2D Yang-Mills Theories, Gauge Orbit Spaces and the Path Integral Quantization

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Abstract

The role of a physical phase space structure in a classical and quantum dynamics of gauge theories is emphasized. In particular, the gauge orbit space of Yang-Mills theories on a cylindrical spacetime (space is compactified to a circle) is shown to be the Weyl cell for a semisimple compact gauge group, while the physical phase space coincides with the quotient $\mathbb{R}^{2r}/W_A$, $r$ a rank of a gauge group, $W_A$ the affine Weyl group. The transition amplitude between two points of the gauge orbit space (between two Wilson loops) is represented via a Hamiltonian path integral over the physical phase space and explicitly calculated. The path integral formula appears to be modified by including trajectories reflected from the boundary of the physical configuration space (of the Weyl cell) into the sum over paths.

The Gribov problem of gauge fixing ambiguities is considered and its solution is proposed in the framework of the path integral modified. Artifacts of gauge fixing are qualitatively analyzed with a simple mechanical example. A relation between a gauge-invariant description and a gauge fixing procedure is established.

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1 Introduction

This paper is devoted to an analysis of quantum dynamics of gauge theories on a gauge orbit space. Its main aim is to establish the path integral representation of the transition amplitude on the gauge orbit space which results from the Dirac’s operator method of quantizing first-class constrained systems [1].

The whole configuration space of any gauge theory is splinted into a set of gauge group orbits so that gauge transformations generated by constraints are shifts along those orbits. Any motion of a system along a gauge orbit is physically irrelevant and, therefore, only transitions between distinct gauge orbits carry physical information. So, dynamics of physical degrees of freedom occurs in the space of gauge orbits which serves as a physical configuration space.

In Sec. 2, we show with trivial examples of a particle in a box, on a circles and on a line that a configuration space structure affects quantum dynamics. Then we give a definition of the physical phase space in gauge theories. In Sec. 3, we analyze a physical configuration (and phase) space structure in 2D Yang-Mills theories (QCD) with a semisimple compact gauge group and show that it differs from an Euclidean space. Due to a solvability of the model [2], [3], it is intensively investigated [4] and regarded as a good toy model for verifying various ideas and methods proposed for 4D gauge theories.

Recently, we have proposed a path integral formula for a transition amplitude on the gauge orbit space for 4D Yang-Mills theories [5]. Because of the Gribov’s ambiguity [6], [7], the gauge orbit space of Yang-Mills potentials cannot be uniquely parametrized by potentials satisfying a gauge condition chosen. A gauge condition surface in the total configuration space contains gauge-equivalent configurations. The gauge orbit space turns out to be isomorphic only to a certain domain on it which is called a modular domain [8]. It was shown in [5] that the Dirac quantization method leads to a modified path integral formula. One should include contributions of trajectories reflected from the modular domain boundary into the Feynman sum over paths in order to obtain a gauge invariant transition amplitude. In Sec. 5, we apply this recipe to the two dimensional case and obtain an exact quantum amplitude for a transition between two gauge orbits [9].

When describing quantum dynamics on the gauge orbit space, one needs to introduce coordinates or a certain set of parameters which span the orbit space. A choice of the orbit space parametrization is not unique and implies, actually, fixing a gauge. Some gauge fixing artifacts which might occur through an inappropriate gauge condition choice are discussed in Sec. 4.

For the Yang-Mills theory on a cylindrical spacetime, physical degrees of freedom can be described by spatially homogeneous connections belonging to the Cartan subalgebra, i.e. the Cartan subalgebra is chosen to be a space of physical coordinates. However, this space is over complete in a sense that there are configurations in it which correspond to the same gauge orbit. The latter does not lead to decreasing a number of physical degrees of freedom, but it does reduce a ”volume” of the physical configuration space. It turns out that physically distinct configurations constitute a compact domain in the Cartan subalgebra, namely, the Weyl cell [9].
The path integral formula resulting from Dirac’s operator method of quantizing first-
class constrained system does not contain an integration over the Weyl cell $K_W^+$. A correct
transition amplitude is given by a sum over all trajectories reflected from the Weyl cell boundary,
which is technically equivalent to carrying out functional integration over the Cartan subalgebra $H$
with a sequent symmetrization the result with respect to the affine Weyl group $W_A$, $K_W^+ = H/W_A$.

The paper also contains a few Appendices where some technical and mathematical
points are explained.

2 Physical phase space in gauge theories

There is no doubt that phase-space (PS) geometry is one of the main characteristics of
Hamiltonian systems, and classical as well as quantum dynamics strongly depend on it.
Consider, for example, a free particle on a line. The Hamiltonian reads $H = p^2/2$. If PS
is assumed to be a plane $\mathbb{R}^2$, then classical trajectories are straight lines perpendicular
to the momentum axis in PS and outgoing to infinity. In quantum theory we have the
Hamiltonian spectrum $E = p^2/2$, $p \in \mathbb{R}$, and wave functions being plane waves $\psi \sim \exp ipx$.

Let us change the PS topology by compactification of the configuration space to a
finite size $L$. One can do it in different ways. We may identify the boundary points of the
configuration space to turn it into a circle $S^1$ of length $L$, then the phase space becomes
a cylinder $PS = \mathbb{R} \otimes S^1$. Another way is to install infinite walls at the boundary points
$x = 0, L$ to prevent a particle from penetrating outside of the interval, then the phase space
is a strip $\mathbb{R} \otimes (0, L)$. The classical motion becomes periodical. The system returns to its
initial state (a PS point) via time $T = L/p_0$, $p_0$ is a particle momentum, if $PS = \mathbb{R} \otimes S^1$.

For a particle moving between two walls the period is equal to $T = 2L/p_0$ because a
particle has to reflect from both walls to reach an initial PS point (notice, each reflection
changes a sign of a particle momentum). The quantum theories are also different. For
the cylindrical PS the spectrum and wave functions read $E_n = 2\pi^2 n^2/L^2$, $n \in \mathbb{Z}$, and
$\psi_n \sim \exp(2\pi inx/L)$, respectively, while for PS being a strip they are $E_n = \pi^2 n^2/(2L^2)$,
and $\psi_n \sim \sin(\pi xn/L)$, $n = 0, 1, 2, \ldots$.

A main feature of gauge theories is the existence of unphysical variables whose evolu-
tion is determined by arbitrary functions of time $[1]$, while physical quantities appear
to be independent of the gauge arbitrariness. A non-trivial geometry of the physical PS
(denoted below as $PS_{ph}$) may occurs in gauge theories, even if the total PS is assumed to
be an even-dimensional Euclidean space $[10]-[13]$.

Let a system with $N$ degrees of freedom have $M$ independent first-class constraints
$[1] \sigma_a(q, p) = 0$, $a = 1, 2, \ldots, M$. Let $H = H(q, p)$ be a Hamiltonian of the system
such that its Poisson bracket with the constraints vanishes on the constraint surface,
\[ \{H, \sigma_a\} = c_{ab} \sigma_b = 0. \]
The latter means that the system never leaves the constraint surface in due course since $\dot{\sigma}_a = \{\sigma_a, H\} = 0$. Then our system admits a generalized dynamical description, namely, on the constraint surface the time evolutions generated
by the Hamiltonians $H$ and $H_E = H + \lambda_a \sigma_a$, where $\lambda_a$ are arbitrary functions of time, are physically indistinguishable [1]. Indeed, the first class constraints generate gauge transformations of canonical variables $\delta q = \omega_a \{\sigma_a, q\}$ and $\delta p = \omega_a \{\sigma_a, p\}$, $\omega_a$ are arbitrary infinitesimal functions of time[1]. Any physical quantity $F = F(q, p)$ must be gauge invariant and, therefore, $\delta F = \omega_a \{\sigma_a, F\} = 0$, which leads to the equality $\dot{F} = \{F, H_E\} = \{F, H\}$ on the surface $\sigma_a = 0$.

A general solution of the constraints and equations of motion induced by $H_E$ depends on $M$ arbitrary functions of time $\lambda_a$, while any variation of those functions means no change of a physical state of the system [1]. Such a variation is nothing but a gauge transformation of canonical variables. If we require that any physical state corresponds just to one point of $PS_{ph}$, then we are led to the following definition of $PS_{ph}$. The physical phase space is the quotient of the constraint surface in the whole $PS$ by gauge transformations $G$ generated by all independent first class constraints $\sigma_a$,

$$PS_{ph} = PS|_{\sigma_a=0}/G.$$  \hspace{1cm} (2.1)

Let us apply (2.1) to a simple gauge system. Consider a particle moving in a plane. Let the only constraint be its angular momentum $\sigma = pT x = 0$, $T = -i\tau_2$, $\tau_2$ the Pauli matrix, $x$ and $p$ are position and momentum vectors of the particle. The constraint generates simultaneous rotations of $p$ and $x$ since $\delta p = \omega \{p, \sigma\} = \omega T p$ and $\delta x = \omega \{x, \sigma\} = \omega T x$ and $T$ is a generator of SO(2). So, the angular variable turns out to be unphysical, and only the radial motion should be of physical interest. Dynamics of the radial degree of freedom occurs on the constraint surface $p \sim x$ in the total $PS = \mathbb{R}^4$. By means of gauge SO(2)-rotations we can identify any vector $x$ with a particular vector $(x, 0)$, $x^2 = x^2$. A momentum vector is simultaneously reduced to $(p, 0)$ by the same gauge transformation since $p \sim x$. The continuous gauge arbitrariness is exhausted. Notice that the relation $p \sim x$ implies $p \sim x$. However, the variables $p$ and $x$ are regarded as independent because the proportionality coefficient is an arbitrary function of time (it can be found only by solving Hamiltonian equations of motion).

One would assume $PS_{ph}$ spanned by $p$ and $x$ to be a plane $\mathbb{R}^2$. But it is not the case. There remain discrete gauge transformations, the SO(2)-rotations of the vector $(x, 0)$ through the angle $\pi$ which identify the phase-space points $p, x$ and $-p, -x$ on the phase plane $\mathbb{R}^2$ (as $p \sim x$, the gauge transformations $x \rightarrow \pm x$ imply the simultaneous change of the momentum sign, $p \rightarrow \pm p$). Therefore, $PS_{ph}$ in the model is a cone unfoldable into a half-plane [10],

$$PS_{ph} = \mathbb{R}^2/\mathbb{Z}^2 = \text{cone}(\pi).$$  \hspace{1cm} (2.2)

A manifestation of the $PS_{ph}$ structure (2.2) can be observed in classical and quantum dynamics. Let the Hamiltonian be $H = (p^2 + \omega^2 x^2)/2$. After eliminating an unphysical degree of freedom we have a one-dimensional oscillator with a conic PS. A classical phase-space trajectory of a harmonic oscillator is an ellipse with its center at the origin. As the points $p, x$ and $-p, -x$ are gauge equivalent, the system returns to its initial state (an initial phase-space point) via time $T_{ph} = \pi/\omega$ rather than the period $T = 2\pi/\omega$, i.e. the physical frequency is doubled [10] \omega_{ph} = 2\pi/T_{ph} = 2\omega.
In quantum theory, the operator $\hat{\sigma}$ must annihilate physical states [1]. Rewriting it via the destruction and creation operators $\hat{a} = (\sqrt{\omega} \hat{x} + i \hat{p} / \sqrt{\omega}) / \sqrt{2}$ and $\hat{a}^+$, respectively, we have $\{\hat{\sigma}, \hat{a} \} = -i \hat{a}^+ T \hat{a} |ph\rangle = 0$. Therefore $|ph\rangle = \Phi(\hat{a}^+) |0\rangle$, $\hat{a} |0\rangle \equiv 0$, where $[\hat{\sigma}, \Phi] = 0$. The latter means that the operator $\Phi$ is a function invariant under the SO(2)-rotations of its argument $\hat{a}^+$. The only independent invariant which can be built of the vector $\hat{a}^+$ is its square $\hat{a}^+ \hat{a}^+$; all invariants are functions of it. Thus, an arbitrary physical Fock state reads

$$|ph\rangle = \sum_{n=0}^{\infty} \Phi_n(\hat{a}^+)^n |0\rangle,$$

(2.3)

which yields the physical spectrum $E_{ph}^n = 2\omega(n+1/2) = \omega_{ph}(n+1/2)$, $n = 0, 1, \ldots$, of the Hamiltonian $H = \omega(\hat{a}^+ \hat{a}^+ + 1)$. The distance between energy levels is doubled. This result can be also recovered by quantizing the conic PS (2.2) in the framework of the WKB method [10], [12].

A further consideration of the $PS_{ph}$ geometry in mechanical gauge systems can be found in a review [12] or in a monograph [13], an application to minisuperspace cosmology [14] is given in [15], a relation between the $PS_{ph}$ geometry and quantum Green function in gauge theories are discussed in [5] (see also for a review [16]).

3 Phase space structure in Yang-Mills theory on a cylinder

The $PS_{ph}$ definition (2.1) holds for gauge field theories, i.e. for systems with an infinite number of degrees of freedom. PS in a field theory is a functional space, which gives rise to some technical difficulties when applying (2.1). Nevertheless, for some particular gauge field theories, the quotient (2.1) can be explicitly calculated. The 2D Yang-Mills theory on a cylindrical spacetime (space is compactified to a circle $S^1$) [2], [3] exhibits a nice example of that kind. We shall establish the $PS_{ph}$ structure of this theory in the case of an arbitrary compact semisimple gauge group.

The Lagrangian reads

$$L = -\frac{1}{4} \int_0^{2\pi l} dx (F_{\mu\nu}, F_{\mu\nu}) \equiv -\frac{1}{4} \langle F_{\mu\nu}, F_{\mu\nu} \rangle,$$

(3.1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$, $g$ a coupling constant, $\mu, \nu = 0, 1$; the Yang-Mills potentials $A_\mu$ being elements of a Lie algebra $X$ are periodic functions of a spatial coordinate, $A_\mu(t, x + 2\pi l) = A_\mu(t, x)$, i.e. $l$ is the space radius; the brackets $(,)$ in the integrand (3.1) stand for the invariant inner product in $X$. We assume it to be the Killing form (see Appendix A) and suppose also that the orthonormal basis (A.5) is introduced in $X$. The spatial coordinate $x \in S^1$, the Lorentz and Lie algebra suffices of $A_\mu$ label degrees of freedom in the theory.

To go over the Hamiltonian formalism, we determine the canonical momenta $E_\mu = \delta L / \delta \dot{A}_\mu = F_{0\mu}$, the overdot denotes the time derivative. The momentum conjugated to
$A_0$ vanishes, $E_0 = 0$, forming the primary constraints. The canonical Hamiltonian has the form $H = \langle E_\mu, A_\mu \rangle - L = \langle E_1, E_1 \rangle/2 - \langle A_0, \sigma \rangle$ where $\sigma = \nabla(A_1)E_1$ with $\nabla(A_1) = \partial_1 - ig[A_1,]$ being the covariant derivative in the adjoint representation. The primary constraints must be satisfied during the time evolution [1]. This yields the secondary constraints

$$E_0 = \{ E_0, H \} = \partial_1 E_1 - ig[A_1, E_1] = \sigma = 0 , \quad (3.2)$$

where the standard symplectic structure $\{ A^a_\mu(x), E^b_\nu(y) \} = \delta^{ab}\delta_{\mu\nu}\delta(x-y)$, $x, y \in S^1$, has been introduced, the suffices $a, b$ enumerate Lie algebra components. Since $\{ \sigma^a(x), \sigma^b(y) \} = if^{abc}(x-y)\sigma^c(x)$, $f^{abc}$ are structure constants of $X$, and $\{ \sigma_a, H \} = -f^{abc}A^b_0\sigma^c$, we conclude that there is no more constraints in the theory, and all constraints are of the first class [1].

The primary and secondary (first-class) constraints are treated as independent generators of gauge transformations (see Sec.2). It is readily to see that the primary constraints $E_0 = 0$ generate shifts of $A^a_0 \ (\delta A^a_0(x) = \{ A^a_0, \langle \omega_0, E_0 \rangle \} = \omega^a_0(x))$ and leave the phase space variables $E^a_0$ and $A^a_0$ untouched. Therefore the hyperplane $E_0 = 0$ being the constraint surface in the total PS is a gauge orbit. In accordance with (2.1) it contributes just a point $E_0 = A_0 = 0$ to $PS_{ph}$. Thus, we can ignore those pure unphysical degrees of freedom and concentrate our attention just on the remaining variables.

The constraints (3.2) generate following gauge transformations

$$E_1 \rightarrow \Omega E_1 \Omega^{-1} = E^\Omega_1 , \quad A_1 \rightarrow \Omega A_1 \Omega^{-1} + ig(\Omega \partial \Omega^{-1} = A^\Omega_1 ; \quad (3.3)$$

here and below $\partial_1 \equiv \partial$, $\Omega = \Omega(x)$ takes its values in a semisimple compact group $G \ (X$ is its Lie algebra). The gauge transformed variables $E^\Omega_1$ and $A^\Omega_1$ must be also periodic functions of $x$. This results in the periodicity of $\Omega$ modulo the center $Z_G$ of $G$

$$\Omega(x + 2\pi l) = z\Omega(x) , \quad z \in Z_G . \quad (3.4)$$

Indeed, by definition $z$ commutes with any element of $X$ and, therefore, $E^\Omega_1$ and $A^\Omega_1$ are invariant under the shift $x \rightarrow x + 2\pi l$.

The relation (3.4) is called a twisted boundary condition [17]. As has been pointed out in [17], twisted gauge transformations (i.e. satisfying (3.4) with $z \neq e$, $e$ a group unit) form distinct homotopy classes. Therefore they cannot be continuously deformed towards the identity. On the other hand, gauge transformations generated by the constraints (3.2) are homotopically trivial because they are built up by iterating the infinitesimal transformations [15] $\delta E_1 = \{ E_1, \langle \omega, \sigma \rangle \} = ig[E_1, \omega]$ and $\delta A_1 = \{ A_1, \langle \omega, \sigma \rangle \} = -\nabla(A)\omega$ with $\omega$ being an $X$-valued periodic function of $x$. Thus, we are led to the following conclusion. When determining $PS_{ph}$ by means of (2.1), one should restrict oneself by periodic (i.e. homotopically trivial) gauge transformations [3]. Such transformations determine a mapping $S^1 \rightarrow G$. Yet we shall see that quantum states annihilated by the operators of the constraints (i.e. the physical states [1]) are not invariant under twisted gauge transformations (see Appendix D) which confirms that twisted gauge transformations belong to a homotopically non-trivial class [18].
Consider a periodic function $f(x)$ taking its values in $X$. It is expanded into a Fourier series
\[ f(x) = f_0 + \sum_{n=1}^{\infty} \left( f_{s,n} \sin \frac{n\pi x}{l} + f_{c,n} \cos \frac{n\pi x}{l} \right). \] (3.5)

We denote a space of functions (3.5) $\mathcal{F}$ and its finite dimensional subspace formed by constant functions $\mathcal{F}_0$ so that $A_1 = A_{10} + \hat{A}_1$, where $A_{10} \in \mathcal{F}_0$ and $\hat{A}_1 \in \mathcal{F} \cap \mathcal{F}_0$.

