Stackelberg stochastic differential game with asymmetric noisy observations

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ABSTRACT
This paper is concerned with a Stackelberg stochastic differential game with asymmetric noisy observation. In our model, the follower cannot observe the state process directly, but could observe a noisy observation process, while the leader can completely observe the state process. Open-loop Stackelberg equilibrium is considered. The follower first solve a stochastic optimal control problem with partial observation, the maximum principle and verification theorem are obtained. Then the leader turns to solve an optimal control problem for a conditional mean-field forward–backward stochastic differential equation, and both maximum principle and verification theorem are proved. A linear-quadratic Stackelberg stochastic differential game with asymmetric noisy observation is discussed to illustrate the theoretical results in this paper. With the aid of some new Riccati equations, the open-loop Stackelberg equilibrium admits its state estimate feedback representation. Finally, an application to the resource allocation and its numerical simulation are given to show the effectiveness of the proposed results.

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1. Introduction
The Stackelberg game is an important type of hierarchical non-cooperative games (Başar & Olsder, 1998), whose study can be traced back to the pioneering work by von Stackelberg (1952). The Stackelberg game is usually known as the leader–follower game, whose economic background comes from markets where some firms have power of domination over others. The solutions of the Stackelberg game, are called Stackelberg equilibrium points in which there are usually two players with asymmetric roles, one leader and one follower. In order to obtain the Stackelberg equilibrium points, it is usual to divide the game problem into two parts. In the first part – the follower’s problem, firstly, the leader announces his strategy, then the follower will make an instantaneous response, and choose an optimal strategy corresponding to the given leader’s strategy to optimise his/her cost functional. In the second part – the leader’s problem, knowing the follower would take such an optimal strategy, the leader will choose an optimal strategy to optimise his/her cost functional. In a word, a distinctive feature of the Stackelberg game is that, the decisions must be made by two players and one of them is subordinate to the other because of the asymmetric roles, therefore one player must make a decision after the other player’s decision is made. The Stackelberg game has been widely applied in the principal-agent/optimal contract problems (Cvitanić & Zhang, 2013), the news-vendor/wholesaler problems (Øksendal et al., 2013) and optimal reinsurance problems (Chen & Shen, 2018).

There exist some literatures about the Stackelberg differential game for Itô’s stochastic differential equations (SDEs for short) in the past decades. Let us mention a few. Yong (2002) studied the indefinite linear-quadratic (LQ for short) leader–follower differential game with random coefficients and control-dependent diffusion. Forward–backward stochastic differential equations (FBSDEs for short) and Riccati equations are applied to obtain the state feedback representation of the open-loop Stackelberg equilibrium points. Bensoussan et al. (2015) introduced several solution concepts in terms of the players’ information sets, and studied LQ Stackelberg games under both adapted open-loop and closed-loop (memoryless) information structures, whereas the control variables do not enter the diffusion coefficient of the state equation. Mukaidani and Xu (2015) studied the Stackelberg game with one leader and multiple followers, in an infinite time horizon. The Stackelberg equilibrium points are developed, by cross-coupled algebraic Riccati equations, under both cooperative and non-cooperative settings of the followers, to attain Pareto optimality and Nash equilibrium, respectively. Xu and Zhang (2016) and Xu et al. (2018) investigated the LQ Stackelberg differential games with time delay. Li and Yu (2018) proved the solvability of a kind of coupled FBSDEs with a multilevel self-similar domination-monotonicity structure, then it is used to characterise the unique equilibrium of an LQ generalised Stackelberg stochastic differential game with hierarchy in a closed form. Moon and Başar (2018) and Lin et al. (2019) studied the LQ mean-field Stackelberg stochastic differential games.
Du and Wu (2019) investigated an LQ Stackelberg game of mean-field backward stochastic differential equations (BSDEs for short). Zheng and Shi (2020) researched the Stackelberg game of BSDEs with complete information. Feng (2020) considered the LQ Stackelberg game of BSDEs with constraints.

However, in all the above literatures about Stackelberg stochastic differential games, the authors assume that both the leader and the follower could fully observe the state of the controlled stochastic systems. Obviously, this is not practical in reality. Generally speaking, the players in the games can only obtain partial information in most cases. Then it is very natural to study the Stackelberg stochastic differential game under partial information. In fact, some efforts have been made such as the following. Shi et al. (2016) and Shi et al. (2017) studied the Stackelberg stochastic differential game and introduced a new explanation for the asymmetric information feature, that the information available to the follower is based on the some sub-$\sigma$-algebra of that available to the leader. Shi et al. (2020) investigated the LQ Stackelberg stochastic differential game with overlapping information, where the follower’s and the leader’s information have some joint part, while they have no inclusion relations. Wang et al. (2020) discussed an asymmetric information mean-field type LQ Stackelberg stochastic differential game with one leader and two followers.

Noting that in the game frameworks of papers (Chang & Xiao, 2014; Shi et al., 2016, 2017, 2020; Wang et al., 2020, 2018; Wang & Yu, 2012), the information available to the players are described by the filtration generated by standard Brownian motions. In fact, in reality there exists many situations, where only some observation processes could be observed by the players. For example, in the financial market, the investors can only observe the security prices. Thus the portfolio process is required to be adapted to the natural filtration of the security price process (Xiong & Zhou, 2007). In general, partially observed problems are related with filtering theory (Bensoussan, 1992; Liptser & Shiryaev, 1977; Xiong, 2008). Partially observed stochastic optimal control and differential games have been researched by many authors, such as Huang et al. (2009), Li and Tang (1995), Shi and Wu (2010), Tang (1998), Wang and Wu (2009), Wang et al. (2013), Wang et al. (2015), Wang et al. (2018), Wu (2010) and Wu and Zhuang (2018).

Inspired by the above literatures, in this paper we study the Stackelberg differential game with asymmetric noisy observation, with deterministic coefficients and convex control domains. To the best of our knowledge, papers on the topic about partially observed Stackelberg differential games are quite lacking, except (Li et al., 2019, July 27–30). Note that in Li et al. (2019, July 27–30), the leader–follower Stackelberg stochastic differential game under a symmetric, partial observed information is researched. However, compared to Li et al. (2019, July 27–30), a deeper understanding of the observation equation and a completely different technical method are considered by us in this paper. The novelty of the formulation and the contribution in this paper is the following.

1. A new kind of Stackelberg stochastic differential game with asymmetric noisy observation is introduced. In our framework, the control processes of the follower are required to be adapted to the information filtration generated by the observation process, which is a Brownian motion in the original probability space, while the information filtration available to the leader is generated by both the Brownian noise and the observation process.

2. For the follower’s problem, a stochastic optimal control problem with partial observation is solved. The partial information maximum principle (Theorem 3.1) similar as Bensoussan (1992) and Li and Tang (1995). Thanks for a mild assumption motivated by Huang et al. (2010), the partial information verification theorem (Theorem 3.2) is proved. It is remarkable that the Hamiltonian function (13) and adjoint Equations (14), (15) are different from those in Li et al. (2019, July 27–30), but similar as Li and Tang (1995).

3. For the leader’s problem, a stochastic optimal control problem of FBSDE is solved. Since the control processes are required to be adapted to the information filtration generated by both the Brownian motion and the observation process, and more importantly, the leader’s ‘state’ equation contains conditional mean-field terms in the form of controlled conditional expectation $\mathbb{E}[\cdot|$ $\mathcal{F}_T]$, we encounter a difficulty when applying the techniques in Wu (2010) and Wang et al. (2013). We overcome this difficulty by again the mild assumption used in Theorem 3.2 and Kallianpur–Striebel formula (Xiong, 2008), to obtain the maximum principle of the leader (Theorem 3.3). However, by Clarke’s generalised gradient, we could prove the verification theorem of the leader (Theorem 3.4) only in a special case, since the difficulty is fatal.

4. For the LQ case, it consists of an LQ stochastic optimal control problem with partial observation for the follower, and followed by an LQ stochastic optimal control problem of the coupled conditional mean-field FBSDE with complete observation information for the leader. The state estimate feedback representation of the Stackelberg equilibrium is obtained (Theorem 4.1 and Theorem 4.2), via some new Riccati equations, and the technique of Yong (2002). The solvability of the Riccati equations is discussed.

5. A motivating and inspiring application to the resource allocation and its numerical simulation are given to show the effectiveness of the proposed results.

The rest of this paper is organised as follows. In Section 2, the Stackelberg stochastic differential game with asymmetric noisy observation is formulated. In Section 3, maximum principles and verification theorems are proved, for the problems of the follower and the leader, respectively. Then the LQ Stackelberg stochastic differential game with asymmetric noisy observation is investigated in Section 4. Specially, Section 4.1 is devoted to the solution to an LQ stochastic optimal control problem with partial observation of the follower. Section 4.2 is devoted to the solution to an LQ stochastic optimal control problem of coupled conditional mean-field FBSDE with complete observation information of the leader. The open-loop Stackelberg equilibrium is represented as its state estimate feedback form. In Section 5, the theoretic results in the previous sections are applied to the resource allocation problem and are illustrated by a simulating example. Finally, Section 6 gives some concluding remarks.
2. Problem formulation

Let $T > 0$ be a finite time duration. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which two independent standard Brownian motions $W(\cdot)$ and $Y(\cdot)$ valued in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ are defined. For $t \geq 0$, $\mathcal{F}^W_t$ and $\mathcal{F}^Y_t$ are the natural filtration generated by $W(\cdot)$ and $Y(\cdot)$, respectively, and we set $\mathcal{F}_t = \mathcal{F}^W_t \times \mathcal{F}^Y_t$. $\mathbb{E}$ denotes the expectation under probability $P$. In this paper, $L^2(\mathbb{R}^n)$ denotes the set of square-integrable random variables, $\mathcal{F}_t$-measurable, square-integrable random variables, $L^2(0, T; \mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued, $\mathcal{F}_t$-adapted, square integrable processes on $[0, T]$, and $L^\infty(0, T; \mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued, bounded functions on $[0, T]$. Let us consider the following controlled SDE:

$$
\begin{align*}
\frac{dx^{u_1,u_2}(t)}{dt} &= b(t, x^{u_1,u_2}(t), u_1(t), u_2(t)) dt + \sigma(t, x^{u_1,u_2}(t), u_1(t), u_2(t)) dW(t), \quad t \in [0, T], \\
x^{u_1,u_2}(0) &= x_0,
\end{align*}
$$

(1)

where $u_1(\cdot)$ and $u_2(\cdot)$ are control processes taken by player 1 (the follower) and player 2 (the leader) with values in non-empty convex sets $U_1 \subseteq \mathbb{R}^{d_1}$ and $U_2 \subseteq \mathbb{R}^{d_2}$, respectively. Here, $x_0 \in \mathbb{R}^n$, $b : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \rightarrow \mathbb{R}^{n \times d_1}$ are given functions.

We assume that the state process $x^{u_1,u_2}(\cdot)$ cannot be observed by the follower directly, but he/she can observe a related process $Y(\cdot)$, which satisfies the following controlled SDE:

$$
Y(t) = \int_0^t h(s, x^{u_1,u_2}(s), u_1(s), u_2(s)) ds + W^{u_1,u_2}(t),
$$

(2)

where $h(t, x, u_1, u_2) : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \rightarrow \mathbb{R}^{d_2}$ are given functions, and $W^{u_1,u_2}(\cdot)$ denotes a stochastic process depending on the control process pair $(u_1(\cdot), u_2(\cdot))$.

The following hypotheses are assumed.

(A1) The functions $b, \sigma$ are linear growth and continuously differentiable with respect to $u_1, u_2$ and $x$, and their partial derivatives with respect to $u_1, u_2$ and $x$ are all uniformly bounded. Moreover, the function $h$ is continuously differentiable with respect to $u_1, u_2$ and $x$, and there exists some constant $K > 0$, such that for any $t \in [0, T], x \in \mathbb{R}^n, u_1 \in \mathbb{R}^{d_1}, u_2 \in \mathbb{R}^{d_2}$,

$$
|h(t, x, u_1, u_2)| + |h_x(t, x, u_1, u_2)| + |h_{u_1}(t, x, u_1, u_2)| + |h_{u_2}(t, x, u_1, u_2)| \leq K.
$$

Motivated by some interesting random phenomena in reality, we begin to explain the asymmetric information between the follower and the leader, in our Stackelberg game problem. In the follower’s problem, a stochastic optimal control problem with partial information need to be solved, since the information available to him/her at time $t$ is based on the filtration generated by the noisy observation process $\mathcal{F}^Y_t = \sigma(Y(s), 0 \leq s \leq t)$. However, in the leader’s problem, a stochastic optimal control problem with complete information is required to be solved, since the information available to him/her at time $t$ is based on the complete information/filtration $\mathcal{F}_t$. Obviously, we have $\mathcal{F}^Y_t \subseteq \mathcal{F}_t$ and the information of the follower and the leader has the asymmetric feature and structure.

Next, we define the admissible control sets of the follower and the leader, respectively, as follows:

$$
\begin{align*}
\mathcal{U}_1 &= \{ u_1 | u_1 : \Omega \times [0, T] \rightarrow U_1 \text{ is } \mathcal{F}^Y_t \text{-adapted and} \\
&\quad \quad \quad \quad \quad \quad \quad \sup_{0 \leq t \leq T} \mathbb{E}|u_1(t)|^2 < \infty, i = 1, 2, \ldots \}, \\
\mathcal{U}_2 &= \{ u_2 | u_2 : \Omega \times [0, T] \rightarrow U_2 \text{ is } \mathcal{F}_t \text{-adapted and} \\
&\quad \quad \quad \quad \quad \quad \quad \sup_{0 \leq t \leq T} \mathbb{E}|u_2(t)|^2 < \infty, i = 1, 2, \ldots \}.
\end{align*}
$$

(3)

For any $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, we know that (1) admits a unique solution under hypothesis (A1), which is denoted by $x^{u_1,u_2}(\cdot) \in L^2(\mathcal{F}_T, 0, T; \mathbb{R}^n)$.

From Girsanov’s theorem, it follows that if we define

$$
Z^{u_1,u_2}(t) = \exp \left\{ \int_0^t h^\top(s, x^{u_1,u_2}(s), u_1(s), u_2(s)) dY(s) - \frac{1}{2} \int_0^t |h(s, x^{u_1,u_2}(s), u_1(s), u_2(s))|^2 ds \right\},
$$

(4)

i.e.

$$
\begin{align*}
\frac{dZ^{u_1,u_2}(t)}{Z^{u_1,u_2}(t)} &= h^\top(t, x^{u_1,u_2}(t), u_1(t), u_2(t)) dY(t), \\
Z^{u_1,u_2}(0) &= 1,
\end{align*}
$$

(5)

and if $d\mathbb{P}^{u_1,u_2} = Z^{u_1,u_2}(T) d\mathbb{P}$, then $\mathbb{P}^{u_1,u_2}$ is a new probability and $(W(\cdot), W^{u_1,u_2}(\cdot))$ is an $\mathbb{R}^{d_1+d_2}$-valued Brownian motion under $\mathbb{P}^{u_1,u_2}$.

In our Stackelberg game problem, knowing that the leader has chosen $u_2(\cdot) \in \mathcal{U}_2$, the follower would like to choose an $\mathcal{F}_T$-adapted control $\tilde{u}_1(\cdot) \equiv \tilde{u}_1(\cdot; u_2(\cdot))$ to minimise his cost functional

$$
J_1(u_1(\cdot), u_2(\cdot)) = \mathbb{E}^{u_1,u_2} \left[ \int_0^T l_1(t, x^{u_1,u_2}(t), u_1(t), u_2(t)) dt + G_1(x^{u_1,u_2}(T)) \right],
$$

(6)

where $\mathbb{E}^{u_1,u_2}$ denotes the expectation under the probability $\mathbb{P}^{u_1,u_2}$. Here functions $l_1 : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \rightarrow \mathbb{R}$ and $G_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ are given.

