Darboux points and integrability of homogeneous Hamiltonian systems with three and more degrees of freedom

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Abstract. We consider natural complex Hamiltonian systems with \( n \) degrees of freedom given by a Hamiltonian function which is a sum of the standard kinetic energy and a homogeneous polynomial potential \( V \) of degree \( k > 2 \). The well known Morales-Ramis theorem gives the strongest known necessary conditions for the Liouville integrability of such systems. It states that for each \( k \) there exists an explicitly known infinite set \( M_k \subset \mathbb{Q} \) such that if the system is integrable, then all eigenvalues of the Hessian matrix \( V''(d) \) calculated at a non-zero \( d \in \mathbb{C}^n \) satisfying \( V'(d) = d \), belong to \( M_k \).

The aim of this paper is, among others, to sharpen this result. Under certain genericity assumption concerning \( V \) we prove the following fact. For each \( k \) and \( n \) there exists a finite set \( I_{n,k} \subset M_k \) such that if the system is integrable, then all eigenvalues of the Hessian matrix \( V''(d) \) belong to \( I_{n,k} \). We give an algorithm which allows to find sets \( I_{n,k} \).

We applied this results for the case \( n = k = 3 \) and we found all integrable potentials satisfying the genericity assumption. Among them several are new and they are integrable in a highly non-trivial way. We found three potentials for which the additional first integrals are of degree 4 and 6 with respect to the momenta.

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1 Introduction

The problem of the integrability of low-dimensional Hamiltonian systems has been a very active field of research in the last half century. Let us recall that a Hamiltonian system with \( n \) degrees of freedom defined by Hamilton’s function \( H = H(q,p) \), where \( q = (q_1,\ldots,q_n) \) and \( p = (p_1,\ldots,p_n) \), is integrable in the Liouville sense if the canonical equations

\[
\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i}, \quad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i}, \quad i = 1,\ldots,n,
\]

admit \( n \) functionally independent and commuting first integrals. Investigations of physically important systems and more or less systematic analysis of systems depending on a certain number of parameters gave rise to several new integrable systems. Among the tools used for such investigations two are most popular: the direct method and the Painlevé test. In spite of their great successes, these techniques are rather limited and are not appropriate
for a more ambitious program which can be formulated in the following way:

Given a complete list of integrable Hamiltonian systems of a given class. In other words: formulate necessary and sufficient conditions for the integrability of a given class of Hamiltonian systems.

Such a program could be attractive if the chosen class contains, on the one hand, physically interesting systems, and on the other hand, it is large enough, and moreover is, in some sense, naturally distinguished. In this paper we consider natural Hamiltonian systems given by the following Hamiltonian

\[ H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q), \]  

where \( V(q) \) is a homogeneous polynomial of degree \( k > 2 \). The canonical equations corresponding to the above Hamiltonian have the form

\[ \frac{dq}{dt} = p, \quad \frac{dp}{dt} = -V'(q), \]  

where \( V'(q) \) denotes the gradient of \( V(q) \). We say that a potential \( V \) is integrable if the above canonical equations are integrable in the Liouville sense.

The class of systems given by (1.2) satisfies the imposed requirements. Among homogeneous potentials there are many important systems which come from physics, astronomy and other natural sciences, see examples in paper [2] and references therein. Furthermore, thanks to works [16; 38] we know that if a non-homogeneous polynomial potential \( V = V_{\text{min}} + \cdots + V_{\text{max}} \), where \( V_{\text{min}} \) and \( V_{\text{max}} \) are homogeneous terms of the lowest and the highest degree, respectively, is integrable, then potentials \( V_{\text{min}} \) and \( V_{\text{max}} \) are also integrable. Thus the integrable homogeneous potentials can be considered as building blocks of more complicated non-homogeneous integrable models.

Obviously, having in mind the described program, we must be equipped with tools strong enough to guarantee its realisation. Fortunately, we have at our disposal two methods: the Ziglin [39; 40] and Morales-Ramis [27; 23] theories which are very strong and effective. In both theories obstructions to the integrability are derived from an analysis of the variational equations along a non-equilibrium particular solution of the considered system. In this paper we will use the Morales-Ramis theory that can be considered as algebraic extension of the Ziglin method and its detailed description with many examples one can find in books [23; 3], see also [25; 26; 28; 29].

The main theorem of the Morales-Ramis theory is following.

**Theorem 1.1** (Morales, 1999). *If a complex Hamiltonian system is integrable in the Liouville sense in a neighbourhood of a particular non-equilibrium solution, then the identity component of the differential Galois group of the variational equations along this solution is Abelian.*

For a proof, examples and the discussion, see [23].

**Remark 1.1** In the above theorem we have to distinguish two cases. If the variational equations are Fuchsian, or its irregular singularities correspond to finite points in the phase space, then we assume the integrability with first integrals which are meromorphic in a connected neighbourhood of the phase curve. If the variational equations have an irregular singularity corresponding to a point in the phase space lying at the infinity, then we assume the integrability with first integrals which are meromorphic in a connected neighbourhood of the phase curve and additionally have a meromorphic growth at the infinity. In the case of Hamiltonian systems considered in this paper it means that first integrals are rational.
Here we underline that the above theorem is very well suited to a study of the integrability of homogeneous potentials, because system (1.3) admits, at least in a generic case, a certain number of particular solutions. For general Hamiltonian systems (1.1) it is not completely obvious how to find particular solutions in order to have a chance to apply the Ziglin or Morales-Ramis method.

In [23, 27] Morales and Ramis obtained very nice general integrability obstructions for homogeneous potentials with an arbitrary number \( n \geq 2 \) of degrees of freedom and an arbitrary degree of homogeneity \( k \). As it was observed by H. Yoshida, system (1.3) has a particular solution of the form

\[
q(t) = \varphi(t)d, \quad p(t) = \dot{\varphi}(t)d,
\]  
(1.4)

provided that the scalar function \( \varphi(t) \) satisfies the following equation

\[
\ddot{\varphi} = -\varphi^{k-1},
\]

and \( d \in \mathbb{C}^n \) is a solution of polynomial equations

\[
V'(d) = d.
\]  
(1.5)

A vector \( d \) satisfying the above equations is called the proper Darboux point of the potential.

The variational equations along solution (1.4) have the form

\[
\dot{x} = y, \quad \dot{y} = -\varphi(t)^{k-2}V''(d)x.
\]  
(1.6)

Hessian \( V''(d) \) of the potential \( V \) calculated at a proper Darboux point \( d \) is a symmetric matrix. Let us assume that \( V''(d) \) is diagonalisable. Then there exists a complex orthogonal \( n \times n \) matrix \( A \) such that the canonical transformation

\[
x = A\eta, \quad y = A\xi,
\]
transforms system (1.6) to the form

\[
\dot{\eta}_i = \xi_i, \quad \dot{\xi}_i = -\lambda_i\varphi(t)^{k-2}\eta_i, \quad i = 1, \ldots, n,
\]
or simply

\[
\ddot{\eta}_i = -\lambda_i\varphi(t)^{k-2}\eta_i, \quad i = 1, \ldots, n,
\]  
(1.7)

where \( (\lambda_1, \ldots, \lambda_n) \) are eigenvalues of \( V''(d) \). It can be shown that if the identity component of the differential Galois group of the above system is Abelian, then the identity component of the differential Galois group of each equation in the system is Abelian.

**Remark 1.2** As it was explained in [10] the assumption that \( V''(d) \) is diagonalisable is irrelevant. That is, the necessary conditions for the integrability are the same: if the potential is integrable, then for each \( \lambda \in \text{spectr} V''(d) \) the identity component of the differential Galois group of equation \( \ddot{\eta} = -\lambda\varphi(t)^{k-2}\eta \) is Abelian.

For a given energy \( e \in \mathbb{C}^* \) the phase curve \( \Gamma_e \) associated with the particular solution \( \varphi(t) \) is, for \( k = 3,4 \) an elliptic, or for \( k > 4 \) a hyperelliptic curve given by

\[
\dot{\varphi}^2 = \frac{2}{k}\left(\epsilon - \varphi^k\right), \quad e = \frac{1}{k}\varepsilon.
\]  
(1.8)
As it was observed by H. Yoshida [37], each of equations (1.7) can be transformed to the hypergeometric equation. It can be done by the following change of the independent variable

\[ t \rightarrow z := \frac{1}{\epsilon} \varphi(t)^k. \]  

(1.9)

After this transformation equations (1.7) read

\[
(z(1-z))\eta''_i + \left(\frac{k-1}{k} - \frac{3k-2}{2k}z\right)\eta'_i + \frac{\lambda_i}{2k}\eta_i = 0,
\]

(1.10)

where \( i = 1, \ldots, n \). Equation (1.10) is a special case of the hypergeometric differential equation, for which, thanks to works of Schwarz [35], Kimura [18] and others, the differential Galois group is well known.

This fact combined with Theorem 1.1 has allowed J. J. Morales-Ruiz and J. P. Ramis to formulate in [27] a general theorem concerning the integrability of Hamiltonian systems with a homogeneous potential. Here, we formulate this theorem for a polynomial homogeneous potential.

**Theorem 1.2.** If Hamiltonian system (1.3) with polynomial homogeneous potential \( V(q) \) of degree \( k > 2 \) is meromorphically integrable in the Liouville sense, then for a proper Darboux point the values of \( (k, \lambda_i) \) for \( i = 1, \ldots, n \) belong to the following list

1. \( \left(k, p + \frac{k}{2}p(p-1)\right) \),
2. \( \left(\frac{k-1}{k} + p(p+1)k\right) \),
3. \( \left(3, -\frac{1}{24} + \frac{1}{6}(1+3p)^2\right) \),
4. \( \left(3, -\frac{1}{12} + \frac{3}{16}(1+4p)^2\right) \),
5. \( \left(3, -\frac{1}{24} + \frac{3}{50}(1+5p)^2\right) \),
6. \( \left(3, -\frac{1}{12} + \frac{3}{50}(2+5p)^2\right) \),
7. \( \left(4, -\frac{1}{8} + \frac{2}{9}(1+3p)^2\right) \),
8. \( \left(5, -\frac{9}{40} + \frac{5}{18}(1+3p)^2\right) \),
9. \( \left(5, -\frac{9}{40} + \frac{1}{10}(2+5p)^2\right) \),

(1.11)

where \( p \) is an integer.

**Remark 1.3** By Remark 1.2 Theorem 1.2 is true independently of the fact whether matrix \( V'(d) \) is diagonalisable or not. However, if \( V''(d) \) is not diagonalisable, new obstacles for the integrability appear. Namely, if the Jordan form of \( V''(d) \) has a block

\[
J_2(\lambda) := \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{bmatrix},
\]

then the system is not integrable. Moreover, if the Jordan form of \( V''(d) \) has a two dimensional block \( J_2(\lambda) \), and \( \lambda \) belongs to the first item of table (1.11), then the system is not integrable. This fact was proved in [10].

We denote by \( M_k \) a subset of rational numbers \( \lambda \) specified by the table in the above theorem for a given \( k \), e.g., for \( k > 5 \) we have

\[
M_k = \left\{ p + \frac{k}{2}p(p-1) \mid p \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2} \left(\frac{k-1}{k} + p(p+1)k\right) \mid p \in \mathbb{Z} \right\}.
\]  

(1.12)
Let us make some remarks about practical aspects of applications of Theorem 1.2 for a study of the integrability of a potential depending on several parameters. At first, we have to find a proper Darboux point, i.e., to solve nonlinear equations (1.5). And just at this very initial step we meet a very fundamental problem. Namely, even if equations (1.5) have solutions, in most cases we cannot find their explicit form. Unfortunately, it is not the only problem that we have. Even if we find the explicit form of a Darboux point \( d \), its coordinates depend on parameters in a very complicated way and, moreover, we obtain much more intricate expressions for eigenvalues \( \lambda_i \) of \( V''(d) \). So, it is at least doubtful if we can extract useful information from the restriction \( \lambda_i \in \mathcal{M}_k \), and this is not only a technical problem. In fact, a restriction of the form \( \lambda_i \in \mathcal{M}_k \) gives rise to an infinite number of potentials which form co-dimension one families in the parameters space. For each of these families, the necessary conditions for the integrability are satisfied, so we need to apply stronger theorems to check if within these families there are integrable potentials. Hence, it seems that we have to restrict applications of Theorem 1.2 to very limited classes of homogeneous potentials.

Our aim in this paper is to show how to overcome the above mentioned difficulties. The basic observation is following. The existence of a proper Darboux point together with the necessary conditions given by Theorem 1.2 put restrictions on the potential no matter if we know explicitly the proper Darboux point or not. The question is, if we can deduce from this fact a kind of a global restriction on \( V \). We give a positive answer to this question. But the most amazing fact is that this answer implies the following statement which is valid for all homogeneous polynomial potentials of degree \( k > 2 \) which satisfy certain genericity conditions:

For each \( n \) and \( k \) there exists a finite set \( \mathcal{I}_{n,k} \subset \mathcal{M}_k \) such that if \( V \) is integrable, then for each Darboux point \( d \) of \( V \) all eigenvalues of \( V''(d) \) belong to \( \mathcal{I}_{n,k} \). Moreover there is a constructive algorithm which allows to find sets \( \mathcal{I}_{n,k} \).

The described general idea has found successful applications for systems with two degrees of freedom, see [21; 30; 32]. So nowadays all integrable homogeneous potentials of degree 3 and 4 are known\(^1\). In this paper we show that the above idea has a nice generalisation for systems with \( n \geq 2 \) degrees of freedom. Moreover, we apply the obtained results to a study of the integrability of homogeneous potentials with three degrees of freedom of the homogeneity degree equal to 3.

The methods applied in this paper force us to work with complex Hamiltonian systems. So, we assume that our system given by Hamiltonian (1.2) is defined on a complex linear symplectic space \((\mathbb{C}^{2n}, \omega)\) equipped with the canonical symplectic form

\[
\omega = \sum_{i=1}^n dq_i \wedge dp_i. \tag{1.13}
\]

Moreover, we assume that potential \( V \) is also complex, i.e., \( V \in \mathbb{C}_k[q] \), where \( \mathbb{C}_k[q] \) denotes \( \mathbb{C} \)-linear space of homogeneous polynomials of degree \( k \).

In order to make a reasonable classification, we have to divide all considered potentials into equivalent classes. To this end we proceed as in [22]. Namely, let \( \text{PO}(n, \mathbb{C}) \) be the complex projective orthogonal subgroup of \( \text{GL}(n, \mathbb{C}) \), i.e.,

\[
\text{PO}(n, \mathbb{C}) = \{ A \in \text{GL}(n, \mathbb{C}), \ | \ A A^T = \alpha E_n, \ \alpha \in \mathbb{C}^* \}, \tag{1.14}
\]

where \( E_n \) is \( n \)-dimensional identity matrix. We say that \( V \) and \( \tilde{V} \) are equivalent if there exists \( A \in \text{PO}(n, \mathbb{C}) \) such that \( \tilde{V}(q) = V_A(q) := V(Aq) \). Later a potential means a class of equivalent potentials in the above sense.

\(^1\)For \( k = 4 \) there exists a family of potentials depending on one discrete parameter and it is unknown if this family contains an integrable case or not, see [22] for details.
2 Darboux points

In the Morales-Ramis Theorem [12] the considered particular solution has a very special form (1.4). To find it, we have to solve a system of polynomial equations $V'(d) = d$. In our considerations it is important to take into account all particular solutions. An analysis of this problem leads us to a definition of a geometrical notion of a Darboux point of a homogeneous potential. It was introduced in [22] for systems with two degrees of freedom. However, the passage from two to three and more degrees of freedom introduces essential difficulties which force us to modify slightly this concept. In this section we give basic definitions and prove several general facts concerning Darboux points.

Let us remark that the definition of a Darboux point given in this section is closely related to the notion of the proper direction (radial orbits) introduced by Guillot [11, 12] for polynomial differential equations with homogeneous right hand sides.

2.1 Basic notions of algebraic geometry

At first we recall very basic notions from the algebraic geometry which are necessary for our further considerations. The main purpose is just to fix the notation. We refer the reader to [34] for a more general and formal exposition.

The set of common zeros of polynomials $f_1, \ldots, f_s \in \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_m]$ is denoted by $\mathcal{V}(f_1, \ldots, f_s)$. It is called the affine algebraic set. An algebraic set is a finite sum of disjoint connected components. We say that point $p \in X := \mathcal{V}(f_1, \ldots, f_s)$ is a simple point of $X$, iff

$$\text{rank } \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_1}(p) & \cdots & \frac{\partial f_s}{\partial x_m}(p) \end{bmatrix} = m - \dim X_p, \quad (2.1)$$

where $X_p$ is the component of $X$ containing $p$. In particular, if $X$ is a finite set, then $p \in X$ is a simple point, iff the above Jacobi matrix has the maximal rank.

A polynomial $f \in \mathbb{C}[x]$ can be uniquely written as a sum of homogeneous terms. The highest degree homogeneous term of $f$ is denoted by $f^+$. We say that $f_1, \ldots, f_m \in \mathbb{C}[x]$ do not intersect at the infinity, iff the only solution of $f_1^+ = \cdots = f_m^+ = 0$ is $x = 0$. The following fact is well known, see e.g. [9, 19, 31].

Proposition 2.1. If $f_1, \ldots, f_m \in \mathbb{C}[x]$ do not intersect at the infinity, then set $\mathcal{V}(f_1, \ldots, f_m)$ is finite.

A point in the $m$ dimensional complex projective space $\mathbb{CP}^m$ is specified by its homogeneous coordinates $[z] = [z_0 : \cdots : z_m]$, where $z = (z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$. If

$$U_i := \{[z_0 : \cdots : z_m] \in \mathbb{CP}^m \mid z_i \neq 0\} \quad \text{for} \quad i = 0, \ldots, m, \quad (2.2)$$

then

$$\mathbb{CP}^m = \bigcup_{i=0}^m U_i, \quad (2.3)$$

and we have natural coordinate maps

$$\theta_i : \mathbb{CP}^m \supset U_i \to \mathbb{C}^m, \quad \theta_i([z]) = (x_1, \ldots, x_m),$$

where

$$(x_1, \ldots, x_m) = \left( \frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i} \right). \quad (2.4)$$
Each $U_i$ is homeomorphic to $\mathbb{C}^m$. It is easy to check that charts $(U_i, \theta_i)$, $i = 0, \ldots, m$ form an atlas which makes $\mathbb{CP}^m$ an holomorphic $m$-dimensional manifold. It is customary to choose one $U_i$, e.g., $U_0$, and call it the affine part of $\mathbb{CP}^m$. Then we define the hyperplane at the infinity

$$H_\infty := \{ [z_0 : \cdots : z_m] \in \mathbb{CP}^m \ | \ z_0 = 0 \}. \tag{2.5}$$

It is clear that $\mathbb{CP}^m = U_0 \cup H_\infty$ and $U_0 \cap H_\infty = \emptyset$. The following map

$$\theta_\infty : H_\infty \to \mathbb{CP}^{m-1}, \quad \theta_\infty([0 : z_1 : \cdots : z_m]) = [z_1 : \cdots : z_m], \tag{2.6}$$

is the bijection, so $\theta_\infty(H_\infty) = \mathbb{CP}^{m-1}$. The above consideration shows that $\mathbb{CP}^m$ is homeomorphic to $\mathbb{C}^m \cup \mathbb{CP}^{m-1}$.

Let $F_1, \ldots, F_s \in \mathbb{C}[z]$ be homogeneous polynomials. Then their common zero locus in $\mathbb{CP}^m$, i.e.,

$$\mathcal{V}(F_1, \ldots, F_s) := \{ [z] \in \mathbb{CP}^m \ | \ F_i(z) = 0, \ i = 1, \ldots, s \}, \tag{2.7}$$

is called the projective algebraic set. The affine part of $X := \mathcal{V}(F_1, \ldots, F_s)$ is, by definition $X \cap U_0$. It is homeomorphic to the algebraic set $\mathcal{V}(f_1, \ldots, f_s)$, where $f_i$ is a dehomogenisation of $F_i$, that is

$$f_i(x_1, \ldots, x_m) = F_i(1, x_1, \ldots, x_m) \quad \text{for} \quad i = 1, \ldots, s. \tag{2.8}$$

Let us show this. Notice that if $F \in \mathbb{C}[z]$ is a homogeneous polynomial, then $F(z)$ is not a function of point $[z] \in \mathbb{CP}^m$. In order to describe the hypersurface $\mathcal{V}(F) \subset \mathbb{CP}^m$ as a zero set of a certain function defined on a subset of $\mathbb{CP}^m$, we define on $U_0$ the following function

$$\tilde{F} : U_0 \to \mathbb{C}, \quad [z_0 : \cdots : z_m] \mapsto \tilde{F}([z]) := z_0^{-\deg F} F(z), \tag{2.9}$$

and then

$$\mathcal{V}(F) \cap U_0 = \{ [z] \in U_0 \ | \ \tilde{F}([z]) = 0 \}. \tag{2.10}$$

The function $\tilde{F}$ is called the function defining $\mathcal{V}(F)$ on $U_0$. If we set

$$f = \tilde{F} \circ \theta_0^{-1} : \mathbb{C}^m \to \mathbb{C}, \quad f(x_1, \ldots, x_n) := F(1, x_1, \ldots, x_m),$$

then

$$\theta_0(\mathcal{V}(F) \cap U_0) = \mathcal{V}(f).$$

Polynomial $f$ is called the dehomogenisation of $F$ with respect to the first variable. The above shows that

$$\theta_0(\mathcal{V}(F_1, \ldots, F_s) \cap U_0) = \mathcal{V}(f_1, \ldots, f_s)$$

as we wanted to show.

Let $[z] \in U_i$ for a certain $0 \leq i \leq n$. We say that $[z]$ is a simple point of $\mathcal{V}(F_1, \ldots, F_s)$ iff $\theta_i([z])$ is a simple point of $\mathcal{V}(f_1, \ldots, f_s)$, where $f_j$ is the dehomogenisation of $F_j$ with respect to the $j$-th variable, for $j = 1, \ldots, s$.

For the later use we will need explicit relations between representations of $\mathcal{V}(F_1, \ldots, F_s)$ on two different charts. Let

$$\theta_1(\mathcal{V}(F_1, \ldots, F_s) \cap U_1) = \mathcal{V}(h_1, \ldots, h_s),$$

where $h_i$ is the dehomogenisation of $F_i$ with respect to the second variable. A relation between polynomials $h_i$ and $f_i$ can be found with a help of the transition map $\psi = \theta_0 \circ \theta_1^{-1}$. Namely, if

$$\theta_0([z_0 : \cdots : z_m]) = (x_1, \ldots, x_m), \quad \text{for} \quad [z_0 : \cdots : z_m] \in U_0, \tag{2.10}$$
and
\[ \theta_1([z_0 : \cdots : z_m]) = (y_1, \ldots, y_m), \quad \text{for} \quad [z_0 : \cdots : z_m] \in U_1, \]  
then
\[ (x_1, \ldots, x_m) = \psi(y_1, \ldots, y_m) = \left( \frac{1}{y_1}, \frac{y_2}{y_1}, \ldots, \frac{y_m}{y_1} \right). \]  

Then, one can check that
\[ h_i(y) := y_1^{\deg f_i} f_i(\psi(y)) = y_1^{\deg f_i} f_i \left( \frac{1}{y_1}, \frac{y_2}{y_1}, \ldots, \frac{y_m}{y_1} \right), \]
for \( i = 1, \ldots, s. \)

In our considerations we invoke several times the Bézout Theorem. Let us recall its formulation, as the same name have several different theorems.

**Theorem 2.1 (Bézout).** The number of solutions \( N \) of a system of \( m \) homogeneous equations
\[ F_i(z) = 0, \quad i = 1, \ldots, m, \]
in \( m + 1 \) unknowns \( z := (z_0, \ldots, z_m) \) is either infinite, or is equal
\[ N = \prod_{i=1}^{m} \deg F_i, \]
provided that solutions considered as points in \( \mathbb{C}P^m \) are counted with their multiplicities.

