When is Wegner’s flow geodesic?

Yuichi Itto\textsuperscript{1} and Sumiyoshi Abe\textsuperscript{1,2,3}

\textsuperscript{1}Department of Physical Engineering, Mie University, Mie 514-8507, Japan
\textsuperscript{2}Institut Supérieur des Matériaux et Mécaniques Avancés, 44 F. A. Bartholdi,
72000 Le Mans, France
\textsuperscript{3}Inspire Institute Inc., McLean, Virginia 22101, USA

Abstract. Wegner’s method of flow equations offers a powerful tool for diagonalizing a given Hamiltonian and is widely used in various branches of quantum physics. Here, generalizing this method, a condition is found, under which the corresponding flow of a quantum state becomes geodesic in a submanifold of the projective Hilbert space. This implies the geometric optimality of the present method as an algorithm of generating stationary states. The result is illustrated by analyzing some physical systems.

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In recent years, much attention has been focused on Wegner’s method of flow equations [1,2]. This method offers a powerful tool for diagonalizing a given quantum-mechanical Hamiltonian, which has been devised in analogy with the theory of renormalization groups and has widely been applied to a variety of problems in condensed matter physics, nuclear and particle physics, and quantum information. Examples include the effective Hamiltonian of the Anderson impurity model [3], a Dirac particle in an external electromagnetic field [4], an effective spin-spin coupling arising from spin-phonon chains [5], the $t-t'$ Hubbard model for high-temperature superconductivity [6], the Tomonaga-Luttinger model for interacting spinless electrons in one dimension [7], localized superfluidity [8], the Lipkin-Meshkov-Glick model in nuclear physics [9], quantum phase transition in the interacting boson model [10], light-cone quantum chromodynamics [11], electron-electron and electron-phonon interactions in the Hubbard-Holstein model [12], and stationary photon-atom entanglement [13].

Wegner’s method employs a continuous unitary transformation represented by the operator, $U(l)$, where $l \in [0, \infty)$ is referred to as the flow parameter. $U(l)$ transforms the original Hamiltonian $H = H(0)$ to $H(l) = U(l) H U^\dagger(l)$, which satisfies $d H(l) / d l = [\eta(l), H(l)]$, where $\eta(l)$ is the anti-Hermitian generator given by $\eta(l) = [d U(l) / d l] U^\dagger(l)$. Wegner’s choice for $\eta(l)$ to diagonalize (or, block-diagonalize) the Hamiltonian reads

$$\eta(l) = \eta^w(l) \equiv [H_d(l), H_{\alpha-d}(l)],$$

where $H_d(l)$ and $H_{\alpha-d}(l)$ stand for the diagonal and off-diagonal parts of $H(l)$,
respectively. It can be shown [1,2] that $H_{o-d}(l)$ tends to vanish in the limit $l \to \infty$. Mathematically, this procedure corresponds to the Jacobi algorithm for eigenvalue problems, which defines a steepest-descent flow in the space of matrices with the Frobenius norm [14].

Although Wegner’s method has been applied to many problems as mentioned above, its inherent properties do not seem to be explored well. It is our opinion that this method is more than just a mathematical tool for diagonalizing a Hamiltonian. The purpose of this paper is to reveal an interesting geometric property hidden behind Wegner’s flow equations. For it, we first generalize Wegner’s method itself. The original method of Wegner is a special case within this generalized framework. Next, we consider the flow of a given quantum state through a submanifold of the projective Hilbert space composed of rays explained below. Then, we find a condition, under which the flow becomes geodesic in the submanifold. This remarkable property is illustrated by analyzing some physical examples. Our discussion may also have significance for laboratory experiments, since one often need realize a stationary state from a certain initially prepared state via quantum operations [13].

