in \prod_{v \in S} H^0_{\text{fl}}(F_v, A^\ell')$. Now, if $x \in \text{Sel}(U, \mathcal{B}^\Gamma)'_\ell$, then $\vartheta(x) \in D^1(U, \mathcal{B}^\Gamma)'_\ell$ by (4.6), where $\vartheta$ is the map (4.9). Therefore, by (4.14), the definitions of $\delta$ and $\{ -, -\}$ (see (4.5) and (4.13)) and (4.11),

$$(c, \delta^D(x)) = [\delta(c), x] = [\eta \delta'(c), x] = [\eta \partial_c(\overline{a}), x] = (\overline{a}, \vartheta(x)) = \{a, \vartheta(x)\} = 0.$$

Consequently, by Lemma 4.4, we may write $c = \varphi(c'_1) + c_2$ with $c'_1 \in \prod_{v \in S} H^0_{\text{fl}}(F_v, A)$ and $c_2 \in \text{Ker} \delta$. Now we have

$$\eta \partial_c(\overline{a} - \theta(c'_1)) = \eta \delta'(c) - \eta \delta' \varphi(c'_1) = \eta \delta'(c - \varphi(c'_1)) = \eta \delta'(c'_2) = \delta(c_2) = 0.$$

Thus the commutative diagram

$$
\begin{array}{ccc}
H^1_{\text{fl}}(U, \mathcal{A}^\Gamma) & \longrightarrow & H^2_{\text{fl}}(U, \mathcal{A}^\Gamma) \\
\bigoplus_{v \in S} \text{H}^1_{\text{fl}}(U, \mathcal{B}^\Gamma)'_\ell & \longrightarrow & \text{H}^2_{\text{fl}}(U, \mathcal{B}^\Gamma)'_\ell \\
\text{H}^1_{\text{fl}}(U, \mathcal{B}^\Gamma)'_\ell \times \text{H}^1_{\text{fl}}(U, \mathcal{B}^\Gamma)'_\ell & \longrightarrow & \text{H}^1_{\text{fl}}(U, \mathcal{B}^\Gamma)'_\ell \\
\end{array}
$$

shows that the element $\overline{a} - \theta(c'_1) \in H^1_{\text{fl}}(U, \mathcal{A}^\Gamma)$ lies in the kernel of the left-hand vertical map. Consequently, by Proposition 4.3 (with the roles of $A$ and $B$ exchanged), $\overline{a} - \theta(c'_1) \in \ell^r H^1_{\text{fl}}(U, \mathcal{A}^\Gamma)$. Thus $a = \psi(\overline{a}) = \psi(\overline{a} - \theta(c'_1)) \in \ell^r \text{Im} \psi = \ell^r D^1(U, \mathcal{A}^\Gamma)$. \qed

Lemma 4.6. If $\text{III}^1(A)(\ell)$ is finite, then so also is $D^1(U, \mathcal{A}^\Gamma)(\ell)$. 
Proof. Proposition 3.4 shows that the lemma is valid if $\Gamma = 0$. Now the exact commutative diagram

$$
\bigoplus_{v \in U} \Gamma_v(k(v)) \to H^1_{fl}(U, \mathcal{A}^\circ) \to H^1_{fl}(U, \mathcal{A}^\Gamma) \to \bigoplus_{v \in U} H^1_{fl}(k(v), \Gamma_v)$$

whose top row is induced by (4.2), yields an exact sequence

$$\bigoplus_{v \in U} \Gamma_v(k(v)) \to D^1(U, \mathcal{A}^\circ) \to D^1(U, \mathcal{A}^\Gamma) \to \bigoplus_{v \in U} H^1_{fl}(k(v), \Gamma_v).$$

For any $i$, $H^i(k(v), \Gamma_v)$ is finite and equal to zero for all but finitely many primes $v \in U$ [19], Chapter XIII, §1, Proposition 1, p.189. It follows that $D^1(U, \mathcal{A}^\Gamma)(\ell)$ is finite if, and only if, $D^1(U, \mathcal{A}^\circ)(\ell)$ is finite. This completes the proof. □

Theorem 4.7. Let $\ell$ be any prime number such that $\mathbb{I}^1(A)(\ell)$ and $\mathbb{I}^1(B)(\ell)$ are finite. Then there exists a perfect pairing of finite groups

$$D^1(U, \mathcal{A}^\Gamma)(\ell) \times D^1(U, \mathcal{B}^\Gamma')(\ell) \to \mathbb{Q}/\mathbb{Z}.$$ 

