Quantum Jacobi-Trudi Formula and $E_8$ Structure in the Ising Model in a Field

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Abstract

We investigate a 1D quantum system associated with the Ising model in a field (the dilute $A_3$ model) by the recently developed quantum transfer matrix (QTM) approach. A closed set of functional relations is found among variants of fusion QTMs which are characterized by skew Young tableaux. These relations are proved by using a quantum analogue of Jacobi-Trudi formula, together with special features at "root of unity". The numerical analysis on their eigenvalues shows a remarkable coincidence with exponents characteristic to $E_8$. From these findings, we have successfully recovered the $E_8$ Thermodynamic Bethe ansatz equation by Bazhanov et al, however, without specific choice of strings solutions.

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1 Introduction

Perhaps the most striking prediction from the deformation of conformal field theory to statistical mechanics is the hidden $E_8$ universality structure in the Ising model in a field \[ l \leq 3 \]. This triggers active researches in both field theory \cite{3-8} and statistical mechanics \cite{3-13}. In this report, we shall attack the problem in view of a solvable lattice model.

In \cite{9}, a $L-$ state RSOS model is proposed of which Boltzmann weights are parameterized by elliptic functions which satisfy the Yang-Baxter relations \cite{17}-\cite{19}. The model, hereafter referred to as the dilute $A_L$ model, has the outstanding property. For odd $L$’s, the up-down symmetry of the local state is broken for nonzero elliptic nome. One may thus identify the magnetic field with elliptic nome. Actually, this identification leads to the desired critical exponent $\delta = 15$ of the Ising model for $L = 3$ \cite{9,10,11}. There are further supports for the conjecture that the dilute $A_3$ model belongs to the same universality class with the Ising model in a field, say, the central charge, critical exponents for bulk and a system with boundaries and so on \cite{9}-\cite{14}.

From these two observations, one may naturally expect the $E_8$ structure behind the dilute $A_3$ model. As an evidence, we refer to the Thermodynamic Bethe Ansatz(TBA) approach in \cite{16}. These authors deal with the one dimensional counterpart to the dilute $A_L$ model, and discuss the finite temperature problem. They claim the dominating nine patterns (strings) of Bethe ansatz roots for $L = 3$ in the thermodynamics limit, and show that the string hypothesis leads to the $E_8$ structure for the TBA equation. This observation originates from the numerical investigation near the zero elliptic nome. These strings are, however, quite involved and could lead to numerical difficulty with increase in the elliptic nome \cite{20,21}. It may be preferable to have the derivation of $E_8$ structure free from specific choice of strings.

In this report, we analyze the same problem through a different route based on the quantum transfer matrix (QTM) \cite{23,24}. The QTM approach has been developed recently as an alternative method in studies of 1D quantum systems at finite temperatures \cite{23}-\cite{40}. The formulation yields the expression of the free energy by the largest eigenvalue of the object called QTM. The method has some advantages over the standard approach \cite{11}-\cite{13} based on the string hypothesis. For example, it yields a transparent derivation of the correlation lengths at finite temperatures \cite{10,44,45}. Once combined with the Yang-Baxter integrability structure, the evaluation of physical quantities at $T > 0$ reduces to the study on analyticity of some auxiliary functions \cite{24,31,34}-\cite{40}. The crucial point in this approach lies in the choice of the auxiliary function, which is not necessary unique. One may adopt convenient functions having nice analytic properties. Here we choose them utilizing a family of commuting transfer matrices which includes QTM as the most fundamental one. We call them fusion QTMs. The functional relations among them allow the explicit evaluation of their eigenvalues. This strategy has been successfully applied to several models \cite{24,38,40}. There fusion QTMs come into play which corresponds to Yangian modules specified by only Young tableaux of rectangle shape. For the present model, this is no longer true. They constitute closed functional relations, however, lack nice analytic property. The absence of the property prevents the derivation of integral.
equations (TBA). From trials and errors, we find it necessary to introduce variants of QTMs corresponding to "skew shape" Young tableaux. A quantum analogue of Jacobi-Trudi formula \([16, 18, 19, 20]\) as well as special features at "root of unity" play fundamental roles in the derivation of the functional relations. Numerical investigations on analyticity of "modified" QTMs show a remarkable coincidence of imaginary parts of locations of zeros with the exponents characterizing massive particles in \(E_8\) scattering theory \([21]\). This validates a modest assumption on the existence of a strip in the complex plane where "modified" QTMs are free from zeros and singularities. Based on these, we successfully recover the TBA equation from the functional relations, namely, totally independent of string hypothesis.

This paper is organized as follows. In the next section, we give a brief review on the dilute \(A_L\) models. A minimal guide to the QTM approach is sketched in section 3. The \(sl_3\) fusion structure found in \([52]\) will be explained in section 4. Based on these preparations, we introduce fusion QTMs parameterized by skew Young tableaux and their variants in section 5. The latter are found to satisfy closed functional relations. Section 6 is devoted to the observation of the analyticity of these functions. Assuming Conjecture 1 and 2, the \(E_8\) TBA equation is derived from the functional relations. We conclude the paper with brief summary and discussion in section 7.

## 2 dilute \(A_L\) model

The dilute \(A_L\) model is proposed in \([4]\) as an elliptic extension of the Izergin-Korepin model \([54]\). The model is of the restricted SOS type with local variables \(\in \{1, 2, \ldots, L\}\). The variables \(\{a, b\}\) on neighboring sites should satisfy adjacency condition, \(|a - b| \leq 1\). The solvable weights are given by:

\[
\begin{align*}
\frac{a}{a} & = \frac{a}{a} a a a = \frac{\theta_1(6 - u) \theta_1(3 + u)}{\theta_1(6) \theta_1(3)} - \frac{\theta_1(u) \theta_1(3 - u)}{\theta_1(6) \theta_1(3)} \\
\frac{a \pm 1}{a} & = \frac{a \pm 1}{a} a a = \frac{\theta_1(3 - u) \theta_1(\pm 2a + 1 - u)}{\theta_1(3) \theta_1(\pm 2a + 1)}, \\
\frac{a}{a} & = \frac{a}{a} a a \pm 1 = \frac{\left(\frac{S_{a+1}}{S_a} \theta_4(2a - 5)ight) \theta_1(2) \theta_1(3)}{\theta_1(2) \theta_1(3)}, \\
\frac{a}{a} & = \frac{a}{a} a a = \frac{\theta_1(2 - u) \theta_1(3 - u)}{\theta_1(2) \theta_1(3)}, \\
\frac{a \pm 1}{a} & = \frac{a \pm 1}{a} a a = \left(\frac{S_{a-1} S_{a+1}}{S_a^2}\right) \frac{1/2 \theta_1(u) \theta_1(\pm 2a - 1 - u)}{\theta_1(2) \theta_1(3)}, \\
\frac{a}{a} & = \frac{a}{a} a a = \frac{\theta_1(3 - u) \theta_1(\pm 2a + 2 + u)}{\theta_1(3) \theta_1(\pm 4a + 2)}.
\end{align*}
\]
\[\begin{align*}
\frac{S_{a+1} \theta_1(u) \theta_1(\pm 4a - 1 + u)}{S_a \theta_1(3) \theta_1(\pm 4a + 2)}, \text{ for } \theta_1(\pm 4a + 2) \neq 0,
\theta_1(3 + u) \theta_1(\pm 4a - 4 + u) \theta_1(3) \theta_1(\pm 4a - 4)
\end{align*}\]

\[\left(\frac{S_{a+1} \theta_1(4)}{S_a \theta_1(2)} - \frac{\theta_4(\pm 2a - 5)}{\theta_4(\pm 2a + 1)} \frac{\theta_1(u) \theta_1(\pm 4a - 1 + u)}{\theta_1(3) \theta_1(\pm 4a - 4)}\right), \text{ otherwise . (1)}\]

Here \(\theta_{1,4}(x) = \vartheta_{1,4}(\lambda x, \tau)\),
\[\begin{align*}
\vartheta_1(x, \tau) &= 2q^{1/4} \sin x \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2x + q^{4n})(1 - q^{2n}),
\vartheta_4(x, \tau) &= \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2x + q^{4n-2})(1 - q^{2n}),
\end{align*}\]

and \(q = \exp(-\tau)\). \(\lambda\) is a parameter of the model specified below and \(S_a\) denotes
\[S_a = (-1)^a \frac{\theta_1(4a)}{\theta_1(2a)}.\]

The model exhibits four different physical regimes depending on parameters,

- **regime 1.** \(0 < u < 3\), \(\lambda = \frac{\pi L}{4(L+1)}\), \(L \geq 2\)
- **regime 2.** \(0 < u < 3\), \(\lambda = \frac{\pi(L+2)}{4(L+1)}\), \(L \geq 3\)
- **regime 3.** \(3 - \frac{\pi}{2} < u < 0\), \(\lambda = \frac{\pi(L+2)}{4(L+1)}\), \(L \geq 3\)
- **regime 4.** \(3 - \frac{\pi}{2} < u < 0\), \(\lambda = \frac{\pi L}{4(L+1)}\), \(L \geq 2\)

We are interested in regimes 2 and 3. The central charges and the scaling dimensions of the leading perturbation have been evaluated in [9, 10, 11]. Specializing to \(L = 3\), they read

- **regime 2.** \(c = \frac{1}{2}, \Delta = \frac{1}{16}\)
- **regime 3.** \(c = \frac{6}{5}, \Delta = \frac{15}{16}\).

Obviously, in connection with the Zamolodchikov’s argument, regime 2 attracts more attention. We will see, however, that both of them can be treated on a same footing.

The one particle excitations are examined in [15]. Eight particles are identified and their masses are summarized by a single formula,
\[m_j = \sum_a \sin(\frac{a \pi}{30}),\]
where

| $j$   | set of allowed $a$’s |
|-------|---------------------|
| 1(1)  | $\{1,11\}$         |
| 2(3)  | $\{2,10,12\}$      |
| 3(5)  | $\{3,9,11,13\}$    |
| 4(7)  | $\{4,8,10,12,14\}$ |
| 5(8)  | $\{5,7,9,11,13,15\}$ |
| 6(6)  | $\{6,8,12,14\}$    |
| 7(2)  | $\{7,13\}$         |
| 8(4)  | $\{6,10,14\}$      |

Table 1.

A number $k$ in the bracket means that it corresponds to the $k$-th light particle. These exponents appear in many contexts, diagonal scattering matrices, Weyl reflections with respect to Coxeter element and so on [51]. They will re-appear in a novel context later.

### 3 Quantum Transfer Matrix

The row to row transfer matrix $T_{RTR}(u)$ is defined by

$$(T_{RTR}(u))_{\{b\}}^{\{a\}} = \prod_{j=1}^{M} b_{j} \underbrace{a_{j}}_{a_{j+1}} b_{j+1}.$$ 

From the commutativity of $T_{RTR}(u)$ with different spectral parameters, it is natural to define the Hamiltonian of associated 1D quantum chain by

$$\mathcal{H}_{e} = \epsilon \frac{\partial}{\partial u} \ln T_{RTR}(u)|_{u=0}$$

as in [16]. Here $\epsilon = -1$, (1) labels regimes 2 (3). In order to evaluate the free energy, we adopt the method of the quantum transfer matrix which is defined in the following "staggered" manner,

$$(T_{QTM}(u,v))_{\{b\}}^{\{a\}} = \prod_{j=1}^{N/2} \underbrace{\begin{array}{c} \begin{array}{c} u+iv \end{array} \end{array}}_{b_{2j}} \underbrace{\begin{array}{c} a_{2j} \end{array}}_{a_{2j+1}} \underbrace{\begin{array}{c} u-iv \end{array}}_{b_{2j+1}}.$$

