On A Stein Method Based Approximation for A Two-Dimensional Markov Chain

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Abstract

We study an approximation method of stationary characters of a two-dimensional Markov chain via the Stein method. For this purpose, innovative methods are developed to estimate the moments of the Markov chain, as well as the solution to the Poisson equation with a partial differential operator.

Keywords: Markov chain, Stein method

1 Introduction

Computing expected function of ergodic Markov chains defined on multidimensional spaces that are not compact, with respect to their stationary distributions, is always a difficult problem. Analytic and algebraic methods are developed for some special cases, such as those Markov chains whose transition probabilities takes only a few possible values, see e.g. Fayolle et al (1999). A popular approach of approximation is to calculated related quantities for a derived Markov chain on a finite state space, which can be calculated efficiently, see, e.g. for the studies in Mazalov and Gurtov (2012); Bhandari et al (2008) in this nature.

We consider an approximation method of evaluating, through known differential equations techniques, the function against a diffusion process whose generator preserve the main characters of the generator of the Markov chain under a proper scaling. This method is elaborated through a two dimensional Markov chain motivated by a queueing application. Using the Stein method,
coupled with estimation by differential equation methods, we are able to quantify the error of this approximations through a comparison analysis of the generators. The Stein method Stein (1986) is a versatile technique in probability theory, rooted in the studies of the concentration of measures, such as the central limit theorems. Recent developments in Gurvich (2014b,a); Braverman and Dai (2017), utilize the Stein method to estimate the stationary distribution of a Markov chain by that of a diffusion process, which is usually mathematical more tractable, by comparing the generators and the solution to the Poisson (Stein) equation. While our overall approach follow the same logic, the bounds on derivatives are different and innovative.

The rest of paper will be organized as follows, in Sec. 2, we provide detailed description of the Markov chain; In Sec. 3 we provide a generator expansion; and in Sec. 4, the main results are discussed and proved.

2 A Two Dimensional Markov Chain

2.1 Definition of the Markov Chain

The Markov chain model is motivated by the following queueing model. The job arrivals follow a Poisson process with rate $\lambda$, and service time is independently drawn after an exponential distribution with rate $\mu$. Meanwhile, a stream of servers arrive, also following an independent Poisson process with rate $\gamma$. When a job arrives, it will be served immediately if there are any idle servers, otherwise it will join a single queue in front of all the servers. Whenever there is a server becomes available, due to either the departure of a job or the arrival of a new server, the jobs in queue will be served in a first-come-first-serve (FCFS) fashion. Meanwhile, each server that becomes idle will start an independent departure clock, which follows an exponential distribution with rate $\nu$, the server will depart if the clock expires before it takes on a job. In other words, a server will leave the system after staying idle for a random time period (exponential with rate $\nu$).

The system can be characterized with a two-dimensional continuous time Markov chain (CTMC). The state space is $\mathbb{Z}_+ \times \mathbb{Z}_+^2$ with $\mathbb{Z}_+$ denoting the set of all nonnegative integers. A state $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ consists of the number of jobs in the system, $i$, and the number of servers in the system, $j$. The transition rates are in the following form,

$$
\begin{align*}
(i, j) &\rightarrow (i+1, j) & \lambda \\
(i, j) &\rightarrow (i-1, j) & (i \wedge j)\mu \\
(i, j) &\rightarrow (i, j+1) & \gamma \\
(i, j) &\rightarrow (i, j-1) & (j-i)^+\nu
\end{align*}
$$

(1)

where $x \wedge y := \min\{x, y\}$ and $(x-y)^+ := \max\{x-y, 0\}$. From (1), we can write the transition rate matrix (which is of infinite dimension) in the following form.
Therefore, it is easy to see that there exist a \( (i, 0) \), there are only two events can happen, the arrival of a job and the arrival of a server, and with rate \( \lambda \) and \( \gamma \) respectively. Hence, 
\[
q(i, 0)\rightarrow(i+1, 0) = \frac{\lambda}{\lambda+\gamma}, \quad q(i, 0)\rightarrow(i, 1) = \frac{\gamma}{\lambda+\gamma}.
\]
• For any states in the form of \((0, j)\), for \( j \geq 1 \), three events can happen, job arrival, server arrival and departure. Hence, 
\[
q(0, j)\rightarrow(0, j+1) = \frac{\gamma}{\lambda+\gamma+j\nu}, \quad \text{and } q(0, j)\rightarrow(0, j-1) = \frac{j\nu}{\lambda+\gamma+j\nu}.
\]
• For any states in the form of \((i, j)\), for \( i, j \geq 1 \), the transition probabilities are 
\[
q(i, j)\rightarrow(i+1, j) = \frac{\lambda}{\lambda+(i+1)\mu+\gamma+(j-1)\nu}, \quad q(i, j)\rightarrow(i-1, j) = \frac{(i\wedge j)\mu}{\lambda+(i\wedge j)\mu+\gamma+(j-i)\nu},
\]
\[
q(i, j)\rightarrow(i, j+1) = \frac{\gamma}{\lambda+(i\wedge j)\mu+\gamma+(j-1)\nu}, \quad \text{and } q(i, j)\rightarrow(i, j-1) = \frac{(j-i)^+\nu}{\lambda+(i\wedge j)\mu+\gamma+(j-i)\nu}.
\]
Note that \( q(i, j)\rightarrow(i, j-1) \) could be zero when \( j \leq i \).

