DELAUNAY ENDS OF CONSTANT MEAN CURVATURE SURFACES

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Abstract. The generalized Weierstrass representation is used to analyze the asymptotic behavior of a constant mean curvature surface that arises locally from an ordinary differential equation with a regular singularity. We prove that a holomorphic perturbation of an ODE that represents a Delaunay surface generates a constant mean curvature surface which has a properly immersed end that is asymptotically Delaunay. Furthermore, that end is embedded if the Delaunay surface is unduloidal.

Introduction

Delaunay surfaces play a prominent role in the theory of non-compact complete constant mean curvature (CMC) surfaces because they constitute the simplest possible end behavior. A famous result by Korevaar, Kusner and Solomon [17], building on results of Meeks [20], asserts that a properly embedded annular end of a CMC surface is a Delaunay end. The study of Delaunay ends by the conjugate surface methods of Grosse-Brauckmann, Kusner, and Sullivan [8, 9] require the additional assumption of Alexandrov embeddedness, and are limited to embedded (unduloidal) Delaunay ends. The gluing techniques of Mazzeo and Pacard [18] are limited to attaching Delaunay ends with small asymptotic necksizes. The methods used in this paper, based on the generalized Weierstrass representation of Dorfmeister, Pedit and Wu [5], provide a means to study both embedded unduloidal and non-Alexandrov-embedded nodoidal type ends of arbitrary asymptotic necksize.

The generalized Weierstrass representation describes constant mean curvature immersions locally via holomorphic potentials. The relation between the potential and the immersion involves a loop group valued differential equation, a loop group factorization and a Sym-Bobenko type formula [26, 2]. Hence the method provides only an indirect relation between the geometric properties of the induced immersion and its potential. Nevertheless, the method has been useful in proving the existence of many new classes of non-compact constant mean curvature surfaces with non-trivial topology. Particular progress has been made when the surface is homeomorphic to an \( n \)-punctured Riemann sphere [13, 14, 24, 25]. In this case, the punctures correspond to poles of the potential. Graphics of these surfaces [23] have long suggested that simple poles with appropriate residues yield the asymptotic end behavior of a Delaunay surface. We prove this correlation between simple poles and Delaunay ends.

More specifically, given a holomorphic potential \( A \frac{dz}{z} \) of a Delaunay surface, consider a holomorphic perturbation \( \xi = A \frac{dz}{z} + O(z^0)dz \). Our main result, stated precisely as Theorem 3.5 and generalized in Theorem 5.9 is:

**Theorem.** An annular constant mean curvature immersion induced by a holomorphic perturbation of a Delaunay potential is \( C^\infty \)-asymptotic to a half-Delaunay surface. In particular, it is properly immersed. Moreover, if the half-Delaunay surface is embedded, then the end of the immersion is properly embedded.

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The surface induced by a perturbed Delaunay potential may gain topology or geometric complexity — see for example the $n$-noids [25, 21, 24] and higher genus examples with ends [12]. Nonetheless at $z = 0$, the perturbed surface is asymptotic to the underlying Delaunay surface.

The convergence of the surfaces is obtained by showing that their moving frames and metrics converge. More specifically, let $\Phi_0$ and $\Phi$ be the respective solutions to the ODEs $d\Phi_0 = \Phi_0 A dz/z$ and its perturbation $d\Phi = \Phi \xi$. The convergence of the ratio of $\Phi$ to $\Phi_0$ is shown using the holomorphic gauge relating them at the regular singularity $z = 0$. The periodicity of the Delaunay surface provides growth rate estimates on the positive part of $\Phi_0$ by Floquet analysis. This leads to the convergence of their unitary and positive factors, in turn implying $C^1$-convergence of the surfaces. A bootstrap argument strengthens this to $C^\infty$-convergence.

In Part 2 we deal with the situation in which the initial condition to the ODE does not extend holomorphically from the $r$-circle to the unit circle, even though the monodromy of the solution is still unitary. We show that in this setting the solution has acquired singularities that arise from Bianchi-Bäcklund transforms. Thus the second part accommodates the additional singularities that appear from dressing by simple factors, and proves that adding bubbles to a surface with a Delaunay end preserves this Delaunay end.

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**Figure 1.** A CMC immersion of the six-punctured sphere with asymptotically Delaunay ends and pyramidal symmetry [24]. Five of the six ends are unduloidal; the sixth is nodoidal with large negative weight. This image was created with CMCLab [22], a freely available software implementation of the generalized Weierstrass representation.
The generalized Weierstrass representation

The \( r \)-Iwasawa factorization. We will use the following subsets of \( \mathbb{C} \):

\[
\begin{align*}
S^1 &= \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} \quad \text{and} \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \\
C_r &= \{ \lambda \in \mathbb{C} \mid |\lambda| = r \}, \quad r \in (0, 1], \\
D_r &= \{ \lambda \in \mathbb{C} \mid |\lambda| < r \}, \quad D_r^* = D_r \setminus \{0\}, \quad r \in (0, 1], \\
\mathcal{A}_{s,r} &= \{ \lambda \in \mathbb{C} \mid s < |\lambda| < r \}, \quad 0 < s < r, \\
\mathcal{A}_r &= \mathcal{A}_{r,1/r}, \quad r \in (0, 1).
\end{align*}
\]

Given a domain \( \mathcal{U} \subset \mathbb{C} \setminus \{0\} \) which is invariant under the map \( \lambda \mapsto 1/\lambda \), and a holomorphic map \( X: \mathcal{U} \to M_{2 \times 2} \mathbb{C} \), define the holomorphic map \( X^*: \mathcal{U} \to M_{2 \times 2} \mathbb{C} \) by

\[
X^*(\lambda) := X(1/\lambda)^t.
\]

We will use the following loop groups:

- For \( r \in (0, 1] \), \( \Lambda_r \mathbb{SL}_2 \mathbb{C} \) is the group of analytic maps \( \mathcal{C}_r \to \mathbb{SL}_2 \mathbb{C} \).
- For \( r \in (0, 1) \), \( \Lambda^r_1 \mathbb{SL}_2 \mathbb{C} \subset \Lambda_r \mathbb{SL}_2 \mathbb{C} \) is the subgroup of loops each of which is the boundary of a holomorphic map \( Y: \mathcal{A}_r \to \mathbb{SL}_2 \mathbb{C} \) satisfying the condition \( Y^* = Y^{-1} \).
- For \( r \in (0, 1) \), \( \Lambda^0_1 \mathbb{SL}_2 \mathbb{C} \subset \Lambda_1 \mathbb{SL}_2 \mathbb{C} \) is the subgroup of loops \( X \) satisfying the condition \( X^* = X^{-1} \).
- For \( r \in (0, 1) \), \( \Lambda^r_{\text{pos}} \mathbb{SL}_2 \mathbb{C} \subset \Lambda_r \mathbb{SL}_2 \mathbb{C} \) is the subgroup of loops for which each loop is the boundary of a holomorphic map \( \mathcal{D}_r \to \mathbb{SL}_2 \mathbb{C} \).
- Let \( \mathcal{T} \subset \mathbb{SL}_2 \mathbb{C} \) denote the group of upper triangular matrices whose diagonal elements are in \( \mathbb{R}_{>0} \). For \( r \in (0, 1) \), \( \Lambda^r_1 \mathbb{SL}_2 \mathbb{C} \subset \Lambda^0_1 \mathbb{SL}_2 \mathbb{C} \) is the subgroup of loops \( X \) such that \( X(0) \in \mathcal{T} \).
- For \( r \in (0, 1) \), \( \Lambda^r_1 \mathbb{SL}_2 \mathbb{C} \subset \Lambda_r \mathbb{SL}_2 \mathbb{C} \) is the group of analytic maps \( X \in \Lambda_r \mathbb{SL}_2 \mathbb{C} \) such that \( X \) is the boundary of a holomorphic map \( \mathcal{A}_r \to \mathbb{SL}_2 \mathbb{C} \).
- For \( r \in (0, 1) \), \( \Lambda^r_{\text{pos}} \mathbb{SL}_2 \mathbb{C} \subset \Lambda^r_1 \mathbb{SL}_2 \mathbb{C} \) is the group of analytic maps \( X \in \Lambda_r \mathbb{SL}_2 \mathbb{C} \) such that \( X \) is the boundary of a meromorphic map \( \mathcal{A}_r \to \mathbb{SL}_2 \mathbb{C} \) satisfying \( X^* = X^{-1} \) away from its poles. For \( r = 1 \), \( \Lambda^1_{\text{pos}} \mathbb{SL}_2 \mathbb{C} = \Lambda^1_1 \mathbb{SL}_2 \mathbb{C} \).

For \( k \in \mathbb{Z}_{\geq 0} \), we define the \( C^k \)-topology on each of these loop groups with respect to the loop parameter \( \lambda \) in their respective domains \( \mathcal{C}_r \), \( \mathcal{A}_r \) or \( \mathcal{D}_r \). The \( C^\infty \)-topology is the intersection of the \( C^k \)-topologies. The multiplication map

\[
\Lambda^r_1 \mathbb{SL}_2 \mathbb{C} \times \Lambda^r_1 \mathbb{SL}_2 \mathbb{C} \to \Lambda_r \mathbb{SL}_2 \mathbb{C}
\]

is a \( C^\infty \) diffeomorphism \cite{19} \cite{4}. The unique factorization of a loop \( \Phi \in \Lambda_r \mathbb{SL}_2 \mathbb{C} \) into

\[
\Phi = \text{Uni}_r[\Phi] \cdot \text{Pos}_r[\Phi]
\]

is the \( r \)-Iwasawa factorization of \( \Phi \). We call \( \text{Uni}_r[\Phi] \) the \( r \)-unitary factor and \( \text{Pos}_r[\Phi] \) the \( r \)-positive factor of \( \Phi \). The QR-factorization is the Iwasawa factorization of constant loops \( \mathbb{SL}_2 \mathbb{C} \to \mathbb{SU}_2 \times \mathcal{T} \).

Dressing a loop \( X \in \Lambda_r \mathbb{SL}_2 \mathbb{C} \) by \( C \in \Lambda_r \mathbb{SL}_2 \mathbb{C} \) is left-multiplication of \( X \) by \( C \), followed by projection of the \( r \)-Iwasawa factorization to the unitary group. We denote the dressed loop by \( C \#_r X \) or \( \text{Uni}_r[CX] \).

The generalized Weierstrass representation. The generalized Weierstrass representation \cite{5} represents harmonic maps in terms of certain holomorphic 1-forms with values in a loop algebra (holomorphic potentials). This representation constructs all constant mean curvature (CMC) surfaces in the 3-dimensional Euclidean, spherical and hyperbolic spaceforms \cite{25}, and is as follows for Euclidean 3-space:

1. Let \( \Sigma \) be a Riemann surface. With \( r \in (0, 1] \), let \( \xi \) be an \( r \)-potential, that is, a \( \Lambda_r \mathbb{SL}_2 \mathbb{C} \)-valued differential form on \( \Sigma \) which is the boundary of a meromorphic differential
on $\mathcal{D}_r$, with a pole only at $\lambda = 0$, which is simple and is only in the upper-right entry. To avoid branch points in the induced surface, assume also that the coefficient of $\lambda^{-1}$ in the series expansion of the upper-right entry of $\xi$ in $\lambda = 0$ is never zero on $\Sigma$.

2. Let $\Phi$ be a solution to the ordinary differential equation $d\Phi = \Phi \xi$ on the universal cover $\tilde{\Sigma}$ of $\Sigma$. We call $\Phi$ the holomorphic frame.

3. Then $F = \text{Uni}_r[\Phi]$ is the extended frame of some CMC immersion $f : \tilde{\Sigma} \to \mathbb{R}^3$.

4. The Sym formula

$$f = \text{Sym}_r[\Phi] = -2H^{-1}F'F^{-1}$$

computes the associate family of CMC immersion $f$ with constant mean curvature $H \in \mathbb{R}^*$ from the frame $F$, with associate family parameter $\lambda \in S^1$. Here, the prime denotes differentiation with respect to $\theta$, where $\lambda = e^{i\theta} \in S^1$.

The immersion $f$ of $\tilde{\Sigma}$ descends to an immersion of $\Sigma$ at $\lambda = 1$ if and only if every element $M_F$ of the monodromy group of the extended frame $F$ satisfies

$$M_F(1) = \pm I \quad \text{and} \quad M_F'(1) = 0.$$

Given $d\Phi = \Phi \xi$ and an analytic gauge $g = g(z, \lambda) : \Sigma \to \Lambda_r\text{SL}_2\mathbb{C}$, then $\Psi := \Phi g$ satisfies the equation $d\Psi = \Psi \eta$, where

$$\eta = \xi g := g^{-1}\xi g + g^{-1}dg.$$

**Part 1. Delaunay asymptotics**

In Part 1 we show that under the assumption of unitary monodromy, an immersion constructed from a perturbed Delaunay potential is asymptotic to a Delaunay immersion.

In Part 2 we generalize this result to the setting of $\Lambda_r\text{SL}_2\mathbb{C}$ for arbitrary $r$.

**Outline of results**

Section 1 discusses the construction of the family of Delaunay immersions via the generalized Weierstrass representation. The generalized Weierstrass potential for the Delaunay immersion is of the form $Adz/z$, where $A$ is an $\text{sl}_2\mathbb{C}$-valued Delaunay residue given in Proposition 1.1. We compute the Iwasawa factorization of the holomorphic Delaunay frame $\exp(A\log z)$ in Theorem 1.5. This is an extension of a result in [25].

These factors are used in subsection 1.4 to compute the growth rate of the positive Iwasawa factor, and in section 4 to compute asymptotics of dressed Delaunay frames.

In subsection 1.4 we estimate the growth rate $\tau$ of the positive Iwasawa factor of $\exp(A\log z)$ as $z \to 0$ (Theorem 1.13). This growth rate result is used in subsection 2.4.

Given a holomorphic perturbation

$$\xi = A\frac{dz}{z} + O(z^0)dz$$

of the Delaunay potential $Adz/z$ producing a closed once-wrapped Delaunay surface, let $\Phi$ satisfy $d\Phi = \Phi \xi$ on an $r$-circle with unitary monodromy around $z = 0$.

In subsection 2.4 we show that the unitary and positive factors of $\Phi$ are asymptotic to those of a holomorphic Delaunay frame (Theorem 2.11).

In subsection 3.2 we use this convergence to obtain $C^\infty$-convergence of the CMC end to a Delaunay surface. The convergence of the positive part implies that of the metric, and together with the frame convergence, a bootstrap argument on the Gauss equation gives the $C^\infty$-convergence. If the base Delaunay surface is embedded, then the asymptotic end is embedded and has exponential convergence. These results are summarized in Theorem 3.5.

In Part 2 we generalize these results to the case of dressed holomorphic frames.
1. The Delaunay frame and its growth

1.1. The Delaunay residue. A Delaunay surface is described by a holomorphic potential on $\mathbb{C}^*$ of a very simple kind. Our description of these potentials is in the setting of $[25]$.

**Proposition 1.1.** Let $A : \mathbb{S}^1 \to \text{sl}_2\mathbb{C}$ be analytic. Then the following are equivalent:

(i) $A^* = A$, and $A$ is the boundary of a meromorphic map $A : \mathcal{D}_1 \to \text{sl}_2\mathbb{C}$ such that $A$ is holomorphic on $\mathcal{D}_1 \setminus \{0\}$, the upper-right entry of $A$ has a simple pole or no pole at 0, and the other entries of $A$ do not have poles at 0.

(ii) There exist $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$ such that

\[
A = \begin{pmatrix} c & a\lambda^{-1} + b \\ b + \pi\lambda & -c \end{pmatrix}.
\]

**Proof.** If $A$ is of the form (1.1), then (i) clearly holds.

Conversely, suppose (i) holds. Since $A$ extends meromorphically to $\mathcal{D}_1$, and $A^* = A$, then $A$ extends meromorphically to $\mathbb{CP}^1$. The entries of $A$ are then meromorphic functions on $\mathbb{CP}^1$, from which it follows that $A$ must be of the form (1.1). □

After a rigid motion, the harmonic Gauss map of any Delaunay surface is framed by the $r$-unitary part of the solution $\Phi(z) = \exp(z A)$ of the ODE $d\Phi = \Phi Adz$, for some $A$ as in (1.1) (see [3, 25]). This prompts us to make the following

**Definition 1.2.** A Delaunay residue is a meromorphic $\text{sl}_2\mathbb{C}$-valued matrix map as in (1.1), with $a, b \in \mathbb{C}^*$ and $c \in \mathbb{R}$. Let $\mu : \mathbb{C}^* \to \mathbb{C}$ be an eigenvalue of $A$ satisfying $\text{Re} \mu \geq 0$.

It can be shown [13] that up to rigid motions, any Delaunay surface in $\mathbb{R}^3$ can be obtained by an off-diagonal Delaunay residue with real non-zero parameters $a, b \in \mathbb{R}^*$ satisfying the closing condition $a + b = 1/2$, and that the resulting necksize of the surface depends on the product $ab$: when $ab > 0$, the resulting surface is an unduloid, when $ab < 0$, it is a nodoid, and when $a = b$, the resulting surface is a round cylinder (see [11, 25]).

For $a, b \in \mathbb{C}^*$, we denote as the vacuum the case $|a| = |b|$ and $c = 0$.

1.2. The Delaunay frame. The unitary frames for all CMC tori are computed in [2, 11] in terms of theta functions. In the case of spectral genus 1, the Iwasawa factorization of $\exp((x + iy)A))$ can be expressed in terms of elliptic functions and elliptic integrals, and has a Floquet form whose period is that of the induced Delaunay surface. The computation of this factorization in Theorem 1.5 requires the following sets and functions.

**Notation 1.3.** Given a Delaunay residue $A$, let $\nu_1, \nu_2 \in \mathbb{C}$ be the zeros of $\det A$, with $|\nu_1| \leq |\nu_2|$. Let $p \in \mathbb{S}^1$ be the point on the straight line segment with endpoints $\nu_1$ and $\nu_2$, or $p = \nu_1 = \nu_2$ in the case of the vacuum. Let $\alpha = -p$. Define the following subsets of $\mathbb{C}$ (see 2(a)):

$\mathcal{I}_A = \{rp \in \mathbb{C} \mid 0 \leq r < \infty\}$, \hspace{1em} $\mathcal{J}_A = \{rp \in \mathbb{C} \mid |\nu_1| \leq r \leq |\nu_2|\}$, \hspace{1em} $\mathcal{K}_A = (\mathcal{I}_A \setminus \mathcal{J}_A) \cup \{\nu_1, \nu_2\}$.

The set of resonance points for $A$ is

\[
\mathcal{S}_A = \{\lambda \in \mathbb{C}^* \mid \mu(\lambda) \in \frac{1}{2}\mathbb{Z}^*\}.
\]

**Notation 1.4.** Let $A$ be a Delaunay residue and let $a, b, c$ be its coefficients as in Proposition 1.1. Define $v : \mathbb{R} \to \mathbb{R}_{>0}$ as the elliptic function satisfying

\[
(v')^2 = -v^4 + 4(|a|^2 + |b|^2 + c^2)v^2 - 16|ab|^2, \hspace{1em} v(0) = 2|b|,
\]

taking the non-constant solution except in the case of the vacuum. When $c \neq 0$, $v'(0)$ is taken to have the same sign as $-c$. The function $v$ is the restriction to $\mathbb{R}$ of an elliptic function on $\mathbb{C}$ with a real and a pure imaginary period; let $\rho \in \mathbb{R}_+$ be its real period.
Define the elliptic integral of the third kind \( \psi : \mathbb{R} \times (\mathbb{C} \setminus \mathcal{J}_A) \to \mathbb{C} \) by

\[
(1.4) \quad \psi(x, \lambda) = \int_0^x \frac{2 \, dt}{1 + (4\bar{a}\lambda)^{-1}v^2(t)},
\]

and define \( \sigma : \mathbb{C} \setminus \mathcal{J}_A \to \mathbb{C} \) by \( \sigma(\lambda) = \psi(\rho, \lambda) \).

