Fekete-Szegö Problem for Certain Subclass of Analytic Functions with Complex Order Defined by $q$–Analogue of Ruscheweyh Operator

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ABSTRACT. In this paper, we study Fekete-Szegö problem for certain subclass of analytic functions with complex order in the open unit disk by applying the $q$–analogue of Ruscheweyh operator in conjunction with the principle of subordination between analytic functions.

Keywords: Analytic functions, univalent functions, $q$–derivative operator, $q$–analogue of Ruscheweyh operator, Fekete-Szego problem, subordination.

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1. INTRODUCTION

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written as $f \prec g$ in $U$ or $f(z) \prec g(z)$ ($z \in U$), if there exists a Schwarz function $\omega$, which (by definition) is analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in U$) such that $f(z) = g(\omega(z))$ ($z \in U$). Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence holds (see [12] and [7]):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For function $f \in A$ given by (1.1) and $0 < q < 1$, the $q$–derivative of a function $f$ is defined by (see [10, 9] and [6])

$$(1.2) \quad D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z} & , z \neq 0 \\ f'(0) & , z = 0 \end{cases}$$

provided that $f'(0)$ exists and $D_q^2 f(z) = D_q(D_q f(z))$. We note from (1.2) that

$$\lim_{q \to 1^-} D_q f(z) = f'(z) \quad \text{and} \quad \lim_{q \to 1^-} D_q^2 f(z) = f''(z).$$
It is readily deduced from (1.1) and (1.2) that

\begin{equation}
D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},
\end{equation}

where

\begin{equation}
[k]_q = \frac{q^k - 1}{q - 1}.
\end{equation}

Aldweby and Darus [1] defined $q$–analogue of Ruscheweyh operator $R_\delta^q : A \rightarrow A$ as follows:

\[ R_\delta^q f(z) = z + \sum_{k=2}^{\infty} \frac{(k + \delta - 1)!}{k! [k - 1]_q!} a_k z^{k} \quad (\delta \geq -1), \]

where $[i]_q!$ is given by

\[ [i]_q! = \begin{cases} [i]_q [i - 1]_q \ldots [1]_q, & i \in \mathbb{N} = \{1, 2, 3, \ldots\} \\ \frac{1}{i}, & i = 0 \end{cases}. \]

We note that

\[ R_0^q f(z) = f(z) \quad \text{and} \quad R_1^q f(z) = zD_q f(z). \]

From the definition of $R_\delta^q$, we observe that if $q \rightarrow 1^-$, we have

\[ \lim_{q \rightarrow 1^-} R_\delta^q f(z) = R_\delta f(z) = z + \sum_{k=2}^{\infty} \frac{(k + \delta - 1)!}{\delta! (k - 1)!} a_k z^{k}, \]

where $R_\delta$ is Ruscheweyh differential operator defined by Ruscheweyh [16].

It is easy to check that

\begin{equation}
zD_q (R_\delta^q f(z)) = \left(1 + \frac{\delta}{q^\delta}\right) R_\delta^{q+1} f(z) - \frac{\delta}{q^\delta} R_\delta^q f(z).
\end{equation}

If $q \rightarrow 1^-$, the equality (1.5) implies

\[ z \left(R_\delta^q f(z)\right)' = (1 + \delta) R_\delta^{q+1} f(z) - \delta R_\delta^q f(z) \]

which is the well known recurrence formula for Ruscheweyh differential operator.

By making use of the $q$–analogue of Ruscheweyh operator $R_\delta^q$ and the principle of subordination, we now introduce the following subclass of analytic functions of complex order.

**Definition 1.1.** Let $\mathcal{P}$ be the class of all functions $\phi$ which are analytic and univalent in $U$ and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $R\phi(z) > 0$ for $z \in U$. A function $f \in A$ is said to be in the class $\mathcal{K}_\delta^q (\gamma, \phi)$ if it satisfies the following subordination condition:

\begin{equation}
1 + \frac{1}{b} \left[ \frac{(1 - \gamma) zD_q R_\delta^q f(z) + \gamma zD_q (zD_q R_\delta^q f(z))}{(1 - \gamma) R_\delta^q f(z) + \gamma zD_q R_\delta^q f(z)} - 1 \right] < \phi(z) \quad (b \in \mathbb{C}^*).
\end{equation}

We note that:

