PARA-QUATERNIONIC REDUCTION

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Abstract. The pseudo-Riemannian manifold $M = (M^{4n}, g)$, $n ≥ 2$ is para-quaternionic Kähler if $hol(M) ⊂ sp(n, \mathbb{R}) ⊕ sp(1, \mathbb{R})$. If $hol(M) ⊂ sp(n, \mathbb{R})$, then the manifold $M$ is called para-hyperKähler. The other possible definitions of these manifolds use certain parallel para-quaternionic structures in $\text{End}(TM)$, similarly to the quaternionic case. In order to relate these different definitions we study para-quaternionic algebras in details. We describe the reduction method for the para-quaternionic Kähler and para-hyperKähler manifolds and give some examples. The decomposition of a curvature tensor of the para-quaternionic type is also described.

1. Introduction

In the paper we try to develop theory of the para-quaternionic structures on a manifold. Algebra of para-quaternions is known but rarely used so far. A nice overview of para-quaternions and related classical geometries is given in [19] (para-quaternions were referred as generalized quaternions). The para-quaternionic sectional curvature of the para-quaternionic projective space $\mathbb{H}P^n$ has been studied in [6]. The para-hyperKähler manifolds have been studied in [11] [16] [17], where they are referred as neutral hyperKähler manifolds. Quite recently, the author have learnt that, independently, the notion of a para-quaternionic Kähler manifold has been developed in [14]. Although the basic definitions and conclusions are the same, the investigations in [14] and this paper go in different directions. In [14] the para-quaternionic sectional curvature of the para-quaternionic Kähler manifold and its relations to the Osserman condition have been studied.

In this paper we systematically study the para-hyperKähler and para-quaternionic Kähler manifolds. We characterize them as the manifolds with holonomies contained in $sp(n, \mathbb{R}) ⊕ sp(1, \mathbb{R})$ and $sp(n, \mathbb{R})$, respectively. They are necessarily of the dimension $4n$, $n ≥ 2$ and of the signature $(2n, 2n)$. Since $sp(n, \mathbb{R})$ are $sp(n)$ are real forms of the same complex Lie algebra, para-hyperKähler and para-quaternionic Kähler manifolds enjoy similar properties to the quaternionic counterparts: hyperKähler and quaternionic Kähler manifolds. Some facts concerning the quaternionic geometry given in [20] can be carried in the para-quaternionic setting by means of the complexification. The classification of para-hyperKähler symmetric spaces is given in [14]. The decomposition of the space of curvature tensors of a para-quaternionic type, given here, is analogous to the decomposition in the quaternionic case (see [14]). The reduction methods for the hyperKähler and quaternion Kähler
manifolds are well known (see \cite{13, 15}). Here we describe the reduction methods for the para-hyperKähler and para-quaternionic Kähler manifolds.

However, many questions remained open in the para-quaternionic case. It is known (see \cite{8, 15}) that the cotangent bundle $T^*M$ of a Kähler manifold $M$ is at least locally a hyperKähler manifold. It is an interesting question what the natural space, on which para-hyperKähler structures can arise, is. In \cite{5} quaternionic Kähler manifolds have been studied as certain quotients of special types of the hyper-Kähler manifolds. Does similar relation between para-hyperKähler and para-quaternionic Kähler manifolds exist?

The structure of the paper is as follows. In Section 2 we establish a basic notation and give basic properties of the para-quaternions and Lie algebras related to them. Subsection 2.5 enables us to use the Grassman formalism in the para-quaternionic vector space. In Section 3 we define the notions of the para-hyperKähler and para-quaternionic Kähler manifold and give their characterizations in terms of the algebra of holonomy (Theorems 3.2 and 3.4). A detailed analysis of the para-quaternionic projective space is given in Subsection 3.3. In Section 4 we give the decompositions of spaces of the curvature tensors of the type $\mathfrak{gl}_{2n}(\mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ (Theorem 4.1) and $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ (Theorem 4.2). Finally, in Section 5 we describe the reduction techniques for the para-hyperKähler manifolds (Theorem 5.1) and for the para-quaternionic Kähler manifolds (Theorem 5.2) and give some examples.

2. The linear algebra of para-quaternions

2.1. The algebra of para-quaternions. In this section we will define the algebra $\mathbb{H}$ of para-quaternions and review its basic properties. Para-quaternions enjoy similar properties to quaternions. The most important difference is the existence of zero divisors in the algebra of para-quaternions.

Both quaternions and para-quaternions are real Clifford algebras. Let $\mathbb{R}^{p,q}$ be the Euclidean space with an inner product $\eta$ of signature $(p, q)$, i.e. there exists a basis $e_1, \ldots, e_{p+q}$ of $\mathbb{R}^{p,q}$ such that $\eta(e_i, e_j) = 0$ for $i \neq j$, $\eta(e_i, e_i) = -1$ for $i = 1, \ldots, p$, and $\eta(e_i, e_i) = +1$ for $i = p+1, \ldots, p+q$. The real Clifford algebra $C(p, q)$ is the universal unital real algebra which is generated by $\mathbb{R}^{p,q}$ subject to the relations

$$v \cdot w + w \cdot v = 2\eta(v, w), \quad v, w \in \mathbb{R}^{p,q}.$$ 

In particular,

$$\mathbb{H} := C(2, 0) \quad \text{and} \quad \tilde{\mathbb{H}} := C(1, 1) \cong C(1, 2)$$

are algebras of quaternions and para-quaternions, respectively. Usually, we do not write the product sign. In other words, the algebra $\mathbb{H}$ of para-quaternions is generated by unity $1$ and generators $i, j, k$ satisfying

$$i^2 = -1, \quad j^2 = 1 = k^2, \quad ij = -ji = -k. \quad (1)$$

Using the notation $J_1 = i, J_2 = j, J_3 = k$ and constants $\epsilon_1 := 1, \epsilon_2 := -1 =: \epsilon_3$, we can write the relations $\mathbb{H}$ as

$$J^2 = -\epsilon_\alpha, \quad J_\alpha J_\beta = -\epsilon_\gamma J_\gamma,$$

where $(\alpha, \beta, \gamma)$ is a cyclic permutation of $(1, 2, 3)$.

Notice that the relation

$$q = a + bi + cj + dk = a + bi + j(c - di) =: z_1(q) + jz_2(q)$$

is
allows us to identify \( \mathbb{H} \) with \( \mathbb{C}^2 \) using the map

\[
\mathbb{H} \ni q \rightarrow z(q) := \begin{pmatrix} z_1(q) \\ z_2(q) \end{pmatrix} \in \mathbb{C}^2.
\]

For a para-quaternion

\[ q = a + bi + cj + dk, \]

with \( a, b, c, d \in \mathbb{R} \), we define its conjugate, real and imaginary part, by

\[ \bar{q} := a - bi - cj - dk, \quad Rq := a, \quad \mathcal{I}q := bi + cj + dk, \]

respectively. We define the scalar product on \( \mathbb{H} \) by

\[
\langle q, q' \rangle := R(q\bar{q}') = aa' + bb' - cc' - dd',
\]

where \( q = a + bi + cj + dk \) and \( q' = a' + b'i + c'j + d'k \). The corresponding square norm is multiplicative, i.e.

\[
|qq'|^2 = |q|^2|q'|^2, \quad q, q' \in \mathbb{H}.
\]

Together with the scalar product, \( \mathbb{H} \) para-quaternions can be naturally identified with \( \mathbb{R}^{2,2} \), a 4-dimensional real vector space with scalar product of signature \( (2, 2) \). The commutator

\[[q, q'] = qq' - q'q, \quad q, q' \in \mathbb{H},\]

defines a Lie algebra structure on the vector space \( \mathbb{H} \). Moreover, there is a Lie algebra decomposition

\[\mathbb{H} = \mathbb{R}1 \oplus \mathcal{I}\mathbb{H},\]

where \( \mathbb{R}1 \) is the center and

\[\mathcal{I}\mathbb{H} \cong su(1, 1) \cong so(2, 1) \cong sl_2(\mathbb{R}) \cong sp(1, \mathbb{R})\]

is the semisimple part. The Lie groups corresponding to the algebras \( \mathbb{H} \) and \( \mathcal{I}\mathbb{H} \) are

\[\mathbb{H}^+ := \{ q \in \mathbb{H} \mid |q|^2 > 0 \}, \quad \mathbb{H}_1 := \{ q \in \mathbb{H} \mid |q|^2 = 1 \},\]

respectively. The group \( \mathbb{H}_1 \) of unit para-quaternions is isomorphic to \( SU(1, 1) \). Geometrically, \( \mathbb{H}_1 \) is a pseudoisometry \( S^{2,1} \subset \mathbb{R}^{2,2} \), diffeomorphic to \( S^1 \times \mathbb{R}^2 \).

### 2.2. Vector spaces over para-quaternions.

Consider a right module \( \mathbb{H}^n \cong \mathbb{R}^{4n}, n \geq 1 \) over the algebra \( \mathbb{H} \), where the multiplication of \( (h_1, \ldots, h_n) \in \mathbb{H}^n \) and \( q \in \mathbb{H} \) is given by

\[(h_1, \ldots, h_n)q := (h_1q, \ldots, h_nq).\]

Right multiplications by \( i, j, k \), respectively, induce endomorphisms \( J_1, J_2, J_3 \) of \( \mathbb{R}^{4n} \) satisfying

\[J_\beta J_\gamma = -\epsilon_\gamma J_\alpha, \quad J_\alpha^2 = -\epsilon_\alpha \text{Id},\]

where \( (\alpha, \beta, \gamma) \) is a cyclic permutation of \( (1, 2, 3) \).