A precise definition of the multiplicity notion used in this theorem is highly technical and needs involved language of algebraic geometry. Here we refer the reader to Chapter 4 in [9]. For the purpose of this paper it is sufficient to know that a simple point has multiplicity one.

### 2.2 General properties of Darboux points

Let \( V \) be a homogeneous polynomial potential of degree \( k > 2 \), i.e., \( V \in \mathbb{C}[q] \). A direction, i.e., a non-zero \( d \in \mathbb{C}^n \), is called a **Darboux point** of \( V \) iff the gradient \( V'(d) \) of \( V \) at \( d \) is parallel to \( d \). Hence, \( d \) is a Darboux point of \( V \) iff
\[ d \wedge V'(d) = 0, \quad d \neq 0, \]  
or
\[ V'(d) = \gamma d, \quad d \neq 0, \]  
for a certain \( \gamma \in \mathbb{C} \). Obviously, if \( d \) satisfies one of the above conditions, then \( \tilde{d} = \alpha d \) for any \( \alpha \in \mathbb{C}^* \) satisfies them. However, we do not want to distinguish between \( d \) and \( \tilde{d} \). Hence we consider a Darboux point \( d = (d_1, \ldots, d_n) \in \mathbb{C}^n \) as a point \([d] := [d_1 : \cdots : d_n]\) in the projective space \( \mathbb{C}P^{n-1} \).

The set \( \mathcal{D}(V) \subset \mathbb{C}P^{n-1} \) of all Darboux points of a potential \( V \) is a projective algebraic set. In fact, \( \mathcal{D}(V) \) is the zero locus in \( \mathbb{C}P^{n-1} \) of homogeneous polynomials \( R_{ij} \in \mathbb{C}[q] \) which are components of \( q \wedge V'(q) \), i.e.,
\[ R_{ij} := q_i \frac{\partial V}{\partial q_j} - q_j \frac{\partial V}{\partial q_i}, \quad \text{where} \quad 1 \leq i < j \leq n. \]  

We say that a Darboux point \([d] \in \mathcal{D}(V)\) is a **proper Darboux point** of \( V \), iff \( V'(d) \neq 0 \). The set of all proper Darboux points of \( V \) is denoted by \( \mathcal{D}^*(V) \). If \([d] \in \mathcal{D}(V) \setminus \mathcal{D}^*(V)\), then \([d]\) is called an **improper Darboux point** of potential \( V \).
We say that \([d]\) is an isotropic Darboux point, if
\[
d_1^2 + \cdots + d_n^2 = 0. \tag{2.17}
\]
The set of all isotropic Darboux points of potential \(V\) is denoted by \(\mathcal{D}_0(V)\).
We consider also two additional subsets of \(\mathbb{C}P^{n-1}\)
\[
\mathcal{Z}(V) := \mathcal{V}(V), \quad \mathcal{S}(V) := \mathcal{V}\left(\frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_n}\right), \tag{2.18}
\]
and we can prove the following relations.

**Proposition 2.2.** For a non-zero \(V \in \mathbb{C}_k[q]\) we have

1. \(\mathcal{D}^*(V) = \mathcal{D}(V) \setminus \mathcal{S}(V)\),
2. \(\mathcal{S}(V) \subset \mathcal{Z}(V)\).

The first of the above statements follows just from the definition of \(\mathcal{D}^*(V)\), and the second from the Euler identity for the homogeneous polynomial \(V\).

To describe the set \(\mathcal{D}(V)\) in the affine coordinates we put, according to our convention, \(U_1 := \mathbb{C}P^{n-1} \setminus \{[q] \in \mathbb{C}P^{n-1} \mid q_1 = 0\}\), and
\[
\theta_1 : U_1 \to \mathbb{C}^{n-1}, \quad \bar{x} := (x_1, \ldots, x_{n-1}) = \theta_1([q]), \tag{2.19}
\]
where
\[
x_i = \frac{q_{i+1}}{q_1}, \quad \text{for} \quad i = 1, \ldots, n-1. \tag{2.20}
\]

**Lemma 2.1.** On the affine chart \((U_1, \theta_1)\) we have
\[
\theta_1(\mathcal{D}(V) \cap U_1) = \mathcal{V}(g_1, \ldots, g_{n-1}), \tag{2.21}
\]
where polynomials \(g_1, \ldots, g_{n-1} \in \mathbb{C}[\bar{x}]\) are given by
\[
v(\bar{x}) := V(1, x_1, \ldots, x_{n-1}), \quad g_0 := k v - \sum_{i=1}^{n-1} x_i \frac{\partial v}{\partial x_i}, \tag{2.22}
\]
and
\[
g_i := \frac{\partial v}{\partial x_i} - x_i g_0, \quad \text{for} \quad i = 1, \ldots, n-1. \tag{2.23}
\]

Moreover, \([d] \in \mathcal{D}(V) \cap U_1\) is an improper Darboux iff its affine coordinates \(\bar{a} := \theta_1([d])\) satisfy \(g_0(\bar{a}) = 0\).

**Proof.** We know that
\[
\mathcal{D}(V) = \mathcal{V}(R_{1,2}, \ldots, R_{n-1,n}), \tag{2.24}
\]
where \(R_{i,j}\), for \(1 \leq i < j \leq n\) are given by \((2.16)\). Hence, as it was explained in the previous subsection, we have
\[
\theta_1(\mathcal{D}(V) \cap U_1) = \mathcal{V}(r_{1,2}, \ldots, r_{n-1,n}), \tag{2.25}
\]
where
\[
r_{i,j}(x_1, \ldots, x_{n-1}) := R_{i,j}(1, x_1, \ldots, x_{n-1}), \quad \text{for} \quad 1 \leq i < j \leq n.
\]
Direct calculations give
\[
\frac{\partial V}{\partial q_1}(q) = q_1^{k-1} \left[ k v(\bar{x}) - \sum_{i=1}^{n-1} x_i \frac{\partial v}{\partial x_i}(\bar{x}) \right] = q_1^{k-1} g_0(\bar{x}), \tag{2.26}
\]
and
\[
\frac{\partial V}{\partial q_i}(q) = q_1^{k-1} \frac{\partial v}{\partial x_{i-1}}(\bar{x}) = q_1^{k-1}(g_{i-1}(\bar{x}) + x_{i-1}g_0(\bar{x})), \quad \text{for } i = 2, \ldots, n. \tag{2.27}
\]

Using the above formulae we easily find that
\[
r_{1,i+1} = g_i \quad \text{for } i = 1, \ldots, n-1,
\]
and
\[
r_{i+1,j+1} = x_ig_j - x_ig_i \quad \text{for } 1 \leq i < j \leq n-1.
\]
It shows that
\[
V(r_{1,2}, \ldots, r_{n-1,n}) = V(g_1, \ldots, g_{n-1}), \tag{2.28}
\]
and thus
\[
\theta_1(D(V) \cap U_1) = V(g_1, \ldots, g_{n-1}). \tag{2.29}
\]

To prove the last statement of the lemma let us assume that \([d]\) is a Darboux point such that \([d] \in U_1\) with affine coordinates \(\bar{a} = \theta_1([d])\). Then, \(g_i(\bar{a}) = 0\), for \(i = 1, \ldots, n-1\). If \([d]\) is an improper Darboux point, then all partial derivatives of \(V\) vanish at \(d\). From (2.26) it follows that if
\[
\frac{\partial V}{\partial q_1}(d) = d_1^{k-1}g_0(\bar{a}) = 0,
\]
then necessarily \(g_0(\bar{a}) = 0\), because \(d_1 \neq 0\). On the other hand, if \(g_i(\bar{a}) = 0\), for \(i = 0, \ldots, n-1\, and\, \theta_1^{-1}(\bar{a}) = [a]\), with \(a := (1, a_1, \ldots, a_{n-1})\), then from (2.26) and (2.27) we have \(V'(a) = 0\). Since \([a] = [d]\), this shows that \([d]\) is an improper Darboux point.

The construction which we are going to describe now plays a very important role in our considerations. Roughly speaking, the idea is to associate with a Darboux point which is located in \(\mathbb{CP}^{n-1}\), a finite set of points in \(\mathbb{CP}^n\). This procedure is a kind of blowup.

Let us define the following \(n\) homogeneous polynomials of \(n+1\) variables \(\bar{q} := (q_0, q_1, \ldots, q_n)\)
\[
F_i := \frac{\partial V}{\partial q_i} - q_0^{k-1}q_i, \quad i = 1, \ldots, n, \tag{2.30}
\]
and an algebraic set \(\hat{D}(V) = \mathcal{V}(F_1, \ldots, F_n) \subset \mathbb{CP}^n\).

Assume that \([d] \in D^*(V)\). Then there exists \(\gamma \in \mathbb{C}^*\), such that \(V'(d) = \gamma d\), so \(k-2\) points \(i\sqrt{\gamma} : d_1 : \ldots : d_n \in \mathbb{CP}^n\) belong to \(\hat{D}(V)\). These points are well defined as they do not depend on a representative for \([d]\). If \([d]\) is an improper Darboux point, then it defines just one point \([0 : d_1 : \ldots : d_n] \in \mathbb{CP}^n\), which is a point of \(\hat{D}(V)\).

Set \(\hat{D}(V)\) is not empty because it contains point \([d_0] := [1 : 0 : \ldots : 0]\). If \([d] = [d_0 : d_1 : \ldots : d_n] \in \mathcal{D}(V) \setminus \{[d_0]\}\), then \([d] = [d_1 : \ldots : d_n]\) is a Darboux point of \(V\). Moreover, if \(d_0 \neq 0\), then \([d]\) is a proper Darboux point.

The natural projection
\[
\pi : \mathbb{CP}^n \setminus \{[d_0]\} \to \mathbb{CP}^{n-1}, \quad \pi([q_0 : q_1 : \ldots : q_n]) = [q_1 : \ldots : q_n], \tag{2.31}
\]
maps \(\hat{D}(V) \setminus \{[d_0]\}\) onto \(\mathcal{D}(V)\), and the intersection of the inverse image \(\pi^{-1}([d])\) of a Darboux point \([d] \in \mathcal{D}(V)\) is a finite set. We define also
\[
\hat{\pi} : \hat{D}(V) \setminus \{[d_0]\} \to \mathcal{D}(V), \tag{2.32}
\]
putting \(\hat{\pi}([d]) := \pi([\tilde{d}])\) for \([\tilde{d}] \in \hat{D}(V) \setminus \{[d_0]\}\). That is, \(\hat{\pi}\) is the restriction of \(\pi\) to \(\hat{D}(V) \setminus \{[d_0]\}\). This construction is illustrated in the Figure 2.1.
Example 2.1 For potential
\[ V = \frac{1}{k} \sum_{i=1}^{n} q_i^k, \quad k > 2, \]  
(2.33)

\( S(V) = \emptyset, \) so \( D^*(V) = D(V). \) Let us calculate the number of points of \( \hat{D}(V). \) Since \( F_i = q_i^{k-1} - q_i^{k-2} q_i \) for \( i = 1, \ldots, n, \) it is easy to find that \( \text{card} \hat{D}(V) = (k-1)^n. \) Hence, \( \text{card} D(V) = \text{card} D^*(V) := D(n,k) \) where
\[ D(n,k) = \frac{(k-1)^n - 1}{k - 2}. \]  
(2.34)

Next lemmas show how to use the above construction for proving several facts which are not so obvious.

At first we distinguish generic potentials. We say that potential \( V \) is generic iff all its Darboux points are proper and simple. This definition is justified be the following lemma.

Lemma 2.2. The set of generic potentials \( G_{n,k} \subset \mathbb{C}_k[q] \) of degree \( k \) is a non-empty open set in \( \mathbb{C}_k[q]. \) A generic \( V \in \mathbb{C}_k[q] \) has \( D(n,k) \) proper Darboux points.

Proof. We proceed as in the proof of Proposition 4 in [12], see also [11]. Except for point \( [d_0], \) every point of \( \hat{D}(V) \) is a Darboux point. Moreover, every proper Darboux point gives \( k - 2 \) points of \( \hat{D}(V). \) Hence, if hypersurfaces \( V(F_i) \subset \mathbb{C}P^n \) intersect at a finite number of points in \( \mathbb{C}P^n \) and all these intersections are transversal, then by the projective Bézout Theorem 2.1 we have \( (k-1)^n \) of such intersections which give \( D(n,k) \) Darboux points. Example 2.1 shows that for \( n \geq 2 \) and \( k > 2 \) set \( G_{n,k} \) is not empty. Fact that \( G_{n,k} \) is open in \( \mathbb{C}_k[q] \) considered as a finite dimensional \( \mathbb{C} \)-vector space with the standard topology, was proved in [33], Lemma 2. \( \square \)

Let us note that this lemma is a generalisation of Corollary 2.1 in [22], when the case \( n = 2 \) was considered and then a generic potential has \( D(2,k) = k \) proper Darboux points.

\[ \bullet \quad \hat{D}(V) \subset \mathbb{C}P^n \quad \text{and} \quad D(V) \subset \mathbb{C}P^{n-1} \]

Figure 2.1: Sets \( \hat{D}(V) \subset \mathbb{C}P^n \) and \( D(V) \subset \mathbb{C}P^{n-1} \)

Lemma 2.3. For an arbitrary \( V \in \mathbb{C}_k[q] \) the set of Darboux points \( D(V) \) is not empty.
Proof. If \( \mathcal{D}(V) = \emptyset \), then \( \hat{\mathcal{D}}(V) = \{ [d_0] \} \). Then, by the Bézout theorem the multiplicity of \([d_0]\) is greater than one. However, as it is easy to notice, \([d_0]\) is a simple point of \(\hat{\mathcal{D}}(V)\). A contradiction finishes the proof.

On the affine chart \((U_0, \theta_0)\), the affine part of \(\hat{\mathcal{D}}(V)\) is an algebraic set \(\mathcal{V}(f_1, \ldots, f_n)\), where \(f_i\) is the dehomogenisation of \(F_i\), i.e.,

\[
  f_i(q_1, \ldots, q_n) := F_i(1, q_1, \ldots, q_n) = \frac{\partial V}{\partial q_i} - q_i, \quad \text{for} \quad i = 1, \ldots, n. \tag{2.35}
\]

Notice that the polynomials \(f_1, \ldots, f_n\) intersect at the infinity iff the gradient \(V'\) vanishes at a certain \(d \neq 0\). But such \(d\) gives an improper Darboux point \([d]\). On the other hand, the points of intersections of \(\hat{\mathcal{D}}(V)\) with the hypersurface at the infinity \(H_\infty\) correspond exactly to the improper Darboux points, more precisely

\[
  \theta_\infty(\hat{\mathcal{D}}(V) \cap H_\infty) = \mathcal{D}(V) \setminus \mathcal{D}^*(V).
\]

The lemma below gives a necessary condition for the finiteness of set \(\mathcal{D}(V)\).

**Lemma 2.4.** If a potential \(V\) does not have the improper Darboux points, then it has a finite number of Darboux points.

**Proof.** Under assumption of the lemma we have \(\hat{\mathcal{D}}(V) \cap H_\infty = \emptyset\). So, all points of \(\hat{\mathcal{D}}(V)\) are located in the affine part of \(\mathbb{C}P^n\), and their affine coordinates are points of \(\mathcal{V}(f_1, \ldots, f_n)\). Since, by the assumption, \(f_1, \ldots, f_n\) do not intersect at the infinity, by Proposition \(2.1\) set \(\mathcal{V}(f_1, \ldots, f_n)\) is finite. Hence, \(\hat{\mathcal{D}}(V)\) as well as \(\mathcal{D}(V)\) are finite. \(\square\)

**Remark 2.1** Let \([d] \in \mathcal{D}(V)\) be a Darboux point of potential \(V\). Then for each \(A \in \text{PO}(n, \mathbb{C})\), the potential \(V_A\) equivalent to \(V\) has Darboux point \([Ad]\). This fact allows us to fix coordinates of one Darboux point. Namely, if \([d]\) is not isotropic, then we can assume that \(d = (0, \ldots, 0, 1)\). If \([d]\) is isotropic, then we can assume that \(d = (0, \ldots, 0, i, 1)\).

For a Darboux point \([d] \in \mathcal{D}(V)\) we can calculate eigenvalues \(\lambda_1(d), \ldots, \lambda_n(d)\) of the Hessian matrix \(V''(d)\). However, numbers \(\lambda_i(d)\) are not well defined, as they depend on a representative \(d\) of the Darboux point \([d]\). There are several possibilities to define properly the quantities related to the eigenvalues of \(V''(d)\) which do not depend on a choice of a representative of \([d]\). However, because of some historical reasons and the convention widely accepted in the literature, we choose the one which is a simple normalisation. Namely, if it is not otherwise stated, we always assume the following

**Assumption 2.1** If \([d]\) is a proper Darboux point, then the chosen representative is such that it satisfies \(V'(d) = d\).

If \([d]\) is an improper Darboux point, then the representative of \([d]\) can be chosen arbitrarily.

Before we formulate our next lemma let us introduce the following auxiliary system

\[
  \frac{d}{dt}q = f(q) := V'(q) - \gamma q, \tag{2.36}
\]

where \(\gamma \in \mathbb{C}\). If \([d]\) is a Darboux point, then \(f(d) = 0\) for a certain \(\gamma\), i.e., \(d\) is an equilibrium of (2.36). If \(\gamma \neq 0\), then \([d]\) is a proper Darboux point and for \(\gamma = 0\) is an improper one. On the other hand, if \(d\) is an equilibrium of (2.36) for a certain \(\gamma\), then \([d]\) is a Darboux point. Under Assumption \(2.1\) if \([d]\) is a proper Darboux point, then \(d\) is the
equilibrium of (2.36) with \( \gamma = 1 \), and the Jacobi matrix of the right hand sides of (2.36) at \( \bar{d} \) is \( f'(\bar{d}) = V''(\bar{d}) - E_n \), so its eigenvalues are \( \Lambda_i(\bar{d}) := \lambda_i(\bar{d}) - 1 \), for \( i = 1, \ldots, n \).

We make a change of coordinates in the system (2.36). Namely, assuming that \( q_1 \neq 0 \), we put

\[
    x_0 = q_1, \quad x_i = \frac{q_{i+1}}{q_1} \quad \text{for} \quad i = 1, \ldots, n - 1.
\]

Then \( x := (x_0, x_1, \ldots, x_{n-1}) \) are coordinates on \( \mathbb{C}^n \setminus \{ q \in \mathbb{C}^n \mid q_1 = 0 \} \). Notice that \( \tilde{x} := (x_1, \ldots, x_{n-1}) = \theta_1([q]) \). In these coordinates system (2.36) reads

\[
    \begin{align*}
    \dot{x}_0 &= -\gamma x_0 + x_0^{k-1} \bar{g}_0, \\
    \dot{x}_i &= x_i^{k-2} \bar{g}_i, & \text{for} & \quad i = 1, \ldots, n - 1,
    \end{align*}
\]

where \( \bar{g}_0, \ldots, \bar{g}_{n-1} \in \mathbb{C}[\bar{x}] \) are given by (2.22) and (2.23). It is clear that the Jacobi matrix \( f'(x) \) has the same spectral properties as

\[
J(x) = \frac{\partial (w_0, \ldots, w_{n-1})}{\partial (x_0, \ldots, x_{n-1})}(x).
\]

However, it appears that using \( J(x) \) we obtain simpler proofs of some facts.

Under Assumption 2.4 we have the following simple but important consequences.

**Lemma 2.5.** If \([d]\) is a proper Darboux point of homogeneous potential \( V \) of degree \( k > 2 \), then

\[
    q(t) := \phi(t)d, \quad p(t) := \dot{\phi}(t)d,
\]

is a solution of Hamilton's equation (1.3) provided \( \dot{\phi} = -\phi^{k-1} \). Moreover, \( V''(d) \cdot d = \lambda_n d \) with \( \lambda_n = k - 1 \), and if additionally \([d]\) is isotropic, then \( \lambda_n \) is a multiple eigenvalue of \( V''(d) \).

If \([d]\) is an improper Darboux point, then

\[
    q(t) := td, \quad p(t) := d,
\]

is a solution of Hamilton's equations (1.3). Moreover, \( V''(d) \cdot d = \lambda_n d \), with \( \lambda_n = 0 \), and if additionally \([d]\) is isotropic, then \( \lambda_n \) is a multiple eigenvalue of \( V''(d) \).

**Proof.** The fact that (2.40) and (2.41) are solutions of (1.3) can be checked directly. Moreover, the fact that \( d \) is an eigenvector of \( V''(d) \) with the prescribed eigenvalue follows directly from the Euler identity. Hence, we have only to show that if \([d]\) is isotropic, then \( \lambda_n \) is a multiple eigenvalue of \( V''(d) \).

Let us assume that \([d]\) is a proper and isotropic Darboux point. We show that matrix \( V''(d) - E_n \) has the multiple eigenvalue \( \Lambda_n = k - 2 \). Here we notice that this matrix is, under Assumption 2.4 exactly the Jacobi matrix of the right hand sides of the auxiliary system (2.36) calculated at equilibrium point \( d \). Hence eigenvalues of \( V''(d) - E_n \) and their multiplicities are the same as eigenvalues of the Jacobi matrix of system (2.38) calculated at an appropriate point. Direct calculations give

\[
J(x) = \begin{bmatrix}
-\gamma + (k - 1)x_0^{k-2}g_0 & x_0^{k-1}d_1g_0 & \cdots & x_0^{k-1}d_{n-1}g_0 \\
(k - 2)x_0^{k-3}g_1 & x_0^{k-2}d_1g_1 & \cdots & x_0^{k-2}d_{n-1}g_1 \\
& \vdots & \vdots & \vdots \\
(k - 2)x_0^{k-3}g_{n-1} & x_0^{k-2}d_1g_{n-1} & \cdots & x_0^{k-2}d_{n-1}g_{n-1}
\end{bmatrix}(x).
\]

At first we assume that \([d] := [1 : i : 0 : \cdots : 0]\) is an isotropic Darboux point, so \( x_0 = \gamma = 1 \) and \( \bar{d} = \theta_1([d]) = (i, 0, \ldots, 0) \). Thus we have

\[
g_0(\bar{d}) = kv(\bar{d}) - i \frac{\partial v}{\partial x_1}(\bar{d}) = 1, \quad g_1(\bar{d}) = -ikv(\bar{d}),
\]
and
\[ g_j = \frac{\partial \nu}{\partial x_j}(\tilde{d}), \quad \text{for} \quad j = 2, \ldots, n - 1. \] (2.44)

Since at a proper Darboux point we have \( g_i(\tilde{d}) = 0 \) for \( i = 1, \ldots, n - 1 \), we obtain also the following conditions
\[ v(\tilde{d}) = 0, \quad \frac{\partial \nu}{\partial x_j}(\tilde{d}) = 0, \quad \text{for} \quad j = 2, \ldots, n - 1, \] (2.45)
as well as
\[ g_0(\tilde{d}) = -i \frac{\partial \nu}{\partial x_1}(\tilde{d}) = 1, \] (2.46)
and finally
\[ J(d) = \begin{bmatrix}
-k - 2 & \partial_1 g_0 & \cdots & \partial_{n-1} g_0 \\
0 & \partial_1 g_1 & \cdots & \partial_{n-1} g_1 \\
0 & \cdots & \cdots & \cdots \\
\partial_1 g_{n-1} & \cdots & \partial_{n-1} g_{n-1} & \cdots
\end{bmatrix} (d). \] (2.47)

But from conditions (2.45) it follows that
\[ \frac{\partial g_1}{\partial x_1}(\tilde{d}) = -(k - 2)i \frac{\partial \nu}{\partial x_1}(\tilde{d}) = k - 2, \]
and
\[ \frac{\partial g_1}{\partial x_j}(\tilde{d}) = 0, \quad \text{for} \quad j = 2, \ldots, n - 1. \] (2.48)

Thus, in the second row of \( J(d) \) all but the second element are zero, and the second element is \( k - 2 \), so \( k - 2 \) is an eigenvalue of \( J(d) \) with the multiplicity at least two.

If \([d]\) is an arbitrary proper and isotropic Darboux point, then we can always change the reference frame with the help of an element of PO\((n, \mathbb{C})\) in such a way that in the new frame \( d = (1, i, 0, \ldots, 0) \in \mathbb{C}^{n+1} \).

