Fermionic Quasinormal Spectrum of the Kerr Black Hole

Shahar Hod

The Ruppin Academic Center, Emek Hefer 40250, Israel

(February 8, 2020)

We study analytically the asymptotic quasinormal spectrum of fermionic fields in the Kerr spacetime. We find an analytic expression for these black-hole resonances in terms of the black-hole physical parameters: its Bekenstein-Hawking temperature $T_{BH}$, and its horizon’s angular velocity $\Omega$, which is valid in the asymptotic limit $1 \ll \omega \ll \omega_R$. It is shown that according to Bohr’s correspondence principle, the emission of a Rarita-Schwinger quantum ($s = 3/2$) corresponds to a fundamental black-hole area change $\Delta A = 4\hbar \ln 2$, while the emission of a Weyl neutrino field ($s = 1/2$) corresponds to an adiabatic quantum transition with $\Delta A = 0$.

Gravitational waves emitted by a perturbed black hole are dominated by ‘quasinormal ringing’, damped oscillations with a discrete spectrum (see e.g., [1] for a detailed review). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dinking tones of a ringing bell [2–5]. Being the characteristic ‘sound’ of the black hole itself, these free oscillations are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters (especially, the mass and angular momentum of the black hole). This has motivated a flurry of activity with the aim of computing the spectrum of black-hole oscillations.

It turns out that for a Schwarzschild black hole, for a given angular harmonic index $l$ there exist an infinite number of quasinormal modes, for $n = 0, 1, 2, \ldots$, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [6,7]. On the other hand, the real part of the Schwarzschild black-hole frequencies approaches an asymptotic constant value $[8,9]$.

The quasinormal modes (QNMs) have been the subject of much recent attention (see [10] for a detailed list of references), with the hope that these classical resonances may shed some light on the elusive theory of quantum gravity. These recent studies are motivated by an earlier work [11], in which a possible correspondence between the black-hole classical resonances and the quantum properties of its surface area was suggested.

The quantization of black holes was proposed long ago by Bekenstein [12,13], based on the remarkable observation that the horizon area of a non-extremal black hole behaves as a classical adiabatic invariant. In the spirit of the Ehrenfest principle [14] – any classical adiabatic invariant corresponds to a quantum entity with a discrete spectrum, and based on the idea of a minimal increase in black-hole surface area [12], it was conjectured that the horizon area of a quantum black hole should have a discrete spectrum of the form

$$A_n = \gamma \ell_P^2 \cdot n \quad ; \quad n = 1, 2, 3, \ldots ,$$

(1)

where $\gamma$ is a dimensionless constant, and $\ell_P = (G\hbar/c^3)^{1/2}$ is the Planck length (we use units in which $G = c = \hbar = 1$ henceforth). This type of area quantization has since been reproduced based on various other considerations (see e.g., [15] for a detailed list of references).

In order to determine the value of the coefficient $\gamma$, Mukhanov and Bekenstein [16–18] have suggested, in the spirit of the Boltzmann-Einstein formula in statistical physics, to relate $g_n \equiv \exp[S_{BH}(n)]$ to the number of the black hole microstates that correspond to a particular external macro-state, where $S_{BH}$ is the black-hole entropy. In other words, $g_n$ is the degeneracy of the $n$th area eigenvalue. Now, the thermodynamic relation between black-hole surface area and entropy, $S_{BH} = A/4\hbar$, can be met with the requirement that $g_n$ has to be an integer for every $n$ only when

$$\gamma = 4 \ln k \quad ,$$

(2)

where $k$ is some natural number.

Identifying the value of $k$ requires further input. This information may emerge by applying Bohr’s correspondence principle to the (discrete) quasinormal mode spectrum of black holes. Based on the correspondence principle, it was argued [11] that the asymptotic resonances are given by (we assume a time dependence of the form $e^{-i\omega t}$),

$$\omega = T_{BH}^s \ln 3 - i2\pi T_{BH}^s (n + \frac{1}{2}) \quad ,$$

(3)

where $T_{BH}^s = 1/8\pi M$ is the Bekenstein-Hawking temperature of the Schwarzschild black hole. An analytical proof of this equality was later given in [19].

