PSEUDO-EINSTEIN STRUCTURE, EIGENVALUE ESTIMATE FOR THE CR PANEITZ OPERATOR AND ITS APPLICATIONS TO UNIFORMIZATION THEOREM

SHU-CHENG CHANG, TING-JUNG KUO, AND CHIEN LIN

Abstract. In this note, we mainly focus on the existence of pseudo-Einstein contact forms, an upper bound eigenvalue estimate for the CR Paneitz operator and its applications to the uniformization theorem for Sasakian space form in a closed strictly pseudoconvex CR 3-manifold. Firstly, the existence of pseudo-Einstein contact form is confirmed if the CR 3-manifold is Sasakian. Secondly, we derive an eigenvalue upper bound estimate for the CR Paneitz operator and obtain the CR uniformization theorem for a class of CR 3-manifolds. At the end, under the positivity assumption of the pseudohermitian curvature, we derive the existence theorem for pseudo-Einstein contact forms and uniformization theorems in a closed strictly pseudoconvex CR 3-manifold of nonnegative CR Paneitz operator with kernel consisting of the CR pluriharmonic functions and the CR $Q$-curvature is pluriharmonic.

1. Introduction

In Riemannian geometry, a Riemannian manifold is called Einstein if the Ricci curvature tensor is function-proportional to its Riemannian metric. For dimension greater than 2, it is equivalent to the constant-proportional case. In contrast to the Riemannian geometry situation, there is a resembling notion that a strictly pseudoconvex CR $(2n+1)$-manifold is called pseudo-Einstein if the pseudohermitian Ricci curvature tensor is function-proportional to its Levi metric. The pseudo-Einstein condition is less rigid than the Einstein condition in Riemannian geometry. Indeed, the CR contracted Bianchi identity no longer implies the pseudohermitian scalar curvature $R$ to be a constant due to the presence of pseudohermitian torsion for $n \geq 2$

$$R_{\alpha \beta \gamma \delta} = R_{\alpha} - i(n-1)A_{\alpha \beta \gamma \delta}.$$ 

Note that any contact form on a closed strictly pseudoconvex 3-manifold is actually pseudo-Einstein since the pseudohermitian Ricci tensor has only one component $R_{1\overline{1}}$.

In [Lee], J. Lee showed that the obstruction to the existence of a pseudo-Einstein contact form $\theta$ is that its first Chern class $c_1(T_{1,0}M)$ vanishes. Indeed, for a closed strictly pseudoconvex $(2n+1)$-manifold $(M, J, \theta)\ldots$
with $c_1(T_{1,0}M) = 0$ and $n \geq 2$, he proved that $M$ admits a globally defined pseudo-Einstein contact form if either $M$ admits a contact form $\theta$ with nonnegative pseudohermitian Ricci curvature tensor or the vanishing pseudohermitian torsion. However, his method couldn’t be applied to the case $n = 1$ directly.

Then it is natural to focus on such existence theorem of pseudo-Einstein contact forms for $n = 1$. Of course, we must find another appropriate definition for the pseudo-Einstein contact form. In fact, by Lemma 2.2 below, it is reasonable to view

$$W_1 \equiv (R,_{1,-iA_{11},\tau}) = 0$$

as the pseudo-Einstein contact form in a closed strictly pseudoconvex CR 3-manifold $(M, J, \theta)$.

Before we start to work on the existence of pseudo-Einstein contact forms in a closed strictly pseudoconvex CR 3-manifold, we make the following observation in a closed strictly pseudoconvex CR $(2n + 1)$-manifold $(M, J)$ with a choice of pseudohermitian contact form $\theta$.

(i) For $n \geq 2$ : Assume that the pseudohermitian Ricci curvature is positive, it is well-known ([K], [Lee]) that we have the solvability of the inhomogeneous tangential Cauchy-Riemann equation

$$(1.1a) \quad \overline{\partial}_b \varphi = \eta$$

for any $\overline{\partial}_b$-closed $(0,1)$-form $\eta$. That is to say that

$$H^{0,1}_{\overline{\partial}_b}(M) = 0.$$

(ii) For $n = 1$ : We consider a closed strictly pseudoconvex CR 3-manifold $(M, \theta)$ with $c_1(T_{1,0}M) = 0$. There is a pure imaginary 1-form

$$(1.2) \quad \sigma = \sigma_\tau \theta^\tau - \sigma_1 \theta^1 + i \sigma_0 \theta$$

such that

$$d\omega^1_1 = d\sigma.$$

Due to J. J. Kohn’s result (Lemma 3.2), we observe that there is a complex function

$$\varphi = u + iv \in C^\infty_c(M)$$

and $\gamma = \gamma_\tau \theta^\tau \in \Omega^{0,1}(M) \cap \ker(\Box_b)$ such that

$$(1.3) \quad \overline{\partial}_b \varphi = \sigma_\tau \theta^\tau - \gamma_\tau \theta^\tau.$$
with
\[ \Box_b(\sigma_\tau \theta^\tau) = 0. \]

Here \( \Box_b = 2 \left( \partial_b \partial_b^* + \partial_b^* \partial_b \right) \) is the Kohn-Rossi Laplacian. Then it is the natural question when we have the solvability of the inhomogeneous tangential Cauchy-Riemann equation (i.e. \( \gamma = 0 \))

\[ (1.4) \quad \partial_b \varphi = \sigma_\tau \theta^\tau. \]

In this paper, we mainly focus on the existence of pseudo-Einstein contact forms as in Corollary 1.1, Theorem 1.2, Theorem 1.4 and Theorem 1.5, an upper bound eigenvalue estimate for the CR Paneitz operator as in Theorem 1.3, Corollary 1.3 and its applications to the uniformization theorem for Sasakian space form as in Corollary 1.3, Corollary 1.4 and Corollary 1.5 in a closed strictly pseudoconvex CR 3-manifold.

We first state one of the main theorems as follows:

**Theorem 1.1.** If \((M,J,\theta)\) is a closed strictly pseudoconvex CR 3-manifold with \( c_1(T_{1,0} M) = 0. \) Then

(i) \( \widetilde{\theta} = e^{(f + 2u)/3} \theta \) is a pseudo-Einstein contact form if and only if \( f \) satisfies the third-order partial differential equation

\[ (1.5) \quad P_1 f = i (A_{11} \gamma^\tau_\tau - \gamma_{1,0}). \]

Here \( P_1 \) is a third-order CR pluriharmonic operator

\[ P_1 f = f_{T_{11}} + iA_{11}f_{\tau}. \]

(ii) In particular, \( \widetilde{\theta} = e^{(f + 2u)/3} \theta \) is a pseudo-Einstein contact form for a CR-pluriharmonic function \( f \) if and only if the equality holds

\[ (1.6) \quad (A_{11} \gamma^\tau_\tau - \gamma_{1,0}) = 0. \]

As a consequence, we are able to show that one of existence theorems for the pseudo-Einstein contact form in this paper.

**Corollary 1.1.** Let \((M,J,\theta)\) be a closed strictly pseudoconvex CR 3-manifold with \( c_1(T_{1,0} M) = 0. \) Then \( M \) admits a globally defined pseudo-Einstein contact form \( e^{(f + 2u)/3} \theta \) for any CR-pluriharmonic function \( f \) if the pseudohermitian torsion is vanishing (Sasakian). More precisely, we have

\[ \gamma_{1,0} = 0. \]
Note that we do not know whether it holds that $\gamma = 0$ in the situation as in Corollary 1.1. However, by deriving the Bochner-type estimate as in the Lemma 3.4, we have

$$\int_M (2R - \text{Tor}) (\gamma, \gamma) \, d\mu + 2 \int_M |\gamma_{1,1}|^2 \, d\mu + \frac{1}{2} \int_M (P_0 f) \, f \, d\mu = 0$$

if $\tilde{\theta} = e^{\frac{(f+2u)}{3}} \theta$ is a pseudo-Einstein contact form. It will conclude

$$\gamma = 0$$

under certain pseudohermitian geometric assumptions and obtain the solvability of the inhomogeneous tangential Cauchy-Riemann equation (1.4).

