Doubly Periodic Self-Dual Vortices for a Relativistic Non-Abelian Chern–Simons Model

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Abstract. In this paper we establish a multiplicity result concerning the existence of doubly periodic solutions for a $2 \times 2$ nonlinear elliptic system arising in the study of self-dual non-Abelian Chern–Simons vortices. We show that the given system admits at least two solutions when the Chern–Simons coupling parameter $\kappa > 0$ is sufficiently small; while no solutions exist for $\kappa > 0$ sufficiently large. As in [36], we use a variational formulation of the problem. Thus, we obtain a first solution via a (local) minimization method and show that it is asymptotically gauge-equivalent to the (broken) principal embedding vacuum of the system, as $\kappa \to 0$. Then we obtain the second solution by a min-max procedure of “mountain pass” type.

1 Introduction

As well known vortices play an important role in many areas of physics, including superconductivity [1,19,27], optics [5], cosmology [21,28,50], the quantum Hall effect [40], and quark confinement [23,24,33–35]. After the pioneer work of Bogomol’nyi [6] and Prasad–Sommerfield [39], rigorous mathematical results about the existence of vortices have been pursued in various self-dual gauge field theories on the basis of an analytical approach that Taubes introduced in [49] to treat the Abelian–Higgs model. Indeed following [49], one is able to reduce the vortex problem to second order elliptic equations with exponential nonlinearity and Dirac source terms. Within this framework we mention for example the $(2 + 1)$-dimensional abelian Chern–Simons model of Hong–Kim–Pac [22] and Jackiw–Weinberg [26], for which Taubes’ approach has lead to the existence of topological multivortices (as described in [41,51]), non-topological multivortics (as constructed in [8,9,11,13,42]) and doubly periodic vortices (as given in [7,14,15,30,37,46,48]). In the non-Abelian context, rigorous existence results are established in [4,10,13,44], while a series of sharp existence results have been obtained in [12,29,31,32,47] for non-Abelian models proposed in connection with the quark confinement phenomenon [23,24,33,34]. For more results about self-dual vortices, we refer the readers to the monographs [45,54].

Here, we are going to analyze a relativistic (self-dual) non-Abelian Chern–Simons model proposed by Dunne in [16,17]. For this model, Yang [53] first established the existence of topological solutions in a very general situation. Subsequently, for the gauge group $SU(3)$, Nolasco and Taran-
tello \cite{36} proved a multiplicity result about the existence of doubly periodic vortices. The purpose of this paper is to establish analogous multiplicity results for theories that involve more general gauge groups. More precisely, we focus on gauge groups with a semi-simple Lie algebra of rank 2.

From the technical point of view, we need to handle a $2 \times 2$ nonlinear elliptic system on the flat 2–torus, with coupling matrix given by the Cartan matrix associated to the gauge group. Clearly, this more general situation poses new analytical difficulties compared to the (already nontrivial) case analyzed in \cite{36}, where the authors handle a (specific) symmetric $2 \times 2$ system. Actually, we manage to resolve such difficulties for a larger class of $2 \times 2$ systems, where our vortex problem is included as a particular case.

2 Derivation of a general $2 \times 2$ nonlinear elliptic system and statement of the main results

The non-Abelian Chern–Simons model introduced by Dunne in \cite{16, 17}, is formulated over the $\mathbb{R}^{1+2}$-Minkowski space with metric tensor: diag$(1, -1, -1)$, that will be used in the usual way to rise and lower indices. Using the summation convention over repeated lower and upper indices (ranging over 0, 1, 2), we consider the Lagrangian density:

$$L = -\kappa \frac{1}{2} \text{Tr} \epsilon^{\mu \nu \alpha} \left( \partial_\mu A_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha \right) + \text{Tr} \left( [D_\mu \phi]^\dagger [D^\mu \phi] \right) - V(\phi, \phi^\dagger), \quad (2.1)$$

where $D_\mu = \partial_\mu + [A_\mu, \cdot]$ is the gauge-covariant derivative applied to the Higgs field $\phi$ in the adjoint representation of the gauge group $G$. The associated semi-simple Lie algebra is denoted by $\mathcal{G}$, with $[\cdot, \cdot]$ the corresponding Lie bracket. Moreover, $(A_\mu)_{\mu=0,1,2}$ denotes the $\mathcal{G}$-valued gauge fields and $\text{Tr}$ refers to the trace in the matrix representation of $\mathcal{G}$. As usual, we denote by $\kappa > 0$ the Chern–Simons coupling parameter, $\epsilon^{\mu \nu \alpha}$ the Levi–Civita totally skew-symmetric tensor with $\epsilon^{012} = 1$ and we let $V$ be the Higgs potential.

The Euler-Lagrange equations corresponding to (2.1) are given by

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger}, \quad (2.2)$$

$$\kappa F_{\mu \nu} = \epsilon_{\mu \nu \alpha} J^\alpha, \quad (2.3)$$

with the strength tensor:

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad (2.4)$$

and covariant current density:

$$J^\mu = [\phi^\dagger, (D^\mu \phi)] - [(D^\mu \phi)^\dagger, \phi], \quad (2.5)$$

which is conserved, by satisfying:

$$D_\mu J^\mu = 0.$$

The system also admits a conserved Abelian current:

$$Q^\mu = -i \text{Tr} \left( \phi^\dagger D^\mu \phi - (D^\mu \phi)^\dagger \phi \right), \quad \mu = 0, 1, 2;$$

that satisfies: $\partial_\mu Q^\mu = 0$, and it is due to the global $U(1)$-invariance of the system.
Note that the energy density associated to (2.1) is given by:
\[ \mathcal{E} = \text{Tr}(D_0\phi^\dagger D_0\phi) + \text{Tr}(D_i\phi^\dagger D_i\phi) + V(\phi,\phi^\dagger), \] (2.6)
that we consider together with the following Gauss law of the system:
\[ \kappa F_{12} = J^0 = [\phi^\dagger, (D_0\phi)] - [(D_0\phi)^\dagger, \phi] \] (2.7)
(corresponding to the \( \alpha = 0 \) component of (2.3)). Then, with the choice of the Higgs potential:
\[ V(\phi,\phi^\dagger) = \frac{1}{\kappa^2} \text{Tr} \left\{ \left( [[\phi,\phi^\dagger],\phi] - v^2 \phi \right)^\dagger \left( [[\phi,\phi^\dagger],\phi] - v^2 \phi \right) \right\}, \]
(\( v > 0 \) is a constant which measures the scale of the broken symmetry) we see that the energy density \( \mathcal{E} \) can be shown to satisfy (16, 17, 54)
\[ \mathcal{E} \geq \frac{v^2}{\kappa} Q_0 \]
(neglecting divergence terms). Moreover, the above lower bound is saturated by field configurations satisfying the following relativistic Chern–Simons self-dual equations:
\[ D_1\phi \pm iD_2\phi = 0, \] (2.8)
\[ iF_{12} = \frac{2}{\kappa^2} \left[[[\phi,\phi^\dagger],\phi] - v^2 \phi, \phi^\dagger\right] = 0. \] (2.9)
See [16, 17, 54] for details. It is not difficult to see that the solutions of (2.8) and (2.9) also satisfy the Euler–Lagrange equations (2.2) and (2.3).

To handle the self-dual equations (2.8) and (2.9), we follow [16], and use the following decomposition:
\[ A_\mu = i \sum_{j=1}^{r} A_{\mu}^j H_j, \quad \phi = \sum_{i=1}^{r} \phi^i E_i, \] (2.10)
where \( A_{\mu}^i \) are real-valued vector fields, \( \phi^i \) are complex valued scalar fields \( (i = 1, \ldots, r) \), \( r \) is the rank of the semi-simple Lie algebra \( \mathcal{G} \), \( \{ H_i \}_{1 \leq i \leq r} \) and \( \{ E_i \}_{1 \leq i \leq r} \) (with \( E_i^\dagger = E_{-i} \)) are the generators of the Cartan subalgebra and the family of simple ladder operators of the semi-simple Lie algebra \( \mathcal{G} \), respectively. The consistency of (2.10) can be checked on the basis of the following commutation and trace relation,
\[ [H_i, H_j] = 0, \]
\[ [E_i, E_{-j}] = \delta_{ij} H_i, \]
\[ [H_i, E_{\pm j}] = \pm K_{ij} E_{\mp j}, \]
\[ \text{Tr}(E_i E_{-j}) = \delta_{ij}, \]
\[ \text{Tr}(H_i H_j) = K_{ij}, \]
\[ \text{Tr}(H_i E_{\pm j}) = 0, \quad i, j = 1, \ldots, r, \]
where \( K = (K_{ij})_{i,j=1,\ldots,r} \) is the Cartan matrix [20] of the semi-simple Lie algebra \( \mathcal{G} \). It is well-known that the entries \( K_{ij} \) of the Cartan matrix \( K \), satisfy the following properties:
If \( i = j \in \{1, \ldots, r\} \) then \( K_{jj} = 2 \),

ii) If \( i \neq j \in \{1, \ldots, r\} \) then \( K_{ij} \in \mathbb{Z}^{-} \) and \( K_{ij} = 0 \iff K_{ji} = 0 \).

We also know that for a semisimple Lie algebra,

\[
\det K > 0, \quad (2.11)
\]

(in fact all its principal diagonal minors are positive), and so \( K \) is non-degenerate. Actually, i) and ii) also imply that the entries of the inverse matrix \( K^{-1} \) are all non-negative, see [20] for details. Going back to \((2.10)\), we observe that it always admits a (trivial) zero-energy configuration for which all the gauge fields vanish, while the Higgs field \( \phi \) satisfies:

\[
[[\phi, \phi^\dagger], \phi] - v^2 \phi = 0. \quad (2.12)
\]

All such vacua configurations correspond to minima for the given potential.

In particular, using the decomposition \((2.10)\), we can identify the so-called principal embedding vacuum: \( \phi(0) = \sum_{j=1}^{r} \phi_j(0) E_j \) whose components \( \phi_j(0) \) satisfy:

\[
|\phi_j(0)|^2 = v^2 \sum_{j=1}^{r} (K^{-1})_{ij}, \quad i = 1, \ldots, r. \quad (2.13)
\]

To obtain non-trivial (self-dual) vortex configurations, we use the following standard notations \[16, 54\]:

\[
\partial_\pm = \partial_1 \pm i \partial_2, \quad A_\pm^i = A_1^i \pm i A_2^i, \quad i = 1, 2
\]

and observe that, in the static case, the self-dual equations \((2.8)-(2.9)\) can be expressed componentwise as follows:

\[
\partial_\pm \ln |\phi^a| = -i \sum_{b=1}^{r} A_\pm^b K_{ba}, \quad (2.14)
\]

\[
F_{12}^a = \pm \frac{2}{\kappa^2} \left( \sum_{b=1}^{r} |\phi^a| |\phi^b| K_{ba} - v^2 |\phi^a|^2 \right), \quad (2.15)
\]

away from the zeros of \( \phi^a \), and with

\[
F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a, \quad a = 1, \ldots, r.
\]

Following \[49\], we can combine equations \((2.14)-(2.15)\) into the following \( r \times r \) system (so called Master equations):

\[
\Delta \ln |\phi^a|^2 = \pm 2 \sum_{b=1}^{r} F_{12}^b K_{ba} = \frac{4}{\kappa^2} \left( \sum_{b=1}^{r} \sum_{c=1}^{r} |\phi^b|^2 |\phi^c|^2 K_{cb} K_{ba} - v^2 \sum_{b=1}^{r} |\phi^b|^2 K_{ba} \right), \quad (2.16)
\]

(away from the zero points of \( \phi^a \)) \( a = 1, \ldots, r \), that we need to solve in combination with the following componentwise expression of the Gauss law \((2.7)\):

\[
\kappa F_{12}^a = J_0^a, \quad a = 1, \ldots, r, \quad (2.17)
\]

with \( J_0^a \) the component relative to the Cartan subalgebra of the current \( J_0 \) in \((2.5)\).
The corresponding energy density takes the form:

\[ \mathcal{E} = v^2 \sum_{a=1}^{r} F_{12}^a. \]  

(2.18)

While, the gauge invariance of the theory is expressed by the following transformation laws:

\[
A_{\mu}^a \rightarrow A_{\mu}^a + \partial_{\mu} \omega_a \quad \mu = 0, 1, 2, \quad \phi^a \rightarrow e^{i \sum_{b=1}^{r} K_{ba} \omega_b} \phi^a
\]  

(2.19)

with \( \omega_a \) a smooth real function, that in the static case depends only on the state variables \( x = (x_1, x_2) \in \mathbb{R}^2, a = 1, \ldots, r \).

