TOWARDS A COULOMB GAS OF INSTANTONS OF THE $SO(4) \times U(1)$ HIGGS MODEL ON $\mathbb{R}^4$

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Abstract

The $SO(4) \times U(1)$ Higgs model on $\mathbb{R}^4$ is extended by a $F^3$ term so that the action receives a nonvanishing contribution from the interactions of 2-instantons and 3-instantons, and can be expressed as the inverse of the Laplacian on $\mathbb{R}^4$ in terms of the mutual distances of the instantons. The one-instanton solutions of both the basic and the extended models have been studied in detail numerically.

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1 Introduction

Instantons are expected to play a central role in the semiclassical analysis of non-perturbative phenomena in non-linear field theories. Both the Electroweak and the Strong interactions are described by the dynamics of the Yang-Mills(YM) model and hence the $SU(2)$ instantons[1] of the relevant YM models on $\mathbb{R}^4$ in each case are expected to be employed. Unfortunately, this programme turned out to be less successful than it was hoped. One of the main obstacles encountered was the infrared problem arising from large instanton effects in the tunneling between topologically distinct vacua. This problem arises directly as a consequence of the scale invariance of the YM model on $\mathbb{R}^4$, which results in the dependence of the instanton solution on an arbitrary scale parameter. The introduction of an absolute scale would overcome this problem, and this is the original motivation of the present work.

The most natural way of introducing an absolute scale is by the introduction of Higgs fields, so that the dimensional parameter representing the non-zero vacuum expectation value in the Higgs self-interaction potential can set the absolute scale of the would be theory. A related but additional feature of the presence of a Higgs field is the exponential localisation of the instanton solution, which results in the possibility of constructing a multi-instanton field configuration in which the individual instantons overlap only asymptotically. This is what would enable the construction of a dilute gas of instantons. Furthermore with the appropriate asymptotic behaviour of these instantons, it may become possible to construct a dilute gas of instantons which interact non-trivially to yield a Coulomb gas in the appropriate dimensions. By a Coulomb gas we understand a gas where the contribution to the action coming from the interactions is given by the inverse of the Laplacian in terms of the mutual distances of the constituents of the gas. It is our aim in the present work to propose a gauged Higgs model on $\mathbb{R}^4$ which achieves the objectives just stated.

The construction of such a Coulomb gas for a Higgs model on $\mathbb{R}^3$ was performed long ago by Polyakov[2]. The Higgs model there[2] consisted of the $SU(2)$ YM field interacting with an $su(2)$ valued Higgs field and the usual
symmetry breaking Higgs potential. Firstly, the well known monopole solution of this model was taken to be the exponentially localised instantons in 2 + 1 dimensions. Secondly, a Coulomb gas of these instantons was constructed using the asymptotic fields of the monopole. Our plan in the present work is to propose a gauged Higgs model which supports instanton solutions with the requisite asymptotic properties capable of describing a Coulomb gas on $\mathbb{R}^4$. We shall restrict ourselves to this first task here, and defer the second technically complicated task of constructing the resulting dilute gas action to a future work. Before proceeding we make a remark aimed at putting the task at hand in perspective: Polyakov’s construction of the Coulomb gas of instantons on $\mathbb{R}^3$ is the 3 dimensional analogue of the 2 dimensional Coulomb gas of instantons contructed previously by Berezinsky and by Kosterlitz and Thouless employing the $O(2)$ model on $\mathbb{R}^2$, while the present work proposes the corresponding instanton field configuration on $\mathbb{R}^4$.

Our sole task in this paper is to find the instanton solution of a particular Higgs model satisfying the above stated criteria. We shall not attempt to compute the action of the corresponding Coulomb gas. Our plan is based on the 3 dimensional example, namely to construct an appropriate gauged Higgs model in contrast with the non-gauged model employed in the 2 dimensional example. Section 2 will be concerned with the instanton solution of the basic $SO(4) \times U(1)$ Higgs model proposed in Ref. We will make a detailed numerical study of this solution and find out that inspite of the instanton in question being exponentially localised, the asymptotic properties do not permit the construction of a dilute gas whose inter-instanton interactions support a Coulomb gas. Then in Section 3 we shall propose two extended versions of the basic model and will verify the existence of unit topological charge instantons by numerical techniques. These extended models are designed specially to support a non-trivial dilute Coulomb gas on $\mathbb{R}^4$. Finally in Section 4 we will summarise our results and give a brief discussion of the outlook for the extended models.
2 The basic model

First we briefly describe the $SO(4) \times U(1)$ model introduced in Ref.\[5\] and then proceed to subsections 2.1, 2.2 and 2.3 where we give the asymptotic solutions, the numerical solutions and, an analysis of the inter-instanton interactions, respectively.

The basic model on $\mathbb{R}_4$ is described by the antihermitian $SO(4) \times U(1)$ gauge connection $\hat{A}_\mu = A_\mu + i\gamma_5 a_\mu$ and the antihermitian Higgs multiplet $\Phi = \gamma_5 \gamma_a \phi_a$, $A_\mu$ being the $so(4)$ connection, $a_\mu$ the $U(1)$ connection and $\phi_a$ a 4-vector real Higgs field. The model, which is derived from the 8-dimensional member of the scale invariant Yang- Mills hierarchy\[6\] on $\mathbb{R}_4 \times S^4$ by dimensional descent\[7\], has the following Lagrange density

$$
\mathcal{L}_4 = \text{Tr}[\hat{F}_{\mu\nu\rho\sigma}^2 + 4\lambda_1 \{\hat{F}_{\mu\nu}, \hat{D}_\rho \Phi\}^2 - 18\lambda_2 (\{(\eta^2 + \Phi^2), \hat{F}_{\mu\nu}\} - [\hat{D}_\mu \Phi, \hat{D}_\nu \Phi])^2$$

$$- 54\lambda_3 (\{(\eta^2 + \Phi^2), \hat{D}_\mu \Phi\}^2 + 54\lambda_4 (\eta^2 + \Phi^2)^4] \quad (1)
$$

In (1) and everywhere below, the brackets $[,]$ and $\{,\}$ denote commutators and anticommutators respectively, and square brackets around indices signify total antisymmetrisation. Also the curvature $\hat{F}$ and the covariant derivative $\hat{D}$ pertain to the $SO(4) \times U(1)$ connection $\hat{A} = A + i\gamma_5 a$, while below we shall use $F$ and $D$ pertaining to the $SO(4)$ connection $A$. The 4-form curvature field $F_{\mu\nu\rho\sigma}$, in this notation, is defined as $F_{\mu\nu\rho\sigma} = \{F_{\mu\nu}, F_{\rho\sigma}\}$. The last term in (1) multiplying $\lambda_4$ is the Higgs symmetry breaking potential. It is clear that any one of the four dimensionless coupling constants $\lambda_a$, $a = 1, 2, 3, 4$, which we shall denote with a vector notation $\vec{\lambda}$, may be set equal to 1 by rescaling the $\Phi$ field, but we do not do that here.

A physically very important property of the system (1) is, that at high momenta or small distances when the magnitude of the Higgs field is negligible with respect to the dimensional constant $\eta$, the system is dominated by the term with the highest power of $\eta$. This happens to be

$$
\mathcal{L}_{\text{high}} \sim \text{Tr}[-\frac{\lambda_2}{3} \hat{F}_{\mu\nu}^2 - \lambda_3 (\hat{D}_\mu \Phi)^2 + \frac{\lambda_4}{4} (\eta^2 + \Phi^2)^2], \quad (2)
$$
namely the usual Yang-Mills-Maxwell-Higgs system which appears as the coefficient of the $\eta^4$ term in (1), which is its dominant contribution at high energy leading to a perturbative action as expected. Thus the model (1) can be regarded as the corresponding effective action density.

There are two other physically relevant properties of the system (1). The first is that the gauge group $SO(4)$ leads to a chirally symmetric dynamics, hence does not fit in with Electroweak theory. The other is that at low momentum or large distances, the gauge field dynamics is dominated by the 4-form curvature $\hat{F}^{2}_{\mu\nu\rho\sigma}$ whose propagation properties again do not fit in with Electroweak dynamics. We must therefore conclude that the model (1) is not suited to be a prototype for an Electroweak theory, but it is consistent with the features Strong interaction dynamics both at high and low momenta. In this respect, the $U(1)$ connection $a_\mu$ in the gauge field multiplet $\hat{A} = (A_\mu + i\gamma_5 a_\mu)$ is irrelevant and can consistently be suppressed. We have left $a_\mu$ in the definition of $L_4$ in (1), because that is the most general model descending from the 4$p$ dimensional member of the Yang-Mills hierarchy that can support instantons, and also because when we restrict to the spherically symmetric configuration below the imposition of this symmetry will eliminate this $U(1)$ field.

