ABSTRACT. Let $A$ be a non-commutative, non-unital $C^*$-algebra. Given a set of commuting positive elements in the corona algebra $Q(A)$, we study some obstructions to the existence of a commutative lifting of such set to the multiplier algebra $M(A)$. Our focus are the obstructions caused by the size of the collection we want to lift. It is known that no obstacles show up when lifting a countable family of commuting projections, or of pairwise orthogonal positive elements. However, this is not the case for larger collections. We prove in fact that for every primitive, non-unital, $\sigma$-unital $C^*$-algebra $A$, there exists an uncountable set of pairwise orthogonal positive elements in $Q(A)$ such that no uncountable subset of it can be lifted to a set of commuting elements of $M(A)$. Moreover, the positive elements in $Q(A)$ can be chosen to be projections if $A$ has real rank zero.

1. Introduction

Let $A$ be a non-unital $C^*$-algebra, denote its multiplier algebra by $M(A)$, its corona algebra (namely $M(A)/A$) by $Q(A)$, and the quotient map from $M(A)$ onto $Q(A)$ by $\pi$. A lifting in $M(A)$ of a set $B \subseteq Q(A)$ is a set $C \subseteq M(A)$ such that $\pi[C] = B$. The study of which properties of $B \subseteq Q(A)$ can be preserved in a lifting, as well as the analysis of the relations between $B$ and its preimage $\pi^{-1}[B]$, has produced a rich theory with strong connections to the study of stable relations in $C^*$-algebras. A general introduction to this subject can be found in [Lor97].

This note focuses on liftings of abelian subalgebras of corona algebras. This topic has been widely studied, for instance, as a mean to produce interesting examples of $*$-algebras, and in the investigation of the masas (maximal abelian subalgebras) of the Calkin algebra $Q(H)$. In [AD79], for example, the authors produce, by means of a lifting, a nonseparable $C^*$-algebra whose abelian subalgebras are all separable. Their proof assumes the Continuum Hypothesis, which was later shown to be not necessary (see [Pop83, Corollary 6.7] and also [BK16]). Another application of the Continuum Hypothesis to liftings of abelian subalgebras of corona algebras can be found in [And79]. Here the author builds a masa of $Q(H)$ which is generated by its projections and does not lift to a masa in $B(H)$. In this case it is not known whether the Continuum Hypothesis can be dropped (see [SST1]).

More recently, the study of liftings led to the first example of an amenable nonseparable Banach algebra which is not isomorphic to a $C^*$-algebra (see [CFO14]; see also [Vig15]).

In this paper we focus on the following problem. Let $A$ be a non-commutative, non-unital $C^*$-algebra, and let $B$ be a commutative family in $Q(A)$. What kind of obstructions could prevent the existence of a commutative lifting of $B$ in $M(A)$? We consider collections with various properties, but our main concern and focus is the role played by the cardinality of the set that we want to lift. The following table summarizes all the cases that we are going to analyze. The symbols “✓” and “×” indicate whether it is possible or not to have a lifting for collections on the left column whose size is the cardinal in the top line.

\begin{center}

\begin{tabular}{|c|c|}
\hline
Collection & Lifting Possible? \\
\hline
Countable & ✓ \\
Uncountable & × \\
\hline
\end{tabular}

\end{center}
| $Q(A) \to M(A)$ | $< \aleph_0$ | $\aleph_0$ | $\aleph_1$ |
|------------------|-------------|-------------|-------------|
| Commuting self-adjoint $\to$ Commuting self-adjoint | $\times$ | $\times$ | $\times$ |
| Commuting projections $\to$ Commuting projections | $\checkmark$ in $Q(H)$ | $\checkmark$ in $Q(H)$ | $\times$ |
| Commuting projections $\to$ Commuting positive | $\checkmark$ | $\checkmark$ | $\times$ |
| Orthogonal positive $\to$ Orthogonal positive | $\checkmark$ | $\checkmark$ | $\times$ |
| Orthogonal positive $\to$ Commuting positive | $\checkmark$ | $\checkmark$ | $\times$ |

It is clear from the table that starting with an uncountable collection is a fatal obstruction. We also remark that the two columns in the middle, representing the lifting problem for finite and countable collections, have the same values. One reason for this phenomenon is that the only obstructions in this scenario are of K-theoretic nature and involve only a finite number of elements, as we shall see in the next paragraph (see also [Dav85]). This situation also relates to other compactness phenomena (at least at the countable level) that corona algebras of $\sigma$-unital algebras satisfy, due to their partial countable saturation (see [FH13]). Most of the results in the table about finite and countable families are already known ([Lor97, FW12 Lemma 5.34], [Lor97 Lemma 10.1.12]). The main contribution of this paper concerns the right column, for which some theorems about projections in the Calkin algebra have already been proved ([FW12 Theorem 5.35], [BK16]).

Let $A$ be $K(H)$, the algebra of the compact operators on a separable Hilbert space $H$, so that $M(A) = B(H)$ and $Q(A) = Q(H)$. By a well-known K-theoretic obstruction, the unilateral shift is a normal element in $Q(H)$ which does not lift to a normal element in $B(H)$ (more on this in [BDF77] and [Dav10]). An element is normal if and only if its real and imaginary part commute. This proves that it is not always possible to lift a couple of commuting self-adjoint elements in a corona algebra to commuting self-adjoint elements in the multiplier algebra.