The map \( z : \mathbb{H}^n \rightarrow \mathbb{C}^2 \) given by relation \( (1) \) enables us to define a real isomorphism \( z : \mathbb{H}^n \rightarrow \mathbb{C}^{2n} \) by the formula

\[
\mathbb{H}^n \ni \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = h \rightarrow z(h) := \begin{pmatrix} z(h_1) \\ \vdots \\ z(h_n) \end{pmatrix} \in \mathbb{C}^{2n}.
\]

If we regard \( \mathbb{H}^n \) as a right vector space over \( \mathbb{C} \), then \( z \) is a complex isomorphism.
In $\mathbb{H}^n$ we define the scalar product of real signature $(2n, 2n)$ by

$$
\langle h, h' \rangle := \Re(h h') = h_1 \bar{h}'_1 + \cdots + h_n \bar{h}'_n,
$$

where $h = (h_1, \ldots, h_n)$, $h' = (h'_1, \ldots, h'_n)$. Hence we can identify $\mathbb{H}^n$ with $\mathbb{R}^{2n, 2n}$. Notice that the scalar product Eq. (5) can be written in terms of the complex representation $\mathbb{C}$ by

$$
\langle h, h' \rangle = \sum_{i=1}^{n} (\Re(z_1(h_i) \bar{z}_1(h'_i)) - \Re(z_2(h_i) \bar{z}_2(h'_i))).
$$

It is a hermitian scalar product of real signature $(2n, 2n)$ on $\mathbb{C}^{2n}$, represented by the following diagonal block matrix

$$
E_n := \text{diag}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \subset \text{gl}_{2n}(\mathbb{C}).
$$

With respect to the scalar product $\langle \cdot, \cdot \rangle$, $J_1$ is an isometry, while $J_2$ and $J_3$ are anti-isometries of $\mathbb{H}^n$. All three endomorphisms $J_1, J_2, J_3$ are skew-symmetric with respect to the metric, i.e.

$$
\langle J_\alpha \cdot, \cdot \rangle = -\langle \cdot, J_\alpha \cdot \rangle, \quad \alpha = 1, 2, 3.
$$

2.3. Para-quaternionic structures in a real vector space $V$.

**Definition 2.1.** Let $V$ be a $4n$-dimensional real vector space with pseudo-Riemannian scalar product $g$ of signature $(2n, 2n), n \geq 1$.

i) A triple $(J_1, J_2, J_3)$ of endomorphisms of $V$ satisfying the relations

$$
J_\beta J_\gamma = -\epsilon_\alpha J_\alpha, \quad J_\alpha^2 = -\epsilon_\alpha \text{Id},
$$

where $(\alpha, \beta, \gamma)$ is a cyclic permutation of $(1, 2, 3)$, is called a para-hypercomplex structure on $V$.

ii) A subalgebra $\mathfrak{G} \subset \text{End}(V)$ is called a para-quaternionic structure on $V$ if there exists its basis $J_1, J_2, J_3$, satisfying the relations Eq. (8). We say that the para-quaternionic structure $\mathfrak{G}$ and the para-hypercomplex structure $(J_1, J_2, J_3)$ correspond to each other.

iii) A para-hypercomplex structure $(J_1, J_2, J_3)$ is called hermitian with respect to $g$ if its endomorphisms are skew-symmetric with respect to $g$.

iv) A para-quaternionic structure $\mathfrak{G}$ on $V$ is called hermitian with respect to $g$ if some (and hence any) corresponding para-hypercomplex structure is hermitian with respect to $g$.

Any two bases $(J_1, J_2, J_3)$ and $(J'_1, J'_2, J'_3)$ of $\mathfrak{G}$ are related by a $SO(2, 1)$ transformation. The existence of a para-hypercomplex structure $(J_1, J_2, J_3)$ allows us to equip $V$ with the structure of a right module over $\mathbb{H}$. Namely, for $e \in V$ and $q = a + bi + cj + dk \in \mathbb{H}$ we define multiplication, i.e. a right module structure on $V$, by

$$
eq := ae + bJ_1 e + cJ_2 e + dJ_3 e.
$$

One can check that there always exist vectors $e_1, \ldots, e_n \in V$ such that the vectors

$$
eq e_1, \ldots, e_n, J_1 e_1, \ldots, J_1 e_n, J_2 e_1, \ldots, J_2 e_n, J_3 e_1, \ldots, J_3 e_n
$$

are linearly independent. Hence we can make the identification

$$
V \cong \mathbb{R}^{4n} \cong \mathbb{H}^n.
$$
The basis $e_1$ is called the basis adopted with the para-hypercomplex structure $(J_1, J_2, J_3)$. Clearly, if the vectors $e_1, \ldots, e_n$ are orthonormal then the adopted basis $e_1$ is a pseudo-orthonormal basis of $V$ with respect to the scalar product $g$. This implies that the signature of $g$ is $(2n, 2n)$ and that this condition was not necessary in Definition 2.1.

To each endomorphism $J \in \mathfrak{g}$ corresponds a nondegenerate 2-form $\omega_J$ on $V$ by the formula

$$\omega_J(\cdot, \cdot) := g(J \cdot, \cdot).$$

The above correspondence is an isometry of the subspace $\mathfrak{g} \subset \text{End} V$ and its image in $\Lambda^2 V$. Let $\omega_1 := \omega_{J_1}$, $\omega_2 := \omega_{J_2}$, $\omega_3 := \omega_{J_3}$.

One can prove that the 4-form

$$(11) \quad \Omega = \Omega(\mathfrak{g}) := \omega_1 \wedge \omega_1 - \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3$$

is invariant under the action of the group $SO(2,1)$ and hence independent of the choice of basis $J_1, J_2, J_3$ of $\mathfrak{g}$. The following lemma is a consequence of the Riemannian version by means of complexification.

**Lemma 2.0.1.** The maximal subalgebra of $so(2n, 2n)$ preserving the 4-form $\Omega$, given by (11), is $sp(n, \mathbb{R}) \oplus sp(1, \mathbb{R})$.

### 2.4. Para-quaternionic linear maps.

Let $V$ be a 4n dimensional real vector space, $\mathfrak{g}$ a para-quaternionic structure on $V$ and $\mathfrak{g} = (J_1, J_2, J_3)$ a corresponding para-hypercomplex structure.

**Definition 2.2.** Let $L \in \text{End} V$ be a $\mathbb{R}$-linear map. $L$ is para-quaternionic if for any $J \in \mathfrak{g}$ there exists $J' \in \mathfrak{g}$ such that $[L, J] = J'$. $L$ is para-hypercomplex if $[L, J_\alpha] = 0, \ \alpha = 1, 2, 3$.

We denote the algebra of all para-quaternionic (resp. para-hypercomplex) linear maps with respect to $\mathfrak{g}$ (resp. $\mathfrak{f}$) by $aut(\mathfrak{g})$ (resp. $aut(\mathfrak{f})$). By the very definitions, $aut(\mathfrak{g})$ and $aut(\mathfrak{f})$ are the normalizer and centralizer of $\mathfrak{g}$ in $\text{End} V$. There is a decomposition

$$aut(\mathfrak{g}) = aut(\mathfrak{f}) \oplus \mathfrak{g}.$$ 

Because of the identification $V \cong \mathbb{H}^n$ we sometimes use the notation $aut(\mathfrak{f}) = gl_n(\mathbb{H})$. The algebra $gl_n(\mathbb{H})$ can be identified with the algebra of all $n \times n$ matrices with para-quaternionic entries, acting by left multiplication on column vectors of $\mathbb{H}^n$.

Let $\mathfrak{g}$ be a hermitian para-quaternionic and $\mathfrak{f}$ the corresponding para-hypercomplex structure with respect to the scalar product $g$ on $V$. We define subalgebras of $aut(\mathfrak{g})$ and $aut(\mathfrak{f})$ preserving the metric $g$ by

$$sp(\mathfrak{g}) := \{ L \in aut(\mathfrak{g}) \mid g(L \cdot, \cdot) = g(\cdot, L \cdot) \}, \quad sp(\mathfrak{f}) := aut(\mathfrak{f}) \cap sp(\mathfrak{g}).$$

Since $\mathfrak{g} \subset sp(\mathfrak{g})$, we have the decomposition

$$sp(\mathfrak{g}) = sp(\mathfrak{f}) \oplus \mathfrak{g}.$$ 

We use the notation $sp_n(\mathbb{H}) = sp(\mathfrak{f})$.

**Lemma 2.0.2.** The following isomorphisms hold:

1. $gl_n(\mathbb{H}) = aut(\mathfrak{f}) \cong gl_{2n}(\mathbb{R})$,
2. $sp_n(\mathbb{H}) = sp(\mathfrak{f}) \cong sp(n, \mathbb{R})$. 

Theorem 2.1. As an immediate consequence of the previous theorem we have the following:

Statement i) is proved.

The matrix $\text{gl}_n(\mathbb{H})$ Responding para-hypercomplex structure hermitian with respect to a scalar product $\mu$.

