On rainbow antimagic coloring of special graphs

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Abstract. Let \( G(V, E) \) be a connected, undirected and simple graph with vertex set \( V(G) \) and edge set \( E(G) \). A labeling of a graph \( G \) is a bijection \( f \) from \( V(G) \) to the set \( \{1, 2, ..., |V(G)|\} \). The bijection \( f \) is called rainbow antimagic vertex labeling if for any two edge \( uv \) and \( u'v' \) in path \( x - y \), \( w(uv) \neq w(u'v') \) \( w(uv) = f(u) + f(v) \) and \( x, y \in V(G) \). A graph \( G \) is a rainbow antimagic connection if \( G \) has a rainbow antimagic labeling. Thus any rainbow antimagic labeling induces a rainbow coloring of \( G \) where the edge \( uv \) is assigned with the color \( w(uv) \). The rainbow antimagic connection number of \( G \), denoted by \( rac(G) \), is the smallest number of colors taken over all rainbow colorings induced by rainbow antimagic labeling of \( G \).

In this paper, we show the exact value of the rainbow antimagic connection number of jahangir graph \( J_{2,m} \), lemon graph \( L_e m \), firecracker graph \( F_{m,3} \), complete bipartite graph \( K_{2,m} \), and double star graph \( (S_{m,m}) \).

1. Introduction

Graph \( G \) is an infinite set \( V(G) \) whose elements are called vertices, along with a set \( E(G) \) whose elements are called edges, which are an unsorted pair of two different elements from \( V(G) \). We follow Chartrand et al. [1] for graph theory terminology. The graph used in this paper is a connected, undirected and simple graph with the set of vertices \( V(G) \) and the set of edges \( E(G) \).

The most basic thing in graph theory is connectivity. In graph theory many useful results have been found about connectivity. One of which was discovered by Chartrand et al. [2] that is rainbow connection. Rainbow connections can be applied in the field of sending secret messages between agents. Suppose that the graph \( G \) is a nontrivially connected graph and the edge coloring in \( G \) is defined as \( c : E(G) \rightarrow \{1, 2, ..., k\} \) so that two adjacent edges may have the same color. An \( x - y \) path is called a rainbow path on \( G \), if no two edge of the \( x - y \) path are the same color. Graph \( G \) is called a rainbow connection, if for every vertex \( x, y \in V(G) \) there is a rainbow path \( x - y \). The edge coloring that causes \( G \) to have a rainbow connection is called rainbow coloring. The minimum integer \( k \) in order to make \( G \) rainbow-connected is called the rainbow connection number of \( G \) and denoted by \( rc(G) \).

Chartrand et al. [2] conducted research on rainbow connections on several graphs, namely cycle graphs, complete graphs, tree graphs and wheel graphs. Kemnitz et al. [3] conducted research on Graph with Rainbow connection number two. Li et al. [4] found rainbow connection...
results on graphs with diameter 2. Syafrizal et al. [5] conducted rainbow connection research on several graphs, namely gear graphs, book graphs, fan graphs, and sun graphs. Li et al. [6] conducted research on Rainbow connection in 3-connected graphs and Li et al. [7] found rainbow connections on graphs with diameter 3.

The graph $G$ is strongly rainbow connected if there exists a rainbow $u - v$ geodesic for any two vertices $u$ and $v$ in $G$. In this case, the coloring $c$ is called a strong rainbow coloring of $G$. Similarly, we define the strong rainbow connection number of a connected graph $G$, denoted $src(G)$, as the smallest number of colors that are needed in order to make $G$ strong rainbow connected. Results of strong rainbow connection have been found by X. Li et al. [8] on $m$-splitting of the complete graph $K_n$ and Shulhany et al. [9] have found results in stellar graphs.

Chartrand et al. [10] defined the rainbow $k$-connectivity $rc_k(G)$ of $G$ to be the minimum integer $j$ for which there exists a $j$-edge-coloring of $G$ such that for every two distinct vertices $u$ and $v$ of $G$, there exist at least $k$ internally disjoint $u - v$ rainbow paths. The results of rainbow $k$-connection have been found by X. Li et al. [11] in complete graphs. Fujita et al. [12] found results on dense graphs and Agustin et al. [13] found results on special graphs and it was sharp lower bound.

