Symmetry Breaking In
Twisted Eguchi-Kawai Models

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Abstract

We present numerical evidence for the spontaneous breaking of the $Z_N^4$ symmetry of four-dimensional twisted Eguchi-Kawai models with $SU(N)$ gauge group and symmetric twist, for sufficiently large $N$. We find that for $N \geq 100$ this occurs for a wide range of bare couplings. Moreover for $N \leq 144$, where we have been able to perform detailed calculations, there is no window of couplings where the physically interesting confined and deconfined phases appear in the reduced model. We provide a possible interpretation for this in terms of generalised ‘fluxon’ configurations. We discuss the implications of our findings for the validity and utility of space-time reduced models as $N \to \infty$. 
1 Introduction

The conventional way of calculating the properties of the large $N$ limit of SU($N$) gauge theories using lattice methods follows three main steps [1]: the calculation of some dimensionless ratios of physical quantities at fixed lattice spacing $a$ and fixed $N$, for several values of $a$ and $N (= 2, 3, 4, 5, \ldots$); the continuum extrapolation ($a \to 0$) of the resulting lattice values at fixed $N$; the large $N$ extrapolation ($N \to \infty$) of the resulting continuum values, assuming a leading correction of order $1/N^2$ (for pure gauge theories). This way of studying the physics of the SU($\infty$) theory has been the subject of intense research in the past decade [1], and has allowed the accurate calculation of a number of nonperturbative properties of the pure gauge theory, such as the string tension [2], some glueball masses [3], and the deconfining temperature [4]. However, the numerical simulations involved become very costly when we consider gauge groups of size $N \geq 10$ on reasonably large volumes. So, if one wishes to check by explicit calculation, at very large $N$, that these extrapolations are indeed under control, one needs another approach. Fortunately such an alternative approach exists: it uses an old idea introduced by Eguchi and Kawai that goes by the name of reduced models [5, 6].

In their original proposal [5], Eguchi and Kawai showed that an exact correspondence can be formally constructed between the $N = \infty$ limit of SU($N$) gauge theories living on an infinite $d$-dimensional lattice (the original theory) and the same theory living on a $1^d$ periodic lattice (its reduced model). The correspondence states that Wilson loops in the original theory and its reduced model obey the same Dyson-Schwinger equations, and therefore it is reasonable to deduce that their numerical values should coincide (ignoring the possibilities raised by multiple solutions). However, these loop equations are only identical if we make two assumptions: 1) the large $N$ factorisation of the physical observables, and 2) an unbroken $Z_d^N$ symmetry of the reduced model. While it is reasonable to believe that factorisation is a universal property of a large class of physical operators in the large $N$ limit [7], nothing prevents the $Z_d^N$ symmetry of the reduced model from being spontaneously broken. Indeed, the symmetry was found to be spontaneously broken in the continuum limit of the original Eguchi-Kawai model [8], so the Eguchi-Kawai correspondence is spoiled in that case. But there are alternative ways of constructing reduced models of SU($N$) gauge theories that keep the $Z_d^N$ symmetry unbroken for arbitrarily weak couplings: 1) by quenching the eigenvalues of the reduced link matrices (quenched Eguchi-Kawai model, or QEK model) [8], or 2) by imposing twisted boundary conditions on the reduced lattice (twisted Eguchi-Kawai model, or TEK model) [9].

A great amount of work was done in the first half of the 1980s in trying to extract physical properties of large $N$ theories from numerical simulations of (mainly) TEK models [6]. A drawback of these models, however, is that the volume of the lattice on which the original theory is defined is directly related to the number of internal degrees of freedom of the reduced model, so very large gauge groups must be used in order to obtain information that is sufficiently near the continuum and large volume limits to be physically relevant. Consequently, due to obvious computational constraints, the results obtained in the early days lacked accuracy and, in general, did not allow one to reach reliable conclusions about the physics of the $N = \infty$ theory (for example, we are not aware of any finite-temperature study using this framework that demonstrated, in a single calculation, a clear separation between the bulk
and deconfining phase transitions). With present day computational resources, one can easily perform more precise calculations and for larger gauge groups than before, and this is one motivation for returning to a careful study of large $N$ physics using reduced models. There are other motivations for such a study: first, it allows us to do simulations with much larger $N$ than with the traditional method, since the number of external degrees of freedom (volume) is reduced to its minimum; and second, it provides an alternative method to approach large $N$ physics, with which we can compare (and hopefully confirm) the numerous results already obtained via the conventional method. Finally we point to the importance of reduced models, in particular the twisted version, in matrix models of noncommutative field theory \[10\] and M-theory \[11\].

In this paper, we will summarise the results of a detailed numerical study of the properties of symmetric-twist SU$(N)$ TEK models in four dimensions. In the next section, we provide a short review of TEK models. Next, we present the details of our numerical simulations and discuss the results obtained from them. The surprise is that we find that the $Z_N^4$ symmetry of the TEK model, usually assumed to be unbroken for all couplings, is in fact broken spontaneously at intermediate couplings when $N$ is sufficiently large. At the same time there appears to be a persistent metastability at weak coupling that prevents the appearance of a physically interesting phase in the reduced model. We conclude with a discussion about the possible origin and the implications of this symmetry breaking. Some of these results have been presented at SMFT 2006 \[12\].