(i) $\lim_{q \rightarrow 1^-} \mathcal{K}_\delta^q (\gamma, \phi) = \mathcal{K}_b (\gamma, \phi) \quad (b \in \mathbb{C}^*)$

\[ = \left\{ f \in A : 1 + \frac{1}{b} \left[ \frac{zf'(z) + \gamma z^2 f''(z)}{(1 - \gamma) f(z) + \gamma zf'(z)} - 1 \right] < \phi(z) \right\}, \]
(ii) $K_{q,(1-\alpha)e^{-i\theta}}^0 (0, \phi) = S_q^\theta (\alpha; \phi) \ (|\theta| \leq \frac{\pi}{2}, 0 \leq \alpha < 1)$
\[= \left\{ f \in A : \frac{e^{i\theta} z D_q f(z)}{D_q f(z)} - \alpha \cos \theta - i \sin \theta \right\} < \phi (z) \]

(iii) $K_{q,(1-\alpha)e^{-i\theta}}^0 (1, \phi) = C_q^\theta (\alpha; \phi) \ (|\theta| \leq \frac{\pi}{2}, 0 \leq \alpha < 1)$
\[= \left\{ f \in A : \frac{e^{i\theta} z D_q f(z)}{D_q f(z)} - \alpha \cos \theta - i \sin \theta \right\} < \phi (z) \]

(iv) $K_{q,(1-\alpha)e^{-i\theta}}^\delta (0, \phi) = S_q^\delta (\phi)$ and $K_{q,1}^\delta (1, \phi) = C_q^\delta (\phi)$ (Alweby and Darus [3]),
(v) $K_{q,b}^0 (0, \phi) = S_{q,b} (\phi)$ and $K_{q,1}^0 (1, \phi) = C_{q,b} (\phi)$ (Seoudy and Aouf [18]),
(vi) $K_{q,1}^0 (0, \phi) = S_q (\phi)$ and $K_{q,1}^0 (1, \phi) = C_q (\phi)$ (Alweby and Darus [2]),
(vii) $\lim_{q \to 1} K_{q,b}^0 (0, \phi) = S_b (\phi)$ and $\lim_{q \to 1} K_{q,b}^0 (1, \phi) = C_b (\phi)$ (Ravichandran et al. [15]),
(viii) $\lim_{q \to 1} K_{q,1}^0 (0, \phi) = S^* (\phi)$ and $\lim_{q \to 1} K_{q,1}^0 (1, \phi) = C (\phi)$ (Ma and Minda [11]),
(ix) $\lim_{q \to 1} K_{q,b}^0 \left( 0, \frac{1 + (1 - 2\alpha) z}{1 - z} \right) = S^*_\alpha (b)$ and $\lim_{q \to 1} K_{q,b}^0 \left( 1, \frac{1 + (1 - 2\alpha) z}{1 - z} \right) = C_\alpha (b)$
$(0 \leq \alpha < 1)$ (Frasin [8]),

(x) $\lim_{q \to 1} K_{q,b}^0 \left( 0, \frac{1 + z}{1 - z} \right) = S^* (b)$ (Nasr and Aouf [14]),

(xi) $\lim_{q \to 1} K_{q,b}^0 \left( 1, \frac{1 + z}{1 - z} \right) = C (b) \in C^*$ (Nasr and Aouf [13] and Wiatrowski [19]),

(xii) $\lim_{q \to 1} K_{q,1-\alpha}^0 \left( 0, \frac{1 + z}{1 - z} \right) = S^* (\alpha)$ and $\lim_{q \to 1} K_{q,1-\alpha}^0 \left( 1, \frac{1 + z}{1 - z} \right) = C (\alpha) \ (0 \leq \alpha < 1)$ (Robertson [17]),

(xiii) $\lim_{q \to 1} K_{q,be^{-i\theta}}^0 \left( 0, \frac{1 + z}{1 - z} \right) = S^\theta (b)$ and $\lim_{q \to 1} K_{q,be^{-i\theta}}^0 \left( 1, \frac{1 + z}{1 - z} \right) = C^\theta (b)$
$(|\theta| < \frac{\pi}{2})$ (Al-Oboudi and Haidan [4] and Aouf et al. [5]).

In order to establish our main results, we need the following lemma.