In this paper, we are interested in obtaining static solutions of (2.16) subject to suitable 't Hooft boundary conditions over a doubly periodic domain \( \Omega \). To be more precise, we let the periodic cell domain \( \Omega \) to be generated by two linearly independently vectors \( e_1, e_2 \in \mathbb{R}^2 \),

\[ \Omega = \{ x = s_1 e_1 + s_2 e_2 \in \mathbb{R}^2 : 0 < s_j < 1, \ j = 1, 2 \}, \]

and set

\[ \Gamma_j = \{ x = s_j e_j, \ 0 < s_j < 1 \}, \ j = 1, 2 \]

so that

\[ \partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \{ e_1 + \Gamma_2 \} \cup \{ e_2 + \Gamma_1 \} \cup \{ 0, e_1, e_2, e_1 + e_2 \}. \]

In view of (2.19), we require \( (A_{\mu}^a)_{\mu=0,1,2} \) and \( \phi^a \) to satisfy the boundary conditions

\[
\begin{aligned}
( e^{i \sum_{b=1}^{r} K_{ba} \omega_b} e^{\phi^a} ) ( x + e_k ) &= ( e^{i \sum_{b=1}^{r} K_{ba} \omega_b} e^{\phi^a} ) ( x ), \\
( A_{\mu}^a + \partial_{\mu} \omega_k^a ) ( x + e_k ) &= ( A_{\mu}^a + \partial_{\mu} \omega_k^a ) ( x ), \quad \mu = 0, 1, 2, \\
x \in \Gamma_j \cup \Gamma_k \setminus \Gamma_k, \ k = 1, 2, \ a = 1, \ldots, r,
\end{aligned}
\]  

(2.20)

where \( \omega_k^a \) is a smooth function defined in a neighborhood of \( \Gamma_j \cup \Gamma_k \) with \( j \neq k \in \{1, 2\}, a = 1, \ldots, r \).

As explicitly derived in [36], solutions of (2.16) and (2.20) carry “quantized” electric and magnetic charges, in the sense that the following hold:

\[
\begin{aligned}
\Phi_a : &= \int F_{12}^a = 2\pi \sum_{b=1}^{r} (K^{-1})_{ba} N_b, \\
Q_a : &= \int J^0 = \kappa \Phi_a = 2\pi \kappa \sum_{b=1}^{r} (K^{-1})_{ba} N_b, \quad a = 1, \ldots, r
\end{aligned}
\]

(2.21)  

(2.22)

with \( N_a \) a suitable integer, that actually counts the zeros of \( \phi^a \) in \( \Omega \) (with multiplicity) \( a = 1, \ldots, r \).

In addition, from (2.18), (2.21) and (2.22), we obtain the following “quantization” formula for the total energy:

\[
E = \int_{\Omega} \mathcal{E} = 2\pi v^2 \sum_{a,b=1}^{r} (K^{-1})_{ab} N_b = 2\pi \sum_{b=1}^{r} |\phi^b_{(0)}|^2 N_b,
\]

(2.23)

where the last identity follows by (2.13), with \( \phi^b_{(0)} \) the component of the principal embedding vacuum.
Here we shall focus on the solvability of (2.16) and (2.20) with gauge groups rank \( r = 2 \). Besides the group \( SU(3) \), with Cartan matrix \( K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \), examples of this situation include the exceptional gauge group \( B_2 (= C_2) \) with Cartan matrix \( K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \) and \( G_2 \) with Cartan matrix \( K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \).

More generally, in the rank \( r = 2 \) case, the Cartan matrix takes the form

\[
K = \begin{pmatrix} 2 & -a_{12} \\ -a_{21} & 2 \end{pmatrix}
\]  

with \( a_{jk} \in \mathbb{Z}^+ \) for \( j \neq k \in \{1, 2\} \) and \( 4 - a_{12}a_{21} > 0 \).

In case \( a_{12} = 0 = a_{21} \) (i.e. \( G = A_1 \times A_1 \)) then the Cartan matrix diagonalizes, and the system (2.16) decouples into two abelian Chern–Simons vortex problems, for which the existence of (at least) two gauge-distinct periodic static configurations has been established in [46], provided \( \kappa > 0 \) is sufficiently small. Our main goal is to extend such multiplicity result to any gauge group of rank 2. More precisely, we prove:

**Theorem 2.1** Let the gauge group \( G \) admit a semisimple Lie algebra \( \mathfrak{g} \) of rank \( r = 2 \) and Cartan matrix \( K \) specified in (2.24). For \( N_a \in \mathbb{N} \), let \( Z_a = \{p_{a,1}, \ldots, p_{a,N_a}\} \subset \Omega \) be a set of \( N_a \) points (not necessarily distinct) \( a = 1, 2 \). For \( \kappa > 0 \) sufficiently small, there exist at least two gauge distinct static solutions of (2.8)–(2.9) subject to the ansatz (2.10) and the boundary condition (2.20) such that:

(i) the component \( \phi^a \) of the Higgs field satisfies: \( |\phi^a| < |\phi^a(0)| \) in \( \Omega \), with \( \phi^a(0) \) the component of the principal embedding vacuum in (2.13); and \( \phi^a \) vanishes exactly at each point \( p_{a,j} \in Z_a \) with the same multiplicity, \( a = 1, 2 \);

(ii) the corresponding magnetic flux \( \Phi_a \), electric charge \( Q_a (a = 1, 2) \) and total energy \( E \), satisfy the “quantization” identity (2.21), (2.22) and (2.23) respectively;

(iii) for at least one of the given solutions the following holds:

\[
|\phi^a| \to |\phi^a(0)| \quad \text{as} \quad \kappa \to 0,
\]

pointwise a.e. in \( \Omega \) and strongly in \( L^p(\Omega) \), for \( p \geq 1 \).

(iv) If

\[
\kappa > v \sqrt{\frac{|\Omega|}{4\pi (\det K) \max \left\{ \frac{2N_a + a_{12}N_a}{2(a_{12} + 2)^2}, \frac{2N_a + a_{21}N_a}{2(a_{21} + 2)^2} \right\} }},
\]

with \( a_{ij} \geq 0, i \neq j \in \{1, 2\} \), the off-diagonal entries of the Cartan matrix \( K \) in (2.24), then problem (2.8)–(2.10) and (2.20) admits no such solutions.

As already mentioned, Theorem 2.1 provides a natural extension of the multiplicity result of Nolasco–Tarantello in [36], concerning the group \( G = SU(3) \), for which (2.10) enjoys additional symmetries. In fact, to establish Theorem 2.1 we adopt the same variational viewpoint. However we are able to handle systems of the type (2.16) with a more general coupling matrix.
More precisely, we take a $2 \times 2$ matrix $K$ of the form:

$$K = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

and assume that $a, b, c, d > 0$ and $ad - bc > 0$. Notice that the case $a, d > 0$ and $b = c = 0$, is already covered in [46].

We denote the zero set of $\phi^i$ by

$$Z_i = \{p_{i,1}, \ldots, p_{i,N_i}\}, \quad i = 1, 2 \quad (2.25)$$

(repeated with multiplicity) and set,

$$|\phi^1|^2 = v^2 \frac{b+d}{ad-bc} e^{u_1}, \quad |\phi^2|^2 = v^2 \frac{a+c}{ad-bc} e^{u_2}, \quad \lambda = \frac{4v^4}{\kappa^2} \quad (2.26)$$

By straightforward calculations, we see that the equations in (2.16) subject to the boundary conditions (2.20) take the form:

$$\begin{align*}
\Delta u_1 &= \lambda \left\{ \frac{1}{ad-bc} \left[ -a(b+d)e^{u_1} + b(a+c)e^{u_2} \right] \\
&\quad + \frac{1}{(ad-bc)^2} \left[ a^2(b+d)^2 e^{2u_1} - b(b+d)(a^2-c^2)e^{u_1+u_2} - bd(a+c)^2 e^{2u_2} \right] \\
&\quad + 4\pi \sum_{j=1}^{N_1} \delta_{p_{1,j}}, \ x \in \Omega, \right. \\
\Delta u_2 &= \lambda \left\{ \frac{1}{ad-bc} \left[ c(b+d)e^{u_1} - d(a+c)e^{u_2} \right] \\
&\quad + \frac{1}{(ad-bc)^2} \left[ -ac(b+d)^2 e^{2u_1} - c(a+c)(d^2-b^2)e^{u_1+u_2} + d^2(a+c)^2 e^{2u_2} \right] \\
&\quad + 4\pi \sum_{j=1}^{N_2} \delta_{p_{2,j}}, \ x \in \Omega, \right. \\
&\quad \text{such that for } \lambda > \lambda_0, \text{ satisfies } \\
&\quad e^{u_1} < 1, \quad e^{u_2} < 1 \quad \text{in } \Omega. \quad (2.27)
\end{align*}$$

Concerning (2.27), we establish the following:

**Theorem 2.2** Assume that $a, b, c, d > 0$ and $ad - bc > 0$. Given $N_j \in \mathbb{N}$ and $Z_j = \{p_{j,1}, \ldots, p_{j,N_j}\} \subset \Omega$ (a set of $N_j$-point repeated with multiplicity), $j = 1, 2$, the following holds:

1. Every solution $(u_1, u_2)$ of (2.27) satisfies

$$e^{u_1} < 1, \quad e^{u_2} < 1 \quad \text{in } \Omega. \quad (2.28)$$

2. If

$$\lambda < \frac{16\pi(ad-bc)}{\Omega} \max \left\{ \frac{dN_1 + bN_2}{a(b+d)^2}, \frac{cN_1 + aN_2}{d(a+c)^2} \right\},$$

then problem (2.27) admits no solutions.

3. There exist $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ problem (2.27) admits at least two distinct solutions, one of which satisfying:

$$e^{u_1} \to 1, \quad e^{u_2} \to 1, \quad \text{as } \lambda \to +\infty \quad (2.29)$$

pointwise a.e. in $\Omega$ and strongly in $L^p(\Omega)$ for any $p \geq 1$. 

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Remark 2.1 As already noticed, when $b = c = 0$, problem (2.27) decouples in two abelian self-dual Chern-Simons equations:

$$
\begin{align*}
\Delta u_i &= \lambda e^{u_i} (e^{u_i} - 1) + 4\pi \sum_{j=1}^{N_i} \delta_{p_{i,j}}, \ x \in \Omega, \\
0 &= \text{doubly periodic on } \partial\Omega, \quad i = 1, 2
\end{align*}
$$

for which the existence and multiplicity results claimed above have been established in [40].

Thus in view of (2.26), [40] and Theorem 2.2 we deduce (by standard arguments [49]) the statement of Theorem 2.1. Hence we devote the following section to the proof of Theorem 2.2.