## 2 TEK model

The SU$(N)$ TEK model in four dimensions is defined by the partition function on a periodic lattice with a single site,

$$Z_{TEK} = \int [dU] \ e^{-b S_{TEK}}$$

where $[dU] \equiv \prod_{\mu=1}^{4} dU_\mu$ is a product of SU$(N)$ Haar measures, $U_\mu \in \text{SU}(N)$ are the link variables on this reduced lattice, $b = 1/g^2 N$ is the inverse bare 't Hooft coupling and $S_{TEK}$ is the action of the TEK model,

$$S_{TEK} \equiv S_{TEK}(n; [U]) = \sum_{\mu \neq \nu}^4 \text{Tr} \left( I - z_{\mu\nu}(n) U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger \right) \geq 0.$$ 

$S_{TEK}$ depends on the reduced link variables $U_\mu$ and on the twist-tensor, $n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z}_N$, that labels the twist factor $z_{\mu\nu}(n) = e^{-i\frac{2\pi}{N}n_{\mu\nu}} \in \mathbb{Z}_N$. The twist-tensor defines uniquely (modulo $\text{SL}(4, \mathbb{Z})$ transformations) the geometry of the effective lattice on which the original theory is defined. We choose to use the standard symmetric twist, $n_{\mu\nu} = \sqrt{N}$, $\forall \mu > \nu$, of González-Arroyo and Okawa \[9\]. With this twist the TEK model corresponds to an effective symmetric $L^4$ lattice with $N = L^2$. Under certain conditions, as discussed below, the TEK model has the same planar limit as a conventional SU$(N)$ lattice theory with a plaquette action, at the same
value of the 't Hooft bare coupling. This restricts the physical value of $N$ to be the square of an integer.

The lattice operators on the reduced $1^4$ lattice are obtained by replacing the link variables in their original definition with the corresponding reduced link variables, and then introducing the appropriate twists. The $I \times J$ Wilson loop in the $\{\mu, \nu\}$ plane in the TEK model is then given by:

$$W_{\mu \nu}(I, J) = z_{\mu \nu}(n)^{IJ}U^I_\mu U^J_\nu U^I_\mu U^J_\nu$$

and the Polyakov loop in the $\mu$ direction (on the equivalent $L = \sqrt{N}$ lattice) is given by:

$$P_\mu = U^L_\mu$$

The action of the TEK model in (2) has two distinct symmetries acting on the link variables: the usual gauge symmetry,

$$U_\mu \mapsto \Omega U_\mu \Omega^\dagger, \quad \Omega \in SU(N)$$

and an additional $Z_N^4$ symmetry,

$$U_\mu \mapsto z_\mu U_\mu, \quad z_\mu \in Z_N$$

In order for the TEK model to be able to reproduce the same large $N$ physics as the original theory, a necessary condition is to have vanishing expectation values for the traces of reduced open lines, such as $U_\mu^\alpha$ with $\alpha$ not an integer multiple of $L = \sqrt{N}$. In the original theory this is guaranteed by the fact that such an open line is not gauge invariant, but in the single-site reduced model any such operator is trivially gauge-invariant and could be non-zero. However, unlike Wilson loops (3) and Polyakov loops (4), we note that reduced open lines are not invariant under the $Z_N^4$ symmetry (6). So, if the $Z_N^4$ symmetry is not spontaneously broken, the expectation value of open lines will indeed be zero, and the loop equations of the TEK and conventional theories will be identical in the planar limit.

In the strong coupling limit, $b \to 0$, the partition function in eqn(1) is dominated by the Haar measure $[dU]$; consequently, the expectation value of the trace of a link variable (and in general of any open line) is automatically zero, so the model has an unbroken $Z_N^4$ symmetry in that limit. We call this phase the random phase and denote it by $Z_N^{4(r)}$. In the weak coupling limit, $b \to \infty$, the partition function in eqn(1) is dominated by the classical vacuum of the TEK model, i.e. by the field configuration $U_\mu = \Gamma_\mu$ that solves the equation:

$$z_{\mu \nu}(n)\Gamma_\mu \Gamma_\nu \Gamma^\dagger_\mu \Gamma^\dagger_\nu = I.$$  

The $\Gamma_\mu$ matrices defined above, also known as twist-eaters, saturate the lower bound of the TEK action in eqn(2). They have the property of being traceless, as do most of its powers: $\text{Tr}(\Gamma^\alpha) = \delta_{\alpha, (\alpha \text{ mod } L)}$. In other words, the classical vacuum of the TEK model has a $Z_N^4$ symmetry that ensures that all lines (except Polyakov loops) vanish. Small quantum fluctuations around the twist-eating configuration also keep the $Z_N^4$ symmetry unbroken. We call this phase the twist-eater phase and denote it by $Z_N^{4(t)}$. Extrapolating from the fact that the symmetry
is unbroken in the \( b \rightarrow 0 \) and \( b \rightarrow \infty \) limits, it is generally claimed that the \( Z_N^4 \) symmetry of the TEK model should be unbroken for all couplings. There is, however, no compelling theoretical argument justifying such a claim for the intermediate-coupling regime, although it receives some support from the fact that previous numerical simulations of the model \cite{6} have not shown any sign of symmetry breaking.

3 Results

In this section we provide numerical evidence that for sufficiently large \( N \), namely \( N \geq 100 \), the \( Z_N^4 \) symmetry of the TEK model is spontaneously broken. For \( N \leq 81 \) we see no sign of the symmetry breaking, which is consistent with the results from numerical simulations done in the past \cite{6}.

3.1 Phase structure

In the large \( N \) limit, SU\((N)\) lattice gauge theories possess a set of phase transitions that we would expect to observe in our simulations of the TEK model, if the Eguchi-Kawai correspondence between the planar limits of the reduced and parent lattice gauge theories is in fact valid. First, as we increase \( b \) we expect to encounter a first-order phase transition between the strong and weak coupling phases of the theory. If the lattice is large enough the transition is between two confining phases. As we increase \( b \) further, the lattice will encounter a deconfining phase transition. This will occur when the smallest lattice length \( L_{\mu} \) satisfies \( a(b)L_{\mu} = 1/T_c \), and it is characterised by the Polyakov loop in that direction acquiring a non-zero vacuum expectation value \cite{13}. As we increase \( b \) further there will be a sequence of three further transitions corresponding to each of the other three Polyakov loops acquiring a non-zero expectation value. Finally we are in a phase where the volume is small and the physics on all length scales is perturbative and there are no further transitions as \( a \rightarrow 0 \). Although it is not obvious that it should be so, it has been observed \cite{14} that exactly the same phase structure persists on a symmetric \( L^4 \) lattice, and hence should be observed in the TEK model. We now discuss these transitions in more detail.