**Theorem 1.2.** If $(M, J, \theta)$ is a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$ and non-negative CR Paneitz operator $P_0$. Assume that the pseudohermitian curvature is $(\frac{1}{2}, 0)$-positive

$$R > |A_{11}|.$$

Then $\tilde{\theta} = e^{\frac{(f+2u)}{3}} \theta$ is a pseudo-Einstein contact form for any CR-pluriharmonic function $f$ if and only if the solvability of the inhomogeneous tangential Cauchy-Riemann equation (1.4).

We observe that, for a strictly pseudoconvex 3-manifold $(M^3, J, \theta)$, we have the invariance property for the CR pluriharmonic operator $P_1$ and CR Paneitz operator $P_0$. It is to say that, for rescaled contact form $\tilde{\theta} = e^{2\theta} \theta$, we have

$$P_1 = e^{-3\theta} P_1 \quad \text{and} \quad P_0 = e^{-4\theta} P_0.$$  

Then the nonnegativity of CR Paneitz operator $P_0$ is CR conformal invariant (H).

Since the CR Paneitz operator $P_0$ is nonnegative (CCC) if the pseudohermitian torsion is vanishing, it follows from Theorem 1.2 and Corollary 1.1 that

**Corollary 1.2.** If $(M, J, \theta)$ is a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$. Assume that the manifold is Sasakian and the Tanaka-Webster scalar curvature is positive, then we have the solvability of the inhomogeneous tangential Cauchy-Riemann equation (1.4). That is to say that the Kohn–Rossi cohomology class of $\sigma_1^T \theta^T$ is vanishing.
Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold with \(c_1(T_{1,0}M) = 0\). With the notations as in section 2 and section 3, another Bochner-type equality holds

\[
\int_M \left( R - \frac{1}{2} Tor - \frac{1}{2} Tor' \right) \gamma, \gamma \, d\mu + \int_M |\gamma_{1,1}|^2 \, d\mu + \int_M Q u \, d\mu + \int_M (P_0 u) \, u \, d\mu = 0.
\]

With the help of the notion of \((C_0, C_1)\)-convexity, we have the eigenvalue estimate for the CR Paneitz operator \(P_0\) in terms of the CR \(Q\)-curvature.

**Theorem 1.3.** Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\) and nonnegative CR Paneitz operator \(P_0\) with kernel consisting of the CR pluriharmonic functions. Assume that the pseudohermitian curvature is \((\frac{1}{2}, \frac{1}{2})\)-positive

\[
R > (|A_{11}| + |A_{11,\overline{1}}|).
\]

If \(\tilde{\theta} = e^{\left(\frac{f+2u}{3}\right)} \theta\) is a pseudo-Einstein contact form for any CR-pluriharmonic function \(f\), then one can derive the upper bound estimate for the first eigenvalue of the CR Paneitz operator \(P_0\)

\[
\Lambda^2 \int_M (u^+)^2 \, d\mu \leq \int_M (Q^+)^2 \, d\mu
\]

with the decomposition \(Q = Q_{\ker} + Q^\perp\) and \(u = u_{\ker} + u^\perp\). Here \(\Lambda\) is the positive constant as in (2.6).

**Remark 1.1.** 1. ([H]) For a closed strictly pseudoconvex CR 3-manifold of vanishing pseudohermitian torsion (Sasakian), we have

\[
ker P_1 = ker P_0.
\]

Furthermore, the real ellipsoids in \(\mathbb{C}^2\) are such that the CR Paneitz operators are nonnegative with kernel consisting of the CR pluriharmonic functions ([CCYa]). In general for non-embeddable CR 3-manifolds, we only have

\[
ker P_1 \subsetneq ker P_0.
\]

2. As in the Remark 2.1, \(M\) admits a Riemannian metric of positive scalar curvature if the pseudohermitian curvature is \((\frac{1}{2}, \frac{1}{2})\)-positive. Then the properties of pseudohermitian curvature positivity and nonnegative CR Paneitz operator \(P_0\) imply the embeddability of \((M, J, \theta_0)\) in the complex Euclidean space \(\mathbb{C}^N\) ([CCY]). Furthermore, the Paneitz operator \(P_0\) with respect to \((J, \theta)\) is essentially positive in this special case as well.
Combining Theorem 1.3, Corollary 1.2, (3.2) and the above remark, we have the following CR uniformization theorem ([T]) in a Sasakian manifold due to the eigenvalue estimate of the CR Paneitz operator (1.8).

**Corollary 1.3.** If \((M,J,\theta)\) is a closed strictly pseudoconvex CR 3-manifold with \(c_1(T_{1,0}M) = 0\). Assume that the manifold is Sasakian and the Tanaka-Webster scalar curvature is positive, then

\[
\Lambda^2 \int_M (u^\perp)^2 d\mu \leq \int_M (Q^\perp)^2 d\mu.
\]

In additional, if the CR \(Q\)-curvature is pluriharmonic (i.e. \(Q^\perp = 0\)), then \((M,J,\theta)\) is the Sasakian space form with the positive constant Tanaka-Webster scalar curvature and vanishing torsion.

Finally, if we do not assume the torsion is vanishing (non-Sasakian), we can derive another existence theorem for the pseudo-Einstein contact form with the stronger condition.

**Theorem 1.4.** Let \((M,J,\theta)\) be a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\) and nonnegative CR Paneitz operator \(P_0\) with kernel consisting of the CR pluriharmonic functions. Assume that the pseudohermitian curvature is \((\frac{1}{2}, \frac{1}{2})\)-positive

\[
R > (|A_{11}| + |A_{11,\tau}|).
\]

If the CR \(Q\)-curvature is pluriharmonic, then

\[
\gamma = 0;
\]

hence \(\tilde{\theta} = e^{\frac{f+2a}{3}} \theta\) is a pseudo-Einstein contact form for any CR-pluriharmonic function \(f\).

As a consequence, we have the CR uniformization theorem ([T]) in a spherical CR 3-manifold.

**Corollary 1.4.** Let \((M,J,\theta)\) be a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\) and nonnegative CR Paneitz operator \(P_0\) with kernel consisting of the CR pluriharmonic functions. Assume that the pseudohermitian curvature is \((\frac{1}{2}, \frac{1}{2})\)-positive and the CR \(Q\)-curvature is pluriharmonic, then \((M,J,\theta)\) is a pseudo-Einstein CR manifold. In additional, if it is spherical with positive constant Tanaka-Webster scalar curvature, then \((M,J,\theta)\) is the Sasakian space form with the positive constant Tanaka-Webster scalar curvature and vanishing torsion.

In the course of the proof of Theorem 1.4, we assume that the CR \(Q\)-curvature is pluriharmonic. However, in particular for assuming vanishing of the CR \(Q\)-curvature (see section 2), we can drop the condition of (1.9) as the following:
Theorem 1.5. Let $(M,J,\theta)$ is a closed strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0}M) = 0$ with non-negative CR Paneitz operator $P_0$. Assume that the pseudohermitian curvature is $\left(\frac{1}{2}, \frac{1}{2}\right)$-positive

$$R > (|A_{11}| + |A_{11,\overline{1}}|)$$

and the CR $Q$-curvature is vanishing

$$\Delta_b R - i(A_{11,\overline{11}} - A_{\overline{11},11}) = 0.$$

Then $\tilde{\theta} = e^{\left(\frac{f + 2u}{3}\right)} \theta$ is a pseudo-Einstein contact form for any CR-pluriharmonic function $f$.

Corollary 1.5. Let $(M,J,\theta)$ is a closed strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0}M) = 0$ with non-negative CR Paneitz operator $P_0$. Assume that the pseudohermitian curvature is $\left(\frac{1}{2}, \frac{1}{2}\right)$-positive and the CR $Q$-curvature is vanishing. Then $(M,J,\theta)$ is a pseudo-Einstein CR manifold. In addition, if it is spherical with positive constant Tanaka-Webster scalar curvature, then $(M,J,\theta)$ is the Sasakian space form with positive constant Tanaka-Webster scalar curvature and vanishing pseudohermitian torsion.

