Estimation of nuclear spin state in a double quantum dot via hyperfine interaction

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Abstract. Hyperfine interaction of electron spins with nuclear spins, in coupled double quantum dots is studied. Results of successive electron spin measurements exhibit bunching due to correlations induced via the nuclear spins. Further nuclear spins can be purified via conditional electron spin measurements which lead to electron spin revivals in the conditional probabilities. The electron spin coherence time can be extended via conditional measurements. The results are extended to a single electron on a single QD.

Electron spins in semiconductor quantum dots are promising candidates as building blocks for quantum information processing[1, 2, 3] due to their long coherence times[4, 5]. The dominating decoherence mechanism of electron spins is the hyperfine(HF) interaction with the spins of the host nuclei[6, 7, 8, 9, 10]. Coherent manipulation of two-electron spin states has been achieved in double quantum dots(QD) in recent experiments[11], and a detailed study of various aspects of hyperfine(HF) interaction and electron spin decoherence became possible[12].

In section-1, we are going to introduce electron nuclei coupled system in an electrically gated double QD occupied by two electrons, then discuss bunching and revival in the results of electron spin measurements. In the section-2 the results will be briefly extended to a single QD occupied by a single electron.

1. Electron nuclei coupled system in a coupled double QD
We consider an electrically gated double quantum dot(QD) occupied by two electrons. Under a high magnetic field, s.t. the electron Zeeman splitting is much greater than the hyperfine fields and the exchange energy, dynamics takes place in the spin singlet ground state $|S\rangle$ and triplet state of zero magnetic quantum number $|T\rangle$,

$$H_e = JS_z + r\delta h_z S_x,$$

where $S$ is the pseudospin operator with $|T\rangle$ and $|S\rangle$ forming the $S_z$ basis. $\delta h_z = h_{1z} - h_{2z}$, where $h_{1z}$ and $h_{2z}$ are the components of nuclear HF field along the external magnetic field in the first and second dot, respectively[6, 9]. $r = t/\sqrt{t^2 + (\delta/2 + \sqrt{\delta^2/4 + t^2})^2}$ is the amplitude of the hyperfine coupling, which is determined by the gate voltages. $\delta$ is the detuning which is a linear function of gate voltage differences, and $t$ is the tunneling coefficient. When $\delta \gg t$, $r \approx 1/\sqrt{1 + (\delta/t)^2}$.
the ground state singlet state corresponds to the case where both electrons are localized in the
dot and HF coupling is switched off, \( r \to 0 \). The opposite limit \( \delta \ll -t \) corresponds to the
singlet state where electrons are located in different dots, and HF coupling is maximized \( r \to 1 \).

1.1. Bunching in electron spin measurements

Now we show that by electron spin measurements the coherent behavior of nuclear spins can be
demonstrated. Electron spins are initialized in the singlet state and the nuclear spin states are
initially in a mixture of \( \delta h_z \) eigenstates, \( \rho(t = 0) = \sum_n p_n |\Psi_n\rangle\langle\Psi_n| \), where \( p_n \)
is a nuclear state with an eigenvalue of \( \delta h_z = h_n \) and satisfies \( Tr(p_n) = 1 \). \( p_n \) is the probability of the hyperfine
field \( \delta h_z \) having the value \( h_n \).

In the unbiased regime \( \delta \ll -t \), the nuclear spins and the electron spins interact for a time
span of \( \tau \). Then the gate voltage is swept adiabatically to a high value(s.t. \( \delta \gg t \)), in a time
scale much shorter than HF interaction time, leading to the state, \( \rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n| \), where
\( |\Psi_n\rangle = \alpha_n |S\rangle + \beta_n |T\rangle \), with \( \alpha_n = \cos(\Omega_n \tau/2) + iJ/\Omega_n \sin(\Omega_n \tau/2) \), \( \beta_n = -ih_n/\Omega_n \sin(\Omega_n \tau/2) \) and
\( \Omega_n = \sqrt{J^2 + h_n^2} \) is the Rabi frequency.

Next a charge state measurement is performed which detects a singlet or triplet state[11].
Probability to detect the singlet state is \( \sum_n p_n |\alpha_n|^2 \), and the triplet state is \( \sum_n p_n |\beta_n|^2 \).
Subsequently one can again initialize the system in the singlet state of electron spins, and
turn on the hyperfine interaction for a time span of \( \tau \), and perform a second measurement. In
general over \( N \) measurements, the probability of \( k \) times singlet outcomes is

\[
P_{N,k} = \binom{N}{k} \langle |\alpha|^2 \rangle^k \langle |\beta|^2 \rangle^{N-k}.
\]  

(2)

where \( \langle ... \rangle \) is the ensemble averaging over the hyperfine field \( h_n \)[9]. Here the key assumption
is that nuclear states preserve their coherence over \( N \) measurements, thus the measurements
are not independent due to nuclear memory. One can easily contrast this result with the
semiclassical(SC) result for which nuclear spins are assumed to be purely classical, whereas
electron spins are taken to obey quantum mechanics[9]. In SC case results of successive
measurements are independent and the probability for obtaining \( k \) times singlet results over
\( N \) measurements is given by,

\[
P'_{N,k} = \binom{N}{k} \langle |\alpha|^2 \rangle^k \langle |\beta|^2 \rangle^{N-k}.
\]  