3 Existence of doubly periodic solutions

In this section we analyze problem (2.27), and for convenience we rewrite it as follows:

$$
\begin{align*}
\Delta u_1 &= \frac{\lambda}{a d - b c} [a (b + d) e^{u_1} (e^{u_1} - 1) + b (a + c) e^{u_2} (1 - e^{u_2})] \\
&\quad + \frac{\lambda (a + c) (b + d) b}{(a d - b c)^2} (a e^{u_1} + c e^{u_2}) (e^{u_1} - e^{u_2}) + 4\pi \sum_{j=1}^{N_1} \delta_{p_{1,j}}, \ x \in \Omega,
\end{align*}
$$

$$
\begin{align*}
\Delta u_2 &= \frac{\lambda}{a d - b c} [d (a + c) e^{u_2} (e^{u_2} - 1) + c (b + d) e^{u_1} (1 - e^{u_1})] \\
&\quad + \frac{\lambda (a + c) (b + d) c}{(a d - b c)^2} (b e^{u_1} + d e^{u_2}) (e^{u_2} - e^{u_1}) + 4\pi \sum_{j=1}^{N_2} \delta_{p_{2,j}}, \ x \in \Omega,
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
\Delta u_1 &= 0 \quad \text{in } \Omega, \quad \text{and } u_2 \text{ doubly periodic on } \partial\Omega.
\end{cases}
\end{align*}
$$

We start to establish the following:

Proposition 3.1 Let $(u_1, u_2)$ satisfy (3.1). Then $u_i < 0$ in $\Omega$, $i = 1, 2$.

Proof. Notice that $u_i$ attains its maximum value at a point $\tilde{x}_i \in \overline{\Omega} \setminus Z_i$, so that $\tilde{u}_i \equiv \max_{\overline{\Omega}} u_i = u_i(\tilde{x}_i), \ i = 1, 2$. We start by showing $\tilde{u}_i \leq 0$, for $i = 1, 2$. Indeed, in case $\tilde{u}_1 \geq \tilde{u}_2$, then we use the first equation in (3.1) to obtain:

$$
0 \geq \Delta u_1(\tilde{x}_1) = \frac{\lambda}{a d - b c} [a (b + d) e^{\tilde{u}_1} (e^{\tilde{u}_1} - 1) + b (a + c) e^{u_2(\tilde{x}_1)} (1 - e^{u_2(\tilde{x}_1)})] \\
+ \frac{\lambda (a + c) (b + d) b}{(a d - b c)^2} (a e^{\tilde{u}_1} + c e^{u_2(\tilde{x}_1)}) (e^{\tilde{u}_1} - e^{u_2(\tilde{x}_1)})
$$

$$
\geq \frac{\lambda}{a d - b c} [a (b + d) e^{\tilde{u}_1} - b (a + c) e^{u_2(\tilde{x}_1)}] (e^{\tilde{u}_1} - 1).
$$

Since $a (b + d) e^{\tilde{u}_1} - b (a + c) e^{u_2(\tilde{x}_1)} \geq (a d - b c) e^{\tilde{u}_1} > 0$

we find that necessarily, $\tilde{u}_i \leq 0$, and the desired conclusion follows in this case. On the other hand if $\tilde{u}_2 \geq \tilde{u}_1$, then we can use a similar argument for the second equation in (3.1) to deduce that $\tilde{u}_2 \leq 0$. Thus, in any case, we have: $\tilde{u}_i \leq 0, \ i = 1, 2$. To obtain that actually the strict inequality holds, we use the strong maximum principle. It can be applied, since for example we see that $u_1$ satisfies:

$$
\begin{align*}
\Delta u_1 + c(x) u_1 &= \frac{\lambda}{a d - b c} [b (a + c) e^{u_2} + \frac{(a + c) (b + d) b}{a d - b c} (a e^{u_1} + c e^{u_2})] (1 - e^{u_2}) \\
&\geq 0 \quad \text{in } \Omega
\end{align*}
$$

for all $x \in \Omega$. Thus, in any case, we have $\tilde{u}_i \leq 0, \ i = 1, 2$. Hence we deduce that $\tilde{u}_i \leq 0, \ i = 1, 2$.
with
\[ c_1(x) = \frac{\lambda}{ad - bc} \left[ a(b + d)e^{u_1} + \frac{(a + c)(b + d)b}{ad - bc} (ae^{u_1} + ce^{u_2}) \right] \frac{1 - e^{u_1}}{u_1}. \]

Similarly for \( u_2 \).

Therefore we conclude that \( u_i < 0 \) in \( \Omega, \ i = 1, 2 \). In particular we have established the first conclusion of Theorem 2.2. □

To proceed further, we let \( u_i = u_0^i + v_i, i = 1, 2 \) with \( u_0^i \) being the unique solution of the problem (see [3])

\[
\begin{align*}
\Delta u_0^i &= 4\pi \sum_{s=1}^{N_i} \delta_{p_i,s} - \frac{4\pi N_i}{|\Omega|}, \\
\int_{\Omega} u_0^i dx &= 0, \quad u_0^i \text{ doubly periodic on} \quad \partial \Omega \quad i = 1, 2.
\end{align*}
\]

Consequently, problem (2.27) (or (3.1)) can be formulated in terms of the unknown \( (v_1, v_2) \) as follows:

\[
\begin{align*}
\Delta v_1 &= \frac{\lambda}{ad - bc} \left[ -a(b + d)e^{u_0^1 + v_1} + b(a + c)e^{u_0^2 + v_2} \right] + \frac{\lambda}{(ad - bc)^2} \left[ a^2(b + d)^2e^{2u_0^1 + 2v_1} \right. \\
&\quad -b(b + d)(a^2 - c^2)e^{u_0^1 + u_0^2 + v_1 + v_2} - bd(a + c)^2e^{2u_0^1 + 2v_2} \left. \right] + \frac{4\pi N_i}{|\Omega|}, \\
\Delta v_2 &= \frac{\lambda}{ad - bc} \left[ c(b + d)e^{u_0^1 + v_1} - d(a + c)e^{u_0^2 + v_2} \right] + \frac{\lambda}{(ad - bc)^2} \left[ -ac(b + d)^2e^{2u_0^1 + 2v_1} \right. \\
&\quad -c(a + c)(d^2 - b^2)e^{u_0^1 + u_0^2 + v_1 + v_2} + d^2(a + c)^2e^{2u_0^2 + 2v_2} \left. \right] + \frac{4\pi N_i}{|\Omega|}, \\
v_1, v_2 \text{ doubly periodic on} \quad \partial \Omega.
\end{align*}
\]

Actually, to emphasize the variational structure of (3.2), we shall use the following equivalent formulation:

\[
\begin{align*}
\frac{ad - bc}{b + d} \Delta (dv_1 + bv_2) &= \lambda \left[ a(b + d)e^{2u_0^1 + 2v_1} - (ad - bc)e^{u_0^1 + v_1} - b(a + c)e^{u_0^1 + u_0^2 + v_1 + v_2} \right] \\
&\quad + \frac{4\pi(ad - bc)(dN_1 + bN_2)}{(b + d)|\Omega|}, \\
\frac{ad - bc}{a + c} \Delta (cv_1 + av_2) &= \lambda \left[ d(a + c)e^{2u_0^2 + 2v_2} - (ad - bc)e^{u_0^2 + v_2} - c(b + d)e^{u_0^1 + u_0^2 + v_1 + v_2} \right] \\
&\quad + \frac{4\pi(ad - bc)(cN_1 + aN_2)}{(a + c)|\Omega|}, \\
v_1, v_2 \text{ doubly periodic on} \quad \partial \Omega.
\end{align*}
\]

We introduce the Hilbert space: \( H(\Omega) \equiv W^{1,2}(\frac{\mathbb{R} \times (0,2\pi)}{\mathbb{Z} \times \mathbb{Z}}) \) of \( \Omega \)-periodic \( L^2 \)-functions whose derivatives also belong to \( L^2(\Omega) \), equipped with the usual norm: \( \|w\| = \|w\|_2 + \|
abla w\|_2 = \int_{\Omega} w^2 dx + \int_{\Omega} |\nabla w|^2 dx, \ w \in H(\Omega) \).

It is not difficult to check that weak solutions to (3.3) are critical points in \( H(\Omega) \times H(\Omega) \) of the functional:

\[
I_\lambda(v_1, v_2) = \frac{d}{2b} \|
abla v_1\|_2^2 + \frac{a}{2c} \|
abla v_2\|_2^2 + \int_{\Omega} \nabla v_1 \cdot \nabla v_2 dx + \lambda \int_{\Omega} Q(v_1, v_2) dx \\
+ \frac{\alpha_1}{|\Omega|} \int_{\Omega} v_1 dx + \frac{\alpha_2}{|\Omega|} \int_{\Omega} v_2 dx.
\]
where

\[
\begin{align*}
Q(v_1, v_2) &= \frac{1}{2ab(ad-bc)} Q_1^2(v_1, v_2) + \frac{(a+c)^2}{2ac} Q_2^2(v_2), \\
Q_1(v_1, v_2) &= \left[ a(b+d)e^{u_1^0+v_1} - b(a+c)e^{u_2^0+v_2} - (ad-bc) \right], \\
Q_2(v_2) &= \left( e^{u_2^0+v_2} - 1 \right), \\
\alpha_1 &= 4\pi \left( \frac{d}{b} N_1 + N_2 \right), \quad \alpha_2 = 4\pi \left( N_1 + \frac{a}{b} N_2 \right).
\end{align*}
\]

(3.5)

In view of our assumption, notice that the quadratic part of \(I_\lambda\) is positive definite. In fact we obtain a first critical point for \(I_\lambda\) via (local) minimization.

### 3.1 Constrained minimization

For a solution \((v_1, v_2)\) of (3.3), after integration over \(\Omega\), we find the following natural constraints:

\[
a(b+d) \int_{\Omega} e^{2u_1^0+2v_1} dx - (ad-bc) \int_{\Omega} e^{u_1^0+v_1} dx - b(a+c) \int_{\Omega} e^{u_1^0+u_2^0+v_1+v_2} dx \\
+ \frac{4\pi(ad-bc)(dN_1 + bN_2)}{\lambda(b+d)} = 0, \tag{3.6}
\]

\[
d(a+c) \int_{\Omega} e^{2u_2^0+2v_2} dx - (ad-bc) \int_{\Omega} e^{u_2^0+v_2} dx - c(b+d) \int_{\Omega} e^{u_1^0+u_2^0+v_1+v_2} dx \\
+ \frac{4\pi(ad-bc)(cN_1 + aN_2)}{\lambda(a+c)} = 0. \tag{3.7}
\]

From (3.5)-(3.7), we obtain

\[
\int_{\Omega} Q(v_1, v_2) dx = \frac{1}{2} \left[ \left( 1 + \frac{d}{b} \right) \int_{\Omega} (1 - e^{u_1^0+v_1}) dx + \left( 1 + \frac{a}{c} \right) \int_{\Omega} (1 - e^{u_2^0+v_2}) dx \right] \\
- \frac{\alpha_1 + \alpha_2}{2\lambda}. \tag{3.8}
\]

