Smooth approximations of the conical Kähler–Ricci flows

Yuanqi Wang

Abstract In this note, we show that the conical Kähler–Ricci flows introduced in Chen and Wang (Bessel functions, Heat kernel and the conical Kähler–Ricci flow. J. Funct. Anal. 269(2), 2013) exist for all time $t \in [0, \infty)$ in the weak sense as in Definition 1.2. As a key ingredient of the proof, we show that a conical Kähler–Ricci flow is actually the limit of a sequence of smooth Kähler–Ricci flows.

Contents

1 Introduction ............................................. 835
2 Approximating the initial metric .......................... 839
3 Construction of the approximation flows and proofs of Theorem 1.9 ................ 841
4 Local Harnack inequality ...................................... 846
5 Bootstrapping of the conical Kähler–Ricci flow and proofs of the main results in the introduction 853
References ................................................ 855

1 Introduction

This is a following up note of [8]. Let $M$ be a polarized Kähler manifold and $D$ is a smooth divisor of the anti-canonical line bundle. Suppose the “twisted” first Chern class ($\beta \in (0, 1]$)

$$C_{1,\beta} = C_1(M) - 2\pi (1 - \beta)[D]$$
has a definite sign. Suppose \( \omega_0 \) is a smooth Kähler metric in \( C_{1,\beta} \) (if \( C_{1,\beta} > 0 \)) or \(-C_{1,\beta}\) (if \( C_{1,\beta} < 0 \)). One important question is to study the existence of the conical Kähler–Einstein metric in \((M, [\omega_0], (1 - \beta)[D])\). A metric \( \omega_\phi \) (cohomologous to \( \omega_0 \)) is said to be Kähler–Einstein if

\[
Ric(\omega_\phi) = \beta \omega_\phi + 2\pi (1 - \beta)[D].
\]

This problem has been studied carefully by many authors, for instance [1–3,11,15,18,23], etc. In particular, “conical Kähler–Einstein metric” is a key ingredient in the recent solution of existence problem for Kähler–Einstein metric with positive scalar curvature [5–7]. In light of these exciting development, we introduce the notion of conical Kähler–Ricci flow \([8]\)

\[
\frac{\partial \omega_g}{\partial t} = \beta \omega_g - Ric(g) + 2\pi (1 - \beta)[D],
\]

(1)

to attack the existence problem of conical Kähler–Einstein metrics and conical Kähler–Ricci solitons. With respect to the potential \( \phi \), Eq. (1) is written as

\[
\frac{\partial \phi}{\partial t} = \log \frac{\omega^n_\phi}{\omega^n_0} + \beta \phi + h + (1 - \beta) \log \|S\|^2,
\]

(2)

where \( h \) is a smooth function.

In [8], we establish short time existence for (1) and (2), initiated from any \((\alpha, \beta)\)-conical Kähler metric (see Definition 1.1 for the definition of \((\alpha, \beta)\)-metrics). In this paper we want to establish the long time existence of this flow in a weaker sense.

Most of the notations in this article follows those of [8]. For the readers’ convenience, we introduce some key definitions from [8] here.

**Definition 1.1** \((\alpha, \beta)\) conical Kähler metric: For any \( \alpha \in (0, \min\{1, \beta - 1, 1\}) \), a Kähler form \( \omega \) is said to be an \((\alpha, \beta)\) conical Kähler metric on \((M, (1 - \beta)D)\) if it satisfies the following conditions.

1. \( \omega \) is a closed positive \((1, 1)\)-current over \( M \).
2. For any point \( p \in D \), there exists a holomorphic chart \( \{z, u_i, \ i = 1, \ldots, n - 1\} \) such that in this chart, \( \omega \) is quasi-isometric to the standard cone metric

\[
\omega_\beta = \beta^2 |z|^{2\beta - 2} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} + \frac{\sqrt{-1}}{2} \sum_{j=1}^{n-1} du_j \wedge d\bar{u}_j.
\]

3. There is a \( \phi \in C^{2,\alpha,\beta}(M) \) such that

\[
\omega = \omega_0 + i \partial \bar{\partial} \phi.
\]

The definition of the function space \( C^{2,\alpha,\beta}(M) \) is due to Donaldson in [10]. One could also see section 2 of [8].
The following model metric defined in [10] satisfies the above definition.

\[ \omega_D = \omega + \delta i \partial \bar{\partial} |S|^{2\beta}, \] where \( \delta \) is a small enough number.

Next we define the notion of weak flows, following the definitions of weak conical Kähler–Einstein metrics by Gunancia and Paun [12] and Yao [26].

**Definition 1.2** A solution \( \phi(t) \) for \( t \in [0, T) \) with \( \phi(0) = \phi_0 \) to (1) is called a weak conical Kähler–Ricci flow if for all \( \tilde{T} < T \), the following hold.

- \[ \phi(t) \in C^{2+\alpha,1+\frac{\alpha}{2}}((M \setminus D) \times (0, \tilde{T})) \cap C^{\alpha,\frac{\alpha}{2}}(M \times (0, \tilde{T})), \phi_0 \in C^{\alpha}(M). \]
- \[ \lim_{t \to 0} |\phi - \phi_0|_\alpha = 0 \text{ over } M. \]
- \[ |\frac{\partial \phi}{\partial t}|_{0,[0,\tilde{T}]} + |\text{tr}_{\omega_D} \omega \phi|_{0,[0,\tilde{T}]} + |\text{tr}_{\omega_D} \omega D|_{0,[0,\tilde{T}]} \leq C(\omega \phi_0, \tilde{T}) \text{ over } M \setminus D. \]
- \[ \omega \phi \geq C(T) \omega_D \text{ over } M \setminus D. \]

In particular, a closed positive \((1,1)\)-current \( \omega = \omega_D + i \partial \bar{\partial} \phi \) is called a weak \((\alpha, \beta)\)-metric if the following holds.

- \[ |\phi|_\alpha \leq C \text{ over } M, \phi \text{ is } C^{2,\alpha} \text{ away from } D. \]
- \[ \omega_\phi \geq C(\tilde{T}) \omega_D \text{ over } M \setminus D. \]
- \[ |\text{tr}_{\omega_D} \omega \phi|_0 + |\text{tr}_{\omega_D} \omega D|_0 \leq C \text{ over } M \setminus D. \]

**Remark 1.3** For convention on Hölder norms, see Definition 2.2. From now on whenever we say “conical” or “\((\alpha, \beta)\)” we mean strong conical or strong \((\alpha, \beta)\), to differ from the weak conical cases. Notice a random weak \((\alpha, \beta)\)-metric does not properly possess the correct cone structure along the singular divisor.

Now we state our main theorem on the long-time existence of the weak flow.

**Theorem 1.4** Suppose \( \omega_{\phi_0} \) is a \((\alpha, \beta)\)-conical metric. Then there exists a weak conical Kähler–Ricci flow \( \omega_\phi(t), t \in [0, \infty) \) initiated from \( \phi_0 \). Moreover, the weak flow above coincides with the strong flow given by Theorem 1.2 in [8] till whenever the strong flow exists.

**Remark 1.5** For smooth Kähler–Ricci flows, the global existence is proved by Cao [4]. For conical Kähler–Ricci flow, when \( n = 1 \), this is recently proved by Yin [27], and also by Mazzeo et al. [20] with different function spaces. For higher dimensions, we believe the parabolic version of Brendle’s work can solve the long time existence problem when \( \beta \leq \frac{1}{2} \).

**Remark 1.6** Recently, the author learned Liu and Zhang also consider the conical Kähler–Ricci flows on Fano manifolds. In [19], they also obtain long time solutions for conical Kähler–Ricci flow on Fano manifolds. Moreover, they obtain convergence of their conical Kähler–Ricci flows in some cases.

**Remark 1.7** For simplicity, we only present the case with one smooth divisor. Our proof certainly works with multiple smooth divisors with no intersections and with possibly different angles along each component.

**Remark 1.8** If the manifold is not Fano or the twisted first Chern has mixed sign, Theorem 1.4 still holds as long as the Kähler class remains to be fixed along the flow.
The next theorem is almost equivalent to Theorem 1.4.

