Generalized Linear Gaussian Cluster-Weighted Modeling

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Abstract

Cluster-Weighted Modeling (CWM) is a flexible mixture approach for modeling the joint probability of data coming from a heterogeneous population as a weighted sum of the products of marginal distributions and conditional distributions. In this paper, we introduce a wide family of Cluster Weighted models in which the conditional distributions are assumed to belong to the exponential family with canonical links which will be referred to as Generalized Linear Gaussian Cluster Weighted Models. Moreover, we show that, in a suitable sense, mixtures of generalized linear models can be considered as nested in Generalized Linear Gaussian Cluster Weighted Models. The proposal is illustrated through many numerical studies based on both simulated and real data sets.

Keywords: Cluster-Weighted Modeling, Mixture Models, Model-Based Clustering, Generalized Linear models.

1. Introduction

Let \((X', Y)\)' be a random vector, defined on \(\Omega\), composed by a \(d\)-dimensional explanatory variable \(X\) and a unidimensional response variable \(Y\). Let \(p(x, y)\) be the joint distribution of \((X', Y)\)'\textsuperscript{'}\textsuperscript{'.} Suppose that \(\Omega\) can be partitioned into \(G\) groups, say \(\Omega_1, \ldots, \Omega_G\). Cluster-Weighted Modeling (CWM) represents a convenient mixture approach for modeling data of this type. Indeed, in this approach, the joint probability of \((X', Y)\)' can be written as

\[
p(x, y; \theta) = \sum_{g=1}^{G} p(y|x, \Omega_g)p(x|\Omega_g)p(\Omega_g) = \sum_{g=1}^{G} p(y|x; \xi_g)p(x; \psi_g)\pi_g, \quad (1)
\]
where \( p(y|x, \Omega_g) = p(y|x; \xi_g) \) is the conditional distribution of \( Y \) given \( x \) in \( \Omega_g \) (depending on some parameters \( \xi_g \)), \( p(x|\Omega_g) = p(x; \psi_g) \) is the probability distribution of \( x \) in \( \Omega_g \) (depending on some parameters \( \psi_g \)), \( \pi_g = p(\Omega_g) \) is the mixing weight of \( \Omega_g \) (\( \pi_g > 0 \) and \( \sum_g \pi_g = 1 \)), and \( \theta = \{ \xi_g, \psi_g, \pi_g; g = 1, \ldots, G \} \) denotes the set of all model parameters.

In the original formulation of Cluster Weighted Modeling (CWM), the random vector \( (X', Y)' \) is assumed to be real-valued, see Gershenfeld (1997) and also Gershenfeld (1999), Schöner (2000) for details). Quite recently, Ingrassia et al. (2012a) reformulated CWM in a statistical setting showing that it is a quite general and flexible family of mixture models. Also, in the majority of the existence literature about CWM, a linear relationship of \( Y \) on \( x \) is considered for each \( \Omega_g \), that is

\[
Y = \mathbb{E}(Y|x, \Omega_g) + \varepsilon_g = \mu(x; \beta_g) + \varepsilon_g = \beta_{0g} + \beta_{1g}'x + \varepsilon_g, \tag{2}
\]

where \( \beta_g = (\beta_{0g}, \beta_{1g})' \), with \( \beta_g \in \mathbb{R}^{d+1} \), and \( \varepsilon_g \) is assumed to have zero mean and a finite variance. If in each \( \Omega_g \), a Gaussian distribution is assumed for both \( Y|x \) and \( X \), say \( Y|x \sim N(0, \sigma^2_g) \) and \( X \sim N(d(\mu_g, \Sigma_g) \), then the original linear Gaussian CWM model is obtained in Gershenfeld (1997). For robustness sake, Ingrassia et al. (2012a) also introduce linear \( t \) CWM, which is based on the assumption of a (multivariate) \( t \) distribution for both \( Y|x \) and \( X \) in each mixture component. Starting from this result, Ingrassia et al. (2012b) define a family of twelve linear \( t \) CWMs for model-based clustering and classification. Finally, by generalizing (2) via a polynomial relationship, Punzo (2012) defines the polynomial Gaussian CWM.

However, in many cases we have to face with modeling categorical variables depending on numerical covariates based on data coming from a heterogeneous population. For example, in effectiveness studies based on administrative data, one may be interested to estimate many different effectiveness models for certain groups of users, taking into account the characteristics of each group; in the healthcare context, one typical outcomes are mortality rate or length of stay, while in the education framework an outcome may be the success in an exam. In statistical literature, such problems are usually approached considering finite mixtures of generalized linear models (FMGLM), see e.g. McLachlan (1997), McLachlan and Peel (2000), Wedel and De Sarbo (1995).
This paper generalizes the linear Gaussian CWM by considering a generalized linear model (with canonical link) for the relationship of $Y$ on $x$ in each $\Omega_g$, $g = 1, \ldots, G$. In particular, we introduce a wide family of Cluster Weighted (CW) models in which the conditional distributions are assumed to belong to the exponential family with canonical links which will be referred to as Generalized Linear Gaussian Cluster Weighted Models (GLGCWM). We remark that while FMGLM model conditional distributions, Cluster Weighted approaches model the joint distribution as the product of marginal and conditional distributions. In this paper, we show also that, in some sense, FMGLM can be considered as nested models of GLGCWM.

Generalized linear models (with canonical link) are regression models where $Y$ is specified to be distributed according to one of the members of the exponential family. Accordingly, those models deal with dependent variables $Y$ that can be either continuous with, for example, Gaussian, gamma or inverse Gaussian distributions, or discrete with, for example, binomial or Poisson distributions. The exponential family is a very useful class of distributions and the common properties of the distributions in the class enable them to be studied simultaneously rather than as a collection of unrelated cases. GLGCWM thus affords a general framework that, in addition to encompass the linear Gaussian CWM as a special case, allow us to use the CW principle also for discrete dependent variables, which are very common in real applications (see, e.g., Wedel and De Sarbo 1993, Wedel et al. 1993, Wang and Puterman 1998, Wang et al. 1998, 2002, Yang and Lai 2005, Xiang et al. 2005).

We remark that also Gershenfeld (1999) coped with the problem of discrete set of values such as events, patterns or conditions, but they did not really model the joint probability of the dependent variable and the covariates; what they introduced in the CWM is the histogram of the probabilities of each state in the clusters (without explicit dependence on the covariates).

The remainder of the paper is organized as follows. In Section 2 we introduce the Generalized Linear Gaussian CWM (GLGCWM); in Section 3 we show some results about the relationships of the GLGCWM with mixture of generalized linear models (with and without concomitants); in Section 4 we present parameter estimation according to likelihood approach; in Section 5 we describe the EM algorithm
for parameter estimation in GLCWM; in Section 6 we introduce some measures for performance evaluation; in Section 7 we illustrate numerical studies based on both simulated and real data. Finally, conclusions and perspectives for future research are presented in Section 8.