In order to bypass this obstruction, we add some conditions to the collection which we want to lift. In [FW12 Lemma 5.34] it is proved that any countable family of commuting projections in the Calkin algebra can be lifted to a family of commuting projections in $B(H)$. Moreover, the authors provide a lifting of simultaneously diagonalizable projections. Proving a more general statement about liftings, in Section 2 we show that any countable collection of commuting projections in a corona algebra can be lifted to a commutative family of positive elements in the multiplier algebra.

Two elements in a $C^*$-algebra are orthogonal if their product is zero. Any countable family of orthogonal positive elements in a corona algebra admits a commutative lifting. This is a consequence of the more general result [Lor97 Lemma 10.1.12], which is relaid in this paper as Proposition 2.2.

In general, we cannot expect to be able to generalize verbatim the above result for uncountable families of orthogonal positive elements. This is the case since, by a cardinality obstruction, a multiplier algebra $M(A)$ which can be faithfully represented on a separable Hilbert space $H$, cannot contain an uncountable collection of orthogonal positive elements. The existence of such a collection in $M(A)$ (and thus in $B(H)$) would in fact imply the existence of an uncountable set of orthogonal vectors in $H$, contradicting the separability of $H$.

We could still ask whether it is possible to lift an uncountable family of orthogonal positive elements to a family of commuting positive elements. This leads to an obstruction of set-theoretic nature. In Theorem 5.35 of [FW12], it is shown that there exists an $\aleph_1$-sized collection of orthogonal projections in the Calkin algebra whose uncountable subsets cannot be lifted to families of simultaneously diagonalizable projections in $B(H)$.

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1We remark that it is not always possible to lift projections in a corona algebra to projections in the multiplier algebra. Such lifting is not possible for instance when $Q(A)$ has real rank zero but $M(A)$ has not, which is the case for $A = Q(H) \otimes K(H)$ (see [Zha92 Example 2.7(iii)]) or $A = \mathbb{Z} \otimes K(H)$, where $\mathbb{Z}$ is the Jiang-Su algebra (see [LN16]).
This result is refined in Theorem 7 of [BK16], where the authors provide an $\aleph_1$-sized set of orthogonal projections in $Q(H)$ which contains no uncountable subset that lifts to a collection of commuting operators in $B(H)$. The main result of this paper is a generalization of Theorem 7 in [BK16]. A C*-algebra is $\sigma$-unital if it has a countable approximate unit, and it is primitive if it admits a faithful irreducible representation.

**Theorem 1.1.** Assume $A$ is a primitive, non-unital, $\sigma$-unital C*-algebra. Then there is a collection of $\aleph_1$ pairwise orthogonal positive elements of $Q(A)$ containing no uncountable subset that simultaneously lifts to commuting elements in $M(A)$.

**Corollary 1.2.** Assume $A$ is a primitive, real rank zero, non-unital, $\sigma$-unital C*-algebra. Then there is a collection of $\aleph_1$ pairwise orthogonal projections of $Q(A)$ containing no uncountable subset that simultaneously lifts to commuting elements in $M(A)$.

The proof of Theorem 1.1 is inspired by the combinatorics used in [BK16] and [FW12], which goes back to Luzin and Hausdorff, and to the study of uncountable almost disjoint families of subsets of $\mathbb{N}$ and Luzin’s families (see [Luz47]). We remark that no additional set theoretic assumption (such as the Continuum Hypothesis) is required in our proof.

The paper is structured as follows: in Section 2 we outline the results needed to settle the problem of liftings of countable families of commuting projections and of orthogonal positive elements. Section 3 is devoted to the proof of Theorem 1.1 while concluding remarks and questions can be found in Section 4.

## 2. Countable collections

Denote the set of self-adjoint and of positive elements of a C*-algebra $A$ by $A_{sa}$ and $A_+$, respectively. Given a compact Hausdorff space $X$, $C(X)$ is the C*-algebra of the continuous functions from $X$ into $\mathbb{C}$.

In [FW12] Lemma 5.34 Farah and Wofsey prove that any countable set of commuting projections in the Calkin algebra can be lifted to a set of simultaneously diagonalizable projections in $B(H)$. The thesis of the following proposition is weaker, but it holds in a more general context.

**Proposition 2.1.** Let $\varphi : A \to B$ be a surjective *-homomorphism between two C*-algebras and let $\{p_n\}_{n \in \mathbb{N}}$ be a collection of commuting projections of $B$. Then there exists a set $\{q_n\}_{n \in \mathbb{N}}$ of commuting positive elements of $A$ such that $\varphi(q_n) = p_n$.

