Solutions for real dispersionless Veselov-Novikov hierarchy

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Abstract

We investigate the dispersionless Veselov-Novikov (dVN) equation based on the framework of dispersionless two-component BKP hierarchy. Symmetry constraints for real dVN system are considered. It is shown that under symmetry reductions, the conserved densities are therefore related to the associated Faber polynomials and can be solved recursively. Moreover, the method of hodograph transformation as well as the expressions of Faber polynomials are used to find exact real solutions of the dVN hierarchy.

1 Introduction

The Veselov-Novikov equation

\[ u_\tau = (uV)_z + (u\bar{V})_{\bar{z}} + u_{zzz} + u_{\bar{z}\bar{z}\bar{z}} , \quad V_{\bar{z}} = -3u_z \]  

was invented in [1] as a certain two-dimensional integrable extension of the KdV equation. Here \( z = x + iy \) and the subscripts \( z, \bar{z}, \tau \) denote partial derivatives. The important subclass of this equation is the so-called dispersionless Veselov-Novikov (dVN) equation that has been considered [2, 3, 4] by taking the quasi-classical limit of (1), in which the dispersion effect had been dropped. Namely,

\[ u_\tau = (uV)_z + (u\bar{V})_{\bar{z}} , \quad V_{\bar{z}} = -3u_z . \]  

Recently, it was demonstrated that in [3, 5, 6, 7] the dVN hierarchy is amenable to the semiclassical \( \partial \)-dressing method. Also, the dVN equation and dVN hierarchy have appeared in aspects of symmetries and relevant in the description of geometrical optics phenomena [4, 5, 7, 6]. In [4], some symmetry constraints for dVN equations were proposed to be efficient ways of construction of reductions (see also [8, 9, 10] for symmetry constraints of dispersionless integrable equations). It was also shown that the dVN equation
can be reduced into (1+1)-dimensional hydrodynamic type systems under the symmetry constraint.

The dispersionless Hirota equations for the two-component BKP system was first derived by Takasaki [11] as the dispersionless limit of the differential Fay identity. Later, the Hirota equations was rederived [12] from the method of kernel formulas provided by Carroll and Kodama [13]. As observed in [5, 6, 11], the Hamilton-Jacobi equation arising from extra equation of these Hirota equations can be related to the dVN equation as well as the Eikonal equation in the geometrical optics limit of Maxwell equations. Inspire by these observations, we are interested in connecting dVN hierarchy to the dispersionless two-component BKP (2-dBKP) hierarchy. In this paper, we study the dVN equation based on the framework of the 2-dBKP hierarchy [11] (see also [12] for the extended dBKP hierarchy). The Hirota equations of 2-dBKP provide an effective way of constructing Faber polynomials [14, 15] and the associated Hamilton-Jacobi equations. Real symmetry constraints will be imposed to find the real solutions of the dVN hierarchy.

This paper is organized as follows. In section 2, we recall the 2-dBKP hierarchy and give an identification of hierarchy flows for the 2-dBKP system and the dVN hierarchy. In section 3, we derive Faber polynomials of the dVN hierarchy by means of its Hirota equations. In particular, we obtain the recursion formulas of the Faber polynomials. In section 4 under symmetry constraint for the dVN hierarchy, we show that the corresponding Faber polynomials characterize the second derivatives of free energy and the conserved densities. In section 5, we present the hodograph solutions for the dVN hierarchy by choosing some suitable initial data. The solutions for S function are also given as examples. In section 6, we discuss 2N-component symmetry constraint for the dVN hierarchy and derive the corresponding conserved densities. Section 7 is devoted to the concluding remarks.

2 The dVN hierarchy

The 2-dBKP system can be characterized by the following Hirota equations [11, 12]

\[
p(\lambda) - p(\mu) \over p(\lambda) + p(\mu) = \exp(-D(\lambda)S(\mu)),
\]

\[
\tilde{p}(\lambda) - \tilde{p}(\mu) \over \tilde{p}(\lambda) + \tilde{p}(\mu) = \exp(-\tilde{D}(\lambda)\tilde{S}(\mu)),
\]

\[
p(\lambda) - \tilde{q}(\mu) \over p(\lambda) + \tilde{q}(\mu) = \exp(-D(\lambda)\tilde{S}(\mu)) = \exp(D(\lambda)\tilde{D}(\mu)\mathcal{F}),
\]

\[
\tilde{p}(\mu) - q(\lambda) \over \tilde{p}(\mu) + q(\lambda) = \exp(-\tilde{D}(\mu)S(\lambda)) = \exp(\tilde{D}(\mu)D(\lambda)\mathcal{F}),
\]

where the generating functions \(S(\lambda), \tilde{S}(\lambda)\) are defined by

\[
S(\lambda) = \sum_{n=0}^{\infty} t_{2n+1} \lambda^{2n+1} - D(\lambda)\mathcal{F}, \quad \tilde{S}(\lambda) = \sum_{n=0}^{\infty} \tilde{t}_{2n+1} \lambda^{2n+1} - \tilde{D}(\lambda)\mathcal{F};
\]
and $D(\lambda) = \sum_{n=0}^{\infty} \frac{2\lambda - 2n - 1}{2n + 1} \partial_{t_{2n+1}}, \quad \tilde{D}(\lambda) = \sum_{n=0}^{\infty} \frac{2\lambda - 2n - 1}{2n + 1} \partial_{\tilde{t}_{2n+1}}$ denote the vertex operators \[ \text{[9]} \]; moreover, $p(\lambda), q(\lambda), \tilde{p}(\lambda), \tilde{q}(\lambda)$ are defined by

$$p(\lambda) = \frac{\partial S(\lambda)}{\partial t_{1}}, \quad \tilde{p}(\lambda) = \frac{\partial \tilde{S}(\lambda)}{\partial \tilde{t}_{1}}, \quad q(\lambda) = \frac{\partial S(\lambda)}{\partial t_{1}}, \quad \tilde{q}(\lambda) = \frac{\partial \tilde{S}(\lambda)}{\partial \tilde{t}_{1}}.$$  

(7)

(8)