Problem of the follower. For any chosen $u_2(\cdot) \in \mathcal{U}_2$ by the leader, choose an $\mathcal{F}_T$-adapted control $\tilde{u}_1(\cdot) = \tilde{u}_1(\cdot; u_2(\cdot)) \in \mathcal{U}_1$ such that

$$
J_1(\tilde{u}_1(\cdot), u_2(\cdot)) \equiv J_1(\tilde{u}_1(\cdot; u_2(\cdot)), u_2(\cdot)) = \inf_{u_1 \in \mathcal{U}_1} J_1(u_1(\cdot), u_2(\cdot)),
$$

(7)

subject to (1) and (6). Such a $\tilde{u}_1(\cdot) = \tilde{u}_1(\cdot; u_2(\cdot))$ is called an optimal control, and the corresponding solution $x^{\tilde{u}_1,u_2}(\cdot)$ to (1) is called an optimal state process, for the follower.

In the following procedure of the game problem, once knowing that the follower would take such an optimal control $\tilde{u}_1(\cdot) = \tilde{u}_1(\cdot; u_2(\cdot))$, the leader will choose $u_2(\cdot) \in \mathcal{U}_2$ to maximise the cost functional $J_2(u_1(\cdot), u_2(\cdot))$.
\( \tilde{u}_1(\cdot), \tilde{u}_2(\cdot) \), the leader would like to choose an \( \mathcal{F}_t \)-adapted control \( \tilde{u}_2(\cdot) \) to minimise its cost functional
\[
J_2(\tilde{u}_1, \tilde{u}_2) = \mathbb{E}[\tilde{u}_1, \tilde{u}_2] \left[ \int_0^T l_2(t, x^{\tilde{u}_1, \tilde{u}_2}(t), \tilde{u}_1(t), \tilde{u}_2(t)) \, dt + G_2(x^{\tilde{u}_1, \tilde{u}_2}(T)) \right].
\] (8)

Here functions \( l_2 : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \rightarrow \mathbb{R} \) and \( G_2 : \mathbb{R}^n \rightarrow \mathbb{R} \) are given.

**Problem of the leader.** Find an \( \mathcal{F}_t \)-adapted control \( \tilde{u}_2(\cdot) \in U_2 \) such that
\[
J_2(\tilde{u}_1, \tilde{u}_2) = \inf_{u_2 \in U_2} J_2(\tilde{u}_1, \tilde{u}_2),
\] (9)
subject to (1) and (8). Such a \( \tilde{u}_2(\cdot) \) is called an optimal control, and the corresponding solution \( x^{\tilde{u}_1, \tilde{u}_2}(\cdot) \) to (1) is called an optimal state process for the leader. We will restate the problem for the leader in more detail, since its precise description has to involve the solution to *Problem of the follower*.

We refer to the problem mentioned above as a Stackelberg stochastic differential game with asymmetric noisy observations. If there exists a control process pair \( (\tilde{u}_1, \tilde{u}_2) \) satisfying (7) and (9), we refer to it as an open-loop Stackelberg equilibrium.

We also introduce the following assumption.

(A2) For \( i = 1, 2 \), the functions \( l_i, G_i \) are continuously differentiable with respect to \( x, u_1, u_2 \), and there exists a constant \( C > 0 \) such that for any \( t \in [0, T], x \in \mathbb{R}^n, u_1 \in \mathbb{R}^{d_1}, u_2 \in \mathbb{R}^{d_2}, \)
\[
1 + |x|^2 + |u_1|^2 + |u_2|^2 + l_i(t, x, u_1, u_2)
+ 1 + |x| + |u_1| + |u_2| \leq C,
\]
\[
|x|^2 + |G_i(x)| + 1 + |x|^{-1} |G_i(x)| \leq C.
\]

### 3. Maximum principle and verification theorem for Stackelberg equilibrium

In this paper, we frequently omit some time variable \( t \) in some mathematical formula for simplicity, if there exists not ambiguity.

#### 3.1 The problem of the follower

For any chosen \( u_2(\cdot) \in U_2 \), we first consider *Problem of the follower* which is a partially observed stochastic optimal control problem.

By Bayes’s formula, *Problem of the follower* is equivalent to minimise
\[
J_1(u_1, u_2) = \mathbb{E} \left[ \int_0^T Z^{u_1, u_2}(t) l_1(t, x^{u_1, u_2}(t), u_1(t), u_2(t)) \, dt + Z^{u_1, u_2}(T) G_1(x^{u_1, u_2}(T)) \right],
\] (10)
over \( U_1 \), subject to (1) and (5). We first present the following lemma about some estimates for \( x^{u_1, u_2}(\cdot) \) and \( Z(\cdot) \), which belong to Li and Tang (1995).

**Lemma 3.1:** For any \( u_1(\cdot), u_2(\cdot) \in U_1 \times U_2 \), let \( x^{u_1, u_2}(\cdot) \) be the corresponding solution to (1). Then there exists some constant \( C > 0 \), such that
\[
\sup_{0 \leq t \leq T} \mathbb{E}|x^{u_1, u_2}(t)|^2 \leq C \left[ 1 + \sup_{0 \leq t \leq T} \mathbb{E}|u_1(t)|^2 + |u_2(t)|^2 \right],
\]
\[
\sup_{0 \leq t \leq T} \mathbb{E}|Z^{u_1, u_2}(t)|^2 \leq C.
\] (11)

The following maximum principle for *Problem of the follower* can be obtained by classical results in Bensoussan (1992) and Li and Tang (1995). We omit the details.

**Theorem 3.1:** Let (A1) and (A2) hold. For any given \( u_2(\cdot) \in U_2 \), if \( \tilde{u}_1(\cdot) \) is an optimal control of Problem of the follower, then the maximum condition
\[
\mathbb{E}_0^{\tilde{u}_1, u_2} \left[ (H_1(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2, p, k, k), v_1 - \tilde{u}_1(t)) \big| \mathcal{F}_t^Y \right] \geq 0,
\]
\[
\forall v_1 \in U_1,
\] (12)
holds for a.e. \( t \in [0, T] \), \( \mathbb{P}^{\tilde{u}_1, u_2} - a.s. \), where the Hamiltonian function \( H_1 : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \times \mathbb{R}^{n \times d_1} \times \mathbb{R}^{n \times d_2} \rightarrow \mathbb{R} \) is defined by
\[
H_1(t, x^{u_1, u_2}, u_1, u_2, p, k, k) \triangleq \left[ p(t), b(t, x^{u_1, u_2}, u_1, u_2) \right] + \mathbf{tr} \left\{ k(t) \mathbf{J} \sigma(t, x^{u_1, u_2}, u_1, u_2) \right\} + \left\{ K(t), h(t, x^{u_1, u_2}, u_1, u_2) \right\} + l_1(t, x^{u_1, u_2}, u_1, u_2),
\] (13)
the adjoint process pairs \( (P(\cdot), K(\cdot)) \in L^2_{\mathbb{P}}(0, T; \mathbb{R}) \times L^2_{\mathbb{P}}(0, T; \mathbb{R}^{d_1}) \) and \( (p(\cdot), k(\cdot)) \in L^2_{\mathbb{P}}(0, T; \mathbb{R}^n) \times L^2_{\mathbb{P}}(0, T; \mathbb{R}^{n \times d_1}) \) satisfy the following two BSDEs, respectively:
\[
\begin{align*}
-dP(t) &= l_1(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2) \, dt + \mathbf{J} k(t) \, dW(t), \\
&P(T) = p(T) = G_1(x^{\tilde{u}_1, u_2}(T)),
\end{align*}
\] (14)
\[
\begin{align*}
-dp(t) &= \left[ l_1(t, x^{\tilde{u}_1, \tilde{u}_2}, \tilde{u}_1, u_2) + h(t, x^{\tilde{u}_1, \tilde{u}_2}, \tilde{u}_1, u_2) \right] + b(t, x^{\tilde{u}_1, \tilde{u}_2}, \tilde{u}_1, u_2) \, dt - k(t) \, dW(t), \\
p(t) &= G_1(t, x^{\tilde{u}_1, \tilde{u}_2}(t)).
\end{align*}
\] (15)

Then we continue to give the sufficient condition (that is, verification theorem) to guarantee the optimality for control \( \tilde{u}_1(\cdot) \) of Problem of the follower.

**Theorem 3.2:** Let (A1) and (A2) hold. For any given \( u_2(\cdot) \in U_2 \), let \( \tilde{u}_1(\cdot) \in U_1 \) and \( x^{\tilde{u}_1, u_2}(\cdot) \) be the corresponding state. Let \( (P(\cdot), K(\cdot)) \) be the adjoint process pairs satisfying (14) and (15), respectively. Suppose for all \( (t, x, u_1, u_2) \in [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \), \( Z^{u_1, u_2}(t) \) is \( \mathcal{F}_t^Y \)-adapted, maps \( (x, u_1) \rightarrow H_1(t, x, u_1, u_2, p, k, k) \) and \( x \rightarrow G_1(x) \) are both convex, and
\[
\mathbb{E}
\left[
H_1(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2, p, k, k) \big| \mathcal{F}_t^Y
\right]
= \min_{u_1 \in U_1}
\mathbb{E}
\left[
H_1(t, x^{u_1, u_2}, u_1, u_2, p, k, k) \big| \mathcal{F}_t^Y
\right]
\] (16)
holds for a.e. \( t \in [0, T] \), \( \mathbb{P} - a.s. \). Then \( \tilde{u}_1(\cdot) \) is an optimal control of Problem of the follower.
Proof: For any \( u_1(\cdot) \in U_1 \), we have
\[
J_1(u_1(\cdot), u_2(\cdot)) = J_1(\tilde{u}_1(\cdot), u_2(\cdot))
\]
\[
= \mathbb{E} \left[ \int_0^T [Z^{u_1, u_2}(t)l_1(t, \tilde{x}^{u_1, u_2}, \tilde{u}_1, u_2) - Z^{\tilde{u}_1, u_2}(t)l_1(t, \tilde{x}^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2)] dt 
+ Z^{u_1, u_2}(T)G_1(x^{u_1, u_2}(T)) - Z^{\tilde{u}_1, u_2}(T)G_1(x^{\tilde{u}_1, u_2}(T)) \right]
\]
\[
= \mathbb{E} \left[ \int_0^T \left( [Z^{u_1, u_2}(t) - Z^{\tilde{u}_1, u_2}(t)]l_1(t, \tilde{x}^{u_1, u_2}, \tilde{u}_1, u_2) \right) dt 
+ [Z^{u_1, u_2}(T) - Z^{\tilde{u}_1, u_2}(T)]G_1(x^{u_1, u_2}(T)) \right] 
\]
\[
\geq \mathbb{E} \left[ \int_0^T \left( [Z^{u_1, u_2}(t) - Z^{\tilde{u}_1, u_2}(t)]l_1(t, \tilde{x}^{u_1, u_2}, \tilde{u}_1, u_2) \right) dt 
+ [Z^{u_1, u_2}(T) - Z^{\tilde{u}_1, u_2}(T)]G_1(x^{u_1, u_2}(T)) \right] \equiv I + II.
\]
Due to the convexity of \( G_1(\cdot) \), we have
\[
II \geq \mathbb{E}^{u_1, u_2} \left[ \int_0^T \left( I_1(t, x^{u_1, u_2}, u_1, u_2) - I_1(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2) \right) dt 
+ G_{1x}(x^{u_1, u_2}(T))(x^{u_1, u_2}(T) - x^{\tilde{u}_1, u_2}(T)) \right].
\]
Applying Itô’s formula to \( Z^{u_1, u_2}(\cdot) - Z^{\tilde{u}_1, u_2}(\cdot)P(\cdot) \), it is easy to get
\[
I = \mathbb{E} \left[ \int_0^T Z^{u_1, u_2}(t) \left( \hat{h}(t, x^{u_1, u_2}, u_1, u_2) 
- \hat{h}(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2) \right) dt 
+ Z^{u_1, u_2}(T)G_1(x^{u_1, u_2}(T)) \right]
\]
\[
= \mathbb{E}^{u_1, u_2} \left[ \int_0^T \left( K(t)h(t, x^{u_1, u_2}, u_1, u_2) 
- (h(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2)) \right) dt \right].
\]
Similarly, applying Itô’s formula to \( p(\cdot)(x^{u_1, u_2}(\cdot) - x^{\tilde{u}_1, u_2}(\cdot)) \), by (18), we have
\[
II \geq \mathbb{E}^{u_1, u_2} \left[ \int_0^T \left( H_1(t, x^{u_1, u_2}, u_1, u_2, p, k, K) 
- H_1(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2, p, k, K) 
- \{K(t)h(t, x^{u_1, u_2}, u_1, u_2) 
- (h(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2)) \} \right) dt \right].
\]
Using the convexity of \( H_1(t, \cdot, \cdot, u_2, p, k, K) \), by (19) and (20), we obtain
\[
I + II \geq \mathbb{E}^{u_1, u_2} \left[ \int_0^T \left( H_{1u_1}(t, x^{u_1, u_2}, \tilde{u}_1, u_2, p, k, K) 
- \{K(t)h(t, x^{u_1, u_2}, u_1, u_2) 
- (h(t, x^{\tilde{u}_1, u_2}, \tilde{u}_1, u_2)) \} \right) dt \right].
\]
Noticing that \( Z^{u_1, u_2}(\cdot) > 0 \) is \( \mathcal{F}_t^Y \)-adapted, by the condition (16), we get
\[
0 \leq \mathbb{E} \left[ \int_0^T \left( H_{1u_1}(t, x^{u_1, u_2}, \tilde{u}_1, u_2, p, k, K) \right) dt \right] 
\]
\[
= \mathbb{E} \left[ \int_0^T \left( H_{1u_1}(t, x^{u_1, u_2}, \tilde{u}_1, u_2, p, k, K) \right) dt \right].
\]
Thus from (17) we have \( J_1(u_1(\cdot), u_2(\cdot)) \leq J_1(\tilde{u}_1(\cdot), u_2(\cdot)) \geq 0 \), for any \( u_1(\cdot) \in U_1 \). Then we complete our proof.

Remark 3.1: We point out that, in the above theorem, the assumption ‘for all \( (t, x, u_1, u_2) \in [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \), \( Z^{u_1, u_2}(t) \) is \( \mathcal{F}_t^Y \)-adapted’, is referred to the condition at page 1442 in Theorem 2 for sufficient conditions of optimality in a partial information risk-sensitive optimal control problem (Huang et al., 2010). And we believe that the condition is inspired by the idea of An et al. (2008). Specifically, we can give two kinds of functions of \( h \) satisfying this assumption: (1) \( h \) is independent of \( x^{u_1, u_2} \) and \( u_2 \), that is, \( h = h(t, u_1(t)) \). In particular, when \( h(t, u_1(t)) = u_1(t) \), (4) is reduced to
\[
Z^{u_1}(t) = \exp \left\{ \int_0^t u_1(s) dY(s) - \frac{1}{2} \int_0^t |u_1(s)|^2 ds \right\},
\]
which is exactly the setting in An et al. (2008), where \( Z^{u_1}(\cdot) \) is adapted to \( \mathcal{F}_t^Y \). Therefore, the condition given in Huang et al. (2010) is an extension of that in An et al. (2008), that is why we directly give a more general case that \( h(t, x^{u_1, u_2}, u_1, u_2) \) is also adapted to \( \mathcal{F}_t^Y \). In this case, it is naturally applied to our problem and can overcome the difficulties encountered in the following discussion. (2) \( h(t, x^{u_1, u_2}, u_1, u_2) = \mathbb{E} [a(t)x^{u_1, u_2} + b(t)u_1 + c(t)u_2 | \mathcal{F}_t^Y] \) or \( h \triangleq h(t, x^{u_1, u_2}, u_1, u_2) \), where \( a(t), b(t), c(t) \) are some deterministic functions, \( x^{u_1, u_2}(t) = \mathbb{E}[x^{u_1, u_2} | \mathcal{F}_t^Y] \) and \( \tilde{u}_2(t) = \mathbb{E}[u_2 | \mathcal{F}_t^Y] \). This case is reasonable because, in the observation Equation (2), the information related to \( x, u_2 \) is only based on the known information generated by the observation process \( Y(\cdot) \), then \( h \) can be regarded as some kind of function of the optimal filters of \( x, u_2 \) based on \( \mathcal{F}_t^Y \). In this framework, we can also guarantee that \( z^{u_1, u_2}(\cdot) \) is \( \mathcal{F}_t^Y \)-adapted. The difference is that the problem becomes a special partially observed one with conditional mean-field term in the drift term of the observation equation, and it can also be solved by the similar method in our paper.

