Projection evolution of quantum states.
Part I.

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Abstract

We discuss the problem of time in quantum mechanics. In the traditional formulation of quantum mechanics time enters the model as a parameter, not an observable, as was suggested by the famous Pauli theorem. It is now known, that Pauli’s assumptions were too strong and that by removing some of them time can be represented as a quantum observable. In this case, instead of the unitary time evolution, other operators which map the initial space of states into the final space of states at each step of the evolution can be used. This allows to treat time as a quantum observable in a consistent way.

Keywords: quantum mechanics, time evolution, interference in time

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1 Introduction

For many years time has been treated in physics in a very simple way. It is regarded as a universal parameter which allows the observer to divide the reality into past, present, and future. What is more, time flows always in one direction, called the arrow of time. This direction implies also the direction of changes that may spontaneously happen to any physical system, which ultimately leads to the notion of causality. We are used to the fact that past affects future, but future cannot affect the past, as this will act against the arrow of time.

The development of relativity theory changed this picture in a substantial way. To obtain a consistent model, time had to be treated as a coordinate, forming with the spatial coordinates the spacetime. The metric and other tensors gained their time components which were transforming during the change of the coordinate system along with the spatial coordinates. For example, the four-momentum $p^\mu$, $\mu = 0, 1, 2, 3$, takes the form $p^\mu = (p^0, \vec{p})$, where $p^0 = E/c$, $E$ being the total energy. This feature is absent in the classical approach in which the nature of time is not discussed.

One may ask if time behaves in the same way in the macroscopic and microscopic scales? We know that both classical and relativistic physics agree upon the properties of time, so if one expects any deviations from the standard picture, one should look at the quantum mechanics.

1.1 Time in the quantum theory

In the standard formulation of the quantum theory, any physical quantity is represented by a hermitian operator whose eigenvalues are the possible outcomes of its measurement. Motivated by the so-called Pauli theorem \[1, 2, 3\], there is no time operator in the standard formulation of quantum mechanics implying, that time is not a physical observable. Instead, it is introduced as a universal numerical parameter. We have therefore very limited means to discuss quantum events in the time domain.

The Pauli theorem is an observation that it is in general impossible to construct a self-adjoint time operator $\hat{t}$, which would be canonically conjugate to a generic Hamiltonian $\hat{H}$. The justification of this claim is as follows: assume that we have such a pair of canonically conjugated hermitian operators,

$$[\hat{t}, \hat{H}] = i. \quad (1)$$

Since $\hat{t}^\dagger = \hat{t}$, one may construct a unitary operator $U = \exp(-i\beta\hat{t})$, with $\beta \in \mathbb{R}$ being an arbitrary parameter. The commutator $[U, \hat{H}]$ can now be
computed using the power expansion of $U$, yielding $[U, \hat{H}] = \beta U$. It immediately follows that for a Hamiltonian eigenvector $\phi$, for which $\hat{H}\phi = E\phi$, we have

$$\hat{H}U\phi = (E - \beta)U\phi.$$  

Because $\beta$ is an arbitrary real number, Eq. (2) implies that the spectrum of the Hamiltonian must always be continuous and unbounded from below, which is obviously not true. Pauli stated: “We conclude that the introduction of an operator $t$ must fundamentally be abandoned and that the time in quantum mechanics has to be regarded as an ordinary number.” This strong conclusion means that it is impossible to discuss the time structure of events within the quantum theory.

A careful mathematical analysis of this problem was presented by E. A. Galapon in Ref. [4]. By considering the domains of the operators $t$ and $\hat{H}$ the author showed, that the domain of their commutator does not contain the domain of $\hat{H}$, contrary to what was silently assumed by Pauli. This implies that the $\beta$ parameter can only have values which appropriately correspond to the eigenvalues of $\hat{H}$ and therefore, as it is not arbitrary, the Pauli conclusion does not hold. It follows that a hermitian time operator canonically conjugated to $\hat{H}$ can in principle be constructed. What is more, observables do not need to be hermitian operators, but are in general represented by positive operator valued measures (POVM). There are no arguments against the construction of a time operator in terms of a POVM – a possibility that was not considered by Pauli.

1.2 Experimental work

The important role of time in quantum theories is suggested by some experiments. In Refs. [5, 6] J.A. Wheeler proposed a Gedankenexperiment based on the Mach–Zehnder interferometer, consisting of two beamsplitters and two mirrors. A single photon was traveling through the interferometer. During the particle’s flight some changes to the setup were introduced, including the removal or insertion of the first or the second beamsplitter, even after the photon has classically passed that part of the machine. Wheeler argued that the final detection of the photon should be sensitive to these changes, mainly due to the spatial width of the photon wave function. This idea has been experimentally tested. An analogue of the Mach–Zehnder interferometer was used by the group of A. Aspect with the primary intention to test Bell’s inequalities [7] showing that Wheeler’s predictions were correct. Other groups [8, 9, 10, 11, 12, 13] arrived at similar results. In order to investigate the problem further, a quantum eraser was used. Its purpose
was to remove the information about an additional measurement, which was
done on the particle during the experiment. It turned out that erasing the
information recreates the quantum behavior of the system even in the case
when the eraser worked after the final detection has been performed [13].
This setup has been called the delayed choice quantum eraser. In another
experiment [15] the interferometer was built between an earth-based station
and a satellite. The photons behaved like particles or waves depending on
the choices made by the investigators on earth. The effect was visible even
when the changes introduced to the setup were causally disconnected from
the particles.

Another experiment was conducted using entangled pairs of photons [16,
17]. The pair was created in one laboratory and one of the particles stayed
there, while the other was sent to the second laboratory. The transmission
took place between two islands, La Palma and Tenerife, with the distance
between them around 144 km. Even though the particles were causally dis-
connected, the changes made in the first laboratory were affecting the second
particle suggesting, that either we accept a faster-than-light communication,
or the notion of the spatial and temporal localization of a quantum object
should be reformulated.

If time in the quantum regime should be treated as a coordinate, and in
fact a quantum observable, all physical objects have to have some “width” in
the time direction, which follows directly from the energy-time uncertainty
relation. This means that it should be possible to observe the interference of
quantum objects through their overlap in time. One of the first experiment
in which such behavior has been observed, was reported in Ref. [18], followed
later by [19]. In [18] a single photon was emitted and a spinning chopper in
the form of a wheel with slits was placed between the source and the detect-
or. The energy spectrum of the detected photons was recorded as a function
of the disc’s rotation frequency. The spectrum clearly showed minima and
maxima in a way very similar to the interference pattern. The authors were
unable to fully explain this behavior, even though they presented a sim-
ple analysis based on the Fourier transform of the energy spectrum. It was
pointed out in Refs. [20, 21] that after the Fourier transform of the energy-
dependent function the authors worked with the time-dependent one and all
what they got could be interpreted as an interference pattern between the
temporal parts of the photon wave function. So the most obvious explana-
tion suggests the observation of the interference of the photon with itself in
different time instances.

It seems to be very difficult to answer the fundamental question: What
is time? In Ref. [22] the authors propose, that time is a consequence of
the entanglement between particles in the universe. If that were true, it
would be impossible to incorporate this notion in any classical theory, as entanglement is a purely quantum effect. To completely destroy the classical picture, a recent paper reports that the thermodynamical arrow of time has been reversed in a quantum system, in which the heat was spontaneously transferred from the colder to the hotter subsystem [23].

