ON CIRCULAR-ARC GRAPHS WITH ASSOCIATION SCHEMES

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Abstract. In this paper, we give a characterization of the class of all circular-arc graphs whose schemes are association. Moreover, all association schemes which are the scheme of a circular-arc graph are characterized, specially it is proved that they are Schurian.

1. Introduction

In [10], B. Weisfeiler and A. Leman have shown that a special matrix algebra is assigned to a given graph which contains the adjacency matrix of the graph. In fact, this algebra is the adjacency algebra of a scheme. The scheme of a graph is the smallest scheme on the vertex set of the graph such that the edge set of which is the union of some basic relations of the scheme. The scheme of forests, interval graphs and some special classes of graphs have been studied in [3]. In this paper, we study the scheme of a circular-arc graph which is the intersection graph of a family of arcs of a circle.

Circular-arc graphs received considerable attention since a series of papers by Tucker in [7, 8, 9], and by Durán, Lin and McConnell in [2, 5, 6]. Various subclasses of circular-arc graphs have been also studied. Among these are the proper circular-arc graphs, unit circular-arc graphs, Helly circular-arc graphs and co-bipartite circular-arc graphs. Several characterizations and recognition algorithms have been formulated for circular-arc graphs and its subclasses. But in this paper, we correspond a finite or algebraic description for circular arc graphs instead of description according arcs of a circle.

Let $n$ be a positive integer and let $S$ be a subset of $\mathbb{Z}_n$ such that $S = \{-1, \ldots, \pm k\}$ for $0 \leq 2k + 1 < n$. Then the Cayley graph $\text{Cay}(\mathbb{Z}_n, S)$ is a circular-arc graph (see Sec. 3), and we call it elementary circular-arc graph. For $k = 0$ it is empty graph and for $k = 1$ it is an undirected cycle.

We say that $\Gamma$ is a graph with association scheme if the scheme of $\Gamma$ is association. Our main results give a characterization of circular-arc graphs with association schemes in the terms of lexicographic product of graphs.

In graph theory, what we have called the lexicographic product or composition of graphs is also often called the wreath product. The term wreath product comes from group theory, and it is also defined in scheme theory.

In fact, analysis duplicating the vertices of an elementary circular-arc graph shows that the lexicographic product of an elementary circular-arc graph and a complete graph is a circular-arc graph.

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The rest of this section is to state our results. The following theorem provides a necessary and sufficient condition for a circular-arc graph whose scheme is association.

**Theorem 1.1.** A graph is a circular-arc graph with association scheme if and only if it is isomorphic to the lexicographic product of an elementary circular-arc graph and a complete graph.

One can associate to any finite permutation group $G$ a scheme, denoted by $\text{Inv}(G)$. The scheme associated to the dihedral group $D_{2n}$ is called *dihedral* scheme.

The class of *forestal schemes* have been defined inductively by means of direct sums and wreath products in [3]. The scheme of cographs, trees and interval graphs are forestal (see [3]).

A scheme is said to be *circular-arc scheme* if it is the scheme of a circular-arc graph. In the following theorem we give a characterization of association circular-arc schemes.

**Theorem 1.2.** A circular-arc scheme is association if and only if it is isomorphic to the wreath product of a rank 2 scheme and a scheme which is either forestal or dihedral.

**Remark 1.3.** Forestal scheme in Theorem 1.2 occurs as a wreath product of the scheme of at most on 2 points and a rank 2 scheme. For example: the scheme of a disjoint union of copies of a complete graph, namely $mK_n$, or the scheme of the lexicographic product of a complete graph and a complete graph without perfect matching.

A scheme $X$ is said to be *Schurian* if $X = \text{Inv}(G)$ for some permutation group $G$, see [11]. In fact, any rank 2 scheme and any dihedral scheme are Schurian. Moreover, the wreath product of two Schurian schemes is Schurian, see [11]. Thus we have the following corollary:

**Corollary 1.4.** Any association circular-arc scheme is Schurian.

It is known that the automorphism group of each graph is equal to automorphism group of its scheme, see [10]. Moreover, it is well-known that the automorphism group of wreath product of two schemes is equal to the wreath product of their automorphism groups. Denote by $\text{Sym}(n)$ the symmetric group on $n$ points, and denote by $G \wr H$ the wreath product of two groups $G$ and $H$. The following corollary is an immediate consequence of Theorems 1.1 and 1.2.

**Corollary 1.5.** Let $\Gamma$ be a circular-arc graph with association scheme on $n$ vertices. Then there is an even integer $k$, $k|n$, such that $\text{Aut}(\Gamma)$ is isomorphic to $\text{Sym}(\frac{n}{k}) \wr G$, where $G$ is $\text{Sym}(k)$ or $\text{Sym}(2) \wr \text{Sym}(\frac{n}{2})$ or $D_{2k}$.

This paper is organized as follows. In Section 2, we present some preliminaries on graph theory and scheme theory. In Section 3, we first remind the concept of circular-arc graphs and then we introduce arc-function and reduced arc-function of a circular-arc graph. Moreover, we characterize non-empty regular circular-arc graphs without twins. Section 4 contains relationship between lexicographic product of graphs and wreath product of their schemes. In Section 5, we define elementary circular-arc graphs. Then, we characterize the scheme of graphs which
belong to this class. Finally, in Section 6 we give the proof of Theorem 1.1 and Theorem 1.2.

**Notation.** Throughout the paper, $V$ denotes a finite set. The diagonal of the Cartesian product $V^2$ is denoted by $1_V$.

For $r,s \subset V^2$ and $X,Y \subset V$ we have the following notations:

$$r^* = \{(u,v) \in V^2 : (v,u) \in r\},$$
$$r \cap X = r \cap (X \times Y), r_X = r_{X,Y},$$
$$r \cdot s = \{(v,u) \in V^2 : (v,w) \in r, (w,u) \in s \text{ for some } w \in V\},$$
$$r \otimes s = \{((v_1,v_2),(u_1,u_2)) \in V^2 \times V^2 : (v_1,u_1) \in r \text{ and } (v_2,u_2) \in s\}.$$

Also for any $v \in V$, set $vr = \{u \in V : (v,u) \in r\}$ and $n_r(v) = |vr|$.

For $S \in 2^{V^2}$ denote by $S^\cup$ the set of all unions of the elements of $S$, and set $S^* = \{s^* : s \in S\}$ and $vS = \cup_{s \in S^*} sv$. For $T \in 2^{V^2}$ set $S \cdot T = \{s \cdot t : s \in S, t \in T\}$.

For an integer $n$, let $\mathbb{Z}_n$ be the ring of integer numbers modulo $n$. Set $AZ_n := \{\{i,i+1,\ldots,i+k\} : i,k \in \mathbb{Z}_n \text{ and } k \neq n-1\}$.

For each set $\{i,i+1,\ldots,i+k\}$, the points $i$ and $i+k$ are called the end-points of the set.