3.2 The problem of the leader

In this subsection, we firstly state the stochastic optimal control problem with complete information of the leader in detail, then give the maximum principle and verification theorem for it. For any \( u_2(\cdot) \in U_2 \), by the maximum condition (12), we assume that a functional \( \tilde{u}_1(t) = \tilde{u}_1(t; x^{\tilde{u}_1, u_2}, \tilde{u}_2, \tilde{p}, \tilde{k}) \) is uniquely defined, where we set
\[
\tilde{x}^{\tilde{u}_1, u_2}(t) \triangleq \mathbb{E}^{\tilde{u}_1, u_2}[x^{\tilde{u}_1, u_2}(t) | \mathcal{F}_t^Y],
\]
\[ \phi(t) \triangleq \mathbb{E}^{\tilde{u}_1,u_2} \left[ \phi(t) \mid F_t^1 \right], \quad t \in [0, T], \tag{24} \]

for \( \phi = u_2, p, k, K \). Firstly, the leader encounters the controlled system of FBSDEs:

\[
\begin{aligned}
\dot{x}^{\tilde{u}_1,u_2}(t) &= b(t, x^{\tilde{u}_1,u_2}, \tilde{u}_1, u_2) \, dt \\
&+ \sigma(t, x^{\tilde{u}_1,u_2}, \tilde{u}_1, u_2) \, dW(t), \\
\dot{Z}^{\tilde{u}_1,u_2}(t) &= Z^{\tilde{u}_1,u_2}(t) h^T(t, x^{\tilde{u}_1,u_2}, \tilde{u}_1, u_2) \, dY(t), \\
-dp(t) &= \left[ l_1(t, x^{\tilde{u}_1,u_2}, \tilde{u}_1, u_2) + \tilde{h}_1^x(t, x^{\tilde{u}_1,u_2}, \tilde{u}_1, u_2) \right] k(t) dt - k(t) \, dW(t), \\
-dP(t) &= \left[ l_2(t, x^{\tilde{u}_1,u_2}, \tilde{u}_1, u_2) + k(t) h(t, x^{\tilde{u}_1,u_2}, \tilde{u}_1, u_2) \right] dt - k(t) \, dY(t), \\
x^{\tilde{u}_1,u_2}(0) &= x_0, \quad Z^{\tilde{u}_1,u_2}(0) = 1, \quad p(T) = G_1(x^{\tilde{u}_1,u_2}(T)), \quad P(T) = G_1(x^{u_2}(T)).
\end{aligned}
\] (25)

For the simplicity of notations, we denote \( x^{u_2}(\cdot) \equiv x^{\tilde{u}_1,u_2}(\cdot), Z^{u_2}(\cdot) \equiv Z^{\tilde{u}_1,u_2}(\cdot) \) and define \( \Phi^L \) on \([0, T] \times \mathbb{R}^n \times U_2 \) as \( \Phi^L(t, x^{u_2}, \tilde{u}_1, \tilde{u}_2, \tilde{p}, \tilde{k}, \tilde{K}) \), for \( \Phi = b, \sigma, h, l_1, l_2 \). Thus the leader’s state Equation (25) can be rewritten as:

\[
\begin{aligned}
\dot{x}^{u_2}(t) &= b^L(t, x^{u_2}, u_2) \, dt + \sigma^L(t, x^{u_2}, u_2) \, dW(t), \\
\dot{Z}^{u_2}(t) &= Z^{u_2}(t) h^L(t, x^{u_2}, u_2) \, dY(t), \\
-dp(t) &= f^L_1(t, x^{u_2}, u_2, p, k, K) dt - k(t) \, dW(t), \\
-dP(t) &= f^L_2(t, x^{u_2}, u_2, K) \quad dt - k(t) \, dY(t), \quad t \in [0, T], \\
x^{u_2}(0) &= x_0, \quad Z^{u_2}(0) = 1, \quad p(T) = G_1(x^{u_2}(T)), \quad P(T) = G_1(x^{u_2}(T)).
\end{aligned}
\] (26)

where we define

\[
\begin{aligned}
f^L_1(t, x^{u_2}, u_2, p, k, K) &\triangleq f^L_1(t, x^{u_2}, u_2) + h^L_1(x^{u_2}, u_2) \, \tilde{K}^T(t) \\
+ b^L_1(x^{u_2}, u_2) p(t) + \sigma^L_1(x^{u_2}, u_2) k(t), \\
f^L_2(t, x^{u_2}, u_2, K) &\triangleq f^L_2(t, x^{u_2}, u_2, K) + k(t) h^L(t, x^{u_2}, u_2).
\end{aligned}
\]

We note that (26) is a controlled conditional mean-field FBSDEs, which now is regarded as the ‘state’ equation of the leader. That is to say, the state of the leader is the six-tuple \((x^{u_2}(\cdot), Z^{u_2}(\cdot), p(\cdot), k(\cdot), P(\cdot), K(\cdot))\). By (8), we define

\[
\begin{aligned}
f^L_1(u_2(\cdot)) &\triangleq f^L_1(\tilde{u}_1(\cdot), u_2(\cdot)) \\
&= \mathbb{E}^{\tilde{u}_1,u_2} \left[ \int_0^T l_1(t, x^{\tilde{u}_1,u_2}(t), \tilde{u}_1(t), u_2(t)) \, dt \right] + G_2(x^{u_2}(T)) \\
&= \mathbb{E}^{u_2} \left[ \int_0^T l_1^L(t, x^{u_2}(t), u_2(t)) \, dt + G_2(x^{u_2}(T)) \right].
\end{aligned}
\]

Suppose \( \tilde{u}_2(\cdot) \) is an optimal control of Problem of the leader, and the associated optimal state \((x^{u_2}(\cdot), Z^{u_2}(\cdot), \tilde{p}(\cdot), \tilde{k}(\cdot), \tilde{P}(\cdot), \tilde{K}(\cdot))\) satisfies (26) with respect to \( \tilde{u}_2(\cdot) \). In order to derive the maximum principle, we define the perturbed control \( u^p_2(t) \triangleq \tilde{u}_2(t) + \theta (v_2(t) - \tilde{u}_2(t)) \), for \( t \in [0, T] \), where \( \theta > 0 \) is sufficiently small and \( v_2(\cdot) \) is an arbitrary element of \( U_2 \). The convexity of \( U_2 \) implies that \( u^p_2(\cdot) \in U_2 \). Let \((x^{u_2}(\cdot), Z^{u_2}(\cdot), \tilde{p}(\cdot), \tilde{k}(\cdot), \tilde{P}(\cdot), \tilde{K}(\cdot))\) be the state corresponding to \( u^p_2(\cdot) \). Keeping in mind that \( b^L, \sigma^L, h^L, f^L_1, f^L_2 \) depend on not only \((x^{u_2}, u_2)\) but also \((x^{\tilde{u}_1,u_2}, \tilde{u}_1, \tilde{u}_2)\). Then we introduce the following system of variational equations whose solution is the six-tuple \((x^{\cdot}(\cdot), Z^{\cdot}(\cdot), p^{\cdot}(\cdot), k^{\cdot}(\cdot), P^{\cdot}(\cdot), K^{\cdot}(\cdot))\):

\[
\begin{aligned}
\dot{x}^{\cdot}(t) &= \left[ \tilde{b}_x^L(t, x^{\cdot}, u_2) + \tilde{b}_p^L(t, x^{\cdot}, u_2) + \tilde{b}_k^L \cdot \tilde{K}^L \right] dt \\
&+ \left[ \tilde{\sigma}_x^L(t, x^{\cdot}, u_2) + \tilde{\sigma}_p^L(t, x^{\cdot}, u_2) + \tilde{\sigma}_k^L \cdot \tilde{K}^L \right] \, dW(t), \\
\dot{Z}^{\cdot}(t) &= \left[ \tilde{h}_x^L(t, x^{\cdot}, u_2) + \tilde{h}_p^L(t, x^{\cdot}, u_2) + \tilde{h}_k^L \cdot \tilde{K}^L \right] dt \\
&+ \left[ \tilde{\tilde{h}}_x^L(t, x^{\cdot}, u_2) + \tilde{\tilde{h}}_p^L(t, x^{\cdot}, u_2) \right] \, dY(t), \\
-dp^{\cdot}(t) &= \left[ \tilde{f}_1^{\cdot}(t, x^{\cdot}, u_2, p, k, K) \right] dt - k^{\cdot}(t) \, dW(t), \\
-dP^{\cdot}(t) &= \left[ \tilde{f}_2^{\cdot}(t, x^{\cdot}, u_2, K) \right] \, dt - k^{\cdot}(t) \, dY(t), \quad t \in [0, T], \\
x^{\cdot}(0) &= x_0, \quad Z^{\cdot}(0) = 1, \quad p^{\cdot}(T) = G_1(x^{\cdot}(T)), \quad P^{\cdot}(T) = G_1(x^{u_2}(T)) x^{\cdot}(T),
\end{aligned}
\] (28)

where we have used \( \tilde{\Lambda}^L(t) \equiv \tilde{\Lambda}^L(t, x^{\cdot}, \tilde{u}_2) \) for \( \Lambda = b, \sigma, h, f_1, f_2, l_1, l_2 \) and all their partial derivatives.

**Remark 3.2:** It is necessary for us to analyse the system of variational Equation (28), which is non-trivial to derive, and we should notice that \( k_1 \triangleq \mathbb{E}^{u_2} [k_1 \mid F_1^1] \), for \( \kappa_1 = x^1, Z^1, p^1, k_1^1, P^1, K^1, v_2 - \tilde{u}_2 \). Actually, for example, the part \( \tilde{b}_x^L \mathbb{E}^{u_2} [k_2^u \mid F_1^1] - \mathbb{E}^{u_2} [k_2 \tilde{u}_2 \mid F_1^1] \), for \( \kappa_2 = x, Z, p, k, P, K \), will appear when we apply the convex variation, which adds difficulty for deduction due to the expectation \( \mathbb{E}^{u_2} \) depending on the control variable \( u_2 \). However, we could overcome
this difficulty under some assumptions, by converting it to the expectation $\mathbb{E}$ independent of $u_2$. For this target, we use Kallianpur–Striebel formula (Bayes’ rule in the filtering setup) in Xiong (2008) to get

$$
\vec{L}_{t_2} \left( \frac{\mathbb{E} u_2^\theta \left[ \kappa_2^\theta \mid \mathcal{F}_t^Y \right]}{2 \kappa_2^\theta \mid \mathcal{F}_t^Y} \right) = \vec{L}_{t_2} \left( \frac{\mathbb{E} \left[ Z_{u_2^\theta k_2^\theta} - Z_{u_2^\theta} \mid \mathcal{F}_t^Y \right]}{2 \kappa_2^\theta \mid \mathcal{F}_t^Y} \right)
$$

Thus,

$$
\int_0^T \left( J_2^\theta (u_2^\theta) - J_2^\theta (\tilde{u}_2^\theta) \right) \ dt = \mathbb{E} \left[ \int_0^T \left( Z_0^\theta (t) \tilde{L}_2^\theta (t, x^\theta, u_2^\theta) - Z_{u_2^\theta} (t) \tilde{L}_2^\theta (t, x^\tilde{u}_2^\theta, \tilde{u}_2^\theta) \right) \ dt \right] + Z_0^\theta (T) G_{x_2^\theta} (x_2^\theta (T)) + Z_{u_2^\theta} (T) G_{x_2^\theta} (x_{u_2^\theta} (T)) \geq 0.
$$

From Lemma 3.2, when $\theta \to 0$, it follows that

$$
\frac{1}{\theta} \left[ J_2^\theta (u_2^\theta) - J_2^\theta (\tilde{u}_2^\theta) \right] \to \mathbb{E} \left[ \int_0^T \left( Z_0^\theta (t) \tilde{L}_2^\theta (t, x^\theta, u_2^\theta) + Z_{u_2^\theta} (t) \tilde{L}_2^\theta (t, x^\theta, \tilde{u}_2^\theta) \right) \ dt \right] + Z_0^\theta (T) G_{x_2^\theta} (x_2^\theta (T)) + Z_{u_2^\theta} (T) G_{x_2^\theta} (x_{u_2^\theta} (T)) \geq 0.
$$

i.e.

$$
\mathbb{E} \left[ \int_0^T \left( Z_{u_2^\theta} (t) - Z_{u_2^\theta} (t) \tilde{L}_2^\theta (t, x^\theta, \tilde{u}_2^\theta) \right) \ dt \right] + Z_{u_2^\theta} (T) G_{x_2^\theta} (x_{u_2^\theta} (T)) \geq 0.
$$

For any $t \in [0, T]$, we set $\tilde{\lambda}^\theta (t) = \tilde{\lambda}^\theta (t) - \lambda^\theta (t)$, for $\lambda = x, Z, p, k, P, K$. By some classical technique (see Wu (2010) and Wang et al. (2013)), we have the following lemma.

