The Cop Number of the One-Cop-Moves Game on Planar Graphs

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Abstract. Cops and robbers is a vertex-pursuit game played on graphs. In the classical cops-and-robbers game, a set of cops and a robber occupy the vertices of the graph and move alternately along the graph’s edges with perfect information about each other’s positions. If a cop eventually occupies the same vertex as the robber, then the cops win; the robber wins if she can indefinitely evade capture. Aigner and Frommer established that in every connected planar graph, three cops are sufficient to capture a single robber. In this paper, we consider a recently studied variant of the cops-and-robbers game, alternately called the one-active-cop game, one-cop-moves game or the lazy-cops-and-robbers game, where at most one cop can move during any round. We show that Aigner and Frommer’s result does not generalise to this game variant by constructing a connected planar graph on which a robber can indefinitely evade three cops in the one-cop-moves game. This answers a question recently raised by Sullivan, Townsend and Werzanski.

1 Introduction

Cops and Robbers, introduced by Nowakowski and Winkler \cite{15} in 1983 and independently by Quillot \cite{17} in 1978, is a game played on graphs, where a cop tries to capture a robber. The cop is first placed on any vertex of the graph $G$, after which the robber chooses a starting vertex in $G$. The cop and robber then move in alternate turns, with the robber moving on odd turns and the cop moving on even turns. A round of the game consists of a robber’s turn and the cop’s subsequent turn. During every turn, each cop or robber either moves along an edge of $G$ to a neighbouring vertex or stays put on his or her current vertex. Furthermore, both the cop and robber have perfect information about each other’s positions at any point in the game. The cop wins the game if he eventually occupies the same vertex as the robber at some moment in the game; the robber wins if she can indefinitely avoid occupying any vertex containing the cop. A winning strategy for the cop on $G$ is a set of instructions that, if followed, guarantees that the cop can win any game played on $G$, regardless of how the robber moves throughout the game. A winning strategy for the robber on $G$ is defined analogously.
Aigner and Frommer [2] generalised the original Cops and Robbers game by allowing more than one cop to play; we shall henceforth refer to this version of the game as the classical cops-and-robbers game. They associated to every finite graph $G$ a parameter known as the cop number of $G$, denoted by $c(G)$, which is the minimum number of cops needed for a cop winning strategy on $G$, and they showed that the cop number of every connected planar graph is at most 3. Nowakowski and Winkler [15] gave a structural characterisation of the class of graphs with cop number one. In the same vein, Clarke and MacGillivray [8] characterised the class of graphs with any given cop number. The cops-and-robbers game has attracted considerable attention from the graph theory community, owing in part to its connections to various graph parameters, as well as the large number of interesting combinatorial problems arising from the study of the cop number such as Meyniel’s conjecture [5,6], which states that for any graph $G$ of order $n$, $c(G) = O(\sqrt{n})$. In addition, due to the relative simplicity and naturalness of the cops-and-robbers game, it has served as a model for studying problems in areas of applied computer science such as artificial intelligence, robotics and the theory of optimal search [7,10,14,20].

This paper examines a variant of the classical cops-and-robbers game, known alternately as the one-active-cop game [16], lazy-cops-and-robbers game [3,4,21] or the one-cop-moves game [23]. The corresponding cop number of a graph $G$ in this game variant is called the one-cop-moves cop number of $G$, and is denoted by $c_1(G)$. One motivation for studying the one-cop-moves cop number comes from Meyniel’s conjecture: it is hoped that an analogue of Meyniel’s conjecture holds in the one-cop-moves game, and it would be easier to prove than the original conjecture (or at least lead to new insights into how Meyniel’s conjecture may be proven). The one-cop-moves cop number has been studied for various special families of graphs such as hypercubes [3,16], generalised hypercubes [19], random graphs [4] and Rook’s graphs [21]. On the other hand, relatively little is known about the behaviour of the one-cop-moves cop number of connected planar graphs [6]. In particular, it is still open at present whether or not there exists an absolute constant $k$ such that $c_1(G) \leq k$ for all connected planar graphs $G$. Instead of attacking this problem directly, one may try to establish lower bounds on $\sup\{c_1(G) : G \text{ is a connected planar graph}\}$ as a stepping stone. Note that the dodecahedron $D$ is a connected planar graph with classical cop number equal to 3 [2]. Since any winning strategy for the robber on $D$ in the classical cops-and-robbers game can also be applied to $D$ in the one-cop-moves game, it follows that $c_1(D) \geq 3$, and this immediately gives a lower bound of 3 on $\sup\{c_1(G) : G \text{ is a connected planar graph}\}$.

To the best of our knowledge, there has hitherto been no improvement on this lower bound. Sullivan, Townsend and Werzanski [21] recently asked whether or not $\sup\{c_1(G) : G \text{ is a connected planar graph}\} \geq 4$. Many prominent planar graphs have a one-cop-moves cop number of at most 3 (such as the truncated icosahedron, known colloquially as the “soccer ball graph”) or at most 2 (such as
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and so the study of such graphs unfortunately does not shed new light on the question. The goal of the present work is to construct a connected planar graph whose structure is specifically designed for a robber to easily evade 3 cops indefinitely, thereby settling the question posed by Sullivan, Townsend and Wezanski affirmatively.

2 Preliminaries

Any unexplained graph terminology is from [22]. The book by Bonato and Nowakowski [5] gives a survey of some proof techniques and important results in the cops-and-robbers game. All graphs in this paper are simple, finite and connected. Let $G$ be a graph with $n$ vertices. For any vertex $u$, a cop $\lambda$ is said to be $k$ edges away from $u$ iff the distance between the position of $\lambda$ and $u$ is $k$; similarly, a vertex $v$ is said to be $k$ edges away from $u$ iff the distance between $v$ and $u$ is $k$. A path $\pi$ is defined to be a sequence $(v_0, \ldots, v_k)$ of distinct vertices such that for $0 \leq i \leq k - 1$, $v_i$ and $v_{i+1}$ are adjacent; the length of $\pi$ is the number of vertices of $\pi$ minus one. Let $u$ and $v$ be any two distinct vertices of $D$, and let $H$ be any subgraph of $D$. If there is a unique shortest path in $H$ from $u$ to $v$, then this path will be denoted by $u \overset{H}{\rightarrow} v$. The concatenation of the paths $u_0 \overset{H_1}{\rightarrow} u_1 \overset{H_2}{\rightarrow} u_2, \ldots, u_{k-1} \overset{H_k}{\rightarrow} u_k$ will be denoted by $u_0 \overset{H_1}{\rightarrow} u_1 \overset{H_2}{\rightarrow} u_2 \overset{H_3}{\rightarrow} \ldots \overset{H_k}{\rightarrow} u_k$. The concatenation of $u_0 \sim u_1, u_1 \sim u_2, \ldots, u_{k-1} \sim u_k$ will be denoted analogously by $u_0 \sim u_1 \sim u_2 \sim \ldots \sim u_k$.

