Interplay between cosmological expansion and massive objects

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Abstract. We have derived a metric for a point mass in an expanding universe. In the spatially flat case, a simple coordinate transformation relates our metric to that derived by McVittie. Nonetheless, our use of non-comoving (physical) coordinates greatly facilitates physical interpretation. We have also derived a coordinate-free expression for the force required to keep a particle at rest in this spacetime. For redshift $z > 0.67$, we have identified two important time-dependent physical radii; the largest possible circular orbit $r_F(t)$, and the largest stable circular orbit $r_S(t)$, which lies inside $r_F(t)$, either of which could be interpreted the edge of an object of mass $m$. In the case of a dynamical background dominated by a phantom fluid, we can use our force expression to predict the time of the ‘Big Rip’ when the universe ends. For open and closed universes, our metrics describe different spacetimes to McVittie’s metrics; we believe the latter to be incorrect. In the closed case, our metric possesses an image mass at the antipodal point of the universe.

1. Introduction

Galaxies and galaxy clusters are not usually considered to have definite edges. Indeed, their densities $\rho(r)$ are often modelled using a Navarro-Frenk-White profile, which falls off gradually with radial distance from the centre. From a Newtonian perspective, the force on a constituent particle falls off as $1/r^2$, and tends to zero as $r \to \infty$ rather than at some finite value of $r$.

There is recent observational evidence, however, for a definite cut-off within objects such as galaxies. Since this separates bound and unbound material, one could interpret it as the object’s edge. One piece of evidence has come from an analysis conducted by Peirani and de Freitas Pacheco [1]. By plotting the radial velocity profiles of constituent stars of nearby galaxies, they have observed a zero velocity surface in each case. For example, for M81 plotted in Figure 1, this appears to be at $\sim 0.8$ Mpc, around 3 times the extent of a typical $\sim 0.26$ Mpc galaxy. We have now aimed to explain this phenomenon using models for a massive object in an expanding universe.
2. Newtonian verses general relativistic analysis with $\Lambda$

The Newtonian force on a particle near some object, including a cosmological constant $\Lambda$, is given by:

$$F(r) \propto \frac{v^2(r)}{r} = -\frac{M(r)}{r^2} + \frac{1}{3}\Lambda,$$

where $v(r)$ is the circular velocity and the object’s mass is

$$M(r) = \int_0^r 4\pi r^2 \rho(\bar{r}) d\bar{r}.$$  

The first term points inwards and is gravitational, and the second term points outwards due to the universe’s de Sitter expansion. For any object, $F(r) = v(r) = 0$ at a finite $r$. Indeed, even for a point mass with $\Lambda = 10^{-35} \text{s}^{-2}$, one obtains results in close agreement with those observed [2].

Here, we have shown the general relativistic circular velocity function to be given by:

$$v(r) = \frac{1}{g_{rr}} \sqrt{\frac{M(r)}{r} - \frac{\Lambda}{3} + 4\pi p(r)},$$

where $g_{rr}$ represents the radial part of the spacetime metric with pressure $p(r)$. The edge of the object occurs at $v = 0$, and the pressure also vanishes there; therefore, the edge is located as the solution to $M(r)/r - \Lambda/3 = 0$. This is equivalent to where $F(r) = 0$ in the Newtonian expression given in Eq. (1). Remarkably, no general relativistic correction is required. The circular velocity profile for a typical galaxy is shown in Figure 2. The cut-off is found to lie at $\sim 1.06 \text{Mpc}$, just over 4 times the typical extent of a galaxy.

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Figure 1. Radial velocity vs. distance of stars in galaxy M81 with mass $9.2 \times 10^{11} \text{M}_\odot$, and the best-fit line (reproduced from [1]).

Figure 2. Circular velocity $v(r)$ (km/s) for a galaxy of mass $1 \times 10^{12} \text{M}_\odot$ with exponentially-decreasing radial density profile, and $r_0 = 3 \text{kpc}$. The dotted line is for $\Lambda = 0$.

3. A mass particle in a spatially flat expanding universe

In reality, the universe is not de Sitter; rather it expands at the Hubble rate $H(t) = R'(t)/R(t)$, where $R(t)$ is the scale factor for expansion. We have now considered a mass embedded in a Friedmann-Robertson-Walker (FRW) background with density $\rho(t)$, albeit at the expense of modelling the mass as pointlike. Working in ‘physical’ (non-comoving) coordinates, the energy contained within a radius $r$ is $M(r,t) = m + (4/3)\pi r^3 \rho(t)$. Omitting the details of the tetrad-based derivation conducted in [3], this leads to the spatially flat spacetime metric:

$$ds^2 = \left[1 - \frac{2m}{r} - r^2 H^2(t)\right] dt^2 + 2rH(t) \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} dr dt - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. This describes the same spacetime as McVittie’s metric via the coordinate transformation $r \rightarrow r R(t)(1 + m/(2r R(t)))^2$. Therefore, McVittie’s metric does...
describe a mass particle in an FRW background, an issue which has been previously under much debate.

3.1. Dynamical apparent horizons
We have shown in [4] that there are two apparent horizons associated with this metric: The ‘black hole horizon’ and the ‘cosmological horizon’. Since, the metric is time-dependent, these horizons are dynamical. In fact, they did not always exist. Their evolution is depicted in Figure 3.