By equating Eqs. (5) and (6), one has the equation \[ \text{[11]} \]: $p(\lambda)q(\lambda) = \tilde{p}(\mu)\tilde{q}(\mu)$, from which, after letting $\lambda, \mu \to \infty$ one obtains

$$-2F_{t_{1}, \tilde{t}_{1}} = -2F_{t_{1}, \tilde{t}_{1}} \equiv u,$$

(9)

where $u = u(t_{1}, t_{2}, \ldots; \tilde{t}_{1}, \tilde{t}_{2}, \ldots)$ is a scalar function. Moreover, for arbitrary $\lambda$, one has

$$p(\lambda)q(\lambda) = \tilde{p}(\lambda)\tilde{q}(\lambda) = u.$$  

(10)

Denoting $H_{2n+1} = 2\partial_{t_{2n+1}} \partial_{t_{1}} F$, $\tilde{H}_{2n+1} = 2\partial_{\tilde{t}_{2n+1}} \partial_{\tilde{t}_{1}} F$, $\tilde{H}_{2n+1} = 2\partial_{\tilde{t}_{2n+1}} \partial_{\tilde{t}_{1}} F$ and $\tilde{H}_{2n+1} = 2\partial_{\tilde{t}_{2n+1}} \partial_{\tilde{t}_{1}} F$, the Eq. (9) can be treated as the evolution of $u$ with respect to $t_{2n+1}$ and $\tilde{t}_{2n+1}$, respectively, namely

$$\frac{\partial u}{\partial t_{2n+1}} = -(H_{2n+1})_{t_{1}} = -(\tilde{H}_{2n+1})_{t_{1}},$$

(11)

$$\frac{\partial u}{\partial \tilde{t}_{2n+1}} = -(\tilde{H}_{2n+1})_{\tilde{t}_{1}} = -(\tilde{\tilde{H}}_{2n+1})_{\tilde{t}_{1}}.$$  

(12)

Now we define the $\tau_{2n+1}$-flow by setting $\partial_{\tau_{2n+1}} = \partial_{t_{2n+1}} + \partial_{\tilde{t}_{2n+1}}$ and identify $\tilde{t}_{2n+1}$ as the complex conjugate of $t_{2n+1}$, in particular, $t_{1} := z$ and $\tilde{t}_{1} := \tilde{z}$, where $z = x + iy$. From now on, the functions $\tilde{H}_{2n+1}, \tilde{\tilde{H}}_{2n+1}$ can also be taken as the complex conjugate of $H, \tilde{H}$:

$$\tilde{H}_{2n+1} = H_{2n+1}, \quad \tilde{\tilde{H}}_{2n+1} = \tilde{H}_{2n+1}.$$  

Thus, incorporating (11) and (12) for $n \geq 1$ together with $n = 0$ in (11) (or (12)) we obtain $\tau_{2n+1}$-flow of $u$

$$u_{\tau_{2n+1}} = -(H_{2n+1})_{\tilde{z}} - (\tilde{H}_{2n+1})_{z} = -(\tilde{H}_{2n+1})_{z} - (\tilde{\tilde{H}}_{2n+1})_{\tilde{z}}, \quad u_{z} = -(H_{1})_{\tilde{z}},$$

(13)

which is what we call the dVN hierarchy. As in what follows, we shall show that for $n = 1$, the corresponding equation reduces to the dVN equation \[ \text{[2]} \].

### 3 Faber polynomials of 2-dBKP

According to Takasaki’s observations \[ \text{[11]} \], the left hand side of (3) (or (5)) can be expanded as the form

$$\log \frac{p(\lambda) - w}{p(\lambda) + w} = -\sum_{n=0}^{\infty} \frac{2\Phi_{2n+1}(w)}{2n + 1} \lambda^{2n-1},$$

(14)

where $w = p(\mu)$ in (3) (or $w = \tilde{q}(\mu)$ in (5)) and $\Phi_{n}(w)$ is the $n$-th Faber polynomial of $p(\lambda)$ defined by

$$\log \frac{p(\lambda) - w}{\lambda} = -\sum_{n=1}^{\infty} \frac{\Phi_{n}(w)}{n} \lambda^{-n}.$$  

(15)
Eq. (15) is analytic for large \( \lambda \) for fixed \( w \in \mathbb{C} \) and, (14) is obtained from (15) by taking into account the symmetry conditions

\[
p(-\lambda) = -p(\lambda), \quad \Phi_n(-w) = (-1)^n \Phi_n(w).
\]

If we replace \( w \) in (14) by \( p(\mu) \), then Eq. (3) can be reduced to the following system of Hamilton-Jacobi equation

\[
\frac{\partial S(\mu)}{\partial t_{2n+1}} = \Phi_{2n+1}(p(\mu)). \tag{16}
\]

Likewise, replacing \( w \) by \( \tilde{q}(\mu) \), Eq. (5) reduces to the following Hamilton-Jacobi equation

\[
\frac{\partial \tilde{S}(\mu)}{\partial t_{2n+1}} = \Phi_{2n+1}(\tilde{q}(\mu)). \tag{17}
\]

After differentiating Eqs. (16), (17) with respect to \( z, \bar{z} \), we have time evolutions of \( p(\mu), q(\mu), \tilde{p}(\mu) \) and \( \tilde{q}(\mu) \) in \( t_{2n+1} \)-flow in the following form

\[
\frac{\partial p(\mu)}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(p(\mu)), \quad \frac{\partial q(\mu)}{\partial t_{2n+1}} = \partial_{\bar{z}} \Phi_{2n+1}(p(\mu)), \tag{18}
\]

\[
\frac{\partial \tilde{p}(\mu)}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(\tilde{q}(\mu)), \quad \frac{\partial \tilde{q}(\mu)}{\partial t_{2n+1}} = \partial_{\bar{z}} \Phi_{2n+1}(\tilde{q}(\mu)). \tag{19}
\]

In the same way, for the Hirota equations (4) and (6), one can derive the corresponding Hamilton-Jacobi equations via the expression of Faber polynomials as

\[
\log \frac{\tilde{p}(\lambda) - w}{\tilde{p}(\lambda) + w} = -\sum_{n=0}^{\infty} \frac{2\Phi_{2n+1}(w)}{2n+1} \lambda^{-2n-1}. \tag{20}
\]

