EXPANDERS HAVE A SPANNING LIPSCHITZ SUBGRAPH WITH LARGE GIRTH

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Abstract. We show that every regular graph with good local expansion has a spanning Lipschitz subgraph with large girth and minimum degree. In particular, this gives a finite analogue of the dynamical solution to the von Neumann problem by Gaboriau and Lyons. We give a new proof and strengthen the Gaboriau-Lyons result, that allows us to answer two questions of Monod about geometric random subgroups. Our finite theorems are kind of converse to the theorem of Bourgain and Gamburd showing that large girth implies expansion for Cayley graphs of $SL_2(\mathbb{F}_p)$. We apply these to the regular case of Thomassen’s conjecture stating that every finite graph with large average degree has a subgraph with large girth and average degree. Our main tool is an infinite version of the Lovász Local Lemma developed in this paper.

1. Introduction

The so-called von Neumann problem asked if every non-amenable group has a non-commutative free subgroup. The question did arise in the twenties, when the notion of amenable groups was introduced by von Neumann [34] in order to explain the Banach-Tarski paradox. The original, algebraic version was disproved by Olshanskiy [35]. On the other hand, K. Whyte found a satisfactory geometric group theoretical solution [40]: a finitely generated group is non-amenable if and only if it has a 4-regular tree as a Lipschitz subgraph. (Benjamini and Schramm [6] proved independently a more general result for arbitrary graphs solving a problem of Deuber, T. Sós and Simonovits [12], and Elek [13] also gave an independent proof.) Gaboriau and Lyons found a dynamical version in terms of measured group theory [22].

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**Theorem 1.** Any finitely generated non-amenable group $\Gamma$ admits a measurable ergodic essentially free action of $F_2$ on $([0; 1]^\Gamma, \nu)$ such that almost every $\Gamma$-orbit of the shift decomposes into $F_2$-orbits.

Theorem 1 connects many fields and has many different formulations and interesting corollaries in terms of the Bernoulli shift, von Neumann factors, random subgroups, factor of IID processes, cost and treeings etc., see Houdayer’s Bourbaki seminar paper [24]. Theorem 1 was applied to extend results about groups containing $F_2$ as a subgroup to every non-amenable group in the work of Epstein [15], and works related to Dixmier’s unitarizibility problem by Epstein and Monod [16], and Monod and Ozawa [31]. In this paper we reprove Theorem 1 with an extra Lipschitz condition on the $F_2$-action.

**Theorem 2.** Any finitely generated non-amenable group $\Gamma$ admits a measurable essentially free action of $F_2$ on $([0; 1]^\Gamma, \nu)$ such that almost every $\Gamma$-orbit of the shift decomposes into $F_2$-orbits. Moreover, given a Cayley graph $G$ of $\Gamma$ there is a constant $L$ such that $\text{dist}_G(x, \alpha(x)) < L$ holds for the generators of the free $F_2$-action $\alpha, \beta$ and almost every $x \in [0; 1]^\Gamma$.

Our methods are from probabilistic combinatorics. Hence we will rather prove the following, equivalent version of this theorem in terms of factor of IID processes. The equivalence of the two theorems is a straightforward consequence of the definitions.

**Theorem 3.** Any finitely generated non-amenable group $\Gamma$ admits a factor of IID almost surely free action of $F_2(\alpha, \beta)$ on $\Gamma$, moreover, there is an $L$ such that $\text{dist}(x, \alpha(x)), \text{dist}(x, \beta(x)) < L$ holds for every $x \in \Gamma$.

As an immediate consequence of the Lipschitz condition we solve Problem K of Monod [30] extending the Gaboriau-Lyons result for random subgroups to geometric random subgroups (in the sense of Monod).

**Corollary 4.** A finitely generated group admits $F_2$ as a geometric random subgroup if and only if it is not amenable.

The free $F_2$-action provided by Theorem 3 induces a $\Gamma$-invariant probability distribution on the space of 4-forests that answers Problem K’ of Monod [30], too.

We compare our methods: The Gaboriau-Lyons proof goes in two steps. They find a factor of IID spanning forest with cost $> 1$ in the first step: moreover, this forest (treeing) is ergodic. They give two general methods for constructing such forests.

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Gaboriau and Lyons had another formulation in terms of countable discrete non-amenable groups.
different proofs of this difficult problem: one uses random forests and the other uses Bernoulli percolation. (Houdayer’s alternative proof uses the free minimal spanning forest \cite{24}, see also Thom \cite{37}.) The second step is based on the theory of cost introduced by Gaboriau \cite{20}. First they construct a factor of IID, ergodic spanning forest with cost \( > 2 \) and then they apply Hjorth’s theorem \cite{23} to get the desired \( F_2 \)-action as a factor of IID.

Our main tool is a version of the Lovász Local Lemma \cite{19}, a very efficient tool in probabilistic combinatorics. We build on the constructive proof of the Local Lemma by Moser and Tardos \cite{33}. We develop a countably infinite version in order to use the local lemma for infinite graphs. First we find a factor of IID spanning forest with arbitrarily large minimum degree using the local lemma: This satisfies the Lipschitz condition of Theorem 3, but we do not know if it is ergodic. So we can not use Hjorth’s result (this would not give a Lipschitz constant anyway), but we find the \( F_2 \)-action ourselves. The large expansion implied by the large minimum degree makes this task easier. The advantage of our approach is that it preserves the Lipschitz condition unlike Hjorth’s proof. In the second step we apply Lyons-Nazarov \cite{29} and Csóka-Lippner \cite{10} type arguments used to find a perfect matching as a factor of IID in a non-amenable graph. We also need to use the local lemma in this second part, too.

