Generalization of Fixed Point Results via Iterative Process of \( F \)-Contraction

Aftab Hussain\(^1\)*, Arshad Muhammad\(^2\)

\(^1\)Department of Basic Sciences, Khwaja Fareed University of Engineering & Information Technology, Rahim Yar Khan Pakistan
\(^2\)Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan
*Corresponding author: aftabshh@gmail.com

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Abstract
The aim of this paper to discuss generalized iterative process of \( F \)-contraction and establish new fixed point theorems in complete metric spaces. As an application of our results, we prove existence and uniqueness of functional equations and system of differential equations. Our results provide extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Keywords: metric space, fixed point, \( F \) contraction, iterative process

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1. Introduction
In 2012, Wardowski [29] introduce a new type of contractions called \( F \)-contraction and proved new fixed point theorems concerning \( F \)-contraction. Afterwards Se-celean [28], proved fixed point theorems consisting of \( F \)-contractions by Iterated function systems. He generalized the Banach contraction principle in a different way than as it was done by different investigators.

Cosentino et al. [11] established some fixed point results of Hardy-Rogers-type for self-mappings on complete metric spaces or complete ordered metric spaces. Lately, Acar et al. [1] introduced the concept of generalized multivalued \( F \)-contraction mappings further Altun et al. [2] extended multivalued mappings with \( \delta \)-Distance and established fixed point results in complete metric space. Sgroi et al. [24] established fixed point theorems for multivalued \( F \)-contractions and obtained the solution of certain functional and integral equations, which was a proper generalization of some multivalued fixed point theorems including Nadler’s. Thereafter, many papers have published on \( F \)-contractive mappings in various spaces. For more detail see [1-20] and references therein.

Definition 1 [16] Let \((X, d)\) be a metric space. A mapping \( T: X \to X \) is said to be an \( F \) contraction if there exists \( \tau > 0 \) such that
\[
\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),
\] (1.1)
where \( F: \mathbb{R}_+ \to \mathbb{R} \) is a mapping satisfying the following conditions:

\( F(1) \) \( F \) is strictly increasing, i.e. for all \( x, y \in \mathbb{R} \) such that \( x < y \), \( F(x) < F(y) \);

\( F(2) \) For each sequence \( \{a_n\}_{n=1}^\infty \) of positive numbers, \( \lim_{n \to \infty} a_n = 0 \) if and only if \( \lim_{n \to \infty} F(a_n) = -\infty \);

\( F(3) \) There exists \( k \in (0,1) \) such that \( \lim_{a \to 0^+} a^k F(a) = 0 \).

Remark 4 From (F1) and (1.1) it is easy to conclude that every \( F \)-contraction is necessarily continuous.

Wardowski [29] stated a modified version of the Banach contraction principle as follows.

Theorem 5 [29] Let \((X, d)\) be a complete metric space and let \( T: X \to X \) be an \( F \) contraction. Then \( T \) has a
unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\left\{ T^n x \right\}_{n \in \mathbb{N}}$ converges to $x^*$.

### 2. Preliminaries

We recollect some essential notations, required definitions, and primary results coherent with the literature.

Let $(X, d)$ be a metric space. We denote by $CL(X)$ the class of all nonempty closed subsets of $X$. For a nonempty set $X$, we denote by $N(X)$ the class of all nonempty subsets of $X$. For $A, B \in CL(X)$, define a set

$$E_{A,B} = \left\{ \varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A) \right\}.$$ 

The Hausdorff metric $H$ on $CL(X)$ induced by metric $d$ is given as:

$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset \\ \infty & \text{if } E_{A,B} = \emptyset \end{cases}.$$ 

Let $f : X \to X$ and $T : X \to CL(X)$. A point $x$ in $X$ is called a fixed point of $T$ if $x \in Tx$. The set of all fixed points of $T$ is denoted by $F(T)$. Furthermore, a point $x$ in $X$ is called a coincidence point of $f$ and $T$ if $fx \in Tx$. The set of all such points is denoted by $C(f, T)$. If for some point $x$ in $X$, we have $x = fx \in Tx$, then a point $x$ is called a common fixed point of $f$ and $T$. We denote set of all common fixed points of $f$ and $T$ by $F(f, T)$.