Let $\{\lambda_1, \ldots, \lambda_k\}$ be a set of $k$ cops, and let $\gamma$ be a robber. The one-cop-moves game is defined as follows. Initially, each of the $k$ cops chooses a starting vertex in $G$ (any two cops may occupy the same vertex); after each cop has chosen his initial position, $\gamma$ chooses her starting vertex in $G$. A game configuration (or simply configuration) is a $(k + 2)$-tuple $(G, v_1, \ldots, v_k; r)$ such that at the end of some turn of the game, $r$ is the vertex occupied by $\gamma$ and for $i \in \{1, \ldots, k\}$, $u_i$ is the vertex occupied by $\lambda_i$. $\gamma$ is said to be captured (or caught) if, at any point in the game, $\gamma$ occupies the same vertex as a cop. The 1-st turn of the game starts after the robber has chosen her starting vertex. During each odd turn $\{1, 3, \ldots\}$, the robber $\gamma$ either stays put or moves to an adjacent vertex, and during each even turn $\{2, 4, \ldots\}$, exactly one of the cops moves to an adjacent vertex. For any $i \in \mathbb{N}$, the $(2i - 1)$-st turn and $2i$-th turn together constitute the $i$-th round of the game.

3 The Classical Cops-and-Robbers Game Versus the One-cop-moves Game on Planar Graphs

Before presenting the main result, we show that for planar graphs, the one-cop-moves cop number can in general be larger than the classical cop number. Recall
that the cube has domination number 2, so it has cop number (the classical version as well as the one-cop-moves version) at most 2. Now let $Q'$ be the graph obtained by subdividing each edge of a cube with one vertex (see Figure 1). Then we have the following result.

**Proposition 1** $c(Q') = 2$ and $c_1(Q') = 3$.

![Subdivided cube $Q'$](image)

**Fig. 1. Subdivided cube $Q'$**

**Proof.** Let $\gamma$ denote the robber. We first show that 2 cops can capture $\gamma$ in the classical cops and robber game. Initially, we place the cops at positions 9 and 5. By symmetry, one may assume that $\gamma$ starts at one of the following positions: 1, 2, or 3. The following list shows all the possible moves of the game before $\gamma$ is caught. A triple $\langle p_1, p_2; p_3 \rangle$ denotes the set of positions of the robber and cops at the end of some turn of the game; $p_1$ and $p_2$ denote the positions of the first cop and second cop respectively, while $p_3$ denotes the position of $\gamma$. An arrow $\rightarrow$ denotes a transition from one turn to the next turn of the game. The first triple in each sequence denotes the set of positions of the robber and cops at the end of the 1-st turn. It is assumed that whenever the robber is adjacent to a cop at the end of the cops' turn, she will try to escape by moving to an adjacent vertex during the next turn.

1. $\langle 9, 5; 1 \rangle \rightarrow \langle 17, 5; 1 \rangle \rightarrow \langle 17, 5; 8 \rangle \rightarrow \langle 1, 6; 8 \rangle$.
2. $\langle 9, 5; 1 \rangle \rightarrow \langle 17, 5; 1 \rangle \rightarrow \langle 17, 5; 2 \rangle \rightarrow \langle 1, 4; 2 \rangle$.
3. $\langle 9, 5; 2 \rangle \rightarrow \langle 17, 4; 2 \rangle \rightarrow \langle 1, 3; 2 \rangle$.
4. $\langle 9, 5; 3 \rangle \rightarrow \langle 9, 4; 3 \rangle \rightarrow \langle 9, 4; 2 \rangle \rightarrow \langle 17, 3; 2 \rangle$.
5. $\langle 9, 5; 3 \rangle \rightarrow \langle 9, 4; 3 \rangle \rightarrow \langle 9, 4; 18 \rangle \rightarrow \langle 10, 3; 18 \rangle$.

To show that two cops cannot capture the robber on $Q'$ in the one-cop-moves game, we show that $\gamma$ can evade capture if she avoids the middle positions 2, 4, 6, 8, 10, 12, 14, 16; and if she is forced to move to at least one of these positions, then she will choose the position that maximises her total distance from the cops. Assume otherwise. By symmetry, it is enough to show that if $\gamma$ were eventually
caught, then her last position is 8 while the two cops are at positions (i) 2 and 6 or (ii) 6 and 17.

If (i) holds, then, since 8 cannot be starting position of $\gamma$, the previous position of $\gamma$ must have been either 7 or 1. Again by symmetry, it suffices to assume that the previous position of $\gamma$ is 7. But if $\gamma$ is at position 7 while the cops are at 2 and 6, then $\gamma$ would move to position 20 on her next turn to maximise her total distance from the cops, a contradiction. For similar reasons, if (ii) holds and $\gamma$’s previous position is 7, then she would move to position 20 on her next turn.

We next show that $c_1(Q') \leq 3$. Start by placing the cops at positions 1, 5, and 11. Note that $\gamma$ cannot start at any one of the following positions: 2, 17, 8, 10, 18, 12, 4, 6, 19, 1, 5, 11. She also cannot start at 3 because each of the 3 escape paths from 3 is guarded by a cop. This leaves the following possible starting positions of $\gamma$: 7, 9, 13, 14, 15, 16, 20. Denote the cops at positions 1, 5, and 11 by $\lambda_1, \lambda_2$ and $\lambda_3$ respectively. Suppose $\gamma$ starts at 9. $\gamma$ cannot escape to 1 or to 11 so long as the cops at 1 and 11 stay put. $\lambda_2$ then moves to 6. Before $\lambda_2$ can block off $\gamma$’s last escape path, $\gamma$ must move to 16. $\lambda_2$ then moves to 7. $\gamma$ must move again before her escape path (9, 16, 15) is cut off by $\lambda_2$, this time moving to 15. $\lambda_3$ then moves to 12, preventing $\gamma$ from escaping along (15, 14, 13). If $\gamma$ moves back to 16, then $\lambda_1$ moves to 17. $\lambda_2$ can then move to 20, thus trapping $\gamma$ in the path (17, 16, 15).

Now suppose $\gamma$ starts at 7. $\lambda_3$ then moves to 12. If $\gamma$ tries to move along (7, 20, 15), then $\lambda_3$ moves to block the path (15, 14, 13), first moving to 13. $\gamma$ also cannot advance along (15, 16, 9) because $\lambda_1$ can move to 17 before $\gamma$ reaches 9. If $\gamma$ returns along (15, 20, 7), then $\lambda_2$ can move to 6, thereby trapping $\gamma$ along (7, 20, 15). If $\gamma$ starts at 20, 15 or 14 then $\lambda_3$ moves to 12 and the cops can follow up with a winning strategy similar to that in the case when $\gamma$ starts at 7.