Any configuration $A_1$ belongs to the same gauge orbit as its homogeneous component $A_1 = A_{10}$ does. Set $\Omega^{-1} = P \exp i g \int_x^x \omega dy$, $\omega \in \mathcal{F} \cap \mathcal{F}_0$, in (3.3) and require $\partial_1 A_1^\Omega = 0$ (the Coulomb gauge $A_1^\Omega \in \mathcal{F}_0$). Using the relations $\partial \Omega^{-1} = ig \omega \Omega^{-1}$ and $\partial \Omega = -ig \Omega \omega$ we derive the equation for $\omega$
\[ \nabla(A_1) \omega = \partial \omega - ig[A_1, \omega] = \partial \hat{A}_1. \] (3.6)

The operator $\nabla(A_1)$ is invertible for $\omega \in \mathcal{F} \cap \mathcal{F}_0$. Indeed, a general solution to the homogeneous equation $\nabla(A_1) \omega = 0$ can be written as $\omega^{\text{hom}}(x) = W \omega(x_0) W^{-1}$, $W = P \exp ig \int_{x_0}^x dy A_1$. For $\omega^{\text{hom}} \in \mathcal{F} \cap \mathcal{F}_0$ there always exists a point $x_0 \in (0, 2\pi l)$ such that $\omega^{\text{hom}}(x_0) = 0$ because $\int_0^{2\pi l} dx \omega^{\text{hom}} = 0$, and, hence, $\omega^{\text{hom}}(x) \equiv 0$ since $W \neq 0$. So, any configuration $A_1 \in \mathcal{F}$ can be reduced towards a spatially homogeneous configuration $A_{10}$ by means of a gauge transformation.

Now we shall prove that the gauge reduction of $A_1$ to $A_{10}$ leads to a simultaneous gauge reduction of the momentum $E_1$ to $E_{10} \in \mathcal{F}_0$ on the constraint surface (3.2). Substituting (3.3) into (3.2) and assuming $A_1^\Omega \equiv A_{10}$ we get $\partial E_1^\Omega - ig[A_{10}, E_1^\Omega] = 0$. Putting $E_1^\Omega = E_{10} + \tilde{E}_1^\Omega$, $\tilde{E}_1^\Omega \in \mathcal{F} \cap \mathcal{F}_0$ we obtain two equations
\[ \sigma_0 \equiv [A_{10}, E_{10}] = 0, \] (3.7)
\[ \partial \tilde{E}_1^\Omega - ig[A_{10}, \tilde{E}_1^\Omega] = 0. \] (3.8)

Equation (3.8) has only trivial solution $\tilde{E}_1^\Omega = 0$ (see the above discussion of the solution $\omega^{\text{hom}} = 0$ to Eq. (3.6)).

Thus, we are led to a system with $N = \dim X$ degrees of freedom and the constraint (3.7) which generates homogeneous gauge transformations of the phase-space variables $A_{10}$ and $E_{10}$ ($\partial \Omega \equiv 0$ in (3.3)). This mechanical system has been studied in [14]. The system is shown to have $r = \text{rank } X$ physical degrees of freedom which can be described by Cartan subalgebra components of $A_{10}$ and $E_{10}$.

Any element of $X$ can be represented in the form $[a] A_{10} = \Omega_A a \Omega_A^{-1}$, $a$ an element of the Cartan subalgebra $H$ (see Appendix A), $\Omega_A \in G$. Therefore configurations $A_{10}$ and $a$ belong to the same gauge orbit. Moreover, a spatially homogeneous gauge transformation with $\Omega = \Omega_A^{-1}$ brings the momentum $E_{10}$ on the constraint surface (3.7) to the Cartan subalgebra. Indeed, from (3.7) we derive $[a, \Omega_A^{-1} E_{10} \Omega_A] = 0$ and conclude that $p_a = \Omega_A^{-1} E_{10} \Omega_A \in H$ by the definition of $H$. The element $a$ has a stationary group being the Cartan subgroup of $G$. This means that not all of the constraints (3.7) are independent. Namely, there are just $N - r$, $r = \dim H$, independent constraints amongst (3.7). The continuous gauge arbitrariness is exhausted in the theory.
One would assume $PS_{ph}$ to be $\mathbb{R}^{2r}$ (meaning $H \sim \mathbb{R}^r$) but this is wrong. There remain discrete gauge transformations which cannot decrease a number of physical degrees of freedom, while they do reduce their PS.

As has been pointed out in [11], the mechanical system with the constraint (3.7) possesses a non-trivial $PS_{ph}$ geometry due to the Weyl group $W$ [19] being a subgroup of the gauge group and acting in the reduced PS spanned by $a$ and $p_a$. Any element of $W$ is a composition of reflections $\hat{s}_\omega$ in hyperplanes orthogonal to simple roots $\omega$, $(a, \omega) = 0$ (see Appendix A),

$$\hat{s}_\omega a = \Omega_\omega a \Omega_\omega^{-1} = a - \frac{2(a, \omega)}{(\omega, \omega)} \omega, \quad \Omega_\omega \in G.$$  \hfill (3.9)

The Weyl group preserves the root system of $X$ [19], p.456. The group elements $\Omega_\omega$ can be easily found in the orthogonal basis (A.5),

$$\Omega_\omega = \Omega_s = \exp \frac{i\pi}{(\omega, \omega)^{1/2}} s_\omega, \quad \text{or} \quad \Omega_\omega = \Omega_c = \exp \frac{i\pi}{(\omega, \omega)^{1/2}} c_\omega.$$  \hfill (3.10)

Using the commutation relations

$$[s_\omega, c_\omega] = i\omega, \quad [\omega, s_\omega] = i(\omega, \omega)c_\omega, \quad [\omega, c_\omega] = -i(\omega, \omega)s_\omega$$  \hfill (3.11)

one can be convinced that the group elements (3.10) satisfy (3.9). The existence of two different representations (3.10) of the reflection operators (3.9) plays the crucial role in gauge dynamics with fermions [20].

Let $G = SU(2)$. The representation of the Cartan-Weyl basis via the Pauli matrices is given in Appendix A. Using it we get $\Omega_s = i\tau_1$ and $\Omega_c = i\tau_2$, $\tau_3$ is the only basis element of $H_{su(2)}$. Then $(i\tau_1) \tau_3 (-i\tau_1) = -\tau_3 = (i\tau_2) \tau_3 (-i\tau_2) = \hat{s}_\omega \tau_3$.

Thus, we conclude that the points $\hat{sp}_a, \hat{sa}, \hat{s} \in \mathbb{R}^{2r}$ in $2D$ are gauge equivalent, i.e. they belong to the same gauge orbit and, therefore, should be identified in accordance with (2.1). The Weyl group simply transitively acts on the set of Weyl chambers [19], p.458. Any element of $H$ can be obtained from an element of the positive Weyl chamber $K^+$ ($a \in K^+$ if $(a, \omega) > 0$, $\omega$ ranges all simple roots) by a certain transformation from $W$. In other words, the chamber $K^+$ is the quotient $H/W$.

In contrast with the mechanical model studied in [11], the Weyl group does not cover the whole admissible discrete gauge arbitrariness in the $2D$ Yang-Mills theory. Put $E_1 = p_a$ and $A_1 = a$ in (3.3) and consider such gauge transformations $\Omega$ which do not transfer $p_a$ and $a$ out of the Cartan subalgebra and preserve the conditions $\partial a = \partial p_a = 0$. If $\Omega$ does not belong to the Cartan subgroup $G_H$ and $\partial \Omega = 0$, then it must be an element of $W$ as we have seen above. If $\Omega \in G_H$ and $\partial_1 \Omega = 0$, then it is an element of the stationary group of $a$. Let now $\Omega \in G_H$ and depend on $x$ such that $\Omega \partial \Omega^{-1}$ is independent of $x$. Obviously, $\Omega = \Omega_\eta = \exp(i\eta/\ell)$ where $\eta \in H$. This transformation transfers $a$ to $a + a_0 \eta$, $a_0 = (g\ell)^{-1}$, and leaves $p_a$ untouched. The group element $\Omega_\eta$ has to obey the boundary condition (3.4) with $z = e$ as has been argued above. This yields the equation for $\eta$

$$\exp(2\pi i \eta) = e.$$  \hfill (3.12)
The set of elements $\eta$ obeying (3.12) is called the unit lattice in the Cartan subalgebra [19], p.305.

Consider the diagram $D(X)$ being a union of a finite number of families of equispaced hyperplanes in $H$ determined by $(\alpha, a) \in a_0 \mathbb{Z}$, $\alpha$ ranges over the root system. Consider then a group $T_e$ of translations in $H$, $a \to a + a_0 \eta$, where $\eta$ belongs to the unit lattice. The group $T_e$ leaves the diagram $D(X)$ invariant [19], p.305. The diagram $D(X)$ is also invariant with respect to Weyl group transformations. Since $W$ is generated by the reflections (3.9), it is sufficient to prove the invariance of $D(X)$ under them. We have $(\alpha, \hat{s}_w a) = a_0 n_\omega$ where $n_\omega = n - 2k_\omega(\omega, \alpha)/(\omega, \omega)$ is an integer (see Appendix A), and $(a, \omega) = k_\omega a_0$, $k_\omega \in \mathbb{Z}$ because $a \in D(X)$. So, $\hat{s}_w D(X) = D(X)$.

Consider the complement $H \supset D(X)$. It consists of equal polyhedrons whose walls form the diagram $D(X)$. Each polyhedron is called a cell. A cell inside of the positive Weyl chamber $K^+$ such that its closure contains the origin is called the Weyl cell $K^+_W$. For instance, $D(su(2))$ consists of points $n a_0 \omega/(\omega, \omega)$, $n \in \mathbb{Z}$ with $\omega$ being the only positive root of $su(2)$, $(\omega, \omega) = 1/2$ ($\omega = \tau_3/4$, see Appendix A). A cell of $H_{su(2)} \supset D(su(2))$ is an open interval between two neighbor points of $D(su(2))$. Since $K^+ \sim \mathbb{R}^+$, we conclude that $a \in K^+_W$ if $a_3 \in (0, \sqrt{2} a_0)$ where $a = \sqrt{2} a_3 \omega$, $(a, a) = a_3^2$ in the orthonormal basis. The translations $a \to a + 2 n a_0 \omega/(\omega, \omega)$, $n \in \mathbb{Z}$ form the group $T_e$, and $W = \mathbb{Z}_2$, $\hat{s}_w a = -a$. So, $D(su(2))$ is invariant under translations from $T_e$ and the reflection from $W$.

For $X = su(3)$ we have three positive roots, $\omega_1$, $\omega_2$ and $\omega_1 + \omega_2$ which have the same norms. The angle between any two neighbor roots is equal to $\pi/3$. The diagram $D(su(3))$ consists of three families of equispaced straight lines $(\omega_{1,2,12}, a) = a_n(1,1,2)$, $n \in \mathbb{Z}$ on the plane $H_{su(3)} \sim \mathbb{R}^2$. The lines are orthogonal to the roots $\omega_{1,2,12}$, respectively. The complement $H_{su(3)} \supset D(su(3))$ is a set of equal-side triangles covering the plane $H_{su(3)}$. The Weyl cell $K^+_W$ is the triangle bounded by lines $(\omega_{1,2}, a) = 0$ (being $\partial K^+$) and $(\omega_{12}, a) = a_0$. The group $T_e$ is generated by integral translations through the vectors $2 a_0 \alpha/(\alpha, \alpha)$, $\alpha$ ranges $\omega_{1,2,12}$, $(\alpha, \alpha) = 1/3$ (see Appendix A).

Since all residual discrete gauge arbitrariness is exhausted by $T_e$ and $W$, we conclude that there are no gauge equivalent points in $K^+_W$, i.e. $H \supset CS_{ph} \supset K^+_W$ where $CS_{ph}$ denotes the physical configuration space. It can be defined analogously to (2.1), $CS_{ph} = CS/G$ if gauge transformations generated by constraints do not mix generalized coordinates spanning the total configuration space $CS$ and generalized momenta. In the case of the 2D Yang-Mills theory, $CP_{ph} = [A_1]/G$ where $G$ is composed of the homotopically trivial transformations (3.3). Actually, we shall prove the equality $CS_{ph} = K^+_W$.

Let $W_A$ denote the group of linear transformations of $H$ generated by the reflections in all the hyperplanes in the diagram $D(X)$. This group is called the affine Weyl group [19], p.314. $W_A$ preserves $D(X)$ and, hence,

$$K^+_W = H/W_A,$$  \hspace{1cm} (3.13)

i.e. the Weyl cell is a quotient of the Cartan subalgebra by the affine Weyl group. Consider a group $T_r$ of translations $a \to a + 2a_0 \sum_{\alpha > 0} n_\alpha \alpha/(\alpha, \alpha)$, $n_\alpha \in \mathbb{Z}$. Then $W_A$ is semidirect product of $T_r$ and $W$ [19], p.315. The fact that $CS_{ph}$ coincides with $K^+_W$ follows from the
equality \[19\], p.317,

\[
\exp \frac{4\pi i \alpha}{(\alpha, \alpha)} = e ,
\]

i.e. \( T_e \supseteq T_r \).

Thus, the residual discrete gauge transformations form the affine Weyl group. The physical phase space is the quotient \[9\]

\[
PS_{ph} = \mathbb{R}^{2r}/W_A ,
\]

where the action of \( W_A \) on \( H \otimes H \sim \mathbb{R}^{2r} \) is determined by all possible compositions of the following transformations

\[
\begin{align*}
\hat{s}_{\alpha,n} p_a &= \hat{s}_a p_a = p_a - \frac{2(\alpha, p_a)}{\alpha, \alpha} \alpha , \\
\hat{s}_{\alpha,n} a &= \hat{s}_a a + \frac{2n_\alpha a_0}{(\alpha, \alpha)} \alpha ,
\end{align*}
\]

where the element \( \hat{s}_{\alpha,n} \in W_A \) acts on \( a \) as a reflection in the hyperplane \((\alpha, a) = n_\alpha a_0, \; n_\alpha \in \mathbb{Z}\), and \( \alpha \) is any root.

To illustrate the formula (3.15), let us first construct \( PS_{ph} \) for the simplest case \( X = su(2) \). We have \( r = 1, \; W = \mathbb{Z}_2, \; (\omega, \omega) = 1/2 \). The group \( T_r = T_e \) acts on the phase plane \( \mathbb{R}^2 \) spanned by the coordinates \( p_3, a_3 \) (we have introduced the orthonormal basis in \( H_{su(2)} \); see the discussion of \( D(su(2)) \) above) as \( p_3, a_3 \rightarrow p_3, a_3 + 2\sqrt{2}n a_0 \). So, \( \mathbb{R}^2/T_r \) is a cylinder or the strip \( p_3 \in \mathbb{R}, \; a_3 \in (-\sqrt{2}a_0, \sqrt{2}a_0) \) with the identified boundary lines \( a_3 = \pm \sqrt{2}a_0 \). On this strip one should stick together the points \( p_3, a_3 \) and \( -p_3, -a_3 \) connected by the reflection from the Weyl group. This converts the cylinder into a half-cylinder ended by two conic horns at the points \( p_3 = 0, a_3 = 0, \sqrt{2}a_0 \). In neighborhoods of these points \( PS_{ph} \) looks locally like \( cone(\pi) \) (cf. (2.2)) because \( W_A \) acts as the \( \mathbb{Z}_2 \)-reflections (3.16) and (3.17) with \( \alpha = \omega \) and \( n = 0,1 \) near \( a_3 = 0, \sqrt{2}a_0 \), respectively.

For groups of rank 2, all conic (singular) points of \( PS_{ph} \) are concentrated on a triangle being the boundary \( \partial K^+ \) of the Weyl cell (if \( X = su(3) \), \( \partial K^+ \) is an equal-side triangle with side length \( \sqrt{3}a_0 \) in the orthonormal basis defined in Appendix A). Let us introduce local symplectic coordinates \( p^\perp_3, a^\perp \) and \( p^\parallel_3, a^\parallel \) in a neighborhood of a point of \( \partial K^+_W \) (except the triangle vertices) which vary along lines perpendicular and parallel to \( \partial K^+_W \), respectively. The \( W_A \)-reflection in the wall of \( \partial K^+_W \) going through this neighborhood leaves \( p^\parallel_3, a^\parallel \) untouched, while it changes the sign of the other symplectic pair, \( p^\perp_3, a^\perp \rightarrow -p^\perp_3, -a^\perp \). Therefore \( PS_{ph} \) locally coincides with \( \mathbb{R}^2 \otimes cone(\pi) \). At the triangle vertices, two conic singularities going along two triangle edges stick together. If those edges are perpendicular, \( PS_{ph} \) is locally \( cone(\pi) \otimes cone(\pi) \). If not, \( PS_{ph} \) is a \( 4D-\)hypercone. The point of the \( 4D-\)hypercone is “sharper” than the point of \( cone(\pi) \otimes cone(\pi) \), meaning that the \( 4D-\)hypercone can be always put inside of \( cone(\pi) \otimes cone(\pi) \). Obviously, a less angle between the triangle edges corresponds to a “sharper” hypercone.

A generalization of this pattern of singular points in \( PS_{ph} \) to gauge groups of an arbitrary rank is trivial. The Weyl cell is an \( rD \)-polyhedron. \( PS_{ph} \) at the polyhedron
vertices has the most singular local $2rD - \text{hypercone}$ structure. On the polyhedron edges it is locally viewed as $\mathbb{R}^2 \otimes 2(r - 1)D - \text{hypercone}$. Then on the polyhedron faces, being polygons, the local $PS_{ph}$ structure is $\mathbb{R}^4 \otimes 2(r - 2)D - \text{hypercone}$, etc.