**Lemma 1.1.** [11] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$ and $\mu$ is a complex number, then

$$|c_2 - \mu c_1^2| \leq 2 \max\{1; |2\mu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}.$$

**Lemma 1.2.** [11] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with a positive real part in $\mathbb{U}$, then

$$|c_2 - \nu c_1^2| \leq \left\{ \begin{array}{ll}
-4\nu + 2 & \text{if } \nu \leq 0 \\
2 & \text{if } 0 \leq \nu \leq 1, \\
4\nu - 2 & \text{if } \nu \geq 1
\end{array} \right.$$  

when $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z)$ is $(1 + z)/(1 - z)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z)$ is $(1 + z^2)/(1 - z^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = \left( \frac{1 + \lambda}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \lambda}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \lambda \leq 1).$$
or one of its rotations. If \( \nu = 1 \), the equality holds if and only if \( p \) is the reciprocal of one of the functions such that equality holds in the case of \( \nu = 0 \).

Also the above upper bound is sharp, and it can be improved as follows when \( 0 < \nu < 1 \):

\[
|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \left( 0 \leq \nu \leq \frac{1}{2} \right)
\]

and

\[
|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \left( \frac{1}{2} \leq \nu \leq 1 \right).
\]

In the present paper, we obtain the Fekete-Szegö inequalities for the class \( K_{q,b} (\gamma, \phi) \). The motivation of this paper is to generalize previously results. Unless otherwise mentioned, we assume throughout this paper that the function \( 0 < q < 1, b \in \mathbb{C}^*, 0 \leq \gamma \leq 1, \phi \in \mathcal{P}, \) \( [k]_q \) is given by (1.4) and \( z \in \mathbb{U} \).

**Theorem 1.1.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( K_{q,b} (\gamma, \phi) \), then

\[
|a_3 - \mu a_2^3| \leq \frac{|b B_1|}{q[1 + \gamma q(q + 1)]^\nu} \max \left\{ 1; \left( \frac{B_2}{B_1} + \left( 1 - \frac{(1 + \gamma q(q + 1)) \min \left( \delta, \min \left( \gamma, \frac{1 - \gamma}{\gamma} \right) \right)}{q[1 + \gamma q(q + 1)]^\nu} \mu \right) B_1 b \right\}.
\]

The result is sharp.

**Proof.** If \( f \in K_{q,b}^\delta (\gamma, \phi) \), then there is a Schwarz function \( \omega \), analytic in \( \mathbb{U} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( \mathbb{U} \) such that

\[
1 + \frac{1}{b} \frac{(1 - \gamma) D_q \mathcal{R}_q^\delta f(z) + \gamma z D_q (z D_q \mathcal{R}_q^\delta f(z))}{(1 - \gamma) \mathcal{R}_q^\delta f(z) + \gamma z D_q \mathcal{R}_q^\delta f(z)} - 1 = \phi(\omega(z)).
\]

Define the function \( p(z) \) by

\[
p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + \ldots.
\]

Since \( \omega \) is a Schwarz function, we see that \( \Re p(z) > 0 \) and \( p(0) = 1 \). Therefore,

\[
\phi(\omega(z)) = \frac{p(z) - 1}{p(z) + 1} = \phi \left( \frac{\left( c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) z^3 + \ldots \right)}{1} \right)
\]

\[
= 1 + \frac{B_1 c_1}{2} z + \left[ \frac{B_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right] z^2 + \ldots.
\]

(1.10)

Now, by substituting (1.10) in (1.8), we have

\[
1 + \frac{1}{b} \frac{(1 - \gamma) D_q \mathcal{R}_q^\delta f(z) + \gamma z D_q (z D_q \mathcal{R}_q^\delta f(z))}{(1 - \gamma) \mathcal{R}_q^\delta f(z) + \gamma z D_q \mathcal{R}_q^\delta f(z)} - 1 = 1 + \frac{B_1 c_1}{2} z + \left[ \frac{B_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right] z^2 + \ldots.
\]