**Lemma 3.2:**

$$
\lim_{\theta \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \tilde{\lambda}^\theta (t) \right|^2 = 0, \quad \lim_{\theta \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \tilde{\lambda}^\theta (t) \right|^2 = 0,
$$

$$
\lim_{\theta \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \tilde{\lambda}^\theta (t) \right|^2 = 0, \quad \lim_{\theta \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \tilde{\lambda}^\theta (t) \right|^2 = 0,
$$

$$
\lim_{\theta \to 0} \mathbb{E} \left[ \int_0^T \left| \tilde{\lambda}^\theta (t) \right|^2 \ dt \right] = 0, \quad \lim_{\theta \to 0} \mathbb{E} \left[ \int_0^T \left| \tilde{\lambda}^\theta (t) \right|^2 \ dt \right] = 0.
$$

(32)

Then, we derive the variational inequality. Since $\tilde{u}_2^\theta (\cdot)$ is an optimal control, we have

$$
\frac{1}{\theta} \left[ J_2^\theta (u_2^\theta (\cdot)) - J_2^\theta (\tilde{u}_2^\theta (\cdot)) \right] \geq 0.
$$

(33)
\[ + \tilde{H}_{\tilde{u}_2}^T (t) (v_2 - \tilde{u}_2(t)) + \tilde{H}_{\tilde{u}_2}^T (t) (\tilde{v}_2 - \tilde{u}_2(t)) \] 
\times dW^{\tilde{u}_2}(t), \ t \in [0, T],
\]
\[ \Gamma^1 (0) = 0, \]
(36)

where \( W(\cdot) \) and \( W^{\tilde{u}_2}(\cdot) \) are two independent standard Brownian motions under the probability \( \mathbb{P}^{\tilde{u}_2} \), \( dW^{\tilde{u}_2} \triangleq Z^{\tilde{u}_2}(T) d\mathbb{P}. \)

Next, we introduce the following system of adjoint equations, consisting of two SDEs and two BSDEs, whose solution is the six-tuple \((\alpha(\cdot), \beta(\cdot), \sigma(\cdot), \varphi(\cdot), \delta(\cdot), \gamma(\cdot))\):

\[
dq(t) = \left[ f_{1q}^T \bar{q} + \Xi^{\tilde{u}_2} \left[ f_{2p}^T \bar{q} + \tilde{H}_{\tilde{u}_2}^T \beta + \tilde{b}_{\tilde{u}_2}^T \varphi + \tilde{\sigma}_{\tilde{u}_2}^T \delta \right. \right. \\
+ \left. \left. \tilde{f}_{1k}^T \bar{k} + \tilde{f}_{1k}^T \bar{q} \right] dW^{\tilde{u}_2}(t), \ t \in [0, T], \right.
\]
\[ -d\varphi(t) = \left[ f_{1k}^T \bar{k} + \Omega^{\tilde{u}_2} \left[ f_{1k}^T \bar{q} + \tilde{H}_{\tilde{u}_2}^T \beta + \tilde{b}_{\tilde{u}_2}^T \varphi + \tilde{\sigma}_{\tilde{u}_2}^T \delta \right. \right. \\
+ \left. \left. \tilde{f}_{1k}^T \bar{k} + \tilde{f}_{1k}^T \bar{q} \right] dW^{\tilde{u}_2}(t), \right. \]
\[ -d\alpha(t) = \left[ \tilde{f}_{1k}^T (t) dt - \beta(t) \right] dW^{\tilde{u}_2}(t), \ t \in [0, T], \]
\[ q(0) = 0, \ Q(0) = 0, \alpha(T) = G_2(\tilde{x}^{\tilde{u}_2}(T)), \]
\[ \varphi(T) = G_{2x}(\tilde{x}^{\tilde{u}_2}(T)) + G_{1x}(\tilde{x}^{\tilde{u}_2}(T)) Q(T) \]
(37)

Then Equation (37) is equivalent to:

\[
dq(t) = \{ \tilde{H}_2 p(t) + E^{\tilde{u}_2} [\tilde{H}_2^T p(t) | \mathcal{F}_t^y] \} dt \\
+ \{ \tilde{H}_2 k(t) + E^{\tilde{u}_2} [\tilde{H}_2^T k(t) | \mathcal{F}_t^y] \} dW^{\tilde{u}_2}(t), \\
-d\varphi(t) = \{ \tilde{H}_2 x(t) + E^{\tilde{u}_2} [\tilde{H}_2^T x(t) | \mathcal{F}_t^y] \} dt - \delta(t) dW^{\tilde{u}_2}(t), \\
-d\alpha(t) = \tilde{f}_2 (t) dt - \beta(t) dW^{\tilde{u}_2}(t), \ t \in [0, T], \\
q(0) = 0, \ Q(0) = 0, \alpha(T) = G_2(\tilde{x}^{\tilde{u}_2}(T)), \]
\[ \varphi(T) = G_{2x}(\tilde{x}^{\tilde{u}_2}(T)) + G_{1x}(\tilde{x}^{\tilde{u}_2}(T)) Q(T) \]
(40)

where we set \( \tilde{H}_2 x(t) \equiv H_2 x(t, \tilde{x}^{\tilde{u}_2}, \tilde{u}_2, \tilde{p}, \tilde{K}, \hat{K}; q, \varphi, \delta, \beta) \) for \( \lambda = x, \tilde{x}, k, \tilde{k}, \hat{k}, \tilde{K}. \)

From (38) and (39), it is easy to obtain the following maximum principle of the leader.

\textbf{Theorem 3.3:} Let (A1) and (A2) hold, and \( \tilde{u}_2(\cdot) \in U_2 \) be an optimal control of Problem of the leader and \((x^{\tilde{u}_2}(\cdot), \tilde{z}^{\tilde{u}_2}(\cdot), \tilde{p}(\cdot), \tilde{k}(\cdot), \hat{K}(\cdot)) \) be the corresponding optimal state. Let \( q(\cdot), Q(\cdot), \varphi(\cdot), \delta(\cdot), \alpha(\cdot), \beta(\cdot) \) be the adjoint six-tuple satisfying (40), then we have

\[
\begin{aligned}
&\{ H_{2x}(t, \tilde{x}^{\tilde{u}_2}, \tilde{u}_2, \tilde{p}, \tilde{K}; q, \varphi, \delta, \beta), \ v_2 - \tilde{u}_2(t) \} \\
&+ [E^{\tilde{u}_2} \{ H_{2k}(t, \tilde{x}^{\tilde{u}_2}, \tilde{u}_2, \tilde{p}, \tilde{K}; q, \varphi, \delta, \beta) | \mathcal{F}_t^y \}] \\
&- \tilde{u}_2 - \tilde{u}_2(t) \geq 0,
\end{aligned}
\]
(41)
a.e. \( t \in [0, T], \) \( E^{\tilde{u}_2} - a.s. \) holds for any \( v_2 \in U_2. \)

In the following, we wish to establish the verification theorem for Problem of the leader. We aim to prove that, under some conditions, for any \( v_2(\cdot) \in U_2, \) \( H_2^L(v_2(\cdot)) - H_2^L(\tilde{u}_2(\cdot)) \geq 0 \) holds. However, we find that, during the duality procedure, when applying Itô’s formula, taking integral and expectation, we cannot guarantee that

\[
E^{\tilde{u}_2} \left[ \int_0^T \{ \cdots \} dW^{\tilde{u}_2}(t) \right] = 0
\]
holds for any \( u_2(\cdot) \in U_2 \) where \( \tilde{u}_2(\cdot) \) is a candidate optimal control. The reason is that it is not sure that \( W^{\tilde{u}_2}(\cdot) \) is a Brownian motion under the expectation \( E^{\tilde{u}_2}. \) This is the main challenging difficulty which is not easy to solve for us up to now. Therefore, in the following of this paper we consider \( h(t, x^{u_2}; u_1, u_2) \equiv h(t). \) In this special case, \( Y(\cdot) \) and \( W^{u_1, u_2}(\cdot) \) are not controlled by \((u_1(\cdot), u_2(\cdot)) \) any more. Thus we could write \( W^{u_1, u_2}(\cdot) \equiv W(\cdot) \) to be a Brownian motion under the probability \( \mathbb{P}^{u_1, u_2} \triangleq \mathbb{P} \) directly. Moreover, the adjoint process \((P(\cdot), K(\cdot)) \) is needless in the follower’s problem, therefore it causes the disappearance of the adjoint processes \((\alpha(\cdot), \beta(\cdot), Q(\cdot)) \) in (40) of Problem of the leader.

In this case, (41) in Theorem 3.3 becomes

\[
\begin{aligned}
&\{ H_{2x}(t, \tilde{x}^{\tilde{u}_2}, \tilde{u}_2, \tilde{p}, \tilde{K}; q, \varphi, \delta), \ v_2 - \tilde{u}_2(t) \} \\
&+ [E^{\tilde{u}_2} \{ H_{2k}(t, \tilde{x}^{\tilde{u}_2}, \tilde{u}_2, \tilde{p}, \tilde{K}; q, \varphi, \delta) | \mathcal{F}_t^y \}] \\
&- \tilde{u}_2 - \tilde{u}_2(t) \geq 0,
\end{aligned}
\]
(42)
holds for a.e. \( t \in [0, T] \), \( \bar{\mathbb{P}} \)-a.s., and for any \( v_2 \in U_2 \). Here, the expectation \( \mathbb{E} \) corresponds to the uncontrolled probability measure \( \mathbb{P}^{u_1,u_2} \). The Hamiltonian function (39) becomes

\[
H_2(t, x^{u_2}, u_2, p, k; q, \varphi, \delta) := \langle \varphi(t), b^L(t, x^{u_2}, u_2) \rangle + \langle \delta(t), a^L(t, x^{u_2}, u_2) \rangle
+ \{ q(t), j^L_1(t, x^{u_2}, u_2, p, k) \} + \int_0^T \{ q(t), j^L_2(t, x^{u_2}, u_2, p, k) \} dt,
\]
and the adjoint FBSDE (40) for \( (q(t), \varphi(t), \delta(t)) \) reduces to

\[
dq(t) = \left[ H_2^p(t) + \mathbb{E}[H_2^q(t) | \mathcal{F}_t^Y] \right] dt
+ \left[ H_2^k(t) + \mathbb{E}[H_2^\gamma(t) | \mathcal{F}_t^Y] \right] dW(t),
\]
\[-d\varphi(t) = \left[ H_2^t(t) + \mathbb{E}[H_2^s(t) | \mathcal{F}_t^Y] \right] dt
- \delta(t) dW(t), \quad t \in [0, T],
\]
\[q(0) = 0, \quad \varphi(T) = G_2(x^\tilde{u}_2(T)) + G_{1x}(x^\tilde{u}_2(T))q(T).
\]

We have the following result. The detailed proof is inspired by Yong and Zhou (1999), by Clarke’s generalised gradient. We omit it and let it to the interested readers.

**Theorem 3.4:** Suppose that (A1) and (A2) hold. Let \( \tilde{u}_2(\cdot) \in U_2 \), \((x^{\tilde{u}_2}(\cdot), \tilde{p}(\cdot), \tilde{k}(\cdot))\) be the corresponding state processes and \( G_{1x} = M_{x}, \ M \in \mathbb{R}^n \). Let the adjoint Equation (44) admits a solution triple \((q(\cdot), \varphi(\cdot), \delta(\cdot))\) and suppose that \((x^{\tilde{u}_2}, u_2, p, k) \rightarrow H_2(t, x^{\tilde{u}_2}, u_2, p, k; q, \varphi, \delta) \) and \( x \rightarrow G_2(x) \) are convex. Suppose

\[
H_2(t, x^{\tilde{u}_2}, u_2, p, k; q, \varphi, \delta)
= \min_{u_2 \in U_2} \left\{ H_2(t, x^{u_2}, u_2, p, k; q, \varphi, \delta)
+ \mathbb{E}[H_2(t, x^{u_2}, u_2, p, k; q, \varphi, \delta) | \mathcal{F}_t^Y] \right\}
\]
holds for a.e. \( t \in [0, T] \), \( \bar{\mathbb{P}} \)-a.s. Then \( \tilde{u}_2(\cdot) \) is an optimal control of Problem of the leader.

### 4. An LQ Stackelberg stochastic differential game with asymmetric noisy observations

In this section, we deal with an LQ Stackelberg stochastic differential game with asymmetric noisy observations, where the maximum principle and verification theorem developed in the previous section will be useful tools. For notational simplicity, we only consider the case for \( n = d_1 = d_2 = m_1 = m_2 = 1 \).

#### 4.1 The problem of the follower

Let us consider the following controlled SDE:

\[
\begin{align*}
dx^{u_1,u_2}(t) &= \left[ A(t)x^{u_1,u_2}(t) + B_1(t)u_1(t) + B_2(t)u_2(t) \right] dt
+ \left[ C(t)x^{u_1,u_2}(t) + D_1(t)u_1(t) + D_2(t)u_2(t) \right] dW(t), \quad t \in [0, T],
\end{align*}
\]
\[x^{u_1,u_2}(0) = x_0,
\]
and the observation equation:

\[
\begin{align*}
dY(t) &= h(t) dt + d\bar{W}(t), \quad t \in [0, T],
Y(0) = 0,
\end{align*}
\]
where \( x_0 \in \mathbb{R} \) and \( A(\cdot), B_1(\cdot), B_2(\cdot), C(\cdot), D_1(\cdot), D_2(\cdot) \) and \( h(\cdot) \) are given deterministic functions. We introduce the following assumption:

\((H1)\) \( A(\cdot), B_1(\cdot), B_2(\cdot), C(\cdot), D_1(\cdot) \) and \( D_2(\cdot) \in L^\infty(0, T; \mathbb{R}) \).

Firstly, for any chosen \( u_2(\cdot) \in U_2 \), the follower would like to choose an \( F_T^Y \)-adapted control \( \tilde{u}_1(\cdot) \) to minimise his cost functional

\[
\begin{align*}
J_1(u_1(\cdot), u_2(\cdot)) &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \right.
\end{align*}
\]
\[\left. \left\{ Q_1(t) |x^{u_1,u_2}(t)|^2 + R_1(t) |u_1(t)|^2 \right\} dt
+ G_1|x^{u_1,u_2}(T)|^2,\right]
\]
where the expectation \( \mathbb{E} \) is corresponding to the probability measure \( \mathbb{P}^{u_1,u_2} = \bar{\mathbb{P}} \) under which \( W(\cdot) \) and \( \bar{W}(\cdot) \) are independent standard Brownian motion mentioned in the previous section. And we also suppose that \( (H2)\) \( Q_1(\cdot), R_1(\cdot) \) and \( G_1 \) are bounded and deterministic, \( R_1^{-1}(\cdot) \) exists and is also bounded.

We write the follower’s Hamiltonian function

\[
\begin{align*}
H_1(t, x, u_1, u_2, p, k) &= \left[ A(t)x + B_1(t)u_1 + B_2(t)u_2 \right] p(t)
+ \left[ C(t)x + D_1(t)u_1 + D_2(t)u_2 \right] k(t)
+ \frac{1}{2} Q_1(t)x^2 + \frac{1}{2} R_1(t)u_1^2.
\end{align*}
\]

From Theorem 3.1, if \( \tilde{u}_1(\cdot) \) is the optimal control, then we have

\[
\tilde{u}_1(t) = - R_1^{-1}(t) \left[ B_1(t)\tilde{p}(t) + D_1(t)\tilde{k}(t) \right],
\]
\[\text{a.e. } t \in [0, T], \ \bar{\mathbb{P}}-a.s.,
\]
with \( \tilde{p}(t) \equiv \mathbb{E}[p(t) | \mathcal{F}_t^Y] \) and \( \tilde{k}(t) \equiv \mathbb{E}[k(t) | \mathcal{F}_t^Y] \), where \( (p(\cdot), k(\cdot)) \) is the \( F_T \)-adapted solution to the following adjoint BSDE:

\[
\begin{align*}
-dp(t) &= \left[ Q_1(t)\tilde{x}^{u_1,u_2}(t) + A(t)p(t) + C(t)k(t) \right] dt
- k(t) d\bar{W}(t), \quad t \in [0, T],
\end{align*}
\]
\[p(T) = G_1\tilde{x}^{u_1,u_2}(T).
\]
Noticing that the representation of \( \tilde{u}_1(\cdot) \) contains the filtering estimate of the second component \( \tilde{k}(\cdot) \) of the solution to (51), since the control variables enter into the diffusion term of (46).

Observing the terminal condition in Equation (51), and the appearance of \( u_2(\cdot) \), we set

\[
p(t) = P(t)\tilde{x}^{u_1,u_2}(t) + \Theta(t), \ \text{a.e. } t \in [0, T],
\]
for some deterministic and differentiable \( \mathbb{R} \)-valued function \( P(\cdot) \) with \( P(T) = G_1 \), and \( \mathbb{R} \)-valued, \( F_T \)-adapted process pair \((\Theta(\cdot), \Gamma(\cdot))\) satisfying the BSDE:

\[
\begin{align*}
d\Theta(t) &= \Xi(t) dt + \Gamma(t) dY(t), \quad t \in [0, T],
\end{align*}
\]
\[\Theta(T) = 0.
\]
In the above equation, $\Xi(\cdot)$ is an $F_t$-adapted process to be determined later. Applying Itô’s formula to (52) and noting (47), (53), we have
\[
dp(t) = \left[ \frac{\partial x}{\partial t} + A P x + \frac{\partial x}{\partial t} x + B t u + \frac{\partial x}{\partial t} + \frac{\partial x}{\partial t} + \Xi + \Gamma h \right] dt + \frac{\partial x}{\partial t} X^2 d W(t) + \nu \Gamma d W(t) + \nu \nu \nu.
\]

Comparing (54) and (51), we get
\[
\frac{\partial x}{\partial t} + A P x + \frac{\partial x}{\partial t} x + B t u + \frac{\partial x}{\partial t} + \frac{\partial x}{\partial t} + \Xi + \Gamma h = \nu \Gamma \nu \nu.
\]

Thus (53) has the unique $F_t$-adapted solution $(\Theta(\cdot), 0)$, which in fact reduces to a backward random differential equation (BRDE for short).

