Derivation of the Hall-MHD Equations from the Navier–Stokes–Maxwell Equations

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Abstract
By using a set of scaling limits, the authors in Acheritogaray et al. (Kinet Relat Models 4:901–918, 2011) and Srinivasan and Shumlak (Phys Plasmas 18(9):620, 2011) proposed a framework of deriving the Hall-MHD equations from the two-fluids Euler–Maxwell equations for electrons and ions. In this paper, we derive the Hall-MHD equations from the Navier–Stokes–Maxwell equations with generalized Ohm’s law in a mathematically rigorous way via the spectral analysis and energy methods.

Keywords Hall-MHD equations · Navier–Stokes–Maxwell equations · Whole space · Spectral analysis · Energy estimates

Mathematics Subject Classification 76W05 · 35Q35 · 35D30

1 Introduction

The Hall-MHD equations have the following form in \([0, T] \times \mathbb{R}^3\):

\[
\begin{align*}
\partial_t u + \text{div}(u \otimes u) - \Delta u + \nabla p &= (\nabla \times B) \times B, \\
\partial_t B + \eta \nabla \times ((\nabla \times B) \times B) - \nabla \times (u \times B) &= \beta \eta^2 \Delta B, \\
\text{div} u &= 0, \\
\text{div} B &= 0.
\end{align*}
\] (1.1)
Here, $u(t, x)$ and $B(t, x)$ are the fluid velocity and the magnetic field, and $p$ is the pressure. The specific physical meaning of $\beta$ and $\eta$ is shown below. The Eq. (1.1) have been studied in physics for a long time and have many applications in the field of physics such as earth generators, etc. Under strong magnetic field or when the plasma density is relatively small, such as the magnetic field reconnection in space plasmas (Forbes 1991; Homann and Grauer 2005), star formation (Balbus and Terquem 2001; Wardle 2004), geo-dynamo (Mininni et al. 2003) and so on, the Hall effect cannot be ignored. Lighthill (1960) first studied the Hall effects and physically derived the Hall terms. Jiang and Masmoudi (2012) established a formally derivation of the so-called Hall effect by using the asymptotic analysis. Next, we sketch the formal derivation in Lighthill (1960). Since the fluids are composed of charged particles, their flow generates electric field $E$ and magnetic field $B$. Here the magnetic field $B$ satisfies Ampere’s law:

$$j = \nabla \times B,$$

where $j$ is the current density, while the electric field $E$ satisfies Faraday’s law:

$$\partial_t B = -\nabla \times E.$$

If $n_e$ and $n_i$ are the number densities of ions and electrons, we have

$$\rho = n_im_i + n_em_e, \quad u = \frac{n_im_iu_i + n_em_enu_e}{n_im_i + n_em_e},$$

where $m_i, m_e$ are the masses of ion and electron, $u_i$ and $u_e$ denote the velocities, and $\rho$ stands for the charge density. Since $m_i$ is much larger than $m_e$ (more than 1800 times), $u$ is very close to $u_i$, and $j$ can be approximated as

$$j = en_e(u - u_e),$$

where $-en_e$ is the electron charge density. The electrons and ions momentum equations have the following form:

$$n_em_e(\partial_t u_e + u_e \cdot \nabla u_e) = -\nabla p_e - M - n_e e(E + u_e \times B),$$
$$n_im_i(\partial_t u_i + u_i \cdot \nabla u_i) = -\nabla p_i + M + n_iZe(E + u_i \times B),$$

where $Z$ is the average charge of the ions and $n_e = n_iZ$. $p_i$ and $p_e$ are the pressures of ions and electrons, $M = -n_eek_j$ denotes the momentum loss of electrons per unit volume during the collision and $\kappa$ represents the resistivity. Since the ions are more massive than the electrons, the electron inertia [the left-hand side of (1.5)] can be neglected. Thus, (1.5) has the following approximation:

$$E = \kappa j + \frac{\nabla p_e}{n_e e} - u_e \times B.$$
Combining (1.2), (1.3), (1.4) and (1.7), we can formally derive the second equation of (1.1) when \( en_e = 1 \) and \( \kappa = 1 \) (see Campos 1998; Lighthill 1960 and the references therein for more details).

However, the above formal derivation about Hall term may not be satisfactory since, in general, Ampere’s law with Maxwell’s correction

\[
j + \partial_t E = \nabla \times B
\]

is required in electromagnetic effects.

Notice that from the two-phase flow model or dynamic model, the Hall-MHD system is derived through the two-layer scale limit (scaling limits) in Acheritogaray et al. (2011) and Srinivasan and Shumlak (2011). However, in these literature, only a framework is provided and rigorous mathematical proof is not given. In this paper, we want to rigorously derive Hall-MHD system (1.1) from the incompressible Navier–Stokes–Maxwell equations with generalized Ohm’s law, which takes into account the full electromagnetic phenomenon described by the Maxwell system. We first consider the following scaled two-fluid incompressible Navier–Stokes–Maxwell system stemming from Acheritogaray et al. (2011) with parameters \( \varepsilon \) and \( \gamma \):

\[
\begin{align*}
\varepsilon^2 (\partial_t u_e + \text{div} (u_e \otimes u_e) - \Delta u_e) + \nabla p_e &= -\alpha^2 (E + u_e \times B) - \beta (u_e - u_i), & \text{div} u_e &= 0, \\
\gamma \partial_t E - \nabla \times B &= -j, & \partial_t B + \nabla \times E &= 0, \\
\gamma^2 \partial_t E - \nabla \times B &= -j, & \text{div} B &= 0,
\end{align*}
\]

where \( 0 < \varepsilon^2 \ll 1 \) denotes the mass ratio of the electron to the ion, \( \alpha \) denotes the ratio of the electric energy to the thermal energy, and \( \beta \) stands for the relaxation frequency of the electron and ion velocities due to collisions. \( \eta \) denotes the ratio of the charge current scale to the electron or ion current scales. \( \gamma > 0 \) is the ratio of the fluid velocity to the speed of light.

Next, we will give a formal asymptotic analysis of Eq. (1.8). For fixed \( \gamma \), setting \( \alpha^2 \eta = 1 \), and letting \( u_i = u, p = p_i + p_e, \) and \( \varepsilon \to 0 \), we can formally get the following incompressible Navier–Stokes–Maxwell system with generalized Ohm’s law:

\[
\begin{align*}
\frac{1}{\eta} (E + u \times B) - \beta \eta j + \nabla p_e &= j \times B, & \text{div} j &= 0, \\
\partial_t u + \text{div} (u \otimes u) - \Delta u + \nabla p &= j \times B, & \text{div} u &= 0, \\
\gamma^2 \partial_t E - \nabla \times B &= -j, & \text{div} B &= 0,
\end{align*}
\]

which enjoys the formal energy conservation law:

\[
\frac{1}{2} \frac{d}{dt} \left( \| u \|^2_{L^2_x} + \| B \|^2_{L^2_x} + \| \gamma E \|^2_{L^2_x} \right) + \| \nabla u \|^2_{L^2_x} + \beta \eta^2 \| j \|^2_{L^2_x} = 0.
\]
Notice that if \( \text{div} E^0 = 0 \), we can immediately conclude that \( \text{div} E = 0 \).

Furthermore, letting \( \gamma \to 0 \), and then cancelling out the terms \( E \) and \( j \), the Hall-MHD system (1.1) is obtained formally. We know that for any initial data \((u^0, B^0) \in L^2(\mathbb{R}^3)\) such that \( \text{div} u^0 = 0 \), there is a global weak solution \((u, B) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \) to the system (1.1) (for example, see Chae et al. 2014).

Now, we begin to introduce the research history of the above systems. The Navier–Stokes–Maxwell system with ideal Ohm’s law can be written as

\[
\begin{align*}
\partial_t u + \text{div}(u \otimes u) - \Delta u &= -\nabla p + j \times B, \quad \text{div} u = 0, \\
\partial_t E - \nabla \times B &= -j, \\
\partial_t B + \nabla \times E &= 0,
\end{align*}
\tag{1.11}
\]

where \( \sigma > 0 \) is the electrical conductivity of the fluid. The global existence of finite energy weak solutions remains an interesting open problem, in both the dimensions \( d = 2, 3 \), for lack of compactness in the magnetic field \( B \), which prevents taking limits in the Lorentz force \( j \times B \). However, existence results are available when more regularity on the initial data is imposed. Masmoudi (2010) built up the global existence and uniqueness of the strong solutions to (1.11) in two dimensions with any large initial data \((u^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times (H^s)^2, \ 0 < s < 1 \). Later, Ibrahim and Keraani (2011) established the global existence of strong solutions to (1.11) for the small initial data \((u^0, E^0, B^0) \in \dot{B}_{2,1}^{1/2} \times (H^{1/2})^2 \) in three dimensions, and the initial data \((u^0, E^0, B^0) \in \dot{B}_2^0 \times \left( \frac{L^2_{\log}}{\gamma} \right)^2 \) in two dimensions. These results are extended in Germain et al. (2014) to small initial data \((u^0, E^0, B^0) \in (H^{1/2})^3 \) in three dimensions, and \((u^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times \left( \frac{L^2_{\log}}{\gamma} \right)^2 \) in two dimensions by applying fixed point arguments. Here the space \( L^2_{\log} \) is the set of tempered distributions with the norm \( \| \phi \|_{L^2_{\log}}^2 := \sum_{q \leq 0} \| \Delta_q \phi \|_{L^2}^2 + \sum_{q > 0} q \| \Delta_q \phi \|_{L^2}^2 < \infty \), where \( \Delta_q \) denotes the dyadic frequency localization operator (for more details, see Germain et al. 2014). Recently, Arsénie and Gallagher (2020) studied global existence of weak solutions in largest possible functional spaces based on a new maximal estimate on the heat equation in Besov spaces, under the hypothesis that the initial data \((u^0, E^0) \) lies in the energy space and that the initial magnetic field \( B_0 \) is sufficient small in \( H^s(\mathbb{R}^3) \) with \( s \in \left[ \frac{1}{2}, \frac{3}{4} \right] \). For the initial data without any restriction, they also built up the global existence of weak solutions to (1.11) in \( \mathbb{R}^2 \). In particular, the result in two-dimensional space improved the asymptotic limit established in Arsénie et al. (2015) and the global well-posedness result obtained in Masmoudi (2010). See also Ibrahim and Yoneda (2012) and Ibrahim et al. (2018) and the references therein for more well-posedness results of the system (1.11).

There have been many research results for the Hall-MHD equations. Acheritog-}

aray et al. (2011) proposed a framework of deriving the Hall-MHD equations from the two-fluids Euler–Maxwell system for electrons and ions (see also Srinivasan and Shumlak 2011) and proved the global existence of weak solutions in the periodic domain. Chae et al. (2014) obtained the global existence of weak solutions and the local well-posedness of classical solutions in the three-dimensional whole space. They
also established the blow-up criterion and the global existence of classical solutions for small initial data. Later, Chae and Lee (2014) generalized the results of Chae et al. (2014). Chae and Wolf (2017) investigated the partial regularity of the weak solutions of the three-dimensional Hall-MHD equations on the plane and obtained that the space–time Hausdorff dimension of the set of possible singularities for a weak solution is at most 2. Chae and Schonbek (2013) established the optimal time decay rate of weak solutions in the three-dimensional whole space. Chae and Weng (2016) considered the singularity formation for the Hall-MHD equations without resistivity in \( \mathbb{R}^3 \). Recently, Dumas and Sueur (2014) investigated energy conservation of weak solutions to the three-dimensional Hall-MHD equations. Dai (2016) established the regularity criterion of the 3D incompressible resistive viscous Hall-MHD equations. Benvenutti and Ferreira (2016) considered the stability of global large strong solutions. Very recently, Dai (2021) obtained the non-uniqueness of the Leray–Hopf weak solution by using the convex integration scheme. For more results on the Hall-MHD equations, see Dreher et al. (2005), Fan et al. (2013), Polygiannakis and Moussas (2001) and Wan and Zhou (2015). For the generalized Hall-MHD equations, Dai and Liu (2022) established the local well-posedness in the Besov space. Chae et al. (2015) obtained the local well-posedness of the smooth solution of the Hall-MHD equations with fractional magnetic diffusion. We refer the readers to Dai and Liu (2019), Jiang and Zhu (2018), Pan and Zhu (2017), Wan (2015), Wan and Zhou (2017), Wu et al. (2017), Ye (2015) and the references therein for the well-posedness results and the regularity criteria of the generalized Hall-MHD equations.