**Theorem 1.5.** Let \( A \) be a Delaunay residue and let \( \Phi = \exp((x + iy)A) \). Then there exists an analytic map \( R : \mathbb{R} \times (\mathbb{C} \setminus \mathcal{J}_A) \to \text{SL}_2\mathbb{C} \) satisfying \( R(x + \rho, \lambda) = R(x, \lambda) \), such that, restricting to \( \mathbb{C} \setminus \mathcal{J}_A \),

\[
(1.5a) \quad \text{Uni}_1[\Phi] = \exp((x + iy - \rho^{-1}\sigma x)A)R,
\]

\[
(1.5b) \quad \text{Pos}_1[\Phi] = R^{-1}\exp(\rho^{-1}\sigma xA).
\]

**Proof.** Let prime denote the derivative with respect to \( x \). With \( h = \text{diag}((b/|b|)^{1/2}, (b/|b|)^{-1/2}) \), define \( R(x, \lambda) : \mathbb{R} \times (\mathbb{C} \setminus \mathcal{J}_A) \to \text{SL}_2\mathbb{C} \) by

\[
R(x, \lambda) = \exp((\rho^{-1}\sigma(\lambda)x - \psi(x, \lambda))A)S(x, \lambda),
\]

\[
S_1(x, \lambda) = \begin{pmatrix} v^2 + 4\bar{a}\lambda & v' + 2cv \\ 0 & 2(b + \bar{a}\lambda)v \end{pmatrix} h, \quad S(x, \lambda) = (\det S_1)^{-1/2}S_1.
\]

The square root \((\det S_1)^{1/2}\) can be taken to be a single-valued analytic function in \( \lambda \) on \( \mathbb{R} \times (\mathbb{C} \setminus \mathcal{J}_A) \), its sign chosen so that \( S(0, \lambda) = I \). Then \( R \) is periodic in \( x \) with period \( \rho \) because each of the two factors defining it are.

On \( \mathbb{C} \setminus \mathcal{J}_A \), define \( F = \exp((1 - \rho^{-1}\sigma)xA)R \) and \( B = R^{-1}\exp(\rho^{-1}\sigma xA) \). Then

\[
(1.6a) \quad F^{-1}F' = ((1 - \psi')A).S = \begin{pmatrix} 0 & -\frac{v'}{2} + \frac{2ab}{\bar{a}v} \\ \frac{v'}{2} - \frac{2ab}{\bar{a}v} & 0 \end{pmatrix}. h =: \theta, \quad F(0, \lambda) = I,
\]

\[
(1.6b) \quad B'B^{-1} = ((-\psi')A).S = \begin{pmatrix} -\frac{v'}{2\bar{a}v} & v' \\ \frac{v'}{2\bar{a}v} & \bar{a}v \end{pmatrix}. h =: \eta, \quad B(0, \lambda) = I.
\]

Since \( \theta \) is analytic in \( \lambda \) on \( \mathbb{C}^* \), and \( \theta^* = -\theta \), then \( F \) extends to a map \( \mathbb{R} \to \Lambda_1^+\text{SL}_2\mathbb{C} \). Similarly, \( \eta \) is analytic in \( \lambda \) on \( \mathbb{C} \), and \( \eta|_{\lambda = 0} \) is upper-triangular with real diagonal entries, so \( B \) extends to a map \( \mathbb{R} \to \Lambda_1^+\text{SL}_2\mathbb{C} \). Hence \( \Phi = \exp(iyA)F \cdot B \) is the 1-Iwasawa factorization of \( \Phi \).

**Corollary 1.6.** Let \( A \) be a Delaunay residue and let \( \sigma \) be as in **Notation 1.2** for this \( A \). Let \( \exp((x + iy)A) = F \cdot B \) be the 1-Iwasawa factorization. Then \( \exp(\sigma A) \) extends to a holomorphic function of \( \lambda \) on \( \mathbb{C} \), and the following quasiperiodicity formulas hold for all \( x \in \mathbb{R} \) and \( n \in \mathbb{Z} \):

\[
(1.7a) \quad F(x + n\rho, y) = \exp(n(\rho - \sigma)A)F(x, y), \quad \lambda \in \mathbb{C}^*,
\]

\[
(1.7b) \quad B(x + n\rho) = B(x) \exp(n\sigma A), \quad \lambda \in \mathbb{C}.
\]

**Proof.** Since \( \Phi = FB \), and \( \Phi \) and \( B \) are holomorphic in \( \lambda \) on \( \mathbb{D}_1^* \), then \( F \) on \( \mathbb{S}^1 \) extends holomorphically in \( \lambda \) to \( \mathbb{D}_1^* \). Since \( F^* = F^{-1} \), then \( F \) extends holomorphically in \( \lambda \) to \( \mathbb{C}^* \). Since \( \Phi \) is holomorphic in \( \lambda \) on \( \mathbb{C}^* \), then \( B \) extends holomorphically in \( \lambda \) on \( \mathbb{C} \). (This can also be seen from the ODE \((1.6b)\) satisfied by \( B \).) But by \((1.5b)\), \( B(\rho) \) and \( \exp(\sigma A) \) are equal on \( \mathbb{C} \setminus \mathcal{J}_A \). The quasiperiodicity formulas for \( F \) and \( B \) follow by \((1.5)\). \( \square \)
1.3. The Delaunay growth exponent. Our proof of the convergence of the \( r \)-Iwasawa factors of a holomorphic Delaunay frame \( \Phi = \exp(A \log z) \) and a perturbation of it requires growth bounds on \( \text{Pos}_r[\Phi] \). Being an exponential, \( \Phi \) grows exponentially, and its growth rate is the absolute value of the real part of an eigenvalue \( \mu \) of the Delaunay residue \( A \). On \( S^1 \) (\( r = 1 \)), the factor \( \text{Pos}_r[\Phi] \) has the same growth behavior as \( \Phi \), because \( \text{Unif}[\Phi] \) does not grow. For \( r < 1 \), the Floquet behavior of \( \text{Pos}_r[\Phi] \) again implies exponential growth, its rate determined by the eigenvalues of its value after one period.

We begin by studying the real part \( \tau \) of these eigenvalues, showing it is less than the real part of the eigenvalue \( \mu \) of \( A \).

**Lemma 1.7.** Let \( A \) be a Delaunay residue and let \( \mu \) be its eigenvalue as in \([1.2]\). Let \( \sigma \) be as in \([\text{Notation 1.3}]\) for this \( A \). Then the function \( \tau = \rho^{-1} \Re \mu \sigma \) extends to a single-valued continuous function on \( C \), which is real analytic and harmonic on \( C \setminus K_A \). Moreover, with \( K_A \) as \([\text{Notation 1.3}]\), \( \tau = 0 \) on \( K_A \), \( \tau > 0 \) on \( D_1 \setminus K_A \), and \( \tau = \mu \) on \( S^1 \).

**Proof.** We first consider the nonvacuum case.

**Step 1.** We show that \( \sigma \) extends holomorphically as a function of \( \lambda \) along any curve in \( C \setminus \{\nu_1, \nu_2\} \). Let \( J(t, \lambda) dt \) be the integrand defining \( \psi \) in \([1.3]\). Then \( J \) can be extended to a meromorphic function of \( t \) on \( C \), by considering \( v \) as an elliptic function on \( C \). Let \( \gamma \) be a curve with endpoints \( 0 \) and \( \rho \) which is homotopic to the straight line \([0, \rho]\) in \( C \setminus \text{sing}(v) \). Let \( \Lambda = \{ \lambda \in C \mid J(t, \lambda) \neq \infty \text{ along } \gamma \} \). Then we can define \( \tilde{\sigma}(\lambda) \) on \( \Lambda \) as the integral of \( J \) along \( \gamma \). Since \( \tilde{\sigma} \) is analytic, and is equal to \( \sigma \) on the intersections of their domains \( \Lambda \cap (C \setminus J_A) \), then \( \tilde{\sigma} \) is an analytic extension of \( \sigma \). This provides a construction for analytically extending \( \sigma \) along any curve in \( C \setminus \{\nu_1, \nu_2\} \).

**Step 2.** Since for all \( x \in \mathbb{R}, B(x, \lambda) = \text{Pos}_1[\exp(x A(\lambda))] \) is a holomorphic function of \( \lambda \) on \( C \), then \( B(\rho, \lambda) \) is holomorphic in \( \lambda \). Hence \( \cosh(\mu \sigma) = \frac{1}{2} \text{tr} B(\rho, \lambda) \) is analytic on \( C \). An argument shows that at \( \lambda \in C \), if \( \cosh(\mu \sigma) \notin \{\pm 1\}, \mu \sigma \) is analytic at \( \lambda \), and if \( \cosh(\mu \sigma) \in \{\pm 1\}, \mu \sigma \) is of the form \( \pi i k + \mu g \) for some \( k \in \mathbb{Z} \) and some holomorphic function \( g \) near \( \lambda \).

**Step 3.** Near \( \lambda = 0, \sigma \) is analytic. Hence if any branch of \( \mu \sigma \) is analytically extended along a closed once-wrapped curve around \( 0 \), with respective values \( p_0 \) and \( p_1 \) at the beginning and end of the curve, then \( p_1 = -p_0 \). Likewise, if any branch of \( \mu \) is analytically extended along a closed curve around \( \nu_1 \), with respective values \( q_0 \) and \( q_1 \) at the beginning and end of the curve, it follows from step 2 that \( q_1 \equiv -q_0 \mod 2\pi i \).

Putting these together, if \( \mu \sigma \) is analytically extended along a closed once-wrapped curve around \( 0 \) and \( \nu_1 \), with respective values \( r_0 \) and \( r_1 \) at the beginning and end of the curve, then \( r_1 \equiv r_0 \mod 2\pi i \). Hence \( \tau \) is a single-valued real analytic function on \( C \setminus K_A \). It is harmonic there because it is locally the real part of an analytic function.

**Step 4.** To show that \( \tau \) is continuous on \( K_A \setminus \{\nu_1, \nu_2\} \) with value \( 0 \), write \( \mu \) and \( \sigma \) in terms of their real and imaginary parts \( \mu = \mu_1 + i\mu_2 \) and \( \sigma = \sigma_1 + i\sigma_2 \). Then \( \Re \mu \sigma = \mu_1 \sigma_1 - \mu_2 \sigma_2 \). Since \( \sigma_2 \) and \( \mu_1 \) are 0 on \( K_A \), \( \tau \) is continuous on \( K_A \setminus \{\nu_1, \nu_2\} \) with value \( 0 \) there.

**Step 5.** We now show that \( \tau \) is continuous at \( \nu_1 \) and \( \nu_2 \) with value \( 0 \). By step 2, with \( k \in \{1, 2\} \), we have \( \tau = \Re(\mu + \mu g) = \Re(\mu g) \) near \( \nu_k \), for some \( c \in \pi i \mathbb{Z} \) and holomorphic function \( g \). Since \( \mu(\nu_k) \) is continuous at \( \nu_k \) and \( \mu(\nu_k) = 0 \), then \( \tau \) extends continuously to \( \nu_k \) with value \( 0 \) there.

**Step 6.** To show that \( \tau = \mu \) on \( S^1 \), since \( F \) in \([1.7a]\) takes values in \( \text{SU}_2 \) on \( S^1 \), then by \([1.7a]\), so does \( \exp((\rho - \sigma) A) \). Since \( A \) is tracefree and is hermitian on \( S^1 \), then \( \rho - \sigma \) is pure imaginary on \( S^1 \). Hence \( \tau = \mu \) on \( S^1 \).

**Step 7.** Since \( \tau \) is continuous on \( \partial D_1 \) and harmonic on \( D_1 \setminus K_A \), then by the maximum principle for harmonic functions, \( \tau \) on \( D_1 \) attains its minimum on the boundary \( S^1 \cup K_A \) of \( D_1 \setminus K_A \). Since \( \tau = 0 \) on \( K_A \), then \( \tau \) is strictly positive on \( D_1 \setminus K_A \).
For the vacuum case, let $a, b$ be the coefficients of the Delaunay residue as in Definition 1.2 and let $\alpha$ be as in Notation 1.3. Then $\tau = \text{Re} \rho^{-1} \mu \sigma$ satisfies all the properties of the theorem, where
\begin{equation}
\mu = |b| \alpha^{-1/2} \lambda^{-1/2} (\lambda + \alpha) \quad \text{and} \quad \rho^{-1} \mu \sigma = 2 |b| \alpha^{-1/2} \lambda^{1/2} .
\end{equation}

**Lemma 1.8.** Let $A$ be a Delaunay residue, let $\mu$ be its eigenvalue as in Definition 1.2 and let $\tau$ be as in Lemma 1.7 for this $A$. Then $\tau \in (0, \text{Re} \mu]$ on $\mathcal{D}_1 \setminus \mathcal{K}_A$.

**Proof.** In the case of the vacuum, the result follows by (1.8); hence we assume the nonvacuum case. (See 2(b) for the graphs of $\text{Re}(\mu)$ and $\tau$ for a typical Delaunay residue.)

We first show that $\text{Re}(\mu) - \tau$ is nonnegative on $\mathcal{D}_r^*$ for some $r$ near zero, and then apply the maximum principle for harmonic functions to conclude it is strictly positive on $\mathcal{D}_1 \setminus \mathcal{K}_A$.

**Step 1.** We prove the following claim: Let $r > 0$ and let $f : \mathcal{D}_r \to \mathbb{C}$ be a holomorphic function. If $f$ has the symmetry $f(\overline{z}) = \overline{f(z)}$, and $f(0) \in \mathbb{R}_+$, then there exists $r_1 \in (0, r]$ such that $g := \text{Re}(\lambda^{-1/2} f)$ is nonnegative on $\mathcal{D}_{r_1}^*$, where the square root is chosen to have nonnegative real part.

To prove the claim, let $z = x + iy = \lambda^{1/2}$, and let
\[ h := \frac{g}{\text{Re} \lambda^{-1/2}} = \text{Re} f(z^2) + \frac{y}{x} \text{Im} f(z^2) . \]

The function $k(x, y) = \text{Im} f(z^2)$ is real analytic in $x$ and $y$. By the symmetry of $f$, $f(-y^2)$ is real, so for all $y$, $k(0, y) = \text{Im} f(-y^2) = 0$. It follows that $k(x, y)/x$ is real analytic in $x$ and $y$. Then $\lim_{z \to 0} y k(x, y)/x = 0$, so $\lim_{z \to 0} h = f(0) \in \mathbb{R}_+$. Hence there exists $r_1 \in (0, r]$ such that $h$ is strictly positive in $\mathcal{D}_{r_1}$. Since $\text{Re} \lambda^{-1/2}$ is strictly positive on $\mathcal{D}_{r_1} \setminus \mathbb{R} \leq 0$, then so is $g$. But $g$ is 0 on $\mathbb{R}_{-}$, so $g$ is nonnegative on $\mathcal{D}_{r_1}^*$.

**Step 2.** Let $\sigma$ be as in Lemma 1.7 on $\mathcal{D}_{|\nu_1|}$, so $\tau = \text{Re} \sigma$. With $\alpha$ as in Notation 1.3 define $f : \mathcal{D}_{|\nu_1|} \to \mathbb{C}$ by
\[ f = \lambda^{1/2} (\mu(\alpha \lambda) - \sigma(\alpha \lambda)) . \]

Then $f$ is holomorphic on $\mathcal{D}_{|\nu_1|}$, $f(\overline{z}) = \overline{f(z)}$ and $f(0) \in \mathbb{R}_+$. By the above claim applied to $f$, there exists $r_1 \in (0, |\nu_1|]$ such that $\text{Re} \lambda^{-1/2} f = \text{Re}(\mu(\alpha \lambda)) - \tau(\alpha \lambda)$ is nonnegative on $\mathcal{D}_{r_1}^*$. Hence $\text{Re}(\mu) - \tau$ is nonnegative on $\mathcal{D}_{r_1}^*$.

**Figure 2.**

(a) The $\lambda$-plane with the zeros $\nu_1$ and $\nu_2 = 1/4\pi$ of the eigenvalues of a typical Delaunay residue.

(b) Graphs of $\text{Re} \mu$ (above) and $\tau$ (below) over a half-disk for a typical Delaunay frame. $\text{Re} \mu$ represents the growth of $\exp(A \log z)$, $\tau$ the growth of its positive factor, and their difference the growth of its unitary factor.
Step 3. For any \( s \in (0, r_1] \), define \( V_s \) as the union of \( C_s \) and the straight line segment along \( K_A \) from \( v_t \) to this circle. Then for all \( s < r_1 \), \( \text{Re}(\mu) - \tau \) is harmonic on the open region \( R_s \) between \( V_s \) and \( S^1 \), is 0 on \( S^1 \) and \( V_s \cap K_A \), and is nonnegative on \( C_s \) by step 2. By the maximum principle for harmonic functions, \( \text{Re}(\mu) - \tau \) is strictly positive on \( R_s \). Since this is true for any \( s \in (0, r_1) \), then \( \text{Re}(\mu) - \tau \) is strictly positive on \( D_1 \setminus K_A \). \( \square \)

1.4. Growth of the Delaunay positive part. Above we investigated the function \( \tau \), the real part of the eigenvalue of the value of \( \text{Pos}_\tau[\exp(A \log z)] \) after one period, showing that it is less than the real part of the eigenvalue \( \mu \) of the corresponding Delaunay residue. The Floquet behavior of \( \text{Pos}_\tau[\exp(A \log z)] \), detailed in Theorem 1.5, implies that \( \tau \), and hence \( \mu \), bounds the exponential growth of \( \text{Pos}_\tau[\exp(A \log z)] \) [Lemma 1.11].

Notation 1.9. For \( v \in \mathbb{C}^2 \), we denote the vector norm by \( |v| = \sqrt{v^* v} \), and for \( M \in M_{2 \times 2}(\mathbb{C}) \), we set
\[
|M| = \max_{|v|=1} |Mv| .
\]

For a map \( M : \mathcal{R} \to M_{2 \times 2}(\mathbb{C}) \) on a subset \( \mathcal{R} \subset \mathbb{C} \) we set
\[
|M|_D = \sup_{\lambda \in \mathcal{R}} \|M(\lambda)\| .
\]

Given \( r \in (0, 1] \), a subset \( \mathcal{R} \subset \mathbb{C} \) containing \( C_r \), and a loop \( X \in \Lambda_r \text{SL}_2 \mathbb{C} \) which extends to a map \( Y : \mathcal{R} \to \text{SL}_2 \mathbb{C} \), by an abuse of notation we will write \( |X|_\mathcal{R} \) for \( \|Y\|_\mathcal{R} \).

Lemma 1.10. Let \( X : \mathcal{R} \to \text{SL}_2 \mathbb{C} \) be a continuous map on a domain \( \mathcal{R} \subset \mathbb{C} \), and let \( \mu : \mathcal{R} \to \mathbb{C} \) be any eigenvalue function of \( X \). Then there exists a continuous function \( c : \mathcal{R} \to \mathbb{R}_+ \) such that \( \|\exp X\| \leq e^{\|\text{Re}\mu\|} \).

Proof. The result follows from the formula \( \exp X = \cosh(\mu) I + \mu^{-1} \sinh(\mu) X \) and the estimates \( |\cosh(\mu)| \leq e^{\|\text{Re}\mu\|} \) and \( |\mu^{-1} \sinh(\mu)| \leq e^{\|\text{Re}\mu\|} \).

Lemma 1.11. Let \( A \) be a Delaunay residue, and let \( \mu \) be its eigenvalue as in Definition 1.2. Let \( \tau \) be as in Lemma 1.7. Then there exists a continuous function \( c : D_1 \setminus \{0\} \to \mathbb{R}_+ \) such that for all \( (z, \lambda) \in \{0 < |z| < 1\} \times D_1 \setminus \{0\} \),
\[
|\text{Pos}_\tau[\exp(A \log z)]| \leq c |z|^{-\tau} \leq c |z|^{-\text{Re}\mu} .
\]

Proof. For \( x + iy \in \mathbb{C} \), define \( B(x+iy, \lambda) = \text{Pos}_1[\exp((x+iy)A)] \). With \( \rho \) as in Corollary 1.6 define \( x_0 : \mathbb{R} \to [0, \rho) \) and \( n : \mathbb{R} \to \mathbb{Z} \) as the unique functions such that \( x = x_0 + \rho n \). Let \( \sigma \) be as in Notation 1.4 for \( A \). By Corollary 1.6
\[
(1.10) \quad B(x + iy) = B(x_0 + \rho n) = B(x_0) \exp(n \sigma A) .
\]

This quasiperiodicity of \( B \) determines its growth rate as follows.