4 Artifacts of gauge fixing and the Gribov problem

The definition (2.1) of $PS_{ph}$ is independent of choosing symplectic coordinates and explicitly gauge-invariant. However, upon a dynamical description (quantum or classical) of a constrained system, we need to introduce coordinates on $PS_{ph}$, which means fixing a gauge or choosing a $PS_{ph}$ parametrization. This parametrization is usually motivated by physical reasons. If we deal with gauge fields, one may describe physical degrees of freedom by transverse components $A^\perp$ of the vector potential and their canonically conjugated momenta $E^\perp$, i.e. the Coulomb gauge $\partial A = 0$ is imposed to remove unphysical degrees of freedom. This choice comes from by our experience in QED where two independent polarizations of a photon are naturally described by the transverse vector-potential. Apparently, for QED $PS_{ph} \sim [A^\perp] \otimes [E^\perp]$ where $[A^\perp]$ implies the functional space of all configurations $A^\perp$.

Transverse fields cannot serve as good variables parametrizing $PS_{ph}$ in the non-Abelian case because there are gauge-equivalent configurations in $[A^\perp]$, Gribov’s copies [6]. Moreover, this gauge fixing ambiguity always arises and has a geometric nature [7] related to topological properties of the gauge orbit space and cannot be avoided if gauge potentials are assumed to vanish at the spatial infinity. So, one should develop a formalism taking into account a true geometric structure of $PS_{ph}$ in a quantum dynamical description [5].

In the 2D Yang-Mills theory considered in Sec.3, spatially homogeneous Cartan subalgebra components of the vector potential $A = a$ and field strength $E = p_a$ can be regarded as symplectic coordinates on $PS_{ph}$. In fact, this implies the Coulomb gauge condition $\partial A = 0$ which is not complete in this case because there are some unphysical degrees of freedom left. They are removed by imposing the additional gauge condition $(e_\pm A) = 0$, i.e. $A \in H$. Gribov’s copies of a configuration $A = a \in [a] = H \sim \mathbb{R}^r$ are obtained by applying elements of the affine Weyl group $W_A$ to $a$. The modular domain (see Sec.1) obviously coincides with the Weyl cell. If $a$ belongs to the modular domain boundary $\partial K^+_W$, the residual gauge arbitrariness contains even continuous transformations. However, such configurations form a set of zero measure in $[a] = H$ and play no role in quantum dynamics.

Gribov’s copies themselves do not have much physical meaning because they strongly depend on a concrete choice of a gauge fixing condition that is rather arbitrary. To illustrate this, let us return back to a simple mechanical gauge model of Sec.2. The unitary gauge $x_2 = 0$ is most convenient to describe the physical configuration space being a space of concentric circles. Suppose for a moment that we do not know the structure of the gauge orbit space. Then all gauge conditions have to be treated on equal footing.

Any gauge condition $F(x) = 0$ determines a curve on a plane $\mathbb{R}^2$ over which a physical
variable ranges. The curve $F(x) = 0$ must cross each orbit at least once because a gauge choice is nothing but a parametrization of the gauge orbit space. In the model under consideration, this yields that the curve has to go through the origin to infinity. Let us introduce a parametrization of the gauge condition curve

$$x = x(u) = f(u), \quad u \in \mathbb{R},$$

where $f(0) = 0$ and $|f| \to \infty$ as $u \to \infty$ so that $u$ serves as a physical variable. If $f_2 = 0$ and $f_1 = u$, we recover the unitary gauge considered above.

Now we can easily see that an inappropriate choice of $f$ might make a dynamical description very complicated. Let points $x$ and $x_s$ belong to the same gauge orbit, then $x_s = \Omega_s x$, $\Omega_s \in SO(2)$. Suppose the curve (4.1) intersects a gauge orbit at points $x = f(u)$ and $x_s = f(u_s)$. We have also $u_s = u_s(u)$ because $f(u_s) = \Omega_s f(u)$. If the structure of gauge orbits is supposed to be unknown, the function $u_s(u)$ can be found by solving the following equations

$$F(\Omega_s f) = 0,$$

$$\Omega_s(u)f(u) = f(u_s(u)).$$

Solutions of (4.2) (the trivial solution, $\Omega_s = 1$, always exists by the definition of $f$) form a set $S_F$ of discrete residual gauge transformations. Eq. (4.3) determines an induced action of $S_F$ on the physical variable $u$ (a representation of $S_F$ in the space of $u \in \mathbb{R}$). The set $S_F$ is not a group because for an arbitrary $F$ a composition $\Omega_s \Omega_s'$ of two elements from $S_F$ might not belong to $S_F$ (if it does not satisfy (4.2)), while for each $\Omega_s$ there exists $\Omega_s^{-1}$ such that $\Omega_s^{-1} \Omega_s = 1$.

Suppose we have two different solutions $\Omega_s$ and $\Omega_s'$ to the system (4.2)-(4.3). Then, in general case, the composition $\Omega_s \Omega_s'$ is not a solution to (4.2), i.e. $F(\Omega_s \Omega_s' f(u)) = F(\Omega_s f(u_s')) \neq 0$ because $u_s' \neq u$ whereas $F(\Omega_s f(u)) = 0$. The functions $u_s(u)$ determined by (4.3) do not have a unique analytical continuation to the whole covering space $u \in \mathbb{R}$, otherwise the composition $u_s \circ u_s' = u_{ss'}(u)$ would be uniquely defined and, hence, one could always find an element $\Omega_{ss'} = \Omega_s \Omega_s'$ being a solution to (4.2), which is not the case. Moreover, a number of elements in $S_F$ can depend on $u$.

To exhibit these artifacts of gauge fixing, we consider a concrete choice of $f$. A general analysis can be found in [10], [13]. Set $f_1 = -u_0$, $f_2 = -\gamma(2u_0 + u)$ for $u < -u_0$ and $f_1 = u$, $f_2 = \gamma u$ for $u > -u_0$ where $\gamma$ and $u_0$ are positive constants. The curve (4.1) touches circles (gauge orbits) of radii $r = u_0$ and $r = u_0\gamma_0$, $\gamma_0 = \sqrt{1 + \gamma^2}$. It intersects twice all circles with radii $r < u_0$ and $r > u_0\gamma_0$, whereas any circle with a radius from the interval $r \in (u_0, u_0\gamma_0)$ has four common points with the gauge condition curve. Therefore, $S_F$ has one nontrivial element for $u \in \mathbb{R}_1 \cup \mathbb{R}_3$, $\mathbb{R}_1 = (-u_0/\gamma_0, u_0/\gamma_0)$, $\mathbb{R}_3 = (-\infty, -3u_0) \cup (u_0, \infty)$ and three nontrivial elements for $u \in \mathbb{R}_2 = (-3u_0, -u_0/\gamma_0) \cup (u_0/\gamma_0, u_0)$. Since points $f(u_s)$ and $f(u)$ belong to the same circle (gauge orbit), the functions $u_s$ have to obey the following equation

$$f^2(u_s) = f^2(u).$$

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Denoting $S_F = S_\alpha$ for $u \in \mathbb{R}_\alpha$, $\alpha = 1, 2, 3$, we have $S_1 = \mathbb{Z}_2$, $u_s(u) = -u$; $S_2$ is determined by the following mappings of the interval $K_2 = (u_0/\gamma_0, u_0)$

\[
\begin{align*}
  u_{s_1}(u) &= -u , & u_{s_1} : K_2 \to (-u_0, -u_0/\gamma_0) ; \\
  u_{s_2}(u) &= -2u_0 + \gamma_0(u^2 - u_0^2/\gamma_0^2)^{1/2}/\gamma , & u_{s_2} : K_2 \to (-u_0, -2u_0) ; \\
  u_{s_3}(u) &= -2u_0 - \gamma_0(u^2 - u_0^2/\gamma_0^2)^{1/2}/\gamma , & u_{s_3} : K_2 \to (-2u_0, -3u_0) ;
\end{align*}
\]

and for $S_3$ we get

\[
  u_s(u) = -2u_0 - \gamma_0(u^2 - u_0^2/\gamma_0^2)^{1/2}/\gamma : (u_0, \infty) \to (-3u_0, -\infty) .
\]

The functions (4.6-7) do not have a unique analytical continuation to the whole domain $\mathbb{R}_2$ and, hence, their composition is ill-defined. The mappings (4.5-7) do not form a group. Since they realize a representation of $S_\alpha$, $S_\alpha$ is not a group.

The physical configuration space is, obviously, isomorphic to $K = \cup K_\alpha$, $K_\alpha = \mathbb{R}_\alpha/S_\alpha$, i.e. $K_\alpha$ is a fundamental domain of $\mathbb{R}_\alpha$ with respect to the action of $S_F = S_\alpha$ in $\mathbb{R}_\alpha$, $\mathbb{R}_\alpha = \cup s K_\alpha$. $s$ ranges over $S_\alpha$. Upon solving (4.4) (or (4.2-3)) we have to choose a particular interval as the fundamental domain. We have put $K_2 = (u_0/\gamma_0, u_0)$ in (4.5-7).

Another choice would lead to another form of the functions $u_s$ (to another representation of $S_F$ in $\mathbb{R}_2$). Setting, for example, $K_2 = (-2u_0, -u_0)$ we obtain from (4.4)

\[
\begin{align*}
  u_{s_1}(u) &= -4u_0 - u , & u_{s_1} : K_2 \to (-3u_0, -2u_0) ; \\
  u_{s_2}(u) &= -(u_0^2 + \gamma^2(2u_0 + u)^2)^{1/2}/\gamma_0 , & u_{s_2} : K_2 \to (-u_0, -u_0/\gamma_0) ; \\
  u_{s_3}(u) &= (u_0^2 + \gamma^2(2u_0 + u)^2)^{1/2}/\gamma_0 , & u_{s_3} : K_2 \to (u_0/\gamma_0, u_0) .
\end{align*}
\]

To find group elements $\Omega_s(u)$ corresponding to $u_s(u)$, one should solve Eq.(4.3). Setting $\Omega_s = \exp(-T \omega_s)$ and substituting (4.5-7) into (4.3), we find

\[
\begin{align*}
  \omega_{s_1}(u) &= \pi ; \\
  \omega_{s_2}(u) &= \frac{3\pi}{2} - \sin^{-1}\left(\frac{u_0}{\gamma_0 u}\right) - \tan^{-1}\gamma ; \\
  \omega_{s_3}(u) &= \frac{\pi}{2} + \sin^{-1}\left(\frac{u_0}{\gamma_0 u}\right) - \tan^{-1}\gamma ,
\end{align*}
\]

where $u \in K_2 = (u_0/\gamma_0, u_0)$. Elements of $S_{1,3}$ are obtained analogously. It is readily seen that $\Omega_{s_1} \Omega_{s_2} \neq \Omega_{s_3}$, etc., i.e. the elements $\Omega_s$ do not form a group. An alternative choice of $K_2$ results in a modification of the functions (4.12-14).

One would assume that all complications of the $C_{S_{ph}}$ structure, $C_{S_{ph}} \sim K$, found above are caused by using non-invariant variables for describing physical degrees of freedom. Indeed, we have fixed a “crazy” gauge $F(x) = 0$ and gained a complicated set of residual gauge transformations (Gribov’s problem). However, one can easily turn the variable $u$ into the gauge-invariant one by means of a special canonical transformation. The set $S_F$ will appear again due to topological properties of such a canonical transformation.
rather than due to gauge fixing ambiguities. It will determine the phase space structure of gauge-invariant canonical variables.

Consider the following canonical transformation of $x$ and $p$

$$
\begin{align*}
    x &= \exp(T\theta)f(u) ; \\
    p_\theta &= p_Tx = \sigma , \quad p_u = \frac{1}{2}(p, x) \frac{d}{du} \ln x^2 , \quad (4.15)
\end{align*}
$$

where $\{\theta, p_\theta\} = \{u, p_u\} = 1$ (if $x_i, p_j = \delta_{ij}$) all other Poisson brackets vanish. Equality (4.15) represents a generalization of the polar coordinates ($f_1 = u$, $f_2 = 0$). Since $p_\theta$ coincides with the constraint, we conclude that $\theta$ is the unphysical variable in the model; $\sigma = p_\theta$ generates its shifts, whereas $\{\sigma, u\} = \{\sigma, p_u\} = 0$ and, hence, $u$ and $p_u$ are gauge-invariant. Using the decomposition

$$
    p = p_\theta \frac{T x}{x^2} + p_u \frac{x}{\mu(u)} , \quad (4.17)
$$

where $\mu(u) = (df/du, f)$, and the constraints $p_\theta = 0$ we derive the physical Hamiltonian

$$
    H_{ph} = \left( \frac{1}{2} p^2 + V(x^2) \right) |_{p_\theta = 0} = \frac{1}{2} \frac{f_2^2(u)}{\mu^2(u)} p_u^2 + V(f^2(u)) . \quad (4.18)
$$

Hamiltonian equations of motion generated by (4.18) provide a gauge-invariant dynamical description.

Let turn now directly to seeking the hidden set of transformations $S_F$. As we have pointed out above, dynamics is very sensitive to a phase space structure. So, to complete our gauge-invariant description, one should establish a structure of the phase space spanned by $u$ and $p_u$. Let us forget for a moment about the gauge symmetry and the constraint $p_\theta = 0$ induced by it. Since (4.15) is a change of variables there should be a one-to-one correspondence between points $x \in \mathbb{R}^2$ and $\theta, u$. The latter yields a restriction on admissible values of $\theta$ and $u$, $\theta \in [0, 2\pi)$ and $u \in K \subset \mathbb{R}$. To see this, we allow the variables $\theta$ and $u$ to have their values on the whole real axis and consider transformations $\theta, u \to \theta + \theta_s = \hat{s}\theta, \quad u_s = \hat{s}u$ such that

$$
    x(\hat{s}\theta, \hat{s}u) = x(\theta, u) , \quad (4.19)
$$

i.e. we assume $f(u)$ to be a real analytical function on $\mathbb{R}$. Points $\hat{s}\theta, \hat{s}u$ of the $(u, \theta)$-plane are mapped to one point on the $x$-plane. To provide the mapping (4.15) to be one-to-one (otherwise it is not a change of variables), one should restrict values of $\theta$ and $u$ by the modular domain $\hat{K} = \mathbb{R}^2/\hat{S}$ where transformations from $\hat{S}$ are defined by (4.19). The set $\hat{S}$ is decomposed into the product $T_e \times S_F$ where elements of $T_e$ are translations of $\theta$ through the group manifold period,

$$
    T_e : \quad \theta \to \theta + 2\pi n , \quad u \to u , \quad n \in \mathbb{Z} , \quad (4.20)
$$

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and $S_F$ coincides with the set of residual gauge transformations described above. Let
\[ \Omega_s = \exp(-T\omega_s(u)), \]
then it is easily to be convinced that transformations
\[ S_F : \quad \theta \to \theta + \omega_s(u), \quad u \to u_s(u) \]  
(4.21)
satisfy (4.19) by $\theta_s = \omega_s$. So, $\hat{K} \sim [0, 2\pi] \cup \hat{K}$. In the case of the polar coordinates, $S_F = \mathbb{Z}_2$, $\omega_s = \pi$ and $u_s = -u$, hence $K \sim \mathbb{R}_+$ (a positive semiaxis).

Under the transformations (4.20), the canonical momenta (4.16) remain untouched, while
\[ p_{\theta} \to p_{\theta}, \quad p_u \to \left(\frac{du_s}{du}\right)^{-1} p_u \equiv p_{u_s} = \hat{s}p_u \]  
(4.22)
under (4.21). In the new canonical variables, a state $p, x$ corresponds to phase-space points $(p_\theta, \hat{s}\theta, \hat{s}p_u, \hat{s}u)$, $\hat{s}$ runs over $S_F$ (assuming $\theta \in [0, 2\pi]$). Therefore, configurations of new canonical variables connected with each other by $S_F$-transformations are not physically distinguishable.

Consider a phase-space plane, where $p_\theta = 0$ and $\theta$ has a fixed value, and states $(p_\theta = 0, \theta, \hat{s}p_u, \hat{s}u)$ on it. These states differ from each other by values of the angular variable $(p_\theta, \theta, \hat{s}p_u, \hat{s}u) \sim (p_\theta, \hat{s}^{-1}\theta, p_u, u)$ where $\hat{s}^{-1}\theta = \theta - \omega_s(u)$. If we switch on the gauge symmetry, the angular variable becomes unphysical and, hence, the difference between all those states disappears. They correspond to the same physical state. Thus, the transformations $u, p_u \to u_s, p_{u_s}$ of the phase plane turn into the gauge ones so that we have to stick together all points $\hat{s}u, \hat{s}p_u, \hat{s} \in S_F$, to obtain $PS_{ph}$ spanned by $u$ and $p_u$. For the polar coordinates, we obviously get $PS_{ph} = cone(\pi)$. If we do not care whether our gauge-invariant variables naturally span the gauge orbit space, we can gain a very complicated structure of $PS_{ph}$, which can make a dynamical analysis hopeless.

One should emphasize that in our approach transformations $\hat{s} \in S_F$ in the $(u, p_u)$-plane cannot be regarded as the ones generated by the constraint $\sigma = p_\theta$ since $\{\sigma, u\} = \{\sigma, p_u\} = 0$ in contrast with the gauge fixing description considered above. Physical variables are chosen so that the set $S_F$ determining their phase space coincides formally with the set of residual gauge transformations in the gauge fixing approach. Thus, one can always construct gauge-invariant variables such that their configuration (phase) space coincides with a surface determined by a gauge condition chosen and therefore, all artifacts inherent to an “inappropriate” gauge fixing may be emerged in a gauge-invariant approach.

To demonstrate qualitatively what kind of artifacts might occur through the “inappropriate” parametrizing $PS_{ph}$, we compare phase-space trajectories in the canonical variables $r = |x|, \ p_r = (x, p)/r$ and $u, \ p_u$. They are connected by the canonical transformation $r = r(u) = |f(u)|, \ p_r = r p_u / \mu = p_u (dr/du)^{-1}$. We also assume the function $f$ to be differentiable such that $dr/du = 0$ only at two points $u = u_{1,2}$ and $dr/du > 0$ as $u < u_2$ and $u > u_1$, while $dr/du < 0$ if $u \in (u_2, u_1)$. Our assumption means that the curve $x = f(u), \ u \in \mathbb{R}_+$, goes from the origin, crosses the circle $|x| = r_1 = r(u_1)$ at $x = f(u'_1)$ and reaches the circle $|x| = r_2 = r(u_2)$, touches it at $x = f(u_2)$ and turns back to the circle $|x| = r_1$ and after touching it at the point $x = f(u_1)$ tends to infinity, crossing the circle $|x| = r_2$ at $x = f(u'_2)$.
In a neighborhood of the origin, $PS(p_r, r)$ has the conic structure as we have shown above. This local structure is preserved upon the canonical transformation to the variables $u, p_u$ because it is a smooth and one-to-one mapping of the strip $r \in (0, r_1)$ on $u \in (0, u'_1)$.