Proof. The finiteness assertion is contained in the previous lemma. Now there exists a canonical exact commutative diagram

$$0 \to D^1(U, \mathcal{A}^\Gamma)(\ell) \to H^1_{fl}(U, \mathcal{A}^\Gamma)(\ell) \to \bigoplus_{v \in S} H^1_{fl}(F_v, A)(\ell)$$

where the middle vertical map is induced by (4.4) and the right-hand vertical map is injective by local duality. Proposition 4.3 implies that the kernel of the middle vertical map is contained in $H^1_{fl}(U, \mathcal{A}^\Gamma)(\ell)_{\ell-\text{div}}$. Now Lemma 4.5 and the finiteness of $D^1(U, \mathcal{A}^\Gamma)(\ell)$ show that the left-hand vertical map in the above diagram is injective. To complete the proof, exchange the roles of $A$ and $B$. □

Corollary 4.8. Assume that $\mathbb{I}^1(A)$ and $\mathbb{I}^1(B)$ are finite. Then there exists a perfect pairing of finite groups

$$D^1(U, \mathcal{A}^\circ) \times D^1(U, \mathcal{B}) \to \mathbb{Q}/\mathbb{Z}.$$ 

Proof. Take $\Gamma = 0$ and $\Gamma' = \Phi'$ in the theorem and let $\ell$ vary. □
Theorem 4.9. Assume that $\mathfrak{III}^1(A)$ and $\mathfrak{III}^1(B)$ are finite. Then there exists a perfect pairing of finite groups

$$C_{A,F,S} \times C_{B,F,S}^1 \to \mathbb{Q}/\mathbb{Z},$$

where $C_{B,F,S}^1$ is the group (3.7) associated to $B$.

Proof. By the previous corollary and the existence of the perfect “Cassels-Tate pairing” $\mathfrak{III}^1(A) \times \mathfrak{III}^1(B) \to \mathbb{Q}/\mathbb{Z}$ (H, Corollary 4.9, and 8, Corollary 6.7), the dual of the second exact sequence appearing in Proposition 3.4 for $B$ is an exact sequence

$$0 \to (C_{B,F,S}^1)^D \to D^1(U, \mathcal{A}^\circ) \to \mathfrak{III}^1(A) \to 0.$$ 

The compatibility of the Cassels-Tate pairing with the pairings (4.3) and (4.4) for $\Gamma = 0$ and $\Gamma' = \Phi'$ (see 5, Appendix) implies that the third map in the previous exact sequence is the same as that appearing in the first exact sequence of Proposition 3.4 for $A$. Thus there exists an exact commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & C_{A,F,S} & \longrightarrow & D^1(U, \mathcal{A}^\circ) & \longrightarrow & \mathfrak{III}^1(A) & \longrightarrow & 0 \\
0 & \longrightarrow & (C_{B,F,S}^1)^D & \longrightarrow & D^1(U, \mathcal{A}^\circ) & \longrightarrow & \mathfrak{III}^1(A) & \longrightarrow & 0,
\end{array}$$

and this yields the result. \hfill \Box

Remarks 4.10. (a) Assume that $\mathfrak{III}^1(A)$, and therefore also $\mathfrak{III}^1(B)$, is finite. Then Proposition 3.4 yields an exact sequence of finite groups

$$0 \to C_{A,F,S} \to D^1(U, \mathcal{A}^\circ) \to D^1(U, \mathcal{A}) \to C_{A,F,S}^1 \to 0$$

whose dual, by Corollary 4.8 and the theorem, is the analogous exact sequence for $B$, i.e.,

$$0 \to C_{B,F,S} \to D^1(U, \mathcal{B}^\circ) \to D^1(U, \mathcal{B}) \to C_{B,F,S}^1 \to 0.$$ 

(b) Since $C_{A,F,S}$ is known to be finite, it is reasonable to question the necessity of the finiteness assumption in the theorem. In connection with this, W.McCallum [14], Proposition 5.7, showed (independently of the finiteness assumption on $\mathfrak{III}^1(A)$) the existence of a nondegenerate pairing

$$H^1_d(U, \mathcal{A}^\Gamma)/\text{div} \times H_c^1(U, \mathcal{B}^\Gamma_{\text{tors}})/\text{div} \to \mathbb{Q}/\mathbb{Z}.$$ 