**Theorem 1.9** Suppose $\omega_{\phi_0}$ is a weak $(\alpha, \beta)$-metric such that

$$\omega_{\phi_0} \in C_{1,\beta}(M), \quad F_{\phi_0} = \log \frac{|S|^{2-2\beta} \omega^n_{\phi_0}}{\omega^n_0} \in C^{2,\alpha,\beta}.$$

Then there exists a weak conical Kähler–Ricci flow $\omega_{\phi}(t)$, $t \in [0, \infty)$ initiated from $\phi_0$.

Moreover, if in addition $\phi_0 \in C^{2,\alpha,\beta}$, then the weak flow above is strong and coincides with the strong flow given by Theorem 1.2 in [8] till whenever the strong flow exists.

**Remark 1.10** Though Theorem 1.9 does not require the initial metric to be (strongly) $(\alpha, \beta)$, it needs the volume form of the initial metric to have $C^{2,\alpha,\beta}$ regularity, which is a relatively strong condition. Nevertheless, Theorem 1.14 guarantees that the conical flow (1) “smoothes” a $(\alpha, \beta)$-metric immediately to possess maximal regularity, so the requirements of Theorem 1.9 are satisfied.

Next we state our result on the smooth approximations of the conical flows.

**Theorem 1.11** Suppose the conical flow $\omega_{\phi(t)}$ (solution to (1)) exists for $t \in [0, T)$. Then, for any $0 < a_0 < T$, the smooth flows $\omega_{\phi_\varepsilon(t)}$, $t \in [a_0, T)$ in (15) approximate $\omega_{\phi(t)}$ in $C^{\alpha,\frac{\alpha}{2}}$-sense away from $D$ over $[a_0, T)$ i.e.

For all $\delta$, $\lim_{\varepsilon \to 0} |\omega_{\phi_\varepsilon(t)} - \omega_{\phi(t)}|_{C^{\alpha,\frac{\alpha}{2}}[M\setminus T_\delta(D)] \times [a_0, T)} = 0.$

Consequently, for any $t \in [a_0, T)$, $\omega_{\phi_\varepsilon(t)}$ approximates $\omega_{\phi(t)}$ in the Gromov–Hausdorff sense.

**Remark 1.12** The $T_\delta(D)$ is the tubular neighborhood of the divisor $D$ of radius $\delta$ (with respect to the smooth reference metric $\omega_0$). Actually it doesn’t matter whether we use $\omega_0$ or the conic metric $\omega_D$.

**Remark 1.13** Theorem 1.11 indicates a phenomenon which we never expected: the cone singularity structure is somehow stable even under smooth Kähler–Ricci flow. Namely, starting from the smooth metric $\omega_{\phi_\varepsilon(\frac{T}{2})}$ which is close to a conical metric in Gromov–Hausdorff sense, the smooth flow $\omega_{\phi_\varepsilon(t)}$ stays close to conical metrics at least up to finite time. A natural and interesting question to ask is: what’s the behaviour of the flow $\omega_{\phi_\varepsilon(t)}$ (solution to 15) as $t \to \infty$?

The next theorem indicates that the conical flow constructed in Theorem 1.2 in [8] has the “smoothing” property.

**Theorem 1.14** There exists a uniform $C$ as in Definition 2.1 with the following properties. Assumptions as in Theorem 1.2 of [8]. The Ricci curvature of the (strong) CKRF satisfies

$$|Ric| \leq Ct^{-1} \text{ over } M \setminus D \text{ for all } t \in [0, t_0],$$
where $t_0$ is a lower bound of the existence time for the conical flow (1) and $t_0$ is small enough with respect to the initial metric $\omega_{0\phi}$. Moreover, we have the following weighted Schauder estimate for $\frac{\partial \phi}{\partial t}$.

$$\left| \frac{\partial \phi}{\partial t} \right|_{2+\alpha, 1+\frac{\alpha}{2}, \beta, M \times [0, t_0]} \leq C.$$

The definition of the norm $|\ast|_{2+\alpha, 1+\frac{\alpha}{2}, \beta, M \times [0, t_0]}$ can be found in section 2 of [8].

For the proof of Theorems 1.4, 1.9, and 1.11, we notice two beautiful recent work by Guenancia and Paun [12] and Yao [26], where they prove independently the existence of weak conical Kähler–Einstein metrics under appropriate assumptions—these approaches have been taken up by others before, the new feature in their work is that the approximation stays uniformly quasi-isometric to the approximated model metrics. While we initially plan to use ideas from Yao where we need to do local cutting and pasting, we notice the beautiful construction of global barrier function in Guenancia–Paun which fits into what we want very nicely. So we end up adopting Guenancia–Paun’s method, although we believe Yao’s idea can be made to work as well.

2 Approximating the initial metric

We would like to make following convention on the constants and Hölder norms in this paper, similar to that of [8].

**Definition 2.1** Without further notice, the “C” in each estimate means a constant depending on the dimension $n$, the angle $\beta$, the background objects $(M, \omega_0, L, h, D, \omega_D)$, the $\alpha$ (and $\dot{\alpha}$ if any) in the same estimate or in the corresponding theorem (proposition, corollary, lemma), and finally the time $T$. Moreover, the “C” in different places might be different.

**Definition 2.2** *(Convention on Hölder norms)* For the various intrinsic Hölder norms with respect to $\omega_\beta$ (for example the $C^{\alpha, \beta}$-norm and its parabolic counterpart $C^{\alpha, \frac{\alpha}{2}, \beta}$), we mainly refer to section 2 of [8] for the full definitions. The point is that, in this article we mainly consider usual Hölder spaces and norms (without any additional “$\beta$” in the notations). The reason is that Hölder continuity with respect to $\omega_\beta$ is equivalent to Hölder continuity in the usual sense (apart from a difference of Hölder exponents). For a precise statement, see Lemma 4.4.

It’s helpful to recall the definition of the parabolic Hölder norm. For any parabolic cylinder $B \times [T_1, T_2]$, the $C^{\alpha, \frac{\alpha}{2}}(B \times [T_1, T_2])$-norm is defined as

$$|u|_{\alpha, \frac{\alpha}{2}, B \times [T_1, T_2]} = \sup_{(x, t_1), (y, t_2) \in B \times [T_1, T_2]} \frac{|u(x, t_1) - u(y, t_2)|}{|x - y|^\alpha + |t_1 - t_2|^\frac{\alpha}{2}} + |u|_{0, B \times [T_1, T_2]}.$$
where \(|\mu|_{0, B \times [T_1, T_2]}\) is the usual \(C^0\)-norm. The \(C^{\alpha, \frac{\gamma}{2}}(B \times [T_1, T_2])\)-space contains exactly those functions with finite \(|\alpha, \frac{\gamma}{2}, B \times [T_1, T_2]|\)-norm. The global norms over \(M\) or \(M \times [T_1, T_2]\) are defined by summing up the norms in each coordinate chart. This is very flexible: if we use the intrinsic coordinates of \(\omega\) near \(D\), we obtain the \(C^{\alpha, \frac{\gamma}{2}}(M \times [T_1, T_2])\)-space; if we use the usual smooth coordinates near \(D\), then we obtain the usual Hölder space \(C^{\alpha, \frac{\gamma}{2}}(M \times [T_1, T_2])\).

To construct the approximation flows, the first step is to construct an approximation of the initial metric in Theorem 1.9. From now on we will repeatedly apply Theorem 4.1.

The initial metric \(\omega_{\phi(0)}\) satisfies

\[
\omega^n_{\phi(0)} = \frac{e^{F(0)}}{|S|^{2-2\beta}} \omega^n_0. \tag{3}
\]

We try to smooth \(F(0)\) first. We consider the reference metric \(\omega_{\epsilon}\), introduced in section 3.1 of [12] as

\[
\omega_{\epsilon} = \omega_0 + \frac{1}{N} i \partial \bar{\partial} \chi_\beta(\epsilon + |S|^2), \quad \text{where} \quad \chi_\beta(\epsilon + y) = \beta \int_0^y \frac{(\epsilon + x)^\beta - \epsilon^\beta}{x} \, dx \tag{4}
\]

and \(N\) is a big enough number. As in [12], We also denote

\[
\Psi_{\epsilon, \rho} = \chi_\rho(\epsilon + |S|^2).
\]

In application we always let \(\rho\) to be small with respect to \(\beta\), as in [12].