**Proof.** We can assume that both $A$ and $B$ are unital, that $\varphi(1_A) = 1_B$ and that $1_B \in \{p_n\}_{n \in \mathbb{N}}$. Let $C \subseteq B$ be the abelian C*-algebra generated by the set $\{p_n\}_{n \in \mathbb{N}}$. Consider the element

$$b = \sum_{n \in \mathbb{N}} \frac{2p_n - 1}{3^n}.$$ 

Let $X$ be the spectrum of $b$ in $A$. The algebra $C$ is generated by $b$ (see [Ric60] p. 293 for a proof), thus $C \cong C(X)$. Fix $a \in A$ such that $\varphi(a) = b$. The element $(a + a^*)/2$ is still in the preimage of $b$ since $b$ is self-adjoint, thus we can assume $a \in A_{sa}$. If $Y$ is the spectrum of $a$, we have in general that $X \subseteq Y$. Fix $f_n \in C(X)_+$ such that $f_n(b) = p_n$. Since the range of $f_n$ is contained in $[0, 1]$ and the spaces $Y$ and $X$ are compact and Hausdorff, by the Tietze Extension Theorem ([Wil70], Theorem 15.8]), for every $n \in \mathbb{N}$, there is a continuous $F_n : Y \to [0, 1]$ such that $F_n \circ f_n$. Set $q_n = F_n(a)$. The map $\varphi$ acts on $C(Y)$ as the restriction on $X$ (here we identify $C^*(a)$ and $C^*(b)$ with $C(Y)$ and $C(X)$ respectively), therefore $\varphi(q_n) = p_n$ for every $n \in \mathbb{N}$.

The $q_n$’s can be chosen to be projections if there is a self-adjoint $a$ in the preimage of $b$ whose spectrum is $X$. By the Weyl-von Neumann theorem, this is the case when $\varphi$ is the quotient map from $B(H)$ onto the Calkin algebra (see [Day96], Theorem II.4.4]).
We focus now on lifting sets of positive orthogonal elements, starting with a set of size two. Let therefore \( \varphi : A \to B \) be a surjective \(*\)-homomorphism of \( C^* \)-algebras, and let \( b_1, b_2 \in B_+ \) be such that \( b_1 b_2 = 0 \). Consider the self-adjoint \( b = b_1 - b_2 \) and let \( a \in A_\sigma \) be such that \( \varphi(a) = b \). The positive and the negative part of \( a \) are two orthogonal positive elements of \( A \) such that \( \varphi(a_+) = b_1, \varphi(a_-) = b_2 \). The situation is analogous when dealing with countable collections, as shown in Lemma 10.1.12 of \([Lor97]\).

**Proposition 2.2** ([Lor97] Lemma 10.1.12). Assume \( \varphi : A \to B \) is a surjective \(*\)-homomorphism between two \( C^* \)-algebras. Let \( \{b_n\}_{n \in \mathbb{N}} \) be a collection of orthogonal positive elements in \( B \). Then there exists a set \( \{a_n\}_{n \in \mathbb{N}} \) of orthogonal positive elements in \( A \) such that \( \varphi(a_n) = b_n \).

### 3. Uncountable collections

Throughout this section, let \( A \) be a primitive, non-unital, \( \sigma \)-unital \( C^* \)-algebra. We can thus assume that \( A \) is a non-commutative strongly dense \( C^* \)-subalgebra of \( B(H) \) for a certain Hilbert space \( H \). A sequence of operators \( \{x_n\}_{n \in \mathbb{N}} \) strictly converges to \( x \in B(H) \) if and only if \( x_n a \to xa \) and \( ax_n \to ax \) in norm for all \( a \in A \). In this scenario \( M(A) \) can be identified with the idealizer

\[
\{x \in B(H) : xA \subseteq A, Ax \subseteq A\}
\]

or with the strict closure of \( A \) in \( B(H) \). Given two elements \( a, b \) in a \( C^* \)-algebra \( A \), we denote the commutator \( ab - ba \) by \([a, b]\). From now on, let \( \{e_n\}_{n \in \mathbb{N}} \) be an approximate unit of \( A \) such that:

1. \( e_0 = 0 \);
2. \( \|e_i - e_j\| = 1 \) for \( i \neq j \);
3. \( e_i e_j = e_i \) for every \( i < j \).

Such an approximate unit exists since \( A \) is \( \sigma \)-unital, as proved in Section 2 of \([Ped90]\). The proof of Theorem 1.1 follows closely the one given by Bice and Koszmider for \([BK16]\) Theorem 7, and a lemma similar to \([BK16]\) Lemma 6 is required.

**Lemma 3.1.** Let \( A \) be a primitive, non-unital, \( \sigma \)-unital \( C^* \)-algebra. There exists a family \( \{a_\beta\}_{\beta \in \mathbb{N}_1} \subseteq M(A)_+ \setminus A \) such that:

1. \( \|a_\beta\| = 1 \) for all \( \beta \in \mathbb{N}_1 \);
2. \( a_\alpha a_\beta \in A \) for all distinct \( \alpha, \beta \in \mathbb{N}_1 \);
3. given \( d_1, d_2 \in M(A) \), for all \( \beta \in \mathbb{N}_1 \), all \( n \in \mathbb{N} \), and all but finitely many \( \alpha < \beta \):

\[
\|(a_\alpha + d_1 e_n), (a_\beta + d_2 e_n)\| \geq \frac{1}{8}.
\]

The rough idea to prove this lemma is to build, for every \( \beta \in \mathbb{N}_1 \), a strictly increasing function \( f_\beta : \mathbb{N} \to \mathbb{N} \) and a norm-bounded sequence \( \{c_k^\beta\}_{k \in \mathbb{N}} \subseteq A_+ \) to define

\[
a_\beta = \sum_{k \in \mathbb{N}} (e_{f_\beta(2k+1)} - e_{f_\beta(2k)}) \frac{1}{2} c_k^\beta (e_{f_\beta(2k+1)} - e_{f_\beta(2k)})^\frac{1}{2}.
\]

Note that this series belongs to \( M(A) \) by Theorem 4.1 in \([Ped90]\) (see also \([FH13]\) Item (10) p.48)). In order to satisfy the thesis of the lemma, we will build each \( c_k^\beta \) so that, for some \( \alpha < \beta \) and some \( n \in \mathbb{N} \), the following holds

\[
\|(a_\alpha + e_n), (c_k^\beta + e_n)\| \geq \frac{1}{8}.
\]

The choice of \( f_\beta \) will guarantee orthogonality in \( Q(A) \) exploiting, for \( n_2 < n_1 < m_2 < m_1 \), the following fact:

\[
(e_{m_1} - e_{m_2})(e_{n_1} - e_{n_2}) = 0.
\]
The main ingredient used to build $c_k^β$ is Kadison’s Transitivity Theorem, which we are allowed to use since $A$ is primitive.