Compared to the system (1.11), there are very little researches have been done on (1.9). The existence of weak solutions to (1.9) in energy space seems very challenging for lack of weak convergence of the Lorentz force \( j \times B \) included both in generalized Ohm’s law and the momentum conservation equation. The lack of weak convergence also reflects the difficulties we encounter in the asymptotic analysis of (1.9). In the present paper, we impose the hypothesis of existence of the global and finite energy weak solutions to system (1.9).

Before starting the main result of this paper, let us first introduce the notations and conventions used throughout this paper.

**Notations:**

1. For any positive \( A \) and \( B \), we use the notation \( A \lesssim B \) to mean that there exists a positive constant \( C \) such that \( A \leq CB \).
2. The Fourier transform of \( f \in L^1(\mathbb{R}^3) \) is defined by

\[
\mathcal{F} f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x \cdot \xi} f(x) dx,
\]

and its inverse is defined by

\[
\mathcal{F}^{-1} g(x) = \hat{g}(x) := \int_{\mathbb{R}^3} e^{2\pi i x \cdot \xi} g(\xi) d\xi.
\]
Let $S(\mathbb{R}^3)$ be the Schwartz space, and then the Fourier transform of $f \in S'$ is the tempered distribution $\hat{f}$ given by

$$<\hat{f}, \varphi> = <f, \hat{\varphi}>, \quad \varphi \in S.$$ 

In particular, for any $f \in L^2(\mathbb{R}^3)$, we have $\hat{f} \in L^2(\mathbb{R}^3)$ with

$$\hat{f}(\xi) = \lim_{M \to \infty} \int_{|x| < M} e^{-2\pi i x \cdot \xi} f(x) dx.$$ 

For more details, see Duoandikoetxea and Zuazo (2001, Chapter 1).

(3) For every $p \in [1, \infty]$, let $\|\cdot\|_{L^p}$ denote the norm in the Lebesgue space $L^p$. We denote $\|\cdot\|_{L^2_{t,x,loc}} := \|\cdot\|_{L^2_{t,loc} \times L^2_{x,loc}}$ for simplicity.

(4) For any normed space $X$, we employ the notation $L^p([0, T], X)$ to denote the space of functions $f$ such that for almost all $t \in (0, T)$, $f(t) \in X$ and $\|f(t)\|_X \in L^p(0, T)$. We simply denote the notation $L^p_T X := L^p([0, T], X)$.

(5) The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$, for any $s \in \mathbb{R}$, as the subspace of tempered distributions whose Fourier transform is locally integrable and the following norm is finite:

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 d\xi \right)^{1/2}.$$ 

Now, we are ready to state our main result in this paper.

**Theorem 1.1** Let $s \in \left(\frac{1}{2}, 1\right)$ be fixed. For any $\gamma > 0$, we assume that $(u^\gamma, E^\gamma, B^\gamma)$ is the global and finite energy weak solution to the incompressible Navier–Stokes–Maxwell system with generalized Ohm’s law (1.9) for some uniformly bounded initial data

$$(u^0, E^0, B^0) \in \left(H^s \times (L^2)^2\right)(\mathbb{R}^3),$$

with $\text{div} u^0 = \text{div} B^0 = \text{div} E^0 = 0$, and suppose that the initial data converges weakly in $H^s \times (L^2)^2$, as $\gamma \to 0$, to some quantity

$$(u^0, E^0, B^0) \in \left(H^s \times (L^2)^2\right)(\mathbb{R}^3),$$

with $\text{div} u^0 = \text{div} B^0 = \text{div} E^0 = 0$. Besides, we also assume that

$$u^\gamma \in L^2_t H^{1+s}_x$$

are uniformly bounded, \hspace{1cm} (1.12)

and that for any $\delta > 0$,

$$\lim_{\gamma \to 0} \sup_{j \gg} \|j^\gamma\|_{L^2_{t,x,loc}} = 0, \hspace{1cm} (1.13)$$

\hspace{1cm} \square
where
\[
j_{\gamma}^{\nu} = \mathcal{F}^{-1} \left( \chi_{\{ |\xi| \geq \phi(\frac{\gamma}{\delta}) \}} \mathcal{F} j^{\nu} \right), \quad \phi(\gamma) = \gamma^{\frac{2}{s-3}}.
\] (1.14)

Then, taking $\gamma \to 0$, up to extraction of a subsequence, $(u^{\nu}, B^{\nu})$ converges weakly to a global and finite energy weak solution $(u, B)$ of the Hall-MHD equations (1.1) with initial data $(u^0, B^0)$.

**Remark 1.1** We say that $(u, B) \in L_1^\infty L_2^2 \cap L_1^2 H_1^1$, $t \in [0, T]$ is a weak solution of the Hall-MHD equation (1.1) provided $(u, B)$ satisfies (1.1) in the sense of distribution for any given time $T < \infty$. Similarly, $u \in L_1^\infty L_2^2 \cap L_1^2 H_1^1$, $(E, B) \in L_1^\infty L_2^2$, $j \in L_1^2 L_2^2$ is a weak solution of the incompressible Navier–Stokes–Maxwell system (1.9) provided $(u, E, B, j)$ satisfies (1.9) in the sense of distribution.

Now, we sketch the strategy of proving this theorem and point out some of the main difficult and techniques involved in the process. The convergence of the linear terms in (1.9) is easy to handle with thanks to the formal energy conservation law. Furthermore, the dissipation on $u$ is clearly sufficient to establish the weak convergence of the nonlinear terms $u \times B$ and $\text{div}(u \otimes u)$. Thus, in order to prove the asymptotic limit of (1.9) rigorously, it is sufficient to establish the convergence of the Lorentz force $j^{\nu} \times B^{\nu}$ towards $j \times B$, in the sense of distribution. To this end, inspired by Arsenio et al. (2015), we will take a careful analysis of the frequency distribution of magnetic field $B^{\nu}$, which is based on the spectral properties of Maxwell operator. Unfortunately, it seems hopeless to prove that the operator
\[
\left( \begin{array}{cc}
\frac{1}{\beta \eta^2} & \frac{1}{\gamma^2} \nabla \times \\
-\frac{1}{\gamma^2} \nabla \times & 0
\end{array} \right)
\]
is the infinitesimal generator of some contraction semi-group, which means that applying Duhamel’s formula may be impossible. Moreover, the modified antisymmetric operators given by
\[
\left( \begin{array}{cc}
\frac{1}{\beta \eta^2} & \frac{1}{\gamma^2} \nabla \times \\
-\frac{1}{\gamma^2} \nabla \times & 0
\end{array} \right) \text{ and } \left( \begin{array}{cc}
\frac{1}{\beta \eta^2} & \frac{1}{\gamma} \nabla \times \\
-\frac{1}{\gamma} \nabla \times & 0
\end{array} \right),
\]
can be the infinitesimal generator of some contraction semi-group. However, the analysis of the frequencies of $B^{\nu}$ seems difficult since the tricky term $B^{\nu}$ in the right-hand side cannot be cancelled out according to the Duhamel’s formula. Luckily, the same problem does not trouble us when we choose the modified antisymmetric operator as
\[
\left( \begin{array}{cc}
\frac{1}{\beta \eta^2} & \frac{1}{\gamma} \nabla \times \\
-\frac{1}{\gamma} \nabla \times & 0
\end{array} \right)
\] [see (2.4) in Sect. 2], since the term $B^{\nu}$ in the right-hand side will vanish after integrating by parts [see (3.13) in Sect. 3]. Furthermore, the systems we are dealing with are more complex and require more sophisticated analysis and energy estimates.
The rest of the paper is arranged as follows. In Sect. 2, we establish spectral analysis of linear Maxwell system in (1.9) and two useful lemmas. We provide a rigorous proof of Theorem 1.1 in Sect. 3.

2 Spectral Analysis

Now, let’s move on to the detailed spectral analysis on the Maxwell operator. Define

\[
\begin{aligned}
G_1 &:= \left( \frac{1}{\nu^2} - \frac{1}{\nu} \right) \nabla \times B + \frac{1}{\beta \eta \nu^2} P \left( \partial_t u + \text{div} G_4 - \Delta u - \frac{1}{\eta} G_3 \right), \\
G_2 &:= \left( \frac{1}{\nu} - 1 \right) \nabla \times E = \left( 1 - \frac{1}{\nu} \right) \partial_t B, \\
G_3 &:= u \times B, \\
G_4 &:= u \otimes u,
\end{aligned}
\]

where \( P \) is the Leray projector onto divergence-free vector fields. Clearly, the Maxwell’s system in (1.9) can be rewritten as

\[
\begin{aligned}
\partial_t E &= -\frac{1}{\beta \eta \nu^2} E + \frac{1}{\nu} \nabla \times B + G_1, \\
\partial_t B &= -\frac{1}{\nu} \nabla \times E + G_2, \\
\text{div} B &= 0,
\end{aligned}
\]

or, equivalently,

\[
\partial_t \begin{pmatrix} E \\ B \end{pmatrix} = A \begin{pmatrix} E \\ B \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},
\]

where Maxwell’s operator \( A \) is given by

\[
A := \begin{pmatrix}
-\frac{\text{Id}}{\beta \eta \nu^2} & \frac{1}{\nu} \nabla \times \\
-\frac{1}{\nu} \nabla \times & 0
\end{pmatrix}.
\]

More precisely, define

\[
\begin{aligned}
X &:= \left\{ (E, B) \in \left( L^2 \left( \mathbb{R}^3 \right) \right)^2 : \text{div} B = 0 \right\}, \\
\mathcal{D}(A) &:= \left\{ (E, B) \in X : (PE, B) \in \left( H^1 \left( \mathbb{R}^3 \right) \right)^2 \right\} \subset X.
\end{aligned}
\]

Then, one can verify that the unbounded linear operator \( A : \mathcal{D}(A) \to X \) is closed and that \( \mathcal{D}(A) \) is dense in \( X \). Moreover, for any \( \lambda > 0 \), a direct computation shows that the operator

\[
\lambda \text{Id} - A : \mathcal{D}(A) \to X
\]
is bijective and that \( \| (\lambda \mathrm{Id} - \mathbf{A})^{-1} \| \leq \frac{1}{\chi} \). In view of the Hille–Yosida theorem, there exists a contraction semi-group denoted by \( \{ e^{t\mathbf{A}} \}_{t \geq 0} \) with \( \mathbf{A} \) being the infinitesimal generator of it. Therefore, for any initial data \((E^0, B^0) \in \mathcal{D}(\mathbf{A})\) and any inhomogeneous term \((G_1, G_2) \in C^1(\mathbb{R}^+, X)\), the unique strong solution to (2.3)

\[
(E, B) \in C\left(\mathbb{R}^+, \mathcal{D}(A)\right) \cap C^1(\mathbb{R}^+, X),
\]

is given by Duhamel’s formula:

\[
\begin{pmatrix}
E

B
\end{pmatrix}
=

\int_0^t e^{(t-\tau)\mathbf{A}} \begin{pmatrix}
G_1

G_2
\end{pmatrix}(\tau) \, d\tau.
\tag{2.5}
\]

Moreover, Duhamel’s formula (2.5) is well defined if \((G_1, G_2) \in L^1(0, T, X)\), \((E^0, B^0) \in X\). In fact, \(e^{(t-\tau)\mathbf{A}} \begin{pmatrix}
G_1

G_2
\end{pmatrix}(\tau)\) is strongly measurable, since it is weakly measurable for any \(t > 0\), and \(L^2(\mathbb{R}^3)\) is separable. Notice that

\[
\tau \rightarrow \left\| e^{(t-\tau)\mathbf{A}} \begin{pmatrix}
G_1

G_2
\end{pmatrix}(\tau) \right\|_X
\]

is integrable, therefore

\[
e^{(t-\tau)\mathbf{A}} \begin{pmatrix}
G_1

G_2
\end{pmatrix}(\tau)
\]

is Bochner integrable. Furthermore, one can easily verify that (2.5) solves (2.3) in the sense of distribution by employing the approximation arguments in this setting. Thus, the weak solution \((E, B)(t) \in C\left(\mathbb{R}^+, X\right)\) of (2.3) which can be given by (2.5) is unique.