The bulk transition is a lattice artifact that occurs, for the standard plaquette action, at \( b = b_B \approx 0.36 \) in the large \( N \) limit. It separates the unphysical phase at strong lattice coupling (known as the bulk phase) from the physical phase at weak lattice coupling, through which one approaches the continuum limit of the theory. The bulk transition also occurs in the TEK model, in this case separating the random phase, \( Z_N^{4(r)} \), from the physically relevant twist-eater phase, \( Z_N^{4(t)} \); it was accurately determined in \cite{15} to have the value \( b_B = 0.3596(2) \) in the \( N = \infty \) limit.

For large enough lattices, the deconfining transition is located on the physical side of the bulk transition, separating the low \( T \) confined phase (with non-zero string tension, \( \sigma \neq 0 \)) from the high \( T \) deconfined phase (with \( \sigma = 0 \)). It is signaled by the spontaneous breaking of the \( Z_N \) symmetry of the Polyakov loop \( P_\mu \) along the direction of the 4-torus with the shortest physical length, \( aL = T^{-1} \), when the temperature \( T \) rises above a critical value.
it is also accompanied by a small discontinuity in the average value of the plaquette (reflecting the latent heat of this first-order transition \[4\]). The critical coupling \(b_c\) at which the deconfining transition occurs scales with the lattice size \(L\), so it is natural to expect that there is a critical size \(L = L_B\) below which the deconfining transition is overtaken by the bulk transition, and only for \(L > L_B\) are the two transitions decoupled. From the calculations of \(T_c\) versus \(N\) in \[4\] we can infer that \(L_B \approx 9\). In the TEK model, we can therefore expect to observe a corresponding deconfining transition, signaled by the spontaneous breaking of the \(Z_L\) symmetry of the reduced Polyakov loops \[17\], once \(N = L^2 > L_B^2 \sim 81\). There is some uncertainty in this value, that will be enhanced in practice by any metastabilities associated with these first-order transitions, so it is only at very large \(N\) that we can expect the symmetric-twist TEK model to exhibit a genuine weak coupling confined phase.

The TEK model is equivalent to a symmetric \(L^4\) lattice, with \(L = \sqrt{N}\), and so there is no direction that naturally plays the role of an inverse temperature. This system has been studied in \[14\]. One finds that as one increases \(b\) from the confining phase one encounters a sequence of four transitions at each of which Polyakov loops along one more direction acquire non-zero expectation values. The first of these transitions occurs when \(a(b)L = T_c^{-1}\) and is indeed just the deconfining transition (as follows from the observed vanishing of finite volume corrections when \(N \to \infty\) \[4\]). The phases are labelled in \[14\] by \(X_c\) (\(X = 0, \ldots, 4\)), with \(0c\) the confining phase and \(4c\) the small volume phase. Each of these transitions has a continuum limit and hence a specific physical interpretation. Therefore, if the Eguchi-Kawai correspondence is correct, we should also be able to observe these \(X_c\) phases in the symmetric-twist TEK model, on the physical weak coupling side of the bulk transition.

### 3.2 Simulation details

We have performed Monte Carlo simulations of the four-dimensional SU(\(N\)) TEK model with the conventional symmetric twist tensor \((n_{\mu\nu} = L = \sqrt{N}, \text{ for } \mu > \nu)\). The thermalised configurations at fixed \(b\) were generated using the heatbath algorithm constructed by Fabricius and Haan \[18\]. As a check, a subset of calculations were performed with a straightforward (albeit less efficient) Metropolis algorithm, specifically a modified version of the algorithm originally constructed by Okawa for the untwisted model \[19\]. The simulations started either from a classical twist-eater configuration, referred to as a cold start, or from a completely randomised configuration (in practice one that was thermalised at \(b = 10^{-3}\)) referred to as a hot start.

We considered gauge groups \(N = L^2 = 25, 36, \ldots, 144\) corresponding to effective lattice sizes between \(L = 5\) and \(L = 12\). For each case, we performed two different simulations differing in the range of values of \(b\) spanned: in the first run (run A), the interval \(b \in [0.10, 2.00]\) was spanned with large steps (varying between \(|\Delta b| = 0.01\) and \(|\Delta b| = 0.02\)); in the second run (run B) we wanted to analyse in more detail the region where the phase transitions occur, so we spanned the interval \(b \in [0.200, 0.500]\) with smaller steps, \(|\Delta b| = 0.005\). The other parameters of the simulations, like the number of sweeps or the number of measurements, varied with each case and are summarised in Table \(4\) for the larger values of \(N\).

We calculated the expectation values of several reduced observables, namely the traces and
eigenvalue densities of Wilson loops of several sizes, of Polyakov loops, of link variables \((U_\mu)\) and of some other open lines \((U_\alpha^\mu, \text{for } \alpha = 2, \ldots, 5)\).

### 3.3 Results for \(N \leq 81\)

The phase structure of the TEK model for \(N \leq 81\) is very simple. Essentially, there is only one phase transition, at \(b \approx 0.36\) [15]. The transition is strongly first-order, as demonstrated by its relatively large hysteresis cycle, and in each one of the two phases the average plaquette fits very well the strong- and weak-coupling expansions of both the TEK model and Wilson’s lattice gauge theory, as shown in Fig. 1. Moreover the location of the transition is consistent with that of the bulk transition in usual lattice calculations at large \(N\). So we can confidently identify it as being the bulk transition.

We see in Fig. 2 that the average traces of the link variables vanish in the whole range of lattice couplings (although the fluctuations at intermediate couplings are relatively large). Therefore, we can conclude that the Eguchi-Kawai correspondence between the planar limits of the reduced model and the parent (Wilson) lattice gauge theory is valid at least up to \(N = 81\). As an example of this, we show in Fig. 1 not only the 3-loop perturbative result for the plaquette in Wilson lattice gauge theory at \(N = \infty\), given by [16]:

\[
\langle u_p \rangle_W \approx 1 - \frac{1}{8b} - \frac{0.653687}{128b^2} - \frac{0.4066406}{512b^3} + O(b^4) \tag{8}
\]

but also some values that we obtained from a direct simulation of Wilson lattice gauge theory for \(4 \leq N \leq 16\), extrapolated to \(N = \infty\). We see from Fig. 1 that these extrapolated values coincide with the TEK values, within small errors that are consistent with the various corrections and systematic errors.