**Proof of Lemma 3.7** Since the C*-algebra $A$ is primitive, we can assume that there is a Hilbert space $H$ such that $A \subseteq B(H)$ and $A$ acts irreducibly on $H$. For each $n < m$, denote the space $(e_m - e_n)H$ by $S_{n,m}$. We start by building $a_0$. Let $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(n) = \begin{cases} 2^{n+1} - 1 & \text{if } n \text{ is even} \\ 2^n & \text{if } n \text{ is odd.} \end{cases}$$

For every $k \in \mathbb{N}$ there is a unit vector $ξ$ in the range of $e_f(2k+1) - e_f(2k)$. By the definition of the approximate unit $(e_n)_{n \in \mathbb{N}}$, the vector $ξ$ is a 1-eigenvector of $e_f(2k+2)$. This, along with the (algebraic) irreducibility of $A \subseteq B(H)$, entails that $A S_{f(2k+1), f(2k)} = H$.

Denote the algebra $(e_f(2k+1) - e_f(2k))A(e_f(2k+1) - e_f(2k))$ by $A_k$. We have that

$$A_k H \supseteq S_{f(2k), f(2k+1)}.$$ 

Let $ξ_k^0, η_k^0 \in S_{f(2k), f(2k+1)}$ be two orthogonal norm one vectors. Since $A$ acts irreducibly on $H$ and $A_k$ is a hereditary subalgebra of $A$, it follows that $A_k$ acts irreducibly on $B(A_k H)$ (see [Mur90, Theorem 5.5.2]). Therefore, by Kadison’s Transitivity Theorem, we can find a self-adjoint $c_k^0 \in A_k$ such that

$$c_k^0(ξ_k^0) = ξ_k^0, \quad c_k^0(η_k^0) = 0,$$

and $\|c_k^0\| = 1$. We can suppose that $c_k^0$ is positive by taking its square, doing so will not change its norm nor the image of $ξ_k^0$ and $η_k^0$. Consider the function

$$f_0(n) = \begin{cases} f(n) - 1 & \text{if } n \text{ is even} \\ f(n) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

We have that

$$e_{f_0(2k+1)}c_k^0 = c_k^0 e_{f_0(2k+1)} = c_k^0,$$

and therefore also

$$(e_{f_0(2k+1)} - e_{f_0(2k)})c_k^0 = c_k^0(e_{f_0(2k+1)} - e_{f_0(2k)}).$$

This entails

$$\|c_k^0\| = \|e_{f_0(2k+1)} - e_{f_0(2k)}\|^{1/2} c_k^0(e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2}.$$ 

The norm $\|c_k^0\|$ is bounded by 1 for every $k \in \mathbb{N}$, therefore the sum

$$a_0 = \sum_{k \in \mathbb{N}} c_k^0 = \sum_{k \in \mathbb{N}} (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2} c_k^0(e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2}$$

is strictly convergent (see [Ped90, Theorem 4.1] or [FH13, Item (10) p.48]), hence $a_0 \in M(A)_+$. Furthermore:

$$\|a_0\| = \| \sum_{k \in \mathbb{N}} (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2} c_k^0(e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2} \| \leq$$

$$\leq \| \sum_{k \in \mathbb{N}} e_{f_0(2k+1)} - e_{f_0(2k)} \| \leq 1.$$

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2 We can always assume $S_{a,n+1}$ has at least 2 linearly independent vectors for each $n \in \mathbb{N}$ by taking, if necessary, a subsequence $(e_{k_j})_{j \in \mathbb{N}}$ from the original approximate unit.
In order to show that \( a_0 \notin A \), first observe that

\[
a_0(\xi_0) = \sum_{m < k} c^0_m(\xi_0) + c^0_k(\xi_0) + \sum_{m > k} c^0_m(\xi_0) = c^0_k(\xi_0) = \xi_0.
\]

The first sum annihilates since \( \xi^0_k \in S_{f(2k),f(2k+1)} \) implies \( c^0_k = (e_{f_0(2k+1)} - e_{f_0(2k)})(\xi^0_k) \), and for \( m < k \)

\[
c^0_m(e_{f_0(2k+1)} - e_{f_0(2k)})(\xi^0_k) = c^0_m e_{f_0(2m+1)}(e_{f_0(2k+1)} - e_{f_0(2k)})(\xi^0_k) = 0,
\]

which follows by \( f_0(2m+1) < f_0(2k) < f_0(2k+1) \). The second series also annihilates, indeed for \( m > k \) we have \( c^0_m e_{f_0(2k+1)} = c^0_m e_{f_0(2m)} c^0_{f(2k+1)} = 0 \) (the same equation also holds for \( e_{f_0(2k)} \)). Using the same argument, it can be proved that

\[
a_0(\xi) = c^0_n(\xi)
\]

for every \( \xi \in S_{f_0(2n),f_0(2n+1)} \). Observe that \( \|(a_0 - e_{f_0(2m+1)a_0})(\xi^0_k)\| = 1 \) for \( k > m \), thus \( a_0 \notin A \).

