Mixed convection in inclined lid driven cavity by Lattice Boltzmann Method and heat flux boundary condition

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Abstract. Laminar mixed convective heat transfer in two-dimensional rectangular inclined driven cavity is studied numerically by means of a double population thermal Lattice Boltzmann method. Through the top moving lid the heat flux enters the cavity whereas it leaves the system through the bottom wall; side walls are adiabatic. The counter-slip internal energy density boundary condition, able to simulate an imposed non zero heat flux at the wall, is applied, in order to demonstrate that it can be effectively used to simulate heat transfer phenomena also in case of moving walls. Results are analyzed over a range of the Richardson numbers and tilting angles of the enclosure, encompassing the dominating forced convection, mixed convection, and dominating natural convection flow regimes. As expected, heat transfer rate increases as increases the inclination angle, but this effect is significant for higher Richardson numbers, when buoyancy forces dominate the problem; for horizontal cavity, average Nusselt number decreases with the increase of Richardson number because of the stratified field configuration.
1. Introduction

The lattice Boltzmann Equation (LBE) is a minimal form of the Boltzmann kinetic equation, which is the evolution equation for a continuous one-body distribution function, wherein all details of molecular motion are removed except those that are strictly needed to represent the hydrodynamic behaviour at the macroscopic scale; it has gained much attention for its ability to simulate fluid flows, and for its potential advantages over conventional numerical solution of the Navier–Stokes equations. The kinetic nature of Lattice Boltzmann Method (LBM) introduces key advantages, including easy implementation of boundary conditions and fully parallel algorithms. In addition, the convection operator is linear, no Poisson equation for the pressure must be resolved and the translation of the microscopic distribution function into the macroscopic quantities consists of simple arithmetic calculations [1]. LBM have met with significant success for the numerical simulation of a large variety of fluid flows, including real-world engineering applications and physical phenomena of various complexities as multiphase flows, complex geometries and interfacial flows [2-4]. The application to fluid flow coupled with non negligible heat transfer turned out to be much more difficult. The LBE thermal models fall into three categories: the multi-speed approach, the passive scalar approach and the doubled populations approach. In this latter, successful, strategy [5] thermal energy density and heat flux are expressed as kinetic moments of a thermal distribution function, so that no kinetic moment beyond the first order is ever required, thus providing numerical stability, also in case of significant temperature gradient [6]; in addition, with respect to the previous approaches, viscous heat dissipation and compression work done by the pressure were naturally incorporated and the boundary conditions are easily implemented because both populations live in the same lattice, where additional speeds are not necessary.

Mixed convection in an inclined cavity has not been investigated by LBM, to the authors’ knowledge, except for a recent previous work by Karimipour et al. [7], in which laminar mixed convection in a two-dimensional rectangular inclined cavity with moving top lid is investigated numerically for Richardson number ranging from 0.1 to 10 and inclination angle ranging from 0° to 90° in case of imposed temperature at top and bottom walls. Mixed convection in an inclined cavity has not been investigated by LBM in case of imposed non zero heat flux. This type of boundary condition, representing very usual situations in physical world, is not simple to model in lattice Boltzmann schemes. In effect, the only boundary condition able to simulate imposed temperature and imposed heat flux at a boundary has been presented by D’Orazio et al. in [8] where the boundary were at rest. In this work, laminar mixed convection in a two-dimensional rectangular inclined cavity with moving top lid is investigated numerically. Top lid motion results in fluid motion inside. Inclination of the cavity causes horizontal and vertical components of velocity to be affected by buoyancy force, since the heat flux enters the cavity through the top moving lid and it leaves the system through the bottom wall, side walls being adiabatic. To include this effect, collision term of Boltzmann equation is modified and the counter-slip internal energy density boundary condition by [6, 8, 9], able to simulate an imposed heat flux at the wall, is applied. The effects of the variations of Richardson number and inclination angle on the thermal and flow behaviour of the fluid inside the cavity are investigated. The results are presented as stream function contours and isotherms; Nusselt number behavior as a function of Richardson number and inclination angle is discussed. Simulations are performed for Pr = 0.7 and Re = 200 and for three value of the Richardson number Ri = 0.1, Ri = 1, Ri = 10; the effects of the inclination angle γ = 0°, 30°, 60°, 90° on fluid flow and heat transfer are studied in case of positive and negative moving lid velocity.