**Theorem 2.3** Suppose \(\phi\) is a \(C^{1, 1, \beta}\) solution to the following equation

\[
\omega^n_{\phi} = \frac{e^F}{|S|^{2-2\beta}} \omega^n_0.
\]

Suppose \(F \in C^{2, \alpha, \beta}\). Then there exists an approximation sequence of \(C^{4, \alpha'}\) functions \(\hat{\phi}_\epsilon\) such that

- \(|\hat{\phi}_\epsilon|_{\alpha'} \leq C_F\),
- \(\frac{1}{\epsilon^F} \omega_\epsilon \leq \omega_{\hat{\phi}_\epsilon} \leq C_F \omega_\epsilon\),
- \(\lim_{\epsilon \to 0} |\hat{\phi}_\epsilon - \phi|_{\alpha'} = 0\),

where \(C_F\) only depends on \(|F|_{2, \alpha, \beta}\) and the data in Definition (2.1).

**Proof of Theorem 2.3** We smooth \(F\) out by solving the following equation

\[
\Delta_{\omega_{\epsilon}} F_\epsilon = \Delta_{\omega_D} F + a_\epsilon, \tag{5}
\]

\(a_\epsilon\) is chosen such that

\[
\int_M (\Delta_{\omega_D} F + a_\epsilon) \omega^n_{\epsilon} = 0, \tag{6}
\]
and $F_\epsilon$ is normalized so that

$$\int_M \frac{e^{F_\epsilon}}{(|S|^2 + \epsilon)^{1-\beta}} \omega_0^n = 1. \quad (7)$$

Notice that (6) directly implies $\lim_{\epsilon \to a} a_\epsilon = 0$. Using the $\epsilon$-independent bounds on the global Sobolev and Poincare constants in Remark 4.6, we deduce the follow $L^\infty$ bound via Moser’s iteration.

$$|F_\epsilon|_{L^\infty} \leq C. \quad (8)$$

Using (8) and the elliptic Harnack-inequality in Theorem 4.1, we estimate

$$[F_\epsilon]_{\alpha} \leq C. \quad (9)$$

Therefore, $F_\epsilon$ subconverge to some $F_\infty$ in $C^\alpha$, $\alpha < \alpha$. Moreover

$$\Delta_\omega F_\infty = \Delta_\omega F \text{ over } M \setminus D. \quad (10)$$

Since $F_\infty \in C^\alpha$, then by Jeffres’ trick in [14], we have $F_\infty = F$. The advantage of smoothing $F$ using Eq. (5) is that by Guenancia–Paun, the condition

$$\Delta_\omega F_\epsilon \geq -C \quad (11)$$

gives us the Laplacian estimate for the smoothing of the initial metric. Namely, we smooth $\omega_\phi(0)$ by considering the following equation

$$\omega_\phi^n = \frac{e^{F_\epsilon}}{(|S|^2 + \epsilon)^{1-\beta}} \omega_0^n, \quad \sup_M \phi_\epsilon = 0. \quad (12)$$

By Yau, Eq. (12) admits a solution $\hat{\phi}_\epsilon \in C^{4,\alpha}$. By the work of Guenancia–Paun in section 5.2 of [12], Kolodziej’s $L^\infty$-estimate in [16] (also see Theorem 1.1 in [9] for a general statement), Theorem 4.1, and the condition (11), we obtain

$$|\hat{\phi}_\epsilon|_{\alpha} \leq C_F \quad (13)$$

$$\frac{1}{C_F} \omega_\epsilon \leq \omega_\hat{\phi}_\epsilon \leq C_F \omega_\epsilon. \quad (14)$$

The proof is thus completed. \Box

3 Construction of the approximation flows and proofs of Theorem 1.9

To construct the weak flow and approximate the CKRF, we first apply Theorem 2.3 to perturb the initial cone metric to $\omega_\phi^n$. Then we consider $\omega_\hat{\phi}_\epsilon$ as our new initial metric and consider the following approximation flows.
\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= \log \frac{\omega^n_\epsilon}{\omega^n_0} + \beta \phi_\epsilon + h + (1 - \beta) \log(|S|^2 + \epsilon), \quad t \in [0, T]. \\
\phi_\epsilon(0) &= \hat{\phi}_\epsilon \quad \text{when} \quad t = 0.
\end{align*}
\]

Now we would like to change the reference metric to \(\omega_\epsilon\). Writing

\[
\hat{V}_\epsilon = h + \log \frac{\omega^n_\epsilon (|S|^2 + \epsilon)^{1-\beta}}{\omega^n_0} + \frac{\beta}{N} \Psi_{\epsilon, \beta},
\]

we change the flow equation to the following.

\[
\begin{align*}
\frac{\partial \phi_\epsilon}{\partial t} &= \log \frac{\omega^n_\epsilon}{\omega^n_0} + \beta \phi_\epsilon + \hat{V}_\epsilon, \quad t \in [0, T]. \\
\phi_\epsilon(0) &= \hat{\phi}_\epsilon - \frac{\beta}{N} \Psi_{\epsilon, \beta} \quad \text{when} \quad t = 0.
\end{align*}
\]

**Lemma 3.1** There exists a constant \(C\) in the sense of Definition 2.1 with the following properties. On the perturbed flow (16), the following estimates hold.

- \(|\phi_\epsilon|_{\alpha, \frac{2}{3}} \leq C\),
- \(\frac{1}{C} \omega_\epsilon \leq \omega_{\phi_\epsilon} \leq C \omega_\epsilon, \quad \left\| \frac{\partial \phi_\epsilon}{\partial t} \right\|_{L^\infty} \leq C\).

**Proof of Lemma 3.1** Step 1 First we show that \(\left\| \frac{\partial \phi_\epsilon}{\partial t} \right\|_{L^\infty} \leq C\). This is directly implied by the maximal principle and the bound on \(\left\| \frac{\partial \phi_\epsilon}{\partial t} \right\|_{t=0}\). The bound on \(\left\| \frac{\partial \phi_\epsilon}{\partial t} \right\|_{t=0}\) directly follows from the properties of our approximating initial metrics. Namely from (12) we have

\[
\left. \frac{\partial \phi_\epsilon}{\partial t} \right|_{t=0} = \log \frac{\omega^n_\epsilon (|S|^2 + \epsilon)^{1-\beta}}{\omega^n_0} + \beta \phi_\epsilon(0) + \hat{V}_\epsilon(0)
\]

\[
= \log \left( e^{F_\epsilon} \omega^n_0 \right)^{|S|^2 + \epsilon} + \beta \phi_\epsilon(0) + \hat{V}_\epsilon(0).
\]

Thus by Theorem 2.3 we obtain

\[
\left. \left\| \frac{\partial \phi_\epsilon}{\partial t} \right\|_{0, M, t=0} \leq C. \right.
\]

Therefore by maximal principle we have

\[
\left. \left\| \frac{\partial \phi_\epsilon}{\partial t} \right\|_{0, M} \leq C. \right.
\]

Step 2 Now we turn to the spacewise second order estimate. By the Siu–Bochner technique in [22] (the reader could also see [12]) and the flow equation (16), denote

\[
h_\epsilon = -\beta \phi_\epsilon - \hat{V}_\epsilon,
\]
we derive the following parabolic Siu–Bochner formula.

\[
\left( \Delta \phi_\epsilon - \frac{\partial}{\partial t} \right) \log tr_{\omega_\epsilon \omega_{\phi_\epsilon}} \\
\geq \frac{1}{tr_{\omega_\epsilon \omega_{\phi_\epsilon}}} \left\{ \Delta_{\omega_\epsilon} h_\epsilon + \Sigma_{i \leq l} \left( \frac{\lambda_i}{\lambda_l} + \frac{\lambda_l}{\lambda_i} - 2 \right) R_{i i l l}(w) \right\},
\]  

\hspace{1cm} (19)

where \( w \) is the geodesic coordinates of \( \omega_\epsilon \) which diagonalize \( \omega_{\phi_\epsilon} \) with respect to \( \omega_\epsilon \), and \( \lambda_i \) are the eigenvalues of \( \omega_{\phi_\epsilon} \) with respect to \( \omega_\epsilon \).