See also 16, Theorems III.3.7, p.317, and III.9.4, p.370. Now, passing to the inverse limit over $r$ in the proof of Lemma 4.5 and using McCallum’s result, one can show (again, independently of the finiteness assumption on $\mathfrak{III}^1(A)$) that there exists a nondegenerate pairing of finite groups

$$D^1(U, \mathcal{A}^\Gamma)(\ell)/\ell\text{-div} \times D^1(U, \mathcal{B}^\Gamma_{\text{tors}})(\ell)/\ell\text{-div} \to \mathbb{Q}/\mathbb{Z}$$

provided $\ell \neq p = \text{char } F$ in the function field case (the problem is that the inverse limit over $r$ of the bottom row of the big diagram in the proof of
Lemma 4.5 might not be exact if $\ell = p$. However, even if the above statement were verified for $\ell = p$ as well, one would still need to check that $C_{A,F,S} \cap D^1(U, \omega^0)_{\mathrm{div}} = 0$. So far, we have been unable to make any progress on this problem.

References

[1] Bosch, S., Lütkebohmert, W. and Raynaud, M. Néron Models. Springer Verlag, Berlin 1989.
[2] Conrad, C. Finiteness theorems for algebraic groups over function fields. Available at http://math.stanford.edu/~conrad/papers/cosetfinite.pdf
[3] Conrad, C. Weil and Grothendieck approaches to adelic points. Available at http://math.stanford.edu/~conrad/papers/adelictop.pdf
[4] Gamst, J. and Hoechsmann, K. Products in sheaf cohomology. Tôhoku Math. J. 22 (1970), 143-162.
[5] González-Avilés, C.D. Brauer groups and Tate-Shafarevich groups. J. Math. Sci. Univ. Tokyo 10 (2003), 391-419.
[6] González-Avilés, C.D. Chevalley’s ambiguous class number formula for an arbitrary torus. Math. Res. Lett. 15 (2008), no. 6, 1149-1165.
[7] González-Avilés, C.D. On Néron-Raynaud class groups of tori and the Capitulation Problem. Available at http://arxiv.org/abs/0903.3221
[8] González-Avilés, C.D. Arithmetic duality theorems for 1-motives over function fields. J. reine angew. Math. 632 (2009), 203-231.
[9] Grothendieck, A. Groupes de Monodromie en Géométrie Algébrique I. Séminaire de Géométrie Algébrique du Bois Marie 1967-69 (SGA 7 I). Lecture Notes in Math. 288, Springer, Berlin-Heidelberg-New York, 1972.
[10] Grothendieck, A., et al. Schémas en groupes I. Séminaire de Géométrie Algébrique du Bois Marie 1962-64 (SGA 3). Lecture Notes in Math. 151, Springer, Berlin-Heidelberg-New York, 1970.
[11] Harari, D. and Szamuely, T.: Arithmetic duality theorems for 1-motives. J. reine angew. Math. 578 (2005), 93-128, and Errata: J. reine angew. Math. 632 (2009), 233-236.
[12] Hewitt, E. and Ross, K.: Abstract Harmonic Analysis, vol. I. Academic Press, Inc., New York, 1963.
[13] Lang, S.: Algebraic groups over finite fields. Amer. J. Math. 78 (1956), 555-563.
[14] McCallum, W.: Duality theorems for Néron models. Duke Math. J. 53 (1986), 1093-1124.
[15] Milne, J.S.: Étale cohomology. Princeton University Press, Princeton, 1980.
[16] Milne, J.S.: Arithmetic Duality Theorems. Academic Press Inc., Orlando, 1986.
[17] Nisnevich, Ye. Étale Cohomology and Arithmetic of Semisimple Groups. Thesis, Harvard University, 1982.
[18] Nisnevich, Ye. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987). NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 279, pp. 241–342. Kluwer Acad. Publ., Dordrecht, 1989.
[19] Serre, J.-P. Local Fields. Grad. Texts in Math. 67, Springer-Verlag, 1979.
[20] Tamme, G. Introduction to Étale Cohomology. Springer-Verlag, Berlin, 1994.
[21] Weibel, C. An introduction to homological algebra. Cambridge Univ. Press, 1994.

*Of course, we believe that the above statement remains valid if $l = p$, but we see no obvious variant of the proof of Lemma 4.5 that will prove this.