2. The model

We consider a broad family of models in which, for each $\Omega_g$, the conditional distributions are assumed to belong to the exponential family with canonical links, that is

$$p(y|x; \xi_g) = q(y|x; \beta_g, \lambda_g) = \exp \left\{ \frac{y \eta(x; \beta_g) - b[\eta(x; \beta_g)]}{a(\lambda_g)} + c(y, \lambda_g) \right\},$$

(3)

where $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are specified functions, $\lambda_g$ is the dispersion parameter with $a(\lambda_g) > 0$, and $\eta(x; \beta_g) = \eta_g = \beta_0 + \beta_1^T x$ is the canonical function (see, e.g., Wedel and De Sarbo 1995 and McCullagh and Nelder 2000). In particular, $b(\cdot)$ is the cumulant function, while $a(\cdot)$ and $b(\cdot)$ satisfy

$$\mu_g = \mathbb{E}(Y|x; \beta_g, \lambda_g) = b'(\eta_g) \quad \text{and} \quad \sigma^2_g = \mathbb{V}(Y|x; \beta_g, \lambda_g) = a(\lambda_g)b''(\eta_g),$$

where $b'(\eta_g)$ and $b''(\eta_g)$ are the first and second derivatives of $b(\eta_g)$ with respect to $\eta_g$, respectively. Moreover, $\eta_g$ is related to the expected value $\mu(x; \beta_g) = \mu_g$, through a link function $h(\cdot)$, in the following way

$$\eta_g = h(\mu_g).$$

For sake of simplicity, we assume Gaussian distributions for marginals, i.e. $X|\Omega_g \sim N_d(\mu_g, \Sigma_g)$. Thus, equation (1) becomes

$$p(x, y; \theta) = \sum_{g=1}^{G} q(y|x; \beta_g, \lambda_g) \phi_d(x; \mu_g, \Sigma_g) \pi_g.$$  

(4)

Model (4) will be referred to as generalized linear Gaussian CWM (GLGCWM) hereafter. Canonical links are the identity, log, logit, inverse and squared inverse functions for the normal, Poisson, binomial, gamma and inverse Gaussian distributions (see McCullagh and Nelder 2000, Table 2.1). Thus, all these distributions can be taken into
account for modeling the $Y$ variable in each $\Omega_g$. In particular, the Poisson and the Binomial distributions are quite useful because they allow the CW principle to be applied also for discrete response variables $Y$; this is the reason why, in the following, we shall focus mainly on such distributions.

It is interesting to note that, from a clustering/classification point of view, the posterior probabilities of group membership $p(\Omega_g|x, y; \theta) = p(y|x; \beta_g, \lambda_g)p(x; \mu_g, \Sigma_g)p_g$, $g = 1, \ldots, G$, (5) depend on both the marginal and conditional component distributions, differently from the standard mixture models.

### 2.1. The Binomial Gaussian CWM

Assume that $Y$ takes values in $\{0, 1, \ldots, M\}$, for some $M \in \mathbb{N}$. Moreover, assume that the probability mass function of $Y|x$ in $\Omega_g$ is binomial with parameters $(M, \gamma_g(x))$, that is $Y|x, \Omega_g \sim \text{Bin}(M, \gamma_g(x))$. In this case

$$q(y|x; M, \beta_0g, \beta_g) = \binom{M}{y} \gamma_g(x)^y [1 - \gamma_g(x)]^{M-y},$$

where the probability $\gamma_g(x)$ is postulated to depend on $x$ through the function

$$\gamma_g(x) = \frac{\exp(\beta_0g + \beta_g'x)}{1 + \exp(\beta_0g + \beta_g'x)} \quad \text{or, equivalently,} \quad \ln \left( \frac{\gamma_g(x)}{1 - \gamma_g(x)} \right) = \beta_0g + \beta_g'x,$$

(7)

with $\beta_0g \in \mathbb{R}$ and $\beta_g \in \mathbb{R}^d$ being parameters to be estimated. When (6) is considered in (5), we have Binomial Gaussian CWM or, more simply, Binomial CWM. Then, Bernoulli CWM results, as a special case, when $M = 1$.

### 2.2. The Poisson Gaussian CWM

Assume that $Y$ takes values in $\mathbb{N}$. Moreover, assume that the probability mass function of $Y|x$ in $\Omega_g$ is Poisson with parameter $\lambda_g(x)$, that is $Y|x, \Omega_g \sim \text{Poi}(\lambda_g(x))$. In this case

$$q(y|x; \beta_0g, \beta_g) = \exp [-\lambda_g(x)] \frac{[\lambda_g(x)]^y}{y!},$$

(8)
where $\lambda_g(x)$ is postulated to depend on $x$ through the function

$$
\lambda_g(x) = \exp(\beta_{0g} + \beta_g' x) \quad \text{or, equivalently,} \quad \ln[\lambda_g(x)] = \beta_{0g} + \beta_g' x, \quad (9)
$$

with $\beta_{0g} \in \mathbb{R}$ and $\beta_g \in \mathbb{R}^d$ being parameters to be estimated. When (8) is considered in (1), we have Poisson Gaussian CWM or, more simply, Poisson CWM.

3. Relationships with finite mixtures of generalized linear models

In this section, we extend results given in Ingrassia et al. (2012a), Ingrassia et al. (2012b) to the generalized linear Gaussian CWM. Proofs are given in the Appendix.

Let

$$
f(y|x; \kappa) = \sum_{g=1}^{G} q(y|x; \beta_g, \lambda_g) \pi_g \quad (10)
$$

be a finite mixture of generalized linear model (FMGLM), see e.g. McLachlan (1997), McLachlan and Peel (2000), Wedel and De Sarbo (1995) where $\kappa = \{\beta_g, \lambda_g, \pi_g; g = 1, \ldots, G\}$ denotes the overall parameters of the model. According to this model, the posterior probability that the generic observation $(x', y')$ belongs to $\Omega_g$ is

$$
p(\Omega_g|x, y) = \frac{q(y|x; \beta_g, \lambda_g) \pi_g}{f(y|x; \kappa)}, \quad g = 1, \ldots, G. \quad (11)
$$

**Proposition 1.** If the marginal distributions $\phi_d(x; \mu_g, \Sigma_g)$ of $X|\Omega_g$ do not depend on $\Omega_g$, i.e. $\mu_g, \Sigma_g = \mu, \Sigma, g = 1, \ldots, G$, then model (4) becomes

$$
p(x, y; \theta) = \phi_d(x; \mu, \Sigma) f(y|x; \kappa),
$$

where $f(y|x; \kappa)$ was defined in (10).

**Corollary 2 (from Proposition 1).** Under the assumptions of Proposition 1, the posterior probability that the generic observation $(x', y')$ belongs to $\Omega_g$ from model (4) coincides with (11).

**Remark.** We remark that result in Proposition 1 holds in a quite more general context. As a matter of fact, the proof does not require any distributional assumption on the
marginal distributions, but it needs only that the marginal distributions \( X \mid \Omega_g \) do not depend on group \( g \), i.e. \( \psi_g = \psi \) for every \( g = 1, \ldots, G \). Thus, the model (1) yields:

\[
p(x, y; \theta) = p(x; \psi) f(y|x; \kappa),
\]

where \( f(y|x; \phi) \) has been defined in (10).

An extension of model (10) concerns the case with concomitant variable; this leads to finite mixtures of GLMs with concomitant variables (FMGLMC), see Grün and Leisch (2008)

\[
f(y|x; \varphi) = \sum_{g=1}^{G} q(y|x; \beta_g, \lambda_g) p(\Omega_g|x; \alpha_g), \tag{12}
\]

where the mixing weight \( p(\Omega_g|x; \alpha_g) \) is now a function of \( x \) through some parameters \( \alpha \), and \( \varphi \) denotes the overall parameters of the model. Note that, in the general formulation of model (12) given by Grün and Leisch (2008, p. 3), the concomitant variables could be also exogenous to \( X \). The probability \( p(\Omega_g|x; \alpha_g) \) is usually modeled by a multinomial logistic distribution with the first component as baseline, that is

\[
p(\Omega_g|x; \alpha_g) = \frac{\exp(\alpha_{g0} + \alpha'_{g1} x)}{\sum_{j=1}^{G} \exp(\alpha_{j0} + \alpha'_{j1} x)}, \tag{13}
\]

where \( \alpha_g = (\alpha_{g0}, \alpha'_{g1})' \). According to model (12), with the specification of \( p(\Omega_g|x; \alpha_g) \) given in (13), the posterior probability that the generic observation \((x', y')\) belongs to \( \Omega_g \) is

\[
p(\Omega_g|x, y) = \frac{q(y|x; \beta_g, \lambda_g) p(\Omega_g|x; \alpha_g)}{\sum_{j=1}^{G} q(y|x; \beta_j, \lambda_j) p(\Omega_j|x; \alpha_j)} = \frac{q(y|x; \beta_g, \lambda_g) \exp(\alpha_{g0} + \alpha'_{g1} x)}{\sum_{j=1}^{G} q(y|x; \beta_j, \lambda_j) \exp(\alpha_{j0} + \alpha'_{j1} x)}. \tag{14}
\]