If $\gamma$ starts at 16, then $\lambda_1$ moves to 17, preventing $\gamma$ from escaping to 9. If $\gamma$ moves to 15, then $\lambda_2$ moves to 6. Wherever $\gamma$ moves to on her next turn, $\lambda_3$ moves to 12. $\gamma$ is then trapped on the set of positions {16, 15, 20, 14}, and the cops can advance along (9, 16, 15), (13, 14, 15) and (7, 20, 15) to capture $\gamma$. □

Having achieved separation between the classical cops-and-robbers game and the one-cop-moves game on planar graphs, a question that follows quite naturally is: how large can the gap between $c(G)$ and $c_1(G)$ be when $G$ is planar? This question is somewhat more difficult. Although we do not directly address the question in this work, the main result shows that for connected planar graphs, the one-cop-moves cop number can break through the upper bound of 3 for the classical cop number.

**Theorem 2.** There is a connected planar graph $D$ such that $c_1(D) \geq 4$.

We organise the proof of Theorem 2 into three main sections. Section 4 details the construction of the planar graph $D$ with a one-cop-moves cop number of at least 4. Section 5 establishes some preparatory lemmas for the proof that $c_1(D) \geq 4$. Section 6 describes a winning strategy for a single robber against three cops in the one-cop-moves game played on $D$. 
4 The Construction of the Planar Graph $\mathcal{D}$

The construction of $\mathcal{D}$ starts with a dodecahedron $D$. Each vertex of $D$ is called a \textit{corner} of $D$. We will add straight line segments on the surface of $D$ to partition each pentagonal face of $D$ into small polygons. For each pentagonal face $U$ of $D$, we add 48 nested nonintersecting closed pentagonal chains, which are called \textit{pentagonal layers}, such that each side of a layer is parallel to the corresponding side of $U$. Each vertex of a layer is called a \textit{corner} of that layer. For convenience, the innermost layer is also called the 1-st layer in $U$ and the boundary of $U$ is also called the outermost layer of $U$ or the 49-th layer of $U$. We add a vertex $o$ in the centre of $U$ and connect it to each corner of $U$ using a straight line segment which passes through the corresponding corners of the 48 inner layers. For each side of the $n$-th layer ($1 \leq n \leq 49$), we add $2n + 1$ internal vertices to partition the side path into $2n + 2$ edges of equal length. Add a path of length 2 from the centre vertex $o$ to every vertex of the innermost layer to partition the region inside the 1-st layer into 20 pentagons. Further, for each pair of consecutive pentagonal layers, say the $n$-th layer and the $(n+1)$-st layer ($1 \leq n \leq 48$), add paths of length 2 from vertices of the $n$-th layer to vertices of the $(n+1)$-st layer such that the region between the two layers is partitioned into $5(2n + 2)$ hexagons and 10 pentagons as illustrated in Figure 2. Let $\mathcal{D}$ be the graph consisting of all vertices and edges current on the surface of the dodecahedron $D$ (including all added vertices and edges). Since $\mathcal{D}$ is constructed on the surface of a dodecahedron without any edge-crossing, $\mathcal{D}$ must be a planar graph.

\textbf{Note on terminology.} We will treat $\mathcal{D}$ as an embedding of the graph on the surface of $D$ because it is quite convenient and natural to express features of $\mathcal{D}$ in geometric terms. Thus we will often employ geometric terms such as \textit{midpoint}, \textit{parallel}, and \textit{side}; the corresponding graph-theoretic meaning of these terms will be clear from the context. The \textit{distance} between any two vertices $u$ and $v$ in a graph $G$, denoted $d_G(u,v)$, will always mean the number of edges in a shortest path connecting $u$ and $v$. Let $v \in V(\mathcal{D})$ and $H$ be any subgraph of $\mathcal{D}$. By abuse of notation, we will write $d_{\mathcal{D}}(\gamma, v)$ (resp. $d_{\mathcal{D}}(\lambda_i, v)$) to denote the distance between $\gamma$ and $v$ (resp. between $\lambda_i$ and $v$) at the point of consideration. $d_{\mathcal{D}}(\gamma, H)$ and $d_{\mathcal{D}}(\lambda_i, H)$ are defined analogously.

For $n \in \{1, \ldots, 49\}$, let $L_{U',n}$ denote the $n$-th pentagonal layer of a pentagonal face $U'$, starting from the innermost layer. Define a \textit{side path} of $L_{U',n}$ to be one of the 5 paths of length $2n + 2$ connecting two corner vertices of $L_{U',n}$. $L_{U',n}$ will often simply be written as $L_n$ whenever it is clear from the context which pentagonal face $L_n$ belongs to.

The pentagonal faces of $\mathcal{D}$ will be denoted by $U, U_1, U_2, \ldots, U_{10}, U_{11}$ (see Figure 3). For $i \in \{1, \ldots, 15\}$, $B_i$ will denote a side path of $L_{U_{11},49}$ for some
Fig. 2. Two innermost and two outermost pentagonal layers of a pentagonal face.

Fig. 3. 12 pentagonal faces of $D$, labelled $U, U_1, \ldots, U_{11}$. $v_1, \ldots, v_5$ denote the 5 corner vertices of $U$. The side paths of $U$ are labelled $B_6, B_7, B_8, B_9, B_{10}$; the side paths $B_1, B_2, B_3, B_4, B_5$ connect $U$ to $U_6, U_7, U_8, U_9$ and $U_{10}$ respectively. The side paths $B_{11}, B_{12}, B_{13}, B_{14}$ and $B_{15}$ connect $U_{11}$ to $U_3, U_4, U_5, U_1$ and $U_2$ respectively. $m$ is the middle vertex of $B_1$. 
pentagonal face \(U'\). The centre vertex of \(U\) will be denoted by \(o\), and for \(i \in \{1, \ldots, 11\}\), the centre vertex of \(U_i\) will be denoted by \(o_i\). Given a pentagonal face \(U\), we shall often abuse notation and write \(U\) to denote the subgraph of \(D\) that is embedded on the face \(U\).

For any \(n \in \{1, \ldots, 49\}\), a middle vertex of \(L_n\) is a vertex that is \(n + 1\) edges away from two corners of \(L_n\), which are end vertices of some side path of \(L_n\). The middle vertex of a side path \(B\) of \(L_n\) is the vertex of \(L_n\) that lies at the midpoint of \(B\). Given any pentagonal face \(U\), a spoke of \(U\) is a path of length 98 connecting a vertex on \(L_{49}\) and the centre of \(U\). Given any \(A, B \subseteq V(D)\) and any \(v \in V(D)\), define \(d_D(v, A) = \min\{d_D(v, x) : x \in A\}\) and \(d_D(A, B) = \min\{d_D(x, y) : x \in A \land y \in B\}\).

Let \(U\) and \(U'\) be any two pentagonal faces of \(D\). Define \(U \cup U'\) to be the subgraph \((V(U) \cup V(U'), E(U) \cup E(U'))\) of \(D\) and \(U \cap U'\) to be the subgraph \((V(U) \cap V(U'), E(U) \cap E(U'))\) of \(D\); these definitions naturally extend to any finite union or finite intersection of pentagonal faces.