Let us denote the Markov chain \((X(t), Y(t))\), and its generator \( \mathcal{G}_0 \). For any bounded function \( f : \mathbb{Z}^2_+ \rightarrow \mathbb{R} \),
\[
\mathcal{G}_0 f(i, j) = \lambda[f(i+1, j) - f(i, j)] + (i \wedge j)\mu[f(i-1, j) - f(i, j)] + \gamma[f(i, j+1) - f(i, j)] + (j-i)^+\nu[f(i, j-1) - f(i, j)].
\]

**Lemma 1** \((X(t), Y(t))\) has a stationary distribution, and more importantly, the stationary distribution has finite third moment.

**Proof** Apply the generator \( \mathcal{G}_0 \) to function \( f(x, y) = x^4 + y^4 \), we have,
\[
\mathcal{G}_0 f(x, y) = \lambda[(x+1)^4 - x^4] + (x \wedge y)[(x-1)^4 - x^4] + \gamma[(y+1)^4 - y^4] + (y-x)^+[(y-1)^4 - y^4]
\]
\[
= \lambda(4x^3 + 6x^2 + 4x + 1) + (x \wedge y)(-4x^3 + 6x^2 - 4x + 1)
\]
\[
+ \gamma(4y^3 + 6y^2 + 4y + 1) + (y-x)^+(-4y^3 + 6y^2 - 4y + 1).
\]
Therefore, it is easy to see that there exist a \( (x_0, y_0) \), and \( C > 0 \), such that when \( (x, y) \geq (x_0, y_0) \), \( \mathcal{G}_0 f(x, y) \leq -C(x^3 + y^3) \). Thus, \( \mathcal{G}_0 f(x, y) \leq -C(x^3 + y^3) + C' \mathbf{1}(\{(x, y) \geq (x_0, y_0)\}), \) and by Theorem 4.2 in Meyn and Tweedie (1993), the Markov chain \((X(t), Y(t))\) has a stationary distribution, and it has finite third moment.

\[\square\]

### 2.2 Centering, Scaling and the Scaled Processes

To facilitate our analysis, we will consider the following ”centered” and ”scaled” Markov chain through translation and scaling. Any stationary function calculations for the original process can be readily transformed to the ones for the centered and scaled Markov chain.

#### 2.2.1 Centering

First, to find the equilibrium point \((x(\infty), y(\infty))\), the pair that represents the equilibrium number of the jobs and servers, consider the following system of
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flow balance equations,

\[ \lambda = [x(\infty) \wedge y(\infty)] \mu, \quad \gamma = [y(\infty) - x(\infty)]^+ \nu. \]

Since, under our assumptions, \(\gamma > 0\) and \(\nu > 0\), the second equation implies \(y(\infty) - x(\infty) > 0\) and \(y(\infty) - x(\infty) = \frac{\gamma}{\nu}\). Combined with the first equation, we have,

\[ x(\infty) = \frac{\lambda}{\mu}, \quad y(\infty) = \frac{\lambda}{\mu} + \frac{\gamma}{\nu}. \] (2)

2.2.2 Scaling

The solutions in (2) indicates that given the parameter \((\lambda, \mu, \gamma, \nu)\), the queue length and number of servers will be in essence approaching the above equilibrium point. Consider a sequence of systems, indexed by \(n\), such that,

\[ \lambda_n = n \mu, \quad \mu_n = \mu, \quad \gamma_n = \kappa n^\alpha \nu, \quad \nu_n = \nu, \] (3)

for some positive real number \(\mu, \kappa\) and \(\alpha\). Hence, the equilibrium states are \(x_n(\infty) = n\) and \(y_n(\infty) = n + \kappa n^\alpha\). Note that \(\alpha = \frac{1}{2}\) represents the famed Halfin-Whitt scaling Halfin and Whitt (1981).

2.2.3 Stationary Function Calculations

For approximating the stationary performance, it is more convenient to consider the "centered" and "scaled" version of the Markov chain \((X(t), Y(t))\). Define,

\[ \tilde{X}^n(t) = \delta [X^n(t) - n], \quad \tilde{Y}^n(t) = \eta [Y^n(t) - n - \kappa n^\alpha] \]

for some scaling factors \(\delta\) and \(\eta\) that tends to zero as \(n\) grows. For example, in the case of Halfin-Whitt scaling \((\alpha = \frac{1}{2})\), \(\delta\) and \(\eta\) can also be choose to be \(\frac{1}{2}\).

The generator \(\mathcal{G}_n\) for \((\tilde{X}^n, \tilde{Y}^n)\) can be written in the following form, for any bounded smooth function \(u : \mathbb{R}^2 \to \mathbb{R}\), with \(x = \delta (i - n), y = \eta (j - n - \kappa n^\alpha)\) (hence, \(i = \frac{x}{\delta} + n\), and \(j = \frac{y}{\eta} + n + \kappa n^\alpha\)),

\[ \mathcal{G}_n u(x, y) = \lambda_n [u(x + \delta, y) - u(x, y)] + b^1_n(x, y) \mu [u(x - \delta, y) - u(x, y)] + \gamma_n [u(x, y + \delta) - u(x, y)] + b^2_n(x, y) \nu [u(x, y - \delta) - u(x, y)], \] (4)

with

\[ b^1_n(x, y) := \left[ \left( \frac{x}{\delta} + n \right) \wedge \left( \frac{y}{\eta} + (n + \kappa n^\alpha) \right) \right], \]

\[ b^2_n(x, y) := \left( \frac{y}{\eta} - \frac{x}{\delta} + \kappa n^\alpha \right)^+. \]
Let function $h(x, y)$ be the quantity of interest, for example, in the motivating queueing system, it can represent the performance of the system that depends on both the number of jobs and the number of servers. The stationary function calculation takes the form of $\mathbb{E}[h(\tilde{X}^n(\infty), \tilde{Y}^n(\infty))]$ with $(\tilde{X}^n(\infty), \tilde{Y}^n(\infty))$ denoting the stationary distribution of the process $(\tilde{X}^n(t), \tilde{Y}^n(t))$.