**Proposition 3.** If in (4) it results \( X \mid \Omega_g \sim N_d(\mu_g, \Sigma) \) and \( \pi_g = \pi = 1/G, g = 1, \ldots, G \), then

\[
p(x, y; \theta) = p(x; \psi) f(y|x; \varphi),
\]

where \( f(y|x; \varphi) \) is defined in (12) through (13) and \( p(x; \psi) = 1/G \sum_{g=1}^{G} \phi_d(x; \mu_g, \Sigma) \).
Corollary 4 (from Proposition 4). Under the assumptions of Proposition 4, the posterior probability that the generic observation \((x', y')\)' belongs to \(\Omega_g\) from model (4) coincides with (14).

4. Likelihood function and parameter estimation

Let \((x_1, y_1), \ldots, (x_N, y_N)\) be a sample of \(N\) independent observation pairs drawn from model in (1) and set \(X = (x'_1, \ldots, x'_n), Y = (y_1, \ldots, y_n)\). The likelihood function of the generalized linear Gaussian CWM (4) is given by:

\[
L_0(\theta; X, y) = \prod_{n=1}^{N} p(x_n, y_n; \theta) = \prod_{n=1}^{N} \left[ \sum_{g=1}^{G} q(y_n | x_n; \beta_g, \lambda_g) \phi_d(x_n; \mu_g, \Sigma_g) \pi_g \right].
\]

Maximization of \(L_0(\theta; X, y)\) with respect to \(\theta\) yields the maximum likelihood estimate of \(\theta\). Previous results under Gaussian assumptions have been presented in Ingrassia and Minotti (2012). Let us consider fully categorized data:

\[
\{w_n : n = 1, \ldots, N\} = \{(x_n, y_n, z_n) : n = 1, \ldots, N\},
\]

where \(z_n = (z_{n1}, \ldots, z_{ng})'\), with \(z_{ng} = 1\) if \((x_n, y_n)\) comes from the \(g\)-th population and \(z_{ng} = 0\) otherwise. Then, the complete-data likelihood function corresponding to \(W = (w_1, \ldots, w_N)\) can be written in the form:

\[
L_c(\theta; X, y) = \prod_{n=1}^{N} \prod_{g=1}^{G} \left[ q(y_n | x_n; \beta_g, \lambda_g) \right]^{z_{ng}} \left[ \phi_d(x_n; \mu_g, \Sigma_g) \right]^{z_{ng} \pi_g}.
\] (15)

Taking the logarithm of (15) after some algebra we get:

\[
\mathcal{L}_c(\theta; X, y) = \ln L_c(\theta; X, y)
\]

\[
= \sum_{n=1}^{N} \sum_{g=1}^{G} [z_{ng} \ln q(y_n | x_n; \beta_g, \lambda_g) + z_{ng} \ln \phi_d(x_n; \mu_g, \Sigma_g) + z_{ng} \ln \pi_g]
\]

\[
= \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \ln q(y_n | x_n; \beta_g, \lambda_g) + \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \ln \phi_d(x_n; \mu_g, \Sigma_g) + \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \ln \pi_g
\]

\[
= \mathcal{L}_1(\xi) + \mathcal{L}_2(\psi) + \mathcal{L}_3(\pi),
\] (16)
where $\xi = \{\beta_g, \lambda_g : g = 1, \ldots, G\}$ and $\psi = \{\mu_g, \Sigma_g : g = 1, \ldots, G\}$. We remark that in (16) the weights $\pi$ are estimated through the posterior probability (5), see Section 5 for details.

**Relationship with the log-likelihood function of FMGLM.** In the following we show that, under suitable hypotheses, the maximization of the likelihood function of GLGCWM in (4) leads to the same parameter estimates of FMGLM in (10). Indeed, based on (10), the log-likelihood is given by

$$
\mathcal{L}_c(\theta; X, y) = \sum_{n=1}^{N} \sum_{g=1}^{G} (z_{ng} \ln q(y_n|x_n; \beta_g, \lambda_g) + z_{ng} \ln \pi_g)
$$

$$
= \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \ln q(y_n|x_n; \beta_g, \lambda_g) + \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \ln \pi_g
$$

$$
= \mathcal{L}_{1c}(\xi) + \mathcal{L}_{3c}(\pi).
$$

(17)

**Proposition 5.** In model (4), if the local densities $\phi_d(x; \mu_g, \Sigma_g)$ have the same parameters $(\mu_g, \Sigma_g) = (\mu, \Sigma)$ for $g = 1, \ldots, G$, then maximum likelihood estimate of $(\xi, \pi)$ in (17) coincides with the corresponding estimate in (16).

**Remark.** We remark that the above result can be generalized like in the previous case. As a matter of fact, the proof of Proposition 5 does not require the Gaussian assumption on the marginal distribution. For any marginal distribution $p(\cdot; \psi_g)$ (depending on some parameter $\psi_g$), if the local densities $p(x_n; \psi_g)$ have the same parameters $\psi_g = \psi$ for $g = 1, \ldots, G$, then maximum likelihood estimate of $(\xi, \pi)$ in (17) coincides with the corresponding estimate in (16).

**Relationship with the log-likelihood function of FMGLMC.** Now let us analyze the relationships with FMGLMC. In particular, we show that, under suitable hypotheses, the maximization of the likelihood function of GLGCWM in (4) leads to the same parameter estimates of FMGLMC in (12). Indeed, let us consider the density function in (12), where the mixing weights $p(\Omega_g|x, \alpha_g)$ are given in (13). The corresponding
complete-data log-likelihood function is given by:

\[
L_c(\theta; X, y) = \sum_{n=1}^{N} \sum_{g=1}^{G} \left[ z_{ng} \ln q(y_n|x_n; \beta_g, \lambda_g) + z_{ng} \ln p(\Omega_g|x, \alpha_g) \right]
\]

\[
= L_{1c}(\xi) + L_{3c}(\zeta),
\]

(18)

where \( \zeta = \{ \alpha_g ; g = 1, \ldots, G \} \).

**Proposition 6.** In model (4), assume that the local densities have the same covariance matrices, i.e. \( \phi_d(x; \mu_g, \Sigma_g) = \phi_d(x; \mu_g, \Sigma) \) and the prior probabilities be equal, i.e. \( \pi_g = 1/G \) for \( g = 1, \ldots, G \). Then, the maximum likelihood estimate of \( (\xi, \zeta) \) in (18) can be derived from the estimate of \( (\xi, \psi) \) in (16).

### 4.1. Discussion

The above results show that both models FMGLM and FMGLMC can be considered as nested models of CWM in (1) even if they have a different structure. As a matter of fact, model (10) and (12) consider only conditional distributions, while CWM considers joint distributions (as a product of marginal and conditional distributions). However, we remark that:

- if the marginal distributions \( p(x; \mu_g, \Sigma_g) \) in (1) do not depend on the \( g \)th group, i.e. \( p(x; \mu_g, \Sigma_g) = p(x; \mu, \Sigma) (g = 1, \ldots, G) \), then the estimates \( \{ \beta_g, \lambda_g, \pi_g, g = 1, \ldots, G \} \) in (1) and (10) are the same according to Proposition 5 in other words, if \( \mu_g, \Sigma_g = \mu, \Sigma (g = 1, \ldots, G) \), then the parameters of the conditional distributions in GLGCWM and FMGLM are the same. Moreover, the posterior probability (5) reduces to (11). An empirical analysis based on simulated data will be presented in Example 1 of Section 7.