**Remark 3** The exact number of pentagonal layers in each face of \(D\) is not important so long as it is large enough to allow the robber’s winning strategy to be implemented. One could increase the number of pentagonal layers in each face and adjust the robber’s strategy accordingly. This will become clearer when we describe the robber’s winning strategy in Section 6.

## 5 Some Preparatory Lemmas

In this section, we shall outline the main types of strategies employed by the robber to evade the three cops. Let \(\gamma\) denote the robber and \(\lambda_1, \lambda_2\) and \(\lambda_3\) denote the three cops. We first state a lemma for determining the distance between any two vertices of a pentagonal face.

**Lemma 4** Let \(U\) be a pentagonal face of \(D\). Let \(x\) be a vertex of \(L_r\) and \(y\) be a vertex of \(L_s\), where \(L_r\) and \(L_s\) are pentagonal layers of \(U\) and \(s \geq r\). Then \(d_D(x, y) = \min\{2r + 2s, (2s - 2r) + d_{L_r}(w, x)\}\), where \(w\) is the intersection vertex of \(L_r\) and the shortest path between \(y\) and the center of \(U\). (See Figure 4 for an illustration.)

**Proof.** We construct a shortest path from \(y\) to \(x\) using any given path from \(y\) to \(x\). The construction is based on the main ideas of the Floyd-Marshall algorithm [9].

First, consider any path from \(y\) to \(x\) that passes through \(o\). It may be directly verified that a shortest path from \(y\) to \(o\) has length \(2s\) while a shortest path from \(o\) to \(x\) has length \(2r\). Thus any shortest path from \(y\) to \(x\) that passes through \(o\) has length \(2s + 2r\).

Second, consider any path from \(y\) to \(x\) that does not pass through \(o\). Define \(x'\) to be the unique vertex on \(L_s\) such that \(d_{L_s}(x', y) = \min\{d_{L_s}(x'', y) : x'' \text{ lies on } L_s \land d_W(x'', x) = s - r\}\).
Suppose $d_{L_s}(x', y) \geq 4s + 4$. Then any shortest path from $y$ to $x$ that does not pass through $o$ covers a distance of at least $4r' + 4$ along a layer $L_{r'}$ for some least $r'$. Since any path along $L_{r'}$ of length at least $4r' + 4$ can be replaced by a shorter path of length $4r'$ passing through $o$, the length of any path from $y$ to $x$ is at least $2r + 2s$.

Now suppose that $d_{L_s}(x', y) \leq 4s + 3$. Observe that any path $\pi$ that starts at a vertex $z$ in a pentagonal layer $L_{r_1}$, goes to a neighbouring layer $L'$ — which includes $L_{r_1+1}$ if $r_1 \leq 48$, $L_{r_1-1}$ if $r_1 \geq 2$, and the 48th layer of a neighbouring face if $r_1 = 49$, and then passes along $L'$, covering a distance equal to at most twice the length of a side path of $L'$ when traversing $L'$, before returning to a vertex $z'$ in $L_{r_1}$, may be replaced with a path $\pi'$ that goes directly from $z$ to $z'$ along $L_{r_1}$ such that the length of $\pi'$ is not more than that of $\pi$. Thus any shortest path from $y$ to $x$ that does not pass through $o$ may be replaced with one that goes from $L_s$ to $L_r$, passing in succession the intermediate pentagonal layers $L_i$ with $r < i < s$ (and possibly passing along each layer). Next, observe that for any $r_2 \geq 2$, any path $\theta$ that starts at a vertex $z$ in $L_{r_2}$, passes along $L_{r_2}$, and then goes directly to a vertex $z'$ in $L_{r_2-1}$, may be replaced with a path $\theta'$ that starts at $z$, goes directly to $L_{r_2-1}$ in 2 rounds, and then passes along $L_{r_2-1}$ before ending at $z'$; in addition, the length of $\theta'$ does not exceed that of $\theta$. Applying this observation iteratively and combining it with the earlier observation that any shortest path from $y$ to $x$ that does not pass through $o$ may be replaced with one that starts by going directly from $L_s$ to $L_r$ in $s - r$ steps, one obtains a path from $y$ to $x$ that starts from $y$, goes directly to a vertex $w$ belonging to $L_r$ in $2s - 2r$ rounds, and then slides along the shortest path in $L_r$ from $w$ to $x$ before ending at $x$; furthermore, the length of this path is not more than that of any other path from $y$ to $x$ that does not pass through $o$.

The following observation will often be used implicitly to simplify subsequent arguments.
Lemma 5 Suppose that γ is currently at vertex $a_1$ of $\mathcal{D}$ and a cop $\lambda$ is currently at vertex $u$. Suppose γ starts moving towards vertex $a_{n+1}$ via the path $(a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{n+1})$. Then, by the 2n-th turn of the game (starting at the turn when γ moves from $a_1$ to $a_2$), γ can reach $a_{n+1}$ without being caught by $\lambda$ if $d_\mathcal{D}(u, a_{n+1}) > n$. 

Proof. Suppose that $\lambda$ catches γ on the 2k-th turn of the game for some $k \in \{1, \ldots, n\}$. It follows that $d_\mathcal{D}(u, a_{k+1}) \leq k$. Since $d_\mathcal{D}(a_{k+1}, a_{n+1}) \leq n - k$, one has $d_\mathcal{D}(u, a_{n+1}) \leq d_\mathcal{D}(u, a_{k+1}) + d_\mathcal{D}(a_{k+1}, a_{n+1}) \leq k + (n - k) = n$. This is a contradiction. 

Suppose that the robber $\lambda$ currently occupies $o$. Consider any set $A \subseteq V(\mathcal{D})$ of vertices. For every $v \in A$, if there is a cop $\lambda$ such that the current distance between $\lambda$ and $v$ is less than $d_\mathcal{D}(o, v)$, then by Lemma 5 $\lambda$ can capture γ if γ tries moving to $v$ (assuming that γ starts the game).

Corollary 6 Suppose that γ is currently at the centre $o$ of a pentagonal face $U$ and there is a centre $o' \neq o$ such that $d_\mathcal{D}(o, o') < d_\mathcal{D}(\lambda_j, o')$ for all $j \in \{1, 2, 3\}$. Then γ can reach $o'$ without being caught.

The following lemma is a direct consequence of Lemma 4.

Lemma 7 Suppose a cop $\lambda$ lies at a vertex $u$ in a pentagonal face $U$ of $\mathcal{D}$ and is not at the centre of $U$. Let $A$ be the set of 5 corners of $L_{49}$. If, for some set $A' \subseteq A$, $d_U(u, v) \leq 98$ whenever $v \in A'$, then $|A'| \leq 2$. Furthermore, if there are two corners $v', v''$ of $L_{49}$ such that $d_U(u, v') \leq 98$ and $d_U(u, v'') \leq 98$, then $d_U(v', v'') = 100$. Let $M$ be the set of 5 middle vertices of $L_{49}$. If, for some set $M' \subseteq M$, $d_U(u, v) \leq 98$ whenever $v \in M'$, then $|M'| \leq 2$.

