Abstract. Affine type systems are substructural type systems where copying of information is restricted, but discarding of information is permissible at all types. Such type systems are well-suited for describing quantum programming languages, because copying of quantum information violates the laws of quantum mechanics. In this paper, we consider a first-order affine type system with inductive data types and present a novel categorical semantics for it. The most challenging aspect of this interpretation comes from the requirement to construct appropriate discarding maps for our data types which might be defined by mutual/nested recursion. We show how to achieve this for all types by taking models of a first-order linear type system whose atomic types are discardable and then presenting an additional affine interpretation of types within the slice category of the model with the tensor unit. We present some concrete categorical models for the language ranging from classical to quantum. Finally, we discuss potential ways of dualising and extending our methods and using them for interpreting coalgebraic and lazy data types.

Keywords: Inductive data types · Categorical Semantics · Affine Types

1 Introduction

Linear Logic [4] is a substructural logic where the rules for weakening and contraction are restricted. Linear logic has been very influential in computer science and has lead to the development of linear type systems where discarding and copying of variables is restricted. Linear logic has also inspired the development of affine type systems, which are substructural type systems where only the rule for contraction (copying of variables) is restricted, but weakening (discarding of variables) is completely unrestricted. Affine type systems are a natural choice for quantum programming languages [14][12][15][1], because they can be used to enforce compliance with the laws of quantum mechanics, where copying of quantum information is impossible [13].

In this paper we consider a first-order affine type system with inductive data types, called Aff, and we present a categorical semantics for it. The main focus of the present paper is on the construction of the required discarding maps that are necessary for the interpretation of the type system. Our semantics is novel in that we assume very little structure on the model side: we do not assume the existence...
of any (sub)category where the tensor unit $I$ is a terminal object. Instead, we merely assume that the interpretation of every atomic type is equipped with some discarding map (which is clearly necessary) and we then show how to construct all other discarding maps by providing an affine interpretation of types within the slice category of the model with the tensor unit. Thus, by taking a categorical model of a first-order linear type system, we construct all the discarding maps we need by performing a careful semantic analysis, instead of assuming additional structure within the categorical model.

Outline. We begin by recalling some background about parameterised initial algebras in Section 2. Next, we describe the syntax of Aff, which is a fragment of the quantum programming language QPL [14][12], in Section 3. In Section 4, we present the operational semantics of Aff. One of our main contributions is in Section 5, where we show how parameterised initial algebras for suitable functors may be reflected into slice categories. Our contributions continue in Section 6, where we describe a categorical model for our language, and with Section 7, where we present a novel categorical semantics for the affine structure of types by providing a non-standard type interpretation within a slice category. In Section 8, we discuss future work and possible extensions and in Section 9, we discuss related work and present some concluding remarks.

2 Parameterised Initial Algebras

Simple inductive data types, like lists and natural numbers, may be interpreted by initial algebras. However, the interpretation of inductive data types defined by mutual/nested induction requires a more general notion called parameterised initial algebra, which we shall now recall.

**Definition 1 (cf. [3], §6.1).** Let $A$ and $B$ be categories and $T : A \times B \to B$ a functor. A parameterised initial algebra for $T$ is a pair $(T^\dagger, \tau)$, such that:

- $T^\dagger : A \to B$ is a functor;
- $\tau : T \circ (Id, T^\dagger) \Rightarrow T^\dagger : A \to B$ is a natural isomorphism;
- For every $A \in \text{Ob}(A)$, the pair $(T^\dagger A, \tau_A)$ is an initial $T(A, -)$-algebra.

Note that by trivialising $A$, we get the well-known notion of initial algebra. Next, we recall a theorem which provides sufficient conditions for the existence of parameterised initial algebras.

**Theorem 2 ([10], Theorem 4.12).** Let $B$ be a category with an initial object and all $\omega$-colimits. Let $T : A \times B \to B$ be an $\omega$-cocontinuous functor. Then $T$ has a parameterised initial algebra $(T^\dagger, \tau)$ and the functor $T^\dagger$ is also $\omega$-cocontinuous.

In particular, the above theorem shows that $\omega$-cocontinuous functors are closed under formation of parameterised initial algebras.
In this section we describe the syntax of **Aff**, which is the language on which we will base the development of our ideas. **Aff** is a fragment of the quantum programming language QPL [12] which is obtained from QPL by removing procedures, quantum resources and copying of classical information. The reason for considering this fragment is just simplicity and brevity of the presentation.

**Remark 3.** In fact, the methods we describe can handle the addition of procedures and the copying of non-linear information with no further effort. The addition of quantum resources can also be handled by our methods, but this requires identifying a suitable category of quantum computation with \(\omega\)-colimits.

The syntax of **Aff** is summarised in Figure 1. A type context \(\Theta\) is well-formed, denoted \(\Theta \vdash \), if \(\Theta\) is simply a list of distinct type variables. Well-formed types, denoted \(\Theta \vdash A\), are specified by the following rules:

\[
\begin{align*}
\Theta \vdash A & \quad \Theta \vdash B & \quad \Theta \vdash \mu X.A & \quad \Theta, X \vdash A \quad \Theta \vdash A + B & \quad \Theta \vdash A \otimes B \quad \Theta \vdash I
\end{align*}
\]

where we assume that there is some set of atomic types \(A\), which we will leave unspecified in this paper (for generality). For example, in quantum programming, it suffices to assume \(A = \{\text{qubit}\}\). This is the case for QPL.

A type \(A\) is closed whenever \(\cdot \vdash A\). Note that nested type induction (also known as mutual induction) is allowed, i.e., it is possible to form inductive data types which have more than one free variable in their type contexts. Henceforth, we implicitly assume that all types we are dealing with are well-formed.

**Example 4.** Natural numbers can be defined as \(\text{Nat} \equiv \mu X.I + X\). A list of a closed type \(\cdot \vdash A\) is defined by \(\text{List}(A) \equiv \mu Y.I + A \otimes Y\).

