LIMITING SOLUTIONS OF SEQUENCES OF GLOBALLY
REGULAR AND BLACK HOLE SOLUTIONS IN SU(N)-EYMD
THEORIES

A. SOOD, B. KLEIHAUS, J. KUNZ
Fachbereich Physik, Universität Oldenburg, Postfach 2503
D-26111 Oldenburg, Germany

Static spherically symmetric solutions in SU(N)-EYM and EYMD theories are classified
by the node numbers of their non-trivial gauge field functions. With increasing node
numbers, the solutions form sequences, tending to limiting solutions. The limiting solu-
tions are based on subalgebras of $su(N)$, consisting of a neutral non-abelian part and a
charged abelian part, belonging to the Cartan subalgebra.

We consider the $SU(N)$ Einstein-Yang-Mills action

$$S = S_G + S_M = \int L_G \sqrt{-g} d^4x + \int L_M \sqrt{-g} d^4x$$  \hspace{1cm} (1)

with

$$L_G = \frac{1}{16\pi G} R, \quad L_M = -\frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}),$$  \hspace{1cm} (2)

and with field strength tensor $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ie[A_{\mu}, A_{\nu}]$, gauge field $A_{\mu} = \frac{1}{2} \lambda^a A_{\mu}^a$ and gauge coupling constant $e$.

We employ Schwarzschild-like coordinates and adopt the spherically symmetric
metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -A^2 dt^2 + N^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (3)

with the metric functions $A(r)$ and $N(r) = 1 - \frac{2m(r)}{r}$.

The static spherically symmetric ansätze for the gauge field $A_{\mu}$ of $SU(N)$ EYM
theory are based on the $su(2)$ subalgebras of $su(N)$. Considering only the embed-
ding of the $N$-dimensional representation of $su(2)$, the gauge field ansatz is

$$A_{\mu}^{(N)} dx^\mu = \frac{1}{2e} \begin{pmatrix} (N-1) \cos \theta d\phi & \omega_1 \Theta & 0 & \cdots & 0 \\ \omega_1 \Theta & (N-3) \cos \theta d\phi & \omega_2 \Theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \omega_{N-1} \Theta & (1-N) \cos \theta d\phi \end{pmatrix}$$  \hspace{1cm} (4)

with $\Theta = i d\theta + \sin \theta d\phi$, and $A_0 = A_r = 0$. The ansatz contains $N-1$ matter field
functions $\omega_j(r)$.

We introduce the dimensionless coordinate $x = er/\sqrt{4\pi G}$, the dimensionless
mass function $\mu = em/\sqrt{4\pi G}$, and the scaled matter field functions $u_j = \omega_j/\sqrt{\gamma_j}$
with $\gamma_j = j(N-j)$. Considering the boundary conditions, globally regular solutions
must satisfy at the origin $\mu(0) = 0$ and $u_j(0) = \pm 1$. Black hole solutions with a
regular horizon with radius $x_H$ must satisfy at the horizon $N(x_H) = 0$ and

$$N' u_j' + \frac{1}{2x^2}(\gamma_{j+1} u_{j+1}^2 - 2\gamma_j u_j^2 + \gamma_{j-1} u_{j-1}^2 + 2)u_j \bigg|_{x_H} = 0.$$  \hspace{1cm} (5)
For asymptotically flat solutions the metric functions \( A \) and \( \mu \) must approach constants at infinity. We choose \( A(\infty) = 1, \mu(\infty) \) is the mass of the solutions. In neutral solutions the gauge field functions \( u_j \) satisfy \( u_j(\infty) = \pm 1 \), i.e. the gauge field approaches a vacuum configuration.

All \( N-1 \) gauge field functions are non-trivial for neutral globally regular and black hole solutions. The solutions are labelled by the node numbers \( n_j \) of the functions \( u_j \). When the node numbers of one or more gauge field functions tend to infinity, the solutions approach limiting solutions, which carry a magnetic charge \( P \).

Considering black hole solutions with \( x_H > P \) (or the exterior part of the solutions with \( x_H < P \)), we observe that in these limiting solutions the corresponding gauge field functions are identically zero. When one or more of the \( N-1 \) gauge field functions are identically zero, magnetically charged solutions are obtained. To classify the charged black hole solutions obtained within the ansatz (4), let us first assume that precisely one gauge field function is identically zero, \( \omega_{j_1} \equiv 0 \). The ansatz then reduces to

\[
A_\mu^{(N)} \ dx^\mu = \begin{pmatrix} A_{\mu}^{(j_1)} \ dx^\mu \\ A_\mu^{(N-j_1)} \ dx^\mu \end{pmatrix} + \frac{\cos \theta d\phi}{2e} \begin{pmatrix} (N-j_1) \mathbf{1}_{j_1} \\ -(j_1-1)(N-j_1) \mathbf{1}_{j_1} \end{pmatrix}
\]