Therefore, if we decompose \(v_1, v_2\) as follows:

\[
v_i = w_i + c_i, \quad \int_{\Omega} w_i dx = 0, \quad c_i = \frac{1}{|\Omega|} \int_{\Omega} v_i dx, \quad i = 1, 2,
\]

then form (3.6) and (3.7) we find:

\[
e^{2c_1} \int_{\Omega} e^{2u_1^0+2w_1} dx - e^{c_1} R_1(w_1, w_2, e^{c_2}) + \frac{4\pi(ad-bc)(dN_1 + bN_2)}{\lambda a(b+d)^2} = 0, \tag{3.9}
\]

\[
e^{2c_2} \int_{\Omega} e^{2u_2^0+2w_2} dx - e^{c_2} R_2(w_1, w_2, e^{c_1}) + \frac{4\pi(ad-bc)(cN_1 + aN_2)}{\lambda d(a+c)^2} = 0, \tag{3.10}
\]

with

\[
R_1(w_1, w_2, e^{c_2}) = \frac{ad-bc}{a(b+d)} \int_{\Omega} e^{u_1^0+w_1} dx + \frac{b(a+c)}{a(b+d)} e^{c_2} \int_{\Omega} e^{u_1^0+u_2^0+w_1+w_2} dx, \tag{3.11}
\]

\[
R_2(w_1, w_2, e^{c_1}) = \frac{ad-bc}{d(a+c)} \int_{\Omega} e^{u_2^0+w_2} dx + \frac{c(b+d)}{d(a+c)} e^{c_1} \int_{\Omega} e^{u_1^0+u_2^0+w_1+w_2} dx. \tag{3.12}
\]
A necessary condition for the solvability of (3.9) and (3.10) with respect to $c_1$ and $c_2$ is that,

$$
(R_1(w_1, w_2, e^{c_2}))^2 \geq \frac{16\pi(ad - bc)(dN_1 + bN_2)}{\lambda(a + d)^2} \int_{\Omega} e^{2u_0^1 + 2w_1} dx, \quad (3.13)
$$

$$
(R_2(w_1, w_2, e^{c_1}))^2 \geq \frac{16\pi(ad - bc)(cN_1 + aN_2)}{\lambda(a + c)^2} \int_{\Omega} e^{2u_0^2 + 2w_2} dx. \quad (3.14)
$$

On the other hand, from Proposition 3.1, we see that $u_0^1 + v_1 + c_1 < 0$ and $u_0^2 + v_2 + c_2 < 0$ in $\Omega$. Therefore, as a consequence of (3.13)-(3.14), we obtain:

$$
\frac{16\pi(ad - bc)(dN_1 + bN_2)}{\lambda(a + d)^2} \int_{\Omega} e^{2u_0^1 + 2w_1} dx \leq |\Omega| \int_{\Omega} e^{2u_0^1 + 2w_1} dx,
$$

$$
\frac{16\pi(ad - bc)(cN_1 + aN_2)}{\lambda(a + c)^2} \int_{\Omega} e^{2u_0^2 + 2w_2} dx \leq |\Omega| \int_{\Omega} e^{2u_0^2 + 2w_2} dx.
$$

Thus, we obtain the following necessary condition for the solvability of (3.3),

$$
\lambda \geq \frac{16\pi(ad - bc)}{|\Omega| \max \left\{ \frac{dN_1 + bN_2}{a(b + d)^2}, \frac{cN_1 + aN_2}{d(a + c)^2} \right\}}, \quad (3.15)
$$

and deduce part 2. of Theorem 2.2.

Conditions (3.13) and (3.14) suggest to focus only with pairs $(v_1, v_2)$ that, under the decomposition: $v_i = w_i + c_i$, $i = 1, 2$, satisfy:

$$
\int_{\Omega} w_1 dx = 0 \quad \text{and} \quad \left( \int_{\Omega} e^{u_0^1 + w_1} dx \right)^2 \geq \frac{16\pi a(dN_1 + bN_2)}{\lambda(ad - bc)} \int_{\Omega} e^{2u_0^1 + 2w_1} dx, \quad (3.16)
$$

$$
\int_{\Omega} w_2 dx = 0 \quad \text{and} \quad \left( \int_{\Omega} e^{u_0^2 + w_2} dx \right)^2 \geq \frac{16\pi d(cN_1 + aN_2)}{\lambda(ad - bc)} \int_{\Omega} e^{2u_0^2 + 2w_2} dx; \quad (3.17)
$$

and where $(c_1, c_2)$ satisfy (3.9) and (3.10).

Hence we define the admissible set:

$$
\mathcal{A} = \{(w_1, w_2) \in H(\Omega) \times H(\Omega) \text{ such that } (3.16) \text{ and } (3.17) \text{ hold}\}. \quad (3.18)
$$

On the basis of (3.9) and (3.10), we aim to obtain $(c_1, c_2)$ from the equations:

$$
e^{c_1} = R_1(w_1, w_2, e^{c_2}) \pm \sqrt{[R_1(w_1, w_2, e^{c_2})]^2 - \frac{16\pi(ad - bc)(dN_1 + bN_2)}{\lambda(a + d)^2} \int_{\Omega} e^{2u_0^1 + 2w_1} dx} \quad \frac{2 \int_{\Omega} e^{2u_0^1 + 2w_1} dx}{2 \int_{\Omega} e^{2u_0^1 + 2w_1} dx} \quad (3.19)
$$

$$
e^{c_2} = R_2(w_1, w_2, e^{c_1}) \pm \sqrt{[R_2(w_1, w_2, e^{c_1})]^2 - \frac{16\pi(ad - bc)(cN_1 + aN_2)}{\lambda(a + c)^2} \int_{\Omega} e^{2u_0^2 + 2w_2} dx} \quad \frac{2 \int_{\Omega} e^{2u_0^2 + 2w_2} dx}{2 \int_{\Omega} e^{2u_0^2 + 2w_2} dx} \quad (3.20)
$$

To this end, we follow [36] and set

$$
F^+(X) \equiv X - g_1^+(g_2^+(X)), \quad F^-(X) \equiv X - g_1^- g_2^-(X)),
$$

$$
F^\pm(X) \equiv X - g_1^+(g_2^-(X)), \quad F^\mp(X) \equiv X - g_1^- (g_2^+(X)),
$$

where

$$
g_i^\pm = \begin{cases} g_i^+ & \text{for } i = 1, 2 \\
\end{cases}
$$
so that the solutions of (3.19) and (3.20), with all possible choices of signs: $* = +, -, \pm, \mp$ corresponds to the zeros of the function:

$$F^* : (0, +\infty) \mapsto \mathbb{R}.$$  

At this point, as in [36], it suffices to check the following claims.

**Claim 1.** The functions $g_i^\pm(X)$ is strictly monotonic with respect to $X > 0$, $i = 1, 2$.

In fact, by direct computation we have:

$$\frac{dg_i^\pm(X)}{dX} = \pm g_i^\pm(X) \sqrt{\frac{b(a+c)}{a(b+d)} \int_\Omega e^{u_0^1+u_0^2+w_1+w_2} dx - \frac{16\sigma(b-d)}{x_0(b+d)^2} \int_\Omega e^{2u_0^1+2w_1} dx},$$  \hspace{1cm} (3.21)

$$\frac{dg_2^\pm(X)}{dX} = \pm g_2^\pm(X) \sqrt{\frac{c(b+d)}{d(a+c)} \int_\Omega e^{u_0^1+u_0^2+w_1+w_2} dx - \frac{16\sigma(b-d)}{x_0(a+d)^2} \int_\Omega e^{2u_0^1+2w_1} dx},$$  \hspace{1cm} (3.22)

and by definition (see (3.19) and (3.20))

$$g_i^\pm(X) > 0, \quad \forall X > 0, \quad i = 1, 2.$$  \hspace{1cm} (3.23)

**Claim 2.** For any $(w_1, w_2) \in A$, there exits a unique $X^*(w_1, w_2) > 0$ such that $F^*(X^*(w_1, w_2)) = 0$; with $* = +, -, \pm, \mp$.

To prove Claim 2, observe that $F^*(0) < 0$, with $* = +, -, \pm, \mp$. Next, we check easily that,

$$\lim_{x \to +\infty} g_i^-(X) = 0, \quad i = 1, 2,$$

$$\lim_{x \to +\infty} \frac{g_i^+(X)}{X} = \frac{b(a+c)}{a(b+d)} \int_\Omega e^{u_0^1+u_0^2+w_1+w_2} dx,$$

$$\lim_{x \to +\infty} \frac{g_2^+(X)}{X} = \frac{c(b+d)}{d(a+c)} \int_\Omega e^{u_0^1+u_0^2+w_1+w_2} dx,$$

and consequently:

$$\lim_{x \to +\infty} \frac{F^+(X)}{X} = 1 - \frac{bc}{ad} \left( \int_\Omega e^{u_0^1+u_0^2+w_1+w_2} dx \right)^2 \ge \frac{ad - bc}{ad} > 0,$$

$$\lim_{x \to +\infty} \frac{F^*(X)}{X} = 1, \quad * = -, \pm, \mp.$$

In particular,

$$\lim_{x \to +\infty} F^*(X) = +\infty, \quad * = +, -, \pm, \mp,$$

and from (3.21) and (3.22), we see that

$$\frac{dF^*(X)}{dX} > 0, \quad * = \pm, \mp,$$

hence we deduce the statement of Claim 2 for $* = \pm, \mp$. 

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On the other hand, from (3.21)–(3.22) and (3.16)–(3.17) we obtain

\[
\frac{dF^+(X)}{dX} = 1 - \frac{b^2(a+c)^2}{a^2(b+d)^2} \frac{g_1^+(g_2^+(X))g_2^+(X)\left(\int_{\Omega} e^{u_0^++u_0^1+w_1+w_2}dx\right)^2}{\sqrt{\left[R_1(w_1, w_2, g_2^+(X))\right]^2 - \frac{16\pi(ad-bc)|dN_1+bN_2|}{\lambda_4(a+b+d)^2} \int_{\Omega} e^{2u_0^++2w_1}dx}}
\times \frac{1}{\sqrt{\left[R_2(w_1, w_2, X)\right]^2 - \frac{16\pi(ad-bc)|cN_1+aN_2|}{\lambda_2(a+c)^2} \int_{\Omega} e^{2u_0^++2w_2}dx}}
\]

\[
> 1 - \frac{g_1^+(g_2^+(X))}{X} = \frac{F^+(X)}{X}.
\]

Similarly, for \(* = -\), we have:

\[
\frac{dF^-(X)}{dX} > 1 - \frac{g_1^-(g_2^-(X))}{X} = \frac{F^-(X)}{X}.
\]

Thus, for \(X > 0\) the function \(\frac{F^*(X)}{X}\) is strictly increasing, with \(* = +, -\), and Claim 2 follows in this case as well. \(\square\)

From the above discussion we see that, for any \((w_1, w_2) \in \mathcal{A}\), there exists a unique \(c^*_j = c^*_j(w_1, w_2)\) for \(j = 1, 2\) and \(* = +, -, \pm, \mp\) such that

\[
v^*_1 = w_1 + c^*_1(w_1, w_2), \quad v^*_2 = w_2 + c^*_2(w_1, w_2), \quad * = +, -, \pm, \mp
\]

satisfy (3.6)–(3.7). Notice also that, by the above property, \(c^*_1\) and \(c^*_2\) depend smoothly on \((w_1, w_2) \in \mathcal{A}\).

We shall use those properties only for \(* = +\), although it is reasonable to expect that other choices may lead to stronger multiplicity results, as in [36].

Thus, in what follows we consider the functional

\[
J^*_\lambda(w_1, w_2) = I_\lambda(w_1 + c^*_1(w_1, w_2), w_2 + c^*_2(w_1, w_2)), \quad (w_1, w_2) \in \mathcal{A}.
\]

From (3.4) and (3.8), we see that

\[
J^*_\lambda(w_1, w_2) = \frac{d}{d\lambda} \|\nabla w_1\|^2 + \frac{a}{2c} \|\nabla w_2\|^2 + \int_{\Omega} \nabla w_1 \cdot \nabla w_2 dx
+ \frac{\lambda}{2} \left[ \left(1 + \frac{d}{b}\right) \int_{\Omega} (1 - e^{c^*_1 e^{u_0^1+w_1}}) dx + \left(1 + \frac{a}{c}\right) \int_{\Omega} (1 - e^{c^*_2 e^{u_0^2+w_2}}) dx \right]
+ \alpha_1 c^*_1 + \alpha_2 c^*_2 - \frac{\alpha_1 + \alpha_2}{2}
\]

(3.24)

with \(\alpha_1, \alpha_2\) defined in (3.5).