Let us start our discussion with considering a quantum system in a $d$-dimensional Hilbert space, where $d$ is either finite or infinite. Its unitary-transformed Hamiltonian, $H(l)$, is decomposed into two parts: $H(l) = H_d(l) + H_{o-d}(l)$, where $H_d(l)$ and $H_{o-d}(l) = \sum_{a=1}^{A} H_{o-d}^{(a)}(l) \ (1 \leq A \leq d-1)$ are the diagonal and off-diagonal parts, respectively. Using the complete set of the normalized eigenstates of $H_d(l)$,
where $\epsilon_n(l)$ is the $n$th eigenvalue of $H_d(l)$ and $C^{(a)}_{n+i_a}(l)$’s are the complex expansion coefficients. The nonzero index $i_a$ describes the off-diagonality and is ordered without loss of generality as follows: $0 < i_1 < i_2 < \cdots < i_A$. It is understood that $C^{(a)}_{n+i_a}(l) = 0$ ($C^{(a)}_{n-i_a}(l) = 0$), if $n + i_a > d$ ($n - i_a < 1$).

Here, we generalize Wegner’s method as follows. Instead of taking the whole of $H_{o-d}(l)$, we employ only a single term, say $H^{(a)}_{o-d}(l)$:

$$
\eta^{(a)}(l) = \left[H_d(l), \ H^{(a)}_{o-d}(l)\right].
$$

Clearly, Wegner’s choice is $\eta^W(l) = \sum_{a=1}^A \eta^{(a)}(l)$. It can be found after a straightforward calculation that the corresponding generalized flow equation,

$$
\frac{d}{dl}H(l) = [\eta^{(a)}(l), H(l)],
$$

leads to

$$
(d/l)(d/l)\sum_{m,n=1,m\neq n}^d \left| \langle u_m | H(l) | u_n \rangle \right|^2 = -4\sum_{n=1}^d \left[ \epsilon_{n+i_a}(l) - \epsilon_n(l) \right]^2 \left| C^{(a)}_{n+i_a}(l) \right|^2,
$$

implying that the off-diagonal elements of $H(l)$ tend to decay as $l$ increases. Therefore,
the present method generalizes Wegner’s in the sense that it uses only a single off-diagonal term.

Next, let us translate the flow of the Hamiltonian into the flow of a state. Given a normalized state $|\psi\rangle$, the stationary Schrödinger equation reads $H|\psi(\infty)\rangle = E|\psi(\infty)\rangle$, where $|\psi(l)\rangle = U(l)|\psi\rangle$. As well known, two states different from each other only by total phases are equivalent in quantum mechanics, and therefore a physical state is represented by a ray. Accordingly, the quantum-state space is the projective Hilbert space. Eq. (5) determines the flows of the physical coefficients contained in the Hamiltonian, which depend on the parameters appearing in the unitary operator. Let us explicitly write as follows: $U = U(\alpha)$, $|\psi(\alpha)\rangle = U^\dagger(\alpha)|\psi\rangle$, where $\alpha \equiv (\alpha^1, \alpha^2, ..., \alpha^k)$. The set of parameters, $\alpha$, defines a local coordinate on the submanifold of the projective Hilbert space, which is referred to as the quantum evolution submanifold [15]. Then, the Fubini-Study metric [15-17] induced on this submanifold is, up to the second-order infinitesimals, given by

$$ds^2 = 1 - |\langle \psi(\alpha) | \psi(\alpha + d\alpha) \rangle|^2 \equiv g_{ij}(\alpha) \, d\alpha^i \, d\alpha^j$$

where the metric tensor is expressed in terms of the anti-Hermitian operator $G_i(\alpha) = \left(\partial_i U(\alpha)\right) U^\dagger(\alpha)$ ($\partial_i = \partial / \partial \alpha^i$; $i = 1, 2, ..., k$) as follows: $g_{ij}(\alpha) = -(1/2) \langle \psi | G_i(\alpha) G_j(\alpha) + G_j(\alpha) G_i(\alpha) | \psi \rangle + \langle \psi | G_i(\alpha) | \psi \rangle \langle \psi | G_j(\alpha) | \psi \rangle$. Here and hereafter, Einstein’s convention is understood for the repeated indices. The equation for a geodesic curve parametrized by the arc length, $\alpha(s)$, is given by

$$d^2\alpha^h / ds^2 + \Gamma^h_{ij}(d\alpha^i / ds)(d\alpha^j / ds) = 0,$$

where $\Gamma^h_{ij}$ is Christoffel’s symbol defined
by \( \Gamma_{hi} = g_{hi} \Gamma'_{ij} = (1/2) \left( \partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij} \right) \).