In order to conduct a spectral analysis of \(\mathbf{A}\), taking the Fourier transform on (2.5), we get

\[
\begin{aligned}
\partial_t \begin{pmatrix}
\hat{E}

\hat{B}
\end{pmatrix} &= \hat{\mathbf{A}}(\xi) \begin{pmatrix}
\hat{E}

\hat{B}
\end{pmatrix} + \begin{pmatrix}
\hat{G}_1

\hat{G}_2
\end{pmatrix},
\end{aligned}
\tag{2.6}
\]

where

\[
\hat{\mathbf{A}}(\xi) = \begin{pmatrix}
\frac{-\mathrm{Id}}{\beta \eta \gamma z} & \frac{1}{\gamma} i \xi \times \\
\frac{-1}{\gamma} i \xi \times & 0
\end{pmatrix}.
\]

More precisely, for every \(\xi \in \mathbb{R}^3 \setminus \{0\}\), following (Arsénio et al. 2015), we define the 5-dimensional vector subspace of \(\mathbb{C}^3 \times \mathbb{C}^3\) as:

\[
\mathcal{E}(\xi) := \left\{ (e, b) \in \mathbb{C}^3 \times \mathbb{C}^3 : \xi \cdot b = 0 \right\}.
\]
then \( \hat{\mathbf{A}}(\xi) : \mathcal{E}(\xi) \to \mathcal{E}(\xi) \) is a linear finite-dimensional operator.

Now, we give a detailed analysis on the properties of the semi-group inspired by Arsenio et al. (2015).

**Lemma 2.1** For any \( \xi \in \mathbb{R}^3 \setminus \{0\} \), such that \( |\xi| \neq \frac{1}{2\beta \eta^2 \gamma} \), there are three different eigenvalues of \( \hat{\mathbf{A}}(\xi) \) denoted by \( \lambda_0 = -\frac{1}{\beta \eta^2 \gamma} \), \( \lambda_+ (\xi) \), \( \lambda_- (\xi) \) with

\[
\lambda_{\pm} (\xi) = -1 \pm \sqrt{1 - 4\beta^2 \eta^4 \gamma^2 |\xi|^2} \quad \frac{2 \beta \eta^2 \gamma^2}{2 \beta \eta^2 \gamma^2}.
\]  

Moreover, the basis of the subspace \( \mathcal{E}(\xi) \) can be made up by eigenvectors, which means \( \hat{\mathbf{A}}(\xi) \) is diagonalizable, and the eigenspaces corresponding to \( \lambda_0 \), \( \lambda_+ (\xi) \) and \( \lambda_- (\xi) \) are, respectively, given by

\[
\mathcal{E}_0(\xi) = \text{span} \left\{ \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\},
\]

\[
\mathcal{E}_+(\xi) = \left\{ \begin{pmatrix} \frac{1}{\gamma \lambda_+} e \times \xi \\ e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 : e \in \mathbb{C}^3, \xi \cdot e = 0 \right\},
\]

\[
= \left\{ \begin{pmatrix} \frac{1}{\gamma \lambda_-} e \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 : b \in \mathbb{C}^3, \xi \cdot b = 0 \right\}, \tag{2.8}
\]

\[
\mathcal{E}_-(\xi) = \left\{ \begin{pmatrix} \frac{1}{\gamma \lambda_-} e \times \xi \\ e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 : e \in \mathbb{C}^3, \xi \cdot e = 0 \right\},
\]

\[
= \left\{ \begin{pmatrix} \frac{1}{\gamma \lambda_+} e \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 : b \in \mathbb{C}^3, \xi \cdot b = 0 \right\}. \tag{2.9}
\]

For any \( \xi \in \mathbb{R}^3 \setminus \{0\} \) with \( |\xi| = \frac{1}{2\beta \eta^2 \gamma^2} \), there are two different eigenvalues of \( \hat{\mathbf{A}}(\xi) \) denoted by \( \lambda_0 = -\frac{1}{2\beta \eta^2 \gamma^2} \), \( \lambda_1 = -\frac{1}{2\beta \eta^2 \gamma^2} \), and the corresponding eigenspaces are, respectively, given by

\[
\mathcal{E}_0(\xi) = \text{span} \left\{ \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\},
\]

\[
\mathcal{E}_1(\xi) = \left\{ \begin{pmatrix} \frac{1}{\lambda_1 \gamma} e \times \xi \\ e \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 : e \in \mathbb{C}^3, \xi \cdot e = 0 \right\},
\]

\[
= \left\{ \begin{pmatrix} \frac{1}{\lambda_1 \gamma} e \times b \\ b \end{pmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3 : b \in \mathbb{C}^3, \xi \cdot b = 0 \right\}. \tag{2.10}
\]

In this case, \( \hat{\mathbf{A}}(\xi) \) is not diagonalizable.
**Proof** Suppose that
\[ \lambda \begin{pmatrix} e \\ b \end{pmatrix} = \hat{A}(\xi) \begin{pmatrix} e \\ b \end{pmatrix}, \]
where \( \lambda \) and \( \begin{pmatrix} e \\ b \end{pmatrix} \) are eigenvalue and eigenvector of \( \hat{A}(\xi) \), respectively. Then we have
\[ \begin{cases} (\lambda + \frac{1}{\beta \eta^2 \gamma^2}) e = \frac{i}{\gamma} \xi \times b, \\ \lambda b = -\frac{i}{\gamma} \xi \times e. \end{cases} \tag{2.11} \]

Firstly, \( e \neq 0 \), otherwise \( b = 0 \). Notice that \( \xi \cdot b = 0 \), then \( b = 0 \) and \( \xi \times e = 0 \) provided \( \lambda = -\frac{1}{\beta \eta^2 \gamma^2} \). Thus \( \lambda = -\frac{1}{\beta \eta^2 \gamma^2} \) is an eigenvalue of \( \hat{A}(\xi) \) with the corresponding eigenspace given by \( \mathcal{E}_0(\xi) = \text{span} \left\{ \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\} \).

Now, setting \( \lambda \neq -\frac{1}{\beta \eta^2 \gamma^2} \), multiplying (2.11) by \( \xi \), we have
\[ \xi \cdot e = 0, \quad \beta \eta^2 \gamma^2 \lambda^2 + \lambda + \beta \eta^2 \xi^2 = 0, \]
whose roots \( \lambda_+(\xi) \) and \( \lambda_-\)(\xi) \) are exactly given by (2.7).

While \( |\xi| = \frac{1}{2 \beta \eta^2 \gamma^2} \), \( \lambda_+(\xi) = \lambda_-\)(\xi) = -\frac{1}{2 \beta \eta^2 \gamma^2} \) and the corresponding eigenspace \( \mathcal{E}_1(\xi) \) is defined by (2.10), one can verify that the operator \( \hat{A}(\xi) \) is not diagonalizable in this condition.

Next, setting \( |\xi| \neq \frac{1}{2 \beta \eta^2 \gamma^2} \), \( \lambda_+(\xi) \) and \( \lambda_-\)(\xi) \) are different. The corresponding eigenspaces \( \mathcal{E}_+(\xi) \) and \( \mathcal{E}_-(\xi) \) are given by (2.8) and (2.9), respectively. Therefore, \( \hat{A}(\xi) \) is diagonalizable in this case. Furthermore, these eigenspaces are not orthogonal to each other since the operator \( \hat{A}(\xi) \) is not symmetric. We complete the proof of Lemma 2.1. \( \square \)

**Lemma 2.2** Suppose that \( 1 < K < 2 \) is a fixed constant. For any \( \xi \in \mathbb{R}^3 \setminus \{0\} \), \( \lambda_{\pm}(\xi) \) are the eigenvalues of \( \hat{A}(\xi) \) defined by (2.7), then we have the following estimates.
If \( |\xi| \leq \frac{1}{2 \beta \eta^2 \gamma^2} \),
\[ -2 \beta \eta^2 |\xi|^2 \leq \lambda_+(\xi) \leq -\beta \eta^2 |\xi|^2, \quad -\frac{1}{\beta \eta^2 \gamma^2} \leq \lambda_-\)(\xi) \leq -\frac{1}{2 \beta \eta^2 \gamma^2}, \]
moreover, if \( |\xi| \leq \frac{1}{2K \beta \eta^2 \gamma^2} \),
\[ \left| \frac{\lambda_-\)(\xi)}{\lambda_-\)(\xi) - \lambda_+(\xi)} \right| \leq \frac{K}{\sqrt{K^2 - 1}}, \quad \left| \frac{\lambda_+(\xi)}{\lambda_-\)(\xi) - \lambda_+(\xi)} \right| \lesssim \gamma^2 |\xi|^2. \]
If $|\xi| > \frac{1}{2\beta\eta^2\gamma}$,

$$|\lambda_+ (\xi)| = |\lambda_- (\xi)| = \frac{|\xi|}{\gamma}, \quad \Re (\lambda_+ (\xi)) = \Re (\lambda_- (\xi)) = -\frac{1}{2\beta\eta^2\gamma^2},$$

moreover, if $|\xi| \geq \frac{K}{2\beta\eta^2\gamma}$,

$$\left|\frac{\lambda_+ (\xi)}{\lambda_- (\xi) - \lambda_+ (\xi)}\right| \leq \frac{K}{2\sqrt{K^2 - 1}}, \quad \left|\frac{1}{\lambda_- (\xi) - \lambda_+ (\xi)}\right| \leq \frac{K}{2\sqrt{K^2 - 1}}\frac{\gamma}{|\xi|}.$$

**Proof** We first consider the case $|\xi| \leq \frac{1}{2\beta\eta^2\gamma}$. Note that

$$
\lambda_+ (\xi) = -1 + \sqrt{1 - 4\beta^2\eta^4\gamma^2|\xi|^2} = -\frac{4\beta^2\eta^4\gamma^2|\xi|^2}{2\beta^2\eta^2\gamma^2 (1 + \sqrt{1 - 4\beta^2\eta^4\gamma^2|\xi|^2})} = \frac{-2\beta^2\eta^4\gamma^2|\xi|^2}{1 + \sqrt{1 - 4\beta^2\eta^4\gamma^2|\xi|^2}},
$$

then we have $-2\beta^2\eta^2|\xi|^2 \leq \lambda_+(\xi) \leq -\beta^2\eta^2|\xi|^2$. The similar estimate on $\lambda_- (\xi)$ is trivial. Furthermore, one has

$$
\left|\frac{\lambda_- (\xi)}{\lambda_- (\xi) - \lambda_+ (\xi)}\right| = \frac{1 + \sqrt{1 - 4\beta^2\eta^4\gamma^2|\xi|^2}}{2\sqrt{1 - 4\beta^2\eta^4\gamma^2|\xi|^2}} \leq \frac{1}{\sqrt{1 - 4\beta^2\eta^4\gamma^2|\xi|^2}} \leq \frac{K}{\sqrt{K^2 - 1}},
$$

provided $|\xi| \leq \frac{1}{2K\beta\eta^2\gamma}$.

