A DIVIDE-AND-CONQUER ALGORITHM FOR DISTRIBUTED OPTIMIZATION ON NETWORKS

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Abstract. In this paper, we consider networks with topologies described by some connected undirected graph \( G = (V, E) \) and with some agents (fusion centers) equipped with processing power and local peer-to-peer communication, and optimization problem \( \min_x \{ F(x) = \sum_{i \in V} f_i(x) \} \) with local objective functions \( f_i \) depending only on neighboring variables of the vertex \( i \in V \). We introduce a divide-and-conquer algorithm to solve the above optimization problem in a distributed and decentralized manner. The proposed divide-and-conquer algorithm has exponential convergence, its computational cost is almost linear with respect to the size of the network, and it can be fully implemented at fusion centers of the network. Our numerical demonstrations also indicate that the proposed divide-and-conquer algorithm has superior performance than popular decentralized optimization methods do for the least squares problem with/without \( \ell^1 \) penalty.

1. Introduction

Networks have been widely used in many real world applications, including (wireless) sensor networks, smart grids, social networks and epidemic spreading [1, 12, 18, 25, 29, 39, 47]. Their complicated topological structures could be described by some graphs with vertices representing agents and edges between two vertices indicating the availability of a peer-to-peer communication between agents, or the functional connectivity between neural regions in brain, or the correlation between temperature records of neighboring weather stations. Graph signal processing and graph machine learning provide innovative frameworks to process and learn data on networks. By leveraging graph spectral theory and applied harmonic analysis, many concepts in the classical Euclidean setting have been extended to the graph setting, such as graph Fourier transform, graph wavelet transform and graph filter banks, in recent years [9–11, 15, 20, 23, 31]. Graph machine learning has also developed new tools, including graph representation learning, graph neural networks and geometric deep learning, to process data on networks [5, 6, 17, 51]. In this paper, we introduce a divide-and-conquer algorithm, DAC for abbreviation, to solve the following optimization problem

\[
\min_{x \in \mathbb{R}^N} \left\{ F(x) = \sum_{i \in V} f_i(x) \right\}
\]

on a connected undirected graph \( G := (V, E) \) of order \( N \geq 1 \), where \( x = (x(i))_{i \in V} \) and local objective functions \( f_i \) depend only on neighboring variables of the vertex \( i \in V \), see Assumption 2.2. The formulation (1.1) can be seen in various applications such as wireless communication, power systems, control problems, empirical risk minimization, binary classification, etc [15, 20, 23, 25].

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Many networks in modern infrastructure have large amount of agents correlated with each other and the size of data set on the network required to process is also huge. It is often impractical or even impossible to have a central server to collect and process the whole data set. Hence it is of great importance to design distributed and decentralized algorithms to solve the global optimization problems, that is, the storage and processing of data need to be distributed among agents and the local peer-to-peer communication should be employed to handle the coordination of results from each agent instead of a central server [2, 4, 26, 28, 32, 46]. In this paper, we consider the optimization problems on networks with a two-layer hierarchical structure: 1) Most of agents of the network have no or very little computational power and their communication ability is only limited to transmit the data within certain range; 2) Some agents, called fusion centers, have greater computational power and could communicate with nearby agents and neighboring fusion centers within certain distance. In other words, each fusion center is a “central” server for its neighboring subnetwork of small size and there is no central server for the whole network, see Section 2.3, Assumption 3.1 and an illustrate example displayed in Figure 1. The above two-layer hierarchical structure on networks is between two extremes: centralized networks and fully decentralized networks, where the centralized network usually assumes only the central server has data processing ability and each agent sends the data to the central server for processing, while the fully decentralized network assumes that each agent has both communication and computation ability to store the data and perform the computation on their own. For a centralized network, it might require an extremely powerful central server and a high demand for each agent to transmit data to the central server. On the other hand, fully decentralized networks might not be practical in some applications in which every agent is not powerful enough to implement the local computation. Our DAC algorithm to solve the distributed optimization problem (1.1) is designed to be completely implemented on the two-layer hierarchical structure, see Algorithm 2.

Most of the existing decentralized methods are based on the consensus technique, where each agent holds a local estimate of the global variable and these local estimates reach a consensus after certain iterations [4, 14, 23, 34, 36, 41, 44, 50]. Each agent of the network would evaluate its local gradient independently and communicates only with its neighbors to update local estimate of the global variable, which makes it amenable to parallel computation and eliminates the large communication bandwidth of a central server. However, the computational cost at each agent still depends on the size of the local variable (the same size of the global variable), which indicates that the local computation cost at each agent might still be quite expensive. In the decentralized implementation of our DAC algorithm, each agent will only update/estimate a few components of the global variable and communicate with its neighbors to have an aggregate combination of the local solutions, and hence the subproblem for each agent has a much smaller size than the global problem, see (3.2) and Algorithm 2.

A key difference of the proposed DAC algorithm from other decentralized methods is the size reduction of the subproblems. Most existing decentralized methods distribute the computation of gradients to each node, but the size of the subproblem at each node usually depends on the dimension of the global variable and remains the same as the global problem. The proposed DAC further distribute the estimates of the components of the global variable to each fusion center, and hence the subproblem would have a much smaller size than the global problem, see (3.1a). This reduces
both the storage and the computation requirements of each fusion center, and is particularly useful when the global variable has a huge number of components. Another difference is the hierarchical network structure to implement DAC. Even though we will focus on a two-layer hierarchical network structure in this paper, the proposed DAC method would be readily extended to multiple layers. This provides flexibility in computation/communication and also makes it more scalable than many other decentralized methods. Moreover, the exponential convergence of the proposed DAC method is ensured by a thorough theoretical analysis.

The paper is organized as follows. In Section 2, we recall some preliminaries on the underlying graph $G$ and the objective function $F$ of the optimization problem (1.1), and we introduce fusion centers to implement the proposed DAC algorithm. In Section 3, we introduce a novel DAC algorithm (3.1) to solve the optimization problem (1.1) in a distributed and decentralized manner and present a scalable implementation at fusion centers, see Algorithm 2. In Section 4, we discuss the exponential convergence of the iterative DAC algorithm and provide an error estimate when the local minimization in the DAC algorithm is inexact, see Theorems 4.1 and 4.2. In Section 5, we demonstrate the performance of the proposed DAC algorithm and make comparisons with some popular decentralized optimization methods. In Section 6, we collect the proofs of all theorems.

2. Preliminaries

In our proposed DAC algorithm to solve the optimization problem (1.1), the underlying graph $G = (V, E)$ is connected and undirected and it has polynomial growth property (see Section 2.1), the global objective function $F$ is strongly convex and each local objective function $f_i$ is smooth and dependent only on neighboring variables of the vertex $i \in V$ (see Section 2.2), and fusion centers to implement the DAC algorithm are appropriately located (see Section 2.3) and equipped with enough processing power and resources (see Assumption 3.1 in Section 3).

2.1. Graphs with polynomial growth property. For a connected undirected graph $G := (V, E)$, let the geodesic distance $\rho(i, j)$ between vertices $i$ and $j \in V$ be the number of edges in the shortest path to connect $i$ and $j$. Using the geodesic distance $\rho$, we denote the set of all $R$-neighbors of a vertex $i \in V$ by

$$B(i, R) = \{ j \in V, \rho(i, j) \leq R \}, \quad R \geq 0.$$ 

Let $\mu_G$ be the counting measure on the graph $G$ defined by $\mu_G(W) = \#W$, the cardinality of a subset $W \subset V$. We say a connected undirected graph $G = (V, E)$ has polynomial growth property if there exist positive constants $d(G)$ and $D_1(G)$ such that

$$\mu_G(B(i, R)) \leq D_1(G)(R + 1)^{d(G)} \quad \text{for all } i \in V \text{ and } R \geq 0. \quad (2.1)$$

The minimal positive constants $d(G)$ and $D_1(G)$ in (2.1) are known as Beurling dimension and density of the graph $G$ respectively [11].

In this paper, we make the following assumption on the underlying graph of our optimization problem (1.1).