Remark 1.2. As in Proposition 2.1 if $M$ is embeddable and the CR Paneitz operator $P_0$ is nonnegative, there always exists a contact form $\theta$ in $[\theta_0]$ with vanishing CR $Q$-curvature

$$\Delta_b R - i(A_{11,\overline{11}} - A_{\overline{11},11}) = 0$$

in a closed CR 3-manifold $(M,J,[\theta_0])$ if the kernel part of $Q_0$-curvature is vanishing w.r.t. $\theta_0$

$$(Q_0)_{\text{ker}} = 0$$

which is a CR conformal invariant due to the transformation law (2.8).

We conclude this introduction with a brief plan of the paper. In Section 2, we derive some preliminary results and indicate the geometry and topology of CR 3-manifolds with the positivity of pseudohermitian curvature. In Section 3, we prove main Theorems. In Appendix, we survey basic notions in the pseudohermitian (strictly pseudoconvex CR) geometry.

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2. Preliminaries

In this section, we derive some necessary ingredients for the proof of main results in this paper. In particular, we define the positivity of pseudohermitian curvature and indicate the geometry and topology of strictly pseudoconvex CR 3-manifolds. Let $C_0, C_1$ be both nonnegative numbers.

**Definition 2.1.** We say that a strictly pseudoconvex CR 3-manifolds is $(C_0, C_1)$-convex if the pseudohermitian curvature is $(C_0, C_1)$-positive. That is

$$\left(R - C_0 \text{Tor} - C_1 \text{Tor'}\right)(X, X) = Rx^1x^\bar{1} - 2C_0 \text{Re}[i(A_{1\bar{1}}x^1x^\bar{1})]$$

$$- 2C_1 \text{Re}[iA_{1\bar{1},1}x^\bar{1}]] > 0$$

for any $X = x^1Z_1 \in T_{1,0}(M)$.

Before giving the proof of Theorem 1.3, we explain why we introduce the notion of $(C_0, C_1)$-convexity as follows:

**Lemma 2.1.** Let $M$ be a closed strictly pseudoconvex CR 3-manifold. For any nonnegative constant $C_0, C_1$; $(C_0, C_1)$-convexity is equivalent to the curvature-torsion pinching condition

$$R(x) > 2 \left(C_0 |A_{1\bar{1}}| + C_1 |A_{1\bar{1},\bar{1}}|\right)(x)$$

for all $x \in M$.

**Remark 2.1.** Let $(M^3, J, \theta)$ be a closed strictly pseudoconvex CR 3-manifold with

$$R(x) > |A_{1\bar{1}}(x)|$$

which is $(C_0, 0)$-positive. Then $M$ admits a Riemannian metric of positive scalar curvature.

**Proof.** Fix a point $x \in M$ and denote

$$A_{1\bar{1}}(x) = a(x) + b(x)i$$

$$A_{1\bar{1},\bar{1}}(x) = c(x) + d(x)i.$$

Without loss of generality, one may consider $X = x^1Z_1 = (1 + si)Z_1$ for some $s \in \mathbb{R}$. The convexity condition (2.1) reads as

$$R(1 + si)(1 - si) > i[C_0(a - bi)(1 - si)^2 + C_1(c - di)(1 - si) - \text{conj}.]$$

$$= -2 \left[(C_0b) s^2 - (2C_0a + C_1c) s + (-C_1d - C_0b)\right].$$
i.e.,

\[
R > f(s) := -2 \left[ (C_0 b) - \frac{(2C_0 b + C_1 d) + (2C_0 a + C_1 c) s}{1 + s^2} \right].
\]

Since \(X\) is arbitrary, this inequality holds for all \(s \in \mathbb{R}\). We have

\[
f(s) \leq \max_{s \in \mathbb{R}} f(s) = f(s_0) = 2 \sqrt{(C_0 a + C_1 \frac{d}{2})^2 + (C_0 b + C_1 \frac{d}{2})^2 + C_1 d}
\leq 2 \left( C_0 |A_{11}| + C_1 |A_{11,1}| \right)
\]

where

\[
s_0 = -\frac{(C_0 b + C_1 \frac{d}{2}) + \sqrt{(C_0 a + C_1 \frac{d}{2})^2 + (C_0 b + C_1 \frac{d}{2})^2}}{(C_0 a + C_1 \frac{d}{2})},
\]

for \(C_0 a + C_1 \frac{d}{2} \neq 0\), is a critical number of \(f\). Thus

\[
R(x) > 2 \left( C_0 |A_{11}| + C_1 |A_{11,1}| \right)(x)
\]

for all \(x \in M\). As for the case of \(C_0 a + C_1 \frac{d}{2} = 0\), the same deduction could be applied. Therefore, \((C_0, C_1)\)-positivity is equivalent to

\[
R > 2 \left( C_0 |A_{11}| + C_1 |A_{11,1}| \right).
\]

□

**Definition 2.2.** ([Lee]) (i) A contact form \(\theta\) on a closed strictly pseudoconvex CR \((2n+1)\)-manifold \((M, \theta)\) is said to be pseudo-Einstein for \(n \geq 2\) if the pseudohermitian Ricci tensor \(R_{\alpha\beta}\) is proportional to the Levi form \(h_{\alpha\beta}\), i.e.,

\[
R_{\alpha\beta} = \frac{R}{n} h_{\alpha\beta},
\]

where \(R = h^{\alpha\beta} R_{\alpha\beta}\) is the Tanaka-Webster scalar curvature of \((J, \theta)\).

(ii) (Lemma 2.2) Note that any contact form on a closed strictly pseudoconvex 3-manifold is actually pseudo-Einstein (since the pseudohermitian Ricci tensor has only one component \(R_{11}\)). Then we define a contact form \(\theta\) on a closed strictly pseudoconvex CR 3-manifold \((M, \theta)\) is said to be pseudo-Einstein if the following tensor is vanishing

\[
W_1 \equiv \left( R_1 - i A_{11,1} \right) = 0.
\]
(iii) We define the first Chern class $c_1(T_{1,0}M) \in H^2(M, \mathbb{R})$ for the holomorphic tangent bundle $T^{1,0}M$ by

$$c_1(T^{1,0}M) = \frac{i}{2\pi} [d\omega^a -]$$

$$= \frac{i}{2\pi} [R_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^\bar{\beta} + A_{\alpha\mu, \pi} \theta^\mu \wedge \theta - A_{\alpha\mu, \bar{\pi}} \theta^\mu \wedge \theta].$$

(iv) Note that any pseudo-Einstein manifold $(M^{2n+1}, \theta)$, the first Chern class $c_1(T_{1,0}M)$ of $T_{1,0}(M)$ is vanishing (Lee).

Next let us recall the equivalent definitions of the pseudo-Einsteinian $(2n+1)$-manifold for $n \geq 2$ and $n = 1$ as well.

**Lemma 2.2.** (i) If $(M, J, \theta)$ is a strictly pseudoconvex CR $(2n+1)$-manifold for $n \geq 2$, then the following propositions are all equivalent:

1. $R_{\alpha\bar{\beta}} = \frac{\mathbb{R}}{n} h_{\alpha\beta}$.
2. $(\omega^\alpha + \frac{i}{n} R \theta)$ is closed,
3. $W_{\alpha} \div (R_{\alpha} - i A_{\alpha\beta}, \beta) = 0$.

As for $n = 1$, we still have the equivalent between (2) and (3).

(ii) By the equivalence of (2) and (3), we see the first Chern class $c_1(T_{1,0}M)$ is vanishing if $(M, J, \theta)$ is a pseudo-Einsteinian 3-manifold.

**Proof.** The equivalence of (1) and (2) could be found in [Lee] for $n \geq 2$. The proof of (2) $\iff$ (3) for $n \geq 2$ is the same with $n = 1$. So, for simplification, we just give the proof of the equivalence of (2) and (3) for $n = 1$.