### 3 Generator Expansion

For any $(x, y) \in \mathbb{R}^2$, the Taylor expansion of the function $u(x, y)$ at $(x, y)$ will help us in expanding the generator $\mathcal{G}_n$ in (4), and identifying the approximating diffusion process. More specifically, we have,

$$
\lambda_n [u(x + \delta, y) - u(x, y)] = \lambda_n \delta u_x(x, y) + \frac{\lambda_n \delta^2}{2} u_{xx}(\xi_1, y)
$$

$$
= \lambda_n \delta u_x(x, y) + \frac{\lambda_n \delta^2}{2} u_{xx}(x, y)
$$

$$
+ \frac{\lambda_n \delta^2}{2} [u_{xx}(\xi_1, y) - u_{xx}(x, y)],
$$

with some $\xi_1 \in [x, x + \delta]$, and

$$
u u_x(x, y) := \frac{\partial}{\partial x} u(x, y), \quad \nu u_y(x, y) := \frac{\partial}{\partial y} u(x, y),
$$

$$
u u_{xx}(x, y) := \frac{\partial^2}{\partial x^2} u(x, y), \quad \nu u_{yy}(x, y) := \frac{\partial^2}{\partial y^2} u(x, y).
$$

Next,

$$
b^{1}_{n}(x, y) \mu[u(x - \delta, y) - f(x, y)]
$$

$$
= b^{1}_{n}(x, y) \mu \left[ -\delta u_x(x, y) + \frac{\delta^2}{2} u_{xx}(\xi_2, y) \right]
$$

$$
= - b^{1}_{n}(x, y) \mu \delta u_x(x, y) + \frac{b^{1}_{n}(x, y) \mu \delta^2}{2} u_{xx}(\xi_2, y)
$$

with some $\xi_2 \in [x - \delta, x]$. Similarly, we have,

$$
\gamma_n [u(x, y + \eta) - u(x, y)] = \gamma_n \eta u_y(x, y) + \gamma_n \eta [u_y(x, \xi_3) - u_y(x, y)],
$$

with some $\xi_3 \in [y, y + \eta]$. And

$$
b^{2}_{n}(x, y) [u(x - \eta, y) - u(x, y)]
$$

$$
= - b^{2}_{n}(x, y) \nu \eta u_y(x, y) - b^{2}_{n}(x, y) \nu \eta [u_y(x, \xi_4) - u_y(x, y)],
$$

with some $\xi_4 \in [y - \eta, y]$.
Now, let us examine the behavior of each term we obtained through above expansion, and explain the rational of the selection of the generator below. First, let us look at the terms with the first order terms in the generator.

### 3.0.1 Terms of the first order \((u_x\) and \(u_y\))

The coefficient for \(u_x\) is \(δ[λ_n - b^1_n(x, y)μ]\), which equals to,

\[
δλ_n - \left[(x + nδ) ∧ \left(\frac{yδ}{η} + (n + κn^α)δ\right)\right]μ
\]

\[
= (nδ)μ - \left[(x + nδ) ∧ \left(\frac{yδ}{η} + (n + κn^α)δ\right)\right]μ
\]

\[
= - \left[x ∧ \left(\frac{yδ}{η} + δκn^α\right)\right]μ.
\]

Meanwhile, the coefficient for \(u_y\) is

\[
η[γ - b^2_n(x, y)ν] = 2nκn^αν - \left(y - \frac{xη}{δ} + κn^α\right)ν.
\]

Observe that, both terms of \(\frac{y}{δ}\) and \(\frac{y}{η}\) are present.

### 3.0.2 Terms of the second order \(u_{xx}\) and \(u_{yy}\)

The coefficient for \(u_{xx}\) is \(\frac{δ^2}{2}(λ_n + b^1_n(x, y)μ)\). Apply the above scaling, we can see that,

\[
\frac{δ^2}{2}[λ_n + b^1_n(x, y)μ] = \frac{δ^2}{2} \left[n + (x + nδ) ∧ \left(\frac{yδ}{η} + (n + κn^α)δ\right)\right]μ
\]

It can be seen that the first term is \(δ^2n/2\), meanwhile,

\[
\frac{δ^2}{2}[(x + nδ) ∧ (y + nδ + κn^αδ)]μ,
\]

will be of the order of \(O(nδ^3)\), and can be treated as an error term. To bound this error term, we need the moment bound (i.e. the first moment bound), more specifically, we need to show that the Markov chain has finite first moment.