First we prove a finite, graph theoretical analogue of the theorem:

**Theorem 5.** Let \( G \) be a finite \( d \)-regular graph, \( 0 < \lambda < 1 \) and \( g \in \mathbb{N} \). Assume that for every \( k < g \) and every vertex \( x \in V(G) \) the number of cycles with length \( k \) containing \( x \) is at most \( (\lambda d)^k \). Then there is a constant \( L = O(1/\lambda) \) and bijections \( \alpha, \beta : V(G) \to V(G) \) such that every nontrivial word \( w \in F_2(\alpha, \beta) \) with length \( < g/L \) has no fixed point on \( V(G) \), and \( \text{dist}(x, \alpha(x)), \text{dist}(x, \beta(x)) < L \) holds for every \( x \in V(G) \).

Note that the conditions of the theorem hold for expanders if \( g = O(\log(|V(G)|)) \). This theorem allows us to give an alternative dynamical solution to the von Neumann problem for so-called sofic groups introduced by Gromov \cite{21} and Weiss \cite{39}: These groups can be approximated by finite labelled graphs, and the ultraproduct of these finite graphs will be a probability space that admits an essentially free action of the group (see Elek, Szegedy \cite{14} for basics on ultraproducts of finite graphs). The \( \Gamma \)-orbits decompose into orbits of a free \( F_2 \)-action by Theorem 5. Our approach works for arbitrary “non-amenable graphs”, but we only find an almost regular forest as spanning Lipschitz subgraph with large minimum degree instead of the free \( F_2 \)-action.
Theorem 6. Let $G$ be a countable $d$-regular graph, $\delta$ a positive, odd integer, $0 < \lambda < 1$ and $g \in \mathbb{N} \cup \{\infty\}$. Assume that for every $k < g$ and every vertex $x \in V(G)$ the number of cycles of length $k$ containing $x$ is at most $(\lambda d)^k$. Then $G$ has a spanning $L$-Lipschitz subgraph $H$, where $L = \max\{2^{\frac{\log(12\delta)}{-\log(\lambda)}} + 2; 2^{\frac{\log(\delta)}{2\log(d)}} + 2\}$, with girth at least $g/L$, minimum degree at least $\delta$ and maximum degree at most $(\delta + 1)$. Moreover, there is a randomized local Borel function on this class of graphs that gives almost surely such an $H$.

Our key theorem is the case of graphs with large expansion, when the Lipschitz constant can be 1, so we get actually a spanning subgraph.

Theorem 7. Let $G$ be a countable, $d$-regular graph, $\delta \leq d$ a positive integer and $g \in \mathbb{N} \cup \{\infty\}$. Assume that for every $k < g$ and every vertex $x \in V(G)$ the number of cycles of length $k$ containing $x$ is at most $(\frac{d}{12\delta})^k$. Then $G$ has a spanning subgraph $H$ with girth at least $g$ and minimum degree at least $\delta$. Moreover, there is a randomized local Borel function on this class of graphs that gives almost surely such an $H$.

Bourgain and Gamburd [8] proved that Cayley graphs of $SL_2(F_p)$ with girth $\Omega(\log(p))$ are actually expanders: Theorems 6 and 7 provide a kind of converse to this.

The theorem gives a strong solution to the regular case of the following conjecture of Thomassen [38]:

Conjecture 8. For every $d$ and $g$ there exists a $D = D(d, g)$ such that every finite graph with average degree at least $D$ contains a subgraph with average degree at least $d$ and girth at least $g$.

Thomassen’s conjecture is a “relaxation” of an influential conjecture of Erdős and Hajnal [17, 18] in the seventies, who asked the same for chromatic numbers instead of average degrees. The case of regular graphs is handled by the straight approach of Alon [2], see Kühn and Osthus [27]. However, the general case can not be reduced to this as proved by Pyber, Rödl and Szemerédi [36]. Kühn and Osthus [27] settled the case $g = 6$, while Dellamonica, Koube, Martin and Rödl [11] proved a directed version of the conjecture. Theorem 7 implies a strengthening of Thomassen’s conjecture for regular graphs: we find a spanning subgraph with the required properties instead of an arbitrary subgraph.

\footnote{It is generally believed that $\Omega(1)$ is enough.}
Corollary 9. Let $d, D, g$ be positive integers, and $G$ be a $D$-regular graph. Assume that $D > (12d)^g$. Then $G$ has a spanning subgraph with minimum degree at least $d$ and girth at least $g$.

Proof. The number of cycles of length $k$ can be at most $D^{k-1}$ at any vertex. This is less than $(\frac{D}{12d})^k$, hence the condition of Theorem 7 holds. The Corollary follows. \hfill \Box

Future work. In [26] the author proves a measurable version of the Lovász Local Lemma. This allows to extend our theorems to graphings (and arbitrary free actions of a non-amenable group) if $g < \infty$. Breuillard and Gelander [9] proved a uniform version of the Tits alternative, showing that for every non-virtually solvable finitely generated group of matrices one can find two elements that are free generators of a free group and are the products of at most $m$ generators, where $m$ depends on the dimension only. We hope to reprove this theorem with our methods: The extension of the Lovász Local Lemma in [26] is the first step in this direction.