We then consider the barrier function \( \Psi_{\epsilon, \rho} \) (for sufficiently small \( \rho \)) of Guenancia and Paun in [12]. Namely, for the sake of a self-contained proof, we quote in the following two beautiful identities from [12] at any point \( p \) near \( D \) (which do not depend on the flow).

- Equation \((\star)\) in page 8 of [12]:

\[
\Delta_{\omega_{\phi_\epsilon}} \Psi_{\epsilon, \rho} \geq -C tr_{\omega_{\phi_\epsilon} \omega_\epsilon} + C \Sigma_{i=1}^n \frac{1}{(|S|^2 + \epsilon)^{1-\rho}} \left| \frac{\partial z}{\partial w_i} \right|^2 \frac{1}{\lambda_i}.
\]  

\hspace{1cm} (20)

- Curvature estimate in page 8 of [12]:

\[
\frac{1}{\Sigma_k \lambda_k} \left\{ \Delta_{\omega_\epsilon} h_\epsilon + \Sigma_{i \leq l} \left( \frac{\lambda_i}{\lambda_l} + \frac{\lambda_l}{\lambda_i} - 2 \right) R_{i i l l}(w) \right\} \\
\geq -C \Sigma_{i=1}^n \frac{1}{(|S|^2 + \epsilon)^{1-\rho}} \left| \frac{\partial z}{\partial w_i} \right|^2 \frac{1}{\lambda_i} \\
- \frac{1}{\Sigma_k \lambda_k} \left\{ \Sigma_{i \leq l} \left( \frac{\lambda_i}{\lambda_l} + \frac{\lambda_l}{\lambda_i} \right) \right\} - C.
\]  

\hspace{1cm} (21)

Then, (19), (20), and (21) imply the following estimate for sufficiently big numbers \( A \) and \( B \) over the whole \( M \).

\[
\left( \Delta \phi_\epsilon - \frac{\partial}{\partial t} \right) \{ \log tr_{\omega_\epsilon \omega_{\phi_\epsilon}} + B \Psi_{\epsilon, \rho} - A \phi_\epsilon \} \\
\geq tr_{\omega_{\phi_\epsilon} \omega_\epsilon} + A \frac{\partial \phi_\epsilon}{\partial t} - C.
\]  

\hspace{1cm} (22)

Equations (18) and (22) indicate that, at the interior maximum of

\[
\log tr_{\omega_\epsilon \omega_{\phi_\epsilon}} + B \Psi_{\epsilon, \rho} - A \phi_\epsilon, \text{ we have } tr_{\omega_{\phi_\epsilon} \omega_\epsilon} \leq C.
\]  

\hspace{1cm} (23)

In terms of the eigenvalues, we have

\[
\Sigma_i \frac{1}{\lambda_i} \leq C.
\]  

\hspace{1cm} (24)
By (18) we have
\[
\frac{1}{C} \leq \prod_{k} \lambda_k \leq C. \tag{25}
\]
Hence
\[
\left( \Sigma_i \frac{1}{\lambda_i} \right)^{n-1} \geq \Sigma_i \frac{1}{\lambda_1 \ldots \lambda_i \ldots \lambda_n} = \frac{\Sigma_i \lambda_i}{\prod_k \lambda_k} \geq C \Sigma_i \lambda_i. \tag{26}
\]
Combining (26), (23), and (24), we end up with
\[
\sup tr \omega \omega \varphi \leq C. \tag{27}
\]
Using (25) again, we get \( \sup tr \omega \omega \varphi < C \).

**Step 3** In Step 2 we assume that \( tr \omega \omega \varphi \) attains interior maximum. On the other hand, suppose \( tr \omega \omega \varphi \) attain maximum when \( t = 0 \), then the second item in Theorem 2.3 implies our desired bound. Thus item 2 in Lemma 3.1 is proved.

**Step 4** To prove Item 1, it suffices to use item 2 and the Harnack inequality in Theorem 4.1. Notice that we automatically have the \( C^0 \)-estimate via the bound on \( |\frac{\partial \varphi}{\partial t}|_0 \) and the \( C^0 \)-bound on the initial potential \( \varphi (0) \).

The proof is complete. \( \square \)

Using Lemma 3.1 and Theorem 2.3, \( \varphi \) sub converges to a \( \varphi_\infty \) such that
\[
\begin{align*}
\frac{\partial \varphi_\infty}{\partial t} &= \log \frac{\omega_\varphi}{\omega_0} + \beta \varphi_\infty + h + (1 - \beta) \log(|S|), \quad t \in [0, T]. \\
\varphi_\infty(0) &= \varphi(0) \text{ when } t = 0. \\
\varphi_\infty &\in C^{\alpha, \frac{\alpha}{2}} [0, T]; \quad \frac{1}{C} \omega_D \leq \omega_{\varphi_\infty} \leq C \omega_D.
\end{align*}
\]

To show that the perturbation really converges back to the original conical flow, we need the following lemma on the uniqueness of the weak conical flows.

**Lemma 3.2** Suppose \( \phi_i, i = 1, 2 \) are two weak conical flows:
\[
\begin{align*}
\frac{\partial \phi_i}{\partial t} &= \log \frac{\omega_\phi}{\omega_0} + \beta \phi_i + h + (1 - \beta) \log(|S|) \text{ over } M \setminus D, \\
\phi_i(0) &= \varphi(0) \text{ when } t = 0, \\
\phi_i &\in C^{\alpha, \frac{\alpha}{2}} (M \times [0, T]).
\end{align*}
\]
\[
\frac{1}{C} \omega_D \leq \omega_{\phi_i} \leq C \omega_D, \quad |\frac{\partial \phi_i}{\partial t}|_{L^\infty} \leq C \text{ over } M \setminus D.
\]

Then \( \phi_1 = \phi_2 \).
Proof We again employ Jeffres’ trick in the parabolic case, adapted to our setting. Consider \( \hat{\phi}_1 = \phi_1 + a|S|^{2p} \). Then we compute

\[
\frac{\partial \hat{\phi}_1}{\partial t} = \log \left( \frac{(\omega + i\partial \bar{\partial} \phi_1)^n}{\omega_0^n} \right) + \beta \hat{\phi}_1 - a\beta |S|^{2p} + h + (1 - \beta) \log(|S|^2).
\]

Denote \( v = \hat{\phi}_1 - \phi_2 \) and \( \Delta = \int_0^1 \sum_{i,j} g_{i\bar{j}} (1-b) \partial_{\bar{z}_i} \partial_{z_j} db \), we compute from (28) that

\[
\frac{\partial v}{\partial t} = \Delta v - a \Delta |S|^{2p} + \beta v - a\beta |S|^{2p}.
\] (28)

The following is due to Jeffres [14].

Claim 3.3 When \( p < \frac{\alpha \beta}{2} \), the spacewise maximum of \( \hat{\phi}_1 \) and \( v \) are attained away from \( D \).