It is easy to check that the functional \(J^*_\lambda\) is Fréchet differentiable in the interior of \(\mathcal{A}\). Moreover, if \((w_1, w_2)\) is a critical point of \(J^*_\lambda\) and lies in the interior of \(\mathcal{A}\), then \((w_1 + c^*_1(w_1, w_2), w_2 + c^*_2(w_1, w_2))\) gives a critical point of \(I_\lambda\).

In the sequel, we show that \(J^*_\lambda\) is bounded from below and admits an interior minimum.

**Lemma 3.1** For any \((w_1, w_2) \in \mathcal{A}\), there holds:

\[
e^{c^*_i} \int_{\Omega} e^{u_0^i+w_i} dx \leq |\Omega|, \quad i = 1, 2.
\]

(3.25)
Remark 3.1 Using Jensen’s inequality, from (3.25) follows that
\[ e_i^\tau \leq 1, \quad i = 1, 2. \]

Proof. Using (3.19)–(3.20), we have
\[
e_i^\tau \leq \frac{a \int \Omega e^{u_0^1 + u_1^1} dx + b \int \Omega e^{u_0^2 + u_1^2 + w_1 + w_2} dx}{\int \Omega e^{2u_0^1 + 2w_1} dx},
\]
and so, by using (3.26)–(3.27) and Hölder inequality, we find
\[
e_i^\tau \leq \frac{\int \Omega e^{u_0^1 + u_1^1} dx + \int \Omega e^{u_0^2 + u_1^2 + w_1 + w_2} dx}{\int \Omega e^{2u_0^1 + 2w_1} dx}.
\]

Consequently,
\[
e_i^\tau \leq \frac{d}{b + d} \int \Omega e^{u_0^1 + u_1^1} dx + \frac{b}{b + d} \int \Omega e^{u_0^2 + u_1^2 + w_1 + w_2} dx.
\]

Similarly, we obtain
\[
e_i^\tau \leq \frac{\int \Omega e^{u_0^2 + w_2} dx}{\int \Omega e^{2u_0^1 + 2w_1} dx}.
\]

Next, we can use (3.28)–(3.29) and Hölder inequality to deduce that
\[
e_i^\tau \int \Omega e^{u_0^1 + u_1^1} dx \leq \int \Omega e^{u_0^1 + u_1^1} dx \leq |\Omega|,
\]
and (3.25) is established. □

The following property of functions in \( A \) was pointed out first in [38] and used in [36]. In our context, it takes the following form:

Lemma 3.2 For any \((w_1, w_2) \in A\) and \(s \in (0, 1)\), we have
\[
\int \Omega e^{u_0^1 + w_1} dx \leq \left( \frac{\lambda (a + c)}{16 \pi a (bN_1 + bN_2)} \right)^{\frac{1}{1 + s}} \left( \int \Omega e^{s_0^1 + s_1^1} dx \right)^{\frac{1}{1 + s}},
\]
and
\[
\int \Omega e^{u_0^2 + w_2} dx \leq \left( \frac{\lambda (a + c)}{16 \pi a (cN_1 + aN_2)} \right)^{\frac{1}{1 + s}} \left( \int \Omega e^{s_0^2 + s_2} dx \right)^{\frac{1}{1 + s}}.
\]
Proof. Although the proof of (3.30), (3.31) follows exactly as in [36,38], here we give the proof for completeness. Let \( s \in (0, 1) \), \( \gamma = \frac{1}{2-s} \) such that \( s\gamma + 2(1-\gamma) = 1 \). Then using Hölder inequality and (3.16) we have

\[
\int_{\Omega} e^{u_1 + w_1} dx \leq \left( \int_{\Omega} e^{s u_1 + s w_1} dx \right)^{\gamma} \left( \int_{\Omega} e^{2 u_1 + 2 w_1} dx \right)^{1-\gamma} \\
\leq \left( \frac{\lambda(ad - bc)}{16\pi a(dN_1 + bN_2)} \right)^{1-\gamma} \left( \int_{\Omega} e^{s u_1 + s w_1} dx \right)^{\gamma} \left( \int_{\Omega} e^{u_0 + w_1} dx \right)^{2(1-\gamma)},
\]

which implies

\[
\int_{\Omega} e^{u_1 + w_1} dx \leq \left( \frac{\lambda(ad - bc)}{16\pi a(dN_1 + bN_2)} \right)^{\frac{1-\gamma}{\gamma}} \left( \int_{\Omega} e^{s u_1 + s w_1} dx \right)^{\frac{\gamma}{\gamma'}} \left( \int_{\Omega} e^{u_0 + w_1} dx \right)^{\frac{1}{\gamma'}}.
\]

Analogously, using Hölder inequality and (3.17), we can get (3.31). □

Lemma 3.2 will allow us to show that the functional \( J_1^\lambda \) is coercive on \( A \). To this purpose, we need the following well-known Moser–Trudinger inequality (see [3]):

\[
\int_{\Omega} e^{w} dx \leq C_1 \exp \left( \frac{1}{16\pi} \| \nabla w \|_2^2 \right), \quad \forall w \in H(\Omega) : \int_{\Omega} wx dx = 0, \tag{3.32}
\]

where \( C_1 \) is a positive constant depending on \( \Omega \) only.

**Lemma 3.3** There exist suitable constants \( C_2 > 0 \) and \( C_3 > 0 \) independent of \( \lambda \) such that, for every \( (w_1, w_2) \in A \) there holds:

\[
J_1^\lambda(w_1, w_2) \geq C_2 \left( \| \nabla w_1 \|_2^2 + \| \nabla w_2 \|_2^2 \right) - C_3(\ln \lambda + 1). \tag{3.33}
\]

**Proof.** From (3.19), (3.20), we see that:

\[
e^{c^+_1} \geq \frac{ad - bc}{2\pi(a + c)} \int_{\Omega} e^{u_1 + w_1} dx, \quad e^{c^+_2} \geq \frac{ad - bc}{2\pi(a + c)} \int_{\Omega} e^{u_2 + w_2} dx.
\]

Thus, by the constraints (3.16)-(3.17), we find:

\[
e^{c^+_1} \geq \frac{8\pi(dN_1 + bN_2)}{\lambda(b + d)} \int_{\Omega} e^{u_0 + w_1} dx, \quad e^{c^+_2} \geq \frac{8\pi(cN_1 + aN_2)}{\lambda(a + c)} \int_{\Omega} e^{u_0 + w_2} dx.
\]

that is,

\[
c^+_1 \geq \ln \frac{8\pi(bN_1 + dN_2)}{b + d} - \ln \lambda - \ln \int_{\Omega} e^{u_0 + w_1} dx, \tag{3.34}
\]

\[
c^+_2 \geq \ln \frac{8\pi(cN_1 + aN_2)}{a + c} - \ln \lambda - \ln \int_{\Omega} e^{u_0 + w_2} dx. \tag{3.35}
\]

For any \( s \in (0, 1) \), using Lemma 3.2 and Moser–Trudinger inequality (3.32), we have

\[
\ln \int_{\Omega} e^{su_1 + w_1} dx \leq \frac{1 - s}{s} \left[ \ln \lambda + \ln \frac{ad - bc}{16\pi a(dN_1 + bN_2)} \right] + \frac{1}{s} \ln \int_{\Omega} e^{su_0 + w_1} dx \leq \frac{s}{16\pi} \| \nabla w_1 \|_2^2 + \frac{1 - s}{s} \left[ \ln \lambda + \ln \frac{ad - bc}{16\pi a(dN_1 + bN_2)} \right] + \max_{x \in \Omega} u_0^1 + \frac{1}{s} \ln C_1; \tag{3.36}
\]
and similarly:
\[
\ln \int_{\Omega} e^{u_0^2 + w_2} \, dx \leq \frac{s}{16\pi} \| \nabla w_2 \|_2^2 + \frac{1 - s}{s} \left( \ln \lambda + \frac{ad - bc}{16\pi d(cN_1 + aN_2)} \right) + \max_{x \in \Omega} u_0^2 + \frac{1}{s} \ln C_1. \quad (3.37)
\]
Therefore from (3.24), (3.8), and (3.25), for any given \( \varepsilon > 0 \), we obtain:
\[
J_\lambda^+(w_1, w_2) \geq \left( \frac{d}{2b} - \frac{\varepsilon}{2} \right) \| \nabla w_1 \|_2^2 + \left( \frac{a}{2c} - \frac{1}{2\varepsilon} \right) \| \nabla w_2 \|_2^2 + \alpha_1 c_1 + \alpha_2 c_2^+, \quad (3.38)
\]
where \( \alpha_1 \) and \( \alpha_2 \) are given in (3.5). So, with the optimal choice
\[
\varepsilon = \frac{1}{2} \left( \frac{d}{b} + \frac{c}{a} \right),
\]
we deduce:
\[
J_\lambda^+(w_1, w_2) \geq \frac{ad - bc}{4ab} \| \nabla w_1 \|_2^2 + \frac{a(ad - bc)}{2c(ad + bc)} \| \nabla w_2 \|_2^2 + \alpha_1 c_1 + \alpha_2 c_2^+. \quad (3.39)
\]
Then, from (3.39) and (3.54)–(3.57) for \( s \in (0, 1) \) we conclude
\[
J_\lambda^+(w_1, w_2) \geq \left( \frac{ad - bc}{4a} - \frac{s\alpha_1}{16\pi} \right) \| \nabla w_1 \|_2^2 + \left( \frac{a(ad - bc)}{2c(ad + bc)} - \frac{s\alpha_2}{16\pi} \right) \| \nabla w_2 \|_2^2
- \frac{\alpha_1 + \alpha_2}{s} \ln \lambda - C,
\quad (3.40)
\]
with \( C \) a positive constant independent of \( \lambda \) and \( \alpha_1 \) and \( \alpha_2 \) given in (3.5). At this point, by choosing \( s > 0 \) sufficiently small, the statement of Lemma 3.3 follows. □

Since \( J_\lambda^+(w_1, w_2) \) is weakly lower semicontinuous in \( \mathcal{A} \), by lemma 3.3 we conclude that \( J_\lambda^+(w_1, w_2) \) attains the infimum in \( \mathcal{A} \).