- If the marginal distributions \( p(x; \mu_g, \Sigma_g) \) in (1) are Gaussian with the same covariance matrices, i.e. \( p(x; \mu_g, \Sigma_g) = \phi_d(x; \mu_g, \Sigma) \) and the mixing weights are equal, i.e. \( \pi_g = 1/G (g = 1, \ldots, G) \), then the estimates \( \{ \beta_g, \lambda_g, g = 1, \ldots, G \} \) in (1) and (12) are the same according to Proposition 6 moreover, the posterior probability (5) reduces to (14).
5. The EM algorithm for the Generalized Linear CWM

In this section, we present the main steps of the EM algorithm for parameter estimation of the Generalized Linear CWM in (4). In this case, the three terms of the complete-data log-likelihood function (16) are given by:

\[ L_{1c}(\xi) = \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \left[ \frac{y_n \eta(x_n; \beta_g) - b(\eta(x_n; \beta_g))}{a(\lambda_g)} + c(y, \lambda_g) \right] \]

\[ L_{2c}(\psi) = \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \frac{1}{2} \left[ -p \ln 2\pi - \ln |\Sigma_g| - (x_n - \mu_g)\Sigma_g^{-1}(x_n - \mu_g) \right] \]

\[ L_{3c}(\pi) = \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} [\ln \pi_g] \]

The E-step on the \((k+1)\)-th iteration of the EM algorithm requires the calculation of the conditional expectation of the complete-data log-likelihood function \(L_c(\theta; X, y)\) in (16), say \(Q(\theta, \theta^{(k)})\), evaluated using the current fit \(\theta^{(k)}\) for \(\theta\). Since \(L_c(\theta; X, y)\) is linear in the unobservable data \(z_{ng}\), this means calculating the current conditional expectation of \(Z_{ng}\) given \(X\) and \(y\), where \(Z_{ng}\) is the random variable corresponding to \(z_{ng}\), that is

\[ Q(\theta, \theta^{(k)}) = E_{\theta^{(k)}} \{ L_c(\theta; X, y) \} \]

\[ = \sum_{n=1}^{N} \sum_{g=1}^{G} E_{\theta^{(k)}} \{ Z_{ng} | x_n, y_n \} \left[ Q_1(\beta_g, \lambda_g; \theta^{(k)}) + Q_2(\mu_g, \Sigma_g; \theta^{(k)}) + \ln \pi_g \right] \]

\[ = \sum_{n=1}^{N} \sum_{g=1}^{G} \tau_{ng}^{(k)} \left[ Q_1(\beta_g, \lambda_g; \theta^{(k)}) + Q_2(\mu_g, \Sigma_g; \theta^{(k)}) + \ln \pi_g \right], \quad (19) \]

where \(Q_1(\beta_g, \lambda_g; \theta^{(k)})\) and \(Q_2(\mu_g, \Sigma_g; \theta^{(k)})\) depend on the functional form of densities \(p(y_n | x_n; \beta_g, \lambda_g)\) and \(p(x_n; \mu_g, \Sigma_g)\), respectively:

\[ Q_1(\beta_g, \lambda_g; \theta^{(k)}) = \frac{y_n \eta(x_n; \beta_g) - b(\eta(x_n; \beta_g))}{a(\lambda_g)} + c(y_n, \lambda_g), \quad (20) \]

\[ Q_2(\mu_g, \Sigma_g; \theta^{(k)}) = \frac{1}{2} \left[ -p \ln 2\pi - \ln |\Sigma_g| - (x_n - \mu_g)\Sigma_g^{-1}(x_n - \mu_g) \right]. \quad (21) \]

The M-step on the \((k+1)\)-th iteration of the EM algorithm requires the maximization of the conditional expectation of the complete-data log-likelihood \(Q(\theta, \theta^{(k)})\) with re-
spect to $\theta$. The maximization of the quantity in (20) is equivalent to the maximization problem of the generalized linear model for the complete data, except that each observation $y_n$ contributes to the log-likelihood for each group $g$ with a known weight $\tau_{ng}^{(k)}$.

The stationary equations are obtained by equating the first order partial derivatives of

$$\sum_{n=1}^{N} \sum_{g=1}^{G} \tau_{ng}^{(k)} \frac{Q_1(\beta_g, \lambda_g; \theta^{(k)})}{a(\lambda_g)}$$

and

$$\sum_{n=1}^{N} \sum_{g=1}^{G} \tau_{ng}^{(k)} \frac{Q_1(\beta_g, \lambda_g; \theta^{(k)})}{a(\lambda_g)}$$

which after some algebra yield

$$\sum_{n=1}^{N} \sum_{g=1}^{G} \tau_{ng}^{(k)} \frac{y_n - b'(\eta(x_n; \beta_g))}{a(\lambda_g)} V_{ng} \mu_g = 0$$

and

$$\sum_{n=1}^{N} \sum_{g=1}^{G} \tau_{ng}^{(k)} \frac{y_n - b'(\eta(x_n; \beta_g))}{a(\lambda_g)} \frac{\partial}{\partial \lambda_g} c(y_n, \lambda_g) = 0.$$
e.g., McLachlan and Peel (2000).

**Convergence criterion.** The convergence criterion is based on the Aitken acceleration procedure (Aitken 1926) which is used to estimate the asymptotic maximum of the log-likelihood at each iteration of the EM algorithm. Based on this estimate, a decision can be made regarding whether or not the algorithm has reached convergence; that is, whether or not the log-likelihood is sufficiently close to its estimated asymptotic value. The Aitken acceleration at iteration $k$ is given by

$$a^{(k)} = \frac{l^{(k+1)} - l^{(k)}}{l^{(k)} - l^{(k-1)}},$$

where $l^{(k+1)}$, $l^{(k)}$, and $l^{(k-1)}$ are the log-likelihood values from iterations $k + 1$, $k$, and $k - 1$, respectively. Then, the asymptotic estimate of the log-likelihood at iteration $k + 1$ is given by

$$l^{(k+1)}_\infty = l^{(k)} + \frac{1}{1 - a^{(k)}} (l^{(k+1)} - l^{(k)}),$$

see Böning et al. (1994). In the analyses in Section 7, the algorithms stopped when $l^{(k+1)}_\infty - l^{(k)} < \epsilon$, with $\epsilon = 0.05$.

### 6. Performance evaluation

In order to evaluate the performance of the models introduced in the paper, some different indices will be taken into account. They are classified according to the type of the available data. In general, model will be selected according to the BIC, see 6.1. Moreover, when data are simulated and the data labeling is known we can use indices which compare the true partition with that arising from the application of a particular model (like the misclassification error and indices described in Section 6.2). On the contrary, when we do not know the true labels, we limit our attention to goodness-of-fit indices (see Section 6.3 and Section 6.4).

#### 6.1. Model selection and performance

In our numerical studies, we considered the Bayesian information criterion (BIC) of Schwarz (1978): \[
\text{BIC} = 2l(\hat{\theta}) - m \ln N.
\]
where $\hat{\theta}$ is the ML estimate of $\theta$, $l(\hat{\theta})$ is the maximized observed-data log-likelihood, and $m$ is the overall number of free parameters in the model.

6.2. The Rand index and the adjusted Rand index

When the true classification is known, often the adjusted Rand index (ARI; Hubert and Arabie 1985) is considered as a measure of class agreement. The original Rand index (RI; Rand 1971) can be expressed as

$$RI = \frac{\text{number of agreements}}{\text{number of agreements} + \text{number of disagreements}},$$

where the number of agreements (observations that should be in the same group and are, plus those that should not be in the same group and are not) and the number of disagreements are based on pairwise comparisons. The RI assumes values between 0 and 1, where 0 indicates no pairwise agreement between the MAP classification and true group membership and 1 indicates perfect agreement.