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2. Definitions

We follow the terminology of the book of Lovász [28] and Kechris [25]. We say that a graph is $d$-regular if every vertex has degree $d$. The girth of a graph $G$ denoted by $g(G)$ is the length of the shortest cycle, or infinite if the graph is acyclic. An acyclic graph is called a forest, a connected forest is called a tree. The minimum degree of $G$ is denoted by $\delta(G)$. A matching is a set of edges that covers every vertex at most once. The matching is perfect if it covers every vertex exactly once. An Eulerian orientation of an undirected graph is an orientation of the edges such that the in-degree equals the out-degree for every vertex.

Definition 10. We say that the graph $H$ is an $L$-Lipschitz subgraph of the graph $G$ if $V(H) \subseteq V(G)$, and for every edge $(xy) \in E(H)$ the distance of $x$ and $y$ is at most $L$ in the graph $G$. We say that $H$ is a spanning $L$-Lipschitz subgraph of $H$ if it is an $L$-Lipschitz subgraph and $V(H) = V(G)$.
Note that the (spanning) 1-Lipschitz subgraphs of a graph are exactly the (spanning) subgraphs. The Cayley graph of the group $\Gamma$ generated by $S \subseteq \Gamma$ is a graph $G$ where $V(G) = \Gamma$ and $E(G) = \{(x, y) : x, y \in \Gamma, x^{-1}y \in S\}$. We will denote this $S$-colored graph by $Cay(\Gamma, S)$. We will assume that $S = S^{-1}$, so the graph will be undirected. We will sometimes consider the (directed) labeling of the vertices by the elements of $S$.

A graphing is a graph on a standard probability measure space, in which all degrees are bounded and $\int_A e(x, B)d\mu(x) = \int_B e(x, A)d\mu(x)$ for all measurable sets $A, B$, where $\mu$ is the probability measure and $e(x, A)$ denotes the number of neighbors of $x$ in $A$. The normalized spectral radius of (the self-adjoint operator corresponding to) a graphing will be denoted by $\rho$. An acyclic graphing is called a treeing.

We say - following Kesten - that a (finitely generated) group is non-amenable if given its $d$-regular Cayley graph there is a $\lambda < 1$ such that the number of $k$-walks at any vertex is at most $(\lambda d)^k$. (The existence of $\lambda$ is independent of the choice of the Cayley graph.) Given a group $G$ and the Lebesgue measure on the interval $[0; 1]$ we will consider the Bernoulli shift $[0; 1]^G$ with the product measure and the natural $G$-action on it. Note that this action is measure-preserving and essentially free.

We do not want to use the terminology of randomized local algorithms to avoid any confusion. (The literature often restricts to local algorithms that depend on a constant neighborhood of a vertex only.) We introduce the notion of randomized local Borel functions instead. Consider the space $G_b$ of connected, rooted graphs with maximum degree at most $b$. Set $F_b = \{(G, f) : G \in G_b, f \in [0; 1]^{V(G)}\}$. Consider the $\sigma$-algebra generated by the following sets: given a finite, connected, rooted graph $G \in G_b$ and $B \subseteq [0; 1]^{V(G)}$ Borel consider the set $\{(H, f) \in F_b : \exists r \text{ s.t. the rooted } r\text{-ball of } H \text{ is isomorphic to } G \text{ and the restriction of } h \text{ to the ball is in } B\}$. Let us call the elements of the $\sigma$-algebra generated by these sets Borel. Given a degree bound $b$ and a topological space $C$ we call a Borel mapping $\varphi : F_b \to C$ a randomized local Borel function. $C$ will be always finite in this paper. A randomized local Borel function induces a random function on every graph (with maximum degree at most $b$). We say that a randomized local Borel function has a property almost surely if it has this property for every graph $G$ for almost every choice of $f \in [0; 1]^{V(G)}$. We say that a randomized local Borel function has a property almost surely with respect to a class of graphs if it has this property for every graph $G$ in this class for almost every choice of $f \in [0; 1]^{V(G)}$. In case of Cayley
graphs a randomized local Borel function is invariant under the natural group action, i. e. it will be a factor of IID.

3. The Lovász Local Lemma

One of the most useful basic facts in probability is the following: If there is a finite set of mutually independent events that each of them holds with positive probability then the probability that all events hold simultaneously is still positive, although small. The Lovász Local Lemma allows one to show that this statement still holds in case of rare dependencies.

We will use the so-called variable version of the lemma: We will consider a set of mutually independent random variables. Given an event $A$ determined by these variables we will denote by $vbl(A)$ the unique minimal set of variables that determines the event $A$: such a set clearly exists. Note that given the events $A, B_1, \ldots, B_m$ if $vbl(A) \cap vbl(B_i) = \emptyset$ for every $1 \leq i \leq m$ then $A$ is mutually independent of all the events $B_1, \ldots, B_m$.

**Lemma 11.** Let $\mathcal{V}$ be a finite set of mutually independent random variables in a probability space. Let $\mathcal{A}$ be a finite set of events determined by these variables. If there exists an assignment $x : \mathcal{A} \rightarrow (0;1)$ such that

$$\forall A \in \mathcal{A} \ Pr[A] \leq x(A)\Pi_{vbl(A) \cap vbl(B) \neq \emptyset}(1 - x(B))$$

then $Pr(A$ holds for every $A \in \mathcal{A})$ is at least $\Pi_{A \in \mathcal{A}}(1 - x(A))$.

The lemma was originally proved by Erdős and Lovász [19]. It has had numerous applications [4], but no effective algorithm was found for more than a decade. The first constructive version was proved by Beck [5] followed by Alon [3]. Recently Moser and Tardos proved an optimal constructive version [33] (following the breakthrough result of Moser [32]).