The construction proceeds by transfinite induction on \( \aleph_1 \), the first uncountable cardinal. At step \( \beta < \aleph_1 \) we assume to have a sequence of elements \( (a_\alpha)_{\alpha < \beta} \) in \( (A_+) \) and functions \( (f_\alpha)_{\alpha < \beta} \) such that:

(i) For all \( \alpha < \beta \) the function \( f_\alpha : \mathbb{N} \to \mathbb{N} \) is strictly increasing and, given any other \( \gamma < \alpha \), for all \( k \in \mathbb{N} \) there exists \( N \in \mathbb{N} \) such that for all \( j > N \) and all \( i \in \mathbb{N} \) the following holds

\[
|f_\alpha(j) - f_\gamma(i)| > 2^k.
\]

Furthermore, we ask that for all \( \alpha < \beta \) and all \( k \in \mathbb{N} \):

\[
f_\alpha(2(k+1)) - f_\alpha(2k+1) > 2^{2k+1}.
\]

(ii) For each \( \alpha < \beta \) there exists a sequence \( (c^\alpha_k)_{k \in \mathbb{N}} \) of positive norm 1 elements in \( A \) such that

\[
a_\alpha = \sum_{k \in \mathbb{N}} c^\alpha_k.
\]

Moreover we require that

\[
e_{f_\alpha(2k+1)} c^\alpha_k = c^\alpha_k e_{f_\alpha(2k+1)} = c^\alpha_k,
\]

\[
e_{f_\alpha(2k)} c^\alpha_k = c^\alpha_k e_{f_\alpha(2k)} = 0,
\]

and that there exist \( \xi^0_k, \eta^0_k \in S_{f_\alpha(2k),f_\alpha(2k+1)} \), two norm one orthogonal vectors, such that \( c^\alpha_k(\xi^0_k) = \xi^0_k \) and \( c^\alpha_k(\eta^0_k) = 0 \).

(iii) Given \( \alpha < \beta \) and \( d_1, d_2 \in (M(A))_+ \), for all \( l \in \mathbb{N} \), and for all but possibly \( l \) many \( \gamma < \alpha \) the following holds:

\[
\left\| \left( a_\alpha + d_1 e_\gamma \right), (a_\gamma + d_2 e_\gamma) \right\| \geq \frac{1}{2}.
\]

It can be shown, as we already did for \( a_0 \), that for all \( \alpha < \beta \):

a. \( a_\alpha \in (M(A))_+ \setminus A \);

b. \( \|a_\alpha\| = 1 \);

c. \( a_\alpha(\xi) = c^\alpha_k(\xi) \in S_{f_\alpha(2k),f_\alpha(2k+1)} \) for every \( \xi \in S_{f_\alpha(2k),f_\alpha(2k+1)} \).

Moreover, by items (ii, iii), along with the fact that for \( n_2 < n_1 < m_2 < m_1 \)

\[
(e_{m_1} - e_{m_2})(e_{n_1} - e_{n_2}) = 0,
\]

we have that \( a_\alpha a_\gamma \in A \) for all \( \alpha, \gamma < \beta \).

We want to find \( f_\beta \) and \( a_\beta \) such that the families \( \{a_\alpha\}_{\alpha < \beta+1} \) and \( \{f_\alpha\}_{\alpha < \beta+1} \) satisfy the three inductive hypotheses. This will be sufficient to continue the induction and to obtain the thesis of the lemma. Since \( \beta \) is a countable ordinal, the sequence \( (a_\alpha)_{\alpha < \beta} \) is either finite or can be written as \( (a_\alpha)_{n < N} \), where \( n \mapsto a_n \) is a bijection between \( \mathbb{N} \) and \( \beta \). We
assume that $\beta$ is infinite, since the finite case is easier. In order to ease the notation, we shall denote $a_{n\alpha}$ by $a_n$ (and similarly $f_{n\alpha}$ by $f_n$, $c_{n\alpha}^k$ by $c_n^k$, etc.).

The construction of $a_\beta$ proceeds inductively on the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}$ ordered along with any well-ordering of type $\omega$ such that $(i, j) \leq (i', j')$ implies $j \leq j'$, like for example

$$(i, j) \leq (i', j') \iff j \leq j' \text{ or } j = j', i \leq i'.$$

Suppose we are at step $M$, which corresponds to a certain couple $(i, j)$. At step $M$ we provide a $c_M^\beta \in A_\beta$ such that, for every $d_1, d_2 \in M(A)$

$$\|[(a_j + d_1 e_i), (c_M^\beta + d_2 e_i)]\| \geq \frac{1}{2}$$

and we define two values of $f_\beta$. Assume that $f_\beta(n)$ has been defined for $n \leq 2M - 1$. Let $m \in \mathbb{N}$ be the smallest natural number such that