The emission of a quantum of frequency $\omega$ results in a change $\Delta M = \hbar \omega R$ in the black-hole mass. Assuming that $\omega$ corresponds to the asymptotic limit Eq. (3), and using the first-law of black-hole thermodynamics $\Delta M = \frac{1}{4} T_{BH}^s \Delta A$, this implies a change $\Delta A = 4\hbar \ln 3$ in the black hole surface area. Remarkably, this value is in accord with the Bekenstein-Mukhanov prediction, Eq. (2).

The possible correspondence between the black-hole classical resonances and the quantum properties of its surface area has triggered a flurry of research attempting to calculate the asymptotic ringing frequencies of various
types of black holes (for a detailed list of references see, e.g., [10]). For instance, the asymptotic QNM spectrum of the Reissner-Nordström (RN) black hole was calculated in [20]. This was followed by an analytical calculation of the asymptotic QNM frequencies of a charged field in the RN spacetime [21]. The results presented in [20,21] indicate that the asymptotic resonances of a charged field correspond to a fundamental black-hole area change of ∆A = 4ℏ ln 2, in accord with the general prediction Eq. (2).

It should be emphasized, however, that less is known about the corresponding QNM spectrum of the generic (rotating) Kerr black hole, which is the most interesting from a physical point of view. Former studies of the Kerr asymptotic spectrum [6,22–26] used numerical tools, and there are in fact no analytical results for the asymptotic QNM frequencies of the Kerr black hole. In this work we provide analytical formulae for the Kerr QNM spectrum in the asymptotic limit 1 ≪ ω ≪ ω_R. We consider fermionic modes, characterized by half-integer spins.

The dynamics of fermionic fields in the Kerr spacetime is governed by the Teukolsky equation [27]

\[ \Delta \frac{d^2 R_{lm}}{dr^2} + (s + 1)(2r - 2M) \frac{dR_{lm}}{dr} + V(r)R_{lm} = 0, \]  (4)

with

\[ V(r; a, \omega, s, l, m) = \Delta^{-1}[(r^2 + a^2ω^2 - 2maωr + ℏ^2)] + is\Delta^{-1}[ma(2r - 2M) - ω(r^2 - a^2)] + i2sωr - (ωa)^2 - A_{slm}, \]  (5)

where ∆ ≡ (r + r_+)(r - r_-) [r_± = M ± (M^2 - a^2)1/2] are the black hole (event and inner) horizons, and a ≡ J/M is the black hole angular momentum per unit mass. The field spin-weight parameter s takes the values ±1/2 and ±3/2 for a two-component Weyl neutrino field, and for a Rarita-Schwinger field, respectively. The angular separation constants A_{slm}(aω) are determined from an independent differential equation [6], which governs the angular dependence of the field.

The black hole QNMs correspond to solutions of the wave equation with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [28]. Such boundary conditions single out a discrete set of resonances \{ω_n\}. The solution to the radial Teukolsky equation may be expressed as [6] (assuming an azimuthal dependence of the form \( e^{imφ} \))

\[ R_{lm} = e^{iωr}(r - r_-)^{-1-s+2MW+is_+}(r - r_+)^{-s-is_+} \sum_{n=0}^{∞} a_n \left( \frac{r - r_+}{r - r_-} \right)^n, \]  (6)

where \( σ_+ ≡ (ωr_+ - ma)/(r_+ - r_-). \)

The sequence of expansion coefficients \{a_n : n = 1, 2, … \} is determined by a recurrence relation of the form [6]

\[ α_n a_{n+1} + β_n a_n + γ_n a_{n-1} = 0, \]  (7)

with initial conditions \( a_0 = 1 \) and \( α_0d_1 + β_0d_0 = 0 \). The recursion coefficients \( α_n, β_n, \) and \( γ_n \) are given in [6]. The quasinormal frequencies are determined by the requirement that the series in Eq. (6) is convergent, that is \( Σa_n \) exists and is finite [6].