The phases separated by the bulk transition can be identified with the random phase, \(Z_N^{4(r)}\), and the twist-eater phase, \(Z_N^{4(t)}\). The justification comes from an analysis of the eigenvalue densities of the link variables: in the random phase, the eigenvalues of the link variables are uniformly distributed over the unit circle\(^1\) while in the twist-eater phase the eigenvalue density of link variables is composed of \(L = \sqrt{N}\) separated lumps spread around the elements of \(Z_L\) on the unit circle (modulo a global \(Z_N\) rotation of the whole spectrum). The latter is due to fluctuations around a twist-eating configuration (modulo a \(Z_N\) rotation), whose eigenvalues are the elements of \(Z_L\) [20].

However, for \(N \leq 81\), it is not possible to pinpoint a deconfining transition in the twist-eater phase, since the Polyakov loop and the plaquette suffer a discontinuity only at the bulk transition. This result should not be too surprising, since, as remarked earlier, the sizes of the effective lattices involved, \(L \leq 9\), are probably smaller than the critical size \(L_B\) needed to have a clear separation between the deconfining and bulk transitions. Since Polyakov loop expectation values are all non-zero in the weak coupling twist-eater phase, while the strong coupling phase is confining, the bulk transition can easily be misinterpreted as being a deconfining transition, particularly in calculations with low statistical accuracy. The only way to be sure that one

\(^1\)Due to unitarity, the eigenvalues of \(U(N)\) or \(SU(N)\) matrices are constrained to be elements of \(U(1)\).
has not confused the two transitions is to vary $b$ so as to identify both transitions explicitly in a single calculation. As we have remarked earlier, none of the old TEK calculations that claimed to find a deconfining transition, displayed such a result. We are therefore forced to discount those early claims.

This negative result motivated us to extend our investigation to $L \geq 10$, whose results are summarised next.

### 3.4 Results for $N \geq 100$

The phase structure of the TEK model for $N \geq 100$ turns out to be much richer than for $N \leq 81$, involving several transitions to new phases. The main characteristic common to all those new phases is that the $Z^4_N$ symmetry is spontaneously broken, thus violating the Eguchi-Kawai correspondence.

In the $N = 100$ case, the average value of the real part of the plaquette undergoes two distinct transitions as we decrease $b$ from a cold-start, but only one transition occurs when we increase $b$ from a hot-start, as shown in Fig. 3. In the case of the cold-start scan, the phase in between the two transitions is characterised by a non-zero expectation value for the trace of one of the link variables as shown in Fig. 4 (and also by an imaginary part of the plaquettes containing that link variable). This implies that the $Z^4_N$ symmetry of the TEK model is spontaneously broken to a $Z^3_N$ symmetry in the intermediate-coupling phase (the traces of the link variables still have zero expectation value in the other three directions). As discussed at the end of section 2, this effect should not come as a complete surprise, since there are no compelling theoretical arguments that prohibit the breaking of the $Z^4_N$ symmetry at intermediate couplings.

The transitions to the regime with $Z^3_N$ symmetry have a discontinuous character, indicating that these are first-order phase transitions (or would be if the number of degrees of freedom were infinite). Increasing $b$ from a hot-start only one transition is observed, and that is the one from the random phase to the $Z^3_N$-symmetric phase. As we see in Fig. 5 there is no transition from the $Z^3_N$ phase to the expected weak coupling twist-eater phase, even at very large values of the inverse lattice coupling. This implies the existence of stable extrema of the TEK action that are $Z^4_N$ non-preserving and from which we are unable to tunnel because of large barriers. This immediately raises the question whether this effect persists for larger $N$ and what consequences it has for the Eguchi-Kawai correspondence.

As at lower $N$, there is no sign of a deconfining transition in the physically relevant phase of the SU(100) TEK model. By `physically relevant’ we mean the twist-eater phase, since it is the only phase that we observe having the necessary $Z^4_N$ symmetry while being continuously connected to the continuum limit.

For $N = 121$ and $N = 144$ we find, at intermediate lattice couplings, a sequence of transitions, as illustrated in Fig. 6 and in more detail in Fig. 7. These transitions are similar to the one we saw for $N = 100$, in that each is associated with the breaking or restoration of one or

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2 That the imaginary part of the plaquettes can be non-zero follows from the fact that the action of the TEK model in (2) is not $CP$-invariant. At small $b$ it can be calculated in a strong coupling expansion and one finds that the contribution decreases with $N$, so that it is real in the planar limit and is consistent with the Eguchi-Kawai correspondence.
more of the $Z_N$ symmetries. Thus they lead to non-zero expectation values for the corresponding link variables, as shown in Fig.8. We can label the phases by the number $k$ of preserved $Z_N$ symmetries, as $Z_N^k$ ($k = 0, \ldots, 3$). In the $N = 144$ case, all transitions between different $Z_N^k$ phases are clearly separated, while for $N = 121$ the simultaneous breaking/restoration of several $Z_N$ symmetries can occur. Tables 2 and 3 summarise the critical couplings at which the observed transitions occur.

Just as for $N = 100$, when we increase $b$ from strong coupling for the SU(121) and SU(144) TEK models, we observe no tunnelling from the final $Z_N^0$ phase to a twist-eater phase. This reinforces the idea that some stable extrema of the TEK action are contributing to the breaking of the $Z_N^4$ symmetry. The nature of these extrema can be inferred from the eigenvalue spectrum of the link variables. In a $Z_N^k$ phase, the $k$ link variables of the $Z_N$-preserved directions have a uniform eigenvalue density, while in the other $N - k$ broken directions their eigenvalue density consists of a bell-shaped lump with finite support, spread around an element of $Z_N$, as illustrated in Fig.9. At larger $b$ the lump becomes narrower, as shown in Fig.10, indicating that in the limit $b \to \infty$ all eigenvalues collapse to one value which is an element of $Z_N$. In sum, the numerical simulations suggest that the breaking of the $Z_N^4$ symmetry in the TEK model for $N \geq 100$ might be caused by fluctuations around (stable) configurations of the form $U_\mu \in Z_N$.