**Definition 6** Let $f : X \to X$ and $T : X \to CL(X)$. Let $x_0$ be an arbitrary but fixed element in $X$. A sequence $(x_n)$ in $X$ is called $f$ iterative sequence of $T$ starting at $x_0$ if $fx_n \in Tx_{n-1}, \forall n \geq 1$. Then $D(f, T, x_0)$ is defined as $\left\{ (fx_n) : fx_n \in Tx_{n-1}, \forall n \geq 1 \right\}$ is called a generalized iterative process of $f$ and $T$ starting at $x_0$. Note that $D(f, T, x_0)$ reduces to dynamic process of $T$ starting at $x_0$ if $f = I_X$ (an identity map on $X$) [16]. The generalized iterative process $D(f, T, x_0)$ will simply be written as $(fx_n)$.

**Example 7** [16] Consider a Banach space $X = C(I)$ with a norm $\left\| x \right\| = \sup_{t \in I} |x(t)|$, $x \in X$ where $I = [0,1]$. Let $T : X \to 2^X$ be such that for any $x \in X$, $Tx$ is a family of the functions $t \mapsto e^{\int_0^t x(s) ds}$, where $e \in [0,1]$ i.e.,

$$(Tx)(t) = \left\{ e^{\int_0^t x(s) ds} : e \in [0,1] \right\}, x \in X$$

and let $x_0(t) = t$, $t \in [0,1]$. Then the sequence $\left\{ \frac{1}{n!(n+1)!} T^{n+1} \right\}$ is a dynamic process of the operator $T$ starting at $x_0$.

**Definition 8** Let $f$ be a self map on a metric space $X$. A multivalued mapping $T : X \to CL(X)$ is called generalized multivalued Ciric type F-contraction with respect to a iterative process $D(f, T, x_0)$ if there exist $F \in \Delta_F$ and $\tau : \mathbb{R} \to \mathbb{R}_+$ such that

$$\forall x_0 \in \text{dom}(F) \quad \begin{cases} \tau(M(x_{n-1}, x_n)) + F(d(fx_n, fx_{n+1})) \\ \leq F(d(M(x_{n-1}, x_n))) \end{cases}$$

where

$$M(x_{n-1}, x_n) = \max \left\{ d(fx_{n-1}, fx_n), d(fx_{n+1}, Tx_{n-1}) \right\}.$$ 

and $\forall s \geq 0 \lim_{s \to r} \tau(s) > 0$.

**Example 9** Let $X = [0, \infty)$, $f : X \to X$ and $T : X \to CL(X)$ be defined as $f(x) = \frac{x-1}{2}$, $T = \left[ \frac{0}{2} \right]$ for all $x \in X$, we obtain a sequence $(x_n) \in T(x_{n-1})$ for all $n \geq 1$, these can be many $f$ iterative sequence of $T$ starting at $x_0$.

$x_n = x_{n-1} + 1$.

Let $x_0 = 1$ be an arbitrary point in $X$. Then

$x_1 = 1 + x_0 = 2$

$f(x_1) = \frac{3}{2} - \frac{1}{2} = 1 \in Tx_1 = \left[ \frac{0}{2} \right]$

$x_2 = 1 + x_1 = 3$

$f(x_2) = \frac{5}{2} - \frac{1}{2} = 1 \in Tx_1 = [0,1]$

$x_3 = 1 + x_2 = 4$

$f(x_3) = \frac{7}{2} - \frac{1}{2} = 3 \in Tx_2 = \left[ \frac{0}{2} \right]$

$x_4 = 1 + x_3 = 5$

$f(x_4) = \frac{9}{2} - \frac{1}{2} = 4 \in Tx_3 = \left[ \frac{0}{2} \right]$

$x_5 = 1 + x_4 = 6$

$f(x_4) = \frac{11}{2} - \frac{1}{2} = 5 \in Tx_4 = \left[ \frac{0}{2} \right]$.

Then clearly $f(x_n) \in T(x_{n-1})$. We obtain a generalized iterative process
is called a generalized dynamic process of \( f \) and \( T \) starting at \( x_0 = 1 \). So you can construct many \( f \) iterative sequences of \( T \) starting at \( x_0 \) for different values.