The composition $\mu \circ z : z(\text{gl}_n(\mathbb{H})) \to \text{gl}_2n(\mathbb{C})$ by

\[
\begin{pmatrix} a_{11} + jb_{11} & \cdots & a_{1n} + jb_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} + jb_{n1} & \cdots & a_{nn} + jb_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & b_{11} & \cdots & a_{1n} & b_{1n} \\ b_{11} & \bar{a}_{11} & \cdots & b_{1n} & \bar{a}_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & b_{n1} & \cdots & a_{nn} & b_{nn} \\ b_{n1} & \bar{a}_{n1} & \cdots & b_{nn} & \bar{a}_{nn} \end{pmatrix} = \begin{pmatrix} a_{pq} & b_{pq} \\ b_{pq} & \bar{a}_{pq} \end{pmatrix}.
\]

We use the above notation for block matrices. Hence we have the following complex representation of $\text{gl}_n(\mathbb{H})$:

\[z(\text{gl}_n(\mathbb{H})) = \left\{ \begin{pmatrix} a_{pq} & b_{pq} \\ b_{pq} & \bar{a}_{pq} \end{pmatrix} \mid a_{pq}, b_{pq} \in \mathbb{C}, \ p, q = 1, \ldots, n \right\} \subset \text{M}_2n(\mathbb{C}).\]

Define the map $\mu : z(\text{gl}_n(\mathbb{H})) \to \text{gl}_2n(\mathbb{C})$ by

\[\mu(Q) = MQM^{-1},\]

where the matrix $M \in \text{gl}_2n(\mathbb{C})$ is the block diagonal matrix

\[M = \frac{\sqrt{2}}{2} \text{diag}(1, i, i, \bar{i}).\]

One can check directly that

\[\mu\left( \begin{pmatrix} a_{pq} & b_{pq} \\ b_{pq} & \bar{a}_{pq} \end{pmatrix} \right) = \begin{pmatrix} \Re a_{pq} - \Im b_{pq} & \Im a_{pq} + \Re b_{pq} \\ \Re b_{pq} - \Im a_{pq} & \Re a_{pq} + \Im b_{pq} \end{pmatrix} \in \text{gl}_2n(\mathbb{R}).\]

The composition $\mu \circ z : z(\text{gl}_n(\mathbb{H})) \to \text{gl}_2n(\mathbb{R})$ is a Lie algebra isomorphism, so that statement i) is proved.

According to the relation $\text{sp}_n(\mathbb{H})$, we can represent the algebra $\text{sp}_n(\mathbb{H})$ in terms of the matrix $E_n$, given by $\mathbb{R}$, in the following way:

\[(\mu \circ z)(\text{sp}_n(\mathbb{H})) = \{ \mu(Q) \mid Q \in z(\text{gl}_n(\mathbb{H})), \mu(Q)^* \mu(E_n) = -\mu(E_n)\mu(Q) \} = \{ A \in \text{gl}_2n(\mathbb{R}) \mid A^TF_n = -F_nA \},\]

where

\[F_n = \text{diag}(0, 1, -1, 0) = \frac{i}{2} \mu(E_n).\]

The matrix $F_n$ represents the nondegenerate two form

\[\omega = e_1 \wedge f_1 + \cdots + e_n \wedge f_n\]

in $\mathbb{R}^{2n}$, where $e_1, f_1, \ldots, e_n, f_n$ is a basis of $\mathbb{R}^{2n}$. Hence $\text{sp}_n(\mathbb{H}) \cong \text{sp}(n, \mathbb{R})$. □

As an immediate consequence of the previous theorem we have the following:

**Theorem 2.1.** Let $\mathfrak{g}$ be a para-quaternionic structure and $\mathfrak{n} = (J_1, J_2, J_3)$ a corresponding para-hypercomplex structure hermitian with respect to a scalar product $g$ on a 4n-dimensional ($n \geq 1$) real vector space $V$. The following isomorphisms hold:

i) $\text{aut}(\mathfrak{n}) \cong \text{gl}_2n(\mathbb{R})$, $\text{aut}(\mathfrak{g}) \cong \text{gl}_2n(\mathbb{R}) \oplus \text{sp}(1, \mathbb{R})$,

ii) $\text{sp}(\mathfrak{n}) \cong \text{sp}(n, \mathbb{R})$, $\text{sp}(\mathfrak{g}) \cong \text{sp}(n, \mathbb{R}) \oplus \text{sp}(1, \mathbb{R})$. 


Remark 2.1.1. The case \( V \cong \mathbb{R}^4 \) is covered by Theorem 2.1, but deserves attention. In fact, the algebra \( \text{aut}(\mathfrak{G}) \cong \mathbb{R} \text{Id} \oplus \mathfrak{so}(2, 2) \) is the algebra of conformal transformations of \( V \). Moreover, by splitting \( \mathfrak{so}(2, 2) = \mathfrak{sp}(1, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) \) and \( \mathfrak{sp}(\mathfrak{G}) \cong \mathfrak{G} \) we have

\[
\text{aut}(\mathfrak{G}) = \mathbb{R} \text{Id} \oplus \mathfrak{G} \oplus \mathfrak{G}',
\]

where \( \mathfrak{G}' \) is another para-quaternionic structure on \( V \).

2.5. Grassman description of a para-quaternionic vector space.

Lemma 2.1.1. Let \( V \) be a real, 4n-dimensional vector space. If we represent \( V \) as a tensor product

\[
V = E \otimes H,
\]

where \( E \cong \mathbb{R}^{2n} \), \( H \cong \mathbb{R}^2 \), then the endomorphisms

\[
J_1 = \text{Id} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \text{Id} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \text{Id} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

of \( V \) form a para-hyperKähler structure on \( V \). Conversely, for any para-hyperKähler structure \( (J_1, J_2, J_3) \) on \( V \) there exists a decomposition (13) of \( V \) and a basis of \( H \) in which endomorphisms \( J_1, J_2, J_3 \) are of the form (14).

Proof: The first statement is obvious. To check the second let \( \mathcal{J} = (J_1, J_2, J_3) \) be a para-hypercomplex structure on \( V \) and let \( H := \mathbb{R}(\text{Id} - J_3, J_1 + J_2) \). The space \( H \) is a subalgebra of \( \mathbb{R}(\text{Id}, J_1, J_2, J_3) \cong \mathbb{R}^4 \). Let

\[
E := \mathbb{R}\langle e_1, \ldots, e_n, J_2e_1, \ldots, J_2e_n \rangle,
\]

for vectors \( e_1, \ldots, e_n \in V \), such that (14) is a basis of \( V \). Then the map \( \phi : E \otimes H \rightarrow V \) defined by

\[
\phi(e \otimes h) := h(e), \quad e \otimes h \in E \otimes H = V,
\]

is a vector space isomorphism. It is easy to check that the endomorphisms \( J_1, J_2, J_3 \) have the form (14) in the basis \( h = \text{Id} - J_3, h_2 = J_1 + J_2 \) of \( H \).

It is clear that Lemma 2.1.1 holds for the para-quaternionic structure \( \mathfrak{G} = \mathbb{R}(J_1, J_2, J_3) \).

The algebra \( \text{aut}(\mathfrak{G}) \cong \mathfrak{gl}_{2n}(\mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) \) acts irreducibly on \( V = E \otimes H \) by

\[
(A, J)(e \otimes h) = Ae \otimes h + e \otimes Jh,
\]

for any \( (A, J) \in \mathfrak{gl}_{2n}(\mathbb{R}) \oplus \mathfrak{sl}(1, \mathbb{R}) \), \( e \otimes h \in V \). The algebra \( \mathfrak{sp}(\mathfrak{G}) = \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) \) acts on \( V \) by the formula (14) preserving some nondegenerate two forms \( \omega^E \) and \( \omega^H \) on \( E \) and \( H \), respectively. The scalar product

\[
g = \omega^E \otimes \omega^H
\]

on \( V \) of signature \( (2n, 2n) \) is invariant under the action of \( \mathfrak{sp}(\mathfrak{G}) \). The para-quaternionic structure \( \mathfrak{G} = \mathbb{R}(J_1, J_2, J_3) \) is hermitian with respect to the scalar product (10).

Conversely, for any para-quaternionic structure \( \mathfrak{G} \), hermitian with respect to a given metric \( g \) on \( V \), there exist forms \( \omega^E \) and \( \omega^H \) such that relation (10) holds.

We conclude this section with the following interesting lemma:

Lemma 2.1.2. Let \( \mathfrak{G} \) and \( \mathcal{J} = (J_1, J_2, J_3) \) be a para-quaternionic and corresponding para-hypercomplex structure hermitian with respect to a scalar product \( g \) on a vector space \( V \). Then

i) The algebra \( \mathfrak{sp}(\mathfrak{G}) = \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) \) acts on \( V \) irreducibly.
The algebra \( sp(\mathfrak{f}) = sp(n, \mathbb{R}) \) acts on \( V \) reducibly. Moreover, there exist a \( S^1 \) family of isotropic \( 2n \)-dimensional subspaces \( V_\phi, \phi \in S^1 \), invariant with respect to the action of \( sp(\mathfrak{f}) \).

**Proof:** The first statement is obvious since the action (15) is a tensor product of irreducible actions. To prove ii), let \( h_1, h_2 \) be a basis of \( H \). For any \( \phi \in [0, \pi] \) the \( 2n \)-dimensional subspace

\[
V_\phi := \{ e \otimes (h_1 \cos \phi + h_2 \sin \phi) \mid e \in E \}
\]

is \( sp(\mathfrak{f}) \) invariant and isotropic.

\( \Box \)

3. Para-quaternionic structures on manifolds

3.1. Para-hyperKähler manifolds.

**Definition 3.1.**

i) A pseudo-Riemannian manifold \( (M^{4n}, g), n \geq 1 \) is called **almost hermitian para-hypercomplex** if there exist three global sections \( J_1, J_2, J_3 \) of \( \text{End}(TM) \) with the following property: for each point \( p \in M \) the triple \( (J_1, J_2, J_3) \) is a hermitian para-hypercomplex structure with respect to the metric \( g \) of \( T_p M \).

ii) An almost hermitian para-hypercomplex manifold \( (M^{4n}, g), n \geq 1 \) is **para-hyperKähler** if \( \nabla J_\alpha = 0, \alpha = 1, 2, 3, \) and \( \nabla \) is the Levi-Civita connection with respect to the metric \( g \).

Note that the definition implies that the signature of the metric \( g \) is \((2n, 2n)\).