If \([d]\) is an improper and isotropic Darboux point, then we can assume also that it is \([d] = [1 : i : 0 : \cdots : 0]\), and then we proceed as in the case of the proper Darboux point. For an improper Darboux point \( \gamma = 0 \), and \( g_i(\tilde{d}) = 0 \) for \( i = 0, \ldots, n - 1 \), where \( d = \theta_1([d]) = (1, 0, \ldots, 0) \). Thus we have
\[ v(\tilde{d}) = 0, \quad \text{and} \quad \frac{\partial \nu}{\partial x_i}(\tilde{d}) = 0, \quad \text{for} \quad i = 1, \ldots, n - 1, \] (2.49)
and
\[ J(d) = \begin{bmatrix}
0 & \partial_1 g_0 & \cdots & \partial_{n-1} g_0 \\
0 & \partial_1 g_1 & \cdots & \partial_{n-1} g_1 \\
0 & \cdots & \cdots & \cdots \\
\partial_1 g_{n-1} & \cdots & \partial_{n-1} g_{n-1} & \cdots
\end{bmatrix} (d), \] (2.50)
as \( x_0 = 1 \). But using (2.49) and definition of \( g_1 \), one can easily find out that
\[ \frac{\partial g_1}{\partial x_i}(\tilde{d}) = 0, \quad \text{for} \quad i = 1, \ldots, n - 1, \]
so indeed 0 is a multiple eigenvalue of \( J(d) \).

\( \square \)
We underline that the presence of more than one eigenvalue equal to \( k - 2 \) does not imply automatically that the Darboux point is isotropic.

Next we explain relations between the properties of a Darboux point \([d] \in \mathcal{D}(V)\), and properties of points \( \tilde{\pi}^{-1}([d]) \subset \tilde{\mathcal{D}}(V)\).

**Lemma 2.6.** Assume that set \( \mathcal{D}(V) \) is finite. Then point \([d] \in \mathcal{D}^*(V)\) is a simple point of \( \mathcal{D}(V) \) iff each point \([\tilde{d}] \in \tilde{\pi}^{-1}([d])\) is a simple point of \( \tilde{\mathcal{D}}(V)\).

**Proof.** Let \([d] \in \mathcal{D}^*(V)\) be a proper Darboux point satisfying \( V'(d) = d \). Without loss of generality we can assume that \( d_1 \neq 0 \). Then the affine coordinates of \([d]\) on the chart \((U_1, \theta_1)\) are \( \tilde{a} := (a_1, \ldots, a_{n-1}) = \theta_1([d]) \in \mathbb{C}^{n-1}\), where

\[
a_i = \frac{d_{i+1}}{d_1}, \quad \text{for} \quad i = 1, \ldots, n-1,
\]

see formulae (2.19) and (2.20). Thus, \([d]\) is simple iff

\[
\det g'(\tilde{a}) := \det \frac{\partial (g_1, \ldots, g_{n-1})}{\partial (x_1, \ldots, x_{n-1})}(\tilde{a}) \neq 0.
\]

Now, let us take point \([\tilde{d}] := [1 : d_1 : \cdots : d_n] \in \mathbb{C}P^n\). Of course, \([\tilde{d}] \in \tilde{\pi}^{-1}([d])\). Moreover, the affine coordinates of \([\tilde{d}]\) on the chart \((U_0, \theta_0)\) in \( \mathbb{C}P^n\) are \( d \). On this chart \( \tilde{\mathcal{D}}(V) \) is given by \( V(f_1, \ldots, f_n) \), see (2.35). Thus \([\tilde{d}]\) is simple iff

\[
\det f'(d) = \det (V''(d) - E_n) \neq 0.
\]

Notice, that \( f(q) \) coincides with the right hand sides of system (2.36) with \( \gamma = 1 \). Thus, matrix \( f'(q) \) is similar to the matrix \( J(x) \) defined by (2.39), with \( x = (x_0, \ldots, x_{n-1}) \) given by (2.37). For \( q = d \), we have \( x = a := (a_0, a_1, \ldots, a_{n-1}) \), where \( a_0 = d_1 \) and \( a_i \) for \( i > 0 \) are given by (2.51). Matrix \( J(x) \) is determined by (2.42) (with \( \gamma = 1 \)), so we have

\[
J(a) = \begin{bmatrix}
-1 + (k - 1)a_0^{k-2}g_0 & a_0^{k-1}\partial_1g_0 & \cdots & a_0^{k-1}\partial_{n-1}g_0 \\
(k-2)a_0^{k-3}g_1 & a_0^{k-2}\partial_1g_1 & \cdots & a_0^{k-2}\partial_{n-1}g_1 \\
\vdots & \vdots & \ddots & \vdots \\
(k-2)a_0^{k-3}g_{n-1} & a_0^{k-2}\partial_1g_{n-1} & \cdots & a_0^{k-2}\partial_{n-1}g_{n-1}
\end{bmatrix},
\]

where \( g_i = g_i(\tilde{a}) \), for \( i = 0, \ldots, n-1 \), see (2.38) and below. But because \([d]\) is a Darboux point, \( a \) is an equilibrium of system (2.38), so \( g_1(\tilde{a}) = \cdots = g_{n-1}(\tilde{a}) = 0 \), and \( a_0^{k-2}g_0(\tilde{a}) = 1 \). Thus we obtain

\[
J(a) = \begin{bmatrix}
(k-2) & a_0^{k-1}\partial_1g_0 & \cdots & a_0^{k-1}\partial_{n-1}g_0 \\
0 & a_0^{k-2}\partial_1g_1 & \cdots & a_0^{k-2}\partial_{n-1}g_1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_0^{k-2}\partial_1g_{n-1} & \cdots & a_0^{k-2}\partial_{n-1}g_{n-1}
\end{bmatrix},
\]

and

\[
\det f'(d) = \det J(a) = \frac{k - 2}{a_0^{(k-1)(n-1)}} \det g'(\tilde{a}).
\]

Thus we show that \([d]\) is simple if and only if its one preimage, namely \([\tilde{d}]\), is simple. Let

\[
[\tilde{d}_j] := [\varepsilon^j : d_1 : \cdots : d_n], \quad \text{for} \quad j = 0, \ldots, k - 3,
\]
where \( \varepsilon \) is a primitive \((k-2)\)-root of the unity. Then
\[
\hat{\pi}^{-1}(d) = \{ [d_0], \ldots, [d_{k-3}] \}.
\]
Because the affine coordinates of \([d_i]\) are \(d_i := \varepsilon^{-i}d\), we have
\[
f'(d_i) = V''(\varepsilon^{-i}d) - E_n = \varepsilon^{-i(k-2)}V''(d) - E_n = f'(d).
\]
Hence our conclusion is valid for an arbitrary element of the preimage of \([d]\), and this finishes the proof.

If \([d]\) is an improper Darboux point, then the situation is more complicated. We can always assume that \(d_1 \neq 0\). Then the affine coordinates of \([d]\) are \(\bar{a} = \theta_1([d])\), where \(\bar{a}\) is given by (2.51). Since \([d]\) is improper Darboux point \(g_i(\bar{a}) = 0\) for \(i = 0, \ldots, n-1\), and thus, as direct calculations show, we have
\[
g'(\bar{a}) = v''(\bar{a}) \cdot \left( E_{n-1} + \bar{a} \cdot \bar{a}^T \right),
\]
where we consider \(\bar{a}\) as a one column matrix. One can show that matrix \(\bar{a} \cdot \bar{a}^T\) is diagonalizable, its one eigenvalue is \(\lambda = a_1^2 + \cdots + a_{n-1}^2\), and the remaining ones are zero. Thus
\[
\det g'(\bar{a}) = (1 + a_1^2 + \cdots + a_{n-1}^2) \det v''(\bar{a}).
\]
Note that \(1 + a_1^2 + \cdots + a_{n-1}^2 = 0\) iff \([d]\) is isotropic. Hence, we have the following.

**Proposition 2.3.** Assume that set \(D(V)\) is finite, and let \([d] \in S(V)\) be an improper Darboux point. Then it is a simple point of \(D(V)\) iff it is not isotropic and \(\det v''(\bar{a}) \neq 0\).

The preimage \(\hat{\pi}^{-1}([d])\) of an improper Darboux point \([d] = [d_1 : \cdots : d_n]\) is just one point, namely \([d] := [0 : d_1 : \cdots : d_n]\). Again we assume that \(d_1 \neq 0\). Now, we cannot use chart \((U_0, \theta_0)\) in order to check whether \([d]\) is simple. We will use chart \((U_1, \theta_1)\). Hence, on the chart \((U_1, \theta_1)\), set \(\hat{D}(V) = V(h_1, \ldots, h_n)\) where polynomials \(h = (h_1, \ldots, h_n)\), according to (2.13), are
\[
h_i(y) := y_1^{\deg f_i} f_i(\psi(y)) = y_1^{\deg f_i} f_i \left( \frac{1}{y_1}, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1} \right),
\]
for \(i = 1, \ldots, n\). Direct calculations show that
\[
h_1(y) := kv(y) - \sum_{i=2}^{n} y_i \frac{\partial \nu}{\partial y_i}(\bar{y}) = y_2^{k-2}, \quad h_i(y) := \frac{\partial \nu}{\partial y_i}(\bar{y}) = y_i y_1^{k-2},
\]
for \(2 \leq i \leq n\), where \(\bar{y} := (y_2, \ldots, y_n)\), and
\[
\nu(\bar{y}) := V(1, y_2, \ldots, y_n).
\]

With the above formulae, we can calculate the Jacobi matrix \(h'(y)\). It has the following form
\[
h'(y) = \begin{bmatrix}
-(k-2)y_1^{k-3} & \frac{\partial h_1}{\partial y_2}(y) & \cdots & \frac{\partial h_1}{\partial y_n}(y) \\
-(k-2)y_2^{k-3} & \frac{\partial^2 \nu}{\partial y_2^2}(y) - y_1^{k-2} & \cdots & \frac{\partial^2 \nu}{\partial y_2 \partial y_n}(y) \\
-(k-2)y_3^{k-3} & \frac{\partial^2 \nu}{\partial y_3^2}(y) & \cdots & \frac{\partial^2 \nu}{\partial y_3 \partial y_n}(y) \\
\vdots & \vdots & \ddots & \vdots \\
-(k-2)y_n^{k-3} & \frac{\partial^2 \nu}{\partial y_n \partial y_2}(y) & \cdots & \frac{\partial^2 \nu}{\partial y_n \partial y_n}(y) - y_1^{k-2}
\end{bmatrix},
\]

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where
\[
\frac{\partial h_1}{\partial y_j}(y) = (k - 1) \frac{\partial v}{\partial y_j} - \sum_{i=2}^{n} y_i \frac{\partial^2 v}{\partial y_i \partial y_j} \quad \text{for} \quad 2 \leq j \leq n. \tag{2.61}
\]

Notice that \( \theta_1([\bar{d}]) = a := (0, a_2, \ldots, a_n) \), where
\[
a_i = \frac{d_i}{d_1}, \quad \text{for} \quad i = 2, \ldots, n.
\]

Moreover, because \([d]\) is an improper Darboux point, we have
\[
\frac{\partial v}{\partial y_j}([\bar{a}]) = 0, \quad \text{for} \quad j = 2, \ldots, n,
\]
where we denoted \( \bar{a} := (a_2, \ldots, a_n) \). This gives
\[
\frac{\partial h_1}{\partial y_j}(a) = -\sum_{i=2}^{n} a_i \frac{\partial^2 v}{\partial y_i \partial y_j}([\bar{a}]), \quad \text{for} \quad 2 \leq j \leq n.
\]

Let \( k = 3 \). Using the above formulae we find that
\[
\begin{pmatrix}
-1 & -v''([\bar{a}]) \cdot \bar{a} & \cdots & -v''([\bar{a}]) \cdot \bar{a} \\
-a_2 & \frac{\partial^2 v}{\partial y_2}([\bar{a}]) & \cdots & \frac{\partial^2 v}{\partial y_2 \partial y_n}([\bar{a}]) \\
\vdots & \vdots & \vdots & \vdots \\
-a_n & \frac{\partial^2 v}{\partial y_n}([\bar{a}]) & \cdots & \frac{\partial^2 v}{\partial y_n^2}([\bar{a}])
\end{pmatrix}.
\tag{2.62}
\]

Hence, as it is easy to see, we have
\[
\det h'(a) = -(1 + a_2^2 + \cdots + a_n^2) \det v''([\bar{a}]). \tag{2.63}
\]

The above calculations show the following.

**Proposition 2.4.** Assume that \( k = 3 \). Then the preimage \( \bar{d} := \pi^{-1}([d]) \) of an improper Darboux point \([d] \in \mathcal{D}(V)\) is a simple point of \( \bar{\mathcal{D}}(V) \) iff \([d] \) is a simple point of \( \mathcal{D}(V) \).

For \( k > 3 \), the point \( \bar{d} \) is not a simple point of \( \bar{\mathcal{D}}(V) \), as all elements of the first column of matrix \( h'(a) \) vanish. For such a case we have the following.

**Proposition 2.5.** Assume that \( k > 3 \). Then
\[
\text{rank } h'(a) = \text{rank } v''([\bar{a}]) \leq n - 1.
\]

**Proof.** For \( k > 3 \) we have
\[
\begin{pmatrix}
0 & -v''([\bar{a}]) \cdot \bar{a} & \cdots & -v''([\bar{a}]) \cdot \bar{a} \\
0 & \frac{\partial^2 v}{\partial y_2^2}([\bar{a}]) & \cdots & \frac{\partial^2 v}{\partial y_2 \partial y_n}([\bar{a}]) \\
\vdots & \vdots & \vdots & \vdots \\
0 & \frac{\partial^2 v}{\partial y_n \partial y_2}([\bar{a}]) & \cdots & \frac{\partial^2 v}{\partial y_n^2}([\bar{a}])
\end{pmatrix}. \tag{2.64}
\]

The first row of this matrix is a linear combination of the remaining ones, so our claim easily follows. \( \Box \)

With the above notation, we say that \([d] \in \mathcal{S}(V) = \mathcal{D}(V) \setminus \mathcal{D}^*(V)\) is **minimally degenerated** iff \( \text{rank } v''([\bar{a}]) = n - 1 \).
2.3 Obstruction for the integrability due to improper Darboux point

As we have already explained, the proper Darboux points are important because they give particular solutions of the considered canonical equations (1.3) and, thanks to this, we have the necessary conditions for the integrability given by Theorem 1.2. The question is if we can obtain any obstruction for the integrability if we are using the proper Darboux points. Lemma 2.5 shows that the improper Darboux points give a particular solution (2.41) of the considered canonical equations (1.3). However, this solution has an extremely simple form and one can doubt if using it we can obtain any obstruction for the integrability. In fact, it is easy to notice that the monodromy group of the variational equations along solution (2.41) is trivial. Thus, in the frame of the Ziglin theory we do not obtain any obstruction for the integrability from the existence of an improper Darboux point. Lemma 2.5 shows that the improper Darboux points give, in some sense, stronger obstacles for the integrability than the proper ones.

**Theorem 2.2.** Assume that a homogeneous potential \( V \in \mathbb{C}_k[q] \) of degree \( k > 2 \) admits an improper Darboux point \([d]\) \(\in \mathbb{C}^{n-1}\). If \( V \) is integrable with rational first integrals, then matrix \( V''(d) \) is nilpotent, i.e., all its eigenvalues vanish.

**Proof.** We prove the theorem by contradiction. We assume that there exists a non-zero \( \lambda \in \text{spectr} \ V''(d) \). Thus, in an appropriate base, the variational equations along solution (2.41) contain equation \( \eta_1 = -\lambda t^{-2}\eta \). As it was shown in [22] the differential Galois group of this equation is \( SL(2, \mathbb{C}) \) and hence the identity component of the differential Galois group of all variational equations is not Abelian. Thus, by Morales-Ramis Theorem 1.1 the system is not integrable. A contradiction finishes the proof.

The above theorem gives a very strong generalisation of Theorem 2.4 in [22] for systems with an arbitrary number of degrees of freedom.

3 Basic Theorems

3.1 Formulation

Let \([d]\) be a proper Darboux point of potential \( V \). Then, thanks to Assumption 2.1 we have well defined eigenvalues \( \lambda_1(d), \ldots, \lambda_n(d) \) of the Hessian matrix \( V''(d) \). According to our convention \( \lambda_n(d) = k - 1 \) is the trivial eigenvalue. Let \( \lambda(d) = (\lambda_1(d), \ldots, \lambda_{n-1}(d)) \). Hence we have the following mapping

\[
\mathcal{D}^*(V) \ni [d] \mapsto \lambda(d) \in \mathbb{C}^{n-1}. \tag{3.1}
\]

Assume that \( \mathcal{D}^*(V) \) is finite. Then the image of \( \mathcal{D}^*(V) \) under the above map is a finite subset of \( \mathbb{C}^{n-1} \). The question is if we can find a potential \( V \) of degree \( k \) such that the elements in the image have values prescribed in advance. We show that the answer to this question is negative. More precisely, we prove that among \( \lambda(d) \) taken at all proper Darboux points \([d] \in \mathcal{D}^*(V)\) a certain number of universal relations exists. These relations play the fundamental and central role in our considerations.

To formulate our first theorem we define \( \Lambda(d) = (\Lambda_1(d), \ldots, \Lambda_{n-1}(d)) \), where \( \Lambda_i(d) := \lambda_i(d) - 1 \) for \( i = 1, \ldots, n - 1 \). By \( \tau_r \) for \( 0 \leq r \leq n - 1 \), we denote the elementary symmetric polynomials in \((n - 1)\) variables of degree \( r \), i.e.,

\[
\tau_r(x) := \tau_r(x_1, \ldots, x_{n-1}) = \sum_{1 \leq i_1 < \cdots < i_r \leq n-1} \prod_{s=1}^{r} x_{i_s}, \quad 1 \leq r \leq n - 1,
\]
and $\tau_0(x) := 1$.

Our first theorem gives the explicit form of the above mentioned relations among $\Lambda(d)$, $[d] \in \mathcal{D}(V)$ for a generic potential $V$.

**Theorem 3.1.** Let $V \in \mathbb{C}_k[q]$ be a homogeneous potential of degree $k > 2$ and let all its Darboux points be proper and simple. Then

$$\sum_{[d] \in \mathcal{D}^\ast(V)} \frac{\tau_1(\Lambda(d))^r}{\tau_{n-1}(\Lambda(d))} = (-1)^{n-1}(-n - (k - 2))^r,$$

(3.2)

and

$$\sum_{[d] \in \mathcal{D}^\ast(V)} \frac{\tau_r(\Lambda(d))}{\tau_{n-1}(\Lambda(d))} = (-1)^{r+n-1} \sum_{i=0}^{r} \binom{n-i-1}{r-i} (k-1)^i,$$

(3.3)

for $r = 0, \ldots, n-1$.

The proof of the above theorem is given in the next subsection. We prove at first the existence of relations (3.2) and later we show that from them relations (3.3) follow immediately. In effect Theorem 3.1 gives $n$ independent relations.

Let us explain the importance of Theorem 3.1. To do this we need more definitions.

Let $\mathcal{C}_m$ denote the set of all unordered tuples $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$, where $\Lambda_i \in \mathbb{C}$ for $i = 1, \ldots, m$. For $M > 0$, the symbol $\mathcal{C}_M^m$ denotes the set of all unordered tuples $(\Lambda_1, \ldots, \Lambda_M)$, where $\Lambda_i \in \mathcal{C}_M$, for $i = 1, \ldots, M$.

We fix $k > 2$ and $n \geq 2$, and say that a tuple $\Lambda \in \mathcal{C}_{n-1}^n$ is admissible iff $\lambda_i = \Lambda_i + 1 \in \mathcal{M}_k$ for $i = 1, \ldots, n-1$. In other words, $\Lambda_i$ is admissible iff $\Lambda_i + 1$ belongs to items, appropriate for a given $k$, in the table of the Morales-Ramis Theorem 1.2 for $i = 1, \ldots, n-1$. We denote the set of all admissible tuples by $\mathcal{A}_{n,k}$. If the potential $V$ is integrable, then for each $[d] \in \mathcal{D}^\ast(V)$, the tuple $\Lambda(d)$ is admissible. The set of all admissible elements $\mathcal{A}_{n,k}$ is countable but infinite.

The set of proper Darboux points of a potential $V$ is non-empty, and $N = \text{card } \mathcal{D}^\ast(V)$, then the $N$-tuples

$$\mathcal{L}(V) := (\Lambda(d) \mid [d] \in \mathcal{D}^\ast(V)) \in \mathcal{C}_n^N,$$

(3.4)

is called the spectrum of $V$. Let $\mathcal{A}_{n,k}^N$ be the subset of $\mathcal{C}_n^N$ consisting of $N$-tuples $(\Lambda_1, \ldots, \Lambda_N)$, such that $\Lambda_i$ is admissible, i.e., $\Lambda_i \in \mathcal{A}_{n,k}$, for $i = 1, \ldots, N$. We say that the spectrum $\mathcal{L}(V)$ of a potential $V$ is admissible iff $\mathcal{L}(V) \in \mathcal{A}_{n,k}^N$. The Morales-Ramis Theorem 1.2 says that if potential $V$ is integrable, then its spectrum $\mathcal{L}(V)$ is admissible. However, the problem is that the set of admissible spectra $\mathcal{A}_{n,k}^N$ is infinite. We show that from Theorem 3.1 it follows that, in fact, if $V$ is integrable, then its spectrum $\mathcal{L}(V)$ belongs to a certain finite subset $\mathcal{I}_{n,k}^N$ of $\mathcal{A}_{n,k}^N$. We call this set distinguished one, and its elements distinguished spectra.

**Theorem 3.2.** Let potential $V$ satisfy assumptions of Theorem 3.1. If $V$ is integrable, then there exists a finite subset $\mathcal{I}_{n,k}^N \subset \mathcal{A}_{n,k}^N$, where $N = \text{card } \mathcal{D}^\ast(V)$, such that $\mathcal{L}(V) \in \mathcal{I}_{n,k}^N$.

Informally speaking, for fixed $k$ and $n$, we restrict the infinite number of possibilities in each line of the Morales-Ramis table to a finite set of choices.

The importance of the above theorem immediately forces us to ask if it is possible to relax assumptions of Theorem 3.1 and preserves its conclusions. Below we give two generalisations of Theorem 3.1. In the first one we restrict ourselves to the case $k = 3$, but we do not assume that all Darboux points are proper. However, for this generalisation we obtain a smaller number of relations.

**Theorem 3.3.** Let $V \in \mathbb{C}_3[q]$ be a homogeneous potential of degree $k = 3$ and let all its Darboux points be simple. Then relations (3.2) and (3.3) with $r = 0, \ldots, n-2$, are satisfied.
In the second generalisation of Theorem 3.1 we do not assume that all Darboux points are proper and \( k > 3 \) can be arbitrary.

**Theorem 3.4.** Let \( V \in \mathbb{C}_k[q] \), be a homogeneous potential of degree \( k > 3 \) which satisfies the following conditions

1. if \( [d] \in \mathcal{D}^*(V) \), then \( [d] \) is simple,
2. if \( [d] \in \mathcal{D}(V) \setminus \mathcal{D}^*(V) \) then \( [d] \) is minimally degenerated.

Then relations (3.2) and (3.3) with \( r = 0, \ldots, n - 2 \), are satisfied.

The fact that Theorems 3.3 and 3.4 give a smaller number of relations than Theorem 3.1 does not weaken their strength. Namely, we show that the following result holds true.

**Theorem 3.5.** Let potential \( V \) satisfy assumptions of either Theorem 3.3 or Theorem 3.4. If \( V \) is integrable, then there exists a finite subset \( \mathcal{I}^N_{n,k} \subset A^N_{n,k} \), where \( N = \text{card} \mathcal{D}^*(V) \), such that \( \mathcal{L}(V) \in \mathcal{I}^N_{n,k} \).