![Figure 3](image)

**Figure 3.** The existence and locations of the apparent horizons around $m$.

3.2. Force required to keep a test particle at rest
In [3], we have derived a new, coordinate-free, fully general relativistic expression for the force required to keep a particle at rest in this spacetime as:

$$f = \frac{-m}{r - rH^2(t)} \frac{rH^2(t)(q(t) + 1)}{1 - 2m} \left(1 - \frac{2m}{r} - r^2H^2(t)\right)^{3/2} \approx m \frac{m}{r^2} + q(t)H^2(t)r,$$

where $q(t) = -H'(t)/H^2(t) - 1$ is the deceleration parameter. Notice that the cosmological component of the Newtonian expression is directed outwards (inwards) when the expansion of the universe is accelerating (decelerating). This highlights the common misconception that there is a force associated simply with the expansion of space; instead it is better associated with the acceleration/deceleration of the expansion. The Newtonian expression does appear in the GADGET cosmological simulation codes. However, the cosmological force is thought to have reversed in the direction for redshift $z < 0.67$. It would, therefore, be interesting to study structure formation in the early universe using the full general relativistic expression for the force.

3.3. Physical radii around point mass
The largest possible $r_F(t)$ and largest stable $r_S(t)$ orbits around the point mass $m$ may be calculated as the minimum in the effective potential of an orbiting particle, equivalent to the zero of its physical acceleration $d^2r/dt^2$. In [2], we have shown, using the Lagrangian method, that:

$$r_F(t) \approx \left[-\frac{m}{q(t)H^2(t)}\right]^{1/3}, \quad \text{and} \quad r_S(t) \approx \left[-\frac{m}{4q(t)H^2(t)}\right]^{1/3}.$$  

(5)
Physically, $r_F(t)$ corresponds to where $f = 0$ in Eq. (4), so it separates bound and unbound material. Since, orbits here may not be stable, it might make more sense to interpret $r_S(t)$ as the object’s edge. These cut-offs only exist when the universe’s expansion is accelerating ($q(t) < 0$). Despite both being time-dependent, $r_S(t)$ lies a constant factor of $\sim 1.6$ inside $r_F(t)$ at these times. Table 1 compares some $r_F(t)$ and $r_S(t)$ values with those obtained by Peirani and de Freitas Pacheco, for $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$, and $q(t) = -0.55$. All three cut-offs are in close agreement.

### 3.4. The Big Rip

There has been recent renewed interest in a phantom background fluid, with equation of state $w = p/\rho < -1$. This has led to the prediction of a Big Rip at a finite time $t_{\text{rip}}$ in the future [5]. If $w = -\frac{3}{2}$, the time remaining for the universe before the Big Rip, $\tilde{t}$, is calculated as $\sim 22$ Gys.

We may use our approximate force expression in Eq. (4) to calculate the time before $t_{\text{rip}}$ at which a particle at the edge of an object becomes gravitationally unbound from it (when $f = 0$). This is interpreted as the time at which specific objects of mass $m$ and size $r$ are ripped apart. We have shown in [2] that at late times $H(t) \to 2/(3|1+w|\tilde{t})$, and $q(t) \to -(1/2)[1+3w]$. Therefore,

$$\tilde{t} \approx \frac{\sqrt{2|1+3w|}}{3|1+w|} \sqrt{\frac{r^3}{m}}.\quad (6)$$

Although approximate, this result matches Caldwell’s precisely, despite having been derived using a different method. Actual values of $\tilde{t}$ for specific objects are given in Table 2.

### 4. A mass particle in a spatially curved expanding universe

The metrics for spatially curved expanding universes are found to involve elliptic integrals. To avoid these, we have instead, obtained series solutions for $m \ll r \ll R(t)$. Unlike in the spatially open case ($k = -1$), there is no notion of spatial infinity in a closed universe ($k = 1$), so we have, instead, specified a boundary condition at the outer radius of pressure divergence $r_F^p(t)$. This is determined to be merely a coordinate singularity, marking a point of reflection for the entire spacetime geometry. Thus, for $k = 1$, there is a real image mass at the anti-portal point of the universe, as depicted in Figure 4. To leading order, the force required to hold a particle at rest in both cases is the same as in the spatially flat case, but an extra term appears at the next order as:

$$f \approx \frac{m}{r^2} + \frac{m^2}{r^3} + \frac{3m^3}{2r^4} + q(t)H^2(t)r + \left(2q(t) + \frac{3}{2}\right)H^2(t)m + \frac{3k}{2} \frac{m}{R^2(t)}\quad (7)$$

For $k = -1$ ($k = 1$), this can be interpreted as an ‘anti-gravitational’ (gravitational) force due to the virtual (real) ‘image’ mass being dragged out to the curvature scale of the universe.

### Table 1. Comparison between our theoretical physical radii and those cut-offs found from a best-fit numerical analysis for galaxies (P&P).

| Galaxy   | Mass ($10^{11} M_\odot$) | P&P  | $r_F(t_0)$ | $r_S(t_0)$ |
|----------|--------------------------|------|------------|------------|
| M83      | 2.1                      | 1.10 | 1.47       | 0.93       |
| M81      | 9.2                      | 0.75 | 1.12       | 0.70       |
| IC 342   | 2.0                      | 0.51 | 0.67       | 0.42       |
| NGC 253  | 1.3                      | 0.40 | 0.58       | 0.36       |

### Table 2. The history and future of the Universe with $w = -3/2$.

| Time (Gyrs) | Event                                |
|------------|--------------------------------------|
| $t_{\text{rip}} - 70$ Myr | Destroy Milky Way                     |
| $t_{\text{rip}} - 4$ months | Unbind Solar System                   |
| $t_{\text{rip}} - 25$ mins | Earth explodes                        |
| $t_{\text{rip}} = 35$ Gyr  | Big Rip                               |
5. Further remarks
In reality, objects are neither pointlike nor fixed. In our subsequent paper, we will investigate the model for a massive object with finite size, and constant spatial density embedded in an otherwise FRW universe. This accounts for mass accretion by construction. We will look at two special cases, which offer elegant analytical solutions: The case that the boundary of the object is fixed, and the case that its mass is fixed. Further numerical results which emerge are left for future research.

References
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