In this paper we present a consistent formulation of the quantum theory with time being an observable. This allows to interpret the zeroth component of the four-momentum as the time translation operator, as well as to define the time operator. In this approach the evolution of quantum states has to be reformulated, as time, being a coordinate, cannot act as a universal ordering parameter any longer.

2 Projection evolution of quantum systems

The probabilistic structure of quantum mechanics allows time to be, under certain conditions, a quantum observable. First of all it cannot be considered as a parameter which enumerates subsequent events but it has to be represented by an operator similar to the position operators. It means that different time characteristics of a given quantum system can be calculated. In general, they are dependent on the state of this system. We start from the assumption that the quantum time, and generally the spacetime, is “created” by changes of the Universe. This requires a modification of some parts of the paradigm of science related to the causality and the ordering of quantum events.

2.1 The changes principle

We start by formulating the fundamental assumption of the projection evolution (PEv) principle: the evolution of a system is a random process caused by some spontaneous changes in the Universe. We call it the changes principle. It means that we treat the change as the primary process, which allows to define time. This is in contradiction with the usual thinking in which the existence of time allows the changes to happen. In our approach the changes happen spontaneously, according to the probability distribution, which is dictated by many factors describing the system and its environment. It does not mean that the changes of a quantum state of the subsystems of the Universe are totally stochastic. They are obviously not deterministic, but because of the interactions, symmetries which have to be conserved, EPR correlations etc., they are related to each other and bound by the rules of their behavior known from our experience.
As a consequence one may expect the existence of a kind of pseudo-causality based on the ordering of the quantum events, which leads to the causality principle in the case of macroscopic physical systems. In order to describe this property we introduce a parameter $\tau$ which orders quantum events. This parameter should be common for the whole Universe. It should take values from an ordered set but it does not need to have any metric structure. The parameter $\tau$ is not an additional dimension of our space and it is not a replacement of time. It serves only to enumerate the subsequent steps of the evolution of the Universe and any of its physical subsystems. The most natural linearly ordered set is any subset of the real numbers.

In what follows we assume that the domain of the evolution parameter $\tau$ is isomorphic to integers or their subset. In this case we can always use the notion of “the next step of the evolution,” which may be problematic for the real numbers. In the situation of a continuous or dense subset of the real numbers as the domain for $\tau$, there are some conceptual difficulties which should be, if needed, solved in the future.

An additional, very important feature of this approach is that this idea does not need the spacetime as the background. Nearly all physical theories constructed till now use the spacetime as the primary object, with the dynamics built on top of it. The projection evolution approach is a background free theory.

### 2.2 Projection evolution operators

In the standard formulation of quantum physics, there are two kinds of time evolution: (i) the unitary evolution, which is a deterministic evolution of the actual quantum state, and (ii) the stochastic evolution, which takes place during a measurement. The latter process involves the projection of the quantum state onto the measured state and can be described by one of the projection postulates. There is a common belief that this process can be described by means of the unitary evolution of a larger system. This approach leads, however, to the known quantum measurement problems [24].

The changes principle is incompatible with the unitary evolution, where time is considered to be a parameter. The idea of the changes principle suggests the opposite scenario – the primary evolution is the stochastic evolution offered by a projection postulate. We propose to use the generalized form of the Lüders [25] type of the projection postulate. We show later that the Schrödinger type evolution can be obtained as the special case of the projection evolution.

In the following, we introduce the projection operators which are formally responsible for the quantum evolution of a physical object. We expect that
in general these operators will be different for different systems, similarly to the Hamiltonian, which is a characteristic object for a given quantum system. On the other hand, one should in principle be able to construct the projection evolution operators for the whole Universe which will contain the operators for any smaller subsystem. It is due to the fact that the proposed formalism does not require any external observer for the evolution.

The projection evolution operator at the evolution step \( \tau_n \), where \( n \in \mathbb{Z} \), is a family of mappings between the space of quantum states at the evolution step \( \tau_n \) and the space of quantum states at the evolution step \( \tau_{n+1} \). The state spaces are assumed to be some subspaces of the trace class operators (the space of operators with finite trace, i.e., the set of density operators) defined on a given Hilbert space \( \mathcal{K} \). In this case the Hilbert space of a single, nonrelativistic spinless particle is not contained in \( L^2(\mathbb{R}^3, d^3x) \) but rather in \( L^2(\mathbb{R}^4, d^4x) \), where the fourth dimension is time, treated here on the same footing as the positions in the 3D-space.

These mappings can always be written in terms of the so-called quantum operations or their generalizations. The formalism of quantum operations was invented around 1983 by Krauss [26], who relied on the earlier mathematical works of Choi [27].

This approach has the advantage that the quantum operation allows to construct operators which transform quantum states into other quantum states or, in general, density operators into density operators.

Let us denote by \( \mathcal{T}_1^+(\mathcal{K}) \) the space of positive, trace one operators in \( \mathcal{K} \), and by \( \mathcal{T}^+(\mathcal{K}) \) the space of finite trace positive operators in \( \mathcal{K} \). Here, \( \mathcal{K} = \mathcal{K}(\tau_n) \) denotes the space of states at the evolution step \( \tau_n \).

The projection evolution operators at the evolution step \( \tau_n \) are formally defined as a family of transformations from the quantum state space (density operators space) \( \mathcal{T}_1^+(\mathcal{K}(\tau_n)) \) to the space \( \mathcal{T}^+(\mathcal{K}(\tau_{n+1})) \),

\[
\mathcal{F}(\tau_n; \nu, \cdot) : \mathcal{T}_1^+(\mathcal{K}(\tau_n)) \to \mathcal{T}^+(\mathcal{K}(\tau_{n+1})),
\]

where \( \nu \in \mathcal{Q}_{n+1} \), with \( \mathcal{Q}_{n+1} = \mathcal{Q}(\tau_{n+1}) \) being a family of sets of quantum numbers defining potentially available new states for the evolution from \( \tau_n \) to \( \tau_{n+1} \). We denote by \( \mathcal{F}(\tau_n; \nu, \rho) \) the action of the operator \( \mathcal{F}(\tau_n; \nu, \cdot) \) on the density operator \( \rho \), such that \( \mathcal{F}(\tau_n; \nu, \cdot)\rho = \mathcal{F}(\tau_n; \nu, \rho) \). The notation \( \mathcal{F}(\tau_n; \nu, \rho) \) is in many cases more appropriate because, in general, the evolution operators do not need to be linear. One needs to remember that we assumed that \( \tau_n \) can be enumerated by integers.

To use the generalized Lüders projection postulate as the principle for the
evolution, the operators \( F(\tau; \nu, \rho) \) have to be hermitian, non-negative, and finite:

\[
F(\tau; \nu, \rho) = F(\tau; \nu, \rho), \quad (4)
\]

\[
F(\tau; \nu, \rho) \geq 0, \quad (5)
\]

\[
\sum_{\nu \in Q} \text{Tr}(F(\tau; \nu, \rho)) < \infty. \quad (6)
\]

All these three conditions form the minimal requirement implied by the properties of the density operators.