The coefficient for \(u_{yy}\) is \(\frac{η^2}{2}[γ_n + b^2_n(x, y)ν]\).

\[
\frac{η^2}{2}[γ_n + b^2_n(x, y)ν] = \frac{η^2}{2} \left[κn^α + \left(y - \frac{xη}{δ} + κn^α\right)ν\right]
\]

Again, to ensure that this error term is small, we need an estimate of the first order quantities.
Therefore, (4) can be written as,

\[ G_n u(x, y) = G u(x, y) + E_1 + E_2 + E_3 + E_4, \]  

(5)

where \( G \) is given by,

\[ G u(x, y) = \delta[\lambda_n - b_n^1(x, y)\mu] \frac{\partial}{\partial x} u(x, y) + \delta[\gamma_n - b_n^2(x, y)\nu] \frac{\partial}{\partial x} u(x, y) + \frac{1}{2} \frac{\partial^2}{\partial x^2} u(x, y), \]

representing the generator of a diffusion process defined as

\[
\begin{cases}
    dX_t = \delta[\lambda_n - b_n^1(X_t, Y_t)\mu] dX_t + \frac{1}{2} dW_t, \\
    dY_t = \delta[\gamma_n - b_n^2(X_t, Y_t)\nu] dY_t,
\end{cases}
\]  

(6)

with \( W_t \) being a standard Brownian motion. \( E_i, i = 1, \ldots, 4 \) are error terms that will be estimated below,

\[ E_1 = \frac{\lambda \delta^2}{2} [u_{xx}(\xi_1, y) - u_{xx}(x, y)], \quad E_2 = \frac{b_n^1(x, y)\mu \delta^2}{2} u_{xx}(\xi_2, y), \]
\[ E_3 = \gamma_n \delta[u_y(x, \xi_3) - u_y(x, y)], \quad E_4 = -b_n^2(x, y)\nu \delta[u_y(x, \xi_4) - u_y(x, y)]. \]

The diffusion process (6) can be viewed as a stochastic Hamiltonian system, a general overview can be founded in e.g. Soize (1994), and a detailed analysis on its stationary behavior are presented in Talay (2002). In Talay (2002), numerical methods are also discussed in the cases that exact form of the stationary distribution can not be obtained.

4 Stein Method for Error Estimation

In this section, we present a detailed analysis on the approximation error via the Stein method. Especially, we will quantify the four error terms identified in the above analysis, which guide the derivations of the moment and derivative bound in the sections below.

4.1 Main Results

**Lemma 2** Let \( f(x, y) \) be a function such that \( f(x,y) \leq C(1+x^3+y^3) \) for some \( C > 0 \), then \( \mathbb{E}[G_n f(\tilde{X}^n(\infty), \tilde{X}^n(\infty))] = 0. \)

**Proof** As indicated in Gurvich (2014a); Braverman and Dai (2017), it suffices to know that \( (X(t), Y(t)) \) is positive recurrent and the stationary distribution has finite third moment, and that is established in Lemma 1. \( \square \)

Meanwhile, the following order estimations of the error terms will be proved in Sec. 4.2.
Lemma 3
\[ E[E_i] = O(n\sigma^3), i = 1, 2, \quad E[E_i] = O(n^\alpha \eta^2), i = 3, 4. \]

Thus,

**Theorem 4** For a performance metric function satisfies that \((x^p + y^p)h(x, y)\) is integrable for \(p \leq 5\), we can conclude that, there exists a constant \(C > 0\), such that
\[ |E[h(\bar{X}^n(\infty), \bar{Y}^n(\infty)) - E[h(\bar{X}_\infty, \bar{Y}_\infty)]| \leq C(n^\delta + n^\alpha \eta^2), \]
with \((\bar{X}^n(\infty), \bar{Y}^n(\infty))\) and \((\bar{X}_\infty, \bar{Y}_\infty)\) represent the stationary distribution of the centered and scaled Markov chain \((\bar{X}^n(t), \bar{Y}^n(t))\), and the diffusion process defined in (6), respectively.

**Proof** Recall that for each \(n\), the centered and scaled process is \((\bar{X}^n(t), \bar{Y}^n(t))\) with generator \(G_n\). The goal is to estimates the average difference of performance,
\[ |E[h(\bar{X}^n(\infty), \bar{Y}^n(\infty))] - E[h(\bar{X}_\infty, \bar{Y}_\infty)]|. \]  
with \((\bar{X}(t), \bar{Y}(t))\) denotes the approximating process, and \(h(x, y)\) a general performance metric function. This function can be very general, could cover probability based performance as seen in many applications. Let us denote \(u^h\) be the solution to the Stein equation,
\[ Gu = h(x, y) - E[h(\bar{X}_\infty, \bar{Y}_\infty)]. \]  
Apply the expectation with the stationary distribution for the \(n\)-th system, we have,
\[ E[G_n u^h(\bar{X}^n(\infty), \bar{Y}^n(\infty))] = E[h(\bar{X}^n(\infty), \bar{Y}^n(\infty))] - E[h(\bar{X}_\infty, \bar{Y}_\infty)]. \]  
This can be written as,
\[ E[(G_n - G_n)u^h(\bar{X}^n(\infty), \bar{Y}^n(\infty))] + E[G_n u^h(\bar{X}^n(\infty), \bar{Y}^n(\infty))] = E[h(\bar{X}^n(\infty), \bar{Y}^n(\infty))] - E[h(\bar{X}_\infty, \bar{Y}_\infty)]. \]
Basic property of the generator, Lemma 2, implies that the term \(E[G_n u^h(\bar{X}^n(\infty), \bar{Y}^n(\infty))]\) vanishes. Hence, we have,
\[ E[h(\bar{X}^n(\infty), \bar{Y}^n(\infty))] - E[h(\bar{X}_\infty, \bar{Y}_\infty)] = E[(G_n - G_n)u^h(\bar{X}^n(\infty), \bar{Y}^n(\infty)). \]
Thus, we only need to estimate the right hand side. The expression in (5) affirms that we only need to estimate \(E[E_1] + E[E_2] + E[E_3] + E[E_4]\), which is provided in the Lemma 3.