Next we show that, for \( \lambda \) is sufficiently large, the minimizer of \( J_\lambda^+ \) belongs to the interior of \( \mathcal{A} \). To this end, we observe the following:

**Lemma 3.4** There exists a positive constant \( C_4 \), independent of \( \lambda \), such that,
\[
\inf_{(w_1, w_2) \in \partial \mathcal{A}} J_\lambda^+(w_1, w_2) \geq \frac{\| \Omega \|}{2} \min \left\{ 1 + \frac{d}{b}, 1 + \frac{a}{c} \right\} \lambda - C_4 (\ln \lambda + \sqrt{\lambda} + 1). \quad (3.41)
\]

**Proof.** On the boundary of \( \mathcal{A} \), we have
\[
\left( \int_{\Omega} e^{u_0^2 + w_1} \, dx \right)^2 = \frac{16\pi a(dN_1 + bN_2)}{\lambda(ad - bc)} \int_{\Omega} e^{2u_0^2 + 2w_1} \, dx \quad (3.42)
\]
or
\[
\left( \int_{\Omega} e^{u_0^2 + w_2} \, dx \right)^2 = \frac{16\pi d(cN_1 + aN_2)}{\lambda(ad - bc)} \int_{\Omega} e^{2u_0^2 + 2w_2} \, dx. \quad (3.43)
\]
Suppose for example that (3.42) holds. Then using (3.28) and Hölder inequality we obtain
\[
e^{c^+} \int_{\Omega} e^{u_0^2 + w_1} \, dx \leq \frac{d}{b + d} \left( \int_{\Omega} e^{u_0^2 + w_1} \, dx \right)^2 + \frac{b}{b + d} \int_{\Omega} e^{u_0^2 + w_1} \, dx \int_{\Omega} e^{u_0^2 + w_2} \, dx \int_{\Omega} e^{u_0^2 + u_0^2 + w_1 + w_2} \, dx
\leq \frac{16\pi d(dN_1 + bN_2)}{\lambda(b + d)(ad - bc)} + \frac{4b\sqrt{\pi a(dN_1 + bN_2)\| \Omega \|}}{\sqrt{\lambda(ad - bc)(b + d)}}.
which implies:
\[
\frac{\lambda}{2} \left[ \left(1 + \frac{d}{b} \right) \int_{\Omega} \left(1 - e^{c_1^+ u_0^1 + w_1^1}\right) dx + \left(1 + \frac{a}{c} \right) \int_{\Omega} \left(1 - e^{c_2^+ u_0^2 + w_2^2}\right) dx \right] \\
\geq \frac{\Omega}{2} \left(1 + \frac{d}{b} \right) \lambda - C_5 \sqrt{\lambda} - C_6
\]
with $C_5 > 0$ and $C_6 > 0$ suitable constants independent of $\lambda$.

Now, estimating $c_1^+, c_2^+$ as in Lemma 3.3, we arrive at the estimate (3.31). □

At this point, we need to test $J_{\lambda}^+$ over a suitable function in the interior of $A$, for which the opposite inequality in (3.41) holds. To this end, we follow [36] and recall that, for $\mu > 0$ sufficiently large, there exist periodic solutions $v_\mu^i$ $(i = 1, 2)$ for the problem:
\[
\Delta v = \mu e^{u_0^i + v} \left(e^{u_0^i + v} - 1\right) + \frac{4\pi N}{|\Omega|} \quad \text{in} \quad \Omega
\]
(3.44)
such that $u_0^i + v_0^i < 0$ in $\Omega$, $c_i^+ := \frac{1}{|\Omega|} \int_{\Omega} v_0^i dx \to 0$ and $w_\mu^i := v_\mu^i - c_i^+ \to -u_0^i$ pointwise a.e. as $\mu \to +\infty$. Those facts were proved in [16].

Since $e^{u_0^i} \in L^\infty(\Omega)$ $(i = 1, 2)$, by the dominated convergence theorem, we have
\[
e^{u_0^i + w_\mu^i} \to 1 \quad \text{strongly in} \quad L^p(\Omega) \quad \text{for any} \quad p \geq 1
\]
as $\mu \to +\infty$. In particular,
\[
\int_{\Omega} e^{2u_0^i + 2w_\mu^i} dx \to |\Omega|, \quad i = 1, 2
\]
as $\mu \to +\infty$. Therefore, for $\lambda_0$ large and for fixed $\varepsilon \in (0, 1)$, we can find $\mu_\varepsilon > 1$, so that $(w_1, w_2)$ lies in the interior of $A$ for every $\lambda > \lambda_0$, and the following holds:
\[
\frac{(ad-bc)(\Omega)}{a+c} \left[a \int_{\Omega} e^{2u_0^1 + 2w_\mu^1} dx + c|\Omega| \right] \\
\geq \frac{(ad-bc)(\Omega)}{b+d} \left[d \int_{\Omega} e^{2u_0^2 + 2w_\mu^2} dx + b|\Omega| \right] \\
\frac{1}{1 - \varepsilon},
\]
(3.45)
\[
\frac{(ad-bc)(\Omega)}{a+b-d} \left[a \int_{\Omega} e^{2u_0^1 + 2w_\mu^1} dx + b|\Omega| \right] \\
\geq \frac{(ad-bc)(\Omega)}{b+d} \left[d \int_{\Omega} e^{2u_0^2 + 2w_\mu^2} dx + b|\Omega| \right] \\
\frac{1}{1 - \varepsilon},
\]
(3.46)

Using Jensen’s inequality, and Remark 3.1 in view of (3.19)-(3.20) by a straightforward calculation we get
\[
e^{c_1^+(w_{1\mu}, w_{2\mu})} \geq \frac{ad-bc}{d(a+b-d)} \int_{\Omega} e^{u_0^1 + w_{1\mu}} dx + \frac{b(a+c)}{a(b+d)} e^{c_2^+(w_{1\mu}, w_{2\mu})} \int_{\Omega} e^{u_0^1 + u_0^2 + w_{1\mu} + w_{2\mu}} dx
\]
\[
\times \left[ 1 + \sqrt{1 - \frac{16\pi a(dN_1 + bN_2)}{\lambda(ad-bc)} \int_{\Omega} e^{2u_0^1 + 2w_\mu^1} dx} \right] \left(\int_{\Omega} e^{u_0^1 + w_\mu^1} dx\right)^2
\]
\[
\geq \frac{ad-bc}{d(a+b-d)} |\Omega| + \frac{b(a+c)}{d(a+b-d)} |\Omega| e^{c_2^+(w_{1\mu}, w_{2\mu})} \int_{\Omega} e^{2u_0^1 + 2w_\mu^1} dx
\]
\[
\geq \frac{8\pi a(dN_1 + bN_2)}{\lambda(ad-bc)|\Omega|}.
\]
(3.47)

and similarly
\[
e^{c_2^+(w_{1\mu}, w_{2\mu})} \geq \frac{ad-bc}{d(a+c)} |\Omega| + \frac{c(b+d)}{d(a+c)} |\Omega| e^{c_1^+(w_{1\mu}, w_{2\mu})} \int_{\Omega} e^{2u_0^1 + 2w_\mu^1} dx
\]
\[
\geq \frac{8\pi d(cN_1 + aN_2)}{\lambda(ad-bc)|\Omega|}.
\]
(3.48)
Then inserting (3.48) into (3.47) we find,
\[
e^{c_i^+(w_{\mu c}, w_{\mu c}^2)} \geq \frac{ad-bc}{a+b+d} \frac{\text{Vol}(\Omega)}{b+d} \left[ \int_{\Omega} e^{2u_{\mu c} + 2w_{\mu c}^2} \, dx + b|\Omega| \right] + \frac{b(a+c)}{a(b+d)} \frac{\text{Vol}(\Omega)}{a+b} \left[ \int_{\Omega} e^{2u_{\mu c} + 2w_{\mu c}^2} \, dx \right] - \frac{8\pi d}{\lambda(ad - bc)|\Omega|} \left[ a(dN_1 + bN_2) + \frac{bd(a + c)(cN_1 + aN_2)}{a(b + d)} \right],
\]
which implies
\[
e^{c_i^+(w_{\mu c}, w_{\mu c}^2)} \geq \frac{(ad-bc)|\Omega|}{a+b+d} \left[ \int_{\Omega} e^{2u_{\mu c} + 2w_{\mu c}^2} \, dx + b|\Omega| \right] + \frac{b(a+c)}{a(b+d)} \frac{(ad-bc)|\Omega|^2}{a+b} \left[ \int_{\Omega} e^{2u_{\mu c} + 2w_{\mu c}^2} \, dx \right] - \frac{8\pi d}{\lambda(ad - bc)|\Omega|} \left[ a(dN_1 + bN_2) + \frac{bd(a + c)(cN_1 + aN_2)}{a(b + d)} \right].
\]
Similarly, we get
\[
e^{c_2^+(w_{\mu c}, w_{\mu c}^2)} \geq \frac{(ad-bc)|\Omega|}{a+b+d} \left[ \int_{\Omega} e^{2u_{\mu c} + 2w_{\mu c}^2} \, dx + c|\Omega| \right] + \frac{b(a+c)}{a(b+d)} \frac{(ad-bc)|\Omega|^2}{a+b} \left[ \int_{\Omega} e^{2u_{\mu c} + 2w_{\mu c}^2} \, dx \right] - \frac{8\pi d}{\lambda(ad - bc)|\Omega|} \left[ d(cN_1 + aN_2) + \frac{ac(b + d)(dN_1 + bN_2)}{d(a + c)} \right].
\]
Then, by combining (3.49)–(3.50) and (3.45)–(3.46) we conclude that,
\[
e^{c_i^+(w_{\mu c}, w_{\mu c}^2)} \geq 1 - \varepsilon - \frac{8\pi a d}{\lambda(ad - bc)^2|\Omega|} \left[ a(dN_1 + bN_2) + \frac{bd(a + c)(cN_1 + aN_2)}{a(b + d)} \right],
\]
\[
e^{c_2^+(w_{\mu c}, w_{\mu c}^2)} \geq 1 - \varepsilon - \frac{8\pi a d}{\lambda(ad - bc)^2|\Omega|} \left[ d(cN_1 + aN_2) + \frac{ac(b + d)(dN_1 + bN_2)}{d(a + c)} \right];
\]
for all \( \lambda > \lambda_0 \).

As a consequence, for all \( \lambda > \lambda_0 \), we obtain that,
\[
\int_{\Omega} \left( 1 - e^{c_i^+(w_{\mu c}, w_{\mu c}^2)} e^{u_{\mu c} + 2w_{\mu c}^2} \right) \, dx \leq |\Omega|\varepsilon + \frac{8\pi a d}{\lambda(ad - bc)^2} \left[ a(dN_1 + bN_2) + \frac{bd(a + c)(cN_1 + aN_2)}{a(b + d)} \right],
\]
\[
\int_{\Omega} \left( 1 - e^{c_2^+(w_{\mu c}, w_{\mu c}^2)} e^{u_{\mu c}^2 + 2w_{\mu c}^2} \right) \, dx \leq |\Omega|\varepsilon + \frac{8\pi a d}{\lambda(ad - bc)^2} \left[ d(cN_1 + aN_2) + \frac{ac(b + d)(dN_1 + bN_2)}{d(a + c)} \right].
\]

**Lemma 3.5** For \( \lambda > 0 \) sufficiently large, there holds:
\[
J^+_\lambda(w_{\mu c}^1, w_{\mu c}^2) - \inf_{(w_1, w_2) \in \partial A} J^+_\lambda(w_1, w_2) < -1.
\]
Proof. Using (3.51)-(3.52) and the fact that \(c_1^+ \leq 0, c_2^+ \leq 0\), we conclude that, for any small \(\varepsilon \in (0,1)\), there exists a constant \(C_\varepsilon > 0\) such that,

\[
J_\lambda^+(w_{\mu c}^1, w_{\mu c}^2) \leq \frac{[\Omega]}{2} \left(2 + \frac{d}{b} + \frac{a}{c}\right) \varepsilon \lambda + C_\varepsilon.
\] (3.54)

So by virtue of Lemma 3.4, we get

\[
J_\lambda^+(w_{\mu c}^1, w_{\mu c}^2) - \inf_{(w_1, w_2) \in \partial A} J_\lambda^+(w_1, w_2)
\leq \frac{[\Omega]}{2} \min \left\{1 + \frac{d}{b}, 1 + \frac{a}{c}\right\} \left[-1 + \left(2 + \frac{d}{b} + \frac{a}{c}\right) \varepsilon \right] \lambda + C(\ln \lambda + \sqrt{\lambda} + 1),
\] (3.55)

with \(C > 0\) independent of \(\lambda\). Clearly, (3.53) easily follows from (3.55) by choosing \(\varepsilon > 0\) sufficiently small and \(\lambda > 0\) sufficiently large. \(\square\)

From Lemma 3.3 and 3.5, we easily conclude:

Corollary 3.1 There exists \(\tilde{\lambda} > 0\) such that for every \(\lambda > \tilde{\lambda}\), the functional \(J_\lambda^+\) attains its minimum at a point \((w_{1,\lambda}, w_{2,\lambda})\), which lies in the interior of \(A\). Furthermore,

\[
v_{1,\lambda}^+ = w_{1,\lambda} + c_1^+(w_{1,\lambda}, w_{2,\lambda}), \quad v_{2,\lambda}^+ = w_{2,\lambda} + c_2^+(w_{1,\lambda}, w_{2,\lambda})
\] (3.56)

defines a critical point for the functional \(I_\lambda\) in \(H(\Omega) \times H(\Omega)\), namely a (weak) solution for (3.2).