**Algorithm 1:**

1. Evaluate every variable $v \in \mathcal{V}$ independently at random.
2. If there is an event $A \in \mathcal{A}$ not violated then pick an arbitrary $A \in \mathcal{A}$ and re-evaluate the variables in $vbl(A)$ independently (of the former evaluations and each other) at random.
3. Stop if every event $A \in \mathcal{A}$ is violated.

This allows a freedom in the choice of the resampled event, so we might call this algorithm rather a class of algorithms. The expected number of resamples can be bounded for this class of algorithms.
Theorem 12. Let $\mathcal{V}$ be a finite set of mutually independent random variables in a probability space. Let $\mathcal{A}$ be a finite set of events determined by these variables. Assume that there exists an assignment $x : \mathcal{A} \to (0; 1)$ such that

$$\forall A \in \mathcal{A} \Pr[A] \leq x(A)\Pi_{v \in \operatorname{vbl}(A) \cap \operatorname{vbl}(B) \neq \emptyset}(1 - x(B)).$$

Then the expected number of resamples of an event $A \in \mathcal{A}$ in Algorithm 1 is at most $\frac{x(A)}{1 - x(A)}$.

This bound is enough to give an effective (randomized, polynomial time) algorithm. However, we will need a version that works for infinitely many events: In this case we might not get a positive probability to violate all events in $\mathcal{A}$ simultaneously, but an element in $\Pi_{v \in \mathcal{V}} \operatorname{range}(v)$: this shows that it is possible to violate all events in $\mathcal{A}$ simultaneously.

Lemma 13. Let $\mathcal{V}$ be a set of mutually independent random variables in a probability space. Let $\mathcal{A}$ be a set of events determined by these variables. Assume that $\operatorname{vbl}(A)$ is finite for every $A \in \mathcal{A}$. If there exists an assignment $x : \mathcal{A} \to (0; 1)$ such that

$$(*) \forall A \in \mathcal{A} \Pr[A] \leq x(A)\Pi_{v \in \operatorname{vbl}(A) \cap \operatorname{vbl}(B) \neq \emptyset}(1 - x(B))$$

then there is a possible evaluation of the variables in $\Pi_{v \in \mathcal{V}} \operatorname{range}(v)$ that violates all $A \in \mathcal{A}$. Moreover, Algorithm 2 finds an evaluation such that any of the events will be almost surely violated.

Algorithm 2 will be a refinement of Algorithm 1 with a more specific choice of the re-evaluated events. It won’t be an algorithm in the classical sense: it won’t stop after finitely many steps, but the expected number of resamples of every variable will be finite. It will stabilize for every variable almost surely, and any event in $\mathcal{A}$ will be almost surely violated. We will consider a sequence $I_1, I_2, \cdots \subseteq \mathcal{A}$ such that $\operatorname{vbl}(A) \cap \operatorname{vbl}(B) = \emptyset$ for every $j$ and $A, B \in I_j$, and every $A \in \mathcal{A}$ appears in infinitely many different sets $I_j$.

Algorithm 2: In the initial step the algorithm will sample the value of every variable in $\mathcal{V}$ independently. In step $(j + 1)$ the algorithm resamples the value of every variable $v \in \operatorname{vbl}(A)$ if $A$ is violated after Step $j$ and $A \in I_j$. The algorithm stops if every $A \in \mathcal{A}$ is violated.
Proof. (of Lemma 13) The expected number of resamples of an event $A \in \mathcal{A}$ is at most $\frac{x(A)}{1-x(A)}$: If we bound the number of steps then Algorithm 2 performs in the same way as Algorithm 1, and the number of resamples at $A$ depends only on the history of finitely many variables, hence we can use the Moser-Tardos estimate. ($\ast$) implies that the expected number of sampling steps is finite for every $v \in \mathcal{V}$. Hence the value of every random variable will almost surely stabilize. Every event in $\mathcal{A}$ will be almost surely violated if it appears in infinitely many sets $I_j$.

Finally, we have to find the sequence $I_j$ such that every event will appear almost surely infinitely many times. Consider a sequence $\{a_n\}_{n=1}^{\infty}$ of natural numbers such that every number appears infinitely many times. Let $I_n$ be the following random subset of $\mathcal{A}$: Consider the set of events $S_n = \{A \in \mathcal{A} : x(A) > 1/a(n), |vbl(A)| < a(n)\}$. Note that for every $A \in S_n$ the set $vbl(A)$ is disjoint of all but finitely many $vbl(B)$, where $B \in \mathcal{A}$. Generate a random number $r(A) \in [0;1]$ independently for every $A \in S_n$ and set $I_n = \{A \in S_n : vbl(A) \cap vbl(B) = \emptyset, A \neq B \in S_n \}$. This sequence will work almost surely. \hfill $\square$

4. The proof of Theorem 7

Consider the following probability distribution on the subsets of $E(G)$: choose $\delta$ distinct edges at every vertex independently, uniformly at random, and let $E(H)$ consist of these edges. We will use the Lovász Local Lemma to prove that $H$ can satisfy the conditions of the theorem: in case of finite graphs this will happen with positive probability. The set of variables $\mathcal{V}$ will correspond to the vertices of $G$. We will call a cycle short if it is shorter than $g$. The ”bad events” of $\mathcal{A}$ correspond to short cycles: for every short cycle $C$ consider the bad event that $H$ contains this cycle. We will write “$C - C'$“ to indicate that $vbl(C) \cap vbl(C') \neq \emptyset$.