2. Preliminaries

2.1. Graphs. All terminologies and definitions about graph theory have been adapted from [I]. In this paper, we consider finite and undirected graphs which contains no loops and multiple edges. We denote complete graph on $n$ vertices by $K_n$, and an undirected cycle on $n$ vertices by $C_n$.

Let $\Gamma = (V,R)$ be a graph with vertex set $V$ and edge set $R$. Let $E$ be an equivalence relation on $V$, then $\Gamma_{V/E}$ is a graph with vertex set $V/E$ in which distinct vertices $X$ and $Y$ are adjacent if and only if at least one vertex in $X$ is adjacent in $\Gamma$ with some vertex in $Y$. For every subset $X$ of $V$, the graph $\Gamma_X$ is the subgraph of $\Gamma$ induced by $X$.

Let $\Gamma_i$ be a graph on $V_i$, for $i = 1,2$. The graphs $\Gamma_1$ and $\Gamma_2$ are isomorphic if there is a bijection $f : V_1 \to V_2$, such that two vertices $u$ and $v$ in $V_1$ are adjacent in $\Gamma_1$ if and only if their images $f(u)$ and $f(v)$ are adjacent in $\Gamma_2$. Such a bijection is called an isomorphism between $\Gamma_1$ and $\Gamma_2$. The set of all isomorphism between $\Gamma_1$ and $\Gamma_2$ is denoted by $\text{Iso}(\Gamma_1,\Gamma_2)$. An isomorphism from a graph to itself is called an automorphism. The set of all automorphisms of a graph $\Gamma$ is the automorphism group of $\Gamma$, and denoted by $\text{Aut}(\Gamma)$.

The lexicographic product or composition of graphs $\Gamma_1$ and $\Gamma_2$ is the graph $\Gamma_1[\Gamma_2]$ with vertex set $V_1 \times V_2$ in which $(u_1,u_2)$ is adjacent to $(v_1,v_2)$ if and only if either $u_1$ and $v_1$ are adjacent in $\Gamma_1$ or $u_1 = v_1$ and also $u_2$ and $v_2$ are adjacent in $\Gamma_2$.

Let $\Gamma = (V,R)$ be a graph. Two vertices $u,v \in V$ are twins if $u$ and $v$ are adjacent in $\Gamma$ and $\Gamma \backslash \{u\} = u \Gamma \backslash \{v\}$, where the set of neighbors of a vertex $v$ in the graph $\Gamma$ is denoted by $v \Gamma$. 

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2.2. Schemes. In this part all terminologies and notations are based on [4].

**Definition 2.1.** A pair $\mathcal{X} = (V, S)$, where $V$ is a finite set and $S$ a partition of $V^2$, is called a *scheme* on $V$ if $1_V \in S^{\cup}$, $S^* = S$, and for any $r, t \in S$ the number

$$c^t_{rs} := |vr \cap us^*|$$

does not depend on the choice of $(v, u) \in t$. The scheme $\mathcal{X}$ is called an *isomorphism* if $1_V \in S$.

The elements of $V$, $S$, $S^{\cup}$ and the numbers $c^t_{rs}$ are called the *points*, the *basic relations*, the *relations* and the *intersection numbers* of the scheme $\mathcal{X}$, respectively. The numbers $|V|$ and $|S|$ are called the *degree* and the *rank* of $\mathcal{X}$. A unique basic relation containing a pair $(v, u) \in V^2$ is denoted by $r_X(v, u)$ or $r(v, u)$.

An equivalence relation on a subset of $V$ belonging to $S^{\cup}$ is called an equivalence relation of the scheme $\mathcal{X}$. Any scheme has trivial equivalence relations: $1_V$ and $V^2$. Let $e \in S^{\cup}$ be an equivalence relation. For a given $X \in V/e$ the restriction of the scheme $\mathcal{X}$ to $X$ is the scheme

$$\mathcal{X}_X = (X, S_X),$$

where $S_X$ is the set of all non-empty relations $r_X$ with $r \in S$. The *quotient* of the scheme $\mathcal{X}$ modulo $e$ is defined to be the scheme

$$\mathcal{X}_{V/e} = (V/e, S_{V/e}),$$

where $S_{V/e}$ is the set of all non-empty relations of the form $\{(X, Y) : s_X, Y \neq \emptyset\}$, with $s \in S$.

Two schemes $\mathcal{X}_1$ and $\mathcal{X}_2$ are called *isomorphic* if there exists a bijection between their point sets in such a way that induces a bijection between their sets of basic relations. Such a bijection is called an *isomorphism* between $\mathcal{X}_1$ and $\mathcal{X}_2$. The set of all isomorphism between $\mathcal{X}_1$ and $\mathcal{X}_2$ is denoted by $\text{Iso}(\mathcal{X}_1, \mathcal{X}_2)$. The group of all isomorphisms of a scheme $\mathcal{X}$ to itself contains a normal subgroup

$$\text{Aut}(\mathcal{X}) = \{f \in \text{Sym}(V) : s^f = s, s \in S\}$$

called the *automorphism group* of $\mathcal{X}$, where $s^f = \{(u^f, v^f) : (u, v) \in s\}$.

The *wreath product* $\mathcal{X}_1 \wr \mathcal{X}_2$ of two schemes $\mathcal{X}_1 = (V_1, S_1)$ and $\mathcal{X}_2 = (V_2, S_2)$ is a scheme on $V_1 \otimes V_2$ with the following basic relations

$$V_1^2 \otimes r, \text{ for } r \in S_2 \setminus 1_{V_2} \text{ and } s \otimes 1_{V_2}, \text{ for } s \in S_1.$$

2.3. The scheme of a graph. There is a natural partial order “$\leq$” on the set of all schemes on $V$. Namely, given two schemes $\mathcal{X} = (V, S)$ and $\mathcal{X}' = (V, S')$ we set

$$\mathcal{X} \leq \mathcal{X}' \iff S^{\cup} \subseteq (S')^{\cup}.$$

In this case $\mathcal{X}$ is called a *fusion* (subscheme) of $\mathcal{X}'$ and $\mathcal{X}'$ is called a *fission* (extension) of $\mathcal{X}$. The minimal and maximal elements with respect to “$\leq$” are the trivial and the complete schemes on $V$ respectively: the basic relations of the former one are the reflexive relation $1_V$ and (if $|V| > 1$) its complement in $V^2$, whereas the relations of the later one are all binary relations on $V$.

Let $R$ be an arbitrary relation on the set $V$. Denote by $\text{Fis}(R)$ the smallest scheme with respect to “$\leq$” such that $R$ is a union of its basic relations. Let $\Gamma = (V, R)$ be a graph with vertex set $V$ and edge set $R$. By the scheme of $\Gamma$ we
mean \( \text{Fis}(\Gamma) = \text{Fis}(R) \). For example, if \( \Gamma \) is a complete graph or empty graph with at least 2 vertices, then its scheme is the trivial scheme of rank 2. One can check that if \( \text{Fis}(\Gamma) \) is an association scheme, then \( \Gamma \) is a regular graph.