Assume that at the evolution step \( \tau_{n-1} \) the actual quantum state of a physical system is given by the density operator \( \rho(\tau_{n-1}; \nu_{n-1}) \), \( \nu_{n-1} \in Q_{n-1} \). The changes principle implies that every step of the evolution is similar to the measurement process. We can say that there exists in the Universe a mechanism, the chooser, which chooses randomly from the set of states determined by the projection postulates the next state of the system for \( \tau = \tau_n \). With these assumptions, following Ref. [25], we postulate \( \rho(\tau_n; \nu_n), \nu_n \in Q_n \), in the form

\[
\rho(\tau_n; \nu_n) = \frac{F(\tau_n; \nu_n, \rho(\tau_{n-1}; \nu_{n-1}))}{\text{Tr}(F(\tau_n; \nu_n, \rho(\tau_{n-1}; \nu_{n-1}))}. \quad (7)
\]

Because the chooser represents a stochastic process, to fully describe it one needs to determine the probability distribution for getting a given state in the next step of the evolution.

In general, the probability distribution for the chooser is given by the quantum mechanical transition probability from the previous to the next state, i.e., \( \text{Prob}(\rho(\tau_{n-1}; \nu_{n-1}) \rightarrow \rho(\tau_n; \nu_n)) \). This probability for pure quantum states is determined by the appropriate probability amplitudes in the form of scalar products. The transition probability among mixed states, in general, remains an open problem. The transition probability (or the transition probability density) from the state labelled by \( \nu_{n-1} \) at \( \tau_{n-1} \) to the state labelled by \( \nu_n \) at \( \tau_n \) for a given evolution process we denote as: \( \text{pev}(\tau_{n-1}, \nu_{n-1} \rightarrow \tau_n, \nu_n) \).

The most important realization of the evolution operators \( F(\tau_n; \nu_n, \rho) \) can be constructed in the form of Krauss–like operators. For every \( \nu_n \in Q_n \) we have

\[
F(\tau_n; \nu_n, \rho) = \sum_k \mathbb{E}(\tau_n; \nu_n, k) \rho \mathbb{E}(\tau_n; \nu_n, k)\dagger, \quad (8)
\]

where the summation over \( k \) is dependent on the quantum numbers \( \nu_n \). It is easy to check that the conditions (4) and (5) are automatically fulfilled, namely:

\[
F(\tau_n; \nu_n, \rho)\dagger = \sum_k \mathbb{E}(\tau_n; \nu_n, k) \rho \mathbb{E}(\tau_n; \nu_n, k)\dagger = F(\tau_n; \nu_n, \rho) \quad (9)
\]
and, since $\rho \geq 0$, we have for all $\phi \in \mathcal{K}$

$$
\langle \phi | \sum_k E(\tau_n; \nu_n, k) \rho E(\tau_n; \nu_n, k) \dagger | \phi \rangle 
= \sum_k (\langle \phi | E(\tau_n; \nu_n, k) \rangle \rho (E(\tau_n; \nu_n, k) \dagger | \phi \rangle) \geq 0.
$$

(10)

Using Eq. (8) and the fact that trace is cyclic, the left hand side of the condition (6) takes the form

$$
\sum_{\nu_n \in \mathbb{Q}_n} \sum_k \text{Tr}(E(\tau_n; \nu_n, k) \rho E(\tau_n; \nu_n, k) \dagger)
= \sum_{\nu_n \in \mathbb{Q}_n} \text{Tr} \left( \sum_k E(\tau_n; \nu_n, k) \dagger E(\tau_n; \nu_n, k) \rho \right).
$$

(11)

It follows that the relation (10) is fulfilled if the transformation

$$
\sum_k E(\tau_n; \nu_n, k) \dagger E(\tau_n; \nu_n, k)
$$

does not lead outside the space $\mathcal{T}^+$, i.e.,

$$
\sum_{\nu_n \in \mathbb{Q}_n} \sum_k E(\tau_n; \nu_n, k) \dagger E(\tau_n; \nu_n, k) : \mathcal{T}^+(\mathcal{K}(\tau_n)) \rightarrow \mathcal{T}^+(\mathcal{K}(\tau_{n+1})).
$$

(12)

This condition is weaker than the conditions usually assumed for the Krauss operators. Especially, in our case, the strong requirement of completely positive map is relaxed, which is in agreement with the suggestion from Ref. [28].

Typical and useful examples of the $E$ operators are connected with the unitary evolution and the orthogonal resolution of unity. In the first case the operator is

$$
E(\tau_n; \nu_n = 1, k = 1) = U(\tau_n),
$$

(13)

where $\nu_n$ and $k$ are fixed and $U(\tau_n)$ is a unitary operator. In this case the next step of the evolution is chosen uniquely with the probability equal to 1. One needs to note that the unitary operator (13) is not parametrized by time but by the evolution parameter $\tau$, even though, in general, it is time dependent.

In the case of the orthogonal resolution of unity with respect to the quantum numbers $\nu$ the following conditions hold ($k = 1$ is fixed, the more general case can be written similarly):

$$
E(\tau; \nu) \dagger = E(\tau; \nu),
E(\tau; \nu) E(\tau; \nu') = \delta_{\nu\nu'} E(\tau; \nu),
\sum_{\nu \in \mathbb{Q}_n} E(\tau; \nu) = I,
$$

(14)
where $E(\tau; \nu) \equiv \mathbb{E}(\tau; \nu, k = 1)$, and the operator $\mathbb{I}$ denotes the unit operator.

Different alternatives are described by different sets of quantum numbers $\nu$.

The probability distribution of choosing the next state of the evolution generated by (14) is now given by the known quantum mechanical formula:

$$p_{\text{ev}}(\tau_n, \nu_n \rightarrow \tau_{n-1}, \nu_{n-1}) = \text{Tr} \left( E(\tau_n; \nu_n) \rho(\tau_{n-1}, \nu_{n-1}) E(\tau_n; \nu_n)^\dagger \right).$$

(15)

The above discussed examples, even though generic for many quantum mechanical systems, are only special cases of more general evolution operators.

3 The time operator

In order to introduce the time operator, let us consider, without loss of generality, a single particle described by the state space $\mathcal{K} \subset L^2(\mathbb{R}^4, d^4x)$ of pure states, i.e., vectors, instead of density operators. The scalar product in the state space $\mathcal{K}$ is given by

$$\langle \Phi_2 | \Phi_1 \rangle = \int_{\mathbb{R}^4} d^4x \Phi_2(x)^* \Phi_1(x),$$

(16)

where $x$ denotes the spacetime coordinates $x = (x^0, \vec{x})$. As usually, $x^0$ is the time coordinate and $\vec{x} = (x^1, x^2, x^3)$ are spatial coordinates. The scalar product (16) has the following probabilistic interpretation: the spacetime realization $\Psi(x)$ of any pure state $|\Psi\rangle \in \mathcal{K}$ represents the probability amplitude of finding the particle in the spacetime point $x$, i.e., $\langle \Psi(x) | \Psi(x) \rangle = |\Psi(x)|^2$ is the density probability of finding this particle at $x$.