2. Thermal and hydrodynamic Lattice Boltzmann Method

The lattice Boltzmann equation with a single relaxation time from the BGK model can be expressed as
\[
\begin{align*}
\dot{f}_i(x + \tilde{c}_idt, t + dt) - \dot{f}_i(x, t) &= -\frac{dt}{\tau_f + 0.5dt}[\dot{f}_i - \dot{f}_i^e] \\
\dot{g}_i(x + \tilde{c}_idt, t + dt) - \dot{g}_i(x, t) &= -\frac{dt}{\tau_g + 0.5dt}[\dot{g}_i - \dot{g}_i^e] - \frac{\tau_g dt}{\tau_g + 0.5dt}f_iZ_i
\end{align*}
\]

where the populations \(\dot{f}_i\) carry mass and momentum and the populations \(\dot{g}_i\) carry internal energy and heat flux. The discrete distribution functions \(\dot{f}_i\) and \(\dot{g}_i\) are introduced as in [5]:

\[
\begin{align*}
\dot{f}_i &= f_i + \frac{dt}{2\tau_f}(f_i - f_i^e) \\
\dot{g}_i &= g_i + \frac{dt}{2\tau_g}(g_i - g_i^e) + \frac{dt}{2}f_iZ_i
\end{align*}
\]

\[
Z_i = (\tilde{c}_i - \tilde{u})D_i\tilde{u}, \quad D_i = \partial_t + \tilde{c}_i \cdot \nabla
\]

where \(f_i\) and \(g_i\) are the discrete populations which evolve when a standard first order integration strategy is adopted, the term

\[
\dot{Z}_i = (\tilde{c}_i - \tilde{u})D_i\tilde{u}
\]

represents the effects of viscous heating,

\[
D_i = \partial_t + \tilde{c}_i \cdot \nabla
\]

is the material derivative along direction \(c_i\), \(\tau_f\) and \(\tau_g\) are relaxations times and \(f_i^e\) and \(g_i^e\) are the equilibrium distribution functions. Throughout of this work, two-dimensional square lattice with the nine speeds, as shown in figure 1, is used.

![Figure 1. Nine-speed square lattice.](image)

The discrete particle lattice speeds are:

\[
\begin{align*}
c_i &= \left(\cos\frac{i-1}{2}\pi, \sin\frac{i-1}{2}\pi\right) c, \quad i = 1, 2, 3, 4 \\
c_i &= \sqrt{2}\left[\cos\left(\frac{i-5}{2}\pi + \frac{\pi}{4}\right), \sin\left(\frac{i-5}{2}\pi + \frac{\pi}{4}\right)\right] c, \quad i = 5, 6, 7, 8 \\
c_0 &= (0, 0)
\end{align*}
\]

where \(c^2 = 3RT\) and \(T\) is the temperature. The equilibrium density distributions are chosen as follows:

\[
f_i^e = \omega_i\rho \left[1 + \frac{3\tilde{c}_i \cdot \tilde{u}}{c^2} + \frac{9(\tilde{c}_i \cdot \tilde{u})^2}{2c^4} - \frac{3(u^2 + v^2)}{2c^2}\right]
\]
\[ g_0^e = -\omega_0 \left[ \frac{3\rho e(u^2 + v^2)}{2c^2} \right] \]

\[ g_{1,2,3,4}^e = \omega_1 \rho e \left[ 1.5 + 1.5 \frac{\hat{c}_i \cdot \hat{u}}{c^2} + 4.5 \frac{(\hat{c}_i \cdot \hat{u})^2}{c^4} - 1.5 \frac{(u^2 + v^2)}{c^2} \right] \]

\[ g_{5,6,7,8}^e = \omega_2 \rho e \left[ 3 + 6 \frac{\hat{c}_i \cdot \hat{u}}{c^2} + 4.5 \frac{(\hat{c}_i \cdot \hat{u})^2}{c^4} - 1.5 \frac{(u^2 + v^2)}{c^2} \right] \]

where \( \hat{u} = (u, v) \) and \( \rho e = \rho RT \) (in two-dimensional geometry). The weights of the different populations are

\[ \omega_0 = \frac{4}{9} \quad \omega_i = \frac{1}{9} \quad i = 1,2,3,4 \quad \omega_i = \frac{1}{36} \quad i = 5,6,7,8 \]

The terms enclosed by the square bracket, multiplied by the corresponding weights \( \omega_i \), will be called corresponding form for equilibrium.