In view of the above reasoning, we will study the properties of this model in the context of Strong interaction dynamics at low momentum, and in particular will attempt to construct a dilute gas of instantons that satisfies the properties of a Coulomb gas in $\mathbb{R}_4$, analogously to Polyakov’s work in $\mathbb{R}_3$. As the title states however, this programme will not be completed but we will restrict henceforth to the study of the classical instanton solutions of this model and its extensions.

When the values of $\lambda_a$ respect the following restrictions: $\lambda_a > 1$ for each $a$, then the action density (1) is bounded from below by the topological charge density $\rho = \partial_\mu \Omega_\mu$, expressed as the divergence of the Chern-Simons form $\Omega_\mu$ given by[\text{5}]

$$
\Omega_\mu = \frac{1}{8\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \gamma_5 [\eta^4 \hat{A}_\nu (\hat{F}_{\rho\sigma} - \frac{2}{3} \hat{A}_\rho \hat{A}_\sigma) - \frac{1}{6} \eta^2 \phi \{ \hat{F}_{[\rho\sigma]}, \hat{D}_\nu \Phi \} - \frac{1}{6} \phi \{ (\eta^2 + \Phi^2), \hat{F}_{\rho\sigma} \} + \hat{D}_\rho \Phi \hat{D}_\sigma \Phi \hat{D}_\nu \Phi].
$$

(3)
The surface integral of this density is the topological charge \( N \), and the spherically symmetric solutions with which we will be concerned in this paper yield \( N = 1 \) for this integral.

In the limit where \( \lambda_a = 1 \) for all \( a \), or \( \vec{\lambda} = (1, 1, 1, 1) \), this bound is saturated and the action is equal to the topological charge \( N \). If however any of the \( \lambda_a \) take positive values smaller than 1, then the action is not bounded any more by \( N \) but by \( \lambda_{\text{min}} N \), where \( \lambda_{\text{min}} \) is the smallest of those \( \lambda_a \) which are smaller than 1. This can be shown using similar arguments to those used in Ref.[9] in the case of the Abelian Higgs model. When \( \lambda_a < 0 \) for any \( a \), we lose the topological lower bound.

In the case where the topological lower bound is saturated we have the following self-duality, or Bogomol’nyi, equations:

\[
\frac{1}{36} \varepsilon_{\mu\nu\rho\sigma} \gamma_5 \hat{F}_{\mu\nu\rho\sigma} - (\eta^2 + \Phi^2)^2 = 0 \tag{4}
\]

\[
\frac{1}{9} \varepsilon_{\mu\nu\rho\sigma} \gamma_5 \{\hat{F}_{[\rho\sigma]}, \hat{D}_{[\mu]} \Phi\} - \{(\eta^2 + \Phi^2), \hat{D}_{\nu} \Phi\} = 0 \tag{5}
\]

\[
\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \gamma_5 (\{(\eta^2 + \Phi^2), \hat{F}_{\rho\sigma}\} + \hat{D}_{[\mu} \Phi \hat{D}_{\nu]} \Phi) - (\{(\eta^2 + \Phi^2), \hat{F}_{\mu\nu}\} + \hat{D}_{[\mu} \Phi \hat{D}_{\nu]} \Phi) = 0. \tag{6}
\]

The system (4) shares an important property with the Skyrme[10] model on \( \mathbb{R}_3 \), namely that all but the second power of the derivative of any field component are excluded. This has been a criterion in the construction of (4). Like the Skyrme model also, the self-duality equations saturating the topological bound are overdetermined[5][11]. Accordingly the only solutions of the equations (4)-(6) are the trivial constant solutions we shall see below, and any non-trivial solutions we find will satisfy the second order Euler-Lagrange equations and not (4)-(6) even for \( \lambda_i = 1 \) for all \( i \).

The most important property we shall require of the instanton solution is the large \( r \) limit of the magnitude of the Higgs field \( |\Phi| = \sqrt{-\text{Tr} \Phi^2} \)

\[
\lim_{r \to \infty} |\Phi| = \eta \tag{7}
\]

in terms of \( \eta \), the vacuum expectation value of the Higgs field. Requiring this asymptotic condition, we shall find out that the corresponding behaviours of
the $SO(4)$ gauge connection $A_\mu$ and the full Higgs multiplet are
\[
\lim_{r \to \infty} A_\mu = \frac{1}{2} q^{-1} \partial_\mu q, \quad \lim_{r \to \infty} \Phi = \eta g, \tag{8}
\]
in terms of the $SO(4)$ group element $\gamma_5 \gamma_\mu \hat{x}_\mu$, with $\hat{x} = \frac{x}{r}$. We now adapt the arguments in Ref. [12] pertaining to the $SU(2)$ YM instanton to the solutions satisfying (8). In the temporal gauge $A_0 = 0$ each component of $A_\mu$ and $\Phi$ become time independent, thus enabling the identification of our solutions as instantons.

The solutions we shall seek below are restricted to the spherically symmetric restriction of the Euler-Lagrange equations, namely the unit topological charge instantons. This restriction is quite adequate since instanton configurations of all charges can be attained by considering the collection of arbitrarily many $N = 1$ instantons, since as we shall see below these overlap only asymptotically. The reason is the presence of the Higgs field which introduces the absolute scale with respect to which the instantons will be localised. Indeed, we shall be able to achieve an exponential decay yielding this localisation, similar to the instantons of the $\phi^4$ model [13] on $\mathbb{R}^2$ and 'compact electrodynamics' [12][13] on $\mathbb{R}^3$.

### 2.1 Asymptotic solutions

Since we are unable to solve analytically the Euler-Lagrange equations of the one-dimensional subsystem of (1) arising from the imposition of spherical symmetry, we will give explicit solutions in the $r >> 1$ and $r << 1$ asymptotic regions only.

Under the imposition of spherical symmetry on $\mathbb{R}_4$, the Abelian field $f_{\mu\nu} = \partial_\mu a_\nu$ defining the system (1) vanishes and we are left with only the $SO(4)$ field $A_\mu = A_\mu$, for which we employ the following spherically symmetric Ansatz
\[
A_\mu = \frac{1 + f(r)}{r} \gamma_\mu \hat{x}_\mu, \quad \Phi = h(r) \gamma_5 \gamma_\mu \hat{x}_\mu, \tag{9}
\]
where \( f(r) \) is a dimensionless function and \( h(r) \) has the same dimensions as \( \eta \). \( \gamma_{\mu\nu} = -\frac{1}{4}[\gamma_\mu, \gamma_\nu] \) is the (Dirac) spinor representation of \( SO(4) \).

Substituting (9) into (1), performing the angular integrations and with a convenient rescaling, we have the following one-dimensional subsystem

\[
L_4 = \frac{1}{r^3}(1 - f^2)^2 f_r^2 + \frac{4\lambda_3}{3} r^3 (\eta^2 - h^2)^2 h_r^2 \\
+ \frac{\lambda_1}{r} \left( [1 - f^2] h_r \right)^2 + \lambda_2 r \left( [\eta^2 - h^2] f_r \right)^2 \\
+ \frac{3\lambda_1}{r^3} \left( fh(1 - f^2) \right)^2 + 4\lambda_3 r \left( fh(\eta^2 - h^2) \right)^2 \\
+ \frac{\lambda_2}{r} \left( (1 - f^2)(\eta^2 - h^2) + 2f^2 h^2 \right)^2 + \frac{4\lambda_1}{4} r^3 (\eta^2 - h^2)^4, \tag{10}
\]

having suppressed the overall constant factor coming from the angular volume. The subscript \( r \) in (10) denotes the differentiation \( f_r = \frac{df}{dr} \).