From the above equation, we obtain

\[
\frac{1}{b} q (1 + \gamma q) [\delta + 1]_q a_2 = \frac{B_1 c_1}{2}.
\]
and
\[
q \frac{a_3 - (1 + \gamma q)^2}{b} \left[ 1 + \gamma q (q + 1) \right] [\delta + 2]_q [\delta + 1]_q \frac{a_3 - (1 + \gamma q)^2}{b} \left[ 1 + \gamma q (q + 1) \right] [\delta + 1]_q \frac{a_3 - (1 + \gamma q)^2}{b} \left[ 1 + \gamma q (q + 1) \right] [\delta + 1]_q \frac{a_3 - (1 + \gamma q)^2}{b} \left[ 1 + \gamma q (q + 1) \right] [\delta + 1]_q
\]
\[
\frac{B_1 c_2}{2} - \frac{B_1 c_1^2}{4} + \frac{B_2 c_1}{4} \frac{B_2 c_1}{4}
\]
or, equivalently,
\[
a_2 = \frac{B_1 c_1 b}{2q (1 + \gamma q) [\delta + 1]_q}
\]
and
\[
a_3 = \frac{b B_1}{2q [1 + \gamma q (q + 1)] [\delta + 2]_q [\delta + 1]_q} \left\{ c_2 - \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} - \frac{B_2 b}{q} \right] c_1^2 \right\}.
\]
Therefore, we have
\[
a_3 - \mu a_2^2 = \frac{b B_1}{2q [1 + \gamma q (q + 1)] [\delta + 2]_q [\delta + 1]_q} \left\{ c_2 - \nu c_1^2 \right\},
\]
where
\[
(1.12) \quad \nu = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} - \frac{B_2 b}{q} \left( 1 - \frac{[1 + \gamma q (q + 1)] [\delta + 2]_q \mu}{(1 + \gamma q)^2 [\delta + 1]_q} \right) \right].
\]
Our result now follows from Lemma 1.1. The result is sharp for the functions
\[
1 + \frac{1}{b} \left[ (1 - \gamma) z D_q R_q^\delta f(z) + \gamma z D_q (z D_q R_q^\delta f(z)) \frac{(1 - \gamma) R_q^\delta f(z)}{(1 - \gamma) R_q^\delta f(z) + \gamma z D_q R_q^\delta f(z)} - 1 \right] = \phi (z^2)
\]
and
\[
1 + \frac{1}{b} \left[ (1 - \gamma) z D_q R_q^\delta f(z) + \gamma z D_q (z D_q R_q^\delta f(z)) \frac{(1 - \gamma) R_q^\delta f(z)}{(1 - \gamma) R_q^\delta f(z) + \gamma z D_q R_q^\delta f(z)} - 1 \right] = \phi (z).
\]
This completes the proof of Theorem 1.1.

Taking \( \gamma = 0 \) and \( b = 1 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 6].

**Corollary 1.1.** Let \( \phi (z) = 1 + B_1 z + B_2 z^2 + ... \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( S_q^\delta (\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q[\delta + 2]_q [\delta + 1]_q} \max \left\{ 1; \left| \frac{B_2}{B_1} + \left( 1 - \frac{[\delta + 2]_q \mu}{[\delta + 1]_q} \right) \frac{B_1}{q} \right| \right\}.
\]
The result is sharp.

Taking \( \gamma = b = 1 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 7].

**Corollary 1.2.** Let \( \phi (z) = 1 + B_1 z + B_2 z^2 + ... \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( K_q^\delta (\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q[1 + q(q + 1)] [\delta + 2]_q [\delta + 1]_q} \max \left\{ 1; \left| \frac{B_2}{B_1} + \left( 1 - \frac{[1 + q(q + 1)][\delta + 2]_q \mu}{[\delta + 1]_q (1 + q)^2} \right) \frac{B_1 b}{q} \right| \right\}.
\]
The result is sharp.

Taking \( \gamma = \delta = 0 \) and \( b = 1 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.1].
Corollary 1.3. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( S_\phi(\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q(q+1)} \max \left\{ 1; \frac{B_2}{B_1} + \left( 1 - (q+1) \mu \right) \frac{B_1}{q} \right\}.
\]
The result is sharp.

Taking \( \gamma = b = 1 \) and \( \delta = 0 \) in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.2].

Corollary 1.4. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( K_\phi(\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \frac{|B_1|}{q(q+1)[1+q(q+1)]} \max \left\{ 1; \frac{B_2}{B_1} + \left( 1 - \frac{[1+q(q+1)]}{(1+q)} \mu \right) \frac{B_1}{q} \right\}.
\]
The result is sharp.

