Non-fermi liquid as passive scalar fluid

Jonathan Miller
NEC Research Institute, Princeton, NJ 08540

I suggest that electron localization by random flux and passive transport in quenched velocity fields in two dimensions be studied as perturbations of the simple operator $K = \mathbf{A} \cdot \nabla$, with incompressible velocity field/vector potential $\mathbf{A} = \nabla \times \phi = (-\partial_y, \partial_x) \phi$. This operator has an infinitely degenerate subspace of zero energy eigenstates, arising from incompressibility, that are extended for generic $\phi(x)$ and are expected to remain so under perturbation. I propose that an anomaly accounts qualitatively for properties of the spectrum and eigenstates of $K$ and its perturbations.

In two dimensions, all eigenstates of the hermitian operator $H = -\nabla^2 + V(x)$ are localized for random $V$, no matter how small the randomness (Anderson localization). Over the years it has emerged that certain kinds of fields can suppress localization. In particular, a strong uniform magnetic field $B = \nabla \times \mathbf{A} = \partial_x A_y - \partial_y A_x$ produces a dramatic change in the character of the spectrum of $H_B = (-i\nabla - \mathbf{A})^2/2m$, yielding Landau levels consisting of degenerate extended eigenstates that remain extended on perturbation by random $V$.

The underlying physics of this phenomenon, the quantum Hall effect, can be thought of as arising from incompressibility. In brief, taking $m \to 0$ in the Lagrangian $\mathcal{L}_B = m\dot{x}^2/2 + \mathbf{A} \cdot \dot{x}$, one obtains the action $\int dt \mathbf{A} \cdot \dot{x} = \int_0^\beta d\omega A_i$, which is manifestly invariant under area-preserving deformations of $\Omega$. The invariance persists after quantization - albeit in a different form - and is the origin of ‘quantum incompressibility’.

This version of incompressibility is expected to apply when there are spatial fluctuations of magnetic field, provided the mean magnetic field is non-vanishing. When the magnetic field has zero mean one enters a different regime that is perturbatively incompressible from Landau levels; rather, a different kind of incompressibility may provide a more suitable starting point for perturbative analysis. Such a regime can be achieved in two-dimensional systems with quenched spatial disorder, such as the random flux (RF) model $H_{rf} = (-i\nabla + \mathbf{A})^2$ (in gauge $\nabla \cdot \mathbf{A} = 0$ for our purposes), and the non-hermitian passive scalar advection-diffusion model (PS) $H_{ps} = \nabla^2 + \nabla \cdot (\mathbf{v})$, with velocity field $\mathbf{v} = \mathbf{A} + \mathbf{u}$ and vorticity $\omega = \nabla \times \mathbf{A}$, where $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{u} = 0$.

Common to both these models is a component of the form $K = \mathbf{A} \cdot \nabla$; for example, $H_{ps} = \nabla^2 + \nabla \cdot (\mathbf{v}) \to K$ as $m \to \infty$. As shown in this Letter, incompressibility determines the character of eigenstates of $K$ even when $B$ (or $\omega$) $= -\nabla^2 \phi$ is not uniform. Models containing $K$ are ubiquitous, and basic features of the spectrum and eigenstates of $K$ can be expected to survive perturbation by $-\nabla^2$ and $\mathbf{A}^2$ or $V$ provided the perturbations are sufficiently weak.

The quenched random fields of the models fall into two classes. The vorticity $\omega = \nabla \times \mathbf{A}$ in $H_{ps}$ is, like magnetic field, a pseudo-scalar quantity, in contrast to the scalar $\nabla \cdot \mathbf{u}$. Similarly, the randomness in $H_{rf}$ can be decomposed into parity-odd and -even contributions $i\mathbf{A} \cdot \nabla$ and $\mathbf{A}^2/4$. This distinction governs the contribution to scattering, and suggests the study of models, which I call RFI and PSI, that retain only the parity–odd random components of RF and PS respectively, with equations of motion:

$$(i)\partial_t \psi = \nabla \cdot \mathbf{J}_\psi = \nabla \cdot \{(-)\nabla \psi + (i)\mathbf{A} \psi\} \quad (1)$$

(The factors of $i$ and $-1$ should be chosen appropriately for each model, but with respect to the symmetry addressed in this paper these factors are inconsequential.)