**Remark 5** When \(\alpha = \frac{1}{2}, \delta = \eta = n^{-1/2}\), the result in Theorem 4 is consist with the Halfin-Whitt type of results that are well-known in the queueing literature. In general, we can see that the approximation depends on the rate of server arrival.

### 4.2 Error Estimates

In this section, we will provide the basic estimation of the error terms. This consists of two parts, In Sec. 4.2.1, we will discuss the bounds related to the first and second moments of the variable \((\bar{X}^n(\infty), \bar{Y}^n(\infty))\); in Sec. 4.2.2, we present arguments for bounding the derivatives of the Stein equation (8).
4.2.1 Moment Bounds

Recall the generator for the Markov chain indexed by $n$, $G_n$,

$$G_n u(x, y) = \lambda_n [u(x + \delta, y) - u(x, y)] + b_1^*(x, y) \mu [u(x - \delta, y) - u(x, y)] + \gamma_n [u(x, y + \delta) - u(x, y)] + b_2^*(x, y) \nu [u(x, y - \delta) - u(x, y)].$$

Lemma 6 $E[b_1^*(\tilde{X}_n^\infty, \tilde{Y}_n^\infty)] = \frac{\lambda_n}{\mu}.$

Proof Let $u(x, y) = x$, Lemma 2 implies,

$$E \left[ (\tilde{X}_n^\infty + n\delta) \wedge \left( \frac{\tilde{Y}_n^\infty(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) \right] = \frac{\lambda_n}{\mu}. \quad (9)$$

That gives the desired expression of $E[b_1^*(\tilde{X}_n^\infty, \tilde{Y}_n^\infty)]$. □

Lemma 7 $E[b_2^*(\tilde{X}_n^\infty, \tilde{Y}_n^\infty)] = \frac{\gamma}{\nu}$

Proof Let $u(x, y) = y$, Lemma 2 implies,

$$E \left[ \left( \frac{\tilde{Y}_n^\infty(\infty)}{\eta} - \frac{\tilde{X}_n^\infty(\infty)}{\delta} + \delta \kappa n^\alpha \right)^+ \right] = \frac{\gamma}{\nu}. \quad (10)$$

That gives the desired expression of $E[b_2^*(\tilde{X}_n^\infty, \tilde{Y}_n^\infty)]$. □

Furthermore,

Lemma 8

$$E[\tilde{Y}_n^\infty(\infty)] \leq \left( \frac{\lambda_n \delta}{\mu} + \frac{\gamma \delta}{\nu} \right) \eta. \quad (11)$$

Proof It is easy to verify that the following inequality holds (due to an elementary inequality $x \vee y + (y + a - x)^+ \geq y + (a \wedge 0)$),

$$\left( \frac{\tilde{X}_n^\infty(\infty) + n\delta}{\eta} \wedge \left( \frac{\tilde{Y}_n^\infty(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) \right) + \left( \frac{\tilde{Y}_n^\infty(\infty)}{\eta} - \frac{\tilde{X}_n^\infty(\infty)}{\delta} + \delta \kappa n^\alpha \right)^+ \geq \frac{\tilde{Y}_n^\infty(\infty)}{\eta}.$$

Therefore, (11) follows immediately from (9) and (10). □

Lemma 9

$$E \left[ (\tilde{X}_n^\infty(\infty) + n\delta)1\left\{ \frac{\tilde{X}_n^\infty(\infty)}{\delta} + n \leq \frac{\tilde{Y}_n^\infty(\infty)}{\eta} + (n + \kappa n^\alpha) \right\} \right] \leq \frac{\delta \lambda_n}{\mu}.$$
Proof of Lemma 9  Let \( u(x, y) = x \mathbf{1}\{ x/\delta + n < y/\eta + (n + \kappa n^\alpha) \} \), to apply Lemma 2, we need the following calculations.

\[
\begin{align*}
\frac{u(x + \delta, y) - u(x, y)}{u(x - \delta, y) - u(x, y)} & = (x + \delta) \mathbf{1}\{ x/\delta + (n + 1) < y/\eta + (n + \kappa n^\alpha) \} \\
& \quad - x \mathbf{1}\{ x/\delta + n < y/\eta + (n + \kappa n^\alpha) \} \\
& = \delta \mathbf{1}\{ x/\delta + (n + 1) < y/\eta + (n + \kappa n^\alpha) \} \\
& \quad + x \mathbf{1}\{ y/\eta + (n + \kappa n^\alpha) - \delta < x/\delta + n \leq y/\eta + (n + \kappa n^\alpha) \}.
\end{align*}
\]

\[
\begin{align*}
\frac{u(x, y + \delta) - u(x, y)}{u(x, y - \delta) - u(x, y)} & = x \mathbf{1}\{ x/\delta + (n - 1) \leq \frac{y}{\eta} + (n + \kappa n^\alpha) \} \\
& \quad - x \mathbf{1}\{ x/\delta + n \leq \frac{y}{\eta} + (n + \kappa n^\alpha) \} \\
& = x \mathbf{1}\{ \frac{y}{\eta} + (n + \kappa n^\alpha) < \frac{x}{\delta} + n \leq \frac{y}{\eta} + (n + \kappa n^\alpha) + 1 \}.
\end{align*}
\]

