SMALL GAPS OF GOE

RENJIE FENG, GANG TIAN AND DONGYI WEI

Abstract. In this article, we study the smallest gaps between eigenvalues of the Gaussian orthogonal ensemble (GOE). The main result is that the smallest gaps, after being normalized by \( n \), will converge to a Poisson distribution, and the limiting density of the \( k \)th normalized smallest gap is \( \frac{2x^{2k-1}}{(k-1)!}e^{-x^2/(k-1)} \). The proof is based on the method developed in Feng and Wei (Small gaps of circular \( \beta \)-ensemble. arXiv:1806.01555). We need to prove the convergence of the factorial moments of the smallest gaps, which makes use of the Pfaffian structure of GOE and some comparison results between the one-component log-gas and the two-component log-gas.

1 Introduction

The problem regarding the spacings between eigenvalues is one of the most important problems in the random matrix theory. The gap probability of eigenvalues for the classical random matrices GOE, GUE, GSE and its universality for more general ensembles such as Wigner matrices have been studied intensively and pretty well-known [AGZ10, BEY14, DG09, EY15, Meh91, TV11]. There are also results on the single spacing between consecutive eigenvalues for the classical matrices and some universal ensembles [EY15, NTV17, Tao13, TV11]. But there are only a few results regarding the extreme gaps. The motivations to study the extreme gaps between consecutive eigenvalues of random matrices come from many different areas such as conjectures regarding the extreme gaps between zeros of Riemann zeta function [Dia03, KS99], quantum chaos [BBRR17, BGS86] and quantum information theory [STKZ]. Now we give a brief review of the existing results.

The way to derive the limit distributions of the smallest gaps for the determinantal point processes is basically well established. The results for the smallest gaps of CUE and GUE were first obtained by Vinson using the moment method [Vin01]. In [Sos], Soshnikov investigated the smallest gaps for any determinantal point process on the real line with a translation invariant kernel and proved that some Poisson distribution can be observed in the limit. Then Ben Arous-Bourgade in [BB13] applied Soshnikov’s method to derive the joint density of the smallest gaps of CUE and GUE, and they proved that the \( k \)th smallest gaps of CUE (after being normalized by \( n^{4/3} \))
and GUE (after being normalized by $n^{5/6}$), converge to some random variables with densities proportional to

$$x^{3k-1}e^{-x^3},$$

(1)

here, the joint density of GUE is

$$\frac{1}{Z_{n,2}}e^{-\sum_{i=1}^{n}\lambda_i^2}\prod_{1\leq i<j\leq n}|\lambda_i-\lambda_j|^2,$$

(2)

where $Z_{n,2}$ is the normalization constant. Later on, Figalli-Guionnet derived the limit distributions of the smallest gaps for some invariant multimatrix Hermitian ensembles [FG16]. In these works [BB13, FG16, Sos, Vin01], the determinantal structure plays an essential role in the proof.

In [FWa], we derived the limit distribution of the smallest gaps between eigenangles of $C^{\beta}E$ beyond the determinantal case for any positive integer $\beta$. To be more precise, given the eigenangles $\theta_1 < \cdots < \theta_n < \theta_{n+1} = \theta_1 + 2\pi$ of $C^{\beta}E$, we considered the following two-dimensional point process

$$\chi^{(n)} = \sum_{i=1}^{n}\delta\left(n^{\frac{\beta+2}{\beta}}(\theta_{i+1}-\theta_i),\theta_i\right),$$

and we proved that $\chi^{(n)}$ converges to a Poisson point process $\chi$ as $n \to +\infty$ with intensity

$$\mathbb{E}_\chi(A \times I) = \frac{A_\beta|I|}{2\pi} \int_A u^\beta du,$$

where $A \subset \mathbb{R}_+$ is any bounded Borel set, $I \subset (-\pi, \pi)$ is an interval, $|I|$ is the Lebesgue measure of $I$ and

$$A_\beta = (2\pi)^{-1}(\frac{\beta}{2})^\beta(\Gamma(\beta/2+1))^3\frac{\Gamma(3\beta/2+1)\Gamma(\beta+1)}{\Gamma(\beta/2+1)\Gamma(\beta+1)}.$$ (3)

In particular, the result holds for COE, CUE and CSE with

$$A_1 = \frac{1}{24}, \quad A_2 = \frac{1}{24\pi}, \quad A_4 = \frac{1}{270\pi},$$

respectively.

As a direct consequence, let $t_{n,k,\beta}$ be the $k$th smallest gap of $C^{\beta}E$:

$$t_{1,\beta} < t_{2,\beta} < t_{3,\beta}, \ldots$$

We define

$$\tau_{k,\beta} := n^{(\beta+2)/(\beta+1)} \times (A_\beta/(\beta+1))^{1/(\beta+1)}t_{n,k,\beta},$$ (4)
then we have
\[
\lim_{n \to +\infty} \mathbb{P}(\tau_{k, \beta}^n \in A) = \int_A \frac{\beta + 1}{(k - 1)!} x^{k(\beta + 1)} e^{-x^{\beta + 1}} dx.
\] (5)

For general C\( \beta \)E, there is no determinantal structure as there is for CUE and the whole proof in [FWa] is based on the Selberg integral.

The decay order \( \sqrt{32 \log n / n} \) of the largest gaps of CUE and GUE was predicted by Vinson in [Vin01], and the proof is given by Ben Arous-Bourgade in [BB13]. The same decay order for the largest gaps of some invariant multimatrix Hermitian matrices is derived by Figalli-Guionnet in [FG16]. Recently, the fluctuations of the largest gaps of CUE and GUE have been derived in [FWb].

But there is no previous result on the extreme gaps for GOE. The proof has some essential difficulties compared with GUE. For GUE, it is a determinantal point process so that one can express the point correlation functions explicitly and the negative correlation property of the determinantal point process ensures that one can apply the Hadamard-Fisher inequality to control the estimates. This is not the case for GOE even though it has a Pfaffian structure. One can only express the point correlation functions as integrals of the joint density, and this causes many difficulties and all the proofs require delicate estimates of the integrals. In this paper, we will derive the limit distribution of the smallest gaps of GOE and this is the first result regarding the extreme gaps for this specific case. The whole proof is based on the approach developed in [FWa].

For GOE, the joint density of the eigenvalues is
\[
\frac{1}{G_n} e^{-\sum_{i=1}^n \lambda_i^2/2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|
\] (6)

with respect to the Lebesgue measure on \( \mathbb{R}^n \). Here, the normalization constant
\[
G_n := \int_{\mathbb{R}^n} d\lambda_1 \cdots d\lambda_n e^{-\sum_{i=1}^n \lambda_i^2/2} \prod_{i<j} |\lambda_i - \lambda_j|
\] (7)
is (Proposition 4.7.1 in [For])
\[
G_n = (2\pi)^{n/2} \prod_{j=0}^{n-1} \frac{\Gamma(1 + (j + 1)/2)}{\Gamma(3/2)}.
\] (8)

In fact, one may view the above joint density as the one-component log-gas of \( n \) particles with charge \( q = 1 \) on the real line and the Hamiltonian is
\[
H(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i^2/2 - \sum_{i<j} \log |\lambda_i - \lambda_j|.
\]
Now we consider the following point process on $\mathbb{R}_+$

$$
\chi^{(n)} := \sum_{i=1}^{n-1} \delta_n(\lambda_{(i+1)} - \lambda_{(i)}),
$$

(9)

where $\lambda_{(i)}$ $(1 \leq i \leq n)$ is the increasing rearrangement of $\lambda_i$ $(1 \leq i \leq n)$. The main result of this article is

**Theorem 1.** Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of GOE, then the point process $\chi^{(n)}$ will converge to a Poisson point process $\chi$ as $n \to +\infty$ with intensity

$$
\mathbb{E}\chi(A) = \frac{1}{4} \int_A u \, du,
$$

where $A \subset \mathbb{R}_+$ is any bounded Borel set.

As a direct consequence of Theorem 1, we will have

**Corollary 1.** Let $t^n_k$ be the $k$th smallest gap and we define $\tau^n_k := 2^{-3/2}nt^n_k$, then we have

$$
\lim_{n \to +\infty} \mathbb{P}(\tau^n_k \in A) = \int_A \frac{2}{(k-1)!} x^{2k-1}e^{-x^2} \, dx
$$

(10)

for any bounded interval $A \subset \mathbb{R}_+$.

As a remark, the factor $1/4$ in Theorem 1 is quite meaningful. In fact, the main observation in Lemma 1 is that

$$
1/4 = (G_{n-2k,k}/G_n)^{1/k},
$$

(11)

i.e., its $k$th power is the quotient of the generalized partition function $G_{n-2k,k}$ of the two-component log-gas (where the system consists of $n - 2k$ particles with charge $q = 1$ and $k$ particles with charge $q = 2$) and the partition function $G_n$ of the one-component log-gas (see Section 2.1 for these definitions). The proof of Lemma 1 is based on the Pfaffian structure of GOE and some integration techniques obtained by Forrester in [For]. Therefore, to prove the main result, we need to prove the following convergence of the $k$th factorial moment

$$
\mathbb{E}\left(\frac{\chi^{(n)}(A)}{(\chi^{(n)}(A) - k)!}\right) - \left(\int_A u \, du\right)^k \frac{G_{n-2k,k}}{G_n} \to 0
$$

as $n \to +\infty$ for any fixed positive integer $k$. We will not prove the above convergence directly. We will introduce an auxiliary process

$$
\tilde{\chi}^{(n)} := \sum_{\lambda_i > \lambda_j} \delta_n(\lambda_i - \lambda_j)
$$
and prove in Lemma 8 that
\[ \tilde{\chi}^{(n)}(A) - \chi^{(n)}(A) \to 0 \] in probability

as \( n \to +\infty \), which indicates that there are no successive smallest gaps. Such result is also proved in \([\text{BB13, Sos}]\) for the smallest gaps of some determinantal point processes. The significance of such result is that instead of proving the convergence of the factorial moments of the smallest gaps \( \chi^{(n)} \), one may only prove those of \( \tilde{\chi}^{(n)} \) which are much easier to work on, i.e., it is enough to prove

\[ \mathbb{E}\left( \frac{\tilde{\chi}^{(n)}(A)!}{(\tilde{\chi}^{(n)}(A) - k)!} \right) - \left( \int_A u \, du \right)^k \frac{G_{n-2k,k}}{G_n} \to 0 \]

as \( n \to +\infty \). We will then introduce another auxiliary process

\[ \rho^{(k,n)} := \sum_{i_1, \ldots, i_{2k}}, \text{all distinct, } i_{2j-1} < i_{2j} \delta(n|\lambda_{i_1} - \lambda_{i_2}|, \ldots, n|\lambda_{i_{2k-1}} - \lambda_{i_{2k}}|) \]

and prove that

\[ \lim_{n \to +\infty} \left( \mathbb{E}\left( \frac{\tilde{\chi}^{(n)}(A)!}{(\tilde{\chi}^{(n)}(A) - k)!} \right) - \mathbb{E}\left( \rho^{(k,n)}(A^k) \right) \right) = 0. \]

Therefore, we only need to prove the following convergence

\[ \mathbb{E}\left( \rho^{(n,k)}(A^k) \right) - \left( \int_A u \, du \right)^k \frac{G_{n-2k,k}}{G_n} \to 0 \]

as \( n \to +\infty \). In order to prove this, one of the crucial ideas is that one can bound \( \mathbb{E}\left( \rho^{(n,k)}(A^k) \right) \) which is expressed in terms of the integral of the joint density of the one-component log-gas on the set \( \Sigma_{n,k,A/n} \) (see (65)) by the generalized partition function of the two-component log-gas (see Lemma 11 and the remark after its proof).

1.1 Remarks. One may consider the smallest gaps for \( G^{\beta}E \) with the joint density

\[ \frac{1}{Z_{n,\beta}} e^{-\beta \frac{1}{2} \sum_{i=1}^n \lambda_i^2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta \] (12)

where \( \beta > 0 \) and \( Z_{n,\beta} \) is the normalization constant.