From which, substitutions of \( w = \tilde{p}(\mu) \) and \( w = q(\mu) \) yielding the following systems of Hamilton-Jacobi equations

\[
\frac{\partial \tilde{S}(\mu)}{\partial t_{2n+1}} = \Phi_{2n+1}(\tilde{p}(\mu)), \quad \frac{\partial S(\mu)}{\partial t_{2n+1}} = \Phi_{2n+1}(q(\mu)).
\]

Therefore, we have the following time evolutions of \( p(\mu), q(\mu), \tilde{p}(\mu) \) and \( \tilde{q}(\mu) \) with respect to \( t_{2n+1} \)-flow

\[
\frac{\partial \tilde{p}(\mu)}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(\tilde{p}(\mu)), \quad \frac{\partial \tilde{q}(\mu)}{\partial t_{2n+1}} = \partial_{\bar{z}} \Phi_{2n+1}(\tilde{p}(\mu)), \tag{21}
\]

\[
\frac{\partial p(\mu)}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(q(\mu)), \quad \frac{\partial q(\mu)}{\partial t_{2n+1}} = \partial_{\bar{z}} \Phi_{2n+1}(q(\mu)). \tag{22}
\]

To see how the Faber polynomials will generate functions \( H_{2n+1}, \tilde{H}_{2n+1}, \tilde{\tilde{H}}_{2n+1} \) and \( \tilde{\tilde{H}}_{2n+1} \) shown in Eqs. (11), (12), similar derivations in Teo’s paper [15] (see also [14]), we differentiate (14) to the both sides with respect to \( \lambda \) and obtain

\[
\frac{wp'(\lambda)}{p^2(\lambda) - w^2} = \sum_{n=0}^{\infty} \Phi_{2n+1}(w) \lambda^{-2n-2}.
\]
Putting $p(\lambda) = \lambda - \sum_{n=0}^{\infty} \frac{H_{2n+1}}{2n+1} \lambda^{-2n-1}$ into this expression, then we have
\[
w + w \sum_{n=0}^{\infty} H_{2n+1} \lambda^{-2n-2} = \left[ \left( \lambda - \sum_{n=0}^{\infty} \frac{H_{2n+1}}{2n+1} \lambda^{-2n-1} \right)^2 - w^2 \right] \left( \sum_{n=0}^{\infty} \Phi_{2n+1}(w) \lambda^{-2n-2} \right).
\]
Comparing coefficients of all powers of $\lambda$ on both sides, we have
\[
\begin{align*}
\Phi_1(w) &= w, \\
\Phi_3(w) &= w^3 + 3H_1w
\end{align*}
\]and the recursion formula
\[
\Phi_{2n+5}(w) = w^2 \Phi_{2n+3}(w) - \sum_{m=0}^{n} \sum_{k=0}^{n-m} \frac{H_{2n-2m-2k+1}H_{2k+1}}{(2n-2m-2k+1)(2k+1)} \Phi_{2m+1}(w)
\]
\[
+ 2 \sum_{m=0}^{n+1} \frac{H_{2n-2m+3}}{2n-2m+3} \Phi_{2m+1} + wH_{2n+3}, \quad n = 0, 1, 2, \ldots,
\]
which can be used to solve for $\Phi_n$. The first few of $\Phi_n(w)$ are given by
\[
\begin{align*}
\Phi_5(w) &= w^5 + 5H_1w^3 + 5(H_1^2 + H_3/3)w, \\
\Phi_7(w) &= w^7 + 7H_1w^5 + 7(2H_1^2 + H_3/3)w^3 + 7(H_1^3 + (2/3)H_1H_3 + H_5/5)w, \\
\Phi_9(w) &= w^9 + 9H_1w^7 + 9(3H_1^2 + H_3/3)w^5 + 3(10H_1^3 + 4H_1H_3 + (3/5)H_5)w^3 \\
&\quad + 9(H_1^4 + H_1^2H_3 + H_3^2/9 + (2/5)H_1H_5 + H_7/7)w.
\end{align*}
\]Similarly, differentiating (20) with respect to $\lambda$, we have
\[
\frac{w p'(\lambda)}{p^2(\lambda) - w^2} = \sum_{n=0}^{\infty} \tilde{\Phi}_{2n+1}(w) \lambda^{-2n-2}.
\]
Now putting $\tilde{p}(\lambda) = \lambda - \sum_{n=0}^{\infty} \frac{H_{2n+1}}{2n+1} \lambda^{-2n-1}$ into this expression and comparing coefficients of powers of $\lambda$, we derive the first few expressions of Faber polynomials
\[
\begin{align*}
\tilde{\Phi}_1(w) &= w, \\
\tilde{\Phi}_3(w) &= w^3 + 3\tilde{H}_1w, \\
\tilde{\Phi}_5(w) &= w^5 + 5\tilde{H}_1w^3 + 5(\tilde{H}_1^2 + \tilde{H}_3/3)w, \\
\tilde{\Phi}_7(w) &= w^7 + 7\tilde{H}_1w^5 + 7(2\tilde{H}_1^2 + \tilde{H}_3/3)w^3 + 7(\tilde{H}_1^3 + (2/3)\tilde{H}_1\tilde{H}_3 + \tilde{H}_5/5)w, \\
\tilde{\Phi}_9(w) &= w^9 + 9\tilde{H}_1w^7 + 9(3\tilde{H}_1^2 + \tilde{H}_3/3)w^5 + 3(10\tilde{H}_1^3 + 4\tilde{H}_1\tilde{H}_3 + (3/5)\tilde{H}_5)w^3 \\
&\quad + 9(\tilde{H}_1^4 + \tilde{H}_1^2\tilde{H}_3 + \tilde{H}_3^2/9 + (2/5)\tilde{H}_1\tilde{H}_5 + \tilde{H}_7/7)w,
\end{align*}
\]in which $\tilde{\Phi}_{2n+1}$ obey the recurrence relations
\[
\begin{align*}
\tilde{\Phi}_{2n+5}(w) &= w^2 \tilde{\Phi}_{2n+3}(w) - \sum_{m=0}^{n} \sum_{k=0}^{n-m} \frac{H_{2n-2m-2k+1}\tilde{H}_{2k+1}}{(2n-2m-2k+1)(2k+1)} \tilde{\Phi}_{2m+1}(w)
\]
\[
+ 2 \sum_{m=0}^{n+1} \frac{\tilde{H}_{2n-2m+3}}{2n-2m+3} \tilde{\Phi}_{2m+1} + w\tilde{H}_{2n+3}, \quad n = 0, 1, 2, \ldots.
\]
4 Symmetry constraint of dVN hierarchy and conserved densities

One way of determine the conserved densities of the dVN hierarchy is to impose the desired symmetry constraints [4], so that the explicit formulas of these densities can be connected to the corresponding Faber polynomials and be solved recursively. The main symmetry constraint we consider is of the form

\[ u_x = (S^i)_{z\bar{z}}, \]  

(26)

where \( S^i = S(\mu_i) \) is evaluated at some point \( \mu_i \) and we assume \( S^i \) is real number. In this section, we would like to show that all of the conserved densities can be derived by means of the associated Faber polynomials under this symmetry reduction. We discuss these relations along the following two ways.