One criticism of the RI is that its expected value is greater than 0, making smaller values difficult to interpret. The ARI corrects the RI for chance by allowing for the possibility that classification performed randomly will correctly classify some observations, see Hubert and Arabie (1985). The ARI can be expressed as

$$ARI = \frac{RI - E(RI)}{\max(RI) - E(RI)},$$

where the expected value $E(RI)$ of the Rand Index is computed considering all pairs of distinct partitions picked at random, subject to having the original number of classes and objects in each. Thus, the ARI has an expected value of 0 and perfect classification would result in a value equal to 1.

6.3. The index of conditional goodness-of-fit

We introduce a descriptive measure of goodness-of-fit which is directly related to the generalized Pearson $\chi^2$ statistic commonly used for generalized linear models. In detail, our index of conditional goodness-of-fit (CGOF) is defined as

$$CGOF = \frac{\chi^2_w}{N} = \frac{1}{N} \sum_{n=1}^{N} \sum_{g=1}^{G} \hat{z}_{ng} \left[ \frac{y_n - E(Y|x_n, \Omega_g)}{\sqrt{V(Y|x_n, \Omega_g)}} \right]^2,$$

(22)
As a special case, when $G = 1$, $\chi^2_w$ corresponds to the classical generalized Pearson $\chi^2$ statistic (see, e.g., McCullagh and Nelder 2000, p. 34 and Olsson 2006, p. 46). In (22) the quantity $\chi^2_w$ is divided by $N$ in order to remove the impact of the sample size. Furthermore, the division by the conditional variance $\mathbb{V}(Y|x_n, \Omega_g)$ makes the squared residuals $[y_n - \mathbb{E}(Y|x_n, \Omega_g)]^2$ comparable between groups.

6.4. An index of goodness-of-fit based on scaled deviance

Another goodness-of-fit criterion consists of an extension of the traditional scaled deviance for GLM (see McCullagh and Nelder 2000, p.34), which is defined as:

$$SD = D^*(y; \mu) = 2l(y; y) - 2l(\mu, y), \quad (23)$$

that is the difference between the maximum likelihood achievable in a full model and that achieved by the model under investigation. Note that

$$D^*(y; \hat{\mu}) = D(y; \hat{\mu})/\phi, \quad (24)$$

where $D(y; \hat{y})$ is the deviance for the current model, so that the scaled deviance is the deviance expressed as a multiple of the dispersion parameter.

Given the forms of the deviances for Binomial and Poisson distributions, respectively (see McCullagh and Nelder 2000, p.34), the corresponding measures for GLGCWM, called generalized deviances, can be defined as:

$$GD_{\text{Bin}} = 2 \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \left\{ y_n \log \left( \frac{y_n}{\mathbb{E}(Y|x_n, \Omega_g)} \right) + (K - y_n) \log \left( \frac{M - y_n}{M - \mathbb{E}(Y|x_n, \Omega_g)} \right) \right\} \right.$$

$$GD_{\text{Poi}} = 2 \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \left\{ y_n \log \left( \frac{y_n}{\mathbb{E}(Y|x_n, \Omega_g)} \right) - [y_n - \mathbb{E}(Y|x_n, \Omega_g)] \right\}.$$

Analogous forms are obtained for the generalized scaled deviances, by dividing the general deviances for the dispersion parameter, which is $1/M$ for the Binomial distribution and $1$ for the Poisson distribution, respectively, that is

$$GSD_{\text{Bin}} = GD_{\text{Bin}}/M$$

$$GSD_{\text{Poi}} = GD_{\text{Poi}}.$$
7. Numerical studies

In this section we illustrate some features of the models we proposed above, on the ground of numerical studies based on both artificial and real data. We aim also at a comparison with the competitive models, that is FMGLM and FMGLMC. Code for all of the analyses presented herein was written in the R computing environment (R Development Core Team 2011). We remark that the parameter estimation of FMGLM and FMGLMC has been carried out by means of the R-package flexmix, see Leisch (2004), Grün and Leisch (2008).

7.1. Simulated data

Example 1 (Estimates comparison between FMGLM and constrained GLGCWM).

In Section 4 we stated that FMGLM can be considered as nested models of GLGCWM. In particular, we showed that if \( \mu_g, \Sigma_g = \mu, \Sigma (g = 1, \ldots, G) \), then the conditional distributions in GLGCWM and FMGLM have the same estimates of \( \{ \beta_g, \lambda_g, \pi_g; g = 1, \ldots, G \} \).

Thus, first numerical simulations concern the comparison between such estimates in data modeling according to either Poisson Gaussian CWM and mixture of Poisson regressions when the marginal distributions do not depend on the \( g \)th group (in the following, this case will be referred to as constrained Poisson Gaussian CWM). Here we considered \( G = 2 \) groups.

The data have been generated as follows. As for the marginal distributions, we considered three cases: \( X \sim N(5, 0.8), X \sim \text{Unif}(4.4, 5.5), \) and \( X \sim \text{Unif}(4, 5) \). In particular, we remark that two out three cases concern non Gaussian marginal distributions. As for the conditional distributions, data have been generated according to a Poisson distribution \( \lambda_g \) with the following parameters: \( \beta_{01} = 1 \) and \( \beta_{11} = 0.2 \) in the first group, \( \beta_{02} = 0 \) and \( \beta_{12} = 0.6 \) in the second group. For each combination of the parameters, we have generated 120 random samples with sample sizes \( N_1 = 250 \) and \( N_2 = 350 \).

In each replication, once both the models were fitted to the generated data, the following index was computed to evaluate the discrepancy

\[
\sum_{i=1}^{2} \sum_{j=0}^{1} \left| \hat{\beta}_{ij}^{\text{GLGCWM}} - \hat{\beta}_{ij}^{\text{FMGLM}} \right| / 4,
\]

(25)
where $\hat{\beta}_{GLGCWM}^{ij}$ and $\hat{\beta}_{FMGLM}^{ij}$ are the ML estimates of $\beta_{ij}$ for the Poisson GCWM and the Poisson GFMR, respectively. For each of the three considered $X$-distributions, the average values, with respect to the 120 replications, of the index in (25) were respectively: 0.00131 when $X \sim N(5, 0.8)$, 0.00386 when $X \sim \text{Unif}(4.4, 5.5)$, and 0.00259 when $X \sim \text{Unif}(4, 5)$. These results underline as the constrained Poisson GCWM reveals to be a very good approximation for the Poisson GFMR, regardless from the distribution of $X$.