Define the continuous function \( c_1 : \mathbb{C}^* \to \mathbb{R}_+ \) by \( c_1(\lambda) = \max_{x \in [0, \rho)} \|B(x, \lambda)\| \). Then \( \|B(x_0, \lambda)\| \leq c_1(\lambda) \) on \( [0, \rho) \times \mathbb{C}^* \). With \( \tau \) as in Lemma 1.7 for \( A \), by Lemma 1.10 there exists a continuous function \( c_2 : \mathbb{C}^* \to \mathbb{R}_+ \) such that on \( \mathbb{C}^* \),
\[
\|\exp(n \rho \sigma(\lambda) A(\lambda))\| \leq c_2 e^{-n \rho \sigma(\lambda)} \leq c_2 e^{-x_0 \sigma(\lambda)} e^{-|x \sigma(\lambda)|} \leq c_2 e^{-|x \sigma(\lambda)|} e^{-|x \sigma(\lambda)|} .
\]

Define the continuous function \( c_3 : \mathbb{C}^* \to \mathbb{R}_+ \) by \( c_3(\lambda) = \max_{x \in [0, \rho)} e^{-|x \sigma(\lambda)|} \). Then
\[
(1.11) \quad \|\exp(n \rho \sigma(\lambda) A(\lambda))\| \leq c_2 c_3(\lambda) e^{-|x \sigma(\lambda)|} \}
\]

The quasiperiodicity \( (1.10) \), the choice of \( c_1 \), and \( (1.11) \) yield the estimate
\[
\|B(x + iy, \lambda)\| \leq c_1(\lambda) c_2(\lambda) c_3(\lambda) e^{-|x \sigma(\lambda)|} .
\]

Since \( 0 < |z| < 1 \), then \( x < 0 \). But \( |z| = e^x \), so \( |z|^{-\tau} = e^{-\tau x} = e^{\tau x} \). The growth estimate \( (1.9) \) follows with \( c = c_1 c_2 c_3 \).
The second inequality in (1.9) follows by Lemma 1.12. \qed

1.5. Dressed Delaunay frames. The growth bound on the positive \( r \)-Iwasawa factor of the holomorphic Delaunay frame \( \Phi = \exp(A \log z) \), computed above in Lemma 1.11 is preserved by dressing (right-multiplying) \( \Phi \) by an \( r \)-loop \( C \), provided that the unitarity of its monodromy is preserved. In the special case that \( C \) can be extended analytically to \( \mathcal{A}_{r,1} \), the resulting frame \( C\Phi \) again induces a Delaunay immersion (Lemma 1.12), and the growth of its positive factor is bounded by the same bound as that for \( \text{Pos}_r[\Phi] \) (Theorem 1.13).

The general case of dressing by an arbitrary loop \( C \) which preserves the unitarity of the monodromy is characterized in subsection 4.2, where it is shown that \( C\Phi \) induces a multibublleton, a Bianchi-Bäcklund transformed Delaunay immersion. The same growth bound applies to the positive part of \( C\Phi \) for this larger class of dressing matrices.

**Lemma 1.12.** Let \( A \) be a Delaunay residue. Let \( r \in (0, 1] \). Let \( C \in \Lambda^+_{r}SL_2\mathbb{C} \), and assume \( C \exp(2\pi i A)C^{-1} \in \Lambda^+_{r}SL_2\mathbb{C} \). Let \( C = C_u \cdot C_+ \) be the \( r \)-Iwasawa factorization of \( C \). Then

(i) \( C_+AC_+^{-1} \) extends meromorphically to \( \mathbb{C}P^1 \) and is a Delaunay residue.

(ii) \( \text{Sym}_r[C \exp(A \log z)] \) and \( \text{Sym}_r[\exp(A \log z)] \) are Delaunay surfaces differing by a rigid motion.

**Proof.** Let \( M = \exp(2\pi i A) \) and let \( A_1 = C_+AC_+^{-1} \). Since \( CMC^{-1} \in \Lambda^+_{r}SL_2\mathbb{C} \), then \( C_+MC_+^{-1} = \exp(2\pi i A_1) \in \Lambda^+_{r}SL_2\mathbb{C} \). With \( \mu \) a local analytic eigenvalue of \( A \) or \( A_1 \), the formula

\[
\exp(2\pi i A_1) = \cos(2\pi \mu) \text{I} + \mu^{-1} \sin(2\pi \mu) A_1
\]

shows that \( A_1 \) extends meromorphically to \( S^1 \). Since \( \exp(2\pi i A_1) \in \Lambda^+_{r}SL_2\mathbb{C} \), then \( A_1^* = A_1 \) away from its poles. Write

\[
A_1 = \begin{pmatrix}
  x & y \\
  y^* & -x
\end{pmatrix}
\]

for some meromorphic functions \( x \) and \( y \) in a neighborhood of \( S^1 \) satisfying \( x = x^* \). Since \( A \) is holomorphic on \( S^1 \), then so is

\[-\det A = -\det A_1 = xx^* + yy^* .\]

Hence \( xx^* + yy^* \) is bounded on \( S^1 \). Since \( xx^* \) and \( yy^* \) are each nonnegative on \( S^1 \), then each is bounded on \( S^1 \), so each is holomorphic on \( S^1 \). It follows that \( x, y \) and \( y^* \) are holomorphic on \( S^1 \), so \( A_1 \) is holomorphic on \( S^1 \). Since \( A_1 \) is holomorphic on \( D_1 \) and satisfies \( A_1^* = A_1 \), then \( A_1 \) is a Delaunay residue by Proposition 1.1

To show that \( C \exp(A \log z) \) induces a Delaunay immersion, note that

\[
C \exp(A \log z) = C_u \exp(A_1 \log z)C_+ ,
\]

so

\[
\text{Uni}_r[C \exp(A \log z)] = C_u \text{Uni}_r[\exp(A_1 \log z)] .
\]

Hence \( \text{Sym}_r[C \exp(A \log z)] \) and \( \text{Sym}_r[\exp(A_1 \log z)] \) differ by a rigid motion. By Lemma 6 in [25], \( \text{Sym}_r[\exp(A_1 \log z)] \) and \( \text{Sym}_r[\exp(A \log z)] \) are Delaunay surfaces differing by a rigid motion. \( \square \)

**Theorem 1.13.** Let \( A \) be a Delaunay residue, and let \( \mu \) be its eigenvalue as in Definition 1.2. Let \( \tau \) be as in Lemma 1.7. Let \( r \in (0, 1] \). Let \( C \in \Lambda^+_{r}SL_2\mathbb{C} \) be such that the dressed monodromy \( C \exp(2\pi i A)C^{-1} \) around \( z = 0 \) is in \( \Lambda^+_{r}SL_2\mathbb{C} \). Then there exists a continuous function \( c : D_1^* \to \mathbb{R}_+ \) such that for all \( (z, \lambda) \in \{0 < |z| < 1\} \times D_1^* \),

\[
|\text{Pos}_r[C \exp(A \log z)]| \leq c|z|^{-\tau} \leq c|z|^{-\text{Re} \mu} .
\]
Proof. Let \( C = C_u \cdot C_v \) be the \( r \)-Iwasawa factorization of \( C \). By Lemma 1.12, \( A_1 = C_+ A C_+^{-1} \) is a Delaunay residue. Then

\[
C \exp(A \log z) = C_u \exp(A_1 \log z) C_+ ,
\]

so

\[
(1.12) \quad \text{Pos}_r[C \exp(A \log z)] = \text{Pos}_r[\exp(A_1 \log z)] C_+ .
\]

It follows from \( \det A = \det A_1 \) that the function \( \tau \) in Lemma 1.7 for \( A \) is the same as that for \( A_1 \). By Lemma 1.11 there exists a continuous function \( c_1 : \mathbb{D} \setminus \{0\} \to \mathbb{R}_+ \) such that on \( \{0 < |z| < 1\} \times \mathbb{D}^*_1 \),

\[
|\text{Pos}_r[\exp(A_1 \log z)]| \leq c_1 |z|^{-\tau} \leq c_1 |z|^{-\text{Re}\mu} .
\]

Let \( c_2 = \|C_+\| \text{ on } \mathbb{D}_1 \). Then by (1.12) and (1.13),

\[
\|\text{Pos}_r[C \exp(A \log z)]\| \leq c_1 c_2 \|\text{Pos}_r[\exp(A_1 \log z)]\| \leq c_1 c_2 |z|^{-\tau} \leq c_1 c_2 |z|^{-\text{Re}\mu} .
\]

The result follows with \( c = c_1 c_2 \). \qed

2. The perturbed Delaunay frame

We will use the following notation throughout the next several sections. Let \( A \) be a Delaunay residue and let \( \mu \) be an eigenvalue of \( A \) with nonnegative real part. Let \( S_A \) be the set of resonance points for \( A \), defined in (1.2). Let \( \Sigma \subset \mathbb{C} \) be a neighborhood of \( 0 \in \mathbb{C} \) and let \( \Sigma^* = \Sigma \setminus \{0\} \). Choose \( r \in (0, 1] \).

Definition 2.1.

(i) A perturbed Delaunay \( r \)-potential is an \( r \)-potential \( \xi \) on \( \Sigma^* \) of the form

\[
\xi = A z^{-1} dz + O(z^0) dz .
\]

(ii) An \( r \)-gauge is an analytic map \( \Sigma \to A_p \text{orSL}_2 \mathbb{C} \).

(iii) Given neighborhoods \( \Sigma, \Sigma^0 \subset \mathbb{C} \), a coordinate change \( \vartheta \) is a holomorphic map \( \vartheta : \Sigma^0 \to \Sigma \) which satisfies \( \vartheta(0) = 0 \) and has a holomorphic inverse \( \vartheta(\Sigma) \to \Sigma^0 \).

2.1. The \( z^A P \) lemma. Given a linear matrix ODE \( d \Phi = \Phi \xi \) for which \( \xi \) has a simple pole at \( z = 0 \) and residue \( A \), a standard result in the theory of regular singularities states that under certain conditions on the eigenvalues of \( A \), there exists a solution of the form \( \Phi = z^A P = \exp(A \log z) P \), where \( P \) extends holomorphically to \( z = 0 \). Lemma 2.3 summarizes these results for our context, in which \( \xi \) depends analytically on a parameter \( \lambda \). We call the decomposition (2.3) the \( z^A P \) decomposition.

The coefficients of the \( z^A P \) gauge \( P \) can be computed in terms of a linear map \( \mathcal{L}_n \), whose definition and properties are given in the next lemma.

Lemma 2.2. Let \( A \in \text{gl}_2 \mathbb{C} \) and let \( \mu_1, \mu_2 \in \mathbb{C} \) be the eigenvalues of \( A \). For \( n \in \mathbb{Z}_{\geq 0} \), define the linear map \( \mathcal{L}_n : \text{gl}_2 \mathbb{C} \to \text{gl}_2 \mathbb{C} \) by

\[
\mathcal{L}_n(X) = nX + [A, X] .
\]

Then:

(i) The eigenvalues of \( \mathcal{L}_n \) are \( n, n, n + \mu_1 - \mu_2, n - \mu_1 + \mu_2 \). Hence \( \mathcal{L}_n \) is invertible if and only if \( n \neq 0 \) and \( \mu_1 - \mu_2 \notin \{n, -n\} \).

(ii) Suppose \( n \in \mathbb{Z}_+ \). Let \( R = n I + A - \tilde{A} \), where \( \tilde{A} \) denotes the adjugate of \( A \). Then \( \mathcal{L}_n \) is invertible if and only if \( R \) is invertible, and in this case, \( \mathcal{L}_n^{-1} \) is given by

\[
(2.1) \quad n \mathcal{L}_n^{-1}(X) = X - R^{-1}[A, X] .
\]

(iii) For any \( X \in \text{gl}_2 \mathbb{C} \), \( \text{tr}(\mathcal{L}_n X) = n \text{tr} X \). Hence if \( \mathcal{L}_n \) is invertible, then for any \( Y \in \text{gl}_2 \mathbb{C} \), \( \text{tr} Y = n \text{tr}(\mathcal{L}_n^{-1} Y) \).
(iv) Let \( n \in \mathbb{Z}_+ \). If \( A \) is holomorphic (respectively meromorphic) on some domain \( \mathcal{R} \), then \( \mathcal{L}_n \) is holomorphic (respectively meromorphic) on \( \mathcal{R} \). In this case, if \( \mu_1 - \mu_2 \) is not identically \( n \) or \(-n\), then \( \mathcal{L}_n^{-1} \) extends meromorphically to \( \mathcal{R} \).

**Proof.** Statement (i) follows from the fact that the eigenvalues of \( \text{ad}_A \) are 0, 0, \( \mu_1 - \mu_2 \), \( -\mu_1 + \mu_2 \).

To prove (ii), since the eigenvalues of \( R \) are \( n + \mu_1 - \mu_2 \), \( n - \mu_1 + \mu_2 \), the by (i), \( R \) is invertible if and only if \( \mathcal{L}_n \) is invertible. In this case,

\[
\mathcal{L}_n (X - R^{-1}[A, X]) = nX + (I - nR^{-1} - AR^{-1}) [A, X] + nX[A, X]A
\]

\[
= nX + R^{-1} ((R - nI - RAR^{-1}) [A, X] + [A, X]A)
\]

\[
= nX + R^{-1} (-\hat{A}[A, X] + [A, X]A) = nX.
\]

Statement (iii) is clear from the definition of \( \mathcal{L}_n \) and the fact that \( \text{tr}[A, X] = 0 \).

Identifying \( \mathfrak{gl}_2 \mathbb{C} \) with \( \mathbb{C}^4 \), the entries of the \( 4 \times 4 \) matrices for \( \mathcal{L}_n \) and \( \mathcal{L}_n^{-1} \) are rational functions of the entries of \( A \), and hence are meromorphic. This proves statement (iv). \( \square \)

**Lemma 2.3.** Let \( \mathcal{R} = \mathbb{C}^* \setminus \mathcal{S}_A \).

(i) There exists a holomorphic solution \( P : \Sigma \times \mathcal{R} \to \text{SL}_2 \mathbb{C} \) to the gauge equation

\[
( Az^{-1} dz ) P = \xi , \quad P(0, \lambda) = I .
\]

(ii) Let \( r \in (0, 1] \) and assume \( \mathcal{C}_r \cap \mathcal{S}_A = \emptyset \). Let \( \Phi : \Sigma^* \to \Lambda \), \( \text{SL}_2 \mathbb{C} \) be a holomorphic solution to the equation \( d\Phi = \Phi \xi \) on the universal cover \( \Sigma^* \to \Sigma^* \) of \( \Sigma^* \). Then there exists \( C \in \Lambda \), \( \text{SL}_2 \mathbb{C} \) such that

\[
\Phi(z, \lambda) = C(\lambda) \exp(A(\lambda) \log z)P(z, \lambda) .
\]

(iii) Moreover, if \( \xi \) satisfies \( \xi = Az^{-1} dz + O(z^n)dz \), then \( P \) satisfies \( P = I + O(z^{n+1}) \).

**Proof.** (i). By the pointwise version of this lemma (see Theorem 10.1 in [10]), at each \( \lambda_0 \in \mathcal{R} \), there exists a unique solution \( P(z, \lambda_0) \) to the gauge equation (2.2). To show that \( P \) is holomorphic in \( \lambda \) on \( \mathcal{R} \), let

\[
\xi = Az^{-1} dz + \sum_{k=0}^{\infty} B_k z^k dz
\]

and

\[
P = \sum_{k=0}^{\infty} P_k z^k , \quad P_0 = I
\]

be the respective series expansions for \( \xi \) and \( P \) in \( z \) at \( z = 0 \). Then for all \( k \in \mathbb{Z}_+ \), \( \mathcal{L}_k \) is invertible at \( \lambda_0 \) by Lemma 2.2(i) and the assumption that \( \mathcal{R} \cap \mathcal{S}_A = \emptyset \). The coefficients \( P_k \) (see [10]) are given by \( P_k = \mathcal{L}_k^{-1}(C_k) \), where

\[
C_k = \sum_{i+j=k-1} P_i B_j , \quad k \in \mathbb{Z}_+ .
\]

By Lemma 2.2(iv), each \( P_k \) is holomorphic in \( \lambda \) at \( \lambda_0 \). It follows by the absolute convergence of power series that \( P \) is holomorphic in \( \lambda \) at \( \lambda_0 \).

(ii) Since \( P \) satisfies (2.2), then the map \( \Phi_0 : \Sigma^* \times \mathcal{C}_r \to \text{SL}_2 \mathbb{C} \) defined by

\[
\Phi_0(z, \lambda) = \exp(A(\lambda) \log z)P(z, \lambda)
\]

satisfies \( d\Phi_0 = \Phi_0 \xi \). Since \( \Phi \) satisfies the same linear ODE as \( \Phi_0 \), then the map \( C = \Phi \Phi_0^{-1} \) is \( z \)-independent, and is an element of \( \Lambda \), \( \text{SL}_2 \mathbb{C} \).
Hence \( Y \) at \( p \)

Let Lemma 2.5.

\( \min \mathcal{L}_k \)

Hence

\[ \text{(2.5)} \]

\( \lambda \) for some of the entries of \( Y \)

By (2.4), we have ord

Proof. \( \) Let Lemma 2.4.

\( \xi \) extends holomorphically to \( \mathcal{L}_k \), which extends meromorphically to \( \lambda = 0 \).

The entries of a matrix \( X \in M_{2 \times 2}(\mathbb{C}) \) are denoted by \( X_{ij} \) with \( i, j \in \{1, 2\} \).

Lemma 2.4. Let \( \mathcal{R} \subset \mathbb{C} \) be a neighborhood of \( p \in \mathbb{C} \). Let \( A : \mathcal{R} \to gl_2 \mathbb{C} \) and \( X : \mathcal{R} \to gl_2 \mathbb{C} \) be meromorphic on \( \mathcal{R} \), with orders of entries

\[ \text{(2.4)} \]

\( \) Let \( n \in \mathbb{Z}_+ \) and let \( \mathcal{L}_n \) as in Lemma 2.2 defined with respect to \( A \).

Then \( \mathcal{L}_n^{-1}(X) \), which extends meromorphically to \( p \) by Lemma 2.2(iv), is holomorphic at \( p \) if and only if \( A_{12}X_{21} + A_{21}X_{12} \) is holomorphic at \( p \).

Proof. By (2.4), we have ord det \( A = -1 \).

A calculation using (2.4) shows that the orders of the entries of \( Y = \mathcal{L}_n^{-1}(X) \) satisfy

\[ \text{ord} Y_{11} \geq 0, \quad \text{ord} Y_{12} \geq -1, \quad \text{ord} Y_{21} \geq 0, \quad \text{ord} Y_{22} \geq 0. \]

Hence \( \mathcal{L}_n^{-1}(X) \) is holomorphic at \( p \) if and only if its upper-right entry \( Y_{12} \) is holomorphic at \( p \).

A calculation shows

\[ \text{ord} Y_{12} = \text{ord}(A_{12}X_{21} + A_{21}X_{12}). \]

Hence \( Y \) is holomorphic at \( p \) if and only if \( A_{12}X_{21} + A_{21}X_{12} \) is.

\( \)

Lemma 2.5. Let \( \xi \) be a perturbed Delaunay \( r \)-potential

\[ \xi = Az^{-1}dz + O(z^{n-1})dz \]

for some \( n \in \mathbb{Z}_+ \), and suppose

\[ \text{(2.5)} \]

\( \min Re \mu(\lambda) \leq \frac{n}{2} < \max Re \mu(\lambda) \).

Then there exists a neighborhood \( \Sigma' \subset \mathbb{C} \) of \( 0 \in \mathbb{C} \), an \( r \)-gauge \( g : \Sigma' \to \Lambda^{\text{pos}}_{\mathbb{C}} \), and \( \kappa \in \mathbb{C} \) such that

\[ \xi.g = Az^{-1}dz + \kappa Az^{-n}dz + O(z^n)dz. \]
Proof. Let $\mathcal{L}_n$ be as in Lemma 2.2, defined with respect to $A$ on $\mathcal{D}_r^*$. The inequalities (2.4) imply $\mathcal{D}_r^* \cap \{ \lambda \in \mathbb{C}^* \mid \mu(\lambda) = n/2 \} = \emptyset$. Hence by Lemma 2.4, $\mathcal{L}_n$ is invertible on $\mathcal{D}_r^*$.

Define $B$ by

$$
\xi = Az^{-1}dz + Bz^{-1}dz + O(z^n)dz .
$$

Then

$$
\operatorname{ord}_{\lambda=0} A_{12} = -1, \quad \operatorname{ord}_{\lambda=0} A_{21} = 0, \quad \operatorname{ord}_{\lambda=0} B_{12} \geq -1, \quad \operatorname{ord}_{\lambda=0} B_{21} \geq 0
$$

so

$$
\operatorname{ord}_{\lambda=0} B_{12}/A_{12} \geq 0, \quad \operatorname{ord}_{\lambda=0} B_{21}/A_{21} \geq 0 .
$$

Hence the limit

$$
\kappa := \frac{1}{2} \lim_{\lambda \to 0} \left( \frac{B_{12}}{A_{12}} + \frac{B_{21}}{A_{21}} \right)
$$

exists in $\mathbb{C}$.