Claim: Let $x_1, \ldots, x_k$ be a cycle in $G$. Then
$$Pr\left( (x_i, x_{i+1}) \in E(H) \text{ for } i = 1, \ldots, k \right) \leq \left(\frac{2\delta}{d}\right)^k.$$

Proof. We suffice to show that for every $i$ the conditional probability $Pr((x_i, x_{i+1}) \in E(H) | (x_1, x_2), \ldots, (x_{i-1}, x_i) \in E(H))$ is at most $Pr((x_i, x_{i+1}) \in E(H)) = 2\delta^2 - \frac{\delta^2}{d} < \frac{2\delta}{d}$. We will prove the following, equivalent inequality:

$$Pr\left( (x_1, x_2), \ldots, (x_{i-1}, x_i) \in E(H) | (x_i, x_{i+1}) \notin E(H) \right) \geq Pr\left( (x_1, x_2), \ldots, (x_{i-1}, x_i) \in E(H) \right).$$

Consider the following distribution on the subsets of $E(G) \setminus (x_i, x_{i+1})$: choose $\delta$ edges at every vertex independently, and let $L$ be the union...
of these edges. The probability that the edges \((x_1, x_2), \ldots, (x_{i-1}, x_i)\)
are in \(L\) equals to the left hand side, while the probability that \(E(H)\)
contains these edges is on the right hand side. The Claim follows. □

Given a short cycle \(C\) in \(G\) let \(A_C\) denote the event that \(E(H)\)
contains the edges of \(C\). Set \(x(A_C) = (\frac{3\delta}{d})^k\), where \(k\) is the length of \(C\).
We will show that condition (*) holds in the Local Lemma. According
to the Claim we suffice to show for every short cycle \(C\) that
\((\frac{2\delta}{d})^k \leq x(A_C)\Pi_{C-C'}(1 - x(A_{C'}))\),
what is the upper bound required by the Local Lemma. We use the
bound on the number of cycles sharing a vertex:

\[\Pi_{C-C'}(1 - x_{C'}) \geq \Pi_{1 \leq i < g}^k(1 - (3\delta/d)^i)(\frac{d}{12k})^i.\]

On the other hand,

\[\Pi_{1 \leq i < g}(1 - (\frac{3\delta}{d})^i)(\frac{d}{12k})^i = \exp\left(\sum_{1 \leq i < g}(\frac{d}{12k})^i \log(1 - (\frac{3\delta}{d})^i)\right) \geq\]
\[\exp\left(\log(1 - \sum_{1 \leq i < g}(\frac{d}{12k})^i (\frac{3\delta}{d})^i)\right) = 1 - \sum_{1 \leq i < g}(\frac{3\delta}{d})^i (\frac{d}{12k})^i \geq\]
\[1 - \sum_{i=1}^{\infty}(\frac{3\delta}{d})^i (\frac{d}{12k})^i = 1 - \sum_{i=1}^{\infty} 4^{-i} = 1 - 1/3 = 2/3.\]
The first inequality holds, since \(f(x) = \frac{\log(1-x)}{x}\) is monotone decreasing
on the interval \((0; 1)\), and \(\sum_{i=1}^{\infty} 4^{-i} < 1\). Hence
\((\frac{2\delta}{d})^k = (\frac{3\delta}{d})^k (\frac{2}{3})^k \leq x(A_C)\Pi_{1 \leq i < g}^k(1 - (\frac{3\delta}{d})^i)(\frac{d}{12k})^i \leq x(A_C)\Pi_{C-C'}(1 - x(A_{C'})).\]
This completes the proof of the theorem.

5. The proof of Theorem [6]

Consider the following (power) graph \(G^{(L/2)}\): \(V(G^{(L/2)}) = V(G)\),
and the multiplicity of the edge \((x, y)\) is the number of walks with
length \(L/2\) from \(x\) to \(y\). The graph \(G\) is \(d^{L/2}\)-regular. The number
of walks with length \(k < 2g/L\) is at most \((\lambda d)^{kL/2} < (\frac{d^{L/2}}{12k})^k\) at every
vertex. This is an upper bound on the number of cycles, too, so we
can apply Theorem [7] in order to get a spanning subgraph \(H'\) of \(G^{(L/2)}\)
with minimum degree \(\geq \delta\) and girth \(> 2g/L\). This graph \(H'\) will be
a spanning \(L/2\)-Lipshitz subgraph of \(V(G)\). We will use the following
lemma in order to get an almost regular Lipschitz subgraph.

**Lemma 14.** Let \(G\) be a countable, loopless, undirected graph with
bounded maximum degree and minimum degree at least \(\delta \in \mathbb{N}\). Then \(G\)
has a spanning \(2\)-Lipschitz subgraph \(H\) with girth at least \(\frac{g(G)}{2}\), mini-
mum degree at least \(\delta\) and maximum degree at most \((\delta + 1)\). Moreover,
there is a randomized local Borel function that gives such an \(H\) almost
surely.
Proof. First we find a spanning subgraph $G_1$ of $G$ such that $\delta(G_1) \geq \delta$ and $G_1$ has no distinct, adjacent pair of vertices with degree $> \delta$. If there is an edge connecting vertices with degree $> \delta$ then we remove this edge. We iterate this process until we get the desired subgraph $G_1$. (We realize this with the following local algorithm: We generate a random number for every edge connecting vertices with degree $> \delta$, and if the number of an edge is larger than the numbers of its neighbors we remove this edge. We iterate this process so all edges connecting vertices with large degree will be almost surely removed.)