**Example 2 (Data from a Binomial Gaussian CWM).** The $N = 600$ artificial bivariate data of this example are referred to $G = 2$ groups of size $N_1 = 250$ and $N_2 = 350$. They are randomly generated from a Binomial Gaussian CWM, with $M = 30$, with parameters given in the first row of Table 1 (here “true parameters” means the parameters we used for data generation). Figure 1 displays the scatterplot of the data and the fitted models. Figure 2 displays the joint density from a Binomial CWM for different values of $y$. In Table 2 we present the ARIs and the misclassification errors we obtained in the three data modeling. Here, it is possible to see as the Binomial Gaussian CWM outperforms the other approaches; in particular, we remark that the Binomial MR is not able to capture the underlying group-structure of the data.

| $\pi_1$ | $\pi_2$ | $\mu_1$ | $\mu_2$ | $\sigma_1$ | $\sigma_2$ | $\beta_{01}$ | $\beta_{02}$ | $\beta_{11}$ | $\beta_{12}$ |
|-------|-------|--------|--------|----------|----------|-----------|-----------|-----------|-----------|
| true parameters | 0.417 | 0.583 | 2.000 | -2.000 | 1.000 | 1.000 | 0.000 | 0.000 | 1.000 | 1.000 |
| estimates | 0.427 | 0.573 | 1.933 | -1.986 | 0.996 | 0.936 | 0.082 | -0.059 | 0.969 | 0.941 |
| (0.028) | (0.032) | (0.074) | (0.059) | (0.114) | (0.088) | (0.081) | (0.071) | (0.047) | (0.040) |

**Table 2: Example 2. Clustering performance of the fitted Binomial-based models**

| misclassError | FMR | FMRC | CWM |
|----------------|-----|------|-----|
| 41.67% | 2.67% | 2.00% |
| ARI | -0.00078 | 0.89595 | 0.92143 |

**Example 3 (Data from a Binomial Gaussian CWM).** The $N = 250$ artificial bivariate data of this example are referred to $G = 2$ groups of size $N_1 = 100$ and
$N_2 = 150$. They are randomly generated from a Binomial Gaussian CWM, with $M = 30$, with parameters given in the first row of Table 3. Figure 1 displays the scatterplot of the data and the fitted models. In Table 4 we list the ARIs and the misclassification errors we obtained in the three data modeling. Like in Example 3, the Binomial Gaussian CWM outperforms the other approaches but, differently from the previous case, here the Binomial Gaussian CWM clearly outperforms the FMRC. Moreover, it is possible to see like the clustering results for the FMRC are the worse than those
Figure 2: Example Joint density from the fitted Binomial CWM.

Table 3: Example Parameters and estimates according to the Binomial Gaussian CWM. Standard errors are given in round brackets

|       | $\pi_1$ | $\pi_2$ | $\mu_1$ | $\mu_2$ | $\sigma_1$ | $\sigma_2$ | $\beta_{01}$ | $\beta_{02}$ | $\beta_{11}$ | $\beta_{12}$ |
|-------|---------|---------|---------|---------|------------|------------|--------------|--------------|--------------|--------------|
| true  | 0.400   | 0.600   | 0.000   | 0.000   | 1.000      | 16.000     | 0.000        | 0.000        | 2.000        | 0.500        |
| estimates | 0.388 | 0.612   | 0.069   | -0.277  | 0.969      | 14.252     | 0.081        | -0.059       | 1.952        | 0.494        |
|       | (0.044) | (0.053) | (0.106) | (0.305) | (0.159)    | (1.688)    | (0.058)      | (0.042)      | (0.089)      | (0.015)      |

obtained via FMR.

Table 4: Example Clustering performance of the fitted Binomial-based models

|       | FMR | FMRC | CWM |
|-------|-----|------|-----|
| misclassError | 19.60% | 20.80% | 7.60% |
| ARI   | 0.36436 | 0.33591 | 0.71775 |
Example 4 (Data from a Poisson GCWM). The $N = 400$ artificial bivariate data of this example are referred to $G = 2$ groups of size $N_1 = 150$ and $N_2 = 250$, respectively. They are randomly generated from a Poisson CWM with parameters specified in the first row of Table 5. Figure 4 displays the scatterplot of the data and the fitted Poisson-based models described in the paper. Figure 5 displays the joint density from a Poisson CWM. In Table 6 we get the ARI and the misclassification error we obtained in the three data modeling. Here, it is possible to see as the Poisson CWM outper-
Table 5: Example 4. Parameters and estimates according to Poisson CWM. Standard errors are given in round brackets.

| true parameters | estimates  |
|-----------------|------------|
| $\pi_1$ | 0.375 | 0.370 |
| $\pi_2$ | 0.625 | 0.630 |
| $\mu_1$ | 0.000 | -0.061 |
| $\mu_2$ | 5.000 | 5.016 |
| $\sigma_1$ | 1.500 | 1.291 |
| $\sigma_2$ | 0.800 | 0.817 |
| $\beta_{01}$ | 1.000 | 0.985 |
| $\beta_{02}$ | 0.000 | 0.005 |
| $\beta_{11}$ | 0.200 | 0.203 |
| $\beta_{12}$ | 0.500 | 0.497 |
| (true) | (estimates) |
| (0.031) | (0.040) |
| (0.100) | (0.060) |
| (0.174) | (0.082) |
| (0.051) | (0.103) |
| (0.047) | (0.019) |

Figure 4: Example 4. Scatter plots and Poisson-based models.
forms the other approaches; in particular, the Poisson FMR is not able to capture the underlying group-structure of the data.

Table 6: Clustering performance of the fitted Poisson-based models

|               | FMR  | FMRC | CWM  |
|---------------|------|------|------|
| misclassError | 37.50% | 6.50% | 0.50% |
| ARI           | -0.00378 | 0.75612 | 0.97997 |

7.2. Real data

Example 5 (Coupon redemption data). The following example is based on the coupon redemption data analyzed in Wedel and De Sarbo (1995), see also Wedel (2000). The sample size is $N = 270$. The response variable $Y$ is the number of yogurt coupons
redeemed by consumers during a period of 104 weeks and the predictor variable \( X \) is
the average price paid for ounce. We ran both constrained and (unconstrained) Poisson
Gaussian CWM, Poisson FMR and Poisson FMRC. The number of groups associated
with the largest BIC is \( G = 2 \).

Constrained Poisson Gaussian CWM outperforms the unconstrained model (BIC resulted
\(-1952.12 \) and \(-1992.64 \), respectively) while Poisson FMR slightly outper-
forms Poisson FMRC (BIC resulted \(-783.71 \) and \(-788.55 \), respectively), where we
recall that the BIC values cannot be compared directly between the two classes of mod-
els, due their different nature (see Section 4.1). The constrained Poisson CWM and the
Poisson FMR attained very similar values of GCOF and GSD measures of goodness
of fit (see Table 8). Moreover, they substantially provide the same parameter estimates
and the same classification (see Table 7 and Figure 6 This confirms again theoretical
results proved in Section 4.

| group | coefficients | Constrained Poisson Gaussian CWM | Poisson FMR |
|-------|--------------|----------------------------------|------------|
| 1     | \( \beta_{01} \) | -1.122                           | -1.134     |
|       | \( \beta_{11} \) | 0.025                            | 0.026      |
| 2     | \( \beta_{02} \) | -0.435                           | -0.445     |
|       | \( \beta_{12} \) | 0.250                            | 0.250      |

Table 7: Coupon redemption data. Coefficients of Constrained Poisson Gaussian CWM and Poisson FMR.

Example 6 (Patent data.). Patent data have been studied in [Wang et al. (1998)] and
are available in the R Flexmix library. They consist of \( N = 70 \) observations on

|                         | Constrained Poisson Gaussian CWM | Poisson FMR |
|-------------------------|----------------------------------|------------|
| \( GGOF \)              | 1.126                            | 1.127      |
| \( GSD \)               | 279.268                          | 278.886    |

Table 8: Coupon redemption data. Generalized weighted Pearson chi-square statistic and generalized scaled deviance in Constrained Poisson Gaussian CWM and Poisson FMRC.
patent applications and R&D spending in millions of dollars from pharmaceutical and biomedical companies in 1976. The number of patent applications is the response variable and the R&D spending is the continuous predictor variable. The number of groups associated with the largest BIC is $G = 3$.

Here, the unconstrained Poisson Gaussian CWM outperforms the constrained model (BIC resulted -774.57 and -793.36, respectively), while Poisson FMRC slightly outperforms Poisson FMR (BIC resulted -438.19 and -441.06, respectively). The unconstrained Poisson CWM and the Poisson FMRC attained very similar values of GCOF and GSD measures of goodness of fit (see Table 10). Again, they substantially provide the same parameter estimates and the same classification (see Table 9 and Figure 7).