Taking \( \gamma = \delta = 0 \) and \( q \to 1^- \) in Theorem 1.1, we obtain the following corollary which improves the result of Ravichandran et al. [15, Theorem 4.1].

Corollary 1.5. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f \) given by (1.1) belongs to the class \( S_\phi(\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \frac{|B_1 b|}{2} \max \left\{ 1; \frac{B_2}{B_1} + (1 - 2\mu) B_1 b \right\}.
\]
The result is sharp.

Theorem 1.2. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let
\[
(1.13) \quad \sigma_1 = \frac{(1 + \gamma q)^2 [\delta + 1]_q [b B_1^2 + q (B_2 - B_1)]}{[1 + \gamma q (q + 1)] [\delta + 2]_q b B_1^2},
\]
(1.14) \quad \sigma_2 = \frac{(1 + \gamma q)^2 [\delta + 1]_q [b B_1^2 + q (B_2 + B_1)]}{[1 + \gamma q (q + 1)] [\delta + 2]_q b B_1^2},
(1.15) \quad \sigma_3 = \frac{(1 + \gamma q)^2 [\delta + 1]_q (b B_1^2 + q B_2)}{[1 + \gamma q (q + 1)] [\delta + 2]_q b B_1^2}.

If \( f \) given by (1.1) belongs to the class \( K^\delta_{q,b}(\gamma, \phi) \) with \( b > 0 \), then
\[
(1.16) \quad |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} \left[ B_2 + \frac{B_1^2 b}{q} \left( 1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q} \mu \right) \right], & \mu \leq \sigma_1 \\
\frac{q(1+\gamma q)^2[\delta+1]_q}{[1+\gamma q(q+1)][\delta+2]_q B_1^2 b} \left[ B_1 - B_2 - \frac{B_1^2 b}{q} \left( 1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q} \mu \right) \right], & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{q[\gamma q(q+1)][\delta+2]_q B_1}{[1+\gamma q(q+1)][\delta+1]_q}, & \mu \geq \sigma_2 \end{cases}
\]
Further, if \( \sigma_1 \leq \mu \leq \sigma_3 \), then
\[
|a_3 - \mu a_2^2| \leq \frac{b B_1}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}.
\]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then
\[
|a_3 - \mu a_2^2| \leq \frac{q(1+\gamma q)^2[\delta+1]_q}{[1+\gamma q(q+1)][\delta+2]_q B_1^2 b} \left[ B_1 + B_2 + \frac{B_1^2 b}{q} \left( 1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma q)^2[\delta+1]_q} \mu \right) \right] |a_2|^2.
\]
Let Corollary 1.6.

(1.18)

\[
\frac{bB_1}{q \left[ 1 + \gamma q (q + 1) \right] [\delta + 2]_q [\delta + 1]_q}.
\]

The result is sharp.

Proof. Applying Lemma 1.2 to (1.11) and (1.12), we can obtain our results asserted by Theorem 1.2.

Taking \( \gamma = 0 \) and \( b = 1 \) in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 10].

**Corollary 1.6.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let

\[
\chi_1 = \frac{[\delta + 1]_q \left[ B_1^2 + q (B_2 - B_1) \right]}{[\delta + 2]_q B_1^2},
\]

\[
\chi_2 = \frac{[\delta + 1]_q \left[ B_1^2 + q (B_2 + B_1) \right]}{[\delta + 2]_q B_1^2},
\]

\[
\chi_3 = \frac{[\delta + 1]_q \left( B_1^2 + q B_2 \right)}{[\delta + 2]_q B_1^2}.
\]

If \( f \) given by (1.1) belongs to the class \( S^\delta_q(\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{q[\delta+2]_q[\delta+1]_q} \left[ B_2 + \frac{B_1^2}{q} \left( 1 - \frac{[\delta+2]_q}{[\delta+1]_q} \mu \right) \right], & \mu \leq \chi_1 \\
\frac{B_1}{q[\delta+2]_q[\delta+1]_q} \left[ -B_2 - \frac{B_1^2}{q} \left( 1 - \frac{[\delta+2]_q}{[\delta+1]_q} \mu \right) \right], & \chi_1 \leq \mu \leq \chi_2. 
\end{cases}
\]

Further, if \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{q[\delta+1]_q}{[\delta+2]_q B_1^2} \left[ B_1 - B_2 - \frac{B_1^2}{q} \left( 1 - \frac{[\delta+2]_q}{[\delta+1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_1}{q[\delta+2]_q[\delta+1]_q}.
\]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_3 - \mu a_2^2| + \frac{q[\delta+1]_q}{[\delta+2]_q B_1^2} \left[ B_1 + B_2 + \frac{B_1^2}{q} \left( 1 - \frac{[\delta+2]_q}{[\delta+1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_1}{q[\delta+2]_q[\delta+1]_q}.
\]

The result is sharp.