Next, focusing on the case $|\xi| > \frac{1}{2\beta\eta^2\gamma}$, writing

$$
\lambda_\pm (\xi) = \frac{1}{2\beta\eta^2\gamma^2} \left( -1 \pm i \sqrt{4\beta^2\eta^4\gamma^2|\xi|^2 - 1} \right),
$$

we get

$$|\lambda_+ (\xi)| = |\lambda_- (\xi)| = \frac{|\xi|}{\gamma}, \quad \Re (\lambda_+ (\xi)) = \Re (\lambda_- (\xi)) = -\frac{1}{2\beta\eta^2\gamma^2}.$$

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Furthermore, assuming that $|\xi| \geq \frac{K}{2\beta^2\eta^2\gamma}$, one has

$$
\left| \frac{\lambda_\pm(\xi)}{\lambda_-(\xi) - \lambda_+(\xi)} \right| = \frac{\beta\eta^2\gamma^2}{\sqrt{4\beta^2\eta^4\gamma^2|\xi|^2 - 1}} |\xi| \leq \frac{1}{\sqrt{4 - \frac{1}{\beta^2\eta^4\gamma^2|\xi|^2}}} \leq \frac{K}{2\sqrt{K^2 - 1}},
$$

and the last estimate is followed. \hfill \Box

### 3 Proof of the Main Result

In this section, we study the asymptotic limit of the incompressible Naiver–Stokes–Maxwell system with generalized Ohm’s law (1.9) based on the spectral analysis established in Sect. 2.

Here, we consider a family of global and finite energy weak solutions $(u^\gamma, E^\gamma, B^\gamma)$ to (1.9) with uniformly bounded initial data:

$$(u^0, E^0, B^0) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3).$$

Recall that the formal energy conservation law (1.10) implies that the following uniform bounds of the weak solutions $(u^\gamma, E^\gamma, B^\gamma)$:

$$u^\gamma \in L^\infty_t L^2_x \cap L^2 \dot{H}^{\frac{1}{2}}_x, \quad (B^\gamma, \gamma E^\gamma) \in L^\infty_t L^2_x, \quad j^\gamma \in L^2_{t,x}.$$

Therefore, up to extraction of subsequences, we have the following weak convergences as $\gamma \to 0$:

$$(u^0, E^0, B^0) \rightharpoonup (u, E, B) \text{ in } L^2_x,$n
$$(u^\gamma, B^\gamma) \rightharpoonup^* (u, B) \text{ in } L^\infty_t L^2_x,$n
$$j^\gamma \rightharpoonup j \text{ in } L^2_{t,x}.$n

From the momentum conservation equation in (1.9), we find that for any $\varphi \in C^\infty_c(\mathbb{R}^3),$

$$(\partial_t u^\gamma, \varphi)_{L^2_x} = \left\langle P \left( \text{div}(u^\gamma \otimes u^\gamma) + \Delta u^\gamma + j^\gamma \times B^\gamma \right), \varphi \right\rangle_{L^2_x} \lesssim \|\varphi\|_{H^3} \left( \|u^\gamma\|_{L^2_x}^2 + \|u^\gamma\|_{L^2_x} + \|j^\gamma\|_{L^2_x} \|B^\gamma\|_{L^2_x} \right),$$

which yields that

$$\|\partial_t u^\gamma\|_{H^{-3}} \lesssim \left( \|u^\gamma\|_{L^2_x}^2 + \|u^\gamma\|_{L^2_x} + \|j^\gamma\|_{L^2_x} \|B^\gamma\|_{L^2_x} \right).$$

Therefore, we have

$$\partial_t u^\gamma \text{ is uniformly bounded in } L^2_{i,\text{loc}}H^{-3}_x.$$ (3.1)
By virtue of the Sobolev embedding and the regularity hypothesis (1.12), $u^\gamma$ is uniformly bounded in

$$L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1 \cap L_t^2 \dot{H}_x^{1+s} \subset L_t^\infty L_x^2 \cap L_t^2 L_x^6 \cap L_t^2 L_x^\infty,$$

then a routine application of classical compactness result by Simon (1986) implies that $u$ is relatively compact in the strong topology of $L_{t,x}^2$. Furthermore, the regularity hypothesis (1.12) and Sobolev embedding also imply that $\nabla u^\gamma$ is uniformly bounded in $L_t^\infty L_x^6$. Inserting the bounds on $B^\gamma$, we have

$$u^\gamma \times B^\gamma \in L_t^2 L_x^2, \quad u^\gamma \otimes u^\gamma \in L_t^2 L_x^2, \quad \nabla (u^\gamma \otimes u^\gamma) \in L_t^2 L_x^{\frac{3}{s}}. \quad (3.2)$$

whence,

$$G_3^\gamma := u^\gamma \times B^\gamma \to G_3 := u \times B \quad \text{in} \quad L_{t,x}^2,$$

$$G_4^\gamma := u^\gamma \otimes u^\gamma \to G_4 := u \otimes u \quad \text{in} \quad L_{t,x}^2. \quad (3.3)$$

From the Faraday’s equation in (1.9), we get

$$E^\gamma = -\partial_t A^\gamma, \quad \text{where} \quad \nabla \times A^\gamma = B^\gamma \quad \text{and} \quad \text{div} A^\gamma = 0,$$

which implies $E^\gamma$ convergence to some function $\tilde{E}$ in the sense of distribution. More precisely, from Ohm’s law and momentum equation, we have

$$E^\gamma = P \left( \eta \left( \partial_t u^\gamma + \text{div}(u^\gamma \otimes u^\gamma) - \Delta u^\gamma \right) + \beta \eta^2 j^\gamma - u^\gamma \times B^\gamma \right).$$

This implies that

$$E^\gamma \to \tilde{E} \quad \text{in} \quad L_{t,\text{loc}}^2 H_x^{-3} + L_t^2 L_x^{\frac{3}{s}} + L_t^2 H_x^{s-1} + L_{t,x}^2.$$ 

Thus, the weak convergence of linear terms in (1.9) is deduced. From the Ampere’s law in (1.9), by employing the approximation arguments, one has

$$\int_0^t \nabla \times B^\gamma (\tau) d\tau = \gamma^2 \left( E^\gamma (t) - E^\gamma (0) \right) + \int_0^t j^\gamma (\tau) d\tau. \quad (3.4)$$

Then

$$\left\| \nabla \times B^\gamma \right\|_{L_{t,\text{loc}}^1 L_x^2} \leq \gamma^2 \left( \left\| E^\gamma \right\|_{L_t^\infty L_x^2} + \left\| E^0 \right\|_{L_t^2} \right) + \left\| j^\gamma \right\|_{L_{t,\text{loc}}^1 L_x^2}.$$
from which we infer that \( \nabla \times B^\gamma \in L^1_{t, \text{loc}} L^2_x \). Similarly, we have \( \nabla \times E^\gamma \in L^1_{t, \text{loc}} L^2_x \) from the Faraday’s law in (1.9). Therefore, we conclude that

\[
G_1^\gamma = \left( \frac{1}{\gamma^2} - \frac{1}{\gamma} \right) \nabla \times B^\gamma + \frac{E^\gamma}{\beta \eta^2 \gamma^2} - \frac{1}{\gamma^2} j^\gamma \in L^1_{t, \text{loc}} L^2_x,
\]

\[
G_2^\gamma = \left( \frac{1}{\gamma} - 1 \right) \nabla \times B^\gamma \in L^1_{t, \text{loc}} L^2_x,
\]

for any fixed \( \gamma > 0 \).

Thus, in order to establish the asymptotic limit of the Navier–Stokes–Maxwell system (1.9), there only remains to establish the weak convergence of the Lorentz force \( j^\gamma \times B^\gamma \), which will follow from a careful combination of frequency analysis on \( B^\gamma \) and the hypothesis (1.13) on the very high frequencies of \( j^\gamma \). Therefore, we decompose the initial data \( (\hat{E}^0_\gamma, \hat{B}^0_\gamma) \in X \) and the inhomogeneous terms \( (\hat{G}^1_\gamma, \hat{G}^2_\gamma) \in L^1(0, T, X) \) based on the spectral analysis in Sect. 2.

Applying Lemma 2.1, for almost every \( \xi \in \mathbb{R}^3 \), one has

\[
\left( \hat{\bar{E}}^{0\gamma}, \hat{\bar{B}}^{0\gamma} \right) = \left( \hat{\bar{E}}^{0\gamma} \cdot \frac{\xi}{|\xi|^2}, 0 \right) + \left( -i \frac{\xi \cdot b^{0\gamma}}{\gamma \lambda_-} \right) + \left( -i \frac{e^{0\gamma}}{\gamma \lambda_-} \right),
\]

or, equivalently,

\[
\left\{ \begin{array}{l}
\hat{E}^{0\gamma} = -i \frac{\xi \cdot b^{0\gamma}}{\gamma \lambda_-} + e^{0\gamma}, \\
\hat{B}^{0\gamma} = b^{0\gamma} - i \frac{\xi \cdot e^{0\gamma}}{\gamma \lambda_-}.
\end{array} \right.
\]

Assuming that \( e^{0\gamma}, b^{0\gamma} \neq 0 \), taking \( \xi \times \) on the first equation in (3.5), we get

\[
\xi \times e^{0\gamma} = \xi \times \hat{E}^{0\gamma} - \frac{i |\xi|^2}{\gamma \lambda_-} b^{0\gamma}.
\]

Combining the second equation in (3.5) and the fact that \( \lambda_+ = \frac{\lambda_-(|\xi|^2)}{\gamma^2} \), we have

\[
\tilde{s} b^{0\gamma} = \hat{B}^{0\gamma} + \frac{i}{\gamma \lambda_-} \xi \times \hat{E}^{0\gamma},
\]

where \( \tilde{s} = \frac{\lambda_-(\xi) - \lambda_+(\xi)}{\lambda_-(\xi)} \), \( \xi \cdot e^{0\gamma} = \xi \cdot b^{0\gamma} = 0 \). Similarly, we have

\[
\tilde{s} e^{0\gamma} = \hat{E}^{0\gamma} + \frac{i}{\gamma \lambda_-} \xi \times \hat{B}^{0\gamma}.
\]

The cases of \( e^{0\gamma} = 0 \) or \( b^{0\gamma} = 0 \) are contained in (3.6) and (3.7). Therefore, according to Lemma 2.2, we get

\[
|\tilde{s} e^{0\gamma}| \leq |\hat{E}^{0\gamma}| + |\hat{B}^{0\gamma}|, \quad |\tilde{s} b^{0\gamma}| \leq |\hat{E}^{0\gamma}| + |\hat{B}^{0\gamma}|.
\]