In general, it is quite difficult to find the scheme of an arbitrary graph. In \([3]\), the scheme of a graph has been studied for some classes of graphs.

3. Circular-arc graphs

From \([1]\), for a given family \( \mathcal{F} \) of subsets of \( V \), one may associate an intersection graph. This is the graph whose vertex set is \( \mathcal{F} \), two different sets in \( \mathcal{F} \) being adjacent if their intersection is non-empty. Circular-arc graph is the intersection graph of a family of arcs of a circle.

**Lemma 3.1.** Let \( \Gamma \) be a graph on \( V \) with \( n \) vertices. Then \( \Gamma \) is a circular-arc graph if and only if there exists a function \( f : V \to \mathbb{AZ}_m \), for some \( m \), such that \( \Gamma \) is the intersection graph of the family \( \text{Im}(f) = \{ f(v) : v \in V \} \). Moreover, this function can be chosen such that

1. any element of \( \mathbb{AZ}_m \) is the end-point of at least one set in \( \text{Im}(f) \),
2. each set in \( \text{Im}(f) \) contains at least two elements.

**Proof.** To prove sufficient part let \( \Gamma \) be a circular-arc graph. Then by the definition it is the intersection graph of some arcs of a circle \( C \). Without loss of generality, we may assume that the end-points of any of these arcs are distinct. Let \( m \) be the number of these end-points and let \( A = \{ a_0, a_1, \ldots, a_{m-1} \} \) be the set of all of them. It is clear that \( m \leq 2n \). Here the indices of the points \( a_i \) are the elements of \( \mathbb{AZ}_m \); they are chosen in such a way that the point \( a_i \) precedes the point \( a_{i+1} \) in the clockwise order of the circle \( C \). Then for any vertex \( v \in V \) there exist uniquely determined elements \( i_v, j_v \in \mathbb{AZ}_m \) such that

\[
A_v := C_v \cap A = \{ a_{i_v}, a_{i_v+1}, \ldots, a_{j_v} \},
\]

where \( C_v \) is a subset of \( C \) which is the arc corresponding to \( v \). Moreover, it is easily seen that \( C_u \cap C_v \) is not empty if and only if \( i_u \in A_u \) or \( j_u \in A_u \) or \( i_u, j_u \in A_u \). Therefore, the vertices \( u \) and \( v \) are adjacent if and only if the set \( A_u \cap A_v \) is not empty. Now define a function \( f : V \to \mathbb{AZ}_m \) by

\[
f(v) = \{ i_v, i_v + 1, \ldots, j_v \}.
\]

Then \( \Gamma \) is the intersection graph of the family \( \text{Im}(f) \). Moreover, statements (1) and (2) immediately follow from the definition of \( f \).

Conversely, let \( m \leq 2n \) and \( f : V \to \mathbb{AZ}_m \) be a function such that \( \Gamma \) is the intersection graph of \( \text{Im}(f) \). Consider a circle \( C \) and choose \( m \) distinct points on it. We may label these points by the elements of \( \mathbb{AZ}_m \) such that these points appears in \( C \) in clockwise order. Since \( \text{Im}(f) \subset \mathbb{AZ}_m \) for each vertex \( v \in V \) there exist \( i_v, j_v \in \mathbb{AZ}_m \) such that \( f(v) = \{ i_v, i_v + 1, \ldots, j_v \} \). We correspond an arc \( C_v \subset C \), from \( i_v \) to \( j_v \) in clockwise order to the vertex \( v \). It is clear that the set \( f(u) \cap f(v) \) is not empty if and only if \( C_u \cap C_v \) is not empty. It follows that \( \Gamma \) is the intersection graph of the set \( \{ C_v : v \in V \} \). So it is a circular-arc graph. This completes the proof of the lemma.

The function \( f : V \to \mathbb{AZ}_m \) satisfying statements (1) and (2) and conditions of Lemma 3.1 is called the arc-function of the graph \( \Gamma \) and the number \( m \) is called
The length of $f$.

**Theorem 3.2.** Let $\Gamma = (V, R)$ be a non-empty circular-arc graph on $n$ vertices. Suppose that for any two vertices $u$ and $v$ in $V$ we have:

\[(3.1)\quad v \in uR \Rightarrow uR \not\subset \{v\} \cup vR.\]

Then there exists an arc-function $f$ of $\Gamma$ such that the following statements hold:

(i) no set of $\text{Im}(f)$ is a subset of another set of $\text{Im}(f)$,
(ii) the length of $f$ is equal to $n$,
(iii) any element $i \in \mathbb{Z}_n$ is the end-point of exactly two sets in $\text{Im}(f)$.

**Remark 3.3.** The graph $\Gamma$ satisfying condition (3.1) is a connected graph. Indeed, otherwise, it is easily seen that $\Gamma$ is an interval graph. On the other hand, each interval graph is chordal, and so it has a vertex whose neighborhood is a complete graph (see [1]) which contradicts the condition (3.1).

**Proof.** By Lemma 3.1 there exists an arc-function $f'$ of $\Gamma$ of length $m' \leq 2n$. Denote by $\sim$ the binary relation on $\mathbb{Z}_{m'}$ defined by $i \sim j$ if and only if for any $v \in V$, $i, j \in f'(v)$ or $i, j \not\in f'(v)$.

One can check that $\sim$ is an equivalence relation, and its equivalence classes belong to $\mathbb{Z}_{m'}$. By the definition of $\sim$ any element of $\text{Im}(f')$ is a disjoint union of some classes. Let us define a function $f$ such that for each $v \in V$, $f(v)$ is the set of $\sim$-classes contained in $f'(v)$. The equivalence classes of $\sim$ can be identified by $\mathbb{Z}_m$. By this identification we have $f(v) \in \mathbb{Z}_m$.

We claim that $f$ is an arc-function of $\Gamma$. Indeed, from the definition of $f$ it follows that for each two vertices $u$ and $v$, the set $f(u) \cap f(v)$ is not empty if and only if the set $f'(u) \cap f'(v)$ is not empty. Moreover, statement (1) of Lemma 3.1 is obvious. Thus, it is sufficient to verify that statement (2) of Lemma 3.1 occurs. Suppose on the contrary that there is a vertex $v \in V$ such that $f(v)$ contains exactly one element. Then $f'(v)$ is a class of the equivalence $\sim$. By condition (3.1) this implies that $v$ is an isolated vertex in $\Gamma$, which is impossible by Remark 3.3. Thus $f$ is an arc-function of $\Gamma$.

By Lemma 3.1 the graph $\Gamma$ is isomorphic to the intersection graph of the family $\text{Im}(f')$. Thus for two adjacent vertices $u$ and $v$ in $V$, if $f'(u) \subseteq f'(v)$ then any vertex in $V \setminus \{v\}$ which is adjacent to $u$ in $\Gamma$ is also adjacent to $v$. On the other hand, it is easy to see that $f'(u) \subseteq f'(v)$ is equivalent to $f(u) \subseteq f(v)$. Therefore, we have

\[(3.2)\quad f(u) \subseteq f(v) \Rightarrow uR \subset \{v\} \cup vR.\]

Hence, statement (i) follows from condition (3.1).