4 Interpretation

Our numerical calculations suggest that the gauge fields responsible for the spontaneous breaking of the $Z_N^4$ symmetry are of the form of fluctuations around $U_\mu \in Z_N$. These field configurations appear to be able to survive reasonably large fluctuations at intermediate couplings, and to become increasingly stable as we increase $b$ to arbitrarily large values. This requires the existence of stable extrema of the TEK action that break the $Z_N^4$ symmetry. At the same time we see explicitly from Figs. 3 and 6 that their action is substantially larger than that of the ‘twist-eating’ fields obtained at the same $b$ from a cold start. Thus the (infinite?) metastability requires large barriers, which implies the existence of corresponding unstable extrema of $S_{TEK}$. We will now argue that the elements of $Z_N$ may provide such configurations. Our arguments below will follow very closely the calculations of van Baal in his study of surviving extrema of the TEK model [21].

Consider a small fluctuation around a general configuration $\Omega_\mu$, parameterised by an (anti-hermitian) element $X_\mu$ of the algebra of SU($N$):

$$U_\mu = \Omega_\mu e^{-X_\mu}$$  (9)

The expansion of the TEK action (2) around $\Omega_\mu$ up to second-order in $X_\mu$ is given by:

$$S_{TEK} \approx \sum_{\mu \neq \nu}^4 \text{Tr} \left\{ I - P_{\{\mu\nu\}} - P_{[\mu\nu]} F_{\mu\nu} - \frac{1}{2} P_{[\mu\nu]} ([D_{\mu} X_\nu + X_\mu, D_{\nu} X_\mu + X_\nu] - [X_\mu, X_\nu]) - \frac{1}{2} P_{[\mu\nu]} F_{\mu\nu}^2 \right\} + O(X^3)$$  (10)
where \( P_{\mu\nu} = z_{\mu\nu}(n)\Omega^\dagger_{\mu\nu} \Omega_{\mu\nu} \) is the reduced plaquette, \( P_{(\mu\nu)} = \frac{1}{4} (P_{\mu\nu} + P_{\nu\mu}) \) is its symmetrisation, \( P_{[\mu\nu]} = \frac{1}{2} (P_{\mu\nu} - P_{\nu\mu}) \) is its antisymmetrisation, \( D_\mu X_\nu = \Omega^\dagger_\mu X_\nu \Omega_\mu - X_\nu \) is a discretised covariant derivative, and \( F_{\mu\nu} = D_\mu X_\nu - D_\nu X_\mu \) is a discretised field strength tensor. The stationarity condition for the configuration \( \Omega_\mu \) to be an extremum, \( \delta S_{\text{TEK}} = 0 \), then implies the equation:

\[
\delta S_{\text{TEK}} = - \sum_{\mu \neq \nu}^4 \text{Tr} (P_{\mu\nu} F_{\mu\nu}) = 0 \implies \sum_{\mu \neq \nu}^4 (\Omega_\nu P_{[\mu\nu]} \Omega^\dagger_\nu - P_{[\mu\nu]}) = 0 \tag{11}
\]

An obvious class of solutions to the stationarity condition above are the configurations satisfying \( P_{\mu\nu} \in Z_N \). In fact, van Baal gave strong arguments [21] for the conjecture that all stable extrema of the TEK action have this form. So let \( P_{\mu\nu} = z_{\mu\nu}(n - m) \), where \( n = n_{\mu\nu} \) is the twist-tensor and \( m \equiv m_{\mu\nu} \) is another integer-valued matrix. Expanding the TEK action around a configuration from this class reduces (10) to:

\[
S_{\text{TEK}} \approx 2N \sum_{\mu \neq \nu}^4 \sin^2 \left( \frac{\pi}{N} (n_{\mu\nu} - m_{\mu\nu}) \right) + \frac{1}{2} \sum_{\mu \neq \nu}^4 \cos \left( \frac{2\pi}{N} (n_{\mu\nu} - m_{\mu\nu}) \right) \text{Tr}(-F^2_{\mu\nu}) + O(X^3) \tag{12}
\]

The stability condition, \( \delta^2 S_{\text{TEK}} \geq 0 \), is equivalent to:

\[
\cos \left( \frac{2\pi}{N} (n_{\mu\nu} - m_{\mu\nu}) \right) \geq 0, \quad \forall \mu, \nu \tag{13}
\]

since \( \text{Tr}(-F^2_{\mu\nu}) = O(X^2) \) is always non-negative.