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Similarly, define
\[
\begin{align*}
\hat{G}_1^\gamma &:= \frac{1}{\gamma} \left[ (1 - \gamma) i \xi \times \hat{B}^\gamma + \frac{1}{\beta \eta} \left[ \partial_t \hat{u}^\gamma + i \xi \cdot \hat{G}_4^\gamma + |\xi|^2 \hat{u}^\gamma - \frac{1}{\eta} \hat{G}_3^\gamma \right] \right], \\
\hat{G}_2^\gamma &:= (1 - \frac{1}{\gamma}) \partial_t \hat{B}^\gamma,
\end{align*}
\]
where $\hat{G}_3^\gamma, \hat{G}_4^\gamma$ are the Fourier transform of $G_3^\gamma, G_4^\gamma$, which are given by (3.3). Applying Lemma 2.1, one has
\[
\begin{align*}
(\hat{G}_1^\gamma, \hat{G}_2^\gamma) &= \left( \frac{\hat{G}_1^\gamma}{|\xi|^2}, 0 \right) + \left( -\frac{i}{\gamma \lambda} \xi \times b^\gamma \right) + \left( -\frac{e^\gamma}{\gamma \lambda} \xi \times e^\gamma \right),
\end{align*}
\]
for almost every $\xi \in \mathbb{R}^3$, or, equivalently,
\[
\begin{align*}
\tilde{e}^\gamma = \hat{G}_1^\gamma + \frac{1}{\gamma \lambda} \xi \times \hat{G}_2^\gamma, \\
\tilde{b}^\gamma = \hat{G}_2^\gamma + \frac{1}{\gamma \lambda} \xi \times \hat{G}_1^\gamma,
\end{align*}
\]
where $\xi \cdot e^\gamma = \xi \cdot b^\gamma = 0$. Hence, from Lemma 2.2, we obtain
\[
|\tilde{e}^\gamma| \leq |\hat{G}_1^\gamma| + |\hat{G}_2^\gamma|, \quad |\tilde{b}^\gamma| \leq |\hat{G}_1^\gamma| + |\hat{G}_2^\gamma|.
\]
Letting the operator $e^{t\hat{A}}$ acts on (3.4) and on (3.10) respectively, combining Lemma 2.1, then we have
\[
\begin{align*}
e^{t\hat{A}} \left( \frac{\hat{E}_0^\gamma}{b_0^\gamma} \right) &= e^{-\frac{1}{\beta \eta} t} \left( \frac{\hat{E}_0^\gamma}{|\xi|^2}, 0 \right) + e^{t\lambda^+} \left( -\frac{1}{\gamma \lambda^+} \xi \times b_0^\gamma \right) + e^{t\lambda^-} \left( -\frac{1}{\gamma \lambda^-} \xi \times e_0^\gamma \right),
\end{align*}
\]
and
\[
\begin{align*}
e^{t\hat{A}} \left( \frac{\hat{G}_1^\gamma}{\hat{G}_2^\gamma} \right) &= e^{-\frac{1}{\beta \eta} t} \left( \frac{\hat{G}_1^\gamma}{|\xi|^2}, 0 \right) + e^{t\lambda^+} \left( -\frac{i}{\gamma \lambda^+} \xi \times b_0^\gamma \right) + e^{t\lambda^-} \left( -\frac{i}{\gamma \lambda^-} \xi \times e^\gamma \right).
\end{align*}
\]
Substituting them into (2.5) implies that
\[
\hat{B}^\gamma = e^{t\lambda^+} b_0^\gamma - \frac{i}{\gamma \lambda^-} e^{t\lambda^-} \xi \times e_0^\gamma + \int_0^t (t-t') e^{(t-t')\lambda^+} b_0^\gamma - \frac{i}{\gamma \lambda^-} e^{(t-t')\lambda^-} \xi \times e^\gamma \, dt.
\]
Notice that
\[
-\frac{i}{\gamma \lambda^-} \xi \times e_0^\gamma = \hat{B}_0^\gamma - b_0^\gamma,
\]
and
\[
-\frac{i}{\gamma \lambda^-} \xi \times e^\gamma = \hat{G}_2^\gamma - b^\gamma.
\]
\[ \tilde{s} b^\gamma = \hat{G}_2^\gamma + \frac{i}{\gamma \lambda_-} \xi \times \hat{G}_1^\gamma, \]
	hen one has
\[ \begin{align*}
\hat{B}^\gamma &= e^{\lambda_-} \hat{B}_0^\gamma + \left( e^{\lambda_+} - e^{\lambda_-} \right) b_0^\gamma + \int_0^t \left( e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-} \right) b^\gamma + e^{(t-\tau)\lambda_-} \hat{G}_2^\gamma \, d\tau \\
&= e^{\lambda_-} \hat{B}_0^\gamma + \left( e^{\lambda_+} - e^{\lambda_-} \right) b_0^\gamma + \int_0^t \left[ \frac{\lambda_-}{\lambda_- - \lambda_+} e^{(t-\tau)\lambda_+} - \frac{\lambda_+}{\lambda_- - \lambda_+} e^{(t-\tau)\lambda_-} \right] \hat{G}_2^\gamma \, d\tau \\
&\quad + \frac{i}{\gamma (\lambda_- - \lambda_+)} \int_0^t \left( e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-} \right) \xi \times \hat{G}_1^\gamma \, d\tau.
\end{align*} \]

Combining (3.9) and integrating by parts in \( t \) yields that
\[ \begin{align*}
\hat{B}^\gamma &= e^{\lambda_-} \hat{B}_0^\gamma + \left( e^{\lambda_+} - e^{\lambda_-} \right) b_0^\gamma + \left( 1 - \frac{1}{\gamma} \right) \hat{B}^\gamma + \frac{1 - \gamma}{\gamma (\lambda_- - \lambda_+)} \left( \lambda_- e^{\lambda_+} - \lambda_+ e^{\lambda_-} \right) \hat{B}_0^\gamma \\
&\quad + \frac{1 - \gamma}{\gamma (\lambda_- - \lambda_+)} \int_0^t \left( e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-} \right) \hat{B}^\gamma \, d\tau \\
&\quad + \frac{1}{\beta \eta \gamma^2 (\lambda_- - \lambda_+)} \int_0^t \left( e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-} \right) \xi \times \left( \partial_\tau \hat{u}^\gamma + i \xi \cdot \hat{G}_4^\gamma + |\xi|^2 \hat{u}^\gamma - \frac{1}{\eta} \hat{G}_3^\gamma \right) \, d\tau,
\end{align*} \]

which together with the fact that \( \lambda_+ \lambda_- = \frac{|\xi|^2}{\gamma^2} \) implies the sum of the fifth and sixth terms is 0. Putting the third term on the left-hand side of the above equality, thus we have
\[ \begin{align*}
\hat{B}^\gamma &= \gamma e^{\lambda_-} \hat{B}_0^\gamma + \gamma \left( e^{\lambda_+} - e^{\lambda_-} \right) b_0^\gamma + \frac{1 - \gamma}{\lambda_- - \lambda_+} \left( \lambda_- e^{\lambda_+} - \lambda_+ e^{\lambda_-} \right) \hat{B}_0^\gamma \\
&\quad + \frac{i}{\beta \eta \gamma^2 (\lambda_- - \lambda_+)} \int_0^t \left( e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-} \right) \xi \times \left( \partial_\tau \hat{u}^\gamma + i \xi \cdot \hat{G}_4^\gamma + |\xi|^2 \hat{u}^\gamma - \frac{1}{\eta} \hat{G}_3^\gamma \right) \, d\tau,
\end{align*} \]

(3.13)

where the term \( \hat{B}^\gamma \) on the right-hand side vanishes. In order to cancel out the term \( \partial_\tau \hat{u}^\gamma \), we integrate by parts in \( t \) again to get
\[ \begin{align*}
\hat{B}^\gamma &= \gamma e^{\lambda_-} \hat{B}_0^\gamma + \frac{1 - \gamma}{\lambda_- - \lambda_+} \left( \lambda_- e^{\lambda_+} - \lambda_+ e^{\lambda_-} \right) \hat{B}_0^\gamma \\
&\quad + \gamma \left( e^{\lambda_+} - e^{\lambda_-} \right) b_0^\gamma - \frac{i}{\beta \eta \gamma^2 (\lambda_- - \lambda_+)} \left( e^{\lambda_+} - e^{\lambda_-} \right) \xi \times \hat{u}^\gamma \\
&\quad - \frac{i}{\beta \eta \gamma^2 (\lambda_- - \lambda_+)} \int_0^t \left( \lambda_- e^{(t-\tau)\lambda_+} - \lambda_+ e^{(t-\tau)\lambda_-} \right) \xi \times \hat{u}^\gamma \, d\tau \\
&\quad + \frac{i}{\beta \eta \gamma^2 (\lambda_- - \lambda_+)} \int_0^t \left( e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-} \right) \xi \times \left( i \xi \cdot \hat{G}_4^\gamma + |\xi|^2 \hat{u}^\gamma - \frac{1}{\eta} \hat{G}_3^\gamma \right) \, d\tau.
\end{align*} \]

(3.14)

Recall that we want to establish the convergence of Lorentz force \( j^\gamma \times B^\gamma \) towards \( j \times B \), in the sense of distribution. To this end, we strengthen the convergence of \( B^\gamma \) towards \( B \) through a precise analysis of the frequency distribution of \( B^\gamma \).
Next, we give a rough formal asymptotic analysis of $B^\gamma$. From the Faraday’s law, one has

$$\partial_t B^\gamma = \nabla \times G_2^\gamma - \beta \eta^2 \nabla \times j^\gamma - \eta \nabla \times (j^\gamma \times B^\gamma),$$

then $\partial_t B^\gamma$ is uniformly bounded in $L^2_{t,\text{loc}} H^{-3}_x$. Similarly, in view of the uniform bounds on $B^\gamma$ in $L^2_{t,\text{loc}} L^2_x$ and embedding $L^2 \hookrightarrow H^{-1} \hookrightarrow H^{-3}$, $B^\gamma$ is relatively compact in $C([0, T], H^{-1})$ through standard compactness result by Simon (1986). Therefore, taking the weak limit in the Faraday’s law, one obtains that $B \in L^\infty_t L^2_x$ solves

$$\partial_t B = \beta \eta^2 \Delta B + \nabla \times G_3 - \eta \nabla \times (\partial_t u + \text{div} G_4 - \Delta u)$$

with the initial data $B^0 \in L^2_x$ in the sense of distribution. Thanks to $\nabla \times G_3 - \eta \nabla \times (\partial_t u + \text{div} G_4 - \Delta u) \in L^1_{t,\text{loc}}(\mathbb{R}^+; S'(\mathbb{R}^3))$, we conclude that Eq. (3.15) has a unique tempered distribution solution, given by Duhamel’s formula (see Danchin and Tan (2021), A.2), for instance)

$$B = e^{\theta \eta^2 t} B^0 + \int_0^t e^{\theta \eta^2 (t - \tau)} [\nabla \times G_3 - \eta \nabla \times (\partial_t u + \text{div} G_4 - \Delta u)] d\tau.$$  