Statement (ii) is a consequence of statement (iii). First we will show that any element $i \in \mathbb{Z}_m$ is the end-point of exactly two sets in $\text{Im}(f)$. Suppose on the contrary that there is an element $i \in \mathbb{Z}_m$ which is an end-point of at least three sets of $\text{Im}(f)$. By statement (2) of Lemma 3.1 there are at least two sets $f(u)$ and $f(v)$ such that

\[i + 1 \in f(v) \cap f(u) \text{ or } i - 1 \in f(v) \cap f(u).\]
It follows that in any case \( f(v) \subseteq f(u) \) or \( f(u) \subseteq f(v) \). From (3.2), this contradicts condition (3.1). Thus any element \( i \in Z_m \) is the end-point of at most two sets in \( \text{Im}(f) \).

To complete the proof, suppose that there exists \( i \in Z_m \) which is an end-point of exactly one set, say \( f(v) \), in \( \text{Im}(f) \). By statement (2) of Lemma 3.1, we have \( i + 1 \in f(v) \) or \( i - 1 \in f(v) \). Suppose that the former inclusion holds. Then by (3.2) we have

\[
(3.3) \quad u \in vR \implies i \notin f(v) \cap f(u).
\]

If \( i + 1 \) is an end-point of \( f(v) \) then, since \( \Gamma \) does not contain any isolated vertex, so there is a vertex \( u \in V \) such that \( f(u) \cap f(v) \) is not empty. From (3.3) it follows that \( i + 1 \) is an end-point of \( f(u) \). Let \( w \in vR \), then the set \( f(v) \cap f(w) \) is not empty. From (3.3) we have \( i + 1 \in f(w) \) and so \( f(w) \cap f(u) \) is not empty. So, \( w \in uR \).

Therefore, in this case \( vR \subseteq \{u\} \cup uR \), that contradicts the condition (3.1). Thus we suppose \( i + 1 \) is not an end-point of \( f(v) \). In this case, statement (1) of Lemma 3.1 implies that \( i + 1 \) is an end-point of a set in \( \text{Im}(f) \), say \( f(u) \). From Lemma 3.1 we have \( f(u) \not\subseteq f(v) \) and it follows that \( f(v) \setminus \{i\} \subseteq f(u) \). Now from (3.3), we have \( vR \subseteq \{u\} \cup uR \), which contradicts the condition (3.1). If \( i - 1 \in f(v) \), by the same argument we have a contradiction again. Thus, any element \( i \in Z_m \) is the end-point of exactly two sets in \( \text{Im}(f) \). This completes the proof of the theorem.

\[ \square \]

Any arc-function \( f : V \to AZ_m \), satisfying conditions (i), (ii) and (iii) of Theorem 3.2 is called the reduced arc-function of the graph \( \Gamma \).

**Corollary 3.4.** Let \( \Gamma = (V, R) \) be a graph which satisfies the conditions of Theorem 3.2. Then \( n_R(v) = 2|f(v)| - 2 \) for each vertex \( v \in V \), where \( f \) is the reduced arc-function of \( \Gamma \).

**Proof.** Let \( v \in V \). From statement (iii) of Theorem 3.2 any \( i \in f(v) \) is the end-point of exactly two elements in \( \text{Im}(f) \). Therefore, we get

\[
(3.4) \quad n_R(v) \leq 2|f(v)| - 2.
\]

In fact, by statement (i) of Theorem 3.2 for each \( u \in vR \), exactly one of the endpoints of \( f(u) \) belongs to \( f(v) \). Thus we have equality in (3.4), and we are done.

\[ \square \]

**Proposition 3.5.** Let \( \Gamma = (V, R) \) be a non-empty circular-arc graph without twins. Then \( \Gamma \) is regular if and only if for any two vertices \( u \) and \( v \) in \( V \) we have:

\[
(3.5) \quad v \in uR \implies uR \not\subseteq \{v\} \cup vR.
\]

**Proof.** Suppose that \( \Gamma \) is regular and \( u \) and \( v \) are two adjacent vertices of the graph \( \Gamma \). Then \( |\{v\} \cup vR| = |\{u\} \cup uR| \). However, \( \{v\} \cup vR \neq \{u\} \cup uR \) because \( u \) and \( v \) are not twins. It follows that there exists a vertex in \( uR \), different from \( v \), which is not in \( vR \); and there exists a vertex in \( vR \), different from \( u \), which is not in \( uR \). Therefore, the condition (3.5) holds.

Conversely, suppose that \( \Gamma \) satisfies condition (3.5). By the same argument as Remark 3.3 the graph \( \Gamma \) is connected. Thus, it is sufficient to show that any two adjacent vertices \( u \) and \( v \) have the same degree. On the other hand, by Theorem 3.2...
there is a reduced arc-function $f : V \rightarrow AZ_n$, where $n = |V|$. So by Corollary 3.4 it is sufficient to show that $|f(v)| = |f(u)|$, or equivalently

$$|f(u) \setminus f(v)| = |f(v) \setminus f(u)|.$$  

(3.6)

Note that the set $f(u) \cap f(v)$ is not empty because $u$ and $v$ are adjacent. Moreover, by hypothesis, $u$ and $v$ are not twins so $f(v) \neq f(u)$. We may assume that

$$f(u) \cup f(v) \neq Z_n.$$  

(3.7)

Indeed, otherwise, we would have $f(u) \cup f(v) = Z_n$ and then from statement (i) of Theorem 3.2 any set in $\text{Im}(f)$ different from both $f(u)$ and $f(v)$ has one end-point in $f(v)$ and one end-point in $f(u)$. This implies that any vertex in $V \setminus \{u, v\}$ is adjacent to both $u$ and $v$, which is impossible because $u$ and $v$ are not twins.

Let $i \in f(u) \setminus f(v)$. Then by statement (iii) of Theorem 3.2 there are exactly two vertices $u_i, v_i \in V$, for which $i$ is the end-point of both $f(u_i)$ and $f(v_i)$, or equivalently due to (3.7) we have

$$\{i\} = f(u_i) \cap f(v_i).$$

Moreover, by statement (i) of Theorem 3.2 neither $f(u_i)$ nor $f(v_i)$ is a subset of $f(u)$. Now, from (3.7) it follows that the end-points of $f(u_i)$ and $f(v_i)$ different from $i$ is not in the set $f(u)$. Thus, exactly one of $f(u_i) \cup f(v_i)$ contains the set $f(v) \cap f(u)$. Assume that $f(u) \cap f(v) \subset f(u_i)$. Since, by statement (i) of Theorem 3.2 $f(v)$ is not a subset of $f(u_i)$, it follows that the end-point of $f(u_i)$, different from $i$, is in the set $f(v) \setminus f(u)$, denote this end-point by $j_i$, (see Fig. 1).