By comparing the limiting densities (2),(10) with (5) with \( \beta = 1, 2 \), it is believed that the smallest gaps of \( G^{\beta}E \) have the same limiting behaviors as \( C^{\beta}E \) after suitable normalizations, and we propose the following conjecture.
Conjecture 1. Let $t^n_{k, \beta}$ be the $k$th smallest gap of $G\beta E$ with the joint density (12), then there exists some constant $c_{\beta}$ depending on $\beta$ such that

$$\tau^n_{k, \beta} = c_{\beta} n^{(\beta+2)/(\beta+1)-1/2} t^n_{k, \beta}$$

converges to some random variable with density

$$\frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}}$$

as $n \to +\infty$.

Note that (13) has an extra factor $1/2$ compared with (4) since $\lambda_i$ is expanding with order $\sqrt{n}$ under the joint density (12).

It seems that our strategy to get the limit distribution of the smallest gaps for GOE can be used to prove that of $G\beta E$ and more general ensembles with the joint density

$$\frac{1}{Z_{n, \beta, V}} e^{-\beta \sum_{i=1}^{n} n V(\lambda_i/\sqrt{n})} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta.$$  

One of the difficulties to study the smallest gaps of the general ensembles is to prove some identity as (11) or some asymptotic limit as in Lemma 4 in [FWa]. Actually, there are results only in the case of $\beta = 2$, for example, Vinson derived the limit distribution of the smallest gaps when the potential $V(x)$ is a real analytic potential which is regular and whose equilibrium measure supported on a single interval [Vin01]; while in [FG16], Figalli-Guionnet derived some universal results for the smallest gaps of some invariant multimatrix Hermitian matrices.

Recently, in [Bou, LLM], Bourgade and Landon-Lopatto-Marcinek proved the universality for the extreme gaps in the bulk of the general Hermitian and symmetric Wigner matrices with assumptions.

There are some conjectures and results regarding the local statistics of many other important point processes that are related to the classical random matrix models. The local statistics of eigenvalues of the Laplacian of several integrable systems are believed to follow Poisson statistics [BT77], while for generic chaotic systems, such as non-arithmetic surfaces of negative curvature, they are expected to follow the GOE [BGS86] (see [BBRR17] for the results about the smallest gaps between the first $N$ eigenvalues of the Laplacian on a rectangular billiard as $N$ large enough). In number theory, the local statistics of zeros of Riemann zeta function are expected to follow the GUE [Dia03, KS99]. In high energy physics, the numerical results in [GV16, GV17] indicate that the local behaviors of the SYK model, which describes $n$ (an even integer) random interacting Majorana modes on a quantum dot [CGHPSSST17], are similar to GOE ($n = 0 \mod 8$), GUE($n = 2, 6 \mod 8$) and GSE($n = 4 \mod 8$), i.e., the single SYK model encodes the three classical random matrix models. We also refer to [FTWa, FTWb, FTWc] for the mathematical results on the SYK model.
The organization of this article is as follows. In Section 2, we prove an important identity for the generalized partition functions of the two-component log-gas of GOE. In Section 3, we introduce and discuss two more auxiliary point processes and prove the non-existence of successive small gaps. In Section 4, we will establish certain integral inequalities for the two-component log-gas and complete the proof of Theorem 1.

2 Partition Functions of Two-component Log-gas

In this section, we will prove Lemma 1 regarding the partition functions of the two-component log-gas of GOE. The proof is based on the Pfaffian structure of GOE (see [DG09, For, Meh91, RSX13] for more details) and some integration techniques from Chapter 6 of [For].

2.1 Preliminaries. As explained in [Meh91] (see (5.2.9) and (6.1.2)–(6.1.5) in [Meh91]), we can rewrite the joint density (6) as

$$|J_n(x_1, \ldots, x_n)|/G_n,$$

where $J_n(x_1, \ldots, x_n)$ can be expressed in terms of a determinant as

$$J_n(x_1, \ldots, x_n) := e^{-\sum_{i=1}^n x_i^2/2} \prod_{j<i}(x_i - x_j) = c_n \det[\varphi_{i-1}(x_j)]_{i,j=1,...,n},$$

and the partition function of the integration constant is

$$G_n = \int_{\mathbb{R}^n} dx_1 \cdots dx_n |J_n(x_1, \ldots, x_n)|$$

$$= n!c_n \int_{x_1 < \cdots < x_n} dx_1 \cdots dx_n \det[\varphi_{i-1}(x_j)]_{i,j=1,...,n},$$

where $c_n > 0$ is a constant depending only on $n$ and

$$\varphi_j(x) = (2^j j! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_j(x) = (2^j j! \sqrt{\pi})^{-1/2} e^{-x^2/2} (-d/dx)^j e^{-x^2}$$

are the “oscillator wave functions” orthogonal over $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \varphi_j(x) \varphi_k(x) dx = \delta_{jk} = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\{H_j(x)\}$ are Hermite polynomials. From the following recurrence relations of Hermite polynomials

$$H_{j+1}(x) = 2x H_j(x) - 2j H_{j-1}(x), \quad H'_j(x) = 2j H_{j-1}(x),$$

one deduces

$$\sqrt{2} \varphi'_j(x) = \sqrt{j} \varphi_{j-1}(x) - \sqrt{j+1} \varphi_{j+1}(x), \quad j \geq 0,$$
where we denote $\varphi_{-1}(x) = 0$. Moreover, we have (see (5.47) in [For])

$$H_j(x) = \sum_{m=0}^{[j/2]} (-1)^m 2^{-m} \binom{j}{2m} \frac{(2m)!}{2^m m!} x^{j-2m}, \quad (22)$$

and $H_n(x)$ is uniquely determined by the first equation of (20) and the initial condition $H_0(x) = 1$, $H_1(x) = 2x$. From the expression of $H_j(x)$, we also have

$$\text{span}\left\{x^j; j \in \mathbb{Z} \cap [0, n]\right\} = \text{span}\left\{H_j(x); j \in \mathbb{Z} \cap [0, n]\right\} \quad (23)$$

and

$$V_n := \text{span}\left\{x^j e^{-x^2/2}; j \in \mathbb{Z} \cap [0, n]\right\} = \text{span}\left\{\varphi_j(x); j \in \mathbb{Z} \cap [0, n]\right\}. \quad (24)$$

Actually, the joint density (6) can be identified with the Boltzmann factor of a particular one-component log-gas (see §1.4 in [For]). One can also define the two-component log-gas for the system that consists of $n_1$ particles with charge $q = 1$ and $n_2$ particles with charge $q = 2$. The two-component log-gas provides an interpolation between GOE ($\beta = 1$) and GSE ($\beta = 4$) (see [RSX13] and §6.7 in [For]). For the two-component log-gas, the generalized partition function of the integration constant is

$$G_{n_1, n_2} := \int_{\mathbb{R}^{n_1+n_2}} d\lambda_1 \cdots d\lambda_{n_1+n_2} e^{-\sum_{i=1}^{n_1+n_2} q_i \lambda_i^2 / 2} \prod_{j<k} |\lambda_j - \lambda_k|^{q_j q_k}, \quad (25)$$

where $q_j = 1$ for $1 \leq j \leq n_1$ and $q_j = 2$ for $n_1 + 1 \leq j \leq n_1 + n_2$.

Let

$$J_{n_1, n_2}(x_1, \ldots, x_{n_1+n_2}) := e^{-\sum_{i=1}^{n_1+n_2} q_i x_i^2 / 2} \prod_{j<i} (x_i - x_j)^{q_i q_j},$$

where $q_j = 1$ for $1 \leq j \leq n_1$ and $q_j = 2$ for $n_1 + 1 \leq j \leq n_1 + n_2$. Then we have

$$J_{n_1, n_2}(x_1, \ldots, x_{n_1+n_2}) = \prod_{j=1}^{n_2} \frac{\partial}{\partial y_{n_1+2j}} J_{n_1, n_2}(y_1, \ldots, y_{n_1+n_2}),$$

where the right hand side is evaluated at $y_j = x_j$ for $j \in \mathbb{Z} \cap [1, n_1]$ and $y_{n_1+2j} = y_{n_1+2j-1} = x_{n_1+j}$ for $j \in \mathbb{Z} \cap [1, n_2]$. Therefore, differentiating (16), we have

$$J_{n_1, n_2}(x_1, \ldots, x_{n_1+n_2}) = c_{n_1+n_2} \det \begin{bmatrix} [\varphi_{i-1}(x_j)]_{i=1,\ldots,n_1; j=1,\ldots,n_1} \\
[\varphi_{i-1}'(x_j)]_{i=1,\ldots,n_1+2n_2; j=n_1+1,\ldots,n_1+n_2} \end{bmatrix}.$$
\[ G_{n_1,n_2} = \int_{\mathbb{R}^{n_1+n_2}} dx_1 \cdots dx_{n_1+n_2} |J_{n_1,n_2}(x_1, \ldots, x_{n_1+n_2})| \]
\[ = (n_1!)c_{n_1+2n_2} \int_{\Delta_j \times \mathbb{R}^{n_2}} dx_1 \cdots dx_{n_1+n_2} \det \begin{bmatrix} [\varphi_{i-1}(x_j)]_{i=1,\ldots,n_1+2n_2; \ j=1,\ldots,n_1} \\ [\varphi'_{i-1}(x_j)]_{i=1,\ldots,n_1+2n_2; \ j=n_1+1,\ldots,n_1+n_2} \end{bmatrix}. \]  

(26)

Here, \( \Delta_j = \{x_1 < \cdots < x_j\} \subset \mathbb{R}^j \) is a simplex. We also have
\[ G_n = G_{n,0}. \]  

(27)

Now we recall the definition of the Pfaffian of an antisymmetric matrix of even size (see Definition 6.1.4 in [For]): Let \( X = [\alpha_{ij}]_{i,j=1,\ldots,2N} \) be an antisymmetric matrix. Then the Pfaffian of \( X \) is defined by
\[ \text{Pf} X = \sum^* \varepsilon(\sigma) \prod_{l=1}^N \alpha_{\sigma(2l-1),\sigma(2l)} \]
\[ = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \varepsilon(\sigma) \prod_{l=1}^N \alpha_{\sigma(2l-1),\sigma(2l)}, \]  

(28)

where in the first summation the * denotes that the sum is restricted to distinct terms only and \( \varepsilon(\sigma) \) is the signature of the permutation \( \sigma \).

When \( X \) is a \( 2N \times 2N \) antisymmetric matrix and \( B \) is a general \( 2N \times 2N \) matrix, then we have (see (6.12) and (6.35) in [For])
\[ (\text{Pf} X)^2 = \det X, \ Pf(B^T XB) = (\det B)(\text{Pf} X), \ Pf(\lambda X) = \lambda^N \text{Pf} X. \]  

(29)

Here, the third identity follows from the definition (28).

2.2 Partition functions of two-component log-gas. In this section, we will prove the following lemma for the two-component log-gas of GOE.

Lemma 1. For any positive integers \( n, k, n \geq 2k \), we have \( G_{n-2k,k} = 2^{-2k}G_n \).

The following Lemmas 2 and 3 give the expressions of \( G_{n_1,n_2} \) for the cases \( n_1 \) even and \( n_1 \) odd separately, where one can express the generalized partition functions \( G_{n_1,n_2} \) in terms of Pfaffians via the method of integration over alternate variables (see §6.3.2 in [For]).