**Theorem 3.1.** The following conditions are equivalent for an almost hermitian para-hypercomplex manifold \( (M^{4n}, g, (J_1, J_2, J_3)) \) (\( \nabla \) denotes the Levi-Civita connection):

i) \( \nabla J_\alpha = 0, \alpha = 1, 2, 3, \)

ii) \( \nabla \omega_\alpha = 0, \alpha = 1, 2, 3, \)

iii) \( d\omega_\alpha = 0, \alpha = 1, 2, 3. \)

The proof is similar as in the Riemannian case.

**Remark 3.1.1.** The para-hyperKähler manifold is Kähler since \( J_1 \) is an integrable complex structure and \( d\omega_1 = 0 \). The structures \( J_2 \) and \( J_3 \) are not complex, but they are product structures.

The following characterization of para-hyperKähler manifolds follows from the definition of the algebra \( sp(\mathfrak{f}) \cong sp(n, \mathbb{R}) \) and the discussion in Section 2.5.

**Theorem 3.2.** A pseudo-Riemannian manifold \( (M^{4n}, g) \) is para-hyperKähler if and only if \( \text{hol}(M) \subset sp(n, \mathbb{R}) \).

According to the results of Section 2.5 we can decompose the tangent space of a para-hyperKähler manifold \( M \) as \( TM = E \otimes H \), where \( E \) and \( H \) are parallel \( 2n \)-dimensional and 2-dimensional vector bundles, respectively. The metric \( g \) of the manifold \( M \) can be written in the form

\[
g = \omega^E \otimes \omega^H
\]

where \( \omega^E \) and \( \omega^H \) are certain nondegenerate forms on \( E \) and \( H \), respectively. The Levi-Civita connection \( \nabla \) is \( sp(n, \mathbb{R}) \otimes T^*M \) valued and is given by

\[
\nabla(e \otimes h) = \nabla^E e \otimes h + e \otimes \nabla^H h
\]
for any \( e \otimes h \in \Gamma(TM) \). The conditions that \( \nabla \) is Levi Civita and commutes with the hypercomplex structure are equivalent to
\[
\nabla^H \omega^H = 0 = \nabla^E \omega^E, \quad \nabla^H \text{ is flat.}
\]
An interesting consequence of the flatness of \( \nabla^H \) and Lemma 2.1.2 b) is the following:

**Theorem 3.3.** Let \( M = (M^{4n}, g) \) be a para-hyperKähler manifold. There exists a \( S^1 \) family \( V_\phi, \phi \in S^1 \), of isotropic, parallel, \( 2n \)-dimensional distributions of \( TM \).

However, since the distributions \( V_\phi \) are isotropic, this does not imply decomposability of the manifold \( M \).

**Example 3.1.** The para-quaternionic vector space \( \tilde{H}^n \) with the metric \( g \) given by formula (5) is a para-hyperKähler manifold. The required endomorphisms \( J_1, J_2, J_3 \) of \( T\tilde{H}^n \cong \mathbb{H}^n \) can be defined as right multiplications by \( i, j \) and \( k \), respectively.

Using the method of para-hyperKähler reduction from this flat example Section 5.1 we will obtain another example of para-hyperKähler manifold.

**Example 3.2.** All Riemannian homogenous hyperKähler manifolds are flat. In particular, all Riemannian symmetric hyperKähler spaces are flat. The indefinite hyperKähler symmetric spaces are not necessarily flat. They are classified in [3]. Indefinite hyperKähler and para-hyperKähler symmetric spaces have the same complexification. Accordingly, classification of complex hyperKähler symmetric spaces from [3] lead to classification of para-hyperKähler symmetric spaces in [2]. It is interesting fact that para-hyperKähler symmetric spaces exist in all dimensions \( 4n \) unlike the indefinite hyperKähler symmetric spaces which exist only in dimensions \( 8n \). For example, there are up to the isomorphism only two (mutually anti-isometric) non-flat para-hyperKähler symmetric spaces given as follows.

Let us take the matrix \( A \)
\[
A = \begin{bmatrix}
0 & -1/2 & 1/2 & 0 \\
1/2 & 0 & 0 & 1/2 \\
1/2 & 0 & 0 & 1/2 \\
0 & 1/2 & -1/2 & 0
\end{bmatrix}
\]
Define a subalgebra \( f = \mathbb{R}A \subset so(2, 2) \) acting on the vector space \( m = \mathbb{R}^{2, 2} \) from the left. Let \( E_1, E_2, E_3, E_4 \) be a pseudo-orthonormal basis of \( m \). Define the Lie algebras \( g_\pm = m + f \) by the nonzero commutators
\[
[A, M] = A(M), \quad M \in m,
\]
\[
[E_1, E_2] = [E_3, E_1] = [E_4, E_2] = [E_3, E_4] = \pm A
\]
The symmetric spaces corresponding to the symmetric decompositions \( g_\pm = m + f \) are required para-hyperKähler spaces. These spaces appeared in [7] as examples of rank 2 Osserman symmetric spaces. Note that the algebras \( g_\pm \) are solvable.

### 3.2. Para-quaternionic Kähler manifolds.

**Definition 3.2.** A pseudo-Riemannian manifold \( M = (M^{4n}, g) \), \( n \geq 2 \), is almost hermitian para-quaternionic if there exists a subbundle \( \mathcal{G} \) of \( \text{End}(TM) \) with the following property: for each point \( p \in M \) the fiber \( \mathcal{G}_p \) of \( \mathcal{G} \) is a hermitian para-quaternionic structure of \( T_pM \) with respect to the metric \( g \).
We can locally choose a para-hypercomplex structure \((J_1, J_2, J_3)\) which is a basis of \(\mathfrak{g}\). Hence any almost hermitian para-quaternionic manifold is locally almost hermitian para-hypercomplex.

As in Section 2.3 we can locally define nondegenerate two forms \(\omega_1, \omega_2, \omega_3\). However the 4-form
\[
\Omega = \Omega(\mathfrak{g}) = \omega_1 \wedge \omega_1 - \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3
\]
is defined globally on \(M\).

**Definition 3.3.**

i) An almost hermitian para-quaternionic manifold \(M = (\mathbb{M}^{4n}, g), n \geq 2\), is called para-quaternionic Kähler if \(\nabla \Omega = 0\) and \(M\) is not para-hyperKähler.

ii) A pseudo-Riemannian 4-dimensional manifold \((\mathbb{M}^4, g)\) with a metric \(g\) of signature \((2, 2)\) is called para-quaternionic Kähler if it is self-dual and Einstein.

Clearly, the condition \(\nabla \Omega = 0\) implies \(d\Omega = 0\) which has strong consequences on the topology of \(M\).

According to Lemma 2.0.1 and Theorem 3.2 for a para-quaternionic Kähler manifold we have \(\text{hol}(M) \subset \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})\), \(\text{hol}(M) \not\subset \mathfrak{sp}(n, \mathbb{R})\). The converse statement follows from the results of Section 2.5, so we have the following characterization.

**Theorem 3.4.** A pseudo-Riemannian manifold \(M = (\mathbb{M}^{4n}, g), n \geq 2\), is para-quaternionic Kähler if and only if
\[
\text{hol}(M) \subset \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}), \ \text{hol}(M) \not\subset \mathfrak{sp}(n, \mathbb{R}).
\]

Note that for a 4-dimensional manifold \(M\) with a metric \(g\) of signature \((2, 2)\) the condition \(\text{hol}(M) \subset \mathfrak{so}(2, 2) = \mathfrak{sp}(1, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})\) is always satisfied. On the other hand, the condition \(\nabla \Omega = 0\) is satisfied since \(\Omega\) is a 4-form. These were the reasons for the separate definition of a 4-dimensional para-quaternionic Kähler manifold. Similarly to the para-hyperKähler case we can write \(TM = E \otimes H\) where \(E\) and \(H\) are certain \(2n\)-dimensional and 2-dimensional vector bundles, respectively. If we write the metric in the form
\[
g = \omega^E \otimes \omega^H
\]
for some 2-forms \(\omega^E\) and \(\omega^H\) then the Levi-Civita connection \(\nabla\) is given by
\[
\nabla(e \otimes h) = \nabla^E e \otimes h + e \otimes \nabla^H h
\]
for any \(e \otimes h \in \Gamma(TM)\). The condition that \(\nabla\) is Levi-Civita is equivalent to
\[
\nabla^H \omega^H = 0 = \nabla^E \omega^E.
\]
The connection \(\nabla\) preserves the subbundle \(\mathfrak{g} \subset \text{End}(TM)\). One can easily check that the structure equations are
\[
\nabla J_\alpha = \epsilon_\alpha (\mu_\beta J_\gamma - \mu_\gamma J_\beta),
\]
where \((\alpha, \beta, \gamma)\) is a cyclic permutation of \((1, 2, 3)\) and \(\mu_1, \mu_2, \mu_3\) are structure 1-forms.

**Example 3.3.** We shall show that the quotient \(\mathfrak{sl}_{n+2}(\mathbb{R})/\mathfrak{sl}_n(\mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})\) is a para-quaternionic Kähler, pseudo-Riemannian symmetric space. Its holonomy \(\mathfrak{sl}_n(\mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) \subset \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})\) is not full. An example of a para-quaternionic symmetric space with full holonomy is given in Section 3.3.
Consider the symmetric decomposition $\mathfrak{sl}_{n+2}(\mathbb{R}) = m + f$ of the algebra $\mathfrak{sl}_{n+2}(\mathbb{R})$, where

$$f = \{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{sl}_2(\mathbb{R}), B \in \mathfrak{sl}_n(\mathbb{R}) \} \cong \mathfrak{sl}_n(\mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}).$$

One can easily check that the adjoint action of the algebra $\mathfrak{sp}(1, \mathbb{R})$ defines a para-quaternionic structure $\mathcal{G}$ on the vector space $m \cong \mathbb{R}^{4n}$ invariant under holonomy. Hence the pseudo-Riemannian symmetric space $\mathfrak{sl}_{n+2}(\mathbb{R})/(\mathfrak{sl}_n(\mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ is para-quaternionic Kähler. One can find a basis of $\mathcal{G}$ and then decompose $m = E \otimes H$ using the proof of Theorem 2.1.1.