Let $X = \kappa A - B$. A calculation shows that with this choice of $\kappa$, $A_{12}X_{21} + A_{21}X_{12}$ is holomorphic at 0. Hence by Lemma 2.4, $\mathcal{L}_n^{-1}(\kappa A - B)$ on $\mathcal{D}_r^*$ extends holomorphically to $\lambda = 0$. Hence we have the holomorphic map $C : \mathcal{D}_r \to \text{SL}_2 \mathbb{C}$ defined by

$$
C = \mathcal{L}_n^{-1}(\kappa A - B) .
$$

Define the holomorphic map $h : \Sigma \times \mathcal{D}_r \to \text{GL}_2 \mathbb{C}$ by $h = I + C z^n$. Since $\det h$ will not in general be identically 1, we define $g = h + O(z^{n+1})$ with $\det g = 1$ as follows. Since $h_{22} = C_{22} z^n$, and $\mathcal{D}_r$ is bounded away from the poles of $C_{22}$, then we have the uniform convergence $h_{22} \to 1$ on $\mathcal{D}_r$ as $z \to 0$. Hence there exists a neighborhood $\Sigma' \subset C$ of $0 \in \Sigma$ on which $h_{22} > 0$. Define the holomorphic map $g : \Sigma' \to \Lambda_{\text{pos}}^1 \text{SL}_2 \mathbb{C}$ by

$$
g = h + \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} , \quad q = \frac{1 - \det h}{h_{22}} .
$$

A calculation shows $\det g = 1$.

In order to show that $g$ satisfies the gauge equation (2.6), we show that $g = h + O(z^{n+1})$. We have $\det h = 1 + \text{tr}(C)z^n + O(z^{n+1})$. Since $\phi A - B$ is tracefree, then $C$ is tracefree by Lemma 2.2(iii). Hence $\det h = 1 + O(z^{n+1})$. Since $h_{22} = 1 + O(z^n)$, then $q = O(z^{n+1})$. Hence $g = h + O(z^{n+1})$.

To complete the proof, we will use the following formula. For any $n \in \mathbb{Z}_+$, $r$-gauge $g = I + g_n z^n + O(z^{n+1})$ and $r$-potential $\xi = Az^{-1}dz + B_{n-1} z^{n-1} + O(z^n)$, we have

$$
\xi.g = Az^{-1}dz + (\mathcal{L}_n(g_n) + B_{n-1}) z^{n-1}dz + O(z^n)dz .
$$

The formula can be shown by expanding $\xi.g = g^{-1}\xi g + g^{-1}dg$ in $z$.

The result (2.6) follows from (2.7), together with formula (2.8) replacing $g_n$ with $C$ and $B_{n-1}$ with $B$.

\begin{lemma}
Let $\xi$ be a perturbed Delaunay $r$-potential Definition 2.7]
(2.9)
$$
\xi = Az^{-1}dz + \kappa A z^{-1}dz + O(z^n)dz
$$
for some $n \in \mathbb{Z}_+$ and $\kappa \in \mathbb{C}$. Then there exists a coordinate change $\vartheta : \Sigma' \to \mathbb{C}$ on a neighborhood $\Sigma' \subset \mathbb{C}$ of $0 \in \Sigma$ with conformal coordinate $w$, such that on $\Sigma' \setminus \{0\}$,

$$
\vartheta^* \xi = Aw^{-1}dw + O(w^n)dw .
$$
\end{lemma}

\begin{proof}
Let $\sigma : \Sigma \to \mathbb{C}$ be any analytic function satisfying

$$
\sigma(z) = z + (\kappa/n) z^{n+1} + O(z^{n+2}) .
$$

Then $\sigma(0) = 0$ and $\sigma'(0) = 1$, so there exists a coordinate change $\vartheta : \Sigma' \to \Sigma$ on a neighborhood $\Sigma' \subset \mathbb{C}$ of 0 which is the inverse of $\sigma$. Because $\vartheta$ is a conformal diffeomorphism,
for any \( j \in \mathbb{Z} \) and any differential \( \omega = O(z^j)dz \), we have \( \vartheta^* \omega = O(w^j)dw \). A calculation shows that on \( \vartheta(\Sigma') \),
\[
(z^{-1} + \kappa z^{n-1}) \, dz = (\sigma^{-1} + O(\sigma^n)) \, d\sigma .
\]
Hence
\[
\vartheta^*((z^{-1} + \kappa z^{n-1}) \, dz) = w^{-1}dw + O(w^n)dw .
\]
This with (2.10) implies the result (2.11). \( \square \)

We now apply Lemma 2.5 and Lemma 2.6 iteratively to transform \( \xi \) to the form \( Aw^{-1}dw + O(w^n)dw \).

**Theorem 2.7.** Let \( \xi \) be a perturbed Delaunay \( r \)-potential (Definition 2.1)
\[
\xi = Az^{-1}dz + O(z^0)dz .
\]
Assume for some \( n \in \mathbb{Z}_+ \) that
\[
(2.11) \quad \min_{\lambda \in C_r} \text{Re} \mu(\lambda) \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{2} < \max_{\lambda \in C_r} \text{Re} \mu(\lambda) .
\]
Then there exists a coordinate change \( \vartheta : \Sigma' \to \Sigma \) on a neighborhood \( \Sigma' \subset \mathbb{C} \) of \( 0 \in \mathbb{C} \) with coordinate \( w \), and an \( r \)-gauge \( g : \Sigma' \to \Lambda_r^{\text{pos}}SL_2 \mathbb{C}, \) such that
\[
\vartheta^*(\xi, g) = Aw^{-1}dw + O(w^n)dw .
\]
Moreover, let \( \Phi \) satisfy \( d\Phi = \Phi \xi \). Then \( \Psi = \vartheta^*(\Phi g) \) satisfies \( d\Psi = \Psi (\vartheta^*(\xi, g)) \), and the monodromies of \( \Phi \) and \( \Psi \) on a loop around \( z = 0 \) are equal.

**Proof.** For any conformal maps \( \vartheta_1 : \Sigma' \to \Sigma, \vartheta_2 : \Sigma'' \to \Sigma' \) and \( r \)-gauges \( g_1 : \Sigma \to \Lambda_r^{\text{pos}}SL_2 \mathbb{C}, \) \( g_2 : \Sigma' \to \Lambda_r^{\text{pos}}SL_2 \mathbb{C} \), we have the composition formula
\[
(2.12) \quad \vartheta_2^* ((\vartheta_1^*(\xi_1, g_1)), g_2) = \vartheta^*(\xi, g) ,
\]
where \( \vartheta : \Sigma'' \to \Sigma \) is the conformal map and \( g : \Sigma \to \Lambda_r^{\text{pos}}SL_2 \mathbb{C} \) is the \( r \)-gauge defined by
\[
\vartheta = \vartheta_1 \circ \vartheta_2 \quad \text{and} \quad g = g_1 \cdot ((\vartheta_1^{-1})^* g_2) .
\]
The result then follows using the composition formula (2.12) by induction on \( n \), applying Lemma 2.5 and Lemma 2.6 repeatedly. The conditions (2.11) insure that condition (2.5) holds for each step. \( \square \)

**Lemma 2.8.** Let \( r \in (0, 1], X \in \Lambda_r SL_2 \mathbb{C}, \) and \( Y \in \Lambda_r^{\text{pos}}SL_2 \mathbb{C}. \) Then
(i) There exists \( U \in SU_2 \) such that
\[
(2.13) \quad \text{Uni}_r[XY] = \text{Uni}_r[X]U \quad \text{and} \quad \text{Pos}_r[XY] = U^{-1}\text{Pos}_r[X]Y .
\]
(ii) Let \( \xi \) be an \( r \)-potential on a Riemann surface \( \Sigma \). Let \( \Phi \) satisfy \( d\Phi = \Phi \xi \). Let \( g : \Sigma \to \Lambda_r^{\text{pos}}SL_2 \mathbb{C} \) be a gauge. Then \( \text{Sym}_r[\Phi] = \text{Sym}_r[\Phi g] \).

**Proof.** To prove (i), let \( Y_+ = \text{Pos}_r[Y] \) and let \( U_1 \) be the unitary factor in the QR-decomposition of \( Y(0) \). Then \( Y = U_1 Y_+ \). Let \( X_+ = \text{Uni}_r[X] \) and \( X_+ = \text{Pos}_r[X] \), so \( X = X_uX_+ \). Let \( U \) be the unitary factor in the QR-decomposition of \( X_+(0)U_1 \). Then \( X_+ U_1 = U \text{Pos}_r[X_+ U_1] \). Hence
\[
\text{Pos}_r[XY] = \text{Pos}_r[X_+ Y] = \text{Pos}_r[X_+ U_1 Y_+] = \text{Pos}_r[X_+ U_1]Y_+ + U^{-1}X_+ U_1 Y_+ = U^{-1}X_+ Y .
\]
This proves the second equality in (2.13), and the first equality follows from it.

To prove (ii), by (i) there exists \( U \in SU_2 \) such that \( \text{Uni}_r[\Phi g] = \text{Uni}_r[\Phi]U \). It follows that \( \text{Sym}_r[\Phi] = \text{Sym}_r[\Phi g] \). \( \square \)
2.3. Cauchy integral formula for vector-valued maps. The following technical lemma shows from the convergence of a family of holomorphic functions the convergence of all its derivatives in a strictly contained subdomain. The proof is standard and uses the Cauchy integral formula. We use the notation \( \mathcal{R}' \subset \subset \mathcal{R} \) to mean that \( \mathcal{R}' \) is bounded away from \( \partial \mathcal{R} \). Let \( X^{(n)} \) denote the \( n \)th derivative of \( X \) with respect to \( \lambda \).

**Lemma 2.9.** Let \( V \) be a finite dimensional vector space over \( \mathbb{C} \), and let \( | \cdot | \) be a vector norm on \( V \). With \( \Sigma \subset \mathbb{C} \) a punctured neighborhood of \( 0 \in \mathbb{C} \), and \( \mathcal{R} \subset \subset \mathcal{R} \) a bounded domain with smooth boundary, let \( X = X(z, \lambda) : \Sigma \times \mathcal{R} \to V \) be continuous in \( z \) and holomorphic in \( \lambda \). Suppose \( \lim_{z \to 0} \sup_{\lambda \in \mathcal{R}'} |X| = 0 \). Then for all \( n \in \mathbb{Z}_{\geq 0} \), and every subset \( \mathcal{R}' \subset \subset \mathcal{R} \),

\[
\lim_{z \to 0} \sup_{\lambda \in \mathcal{R}'} |X^{(n)}| = 0 .
\]

**Proof.** With \( n = \dim V \), we may assume that \( V = \mathbb{C}^n \). Since all vector norms on a finite dimensional vector space are equivalent, we may assume that the norm \( | \cdot | \) is given by \( |(Y_1, \ldots, Y_n)| = \max_j |Y_j| \). Let \( \mathcal{R}' \subset \subset \mathcal{R} \). By the Cauchy integral formula, for all \( n \in \mathbb{Z}_{\geq 0} \) and all \( \lambda \in \mathcal{R}' \),

\[
X^{(n)}(z, \lambda) = \frac{n!}{2\pi i} \int_{\partial \mathcal{R}} \frac{X(z, \nu)}{\nu - \lambda}^{n+1} d\nu ,
\]

so

\[
|X^{(n)}(z, \lambda)| \leq \frac{n!}{2\pi} \max_j \int_{\partial \mathcal{R}} \left| \frac{X_j(z, \nu)}{\nu - \lambda}^{n+1} \right| d\nu \leq \frac{n!}{2\pi} \int_{\partial \mathcal{R}} \left| \frac{X(z, \nu)}{\nu - \lambda}^{n+1} \right| d\nu .
\]

Since \( \lambda \in \mathcal{R}' \) and \( \mathcal{R}' \) is bounded away from \( \partial \mathcal{R} \), then there exists \( c_1 \in \mathbb{R}_{>0} \) such that for all \( \nu \in \partial \mathcal{R} \), \( |\nu - \lambda|^{-(n+1)} < c_1 \). Hence with \( c_2 = c_1 n!/(2\pi) \),

\[
|X^{(n)}(z, \lambda)| \leq c_2 \int_{\partial \mathcal{R}} |X(z, \nu)||d\nu| .
\]

Since this holds for all \( \lambda \in \mathcal{R}' \), and the right-hand side is independent of \( \lambda \), so

\[
\sup_{\lambda \in \mathcal{R}'} |X^{(n)}(z, \lambda)| \leq c_2 \int_{\partial \mathcal{R}} |X(z, \nu)||d\nu| .
\]

The result follows by taking limits. \( \square \)

2.4. Asymptotics of the perturbed Delaunay frame. We now have the tools to show that the \( r \)-Iwasawa factors of a holomorphic frame constructed from a perturbed Delaunay potential converge to the respective factors of a holomorphic frame constructed from a Delaunay potential ([Theorem 2.11](#)). This convergence implies that the immersions and their metrics converge ([Theorem 3.2](#)). A bootstrap argument in [section 3](#) strengthens this to \( C^\infty \)-convergence of the immersions.

**Lemma 2.10.** Let \( \xi \) be a perturbed Delaunay potential, let \( d\Phi = \Phi \xi \), and assume that the monodromy \( M \) of \( \Phi \) around \( z = 0 \) satisfies \( M \in \Lambda^*_s \text{SL}_2 \mathbb{C} \). With \( \Phi_0 = \exp(A \log z) \), let \( \Phi = C \Phi_0 P \) be the decomposition as in [Lemma 2.3](#). Let \( C \Phi_0 = F_0 \cdot B_0 \) and \( \Phi = F \cdot B \) be the respective \( r \)-Iwasawa factorizations on \( \Sigma^* \). Then \( F_0^{-1} F \), \( B \) and \( B_0 \) are lifts of unique single-valued maps on \( \Sigma^* \).

**Proof.** Let \( \tau : \hat{\Sigma}^* \to \hat{\Sigma}^* \) be a deck transformation corresponding to the curve defining the monodromy \( M \) of \( \Phi \). Then by the definition of \( M \), for all \( z \in \hat{\Sigma}^* \) we have \( \tau^* \Phi = M \Phi \), so

\[
(\tau^* F)(\tau^* B) = M F B ,
\]

or

\[
(\tau^* B) B^{-1} = (\tau^* F)^{-1} M F .
\]

(2.14)
The left hand side of (2.14) is in $\Lambda^*_+\text{SL}_2\mathbb{C}$, and since by hypothesis, $M \in \Lambda^*_+\text{SL}_2\mathbb{C}$, the right hand side of (2.14) is in $\Lambda^*_+\text{SL}_2\mathbb{C}$. Hence each side is equal to $I$, so $\tau^*B = B$, and $B$ is the lift of a single-valued map on $\Sigma^*$.

Since $\Phi = C \exp(A \log z)P$, and $\tau^*P = P$, then
\[ M = C \exp(2\pi i A)C^{-1}. \]

Since $\Phi_0 = C \exp(A \log z)$, then $M$ is also the monodromy of $\Phi_0$. Hence $\tau^*\Phi_0 = M\Phi_0$.

The same argument as above, with $F$ and $B_0$ replacing $F$ and $B$ respectively, shows that $\tau^*B_0 = B_0$, so $B_0$ is the lift of a single-valued map on $\Sigma^*$.

Since (2.15)
\[ F_0^{-1}F = B_0PB^{-1}. \]
and the right hand side of (2.15) is invariant under the action of $\tau^*$, then so is the left hand side of (2.15), so $F_0^{-1}F$ is the lift of a single-valued map on $\Sigma^*$.

**Theorem 2.11.** Let $r \in (0, 1)$ and assume $C_r \cap \mathcal{S}_A = \emptyset$. Let $\xi$ be a perturbed Delaunay $r$-potential
\[ \xi = Az^{-1}dz + O(z^n)dz. \]

Suppose
\[ \max_{\lambda \in C_r} \text{Re } \mu(\lambda) < \frac{n+1}{2}. \]

Let $\Phi : \tilde{\Sigma}^* \to \Lambda^*_r\text{SL}_2\mathbb{C}$ satisfy $d\Phi = \Phi \xi$ on the universal cover $\tilde{\Sigma}^* \to \Sigma^*$ of $\Sigma^*$, and assume that the monodromy $M$ of $\Phi$ around $z = 0$ satisfies $M \in \Lambda^*_+\text{SL}_2\mathbb{C}$. Let $\Phi_0 = \exp(A \log z)$ and let $\Phi = C\Phi_0P$ be the $z^4P$-decomposition of $\Phi$ as in Lemma 2.3. Assume for some $0 < s_1 < r < s_2 < 1$ there exists a continuous function $c : A_{s_1, s_2} \to \mathbb{R}_+$ such that $\text{Pos}_r[C\Phi_0]$ extends analytically to $A_{s_1, s_2}$ and
\[ \|\text{Pos}_r[C\Phi_0]\|_{A_{s_1, s_2}} \leq c|z|^{-\text{Re } \mu}. \]

Then $\Phi$ and $C\Phi_0$ have the convergence
\[ \lim_{x \to 0} \left\| (\text{Uni}_r[C\Phi_0])^{-1} \text{Uni}_r[\Phi] - 1 \right\|_{A_r} = 0, \]
\[ \lim_{x \to 0} \left\| \text{Pos}_r[\Phi] (\text{Pos}_r[C\Phi_0])^{-1} - 1 \right\|_{D_r} = 0. \]

**Proof.** Step 1. Let $v_1 = \max_{\lambda \in C_r} \text{Re } \mu(\lambda)$. There exist $r_1, r_2 \in \mathbb{R}_+$ such that $s_1 \leq r_1 < r < r_2 \leq s_2$, $A_{r_1, r_2} \cap \mathcal{S}_A = \emptyset$ and $\text{Pos}_r[C\Phi_0]$ extends analytically to $A := A_{r_1, r_2}$. By (2.18),
\[ \|\text{Pos}_r[C\Phi_0]\|_{A_r} \leq c|z|^{-v_1}. \]

Step 2. Let $C\Phi_0 = F_0 \cdot B_0$ and $\Phi = C\Phi_0P = F \cdot B$ and be the respective $r$-Iwasawa factorizations on $\tilde{\Sigma}^*$. Since by hypothesis $M \in \Lambda^*_+\text{SL}_2\mathbb{C}$, by Lemma 2.10 $B, B_0$ and $F_0^{-1}F$ on $\tilde{\Sigma}^*$ descend to single-valued analytic maps on $\Sigma^*$.

Let $m = n + 1$. By Lemma 2.3(iii), $P$ has the expansion on $\Sigma'$
\[ P(z, \lambda) = I + \sum_{k=m}^{\infty} P_k(\lambda)z^k. \]

Then
\[ B_0PB_0^{-1} - I = \sum_{k=m}^{\infty} B_0P_kB_0^{-1}z^k. \]

By (2.20),
\[ \|B_0\|_A \leq c|z|^{-v_1} \quad \text{and} \quad \|B_0^{-1}\|_A \leq c|z|^{-v_1}, \]
For $k$ entry $r$

Suppose (3.2)

Since by (3.10), $v_1 < m/2$, then $k - 2v_1 > 0$ for all $k \in \mathbb{Z}_{\geq m}$. Hence

\[
\lim_{z \to 0} \|B_0 P B_0^{-1} - 1\|_A = 0.
\]

By Lemma 2.9 $B_0 P B_0^{-1} \to I$ as $z \to 0$ as a map in $C^\infty(C_r, SL_2 \mathbb{C})$.

Step 3. We have the $r$-Iwasawa factorization

\[
B_0 P B_0^{-1} = (F_0^{-1} F) \cdot (B B_0^{-1}).
\]

As noted in the preliminary section, the $r$-Iwasawa factorization is a homeomorphism in the $C^\infty$-topologies on its domain and target spaces. The result (2.19) follows. \qed

3. Asymptotics of immersions

3.1. Convergence of geometric data. In Theorem 2.11 we showed that the unitary and positive part of a holomorphic frame constructed from a perturbed Delaunay potential converge to those of a Delaunay holomorphic frame. From this we obtain convergence of the immersions, metrics, and moving frames and normals. This convergence is strengthened to $C^\infty$-convergence in subsection 3.2.