### 3.2 The Euler-Jacobi-Kronecker formula and multi-dimensional residues

In proofs of Theorems 3.1, 3.3 and 3.4 formulated in the previous subsection, we use the classical Euler-Jacobi-Kronecker formula and its known generalisations. In these generalisations the multi-dimensional residues are used. We need to calculate these residues effectively. In this subsection we recall basic facts about the multi-dimensional residues and the Euler-Jacobi-Kronecker formula. For details the reader is referred to \([1; 14; 13; 36; 17]\).

Let \( f_i : \mathbb{C}^n \supset U \to \mathbb{C} \), where \( U \) is an open neighbourhood of the origin, be holomorphic functions for \( i = 1, \ldots, n \), and \( x = 0 \) be an isolated common zero of \( f_i \). We consider differential \( n \)-form

\[
\omega := \frac{p(x)}{f_1(x) \cdots f_n(x)} \, dx_1 \wedge \cdots \wedge dx_n,
\]

where \( p : U \to \mathbb{C} \) is a holomorphic function. The residue of the form \( \omega \) at \( x = 0 \) can be defined as

\[
\text{res}(\omega, 0) := \frac{1}{(2\pi i)^n} \int_{\Gamma} \omega,
\]

where

\[
\Gamma := \{ x \in U \mid |f_1(x)| = \epsilon_1, \ldots, |f_n(x)| = \epsilon_n \},
\]

and \( \epsilon_1, \ldots, \epsilon_n \) are sufficiently small positive numbers. The orientation of \( \Gamma \) is fixed by

\[
d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geq 0.
\]

Let us denote \( f := (f_1, \ldots, f_n) \). It can be shown that if the Jacobian \( \det f'(0) \neq 0 \), then

\[
\text{res}(\omega, 0) = \frac{p(0)}{\det f'(0)}.
\]

The following theorem gives the classical Euler-Jacobi-Kronecker formula, see e.g. \([13]\).

**Theorem 3.6 (Euler-Jacobi-Kronecker).** Let \( f_1, \ldots, f_n \in \mathbb{C}[x] \) be non-constant polynomials such that \( \mathcal{V}(f) := \mathcal{V}(f_1, \ldots, f_n) \) is finite and all points of this set are simple. If \( f_1, \ldots, f_n \) do not intersect at infinity, then for each \( p \in \mathbb{C}[z] \) such that

\[
\deg p \leq \sum_{i=1}^n \deg f_i - (n + 1),
\]

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we have
\[ \sum_{d \in \mathcal{V}(f)} \text{res}(\omega, d) = \sum_{d \in \mathcal{V}(f)} \frac{p(d)}{\det f'(d)} = 0. \quad (3.11) \]

The above theorem is not sufficient for our investigations. We have to consider cases when \( f_1, \ldots, f_n \) have intersections at the infinity as well as cases when intersections of \( f_1, \ldots, f_n \) are not simple.

The homogenisations of \( f_i \) are given by
\[ F_i(z_0, z_1, \ldots, z_n) := z_0^{\deg f_i} f_i \left( \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right), \quad \text{for} \quad i = 1, \ldots, n. \quad (3.12) \]

They define the projective algebraic set \( \mathcal{V}(F) := \mathcal{V}(F_1, \ldots, F_n) \subset \mathbb{CP}^n \) whose affine part is homeomorphic to \( \mathcal{V}(f) \). Next we extend the form \( \omega \) to a rational form \( \Omega \) defined on \( \mathbb{CP}^n \).

To this end we consider \( \omega \) as the expression of \( \Omega \) on the chart \((U_0, \theta_0)\). In order to express \( \Omega \) on other charts we use the standard coordinate transformations. For example, if we are using formulae from Section 2.1 on \((U_1, \theta_1)\), the form \( \Omega \) is given by the pullback \( \tilde{\omega} := \psi^* \omega \).

To write down \( \tilde{\omega} \) explicitly we put
\[ h_i(y) := y_1^{\deg f_i} f_i(\psi(y)) = y_1^{\deg f_i} f_i \left( \frac{1}{y_1}, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1} \right), \quad (3.13) \]
for \( i = 1, \ldots, n \), and
\[ r(y) := y_1^{\deg p} p(\psi(y)) = y_1^{\deg p} p \left( \frac{1}{y_1}, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1} \right). \quad (3.14) \]

Then we obtain
\[ \tilde{\omega} = -\frac{r(y)y_1^s}{h_1(y) \cdots h_n(y)} dy_1 \wedge \cdots \wedge dy_n, \quad (3.15) \]
where
\[ s := \sum_{i=1}^n \deg f_i - \deg p - (n + 1). \quad (3.16) \]

Notice that \( r, h_1, \ldots, h_n \in \mathbb{C}[y] \).

Let \([p] = [p_0 : \cdots : p_n] \in U_i \cap \mathcal{V}(F_1, \ldots, F_n)\). We can define the residue of the form \( \Omega \) at point \([p]\) as
\[ \text{res}(\Omega, [p]) := \text{res}(\tilde{\omega}, \theta_i([p])) \], \quad (3.17) \]
where \( \tilde{\omega} \) denotes form \( \Omega \) expressed in the chart \((U_i, \theta_i)\).

The form \( \Omega \) is defined by homogeneous polynomials \( F_1, \ldots, F_n \) and
\[ P(z_0, z_1, \ldots, z_n) := z_0^{\deg p} p \left( \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right). \quad (3.18) \]

To underline the explicit dependence of \( \Omega \) on \( F_i \) and \( P \) we write symbolically \( \Omega = P/F \).

The following theorem is a special version of the global residue theorem.

**Theorem 3.7.** Let \( \mathcal{V}(F) := \mathcal{V}(F_1, \ldots, F_n) \) be a finite set. Then for each polynomial \( P \) such that
\[ \deg P \leq \sum_{i=1}^n \deg F_i - (n + 1), \quad (3.19) \]
we have
\[ \sum_{[s] \in \mathcal{V}(F)} \text{res}(P/F, [s]) = 0. \quad (3.20) \]
For the proof and the more detailed exposition we refer the reader to [13; 7].

If \( 0 \in \mathcal{V}(f) \) is an isolated but not simple point, then we cannot use formula (3.9) to calculate the residue of the form \( \omega \) at this point. In such a case we can apply a very nice method developed by Biernat in [3; 4] that reduces the calculation of multi-dimensional residue to a one dimensional case. We describe it shortly below.

Let us consider the following analytic set
\[
\mathcal{A} := \{ x \in U \mid f_2(x) = \cdots = f_n(x) = 0 \}, \tag{3.21}
\]
where \( U \subset \mathbb{C}^n \) is a neighbourhood of the origin. Set \( \mathcal{A} \) is a sum of irreducible one dimensional components \( \mathcal{A} = A_1 \cup \cdots \cup A_m \). Let \( t \mapsto \varphi_i(t) \in A_i, \varphi_i(0) = 0 \), be an injective parametrisation of \( A_i \).

Then we define the following forms
\[
\omega_i = p(\varphi_i(t)) \frac{f_1'(\varphi_i(t)) \cdot \varphi_i(t)}{f'(\varphi_i(t))} \, dt. \tag{3.22}
\]

As it was shown in [6] we have
\[
\text{res}(\omega, 0) = \sum_{i=1}^{m} \text{res}(\omega_i, 0). \tag{3.23}
\]

### 3.3 Proofs

Let \( V \in \mathbb{C}_k[x] \), where \( x = (x_1, \ldots, x_n) \), be a homogeneous polynomial of degree \( k > 2 \). We define \( n \) polynomials \( f = (f_1, \ldots, f_n) \) putting
\[
f_i = \frac{\partial V}{\partial x_i} - x_i, \quad \text{for} \quad i = 1, \ldots, n, \tag{3.24}
\]
i.e., \( f(x) = V'(x) - x \). In our considerations the basic role plays the following \( n \) form
\[
\omega = \frac{p(x)}{f_1(x) \cdots f_n(x)} \, dx_1 \wedge \cdots \wedge dx_n, \tag{3.25}
\]
where \( p \) is a polynomial. The polar loci of this form are points of \( \mathcal{V}(f) := \mathcal{V}(f_1, \ldots, f_n) \). Let us notice the following facts.

**Proposition 3.1.** Point \( 0 \in \mathcal{V}(f) \) is a simple point and \( f'(0) = -E_n \). Thus we have
\[
\text{res}(\omega, 0) = (-1)^n p(0). \tag{3.26}
\]

If \( d \in \mathcal{V}(f) \) and \( d \neq 0 \), then
1. point \( [d] \in \mathbb{C}P^{n-1} \) is a proper Darboux point of \( V \), i.e., \( [d] \in D^*(V) \),
2. the Jacobi matrix \( f'(d) \) has eigenvalues \( \Lambda_1(d), \ldots, \Lambda_{n-1}(d), \Lambda_n(d) = k - 2 \),
3. if \( \det f'(d) \neq 0 \), then
\[
\text{res}(\omega, d) = \frac{p(d)}{(k - 2)\Lambda_1(d) \cdots \Lambda_{n-1}(d)}, \tag{3.27}
\]
4. points \( d_j := \epsilon^j d \in \mathcal{V}(f), \) where \( \epsilon \) is a primitive \( (k - 2) \)-root of the unity, satisfy \( f'(d_j) = f'(d), \) for \( j = 0, \ldots, k - 3 \).
An easy proof of the above proposition we left to the reader.

Let us notice that \( f_i^+ = \partial_i V \), so \( f_i \) intersect at the infinity at a point \( d \neq 0 \) iff \( V'(d) = 0 \). For such \( d \), point \([d] \in \mathbb{CP}^{n-1}\) is an improper Darboux point. Hence, in order to investigate the intersection of \( f \) at the infinity we extend the form \( \omega \) into a rational form \( \Omega \) on \( \mathbb{CP}^n \). We proceed exactly as in the previous subsection. Namely, let

\[
F_i(z_0, z_1, \ldots, z_n) := z_0^{\deg f_i} f_i \left( \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right) = \frac{\partial V}{\partial z_i} - z_0^{k-2} z_i,
\]

be the homogenisation of \( f_i \), for \( i = 1, \ldots, n \), and

\[P(z_0, z_1, \ldots, z_n) := z_0^{\deg p} p \left( \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right),\]

the homogenisation of \( p \). The affine part of the projective algebraic set \( \mathcal{V}(F) := \mathcal{V}(F_1, \ldots, F_n) \) is homeomorphic to \( \mathcal{V}(f) \). Notice that \( \mathcal{V}(F) = \hat{D}(V) \). Hence, the form \( \Omega = P/F \) on the chart \((U_0, \theta_0)\) is given by (3.25). According to the previous subsection on the chart \((U_1, \theta_1)\) it reads

\[
\hat{\omega} = -\frac{r(y)y_1^{s}}{h_1(y) \cdots h_n(y)} dy_1 \wedge \cdots \wedge dy_n,
\]

where

\[
s := n(k - 1) - \deg p - (n + 1),
\]

and polynomials \( h_i \) are defined by (2.58) and (2.59), and \( r(y) \) is given by (3.14).

In the proofs of Theorems 3.1, 3.3 and 3.4 we use the same idea: we calculate residues of the form \( \Omega = P/F \) with appropriately chosen \( P \) at all points of \( \mathcal{V}(F) = \hat{D}(V) \).

### 3.3.1 Proof of Theorem 3.1

At first we prove the following lemma.

**Lemma 3.1.** If assumptions of Theorem 3.1 are fulfilled, then for a polynomial \( P \in \mathbb{C}[q] \), with degree \( \deg P = m \leq n(k - 2) - 1 \), satisfying

\[
P(x) = P(\varepsilon^i x) \quad \text{for all} \quad x \in \mathbb{C}^n \quad \text{and} \quad i = 0, \ldots, k - 3,
\]

where \( \varepsilon \) is a primitive \((k - 2)\)-root of unity, we have

\[
\sum_{[d] \in \mathcal{V}} \frac{P(d)}{\tau_{n-1}(\Lambda(d))} = (-1)^{n-1} P(0).
\]

**Proof.** We take the form \( \Omega = P/F \) and we calculate residues of this form at all points of \( \mathcal{V}(F) = \hat{D}(V) \). In the affine part of \( \mathbb{CP}^n \), form \( \Omega \) is given by (3.25).

Assumptions of Theorem 3.1 imply that \( V'(x) = 0 \) only for \( x = 0 \). Thus, polynomials \( f_1, \ldots, f_n \) do not intersect at infinity. So all points of \( \mathcal{V}(F) \) lie in the affine part of \( \mathbb{CP}^n \).

If \( d \in \mathcal{V}(f_1, \ldots, f_n) \), and \( d \neq 0 \), then \([d]\) is a proper Darboux point of \( V \). By our assumption, \([d]\) is a simple point of \( \hat{D}(V) \). By Lemma 2.6 all points of the preimage \( \hat{n}^{-1}([d]) \) are simple. This is why the inequality holds

\[
\det f'(d) = (k - 2) \prod_{i=1}^{n-1} \Lambda_i(d) \neq 0.
\]

Moreover, by Proposition 3.1 \( d_j := \varepsilon^j d \in \mathcal{V}(f_1, \ldots, f_n) \), for \( j = 0, \ldots, k - 3 \), where \( \varepsilon \) is a primitive \((k - 2)\)-root of the unity, and \( f'(d_j) = f'(d) \) for \( j = 0, \ldots, k - 3 \), so all Jacobi
matrices $f'(d_i)$ have the same eigenvalues $\Lambda_1(d), \ldots, \Lambda_{n-1}(d), \Lambda_n(d) = k - 2$. Additionally, polynomial $p$ satisfies restriction (3.10) of Theorem 3.6 and $p(d_j) = p(d)$ for $j = 0, \ldots, k - 3$.

The above considerations show that all assumptions of Theorem 3.6 are satisfied. For a given $d \in V(f_1, \ldots, f_n)$, $d \neq 0$, we have

$$\sum_{j=0}^{k-3} \text{res}(\omega, d_j) = \frac{p(d)}{\Lambda_1(d) \cdots \Lambda_{n-1}(d)},$$

(3.33)

and for $0 \in V(f_1, \ldots, f_n)$ we obtain

$$\text{res}(\omega, 0) = (-1)^n p(0),$$

(3.34)
as $f'(0) = -E_n$. Taking the sum of (3.33) over all Darboux points and adding (3.34) we obtain zero.

The elements of matrix $f'(x)$ are polynomials of degree $k - 2$. Thus, we can take

$$p(x) = (\text{Tr} f'(x) - (k - 2))r, \quad \text{with} \quad r \in \{0, \ldots, n - 1\},$$

(3.35)

and apply Lemma 3.1 For this choice of $p(x)$ we have

$$p(d) = \tau_1(\Lambda(d))^r \quad \text{for} \quad d \in D^r(V),$$

(3.36)

and

$$p(0) = (-n - (k - 2))^r.$$  

(3.37)

Thus, making the above choice for $p(x)$ we obtain relations (3.2) in Theorem 3.1.

In order to prove that relations (3.2) are valid we define polynomials $p_i(x)$ which are coefficients of the characteristic polynomial of matrix $f'(x)$, namely

$$\det(f'(x) - \lambda E_n) = (-1)^n \left( \lambda^n - p_1(x)\lambda^{n-1} + \cdots + (-1)^np_n(x) \right).$$

(3.38)

Obviously $\text{deg } p_i = i(k - 2)$, and $p_i(e^ix) = p_i(x)$ for all $x \in C^n$, $j = 0, \ldots, k - 3$, and $i = 0, \ldots, n$. For $d \in D^r(V)$ we have $p_i(d) = \tau_i(\Lambda(d))$, where

$$\tilde{\Lambda}(d) = (\Lambda_1(d), \ldots, \Lambda_{n-1}(d), k - 2),$$

and

$$p_i(0) = (-1)^i \left( \begin{array}{c} n \\ i \end{array} \right).$$

Applying Lemma 3.1 for $p(x) = p_i(x)$ with $0 \leq i \leq n - 1$, we obtain the following relations

$$\sum_{|d| \in D^r(V)} \frac{\tau_i(\tilde{\Lambda}(d))}{\tau_{n-1}(\Lambda(d))} = (-1)^{n+r-1} \left( \begin{array}{c} n \\ r \end{array} \right).$$

(3.39)

Since

$$\tau_i(\tilde{\Lambda}(d)) = \tau_i(\Lambda(d)) + (k - 2)\tau_{i-1}(\tilde{\Lambda}(d)), \quad i = 0, \ldots, n - 1,$$

(3.40)

where $\tau_{-1}(\Lambda(d)) = 0$, we have the following system of linear equations

$$T_r + (k - 2)T_{r-1} = (-1)^{n+r-1} \left( \begin{array}{c} n \\ r \end{array} \right), \quad T_0 = (-1)^n - 1,$$

(3.41)
where $r = 1, \ldots, n - 1$, and
\[
T_r = \sum_{[d] \in \mathbb{D}^*(V)} \frac{\tau_r(\Lambda(d))}{\tau_{n-1}(\Lambda(d))}.
\] (3.42)

Now using induction with respect to $r$ and the relation
\[
\binom{n-i}{r-i+1} = \binom{n-i-1}{r-i} + \binom{n-i-1}{r-i+1},
\]
it is easy to show that
\[
T_r = (-1)^{r+n-1} \sum_{i=0}^r \binom{n-i-1}{r-i}(k-1)^i, \quad \text{for} \quad r = 0, \ldots, n-1.
\] (3.43)

This shows that relations (3.2) are satisfied and the proof is finished.

### 3.3.2 Proof of Theorem 3.3

Our proof is based on the following lemma.

**Lemma 3.2.** If assumptions of Theorem 3.3 are satisfied, then for a polynomial $p \in \mathbb{C}[q]$, with degree $\deg p = m \leq n(k-2) - 2 = n - 2$, we have
\[
\sum_{[d] \in \mathbb{D}(V)} \frac{p(d)}{\tau_{n-1}(\Lambda(d))} = (-1)^{n-1}p(0).
\] (3.44)

**Proof.** We proceed in a similar way as in the proof of Theorem 3.1. However, now polynomials $f_1, \ldots, f_n$ can have intersections at the infinity, and because of this we have to apply Theorem 3.7. Thus, we have to calculate residues of the form $\Omega = P/F$ at all points of $\mathbb{V}(F) = \hat{\mathbb{D}}(V)$. Points of $\hat{\mathbb{D}}(V)$ which lie in the affine part of $\mathbb{C}P^n$ are given by $\mathbb{V}(f)$. Since, by assumption, $k = 3$, there is one correspondence between points of $\mathbb{D}^*(V)$ and $\mathbb{V}(f) \setminus \{0\}$. Thus, for each $[d] \in \mathbb{D}^*(V)$, by assumption, $[d]$ is a simple point of $\hat{\mathbb{D}}(V)$, so by Lemma 2.6 $\hat{\pi}^{-1}([d])$ is a simple point of $\mathbb{D}(V)$. As a result for each non-zero $d \in \mathbb{V}(f)$ we have
\[
\det f'(d) = \Lambda_1(d) \cdots \Lambda_{n-1}(d) \neq 0,
\]
and
\[
\text{res}(\omega, d) = \frac{p(d)}{\Lambda_1(d) \cdots \Lambda_{n-1}(d)}.
\] (3.45)

Moreover, the local residue of the form $\omega$ at $x = 0$ is
\[
\text{res}(\omega, 0) = (-1)^n p(0).
\] (3.46)

Let us assume that $[\tilde{d}] \in \mathbb{V}(F)$ lies in the hyperplane at the infinity. Without any loss of the generality we can assume that $[\tilde{d}] = [0 : d_1 : \cdots : d_n]$ with $d_1 = 1$. We know that $[d] = [d_1 : \cdots : d_n]$ is an improper Darboux point which is, by assumption, a simple point of $\mathbb{D}(V)$, and thus, by Proposition 2.4 $[\tilde{d}]$ is a simple point of $\mathbb{D}(V)$. It turns out that we can calculate the residue of the form $\Omega$ using the local residue formula on the chart $(U_1, \theta_1)$, where form $\Omega$ is given by (3.29). Since coordinates of $[\tilde{d}]$ on $(U_1, \theta_1)$ are $a = (0, d_2, \ldots, d_n) = \theta_1([\tilde{d}])$, and $s$ given by formula (3.30) is greater than zero, we have
\[
\text{res}(\tilde{\omega}, a) = 0.
\]

The above shows that points of $\mathbb{V}(F)$ which lie at infinity do not enter in the total sum of the residues.
To prove the theorem we apply Lemma 3.2 taking \( p(x) = (\text{Tr} f'(x) - 1)^r \) and \( p(x) = p_r(x) \) with \( r = 0, \ldots, n - 2 \). Then repeating arguments used in the proof of Theorem 3.1 we show that relations (3.2) and (3.3) are satisfied.

### 3.3.3 Proof of Theorem 3.4

The proof will follow directly from the following lemma.

**Lemma 3.3.** If assumptions of Theorem 3.4 are satisfied, then for a polynomial \( p \in \mathbb{C}[q] \), with degree \( \deg p = m \leq (n - 1)(k - 2) - 1 \), satisfying

\[
P(x) = P(e^i x) \quad \text{for all} \quad x \in \mathbb{C}^n \quad \text{and} \quad i = 0, \ldots, k - 3,
\]

where \( e \) is a primitive \((k - 2)\)-root of unity, we have

\[
\sum_{[d] \in D(V)} \frac{P(d)}{\tau_{n-1}(\Lambda(d))} = (-1)^{n-1}P(0).
\]

**Proof.** As in the proof of Lemma 3.2 we have to show that residues of the form \( \Omega = P/F \) at points \( [d] \in V(F) \) which lie on the hypersurface at the infinity, vanish. The difficulty that we meet here is that now these points, by Proposition 2.5, are not simple and we cannot use the local residue formula.

Let us assume that \( [d] \in V(F) \cap H_{\infty} \). Without any loss of generality we can assume that \( [d] = [0:1:d_2: \cdots : d_n] \). The coordinates of \([d]\) on the chart \((U_1, \theta_1)\) are \( a = (0,d_2, \ldots, d_n) = \theta_1([d])\). We have to calculate the residue of the form \( \bar{\omega} \) given by (3.29) at point \( a \). We know that \( [1:d_2: \cdots : d_n] \) is an improper Darboux point. By assumption it is weakly degenerated. To calculate \( \text{res}(\bar{\omega}, a) \) we apply the Biernat formula, see the end of Section 3.2 To this end we need to determine all the branches of the analytic set

\[
A := \{ y \in U_a \mid h_2(y) = \cdots = h_n(y) = 0 \},
\]

where \( U_a \) is a neighbourhood of \( a \). Since \( a \) are the coordinates of an improper Darboux point that is minimally degenerated, by Proposition 2.5 we have

\[
\text{rank} \frac{\partial(h_2, \ldots, h_n)}{\partial(y_1, \ldots, y_n)}(a) = n - 1,
\]

and moreover

\[
\text{det} \frac{\partial(h_2, \ldots, h_n)}{\partial(y_2, \ldots, y_n)}(a) \neq 0.
\]

Hence, by the implicit function theorem there is only one analytic branch passing by point \( a \). We can always choose coordinates such that \( a = 0 \). Then, because

\[
h_i(y) = \frac{\partial v}{\partial y_i} - y_i y_1^{k-2}, \quad \text{for} \quad i = 2, \ldots, n,
\]

and

\[
\frac{\partial v}{\partial y_i}(0) = 0, \quad \text{for} \quad i = 2, \ldots, n,
\]

we can put

\[
y_1(t) = t, \quad \text{and} \quad y_i(t) = 0, \quad \text{for} \quad 2 \leq i \leq n,
\]

(3.50)
as the parametrisation of the unique branch of $A$. Then, the Biernat formula \((3.22)\), applied to the form $\tilde{\omega}$ gives the differential form $w(t)dt$, where

$$w(t) = -\frac{r(y(t))t^s}{h_1(y(t)) \det h'(y(t))} \frac{d}{dt}h_1(y(t)). \quad (3.51)$$

Calculations show that

$$\det h'(y(t)) = -(k-2)t^{k-3}h(t), \quad (3.52)$$

where

$$h(t) = \det \left[ \frac{\partial^2 \psi}{\partial y_i \partial y_j}(0) - t^{k-2} \delta_{ij} \right]_{2 \leq i,j \leq n}. \quad (3.53)$$

As an effect we have

$$w(t) = \frac{\tilde{r}(t)t^{s-(k-2)}}{h(t)}, \quad (3.54)$$

where $\tilde{r}(t) = r(y(t))$ is a polynomial and moreover

$$h(0) = \det \left[ \frac{\partial^2 \psi}{\partial y_i \partial y_j}(0) \right]_{2 \leq i,j \leq n} \neq 0, \quad (3.55)$$

by assumption that the considered point is weakly degenerated. Thus, $w(t)$ is regular at $t = 0$, iff $s \geq k - 2$. Thus, if $m = \deg p \leq (n-1)(k-2) - 1$, then $w(t)$ is regular at $t = 0$. But this is the assumption of our lemma, so all the residues at points vanish at infinity.