**Term variables** are denoted by small Latin characters (e.g. \(x, y, u, b\)). In particular, \(u\) ranges over variables of unit type \(I\), \(b\) over variables of type \(\text{bit} := I + I\).
and $x, y$ range over arbitrary variables. Variable contexts are denoted by capital Greek letters, such as $\Gamma$ and $\Sigma$. Variable contexts contain only variables of closed types and are written as $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.

A term judgement $\vdash \langle \Gamma \rangle \ M \langle \Sigma \rangle$ indicates that term $M$ is well-formed assuming an input variable context $\Gamma$ and an output variable context $\Sigma$. All types within it are necessarily closed. The formation rules are shown in Figure 2.

Remark 5. Because we are not concerned with any domain-specific applications in this paper, we leave the atomic types uninhabited. Of course, any domain-specific extension should add suitable introduction and elimination rules for each atomic type. In the case of QPL, the term language has to be extended with three terms – one each for preparing a qubit in state $|0\rangle$, applying a unitary gate to a term and finally measuring a qubit. See [12] for more information.

4 Operational Semantics of Aff

The purpose of this section is to present the operational semantics of Aff. We begin by introducing program configurations which completely and formally describe the current state of program execution. A program configuration is a pair
Value Assignments. Values are expressions defined by the following grammar:

\[
v, w ::= * \mid \text{left}_{A,B} v \mid \text{right}_{A,B} v \mid (v, w) \mid \text{fold}_{\mu X . A} v.
\]

The expression * represents the unique value of unit type \( I \). Other particular values of interest are the canonical values of type \( \text{bit} \), called \textit{false} and \textit{true}, which are formally defined by \( \text{ff} := \text{left}_{I,I} * \) and \( \text{tt} := \text{right}_{I,I} * \). They play an important role in the operational semantics.

The well-formed values, denoted \( \vdash v : A \), are specified by the following rules:

\[
\begin{align*}
\vdash v : I & \quad \vdash v : A \quad \vdash v : B \\
\vdash (v, w) : A \otimes B & \quad \vdash \text{fold}_{\mu X . A} v : \mu X . A
\end{align*}
\]

A \textit{value assignment} is simply a function from term variables to values. We write value assignments as \( \vdash A_i \) for each \( i \in \{1, \ldots, n\} \).

Program configurations. A \textit{program configuration} is a couple \( (M \ | \ V) \), where \( M \) is a term and \( V \) is a value assignment. A \textit{well-formed} program configuration, denoted \( \Gamma ; \Sigma \vdash (M \ | \ V) \), is a program configuration \( (M \ | \ V) \), such that there exist (necessarily unique) \( \Gamma, \Sigma \) with: (1) \( \vdash (\Gamma) M (\Sigma) \) is a well-formed term; and (2) \( \Gamma \vdash V \) is a well-formed value assignment.

The (small step) operational semantics is defined as a function \( \vdash \) on program configurations \( (M \ | \ V) \) by induction on the structure of \( M \) in Figure 3. Note that, in the rule for while loops, the term \( \text{if } b \text{ then } \{ M \} \) is just syntactic sugar for \( \text{case } b \text{ of } \{ \text{left } u \rightarrow b = \text{left } u \mid \text{right } u \rightarrow b = \text{right } u ; M \} \).

Theorem 6 (Subject reduction [12]). If \( \Gamma ; \Sigma \vdash (M \ | \ V) \) and \( (M \ | \ V) \vdash (M', V') \), then \( \Gamma ; \Sigma \vdash (M', V') \), for some (necessarily unique) context \( \Gamma' \).

Assumption 7. Henceforth, all configurations are assumed to be well-formed.

We shall use calligraphic letters (\( C, D \ldots \)) to denote configurations. A \textit{terminal} configuration is a configuration \( C \), such that \( C = (\text{skip}, V) \).

Theorem 8 (Progress [12]). If \( C \) is a configuration, then either \( C \) is terminal or there exists a configuration \( D \), such that \( C \vdash D \).

Remark 9. Any domain-specific extension should, of course, also adapt the operational semantics as necessary. In the case of QPL, this requires introducing new reduction rules for the additional terms and extending the notion of configuration with an extra component that stores the quantum data.
\[ \text{(new unit } u \mid V) \leadsto (\text{skip } \mid V, u = \ast) \]

\[ \text{(discard } x \mid V, x = v) \leadsto (\text{skip } \mid V) \]

\[ (\text{skip; } P \mid V) \leadsto (P \mid V) \]

\[ (P \mid V) \leadsto (P' \mid V') \]

\[ (P; Q \mid V) \leadsto (P'; Q \mid V') \]

\[ (\text{while } b \text{ do } M \mid V, b = v) \leadsto (\text{if } b \text{ then } \{ M; \text{ while } b \text{ do } M \} \mid V, b = v) \]

\[ (y = \text{left } x \mid V, x = v) \leadsto (\text{skip } \mid V, y = \text{left } v) \]

\[ (y = \text{right } x \mid V, x = v) \leadsto (\text{skip } \mid V, y = \text{right } v) \]

\[ (\text{case } y \text{ of } \{ \text{left } x_1 \to M_1 \mid \text{right } x_2 \to M_2 \} \mid V, y = \text{left } v) \leadsto (M_1 \mid V, x_1 = v) \]

\[ (\text{case } y \text{ of } \{ \text{left } x_1 \to M_1 \mid \text{right } x_2 \to M_2 \} \mid V, y = \text{right } v) \leadsto (M_2 \mid V, x_2 = v) \]

\[ ((x_1, x_2) = x \mid V, x = (v_1, v_2)) \leadsto (\text{skip } \mid V, x = (v_1, v_2)) \]

\[ (y = \text{fold } x \mid V, x = v) \leadsto (\text{skip } \mid V, y = \text{fold } v) \]

\[ (y = \text{unfold } x \mid V, x = \text{fold } v) \leadsto (\text{skip } \mid V, y = v) \]

Fig. 3: Small Step Operational semantics of \textbf{Aff}.