Lemma 3.1. Let $\Sigma^* \subset \mathbb{C}$ be a punctured neighborhood of $0 \in \mathbb{C}$ with coordinate $z$. Let $r \in (0, 1]$. Let $\xi_1$ and $\xi_2$ be $r$-potentials. Let $\alpha_k$ be the coefficient of $\lambda^{-1}$ in the upper-right entry $\xi_k$ in its $\lambda$ expansion, and suppose

\[
\lim_{z \to 0} (\alpha_1/\alpha_2) = 1.
\]

For $k \in \{1, 2\}$, let $\Phi_k$ satisfy $d\Phi_k = \Phi_k \xi_k$. Let $R \subset \mathbb{C}$ be a neighborhood of $\lambda_0 \in \mathbb{S}^1$. Suppose

\[
\lim_{z \to 0} \|\text{Uni}_r[\Phi_1]^{-1}\text{Uni}_r[\Phi_2] - I\|_R = 0
\]

(3.2)

\[
\lim_{z \to 0} \text{Pos}_r[\Phi_1](\text{Pos}_r[\Phi_2])^{-1}|_{\lambda=0} = I.
\]

Let $f_k = \text{Sym}_r[\Phi_k]$ be the immersions obtained from $\Phi_k$, and let $v_k^2|dz|^2$ be the respective metrics of $f_k$. Then:

(i) The immersions converge, that is, \(\lim_{z \to 0} (f_2 - f_1)|_{\lambda_0} = 0\).

(ii) The metrics converge in ratio; that is, \(\lim_{z \to 0} (v_2/v_1)|_{\lambda_0} = 1\).

(iii) The moving frames $G_k$ for $f_k$ converge; that is, \(\lim_{z \to 0} (G_1^{-1} G_2)|_{\lambda_0} = I\).

(iv) The difference of the normals of $f_1$ and $f_2$ converges to $0$ as $z \to 0$.

Moreover, let $x + iy = \log z$ and let $w_k^2(dx^2 + dy^2)$ be the metric for $f_k$ in the $x$ and $y$ coordinates. If $w_1$ is bounded and bounded away from $0$ on $\Sigma$, then the immersions have $C^1$-convergence in the coordinates $x$ and $y$; that is,

\[
\lim_{z \to 0} \left( \frac{d}{dx} f_2 - \frac{d}{dx} f_1 \right)|_{\lambda_0} = 0 \quad \text{and} \quad \lim_{z \to 0} \left( \frac{d}{dy} f_2 - \frac{d}{dy} f_1 \right)|_{\lambda_0} = 0.
\]

Proof. For $k \in \{1, 2\}$, let $F_k = \text{Uni}_r[\Phi_k]$ and $B_k = \text{Pos}_r[\Phi_k]$. Let prime denote differentiation with respect to $\theta$, where $\lambda = e^{i\theta}$.

Step 1. To show the convergence of the immersions, by the Sym formula for $\mathbb{R}^3$ in the preliminary section, we have

\[
f_k = -2H^{-1} F_k F_k^{-1}, \quad k \in \{1, 2\},
\]
where $H \in \mathbb{R}_+$ is the constant mean curvature. Hence

$$f_2 - f_1 = -2H^{-1}F_{1}(F_1^{-1}F_2)'F_2^{-1}.$$  

Let $\mathcal{R}' \subset \subset \mathcal{R}$ be a neighborhood of $\lambda_0$. Since $F_1^{-1}F_2 - I$ is holomorphic on $\mathcal{R}$, by Lemma 2.9

$$\lim_{z \to 0} \| (F_1^{-1}F_2)' \|_{\mathcal{R}'} = 0.$$  

The convergence of the immersions $\lim_{z \to 0} (f_2 - f_1)|_{\lambda_0} = 0$ follows from (3.4) and (3.5) and the fact that $\| F_1(\lambda_0) \| = 1 = \| F_2^{-1}(\lambda_0) \|$.

**Step 2.** To show the convergence of the metrics, since $F_kB_k = \Phi_k$, then $B_k$ satisfies the gauge equation $(F_k^{-1}dF_k)B_k = \xi_k$. An examination of the coefficient of $\lambda^{-1}$ in this gauge equation above shows that

$$v_k = 2|H^{-1}\alpha_k|\rho_k^2,$$

where $\rho_k$ is the constant term of the upper-right entry of $B_k$ in its $\lambda$ expansion. Equation (3.1) implies $\lim_{z \to 0} (\rho_1/\rho_2) = 1$, and the result follows.

**Step 3.** The moving frames $G_k$ for $f_k$ are defined by the equations

$$\left. (f_k)_{x} = v_kG_k \left( \begin{array}{c} 0 \\ -z \\ 0 \end{array} \right) \right) \quad k \in \{1, 2\}.$$  

The extended frames $F_k$ and the moving frames $G_k$ are related by a gauge $g_k$ defined by

$$G_k = F_kg_k, \quad g_k = \text{diag}(p_k, p_k^{-1}), \quad p_k^2 = iH\alpha_k/(|\lambda|H\alpha_k) \quad k \in \{1, 2\}.$$  

Then

$$G_1^{-1}G_2 - I = g_1^{-1}(F_1^{-1}F_2 - g_1g_2^{-1})g_2.$$  

By (3.1), it follows that $\lim_{z \to 0} g_1(\lambda_0)g_2^{-1}(\lambda_0) = 1$. Since $|g_k(\lambda_0)|$ is finite, the convergence of the moving frames (iii) follows from (3.8). The convergence of the normals follows from the convergence of the moving frames.

**Step 4.** To show the convergence of the derivatives of the immersions (3.3), let

$$e_1 = \left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right) \quad \text{and} \quad e_2 = \left( \begin{array}{c} 0 \\ i \\ 0 \end{array} \right).$$  

Writing $x = x_1$ and $y = x_2$, by (3.7) we have $(f_k)_{x_1} = w_kG_ke_jG_1^{-1}$. Hence

$$\frac{d}{dx_1}(f_2 - f_1) = (w_2 - w_1)G_2e_jG_2^{-1} + w_1G_1G_2^{-1}G_2, \quad e_jG_2^{-1}.$$  

Since $\lim_{z \to 0}(v_2/v_1) = 1$ and $v_k = w_ke^{-v_2/2}$, then $\lim_{z \to 0}(w_2/w_1) = 1$. Since by assumption, $w_1$ or $w_2$ is bounded away from 0, then $\lim_{z \to 0}(w_2 - w_1) = 0$. The first equation of (3.3) follows, since $\| G_1(\lambda_0)e_jG_1^{-1}(\lambda_0) \|$ is finite, $w_1$ is bounded, $\| G_1(\lambda_0) \|$ and $\| G_2^{-1}(\lambda_0) \|$ are finite, and $\lim_{z \to 0} \| G_1^{-1}(\lambda_0)G_2(\lambda_0) \| = 0$ by (3.5). \qed

**Theorem 3.2.** Let $A$ be a Delaunay residue satisfying

$$\min_{\lambda \in \mathcal{C}_r} \text{Re } \mu(\lambda) \leq \frac{1}{2}.$$  

Let $\xi$ be a Delaunay $r$-perturbation of $A$. Let $\Phi$ satisfy $d:\Phi = \Phi \xi$ with unitary monodromy at $z = 0$. Assume $\Phi$ is in $\Lambda^0_{\xi}S_L$, and $(\mathcal{C}_r \cup \mathcal{A}_r, 1) \cap \mathcal{S}_A = \emptyset$. Let $f$ be the CMC immersion induced by $\Phi$. Then there exists a Delaunay immersion $f_0$ with the same necksize as that induced by $\Phi_0 := \exp(A \log z)$, such that as $z \to 0$, in the coordinates $x + iy = \log z$, the ratio of their metrics converges to 0, and $f - f_0$ converges $C^1$ to 0.

**Proof.** We shall require the following fact: as functions of $r$, max$_{\lambda \in \mathcal{C}_r} \text{Re } \mu$ is strictly decreasing on $(0, 1]$, and min$_{\lambda \in \mathcal{C}_r} \text{Re } \mu$ is 0 on $(0, |\nu_1|]$ and is strictly increasing on $[|\nu_1|, 1]$. (The boundary of the upper graph in $2(b)$ illustrates this behavior.)
Let \( v_0 = \min_{\lambda \in \mathbb{C}} \text{Re}\, \mu(\lambda) \) and \( v_1 = \max_{\lambda \in \mathbb{C}} \text{Re}\, \mu(\lambda) \). Since \( v_1 \geq v_0 \geq 1/2 \), there exists \( n \in \mathbb{Z}_+ \) such that
\[
\frac{n}{2} \leq v_1 < \frac{n+1}{2}.
\]
If \( v_1 = n/2 \), by the above fact, \( r \) can be replaced with a smaller value and \( v_0 \) and \( v_1 \) by the corresponding values so that this new \( v_1 \) satisfies
\[
\frac{n}{2} < v_1 < \frac{n+1}{2}.
\]
Then \( v_0 \) is still \( \leq 1/2 \) by the above fact. By Theorem 2.7 and Lemma 2.8 (ii), we may assume after a gauge and coordinate change that \( \xi \) has the form \((2.16)\).

Since \( \Phi \) is in \( \mathbb{A}^2 \mathbb{L} \mathbb{P} \mathbb{C} \), we have the \( z^A P \) decomposition \( \Phi = C \exp(A \log z)P \) on \( A_{r,1} \setminus S_A \). Since by assumption \((C_r \cup A_{r,1}) \cap S_A = \emptyset \), then \( C \) and \( P \) extend to \( C_r \cup A_{r,1} \).

By Theorem 1.13 there exists a continuous function \( c : D_1^* \to \mathbb{R}_+ \) such that for all \( (z, \lambda) \in \{0 < |z| < 1\} \times D_1^* \),
\[
|\text{Pos}_r[C\Phi_0]| \leq c|z|^{-\text{Re}\mu}.
\]
By Theorem 2.11 \( \Phi \) and \( C\Phi_0 \) have the convergence \((2.19)\). Hence taking \( \Phi_1 = \Phi_0 \) and \( \Phi_2 = \Phi \) in Lemma 3.1 condition \((3.2)\) holds. By Lemma 1.12 \( C\Phi_0 \) induces a Delaunay immersion which differs from \( \Phi_0 \) by a rigid motion. Taking \( \xi_1 = Adz/z \) and \( \xi_2 = \xi \) in Lemma 3.1 condition \((3.1)\) of Lemma 3.1 holds because \( \xi_2 \) is a holomorphic perturbation of \( \xi_1 \).

With \( w_1 \) and \( w_2 \) as in Lemma 3.1 a computation shows that with \( v \) as in Theorem 1.5 by \((3.6)\), the metric of the Delaunay immersion is \( w_1 = 4|abH^{-1}|v^{-1} \). Hence \( w_1 \) is periodic and nonzero in the coordinate \( x = \log |z| \), and is hence bounded and bounded away from 0 on \( \Sigma \). The assertion follows by Lemma 3.1. \( \square \)

### 3.2. Asymptotics of conformal immersions

Now that we have shown that the frames and metrics of the immersion produced by a perturbed Delaunay potential converge to those of a Delaunay immersion, we apply a bootstrap argument on the Gauss equation to obtain \( C^\infty \)-convergence of the immersions up to rigid motion. For details and formulas, see [2].

The notation \( f \to C^0 \) means \( \lim_{z \to 0} |f - g|_{C^0} = 0 \), where the \( C^0 \) norm is taken over some implied compact domain.

**Theorem 3.3.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain, and let \( \Omega' \subset \subset \Omega \) be a strictly contained subset. Let \( f : \overline{\Omega} \to \mathbb{R}^3 \) and the sequence \( f_j : \overline{\Omega} \to \mathbb{R}^3 \), \( j \in \mathbb{Z}_+ \) be a sequence of conformal immersions in \( C^\infty(\Omega) \cap C^0(\overline{\Omega}) \). Let \( e^\mu, H, Q dz^2 \) and \( e^\nu, H, Q dz^2 \) be the respective conformal factors, mean curvature functions, and Hopf differential of \( f \) and the \( f_j \). Suppose that \( f_j \to f, H_j \to H, Q_j \to Q \) and \( u_j \to u \) on \( \Omega \). Then \( f_j \to f \) on \( \Omega' \).

**Proof.** We will use the following gradient estimate for Poisson’s equation [7]. Let \( \Sigma \) be a bounded domain in \( \mathbb{R}^2 \) and \( \Sigma' \subset \subset \Sigma \). Then there exists \( c \in \mathbb{R}_+ \) depending only on \( \text{dist}(\Sigma', \partial\Sigma) \) such that for all \( w \in C^2(\Sigma) \cap C^0(\overline{\Sigma}) \),
\[
\sup_{\Sigma'} |\partial_x w| + \sup_{\Sigma'} |\partial_y w| \leq c(\sup_{\Sigma} |w| + \sup_{\Sigma} |\Delta w|).
\]
Fix \( k \in \mathbb{N}_{\geq 1} \) and choose domains \( \Omega_1, \ldots, \Omega_k \) with \( \Omega' = \Omega_k \subset \subset \cdots \subset \subset \Omega_1 \subset \subset \Omega \).

**Step 1.** We apply the estimate \((3.11)\) \( k \) times to show that \( u_j \to u \) and \( H_j \to H \) on \( \Omega' \).

The first iteration is as follows. For the functions \( u, H, Q \), define
\[
\Psi[u, H, Q] = -\frac{1}{2} H^2 e^w + 2|Q|^2 e^{-w}.
\]
Then the Gauss equations for \( f \) and the \( f_j \) are respectively
\[
\Delta u = \Psi[u, H, Q], \quad \Delta u_j = \Psi[u_j, H, Q_j].
\]
Hence
\[ \Delta(u_j - u) = \Phi[u_j, H_j, Q_j] - \Phi[u, H, Q]. \]

Since \( u_j \xrightarrow{C^0} u, H_j \xrightarrow{C^0} H \) and \( Q_j \xrightarrow{C^0} Q \) on \( \Omega \) then
\[ \Phi[u_j, H_j, Q_j] - \Phi[u, H, Q] \xrightarrow{C^0} 0, \]
so \( \Delta(u_j - u) \xrightarrow{C^0} 0 \) on \( \Omega \). Applying the estimate (3.11) with \( \Sigma = \Omega, \Sigma' = \Omega_1 \), and \( w = u_j - u \), we obtain \( u_j \xrightarrow{C^1} u \) on \( \Omega_1 \).

The Codazzi equations for \( f \) and the \( f_j \) are respectively
\[ H_z = 2e^{-u}Q_\tau, \quad (H_j)_z = 2e^{-u_j}(Q_j)_\tau. \]

Since \( u_j \xrightarrow{C^0} u \) and \( Q_j \xrightarrow{C^1} Q \) on \( \Omega \), it follows that \( H_j \xrightarrow{C^1} H \) on \( \Omega \).

Using the Gauss equation again in the same way, since \( u_j \xrightarrow{C^1} u, H_j \xrightarrow{C^1} H \) and \( Q \xrightarrow{C^1} Q \) on \( \Omega_1 \), then \( \Delta(u_j - u) \xrightarrow{C^1} 0 \). Hence \( \Delta \partial_x(u_j - u) \xrightarrow{C^0} 0 \) and \( \Delta \partial_y(u_j - u) \xrightarrow{C^0} 0 \). Applying the estimate (3.11) again, with \( \Sigma = \Omega_1, \Sigma' = \Omega_2 \), and \( w \) take to be first \( \partial_x(u_j - u) \) and then \( \partial_y(u_j - u) \), we obtain \( u_j \xrightarrow{C^2} u \) on \( \Omega_2 \).

Repeating the argument \( k - 2 \) more times shows that \( u_j \xrightarrow{C^k} u \) and \( H_j \xrightarrow{C^{k+1}} H \) on \( \Omega_k = \Omega' \).

Since \( k \) is arbitrary, the above argument shows \( u_j \xrightarrow{C^\infty} u \) and \( H_j \xrightarrow{C^\infty} H \) on \( \Omega' \).

Step 2. Let \( e_1, e_2, e_3 \) be a positively oriented orthonormal basis for \( \mathfrak{su}_2 \). Let \( F \) and \( F_j \) be the respective moving frames for \( f \) and \( f_j \) with values in \( \mathfrak{su}_2 \), defined respectively by (3.12)
\[ \partial_x f = e^u F e_1 F^{-1}, \quad \partial_y f = e^u F e_2 F^{-1}, \quad \partial_x f_j = e^u F_j e_1 F_j^{-1}, \quad \partial_y f_j = e^u F_j e_2 F_j^{-1}. \]

By assumption, \( f_j \xrightarrow{C^1} f \), from which it follows that \( F_j \xrightarrow{C^0} F \).

Write the Lax pairs for \( F \) and \( F_j \) respectively
\[ \partial_x F = FU, \quad \partial_y F = FV, \quad \partial_x F_j = F_j U_j, \quad \partial_y F_j = F_j V_j. \]

Since \( u_j \xrightarrow{C^\infty} u, H_j \xrightarrow{C^\infty} H \) and \( Q_j \xrightarrow{C^\infty} Q \), then \( U_j \xrightarrow{C^\infty} U \) and \( V_j \xrightarrow{C^\infty} V \). Since the Lax pair equations are linear and \( F_k \xrightarrow{C^0} F \), it follows that \( F_j \xrightarrow{C^\infty} F \) by smooth dependence of solutions on parameters. By (3.12), we obtain the convergence \( f_j \xrightarrow{C^\infty} f \). \( \square \)

3.3. Embedded ends. Under appropriate assumptions, if an immersion \( f \) of a cylinder converges \( C^2 \) to an embedding of the cylinder, then \( f \) is embedded.

Let \( \Sigma_{Cyl} \) be the cylinder
\[ \Sigma_{Cyl} = \{ (x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } 0 \leq y \leq 2\pi \}, \]
with the boundary lines \( y = 0 \) and \( y = 2\pi \) identified.

Lemma 3.4. With \( \Sigma_{Cyl} \) as in (3.13), let \( f : \Sigma_{Cyl} \rightarrow \mathbb{R}^3 \) be a properly embedded surface with bounded curvature which satisfies the following condition:
\[ \text{For all } \delta > 0 \text{ there exists } \epsilon > 0 \text{ such that for all } p, q \in \Sigma_{Cyl}, \]
\[ \text{if } |p - q|_{\Sigma_{Cyl}} > \delta, \text{ then } |f(p) - f(q)|_{\mathbb{R}^3} > \epsilon. \]

Let \( g : \Sigma_{Cyl} \rightarrow \mathbb{R}^3 \) be an immersion which converges to \( f \) as \( x \rightarrow +\infty \), in the \( C^2 \)-topology of \( \Sigma_{Cyl} \). Then there exists \( x_0 \in \mathbb{R}_+ \) such that \( g \) restricted to \( \{ x > x_0 \} \) is properly embedded.
Remark 3.6. When \( f \) is embedded, the rigid motion of \( f \) converges exponentially to \( f_0 \) in the sense of \([17]\), as proven there.

Proof. The \( C^0 \)-convergence of \( g \) to \( f \) and the properness of \( f \) implies that \( f \) is proper.

Since \( f \) has bounded curvature, there exists \( \delta > 0 \) such that for any two points \( p, q \in \mathbb{R}^2 \), if \( |p - q|_{\Sigma_{Cyl}} < \delta \), then the total curvature of the image under \( f \) of the shortest straight line from \( p \) to \( q \) in \( \Sigma_{Cyl} \) is at most \( \pi/4 \). By the \( C^2 \)-convergence of \( g \) to \( f \), we can choose \( x_1 > 0 \) so that the total curvature of the image under \( g \) of those straight lines is less than \( \pi/2 \) when the \( x \) coordinates of \( p \) and \( q \) are greater than \( x_1 \).

With the above choice of \( \delta \), let \( \epsilon \) be as in \((3.13)\). By the \( C^0 \)-convergence of \( g \) to \( f \), we can choose an \( x_2 \) so that \( |f(x) - f_0(x)| < \epsilon/4 \) for any point for which \( x > x_2 \). Set \( x_0 = \max\{x_1, x_2\} \).

Now suppose that \( g \) is not embedded on \( \Sigma_{Cyl} \cap \{x > x_0\} \), so there exist distinct \( p, q \in \Sigma_{Cyl} \cap \{x > x_0\} \) such that \( g(p) = g(q) \). Then the proof divides into two cases.

Case 1: \( |p - q|_{\Sigma_{Cyl}} \geq \delta \). Then \( |f(p) - f(q)|_{\mathbb{R}^3} > \epsilon \). Thus \( |g(p) - g(q)|_{\mathbb{R}^3} > \epsilon/2 \), which contradicts \( g(p) = g(q) \).