Let us consider particular solutions to the stationarity condition \( P_{\mu\nu} = z_{\mu\nu}(n - m) \), i.e. \( z_{\mu\nu}(m)\Omega^\dagger_{\mu\nu} \Omega_{\mu\nu} = I \). For the case \( m_{\mu\nu} = n_{\mu\nu} \), the unique solution to \( P_{\mu\nu} = I \) is obviously the twist-eating configuration, the classical vacuum of the TEK model. Now consider the case when \( m_{\mu\nu} \sim O(\sqrt{N}) \) and (13) is satisfied. In this case the solutions to \( P_{\mu\nu} = z_{\mu\nu}(n - m) \) (if they exist) are called fluxons [21]. Fluxons survive the large \( N \) limit, being stable minima of \( S_{\text{TEK}} \), and share some properties with twist-eaters, like tracelessness; therefore, fluxons are not expected to contribute to the breaking of the \( Z_N^4 \) symmetry. Next consider the case \( m_{\mu\nu} = 0 \). The solution to \( P_{\mu\nu} = z_{\mu\nu}(n) \) is the set of all \( SU(N) \) diagonal matrices. In general, these matrices have non-zero trace, and so they are good candidates for being the stable extrema that break the \( Z_N^4 \) symmetry. As a consequence of the first term on the l.h.s. of (12), the action of the diagonal matrices grows proportionally with \( N \), i.e. in the partition function such diagonal matrices appear to be suppressed relative to twist eaters by a factor of \( e^{-bN} \). While naively this appears to be a large suppression, in a system with \( O(N^2) \) degrees of freedom this is not so. Quantum fluctuations will naturally contribute terms of \( O(N^2) \) to the effective action which can easily overwhelm the difference in the classical action. At fixed \( N \), for large enough \( b \), the action difference eventually dominates; but if the barrier between these diagonal fields and the twist-eating fields has a natural \( O(N^2) \) value, then the tunnelling to the twist-eater phase would be suppressed by a factor like \( e^{-bN^2} \) and would essentially never occur in a finite simulation – just as we appear to see in Figs. 3, 6 and 7. Such a barrier
will be related to a maximum or saddle-point of the TEK action whose action grows $\propto N^2$. Possible candidates for such maxima would be configurations satisfying $P_{\mu\nu} = z_{\mu\nu}(n - m)$, with $m_{\mu\nu} \sim O(N)$ in such a way that the condition (13) is violated.

At the classical level, the set of all diagonal SU($N$) matrices form a degenerate set of extrema of the TEK action. However, if we consider the effect of quantum fluctuations, we can easily see that the degeneracy is lifted and only the elements of $Z_N$ become the true extrema. This is so because, for arbitrary $X_\mu$, $D_\mu X_\mu = 0$ for elements of $Z_N$ while $D_\mu X_\mu \neq 0$ for other diagonal matrices. Consequently the $O(X^2)$ term in (12), which is $\propto \text{Tr}(-F^2_{\mu\nu})$, vanishes in the former case and is always positive in the latter. Therefore, when quantum fluctuations $X_\mu$ are taken into account, the action of a general diagonal matrix becomes larger than the action of an element of $Z_N$, if $b$ is large enough for $O(X^3) \ll O(X^2)$, and the degeneracy is lifted. Therefore, the elements of $Z_N$ provide stable vacua that break the $Z^4_N$ symmetry. This mechanism is in fact the same as that which leads to the $Z^4_N$ symmetry breaking in the original untwisted Eguchi-Kawai model [22]; it is also similar to mechanisms that generate ‘order from disorder’ in condensed matter physics.

Let us therefore compare the contributions to the TEK action of the fluctuations $X_\mu$ around twist-eaters ($\Gamma_\mu$) and around elements of $Z_N$ ($z_\mu$),

$$S_{\text{TEK}}(n; [\Gamma]) = \sum_{\mu > \nu} \text{Tr}(-F^2_{\mu\nu}) + O(X^3) = O(X^2) + O(X^3) \quad (14)$$

$$S_{\text{TEK}}(n; [z]) = 24N \sin^2\left(\frac{\pi}{\sqrt{N}}\right) + O(X^3) = O(X^0) + O(X^3) \quad (15)$$

We might speculate that the different sensitivities with respect to the fluctuations $X_\mu$ might be at the origin of the spontaneous breaking of the $Z^4_N$ symmetry at intermediate couplings. In this regime the fluctuations are reasonably large, but the Haar measure does not dominate yet. The contribution to the action of the fluctuations around twist-eaters is larger ($\sim O(X^2)$) than for elements of $Z_N$ ($\sim O(X^3)$), so it might be possible that at a certain critical value of the coupling $b$ the fluctuations around elements of $Z_N$ will cost less action than the fluctuations around twist-eaters, and consequently the elements of $Z_N$ become preferred over the twist-eaters. This argument is, however, very speculative, since it uses perturbation theory to justify effects that are observed at intermediate-to-strong couplings, where phase transitions are also involved.

Returning to the sequence of transitions observed in the large $N$ TEK model, as shown in Fig. 7 for $N = 144$, we note their qualitative resemblance to the transitions described in [14] that occur on conventional symmetric lattices for large $N$ as $b \to \infty$. This might suggest a natural identification between the $Xc$ phases and our $Z^k_N$ phases, namely $Xc \equiv Z^4_N-X$. However, there are at least two fundamental differences that undermine such an identification. First, the reduced model version of the so-called 0c phase (the confined phase on the physical, weak coupling side of the bulk transition, which is situated between the strong coupling random phase and the 1c phase where Polyakov loops in one direction have a non-zero expectation value) was never observed in any of the simulations. Second, and more importantly, our transitions do not seem to scale with $L = \sqrt{N}$ in the same way as do the ones in [13]. That is
to say, the critical couplings of the latter increase with the lattice size $L$ in such a way that the $Xc$ phases have a proper continuum limit. However, according to Tables 2 and 3 our transitions do not show any sign of the scaling with $N$ that would be expected if we were seeing a physical transition on the equivalent $L = \sqrt{N}$ lattice that occurred at a fixed value of the physical length $l = aL = a\sqrt{N}$.

That is to say, they appear to be lattice transitions rather than physical transitions. Moreover all our $Z_N^k$ phases are situated below the value of $b \approx 0.36$ expected for the bulk transition, and not above (as in the case of the $Xc$ phases), and finally their breaking of the centre symmetry undermines any connection with a supposedly equivalent conventional lattice. On the other hand, if we forget the Eguchi-Kawai correspondence (which, after all, we have shown is invalid in this range of $b$) and consider the TEK model simply as a $1^4$ lattice, then the links are Polyakov loops, and the identification with the $Xc$ phases of [14] is essentially trivial. This suggests looking at the untwisted $1^4$ lattice, where such phases are known to occur [14]. In Fig.11 we show a plot of this phase structure that we have calculated for $N = 81$. The phase structure we see is essentially identical to that obtained in the large $N$ TEK model with a hot start, as shown in Fig. 7, and the interpretation of the phases appears to be identical. It appears that for large enough $N$, once quantum fluctuations are sufficiently important, the effect of the twist factor in the symmetric TEK model, $z_{\mu\nu}(n) = e^{2\pi i/\sqrt{N}}$, vanishes for all practical purposes.