Let us parametrize the curve by the flow parameter, instead of the arc length, i.e., \( \alpha(l) \), and consider the functional, \( S = \int_{l_1}^{l_2} dl \, L \) with \( L(\alpha, \dot{\alpha}) \equiv ds / dl = \sqrt{g_{ij} \dot{\alpha}^i \dot{\alpha}^j} \) \( (\dot{\alpha}^i \equiv d\alpha^i(l) / dl) \), which is the arc length in the interval \([l_1, l_2]\). The variation of \( S \) with respect to \( \alpha(l) \) is calculated to be \( \delta S / \delta \dot{\alpha}^i = -(1/4) L^{-3} X_i \), where

\[
X_i = \left( 2 \langle \psi | \eta | \psi \rangle \frac{\partial \eta}{\partial \dot{\alpha}^i} \frac{\partial \eta}{\partial \dot{\alpha}^i} \frac{\partial \eta}{\partial \dot{\alpha}^i} | \psi \rangle - \langle \psi \frac{\partial \eta^2}{\partial \dot{\alpha}^i} | \psi \rangle \frac{d}{dl} \left( \langle \psi | \eta^2 | \psi \rangle - \langle \psi | \eta | \psi \rangle^2 \right) \right)
\]

\[
+ 2 \left( \langle \psi | \frac{d \eta}{dl} \frac{\partial \eta}{\partial \dot{\alpha}^i} | \psi \rangle + \left[ \eta^2, \frac{\partial \eta}{\partial \dot{\alpha}^i} \right] | \psi \rangle - 2 \langle \psi | \frac{d \eta}{dl} | \psi \rangle \langle \psi | \frac{\partial \eta}{\partial \dot{\alpha}^i} | \psi \rangle \right) \\
- 2 \langle \psi | \eta | \psi \rangle \left( \eta \frac{\partial \eta}{\partial \dot{\alpha}^i} \right) \left( \langle \psi | \eta^2 | \psi \rangle - \langle \psi | \eta | \psi \rangle^2 \right) \tag{7}
\]

with \( \eta = \eta(\alpha(l)) = \left[ dU(\alpha(l)) / dl \right] U^\dagger(\alpha(l)) = G_i(\alpha(l)) \dot{\alpha}^i(l) \), provided that the following relation has been used: \( (d / dl) \left( \partial \eta / \partial \dot{\alpha}^i \right) = \partial \eta / \partial \dot{\alpha}^i + [\eta, \partial \eta / \partial \dot{\alpha}^i] \).

We wish to prove that under a certain condition the generalized flow equation (5) defines a geodesic curve associated with the “initial state” \( | \psi \rangle = | u_n \rangle \), which is an eigenstate of \( H_l(l) \). (This is the state of relevance, because we are considering the flow to an exact stationary state.) That is, we are going to show that, under the condition found later, the above \( X_i \) vanishes for \( \eta = \eta^{(a)} \) in Eq. (4).

Clearly, \( \langle u_n | \eta^{(a)} | u_n \rangle \) and its derivatives with respect to \( l \) and \( \dot{\alpha}^i \) vanish. Furthermore, it can be shown by using Eqs. (2)-(4) that \( \langle u_n \left[ \eta^{(a)}^2, \partial \eta^{(a)} / \partial \dot{\alpha}^i \right] u_n \rangle \) also vanishes.
Therefore, $X_i$ for $|\psi\rangle = |u_n\rangle$ and $\eta = \eta^{(a)}$ is reduced to

$$X_i = 2\langle u_n | \eta^{(a)2} | u_n \rangle \left\{ \frac{d \eta^{(a)}}{dl} \frac{\partial \eta^{(a)}}{\partial \alpha^i} | u_n \rangle \right\}$$

$$- \langle u_n | \frac{d \eta^{(a)2}}{dl} | u_n \rangle \langle u_n | \frac{\partial \eta^{(a)2}}{\partial \alpha^i} | u_n \rangle. \quad (8)$$