Next we will find a spanning 2-Lipschitz subgraph $H$ of $G_1$ such that $\delta(H) = \delta$, $\Delta(H) \leq \delta + 1$ and the degree of every vertex is at most its degree in $H$: For every vertex $x$ of $G_1$ of degree $> \delta$ let $v_{x,1}, \ldots, v_{x,\deg(x)}$ denote the neighbors of $x$, and set

$$E(H) = \bigcup_{x: \deg(x) > \delta} \{ (v_{x,i-1}, v_{x,2i}), 1 \leq i \leq \left[ \frac{\deg(x) - \delta}{2} \right] \} \cup \{ (v_{x,i}, v_{x,i+2}) : 1 \leq i \leq 2\left[ \frac{\deg(x) - \delta}{2} \right] \} \cup E(G_1) \setminus \bigcup_{x: \deg(x) > \delta} \{ (x, v_{x,i}) : \deg(x) > \delta, 1 \leq i \leq 2\left[ \frac{\deg(x) - \delta}{2} \right] \}.$$ 

□

6. The proof of Theorem 5

Lemma 15. Let $\delta \geq 4$ an even integer and $G$ a finite graph with minimum degree $\delta$ and maximum degree at most $(\delta + 1)$. Assume that $G$ has no adjacent pair of vertices with degree $(\delta + 1)$. Then $G$ has a $\delta$-regular spanning 3-Lipschitz subgraph $H$ with girth at least $g(G)/3$.

Proof. Call the vertices with degree $(\delta + 1)$ special. We remove the cycles of $G$ iteratively in order to end up at a forest $F$ as spanning subgraph. Special vertices will still have odd degree, and the other vertices will have even degree. We will use the subgraph $F$ to make surgeries on the graph $G$.

We will do the following for a well chosen path $x_1, \ldots, x_{k-1}$ connecting special vertices: We add an extra vertex to both ends of the path so we get a new path $x_0, \ldots, x_k$, where $x_1$ and $x_{k-1}$ are special vertices. Remove all edges of the path from $E(G)$ and add edges of the form $(x_i, x_{i+2})$, where $i = 0, \ldots, (k - 2)$. The degree of the special vertices, $x_1$ and $x_{k-1}$ has decreased by one. The degree of the other vertices has not changed.

There are vertices with degree one in the graph $F$ connected by a path in $F$ that has at most one vertex with degree $> 2$ (in particular at most one special vertex) in its interior. We do the surgery on $G$ for such a path: we can choose the extra edge at the endpoint so that our paths will be edge-disjoint. We remove this path from $E(F)$ and iterate the process for the remaining forest until we match all special vertices and do the corresponding surgery.
We claim that we will have a 3-Lipschitz subgraph of $G$ in the end. The danger is that edges might get longer and longer in the iteration. But the edges we use are only the edges of the forest plus the edges added at the ends of the paths. If we use an edge of the forest then it will be removed from the forest and glued together with another edge. If we use an additional edge not in the actual forest then it will be removed and replaced by a new edge glued from this edge and an edge of the forest, i.e. an original edge of $G$. However, this new edge won’t be adjacent to any other edges of the new forest, since special vertices are not adjacent. So we won’t touch this new edge anymore. The girth condition is easy to check.

Theorem 6 and the lemma give a 4-regular spanning Lipschitz subgraph with large girth. This can be partitioned into two 2-regular spanning subgraphs. The edges of these 2-regular graphs have an Eulerian orientation, and these two digraphs could be actually the graphs of the functions $\alpha$ and $\beta$, respectively. This completes the proof of Theorem 5.

7. A regular spanning Lipschitz subforest

We have found a factor of IID forest $F$ with minimum degree $\delta$ and maximum degree at most $(\delta+1)$ such that there are no adjacent vertices with degree $(\delta+1)$. Call the vertices with degree $(\delta+1)$ special. We will do similar surgeries to the ones in the last section, but we should be more careful, since we want to realize these as a randomized local Borel function. On the other hand, surgeries will be simpler in this section, since special vertices have even degree.

First we will find an edge-disjoint set of paths $P$ connecting special vertices such that every special vertex is the endpoint of exactly one path. Then we will make the following surgery. Add an extra vertex to both ends of every path: these new paths can be still edge-disjoint, since special vertices have even degree. For every new path $x_0, \ldots, x_k$ remove all edges of the path and add edges of the form $(x_{2i}, x_{2i+2})$, where $i = 0, \ldots, (k - 2)$. The degree of the special vertices, $x_1$ and $x_{k-1}$ has decreased by one. The degree of the other vertices has not changed, hence $H$ is $\delta$-regular.

Altogether, we suffice to find an edge-disjoint set of paths connecting special vertices such that every special vertex is the endpoint of exactly one path. We will proceed with the following local algorithm:

We start with the emptyset. Assume that we got a set of paths after Step $(k - 1)$. At Step $k$ we consider every path of length $k$ connecting unmatched special vertices. We generate a random number in $[0; 1]$ for every such path independently. We add a path to our set of paths if its
number is larger than the number of the paths sharing a vertex with it. Then we remove double edges. We iterate this infinitely many times.

**Claim 1:** Almost surely there are no pairs of unmatched special vertices at distance \( \leq k \) after Step \( k \).

*Proof.* For every pair of unmatched vertices at distance \( j \leq k \) there are infinitely many independent possibilities at Step \( j \) to get matched, so at least one vertex in such a pair of special vertices will be almost surely matched. \( \square \)

**Claim 2:** The probability that a vertex will be a special, unmatched vertex after Step \( k \) is at most \( \delta^{-\frac{k-1}{2}} \).