5 Slice Categories for Affine Types

Our type system is affine and in order to provide a denotational interpretation we have to construct discarding maps in our model at every type. This is achieved in the following way: (1) for every closed type \( A \) we provide a standard interpretation \([A] \in \text{Ob}(C)\) in our model \( C \); (2) in addition, we provide a affine type interpretation \([A] \in \text{Ob}(C/I)\) within the slice category with the tensor unit, that is, for every type we carefully pick out a specific discarding map; (3) we prove \([A] = ([A], \circ_A : [A] \to I)\) and show that our choice of discarding map \( \circ_A \) can discard all values of our language, as required.

The purpose of this section is to show the slice category \( C/I \) has sufficient categorical structure for the affine interpretation of types. Our analysis is quite
general and works for many affine scenarios. Under some basic assumptions on $\mathbf{C}$ we show that $\mathbf{C}/I$ inherits from $\mathbf{C}$: a symmetric monoidal structure (Proposition 13), finite coproducts (Proposition 14) and (parameterised) initial algebras for a sufficiently large class of functors (Theorem 18).

**Assumption 10.** Throughout the remainder of the section we assume we are given a category $\mathbf{C}$ and we fix an object $I \in \text{Ob}(\mathbf{C})$. Let $\mathbf{C}_a := \mathbf{C}/I$ be the slice category of $\mathbf{C}$ with the fixed object $I$.

Thus, the objects of $\mathbf{C}_a$ are pairs $(A, \diamond_A)$, where $A \in \text{Ob}(\mathbf{C})$ and $\diamond_A : A \to I$ is a morphism of $\mathbf{C}$. Then, a morphism $f : (A, \diamond_A) \to (B, \diamond_B)$ of $\mathbf{C}_a$ is a morphism $f : A \to B$ of $\mathbf{C}$, such that $\diamond_B \circ f = \diamond_A$. Composition and identities are the same as in $\mathbf{C}$. We refer to the maps $\diamond_A$ as the **discarding** maps and to the morphisms of $\mathbf{C}_a$ as **affine** maps.

**Notation 11.** There exists an obvious forgetful functor $U : \mathbf{C}_a \to \mathbf{C}$ given by $U(A, \diamond_A) = A$ and $U(f) = f$.

The following (well-known) proposition will be used to show the existence of certain initial algebras in $\mathbf{C}_a$. For completeness, we provide a proof.

**Proposition 12.** The functor $U : \mathbf{C}_a \to \mathbf{C}$ reflects small colimits.

*Proof.* In Appendix A.1. 

Next, we show how a symmetric monoidal structure on $\mathbf{C}$ induces one on $\mathbf{C}_a$.

**Proposition 13.** Assume that $\mathbf{C}$ is equipped with a (symmetric) monoidal structure $(\mathbf{C}, \otimes, I, \alpha, \rho, (\sigma))$. Then, the tuple $(\mathbf{C}_a, \otimes_a, (I, \text{id}_I), \alpha_a, \lambda_a, \rho_a, (\sigma_a))$ is a (symmetric) monoidal category, where $\otimes_a : \mathbf{C}_a \times \mathbf{C}_a \to \mathbf{C}_a$ is defined by:

$$(A, \diamond_A) \otimes_a (B, \diamond_B) := (A \otimes B, \lambda_I \circ (\diamond_A \otimes \diamond_B))$$

and where the natural isomorphisms $\alpha_a, \lambda_a, \rho_a, (\sigma_a)$ are componentwise equal to $\alpha, \lambda, \rho, (\sigma)$ respectively. Moreover, this data makes $U : \mathbf{C}_a \to \mathbf{C}$ a strict monoidal functor and we also have:

$$\otimes \circ (U \times U) = U \circ \otimes_a : \mathbf{C}_a \times \mathbf{C}_a \to \mathbf{C}.$$

*Proof.* Straightforward verification. 

Next, we show how coproducts on $\mathbf{C}$ induce coproducts on $\mathbf{C}_a$.

**Proposition 14.** Assume that $\mathbf{C}$ has finite coproducts with initial object denoted $\varnothing$ and binary coproducts by $(A + B, \text{left}_A, \text{right}_B)$. Then, the category $\mathbf{C}_a$ has finite coproducts. Its initial object is $(\varnothing, \varnothing)$ and binary coproducts are given by $(A, \diamond_A) +_a (B, \diamond_B) := (A + B, [\diamond_A, \diamond_B])$. Moreover, we have:

$$+_a (U \times U) = U \circ +_a : \mathbf{C}_a \times \mathbf{C}_a \to \mathbf{C}.$$

*Proof.* Straightforward verification.
5.1 (Parameterised) initial algebras in \( C_a \)

In this subsection we will show how (parameterised) initial algebras from \( C \) may be reflected into \( C_a \) by using methods from [10,8]. Towards this end, we assume that \( C \) has some additional structure, so that parameterised initial algebras may be formed within it.

**Assumption 15.** Throughout the remainder of the section, we assume that \( C \) has an initial object \( \emptyset \) and all \( \omega \)-colimits.

**Proposition 16.** The category \( C_a \) has an initial object and all \( \omega \)-colimits. Moreover, the forgetful functor \( U : C_a \to C \) preserves and reflects \( \omega \)-colimits.