Finally, using \( f_i^p \) and \( g_i^p \), hydrodynamic and thermal variables are calculated as follows:

\[ \rho = \sum_i f_i \quad \rho e = \sum_i \hat{g}_i - \frac{dt}{2} \sum_i f_i \ Z_i \]

\[ \rho \hat{u} = \sum_i \hat{c}_i \hat{f}_i \quad \hat{q} = \left[ \sum_i \hat{c}_i \hat{g}_i - \rho e \hat{u} - \frac{dt}{2} \sum_i \hat{c}_i f_i \ Z_i \right] \frac{\tau g}{\tau_g + 0.5 \ dt} \]

The kinematic viscosity and the thermal diffusivity in the two-dimensional geometry are given by:

\[ \nu = \tau_f R \bar{T}, \quad \chi = 2 \tau_g R \bar{T} \]

**Figure 2.** Geometry and coordinates axis in the inclined cavity.

In this problem shear stress applied by moving lid on the fluid layers results in fluid motion, thus creating suitable temperature gradient that enhances buoyancy forces. Therefore, mixed convection is produced in the fluid confined in the cavity. With the Boussinesq approximation, all the fluid properties are considered as constant, except in the body force term in the Navier–Stokes equations,
where the fluid density is assumed \( \rho = \bar{\rho}[1 - \beta (T - \bar{T})] \), in which \( \beta \) is volumetric expansion coefficient, \( \bar{\rho} \) and \( \bar{T} \) are the average fluid density and temperature. In order to simulate the mixed convection of nearly incompressible flows, buoyancy force per unit mass is defined as \( G = \beta g(T - \bar{T}) \) and this is used to drive the flow. By considering inclination angle, coordinate axis and gravity acceleration direction, as shown in figure 2, all of the aforementioned relations are maintained and used except those that are modified below:

\[
\tilde{f}_i (\bar{x} + \bar{c}_i dt, t + dt) - \tilde{f}_i (\bar{x}, t) =
\]

\[
= - \frac{dt}{\tau_f + 0.5dt} \left[ \tilde{f}_i - \tilde{f}_i^e \right] + \left[ \frac{dt \tau_f}{\tau_f + 0.5dt} \frac{3G(c_{ix} - u)}{c^2} \tilde{f}_i^e \sin y + \left[ \frac{dt \tau_f}{\tau_f + 0.5dt} \frac{3G(c_{iy} - v)}{c^2} \tilde{f}_i^e \right] \cos y \right]
\]

with:

\[
\tilde{f}_i =
\]

\[
= \frac{\tau_f \tilde{f}_i + 0.5dt \tilde{f}_i^e}{\tau_f + 0.5dt} + \left[ \frac{0.5dt \tau_f}{\tau_f + 0.5dt} \frac{3G(c_{ix} - u)f_i^e}{c^2} \sin y + \left[ \frac{0.5dt \tau_f}{\tau_f + 0.5dt} \frac{3G(c_{iy} - v)f_i^e}{c^2} \right] \cos y \right]
\]

In this case hydrodynamic macroscopic variables are calculated as follows:

\[
\rho = \sum_i \tilde{f}_i, \quad u = \frac{1}{\rho} \sum_i \tilde{f}_i c_{ix} + \frac{dt}{2} G \sin y, \quad v = \frac{1}{\rho} \sum_i \tilde{f}_i c_{iy} + \frac{dt}{2} G \cos y
\]

where \( \bar{c}_i = (c_{ix}, c_{iy}) \) denote discrete particle speeds.