Assumption 2.1. The underlying graph $G$ is connected and undirected with order $N$, and it has polynomial growth property with Beurling dimension and density denoted by $d(G)$ and $D_1(G)$ respectively.
Figure 1. Plotted on the left is a random geometric graph $G_N$ of the order $N = 2048$ with Beurling dimension $d(G_{2048}) = 2$ and Beurling density $D_1(G_{2048}) = 7.6116$. In the middle is a family of fusion centers marked in stars, and corresponding Voronoi diagrams $D_\lambda$, $\lambda \in \Lambda$, marked with various colors. On the right is the communication network for the fusion centers, marked by black solid line, where fusion center $\lambda$ communicates with $\lambda'$ if $D_\lambda \cap D_{\lambda', R, 2m} \neq \emptyset$ with $R = 3$ and $m = 1$, see Assumption 3.1.

A stronger assumption on the graph $G$ than its polynomial growth property is that the counting measure $\mu_G$ is a doubling measure, i.e., there exists a positive number $D_0(G)$ such that

$$\mu_G(B(i, 2R)) \leq D_0(G) \mu_G(B(i, R)) \quad \text{for all } i \in V \text{ and } R \geq 0$$

\cite{11, 38, 45}. The smallest constant $D_0(G)$ in (2.2) is known as the doubling constant of the counting measure. In other words, the doubling property for the counting measure indicates that for any vertex $i$, the number of $(2R)$-neighbors is at most a multiple of the number of $R$-neighbors. It is direct to observe that the doubling property (2.2) for the counting measure $\mu_G$ implies the polynomial growth property (2.1) with $d(G) \leq \log_2(D_0(G))$ and $D_1(G) \leq D_0(G)$, since for all $R \geq 1$,

$$\mu_G(B(i, R)) \leq (D_0(G))^l \mu_G(B(i, R/2^l)) = (D_0(G))^l \leq D_0(G)(R + 1)^{\log_2(D_0(G))},$$

where $l$ is the integer satisfying $2^{l-1} \leq R < 2^l$.

Our illustrative examples of connected undirected graphs are random geometric graphs generated by the GSPToolbox, where $N$ vertices are randomly deployed in the unit square $[0, 1]^2$ and an edge existing between two vertices if their Euclidean distance is not larger than $\sqrt{2/N}$ \cite{19, 30}, see Figure 1.

2.2. **Local and global objective functions.** In this paper, the local objective functions $f_i$, $i \in V$, in the optimization problem (1.1) are assumed to be smooth and dependent only on neighboring variables.
Assumption 2.2. For each \( i \in V \), the local objective function \( f_i(x) \) is continuously differentiable and it depends only on \( x(j) \), \( j \in B(i,m) \), where \( m \geq 1 \) and \( x = (x(j))_{j \in V} \).

For a matrix \( A = (a(i,j))_{i,j \in V} \) on the connected undirected graph \( G = (V,E) \), we define its \textit{geodesic-width} \( \omega(A) \) to be the smallest nonnegative integer such that
\[
a(i,j) = 0 \quad \text{for all} \quad i,j \in V \quad \text{with} \quad \rho(i,j) > \omega(A).
\]

Then the neighboring variable dependence of the objective functions \( f_i, i \in V \), in Assumption 2.2 can be described by a geodesic-width requirement for the gradient of the vector-valued function \( x \rightarrow (f_i(x))_{i \in V} \) on \( \mathbb{R}^N \),
\[
\omega\left(\left(\frac{\partial f_i(x)}{\partial x(j)}\right)_{i,j \in V}\right) \leq m \quad \text{for all} \quad x \in \mathbb{R}^N.
\]

Due to the above characterization, we use \( m \) to denote the \textit{neighboring radius} of the local objective functions \( f_i, i \in V \). In the classical least squares problem \( \min_{x \in \mathbb{R}^N} \|Ax - b\|^2 \) associated with the measurement matrix \( A = (a(i,j))_{i,j \in V} \) having geodesic-width \( m \) and the noisy observation data \( b = (b(i))_{i \in V} \in \mathbb{R}^N \), one may verify that the local objective functions
\[
f_i(x) = \left(\sum_{j \in B(i,m)} a(i,j)x(j) - b(i)\right)^2, \quad i \in V,
\]

satisfy Assumption 2.2 and have neighboring radius \( m \).

For two square matrices \( A \) and \( B \) on the graph \( G \), we say that \( A \preceq B \) if \( B - A \) is positive semi-definite. In this paper, the global objective function \( F = \sum_{i \in V} f_i \) in the optimization problem (1.1) are assumed to be smooth and strongly convex.

Assumption 2.3. There exist positive constants \( 0 < c < L < \infty \) and positive definite matrices \( \Phi(x,y), x, y \in \mathbb{R}^N \), with geodesic-width \( 2m \) such that
\[
\omega(\Phi(x,y)) \leq 2m \quad \text{and} \quad cI \preceq \Phi(x,y) \preceq LI,
\]
and
\[
\nabla F(x) - \nabla F(y) = \Phi(x,y)(x - y) \quad \text{for all} \quad x, y \in \mathbb{R}^N.
\]

The requirement (2.5) can be considered as a strong version of its \textit{strictly monotonicity} for the gradient \( \nabla F \),
\[
(x - y)^T(\nabla F(x) - \nabla F(y)) \geq c\|x - y\|^2 \quad \text{for all} \quad x, y \in \mathbb{R}^N,
\]
where \( c \) is an absolute constant [40, 49]. On the other hand, Assumption 2.3 is satisfied when the local objective functions \( f_i, i \in V \), are twice differentiable and satisfy Assumption 2.2, and the global objective function \( F \) satisfies the classical strict convexity condition
\[
cI \preceq \nabla^2 F(x) \preceq LI \quad \text{for all} \quad x \in \mathbb{R}^N.
\]

In particular, we have
\[
\nabla F(x) - \nabla F(y) = \left(\int_0^1 \nabla^2 F(tx + (1-t)y)dt\right)(x - y) \quad \text{for all} \quad x, y \in \mathbb{R}^N,
\]
and
\[
\omega(\nabla^2 F(x)) \leq 2m \quad \text{for all} \quad x \in \mathbb{R}^N,
\]
where
\[ \frac{\partial^2 F(x)}{\partial x(i) \partial x(j)} = \sum_{k \in V} \frac{\partial^2 f_k(x)}{\partial x(i) \partial x(j)} = 0, \quad x \in \mathbb{R}^N, \]
hold for any \( i, j \in V \) with \( \rho(i, j) > 2m \), since for any \( k \in V \), either \( \frac{\partial f_k(x)}{\partial x(i)} = 0 \) or \( \frac{\partial f_k(x)}{\partial x(j)} = 0 \) by (2.3).

2.3. Fusion centers for distributed and decentralized implementation. In our distributed and decentralized implementation of the proposed DAC algorithm, all processing of data storage, data exchange and numerical computation are conducted on fusion centers located at some vertices of the graph \( G \). Denote the location set of fusion centers by \( \Lambda \), a subset of the vertex set \( V \) of the underlying graph \( G \). Associated with each fusion center \( \lambda \in \Lambda \), we divide the whole set of vertices \( V \) into a family of governing vertex sets \( D_\lambda \subset V, \lambda \in \Lambda \) such that
\[ \cup_{\lambda \in \Lambda} D_\lambda = V \quad \text{and} \quad D_\lambda \cap D_{\lambda'} = \emptyset \quad \text{for distinct} \quad \lambda, \lambda' \in \Lambda. \tag{2.6}\]
A common selection of the governing vertex sets is the Voronoi diagram of \( G \) with respect to \( \Lambda \), which satisfies
\[ \{ i \in V, \rho(i, \lambda) < \rho(i, \lambda') \} \subset D_\lambda \subset \{ i \in V, \rho(i, \lambda) \leq \rho(i, \lambda') \} \quad \text{for all} \quad \lambda' \neq \lambda, \quad \lambda \in \Lambda. \]

In our setting, we do not have any restriction on the sizes of the governing vertices \( D_\lambda, \lambda \in \Lambda \), however it is more reasonable to assume that \( D_\lambda, \lambda \in \Lambda \), have similar sizes and are located in some “neighborhood” of fusion centers, see our illustrative Example 2.4 below.