Note that the number of sites $N$, sometimes referred to as the Trotter number, has no relation to the real system size $M$ and is even by the construction. We further remark the commutative property of QTMs,

$$[T_{QTM}(u,v), T_{QTM}(u,v')] = 0.$$ 

The free energy per site is represented only by the largest eigenvalue of $T_{QTM}$ at $v = 0$ and $u = -\epsilon \frac{\beta}{N}$,

$$\beta f = -\lim_{M \to \infty} \frac{1}{M} \ln \text{Tr} \exp(-\beta \mathcal{H}_{e})$$

$$= -\lim_{N \to \infty} \ln \text{(the largest eigenvalue of } T_{QTM}(u = -\epsilon \frac{\beta}{N}, v = 0)).$$
Explicitly, the eigenvalue $T_1(u, v)$ of $T_{QTM}(u, v)$ is given by

\[
T_1(u, v) = w \phi(v + \frac{3}{2}i) \phi(v + \frac{1}{2}i) Q(v - \frac{5}{2}i) Q(v - 1/2i) + \phi(v + \frac{3}{2}i) \phi(v - \frac{3}{2}i) \frac{Q(v - 3/2i)}{Q(v - 1/2i)} Q(v + 3/2i) + w^{-1} \phi(v - \frac{3}{2}i) \phi(v - \frac{1}{2}i) \frac{Q(v + 5/2i)}{Q(v + 1/2i)}.
\]

(2)

\[
Q(v) := \prod_{j=1}^{N/2} h[v - v_j]
\]

\[
\phi(v) := \left( \frac{h[v + (3/2 - u)i] h[v - (3/2 - u)i]}{h[2i] h[3i]} \right)^{N/2}, \quad h[v] := \theta_1(iv),
\]

(3)

where $w = \exp(i \pi \ell/(L + 1)$ ($\ell = 1$ for the largest eigenvalue sector). The parameters, \{\(v_j\}\} are solutions to "Bethe ansatz equation" (BAE),

\[
w \frac{\phi(v_j + i)}{\phi(v_j - i)} = -\frac{Q(v_j - i) Q(v_j + 2i)}{Q(v_j + i) Q(v_j - 2i)}, \quad j = 1, \ldots, \frac{N}{2}.
\]

(4)

Now the difficulty arises. $N$ comes into the expression of the free energy through $u$, as well as the rhs of (4). Thus a simple-minded transformation of BAE into an integral equation using the root density function is not legitimate in contrast to $T = 0$ problems. A naive extrapolation to $N \to \infty$ by numerics may bring about errors. One must thus devise other methods in encoding the information of BAE roots. In this report, we make use of the commuting family of the transfer matrices (fusion hierarchy). Indeed, it has been known that the functional relations among them, which hold for arbitrary $N$, can be an alternative to BAE [54, 55]. In the next section, we shall argue the $sl_3$ fusion structure in the dilute $A_L$ model.

4 $sl_3$ fusion structure

The $sl_3$ type fusion structure in the dilute $A_L$ model has been discussed in details [52]. This comes from the singularity of the RSOS weights at $u = \pm 3$ where the face operator becomes projectors. A desired subspace can be picked up from tensor products of spaces by using these projectors. (For explicit procedure, see [52].) The adjacency matrices are identified with Young diagrams, and combinatorics of tableaux describe their tensor decomposition.

We are interested in eigenvalues of fusion QTMs. In this view point, the most relevant fact is that these eigenvalues are again expressible in terms of "Young tableaux" depending on spectral parameters. Let three boxes with letters 1,2 and 3 represent the three terms in eigenvalue of the quantum transfer matrix;

\[
T_1(u, v) = \begin{array}{c} \boxed{1} \\ \boxed{2} \end{array} + \begin{array}{c} \boxed{3} \end{array}.
\]

\footnote{Results on other choices of auxiliary functions for several models, see [31, 33, 36, 39]}


Obviously, the eigenvalue of a fusion QTM can be represented by sum over products of "boxes" with different letters and spectral parameters. The point is that the assembly of such boxes can be identified with semi-standard Young tableaux (SST) for $sl_3$. We present a simple example. (See [46, 47, 48, 49, 50, 52] for details.) By the fusion procedure, one can construct a transfer matrix of which auxiliary space acts on a symmetric subspace of $V \times V$. The set of the SST, $[i_1, i_2]$ ($i_1 \leq i_2$) is associated with this subspace. The eigenvalue of the transfer matrix is then represented by,

$$
\sum_{i_1 \leq i_2} \begin{bmatrix} i_1 \\ i_2 \\ \cdots \\ i_m \end{bmatrix} v^{i_1 - i_2} v^{i_3 - i_2} + \cdots + v^{i_m - i_2}.
$$

(5)

Similar construction leads to fusion models based on general Young tableaux, of which eigenvalues are expressed by their shapes. On each diagrams, the spectral parameter changes $+2i$ from left to right and $-2i$ from top to the bottom.

We restrict ourselves to rectangular shapes in this section. There are still interesting properties, some of which are crucial for the discussion in the next section. First, due to identities,

$$
\begin{bmatrix} 1 \\ 2 \\ \cdots \\ m \end{bmatrix}^{v+i} = \phi(v + \frac{5}{2}i)\phi(v - \frac{5}{2}i) \begin{bmatrix} 1 \\ 2 \\ \cdots \\ m \end{bmatrix},
$$

$$
\begin{bmatrix} 1 \\ 2 \\ \cdots \\ m \end{bmatrix}^{v-i} = \phi(v + \frac{5}{2}i)\phi(v - \frac{5}{2}i) \begin{bmatrix} 1 \\ 2 \\ \cdots \\ m \end{bmatrix},
$$

$$
\begin{bmatrix} 2 \\ 3 \\ \cdots \\ m \end{bmatrix}^{v+i} = \phi(v + \frac{5}{2}i)\phi(v - \frac{5}{2}i) \begin{bmatrix} 2 \\ 3 \\ \cdots \\ m \end{bmatrix},
$$

$$
\begin{bmatrix} 2 \\ 3 \\ \cdots \\ m \end{bmatrix}^{v-i} = \phi(v + \frac{5}{2}i)\phi(v - \frac{5}{2}i) \begin{bmatrix} 2 \\ 3 \\ \cdots \\ m \end{bmatrix},
$$