Then, after taking Fourier transformation, we have

$$\hat{B} = e^{-\theta \eta^2 |\xi|^2} \hat{B}^0 + \int_0^t e^{-\theta \eta^2 |\xi|^2 (t - \tau)} \left[i \xi \times \hat{G}_3 - \eta i \xi \times \left(\partial_t \hat{u} + i \xi \cdot \hat{G}_4 + |\xi|^2 \hat{u}\right)\right] d\tau.$$ (3.16)

In view of Lemma 2.2, for any fixed $|\xi| \leq \frac{1}{2K \beta \eta \gamma}$, one has

$$e^{t \lambda-} \leq e^{-\frac{t \beta \eta \gamma^2}{2}} \to 0, \quad e^{t \lambda+} \to e^{-\beta \eta^2 |\xi|^2 t}, \quad \frac{i}{\beta \eta \gamma^2 (\lambda_+ - \lambda_-)} \to -i \eta, \quad \bar{s} \to 1,$$

as $\gamma \to 0$. And (3.13) can be written as

$$\hat{B}^\gamma = \gamma e^{i \lambda-} \left(\hat{B}_0^\gamma - b_0^\gamma\right) + \frac{\lambda_+ (1 - \gamma)}{\lambda_+ - \lambda_-} e^{i \lambda-} \hat{B}_0^\gamma + \frac{i}{\bar{s} \lambda_-} e^{i \lambda+} \xi \times \hat{E}_0^\gamma + \frac{1}{\bar{s}} e^{i \lambda+} \hat{B}_0^\gamma + \frac{i}{\beta \eta \gamma^2 (\lambda_+ - \lambda_-)} \int_0^t \left(e^{(u-r)\lambda+} - e^{(u-r)\lambda-}\right) \xi \times \left(\partial_t \hat{u}^\gamma + i \xi \cdot \hat{G}_4^\gamma + |\xi|^2 \hat{u}^\gamma - \frac{1}{\eta} \hat{G}_3^\gamma\right) d\tau,$$

then we expect that $\hat{B}^\gamma \to \hat{B}$ in $L^2_{t,\xi,\text{loc}}$ in low frequencies. Similarly, if $|\xi| \geq \frac{K}{2\beta \eta \gamma}$,

$$|e^{t \lambda+}| = |e^{t \lambda-}| = e^{-\frac{t \beta \eta \gamma^2}{2}} \to 0,$$

as $\gamma \to 0$ and $\frac{i}{\beta \eta \gamma^2 (\lambda_+ - \lambda_-)}$ and $\bar{s}$ is bounded. So we may expect that the high frequencies of $\hat{B}^\gamma$ will vanish after taking $\gamma \to 0$.  

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Whence, motivated by Arsénio et al. (2015), we decompose $B^Y$ into five parts

\[ B^Y = B^Y_{\ll} + B^Y_{<} + B^Y_{\sim} + B^Y_{>} + B^Y_{\gg}, \]

where

\[ B^Y_{\ll} = \mathcal{F}^{-1} \left( \chi_{[0 \leq |\xi| \leq R]} \hat{B}^Y \right), \]
\[ B^Y_{<} = \mathcal{F}^{-1} \left( \chi_{\left\{ R < |\xi| \leq \frac{1}{2K_{\beta n^2}} \right\}} \hat{B}^Y \right), \]
\[ B^Y_{\sim} = \mathcal{F}^{-1} \left( \chi_{\left\{ \frac{1}{2K_{\beta n^2}} < |\xi| \leq \frac{K_{\beta n^2}}{2} \right\}} \hat{B}^Y \right), \]
\[ B^Y_{>} = \mathcal{F}^{-1} \left( \chi_{\left\{ \frac{K_{\beta n^2}}{2} < |\xi| \leq 1/2 \phi(\gamma) \right\}} \hat{B}^Y \right), \]
\[ B^Y_{\gg} = \mathcal{F}^{-1} \left( \chi_{|\xi| > 2 \phi(\gamma)} \hat{B}^Y \right), \]

for some fixed constant $1 < K < 2$, which will be determined later, for any large radius $0 < R < \frac{1}{2K_{\beta n^2}}$ and for some small parameter $\delta > 0$. Hence, for any $\varphi \in C_c^\infty (\mathbb{R}^+ \times \mathbb{R}^3)$, we have

\[
\int_{\mathbb{R}^+ \times \mathbb{R}^3} (j^Y \times B^Y - j \times B) \cdot \varphi \ dt \ dx \\
= \int_{\mathbb{R}^+ \times \mathbb{R}^3} j^Y \times (B^Y_{\ll} - B_{\ll}) \cdot \varphi \ dt \ dx + \int_{\mathbb{R}^+ \times \mathbb{R}^3} j^Y \times (B_{<} - B) \cdot \varphi \ dt \ dx \\
+ \int_{\mathbb{R}^+ \times \mathbb{R}^3} (j^Y - j) \times B \cdot \varphi \ dt \ dx + \int_{\mathbb{R}^+ \times \mathbb{R}^3} j^Y \times (B^Y_{<} + B^Y_{\sim} + B^Y_{>}) \cdot \varphi \ dt \ dx \\
+ \int_{\mathbb{R}^+ \times \mathbb{R}^3} (j^Y \times B^Y_{\gg}) \cdot \varphi \ dt \ dx. \tag{3.17}
\]

Firstly, we handle with the last term in (3.17), which can be recast as

\[
\int_{\mathbb{R}^+ \times \mathbb{R}^3} (j^Y \varphi)_{\gg} \times B^Y \ dt \ dx \\
= \int_{\mathbb{R}^+ \times \mathbb{R}^3} j^Y \varphi \times \mathcal{F}^{-1} \left( \chi_{|\xi| > 2 \phi(\gamma)} \right) \hat{B}^Y \ dt \ dx \\
= \int_{\mathbb{R}^+ \times \mathbb{R}^3} j^Y \varphi \times \mathcal{F} \left( \chi_{|\xi| > 2 \phi(\gamma)} \right) \times \hat{B}^Y \ dt \ dx \\
= \int_{\mathbb{R}^+ \times \mathbb{R}^3} (j^Y \varphi)_{\gg} \times B^Y \ dt \ dx,
\]

where

\[
(j^Y \varphi)_{\gg} = \mathcal{F}^{-1} \left( \chi_{|\xi| > 2 \phi(\gamma)} \right) \mathcal{F} \left( j^Y \varphi \right).
\]
One can infer that
\[
\limsup_{\gamma \to 0} \left\| (j^\gamma \varphi)_{>\gamma} \right\|_{L^2_{t,x}} = 0,
\] (3.18)
from the assumption (1.13) on the very high frequencies of \( j^\gamma \). Indeed, for any \( \delta > 0 \) and \( \varphi \in C_\infty_c (\mathbb{R}^+ \times \mathbb{R}^3) \),
\[
(j^\gamma \varphi)_{>\gamma} = (j^\gamma \varphi + j^\gamma \varphi)_{>\gamma}
= (j^\gamma \varphi)_{>\gamma} + (j^\gamma \varphi_{<\gamma})_{>\gamma} + (j^\gamma \varphi_{<\gamma})_{>\gamma},
\] (3.19)
where \( j^{\gamma}_{>\gamma} \) is defined in (1.14) and
\[
j^\gamma_{<\gamma} = \mathcal{F}^{-1} \left( \chi_{\{ \| \xi \| \leq \phi(\frac{\gamma}{\delta}) \}} j^\gamma \right);
\varphi_{<\gamma} = \mathcal{F}^{-1} \left( \chi_{\{ \| \xi \| \leq \phi(\frac{\gamma}{\delta}) \}} \hat{\varphi} \right);
\varphi_{>\gamma} = \mathcal{F}^{-1} \left( \chi_{\{ \| \xi \| > \phi(\frac{\gamma}{\delta}) \}} \hat{\varphi} \right).
\]
We will estimate terms on the right-hand side of (3.19) one by one. By the assumption (1.13), we have
\[
\limsup_{\gamma \to 0} \left\| (j^\gamma \varphi_{>\gamma}) \right\|_{L^2_{t,x}} \leq \limsup_{\gamma \to 0} \| j^\gamma \varphi \|_{L^2_{t,x,loc}} \leq \limsup_{\gamma \to 0} \| j_{>\gamma} \|_{L^2_{t,x,loc}} \| \varphi \|_{L^\infty_{t,x}} = 0.
\]
Since
\[
j^\gamma_{<\gamma} \varphi_{<\gamma} = \mathcal{F}^{-1} \left( \chi_{\{ \| \xi \| \leq \phi(\frac{\gamma}{\delta}) \}} j^\gamma \right) \mathcal{F}^{-1} \left( \chi_{\{ \| \xi \| \leq \phi(\frac{\gamma}{\delta}) \}} \hat{\varphi} \right)
= \mathcal{F}^{-1} \left( \left( \chi_{\{ \| \xi \| \leq \phi(\frac{\gamma}{\delta}) \}} j^\gamma \right) * \left( \chi_{\{ \| \xi \| \leq \phi(\frac{\gamma}{\delta}) \}} \hat{\varphi} \right) \right),
\]
\( \mathcal{F} \left( j^\gamma_{<\gamma} \varphi_{<\gamma} \right) \) is supported in \( \{ \xi : \| \xi \| \leq 2 \phi \left( \frac{\gamma}{\delta} \right) \} \), which implies that the second term on the right-hand side of (3.19) vanishes. Furthermore, we have
\[
\left\| (j^\gamma_{<\gamma} \varphi_{<\gamma}) \right\|_{L^2_{t,x}} \leq \left\| j^\gamma_{<\gamma} \varphi \right\|_{L^2_{t,x}} \ll \left\| j^\gamma_{>\gamma} \right\|_{L^2_{t,x}} \left\| \varphi \right\|_{L^\infty_{t,x}} \ll \int_{\mathbb{R}^3} \sup_t |\xi|^2 \hat{\varphi}^2 \, d\xi.
\]
Noting that \( \int_{\mathbb{R}^3} \sup_t |\xi|^2 \hat{\varphi}^2 \, d\xi = \| \varphi \|_{L^\infty_{t,x}}^2 \ll \infty \) and \( \phi \left( \frac{\gamma}{\delta} \right) \to \infty \) as \( \gamma \to 0 \), then we conclude
\[
\limsup_{\gamma \to 0} \left\| (j^\gamma_{<\gamma} \varphi_{<\gamma}) \right\|_{L^2_{t,x}} = 0.
\]
Now, we deal with the term $B_\|\gamma\| - B_\|\gamma\|$. For any small $h > 0$, applying Lemma 2.2 and the estimates (3.8) and (3.12), we find from (3.13) that

$$
\left| \dot{B}_\|\gamma\|(t + h) - \dot{B}_\|\gamma\|(t) \right| 
\lesssim e^{-\frac{1}{2\beta_\|\gamma\|^2} t} \left( |\gamma B_0^\|\gamma\| + |\gamma b_0^\|\gamma\| + \gamma^2 |\xi|^2 |\dot{B}_0^\|\gamma\| |\right) \chi_{\|\xi\| \leq R} 
+ e^{-\beta_\|\gamma\|^2 |\xi|^2} h|\xi|^2 \left( |\gamma b_0^\|\gamma\| + h |\dot{B}_0^\|\gamma\| |\right) \chi_{\|\xi\| \leq R} 
+ \int_0^t \left[ e^{-\beta_\|\gamma\|^2 |\xi|^2 (t-\tau)} h|\xi|^2 + e^{-\frac{1}{2\beta_\|\gamma\|^2} (t-\tau)} \right] 
\left( |\xi| |\partial_\tau u_\|\gamma\| + |\xi|^2 \left| \dot{G}_4^\|\gamma\| + |\xi|^3 |\dot{u}_\|\gamma\| \right| + |\xi| \left| \dot{G}_3^\|\gamma\| \right| \right) \chi_{\|\xi\| \leq R} \mathrm{d}\tau 
+ \int_0^\infty \chi_{[0,h]}(v) \left( e^{-\beta_\|\gamma\|^2 |\xi|^2 v + e^{-\frac{1}{2\beta_\|\gamma\|^2} v}} \right) 
\left( |\xi| |\partial_\tau u_\|\gamma\| + |\xi|^2 \left| \dot{G}_4^\|\gamma\| + |\xi|^3 |\dot{u}_\|\gamma\| \right| + |\xi| \left| \dot{G}_3^\|\gamma\| \right| \right) (t + h - v) \chi_{\|\xi\| \leq R} \mathrm{d}v.
$$