Concerning such a solution we prove:

Proposition 3.2 Let \((v_{1,\lambda}^+, v_{2,\lambda}^+)\) be the solution of (3.2) found above and defined by (3.56). We have

i) \(e^{u_0^i + v_{i,\lambda}^+} \to 1\), as \(\lambda \to +\infty\) \((i = 1, 2)\)

pointwise a.e. in \(\Omega\) and in \(L^p(\Omega)\) for any \(p \geq 1\).

ii) \((v_{1,\lambda}^+, v_{2,\lambda}^+)\) defines a local minimum for \(I_\lambda\) in \(H(\Omega) \times H(\Omega)\).

Proof. If we use (3.33) together with (3.54) we readily find that,

\[
\int_{\Omega} \left(e^{u_0^i + v_{i,\lambda}^+} - 1\right)^2 dx \to 0, \quad \text{as} \quad \lambda \to +\infty, \quad i = 1, 2.
\] (3.58)

Since, by Proposition 3.1, we know that \(e^{u_0^i + v_{i,\lambda}^+} < 1, e^{u_0^i + v_{i,\lambda}^+} < 1\) in \(\Omega\), so by the dominated convergence theorem we conclude:

\[
e^{u_0^i + v_{i,\lambda}^+} \to 1, \quad e^{u_0^i + v_{i,\lambda}^+} \to 1 \quad \text{as} \quad \lambda \to +\infty
\]

pointwise a.e. in \(\Omega\) and in \(L^p(\Omega), \forall p \geq 1\).

To establish ii), we check that for any \((w_1, w_2) \in A\) and corresponding \((c_1, c_2)\) given by (3.9), (3.10), we have:

\[
\partial_{c_1} I_\lambda(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) = 0 = \partial_{c_2} I_\lambda(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2))
\]
and

\[
\partial_{c_1}^2 I_{\lambda}(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) = \frac{a(b + d)^2}{b(ad - bc)} \left[ 2e^{2c_1} \int_{\Omega} e^{2u_0 + 2w_1} \, dx - e^{c_1} R_1(w_1, w_2, e^{c_2}) \right], \tag{3.59}
\]

\[
\partial_{c_2}^2 I_{\lambda}(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) = \frac{d(a + c)^2}{c(ad - bc)} \left[ 2e^{2c_2} \int_{\Omega} e^{2u_0 + 2w_2} \, dx - e^{c_2} R_2(w_1, w_2, e^{c_1}) \right], \tag{3.60}
\]

\[
\partial_{c_1 c_2}^2 I_{\lambda}(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2)) = -\frac{(a + c)(b + d)\lambda}{(ad - bc)} \int_{\Omega} e^{u_0 + u_0^2 + w_1 + w_2} \, dx. \tag{3.61}
\]

Next we use (3.59)-(3.61) with \((c_1, c_2) = (c_1^+, c_2^+)\), so that (3.19) and (3.20) hold with + sign. Thus, for \(v_i^+ = w_i + c_i^+, i = 1, 2\), after straightforward calculation we find:

\[
\partial_{c_1}^2 I_{\lambda}(v_1^+, v_2^+) = \frac{a(b + d)^2}{b(ad - bc)} \left\{ \left[ \frac{ad - bc}{a(b + d)} \int_{\Omega} e^{u_0 + v_1^+} \, dx + \frac{b(a + c)}{a(b + d)} \int_{\Omega} e^{u_0 + u_0^2 + v_1^+ + v_2^+} \, dx \right]^2 - \frac{16\pi(ad - bc)(dN_1 + bN_2)}{\lambda a(b + d)^2} \int_{\Omega} e^{2u_0 + 2v_1^+} \, dx \right\}^{\frac{1}{2}},
\]

\[
\partial_{c_2}^2 I_{\lambda}(v_1^+, v_2^+) = \frac{d(a + c)^2}{c(ad - bc)} \left\{ \left[ \frac{ad - bc}{d(a + c)} \int_{\Omega} e^{u_0 + v_2^+} \, dx + \frac{c(b + d)}{d(a + c)} \int_{\Omega} e^{u_0 + u_0^2 + v_1^+ + v_2^+} \, dx \right]^2 - \frac{4\pi(ad - bc)(cN_1 + aN_2)}{\lambda d(a + c)^2} \int_{\Omega} e^{2u_0 + 2v_2^+} \, dx \right\}^{\frac{1}{2}}.
\]

In case \((w_1, w_2)\) lies in the interior of \(A\), then we can use the strict inequality in (3.16), (3.17) and obtain

\[
\partial_{c_1}^2 I_{\lambda}(v_1^+, v_2^+) > \frac{(a + c)(b + d)\lambda}{(ad - bc)} \int_{\Omega} e^{u_0 + u_0^2 + v_1^+ + v_2^+} \, dx,
\]

\[
\partial_{c_2}^2 I_{\lambda}(v_1^+, v_2^+) > \frac{(a + c)(b + d)\lambda}{(ad - bc)} \int_{\Omega} e^{u_0 + u_0^2 + v_1^+ + v_2^+} \, dx.
\]

Therefore we have checked that, if \((w_1, w_2)\) is an interior point of \(A\) then the Hessian matrix of \(I_{\lambda}(w_1 + c_1(w_1, w_2), w_2 + c_2(w_1, w_2))\) with respect to \((c_1, c_2)\) is strictly positive definite at \((c_1^+(w_1, w_2), c_2^+(w_1, w_2))\). We apply such property, near the critical point \((v_{1,\lambda}^+, v_{2,\lambda}^+)\). Indeed, by continuity, for \(\delta > 0\) sufficiently small, we can ensure that, if \((v_1, v_2) = (w_1 + c_1, w_2 + c_2)\) satisfies:

\[
\|v_1 - v_{1,\lambda}^+\| + \|v_2 - v_{2,\lambda}^+\| \leq \delta,
\]

then \((w_1, w_2)\) belongs to the interior of \(A\) and

\[
I_{\lambda}(v_1, v_2) = I_{\lambda}(w_1 + c_1, w_2 + c_2) \geq I_{\lambda}(w_1 + c_1^+(w_1, w_2), w_2 + c_2^+(w_1, w_2)) = J_{\lambda}(w_1, w_2) \geq I_{\lambda}(v_{1,\lambda}^+, v_{2,\lambda}^+).
\]

Consequently, \((v_{1,\lambda}^+, v_{2,\lambda}^+)\) defines a local minimizer for \(I_{\lambda}\) in \(H(\Omega) \times H(\Omega)\), as desired. \(\square\)
3.2 Mountain-Pass solution

To complete the proof of Theorem 2.2 it remains to establish the existence of a second solution. Again we use the variational approach and show that the functional $I_{\lambda}$ admits also a “saddle” critical point of “mountain-pass” type. We start to establish the following:

**Lemma 3.6** The functional $I_{\lambda}$ satisfies the (P.S.) condition in $H(\Omega) \times H(\Omega)$. Namely, every sequence $(v_{1,n}, v_{2,n}) \in H(\Omega) \times H(\Omega)$ satisfying:

\[ I_{\lambda}(v_{1,n}, v_{2,n}) \to m_0 \quad \text{as} \quad n \to +\infty, \quad (3.62) \]

\[ \|I'_{\lambda}(v_{1,n}, v_{2,n})\| \to 0 \quad \text{as} \quad n \to +\infty, \quad (3.63) \]

admits a strongly convergent subsequence in $H(\Omega) \times H(\Omega)$, where $m_0$ is a constant and $\|\cdot\|$ denotes the norm of dual space of $H(\Omega) \times H(\Omega)$.

**Proof.** Let $\varepsilon_n = \|I'_{\lambda}(v_{1,n}, v_{2,n})\| \to 0$, $n \to +\infty$, and observe that $\forall (\psi_1, \psi_2) \in H(\Omega) \times H(\Omega)$, we have:

\[
I'_{\lambda}(v_{1,n}, v_{2,n})[(\psi_1, \psi_2)] = \frac{d}{b} \int_\Omega \nabla v_{1,n} \cdot \nabla \psi_1 dx + \frac{a}{c} \int_\Omega \nabla v_{2,n} \cdot \nabla \psi_2 dx + \int_\Omega \nabla v_{1,n} \cdot \nabla v_{2,n} dx + \int_\Omega \nabla v_{2,n} \cdot \nabla \psi_1 dx \\
+ \frac{\lambda}{ab(ad - bc)} \int_\Omega Q_1(v_{1,n}, v_{2,n}) \left[ a(b + d)e^{a_1 + v_{1,n}} - b(a + c)e^{a_2 + v_{2,n}} \right] \psi_1 dx \\
+ \frac{\lambda (a + c)^2}{ac} \int_\Omega Q_2(v_{2,n})e^{a_2 + v_{2,n}} \psi_2 dx + \frac{\alpha_1}{|\Omega|} \int_\Omega \psi_1 dx + \frac{\alpha_2}{|\Omega|} \int_\Omega \psi_2 dx 
\]

(3.64)

and

\[ |I'_{\lambda}(v_{1,n}, v_{2,n})(\psi_1, \psi_2)| \leq \varepsilon_n (\|\psi_1\| + \|\psi_2\|). \quad (3.65) \]

In particular, if we take $\psi_1 = \psi_2 = \psi \in H(\Omega)$ in (3.64) we find:

\[
I'_{\lambda}(v_{1,n}, v_{2,n})[(\psi, \psi)] = \int_\Omega \nabla \left[ \frac{b + d}{b} v_{1,n} + \frac{a + c}{c} v_{2,n} \right] \cdot \nabla \psi \\
+ 2\lambda \int_\Omega Q(v_{1,n}, v_{2,n}) \psi dx + \frac{\lambda}{ab} \int_\Omega Q_1(v_{1,n}, v_{2,n}) \psi dx \\
+ \frac{\lambda (a + c)^2}{ac} \int_\Omega Q_2(v_{2,n}) \psi dx + \frac{\alpha_1 + \alpha_2}{|\Omega|} \int_\Omega \psi dx, \quad (3.66) \]

where we recall that $Q, Q_1, Q_2$ and $\alpha_1, \alpha_2$ are defined in (3.5).

As a consequence for $\psi \equiv 1$, we deduce that,

\[
\int_\Omega Q_1^2(v_{1,n}, v_{2,n}) dx + \int_\Omega Q_2^2(v_{2,n}) dx \leq C
\]

for some suitable constant $C > 0$. In particular,

\[
\int_\Omega e^{2(v_{1,n}^0 + v_{1,n})} dx + \int_\Omega e^{2(v_{2,n}^0 + v_{2,n})} dx \leq C \quad (3.67)
\]

with a (possible different) $C > 0$. 