We only prove it for \( \hat{\phi}_1 \), the others are similar. Was this claim not true, let the spacewise maximum of \( \hat{\phi}_1 \) be attained at \( q \in D \), near \( q \) we have a holomorphic chart such that \( q \) corresponds to the origin 0, and \( |S|^2 = h|z|^2 \), \( h \) is the metric of the line bundle of \( D \) ( \( \frac{1}{C} \leq h \leq C \)). Suppose in this chart we have

\[
\phi_1(z, 0) + \epsilon |S|^{2p}(z, 0) - \phi_1(0, 0) \leq 0.
\] (29)

Since \( \phi_1 \in C^\alpha \) spacewisely, we compute,

\[
\frac{\phi_1(z, 0) + \epsilon |S|^{2p}(z, 0) - \phi_1(0, 0)}{|z|^\alpha} = \frac{\phi_1(z, 0) - \phi_1(0, 0)}{|z|^\alpha} + \frac{\epsilon h^p |z|^{2p}}{|z|^\alpha} \geq -[\phi_1]_\alpha + \epsilon h^p |z|^{-(\alpha - 2p)}
\]

\[
\geq 1 \text{ when } z \text{ is sufficiently close to } 0, \ p < \frac{\alpha \beta}{2}.
\]

This contradicts (29). The proof of Claim 3.3 is complete. Actually it suffices to require \( 2p < \alpha \). The reason of requiring the stronger condition \( 2p < \alpha \beta \) is that it even works more generally for \( \phi_1 \in C^{\alpha, \beta} \) (the intrinsic Hölder space of \( \omega_{\beta} \)), thus we can avoid any confusion.

Furthermore, we have

\[
i \partial \bar{\partial} |S|^{2p} = p^2 |S|^{2p} \partial \log |S|^2 + \bar{\partial} \log |S|^2 + p |S|^{2p} i \partial \bar{\partial} \log |S|^2.
\] (30)

By the second-order estimate in the assumptions, we have the following estimate.

\[
\left| \sum_{i,j} g_{i\bar{j}} (1-b) \partial_{\bar{z}_i} \partial_{z_j} \right| \leq C, \text{ where the basis is } \left( \frac{\partial}{\partial z_i}, i = 1, \ldots, n \right).
\] (31)
Then away from the divisor, $g_{b\phi_1+(1-b)\phi_2}$ is at least $C^\alpha$. From (30) we compute over $M\setminus D$ that

$$
\begin{align*}
\Delta |S|^{2p} &\geq p|S|^{2p} g_{b\phi_1+(1-b)\phi_2} \frac{\partial^2 \log |S|^2}{\partial z_i \partial \bar{z}_j} \\
&\geq -C \left| pg_{b\phi_1+(1-b)\phi_2} \Theta_h,ij \right| \\
&\geq -C,
\end{align*}
$$

(32)

where $\Theta_h$ is the smooth curvature form of $(L, h)$.

Then, by (28) and Proposition 2.23 in [21], we deduce

$$
\frac{\partial \sup v}{\partial t} \leq aC + \beta \sup v,
$$

(33)

in the sense of forward difference quotients. Using $v(0) = a|S|^{2p} \leq aC$ and Proposition 2.23 in [21] again, we obtain

$$
\sup v \leq [\sup v(0) + aC]e^{\beta t} - aC \leq aCe^{\beta T}.
$$

(34)

Thus let $a$ tend to 0, we end up with $\phi_1 \leq \phi_2$. By the same reason we have $\phi_2 \leq \phi_1$, then $\phi_1 = \phi_2$. □

**Proof of Theorem 1.9** By letting $\epsilon \to 0$ in Lemma 3.1 and flow (15), notice that our time $T$ can be arbitrarily large, then Theorem 1.9 is a direct corollary of Lemma 3.1 and 3.2. □

4 Local Harnack inequality

In this section we prove Theorem 4.1 by proving the harder Theorem 4.7. Theorem 4.1 is all we need to prove the results in the introduction.

**Theorem 4.1** Suppose $\omega$ is a weak conical-Kähler metric and $\sup C \leq \omega \leq C\omega_D$, or $\omega$ is $C^\alpha$ over the whole $M$ (across $D$) in the usual sense and $\frac{\alpha \omega}{C} \leq \omega \leq C\omega_\epsilon$ for some $0 < \epsilon \leq 1$.

Suppose $u$ is a bounded weak solution to

$$
\Delta_\omega u = f \text{ over } M,
$$

then there exists a $\alpha' > 0$ such that

$$
[u]_{\alpha',M} \leq C(|u|_{0,M} + |f|_{0,M}).
$$
Theorem 4.2 Suppose $\omega_t$ is a (strong) conical Kähler–Ricci flow, or the perturbed smooth flow (15) over $[0, T]$. Suppose $u$ is a bounded weak solution to

$$\frac{\partial}{\partial t} u = \Delta_{\omega_t} u + f$$ over $M \times [0, T]$. Then for all $\delta \in (0, T)$, there exists a $\alpha' > 0$ and a $C(\delta)$ in the sense of Definition 2.1 such that

$$[u]_{\alpha', \alpha, M \times [\delta, T]} \leq C(\delta)(|u|_{0, M \times [0, T]} + |f|_{0, M \times [0, T]}).$$

Proof of Theorem 4.2 and 4.1 Theorem 4.2 is directly implied by Theorem 4.7 and Lemma 4.9. Theorem 4.1 is implied by Theorem 4.7 directly. □

Lemma 4.3 For any $\epsilon > 0$ and any point $p \in D$, there exists a canonical polar coordinate $i_\epsilon$ such that

$$i_\epsilon^* \left\{ \frac{\beta^2}{(|z|^2 + \epsilon)^{1-\beta}} dz \otimes d\bar{z} \right\} = ds^2 + a_\epsilon s^2 d\theta^2, \ p \text{ is the origin in these coordinates},$$

where $a_\epsilon$ is a smooth function of $s$ ($s \in [0, r_0]$ for $r_0$ sufficiently small), and $\beta^2 < a_\epsilon \leq 1$. In particular, we have

$$\beta^2 \omega_{E, \epsilon} < i_\epsilon^* \omega_{\beta, \epsilon} \leq \omega_{E, \epsilon},$$

where $\omega_{E, \epsilon} = ds^2 + s^2 d\theta^2 + \sum_{i=1}^{n-1} du_i \otimes d\bar{u}_i$ is the Euclidean metric in the coordinate $i_\epsilon$.

Proof Actually the proof is quite elementary, since this fact is very important we include the full detail here. Let $\rho = |z|$. Define $s$ as

$$ds = \frac{\beta}{(\rho^2 + \epsilon)^{1-\beta}}, \ s(0) = 0.$$ (37)

Then

$$\frac{\beta^2}{(|z|^2 + \epsilon)^{1-\beta}} dz \otimes d\bar{z} = ds^2 + a_\epsilon s^2 d\theta^2,$$

where

$$a_\epsilon = \frac{\beta^2 \rho^2}{(\rho^2 + \epsilon)^{1-\beta}s^2}. (38)$$
From (37) we have

\[
\frac{d[(\rho^2 + \epsilon)^{\frac{1-\beta}{2}} s]}{d\rho} = \beta + \frac{(1 - \beta)\rho s}{(\rho^2 + \epsilon)^{\frac{1+\beta}{2}}} \geq \beta. \tag{39}
\]

Then

\[
\beta \rho \leq (\rho^2 + \epsilon)^{\frac{1-\beta}{2}} s.
\]

Hence

\[
a_\epsilon = \frac{\beta^2 \rho^2}{(\rho^2 + \epsilon)^{1-\beta} s^2} \leq 1.
\]

Now we would like to study the uniform lower bound of \(a_\epsilon\). Using (39), denote \(v = (\rho^2 + \epsilon)^{\frac{1-\beta}{2}} s\), we compute

\[
\frac{dv}{d\rho} < \beta + \frac{(1 - \beta)(\rho^2 + \epsilon)^{\frac{1}{2}} s}{(\rho^2 + \epsilon)^{\frac{1+\beta}{2}}}
\]

\[
= \beta + (1 - \beta) v (\rho^2 + \epsilon)^{-\frac{1}{2}}
\]

\[
< \beta + (1 - \beta) v \frac{\rho}{\rho}.
\]

Then

\[
\frac{d}{d\rho} \left( \frac{v}{\rho} \right) = -\frac{v}{\rho^2} + \frac{1}{\rho} \frac{dv}{d\rho}
\]

\[
\leq -\frac{v}{\rho^2} + \frac{\beta}{\rho} + (1 - \beta) \frac{v}{\rho^2}
\]

\[
= \frac{\beta - \beta v}{\rho} - \frac{\beta v}{\rho^2}
\]

\[
= \frac{\beta}{\rho} \left( 1 - \frac{v}{\rho} \right).
\]

Denote \(u = \frac{v}{\rho}\), we get

\[
\frac{du}{d\rho} < \frac{\beta}{\rho} (1 - u). \tag{40}
\]

Furthermore, from (37) we have \(u(0) = \beta\). Then simple comparison implies

\[
u < 1 \quad \text{for all } \rho. \tag{41}
\]

To be precise, if \(u(\rho_0) = 1\) for some \(\rho_0\), then take \(\rho_0\) as the first one among those \(\rho\) of which \(u(\rho) = 1\), then we deduce \(\frac{du}{d\rho}(\rho_0) \geq 0\), which contradicts (40).
Then it’s easy to see from (41) that
\[ a_\epsilon = \frac{\beta^2}{u^2} > \beta^2. \]
\[ \Box \]

When \( \epsilon = 0 \), the coordinate in Lemma 4.3 is exactly the polar coordinate of \( \omega_\beta \).