The same is related to the half-planes $r > r_2$ and $u > u'_2$. Troubles rise up on the domain $r \in (r_1, r_2)$ where the inverse function $u = u(r)$ becomes multi-valued; it has three branches in our particular case. States belonging to the strips $u \in (u'_1, u'_2)$, $u \in (u_2, u_1)$ and $u \in (u_1, u'_2)$ are physically equivalent because there are transformations from $S_F$ mapping the strips on each other and leaving points $p_r, r \in (r_1, r_2)$ untouched.

To understand what might happen to phase-space trajectories in the PS region $u \in (u'_1, u'_2)$, consider a motion with a constant momentum $p_r$ and suppose that a particle is outgoing from the origin $r = 0$. On the $(p_u, u)$-plane, the particle motion corresponds to a point running along a curve going from the origin $u = 0$. As soon as the PS point reaches the line $u = u'_1$, there appear two “phantom” PS trajectories outcoming from the point $p_u = 0, u = u_1$ (notice, $p_{u'_1} = p_{u_2} \equiv 0$ since $dr/du = 0$ at $u = u_1, u_2$, and $u'_1$ is $S_F$-equivalent to $u_1$).

If $u_{s_1}$ and $u_{s_2}$ map $(u'_1, u_2)$ onto $(u_2, u_1)$ and $(u_1, u'_2)$, respectively, such that $r(u) = r(u_{s_1}) = r(u_{s_2})$, $u \in (u'_1, u_2)$, then the “phantom” trajectories are described by the pairs $s_{1,2}p_u, s_{1,2}u$ (cf. (4.22)) where $p_u, u$ range the trajectory in the PS region $u \in (u'_1, u_2)$. Since $du_{s_1}/du < 0$ and $du_{s_2}/du > 0$, the “phantom” trajectory $s_{2}p_u, s_{2}u$ goes to the infinity, while the point $s_{1}p_u, s_{1}u$ runs in the opposite direction. The points $p_u, u$ and $s_{1}p_u, s_{1}u$ arrive at $p_u = p_{u_2} = 0, u = u_2$ in the same time and annihilate each other, whereas a “phantom” particle moving along the branch $s_{2}p_u, s_{2}u$ approaches the line $u = u'_2$. In the next time moment a particle leaves the interval $r \in (r_1, r_2)$ (or $u \in (u'_1, u'_2)$).

Such “branching” of classical PS trajectories is a pure artifact of an “inappropriate” parametrization of $PS_{ph}$ (or, as we have argued above, of gauge fixing). It has to be removed by gluing all the “phantom” trajectories (branches). In so doing, we cannot however avoid breaking PS trajectories at the singular points $u = u_{1,2}$. Indeed, consider trajectories approaching the line $u = u'_1$ with different momenta $p_u$ from the origin and crossing it. The element $s_{1}$ maps these trajectories with $u > u'_1$ onto trajectories outcoming from just one PS point $p_u = 0, u = u'_1$, but $s_{1}$ does not touch these trajectories with $u < u'_1$ at all. So, we will gain breaking of PS trajectories at $u = u'_1$ after gluing points $p_u, u$ and $s_{1}p_u, s_{1}u$. The same occurs at the line $u = u'_2$ and the singular point $p_u = 0, u = u_2$.

Notice also that points $p_u = 0, u_1$ and $p_u = 0, u'_1$ are stuck together as well as $p_u = 0, u_2$ and $p_u = 0, u'_2$, i.e. trajectories in the PS domain $u \in (u_2, u_1)$ change their orientation when gluing.

To construct $PS(p_u, u)$, we take the half-plane $u \geq 0, p_u \in \mathbb{R}$, cut it along the lines $u \in u_{1,2}$ and $u \in u'_{1,2}$, remove the strips $u \in (u'_1, u_2)$ and $u \in (u_1, u'_2)$. Then we glue the semi-axes $u = 0, p_u > 0$ and $u = 0, p_u < 0$ (the conic structure at the origin is preserved!). To complete constructing $PS_{ph}(p_u, u)$, we have to identify the points $p_u = 0, u = u'_{1,2}$ with $p_u = 0, u = u_{1,2}$, respectively, so that the phase space looks like a chain of the “cut” cone, the strip $u \in (u_1, u_2)$ and the half-plane $u > u'_2$ coupled just by a point-like bridges.

We cannot identify the lines $u = u_1$ and $u = u'_1$ (as well as $u = u_2$ and $u = u'_2$) because there is no PS trajectory in the PS region $u \in (u_1, u_2)$ reaching the lines $u = u_{1,2}$ with a
non-zero momentum $p_u$; all trajectories inside the strip $u \in (u_1, u_2)$ fall on the singular points $p_u = 0, u = u_{1,2}$ being artificial attractors created by gauge fixing. These attractors correspond to zeros of the Faddev-Popov determinant $\mu(u_{1,2}) = 0$ [16].

We conclude this section with a few brief remarks.

1. Our analysis of gauge fixing artifacts can be generalized to gauge systems with many degrees of freedom [12], [13] (even to gauge field theories [16], [5]). It can be achieved by treating $u$ and $f(u)$ as elements of (in)finite dimensional Euclidean spaces; $\Omega$ becomes an element of a gauge group generated by all independent first-class constraints; the condition $F = 0$ has to fix a gauge completely, i.e., it has to remove all unphysical degrees of freedom.

2. A construction of gauge-invariant symplectic variables (meaning that they commute with constraints) might turn out to be not very helpful for analyzing dynamics if the gauge orbit space is unknown. A phase space of those variables may have artificial attractors and “branching” trajectories, i.e. all artifacts inherent to the gauge fixing approach.

3. Nevertheless, if the analysis of the gauge orbit space is complicated (or there is a reason to prefer a particular gauge condition to the others), one can develop a quantum theory in the curvilinear coordinates (4.15) or in their generalization to gauge systems with a higher number of degrees of freedom [16], [13]. A summary of this approach is given in Appendix B.

4. The variables $a$ and $p_a$ we constructed in Sec. 3 look gauge non-invariant (they are related to the Coulomb gauge). In virtue of a canonical transformation analogous to (4.15)-(4.16), one can always turn $a$ and $p_a$ into gauge-invariant symplectic variables. Dynamics in these invariant variables occurs on the phase space (3.15) (see Sec. 6).

5 The Hamiltonian path integral and a configuration space topology

In the previous sections we have seen that the $PS_{ph}$ structure plays an important role in classical mechanics of gauge theories. The path integral formalism [24] is a natural bridge between classical and quantum mechanics because it allows us to formulate a quantum theory in terms of classical quantities. A transition of a classical system from an initial state to the final one is described by a phase-space trajectory connecting two points on PS (the initial and final states) and satisfying Hamiltonian equations of motion. After quantization, a transition amplitude is determined by a sum over all trajectories connecting initial and final configuration space points, i.e. by the path integral [24]. Therefore one might expect that such a sum depends on a topological structure of a space formed by these trajectories.

To elucidate what happens to the path integral (PI) representation of the transition amplitude upon changing a topology of PS or CS, we first analyze the problem in the framework of the operator formalism. A reason for this is rather obvious. The operator and PI approaches are two languages for describing the same theory. If we have a solution of a problem in one of them, one can always ”translate” it to the other language by means
of some basic rules [24], [25]. In the operator formalism, the transition amplitude obeys the Schroedinger equation, while topological properties of CS are taken into account by imposing some boundary conditions. Thus, the problem is reduced to solving a standard mathematical task.

Consider, for example, a free one-dimensional motion. The transition amplitude
\[ U_t(x, x') = \langle x | \exp(-itH|x') \rangle, \]
where \( H = -\frac{\partial^2}{2} \) being the Hamiltonian, satisfies the Schroedinger equation
\[ i\partial_t U(x, x') = -\frac{1}{2} \partial^2_x U_t(x, x') \] (5.1)
with the initial condition
\[ U_{t=0}(x, x') = \langle x | x' \rangle = \delta(x - x'). \] (5.2)

If \( CS = \mathbb{R} \), a solution to (5.1-2) is well-known [24]
\[ U_t(x, x') = (2\pi it)^{-1/2} \exp i(x - x')^2 = \]
\[ = \int \prod_{\mathbb{R}^2} \left( \frac{dp(\tau)dx(\tau)}{2\pi} \right) \exp i \int_0^t d\tau \left( p\dot{x} - \frac{1}{2} p^2 \right) , \] (5.4)
where the measure in (5.4) implies a sum over all trajectories \( x(\tau) \) going from \( x' = x(0) \) to \( x = x(t) \).

Let now CS be compactified to a circle or to a strip (a particle on a circle or in a box). Then in addition to Eqs.(5.1-2), the transition amplitude has to obey the boundary conditions
\[ U^c_t(x + L, x') = U^c_t(x, x' + L) = U^c_t(x, x') , \]
\[ U^b_t(0, x') = U^b_t(L, x') = U^b_t(x, 0) = U^b_t(x, L) = 0 , \] (5.5) (5.6)
with \( L \) being a configuration space volume, for a particle on a circle and in a box, respectively. To obtain a solution to (5.1-2) and (5.5) or (5.6), one should take the following linear combinations of (5.3) [24], [25], [13]
\[ U^{c,b}_t(x, x') = \int dx''U_t(x, x'')Q^{c,b}(x'', x') , \] (5.7)
\[ Q^b(x'', x') = \sum_{n=-\infty}^{\infty} \left[ \delta(x'' - x' - 2Ln) - \delta(x'' + x' - 2Ln) \right] , \] (5.8)
\[ Q^c(x'', x') = \sum_{n=-\infty}^{\infty} \delta(x'' - x' - Ln) , \] (5.9)
where \( x'' \in \mathbb{R} \) and \( x, x' \in (0, L) \). From the technical point of view, the kernels (5.8-9) are analytical continuations of the unit operator kernel (5.2) to the whole axis. They can
be obtained by straightforward summing the spectral decomposition of the unit operator kernel \( \langle x''|x' \rangle \) assuming \( x'' \in \mathbb{R} \). For example, in the case of a particle on a circle we have

\[
\langle x''|x' \rangle = \sum_E \psi_E(x'') \psi_E^*(x') = L^{-1} \sum_{n=-\infty}^{\infty} \exp \left( \frac{2\pi i n}{L} (x'' - x') \right) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{L} \delta \left( \frac{2\pi}{L} (x'' - x') - 2\pi n \right) = Q^c(x'', x')
\]  

(5.10)

for \( x'' \in \mathbb{R} \) and \( x' \in (0, L) \). The kernels (5.8-9) contain all information about symmetry properties of wave functions satisfying either periodic or zero boundary conditions on the interval \((0, L)\). The representation (5.7) provides keeping these properties during the time evolution of any state because the evolution operator kernel (5.7) possesses these properties for its left argument \( x \) and, hence, its application to any state \( \psi_0(x) \) always produces a state with the same symmetry properties, \( \psi_t(x) = \int_0^L dx' U_{t,b}(x,x') \psi_0(x') \).

Doing the integral over \( x'' \) in (5.7) we see that the transition amplitude in CS with a non-trivial topology is given by a sum over paths outgoing from not the only initial point. For a particle on a circle, the sum contains trajectories going from \( x' + Ln, n \in \mathbb{Z} \) to \( x \). Because of the periodicity of the action, we can interpret such a sum as a sum over all trajectories with all possible winding numbers \( n \). The latter provides fulfilling the boundary conditions (5.5) or incorporating the CS topology into the path integral formalism.

In the case of a particle in a box, contributions of trajectories going from \( x' + 2Ln \) to \( x \) and of those going from \(-x' + 2Ln\) to \( x \) have opposite signs, which results from the boundary conditions (5.6). The straight trajectories \( x' + 2Ln \to x \) can be interpreted as continuous trajectories connecting \( x', x \in (0, L) \) with \( 2n \) reflections from the walls at the interval boundaries because they have the same action. Contributions of the trajectories \(-x' + 2Ln \to x \) are equivalent to contributions of trajectories inside of the box with an odd number of reflections \( 2n + 1 \).

A general analysis of boundary conditions (or CS topology) in the path integral formalism is given in [13]. A conclusion is similar to that we have found above – a topology of CS (or PS) can be taken into account by including additional "reflected" trajectories into a sum over paths. For gauge theories with a non-trivial \( PS_{ph} \), this statement is proved in Appendix B.

6 The Hamiltonian PI on the gauge orbit space for 2D Yang-Mills theory

In this section we show that the idea of including reflected trajectories into a sum over paths can be successfully applied to 2D Yang-Mills theories to construct PI on the gauge orbit space. PI for the 4D-case has been obtained in [3],[11]. An advantage of considering 2D gauge field theories is that they are exactly solvable. Therefore, they give nice toy models for verifying ideas invented for gauge field theories of a general type.
Following the Dirac method we change the canonical variables \( E_1(x) \to -i\hbar \delta /\delta A(x), \ A_1(x) \to A(x), \ A(x) \in \mathcal{F} \), by operators \( \hat{\mathbf{1}} \) and get the quantum theory in the Shroedinger functional representation \( [18] \)

\[
\hat{H} \Phi_n[A] = -\frac{\hbar}{2} \left( \frac{\delta}{\delta A} A \right) \Phi_n[A] = E_n \Phi_n[A], \tag{6.1}
\]

\[
\hat{\sigma} \Phi_n[A] = \nabla(A) \left( \frac{\delta}{\delta A} \Phi_n[A] \right) = 0. \tag{6.2}
\]

In accordance with the general method proposed in Appendix B, to solve Eq.(6.2) and to project the Hamiltonian in (6.1) onto the gauge orbit space, one should introduce curvilinear coordinates associated with both a gauge transformation law and a gauge condition chosen \( [9] \) (cf.(4.15) and (B.4))

\[
A(x) = \Omega(x) a \Omega^{-1}(x) + \frac{i}{g} \Omega(x) \partial \Omega^{-1}(x), \quad \Omega(x) \in \mathcal{G}/\mathcal{G}_H^0, \quad a \in K^+_W, \tag{6.3}
\]

where \( \mathcal{G} \) is a gauge group, \( \mathcal{G}_H^0 \) is spatially homogeneous Cartan subgroup of \( \mathcal{G} \), \( \mathcal{G}_f = \mathcal{G}_H^0 \).

In Sec.3, any configuration \( A_1(x), \ x \in \mathbf{S}^1 \), is shown to be uniquely represented in the form (6.3). The condition \( a \in K^+_W \) is imposed to provide a one-to-one correspondence between the "old" and "new" variables. If we assume \( \Omega(x) \in \mathcal{G}/\mathcal{G}_H^0 \) and \( a \in H \sim \mathbb{R}^r \), then configurations \( \Omega(x) \Omega_s^{-1}, \ s a, \ s \) ranges the affine Weyl group \( W_A \) (see (3.17), (3.10) and (3.9)), are mapped to the same configuration \( A(x) \) by (3.6). That is why admissible values of \( a \) in (6.3) have to be restricted by the Weyl cell \( K^+_W \), otherwise equality (6.3) does not determine a change of variables. Thus, we parametrize the gauge orbit space \( [A_1]/\mathcal{G} = \mathcal{F}/\mathcal{G} \sim K^+_W \) by the variables \( a \). We show below (see (6.20)) that the constraint operator \( \hat{\sigma} \) commutes with \( a \) and, therefore, \( a \) is a gauge-invariant variable.

For sequential calculations we introduce the following decomposition of the functional space (3.5)

\[
\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n = \bigoplus_{n=0}^{\infty} (\mathcal{F}_n^H \oplus \tilde{\mathcal{F}}_n), \tag{6.4}
\]

where \( \mathcal{F}_0 \) is a space of constant Lie algebra-valued functions (the first term in the series (3.5)), \( \mathcal{F}_n, n \neq 0, \) is a space of functions with the fixed period \( 2\pi l/n \) (a term in the sum (3.5) with a fixed \( n \)). Each subspace of \( \mathcal{F} \) is finite-dimensional, \( \dim \mathcal{F}_0 = \dim X, \dim \mathcal{F}_n = 2 \dim X, \ n \neq 0 \) (we recall that Lie algebra-valued functions are considered). Functions belonging to \( \mathcal{F}_n^H \) take their values in the Cartan subalgebra \( H \), while \( \tilde{\mathcal{F}}_n \) is composed of \( 2\pi l/n \)-periodic functions with values in \( X \ominus H \). All subspaces introduced are orthogonal with respect to the scalar product \( \langle \cdot, \cdot \rangle = \int_{0}^{2\pi l} dx \langle \cdot, \cdot \rangle \).

The differential \( \delta A \in \mathcal{F} \) can be represented in the form

\[
\delta A = \Omega \left( da - \frac{i}{g} (\partial \delta w - ig[a, \delta w]) \right) \Omega^{-1} = \Omega \left( da - \frac{i}{g} \nabla(a) \delta w \right) \Omega^{-1}, \tag{6.5}
\]

\(^1\)We ignore the pure unphysical degree of freedom \( A_0, E_0 \) \([8]\).
where by the definition of the change of variable \( da \in \mathcal{F}_0^H \) and \( \delta w(x) = i\Omega^{-1}\delta \Omega \in \mathcal{F} \otimes \mathcal{F}_0^H \). Therefore the metric tensor (B.5) reads

\[
\langle \delta A, \delta A \rangle = 2\pi l(da, da) - g^{-2}\langle \delta w, \nabla^2(a)\delta w \rangle. \tag{6.6}
\]

Equality (6.6) results from the trivial observation \( \langle da, \nabla(a)\delta w \rangle = -\langle \nabla(a)da, \delta w \rangle = 0 \) which is due to \( \partial da = 0 \) and \([da, a] = 0\). In the notations of Appendix B, we have \( g_{11} = 2\pi l, \ g_{12} = g_{21} = 0, \ g_{22} = -g^{-2}\nabla^2(a) \), where the operator \( \nabla(a) \) acts in the space \( \mathcal{F} \otimes \mathcal{F}_0^H \). It has no zero mode on this space if \( a \in K_W^+ \) and, hence, is invertible.