We calculate the residues at points in the affine part of $V(F)$ as in the proof of Lemma \[3.1\] \]

To prove the theorem we have only to check if we can apply Lemma \[3.3\] for $p(x) = (\text{Tr} f'(x) - (k-2))^r$ with $r = 0, \ldots, n - 2$. But $\deg (\text{Tr} f'(x) - (k-2))^r = r(k-2)$, so $\deg p \leq (n-1)(k-2) - 1$ for $r = 0, \ldots, n - 2$.

### 3.3.4 Proof of Theorem \[3.2\] and Theorem \[3.5\]

Let us recall that $A_{n,k}$ denotes the set of all admissible $(n-1)$-tuples for a given $k$. We define the set of distinguished spectra $\mathcal{A}^N_{n,k}$, which appears in Theorem \[3.2\] and \[3.5\] as the set of admissible $(\Lambda_1, \ldots, \Lambda_N) \in A^N_{n,k}$ satisfying all relations \[3.2\] and \[3.3\]. In particular, $(\Lambda_1, \ldots, \Lambda_N) \in A^N_{n,k}$ satisfies relation \[3.3\] with $r = n - 2$, i.e.,

$$\sum_{i=1}^N \frac{\tau_{n-2}(\Lambda_i)}{\tau_{n-1}(\Lambda_i)} = -\frac{(k-1)^n - n(k-2) - 1}{(k-2)^2}. \quad (3.55)$$

If we denote $(\Lambda_1, \ldots, \Lambda_N) = (\Lambda_1, \ldots, \Lambda_M)$, where $M = (n-1)N$, then \[3.55\] reads

$$\sum_{i=1}^M \frac{1}{\Lambda_i} = -\frac{(k-1)^n - n(k-2) - 1}{(k-2)^2}. \quad (3.56)$$

We need the following technical lemma.

**Lemma 3.4.** For $k > 2$, $n \geq 2$ and $N \geq 1$, the set of solutions of \[3.55\] in $A^N_{n,k}$ is finite.
Proof. We rewrite relation (3.56) in the form
\[
\sum_{i=1}^{M} X_i = -c, \quad \text{where} \quad c > 0. \tag{3.57}
\]
As it is easy to check, the admissible set \(X_k \subset \mathbb{Q}\) of values of \(X_i\) for a given \(k > 2\), has the following properties: \(X_k = X_k^- \cup X_k^+\), where \(X_k^-\) is finite and its elements are negative; set \(X_k^+\) is infinite and its elements are positive. From (3.57) it follows that a certain number of \(X_i\) must be negative, and we have only a finite number of choices of these negative \(X_i\). For each of them we have to investigate equation
\[
\sum_{i=1}^{p} X_i = c', \quad \text{where} \quad c' > 0, \tag{3.58}
\]
where \(X_i \in X_k^+\), for \(i = 1, \ldots, p < M.\) But the set \(X_k^+\) has an important property. Namely, its only accumulation point is zero, i.e., \(\overline{X_k^-} \setminus X_k^+ = \{0\}\). Hence, applying Lemma B.1 from [22] we obtain that the set of solutions \((X_1, \ldots, X_p) \in (X_k^+)^p\) is finite.

Now, we pass to the proofs of Theorems 3.2 and 3.5. If an integrable potential \(V\) satisfies assumptions of one of Theorems 3.1, 3.3 or 3.3, then
\[
\mathcal{L}(V) := (\mathcal{A}(d) \mid [d] \in \mathcal{D}^*(V)) \in \mathcal{A}_{n,k}^N, \quad \text{where} \quad N = \text{card} \mathcal{D}^*(V), \tag{3.59}
\]
fulfils, among others, relation (3.3) with \(r = n - 2\). By Lemma 3.4, this relation is fulfilled only by a finite number of admissible elements.

3.4 Discussion

We know that, for a given \(k > 2\), and \(n \geq 2\), the set of distinguished spectra \(I_{n,k}^N\) is finite but we do not know how many elements it has. We show that this set is not empty.

**Proposition 3.2.** For each \(k > 2\), \(n \geq 2\), and each \(N > 1\), set \(I_{n,k}^N\) is not empty and it contains at least two elements.

**Proof.** We will use the following convention. If a collection \(J \in \mathcal{C}_{n,k}^N\) contains \(n_1\) copies of the element \(e_1, \ldots,\) and \(n_j\) copies of the element \(e_j\), then we write
\[
J = (n_1 \times e_1, \ldots, n_j \times e_j) = (n_i \times e_i \mid i = 1, \ldots, j),
\]
instead of
\[
J = (e_1, \overbrace{e_1 \ldots e_1}^{n_1 \text{ times}}, \ldots, e_j, \overbrace{e_j \ldots e_j}^{n_j \text{ times}}).
\]

For given \(k\) and \(n\) we define
\[
a_i := (-1, \ldots, -1, k - 2, \ldots, k - 2), \quad \text{for} \quad i = 0, \ldots, n - 1,
\]
\[
b_i := \left(-1, \ldots, -1, k - 2, \ldots, k - 2, \frac{k + 1}{2k}\right), \quad \text{for} \quad i = 0, \ldots, n - 2,
\]
\[
c_i := (-1, \ldots, -1, k - 2, \ldots, k - 2, k + 1), \quad \text{for} \quad i = 0, \ldots, n - 2,
\]
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and

\[ \mathcal{J}_1 := \left( (k-2)^i \binom{n}{i+1} \times a_i \mid i = 0, \ldots, n-1 \right), \]  
\[ \mathcal{J}_2 := (a_i \times a_i, \beta_i \times b_i, \gamma_i \times c_i \mid l = 1, \ldots, n-2; i = 0, \ldots, n-2), \]  
where

\[ \alpha_i := (k-2)^{n-2-2i} \binom{n-2}{1-i}, \quad \beta_i := (k-2)^{n-2-i} \binom{n-2}{i}, \]
\[ \gamma_i = (k-1)(k-2)^{n-i-2} \binom{n-2}{i}. \]

Now, it is only the matter of simple but lengthy calculations to check that \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are elements of \( \mathcal{J}_{n,k}^N \) with \( N = D(n,k) \).

One can wonder whether there exist potentials \( V \) with spectrum given by \( \mathcal{J}_1 \) or \( \mathcal{J}_2 \). Below we give examples of potentials satisfying this condition.

Let us take

\[ V_1 = \sum_{i=1}^{n} v_i q_i^k, \quad \text{where} \quad (v_1, \ldots, v_n) \in \mathbb{C}^n. \]  

For generic values of parameters \( v_i \) this potential possesses \( D(n,k) \) Darboux points and one can check that \( (\Lambda(d) \mid [d] \in \mathcal{D}(V_1) ) = \mathcal{J}_1. \)

Potential

\[ V_2 = \sum_{i=0}^{[k/2]} 2^{-2i} \binom{k-i}{i} q_{1}^{2i} q_{2}^{k-2i} + \sum_{i=3}^{n} c_i q_i^k, \quad \text{where} \quad (c_3, \ldots, c_n) \in \mathbb{C}^{n-3}, \]  

possesses also \( D(n,k) \) Darboux points (for generic values of parameters). Moreover now \( (\Lambda(d) \mid [d] \in \mathcal{D}(V_2) ) = \mathcal{J}_2. \)

It is obvious that one would like to have theorems more general than those proved in the previous subsection. Having relations of the form given in the theorems, we have to keep the assumption that all proper Darboux points are simple. Hence the only possibility is to weaken assumptions concerning improper Darboux points. However, due to theoretical difficulties, we were not able to find good generalisations of the theorems. For example, we tried to relax the second assumption of Theorem 3.4 and proceed in the same way as in its proof. However we meet a serious difficulty. We did not find a general enough method to distinguish and parametrise branches of the analytic set (3.49).

Examples suggest that it is possible to find the desired generalisation.

**Example 3.1 Potential**

\[ V = q_1(aq_1^2 + bq_2q_3), \quad \text{with} \quad a \neq 0, \quad \text{and} \quad b \neq 0, \]  

has five proper Darboux points \( [d_i], i = 1, \ldots, 5 \) and two improper Darboux points. All of them are simple, and

\[ \Lambda(d_1) = \left( -\frac{3a+b}{3a}, -\frac{3a-b}{3a} \right), \quad \Lambda(d_2) = \Lambda(d_3) = \left( -2, -\frac{2(3a-b)}{b} \right), \]  
\[ \Lambda(d_4) = \Lambda(d_5) = \left( -2, -\frac{2(3a+b)}{b} \right). \]

Although \( V \) does not satisfy assumptions of Theorem 3.1 all relations (3.2) are fulfilled. Of course (3.64) satisfies assumptions of Theorem 3.3.
Example 3.2 Potential

\[ V = q_1(q_1^2 + i q_1 q_2 + a q_3^2), \]

has, for generic values of parameter \( a \), four simple proper Darboux points and one improper Darboux point which has multiplicity 3. Although assumptions of Theorem 3.3 are not satisfied, relations (3.2) and (3.3) for \( r = 0, 1 \) are fulfilled. But

\[
\sum_{[d] \in D^s(V)} \frac{(\Lambda_1(d) + \Lambda_2(d))^2}{\Lambda_1(d) \Lambda_2(d)} = \frac{1 + a(28a - 9)}{2a^2}.
\]

Example 3.3 Potential

\[ V = (q_2 - i q_1)^2(b_1 q_1^2 + b_2 q_2^2 + b_3 q_3^2 + b_4 q_1 q_2), \]

has, for generic values of parameters, three simple proper Darboux points and a curve of improper Darboux points \([1 : i : \alpha]\), where \( \alpha \in \mathbb{C} \). None of our theorems can be applied for this potential. In fact, we have

\[
\sum_{[d] \in D^s(V)} \frac{1}{\Lambda_1(d) \Lambda_2(d)} = -\frac{4b_1b_2 + 2(b_1 + b_2)b_3 - b_2^2}{2(b_1 + b_2)^2},
\]

\[
\sum_{[d] \in D^s(V)} \frac{\Lambda_1(d) + \Lambda_2(d)}{\Lambda_1(d) \Lambda_2(d)} = -\frac{2(b_1^2 + b_2^2) + b_3^2}{2(b_1 + b_2)^2},
\]

\[
\sum_{[d] \in D^s(V)} \frac{(\Lambda_1(d) + \Lambda_2(d))^2}{\Lambda_1(d) \Lambda_2(d)} = \frac{4(b_1^2 + b_1 b_2 + b_2^2) + 2(b_1 + b_2)b_3 + b_4^2}{2(b_1 + b_2)^2},
\]

so none of the relations of the form (3.2) is satisfied. Nevertheless, we have one universal relation for the considered potential. Namely combining the above formulae we obtain the equality

\[
\sum_{[d] \in D^s(V)} \frac{(1 + \Lambda_1(d) + \Lambda_2(d))^2}{\Lambda_1(d) \Lambda_2(d)} = 0.
\]

4 Three degrees of freedom and potential of third degree

In this section we consider systems with three degrees of freedom with a homogeneous potential of the third degree. The aim is to show how to apply our theorems to distinguish integrable potentials. In fact, we consider here only generic potentials having a maximal number of simple Darboux points. Investigations of non-generic potentials will be presented in a separate publication.

The whole analysis can be divided roughly into two steps:

1. determination of the set of distinguished spectra \( J_{n,k}^N \),

2. reconstruction of the potential. Knowing a distinguished spectrum \( (\Lambda_1, \ldots, \Lambda_N) \in J_{n,k}^N \) we have to find all potentials \( V \) with \( N \) proper Darboux points \([d_i]\), such that \( \Lambda_i = \Lambda(d_i) \) for \( i = 1, \ldots, N \).
4.1 Distinguished spectra

For \( n = k = 3 \) we have \( N = D(n,k) = 7 \), so we look for seven admissible pairs \( \Lambda_i = (\Lambda_1^{(i)}, \Lambda_2^{(i)}) \in A_{3,3}, i = 1, \ldots, 7 \), which satisfy the following three relations

\[
\begin{aligned}
\sum_{i=1}^{7} \frac{1}{\Lambda_1^{(i)} \Lambda_2^{(i)}} &= 1, \\
\sum_{i=1}^{7} \frac{\Lambda_1^{(i)} + \Lambda_2^{(i)}}{\Lambda_1^{(i)} \Lambda_2^{(i)}} &= -4, \\
\sum_{i=1}^{7} \frac{(\Lambda_1^{(i)} + \Lambda_2^{(i)})^2}{\Lambda_1^{(i)} \Lambda_2^{(i)}} &= 16.
\end{aligned}
\]  

(4.1)

Numbers \( \Lambda_j^{(i)} + 1 \) belong to the set \( M_3 \) which corresponds to items 1–6 in the table of the Morales-Ramis Theorem \([1,2]\), i.e.,

\[
M_3 = \left\{ p + \frac{3}{2} p(p-1) \mid p \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2} \left[ \frac{2}{3} + 3p(p+1) \right] \mid p \in \mathbb{Z} \right\} \\
\cup \left\{ -\frac{1}{24} + \frac{1}{6} (1+3p)^2 \mid p \in \mathbb{Z} \right\} \cup \left\{ -\frac{1}{24} + \frac{3}{32} (1+4p)^2 \mid p \in \mathbb{Z} \right\} \\
\cup \left\{ -\frac{1}{24} + \frac{3}{50} (1+5p)^2 \mid p \in \mathbb{Z} \right\} \cup \left\{ -\frac{1}{24} + \frac{3}{50} (2+5p)^2 \mid p \in \mathbb{Z} \right\}.
\]

The first step in solving (4.1) is to find all solutions of the second relation which we write in the form

\[
\sum_{i=1}^{14} X_i = -4,
\]  

(4.2)

where \( X_i = 1/\Lambda_j^{(i)} \). Knowing \( M_3 \) we easily determine the set \( X_3 \) of admissible values of \( X_i \). It has, as it is easy to check, the following properties:

1. \( X_3 = X_3^- \cup X_3^+ \), and \( X_3^- \) is finite,
2. if \( x \in X_3^- \), then \( x \leq -1 \),
3. if \( x \in X_3^+ \), then \( x \leq 1 \).

It follows that among \( X_i \) satisfying (4.2), there is a certain number of negative ones. Moreover, at most nine of \( X_i \) are negative and we have a finite number of choices of them.

**Example 4.1** Assume that \( X_1, \ldots, X_7 \) are negative. Taking \( X_1, \ldots, X_7 \in X_3^- \) we obtain 2426 different choices satisfying \( X_1 + \cdots + X_7 < -4 \).

Thus we reduce the problem to finding solutions of

\[
\sum_{i=1}^{p} X_i = c, \quad \text{where} \quad c > 0, \quad p < 14, \quad X_i \in X_3^+.
\]  

(4.3)

If \((X_1, \ldots, X_p)\) is a solution of the above equation, then we can assume that \( X_1 \leq X_2 \leq \cdots \leq X_p \). Hence \( X_p \geq c/p \), and we have only a finite number of choices for \( X_p \). For each of this choices we have to find the solution of equation (4.3) where we replace \( p \) by \( p - 1 \). Repeating this reasoning we end up with equation

\[
X_1 + X_2 = \epsilon,
\]  

(4.4)
4.2 Normalisation of the potential

Starting from the general potential of degree three

\[ V = a_1 q_1^3 + a_2 q_1^2 q_2 + a_3 q_1^2 q_3 + a_4 q_1 q_2^2 + a_5 q_1^3 + a_6 q_2^2 q_3 + a_7 q_1 q_2 q_3 + a_8 q_1 q_2 + a_9 q_2 q_3 + a_{10} q_1 q_2 q_3, \]  

so, \( \varepsilon > X_2 \geq \varepsilon / 2 \), and we have only a finite number of choices for \( X_2 \). This procedure proves that we have only a finite number of solutions of (4.3), however a serious practical problem appears because we do not know \( \text{a priori} \) the lower bound for \( X_1 \). We can order elements of \( X_3^+ \) into decreasing sequence \( \{ x_i \} \). Then \( x_i \) decreases with \( i \) as \( i^{-2} \). For a small \( \varepsilon \) interval \( [\varepsilon / 2, \varepsilon] \) contains approximately \( [1/\varepsilon] \) admissible elements \( X_2 \). For each of them we have to check if \( X_1 = \varepsilon - X_2 \) is admissible, i.e., if \( X_1 \in X_3^+ \). Examples show that \( \varepsilon \) can be of order \( 10^{-15} \) and smaller. Very small values of \( \varepsilon \) in (4.4) appear when the number of negative elements is smaller than seven. For such cases, even using a dedicated software running on the quickest accessible computers, we were unable to perform the computations up to the end in a reasonable period of the time (one month). This is a reason why we restrict our searches giving \( \text{a priori} \) lower bound for the smallest element \( X_1 \).

A solution of (4.2) gives 14 numbers \( L_1, \ldots, L_{14} \), where \( L_i = 1 / X_i \). We divide them into all possible seven unordered pairs \( \Lambda_i = (\Lambda_1^{(i)}, \Lambda_2^{(i)}) \) and check if these pairs satisfy the remaining two relations (4.1).

Using the described algorithm we have found the distinguished spectra listed in Table 4.1.

Remark 4.1 The described algorithm has an obvious generalisation for arbitrary \( k \) and \( n \).
we have to select its ‘good’ equivalent representative with the minimal number of free parameters. The group PO(3, C) is four dimensional, so a typical orbit of its action on C^3[q] has dimension 6. However, the problem is that there is no good parametrisation of these orbits.

Here we consider only a generic case and this simplifies the analysis a lot. In our further considerations we will need the following fact.

**Proposition 4.1.** Assume that a potential \( V \in C^3[q] \) has a proper non-isotropic Darboux point \([d]\) such that matrix \( V''(d) \) is semi-simple. Then it is equivalent to

\[
V = a_1 q_1^3 + a_2 q_1^2 q_2 + a_3 q_1^2 q_3 + a_4 q_1 q_2^2 + a_5 q_2^3 + a_6 q_2 q_3 + \frac{1}{3} q_3^3. \tag{4.6}
\]

Moreover, \( d = (0, 0, 1) \) and matrix \( V''(d) \) has eigenvalues \( \lambda_1 = 2a_3, \lambda_2 = 2a_6 \) and \( \lambda_3 = 2 \).

**Proof.** By Remark 2.1 we can assume that \( d = (0, 0, 1) \). This implies that in (4.5) we have \( a_8 = a_9 = 0 \) and \( a_7 = 1/3 \). Vector \( d \) is an eigenvector of \( V''(d) \) with eigenvalue \( \lambda_3 = k - 1 = 2 \). Hence \( V''(d) \) has the following form

\[
V''(d) = \begin{bmatrix}
2a_3 & a_{10} & 0 \\
a_{10} & 2a_6 & 0 \\
0 & 0 & 2
\end{bmatrix}, \tag{4.7}
\]

and its remaining eigenvalues are \( \lambda_{1,2} = a_3 + a_6 \pm \sqrt{a_{10} + (a_3 - a_6)^2} \). If

\[
a_{10} \neq \pm i (a_3 - a_6), \tag{4.8}
\]

then we can make a rotation around \( d \) such that in the new frame \( a_{10} = 0 \). Under this assumption the normalised potential reads

\[
V = a_1 q_1^3 + a_2 q_1^2 q_2 + a_3 q_1^2 q_3 + a_4 q_1 q_2^2 + a_5 q_2^3 + a_6 q_2 q_3 + \frac{1}{3} q_3^3. \tag{4.9}
\]

If condition (4.8) is not fulfilled, then \( V''(d) \) has a double eigenvalue \( \lambda_{1,2} = a_3 + a_6 \). In this case, because \( V''(d) \) is semi-simple, we have

\[
\text{rank}(V''(d) - (a_3 + a_6)E_3) < 2. \tag{4.10}
\]

This condition gives

\[
(a_3 + a_6 - 2)(a_3 - a_6) = 0. \tag{4.11}
\]

If \( a_3 + a_6 - 2 = 0 \), then all eigenvalues of \( V''(d) \) are 2. But now, if \( a_3 \neq 1 \), then \( \text{rank}(V''(d) - 2E_3) = 1 \). Hence, \( a_3 = a_6 = 1 \), and so \( a_{10} = 0 \). In the alternative subcase \( a_3 = a_6 \), so again \( a_{10} = 0 \), and this finishes the proof.

Notice that for the normalised potential (4.6) the proper Darboux point \([d] = [0 : 0 : 1]\) lies in the line at infinity \( H_\infty \). We called it the Darboux point at the infinity, although, as the following proposition shows, we can have more Darboux points in this line.

**Proposition 4.2.** Potential (4.6) has at most 3 proper Darboux points in the line at infinity \( H_\infty \). If \( a_4 \neq 0 \), then it has exactly one proper Darboux point at infinity.

**Proof.** If \([q]\) is a proper Darboux point of \( V \) which lies in \( H_\infty \), then \( V'(q) = q \) and \( q_1 = 0 \). For the potential (4.6) this gives the following system of equations

\[
a_4 q_2^3 = 0, \quad q_2(-1 + 3a_5 q_2 + 2a_6 q_3) = 0, \quad a_6 q_2^3 - q_3 + q_3^2 = 0. \tag{4.12}
\]

It is easy to see that if \( a_4 \neq 0 \), then the above system has only one non-zero solution corresponding to \([d] = [0 : 0 : 1] \). Thus, \( a_4 = 0 \). Now, we have equations

\[
-1 + 3a_5 q_2 + 2a_6 q_3 = 0, \quad a_6 q_2^3 - q_3 + q_3^2 = 0, \tag{4.13}
\]

that admit at most 2 solutions.
4.3 Reconstruction of potential

At this point the problem is following. For each distinguished spectrum given in Table 4.1 we have to find all non-equivalent potentials having such a spectrum. This problem is difficult. We have to determine the coefficients of the potential knowing only the eigenvalues of its Hessian matrix calculated at certain points with unknown coordinates.

For each distinguished spectrum from Table 4.1 we can assume that the potential has the form (4.6). So, we assign a pair \((\Lambda_1, \Lambda_2)\) from the spectrum to the Darboux point at infinity. By Proposition 4.1 we have \(\Lambda_1 = 2a_3 - 1\) and \(\Lambda_2 = 2a_6 - 1\). In this way, we fix values of two unknown parameters \(a_3\) and \(a_6\). Hence, the unknown parameters are \(a = (a_1, a_2, a_4, a_5) \in \mathbb{C}^4\).

We do not know the coordinates of the remaining Darboux points. So we have the following problem. Let us assume that we know eigenvalues \(\Lambda_1, \Lambda_2\) and \(\Lambda_3 = k - 2\) of matrix \(f'(q) = V''(q) - E_3\), at an unknown point \(q\) satisfying \(f(q) := V'(q) - q = 0\). We have to derive restrictions on the coefficients of the potential. A direct approach to this problem is based on a simple idea. On the one hand, we know numerical values of symmetric polynomials \(s_i = \tau_i(\Lambda_1, \Lambda_2, \Lambda_3)\) for \(i = 1, 2, 3\). On the other hand, we can express these values as coefficients \(\tilde{p}_i(q)\) of the characteristic polynomial of matrix \(f'(q)\), namely

\[
\det(f'(q) - \lambda E_3) = - (\lambda^3 - \lambda^2 \tilde{p}_1(q) + \lambda \tilde{p}_2(q) - \tilde{p}_3(q)).
\]

As a result we obtain the following system of polynomial equations

\[
f_i(q) = 0, \quad p_i(q) := \tilde{p}_i(q) - s_i = 0, \quad i = 1, 2, 3,
\]

where polynomials \(f_i\) and \(p_i\) are considered as elements of the ring \(\mathbb{C}[a][q]\). An elimination of variables \(q\) from the above system gives rise to a set of polynomial equations for the coefficients of potential \(V\). We repeat the calculations for each Darboux point and then find solutions of all obtained equations.