Using the notation \( \delta_f L = \frac{\partial L}{\partial f} - \frac{d}{dr} \left( \frac{\partial L}{\partial f_r} \right) \), we express the two Euler-Lagrange equations of (11). The equation \( \delta_f L = 0 \) arising from the arbitrary variations \( \delta f \) is

\[
\delta_f L_4 = \frac{2}{r^3}(1 - f^2) \left( 2ff_r^2 + \frac{3}{r}(1 - f^2)f_r - (1 - f^2)f_{rr} \right) \\
+ \frac{4\lambda_1}{r^2} fh \left( [f^2 - 1] h_r \right) - r \left( 2f^2 h + 2ff_r h + 4ff_r h_r + (f^2 - 1)h_{rr} \right) \\
- 2\lambda_2 (h^2 - \eta^2) \left( [h^2 - \eta^2] f_r + r \left( 2h_r^2 f + 2hh_r f + 4hh_r f_r + (h^2 - \eta^2)f_{rr} \right) \right) \\
+ \frac{6\lambda_1}{r^3} fh^2 (f^2 - 1)(3f^2 - 1) + 8\lambda_3 rf h^2 (h^2 - \eta^2)^2 \\
+ \frac{4\lambda_2}{r} f (3h^2 - \eta^2) \left( (h^2 - \eta^2)(f^2 - 1) + 2f^2 h^2 \right), \tag{11}
\]

and using the similar notation \( \delta h L = \frac{\partial L}{\partial h} - \frac{d}{dr} \left( \frac{\partial L}{\partial h_r} \right) \), the equation \( \delta h L = 0 \) arising from \( \delta h \) is given by

\[
\delta_h L_4 = -\frac{8\lambda_3}{3} r^2 (h^2 - \eta^2) \left( 2rh h_r^2 + (h^2 - \eta^2)(rh_{rr} + 3h_r) \right)
\]
\[ + \frac{\lambda_1}{r^2} (f^2 - 1) \left[ [(f^2 - 1) h]_r - r [2 f_r^2 h + 2 f f_r h + 4 f f_r h_r (f^2 - 1) h_{rr}] \right] \]

\[ - 4 \lambda_2 h f \left[ [(h^2 - \eta^2) f]_r + r [2 h^2 f + 2 hh_r f + 4 h h_r f_r + (h^2 - \eta^2) f_{rr}] \right] \]

\[ + \frac{6 \lambda_1}{r^3} (f^2 - 1)^2 f^2 h + 24 \lambda_3 r f^2 h (h^2 - \eta^2)(3 h^2 - \eta^2) \]

\[ + \frac{4 \lambda_2}{r} h (3 f^2 - 1) \left[ (h^2 - \eta^2)(f^2 - 1) + 2 h^2 f^2 \right] + 2 \lambda_4 r^3 (h^2 - \eta^2)^3 h \quad (12) \]

Since the system (11) is a non-Abelian Higgs model like the models employed in Refs. [3], we expect the solution to have asymptotic behaviour similar to that of the monopole [3]. Accordingly we seek solutions with the following properties

\[ \lim_{r \to 0} f(r) = -1 \quad \lim_{r \to \infty} f(r) = 0 \quad (13) \]

\[ \lim_{r \to 0} h(r) = 0 \quad \lim_{r \to \infty} h(r) = \eta. \quad (14) \]

The first members of both (13) and (14) guarantee that the solution is regular at \( r = 0 \), while the second members satisfy criteria of finite action. Moreover, the second members of these equations result in the asymptotic behaviour anticipated in (8), which is essential for the interpretation of the finite action topologically stable solution as the instanton of the dynamical model.

Expanding around their asymptotic values \( f(r) = 0 + F(r) \) and \( h(r) = \eta - H(r) \) in the region \( r >> 1 \) in terms of the small functions \( F(r) \) and \( H(r) \), and retaining only linear terms in these functions, (11) and (12) reduce respectively to

\[ \rho^2 F_{\rho\rho} - 3 \rho F_{\rho} - 3 \rho^2 F = 0 \quad (15) \]

\[ \sigma^2 H_{\sigma\sigma} - \sigma H_{\sigma} - 8 \sigma^2 H = 0, \quad (16) \]

in which we have used the following two dimensionless rescalings of \( r \): \( \rho = \sqrt{\lambda_1} \eta r, \sigma = \sqrt{\frac{2 \lambda_2}{\lambda_1}} \eta r. \)

The solutions of (15) and (16) respectively are expressed in terms of modified Bessel functions and lead to the following \textit{exponentially} decaying solutions

\[ f(r) = \lambda_1 \eta^2 r^2 K_2(\sqrt{3 \lambda_1} \eta r) \quad (17) \]
\[ h(r) = \eta \left( 1 - \sqrt{\frac{\lambda_2}{2\lambda_1}} \eta r K_1(2\sqrt{\frac{\lambda_2}{\lambda_1}} \eta r) \right), \] (18)

in the \( r \gg 1 \) region.

In the \( r \ll 1 \) region, we tried a solution in powers of the rescaled radial variable \( \rho = \eta r \) and found

\[
\begin{align*}
  f(\rho) &= A\rho^2 + o(\rho^4) \\
  h(\rho) &= B\rho + o(\rho^3)
\end{align*}
\] (19) (20)

where the dimensionless constant \( A \) and the constant \( B \) with the dimensions of \( \eta \) are arbitrary and will be determined by the numerical integrations immediately below. Note that the behaviours (19) and (20) lead to fields \((A\mu, \Phi)\) that are regular and differentiable at \( r = 0 \).

2.2 Numerical solutions

We have integrated the equations \( \delta_f L_4 = 0 \) and \( \delta_h L_4 = 0 \) given respectively by (11) and (12) numerically starting with the functions (19) and (20) near \( r = 0 \) and choosing the numerical values of the constants \( A \) and \( B \) so that \( f(r) \) and \( h(r) \) reach their asymptotic values given by the second members of (13) and (14). We have performed this integration for the following values of the dimensionless coupling constants \( \vec{\lambda}(i) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \), with \( \vec{\lambda}^{(1)} = (0.5, 0.5, 0.5, 0.5) \), \( \vec{\lambda}^{(2)} = (0.8, 0.8, 0.8, 0.8) \), \( \vec{\lambda}^{(3)} = (1, 1, 1, 1) \), and \( \vec{\lambda}^{(4)} = (1.2, 1.2, 1.2, 1.2) \). The numerical values of the pair of constants \( \{A, B\} \) are fixed by each of these the numerical integrations, and are listed respectively in the first and second columns of Table 1. The profiles of the functions \( f(r) \) are given in Figure 1, and those of \( h(r) \) are given in Figure 2. We do not exhibit the profiles of the action densities corresponding to these solutions since they are all ball shaped as expected of spherically symmetric lumps. The values of the action integrals \( S(\vec{\lambda}^{(i)}), i = 1, 2, 3, 4 \) for each of
these solutions are listed in the third column of Table 1. We defer a detailed discussion of these quantitative results to Section 4.

The above integrations demonstrate the existence of the spherically symmetric solutions of the basic model 5, but we find it interesting to pursue our numerical studies somewhat further because of an unusual feature of the model at hand. This concerns the overdetermined nature of the Bogomol’nyi equations (4)-(6), a feature shared with the Skyrme model 10 on $\mathbb{R}^3$. While in the usual Skyrme model the overdetermined self-duality equations are parametrised only by one dimensional constant which sets the absolute scale, here in addition to the corresponding dimensional constant $\eta$ we have the three independent components of the four dimensionless coupling constants $\vec{\lambda}$. It is therefore interesting to investigate numerically the quantitative departure from the ‘minimal’ configuration $\vec{\lambda} = (1, 1, 1, 1)$.