**Example 7 (Healthcare data).** This case study is based on administrative data concerning the healthcare system in the Italian Lombardy region. The sample consists of $N = 332$ patients on which “length of stay” (in days, count response variable $Y$) and “age” (covariate $X$) are measured. Only living persons at the end of the hospital-
| group | coefficients | Poisson Gaussian CWM | Poisson FMRC |
|-------|--------------|----------------------|--------------|
| 1     | $\beta_{01}$ | 2.383                | 2.396        |
|       | $\beta_{11}$ | 0.569                | 0.567        |
| 2     | $\beta_{02}$ | 0.782                | 0.803        |
|       | $\beta_{12}$ | 0.825                | 0.821        |
| 3     | $\beta_{03}$ | -1.812               | -1.801       |
|       | $\beta_{13}$ | 1.410                | 1.404        |

Table 9: Patent data. Coefficients estimates of Poisson Gaussian CWM and Poisson FMRC.

Figure 7: Patent data. Reclassified data (Poisson Gaussian CWM and Poisson FMRC).

Figure 8(a) displays the scatterplot of the data; more-

...
over, in Figures 8(b)-(d) the fitted Poisson-based models FMR, FMRC and CWM and the corresponding data clustering are plotted. The ARI and the misclassification error obtained in the three data modeling are given in Table 11. The results show that the Poisson CWM clearly outperforms both FMR and FMRC, and it yields a perfect separation between the two classes. On the contrary, the Poisson FMR and FMRC are not able to capture the evident underlying group-structure of the data. We remark that we carried out other analyses considering patients based on other pairs DRGs. Here, the Poisson CWM outperformed both FMR and FMRC, even if in many cases it didn’t yield a perfect separation between classes. The results obtained in clustering data coming from other pairs of DRGs are given in Table 12.

Table 10: Patent data. Generalized weighted Pearson chi-square statistic and generalized scaled deviance in Poisson Gaussian CWM and Poisson FMRC.

|               | Poisson Gaussian CWM | Poisson FMRC |
|---------------|----------------------|--------------|
| GGOF          | 1.283                | 1.277        |
| GSD           | 89.659               | 87.339       |

Table 11: Example 7. DRG: 348 and 396. Clustering performance of the fitted Poisson-based models on the Healthcare data

|               | FMR | FMRC | CWM |
|---------------|-----|------|-----|
| misclassError | 45.48% | 28.01% | 0.00% |
| ARI          | 0.00531 | 0.19098 | 1.00000 |

Table 12: Example 7. Clustering performance of the fitted Poisson-based models on the Healthcare data (other pairs of DRG).

| Couples of DRGs | FMR   | FMRC   | CWM     |
|----------------|-------|--------|---------|
| (68, 317)      | 34.67% | 37.00% | 30.00%  |
| (345, 68)      | 41.97% | 43.00% | 34.47%  |
| (345, 168)     | 43.81% | 44.48% | 36.12%  |
7.3. Discussion

The results of the numerical studies given in this section emphasize the effectiveness of the GLGCWM in comparison with some finite mixtures of regression models. Indeed the results of the above examples can be summarized as follows:

Example 1: Here the theoretical results given in Section 4 have been investigated from the numerical point of view and the constrained Poisson GCWM reveals to be a very good approximation for the Poisson GFMR, regardless from the distribution...
of $X$.

**Example 2**: here FMRC strongly outperformed FMR and CWM gave comparable results obtained using FMRC.

**Example 3**: here FMR and FMRC gave comparable results while CWM strongly outperformed both FMR and FMRC.

**Example 4**: here FMRC strongly outperformed FMR and CWM outperformed FMR.

**Example 5**: here FMR slightly outperformed FMRC and CWM gave comparable results obtained using FMR.

**Example 6**: here FMRC slightly outperformed FMR and CWM gave comparable results obtained using FMRC.

**Example 7**: here CWM outperformed both FMR and FMRC.

In order to deepen such results, in Figures 9 and 10 we show the distribution along the covariate $X$ for both simulated and real data. Figures 9(a, c) (Examples 2 and 4) and Figure 10(c) (Example 7) highlight that CWM yields better performance than FMR when the covariates present a clear group-structure; Figure 9(b) (Examples 3) highlights that CWM yields better performance than FMRC when distributions of the covariates are overlapped with different ranges. This confirms the previous results obtained in Ingrassia et al. (2012a). On the contrary, when there is no a group-structure in the covariates, then CWM, FMR and FMRC give comparable results, see Figures 10(a,b) (Examples 5 and 6).

8. **Concluding remarks**

In this paper we have introduced the generalized linear Gaussian CWM (GLGCWM) and we have shown that they are quite flexible statistical models. Such models allow modeling categorical variables depending on numerical covariates based on data coming from a heterogeneous population. Some relationships with finite mixtures of generalized mixture models (with and without concomitants) have also been investigated; in particular, we have shown that mixtures of generalized linear models can be
We remark that in this paper we have considered Gaussian marginal distributions. However, the extension to multivariate Student-$t$ is straightforward, and in this case the parameters of Student-$t$ local densities can be estimated according to mixtures of multivariate $t$ distributions, see e.g Section 7.5 in [McLachlan and Peel (2000)]. In this direction, we are currently working on further extensions of the models presented here. In this framework, an important issue concern data modeling by Cluster Weighted approaches when the dimension of the input vector $X$ is large. First results, are given in [Subedi et al. (2012)].
An issue which deserves attention for future research concerns the behaviour of the EM algorithm. It is well known that the EM algorithm suffers for local maxima and singularities and our first simulation studies confirmed previous results given in literature, see e.g. Faria and Soromenho (2010), Ingrassia et al. (2012a), Ingrassia et al. (2012b), showing that the performance of the algorithm strongly depend on the choice of initial values. Moreover, suitable constraints on the eigenvalues of the covariance matrices of the marginal distributions can be imposed in order to run the algorithm in a parameter space with no singularities and a reduced number of local maxima, see e.g. Ingrassia and Rocci (2007) for details.

Finally, we point out that the above results open the perspective of model based
clustering of mixed type data coming from distributions with density $p(x, y)$, where $X$ is a mixed continuous variable and $Y$ is either continuous or discrete.

**Appendix: Proofs**

**Proof of Proposition 1.** If $\mu_g, \Sigma_g = \mu, \Sigma, g = 1, \ldots, G$, then (4) becomes

$$p(x, y; \theta) = \sum_{g=1}^{G} q(y|x; \beta_g, \lambda_g) \phi_d(x; \mu, \Sigma) \pi_g = \phi_d(x; \mu, \Sigma) \sum_{g=1}^{G} q(y|x; \beta_g, \lambda_g) \pi_g$$

$$= \phi_d(x; \mu, \Sigma) f(y|x; \kappa).$$

\[\square\]

**Proof of Corollary 2.** If $(\mu_g, \Sigma_g) = (\mu, \Sigma), g = 1, \ldots, G$, then the posterior probability that the generic observation $(x', y')$ belongs to $\Omega_g$ from model (4) specifies as

$$p(\Omega_g|x, y) = \frac{q(y|x; \beta_g, \lambda_g) \phi_d(x; \mu, \Sigma) \pi_g}{\sum_{j=1}^{G} q(y|x; \beta_j, \lambda_j) \phi_d(x; \mu, \Sigma) \pi_j}$$

$$= \frac{q(y|x; \beta_g, \lambda_g) \pi_g}{\sum_{j=1}^{G} q(y|x; \beta_j, \lambda_j) \pi_j}, \quad g = 1, \ldots, G,$$

which coincides with (11).

\[\square\]