This equation can be rewritten by using Eqs. (2)-(4) again as follows:

$$X_i = 4 \left( |D_{n+i_a}^{(a)}|^2 + D_n^{(a)} \sum_{m=1}^{d} \delta_{n,m+i_a} \right) \text{Re} \left[ \frac{d D_{n+i_a}^{(a)}}{dl} \frac{\partial D_{n+i_a}^{(a)*}}{\partial \alpha^i} + \frac{d D_n^{(a)}}{dl} \frac{\partial D_n^{(a)*}}{\partial \alpha^i} \sum_{m=1}^{d} \delta_{n,m+i_a} \right]$$

$$- \left( \frac{d |D_{n+i_a}^{(a)}|^2}{dl} + D_n^{(a)} \sum_{m=1}^{d} \delta_{n,m+i_a} \right) \left[ \frac{\partial |D_{n+i_a}^{(a)}|^2}{\partial \alpha^i} + \frac{\partial |D_n^{(a)}|^2}{\partial \alpha^i} \sum_{m=1}^{d} \delta_{n,m+i_a} \right], \quad (9)$$

where $D_{n+i_a}^{(a)} = (e_{n+i_a}(l) - e_{n}(l))c_{n+i_a}^{(a)}(l)$. The $l$-derivatives appearing in this expression should be calculated from the generalized flow equation in Eq. (5). It is also noticed that $\alpha^i$ and $\dot{\alpha}^i$ are not independent in Eqs. (8) and (9) any more, since Eq. (4) has already been used.

Now, we present our finding. If the label $a$ satisfying

$$i_a' \neq 2i_a, 3i_a \quad (1 \leq a' \leq A) \quad (10)$$

can be taken for $\eta^{(a)}(l)$, then sandwiching the flow equation in Eq. (5) by three sets,

$$\langle u_n \rangle \text{ and } |u_{n-i_a}\rangle, \quad \langle u_{n+i_a} \rangle \text{ and } |u_n\rangle, \quad \langle u_{n+i_a} \rangle \text{ and } |u_{n-i_a}\rangle,$$

we find
\[ \langle u_n \l| \eta^{(a)}(l), \sum_{d=1}^{A} H_{\alpha d}^{(a)}(l) \r| u_{n-i} \rangle = 0, \quad \langle u_{n+i} \l| \eta^{(a)}(l), \sum_{d=1}^{A} H_{\alpha d}^{(a)}(l) \r| u_{n-i} \rangle = 0, \quad \text{and} \]

\[ \langle u_{n+i} \l| \eta^{(a)}(l), \sum_{d=1}^{A} H_{\alpha d}^{(a)}(l) \r| u_{n-i} \rangle = 0, \quad \text{respectively. Then, the sandwiched flow equations are} \]

(i) \[ \frac{d C_n^{(a)}(l)}{dl} = - \left( \varepsilon_n(l) - \varepsilon_{n-1}(l) \right) C_n^{(a)}(l), \]

(ii) \[ \frac{d C_{n+i}^{(a)}(l)}{dl} = - \left( \varepsilon_{n+i}(l) - \varepsilon_n(l) \right) C_{n+i}^{(a)}(l), \]

and (iii) \[ \left( \varepsilon_{n+i}(l) + \varepsilon_{n-1}(l) - 2 \varepsilon_n(l) \right) C_{n+i}^{(a)}(l) C_n^{(a)}(l) = 0, \]