the QTMs from $2 \times m$ (or just scalars). Second, the eigenvalues of $1 \times m$ fusion QTMs have the "duality" in the sense of eq.(7). For the explanation, we introduce renormalized $1 \times m$ fusion QTMs $T_m(v)$ by

$$
T_m(v) = \frac{1}{f_m(v)} \sum \begin{bmatrix} i_1 \\ i_2 \\ \cdots \\ i_m \end{bmatrix}
$$

where the semi-standard (SS) condition $i_1 \leq i_2 \leq \cdots \leq i_m$ is imposed on the summation. (From now on we suppress the dependency on $u$ which must be kept identical for all fusion QTMs.) The spectral parameters, $v - i(m - 1) \cdots v + i(m - 1)$, are assigned from left to right. A renormalization factor, which is the common factor of the expressions from tableaux of length $m$, is given by

$$
f_m(v) := \prod_{j=1}^{m-1} \phi(v + i(\frac{2m-1}{2} - j))\phi(v - i(\frac{2m-1}{2} - j)).
$$

Then the resultant $T_m$'s are all degree $2N$ w.r.t. $h[v + \text{shift}]$, and have a periodicity due to Boltzmann weights;

$$
T_m(v + 16/5i) = T_m(v).
$$
From the \(sl_3\) structure, together with the above property, the following functional relations are valid,

\[ T_m(v - i)T_m(v + i) = g_m(v)T_m(v) + T_{m+1}(v)T_{m-1}(v), \quad m \geq 1 \]
\[ g_m(v) = \phi(v + i(m + 3/2))\phi(v - i(m + 3/2)), \]
\[ T_{-1}(v) := 0 \]
\[ T_0(v) := f_2(v). \quad (6) \]

The periodicity, \(\phi(v + 16/5i) = \phi(v)\), leads to \(g_{m+16}(v) = g_m(v), (g_8 + m(v) = g_m(v + 8/5i))\) \(m \geq 0\) and \(g_{5-m}(v) = g_m(v \pm 8/5i), (0 \leq m \leq 5)\). As the adjacency matrices are vanishing, \(T_6(v) = T_7(v) = 0\). Thus one deduces the duality relations,

\[ T_m(v) = T_{5-m}(v + \frac{8i}{5}), \quad m = 0, \ldots, 5 \quad (7) \]

and \(T_{m+8}(v) = T_m(v + \frac{8i}{5}), m \geq -1\) for the solutions to eq.(6). These relations can be in principle proved by using explicit fusion weights. We have at least checked the validity numerically. Thus, for the dilute \(A_3\) case, only 5 “one-row” fusion QTMs need consideration.

Finally we discuss the closed set of functional relations among rectangular types, motivated by success of several other cases. From \(T-\) system (6), we are led to introduce the ”\(Y-\)” functions as in [55, 56, 57]

\[ Y_m(v) := \frac{T_{m+1}(v)T_{m-1}(v)}{g_m(v)T_m(v)}, (1 \leq m \leq 4) \]

which satisfy closed set of equation,

\[ Y_{m}(v - i)Y_{m}(v + i) = (1 + Y_{m+1}(v))(1 + Y_{m-1}(v)) \frac{1}{1 + Y_{m}(v)} (1 \leq m \leq 4), \quad (8) \]

where \(Y_{5}(v) = 0\).

In [24, 58, 59], TBA equations are re-derived from such subsets of fusion hierarchy, as \(Y\) equations possess good analytic properties; both sides of equations are analytic and nonzero and have constant asymptotic behavior as \(|v| \to \infty\) (ANZC) in certain strips. By taking the logarithmic derivative of both sides, and applying Cauchy’s theorem, one can directly rewrite \(Y\) equations by coupled integral equations which are nothing but TBA equations.

The situation is different for the present model; numerically, we find ”intrinsic” zeros and singularities of \(Y_m\)’s in the strips \(3v = [-1, 1]\) and \(3v = [-1.6, -1] \cup [1, 1.6]\). Therefore the above argument can not be applied to (8).

Summarizing, the restriction to rectangle shapes leads to a closed set of equations, but with poor analytic properties.

In the next section, we thus consider a wider class of skew Young tableaux which does lead to finitely closed functional relations having the desired property.
5 quantum Jacobi-Trudi formula and $E_8$ fusion hierarchy

Let $\mu$ and $\lambda$ be a pair of Young tableaux satisfying $\mu_i \geq \lambda_i, \forall i$. We subtract a diagram $\lambda$ from $\mu$. The resultant "narrower" one, consisted of $(\mu_1 - \lambda_1, \mu_2 - \lambda_2, \cdots)$ boxes is called a skew Young diagram $\mu - \lambda$. (The usual Young diagram is the special case that $\lambda$ is empty, and we will omit $\lambda$ in the case hereafter.) In the theory of symmetric polynomials, the

![Figure 1: An example of a skew Young Tableau, (4,4)-(2). In the lhs, the Young diagram (4,4) with the shaded part (2) is given, while we have the corresponding skew diagram in the rhs.](image)

Schur function deserves the one of the fundamental objects. Generally, a Schur function is defined in correspondence to a skew Young diagram. The Jacobi-Trudi formula asserts its representation by a determinant of a matrix of which elements are given by those related to "one-row" diagrams or "one-column" ones. An analogous formula holds for the present situation [46, 48, 49, 50].

Consider a set of semi-standard skew Young tableaux of the shape $\mu - \lambda$. As remarked in the previous section, spectral parameters are assigned to boxes in a tableau in such a manner that they change by $+2i$ from left to right and $-2i$ from top to the bottom. We fix the spectral parameter of the "top-left" box by $v + i(\mu_1 - \mu'_1)$ where $\mu'_1$ denotes the depth of the tableaux. One regards each box in a tableau as an expression under the rule

![Figure 2: The spectral parameter $v + i(\mu_1 - \mu'_1)$ is assigned to the hatched place](image)

with the shift of the spectral parameter. Then the product over all constituting boxes yields an expression for a tableau.