Lemma 2. For the case \( n_1 \) even, we have
\[ G_{n_1,n_2} = (n_1!n_2!)c_{n_1+2n_2} [\zeta^{n_1/2}] \text{Pf} [\beta_{j,k} + \zeta \alpha_{j,k}]_{j,k=1,\ldots,n_1+2n_2}, \]
where \([\zeta^j]f\) denotes the coefficient of \(\zeta^j\) in the power series expansion of \(f\) and

\[
\alpha_{j,k} = \int_{\mathbb{R}^2} \varphi_{k-1}(x) \varphi_{j-1}(y) \text{sgn}(x - y) \, dx \, dy
\]

and

\[
\beta_{j,k} = \int_{\mathbb{R}} (\varphi'_{k-1}(x) \varphi_{j-1}(x) - \varphi_{k-1}(x) \varphi'_j(x)) \, dx.
\]

**Proof.** According to (26), as in the proof of Proposition 6.3.4 in [For], applying the method of integration over alternate variables to integrate over \(x_1, x_3, \ldots, x_{n_1 - 1}\), and expanding the resulting determinant to integrate over all other variables gives

\[
G_{n_1, n_2} = \frac{(n_1!) c_{n_1 + 2n_2}}{(n_1/2)!} \sum_{\sigma \in S_{n_1 + 2n_2}} \varepsilon(\sigma) \prod_{l=1}^{n_1/2} a_{\sigma(2l-1), \sigma(2l)} \prod_{l=n_1/2+1} b_{\sigma(2l-1), \sigma(2l)}
\]

where

\[
a_{j,k} = \int_{\mathbb{R}} dx \varphi_{k-1}(x) \int_{-\infty}^x dy \varphi_{j-1}(y)
\]

and

\[
b_{j,k} = \int_{\mathbb{R}} \varphi'_{k-1}(x) \varphi_{j-1}(x) \, dx.
\]

Making the restriction \(\sigma(2l) > \sigma(2l - 1)\), we further have

\[
G_{n_1, n_2} = \frac{(n_1!) c_{n_1 + 2n_2}}{(n_1/2)!} \sum_{\sigma(2l) > \sigma(2l - 1)} \varepsilon(\sigma) \prod_{l=1}^{n_1/2} \alpha_{\sigma(2l-1), \sigma(2l)} \prod_{l=n_1/2+1} \beta_{\sigma(2l-1), \sigma(2l)}.
\]

Then the result is a consequence of the definition of a Pfaffian. \(\square\)

**Lemma 3.** For the case \(n_1\) odd, let \(n = n_1 + 2n_2\), then we have

\[
\frac{G_{n_1, n_2}}{(n_1! n_2!) c_n} = \left[\zeta^{(n_1-1)/2}\right] \text{Pf} \begin{bmatrix}
\beta_{j,k} + \zeta \alpha_{j,k} & \nu_j & 0 \\
-\nu_k & j, k = 1, \ldots, n \\
0 & j, k = 1, \ldots, n
\end{bmatrix},
\]

where \(\alpha_{j,k}, \beta_{j,k}\) are defined in Lemma 2 and

\[
\nu_k = \int_{\mathbb{R}} \varphi_{k-1}(x) \, dx.
\]

**Proof.** With the same definitions of \(a_{j,k}\) and \(b_{j,k}\) as in the proof of Lemma 2, we apply the method of integration over alternate variables again to integrate over \(x_1, x_3, \ldots, x_{n_1}\) first, then we expand the resulting determinant and integrate over all other variables to get

\[
G_{n_1, n_2} = \frac{(n_1!) c_{n_1 + 2n_2}}{((n_1 - 1)/2)!} \sum_{\sigma \in S_{n_1 + 2n_2}} \varepsilon(\sigma) \nu_{\sigma(n_1)}
\]
\[
\prod_{l=1}^{n_1/2} a_{\sigma(2l-1) \sigma(2l)} \prod_{l=1}^{n_2} b_{\sigma(n_1+2l-1) \sigma(n_1+2l)} \\
= \frac{(n_1!)c_{n_1+2n_2}}{((n_1-1)/2)!} \sum_{\sigma(2l) > \sigma(2l-1); l=1, \ldots, (n_1-1)/2+n_2} \varepsilon(\sigma) \nu_{\sigma(n_1+2n_2)} \\
\times \prod_{l=1}^{(n_1-1)/2} a_{\sigma(2l-1) \sigma(2l)} \prod_{l=(n_1-1)/2} \beta_{\sigma(2l-1) \sigma(2l)}.
\]

Here, we changed the order
\[
\sigma(n_1), \sigma(n_1+1), \ldots, \sigma(n_1+2n_2) \rightarrow \sigma(n_1+2n_2), \sigma(n_1), \ldots, \sigma(n_1+2n_2-1),
\]
and made the restriction \(\sigma(2l) > \sigma(2l-1)\). Now we write
\[
\nu_{\sigma(n_1+2n_2)} = \nu_{\sigma(n)} := \nu_{\sigma(n), n+1} = -\nu_{n+1, \sigma(n)}
\]
in the above expression, then the result is again a consequence of the definition of a Pfaffian. \(\square\)

Now we need several properties of \(\alpha_{j,k}, \beta_{j,k}\) and \(\nu_k\). By (19) and (21), we first have
\[
\beta_{k,k+1} = -\beta_{k+1,k} = \sqrt{2k}, \beta_{j,k} = 0 \text{ for } |j-k| \neq 1. \tag{30}
\]
We also have the following

**Lemma 4.** Let \(\alpha_{j,k}, \beta_{j,k}\) be defined in Lemma 2, \(\nu_k\) be defined in Lemma 3, and we define \(\alpha_{0,k} = \alpha_{j,0} = \nu_0 = 0\). Then we have

(a) for positive integers \(j, k\), we have
\[
\sqrt{j-1} \alpha_{j-1,k} - \sqrt{j} \alpha_{j+1,k} = 2\sqrt{2} \delta_{jk}, \sqrt{j-1} \nu_{j-1} - \sqrt{j} \nu_{j+1} = 0.
\]

(b) \(\nu_j = 0\) for \(j\) even; \(\nu_j > 0\) for \(j\) odd.

(c) \(\alpha_{j,k} = \alpha_{1,k} \nu_j / \nu_1\) for \(0 < j \leq k\).

(d) If \(k\) is odd, then \(\alpha_{j,k} = 0\) for \(0 < j \leq k\); if \(k\) is even \((k > 0)\), then \(\alpha_{1,k} > 0\).

(e) If \(n\) is even, \(n > 0\), \(j, k \in \mathbb{Z} \cap [1, n]\), then
\[
\sum_{l=1}^{n} \beta_{j,l} \alpha_{l,k} = -4\delta_{jk}.
\]

**Proof.** We define the following skew symmetric inner product
\[
\langle f | g \rangle_1 := \int_{\mathbb{R}^2} g(x) f(y) \text{sgn}(x - y) dxdy,
\]
then we have
\[
\alpha_{j,k} = \langle \varphi_{j-1} | \varphi_{k-1} \rangle_1 = -\alpha_{k,j}.
\]
Thanks to (19) and \( \lim_{x \to \pm \infty} \varphi_j(x) = 0 \), we have
\[
\langle \varphi'_j | \varphi_k \rangle_1 = \int_{\mathbb{R}} dx \varphi_k(x) \left( \int_{-\infty}^{x} \varphi'_j(y) dy - \int_{x}^{+\infty} \varphi'_j(y) dy \right)
= \int_{\mathbb{R}} dx \varphi_k(x) (2\varphi_j(x)) = 2\delta_{jk}.
\]

Hence, by (21), we will have
\[
2\sqrt{2} \delta_{jk} = \langle \sqrt{2} \varphi'_j | \varphi_k \rangle_1 = \sqrt{j} \langle \varphi_{j-1} | \varphi_k \rangle - \sqrt{j+1} \langle \varphi_{j+1} | \varphi_k \rangle_1
= \sqrt{j} \alpha_{j,k+1} - \sqrt{j+1} \alpha_{j+2,k+1},
\]
and thus we conclude the first identity of (a).

Similarly, we have
\[
0 = \int_{\mathbb{R}} \sqrt{2} \varphi'_j(x) dx = \int_{\mathbb{R}} (\sqrt{j} \varphi_{j-1}(x) - \sqrt{j+1} \varphi_{j+1}(x)) dx = \sqrt{j} \nu_j - \sqrt{j+1} \nu_{j+2},
\]
which implies the second identity of (a).

If \( j \) is even, we have
\[
\nu_j = \nu_0 \prod_{l=0}^{(j-2)/2} \frac{\sqrt{2l}}{\sqrt{2l+1}} = 0.
\]

By (18), we have \( \varphi_0(x) > 0 \), and thus
\[
\nu_1 = \int_{\mathbb{R}} \varphi_0(x) dx > 0,
\]
therefore, for \( j \) odd, we have
\[
\nu_j = \nu_1 \prod_{l=1}^{(j-1)/2} \frac{\sqrt{2l-1}}{\sqrt{2l}} > 0.
\]

This shows that (b) is true.

By (a) where \( \sqrt{j} - 1 \alpha_{j-1,k} - \sqrt{j} \alpha_{j+1,k} = 0 \) for \( 0 < j < k \), we will have
\[
\alpha_{j,k} = \alpha_{0,k} \prod_{l=0}^{(j-2)/2} \frac{\sqrt{2l}}{\sqrt{2l+1}} = 0 = \alpha_{1,k} \nu_j / \nu_1 \text{ for } j \text{ even}, \quad 0 < j \leq k
\]
and
\[
\alpha_{j,k} = \alpha_{1,k} \prod_{l=1}^{(j-1)/2} \frac{\sqrt{2l-1}}{\sqrt{2l}} = \alpha_{1,k} \nu_j / \nu_1 \text{ for } j \text{ odd}, \quad 0 < j \leq k,
\]
and thus (c) is true.
Since $\alpha_{j,k} = -\alpha_{k,j}$, we have $\alpha_{k,k} = 0$. If $k$ is odd, then $0 = \alpha_{k,k} = \alpha_{1,k}\nu_k/\nu_1$ and $\nu_1 > 0, \nu_k > 0$, then we must have $\alpha_{1,k} = 0$ and $\alpha_{j,k} = \alpha_{1,k}\nu_j/\nu_1 = 0$ for $0 < j \leq k$. If $k$ is even ($k > 0$), then $k \pm 1$ are odd and thus $\alpha_{k,k+1} = 0 = -\alpha_{k+1,k}$. By (a), we have

$$\sqrt{k-1}\alpha_{k-1,k} = \sqrt{k-1}\alpha_{k-1,k} - \sqrt{k}\alpha_{k+1,k} = 2\sqrt{2}\delta_{kk} = 2\sqrt{2}$$

and $\alpha_{1,k}\nu_{k-1}/\nu_1 = \alpha_{k-1,k} > 0$. Thus we must have $\alpha_{1,k} > 0$, which completes (d).

Now we assume that $n$ is even, $n > 0$, $j, k \in \mathbb{Z} \cap [1,n]$, then $n + 1 > k$ and $n + 1$ is odd. By (d), we have $\alpha_{n+1,k} = -\alpha_{k,n+1} = 0$. Thus by (30) and (a), we have

$$\sum_{l=1}^{n} \beta_{j,l}\alpha_{l,k} = \sum_{l=1}^{n+1} \beta_{j,l}\alpha_{l,k} = -\sqrt{2(j-1)}\alpha_{j-1,k} + \sqrt{2j}\alpha_{j+1,k}$$

$$= (-\sqrt{2}) \cdot 2\sqrt{2}\delta_{jk} = -4\delta_{jk},$$

which is (e). \qed

For the evaluation of Pfaffians, we need the following abstract result.