### 3. Boundary conditions

With regard to hydrodynamic boundary condition, no slip boundary condition is applied; It is obtained by means of the non-equilibrium bounce back rule as applied to the population perpendicular to the boundary. With regard to thermal boundary conditions, the top cavity lid and bottom wall are heated and cooled respectively by an uniform and constant heat flux entering and leaving respectively, and the sidewalls are insulated. These boundary conditions are obtained by means a thermal counter-slip approach as proposed by [6, 8, 9], in which the incoming unknown thermal populations are assumed to be equilibrium distribution functions with a counter slip thermal energy density \( e' \), which is determined so that suitable constraints are verified. For the top wall of the cavity, named as “north wall”, in which entering heat flux is constant and equal to \( q_N \), the unknown \( \tilde{g}_4, \tilde{g}_5 \) and \( \tilde{g}_6 \) are chosen as follows.

\[
\tilde{g}_i = \rho(e_N + e') \times [\text{corresponding form for equilibrium}] \quad i = 4, 7, 8
\]

By definition:

\[
\sum_i c_{iy} \tilde{g}_i = \frac{dt}{2} \sum_i c_{iy} f_i Z_i + \rho c e N V_N + \frac{\tau_g + 0.5dt}{\tau_g} q_y
\]

which yields to:

\[
\rho e_N + pe' = \left[ K - \frac{dt}{2} \sum_i c_{iy} f_i Z_i - \rho c e N \frac{V_N}{c} - \frac{\tau_g + 0.5dt}{\tau_g} q_y \right] \left[ \frac{1}{3} - \frac{1}{2} \frac{V_N}{c} + \frac{1}{2} \frac{V_N^2}{c^2} \right]
\]

where \( V_N \) is the component normal to the wall of flow velocity at the wall, \( K \) is the sum of the three known populations (\( \tilde{g}_2, \tilde{g}_5 \) and \( \tilde{g}_6 \)) \( e_N \) denotes the current value of thermal energy density at the
north wall. At the insulated walls, the constraint on the heat flux is obtained by imposing \( q_x = 0 \) on the corresponding equation, for example for the west wall, so that we have [8]

\[
\sum_i c_{ix} \bar{g}_i = 0.5 \sum_i c_{ix} f_i Z_i + \rho e_w U_w
\]

where \( e_w \) denotes the thermal energy density current value (coming from the run) at the West wall.

The unknown populations \( \bar{g}_1, \bar{g}_5 \) and \( \bar{g}_8 \) are chosen as

\[
\bar{g}_i = \rho(e_w + e^* \times [\text{corresponding form for equilibrium}]) \quad i = 1, 5, 8
\]

and become

\[
\bar{g}_i = \left[ K + \frac{dt}{2} \sum_i c_{ix} f_i Z_i + \rho e_w U_w \frac{u_{w}}{c} \right] \times [\text{corresponding form for equilibrium}] \quad i = 1, 5, 8
\]

where \( U_w \) is the component normal to the wall of the flow velocity at the wall and \( K \) is the sum of the known populations (\( \bar{g}_3, \bar{g}_6 \) and \( \bar{g}_7 \) in this case). The corners nodes are treated similarly and the counter-slip procedure can be applied to the five unknown incoming populations at the corner. The same relations are used for the bottom wall (named as “south wall”), in which leaving heat flux is constant and equal to \( q_S \) and the unknown populations are \( \bar{g}_2, \bar{g}_5 \) and \( \bar{g}_6 \). With regard to the initial conditions, the velocities of all nodes inside the cavity are taken as zero initially. The initial density is set to a value of 2.7. The initial equilibrium distribution functions are evaluated correspondingly. The initial distribution functions are taken as the corresponding equilibrium values.