respectively. It is noticed that, from (i) and (ii), the phases of \( C_n^{(a)}(l) \) and \( C_{n+i}^{(a)}(l) \) are found to be independent of \( l \). (Case-A): If \( C_n^{(a)}(l) = C_{n+i}^{(a)}(l) = 0 \), then \( X_i \) in Eq. (9) obviously vanishes. (Case-B): If \( C_n^{(a)}(l) = 0 \) and \( C_{n+i}^{(a)}(l) \neq 0 \) (or, \( C_n^{(a)}(l) \neq 0 \) and \( C_{n+i}^{(a)}(l) = 0 \)), then \( X_i \) vanishes, due to the fact that the phases of \( D_n^{(a)}(l) \) and \( D_{n+i}^{(a)}(l) \) are independent of \( l \). (Case-C): If both \( C_n^{(a)}(l) \) and \( C_{n+i}^{(a)}(l) \) are nonzero, then (iii) yields

\[ \varepsilon_{n+i}(l) + \varepsilon_{n-i}(l) - 2 \varepsilon_n(l) = 0. \]

Combining this relation with (i) and (ii), we obtain a crucial relation that \( \left| C_{n+i}^{(a)}(l) \right| \) is proportional to \( \left| C_n^{(a)}(l) \right| \). And, such a relation makes \( X_i \) vanish again. Therefore, we conclude that the flow equation with the generator \( \eta^{(a)}(l) \) with the label \( a \) satisfying the condition in Eq. (10) gives rise to the geodesic flow of \( \left| u_n \right| \). This is the main result of the present work.

In what follows, we illustrate our result by analyzing some physical examples.

The first example we consider is the generalized harmonic oscillator. The Hamiltonian reads \( H = \omega a^\dagger a + \lambda a^2 + \lambda^* a^\dagger a^2 + \nu \). \( a^\dagger \) and \( a \) are the creation and annihilation operators obeying the algebra: \( [a, a^\dagger] = 1, \ [a^\dagger, a] = [a, a] = 0. \ \omega (> 0) \) and \( \nu \) are real constants, whereas \( \lambda \) is a complex constant. The relevant unitary operator for this system is the squeezing operator: \( U(l) = \exp \left[ \left( \xi(l) a^\dagger - \xi^*(l) a^2 \right) / 2 \right] \), where the
complex coefficient $\xi(l)$ is parametrized as $\xi(l) = r(l) e^{-2i\phi(l)} \ (0 \leq r(l), 0 \leq \phi(l) < 2\pi)$. Accordingly, $(\alpha^1, \alpha^2) \equiv (r, \phi)$. The condition $U(0) = I$ (with the identity operator $I$) leads to $r(0) = 0$. The “initial state” is taken to be the number state, $|n\rangle = (n!)^{-1/2} a^\dagger n |0\rangle$, with the normalized ground state satisfying $a|0\rangle = 0$. The corresponding metric is [15]:

ds^2 = (1/2) (n^2 + n + 1) \left[ dr^2 + (\sinh^2 2r) d\phi^2 \right],

which shows that the manifold is the Lobachevsky space. The transformed Hamiltonian is written as follows:

$H(l) = \omega(l) a^\dagger a + \lambda(l) a^\dagger a^2 + \lambda'(l) a^2 + \nu(l)$. The coefficients appearing here depend not only on their original values at $l = 0$ but also on $\xi(l)$. In this case, $H_{o-d}(l) = \lambda(l) a^\dagger a + \lambda'(l) a^2$ is the one and only off-diagonal part. Therefore, the generator in Eq. (4) is identical to Wegner’s choice, $\eta^W(l) = 2 \omega(l) \left[ \lambda(l) a^\dagger a - \lambda'(l) a^2 \right]$, and the condition in Eq. (10) is automatically fulfilled. The flow equation for $\lambda(l)$ is given by $d \lambda(l) / dl = -4 \omega^2(l) \lambda(l)$, showing that the phase of $\lambda(l)$ does not depend on $l$. Then, one can explicitly find that Eq. (8) vanishes. Comparing $[dU(l) / dl] U^\dagger(l)$ with $\eta^W(l)$, we obtain $\phi(l) = \text{const}$, which in fact turns out to make both $\delta S / \delta r(l)$ and $\delta S / \delta \phi(l)$ vanish, where $S$ is the arc-length functional defined in terms of the above metric. We also mention that the spectrum of $H_d(l) = \omega(l) a^\dagger a + \nu(l)$ is equally spaced, and accordingly Case-C is realized. Thus, Wegner’s flow is geodesic.