**Lemma 5.** Let $\alpha_{j,k}$, $\beta_{j,k}$ be defined for positive integers $j, k$ such that $\alpha_{k,j} = -\alpha_{j,k}$, $\beta_{k,j} = -\beta_{j,k}$ and $\beta_{j,k} = 0$ for $|j-k| \neq 1$. Let

$$A_n = [\alpha_{j,k}]_{j,k=1,\ldots,n}, \quad B_n = [\beta_{j,k}]_{j,k=1,\ldots,n}, \quad B_n' = \text{diag}(B_{n-1}, 0)$$

(31)

be $n \times n$ antisymmetric matrices. We define

$$D_n(\lambda) := \det(B_n + 2\lambda I_n), \quad D_0(\lambda) := 1,$$

where $I_n$ is the identity matrix, then we have

$$D_{n+1}(\lambda) = 2\lambda D_n(\lambda) + \beta_{n,n+1}^2 D_{n-1}(\lambda) \text{ for } n \in \mathbb{Z}, \ n > 0.$$  \hspace{1cm} (32)

If $n > 0$ is even, then we have (we define $\text{Pf}(B_0 + \lambda A_0) := 1$)

$$\text{Pf}(B_n + \lambda A_n) = \text{Pf}(B_n' + \lambda A_n) + \beta_{n-1,n} \text{Pf}(B_{n-2} + \lambda A_{n-2}).$$

(33)

Moreover, if $n > 0$ is even and $B_n A_n = -4I_n$, then we have

$$\text{Pf}(B_n + \lambda^2 A_n) = D_n(\lambda)/\text{Pf} \ B_n$$

(34)

and

$$\text{Pf}(B_n' + \lambda^2 A_n) = 2\lambda D_{n-1}(\lambda)/\text{Pf} \ B_n.$$  \hspace{1cm} (35)
Proof. The formula (32) follows from the Laplace expansion of the determinant in the \((n+1)\)th row of \(B_{n+1} + 2\lambda I_{n+1}\). The formula (33) follows from the Laplace expansion of the Pfaffian (see (6.36) in [For]). Now we assume that \(n > 0\) is even and \(B_nA_n = -4I_n\), then \(B_n\) is invertible, \(A_n = -4B_n^{-1}\) and

\[
B_n + \lambda^2 A_n = B_n - 4\lambda^2 B_n^{-1} = (B_n - 2\lambda I_n)B_n^{-1}(B_n + 2\lambda I_n)
\]

\[
= -(B_n + 2\lambda I_n)^T B_n^{-1}(B_n + 2\lambda I_n),
\]

here we used the fact that \(B_n\) is antisymmetric. By (29) we have

\[
\text{Pf}(B_n + \lambda^2 A_n) = (-1)^{n/2} \det(B_n + 2\lambda I_n) \text{Pf}(B_n^{-1}).
\]

Taking \(\lambda = 0\), we have \(\text{Pf}(B_n) = (-1)^{n/2} \det(B_n) \text{Pf}(B_n^{-1})\). Since \(B_n\) is invertible, by (29) again, we have \(\det(B_n) = (\text{Pf} B_n)^2 \neq 0\), and thus \((-1)^{n/2} \text{Pf}(B_n^{-1}) = (\text{Pf} B_n)^{-1}\). Therefore, we have

\[
\text{Pf}(B_n + \lambda^2 A_n) = \det(B_n + 2\lambda I_n)(\text{Pf} B_n)^{-1} = D_n(\lambda)/(\text{Pf} B_n),
\]

which is (34). By definition, the above result is also true for \(n = 0\). By definition of a Pfaffian and the fact that \(\beta_{j,k} = 0\) for \(|j - k| \neq 1\), we have

\[
\text{Pf} B_n = \prod_{j=1}^{\lfloor n/2 \rfloor} \beta_{2j-1,2j}, \ \text{Pf} B_n = \beta_{n-1,n} \text{Pf} B_{n-2}.
\]

Combining this with (32), (33) and (34), we have

\[
\text{Pf}(B_n' + \lambda^2 A_n) = \text{Pf}(B_n + \lambda^2 A_n) - \beta_{n-1,n} \text{Pf}(B_{n-2} + \lambda^2 A_{n-2})
\]

\[
= D_n(\lambda)/(\text{Pf} B_n) - \beta_{n-1,n} D_{n-2}(\lambda)/(\text{Pf} B_{n-2})
\]

\[
= D_n(\lambda)/(\text{Pf} B_n) - \beta_{n-1,n}^2 D_{n-2}(\lambda)/(\text{Pf} B_n)
\]

\[
= 2\lambda D_{n-1}(\lambda)/(\text{Pf} B_n),
\]

which is (35). This completes the proof. 

We also need to evaluate the determinant \(D_n(\lambda)\).

**Lemma 6.** Let \(\beta_{j,k}\) be defined in Lemma 2, i.e., \(\beta_{j,k}\) satisfies (30). We define \(B_n = [\beta_{j,k}]_{j,k=1}^n\) and \(D_n(\lambda) = \det(B_n + 2\lambda I_n)\) with \(D_0(\lambda) = 1\), then we have

\[
D_n(\lambda) = \sum_{m=0}^{\lfloor n/2 \rfloor} 2^{n-m} \binom{n}{2m} \frac{(2m)!}{2^m m!} \lambda^{n-2m}.
\]

**Proof.** By (30) and (32), we have

\[
D_{n+1}(\lambda) = 2\lambda D_n(\lambda) + 2n D_{n-1}(\lambda) \text{ for } n \in \mathbb{Z}, n > 0.
\]
Let $\tilde{H}_n(x) = i^{-n}D_n(ix)$, then we have
\[
\tilde{H}_{n+1}(x) = 2x\tilde{H}_n(x) - 2n\tilde{H}_{n-1}(x) \quad \text{for } n \in \mathbb{Z}, \ n > 0.
\]
Moreover, we have $D_0(\lambda) = 1, \ B_1 = (0), \ D_1(\lambda) = 2\lambda; \ \tilde{H}_0(x) = 1, \ \tilde{H}_1(x) = 2x$.
Thus $\tilde{H}_n$ satisfy the same iteration formula and initial condition as the Hermite polynomials $H_n$ (recall (20)), which implies that $\tilde{H}_n(x) = H_n(x)$.

By (22) we have
\[
D_n(\lambda) = i^n\tilde{H}_n(-i\lambda) = i^nH_n(-i\lambda)
\]
\[
= \sum_{m=0}^{[n/2]} i^n (-1)^m 2^{n-m} \binom{n}{2m} (2m)! (-i\lambda)^{n-2m}
\]
\[
= \sum_{m=0}^{[n/2]} 2^{n-m} \binom{n}{2m} (2m)! \lambda^{n-2m},
\]
which completes the proof.

Now we give the proof of Lemma 1.

Proof. Let $\alpha_{j,k}, \ \beta_{j,k}$ be defined in Lemma 2, $\nu_k$ be defined in Lemma 3, and $A_n, \ B_n, \ B'_n$ be defined in (31). If $n$ is even, then by (e) of Lemma 4, we have
\[
B_n A_n = -4I_n.
\]
By Lemma 2, we have
\[
G_{n-2k,k} = (n-2k)!k!c_n [\zeta^{n/2-k}] \text{Pf}[\beta_{j,l} + \zeta \alpha_{j,l}, j,l=1,\ldots,n]
\]
\[
= (n-2k)!k!c_n [\zeta^{n/2-k}] \text{Pf}(B_n + \zeta A_n),
\]
by Lemma 5, we have
\[
[\zeta^{n/2-k}] \text{Pf}(B_n + \zeta A_n) = [\zeta^{n-2k}] \text{Pf}(B_n + \zeta^2 A_n) = [\zeta^{n-2k}] \text{D}_n(\zeta)/(\text{Pf } B_n),
\]
by Lemma 6, we have
\[
(n-2k)!k!c_n [\zeta^{n-2k}] \text{D}_n(\zeta)/(\text{Pf } B_n)
\]
\[
= (n-2k)!k!c_n 2^{n-k} \binom{n}{2k} (2k)! (\text{Pf } B_n)^{-1}
\]
\[
= c_n 2^{n-2k} n! (\text{Pf } B_n)^{-1},
\]
and thus
\[
G_{n-2k,k} = c_n 2^{n-2k} n! (\text{Pf } B_n)^{-1} = 2^{-2k} G_{n,0} = 2^{-2k} G_n.
\]

If $n$ is odd, by Lemma 3, we first have
\[
\frac{G_{n-2k,k}}{(n-2k)!k!c_n} = [\zeta^{(n-2k-1)/2}] \text{Pf} \left[\begin{array}{c}
[\beta_{j,l} + \zeta \alpha_{j,l}, j,l=1,\ldots,n] \\
-\nu_l, l=1,\ldots,n
\end{array}\right].
\]
Therefore, we have \( B_{n+1}A_{n+1} = -4I_{n+1} \), \( \alpha_{j,n+1} = \alpha_{1,n+1} \nu_j / \nu_1 = -\alpha_{n+1,j} \) for \( 0 < j \leq n \), and \( \alpha_{1,n+1} > 0, \nu_1 > 0 \). By definition, \( Pf \) is linear with respect to the last row of \( X \), thus for \( \lambda := \zeta \alpha_{1,n+1}/\nu_1 \), we have

\[
\lambda \text{Pf} \left[ \begin{array}{c|c}
[\beta_{j,l} + \zeta \alpha_{j,l}]_{j,l=1,\ldots,n} & [\nu_j]_{j=1,\ldots,n} \\
\hline
-\nu_l & 0
\end{array} \right] = \text{Pf} \left[ \begin{array}{c|c}
[\beta_{j,l} + \zeta \alpha_{j,l}]_{j,l=1,\ldots,n} & [\nu_j]_{j=1,\ldots,n} \\
\hline
-\lambda \nu_l & 0
\end{array} \right] = \text{Pf} \left[ \begin{array}{c|c}
[\beta_{j,l} + \zeta \alpha_{j,l}]_{j,l=1,\ldots,n} & [\zeta \alpha_{j,n+1}]_{j=1,\ldots,n} \\
\hline
\zeta \alpha_{n+1,l} & 0
\end{array} \right] = \text{Pf}(B'_{n+1} + \zeta A_{n+1}),
\]

where \( B'_{n+1} = \text{diag}(B_n,0) \). Hence,

\[
\frac{\alpha_{1,n+1}G_{n-2k,k}}{\nu_1(n-2k)!k!c_n} = \frac{\alpha_{1,n+1}(\zeta^{(n-2k-1)/2+1})}{\nu_1} \text{Pf} \left[ \begin{array}{c|c}
[\beta_{j,l} + \zeta \alpha_{j,l}]_{j,l=1,\ldots,n} & [\nu_j]_{j=1,\ldots,n} \\
\hline
\zeta^{(n-2k-1)/2+1} & 0
\end{array} \right] = \left[ \zeta^{(n-2k-1)/2+1} \right] \text{Pf}(B'_{n+1} + \zeta A_{n+1}),
\]

by Lemma 5, we have

\[
\left[ \zeta^{(n-2k-1)/2+1} \right] \text{Pf}(B'_{n+1} + \zeta A_{n+1}) = \left[ \zeta^{n-2k+1} \right] \text{Pf}(B'_{n+1} + \zeta^2 A_{n+1}) = \left[ \zeta^{n-2k+1} \right] (2 \zeta D_n(\zeta)/(\text{Pf} B_{n+1})) = 2\zeta^{n-2k} D_n(\zeta)/(\text{Pf} B_{n+1}),
\]

by Lemma 6, we have

\[
2\zeta^{n-2k} D_n(\zeta)/(\text{Pf} B_{n+1}) = \frac{2^{n-2k+1}}{\text{Pf} B_{n+1}} \left( \frac{n}{2k} \right) (2k)! = \frac{2^{n-2k+1}}{\text{Pf} B_{n+1}} \frac{n!}{(n-2k)!k!}.
\]

Therefore, we have

\[
G_{n-2k,k} = \frac{2^{n-2k+1}n!\nu_1c_n}{\alpha_{1,n+1} \text{Pf} B_{n+1}},
\]

which implies

\[
G_{n-2k,k} = 2^{-2k}G_{n,0} = 2^{-2k}G_n.
\]

This completes the whole proof of Lemma 1. \( \square \)

**3 No Successive Small Gaps**

In this section, we will prove that there are no successive small gaps.
3.1 Auxiliary point processes. We now introduce two more auxiliary point processes. First, instead of $\chi(n)$, it is more convenient to consider the following point process

$$\tilde{\chi}(n) := \sum_{i < j} \delta_{n|\lambda_i - \lambda_j|} = \sum_{\lambda_i > \lambda_j} \delta_{n(\lambda_i - \lambda_j)}.$$  \hspace{1cm} (36)

Then we have

$$\chi(n) \leq \tilde{\chi}(n),$$

in fact, we can write

$$\tilde{\chi}(n) = \sum_{j=1}^{n-1} \tilde{\chi}^{(n,j)},$$

such that

$$\tilde{\chi}^{(n,j)} = \sum_{i=1}^{n-j} \delta_{n(\lambda_{i+j} - \lambda_i)}.$$  \hspace{1cm} (37)

For any Borel set $B \subset \mathbb{R}$, we have

$$\tilde{\chi}^{(n,1)} = \chi(n)$$ and $0 \leq \tilde{\chi}^{(n,j)}(B) \leq n.$

For the auxiliary point process $\tilde{\chi}(n) \geq \chi(n)$, the main task in this section is to prove that $\tilde{\chi}(n)(A) - \chi(n)(A) \to 0$ as $n \to \infty$ in probability for any bounded interval $A \subset \mathbb{R}_+$, this implies that there are no successive small gaps.