**Proof.** The initial object is \( (\emptyset, \bot, I) \), because \( U \) reflects colimits (Proposition 12).

To show that \( C_a \) has all \( \omega \)-colimits, let \( D : \omega \to C_a \) be an arbitrary \( \omega \)-diagram of \( C_a \) with \( D = \left( (D_0, \circ_0) \xrightarrow{d_0} (D_1, \circ_1) \xrightarrow{d_1} \cdots \right) \). Let \( \mu = (M, \mu_i : D_i \to M) \) be the colimiting cocone of \( UD \) in \( C \). Using the discarding maps \( \circ_i \), we can now form a cocone \( \circ = (I, \circ_i : D_i \to I) \) of \( UD \) in \( C \). Let \( \circ_M : M \to I \) be the unique cocone morphism from \( \mu \) to \( \circ \) induced by the colimit. It is now easy to see that we have a cocone \( \tau = ((M, \circ_M), \mu_i : (D_i, \circ_i) \to (M, \circ_M)) \) of \( D \) in \( C_a \). Clearly, \( \mu = U\tau \) and since \( U \) reflects colimits (Proposition 12), it follows that \( \tau \) is the colimiting cocone of \( D \) in \( C_a \). Therefore, \( C_a \) has \( \omega \)-colimits and by construction of the colimits, we see that \( U \) preserves (and reflects) them. \( \square \)

Next, we show that the functor \( U \) may be used to reflect \( \omega \)-cocontinuity of functors on \( C \) to functors on \( C_a \).

**Theorem 17.** Let \( H : C^n_a \to C_a \) be a functor and \( T : C^n \to C \) an \( \omega \)-cocontinuous functor, such that the diagram:

\[
\begin{array}{ccc}
C^n_a & \xrightarrow{U^n} & C^n \\
H \downarrow & & \downarrow T \\
C_a & \xrightarrow{U} & C
\end{array}
\]

commutes. Then, \( H \) is also \( \omega \)-cocontinuous.

**Proof.** Let \( D : \omega \to C^n_a \) be an arbitrary \( \omega \)-diagram in \( C^n_a \) and let \( \mu \) be its colimiting cocone. Since \( U \) preserves \( \omega \)-colimits (Proposition 10), it follows that \( U^n \mu \) is a colimiting cocone of \( U \times_n D \) in \( C^n \). By assumption \( T \) is \( \omega \)-cocontinuous, so \( TU^n \mu \) is a colimiting cocone of \( TU \times_n D \) in \( C \). By commutativity of the above diagram, it follows \( UH \mu \) is a colimiting cocone of \( UHD \) in \( C \). But \( U \) reflects colimits, so this means that \( H \mu \) is a colimiting cocone of \( HD \), as required. \( \square \)

Therefore, in the situation of the above theorem, both functors \( H \) and \( T \) have parameterised initial algebras by Theorem 2. This brings us to our next theorem.
Theorem 18. Let $H$ and $T$ be $\omega$-cocontinuous functors, such that the diagram

\[
\begin{array}{ccc}
C^{n+1}_a & \xrightarrow{\times(n+1)} & C^{n+1}_a \\
\downarrow H & & \downarrow T \\
C_a & \xrightarrow{U} & C
\end{array}
\]

commutes. Let $(T^\dagger, \phi)$ and $(H^\dagger, \psi)$ be their parameterised initial algebras. Then:

1. The following diagram:

\[
\begin{array}{ccc}
C^n_a & \xrightarrow{\times n} & C^n \\
\downarrow H^\dagger & & \downarrow T^\dagger \\
C_a & \xrightarrow{U} & C
\end{array}
\]

commutes.

2. The following (2-categorical) diagram:

\[
\begin{array}{ccc}
C^n_a & \xrightarrow{\times n} & C^n \\
\Downarrow H \circ \langle \text{Id}, H^\dagger \rangle & & \Downarrow T \circ \langle \text{Id}, T^\dagger \rangle \\
C_a & \xrightarrow{U} & C
\end{array}
\]

\[
\begin{array}{ccc}
\psi & \Downarrow H^\dagger & \phi \\
\Downarrow T \circ \langle \text{Id}, T^\dagger \rangle & & \Downarrow
\end{array}
\]

commutes.

Proof. The first statement follows by [10, Corollary 4.21] and the second statement follows by [10, Corollary 4.27].

Remark 19. The above theorem shows that the parameterised initial algebras of $H$ and $T$ respect the forgetful functor $U$ and are therefore constructed in the same way.

Remark 20. If one is not interested in interpreting inductive data types defined by mutual induction, then there is no need to form parameterised initial algebras, but merely initial algebras. In that case, the assumption that $C$ has all $\omega$-colimits may be relaxed and one can assume that $C$ has colimits of the initial sequences of the relevant functors. Then, most of the results presented here can be simplified in a straightforward manner to handle this case.
6 Categorical Model

In this section we formulate our categorical model which we use to interpret \textit{Aff}.

\textbf{Notation 21.} We write DCPO (DCPO\(_{\bot!}\)) for the category of (pointed) dcpo’s and (strict) Scott-continuous maps between them.

\textbf{Definition 22.} A categorical model of \textit{Aff} is given by the following data:
1. A symmetric monoidal category \((\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)\).
2. An initial object \(\emptyset \in \text{Ob}(\mathcal{C})\) and binary coproducts \((A + B, \text{left}_{A,B}, \text{right}_{A,B})\).
3. The tensor product \(\otimes\) distributes over \(+\).
4. For each atomic type \(A \in \mathcal{A}\), an object \(A \in \text{Ob}(\mathcal{C})\) together with a discarding map \(\diamond_A : A \to I\).
5. The category \(\mathcal{C}\) has all \(\omega\)-colimits and \(\otimes\) is an \(\omega\)-cocontinuous functor.
6. The category \(\mathcal{C}\) is DCPO\(_{\bot!}\)-enriched with least morphisms denoted \(\bot_{A,B}\) and such that the symmetric monoidal structure and the coproduct structure are both DCPO-enriched.