4. Results and discussion

Laminar mixed convection of a fluid inside a rectangular cavity with moving top lid and aspect ratio \( AR = L/H = 3 \), in which \( L \) and \( H \) are shown in Figure 2, is studied numerically using Lattice Boltzmann Method previously described. The bottom wall is cooled and the top lid is heated, and side walls are assumed insulated. Top lid moves with constant velocity \( U_0 \). Characteristic dimensionless number in the analysis of mixed convection problems is Richardson number defined as \( Ri = Ra Pr Re^2 \). As stated before, lattice Boltzmann method is used for near-incompressible flows and therefore Mach number is assumed as \( Ma = 0.1 \). More specifically, the characteristic velocity of the flow for both natural regime, \( U^* = \sqrt{\beta g H \Delta T} \), and forced regime \( U^* = v Re/H \) must be small compared with the fluid speed of sound; in present study the velocity \( U_0 \) was selected as \( U_0 = 0.1 \). The Prandtl and Reynolds numbers are assumed as \( Pr = 0.7 \) and \( Re = 200 \) respectively and the effects of the variations of the Richardson number \( Ri \) and angle \( \gamma \) on the flow and heat transfer are studied. Inclination angle varies from zero degree, horizontal cavity, to 90º, vertical cavity. To avoid ambiguity, the cavity’s walls are referred to according to the coordinates shown in Figure 2. In the following the macroscopic variables of fluid flow are made dimensionless as follows, respectively for dimensionless coordinates, velocity components, temperature, time

\[
Y = \frac{y}{H} , \quad X = \frac{x}{H} , \quad U = \frac{u}{U_0} , \quad V = \frac{v}{U_0} , \quad \Theta = \frac{T - T_e}{T^* - T_e} , \quad \tau = \frac{t U_0}{H} \quad T^* = \frac{\chi v Ra^4}{\beta g H^3}
\]

Therefore, local and average Nusselt numbers along the lid and bottom wall are calculated using following relations

\[
Nu_X = \frac{q H}{k \Delta T} = - \left( \frac{\partial \Theta}{\partial Y} \right)_{wall} , \quad Nu_m = \frac{1}{AR} \int_0^{AR} Nu_X dX
\]
The top lid is the hot moving wall at \( Y = 1 \), the bottom wall is the cold wall at \( Y = 0 \) and two insulated side walls are at \( X = 0 \) and \( X = 3 \).

In order to obtain grid independent solution, a grid refinement study is performed for a horizontal cavity \( (\gamma = 0) \). Grid independence of the results has been established in term of average Nusselt number on the lid and dimensionless values of \( x \)-velocity \( U \), \( y \)-velocity \( V \), and temperature \( \Theta \) at \( X = 1.5 \) and \( Y = 0.5 \) (cavity center) for three different grid size, namely \( 300 \times 100 \), \( 450 \times 150 \) and \( 600 \times 200 \) lattice nodes; due to small difference between the results of the last two grid sizes, the \( 450 \times 150 \) grid is chosen as a suitable one in this work.

To validate the computer code, the comparison with the analytical solution given by Kimura and Bejan \[10\] and the value obtained by D’Orazio et al. \[6\] is examined for the Rayleigh number equal to \( 10^5 \) and \( 10^6 \) and the Prandtl number equal to 2.0. With regard to the average Nusselt number \( \bar{N}_\text{Nu} \), results show good agreement with those of Bejan et al. \[10\] and of D’Orazio et al. \[8\], with a maximum error equal to 18.6%.

In order to show the effect of inclination angles on the flow field and heat transfer, in Figures 3, 4 and 5 and 6, 7 and 8 streamlines and isotherms are reported at different inclination angles \( \gamma = 0^\circ, 30^\circ, 60^\circ \) and \( 90^\circ \), for the cases for \( Ri = 0.1, 1 \) and 10, respectively for positive and negative velocity (in this case the inclination angle ranges from \( 30^\circ \) to \( 90^\circ \), being the case \( \gamma = 0^\circ \) not depending on the moving lid direction).

The motion of the cavity lid, with positive lid velocity, causes the fluid motion in the cavity and produces a strong clockwise rotational cell in it. This motion transfers hot fluid to the lower parts and enhances favourable pressure gradient along the vertical direction, leading to the generating buoyancy motions and transferring hot fluid to the upper parts. Therefore, the combination of free and forced convection, called “mixed convection”, is made. In Figure 3, the Richardson number is \( Ri = 0.1 \), with
forced convection dominant with respect to natural convection, and it implies that by increasing \( \gamma \), the rotational power of the cell in the center of the cavity increases slightly and it does not affect significantly the other moving and thermal behaviour of the fluid.