The divergence form of (1) shows explicitly that RFI and PSI conserve three currents: the ordinary probability-current, conserved also when parity-even fields are present, but in addition the $\psi$-current $\mathbf{J}_\psi$ generated by $\psi \to \psi + C$ and its transpose $\psi^T$-current. When second quantized, the currents correspond to three non-commuting operators: the number operator $N_0 = \sum_i c_i^\dagger c_i$, and the $\psi$-current operators $c_i^\dagger = \sum_i c_i^\dagger$ and $c_0 = \sum_i c_i$, the sums being over positions in space. All three operators commute with $\nabla^2$ and parity-odd operators, but only $N_0$ commutes with potential disorder.

The other two fluxes embody a dynamical constraint on relaxation of both the real and imaginary parts of $\psi$: when it is satisfied, $\text{Re} \psi$ and $\text{Im} \psi$-fluctuations leaving one region must be exactly compensated somewhere else; $\text{Re} \mathbf{u}$ and $\mathbf{A}^2/4$ violate the constraint by introducing uncompensated sources and sinks of $\text{Re} \psi$ and $\text{Im} \psi$. When the current $\mathbf{J}_\psi$ and its transpose are conserved, I say that the coupling to the random field is incompressible.

In obtaining RFI from RF by neglecting $\mathbf{A}^2$, I have followed ref.[6], where standard methods are then used to map RFI to a unitary non-linear sigma model (NLSM), for which all eigenfunctions are localized. From the localization of all eigenfunctions of RFI, it is deduced in ref.[6] that all eigenfunctions of RF are localized. In contrast, I show below that an infinite number of eigenstates of RFI are extended, which suggests that when $\mathbf{A}^2$ can be properly treated as a perturbation to RFI, the corresponding
eigenfunctions of RF will be extended. The conservation
laws (1) should be reflected in the correct NLSM for in-
compressible disorder, but have no counterparts in the
unitary NLSM, indicating that the conclusions of ref.[6]
may not be universally valid.

Generically, \( tA \cdot \nabla \) does not commute with \( -\nabla^2 \), and the spectra of these operators differ qualita-
tively. If \( iK \psi = E \psi \), then \( iK \psi^T = -E \psi^T \) and \( iK \psi_m = n \psi_m \), so the spectrum is symmetric and unbounded. It follows
that at zero temperature, whereas the \( k = 0 \) mode of
a free particle is buried at the center of the fermi disc,
for \( iA \cdot \nabla \) this mode sits instead at the center of the
spectrum, yielding a dirac sea rather than a fermi sea.

To further study RFI, I at first examine the spectrum
and eigenfunctions of \( K \) alone, and only later restore
\( -\nabla^2 \). Eigenfunctions of \( iK = i \nabla \times \phi \cdot \nabla \) lie on char-
acteristics: the trajectories of \( \dot{x} = A \) or equivalently,
the streamlines (equipotentials) of \( \phi \). The symmetry of
\( K \) is elucidated by observing that \( K \psi \) is just a Poisson
bracket, if \( \phi \) is viewed as a Hamiltonian and \( y \) as momentum
to conjugate to \( x \): \( K \psi = \{ \psi, \phi \}_{PB} = \partial_x \psi \partial_y \phi - \partial_y \psi \partial_x \phi \), so that the
time evolution of \( \psi \) is given by \( 0 = i E \psi/dt = \partial_t \psi + \{ \psi, \phi \}_{PB} \). Incomplete reparameterizations are the
canonical transformations of \( x, y \). \( K \) is covariant
under this non-abelian group of transformations \( \mathbb{F} \).

Under a subgroup of the canonical transformations,
\( K \) is invariant. This subgroup represents translations
along the streamlines of \( \phi \); in infinitesimal form \( \eta \ll 1 \),
\( x = x' + \eta \nabla \times f(\phi) \), so that \( \phi(x) = \phi(x') \). These translations
are generated by \( f(\phi) \), which satisfies \( \nabla \times f(\phi) = 0 \) and represents an infinite set of conserved charges,
one for each streamline. The Lagrangian correspond-
ing to \( \phi(x, p) \) is not invariant under these translations,
but varies under canonical transformation by a total time
derivative \( \mathbb{F} \), which can lead to an anomaly \( \mathbb{F} \).

Time-independent canonical transformations act isospectrally on \( K \); it is instructive to rewrite the Hamiltonian in action-angle variables. Take \( s \) to be arc-length along \( \phi(x) = \phi_0 \) and set \( \chi = \int_{s}^x ds'/|\nabla \phi| \), where \( x(s) = 0 \) is an arbitrarily chosen point on the
streamline. Defining the period \( \tau(\phi_0) = \int_{\phi_0}^\phi d\chi \), one
obtains as variables the area enclosed by the streamline,
\( J(\phi) = \int_{\phi_0}^\phi y \ dx \) (action), and \( w = \chi/\tau \) (angle). The
eigenstates of \( iK = i \{ \tau(\phi_0) \}^{-1} \partial_w \) on equipotential \( \phi_0 \) are
\[
\psi_{m, \phi_0} = e^{i 2\pi m w} \delta \left( J(\phi) - J(\phi_0) \right) \tag{2}
\]
with eigenvalues \( E_{m, \phi_0} = 2\pi m/\tau(\phi_0) \). In this local pa-
rameterization, the form of the Hamiltonian is (half) a
chiral fermion, yielding the dirac sea obtained above.