Denote \( |x - y|_{(\beta, \epsilon)} \) as the distance between \( x, y \) in the polar coordinate in Lemma 4.3, and \( |x - y|_{holo} \) as the distance in the holomorphic (smooth) coordinates. Comparing the 2 distances gives the equivalence of Hölder continuities with respect to the 2 different model metrics. To be precise, let \( C^{\alpha,(\beta,\epsilon)}(M) \) be the Hölder space of exponent \( \alpha \) with respect to the distance \( | |_{(\beta, \epsilon)} \), and \( C^{\alpha,\frac{\alpha}{2},(\beta,\epsilon)}(M) \) be its parabolic counterpart, the following is true.

**Lemma 4.4** Suppose \( \epsilon \in [0, \frac{1}{100}] \). Given any 2 points \( x, y \) such that \( |x|_{(\beta, \epsilon)}, |y|_{(\beta, \epsilon)} \leq 1 \), the following is true
\[ C|x - y|_{holo} \leq |x - y|_{(\beta, \epsilon)} \leq C|x - y|^\beta_{holo}. \]  
(42)

Consequently

- \( C^{\alpha,(\beta,\epsilon)}(M) \) embeds continuously into \( C^{\alpha\beta}(M) \), \( C^{\alpha}(M) \) embeds continuously into \( C^{\alpha,(\beta,\epsilon)}(M) \)
- \( C^{\alpha,\frac{\alpha}{2},(\beta,\epsilon)}(M) \) embeds continuously into \( C^{\alpha\beta,\frac{\alpha}{2}}(M) \), \( C^{\alpha,\frac{\alpha}{2}}(M) \) embeds continuously into \( C^{\alpha,\frac{\alpha}{2},(\beta,\epsilon)}(M) \)

**Proof of Lemma 4.4** The proof is elementary, and is obvious when \( \epsilon = 0 \). The embedding results are straightforward by Definition 2.2 and (42). To prove (42), it suffices to prove the following claim.

**Claim 4.5**
\[ C|\rho_x - \rho_y| \leq |s_x - s_y| \leq C|\rho_x - \rho_y|^\beta. \]  
(43)

In particular we have
\[ C\rho_x \leq s_x \leq C\rho_x^\beta, \ C\rho_y \leq s_y \leq C\rho_y^\beta. \]  
(44)

Recall the following
\[ |x - y|_{(\beta, \epsilon)} \approx |s_x - s_y| + \left| \sin \frac{\theta_x - \theta_y}{2} \right| \sqrt{s_x s_y} + |x_T - y_T|, \]
\[ |x - y|_{holo} \approx |\rho_x - \rho_y| + \left| \sin \frac{\theta_x - \theta_y}{2} \right| \sqrt{\rho_x \rho_y} + |x_T - y_T|, \]
where \( x_T, y_T \) are the tangential component of \( x, y \) along the singularity. Then Claim 4.5 directly implies (42).
Next we prove Claim 4.5. Since \( \frac{dx}{d\rho} \leq C \rho^{\beta-1} \), we deduce
\[
|s_x - s_y| \leq C|\rho_x^\beta - \rho_y^\beta| \leq C|\rho_x - \rho_y|^\beta.
\]
(45)

To obtain lower bound, notice when \( \rho > 10\sqrt{\epsilon} \), we have \( \frac{ds}{d\rho} \geq C \rho^{\beta-1} \), then
\[
|s_x - s_y| \geq C|\rho_x^\beta - \rho_y^\beta| \geq C|\rho_x - \rho_y|.
\]
(46)

When \( \rho \leq 10\sqrt{\epsilon}, \frac{\epsilon^{\beta-1}}{C} \leq \frac{ds}{d\rho} \leq C \epsilon^{\beta-1} \). Hence
\[
|s_x - s_y| \geq \frac{\epsilon^{\beta-1}}{C}|\rho_x - \rho_y|.
\]
(47)

Thus when \( \rho_x, \rho_y \leq 1 \), we deduce
\[
|s_x - s_y| \geq C|\rho_x - \rho_y|.
\]
(48)

The proof of Claim 4.5 is complete. \( \square \)

**Remark 4.6** Actually Lemma 4.3 implies the bound for the Poincare and Sobolev constants in the following sense. For any model metric \( \omega \), denote
\[
E_{\omega, \lambda} = \left\{ \omega' \right| \mu^\alpha [\omega']_\alpha, M \setminus T_\epsilon(D) \leq \lambda, \frac{\omega'}{\lambda} \leq \omega' \leq \lambda \omega \right\}.
\]

With respect to the global perturbed metric \( \omega_\epsilon \), using Lemma 4.3, it’s quite straightforward to show the global and local Sobolev constants \( C_{S, \epsilon}, C_{S, \epsilon}^* \) for all the metrics in \( E_{\omega_\epsilon, \lambda} \) are uniformly bounded from above independent of \( \epsilon \). On the Poincare inequality, the global and local Poincare constants \( C_{P, 0}, C_{P, 0}^* \) of \( E_{\omega, \lambda} \) are uniformly bounded from above. Moreover, using very simple counter-proofs based on the Rellich–Kondrachov compact-imbedding theorem, we deduce that both the local Poincare constants \( C_{P, \epsilon}^* \) and the global Poincare constants \( C_{P, \epsilon} \) of \( E_{\omega_\epsilon, \lambda} \) are bounded from above independent of \( \epsilon \). These are necessary for doing the Nash–Moser iteration and the proofs of the Harnack inequalities in [13,17].

**Theorem 4.7** There exists a constant \( C \) in the sense of Definition 2.1 with the following properties. Suppose \( \omega_t \) is a time-differentiable family of Kähler metrics which is \( C^\alpha \) away from \( D \). Suppose

1. \( \omega_t \) are weak conical-Kähler metrics and \( \frac{\omega_0}{C} \leq \omega_t \leq C \omega_D \) for all \( t \in [0, T] \), or \( \omega_t \) is \( C^\alpha \) over the whole \( M \) (across \( D \)) in the usual sense and \( \frac{\omega_0}{C} \leq \omega_t \leq C \omega_\epsilon \) for all \( t \in [0, T] \) and some \( 0 < \epsilon \leq 1 \);
2. \( \frac{d}{dt} d\Omega_t \leq C d\Omega_t \).

Suppose \( u \) is a bounded weak solution to
\[
\frac{\partial}{\partial t} u = \Delta_{\omega_t} u + f \text{ over } M \times [0, T].
\]
Then for all $\delta \in (0, T)$, there exists a $\alpha' > 0$ and $C(\delta)$ such that

$$[u]_{\alpha',\frac{T}{2},M \times [\delta,T]} \leq C(\delta)(|u|_{0,M \times [0,T]} + |f|_{0,M \times [0,T]}).$$

**Proof of Theorem 4.7** With Lemma 4.3, actually the proof is quite straightforward. The only possible problem is the Hölder estimate near $D$. Notice that in the coordinate $t_\epsilon$, $\omega_t$ is quasi-isometric to the Euclidean metric.