To obtain the scalar product (B.13), one has to calculate the determinant \( \det g_{AB} = \det g_{11} \det g_{22} = (2\pi l)^r \det(g^{-2}\nabla^2(a)) \). Consider the orthogonal decomposition

\[
\tilde{F}_n = \sum_{\alpha>0} \oplus \mathcal{F}^\alpha_n, \tag{6.7}
\]

where \( \mathcal{F}^\alpha_n \) contains only \( 2\pi l/n \)-periodic functions taking their values in the two-dimensional subspace \( X_+ \otimes X_\alpha \) of the Lie algebra \( X \) (see Appendix A, (A.1-4)). The subspaces \( \mathcal{F}^n_H, \mathcal{F}^\alpha_n \) are invariant subspaces of the operator \( \nabla(a) \); it has a block-diagonal form in the in the decomposition (6.4), (6.7). Indeed, we have \( \nabla(a) = \partial - ig\hat{a} \), where \( \hat{a} = [a, ] \) is the adjoint operator acting in \( X \) (see Appendix A). The operator \( \partial \) is diagonal in the algebra space, and its action leaves periods of functions untouched, i.e. \( \mathcal{F}^n_{H,\alpha} \) are its invariant spaces. Obviously, \( \hat{a}\mathcal{F}^n_H = 0 \) and \( \hat{a}\mathcal{F}^\alpha_n = \mathcal{F}^\alpha_n \) if \( (\alpha, a) \neq 0 \) with accordance with (A.2). Therefore an action of the operator \( \nabla(a) \) on \( \mathcal{F} \otimes \mathcal{F}_0^H \) is given by an infinite-dimensional, block-diagonal matrix. Its blocks have the form

\[
\nabla^H_n(a) \equiv \nabla(a)|_{\mathcal{F}^H_n} = \partial|_{\mathcal{F}^H_n} = \left( \otimes \frac{\pi l}{T} \right)^r, \quad n \neq 0, \quad r = rankX, \tag{6.8}
\]

\[
\nabla^0_n(a) \equiv \nabla(a)|_{\mathcal{F}^0_n} = -ig\hat{a}|_{\mathcal{F}^0_n} = -ig(a, \alpha)T \tag{6.9}
\]

\[
\nabla^\alpha_n(a) \equiv \nabla(a)|_{\mathcal{F}^\alpha_n} = \mathbb{I} \otimes \frac{n}{l} T - ig(a, \alpha) T \otimes \mathbb{I} \tag{6.10}
\]

where \( T = -i\tau_2 \), \( \tau_2 \) the Pauli matrix, \( \mathbb{I} \) is the \( 2 \times 2 \) unit matrix. In (6.10) the first components in the tensor products correspond to the algebra indices, while the second ones determine the action of \( \nabla(a) \) on the functional basis \( \sin nx/l, \; \cos nx/l \). We have

\[
\det g_{AB} = (2\pi l)^r \prod_{\alpha>0} \det(i g^{-1}\nabla^0_n)^2 \prod_{n=1}^\infty \left[ \det(i g^{-1}\nabla^H_n)^2 \prod_{\alpha>0} \det(-i g^{-1}\nabla^\alpha_n)^2 \right] = \tag{6.11}
\]

\[
= (2\pi l)^r \prod_{\alpha>0} (a, \alpha)^4 \prod_{n=1}^\infty \left[ \left( \frac{n}{gl} \right)^4 \prod_{\alpha>0} \left( \frac{n^2}{g^2l^2} - (a, \alpha)^2 \right) \right].
\]

Set \( \det g_{AB}^{1/2} = C(l)\mu(a), \; \mu(a) = \kappa^2(a) \). Including all divergences of the product (6.11) into \( C(l) \) we get

\[
\kappa(a) = \prod_{\alpha>0} \left[ \frac{\pi(a, \alpha)}{a_0} \prod_{n=1}^\infty \left( 1 - \frac{(a, \alpha)^2}{a_0^2} \right) \right] = \prod_{\alpha>0} \sin \frac{\pi(a, \alpha)}{a_0}, \tag{6.12}
\]

\[
C(l) = (2\pi l)^{r/2} \left( \frac{a_0}{\pi} \right)^{N^+} \prod_{n=1}^\infty \left( n^2a_0^2 \right)^{r+2}, \tag{6.13}
\]

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where $a_0 = (gl)^{-1}$, $N_+ = (N - r)/2$ is the number of positive roots in $X$; the last equality in (6.12) results from a product formula given in p.37, [26]. Following the prescription (B.13) we define the scalar product in the model

$$
I \prod_{x \in s_1} dA(x) \Phi_n^*[A] \Phi_{n'}[A] \to \int_{K^+_W} d\kappa^2(a) \Phi_n^*(a) \Phi_{n'}(a) = \delta_{nn'}, \tag{6.14}
$$

where the infinite constant $C(l) \int_G/G_H \prod x \wedge dw(x)$ is removed by a renormalization of physical states, which we denote by the arrow in (6.14). This renormalization is admissible because of the gauge invariance of physical states $\Phi_n[A] = \Phi_n(a)$ (compare (3.3) and (6.3)). Notice that Eq.(6.2) being written via the new variables has the form

$$
\hat{\sigma} \Phi_n[a, \omega] = -ig \hat{\Omega} \frac{\delta}{\delta w} \Phi_n[a, w] = 0; \tag{6.15}
$$

here we have used the notation $\hat{\Omega}y = \Omega y \Omega^{-1} = y \hat{\Omega}^T$, $\hat{\Omega}^T \hat{\Omega} = \hat{\Omega} \hat{\Omega}^T = 1$ for any element $y \in X$. So, $\det \hat{\Omega} \neq 0$ and solutions to (6.15) are given by functionals independent of $w(x)$.

To prove the equivalence of Eqs.(6.2) and (6.15), we derive first the following relations from (6.5)

$$
da = P_0^H \hat{\Omega}^T \delta A, \tag{6.16}
$$

$$
\delta w = ig \nabla^{-1}(a)(1 - P_0^H) \hat{\Omega}^T \delta A \tag{6.17}
$$

with $P_0^H$ being a projector on $\mathcal{F}_0^H$ (the operator $\nabla(a)$ is invertible on $(1 - P_0^H) \mathcal{F}$). The straightforward calculations lead us to the desired result

$$
\nabla(A) \frac{\delta}{\delta A} = \nabla(A) \left[ \left( \frac{\delta}{\delta A}, \frac{\partial}{\partial a} \right)_a + \left( \frac{\delta w}{\delta A}, \frac{\delta}{\delta w} \right)_w \right] = \nabla(A) \left[ (P_0^H \hat{\Omega}^T)^T \frac{\partial}{\partial a} + (ig \nabla^{-1}(a)(1 - P_0^H) \hat{\Omega}^T)^T \frac{\delta}{\delta w} \right] = \hat{\Omega} \nabla(a) P_0^H \frac{\partial}{\partial a} - ig \hat{\Omega} \nabla(a)(1 - P_0^H) \nabla^{-1}(a) \frac{\delta}{\delta w} = -ig \hat{\Omega} \frac{\delta}{\delta w}. \tag{6.20}
$$

In (6.18), the suffix at the scalar product brackets denotes variables over whose indices the scalar product is taken, i.e. all indices of $A(x)$ (the Lie algebra ones and $x \in S^1$) in the scalar products entering into (6.18) are left free. Equality (6.19) results from (6.16-17). In (6.20), we have used $\nabla(a) P_0^H \partial/\partial a \equiv 0$ and $\nabla(A) \hat{\Omega} = \hat{\Omega} \nabla(a)$.

Thus, we have proved the gauge invariance of the variable $a$, $[\hat{\sigma}, \hat{a}] = 0$. In this approach, the Gauss law (6.2) is explicitly solved (even in the 4D case [5]). We do not fix a gauge at all, but we do choose a certain coordinates on the gauge orbit space.

To project the functional Laplace operator in (6.1) on the gauge orbit space spanned by variables $a$, one should calculate the Laplace-Beltrami operator in the new coordinates
(6.3) and drop all terms containing $\delta/\delta w$ in it. A general recipe is given by (B.11). Using it, we arrive at a quantum mechanical problem

$$
\hat{H}^{\text{ph}} \Phi_n(a) = \left[ -\frac{\hbar^2}{4\pi l} \frac{1}{\kappa(a)} (\partial_a, \partial_a) \circ \kappa(a) - E_C \right] \Phi_n(a) = E_n \Phi_n(a) ,
$$

(6.21)

where we have taken into account $g^{11} = (2\pi l)^{-1}$, the quantum potential (B.12) turns out to be independent of $a$,

$$
V_q = \frac{\hbar^2}{4\pi l} \kappa^{-1}(\partial_a, \partial_a) \kappa = -\frac{\pi \hbar^2}{4a_0^2l} \left( \sum_{\alpha>0} \right)^2 = -E_C .
$$

(6.22)

A proof of (6.22) is given in Appendix C. In fact, $E_C$ coincides with the Casimir energy related to the curvature of the group manifold (cf. the SU(N) case considered in [27],[28]). Notice that the Casimir energy naturally appears upon solving the constraint (6.2), which would not occur through quantizing after eliminating all unphysical degrees of freedom by imposing the gauge condition $A_1(x) = a$ (this approach is considered in [29] for $G = SU(N)$). In the case of an arbitrary gauge, a quantization of the gauge-fixed theory might give rise to not only loosing the Casimir energy, but even to more drastic consequences – the gauge dependence of the quantum theory – as we have shown in Appendix B. To ensure the gauge independence, one should quantize before removing gauge degrees of freedom.

As has been proved in Appendix B, any regular solution of (6.21) must be invariant with respect to the discrete transformations (3.17) $S_F = W_A$

$$
\Phi_n(\hat{s}a) = \Phi_n(a) , \quad \hat{s} \in W_A .
$$

(6.23)

In Appendix D we verify (6.23) by explicit solving Eq.(6.21) and show that the eigen functions are given by characters of the irreducible representations of $G$, while the eigen values are proportional to eigen values of the quadratic Casimir operator of $G$. The property (6.23) determines the boundary conditions which we need to construct the PI representation of the transition amplitude by means of the method of “reflected” trajectories. It gives also an analytical continuation of physical wave functions to the unphysical region $a \in H$ (to the covering space of the gauge orbit space) in full accordance with a general analysis in Appendix B (see (B.14)).

To obtain a PI representation of the transition amplitude $U_t^{\text{ph}}(a, a') = \langle a \vert e^{-it\hat{H}^{\text{ph}}/\hbar} \vert a' \rangle$, one should repeat calculations (B.17-30) for this particular model. Due to (6.23) and (6.14), the analytical continuation of the unit operator kernel reads

$$
\langle a \vert a' \rangle = \sum_{\hat{s} \in W_A} (\kappa(a) \kappa(\hat{s}a'))^{-1} \delta^r(a - \hat{s}a') , \quad a \in H , \quad a' \in K^+_W .
$$

(6.24)

The kernel (6.24) is $W_A$-invariant. Consider the reflection (3.17); we have to prove $\langle \hat{s}_{\alpha,n} a \vert a' \rangle = \langle a \vert a' \rangle$. Change the summation over $\hat{s}$ in (6.24) by $\hat{s}_\alpha \hat{s}$ and use relations (C.7-8) $\kappa(\hat{s}_{\alpha,n} a) = \kappa(\hat{s}_\alpha a) = -\kappa(a)$ and $\delta^r(\hat{s}_{\alpha,n} a - \hat{s}_\alpha \hat{s} a') = \delta(\hat{s}_{\alpha,n} a - \hat{s}_\alpha \hat{s} a')$. The element...
\( \hat{s}_\alpha^{-1} \hat{s}_{\alpha,n} \) is a translation through periods of the unit group lattice which are independent of \( a \). Since the measure \( \kappa(a) \) is invariant under these translations (see Appendix C), the shift of \( \hat{s}_\alpha^{-1} \hat{s}_{\alpha,n} a = a - 2n a_\alpha / (\alpha, \alpha), \ n \in \mathbb{Z}, \) can be included into shift of \( a' \) generated by all \( \hat{s} \in W \) by virtue of resummation in the infinite series (6.24), which completes the proof.

Therefore the infinitesimal transition amplitude is given by (B.18) where \((\mu \mu')^{-1/2}\) is changed by \((\kappa \kappa')^{-1}\), \( \mathbb{R}^M \to H \sim \mathbb{R}^r \in a''\),

\[
Q(a'', a') = \sum_{\hat{s} \in W_A} \delta^r(\alpha'' - \hat{s} \alpha'),
\]

(6.25)

and in (B.19) \((u, p) \to (a, p_a), \ p_a \in H \sim \mathbb{R}^r\),

\[
H^{\epsilon \text{eff}}(a, p_a) = \frac{1}{4\pi i}(p_a, p_a) - E_c;
\]

(6.26)

notice that the second term in (B.20) vanishes because \( g_{ph} \) is independent of \( a \). The \( W_A \) – invariance of the infinitesimal kernel

\[
U^{ph}_\epsilon(\hat{s}_{\alpha,n}, a') = U^{ph}_\epsilon(a, a'), \ \hat{s}_{\alpha,n} \in W_A,
\]

(6.27)

results from the representation (B.18) for \( U^{ph}_\epsilon(\hat{s}_{\alpha,n} a, a') \). Indeed, the transformation \( \hat{s}_{\alpha,n} \) is a composition of a translation \( t_n \in T_e \) and a reflection \( \hat{s}_\alpha \) from the Weyl group. The \( T_e \) invariance of \( U^{ph}_\epsilon(a, a') \) is obvious because \( U^{\epsilon \text{eff}}(a, a') \) in (B.18) depends only on \( \Delta = a - a'' \). Indeed, any \( T_e \)-shift of \( a \) can be removed by a shift of the integration variable \( a'' \). The \( T_e \)-invariance follows from the \( T_e \)-invariance of the kernel (6.25) and the function \( \kappa(a) \) (see appendix C). To prove the \( W \) – invariance, one should do the integral over \( a'' \) in (B.18), then change the integration variables in (B.19) \( p_a \to \hat{s}_a p_a \) and use the \( W \)-invariance of the Killing form \( (\hat{s}_a p_a, \hat{s}_a a) = (p_a, a) \).

Proved (6.27) we find for the convolution of infinitesimal evolution operator kernels

\[
U^{ph}_2(\epsilon, a') = \int \limits_{W_A K^+_W} da_1 \kappa^2(a_1) U^{ph}_\epsilon(a, a_1) U^{ph}_\epsilon(a_1, a') =
\]

(6.28)

\[
= \sum_{W_A K^+_W} \int \frac{da_1 \kappa^2(a_1)}{\kappa(a) \kappa(\hat{s}a_1)} U^{\epsilon \text{eff}}(a, \hat{s}a_1) U^{ph}_\epsilon(a_1, a') =
\]

(6.29)

\[
= \int \limits_{H \sim \mathbb{R}^r} \frac{da''}{\kappa(a) \kappa(a'')} \left( \int \limits_{H \sim \mathbb{R}^r} \frac{da_1 U^{\epsilon \text{eff}}(a, a_1) U^{\epsilon \text{eff}}(a_1, a'')}{H^{\epsilon \text{eff}}(a, p_a) - E_c} \right) Q(a'', a') ;
\]

(6.30)

where (6.29) is obtained by substitution the representation (B.18) into (6.28) and doing the integral over \( a'' \), then we have changed the integration variables in (6.29) \( a_1 \to \hat{s}a_1 \) (the Jacobian being \( J_\hat{s}(a) = \det \hat{s} = \pm 1 \)), and used (6.27), the integration rule \( \sum_{W_A} \int_{\hat{s}K^+_W} da_1 = \int_{H} da_1, \ \kappa^2(\hat{s}a) = \kappa^2(a) \) and (B.18) again to derive (6.30). Equalities (6.28-30) are to
illustrate (B.26-28). Thus, for a finite time the convolutions (B.29) and (B.30) yield

\[ U^p_t(a, a') = \int_{\mathbb{R}^r} \frac{da''}{\kappa(a)\kappa(a'')} U^{eff}_t(a, a'') Q(a'', a) \]  

(6.31)

\[ U^{eff}_t(a, a'') = \int_{\mathbb{R}^{2r}} \prod_{\tau=0}^{t} \left( \frac{dp_{\tau}(\tau)da(\tau)}{(2\pi\hbar)^r} \right) \exp \left( \frac{i}{\hbar} \int_0^t d\tau \left( (p, \dot{a}) - \frac{(p, p)}{4\pi l} + E_C \right) \right) = (6.32) \]

\[ = \left( \frac{l}{ihl} \right)^{r/2} \exp \left( \frac{i\pi l(a - a'')^2}{\hbar t} + \frac{itE_C}{\hbar} \right). \]  

(6.33)

The function (6.33) is a transition amplitude for a free \(r\)-dimentional particle of mass \(2\pi l\). In fact, we have proved that the 2D Yang-Mills theory on a cylindrical spacetime is equivalent to a quantum theory of an \(r\)-dimentional free particle moving in a polyhedron being the Weyl cell of the gauge group. Substituting \(\Phi_n = \kappa^{-1}\Psi_n\) into (6.21) we see that \(\Psi_n\) satisfies the Schroedinger equation for a free \(r\)-dimentional particle. But it is not completely free motion because of the boundary condition (6.23). The particle ”interacts” with the boundary of the Weyl cell, which leads to the ”twisted” boundary condition

\[ \Psi_n(\hat{s}a) = \det \hat{s}\Psi_n(a), \; \hat{s} \in W_A . \]  

(6.34)

Any regular function on \(H\) obeying (6.34) vanishes on \(\partial K^+_W\) (even on the whole diagram \(D(X)\) (see Sec.3)) providing a regular behavior of \(\Phi_n\) on \(\partial K^+_W\) (and on the covering space \(H\)). An important point following from our consideration is that the boundary condition (6.23) or (6.34) automatically results from the Dirac method, and we do not need to impose them (or another ones) by hand (in contrast with [28], [29]). Moreover, \(W_A\)–invariant functions (6.23) are regular functions of the Wilson loop (see Appendix D), i.e. they turn out to be explicitly gauge-invariant.

Notice also that (6.31) can be regarded as a transition amplitude between two Wilson loops because a space of Wilson loops is isomorphic to the Weyl cell \(K^+_W\) (cf. (D.6)).