**Theorem 1** Let $T_{\mu/\lambda}(v)$ be the sum of the resultant expressions over the SST divided by a common factor, $\prod_{j=1}^{\mu'_1} f_{\mu_j - \lambda_j}(v + i(\mu'_1 - \mu_1 + \mu_j + \lambda_j - 2j + 1))$. Then the following equality holds.

$$T_{\mu/\lambda}(v) = \det_{1 \leq j, k \leq \mu'_1} (T_{\mu_j - \lambda_k - j + k}(v + i(\mu'_1 - \mu_1 + \mu_j + \lambda_k - j - k + 1)))$$  \hspace{1cm} (9)

where $T_{m<0} := 0$. 


We regard this as a quantum analogue of the Jacobi-Trudi formula. The proof utilizes the decomposition rules for products of two tableaux with spectral parameters \([48, 49, 50]\), which is quite parallel to those for Young tableaux. One must only pay attention that the allowed positions of a box is restricted by its spectral parameter. \(\mathcal{T}_{\mu/\lambda}(v)\) may be naturally identified with the eigenvalue of QTM corresponding to fusion \(\mu - \lambda\). The proof of this is beyond the scope of the present report. The important fact for our purpose is that \(\mathcal{T}_{\mu/\lambda}(v)\) defined in such a manner is an analytic function of \(v\) due to BAE, and contains \(\mathcal{T}_1(v)\) as a special case. The former assertion is not obvious from the original definition by the tableaux, but it is trivial from the quantum Jacobi-Trudi formula.

In the same spirit, we introduce \(\Lambda_{\mu/\lambda}(v)\), which is analytic under BAE, from \(\mathcal{T}_{\mu/\lambda}(v)\) by putting \(T_{m \geq 6}(v) = 0\) in the latter,

\[
\Lambda_{\mu/\lambda}(v) := \mathcal{T}_{\mu/\lambda}(v) / \{T_{m \geq 6}(v) \to 0\}.
\]

The pole-free property of \(\Lambda_{\mu/\lambda}(v)\) is apparent from (9).

The following "dualities" are simple corollaries of eq. (9) and (10).

**Lemma 1**

\[
\Lambda_{(4,4)/(2)}(v) = \Lambda_{(3,1)}(v + \frac{2}{5}i), \quad \text{(10)}
\]

\[
\Lambda_{(4,2)/(1)}(v) = \Lambda_{(4,3)/(2)}(v + \frac{3}{5}i), \quad \text{(11)}
\]

\[
\Lambda_{(4,4)/(3)}(v) = \Lambda_{(4,1)}(v + \frac{7}{5}i), \quad \text{(12)}
\]

\[
\Lambda_{(7,4,4)/(3,3)}(v) = \Lambda_{(4,4,1)/(3)}(v + \frac{8}{5}i), \quad \text{(13)}
\]

\[
\Lambda_{(6,4,3)/(3,2)}(v) = \Lambda_{(5,4,2)/(3,1)}(v + \frac{8}{5}i), \quad \text{(14)}
\]

\[
\Lambda_{(7,7,4,4)/(6,3,3)}(v) = \Lambda_{(7,4,4,1)/(3,3)}(v + \frac{7}{5}i), \quad \text{(15)}
\]

\[
\Lambda_{(7,7,4,4,1)/(6,3,3)}(v) = \Lambda_{(10,7,7,4,4)/(6,6,3,3)}(v + \frac{8}{5}i). \quad \text{(16)}
\]

To establish a closed set of functional equations we need further

**Lemma 2**

The following relations hold,

\[
\Lambda_{(3,1,1)}(v) = \frac{f_5(v - 2i)}{f_2(v - 2i)f_2(v + i)}\Lambda_{(2)}(v + 3i), \quad \text{(17)}
\]

\[
\Lambda_{(6,4,4)/(3,3)}(v) = \frac{f_5(v + i)}{f_2(v + i)} \frac{\Lambda_{(4,2)/(1)}(v - 7i)}{\phi(v - i5/2)\phi(v - i3/2)\phi(v + i7/2)\phi(v + i9/2)}, \quad \text{(18)}
\]

\[
\Lambda_{(10,7,7,4,4)/(6,6,3,3)}(v) = \phi(v - \frac{13}{2}i)\phi(v - \frac{7}{2}i)\phi(v + \frac{7}{2}i)\phi(v + \frac{13}{2}i)\Lambda_{(6,4,3)/(3,2)}(v), \quad \text{(19)}
\]
\[ \Lambda_{(10,7,4,4,1)/(6,6,3,3)}(v) = \phi(v - \frac{11}{2}i)\phi(v - \frac{5}{2}i)\phi(v + \frac{9}{2}i)\phi(v + \frac{15}{2}i) \]
\[ \Lambda_{(3,1)}(v - 11i)\Lambda_{(4,3)/(2)}(v - 2i). \]

(17) can be shown trivially from SS condition. To prove the rest, it is convenient to go back to the original definition of \( T \). For example, we consider (18).

\[ T_{(6,4,4)/(3,3)}(v) = \Lambda_{(6,4,4)/(3,3)}(v) + T_0(v)T_0(v + 2i)T_5(v) \]
\[ = \frac{f_5(v + i)f_3(v - 5i)f_2(v + 6i)}{f_2(v + i)f_3(v + 5i)f_4(v - 4i)}T_3(v - 5i)T_2(v + 6i) \]
\[ = \frac{f_5(v + i)}{f_2(v + i)} \frac{T_3(v - 5i)T_2(v - 10i)}{\phi(v - i5/2)\phi(v - i3/2)\phi(v + i7/2)\phi(v + i9/2)}. \]

We have applied (17) in the first line. The second line follows from SS condition; 3 boxes in a column reduce to a scalar, so that the configurations in other two rows are completely independent.

Figure 3: Decomposition of the skew diagram (6,4,4)/(3,3).