Taking \( \gamma = b = 1 \) in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 11].

**Corollary 1.7.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let

\[
\chi_1 = \frac{[2]_q^2 [\delta + 1]_q \left[ B_1^2 + q (B_2 - B_1) \right]}{[3]_q [\delta + 2]_q B_1^2},
\]

\[
\chi_2 = \frac{[2]_q^2 [\delta + 1]_q \left[ B_1^2 + q (B_2 + B_1) \right]}{[3]_q [\delta + 2]_q B_1^2},
\]

\[
\chi_3 = \frac{[2]_q^2 [\delta + 1]_q \left( B_1^2 + q B_2 \right)}{[3]_q [\delta + 2]_q B_1^2}.
\]
If $f$ given by (1.1) belongs to the class $K^\delta_q (\phi)$, then

$$
|a_3 - \mu a_2^2| \leq \begin{cases}
\frac{1}{q[3]_q[3\delta + 2]_q[\delta + 1]_q} \left[ B_2 + \frac{B_2^2}{q} \left( 1 - \frac{[3]_q[3\delta + 2]_q[\delta + 1]_q}{[2]_q[3\delta + 2]_q[\delta + 1]_q} \mu \right) \right], & \mu \leq \kappa_1 \\
\frac{1}{q[3]_q[3\delta + 2]_q[\delta + 1]_q} \left[ B_2 - \frac{B_2^2}{q} \left( 1 - \frac{[3]_q[3\delta + 2]_q[\delta + 1]_q}{[2]_q[3\delta + 2]_q[\delta + 1]_q} \mu \right) \right], & \kappa_1 \leq \mu \leq \kappa_2.
\end{cases}
$$

Further, if $\kappa_1 \leq \mu \leq \kappa_3$, then

$$
|a_3 - \mu a_2^2| + \frac{q[2]_q[3\delta + 2]_q[\delta + 1]_q}{[3]_q[3\delta + 2]_q[\delta + 1]_q} \left[ B_1 - B_2 - \frac{B_2^2}{q} \left( 1 - \frac{[3]_q[3\delta + 2]_q[\delta + 1]_q}{[2]_q[3\delta + 2]_q[\delta + 1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_1}{q[3]_q[3\delta + 2]_q[\delta + 1]_q}.$$

If $\kappa_3 \leq \mu \leq \kappa_2$, then

$$
|a_3 - \mu a_2^2| + \frac{q[2]_q[3\delta + 2]_q[\delta + 1]_q}{[3]_q[3\delta + 2]_q[\delta + 1]_q} \left[ B_1 + B_2 + \frac{B_2^2}{q} \left( 1 - \frac{[3]_q[3\delta + 2]_q[\delta + 1]_q}{[2]_q[3\delta + 2]_q[\delta + 1]_q} \mu \right) \right] |a_2|^2 \leq \frac{B_3}{q[3]_q[3\delta + 2]_q[\delta + 1]_q}.
$$

The result is sharp.

**Remark 1.1.** Putting $\delta = \gamma = 0$ in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf [18, Theorems 1 and 3, respectively].

**Remark 1.2.** Putting $\delta = 0$ and $\gamma = 1$ in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf [18, Theorems 2 and 4, respectively].

**Remark 1.3.** For different choices of the parameters $b, \delta, q, \gamma$ and $\phi$ in Theorems 1.1 and 1.2, we can deduce some results for the classes $K_b (\gamma, \phi), S^0_\alpha (\alpha; \phi), C^0_b (\gamma; \phi), S_\gamma (\phi), C_\alpha (\phi), S_b (\phi), C_b (\phi), S^* (\phi), C (\phi), S^*_\alpha (b), C_\alpha (b), S^* (b), C (b), S^* (\alpha), C (\alpha), S^0 (b)$ and $C^0 (b)$ which are defined in Section 1.

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