(6)

where \( A_\mu^{(N)} \) denotes the non-abelian spherically symmetric ansatz for the \( su(\bar{N}) \) subalgebra of \( su(N) \) (based on the \( \bar{N} \)-dimensional embedding of \( su(2) \)), and the second term represents the abelian ansatz for the element \( h \) of the Cartan subalgebra of \( su(N) \). The full system of equations thus consists of two non-abelian and one abelian subsystems, coupled only via the metric functions. The non-abelian parts of the solutions corresponding to \( su(\bar{N}) \) subalgebras are neutral with \( \tilde{u}_i(\infty) = \pm 1 \). The charge of the solutions is carried by the abelian subsystem, belonging to the Cartan subalgebra, and given by

\[
P^2 = \frac{1}{2} \Tr \ h^2 .
\]

(7)

By expanding the element \( h \) in terms of the basis \( \{ \lambda_{n^2-1} \mid n = 2, \ldots, N \} \), the charge can also be directly read off the expansion coefficients (see Table 1).

By applying these considerations again to the subalgebras \( su(N) \) of eq. (3), we obtain the general case for \( SU(N) \) EYM theory, where \( M \) gauge field functions are identically zero, \( \omega_{j_m} = 0 \), \( j_m \in \{ j_1, j_2, \ldots, j_M \} \), \( j_1 < j_2 < \cdots < j_M \). The charge is again carried by the abelian subsystem, belonging to the Cartan subalgebra, and given by

\[
P^2 = \frac{1}{2} \sum_{m=1}^{M} (N-j_m)(N-j_{m-1})(j_m-j_{m-1}) \ (j_0 = 0) .
\]

(8)

In the special case where all gauge field functions are identically zero, an embedded RN solution is obtained with charge \( P^2 = (N-1)N(N+1)/6 \). RN black hole solutions exist only for horizon radius \( x_H \geq P \), and the extremal RN solution has \( x_H = P \). The same is true for charged non-abelian black holes. For extremal non-abelian black hole solutions \( N' = 0 \), leading to \( \tilde{u}_i(x_H) = \pm 1 \). There are no such globally regular charged solutions.
We demonstrate the above general considerations for SU(5) EYM theory. Presenting all inequivalent cases (within the ansatz (4)), the classification of the charged SU(5) EYM black hole solutions is given in Table 1.

| #  | $u_j, j = 1-4$ | $P^2$ | non-abelian subalgebra | Cartan subalgebra |
|----|----------------|-------|------------------------|------------------|
| 0  | 0 0 0 0       | 20    | $su(2)$ 0              | $\lambda_3$ $P_3$ $\lambda_8$ $P_8$ $\lambda_{15}$ $P_{15}$ $\lambda_{24}$ $P_{24}$ |
| 1a | $u_1$ 0 0 0   | 19    | $su(2)$ 0              | $\lambda_3$ $P_3$ $\lambda_8$ $P_8$ $\lambda_{15}$ $P_{15}$ $\lambda_{24}$ $P_{24}$ |
| 2a | $u_1$ $u_2$ 0 | 16    | $su(3)$ 0              | $\lambda_3$ $P_3$ $\lambda_8$ $P_8$ $\lambda_{15}$ $P_{15}$ $\lambda_{24}$ $P_{24}$ |
| 2b | $u_1$ 0 $u_3$ | 18    | $su(2) \oplus su(2)$ 0 | $\lambda_3$ $P_3$ $\lambda_8$ $P_8$ $\lambda_{15}$ $P_{15}$ $\lambda_{24}$ $P_{24}$ |
| 3a | $u_1$ $u_2$ $u_3$ | 10 | $su(4)$ 0               | $\lambda_3$ $P_3$ $\lambda_8$ $P_8$ $\lambda_{15}$ $P_{15}$ $\lambda_{24}$ $P_{24}$ |
| 3b | $u_1$ $u_2$ 0 $u_4$ | 15 | $su(3) \oplus su(2)$ 0 | $\lambda_3$ $P_3$ $\lambda_8$ $P_8$ $\lambda_{15}$ $P_{15}$ $\lambda_{24}$ $P_{24}$ |

Table 1
Charged black hole solutions of SU(5) EYM theory ($P_{n^2-1} = \sqrt{n(n-1)/2}$).

Like the neutral solutions of SU(N) EYM theory, the charged solutions are labelled by the node numbers $n_i$ of their (non-vanishing) gauge field functions $\bar{u}_i$. When the node numbers of one or more gauge field functions tend to infinity, the solutions approach limiting solutions with higher charge. In Fig. 1 we present the lowest odd solutions of case 3a for the sequence $(n,0,0)$ and extremal horizon $x_H = \sqrt{10}$, also in the black hole interior ($x < x_H$).

Figure 1: The matter functions for case 3a of Table 1

The above classification remains valid for SU(N) EYMD theory. However, the charged non-abelian EYMD black hole solutions exist for arbitrary horizon radius $x_H > 0$.

References
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