3. Main Result

Throughout this section, we assume that the mapping \( F \) is right continuous. In the following we will consider only the dynamic processes \( (f_n) \) satisfying the following condition:

\((D)\) For any \( n \in \mathbb{N} \),
\[
d(f_n, f_{n+1}) > 0 \Rightarrow d(f_{n-1}, f_n) > 0.
\]

If dynamic processes \( (f_n) \) does not satisfy property \((D)\), then there exists \( n_0 \in \mathbb{N} \) such that \( d(f_{n_0}, f_{n_0+1}) > 0 \) and \( d(f_{n_0-1}, f_{n_0}) = 0 \) which implies that \( f_{n_0-1} = f_{n_0} \in T x_{n_0-1} \), that is, the set of coincidence point of hybrid pair \((f, T)\) is nonempty. Under suitable conditions on hybrid pair \((f, T)\), one obtains the existence of common fixed point of \((f, T)\).

Our main result is the following.

**Theorem 10** Let \( x_0 \in X, \quad f : X \to X \) and \( T : X \to CL(X) \) a generalized multivalued Ciric type \( F\)-contraction with respect to generalized iterative process \( D(f, T, x_0) \). Then \( C(f, T) \neq \emptyset \) provided that \( f(X) \) is complete and \( F \) is continuous or \( T \) is closed multivalued mapping. Moreover \( F(f, T) \neq \emptyset \) if only one of the following conditions holds:

(a) for some \( x \in C(f, T) \), \( f \) is \( T \)-weakly commuting at \( x \), \( f^2 x = f x \). (b) \( C(f, T) \) is a singleton subset of \( C(f, T) \).

**Proof.** Let \( (f_n) \) be a generalized iterative process of the mapping \( f \) and \( T \) starting at \( x_0 \). We observe that if there exists \( n_0 \in \mathbb{N} \) such that \( x_{n_0+1} = x_{n_0} \), then the existence of a fixed point is obvious. Hence we can assume that \( d(x_n, x_{n+1}) > 0 \) for all \( n \in \mathbb{N} \). Since \( T \) is a generalized multivalued Ciric type \( F\)-contraction with respect to a generalized iterative process, it follows that

\[
\tau(M(x_{n-1}, x_n)) + F(d(f_n, f_{n+1})) \leq F(M(x_{n-1}, x_n)).
\] (2.1)

Implies
Consequently
\[\tau(d(f_{n-1}, f_n)) + F(d(f_n, f_{n+1})) \leq F(d(f_{n-1}, f_n)).\]
(2.3)
for all \(n \in \mathbb{N}\). By definition 8, there exists \(b > 0\) such that \(\tau(d(x_n, x_{n+1})) > b\) for all \(n > n_0\). Thus, we obtain
\[F(d(f_n, f_{n+1})) \leq F(d(f_{n-1}, f_n)) - \tau(d(f_{n-1}, f_n)) \leq F(d(f_{n-2}, f_{n-1})) - \tau(d(f_{n-2}, f_{n-1})) - \tau(d(f_{n-1}, f_n)) \leq \cdots \leq F(d(f_0, f_1)) - \tau(d(f_0, f_1)) - \tau(d(f_{n-1}, f_n)) \leq F(d(f_0, f_1)) - (n - n_0)b\]

On taking limit as \(n \to \infty\), we have
\[\lim_{n \to \infty} F(d(f_n, f_{n+1})) = -\infty.\]
By (F1), we get
\[\lim_{n \to \infty} d(f_n, f_{n+1}) = 0.\]
Hence it follows that
\[\lim_{n \to \infty} \left\{d(f_{n-1}, f_n)\right\} = -\infty.\]
(2.4)

Hence, for all \(n \in \mathbb{N}\), we have
\[H(Tx_n, Tu^*) < M(x_n, u^*)\]
for all \(n \in \mathbb{N}\). Therefore
\[d(f_{n+1}, Tu^*) \leq H(Tx_n, Tu^*) < M(x_n, u^*).\]

Since from definition 8, we have
\[\tau(M(x_n, u^*)) + F(d(f_{n+1}, Tu^*)) \leq F(M(x_n, u^*))\]
(2.5)
for all \(n \in \mathbb{N}\).