We now introduce another auxiliary point process

$$\rho^{(k,n)} := \sum_{i_1, \ldots, i_{2k} \text{ all distinct}, \ i_{2j-1} < i_{2j}} \delta_{n|\lambda_{i_1} - \lambda_{i_2}|, \ldots, n|\lambda_{i_{2k-1}} - \lambda_{i_{2k}}|}.$$  \hspace{1cm} (38)

The following lemma gives the estimates of $\rho^{(k,n)}$ in terms of $\tilde{\chi}(n)$, and we will see that the expectation of $\rho^{(k,n)}$ is basically equivalent to the factorial moment of $\tilde{\chi}(n)$ (see (67)), and hence it will be equivalent to the factorial moments of $\chi(n)$ by Lemma 8.

**LEMMA 7.** For any bounded interval $A \subset \mathbb{R}_+$, we have

$$\rho^{(k,n)}(A^k) \leq \frac{(\tilde{\chi}(n)(A))!}{(\tilde{\chi}(n)(A) - k)!}. \hspace{1cm} (38)$$

Given $c_1$ such that $A \subset (0, c_1)$, we denote $c_n = c_1 n^{-1}$ and

$$a = \max \left\{ i - j : i, j \in \mathbb{Z} \cap [1, n], \lambda_{(i)} - \lambda_{(j)} < 2c_n \right\}. \hspace{1cm} (39)$$
If \( c_n \in (0, 1) \), we have
\[
0 \leq \frac{(\tilde{\chi}(n)(A))!}{(\tilde{\chi}(n)(A) - k)!} - \rho^{(k,n)}(A_k) \leq k(k - 1)(a - 1)(\tilde{\chi}(n)(A))^{k-1} 
\]
and
\[
\rho^{(k,n)}(A_k) \geq (\tilde{\chi}(n)(A))^k - k(k - 1)a(\tilde{\chi}(n)(A))^{k-1}.
\]
Moreover, let \( A_1 = (0, 2c_1) \), then we have
\[
\rho^{(k,n)}(A_1^k) \geq \frac{(a + 1)!}{(a + 2k)!2^k}.
\]

**Proof.** We define
\[
X_1 := \left\{ (i_1, \ldots, i_{2k}) : i_j \in \mathbb{Z}, 1 \leq i_j \leq n, \ \forall \ 1 \leq j \leq 2k, \right. \nonumber
\]
\[
i_{2j-1} < i_{2j}, \ \forall \ 1 \leq j \leq k, \ \{i_{2j-1}, i_{2j}\} \neq \{i_{2l-1}, i_{2l}\}, \ \forall \ 1 \leq j < l \leq k, \left. \right\}
\]
\[
X_2 := \left\{ (i_1, \ldots, i_{2k}) : i_j \in \mathbb{Z}, 1 \leq i_j \leq n, \ \forall \ 1 \leq j \leq 2k, \right. \nonumber
\]
\[
i_{2j-1} < i_{2j}, \ \forall \ 1 \leq j \leq k, \ i_j \neq i_l, \ \forall \ 1 \leq j < l \leq 2k, \left. \right\}
\]
\[
Y_{j,l} := \left\{ (i_1, \ldots, i_{2k}) \in X_1 : \{i_{2j-1}, i_{2j}\} \cap \{i_{2l-1}, i_{2l}\} \neq \emptyset \right\},
\]
then we have
\[
X_2 \subseteq X_1 \text{ and } X_1 \setminus X_2 = \cup_{1 \leq j < l \leq k} Y_{j,l}.
\]

Let
\[
X_{m,A} := \left\{ (i_1, \ldots, i_{2k}) \in X_m : n|\lambda_{i_{2j-1}} - \lambda_{i_{2j}}| \in A, \ \forall \ 1 \leq j \leq k \right\}, \quad m = 1, 2,
\]
\[
Y_{m,l,A} := \left\{ (i_1, \ldots, i_{2k}) \in Y_{m,l} : n|\lambda_{i_{2j-1}} - \lambda_{i_{2j}}| \in A, \ \forall \ 1 \leq j \leq k \right\},
\]
then we have
\[
\rho^{(k,n)}(A_k) = |X_{2,A}|, \ \ X_{2,A} \subseteq X_{1,A} \text{ and } |X_{1,A}| = \frac{(\tilde{\chi}(n)(A))!}{(\tilde{\chi}(n)(A) - k)!},
\]
which implies (38), here \(|X|\) is the cardinality of the set \(X\).

We also have \( X_{1,A} \setminus X_{2,A} = \cup_{1 \leq j < l \leq k} Y_{j,l,A} \) and \(|Y_{j,l,A}| = |Y_{1,2,A}|\) for \(1 \leq j < l \leq k\) by symmetry. Therefore, we have
\[
|X_{1,A}| - |X_{2,A}| \leq \sum_{1 \leq j < l \leq k} |Y_{j,l,A}| = k(k - 1)|Y_{1,2,A}|/2.
\]
If $a = 0$, then we have $n|\lambda_j - \lambda_l| \geq n(2c_n) = 2c_1$ for every $1 \leq j < l \leq n$, i.e.,
$n|\lambda_j - \lambda_l| \not\in A$, and thus $\tilde{\chi}^{(n)}(A) = \rho^{(k,n)}(A^k) = 0$; if $k = 1$ then $\tilde{\chi}^{(n)}(A) = \rho^{(1,n)}(A)$
by definitions. In both cases, the inequalities (40) and (41) are clearly true, thus we
only need to consider the case $a > 0, k > 1$. The key point is to estimate $|Y_{1,2,A}|$.

Let $\lambda_{i,j} := n|\lambda_i - \lambda_j|$. For fixed $\lambda_{i_1, i_2} \in A$, we will show that there are at most
$2(a - 1)$ choices of $(i_3, i_4)$ to satisfy $(i_1, \ldots, i_{2k}) \in Y_{1,2,A}$. Let
\[
T_j = \{l : l \neq i_j, n|\lambda_{i_j} - \lambda_l| \in A\},
T_j' = \{l : l \neq i_j, |\lambda_{i_j} - \lambda_l| \in (0, c_n)\}, \quad j = 1, 2.
\]
Then we have $T_j \subseteq T_j'$ because $n|\lambda_{i_j} - \lambda_l| \in A$ implies $|\lambda_{i_j} - \lambda_l| \in n^{-1}A \subset n^{-1}(0, c_1) = (0, c_n)$. We assume $\lambda_{i_1} = \lambda_{(p)}$, then we have
\[
\{\lambda_l : l \in T_j' \cup \{i_1\}\} = \{\lambda_{(q)} : |\lambda_{(q)} - \lambda_{(p)}| < c_n\}
= \{\lambda_{(q)} : r \leq q \leq s\},
\]
for some $r, s \in \mathbb{Z} \cap [1, n]$ such that $|\lambda_{(r)} - \lambda_{(p)}| < c_n$, $|\lambda_{(s)} - \lambda_{(p)}| < c_n$. Therefore, we have $|\lambda_{(r)} - \lambda_{(s)}| < 2c_n$ and $s - r \leq a$ by the definition of $a$. Since $i_1 \not\in T_j'$, we have
\[
|T_j'| + 1 = |\{\lambda_l : l \in T_j' \cup \{i_1\}\}| = |\{\lambda_{(q)} : r \leq q \leq s\}| \\
\leq s - r + 1 \leq a + 1,
\]
and thus $|T_j| \leq |T_j'| \leq a$. Similarly we have $|T_2| \leq |T_2'| \leq a$.

Now for $\lambda_{i_1, i_2} \in A$, by definition we have $i_2 \in T_1$ and $i_1 \in T_2$. If $\lambda_{i_3, i_4} \in A$, $i_3 < i_4$, $\{i_1, i_2\} \cap \{i_3, i_4\} \neq \emptyset$, $\{i_1, i_2\} \neq \{i_3, i_4\}$, then we must have $\{i_3, i_4\} = \{i_1, l\}$, $l \in T_2\{i_1\}$ or $\{i_3, i_4\} = \{i_2, l\}$, $l \in T_1\{i_2\}$. Thus for $\lambda_{i_1, i_2} \in A$, the number of $(i_3, i_4)$ satisfying $\lambda_{i_3, i_4} \in A$, $i_3 < i_4$, $\{i_1, i_2\} \cap \{i_3, i_4\} \neq \emptyset$, $\{i_1, i_2\} \neq \{i_3, i_4\}$ is at most $|T_2\{i_1\}| + |T_1\{i_2\}| = |T_2| - 1 + |T_1| - 1 \leq 2(a - 1)$.

Now there are $\tilde{\chi}^{n}(A)$ choices of $(i_1, i_2)$; for fixed $(i_1, i_2)$, there are at most
$2(a - 1)$ choices of $(i_3, i_4)$ and $\tilde{\chi}^{n}(A)$ choices of $(i_{2l-1}, i_{2l})$ with $3 \leq l \leq k$ to satisfy
$(i_1, \ldots, i_{2k}) \in Y_{1,2,A}$, thus we have
\[
|Y_{1,2,A}| \leq \tilde{\chi}^{n}(A) \times 2(a - 1) \times \tilde{\chi}^{n}(A)^{k-2} = 2(a - 1)\tilde{\chi}^{n}(A)^{k-1}.
\]
Therefore, by (43) we have
\[
0 \leq \frac{\tilde{\chi}^{(n)}(A)!}{(\tilde{\chi}^{n}(A) - k)!} - \rho^{(k,n)}(A^k) \\
= |X_{1,A} - |X_{2,A}| \\
\leq k(k - 1)|Y_{1,2,A}|/2 \\
\leq k(k - 1)(a - 1)(\tilde{\chi}^{n}(A))^{k-1},
\]
which is (40). The inequality (41) follows from (40) and the fact that
\[
\frac{\tilde{\chi}^{(n)}(A)!}{(\tilde{\chi}^{n}(A) - k)!} = \prod_{j=0}^{k-1}(\tilde{\chi}^{n}(A) - j) = (\tilde{\chi}^{n}(A))^k \prod_{j=0}^{k-1}(1 - j/\tilde{\chi}^{n}(A)).
\]
\[
\geq (\tilde{\chi}^{(n)}(A))^k \left( 1 - \sum_{j=0}^{k-1} j/\tilde{\chi}^{(n)}(A) \right)
= (\tilde{\chi}^{(n)}(A))^k - k(k - 1)(\tilde{\chi}^{(n)}(A))^{k-1}/2.
\]

To prove (42), the key point is to construct a subset of \(X_{2,A_1}\) with the cardinality \((a+1)!/(a+1-2k)!2^k\). By the definition of \(a\), there exists \(r,s \in \mathbb{Z} \cap [1,n]\) such that \(|\lambda(r) - \lambda(s)| < 2c_n\), \(s - r = a\). Let

\[
Z = \left\{ j : \lambda_j = \lambda_q, \; r \leq q \leq s \right\}
\]

and

\[
X_3 = \left\{ (i_1, \ldots, i_{2k}) : i_j \in \mathbb{Z}, \; \forall \; 1 \leq j \leq 2k, \right. \\
i_{2j-1} < i_{2j}, \; \forall \; 1 \leq j \leq k, \; i_j \neq i_l, \; \forall \; 1 \leq j < l \leq 2k \right\}.
\]

Then we have \(|Z| = s - r + 1 = a + 1\), \(X_3 \subseteq X_2\) and

\[
|X_3| = \frac{|Z|!}{(|Z| - 2k)!2^k} = \frac{(a+1)!}{(a+1-2k)!2^k}.
\]

Moreover, we have \(|\lambda_j - \lambda_l| \leq |\lambda(r) - \lambda(s)| < 2c_n\) for \(j, l \in Z\). For \((i_1, \ldots, i_{2k}) \in X_3\), we have

\[
0 < n|\lambda_{i_{2j-1}} - \lambda_{i_{2j}}| < 2nc_n = 2c_1 \text{ for } 1 \leq j \leq k,
\]

i.e., \(n|\lambda_{i_{2j-1}} - \lambda_{i_{2j}}| \in (0, 2c_1) = A_1\). Therefore, we have \(X_3 \subseteq X_{2,A_1}\) and thus

\[
\rho^{(k,n)}(A_1^k) = |X_{2,A_1}| \geq |X_3| = \frac{(a+1)!}{(a+1-2k)!2^k},
\]

which is (42). This completes the whole proof. \(\Box\)

3.2 No successive small gaps. In this subsection, we will prove the following

**Lemma 8.** For any bounded interval \(A \subset \mathbb{R}_+\), we have \(\chi^{(n)}(A) - \tilde{\chi}^{(n)}(A) \to 0\) in probability as \(n \to +\infty\).