$$f_j(2m) > \max \{i + 2, f_\beta(2M - 1) + 2^{2M-1} + 1\}$$

and such that, for $l \geq 2m$, the inequality $|f_j(l) - f_k(n)| > 2^{M + 1}$ holds for all $k \in \mathbb{N}$ such that $\alpha_k < \alpha_j$, and all $n \in \mathbb{N}$. By inductive hypothesis there are two norm one orthogonal vectors $\xi_m, \eta_m \in S_{f_j(2m), f_j(2m+1)}$ such that $c_m(\xi_m^\beta) = \xi_m$ and $c_m(\eta_m) = 0$. Set $\xi_M = \frac{1}{\sqrt{2}}(\xi_m + \eta_m)$ and $\eta_M = \frac{1}{\sqrt{2}}(\xi_m - \eta_m)$. Using Kadison’s Transitivity Theorem, fix a positive, norm one element

$$c_M^\beta \in (e_{f_j(2m+1)} - e_{f_j(2m)})A(e_{f_j(2m+1)} - e_{f_j(2m)})$$

such that

$$c_M^\beta(\xi_M^\beta) = \xi_M^\beta,$$

$$c_M^\beta(\eta_M^\beta) = 0.$$

Let $f_\beta(2M) = f_j(2m) - 1$ and $f_\beta(2M + 1) = f_j(2m + 1) + 1$. We have therefore that

$$e_{f_j(2m+1)}c_M^\beta e_{f_j(2m+1)} = c_M^\beta,$$

$$e_{f_j(2m)}c_M^\beta = c_M^\beta e_{f_j(2m)} = 0.$$

Moreover:

$$(*) \quad \| (a_j + d_1 e_i)(c_M^\beta + d_2 e_i)(\xi_M^\beta) - (c_M^\beta + d_2 e_i)(a_j + d_1 e_i)(\xi_M^\beta) \| = \frac{1}{2} \| e_m - \eta_m \| = \frac{1}{2}.$$

This is the case since $c_i(\xi) = 0$ for every $\xi \in S_{f_j(2m), f_j(2m+1)}$ (we chose $m$ so that $f_j(2m) > i + 2$) and $c_M^\beta(\xi_M^\beta), a_j(\xi_M^\beta) = c_m(\xi_M^\beta) \in S_{f_j(2m), f_j(2m+1)}$. Define

$$a_\beta = \sum_{n \in \mathbb{N}} c_n^\beta = \sum_{n \in \mathbb{N}} (e_{f_\beta(2m+1)} - e_{f_\beta(2m)})^\frac{1}{2} c_n^\beta (e_{f_\beta(2m+1)} - e_{f_\beta(2m)})^\frac{1}{2}.$$

This series is strictly convergent since all $c_n^\beta$’s have norm 1. The families $\{f_n\}_{n \in \mathbb{N}} \cup \{f_\beta\}$ and $\{a_n\}_{n \in \mathbb{N}} \cup \{a_\beta\}$ satisfy items (i)-(iii) of the inductive hypothesis.

Finally we verify clause (iii). Notice that, by construction, for every $k \in \mathbb{N}$, given $\xi \in S_{f_\beta(2k), f_\beta(2k+1)}$ we have

$$a_\beta(\xi) = c_k^\beta(\xi).$$

The induction to define $a_\beta$ and $f_\beta$ is on the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}$ ordered with a well-ordering of type $\omega$ such that $(i, j) \leq (i', j')$ implies $j \leq j'$. This is used to show that $f_\beta$ satisfies clause (iii) of the inductive hypothesis.
Let $i \leq j \in \mathbb{N}$, denote the step corresponding to the couple $(i,j)$ by $M$, and let $m \in \mathbb{N}$ be such that $f_{\beta}(2M) = f_{\beta}(2m) - 1$ (by construction we can find such $m$). Remember that $\xi^\beta_M = \frac{1}{\sqrt{2}} (\xi_j^m + \eta_j^m) \in S_{f_{\beta}(2M), f_{\beta}(2M+1)}$. Given $d_1, d_2 \in M(A)$, we have that

$$
\| (a_j + d_1 e_i)(a_{\beta} + d_2 e_i)(\xi^\beta_M) - (a_j + d_2 e_i)(a_j + d_1 e_i)(\xi^\beta_M) \| = \frac{1}{2} \| \xi^m_j - \eta^m_j \| = \frac{1}{2}.
$$

This equation can be shown using the same arguments used to prove $(\ast)$. Notice that if $\beta$ is finite, we only obtain a finite number of $c_n^\alpha$, therefore their sum (which is finite) does not belong to $M(A) \setminus A$. In this case it is sufficient to add an infinite number of addends, as we did for $a_0$. Suppose that $\beta$ is the ordinal corresponding to $N \in \mathbb{N}$, then the previous construction defines $f_N$ only up until $2N + 1$. Let $f_{N}(2(N+1))$ be the smallest integer such that

- $f_{N}(2(N+1)) - f_{N}(2N + 1) > 2^{2N+1}$;
- $|f_{N}(2(N+1)) - f_{j}(n)| > 2^{2(N+1)}$ for all $j < N$; and for all $n \in \mathbb{N}$.