*Proof.* The probability \( Pr(\gamma \text{ is a special, unmatched vertex}) \) does not depend on \( \gamma \in \Gamma \). Consider the graph \( F^* \), where \( V(F^*) = V(F) \), and \( (x, y) \in F^* \) if the distance of \( x \) and \( y \) is at most \( \frac{k-1}{2} \) in \( F \). For every vertex there can be almost surely at most one special, unmatched vertex at distance \( \leq \frac{k-1}{2} \) by Claim 1. Hence the probability for a given \( \gamma \) that there will be a special, unmatched vertex at distance at most \( \frac{k-1}{2} \) is

\[
\sum_{\phi \in \Gamma} Pr(\gamma\phi \text{ is a special, unmatched vertex}, (\gamma, \gamma\phi) \in E(F^*))
\]

Note that the sum is essentially finite, since this probability is zero for all, but finitely many \( \phi \)’s. On the other hand, the process is \( \Gamma \)-invariant, hence

\[
Pr(\gamma\phi \text{ is a special, unmatched vertex}, (\gamma, \gamma\phi) \in E(F^*)) = Pr(\gamma \text{ is a special, unmatched vertex, } (\gamma, \gamma\phi^{-1}) \in E(F^*)) \]

We get

\[
1 \geq \sum_{\phi \in \Gamma} Pr(\gamma\phi \text{ is a special, unmatched vertex}, (\gamma, \gamma\phi) \in E(F^*)) = Pr(\gamma \text{ is a special, unmatched vertex} \sum_{\phi \in \Gamma} Pr((\gamma, \gamma\phi^{-1}) \in E(F^*)) \geq Pr(\gamma \text{ is a special, unmatched vertex}) \delta^{-\frac{k-1}{2}}.
\]

The last inequality holds since for every (unmatched) vertex there are at most \( \delta^{-\frac{k-1}{2}} \) vertices at distance at most \( \frac{k-1}{2} \). The claim follows. \( \square \)

**Claim 3:** The expected number of edges at a vertex to be added to or removed from \( P \) at Step \( k \) is at most \( k\delta^{-\frac{k-3}{2}} \).

*Proof.* We add paths of length \( k \) connecting a pair of special, unmatched vertices, and remove the possible double edges. Every path has length at most \( k \). The probability that a special vertex remains unmatched after Step \( (k-1) \) is at most \( \delta^{-\frac{k-3}{2}} \). The Claim follows from the \( \Gamma \)-invariance and Claim 2:

\[
Pr(\gamma \text{ is a vertex of a path created at Step } k) \leq \sum_{\phi \in \Gamma, 0 \leq i \leq k} Pr(\gamma\phi \text{ is the endpoint of a path created at Step } k, \gamma \text{ is the } i^{th} \text{ vertex of the path}) = \sum_{\pi} Pr(\gamma \text{ is the endpoint of the path } \pi \text{ created at Step } k) *
\]
\[ \sum_{\phi \in \Gamma, 0 \leq i \leq k} Pr(\gamma \phi^{-1} \text{ is the } i^{th} \text{ vertex of the path } \pi) \leq k \sum_{i=1}^{\infty} \frac{k \delta - k - 3}{2} \]

The expected number of edges changed at a vertex is at most
\[ \sum_{k=1}^{\infty} k \delta^{-k-\frac{3}{2}} < \infty. \]
Hence every edge will be almost surely untouched after finitely many steps, i.e. the process will stabilize. And the set of paths in the limit will almost surely satisfy our conditions: every special vertex will be almost surely matched.

8. The proof of Theorem 3

We have already found a regular spanning Lipschitz subforest. We only need to find a free action of \( F_2 \) on it.

Lemma 16. The \( \delta \)-regular infinite tree has a randomized local Borel subgraph with an orientation that is almost surely 2-regular and the orientation is almost surely Eulerian if \( \delta > 10000 \).

Note that such an orientation induces a free \( \mathbb{Z} \)-action \( \alpha \) on the tree \( T \) such that \( (x, \alpha(x)) \) is an edge: we set \( \alpha(x) \) to be the sole out-neighbor of \( x \). We use this lemma another time to find the action of \( \beta \), the other generator of \( F_2 \).

Proof. (of Lemma 16) Lyons and Nazarov [29] proved that the Cayley graph of a countable, non-amenable graph admits a factor of IID perfect matching, while Csóka and Lippner [10] have extended this result to expander graphings. (See also the work of Abért, Csikvári, Frenkel and the author [1].) First we use the Lyons-Nazarov theorem to find a randomized local Borel matching \( \mathcal{M} \) in the tree: this will be the half of the 2-regular subgraph. Then we orient the edges of \( \mathcal{M} \) randomly using the Local Lemma in order to get a randomized local Borel partition of the vertices into two parts such that every vertex has at least \( \frac{2d}{5} \) neighbors in the other part. We will use that the original treeing was a very good expander (in fact Ramanujan) as a Bernoulli shift, so this bipartite graphing will be still an expander. The result of Csóka and Lippner [10] implies that this bipartite graphing obtained from the Bernoulli shift has an almost perfect matching. (The Csóka-Lippner proof is quite involved, but we only use it in the simple case of bipartite graphings. For the sake of completeness we include the proof of this case in the Appendix.) This matching will be a randomized local Borel matching. Every edge of this second perfect matching connects an endpoint of an edge in \( \mathcal{M} \) to a starting point of an edge in \( \mathcal{M} \). This
induces an extension of the orientation of $\mathcal{M}$ to an Eulerian orientation of the union of the two matchings.