The second example is a spin-$s$, $S = (S_x, S_y, S_z)$, in a constant external magnetic field, $\mathbf{B}$. The Hamiltonian reads $H = \mathbf{S} \cdot \mathbf{B}$ in an appropriate unit. The unitary operator to be considered is $U(l) = \exp \left[ \sigma(l) S_x - \sigma^* (l) S_x \right]$, where $S_x = S_x \pm i S_y$ and the complex coefficient $\sigma(l)$ is parametrized as $\sigma(l) = [\theta(l) / 2] e^{-i\phi(l)} \ (0 \leq \theta(l) < \pi, 0 \leq \phi(l) < 2\pi)$
with \( \theta(0) = 0 \). The basic commutation relations satisfied by the spin operators are as follows: 
\[
[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2 S_z. 
\]
The local coordinate is given by \((\alpha^1, \alpha^2) \equiv (\theta, \phi)\). The initial state is taken to be \(|m\rangle_s\) \((m = -s, -s + 1, ..., 0, ..., s - 1, s)\), which satisfies \(S_z|m\rangle_s = m|m\rangle_s\). The metric is found to be given by \([15]\):
\[
ds^2 = (1/2)(s^2 + s - m^2)\left[ d\theta^2 + (\sin^2 \theta) d\phi^2 \right],
\]
which is of a sphere. The transformed Hamiltonian is written as 
\[H(l) = \beta_z(l)S_z + \beta(l)S_+ + \beta^*(l)S_-\]
\(\beta_z(l)\) is real, whereas \(\beta(l)\) is complex. They depend not only on \(B\) but also on \(\sigma(l)\). The one and only off-diagonal part is 
\[H_{o-d}(l) = \beta(l)S_+ + \beta^*(l)S_-\]
and so Wegner’s choice is employed for the generator: 
\[\eta^W(l) = \beta_z(l)\left[ \beta(l)S_+ - \beta^*(l)S_- \right]\]
Therefore, clearly the condition in Eq. (10) is fulfilled. The flow equation for \(\beta(l)\) is 
\[
d\beta(l)/dl = -\beta_z^2(l)\beta(l),
\]
from which the phase of \(\beta(l)\) is seen to be independent of \(l\), and accordingly Eq. (8) vanishes. Comparison of the above generator, \(\eta^W(l)\), with \([dU(l)/dl]U^\dagger(l)\) yields \(\phi(l) = \text{const}\), which leads to the fact that the variations of the arc-length functional \((S, \text{calculated using the above metric})\) with respect to \(\theta(l)\) and \(\phi(l)\) vanish. Also, the spectrum of \(H_\sigma(l) = \beta_z(l)S_z\) with \(s \geq 1\) is equally spaced, and so realized is Case-C. Thus, Wegner’s flow is geodesic (i.e., the great circle).

The third and last example is the Jaynes-Cummings model \([18]\), which describes a two-level atom interacting with a single-mode radiation field. The Hamiltonian is given by 
\[H = \omega_0 \sigma_z / 2 + \omega a^\dagger a + \kappa (\sigma_+ a + \sigma_- a^\dagger)\]. Here, \(\omega_0\), \(\omega\), and \(\kappa\) are the transition frequency of the atom, the frequency of the radiation, and the coupling constant, respectively. \(a^\dagger\) and \(a\) are the creation and annihilation operators of the radiation field.
satisfying the same algebra as in the first example. $\sigma$’s are the operators of the atom, which are given in terms of the orthonormal basis, the ground state $|g\rangle$ and the excited state $|e\rangle$, as follows: $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$, $\sigma_3 = |e\rangle\langle e| - |g\rangle\langle g|$. This model is known to be exactly solvable. The associated unitary operator is $U(l)$