3.3. The para-quaternionic projective space $\mathbb{H}P^n$. Consider the vector space $\mathbb{H}^{n+1}$ with a natural scalar product $\langle \cdot, \cdot \rangle$ of signature $(2n+2, 2n)$ defined in Section 2.2. Denote by

$$S = S^{2(n+1), 2n+1} \subset \mathbb{H}^{n+1}$$

the pseudosphere of unit vectors in $\mathbb{H}^{n+1}$ with the induced metric. Its signature is $(2n+2, 2n+1)$. The group $\mathbb{H}_1$ of unit para-quaternionic numbers acts freely and isometrically on $S$ from the right by the formula

$$x q = (x_1, \ldots, x_{n+1}) q := (x_1q, \ldots, x_{n+1}q),$$

$s \in H$, and

$$x = (x_1, \ldots, x_{n+1}) \in S, \quad q \in \mathbb{H}_1.$$

The para-quaternionic projective space $\mathbb{H}P^n$ is defined as the orbit space

$$\mathbb{H}P^n := S/\mathbb{H}_1.$$ Denote by

$$\pi : S \to \mathbb{H}P^n, \quad \pi((x_1, \ldots, x_{n+1})) := [x_1, \ldots, x_{n+1}]$$

the natural projection.

Since the orbits of $\mathbb{H}_1$ are nondegenerate of signature $(2, 1)$ the metric $\langle \cdot, \cdot \rangle$ pushes down to the pseudo-Riemannian metric $g$ of signature $(2n, 2n)$ on $\mathbb{H}P^n$. The manifold $\mathbb{H}P^n$ has the constant para-quaternionic sectional curvature (see [7]).

Let us show that $\mathbb{H}P^n$ is a para-quaternionic Kähler manifold. The vertical subspace $T_x^v S$ of $T_x S$, $x \in S$, with respect to the action of $\mathbb{H}_1$ is

$$T_x^v S := \text{Ker}(d_x \pi) = \mathbb{R}\langle xi, xj, xk \rangle \subset \mathbb{H}^{n+1}.$$ Moreover, there is an orthogonal decomposition

$$T_x S = T_x^v S \oplus T_x^h S$$

of $T_x S$, $x \in S$, invariant to the action of $\mathbb{H}_1$. The right multiplication by $h \in \text{Im} \mathbb{H}$ preserves $T_x^h S$. Let

$$s : \mathbb{H}P^n \to S$$

be a horizontal section. For any $h \in \text{Im} \mathbb{H}$ define the endomorphism $J_h$ of $T \mathbb{H}P^n$ by

$$J_h(X) := \pi_*(s_*(X)) h,$$

where $X \in T \mathbb{H}P^n$. Since the endomorphisms $J_h$ and $J'_h$ corresponding to different sections $s$ and $s' = sq$, $q \in \mathbb{H}_1$, are $Ad(q)$ related, they generate a global subbundle $\mathcal{G}$ of $\text{End}(T \mathbb{H}P^n)$. One can easily check that $\mathcal{G}$ is an almost para-quaternionic Kähler structure on $\mathbb{H}P^n$ and that the corresponding 4-form $\Omega(\mathcal{G})$ is parallel. Hence $\mathbb{H}P^n$ is a para-quaternionic Kähler manifold.
The para-quaternionic projective space $\tilde{\mathbb{H}}P^n$ is the homogenous space

$$\tilde{\mathbb{H}}P^n = Sp(n+1, \mathbb{R})/Sp(1, \mathbb{R}) \cdot Sp(n, \mathbb{R})$$

(17)

where $Sp(1, \mathbb{R}) \cdot Sp(n, \mathbb{R}) = (Sp(1, \mathbb{R}) \times Sp(n, \mathbb{R}))/\mathbb{Z}_2$ and $Sp(n, \mathbb{R})$ denotes the (connected) group of symplectic transformations.

**Proof:** The map $\mu$ defined by formula (12) is a Lie group isomorphism and

$$Sp(n+1, \mathbb{R}) \cong \mu^{-1}(Sp(n+1, \mathbb{R})) = \{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in SU(n+1, n+1) \}$$

holds. Using the representation $z : \tilde{\mathbb{H}}^{n+1} \to \mathbb{C}^{2(n+1)}$ given by (11), we obtain an isometric action of $\mu^{-1}(Sp(n+1, \mathbb{R}))$ on the pseudosphere $S \subset \tilde{\mathbb{H}}^{n+1}$. This action commutes with the right multiplication by the para-quaternionic numbers defining the space $\tilde{\mathbb{H}}P^n$ and hence the group $\mu^{-1}(Sp(n+1, \mathbb{R}))$ acts on $\tilde{\mathbb{H}}P^n$. The induced action is isometric and preserves the para-quaternionic-structure $\mathcal{G}$ on $\tilde{\mathbb{H}}P^n$. To show its transitivity we will show that the point

$$o = (1, 0, \ldots, 0) + j(0, 0, \ldots, 0) \in S \subset \tilde{\mathbb{H}}^{n+1}$$

can be mapped to any point $a_1 + jb_1 \in S$ by a matrix $M \in \mu^{-1}(Sp(n+1, \mathbb{R}))$. Notice that $|a_1 + jb_1|^2 = 1$ imply

$$|b_1 + j\bar{a}_1|^2 = -1, \quad \langle a_1 + jb_1, \bar{b}_1 + j\bar{a}_1 \rangle = 0$$

and $\langle a_1 + jb_1, \bar{a}_2 + j\bar{b}_2 \rangle = 0$, $|a_2 + jb_2|^2 = 1$ implies

$$\langle \bar{b}_1 + j\bar{a}_1, \bar{b}_2 + j\bar{a}_2 \rangle = 0.$$ 

Hence we can construct a matrix $M \in \mu^{-1}(Sp(n+1, \mathbb{R})) \subset SU(n+1, n+1)$ having the vectors

$$(z(a_1 + jb_1), z(\bar{b}_1 + j\bar{a}_1), \ldots, z(a_{n+1} + jb_{n+1}), z(\bar{b}_{n+1} + j\bar{a}_{n+1}))$$

as columns and mapping the point $o \in S$ to a point $a_1 + jb_1 \in S$. One can easily check that the stabilizer of the point $[o] = \pi(o) \in \tilde{\mathbb{H}}P^n$ is isomorphic to $Sp(1, \mathbb{R}) \cdot Sp(n, \mathbb{R})$. \hfill $\square$

**Theorem 3.6.** The space $\tilde{\mathbb{H}}P^n$ is a para-quaternionic Kähler, pseudo-Riemannian symmetric space

$$\tilde{\mathbb{H}}P^n = sp(n+1, \mathbb{R})/(sp(n, \mathbb{R}) \oplus sp(1, \mathbb{R})).$$

Its curvature tensor is given by the formula

$$R(A, B)C = g(B, C)A - g(A, C)B + \sum_{a} \epsilon_{a} (g(J_{a}B, C)J_{a}A -$$

$$- g(J_{a}A, C)J_{a}B) - 2 \sum_{a} \epsilon_{a} g(J_{a}A, B)J_{a}C.$$ 

(18)

**Proof:** Since the calculations are simple unlike the notation, we give only the idea. The notation is the same as in the proof of Theorem 3.5. First one should find the symmetric algebra decomposition

$$sp(n + 1, \mathbb{R}) \cong \mu^{-1}(sp(n + 1, \mathbb{R})) = m' + f'$$

(19)
by interpreting the proof of the Theorem 3.5. Then one should find expressions for the metric $g'$ and endomorphisms $J_1', J_2', J_3'$ in terms of matrices from $m'$ (representing tangent vectors). The next step is to carry the decomposition together with all corresponding structures into the decomposition

\[(20) \quad sp(n+1, \mathbb{R}) = m + f, \quad f = sp(n, \mathbb{R}) \oplus sp(1, \mathbb{R}),\]

by the isomorphism $\mu$. Using the expression $R(A, B)C = [[A, B], C]$ for the curvature of a symmetric space one easily checks the relation (18).  

Remark 3.6.1. Let $e_1, f_1, \ldots, e^{n+1}, f^{n+1}$ be a basis and $\omega = e^1 \wedge f^1 + \cdots + e^{n+1} \wedge f^{n+1}$ be a nondegenerate 2-form on $\mathbb{R}^{2(n+1)}$. One can identify $sp(n + 1, \mathbb{R})$ and $S^2 \mathbb{R}^{2(n+1)}$ by

\[(a \vee b)(x) = \omega(a, x)b + \omega(b, x)a, \quad x \in \mathbb{R}^{2(n+1)},\]

where $a \vee b \in S^2 \mathbb{R}^{2(n+1)}$ denotes a symmetric product of vectors. The decomposition (20) is obtained after stabilizing $\mathbb{R}(e_0, f_0)$ under that action.

Concerning the topology of para-quaternionic projective spaces the following is known (see [3]).

Theorem 3.7. The space $\tilde{H}P^n$ is homotopically equivalent to the complex projective space $\mathbb{C}P^n$.

4. Space of curvature tensors of the para-quaternionic type

In this we shall just state the basic results. For details the reader is referred to [2] where a similar decomposition has been done in the quaternionic case.