This data suffices to interpret the language in the following way:
1. To interpret pair types.
2. To interpret sum types.
3. Used in the interpretation of \texttt{while} loops.
4. Necessary for the affine interpretation of the language.
5. To interpret inductive data types by forming parameterised initial algebras.
6. Used in the interpretation of \texttt{while} loops.

\textbf{Assumption 23.} Henceforth, \(\mathcal{C}\) refers to an arbitrary, but fixed, model of \textit{Aff} and \(\mathcal{C}_a := \mathcal{C}/I\) refers to the corresponding slice category with the tensor unit \(I\).

By using results from Section 5, we can now easily establish some important properties of the category \(\mathcal{C}_a\). By Proposition 13 it follows \(\mathcal{C}_a\) has a symmetric monoidal structure with tensor product \(\otimes_a\) and by Proposition 14 it follows \(\mathcal{C}_a\) has finite coproducts with coproduct functor \(+_a\). We also know \(\mathcal{C}_a\) has \(\omega\)-colimits by Proposition 16. Finally, the next proposition is crucial for the construction of discarding maps for inductive data types.

\textbf{Proposition 24.} The functors \(\otimes_a : \mathcal{C}_a \times \mathcal{C}_a\) and \(+_a : \mathcal{C}_a \times \mathcal{C}_a \to \mathcal{C}_a\) are both \(\omega\)-cocontinuous.

\textit{Proof.} In the previous section we showed \(\bigcirc \circ (U \times U) = U \circ \bigcirc_a : \mathcal{C}_a \times \mathcal{C}_a \to \mathcal{C}\), for \(\bigcirc \in \{\otimes, +\}\). Then, by Theorem 17 it follows \(\bigcirc\) is also \(\omega\)-cocontinuous. \(\Box\)

Therefore, by Theorem 2 we see that both categories \(\mathcal{C}\) and \(\mathcal{C}_a\) have sufficient structure to form parameterised initial algebras for all functors composed out of tensors, coproducts and constants. The category \(\mathcal{C}_a\) has the additional benefit that its parameterised initial algebras also come equipped with discarding maps.
6.1 Concrete models

In this subsection we consider some concrete models of $\text{Aff}$.

Example 25. The terminal category $1$ is a (completely degenerate) $\text{Aff}$ model.

Next, we consider some non-degenerate models.

Example 26. The category $\text{DCPO}_{\bot}$ is an $\text{Aff}$ model.

However, in this model every object has a canonical comonoid structure, so it is not a truly representative model for an affine type system like ours. In the next example we describe a more representative model which has been studied in the context of circuit description languages and quantum programming.

Example 27. Let $M$ be a small $\text{DCPO}_{\bot}$-symmetric monoidal category and let $\hat{M} = [M^{op}, \text{DCPO}_{\bot}]$ be the indicated $\text{DCPO}_{\bot}$-functor category. Then $\hat{M}$ is an $\text{Aff}$ model when equipped with the Day convolution monoidal structure $[2]$.

By making suitable choices for $M$, the category $\hat{M}$ becomes a model of Proto-Quipper-M $[13]$ and ECLNL $[9]$, which are programming languages for string diagrams that have also been studied in the context of quantum programming.

Next, we discuss how fragments of the language may be interpreted in categories of $W^*$-algebras $[17]$, which are used to study quantum computing.

Example 28. Let $W^*_{\text{NMIU}}$ be the category of $W^*$-algebras and normal unital $*$-homomorphisms between them. Let $V := (W^*_{\text{NMIU}})^{op}$ be its opposite category. Then $V$ is an $\text{Aff}$ model without recursion $[5]$, i.e., one can interpret all $\text{Aff}$ constructs except for while loops within $V$, because $V$ is not $\text{DCPO}_{\bot}$-enriched.

Example 29. Let $W^*_{\text{NCPSU}}$ be the category of $W^*$-algebras and normal completely-positive subunital maps between them. Let $D := (W^*_{\text{NCPSU}})^{op}$ be its opposite category. Then $D$ is an $\text{Aff}$ model which supports simple non-nested type induction, i.e., one can interpret all $\text{Aff}$ constructs within $D$ using the methods described in this paper, provided that inductive data types contain at most one free type variable.

Remark 30. In fact, it is possible to interpret all of QPL (and therefore also $\text{Aff}$ which is a fragment of QPL) by using an adjunction between $V$ and $D$, as was shown in $[12]$. However, this requires considering the specifics of this particular model, which has not been axiomatised yet, and separating the interpretation of types and values (in $V$) from the interpretation of terms (in $D$).

7 Denotational Semantics of $\text{Aff}$

In this section we present the denotational semantics of $\text{Aff}$. First, we show how types are interpreted in §7.1. Since our type system is affine, we construct discarding maps for all types in §7.2. Folding and unfolding of inductive types are shown to be discardable operations in §7.3. The interpretations of terms and configurations are defined in §7.4 and §7.5. Finally, we prove soundness and adequacy in §7.6.
7.1 Interpretation of types

The (standard) interpretation of a type $\Theta \vdash A$ is a functor $\llbracket \Theta \vdash A \rrbracket : C^{(\Theta)} \to C$, defined in Figure 4 (left), where $K_X$ indicates the constant $X$-functor. We begin by showing that this assignment is well-defined, i.e., we have to show that the required parameterised initial algebras exist.

**Proposition 31.** $\llbracket \Theta \vdash A \rrbracket$ is a well-defined $\omega$-cocontinuous functor.

**Proof.** Projection functors and constant functors are obviously $\omega$-cocontinuous. The coproduct functor is $\omega$-cocontinuous, because it is given by a colimiting construction. The tensor product $\otimes$ is $\omega$-cocontinuous by assumption. Also, $\omega$-cocontinuous functors are closed under composition and pairing [7]. By Theorem 2 $\llbracket \Theta, X \vdash A \rrbracket^\dagger$ is well-defined and also an $\omega$-cocontinuous functor.