Denote the distance between two vertex subsets \( A, B \) of \( V \) by \( \rho(A, B) = \inf_{i \in A, j \in B} \rho(i, j) \). In the proposed DAC algorithm, we solve some local minimization problems on extended R-neighbors \( D_{\lambda, R} \) of the governing vertex sets \( D_\lambda, \lambda \in \Lambda \), which satisfy
\[ D_\lambda \subset D_{\lambda, R} \quad \text{and} \quad \rho(D_\lambda, V \setminus D_{\lambda, R}) > R \quad \text{for all} \quad \lambda \in \Lambda, \tag{2.7}\]
where \( R \geq 1 \) is a positive number chosen later, see (4.1). A simple choice of extended R-neighbors are the sets
\[ D_{\lambda, R} = \cup_{i \in D_\lambda} B(i, R) = \{ j \in V, \rho(j, i) \leq R \ \text{for some} \ i \in D_\lambda \}, \]
of all R-neighboring vertices of \( D_\lambda, \lambda \in \Lambda \).

Next we present an example of the location set \( \Lambda \) of fusion centers, the governing vertex sets \( D_\lambda, \lambda \in \Lambda \), and their extended R-neighbors \( D_{\lambda, R}, \lambda \in \Lambda \).

**Example 2.4.** We say that \( \Lambda \subset V \) is a maximal \( R_0 \)-disjoint set if
\[ B(i, R_0) \cap \left( \cup_{\lambda \in \Lambda} B(\lambda, R_0) \right) \neq \emptyset \quad \text{for all} \quad i \in V, \]
and
\[ B(\lambda, R_0) \cap B(\lambda', R_0) = \emptyset \quad \text{for all distinct} \quad \lambda, \lambda' \in \Lambda. \]

Our illustrative example of location/vertex set \( \Lambda \) of fusion centers and the governing vertex sets \( D_\lambda, \lambda \in \Lambda \) is a maximal \( R_0 \)-disjoint set and the corresponding Voronoi diagram, see Figure 1 where \( R_0 = 4 \) and the underlying graph \( G \) is a geometric random graph with \( N = 2048 \) vertices.
For the above governing vertex sets $D_{\lambda}, \lambda \in \Lambda$, we have the following size estimate, 

$$B(\lambda, R_0) \subseteq D_{\lambda} \subseteq B(\lambda, 2R_0) \text{ for all } \lambda \in \Lambda.$$ 

For the case that the sets of all $R$-neighboring vertices of $D_{\lambda}$ are chosen as extended $R$-neighbors $D_{\lambda,R} = \cup_{i \in D_{\lambda}} B(i, R), \lambda \in \Lambda$, we obtain 

$$B(\lambda, R_0 + R) \subseteq D_{\lambda,R} \subseteq B(\lambda, 2R_0 + R) \text{ for all } \lambda \in \Lambda.$$ 

We finish this section with a constructive approach, Algorithm 1, to construct a maximal $R_0$-disjoint set on a connected undirected graph $G = (V, E)$ of order $N \geq 1$, and show that the total computational cost to implement Algorithm 1 is about $O(N^2)$. Here we say that two positive quantities $a$ and $b$ satisfy $b = O(a)$ if $b/a$ is bounded by an absolute constant $C$ independent on the order $N$ of the underlying graph $G$, which could be different at different occurrences and may depend on the radius $R_0$, Beurling dimension $d(G)$ and Beurling density $D_1(G)$. Denote total number of steps used in the Algorithm 1 by $M$ and the sets $U$ and $W$ at step $n$ by $U_n$ and $W_n, 1 \leq n \leq M$ respectively. Then one may verify by induction on $n \geq 1$ that 

$$U_n \subset V \setminus W_n \subset \cup_{i \in U_n} B(i, R_0), \ 1 \leq n \leq M,$$ 

the sequence of cardinalities $\#W_n \in [0, N - 1]$ of the sets $W_n, 1 \leq n \leq M$, is strictly decreasing, 

$$\#W_{n+1} \leq \#W_n - 1, \ 1 \leq n \leq M - 1,$$ 

and the sequence of cardinalities $\#U_n \in [1, N]$ of the sets $U_n, 1 \leq n \leq M$ is increasing and has bounded increment, 

$$\#U_n \leq \#U_{n+1} \leq \#U_n + D_1(G)(R_0 + 1)^{d(G)}, \ 1 \leq n \leq M - 1,$$ 

where the last inequality follows from Assumption 2.1. By (2.8) and Algorithm 1, the computational cost to find $j \in W_n$ and verify whether $B(j, R_0) \cap U_n = \emptyset$ for any given $j \in W_n$ is about $O(\#U_n), 1 \leq n \leq M$. Therefore the total computational cost to implement Algorithm 1 to find a maximal $R_0$-disjoint set is 

$$\sum_{n=1}^{M} O(\#U_n) = \sum_{n=1}^{M} O(n) = O(N^2),$$ 

where the first equality follows from (2.10) and the second estimate hold as $M \leq N$ by (2.9).

### 3. Divide-and-Conquer Algorithm and its Distributed Implementation

Let $G = (V, E)$ be a connected undirected graph, $D_{\lambda}, \lambda \in \Lambda$, be a family of governing vertex sets satisfying (2.6), and $D_{\lambda,R}, \lambda \in \Lambda$ be their extended $R$-neighbors satisfying (2.7), see Section 2.3. In this section, we introduce a novel DAC algorithm (3.1) to solve the optimization problem (1.1), where we break down the global optimization problem (1.1) into local minimization problems (3.1a) on overlapping extended $R$-neighbors $D_{\lambda,R}, \lambda \in \Lambda$, and then we combine the core part of solutions of the above local minimization problems to provide a better approximation to the solution of the original minimization problem in each iteration, see (3.1b). Due to neighboring variable dependence of local objective functions $f_i, i \in V$, we propose a scalable implementation of the DAC algorithm (3.1) at fusion centers equipped with enough processing power and resources, see Algorithm 2 for the implementation and Assumption 3.1 for the equipment requirement at fusion centers.
Algorithm 1 Maximal $R_0$-disjoint subset algorithm

**Initialization:** Pick one vertex $i \in V = \{1, 2, ..., N\}$, and then set $\Lambda = \{i\}$, $U = B(i, R_0)$ and $W = V \setminus B(i, R_0)$.

for $n = 2, ..., N$

Pick minimal $j \in W$

if $B(j, R_0) \cap U = \emptyset$

$\Lambda \leftarrow \Lambda \cup \{j\}; U \leftarrow U \cup B(j, R_0); W \leftarrow W \setminus B(j, R_0)$

else

$W \leftarrow W \setminus \{j\}$

end

if $\#W = 0$

stop

end

end

Output: $\Lambda$

For a vector $x = (x(i))_{i \in V} \in \mathbb{R}^{\#V}$ and a subset $W \subset V$, we use $\chi_W : \mathbb{R}^{\#V} \to \mathbb{R}^{\#W}$ to denote the selection mapping $\chi_W x = (x(i))_{i \in W}$. Its adjoint mapping $\chi_W^* : \mathbb{R}^{\#W} \to \mathbb{R}^{\#V}$ is defined for a vector $u = (u(i))_{i \in W}$ as $\chi_W^* u \in \mathbb{R}^{\#V}$ with the $i$-th component same as $u(i)$ when $i \in W$ and 0 otherwise. Moreover, we let $I_W = \chi_W^* \chi_W$ be the projection operator making the $i$-th block 0 for $i \notin W$.

In this paper, we propose the following iterative divide-and-conquer algorithm, DAC for abbreviation, to solve the optimization problem (1.1),

\[
\begin{align*}
\boldsymbol{w}_\lambda^n &= \arg \min_{u \in \mathbb{R}^{\#D_{\lambda,R}}} F(\chi^*_{D_{\lambda,R}} u + I_{V \setminus D_{\lambda,R}} x^n), \quad \lambda \in \Lambda, \quad (3.1a) \\
x^{n+1} &= \sum_{\lambda \in \Lambda} I_{D_{\lambda,R}} \chi^*_{D_{\lambda,R}} \boldsymbol{w}_\lambda^n, \quad n \geq 0, \quad (3.1b)
\end{align*}
\]

where an initial $x^0$ is arbitrarily or randomly chosen. The iteration step in the DAC algorithm solves a family of the local minimization problems (3.1a) on the overlapping extended $R$-neighbors $D_{\lambda,R}, \lambda \in \Lambda$, and combines the core part $D_{\lambda}$ of solutions $\boldsymbol{w}_\lambda^n, \lambda \in \Lambda$ of those local minimization problems to provide a better approximation to the solution of the original minimization problem when the radius parameter $R \geq 0$ is appropriately chosen, see Theorem 4.1.