The physical content of the recursion coefficients becomes clear when they are expressed in terms of the black-hole physical parameters [10]: the Bekenstein-Hawking temperature \( T_{BH} = (r_+ - r_-)/A \), and the horizon’s angular velocity \( Ω = 4πa/A \), where \( A = 4π(r_+^2 + a^2) \) is the black-hole surface area. The recursion coefficients obtain a simple form in terms of these physical quantities,

\[ α_n = (n + 1)(n + 1 - s - 2iβ_+\omega), \]  (8)

\[ β_n = -2(n + 1)^2 - 2iβ_+\omega)(n + 2 - 2iωr_+), \]  (9)

and

\[ γ_n = (n - 4iM\omega)(n + s - 2iβ_+\omega), \]  (10)

where \( β_+ \equiv (4πT_{BH})^{-1} \) is the black-hole inverse temperature, \( ω ≡ mΩ \), and \( λ_{slm} ≡ A_{slm} + (ωa)^2 - 2maω. \)

The angular separation constants \( A_{slm}(aω) \) are determined by an independent recurrence relation [6]. The asymptotic large \( aω \) limit of the angular separation constants is given by [29–32]

\[ A_{slm} = -(aω)^2 + 2d^{(1)}_{slm}aω + q_{slm}^{(0)} + q_{slm}^{(-1)}(aω)^{-1} + · · · , \]  (11)

where the expansion coefficients \( q_{slm}^{(n)} \) are given in [31,32]. Remarkably, one finds

\[ A_{slm} = -(aω)^2 + 2mω + O(1), \]  (12)

for fermionic modes characterized by \( l = m + |s| - \frac{1}{2} \), thus yielding \( λ_{slm} = O(1) \) in these cases.

The Teukolsky equation also describes the propagation of fermionic fields in the RN spacetime [33] [one should simply replace \( r_± = M ± (M^2 - a^2)1/2 \) by \( r_± = M ± (M^2 - Q^2)1/2 \), and take \( a = 0 \) elsewhere. In addition, \( A_{slm} = l(l + 1) - s(s + 1) \) for the spherically symmetric RN spacetime [6].] The spectrum of the RN QNMs may be found using a recursion relation of the form Eq. (7). The \( \{α_n\} \) and \( \{γ_n\} \) coefficients of the RN perturbations have exactly the same form as in the Kerr case (where for the RN black hole \( ω = ω_R \)). In addition, one finds \( β_{n, RN} = β_{n,Kerr} + λ_{slm}(aω). \) Taking cognizance of the asymptotic limit Eq. (12), one finds that for fermionic modes with \( l = m + |s| - \frac{1}{2} \) the \( \{β_n\} \) coefficients of the Kerr black hole
coincide with those of the RN black hole (in addition to the coincidence of the \( \{ \alpha_n \} \) and \( \{ \gamma_n \} \) terms).

Thus, one finds that the asymptotic quasinormal frequencies of fermionic fields (with \( l = m + |s| - \frac{1}{2} \)) in the Kerr spacetime are related to the corresponding asymptotic frequencies of the RN black hole. The asymptotic \( 1 \ll \omega_I \) behavior of the RN resonances is determined by the equation \([20,34]\)

\[
e^{-4\pi\beta_+\omega} = 1 ,
\]

for the two-component Weyl field, and by \([20,35]\)

\[
2e^{-4\pi\beta_+\omega} + 3e^{-8\pi M\omega} = -1 ,
\]

for the Rarita-Schwinger field.