Substituting (52) and (56) into (50), and supposing that $(D^2 t P + R_1)^{-1}$ exist, we obtain
\[
\frac{\partial x}{\partial t} + A P x + \frac{\partial x}{\partial t} x + B t u + \frac{\partial x}{\partial t} + \frac{\partial x}{\partial t} + \Xi + \Gamma h = \nu \Gamma \nu \nu.
\]

where $\nu \Gamma \nu \nu(t) \equiv \frac{\partial}{\partial x} [x^2 + u(t) \mid F_t^\nu]$, $\Theta(t) \equiv \frac{\partial}{\partial x} [\Theta(t) \mid F_t^\nu]$ and $\nu \Gamma \nu \nu(0) \equiv \frac{\partial}{\partial x}[x^2 + u(t) \mid F_t^\nu]$.

Inserting (52), (56) and (58) into (55), we obtain that if the Riccati equation:
\[
\begin{cases}
\dot{P} + 2 A P + C^T P - (D^2 t P + R_1)^{-1} (B_1 + D_1 C)^2 P^2 + Q_1 = 0, \\
t \in [0, T], \\
P(t) = G_1, \\
D_1^2 P + R_1 > 0,
\end{cases}
\]

admits a unique solution $P(\cdot)$, then we have
\[
\Xi = (B_1 + D_1 C)^2 (D^2 t P + R_1)^{-1} P^2 x^2 + u(t) \nu \nu \nu.
\]

Moreover, for given $u_2(\cdot)$, plugging (58) into (46), we derive
\[
\begin{aligned}
dx(\nu \Gamma \nu \nu(\cdot)) &= \left[ A B_1 + D_1 C \right] (D^2 t P + R_1)^{-1} (B_1 + D_1 C) P x^2 + u(t) \nu \nu \nu.
\end{aligned}
\]

Therefore, from the observation Equation (47) and applying Theorem 8.1 in Liptser and Shirvayev (1977), we can derive the following optimal filtering equation:
\[
\begin{cases}
dx(\nu \Gamma \nu \nu(\cdot)) = \left[ A B_1 + D_1 C \right] (D^2 t P + R_1)^{-1} (B_1 + D_1 C) P x^2 + u(t) \nu \nu \nu, \\
t \in [0, T], \\
\hat{x}(\nu \Gamma \nu \nu(\cdot)) = x_0.
\end{cases}
\]

which admits a unique $F_t^\nu$-adapted solution $\nu \Gamma \nu \nu(\cdot)$, as long as $\hat{x}(\cdot)$ is determined.

In fact, similarly, by (61) we have
\[
\begin{cases}
-\dot{\hat{x}}(\cdot) = \left[ -[(B_1 + D_1 C) (D^2 t P + R_1)^{-1} (B_1 + D_1 C) P] x^2 + u(t) \nu \nu \nu, \\
t \in [0, T], \\
\hat{x}(\cdot) = 0.
\end{cases}
\]
which admits a unique $F^T_t$-adapted solution $(\hat{x}^{u_1,u_2}(\cdot), \hat{\Theta}(\cdot)) \equiv (\hat{x}^{u_1,u_2}(\cdot), \hat{\Theta}(\cdot), 0)$, for given $u_2(\cdot)$.

Now, noting that the conditions in Theorem 3.2 are satisfied, we could summarise the above procedure in the following theorem.

**Theorem 4.1:** Let (H1) and (H2) hold, and $P(\cdot)$ satisfies (59). For given $u_2(\cdot)$ of the leader, $\hat{u}_1(\cdot)$ given by (58) is the state estimate feedback optimal control of the follower, where $(\hat{x}^{u_1,u_2}(\cdot), \hat{\Theta}(\cdot))$ is the unique $F^T_t$-adapted solution to (65).

**Remark 4.1:** In Theorem 4.1, if in addition the following standard assumption (4.23) at page 308 of Yong and Zhou (1999)):

$$G_1 \geq 0, \quad R_1(\cdot) > 0, \quad Q_1(\cdot) \geq 0,$$

holds, we can guarantee that Riccati equation (59) admits a unique solution $P(\cdot)$.

### 4.2 The problem of the leader

Since the leader knows that the follower will take $\hat{u}_1(\cdot)$ by (58), the state equation of the leader can be written as:

$$\left\{ \begin{array}{l}
\text{d}x^{u_1}(t) = \left[ A x^{u_2} - B_1 (D^2_t P + R_1)^{-1} (B_1 + D_1 C) P \hat{x}^{u_2} \\
- (D^2_t P + R_1)^{-1} B_1 \hat{\Theta} \\
- B_1 (D^2_t P + R_1)^{-1} D_1 D_2 P \hat{u}_2 + B_2 u_2 \right] \text{d}t \\
+ \left[ C x^{u_2} - D_1 (D^2_t P + R_1)^{-1} (B_1 + D_1 C) P \hat{x}^{u_2} \\
- D_1 (D^2_t P + R_1)^{-1} B_1 \hat{\Theta} \\
- (D^2_t P + R_1)^{-1} D_1 D_2 P \hat{u}_2 + D_2 u_2 \right] \text{d}W(t), \\
- \text{d}\hat{\Theta}(t) = \left\{ - [(B_1 + D_1 C) (D^2_t P + R_1)^{-1} D_1 D_2 P^2 \\
- (B_1 + D_1 C) P \hat{u}_2 \\
- [(B_1 + D_1 C) (D^2_t P + R_1)^{-1} B_1 P - A] \hat{\Theta} \right\} \text{d}t, \\
t \in [0, T], \\
x^{u_2}(0) = x_0, \quad \hat{\Theta}(T) = 0,
\end{array} \right. \tag{66}$$

where $x^{u_2}(\cdot) \equiv \hat{x}^{u_1,u_2}(\cdot)$ and $\hat{x}^{u_2}(\cdot) \equiv \hat{x}^{u_1,u_2}(\cdot)$. Note that (66) is a decoupled conditional mean-field FBSDE, and its solvability can be easily obtained in this case. The leader would like to choose an $F^T_t$-adapted control $u_2(\cdot)$ to minimise his cost functional

$$J_2(u_2(\cdot)) = \frac{1}{2} E \left[ \int_0^T \left\{ Q_2(t)|x^{u_2}(t)|^2 + R_2(t)|u_2(t)|^2 \right\} \text{d}t \\
+ G_2 |x^{u_2}(T)|^2 \right]. \tag{67}$$

We suppose (H3) $Q_2(\cdot), R_2(\cdot)$ and $G_2$ are bounded and deterministic, $R_2^{-1}(\cdot)$ exists and is also bounded.

Applying Theorems 3.3 and 3.4, we can write the leader’s Hamiltonian function

$$H_2(t,x^{u_2}, u_2, \hat{\Theta}; q, \varphi, \delta)$$

$$= \frac{1}{2} Q_2(x^{u_2})^2 + \frac{1}{2} R_2 u_2^2 + \left[ A x^{u_2} - B_1 (D^2_t P + R_1)^{-1} \right. \\
\times (B_1 + D_1 C) P \hat{x}^{u_2} \\
- (D^2_t P + R_1)^{-1} B_1 \hat{\Theta} \\
- D_1 (D^2_t P + R_1)^{-1} D_1 D_2 P \hat{u}_2 + B_2 u_2 \right] \varphi \\
+ \left[ C x^{u_2} - D_1 (D^2_t P + R_1)^{-1} (B_1 + D_1 C) P \hat{x}^{u_2} \\
- D_1 (D^2_t P + R_1)^{-1} B_1 \hat{\Theta} \\
- (D^2_t P + R_1)^{-1} D_1 D_2 P \hat{u}_2 + D_2 u_2 \right] \delta \\
- \left\{ (B_1 + D_1 C) (D^2_t P + R_1)^{-1} D_1 D_2 P^2 \\
- (B_1 + D_1 C) P \hat{u}_2 + [(B_1 + D_1 C) (D^2_t P + R_1)^{-1} \right. \\
\times B_1 P - A] \hat{\Theta} \left. \right\} q. \tag{68}$$

The optimal control $u_2(\cdot)$ of the leader satisfies:

$$R_2 \hat{u}_2 + B_2 \varphi + D_2 \delta - (B_1 + D_1 C) (D^2_t P + R_1)^{-1} D_1 D_2 \hat{\varphi} \\
- (D^2_t P + R_1)^{-1} D_1 D_2 \hat{\delta} - [(B_1 + D_1 C) (D^2_t P + R_1)^{-1} \right. \\
\times D_1 D_2 P^2 - (B_1 + D_1 C) P] \hat{q} = 0, \quad a.e. \ t \in [0, T], \ \hat{\varphi}, \text{a.s.} \tag{69}$$

with $\hat{\varphi}(t) \triangleq E[\varphi(t) | \mathcal{F}_t], \hat{\delta}(t) \triangleq E[\delta(t) | \mathcal{F}_t^T]$ and $\hat{q}(t) \triangleq E[q(t) | \mathcal{F}_t^T]$, where $(q(\cdot), \varphi(\cdot), \delta(\cdot))$ satisfies the adjoint FBSDE:

$$\text{d}q(t) = \left\{ - (D^2_t P + R_1)^{-1} B_1 \varphi - D_1 (D^2_t P + R_1)^{-1} B_1 \delta \right. \\
- [(B_1 + D_1 C) (D^2_t P + R_1)^{-1} B_1 P - A] \hat{q} \left. \right\} \text{d}t, \tag{69}$$

$$\text{d}\varphi(t) = \left[ Q_2 x^{u_2} + A \varphi + C \delta - B_1 (D^2_t P + R_1)^{-1} \right. \\
\times (B_1 + D_1 C) P \hat{\varphi} - D_1 (D^2_t P + R_1)^{-1} (B_1 + D_1 C) \right. \\
\times \hat{\delta} \left. \right\} \text{d}W(t), \ t \in [0, T],$$

$$q(0) = 0, \ \varphi(T) = G_2 x^{u_2}(T). \tag{70}$$

Next, for obtaining the state feedback representation of $u_2(\cdot)$ via some Riccati equations, let us put (66) (for $u_2(\cdot)$) and (70) together and regard $\left( x^{u_2}(\cdot), q(\cdot) \right)$ as the optimal `state':

$$\text{d}x^{u_2}(t) = \left[ A x^{u_2} - B_1 (D^2_t P + R_1)^{-1} (B_1 + D_1 C) P \hat{x}^{u_2} \\
- (D^2_t P + R_1)^{-1} B_1 \hat{\Theta} \\
- B_1 (D^2_t P + R_1)^{-1} D_1 D_2 P \hat{u}_2 + B_2 u_2 \right] \text{d}t \\
+ \left[ C x^{u_2} - D_1 (D^2_t P + R_1)^{-1} (B_1 + D_1 C) P \hat{x}^{u_2} \\
- D_1 (D^2_t P + R_1)^{-1} B_1 \hat{\Theta} \\
- (D^2_t P + R_1)^{-1} D_1 D_2 P \hat{u}_2 + D_2 u_2 \right] \text{d}W(t),$$

$$\text{d}q(t) = \left\{ - (D^2_t P + R_1)^{-1} B_1 \varphi - D_1 (D^2_t P + R_1)^{-1} B_1 \delta \right. \\
- [(B_1 + D_1 C) (D^2_t P + R_1)^{-1} B_1 P - A] \hat{q} \left. \right\} \text{d}t, \tag{69}$$

$$\text{d}\varphi(t) = \left[ Q_2 x^{u_2} + A \varphi + C \delta - B_1 (D^2_t P + R_1)^{-1} \right. \\
\times (B_1 + D_1 C) P \hat{\varphi} - D_1 (D^2_t P + R_1)^{-1} (B_1 + D_1 C) \right. \\
\times \hat{\delta} \left. \right\} \text{d}W(t), \ t \in [0, T],$$