We know from Refs. 5, 11 that (4)-(6) are only satisfied by the constant asymptotic values of $f(r)$ and $h(r)$, given by (13) and (14), and therefore that the stress tensor $T_{\mu\nu}$ corresponding to these solutions does not vanish. It is interesting to see how the component $T_{44}$ of the stress tensor behaves in detail, quantitatively. Now the component $T_{44}$ of the stress tensor corresponding to the spherically symmetric sub-system equivalent to the one-dimensional Lagrangian (10) is

$$T_{44} = \left( \frac{1}{r^2} (1 - f^2)^2 f_r^2 - \frac{\lambda_4}{4} r^3 (\eta^2 - h^2)^4 \right) + \left( \frac{4\lambda_3}{3} r^3 (\eta^2 - h^2)^2 h_r^2 - \frac{3\lambda_1}{r^3} f^2 h^2 (1 - f^2)^2 \right)$$

$$+ \left( \frac{\lambda_1}{r} [(1 - f^2) h_r^2 - 4\lambda_1 r f^2 h^2 (\eta^2 - h^2)] \right)$$

$$+ \left( \frac{\lambda_2}{r} [(\eta^2 - h^2) f_r^2 - \frac{\lambda_2}{r} [(1 - f^2)(\eta^2 - h^2) + 2 f^2 h^2]^2] \right). \quad (21)$$

Note that each one of the four large brackets in (21) vanishes separately in the (overdetermined) self-dual limit $\vec{\lambda} = (1, 1, 1, 1)$. In that case, the vanishing of each large bracket states the corresponding Bogomol’nyi equation, namely
\[
\frac{1}{r^3}(1 - f^2)f_r + \frac{1}{2}(\eta^2 - h^2) = 0 \quad (22)
\]
\[
2(\eta^2 - h^2)h_r - \frac{3}{r^3}(1 - f^2)f h = 0 \quad (23)
\]
\[
\frac{1}{r^2}[(1 - f^2)h]_r + 2(\eta^2 - h^2)f h = 0 \quad (24)
\]
\[
\frac{1}{r}[(\eta^2 - h^2)f]_r - \frac{1}{r^2}[(1 - f^2)(\eta^2 - h^2) + 2 f^2 h^2] = 0. \quad (25)
\]

Thus \( T_{44} \) gives a quantitative measure of the departure of a solution from the trivial self-dual field configuration satisfying (22)- (25). We have plotted the profiles of \( T_{44} \) for each of the four solutions exhibited in Table 1 and Figures 1-3, in Figure 4.

Equations (22)-(23) are the spherically symmetric restrictions of the Bogomol’nyi equations (4)-(6). Specifically, (22) pertains to (4), (23) and (24) to (5) and (25) to (6). It can be observed immediately that when each of (22)-(25) is squared, the sum of the square terms yields the spherically symmetric restriction of (1), namely the one-dimensional system (10). The corresponding sum \( \sigma \) of the cross terms is then the spherically symmetric restriction of the topological charge density \( \rho = \partial_\mu \Omega^\mu (3) \), divided by \( \frac{3}{8\pi^2\eta^2} \).

Since \( \rho \) is a total divergence, the corresponding one dimensional topological charge density which is this quantity \( \sigma \), turns out to be a total derivative given by

\[
\sigma = -\frac{1}{2}\frac{d}{d\rho}\left(3f(1 - \frac{1}{3}f^2) - 6(1 - f^2)f g^2 + (1 - 3f^2) f^4\right), \quad (26)
\]

which is expressed in terms of the rescaled radius \( \rho = \eta r \) and the rescaled dimensionless function \( g(\rho) = \eta^{-1}h(\rho) \). The integral \( f \sigma d\rho \), the topological charge of the one-dimensional subsystem (11), is immediately evaluated using the limits (13) and (14) to yield \( N = 1 \).

Having achieved a detailed quantitative understanding of our spherically symmetric solutions, we proceed to consider the question of inter-instanton interactions.
2.3 Inter-instanton interaction

As we saw above, even for the coupling constant vector \( \vec{\lambda} = (1, 1, 1, 1) \), the instanton solution is not self-dual and the stress tensor does not vanish. It is therefore expected that the force between such instantons is non-vanishing, and hence that a dilute gas of such instantons can be constructed.

In this paper, we shall adapt Polyakov’s construction of a dilute gas of instantons on \( \mathbb{R}_3 \) to our problem on \( \mathbb{R}_4 \). As in Ref.\cite{2}, we will set out to construct a Coulomb gas of instantons. Given that our solution is expressed by the Ansatz (9) and obeys boundary conditions (13) and (15), we opt to work in the Dirac-string gauge introduced in Refs.\cite{3,4}. In this gauge, the Higgs field in the \( r \gg 1 \) region \( \Phi(\infty) = \eta \gamma_{5} \gamma_{\mu} \hat{x}_{\mu} \) is gauged to be a constant valued field \( \omega \Phi = \eta \gamma_{4} \), by rotating it in the direction of the \( x_4 \) axis under the action of a suitable \( SO(4) \) gauge rotation \( \omega \). As a result the Higgs potential and the covariant derivative of the Higgs field vanish, reducing the (asymptotic \( r \gg 1 \)) action density (1) to the only remaining term \( F_{\mu\nu\rho\sigma}^{2} \).

The gauge field connection and curvature in the Dirac-string gauge are calculated from (9) with \( f(r) = 0 \) according to (13). Just as the \( SU(2) \) symmetry of the connection in the Georgi-Glashow model on \( \mathbb{R}_3 \) breaks to \( U(1) \) for the asymptotic fields in the Dirac string gauge, so the \( SO(4) \) symmetry of the connection (9) breaks to \( SO(3) \) here. This \( SO(3) \) valued asymptotic connection and its curvature are given\cite{3,4}, respectively, by

\[
\omega A_i = \frac{1}{(1 + \hat{x}_4)r} \gamma_{ij} \hat{x}_j, \quad \omega A_4 = 0, \quad (27)
\]

\[
\omega F_{ij} = -\frac{1}{r^2} \left( \gamma_{ij} + \frac{1}{1 + \hat{x}_4} \hat{x}_i \gamma_{jk} \hat{x}_k \right), \quad \omega F_{i4} = \frac{1}{r^2} \gamma_{ij} \hat{x}_j, \quad (28)
\]

where the same notations as above are used, and with the index \( \mu = i, 4 \), \( i = 1, 2, 3 \). The Dirac-string singularity along the negative \( x_4 \) axis is manifest in (27). Indeed the \( SO(d) \) gauge field on \( \mathbb{R}_d \) for arbitrary \( d \) breaks down to an \( SO(d - 1) \) asymptotic field in the Dirac-string gauge, given by (28) in which the subscripts 4 are replaced by \( d \), the gamma matrices are understood to be
the $d$-dimensional gamma matrices and the singularity in the connection is on the negative $x_d$-axis. This is explained in detail in Ref. [14]. Note that the apparent lack of rotational invariance in (28) is just a gauge artifact, and it is easy to check that gauge invariant quantities such as Tr $F^2_{\mu\nu}$, Tr$^* F_{\mu\nu} F_{\mu\nu}$, Tr$F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}$ etc., are SO(4) scalars. Only in the $d = 3$ case does (28) take a manifestly SO(3) invariant form, namely

$$\omega F_{\mu\nu} = \frac{1}{2r^2} \varepsilon_{\mu\nu\lambda\sigma} x^\lambda \sigma_3.$$  

The components of the curvature in (28) are given in Cartesian coordinates and are valid in the region $r >> 1$. It is therefore desirable to express the components of the curvature on the 3-sphere $S^3$ at infinity in polar coordinates like the Wu-Yang monopole [15] whose only non-vanishing component on the 2-sphere $S^2$ at infinity is $\omega F_{\theta\phi} = \frac{1}{2} \sigma_3 \sin \theta$. In terms of the two polar coordinates $\psi$ and $\theta$, and the azimuthal coordinate $\phi$ on $S^3$ at infinity, the non-vanishing components of the curvature $\omega F$ are

$$\omega F_{\psi\theta} = \sin \psi (\gamma_{31} \cos \phi + \gamma_{23} \sin \phi), \quad (29)$$
$$\omega F_{\theta\phi} = -\sin^2 \psi \sin \theta (\gamma_{12} \cos \theta + (\gamma_{31} \sin \phi + \gamma_{23} \cos \phi) \sin \theta), \quad (30)$$
$$\omega F_{ij} = -\sin \psi \sin \theta (-\gamma_{12} \sin \theta + (\gamma_{31} \sin \phi + \gamma_{23} \cos \phi) \cos \theta), \quad (31)$$

Since our instantons are exponentially localised, c.f. (17) and (18), we shall consider that a gas of non overlapping instantons at positions $\{x_a\}$, with $x_{ab} = |x^\mu_a - x^\mu_b| >> \eta^{-1}$ can be described by the linear superposition

$$F_{\mu\nu} = \sum_a q_a \omega F_{\mu\nu}(x - x_a), \quad (q_a = \pm 1). \quad (32)$$

Following Ref.[2], we argue that the first contribution to the action comes from the sum over $a$ of the action integrals of the one-instanton solution. The first, dominant contributions to these integrals comes from the integration of the action density

$$\mathcal{L}_4^{(1)} = \sum_a \text{Tr} \{F_{\mu|\nu}(x - x_a), F_{\rho\sigma}(x - x_a)\}^2, \quad (33)$$
where the integration is restricted to inside the 4-dimensional spheres of radii $R$ around each instanton, such that $\eta^{-1} << R << x_{ab}$. The second contribution comes from the inter-instanton interaction terms in the action density calculated from (32), integrated over the 4-dimensional volume outside these spheres with radii $R$. The second contribution to the action is the integral of the following density

$$L_4^{(2)} = \sum_{a,b,c,d} \text{Tr} \{ F_{\mu[\nu(x-x_a)}(x-x_b)} \{ F_{\mu[\nu(x-x_c)}(x-x_d)} \}, \quad (34)$$

where at most two of the four summation indices $a, b, c$ and $d$ must be different. Otherwise, when $a = b = c = d$, the sum is over one index only and (33) and (34) coincide, and the integral in the region outside $R$ is negligible compared with that in the region inside $R$.