To prove Lemma 8, we first need the upper and lower bounds in the following integral lemma.

**Lemma 9.** We assume \(\lambda_j \in \mathbb{R}\) (not necessarily distinct) for \(1 \leq j \leq m, m \text{ and } n\) are positive integers such that \(m < n\), and \(2nc^2 \in (0,1)\) with \(c > 0\). We define

\[
F(x) := e^{-x^2/2} \prod_{j=1}^{m} (x - \lambda_j),
\] (44)
then we have
\[ \int_{\mathbb{R}} |F'(x)|^2 dx \leq (2n) \int_{\mathbb{R}} |F(x)|^2 dx \]  
(45)

and
\[ (1 - nc^2) c^2 \int_{\mathbb{R}} dx_1 |F(x_1)|^2 \leq \int_{\mathbb{R}} dx_1 \int_{x_1 - c}^{x_1 + c} dx_2 |x_1 - x_2| |F(x_1)| |F(x_2)| \]
\[ \leq c^2 \int_{\mathbb{R}} dx_1 |F(x_1)|^2. \]  
(46)

Moreover, given an interval \( A \subset (0, c) \), we denote \( A_1 = A \cup (-A) \) and
\[ \varphi(A) := \int_A u du, \]  
(47)

then we have
\[ (1 - nc^2) \cdot 2 \varphi(A) \int_{\mathbb{R}} dx_1 |F(x_1)|^2 \leq \int_{\mathbb{R}} dx_1 \int_{x_1 + A_1}^{x_1 + A_1} dx_2 |x_1 - x_2| |F(x_1)| |F(x_2)| \]
\[ \leq 2 \varphi(A) \int_{\mathbb{R}} dx_1 |F(x_1)|^2. \]  
(48)

Given \( B = \cup_{i=1}^m (\lambda_i, \lambda_i + c)^2 \subset \mathbb{R}^2 \), we have
\[ \int_B |x_1 - x_2| |F(x_1)| |F(x_2)| dx_1 dx_2 \leq nc^4 \int_{\mathbb{R}} |F(x)|^2 dx. \]  
(49)

Proof. Note that \( F(x) \in V_m \) (see (24)), therefore, we can write
\[ F(x) = \sum_{j=0}^m a_j \varphi_j(x). \]

By (21) we have
\[ F'(x) = \sum_{j=0}^m \frac{a_j}{\sqrt{2}} (\sqrt{j} \varphi_{j-1}(x) - \sqrt{j + 1} \varphi_{j+1}(x)) \]
\[ = \sum_{j=0}^{m+1} \frac{\sqrt{j+1} a_{j+1} - \sqrt{j} a_j}{\sqrt{2}} \varphi_j(x), \]
where \( \varphi_{-1}(x) = 0, \ a_{-1} = a_{m+1} = a_{m+2} = 0. \) By (19) we have
\[ \int_{\mathbb{R}} |F(x)|^2 dx = \sum_{j=0}^m |a_j|^2 \]
and
\[ \int_{\mathbb{R}} |F'(x)|^2 dx = \sum_{j=0}^{m+1} \left| \sqrt{j+1}a_{j+1} - \sqrt{j}a_{j-1} \right|^2. \]

Using \(|a + b|^2 \leq 2(|a|^2 + |b|^2)| and \(a_{-1} = a_{m+1} = a_{m+2} = 0\), we have
\[
\int_{\mathbb{R}} |F'(x)|^2 dx \leq \sum_{j=0}^{m+1} \left( |\sqrt{j+1}a_{j+1}|^2 + |\sqrt{j}a_{j-1}|^2 \right)
= \sum_{j=1}^{m+2} j|a_j|^2 + \sum_{j=-1}^{m} (j + 1)|a_j|^2 = \sum_{j=0}^{m} (2j + 1)|a_j|^2
\leq \sum_{j=0}^{m} (2m + 1)|a_j|^2 = (2m + 1) \int_{\mathbb{R}} |F(x)|^2 dx
\leq (2n) \int_{\mathbb{R}} |F(x)|^2 dx,
\]
which is the first inequality (45). Here we used the fact that \(m < n, \ n \geq m + 1\).

To prove (46), a change of variables \(x_2 = x_1 + t\) yields
\[
\int_{\mathbb{R}} dx_1 \int_{x_1-c}^{x_1+c} dx_2 |x_1 - x_2||F(x_1)||F(x_2)|
= \int_{-c}^{c} |t| dt \int_{\mathbb{R}} |F(x_1)||F(x_1 + t)| dx_1. \tag{50}
\]

We also have
\[
\int_{\mathbb{R}} ||F(x_1)| - |F(x_1 + t)||^2 dx_1
= \int_{\mathbb{R}} (|F(x_1)|^2 + |F(x_1 + t)|^2) dx_1 - 2 \int_{\mathbb{R}} |F(x_1)||F(x_1 + t)| dx_1 \tag{51}
= 2 \int_{\mathbb{R}} |F(x_1)|^2 dx_1 - 2 \int_{\mathbb{R}} |F(x_1)||F(x_1 + t)| dx_1,
\]
which implies
\[
\int_{\mathbb{R}} |F(x_1)||F(x_1 + t)| dx_1 \leq \int_{\mathbb{R}} |F(x_1)|^2 dx_1. \tag{52}
\]

By (50) and (52), we have
\[
\int_{\mathbb{R}} dx_1 \int_{x_1-c}^{x_1+c} dx_2 |x_1 - x_2||F(x_1)||F(x_2)|
\leq \int_{-c}^{c} |t| dt \int_{\mathbb{R}} |F(x_1)|^2 dx_1 = c^2 \int_{\mathbb{R}} |F(x_1)|^2 dx_1,
which is the upper bound in (46).

On the other hand, we have

$$||F(x_1)| - |F(x_1 + t)||^2 \leq |F(x_1) - F(x_1 + t)|^2$$

and thus by (45) we have

$$\int \mathbb{R} ||F(x_1)| - |F(x_1 + t)||^2 \, dx_1$$

$$\leq |t|^2 \int_0^1 \int \mathbb{R} |F'(x_1 + ts)|^2 ds \, dx_1$$

$$= |t|^2 \int_0^1 \int \mathbb{R} |F'(x_1 + ts)|^2 ds |ds dx_1| = |t|^2 \int_0^1 \int \mathbb{R} |F'(x_1)|^2 \, dx_1 ds$$

$$= |t|^2 \int \mathbb{R} |F'(x_1)|^2 \, dx_1 \leq 2n|t|^2 \int \mathbb{R} |F(x_1)|^2 \, dx_1.$$ Combining this estimate with identity (51), we have

$$\int \mathbb{R} ||F(x_1)||F(x_1 + t)|| \, dx_1 \geq (1 - n|t|^2) \int \mathbb{R} |F(x_1)|^2 \, dx_1, \forall t \in (-c, c),$$

and thus the uniform lower bound

$$\int \mathbb{R} ||F(x_1)||F(x_1 + t)|| \, dx_1 \geq (1 - nc^2) \int \mathbb{R} |F(x_1)|^2 \, dx_1, \forall t \in (-c, c).$$ (53)

Therefore, combining (50) and (53), we can conclude the lower bound in (46).

Notice that

$$\int \mathbb{R} dx_1 \int_{x_1 + A_1} dx_2 |x_1 - x_2||F(x_1)||F(x_2)|$$

$$= \int_{A_1} \int \mathbb{R} |t||t| \int \mathbb{R} |F(x_1)||F(x_1 + t)|| \, dx_1,$$

then (48) follows from (52), (53) and the fact that

$$\int_{A_1} \int \mathbb{R} |t| \, dt = 2 \int \mathbb{R} t \, dt = 2 \varphi(A).$$

Let \( B_1 = \cup_{i=1}^m (\lambda_i, \lambda_i + c) \subset \mathbb{R} \), then for \((x_1, x_2) \in B = \cup_{i=1}^m (\lambda_i, \lambda_i + c)^2 \), we have \(x_1, x_2 \in B_1, (x_2, x_1) \in B, |x_1 - x_2| \leq c\), and thus we first have

$$\int_B |x_1 - x_2||F(x_1)||F(x_2)|dx_1dx_2 \leq \frac{1}{2} \int_B |x_1 - x_2|(\|F(x_1)\|^2 + \|F(x_2)\|^2)dx_1dx_2$$

$$= \int_B |x_1 - x_2||F(x_1)|^2dx_1dx_2.$$
\[
\int_{B_1} dx_1 \int_{x_1-c}^{x_1+c} dx_2 |x_1 - x_2||F(x_1)|^2 \\
\leq c^2 \int_{B_1} |F(x_1)|^2 dx_1.
\]

Without loss of generality we can assume that \(\lambda_1 \leq \cdots \leq \lambda_m\) and we denote \(I_j = (\lambda_j, \lambda_j + c) \cap (\lambda_j, \lambda_j + 1)\) for \(1 \leq j < m\), \(I_m = (\lambda_m, \lambda_m + c)\). Then we have \(B_1 = \bigcup_{j=1}^m I_j\) and \(I_j \cap I_k = \emptyset\) for \(j \neq k\). By definition we have \(F(\lambda_j) = 0\) and

\[
|F(z)|^2 = \left| \int_{\lambda_j}^{z} F'(x) dx \right|^2 \leq |z - \lambda_j| \int_{\lambda_j}^{z} |F'(x)|^2 dx \leq |z - \lambda_j| \int_{I_j} |F'(x)|^2 dx
\]

for \(z \in I_j \subseteq (\lambda_j, \lambda_j + c)\). Thus we have

\[
\int_{I_j} |F(z)|^2 dz \leq \int_{I_j} |z - \lambda_j| dz \int_{I_j} |F'(x)|^2 dx \\
\leq \int_{(\lambda_j, \lambda_j + c)} |z - \lambda_j| dz \int_{I_j} |F'(x)|^2 dx \\
= \left( \frac{c^2}{2} \right) \int_{I_j} |F'(x)|^2 dx.
\]

Combining this with (45), we further have

\[
\int_{B} |x_1 - x_2||F(x_1)||F(x_2)| dx_1 dx_2 \\
\leq c^2 \int_{B_1} |F(x_1)|^2 dx_1 = c^2 \sum_{j=1}^m \int_{I_j} |F(x_1)|^2 dx_1 \\
\leq c^2 \sum_{j=1}^m \left( \frac{c^2}{2} \right) \int_{I_j} |F'(x)|^2 dx = \left( \frac{c^4}{2} \right) \int_{B_1} |F'(x)|^2 dx \\
\leq \left( \frac{c^4}{2} \right) \int_{\mathbb{R}} |F'(x)|^2 dx \leq \left( \frac{c^4}{2} \right) (2n) \int_{\mathbb{R}} |F(x)|^2 dx = n c^4 \int_{\mathbb{R}} |F(x)|^2 dx,
\]

which is (49). This completes the proof. \(\square\)

We also need the following lemma which gives more precise meaning to the claim that there are no successive small gaps.