Suppose the graph $G$ is a nontrivially connected graph and the dot coloring in $G$ is defined as $c : V(G) \rightarrow \{1, 2, ..., k\}$. The point coloring of the graph $G$ is a rainbow vertex-connection if for every two points connected by a path whose internal points have a different color. Coloring the points that cause $G$ to be a rainbow vertex connection is called rainbow vertex coloring [14]. Results of rainbow vertex connection have been found by X. Li et al. [15] on graphs and Simamora et al. [16] found results on pencil graphs.

The vertex-colored graph $G$ is said strong rainbow vertex-connected, if every two vertices of $G$ are connected by at least one shortest rainbow path. The strong rainbow vertex-connection number, denoted by $src(G)$, is the minimum number of colors required to make a strongly rainbow vertex-connected graph $G$ [17]. The results of the strong rainbow connection have been found by Arputhamari et al. [18] in generalized Petersen graphs $G(n, 2)$ and $G(n, 3)$ and Dafik et al. [19] found results of graphs resulting edge comb products. Then found a rainbow vertex $k$-connection by H. Liu et al. [20] and have defined a rainbow vertex $k$-connection. A vertex-coloured path is vertex-rainbow if the internal vertices have distinct colours. A vertex-colouring of a $k$-connected graph $G$, not necessarily proper and possibly with uncoloured vertices, is rainbow vertex $k$-connected if any two vertices of $G$ are connected by $k$ disjoint vertex-rainbow paths. The rainbow vertex $k$-connection number of $G$, denoted by $rvc_k(G)$, is the minimum integer $t$ such that there exists a rainbow vertex $k$-connected colouring of $G$, using $t$ colours. We write $rvc(G)$ for $rvc_1(G)$. Again by Menger’s theorem, $rvc_k(G)$ is well defined if $G$ is $k$-connected.

Suppose the graph $G$ is a nontrivially connected graph and the total coloring in $G$ is defined as $c : V(G) \cup E(G) \rightarrow \{1, 2, ..., k\}$ so that two adjacent vertices, two adjacent edges, and the incident vertex on the edge may be the same color. A $x - y$ path with coloring $c$ connecting two points $x$ and $y$ is called the rainbow total path $x - y$ if all elements are in $V(G) \cup E(G)$ except $x$ and $y$, assigned a different color by $c$. The graph $G$ is called the total rainbow connection if for each pair of points $x, y \in V(G)$ there is a total path of the rainbow $x - y$. The total coloring that causes $G$ to be a total rainbow connection is called total rainbow coloring. The results of the total rainbow connection have been found by Sun et al. [21] on the graph, Hasan et al. [22] have found the results on wheel related graphs and Ma et al. [23] have found the results on some special graphs.

Strong total connection was later found by Chen et al. [24] and has made a definition of strong total connection. The graph $G$ is strongly total rainbow connected if any two vertices of $G$ are connected by a total rainbow geodesic. Such a coloring is called a strong total rainbow...
coloring of \( G \). The strong total rainbow connection number of \( G \), denoted by \( strc(G) \), is defined to be the smallest number of colors needed to make \( G \) strongly total rainbow connected.

A total-coloured path is total-rainbow if its edges and internal vertices have distinct colours. A total colouring of a \( k \)-connected graph \( G \), not necessarily proper, is total-rainbow \( k \)-connected if any two vertices of \( G \) are connected by \( k \) disjoint total-rainbow paths. The total rainbow \( k \)-connection number of \( G \), denoted by \( trc_k(G) \), is the minimum integer \( t \) such that, there exists a total-rainbow \( k \)-connected colouring of \( G \) using \( t \) colours \[24\]. Result of total rainbow \( k \)-connection has been found by W. li et al. \[26\] on a graph with hardness result. Y. sun et al. \[27\] found the results on the graph

Graph labeling is one of the big concepts in graph theory that is of interest to many mathematicians around the world. In Wallis \[25\], Labeling on a graph is a mapping that pairs vertex and edge elements into a positive integer with certain rules. Vertex labeling is labeling with the vertex set domain, if the domain is a set of edges then it is called edge labeling and if the domain is a combination of vertexts and edges, then the labeling is called total labeling. Antimagic labeling was introduced by Hartsfield and Ringel \[29\]. An antimagic labeling of a graph \( G \) with edges is a bijection from \( E(G) \) to \( \{1, 2, ..., |E(G)|\} \) such that for any vertices \( u \) and \( v \), the sum of labels on edges incident to \( u \) differs from that for edge incident to \( v \).