In the last equation, the periodicity brings about a new feature, namely, further decomposition of the products of \( T \),

\[ T_3(v - 5i)T_2(v - 10i) = \Lambda_{(4,2)/(1)}(v - 7i) + \frac{f_5(v - 7i)}{f_2(v - 10i)f_3(v - 5i)}T_5(v - 7i). \]

Now that \( T_3(v) = T_5(v) = T_0(v + \frac{8}{9}i) \) as discussed in the previous section, the proposition reduces to

\[ T_0(v)T_0(v + 2i)T_0(v + \frac{8}{5}i) = \frac{f_5(v - 7i)f_5(v + i)}{f_2(v + i)f_3(v + 5i)f_4(v - 4i)}T_0(v - 7i + \frac{8}{5}i), \]

which is proved by representing \( T_0 \) by \( \phi \). The other two relations are proved similarly.

We define the following fundamental objects (referred to as "modified fusion QTMs" in the introduction);

**Definition 1**

\[ T^{(1)}(v) := \Lambda_{(1)}(v) \]
\[ T^{(2)}(v) := \Lambda_{(4,1)}(v + \frac{7}{10}i)/\phi(v + \frac{8}{9}i) \]
\[ T^{(3)}(v) := \Lambda_{(7,4,4)/(3,3)}(v)/\{\phi(v + 3/2i)\phi(v - 3/2i)\} \]
In terms of \( h[v+\text{ shift }] \), they are of degree, \( 2N, 3N, 4N, 5N, 6N, 4N, 2N, 3N \), respectively.

**Proposition 1** The following "\( E_8 \) type T-system" holds among them,

\[
\begin{align*}
T^{(1)}(v) & := \Lambda_{(7,7,4)/(6,3,3)}(v + \frac{9}{10}i)/\{\phi(v + i\frac{7}{5})\phi(v + i\frac{8}{5})\phi(v + i\frac{9}{5})\} \\
T^{(5)}(v) & := \Lambda_{(5,4,2)/(3,1)}(v) \\
T^{(6)}(v) & := \Lambda_{(4,2)/(1)}(v + \frac{13}{10}i) \\
T^{(7)}(v) & := \Lambda_{(2)}(v + \frac{8}{5}i) \\
T^{(8)}(v) & := \Lambda_{(3,1)}(v - \frac{7}{10}i)/\phi(v + \frac{6}{5}i)
\end{align*}
\]

(21)

Here and in the following we use \( g(x \pm y) \) to mean \( g(x + y)g(x - y) \) for some function \( g \).

Note that the difference of the arguments of lhs is much smaller \( (\frac{1}{5}i) \) than the difference \( (2i) \) in the original \( sl_3 \) type fusion equation. This leads to the analytical property shown in the next section.

Let us sketch the derivation of the above equations. We consider the decomposition of \( T^{(a)} \)'s and their dual partners. The simplest case, (22), follows from consideration on the decomposition of \( T_1(v - 4i)T_4(v + i) \). From (3) we have,

\[ T_1(v - 4i)T_4(v + i) = \Lambda_{(4,1)}(v) + T_5(v)T_0(v - 3i). \]

Duality (5) allows us to rewrite \( T_4(T_5) \) by of \( T_1(T_0) \). After the shift \( v \rightarrow v + 0.7i \) together with periodicity, (22) is established. Eqs. (23), (24), (25), (26), (27) and (28) can be proved
similarly. Next we consider (26) by utilizing,

\[ 
\Lambda(7,7,4,4,1)/(6,3,3)(v)\Lambda(10,7,7,4,4)/(6,6,3,3)(v + 5i) 
= T_0(x + i)T_0(x + 9i)T_0(x + 11i)T_5(x + 4i)T_5(x + 6i)T_5(x + 14i) 
+ \Lambda(10,7,7,4,4)/(6,6,3,3)(v + 4i)\Lambda(7,7,4,4)/(6,3,3)(v + i). 
\]

The eqs. (20) and (11) allow us to rewrite the rhs by \( T^{(4)}, T^{(6)} \) and \( T^{(8)} \). The second factor in the lhs can be rewritten by \( T^{(5)} \) after transformations \( \Lambda(10,7,7,4,4)/(6,6,3,3) \to \Lambda(6,4,3)/(3,2) \) \((19) in Lemma 2) \to \Lambda(5,4,2)/(3,1) \((14) in Lemma 1). The first term can also be reduced to \( T^{(5)} \) with one further step \((19) in Lemma 1). By shifting \( v \to v - 0.1i \) and dividing both sides by the factor \( \phi(v + \frac{5}{10})\phi(v + \frac{9}{10})\phi(v + \frac{6}{10})\phi(v + 2\alpha) \), we obtain (26). (29) can be proved with the aid of \((19), (17) and (18).\)

6 \( E_8 \) Thermodynamic Bethe Ansatz from Y-system

Now we transform the functional relations into ”gauge invariant” forms. Let us define \( Y \)–functions by combinations of \( T^{(a)}; \)

**Definition 2**

\[
Y^{(1)}(v) := \frac{T(v + \frac{5}{10})T^{(2)}(v)}{T_0(v + \frac{10}{10})T_0(v + \frac{5}{10})} \\
Y^{(2)}(v) := \frac{T(v + \frac{5}{10})T^{(3)}(v)}{T_0(v + \frac{10}{10})T_0(v + \frac{5}{10})} \\
Y^{(3)}(v) := \frac{T^{(2)}(v + \frac{5}{10})T^{(4)}(v + \frac{10}{10})}{T_0(v + \frac{10}{10})T_0(v + \frac{5}{10})} \\
Y^{(4)}(v) := \frac{T^{(3)}(v + \frac{5}{10})T^{(5)}(v + \frac{10}{10})}{T_0(v + \frac{10}{10})T_0(v + \frac{5}{10})} \\
Y^{(5)}(v) := \frac{T^{(4)}(v + \frac{10}{10})T^{(6)}(v + \frac{10}{10})T^{(8)}(v + \frac{10}{10})}{T_0(v + \frac{10}{10})T_0(v + \frac{5}{10})T_0(v + \frac{10}{10})} \\
Y^{(6)}(v) := \frac{T^{(5)}(v + \frac{10}{10})T^{(7)}(v + \frac{10}{10})}{T_0(v + \frac{10}{10})T_0(v + \frac{5}{10})T_0(v + \frac{10}{10})} \\
Y^{(7)}(v) := \frac{T^{(6)}(v + \frac{10}{10})}{T_0(v + \frac{10}{10})} \\
Y^{(8)}(v) := \frac{T^{(5)}(v + \frac{10}{10})}{T_0(v + \frac{10}{10})T_0(v + \frac{5}{10})}
\]

then functional relations follow from the \( T \)–system. To write it down neatly, we assign numbers to nodes in the Dynkin diagram of \( E_8 \) as in the following figure.