The above presented approach leads to the breaking of the classically understood causality. The functions $\Psi(x) \in L^2(\mathbb{R}^4, d^4x)$ in their general form connect also events with space-like intervals $(x^0)^2 - \vec{x}^2 < 0$. Obviously, this can be easily removed by assuming that $\mathcal{K}$ consists of functions with time-like and zero-like support, which means that outside of the set $(x^0)^2 - \vec{x}^2 \geq 0$ the functions $\Psi(x)$ are zero. Some experimental works [29] suggest, however, that it is a natural phenomenon that the classical causality is broken in the quantum world. To be more general, we allow for states which break the classical causality to some acceptable extent. Within the PEv approach the quantum causality is realized by keeping the correct sequence of the subsequent steps of the evolution, ordered by the parameter $\tau$.

The PEv formalism allows to construct the time operator, as the Pauli theorem is not applicable in this approach. Let us remind, that all the spacetime components of the position operator $\hat{x} = (\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$ are considered on the same footing within the PEv formalism – they are quantum observables. Using the spacetime representation of the states $\Psi(x) =$
Ψ(x₀, x¹, x², x³) ∈ K, the position operators are usually defined as the multiplication operators:

\[ \hat{x}^\mu \Psi(x) = x^\mu \Psi(x). \]  \hspace{1cm} (17)

Because they commute, they have a common spectral decomposition,

\[ \hat{x}^\mu = \int_{\mathbb{R}^4} d^4x \, x^\mu M_X(x), \]  \hspace{1cm} (18)

where \( M_X(x) = |x\rangle\langle x| \) stands for the orthogonal spectral measure, forming the resolution of unity, of the four-vector position operator \( \hat{x} \). In other words, the kets \( |x\rangle \) are understood as generalized eigenvectors of the four-vector position operator \( \hat{x} \).

In the non-relativistic case the notion of simultaneity is independent of any observer and one can construct a spectral measure \( M_T(x^0) \), which for any fixed time \( t = x^0 \) projects onto the space of simultaneous events:

\[ M_T(x^0) = \int_{\mathbb{R}^3} d^3x \, M_X(x). \]  \hspace{1cm} (19)

This allows to built the time operator (a preliminary attempt can be found in Ref. \[30\]) in the form

\[ \hat{t} \equiv \hat{x}^0 = \int_{\mathbb{R}} dx^0 \, x^0 M_T(x^0), \]  \hspace{1cm} (20)

which, in fact, is the multiplication operator as mentioned above. Using (19) and (20) we get

\[ \hat{t}\Psi(x) \equiv \langle x|\hat{t}|\Psi \rangle = \int_{\mathbb{R}^4} d^4x' t'|x\rangle\langle x'|\Psi \rangle = x^0 \Psi(x), \]  \hspace{1cm} (21)

where the normalization of the position states \( |x\rangle \) is given by \( \langle x|x'\rangle = \delta^4(x-x') \).

In the relativistic case the situation is a bit more complicated because the simultaneity relation is observer dependent. One needs to notice, however, that the time operator is the zero component of the four-vector spacetime position operator (18) which is a covariant quantity with respect to the Poincaré group,

\[ \hat{t} = \int_{\mathbb{R}} dx^0 \, x^0 \left\{ \int_{\mathbb{R}^3} d^3x \, M_X(x) \right\}. \]  \hspace{1cm} (22)

This time operator is well determined for every observer but it cannot be considered as a stand alone observable. It has always to be treated as a part of the four-vector position operator \( \hat{x} \).
As a by-product one can construct the spectral measure which can be used as a measure of causality of a given state $|\Psi\rangle$ at the time $x^0$,

$$M^{(C)}_T(x^0) = \left\{ \int_{C(x^0)} d^3 x M_X(x) \right\}, \tag{23}$$

where $C(x^0) = \{ \vec{x} : (x^0) - \vec{x}^2 \geq 0 \}$. The expectation value of this operator,

$$\text{Prob}_C[\Psi] = \langle \Psi | M^{(C)}_T(x^0) | \Psi \rangle, \tag{24}$$

gives the probability that the particle described by the state $|\Psi\rangle$ is in the light cone, both in the past and in the future directions, with the vertex at $x^0$.

An important operator related to the time operator is the temporal component $\hat{p}_0$ of the four-momentum operator $\hat{p} = (\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3)$. In the space-time representation, the operator which is canonically conjugate to the position operator $\hat{x}$ is the generator of the translations in the spacetime,

$$\hat{p}_\mu = i \frac{\partial}{\partial x^\mu}. \tag{25}$$

The temporal component of the momentum operator measures, similarly to the spatial components, the value of the product “inertia”×“speed” for the particle moving along the time direction. At the same time it also allows to determine the arrow of time: one direction corresponds to $p_0 > 0$, the opposite direction to $p_0 < 0$.

The traditional interpretation of $p_0$ as the energy holds only in the case when the equations of motion relate $p_0$ directly to the energy of the system, like in the Schrödinger equation $\hat{p}_0 = \hat{H}$, $\hat{H}$ being the Hamiltonian. Similar relation is present in the relativistic Klein-Gordon equation, $p_0^2 = m_0^2 + \vec{p}^2$. This type of relations can also be found for other physical systems. In general, one can expect that in the spacetime representation, the equation of motion of a free particle relates its four-position to its four-momenta. Also other degrees of freedom, if present, can be involved.

Both the Schrödinger and the Klein-Gordon equations of motion allow to indirectly measure the temporal component $p_0$ of the four-vector momentum operator $\hat{p}$. It is traditionally expected that $p_0 \geq 0$, even though this feature does not follow from the mathematical structure of the model, as the $\hat{p}_0$ operator has the full spectrum $\mathbb{R}$. The condition $p_0 \geq 0$ can be imposed either by assuming that the equation of motion allows for real motion only if $p_0 \geq 0$, or that this condition is a more fundamental property of our part of the Universe. A simple argument supporting the latter possibility is related.
to the initial state of our Universe. Assuming that the four-momentum is a conserved quantity, the initial chaotic motion of matter should have lead to the situation in which the matter moved in the \( p_0 > 0 \) and \( p_0 < 0 \) directions with the same probability. The spatial components lead to the expansion of matter in the \( \mathbb{R}^3 \) space, the temporal component of the four-momentum, however, lead to the separation of the Universe into two parts: one of which is moving in the positive direction of time, while the other in the negative direction of time. Both subspaces of states are orthogonal and cannot communicate unless an interaction connecting both time directions occurs. This implies that our part of the Universe corresponds, in principle, to one of the directions of the time flow, say, \( p_0 > 0 \). It does not mean, obviously, that in our part of the Universe we do not have the possibility to create particles with \( p_0 < 0 \). According to common interpretation, such objects are antiparticles. This strongly simplified picture requires further analysis but can provide a possible explanation of the \( p_0 > 0 \) phenomenon.