Hartsfield and Ringel \[29\] found the conjecture that every connected graph except \( K_2 \) is antimagic and every tree graph except \( K_2 \) is antimagic. Alone et al. \[30\] found the dense graph are antimagic, Cranston et al. \[31\] found that regular bipartite graphs are antimagic. Liang et al. \[33\] found antimagic labeling in tree graph and cubic graph. Liang and Zhu \[32\] found that cubic graphs are antimagic, then Cranston, et al. \[34\] found that regular graphs with odd degrees are antimagic. Chang et al. \[35\] found that the regular graph is antimagic.

Arunuğan et al. \[36\] introduces the concept of local antimagic vertex coloring of a graph as a combination of the concept of antimagic labeling and vertex coloring of a graph. Given a graph \( G \) with \( q \) edges, a bijection \( f : E(G) \to \{1, 2, ..., q\} \) is said to be a local antimagic labeling if for any two adjacent vertices \( u \) and \( v \), \( w_f(u) \neq w_f(v) \), where \( w_f(u) = \sum_{e \in E(u)} f(e) \). The local antimagic chromatic number of \( G \) is defined as the minimum number of colors taken over all colorings of \( G \) induced by local antimagic labelings of \( G \) and denoted by \( \chi_{la}(G) \).

2. The definition

Based on the description above about rainbow coloring and antimagic labeling, Dafik et al defined a new concept called rainbow antimagic coloring. Let \( G \) be a connected graph. A labeling of a graph \( G \) is a bijection \( f \) from \( V(G) \) to the set \( \{1, 2, ..., |V(G)|\} \). The bijection \( f \) is called rainbow antimagic vertex labeling if for any two edge \( uv \) and \( u'v' \) in path \( x-y \), \( w(uv) \neq w(u'v') \), where \( w(uv) = f(u) + f(v) \) and \( x, y \in V(G) \). A graph \( G \) is a rainbow antimagic connection if \( G \) has a rainbow antimagic labeling. Thus any rainbow antimagic labeling induces a rainbow coloring of \( G \) where the edge \( uv \) is assigned with the color \( w_f(uv) \). The rainbow antimagic connection number of \( G \), denoted by \( rac(G) \), is the smallest number of colors taken over all rainbow colorings induced by rainbow antimagic labeling of \( G \). In dafik et al. \[37\] it has been found that the lower bound for any connected graph and the result of rainbow antimagic coloring.

3. Results and Discussion

We have determined the rainbow antimagic connection number of jahangir graph \( J_{2,m} \), lemon graph \( Le_m \), firecracker \( (F_{m,3}) \), complete bipartite \( (K_{2,m}) \), and double star \( (S_{m,m}) \).

**Proposition 1.** For any connected graph \( G \), then \( rc_A(G) \geq rc(G) \).

**Proposition 2.** Let \( G \) be any connected graph of size \( m \), then \( rc(G) = m \) if and only if \( G \) is a tree.
Based on the rainbow connection, denote by $rc(G)$ and maximum degree of vertices of graphs, denote by $\Delta(G)$. We have the general lower bound of rainbow antimagic connection number for any connected graph.

**Theorem 1.** Let $G$ be any connected graph. Then, $rac(G) \geq \max\{rc(G), \Delta(G)\}$.

*Proof.* Based on Proposition 1, it is found that $rac(G) \geq rc(G)$. Let $x \in V(G)$ where $d(x) = \Delta(G)$ and $f : V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ is a bijection function. Since $f$ is a bijection function, then $f(u) \neq f(x)$, for every $ux \in E(G)$ so that for every $ux, vx \in E(G), w(ux) \neq w(vx)$. Therefore, $rac(G) \geq \Delta(G)$. Based on the description above, it is found that $rac(G) \geq \max\{rc(G), \Delta(G)\}$.

**Theorem 2.** For any integer $2 \leq m \leq 8$, $rac(J_{2,m}) = m$.

*Proof.* Let $V(J_{2,m}) = \{a\} \cup \{y_i; 1 \leq i \leq m\} \cup \{z_i; 1 \leq i \leq m\}$ and $E(J_{2,m}) = \{ay_i; 1 \leq i \leq m\} \cup \{yz_i; 1 \leq i \leq m\} \cup \{yz_{i+1}; 1 \leq i \leq m-1\} \cup \{ymz_1\}$. The cardinality of the vertex set of $J_{2,m}$ is $|V(J_{2,m})| = 2m - 1$ and the cardinality of the edges set $|E(J_{2,m})| = 3m$. From [38] it is found that $rc(J_{2,m}) \leq m$ and based on the definition of graph $J_{2,m}$ it is found that $\Delta(J_{2,m}) = m$. Based on Lemma 1, it is obtained $rac(G) \geq \max\{rc(J_{2,n}), \Delta(J_{2,n})\}$.