(3)

In the SC case the probability distribution (3) obeys simply a Gaussian distribution with mean
\( k = N \langle |\alpha|^2 \rangle \), and variance \( N \langle |\alpha|^2 \rangle \langle |\beta|^2 \rangle \), as \( N \to \infty \). However, in quantum mechanical(QM)
treatment of nuclear spins, the probability distribution (2) may exhibit different statistics
depending on the initial nuclear state. If the SC distribution of \( h_n \) is characterized by the
same distribution as in QM case, the two probability distributions (2) and (3) yield the same
mean value, \( \bar{k} = N \langle |\alpha|^2 \rangle \), however with distinct higher order moments. If the distribution of
initial nuclear state \( p_n \) has a width \( \Delta \), then for HF interaction time \( \tau \geq 1/\Delta \), the SC and QM
distributions start to deviate from each other. They yield the same distribution only when the
initial nuclear state is in a well defined eigenstate of \( \delta h_z \), i.e. when \( \Delta = 0 \).

In particular we are going to consider the case when the nuclear spins are initially randomly
oriented; probability distribution for hyperfine fields obeying a Gaussian distribution \( p_n \to
\) 

\[
p(h) = 1/\sqrt{2\pi} \sigma^2 \exp[-h^2/2\sigma^2].
\]

In Fig. 1, for \( N = 20 \) measurements, \( P_{N,k} \) is shown for HF interaction times \( \sigma \tau = 0.5, 1.5, \infty \). For \( \tau = 0 \), the probability for both SC and QM cases is peaked at \( k = 20 \). However, immediately after the HF interaction is introduced, the probability
distributions show distinct behavior. The SC distribution converges to a Gaussian distribution.
In QM case the probabilities bunch at \( k = 0, 20 \) for \( J = 0 \), and when \( J/\sigma = 0.5 \) those bunch
at \( k = 20 \) only. As \( J \) is increased above some critical value, no bunching takes place at \( k = 0 \)
singlet measurement.
1.2. Electron spin revivals

The modified nuclear spectrum leads to correlations between the successive electron spin measurements. Depending on the results of previous measurement, one may increase the singlet-triplet mixing. As a particular example consider the case: Starting from a random spin configuration, $N$ successive electron spin measurements are performed, each following initialization of electron spins in the spin singlet state and a HF interaction of duration $\{\tau_i, i = 1 \ldots N\}$ and all outcomes turn out to be singlet. Then again HF interaction is switched on for a time $t$, and the $(N+1)$th measurement is carried out. The conditional probability to detect the singlet state is given by

$$P = \frac{\sum(s_1^1)\ldots(s_N^N) e^{-\Delta}[(s_1-1)\tau_1+(s_2-1)\tau_2+\ldots+(s_N-1)\tau_N+(s_N+1-1)t]^2}{\sum(s_1^1)\ldots(s_N^N) e^{-\Delta}[(s_1-1)\tau_1+(s_2-1)\tau_2+\ldots+(s_N-1)\tau_N]^2},$$

where the sums run over $s_i = 0 \ldots 2$. For the particular case $\tau_1 = \tau_2 = \ldots = \tau_N = \tau \gg 1/\sigma$, the initial state is revived at $t = n\tau$, ($n = 1, 2, \ldots, N$) with a decreasing amplitude, $P \simeq 1/2 + \sum_{s=0}^{N} (2^N s^N) e^{-\frac{\Delta}{2}}(t-(N-s)\tau)^2/4(2^N)$. In Fig. 3 the conditional probabilities (4) are shown for $\sigma\tau = 1.0, 3.0, 6.0$ subject to $N = 0, 1, 2, 5, 10$ prior singlet measurements in each. Revivals are observable only for $\sigma\tau > 1$, because the modulation period of the nuclear state spectrum characterized by $1/\tau$ should be smaller than the variance $\sigma$.

From (4), number of revivals can be increased with various choices for the ratios of HF interaction times $\tau_i$. The underlying mechanism of revivals is purification of nuclear spins by the electron spin measurements. The purity of a system characterized by the density matrix $\hat{\rho}$ is given by $P = Tr \hat{\rho}^2$. As an example we are again going to consider the nuclear state prepared by $N$ successive electron spin measurements with singlet outcomes, each followed by HF interaction times $\tau_1 \ldots \tau_N$.

$$P = \frac{\sum_{s_1=0}^{4}(s_1^1)\ldots(s_N^N) e^{-\frac{\Delta}{2}}[(s_1-2)\tau_1+(s_2-2)\tau_2+\ldots+(s_N-2)\tau_N]^2}{[\sum_{s_1=0}^{2}(s_1^1)\ldots(s_N^N) e^{-\frac{\Delta}{2}}[(s_1-1)\tau_1+(s_2-1)\tau_2+\ldots+(s_N-1)\tau_N]^2]^2}.