![Figure 4: The Dynkin diagram for \( E_8 \).](image)

We denote \( a \sim b \) if \( a \) and \( b \) are adjacent in the Dynkin diagram.
Theorem 2  Functional relations among $Y -$ functions exhibit the $E_8$ structure in the following form,

$$Y^{(a)}(v - i)Y^{(a)}(v + i) = \prod_{b \sim a}(1 + Y^{(b)}(v)), \quad a = 1, \cdots, 8.$$  

This coincides with the $E_8$ case of the universal $Y -$ system in [58]. In order to reach our final goal, the TBA equation, further information is required on analytic structures of $Y^{(a)}(v), 1 + Y^{(a)}(v), a = 1, \cdots, 8$. We employ numerical calculations for some fixed values of $q, N$ and $\beta$ for this purpose. Remark that only the largest eigenvalue sector needs examinations. This is a drastic simplification from standard string hypothesis argument in which analyses on all excitation sectors are, in principle, necessary. Though we have performed numerics for relatively small values of $N$, intriguing patterns are already found for zeros of $T^{(a)}(v)$. First, these zeros are symmetric with respect to real axis, which assures that $T^{(a)}(v)$'s are real on the axis. Second, imaginary parts of coordinates of zeros have the remarkable coincidence with the exponents in Table 1. See Table 2 for the example of the case $u = -0.08, q = 0.3, N = 12$. We summarize them as a conjecture for arbitrary $N$.

Conjecture 1  Zeros of $T^{(a)}$ distribute along approximately on the lines, $\Im v \sim \pm 0.1(a_j + 1)$. The set $\{a_j\}$ agrees with $\{a\}$ for the particle $j$ in Table 1.

In addition, asymptotic behaviors are specified as,

**Conjecture 2**  $Y^{(a)}$ 's have bounded asymptotic behaviors in the largest eigenvalue sector of QTM. Explicitly, their limiting values read,

$$
\begin{align*}
Y^{(1)}(\pm \infty) &= 2(1 + \sqrt{2}), \
Y^{(2)}(\pm \infty) &= (1 + \sqrt{2})(5 + 3\sqrt{2}), \
Y^{(3)}(\pm \infty) &= 6(1 + \sqrt{2})(3 + 2\sqrt{2}), \
Y^{(4)}(\pm \infty) &= 4(4 + 3\sqrt{2})(5 + 3\sqrt{2}), \
Y^{(5)}(\pm \infty) &= 3(3 + 2\sqrt{2})^2(5 + 4\sqrt{2}), \
Y^{(6)}(\pm \infty) &= 4(2 + \sqrt{2})(4 + 3\sqrt{2}), \
Y^{(7)}(\pm \infty) &= 5 + 4\sqrt{2}, \
Y^{(8)}(\pm \infty) &= 4(4 + 3\sqrt{2}).
\end{align*}
$$

Combining these with the $T-$ system, we have

**Lemma 3**  Assume that above conjectures are valid. Then $\tilde{Y}^{(a)}(v)$ and $1 + Y^{(a)}(v)$ are Analytic, NonZero and have Constant asymptotic behavior (ANZC) in strips $\Im v \in [-1, 1], [-0^+, 0^+]$, respectively.

$\tilde{Y}^{(a)}(v) = Y^{(a)}(v)$ for $a \neq 1$ and,

$$
\tilde{Y}^{(1)}(v) = Y^{(1)}(v)\{\kappa(v + i(1 + \tilde{u}))\kappa(v - i(1 + \tilde{u}))\}^\epsilon
$$

where $\epsilon = 1(-1)$ for $u > 0(u < 0)$ , $\tilde{u} = u/10$ and

$$
\kappa(v) = \left(\frac{\vartheta_2(i\pi v/4, \tau')}{\vartheta_1(i\pi v/4, \tau')}\right)^{-1/4}.
$$
Here $\tau' = 8\tau$ is introduced so as to respect the periodicity of $Y-$ functions along the real axis. Remark that in deriving Lemma 3 we actually do not need such fine structures as given in Conjecture 1. Lemma 3 is robust in this sense.

Thanks to the simple identity $\kappa(v + i)\kappa(v - i) = \pm 1$, the lhs of the theorem 2 can be replaced by $Y^{(a)}(v - i)\overline{Y^{(a)}(v + i)}$. Then we are in position to apply the Cauchy theorem directly and transform logarithmic derivatives of resultant algebraic functional relations to a set of coupled integral equations. Note that the resultant equations depend on Trotter number only in the renormalization factor of $Y^{(1)}$ through the combination, roughly speaking, $uN$. The limit $N \to \infty$ is thus performed straightforwardly.

Let us introduce Fourier transformations by
\[
\hat{F}(x) = \int_{-16\tau/\pi}^{16\tau/\pi} e^{-ivx} F(v) dv, \quad F(v) = \frac{\delta}{2\pi} \sum_n e^{ivx_n} \hat{F}(x_n)
\]
with $\delta = \pi^2/(16\tau)$, $x_n = n\delta$. We also denote by $\hat{\ln} F(x)$ the FT of logarithm of $F(v)$.

**Theorem 3** By Lemma 3, one can prove a set of nonlinear integral equations among $\ln Y^{(a)}(v)$ for arbitrary $N$.

By taking the limit $N \to \infty$, the resultant equations read in the Fourier space as,
\[
\begin{align*}
\hat{\ln} Y^{(a)}(x) &= -\epsilon \delta_{a,1} \hat{\beta} \hat{s}(x) + \hat{C}_{a,b}(x) \hat{\ln}(1 + Y^{(b)})(x) \\
\hat{s}(x) &= \frac{1}{2 \cosh x} \\
\hat{C}_{a,b}(x) &= \hat{s}(x)(2I - C^{E_8})_{a,b},
\end{align*}
\]
where $\hat{\beta} = 20\pi\beta$, and $C^{E_8}$ denotes the Cartan matrix for $E_8$.