Therefore, $rac(J_{2,n}) \geq m$. Let $f : V(J_{2,m}) \rightarrow \{1, 2, \ldots, 2m - 1\}$ defined as follows.

$$f(a) = \begin{cases} m, & \text{if } m \text{ is odd,} \\ m + 1, & \text{if } m \text{ is even,} \end{cases}$$

$$f(y_i) = \begin{cases} i + 1, & \text{if } i = 1, \\ 2i + 2, & \text{if } 2 \leq i \leq m - 1, \\ 4, & \text{if } i = m, \end{cases}$$

$$f(z_i) = \begin{cases} 2m - 2i + 3, & 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil \\ 2m - 2i - 1, & \left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m - 1, \end{cases}$$

$$f(z_m) = \begin{cases} i - 2, & \text{if } m \text{ is odd,} \\ i - 1, & \text{if } m \text{ is even,} \end{cases}$$

For the edge weights that induces edges coloring of the graph $J_{2,m}$, it is obtained:

$$w_f(ay_1) = \begin{cases} m + i + 1, & \text{if } m \text{ is odd,} \\ m + i + 2, & \text{if } m \text{ is even,} \end{cases}$$

$$w_f(ay_i) = \begin{cases} m + 2i + 2, & \text{if } m \text{ is odd, } 2 \leq i \leq m - 1, \\ m + 2i + 3, & \text{if } m \text{ is even, } 2 \leq i \leq m - 1, \end{cases}$$

$$w_f(ay_m) = \begin{cases} m + 4, & \text{if } m \text{ is odd,} \\ m + 5, & \text{if } m \text{ is even,} \end{cases}$$

$$f(w_f(y_1z_1)) = 2m + 3$$

$$f(w_f(y_i,z_i)) = \begin{cases} 2m + 5, & 2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil, \\ 2m + 1, & \left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m - 1, \end{cases}$$

$$f(w_f(y_mz_m)) = 2m - 3,$$

$$f(w_f(y_i,z_{i+1})) = 2m + 1.$$
\[
f(w_f(y, z_{i+1})) = \begin{cases} 
2m + 3, & \text{if } 2 \leq i \leq m - 2, i \neq \left\lceil \frac{m}{2} \right\rceil \\
2m - 1, & \text{if } i = \left\lceil \frac{m}{2} \right\rceil,
2m + 2i - 5, & \text{if } i = m - 1,
\end{cases}
\]

\[
w_f(y_m z_1) = 2m + 5
\]

Next, it is evaluated that the \( J_{2,n} \) edge coloring is a rainbow antimagic connection. Suppose that, taken \( x, y \in V(J_{2,n}) \), there are two possible \( x, y \), namely: Suppose \( xy \in E(J_{2,n}) \), then based on [2] the path \( xy \) is the rainbow path and \( xy \notin E(J_{2,n}) \) is shown to be 15 cases as shown in Table 1.

| Case | \( y \) | \( z \) | rainbow path | condition |
|------|---------|---------|--------------|-----------|
| 1    | \( a \) | \( z_1 \) | \( a, y_1, y_1 \) | \( m \) is even |
| 2    | \( a \) | \( z_i \) | \( a, y_i, z_i \) | \( m \) is even, \((2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil)\)|
| 3    | \( a \) | \( z_i \) | \( a, y_i, z_i \) | \( m \) is even, \((\left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m - 1)\)|
| 4    | \( a \) | \( z_m \) | \( a, y_m, z_m \) | \( m \) is even |
| 5    | \( a \) | \( z_1 \) | \( a, y_1, z_1 \) | \( m \) is odd |
| 6    | \( a \) | \( z_i \) | \( a, y_i, z_i \) | \( m \) is odd, \((2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil)\)|
| 7    | \( a \) | \( z_i \) | \( a, y_i, z_i \) | \( m \) is odd, \((\left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m - 1)\)|
| 8    | \( a \) | \( z_m \) | \( a, y_m, z_m \) | \( m \) is odd |
| 9    | \( y_1 \) | \( z_i \) | \( y_1, a, y_i, z_i \) | \((3 \leq i \leq \left\lceil \frac{m}{2} \right\rceil)\)|
| 10   | \( y_1 \) | \( z_i \) | \( y_1, a, y_i, z_i \) | \((\left\lceil \frac{m}{2} \right\rceil \leq i \leq m - 1)\)|
| 11   | \( y_1 \) | \( z_m \) | \( y_1, z_1, y_m, z_m \) | - |
| 12   | \( z_i \) | \( z_j \) | \( z_i, y_i, z_j, y_j \) | - |
| 13   | \( z_i \) | \( z_j \) | \( z_i, y_i - 1, a, y_j, z_j \) | - |
| 14   | \( z_i \) | \( z_j \) | \( z_i, y_i - 1, a, y_j, y_j \) | - |
| 15   | \( z_i \) | \( z_j \) | \( z_i, y_i, a, y_j, z_j \) | - |