Next suppose that \(F\) is continuous. Since
\[\lim_{n \to \infty} d(f_n, Tu^*) = d(fu^*, Tu^*)\]
we deduce that
\[\lim_{n \to \infty} M(x_n, u^*) = d(fu^*, Tu^*).\]

so, by continuity of \(F\),
\[\tau(d(fu^*, Tu^*)) + F(d(fu^*, Tu^*)) \leq F(d(fu^*, Tu^*))\]
which provides a contradiction. We conclude that \(d(fu^*, Tu^*) = 0\), and thus \(fu^* \in Tu^*\). Now let (a) holds, that is for \(x \in C(f, T), f\) is \(T\)-weakly commuting at \(x\).

So we get \(f^2x \in Tf_{x}\). By the given hypothesis \(fx = f^2x\) and hence \(fx = f^2x \in Tf_{x}\). Consequently \(fx \in F(f, T)\).
(b) Since \(F(C(f, T)) = \{x\}\) (say) and \(x \in C(f, T),\) this implies that \(x = fx \in T_x\). Thus \(F(1, x) \neq 0\).

**Example 11** Let \(X = [0, 1], f : X \to X \) and \(T : X \to \text{CL}(X)\) be defined as \(f(x) = x, T = \left[0, \frac{x}{2}\right]\) and \(d\) be the usual metric on \(X\). Define \(F : R^+ \to R\) and \(\tau : \mathbb{R}_+ \to \mathbb{R}_+\) by \(\tau(t) = \left\lfloor -\ln(t + 1) \right\rfloor, t \in (0, 1)\)
and \(\tau(t) = 0\) for \(t \in [1, \infty)\). Then for all \(t > 0\), we obtain
\[\tau(M(x, y)) + F(H(Tx, Ty)) = \ln(t) + \ln(H(Tx, Ty)) + H(Tx, Ty) = \ln(t) + \ln\left(\frac{1}{2}|y - x| + \frac{1}{2}|y - x|\right) \leq \ln(t) + \ln\left(\frac{1}{2}|y - x| + \frac{1}{2}|y - x|\right) \leq \ln\left(|y - x| + |y - x|\right) = F(d(x, y)) \leq F(M(x, y)).\]

Thus all conditions of above Theorem 10 is satisfied and 0 is a fixed point of \(T\).

**Example 12** Let \(X = [1, \infty)\) be the usual metric space. Define \(f : X \to X, \tau : \mathbb{R}_+ \to \mathbb{R}_+\) and \(T : X \to \text{CL}(X)\)
by $f(x) = x^2$ and $T(x) = [x + 2, \infty)$ for all $x \in X$ and 
\[ \tau(t) = \begin{cases} \ln \left(1 + \frac{1}{2}\right) & \text{for } t \in (0,1) \\ \ln 3 & \text{for } t \in [1,\infty) \end{cases} \]
$t > 0$. Note that so $f(X)$ is complete. It is easy to check that for all $x,y \in X$ with $T(x) \neq T(y)$ (equivalently with $x \neq y$), one has
\[ \tau(M(x,y)) + F(H(Tx,Ty)) \leq F(M(x,y)) \]
So we can apply Theorem 10.  
**Corollary 13** Let $(X,d)$ be a complete metric space, $x_0$ be an arbitrary point in $X$, and $T: X \to \text{C}(X)$ a multivalued Ciric type F-contraction with respect to dynamic process $D(T,x_0)$, either $F$ is continuous or $T$ is closed multivalued mapping. Then there exists a fixed point of $T$. 