Because

$$d\omega^1 = R\theta^1 \wedge \theta^\bar{\tau} + A_{11, \tau} \theta^1 \wedge \theta - A_{1\tau, 1} \theta^\tau \wedge \theta,$$

we have

$$d (\omega^1 + iR\theta) = d\omega^1 + i\left(R_{1, \tau} \theta^1 + R_{1\tau, \tau} \theta^\tau\right) \wedge \theta - R\theta^1 \wedge \theta^\tau.$$

$$= i \left[(R_{1, \tau} - iA_{11, \tau}) \theta^1 + (R_{1\tau, \tau} + iA_{1\tau, 1}) \theta^\tau\right] \wedge \theta.$$

Hence

$$d (\omega^1 + iR\theta) = 0 \iff R_{1, \tau} - iA_{11, \tau} = 0.$$
We recall some useful notations as well.

**Definition 2.3.** ([Lee]) (i) Let \((M, J, \theta)\) be a three-dimensional strictly pseudoconvex CR manifold. We define

\[
P_\varphi = (P_1 \varphi) \theta^1,
\]

which is an operator that characterizes CR-pluriharmonic functions. Here \(P_1 \varphi = \varphi_{1 \bar{1}} + i A_{11} \varphi^1\) and \(\overline{P} = (\overline{P}_1) \theta^\bar{1}\), the conjugate of \(P\). The CR Paneitz operator \(P_0\) is defined by

\[
P_0 \varphi = (\delta_b (P \varphi) + \overline{\delta_b (P \varphi)})
\]

where \(\delta_b\) is the divergence operator that takes \((1, 0)\)-forms to functions by \(\delta_b (\sigma_1 \theta^1) = \sigma_1\) and, similarly, \(\overline{\delta_b (\sigma_1 \theta^\bar{1})} = \sigma_{1 \bar{1}}\). We observe that

\[
\int_M (P \varphi + \overline{P} \varphi, d_\mu) L^\ast \theta \, d\mu = - \int_M P_0 \varphi \cdot \varphi \, d\mu
\]

with \(d\mu = \theta \wedge d\theta\). One can check that \(P_0\) is self-adjoint, that is, \(\langle P_0 \varphi, \psi \rangle = \langle \varphi, P_0 \psi \rangle\) for all smooth functions \(\varphi\) and \(\psi\). For the details about these operators, the reader can make reference to [GL], [H], [Lee], [GG] and [FH].

(ii) On a complete pseudohermitian 3-manifold \((M, J, \theta)\), we call the Paneitz operator \(P_0\) with respect to \((J, \theta)\) essentially positive if there exists a constant \(\Lambda > 0\) such that

\[
\int_M P_0 \varphi \cdot \varphi \, d\mu \geq \Lambda \int_M \varphi^2 \, d\mu.
\]

for all real smooth functions \(\varphi \in \ker(P_0)^\perp\) (i.e. perpendicular to the kernel of \(P_0\) in the \(L^2\) norm with respect to the volume form \(d\mu = \theta \wedge d\theta\)). We say that \(P_0\) is nonnegative if

\[
\int_M P_0 \varphi \cdot \varphi \, d\mu \geq 0
\]

for all real smooth functions \(\varphi\).

**Remark 2.2.** 1. The notions of Paneitz operator \(P_0\) and \(Q\)-curvature were initially introduced on a Riemannian manifold, and were considered as a kind of generalization of Laplacian and Gaussian curvature on a two-dimensional manifold, respectively ([H]).

2. The kernel of the CR Paneitz operator \(P_0\) is infinite dimensional, containing all \(CR\)-pluriharmonic functions.
3. Let \((M, J, \theta)\) be a closed strictly pseudoconvex 3-manifold with vanishing pseudohermitian torsion. Then the corresponding CR Paneitz operator \(P_0\) is essentially positive (CCC, CaC).

Finally, we define the CR \(Q\)-curvature in a pseudohermitian 3-manifold by

\[
Q := -\text{Re}(R_1 - iA_{11,\bar{i}}) - \text{Re}(R_{1\bar{i}} - iA_{11,\bar{i}\bar{i}}).
\]

Then

\[
Q = -\frac{1}{2}[\Delta_b R - i(A_{11,\bar{i}\bar{i}} - A_{1\bar{i}\bar{i},\bar{i}})].
\]

Now for \(\theta = e^{2\gamma}\theta_0\), under this conformal change, it is known that we have the following transformation laws (H):

\[
Q = e^{-4\gamma}(Q_0 + \frac{3}{4}P_0\gamma)
\]

and

\[
W_1 := (R_1 - iA_{11,\bar{i}}) = e^{-3\gamma}[R_1 - iA_{11,\bar{i}} - 6P_1\gamma],
\]

where \(P_0\) and \(Q_0\) denote the CR Paneitz operator and the CR \(Q\)-curvature with respect to \((M, J, \theta_0)\), respectively.

In the paper of [CS], we consider the fourth-order CR \(Q\)-curvature flow in a closed CR 3-manifold \((M, J, [\theta_0])\):

\[
\begin{aligned}
\frac{\partial v}{\partial t} &= -(Q_0^\perp + \frac{3}{4}P_0 v) + r, \\
\theta &= e^{2\gamma}\theta_0; \quad v(p, 0) = v_0(p), \\
\int_M e^{4v}d\mu_0 &= \int_M d\mu_0.
\end{aligned}
\]

Here

\[
r = \frac{\int_M (Q_0^\perp + \frac{3}{4}P_0 v)d\mu}{\int_M d\mu}
\]

and we decompose

\[
Q_0 = (Q_0)_{\ker} \oplus Q_0^\perp
\]

with respect to \(P_0\), where \(Q_0^\perp\) denotes the component of \(Q_0\) which is perpendicular to \(\ker P_0\) and \((Q_0)_{\ker}\) denotes the component of \(Q_0\) in \(\ker P_0\).
Here we recall the following previous result.

**Proposition 2.1.** ([CS]) Let \((M, J, \theta_0)\) be a closed CR 3-manifold. Suppose that \(M\) is embeddable in \(\mathbb{C}^2\) and the CR Paneitz operator \(P_0\) is nonnegative. Then the solution of (2.10) exists on \(M \times [0, \infty)\) and converges smoothly to \(v_\infty \equiv v(\cdot, \infty)\) as \(t \to \infty\). Moreover, the contact form \(\theta_\infty = e^{2v_\infty} \theta_0\) has CR \(Q\)-curvature

\[ Q_\infty = e^{-4v_\infty} (Q_0)_{\ker}. \]

In particular, if \((Q_0)_{\ker} = 0\), we have

\[ (2.11) \quad Q_\infty = 0. \]

Note that if \(M\) is embeddable in \(\mathbb{C}^2\) and the CR Paneitz operator \(P_0\) is nonnegative (In fact, it is essentially positive). Then we have the subelliptic estimate for the CR Paneitz operator \(P_0\) on the orthogonal complement of \(\ker P_0\) which is one of the key steps in the proof for Proposition 2.1.

Finally, we recall that

**Definition 2.4.** We call a CR structure \(J\) spherical if Cartan curvature tensor \(Q_{11}\) vanishes identically. Here

\[ Q_{11} = \frac{1}{6} R_{111} + \frac{i}{2} RA_{11} - A_{11,0} - \frac{2i}{3} A_{11,11}. \]

Note that \((M, J, \theta)\) is called a spherical CR 3-manifold if \(J\) is a spherical structure. We observe that the spherical structure is CR invariant and a closed spherical CR 3-manifold \((M, J, \theta)\) is locally CR equivalent to the standard CR 3-sphere \((S^3, \hat{J}, \hat{\theta})\). In additional, if \(M\) is simply connected, then \((M, J, \theta)\) is the standard CR 3-sphere.

### 3. Proofs of Main Theorems

In this section, we prove the main theorems. We start from the groundwork for Theorem 1.1.