For a subset $W \subset V$ and $i \in V$, we use $\rho(i, W) = \inf_{j \in W} \rho(i, j)$ to denote the distance between the vertex $i$ and the set $W$. Let $m \geq 1$ be the neighboring radius of the local objective functions $f_i, i \in V$, and define $l$-neighbors $D_{\lambda,R,l}$ of $D_{\lambda,R}, \lambda \in \Lambda$ by

$$D_{\lambda,R,l} = \{i \in V, \rho(i, D_{\lambda,R}) \leq l\}, \quad \lambda \in \Lambda.$$
Algorithm 2 Implementation of the DAC algorithm (3.1) at a fusion center $\lambda \in \Lambda$.

**Initialization:** Maximal iteration number $T$, vertex sets $D_{\lambda}, D_{\lambda,R}$ and $D_{\lambda,R,2m}$, local objective functions $f_i, i \in D_{\lambda,R,m}$, neighboring fusion sets $\Lambda_{\lambda,R,2m}^{\text{out}}$ and $\Lambda_{\lambda,R,2m}^{\text{in}}$ in (3.3), and initial guess $\chi x^0_{D_{\lambda,R,2m}}$, i.e., its components $x_0(i), i \in D_{\lambda,R,2m}$.

**Iteration:**

for $n = 0, 1, \ldots, T$

Solve the local minimization problem

$$w^n_\lambda = \arg\min_{u \in \mathbb{R}^{#D_{\lambda,R}}} \sum_{i \in D_{\lambda,R,m}} f_i(\chi x^*_i u + I_{D_{\lambda,R,2m}\setminus D_{\lambda,R}} x^n)$$

Send $\chi x^*_i u + I_{D_{\lambda,R,2m}\setminus D_{\lambda,R}} x^n$ to neighboring fusion centers $\lambda' \in \Lambda_{\lambda,R,2m}^{\text{out}}$;

Receive $\chi x^*_i u + I_{D_{\lambda,R,2m}\setminus D_{\lambda,R}} x^n$ from neighboring fusion centers $\lambda' \in \Lambda_{\lambda,R,2m}^{\text{in}}$;

Evaluate $\chi x^*_i u + I_{D_{\lambda,R,2m}\setminus D_{\lambda,R}} x^n_{n+1} = \chi x^*_i u + I_{D_{\lambda,R,2m}\setminus D_{\lambda,R}} x^n$.

end

Output: $\chi x_{T+1} := \chi x^*_i u + I_{D_{\lambda,R,2m}\setminus D_{\lambda,R}} x^n$

For any $y \in \mathbb{R}^N$ and $\lambda \in \Lambda$, we obtain from Assumption 2.2 that

$$\arg\min_{u \in \mathbb{R}^{#D_{\lambda,R}}} F(\chi x^*_i u + I_{D_{\lambda,R}} y)$$

$$= \arg\min_{u \in \mathbb{R}^{#D_{\lambda,R}}} \left( \sum_{i \in D_{\lambda,R,m}} f_i(\chi x^*_i u + I_{D_{\lambda,R}} y) \right)$$

$$= \arg\min_{u \in \mathbb{R}^{#D_{\lambda,R}}} \sum_{i \in D_{\lambda,R,m}} f_i(\chi x^*_i u + I_{D_{\lambda,R}} y)$$

$$= \arg\min_{u \in \mathbb{R}^{#D_{\lambda,R}}} \sum_{i \in D_{\lambda,R,m}} f_i(\chi x^*_i u + I_{D_{\lambda,R,2m}\setminus D_{\lambda,R}} y)$$

(3.2)

Based on the above observation, the iterative DAC algorithm (3.1) can be implemented at fusion centers, see Algorithm 2 for the implementation at each fusion center $\lambda \in \Lambda$ where

$$\Lambda_{\lambda,R,2m}^{\text{out}} = \{ \lambda' \in \Lambda, D_{\lambda} \cap D_{\lambda',R,2m} \neq \emptyset \}$$

and

$$\Lambda_{\lambda,R,2m}^{\text{in}} = \{ \lambda' \in \Lambda, D_{\lambda'} \cap D_{\lambda,R,2m} \neq \emptyset \}, \lambda \in \Lambda. \quad (3.3)$$

We consider any fusion center $\lambda' \in \Lambda_{\lambda,R,2m}^{\text{out}}$ and $\Lambda_{\lambda,R,2m}^{\text{in}}$ as an *out-neighbor* and *in-neighbor* of the fusion center $\lambda \in \Lambda$ respectively. For $\lambda, \lambda' \in \Lambda$, one may verify that $\lambda' \in \Lambda_{\lambda,R,2m}^{\text{out}}$ if and only if $\lambda \in \Lambda_{\lambda',R,2m}^{\text{in}}$ and hence in Algorithm (3.1) the data vector transmitted from a fusion center to its out-neighbors will be received.

For the implementation of Algorithm 2 at fusion centers, it is required that each fusion center is equipped with enough memory for data storage, proper communication bandwidth to exchange data with its neighboring fusion centers, and high computing power to solve local minimization problems.
Assumption 3.1. (i) Each fusion center $\lambda \in \Lambda$ can store vertex sets $D_{\lambda}, D_{\lambda,R}, D_{\lambda,R,m}$ and $D_{\lambda,R,2m}$, neighboring fusion sets $\Lambda_{\lambda,R,2m}^\text{out}$ and $\Lambda_{\lambda,R,2m}^\text{in}$, and the vectors $\chi_{D_{\lambda,R,2m}}^n$ and $\mathbf{w}_\lambda^n$ at each iteration, and reserve enough memory used for storing local objective functions $f_i, i \in D_{\lambda,R,m}$ and solving the local minimization problem (3.2) at each iteration;

(ii) each fusion center $\lambda \in \Lambda$ has computing facility to solve the local minimization problem (3.2) quickly; and

(iii) each fusion center $\lambda \in \Lambda$ can send data $\chi_{D_{\lambda}}x_{D_{\lambda}}^\lambda \mathbf{w}_\lambda^n$ to fusion centers $\lambda' \in \Lambda_{\lambda,R,2m}^\text{out}$ and receive data $\chi_{D_{\lambda'}}x_{D_{\lambda'}}^\lambda \mathbf{w}_{\lambda'}^n$ from fusion centers $\lambda' \in \Lambda_{\lambda,R,2m}^\text{in}$ at each iteration.

We finish this section with a remark on the above assumption when location set $\Lambda$ of fusion centers, the governing vertex sets $D_{\lambda}, \lambda \in \Lambda$ and the extended $R$-neighbors $D_{\lambda,R}, \lambda \in \Lambda$ are the maximal $R_0$-disjoint set, its corresponding Voronoi diagram and the set of all $R$-neighboring vertices of the Voronoi diagram respectively, see Example 2.4. In this case, we have

$$D_{\lambda} \subset B(\lambda, 2R_0), \quad D_{\lambda,R} \subset B(\lambda, 2R_0 + R), \quad D_{\lambda,R,l} \subset B(\lambda, 2R_0 + R + l) \text{ for } l = m, 2m,$$

and

$$\Lambda_{\lambda,R,2m}^\text{out} \subset \{\lambda' \in \Lambda, B(\lambda, 2R_0) \cap B(\lambda', 2R_0 + R + 2m) \neq \emptyset\} \subset B(\lambda, 4R_0 + R + 2m) \cap \Lambda, \lambda \in \Lambda,$$

$$\Lambda_{\lambda,R,2m}^\text{in} \subset \{\lambda' \in \Lambda, B(\lambda', 2R_0) \cap B(\lambda, 2R_0 + R + 2m) \neq \emptyset\} \subset B(\lambda, 4R_0 + R + 2m) \cap \Lambda, \lambda \in \Lambda,$$

see Figure 1. This together with Assumption 2.1 implies that the local minimization problem of the form (3.2) is of size at most $D_1(G)(2R_0 + R + 1)^d(G)$, each fusion center has at most $D_1(G)(4R_0 + R + 2m + 1)^d(G)$ neighboring fusion centers (usually it is much smaller), and in addition to memory requirement to solve local minimization problem (3.2) of dimension at most $D_1(G)(4R_0 + R + 1)^d(G)$, each fusion center has memory of size $D_1(G)(4R_0 + R + 2m + 1)^d(G)$ to store some vertex sets and vectors. From the convergence result in Theorem 4.1, the parameter $R$ in the DAC algorithm can be chosen to depend only on the constants in Assumption 2.1, Assumption 2.2 and Assumption 3.1, see (4.1). Therefore to meet the requirements in Assumption 3.1, in the above setting we could equip data storage, communication devices and computing facility at each fusion center independently on the order $N$ of the underlying graph $G$, and hence the distributed implementation of Algorithm 2 at fusion centers is scalable.