$$q(0) = 0, \ \varphi(T) = G_2 x^{u_2}(T). \tag{70}$$
\[
\times (B_1 + D_1 C) \Delta \hat{\Theta}(t) dt - \delta(t) dW(t),
\]
\[
-d\hat{\Theta}(t) = \left\{ -[(B_1 + D_1 C)(D_1^2 P + R_1)^{-1} D_1 D_2 P^2
\]
\[- (B_2 + D_2 C) P] \hat{u}_2
\]-\left[(B_1 + D_1 C)(D_1^2 P + R_1)^{-1} B_1 P - A \right] \hat{\Theta} \right\} dt,
\]
t \in [0, T],
x \hat{u}_2(0) = x_0, \quad q(0) = 0, \quad \varphi(T) = G_2 x \hat{u}_2(T), \quad \hat{\Theta}(T) = 0. \tag{71}
Then, we put
\[
X = \left( \begin{array}{c} x \hat{u}_2 \\ q \end{array} \right), \quad Y = \left( \begin{array}{c} \varphi \\ \delta \end{array} \right), \quad Z = \left( \begin{array}{c} \delta \\ 0 \end{array} \right),
\]
\[
X_0 = \left( \begin{array}{c} x_0 \\ 0 \end{array} \right), \quad \hat{G} = \left( \begin{array}{cc} G_2 & 0 \\ 0 & 0 \end{array} \right), \tag{72}
\]
and
\[
\begin{align*}
A_1 &= \begin{pmatrix} A & 0 \\ 0 & -[(B_1 + D_1 C)(D_1^2 P + R_1)^{-1} B_1 P - A] \end{pmatrix}, \\
A_2 &= \begin{pmatrix} -B_1 (D_1^2 P + R_1)^{-1} (B_1 + D_1 C) P & 0 \\ 0 & 0 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \\
A_4 &= \begin{pmatrix} -D_1 (D_1^2 P + R_1)^{-1} (B_1 + D_1 C) P & 0 \\ 0 & 0 \end{pmatrix}, \\
A_5 &= \begin{pmatrix} Q_2 & 0 \\ 0 & 0 \end{pmatrix}, \\
B_1 &= \begin{pmatrix} 0 & -D_1 (D_1^2 P + R_1)^{-1} B_1^2 \\ -(D_1^2 P + R_1)^{-1} B_1^2 & 0 \end{pmatrix}, \\
C_1 &= \begin{pmatrix} 0 & 0 \\ 0 & -D_1 (D_1^2 P + R_1)^{-1} B_1 \end{pmatrix}, \\
D_1 &= \begin{pmatrix} 0 & -(D_1^2 P + R_1)^{-1} D_1 D_2 P \\ -(D_1^2 P + R_1)^{-1} D_1 D_2 P & 0 \end{pmatrix}, \\
D_3 &= \begin{pmatrix} 0 & 0 \\ 0 & -D_2 (D_2^2 P + R_1)^{-1} D_2^2 P \end{pmatrix}, \\
D_4 &= \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}, \\
D_5 &= \begin{pmatrix} 0 & -[(B_1 + D_1 C)(D_1^2 P + R_1)^{-1} D_1 D_2 P^2 \\ -(B_2 + D_2 C) P \end{pmatrix}.
\end{align*}
\]
Then Equation (71) can be rewritten as
\[
\begin{aligned}
dX(t) &= \left[ A_1 X + A_2 \tilde{X} + B_1 Y + C_1 Z + D_1 \hat{u}_2 + D_2 \hat{u}_2 \right] dt \\
&\quad + \left[ A_3 X + A_4 \tilde{X} + C_1^T Y + D_1 \hat{u}_2 + D_4 \hat{u}_2 \right] dW(t), \\
-dY(t) &= \left[ A_3 X + A_4 \tilde{X} + A_2 \tilde{Y} + A_3 Z + A_4 \tilde{Z} + D_5 \hat{u}_2 \right] dt \\
&\quad - Z(t) dW(t), \quad t \in [0, T], \\
X(0) &= x_0, \quad Y(T) = \hat{G} X(T), \tag{73}
\end{aligned}
\]
and (69) can be represented as
\[
R_2 \hat{u}_2 + D_2^T Y + D_4^T Z + D_1^T \tilde{Y} + D_3^T \tilde{Z} + D_5^T \hat{X} = 0,
\]
a.e. \( t \in [0, T], \quad \hat{\mathbb{P}} \)-a.s.
\tag{74}
Thus we have
\[
\hat{u}_2 = -R_2^{-1} \left[ D_2^T Y + D_4^T Z + D_1^T \tilde{Y} + D_3^T \tilde{Z} + D_5^T \hat{X} \right],
\]
a.e. \( t \in [0, T], \quad \hat{\mathbb{P}} \)-a.s., \tag{75}
and
\[
\tilde{u}_2 = -R_2^{-1} \left[ (D_1 + D_2)^T \tilde{Y} + (D_3 + D_4)^T \tilde{Z} + D_5^T \hat{X} \right],
\]
a.e. \( t \in [0, T], \quad \hat{\mathbb{P}} \)-a.s.. \tag{76}
Inserting (75) and (76) into (73), we get
\[
\begin{aligned}
dX(t) &= \left\{ A_1 X + \left[ A_2 - (D_1 + D_2) R_2^{-1} D_2^T \right] \tilde{X} \\
&\quad + (B_1 - D_2 R_2^{-1} D_2^T) Y \\
&\quad - \left[ D_2 R_2^{-1} (D_1 + D_2)^T + D_2 R_2^{-1} D_1^T \right] \tilde{Y} \\
&\quad + (C_1 - D_2 R_2^{-1} D_2^T) Z \\
&\quad - \left[ D_2 R_2^{-1} (D_1 + D_2)^T + D_2 R_2^{-1} D_1^T \right] \tilde{Z} \right\} dt \\
&\quad + \left\{ A_3 X + \left[ A_4 - (D_3 + D_4) R_2^{-1} D_5^T \right] \hat{X} \\
&\quad + (C_1^T - D_2 R_2^{-1} D_5^T) \tilde{Y} \\
&\quad - \left[ D_3 R_2^{-1} (D_1 + D_2)^T + D_4 R_2^{-1} D_5^T \right] \tilde{Y} \\
&\quad - D_4 R_2^{-1} D_4^T \hat{Y} \\
&\quad - \left[ D_4 R_2^{-1} (D_1 + D_2)^T + D_4 R_2^{-1} D_5^T \right] \tilde{Y} \right\} \hat{X} \\
&\quad - D_5 R_2^{-1} D_5^T \tilde{Z} \\
&\quad - \left[ D_5 R_2^{-1} (D_1 + D_2)^T + D_5 R_2^{-1} D_5^T \right] \tilde{Z} \right\} dW(t), \\
-dY(t) &= \left\{ A_3 X + A_1 Y + \left[ A_2 - D_2 R_2^{-1} (D_1 + D_2)^T \right] \tilde{Y} \\
&\quad + A_3 Z + \left[ A_4 - D_5 R_2^{-1} (D_3 + D_4)^T \right] \tilde{Z} \\
&\quad - D_5 R_2^{-1} D_5^T \hat{X} \right\} dt - Z dW(t), \quad t \in [0, T], \\
X(0) &= x_0, \quad Y(T) = \hat{G} X(T). \tag{77}
\end{aligned}
\]
\[\text{Remark 4.2:} \text{ In general, we need to prove the existence and uniqueness of solution to (77). Actually, since the forward–backward system (77) is not only fully-coupled, but also coupling with its filtering terms, we fails to find a direct method to prove the existence and uniqueness of the solution to it, for its general case. However, we can consider another way to solve it in a special case. In detail, we first find some solutions by decoupling the system (77) in its general case. Then we can prove the solutions are unique inspired by the method in Lim and Zhou (2001) with the special case of } B_1(\cdot) = D_2(\cdot) = 0, \text{ that is, } A_2(\cdot) = B_1(\cdot) = C_1(\cdot) = D_1(\cdot) = D_3(\cdot) = D_4(\cdot) = 0.
\]
In order to decouple the conditional mean-field system (77), we set
\[
Y(t) = \Pi_1(t) X(t) + \Pi_2(t) \tilde{X}(t), \tag{78}
\]
where \( \Pi_1(\cdot) \) and \( \Pi_2(\cdot) \) are both differentiable, deterministic matrix-valued functions with \( \Pi_1(T) = \hat{G} \) and \( \Pi_2(T) = 0. \)
First, from the forward equation of (77), applying again Theorem 8.1 in Liptser and Shiryayev (1977), we obtain

\[
\begin{align*}
d\hat{X}(t) & = \left[ [A_1 + A_2 - (D_1 + D_2)R_2^{-1}D_3^\top] \hat{X} + [B_1 - (D_1 + D_2)R_2^{-1}(D_1 + D_2)^\top] \hat{Y} + [C_1 - (D_1 + D_2)R_2^{-1}(D_3 + D_4)^\top] \hat{Z} \right] dt, \\
\hat{X}(0) & = X_0.
\end{align*}
\]

Applying Itô's formula to (78), we have

\[
\begin{align*}
dY(t) & = \left[ \hat{\Pi}_1 X + \hat{\Pi}_1 A_1 X + \hat{\Pi}_1 [A_2 - (D_1 + D_2)R_2^{-1}D_3^\top] \hat{X} + \hat{\Pi}_1 \{B_1 - (D_1 + D_2)R_2^{-1}\} \hat{Y} \\
& \quad + \hat{\Pi}_1 \{C_1 - (D_1 + D_2)R_2^{-1}D_3^\top\} \hat{Z} \right] dt \\
& \quad + \hat{\Pi}_1 X + \hat{\Pi}_1 A_1 X + \hat{\Pi}_1 [A_2 - (D_1 + D_2)R_2^{-1}] \hat{X} \\
& \quad + \hat{\Pi}_1 \{B_1 - (D_1 + D_2)R_2^{-1}\} \hat{Y} \\
& \quad + \hat{\Pi}_1 \{C_1 - (D_1 + D_2)R_2^{-1}\} \hat{Z} \right] dW(t). \\
\end{align*}
\]
Comparing the diffusion term between the BSDE in (77) and (80), it yields

\[
\begin{align*}
Z & = \hat{\Pi}_1 A_3 X + \hat{\Pi}_1 [A_4 - (D_3 + D_4)R_2^{-1}D_3^\top] \hat{X} \\
& \quad + \hat{\Pi}_1 \{C_1^\top - D_4R_2^{-1}D_3^\top\} \hat{Y} \\
& \quad - \hat{\Pi}_1 [D_3R_2^{-1}(D_1 + D_2)^\top + D_4R_2^{-1}D_3^\top] \hat{Z} \\
& \quad - \hat{\Pi}_1 D_3R_2^{-1}D_3^\top \hat{X} \\
& \quad - \hat{\Pi}_1 [D_3R_2^{-1}(D_3 + D_4)^\top + D_4R_2^{-1}D_3^\top] \hat{Z} \\
& \quad - \hat{\Pi}_1 [D_3R_2^{-1}(D_3 + D_4)^\top + D_4R_2^{-1}D_3^\top] \hat{Z} \\
& = \Sigma_1(\Pi_1, \Pi_2) \hat{X}, \quad \hat{P}\text{-a.s.}, \quad (82)
\end{align*}
\]

Then putting (82) back into (81), and supposing that (H5) \( \mathcal{M}_2 \triangleq \left[ I + \Pi_1 D_4R_2^{-1}D_3^\top \right]^{-1} \) exists, we get

\[
\begin{align*}
Z & = \Sigma_2(\Pi_1) X + \Sigma_3(\Pi_1, \Pi_2) \hat{X}, \quad \hat{P}\text{-a.s.}, \quad (83)
\end{align*}
\]

Next, comparing the drift term between the BSDE in (77) and (80), it leads to

\[
\begin{align*}
\hat{\Pi}_1 X & + \hat{\Pi}_1 A_1 X + \hat{\Pi}_1 [A_2 - (D_1 + D_2)R_2^{-1}D_3^\top] \hat{X} \\
& \quad + \hat{\Pi}_1 \{B_1 - (D_1 + D_2)R_2^{-1}\} \hat{Y} \\
& \quad - \hat{\Pi}_1 [D_1R_2^{-1}(D_1 + D_2)^\top + D_2R_2^{-1}D_1^\top] \hat{Y} \\
& \quad + \hat{\Pi}_1 \{C_1^\top - D_2R_2^{-1}D_1^\top\} \hat{Z} \\
& \quad - \hat{\Pi}_1 [D_1R_2^{-1}(D_3 + D_4)^\top + D_2R_2^{-1}D_3^\top] \hat{Z} \\
& \quad + \hat{\Pi}_1 X + \hat{\Pi}_1 A_1 X + \hat{\Pi}_1 [A_2 - (D_1 + D_2)R_2^{-1}D_3^\top] \hat{X} \\
& \quad + \hat{\Pi}_1 \{B_1 - (D_1 + D_2)R_2^{-1}\} \hat{Y} \\
& \quad - \hat{\Pi}_1 [D_1R_2^{-1}(D_3 + D_4)^\top + D_2R_2^{-1}D_3^\top] \hat{Z} \\
& \quad - \hat{\Pi}_1 D_2R_2^{-1}D_3^\top \hat{X} \\
& \quad - \hat{\Pi}_1 [D_2R_2^{-1}(D_3 + D_4)^\top + D_2R_2^{-1}D_3^\top] \hat{Z} \\
& = 0.
\end{align*}
\]

After inserting (78), (82) and (83) into (84), we derive the following two Riccati equations:

\[
\begin{align*}
\hat{\Pi}_1 + \hat{\Pi}_1 A_1 + \hat{\Pi}_1 A_1 + \hat{\Pi}_1 [B_1 - D_2R_2^{-1}D_2^\top] \Pi_1 \\
& \quad + A_3 + [A_3 + \hat{\Pi}_1 \{C_1 - D_2R_2^{-1}D_1^\top\}] \\
& \quad \times \hat{\Pi}_1 [A_3 + \{C_1^\top - D_2R_2^{-1}D_3^\top\} \Pi_1] = 0, \quad t \in [0, T],
\end{align*}
\]

\[
\begin{align*}
\hat{\Pi}_2 + \hat{\Pi}_2 [A_2 - (D_1 + D_2)R_2^{-1}D_1^\top] \\
& \quad + [A_2 - D_2R_2^{-1}(D_1 + D_2)^\top] \Pi_1 + \Pi_2 \\
& \quad + \Pi_2 A_1 + \Pi_2 A_1 + \Pi_2 [B_1 - (D_1 + D_2)R_2^{-1} \times (D_1 + D_2)^\top] \Pi_1 \\
& \quad - \Pi_1 [B_1 - D_2R_2^{-1}D_1^\top] \Pi_1 + [A_3 + \Pi_1 \{C_1 \\
& \quad - D_2R_2^{-1}D_1^\top\}] \Sigma_3(\Pi_1, \Pi_2) \\
& \quad + [A_4 - D_2R_2^{-1}(D_3 + D_4)^\top] + \Pi_2 \{C_1 \\
& \quad - D_2R_2^{-1}D_1^\top\} = 0. \quad (85)
\end{align*}
\]
Lemma 4.2: The two Riccati equations (85) and (86) are not standard and entirely new, and we cannot obtain their solvability up to now. However, a special case could be dealt with by some existing results.

Remark 4.4: In the special case, if in addition to the following standard assumption (4.23) at page 308 of Yong and Zhou (1999):

\[ G_2 \geq 0, \quad R_2(\cdot) > 0, \quad Q_2(\cdot) \geq 0, \]

holds, it help guarantee that Riccati equations (87) and (88) in the following admit unique solutions \( \Pi_1(\cdot) \) and \( \Pi_2(\cdot) \).

We now consider \( B_1(\cdot) = D_2(\cdot) = 0 \), then we have \( A_2(\cdot) = B_1(\cdot) = C_1(\cdot) = \Pi_1(\cdot) = \Pi_3(\cdot) = \Pi_4(\cdot) = 0 \). In this case, there are no follower’s control \( u_\lambda(\cdot) \) in the drift term and no leader’s control \( u_\kappa(\cdot) \) in the diffusion term of (46), and the two players still affect the game. The Riccati equations (85) and (86) of \( \Pi_1(\cdot) \) and \( \Pi_2(\cdot) \) reduce to:

\[
\begin{align*}
\dot{\Pi}_1 + \Pi_1 A_1 + A_1 \Pi_1 - \Pi_1 D_2 R_2^{-1} \Sigma_1 = 0, & \quad t \in [0, T], \\
\Pi_1(T) = \bar{G},
\end{align*}
\]

\[
\begin{align*}
\dot{\Pi}_2 + \Pi_2 (A_1 - D_2 R_2^{-1} \Sigma_1) - \Pi_1 D_2 R_2^{-1} \Sigma_1 = 0, & \quad t \in [0, T], \\
\Pi_2(0) = \bar{G},
\end{align*}
\]

respectively. The solvability of (87) and (88) can be guaranteed by the sufficient conditions in Chapter 6 of Yong and Zhou (1999) and Theorem 5.3 of Yong (1999), respectively.

In detail, we give the following lemmas.

The solvability of (87) can be guaranteed by the following lemma similar to Theorem 7.2 in Chapter 6 of Yong and Zhou (1999).

Lemma 4.1: Let (H1)–(H5) and the standard assumptions in Remarks 4.1 and 4.4 hold, \( B_1(\cdot) = D_2(\cdot) = 0 \) hold. Then (87) admits a unique solution \( \Pi_1(\cdot) \).

Lemma 4.2: Let (H1)–(H5) and the standard assumptions in Remarks 4.1 and 4.4 hold, \( B_1(\cdot) = D_2(\cdot) = 0 \),

\[
\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} e^{B t} \begin{bmatrix} 0 \\ I \end{bmatrix} > 0, \quad t \in [0, T]
\]

hold, and (87) admits a unique solution \( \Pi_1(\cdot) \). Then (88) admits a unique solution \( \Pi_2(\cdot) \) which has the following representation

\[
\Pi_2(t) = - \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} e^{B(t-t)} \begin{bmatrix} 0 \\ I \end{bmatrix} e^{B(t-t)} \begin{bmatrix} 0 \\ I \end{bmatrix},
\]

where we let

\[
\begin{align*}
\kappa & \triangleq A_1 - D_2 R_2^{-1} \Sigma_1 - D_2 R_2^{-1} \Sigma_1 \Pi_1, \\
\lambda & \triangleq -\Pi_1 D_2 R_2^{-1} \Sigma_1 - D_2 R_2^{-1} \Sigma_1 \Pi_1 - D_2 R_2^{-1} \Sigma_1 \Pi_2 \\
& + A_3 \Pi_1 A_4 + A_4 \Pi_1 A_3 + A_4 A_1 A_4,
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{B} & \triangleq \begin{bmatrix} \kappa & -\lambda \\ -\lambda & -\kappa \end{bmatrix}.
\end{align*}
\]

Substituting (78), (82) and (83) into (75), we get

\[
\begin{align*}
\tilde{u}_2 = -R_2^{-1} \left\{ [D_2^\top \Pi_1 + D_2^\top \Sigma_2 (\Pi_1)] X + [D_2^\top (\Pi_1 + \Pi_2) + D_2^\top (\Pi_2 + \Sigma_2 (\Pi_1, \Pi_2)] + D_2^\top [\tilde{X}] \right\}, \quad \text{a.e. } t \in [0, T], \tilde{D}_n \text{-a.s.,}
\end{align*}
\]

where the optimal ‘state’ \( X(\cdot) \) and its optimal estimate \( \tilde{X}(\cdot) \) satisfy

\[
\begin{align*}
\begin{bmatrix} A_1 + (B_1 - D_2 R_2^{-1} \Sigma_1) & X \\ X & \Sigma_1 (\Pi_1, \Pi_2) \end{bmatrix} dt \\
+ \begin{bmatrix} (A_1 - D_2 R_2^{-1} \Sigma_1) & X \\ X & \Sigma_1 (\Pi_1, \Pi_2) \end{bmatrix} \tilde{X} \\
- \begin{bmatrix} (A_1 - D_2 R_2^{-1} \Sigma_1) & X \\ X & \Sigma_1 (\Pi_1, \Pi_2) \end{bmatrix} \tilde{X} dW(t), \quad t \in [0, T],
\end{align*}
\]

\[
\begin{align*}
X(0) = X_0.
\end{align*}
\]
and
\[
\begin{aligned}
d\tilde{X}(t) &= \left[ A_1 + A_2 + [B_1 - (D_1 + D_2)R_2^{-1}(D_1 + D_2)^\top] \right] X(t) \\
&\quad - (D_1 + D_2)R_2^{-1}D_2^\top X(t) + (C_1 - D_2R_2^{-1}D_2^\top) \Sigma(t) X(t) \\
&\quad + (C_1 - D_2R_2^{-1}D_2^\top) \Sigma(t) X(t) dt, \quad t \in [0, T], \\
\tilde{X}(0) &= X_0,
\end{aligned}
\]
respectively. We summarise the above in the following theorem.