**Proof of Proposition 3.** From (4) we get:

$$p(x, y; \theta) = \sum_{g=1}^{G} q(y|x; \beta_g, \lambda_g) \phi_d(x; \mu_g, \Sigma) \pi_g = p(x; \psi) \sum_{g=1}^{G} p(y|x; \beta_g, \lambda_g) \frac{\phi_d(x; \mu_g, \Sigma) \pi_g}{p(x; \psi)}$$

$$= p(x; \psi) \sum_{g=1}^{G} q(y|x; \beta_g, \lambda_g) \exp \left[ -\frac{1}{2} (x - \mu_g)^T \Sigma^{-1} (x - \mu_g) \right]$$

$$\sum_{j=1}^{G} \exp \left[ -\frac{1}{2} (x - \mu_j)^T \Sigma^{-1} (x - \mu_j) \right].$$

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where

\[
\sum_{j=1}^{G} \exp \left[ -\frac{1}{2} (x - \mu_g)' \Sigma^{-1} (x - \mu_g) \right] = 1 + \sum_{j \neq g} \exp \left[ -\frac{1}{2} (x - \mu_j)' \Sigma^{-1} (x - \mu_j) + \frac{1}{2} (x - \mu_g)' \Sigma^{-1} (x - \mu_g) \right] = 1 + \sum_{j \neq g} \exp \left[ (\mu_j - \mu_g)' \Sigma^{-1} x - \frac{1}{2} (\mu_j - \mu_g)' \Sigma^{-1} (\mu_j - \mu_g) \right] = 1 + \sum_{j \neq g} \exp \left[ \mu_j' \Sigma^{-1} x - \frac{1}{2} \mu_j' \Sigma^{-1} \mu_j - (\mu_g' \Sigma^{-1} x - \frac{1}{2} \mu_g' \Sigma^{-1} \mu_g) \right] = \frac{\exp (-\frac{1}{2} \mu_g' \Sigma^{-1} \mu_g + \mu_g' \Sigma^{-1} x)}{\sum_{j=1}^{G} \exp (-\frac{1}{2} \mu_j' \Sigma^{-1} \mu_j + \mu_j' \Sigma^{-1} x)}. \tag{26}
\]

If we set

\[
\alpha_{g0} = -\frac{1}{2} \mu_g' \Sigma^{-1} \mu_g \quad \text{and} \quad \alpha_{g1} = \mu_g' \Sigma^{-1}, \quad g = 1, \ldots, G, \tag{27}
\]

we recognize that (26) can be written in form (13). This completes the proof. \(\square\)

**Proof of Corollary** If \(X|\Omega_g \sim N_d(\mu_g, \Sigma)\) and \(\pi_g = \pi = 1/G, \ g = 1, \ldots, G,\) then the posterior probability that the generic observation \((x', y)'\) belongs to \(\Omega_g\) from model (4) specifies as

\[
p(\Omega_g|x, y) = \frac{q(y|x; \beta_g, \lambda_g) \phi_d(x; \mu_g, \Sigma)}{\sum_{j=1}^{G} q(y|x; \beta_j, \lambda_j) \phi_d(x; \mu_j, \Sigma)} = \frac{q(y|x; \beta_g, \lambda_g) \exp \left[ -\frac{1}{2} (x - \mu_g)' \Sigma^{-1} (x - \mu_g) \right]}{\sum_{j=1}^{G} q(y|x; \beta_j, \lambda_j) \exp \left[ -\frac{1}{2} (x - \mu_j)' \Sigma^{-1} (x - \mu_j) \right]}
\]
which can be written as
\[ p(\Omega_g | x, y) = \frac{q(y | x; \beta_g, \lambda_g) \exp(\alpha_{g0} + \alpha_g x)}{\sum_{j=1}^{G} q(y | x; \beta_j, \lambda_j) \exp(\alpha_{j0} + \alpha_j x)}, \] (28)

where \( \alpha_{g0} \) and \( \alpha_g \) are specified as in (27).

\[ \text{Proof of Proposition 5.} \]

In order to prove the proposition, it is sufficient to show that, under the assumption that \( \mu_g, \Sigma_g = \mu, \Sigma \), the terms \( L_{1c}(\xi) \) and \( L_{3c}(\pi) \) in (16) do not depend on \( \mu, \Sigma \). Indeed, if \( \mu_g, \Sigma_g = \mu, \Sigma \), then the complete-data log-likelihood function becomes:
\[ L_c(\theta; X, y) = \sum_{n=1}^{N} \left[ z_{ng} \ln q(y_n | x_n; \beta_g, \lambda_g) + z_{ng} \ln \phi_d(x_n; \mu, \Sigma) + z_{ng} \ln \pi_g \right] \]
\[ = L_{1c}(\xi) + L_{2c}(\psi^*) + L_{3c}(\pi), \] (29)

where \( \psi^* = (\mu, \Sigma) \) and \( L_{2c}(\psi) \) in (16) is now replaced by
\[ L_{2c}(\psi^*) = \sum_{n=1}^{N} \ln \phi_d(x_n; \mu, \Sigma), \]

since \( \sum_{g=1}^{G} z_{ng} = 1 \) for \( n = 1, \ldots, N \). Moreover, since \( (\mu_g, \Sigma_g) = (\mu, \Sigma) \) for \( g = 1, \ldots, G \), then the posterior probability (5) reduces to
\[ p(\Omega_g | x, y) = \frac{q(y | x; \beta_g, \lambda_g) \pi_g}{\sum_{j=1}^{G} q(y | x; \beta_j, \lambda_j) \pi_j}. \]

Thus, \( z_{ng} \) does not depend on \( \phi_d(x_n; \mu_g, \Sigma_g) \) and neither does the term \( L_{3c}(\pi) \). In summary, the maximization of (29) can be attained by independently maximizing the three terms \( L_{1c}(\xi) \), \( L_{2c}(\psi^*) \) and \( L_{3c}(\pi) \) and hence, the maximization of (17) and (29) leads to the same estimates of \( (\xi, \pi) \). This completes the proof.

\[ \text{Proof of Proposition 6.} \]

In order to prove the result, it is sufficient to show that if \( \Sigma_g = \Sigma \) and \( \pi_g = 1/G, g = 1, \ldots, G \), then the terms \( L_{1c}(\xi) \) and \( L_{3c}(\pi) \) in (16) do not depend on \( (\mu_g, \Sigma), g = 1, \ldots, G \). Indeed, we have:
\[ L_c(\theta; X, y) = \prod_{n=1}^{N} \prod_{g=1}^{G} q(y_n | x_n; \beta_g, \lambda_g)^{z_{ng} \phi_d(x_n; \mu_g, \Sigma)} \pi_g^{z_{ng} \pi_g} \] (30)
and taking the logarithm of (30), after some algebra we get

\[ \mathcal{L}_c(\theta; \tilde{X}, y) = \ln \mathcal{L}_c(\theta; X, y) = \sum_{n=1}^{N} \sum_{g=1}^{G} \left[ z_{ng} \ln q(y_n|x_n; \beta_g, \lambda_g) + z_{ng} \ln \phi_d(x_n; \mu_g, \Sigma_g) \right] + \pi \]

\[ = \mathcal{L}_1c(\xi) + \mathcal{L}_2c(\psi^{**}) + \pi, \quad (31) \]

where \( \psi^{**} = \{\mu_g, \Sigma; g = 1, \ldots, G\} \) and \( \mathcal{L}_2c(\psi) \) in (16) is now replaced by

\[ \mathcal{L}_2c(\psi^{**}) = \frac{1}{2} \sum_{n=1}^{N} \sum_{g=1}^{G} z_{ng} \left[ -p \ln 2\pi - \ln |\Sigma| - (x_n - \mu_g)\Sigma^{-1}(x_n - \mu_g) \right]. \]

Once the estimates of \( (\mu_g, \Sigma) \) have been obtained, quantity \( p(\Omega_g|x, \xi) \) in (18) can be obtained immediately like in (28). This completes the proof. \( \square \)

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