A High-Order Accurate Numerical Algorithm for the Numerical Solution of Burgers’ Equation

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ABSTRACT

In this study, a high accurate numerical solution of the Burgers’ equation is obtained. For this, the collocation method based on quintic B-spline functions for space discretization and the fourth-order single step method for time discretization are used. In order to see the efficiency of the algorithm, a test problem with an analytical solution is discussed and compared with the numerical solution obtained.

Keywords: Burgers' equation, Collocation method, Fourth order single step method, Quintic B-spline functions

INTRODUCTION

The Burgers equation which is frequently encountered in many sciences such as engineering, theoretical and environmental is as follows:

\[ u_t + uu_x - \nu u_{xx} = 0. \]  (1)

Although this equation was first proposed by Bateman [1], it is known as Burgers because of his work [2]. Numerical solutions of this equation, used in modelling the solitary wave and travelling waves, have been studied by the researchers due to the limitations in analytical solutions. Numerical solutions, which have been investigated recently, have been obtained by least squares, splitting, homotopy perturbation, finite difference and quadrature methods [3-10]. In parallel with these studies, we used the quintic B-spline collocation method for space and the various-order single step methods for time discretization that were not implemented before. The aim of this study is to see the effectiveness of time discretization for the most accurate numerical solution of the Burgers’ equation.

Application of the Methods

The time and space steps are denoted with \( \Delta t \) and \( h \), respectively. The exact solution is represented by

\[ u(x_m, t_n) = u_m^n; \ m = 0, 1, \ldots, N; \ n = 0, 1, 2, \ldots \]

where \( x_m = a + mh \), \( t_n = n\Delta t \) and the numerical value of \( u_m^n \) is shown by \( U_m^n \) at the grid points.

Time discretization

Consider the Burgers’ equation of the form

\[ u_t = \nu u_{xx} - uu_x \]  (2)

and the following fourth order single step method

\[ u^{n+1} = u^n + \theta_1 u^{n+1}_1 + \theta_2 u^{n+1}_2 + \theta_3 u^{n+1}_3 + \theta_4 u^{n+1}_4. \]  (3)

By choosing \( \theta_1 = \theta_2 = \frac{\Delta t}{2}, \theta_3 = 0, \theta_4 = 0 \) in (4), we get the method 1, which is of order 2, known as Crank-Nicolson method. By changing the values as \( \theta_1 = \theta_2 = \frac{\Delta t}{2}, \theta_3 = -\frac{\Delta t}{12}, \theta_4 = \frac{\Delta t}{12} \), the method 2 which is of order 4 is obtained. Using Eqs. (2) and (3) in (4), the discretized Burgers’ equation in time is obtained as

\[ u^{n+1} + [\theta_1 u^{n+1}_1 - 2\theta_3 u^{n+1}_3(u_2)_n^{n+1}] + [4\theta_3 u^{n+1}_3] + \theta_2 u^{n+1}_2 + u^{n+1}_3(u_2)_n^{n+1}] + 2\theta_3 u^{n+1}_3(u_{xxx})_{n+1} - \theta_3 v^2(u_{xxx})_{n+1} = u^n + [-\theta_2 u^n + 2\theta_3 u^n(u_2)_n^1](u_2)_n^n \]  (5)

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+[−4vθ₄(uₓ)ₙ + θ₂v + θ₄uᵣuₙ] + \theta₄vₙ²(uₓₓₓ)ₙ².

Space discretization

The quintic B-spline function is defined as

\[ Q_m(x) = \begin{cases} \frac{1}{6}h^5((x-x_{m-3})^2(x-x_{m-2})^2, & x \in [x_{m-3},x_{m-2}) \\ \frac{1}{6}h^5((x-x_{m-3})^2 - 6(x-x_{m-2})^2, & x \in [x_{m-2},x_{m-1}) \\ \frac{1}{6}h^5((x-x_{m-3})^2 - 6(x-x_{m-2})^2 + 15(x-x_{m-1})^2, & x \in (x_{m-2},x_{m-1}) \\ \frac{1}{6}h^5((x-x_{m-3})^2 - 6(x-x_{m-2})^2 + 15(x-x_{m-1})^2, & x \in [x_{m-3},x_{m-2}) \end{cases} \]

The approximate solutions \( U_N(x,t) \) can be expressed in terms of the quintic B-spline functions as

\[ U_N(x,t) = \sum_{j=2}^{n+2} \delta_j(t) Q_j(x) \]

where \( \delta_j \) is the basis function. The order of convergence is obtained by the following formula:

\[ L_{\infty} \, ||u - U_N||_{\infty} = \max_j |u_j - (U_N)_j| \]

The order of convergence is obtained by the following formula:

\[ \text{order} = \log \left( \frac{u - U_{\Delta t n}}{u - U_{\Delta t n+1}} \right) \]