Thus

$$
\| \hat{B}_\|\gamma\|(t + h) - \hat{B}_\|\gamma\|(t) \|_{L^2_{t,\text{loc}} L^2_\xi} 
\lesssim \gamma \left[ e^{-\frac{1}{2\beta_\|\gamma\|^2} t} \left( \left\| B_0^\|\gamma\| \right\|_{L^2_\xi} + \left\| \dot{B}_0^\|\gamma\| \right\|_{L^2_\xi} \right) + h \left\| e^{-\beta_\|\gamma\|^2 |\xi|^2} t \left\| E_0^\|\gamma\| \right\|_{L^2_\xi} \right\|_{L^2_\xi} \right) \chi_{\|\xi\| \leq R} \right\|_{L^2_\xi} 
+ h \left\| e^{-\beta_\|\gamma\|^2 |\xi|^2} t \left\| |\xi|^3 |\partial_\tau u_\|\gamma\| \right\|_{L^2_\xi} + \left\| |\xi|^4 \left| \dot{G}_4^\|\gamma\| \right| + |\xi|^5 |\dot{u}_\|\gamma\| \right| + |\xi|^3 \left| \dot{G}_3^\|\gamma\| \right| \right\|_{L^2_{t,\text{loc}} L^2_\xi} \chi_{\|\xi\| \leq R} \right\|_{L^2_\xi} 
+ \left\| e^{-\frac{1}{2\beta_\|\gamma\|^2} t} \chi_{[0,h]}(t) \left\| |\xi|^3 |\partial_\tau u_\|\gamma\| \right\|_{L^2_\xi} + \left\| |\xi|^2 \left| \dot{G}_4^\|\gamma\| \right| + |\xi|^3 |\dot{u}_\|\gamma\| \right| + |\xi| \left| \dot{G}_3^\|\gamma\| \right| \right\|_{L^2_{t,\text{loc}} L^2_\xi} \chi_{\|\xi\| \leq R} \right\|_{L^2_\xi} 
+ \left\| e^{-\beta_\|\gamma\|^2 |\xi|^2} \chi_{[0,h]}(t) \left\| |\xi|^3 |\partial_\tau u_\|\gamma\| \right\|_{L^2_\xi} + \left\| |\xi|^2 \left| \dot{G}_4^\|\gamma\| \right| + |\xi|^3 |\dot{u}_\|\gamma\| \right| + |\xi| \left| \dot{G}_3^\|\gamma\| \right| \right\|_{L^2_{t,\text{loc}} L^2_\xi} \chi_{\|\xi\| \leq R} \right\|_{L^2_\xi} 
\lesssim (\gamma^2 + h R) \left( \left\| E_0^\|\gamma\| \right\|_{L^2_\xi} + \left\| B_0^\|\gamma\| \right\|_{L^2_\xi} \right) + (h + \gamma^2) \left( R^4 \left\| \partial_\tau u_\|\gamma\| \right\|_{L^2_{t,\text{loc}} H^{-3}_\xi} + R^2 \left\| u_\|\gamma\| \right\|_{L^2_{t,\text{loc}} H^1_\xi} \right) 
+ (h + \gamma^2) \left( R^2 \left\| \dot{G}_4^\|\gamma\| \chi_{\|\xi\| \leq R} \right\|_{L^2_{t,\text{loc}} L^2_\xi} + R \left\| \dot{G}_3^\|\gamma\| \chi_{\|\xi\| \leq R} \right\|_{L^2_{t,\text{loc}} L^2_\xi} \right).
$$

By virtue of the uniform estimates of $G_3^\|\gamma\|$ and $G_4^\|\gamma\|$ in (3.3) and $\|\partial_\tau u_\|\gamma\| \|_{L^2_{t,\text{loc}} H^{-3}_\xi}$ in (3.1), and applying Plancherel’s theorem, we obtain

$$
\lim \sup_{h \to 0} \gamma \left\| B_\|\gamma\|(t + h) - B_\|\gamma\|(t) \right\|_{L^2_{t,\text{loc}} L^2_\xi} = 0,
$$

whence, utilizing Riesz–Fréchet–Kolmogorov compactness criterion (for more details, see Yosida 1995), the relative compactness of $B_\|\gamma\|$ in the strong topology of $L^2_{t,x,\text{loc}}$.
can be deduced, that is,

$$B_{\ll}^\gamma \to B_{\ll} \text{ in } L^2_t, x, \text{loc}^*, \quad (3.20)$$

where $B_{\ll} = F^{-1}\left(\chi_{\{0 \leq |\xi| \leq R\}} \hat{B}\right)$. 

Next, we handle with the term $B_{\ll}^\gamma$. Employing Lemma 2.2 and the estimates (3.8) and (3.12), from (3.14), we find that

$$\|\hat{B}_{\ll}^\gamma\|_{L^2_t, x, \text{loc}^*} \lesssim e^{-\beta \eta^2 |\xi|^2 t} \left( |\gamma \hat{u}_{0v}| + |\hat{B}_{0v}| + |\xi \times \hat{u}_{0v}| \right) \chi \left\{ R < |\xi| \leq \frac{1}{2K \beta \eta^2} \right\} + \gamma e^{-\frac{1}{2\beta \eta^2} |\xi|^2 t} \left( |\hat{B}_{0v}| + |\int_0^t \left[ \frac{1}{\gamma^2} e^{-\frac{1}{2\beta \eta^2} (t-r)} + |\xi|^2 e^{-\beta \eta^2 |\xi|^2 (t-r)} \right] |\xi \times \hat{u}| X \left\{ R < |\xi| \leq \frac{1}{2K \beta \eta^2} \right\} dr \right) + \int_0^t e^{-\beta \eta^2 |\xi|^2 (t-r)} \left( |\xi|^2 |\hat{G}_{4v}| + |\xi|^3 |\hat{u}| + |\xi| |\hat{\gamma}_{3v}| \right) X \left\{ R < |\xi| \leq \frac{1}{2K \beta \eta^2} \right\} dr.

Then, one has

$$\|\hat{B}_{\ll}^\gamma\|_{L^2_t, x, \text{loc}^*} \lesssim$$

$$\lesssim e^{-\beta \eta^2 |\xi|^2 t} \left( |\gamma \hat{u}_{0v}| + |\hat{B}_{0v}| + |\xi \times \hat{u}_{0v}| \right) \chi \left\{ R < |\xi| \leq \frac{1}{2K \beta \eta^2} \right\} \|L^2_t, x, \text{loc}^*\|_{L^2_{\xi}} + \frac{1}{\gamma^2} e^{-\frac{1}{2\beta \eta^2} |\xi|^2 t} \left| \xi \times \hat{u} \right| \chi \left\{ R < |\xi| \leq \frac{1}{2K \beta \eta^2} \right\} \|L^2_t, x, \text{loc}^*\|_{L^2_{\xi}} + \frac{1}{\gamma^2} e^{-\frac{1}{2\beta \eta^2} |\xi|^2 t} \left| \xi \times \hat{u} \right| \chi \left\{ R < |\xi| \leq \frac{1}{2K \beta \eta^2} \right\} \|L^2_t, x, \text{loc}^*\|_{L^2_{\xi}} + \frac{1}{\gamma^2} e^{-\frac{1}{2\beta \eta^2} |\xi|^2 t} \left| \xi \times \hat{u} \right| \chi \left\{ R < |\xi| \leq \frac{1}{2K \beta \eta^2} \right\} \|L^2_t, x, \text{loc}^*\|_{L^2_{\xi}} + \frac{1}{R} \left( |\hat{B}_{0v}| \right) + \frac{1}{R^2} \left( |u_{0v}| \right) + \frac{1}{R} \left( |u'_{0v}| \right) + \frac{1}{R} \left( |u''_{0v}| \right)$$

$$+ R \frac{1}{s} \left( \|\nabla G_{4v}\|_{L^2_t, x, \text{loc}^*}^{s - \frac{3}{2}} + \frac{1}{R} \left( |G_{3v}| \right) \right)$$

$$\lesssim \frac{1}{R} \left( |\hat{B}_{0v}| \right) + \frac{1}{R} \left( |B_{0v}| \right) + \frac{1}{R} \left( |u_{0v}| \right) + \frac{1}{R} \left( |u'_{0v}| \right) + \frac{1}{R} \left( |u''_{0v}| \right) + R \frac{1}{s} \left( \|\nabla G_{4v}\|_{L^2_t, x, \text{loc}^*}^{s - \frac{3}{2}} + \frac{1}{R} \left( |G_{3v}| \right) \right)$$
\[ \approx \frac{1}{R} \left( \| u^0 \|_{L^2_x} + \| B^{0y} \|_{L^2_x} \right) + \frac{1}{R^2} \left( \| u^0 \|_{H^{s_x}} + \| u^y \|_{L^2_{t, loc} H^{1+s}} \right) \\
+ R^{\frac{1}{2} - s} \| \nabla G^V_4 \|_{L^2_{t, loc} L^{\frac{3}{2 - s}}_x} + \frac{1}{R} \| G^V_3 \|_{L^2_{t, loc} L^2_x}. \]

In view of the uniform bounds (3.2) of \( G^V_3 \) and \( G^V_4 \) and Plancherel’s theorem, we conclude that

\[ \limsup_{\gamma \to 0} \| B^V \|_{L^2_{t, loc} L^2_x} \lesssim R^{\frac{1}{2} - s}. \]

Thirdly, we deal with the term \( B^V \). Clearly, (3.14) can be recast as

\[ \hat{B}^V = \left[ g e^{\lambda_+} + (1 - \gamma) e^{\lambda_-} \right] B^{0y} + \gamma \lambda_- e^{\lambda_+} \frac{1 - e^{(\lambda_- - \lambda_+)} t}{t(\lambda_- - \lambda_+)} B^{0y} + \left( \frac{1 - e^{(\lambda_- - \lambda_+)} t}{t(\lambda_- - \lambda_+)} \right) \xi \times u^0 \]

For fixed \( 1 < K < \frac{\sqrt{2}}{2} \), using the complex mean value theorem (see Pemba et al. 2007), we can infer that

\[ \left| e^{\lambda_-} \frac{1 - e^{(\lambda_- - \lambda_+)} t}{t(\lambda_- - \lambda_+)} \right| \leq e^{-\frac{\omega}{\gamma} t} \]

for some constant \( \omega > 0 \) depending on \( K \), provided \( \frac{1}{2K\beta\gamma} < |\xi| \leq \frac{K}{2\beta\gamma}. \) Combining Lemma 2.2, estimates (3.8) and (3.12), we have

\[ \| \hat{B}^V \| \lesssim \left( \gamma e^{-\frac{1}{2\beta\gamma} \gamma^{-1}} + e^{-\frac{K + \sqrt{K^2 - 1}}{2K \beta \gamma}} + \frac{1}{\gamma^2} e^{-\frac{\omega \gamma}{\gamma^2}} \right) \| \hat{B}^0 \| + \frac{1}{\gamma^2} e^{-\frac{\omega \gamma}{\gamma^2}} \| \hat{E}^0 \| \]