The next technical lemma will be used to devise an evasion tactic for γ in a set of game configurations. More generally, the sort of tactic described in the proof of this lemma will often be used by γ to escape to the centre of a pentagonal face. It may be described informally as follows. γ first tries to move to the centre of a neighbouring face, say $U'$. Then at least one cop (say $\lambda_1$) will be forced to guard the centre of $U'$. Just before $\lambda_1$ can catch γ in $U'$, γ deviates from her original path towards the centre of $U'$ and moves towards the centre of yet another neighbouring face, say $U''$, such that γ is closer to the centre of $U''$ than $\lambda_1$ is. Since at most cop can move during any round, the speed of the remaining two cops ($\lambda_2$ and $\lambda_3$) will be reduced as $\lambda_1$ is chasing γ. Thus all three cops will be sufficiently far away from the centre of $U''$ during the round when γ deviates from her original path, and this will allow γ to successfully reach the centre of $U''$.

Lemma 8 Suppose the one-cop-moves game played on $\mathcal{D}$ starts on γ’s turn with the following configuration (illustrated in Figure 3). γ lies at the centre $o$ of the pentagonal face $U$ and the 3 cops lie in $U$. Let $u_1, u_2$ and $u_3$ denote the vertices
currently occupied by $\lambda_1, \lambda_2$ and $\lambda_3$ respectively. Let $m'$ be any middle vertex of $L_{U,49}$ and let $B$ be the side path of $L_{U,49}$ containing $m'$. Let $p'$ be any vertex in $B$ that is 1 edge away from $m'$. Suppose that $d_D(u_2,m') \geq 99$ and $d_D(u_3,m') \geq 99$ (resp. $d_D(u_2,p') \geq 99$ and $d_D(u_3,p') \geq 99$). Suppose that $d_D(u_1,o) = 1$ and $d_D(u_i,B) + d_D(u_j,B) \geq 104$ (resp. $d_D(u_1,B) + d_D(u_j,B) \geq 110$) for all distinct $i,j \in \{1,2,3\}$. Assume that $d_D(u_1,m') \geq 98$ (resp. $d_D(u_1,p') \geq 98$) and both $d_D(u_2,o) \geq 2$ and $d_D(u_3,o) \geq 2$ hold. Then $\gamma$ can reach the centre of a pentagonal face at some point after the first round of the game without being caught.

Fig. 5. The relative positions of the cops and $\gamma$.

Proof. Suppose that $d_D(u_2,m') \geq 99$ and $d_D(u_3,m') \geq 99$, $d_D(u_i,B) + d_D(u_j,B) \geq 104$ for all distinct $i,j \in \{1,2,3\}$ and $d_D(u_1,m') \geq 98$ (the proof for the other case is entirely similar). The proof of this lemma will be explained with the aid of Figure 3.

Suppose that $m' = m_2$, so that $B = B_7$. $\gamma$ begins by moving towards $m_2$, traversing the middle vertices of the side paths of $L_1, L_2, \ldots, L_{49}$ parallel to $B_7$. Note that if $\lambda_1$ moves 1 step into $L_1$ during the first round of the game, then $\gamma$ can simply move back to $o$ during her next turn without being caught. Now suppose that $\lambda_1$ does not move during the first round of the game. Then $\gamma$ can safely reach $m_2$ in 98 rounds. After the 98-th round of the game, the total distance travelled by $\lambda_1, \lambda_2$ and $\lambda_3$ is at most 98. Suppose that $\gamma$ reaches $m_2$ in the 98-th round. Consider the following case distinction.

Case (a): For each $j \in \{1,2,3\}$, the distance between $\lambda_j$ and $U_2$ at the end of the 98-th round is at least 1. By Lemma 5, $\gamma$ can reach $o_2$ in another 98 rounds without being caught.
\textbf{Case (b):} At least one of \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) occupies a vertex of \( U_2 \) at the end of the 98-th round of the game. Note that since \( d_D(u_1, B_7) + d_D(u_2, B_7) \geq 104 \) for all distinct \( i, j \in \{1, 2, 3\} \), it follows that at least 104 rounds are needed for a minimum of two cops to reach \( B_7 \), and therefore at most one of \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) can occupy a vertex of \( U_2 \) at the end of the 98-th round of the game.

Let \( \lambda_\alpha \) be the first cop that reaches \( U_2 \) and \( s \) be the first vertex of \( U_2 \) that \( \lambda_\alpha \) reaches as \( \gamma \) is moving from \( o \) to \( m_2 \). Without loss of generality, assume that \( s \) lies on \( B_7 \). Note that \( s \) cannot be \( m_2 \) (since \( \gamma \) can safely reach \( m_2 \) in 98 rounds), and therefore either \( d_D(s, B_1) > d_D(s, B_2) \) or \( d_D(s, B_1) < d_D(s, B_2) \) holds. Assume that \( d_D(s, B_1) > d_D(s, B_2) \). For each \( j \in \{1, 2, 3\} \), let \( \ell_j = d_D(u_j, B_7) \). Note that \( \ell_1 \geq 97 \) and for each fixed \( j \in \{2, 3\} \), \( u_j \) is a vertex on \( B_7 \) such that \( d_D(u_j, u_j') = \ell_j \), then \( \ell_j = d_D(u_j, u_j') \geq d_D(u_j, m_2) - d_D(m_2, u_j') \geq 99 - 50 = 49 \).

Let \( k \) be the total distance travelled by \( \lambda_\alpha \) between the 1-st and the 98-th round.

\textbf{Case (b.1):} \( k \geq \ell_\alpha + 46 \). Since \( 46 + \ell_\alpha \leq k \leq 98 \), it holds that \( \ell_\alpha \leq 52 \) and therefore \( \alpha \in \{2, 3\} \). Without loss of generality, assume that \( \alpha = 2 \).

\( \gamma \) moves along the path \( m_2 \xrightarrow{B_7} v_1 \xrightarrow{B_1} m \) (where \( m \) is the midpoint of \( B_7 \)). Since \( s \) lies on \( B_7 \) and \( d_D(s, B_1) > d_D(s, B_2) \) by assumption, an application of Lemma 4 shows that the shortest path from \( s \) to \( m \) passes through \( m_2 \). As \( \gamma \) can reach \( m_2 \) in 98 rounds but \( \lambda_2 \) needs at least 99 rounds to reach \( m_2 \), it follows that \( \lambda_2 \) cannot catch \( \gamma \) before or during the round when \( \gamma \) reaches \( m \). Furthermore, for \( j \in \{1, 3\} \), the distance between \( \lambda_j \) and \( m \) at the end of the 98-th round is at least \( \ell_j + 50 - 98 + k \geq (\ell_j + \ell_2) - 48 + 46 \geq 102 \) (since \( \lambda_j \) could have moved at most \( 98 - k \) steps between the 1-st and the 98-th round and \( d_D(B_7, m) = 50 \)).