**Lemma 17.** Let $T$ be the $d$-regular infinite tree, where $d > 100$ and $\mathcal{M}$ a perfect matching of $T$. Then there is a randomized local Borel orientation of the edges of $\mathcal{M}$ such that for the induced partition of $V(T)$ into in- and out-vertices the following holds: every $v \in V(T)$ has almost surely at least $2^{d/5}$ of its $d$ non-matching neighbors in the other class of the partition. Moreover, the probability that a given vertex will be an in-vertex (out-vertex) is half.

**Proof.** Consider the independent, uniform, random orientation of the edges. We will apply the Lovász Local Lemma to this probability distribution. The Chernoff inequality implies that the probability that the neighbors of a vertex are badly directed is at most $e^{-d^2/200}$. We choose for every bad event $A_v$ (corresponding to a vertex $v$) $x = x(A_v) = 1/(d+1)$. We only need to check the condition of the Local Lemma: $(1 - x)^d > e^{-d^2/200}$, where the second inequality uses that $d$ is large enough. □

**Lemma 18.** Let $G$ be a $d$-regular expander and $\rho > 0$ its normalized spectral radius. Partition $V(G)$ into two disjoint sets $A$ and $B$ with equal measure such that for almost every $x \in V(G)$ at least $\frac{2d}{5}$ neighbors of $x$ will be in the other set of the partition. If $50\rho < 1$ then the bipartite graph will be an expander: for every $S \subseteq A$ ($S \subseteq B$) measurable with $|S| < |A|/2$ we have $|N(S)| > \frac{3|S|}{2}$.

**Proof.** Let $S \subseteq A$ measurable, $|S| \leq \frac{|A|}{2}$. The Expander Mixing Lemma implies $E(S, N(S)) \leq \frac{d|S||N(S)|}{|A|+|B|} + \rho d \sqrt{|S||N(S)|}$. On the other hand, every vertex of $S$ has at least $2d/5$ neighbors in $B$, i.e. in $N(S)$, hence $2d|S|/5 \leq \frac{d|S||N(S)|}{|A|+|B|} + \rho \sqrt{|S||N(S)|}$. Altogether, $\frac{4}{5} \leq \frac{|N(S)|}{|B|} + 2\rho \sqrt{\frac{|N(S)|}{|S|}}$: the lemma follows, since $50\rho < 1$. □

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102, 2006,
We prove the following special case of the Csóka-Lippner theorem:

**Proposition 19.** Let $G$ be a bipartite graphing on $V(G) = A \cup^* B$, where $|A| = |B|$. Assume that for every $S \subseteq A$ ($S \subseteq B$) measurable with $|S| < |A|/2$ the inequality $|N(S)| > \frac{3|S|}{2}$ holds. Then $G$ has a matching covering almost every vertex.

**Proof.** Given a matching an augmenting path is a path with an odd number of edges such that every other edge in the path is in the matching, and the two endpoints of the path are unmatched. Switching the
matching and non-matching edges of an alternating path increases the size of the matching by one: this is the standard way to find a large matching. The following lemma gives us the short augmenting paths.

**Lemma 20.** Let $G$ be a bipartite graphing on $V(G) = A \cup^* B$, where $|A| = |B| = \frac{1}{2}$. Assume that for every $S \subseteq A$ ($S \subseteq B$) measurable with $|S| < |A|/2$ the inequality $|N(S)| > \frac{3|S|}{2}$ holds. Let $\mathcal{M}$ be a matching of $G$ covering all vertices of $G$ but a set of measure $\leq \varepsilon$. Then there is an augmenting path with length $O(\log(\frac{1}{\varepsilon}))$. Moreover, these paths cover at least the quarter of the unmatched vertices.

**Proof.** Let $l$ be the smallest integer such that $(\frac{3}{2})^l > 1/\varepsilon$. Let $U$ denote the set of unmatched vertices, and $U'$ the set of unmatched vertices not covered by augmenting paths with length $\leq (2l - 1)$. Consider the sequence of sets $S_1, \ldots, S_l \subseteq B$ defined recursively: $S_1 = N(A \cap U')$, and set $S_{i+1}$ to be the neighborhood of the set of vertices matched to $S_i$. Note that $S_i \cap U = \emptyset$. The expansion property implies that $|S_{k+1}| \geq \min\{\frac{3|B|}{4}; \frac{3|S_k|}{2}\}$. Hence $|S_i| \geq \min\{\frac{3|B|}{4}; \frac{|A \cap U'|}{\varepsilon}\}$. We define another sequence of sets $T_1, \ldots, T_l \subseteq B$, where $T_1 = N(B \cap U')$, and set $T_{i+1}$ to be the neighborhood of the set of vertices matched to $T_i$. Now $|T_i| \geq \min\{\frac{3|A|}{4}; \frac{|B \cap U'|}{\varepsilon}\}$. There is no matching edge between $S_l$ and $T_l$, otherwise we could get a short augmenting path from $A \cap U'$ to $U$. Hence $|S_l| + |T_l| \leq \frac{1}{2}$, and so $\min\{\frac{|A \cap U'|}{\varepsilon}; \frac{|B \cap U'|}{\varepsilon}\} \leq \frac{1}{2}$. The lemma follows.

We will find our matching iterating the following process. We start with the empty matching. Given a matching $\mathcal{M}$ covering all but at most $\varepsilon$ vertices we can find a set of augmenting paths with length $O(\log(\frac{1}{\varepsilon}))$ covering a positive portion of the unmatched vertices. Moreover, we may assume that these paths are disjoint. We can switch the matching and non-matching edges along these paths increasing the size of our matching. The total length of the paths used in this process is $<< \int_{0}^{1} -\log(x)dx < \infty$, hence the process will stabilize almost everywhere and give a matching covering almost every vertex.

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