Define $f_{N}(2(N+1) + 1) = f_{N}(2(N+1)) + 3$ and continue inductively the definition of $f_N$. For each $n > N$ we can therefore, as we did for $a_0$ using Kadison’s Transitivity Theorem, find a positive element

$$
\xi_N^N \in S_{f_{N}(2n)\cdot f_{N}(2n+1)}
$$

which moves a norm one vector $\xi_N^N$ into itself, and another orthogonal norm one vector $\eta_N^N$ to zero. If we define $a_N$ to be the sum of such $\xi_N^N$’s, it is possible to show, using the same arguments exposed when $\beta$ was assumed to be infinite, that the families $\{f_n\}_{n<N} \cup \{f_\beta\}$ and $\{a_n\}_{n<N+1}$ satisfy items (I)-(III) of the inductive hypothesis. \hfill \Box

The proof of Theorem 1.1 is analogous to the one given in Theorem 7 of [BK16], but it uses our Lemma 3.1 instead of [BK16, Lemma 6].

**Proof of Theorem 1.1.** Let $(e_n)_{n \in \mathbb{N}} \subseteq A$ be the approximate unit defined at the beginning of the current section, and let $(a_{\beta})_{\beta \in \mathbb{N}_1}$ be the $\mathbb{N}_1$-sized collection obtained from Lemma 3.1. Suppose there is an uncountable $U \subseteq \mathbb{N}_1$ and $(d_\beta)_{\beta \in U} \subseteq A$ such that

$$
[(a_\alpha + d_\alpha), (a_{\beta} + d_\beta)] = 0
$$

for all $\alpha, \beta \in U$. By using the pigeonhole principle, we can suppose that $\|d_\beta\| \leq M$ for some $M \in \mathbb{R}$, and that there is a unique $n \in \mathbb{N}$ such that $\|d_\beta - d_\beta e_n\| \leq \frac{1}{\sqrt{16}}$ for all $\beta \in U$.

Therefore, for every $\beta \in U$ and all but finitely many $\alpha \in U$ such that $\alpha < \beta$, we have

$$
0 = \|[(a_\alpha + d_\alpha), (a_{\beta} + d_\beta)]\| \geq \|[(a_\alpha + d_\alpha e_n), (a_{\beta} + d_\beta e_n)]\| - \frac{1}{16} \geq \frac{1}{16}.
$$

This is a contradiction when $\{\alpha \in U : \alpha < \beta\}$ is infinite. \hfill \Box

**Proof of Corollary 1.2.** The proof follows verbatim the one given for Lemma 3.1 and Theorem 1.1. The only difference is that each time Kadison’s Transitivity Theorem is invoked in Lemma 3.1, it is possible to use a stronger version of such theorem for $C^*$-algebras with real rank zero (see for instance Theorem 6.5 of [Bic13]), which allows us to choose a projection at each step. This stronger version of Kadison’s Transitivity Theorem can be used throughout the whole iteration since hereditary subalgebras of real rank zero $C^*$-algebras have real rank zero. \hfill \Box
4. Concluding remarks and questions

If $A$ is a commutative non-unital $C^*$-algebra, then the problem of lifting commuting elements from $Q(A)$ to $M(A)$ is trivial, as both $M(A)$ and $Q(A)$ are abelian. In Section $3$ we ruled out this possibility by asking for $A$ to be primitive.

The other important feature we required to prove Theorem $1.1$ is $\sigma$-unitality. We do not know whether this assumption could be weakened, but it certainly cannot be removed tout-court. Indeed, there are extreme examples of primitive, non-$\sigma$-unital $C^*$-algebras whose corona is finite-dimensional (see [Sak71] and [GKi8]), for which Theorem $1.1$ is trivially false. Our conjecture is that there might be a condition on the order structure of the approximate unit of $A$ which is weaker than $\sigma$-unitality, but still makes Theorem $1.1$ true. For instance, it would be interesting to know whether the techniques used in Theorem $1.1$ could be applied to the algebra of the compact operators on a nonseparable Hilbert space, or more in general to a $C^*$-algebra $A$ with a projection $p \in M(A)$ such that $pA$p is primitive, non-unital and $\sigma$-unital.

We remark that the proof of Theorem $1.1$ we gave can be adapted to any primitive $C^*$-algebra $A$ which admits an increasing approximate unit $\{e_\alpha\}_{\alpha \in \kappa}$, for $\kappa$ regular cardinal, to produce a $\kappa^+$-sized family of orthogonal positive elements in $Q(A)$ which cannot be lifted to a set of commuting elements in $M(A)$.

Another lifting problem that could be considered, is the following.

**Question 4.1.** Assume $F \subseteq Q(A)_{sa}$ is a commutative family such that any smaller (in the sense of cardinality) subset can be lifted to a set of commuting elements in $M(A)_{sa}$. Can $F$ be lifted to a collection of commuting elements in $M(A)_{sa}$?

Theorem $1.1$ and Proposition $2.2$ entail that this is not true in general for primitive, non-unital, $\sigma$-unital $C^*$-algebras if $|F| = \aleph_1$, pointing out the set theoretic incompactness of $\aleph_1$ for this property.

If the family $F$ is infinite and countable, then Question $4.1$ has a positive answer in the Calkin algebra.

**Proposition 4.2.** Suppose that $A$ is a separable abelian $C^*$-subalgebra of $Q(H)$ such that every finitely-generated subalgebra of $A$ has an abelian lift. Then $A$ has an abelian lift.