It’s well known that the integration by parts holds true (cf. [25]). For the reader’s convenience we include the proof of this fact here. We only consider the case when $\omega_t$ is weakly conic i.e $\omega_t^D \leq \omega_t \leq C \omega_t^D$. Suppose $B$ is a ball with nonempty intersection with $D$, and $u$ is locally $C^2$ function defined over $B \setminus D$, $|\nabla u| \in L^2(B)$, Suppose $v \in C^1_c(B) \cap C^1(B \setminus D)$, $|\nabla v| \in L^2(B)$. The integration by parts formula we want to show is:

$$\lim_{\epsilon_k \to 0} \int_{B \setminus T_{\epsilon_k}(D)} v \Delta u \omega^n_i = - \int_B \nabla v \cdot \nabla u \omega^n_i \text{ for some sequence } \epsilon_k \to 0. \quad (49)$$

Proof of (49): We compute for any $\epsilon > 0$ that

$$\int_{B \setminus T_{\epsilon}(D)} v \Delta u \omega^n_i = - \int_{B \setminus T_{\epsilon}(D)} \nabla v \cdot \nabla u \omega^n_i + \int_{\partial T_{\epsilon}(D)} v(\nabla u \cdot n)dA.$$

It suffices to show there exists a sequence $\epsilon_k \to 0$ such that

$$\lim_{k \to \infty} \int_{\partial T_{\epsilon_k}(D)} v(\nabla u \cdot n)dA = 0. \quad (50)$$

Let $F = |\nabla u|$, extend $F$ to be 0 outside $B$. We compute via Hölder inequality and coarea formula that

$$C \geq \int_B F^2 \omega^n_i \geq C \int_0^1 \int_{r=\epsilon} F^2 dA d\epsilon, \quad r = |z|^\beta$$

$$\geq C \int_0^1 \text{Area}^{-1}\{r = \epsilon\} \left(\int_{r=\epsilon} F dA\right)^2 d\epsilon$$

$$= C \int_0^1 \frac{Y}{\epsilon(1 - \log \epsilon)} d\epsilon, \quad (51)$$

where

$$Y(\epsilon) = \left(\int_{r=\epsilon} F dA\right)^2 (1 - \log \epsilon).$$

Thus it’s easy to deduce the following Claim.

**Claim 4.8** There exists a sequence $\epsilon_k \to 0$ such that $Y(\epsilon_k) \to 0$. 

Springer
If not, then there exist $\delta_0$ and $r_0$ such that $Y(\epsilon) \geq \delta_0$ for all $\epsilon < r_0$.

Thus (51) implies $c \geq C \delta_0 \int_0^{r_0} \frac{1}{\epsilon(1-\epsilon)} d\epsilon = \infty$. This is a contradiction. Thus Claim 4.8 is true. Hence $\lim_{k \to \infty} \int_{r_0 \leq \epsilon \leq k} |\nabla u| dA = 0$. This implies (50) is true via the sequence $\epsilon_k$ in Claim 4.8. The proof of (49) is complete.

Then the proof of Theorem 10.1 in section 10 of Chapter III in [17] directly goes through in $B_p(r_0)$, $p \in D$ and $r_0$ is sufficiently small, provided we have $\frac{\partial}{\partial t} dvol_t \leq Cdvol_t$.

Lemma 4.9 Along the conical Kähler–Ricci flow over $[0, t_0]$ ($t_0$ as in Theorem 1.14), the scalar curvature $R$ satisfies $R \geq -\frac{C}{t}$ over $M \setminus D$. In particular, we have

$$\frac{\partial}{\partial t} dvol_t \leq \left( \frac{C}{t} + n\beta \right) dvol_t.$$  (52)

Moreover, along the perturbed smooth flow (15), (52) also holds.

Proof of Lemma 4.9 We have

$$\frac{\partial (R - n\beta)}{\partial t} \geq \Delta (R - n\beta) + \frac{(R - n\beta)^2}{n} + \beta (R - n\beta).$$  (53)

Notice that Proposition 1.14 implies $R(t) \in C^{\alpha, \beta}$ when $t > 0$. It follows from Jeffres’ trick as in the proof of Lemma 3.2 and maximal principles that $R \geq -\frac{C}{t}$. To elaborate how to deal with the conical singularity, we show more detail. By changing notation, the target estimate is

$$\tau R(\tau) > -C, \quad \tau \in [0, t_0].$$  (54)

We consider the flow initiated from time $\frac{\tau}{2}$ by letting $s = t - \frac{\tau}{2}$, $s \in [0, \frac{\tau}{2}]$. It suffices to show

$$sR(s) \geq -C.$$  (55)

The advantage of doing this is that now the regularity of the metric is improved such that $R(s) \in C^{\alpha, \frac{\alpha}{2}, \beta}$. Hence $\lim_{s \to 0} sR(s) = 0$ in Hölder continuous sense. Let

$$u_\delta = s(R - n\beta) - \delta |S|^{2p}, \quad p < \frac{\alpha\beta}{2}$$ as in Claim 3.3.  (56)

(32) and (53) imply

$$\frac{\partial u_\delta}{\partial s} \geq \Delta u_\delta + \frac{u_\delta^2}{ns} + \beta u_\delta + \frac{u_\delta}{s} + \frac{2u_\delta \delta |S|^{2p}}{ns} - \delta C.$$  (57)

Notice that the short existence time $t_0$ is very small in the sense of Definition 2.1. Let $\delta$ be small enough, we deduce when $u_\delta \leq -100n$, the term $\frac{u_\delta^2}{ns}$ is much more positive than the other terms such that

Springer
\[ \frac{\partial u_\delta}{\partial s} \geq \Delta u_\delta + 1. \quad (58) \]

Then

**Claim 4.10** \( u_\delta > -100n \).

If the claim does not hold, since \( u_\delta(0) \geq -1 \) (when \( \delta \) is sufficiently small) and \( u_\delta \) can not attain spacewise minimum on \( D \) (see Claim 3.3 with sign reversed), there exists a space-time point \((x_0, s_0)\) such that

\[
u_\delta(x_0, s_0) = -100n, \quad u_\delta(x, s) \geq -100n \text{ when } s < s_0, \]

\( x_0, s_0 \) is the spacewise minimum of \( u_\delta \), \( x_0 \notin D \).

Then \( \frac{\partial u_\delta}{\partial s}(x_0, s_0) \leq 0 \). But Eq. (58) implies \( \frac{\partial u_\delta}{\partial s}(x_0, s_0) \geq 1 \), a contradiction. The proof of Claim 4.10 is complete. Letting \( \delta \to 0 \), Claim 4.10 implies (55). The proof of the conic flow part of Lemma 4.9 is complete. The crucial point is that we need the perturbation by \( \delta|S|^{2p} \) in (56).

To prove the conclusion for the perturbed flow, we don’t need the perturbation by \( \delta|S|^{2p} \) above. We first denote

\[ v_\epsilon = -\frac{\partial}{\partial t} d\text{vol}_t. \]

By routine computation we have

\[ \frac{\partial v_\epsilon}{\partial t} \geq \Delta v_\epsilon + \frac{v_\epsilon^2}{n} + \beta v_\epsilon, \quad (59) \]

where \( v_\epsilon \) is a smooth function on which the maximal principle directly works. Then (52) is true for the perturbed flow (15).

\[ \square \]

5 Bootstrapping of the conical Kähler–Ricci flow and proofs of the main results in the introduction

In this short section, we show the solutions to the conical Kähler–Ricci flow possess maximal regularity when \( t > 0 \) by proving Theorem (1.14). This shows the requirements in Theorem 1.9 are satisfied when \( t = t_0^2 \) (\( t_0 \) as in Theorem 1.14).