It turns out that when the asymptotic field strengths (28) in the Dirac-string gauge are substituted into (34) the latter vanishes exactly, leading to vanishing contribution to the action due to inter-instanton interactions. This is because the 4-form curvature $F_{\mu\nu\rho\sigma}$ constructed from two distinctly situated 2-form curvatures $F_{\mu\nu}(x-x_a)$ and $F_{\rho\sigma}(x-x_b)$ itself vanishes exactly. This last statement can be understood more succinctly by noting that the asymptotic curvature 2-form given by (29)-(31) is defined with respect to the three coordinates $\{\psi, \theta, \phi\}$ on $S^3$ and to construct a non-vanishing curvature 4-form $\omega F_{ABCD}$ we need at least four coordinates.

### 3 The extended models

The status of the unit charge spherically symmetric instanton solution of the basic $SO(4) \times U(1)$ model (1) is that, while the individual lumps are exponentially localised permitting the construction of a dilute instanton gas, the asymptotic interactions of these instantons are not sufficiently strong to lead to a Coulomb gas, in contrast to the situation in the two well known models, namely the $O(2)$ model in two[4] and the Georgi-Glashow model in three[2] dimensions. Since it is our eventual aim to construct a Coulomb gas of instantons in four dimensions, the basic model (1) must be modified.
The search for such possible modifications is the purpose of this Section. The asymptotic solutions of these will be given in Subsection 3.1 and their numerical solutions in Subsection 3.2.

The modification of (1) will take the form of an extension of (1). The extended model will consist of (1) plus some other gauge invariant terms depending on \((A_\mu, \Phi)\), such that at least one term in it depends only on the curvature \(F_{\mu\nu}\) and is independent of the Higgs field \(\Phi\). This is because in the Dirac-string gauge \(\text{(27)}\) all gauge invariant quantities depending on \(\Phi\) vanish. If the extension consists of a positive definite quantity, then it would be expected that the topological lower bound given by the surface integral of (3) remains valid and hence that the new instanton will be classically stable.

To orient ourselves we reconsider the situation in the basic model. There the curvature decays with an inverse square power like the monopole\(\text{[3]}\) on \(\mathbb{R}^3\) and like the latter the asymptotic connection behaves as

\[ A_\mu^{(\infty)} = \frac{1}{2} g^{-1} \partial_\mu g, \quad g = \hat{x}_\mu \sigma_\mu, \quad g^{-1} = \hat{x}_\mu \tilde{\sigma}_\mu, \]

with \(\sigma_\mu = (i\tilde{\sigma}, 1)\), and \(\tilde{\sigma}_\mu = (-i\tilde{\sigma}, 1)\). After passing to the Dirac-string gauge the only term in (1) which contributes, namely \(\omega F_{\mu\nu\rho\sigma}\), decays with the eighth power of \(r\) and hence the 4-dimensional integral of it decays with the fourth power of \(r\). It is clear why this decay is too strong to result in a Coulomb potential since the latter is characterised by its inverse square behaviour in 4-dimensions. This leads us unambiguously to the criterion that the extension to our basic model must include a term decaying with the sixth power of \(r\), and since we expect the asymptotic properties of the extended model to be the same as those of the basic model (13), i.e. inverse square decaying curvature, the extension must include the third power of the curvature. Inevitably this means that a new dimensional constant different from the Higgs vacuum expectation value \(\eta\) must be introduced. We shall refer to this as criterion (A) below.

The other criterion, (B), which we shall require is that the density in question involve no higher powers of the velocity fields than the second. In other words, for fixed \(\mu\) and \(\nu\), no higher powers of \((\partial_\mu A_\nu)^2\) should occur in the new density. This criterion is respected by all Skyrme-like models, including the
Skyrme model\cite{14} as well as the hierarchy of Yang-Mills models\cite{3} and their descendents\cite{7}. In the context of some more phenomenologically motivated considerations, this criterion could be relaxed, but this is not the case in the present work. The reason for requiring this criterion is that in its absence the definition of the canonical momentum-fields would lead to problems in, say the collective coordinate quantisation and more specifically in the analysis of the fluctuation spectrum when this is eventually performed. While this analysis is beyond the scope of the present work, we envisage that it should in principle be accessible and hence insist here on this criterion (B).

Having stated our main criteria (A) and (B) for the extension, we introduce the other criteria. Obviously this density must be both Lorentz invariant, and gauge invariant. In addition it would be an advantage from the viewpoint of establishing the existence of the instanton of the extended model, if this density was positive definite by construction. As we shall see below, it will not be possible to satisfy this last criterion.

Subject to the constraints of Lorentz and gauge invariances, criterion(A) narrows the choices for candidates down to the following three densities:

\[
\text{Tr} F_{\nu\lambda} F_{\lambda\mu} F_{\mu\nu}, \quad \text{Tr} F_{\mu\lambda} F_{\lambda\nu} F_{\mu\nu}, \quad \text{Tr} (D_{\lambda} F_{\mu\nu})(D_{\lambda} F_{\mu\nu}).
\]

The last of these is equivalent to the non-local effective action density for low momentum gluons\cite{20}\cite{21} $\text{Tr} F_{\mu\nu} D_{\lambda}^2 F_{\mu\nu}$, while the other possible term $\text{Tr}(D_{\mu} F_{\mu\nu})^2$ with the required dimensions is related to the two of the above listed terms through

\[
\text{Tr}(D_{\mu} F_{\mu\nu})^2 = 2\text{Tr} F_{\mu\nu} F_{\nu\lambda} F_{\lambda\mu} \frac{1}{2}\text{Tr}(D_{\lambda} F_{\mu\nu})^2.
\]  (35)

Invoking the criterion (B) eliminates all but the first of these three candidates, namely the extension to our basic model (1) must be uniquely

\[
\mathcal{L}_3 = \kappa_1^2 \text{Tr} F_{\mu\nu} F_{\nu\lambda} F_{\lambda\mu},
\]  (36)

where $\kappa_1$ is the new constant with the same dimensions as $\eta$. This particular dynamical model \cite{16} defined on $\mathbb{R}_6$, has appeared previously
in the literature\textsuperscript{[16, 17]} in a different context where self-dual solutions are found\textsuperscript{[16, 17]} and their stability is examined\textsuperscript{[18]}. In particular, the fact that criterion (A) is satisfied by (36) was discussed in some detail in Ref.\textsuperscript{[16]}. In the present context on $\mathbb{R}_4$, it was first proposed in Ref.\textsuperscript{[19]}.

Anticipating a result of the next Subsection, namely that the extension of (1) by the density (36) will result in a power decay for the gauge connection function $f(r)$ which in its absence had decayed exponentially according to (17), we introduce a second extension to be added to the first one (36) with the aim of rectifying this situation. This second extension, which will introduce yet another dimensional constant, is

$$\mathcal{L}_2 = \kappa_2^4 \text{Tr} \left( (D_\mu \Phi)^2 + \lambda_5 (\eta^2 + \Phi^2)^2 \right),$$

(37)

where $\kappa_2$ is this second constant with the dimensions of $\eta$, and $\lambda_5$ is an unimportant dimensionless constant which will be set equal to 1 below. The density (37) is immediately recognised as the usual quadratic Higgs kinetic term plus the usual symmetry breaking potential. This term, being entirely $\Phi$ dependent, will have no effect on the inter-instanton interactions and is introduced merely to restore exponential decay. Clearly it is not possible to add the usual YM term $F_{\mu \nu}^2$ term to (37) as this will render the action logarithmically divergent and would invalidate the instanton solution.