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**Appendix**

**A. The Cartan-Weyl basis in Lie algebras**

Any simple Lie algebra \(X\) is characterized by a set of linearly independent \(r\)-dimen-

sional vectors \(\vec{\omega}_j, \; j = 1, 2, ..., r = \text{rank } X\), called simple roots. The simple roots form a
basis for the algebra root system. Any root $\bar{\alpha}$ is a linear combination of $\bar{\omega}_j$ with either non-negative integer coefficients ($\bar{\alpha}$ a positive root) or non-positive integer coefficients ($\bar{\alpha}$ a negative root). Obviously, all simple roots are positive. If $\bar{\alpha}$ is a root then $-\bar{\alpha}$ is also a root. The root system is fixed by the Cartan matrix $c_{ij} = -2(\bar{\omega}_i, \bar{\omega}_j) / (\bar{\omega}_j, \bar{\omega}_j)$ (here $(\bar{\omega}_i, \bar{\omega}_j)$ is a usual scalar product of two $r$-vectors) which can be uniquely restored from the Dynkin diagrams $[19]$. Elements of the Cartan matrix are integers. For any two roots $\bar{\alpha}$ from the Dynkin diagrams $[19]$. Elements of the Cartan matrix are integers. For any two roots $\bar{\alpha}$ and $\bar{\beta}$, the cosine of the angle between them can take only following values $(\bar{\alpha}, \bar{\beta})[(\bar{\alpha}, \bar{\alpha})(\bar{\beta}, \bar{\beta})]^{-1/2} = 0, \pm 1/2, \pm 1/\sqrt{2}, \pm \sqrt{3}/2$. By means of this fact and the Cartan matrix, the whole root system can be restored $[19]$, p.460.

For any two elements $x$, $y$ of $X$, the Killing form is defined as $(x, y) = tr(\hat{x}\hat{y}) = (y, x)$ where the operator $\hat{x}$ acts in $X$, $\hat{x}y = [x, y]$. A maximal Abelian subalgebra $H$ in $X$ is called the Cartan subalgebra, dim $H = rank X = r$. There are $r$ linearly independent elements $\omega_j$ in $H$ such that $(\omega_i, \omega_j) = (\bar{\omega}_i, \bar{\omega}_j)$. We shall also call the algebra elements $\omega_i$ simple roots. It will not lead to any confusing in what follows because the root space $\mathbb{R}^r$ and the Cartan subalgebra are isomorphic, but we shall keep arrows over elements of $\mathbb{R}^r$. The corresponding elements of $H$ have no the over-arrow.

A Lie algebra $X$ is decomposed into the direct sum $X = H \oplus \sum_{\alpha>0}(X_\alpha \oplus X_{-\alpha})$, $\alpha$ ranges the positive roots, dim $X_\pm \alpha = 1$. Simple roots form a basis (non-orthogonal) in $H$. Basis elements $e_{\pm \alpha}$ of $X_\pm \alpha$ can be chosen such that $[19]$, p.176,

$$[e_{\alpha}, e_{-\alpha}] = \alpha, \quad (A.1)$$

$$[h, e_{\alpha}] = (h, e_{\alpha}), \quad (A.2)$$

$$[e_{\alpha}, e_{\beta}] = N_{\alpha,\beta} e_{\alpha+\beta}, \quad (A.3)$$

for all $\alpha, \beta$ belonging to the root system and for any $h \in H$, where the constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. For any such choice $N^2_{\alpha,\beta} = 1/2q(1-p)(\alpha, \alpha)$ where $\beta + n\alpha$ ($p \leq n \leq q$) is the $\alpha$-series of roots containing $\beta$; $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root.

Any element $x \in X$ can be decomposed over the Cartan-Weyl basis (A.1-3),

$$x = x_H + \sum_{\alpha>0}(x^\alpha e_{\alpha} + x^{-\alpha} e_{-\alpha}) \quad (A.4)$$

with $x_H$ being the Cartan subalgebra component of $x$.

The commutation relations (A.1-3) imply a definite choice of the norms of the elements $e_{\pm \alpha}$, namely, $(e_{\pm \alpha}, e_{\pm \alpha}) = 0$ and $(e_{\alpha}, e_{-\alpha}) = 1$ $[19]$, p.167. Norms of simple roots are also fixed in (A.1-3). Consider, for instance, the $su(2)$ algebra. There is just one positive root $\omega$. Let its norm be $\gamma = (\omega, \omega)$. The Cartan-Weyl basis reads $[e_\omega, e_{-\omega}] = \omega$ and $[\omega, e_{\pm \omega}] = \pm \gamma e_{\pm \omega}$. Let us calculate $\gamma$ in this basis. By definition $\gamma = tr\hat{\omega}\hat{\omega}$. The operator $\hat{\omega}$ is $3 \times 3$ diagonal matrix with $0, \pm \gamma$ being its diagonal elements as follows from the basis commutation relations and the definition of the operator $\hat{\omega}$. Thus, $tr\hat{\omega}^2 = 2\gamma^2 = \gamma$, i.e. $\gamma = 1/2$.

The $su(3)$ algebra has two equal-norm simple roots $\bar{\omega}_1$ and $\bar{\omega}_2$ with the angle between them equal to $2\pi/3$. For the corresponding Cartan subalgebra elements we have $(\omega_1, \omega_1) = (\omega_2, \omega_2) = \gamma$ and $(\omega_1, \omega_2) = -\gamma/2$. The whole root system is given by six elements
\[ \pm \omega_1, \pm \omega_2 \text{ and } \pm (\omega_1 + \omega_2) \equiv \pm \omega_{12}. \]

It is readily to see \((\omega_{12}, \omega_{12}) = \gamma \) and \((\omega_1, \omega_{12}) = (\omega_2, \omega_{12}) = \gamma / 2\). All the roots have the same norm and the angle between two neighbor roots is equal to \(\pi / 3\). Having obtained the root pattern, we can evaluate the number \(\gamma\).

The basis (non-orthogonal) consists of eight elements \(\omega_{1,2}, e_{\pm 1}, e_{\pm 2}, e_{\pm 12}\) where we have introduced simplified notations \(e_{\pm 1} \equiv e_{\pm 1}\), etc. The operators \(\hat{\omega}_{1,2}\) are \(8 \times 8\) diagonal matrices as follows from (A.2) and \([\omega_1, \omega_2] = 0\). Using (A.2) we find \(\text{tr} \hat{\omega}_{1,2}^2 = 3\gamma^2 = \gamma\) and, therefore, \(\gamma = 1 / 3\). As soon as root norms are established, one can obtain the structure constants \(N_{\alpha,\beta}\). For \(X = su(3)\) we have \(N_{1,2}^2 = N_{2,1}^2 = N_{1,2,2}^2 = 1 / 6\) and all others vanish (notice that \(N_{\alpha,\beta} = -N_{-\alpha,-\beta}\) and \(N_{\alpha,\beta} = -N_{\beta,\alpha}\)). The latter determines the structure constants up to a sign. The transformation \(e_\alpha \rightarrow -e_\alpha\), \(N_{\alpha,\beta} \rightarrow -N_{\alpha,\beta}\) leaves the Cartan-Weyl basis untouched. Therefore only relative signs of the structure constants must be fixed. Fulfilling the Jacobi identity for elements \(e_{-1}, e_1, e_2, e_{-1}, e_1, e_2\) results in \(N_{1,2} = -N_{1,2,-1}\) and \(N_{1,2} = N_{1,2,-2}\), respectively. Now one can set \(N_{1,2} = N_{1,2,-1} = 1 / \sqrt{6}\), which completes determining the structure constants of \(su(3)\).

One can construct a basis orthonormal with respect to the Killing form. With this purpose we introduce the elements \([13]\), p.181,

\[
s_\alpha = i(e_\alpha - e_{-\alpha}) / \sqrt{2}, \quad c_\alpha = (e_\alpha + e_{-\alpha}) / \sqrt{2}. \tag{A.5}\]

Then \((s_\alpha, s_\beta) = (c_\alpha, c_\beta) = \delta_{\alpha\beta}\) and \((c_\alpha, s_\beta) = 0\). Also,

\[
(x, x) = \sum_{\alpha > 0} \left[ (x_\alpha^s)^2 + (x_\alpha^c)^2 \right] + (x_H, x_H), \quad \tag{A.6}
\]

where \(x_\alpha^s, x_\alpha^c\) are real decomposition coefficients of \(x\) in the orthonormal basis (A.5). Supplementing (A.5) by an orthonormal basis \(h_j, (h_j, h_i) = \delta_{ij}\), of the Cartan subalgebra (it might be obtained by orthogonalizing the simple-root basis of \(H\)), we get an orthonormal basis in \(X\); we shall denote it \(\lambda_\alpha = (h_j, s_\alpha, c_\alpha)\).

Suppose we have a matrix representation of \(X\). Then \((x, y) = c_r \text{ tr } xy\) where \(xy\) means a matrix multiplication. The number \(c_r\) depends on \(X\). For classical Lie algebras, the numbers \(c_r\) are listed in \([19]\), pp.187-190. For example, \(c_r = 2(r + 1)\) for \(X = su(r + 1)\). Using this, one can establish a relation of the orthonormal basis constructed above for \(su(2)\) and \(su(3)\) with the Pauli matrices \([21]\), p.13, and the Gell-Mann matrices \([21]\), p.17, respectively. For the Pauli matrices we have \([\tau_a, \tau_b] = 2i \varepsilon_{abc} \tau_c\), \(\varepsilon_{abc}\) a totally antisymmetrical tensor, \(\varepsilon_{123} = 1\), hence, \((\tau_a, \tau_b) = -4 \varepsilon_{ab'c'} \varepsilon_{bc'c''} = 8 \delta_{ab} = 4tr \tau_\alpha \tau_\beta\) in full accordance with \(c_r = 2(r + 1), r = 1\). One can set \(\omega = \tau_3 / 4\), \(s_\omega = \varphi \tau_1\) and \(c_\omega = \varphi \tau_2\) where \(1 / \varphi = 2 \sqrt{2}\). A similar analysis of the structure constants for the Gell-Mann matrices \(\lambda_a\)

\([21]\), p.18, yields \(\omega_1 = \lambda_3 / 6, s_1 = \varphi \lambda_1, c_1 = \varphi \lambda_2, \omega_2 = (\sqrt{3} \lambda_8 - \lambda_3) / 12, s_2 = \varphi \lambda_6, c_2 = \varphi \lambda_7, \omega_12 = (\sqrt{3} \lambda_8 + \lambda_3) / 12, s_12 = \varphi \lambda_5\) and \(c_{12} = -\varphi \lambda_4\) where \(1 / \varphi = 2 \sqrt{3}\). This choice is not unique. Actually, the identification of non-diagonal generators \(\lambda_a, a \neq 3, 8\) with (A.5) depends on a representation of the simple roots \(\omega_{1,2}\) by the diagonal matrices \(\lambda_{3,8}\). One could choose \(\omega_1 = \lambda_3 / 6\) and \(\omega_2 = - (\sqrt{3} \lambda_8 + \lambda_3) / 12\), which would lead to another matrix realization of elements (A.5).

### B. The Hamiltonian path integral in an arbitrary gauge
Consider a quantum theory determined by the Schrödinger equation
\[
\left( -\frac{1}{2} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle + V(x) \right) \psi_E = E \psi_E .
\] (B.1)

The eigen-functions \( \psi_E \) are normalized by the condition
\[
\int_{\mathbb{R}^N} d x \psi^*_E(x) \psi_E(x) = \delta_{EE'} .
\] (B.2)

We assume \( x \) to realize a linear representation of a compact group \( G: x \to \Omega(\omega)x, \ \Omega(\omega) \in G \) and \( V(\Omega x) = V(x) \); \( \left\langle x, y \right\rangle = \sum_1^N x_i y_i = \left\langle \Omega x, \Omega y \right\rangle \) is an invariant scalar product in the representation space that is isomorphic to \( \mathbb{R}^N \). The theory turns into the gauge one if we require that physical states are annihilated by operators \( \hat{\sigma}_a \) generating \( G \)-transformations of \( x, \hat{\sigma}_a \Phi(x) = 0 \). These conditions determine a physical subspace in the Hilbert space. By definition, we have \( \exp(\omega_a \hat{\sigma}_a) \psi(x) = \psi(\Omega(\omega)x) \) where \( \Omega(\omega) \in G \). Therefore, the physical states are \( G \)-invariant
\[
\Phi(\Omega(\omega)x) = \Phi(x) .
\] (B.3)

Let a number of physical degrees of freedom in the system is equal to \( M \), then a number of independent constraints is \( N - M \). Suppose we would like to span the physical configuration space \( K \sim \mathbb{R}^N/G \) by coordinates ranging a gauge condition surface \( F(x) = 0 \). We assume the gauge condition to be complete, meaning that there is no unphysical degree of freedom left. Let \( u \in \mathbb{R}^M \) be a parameter of the gauge condition surface; \( x = f(u) \) such that \( F(f(u)) \) identically vanishes for all \( u \in \mathbb{R}^M \). By analogy with (4.15) we introduce curvilinear coordinates
\[
x = x(\theta, u) = \Omega(\theta)f(u) ,
\] (B.4)
where variables \( \theta \) ran over the manifold \( G/G_f \) with \( G_f \) being a stationary group of the vector \( x = f, \ G_f f = f \).

The metric tensor in the new coordinates reads
\[
\left\langle dx, dx \right\rangle = \left\langle df, df \right\rangle + 2 \left\langle df, d\theta f \right\rangle + \left\langle d\theta f, d\theta f \right\rangle \equiv g_{AB} dy^A dy^B ,
\] (B.5)
where we have put \( d\theta = Ω^+ dΩ \) and \( dy^1 = du, \ dy^2 = d\theta \). Therefore,
\[
\int_{\mathbb{R}^N} dx = \int_{G/G_f} d\theta \int_K d^M u \mu(u) ;
\] (B.6)

here \( \mu(u) = (\det g_{AB})^{1/2} \), \( K \) is a subdomain in \( \mathbb{R}^M \) such that the mapping (B.4), \( K \otimes G/G_f \to \mathbb{R}^N \), is one-to-one. To determine \( K \), one should find transformations \( \theta, u \to \hat{s}\theta, \hat{s}u, \ \hat{s} \in \hat{S}_F \) which leave \( x \) untouched, \( x(\hat{s}\theta, \hat{s}u) = x(\theta, u) \) (cf. (4.19)). Obviously, \( \hat{S}_F = T_e \times S_F \) where \( T_e \) is a group of translations of \( \theta \) through periods of the manifold \( G/G_f \), while the set \( S_F \) is obtained by solving Eqs.(4.2-3) with \( f \to f \in \mathbb{R}^N, \ u \in \mathbb{R}^M, \ \Omega_s \in G, \ \text{so} \ K \sim \mathbb{R}^M/S_F \). Indeed, assuming Eq.(4.2) to have non-trivial solutions (the trivial one \( \Omega_s = 1 \)
always exists by the definition of $f(u)$ we observe that all points $\Omega_s f$ belong to the gauge condition surface and, hence, $\Omega_s f(u) = f(u_s)$, $u_s = u_s(u)$. Consider transformations of $\theta$ generated by the group shift $\Omega(\theta) \to \Omega(\theta)T_s^{-1} = \Omega(\theta_s)$, $\theta_s = \theta(\phi, u)$. Setting $\hat{s}u = u_s$ and $\hat{s}\theta = \theta_s$ we see that the transformations $\hat{s} \in S_F$ leave $x = x(\theta, u)$ untouched. To avoid a "double" counting in the scalar product integral (B.6), one has to restrict the integration domain for $u$ to the quotient $\mathbb{R}^M/S_F = K$. The modular domain $K$ can also be determined by the requirement as a part of the gauge condition surface $x = f(u)$, $u \in K \subset \mathbb{R}^M$, which has just one common point with any gauge orbit.

A choice of the fundamental domain parametrization is not unique as we have seen in Sec.4 (cf. (4.5-7) and (4.9-11)). In (B.6), $\mu \geq 0$ for $u \in K$. Having chosen the parametrization of $K$, we fix a representation of $S_F$ by functions $\hat{s}u = u_s(u)$, $u \in K$, $u_s \in K_s$, $K_s \cap K_{s'} = \emptyset$ for any $\hat{s} \neq \hat{s}'$ and $\mathbb{R}^M = \bigcup K_s$ up to a set of zero measure being a unification of the boundaries $\partial K_s$. We define an orientation of $K_s$ so that for all $\hat{s} \in S_F$, $\int_{K_s} du \phi \geq 0$ if $\phi \geq 0$, so that the following rules hold

$$\int_{\mathbb{R}^M} du = \sum_{S_F K_s} \int_{K_s} du \ , \quad \text{(B.7)}$$
$$\int_{K} du |J_s(u)| = \int_{K_s} du \ , \quad \text{(B.8)}$$

where $J_s(u) = Du_s/ Du$ is the Jacobian, the absolute value of $J_s$ has been inserted into the right-hand side of (B.8) for preserving the positive orientation of the integration domain.

**Remark.** A number of elements in $S_F$ can depend on $u$. We define a region $\mathbb{R}^M_{\alpha} \subseteq \mathbb{R}^M$ such that $S_F = S_k$ has a fixed number of elements for all $u \in \mathbb{R}^M_{\alpha}$. Then $K = \cup_{\alpha} K_{\alpha}$, $K_{\alpha} = \mathbb{R}^M_{\alpha}/S_{\alpha}$, $\mathbb{R}^M = \bigcup_{\alpha} \mathbb{R}^M_{\alpha}$. The sum in (B.7) implies $\sum_{S_F} = \sum_{\alpha} \sum_{S_k}$ and $K_s$ in (B.7-8) carries an additional suffix $\alpha$. In what follows we shall omit it and use the simplified notations (B.7-8) to avoid complications of formulas. The suffix $\alpha$ can be easily restored by means of the rule proposed above.