21
Decompose \( v_{j,n} = w_{j,n} + c_{j,n} \) with \( \int_\Omega w_{j,n} \, dx = 0 \), \( c_{j,n} = \frac{1}{|\Omega|} \int_\Omega v_{j,n} \, dx \) \((j = 1, 2)\) and observe that, (by assumption)

\[
I_\lambda(v_{1,n}, v_{2,n}) = \frac{1}{2} \left[ \left( \frac{d}{b} - \frac{c}{a} \right) \|\nabla v_{1,n}\|^2 + \left\| \nabla \left( \sqrt{\frac{c}{a}} v_{1,n} + \sqrt{\frac{a}{c}} v_{2,n} \right) \right\|^2 \right] \\
+ \lambda \int_\Omega Q(v_{1,n}, v_{2,n}) \, dx + \alpha_1 c_{1,n} + \alpha_2 c_{2,n} \to m_0 \quad \text{as} \quad n \to +\infty. \quad (3.68)
\]

Moreover, from \((3.67)\) and Jensen’s inequality, we find

\[
c_{j,n} \leq c_0, \quad \forall n \in \mathbb{N}, \quad j = 1, 2 \quad (3.69)
\]

with suitable \( c_0 > 0 \).

Next, let

\[
z_n = \frac{b + d}{b} w_{1,n} + \frac{a + c}{c} w_{2,n} \quad (3.70)
\]

so that \( \int_\Omega z_n \, dx = 0 \). If we take in \((3.66)\) \( \psi = z^+_n = \max\{z_n, 0\} \), from \((3.67)\) we find

\[
\left\| \nabla z_n^+ \right\|^2_2 + \frac{\lambda}{ad - bc} \int_\Omega \left[ \sqrt{\frac{a}{b}} (b + d) e^{u_{1,n}^0 + v_{1,n}} - \sqrt{\frac{d}{c}} (a + c) e^{u_{2,n}^0 + v_{2,n}} \right] z_n^+ \, dx \\
+ \frac{2\lambda(a + c)(b + d)}{ad - bc} \left( \sqrt{\frac{ad}{bc}} - 1 \right) \int_\Omega e^{u_{1,n}^0 + v_{1,n}} e^{u_{2,n}^0 + v_{2,n}} z_n^+ \, dx \\
\leq C(\|z_n^+\|_2 + \varepsilon_n \|z_n^+\|) \quad (3.71)
\]

Then from \((3.71)\) and Poincaré inequality we obtain

\[
\int_\Omega e^{u_{1,n}^0 + v_{1,n}} e^{u_{2,n}^0 + v_{2,n}} z_n^+ \, dx \leq C(\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2). \quad (3.72)
\]

To proceed further we choose \( \psi_1 = w_{1,n} \) and \( \psi_2 = w_{2,n} \) in \((3.64)\) and after straightforward calculations we obtain:

\[
I'(v_{1,n}, v_{2,n})[(w_{1,n}, w_{2,n})] = \left( \frac{d}{b} - \frac{c}{a} \right) \|\nabla w_{1,n}\|^2_2 + \left\| \nabla \left( \sqrt{\frac{c}{a}} w_{1,n} + \sqrt{\frac{a}{c}} w_{2,n} \right) \right\|^2_2 \\
+ \frac{\lambda}{ab(ad - bc)} \left[ \int_\Omega a^2(b + d)^2 e^{2(u_{1,n}^0 + v_{1,n})} w_{1,n} + b^2(a + c)^2 e^{2(u_{2,n}^0 + v_{2,n})} w_{2,n} \right. \\
- ab(a + c)(b + d) e^{u_{1,n}^0 + v_{1,n}} e^{u_{2,n}^0 + v_{2,n}} (w_{1,n} + w_{2,n}) \left] + \frac{\lambda(a + c)^2}{ac} \int_\Omega e^{2(u_{2,n}^0 + v_{2,n})} w_{2,n} \, dx \\
- \lambda \left[ \frac{b + d}{b} \int_\Omega e^{u_{1,n}^0 + v_{1,n}} w_{1,n} \, dx + \frac{a + c}{c} \int_\Omega e^{u_{2,n}^0 + v_{2,n}} w_{2,n} \, dx \right]. \quad (3.73)
\]

Clearly, in view of \((3.67)\) we can estimate

\[
\left| \int_\Omega e^{u_{1,n}^0 + v_{j,n}} w_{j,n} \, dx \right| \leq C \|w_{j,n}\|_2.
\]

While from \((3.69)\) we get

\[
\int_\Omega e^{2(u_{1,n}^0 + v_{j,n})} w_{j,n} \, dx = \int_\Omega e^{2(u_{1,n}^0 + v_{j,n})} (w_{j,n} - 1) w_{j,n} \, dx + \int_\Omega e^{2(u_{1,n}^0 + v_{j,n})} w_{j,n} \, dx \\
\geq -e^{c_0} \|e^{2u_{1,n}^0}\|_2 \|w_{j,n}\|_2 \\
\geq -C \|\nabla w_{j,n}\|_2, \quad j = 1, 2.
\]
Furthermore, we see that
\[
\int_{\Omega} e^{u_0^1+v_1,n}e^{u_0^2+v_2,n}(w_{1,n}+w_{2,n})dx \\
\leq \int_{\Omega} e^{u_0^1+v_1,n}e^{u_0^2+v_2,n}(w_{1,n}+w_{2,n})_+dx \\
= \int_{\{w_{1,n}\leq 0\leq w_{2,n}\}} e^{u_0^1+c_1,n}(e^{w_{1,n}}-1)e^{u_0^2+v_2,n}(w_{1,n}+w_{2,n})_+dx \\
+ \int_{\{w_{1,n}\leq 0\leq w_{2,n}\}} e^{u_0^1+c_2,n}(e^{w_{2,n}}-1)e^{u_0^2+v_1,n}(w_{1,n}+w_{2,n})_+dx \\
+ \int_{\{w_{2,n}\leq 0\leq w_{1,n}\}} e^{u_0^1+c_2,n}e^{u_0^2+v_1,n}(w_{1,n}+w_{2,n})_+dx \\
+ \int_{\{w_{2,n}\leq 0\leq w_{1,n}\}} e^{u_0^1+v_1,n}e^{u_0^2+v_2,n}(w_{1,n}+w_{2,n})_+dx \\
\leq C (\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2) + \left(\frac{b}{b+d} + \frac{c}{a+c}\right) \int_{\Omega} e^{u_0^1+v_1,n}e^{u_0^2+v_2,n} w_{1,n}^+ \cdot dx.
\]

Thus from (3.72), we conclude:
\[
\int_{\Omega} e^{u_0^1+v_1,n}e^{u_0^2+v_2,n}(w_{1,n}+w_{2,n})dx \leq C (\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2).
\]

Using the estimates above, together with (3.73) and (3.63), we conclude that
\[
\left(\frac{d}{b} - \frac{c}{a}\right) \|\nabla w_{1,n}\|_2 + \|\nabla \left(\sqrt{\frac{c}{a}}w_{1,n} + \sqrt{\frac{a}{c}}w_{2,n}\right)\|_2^2 \leq C. \tag{3.74}
\]

Now from (3.68), (3.69) and (3.74), we deduce that \{c_{j,n}\} is also uniformly bounded from below, for \(j = 1, 2\).

Consequently, \{v_{j,n}\} is a uniformly bounded sequence in \(H(\Omega)\), for \(j = 1, 2\). So, along a subsequence, (denoted the same way), and for suitable \(v_j \in H(\Omega)\) \((j = 1, 2)\) we have:
\[
v_{j,n} \rightharpoonup v_j \text{ as } n \to +\infty \text{ weakly in } H(\Omega), \text{ and strongly in } L^p(\Omega), \ p \geq 1 \text{ and pointwise a.e. in } \Omega; \\
e^{u_0^1+v_{j,n}} \to e^{u_0^1+v_j} \text{ as } n \to +\infty \text{ in } L^p(\Omega), \ p \geq 1; \ j = 1, 2.
\]

In particular, \((v_1, v_2)\) is a critical point for \(I_\lambda\) and by the above convergence properties we have:
\[
\left(\frac{d}{b} - \frac{c}{a}\right) \|\nabla (v_{1,n} - v_1)\|_2^2 + \|\nabla \left(\sqrt{\frac{c}{a}}(v_{1,n} - v_1) + \sqrt{\frac{a}{c}}(v_{2,n} - v_2)\right)\|_2^2 = (I'_\lambda(v_{1,n}, v_{2,n}) - I'_\lambda(v_1, v_2)) [(v_{1,n} - v_1, v_{2,n} - v_2)] + o(1) \to 0, \text{ as } \ n \to +\infty.
\]

Thus, \(v_{j,n} \rightharpoonup v_j \) strongly in \(H(\Omega)\) as \(n \to +\infty, \ j = 1, 2\); and the proof of Lemma 3.6 is completed. □
To proceed further, we need to use the minimization property of \((v^+_{1,\lambda}, v^+_{2,\lambda})\) as given in Proposition 3.2(ii). In case it defines a degenerate (local) minimum for \(I_\lambda\), in the sense that for every \(\delta > 0\) sufficiently small,
\[
\inf_{\{\|v_1 - v^+_{1,\lambda}\| + \|v_2 - v^+_{2,\lambda}\| = \delta\}} \inf \{I_\lambda(v_1, v_2) = I_\lambda(v^+_{1,\lambda}, v^+_{2,\lambda}).
\]
then we obtain a 1-parameter family of (degenerate) local minima of \(I_\lambda\), (see Corollary 1.6 of [18]), and the conclusion of Theorem 2.2 is obviously established in this case.

Hence, we suppose that \((v^+_{1,\lambda}, v^+_{2,\lambda})\) defines a strict local minimum for \(I_\lambda\). In particular, for \(\delta > 0\) sufficiently small, the following holds:
\[
I_\lambda(v^+_{1,\lambda}, v^+_{2,\lambda}) < \inf_{\{\|v_1 - v^+_{1,\lambda}\| + \|v_2 - v^+_{2,\lambda}\| = \delta\}} I_\lambda(v_1, v_2) := \gamma_0.
\]

In addition, we observe that,
\[
I_\lambda(v^+_{1,\lambda} - \xi, v^+_{2,\lambda} - \xi) \to -\infty, \quad \text{as} \quad \xi \to +\infty.
\]
Hence, for fixed \(\bar{\xi} > 1\) sufficiently large and \(\bar{v}_i = v^+_{i,\lambda} - \bar{\xi}, \ i = 1, 2\), we find
\[
\|v^+_{1,\lambda} - \bar{v}_1\| + \|v^+_{2,\lambda} - \bar{v}_2\| > \delta \quad \text{and} \quad I_\lambda(\bar{v}_1, \bar{v}_2) < I_\lambda(v^+_{1,\lambda}, v^+_{2,\lambda}).
\]

Lemma 3.6 together with (3.75) and (3.76), allow us to use the “mountain-pass” lemma of Ambrosetti-Rabinowitz [2] and conclude the existence of a second critical point \((\bar{v}_{1,\lambda}, \bar{v}_{2,\lambda})\) for \(I_\lambda\) satisfying:
\[
I_\lambda(\bar{v}_{1,\lambda}, \bar{v}_{2,\lambda}) \geq \gamma_0 > I_\lambda(v^+_{1,\lambda}, v^+_{2,\lambda}).
\]
By virtue of (3.77), such critical point yields to a solution for (3.2) distinct from \((v^+_{1,\lambda}, v^+_{2,\lambda})\). This completes the proof of Theorem 2.2.

It would be interesting to see whether, as for the gauge group \(SU(3)\), a stronger multiplicity result holds, in relation to each vacua state of the system.

For example, it is natural to expect that the “mountain-pass” solution is asymptotically gauge equivalent to the unbroken vacuum for \(\lambda \to +\infty\); as it occurs in the Abelian case, see [15].

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