The semantics of terms is defined on closed types, so for brevity we introduce the following notation.

**Notation 32.** For any closed type $\vdash A$, let $[\vdash A] := [\vdash A](*) \in \text{Ob}(C)$, where $*$ indicates the only object of the terminal category $1$.

7.2 Affine Structure of Types

In this subsection we describe the affine structure of our types by constructing an appropriate discarding map for every type. This is achieved by using the results we established in §3 and by providing an **affine interpretation of types** as functors on the slice category $C_a = C/I$. The affine interpretation is related to the standard one via the forgetful functor which results in the construction of the required discarding maps.

The affine interpretation of a type $\Theta \vdash A$ is a functor $\llbracket \Theta \vdash A \rrbracket : C^{(\Theta)}_a \to C_a$, defined in Figure 4 (right).

**Proposition 33.** $\llbracket \Theta \vdash A \rrbracket$ is a well-defined $\omega$-cocontinuous functor.

**Proof.** The tensor product $\otimes_a$ and coproduct functors $+_a$ are $\omega$-cocontinuous by Proposition 24. Using the same arguments as in Proposition 31 we finish the proof. □
**Notation 34.** For any closed type \( \cdot \vdash A \), let \( \llbracket A \rrbracket := \llbracket \cdot \vdash A \rrbracket(*) \in \text{Ob}(C_a) \).  

We proceed by describing the relationship between the standard and affine interpretation of types.

**Theorem 35.** For any type \( \Theta \vdash A \), the following diagram

\[
\begin{array}{ccc}
C_a^{(\Theta)} & \xrightarrow{U \times |\Theta|} & C^{(\Theta)} \\
\llbracket \Theta \vdash A \rrbracket \downarrow & & \downarrow \llbracket \Theta \vdash A \rrbracket \\
C_a & \xrightarrow{U} & C
\end{array}
\]

commutes. Therefore, for any closed type \( \cdot \vdash A \), we have \( \llbracket A \rrbracket = U \llbracket A \rrbracket \).

**Proof.** By induction on \( \Theta \vdash A \) using the established results from §5. \( \square \)

This theorem shows that for any closed type \( A \), we have \( \llbracket A \rrbracket = ([A], \cdot \vdash A) \), where the discarding map \( \cdot \vdash A) : \llbracket A \rrbracket \to I \) is constructed by the affine type interpretation in Figure 4 (right). We will later see (Theorem 39) that the interpretations of our values are discardable morphisms with respect to this choice of discarding maps.

### 7.3 Folding and Unfolding of Inductive Datatypes

The purpose of this subsection is to define folding and unfolding of inductive data types (which we need to define the term semantics) and also to demonstrate that folding/unfolding is a discardable isomorphism with respect to the affine structure of our types.

**Lemma 36 (Type Substitution).** Let \( \Theta, X \vdash A \) and \( \Theta \vdash B \) be types. Then:

1. \( \llbracket A[B/X] \rrbracket = \llbracket \Theta, X \vdash A \rrbracket \circ \langle Id, \llbracket \Theta \vdash B \rrbracket \rangle \).
2. \( \llbracket A[B/X] \rrbracket = \llbracket \Theta, X \vdash A \rrbracket \circ \langle Id, \llbracket \Theta \vdash B \rrbracket \rangle \).

**Proof.** Straightforward induction, essentially the same as [10] Lemma 6.5. \( \square \)

**Definition 37.** For any closed type \( \cdot \vdash \mu X.A \), we define two isomorphisms:

\[
\text{fold}_{\mu X.A} : \llbracket A[\mu X.A/X] \rrbracket = \llbracket X \vdash A \rrbracket \llbracket \mu X.A \rrbracket \cong \llbracket \mu X.A \rrbracket : \text{unfold}_{\mu X.A}
\]

Since type substitution holds up to equality, it follows that folding/unfolding of inductive data types is determined entirely by the initial algebra structure of the corresponding endofunctors. Finally, we show that folding/unfolding of an inductive data type is the same isomorphism for both the standard and affine type interpretations.
In this subsection we explain how to interpret the terms of Aff.

A variable context \( \Gamma = x_1 : A_1, \ldots, x_n : A_n \) is interpreted as the object \( \llbracket \Gamma \rrbracket := \llbracket A_1 \rrbracket \otimes \cdots \otimes \llbracket A_n \rrbracket \in \text{Ob}(C) \). A term judgement \( \llbracket \Gamma \rrbracket \vdash \llbracket M \rrbracket \rightarrow \llbracket \Sigma \rrbracket \) is interpreted as a morphism \( \llbracket \Gamma \rrbracket \rightarrow \llbracket \Sigma \rrbracket \) of \( C \) which is defined in Figure 5. For brevity, we will simply write \( \llbracket M \rrbracket := \llbracket \Pi \vdash \llbracket \Gamma \rrbracket \rightarrow \llbracket \Sigma \rrbracket \rrbracket \) whenever the contexts are clear. Next, we clarify some of the notation used in Figure 5. The map \( \otimes \llbracket A \rrbracket \) is defined in \( \S 7.2 \) as already explained. In order to interpret while loops, we use a