An interesting feature of the pair of the operators \( \hat{x} \) and \( \hat{p} \) is that, since they fulfill the canonical commutation relations

\[
[\hat{p}_\mu, \hat{x}^\nu] = i\delta^\nu_\mu, \tag{26}
\]

they obey the standard Heisenberg uncertainty principle,

\[
\langle (\Delta \hat{p}_\mu)^2 \rangle \langle (\Delta \hat{x}^\mu)^2 \rangle \geq \frac{1}{4} \delta^\mu_\mu. \tag{27}
\]

In the case when \( p_0 \) is related to the energy by means of the equations of motion for a given system, one obtains in a natural way the uncertainty relation between the energy and time. For example, in the case of the Schrödinger type of motion, described by the equation of motion \( \hat{p}_0 = \hat{H} \), the Heisenberg relation (27) can be rewritten as

\[
\langle (\Delta \hat{H})^2 \rangle \langle (\Delta \hat{x}^0)^2 \rangle \geq \frac{1}{4}. \tag{28}
\]

This relation is fulfilled in the space of solutions of the Schrödinger equation. A similar relation between time and energy can always be obtained from appropriate equations of motion of the system under consideration.

4 Generators of the projection evolution

Within the traditional approach, the evolution of a quantum state is driven by a Hamiltonian dependent operator \( e^{-i\hat{H}t} \). In the projection evolution mechanism the changes of the system are spontaneous and independent of
time. We assume that a subset of the evolution operators can be obtained from the appropriate operators $\hat{W}$, the generators of the projection evolution.

For a given evolution step $\tau$, the projection evolution generators $\hat{W}$ are defined as a family of hermitian operators which spectral decompositions give the orthogonal resolution of unity, serving as the evolution operators. The idea of generators for creation of the projection evolution operators can probably be extended to other kind of operators, not only hermitian. Here, however, we restrict ourselves to the hermitian case only.

Let us consider a free single particle with spin equal to zero and no intrinsic degrees of freedom. In this case the generator $\hat{W}$ can be dependent on the spacetime position $\hat{x}$ and the four-momentum $\hat{p}$ operators only.

Taking into account the translational symmetry, the dependence of $\hat{W}$ on the position operators disappears. Imposing the additional requirement of the rotational symmetry for this evolution generator results in the construction of the operator $\hat{W}$ as a function of the rotational invariants of the form $a^\mu \hat{p}_\mu$, $a^{\mu\nu} \hat{p}_\mu \hat{p}_\nu$, . . . , where $a^\mu$, $a^{\mu\nu}$, . . . are appropriate tensors with respect to the SO(3) group. Basing on the experience of classical and quantum physics one can expect that the expansion up to the second order in momenta should be a good approximation, which leaves us with

$$\hat{W} \overset{C}{=} a^\mu \hat{p}_\mu + a^{\mu\nu} \hat{p}_\mu \hat{p}_\nu,$$

where $\overset{C}{=} \overset{C}{=} \overset{C}{=}$ means that $\hat{W}$ is equal to the right-hand side of Eq. (29) only if the additional condition $C$ is fulfilled. This condition depends on the physical properties of the studied case. We will use that in Sec. where the symmetries are discussed.

The additional symmetries expected for a free particle are the space inversion and the time reversal. Assuming that $a^\mu$, $a^{\mu\nu}$, . . . are invariant with respect to both of these symmetries, the linear term in momenta reduces to $a^0 \hat{p}_0$. The quadratic term splits into two parts $a^{00} \hat{p}_0^2 + a^{mn} \hat{p}_m \hat{p}_n$, where $m,n = 1,2,3$. The spatial quadratic term has no preferred direction implying, that it can be written in the form $a^{mn} = B\delta^{mn}$, which casts $\hat{W}$ in the form

$$\hat{W} \overset{C}{=} a^0 \hat{p}_0 + a^{00}(\hat{p}_0)^2 + B(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2).$$

To compare Eq. (30) with the standard quantum mechanics, one can rescale it setting $a^0 = 1$. Then, the first and the third term represent the Schrödinger equation for a free particle with mass $m = \frac{1}{2B}$. The second term is proportional to the second time derivative $(\hat{p}_0)^2 \sim -\frac{\partial^2}{\partial t^2}$ and is not a part of the Schrödinger equation in the standard formulation. It is probably highly suppressed by the $a^{00}$ coefficient. By setting this coefficient to zero we can
remove this term from the equation, recreating the standard Schrödinger evolution.

Imposing in the next step the Lorentz invariance of \( \hat{W} \), one has to reject the first order term completely. Setting \( a^{\mu \nu} = g^{\mu \nu} = \text{diag}(+1,-1,-1,-1) \) we are left with

\[
\hat{W}_{\text{KG}} \overset{C}{=} \hat{p}_\mu \hat{p}^\mu;
\]

which leads to the Klein-Gordon equation \( \hat{p}_\mu \hat{p}^\mu = m^2 \) with potentially additional conditions \( C \). Assuming that \( C \) stands for positive mass \( m > 0 \) and positive temporal component of the momentum operator \( p_0 > 0 \), the generator (31) describes the evolution of a free scalar particle. Changing the set of conditions \( C \), one can generate the evolution of other scalar objects. If \( a^\mu, a^{\mu \nu}, \ldots \) are some tensor operators, one can reproduce other equations of motion. For example, in the case of spin-\( \frac{1}{2} \) particles, assuming \( a^\mu = \gamma^\mu \), where \( \gamma^\mu \) are Dirac matrices, one gets the Dirac equation

\[
\hat{W}_D \overset{C}{=} \gamma^\mu \hat{p}_\mu.
\]

We conclude that the known equations, which describe specific quantum particles, are some special forms of the evolution operator \( \hat{W} \), which allows also to describe much more complicated cases.

### 4.1 The Schrödinger evolution as a special case of PEv

The generator of the Schrödinger evolution can be written as

\[
\hat{W}_S = i \frac{\partial}{\partial t} - \hat{H} = \hat{p}_0 - \hat{H}.
\]

Let us assume that the Hamiltonian \( \hat{H} \) is independent of time. The eigenvalues and the corresponding orthonormal eigenvectors of \( \hat{H} \) will be denoted by \( \epsilon_n \) and \( \phi_{n\mu}(\vec{x}) \), respectively, such that

\[
\hat{H} \phi_{n\mu}(\vec{x}) = \epsilon_n \phi_{n\mu}(\vec{x}).
\]

The action of \( \hat{W}_S \) on the full wave function results in

\[
\hat{W}_S \eta_{k_0}(x^0) \phi_{n\mu}(\vec{x}) = w(k_0, n) \eta_{k_0}(x^0) \phi_{n\mu}(\vec{x}),
\]

where

\[
w(k_0, n) = k_0 - \epsilon_n, \quad \eta_{k_0}(x^0) = \frac{1}{\sqrt{2\pi}} e^{-ik_0x^0}.
\]
The spectral decomposition of the generator $\hat{W}_S$ in the form of Riemann–Stieltjes integral can be written as

$$\hat{W}_S = \int \omega \, dE_{\omega}(\omega),$$

(38)

where $dE_{\omega}(\omega)$ projects onto the eigenspace of $\hat{W}_S$ belonging to the eigenvalue $\omega$. This subspace is spanned by the generalized eigenfunctions of the form

$$\Phi_{\omega}(x^0, \vec{x}) = \frac{1}{\sqrt{2\pi}} \sum_n \sum_\mu c_{n\mu} e^{-i(\epsilon_n + \omega)x^0} \phi_{n\mu}(\vec{x}),$$

(39)

with $c_{n\mu}$ being c-number coefficients. The scalar product in the state space is given by (16). Note that in the traditional three-dimensional scalar product the integration over time is absent,

$$\langle \Phi_2 | \Phi_1 \rangle_3 = \int_{\mathbb{R}^3} d^3x \Phi_2(x^0, \vec{x})^* \Phi_1(x^0, \vec{x}),$$

(40)

because the state space $K_3 = L^2(\mathbb{R}^3)$ does not contain time.