Definition 4.1. Let $V$ be a real vector space and let $\mathfrak{L}$ be any subalgebra of $\text{End}(V)$. The space $\mathfrak{R}(\mathfrak{L})$ of curvature tensors of type $\mathfrak{L}$ is the space of $\mathfrak{L}$ valued 2-forms $R$ on $V$ satisfying the Bianchi identity

\[R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,\]

for all $X, Y, Z \in V$.

The space $\mathfrak{R}(\mathfrak{L})$ is an $\mathfrak{L}$ module. Let $\mathfrak{G} \subset \text{End}(V)$ be a para-quaternionic structure on a $4n$-dimensional real vector space $V$ and $\mathfrak{H} = (J_1, J_2, J_3)$ a corresponding para-hypercomplex structure. To describe the decomposition of the module $\mathfrak{R}(\mathfrak{L})$ on irreducible submodules in case $\mathfrak{L} \in \{\text{aut}(\mathfrak{G}), \text{aut}(\mathfrak{H})\}$ we need some additional notation.

We say that a bilinear form $B \in \text{Bil}(V)$ is hermitian with respect to a para-quaternionic structure $\mathfrak{G}$ if

\[B(JX, Y) = -B(X, JY),\]

for all $J \in \mathfrak{G}$, $X, Y \in V$. Denote by $\text{Bil}_\mathfrak{G}(V)$ the set of all bilinear forms on the space $V$ with respect to $\mathfrak{G}$ and by $\Pi_\mathfrak{G}: \text{Bil}(V) \to \text{Bil}_\mathfrak{G}(V)$ the projector

\[\Pi_\mathfrak{G}(B)(X, Y) := \frac{1}{4} (B(X, Y) + \sum_{a=1}^{3} \epsilon_a B(J_aX, J_aY)),\]

$B \in \text{Bil}(V), X, Y \in V$, which is well defined, i.e. independent of the basis $J_1, J_2, J_3$ of $\mathfrak{G}$. The set of bilinear forms decomposes as follows:

\[\text{Bil}(V) = S^2_\mathfrak{G} + \Lambda^2 + S^2_{\text{mix}} + \Lambda^2_{\text{mix}},\]
where $S^2_G$ and $\Lambda^2_G$ are the sets of symmetric and antisymmetric forms in $\text{Bil}_G(V)$, respectively, and $S^2_{\text{mix}} := (\text{Id} - \Pi_G)(S^2_G)$ and $\Lambda^2_{\text{mix}} := (\text{Id} - \Pi_G)(\Lambda^2_G)$ their complements.

For any $B \in \text{Bil}(V)$ define
\[
R^B(X, Y)Z := (B(Y, X) - B(X, Y))Z + \sum_\alpha \epsilon_\alpha (B(X, J_\alpha Y) - B(Y, J_\alpha X))Z + B(X, J_\alpha Z)J_\alpha Y - B(Y, J_\alpha Z)J_\alpha X,
\]
for all $X, Y, Z \in V$. One can check that the map
\[
\phi : \text{Bil}(V) \to \mathcal{R}(\text{aut}(\mathfrak{G})), \quad \phi(B) := R^B
\]
is a well defined monomorphism. Its image
\[
\mathcal{R}^{\text{Bil}} := \phi(\text{Bil}(V)) \subset \mathcal{R}(\text{aut}(\mathfrak{G}))
\]
is an $\text{aut}(\mathfrak{G})$ module.

**Remark 4.0.1.** Let the para-quaternionic structure $\mathfrak{G}$ be hermitian with respect to the scalar product $g$ on $V$. Comparing the expression (21) for $B = g$ and the expression (18) for the curvature at any point of the para-quaternionic projective space $\mathbb{H}P^n$, we can see that $R^g = R_{\mathbb{H}P^n}$.

For a curvature tensor $R$ we define its Ricci tensor $\text{Ric}(R)$ by
\[
\text{Ric}(R)(Y, Z) := \text{Tr}(X \mapsto R(X, Y)Z),
\]
for $X, Y, Z \in V$. One can check that the Ricci map $\text{Ric} : \mathcal{R}^{\text{Bil}} \to \text{Bil}(V)$ is a monomorphism. Hence a curvature tensor $R \in \mathcal{R}(\text{aut}(\mathfrak{G}))$ has an unique decomposition $R = W(R) + \text{Ric}(R)$ with respect to the decomposition
\[
\mathcal{R}(\text{aut}(\mathfrak{G})) = \mathfrak{W} + \mathcal{R}^{\text{Bil}}, \quad \mathfrak{W} := \text{Ker}(\text{Ric}).
\]
$W(R)$ is called Weil part and $\text{Ric}(R)$ is called Ricci part of the tensor $R$. Now we can state the main theorem:

**Theorem 4.1.** Let $\mathfrak{G}$ be a para-quaternionic structure on a $4n$-dimensional vector space $V$ and $\mathfrak{H} = (J_1, J_2, J_3)$ a corresponding para-hypercomplex structure.

1. For $n \geq 2$ the module $\mathcal{R}(\text{aut}(\mathfrak{G}))$ is the sum of the following irreducible submodules
\[
\mathcal{R}(\text{aut}(\mathfrak{G})) = \mathfrak{W} + \mathcal{R}(S^2_G) + \mathcal{R}(S^2_{\text{mix}}) + \mathcal{R}(\Lambda^2_G) + \mathcal{R}(\Lambda^2_{\text{mix}}),
\]
where $\mathfrak{W}$ is the kernel of the Ricci map and $\mathcal{R}(S^2_G)$, $\mathcal{R}(S^2_{\text{mix}})$, $\mathcal{R}(\Lambda^2_G)$, $\mathcal{R}(\Lambda^2_{\text{mix}})$ are images of $S^2_G$, $\Lambda^2_G$, $S^2_{\text{mix}}$, and $\Lambda^2_{\text{mix}}$, respectively, by the map $\phi$.

For $n = 1$ the above decomposition holds but the module $\mathfrak{W}$ is decomposable as
\[
\mathfrak{W} = \mathfrak{W}_+ + \mathfrak{W}_- = \mathcal{R}(\mathfrak{G}') + \mathcal{R}(\mathfrak{G})
\]
where $\mathfrak{G}'$ is para-quaternionic structure on $V$ described in Remark 2.1.2.

2. $\mathcal{R}(\text{aut}(\mathfrak{H})) = \mathfrak{W} + \mathcal{R}(\Lambda^2_G)$, $n \geq 2$, \quad $\mathcal{R}(\text{aut}(\mathfrak{H})) = \mathfrak{W}_- + \mathcal{R}(\Lambda^2_G)$, $n = 1$.

**Theorem 4.2.** Let $\mathfrak{G}$ be a para-quaternionic hermitian structure with respect to the scalar product $g$ on a $4n$-dimensional, vector space $V$, $n \geq 2$ and let $\mathfrak{H} = (J_1, J_2, J_3)$ be a corresponding para-hypercomplex structure.
i) Any curvature tensor $R \in \mathfrak{R}(\text{sp}(\mathfrak{G}))$ is Einstein i.e.
$$\text{Ric}(R) = \frac{K(R)}{4n}g,$$
holds, where $K(R)$ is the scalar curvature of $R$.

ii) $\mathfrak{R}(\text{sp}(\mathfrak{G})) = \mathfrak{R}R_{\mathbb{H}P^n} + \mathfrak{R}(\text{sp}(\mathfrak{H}))$ where $\mathfrak{R}(\text{sp}(\mathfrak{H})) \subset \mathfrak{W}$ is an irreducible submodule and $R_{\mathbb{H}P^n}$ is the curvature of the para-quaternionic projective space.

We have the following important consequences concerning para-quaternionic Kähler manifolds.

**Theorem 4.3.**

i) A para-quaternionic Kähler manifold is an Einstein manifold.

ii) A para-quaternionic Kähler manifold of dimension $4n$, $n \geq 2$, with zero scalar curvature is a locally para-hyperKähler manifold.

The following lemma is important both for the proofs of the previous theorems and for various applications.

**Lemma 4.3.1.** Let $\mathfrak{G}$ be a para-quaternionic hermitian structure with respect to the scalar product $g$ on a $4n$-dimensional, vector space $V$ and $\mathfrak{H} = (J_1, J_2, J_3)$ a corresponding para-hypercomplex structure.

i) A curvature tensor $R \in \mathfrak{R}(\text{End}(V))$ belongs to $\mathfrak{R}(\text{aut}(\mathfrak{G}))$ if and only if
$$[R, J_\alpha] = \frac{\epsilon_\alpha}{2n}(\text{Tr}(J_\beta R)J_\gamma - \text{Tr}(J_\gamma R)J_\beta),$$
where $(\alpha, \beta, \gamma)$ is a cyclic permutation of $(1, 2, 3)$.

ii) Tensors belonging to the module $\mathfrak{W}$ are traceless.

iii) The following commutator relations hold:
$$[\mathfrak{W}, \mathfrak{G}] = 0, \ n \geq 2 \quad \text{and} \quad [\mathfrak{W}, \mathfrak{G}'] = 0, \ n = 1,$$
where the para-quaternionic structure $\mathfrak{G}'$ is described in Remark 2.1.1.

5. **Reduction techniques**

5.1. **Para-hyperKähler reduction.** Let $M = (M^{4n}, g, (J_1, J_2, J_3))$ be a para-hyperKähler manifold and let $\omega_1, \omega_2, \omega_3$ be the global nondegenerate 2-forms associated with $J_1, J_2, J_3$. Let a Lie group $G$ acts freely on $M$ by isometries and preserves para-hyperKähler structure i.e. $g^* J_\alpha = J_\alpha$, $\alpha = 1, 2, 3$, holds for a $g \in G$. Let $V$ be a Killing vector field on $M$ generated by an element $V^* \in \mathfrak{g} = \text{Lie}(G)$.