**Remark 4.2** If the considered distinguished spectrum contains \(l \leq 7\) different admissible pairs, then, because one of them is attached to the Darboux point at the infinity, we have at our disposal \(l - 1\) pairs. Hence, if \(l < 7\), we have to attach the same pair to two or more Darboux points. As an effect, we have to apply the algorithm only for \(l - 1\) Darboux points with different attached pairs.

The elimination step in the described procedure consists of determination of the so called elimination ideal of a given set of polynomials, see [8, 9]. It can be performed only with the help of a computer algebra system. However, in practice, the known algorithms work effectively only in cases with a relatively small number of variables. Hence, in order to have a chance to obtain the desired result, we have to minimalise the number of unknowns. In the considered problem we can do this introducing the affine coordinates \(x_1\) and \(x_2\) for the unknown Darboux point. They satisfy the following polynomial equations

\[
g_1(x_1, x_2) = 0 \quad \text{and} \quad g_2(x_1, x_2) = 0,
\]

where

\[
g_i := \frac{\partial v}{\partial x_i} - x_i g_0, \quad \text{for} \quad i = 1, 2,
\]

and

\[
g_0 = kv - x_1 \frac{\partial v}{\partial x_1} - x_2 \frac{\partial v}{\partial x_2}, \quad v(x_1, x_2) = V(1, x_1, x_2),
\]

see Lemma 2.1 and its proof. Moreover, we take values of two symmetric polynomials \(S_1 = \tau_1(\Lambda_1, \Lambda_2) = \Lambda_1 + \Lambda_2\) and \(S_2 = \tau_2(\Lambda_1, \Lambda_2) = \Lambda_1 \Lambda_2\). Then, we have to express \(S_1\) and
$S_2$ as functions of $x_1$ and $x_2$. To this end we recall that the eigenvalues of $f'(q)$ coincide with the eigenvalues of Jacobi matrix $J(x)$ defined by (2.39). Performing simple calculations like in the proof of Lemma 2.5 we find that

$$g_0 S_1 = \left( \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} \right) \quad \text{and} \quad g_0^2 S_2 = \left( \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \right). \quad (4.16)$$

Now, eliminating $x_1$ and $x_2$ from equations (4.15) and (4.16) we obtain a set of equations for unknown coefficients of the potential.

**Remark 4.3** The drawback of using the affine coordinates is following. We do not know in advance how many points are located at the infinity and what are their admissible spectra. Thus we have to check all the possibilities. One can perform another normalisation of the potential with the assumption that all Darboux points are located in the affine part of $\mathbb{CP}^2$. However this causes other computational problems.

Using the described algorithm it is always possible to derive a set of polynomial equations which must be satisfied by the coefficients of the potential we are looking for. However, generally the equations are very complicated, and it is impossible to find their explicit solutions even with the help of the strongest tools of the computer algebra. In order to overcome this problem we are forced to apply a different strategy.

In fact we do not want to know the explicit form of all potentials with a given distinguished spectrum, only those which are integrable have to be selected. Our potential satisfies a priori all the necessary integrability conditions of Theorem 1.2. However, the stronger integrability conditions were formulated by Morales, Ramis, and Simó, see [23; 24; 28].

**Theorem 4.1** (Morales, Ramis, Simó, 2006). If a complex Hamiltonian system is integrable in the Liouville sense in a neighbourhood of a particular non-equilibrium solution, then the identity component $G^0_m$ of the differential Galois group $G_m$ of $m$-th order variational equations along this solution is Abelian, for an arbitrary $m \in \mathbb{N}$.

Generally it is difficult to apply the above theorem, because higher order variational equations have a big dimension. However, if the first variational equations are a direct product of Lamé equations of the form

$$\frac{d^2 x}{dt^2} = [n(n+1)\varphi(t) + B]x, \quad n \in \mathbb{Z}, \quad B \in \mathbb{C},$$

then we can use the following local criterion. If $G_1$ is not a finite group and a local solution around $t = 0$ of $m$-th order variational equations contains a logarithmic term, then $G^0_m$ is not Abelian, see Appendix A in the recent survey of Morales and Ramis [29].

For details how to apply this criterion for homogeneous potentials with two degrees of freedom see [21; 22]. In the case of homogeneous Hamiltonian systems with three degrees of freedom the strategy is completely similar to this applied in [21; 22]. We remark only that the criterion is applicable for a solution generated by such a Darboux point $[d]$ that its eigenvalues $\lambda_i$ of $V''(d)$ are admissible and all are non-negative integers. In all cases, when we apply the higher order variational equation, we take a particular solution generated by the Darboux point at the infinity.

### 4.3.1 Families 1–4

Let us take as the distinguished spectrum the first line in Table 4.1. From Lemma 2.5 it follows that a potential with such a spectrum admits a proper and non-isotropic Darboux
point. We can assume that this point is \([d] = [0: 0: 1]\) and assign to it the admissible pair \((\Lambda_1, \Lambda_2) = (-1, -1)\). Moreover, by Remark 1.3 we can assume that matrix \(V''(d)\) is semi-simple. As an effect, we consider potential of the form (4.6). By Proposition 4.1 we have \(\Lambda_1 = 2a_3 - 1 \) and \(\Lambda_2 = 2a_3 - 1\), so \(a_3 = a_5 = 0\). But then in the potential (4.6) variable \(q_3\) is separated from \(q_1\) and \(q_2\). This fact, and the assumption that the potential has seven Darboux points implies that one can make a rotation in the plane \((q_1, q_2)\) and achieve that \(a_4 = 0\). Then the potential simplifies to the form

\[
V = a_1q_1^3 + a_2q_1^2q_2 + a_5q_2^3 + \frac{1}{3}q_3^3,
\]

and in the affine coordinates we have

\[
g_1 = (3a_5 - 2a_2)x_1^2 - 3a_1x_1 + 2a_2, \quad g_2 = x_2^2 - x_2(3a_1 + 2a_2x_1), \quad g_0 = 3a_1 + 2a_2x_1 .
\]

By the Bézout theorem, the number of solutions of \(g_1 = g_2 = 0\) is not bigger than four, so there is at most four Darboux points in the affine part of \(\mathbb{C}P^2\). Since by Proposition 4.2 at most three Darboux points can lie at infinity, and we assumed that the potential has seven Darboux points, four Darboux points must lie in the affine part of \(\mathbb{C}P^2\) and three at infinity. This fact implies that \(a_5 \neq 0\) and \(3a_5 - 2a_2 \neq 0\). For each Darboux point in the affine part of \(\mathbb{C}P^2\) we attach the admissible pair \((\Lambda_1, \Lambda_2) \in \{(-1, -1), (-1, 1), (1, 1)\}\). Next we calculate symmetric functions \(S_1 = S_1(\Lambda_1, \Lambda_2), S_2 = S_2(\Lambda_1, \Lambda_2)\), and then, using a computer algebra system, perform an elimination of \(x_1\) and \(x_2\) from the corresponding equations (4.15) and (4.16). This elimination gives only one restriction on parameters \(a_i\), namely \(a_2 = 0\). But then, from (4.18) we see that \(g_0 = a_1\). By assumption all Darboux points are proper, and we have necessarily \(a_1 \neq 0\). The affine coordinates of four Darboux points are following

\[(x_1, x_2) = (0, 0), \quad (x_1, x_2) = (0, 3a_1), \quad (x_1, x_2) = \left( \frac{a_1}{a_5}, 0 \right), \quad (x_1, x_2) = \left( \frac{a_1}{a_5}, 3a_1 \right) . \]

We can check directly that the non-trivial eigenvalues \((\Lambda_1, \Lambda_2)\) are as we required \((-1, -1), (-1, 1), (-1, 1)\) and \((1, 1)\), respectively.

Assumption \(a_5 \neq 0\) guarantees that three Darboux points are localised at infinity, and their coordinates are following

\[(q_1, q_2, q_3) = (0, 0, 1), \quad (q_1, q_2, q_3) = \left( 0, \frac{1}{3a_5}, 0 \right), \quad (q_1, q_2, q_3) = \left( 0, \frac{1}{3a_5}, 1 \right) . \]

The non-trivial eigenvalues \((\Lambda_1, \Lambda_2)\) at these points are \((-1, -1), (-1, -1)\), and \((-1, 1)\), respectively.

In this way we showed that if a potential admits seven proper Darboux points with a distinguished spectrum given by the first line in Table 4.1 then it is equivalent to the separable potential

\[
V_1 = a_1q_1^3 + a_5q_2^3 + \frac{1}{3}q_3^3, \quad a_1a_5 \neq 0,
\]

with obvious additional first integrals

\[
I_1 = \frac{1}{2}p_1^2 + \frac{1}{3}q_3^3, \quad I_2 = \frac{1}{2}p_2^2 + a_5q_2^3.
\]

An analysis for the next three distinguished spectra is completely analogous to that described above. For the spectrum given by the second line in Table 4.1 we obtained one integrable potential

\[
V_2 = \frac{a_5}{2}q_1^2q_2 + a_5q_2^3 + \frac{1}{3}q_3^3, \quad a_5 \neq 0,
\]

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with first integrals
\[ I_1 = \frac{1}{2} p_3^2 + \frac{1}{3} q_3^3, \quad I_2 = 8p_1(q_1p_2 - q_2p_1) + a_5q_1^2(q_1^2 + 4q_2^2). \]

The third spectrum gives rise to another integrable potential
\[ V_3 = \frac{3a_5}{16} q_1^2 q_2 + a_5q_2^3 + \frac{1}{3} q_3^3, \quad a_5 \neq 0, \quad (4.21) \]
with additional first integrals
\[ I_1 = \frac{1}{2} p_3^2 + \frac{1}{3} q_3^3, \quad I_2 = 128p_1^4 - a_5q_1^2(32p_1p_2q_1 - 96p_1^2q_2 + a_5q_1^2(q_1^2 + 6q_2^2)). \]

The fourth spectrum generates also an integrable potential
\[ V_4 = a_5(i\sqrt{3}q_1^3 + 9q_1^2q_2 + 18q_2^2) + \frac{1}{3} q_3^3, \quad a_5 \neq 0, \quad (4.22) \]
which possesses two additional first integrals
\[ I_1 = \frac{1}{2} p_3^2 + \frac{1}{3} q_3^3, \]
\[ I_2 = 3p_1^4 + 2i\sqrt{3}p_1^2q_2 - 54a_5p_1p_2q_1^2(q_1 + i\sqrt{3}q_2) + 18a_5p_1^2q_1(i\sqrt{3}q_1^2 + 6q_1q_2 + 6i\sqrt{3}q_2^2) \]
\[ + 9ia_5q_1^2(2\sqrt{3}p_2^2 + 3a_5(5iq_1^3 + 6\sqrt{3}q_2^2q_2 + 9iq_1q_2^2 + 12\sqrt{3}q_2^3)). \]

Let us note that potentials \( V_1, V_2, V_3 \) and \( V_4 \) have the form \( V_i(q_1, q_2, q_3) = \tilde{V}_i(q_1, q_2) + q_3^3/3 \), where \( \tilde{V}_i(q_1, q_2) \) are integrable two dimensional homogeneous potentials. They have appeared in the famous Hietarinta table in [15] and later were re-obtained in the differential Galois framework in [21] where it was shown that they are the only integrable potentials with the maximal number of Darboux points for \( k = 3 \).

The reconstruction of the potentials corresponding to the next families is more complicated by reason of the absence of the admissible pair \((-1, -1)\). Still one can easily decode from the pair of \( \Lambda \)s at the infinity two coefficients of the potential but, separation of the part of \( V \) depending on \((q_1, q_2)\) and \( q_3 \) no more appears, thus we cannot put \( a_4 = 0 \). Thus all six Darboux points rest at the affine part of \( \mathbb{C}P^2 \) and the level of computational difficulties increases very quickly.

### 4.3.2 Families 5–7

For the fifth distinguished spectrum in Table 4.1 we assign the pair \((\Lambda_1, \Lambda_2) = (-1, 4)\) to the Darboux point at the infinity. Then we obtain immediately two coefficients \( a_3 = 0 \) and \( a_6 = 5/2 \) of the potential \((4.6)\). Thus, we will analyse the higher order variational equations for the potential
\[ V = a_1q_1^3 + a_2q_1^2q_2 + a_4q_1q_2^2 + a_5q_2^3 + \frac{5}{2} q_2^2q_3 + \frac{1}{3} q_3^3. \quad (4.23) \]
A local solution around \( t = 0 \) of the second order variational equations does not have logarithmic terms. Such terms appear in the solutions of the third order variational equations. Comparing them to zero we obtain the following equations
\[ a_4(35a_2 - 18a_3) = 0, \quad -147 + 28a_4^2 - 27a_5^2 = 0, \quad a_2(9a_1 - a_4) = 0, \]
\[ 84a_3^2 + 168a_1a_4 - 56a_2^2 - 27a_2a_5 = 0, \]

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with solutions

\[ a_2 = 0, \quad a_4 = 0, \quad a_5 = \pm \frac{7i}{3} \]  \hspace{1cm} (4.24)

\[ a_1 = 0, \quad a_2 = \pm \frac{3i}{4}, \quad a_5 = \pm \frac{7i}{3}, \quad a_4 = 0, \]  \hspace{1cm} (4.25)

\[ a_1 = -\frac{i}{2} \sqrt{\frac{21}{269}}, \quad a_2 = \pm 6i \sqrt{\frac{14}{269}}, \quad a_4 = -\frac{9i}{2} \sqrt{\frac{21}{269}}, \quad a_5 = \pm \frac{35i}{3} \sqrt{\frac{14}{269}}, \]  \hspace{1cm} (4.26)

\[ a_1 = \frac{i}{2} \sqrt{\frac{21}{269}}, \quad a_2 = \pm 6i \sqrt{\frac{14}{269}}, \quad a_4 = \frac{9i}{2} \sqrt{\frac{21}{269}}, \quad a_5 = \pm \frac{35i}{3} \sqrt{\frac{14}{269}}, \]  \hspace{1cm} (4.27)

\[ a_1 = \frac{1}{2} \sqrt{\frac{7}{3}}, \quad a_2 = 0, \quad a_4 = \pm \frac{1}{2} \sqrt{21}, \quad a_5 = 0. \]  \hspace{1cm} (4.28)

The first solution \((4.24)\) gives the integrable potential

\[ V = a_1 q_1^3 \pm \frac{7i}{3} q_2^3 + \frac{5}{2} q_2 q_3 + \frac{1}{3} q_3^3, \]

which is equivalent to potential \(V_2\) defined by \((4.20)\).

Solution \((4.25)\) generates the integrable potential

\[ V_5 = \frac{3i}{4} q_1^2 q_2 + \frac{7i}{3} q_3^3 + \frac{5}{2} q_2 q_3 + \frac{1}{3} q_3^3, \]

with first integrals

\[ I_1 = 48 p_1 \left( i p_2 + p_3 \right) q_1 - (3q_1^2 + 2\left(q_2 - 2iq_3\right)^2) \left(3q_1^2 + 2q_2 \left(5q_2 - 4i q_3\right)\right) - 48 p_1^2 \left(i q_2 + q_3\right) + 32 \left( p_2 - 2i p_3 \right) \left(p_2 q_3 - p_3 q_2\right), \]

\[ I_2 = 24 p_1^3 - 8p_1^2 \left(12 \left(p_2 - i p_3\right) \left(p_2 - 2i p_3\right) - 9i q_2^2 q_2 + 2\left(q_2 - i q_3\right) \left(q_2 - 2i q_3\right) \left(5i q_2 + 4q_3\right)\right) + 3q_1^2 \left(3q_1^2 + 16 \left(p_2 + i p_3\right) \left(p_3 q_2 - p_2 q_3\right) + 6q_2^2 \left(3q_2^2 - 4i q_2 q_3 + q_3^2\right) + 4q_2 \left(q_2 - 2i q_3\right) \left(7q_2^2 - 4i q_2 q_3 + 2q_3^2\right) + 24 p_1 q_1 \left(p_3 - 3q_1^2 - 22q_2^2 + 20i q_2 q_3\right) - 1p_2 \left(3q_1^2 + 6q_2^2 + 4i q_2 q_3 + 8q_3^2\right)\right). \]

Solutions \((4.26) - (4.28)\) give potentials

\[ V = -\frac{i}{2} \sqrt{\frac{21}{269}} q_1^3 + 6i \sqrt{\frac{14}{269}} q_1^2 q_2 - \frac{9i}{2} \sqrt{\frac{21}{269}} q_1 q_2^2 + \frac{35i}{3} \sqrt{\frac{14}{269}} q_2^3 + \frac{5}{2} q_2 q_3 + \frac{1}{3} q_3^3, \]

\[ V = \frac{i}{2} \sqrt{\frac{21}{269}} q_1^3 + 6i \sqrt{\frac{14}{269}} q_1^2 q_2 + \frac{9i}{2} \sqrt{\frac{21}{269}} q_1 q_2^2 + \frac{35i}{3} \sqrt{\frac{14}{269}} q_2^3 + \frac{5}{2} q_2 q_3 + \frac{1}{3} q_3^3, \]

\[ V = \pm \frac{1}{2} \sqrt{\frac{7}{3}} q_1^3 \pm \frac{1}{2} \sqrt{\frac{21}{269}} q_1 q_2^2 + \frac{5}{2} q_2 q_3 + \frac{1}{3} q_3^3. \]

All of them are not integrable because the respective solutions of the fifth order variational equations have logarithmic terms.

For the sixth distinguished spectrum we assign pair \((\Lambda_1, \Lambda_2) = (4, 21)\) to the Darboux point at the infinity and we apply the same procedure as in the previous case. Solving conditions obtained from an analysis of solutions of the third order variational equations, we obtained only one integrable potential

\[ V_6 = 364 \sqrt{17} q_1^3 + 2835i \sqrt{17} q_1^2 q_2 + 1560 \sqrt{17} q_1 q_2^2 + 6552i \sqrt{17} q_2^3 + 4335q_1^2 q_3 + 19074q_2 q_3 + 578 q_3^3, \]  \hspace{1cm} (4.29)
with one first integral of degree two in the momenta
\[ I_1 = 34\sqrt{17}p_3^2q_1 + 8p_2p_3(18i_1q_1 + q_2) - 2p_2^2(7\sqrt{17}q_1 + 4q_3) - p_1^2(54i\sqrt{17}q_2 + 26q_3) \\
+ 2p_1(p_2(27i\sqrt{17}q_1 + 7\sqrt{17}q_2 + 9i_1q_3) + p_3(13q_1 - 81i_1q_2 - 17\sqrt{17}q_3)) \\
- 51(-2568q_1^4 + 36i_2^2(695q_1 + 52\sqrt{17}q_3) - 36i_1q_2(151q_1^2 + 241\sqrt{17}q_1q_3 + 221q_3^2) \\
+ 2q_2^2(31351q_1^2 + 5386\sqrt{17}q_1q_3 + 2924q_3^2) + q_1(12205q_1^3 + 1668\sqrt{17}q_1^2q_3 + 3978q_1q_3^2 + 1156\sqrt{17}q_3^3)),
\]
and the second more complicated of degree four in the momenta is given in Appendix.

For the seventh distinguished spectrum in Table 4.1 we assign pair \((\Lambda_1, \Lambda_2) = (-1, 6)\) to the Darboux point at the infinity. The conditions that there are no logarithmic terms in solutions of variational equations of orders 3, 4 and 7 yield one integrable potential
\[ V_7 = 44\sqrt{7}q_1^4 + 240i\sqrt{14}q_1^2q_2 + 330\sqrt{7}q_1q_2^2 + 935i\sqrt{14}q_2^3 + 3087q_2^4q_3 + 294q_3^5.
\]
One its first integral is of degree four with respect the momenta
\[ I_1 = 19p_1^4 + 47p_2^4 - 14i\sqrt{14}p_2^3p_3 + 16p_1^3(8i\sqrt{2}p_2 + 7\sqrt{7}p_3) + 294p_2p_3(−436i\sqrt{2}q_1^3 \\
+ 6q_1q_2(-195i\sqrt{2}q_2 + 56\sqrt{7}q_3) + 6q_1^2(17q_2 - 77i\sqrt{14}q_3) + q_2^2(827q_2 - 399i\sqrt{14}q_3)) \\
+ 252p_2^2(12\sqrt{7}q_3^3 + q_1^2(145i\sqrt{14}q_2 - 98q_3) + 2\sqrt{7}q_1(185q_2^2 - 98q_3^2) + q_2(612i\sqrt{14}q_2) \\
+ 230q_2q_3 - 49i\sqrt{14}q_2^3) + 18p_1^2(-11p_2^2 + 14i\sqrt{14}p_2p_3 + 7(7p_3^2 - 136\sqrt{7}q_1^3 + 480i\sqrt{14}q_1^2q_2 \\
- 2756\sqrt{7}q_1q_3^2 - 27i\sqrt{14}q_3 + 6272i\sqrt{2}q_1q_2q_3 - 9898q_2q_3^2 + 784\sqrt{7}q_1q_3^3 + 3430i\sqrt{14}q_2q_3^2 \\
+ 4116q_3^3) + p_1(-65i\sqrt{2}p_3^3 - 147\sqrt{7}p_2^3p_3 - 147p_3(1216q_1^2 + 24q_1^2(-231i\sqrt{2}q_2 + 28\sqrt{7}q_3) \\
+ 96q_1q_2(172q_2 - 49i\sqrt{14}q_3) + q_2(4189i\sqrt{2}q_2^2 + 5964\sqrt{7}q_2q_3 - 6174i\sqrt{2}q_3^2)) + 21p_2(14i\sqrt{2}p_3^3 \\
- 1664i\sqrt{14}q_3^3 + 6607\sqrt{7}q_3^2 - 28665i\sqrt{2}q_2q_3^2 - 3822\sqrt{7}q_3^2q_3 - 6714i\sqrt{2}q_3^3 \\
+ 384q_3^3(4\sqrt{7}q_2 - 49i\sqrt{7}q_3) + 6q_1(-155i\sqrt{14}q_3^2 - 392q_2q_3 - 1666i\sqrt{14}q_3^2))) + 294(27\sqrt{7}p_3^3(8q_1^3 \\
+ 22i\sqrt{2}q_1q_2 - 24q_1q_2^2 + 9i\sqrt{2}q_3^2) + 112(434q_1^2 + 96q_1^2(-7i\sqrt{2}q_2 + 12\sqrt{7}q_3) + 6q_1^2(119q_2^3 \\
- 324i\sqrt{14}q_2q_3 + 924q_3^2) + 12q_1q_2^2(2044i\sqrt{2}q_2^3 + 1782\sqrt{7}q_2q_3 - 2625i\sqrt{2}q_2q_3^2 - 504\sqrt{7}q_3^3) \\
+ q_1^3(2233i\sqrt{2}q_2^3 + 5931\sqrt{7}q_2q_3 - 14364i\sqrt{2}q_2q_3^2 + 1134\sqrt{7}q_3^3) - 8q_3^3(698q_2^3 - 3492i\sqrt{14}q_2q_3 \\
- 7749q_2q_3^2 + 378i\sqrt{14}q_3^3) - 3q_1q_2(3626q_2^2 + 48i\sqrt{14}q_2q_3 - 3465q_2q_3^2 + 1386i\sqrt{14}q_3^3)),
\]
and the second of degree six with respect to the momenta is given in Appendix.