Therefore, the \( m \)-edge coloring of \( J_{2,m} \) is a rainbow antimagic connection, so we get \( \text{rac}(J_{2,m}) \leq m \). Based on the description above, it is found that \( \text{rac}(J_{2,m}) = m \). □

For an illustration, a rainbow antimagic coloring of \((J_{2,n})\) is depicted in Figure 1.

**Theorem 3.** For any integer \( m \geq 3 \), \( \text{rac}(Le_m) = n \).

**Proof.** Let \( V(Le_m) = \{a\} \cup \{y_i; 1 \leq i \leq m\} \cup \{z_i; 1 \leq i \leq m\} \) and \( E(Le_m) = \{ay_i; 1 \leq i \leq m\} \cup \{az_i; 1 \leq i \leq m\} \cup \{yi; 1 \leq i \leq m\} \cup \{yi, y_{i+1}; 1 \leq i \leq m - 1\} \cup \{ym, y_1\} \cup \{zi, z_{i+1}; 1 \leq i \leq m - 1\} \cup \{z_m, z_1\} \). The cardinality of the vertex set \( Le_m \) is \(|V(Le_m)| = 2m + 1\) and the cardinality of the edges set is \(|E(Le_m)| = 6m\). From [39] it is found that \( \text{rc}(Le_m) \leq 2m \) and based on the definition of graph \( Le_m \) it is found that \( \Delta(Le_m) = 2m \). Based on Lemma 1, it is obtained \( \text{rac}(G) \geq \max\{\text{rc}(Le_m), \Delta(Le_m)\} = 2m \). Therefore, \( \text{rac}(Le_m) \geq 2m \).

Let \( f : V(Le_m) \rightarrow \{1, 2, ..., 2m + 1\} \) defined as follows.

\[
f(a) = m + 1
\]

If \( m \) is odd

\[
f(y_i) = 2m - \left\lceil \frac{i}{2} \right\rceil - 3
\]
Figure 1. Rainbow antimagic coloring of jahangir graph $J_{2,6}$

\[
f(z_i) = i, \\
\text{If } m \text{ is even} \Rightarrow f(y_i) = i, \\
\]

\[
f(z_i) = 2m - \left\lceil \frac{i}{2} \right\rceil + 2
\]

For the edge weights that induces edges coloring of the graph $J_{2,m}$, it is obtained:

\[
\text{If } m \text{ is odd} \\
\]

\[
w_f(ay_i) = 3m - \left\lceil \frac{i}{2} \right\rceil - 2 \\
w_f(az_i) = m + i + 1 \\
w_f(y_iz_i) = 2m - \left\lceil \frac{i}{2} \right\rceil + i - 3 \\
w_f(y_iy_{i+1}) = 2m + \left\lceil \frac{i}{2} \right\rceil - 3 \\
w_f(y_my_1) = 2m + \left\lceil \frac{m}{2} \right\rceil + 1 \\
w_f(z_iz_{i+1}) = 2m + \left\lceil \frac{i}{2} \right\rceil + 1 \\
w_f(z_my_1) = m + 1
\]

\[
\text{If } m \text{ is even} \\
w_f(ay_i) = m + i + 1 \\
w_f(az_i) = 3m - \left\lceil \frac{i}{2} \right\rceil + 3
\]
\[ w_f(y_iz_i) = 2m - \left\lfloor \frac{i}{2} \right\rfloor + i + 2 \]

\[ w_f(y_iy_{i+1}) = 2m + \left\lceil \frac{i}{2} \right\rceil - 4 \]

\[ w_f(y_my_1) = 2m + \left\lceil \frac{m}{2} \right\rceil + 3 \]

\[ w_f(z_iz_{i+1}) = 2m + \left\lceil \frac{i}{2} \right\rceil + 3 \]

\[ w_f(z_mz_1) = m + \left\lceil \frac{m}{2} \right\rceil + 3 \]