**Theorem 4.2:** Let (H1)–(H5) hold, \( \Pi_1(\cdot) \) and \( \Pi_2(\cdot) \) satisfy (85) and (86), respectively, \( X(\cdot) \) be the \( F_t \)-adapted solution to (92), and \( \tilde{X}(\cdot) \) be the \( F^\gamma_t \)-adapted solution to (93). Define \( Y(\cdot), Z(\cdot) \) and \( \tilde{Z}(\cdot) \) by (78), (83) and (82), respectively. Then Equation (77) holds and \( \tilde{u}_3(\cdot) \) given by (91) is the state estimate feedback representation of the leader’s optimal control.

**Remark 4.5:** By Theorem 4.2, via decoupling technique, we can prove the existence of the solution to the conditional mean-field FBSDE (77). In fact, \( \tilde{X}(\cdot) \) is first given by (93) and then \( X(\cdot) \) by (92). Thus \( Y(\cdot) \) and \( Z(\cdot) \) are obtained by (78) and (83), respectively (together with \( \tilde{Y}(\cdot) \) and \( \tilde{Z}(\cdot) \)). Next, as mentioned in Remark 4.2, we will prove the uniqueness of solutions to (77) with the methods of Lim Zhou (2001), in the same special case when \( B_1(\cdot) = D_2(\cdot) = 0 \), that is, \( A_2(\cdot) = B_1(\cdot) = C_1(\cdot) = D_1(\cdot) = D_3(\cdot) = D_4(\cdot) = 0 \).

In this case, (77) is reduced to the following
\[
\begin{aligned}
dX(t) &= \left[ A_1 X - D_2R_2^{-1}D_2^\top \tilde{X} - D_2R_2^{-1}D_2^\top Y \right] dt \\
&\quad + [A_3 X + A_4 X] dW(t), \\
-dY(t) &= \left[ A_3 X + A_4 Y - D_3R_2^{-1}D_2^\top \tilde{Y} + A_3Z \right] dt \\
&\quad + [A_3 \tilde{Z} - D_3R_2^{-1}D_2^\top \tilde{X}] dt \\
X(0) &= X_0, \quad Y(T) = G\tilde{X}(T).
\end{aligned}
\]

Besides the fully-coupled feature, we observe that the filtering \( \tilde{X}(\cdot) \) and \( \tilde{Y}(\cdot) \) also appear in (94). We take the equations of \( \langle \tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot) \rangle \) into account first. By Theorem 8.1 in Liptser Shiryaev (1977), we have
\[
\begin{aligned}
d\tilde{X}(t) &= \left[ (A_1 - D_2R_2^{-1}D_2^\top) \tilde{X} - D_2R_2^{-1}D_2^\top \tilde{Y} \right] dt, \\
-d\tilde{Y}(t) &= \left[ (A_5 - D_3R_2^{-1}D_2^\top) \tilde{X} + (A_1 - D_3R_2^{-1}D_2^\top) \tilde{Y} \right] dt + [A_3 + A_4] \tilde{Z} dt, \quad t \in [0, T], \\
\tilde{X}(0) &= X_0, \quad \tilde{Y}(T) = G\tilde{X}(T).
\end{aligned}
\]

Next, we first obtain the uniqueness of solutions to (95). Suppose that \( (X_1(\cdot), Y_1(\cdot), Z_1(\cdot)) \) and \( (X_2(\cdot), Y_2(\cdot), Z_2(\cdot)) \) are solutions to (94), \( (\tilde{X}_1(\cdot), \tilde{Y}_1(\cdot), \tilde{Z}_1(\cdot)) \) and \( (\tilde{X}_2(\cdot), \tilde{Y}_2(\cdot), \tilde{Z}_2(\cdot)) \) are solutions to (95). It follows that \( (\tilde{X}_{12}(\cdot), \tilde{Y}_{12}(\cdot), \tilde{Z}_{12}(\cdot)) \equiv (\tilde{X}_1(\cdot) - \tilde{X}_2(\cdot), \tilde{Y}_1(\cdot) - \tilde{Y}_2(\cdot), \tilde{Z}_1(\cdot) - \tilde{Z}_2(\cdot)) \) satisfies the following equation
\[
\begin{aligned}
d\tilde{X}_{12}(t) &= \left[ (A_1 - D_2R_2^{-1}D_2^\top) \tilde{X}_{12} - D_2R_2^{-1}D_2^\top \tilde{Y}_{12} \right] dt, \\
-d\tilde{Y}_{12}(t) &= \left[ (A_5 - D_3R_2^{-1}D_2^\top) \tilde{X}_{12} + (A_1 - D_3R_2^{-1}D_2^\top) \tilde{Y}_{12} \right] dt, \quad t \in [0, T], \\
\tilde{X}_{12}(0) &= 0, \quad \tilde{Y}_{12}(T) = G\tilde{X}_{12}(T).
\end{aligned}
\]
And we should note that the relation (78), (82) and (83), derived from the decoupling process, always hold. Then (96) becomes
\[
\begin{aligned}
d\tilde{X}_{12}(t) &= \left[ (A_1 - D_2R_2^{-1}D_2^\top) \tilde{X}_{12} - D_2R_2^{-1}D_2^\top \tilde{Y}_{12} \right] dt, \\
-d\tilde{Y}_{12}(t) &= \left[ (A_5 + (A_3 + A_4) \Pi_1(A_3 + A_4) \right. \\
&\quad - D_3R_2^{-1}D_2^\top) \tilde{X}_{12} + (A_1 - D_3R_2^{-1}D_2^\top) \tilde{Y}_{12} \right] dt, \quad t \in [0, T], \\
\tilde{X}_{12}(0) &= 0, \quad \tilde{Y}_{12}(T) = G\tilde{X}_{12}(T).
\end{aligned}
\]

Applying Itô’s formula to \( \langle \tilde{X}_{12}(\cdot), \tilde{Y}_{12}(\cdot) \rangle \), we have
\[
\begin{aligned}
\mathbb{E}(\tilde{X}_{12}(T), G\tilde{X}_{12}(T)) &= -\mathbb{E} \int_0^T \left[ (A_5 + (A_3 + A_4) \Pi_1(A_3 + A_4) \right. \\
&\quad - D_3R_2^{-1}D_2^\top) \tilde{X}_{12} + (A_1 - D_3R_2^{-1}D_2^\top) \tilde{Y}_{12} \right] dt.
\end{aligned}
\]
Due to the relation (78), we get \( \tilde{Y}_{12} = (\Pi_1 + \Pi_2)\tilde{X}_{12} \). Then we achieve
\[
\begin{aligned}
\mathbb{E}(\tilde{X}_{12}(T), G\tilde{X}_{12}(T)) &= -\mathbb{E} \int_0^T \left[ (A_5 + (A_3 + A_4) \Pi_1(A_3 + A_4) \right. \\
&\quad - D_3R_2^{-1}D_2^\top) \tilde{X}_{12} + (A_1 - D_3R_2^{-1}D_2^\top) \tilde{Y}_{12} \right] dt.
\end{aligned}
\]

Suppose that \( (H6) A_5 + (A_3 + A_4) \Pi_1(A_3 + A_4) - D_3R_2^{-1}D_2^\top + (\Pi_1 + \Pi_2)D_2R_2^{-1}D_2^\top (\Pi_1 + \Pi_2) > 0 \), then we get
\[
\begin{aligned}
\mathbb{E} \int_0^T \left[ (A_5 + (A_3 + A_4) \Pi_1(A_3 + A_4) \right. \\
&\quad - D_3R_2^{-1}D_2^\top) \tilde{X}_{12} + (A_1 - D_3R_2^{-1}D_2^\top) \tilde{Y}_{12} \right] dt = 0.
\end{aligned}
\]
Thus
\[
\tilde{X}_{12}(t) = 0, \quad a.e. \ t \in [0, T], \ \tilde{F}_\text{a.s.}
\]
Substituting (101) into (97) and combining with (78) and (82), we can verify that \( \tilde{X}_{12}(\cdot) = \tilde{Y}_{12}(\cdot) = \tilde{Z}_{12}(\cdot) = 0 \) by the uniqueness of solutions to linear SDEs and BSDEs. Then we obtain the uniqueness of solutions to (95).

Next, we return to prove the uniqueness of solutions to (94). Since \( (X_1(\cdot), Y_1(\cdot), Z_1(\cdot)) \) and \( (X_2(\cdot), Y_2(\cdot), Z_2(\cdot)) \) are
solutions to (94). It follows that \((X_{12}(\cdot), Y_{12}(\cdot), Z_{12}(\cdot)) \equiv (X_1(\cdot) - X_2(\cdot), Y_1(\cdot) - Y_2(\cdot), Z_1(\cdot) - Z_2(\cdot))\) satisfies the following equation
\[
\begin{aligned}
dX_{12}(t) &= \left[ A_1X_{12} - D_2R_2^{-1}D_2^T Y_{12} \right] \mathrm{d}t + A_3X_{12} \mathrm{d}W(t), \\
-dY_{12}(t) &= \left[ A_3X_{12} + A_1Y_{12} + A_3Z_{12} \right] \mathrm{d}t - Z_{12} \mathrm{d}W(t), \\
X_{12}(0) &= 0, \quad Y_{12}(T) = \bar{G}X_{12}(T).
\end{aligned}
\]
(102)

Applying Itô’s formula to \((X_{12}(\cdot), Y_{12}(\cdot))\) and noting that \(\dot{X}_{12}(\cdot) = \dot{Y}_{12}(\cdot) = 0\), we have
\[
\mathbb{E}\{X_{12}(T), \bar{G}X_{12}(T)\} = -\mathbb{E}\int_0^T \langle (A_5 + \Pi_1D_2R_2^{-1}D_2^T \Pi_1)X_{12}, X_{12} \rangle \mathrm{d}t.
\]
(103)

Suppose that \((H7) A_5 + \Pi_1D_2R_2^{-1}D_2^T \Pi_1 > 0\), it follows that
\[
\mathbb{E}\int_0^T \langle (A_5 + \Pi_1D_2R_2^{-1}D_2^T \Pi_1)X_{12}, X_{12} \rangle \mathrm{d}t = 0.
\]
(104)

Thus, we have \(X_{12}(t) = 0\), \(a.e. \ t \in [0, T]\), \(\bar{P}\)-a.s. And substituting it into (102) and combining with (78),(83), it is easy to check that, by the uniqueness of solutions to the linear SDE in (102), we have \(X_{12}(\cdot) = 0\). And by the uniqueness of solutions to the linear BSDE in (102), we have \((Y_{12}(\cdot), Z_{12}(\cdot)) = (0, 0)\). Therefore, we can obtain the uniqueness of solutions to (94). Up to now, we have proved the existence and uniqueness of solutions to the special case of (77), under (H1)–(H7).

Finally, the optimal control \(\bar{u}_1(\cdot)\) of the follower can also be represented in \(\hat{X}(\cdot)\). More precisely, by (58), noting (72) and (75), we derive
\[
\bar{u}_1 = \left\{ A_6 + B_2(\Pi_1 + \Pi_2) + (D_2^2P + R_1)^{-1}D_2D_2^2P_2^{-1} \times \left[ (D_1 + D_2)^T (\Pi_1 + \Pi_2) \right. \right.
\]
\[
\left. + (D_3 + D_4)^T \Sigma_1 (\Pi_1, \Pi_2) + D_3^T \right\} \hat{X},
\]
\(a.e. \ t \in [0, T]\), \(\bar{P}\)-a.s.,
(105)

where
\[
A_6 \triangleq \left( -(D_2^2P + R_1)^{-1} (B_1 + D_1C)P \right. \underset{0}{}, \quad B_2 \triangleq \left( 0 \quad -(D_2^2P + R_1)^{-1} B_1 \right).
\]

Thus, the open-loop Stackelberg equilibrium \((\bar{u}_1(\cdot), \bar{u}_2(\cdot))\) is given by (105) and (91), in its state estimate feedback form.

### 5. Application to resource allocation with partially observed information

In this section, we give a resource allocation example, which motivates us to study the Stackelberg stochastic differential game problem with asymmetric noisy observations. Simulating results are also presented to demonstrate the effectiveness of the theoretical results in the previous sections.

#### 5.1 Resource allocation problem

We consider a simplified model for a dynamic research and development resource allocation problem under rivalry (Chen & Cruz Jr., 1972; Scherer, 1967). However, we improve the dynamic model and extend it to the stochastic case. It is assumed that two firms, labelled as firm 1 and firm 2, are competing with each other for a share of the market for a specific consumer goods. Each firm’s share of the market has a direct relation to the quality of their product and in turn depends on their research and development effort. We assume that the other costs, such as the cost of modifying plants, are negligible compared with the cost of research and development.

Let \(M_1(t)\) and \(M_2(t)\) be the amounts of money invested in the research and development by firms 1 and 2 at time \(t\), respectively. Let \(\gamma(\cdot)\) be a certain kind of measurement of technical gap between these two firms. Different from the deterministic dynamic model in Chen Cruz Jr. (1972), we reformulate the evolution of this gap as the following SDE:
\[
\begin{aligned}
\mathrm{d}y(t) &= \left[ M_1^\gamma(t) - \alpha M_1^\gamma(t) \right] \mathrm{d}t + \beta y(t) \mathrm{d}W(t), \quad t \geq 0, \\
y(0) &= y_0.
\end{aligned}
\]
(106)

The multiplying factors \(\alpha \geq 1\) accounts for the fact that the technically lagging firm 1 (the follower) want to catch up with the technically advanced firm 2 (the leader) as soon as possible, and to innovate its technique. In (106), the first part \((\mathrm{d}t\text{ term})\) represents the difference of the amounts of money invested by these two firms which can be controlled by \(M_1^\gamma(\cdot)\) and \(M_1^\gamma(\cdot)\), and the second part \((\mathrm{d}W\text{ term})\) represents some random interference from external environment which have impacts on the gap, with a constant \(\beta > 0\).