Using the scalar product (16) we see that the eigenfunctions (39) are normalized to the Dirac delta functions,

$$\langle \Phi_{\omega'} | \Phi_{\omega} \rangle_3 = \int_{\mathbb{R}^4} dx^0 dx^1 dx^2 dx^3 \Phi_{\omega'}(x^0, \vec{x})^* \Phi_{\omega}(x^0, \vec{x}) = \delta(\omega' - \omega).$$

(41)

There are a few methods of obtaining vectors belonging to the state space $K$. For example, one can consider the extended Schrödinger equation which contains the temporal part describing the temporal dependencies of the kinetic and potential terms. A possible, but not the most general, such extension is given by the generator

$$\hat{W}_{GS}(\tau) = \hat{p}_0 - \hat{H}(\tau) + \left[ \frac{1}{2} B_T^{-1}(\tau) \hat{p}_0^2 + V_T(\tau, x^0) \right],$$

(42)

where, in agreement with the PEv approach, the temporal parts of the kinetic and potential terms were added. They represent the kinematics and the possible localization of a physical object in the time axis direction. The parameter $B_T^{-1}(\tau)$ represents a kind of temporal inertia of the physical object.

The eigenfunctions (39) considered within the traditional state space $K_3$ are general solutions of the Schrödinger equation, where the eigenvalue $\omega$ determines the zero value of the energy represented by the Hamiltonian $\hat{H}$. It follows from the fact that the eigenvalue for $\hat{W}_S$, from Eq. (33), can be written in the form

$$i \frac{\partial}{\partial t} \phi_{\omega} = (\hat{H} + \omega) \phi_{\omega},$$

(43)
which means that the arbitrary eigenvalue $w$ shifts the energy spectrum. Of course, the physics in $K_3$ is independent of the chosen value of $w$.

We conclude that an important difference between the PEv approach and the traditional formulation of quantum mechanics lies in the interpretation of the wave functions $\Psi(x^0, \vec{x})$. In the PEv formalism the function $|\Psi(x^0, \vec{x})|^2$, where $\Psi \in K$, represents the joined probability density of finding the particle in the four-dimensional spacetime point $(x^0, \vec{x})$. In the traditional form of quantum mechanics with time being a parameter, the function $|\Psi(x^0, \vec{x})|^2$, where $\Psi \in K_3$, represents the conditional probability density of finding the particle in the three-dimensional space point $\vec{x}$, assuming that the particle is localized at time $x^0$.

### 4.2 Relativistic equations of motion

To see that the PEv approach allows to describe the relativistic evolution equations in a more natural way than the $(1+3)$-formalism, it is sufficient to consider the Klein-Gordon equation of motion for a free scalar particle. The generator of the appropriate evolution is given by (31). The mass operator $\hat{m}^2 = \hat{p}_\mu \hat{p}^\mu$ has the following continuous spectrum and generalized eigenvectors:

$$\hat{p}_\mu \hat{p}^\mu \eta_k(x) = w \eta_k(x), \quad (44)$$

where $\eta_k(x) = \exp(-ik_{\mu}x^{\mu})/(4\pi^2)$, $p_\mu = \hbar k_\mu$, and $w \in \mathbb{R}$. Comparing both sides of Eq. (44) one gets the relation $k_\mu k^\mu = w$. This relation determines the subspace which is invariant under the Poincaré group, corresponding to states with definite $w$. This subspace consists of vectors of the form

$$\Phi_w(x) = \int_{\mathbb{R}^4} d^4k \delta^4(k_{\mu}k^{\mu} - w)c(k)\eta_k(x). \quad (45)$$

Using the usual conditions that the space of states is restricted to the states for which $\hat{m}^2 > 0$ and $\hat{p}_{0} > 0$, the eigenvalues $w$ can be interpreted as the invariant mass squared, $m^2$. In this case one gets the known solution for the standard scalar particle of non-zero mass,

$$\Phi_w(x) = \int_{\mathbb{R}^3} \frac{d^3k}{k_0} c(\vec{k})\eta_{(k_0, \vec{k})}(x), \quad (46)$$

where $k = (k_0, \vec{k}) = (\sqrt{m^2 + \vec{k}^2}, \vec{k})$. Note that both vectors (45) and (46) are normalized Dirac delta type distributions.

To get the quantum states belonging to the state space $K$, one may extend the traditional Klein-Gordon equation by including an appropriate vector
field $A_\mu$. Using the minimal coupling scheme one gets

$$\dot{W} = (\dot{p}_\mu - A_\mu)(\dot{p}^\mu - A^\mu).$$

This vector field plays a role similar to the temporal part of the potential in the extended Schrödinger PEv generator \((42)\).

All other relativistic equations of motion can be reproduced in a similar way. One needs, however, to remember that physical consequences of the PEv approach are tremendous. First of all, time becomes a quantum observable and it has to be treated on the same footing as the remaining position coordinates. This makes a lot of new physical phenomena possible, which have to be analyzed.

5 Symmetries

As it is well known, different kinds of symmetries play a fundamental role in physics. They are the most important constraints for structure, interactions and motions of physical objects.

In the case of the PEv formalism one thinks about two distinct types of symmetries:

(A) the symmetries for a fixed step of the evolution, i.e., for a constant evolution parameter $\tau$;

(B) the symmetries related to the transition of the system from one step to another of the evolution, i.e., for the case when the evolution parameter changes, $\tau_{n-1} \rightarrow \tau_n$.

The first type of symmetries (A) describes structural, spacetime and intrinsic properties of a quantum system. An important difference is that the time is now the quantum observable. Taking this into account, symmetry analysis seems to be similar to those performed in relativistic quantum mechanics. Many results remain valid, but most of them require reinterpretation. This problem is still open.

The second type of symmetries (B) is different because the evolution operators are involved in the symmetry analysis. The operators $\mathcal{F}(\tau; \nu, \rho)$ can have different structures, they can be unitary operators, projection operators and other type of operators which allow to transform quantum states into other quantum states. This opens many mathematical and physical (interpretation) problems. In this paper we only introduce the most fundamental properties related to symmetries of the type (B). More extended analysis is shifted to the second part of this paper.
In the following we consider only the evolution operators $F(\tau; \nu)$ which are either a combination of unitary operators or they form an orthogonal resolution of unity.

The problem is to find those properties of a physical object which remain invariant at subsequent steps of the evolution. In other words, we are looking for these transformations from one evolution step to another which do not change this object. One of the most interesting problems for studying such relations are the relations between the symmetries and the conservation laws.

We start by writing the definition of transformations of the evolution operator $F(\tau; \nu, \rho)$. Because $F(\tau; \nu, \rho)$ transforms the quantum state $\rho$ into another quantum state (not normalized), the resulting image of $F(\tau; \nu, \rho)$ has to be consistent with the transformation of its arguments.

To explain this, let us consider the problem of the rotation of some vector function. Let $f : R^3 \rightarrow R^3$. The values of the rotated function $f'$ having the rotated argument $x'$ should be equal to the rotation of the value of the original function having the original argument, $f'(x') = Rf(x)$. In an analogous way the transformation of $F(\tau; \nu, \rho)$ is defined.