5 Conclusions

We have analysed numerically the properties of four-dimensional SU($N$) TEK models with a conventional symmetric twist [9]. The gauge groups used in our simulations ranged from $N = 25$ to $N = 144$.

For $N \leq 81$, the properties of the TEK model are essentially the same as those obtained in earlier simulations. As we vary the inverse bare 't Hooft coupling, $b = 1/g^2 N$, we encounter a single first-order phase transition separating the strong and weak coupling regions and, crucially, the centre symmetry which is needed for the Eguchi-Kawai correspondence to be valid, is indeed maintained at all $b$. However our greater accuracy also makes it clear that there is no confining phase on the physical weak coupling side of the bulk transition, in contrast to some claims in the early literature. We pointed out that this should not occasion great surprise since we know, from conventional calculations of the deconfining temperature $T_c$ in SU($N$) gauge theories [4], that the length of the equivalent lattice, $l = aL = a\sqrt{N}$, is probably less than $1/T_c$ for all $b$ in the weak coupling region.

For $N \geq 100$, on the other hand, we observe a clear breaking of the $Z_N^4$ symmetry over a wide range of intermediate couplings, challenging the conventional wisdom that this symmetry is unbroken for all couplings. We observe that the breaking of the $Z_N^4$ symmetry occurs step by step in a sequence of transitions, with intermediate phases in which one or more of the $Z_N$ symmetries are unbroken. When, for our largest values of $N$, we increase $b$ from $b \sim 0$ the final weak coupling phase appears to be one in which all four centre symmetries are spontaneously broken. At large enough $b$ it becomes clear that the symmetry breaking fields are nothing but fluctuations around centre elements. These are stable minima of the twisted lattice action, as
originally discussed by van Baal [21]. Since the action of these ‘generalised fluxons’ is only $O(N)$ higher than that of the absolute minimum (the twist eater), it is not surprising that it does not tunnel to the latter since the generic barrier height will be $O(N^2)$ in an SU($N$) gauge theory.

The sequence of symmetry breaking transitions in the TEK model at very large $N$ is essentially identical to that which occurs in the original untwisted Eguchi-Kawai model (apart from a shift in the location of the transitions). It is as if the effect of the twist, a phase factor $e^{2\pi i/\sqrt{N}}$ associated with each plaquette, becomes irrelevant for large enough $N$. Naively one might expect this to happen once the phase is small, i.e. for $2\pi/\sqrt{N} \ll 1$ or, equivalently, $N \gg 4\pi^2$. This provides a very large scale for what we mean by $N$ being ‘large’, and might explain why new effects arise for $N > 100$ in the TEK model, in contrast to conventional calculations where finite $N$ corrections are, typically, already very small for $N = 6$ or $N = 8$.

One might still hope to encounter physically interesting phases by beginning at $b = \infty$ with a twist-eating field configuration and then decreasing $b$. If the first transition to strong coupling were to be at $b \simeq 0.360$, as it is for $N \leq 81$, then the $12^4$ lattice that is equivalent to the SU(144) TEK model, should be large enough to be in the physical confining phase before reaching that value of $b$. Frustratingly, however, for SU(144) the critical value of $b$ increases, as we see in Tables 2 and 3, so that there is once again no window for any weak coupling confining physics. Whether this will continue if $N$ is increased further is obviously a very interesting question.

On conventional symmetric $L^4$ lattices at large $N$ there are additional physical phases for $aL \leq 1/T_c$, corresponding to the Polyakov loops acquiring non-zero expectation values one after the other [14]. The first of these, as one increase $b$, is simply the usual deconfining transition and will occur at $aL = 1/T_c$. The subsequent transitions are expected to occur at higher $b$ [14] although there are unfortunately no values in the literature for the corresponding critical values of $b$. It is not clear why we have not observed any of these transitions when we reduce $b$ from the twist-eating start, particularly in the $N \leq 81$ case where the Eguchi-Kawai correspondence should hold for all $b$. It may be that there is a large metastability so that the twist-eating configuration tunnels to the bulk phase (for $N \leq 81$) or to the symmetry broken phase (for larger $N$) before it can tunnel to one of these physical ‘small volume’ phases. Since the Eguchi-Kawai equivalence only holds for the planar limit, one cannot expect it to extend to the details of transitions and metastabilities which are typically driven by effects that are exponentially small in $N$, and so there may be some surprises here.

In this paper we have focused on two particularly interesting results from our study of the twisted Eguchi-Kawai model: the symmetry breaking that invalidates the Eguchi-Kawai planar correspondence over a wide range of couplings, and the apparent inaccessibility of any physical phase even at $N = 144$. We have been able to make substantial analytic progress in understanding these phenomena, using configurations that are a generalisation of the ‘fluxons’ studied long ago by van Baal. Even if from a practical point of view our results undermine the utility of the most conventional TEK models, it would be theoretically interesting to pursue the numerical calculations to somewhat larger $N$, so as to obtain some insight into what happens at asymptotic $N$, and to develop further our analytic understanding of the model.
This we hope to do elsewhere where we will also present our results for TEK models with anisotropic lattice spacings and for the partially reduced models that one would need to use for calculations of the planar mass spectrum.

**Note added:** During the course of this work we became aware in [23] of an unpublished talk [24] in which observations were made about centre symmetry breaking at intermediate couplings in TEK models.