(I) Let us take the derivatives of \( S(\lambda) \) with respect to \( z, \bar{z}, x \) and, noticing that \(-2F_{z\bar{z}} = u\), we have

\[ \frac{\partial^3 S(\lambda)}{\partial z \partial \bar{z} \partial x} = -D(\lambda)F_{z\bar{z}x} = \frac{1}{2} D(\lambda)u_x = \frac{1}{2} D(\lambda)S^i, \]  

(27)

in which \( \partial S^i/\partial z = p(\mu_i) = p^i \), \( \partial S^i/\partial \bar{z} = q(\mu_i) = q^i \). \( \partial S^i/\partial z = p^i(p^\mu_i) \) obey the algebraic relation \( u = p^i q^i = p^\mu_i p^\mu_i \) and then \( u \) is positive real number. We remark here that in the context of nonlinear geometry optics, the quantity \( \sqrt{u} \) is proportional to the refractive index \( u = S^i S^\bar{i} \) is nothing but the standard Eikonal equation arises from the high-frequency limit of Maxwell equations [4 5 7 6].

Integrating (27) with respect to \( z, \bar{z} \) respectively and considering (3), it follows that

\[ (p(\lambda))_x = \left( \frac{1}{2} D(\lambda)S^i \right)_z = -\frac{1}{2} \partial z \left( \log \frac{p(\lambda) - p^i}{p(\lambda) + p^i} \right), \]  

(28)

\[ (q(\lambda))_x = \left( \frac{1}{2} D(\lambda)S^i \right)_{\bar{z}} = -\frac{1}{2} \partial \bar{z} \left( \log \frac{p(\lambda) - p^i}{p(\lambda) + p^i} \right). \]  

(29)

Using (14) with \( w \) replaced by \( p^i \) and the expansions of \( p(\lambda) \) and \( q(\lambda) \), Eqs. (28) and (29) can be rewritten respectively by

\[ \partial_x H_{2n+1} = -\partial_x \Phi_{2n+1}(p^i), \]  

(30)

\[ \partial_x \hat{H}_{2n+1} = -\partial_x \Phi_{2n+1}(p^i), \]  

(31)

where \( H_{2n+1} \equiv 2 \partial_x \partial_{2n+1} F \) and \( \hat{H}_{2n+1} \equiv 2 \partial_x \partial_{2n+1} \bar{F} \). Hence, Eqs. (30), (31) provide the Hamilton-Jacobi equations (18), which can now be read as

\[ \frac{\partial p^i}{\partial t_{2n+1}} = -\frac{\partial H_{2n+1}}{\partial x}, \quad \frac{\partial \bar{p}^i}{\partial t_{2n+1}} = -\frac{\partial \hat{H}_{2n+1}}{\partial x}. \]  

(32)

As the result, the functions \( H_{2n+1} \) and \( \hat{H}_{2n+1} \) appear to be the conserved densities that characterized by the associated Hamilton-Jabobi equations. Furthermore, from (30), (31) we see that \( H_{2n+1} \) and \( \hat{H}_{2n+1} \) are related by the compatibility relations

\[ \partial_x H_{2n+1} = \partial_x \hat{H}_{2n+1}, \]
And can be obtained by solving Eqs. (30) and (31).

(II) In the same way, the differentiation of $S(\lambda)$ with respect to $z, \bar{z}, x$ shows that

$$ \frac{\partial \bar{S}(\lambda)}{\partial z \partial \bar{z} \partial x} = -\bar{D}(\lambda) F_{\bar{z}x} = \frac{1}{2} \bar{D}(\lambda) u_x = \frac{1}{2} \bar{D}(\lambda) S_i^x, $$

Then we get

$$ (\bar{p}(\lambda))_x = \left( \frac{1}{2} \bar{D}(\lambda) S_i^x \right)_z = -\frac{1}{2} \partial_x \left( \log \frac{\bar{p}(\lambda) - \bar{p}^i}{\bar{p}(\lambda) + \bar{p}^i} \right), \quad (33) $$

$$ (\bar{q}(\lambda))_x = \left( \frac{1}{2} \bar{D}(\lambda) S_i^x \right)_z = -\frac{1}{2} \partial_x \left( \log \frac{\bar{p}(\lambda) - \bar{p}^i}{\bar{p}(\lambda) + \bar{p}^i} \right). \quad (34) $$

Using (20) with $w$ replaced by $\bar{p}^i$ and the expansion of $\bar{p}(\lambda)$, we rewrite (33) and (34) as

$$ \partial_x \bar{H}_{2n+1} = -\partial_z \bar{\Phi}_{2n+1}(\bar{p}^i), \quad (35) $$

$$ \partial_z \bar{H}_{2n+1} = -\partial_x \bar{\Phi}_{2n+1}(\bar{p}^i). \quad (36) $$

where $\bar{H}_{2n+1} \equiv 2\partial_z \partial_{t_{2n+1}} F$ and $\bar{\bar{H}}_{2n+1} \equiv 2\partial_x \partial_{t_{2n+1}} F$. Therefore, the Hamilton-Jacobi equations (22) can now be read by the conservation laws:

$$ \frac{\partial \bar{p}^i}{\partial t_{2n+1}} = -\frac{\partial \bar{H}_{2n+1}}{\partial x}, \quad \frac{\partial \bar{p}^i}{\partial t_{2n+1}} = -\frac{\partial \bar{H}_{2n+1}}{\partial x}. \quad (37) $$

Also, the conserved densities $\bar{H}_{2n+1}$ and $\bar{\bar{H}}_{2n+1}$ in (35) and (36) satisfy the compatibilities

$$ \partial_z \bar{H}_{2n+1} = \partial_z \bar{\bar{H}}_{2n+1} $$

and can be solved according to (35), (36). Notice that the Faber polynomials $\bar{\Phi}_{2n+1}(\bar{p}^i)$ have become the complex conjugate of $\Phi_{2n+1}(p^i)$ i.e., $\bar{\Phi}_{2n+1}(\bar{p}^i) = \Phi_{2n+1}(\bar{p}^i)$. We remark here that since $u = p^i \bar{p}^i$ in those $\bar{H}_i$’s, $\Phi_{2n+1}$ is understood as functions of $p^i, \bar{p}^i$.