It is helpful to examine the simpler case when \( \phi \) is a function of only the coordinate \( x \), where \( x \) lies on a
torus \( T^2 \) of side \( L \): \( \phi(x + L) = \phi(x) \). The spectrum is then \( \{ (2\pi m/L) \partial_x \phi(x) : 0 \leq x < L \} \). Zero modes are functions independent of \( y \), and the fourier modes \( \exp (i 2\pi n x/L) \), \( n \) integer, constitute a basis of extended
eigenstates at \( E = 0 \). Each non-zero eigenstate \( \delta(x - x_0) f(x) \exp (i 2\pi m y/L) \), for integer \( m \), corresponds to a
one-dimensional wave-function, confined transversely to
\( x_0 \) and extended longitudinally in \( y \) around the torus.

To see that the non-zero eigenstates are extended in the
\( y \) direction, one applies a phase twist \( \alpha \) at \( y = 0 \). A
unitary transformation by \( \exp iax/L \) then yields \( iK \alpha =
(\partial_x \phi)(i \partial_y + \alpha /L) \). As \( \alpha \) varies from 0 to \( 2\pi \), the eigen-
values for any fixed \( x_0 \) shift up or down an energy level
depending on the sign of \( \partial_x \phi \), whereas the wavefunctions
remain constant. This sensitivity of eigenvalues to pertur-
bation at the boundary is known as ‘spectral flow,’
and is the hallmark of an anomaly \( \mathbb{F} \).

So long as \( \phi \) is independent of \( y \), the model can be
directly rewritten in fourier basis on an \( N \times N \) lattice.
Of the \( N^2 \) modes, \( N^2 - N \) are extended in the \( y \) direction and localized in the \( x \) direction, and the remaining \( N \) zero modes, the \( m = 0 \) fourier mode on each streamline, are extended in both the \( x \) and \( y \) directions.

For generic \( \phi(x, y) \), the non-zero eigenstates remain
one-dimensional and confined to the streamlines of \( \phi(x) \); however, they no longer all traverse the torus. One ex-
perts generally two sets of streamlines with opposite
orientation that circle one of the holes in the torus, but most eigenstates close without circulating the torus;
they are localized. The zero modes are once again com-
posed of functions that are constant along the stream-
lines, but have arbitrary variation perpendicular to the
streamlines; \( e^{i 2\pi n a} \) for integer \( n \) constitute a suitable ba-
sis of extended eigenstates at \( E = 0 \). Eigenstates on streamlines that circulate the torus will once again shift
by an energy level as the boundary phase twist is in-
creased, but the localized closed loops will undergo spec-
tral flow only if a phase twist is imposed somewhere along
the loop.

The eigenstates may be further characterized by defin-
ing translation operators \( T_\mu = e^{i \partial_\mu} \) for \( \mu = w, J \) corresponding
to the pseudomomentum operators \( i \partial_\mu \). \( T_w \) translates wave functions parallel to streamlines; it acts
on zero modes as the identity and commutes with \( K \), re-
vealing a residual translation invariance for arbitrary \( \phi \)
that makes \( K \) locally one-dimensional. \( T_J \) deforms states in
directions \( \perp \) to streamlines: \( T_J f(J) = f(J + \eta) \); it does not commute with \( K \) in general, but does once \( K \) is projected onto the 0 eigenspace, reflect-
ing an enhanced symmetry: the \( E = 0 \) subspace may be
decomposed into eigenstates of \( T_J \) and \( T_w \) simulta-
neously. The deformations represent gapless modes, and
are the counterpart of the area-preserving deformations
that leave the 0 mass limit of \( \mathcal{H}_B \) invariant. The symme-
tries of the continuum model are fully incorporated into
the algebra of Poisson brackets, and a faithful discretiza-
tion ought to preserve this algebra.