Therefore, on the set \((-\infty, \tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta - \delta] \), we have, \( \lambda \delta - b_n^1 \delta \); on the set \((\tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta - \delta, \tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta) \), we have, \( \lambda x - b_n^1 \delta - b_n^2 x \), \((\tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta, \tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta + \delta] \), we have, \((x - \delta) b_n^1 + \nu x \), and beyond \((\tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta + \delta, \) the value is zero. Thus,

\[
\delta \lambda_n \mathbb{P}[\tilde{X}^n(\infty) + n \delta \leq \tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta] \\
\quad - \mu \mathbb{E}[(\tilde{X}^n(\infty) + n \delta)] \mathbf{1}\{ \tilde{X}^n(\infty) + n \delta \leq \tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta \}] = 0.
\]

Thus, we have,

\[
\mathbb{E}[(\tilde{X}^n(\infty) + n \delta)] \mathbf{1}\{ \tilde{X}^n(\infty) + n \delta \leq \tilde{y}^{\infty}(\infty)\delta/\eta + (n + \kappa n^\alpha)\delta \}] \leq \frac{\delta \lambda}{\mu}.
\]

Moreover, we can obtain the second moments of \( b_n^1 \) and \( b_n^2 \).
Lemma 10

\[ E[(b_n^1)^2] \leq \frac{1}{2} E[\lambda (2\delta \tilde{X}^n(\infty) + \delta^2)] + (2n + 1)\delta E \left[ \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) \right] \]

Proof of Lemma 10 Set \( u(x, y) = x^2 \), we have,

\[ E \left[ \lambda_n (2\delta \tilde{X}^n(\infty) + \delta^2) - \frac{1}{\delta} \left( \tilde{X}^n(\infty) + n\delta \right) \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) (2\delta \tilde{X}^n(\infty) - \delta^2) \right] = 0. \]

Thus,

\[ E[\lambda_n (2\delta \tilde{X}^n(\infty) + \delta^2)] = E \left[ \left( \tilde{X}^n(\infty) + n\delta \right) \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) (2\tilde{X}^n(\infty) - \delta) \right] \]

\[ = E \left[ \left( \tilde{X}^n(\infty) + n\delta \right) \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) (2\tilde{X}^n(\infty) + 2n\delta - 2n) \right] \]

Hence,

\[ E[\lambda_n (2\delta \tilde{X}^n(\infty) + \delta^2)] + (2n + 1)\delta E \left[ \left( \tilde{X}^n(\infty) + n\delta \right) \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) \right] \]

\[ \geq 2E \left[ \left( \tilde{X}^n(\infty) + n\delta \right) \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) \right]^2. \]

This produces an upper bound on the second moment of \( b_n^1 \). \( \square \)

Similarly,

Lemma 11

\[ E[(b_n^2)^2] \leq (\delta \gamma + \lambda_n \delta) E[(\tilde{Y}^n(\infty))] - E[E[(b_n^2)]E[(\tilde{Y}^n(\infty))^2] - \lambda_n \delta E[(\tilde{X}^n(\infty))] \]

\[ + \frac{1}{2} [\delta^2 \gamma + (\delta + \delta \kappa n^\alpha) E[b_n^2(\tilde{X}^n(\infty), \tilde{Y}^n(\infty))]] \]

Proof of Lemma 11 Let \( u(x, y) = xy \), we have, from Lemma 2,

\[ E \left[ \lambda_n \delta \tilde{Y}^n(\infty) - \frac{1}{\delta} \left( \tilde{X}^n(\infty) + n\delta \right) \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) \right] \delta \tilde{Y}^n(\infty) \]

\[ + \gamma \delta \tilde{X}^n(\infty) - \frac{1}{\delta} \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} - \tilde{X}^n(\infty) + \delta \kappa n^\alpha \right) \delta \tilde{X}^n(\infty) = 0. \]

Therefore,

\[ E \left[ \frac{1}{\delta} \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} - \tilde{X}^n(\infty) + \delta \kappa n^\alpha \right) \right] \delta \tilde{X}^n(\infty) \]

\[ = E \left[ \lambda_n \delta \tilde{Y}^n(\infty) - \frac{1}{\delta} \left( \tilde{X}^n(\infty) + n\delta \right) \left( \frac{\tilde{Y}^n(\infty)\delta}{\eta} + (n + \kappa n^\alpha)\delta \right) \right] \delta \tilde{Y}^n(\infty) - \gamma \delta \tilde{X}^n(\infty) \]
Two-Dimensional Markov Chain

\[
\geq E \left[ \lambda_n \delta \bar{Y}^n(\infty) - \gamma \delta \bar{X}^n(\infty) \right] - E[b_2^2]E[(\bar{Y}^n(\infty))^2]
\]

where the inequality is due to the Cauchy-Schwartz inequality. Meanwhile, \( u(x, y) = y^2 \) leads to,

\[
E \left[ \gamma [2\delta \bar{Y}^n(\infty) + \delta^2] - \frac{1}{\delta} \left( \frac{\bar{Y}^n(\infty)\delta}{\eta} - \bar{X}^n(\infty) + \delta \kappa n_\alpha \right)^+ \right] [2\delta \bar{Y}^n(\infty) - \theta^2] = 0.
\]

Hence,

\[
E[\gamma [2\delta \bar{Y}^n(\infty) + \delta^2]] = E \left[ \frac{1}{\delta} \left( \frac{\bar{Y}^n(\infty)\delta}{\eta} - \bar{X}^n(\infty) + \delta \kappa n_\alpha \right)^+ \right] [2\delta \bar{Y}^n(\infty) - \theta^2]
\]