**(1) Applications**  
Decision space and a state space are two basic components of dynamic programming problem. State space is a set of states including initial states, action states and transitional states. So a state space is set of parameters representing different states. A decision space is the set of decision spaces respectively. We aim to give the existence and uniqueness of common and bounded solution of functional equations given in (2.6) and (2.7). Let $B(W)$ denotes the set of all bounded real valued functions on $W$. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W),\|\|)$ is a Banach space endowed with the metric $d$ defined as 
\[ d(h,k) = \sup_{x \in W} |h(x) - k(x)|. \]
Suppose that the following conditions hold:  
(C1): $G_1, G_2, g$, and $g'$ are bounded.  
(C2): For $x \in W$, $h \in B(W)$ and $b \geq 0$, define 
\[ K_h(x) = \sup_{x \in D} \{g(x,y) + G_1(x,y,h(\xi(x,y)))\} \]
\[ J_h(x) = \sup_{x \in D} \{g'(x,y) + G_2(x,y,h(\xi(x,y)))\} \]
Moreover assume that $\tau: \mathbb{R} \to \mathbb{R}$ and $L \geq 0$ such that for every $(x,y) \in W \times D$, $h,k \in B(W)$ and $t \in W$ implies 
\[ |G_1(x,y,h(t)) - G_1(x,y,h(t))| \leq e^{-\tau(t)}|M(h(t),k(t))| \]
where 
\[ M(h(t),k(t)) = \max \{d(J_h(t),J_k(t)),d(J_k(t),K_h(t)), \frac{d(J_h(t),K_k(t)) + d(J_k(t),K_k(t))}{2} \} \]
(C3): For any $h \in B(W)$, there exists $k \in B(W)$ such that for every $x \in W$ 
\[ K_h(x) = J_h(x). \]
(C4): There exists $h \in B(W)$ such that 
\[ J_h(x) = J_k(x). \]

**Theorem 14** Assume that the conditions (C1)-(C4) are satisfied. If $J(B(W))$ is a closed convex subspace of $B(W)$, then the functional equations (2.6) and (2.7) have a unique, common and bounded solution.  

**Proof.** Note that $(B(W),d)$ is a complete metric space. By (C1), $J, K$ are self-maps of $B(W)$. The condition (C3) implies that $K(B(W)) \subseteq J(B(W))$. It follows from (C4) that $J$ and $K$ commute at their coincidence points. Let $\lambda$ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Choose $x \in W$ and $y_1, y_2 \in D$ such that 
\[ Kh_j < g(x,y_j) + G_1(x,y_j,h_j(x_j)) + \lambda, \]
where $x_j = \xi(x,y_j), j = 1,2$. Further from (2.9) and (2.10), we have 
\[ Kh_1 \geq g(x,y_2) + G_1(x,y_2,h_1(x_2)) \]
\[ Kh_2 \geq g(x,y_1) + G_1(x,y_1,h_2(x_1)). \]
Then (2.12) and (2.14) together with (2.11) imply


\[ K_{h_{1}}(x) = K_{h_{2}}(x) \]

\[ < G_{1}(x, y, h_{2}(x)) - G_{1}(x, y, h_{1}(x)) + \lambda \]

\[ \leq \left| G_{1}(x, y, h_{2}(x)) - G_{1}(x, y, h_{1}(x)) \right| + \lambda \]

\[ \leq e^{-\tau(t)} \left( \left| M \left( h(t), k(t) \right) \right| + \lambda \right). \]

Then (2.12) and (2.13) together with (2.11) imply

\[ K_{h_{2}}(x) - K_{h_{1}}(x) \leq G_{1}(x, y, h_{2}(x)) - G_{1}(x, y, h_{1}(x)) \]

\[ \leq \left| G_{1}(x, y, h_{2}(x)) - G_{1}(x, y, h_{1}(x)) \right| \leq e^{-\tau(t)} \left( \left| M \left( h(t), k(t) \right) \right| \right). \]

From (2.15) and (2.16), we have

\[ K_{h_{1}}(x) - K_{h_{2}}(x) \leq e^{-\tau(t)} \left( \left| M \left( h(t), k(t) \right) \right| \right). \]

The inequality (2.17) implies

\[ d \left( K_{h_{1}}(x) - K_{h_{2}}(x) \right) \leq e^{-\tau(t)} \left[ \left| M \left( h(t), k(t) \right) \right| \right]. \]

\[ \tau(t) + \ln \left[ d \left( K_{h_{1}}(x) - K_{h_{2}}(x) \right) \right] \leq \ln \left( \left| M \left( h(t), k(t) \right) \right| \right). \]

Therefore, by Theorem 10, the pair \((K, J)\) has a common fixed point \( h^{*} \), that is, \( h^{*}(x) \) is unique, bounded, and common solution of (2.6) and (2.7).