It is assumed that the shares of the market are equal when there is no difference in technical levels. When the gap of technical level of these two firms becomes too large, a part of shares of the market of the firm 1 (the follower) are temporarily taken over by the firm 2 (the leader), and the shares of the market at each instant are determined by the technical levels at that instant only. It is assumed that the part of shares of the market are proportional to the square of the difference in technical levels. Then these two firms aim to minimise their own cost functionals as follows
\[
\begin{aligned}
J_1 &= \mathbb{E} \left\{ \int_0^T \left[ \left( \frac{1}{2} (1 + \gamma^2) \right) V + M_1 \right] \exp(-\gamma t) \mathrm{d}t \\
&\quad + V_1 y^2(T) \right\},
\end{aligned}
\]
(107)
and
\[
\begin{aligned}
J_2 &= \mathbb{E} \left\{ \int_0^T \left[ \left( \frac{1}{2} (1 + \gamma^2) \right) V + M_2 \right] \exp(-\gamma t) \mathrm{d}t \\
&\quad + V_2 y^2(T) \right\},
\end{aligned}
\]
(108)
where \(\gamma\) is some constant such that, when \(y(\cdot)\) reaches \(\gamma\), the market is completely taken over by the technically advanced firm
2 (the leader). $V$ is the quasi-rent, which can be regarded as the rewards for their innovations, techniques and patents, that is assumed to be constant. $\gamma > 0$ is the discount rate, and $V_i$ represents the the cost coefficient of $y^2(T)$ for the firm $i = 1, 2$, respectively, that is also assumed to be constant.

The control variables of the follower and leader are the amount of the money invested in research and development: $M_1(\cdot)$ and $M_2(\cdot)$, respectively. Noticing the drift term in (106), it is obvious that the increase of $M_1(\cdot)$ would reduce the difference and the increase of $M_2(\cdot)$ tends to increase the difference. In the example, we only consider that $y(t) \leq \hat{y}$.

We can rewrite the above problem by the following change of variables:

$$x = \exp \left(-\frac{\gamma t}{2}\right) y,$$

$$u_i = M_i^t \exp \left(-\frac{\gamma t}{2}\right) + u_0, \quad i = 1, 2,$$

$$Q = \frac{V}{2\gamma^2} \geq 0,$$

$$J_i = V_i \exp(\gamma T), \quad i = 1, 2,$$

$$x(0) = y(0).$$

Applying Itô’s formula, then Equations (106), (107) and (108) become

$$\begin{align*}
\mathcal{dx}(t) &= \left[-\frac{\gamma}{2} x(t) + (u_2(t) - u_1(t)) - \alpha(u_1(t) - u_2(t))\right] dt + \beta x(t) dW(t), \quad t \geq 0, \\
J_1 &= \mathbb{E} \left\{ \int_0^T [Qx^2(t) + (u_1(t) - u_1(t))^2] dt + H_1 x^2(T) \right\} \bigg|_{x(0) = y_0} \\
&\quad - \frac{1}{2} \int_0^T V \exp(-\gamma t) dt,
\end{align*}$$

and

$$\begin{align*}
J_2 &= \mathbb{E} \left\{ \int_0^T [ - Qx^2(t) + (u_1(t) - u_1(t))^2] dt + H_1 x^2(T) \right\} \\
&\quad - \frac{1}{2} \int_0^T V \exp(-\gamma t) dt.
\end{align*}$$

Noting that the last terms in (111) and (112) are constant. Then it is equivalent to consider the following cost functionals:

$$J_1 = \mathbb{E} \left\{ \int_0^T [Qx^2(t) + (u_1(t) - u_1(t))^2] dt + H_1 x^2(T) \right\},$$

and

$$J_2 = \mathbb{E} \left\{ \int_0^T [ - Qx^2(t) + (u_1(t) - u_1(t))^2] dt + H_1 x^2(T) \right\}.$$

It is necessary to explain the leader–follower feature with asymmetric noisy observations. We assume that the technically lagging firm (the follower) can not directly observe the complete information about the specific technical gap due to the own technical limitation. More importantly, the follower will suffer from the technological monopoly by its rival, but can observe partial information by the observation process satisfying:

$$Y(t) = \int_0^t h(s) ds + d\tilde{W}(s),$$

where $\tilde{W}(\cdot)$ and $W(\cdot)$ are independent Brownian motions under the probability measure $\bar{P}$.

For the technically advanced firm (the leader), compared with the follower, much more innovative patents and valuable brands are owned, thus more information resources are gathered, which is helpful to attract and introduce many talented and skilled technicians and benefit to improve the technique level and accelerate the research and development of the firm 2. Therefore, we assume that the leader can observe the complete information.

In this system (110), (115), (113) and (114), $x(t)$ stands for the gap of these two firms’ technical levels at time $t$, $x(0)$ stands for the initial gap between these two firms, and $u_1(t), u_2(t)$ denotes the amount of money invested in the research and development by firm 1 and 2 at time $t$, respectively. For the technically lagging firm 1 (the follower), who only knows a part of information about the specific technical gap due to the own technical limitation. More importantly, the follower will suffer from the technological monopoly by its rival, but can observe partial information by the observation process satisfying:

$$Y(t) = \int_0^t h(s) ds + d\tilde{W}(s),$$

where $\tilde{W}(\cdot)$ and $W(\cdot)$ are independent Brownian motions under the probability measure $\bar{P}$.

For the technically advanced firm (the leader), compared with the follower, much more innovative patents and valuable brands are owned, thus more information resources are gathered, which is helpful to attract and introduce many talented and skilled technicians and benefit to improve the technique level and accelerate the research and development of the firm 2. Therefore, we assume that the leader can observe the complete information.

In this system (110), (115), (113) and (114), $x(t)$ stands for the gap of these two firms’ technical levels at time $t$, $x(0)$ stands for the initial gap between these two firms, and $u_1(t), u_2(t)$ denotes the amount of money invested in the research and development by firm 1 and 2 at time $t$, respectively. For the technically lagging firm 1 (the follower), who only knows a part of information by the observation process, the aim is to increase his profit. Thus in $J_1$, the first part, for $Q \geq 0$, the aim is to shorten the technical gap as soon as possible, the second part is to make the deviation of the investment amount to dynamic benchmark $u_1(t)$ be small and the third part is to reduce the cost for maintaining or cutting down the technical gap at terminal time. For the technically advanced firm 2 (the leader), who knows the complete information. Thus in $J_2$, the first part, for $-Q \leq 0$, the aims is to enlarge the technical gap as soon as possible, and similarly, the second and third parts should also be minimised as in $J_1$.

### 5.2 Numerical simulation

To apply the theoretic results in Section 4, compared with (46), (47), (48) and (67), we have $A = -\frac{\gamma}{2}, B_1 = -\alpha, B_2 = 1, C = \beta, D_1 = D_2 = 0, Q_1 = 2Q, R_1 = 2, G_1 = 2H_1, Q_2 = -2Q, R_2 = 2, G_2 = 2H_2$ and $x_0 = y_0$. Then for $t \geq 0$,

$$A_1(t) = \begin{pmatrix}
-\frac{\gamma}{2} & 0 \\
0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2}
\end{pmatrix},$$

$$A_2(t) = \begin{pmatrix}
-\frac{1}{2} \alpha^2 P(t) & 0 \\
0 & 0
\end{pmatrix},$$

$$A_3 = \begin{pmatrix}
\beta & 0 \\
0 & 0
\end{pmatrix},$$

$$A_4 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},$$

$$A_5 = \begin{pmatrix}
-2Q & 0 \\
0 & 0
\end{pmatrix},$$

$$B_1 = \begin{pmatrix}
0 & -\frac{1}{2} \alpha^2 \\
-\frac{1}{2} \alpha^2 & 0
\end{pmatrix},$$

$$C_1 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.$$
\[ D_1 = D_3 = D_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]
\[ D_3(t) = \begin{pmatrix} 0 \\ P(t) \end{pmatrix}, \quad X_0 = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 2H_2 \\ 0 \\ 0 \end{pmatrix}, \]

where \( P(\cdot) \) satisfies the Riccati equation
\[
\begin{cases}
\dot{P}(t) + (\beta^2 - \gamma)P(t) - |\alpha P(t)|^2 + 2Q = 0, & t \in [0, T], \\
P(T) = 2H_1.
\end{cases}
\]

Thus, we derive the pair of Stackelberg equilibrium \((\bar{u}_1(\cdot), \bar{u}_2(\cdot))\) by (105) and (91):
\[
\begin{cases}
\bar{u}_1(t) = \left[ \begin{array}{c} \frac{1}{2} \alpha P(t) \\ 0 \end{array} \right] + \begin{pmatrix} 0 \\ \frac{1}{2} \alpha \end{pmatrix} \\
\times (\Pi_1(t) + \Pi_2(t)) \right]\dot{X}(t), \\
\bar{u}_2(t) = \frac{1}{2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \Pi_1(t)X(t) \\
- \frac{1}{2} \left[ \begin{array}{c} \frac{1}{2} \alpha \end{array} \right] \Pi_2(t) + \left( \begin{array}{c} 0 \\ P(t) \end{array} \right) \right]\dot{X}(t),
\end{cases}
\]

where \( \Pi_1(\cdot), \Pi_2(\cdot) \) satisfy the Riccati equations
\[
\begin{cases}
\dot{\Pi}_1(t) + \Pi_1(t) \left( \begin{array}{cc} -\frac{\gamma}{2} & 0 \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \\
\quad + \left( \begin{array}{cc} -\frac{\gamma}{2} & 0 \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_1(t) \\
\quad + \Pi_1(t) \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & 0 \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_1(t) \\
\quad + \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) \Pi_1(t) \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) + \left( \begin{array}{cc} -2Q & 0 \\ 0 & 0 \end{array} \right) = 0, \\
\quad t \in [0, T], \\
\Pi_1(T) = \begin{pmatrix} 2H_2 & 0 \\ 0 & 0 \end{pmatrix},
\end{cases}
\]
\[
\begin{cases}
\dot{\Pi}_2(t) + \Pi_2(t) \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & -\frac{P(t)}{2} \\ -\frac{P(t)}{2} & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \\
\quad + \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & -\frac{P(t)}{2} \\ -\frac{P(t)}{2} & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_2(t) \\
\quad + \Pi_1(t) \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & -\frac{P(t)}{2} \\ -\frac{P(t)}{2} & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_2(t) \\
\quad + \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) \Pi_1(t) \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) + \left( \begin{array}{cc} -2Q & 0 \\ 0 & 0 \end{array} \right) = 0, \\
\quad t \in [0, T], \\
\Pi_2(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{cases}
\]

respectively. The optimal technical gap between these two firms \( X(\cdot) \) and its optimal estimate \( \dot{X}(\cdot) \) satisfy:
\[
\begin{cases}
dX(t) = \left[ \begin{array}{cc} -\frac{\gamma}{2} & 0 \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right] \\
\quad + \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & 0 \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_1(t) \\
\quad + \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & 0 \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_2(t) \\
\quad + \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) X(t) \, dW(t), & t \in [0, T], \\
X(0) = \begin{pmatrix} y_0 \\ 0 \end{pmatrix},
\end{cases}
\]

and
\[
\begin{cases}
d\dot{X}(t) = \left[ \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & -\frac{P(t)}{2} \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right] \\
\quad + \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & -\frac{P(t)}{2} \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_1(t) \Pi_2(t) \\
\quad + \left( \begin{array}{cc} -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} & -\frac{P(t)}{2} \\ 0 & -\frac{1}{2} \alpha^2 P(t) - \frac{\gamma}{2} \end{array} \right) \Pi_2(t) \\
\quad + \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) \Pi_1(t) \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right) + \left( \begin{array}{cc} -2Q & 0 \\ 0 & 0 \end{array} \right) = 0, \\
\quad t \in [0, T], \\
\dot{X}(0) = \begin{pmatrix} y_0 \\ 0 \end{pmatrix},
\end{cases}
\]

respectively.
To be more intuitive, we give some numerical simulation and plot some figures with the certain particular coefficients to demonstrate our reasonable practical results. Let $\gamma = 1, \alpha = 1.6, \beta = 0.2, Q = 0.15, H_1 = 0.62, H_2 = 0.52, y_0 = 0.6$. Applying Euler’s method, we derive the numerical solutions of $P(\cdot), \Pi_1(\cdot)$ and $\Pi_2(\cdot)$, shown in Figures 1–3, respectively.

In detail, $P(\cdot)$ is one-dimensional function, and $\Pi_1(\cdot), \Pi_2(\cdot)$ are two $2 \times 2$ symmetric matrix-valued functions, and we have $\Pi_i(1,2) = \Pi_i(2,1)$, where $\Pi_i(1,2)$ denotes the value of the 1 row and the 2 column of $\Pi_i(\cdot), i = 1, 2$. Especially, we have $\Pi_1(1,2) = \Pi_1(2,1) = \Pi_1(2,2) = 0$.

Remark 5.1: As we can see that, although the coefficients of $x^2$ in (114) are negative when $Q > 0$, we still can draw the trajectories of $P(\cdot), \Pi_1(\cdot)$ and $\Pi_2(\cdot)$ in Figures 1–3, respectively. In other words, Riccati equations (116), (118) and (119) may admit unique solutions in some special cases, even if, at the same time, $Q$ is not non-negative. It is reasonable to illustrate that the standard assumption above in Remark 4.1 is a sufficient condition, not necessary.

With the same method, we plot the trajectories of $\hat{X}(\cdot), X(\cdot)$ and $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ in Figures 4 and 5, respectively. In detail, $\hat{X}(\cdot)$ is a $2 \times 1$ vector-valued function with the initial value $\hat{X}(0) = \begin{pmatrix} \hat{x}(0) \\ \hat{q}(0) \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$ and $X(\cdot)$ is a $2 \times 1$ vector-valued process with the initial value $X(0) = \frac{x(0)}{q(0)} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$. And, $\hat{u}_1(\cdot), \hat{u}_2(\cdot)$ are one-dimensional function and process, respectively.

From Figure 4, we find the dynamic curves of the first components of $X(\cdot)$ and $X(\cdot)$: $\hat{X}(1)$ and $X(1)$, present both decreasing trends which mean that the technical gap between these two firms is gradually becoming small in the process of the game.

For this interesting phenomenon, the reason can be seen through Figure 5. In Figure 5, at the beginning, the technically lagging firm (the follower) invests more than two times as much as the technically advanced firm (the leader) does in the research and development in order to catch up with the leader as soon as possible. Then from the horizontal perspective,
the investment amounts of these two firms are both reducing, and the variation tendency of the follower is sharper than that of the leader. However, from the longitudinal perspective, we notice that the trajectory of \( u_1(t) \) is always on the above of that of \( u_2(t) \), which means that the follower keeps investing more money than the leader does at every time unit. And we note that the leader has no investment in the research and development at \( t = 1.7 \), moreover the investment amount becomes negative value after \( t = 1.7 \). To some degree, it is reasonable to be understood that the leader, affected by the uncertain external market environment, has stopped investing. On the contrary, the follower, although cutting down his investment amount, maintains his/her investment status all the time.

6. Concluding remarks

Motivated by some interesting phenomena in the resource allocation problem, in this paper, we have discussed the Stackelberg stochastic differential game with asymmetric noisy observation. This kind of game problem has three interesting characteristics worthy of being emphasised. Firstly, the follower could only observe the noisy observation process, while the leader can observe both the state and noisy observation processes. Thus, the information between the follower and the leader has the asymmetric feature. Second, the leader’s problem is solved under some mild assumptions, with the aid of some new Riccati equations. Finally, the optimal control of the leader relies not only on the state but also on its estimate based on the observation process.

Possible extension to the Stackelberg stochastic differential game with correlated state and observation noises, applying state decomposition and backward separation principle (Wang et al., 2013,2015,2018), rather than Girsanov’s measure transformation, are worthy to research. The general solvability of the Riccati equations (87) and (88) requires systematic study. We will consider these topics in the future research.

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