When buoyancy forces dominate the problem, as for \( Ri = 10 \) reported in Figure 5, inclination angle has significant effect on the flow field and heat transfer. For \( \gamma = 0 \), a strong cell in the upper half and a weak cell in the lower half of the cavity, are generated due to the forcing of moving wall. In addition, for \( \gamma = 0 \), isotherms in the lower half of the cavity are straight lines and perpendicular to the side walls, indicating that conduction heat transfer is dominant in this region and that, without the forced convection contribution, no motion occurs, due the stratified fluid. As the inclination angle increases, the two cells merge, so that for \( \gamma = 90 \) a large rotational cell covers the whole space of the cavity, with the central part practically motionless.

When the buoyancy effect is predominant (\( Ri = 10 \)) in case of negative velocity, the effect of the direction of the moving wall, in term of modified flow field, can be detected. While for \( U_0 < 0 \) the two rotational cells lose their individuality already for low inclination angles, for \( U_0 < 0 \) they remain distinct because the hot moving wall has an opposite effect with respect to the buoyancy.

When \( Ri = 1 \), it becomes evident as the contribution of the moving wall with negative velocity and the buoyancy due to gravity acceleration have opposite effects on the fluid field; it gives rise to the formation of two rotational cell into the cavity.

For \( Ri = 0.1 \), when forced convection is the dominant effect on the flow field, streamlines and isotherms for negative velocity, have the same behaviour shown in case of positive wall velocity, although as in a mirror, but the result of the two cited opposing effects is the persistence of the two rotational cells.

In Figure 9, the average Nusselt number on the lid surface is reported as a function of inclination angle for different Richardson number in both case of positive and negative moving wall velocity. It has to be noted that for \( \gamma = 90 \), Nu\(_m\) decreases when \( Ri \) increases, since it implies to go to a stratified field configuration. With regard to \( \gamma \neq 0 \), it is observed that for positive lid velocity when \( Ri = 0.1 \), Nu\(_m\) increases slightly with the increase of \( \gamma \). At \( Ri \geq 1 \), Nu\(_m\) is increased more intensively. In fact at \( Ri = 10 \), it is increased by a factor of 5 when \( \gamma \) varies from 0 to 90, indicating that in this case free convection effect is enhanced and its contribution can be added to the forced convection effect. When \( \gamma = 0 \) (horizontal cavity) Nu\(_m\) is maximum at \( Ri = 0.1 \) (significant forced convection), but for inclined cavity, the maximum Nu\(_m\) occurs at \( Ri = 10 \) (significant natural convection). For negative velocity of moving wall, the opposite effect of buoyancy and forced convection contribution can be observed since the Nusselt number results ever lower than the value corresponding to the case of positive velocity. For low inclination angle, the contribution of natural convection is less important than the hindering effect of the negative moving wall velocity, whereas for increasing \( \gamma \) angle the importance of natural convection becomes predominant.
Figure 9. Average Nusselt number $\text{Nu}_{\text{av}}$ on the hot surface as a function of inclination angle $\gamma$ for different Richardson number and in case of $U_0 > 0$ and $U_0 < 0$

5. Conclusions

A thermal lattice Boltzmann BGK model with a dedicated boundary condition was used to study numerically laminar two-dimensional mixed convection heat transfer inside an inclined rectangular cavity when heat transfer rate is imposed at the boundaries. Since the inclination of the cavity enhances the buoyancy force, which affects the velocity components of the flow, the forcing term simulating the buoyancy effect in the lattice Boltzmann equation were modified. The results show that, as expected, heat transfer rate increases as increases the inclination angle, but this effect is significant for the higher Richardson numbers, when buoyancy forces dominate the problem; for horizontal cavity, average Nusselt number decreases with the increase of the Richardson number because of the stratified field configuration. The effects of forced convection and natural convection can be considered as cooperating for positive velocity of the moving lid, while on the contrary they can be considered as opposite for negative velocity. This study shows that lattice Boltzmann method together with the counter-slip thermal energy density boundary condition can be effectively used to simulate heat transfer phenomena also in case of moving walls. The method can be successfully applied to simulate a wide class of cooling process where a given thermal power must be removed.

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