**Lemma 3.1.** If \((M, J, \theta)\) is a strictly pseudoconvex 3-manifold with \(c_1(T_{1,0}M) = 0\), then there is a pure imaginary 1-form

\[ \sigma = \sigma_{\bar{T}} \theta^\bar{T} - \sigma_1 \theta^1 + i \sigma_0 \theta \]

with \(d\omega_1 = d\sigma\) such that

\[ (3.1) \left\{ \begin{array}{l} R = R_{1\bar{T}} = \sigma_{1,1} + \sigma_{1,\bar{T}} - \sigma_0 \\ A_{11,\bar{T}} = \sigma_{1,0} + i \sigma_{0,1} - A_{11} \sigma_T \end{array} \right. \]
Proof. Because

\[ c_1(T_{1,0}M) = -\frac{1}{2\pi i} [d\omega_1] = 0, \]

we know there is a pure imaginary 1-form

\[ \sigma = \sigma_\tau \theta^\tau - \sigma_1 \theta^1 + i\sigma_0 \theta \]

such that

\[ d\omega_1 = d\sigma. \]

By the structure equation

\[ \begin{cases} 
  d\theta = i\theta^1 \wedge \theta^\tau \\
  d\theta^1 = A_{\tau\tau} \theta \wedge \theta^1 
\end{cases} \]

we have

\[ d\sigma = (\sigma_{\tau,1} \theta^1 + \sigma_{\tau,0} \theta) \wedge \theta^\tau + \sigma_{\tau} d\theta^1 - (\sigma_{1,\tau} \theta^\tau + \sigma_{1,0} \theta) \wedge \theta^1 - \sigma_1 d\theta^1 + i \left( \sigma_{0,1} \theta^1 + \sigma_{0,\tau} \theta^\tau \right) \wedge \theta + i\sigma_0 d\theta 
\]

\[ = (\sigma_{\tau,1} + \sigma_{1,\tau} - \sigma_0) \theta^1 \wedge \theta^\tau - (\sigma_{1,0} + i\sigma_{0,1} - \sigma_{\tau} A_{11}) \theta \wedge \theta^1 + (\sigma_{\tau,0} - i\sigma_{0,\tau} - \sigma_{1} A_{\tau\tau}) \theta \wedge \theta^\tau. \]

Due to

\[ d\sigma = d\omega_1 = R_{\tau\tau} \theta^1 \wedge \theta^\tau + A_{11,\tau} \theta^1 \wedge \theta - A_{\tau\tau,1} \theta^\tau \wedge \theta, \]

we derive

\[ \begin{cases} 
  R_{\tau\tau} = \sigma_{\tau,1} + \sigma_{1,\tau} - \sigma_0 \\
  A_{11,\tau} = \sigma_{1,0} + i\sigma_{0,1} - \sigma_{\tau} A_{\tau\tau} 
\end{cases} \]

\[ \square \]

We would need the J.J. Kohn’s Hodge theory for the $\overline{\partial}_b$ complex (see [K]):

Lemma 3.2. If $(M, J, \theta)$ is a closed strictly pseudoconvex CR $(2n+1)$-manifold and $\eta \in \Omega^{0,1}(M)$, a smooth $(0,1)$-form on $M$ with

\[ \overline{\partial}_b \eta = 0, \]

then there are a smooth complex function $\varphi \in C_0^\infty(M)$ and a smooth $(0,1)$-form $\gamma \in \Omega^{0,1}(M)$ such that

\[ (\eta - \overline{\partial}_b \varphi) = \gamma \in \ker (\square_b), \]

where $\square_b = 2 \left( \overline{\partial}_b \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b \right)$ is the Kohn-Rossi Laplacian.
Subsequently, we deduce the expression for $W_1$. We denote $\gamma_1 := \gamma_T$.

**Lemma 3.3.** If $(M, J, \theta)$ is a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$, then there are $u \in C_\infty^\infty(M)$ and $\gamma = \gamma_T T_1^1 \in \Omega^{0,1}(M)$ with

$$\gamma_{T,1} = \gamma_{1,T} = 0$$

such that

$$W_1 = 2P_1u + i(A_{11} \gamma_T - \gamma_{1,0}).$$

**Proof.** By choosing $\eta = \sigma \theta_T^T$ as in Lemma 3.2 where $\sigma$ is chosen from Lemma 3.1 there are $\varphi = u + iv \in C_\infty^\infty(M)$ and $\gamma = \gamma_T T_1^1 \in \Omega^{0,1}(M) \cap \ker(\Box_b)$ such that

$$\sigma_T = \varphi_T + \gamma_T$$

Note that

$$\Box_b \gamma = 0 \implies \Box_b^\ast \gamma = 0 \implies \gamma_{T,1} = 0$$

and

$$\sigma_1 = (\varphi)_1 + \gamma_1.$$ 

Thus

$$\sigma_{1,T} = (\varphi)_{1,T} + \gamma_{1,T} \quad \text{by (3.4)}$$

$$= (\varphi)_{1,T} \quad \text{by (3.4)}$$

$$= (\varphi)_{1,T} + i(\varphi)_{0,1} \quad \text{by (A.5)}$$

$$= (\varphi)_{1,T} + i[(\varphi)_{1,0} + A_{11}(\varphi)_1] \quad \text{by (A.5)}$$

and

$$\sigma_{T,1} = \varphi_{T,1} \quad \text{from (3.3) and (3.4)}$$
imply
\[
W_1 = R_{11} - iA_{11}\tau
\]
\[
= \sigma_{\tau_{11}} + \sigma_{1,0} - i\sigma_{1,0} + iA_{11}\sigma_{\tau} \quad \text{by (3.1)}
\]
\[
= \varphi_{\tau_{11}} + (\varphi)_{\tau_{11}} + iA_{11}(\varphi)_{\tau} - i\gamma_{1,0} + iA_{11}(\varphi + \gamma) .
\]
\[
= 2\left(\nu_{\tau_{11}} + iA_{11}\nu\right) + i\left(A_{11}\gamma - \gamma_{1,0}\right)
\]
\[
= 2P_1u + i(A_{11}\gamma - \gamma_{1,0})
\]
This completes the proof. □

Now we are ready to give the proof of Theorem 1.1.

Proof. (Proof of Theorem 1.1) Set
\[
\tilde{\theta} = e^{2\lambda}\theta.
\]
By the transformation law (refer to Lemma 5.4 in [H] or Lemma 3.1 in [CW]), we know
\[
(3.6) \quad \tilde{W}_1 = e^{-3\lambda}(W_1 - 6P_1\lambda),
\]
where the notation with "tilde" means such quantity corresponds to the new contact form \( \tilde{\theta} \). With the help of (3.6) and Lemma 3.3 we have
\[
\tilde{W}_1 = 0
\]
if and only if
\[
W_1 = 6P_1\lambda
\]
if and only if
\[
6P_1\lambda = 2P_1u + i(A_{11}\gamma - \gamma_{1,0}).
\]
That is to say
\[
P_1f = i(A_{11}\gamma - \gamma_{1,0})
\]
for
\[
f = (6\lambda - 2u).
\]
□

Remark 3.1. From (1.5) and
\[
\gamma_{1,0}\tau = \gamma_{1,0} + A_{\tau\tau}\gamma_{1,1} + A_{\tau\tau,1}\gamma_1 = A_{\tau\tau}\gamma_{1,1} + A_{\tau\tau,1}\gamma_1,
\]
we could deduce $f$ satisfies the fourth-order partial differential equation

$$P_0 f = 2i \left[ (A_{11} \gamma_1, \overline{T})_{,T} - (A_{TT} \gamma_1)_{,T} \right]$$

where $P_0$ is the CR Paneitz operator (see section 2). This suggests us there is an obstruction to the existence of pseudo-Einstein contact form pertaining to the CR Paneitz operator. See Theorem 1.2 below for more details.