\( \nabla^2 \) represents a singular perturbation to \( K \), and to
proceed further \( K \) must be regularized. On the lattice, the
chain rule, which is needed to show that \( K f(\phi) = 0 \),
breaks down for local discretizations, so that a naive reg-
ularization of \( K \) destroys the conservation laws; at most
a pair of eigenstates with zero energy survive. A simi-
lar problem applies to a naive fourier discretization with momentum cutoff. Nevertheless, in taking the continuum limit, the energies of a subset of the modes approach zero; their eigenfunctions in the discrete theory are smoothly connected to those of the continuum theory as the UV cutoff is varied and their identification with zero modes of the continuum model remains. For this reason, the choice of regularization is not expected to affect physical quantities; however, if one needs to identify the subspace of zero modes explicitly, a symmetry preserving regularization is essential.

The choice of regularization is therefore dictated by a need to preserve the algebraic properties of the continuum model, even after truncating the system to a finite number of degrees of freedom. This requirement is satisfied by the Lie algebras $su(N)$, where $su(N) \to sl(T^2)$ as $N \to \infty$. Setting $p = -i\hbar \partial_x$ in $\phi(x, p)$, one obtains the Heisenberg-Moyal bracket: 
\[ \{\phi(x, -i\hbar \partial_x), \psi(x, -i\hbar \partial_x)\}/h \to \{\phi(x, p), \psi(x, p)\}_{PB} \text{ as } \hbar \to 0. \]
Taking $h = 2\pi/N$ for integer $N$, yields, in Fourier representation, the algebra $su(N)$ with $N^2 - 1$ degrees of freedom. Note that this is the adjoint representation of $su(N)$: $K\psi = [\phi, \psi]$, the wave-function is an operator, and the zero modes of $K$ are equivalent under unitary transformation to the $N - 1$ Casimir elements of $su(N)$ together with the identity.

In Fourier space on the torus $[0, 1] \times [0, 1]$ the Heisenberg-Moyal bracket takes the form:
\[ [\phi, \psi]_k = \sum_l \phi_{k-l}(2\pi/N) \sin((2\pi/N)k \times l)\psi_l \quad (3) \]

where $k, l = 2\pi(m, n)/N$ , $0 \leq m, n < N - 1$. As shown in Fig. 1a, for which $\nabla^2 \phi$ is taken as random and $\delta$-correlated on distances of order $1/N$, this form correctly reproduces the overall spectrum obtained via lattice regularization that is shown in Fig. 1b, except in the neighborhood of 0 energy, where the $N$ zero modes appear, separated from the rest of the spectrum by a pseudogap. Qualitatively, the gap can be thought of as originating in level repulsion; from Eq. (2) it is clear that the low-lying states come from the neighborhood of saddle points of $\phi$. The scaling of the size of the gap with $N$ and its dependence on the correlations of $\phi$ can be estimated by observing that if $\phi(k) \sim |k|^{\ell}$ then $\nabla \times \phi, \nabla \phi \sim N^{-(\ell+2)}$; however, $1/N$ of the spectral weight must be transferred to the zero modes, yielding a gap of order $N^{-(\ell+3)}$; $\ell = -2$ in Fig. 1.

My continuum arguments indicate that the eigenstates corresponding to Casimir elements of $su(N)$ should be, in general, extended. I have confirmed this hypothesis by numerically evaluating their Chern number $[13]$. Chern number is computed by parameterizing the Hamiltonian by periodic variables $\theta_x, \theta_y$ and integrating the Berry flux over the Brillouin zone. The degeneracy of $K$ at 0 energy leads to Berry matrices rather than scalar Berry phases; to avoid this technical problem, I compute the Chern number for eigenstates of the operator $H = -\nabla^2 + iK$. There is no reason to expect that diffusion, or for that matter any other non-random, homogeneous coupling, should localize otherwise delocalized states. It turns out that an extensive number of states with non-zero Chern number is found only within the ‘diffusion band,’ the energy scale set by $-\nabla^2$; these states span the entire diffusion band.

Fig. 2 shows the fraction of eigenstates within the diffusion band having non-zero Chern number for a naive lattice discretization of $H$ on an $N \times N$ grid, as a function of $N$, averaged over many disorder realizations $[3]$. The components of $A$ were chosen uniformly on an interval $[-W, W]$ with $\delta$-correlations in space, and the compressible part was subtracted. $\theta_x$ and $\theta_y$ represent phase twists applied at the boundary of the torus. The constant asymptote is consistent with a band of extended states in the thermodynamic limit.