In the following, under the symmetry constraint (26) we shall give some examples to demonstrate how to solve conserved densities $H_{2n+1}, \bar{H}_{2n+1}$. Then, $\bar{H}_{2n+1}$ and $\bar{\bar{H}}_{2n+1}$ are given automatically by taking the complex conjugate of $H_{2n+1}$ and $\bar{H}_{2n+1}$, respectively. For simplifying calculations, we shall use Faber polynomials (23), (24) and the useful identities

$$ p^i_x = p^i_x - u_x, \quad (38) $$

$$ u_x p^i = u p^i_x - u u_x + u_x (p^i)^2. \quad (39) $$

We determine the relationship of $\bar{H}_{2n+1}$ and $\bar{\bar{H}}_{2n+1}$ from the equation (10). Noting that $p(\lambda)$ and $q(\lambda)$ are defined by

$$ p(\lambda) = \lambda - \sum_{n=0}^{\infty} \frac{H_{2n+1}}{2n+1} \lambda^{-2n-1}, \quad q(\lambda) = - \sum_{n=0}^{\infty} \frac{\bar{H}_{2n+1}}{2n+1} \lambda^{-2n-1}, $$

and putting $p(\lambda)$ and $q(\lambda)$ into (10), we have the expression:

$$ u = - \sum_{n=0}^{\infty} \frac{\bar{H}_{2n+1}}{2n+1} \lambda^{-2n} + \sum_{n,m=0}^{\infty} \frac{H_{2n+1} \bar{H}_{2m+1}}{(2n+1)(2m+1)} \lambda^{-2n-2m-2}. \quad (40) $$

7
Identifying the coefficients of all powers of $\lambda$ at the both sides, we obtain

$$\hat{H}_1 = -u$$

and the recursion relation of $\hat{H}_{2n+1}$ and $H_{2n+1}$ by

$$\hat{H}_{2n+3} = (2n + 3) \sum_{k=0}^{n} \frac{H_{2n-2k+1} \hat{H}_{2k+1}}{(2n - 2k + 1)(2k + 1)}, \quad n = 0, 1, 2, \ldots. \quad (41)$$

Some of them are given by

$$\begin{align*}
\hat{H}_3 &= -3uH_1, \\
\hat{H}_5 &= -\frac{5}{3}u(3H_1^2 + H_3), \\
\hat{H}_7 &= -\frac{7}{3}u(3H_3^3 + 2H_1H_3 + \frac{3}{5}H_5).
\end{align*}$$

Examples of constructing conserved densities

**Example 1.** By (30), for $n = 0$

$$H_{1x} = -\Phi_{1z} = -p_z^i = (u - p^i)_x,$$

where we have used (38). Integrating both sides with respect to $x$ yields

$$H_1 = u - p^i. \quad (42)$$

**Example 2.** By (30), for $n = 1,$

$$\begin{align*}
H_{3x} &= -\Phi_{3z} = -((p^i)^3 + 3(u - p^i)p^i)_z, \\
&= (3(u - p^i)^2 - (p^i)^3)_x = (3H_1^2 - (p^i)^3)_x,
\end{align*}$$

where we have used Eqs. (23), (42) in the first line and (38), (39) to obtain the second line. After integrating both sides with respect to $x,$ we get

$$H_3 = 3H_1^2 - (p^i)^3. \quad (43)$$

**Example 3.** For $n = 2$ in (30), using (24), (42), (43) and the identities (38), (39) we have

$$\begin{align*}
H_{5x} &= -\Phi_{5z} = -\left((p^i)^5 + 5(u - p^i)(p^i)^3 + 5\left(2(u - p^i)^2 - \frac{1}{3}(p^i)^3\right)p^i\right)_z, \\
&= \left(10(u - p^i)^3 - \frac{20}{3}(u - p^i)(p^i)^3 - (p^i)^5\right)_x = \left(10H_1^3 - \frac{20}{3}H_1(p^i)^3 - (p^i)^5\right)_x, \\
&= \left(-10H_1^3 + \frac{20}{3}H_1H_3 - (p^i)^5\right)_x,
\end{align*}$$

where we have used the substitution for $(p^i)^3$ by (43) to obtain the last equality. Integrating both sides with respect to $x,$ we have

$$H_5 = -10H_1^3 + \frac{20}{3}H_1H_3 - (p^i)^5. \quad (44)$$
Example 4. For \( n = 3 \), similarly, we have

\[
H_{7x} = -\Phi_{7z},
\]

\[
= -\left\{ (p^i)^7 + 7(u - p^i)(p^i)^5 + 7\left(3(u - p^i)^2 - \frac{1}{3}(p^i)^3\right)(p^i)^3 \right. \\
\left. +7\left[(u - p^i)^3 + 2(u - p^i)\left((u - p^i)^2 - \frac{1}{3}(p^i)^3\right) \right. \\
\left. -2(u - p^i)^3 + 4(u - p^i)\left((u - p^i)^2 - \frac{1}{3}(p^i)^3\right) - \frac{1}{5}(p^i)^5\right]\right\} \frac{p^i}{\bar{p}^i} \zeta,
\]

\[
= 7\left(5(u - p^i)^4 - 5(u - p^i)^2(p^i)^3 - \frac{6}{5}(u - p^i)(p^i)^5 + \frac{1}{3}(p^i)^6 - \frac{1}{7}(p^i)^7\right)_x.
\]

Again, with substitutions for \((p^i)^3\) and \((p^i)^5\) obtained by (43) and (44) respectively and integrating over \( x \), we solve

\[
H_7 = 7\left(5H_1^4 - 5H_1^2H_3 + \frac{6}{5}H_1H_5 + \frac{1}{3}H_3^2 - (p^i)^7/7\right). 
\tag{45}
\]

5 Hodograph solutions of dVN hierarchy

Having set up the Faber polynomials in terms of \( p^i, \bar{p}^i \) for the dVN hierarchy that underline the imposed symmetry constraint (26), now we would like to use the hodograph method to find the hodograph solutions of \( p^i(z, \bar{z}, \tau_{2n+1}) \) and \( \bar{p}^i(z, \bar{z}, \tau_{2n+1}) \). Hence we shall obtain solutions of the dVN equation.