![Fig. 1: Some arcs of reduced arc-function $f$ of $\Gamma$](image)

So far we could define the mapping, $i \rightarrow j_i$, from $f(u) \setminus f(v)$ to $f(v) \setminus f(u)$. Now we claim that this mapping is bijection. To do so, we first prove that it is injective. Suppose on the contrary that there are $i, i' \in f(u) \setminus f(v)$ such that $j_i = j_{i'}$. Then $f(u_i) \subset f(u_{i'})$ or $f(u_{i'}) \subset f(u_i)$. However, in both cases this contradicts statement (i) of Theorem 3.2. Now, let $j \in f(v) \setminus f(u)$ then in a similar way there is a corresponding element of $f(u) \setminus f(v)$, say $i$. By statement (i) of Theorem 3.2 it is clear that $j_i = j$. This shows that the mapping is surjective.

The same argument can be done for the case $f(u) \cap f(v) \subset f(v_i)$. Thus (3.6) follows and this proves the proposition.

**Corollary 3.6.** Let $\Gamma$ be an $m$-regular circular-arc graph on $n$ vertices. Suppose that $m \geq 1$ and the graph has no twins. Then for each reduced arc-function $f$ of $\Gamma$ and each $v \in V$, we have $|f(v)| = \frac{m+2}{2}$.

**Proof.** By the hypothesis the graph $\Gamma$ is non-empty and without twins. Therefore, the hypothesis of Corollary 3.4 holds and we are done.
4. Graphs and schemes

In this section we prove some results on the scheme of a graph. In particular we will study a relationship between the lexicographic product of two graphs and the wreath product of their schemes.

**Lemma 4.1.** Let $\Gamma = (V, R)$ be a graph. For each integer $k$, let

$$R_k = \{(u, v) \in R : |uR \cap vR| = k\}.$$  

Then $R_k$ is a union of some basic relations of the scheme $\text{Fis}(\Gamma)$.

**Proof.** Let $S$ be the set of basic relations of $\text{Fis}(\Gamma)$. Then $R = \bigcup_{s \in S'} s$ where $S' \subset S$. It is sufficient to show that $R_k$ contains any relation from $S'$ whose intersection with $R_k$ is not empty. To do this, let $s$ be such a relation. Then there exists a pair $(u, v) \in s$ such that $|uR \cap vR| = k$. On the other hand, by the definition of intersection numbers we have

$$|uR \cap vR| = \sum_{r,t \in S'} c^s_{rt}.$$  

Thus the number $|uR \cap vR|$ does not depend on the choice of $(u, v) \in s$. By definition of $R_k$ this implies that $s \subset R_k$ as required. $\square$

**Theorem 4.2.** Let $\Gamma$ be a graph on the vertex set $V$ and let

$$E = \{(u, v) \in V \times V : u \text{ and } v \text{ are twins or } u = v\}.$$  

Then $E$ is an equivalence relation of the scheme $\text{Fis}(\Gamma)$. Moreover, if $\Gamma$ is a graph with association scheme then, it is isomorphic to lexicographic product of the graph $\Gamma_{V/E}$ and a complete graph.

**Proof.** Let $S$ be the set of basic relations of the scheme $\text{Fis}(\Gamma)$ and let $R$ be the edge set of $\Gamma$. Then there exists a subset $S'$ of $S$ such that

$$R = \bigcup_{s \in S'} s.$$  

To prove the first statement, we need to check that any non-reflexive basic relation $r \in S$ such that $r \cap E \neq \emptyset$ is contained in $E$. To do this, let $(u, v) \in r$. We claim that $(u, v) \in E$, or equivalently $u$ and $v$ are twins.

First we show that

$$uR \setminus \{v\} \subseteq vR \setminus \{u\}.  \tag{4.1}$$

If the set $uR \setminus \{v\}$ is empty, then $uR \setminus \{v\}$ is clear. Now, we may assume that there exists an element $w$ in $V$ such that $w \in uR \setminus \{v\}$. It is enough to show that $v$ is adjacent to $w$ in $\Gamma$. By (4.1) there exists a basic relation $s \in S'$ so that $(u, w) \in s$. Denote by $t$ the basic relation which contains $(v, w)$. It is sufficient to show that $t \in S'$.

We have $w \in us \cap vt^*$, thus $|us \cap vt^*| = c^s_{st^*} \neq 0$. Since intersection number does not depend on the choice of $(u, v) \in r$, for $(u', v') \in r$ we have

$$|u's \cap v't^*| = c^s_{st^*} \neq 0.  \tag{4.2}$$

On the other hand, by the choice of $r$ there exists $(u', v') \in r \cap E$. So by (4.3), there exists a vertex $w' \in V$ such that

$$w' \in u's \cap v't^*.  \tag{4.4}$$
Moreover, since \( s \in S' \), we have \( u' \in u'R \setminus \{v'\} \). On the other hand, \( u'R \setminus \{v'\} = v'R \setminus \{u'\} \), since \( u' \) and \( v' \) are twins. It follows that \( u' \in v'R \setminus \{u'\} \), so from (4.4) we conclude that \( u' \) is adjacent to \( v' \) in \( \Gamma \) and so from (4.4) we have \( t \in S' \). The converse inclusion of (4.2) can be proved in a similar way. Thus \( u \) and \( v \) are twins and the first statement follows.

To prove the second statement, suppose that \( \Gamma \) is a graph with association scheme. It is well-known fact that all classes of an equivalence relation of an association scheme have the same size, say \( m \), where \( m \) divides \( n = |V| \). Moreover, for each \( X \in V/E \) we have \( u, v \in X \) if and only if \( u \) and \( v \) are twins. Thus for each \( X \in V/E \) we have

\[
(4.5) \quad \Gamma_X \simeq K_m.
\]

Fix a class \( X_0 \in V/E \). For each \( X \in V/E \) choose an isomorphism \( f_X \in Iso(\Gamma_{X_0}, \Gamma_X) \). Then the mapping

\[
(4.6) \quad f : V \rightarrow V/E \times X_0 \quad v \mapsto (X, f_X^{-1}(v)),
\]

is a bijection, where \( X \) is a class of \( E \) containing \( v \). In order to complete the proof, we show that the above bijection is a required isomorphism:

\[
(4.7) \quad f \in Iso(\Gamma, \Gamma_{V/E}[\Gamma_{X_0}]).
\]

Take two different vertices \( u \) and \( v \) in \( V \), then \( f(u) = (X, u_0) \) and \( f(v) = (Y, v_0) \), where \( X, Y \in V/E \), \( u \in X \), \( v \in Y \) and \( u_0, v_0 \in X_0 \). It is enough to show that \( u \) and \( v \) are adjacent in \( \Gamma \) if and only if \( f(u) \) and \( f(v) \) are adjacent in \( \Gamma_{V/E}[\Gamma_{X_0}] \).