4. Exponential convergence of the divide-and-conquer algorithm

Let $\ell^p, 1 \leq p \leq \infty$, be the space of all $p$-summable sequences on the graph $G$ with its standard norm denoted by $\| \cdot \|_p$. In Theorem 4.1 of this section, we show that $\mathbf{x}^n, n \geq 1$, in the proposed iterative DAC algorithm (3.1) converges exponentially to the solution $\bar{x}$ of the optimization problem (1.1) in $\ell^p, 1 \leq p \leq \infty$, when the parameter $R \geq 1$ in the extended neighbors of governing vertex sets is chosen so that

$$\delta_R := \frac{D_1(G)d(G)|L|^2(2m)^d(G)}{c(L-c)|\ln(1-c/L)|^{d(G)}}(R+2)^{d(G)}\left(1 - \frac{c}{L}\right)^{R/(2m)} < 1. \quad (4.1)$$

In many practical applications, it could be numerically expensive to solve the local minimization problem (3.2) exactly. In Theorem 4.2, we provide an error estimate when the local minimization problems in the DAC algorithm are solved up to certain bounded accuracy.
Theorem 4.1. Let the underlying graph $G = (V, E)$, the local objective functions $f_i, i \in V$ and the global objective function $F$ of the optimization problem (1.1) satisfy Assumptions 2.1, 2.2 and 2.3 respectively, $\hat{x} = \arg\min_{x \in \mathbb{R}^N} F(x)$ be the unique solution of the global minimization problem (1.1), and \{x^n, n \geq 0\} be the sequence generated in the iterative DAC algorithm (3.1). If the parameter $R \geq 1$ in (3.1) is chosen to satisfy \(4.1\), then \{x^n, n \geq 0\} converges to $\hat{x} \in \mathbb{R}^N$ exponentially in $\ell^p, 1 \leq p \leq \infty$, with convergence rate $\delta_R$, 

$$\|x^n - \hat{x}\|_p \leq (\delta_R)^n \|x^0 - \hat{x}\|_p, \quad n \geq 0. \tag{4.2}$$

For a matrix $A = (a(i,j))_{i,j \in V}$ on the graph $G = (V, E)$, define its entrywise bound by $\|A\|_\infty = \sup_{i,j \in V} |a(i,j)|$, its operator norm on $\ell^p, 1 \leq p \leq \infty$, by 

$$\|A\|_{\ell^p} = \sup_{\|x\|_p = 1} \|Ax\|_p,$$

and its Schur norm by 

$$\|A\|_S = \max \left( \sup_{i \in V} \sum_{j \in V} |a(i,j)|, \sup_{j \in V} \sum_{i \in V} |a(i,j)| \right) = \max_{1 \leq p \leq \infty} \|A\|_{\ell^p}.$$ 

As shown in [19, Prop. III.3], a bounded matrix $A$ with limited geodesic-width $\omega(A)$ is a bounded operator on $\ell^p, 1 \leq p \leq \infty$. 

$$\|A\|_\infty \leq \|A\|_{\ell^p} \leq \|A\|_S \leq D_1(G)(\omega(A) + 1)^{d(G)} \|A\|_\infty. \tag{4.3}$$

The crucial step in the proof of Theorem 4.1 is to find matrices $H_n, n \geq 0$, on the graph $G$ such that 

$$x^{n+1} - \hat{x} = H_n(x^n - \hat{x}) \quad \text{and} \quad \|H_n\|_S \leq \delta_R, \quad n \geq 0, \tag{4.4}$$

see (6.11) and (6.15). The detailed argument of Theorem 4.1 will be given in Section 6.1.

It is direct to observe from Assumption 2.3 that the local optimizer $w^n_\lambda$ in (3.1a) satisfies 

$$\chi_{D_{\lambda,R}} \nabla F(\chi_{D_{\lambda,R}}^* w^n_\lambda + I_{V \setminus D_{\lambda,R}} x^n) = 0, \quad \lambda \in \Lambda. \tag{4.5}$$

This motivates us to consider the following inexact DAC algorithm starting from an initial $\tilde{x}^0$ arbitrarily or randomly chosen, and solving a family of the local minimization problems with bounded accuracy $\epsilon_n, n \geq 0$, 

$$\tilde{w}^n_\lambda \quad \text{is selected such that} \quad \|\chi_{D_{\lambda,R}} \nabla F(\chi_{D_{\lambda,R}}^* \tilde{w}^n_\lambda + I_{V \setminus D_{\lambda,R}} x^n)\|_\infty \leq \epsilon_n, \quad \lambda \in \Lambda, \tag{4.6a}$$

and combining the core part of the above inexact solutions 

$$\tilde{x}^{n+1} = \sum_{\lambda \in \Lambda} I_{D_{\lambda}} \chi_{D_{\lambda,R}} \tilde{w}^n_\lambda, \quad n \geq 0, \tag{4.6b}$$

to provide a novel approximation. Shown in the following theorem is the error estimate $\|\tilde{x}^n - \hat{x}\|_\infty, n \geq 0$. 


Theorem 4.2. Let the underlying graph $G = (V,E)$, the global objective function $F$, the optimal point $\hat{x}$, the parameters $R$, and the convergence rate $\delta_R$ be as in Theorem 4.1, and $\tilde{x}^n, n \geq 1$, be the solutions of the inexact DAC algorithm (4.6) with bounded accuracy $\epsilon_n, n \geq 0$. Then
\[
\|\tilde{x}^n - \hat{x}\|_{\infty} \leq (\delta_R)^n \|x^0 - \hat{x}\|_p + \frac{L - c}{L^2} \sum_{m=0}^{n-1} (\delta_R)^{n-m} \epsilon_m, n \geq 1. \tag{4.7}
\]

The crucial step in the proof of Theorem 4.2 is to find matrices $\tilde{H}_n, n \geq 0$, such that
\[
\|\tilde{H}_n\|_S \leq \delta_R \text{ and } \|x^{n+1} - \hat{x} - \tilde{H}_n(x^n - \hat{x})\|_{\infty} \leq \frac{L - c}{L^2} \delta_R \epsilon_n, n \geq 0,
\]
similar to (4.4) where $\epsilon_n = 0$ for all $n \geq 0$. The detailed argument of Theorem 4.2 is given in Section 6.2.

By Theorem 4.2, we conclude that the solutions of the inexact DAC algorithm (4.6) converges to the true optimal point when the accuracy bounds $\epsilon_n, n \geq 0$, have zero limit.

Corollary 4.3. Let the underlying graph $G = (V,E)$, the global objective function $F$, the optimal point $\hat{x}$, the parameters $R$, and the convergence rate $\delta_R$, and the inexact solutions $\tilde{x}^n$ and accuracy bounds $\epsilon_n, n \geq 0$, be as in Theorem 4.2. If $\lim_{n \to \infty} \epsilon_n = 0$, then $\lim_{n \to \infty} \tilde{x}^n = \hat{x}$.

5. Numerical Examples

In this section, we demonstrate the performance of the iterative DAC for the least squares minimization problem and the LASSO model, and we make a comparison with the performance of some popular decentralized optimization methods, including decentralized gradient descent (DGD) [24, 27, 48], Diffusion [7, 8], exact first-order algorithm (EXTRA) [34], proximal gradient exact first-order algorithm (PG-EXTRA) [35], and network-independent step-size algorithm (NIDS) [22]. The numerical experiments show that, comparing with DGD, Diffusion, EXTRA, PD-EXTRA and NIDS, the proposed DAC method has superior performance for the least squares problem with/without $\ell^1$-penalty and has much faster convergence. Moreover, the computational cost of the proposed DAC is almost linear with respect to the graph size, and hence it has a great potential in scalability to work with extremely large networks.