Next, it is evaluated that the \( L_{em} \) edge coloring is a rainbow antimagic connection. Suppose that, taken \( x, y \in V(L_{em}) \), there are two possible \( x, y \), namely: Suppose \( xy \in E(L_{em}) \), then based on [2] the path \( xy \) is the rainbow path and \( xy \notin E(L_{em}) \) is shown to be 15 cases as shown in Table 2.

| Case | \( y \) | \( z \) | Rainbow Path |
|------|------|------|--------------|
| 1    | \( y_i \) | \( y_j \) | \( y_i, a, y_j \) |
| 2    | \( y_i \) | \( z_j \) | \( y_i, a, z_j \) |
| 3    | \( z_i \) | \( z_j \) | \( z_i, a, z_j \) |

Therefore, the \( 2m \)-edge coloring of \( L_{em} \) is a rainbow antimagic connection, so we get \( rac(L_{em}) \leq 2m \). Based on the description above, it is found that \( rac(L_{em}) = 2m \).

For an illustration, a rainbow antimagic coloring of \( L_{em} \) is depicted in Figure 2.

\[ \text{Figure 2. Rainbow antimagic coloring of lemon graph } L_8 \]

**Theorem 4.** For any integer \( m \geq 3 \), \( rac(F_{m,3}) = 3m - 1 \).
Proof. Let $V(F_{m,3}) = \{x_i, y_i, z_i, 1 \leq i \leq m\}$ and $E(F_{m,3}) = \{x_i x_{i+1}, 1 \leq i \leq m-1\} \cup \{x_i y_i, 1 \leq i \leq m\} \cup \{y_i z_i, 1 \leq i \leq m\}$. The cardinality of the vertex set $F_{m,3}$ is $|V(F_{m,3})| = 3m$ and the cardinality of the edge set is $|E(F_{m,3})| = 3m - 1$. Based on the definition of the graph $F_{m,3}$, it is obtained $\Delta(F_{m,3}) = m + 1$, and from [40] it is found that $(F_{m,3}) \leq rc(F_{m,3}) = 3m - 1$. Based on Lemma 1., it is obtained that $rac(G) \geq \max\{rc(F_{m,3}), \Delta(F_{m,3})\} = 3m - 1$. Therefore, $rac(F_{m,3}) \geq 3m - 1$.

Let $f : V(F_{m,3}) \rightarrow \{1, 2, ..., 3m\}$ defined as follows. If $n \equiv 1 \mod 2$

$$f(x_i) = i \quad \text{if } 1 \leq i \leq m$$

$$f(y_i) = 2n + i \quad \text{if } 1 \leq i \leq m$$

$$f(z_i) = n + i \quad \text{if } 1 \leq i \leq m$$

If $n \equiv 0 \mod 2$

$$f(x_i) = i \quad \text{if } 1 \leq i \leq m$$

$$f(y_i) = n + i \quad \text{if } 1 \leq i \leq m$$

$$f(z_i) = 2n + i \quad \text{if } 1 \leq i \leq m$$

For the edge weights that induces edges coloring of the graph $F_{m,3}$, it is obtained:

For $n \equiv 1 \mod 2$

$$w(x_i x_{i+1}) = 2i + 1, \quad \text{for } 1 \leq i \leq m - 1$$

$$w(x_i y_i) = 2n + 2i, \quad \text{for } 1 \leq i \leq m$$

$$w(y_i z_i) = 3n + 2i, \quad \text{for } 1 \leq i \leq m$$

For $n \equiv 0 \mod 2$

$$w(x_i x_{i+1}) = 2i + 1, \quad \text{for } 1 \leq i \leq m - 1$$

$$w(x_i y_i) = n + 2i, \quad \text{for } 1 \leq i \leq m$$

$$w(y_i z_i) = 3n + 2i, \quad \text{for } 1 \leq i \leq m$$

Next, it is evaluated that the $F_{m,3}$ edge coloring is a rainbow antimagic connection. Suppose that, taken $x, y \in V(F_{m,3})$, there are two possible $x, y$, namely: Suppose $xy \in E(F_{m,3})$, then based on [2] the path $xy$ is the rainbow path and $xy \notin E(F_{m,3})$ is shown to be 9 cases as shown in Table 3.