**Lemma 10.** For \(A = (0, c_0)\) and \(n > 2c_0^2 + 2\), we have

\[
P(\tilde{\chi}^{(n, 2)}(A) > 0) \leq \frac{c_0^4}{(8n)}.
\]
Proof. If \( \tilde{\chi}^{(n,2)}(A) > 0 \), then there exist distinct \( i, j, k \) such that \( \lambda_j, \lambda_k \in (\lambda_i, \lambda_i + c_0/n) \). We define

\[
\Lambda_{j,k,c} := \left\{ (\lambda_1, \ldots, \lambda_n) : \exists i \in \mathbb{Z} \cap [1, n] \text{ s.t. } \lambda_j, \lambda_k \in (\lambda_i, \lambda_i + c) \right\},
\]

then we have

\[
\mathbb{P}(\tilde{\chi}^{(n,2)}(A) > 0) \leq \sum_{1 \leq j < k \leq n} \mathbb{P}((\lambda_1, \ldots, \lambda_n) \in \Lambda_{j,k,c_0/n}) = \mathbb{P}((\lambda_1, \ldots, \lambda_n) \in \Lambda_{n-1,n,c_0/n}) n(n - 1)/2.
\]

For fixed \( (\lambda_1, \ldots, \lambda_{n-2}) \in \mathbb{R}^{n-2}, c > 0 \), as in Lemma 9, we define

\[
B(\lambda_1, \ldots, \lambda_{n-2}, c) := \bigcup_{i=1}^{n-2} (\lambda_i, \lambda_i + c)^2 \subset \mathbb{R}^2,
\]

then \( (\lambda_1, \ldots, \lambda_n) \in \Lambda_{n-1,n,c} \) is equivalent to \( (\lambda_{n-1}, \lambda_n) \in B(\lambda_1, \ldots, \lambda_{n-2}, c) \). Hence, we have

\[
\mathbb{P}((\lambda_1, \ldots, \lambda_n) \in \Lambda_{n-1,n,c_0/n}) = \frac{1}{G_n} \int_{\Lambda_{n-1,n,c_0/n}} e^{-\frac{1}{2n} \sum_{i=1}^{n/2} \lambda_i^2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j| d\lambda_1 \cdots d\lambda_n
\]

\[
= \frac{1}{G_n} \int_{\mathbb{R}^{n-2}} d\lambda_1 \cdots d\lambda_{n-2} e^{-\frac{1}{2n} \sum_{i=1}^{n-2} \lambda_i^2} \prod_{1 \leq j < k \leq n-2} |\lambda_j - \lambda_k| \times \int_{B(\lambda_1, \ldots, \lambda_{n-2}, c_0/n)} |x_1 - x_2| e^{-x_1^2/2-x_2^2/2} \prod_{i=1}^{n-2} \prod_{j=1}^{n-2} |x_i - \lambda_j| dx_1 dx_2. \quad (54)
\]

With \( c = c_0/n > 0 \), we have \( 2nc^2 = 2c_0^2/n \in (0, 1) \) by assumption, then by (49) and (44) in Lemma 9 we have

\[
\int_{B(\lambda_1, \ldots, \lambda_{n-2}, c_0/n)} |x_1 - x_2| e^{-x_1^2/2-x_2^2/2} \prod_{i=1}^{n-2} \prod_{j=1}^{n-2} |x_i - \lambda_j| dx_1 dx_2
\]

\[
\leq n(c_0/n)^4 \int_{\mathbb{R}} e^{-x^2} \prod_{j=1}^{n-2} |x - \lambda_j|^2 dx.
\]

Inserting this into (54) we have

\[
\mathbb{P}((\lambda_1, \ldots, \lambda_n) \in \Lambda_{n-1,n,c_0/n}) \leq \frac{n(c_0/n)^4}{G_n} \int_{\mathbb{R}^{n-2}} d\lambda_1 \cdots d\lambda_{n-2} e^{-\frac{1}{2n} \sum_{i=1}^{n-2} \lambda_i^2} \prod_{1 \leq j < k \leq n-2} |\lambda_j - \lambda_k|
\]
\[ \times \int_{\mathbb{R}} e^{-x^2} \prod_{j=1}^{n-2} |x - \lambda_j|^2 \, dx \]
\[ = \frac{n(c_0/n)^4}{G_n} G_{n-2,1} = \frac{n(c_0/n)^4}{4} , \]
where we used Lemma 1 with \( k = 1 \) in the last step. Therefore, we have
\[ \mathbb{P}(\tilde{\chi}^{(n,2)}(A) > 0) \leq \mathbb{P}((\lambda_1, \ldots, \lambda_n) \in \Lambda_{n-1,n,c_0/n}) n(n - 1)/2 \]
\[ \leq \frac{n(c_0/n)^4}{4} \frac{n^2}{2} = \frac{c_0^4}{8n} . \]
This completes the proof. \( \Box \)

Now we can give the proof of Lemma 8 using Lemma 10.

**Proof.** Let \( c_0 \) be such that \( A \subset (0, c_0) \) and \( A_1 = (0, c_0) \). Then \( \chi^{(n)}(A) - \tilde{\chi}^{(n)}(A) \neq 0 \) implies \( \tilde{\chi}^{(n,j)}(A) > 0 \) for some \( j > 1 \) and thus we must have \( \tilde{\chi}^{(n,2)}(A_1) \geq \tilde{\chi}^{(n,j)}(A_1) \geq \tilde{\chi}^{(n,j)}(A) > 0 \). For \( n > 2c_0^2 + 2 \), by Lemma 10 we deduce that
\[ \mathbb{P}(\chi^{(n)}(A) - \tilde{\chi}^{(n)}(A) \neq 0) \leq \mathbb{P}((\tilde{\chi}^{(n,2)}(A_1) > 0) \leq \frac{c_0^4}{(8n)} \to 0 , \]
which completes the proof. \( \Box \)

### 4 Proof of Theorem 1

#### 4.1 Integral inequalities.

In this subsection, we will prove several useful inequalities regarding the two-component log-gas, which is one of the crucial steps in proving the main result.

Let \( A = (0, c_0) \), \( n > 2k \), by the definition of \( \rho^{(k,n)} \), we have
\[ \mathbb{E}(\rho^{(k,n)}(A^k)) = \frac{n!}{(n - 2k)!2^k G_n} \int_{\Sigma_{n,k,c_0/n}} |J_n(\lambda_1, \ldots, \lambda_n)| \, d\lambda_1 \cdots d\lambda_n , \tag{55} \]
where \( J_n \) is defined in (16) and
\[ \Sigma_{n,k,c} = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : |\lambda_j - \lambda_{j-k}| < c, \forall n - k < j \leq n \right\} , \tag{56} \]
i.e., \( \Sigma_{n,k,c} \) is the set \((\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) with \( k \) pairs \((\lambda_j, \lambda_{j-k})\) such that \(|\lambda_j - \lambda_{j-k}| < c\).

For \( 0 \leq l \leq k \), we define the following integral of the two-component log-gas
\[ E_{n,k,l}(c) := \int_{\Sigma_{n-l,k-l,c}} \, d\lambda_1 \cdots d\lambda_{n-l} e^{-\sum_{i=1}^{n-l} q_i \lambda_i^2/2} \prod_{j \leq m} |\lambda_j - \lambda_m|^{q_j q_m} \bigg|_{q_s = 1 + \chi(s \leq l)} , \]
where $\Sigma_{n-l,k-l,c}$ is defined via (56). By definition of $G_{n_1,n_2}$ (recall (25)), we first have

$$E_{n,k,k}(c) = G_{n-2k,k}.$$  

We also have

$$E_{n,k,0}(c) = \int_{\Sigma_{n,k,c}} |J_n(\lambda_1, \ldots, \lambda_n)| d\lambda_1 \cdots d\lambda_n,$$  

which implies

$$E(\rho^{(k,n)}(A^k)) = \frac{n!}{(n-2k)!2^kG_n} E_{n,k,0}(c_0/n).$$  

We will show that (for $0 < 2nc^2 < 1$)

$$(1 - nc^2)e^2 \leq \frac{E_{n,k,l-1}(c)}{E_{n,k,l}(c)} \leq e^2.$$  

In fact, after changing the order of variables, we can rewrite

$$E_{n,k,l-1}(c) = \int_{\Sigma_{n-l,k-l,c}} d\lambda_1 \cdots d\lambda_{n-l-1} e^{-\frac{n-l-1}{2} \sum_{i=1}^{n-l-1} q_i \lambda_i^2} \prod_{1 \leq j < m \leq n-l-1} |\lambda_j - \lambda_m|^{q_j q_m}$$

$$\times \int_{\mathbb{R}} dx_1 \int_{x_1-c}^{x_1+c} dx_2 |x_1 - x_2| e^{-x_1^2/2 - x_2^2/2} \prod_{j=1}^{n-l-1} |x_j - \lambda_m|^{q_m} |q_s = 1 + \chi_{s \leq l-1}|,$$

and

$$E_{n,k,l}(c) = \int_{\Sigma_{n-l,k-l,c}} d\lambda_1 \cdots d\lambda_{n-l-1} e^{-\frac{n-l-1}{2} \sum_{i=1}^{n-l-1} q_i \lambda_i^2} \prod_{1 \leq j < m \leq n-l-1} |\lambda_j - \lambda_m|^{q_j q_m}$$

$$\times \int_{\mathbb{R}} dx_1 e^{-x_1^2} \prod_{m=1}^{n-l-1} |x_1 - \lambda_m|^{2q_m} |q_s = 1 + \chi_{s \leq l-1}|.$$

Then (59) follows from (46) by taking

$$F(x) = e^{-x^2/2} \prod_{j=1}^{n-l-1} |x - \lambda_m|^{q_m}.$$  

By (59) we will have

$$E_{n,k,l}(c) \leq (c^2)^{k-l} E_{n,k,k}(c) = e^{2(k-l)} G_{n-2k,k}.$$  

(61)
For $n > 2k$, given any bounded interval $A$, we define
\[
\Sigma_{n,k,A} = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : |\lambda_j - \lambda_{j-k}| \in A, \forall n-k < j \leq n \right\}.
\] (62)

For $0 \leq l \leq k$, we define
\[
E_{n,k,l}(A) := \int_{\Sigma_{n-k-l,k-l}} d\lambda_1 \cdots d\lambda_{n-l} e^{\frac{-n-l}{2} \sum_{i=1}^{n-l} q_i \lambda_i^2} \prod_{1 \leq j < m \leq n-l} |\lambda_j - \lambda_m|^{q_j q_m}
\]
where $q_s = 1 + \chi_{\{0 < s \leq l\}}$ and $\Sigma_{n-k-l,k-l}$ is defined via (62). Then we have
\[
E_{n,k,0}(A) = \int_{\Sigma_{n,k}} J_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n
\] (63)
and
\[
E_{n,k,k}(A) = G_{n-2k,k}.
\] (64)

We also need the following lemma.