Let $F(\tau; \nu, \rho)$ be the evolution operator defined in (3). Let $G$ be a group with two realizations, $S_1(g) : T_1^+(\mathcal{K}) \rightarrow T_1^+(\mathcal{K})$ and $S(g) : T^+(\mathcal{K}) \rightarrow T^+(\mathcal{K})$. The transformation of the evolution operator $F$ is defined as:

$$F(\tau; \nu, \rho') = S(g) F(\tau; \nu, \rho) S(g^{-1}), \quad (48)$$

i.e., the resultant evolution operator for the transformed state is equal to the appropriately chosen transformation of the evolution operator for the original state. This idea can be expressed in a more convenient form:

$$F(\tau; \nu, \rho) = S(g) F(\tau; \nu, S_i(g^{-1})\rho S_i(g)) S(g^{-1}). \quad (49)$$

The group $G$ provides the same physical interpretation for the operators $S(g)$ and $S_1(g)$ in the above definition. Definition (48) allows to conserve the probability distributions for unitary equivalent images of quantum mechanics.

Let us assume that the transition probability from the evolution step $\tau_{n-1}$ to $\tau_n$ is given by

$$\text{Prob} [\rho(\tau_{n-1}; \nu_{l}^{(n-1)}) \rightarrow \rho(\tau_n; \nu_{k}^{(n)})] = \text{Tr} [F(\tau_n; \nu_{k}^{(n)}, \rho(\tau_{n-1}; \nu_{l}^{(n-1)}))], \quad (50)$$
as it is for the projection evolution operators represented by an orthogonal resolution of the unit operator. Here, $\nu^{(n)}_k$ denotes one of the set of quantum numbers from $\nu_n = \{\nu_1^{(n)}, \nu_2^{(n)}, \ldots, \nu_k^{(n)}, \ldots\} \in Q_n$. Similarly, the transition probability of the transformed by the group $G$ state $\rho'(\tau_{n-1}; \nu_{l}^{(n-1)})$ and, at the same time, the transformed evolution operator $F'$, is given by

$$\text{Prob} [\rho'(\tau_{n-1}; \nu_{l}^{(n-1)}) \rightarrow \rho'(\tau_n; \nu_{k}^{(n)})] = \text{Tr} [F'(\tau_n; \nu_{k}^{(n)}, \rho'(\tau_{n-1}; \nu_{l}^{(n-1)}))]. \quad (51)$$

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It follows from equation (45) that both probabilities are equal:

\[
\text{Prob} [\rho(\tau_{n-1}; \nu^{(n-1)}_l) \rightarrow \rho(\tau_{n}; \nu^{(n)}_k)] = \text{Prob} [\rho'(\tau_{n-1}; \nu^{(n-1)}_l) \rightarrow \rho'(\tau_{n}; \nu^{(n)}_k)].
\]

The consequence of such symmetry for every step of the evolution is the fact that \(g\) does not change the structure of the possible evolution paths, which means that the probability distribution of the potential evolution paths is conserved. In this case, the operations \(g \in G\) can also be interpreted as operations which transform among equivalent descriptions of a given model (covariance), or as the transformation between equivalent observers (a kind of “relativity”).

In the following we consider only the most common case when \(S(g) = S_1(g)\) and the evolution operators are \(E(\tau_n; \nu^{(n)}_k)\). For shortness we use the notation \(S(g) = g\). In this special case the evolution operators \(E(\tau_n; \nu^{(n)}_k)\) transform as:

\[
\left[ E(\tau_n; \nu^{(n)}_k) \rho(\tau_{n-1}; \nu^{(n-1)}_l) E(\tau_n; \nu^{(n)}_k) \right]' = g E(\tau_n; \nu^{(n)}_k) g^{-1} \rho(\tau_{n-1}; \nu^{(n-1)}_l) g E(\tau_n; \nu^{(n)}_k) g^{-1}.
\]

The above equation shows that the transformation (49) for the operator \(E(\tau_n, \nu^{(n)}_k)\) is equivalent to the transformation by the group element \(g\),

\[
\left[ E(\tau_n; \nu^{(n)}_k) \right]' = g E(\tau_n; \nu^{(n)}_k) g^{-1}.
\]

Let us consider the invariance of the transition probabilities when the final states are generated by the group \(G\). In this case, the symmetry group \(G\) is responsible for creating the sets of states which, at the end, are chosen by the evolution in a completely random way. This specific conservation of the transition probability can be defined as:

\[
\text{Prob} [\rho(\tau_{n-1}; \nu^{(n-1)}_l) \rightarrow \rho(\tau_{n}; \nu^{(n)}_k)] = \text{Prob} [\rho(\tau_{n-1}; \nu^{(n-1)}_l) \rightarrow \rho'(\tau_{n}; \nu^{(n)}_k)].
\]

The transition probabilities among the transformed states are:

\[
\text{Prob} \left[ \rho(\tau_{n-1}; \nu^{(n-1)}_l) \rightarrow \rho'(\tau_{n}; \nu^{(n)}_k) \right] = \text{Tr} \left[ g E(\tau_n; \nu^{(n)}_k) g^{-1} \rho(\tau_{n-1}; \nu^{(n-1)}_l) g E(\tau_n; \nu^{(n)}_k) g^{-1} \right].
\]

It is easy to see that in two special cases,

\[
[g, E(\tau_n, \nu^{(n)})] = 0 \quad \text{or} \quad [g, \rho(\tau_{n-1}, \nu^{(n-1)})] = 0,
\]

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this probability is the same as for the original states.

The next important problem is the relation between symmetries and conservation laws. The generally correct formulation of this problem in the projection evolution model is at present not known. Intuitively one can say that we are looking for conditions under which the expectation value of a given observable \( A \) is conserved within the PEv approach:

\[
\left\langle A \right\rangle_{\rho(\tau_0; \nu_k^{(0)})} = \left\langle A \right\rangle_{\rho(\tau_1; \nu_k^{(1)})} = \cdots = \left\langle A \right\rangle_{\rho(\tau_n; \nu_k^{(n)})} = \cdots \tag{58}
\]

The required conditions may involve special relations between the evolution operators, density operators and quantum observables.

In the case when the evolution is described by the operators \( \mathcal{E}(\tau_n; \nu_k^{(n)}) \), the conservation of the expectation value \( \left\langle A \right\rangle \) has the form:

\[
\text{Tr} \left[ A \rho(\tau_{n-1}; \nu_l^{(n-1)}) \right] = \frac{\text{Tr} \left[ A \mathcal{E}(\tau_n; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_n; \nu_k^{(n)})^\dagger \right]}{\text{Tr} \left[ \mathcal{E}(\tau_n; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_n; \nu_k^{(n)})^\dagger \right]} \tag{59}
\]

for a series of \( n \).