Since the distance between \( \gamma \) and \( m \) at the end of the 98-th round is 100, it follows from Lemma 5 that for \( j \in \{1, 3\} \), \( \lambda_j \) cannot catch \( \gamma \) either before or during the round when \( \gamma \) reaches \( m \).

If \( \lambda_3 \) moves at most 96 steps between the 99-th round and the 198-th round, then \( \gamma \) can reach \( o_0 \) via \( m \xrightarrow{B_7} q_1 \xrightarrow{S} o_0 \), where \( S \) is the spoke connecting \( q_1 \) and \( o_0 \). If \( \lambda_2 \) moves at least 97 steps between the 99-th round and the 198-th round, then any \( \lambda \in \{\lambda_1, \lambda_3\} \) can move at most 3 steps between the 99-th round and the 198-th round. \( \gamma \) now starts moving from \( m \) to \( o_1 \) (via the spoke connecting \( m \) and \( o_1 \)).

We claim that for some appropriate choice of \( r \), \( \gamma \) can either reach \( o_1 \) or move to a vertex of \( L_r \) and thence to \( w \) without being caught via the roundabout path \( m \sim p \sim q \sim t \sim w \) shown in Figure 6.

First, note that between the 1-st and the 98-th round, the total distance travelled by \( \lambda_1 \) and \( \lambda_3 \) is at most \( 98 - k \leq 98 - \ell_2 - 46 \leq 3 \). Thus the total distance travelled by \( \lambda_1 \) and \( \lambda_3 \) between the 1-st and the 198-th round is at most 6, so that when \( \gamma \) is at \( m \), the distance between \( w \) and the cop that is nearest to \( w \) (say \( \lambda_3 \)) is at least 190. A direct calculation gives that the length of the path \( m \sim p \sim q \sim t \sim w \) is \( 2(98 - 2r) + (r + 1) + (2r + 2) = 199 - r \), and so by Lemma 5 choosing any \( r \geq 10 \) ensures that \( \lambda_3 \) will not be able to catch \( \gamma \) before or during the round when \( \gamma \) reaches \( w \). In particular, for any \( r \geq 10 \), \( \lambda_3 \) will not
be able to catch $\gamma$ during the round when $\gamma$ reaches $p$. Now suppose that $r \geq 10$. If, between the 198-th round and the round when $\gamma$ reaches $p$, $\lambda_3$ skips at least 7 turns, then $\gamma$ will be closer to $o_1$ than any other cop just after the round when $\gamma$ reaches $p$, and therefore $\gamma$ can reach $o_1$ without being caught.

Suppose, on the other hand, that $\lambda_3$ skips no more than 6 turns as $\gamma$ is moving from $m$ to $p$. Then, just after the round when $\gamma$ reaches $p$, $\lambda_2$ must be at least 141 edges away from $w$. Thus by choosing $r$ so that the distance from $p$ to $w$ (via the path highlighted in Figure 6) is less than 141 steps, $\gamma$ can reach $w$ without being caught by $\lambda_2$. Therefore one requires $3r + 3 + 98 - 2 = 101 + r < 141$, or $r < 40$. Fixing any $r$ in the range of 10 to 39 (inclusive) establishes the claim.

Case (b.2): $k \leq \ell_\alpha + 45$.

$\gamma$ adopts a winning strategy similar to that in Case (b.1), this time moving towards $o_2$. As in Case (b.1), we claim that for some appropriate choice of $r$, $\gamma$ can either reach $o_2$ without being caught or move to $q_1$ and thence to $o_6$ without being caught via the path $m_2 \rightsquigarrow p' \rightsquigarrow q' \rightsquigarrow t' \rightsquigarrow q_1$ highlighted in Figure 6. We shall again assume that $\alpha = 2$; it will become clear below that the following winning strategy for $\gamma$ also works for $\alpha \in \{1, 3\}$.

We briefly explain how Algorithm 1 works. $\gamma$ moves successively through $w_1, w_2, \ldots, w_{k-\ell_2+1}$ until at least one of the following occurs: (i) she reaches some $w_i$ such that the total number of turns $j_i$ that $\lambda_2$ skips between the round when $\gamma$ is at $w_0$ and the round when $\gamma$ is at $w_i$ is exactly equal to $i - 1$, or (ii)
Therefore no shortest path from $i$ to $w_i$ exists. The case $\gamma = o$ is trivial. If $\lambda < \gamma$, then after reaching $o$, $\gamma$ can safely get from $w_0 := m_2$ to $w_i$ in $98 - 2\ell$ rounds. To show that $\gamma$ can safely reach $w_i$, it suffices to show that $\lambda < \gamma$ before or during the round when $\gamma$ reaches $w_i$.

The case $i = 0$ was established earlier. For the inductive step, suppose that for some $i \in \{1, \ldots, \ell\}$, $\gamma$ can safely reach $w_{i-1}$ in $98 - 2\ell_{i-1}$ rounds. We first calculate a lower bound for the distance between $\lambda_2$ and $w_i$ at the end of the round when $\gamma$ reaches $w_{i-1}$. Now, any path from $s$ to $w_i$ that passes through $o_2$ has length at least 99. On the other hand, the path from $s$ to $w_i$ that starts by going directly to $L_{\ell_i}$ in $98 - 2\ell$ rounds and then passing along the side path of $L_{\ell_i}$ parallel to $B_G$ until $w_i$ is reached has length at most $(98 - 2\ell_i) + (\ell_i + 1) = 99 - \ell_i \leq 99 - 4 = 95$.

Therefore no shortest path from $s$ to $w_i$ passes through $o_2$. Hence by Lemma 4, a shortest path from $s$ to $w_i$ starts by going directly to $L_{\ell_i}$ in $98 - 2\ell$ rounds, and then passing along the side path of $L_{\ell_i}$ parallel to $m_2$ until $w_i$ is reached. Denote this shortest path from $s$ to $w_i$ by $P$. Observe that the distance between $s$ and $m_2$ is at least $99 - \ell'_2$, where $\ell'_2 = d_G(2, s)$. Thus the shortest distance between $w_i$ and the first vertex of $P$ on $L_{\ell_i}$ is either $99 - \ell'_2$ or $\ell_i + 1$. Note that $\lambda_2$ can move a distance of at most $k - \ell'_2$ between the round he reaches $s$ and the 98-th round. In addition, $\lambda_2$ moves at most $98 - 2\ell_{i-1} - j_{i-1}$ steps between the round when $\gamma$ is at $w_0$ and the round when $\gamma$ is at $w_{i-1}$. It follows that at the end of the round when $\gamma$ reaches $w_{i-1}$, the distance between $\lambda_2$ and $w_i$ is at least

\[
\min\{(98 - 2\ell_i) + (99 - \ell'_i) - (k - \ell'_2) - (98 - 2\ell_{i-1} - j_{i-1}), (98 - 2\ell_i) + (\ell_i + 1) - (k - \ell'_2) - (98 - 2\ell_{i-1} - j_{i-1})\}
\]

\[
= \min\{2(\ell_{i-1} - \ell_i) + (99 - k + j_{i-1}), 2(\ell_{i-1} - \ell_i) + (\ell_i + 1 - k + \ell'_2 + j_{i-1})\}.
\]

Note that $\gamma$ needs $2(\ell_{i-1} - \ell_i)$ rounds to get from $w_{i-1}$ to $w_i$. Since $k \leq 98, 99 - k + j_{i-1} \geq 1$. Similarly,

\[
r_i + 1 - k + \ell'_2 + j_{i-1} = k - \ell'_2 + 5 - i + 1 - k + \ell'_2 + j_{i-1} \geq_{{i-1}} \ell'_2 + 5 \geq 0.
\]

Consequently, $\gamma$ can move from $w_{i-1}$ to $w_i$ in $2(\ell_{i-1} - \ell_i)$ rounds without being caught by $\lambda_2$, and this completes the inductive step.