$$

For a fixed ratio of $\tau_1 : \tau_2 : \ldots : \tau_N$, purity (5) is a monotonically increasing function of time. For $\sigma\tau_i \gg 1$, one can attain various asymptotic limits for the purity. For instance for $N = 2$,
there are three asymptotic limits; when a) $\tau_1 = 2 \tau_2$ then $P = 11/4$, b) $\tau_1 = \tau_2$ then $P = 35/18$, c) otherwise $P = 9/4$. For $N = 2$ with $\tau_1 = 2 \tau_2 = 2 \tau \gg 1/\sigma$, the conditional probability (4) is given as follows,

$$P \approx \frac{1}{2} + \frac{1}{8} \{e^{-\frac{(2-2\tau)^2}{2}} + 2e^{-\frac{(1-2\tau)^2}{2}} + 3e^{-\frac{(1-\tau)^2}{2}} + 4e^{-\frac{\tau^2}{2}}\}, \quad (6)$$

whereas for $\tau_2 = \tau_1 = \tau \gg 1/\sigma$,

$$P \approx \frac{1}{2} + \frac{1}{12} \{e^{-\frac{(2-2\tau)^2}{2}} + 4e^{-\frac{(1-\tau)^2}{2}} + 6e^{-\frac{\tau^2}{2}}\}. \quad (7)$$

As the purity of nuclear spins increase, more revivals are present with an increased amplitude.

2. Electron spin bunching and revivals in a single QD

Now we are going to consider a single electron on a single QD. Under external field $B$, the system is governed by the Hamiltonian,

$$H = g_e \mu_B B S_z + g_n \mu_n B \sum j_z + h \cdot S. \quad (8)$$

In (8), the first two terms are electron and nuclear Zeeman energies respectively, and the last term is the HF interaction where $h$ is the HF field. When electron Zeeman energy is much greater than rms value of HF fields, viz. $g_e \mu_B B \gg \sqrt{\langle h^2 \rangle}$, flip-flop terms are suppressed and the Hamiltonian (8) becomes, $H \approx g_e \mu_B B S_z + h_z S_z$. Up to a unitary rotation, with $B = 0$, this is equivalent to the Hamiltonian (1) with $J = 0$. $\pm = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ states are coupled by HF interaction with $|\uparrow\rangle$ being the eigenstates of $S_z$. Each time the electron is prepared in $|\uparrow\rangle$. Next it is loaded onto the QD, then removed from the QD after some dwelling time $\tau$. Next spin measurement is performed in $|\pm\rangle$ basis. Essentially the same predictions as that of double QD can be made for this system, namely electron spin bunching and revival.

In Fig. 2, for $N = 20$ measurements, the QM probability distribution of $P_{N,k}$ is shown at electron Zeeman energy $\epsilon = 3\sigma$, for $\sigma = 0.3$, 0.6, 0.9, $\infty$. It is seen that contrary to the double QD, the population bunches at triplet states at times $\tau \sim 1/\sigma$, but then relaxes to the equilibrium distribution cf. Fig. 1.

Next we are going to consider electron spin revivals. For instance after $N$ times HF interaction of duration $\tau \gg 1/\sigma$, each followed by $|\pm\rangle$ measurement, the conditional probability for obtaining $|+\rangle$ in the $(N+1)$th step followed by a HF interaction of duration $t$ is given as,

$$P \approx \frac{1}{2} + \sum_{s=0}^{N} c_{s}^{(2N)} e^{-\sigma^2 (t-(N-s)\tau)^2/2} \cos \epsilon [t - (N-s)\tau]/4^{(2N)}.$$ 

$\epsilon = g_e \mu_B B \sqrt{2}$ is the electron Zeeman energy. This is essentially the same result for that of a double QD discussed in section 1.2.
3. Discussion and conclusion

The randomization of nuclear spins will lead to loss of memory effects described above. The nuclear state conditioned on the electron spin measurements will decohere during time interval between the successive measurements, i.e., when the HF interaction is switched off. Thus, the main decoherence mechanism of nuclear spins is due to intrinsic nuclear dipole-dipole interactions. In double quantum dots the duration of the cycle involving electron spin initialization and measurement is about $10 \mu$s[11]. Since the nuclear spin coherence time determined mostly by the nuclear spin diffusion is longer than about several tens of ms[13, 14, 8], the bunching for $N$ successive measurements up to $N > 1000$ can be observed. The same holds for the number of revivals that can be observed.

We have studied the quantum dynamics of the electron-nuclei coupled system in QD’s. The bunching of results of the electron spin measurements and the revival in the conditional probabilities are emerging features of coherence of nuclear spins. The underlying mechanism is the correlations between successive measurements induced via nuclear spins and the increase in the purity of the nuclear spin state through the electron spin measurements. This mechanism is expected to lead to the extension of the electron spin coherence time.

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