First, we assume that \( u \) and \( v \) are not twins. Then \( X \neq Y \). In this case, by definition of \( E \), if \( u \) and \( v \) are adjacent in \( \Gamma \) then any vertices in \( X \) and any vertices in \( Y \) are adjacent to each other. Also if \( X \) and \( Y \) be adjacent in \( \Gamma_{V/E} \), by definition of \( \Gamma_{V/E} \), there is a vertex in \( X \) and a vertex in \( Y \) which are adjacent. However, since all of the vertices in each class of \( V/E \) are twins, then \( u \) and \( v \) are adjacent in \( \Gamma \). Thus \( u \) and \( v \) are adjacent in \( \Gamma \) if and only if \( X \) and \( Y \) are adjacent in \( \Gamma_{V/E} \). Therefore, \( u \) and \( v \) are adjacent in \( \Gamma \) if and only if \( f(u) \) and \( f(v) \) are adjacent in \( \Gamma_{V/E}[\Gamma_{X_0}] \).

Now, we may assume that \( u \) and \( v \) are twins. Then \( X = Y \). However, \( u \) and \( v \) are twins, so they are adjacent in \( \Gamma \). Since, \( f_X \) is an isomorphism thus \( f_X^{-1}(u) \neq f_X^{-1}(v) \), so \( u_0 \neq v_0 \). From (4.5), it follows that \( u_0 \) and \( v_0 \) are adjacent in \( \Gamma_X \). Therefore, \( f(u) \) and \( f(v) \) are adjacent in \( \Gamma_{V/E}[\Gamma_{X_0}] \). Thus \( f \) is an isomorphism and (4.7) follows, as desired.

In the next theorem we show that the scheme of the lexicographic product of two graphs is smaller than the wreath product of their schemes. In general, we do not have equality here. For example, the scheme of the lexicographic product of two complete graphs is a scheme of rank 2, but the wreath product of their schemes has rank 3.

**Theorem 4.3.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be two graphs. Then the scheme \( \mathcal{X} = \text{Fis}(\Gamma_2[\Gamma_1]) \) is isomorphic to a fusion of the scheme \( \mathcal{Y} = \text{Fis}(\Gamma_1) \triangleright \text{Fis}(\Gamma_2) \). Moreover, if \( \Gamma_1 \) is a complete graph and \( \Gamma_2 \) is a graph without twins such that its scheme is association, then \( \mathcal{X} \) and \( \mathcal{Y} \) are isomorphic.
Proof. Let \( \Gamma_i = (V_i, R_i) \), \( \mathcal{X}_i = \text{Fis}(\Gamma_i) \) and \( S_i \) be the set of basic relations of \( \mathcal{X}_i \) for \( i = 1, 2 \). Then there exists \( S'_i \subset S_i \) such that
\[
(4.9) \quad R_i = \bigcup_{s \in S'_i} s.
\]
Let \( \Gamma \) be the lexicographic product of \( \Gamma_2 \) and \( \Gamma_1 \), and let \( R \) be the edge set of \( \Gamma \). Then we have
\[
R = \{((i, k), (j, l)) \in (V_2 \times V_1)^2 : (i, j) \in R_2 \text{ or } (k, l) \in R_1 \text{ with } i = j \}.
\]
Let \( \Gamma' \) be a graph on vertex set \( V_1 \times V_2 \) such that \( (k, i) \) and \( (l, j) \) are adjacent in \( \Gamma' \) if and only if \( (i, k) \) and \( (j, l) \) are adjacent in \( \Gamma \). Define \( \sigma : V_2 \times V_1 \to V_1 \times V_2 \) such that \( (i, k)^\sigma = (k, i) \). Then \( \sigma \in \text{Iso}(\Gamma, \Gamma') \). Thus \( \Gamma \) and \( \Gamma' \) are isomorphic and it follows that
\[
(4.10) \quad \text{Fis}(\Gamma)^{\sigma} = \text{Fis}(\Gamma').
\]
Let \( R' \) be the edge set of \( \Gamma' \), then
\[
R' = \{((k, i), (l, j)) \in (V_1 \times V_2)^2 : (i, j) \in R_2 \text{ or } (k, l) \in R_1 \text{ with } i = j \}
= \{((k, i), (l, j)) \in (V_1 \times V_2)^2 : (i, j) \in R_2 \} \cup \{((k, i), (l, j)) \in (V_1 \times V_2)^2 : (k, l) \in R_1 \text{ with } i = j \}.
\]
So, by (4.8) we have
\[
(4.11) \quad \text{Fis}(\Gamma') \leq \mathcal{Y},
\]
and from (4.9) the first statement follows.

To prove the second statement, let \( \Gamma_1 \) be a complete graph on \( n \) vertices and let \( \Gamma_2 \) be a graph without twins such that \( \mathcal{X}_2 \) is an association scheme. Then, \( \mathcal{X}_1 \times \mathcal{X}_2 \) is association and from the first statement it follows that \( \text{Fis}(\Gamma') \) is association. Thus \( 1_{V_1} \otimes 1_{V_2} \) is a basic relation of \( \text{Fis}(\Gamma') \).

If \( \Gamma_2 \) is an empty graph, then it is easy to see that we have equality in (4.11). So we may suppose that \( \Gamma_2 \) is a non-empty graph. Since it is a graph with association scheme, there exists a positive integer \( t \) such that \( \Gamma_2 \) is a \( t \)-regular graph. Let \( t_0(i, j) \) be the number of common neighbors of two adjacent vertices \( i \) and \( j \) in \( \Gamma_2 \). Since \( \Gamma_2 \) is without twins, we have
\[
(4.12) \quad t_0(i, j) < t - 1.
\]
Let \( u = (k, i) \) and \( v = (l, j) \) be two adjacent vertices in \( \Gamma' \). Since, for each \( i \in V_2 \) the graph \( \Gamma_{V_2 \times i} \) is isomorphic to \( \Gamma_1 \), and for each two adjacent vertices \( i \) and \( j \) in \( \Gamma_2 \) the set \( (V_1 \times i) \times (V_1 \times j) \) is a subset of \( R' \), thus we have
\[
(4.13) \quad |uR' \cap vR'| = \begin{cases} 
(n - 2) + tn, & i = j \\
2(n - 1) + t_0(i, j)n, & i \neq j.
\end{cases}
\]
Using (4.12), for each \( i \neq j \) we have
\[
(4.14) \quad 2(n - 1) + t_0(i, j)n < (n - 2) + tn.
\]
Define
\[
E := \{(u, v) \in R' : |uR' \cap vR'| = (n - 2) + tn\}.
\]
From (4.13) and (4.14) it follows that
\[ E = \bigcup_{i \in V_2} (V_1 \times i)^2 \setminus (V_1 \otimes V_2). \]
Now, from Lemma 4.1, the set \( E \) is a union of some of the basic relations of \( \text{Fis}(\Gamma') \). On the other hand, \( E = s \otimes V_2 \), where \( s \) is the non-reflexive basic relation of the scheme \( \mathcal{X}_1 \) of rank 2. However, \( s \otimes V_2 \) is a basic relation of the scheme \( \mathcal{X}_1 \wr \mathcal{X}_2 \). Therefore, from (4.11) it is obvious that \( E \) is a basic relation of \( \text{Fis}(\Gamma') \). Hence,
\[ (4.15) \quad F = E \cup (V_1 \otimes V_2) \]
is an equivalence relation of the scheme \( \text{Fis}(\Gamma') \).