If $j_i > k - \ell_2$, then after reaching $p'$, $\gamma$ continues moving towards $o_2$ until she reaches $o_2$ in another $2\ell$ rounds. Suppose $j_i \leq k - \ell_2$. It can be directly verified that in this case, the condition to break out of the loop in Algorithm [1] will eventually be satisfied, and that $r = k - \ell_2 + 4 - j_i$. $\gamma$ now moves along the path
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Note that between the 1-st round and the round when \( \gamma \) reaches \( p' \), \( \lambda_1 \) and \( \lambda_3 \) could have moved a total of at most \( j_\ell + (98 - k) \) steps. Suppose that \( \lambda_3 \) chases \( \gamma \) for the duration of \( \gamma \)'s movement from \( p' \) to \( q_1 \). During the round when \( \gamma \) is at \( p' \), the distance between \( \lambda_3 \) and \( q_1 \) is at least \((100 + \ell) - (98 - k + j_\ell) = \ell_2 + \ell_3 + 2 + (k - \ell_2 - j_\ell) \). The length of the path \( p' \Rightarrow q' \Rightarrow t' \Rightarrow q_1 \) is \( 101 + r = 101 + (k - \ell_2 + 4 - j_\ell) = 105 + k - \ell_2 - j_\ell \). Since \((\ell_2 + \ell_3 + 2 + (k - \ell_2 - j_\ell)) - (105 + (k - \ell_2 - j_\ell)) = \ell_2 + \ell_3 - 103 \geq 1 \), \( \gamma \) can reach \( q_1 \) without being caught by \( \lambda_3 \). One can show in an analogous way that \( \gamma \) can reach \( q_1 \) without being caught by \( \lambda_1 \). If \( \lambda_2 \) chases \( \gamma \) by moving along \( L_r \) (or by any other path that does not pass through \( o_2 \)), then, since \( \gamma \) can safely get from \( m_2 \) to \( p' \) in \( 98 - 2r \) rounds, \( \lambda_2 \) cannot catch \( \gamma \) before or during the round when \( \gamma \) reaches \( q_1 \). Suppose \( \lambda_2 \) chases \( \gamma \) by first moving to \( o_2 \) and then to \( q_1 \). The number of rounds required by \( \lambda_2 \) to move from his position when \( \gamma \) is at \( p' \) to \( q_1 \) by taking a path passing through \( o_2 \) is at least \( 196 - ((98 - 2r - j_\ell) + (k - \ell_2)) = 98 + 2r - k + \ell_2 + j_\ell = 98 + 2(k - \ell_2 + 4 - j_\ell) - k + \ell_2 + j_\ell = 106 + k - \ell_2 - j_\ell \). Thus \( \gamma \) can reach \( q_1 \) without being caught by \( \lambda_2 \). After reaching \( q_1 \), \( \gamma \) can safely reach \( o_6 \) in another 98 rounds.

The following lemma will establish a winning strategy for \( \gamma \) in another specific game configuration. As in Lemma 8, \( \gamma \)'s strategy in Lemma 9 exploits the condition that at most one cop can move during any round. Roughly speaking, the strategy works as follows: when \( \gamma \) is at a corner \( v \), she attempts to lure a cop into a face \( U'' \) containing \( v \) by moving to a neighbour of \( v \) in \( U'' \). If no cop is in \( U'' \) at the end of the next turn, then \( \gamma \) can safely reach the centre of \( U'' \); otherwise, \( \gamma \) safely moves back to \( v \) during the next round and repeats the same strategy.

Fig. 6. The escape path of \( \gamma \) in Case (b.1).
used during the preceding round. Lemma 9 shows that it is advantageous for $\gamma$ to occupy a corner, and this fact underlies $\gamma$’s strategy as described in Section 6.

**Lemma 9**  
Suppose $\gamma$ is currently at a vertex $v$ that lies in two intersecting pentagonal faces $U$ and $U'$ of $\mathcal{D}$, and it is $\gamma$’s turn. Suppose $\lambda_1$ is at some vertex $w$ of $U \cup U'$ such that $d_D(v,w) \geq 2$, $d_D(\lambda_2, U \cup U') \geq 2$ and $d_D(\lambda_3, U \cup U') \geq 2$. Then $\gamma$ can either (i) reach the centre of $U$ or $U'$ without being caught, or (ii) oscillate infinitely often between $v$ and one of its neighbours.

**Proof.** We prove this lemma by induction on the odd rounds of the game. Assume that the 1-st round starts on $\gamma$’s turn. Inductively, suppose that at the start of the $(2n-1)$-st round of the game (for some $n \geq 1$), $\gamma$ is at a vertex $v$ in $U \cap U'$, $\lambda$ is at some vertex $w_n$ of $U \cup U'$ such that $d_D(v,w_n) \geq 2$, the distance between the position of every cop (other than $\lambda$) and $U \cup U'$ is at least 2, and it is currently $\gamma$’s turn. Without loss of generality, assume that $w_n$ lies in $U$. $\gamma$ then moves to a vertex $v'$ in $U'$ such that $v'$ is adjacent to $v$ and the distance between $v'$ and the centre of $U'$ is 97. If $\lambda$ does not move towards the centre of $U'$ during the $(2n-1)$-st round or if $w_n$ does not lie in $U'$, then, since $d_D(v,w_n) \geq 2$ and the distance between every cop (other than $\lambda$) and $U \cup U'$ is at least 2, $\gamma$ can continue moving safely towards the centre of $U'$, reaching this vertex in another 97 rounds. On the other hand, if $\lambda$ does move towards the centre of $U'$ on the $(2n-1)$-st round and $w_n$ lies in $U'$, then $\gamma$ moves back to $v$ during the 2n-th round without being caught. Note that in this case, at the start of the $(2n+1)$-st round, $\lambda$ is at a vertex $w_{n+1}$ in $U \cup U'$ such that $d_D(v,w_{n+1}) \geq 2$, and the distance between every other cop and $U \cup U'$ is still at least 2. This completes the induction step.