Plug it into the previous one, we have,

\[
\geq E \left[ \frac{1}{\delta} \left( \frac{\bar{Y}^n(\infty)\delta}{\eta} - \bar{X}^n(\infty) + \delta \kappa n_\alpha \right)^+ \right] [2\delta \bar{Y}^n(\infty) - 2\delta \bar{X}^n(\infty) - \delta^2]
\]

\[
= \left( \delta + \delta \kappa n_\alpha \right) E \left( \bar{Y}^n(\infty) - \bar{X}^n(\infty) + \delta \kappa n_\alpha \right)^+
\]

\[
= 2E[b_2^2(\bar{X}^n(\infty), \bar{Y}^n(\infty))] - (\delta + \delta \kappa n_\alpha) E[b_2(\bar{X}^n(\infty), \bar{Y}^n(\infty))]
\]

Therefore, we have,

\[
E[b_2^2(\bar{X}^n(\infty), \bar{Y}^n(\infty))] \leq (\delta + \lambda_n \delta) E[(\bar{Y}^n(\infty))] - E[b_2^2]E[(\bar{Y}^n(\infty))^2] - \lambda_n \delta E[(\bar{X}^n(\infty))] + \frac{1}{2}[\delta^2 \gamma + (\delta + \delta \kappa n_\alpha) E[b_2(\bar{X}^n(\infty), \bar{Y}^n(\infty))]].
\]

\[
\Box
\]

4.2.2 Derivative Bounds

Recall that we need to bound terms related to the derivatives of solution to the Stein equation \( G u(x, y) = H(x, y) \), with \( H(x, y) = h(x, y) - Eh(X, Y) \), and

\[
G u(x, y) = \delta [\lambda_n - b_1(x, y) \mu] \frac{\partial}{\partial x} u(x, y) + \theta [\gamma - b_2(x, y) \nu] \frac{\partial}{\partial y} u(x, y) + \frac{1}{2} \frac{\partial^2}{\partial x^2} u(x, y).
\]

Note that the second order derivative is only related to the \( x \) direction, which reflects the fact that the randomness in the two-dimensional diffusion process comes from a one dimensional Brownian motion. This type of equation belongs to the family of degenerated Kolmogorov equations, for background, and detailed analysis, see, e.g. Menozzi (2018); Talay (2002); Soize (1994). Furthermore, the special form of the differential equation in our system allows us to further reduce it to an ordinary differential equation(ODE). More specifically, note that the Stein equation bears the following form,

\[
\delta [\lambda_n - b_1(x, y) \mu] u_x(x, y) + \eta [\gamma - b_2(x, y) \nu] u_y(x, y) + \frac{1}{2} u_{xx}(x, y) = H(x, y).
\]

(12)
Consider two separate domains. On \( \{ \frac{x}{\delta} + n \geq \frac{y}{\eta} + (n + \kappa \eta^\alpha) \} \) (12) becomes,

\[
\delta \left[ \lambda_n - \left( \frac{y}{\eta} + (n + \kappa \eta^\alpha) \right) \mu \right] u_x(x, y) + \eta \gamma u_y(x, y) + \frac{1}{2} u_{xx}(x, y) = H(x, y). \tag{13}
\]
or equivalently,

\[
-\left[ \frac{y \delta}{\eta} \mu + \kappa \eta^\alpha \delta \mu \right] u_x(x, y) + \eta \gamma u_y(x, y) + \frac{1}{2} u_{xx}(x, y) = H(x, y). \tag{14}
\]

When \( \frac{x}{\delta} + n < \frac{y}{\eta} + (n + \kappa \eta^\alpha) \), (12) takes the form,

\[
\delta \left[ \lambda_n - \frac{1}{\delta} (x + n \delta) \mu \right] u_x(x, y) \\
+ \delta \left[ \gamma - \left( \frac{y \delta}{\eta} - x + \kappa \eta^\alpha \right) \nu \right] u_y(x, y) + \frac{1}{2} u_{xx}(x, y) = H(x, y). \tag{15}
\]
or equivalently,

\[
-x \mu u_x(x, y) - \left( \frac{y \delta}{\eta} - x \right) \nu u_y(x, y) + \frac{1}{2} u_{xx}(x, y) = H(x, y). \tag{16}
\]

4.2.3 The solution in Domain I

In Domain I: \( \{ (x, y) : \frac{x}{\delta} + n \geq \frac{y}{\eta} + (n + \kappa \eta^\alpha) \} \), we have the equation (13). From well-known results on linear elliptic equation, see e.g. Krylov (1996), we know that the solution exists, and its Sobolev norm of \( u \) is bounded by that of the \( H(x, y) \) and the boundary condition, that is, \( u_x \) and \( u_y \) are bounded in \( L_p \) space for a proper \( p \). The solution can also be observed to have the following presentation,

\[
-\left[ \frac{y \delta}{\eta} \mu + \kappa \eta^\alpha \delta \mu \right] u_x(x, y) + \frac{1}{2} u_{xx}(x, y) = H(x, y) + Q(x, y) \\
\eta \gamma u_y = -Q(x, y).
\]

for some function \( Q(x, y) \). Thus,

\[
u(x, y) = \int_0^x \left[ \int_0^w \exp \left( -\int_0^z 2(y + \kappa \eta^\alpha) \mu du \right) [H(x, y) + Q(x, y)] dz \right] \\
\cdot \exp \left( \int_0^w 2(y + \kappa \eta^\alpha) \mu du \right) dw.
\]

Since the solution to a linear second order differential equation,

\[f''(x) + a(x)f'(x) + b(x) = 0,
\]
with proper boundary condition will have a solution in the form of
\[ f(x) = \int_0^x \left[ \int_0^w \exp \left( \int_0^z a(u)du \right) b(z)dz \right] \left\{ \exp \left( - \int_0^w a(u)du \right) \right\} dw. \]

and
\[ f'(x) = \left[ \int_0^x \exp \left( \int_0^z a(u)du \right) b(z)dz \right] \left\{ \exp \left( - \int_0^x a(u)du \right) \right\}. \]

Direct calculations, similar to those in Gurvich (2014b,a); Braverman and Dai (2017), thus provides us with the following bound for the solutions in domain I.