From (30)-(32) and (35)-(37), the \( t_{2n+1} \) and \( \bar{t}_{2n+1} \)-flows of \( p^i, \bar{p}^i \) can be written respectively by

\[
\begin{align*}
\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{t_{2n+1}} &= \partial_{p^i}\Phi_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_z + \partial_{\bar{p}^i}\Phi_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_\bar{z}, \\
\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\bar{t}_{2n+1}} &= \partial_{p^i}\Phi_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_z + \partial_{\bar{p}^i}\Phi_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_\bar{z},
\end{align*}
\]

where we have used the fact that \( p^i_z = \bar{p}^i_\bar{z} \). Therefore, the \( \tau_{2n+1} \)-flow of the dVN hierarchy is governed by

\[
\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\tau_{2n+1}} = \partial_{p^i}M_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_z + \partial_{\bar{p}^i}M_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_\bar{z}, \quad n \geq 1 
\tag{46}
\]

where \( M_{2n+1} \equiv \Phi_{2n+1} + \overline{\Phi_{2n+1}} \). Note that the first equation \( n = 0 \) of the hierarchy says that \( p^i \) and \( \bar{p}^i \) depend on \( \tau_1 \) and \( x \) only through the linear combination \( \tau_1 + x \). It is easy to see that the above equation has the following implicit form of hodograph equations

\[
z + \sum_{n=1}^{\infty} f_{2n+1}(p^i, \bar{p}^i)\tau_{2n+1} = F(p^i, \bar{p}^i),
\]

\[
\bar{z} + \sum_{n=1}^{\infty} g_{2n+1}(p^i, \bar{p}^i)\tau_{2n+1} = G(p^i, \bar{p}^i),
\tag{47}
\]

9
where $F$ and $G$ are the initial data at $\tau_{2n+1} = 0$, and $f_{2n+1} = \partial_{\rho}M_{2n+1}$. $g_{2n+1} = \partial_{\rho}M_{2n+1}$.

One can show that, because of commutativity of the $\tau_{2n+1}$-flows of $p^i, \bar{p}^i$, $G$ and $F$ obey the following constraints

\begin{align*}
F_{p^i} &= G_{p^i}, \\
p^iF_{p^i} &= -\bar{p}^iG_{p^i} - (1 - p^i - \bar{p}^i)G_{p^i}.
\end{align*}

(48) (49)

It turns out that in (48) there exists a function $\varphi(p^i, \bar{p}^i)$ such that $F = \partial_{p^i}\varphi$, $G = \partial_{\bar{p}^i}\varphi$. Substituting into (49) we have the defining equation for $\varphi$

\begin{equation}
\label{eq:50}
p^i\varphi_{p^ip^i} + \bar{p}^i\varphi_{\bar{p}^i\bar{p}^i} + (1 - p^i - \bar{p}^i)\varphi_{p^i\bar{p}^i} = 0.
\end{equation}

(50)

Let $p^i = (\rho_1 - i\rho_2)/2, \bar{p}^i = (\rho_1 + i\rho_2)/2$, we have $\partial/\partial p^i = \partial/\partial \rho_1 - i\partial/\partial \rho_2$ and $\partial/\partial \bar{p}^i = \partial/\partial \rho_1 + i\partial/\partial \rho_2$. Then the defining equation (50) becomes

\begin{equation}
\varphi_{\rho_1\rho_1} + 2\rho_2\varphi_{\rho_1\rho_2} + (1 - 2\rho_1)\varphi_{\rho_2\rho_2} = 0. \\
\end{equation}

(51)

In fact, due to the existence of $\varphi$, the functions $F, G$ can be chosen as a natural setting in the linear combination of $f$ and $g$ defined by (47). Namely, $F = \sum_{n\geq 0} \mu_n f_{2n+1}$ and $G = \sum_{n\geq 0} \xi_n g_{2n+1}$ with constraint $\mu_n = \xi_n$. We deduce that $\varphi$ has the polynomial type expansion in $\rho_1, \rho_2$:

\begin{equation}
\varphi = \sum_{n=0}^{\infty} \mu_n M_{2n+1}(\rho_1, \rho_2) \\
\end{equation}

(52)

satisfies (51). For instance, some cases are established as follows.

(i) $\varphi = M_1 = \Phi_1 + \Phi_1 = \rho_1$. It is obvious.

(ii) $\varphi = M_3 = \Phi_3 + \Phi_3 = \rho_1^3 - 3\rho_1^2 + \frac{3}{2}(\rho_1^2 + \rho_2^2)$.

(iii) $\varphi = M_5 = \Phi_5 + \Phi_5 = \rho_1^5 - \frac{29}{3}\rho_1^4 + 10\rho_1^3 + \frac{5}{2}\rho_1^2(\rho_1^2 + \rho_2^2) - \frac{15}{2}\rho_1(\rho_1^2 + \rho_2^2) + \frac{5}{3}(\rho_1^2 + \rho_2^2)^2$.