If it exists, the map $f = (f_1, f_2, f_3) : M \to g^* \otimes g^* \otimes g^*$ such that
$$d(f_\alpha(V)) = \omega_\alpha(V^*, \cdot),$$
$\alpha = 1, 2, 3$, holds for any $V \in \mathfrak{g}$ at any point of the manifold $M$, is called the moment map for the action $G$ on $M$.

**Theorem 5.1** (Para-hyperKähler reduction). Let a Lie group $G$ act freely and isometrically on a para-hyperKähler manifold $M = (M^{4n}, g, (J_1, J_2, J_3))$ and preserve its para-hyperKähler structure. Let $f = (f_1, f_2, f_3) : M \to g^* \otimes g^* \otimes g^*$ be the corresponding equivariant moment map. Suppose that $\xi \in g^* \otimes g^* \otimes g^*$ is such that
$$\mathfrak{R}_\xi := f^{-1}(\xi)$$
(on which $G$ acts by isometries) is a smooth submanifold of $M$. Suppose that the quotient

$$M_{\xi} := \mathcal{R}_{\xi}/G$$

has a smooth manifold structure for which the projection $\pi_{\xi} : \mathcal{R}_{\xi} \to M_{\xi}$ is a smooth pseudo-Riemannian submersion and $g_{\xi}$ is the induced metric on $M_{\xi}$. Then the pseudo-Riemannian manifold $(M_{\xi}, g_{\xi})$ is a para-hyperKähler manifold with respect to the para-hyperKähler structure obtained by projection $\pi$ from the structure on $M$.

**Example 5.1.** The natural para-hyperKähler structure on the para-quaternionic vector space $\mathbb{H}^{n+1}$ is described in Example 6. Let the group $G = S^1$ of unit complex numbers act on $\mathbb{H}^{n+1}$ by left multiplication, i.e.

$$e^{it} \cdot h = e^{it} \cdot (h_1, \ldots, h_{n+1}) := (e^{it} h_1, \ldots, e^{it} h_{n+1}), \quad t \in \mathbb{R},$$

for $h = (h_1, \ldots, h_{n+1}) \in \mathbb{H}^{n+1}$. The action is isometric and preserves the para-hyperKähler structure on $\mathbb{H}^{n+1}$. We identify $\mathbb{H}^{n+1}$ with $\mathbb{C}^{2(n+1)} = \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$ by $h = (z, w)$. After a straightforward calculation one finds that the equivariant moment map of $G$ is

$$f(z, w) = (|z|^2 + |w|^2, -\Re(zw), -\Im(zw)).$$

According to the notation of Theorem 5.2 choose $\xi = (-1, 0, 0)$. Then

$$\mathcal{R}_{\xi} = \{(z, w) \in \mathbb{C}^{2(n+1)} \mid |z|^2 + |w|^2 = 1, zw = 0\} \subset \mathbb{C}^{2(n+1)}.$$

The action of the group $G$ on $\mathcal{R}_{\xi}$ is given by $e^{it} \cdot (z, w) = (e^{it} z, e^{-it} w), \quad t \in \mathbb{R}$. After simple transformations one obtains that the resulting para-hyperKähler manifold is a (real) submanifold of $\mathbb{C}P^{2n+1}$ of real codimension two, given in homogenous coordinates by

$$M_{\xi} = \{(z, w) \in \mathbb{C}P^{2n+1} \mid |z|^2 = |w|^2, \Im(\bar{z}w) = 0\}.$$

**Remark 5.1.1.** It is interesting that the resulting manifold is compact. Analogous action in the quaternionic case results in a hyperKähler structure on the (noncompact) cotangent space $T^*\mathbb{C}P^n$ of the complex projective space $\mathbb{C}P^n$.

5.2. Para-quaternionic Kähler reduction. Let $M = (M^{4n}, g, \mathfrak{g})$ be a para-quaternionic Kähler manifold. Let a Lie group $G$ act freely and isometrically on $M$ and preserve the 4-form $\Omega = \Omega(\mathfrak{g})$, i.e. $g^* \Omega = \Omega$ holds for a $g \in G$.

Denote by $V$ the unique Killing vector field corresponding to a Lie algebra vector $V^* \in \mathfrak{g} = \text{Lie}(G)$. The section $\Theta_V$ of the bundle $\Omega^1(\mathfrak{g})$ of one forms with values in $\mathfrak{g}$ defined by

$$\Theta_V(X) := \sum_{\alpha} \omega_\alpha(V, X) J_\alpha,$$

for $X \in TM$, is well defined globally. Here is the main reduction theorem.

**Theorem 5.2** (Para-quaternionic Kähler reduction). If a Lie group acts freely and isometrically on the para-quaternionic Kähler manifold $M = (M^{4n}, g, \mathfrak{g})$ and preserves the 4-form $\Omega(\mathfrak{g})$ then there exist a unique section $f$ of bundle $\mathfrak{g}^* \otimes \mathfrak{g}$ such that

$$\nabla f^* = \Theta_V^*,$$

for every $V^* \in \mathfrak{g}$. Moreover, the group $G$ acts by isometries on the preimage $\mathcal{R} := f^{-1}(0)$ of the zero-section $0 \in \mathfrak{g}^* \otimes \mathfrak{g}$. Suppose that $\mathcal{R}$ is a smooth submanifold of
M and that the quotient $\tilde{M} := \tilde{\mathcal{R}}/G$ has a smooth structure for which the projection $\pi: \tilde{\mathcal{R}} \to \tilde{M}$ is a pseudo-Riemannian submersion with induced metric $\tilde{\mathcal{g}}$ on $\tilde{M}$. Then the pseudo-Riemannian manifold $(\tilde{M}, \tilde{\mathcal{g}})$ is a para-quaternionic Kähler manifold with respect to the structure $\Theta'$ induced on $\tilde{M}$ from the structure $\Theta$ by the projection $\pi$.

**Remark 5.2.1.** Theorems 4.1 and 4.2 can be applied in case of a locally free action of the group $G$. In that case the resulting para-quaternionic Kähler space may have an orbifold (rather than a manifold) structure (see [13]).

For the proof of the Theorem 5.2 we need several lemmas.

**Lemma 5.2.1.** The unique solution $f_{V^*}, \ V^* \in \mathfrak{g}$ of equation (22) is given by

$$f_{V^*} = \frac{4n}{K} \sum_{\alpha} \text{Tr}(J_\alpha L_V) J_\alpha,$$

where $K$ is the scalar curvature of $M$ and $L_V = \nabla_V - \mathcal{L}_V$ is the Nomizu operator ($\mathcal{L}$ is the Lie derivative).

**Proof:** Notice that the scalar curvature of a para-quaternionic Kähler manifold is different from zero. Since $V$ is a Killing vector field preserving para-quaternionic Kähler structure both operators $\mathcal{L}_V$ and $\nabla_V$, and hence the Nomizu operator $L_V$ take values in the holonomy $sp(n, \mathbb{R}) \oplus sp(1, \mathbb{R})$. Let $e_1, \ldots, e_{4n}$ be a local pseudo-orthonormal reper. Then the relation

$$\nabla_k \text{Tr}(J_\alpha L_V) = \nabla_k \left( \sum_{i,j=1}^{4n} (J_\alpha)_{ij}^k V_i^j \right) =$$

$$= \sum_{i,j=1}^{4n} (\nabla_k J_\alpha)_{ij}^k V_i^j + \sum_{i,j=1}^{4n} (J_\alpha)_{ij}^k (\nabla_k V_i^j) =$$

$$= \epsilon_{\alpha} \sum_{i,j=1}^{4n} (\epsilon_{\gamma} (e_k)_{j}^i J_{\beta} - \mu_{\beta} (e_k)_{j}^i J_{\gamma}) V_i^j + \sum_{i,j=1}^{4n} (J_\alpha)_{ij}^k g(R(V, e_k) e_i, e_j)$$

holds for any $k = 1, \ldots, 4n$ and any $\alpha = 1, 2, 3$. In other words, the relation

(23) \[ \nabla \text{Tr}(J_\alpha L_V) = \epsilon_\alpha (\mu_\gamma \text{Tr}(J_\beta L_V) - \mu_\beta \text{Tr}(J_\gamma L_V)) + \text{Tr}(J_\alpha R(V, \cdot)) \]

holds, where $(\alpha, \beta, \gamma)$ is a cyclic permutation of $(1, 2, 3)$. Summing up over $\alpha = 1, 2, 3$ most of the terms cancel and one obtains

(24) \[ \nabla f_V = \frac{4n}{K} \sum_{\alpha} \epsilon_\alpha \text{Tr}(J_\alpha R(V, \cdot)) J_\alpha. \]

Since the curvature tensor $R$ of the manifold $M$ is of type $sp(n, \mathbb{R}) \oplus sp(1, \mathbb{R})$, according to Theorem 1.2 ii) the only component of the curvature tensor which does not commute with all $J_\alpha$ is its projection $R'$ onto $\Theta$

$$R'(X, Y) = - \frac{K}{(4n)^2} \sum_{\alpha} \epsilon_\alpha g(J_\alpha X, Y) J_\alpha.$$

Substituting $R'$ from the last relation into relation (24) one easily checks (22). \hfill \Box

**Lemma 5.2.2.** The quotient space $\tilde{M} = (\tilde{\mathcal{R}}/G, \tilde{\mathcal{g}})$ from Theorem 5.2 has an almost hermitian para-quaternionic structure.
**Proof:** Let $V$ be a Killing vector field corresponding to a vector $V^* \in \mathfrak{g} = \text{Lie}(G)$. If $e_k$ is a vector tangent to $\mathfrak{R} = f^{-1}(0)$ then using relation (23) one obtains
\[ 0 = \nabla_k f v^* = \sum_{\alpha} \epsilon_\alpha \text{tr}(J_{\alpha} R(V, e_k)) J_{\alpha} = \epsilon_\alpha \frac{K}{4n} g(J_{\alpha} V, e_k) J_{\alpha}, \]
and hence the vectors $J_{\alpha} V$, $\alpha = 1, 2, 3$, are orthogonal to $\mathfrak{R}$.