As for the proof of the case of vanishing pseudohermitian torsion:

**Proof.** (Proof of Corollary 1.1)

Setting $A_{11} = 0$ in (1.6), by Theorem 1.1, it suffices to show that

$$\gamma_{1,0} = 0$$

in order to have a globally defined pseudo-Einstein contact form $\tilde{\theta} = e^{\frac{(f+2\alpha)}{3}} \theta$.

Note that, from (3.2) and $A_{11} = 0$,

$$R_{,1} = 2u_{,T11} - i \gamma_{1,0}.$$  

Utilizing integration by parts, it follows from (3.8) and $\gamma_{T,1} = 0$ that

$$0 \leq \int_M |\gamma_{1,0}|^2 d\mu = - \int_M \gamma_1 \gamma_{T,00} d\mu = - \int_M \gamma_1 ( -i R_{,T} + 2i u_{,T1})_{,0} d\mu = i \int_M \gamma_1 (R_{,0} - 2u_{,T0})_{,T} d\mu = \int_M \gamma_{1,T} (R_{,0} - 2u_{,T0}) d\mu = 0.$$  

The third equality comes from (2.3) and $A_{11} = 0$. Then

$$\gamma_{1,0} = 0.$$  

Before giving the proof of Theorem 1.2 we need the following Bochner-type equality.
Lemma 3.4. Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold and \(\tilde{\theta} = e^{(f + 2u)}\theta\) is a pseudo-Einstein contact form. Then we have

\[ \int_M (2R - \text{Tor}) \langle \gamma, \gamma \rangle d\mu + 2 \int_M |\gamma_{1,1}|^2 d\mu + \frac{1}{2} \int_M (P_0 f) f d\mu = 0. \]  

Proof. From Theorem 1.1 and the commutation formula, it follows that \(\tilde{\theta} = e^{(f + 2u)}\theta\) is a pseudo-Einstein contact form if and only if

\[ P_1 f = iA_{1T}\gamma_T + R\gamma_1 - \gamma_{1,1T}. \]

By the fact that \(\gamma_{1,1T} = 0\), it’s easy to see

\[ \int_M (P_1 f) \gamma_T d\mu = \int_M (f_{1T} + iA_{1T} f_T) \gamma_T d\mu = i \int_M A_{1T} f_T \gamma_T d\mu. \]

Then, substituting (3.10) into the last equality and adding its conjugation, we have

\[ -\int_M \text{Tor} (d_b f, \gamma) d\mu = \int_M (2R - \text{Tor}) \langle \gamma, \gamma \rangle d\mu + 2 \int_M |\gamma_{1,1}|^2 d\mu. \]

On the other hand, the equality (3.10) and the commutation formulas enable us to get

\[ \int_M (P_1 f) f_T d\mu = \int_M (iA_{1T}\gamma_T + R\gamma_1 - \gamma_{1,1T}) f_T d\mu - \int_M \gamma_1 f_{TT} d\mu \]

\[ = \int_M (iA_{1T}\gamma_T + R\gamma_1) f_T d\mu + \int_M \gamma_1 (-f_{TT} + i f_{0T} - R\gamma_1 f_T) d\mu \]

\[ = \int_M i A_{1T} f_T d\mu + \int_M \gamma_1 (-f_{TT} + i f_{0T}) d\mu \]

\[ = i \int_M (A_{1T} f_T - A_{TT} \gamma_1 f_1) d\mu \]

\[ = -\int_M \text{Tor} (d_b f, \gamma) d\mu. \]

By the definition of the CR Paneitz operator, we obtain

\[ \int_M (P_0 f) f d\mu = -\int_M ((P_1 f) f_T + (P_T f) f_1) d\mu = 2 \int_M \text{Tor} (d_b f, \gamma) d\mu. \]
Therefore, it follows from the equalities (3.11) and (3.12) that
\[
\int_M (2R - Tor) (\gamma, \gamma) \, d\mu + 2 \int_M |\gamma_{1,1}|^2 \, d\mu + \frac{1}{2} \int_M (P_0 f) \, f \, d\mu = 0.
\]
Then we are done.  \hfill \Box

Such equality enables us to prove Theorem 1.2 as follows:

Proof. (Proof of Theorem 1.2) From the equality (3.9) and the hypotheses, it is clear that if \( \tilde{\theta} = e^{\frac{i f + 2u}{2}} \theta \) is a pseudo-Einstein contact form, then
\[
\gamma = 0.
\]
Hence we can solve the inhomogeneous tangential Cauchy-Riemann equation
\[
\overline{\partial}_b \phi = \sigma_1 \theta^T
\]
by Lemma 3.2. Note that this implicitly implies \( f \) is CR-pluriharmonic. So the sufficient part is completed.

As for the necessary part, it’s obvious from Theorem 1.1.  \hfill \Box

Before to go further, we need the following key lemma.

Lemma 3.5. Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold with \( c_1(T_{1,0}M) = 0 \). Then, with the notations as above, the following equality holds
\[
(3.13) \quad \int_M \left( R - \frac{1}{2} Tor - \frac{1}{2} Tor' \right) (\gamma, \gamma) \, d\mu + \int_M |\gamma_{1,1}|^2 \, d\mu + \int_M Q \, d\mu + \int_M (P_0 u^+) u^+ \, d\mu = 0.
\]

Proof. From the equality (3.2), we are able to get
\[
(R_1 - i A_{11, T}) \gamma_T = W_1 \gamma_T = 2 (u_{111} + i A_{11} u_T + i A_{11} \gamma_{1 T} - i \gamma_{1,0} \gamma_T - \gamma_{1,1} \gamma_T - R \gamma_{1,1} - R \gamma_{1,1}) \gamma_T.
\]
Taking the integration over \( M \) of both sides and its conjugation, we have, by the fact that \( \gamma_{1,1} = 0 \),
\[
i \int_M (A_{11, T} \gamma_T - A_{TT,1} \gamma_1) \, d\mu + \int_M (2R - Tor) (\gamma, \gamma) \, d\mu + 2 \int_M |\gamma_{1,1}|^2 \, d\mu + 2 \int_M Tor (d_b u, \gamma) \, d\mu = 0.
\]
That is

\[(3.14) \int_M \left( R - \frac{1}{2} Tor - \frac{1}{2} Tor' \right) (\gamma, \gamma) \, d\mu + \int_M |\gamma_{1,1}|^2 \, d\mu - \int_M Tor (d_{b,\mu}, \gamma) \, d\mu = 0.\]

On the other hand, it follows from the equality (3.2) that

\[(3.15) (R_1 - iA_{11,\gamma_1}) u_{\gamma_1} = W_1 u_{\gamma_1} = [P_1 u + i (A_{11} \gamma_{11} - \gamma_{1,0})] u_{\gamma_1}.\]

By the fact that \(\gamma_{1,\gamma_1} = 0\), we see that

\[(3.16) \int_M \gamma_{1,0} u_{\gamma_1} \, d\mu = \int_M \gamma_1 u_{\gamma_0} \, d\mu = -\int_M \gamma_1 (u_{\gamma_1} - A_{\gamma_1} u_{\gamma_1}) \, d\mu = \int_M A_{\gamma_1} u_{\gamma_1} \, d\mu.\]

It follows from (3.15) and (3.16) that

\[2 \int_M Qu \, d\mu + 2 \int_M (P_0 u) \, u \, d\mu = i \int_M [(A_{11} u_{\gamma_1} - A_{\gamma_1} u_{\gamma_1}) - \text{conj}] \, d\mu = -2 \int_M Tor (d_{b,\mu}, \gamma) \, d\mu.\]

Thus by (3.14)

\[\int_M \left( R - \frac{1}{2} Tor - \frac{1}{2} Tor' \right) (\gamma, \gamma) \, d\mu + \int_M |\gamma_{1,1}|^2 \, d\mu + \int_M Qu \, d\mu + \int_M (P_0 u^+) \, u^+ \, d\mu = 0.\]

\[\Box\]

**Proof.** (proof of Theorem 1.3 and Corollary 1.3) If we assume that

\[\ker P_1 = \ker P_0.\]