The preceding discussion indicates that expressions similar to Eqs. (13) and (14) must hold true for the asymptotic QNMs of fermionic fields (with \( l = m + |s| - \frac{1}{2} \)) in the Kerr black-hole spacetime. Equations (13) and (14) suggest that the spectrum depends on the combinations \( \beta_+\omega \) and \( M\omega \) appearing in Eqs. (8)-(10), but does not depend explicitly on \( \omega_+ \). From Eqs. (8)-(10) one learns that the analogy between the asymptotic spectrum of fermionic fields in the Kerr spacetime, and the corresponding spectrum in the RN spacetime is obtained by applying the transformation \( \beta_+\omega \rightarrow \beta_+\bar{\omega} \) in Eqs. (13) and (14). Using this transformation, one finds that the asymptotic \( 1 \ll \omega_I \ll \omega_R \) quasinormal spectrum of neutrino modes with \( l = m \) is determined by the equation

\[
e^{-4\pi\beta_+(\omega-m\Omega)} = 1 ,
\]

thus yielding

\[
\omega = m\Omega - i2\pi T_{BH} n .
\]

The corresponding spectrum of a Rarita-Schwinger field with \( l = m + 1 \) is given by

\[
2e^{-4\pi\beta_+(\omega-m\Omega)} + 3e^{-8\pi M\omega} = -1 ,
\]

thus yielding

\[
\omega = m\Omega + T_{BH} \ln 2 - i2\pi T_{BH}(n + \frac{1}{2}) .
\]

The emission of a quantum of frequency \( \omega \) and azimuthal number \( m \) results in a change \( \Delta M = \hbar \omega_R \) in the black-hole mass, and a change \( \Delta J = m\hbar \) in its angular momentum. Substituting the fundamental resonances, Eqs. (17) and (18), into the first law of black-hole thermodynamics

\[
\Delta M = \frac{1}{4} T_{BH} \Delta A + \Omega \Delta J ,
\]

one obtains

\[
\Delta A = 0 ,
\]

for the Weyl neutrino field, and

\[
\Delta A = 4\hbar \ln 2 ,
\]

for the Rarita-Schwinger field.

Thus, the emission of a neutrino quantum (with \( l = m \) and \( m \gg 1 \)) from a Kerr black hole corresponds to an adiabatic transition, for which there is no net change in the black-hole surface area (though the mass and angular-momentum of the black hole have been changed by the emission). We note that, remarkably this is a generic feature of asymptotic neutrino resonances: they correspond to \( \Delta A = 0 \) for both the Schwarzschild \([19]\), RN \([33]\), and Kerr black holes.

The emission of a Rarita-Schwinger fermion (with \( l = m + 1 \) and \( m \gg 1 \)) results with a fundamental change \( \Delta A = 4\hbar \ln 2 \) in black-hole surface area. Remarkably, this fundamental change in the surface area is in accord with the Bekenstein-Mukhanov general prediction, Eq. (2).

In summary, motivated by novel results in the theory of black-hole quantization, we have studied analytically the QNM spectrum of fermionic fields in the Kerr black-hole spacetime. It was shown that in the asymptotic limit \( 1 \ll \omega_I \ll \omega_R \), these black-hole resonances can be expressed in terms of the black-hole physical parameters: its temperature \( T_{BH} \), and its angular velocity \( \Omega \).

The analysis of the Kerr quasinormal spectrum enabled us to test the applicability of Bohr’s correspondence principle to the quantization of black holes in generalized situations, in which there is no one-to-one correspondence between the energy of the emitted quantum and the resulting change in black-hole surface area.\([1]\) We have shown that according to the correspondence principle, the emission of a Rarita-Schwinger quantum characterized by \( l = m + 1 \) induces a fundamental change in the Kerr black-hole surface area, \( \Delta A = 4\hbar \ln 2 \). Remarkably, this area unit is universal in the sense that it is independent of the black-hole parameters. The emission of a Weyl neutrino with \( l = m \) corresponds to an adiabatic transition, in which there is no change in the black-hole surface area.