(iv) $\varphi = M_7 = \Phi_7 + \Phi_7$ has expression as

\begin{equation}
\varphi = \rho_1^7 - \frac{259}{30}\rho_1^6 + \frac{35}{2}\rho_1^5 + \frac{21}{10}\rho_1^4\rho_2^2 + \frac{7}{5}(\rho_1^2 + \rho_2^2)^2\rho_1^2 - \frac{35}{2}\rho_1^3\rho_2^2 \\
+ 35\rho_1^2\rho_2^2 - \frac{35}{4}(\rho_1^2 + \rho_2^2)^2\rho_1 - \frac{35}{8}(\rho_1^2 + \rho_2^2)^2 + \frac{28}{15}(\rho_1^2 + \rho_2^2)^3.
\end{equation}

Remark. One can also find several simple solutions of the certain PDEs in Eq. (51) in the following ways: (a) $\varphi_{\rho_1\rho_1} = 0, 2\rho_2\varphi_{\rho_1\rho_2} + (1 - 2\rho_1)\varphi_{\rho_2\rho_2} = 0$, (b) $\varphi_{\rho_1\rho_1} = 0, \varphi_{\rho_1\rho_1} + (1 - 2\rho_1)\varphi_{\rho_2\rho_2} = 0$, and (c) $\varphi_{\rho_2\rho_2} = 0, \varphi_{\rho_1\rho_1} + 2\rho_2\varphi_{\rho_1\rho_2} = 0$. It can be shown that cases (a) and (b) have solutions of polynomial type involved in (52), while (c) is not the case. For example, case (c) has solutions of the form: $\varphi = c_0 + c_1\rho_1 + c_2\rho_2 + c_3\rho_2 \exp(-2\rho_1)$.

Example

To find the $(2+1)$-dimensional solutions involving $(z, \bar{z}, \tau)$ that satisfy (46), using $\Phi_3 = (p^i)^3 + 3H_1 p^i$ where $H_1 = p^i\bar{p}^i - p^i$, we expand the hodograph equation (47) up to $\tau_3 = \tau$:

\begin{align*}
F(p^i, \bar{p}^i) &= z + f_3 \tau = z + \left(3(p^i + \bar{p}^i)^2 - 6p^i\right)\tau, \\
G(p^i, \bar{p}^i) &= \bar{z} + g_3 \tau = \bar{z} + \left(3(p^i + \bar{p}^i)^2 - 6\bar{p}^i\right)\tau.
\end{align*}
Choosing \( F = 1, G = 1 \), the above equations can be easily solved by

\[
\begin{align*}
p^i &= \frac{1}{12 \tau} \left( 3 \tau + (z - \bar{z}) \pm \sqrt{9 \tau^2 - 6 \tau (z + \bar{z} - 2)} \right), \\
\bar{p}^i &= \frac{1}{12 \tau} \left( 3 \tau - (z - \bar{z}) \pm \sqrt{9 \tau^2 - 6 \tau (z + \bar{z} - 2)} \right).
\end{align*}
\]

Then \( u \) is read as

\[
u = p^i \bar{p}^i = \frac{1}{144 \tau^2} \left( 18 \tau^2 - 6 \tau (z + \bar{z} - 2) - (z - \bar{z})^2 \pm 6 \tau \sqrt{9 \tau^2 - 6 \tau (z + \bar{z} - 2)} \right). (53)\]

One can verify that (53) satisfies the dVN equation (2) with \( \tau = \tau_3, V = 3H_1 \). Furthermore, if we choose \( F = f_3, G = g_3 \), we get

\[
\begin{align*}
p^i &= \frac{1}{12 (\tau - 1)} \left( 3(\tau - 1) + (z - \bar{z}) \pm \sqrt{9(\tau - 1)^2 - 6(\tau - 1)(z + \bar{z})} \right), \\
\bar{p}^i &= \frac{1}{12 (\tau - 1)} \left( 3(\tau - 1) - (z - \bar{z}) \pm \sqrt{9(\tau - 1)^2 - 6(\tau - 1)(z + \bar{z})} \right).
\end{align*}
\]

Therefore,

\[
u = \frac{18(\tau - 1)^2 - 6(\tau - 1)(z + \bar{z}) - (z - \bar{z})^2 \pm 6(\tau - 1) \sqrt{9(\tau - 1)^2 - 6(\tau - 1)(z + \bar{z})}}{144(\tau - 1)^2}.
\]

More new solutions can be given in this manner, but the main difficulty we have to confront with is to solve higher order algebraic equations. Finally, we want to solve \( S^i \) function of the above example via the partial differentiations \( \partial S^i / \partial z = p^i \) and \( \partial S^i / \partial \bar{z} = \bar{p}^i \). It is easy to obtain that the expression of \( S^i \) is given by

\[
S^i(z, \bar{z}, \tau) = \frac{3(\tau - 1)(z - \bar{z})^2 + 18(\tau - 1)^2(z + \bar{z} + 4C) - 2 \sqrt{3} (3(\tau - 1)^2 - 2(\tau - 1)(z + \bar{z}))^{3/2}}{72(\tau - 1)^2},
\]

where \( C \) is an arbitrary constant.

\section{2N-component case}

In this section, we give an 2\( N \)-component reduction of the dVN hierarchy under a more general symmetry constraint, and construct the corresponding hodograph equation. Let us consider the symmetry constraint of the form [4]

\[
u_x = \sum_{i=1}^{N} \epsilon_i S^i_{z \bar{z}}. \tag{54}\]

Particularly, we impose two assumptions: \( \nu = p^i \bar{p}^i, \forall i = 1, \ldots, N \) and \( \sum_{i=1}^{N} \epsilon_i = 1 \). Similar calculations in Sec.4, we have the following relations between conserved densities and the associated Faber polynomials:

\[
\begin{align*}
(H_{2n+1})_x &= - \sum_{i=1}^{N} \epsilon_i \partial_z \Phi_{2n+1}(p^i), \\
(\bar{H}_{2n+1})_x &= - \sum_{i=1}^{N} \epsilon_i \partial_{\bar{z}} \Phi_{2n+1}(p^i),
\end{align*}
\]
where the Faber polynomials $\Phi_{2n+1}(p^i)$ are defined as before, in which the conserved densities have different forms and can also be determined recursively. Some of $H_{2n+1}$ for the $2N$-reduction system are given by

$$H_1 = u - \sum_{i=1}^{N} \epsilon_i p^i, \quad H_3 = 3H_1^2 - \sum_{i=1}^{N} \epsilon_i (p^i)^3, \quad H_5 = -10H_1^3 + \frac{20}{3} H_1 H_3 - \sum_{i=1}^{N} \epsilon_i (p^i)^5,$$

$$H_7 = 35H_1^4 - 35H_1^2 H_3 + \frac{42}{5} H_1 H_5 + \frac{7}{3} H_3^2 - \sum_{i=1}^{N} \epsilon_i (p^i)^7.$$  