We note here that the optional criterion of positive definiteness is not satisfied by (36). This will necessitate some detailed asymptotic and numerical analysis of the extended systems carried out below, because the lack of positive definiteness of the extension means that the topological stability of the instanton of the basic model (1) does not automatically guarantee the same for the extended model. Indeed, if the coupling constant $\kappa_1$ in (36) is allowed to be large, its lack of positivity will destroy the stability of the instanton in the extended model. Thus it is imperative that this constant $\kappa_1$ be taken to be small. What small means in this context will be determined according to quantitative criteria to be applied in Subsection 3.2 where the numerical integrations are described, and the extent to which these criteria are met will be discussed in detail in Section 4. Here, in justification the
extension (36), we suffice to note that the value of the this density remains positive as long as the asymptotic conditions (13) are respected as we shall find to be the case in the Subsection 3.1 immediately below.

Summarising, we propose two related extensions of (1). The first is obtained by the addition of (36) to (1), namely

\[ L_I = L_4 + L_3, \]

and the second by the addition of (36) and (37) to (1), namely

\[ L_{II} = L_4 + L_3 + L_2. \]

### 3.1 Asymptotic solutions

The one-dimensional subsystems \( L_3 \) and \( L_2 \) arising from the imposition of spherical symmetry on the extensions \( L_3 \) and \( L_2 \), analogous to the one-dimensional subsystem (10) of (1) are, respectively

\[ L_3 = \frac{\kappa^2}{r} (1 - f^2) \left( 3f^2 + \frac{1}{r^2} (1 - f^2)^2 \right), \quad (38) \]
\[ L_2 = \frac{1}{2} \kappa_2^4 r^3 \left( h^2 + \frac{3}{r^2} f^2 h^2 + 2\lambda_5 (\eta^2 - h^2)^2 \right). \quad (39) \]

It is important to note that the one dimensional density \( L_3 \) remains positive provided that the asymptotic conditions (13) are satisfied by the solution, inspite of not being a positive definite quantity by construction. We shall find this to be the case below when we solve the Euler-Lagrange equations \( \delta_f L_I = 0 \) and \( \delta_f L_{II} = 0 \) in the region \( r >> 1 \). This will be the justification of employing (36) as an extension.

The Euler-Lagrange equations \( \delta_f L_3 = 0 \) for (38), and, \( \delta_f L_2 = 0 \) and \( \delta_h L_2 = 0 \) for (39), are given respectively by

\[ \delta_f L_3 = - \frac{6\kappa_1^2}{r} \left( (1 - f^2) f_{rr} - f f_{r} + \frac{1}{r^2} f_{r} + \frac{1}{r^2} (1 - f^2)^2 f \right), \quad (40) \]
\[ \delta_f L_2 = 3\kappa_2^4 r h^2 f. \quad (41) \]
\[ \delta_h L_2 = \kappa^4 r \left( 3 f^2 h - 4 \lambda_5 r^2 (\eta^2 - h^2) h - 3 r h_r - r^2 h_{rr} \right). \]  

(42)

To start with we dispose of the question of the asymptotic solutions in the \( r << 1 \) region, to both the extended models \( \mathcal{L}_I \) and \( \mathcal{L}_{II} \). We have verified that the additional terms do not change the behaviours (19) and (20) for the functions \( f(r) \) and \( h(r) \) respectively. There then just remains to find the asymptotic behaviours in the \( r >> 1 \) region.

Consider first the extended system \( \mathcal{L}_I = \mathcal{L}_4 + \mathcal{L}_3 \). The \( \delta_h L_I = \delta_h L_4 = 0 \) equations of its one-dimensional subsystem are unchanged and are given by (12), while the \( \delta_f L_I = \delta_f L_4 + \delta_f L_3 = 0 \) equations are now given by (11) and (40). Linearising the latter around the asymptotic value of \( f(r) = 0 + F(r) \) we find

\[ r^2 F_{rr} - r F_r + \left( 1 - \left( \frac{\lambda_1 \eta^2}{\kappa_1^2} \right) \right) F = 0, \]  

(43)

which has a power decay solution yielding

\[ f(r) \sim r^{1 - \sqrt{\frac{\eta^2}{\kappa_1^2}}} \]  

(44)

provided that \( \sqrt{\frac{\eta^2}{\kappa_1^2}} > 1 \). This is not physically unreasonable since the constant \( \eta \) is expected to be large, while \( \kappa_1 \) can be taken to be small. In practice we have set \( \eta = 1 \) in all our numerical computations and, taken \( \lambda_1 \) to be of the order of unity, while \( \kappa_1 \) must be taken to be substantially smaller than unity if the lack of positive definiteness of \( \mathcal{L}_3 \) (38) is not to prejudice the instanton solution of the basic model. Thus, for the parameters employed in our numerical computations, the conditions for (44) to imply a rapid decay are met.

Notwithstanding the fact that we expect a rapid decay (44) for the function \( f(r) \), it would be desirable to extend the model in such a way that like the function \( h(r) \), the function \( f(r) \) also decays exponentially. This is because the validity of the approximations that will be made in an eventual application of the instanton solutions found here to the construction of a dilute instanton gas, rely on the strong localisation of the instantons and the best way of meeting this requirement is by exponential localisation.
To this end, we consider the extension $\mathcal{L}_{II} = \mathcal{L}_4 + \mathcal{L}_3 + \mathcal{L}_2$. In this case, the one-dimensional Euler-Lagrange equations $\delta f L_{II} = \delta f L_4 + \delta f L_3 + \delta f L_2 = 0$ and $\delta h L_{II} = \delta h L_4 + \delta h L_2 = 0$ are given by (11), (12), (40) and (42). Linearising these around their asymptotic values $f(r) = 0 + F(r)$ and $h(r) = \eta + H(r)$ we find

$$\rho^2 F_{\rho\rho} - \rho F_{\rho} - \rho^4 F = 0 \quad (45)$$
$$\sigma^2 H_{\sigma\sigma} + 3\sigma H_{\sigma} - \sigma^2 H = 0 \quad (46)$$

where we have used the dimensionless rescalings $\rho = (\frac{\kappa^2}{2\gamma_5 \eta})^{\frac{1}{4}} \eta r$ and $\sigma = 2\sqrt{2\lambda_5 \eta} r$ of the radial variable $r$. Note that the notations $\rho$ and $\sigma$ in (45) and (46) are different from those in (15) and (16). From the fourth power of $\rho$ in the last term of (45) it is clear that this equation cannot be brought to the form of a modified Bessel equation so we evaluate its asymptotic solution directly, yielding $f(r)$ in the $r \gg 1$ region to be

$$f(r) = e^{\kappa^2 \gamma_5 \eta^2 r^2}, \quad (47)$$

which is clearly exponentially localised. Equation (46) on the other hand can be brought to the form of a Bessel equation with the solution $H(\sigma) = \sigma^{-1} K_1(\sigma)$ yielding

$$h(r) = \eta \left( 1 - \frac{1}{2\sqrt{2\lambda_5 \eta} r} K_1(2\sqrt{2\lambda_5 \eta} r) \right). \quad (48)$$

In summary, we see that in the extended model $\mathcal{L}_{I} = \mathcal{L}_4 + \mathcal{L}_3$, the function $h(r)$ is exponentially localised according to (18), while the function $f(r)$ is power localised according to (44). In the extended model $\mathcal{L}_{II} = \mathcal{L}_4 + \mathcal{L}_3 + \mathcal{L}_2$ both functions $f(r)$ and $h(r)$ are exponentially localised according to (47) and (48) respectively.

We see that the function $h(r)$ is exponentially localised according to (18) and (18) for both the extended models $\mathcal{L}_{I} = \mathcal{L}_4 + \mathcal{L}_3$ and $\mathcal{L}_{II} = \mathcal{L}_4 + \mathcal{L}_3 + \mathcal{L}_2$, respectively, while the function $f(r)$ is power localised according to (44) for the model $\mathcal{L}_{I} = \mathcal{L}_4 + \mathcal{L}_3$ and exponentially localised according to (47) for $\mathcal{L}_{II} = \mathcal{L}_4 + \mathcal{L}_3 + \mathcal{L}_2$. 