The proof of this proposition relies on Voiculescu’s Theorem [HR00, Theorem 3.4.6], starting from the following lemma. Given a map $\varphi : A \rightarrow Q(H)$, we say that $\Phi : A \rightarrow B(H)$ lifts $\varphi$ if $\varphi = \pi \circ \Phi$, where $\pi : B(H) \rightarrow Q(H)$ is the quotient map.

**Lemma 4.3.** Let $A$ be a separable unital abelian $C^*$-subalgebra of $Q(H)$. If there exists a unital abelian $C^*$-algebra $B \subseteq B(H)$ lifting $A$, then there is a unital $*$-homomorphism $\Phi : A \rightarrow B(H)$ lifting the identity map on $A$.

**Proof.** Since $B$ is abelian, there exists a masa (maximal abelian subalgebra) of $B(H)$ containing $B$. Massas in $B(H)$ are von Neumann algebras and, as such, they are generated by their projections. This entails that $A$ is contained in a separable unital abelian subalgebra $C(Y)$ of $Q(H)$ which is generated by its projections. By [BDF77, Theorem 1.15] there exists a unital $*$-homomorphism $\Psi : C(Y) \rightarrow B(H)$ lifting the identity on $C(Y)$. Let $\Phi$ be the restriction of $\Psi$ to $C(X)$.

**Proof of Proposition 4.2.** Suppose that $F = \{a_n\}_{n \in \mathbb{N}} \subseteq Q(H)_{sa}$ is an abelian family such that every finite subset of $F$ has a commutative lift. Without loss of generality, we can assume that $a_0 = 1$. By Lemma $4.3$ we can assume that, for every $k \in \mathbb{N}$, there is a unital $*$-homomorphism $\Phi_k : C^*(\{a_n\}_{n \leq k}) \rightarrow B(H)$ lifting the identity map on $C^*(\{a_n\}_{n \leq k})$. By Voiculescu’s Theorem [HR00, Theorem 3.4.6] we can moreover assume that, for every $n \in \mathbb{N}$, the sequence $\{\Phi_k(a_n)\}_{k \geq n}$ converges to some self-adjoint operator $A_n$ in $B(H)$ such that $A_n - \Phi_k(a_n)$ is compact for every $k \in \mathbb{N}$. The family $\{A_n\}_{n \in \mathbb{N}}$ is a commutative lifting of $\{a_n\}_{n \in \mathbb{N}}$. 

□
More general forms of Voiculescu’s Theorem are known to hold for extensions of various separable C*-algebras other than $K(H)$ (see [EK01, Gab16, Sch18 Section 2.2]). Such generalizations could potentially be used to carry out the arguments exposed above for coronas of other separable nuclear stable C*-algebras. We remark however the importance of being able to lift separable abelian subalgebras of $Q(H)$ to abelian algebras in $B(H)$ with the same spectrum, as guaranteed by Lemma 4.3. This is false in general in other coronas, as it happens for instance when $A = Z \otimes K(H)$. In this case, projections in $Q(A)$ do not necessarily lift to projections in $M(A)$, since the former has real rank zero but the latter has not (see [LN16]).

The following example shows that Question 4.1 has negative answer for finite families with an even number of elements.

**Example 4.4.** Let $S^n$ be the $n$-dimensional sphere. The algebra $C(S^n)$ is generated by $n + 1$ self-adjoint elements $\{h_i\}_{0 \leq i \leq n}$ satisfying the relation

$$h_0^2 + \cdots + h_n^2 = 1.$$ 

Let $F = \{h_i\}_{0 \leq i \leq n}$. The relation above implies that the joint spectrum of a subset of $F$ of size $m \leq n$ is the $m$-dimensional ball $B^m$. The space $B^m$ is contractible, therefore the group $\text{Ext}(B^m)$ is trivial (see [HR00 Section 2.6–2.7] for the definition of the functor $\text{Ext}$ and its basic properties). As a consequence, for any $[\tau] \in \text{Ext}(S^n)$, any proper subset of $\tau[F]$ can be lifted to a set of commuting self-adjoint operators in $B(H)$. On the other hand $\text{Ext}(S^{2k+1}) = \mathbb{Z}$ for every $k \in \mathbb{N}$. We conclude that any non-trivial extension $\tau$ of $C(S^{2k+1})$ produces, by Lemma 4.3, a family $\tau[F]$ of size $2k + 2$ in the Calkin algebra for which Question 4.1 has negative answer.

The argument above does not apply to families of odd cardinality, since $\text{Ext}(S^{2k}) = \{0\}$ for every $k \in \mathbb{N}$. However, in [Dav85] (see also [Voi81, Lor88]), the author builds a set of three commuting self-adjoint elements in the corona algebra of $\mathbb{Z} \oplus \bigoplus_{n \in \mathbb{N}} M_n(\mathbb{C})$ with no commutative lifting to the multiplier algebra, whose proper subsets of size two all admit a commutative lifting. The answer to Question 4.1 for larger finite families with an odd number of elements is, to the best of our knowledge, unknown.

**Acknowledgements.** My sincerest gratitude goes to Ilijas Farah, for his patience, and for the impressive amount of crucial suggestions he gave me while working on this problem and on the earlier drafts of this paper. I wish to thank Alessandro Vignati for the stimulating conversations we had, for the knowledge he shared with me, and for the suggestions on the earlier drafts of this paper. I wish to thank George Elliott for the incredible amount of interesting questions related to this topic he posed and for having suggested me a more intuitive way to present this problem and a more readable form of Lemma 8.1.

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