Then we also have

\[(3.17) 0 = \int_M \left( R - \frac{1}{2} Tor - \frac{1}{2} Tor' \right) (\gamma, \gamma) \, d\mu + \int_M |\gamma_{1,1}|^2 \, d\mu + \int_M Qu \, d\mu + \int_M (P_0 u^+) \, u^+ \, d\mu.\]

Here we have used the fact that \(P_0\) is self-adjoint and

\[\int_M Tor (d_{b,\mu_{ker}}, \gamma) \, d\mu = 0.\]
Now if \( \tilde{\theta} = e^{(f+2u)} \theta \) is a pseudo-Einstein contact form for any CR-pluriharmonic function \( f \), it follows from (3.9) that

\[ \gamma = 0 \]

and then

\[
0 = \int_M Q u^+ d\mu + \int_M (P_0 u^+) u^+ d\mu = \int_M Q^+ u^+ d\mu + \int_M (P_0 u^+) u^+ d\mu.
\]

By the Hölder’s inequality and essentially positivity of the CR Paneitz operator, we have

\[
\int_M Q^+ u^+ d\mu + \int_M (P_0 u^+) u^+ d\mu \geq \Lambda \int_M (u^+)^2 d\mu - (\int_M (Q^+)^2 d\mu)^{\frac{1}{2}} (\int_M (u^+)^2 d\mu)^{\frac{1}{2}}
\]

and then

\[
0 \geq \Lambda \left( \int_M (u^+)^2 d\mu \right)^{\frac{1}{2}} - (\int_M (Q^+)^2 d\mu)^{\frac{1}{2}}.
\]

Hence

\[
\int_M (Q^+)^2 d\mu \geq \Lambda^2 \int_M (u^+)^2 d\mu.
\]

Furthermore, if the CR \( Q \)-curvature is pluriharmonic (i.e. \( Q^+ = 0 \)), then

\[ u^+ = 0 \]

and by (3.2)

\[ W_1 = 0. \]

Hence \( \theta \) is also a globally defined pseudo-Einstein contact form. Moreover, if the pseudohermitian torsion is vanishing, then \((M, J, \theta)\) is the Sasakian space form.

Proof. (proof of Theorem 1.4 and Corollary 1.4) As before

\[
\int_M Q u^+ d\mu + \int_M (P_0 u^+) u^+ d\mu = \int_M Q^+ u^+ d\mu + \int_M (P_0 u^+) u^+ d\mu \geq \Lambda \int_M (u^+)^2 d\mu \geq 0
\]

if

\[ Q^+ = 0. \]
It follows from (3.17) that

\begin{equation}
0 \geq \int_M \left( R - \frac{1}{2} 
\text{Tor} - \frac{1}{2} 
\text{Tor}' \right) \langle \gamma, \gamma \rangle \, d\mu + \int_M |\gamma_{1,1}|^2 \, d\mu.
\end{equation}

if (3.18) holds. Hence

\[ \gamma = 0 \]

if the pseudohermitian curvature is \( \left( \frac{1}{2}, \frac{1}{2} \right) \)-positive. It follows from Theorem 1.2 that \( M \) admits a globally defined pseudo-Einstein contact form \( \tilde{\theta} = e^{(f+\alpha)} \theta \).

Furthermore, if the CR \( Q \)-curvature is pluriharmonic (i.e. \( Q^\perp = 0 \)), then

\[ u^\perp = 0 \]

and by (3.3)

\[ W_1 = 0. \]

Hence \( \theta \) is also a globally defined pseudo-Einstein contact form.

Now if (\( M, J, \theta \)) is spherical and pseudo-Einstein, we have

\[ W_1 = R_{1,1} - i A_{11, \mathbf{T}} = 0 \]

and

\[ i R_{11} = 3 R A_{11} + 6 i A_{11,0} - 4 A_{11, \mathbf{T}}. \]

By cancelling \( R_{11} \), one derives

\[ 3 R A_{11} + 6 i A_{11,0} - 3 A_{11, \mathbf{T}} = 0. \]

On the other hand, it follows from the commutation relation (Lee) that

\[ A_{11, \mathbf{T}} - A_{11, \mathbf{T}} = i A_{11,0} + 2 R A_{11}, \]

we obtain

\[ -3 R A_{11} + 2 A_{11, \mathbf{T}} - 3 A_{11, \mathbf{T}} = 0 \]

and then

\[ -2 \int_M |A_{11,1}|^2 \, d\mu + 3 \int_M |A_{11, \mathbf{T}}|^2 \, d\mu = 3 \int_M R |A_{11}|^2 \, d\mu. \]
Moreover, if $R$ is a positive constant, then

$$A_{11,\mathcal{T}} = 0$$

and

$$-2\int_M |A_{11,1}|^2 d\mu = 3\int R |A_{11}|^2$$

which implies

$$A_{11} = 0.$$

It follows that $(M,J,\theta)$ is the Sasakian space form with positive constant Tanaka-Webster scalar curvature and vanishing pseudohermitian torsion. \qed

**Proof.** (proof of Theorem 1.5 and Corollary 1.5)

It follows from (3.13) that

$$\int_M \left( R - \frac{1}{2} Tor - \frac{1}{2} Tor' \right) (\gamma, \gamma) d\mu + \int_M |\gamma_{1,1}|^2 d\mu + \int_M (P_0 u^\perp) u^\perp d\mu = 0$$

if

$$Q = 0.$$

Then again, we are in the line of (3.19) and we have

$$\gamma = 0$$

under the assumptions. Again we have

$$u^\perp = 0$$

and by (3.2)

$$R_1 - iA_{11,1} = 2P_1 u_{\ker}.$$ 

Hence

$$\tilde{\theta} = e^{\frac{(f+2\alpha)}{3}} \theta = e^{\frac{(f+2\alpha)_{\ker}}{3}} \theta$$

is pseudo-Einstein. It follows from (2.9) that

$$P_1 u_{\ker} = 0$$
and then
\[ R_{i1} - iA_{11,1} = 0 \]
which implies \( \tilde{\theta} \) is pseudo-Einstein as well. The argument for CR uniformization theorem as in Corollary 1.5 are easily derived from the previous one. Then we are done. \( \square \)

**Appendix A.**

In this appendix, we introduce some basic notions from pseudohermitian geometry as in [Lee].

**Definition A.1.** Let \( M \) be a smooth manifold and \( \xi \subset TM \) a subbundle. A \textbf{CR structure} on \( \xi \) consists of an endomorphism \( J : \xi \to \xi \) with \( J^2 = -\text{id} \) such that the following integrability condition holds.

1. If \( X, Y \in \xi \), then so is \( [JX, Y] + [X, JY] \).
2. \( J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y] \).

The CR structure \( J \) can be extended to \( \xi \otimes \mathbb{C} \), which we can then decompose into the direct sum of eigenspaces of \( J \). The eigenvalues of \( J \) are \( i \) and \( -i \), and the corresponding eigenspaces will be denoted by \( T^{1,0} \) and \( T^{0,1} \), respectively. The integrability condition can then be reformulated as

\[ X, Y \in T^{1,0} \implies [X, Y] \in T^{1,0}. \]

Now consider a closed \( 2n + 1 \)-manifold \( M \) with a cooriented contact structure \( \xi = \ker \theta \). This means that \( \theta \wedge d\theta^n \neq 0 \). The **Reeb vector field** of \( \theta \) is the vector field \( T \) uniquely determined by the equations

\begin{equation}
(A.1) \quad \theta(T) = 1, \quad \text{and} \quad d\theta(T, \cdot) = 0.
\end{equation}

A **pseudohermitian manifold** is a triple \( (M^{2n+1}, \theta, J) \), where \( \theta \) is a contact form on \( M \) and \( J \) is a CR structure on \( \ker \theta \). The **Levi form** \( \langle \cdot, \cdot \rangle \) is the Hermitian form on \( T^{1,0} \) defined by

\[ H(Z, W) = \langle Z, W \rangle = -i \langle d\theta, Z \wedge \overline{W} \rangle. \]

We can extend this Hermitian form \( \langle \cdot, \cdot \rangle \) to \( T^{0,1} \) by defining \( \langle Z, W \rangle = \overline{\langle Z, W \rangle} \) for all \( Z, W \in T^{1,0} \). Furthermore, the Levi form naturally induces a Hermitian form on the dual bundle of \( T^{1,0} \), and hence on all induced tensor bundles.
We now restrict ourselves to strictly pseudoconvex CR manifolds, or in other words compatible complex structures $J$. This means that the Levi form induces a Hermitian metric $\langle \cdot, \cdot \rangle_{J,\theta}$ by

$$\langle V, U \rangle_{J,\theta} = d\theta(V, JU).$$

The associated norm is defined as usual: $|V|_{J,\theta}^2 = \langle V, V \rangle_{J,\theta}$. It follows that $H$ also gives rise to a Hermitian metric for $T^{1,0}$, and hence we obtain Hermitian metrics on all induced tensor bundles. By integrating this Hermitian metric over $M$ with respect to the volume form $d\mu = \theta \wedge d\theta^n$, we get an $L^2$-inner product on the space of sections of each tensor bundle.