\begin{align*}
\llbracket \Gamma \rrbracket & \vdash \text{new unit } u \llbracket \Gamma, u : I \rrbracket := (\llbracket \Gamma \rrbracket \xrightarrow{\text{new}} \llbracket I \rrbracket \otimes \llbracket I \rrbracket) \\
\llbracket \Gamma \rrbracket & \vdash \text{discard } x \llbracket \Gamma \rrbracket := ((\llbracket \Gamma \rrbracket \otimes \llbracket A \rrbracket) \xrightarrow{id \otimes \text{left}_{A,B}} \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket)) \\
\llbracket \Gamma \rrbracket & \vdash \text{skip } \llbracket \Gamma \rrbracket := \left( \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \xrightarrow{id \otimes \text{left}_{A,B}} \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \right) \\
\llbracket \Gamma, y : A + B \rrbracket & \vdash \text{left}_{A,B} x \llbracket \Gamma, y : A + B \rrbracket := \left( \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \xrightarrow{id \otimes \text{left}_{A,B}} \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \right) \\
\llbracket \Gamma, y : A + B \rrbracket & \vdash \text{right}_{A,B} x \llbracket \Gamma, y : A + B \rrbracket := \left( \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \xrightarrow{id \otimes \text{right}_{A,B}} \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \right) \\
\llbracket \Gamma, y : A + B \rrbracket & \vdash \text{case } y \llbracket \Gamma, y : A + B \rrbracket := \left( \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \xrightarrow{id \otimes \text{left}_{A,B}} \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \right) \\
\llbracket \Gamma, x : A, y : A + B \rrbracket & \vdash \text{fold } x \llbracket \Gamma, x : A, y : A + B \rrbracket := \left( \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \xrightarrow{id \otimes \text{fold}_{A,B}} \llbracket \Gamma \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket B \rrbracket \right) \\
\llbracket \Gamma, x : A, y : A + B \rrbracket & \vdash \text{unfold } x \llbracket \Gamma, x : A, y : A + B \rrbracket := \left( \llbracket \Gamma \rrbracket \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \xrightarrow{id \otimes \text{unfold}_{A,B}} \llbracket \Gamma \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket B \rrbracket \right)
\end{align*}

Fig. 5: Interpretation of Aff terms.

**Theorem 38.** Given a closed type \( \llbracket \Gamma \rrbracket \vdash \mu X.A \), then the following diagram

\[
\begin{array}{ccc}
\llbracket A[\mu X.A/X] \rrbracket & \xrightarrow{\text{fold}} & \llbracket \mu X.A \rrbracket \\
\llbracket A[\mu X.A/X] \rrbracket & \xrightarrow{\text{fold}} & \llbracket \mu X.A \rrbracket
\end{array}
\]

commutes.

**Proof.** This follows immediately by Theorem 15 (2). \( \square \)

Therefore folding/unfolding of types is a discardable isomorphism.

7.4 Interpretation of terms

In this subsection we explain how to interpret the terms of Aff.
\[ [\vdash \ast : I] := \text{id}_I \]
\[ [Q \vdash \text{left}_{A,B}v : A + B] := \text{left} \circ [v] \]
\[ [Q \vdash \text{right}_{A,B}v : A + B] := \text{right} \circ [v] \]
\[ [Q_1, Q_2 \vdash (v, w) : A \otimes B] := ([v] \otimes [w]) \circ \lambda^1 \]
\[ [Q \vdash \text{fold}_{\mu X.A}v : \mu X.A] := \text{fold} \circ [v] \]

Fig. 6: Interpretation of **Aff** values.

Scott-continuous endofunction \( W_f \), which is defined as follows. For a morphism \( f : A \otimes \text{bit} \to A \otimes \text{bit} \), we set:

\[
W_f : \mathbf{C}(A \otimes \text{bit}, A \otimes \text{bit}) \to \mathbf{C}(A \otimes \text{bit}, A \otimes \text{bit})
\]

\[
W_f(g) = [\text{id} \otimes \text{left}_{I,I}, g \circ f \circ (\text{id} \otimes \text{right}_{I,I})] \circ d_{A,I,I},
\]

where \( d_{A,I,I} : A \otimes (I + I) \to (A \otimes I) + (A \otimes I) \) is the isomorphism which is induced by the distributivity of \( \otimes \) over \( + \) (see Definition 22). Finally, for a pointed dcpo \( D \) and Scott-continuous endofunction \( h : D \to D \), the least fixpoint of \( h \) is given by \( \text{lfp}(h) := \bigvee_{i=0}^{\infty} h^i(\bot) \), where \( \bot \) is the least element of \( D \).

### 7.5 Interpretation of configurations

Before we explain how to interpret configurations, we have to show how to interpret values.

**Interpretation of values.** The interpretation of a value \( \vdash v : A \) is a morphism \([\vdash v : A] : I \to [A]\), and we shall simply write \([v]\) if its type is clear from context. The interpretation is defined in Figure 6. In order to prove soundness of our affine type system, we have to show every value is discardable.

**Theorem 39.** For every value \( \vdash v : A \), we have: \( \circ [A] \circ [\vdash v : A] = \text{id}_I \).

**Proof.** By construction, \( \circ [A] \) enjoys all of the properties established in §5. The proof proceeds by induction on the derivation of \( \vdash v : A \). The base case is trivial. Discardable morphisms are closed under composition, because \( \mathbf{C}_a \) is a category. Moreover, discardable maps are closed under tensor products (Proposition 13). Using the induction hypothesis, it suffices to show that the coproduct injections and folding are discardable maps. But this follows by Proposition 14 and Theorem 38 respectively. \( \square \)

Given a variable context \( \Gamma = x_1 : A_1, \ldots, x_n : A_n \), then a value context \( \Gamma \vdash V \) is interpreted by the morphism:

\[
[\Gamma \vdash V] = \left( I \xrightarrow{\cong} I^{\otimes n} \xrightarrow{[v_1] \otimes \cdots \otimes [v_n]} [\Gamma] \right),
\]

where \( V = \{ x_1 = v_1, \ldots, x_n = v_n \} \) and we abbreviate this by writing \([V]\). Note that \([V]\) is also discardable due to Theorem 39.
Interpretation of configurations. A configuration \( \Gamma; \Sigma \vdash (M \mid V) \) is interpreted as the morphism
\[
\llbracket \Gamma; \Sigma \vdash (M \mid V) \rrbracket = \left( I \xrightarrow{\llbracket \Gamma+V \rrbracket} \llbracket \Gamma \rrbracket \xrightarrow{\llbracket \Gamma \vdash (\langle \Gamma \rangle M \langle \Sigma \rangle) \rrbracket} \llbracket \Sigma \rrbracket \right).
\]

We write \( \llbracket (M \mid V) \rrbracket \) for this morphism whenever the contexts are clear.