Assume that the operator \( \mathcal{E}(\tau_n; \nu_k^{(n)}) \) is unitary. Under this assumption the condition (59) takes the following form:

\[
\text{Tr} \left[ A \rho(\tau_n; \nu_k^{(n)}) \right] = \frac{\text{Tr} \left[ A \mathcal{E}(\tau_n; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_n; \nu_k^{(n)})^\dagger \right]}{\text{Tr} \left[ \mathcal{E}(\tau_n; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_n; \nu_k^{(n)})^\dagger \right]}
\]

\[
= \text{Tr} \left[ \mathcal{E}(\tau_n; \nu_k^{(n)})^\dagger A \mathcal{E}(\tau_n; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \right]. \tag{60}
\]

In this case the expectation value of the observable \( A \) is conserved if the operator \( A \) commutes with the evolution operator, i.e., \([A, \mathcal{E}(\tau_n; \nu_k^{(n)})] = 0\). This fact has its counterpart in the standard quantum mechanics – if the Hamiltonian commutes with the operator \( A \), the expectation value \( \left\langle A \right\rangle \) is conserved during the unitary evolution generated by this Hamiltonian.

The PEv approach allows for the generalization of the idea of the unitary evolution. For example, it is possible to consider the case when a few different unitary evolution channels interfere. In this case, the state for the evolution step \( n \) is a linear combination of the products of different unitary evolutions of the previous state,

\[
\rho(\tau_n; \nu_k^{(n)}) = \frac{\sum_{m=1}^N U_m(\tau_n) \rho(\tau_{n-1}; \nu_l^{(m)}) U_m(\tau_n)^\dagger}{\text{Tr} \left[ \sum_{m=1}^N U_m(\tau_n) \rho(\tau_{n-1}; \nu_l^{(m)}) U_m(\tau_n)^\dagger \right]}, \tag{61}
\]

where \( U_m = U_m^{-1} \) for \( m = 1, \ldots, N \).

The second example, which leads to a unitary type of the evolution, is when the evolution generator \( \mathcal{W}(\tau_n) \) evolves unitarily \( \mathcal{W}(\tau_n) = U(\tau_n) \mathcal{W}_0 U(\tau_n)^\dagger \).
Both examples open a new problem in the analysis of symmetries in the PEv model and they require further investigation.

Another problem is the generalization of the situation when both, the evolution operator and the quantum observable, are invariant under a given symmetry.

Let \( G \) be a symmetry group represented by a Lie group, \( C \) be its Casimir operator and the ket \(|\kappa \Gamma a\rangle\) be a basic vector of an irreducible representation of the symmetry group \( G \). Assume that the operators \( \mathcal{E}(\tau_n) \) commute with the group, \([G, \mathcal{E}(\tau_n)] = 0\). Then the Casimir operator and the density operator can be expanded in a basis of irreducible representations of the group \( G \) in the following way:

\[
C = \sum_\Gamma C_\Gamma \sum_{\kappa,a} |\kappa \Gamma a\rangle \langle \kappa \Gamma a| \quad \text{and} \quad \mathcal{E}(\tau_n, \nu^{(n)}) = \sum_{\kappa,a} |\kappa \Gamma a\rangle \langle \kappa \Gamma a|,
\]

(62)

where \( \Gamma \) depends on \( \nu_n \). Then

\[
\text{Tr}[\mathcal{C} \rho(\tau_n; \nu^{(n)}_k)] = \frac{\text{Tr}[\mathcal{C} \mathcal{E}(\tau_n; \nu^{(n)}_k) \rho(\tau_{n-1}; \nu^{(n-1)}_l) \mathcal{E}(\tau_n; \nu^{(n)}_k) \mathcal{E}(\tau_n; \nu^{(n)}_k)]}{\text{Tr}[\mathcal{E}(\tau_n; \nu^{(n)}_k) \rho(\tau_{n-1}; \nu^{(n-1)}_l) \mathcal{E}(\tau_n; \nu^{(n)}_k) \mathcal{E}(\tau_n; \nu^{(n)}_k) \mathcal{E}(\tau_n; \nu^{(n)}_k) \mathcal{E}(\tau_n; \nu^{(n)}_k) \mathcal{E}(\tau_n; \nu^{(n)}_k)]} = C_\Gamma.
\]

(63)

We conclude that if the evolution operators are invariant with respect to the group \( G \), the expectation value of the Casimir operator \( C \) of the group \( G \) is conserved during the evolution.

This fact has its analogy in the standard quantum mechanics. Let us assume that the Hamiltonian \( H \) is invariant with respect to a group \( G \). The eigenvectors of \( H \) belong to the invariant subspaces spanned by the bases of the irreducible representations of the group \( G \). In this case the expectation value of the Casimir operator is conserved during the unitary evolution generated by this Hamiltonian.

This section is only a short outline of some open problems related to the symmetry analysis within the projection evolution approach. PEv opens new areas for applications of symmetries and generally group theoretical methods.

6 Concluding remarks

The discussion about the structure and the role of time is as long as the history of physics. A collection of papers devoted to different aspects of the physical time from the modern perspective can be found, among others, in [21, 31]. In Ref. [21] the paper by P. Busch mentions three types of time. The most popular one is the time considered as a parameter which is measured by an external laboratory clock, uncoupled from the measured
system. This time is called the external time. Time can be defined also through the dynamics of the observed quantum systems, in which case we deal with the dynamical (or intrinsic) time. Lastly, time can be considered on the same footing as other quantum observables, especially as positions in space. This is called by P. Busch the observable (or event) time and it represents the approach discussed in the present paper.

In the experimental practice the external time is usually used. It is introduced by constructing different kinds of clocks uncoupled from the analyzed physical phenomenon. In this case, the clocks are defined by some processes which can be parameterized by the parameter $\theta$ (usually the label $t$ is used instead of $\theta$). One can expect that for the external clocks, $\theta$ is approximately a monotonic function of the evolution parameter $\tau_n$, e.g., $\theta = \text{Tr}(\hat{t}\rho(\tau_n; \nu))$ and $\text{Tr}(\hat{t}\rho(\tau_{n+1}; \nu')) > \text{Tr}(\hat{t}\rho(\tau_n; \nu))$. The trace $\text{Tr}(\hat{t}\rho(\tau_n; \nu))$ denotes, in analogy to the average position of an object in space, the average time on the clock being in the state $\rho(\tau_n; \nu)$ at the step evolution $\tau_n$. Having one clock, one can treat it as the standard clock. All other clocks can be constructed and synchronized to this standard clock. In this context the external time, though very useful, is a conventional rather than physical entity.

The intrinsic time, or times, to be more precise, is determined by any arbitrary set of dynamical variables. It is compatible with our “changes principle”, i.e., that changes of states or observables are more fundamental than the time itself. However, because in our approach the physical time is a quantum observable, the required characteristic times (intrinsic times) for a given physical process can be directly calculated. In this context, the intrinsic times are not fundamental but derivable temporal observables.

The observable time can, in a natural way, account for many quantum mechanical effects regarded as paradoxes. It is also important that it allows to calculate temporal characteristics of a quantum system on the same basis as it can be done for other observables. It introduces through the equations of motion the time-energy uncertainty relation on the same basis as for the position–momentum observables. The time operator and the corresponding conjugate temporal momentum operator are the very natural complements of the covariant relativistic four-position and four-momentum operators. A few examples of processes analyzed in terms of the observable time can be found in [32, 33, 34, 35, 36, 37, 38, 39, 40]. The cited papers show some steps in the historical development of the PEv idea.

The PEv approach opens several fundamental problems for discussion and requires further investigation.
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