In all the experiments below, the underlying graph of our distributed optimization problems is the random geometric graph $G_N = (V_N, E_N)$ of order $N \geq 256$, where vertices are randomly deployed in the unit square $[0,1]^2$ and an edge between two vertices exist if their Euclidean distance is not larger than $\sqrt{2/N}$, see Figure 1. On the random geometric graph $G_N$, we denote its adjacent matrix, degree matrix and symmetric normalized Laplacian matrix by $A, D$ and $L^{sym} = I - D^{-1/2} AD^{-1/2}$ respectively.

All the numerical experiments are implemented using Python 3.8 on a computer server with Intel(R) Xeon(R) Gold 6148 CPU 2.4GHz and 32G memory.

5.1. Least Squares Problem. Consider the following least square problem:
\[
\tilde{x} = \arg \min_{x \in \mathbb{R}^N} \left\{ F(x) = \frac{1}{2} \|Hx - b\|_2^2 \right\}. \tag{5.1}
\]
where $H = I + 5I^{sym}$ and $b = (b(i))_{i \in V} \in \mathbb{R}^N$ is randomly generated from normal distribution with mean 0 and variance 1. Write $H = (H(i,j))_{i,j \in V_N}$ and define

$$f_i(x) = \frac{1}{2} \left( \sum_{j \in V_N} H(i,j)x(j) - b(i) \right)^2, \quad i \in V_N$$

for $x = (x(i))_{i \in V_N}$. As the symmetric normalized Laplacian matrix $L^{sym}$ has geodesic-width one and satisfies $0 \leq L^{sym} \leq 2I$, Assumptions 2.2 and 2.3 are satisfied with

$$m = 1, \quad c = 1, \quad L = 121 \quad \text{and} \quad \Phi(x, y) = H^2 \quad \text{for} \quad x, y \in \mathbb{R}^N.$$ 

To implement the DAC algorithm (3.1), we start from applying Algorithm 1 to find a maximal $R_0$-disjoint set with $R_0 = 1$, use the maximal $R_0$-disjoint set and its corresponding Voronoi diagram as the location set $\Lambda$ of fusion centers and the family of governing vertex sets $D_\Lambda, \lambda \in \Lambda$. Next we select $R = 3$ and use the set of all $R$-neighbors of $D_\Lambda$ as the extended $R$-neighbors $D_{\lambda,R}, \lambda \in \Lambda$. Having the fusion centers selected and extended $R$-neighbors ready, we then follow Algorithm 2 to implement the DAC algorithm (3.1) with zero initial $x^0 = 0$ and use $\|x^{n+1} - x^n\|_2 / \|x^n\|_2 \leq 10^{-14}$ as the stopping criterion.

We will compare the performance of our proposed algorithm with DGD, Diffusion, and EXTRA. In this regard, we will use the partition $\{D_\lambda : \lambda \in \Lambda\}$ as the nodes in DGD, i.e., each block $D_\lambda$ of vertices would be treated as a single node in DGD and two nodes $D_\lambda$ and $D_{\lambda'}$ are connected if the two fusion centers $\lambda$ and $\lambda'$ are within distance $2(R + R_0) = 8$, i.e., $\rho(\lambda, \lambda') \leq 8$. We define the objective function at each fusion center:

$$F_\lambda(x) = \sum_{i \in D_\lambda} f_i(x) = \frac{1}{2} \|H_{D_\lambda}x - b_{D_\lambda}\|^2, \quad \lambda \in \Lambda,$$

where $H_{D_\lambda} = (H(i,j))_{i,j \in V_N}$ and $b_{D_\lambda} = (b(i))_{i \in D_\lambda}$. Denote the stacked local copies $x_\lambda$ and the stacked local gradients $\nabla F_\lambda(x_\lambda)$ held at each fusion center $\lambda \in \Lambda$ by $X$ and $\nabla F(X)$ respectively, i.e.,

$$X = \begin{pmatrix} (x_1)^T \\ \vdots \\ (x_{\#\Lambda})^T \end{pmatrix} \quad \text{and} \quad \nabla F(X) = \begin{pmatrix} \nabla^T F_1(x_1) \\ \vdots \\ \nabla^T F_{\#\Lambda}(x_{\#\Lambda}) \end{pmatrix}.$$

In the DGD, Diffusion and EXTRA, the following iteration schemes

- **DGD** $\quad X^{n+1} = WX^n - \alpha \nabla F(X^n),$
- **Diffusion** $\quad X^{n+1} = WX^n - \beta \nabla F(X^n),$
- **EXTRA** $\quad X^{n+1} = (I + W)X^n - \frac{1}{2} X^{n-1} - \gamma (\nabla F(X^n) - \nabla F(X^{n-1})),$

are used, where the step sizes are set to be $\alpha = 0.99(\max_{\lambda \in \Lambda} \|H_{D_\lambda}\|^2)^{-1}$ and $\beta = \gamma = 2\alpha$. There are several different choices of the mixing matrix $W$ [3, 33, 42, 43, 48]. In our simulations, we use
the Metropolis mixing matrix \[3, 43\] with entries \(w_{\lambda,\lambda'}, \lambda, \lambda' \in \Lambda\) defined by
\[
w_{\lambda,\lambda'} = \begin{cases} 
(\max\{\#N_\lambda, \#N_{\lambda'}\} + 0.1)^{-1} & \text{if } \lambda, \lambda' \text{ are connected} \\
1 - \sum_{\lambda \in N_\lambda} W_{\lambda,\lambda} & \text{if } \lambda' = \lambda \\
0 & \text{otherwise}
\end{cases}
\] (5.2)
as the mixing matrix \(W\), where \(N_\lambda\) is the set of all nodes connected to the vertex \(\lambda \in \Lambda\).

We test our proposed DAC method, DGD, Diffusion, and EXTRA on four geometric random graphs with size 256, 512, 1024, and 2048 respectively and plot in Figure 2 the average logarithm error \(\log_{10}\|x^n - \hat{x}\|\) with respect to different running time (seconds) over 100 random selections of the observation vector \(b\).

We observe from Figure 2 that the proposed DAC method has superior performance to all the other three, it takes much less computational time to reach the machine accuracy and the corresponding computational time is close to a linear dependence on the order of the graph. This strongly suggests that the proposed DAC method has a strong scalability and the great potential to apply distributed algorithms on networks of extremely large size.

5.2. LASSO. In this subsection, we consider solving the following least squares problem with \(\ell^1\) penalty,
\[
\hat{x} = \arg \min_{x \in \mathbb{R}^N} \left\{ F(x) = \frac{1}{2}\|Hx - b\|_2^2 + \mu\|x\|_1 \right\}
\] (5.3)
on the random geometric graph \(G_N\), where \(H, b\) are the same as those in eq. (5.1) and \(\mu = 10\). In the implementation of the proposed DAC algorithm, we use the same \(\Lambda, D\Lambda, D_{\lambda,R}\) and the stopping criterion as the ones in Section 5.1. Our numerical results show that the proposed DAC algorithm is applicable to the above LASSO model and it has a superior performance comparing to some popular decentralized methods, including NIDS and PG-EXTRA. Here we remark that the local objective functions
\[
f_i(x) = \frac{1}{2} \left[ \sum_{j \in B(i,m)} H(i,j)x(j) - b(i) \right]^2 + \mu|x(i)|, \quad i \in V,
\]
in the above LASSO model are not differentiable and hence Theorem 4.1 can not be applied to guarantee the exponential convergence of the proposed DAC algorithm in the above LASSO model.