The inset shows Chern numbers for a typical disorder realization, discretized with Eq. (3) and $N = 24$. For this form of $H$, kinetic energy has been parameterized as $2 \cos(2\pi m/N + \theta_x) + 2 \cos(2\pi n/N + \theta_y) - 4$, $0 \leq m, n \leq N - 1$. Nearly every eigenstate in the diffusion band has a non-zero Chern number; outside the diffusion band, only isolated energies, corresponding presumably to the percolating streamlines, show non-zero Chern number. When $iK$ is replaced by $V$, all eigenstates have zero Chern number. These numerical results confirm extensive transfer matrix studies on the lattice $[3]$ for both RFI and PSL.

The phenomena described in this paper originate in the infinite number of conserved fluxes that, for $K$, are reflected in a huge degeneracy at zero energy. The charges of the degeneracies produce non-zero Chern numbers when $iK$ is perturbed by $-\nabla^2$, showing that the existence of extended states does not rely on the infinite degeneracy itself, but only on the fact that the perturbation doesn’t
lead to cancellation of the charges.

More generally, one might expect that the infinite number of fluxes, and the symmetry that generates them, would lead to an anomaly. This point of view applies naturally to vorticity ω in a 2d incompressible fluid, where \( f \, d a \, f(\omega) \) represents the constants of the motion corresponding to the \( f(\phi) \) discussed here. Vorticity pumped into a flow at large scales can be destroyed at small scales only by local action of \( \nabla^2 \), creating fluxes of conserved charge in momentum space. There is no reason to single out a particular charge, such as enstrophy, for special treatment, as in ref. 3.

I have exhibited an extensive fraction of delocalized states in the diffusion band when \( iK \gg \nabla^2 \), that is, in the strong disorder limit, whereas arbitrarily weak potential disorder in \( d = 2 \) localizes all states of \( -\nabla^2 + V \). Bar- ring some intermediate reentrant localized phase, which seems improbable, it is likely that the states will remain extended into the weak disorder regime \( iK \ll \nabla^2 \). In RG language, one would naively expect fixed points at \( K \approx 0 \) and \( D\nabla^2 = 0 \), both of which display an infinite number of extended eigenstates.

The \( iK \ll \nabla^2 \) regime could be achieved if the random magnetic field \( B \) had strong anti-ferromagnetic correlations so that \( \phi \sim 1 \) and consequently \( A \sim k \). The center of the band sets a momentum scale \( k_f \), and the order of magnitude of the terms in RF may be estimated as \( \nabla^2 \sim k_f^2 \), \( iK \sim k_f k \), and \( A^2 \sim k^2 \), suggesting that it may be reasonable to incorporate the effect of the compressible disorder \( A^2 \) as a perturbation to RFI.

I conclude by addressing the effect of compressible disorder on RFI. While eigenstates of \( \nabla^2 \) with opposite momentum are degenerate, as of course are 0 modes of \( iK \), these degeneracies are broken for the extended eigenstates of \( -\nabla^2 + iK \); consequently, those eigenstates are chiral and small compressible disorder should not localize them. Rather, it can be expected to compete with incompressible disorder, with its localizing effects most strongly felt near the band edges. For example, when \( A^2 \) is negligible compared to \( -\nabla^2 + iK \) then the entire diffusion band (or blob, for PSI) should be extended. When \( \phi \sim 1/|k| \), the critical energy ought to depend on the amplitude of the vector potential randomness; as the relative strength of \( A^2 \) increases, the extended regime should contract to the point of particle-hole symmetry 2. For these reasons, one might imagine that the divergent density of states recently found at the band center of a random-flux model 2 reflects the dirac sea I have identified above.

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1. see e.g. A. Capelli, C.A. Trugenberger, and G.R. Zemba, Ann. Phys. 246, 86 (1996), and references therein.
2. see, e.g., J. Miller and J. Wang, Phys. Rev. Lett. 76, 1461 (1996); A. Furusaki, Phys. Rev. Lett. 82, 604 (1999); A. Altland and B. Simons, cond-mat/9909152.
3. J. Miller, Passive scalars and random flux, 1998 APS March meeting.
4. Conservation of the \( \phi \)-currents does not mean simply that \( A \) vanishes for a constant wave function; consider \( A = i \sum_{i,j} \alpha_{ij} (\psi_i^* \psi_j - \psi_j^* \psi_i) \), \( \alpha_{ij} \) spatially random.
5. The term 'incompressible' is justified since reparameterization by an arbitrary incompressible coordinate transformation preserves all three conserved currents:

\[
\frac{1}{\sqrt{g}}(\partial_i \sqrt{|g|} \psi \partial_j + i \psi^\dagger \partial_i \phi \partial_j)
\]

can be written in divergence form, provided \( g = \det(g_{ij}) \) is constant, \( g_{ij} \) the metric tensor.
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