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Perhaps the most important result of this Subsection is, that the asymptotic behaviours of the function $f(r)$ in both extended models given respectively by (19), (44), and (19), (47), ensure that the one dimensional density (38) does not become negative.

### 3.2 Numerical solutions

For the purposes of numerical integrations we have employed the following two values of the coupling constant $\kappa_1$, $\kappa_1 = 0.1$ and 0.01 in (36), which result in appropriately small quantitative extensions of the basic model.

For the extended model $\mathcal{L}_I = \mathcal{L}_4 + \mathcal{L}_3$ given by (1) and (36), we have integrated equations $\delta f L_1 = \delta f L_4 + \delta f L_3 = 0$ and $\delta h L_1 = \delta h L_4 = 0$ given respectively by (11), (14), and by (12), numerically, starting with the functions (19) and (20) near $r = 0$ and choosing the numerical values of the constants $A$ and $B$ so that $f(r)$ and $h(r)$ reach their asymptotic values given by the second members of (13) and (14). We have performed this integration only for the dimensionless coupling constants $\bar{\lambda}^{(3)} = (1, 1, 1, 1)$. The numerical values of the pair of constants $\{A, B\}$ are fixed by the numerical integrations, and are listed in the first and second columns of Table 2. The profiles of the functions $f(r)$ for this extended model are not given since they are qualitatively the same as those in Figure 1, while the function $h(r)$ is the same as that for the basic model and is given by the $\bar{\lambda} = (1, 1, 1, 1)$ curve in Figure 2. The numerical values of the total actions $S_I(\kappa_1 = 0.1) = \text{and } S_I(\kappa_1 = 0.1) = \text{pertaining to these two solutions are listed in the third column of Table 2.}$

For the extended model $\mathcal{L}_{II} = \mathcal{L}_4 + \mathcal{L}_3 + \mathcal{L}_2$ given by (1), (36) and (37), we have employed the same two values of the coupling constant $\kappa_1 = 0.1$, 0.01 as above and for each of these the values of the coupling constant $\kappa_2 = 0.25$ and $\kappa_2 = 0.5$. We have integrated the two second order equations $\delta f L_{II} = \delta f L_4 + \delta f L_3 + \delta f L_2 = 0$ given by (11) (14) and (14), and, $\delta h L_{II} = \delta h L_4 + \delta_2 L_2 = 0$ given by (12) and (12). Again we have performed this integration only for $\bar{\lambda} = (1, 1, 1, 1)$, and the values of the pair of numerical constants $\{A, B\}$ are fixed by the numerical integration and are listed in the first and second
columns of Table 3. Again the profiles of the functions $f(r)$ and $h(r)$ are not exhibited since they are qualitatively the same as those in Figures 1 and 2. The numerical values of the total actions pertaining to these two solutions are $S_{II}(\kappa_1 = 0.1) = $ and $S_{II}(\kappa_1 = 0.01) = $.

Since the density $L_3$ is not positive definite, the stability of the instantons of both the models described by the densities $L_I$ and $L_{II}$ are not guaranteed by the stability of the instanton of the basic model. We argued above however, that if the coupling constant $\kappa_1$ is small enough, the instantons of the extended models will nevertheless be stable. To this end, the values of the total actions corresponding to each instanton given above are relevant. Quantitatively, we seek to demonstrate that the addition of the extension densities (36) and (37) to the basic density (1) does not result in an appreciable change in the value of the action integrals. The detailed comparisons are deferred to Section 4.

Before proceeding to Section 4 however, we must perform one further numerical operation. From the viewpoint of these quantitative comparisons, the difference of the extended model $L_I$ from the basic model is just the non-positive density $L_3$ and hence the interesting quantity is the difference of their action integrals. The extended model $L_{II}$ however is the result of the addition of the non-positive density $L_3$ to the positive definite density $L_{III} = L_4 + L_2$ and not to the basic density alone. The interesting quantity in this case therefore is the difference of the action integral of $L_{II}$ from that of the system described by $L_{III}$, which we have not studied numerically hitherto. This we now do by integrating the Euler-Lagrange equations $\delta f L_{III} = \delta f L_4 + \delta f L_2 = 0$ given by (11) and (41), and $\delta h L_{III} = \delta h L_4 + \delta h L_2 = 0$ given by (12) and (42). We have performed the numerical integrations for the value of $\vec{\lambda}(3) = (1, 1, 1, 1)$ only, and for the values of the coupling constant $\kappa_2 = 0.25$ and $\kappa_2 = 0.5$. The numerical values of the constants $\{A, B\}$ for these solutions are listed in the first and second columns of Table 4, while the values of the total action are listed in the third column of Table 4. We do not plot the profiles of the functions $f(r)$ and $h(r)$ in this case because they are qualitatively the same as those in Figures 1 and 2.
4 Summary and discussion

We have made a detailed numerical study of unit topological charge instanton solutions of the $SO(4) \times U(1)$ Higgs model[5] and its extensions on $\mathbb{R}^4$. The extensions to the basic model are devised such that the dynamics of the extended model would in principle be capable of supporting a dilute Coulomb gas of interacting instantons. This construction however is deferred to future work, and in the present work we have restricted ourselves to establishing the existence of the classical instanton solutions using numerical methods.

In Section 2, we have studied the spherically symmetric instanton solution of the basic models (1) parametrised by a set of dimensionless constants $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, for four different such sets $\vec{\lambda}_i, i = 1, 2, 3, 4$ listed in Table 1. The results are exhibited in Table 1 and Figures 1 and 2. The unusual feature of the system (1) is that for the case $i = 3$, namely for $\vec{\lambda}_3 = (1, 1, 1, 1)$, where the lower bound on the action is saturated by the Bogomol’nyi equations (4)-(6), the only solution is the trivial constant asymptotic values of the fields. That equations (4)-(6) are overdetermined can clearly be seen by inspection of their spherically symmetric restrictions (22)-(25), which are four constraints on two functions. This feature is shared with the usual Skyrme model[10], in which case however the departure of the soliton solution from the self-dual field configuration is fixed by the absolute scale there. Here by contrast this departure is depends on the dimensionless coupling constant $\vec{\lambda}_i$, and in this sense our model is more akin to the Abelian Higgs model[9] on $\mathbb{R}^2$ which even though it does support self-dual solutions, the departure from self-duality depends on the value of the dimensionless coupling strength $\lambda$ multiplying the Higgs potential. $\lambda$ in that case is the analogue of $\vec{\lambda}_i$ here. In both models the value of the action integral increases with an increase in the value of these dimensionless coupling strength(s). This is seen to be the case from the third column of Table 1, where all the action integrals $S(\vec{\lambda}_i)$ are larger than the topological lower bounds. In particular the action integral $S(\vec{\lambda}_4)$ is larger than the lower bound 1 in that case, while the actions $S(\vec{\lambda}_i)$ for $i = 1, 2, 3$ are each larger than their respective lower bounds $(\lambda_i)_{\text{min}}$. To demonstrate more quantitatively the departute from the self-dual configuration, we have plotted the profiles of the one-dimensional restriction of the stress-tensor $T_{44}$.
for each solution in Figure 4.

In Section 3 we have given some arguments in favour of our choice of extended models and proceeded to study these numerically. It turns out that our choice for an extension of (1) is narrowed down to the cubic density (36). This density being non positive definite, there is no guarantee that its addition to (1) will not invalidate the topological lower bound. In that case the extended model would not support an instanton solution. We have argued that for small enough coupling of this term, the instanton solution persists in the extended model is the dynamics of the basic model is robust enough. The numerical data pertaining to this extended model \( \mathcal{L}_I \) is given in Table 2. We can see that the action of \( \mathcal{L}_I \) does not differ appreciably from the action of the basic model, by comparing the actions \( S_I(\lambda_3, \kappa_1) \) in Table 2 with the action \( S(\tilde{\lambda}_3) \) in Table 1. Indeed, this difference is smaller for the case with the smaller value of the coupling constant \( \kappa_1 \), thus justifying our claim that for small enough modifications of the basic model we can expect to have instanton solutions.