In terms of these $H_n$’s, the expressions of $\hat{H}_{2n+1}$ follow the same as the presented form in (41). Under the symmetry constraint, the Hamilton-Jacobi equations can now be written in the following way:

$$\frac{\partial p^k}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(p^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N), \quad \frac{\partial \bar{p}^k}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(p^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N),$$

$$\frac{\partial p^k}{\partial t_{2n+1}} = \partial_{\bar{p}^i} \bar{\Phi}_{2n+1}(\bar{p}^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N), \quad \frac{\partial \bar{p}^k}{\partial t_{2n+1}} = \partial_{\bar{p}^i} \bar{\Phi}_{2n+1}(\bar{p}^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N),$$

where $k = 1, \ldots, N$. After incorporating the above evolution equations to the $\tau_{2n+1}$-flow of dVN hierarchy and noting that $p^i_z = \bar{p}^i_z$ for $i = 1, \ldots, N$, we arrive the hodograph equation of $2N$-component system

$$\left( \begin{array}{c} p^k \\ \bar{p}^k \end{array} \right)_{\tau_{2n+1}} = \sum_{i=1}^{N} f^{i}_{2n+1} \left( \begin{array}{c} p^i \\ \bar{p}^i \end{array} \right)_z + \sum_{i=1}^{N} g^{i}_{2n+1} \left( \begin{array}{c} p^i \\ \bar{p}^i \end{array} \right)_\bar{z}, \quad k = 1, \ldots, N, \quad n \geq 1, \quad (55)$$

where $f^{i}_{2n+1}(p^k; \bar{p}^k) = \partial_{p^i}(\Phi_{2n+1}(p^k) + \bar{\Phi}_{2n+1}(\bar{p}^k))$ and $g^{i}_{2n+1}(p^k; \bar{p}^k) = \partial_{\bar{p}^i}(\Phi_{2n+1}(p^k) + \bar{\Phi}_{2n+1}(\bar{p}^k))$. For example, in the case of $N = 2$ we have

$$\left( \begin{array}{c} p^1 \\ \bar{p}^1 \\ \bar{p}^2 \\ \bar{p}^2 \end{array} \right)_{\tau_{2n+1}} = \left( \begin{array}{cc} f^1(p^1, \bar{p}^1)I_2 + g^1(p^1, \bar{p}^1)A(p^1) & f^2(p^1, \bar{p}^1)I_2 + g^2(p^1, \bar{p}^1)A(p^2) \\ f^1(p^2, \bar{p}^2)I_2 + g^1(p^2, \bar{p}^2)A(p^1) & f^2(p^2, \bar{p}^2)I_2 + g^2(p^2, \bar{p}^2)A(p^2) \end{array} \right) \left( \begin{array}{c} p^1 \\ \bar{p}^1 \\ p^2 \\ \bar{p}^2 \end{array} \right)_z,$$

where $I_2$ is the $2 \times 2$ identity matrix and

$$A(p^i) = \left( \begin{array}{cc} 0 & 1-p^i-p^\prime \\ -p^\prime & 1-p^i-p^\prime \end{array} \right), \quad i = 1, 2.$$

7 Conclusion remarks

In this paper we have studied dVN hierarchy from the framework of the 2-dBKP system. One demonstrates how to derive the associated Faber polynomials and their recursion relation via the Hirota equations of 2-dBKP hierarchy. Under the symmetry constraint (20), we solve conserved densities by the derived Faber polynomials. Also, we provide a set of hodograph equation of the dVN hierarchy, expanded by the derivatives of its associated Faber polynomials. Explicitly, we obtain the hodograph solutions to the dVN equation as an example.
For the more general symmetry constraint, we construct the $2N$-component reduction system by the generalized Faber polynomials and wrote down the corresponding hodograph equation. However, the main difficulty is to find the explicit solutions of the $2N$-reduction system (55). We hope to address this problem elsewhere.

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References

[1] A. P. Veselov and S. P. Novikov, Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formulas and evolution equations, Soviet Math. Dokl. 30 (1984) 588–591.

[2] I. M. Krichever, Method of averaging for two-dimensional ”integrable” equations, Functional Anal. Appl. 22 (1988) 200–213.

[3] B. G. Konopelchenko and L. Martínez Alonso, Nonlinear dynamics on the plane and integrable hierarchies of infinitesimal deformations, Stud. Appl. Math. 109 (2002) 313–336.

[4] L. V. Bogdanov, B. G. Konopelchenko, and A. Moro, Symmetry constraints for real dispersionless Veselov-Novikov equation, J. Math. Science 136 (2006) 4411–4418.

[5] B. G. Konopelchenko and A. Moro, Geometrical optics in nonlinear media and integrable equations, J. Phys. A: Math. Gen. 37 (2004) L105–L111.

[6] B. G. Konopelchenko and A. Moro, Integrable equations in nonlinear geometrical optics, Stud. Appl. Math. 113 (2004) 325–352.

[7] B. G. Konopelchenko and A. Moro, Light propagation in a Cole-Cole nonlinear medium via the Burgers-Hopf equation, Theoret. Math. Phys. 144 (2005) 968–974.

[8] L. V. Bogdanov and B. G. Konopelchenko, Symmetry constraints for dispersionless integrable equations and systems of hydrodynamic type, Phys. Lett. A 330 (2004) 448–459.

[9] L. V. Bogdanov and B. G. Konopelchenko, On dispersionless BKP hierarchy and its reductions, J. Nonlinear Math. Phys. 12 suppl. 1 (2005) 64–73.

[10] J. H. Chang, On the waterbag model of the dispersionless KP hierarchy (II), J. Phys. A: Math. Theor. 40 (2007) 12973–12985.

[11] K. Takasaki, Dispersionless Hirota equations of two-component BKP hierarchy, SIGMA 2, Paper 057 (2006).
[12] Y. T. Chen and M. H. Tu, On kernel formulas and dispersionless Hirota equations of the extended dispersionless BKP hierarchy, J. Math. Phys. 47 (2006) 102702.

[13] R. Carroll and Y. Kodama, Solution of the dispersionless Hirota equations, J. Phys. A 28 (1995) 6373–6387.

[14] C. Pommerenke, Univalent Functions, Vandenhoek & Ruprecht, Göttingen, 1975, with a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.

[15] L. P. Teo, Analytic functions and integrable hierarchies–characterization of tau functions, Lett. Math. Phys. 64 (2003) 75–92.