For the mechanical model considered in Sec.4, we have $G = SO(2)$, $G_f = 1$, $\det G_{AB} = f'f - (f'Tf)^2 = (f', f)^2 = \mu^2(u)$. Set $K = \cup_{\alpha} K_{\alpha}$, $K_1 = (0, u_0/\gamma_0)$, $K_2 = (u_0/\gamma_0, u_0)$, $K_3 = (u_0, \infty)$, i.e. $K = \mathbb{R}_+$, then $\int_{-\infty}^{\infty} du = \sum_{\alpha} \int_{\mathbb{R}_{\alpha}} du$ and (B.7) means that the upper integral limit is always greater than the lower one, for example,

$$\int_{\mathbb{R}_2} du = \left(\int_{-u_0}^{u_0/\gamma_0} + \int_{-2u_0}^{-u_0} + \int_{-3u_0}^{-2u_0} + \int_{0}^{u_0/\gamma_0} \right) du \ ,$$

where the terms of the sum correspond to integrations over $\hat{s}_3 K_2$, $\hat{s}_2 K_2$, $\hat{s}_1 K_2$ and $K_2$, respectively (cf. (4.5-7)). The following chain of equalities is to illustrate the rule (B.8)

$$\int_{\hat{s}_3 K_2} du_{s_3} = \int_{-2u_0}^{-u_0/\gamma_0} du_{s_3} = \int_{-3u_0}^{-2u_0} du_{s_3} \left(\int_{u_0/\gamma_0}^{u_0} du J_{s_3} = - \int_{u_0/\gamma_0}^{u_0} du J_{s_3} = \int_{K_2} du |J_{s_3}| \right)$$

the last equality results from $J_{s_3} = du_{s_3}/ du < 0$ (cf. (4.7)).
By means of the curvilinear coordinates (B.4) we can naturally incorporate a gauge condition chosen into the Dirac operator method [1] of quantizing first-class constrained systems. Solutions of the equation \( \hat{\sigma} \Phi(x) = 0 \) are given by functions independent of \( \theta \),

\[
\Phi(x) = \Phi(\Omega(\theta)f(u)) = \Phi(f(u)) = \Phi(u),
\]

(B.10)
because \( \hat{\sigma} \) generate shifts of \( \theta \) and leave \( u \) untouched. To obtain a physical Hamiltonian, one has to write the Laplacian in (B.1) via the new variables (B.4) and omit all terms containing derivatives with respect to \( \theta \). In so doing, we get [20], [13]

\[
\hat{H}_{ph}^{f} \Phi_{E}(u) = \left( \frac{1}{2} \hat{p}_{i} g_{ph}^{ij} \hat{p}_{j} + V_{q}(u) + V(f(u)) \right) \Phi_{E}(u) = E \Phi(u);
\]

(B.11)

here we have introduced hermitian momenta \( \hat{p}_{i} = -i\mu^{-1/2} \partial_{i} \circ \mu^{1/2} \), \( \partial_{i} = \partial/\partial u^{i} \); the metric \( g_{ph}^{ij} \) in the physical configuration space is the 11-component of a tensor \( g^{AB} \) inverse to \( g_{AB} \), \( g^{AC} g_{CB} = \delta_{A}^{B} \), \( g_{ph}^{ij} = (g^{11})^{ij} \), \( i, j = 1, 2, ..., M \); a quantum potential

\[
V_{q} = \frac{1}{2\sqrt{\mu}} (\partial_{i} g_{ph}^{ij} \partial_{j} \sqrt{\mu}) + \frac{1}{2\mu} g_{ph}^{ij} \partial_{i} \partial_{j} \sqrt{\mu}
\]

appears due to the chosen ordering of the operators \( \hat{u}^{i} \) and \( \hat{p}_{i} \) in the Laplace-Beltrami operator. The scalar product is reduced to

\[
\int_{\mathbb{R}^{N}} dx \Phi_{E}^{*}(u) \Phi_{E'}(u) \to \int_{K} d^{M}u \mu(u) \Phi_{E}^{*}(u) \Phi_{E'}(u) = \delta_{EE'},
\]

(B.13)

where a gauge orbit volume (integral over \( G/G_{f} \) (see (B.6)) has been included into norms of physical states, which we denoted by the arrow in (B.13). A construction of an operator description of a gauge theory in a given gauge condition is completed.

Notice, in this approach the variables \( u \) appear to be gauge-invariant; they parametrize the physical configuration space \( CS_{ph} = \mathbb{R}^{N}/G \). Two different choices of \( f(u) \) correspond two different parametrizations of \( CS_{ph} \) related to each other by a change of variables \( u = u(\bar{u}) \) in (B.11-13). Therefore quantum theories with different \( f \)'s are unitary equivalent [13].

To illustrate this statement, we consider again the simplest case \( G = SO(2) \), \( M = 1 \), \( g_{ph} = r^{2}(u)/\mu^{2}(u) \), and compare descriptions in the coordinates (4.15) and in the polar ones \( (f_{1} = r, f_{2} = 0) \). With this purpose we change variables \( r = r(u) \) in (B.11-13). For \( u \in K \) the function \( r(u) \) is invertible, \( u = u(r), r \in \mathbb{R}_{+} \). Simple straightforward calculations [3] lead us to the following equalities \( \hat{H}_{ph}^{f} = 1/2 \hat{p}_{r}^{2} + V_{q}(r) + V, \hat{p}_{r} = -ir^{-1/2} \partial_{r} \circ r^{1/2}, V_{q} = -(8r^{2})^{-1}, f_{K} du \mu = f_{0}^{\infty} dr r. \) It is nothing but quantum mechanics of a radial motion on a plane. All theories with different \( f \)'s are unitary equivalent to it and, therefore, to each other. One should stress that the operator ordering we obtained by applying the Dirac method plays the crucial role in providing this unitary equivalence. Another ordering of operators in (B.11) would break this property.

A few observations resulting from our consideration have to be emphasized.
1. All regular solutions of (B.11) have a unique analytical continuation to the whole space $u \in \mathbb{R}^M$, and they are $S_F$–invariant,
\[
\Phi_E(u_s(u)) = \Phi_E(u), \quad u \in K.
\] (B.14)
For a proof, we point out that any regular solution of (B.11) is a projection of a regular $G$-invariant solution of (B.1) on $K$ determined by (B.10). The last equality in (B.10) defines the analytical continuation of $\Phi_E(u)$; (B.14) follows from the second equality in (B.10) and (4.3).

2. Any amplitude, i.e. a scalar product (B.13) of two $S_F$-invariant states, is independent of a $CS_{ph}$ parametrization (of a gauge choice). An $S_F$-invariant regular function of $u \in \mathbb{R}^M$ can be decomposed over the basis $\Phi_E(u)$. Our statement follows from the fact that theories (B.11-13) corresponding different parametrizations of $CS_{ph}$ are unitary equivalent.

3. Quantization before eliminating unphysical variables (the Dirac method) is necessary, otherwise one might gain a gauge-dependence of a quantum theory and, as a result, 1 and 2 do not hold. Indeed, if we would quantize the classical Hamiltonian (4.18), we encounter the operator ordering problem whose solution is not unique. However the gauge (parametrization) independence of the quantum theory can be achieved just at the definite operator ordering in (B.11-12) uniquely resulting from the application of the Dirac method as we have seen above.

4. The physical Hamiltonian in (B.11) is $S_F$-invariant
\[
\hat{H}^f_{ph}(u_s(u)) = \hat{H}^f_{ph}(u), \quad u \in K.
\]
Let us write the Laplace-Beltrami operator $\langle \partial / \partial x, \partial / \partial x \rangle = \Delta(\theta, u)$ in the variables (B.4), push all derivatives $\partial_\theta$ in it to the right by commuting them with $\theta$ and then set $\partial_\theta = 0$. We denote the operator thus obtained $\Delta_{ph} = \Delta(\theta, u)|_{\partial_\theta = 0}$. For any physical state $\Phi = \Phi(u)$, we have $\Delta(\theta, u)\Phi = \Delta_{ph}\Phi$ because $\delta \sim \partial_\theta$. Due to the gauge-invariance, the Hamiltonian in (B.1) commutes with the constraints, $[\hat{H}, \delta] = 0$ and, hence, $[\Delta(\theta, u), \delta] = 0$ (the potential $V$ is $G$-invariant). Gathering the definition of $\Delta_{ph}$ and $G$-invariance of $\Delta$ we conclude that $\Delta_{ph} = \Delta_{ph}(u)$ is independent of $\theta$ (otherwise we would arrive to the contradiction $0 = [\delta, \hat{\Phi}] = \delta \Delta \hat{\Phi} = \delta \Delta_{ph}\Phi \sim \partial_\theta \Delta_{ph}\Phi \neq 0$). Consider now the change of variables $\theta, u \rightarrow \theta_s, u_s$. By its definition $\Delta(\theta_s, u_s) = \Delta(\theta, u)$ and $\partial_\theta \sim \partial_{\theta_s}$ (i.e. $\partial_\theta$ does not contain a term proportional $\partial_{u_s}$ since $\partial u_s/\partial_\theta = 0$). This yields $\Delta_{ph}(u_s) = \Delta(\theta_s, u_s)|_{\partial_{\theta} = 0} = \Delta(\theta, u)|_{\partial_{\theta} = 0} = \Delta_{ph}(u)$, which completes the proof of the $S_F$–invariance of the physical Hamiltonian.

To derive a path integral representation of the quantum theory (B.11-13), we consider a slice approximation of the transition amplitude $U_{t}^{ph}(u, u') = \langle u | \exp(-i\hat{H}_{ph}t)| u' \rangle$,
\[
U_{t}^{ph}(u, u') = \lim_{\epsilon \to 0} \int_{K} \prod_{k=0}^{n} (d^M u_k \mu(u_k)) U_{\epsilon}^{ph}(u, u_n)U_{\epsilon}^{ph}(u_n, u_{n-1}) \cdots U_{\epsilon}^{ph}(u_1, u'),
\] (B.15)
where $(n + 1)\epsilon = t$, the limit is taken so that $n \to \infty$, $\epsilon \to 0$, while $t$ is kept fixed; the infinitesimal evolution operator kernel reads
\[
U_{\epsilon}^{ph}(u, u') = [1 - i\epsilon \hat{H}_{ph}(u)]| u | u' \rangle + O(\epsilon^2).
\] (B.16)
A naive limit in (B.15) gives a formal PI with a restricted integration domain $K \subset \mathbb{R}^M$. A calculation of such a PI meets difficulties because even a finite dimensional Gaussian integral cannot be explicitly done over a part of an Euclidean space. In addition, a restriction of the PI integration domain is meaningless for systems with boundary conditions appearing due to a non-trivial topology of a configuration space like for a particle in a box or on a circle \[22, 23\] (see also Sec.5). Topological properties of a configuration space are taken into account in PI by including additional “reflected” trajectories into the sum over paths \[22\] rather than by restricting the PI integration domain. Technically, a relation taken into account in PI by including additional “reflected” trajectories into the sum over paths \[22\] is obtained by a convolution formula with the infinitesimal evolution operator kernel (B.23). The same form of PI turns out to be valid in gauge theories \[11\]-\[16\]. Below we shall prove this. For deriving a PI formula we shall use a method of an analytical continuation of the unit operator kernel \[1\], \[13\].

The unit operator kernel $\langle u|u' \rangle$ has a natural analytical continuation to the unphysical domain $u \in \mathbb{R}^M$. Indeed, due to the $S_F$-invariance of the basis states (B.14) we have \[1\]-\[16\], \[13\]:

$$
\langle u|u' \rangle = \sum_E \Phi_E(u)\Phi_E^*(u') = \sum_{S_F} (\mu(u)\mu(\hat{u}'))^{-1/2} \delta^M(u - \hat{u}') ,
$$

(B.17)

where $u \in \mathbb{R}^M$, $u' \in K$. Representing $\delta$-functions in (B.17) through the Fourier integral and calculating the action of $\hat{H}_{ph}(u)$ on the unit operator kernel in (B.16), we obtain

$$
U_{ph}^\epsilon(u, u') = \int_{\mathbb{R}^M} \frac{d^M u''}{(\mu'')^{1/2}} U_{eff}^\epsilon(u'', u')Q(u'', u) ,
$$

(B.18)

$$
U_{eff}^\epsilon(u, u'') = \int_{\mathbb{R}^M} \frac{d^M p}{(2\pi)^M} \exp \left[ i \epsilon \left( p_j \frac{\Delta_j}{\epsilon} - H_{eff}(u, p) \right) \right] ,
$$

(B.19)

$$
H_{eff}(u, p) = \frac{1}{2} g_{ji}^\epsilon(u)p_j p^i_j + \frac{i}{2} \partial_i g_{ph}^\epsilon(u)p_j + V_{q}(u) + V ,
$$

(B.20)

$$
Q(u, u'') = \sum_{S_F} \delta^M(u - \hat{u}''), \quad u'' \in \mathbb{R}^M , \quad u \in K ,
$$

(B.21)

where $\mu'' = \mu(u'')$ and $\Delta_j = u_j - u''_j$. So, the infinitesimal evolution operator kernel (B.18) has the desired form $\hat{U}_{ph}^\epsilon = \hat{U}_{eff}^\epsilon \hat{Q}$. A next step is to prove the convolution formula

$$
U_{ph}^{2\epsilon}(u, u') = \int_K d^M u_1 \mu(u_1)U_{ph}^\epsilon(u, u_1)U_{ph}^\epsilon(u_1, u')
$$

(B.22)

$$
U_{eff}^{2\epsilon}(u, u'') = \int_{\mathbb{R}^M} \frac{d^M u''}{(\mu'')^{1/2}} U_{eff}^{2\epsilon}(u, u'')Q(u'', u') ,
$$

(B.23)

$$
U_{eff}^{2\epsilon}(u, u''') = \int_{\mathbb{R}^M} d^M u'''U_{eff}^{2\epsilon}(u, u''')U_{eff}^{2\epsilon}(u'', u') ,
$$

(B.24)
or in the operator form
\[ \hat{U}_{2e}^{ph} = \hat{U}_e^{eff} \hat{Q} \hat{U}_e^{eff} \hat{Q} = \hat{U}_e^{eff} \].

The proof is given by the following chain of equalities

\[ U_{2e}^{ph}(u, u') = \sum_{S_F K} \int_{dF K} \frac{\mu_1}{(\mu(u_1))^{1/2}} U_e^{eff}(u, \hat{s}u_1)U_{\epsilon}^{ph}(u_1, u') = \]

\[ = \sum_{S_F K} \int_{dF K} |J_s(u_1)|^{1/2} \left( \frac{\mu_1}{\mu} \right)^{1/2} U_e^{eff}(u, \hat{s}u_1)U_{\epsilon}^{ph}(\hat{s}u_1, u') = \]

\[ = \int_{\mathbb{R}^M} \frac{du''}{(\mu(u''))^{1/2}} \sum_{S_F K} \int_{dF K} |J_s(u_1)|U_e^{eff}(u, \hat{s}u_1)U_e^{eff}(\hat{s}u_1, u'')Q(u'', u') , \]

where \( \hat{s}u_1 = u_s(u_1), \mu_1 = \mu(u_1), \mu = \mu(u) \) and \( \mu'' = \mu(u'') \). To obtain (B.26), we substitute (B.18) into (B.22) and do the integral over \( u'' \). For the transformation (B.27) we use the relation \( \mu(u_1) = \mu(u)/J_s(u) \) (which follows from the \( S_F \)-invariance of the measure \( d^N x = (\wedge d\theta)d^M u \mu(u) = (\wedge d\theta)d^M u \mu(u) \)) and the \( S_F \)-invariance of the kernel (B.18) (or (B.16)). The latter results from the obvious relation \( \langle \hat{s}u | u' \rangle = \langle u | u' \rangle \) (cf. (B.17)) and \( H^I_{ph}(\hat{s}u) = \overline{H}_{ph}(u) \) (see p.4 above). Equality (B.28) is derived by substituting (B.18) into (B.27) and using the relation \( \mu(\hat{s}u_1) = \mu(u)/J_s(u) \) again. Finally, (B.28) turns into (B.23) after changing variables \( u_1 \rightarrow \hat{s}u_1 \) in each term of the sum in (B.28) by means of the rules (B.8) and (B.7).

For a finite time we get
\[ \hat{U}_t^{ph} = \lim_{\epsilon \to 0} \hat{U}_e^{eff} \hat{Q} \hat{U}_e^{eff} \hat{Q} \cdots \hat{U}_e^{eff} \hat{Q} = \lim_{\epsilon \to 0} \hat{U}_e^{eff} \cdots \hat{U}_e^{eff} \hat{Q} = \hat{U}_t^{eff} \hat{Q} , \]

\[ U_t^{eff}(u, u'') = \lim_{\epsilon \to 0} \int_{\mathbb{R}^M} \left( \prod_{k=1}^n d^M u_k \right) U_e^{eff}(u, u_n)U_e^{eff}(u_n, u_{n-1}) \cdots U_e^{eff}(u_1, u'') . \]

Formulas (B.29-30) and (B.19) with \( \Delta_\epsilon/\epsilon = \hat{u}_k + O(\epsilon) \) solve the problem of the PI construction. Equalities (B.30) and (B.19) determine a standard slice approximation of PI over an Euclidean phase space. Removing the slice regularization in (B.30) we obtain the path integral
\[ U_t^{eff}(u, u'') = \int_{\mathbb{R}^{2M}} \left( \prod_{\tau=0}^{t} \frac{d^M p(\tau)d^M u(\tau)}{(2\pi)^n} \right) \exp i \int_0^t d\tau \left( p_j \hat{u}_j - H^{eff} \right) , \]

where the measure implies a sum over all trajectories \( u(\tau) \) going from the initial point \( u'' = u(0) \) to the final one \( u = u(t) \). The physical transition amplitude is given by (B.18) \( (\epsilon \to 0) \) and implies a sum over trajectories going from initial points \( u_s(u') = \hat{s}u' \), \( u' = u(0), \hat{s} \in S_F \) rather than from the only one \( u' \). A trajectory going from one of these points, say, \( \hat{s}u' \in K_s, \hat{s} = 1 \), to \( u = u(t) \in K \) must cross the boundary \( \partial K \) at a point \( \hat{u} = u(\tau) \). Suppose for simplicity that \( u(\tau) \in K \) if \( \tau \in (\bar{\tau}, t) \) and \( u(\tau) \in K_s \) if \( \tau \in (0, \bar{\tau}) \). Consider a reflected trajectory composed of two pieces \( \hat{s}^{-1}u(\tau), \tau \in (0, \bar{\tau}) \) and \( u(\tau), \tau \in (\bar{\tau}, t) \), i.e.
it connects the initial point \( u' \in K \), the "reflection" point \( \tilde{u} \in \partial K \) and the final point \( u \in K \). Due to the \( S_F \)-invariance of the effective action, the reflected trajectory gives the same contribution into the sum over paths as the "straight" one \( u_s(u') \rightarrow u \). Therefore the PI modification (B.29) due to a non-trivial topology of \( PS_{ph} \) (or \( CS_{ph} \)) means that in addition to "straight" trajectories \( u' \rightarrow u \), the reflected trajectories \( u' \rightarrow \partial K \rightarrow u \) must be included into the sum over paths.