7.6 Soundness and Computational Adequacy

Soundness is the statement that the denotational semantics is invariant under program execution.

**Theorem 40 (Soundness).** If \( \mathcal{C} \rightsquigarrow \mathcal{D} \), then \( \llbracket \mathcal{C} \rrbracket = \llbracket \mathcal{D} \rrbracket \).

**Proof.** Straightforward induction. \(\Box\)

We conclude our technical contributions by proving a computational adequacy result. Towards this end, we have to assume that our categorical model is not degenerate.

**Definition 41.** A computationally adequate \( \texttt{Aff} \) model is an \( \texttt{Aff} \) model, where \( \text{id}_I \neq \bot \).

A program configuration \( \mathcal{C} \) is said to terminate, denoted \( \mathcal{C} \Downarrow \), if there exists a terminal configuration \( \mathcal{T} \), such that \( \mathcal{C} \rightsquigarrow^* \mathcal{T} \), where \( \rightsquigarrow^* \) is the reflexive and transitive closure of \( \rightsquigarrow \).

**Theorem 42 (Adequacy).** Let \( \vdash \langle \cdot \rangle M \langle \Sigma \rangle \) be a closed term. Then:
\[
\llbracket M \rrbracket \neq \bot \iff (M \mid \cdot) \Downarrow.
\]

**Proof.** By simplifying the adequacy proof strategy of QPL \cite{12} in the obvious way. \(\Box\)

8 Future Work

As part of future work it will be interesting to see whether these methods can be adapted to also work with coinductive data types and/or with recursive data types where function types \( (\rightarrow) \) become admissible constructs within the type recursion schemes. This is certainly a more challenging problem which would probably require us to assume additional structure within the model, such as a limit-colimit coincidence \cite{16}, so that we may deal with the contravariance induced by function types. It is also likely that we would have to modify the slice construction to accommodate the addition of limits.
9 Conclusion and Related Work

Since the introduction of Linear Logic [4], there has been a massive amount of research into finding suitable models for (fragments) of Linear Logic (see [11] for an excellent overview). However, there has been less research into models of affine logics and affine type systems. The principle difference between linear and affine logic is that weakening is restricted in the former, but allowed in the latter, so affine models have to contain additional discarding maps. Most models of affine type systems (that I am aware of) use some specific properties of the model, such as finding a suitable (sub)category where the tensor unit $I$ is a terminal object, in order to construct the required discarding maps (e.g. [6,1,12,15]). This means, the solution is provided directly by the model. In this paper, we have taken a different approach, because we present a semantic solution to this problem. The only assumption that we have made for our model is that the interpretations of the atomic types are equipped with suitable discarding maps and we then show how to construct all other required discarding maps by considering an additional and non-standard interpretation of types within a slice category. Overall, the "model" solution is certainly simpler and more concise compared to the "semantic" solution presented here. On the other hand, our solution in this paper is very general and can in principle be applied to models where the required discarding maps are unknown a priori.

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Omitted Proofs from Section

A.1 Proof of Proposition

Proof. Let $D : J \to C_a$ be a diagram and let $\mu = ((M, \diamond_M), \mu_i : (D_i, \diamond_i) \to (M, \diamond_M))$ be a cocone of $D$ in $C_a$, such that $U\mu = (M, \mu_i : D_i \to M)$ is a colimiting cocone of $UD$ in $C$. Let $\chi = ((X, \diamond_X), \chi_i : (D_i, \diamond_i) \to (X, \diamond_X))$ be another cocone of $D$ in $C_a$. We will show there exists a unique cocone morphism $f : \mu \to \chi$ in $C_a$.

By the universal property of the colimit in $C$, we see there exists a unique cocone morphism $f : U\mu \to U\chi$ of $UD$ in $C$, i.e. $f : M \to X$ is the unique morphism of $C$, such that

$$f \circ \mu_i = \chi_i \quad \forall i \in \text{Ob}(J) \quad (1)$$

Since $D : J \to C_a$ is a diagram, then for any $f : i \to j$ in $J$, we have $D_f : (D_i, \diamond_i) \to (D_j, \diamond_j)$ and thus $\diamond_j \circ D_f = \diamond_i$. This means we have a cocone $\diamond = (I, \diamond_i : D_i \to I)$ of $UD$ in $C$. By the universal property of the colimit, there exists a unique cocone morphism $g : U\mu \to \diamond$, i.e. $g : M \to I$ is the unique morphism of $C$, such that

$$g \circ \mu_i = \diamond_i \quad \forall i \in \text{Ob}(J) \quad (2)$$

But $\mu_i : (D_i, \diamond_i) \to (M, \diamond_M)$ in $C_a$, therefore $\diamond_M \circ \mu_i = \diamond_i$ and thus $g = \diamond_M$.

However:

$$(\diamond_X \circ f) \circ \mu_i = \diamond_X \circ (f \circ \mu_i)$$
$$= \diamond_X \circ \chi_i \quad (\chi_i : (D_i, \diamond_i) \to (X, \diamond_X) \text{ in } C_a) \quad (3)$$

And therefore by (2) it follows $\diamond_X \circ f = g = \diamond_M$. But this now shows that $f : (M, \diamond_M) \to (X, \diamond_X)$ is a morphism of $C_a$. Obviously, $f : \mu \to \chi$ is also a cocone morphism of $D$, because composition in $C_a$ coincides with composition in $C$.

Finally, to show $f$ is unique, assume that $h : \mu \to \chi$ is a cocone morphism of $D$. But then $h : U\mu \to U\chi$ is a cocone morphism of $UD$ in $C$ and therefore $f = h$. 