The free energy is expressed via $Y-$ functions with the aid of (22).
\[
-\beta f = -\beta e_0 - \hat{\beta} b_1 * s(0) + s * \ln(1 + Y^{(1)})(0) \quad \epsilon = 1,
\]
\[
\begin{align*}
\epsilon e_0 &:= \lambda [\ln(\vartheta_1(\pi/16)\vartheta_1(3\pi/8))]' \\
b_0(x) &:= \frac{\sinh 6x}{\sinh 16x}, \quad b_1(x) := \frac{\sinh 5x + \sinh 15x}{\sinh 16x}, \\
\hat{\rho}_0(x) &:= \frac{1}{2 \cosh x - 1}, \quad \hat{\alpha}^{E_8}_{1,1}(x) := \hat{s}(x)((1 - \hat{C}(x))^{-1})_{1,1},
\end{align*}
\]
where $A*B := \int_{-16\tau/\pi}^{16\tau/\pi} A(v - v')B(v')dv'$. The final expressions are given in different forms depending on the magnitudes of $Y$ in the vicinity of the origin. ($|Y^{(a)}| \ll 1(\gg 1)$ for $\epsilon = 1(-1)$).

The set of nonlinear integral equations is nothing but the TBA equations obtained in [16], and the free energy also coincides with their result. (Note that the normalization of our hamiltonian is $20\pi$ times the one in [16].)

Therefore, we have reached the same results, however, completely independent of the string hypothesis.
7 Summary and Discussion

In this report we have applied the recent developed QTM method to the 1D quantum chain related to the dilute $A_3$ model. Modified QTMs described by skew Young tableaux have played a role, which has not been observed in former successful applications of the QTM method. The $E_8$ structure of the TBA equation has been recovered without hypothesis on specific forms of dominant solutions. This gives further supports and consistency of the "Trinity" among minimal unitary CFT theory $p = 3$ perturbed by $\phi_{1,2}$, the Ising model in a field and the dilute $A_3$ model.

There may be a direct connection to $E_8$. In the context of the analytic Bethe ansatz, the dressed vacuum form for $E_8$ case, consisted of 249 terms, is already obtained explicitly [59] as shortly remarked in [48]. It would be interesting to show the direct reduction of these 249 terms to only 3 terms.

The method employed here is in principle applicable to dilute $A_4$, $A_6$ model of which underlying symmetries are expected to be of $E_7$ and $E_6$. Partial evidences are found very recently in a different context [60]. We hope to report the study on them, as well as the extensive numerical studies on the $E_8$ case in the near future.

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| exponent | locations of zeros |
|----------|--------------------|
| $T^{(1)}$ |
| 1        | $\pm0.0160536139 \pm 0.194948867i, \pm0.0538522756 \pm 0.192860391i,$ |
|          | $\pm0.124085873 \pm 0.179123531i$ |
| 11       | $\pm0.0161627786 \pm 1.19901092i, \pm0.0544016497 \pm 1.19859179i,$ |
|          | $\pm0.129454048 \pm 1.19555064i$ |
| $T^{(2)}$ |
| 2        | $\pm0.0160900147 \pm 0.295944203i, \pm0.0540354627 \pm 0.294252469i,$ |
|          | $\pm0.125806251 \pm 0.282759557i$ |
| 10       | $\pm0.0161506232 \pm 1.09758994i, \pm0.0543363518 \pm 1.09656338i,$ |
|          | $\pm0.128545339 \pm 1.08913466i$ |
| 12       | $\pm0.0161758534 \pm 1.3002265i, \pm0.0544655721 \pm 1.3003507i,$ |
|          | $\pm0.130002895 \pm 1.30016934i$ |
| $T^{(3)}$ |
| 3        | $\pm0.0160996137 \pm 0.396199839i, \pm0.0540833902 \pm 0.394610881i,$ |
|          | $\pm0.126249978 \pm 0.383723324i$ |
| 9        | $\pm0.0161379988 \pm 0.997143234i, \pm0.0542727134 \pm 0.995931521i,$ |
|          | $\pm0.12791469 \pm 0.987294006i$ |
| 11       | $\pm0.0161709223 \pm 1.19860006i, \pm0.0544390469 \pm 1.19800087i,$ |
|          | $\pm0.129624985 \pm 1.19357709i$ |
| 13       | $\pm0.0161774898 \pm 1.40026185i, \pm0.0544730978 \pm 1.40037631i,$ |
|          | $\pm0.130044745 \pm 1.40126236i$ |
| $T^{(4)}$ |
| 4        | $\pm0.0161030779 \pm 0.496280551i\ast, \pm0.054100446 \pm 0.49472356i\ast,$ |
|          | $\pm0.126398274 \pm 0.484020597i\ast$ |
| 8        | $\pm0.0161379988 \pm 0.896993334i, \pm0.0542433172 \pm 0.895721491i,$ |
|          | $\pm0.127657169 \pm 0.886719531i$ |
| 10       | $\pm0.0161605145 \pm 1.09815223i, \pm0.0543870319 \pm 1.09736514i,$ |
|          | $\pm0.129124614 \pm 1.09163793i$ |
| 12       | $\pm0.0161742694 \pm 1.29883901i, \pm0.0544557686 \pm 1.29834129i,$ |
|          | $\pm0.12979684 \pm 1.2946801i$ |
| 14       | $\pm0.0161781274 \pm 1.50032712i, \pm0.0544760749 \pm 1.50046909i,$ |
|          | $\pm0.130062127 \pm 1.50155269i$ |
The locations of zeros of fusion QTMs for $N = 12, q = 0.3, u = -0.08$. The corresponding exponents of Table 1 are given in the second column.

* For these cases, we have a problem in the convergence by the simple Müller algorithm. These values must be thus taken to be rather approximate.