C. Properties of the measure on the gauge orbit space

Consider properties of the function (6.12) under transformations from \( W_A \). According to (3.17) we have

\[
\kappa(\hat{s}_\beta a) = (-1)^{n_\rho} \kappa(\hat{s}_\beta a) \tag{C.1}
\]

where an integer \( n_\rho \) is given by

\[
n_\rho = \frac{2(\beta, \alpha)}{(\beta, \beta)} = \frac{4(\beta, \rho)}{(\beta, \beta)} , \tag{C.2}
\]

\[
\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha . \tag{C.3}
\]

The half-sum (C.3) of all positive roots possesses remarkable properties \[15\], p.461,

\[
\frac{2(\omega, \rho)}{(\omega, \omega)} = 1 , \tag{C.4}
\]

\[
\hat{s}_\omega \rho = \rho - \omega \tag{C.5}
\]

for any simple root \( \omega \). Since the Weyl group \( W \) preserves the root system and a reflection \( \hat{s}_\beta \) in the hyperplane \( (\beta, a) = 0 \) is a composition of reflections \( \hat{s}_\omega , \omega \) ranges simple roots, there exists an element \( \hat{s} \in W \) and a simple root \( \omega_\beta \) such that \( \hat{s} \omega_\beta = \beta \). Hence, the number \( n_\rho/2 \) is an integer,

\[
\frac{n_\rho}{2} = \frac{2(\beta, \rho)}{(\beta, \beta)} = \frac{2(\omega_\beta, \beta)}{(\omega_\beta, \omega_\beta)} \in \mathbb{Z} . \tag{C.6}
\]

Indeed, representing \( \hat{s}^{-1} \) as a composition of the generating elements \( \hat{s}_\omega \) and applying (C.4) and (C.5) we obtain (C.6) since \( 2(\omega_\beta, \alpha)/(\omega_\beta, \omega_\beta) \in \mathbb{Z} \) for any root \( \alpha \) (Appendix A). Thus, \( n_\rho \) is an even integer, i.e. the factor \((-1)^{n_\rho} \) in (C.1) is equal to 1.

A reflection \( \hat{s}_\beta \) permutes roots and therefore for any positive root \( \alpha \) we have \( \hat{s}_\beta \alpha = \pm \gamma \) where \( \gamma \) is also a positive root, i.e. the function (6.12) may only change its sing under transformations from the Weyl group. Consider positive roots \( \alpha \) and \( \gamma \) such that \( \hat{s}_\beta \alpha = -\gamma \). Then \( \hat{s}_\beta \gamma = -\alpha \) because \( \hat{s}_\beta^2 = 1 \), which implies that a number of positive root changing their sing under the reflection \( \hat{s}_\beta \) is always odd since \( \hat{s}_\beta \beta = -\beta \). This yields

\[
\kappa(\hat{s}_\beta a) = -\kappa(a) \tag{C.7}
\]
for any root $\beta$.

Since any elements of the affine Weyl group $W_A$ is a composition of the reflections (3.17), we conclude

$$\kappa(\hat{s}a) = \det \hat{s}\kappa(a) = \pm \kappa(a), \quad \hat{s} \in W_A,$$

where $\det \hat{s} = -1$ if $\hat{s}$ contains an odd number of the reflections (3.17) and $\det \hat{s} = 1$ for the even one.

The function $\kappa(a)$ is an eigenfunction of the Laplace operator in $\mathbb{R}^r$

$$(\partial_a, \partial_a)\kappa(a) = -\frac{4\pi^2}{a_0^2}(\rho, \rho)\kappa(a).$$

(A.9)

A straightforward calculation of the action of the Laplace operator on $\kappa(a)$ leads to the equality

$$(\partial_a, \partial_a)\kappa(a) = -\frac{4\pi^2}{a_0^2}(\rho, \rho)\kappa(a) + \frac{\pi^2}{a_0^2} \sum_{\alpha \neq \beta > 0} (\alpha, \beta) \left[ \cot \frac{\kappa(a, \alpha)}{a_0} \cot \frac{\kappa(a, \beta)}{a_0} + 1 \right] \kappa(a).$$

(A.10)

Relation (A.9) follows from

$$\sum_{\alpha \neq \beta > 0} (\alpha, \beta) [\cot(b, \beta) \cot(b, \beta) + 1] = \sum_{\text{all } \alpha \neq \beta > 0} \sum_{\text{planes in one plane}} (\alpha, \beta) [\cot(b, \alpha) \cot(b, \beta) + 1] = 0,$$

(A.11)

where $b \in H$. We have divided the sum over $\alpha \neq \beta > 0$ in (A.11) into sum over two dimensional planes; each plane contains at least two roots. A root pattern in each plane coincides with one of the root patterns for algebras of rank 2, $su(3)$, $sp(4) \sim so(5)$ and $g_2$ because the absolute value of cosine of an angle between any two roots $\alpha$ and $\beta$ may take only four values $|\cos \theta_{\alpha \beta}| = 0, 1/\sqrt{2}, 1/2, \sqrt{3}/2$ (see Appendix A). For the algebras of rank 2, equality (A.11) can be verified by explicit calculations, i.e. the sum (A.11) for each plane vanishes, which implies vanishing the whole sum. In the case of the $su(3)$ algebra, the sum (A.11) is proportional to

$$- \cot b_1 \cot b_2 + \cot b_1 \cot(b_1 + b_2) + \cot b_2 \cot(b_1 + b_2) + 1 = 0,$$

where $b_{1,2} = (b, \omega_{1,2})$, and $\omega_1, \omega_2$ and $\omega_1 + \omega_2$ constitute all positive roots (see Appendix A). An explicit form of the quantum potential (6.22) results from (A.9).

The measure (6.12) is proportional to the Weyl determinant [30], p.185

$$(2i)^{N_+} \kappa(a) = \prod_{\alpha > 0} \left( e^{i\pi(a, a)/a_0} - e^{-i\pi(a, a)/a_0} \right) = \sum_{\hat{s} \in W} \det \hat{s} \exp \left( \frac{2\pi i}{a_0} (\hat{s}\rho, a) \right)$$

(C.12)

with $N_+$ being a number of positive roots, $N_+ = (\dim X - r)/2$.

D. Eigenstates in 2D Yang-Mills theories
Substituting $\Phi_n = \kappa^{-1}\Psi_n$ into (6.21) we find that $\Psi_n$ is an $r$-dimensional plane wave, $\exp(2\pi i(\gamma_n/a_0))$. However, not all values of the momentum vector $\gamma_n \in H$ are admissible. As has been shown in Appendix B, only regular solutions to (6.21) have a physical meaning. First of all, $\Psi_n(a)$ should vanish on the hyperplanes orthogonal to positive roots, $(\alpha, a) = 0$, because the function $\kappa^{-1}(a)$ has simple poles on them. Since $(\hat{s}\gamma_n, \hat{s}\gamma_n) = (\gamma_n, \gamma_n)$, $\hat{s} \in W$, the function

$$\Psi_n(a) \sim \sum_{\hat{s} \in W} \det \hat{s} \exp \frac{2\pi i}{a_0}(\hat{s}\gamma_n, a)$$

is an eigenstate of the $r$-dimensional Laplace operator and vanishes as $a$ approaches any of hyperplanes $(\alpha, a) = 0$. To prove the latter, let us decompose $a$ into two parts $a = a^\parallel + a^\perp$, such that $(a^\perp, \alpha) = 0$ for a root $\alpha$ and let $W^{(\alpha)}$ be the quotient $W/\mathbb{Z}_{\alpha}^{(\alpha)}$, $\mathbb{Z}_{\alpha}^{(\alpha)} = \{1, \hat{s}_\alpha\}$, where $\hat{s}_\alpha \alpha = -\alpha$ and, therefore, $\hat{s}_\alpha a^\perp = a^\perp$, $\hat{s}_\alpha a^\parallel = -a^\parallel$, $\det \hat{s}_\alpha = -1$. Then the sum (D.1) can be rewritten as follows

$$\Psi_n(a) \sim \sum_{\hat{s} \in W^{(\alpha)}} \det \hat{s} \exp \frac{2\pi i}{a_0}(\hat{s}\gamma_n, a) + \sum_{\hat{s} \in W^{(\alpha)}} \det \hat{s}_\alpha \hat{s} \exp \frac{2\pi i}{a_0}(\hat{s}_\alpha \hat{s}\gamma_n, a) =$$

$$\sum_{\hat{s} \in W^{(\alpha)}} \det \hat{s} \exp \frac{2\pi i}{a_0}(\hat{s}\gamma_n, a) \left[1 - \exp \left(-\frac{4\pi i}{a_0}(\hat{s}\gamma_n, a^\parallel)\right)\right].$$

As $a^\parallel$ approaches zero (i.e. $a$ approaches the hyperplane $(\alpha, a) = 0$), the function (D.2) vanishes as $(a^\parallel, \alpha)$, therefore $\Phi_n = \Psi_n/\kappa$ has a regular behavior on the hyperplanes $(\alpha, a) = 0$.

In a neighborhood of the hyperplane $(\alpha, a) = n_\alpha a_0$, $n \in \mathbb{Z}$, $n_\alpha \neq 0$, $a^\parallel = n_\alpha a_0 \alpha / (\alpha, \alpha) + c\alpha$ where $c \to 0$. The function (D.2) vanishes as $c \to 0$ if $2(\hat{s}\gamma_n, \alpha) / (\alpha, \alpha)$. $\hat{s} \in W^{(\alpha)}$, is an integer. Since $\hat{s}_\alpha = \beta$ is a root we conclude that the function $\Phi_n = \Psi_n/\kappa$ with $\Psi_n$ given by (D.1) is regular if

$$\frac{2(\gamma_n, \beta)}{(\beta, \beta)} \in \mathbb{Z}$$

(D.3)

for any root $\beta$.

Eigenvalues $E_n$ in (6.21) read

$$E_n = \frac{\pi \hbar^2}{a_0^2} \left[(\gamma_n, \gamma_n) - (\rho, \rho)\right].$$

(D.4)

For any $\gamma_n$ obeying (D.3), a vector $\hat{s}_0 \gamma_n$, $\hat{s}_0 \in W$, also satisfies (D.3) and corresponds to the same energy level (D.4) because the Killing form is $W$-invariant. Exchanging $\gamma_n$ by $\hat{s}_0 \gamma_n$ in (D.1) we have $\Psi_n(a) \to \det \hat{s}_0 \Psi_n(a)$, which means that linearly independent wave functions corresponding to a given energy level (D.4) are determined only by $\gamma_n \in K^+$.

\footnote{Moreover, singular solutions do not satisfy the Schroedinger equation in the whole configuration space (see, for example, [31], pp. 155-156).}
i.e. \((\omega, \gamma_n) > 0\), \(\omega\) ranges simple roots. Moreover, if \(\gamma_n \in \partial K^+\), meaning that \((\gamma_n, \omega) = 0\) for a certain simple root \(\omega\), \(\Psi_n(a) = 0\). Indeed, changing the summation in (D.1) \(\hat{s} \to \hat{s} s_\omega\) and using \(\det \hat{s}_\omega = -1\) with \(\hat{s}_\omega \gamma_n = \gamma_n\) (for \((\gamma_n, \omega) = 0\)) we get \(\Psi_n(a) = -\Psi_n(a)\) and, hence, \(\Psi_n(a) = 0\).

Due to (D.3) the function (D.1) acquires only the factor \(\det \hat{s}_n\) under transformations (3.17) of its argument. Hence, regular solutions of (6.21) are invariant with respect of the affine Weyl group, \(\Phi_n(\hat{s}_a a) = \Psi_n(\hat{s}_a a) / \kappa(\hat{s}_a a) = \Phi_n(a)\) (see (C.8)), which confirms our general analysis given in Appendix B (cf. (6.23)). Thus, we do not need to postulate the invariance of physical states with respect to residual gauge transformation if the Dirac quantization scheme is used. The boundary conditions (6.23) determining the physical configuration space topology are automatically fixed in the Dirac method (in contrast with the reduced phase space quantization where, first, boundary conditions are imposed by hand and, second, they are not unique \cite{28}, \cite{29}).

By means of (C.12), we are convinced that regular solutions to (6.21) are given by the characters \(\chi_{\Lambda_n}\) of the irreducible representations of the gauge group \cite{32}, p.909

\[
\Phi_n(a) = c_n \sum_{\hat{s} \in W} \det \hat{s} \exp \frac{2\pi i}{a_0} (\rho + \Lambda_n, \hat{s} a) = c_n \chi_{\Lambda_n} \left( \exp \frac{2\pi i a}{a_0} \right) \tag{D.5}
\]

where \(c_n\) are normalization constants, \(\gamma_n = \rho + \Lambda_n\), \(\Lambda_n\) labels the irreducible representations (notice that the sum (D.1) should vanish for all \(\gamma_n\) such that \((\gamma_n, \gamma_n) < (\rho, \rho)\) because the function (D.3) must be regular, which is possible only if \((\gamma_n, \gamma_n) \geq (\rho, \rho)\)).

For the character \(\chi_{\Lambda_n}\) we have the following representation (see (6.3))

\[
\chi_{\Lambda_n} \left( \exp \frac{2\pi i a}{a_0} \right) = Tr (\exp 2\pi i g \Lambda_n) = Tr \left( P \exp i g \oint A dx \right)_{\Lambda_n} \tag{D.6}
\]

where \((e^g)_{\Lambda_n}\) implies a group element \(e^g\) in the irreducible representation \(\Lambda_n\). Formula (D.4) shows that solution of (6.1) and (6.2) are given by the Wilson loops in all irreducible representations of the gauge group.

The wave functions are orthogonal with respect to the scalar product (6.14). This follows from the orthogonality of the characters (D.4). For normalization coefficients \(c_n\) we obtain

\[
\delta_{nn'} = \int \limits_{K^+_W} da \kappa^2(a) \Phi_n(a) \Phi_n^*(a) = 2^{2N_c} c_n c_n^* \int \limits_{K^+_W} da \sum_{\hat{s}, \hat{s}' \in W} \det \hat{s} \hat{s}' \exp \frac{2\pi i}{a_0} (a, \hat{s} \gamma_n - \hat{s}' \gamma_{n'}) . \tag{D.7}
\]

The integrand in (D.7) is a periodic function on the Weyl cells covering the Cartan subalgebra, therefore, the integral (D.5) vanishes for all \(\hat{s} \neq \hat{s}'\) because \(\gamma_n\) and \(\gamma_{n'}\) belong to the Weyl chamber and the Weyl group acts simply and transitively on the set of the Weyl chambers. So, there is no Weyl group element \(\hat{s}\) such that \(\hat{s} \gamma_n = \gamma_{n'}\) if \(\gamma_n, \gamma_{n'} \in K^+\). For \(\hat{s} = \hat{s}'\) the integral differs from zero only for \(\gamma_n = \gamma_{n'}\) (due to the periodicity of the

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\( |c_n| = 2^{-N_W^+}(N_W \cdot V_{K_W^+})^{-1/2} \) \( \text{ (D.8)} \)

where \( N_W \) is a number of elements in the Weyl group, \( V_{K_W^+} \) is the volume of the Weyl cell.

Consider \( X = su(2) \). The algebra has one positive root \( \omega \). Solutions to (D.3) are given by \( \gamma_n = \omega n / 2 \) where \( n \) ranges positive integers because \( K^+ = \mathbb{R}_+ \) and \( \partial K^+ \) coincides with the origin \( \gamma_n = 0 \). The spectrum and wave functions respectively read

\[ E_n = \frac{\pi \hbar^2}{4a_0^2} (n^2 - 1)(\omega, \omega), \quad n = 1, 2, \ldots; \text{ (D.9)} \]
\[ \Phi_n = c'_n \frac{\sin n(\omega) / a_0}{\sin \pi(n, \omega) / a_0}. \text{ (D.10)} \]

Substituting \( n = 2j + 1, \ j = 0, 1/2, 1, \ldots \) into (D.9) we observe that \( E_n \) is proportional to eigenvalues of the quadratic Casimir operator of \( su(2) \); \( E_n \sim j(j + 1) \) where the spin \( j \) labels the irreducible representations of \( su(2) \).

Notice, the norm of \( \omega \) in (D.9) cannot be chosen arbitrary and is fixed by structure constants in the Cartan-Weyl basis, \( (\omega, \omega) = 1/2 \) (Appendix A). If we rescale roots by a factor, which means, in fact, rescaling the structure constants in (A.1-3), the physical Hamiltonian in (6.21) is also changed so that its spectrum is independent of the rescaling factor.

If one sets \( a = a_0 \omega \theta / (\omega, \omega), \) then \( a \in K_W^+ \) implies \( \theta \in (0, 1) \). The measure \( da \) is defined in the orthonormal basis in \( H \sim \mathbb{R}^r \). For the \( su(2) \) case we have \( da \equiv da_3, \ a = \sqrt{2} \omega a_3 \) so that \( (a, a) = a_3^2, \ a_3 \in \mathbb{R} \). Hence, the normalization coefficients \( c'_n \) in (D.10) read

\[ c'_n = \left( \sqrt{2} a_0 \int_0^1 d\theta \sin^2 \pi n\theta \right)^{-1/2} = \left( \frac{a_0}{\sqrt{2}} \right)^{-1/2}. \text{ (D.11)} \]

Consider now symmetry properties of eigenstates (D.10) under transformations generated by homotopically non-trivial gauge group elements (3.4). For an arbitrary simple compact gauge group, they are determined by shifts \( a \to a + i/g \Omega \eta \partial \Omega^{-1} \eta \) where \( \Omega = \exp(i \pi \eta / l) \) so that (cf. (3.12))

\[ \exp(2\pi i \eta) = z \in Z_G. \text{ (D.12)} \]

The lattice \( \eta \) is given by integral linear combinations of elements \( \alpha / (\alpha, \alpha) \), \( \alpha \) runs a root systems, because

\[ \exp \frac{2\pi i \alpha}{(\alpha, \alpha)} \in Z_G \text{ (D.13)} \]

for any root \( \alpha \) [13], p.311. Let \( X \) be \( su(2) \). Its only positive root is \( \omega = \tau_3/4 \) (see Appendix A). Then \( \exp(2\pi i \omega / (\omega, \omega)) = \exp i \pi \tau_3 = -1 \in \mathbb{Z}_2 = \mathbb{Z}_{su(2)} \). Thus, homotopically non-trivial gauge transformations are generated by shifts

\[ a \to a + \frac{n\alpha a_0}{(\alpha, \alpha)}, \ n \in \mathbb{Z}. \text{ (D.14)} \]
Substituting (D.15) into (D.10) we get

\[ \Phi_n \left( a + \frac{a_0 \omega n}{(\omega, \omega)} \right) = (-1)^{n+1} \Phi_n(a), \]  

(D.15)

i.e. physical states acquire a phase factor under homotopically non-trivial gauge transformations. The Gauss law (6.2) provides only the invariance of physical states with respect to gauge transformations which can be continuously deformed towards the identity.
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