The scheme \( \mathcal{X}_1 \wr \mathcal{X}_2 \) is the minimal scheme which contains an equivalence \( F \) such that for each class \( X \in V/F \), the scheme \( (\mathcal{X}_1 \wr \mathcal{X}_2)_{V/F} \) is isomorphic to \( \mathcal{X}_1 \). In order to prove equality in (4.11), it is sufficient to show that \( \text{Fis}(\Gamma') \) have the above property.

Let \( X \in V/F \). Then by (4.15), the scheme \( \text{Fis}(\Gamma')_{V/F} \) is isomorphic to the scheme \( \mathcal{X}_1 \). Moreover, from (4.11) it follows that
\[ (4.16) \quad \text{Fis}(\Gamma')_{V/F} \leq \mathcal{X}_2. \]
However, the edge set of \( \Gamma_2 \) is a union of some of the basic relations of \( \text{Fis}(\Gamma')_{V/F} \). Thus we have equality in (4.16), and we are done.

\[ \square \]

**Remark 4.4.** Let \( \Gamma \) and \( \Gamma' \) be two graphs with the same vertex set. Suppose that the edge set of \( \Gamma' \) is a union of some basic relations of the scheme of \( \Gamma \). Then \( \text{Fis}(\Gamma') \leq \text{Fis}(\Gamma) \).

5. **Elementary circular-arc graphs**

Given integers \( n \) and \( k \) such that \( 0 \leq 2k + 1 < n \), set \( C_{n,k} = \text{Cay}(\mathbb{Z}_n, S) \) where \( S = \{ \pm 1, \ldots, \pm k \} \). It immediately follows that \( C_{n,k} \) is a 2\( k \)-regular graph without twins. Note that \( C_{n,0} \) is an empty graph and \( C_{n,1} \) is an undirected cycle on \( n \) vertices.

From definition, one can verify that \( C_{n,k} \) is the graph with vertex set \( V = \mathbb{Z}_n \) in which two vertices \( i \) and \( j \) are adjacent if and only if
\[ \{i, \ldots, k+i\} \cap \{j, \ldots, k+j\} \neq \emptyset. \]
Suppose that \( f : V \to \mathbb{A} \mathbb{Z}_n \), such that \( f(i) = \{i, \ldots, k+i\} \). Then the graph \( C_{n,k} \) is the intersection graph of the family \( \text{Im}(f) \). Thus, by Lemma 3.1 we conclude that \( C_{n,k} \) is a circular-arc graph, and we call it **elementary circular-arc graph**.

**Example 5.1.** For \( n = 2k + 2 \), two different vertices \( i \) and \( j \) are adjacent in \( C_{n,k} \) if and only if \( j \neq i + k + 1 \). Thus \( C_{n,k} \) is isomorphic to a graph on \( n \) vertices which is obtained from a complete graph by removing the edges of a perfect matching. It is easy to check that the scheme of this graph is isomorphic to \( \mathcal{X}_1 \wr \mathcal{X}_2 \), where \( \mathcal{X}_1 \) is a rank 2 scheme on 2 points and \( \mathcal{X}_2 \) is a rank 2 scheme on \( k + 1 \) points.

**Theorem 5.2.** A regular circular-arc graph without twins is elementary.

**Proof.** Let \( \Gamma \) be a circular-arc graph with the vertex set \( V \) where \( n = |V| \). Suppose that it has no twins. Then by Proposition 3.3 and Theorem 3.2 there exists a reduced arc-function \( f \) of \( \Gamma \), such that for each vertex \( v \in V \) we have \( f(v) = \{i_v, \ldots, j_v\} \). Define a bijection from \( V \) to \( \mathbb{Z}_n \), the vertex set of \( C_{n,k} \), such
that \( v \rightarrow i_v \). From Corollary 3.4 we conclude that \( \Gamma \) is a \( 2k \)-regular graph. Then \( k = |f(v)| - 1 \) for any vertex \( v \) by Corollary 3.6. Hence \( j_v = i_v + k \). By Lemma 3.1 two vertices \( u \) and \( v \) in \( V \) are adjacent if and only if \( f(u) \cap f(v) \neq \emptyset \). It follows that \( i_u \) and \( i_v \) are adjacent if and only if \( \{i_u, \ldots, k + i_u\} \cap \{i_v, \ldots, k + i_v\} \neq \emptyset \). Therefore, the bijection defined above gives the required isomorphism.

**Theorem 5.3.** Let \( n \) and \( k \) be two positive integers such that \( 2k + 2 < n \). Then the scheme of the graph \( C_{n,k} \) is isomorphic to a dihedral scheme.

**Proof.** Let \( R \) be the edge set of \( C_{n,k} \). For two vertices \( i, j \in \mathbb{Z}_n \), we define \( d(i, j) \) be the distance of \( i \) and \( j \) in the graph \( C_{n,1} \). Suppose that \( i \) and \( j \) be two adjacent vertices in \( C_{n,k} \). Then by definition of \( C_{n,k} \), we have \( d(i, j) \leq k \). Without loss of generality we may assume that the vertices \( \{i + 1, i + 2, \ldots, j - 1\} \) are between \( i \) and \( j \). Thus they are adjacent to both of \( i \) and \( j \) in the graph \( C_{n,k} \). Moreover, the vertices \( \{j - k, j - k + 1, \ldots, i - 1\} \) and \( \{j + 1, j + 2, \ldots, i + k\} \) are adjacent to both of \( i \) and \( j \) too. Thus the latter three sets are subsets of \( iR \cap jR \), which are of size \( d(i, j) - 1, k - d(i, j) \) and \( k - d(i, j) \) respectively. Moreover, since \( n > 2k + 2 \), they are disjoint. On the other hand, \( B_i := \{i - k, i - k + 1, \ldots, j - k - 1\} \) is the set of all other vertices which are adjacent to \( i \), and \( B_j := \{i + k + 1, i + k + 2, \ldots, j + k\} \) is the set of all other vertices which are adjacent to \( j \), (see Fig. 2).