Case 2: \( |p - q|_{\Sigma_{Cyl}} < \delta \). Let \( \gamma : [0, 1] \to \Sigma_{Cyl} \) be the parametrized straight line from \( p \) to \( q \). Then \( g \circ \gamma \) is a closed curve in \( \mathbb{R}^3 \) which is smooth except at one point. If \( g \circ \gamma \) is smoothed out at that point, it has total curvature \( \geq 2\pi \). But the turning angle of the tangent at the non-smooth point is \( \leq \pi \), hence the total curvature of the smoothed out \( g \circ \gamma \) is \( \leq 3\pi/2 \), a contradiction. \( \square \)

3.4. Asymptotics of Delaunay immersions. We conclude Part 1 with its culminating theorem, showing that a CMC end obtained from a perturbed Delaunay potential is asymptotic to a half-Delaunay surface. This theorem restricts itself to the case that the holomorphic frame extends analytically to \( \mathcal{A}_{r,1} \).

Theorem 3.5. Let \( A \) be a Delaunay residue as in \((1.1)\) satisfying condition \((3.9)\). Let \( r \in (0, 1] \) and assume \( \mathcal{A}_{r,1} \cap \mathcal{S}_A = \emptyset \). On the punctured unit disk \( \Sigma^* = \{z \in \mathbb{C} | 0 < |z| < 1\} \), let

\[
\xi = A \frac{dz}{z} + O(z^0)dz
\]

be a perturbed Delaunay \( r \)-potential. Let \( f_0 \) and \( f \) be the immersions of \( \Sigma^* \) induced by the generalized Weierstrass representation at \( \lambda = 1 \) by \( Adz/z \) and \( \xi \) respectively, so \( f_0 \) is a Delaunay immersion. Assume that \( f \) is obtained from a holomorphic \( r \)-frame with values in \( \Lambda^*_0 SL_2 \mathbb{C} \) whose monodromy at \( z = 0 \) is in \( \Lambda^*_0 SL_2 \mathbb{C} \).

With \( \Sigma_{Cyl} \) as in \((3.13)\), let \( \phi : \Sigma_{Cyl} \to \Sigma^* \) be the map \( \phi(x, y) = e^{x+iy} \). Then some rigid motion of \( f \circ \phi \) converges to \( f_0 \circ \phi \) in the \( C^\infty \)-topology of \( \Sigma_{Cyl} \) as \( x \to -\infty \). Furthermore, if \( f_0 \) is embedded, then \( f \) is properly embedded.

Proof. Let \( \rho \) be the period of the Delaunay surface \( f_0 \). Let \( \Sigma_{Rect} = [0, \rho] \times [0, 2\pi] \subset \mathbb{R}^2 \). Fix a small \( \epsilon > 0 \). Then \( f \circ \phi \) and \( f_0 \circ \phi \) can be lifted to the \( \epsilon \)-neighborhood \( \mathcal{N}_\epsilon(\Sigma_{Rect}) \) of \( \Sigma_{Rect} \). We define a sequence \( g_k \) of \( C^\infty \) functions on \( \Omega \) by

\[
g_k(x, y) = (f \circ \phi)((x + kp) + iy)
\]
on \( \Sigma_{Rect} \) and extend \( g_k \) real analytically to \( \mathcal{N}_\epsilon(\Sigma_{Rect}) \).

Let \( \mathcal{U}, H, Q \) and \( u_k, H_k, Q_k \) be the geometric data for \( f_0 \) and the \( g_k \) respectively. Let \( \Phi_0 \) and \( \Phi \) be the respective solutions to \( d\Phi_0 = \Phi Adz/z \) and \( d\Phi = \Phi \xi \) which respectively induce \( f_0 \) and \( f \) via the generalized Weierstrass representation. By \( \text{Theorem 3.2} \), the metrics are \( v = e^u \) and \( v_k = e^{u_k} \). We have \( C^1 \)-convergence of \( g_k \) to some rigid motion of \( f_0 \circ \phi \) on \( \mathcal{N}_\epsilon(\Sigma_{Rect}) \) as \( x \to \infty \), and \( v/v_k \to 1 \) as \( x \to \infty \). Hence \( u_k \to u \) on \( \mathcal{N}_\epsilon(\Sigma_{Rect}) \). We have that \( Q_k \to Q \), and \( H_k = H \) for all \( k \), so \( \text{Theorem 3.3} \) and \( \text{Lemma 3.4} \) imply the theorem. \( \square \)
Remark 3.7. The above Delaunay asymptotics result for Euclidean space $\mathbb{R}^3$ most likely also holds for the spaceform $S^3$, with appropriate changes in the Sym and moving frame formulas. For hyperbolic space $\mathbb{H}^3$, the value $\lambda_0$ in the Sym formula is in the interior of the unit disk, so if the asymptotics result is to hold, its proof would require the results of Part 2.

Part 2. Dressed Delaunay asymptotics

Part 2 generalizes the results in Part 1 to the setting of $\Lambda_r \text{SL}_2 \mathbb{C}$ for arbitrary $r$. It shows for this larger class of frames that an immersion constructed from a perturbed Delaunay $r$-potential via $r$-dressing is asymptotic to a Delaunay surface.

Outline of results

Dressing by simple factors performs a Bianchi-Bäcklund transformation on a CMC surface. The formula for dressing by simple factors is known explicitly [27]. In subsection 4.2 we show that a dressed Delaunay frame $C \exp(A \log z)$ which has unitary monodromy is a multibubbleton frame, that is, a Delaunay frame dressed by a finite product of simple factors (Theorem 4.4). This result is used in subsection 5.4.

In subsection 5.1 we show that dressing by a finite product $G$ of simple factors with distinct singularities preserves asymptotics (Theorem 5.2).

In Equation 5.1 we show that if an immersion $f$ converges to a Delaunay immersion $f_0$, then $f$ dressed by a product of simple factors with distinct singularities also converges to a Delaunay immersion with the same necksize as $f_0$ (Theorem 5.6). Together with the methods of section 3, this implies that dressing by finitely many simple factors preserves convergence to a Delaunay surface (Theorem 5.7).

Given a holomorphic perturbation

$$\xi = A \frac{dz}{z} + O(z^0)dz$$

of the Delaunay potential $Adz/z$ which represents a closed once-wrapped Delaunay surface, let $\Phi$ satisfy $d\Phi = \Phi \xi$ with unitary monodromy at $z = 0$. In subsection 5.4 we show that the unitary factor of $\Phi$ is asymptotic to a Delaunay frame and the metric of the surface induced by $\Phi$ is asymptotic to the corresponding Delaunay metric (Theorem 5.8).

These asymptotic frames results are applied to construct CMC surfaces in $\mathbb{R}^3$. The surface induced by $\Phi$ is asymptotic to a half-Delaunay surface induced by $Adz/z$ (Theorem 5.9).

4. Simple factor dressing

4.1. Simple factors. We recall the notion of a simple factor [27].

Definition 4.1. Given $r \in (0, 1]$ and $\lambda_0 \in \mathcal{A}_{r, 1}$, let $f$ be the unique rational map on $\mathbb{CP}^1$ with degree one whose pole is at $\lambda_0$, satisfying $f^* = f^{-1}$ and $f(1) = 1$. (Here, $f^{-1}$ denotes the multiplicative inverse). Let $L \in \mathbb{CP}^1$. An unnormalized simple factor $g^0[\lambda_0, L] \in \Lambda^+_{r, \text{SL}_2 \mathbb{C}}$ is a loop

$$g^0[\lambda_0, L] = f^{1/2} \pi_L + f^{-1/2} \pi_{L^\perp},$$

where $\pi_L$ denotes the orthogonal projection to $L$. A normalized simple factor $g[\lambda_0, L] \in \Lambda^+_{r, \text{SL}_2 \mathbb{C}}$ is a map of the form $g[\lambda_0, L] = U^{-1} g^0[\lambda_0, L]$, where $U$ is the unitary factor of the QR-decomposition of $g^0[\lambda_0, L]|_{\lambda = 0}$. A general simple factor is a map of the form $U g \in \Lambda^+_{r, \text{SL}_2 \mathbb{C}}$, where $U \in \text{SU}_2$ and $g$ is a normalized simple factor.
By Proposition 4.2 in [27], dressing by simple factors is explicit: given \( r \in (0, 1) \), an extended \( r \)-unitary frame \( F(z, \lambda) \), and a normalized simple factor \( g[\lambda_0, L] \) with \( \lambda_0 \in \mathcal{A}_{r,1} \), we have the formula [27] (see also [13, 15])

\[
\text{Uni}_r[g[\lambda_0, L] F] = g[\lambda_0, L] F g[\lambda_0, \overline{F(z, \lambda_0)} L]^{-1}.
\]

While simple factors are positive \( r \)-loops, the product of two simple factors with the same singularity extends to a meromorphic map on \( \mathbb{CP}^1 \).

**Lemma 4.2.** For \( k \in \{1, 2\} \), let \( g_k = U_k^{-1}g^\circ[\lambda_0, L_k] \) be general simple factors, where \( \lambda_0 \in \mathcal{D}_r^k \), \( L_k \in \mathbb{CP}^1 \), and \( U_k \in \text{SU}_2 \). Let \( X : \mathcal{R} \to M_{2 \times 2} \mathbb{C} \) be a meromorphic map on a domain \( \mathcal{R} \subset \mathbb{C} \). Then \( g_1 X g_2^{-1} \) extends meromorphically to a map \( Y \) on \( \mathcal{R} \). Moreover, suppose \( \mathcal{R} \) is invariant under the map \( \lambda \to 1/\lambda \). Then

(i) If \( g_1 = g_2 \) and \( X^* = X \), then \( Y^* = Y \) away from its poles.

(ii) If \( X \) is invertible and \( X^* = X^{-1} \), then \( Y^* = Y^{-1} \) away from its poles.

**Proof.** By the definition of simple factors (Definition 4.1), for \( k \in \{1, 2\} \) we can write \( g_k = U_k f_{k}^{-1/2} h_k \), where \( U_k \in \text{SU}_2 \), \( h_k = f \pi_k + \pi_{k}^* \), \( f \) has a simple pole at \( \lambda_0 \), \( f^* = f^{-1} \), and \( h_k^* = h_k^{-1} \). Then \( g_1 X g_2^{-1} = U_1 h_1 X h_2^{-1} U_2^{-1} \) extends meromorphically to \( \mathcal{R} \) because all its components do. Statements (i) and (ii) follow from the symmetries \( f^* = f^{-1} \), \( h_k^* = h_k^{-1} \), and \( U_k^* = U_k^{-1} \).

### 4.2. The dressed Delaunay frame

We shall prove that if a dressing matrix preserves the unitarity of a Delaunay monodromy, then the dressing matrix is a product of simple factors and a loop \( V \) which conjugates the Delaunay residue to a Delaunay residue. The effect of \( V \) is a coordinate change and a rigid motion of the surface. Thus, for Delaunay surfaces, the class of dressing matrices that preserve unitarity of monodromy coincide with the Bianchi-Bäcklund transformations.

**Lemma 4.3.** Let \( A \) be a Delaunay residue and let \( M = \exp(2\pi i A) \). With \( r \in (0, 1) \), let \( C_+ \in A^+_r \text{SL}_2 \mathbb{C} \) and suppose \( C_+ MC_+^{-1} \in A^+_r \text{SL}_2 \mathbb{C} \). Then \( C_+ MC_+^{-1} \) extends to a rational map on \( \mathbb{CP}^1 \) all of whose poles are simple and lie in \( (\mathcal{S}_A \cap \mathcal{A}_r \setminus \{1\}) \cup \{0, \infty\} \).

**Proof.** Let \( \mu \) be an eigenvalue of \( A \) as in [Definition 1.2]. Let \( x, y : \mathbb{C}^* \to \mathbb{C} \) be the holomorphic functions \( x = \cos(2\pi \mu) \) and \( y = \mu^{-1} \sin(2\pi \mu) \). Note that \( x \) and \( y \) are always holomorphic on \( \mathbb{C}^* \), even if \( \mu \) is not. Then \( M = x I + iy A \). Writing \( A_1 = C_+ AC_+^{-1} \), on \( D_r^* \) we have

\[
C_+ MC_+^{-1} = x I + iy A_1.
\]

Since by hypothesis \( C_+ MC_+^{-1} \) extends holomorphically to \( \mathbb{C}^* \), and since \( x I \) is holomorphic on \( \mathbb{C}^* \), then \( y A_1 \) extends holomorphically to \( \mathbb{C}^* \). Since \( y \) is holomorphic on \( \mathbb{C}^* \), then \( A_1 \) extends meromorphically to \( \mathbb{C}^* \), and for all \( \lambda \in \mathbb{C}^* \)

\[
\text{ord}_\lambda A_1 \geq -\text{ord}_\lambda y.
\]

We compute the orders of the poles of \( A_1 \) on \( \mathbb{C}^* \). These occur only at the zeros of \( y \), all of which are in \( \mathcal{S}_A \). Let \( \lambda_0 \in \mathbb{C}^* \) be a zero of \( y \). Then \( \mu(\lambda_0) \in i/\mathbb{Z}^* \).

**Case 1.** If \( \lambda_0 = 0 \), then \( C_+ \) is holomorphic and \( A_1 \) has a simple pole, so \( A_1 \) is meromorphic with at worst a simple pole at 0.

**Case 2.** If \( 0 < |\lambda_0| \leq r \), then \( A_1 \) is holomorphic at \( \lambda_0 \) because \( C_+ \) and \( A \) are.

**Case 3.** \( r < |\lambda_0| < 1 \). It can be shown that \( \{\lambda \in \mathbb{C}^* | \mu'(\lambda) = 0\} \subset \mathbb{S}^1 \). Hence \( \mu'(\lambda_0) \neq 0 \), from which it follows that \( \text{ord}_{\lambda_0} y = 1 \). By (4.1) \( \text{ord}_{\lambda_0} A_1 \geq -1 \), so \( A_1 \) has at most a simple pole at \( \lambda_0 \).

**Case 4.** \( |\lambda_0| = 1 \). In this case, \( A_1 \) is holomorphic at \( \lambda_0 \) as in the proof of Lemma 1.12.
Since \( \exp(2\pi i A_1) \) takes values in \( \text{SU}_2 \) on \( S^1 \), and \( A_1 \) is tracefree, then \( A_1 \) has the hermitian symmetry \( A_1^* = A_1 \). Hence the poles of \( A_1 \) on \( \mathbb{C}^* \) lie in \( (A_r \setminus S^1) \cap S_A \).

**Theorem 4.4.** Let \( A \) be a Delaunay residue and let \( M = \exp(2\pi i A) \). With \( r \in (0, 1) \), let \( C_+ \in \Lambda^+_r \text{SL}_2 \mathbb{C} \) and suppose \( C_+ M C_+^{-1} \in \Lambda^+_r \text{SL}_2 \mathbb{C} \). Then there exists a loop \( G \in \Lambda^+_r \text{SL}_2 \mathbb{C} \) which is a product of normalized simple factors (or \( G = I \)), the singularities of the simple factors are distinct and lie in \( S_A \cap (A_r \setminus S^1) \), and, with \( V = G^{-1} C_+ \), \( A_1 = V A V^{-1} \) is a Delaunay residue.

**Proof.** The proof is by induction on the number of poles of \( C_+ A_1 C_+^{-1} \) in \( A_{r,1} \). We prove the following induction step, which constructs a single simple factor \( g \).

With \( A, M, r \) and \( C_+ \) as in the statement of the theorem, there exists a normalized simple factor \( g \in \Lambda^+_r \text{SL}_2 \mathbb{C} \) such that, with \( V = g^{-1} C_+ \in \Lambda^+_r \text{SL}_2 \mathbb{C} \), we have \( V M V^{-1} \in \Lambda^+_r \text{SL}_2 \mathbb{C} \), and \( V A V^{-1} \) has one fewer pole in \( D_1 \) than \( C_+ A_1 C_+^{-1} \).

Proof of the induction step:

By **Lemma 4.3** \( C_+ A_1 C_+^{-1} \) extends to a rational map on \( \mathbb{CP}^1 \) all of whose poles are simple and lie in \( (S_A \cap (A_r \setminus S^1)) \cup \{0, \infty\} \). Let \( M = C_+ M C_+^{-1} \) and \( A_1 = C_+ A_1 C_+^{-1} \).

Let \( x = \cos(2\pi \mu) \) and \( y = \mu^1 \sin(2\pi \mu) \) be the holomorphic functions on \( \mathbb{C}^* \) as in the proof of **Lemma 4.3** so that \( M = x I + iy A \). Then

\[
C_+ M C_+^{-1} = x I + iy A_1.
\]

Choose a pole \( \lambda_0 \in S_A \cap (A_r \setminus S^1) \) of \( A_1 \), so ord\( \lambda_0 A_1 = -1 \). Since \( \lambda_0 \in S_A \), then \( \mu(\lambda_0) \in \frac{1}{2} \mathbb{Z}^* \), so \( \iota := x(\lambda_0) \in \{1, -1\} \). As in case 3 of the proof of **Lemma 4.3** \( \text{ord}_{\lambda_0} y A_1 = 0 \). Then \( M_1(\lambda_0) \neq \iota I \), by [13]. Hence \( M_1(\lambda_0) \) is not semisimple, so the eigenspace of \( M_1(\lambda_0) \) associated to the eigenvalue \( \iota \) is one-dimensional. Let \( v \in \mathbb{C}^2 \setminus \{0\} \) be an eigenvector of \( M_1(\lambda_0) \) and let \( L = [v] \in \mathbb{CP}^1 \). Define the unnormalized simple factor \( h = g^\lambda_{\lambda_0}(\lambda_0, L) \) on \( D_1 \).

We show that \( h^{-1} A_1 h \) is holomorphic at \( \lambda_0 \). Let \( f \) be as in **Definition 4.1** for \( h \), so \( h = f^{1/2} P L + f^{-1/2} \pi L \) and ord\( \lambda_0 f = -1 \). Working in the basis \( \{v, v^\perp\} \), define the functions \( a, b, c, d \) by \( A_1 v = av + cv^\perp \) and \( A_1 v^\perp = bv + dv^\perp \). Since span\( \mathbb{C}\{v\} \) is the kernel and image of \( y A_1 |_{\lambda_0} \), then

\[
\begin{align*}
\text{ord}_{\lambda_0} a &\geq 0, \\
\text{ord}_{\lambda_0} b &\leq -1, \\
\text{ord}_{\lambda_0} c &\geq 0, \\
\text{ord}_{\lambda_0} d &\geq 0.
\end{align*}
\]

Then det \( A_1 = ad - bc \), so ord\( \lambda_0 (ad - bc) \geq 0 \). Hence ord\( \lambda_0 c \geq 1 \). On the other hand,

\[
h^{-1} A_1 h v = av + fc v^\perp \text{ and } h^{-1} A_1 h v^\perp = f^{-1} bv + dv^\perp.
\]

Hence \( h^{-1} A_1 h v \) and \( h^{-1} A_1 h v^\perp \) are holomorphic at \( \lambda_0 \), so \( h^{-1} A_1 h \) is holomorphic at \( \lambda_0 \).

Let \( U \in \text{SU}_2 \) be the unitary factor in the QR-decomposition of \( h(0) \), and define \( g = h U^{-1} \).

Then \( g \in \Lambda^+_r \text{SL}_2 \mathbb{C} \) is a normalized simple factor. Define \( V = g^{-1} C_+ \). Then \( C_+ \) and \( V \) have the same poles on \( \mathbb{CP}^1 \setminus \{\lambda_0, 1/\lambda_0\} \). Since \( C_+ A_1 C_+^{-1} \) has a pole at \( \lambda_0 \), and as shown above, \( V A V^{-1} \) does not, then \( V A V^{-1} \) has one fewer pole that \( C_+ A_1 C_+^{-1} \) in \( D_1 \).

Since \( A_1 \) is hermitian away from its poles, then so is \( h^{-1} A_1 h \) by **Lemma 4.2** i). By conjugating \([13]\) by \( h^{-1} \), we obtain that \( h^{-1} M_1 h \in \Lambda^+_r \text{SL}_2 \mathbb{C} \). Then \( V M V^{-1} = U h^{-1} M_1 h U^{-1} \in \Lambda^+_r \text{SL}_2 \mathbb{C} \). This proves the induction step.

To prove the theorem, let \( n \) be the number of poles of \( C_+ A_1 C_+^{-1} \) in \( A_{r,1} \). If \( n = 0 \), then by **Lemma 1.12** \( C_+ A_1 C_+^{-1} \) is a Delaunay residue, and the theorem follows with \( G = I \) and \( V = C_+ \). Otherwise, repeated application of the induction step produces \( n \) normalized simple factors \( g_1, \ldots, g_n \in \Lambda^+_r \text{SL}_2 \mathbb{C} \) with distinct singularities in \( S_A \cap (A_r \setminus S^1) \), such that, with \( G = g_1 \cdots g_n \) and \( V = G^{-1} C_+ \in \Lambda^+_r \text{SL}_2 \mathbb{C} \), \( V A V^{-1} \) has no poles in \( A_r \), and hence in

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\(D_1\), and \(V MV^{-1} \in \Lambda^+_{1} \text{SL}_2 \mathbb{C}\). By \textbf{Lemma 1.12}, \(VAV^{-1}\) is a Delaunay residue. This proves the theorem.