For both NIDS and PG-EXTRA, the settings of fusion centers are the same as DGD in the least squares problem, and the local objective function at each fusion center is:
\[
F_\lambda(x) = g_\lambda(x) + h_\lambda(x), \quad \lambda \in \Lambda,
\]
where \(g_\lambda(x) = \frac{1}{2}\|H_{D\lambda}x - b_{D\lambda}\|\) and \(h_\lambda(x) = \frac{\mu}{\#\lambda}\|x\|_1\). The iteration scheme for NIDS is
\[
X^{n+1} = \text{Prox}_{\alpha h}(Z^n)
\]
\[
Z^{n+1} = Z^n - X^{n+1} + \left( I - \frac{\beta}{2}(I - W) \right) \left( 2X^{n+1} - X^n + \alpha(\nabla g(X^n) - \nabla g(X^{n+1})) \right),
\]
Figure 2. The logarithm error over running time (seconds) for least square problems (5.1) on random geometric graphs of size $N$ and the iteration scheme for PG-EXTRA is

$$X^{n+1} = \text{Prox}_{\alpha h}(Z^n)$$
$$Z^{n+1} = Z^n - X^{n+1} + \frac{1}{2} W (2X^{n+1} - X^n) + \gamma (\nabla g(X^n) - \nabla g(X^{n+1})),$$

where $\text{Prox}_{\alpha h}(Z^n) = ((\text{Prox}_{\alpha h}(z^n_\lambda))^T)_{\lambda \in \Lambda}$. $\nabla g(X^n) = (\nabla^T g(\lambda)(X^n))_{\lambda \in \Lambda}$ and $W$ is the same Metropolis mixing matrix as in eq. (5.2). The step sizes for NIDS are $\alpha = 1.99(\max_{\lambda \in \Lambda} ||H_{D\lambda}||^2)^{-1}$ and $\beta = \frac{\alpha}{4}$, while the step size for PG-EXTRA is $\gamma = \alpha/2$. Shown in Figure 3 are the numerical results to solve the LASSO model (5.3) via the DAC, NIDS and PG-EXTRA algorithms. It is observed that our proposed DAC method converges faster than both PG-EXTRA and NIDS do, and the computational cost of DAC is also close to be linear with respect to the size of networks, which makes it more scalable than the other two do.
6. Proofs

In this section, we collect the proofs of Theorems 4.1 and 4.2.

6.1. Proof of Theorem 4.1. To prove Theorem 4.1, we need two technical lemmas. First we follow the argument used in the proof of Theorem IV.4 in [19] to show that the inverse of a positive definite matrix with limited geodesic-width has exponential off-diagonal decay. The well-localization for the inverse of matrices is of great importance in applied harmonic analysis, numerical analysis, distributed optimization and many mathematical and engineering fields, see [13, 16, 21, 37, 38] for historical remarks and recent advances.

Lemma 6.1. Let $G = (V, E)$ be a connected undirected graph, and matrices $A, B$ on the graph $G$ satisfy

$$\omega(A) \leq \omega \quad \text{and} \quad cI \preceq A \preceq L I$$  \hspace{0.5cm} (6.1)
and

$$\omega(B) \leq \omega \quad \text{and} \quad \|B\|_{B_2} \leq M,$$

where $\omega$ is a positive integer and $c, L, M$ are positive constants with $0 < c < L$. Then $A^{-1} = (G_1(i, j))_{i,j \in V}$ and $A^{-1}B = (G_2(i, j))_{i,j \in V}$ have exponential off-diagonal decay,

$$|G_1(i, j)| \leq \frac{1}{c} \left(1 - \frac{c}{L}\right)^{\rho(i,j)/\omega} \quad \text{and} \quad |G_2(i, j)| \leq \frac{LM}{c(L-c)} \left(1 - \frac{c}{L}\right)^{\rho(i,j)/\omega}, \quad i, j \in V.$$

**Proof.** Set $D = I - L^{-1}A$. Then it follows from (6.1) that

$$\omega(D) \leq \omega \quad \text{and} \quad \|D\|_{B_2} \leq 1 - c/L.$$  \hfill (6.2)

Hence

$$A^{-1} = L^{-1} \sum_{n=0}^{\infty} D^n, \quad \hfill (6.3)$$

and

$$\omega(D^n) \leq n\omega \quad \text{and} \quad \omega(D^nB) \leq (n+1)\omega, \quad n \geq 0. \quad \hfill (6.4)$$

Take $i, j \in V$, let $n_0(i, j)$ be the smallest nonnegative integer such that $\rho(i, j)/\omega \leq n_0(i, j)$, and write $D^n = (D_n(i, j))_{i,j \in V}, n \geq 0$. Then we obtain from (4.3), (6.2), (6.3) and (6.4) that

$$|G_1(i, j)| = L^{-1} \left| \sum_{n=0}^{\infty} D_n(i, j) \right| \leq L^{-1} \left| \sum_{n=n_0(i, j)}^{\infty} D_n(i, j) \right|$$

$$\leq L^{-1} \sum_{n=n_0(i, j)}^{\infty} \|D^n\|_\infty \leq L^{-1} \sum_{n=n_0(i, j)}^{\infty} \|D^n\|_{B_2} \leq L^{-1} \sum_{n=n_0(i, j)}^{\infty} \|D\|_{B_2}$$

$$\leq L^{-1} \sum_{n=n_0(i, j)}^{\infty} \left(1 - \frac{c}{L}\right)^n = \frac{1}{c} \left(1 - \frac{c}{L}\right)^{n_0(i, j)} \leq \frac{1}{c} \left(1 - \frac{c}{L}\right)^{\rho(i,j)/\omega},$$

and similarly

$$|G_2(i, j)| \leq L^{-1} \sum_{n=n_0(i, j)-1}^{\infty} \|D^nB\|_{B_2} \leq \frac{LM}{c(L-c)} \left(1 - \frac{c}{L}\right)^{\rho(i,j)/\omega}, \quad i, j \in V.$$

This completes the proof. \hfill \qed

To prove the Theorem 4.1, we also need the following lemma about the summation of an exponential decay sequence.

**Lemma 6.2.** Let $G = (V, E)$ be as in Theorem 4.1. Then for all $R \geq 1$ and $\beta > 0$,

$$\sum_{j \in V \text{ with } \rho(i,j) \geq R} e^{-\beta\rho(i,j)} \leq D_1(G)d(G)!\beta^{-d(G)}(R+1)^d(G)e^{-\beta(R-1)}, \quad i \in V. \quad \hfill (6.5)$$
Proof. We follow the arguments in [11, 19, 38]. Take \(i \in V\). For any \(0 < \sigma < 1\), we have
\[
\sum_{j \in V \text{ with } \rho(i,j) \geq R} e^{-\beta \rho(i,j)} \leq \sum_{k \in \mathbb{Z} \text{ with } k \sigma \geq R} e^{-(k-1)\beta \sigma} \mu G(B(i,k\sigma) \setminus B(i,(k-1)\sigma))
\]
\[
\leq \sum_{k \in \mathbb{Z} \text{ with } k \sigma \geq R} (e^{-(k-1)\beta \sigma} - e^{-k\beta \sigma}) \mu G(B(i,k\sigma))
\]
\[
\leq \beta \sigma D_1(G) \sum_{k \in \mathbb{Z} \text{ with } k \sigma \geq R} e^{-(k-1)\beta \sigma} (k \sigma + 1)^d(G).
\]
Taking limit \(\sigma \to 0\) in the above estimate yields
\[
\sum_{j \in V \text{ with } \rho(i,j) \geq R} e^{-\beta \rho(i,j)} \leq \beta D_1(G) \int_{R}^{\infty} (t + 1)^d(G) \exp(-\beta t) dt
\]
\[
\leq \beta D_1(G)(R + 1)^d(G) e^{-\beta(R-1)} \times \int_{1}^{\infty} s^d(G) e^{-\beta s} ds
\]
\[
\leq D_1(G)d(G)! \beta^{-d(G)} (R + 1)^d(G) e^{-\beta(R-1)},
\]
where we substitute \(t\) by \(s = R + 1\) in the second inequality and apply the inequality \(t + 1 = s + R \leq s(R + 1)\). This proves (6.5). \(\square\)

Proof of Theorem 4.1. Let \(w^n_\lambda, \lambda \in \Lambda\), be as in (3.1a). By (2.5) and (4.5), we have for \(\lambda \in \Lambda\)
\[
\chi_{D_{\lambda,R}} \nabla F(I_{D_{\lambda,R}} \hat{x} + I_{V \setminus D_{\lambda,R}} x^n)
\]
\[
= \chi_{D_{\lambda,R}} \nabla F(I_{D_{\lambda,R}} \hat{x} + I_{V \setminus D_{\lambda,R}} x^n) - \nabla F(\chi_{D_{\lambda,R}} w^n_\lambda + I_{V \setminus D_{\lambda,R}} x^n)
\]
\[
= \chi_{D_{\lambda,R}} \Phi(I_{D_{\lambda,R}} \hat{x} + I_{V \setminus D_{\lambda,R}} x^n, \chi_{D_{\lambda,R}} w^n_\lambda + I_{V \setminus D_{\lambda,R}} x^n) I_{D_{\lambda,R}} (x^n - \chi_{D_{\lambda,R}} w^n_\lambda).
\] (6.6)