**Lemma 11.** If $A \subset (0, c_1), 2nc_1^2 \in (0, 1)$, $n > 2k$, $n, k$ are positive integers, then we have
\[
(1 - nc_1^2)^k \left( 2 \int_A udu \right)^k G_{n-2k,k} \leq E_{n,k,0}(A) \leq \left( 2 \int_A udu \right)^k G_{n-2k,k}.
\]

**Proof.** Let $A_1 = A \cup (-A)$, after changing the order of variables, we can rewrite
\[
E_{n,k,l-1}(A) = \int_{\Sigma_{n-k-l,k-l}} d\lambda_1 \cdots d\lambda_{n-l-1} e^{\frac{-n-l-1}{2} \sum_{i=1}^{n-l-1} q_i \lambda_i^2} \prod_{1 \leq j < m \leq n-l-1} |\lambda_j - \lambda_m|^{q_j q_m}
\]
\[
\times \int_{\mathbb{R}} dx_1 \int_{x_1 + A_1} dx_2 |x_1 - x_2| e^{-x_1^2/2 - x_2^2/2}
\]
\[
\prod_{j=1}^{2} \prod_{m=1}^{n-l-1} |x_j - \lambda_m|^{q_m} \|_{q_s = 1 + \chi_{\{s \leq l-1\}}},
\]
and
\[
E_{n,k,l}(A) = \int_{\Sigma_{n-k-l,k-l}} d\lambda_1 \cdots d\lambda_{n-l-1} \prod_{1 \leq j < m \leq n-l-1} |\lambda_j - \lambda_m|^{q_j q_m}.
\]
\[ \times \int_{\mathbb{R}} dx_1 e^{-x_1^2} \prod_{m=1}^{n-l-1} |x_1 - \lambda_m|^{2q_m} \left| q_{s=1+\chi_{x \leq t-1}} \right|. \]

Taking \( F(x) \) as in (60) again, by (48) we have
\[ (1 - nc_1^2) \cdot 2 \int_A udu \leq \frac{E_{n,k,l-1}(A)}{E_{n,k,l}(A)} \leq 2 \int_A udu, \]
and the result follows by induction and (64).

As a remark, if we combine (65), we can see that the significance of Lemma 11 is that it implies the bounds of \( \mathbb{E}(\rho^{(k,n)}(A^k)) \) by the quotient of the partition functions \( G_{n-2k,k}/G_n \).

4.2 Proof of Theorem 1. By Lemma 8 and the moment method, Theorem 1 will be proved if we can prove the following convergence of the factorial moment
\[ \lim_{n \to +\infty} \mathbb{E} \left( \frac{(\chi^{(n)}(A))!}{(\chi^{(n)}(A) - k)!} \right) = \left( \frac{1}{4} \int_A udu \right)^k \]
for every positive integer \( k \) and bounded interval \( A \subset \mathbb{R}_+ \). Actually, combining Lemma 1, (66) is equivalent to

**Lemma 12.** For any bounded interval \( A \subset \mathbb{R}_+ \) and any positive integer \( k \geq 1 \), we have
\[ \mathbb{E} \left( \frac{(\chi^{(n)}(A))!}{(\chi^{(n)}(A) - k)!} \right) - \left( \frac{1}{4} \int_A udu \right)^k \frac{G_{n-2k,k}}{G_n} \to 0 \]
as \( n \to +\infty \).

We will first use Lemma 7 to prove that
\[ \lim_{n \to +\infty} \left( \mathbb{E} \frac{(\chi^{(n)}(A))!}{(\chi^{(n)}(A) - k)!} - \mathbb{E}(\rho^{(k,n)}(A^k)) \right) = 0, \] (67)
and then use Lemma 11 to prove that
\[ \lim_{n \to +\infty} \left( \mathbb{E}(\rho^{(k,n)}(A^k)) - \left( \int_A udu \right)^k \frac{G_{n-2k,k}}{G_n} \right) = 0, \] (68)
then Lemma 12 follows from (67) and (68), and hence we complete the proof of Theorem 1.

For the rest of the article, for any bounded interval \( A \subset \mathbb{R}_+ \), let \( c_1 \) be such that \( A \subset (0, c_1) \), and \( A_1 = (0, 2c_1) \), then \( A \subset A_1 \); we denote \( c_n = c_1/n \), then \( 2n(2c_n)^2 = 8n^{-1}c_1^2 \in (0, 1) \) for \( n \) large enough. By (58), (61) with \( l = 0 \) and Lemma 1, we have
\[ \mathbb{E}(\rho^{(k,n)}(A_1^k)) = \frac{n!}{(n-2k)!2^k} \frac{E_{n,k,0}(2c_n)}{G_n} \]
\begin{align*}
&\leq \frac{n!}{(n-2k)!2^k} \frac{G_{n-2k,k}}{G_n} (2c_n)^{2k} \leq \frac{n^{2k}}{2^k} 2^{-2k} \left( \frac{2c_1}{n} \right)^{2k} \\
&= 2^{-k} c_1^{2k}.
\end{align*}

Let $a$ be defined in Lemma 7, then we have
\begin{align*}
\rho^{(k,n)}(A_1^k) &\geq \frac{(a+1)!}{(a+1-2k)!2^k} \geq \frac{(a+1-2k)^{2k}}{2^k},
\end{align*}
and hence
\begin{align*}
\mathbb{E}(a+1-2k)^{2k} &\leq 2^k \mathbb{E} \rho^{(k,n)}(A_1^k) \leq c_1^{2k},
\end{align*}
here we denote $f_+ := \max(f, 0)$. Since $a, k \in \mathbb{Z}, a \geq 0, k \geq 1$, by Hölder’s inequality we have
\begin{align*}
\mathbb{E}(a+1-2k)^{k} &\leq (\mathbb{E}(a+1-2k)^{2k})^{\frac{1}{2}} (\mathbb{P}(a \geq 2))^{\frac{1}{2}} \leq c_1^{k} (\mathbb{P}(a \geq 2))^\frac{1}{2}.
\end{align*}
Moreover, we have
\begin{align*}
(a-1)_+ &\leq \max (2(a+1-2k)_+, (4k-4)\chi_{a \geq 2}),
\end{align*}
and thus
\begin{align*}
(a-1)_+^k &\leq 2^k (a+1-2k)_+^k + (4k-4)^k \chi_{a \geq 2},
\end{align*}
hence, we have
\begin{align*}
\mathbb{E}(a-1)_+^k &\leq 2^k \mathbb{E}(a+1-2k)_+^k + (4k-4)^k \mathbb{P}(a \geq 2) \\
&\leq 2^k c_1^{k} (\mathbb{P}(a \geq 2))^\frac{1}{2} + (4k-4)^k \mathbb{P}(a \geq 2).
\end{align*}

On the other hand, $a \geq 2$ is equivalent to $\tilde{\chi}^{(n,2)}(A_1) > 0$, by Lemma 10 we have
\begin{align*}
\mathbb{P}(a \geq 2) = \mathbb{P}(\tilde{\chi}^{(n,2)}(A_1) > 0) \leq 2c_1^4/n \to 0,
\end{align*}
and thus we further have
\begin{align*}
\lim_{n \to +\infty} \mathbb{E}(a-1)_+^k = 0.
\end{align*}

By (41) in Lemma 7 we have
\begin{align*}
(\tilde{\chi}^{(n)}(A))^k &\leq 2\rho^{(k,n)}(A^k) \text{ or } (\tilde{\chi}^{(n)}(A))^k \leq 2k(k-1)a(\tilde{\chi}^{(n)}(A))^{k-1},
\end{align*}
therefore,
\begin{align*}
(\tilde{\chi}^{(n)}(A))^k &\leq \max(2\rho^{(k,n)}(A^k), (2k(k-1)a)^k),
\end{align*}
and thus we have
\begin{align*}
\mathbb{E}(\tilde{\chi}^{(n)}(A))^k &\leq 2\mathbb{E}(\rho^{(k,n)}(A^k)) + (2k(k-1))^{k}\mathbb{E}(a^k).
\end{align*}
By (69), (70) and the fact that $E(\rho^{(k,n)}(A^k)) \leq E(\rho^{(k,n)}(A_1^k))$ since $A \subset A_1$, we further have
\[
\limsup_{n \to +\infty} E(\tilde{\chi}^{(n)}(A)) < +\infty. \tag{71}
\]
Note that (67) is clearly true for $k = 1$ by definitions. For $k \geq 2$, by (40) in Lemma 7, Hölder’s inequality, (70) and (71), we have
\[
0 \leq E\left(\frac{(\tilde{\chi}^{(n)}(A))!}{(\tilde{\chi}^{(n)}(A)) - k}! - \rho^{(k,n)}(A^k)\right)
\leq k(k - 1)E((a - 1)_+ (\tilde{\chi}^{(n)}(A))^{k-1})
\leq k(k - 1)(E((a - 1)^k))^{1/k}(E(\tilde{\chi}^{(n)}(A)^k))^{1 - 1/k} \to 0
\]
as $n \to +\infty$, which finishes the proof of (67).

Now we prove (68). By (65) and changing of variables, we have
\[
E(\rho^{(k,n)}(A)) = \left(\int_A u \, du\right)^k \frac{G_{n-2k,k}}{G_n}
= \frac{n!}{(n - 2k)!2^k} \frac{E_{n,k,0}(A/n)}{G_n} - \left(\int_{A/n} u \, du\right)^k \frac{n^{2k}G_{n-2k,k}}{G_n}
= \frac{n^{2k}}{2^kG_n} \left(E_{n,k,0}(A/n) - \left(2\int_{A/n} u \, du\right)^k G_{n-2k,k}\right)
- \left(n^{2k} - \frac{n!}{(n - 2k)!}\right) \frac{E_{n,k,0}(A/n)}{2^kG_n}.
\]
We first notice that
\[
0 \leq n^{2k} - \frac{n!}{(n - 2k)!} = n^{2k} - \prod_{j=0}^{2k-1} (n - j) = n^{2k} - n^{2k} \prod_{j=0}^{2k-1} (1 - j/n)
\leq n^{2k} - n^{2k} \left(1 - \sum_{j=0}^{2k-1} j/n\right) = n^{2k} \sum_{j=0}^{2k-1} j/n = n^{2k-1}k(2k - 1).
\]
We also have $A/n \subset (0, c_1/n)$ and $2n(c_1/n)^2 \in (0, 1)$ for $n$ large enough, then by (57), (61) and (63), we have
\[
0 \leq E_{n,k,0}(A/n) \leq E_{n,k,0}(c_1/n) \leq G_{n-2k,k}(c_1/n)^{2k}.
\]
Therefore, using Lemma 1 we have
\[
0 \leq \left(n^{2k} - \frac{n!}{(n - 2k)!}\right) \frac{E_{n,k,0}(A/n)}{2^kG_n}.
\]
\[
\leq n^{2k-1}k(2k - 1) \frac{G_{n-2k,k}}{2^k G_n} (c_1/n)^{2k}
\]
\[
= n^{-1}k(2k - 1)2^{-3k}c_1^{2k}.
\]

By Lemmas 1 and 11, we have

\[
\frac{n^{2k}}{2^k G_n} \left| E_{n,k,0}(A/n) - \left(2 \int_{A/n} u du \right)^k G_{n-2k,k}\right|
\]
\[
\leq \frac{n^{2k}}{2^k G_n} (1 - (1 - n(c_1/n)^2)^k) \left(2 \int_{A/n} u du \right)^k G_{n-2k,k}
\]
\[
\leq \frac{n^{2k}}{2^k G_n} (kn(c_1/n)^2) \left(2 \int_{c_1/n}^0 u du \right)^k G_{n-2k,k}
\]
\[
= \frac{n^{2k}}{2^k G_n} (kc_1^2/n)(c_1/n)^{2k} G_{n-2k,k}
\]
\[
= \frac{G_{n-2k,k}}{2^k G_n} (kc_1^{2k+2}/n) = \frac{k c_1^{2k+2}}{2^3 k n}.
\]

Therefore, we have

\[
\left| \mathbb{E} \left( \rho^{(k,n)}(A) \right) - \left( \int_A u du \right)^k \frac{G_{n-2k,k}}{G_n} \right| \leq \frac{k c_1^{2k+2} + k(2k - 1)c_1^{2k}}{2^3 k n},
\]

which implies (68). Therefore, we finish the proof of Lemma 12 and thus the whole proof of Theorem 1.

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