$$U(l) = \sum_{n=0}^{\infty} \left[ a_n(l) \sigma_+ \sigma_- |n\rangle\langle n| + b_{n-1}(l) \sigma_- \sigma_+ |n\rangle\langle n| + c_n(l) \sigma_3 |n\rangle\langle n| + d_n(l) \sigma_- |n+1\rangle\langle n| \right],$$

where $|n\rangle = (n!)^{-1/2} a^n |0\rangle$ is the $n$-photon state (for the details of this unitary operator, see Ref. [13]). The unitarity of this operator leads to the conditions: $|\alpha_n|^2 + |\gamma_n|^2 = 1$, $|\alpha_n| = |\beta_n|$, $|\gamma_n| = |\delta_n|$, and $\alpha_n \delta_n^* + \beta_n^* \gamma_n = 0$. From the polar forms, $\alpha_n = |\alpha_n| \exp(i \theta_n)$ and so on, follows the condition on the phases:

$$\theta_{\alpha_n} + \theta_{\beta_n} - \theta_{\gamma_n} - \theta_{\delta_n} = (2m + 1) \pi \quad (m = 0, \pm 1, \pm 2, \ldots).$$

Here, we choose $|e\rangle|n\rangle$ as the initial state. $|\alpha_n\rangle$, $\theta_{\alpha_n}$, $\theta_{\beta_n}$, and $\theta_{\gamma_n}$ form as a set of independent coordinate variables. The metric is $ds^2 = \left(1 - |\alpha_n|^2\right)^{-1} d|\alpha_n|^2 + |\alpha_n|^2 \left(1 - |\alpha_n|^2\right) d\theta_{\alpha_n}^2$. The $\theta_{\beta_n}$-dependence disappears due to the above choice of the initial state. The transformed Hamiltonian is $H(l) = \sum_{n=0}^{\infty} \left[ a_n(l) \sigma_+ \sigma_- |n\rangle\langle n| + b_{n-1}(l) \sigma_- \sigma_+ |n\rangle\langle n| + c_n(l) \sigma_3 |n\rangle\langle n| + d_n(l) \sigma_- |n+1\rangle\langle n| \right]$, and so $H_{o-d}(l) = \sum_{n=0}^{\infty} \left[ c_n(l) \sigma_3 |n+1\rangle\langle n| + c_n^*(l) \sigma_- |n+1\rangle\langle n| \right]$, where the coefficients, $A_n(l)$’s and $B_{n-1}(l)$’s are real, and $C_n(l)$’s turn out to be also real, as can be seen from the flow equations. They are expressed in terms of the physical coefficients contained in the original Hamiltonian as well as the coefficients appearing in $U(l)$ (for details, see Ref. [13]). Let us consider Wegner’s choice: $\eta^W(l) = \sum_{n=0}^{\infty} \left[ A_n(l) - B_n(l) \right] C_n(l)$.
\times (\sigma_+ |n\rangle\langle n+1| - \sigma_- |n+1\rangle\langle n|). Then, we find that Eq. (8) vanishes. The flow equations give rise to the result, \(d\theta_{\alpha_n}(l)/dl = d\theta_{\gamma_n}(l)/dl = 0\), which explicitly makes the variations of the arc-length functional with respect to \(\alpha_n(l)\), \(\theta_{\alpha_n}(l)\), and \(\theta_{\gamma_n}(l)\) all vanish. In addition, comparing \(\eta(l) = [H_{d}(l), H_{v,d}(l)]\) with \(\eta^w(l)\) above, we see that Case-B is realized. Thus, again Wegner’s flow is geodesic.

In conclusion, generalizing Wegner’s method of flow equations, we have found the condition, under which the corresponding flow of a quantum state becomes geodesic in a quantum evolution submanifold. We have illustrated this by employing the generalized harmonic oscillator, the spin in an external magnetic field, and the Jaynes-Cummings model. The present result implies that the method of flow equations is not just a mathematical tool for diagonalizing a Hamiltonian but provides an optimal strategy in quantum state engineering for realizing a stationary state from a given initial state.

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