| Case | $y$ | $z$ | rainbow path |
|------|-----|-----|--------------|
| 1    | $x_i$ | $x_j$ | $x_i, x_{i+1}, \ldots, x_j$ |
| 2    | $x_i$ | $y_j$ | $x_i, x_{i+1}, \ldots, x_j, y_j$ |
| 3    | $x_i$ | $z_j$ | $x_i, x_{i+1}, \ldots, x_j, y_j, z_j$ |
| 4    | $y_i$ | $x_j$ | $y_i, x_i, x_{i+1}, \ldots, x_j$ |
| 5    | $y_i$ | $y_j$ | $y_i, x_i, x_{i+1}, \ldots, x_j, y_j$ |
| 6    | $y_i$ | $z_j$ | $y_i, x_i, x_{i+1}, \ldots, x_j, y_j, z_j$ |
| 7    | $z_j$ | $x_j$ | $z_i, y_i, x_i, x_{i+1}, \ldots, x_j$ |
| 8    | $z_j$ | $y_j$ | $z_i, y_i, x_i, x_{i+1}, \ldots, x_j, y_j$ |
| 9    | $z_j$ | $z_j$ | $z_i, y_i, x_i, x_{i+1}, \ldots, x_j, y_j, z_j$ |

Therefore, the $3m - 1$-edge coloring of $F_{m,3}$ is a rainbow antimagic connection, so we get $rac(F_{m,3}) \leq 3m - 1$. Based on the description above, it is found that $rac(F_{m,3}) = 3m - 1$. \( \square \)
For an illustration, a rainbow antimagic coloring of \((F_{m,3})\) is depicted in Figure 3.

![Figure 3. Rainbow antimagic coloring of firecraker graph \(F_{2,6}\)](image)

**Theorem 5.** For any integer \(m \geq 3\), \(\text{rac}(K_{2,m}) = m + 1\).

**Proof.** Let \(V(K_{2,m}) = \{a\} \cup \{b\} \cup \{x_i, 1 \leq i \leq m\}\) and \(E(K_{2,m}) = \{ax_i, 1 \leq i \leq m\} \cup \{bx_i, 1 \leq i \leq m\}\). The cardinality of the vertex set \(K_{2,m}\) is \(|V(K_{2,m})| = m + 2\) and the cardinality of the edge set is \(|E(K_{2,m})| = 2m\). Based on the definition of the graph \(K_{2,m}\), we have \(N(a) = N(b) = \{x_i : 1 \leq i \leq m - 1\}, ab \in E(K_{2,m}), d(a) = \Delta(K_{2,m}), d(b) = \Delta(K_{2,m})\) and \(\Delta(K_{2,m}) = m\). The function \(f : V(G) \to \{1, 2, \ldots, |V|\}\) is a bijection function. Let \(f(a) = 1\) and \(f(b) = 2\). Since \(f\) is a bijection function, then \(f(x) \neq f(a)\) and \(f(x) \neq f(b)\), for each \(xa, xb \in E(G)\) so that for every \(xa, yes, xb, yb \in E(G)\), \(w(xa) \neq w(ya), w(xb) \neq w(yb)\) and \(w(xa) \neq w(xb)\). Based on the description above, it is found that \(\text{rac}(K_{2,m}) \geq \Delta(G) + 1\).

Let \(f : V(K_{2,m}) \to \{1, 2, \ldots, m + 2\}\) define as follows.

\[
\begin{align*}
f(a) & = 1 \\
f(b) & = 2 \\
f(x_i) & = i + 2, \quad \text{if } 1 \leq i \leq m
\end{align*}
\]

For the edge weights that induces edge coloring of the graph \(K_{2,m}\), it is obtained:

\[
\begin{align*}
w(ab) & = 3 \\
w(ax_i) & = i + 3, \quad \text{if } 1 \leq i \leq m \\
w(bx_i) & = i + 4, \quad \text{if } 1 \leq i \leq m
\end{align*}
\]

Furthermore, it is evaluated that the \(K_{2,m}\) edge coloring is a rainbow antimagic connection. Suppose that any \(x, y \in V(K_{2,m})\) is taken, there are two possible \(x, y\), namely: Suppose \(xy \in E(K_{2,m})\), then based on [2] the path \(x - y\) is the path of the rainbow and \(xy \notin E(K_{2,m})\), if the path \(x - y = x_i, a, y_i\), then based on [2] the \(x - y\) path is the rainbow path.