Denote by $g' \subset T M$ the distribution spanned by all vector fields $V$ generated by the Lie algebra $\mathfrak{g}$. Then the subbundle $\mathfrak{S} := \mathfrak{R} \langle g', J_1 g', J_2 g', J_3 g' \rangle$ of $T M$ is $\mathfrak{S}$ invariant. Moreover we have an orthogonal, $\mathfrak{S}$ invariant splitting of $T M$
\[ T M = \mathfrak{S} \oplus \mathfrak{S}^\perp \]
which allows us to descend the para-quaternionic structure $\mathfrak{S}$ of $M$ to an almost para-quaternionic structure $\mathfrak{S}$ of $\tilde{M}$, hermitian with respect to $\tilde{g}$. We use the notation $\tilde{J}_{\alpha}$ and $\tilde{\omega}_{\alpha}$, $\alpha = 1, 2, 3$, for the basis of $\mathfrak{S}$ and the corresponding 2-forms. Denote by $\tilde{\Omega}$ the 4-form associated to $\tilde{\mathfrak{S}}$.

In dimension four para-hyperKähler manifolds are characterized as pointwise Osserman manifolds which we are going to define. Hence we need the definition of a (pointwise) Osserman manifold. Let $(M, g)$ be a Riemannian or pseudo-Riemannian manifold of any signature, and $\tilde{R}$ its curvature tensor. We define its *Jacobi operator* in the unit direction $X \in T_p M$ (i.e. $|X|^2 = \pm 1$) by
\[ K_X(Y) := \tilde{R}(X, Y) X. \]
It is a self-adjoint operator with respect to the metric $g$. We say that the manifold $M$ is pointwise Osserman if the Jordan form of $K_X$ is independent of the direction $X$. We say that $M$ is (globally) Osserman if the Jordan form of $K_X$ is independent of both, the direction $X \in T_p M$ and the point $p$ of $M$.

**Theorem 5.3.** (II) A pseudo-Riemannian manifold $(M, g)$ of signature $(2, 2)$ is pointwise Osserman if and only if it is Einstein and self-dual.

**Lemma 5.3.1.** The 4-dimensional manifold $\tilde{M} = \mathfrak{R}/G$ obtained by the para-quaternionic Kähler reduction (Theorem 5.2.3) from a para-quaternionic Kähler manifold $M$ is a pointwise Osserman manifold.

**Proof:** We explain just the main steps. To simplify the notation we suppose that the group $G$ is one-dimensional and that $V$ is the Killing vector field of its action. Denote by $V_X = \nabla_X V$ the covariant derivative of the vector field $V$, where $\nabla$ is the Levi-Civita connection of the metric $g$ on $M$. Let $i : \mathfrak{R} \rightarrow \tilde{M} = \mathfrak{R}/G$ be the natural projection.

At first, using fundamental relations for immersion and submersion one relates the curvature tensors $\tilde{R}$ and $\tilde{R}$ of $M$ and $\tilde{M}$, respectively. Then, having a curvature tensor $\tilde{R}$ of $M$, one can prove that the Jacobi operator $K_X$ of the manifold $\tilde{M}$ in a point $\tilde{\pi}(u)$, $u \in \mathfrak{R}$, and in the unit direction $X \in T_{\tilde{\pi}(u)} M$, is given by
\[ \pi^*(K_X Y) = \pi^*(\tilde{R}(\tilde{X}, \tilde{Y}) \tilde{X}) = h(R(X, Y) X) + \]
\[ + \frac{1}{|V|^2} \sum_{\alpha=1}^3 \epsilon_\alpha (g(X, J_{\alpha}(V_X)) g(Y, h(J_{\alpha}(V_Y))) - g(Y, h(J_{\alpha}(V_X)))^2) + \]
\[ + \frac{1}{|V|^2} 3g(Y, h(V_X)). \]
Here $X = \pi^*(\tilde{X})$, $Y = \pi^*(\tilde{Y})$ are horizontal lifts to $T_u\mathcal{R}$ of unit vectors $\tilde{X}, \tilde{Y} \in T_{\pi(u)}M$ and $h(Z)$ denotes the $\mathfrak{g}^\perp$ part (see (25)) of the vector $Z \in T_uM$.

Using the decomposition of the curvature tensor $\tilde{R}$ from Theorem 5.2(ii) one obtains that

$$h(R(X, Y), X) = \frac{K}{4n} Y,$$

where $K$ is the scalar curvature of $M$. Since the Jacobi operator $\tilde{K}_X$ is self-adjoint and the vector $\tilde{X}$ is an eigenvector with eigenvalue $0$, we are interested only in the restriction of $\tilde{K}_X$ to the orthogonal complement of $\tilde{X}$. One can calculate the matrix of the restriction of $\tilde{R}_X$ (for example in the basis $\tilde{J}_1\tilde{X}, \tilde{J}_2\tilde{X}, \tilde{J}_3\tilde{X}$) and show that it is diagonalizable with eigenvalues

$$\lambda_1 = \lambda_2 = \frac{K}{4n} - \frac{2|h(V_X)|^2}{|V|^2}, \quad \lambda_2 = \frac{K}{4n} + \frac{4|h(V_X)|^2}{|V|^2}.$$

One can show that the ratio $\frac{|h(V_X)|^2}{|V|^2}$ is independent of the unit direction $X = \pi^*(\tilde{X})$ and hence the manifold $\tilde{M}$ is pointwise Osserman.

**Remark 5.3.1.** In general the ratio $\frac{|h(V_X)|^2}{|V|^2}$ may depend of the point $\pi(u), \ u \in \mathcal{R}$, and hence $\tilde{M}$ is not necessarily Osserman as we will see in Example 5.2.

**Proof of Theorem 5.2.** The existence of the unique moment map $f \in \mathfrak{g}^* \otimes \Omega^0(\mathfrak{g})$ is proved in Lemma 5.2.1. The moment map $f$ enjoys an important equivariance property in the following sense. For any $g \in G, V \in \mathfrak{g}$ and any point $p \in M$

$$f_{g_*V}(p) = \tilde{g}(f_V(g^{-1}(p)))$$

holds, where $\tilde{g}$ is the map induced on $\mathfrak{g}$ by $g$. From this equivariance property it follows that the submanifold

$$\mathcal{R} := \{ p \in M \mid f(p) = 0 \} \subset M$$

of $M$, is invariant under the (isometric) action of $G$.

According to Lemma 5.2.2 there exists the natural almost hermitian para-quaternionic structure $\mathfrak{g}$ on $M$ with 4-form $\tilde{\Omega}$. It remains to prove that $\tilde{M}$ is a para-quaternionic Kähler manifold. If $\dim M > 4$ its enough to prove that $\tilde{\Omega}$ is parallel with respect to the Levi-Civita connection of the metric $\tilde{g}$. The proof is the same as in the Riemannian case (see [12]). If $\dim = 4$ the proof follows from Lemma 5.3.1.

**Remark 5.3.2.** In practice $\mathcal{R}$ must not be a differentiable submanifold of $M$. In that case we can take a subset which is a submanifold of $M$ and which is invariant under the action of $G$ as shown in Example 5.2.

**Example 5.2.** This example is from [9] and it is a para-quaternionic version of the example by Galicki and Lawson (see [12]). Let $p, q \in \mathbb{N}$ be distinct, relatively prime natural numbers. We define the action of the Lie group $G := \{ e^{jt} \mid t \in \mathbb{R} \} \cong (\mathbb{R}^+)$ on $\mathbb{H}P^2$ by

$$\phi_{p,q}(t) \cdot [u_0, u_1, u_2] := [e^{jqt}u_0, e^{ipt}u_1, e^{ipt}u_2],$$

where $e^{jt} := \cosh t + j \sinh t$ and $[u_0, u_1, u_2]$ are homogenous coordinates on $\mathbb{H}P^2$. The action is free, isometric and preserves the para-quaternionic structure on $\mathbb{H}P^2$. 

One can show that the preimage of $0 \in \text{Im} \tilde{H}$ by the moment map $f_{p,q} : \tilde{H}P^2 \to \text{Im} \tilde{H}$ is given by
\[ \mathcal{R}_0^{p,q} = \{ [u_0, u_1, u_2] \in \tilde{H}P^2 \mid q \bar{u}_0 j u_0 + p \bar{u}_1 j u_1 + p \bar{u}_2 j u_2 = 0 \}. \]
The set $\mathcal{R}_{p,q}$ of regular points of $\mathcal{R}_0^{p,q}$ is
\[ \mathcal{R}_{p,q} = \{ [u_0, u_1, u_2] \in \mathcal{R}_0^{p,q} \mid q^2 |u_0|^2 + p^2 |u_1|^2 + p^2 |u_2|^2 \neq 0 \}. \]
The group $G$ acts freely and isometrically on $\mathcal{R}_{p,q}$. The manifolds $\tilde{M}_{p,q} = \mathcal{R}_{p,q}/G$ obtained by the para-quaternionic reduction are 4-dimensional, Einstein and self-dual manifolds of signature $(2,2)$. In [10] it is proved by computation of the eigenvalues that the manifolds $\tilde{M}_{p,q}$ are neither globally Osserman nor locally homogenous.

**Remark 5.3.3.** It is interesting that in case of action by $e^{it}$ instead of $e^{jt}$ one obtains an empty set as a preimage of $0 \in \text{Im} \tilde{H}$ by the moment map.

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