Numerical Example

In this part, we applied the proposed methods to one example of nonlinear Burgers’ equation. To compute the maximum error \( L_{\infty} \), we used the following formula:

\[ L_{\infty} \, ||u - U_N||_{\infty} = \max_j |u_j - (U_N)_j| \]

The order of convergence is obtained by the following formula:

\[ \text{order} = \log \left( \frac{u - U_{\Delta t n}}{u - U_{\Delta t n+1}} \right) \]

where \( u \) is the exact solution and \( U_{\Delta t n} \) is the numerical solution with time step \( \Delta t_n \). The exact solution, which models a shock propagation of the Burgers’ equation is as

\[ u(x,t) = \frac{x}{t} \, \left( 1 + \frac{t}{t_0} \, \exp \left( x^2/(4vt) \right) \right), \quad t \geq 1 \]

where \( t_0 = \exp \left( \frac{1}{h^2} \right) \). With this solution of the Burgers’ equation, the sharpness of the shock waves can be simulated by choice of various viscosity values. The initial shock is obtained from Eq. (11) with the choice of \( t = 1 \). The boundary conditions are taken as \( u(0,t) = 0 \) and \( u(1,t) = u_x(1,t) = 0 \). The computations are performed with the parameters \( v = 0.005, 0.0005, h = 0.001 \) and \( \Delta t = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002, 0.001 \) over the solution domain \([-2,2]\]. The propagation of the shock wave is simulated in Fig. 1 for \( v = 0.005, h = 0.001 \) and \( v = 0.0005, h = 0.001 \) by method 2. The figure shows that the obtained wave for the parameter \( v = 0.0005 \) is steeper than the other.
The absolute errors profiles of the proposed methods at time \( t = 3 \) is given in Fig. 2 for various viscosity coefficients. From these figures, as expected, the absolute error increases at the peak of the waves. For this problem, by calculating the \( L_\infty \) error norm and the rate of convergence of algorithms, the accuracy of the proposed methods can be seen and compared each other in Table 1. From this table we conclude that the method 2, for finding the shock wave solution of the Burgers’ equation, gives better accuracy than the method 1. The rate of convergence for time is around 2 for method 1 and around 4 for method 2.

**CONCLUSION**

In this paper, an effective algorithm is proposed for the numerical solution of the Burgers’ equation. This algorithm is constructed by using the quintic B-spline based collocation method for space discretization and fourth order single-step method for the time discretization. To compare the effect of the time discretization methods, the single step methods which has two different order as 2 and 4 is chosen and the obtained solutions are compared with the those of a test problem which has an exact solution. By this study it is concluded that the fourth order single step method which used for the time discretization gives the better solutions than the Crank-Nicolson method.

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**Figure 1.** Solutions for \( v = 0.005, 0.0005; \Delta t = h = 0.001 \).

**Figure 2.** Errors for \( v = 0.005, 0.0005; \Delta t = h = 0.001 \).
Table 1. Comparison of numerical results at different times for $h = 0.001$.

| $\Delta t$ | $L_\infty$ order | $L_\infty$ order |
|---|---|---|
| 0.1 | $5.76 \times 10^{-4}$ | $1.20 \times 10^{-4}$ |
| 0.05 | $1.54 \times 10^{-4}$ | $1.908 \times 10^{-5}$ | 3.123 |
| 0.02 | $2.55 \times 10^{-5}$ | $1.960 \times 10^{-7}$ | 3.521 |
| 0.01 | $6.42 \times 10^{-6}$ | $1.989 \times 10^{-8}$ | 3.748 |
| 0.005 | $1.61 \times 10^{-6}$ | $1.997 \times 10^{-9}$ | 3.820 |
| 0.002 | $2.58 \times 10^{-7}$ | $1.999 \times 10^{-10}$ | 3.590 |
| 0.001 | $6.44 \times 10^{-8}$ | $2.000 \times 10^{-11}$ | 1.637 |

| $\Delta t$ | $L_\infty$ order | $L_\infty$ order |
|---|---|---|
| 0.1 | $2.32 \times 10^{-4}$ | $2.05 \times 10^{-4}$ |
| 0.05 | $1.44 \times 10^{-1}$ | $0.692 \times 10^{-2}$ | 1.132 |
| 0.02 | $2.36 \times 10^{-2}$ | $1.973 \times 10^{-2}$ | 1.974 |
| 0.01 | $3.39 \times 10^{-3}$ | $2.800 \times 10^{-3}$ | 2.725 |
| 0.005 | $4.17 \times 10^{-4}$ | $3.024 \times 10^{-4}$ | 3.203 |
| 0.002 | $3.18 \times 10^{-5}$ | $2.807 \times 10^{-5}$ | 3.544 |
| 0.001 | $6.20 \times 10^{-6}$ | $2.360 \times 10^{-6}$ | 3.352 |

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