**Proof of Theorem 1.14** Denote \( v = e^{-\beta t} \frac{\partial \phi}{\partial t} \), then \( v \) satisfies the following equation

\[ \frac{\partial v}{\partial t} = \Delta \phi v. \quad (60) \]
Suppose \( v \in C^{2+\alpha,1+\frac{\alpha}{2},\beta}(0, T) \) (Claim 5.1), using the parabolic interior Schauder estimate as in the equation (21) in [8], we obtain the following

\[
|i\partial\bar{\partial}v|^{(2)}_{\alpha,\frac{\alpha}{2},\beta,M\times[0,t_0]} + \left|\frac{\partial v}{\partial t}\right|^{(2)}_{\alpha,\frac{\alpha}{2},\beta,M\times[0,t_0]} \leq C|v|_{0,M\times[0,t_0]}.
\]

(61)

Then Eq. (61) directly implies Theorem 1.14, because \( i\partial\bar{\partial}e^\phi = -Ric + \beta\omega \) over \( M\setminus D \).

Thus it suffices to show

**Claim 5.1** \( v \in C^{2+\alpha,1+\frac{\alpha}{2},\beta}(0, T) \).

The claim is proved in a very easy way as follows. For any \( \delta > 0 \), choose a timewise-cutoff function \( \eta(t) \) such that

\[ \eta(t) = 0 \text{ when } t \leq \delta, \quad \eta(t) = 1 \text{ when } t \geq 2\delta. \]

It suffices to prove \( \eta v \in C^{2+\alpha,1+\frac{\alpha}{2},\beta}[0, T] \). First we compute

\[
\frac{\partial (\eta v)}{\partial t} = \Delta\phi (\eta v) + \eta' v.
\]

(62)

Since \( v \in C^{\alpha,\frac{\alpha}{2},\beta}[0, T] \), by Theorem 1.8 in [8], there exists a solution \( U \in C^{2+\alpha,1+\frac{\alpha}{2},\beta}[0, T] \) which solves

\[
\frac{\partial U}{\partial t} = \Delta\phi U + \eta' v, \quad U(x, 0) = 0.
\]

(63)

Consider \( W = U - \eta v \), then

\[
W \in C^{\alpha,\frac{\alpha}{2},\beta}[0, T]
\]

(64)

and

\[
\frac{\partial W}{\partial t} = \Delta\phi W, \quad W(x, 0) = 0 \text{ over } M\setminus D.
\]

(65)

Using Jeffres’s trick as in the proof of Lemma (3.2), (64) and (65) imply \( W = 0 \). Then \( \eta v = U \in C^{2+\alpha,1+\frac{\alpha}{2},\beta}[0, T] \).

Proof of Theorems 1.4 and 1.11 When \( t = \frac{b_0}{2} \), we define

\[
F_{\frac{b_0}{2}} \triangleq \log \frac{|S|^{2-2\beta}\omega_0^n}{\omega_0^n} = -h + \frac{\partial \phi}{\partial t}\bigg|_{\frac{b_0}{2}} - \beta\phi.
\]

(66)

By Theorem 1.14, we directly get

\[
F_{\frac{b_0}{2}} \in C^{2,\alpha,\beta}, \quad \left| F_{\frac{b_0}{2}} \right|_{2,\alpha,\beta} \leq C.
\]

(67)
Theorem 1.9 implies the long time existence of the weak flow for all time \( t \in [\frac{t_0}{2}, \infty) \). Combining the strong conical flow over \([0, \frac{t_0}{2}]\), we end up with a strong flow for all \( t \in [0, \infty) \), for any \( C^{2, \alpha, \beta} \) initial potential \( \phi_0 \).

Lemma 3.2 implies the weak flow coincides with the strong flow given by Theorem 1.2 in [8] till whenever the strong flow exists.

By approximating the conical flow over \([\frac{t_0}{2}, T)\) by the flows (15) as in Sect. 3, Theorem 1.11 follows from the second-order estimates in Proposition 3.1, the parabolic Evans–Krylov estimate over \( M \setminus T_\delta(D) \) (as in [24], \( T_\delta(D) \) defined in Remark 1.12), and the arguments in Proposition 2.5 in [5]. □

Acknowledgments This is a side project which grows out of a joint project with Prof Xiuxiong Chen on the conical KRFs. The author would like to thank Prof Chen for kindly suggesting this project and for his constant support over years. The author also would like to thank Chengjian Yao for related discussions.

References

1. Berman, R.: A thermodynamic formalism for Monge-Ampère equations, Moser-Trudinger inequalities and Kähler-Einstein metrics. Adv. Math. 248, 1254–1297 (2013)
2. Brendle, S.: Ricci flat Kähler metrics with edge singularities. Int. Math. Res. Not. 24, 5727–5766 (2013)
3. Campana, F., Guenancia, H., Paun, M.: Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields
4. Cao, H.-D.: Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. Invent. Math. 81(2), 359–372 (1985)
5. Chen, X.-X., Donaldson, S., Sun, S.: Kähler-Einstein metric on Fano manifolds, I: approximation of metrics with cone singularities. J. Amer. Math. Soc. 28, 183–197 (2015)
6. Chen, X.-X., Donaldson, S., Sun, S.: Kähler-Einstein metric on Fano manifolds, II: limits with cone angle less than 2\( \pi \). J. Amer. Math. Soc. 28, 199–234 (2015)
7. Chen, X.-X., Donaldson, S., Sun, S.: Kähler-Einstein metric on Fano manifolds, III: limits with cone angle approaches 2\( \pi \) and completion of the main proof. J. Amer. Math. Soc. 28, 235–278 (2015)
8. Chen, X.-X., Wang, Y.: Bessel functions, Heat kernel and the conical Kähler-Ricci flow. J. Funct. Anal. 269(2) (2013)
9. Dinew, S., Zhang, Z.: Stability of bounded solutions for degenerate complex Monge-Ampère equations. Adv. Math. 225(1), 367–388 (2010)
10. Donaldson, S.: Kähler metrics with cone singularities along a divisor. In: Essays in mathematics and its applications, pp. 49–79. Springer, Heidelberg (2012)
11. Eyssidieux, P., Guedj, V., Zeriahi, A.: Singular Kähler-Einstein metrics. J. Am. Math. Soc. 22, 607–639 (2009)
12. Guenancia, H., Paun, M.: Conic singularities metrics with perscribed Ricci curvature: the case of general cone angles along normal crossing divisors. arXiv:1307.6375
13. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin
14. Jeffres, T.: Uniqueness of Kähler-Einstein cone metrics. Publ. Math. 44 (2000)
15. Jeffres, T., Mazzeo, R., Rubinstein, Y.: Kähler-Einstein metrics with edge singularities. Ann. Math. (To appear)
16. Kolodziej, S.: Hölder continuity of solutions to the complex Monge-Ampere equation with the right-hand side in \( L^{s,p} \): the case of compact Kähler manifolds. Math. Ann., 379–386 (2008)
17. Ladyzenskaja, S., Ural’ceva, N.N.: Linear and quasi-linear equations of parabolic type. In: Translations of Mathematical Monographs, vol. 23. American Mathematical Society, Providence
18. Li, C., Sun, S.: Conical Kähler-Einstein metric revisited. Comm. Math. Phys. 331(3), 927–973 (2014)
19. Liu, J.W., Zhang, X.: The conical Kähler–Ricci flow on Fano manifolds. arXiv:1402.1832
20. Mazzeo, R., Rubinstein, Y., Sesum, N.: Ricci flow on surfaces with conic singularities. arXiv:1306.6688
21. Morgan, J., Tian, G.: Ricci flow and the poincare conjecture. In: Clay Mathematics Monographs
22. Siu, Y.T.: Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kähler-Einstein Metrics. Birkhäuser, Basel (1987)
23. Song, J., Wang, X.: The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality. arXiv:1207.4839
24. Wang, K.H.: On the regularity theory of fully nonlinear parabolic equations. Bull. Am. Math. Soc. (N.S.). 22(1), 107–114 (1990)
25. Wang, Y.: Notes on the $L^2$-estimates and regularity of parabolic equations over conical manifolds. (Unpublished work)
26. Yao, C.J.: Existence of weak conical Kähler-Einstein metrics along smooth hypersurfaces. Math. Ann. 362(3–4), 1287–1304 (2015)
27. Yin, H.: Ricci flow on surfaces with conical singularities, II. arXiv:1305.4355