### 4.3. Dressed Delaunay growth. \textbf{Lemma 1.11} showed that under the assumption of unitary monodromy, dressing a Delaunay holomorphic frame \(\exp(A \log z)\) by a loop \(C \in \Lambda^+_{1} \text{SL}_2 \mathbb{C}\) does not affect the growth behavior of its positive factor, provided that \(C \in \Lambda^+_{1} \text{SL}_2 \mathbb{C}\). \textbf{Theorem 4.6} generalizes this result by removing this provision.

\textbf{Lemma 4.5}. Let \(h(z) = U(z)^{-1} g^e[\lambda_0, L(z)]\) be a map from a domain \(\Sigma \subset \mathbb{C}\) into the space of general simple factors. Then for every set \(R \subset \mathbb{C}^1\) bounded away from \(\{\lambda_0, 1/\lambda_0\}\), \(|h(z)|_R\) is bounded as a function of \(z\).

\textbf{Proof}. With \(f\) and \(U\) as in \textbf{Definition 4.1}, write

\[h = U^{-1} \left( f(\lambda)^{1/2} \pi_L(z) + f(\lambda)^{-1/2} \pi_L(z) \right).

Then \(|h| = \max\{|f|^{1/2}, |f|^{-1/2}\}\. On any set \(R \subset \mathbb{C}^1\) bounded away from \(\{\lambda_0, 1/\lambda_0\}\), the function \(|f|\) is bounded away from 0 and \(\infty\), and so \(|f|^{1/2}\) and \(|f|^{-1/2}\) are bounded on \(R\). Hence for all \(z \in \Sigma\),

\[|h(z)|_R \leq \max\{|f|^{1/2}|_R, |f|^{-1/2}|_R\}\. \qed

\textbf{Theorem 4.6}. Let \(A\) be a Delaunay residue, and let \(\mu\) be its eigenvalue as in \textbf{Definition 1.2}. Let \(\tau\) be as in \textbf{Lemma 1.7}. Let \(r \in (0, 1)\), and assume \(C_r \cap S_A = \emptyset\), where \(S_A\) is defined by \textbf{(1.2)}\. Let \(C \in \Lambda \text{SL}_2 \mathbb{C}\) and assume the monodromy \(C \exp(2\pi i \Lambda) C^{-1}\) is in \(\Lambda^+_{1} \text{SL}_2 \mathbb{C}\). Then there exists \(r_1, r_2\) with \(0 < r_1 < r < r_2 < 1\), and a continuous function \(c : A_{r_1, r_2} \to \mathbb{R}_+\), such that for all \((z, \lambda) \in \{0 < |z| < 1\} \times A_{r_1, r_2}\),

\[|\text{Pos}_r[C \exp(A \log z)]| \leq c|z|^{-\tau} \leq c|z|^{-\text{Re} \mu}.

\textbf{Proof}. Step 1. Let \(\Sigma^* = \{0 < |z| < 1\}\). Let \(C = C_u \cdot C_+\) be the \(r\)-Iwasawa factorization of \(C\). By \textbf{Theorem 4.4}, there exist loops \(G, V \in \Lambda^+ \text{SL}_2 \mathbb{C}\) such that \(C_+ = GV, G\) is a product of normalized simple factors (or \(G = 1\)), \(A : = VAV^{-1}\) is a Delaunay residue, and

\[X := C \exp(A \log z) = C_u G \exp(A_1 \log z) V.

\textit{Step 2}. Since \(C_r \cap S_A = \emptyset\), and the points of \(S_A\) are isolated on \(\mathbb{C}^*\), there exist \(r_1\) and \(r_2\) satisfying \(0 < r_1 < r < r_2 < 1\) such that \(A_{r_1, r_2} \cap S_A = \emptyset\), and \(V\) extends holomorphically to \(D_{r_2}^c\). Let \(r_2 \in (r, r_1)\) and let \(A := A_{r_1, r_2}\). Since \(V\) extends holomorphically to \(D_{r_2}^c\), then \(c_1 := |V|_{\text{A}}\) is finite on \(\Sigma^* \times A\), so

\[|\text{Pos}_r[X]| = c_1 |\text{Pos}_r[G \exp(A_1 \log z)]|\).

\textit{Step 3}. Write \(G = g_n \cdots g_1\) as a product of normalized simple factors and for \(k \in \{1, \ldots, n\}\), write \(g_k = g[\lambda_k, L_k]\), with \(\lambda_k \in A_{r_1, r_1}\) and \(L_k \in \mathbb{C}^1\). By repeated use of the simple factor dressing formula \textbf{(4.1)}, we obtain maps \(h_1, \ldots, h_n : \{0 < |z| < 1\} \to \Lambda^+_{1} \text{SL}_2 \mathbb{C}\) defined by

\[h_k = g[\lambda_k, F_k(\lambda_k)^{1/2} L_k], \quad F_k = \text{Uni}_{r_1}[g_{k-1} \cdots g_1 X]\]

such that

\[\text{Pos}_r[g_n \cdots g_1] = h_n \cdots h_1 \text{Pos}_r[X].\]

Since the singularities of \(G\) are in \(S_A\), which is bounded away from \(A\), then by \textbf{Lemma 4.5}, \(|h_k|^A\) is finite for each \(k \in \{1, \ldots, n\}\), so let \(c_2\) be their product. Then using \textbf{(4.3)}, we have on \(\Sigma^* \times A\),

\[|\text{Pos}_r[X]| \leq c_1 c_2 |\text{Pos}_r[\exp(A_1 \log z)]|\).


Step 4. It follows from det $A = det A_1$ that the function $\tau$ in Lemma 1.7 for $A$ is the same as that for $A_1$. By Lemma 1.11 there exists a continuous function $c_3 : A \to \mathbb{R}_+$ such that on $\{0 < |z| < 1\} \times A$,

\begin{equation}
|\text{Pos}_r[\exp(A_1 \log z)]| \leq c_3|z|^{-\tau} \leq c_3|z|^{-\text{Re} \mu}.
\end{equation}

The result follows from (4.3) and (4.6) with $c = c_1c_2c_3$.

\section{Delaunay dressing asymptotics}

5.1. Simple factor asymptotics. The next preliminary lemma shows a basic convergence property of simple factors: two simple factors with the same fixed singularities converge if the lines defining them converge.

Lemma 5.1. For $k \in \{1, 2\}$, let $h_k = g^0[\lambda_0, L_k]$, where $\lambda_0 \in \mathcal{D}_1^*$ and $L_k : \Sigma^* \to \mathbb{CP}^1$ are continuous maps on a punctured neighborhood $\Sigma^* \subset \mathbb{C}$ of $0 \in \mathbb{C}$. Note that the map $\tau \lambda_2^{-1}$ extends meromorphically to $\mathbb{CP}^1$ (see Lemma 4.3) and this extension is holomorphic on $\mathbb{CP}^1 \setminus \{\lambda_0, 1/\lambda_0\}$. If $L_1$ and $L_2$ converge as $z \to 0$, then on every region $R \subseteq \mathbb{CP}^1$ bounded away from $\{\lambda_0, 1/\lambda_0\}$,

\begin{equation}
\lim_{z \to 0} \|h_1h_2^{-1} - I\|_R = 0.
\end{equation}

Proof. Let $R \subseteq \mathbb{CP}^1$ be a region bounded away from $\{\lambda_0, 1/\lambda_0\}$.

Step 1. For $k \in \{1, 2\}$, let $h_k = U_k^{-1}g^0[\lambda_0, L_k]$. Since $h_1$ and $h_2$ share the same singularity $\lambda_0$, we have $f$ as in Definition 4.1 such that

\begin{equation}
h_k^0 = f^{1/2}11 + f^{-1/2}11, \quad k \in \{1, 2\}.
\end{equation}

We show that $h_k^0h_2^{-1} \to I$. We have

\begin{equation}
h_1^0h_2^{-1} - I = \left(\frac{\pi L_1}{\pi L_2} + \frac{\pi L_2}{\pi L_1} - I\right) + f\pi L_1\pi L_2 + f^{-1}\pi L_1\pi L_2.
\end{equation}

Then

\begin{equation}
\lim_{z \to 0} \left(\frac{\pi L_1}{\pi L_2} + \frac{\pi L_2}{\pi L_1} - I\right) = 0, \quad \lim_{z \to 0} \pi L_1\pi L_2 = 0, \quad \lim_{z \to 0} \pi L_1\pi L_2 = 0.
\end{equation}

Moreover, since $f$ and $f^{-1}$ are holomorphic on $R$, which is bounded away from $\{\lambda_0, 1/\lambda_0\}$, then $f$ and $f^{-1}$ are bounded on $R$. Then (5.2) and (5.3) imply

\begin{equation}
\lim_{z \to 0} \|h_1^0h_2^{-1} - I\|_R = 0.
\end{equation}

Step 2. We show that $h_1h_2^{-1} \to I$. We have

\begin{equation}
h_1h_2^{-1} - I = U_1^{-1}h_1^0h_2^{-1}U_2 - I = U_1^{-1}(h_1^0h_2^{-1} - U_1U_2^{-1})U_2,
\end{equation}

so

\begin{equation}
\|h_1h_2^{-1} - I\|_R \leq \|h_1^0h_2^{-1} - U_1U_2^{-1}\|_R \leq \|h_1^0h_2^{-1} - I\|_R + \|U_1U_2^{-1} - I\|.
\end{equation}

By (5.4), the first term of the right-hand side of (5.5) converges to 0 as $z \to 0$.

We have $U_1U_2^{-1} = h_1^0(0)h_2^0(0)^{-1}$. Since $\{0\}$ is bounded away from $\{\lambda_0, 1/\lambda_0\}$, then by step 1, (5.4) holds at 0. Hence

\begin{equation}
\lim_{z \to 0} \|h_1^0(x, 0)h_2^0(x, 0)^{-1} - I\| = 0.
\end{equation}

Hence the second term of the right-hand side of (5.5) converges to 0 as $z \to 0$. This implies the result (5.1).
The following theorem shows that dressing by a simple factor preserves convergence. For a meromorphic map $U$, the set of poles of $U$ is denoted by $\text{sing}(U)$.

**Theorem 5.2.** With $r \in (0, 1)$, let $\Phi_1, \Phi_2 : \Sigma^r \to \Lambda_r \text{SL}_2 \mathbb{C}$ be analytic maps on the universal cover $\Sigma^r \to \Sigma^*$ of a punctured neighborhood $\Sigma^* \subset \mathbb{C}$ of $0 \in \mathbb{C}$. Let $U \in \Lambda^+_r \text{SL}_2 \mathbb{C}$ and $V \in \Lambda^+_r \text{SL}_2 \mathbb{C}$. Suppose that for every region $\mathcal{A} \subseteq \mathcal{A}_r$ bounded away from $\text{sing}(U)$, $\Phi_1$ and $\Phi_2$ have the convergence

\begin{align}
(5.6a) \quad & \lim_{z \to 0} \| (\text{Uni}_r[\Phi_1])^{-1} U \text{Uni}_r[\Phi_2] - 1 \|_{\mathcal{A}} = 0, \\
(5.6b) \quad & \lim_{z \to 0} \| \text{Pos}_r[\Phi_2] V^{-1} (\text{Pos}_r[\Phi_1])^{-1} - 1 \|_{\mathcal{D}_r} = 0.
\end{align}

For $k \in \{1, 2\}$, let $g_k = W g[\lambda_0, L_k]$ be general simple factors, where $W \in \text{SU}_2$, $\lambda_0 \in \mathcal{A}_{r,1} \setminus \text{sing}(U)$, and $L_1, L_2 \in \mathbb{CP}^1$ are related by $L_2 = \overline{U(\lambda_0)} L_1$. Then for every region $\mathcal{A}' \subset \mathcal{A}_r$ bounded away from $\text{sing}(U) \cup \{\lambda_0, 1/\lambda_0\}$, $g_1 \Phi_1$ and $g_2 \Phi_2$ have the convergence

\begin{align}
(5.7) \quad & \lim_{z \to 0} \| (\text{Uni}_r[g_1 \Phi_1])^{-1} (g_1 U g_2^{-1}) \text{Uni}_r[g_2 \Phi_2] - 1 \|_{\mathcal{A}'} = 0, \\
& \lim_{z \to 0} \| \text{Pos}_r[g_2 \Phi_2] V^{-1} (\text{Pos}_r[g_1 \Phi_1])^{-1} - 1 \|_{\mathcal{D}_r} = 0.
\end{align}

**Proof.** For $k \in \{1, 2\}$, let $F_k = \text{Uni}_r[\Phi_k]$ and $B_k = \text{Pos}_r[\Phi_k]$. For $k \in \{1, 2\}$ we have by the simple factor formula (4.1)

$$\text{Uni}_r[g_k \Phi_k] = g_k F_k h_k^{-1} \quad \text{and} \quad \text{Pos}_r[g_k \Phi_k] = h_k B_k, \quad k \in \{1, 2\},$$

where the $h_k : \Sigma^* \to \Lambda^+_r \text{SL}_2 \mathbb{C}$ are defined by

$$h_k = g[\lambda_0, F_k(\lambda_0) L_k], \quad k \in \{1, 2\}.$$

Let $\mathcal{A}' \subset \mathcal{A}_r$ be a region bounded away from $\text{sing}(U) \cup \{\lambda_0, 1/\lambda_0\}$. Since $\mathcal{A}'$ is bounded away from $\text{sing}(U)$, then (5.6a) hold for $\mathcal{A}'$:

\begin{align}
(5.8) \quad & \lim_{z \to 0} \| F_1^{-1} UF_2 - 1 \|_{\mathcal{A}'} = 0, \\
& \lim_{z \to 0} \| B_2 V^{-1} B_1 - 1 \|_{\mathcal{D}_r} = 0.
\end{align}

By **Lemma 4.2** $h_1 h_2^{-1}$ extends to a map $\Sigma^* \times \mathcal{A}_r' \to \text{SL}_2 \mathbb{C}$ which is meromorphic in the second variable, and $h_2 h_1^{-1}$ extends to a holomorphic map $\Sigma^* \to \Lambda^+_r \text{SL}_2 \mathbb{C}$. For $k \in \{1, 2\}$, define $M_k = F_k(\lambda_0)$, and define $U_0 = \overline{U(\lambda_0)}$. Since $\lambda_0 \not\in \text{sing}(U)$, by (5.6a) we have

$$\lim_{z \to 0} M_2 U_0 M_1^{-1} = I.$$

Then

$$M_2 L_2 = (M_2 U_0 M_1^{-1}) M_1 (U_0^{-1} L_2) = (M_2 U_0 M_1^{-1}) M_1 L_1,$$

so $M_1 L_1$ and $M_2 L_2$ converge as $z \to 0$. Since $\mathcal{A}'$ is bounded away from $\{\lambda_0, 1/\lambda_0\}$, we have by **Lemma 5.1**

\begin{align}
(5.9) \quad & \lim_{z \to 0} \| h_1 h_2^{-1} - 1 \|_{\mathcal{A}'} = 0, \\
& \lim_{z \to 0} \| h_2 h_1^{-1} - 1 \|_{\mathcal{D}_r} = 0.
\end{align}

Since $\mathcal{A}'$ and $\mathcal{D}_r$ are bounded away from $\{\lambda_0, 1/\lambda_0\}$, then by **Lemma 4.5** each of $h_1^{-1}$, $h_2$ and $h_2^{-1}$ is bounded on $\Sigma^* \times \mathcal{A}'$ and on $\Sigma^* \times \mathcal{D}_r$. By (5.8) and (5.9) it follows that

\begin{align}
& \lim_{z \to 0} \| h_1 (F_1^{-1} UF_2 - I) h_2^{-1} + (h_1 h_2^{-1} - I) \|_{\mathcal{A}'} = 0, \\
& \lim_{z \to 0} \| h_2 (B_2 V^{-1} B_1 - I) h_1^{-1} + (h_2 h_1^{-1} - I) \|_{\mathcal{D}_r} = 0.
\end{align}

This gives the result (5.7). □
5.2. Bubbleton asymptotics. In this section, we show that if an immersion \( f \) converges to a Delaunay immersion \( f_0 \), then \( f \) dressed by a product of simple factors with distinct singularities converges to a Delaunay immersion with the same necksize as \( f_0 \) \[\text{(Theorem 5.6)}\]. The proof of \[\text{(Theorem 5.6)}\] is outlined as follows.

For each singularity \( \lambda_0 \) there is a family of simple factors sharing this singularity. Given a Delaunay residue \( A \), dressing the Delaunay frame \( F = \text{Uni}[\exp((x + iy)A)] \) by this family generically produces a bubbleton frame \( g#F \). However, for one special member \( g_0 \) of this family of simple factors, \( g_0#F \) is not a bubbleton frame as expected, but rather a Delaunay frame \[\text{(Lemma 5.5)}\].

On the other hand, any two Delaunay frames dressed by simple factors are asymptotic \[\text{(Lemma 5.4)}\]. In particular, a generic bubbleton frame \( g#F \) constructed by dressing a Delaunay frame \( F \) by a simple factor \( g \), is asymptotic to the corresponding special Delaunay frame \( g_0#F \). This implies that any bubbleton frame is asymptotic to a Delaunay frame.

By the general result in \[\text{subsection 5.1} \] that dressing by simple factors preserves end convergence \[\text{(Theorem 5.2)}\], step 3 above implies that dressing by a finite product of simple factors with distinct singularities preserves Delaunay asymptotics \[\text{(Theorem 5.6)}\].

The following lemma shows that given a unitary Delaunay frame \( F \) and simple factors \( g_1 \) and \( g_2 \) with the same singularity, then generically, \( g_1#F \) and \( g_2#F \) are asymptotic modulo a unitary factor. A similar result holds for the positive factors.

**Lemma 5.3.** Let \( A \in \text{sl}_2 \mathbb{C} \), let \( \mu \) be an eigenvalue of \( A \) and suppose \( \mu \in \mathbb{R}_+ \). Let \( (\mu, \lambda_+) \), \( (\mu, \lambda_-) \in \mathbb{R}^3 \times \mathbb{C}P^1 \) be the eigenvalue-eigenline pairs for \( A \). Then for all \( L \in \mathbb{C}P^1 \setminus \{L_+\} \), \( \exp(xA)L \to L_\infty \) as \( x \to -\infty \).

\[\text{(5.10)}\]

\[
\lim_{x \to -\infty} \left\| (\text{Uni}_{r}(g_1\Phi_0))^{-1}(g_1g_2^{-1})\text{Uni}_{r}(g_2\Phi_0) - 1 \right\|_A = 0 ,
\]

\[
\lim_{x \to -\infty} \left\| \text{Pos}_{r}(g_2\Phi_0)(\text{Pos}_{r}(g_1\Phi_0))^{-1} - 1 \right\|_{D} = 0 .
\]

**Proof.** Let \( F = \text{Uni}_{r}(\Phi_0) \) and \( B = \text{Pos}_{r}(\Phi_0) \). By the simple factor dressing formula \[\text{(4.1)}\] applied to \( g_1 \) and \( g_2 \), we have

\[
\text{Uni}_{r}[g_k\Phi_0] = g_kFh_k^{-1} \quad \text{and} \quad \text{Pos}_{r}[g_k\Phi_0] = h_kB , \quad k \in \{1, 2\} ,
\]

where

\[
h_k = \text{Pos}_{r}[g_kF] = g[\lambda_0, GL_k] , \quad G = \overline{F(\lambda_0)} .
\]

By \[\text{Lemma 4.2} \] \( h_1h_2^{-1} \) and \( h_2h_1^{-1} \) are holomorphic on \( \mathbb{C}P^1 \setminus \{\lambda_0, 1/\lambda_0\} \). We compute in the coordinates \( x + iy = \log z \). We show that the lines \( GL_1 \) and \( GL_2 \) converge as \( x \to -\infty \).