**1 Application of system of integral equations:**

Now we discuss an application of fixed point theorem we proved in the previous section in solving the system of Volterra-type integral equations. Such system is given by the following equations:

\[ u(t) = \int_{0}^{t} K_{1}(t, s, u(s))ds + g(t), \quad \text{(2.20)} \]

\[ w(t) = \int_{0}^{t} K_{2}(t, s, w(s))ds + f(t). \quad \text{(2.21)} \]

for \( t \in [0, a] \), where \( a > 0 \). We find the solution of the system (2.20) and (2.21). Let \( C([0, a], \mathbb{R}) \) be the space of all continuous functions defined on \([0, a]\). For \( u \in C([0, a], \mathbb{R}) \), define supremum norm as:

\[ \| u \| = \sup_{t \in [0, a]} \left\{ u(t)e^{-\tau(t)} \right\}, \]

where \( \tau: \mathbb{R} \to \mathbb{R} \) is taken as a function. Let \( C([0, a], \mathbb{R}) \) be endowed with the metric

\[ d_{e}(u, v) = \sup_{t \in [0, a]} \left\| u(t) - v(t)e^{-\tau(t)} \right\| \]

\[ \text{for all } u, v \in C([0, a], \mathbb{R}). \]

With these setting \( C([0, a], \mathbb{R}, \| \|_{e}) \) becomes Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations. For more details on such applications we refer the reader to [3, 21].

**Theorem 15** Assume the following conditions are satisfied:

(i) \( K_{1}, K_{2} : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R} \) and \( f, g : [0, a] \to \mathbb{R} \)

(ii) Define

\[ Tu(t) = \int_{0}^{t} K_{1}(t, s, u(s))ds + g(t), \]

\[ Su(t) = \int_{0}^{t} K_{2}(t, s, u(s))ds + f(t). \]

Suppose there exist \( \tau : \mathbb{R}_+ \to \mathbb{R}_+ \) and \( L \geq 0 \) such that

\[ \left\| K_{1}(t, s, u) - K_{1}(t, s, v) \right\| \leq \tau(t)e^{-\tau(t)} \left\| M(u, v) \right\| \]

for all \( t, s \in [0, a] \) and \( u, v \in C([0, a], \mathbb{R}) \), where

\[ M(u, v) = \max \left\{ \left| Su(t) - Sv(t) \right|, \left| Sv(t) - Tv(t) \right|, \left| Tu(t) - Tv(t) \right| + \left| Sv(t) - Tu(t) \right| \right\}. \]

(iii) there exists \( u \in C([0, a], \mathbb{R}) \) such that

\[ Tu(t) = Su(t) \]

remains TSu(t) = STu(t). Then the system of integral equations given in (2.20) and (2.21) has a solution.

**Proof.** By assumption (iii)

\[ \left\| Tu(t) - Tv(t) \right\| \]

\[ = \int_{0}^{t} \left| K_{1}(t, s, u(s)) - K_{1}(t, s, v(s)) \right|ds \]

\[ \leq \int_{0}^{t} \tau(t)e^{-\tau(t)} \left\| M(u, v) \right\|e^{-\tau(t)}ds \]

\[ \leq \tau(t)e^{-\tau(t)} \left\| M(u, v) \right\| \leq e^{-\tau(t)} \left\| M(u, v) \right\|_{e}. \]

This implies

\[ \left\| Tu(t) - Tv(t) \right\|e^{-\tau(t)} \leq e^{-\tau(t)} \left\| M(u, v) \right\|_{e}. \]

That is

\[ \left\| Tu(t) - Tv(t) \right\|_{e} \leq e^{-\tau(t)} \left\| M(u, v) \right\|_{e}, \]
which further implies
\[
\tau(t) + \ln \left\| Tu(t) - Tv(t) \right\| \leq \ln \left\| M(u, v) \right\|.
\]

So all the conditions of Theorem 10 are satisfied. Hence the system of integral equations given in (2.20) and (2.21) has a unique common solution.

4. Conclusion

This paper presents fixed point theorems for generalized iterative process under the improved notion of dynamic process. The presented theorem provide generalized iterative process under the improved notion of several well known results. The present version of these results make significant and useful contribution in the existing literature.

Conflict of Interests

The authors declare that they have no competing interests.

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