For \(\lambda \in \Lambda\), define
\[
\Phi_{\lambda,R,n} = I_{D_{\lambda,R}} \Phi(I_{D_{\lambda,R}} \hat{x} + I_{V \setminus D_{\lambda,R}} x^n, \chi_{D_{\lambda,R}} w^n_\lambda + I_{V \setminus D_{\lambda,R}} x^n) I_{D_{\lambda,R}} + \frac{L + c}{2} I_{V \setminus D_{\lambda,R}}.
\]
Then it follows from (2.4) and (6.6) that
\[
\omega(\Phi_{\lambda,R,n}) \leq 2m, \quad c_1 \leq \Phi_{\lambda,R,n} \leq LI,
\] (6.7)
and
\[
I_{D_{\lambda,R}} (\chi_{D_{\lambda,R}} w^n_\lambda - \hat{x}) = -I_{D_{\lambda,R}} (\Phi_{\lambda,R,n})^{-1} I_{D_{\lambda,R}} \nabla F(I_{D_{\lambda,R}} \hat{x} + I_{V \setminus D_{\lambda,R}} x^n), \lambda \in \Lambda.
\] (6.8)

By (2.4) and (2.5), we conclude that the objective function \(F\) is strictly convex, and hence the solution \(\hat{x} = \arg \min F(\hat{x})\) satisfies
\[
\nabla F(\hat{x}) = 0.
\] (6.9)

Following the argument used in (6.6) and applying (2.5) and (6.9), we obtain
\[
\chi_{D_{\lambda,R}} \nabla F(I_{D_{\lambda,R}} \hat{x} + I_{V \setminus D_{\lambda,R}} x^n) = \chi_{D_{\lambda,R}} \Phi(I_{D_{\lambda,R}} \hat{x} + I_{V \setminus D_{\lambda,R}} x^n, \hat{x}) I_{V \setminus D_{\lambda,R}} (x^n - \hat{x}).
\] (6.10)

Combining (3.1b), (6.8) and (6.10) yields
\[
x^{n+1} - \hat{x} = \sum_{\lambda \in \Lambda} I_{D_{\lambda}} (\chi_{D_{\lambda,R}} w^n_\lambda - \hat{x}) = H_n (x^n - \hat{x}),
\] (6.11)
where
\[ H_n = -\sum_{\lambda \in \Lambda} I_D(\Phi, R, n)^{-1} I_D(\Phi, R) \Phi(I_D(\Phi, R) \hat{x} + I_V \Delta R, x^n, \hat{x}) I_V \Delta R, \ n \geq 0. \] (6.12)

By (2.4) and (4.3), we have
\[ \omega(I_D(\Phi, R) \Phi(I_D(\Phi, R) \hat{x} + I_V \Delta R, x^n, \hat{x})) \leq 2m \quad \text{and} \quad \|I_D(\Phi, R) \Phi(I_D(\Phi, R) \hat{x} + I_V \Delta R, x^n, \hat{x})\|_{B_2} \leq L. \] (6.13)

Write \( H_n = (H_n(i, j))_{i,j \in V} \). Then for any \( i, j \in V \) we get from (2.6), (2.7), (6.7), (6.13) and Lemma 6.1 that
\[ |H_n(i, j)| \leq \frac{L^2}{c(L - c)} \left(1 - \frac{c}{L}\right)^{\rho(i,j)/2m} \times \sum_{\lambda \in \Lambda} I_D(\Phi) I_V \Delta R (j) \]
\[ \leq \begin{cases} \frac{L^2}{c(L - c)} \left(1 - \frac{c}{L}\right)^{\rho(i,j)/2m} & \text{if } \rho(i, j) > R \\ 0 & \text{if } \rho(i, j) \leq R, \end{cases} \] (6.14)

where \( I_E \) is the indicator function on a set \( E \). Let \( \delta_R \) be as in (4.1). Combining (6.5) and (6.14) yields
\[ \|H_n\|_S \leq \sup_{i \in V} \sum_{\rho(i,j) > R} \frac{L^2}{c(L - c)} \left(1 - \frac{c}{L}\right)^{\rho(i,j)/2m} \leq \delta_R. \] (6.15)

By (4.3), (6.11) and (6.15), we obtain
\[ \|x^{n+1} - \tilde{x}\|_p \leq \|H_n\|_S \|x^{n+1} - \hat{x}\|_p \leq \delta_R \|x^n - \hat{x}\|_p, \ n \geq 0. \] (6.16)

Applying (6.16) iteratively proves (4.2) and hence completes the proof. \( \square \)

6.2. Proof of Theorem 4.2. Let \( \tilde{w}_\Lambda^\alpha, \lambda \in \Lambda \), be as in (4.6a). By (2.5) and (6.9), we have
\[ \chi_D(\hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha) \]
\[ = \chi_D(\hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha) - \nabla F(\chi_D(\hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha)) \]
\[ + \chi_D \nabla F(\chi_D \tilde{w}_\Lambda^\alpha + I_V \Delta R, \tilde{w}_\Lambda^\alpha) \]
\[ = \chi_D(\hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha) \]
\[ + \chi_D \nabla F(\chi_D \tilde{w}_\Lambda^\alpha + I_V \Delta R, \tilde{w}_\Lambda^\alpha) \] (6.17)

and
\[ \chi_D \nabla F(\hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha) = \chi_D(\hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha + I_V \Delta R, \tilde{w}_\Lambda^\alpha, \hat{x}) I_V \Delta R, \lambda \in \Lambda. \] (6.18)

Define
\[ \tilde{\Phi}_\Lambda = I_D(\Phi) \Phi(I_D(\Phi, R) \hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha, \hat{x}) I_V \Delta R, \lambda \in \Lambda, \] and
\[ \tilde{H}_n = -\sum_{\lambda \in \Lambda} I_D(\Phi) \Phi(I_D(\Phi, R) \hat{x} + I_V \Delta R, \tilde{w}_\Lambda^\alpha, \hat{x}) I_V \Delta R, n \geq 0. \]
By (4.6b), (6.17) and (6.18), we obtain
\[ \tilde{x}^{n+1} - \hat{x} = \tilde{H}_n(\tilde{x}^n - \hat{x}) + \sum_{\lambda \in \Lambda} I_{D\lambda}(\tilde{\Phi}_{\lambda,R,n})^{-1} I_{D\lambda,R} \nabla F(\chi_{D\lambda,R}^* \tilde{w}_n^\lambda + I_{V \setminus D\lambda,R} \tilde{x}^n) \] (6.19)

Following the argument used in the proof of Theorem 4.1, we can show that
\[ \omega(\tilde{\Phi}_{\lambda,R,n}) \leq 2m \quad \text{and} \quad cI \leq \tilde{\Phi}_{\lambda,R,n} \leq L, \quad \lambda \in \Lambda, \] (6.20)
and
\[ \|H_n\|_S \leq \delta_R. \] (6.21)
Write \( (\tilde{\Phi}_{\lambda,R,n})^{-1} = (G_1(i,j))_{i,j \in V} \). Then from (6.20) and Lemma 6.1 we obtain
\[ |G_1(i,j)| \leq \frac{1}{c} \left( 1 - \frac{c}{L} \right)^{\rho(i,j)/(2m)}, \quad i,j \in V. \]
This together with (2.6), (2.7) and (4.6a) implies that
\[ \left\| \sum_{\lambda \in \Lambda} I_{D\lambda}(\tilde{\Phi}_{\lambda,R,n})^{-1} I_{D\lambda,R} \nabla F(\chi_{D\lambda,R}^* \tilde{w}_n^\lambda + I_{V \setminus D\lambda,R} \tilde{x}^n) \right\|_\infty \leq \frac{\epsilon_n}{c} \sup_{i \in V} \sum_{\lambda \in \Lambda} \sum_{j \in V} 1_{D\lambda}(i) \left( 1 - \frac{c}{L} \right)^{\rho(i,j)/(2m)} 1_{D\lambda,R}(j) \leq \frac{L-c}{L^2} \delta_R \epsilon_n, \quad n \geq 0. \] (6.22)
Combining (4.3), (6.19), (6.21) and (6.22), we get
\[ \|\tilde{x}^{n+1} - \hat{x}\|_\infty \leq \delta_R \|\tilde{x}^n - \hat{x}\|_\infty + \frac{L-c}{L^2} \delta_R \epsilon_n, \quad n \geq 0. \] (6.23)
Applying (6.23) repeatedly proved the desired conclusion (4.7).

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