**Lemma 12**  \( u_{xxx} \) and \( u_{yy} \) are bounded quantities in domain I.

### 4.2.4 The solution in Domain II

In Domain II: \( \{(x, y) : \frac{x}{\delta} + n < \frac{y}{\eta} + (n + \kappa n^\alpha)\} \), we have the equation (16). The solution in domain I provide the values of \( u(x, y) \) on the line \( \{(x, y) : \frac{x}{\delta} + n = \frac{y}{\eta} + (n + \kappa n^\alpha)\} \), this serves as part of the boundary conditions for the solution in domain II, the other part is, of course, the original boundary condition. We will provide the necessary estimation via \textit{a priori} estimation of its solution, that is, obtain those estimation without solving the equation. (16) implies that, for any smooth function \( \phi(x, y) \) of polynomial growth, since \( H(x, y) \) is assumed to be integrable,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{y+\kappa} [ -xu_x - (y-x)u_y + \frac{1}{2}u_{xx} ] \phi(x, y) dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{y+\kappa} \phi(x, y)H(x, y) dxdy. \]

Note that, for the ease of exposition, we only discuss the case \( \eta = \delta \). It is easy to see that the results extend to general case. Integration by part gives us,

\[
- \int_{-\infty}^{\infty} \int_0^{y+\kappa} 2u(x, y)\phi(x, y) + u(x, y)[x\phi_x + (y-x)\phi_y + \frac{1}{2}\phi_{xx}] dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{y+\kappa} \phi(x, y)H(x, y) dxdy. \quad (17)
\]

**Lemma 13** For any bounded set \( \Omega \) in Domain II, there exists a constant \( C_1 \), such that,

\[
\int \int_{\Omega} u(x, y) dxdy \leq C_1.
\]
Proof Let \( \phi(x, y) = \eta u \) with \( \eta \) being a smooth function with suitable growth and \( u \) being the solution. The existence and integrability of itself and its generalized derivatives (regularity in Sobolev spaces) have been established in Menozzi (2018); Talay (2002). Thus,

\[
x \phi_x + (y - x) \phi_y + \phi_{xx} \\
= x \eta_x u + x \eta u_x + (y - x) \eta_y u + (y - x) \eta u_y + \eta u_x + 2 \eta_x u_x + \eta u_{xx} \\
= [x \eta_x + (y - x) \eta_y + \eta_{xx}] u + 2 \eta_x u_x + \eta [x u_x (y - x) u_y - u_{xx}] + 2 \eta u_{xx}.
\]

Plug it into (17). This follows the same approach that is conducted in Bensoussan and Frehse (2013), we can have a cut-off and/or mollifier of \( u \) instead of \( u \) itself if necessary. Apparently, the term \( \int x \eta_x + (y - x) \eta_y u - \eta_{xx} u \) is finite and known. Of course, the above quantity equal to \( \int_{-\infty}^{\infty} y \phi(y, y) u(y, y) dy \), which is known.

The term,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{y+\kappa} \eta_x u_x dx dy = \int_{0}^{\infty} \int_{0}^{y} u_x d\eta dy \\
= \int_{-\infty}^{\infty} u_x (y, y) \eta(y, y) - \int_{-\infty}^{\infty} \int_{-\infty}^{y+\kappa} \eta u_{xx} dx dy.
\]

This will cancel the term \( \eta u_{xx} \), hence, we can conclude that there exists a \( C \) such that,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{y+\kappa} [x \eta_x + (y - x) \eta_y + \eta_{xx}] u dx dy \leq C
\]

Now we can pick a proper \( \eta \) to have the desired result. For example, \( \eta \) is taken as \( \frac{x^2}{2} 1_\Omega \), then we can conclude \( \int_\Omega x + 1^2 u \leq C \), which is sufficient for the desired result.

Lemma 14 For any bounded set \( \Omega \) in Domain II, there exist positive constants \( C_2 \) and \( C_3 \), such that,

\[
\int \int_\Omega |u_{xxx}(x, y)| dx dy \leq C_2, \quad \int \int_\Omega |u_{yy}(x, y)| dx dy \leq C_3.
\]

Proof The above arguments also applies to \( \phi = \eta u_{xxx} \) and \( \phi = \eta u_{yy} \). In fact, this type of estimation falls into the general category of the Bernstein techniques, see, e.g. Oleinik and Kruzhkov (1961). Here, we made use of the solution in domain I, and some explicit calculation in the place of maximum principle that is normally instrumental in applying Bernstein techniques.

Lemmas 13 and 14, in conjunction with one of the moment bounds, implies that,

Proof of Lemma 3 To show that \( \mathbb{E} [E_1] = O(n \delta^3) \), we only need that \( u_{xxx} \) is locally integrable, which is the result of Lemmas 12 and 14. The same lemmas, in conjunction with Lemma 6, will guarantee that \( \mathbb{E} [E_2] = O(n \delta^3) \). Then, Lemma 12 for domain I and Lemma 14 for domain II indicate the boundedness of \( u_{yy} \), together with Lemmas 7, they imply, \( \mathbb{E} [E_i] = O(n^a \delta^2) \) for \( i = 3, 4 \).
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