![Fig. 2: Some vertices of the graph C_{n,1} and the sets B_i and B_j](image)

It is clear that \( B_i \) and \( B_j \) are disjoint from the above three subsets. In addition, we have \( |B_i| = |B_j| = d(i, j) \). Moreover, since \( n > 2k + 2 \), the vertex \( j - k - 1 \) is not in \( B_j \) and the vertex \( i + k + 1 \) is not in \( B_i \). It follows that \( B_i \neq B_j \), and

\[
(5.1) \quad |B_i \cap B_j| < d(i, j).
\]

On the other hand, \( B_i \cap B_j \subset iR \cap jR \) and thus

\[
(5.2) \quad |iR \cap jR| = (d(i, j) - 1) + 2(k - d(i, j)) + |B_i \cap B_j| = 2k - d(i, j) - 1 + |B_i \cap B_j|.
\]

Now, set \( R_{2k-2} := \{(i, j) \in R : |iR \cap jR| = 2k - 2\} \). Then \( R_{2k-2} \) is a symmetric relation. Moreover, from (5.2) we see that

\[
(5.3) \quad (i, j) \in R_{2k-2} \iff |B_i \cap B_j| = d(i, j) - 1.
\]
If \( d(i, j) = 1 \), then by (5.1) we have \( |B_i \cap B_j| = 0 \). Thus, from (5.3) we see that (5.4)
\[
d(i, j) = 1 \Rightarrow (i, j) \in R_{2k-2}.
\]

If \( 1 < d(i, j) \leq k \), then \( j - k - 2 \notin B_i \setminus B_j \) and \( i + k + 2 \notin B_j \setminus B_i \). It follows that \( |B_i \cap B_j| < d(i, j) - 1 \). Thus, from (5.3) we have \( (i, j) \notin R_{2k-2} \). Then using (5.4) we have \( (i, j) \in R_{2k-2} \) if and only if \( d(i, j) = 1 \).

It follows that the graph \((\mathbb{Z}_n, R_{2k-2})\) is isomorphic to an undirected cycle on \( n \) points, say \( C_n \). By Remark 4.1, we conclude that \( R_{2k-2} \) is union of some basic relations of \( \text{Fis}(C_{n,k}) \). Thus by Lemma 4.4 we have
\[
\text{Fis}(C_n) \leq \text{Fis}(C_{n,k}).
\]

It is well-known that \( \text{Fis}(C_n) = \text{Inv}(D_{2n}) \), where \( D_{2n} \) is a dihedral group on \( n \) elements. So, in order to complete the proof of the theorem it is enough to show that
\[
\text{Fis}(C_{n,k}) \leq \text{Fis}(C_n).
\]
Equivalently, it is sufficient to verify that
\[
\text{Aut}(\text{Fis}(C_n)) \leq \text{Aut}(\text{Fis}(C_{n,k})).
\]

Since the automorphism group of a graph is equal to the automorphism group of its scheme, it is sufficient to show that \( \text{Aut}(C_n) \leq \text{Aut}(C_{n,k}) \).

Since \( \text{Aut}(C_n) \) is the dihedral group \( D_{2n} \), and \( D_{2n} \) is generated by automorphisms \( \sigma \) and \( \delta \) where \( i^\sigma = i + 1 \) and \( i^\delta = n - i \) for each \( i \in \mathbb{Z}_n \). It is enough to show that
\[
\sigma, \delta \in \text{Aut}(C_{n,k}).
\]

Let \( i, j \in \mathbb{Z}_n \). Two vertices \( i \) and \( j \) are adjacent in \( C_{n,k} \) if and only if
\[
\{i, \ldots, k + i\} \cap \{j, \ldots, k + j\} \neq \emptyset \Leftrightarrow \{i + 1, \ldots, k + i + 1\} \cap \{j + 1, \ldots, k + j + 1\} \neq \emptyset \Leftrightarrow
\]
(5.5)
\[
\{i^{\sigma}, \ldots, k + i^{\sigma}\} \cap \{j^{\sigma}, \ldots, k + j^{\sigma}\} \neq \emptyset.
\]

Moreover, we have (5.5) if and only if \( i^{\sigma} \) and \( j^{\sigma} \) are adjacent in \( C_{n,k} \). Thus \( \sigma \in \text{Aut}(C_{n,k}) \). In a similar way we can show that \( \delta \in \text{Aut}(C_{n,k}) \), this completes the proof. \( \square \)

6. Proof of the main theorems

Proof of Theorem 1.1. We first prove the necessity condition of the theorem. Let \( \Gamma = (V, R) \) be a circular-arc graph with association scheme and \( |V| = n \). Denote by \( E \) the equivalence relation on \( V \) defined in Theorem 1.2. Then by this theorem the graph \( \Gamma \) is isomorphic to lexicographic product of the graph \( \Gamma_{V/E} \) and a complete graph. In particular, it is easy to see that \( \Gamma_{V/E} \) is also a circular-arc graph and has no twins. So, to complete the proof it is enough to show that \( \Gamma_{V/E} \) is an elementary circular-arc graph.

If \( \Gamma_{V/E} \) is empty, then it is isomorphic to \( C_{n,0} \) with \( m = |V/E| \), and we are done. We suppose that \( \Gamma_{V/E} \) is non-empty. The graph \( \Gamma \) is regular, because it is a graph with association scheme. By the definition of \( E \) this implies that the graph \( \Gamma_{V/E} \) is regular too. From Corollary 3.6 the graph \( \Gamma_{V/E} \) is 2k-regular for some integer \( k > 0 \). Therefore, from Theorem 5.2 the latter graph is isomorphic to the elementary circular-arc graph \( C_{m,k} \). Thus \( \Gamma \) is isomorphic to lexicographic product of an elementary circular-arc graph and a complete graph.
Conversely, let $\Gamma_1$ be a complete graph and let $\Gamma_2$ be an elementary circular-arc graph. If $\Gamma_2$ be an empty graph then it is a graph with association scheme. If it is a non-empty elementary circular-arc graph then from Example 5.1 and Theorem 5.3 we conclude that $\Gamma_2$ is a graph with association scheme. On the other hand, the wreath product of two association scheme is association. Thus, from Theorem 4.3 the scheme of $\Gamma_2[\Gamma_1]$ is association. This completes the proof of the theorem. □

**Proof of Theorem 1.2** We first assume that $\mathcal{X}$ is an association circular-arc scheme. Then there is a circular-arc graph $\Gamma$ such that $\text{Fis}(\Gamma) = \mathcal{X}$. From Theorem 1.1, the graph $\Gamma$ is isomorphic to the lexicographic product of an elementary circular-arc graph and a complete graph. On the other hand, if the elementary circular-arc graph is empty then its scheme is of rank 2. Otherwise, from Example 5.1 and Theorem 5.3 the scheme of an elementary circular-arc graph is isomorphic to the wreath product of a rank 2 scheme on 2 points and a rank 2 scheme, or it is isomorphic to a dihedral scheme. Therefore, in any case the scheme of an elementary circular-arc graph is association. Note that in the first two cases the scheme of an elementary circular-arc graph is forestal. Moreover, any elementary circular-arc graph is without twins. Hence, from Theorem 4.3 it follows that $\text{Fis}(\Gamma)$ is isomorphic to the wreath product of a rank 2 scheme and the scheme of an elementary circular-arc graph which is either forestal or dihedral.

Conversely, assume that $\mathcal{X}$ is a circular-arc scheme such that it is isomorphic to the wreath product of a rank 2 scheme and a scheme which is either forestal or dihedral. Since any rank 2 scheme, any forestal scheme and any dihedral scheme are association, it is enough to note that the wreath product of two association schemes is an association scheme. Thus $\mathcal{X}$ is an association scheme and the proof is complete. □

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