**ACKNOWLEDGMENTS**

The research of SH was supported by G.I.F. Foundation. I thank Uri Keshet for numerous discussions, as well as for a continuing stimulating collaboration.

\[1\] For an excellent review and a detailed list of references see H. P. Nollert, Class. Quantum Grav. 16, R159 (1999).
[2] W. H. Press, Astrophys. J. **170**, L105 (1971).
[3] V. de la Cruz, J. E. Chase and W. Israel, Phys. Rev. Lett. **24**, 423 (1970).
[4] C.V. Vishveshwara, Nature **227**, 936 (1970).
[5] M. Davis, R. Ruffini, W. H. Press and R. H. Price, Phys. Rev. Lett. **27**, 1466 (1971).
[6] E. W. Leaver, Proc. R. Soc. A **402**, 285 (1985).
[7] A. Bachelot and A. Motet-Bachelot, Ann. Inst. H. Poincaré **59**, 3 (1993).
[8] H. P. Nollert, Phys. Rev. D **47**, 5253 (1993).
[9] N. Andersson, Class. Quantum Grav. **10**, L61 (1993).
[10] S. Hod and U. Keshet, Class. Quant. Grav. **22**, L71 (2005).
[11] S. Hod, Phys. Rev. Lett. **81**, 4293 (1998).
[12] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[13] J. D. Bekenstein, Lett. Nuovo Cimento **11**, 467 (1974).
[14] See for example M. Born, Atomic Physics (Blackie, London, 1969), eighth edition.
[15] S. Hod, Class. Quant. Grav., **21**, L97 (2004).
[16] V. Mukhanov, JETP Lett. **44**, 63 (1986).
[17] J. D. Bekenstein and V. F. Mukhanov, Phys. Lett. B **360**, 7 (1995).
[18] J. D. Bekenstein in XVII Brazilian National Meeting on Particles and Fields, eds. A. J. da Silva et. al. (Brazilian Physical Society, Sao Paulo, 1996), J. D. Bekenstein in Proceedings of the VIII Marcel Grossmann Meeting on General Relativity, eds. T. Piran and R. Ruffini (World Scientific, Singapore, 1998).
[19] L. Motl, Adv. Theor. Math. Phys. **6**, 1135 (2003).
[20] L. Motl and A. Neitzke, Adv. Theor. Math. Phys. **7**, 307 (2003).
[21] S. Hod, e-print gr-qc/0511047.
[22] S. Detweiler, Astrophys. J. **239**, 292 (1980).
[23] H. Onozawa, Phys. Rev. D **55**, 3593 (1997).
[24] E. Berti and K. D. Kokkotas, Phys. Rev. D **68**, 044027 (2003).
[25] E. Berti, V. Cardoso, K. D. Kokkotas and H. Onozawa, Phys. Rev. D **68**, 124018 (2003).
[26] E. Berti, V. Cardoso, S. Yoshida, Phys. Rev. D **69**, 124018 (2004).
[27] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972); Astrophys. J. **185**, 635 (1973).
[28] S. L. Detweiler, in Sources of Gravitational Radiation, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).
[29] M. Abramowitz, *Handbook of Mathematical Functions*, (Dover Publications, inc., New York, 1965).
[30] C. Flammer, in *Spheroidal Wave Functions*, (Stanford University Press, Stanford, CA, 1957).
[31] M. Casals and A. C. Ottewill, Phys. Rev. D **71**, 064025 (2005).
[32] E. Berti, V. Cardoso, and M. Casals, e-print gr-qc/0511111.
[33] S. Hod and U. Keshet, Phys. Rev. D, to be published (2006).
[34] The asymptotic spectrum of the Weyl neutrino field in the RN spacetime can be obtained by substituting $j = 1$ in equation (64) of [20]. The corresponding spectrum of the Rarita-Schwinger field is obtained by substituting $j = 1/3$ in equation (64) of [20].
[35] The final result of [20] for the RN spectrum, can be written as Eq. (14) by simply multiplying both sides of equation (66) in [20] by $e^{-\beta + \omega}$.
[36] Note that the second exponent in Eq. (17) is a factor $\exp(4\pi m\Omega)$ smaller than the first exponent.