4.3.3 Families 8–9

For the eighth distinguished spectrum from Table 4.1 we assign pair \((\Lambda_1, \Lambda_2) = (6, 14)\) to the Darboux point at the infinity. The conditions that there are no logarithmic terms in solutions of variational equations of order four are following
\[ a_4(-7291848638025 - 195171588288a_2^3 - 79474355101a_4 + 20220446680a_4^2 \\
+ 315697519440a_2a_5 - 183025477350a_5^2) = 0, \\
a_4(411968789569575 + 2262392768724a_2^2 + 1291180817430a_1a_4 + 339419628940a_4^2 \\
+ 1213154206380a_2a_5 + 4209585979050a_5^2) = 0,
\]
\[
2013340175a_1 + 114593850a_1^3 + 9791320a_1a_2^2 - 6923862a_2a_5 + 0, \\
729732302775a_1 - 633757824a_1a_2^2 + 43836570855a_4 + 8866402020a_2a_4 + 4915882928a_2a_4 \\
+ 1092887040a_1a_2^2 + 217032200a_4^3 + 16783940880a_1a_2a_5 - 5584905348a_2a_4a_5 = 0.
\]
(4.30)
The above system has very complicated solutions. In order to discard those which do not satisfy our assumptions about the number of Darboux points and their spectra, we decided to proceed as follows. At first we assume that there are six Darboux points in the affine part of \( \mathbb{CP}^2 \), i.e., by Proposition 4.2 we assume that \( a_4 \neq 0 \). For three of them we assign \((-\frac{2}{3}, \frac{2}{3}), \ (-1, \frac{2}{3}) \) and \((6, 14)\), as the respective admissible pairs \((\Lambda_1, \Lambda_2)\). For each of these points we considered system of equations (4.15) and (4.16). Then we calculated the elimination ideal with respect to variables \((x_1, x_2)\) and its Groebner basis. Next we join all these three bases and polynomials given by the left hand sides of equations (4.30). The Groebner basis of all these polynomials is very simple. Equating to zero all polynomials from this basis we obtained equations

\[
a_3^4 + 28431/320a_2 + 729/64a_5 = 0, \quad a_2^3 - 125/117a_5^2 - 600/13 = 0, \quad a_2a_5 - 145/117a_5^2 - 540/13 = 0, \quad a_1 = a_4 = 0.
\]

(4.31)

So we obtained a contradiction with our assumption that \( a_4 = 0 \). Hence, only four points are in the affine part of \( \mathbb{CP}^2 \). Assuming that to three of them the attached admissible pairs are as above, we arrive to the same equations (4.31). Their solutions give rise to only one integrable potential

\[
V_8 = \frac{7}{2}q_1^2q_3 - \frac{5i\sqrt{3}}{2}q_1^2q_2 - \frac{9i\sqrt{3}}{2}q_2^3 + \frac{15}{2}q_2q_3^2 + \frac{1}{3}q_3^3,
\]

that admits the following first integrals

\[
I_1 = 26i\sqrt{3}p_1^3 + 3p_1(6i\sqrt{3}p_2^2 - 48p_2p_3 - 32i\sqrt{3}p_3^2 + 27q_3^2 + 69i\sqrt{3}q_2q_3 + 15q_2(13q_1^2 - 8q_3^2) + i\sqrt{3}q_3(q_1^2 + 16q_3) + 3q_1(p_2(-91q_1^2 + 45q_2^2 + 42i\sqrt{3}q_2q_3 + 72q_3) - ip_3(65\sqrt{3}q_1^2 - 99\sqrt{3}q_2^2 - 480q_2q_3 + 112\sqrt{3}q_3^2)),
\]

\[
I_2 = 8619p_1^4 + 459p_2^4 + 9000i\sqrt{3}p_2p_3 + 21519p_3^4 + 43940q_1^6 + 78p_1^2(141p_2^2 + 220i\sqrt{3}p_2p_3 + 41p_3^2 - 369i\sqrt{3}q_3^2 + 1547q_1^2q_3 + 2595q_2q_3 + 214q_3^2 + 85i\sqrt{3}q_2(-13q_1^2 + 4q_3^2)) - 18p_2^2(2649p_3^2 - 791i\sqrt{3}q_3^2 - 6357q_1^2q_3 - 765q_2q_3 + 86q_3^2 - 5i\sqrt{3}q_2(299q_1^2 + 100q_3^2)) + 20p_2p_3(-1568i\sqrt{3}p_3^2 + 6075q_2^2 + 9675i\sqrt{3}q_2q_3 - 189q_2(65q_1^2 + 24q_3^2) + 21i\sqrt{3}q_3(65q_1^2 + 112q_3^2)) + 6p_3^2(21573i\sqrt{3}q_2^2 + 11011q_1^2q_3 - 104085q_2q_3 + 4782q_3^2 - 5i\sqrt{3}q_2(3939q_1^2 + 7840q_3^2)) + 780p_1q_1(p_3(13q_1^2 + 693q_2^2 + 262i\sqrt{3}q_2q_3 - 56q_3^2) + 3i p_2(13\sqrt{3}q_1^2 - 35\sqrt{3}q_2^2 + 32iq_2q_3 - 4\sqrt{3}q_3^2)) + 12(28674q_6^2 + 251110i\sqrt{3}q_2q_3 + 30758q_4q_5^2 + 7917q_3^4q_5^2 + 797q_5^6 + 405q_2^4(-13q_1^2 + 30q_3^2) - 15q_2q_3^2(-6500q_1^2 + 381q_3^2)) + 4i\sqrt{3}q_3^2q_3(-11583q_1^2 + 5959q_3^2) + 10i\sqrt{3}q_3q_3(-2197q_1^4 + 1118q_1^2q_3^2 + 392q_3^4)).
\]

It does not end the analysis, because it is possible that to four Darboux points in the affine part of \( \mathbb{CP}^2 \) only two different admissible pairs are attached. To analyse all these cases we proceed as follows. We know that two additional Darboux points are in line at the infinity. Their coordinates and spectra \((\Lambda_1^{(i)}, \Lambda_2^{(i)})\) can be found explicitly. Since now \( a_4 = 0 \), they can depend only on three parameters \( a_1, a_2, \) and \( a_5 \). In fact they depend only on \( a_2 \), and \( a_5 \). So \((\Lambda_1^{(1)}, \Lambda_2^{(1)})\) and \((\Lambda_1^{(2)}, \Lambda_2^{(2)})\) are not independent, and the following relations hold

\[
(8 + 15\Lambda_2^{(1)})\Lambda_1^{(1)} = 14(\Lambda_2^{(2)} - \Lambda_1^{(1)}), \quad (8 + 15\Lambda_2^{(2)})\Lambda_1^{(1)} = -14(\Lambda_2^{(2)} - \Lambda_1^{(1)}), \quad (14 + 15\Lambda_2^{(1)})\Lambda_1^{(1)} = -14\Lambda_1^{(2)}.
\]

(4.32)
Now, it is easy to check that from the considered distinguished spectrum there is only one choice of two pairs satisfying the above relations, namely

\[(\Lambda_1^{(1)}, \Lambda_2^{(1)}) = \left(-\frac{2}{3}, -\frac{2}{3}\right), \quad (\Lambda_1^{(2)}, \Lambda_2^{(2)}) = \left(\frac{7}{3}, -1\right),\]

and this gives

\[a_2 = \pm \frac{5i}{\sqrt{3}} \frac{\sqrt{2}}{2} \quad \text{and} \quad a_5 = \pm \frac{9i}{\sqrt{3}} \frac{\sqrt{2}}{2}.\]

For both sign choices equations (4.30) imply that \(a_1 = 0\). In this way, we recover a potential equivalent to \(V_8\).

For the ninth distinguished spectrum from Table 4.1 we assign the pair \((\Lambda_1, \Lambda_2) = (14, 39)\) to the Darboux point at the infinity, and then we proceed in the same way as in the previous case. Logarithmic terms appear only in solutions of the seventh order variational equations. The coefficients of these terms are four very complicated polynomials depending on \(a_1, a_2, a_4, a_5\). We join these polynomials and all eliminating ideals obtained for five Darboux points in the affine part of \(\mathbb{C}P^2\). The Groebner basis of this ideal is very simple and it gives the following equations

\[a_2^2 + \frac{4609248}{125} = 0, \quad a_5^2 - \frac{1395372}{125} = 0, \quad a_1 + \frac{9}{152} a_4 = 0, \quad a_2 + \frac{621}{682} a_5 = 0.\]

This system yields the following integrable potential

\[V_9 = 27i \sqrt{3990} q_1^3 + 3726 \sqrt{15} q_1^2 q_2 - 456i \sqrt{3990} q_1 q_2^2 - 4092 \sqrt{15} q_2^3 - 1125 q_1^2 q_3 - 3000 q_2^2 q_3 - 50 q_3^3,\]

with very complicated first integrals of degree 4 and 6 in the momenta which are given in Appendix.

Potential \(V_9\) was obtained under assumption that six Darboux points are localised in the affine part of \(\mathbb{C}P^2\). The cases when a smaller number of points is in the affine part of \(\mathbb{C}P^2\) we analyse in a similar way as for the eighth distinguished spectrum. It appears that such cases for an integrable potential are impossible.

### 4.3.4 Family 10

For the distinguished spectrum from Table 4.1 we assign the pair \((\Lambda_1, \Lambda_2) = (-1, 1)\) to the Darboux point at the infinity. Doing this we additionally assume that the admissible pair \((-1, 1)\) is attached to a non-isotropic Darboux point. The case corresponding to the isotropic Darboux point will be considered separately. The potential has the form

\[V = a_1 q_1^3 + a_2 q_1^2 q_2 + a_4 q_1 q_2^2 + a_5 q_2^3 + q_2^3 + \frac{1}{3} q_3^3.\]

The solutions of the second and fourth order variational equations do not have logarithmic terms iff \(a_2 = 0\) and

\[a_4 (-20 - 144 a_1 a_4 + 160 a_4^2 - 45 a_5^2) = 0, \quad a_4 (-100 - 25 a_1 a_4 + 128 a_4^2 - 225 a_5^2) = 0.\]

The above equations have the following solutions

\[\{a_4 = 0\}, \quad \left\{a_1 = \frac{16}{15} a_4, a_5 = \pm \frac{2}{15} \sqrt{-25 + 8 a_4^2}\right\}.\]
The first solution yields the potential
\[ V = a_1 q_1^3 + a_3 q_2^3 + q_2^2 q_3 + \frac{1}{3} q_3^3, \]
that is equivalent to (4.19).

Let us take the second solution with the sign +. Then fifth order variational equations yield
\[ a_4 = 0 \quad \text{or} \quad a_4 = \pm \frac{5}{2\sqrt{2}}. \]
If \( a_4 = 0 \), then the corresponding potential has only one Darboux point, so we discard this case. The second solution yields the following integrable potential
\[ V_{10} = \frac{4\sqrt{2}q_1^3}{3} + \frac{5q_1 q_2^2}{2\sqrt{2}} + q_2^3 q_3 + \frac{1}{3} q_3^3, \]
with first integrals of degree 4 and 6 in the momenta
\[
I_1 = 12 p_2^2 - 27 q_2^2 - 18 q_2^2 (q_1^2 - 4\sqrt{2}q_1 q_3 + 2q_3^2) + 4(6 p_2^2 - 3p_3^2 + 16\sqrt{2}q_3^3 - 2q_3^2)(3p_3^2 + 2q_3^3) \\
+ 12q_2^2 (3p_3^2 (\sqrt{2}q_1 - 4q_3) + 12p_1 p_3 (q_1 + \sqrt{2}q_3) - 2q_3^2 (12q_1^2 + \sqrt{2}q_1 q_3 + 2q_3^2)) \\
- 12 p_2 q_2 (2p_3 (16q_1^2 + 3q_3^2 + 8\sqrt{2}q_1 q_3 - 4q_3^2) + 3\sqrt{2} p_1 (q_3^2 + 4q_3^3)) \\
- 12 p_2^2 (2p_3 (2\sqrt{2} p_1 + p_3) - 4(q_2 - q_3) q_3 (q_2 + q_3) - \sqrt{2} q_1 (5q_2^2 + 8q_3^2)).
\]
\[
I_2 = 81q_2^2 (2\sqrt{2}q_1 + q_3) + 216 p_2 p_3 q_2^2 (\sqrt{2}q_1 + q_3) + 54q_2^2 (p_3^2 - 3p_3^2 + 4\sqrt{2} q_1^3 - 24q_1^3 q_3) \\
- 6\sqrt{2} q_1 q_3^3 + 384 p_2 p_3 q_2^2 q_3 (3p_3^2 + 8\sqrt{2} q_3^3 + 8q_3^2 q_3 - 2\sqrt{2} q_1 q_3^3) - 72 p_4^2 (3p_3^2 + 2q_3^3) \\
+ 144 p_2 p_3 q_2^2 (p_2^2 + 8q_1^2 (2\sqrt{2} q_1 + 3q_3)) + 144 p_1^2 (\sqrt{2} p_2 p_3 + 3\sqrt{2} p_2 q_2^2 - 3p_3 q_2^2 (q_1 + \sqrt{2} q_3)) \\
- 32(p_2^2 + 12 p_2 q_3^2 - 12 p_2 q_3^2 (\sqrt{2} p_2^2 + 4q_1 q_3^3) + 32 q_2^3 (3p_3^2 + 2q_3^3)) - 12 p_1^2 (4p_2^2) \\
- 6p_2 p_3 q_2 (16q_1^2 + 9q_3^2 + 8\sqrt{2} q_1 q_3 + 4q_3^2) + 9q_2^3 (2q_1^2 + 4\sqrt{2} q_1 q_3 + q_3^2) + 32\sqrt{2} q_1^3 (3p_3^2 + 2q_3^3) \\
+ 12 p_2^2 (p_3^2 + 4q_2^2 q_3 - \sqrt{2} q_1 (q_2^2 - 2q_3^2)) + 6q_2^2 (9\sqrt{2} p_2 q_1 + 2q_3^2 (-6q_1^2 + 2\sqrt{2} q_1 q_3 + q_3^2)) \\
- 144 q_2^3 (p_2^2 (7q_1^2 + 5\sqrt{2} q_1 q_3 + 2q_3^2) + 3q_3^3 (3p_3^2 - 2q_3^2 (-2q_1^2 + 2\sqrt{2} q_1 q_3 + q_3^2))) \\
- 48q_2^2 (p_2^2 (5\sqrt{2} q_1 + 4q_3) + 4p_2 q_1^2 (8q_1^2 + 2\sqrt{2} q_1 q_3 + 3q_3^2) + 8q_3^3 (9p_1^2 q_1 + q_3^2 (-6\sqrt{2} q_1 \\
+ 4q_1 q_3 + \sqrt{2} (3q_2^3 + 4q_3^3)) + 6p_1 (16q_2^3 p_3 + 16q_2^2 p_3 (8q_1^2 - 6q_1 q_3^2 + 3\sqrt{2} q_2^2 q_3) + 4p_2 p_2 q_2 (-16\sqrt{2} q_1^3 \\
+ 32q_1 q_3 + \sqrt{2} (3q_2^3 + 4q_3^3)) + 3p_3 q_2^2 (9\sqrt{2} q_3 - 64q_3^3 (\sqrt{2} q_1 q_3 + 2q_3) - 12q_1^2 (2\sqrt{2} q_1^2 + 8q_1 q_3 + \sqrt{2} q_3)) \\
+ \sqrt{2} q_3^3)) + 12 p_2 q_2 (-3\sqrt{2} p_2 q_3^2 + 9q_1 q_2^2 + 32q_1 q_3^3 + 4q_4^2 (4q_1^3 + 6q_1 q_3^2 + \sqrt{2} q_3))))).}

Let us consider the case when pair \((\Lambda_1, \Lambda_2) = (-1, 1)\) is attached to an isotropic Darboux point. In this case we assign pair \((\Lambda_1, \Lambda_2) = (14, 39)\) to the Darboux point at the infinity and this gives \(a_3 = 15/2\) and \(a_6 = 20\). Let \(a\) be the isotropic Darboux point corresponding to admissible pair \((-1, 1)\). We have two possibilities:

1. either \(d_3 \neq 0\), and then using a rotation around the third axis one can annihilate one component, e.g. \(d_1 = 0\) and \(d = (0, s, is)\), where \(s \neq 0\), or

2. \(d_3 = 0\) and \(d = (s, is, 0)\), where \(s \neq 0\).

In the first case the condition that \(d = (0, s, is)\) is a Darboux point yields
\[ a_4 = 0, \quad a_5 = \frac{59}{3}i, \quad s = \frac{i}{19}. \]
Then the eigenvalues of $V''(d)$ are $((2i\alpha_2 - 15)/19, 2, 2)$ and $V''(d)$ is not semi-simple. Thus by Remark 1.3 this potential is not integrable.

In the second case condition that $d = (s, is, 0)$ satisfies $V'(d) = d$ leads to $s = 0$ but it was assumed that $s \neq 0$.

5 Final remarks

Presented calculations show that, formulated at the beginning, classification programme for homogeneous potentials is very ambitious and meets a lot of non-trivial theoretical, as well as computational difficulties. Natural question arises about possible simplifications and improvements allowing to dream about effective calculations for $n = 3$ and $k > 3$, as well as for $n > 3$. One possibility is to use a full power of the higher order variational equations techniques. Here we used only the local criterion of the existence of logarithmic terms in solutions of higher order variational equations, which implies that their differential Galois groups are not Abelian. In fact it is the only known way to use the higher variational equations. However, it seems that, for systems with homogeneous potentials, stronger obstacles for the integrability can be deduced from an analysis of global differential Galois groups of these equations.

Another question is if we have already detected all possible relations between the eigenvalues of Hessian $V''(d)$ at different Darboux points. In [12] Guillot mentioned that other relations exist, but their form is completely not obvious. If such additional relations exist, then they give new obstacles for the integrability and they limit the number of elements of the distinguished spectra. Thus the answer to this question is of great practical importance.

Let us mention also that relations (3.2) and (3.3) were already presented in [33]. However in [33] only the cases with maximal number of proper Darboux points were analysed, corresponding here to Theorem 3.1. Furthermore, the proof of this result was based on results of Guillot [12] obtained with the help of highly advanced techniques [4] for which there are no obvious ways for generalisations. In present paper Theorem 3.1 was obtained ab initio and its proof is completely elementary. Additionally, using the introduced here approach of multidimensional residue calculus, we were able to analyse also some non-generic cases.

The first integrals given for the presented integrable potentials were found by means of the direct method [16]. We have postulated the form of a first integral as a weight-homogeneous polynomial function with respect to the coordinates and the momenta. We have started from the lowest possible weight degree equal to 2, and increased this degree up to finding the first integral. Thus the presented first integrals are of the lowest possible degrees. All of them are irreducible polynomials. This follows from the results contained in [20]. First integrals for some potentials are very complicated functions. We only mention that theses expressions have already been simplified.

Another problem is that some integrable potentials possess imaginary coefficients. In all cases it is possible to convert them into real potentials, but together with a simultaneous change of the signature of the kinetic energy form. There is an open problem how to find their equivalent real form, or how to prove that such a form does not exist.

Also the problems how to improve the calculations of distinguished spectra that are the starting point to look for new integrable potentials and their reconstruction are worth considering. Our calculations described in the last section show that the one distinguished spectrum gives rise to a finite number of potentials. However, in some cases (the first four distinguished spectra) we obtained families of integrable potentials depending on parameters. Let us notice, however, that this occurs only in cases when potentials separate,
i.e., if they are sums of two integrable potentials depending on different coordinates.

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**Appendix**

For potential

\[ V_6 = 364\sqrt{17}q_1^3 + 2835i\sqrt{17}q_1^2q_2 + 1560\sqrt{17}q_1q_2^2 + 6552i\sqrt{17}q_2^3 + 4335q_1^2q_3 + 19074q_2^2q_3 + 578q_3^3, \]
the second first integral is following

\[ I_2 = 20i p_1^4 (2\sqrt{17}q_2 + 5i q_3) + 120 p_2^4 q_3 + 24 i p_2^3 p_3 (29 q_1 + 5 i q_2 - 14 \sqrt{17} q_3) + 4 p_1^5 (-2 i p_2 (5 \sqrt{17} q_1 - 7 i \sqrt{17} q_2 - 31 q_3) + p_3 (25 q_1 + 247 i q_2 + 103 \sqrt{17} q_3)) + p_2^5 (682193 q_1^4 - 572220 q_2^4 + 8 p_3^3 (134 \sqrt{17} q_1 + 42 i \sqrt{17} q_2 - 713 q_3) + 537856 \sqrt{17} q_1 q_3 - 5080620 q_1^2 q_3 + 1895840 \sqrt{17} q_1 q_3^3 - 122882 q_3^4 + 432 i q_2^3 (8279 q_1 + 2566 \sqrt{17} q_3) + 180 q_2^4 (24259 q_1^2 + 1136 \sqrt{17} q_1 q_3 + 50286 q_3^3) + 72 i q_2 (17153 q_3^3 - 9978 \sqrt{17} q_2 q_3 + 39882 q_1 q_3^2 - 8092 \sqrt{17} q_3^3)) + p_1^2 (590325 q_1^4 - 1186484 q_2^4 + 215380 \sqrt{17} q_1 q_3 - 1647300 q_1^2 q_3 + 714408 \sqrt{17} q_1 q_3^3 + 1268132 q_3^2 + p_2^5 (56 \sqrt{17} q_1 + 96 i \sqrt{17} q_2 + 76 q_3) - 4 p_3^3 (103 \sqrt{17} q_1 + (249 i) \sqrt{17} q_2 + 1600 q_3) + 72 i q_2^3 (60775 q_1 + 28124 \sqrt{17} q_3) - 12 q_2^3 (129965 q_1^3 - 18626 \sqrt{17} q_1 q_3 - 22542 q_3^3) + 36 i q_2^4 (22525 q_1 - 2680 \sqrt{17} q_1 q_3 + 310386 q_1 q_3^3 - 50864 \sqrt{17} q_3^3 + 4 p_2^2 p_3 (439 q_2 - 3 i (103 q_1 + 35 \sqrt{17} q_3))).\]