Thus, we get

\[ \| \hat{B}^Y \|_{L^2_{t, loc} L^2_\xi} \]

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Therefore, we have

\[ \left\| B_{1, \xi}^\gamma \right\|_{L^2_{t, \text{loc}} L^2_\xi} \leq e^{-\frac{1}{2\beta^{-2} + \gamma^2} t} \left( \left| \hat{B}^0 \right| + \left| \hat{E}^0 \right| \right) + \frac{1}{\gamma^2} e^{-\frac{1}{2\beta^{-2} + \gamma^2} t} \left| \hat{u}^0 \right| \chi \left\{ \frac{K}{2\beta^{-2} + \gamma^2} < |\xi| \leq 2\phi \left( \frac{2\phi}{3} \right) \right\} \]

\[ + \frac{1}{\gamma^2} \int_0^t e^{-\frac{1}{2\beta^{-2} + \gamma^2} (t-\tau)} |\xi| \times \hat{u}^\gamma \chi \left\{ \frac{K}{2\beta^{-2} + \gamma^2} < |\xi| \leq 2\phi \left( \frac{2\phi}{3} \right) \right\} d\tau \]

\[ + \frac{1}{\gamma^2 |\xi|} \int_0^t e^{-\frac{1}{2\beta^{-2} + \gamma^2} (t-\tau)} \left( |\xi|^2 \left| \hat{G}^\gamma_4 \right| + |\xi|^3 |\hat{u}^\gamma| + |\xi| \left| \hat{G}^\gamma_3 \right| \right) \chi \left\{ \frac{K}{2\beta^{-2} + \gamma^2} < |\xi| \leq 2\phi \left( \frac{2\phi}{3} \right) \right\} d\tau. \]

Therefore, we have

\[ \left\| B_{1, \xi}^\gamma \right\|_{L^2_{t, \text{loc}} L^2_\xi} \to 0. \]
we are ready to prove the weak convergence of the Lorentz force $j^\gamma \times B^\gamma$. Combining the estimates (3.18), (3.20), (3.21), (3.22), (3.23) and the fact that $j^\gamma \rightharpoonup j$ in $L^2_{t,x}$, we can deduce from (3.17) that

$$\limsup_{\gamma \to 0} \left\| B^\gamma \right\|_{L^2_{t,\text{loc}} L^\infty_x} \lesssim \delta. \quad (3.23)$$

Now, we are ready to prove the weak convergence of the Lorentz force $j^\gamma \times B^\gamma$. Combining the estimates (3.18), (3.20), (3.21), (3.22), (3.23) and the fact that $j^\gamma \rightharpoonup j$ in $L^2_{t,x}$, we can deduce from (3.17) that

$$\limsup_{\gamma \to 0} \left| \int_{\mathbb{R}^+ \times \mathbb{R}^3} \left( j^\gamma \times B^\gamma - j \times B \right) \cdot \varphi \, dt dx \right| \lesssim \int_{\mathbb{R}^+ \times \mathbb{R}^3} j \times (B_{\ll} - B) \cdot \varphi \, dt dx + \delta + R^{1-s}. \quad (3.24)$$

Notice that as $R \to \infty$,

$$B_{\ll} \to B \text{ in } L^2_{t,\text{loc}} L^\infty_x,$$

by the arbitrariness of $R > 0$ (large) and $\delta > 0$ (small), we have

$$\lim_{\gamma \to 0} \int_{\mathbb{R}^+ \times \mathbb{R}^3} j^\gamma \times B^\gamma \cdot \varphi \, dt dx = \int_{\mathbb{R}^+ \times \mathbb{R}^3} j \times B \cdot \varphi \, dt dx,$$

which implies the desired results of Theorem 1.1. \(\square\)
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References

Acheritogaray, M., Degond, P., Frouvelle, A., Liu, J.G.: Kinetic formulation and global existence for the Hall-magnetohydrodynamic system. Kinet. Relat. Models 4, 901–918 (2011)
Arséno, D., Gallagher, I.: Solutions of Navier–Stokes–Maxwell systems in large energy spaces. Trans. Am. Math. Soc. 373, 3853–3884 (2020)
Arséno, D., Ibrahim, S., Masmoudi, N.: A derivation of the magnetohydrodynamic system from Navier–Stokes–Maxwell systems. Arch. Ration. Mech. Anal. 216, 767–812 (2015)
Balbus, S.A., Terquem, C.: Linear analysis of the Hall effect in protostellar disks. Astrophys. J. 552, 235–247 (2001)
Benvenutti, M.J., Ferreira, L.C.F.: Existence and stability of global large strong solutions for the Hall-MHD system. Differ. Integral Equ. 29, 9771000 (2016)
Campos, I.M.B.C.: On hydromagnetic waves in atmospheres with application to the sun. Theor. Comput. Fluid Dyn. 10, 37–70 (1998)
Chae, D., Lee, J.: On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. J. Differ. Equ. 256, 3835–3858 (2014)
Chae, D., Schonbek, M.: On the temporal decay for the Hall-magnetohydrodynamic equations. J. Differ. Equ. 255, 3971–3982 (2013)
Chae, D., Wolf, J.: On partial regularity for the 3D non-stationary Hall magnetohydrodynamics equations on the plane. Commun. Math. Phys. 354, 213–230 (2017)
Chae, D., Degond, P., Liu, J.G.: Well-posedness for Hall-magnetohydrodynamics. Ann. Inst. Henri Poincaré Anal. Non Linéaire 31(3), 555–565 (2014)
Chae, D., Wan, R., Wu, J.: Local well-posedness for the Hall-MHD equations with fractional magnetic diffusion. J. Math. Fluid Mech. 17, 627–638 (2015)
Chae, D., Weng, S.: Singularity formation for the incompressible Hall-MHD equations without resistivity. Ann. Inst. Henri Poincaré Anal. Non Linéaire 33, 1009–1022 (2016)
Dai, M.: Regularity criterion for the 3D Hall-magneto-hydrodynamics. J. Differ. Equ. 261, 573–591 (2016)
Dai, M.: Nonunique weak solutions in Leray–Hopf class for the three-dimensional Hall-MHD system. SIAM J. Math. Anal. 53, 5979–6016 (2021)
Dai, M., Liu, H.: Long time behavior of solutions to the 3D Hall-magneto-hydrodynamic system with one diffusion. J. Differ. Equ. 266, 7658–7677 (2019)
Dai, M., Liu, H.: On well-posedness of generalized Hall-magneto-hydrodynamics. Z. Angew. Math. Phys. 73, 139 (2022)
Danchin, R., Tan, J.: On the well-posedness of the Hall-magnetohydrodynamics system in critical spaces. Commun. Partial Differ. Equ. 46(1), 31–65 (2021)
Dreher, J., Runban, V., Grauer, R.: Axisymmetric flows in Hall-MHD: a tendency towards finite-time singularity formation. Phys. Scr. 72, 451–455 (2005)
Dumas, E., Sueur, F.: On the weak solutions to the Maxwell–Landau–Lifshitz equations and to the Hall-magnetohydrodynamic equations. Commun. Math. Phys. 330, 1179–1225 (2014)
Duoandikoetxea, J., Zuazo, J.D.: Fourier Analysis. American Mathematical Soc., Providence (2001)
Fan, J., Huang, S., Nakamura, G.: Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamic equations. Appl. Math. Lett. 26(9), 963–967 (2013)
Forbes, T.G.: Magnetic reconnection in solar flares. Geophys. Astrophys. Fluid Dyn. 62, 15–36 (1991)
Germain, P., Ibrahim, S., Masmoudi, N.: Well-posedness of the Navier–Stokes–Maxwell equations. Proc. R. Soc. Edinb. Sect. A 144, 71–86 (2014)
Homann, H., Grauer, R.: Bifurcation analysis of magnetic reconnection in Hall-MHD systems. Phys. D 208, 59–72 (2005)

Ibrahim, S., Keraani, S.: Global small solutions for the Navier–Stokes–Maxwell system. SIAM J. Math. Anal. 43, 2275–2295 (2011)

Ibrahim, S., Yoneda, T.: Local solvability and loss of smoothness of the Navier–Stokes–Maxwell equations with large initial data. J. Math. Anal. Appl. 396, 555–561 (2012)

Ibrahim, S., Shen, S., Yoneda, T., Giga, Y.: Global well-posedness for a two-fluid model. Differ. Integral Equ. 31, 187–214 (2018)

Jiang, J., Masmoudi, N.: Derivation of Ohm’s law from the kinetic equations. SIAM J. Math. Anal. 44(5), 3649–3669 (2012)

Jiang, Z., Zhu, M.: Regularity criteria for the 3D generalized MHD and Hall-MHD systems. Bull. Malays. Math. Sci. Soc. 41, 105–122 (2018)

Lighthill, M.J.: Studies on magnetohydrodynamic waves and other anisotropic wave motions. Philos. Trans. R. Soc. Lond. Ser. A 252, 397–430 (1960)

Masmoudi, N.: Global well-posedness for the Maxwell–Navier–Stokes system in 2D. J. Math. Pures Appl. 93, 559–571 (2010)

Mininni, P.D., Gómez, D.O., Mahajan, S.M.: Dynamo action in magnetohydrodynamics and Hall magnetohydrodynamics. Astrophys. J. 587, 472–481 (2003)

Pan, N., Zhu, M.: A new regularity criterion for the 3D generalized Hall-MHD system with $\beta \in (\frac{1}{2}, 1]$. J. Math. Anal. Appl. 445, 604–611 (2017)

Pemba, J.P., Davies, A.R., Muoneke, N.K.: A complexification of Rolle’s theorem. Appl. Appl. Math. Int. J. AAM 2, 28–31 (2007)

Polygiannakis, J.M., Moussas, X.: A review of magneto-vorticity induction in Hall-MHD plasmas. Plasma Phys. Control. Fusion 43, 195–221 (2001)

Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. 146, 65–96 (1986)

Srinivasan, B., Shumlak, U.: Analytical and computational study of the ideal full two-fluid plasma model and asymptotic approximations for Hall-magnetohydrodynamics. Phys. Plasmas 18(9), 620 (2011)

Wan, R.: Global regularity for generalized Hall magnetohydrodynamics systems. Electron. J. Differ. Equ. 179, 1–18 (2015)

Wan, R., Zhou, Y.: On global existence, energy decay and blow-up criteria for the Hall-MHD system. J. Differ. Equ. 259, 5982–6008 (2015)

Wan, R., Zhou, Y.: Low regularity well-posedness for the 3D generalized Hall-MHD system. Acta Appl. Math. 147, 95–111 (2017)

Wardle, M.: Star formation and the Hall effect. Astrophys. Space Sci. 292, 317–323 (2004)

Wu, X., Yu, Y., Tang, Y.: Global existence and asymptotic behavior for the 3D generalized Hall-MHD system. Nonlinear Anal. 151, 41–50 (2017)

Ye, Z.: Regularity criteria and small data global existence to the generalized viscous Hall-magnetohydrodynamics. Comput. Math. Appl. 70, 2137–2154 (2015)

Yosida, K.: Functional Analysis. Classics in Mathematics, Springer, Berlin (1995)

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