The function \( h(r) \) parametrising the instanton of the extended model \( \mathcal{L}_I \) is localised exponentially according to (18), but the function \( f(r) \) is power localised according to (44). This has motivated us to consider yet another extension, for which both functions \( h(r) \) and \( f(r) \) are both exponentially localised. This is the model \( \mathcal{L}_{II} \) which results from the further addition of the density (37) to \( \mathcal{L}_I \) and whose exponentially decaying solutions are given by (47) and (48). The numerical data for this model is given in Table 3. To carry out the corresponding quantitative checks as in the previous case, we need to compare the values of the actions \( S_{II}(\lambda_3, \kappa_1) \) with the action integral of the positive definite density defined by the sum of (11) and the positive definite density \( \mathcal{L}_2 \) defined by (37). We have denoted this system by \( \mathcal{L}_{III} \) and listed the numerical data pertaining to it in Table 4. We must therefore compare the actions \( S_{II}(\tilde{\lambda}_3, \kappa_1) \) in Table 3 with the corresponding actions \( S_{III}(\tilde{\lambda}_3, \kappa_2) \) in Table 4. The qualitative conclusions are exactly the same as those for the extended model \( \mathcal{L}_I \), namely that the action in the extended model \( \mathcal{L}_{II} \) is close enough to that of the basic model, that \( \mathcal{L}_{II} \) also supports instanton solutions.
Quantitatively, the differences $S_I(\vec{\lambda}_3, \kappa_1 = 0.01, \kappa_2 = 0) - S(\vec{\lambda}_3, \kappa_1 = 0, \kappa_2 = 0) = 67 \times 10^{-6}$ and $S_{II}(\vec{\lambda}_3, \kappa_1 = 0.01, \kappa_2 = 0.25) - S_{III}(\lambda_3, \kappa_1 = 0, \kappa_2 = 0.25) = 68 \times 10^{-6}$ are practically equal, implying that the system (1) is equally robust against the addition of (36) as against the addition of (37) and (38), even though in the second case the exponential decay was better satisfied. Evidently, this means that the value chosen for the coupling constant $\kappa_1$ was sufficiently small so that the extended systems arising from the addition of the non-positive definite $F^3$ term (36) result in new systems that also support instanton solutions. To test the validity of this criterion, consider the differences of the corresponding actions for the extended models with a larger value of the coupling constant $\kappa_1$. Thus $S_I(\vec{\lambda}_3, \kappa_1 = 0.1, \kappa_2 = 0) - S(\vec{\lambda}_3, \kappa_1 = 0, \kappa_2 = 0) = 6665 \times 10^{-6}$ and $S_{II}(\vec{\lambda}_3, \kappa_1 = 0.1, \kappa_2 = 0.25) - S_{III}(\lambda_3, \kappa_1 = 0, \kappa_2 = 0.25) = 6685 \times 10^{-6}$, which are again nearly equal to each other but are very considerably larger than the former pair of differences. This completes our justification of the extended models $L_I = L_4 + L_3$ and $L_{II} = L_4 + L_3 + L_2$.

Having established that the extended models also can support instantons with the same asymptotic properties as the instantons of the basic model, the remaining task is to compute the volume integral

$$S_{int} = \int d^4x \sum_{a,b,c} \text{Tr} F_{\mu\nu}(x - x_a) F_{\nu\lambda}(x - x - x_b) F_{\lambda\mu}(x - x_c)$$  \hspace{1cm} (49)$$

in the Dirac string gauge. Clearly $S_{int}$ in (49) is the only nonvanishing contribution to the action due to inter-instanton interactions since in the Dirac string gauge the asymptotic contributions of all Higgs dependent terms vanish and the density of the only other Higgs independent term in either extended action density coming from $F_{\mu\nu\rho\sigma}$ also vanishes according to our arguments at the end of Subsection 3.2, namely (54). The integrand in (49), which is readily calculated by substituting (28) in (49), is a quantity of considerable complexity and since we will not perform the integration here we do not write it down. If one set any two of the summation indices $a, b, c$, in (49) the same, namely restricted to "two neighbour" interactions, then the integrand would become singular at the origin in the usual way. Thus the
interaction term will have to take account of "three neighbour" interactions. This situation may be changed by replacing the cubic extension term \((36)\) in \((19)\) by the alternative density \(F_{\mu\nu}D^2F_{\mu\nu}\) appearing in \((33)\).

\[
S_{\text{int}} = \int d^4x \sum_{a,b} \text{Tr} D_\lambda F_{\mu\nu}(x - x_a)D_\lambda F_{\mu\nu}(x - x - x_b), \quad (50)
\]

which is a "two neighbour" interaction. The details of these contributions to the action as well as their consequences in the resulting effective theory of Strong interactions will be investigated and given elsewhere.

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**TABLES:**

Table 1

| $\lambda_i$ | $A$   | $B$   | $S(\lambda_i)$ |
|-------------|-------|-------|-----------------|
| 0.5         | 0.300044 | 0.653908 | 1.023269 |
| 0.8         | 0.324460 | 0.650557 | 1.611897 |
| 1.0         | 0.335691 | 0.648918 | 2.002486 |
| 1.2         | 0.344599 | 0.647561 | 2.392250 |

Table 2

| $\kappa_2 = 0$ | $\kappa_1$ | $A$   | $B$   | $S_I(\lambda_3, \kappa_1)$ |
|----------------|-------------|-------|-------|---------------------------|
|                | 0.01        | 0.335669 | 0.648919 | 2.002553 |
|                | 0.1         | 0.333488 | 0.648989 | 2.009151 |

Table 3

| $\kappa_2 = 0.25$ | $\kappa_1$ | $A$   | $B$   | $S_{II}(\lambda_3, \kappa_1)$ |
|-------------------|-------------|-------|-------|-----------------------------|
|                   | 0.01        | 0.336761 | 0.651287 | 2.010352 |
|                   | 0.1         | 0.334577 | 0.651353 | 2.016969 |
| $\nu_i$ | $A$   | $B$   | $S_{II}(\lambda_3, \kappa_1)$ |
| 0.01         | 0.350511 | 0.679313 | 2.108078 |
| 0.1          | 0.348347 | 0.679341 | 2.115000 |

Table 4

| $\kappa_1 = 0$ | $\kappa_2$ | $A$   | $B$   | $S_{III}(\lambda_3, \kappa_2)$ |
|----------------|-------------|-------|-------|-----------------------------|
|                | 0.25        | 0.336783 | 0.651286 | 2.010284 |
|                | 0.50        | 0.350533 | 0.679313 | 2.108006 |
FIGURE CAPTIONS:

Figure 1. Profiles of the function $f(r)$ for the systems characterised by $\vec{\lambda}_1 = (0.5, 0.5, 0.5, 0.5), \vec{\lambda}_2 = (0.8, 0.8, 0.8, 0.8), \vec{\lambda}_3 = (1, 1, 1, 1), \vec{\lambda}_3 = (1.2, 1.2, 1.2, 1.2)$ respectively from right to left.

Figure 2. Profiles of the function $h(r)$ for the systems characterised by $\vec{\lambda}_1 = (0.5, 0.5, 0.5, 0.5), \vec{\lambda}_2 = (0.8, 0.8, 0.8, 0.8), \vec{\lambda}_3 = (1, 1, 1, 1), \vec{\lambda}_3 = (1.2, 1.2, 1.2, 1.2)$. All curves strongly overlapping.

Figure 3. Profiles of the action densities of the solutions to the systems characterised by $\vec{\lambda}_3 = (0.5, 0.5, 0.5, 0.5), \vec{\lambda}_2 = (0.8, 0.8, 0.8, 0.8), \vec{\lambda}_3 = (1, 1, 1, 1), \vec{\lambda}_3 = (1.2, 1.2, 1.2, 1.2)$ respectively in order of the heights of the curves.

Figure 4. Profiles of $T_{44}$ given by (21) for the systems characterised by $\vec{\lambda}_1 = (0.5, 0.5, 0.5, 0.5), \vec{\lambda}_2 = (0.8, 0.8, 0.8, 0.8), \vec{\lambda}_3 = (1, 1, 1, 1), \vec{\lambda}_3 = (1.2, 1.2, 1.2, 1.2)$ respectively in order of the lowest to highest curves.
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