Therefore, the \(m + 2\)-edge coloring of \(K_{2,m}\) is a rainbow antimagic connection, so we get \(\text{rac}(K_{2,m}) \leq m + 2\). Based on the description above, it is found that \(\text{rac}(K_{2,m}) = m + 2\).

For an illustration, a rainbow antimagic coloring of \((K_{2,m})\) is depicted in Figure 4.

**Theorem 6.** For any integer \(m \geq 4\), \(\text{rac}(S_{m,m}) = 2m + 1\).
Proof. Let $V(S_{m,m}) = \{p\} \cup \{q\} \cup \{x_i, y_i, 1 \leq i \leq m\}$ and $E(S_{m,m}) = \{pq\} \cup \{px_i, 1 \leq i \leq m\} \cup \{qy_i, 1 \leq i \leq m\}$. Cardinality of the vertex set $S_{m,m}$ is $|V(S_{m,m})| = 2m + 2$ and the cardinality of the edge set is $|E(S_{m,m})| = 2m + 1$. Based on the definition of the graph $S_{m,m}$, it is obtained $\Delta(S_{m,m}) = m + 1$, and from [10] it is found that $\Delta(S_{m,m}) \leq rc(S_{m,m}) = 2m + 1$. Based on Lemma 1., it is obtained that $rac(G) \geq \max\{rc(S_{m,m}), \Delta(S_{m,m})\} = 2m + 1$. Therefore, $rac(S_{m,m}) \geq 2m + 1$.

Let $f : V(S_{m,m}) \rightarrow \{1, 2, ..., 2m + 2\}$ defined as follows.

$$f(p) = 1$$

$$f(q) = 2$$

$$f(x_i) = i + 2, \text{ for } 1 \leq i \leq m$$

$$f(y_i) = m + i + 2, \text{ for } 1 \leq i \leq m$$

For the edge weights that induces edges coloring of the graph $S_{m,m}$, it is obtained:

$$w(px_i) = a + 3, \text{ for } 1 \leq i \leq m$$

$$w(qy_i) = 2a + 3, \text{ for } 1 \leq i \leq m$$

$$w(pq) = 3$$

Next, it is evaluated that the $S_{m,m}$ edge coloring is a rainbow antimagic connection. Suppose that, taken $x, y \in V(S_{m,m})$, there are two possible $x, y$, namely: Suppose $xy \in E(S_{m,m})$, then based on [2] the path $xy$ is the rainbow path and $xy \notin E(S_{m,m})$ is shown to be 9 cases as shown in Table 4.

Therefore, the $2m + 1$-edge coloring of $S_{m,m}$ is a rainbow antimagic connection, so we get $rac(S_{m,m}) \leq 2m + 1$. Based on the description above, it is found that $rac(S_{m,m}) = 2m + 1$. □

For an illustration, a rainbow antimagic coloring of $(S_{m,m})$ is depicted in Figure 5.

![Figure 4](image-url)
Table 4. $y-z$ rainbow path in lemon graph $F_{m,3}$

| Case | $y$  | $z$  | rainbow path          |
|------|------|------|-----------------------|
| 1    | $x_i$| $x_j$| $x_i, p, x_j$         |
| 2    | $x_i$| $y_j$| $x_i, p, q, y_j$      |
| 3    | $y_i$| $y_j$| $y_i, q, y_j$         |

Figure 5. Rainbow antimagic coloring of double star graph graph $S_{7,7}$

4. Conclusion
In this paper we have found the results of rainbow antimagic coloring of several graphs. The rainbow antimagic connection number of jahangir graph $J_{2,m}$ is $rac(J_{2,m}) = m$. The rainbow antimagic connection number of lemon graph $Le_m$ is $rac(Le_m) = m + 2$. The rainbow antimagic connection number of firecracker $(F_{m,3})$ is $rac(F_{m,3}) = 3m - 1$. The rainbow antimagic connection number of complete bipartite $(K_{2,m})$ is $rac(K_{2,m}) = m + 1$. The rainbow antimagic connection number of double star $(S_{m,m})$ is $rac(S_{m,m}) = 2m + 1$.

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