For the next potential

\[ V_7 = 44 \sqrt{7} q_3^3 + 240 i \sqrt{14} q_2 q_3^2 + 330 \sqrt{7} q_1 q_2 q_3 + 935 i \sqrt{14} q_2^3 + 3087 q_2^3 q_3 + 294 q_3^3.\]
the second first integral has the following form

\[ I_2 = 3030p_1^6 + 13140p_2^6 + 522i\sqrt{14}p_2^5p_3 + 96p_1^5(120i\sqrt{2}p_2 + 127\sqrt{7}p_3) + 21p_2^4(2029p_3^2 + 2(69576\sqrt{7}q_1^3 + 1775759i\sqrt{14}q_2^2q_3 + 10962i\sqrt{14}q_2q_3^2 + 880040q_3^3 + 12q_1^2(38327i\sqrt{14}q_2 + 23618q_3 + 24\sqrt{3}q_1(24385q_2^2 + 5257q_3^2))) + 14p_2^3p_3(20i\sqrt{14}p_3^2 + 21(37144i\sqrt{2}q_2^3 + 74257q_2^3 + 52889i\sqrt{14}q_2q_3^2 - 121506q_2q_3^2 - 58730i\sqrt{14}q_3^3 + 12q_1^2(3897q_2 + 1903i\sqrt{14}q_3) + 6q_1(10555i\sqrt{2}q_2^2 + 6008\sqrt{7}q_2q_3 + 39270i\sqrt{2}q_3^2))) + 2058p_2p_3(906048i\sqrt{7}q_1^3 - 864q_1^5(6577\sqrt{7}q_2 + 6237i\sqrt{2}q_3) + 576q_1^4(8213i\sqrt{14}q_2^2 + 30646q_2q_3 + 1232i\sqrt{14}q_3^2) + 12q_1q_2(-3186975\sqrt{7}q_2^2 + 5829390i\sqrt{2}q_2q_3 - 5851272\sqrt{7}q_2q_3^2 + 45i\sqrt{2}q_2(79p_3^2 + 281456q_3^2) + 4\sqrt{7}q_3(-757p_3^2 + 403809q_3^3) + q_2(-7918053i\sqrt{14}q_3^3 - 55828710q_2q_3^3 - 828576i\sqrt{14}q_3q_2q_3^2 - 2q_2(7519p_3^2 - 35492856q_3^3) + 116424q_2^2(p_3^2 - 441q_3^3) - 30i\sqrt{14}q_2q_3(809p_3^2 + 1222746q_3^3) + 12q_1(-99678i\sqrt{14}q_2^2 + 2562546q_2q_3^2 - 248094i\sqrt{14}q_2q_3 + 11i\sqrt{14}q_3(41p_3^2 - 25284q_3^3) + q_2(1313p_3^2 + 3566220q_3^3) - 8q_1(1571i\sqrt{2}p_3^3 + 6(695966\sqrt{7}q_2^3 - 296079i\sqrt{2}q_2q_3 + 604296\sqrt{7}q_2q_3^2 - 3597581i\sqrt{2}q_3^2))) + 147p_2^2(264p_4^4 + 2p_3^2(21824\sqrt{7}q_2^3 + 530687i\sqrt{14}q_2^2 + 24iq_2^2(6465\sqrt{14}q_2 + 3703i) + 1911084q_2^3 + 178710i\sqrt{14}q_2q_3^2 - 116424q_3^3 + 12q_1(16503\sqrt{7}q_2^2 - 39270i\sqrt{2}q_2q_3 + 10598\sqrt{7}q_3^3) + 7(-1915063849q_2^6 + 6i\sqrt{2}q_2^5(107803418q_1 + 149377659\sqrt{7}q_3) - 6q_2^2(150209548q_2^3 - 49807524\sqrt{7}q_1q_3 - 236319531q_3^3) + 8i\sqrt{2}q_2^2(31990100q_2^3 + 33308910\sqrt{7}q_2q_3 - 5276250q_1q_3^2 + 15723981\sqrt{7}q_3^3) - 12q_2^2(9189296q_4^4 - 3549024\sqrt{7}q_1q_3^3 + 6806016q_2q_3 + 2010792\sqrt{7}q_1q_3^3 - 35685867q_3^4) + 24i\sqrt{2}q_2(898328q_1^5 + 534000\sqrt{7}q_2q_3^3 + 936936q_3q_2q_3^3 + 1340420\sqrt{7}q_2q_3^2 + 474010q_1q_4^3 - 616665\sqrt{7}q_3^3) - 8(128528q_2^2 - 229392\sqrt{7}q_3^3 + 720720q_4q_3^2 + 687456\sqrt{7}q_1q_3^3 - 180104q_3q_2q_3^3 - 633276\sqrt{7}q_1q_3^3 - 916893q_3^4)) + 686(-517045760\sqrt{7}q_2^4 + 218437632i\sqrt{14}q_2^5 + 16128q_2^7(1231361\sqrt{7}q_2^2 - 730296i\sqrt{2}q_2q_3 - 116424\sqrt{7}q_3^3) - 9408q_1^6(269p_3^2 + 1626662i\sqrt{14}q_2^3 + 8138214q_2q_3 + 299376i\sqrt{14}q_2q_3^2 + 690396q_3^4) + 403q_4^2(52644172i\sqrt{14}q_2^2 + 375243372q_2q_3^2 - 18652536i\sqrt{14}q_2q_3 + 12i\sqrt{14}q_2q_3(227p_3^2 + 358092q_3^3) + 924q_1^3(5p_3^2 - 588q_3^2) + 1813q_2^2(25p_3^2 + 38064q_3^3) + 2016q_1^5(271811i\sqrt{2}q_2 - 1152\sqrt{7}q_3) - 56(2260588\sqrt{7}q_2^3 - 2543688i\sqrt{2}q_2q_3 - 193725\sqrt{7}q_2q_3^2 + 344190i\sqrt{2}q_2q_3^3 + 56448\sqrt{7}q_3^3) + q_2^3(-1803124370432i\sqrt{14}q_2^2 - 17762238821376q_2q_3 + 4091606410752i\sqrt{14}q_2q_3^2 + 539101332p_3^2q_3^2 - 147q_2^3(20185577p_3^2 - 13484416512q_3^3) + 18144i\sqrt{14}q_2q_3(74785p_3^3 + 62376608q_3^3) + 15876q_2q_3^2(99587p_3^3 + 148533504q_3^3) + 810i\sqrt{14}q_2(153p_3^4 + 374752p_3q_3q_2) + 46)}
admits the following first integrals

\[ \begin{align*}
V & = 27i \sqrt{3990} q_1^3 + 3726 \sqrt{15} q_1 q_2 - 456i \sqrt{3990} q_1 q_2^2 - 4092 \sqrt{15} q_1^3 - 1125 q_1^2 q_3 - 300 q_2 q_3 - 50 q_3^3,
\end{align*} \]

admits the following first integrals

\[ \begin{align*}
I_1 &= 621756 p_1^4 - 171i \sqrt{266} p_1^3 (639 p_2 + 82 \sqrt{15} p_3) - 171 p_1^2 (11366 p_2^2
\end{align*} \]

\[ + 3381 \sqrt{15} p_2 p_3 + 60(21 p_2^2 + 5(1234i \sqrt{3990} q_1^3 + 36 q_2^3(2159 \sqrt{15} q_2 + 1515 q_3)
\end{align*} \]

\[ - i \sqrt{266} q_1 (4861 \sqrt{15} q_2^2 + 12780 q_2 q_3 + 41 \sqrt{15} q_3) - 6(3616 \sqrt{15} q_2^2 + 22665 q_2 q_3
\end{align*} \]

\[ + 966 \sqrt{15} q_2 q_3^2 + 35 q_3^3) \) + 3 (81377 p_2^4 + 40584 \sqrt{15} p_3^2 p_3
\end{align*} \]

\[ + 75 p_2 p_3(558657i \sqrt{266} q_1^3 + 110124 q_2^2 (424 q_1 + 3 \sqrt{15} q_3) - 2i \sqrt{266} q_1 (1842408 q_2
\end{align*} \]

\[ + 162352 \sqrt{15} q_2 q_3 - 65205 q_3^2 - 304 q_2 (72658 q_2^2 + 15753 \sqrt{15} q_2 q_3 + 910 q_3^2)
\end{align*} \]

\[ + 750(-380805543 q_1^4 + 9234i \sqrt{266} q_1^3 (41202 q_1 - 641 \sqrt{15} q_3) + 9234 q_1 (396143 q_2
\end{align*} \]

\[ - 5802 \sqrt{15} q_2 q_3 + 235 q_3^3) - 4560i \sqrt{266} q_1^3 (1322406 q_3^2 + 12177 \sqrt{15} q_2 q_3 - 8505 q_2 q_3
\end{align*} \]

\[ + 65 \sqrt{15} q_3^3) - 3420 q_1^3 (39391416 q_2^4 + 1584720 \sqrt{15} q_2 q_3 + 643960 q_2 q_3^2
\end{align*} \]

\[ - 28980 \sqrt{15} q_2 q_3^2 + 3465 q_3^4) + 304 q_2^2 (80689622 q_2^4 + 9830304 \sqrt{15} q_2 q_3
\end{align*} \]

\[ + 9897000 q_2^2 q_3^2 + 650400 \sqrt{15} q_2 q_3^2 + 206325 q_3^4) + 24i \sqrt{266} q_1 (234137952 q_2
\end{align*} \]

\[ + 18121160 \sqrt{15} q_2 q_3^3 + 13755600 q_2 q_3^3 + 406400 \sqrt{15} q_2 q_3^3 - 217350 q_2 q_3 + 875 \sqrt{15} q_3^3
\end{align*} \]

\[ + p_3^2 (12825i \sqrt{3990} q_1^3 + 14364 q_1^2 (69 \sqrt{15} q_2 + 14 q_3) - 5776 q_2^2 (111 \sqrt{15} q_2 + 196 q_3
\end{align*} \]

\[ - 2i \sqrt{266} q_1 (44031 \sqrt{15} q_2^2 + 34776 q_2 q_3 - 350 \sqrt{15} q_3^2) \) + 10 p_2^2 (14497 q_3
\end{align*} \]

\[ + 15 (42237i \sqrt{3990} q_1^3 + 114 q_1^2 (41736 \sqrt{15} q_2 - 6085 q_3) - 2i \sqrt{266} q_1 (233356 \sqrt{15} q_2
\end{align*} \]

\[ + 157860 q_2 q_3 - 4843 \sqrt{15} q_3^2) - 76 (45750 \sqrt{15} q_2^3 + 85660 q_2 q_3 + 534 \sqrt{15} q_2 q_3
\end{align*} \]

\[ + 665 q_3^3) \) + p_1 (63144i \sqrt{266} p_3^2 + 28596i \sqrt{3990} p_2 p_3 + 135 p_2 (3221 \sqrt{266} p_3^2
\end{align*} \]

\[ + 5 (-1374327 \sqrt{15} q_3^3 + 95i \sqrt{266} q_1^3 (4706 \sqrt{15} q_2 + 639 q_3) + 38 q_1 (247668 \sqrt{15} q_2
\end{align*} \]

\[ + 365320 q_2 q_3 - 8211 \sqrt{15} q_3^2) - 2i \sqrt{266} (105848 \sqrt{15} q_2^3 + 368340 q_2 q_3 + 6406 \sqrt{15} q_2 q_3
\end{align*} \]

\[ - 2415 q_3^3) \) + 50 p_2 (1763 \sqrt{15} q_3) + 342 q_1 (157509 q_2^2 + 18676 \sqrt{15} q_2 q_3 + 210 q_3^2) - 2i \sqrt{266} (576558 q_2
\end{align*} \]

\[ + 140443 \sqrt{15} q_2 q_3 + 57960 q_2 q_3^2 + 350 \sqrt{15} q_3^3) \),

Potential

\[ V_0 = 27i \sqrt{3990} q_1^3 + 3726 \sqrt{15} q_1 q_2 - 456i \sqrt{3990} q_1 q_2^2 - 4092 \sqrt{15} q_1^3 - 1125 q_1^2 q_3 - 300 q_2 q_3 - 50 q_3^3, \]
\[ l_2 = 69896714949 \rho_1^6 - 4410941719 \rho_2^6 - 12178545120 \sqrt{15} \rho_2^5 \rho_3 \\
- 5185404i \sqrt{266} \rho_1^5 (3321 \rho_2 + 1130 \sqrt{15} \rho_3) - 48735p_1^2 (9477831 \rho_2^2 + 6664140 \sqrt{15} \rho_2 \rho_3 \\
+ 17011399 \rho_3^2 + 161435514i \sqrt{3990} \rho_1^3 + 342q_1^2 (31570326 \sqrt{15} \rho_2^2 + 28306975q_3) \\
- 912i \sqrt{266} q_1 (849176 \sqrt{15} \rho_2^3 + 2324700q_2 \rho_3 + 19775 \sqrt{15} \rho_3^3) - 4(1115132226 \sqrt{15} \rho_2^2 \\
+ 600137430 \rho_2^2 q_3 + 230353200 \sqrt{15} \rho_2 \rho_3^2 + 208136575q_3^3)) - 15p_2^2 (11812848011 \rho_3^2 \\
+ 6(3637289871i \sqrt{3990} \rho_1^4 + 57q_1^2 (5057639934 \sqrt{15} \rho_2^2 + 4023927275q_3) \\
- 8i \sqrt{266} q_1 (2599271621 \sqrt{15} \rho_3^3 + 855211770q_2 \rho_3 + 178766525 \sqrt{15} \rho_3^3) \\
- 2(54592416078 \sqrt{15} \rho_3^3 + 441094171900q_2 \rho_3 + 10148787600 \sqrt{15} \rho_2 \rho_3^2 \\
+ 7399814225q_3^3)) - 420p_3^2 \rho_2^3 (200059064 \sqrt{15} \rho_3^4 + 75(391073751i \sqrt{266} \rho_1^3 \\
+ 684q_1^3 (51068432q_2 + 871817 \sqrt{15} q_3) - 2i \sqrt{266} q_1 (1391808024q_2^2 \\
+ 204067408 \sqrt{15} \rho_3 q_3 - 95297139q_3^3) - 16(1053113326q_2^3 + 358832133 \sqrt{15} \rho_2 \rho_3 q_3 \\
+ 26186342q_2 \rho_3^2 + 7756986 \sqrt{15} q_3^3)) - 225p_3^2 (14343103249p_3^4 \\
+ 4p_3^2 (12485855187i \sqrt{3990} \rho_1^4 - 532583214012 \sqrt{15} \rho_3^3 - 2347905250680q_2 q_3 \\
- 28980666960 \sqrt{15} \rho_2 \rho_3^2 + 98949864350q_3^3 + 171q_1^3 (6148072746 \sqrt{15} q_2 \\
+ 1899249050q_3) - 12(2664 q_1 (7036440863 \sqrt{15} q_2^2 + 14574507780q_2 q_3 \\
- 560965895 \sqrt{15} q_3)) - 60(841855524804q_1^4 - 58482i \sqrt{266} q_1^5 (3649122q_2 \\
+ 1662263 \sqrt{15} q_3) + 48735q_1^4 (171927564q_2^2 - 272077812 \sqrt{15} \rho_2 \rho_3 - 109530025q_3^3) \\
- 1140i \sqrt{266} q_1^5 (2587132818q_2^3 - 1736199048 \sqrt{15} \rho_3 q_2^2 - 3090859380q_2 q_3^3 \\
- 9410749 \sqrt{15} q_3^3 - 1140q_1^7 (57246559116q_2^3 - 2857268000 \sqrt{15} q_2 q_3^2 \\
- 109359463740q_2 q_3^2 - 7143208482 \sqrt{15} q_2 \rho_3^3 + 1499017285q_3^3) \\
+ 32i \sqrt{266} q_1 (59926048758q_2^4 - 24985763220 \sqrt{15} q_2 q_3^2 - 210263522175q_2^2 q_3 \\
- 20299324615 \sqrt{15} q_2 q_3^3 - 24297828975q_2 \rho_3^4 + 606252325 \sqrt{15} q_3) \\
+ 4(766529106284q_2^5 - 264155752944 \sqrt{15} q_2 q_3^2 - 932323202280q_4 q_3^2 \\
- 755223062340 \sqrt{15} q_2 q_3^3 - 297832308400q_2 \rho_3^4 - 10859780400 \sqrt{15} q_2 q_3^5 \\
- 48468636375q_3^4)) - 6300p_2 q_3 (6889792 \sqrt{15} \rho_2 \rho_3 + 15p_3^2 (735779997i \sqrt{266} q_1^3 \\
+ 1596q_1^4 (33673656q_2 + 1541923 \sqrt{15} q_3) - 2i \sqrt{266} q_1 (2058576648q_2 \\
+ 331731248 \sqrt{15} q_2 q_3^2 - 56134785q_3^3) - 16(1603777022q_2^3 + 482897263 \sqrt{15} q_2 q_3 \\
+ 304743350q_2 q_3^2 - 21530600 \sqrt{15} q_3)) - 30(54812321982 \sqrt{15} q_1^4 \\
- 87723i \sqrt{266} q_1 (305074 \sqrt{15} \rho_2 + 740505q_3) - 1169640q_4^2 (1118502 \sqrt{15} q_2^2 \\
+ 8577605q_2 q_3 + 98030 \sqrt{15} q_3^3) + 3420i \sqrt{266} q_1^3 (12610437 \sqrt{15} q_3^3 + 705901770q_2 q_3 \\
+ 1901505 \sqrt{15} q_2 q_3^2 + 21915q_3^3) - 6840q_1^4 (273535608 \sqrt{15} q_2^4 - 87960447102q_2 q_3^3 \\
- 85572350q_2 q_3^5 - 131287940q_2 q_3^3 + 1126965 \sqrt{15} q_3^3) \\
+ 4i \sqrt{266} q_1 (37432776912 \sqrt{15} q_3^5 - 569283400800q_2 q_3 - 128920054050 \sqrt{15} q_2 q_3^3 \\
- 26461961100q_2 q_3^3 - 407403200 \sqrt{15} q_2 q_3^4 + 1772465625q_3^5) + 32(24752235324 \sqrt{15} q_2^5)
\[
+ 8249\sqrt{15q_3} - 342q_1 (8389805q_3^2 + 1513572\sqrt{15q_2q_3} + 103404q_3^3) \\
+ 2i\sqrt{266}(32018130q_2^3 + 10832491\sqrt{15q_2q_3} + 7252524q_2q_3^2 + 9322\sqrt{15q_3^3})) \\
+ 855p_1^2(216247837p_2^2 + 340148340\sqrt{15p_2p_3} + 2p_3^2(1362509849p_3^2) \\
+ 6(927498597\sqrt{3990q_1^5} + 57q_1^2(1233825666\sqrt{15q_2} + 815729225q_3) \\
+ 4i\sqrt{266q_1}(1283526583\sqrt{15q_2} + 3508986600q_2q_3 + 10864700\sqrt{15q_3^3}) \\
+ 2(14147249322\sqrt{15q_2^3} + 88856666500q_2q_3 + 1646534400\sqrt{15q_2q_3^3} \\
+ 790136777q_3^5)) + 84p_2p_3(7399217\sqrt{15p_2} + 25(98234883i\sqrt{266q_1} \\
+ 342q_1^2(21342982q_2 + 860271\sqrt{15q_3} - 2i\sqrt{266q_1}(276720192q_2^2 \\
+ 44322832\sqrt{15q_2q_3} - 9351261q_3^3) - 4(834242922q_3^2 + 272614248\sqrt{15q_3^3} \\
+ 96947032q_2q_3^2 - 1221759(15q_3^3))) + 15(53114167p_4^3 + 4p_3^2(1073099271i\sqrt{3990q_1} \\
+ 171q_1^2(412247118\sqrt{15q_2} + 410808415q_3) - 4i\sqrt{266q_1}(1281131112\sqrt{15q_2^3} \\
+ 3566728620q_2q_3 + 33689845\sqrt{15q_3^3} - 2(15405850098\sqrt{15q_2} + 76504861120q_2q_3 \\
+ 4369629240\sqrt{15q_2q_3}^3 - 810863525q_3^3)) - 20(262207731366q_2^5 \\
- 19494i\sqrt{266q_1}^5(20214q_2 + 201130\sqrt{15q_3}) + 48735q_1^4(153003356q_2^2 \\
- 79345668\sqrt{15q_2q_3} - 69945825q_3^2) - 3420i\sqrt{266q_1}(499000974q_2^3 - 161887814\sqrt{15q_2q_3} \\
+ 328039740q_2q_3^2 - 10804817\sqrt{15q_3^3}) - 3420q_1^2(12226096388q_2^4 - 2645440290\sqrt{15q_3q_3^3} \\
- 10784656720q_2q_3^3 - 786696666\sqrt{15q_2q_3} - 218993995q_3^4) + 48i\sqrt{266q_1}(36201520638q_2^2 \\
- 4196311620\sqrt{15q_2q_3}^3 - 404941746550q_2q_3q_3^3 - 4439555265\sqrt{15q_2q_3}^3 - 4418895600q_2q_3^4 \\
- 815675\sqrt{15q_3^3} + 4(1788936608636q_2^5 - 9012180096\sqrt{15q_2q_3}^3 - 2441620201200q_2q_3^4 \\
- 339300426660\sqrt{15q_2q_3}^3 - 6536736048600q_2q_3^4 - 16245381600\sqrt{15q_2q_3}^5 \\
- 3250847735q_3^5)) - 84p_1(32579496i\sqrt{266p_2} + 69384500i\sqrt{3990p_2p_3} \\
+ 5p_2^2(153859518i\sqrt{266p_3} - 110609531073\sqrt{15q_2} + 171i\sqrt{266q_1}(187871254\sqrt{15q_2} \\
+ 137031525q_3) + 342q_1(1782095052\sqrt{15q_2} + 5504929400q_2q_3 + 68484675\sqrt{15q_3^3} \\
- 2i\sqrt{266}(5940190504\sqrt{15q_2} + 45204050700q_2q_3^2 + 887539050\sqrt{15q_2q_3}^3 \\
+ 6030275175q_3^3)) + 10p_2p_3(26762086i\sqrt{3990p_2}^3 + 75(-20781023121q_3^3 \\
+ 513i\sqrt{266q_1}^3(12502546q_2 + 325249\sqrt{15q_3} + 342q_1(390450499q_2^2 + 58084012\sqrt{15q_2q_3} \\
- 22450386q_3^3) - 2i\sqrt{266}(1525002138q_2^3 + 498142221\sqrt{15q_2q_3} + 80902584q_2q_3^2 \\
+ 2878382\sqrt{15q_3^3})) + 75p_2(905686i\sqrt{266p_2} + p_3^2(127897609527\sqrt{15q_1^3} \\
+ 171i\sqrt{266q_1}^3(208667026\sqrt{15q_2} + 128025855q_3) + 342q_1(2125927828\sqrt{15q_2}^3 \\
+ 5043793480q_2q_3 - 84854595\sqrt{15q_2}^3) + 2i\sqrt{266}(8427582736\sqrt{15q_2}^3 \\
+ 39102611940q_2q_3^3 + 1197339030\sqrt{15q_2q_3}^3 - 1363167675q_3^5)) \\
+ 30(-17935756857i\sqrt{266q_1}^5 - 555579q_1^5(926178q_2 + 1164607\sqrt{15q_3} \\
- 48735i\sqrt{266q_1}^2(9819012q_2^2 - 6205816\sqrt{15q_2q_3} - 3190425q_3^3) \\
- 194949q_1^2(173644949q_2^2 - 63035364\sqrt{15q_2q_3} - 107487665q_2q_3^2 - 1185437\sqrt{15q_3^3} \\
+ 1140i\sqrt{266q_1}^2(2832758478q_2^2 - 671800025\sqrt{15q_2q_3} - 2502926370q_2q_3^2 \\
- 153606221\sqrt{15q_2q_3}^3 + 16305030q_3^4) + 228q_1(156415390456q_2^5 \\
+ 51}
\[ -19923354240\sqrt{15}q_3^4q_3^5 - 180078382850q_3^3q_3^5 - 1819769800\sqrt{15}q_2^3q_3^5 \\
-16318512200q_2^4q_3^5 + 417338775\sqrt{15}q_3^5 - 4i\sqrt{266}(135040386672q_2^6 \\
-752761672\sqrt{15}q_2^3q_3^5 - 198974124900q_2^4q_3^5 - 23736446770\sqrt{15}q_2^3q_3^5 \\
-5437197200q_2^5q_3^5 - 418389450\sqrt{15}q_2q_3^5 - 172155375q_3^5) \] + 5p_3(946148i\sqrt{3990p_3^4} \\
+ 5p_3^2(-122280569379q_1^3 + 171i\sqrt{266q_1^2}(172161162q_2 + 12932861\sqrt{15}q_3) \\
+ 342q_1(1655383241q_2^2 + 319116756\sqrt{15}q_2q_3 + 91659330q_3^3) - 2i\sqrt{266}(6445964502q_2^3 \\
+ 210583303\sqrt{15}q_2q_3^3 + 2294558280q_2q_3^3 - 11826850\sqrt{15}q_3^3)) \\
+ 90(-4080172176i\sqrt{3990q_1^6} - 1666737q_1^2(251098\sqrt{15}q_2 + 901885q_3) \\
+ 48735i\sqrt{266q_1^2}(1599714\sqrt{15}q_2^2 + 13818360q_2q_3 + 449075\sqrt{15}q_3^3) \\
+ 97470q_1^3(10497416\sqrt{15}q_2^3 + 409089485q_2q_3^3 + 17776120\sqrt{15}q_2q_3^3 + 7253595q_3^3) \\
+ 570i\sqrt{266q_1^2}(110678928\sqrt{15}q_2^4 - 6424828110q_2q_3^3 - 745058085\sqrt{15}q_2q_3^3 \\
- 2200210q_2q_3^3 - 5745035\sqrt{15}q_3^4) + 228q_1(7552766358\sqrt{15}q_2^5 - 161449719700q_2q_3^3 \\
- 40151682900\sqrt{15}q_2q_3^5 - 15760699275q_2q_3^5 + 17413200\sqrt{15}q_2q_3^5 + 521797500q_3^5) \\
- 4i\sqrt{266}(9011136556\sqrt{15}q_2^6 - 114005567430q_2q_3^6 - 56141394100\sqrt{15}q_2q_3^6 \\
- 52084382850q_2q_3^6 - 2311027175\sqrt{15}q_2q_3^6 + 3038175000q_2q_3^6 + 59134250\sqrt{15}q_3^6))))

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