The pseudohermitian connection or Tanaka-Webster connection ([Ta], [We]) of $(J,\theta)$ is the connection $\nabla$ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $\{Z_\alpha\}$ for $T^{1,0}$ by

$$\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_\bar{\alpha} = \omega_\bar{\alpha}^\bar{\beta} \otimes Z_\bar{\beta}, \quad \nabla T = 0,$$

where $\omega_\alpha^\beta$ is the 1-form uniquely determined by the following equations:

\begin{align*}
  d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta \\
  \tau \wedge \theta^\alpha &= 0 \\
  \omega_\alpha^\beta + \omega^\bar{\alpha}_{\bar{\beta}} &= 0.
\end{align*}

(A.2)

Here $\tau^\alpha$ is called the pseudohermitian torsion, which we can also write as

$$\tau_\alpha = A_{\alpha\beta} \theta^\beta.$$

The components $A_{\alpha\beta}$ satisfy

$$A_{\alpha\beta} = A_{\beta\alpha}.$$

We often consider the torsion tensor given by

$$A_{J,\theta} = A^\alpha_{\beta} Z_\alpha \otimes \theta^\beta + A^\bar{\alpha}_{\bar{\beta}} Z_\bar{\alpha} \otimes \theta^\bar{\beta}.$$
We now consider the curvature of the Tanaka-Webster connection in terms of the coframe \( \{ \theta = \theta^0, \theta^\alpha, \theta^\bar{\beta} \} \).

The second structure equation gives

\[
\Omega^\beta_\alpha = \Omega^\beta_\alpha^\gamma \omega^\gamma, \\
\Omega^0_\alpha = \Omega^0_\alpha^0 = 0.
\]

In [We, Formulas 1.33 and 1.35], Webster showed that the curvature \( \Omega^\beta_\alpha \) can be written as

\[
\Omega^\beta_\alpha = R^\beta_\alpha^\rho_\bar{\sigma} \theta^\rho \wedge \theta^\bar{\sigma} + W^\beta_\alpha^\rho_\bar{\sigma} \theta^\rho \wedge \theta - W^\alpha_\beta^\rho_\bar{\sigma} \theta^\rho \wedge \theta + i \theta_\beta \wedge \tau^\alpha - i \tau_\beta \wedge \theta^\alpha,
\]

where the coefficients satisfy

\[
R^\beta_\alpha^\rho_\bar{\sigma} = R^\alpha_\beta^\rho_\bar{\sigma}, \quad W^\alpha_\beta^\rho_\bar{\sigma} = W^\rho_\alpha_\beta_\bar{\sigma}.
\]

In addition, by [Lee, (2.4)] the coefficients \( W^\beta_\alpha^\rho_\bar{\sigma} \) are determined by the torsion,

\[
W^\beta_\alpha^\rho_\bar{\sigma} = A^\alpha_\beta^\rho_\bar{\sigma}.
\]

Contraction of (A.3) yields

\[
\Omega^\alpha_\alpha = d\omega^\alpha_\alpha = R_\rho_\bar{\sigma} \theta^\rho \wedge \theta^\bar{\sigma} + W_\rho_\alpha^\rho_\bar{\sigma} \theta^\rho \wedge \theta - W_\rho_\bar{\sigma}^\rho_\alpha \theta^\rho \wedge \theta
\]

\[
= R_\rho_\bar{\sigma} \theta^\rho \wedge \theta^\bar{\sigma} + A_\rho_\alpha^\alpha \theta^\rho \wedge \theta - A_\bar{\sigma}^\rho_\alpha \theta^\rho \wedge \theta
\]

(A.4)

We will denote components of covariant derivatives by indices preceded by a comma. For instance, we write \( A_{\alpha,\beta,\gamma} \). Here the indices \( \{0, \alpha, \bar{\beta}\} \) indicate derivatives with respect to \( \{T, Z_\alpha, Z_{\bar{\beta}}\} \). For derivatives of a scalar function, we will often omit the comma. For example, \( \varphi_\alpha = Z_\alpha \varphi, \varphi_{\alpha \bar{\beta}} = \bar{Z}_{\bar{\beta}} Z_\alpha \varphi - \omega^\alpha_{\gamma \bar{\beta}} (Z_{\bar{\beta}}) Z_\gamma \varphi, \varphi_0 = T \varphi \) for a (smooth) function \( \varphi \).

In particular, we define followings for \( n = 1 \) For a real function \( \varphi \), the subgradient \( \nabla_b \) is defined by \( \nabla_b \varphi \in \xi \) and \( \langle Z, \nabla_b \varphi \rangle_{\ell^0} = d\varphi(Z) \) for all vector fields \( Z \) tangent to contact plane. Locally \( \nabla_b \varphi = \varphi_1 Z_1 + \varphi_\bar{1} Z_{\bar{1}} \). We can use the connection to define the subhessian as the complex linear map

\[
(\nabla^H)^2 \varphi : T_{1,0} \oplus T_{0,1} \rightarrow T_{1,0} \oplus T_{0,1},
\]

by

\[
(\nabla^H)^2 \varphi(Z) = \nabla_{Z} \nabla_b \varphi.
\]
Also

\[ \Delta \varphi = \text{Tr} \left( (\nabla^H)^2 \varphi \right) = (\varphi_{1\bar{1}} + \varphi_{\bar{1}1}). \]

For all \( Z = x^1 Z_1 \in T_{1,0} \), we define

\[ \text{Ric}(Z, Z) = W x^1 x^1 = W|Z|_{L^2}^2, \]
\[ \text{Tor}(Z, Z) = 2 \text{Re} i A_{1\bar{1}} x^1 x^1. \]

We also need the following commutation relations (\cite{Lee}).

(A.5)

\[
\begin{align*}
C_{I,01} - C_{I,10} &= C_{I,T} A_{1\bar{1}} - k C_{I,1\bar{1},T}, \\
C_{I,0T} - C_{I,T0} &= C_{I,1T} A_{1\bar{1}} - k C_{I,1\bar{1},1T}, \\
C_{I,1T} - C_{I,T1} &= i C_{I,0} + k W C_{I}.
\end{align*}
\]

Here \( C_I \) denotes a coefficient of a tensor with multi-index \( I \) consisting of only 1 and \( \bar{1} \), and \( k \) is the number of 1’s minus the number of \( \bar{1} \)’s in \( I \).

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DEPARTMENT OF MATHEMATICS AND TAIDA INSTITUTE FOR MATHEMATICAL SCIENCES (TIMS), NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN, R.O.C., CURRENT ADDRESS : YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA

E-mail address: scchang@math.ntu.edu.tw

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN NORMAL UNIVERSITY, TAIPEI 11677, TAIWAN

E-mail address: tjkuo1215@ntnu.edu.tw

DEPARTMENT OF MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, Hsinchu 30013, TAIWAN, R.O.C

E-mail address: r97221009@ntu.edu.tw