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| \( N \) | run | \( n_{\text{heats}} \) | \( n_{\text{sweeps}} \) |
|-----|-----|-----|-----|
| 64  | run A | 200 | 6000 |
| 81  | run A | 200 | 6000 |
|     | run B | 200 | 6000 |
| 100 | run A | 200 | 6000 |
|     | run B | 200 | 6000 |
| 121 | run B | 100 | 2000 |
| 144 | run A | 100 | 1000 |
|     | run B | 100 | 1000 |

Table 1: Parameters of the numerical simulations performed at each value of the inverse ’t Hooft coupling, \( b \); \( N \) is the size of the gauge group; \( n_{\text{heats}} \) is the number of thermalising sweeps; \( n_{\text{sweeps}} \) is the number of sweeps used for measurements.

| \( N \) | \( Z_{N}^{4(r)} \) | \( Z_{N}^{3} \) | \( Z_{N}^{2} \) | \( Z_{N}^{1} \) | \( Z_{N}^{0} \) |
|-----|-----|-----|-----|-----|-----|
| 100 | run A | 0.305(5) | — | — | — |
|     | run B | 0.300(5) | — | — | — |
| 121 | run B | 0.275(5) | 0.350(5) | 0.44(1) | 0.445(5) |
| 144 | run A | 0.26(2) | 0.325(5) | 0.405(15) | 0.44(2) |
|     | run B | 0.2575(25) | 0.3275(25) | 0.3975(25) | 0.425(5) |

Table 2: Critical values of the inverse ’t Hooft coupling, \( b \), associated with the breaking of one (or more) \( Z_{N} \) symmetries of the TEK model. This table refers to simulations that started from randomised configurations. \( Z_{N}^{k} \) refers to a phase of the TEK model with \( k \) unbroken directions, while the arrows mean transitions between those phases; a numerical value spanning multiple columns corresponds to the critical coupling associated with the simultaneous breaking of more than one \( Z_{N} \) symmetry.

| \( N \) | \( Z_{N}^{4(r)} \) | \( Z_{N}^{3} \) | \( Z_{N}^{2} \) | \( Z_{N}^{1} \) | \( Z_{N}^{0} \) |
|-----|-----|-----|-----|-----|-----|
| 100 | run A | 0.27(1) | 0.350(5) | — | — |
|     | run B | 0.275(5) | 0.350(5) | — | — |
| 121 | run B | 0.250(5) | 0.325(5) | 0.360(5) | — |
| 144 | run A | 0.23(1) | 0.27(1) | 0.325(5) | 0.385(10) |
|     | run B | 0.235(5) | 0.275(5) | 0.325(5) | 0.370(5) |

Table 3: Critical values of the inverse ’t Hooft coupling, \( b \), associated with the breaking (or restoration) of one (or more) \( Z_{N} \) symmetries of the TEK model. This table refers to simulations that started from twist-eating configurations. \( Z_{N}^{k} \) refers to a phase of the TEK model with \( k \) unbroken directions, while the arrows mean transitions between those phases; a numerical value spanning multiple columns corresponds to the critical coupling associated with the simultaneous breaking (or restoration) of more than one \( Z_{N} \) symmetry.
Figure 1: Average value of the real part of the plaquette, $\langle \text{Re } u_\mu \rangle$, in the SU(64) TEK model versus the inverse bare 't Hooft coupling, $b$. The squares ($\square$) represent the $N \to \infty$ extrapolation of the average plaquette in Wilson’s lattice gauge theory. The solid line (—) and the dashed line (---) represent the strong-coupling and the 3-loop weak-coupling expansions of Wilson’s lattice gauge theory, respectively.

Figure 2: Average value of the real and imaginary parts of traced link variables, $\frac{1}{N} \text{Tr } U_\mu$, in the SU(64) TEK model versus the inverse bare 't Hooft coupling, $b$ (from a cold start simulation).
Figure 3: Average value of the real part of the plaquette, $\langle \text{Re} \, u_p \rangle$ in the SU(100) TEK model versus the inverse bare 't Hooft coupling, $b$. The squares ($\square$) represent the $N \to \infty$ extrapolation of the average plaquette in Wilson’s lattice gauge theory.

Figure 4: Average value of the real and imaginary parts of traced link variables, $\frac{1}{N} \text{Tr} \, U_\mu$, in the SU(100) TEK model versus the inverse bare 't Hooft coupling, $b$ (from a cold start simulation); the dot-dashed line (\ldots) represents the average real plaquette.
Figure 5: Average value of the real and imaginary parts of traced link variables, $\frac{1}{N} \text{Tr} U_\mu$, in the SU(100) TEK model versus the inverse bare 't Hooft coupling, $b$ (from a hot start simulation); the dot-dashed line (− · − ) represents the average real plaquette.

Figure 6: Average value of the real part of the plaquette, $\langle \text{Re } u_p \rangle$, in the SU(144) TEK model versus the inverse bare 't Hooft coupling, $b$. The squares (□) represent the $N \to \infty$ extrapolation of the average plaquette in Wilson's lattice gauge theory. The solid line (−) and the dashed line (- - -) represent the strong-coupling and the 3-loop weak-coupling expansions of Wilson's lattice gauge theory, respectively.
Figure 7: Average value of the real part of the plaquette, $\langle \text{Re } u_\mu \rangle$, in the SU(144) TEK model versus the inverse bare 't Hooft coupling, $b$ (from a hot start simulation).

Figure 8: Average value of the magnitude of traced link variables, $\langle \frac{1}{N} \text{Tr } U_\mu \rangle$, in the SU(144) TEK model versus the inverse bare 't Hooft coupling, $b$ (from a hot start simulation).
Figure 9: Typical profile of the eigenvalue densities of the link variables, $U_\mu$, in the several $Z_N^4$-breaking phases of the SU(144) TEK model; $\alpha$ labels the eigenvalues of SU($N$) matrices, which are pure phases of the form $e^{i\alpha}$.

Figure 10: Eigenvalue densities of a particular link variable at an inverse bare 't Hooft coupling of $b = 2.00$, sitting at two different phases of the SU(144) TEK model, namely the twist-eater phase ($Z_N^{4(t)}$, in the left) and the completely asymmetric phase ($Z_N^0$, in the right).
Figure 11: Average value of the real part of the plaquette, $\langle \text{Re } u_p \rangle$, in the original untwisted SU(81) Eguchi-Kawai model versus the inverse bare 't Hooft coupling, $b$. The nomenclature used for the different phases is the same as in [14].