By \[\text{(1.7a)}\],

\[
F(x + n\rho, y) = C^nF(x, y) , \quad C = \exp((\rho - \sigma)A) .
\]

Let \( C_0 = \overline{C(\lambda_0)} \). By \[\text{Lemma 5.3} \] for any \( L \in \mathbb{C}P^1 \setminus \{E\} \), the sequence \( C_0^nL \) converges to \( L_0 \) as \( n \to \infty \). Let \( F \) be the space of continuous functions \( [0, \rho] \to \mathbb{C}P^1 \) with the \( C^0 \)-norm. Define the map \( \mathcal{P} : F \to F \) by \( f(x) \mapsto F((x, y, \lambda_0)) f(x) \). Then \( \mathcal{P} \) is continuous, ecause the multiplication map is continuous. so \( \overline{F(x, y, \lambda_0)} C_0^nL \) converges to \( \overline{F(x, y, \lambda_0)} L_0 \) in \( F \) as \( n \to \infty \). Hence

\[
\lim_{x \to -\infty} \text{dist} \left( \overline{F(x, y, \lambda_0)} L_1, \overline{F(x, y, \lambda_0)} L_0 \right) = 0 .
\]
Hence $GL_1$ and $GL_2$ converge as $x \to -\infty$. By Lemma 5.1 for every region $A \subseteq A_r$ bounded away from $\{\lambda_0, 1/\lambda_0\}$,

\[
\lim_{z \to 0} \|h_1 h_2^{-1} - 1\|_A = 0 ,
(5.11)
\]

\[
\lim_{z \to 0} \|h_2 h_1^{-1} - 1\|_{D_r} = 0 .
\]

Since

\[
(\text{Uni}_r[g_1 \Phi_0])^{-1} (g_1 g_2^{-1}) \text{Uni}_r[g_2 \Phi_0] = (g_1 F h_1^{-1})^{-1} (g_1 g_2^{-1}) (g_2 F h_2^{-1}) = h_1 h_2^{-1} ,
(5.12)
\]

Pos$_r[g_2 \Phi_0] (\text{Pos$_r$}[g_1 \Phi_0])^{-1} = h_2 B (h_1 B)^{-1} = h_2 h_1^{-1} ,

the result (5.10) follows from (5.11) and (5.12).

5.3. Delaunay dressing asymptotics. We show that for a certain special simple factor $g$, the dressed Delaunay frame $g \# \exp((x + iy)A)$ is a unitary Delaunay frame, rather than a multibubbleton frame as is generically the case. For each choice of $\lambda_0$, there are generically two special Delaunay dressings, one for each eigenline of $\overline{A(\lambda_0)}$.

Lemma 5.5. With $A$ a Delaunay residue, let $g = g[\lambda_0, L]$ be the normalized simple factor defined with $\lambda_0 \in D_1^+$, and $L$ an eigenline of $\overline{A(\lambda_0)}$. Then $gA^{-1}$ is a Delaunay residue.

Proof. By Lemma 4.2(i), $gA^{-1}$ extends meromorphically to $\mathbb{C}^*$ and this extension satisfies $(gA^{-1})^* = gA^{-1}$ away from its poles.

We show that $gA^{-1}$ is holomorphic at $\lambda_0$. As in Definition 4.1 let $U \subseteq SU_2$ and $f$ with $\text{ord}_{\lambda_0} f = -1$ be such that $g = U^{-1} (f^{1/2} x_L + f^{-1/2} x_{L^1}).$ Then

\[
gA^{-1} = U^{-1} (f^{1/2} x_L + f^{-1/2} x_{L^1}) A (f^{-1/2} x_L + f^{1/2} x_{L^1}) U
\]

\[
= U^{-1} (\tau L_A x_L + f x_L A x_{L^1} + f^{-1} x_L A x_{L^1} + f x_{L^1} A x_{L_{L^1}}) U .
(5.13)
\]

At $\lambda_0$, the image of $\tau L_{L^1}$ is $L^\perp$. Since $L$ is an eigenline of $\overline{A(\lambda_0)}$, it follows that $L^\perp$ is an eigenline of $A(\lambda_0)$. Hence the image of $L^\perp$ under $A(\lambda_0)$ is $L^\perp$. Since the image of $L^\perp$ under $\tau_L$ is 0, then $\text{ord}_{\lambda_0} x_{L} A x_{L_{L^1}} = 1$. Hence $f x_L A x_{L_{L^1}}$ is holomorphic at $\lambda_0$. Since none of the other three terms in (5.13) has a pole at $\lambda_0$, then $gA^{-1}$ is holomorphic at $\lambda_0$. By its hermitian symmetry, $gA^{-1}$ is holomorphic at $1/\lambda_0$, and hence is holomorphic on $\mathbb{C}^*$.

We consider $gA^{-1}$ at $\lambda = 0$. Since $g$ is holomorphic and upper-triangular at $\lambda = 0$, and the only pole of $A$ at $\lambda = 0$ appears in the upper-right entry, with order $-1$, the same is true of $gA^{-1}$. Hence $gA^{-1}$ is a Delaunay residue by Proposition 1.1.

Lemma 5.4 and Theorem 5.2 come together in the following theorem, which shows that if the Iwasawa factors of a holomorphic frame $\Phi$ converge to those of a holomorphic Delaunay frame respectively, then the same holds after dressing $\Phi$ by a finite product of simple factors.

Theorem 5.6. With $r \in (0, 1)$, let $\Phi : \Sigma^* \to \Lambda_r \text{SL}_2 \mathbb{C}$ be an analytic map on a punctured neighborhood $\Sigma^* \subseteq \mathbb{C}$ of $0 \subseteq \mathbb{C}$. Let $A$ be a Delaunay residue. Suppose that $\Phi$ and $\Phi_1 = \exp(A \log z)$ have the convergence

\[
\lim_{z \to 0} \| (\text{Uni}_r[\Phi_1])^{-1} \text{Uni}_r[\Phi] - 1\|_A = 0 ,
(5.14)
\]

\[
\lim_{z \to 0} \| \text{Pos}_r[\Phi] (\text{Pos}_r[\Phi_1])^{-1} - 1\|_{D_r} = 0 .
\]

Let $G \in \Lambda_r^+ \text{SL}_2 \mathbb{C}$ be a product of general simple factors (or $G = 1$) with distinct singularities in $A_r \cap \{ \mu \in \mathbb{R}_+ \}$. Then there exist

* a loop $U_2 \subseteq \Lambda^\text{MSL}_2 \mathbb{C}$ satisfying $\text{sing}(U_2) = \text{sing}(G)$,
• and a loop $V_2 \in \Lambda^+_\ast SL_2 \mathbb{C}$ for which $A_2 := V_2AV_2^{-1}$ is a Delaunay residue, such that, for every region $A \subseteq A_r$ bounded away from $\text{sing}(U_2)$, $G \Phi$ and $\Phi_2 := \exp(A_2 \log z)$ have the convergence

$$\lim_{z \to 0} \left\| (\text{Uni}_r[\Phi_2])^{-1} U_2 \text{Uni}_r[\Phi] - I \right\|_A = 0,$$

$$\lim_{z \to 0} \left\| \text{Pos}_r[G \Phi] V_2^{-1} (\text{Pos}_r[\Phi_2])^{-1} - I \right\|_{D_r} = 0.$$

**Proof.** The proof is by induction on the factors of $G$. We first prove the following induction step, which shows convergence after dressing by a single simple factor $g$.

With $r \in (0, 1)$, let $\Phi : \Sigma^* \to \Lambda_r SL_2 \mathbb{C}$ be an analytic map on a universal cover $\tilde{\Sigma}^* \to \Sigma^*$ of a punctured neighborhood $\Sigma^* \subset \mathbb{C}$ of $0 \in \mathbb{C}$. Let $A$ be a Delaunay residue, and let $\mu$ be its eigenvalue as in [Definition 1.2](#).

• Let $U_1 \in \Lambda^+_r SL_2 \mathbb{C}$.

• Let $V_1 \in \Lambda^+_r SL_2 \mathbb{C}$ for which $A_1 := V_1AV_1^{-1}$ is a Delaunay residue.

Suppose that for every region $A \subseteq A_r$ bounded away from $\text{sing}(U_1)$, $\Phi$ and $\Phi_1 = \exp(A_1 \log z)$ have the convergence

$$\lim_{z \to 0} \left\| (\text{Uni}_r[\Phi])^{-1} U_1 \text{Uni}_r[\Phi] - I \right\|_A = 0,$$

$$\lim_{z \to 0} \left\| \text{Pos}_r[\Phi] V_1^{-1} (\text{Pos}_r[\Phi_1])^{-1} - I \right\|_{D_r} = 0.$$

Let $g$ be a general simple factor with singularity $\lambda_0 \in A_{r,1} \cap \{\mu \in \mathbb{R}^+\}$. Then there exist

• a loop $U_2 \in \Lambda^+_r SL_2 \mathbb{C}$ satisfying $\text{sing}(U_2) = \text{sing}(U_1) \cup \{\lambda_0, 1/\lambda_0\}$,

• and a loop $V_2 \in \Lambda^+_r SL_2 \mathbb{C}$ for which $A_2 := V_2AV_2^{-1}$ is a Delaunay residue, such that for every region $A' \subseteq A_r$ bounded away from $\text{sing}(U_2)$, $g \Phi$ and $\Phi_2 := \exp(A_2 \log z)$ have the convergence

$$\lim_{z \to 0} \left\| (\text{Uni}_r[\Phi_2])^{-1} U_2 \text{Uni}_r[g \Phi] - I \right\|_{A'} = 0,$$

$$\lim_{z \to 0} \left\| \text{Pos}_r[g \Phi] V_2^{-1} (\text{Pos}_r[\Phi_2])^{-1} - I \right\|_{D_r} = 0.$$

**Proof of the induction step:**

Let $A' \subseteq A_r$ be a region bounded away from $\text{sing}(U_1) \cup \{\lambda_0, 1/\lambda_0\}$.

**Step 1.** Write $g = Wg[\lambda_0, L]$ and let $\tilde{g} = Wg[U(\lambda_0)^{-1}, L]$. By [Theorem 5.2](#) applied to (5.15), $g \Phi$ and $\tilde{g} \Phi_1$ have the convergence

$$\lim_{z \to 0} \left\| (\text{Uni}_r[\tilde{g} \Phi_1])^{-1} (\tilde{g} U_1 g^{-1}) \text{Uni}_r[g \Phi] - I \right\|_{A'} = 0,$$

$$\lim_{z \to 0} \left\| \text{Pos}_r[g \Phi] V_1^{-1} (\text{Pos}_r[\tilde{g} \Phi_1])^{-1} - I \right\|_{D_r} = 0.$$

**Step 2.** Let $\mu$ be the eigenvalue function of $A$ as in [Definition 1.2](#). Let $(\mu[\lambda_0], E_+)$ and $(-\mu[\lambda_0], E_-)$ be the eigenvalue-eigenline pairs of $A_1(\lambda_0)$. Let $g' = g[\lambda_0, E_-]$ be the normalized simple factor as in [Lemma 5.5](#). By that lemma, $A_2 := g' A_1 g'^{-1}$ is a Delaunay residue. Since $\mu(\lambda_0) \neq 0$, then $E_- \neq E_+$. Hence by [Lemma 5.4](#) applied to $g'$ and $\tilde{g}$, the loops $g' \Phi_1$ and $g' \Phi_1$ have the convergence

$$\lim_{z \to 0} \left\| (\text{Uni}_r[g' \Phi_1])^{-1} g' \tilde{g} \Phi_1 - I \right\|_{A'} = 0,$$

$$\lim_{z \to 0} \left\| \text{Pos}_r[\tilde{g} \Phi_1] (\text{Pos}_r[g' \Phi_1])^{-1} - I \right\|_{D_r} = 0.$$
Let $\Phi_2 = \exp(A_2 \log z)$. Then $g'\Phi_1 = \Phi_2g'$, so by (5.18), $g\Phi_1$ and $\Phi_2$ have the convergence

$$
\lim_{z \to 0} \left\| (\text{Uni}_A[\Phi_2])^{-1} g'\Phi_1 - I \right\|_{A'} = 0,
$$

(5.19)

$$
\lim_{z \to 0} \left\| \text{Pos}_A[g\Phi_1] g^{-1} (\text{Pos}_A[\Phi_2])^{-1} - I \right\|_{D'} = 0.
$$

Step 3. By Lemma 4.2(ii), $U_2 := g'U_1g^{-1}$ is an element of $\Lambda_r^{\text{MSL}_2}\mathbb{C}$ and $\text{sing}(U_2) = \text{sing}(U_1) \cup \{\lambda_0, 1/\lambda_0\}$. The loop $V_2 := g'V_1$ is an element of $\Lambda_r^{\text{SL}_2}\mathbb{C}$ and $V_2AV_2^{-2} = g'A_1g^{-1}$ is a Delaunay residue. By (5.17) and (5.19), $g\Phi$ and $\Phi_2$ have the convergence given by (5.16). This proves the induction step.

To prove the theorem, first note that if $G = I$, the theorem is trivially true with $U_2 = V_2 = I$. Otherwise, write $G = g_0 \cdots g_1$ as a product of simple factors, and starting with $U_1 = V_1 = I$, apply the induction step to $g_1, \ldots, g_n$ in turn. \qed

By the methods of section 3, Theorem 5.6 implies the following (see also [16]):

**Theorem 5.7.** If a CMC surface $f$ converges to a half Delaunay surface $f_0$, then the surface produced from $f$ by dressing by finitely many simple factors converges to a rigid motion of $f_0$.

### 5.4. Dressed Delaunay asymptotics.

We show that the $r$-Iwasawa factors of a holomorphic frame obtained from a perturbed Delaunay potential converges to those of a Delaunay frame. This theorem is a generalization of Theorem 2.11 removing the restriction that the holomorphic frame extend to $\mathcal{A}_{r,1}$.

**Theorem 5.8.** Let $r \in (0, 1)$ and assume $C_r \cap S_A = \emptyset$. Let $\xi$ be a perturbed Delaunay $r$-potential

$$
\xi = Az^{-1}dz + O(z^n)dz.
$$

(5.20)

Suppose

$$
\max_{\lambda \in C_r} \Re \mu(\lambda) < (n+1)/2.
$$

(5.21)

Let $\Phi : \tilde{\Sigma}^* \to \Lambda_r\text{SL}_2\mathbb{C}$ satisfy $df = \Phi\xi$ on the universal cover $\tilde{\Sigma}^* \to \Sigma^*$ of $\Sigma^*$, and assume that the monodromy $M$ of $\Phi$ around $z = 0$ satisfies $M \in \Lambda_r^{\text{SL}_2}\mathbb{C}$.

Then there exists a loop $U \in \Lambda_r^{\text{MSL}_2}\mathbb{C}$ with $\text{sing}(U) \subset \mathcal{A}_{r,1} \cap S_A$ and only simple poles, and a loop $V \in \Lambda_r^{\text{SL}_2}\mathbb{C}$ for which $A_1 := VA^{-1}$ is a Delaunay residue, such that for every region $A \subseteq \mathcal{A}$ bounded away from $\text{sing}(U)$, $\Phi$ and $\Phi_1 := \exp(A_1 \log z)$ have the convergence

$$
\lim_{z \to 0} \left\| (\text{Uni}_A[\Phi])^{-1} U\text{Uni}_A[\Phi] - I \right\|_{\mathcal{A}} = 0,
$$

(5.22)

$$
\lim_{z \to 0} \left\| \text{Pos}_A[\Phi] V^{-1} (\text{Pos}_A[\Phi_1])^{-1} - I \right\|_{D'} = 0.
$$

**Proof.** Step 1. Let $\Phi_0 = \exp(A \log z)$ and let $\Phi = C\Phi_0P$ be the $z^AP$-decomposition of $\Phi$. By Theorem 4.6, there exist $0 < s_1 < r < s_2 < 1$ and a continuous function $c : \mathcal{A}_{s_1,s_2} \to \mathbb{R}_+$ such that $\text{Pos}_A[C\Phi_0]$ extends continuously to $\mathcal{A}_{s_1,s_2}$ and

$$
\|\text{Pos}_A[C\Phi_0]\|_{\mathcal{A}_{s_1,s_2}} \leq c|z|^{-\Re \mu}.
$$

Step 2. By Theorem 2.11 $\Phi$ and $C\Phi_0$ have the convergence

$$
\lim_{z \to 0} \left\| (\text{Uni}_A[C\Phi_0])^{-1} \text{Uni}_A[\Phi] - I \right\|_{\mathcal{A}} = 0,
$$

(5.23)

$$
\lim_{z \to 0} \left\| \text{Pos}_A[\Phi] (\text{Pos}_A[C\Phi_0])^{-1} - I \right\|_{D'} = 0.
$$
Step 3. Let $C = C_u C_+$ be the $r$-Iwasawa factorization of $C$. Using that $C \exp(2\pi i A) C^{-1} \in \Lambda_r^+ \mathbb{SL}_2 \mathbb{C}$, by Theorem 4.4 there exist loops $G, V_1 \in \Lambda_r^+ \mathbb{SL}_2 \mathbb{C}$ such that $C_+ = GV_1$, $G$ is a product of normalized simple factors (or $G = I$), $A_0 := V_1 AV_1^{-1}$ is a Delaunay residue, and the singularities of the simple factors are distinct and in $\mathcal{S}_A \cap (A_r \setminus S^1)$. Then, with $\Phi_0 = \exp(A_0 \log z)$,

$$C\Phi_0 = C_u GV_1 \exp(A \log z) = C_u G \exp(A_0 \log z) V_1 = C_u G \Phi_0 V_1 .$$

By (5.23) and (5.24), $\Phi$ and $G \Phi_0$ have the convergence

$$\lim_{z \to 0} \left\| (\text{Uni}_r [G\Phi_0])^{-1} C_u \text{Uni}_r [\Phi] - I \right\|_{A_r} = 0 ,$$

(5.25)

$$\lim_{z \to 0} \left\| \text{Pos}_r [\Phi] V_1^{-1} (\text{Pos}_r [G\Phi_0])^{-1} - I \right\|_{D_r} = 0 .$$

Step 4. The singularities of $G$ in $D_1$ are distinct and lie in $A_{r,1} \cap \{ \mu \in \mathbb{R}_+ \}$. By Theorem 5.6 where in (5.14), the $\Phi$ and $G \Phi_0$ are both replaced by $\Phi_0$, so that (5.14) holds vacuously, there exists a loop $V_2 \in \Lambda_r^+ \mathbb{SL}_2 \mathbb{C}$ for which $A_1 := V_2 AV_2^{-1}$ is a Delaunay residue, and a loop $U_2 \in \Lambda_r^m \mathbb{SL}_2 \mathbb{C}$ such that for every region $A \subset A_r$ bounded away from sing$(U_2)$, $G \Phi_0$ and $\Phi_1 := \exp(A_1 \log z)$ have the convergence

$$\lim_{z \to 0} \left\| (\text{Uni}_r [\Phi_1])^{-1} U_2 \text{Uni}_r [G\Phi_0] - I \right\|_{A} = 0 ,$$

(5.26)

$$\lim_{z \to 0} \left\| \text{Pos}_r [G\Phi_0] V_2^{-1} (\text{Pos}_r [\Phi_1])^{-1} - I \right\|_{D_r} = 0 .$$

Step 5. The result (5.22) follows from (5.25) and (5.26), with $U = U_2 C_u$ and $V = V_2 V_1$. □

We conclude with the main theorem of the paper, showing that a CMC end obtained from a perturbed Delaunay potential is asymptotic to a half-Delaunay surface. This theorem is a generalization of Theorem 3.5, removing the conditions $A_{r,1} \cap S_A = \emptyset$ and $\Phi \in \Lambda_1^+ \mathbb{SL}_2 \mathbb{C}$. Hence it applies to surfaces obtained by $r$-dressing, such as the $n$-noids with bubbles constructed in [12]. The proof is similar to that of Theorem 3.5 taking the appearance of the loops $U$ and $V$ into consideration. In the case of embedded ends, we obtain exponential convergence (see Remark 3.6).

**Theorem 5.9.** Let $A$ be a Delaunay residue as in (1.1) satisfying condition (2.17). On the punctured unit disk $\Sigma^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$, let

$$\xi = A \frac{dz}{z} + O(z^0)dz$$

be a perturbed Delaunay $r$-potential, $r \in (0, 1]$. Let $f_0$ and $f$ be the immersions of $\Sigma^*$ induced by the generalized Weierstrass representation at $\lambda = 1$ by $Adz/z$ and $\xi$ respectively, so $f_0$ is a Delaunay immersion. Assume that $f$ is obtained from a holomorphic $r$-frame whose monodromy at $z = 0$ is in $\Lambda_r^+ \mathbb{SL}_2 \mathbb{C}$.

With $\Sigma_{Cyl}$ as in (3.13), let $\phi : \Sigma_{Cyl} \to \Sigma^*$ be the map $\phi(x, y) = e^{x+iy}$. Then some rigid motion of $f \circ \phi$ converges to $f_0 \circ \phi$ in the $C^\infty$-topology of $\Sigma_{Cyl}$ as $x \to -\infty$. Furthermore, if $f_0$ is embedded, then $f$ is properly embedded.

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