The next lemma is the analogue of Lemma 9 when $\gamma$ lies at the intersection of 3 pentagonal faces.
Lemma 10 Suppose \( \gamma \) is currently at a vertex \( v \) that lies in 3 pentagonal faces \( U, U' \) and \( U'' \) of \( D \), and it is \( \gamma \)'s turn. Suppose moreover that there are at most 2 cops, say \( \lambda_1 \) and \( \lambda_2 \), lying in \( U \cup U' \cup U'' \), and \( d_D(\lambda_1, \gamma) \geq 2 \), \( d_D(\lambda_2, \gamma) \geq 2 \), and \( d_D(\lambda_3, U \cup U' \cup U'') \geq 2 \). Then \( \gamma \) can either (i) reach the centre of \( U \) or the centre of \( U' \) or the centre of \( U'' \) without being caught, or (ii) oscillate infinitely often between \( v \) and one of its neighbours.

Proof. For any two faces \( U \) and \( U' \) of \( D \), define \( U \setminus U' \) to be the subgraph of \( D \) induced by \( V(U) \setminus V(U') \) and \( U \triangle U' \) to be the subgraph of \( D \) induced by \( (V(U) \setminus V(U')) \cup (V(U') \setminus V(U)) \). Like the proof of Lemma 9 we use induction on the odd rounds of the game. Inductively, suppose that at the start of the \((2n-1)\)-st round of the game (for some \( n \geq 1 \)), \( \gamma \) is at vertex \( v \) and the three cops are situated as follows: either (1) one cop (say \( \lambda_1 \)) is in \( (U \triangle U') \triangle U'' \) and is at a distance of at least 2 from \( \gamma \), one cop (say \( \lambda_2 \)) is in \( ((U \cap U') \triangle U'') \cup ((U' \cap U') \triangle U') \cup ((U'' \cap U) \triangle U'') \cup ((U' \cap U) \triangle U) \) and is at a distance of at least 1 from \( \gamma \), and the remaining cop (say \( \lambda_3 \)) is at a distance of at least 1 from \( \gamma \), or (2) both \( \lambda_1 \) and \( \lambda_2 \) are in \( ((U \cap U') \triangle U'') \cup ((U' \cap U') \triangle U') \cup ((U'' \cap U) \triangle U'') \cup ((U' \cap U) \triangle U) \) and are each at a distance of at least 2 from \( \gamma \), and the distance between \( \lambda_3 \) and \( U \cup U' \cup U'' \) is at least 2, or (3) both \( \lambda_1 \) and \( \lambda_2 \) are in \( (U \triangle U') \triangle U'' \) and are each at a distance of at least 1 from \( \gamma \), and the distance between \( \lambda_3 \) and \( U \cup U' \cup U'' \) is at least 1.

Suppose (1) holds. Without loss of generality, assume that \( \lambda_1 \) is in \( U \setminus (U' \cup U'') \) and is at a distance of at least 2 from \( \gamma \), and \( \lambda_2 \) is in \( (U' \cap U'') \setminus U \) and is at a distance of at least 1 from \( \gamma \). \( \gamma \) then takes 1 step towards the centre of \( U' \). Suppose \( \lambda_2 \) does not move towards the centre of \( U' \) during the \((2n-1)\)-st round. Then, since both \( \lambda_1 \) and \( \lambda_3 \) are at a distance of at least 99 from the centre of \( U' \) at the start of the \((2n-1)\)-st round, \( \gamma \) can safely reach the centre of \( U' \). Now suppose \( \lambda_2 \) moves towards the centre of \( U' \) during the \((2n-1)\)-st round of the game. \( \gamma \) then moves back to \( v \) during the \(2n\)-th round. If \( \lambda_2 \) does not return to a vertex in \( (U' \cap U'') \setminus U \) during the \(2n\)-th round of the game, then \( \gamma \) can safely reach the centre of \( U'' \) in another 98 rounds. If \( \lambda_2 \) returns to a vertex in \( (U' \cap U'') \setminus U \) during the \(2n\)-th round, then scenario (1) is repeated at the start of the \((2n+1)\)-st round.

Suppose (2) holds. Without loss of generality, assume that \( \lambda_1 \) is in \( (U \cap U') \setminus U'' \) and \( \lambda_2 \) is in \( (U' \cap U'') \setminus U \). \( \gamma \) then moves towards the centre of \( U \). If, during the \((2n-1)\)-st round, \( \lambda_1 \) does not move towards the centre of \( U \), then \( \gamma \) can safely reach the centre of \( U \) in another 98 rounds. If \( \lambda_1 \) moves towards the centre of \( U \) during the \((2n-1)\)-st round, then \( \gamma \) returns to \( v \) during the \(2n\)-th round. If \( \lambda_1 \) does not move back to a vertex in \( (U \cap U') \setminus U'' \) during the \(2n\)-th round, then either (1) or (3) holds at the start of the \((2n+1)\)-st round. If \( \lambda_1 \) does move back to a vertex in \( (U \cap U') \setminus U'' \) during the \(2n\)-th round, then scenario (2) is repeated at the start of the \((2n+1)\)-st round.

Suppose (3) holds. Without loss of generality, suppose \( \lambda_1 \) is in \( U \setminus (U' \cup U'') \) and \( \lambda_2 \) is in \( U' \setminus (U \cup U'') \). \( \gamma \) can then reach safely the centre of \( U'' \) in another 98 rounds. This completes the induction step. \( \square \)
6 The Robber’s Winning Strategy: Proof of Theorem 2

We begin with a high-level description of $\gamma$’s winning strategy; see Algorithm 2.

Algorithm 2: High-level strategy for $\gamma$

1. $\gamma$ picks the centre of a pentagonal face that is free of cops. Let $U$ be this face.
2. $\gamma$ stays at the centre $o$ of $U$ until there is exactly one cop that is 1 edge away from $\gamma$.
3. $\gamma$ does one of the following depending on the cops’ positions and strategy (details will be given in Cases (A), (B) or (C) below; see Sections 6.1, 6.2 and 6.3): (i) she moves to the centre of a pentagonal face $U'$, which may or may not be $U$, without being caught at the end of a round, or (ii) she oscillates back and forth along an edge for the rest of the game without being caught.
4. If, in Step 3, $\gamma$ does (i), then set $U ← U'$ and go back to Step 2.

Since there are 12 pentagonal faces but only 3 cops, Step 1 of Algorithm 2 can be readily achieved. Let $U$ denote the pentagonal face whose centre $o$ is currently occupied by $\gamma$. The precise winning strategy for $\gamma$ in Step 3 will depend on the relative positions of the cops when exactly one cop is 1 edge away from $\gamma$. The details of this phase of $\gamma$’s winning strategy will be described in three cases: (A) when three cops lie in $U$; (B) when exactly one cop lies in $U$; (C) when exactly two cops lie in $U$. These cases reflect three possible strategies for the cops: all three cops may try to encircle $\gamma$, or one cop may try to chase $\gamma$ while the remaining two cops guard the neighbouring faces of $U$, or two cops may try to encircle $\gamma$ while the remaining cop guards the neighbouring faces of $U$.

Remark 11 It will be assumed that the starting game configurations in Cases (A), (B) and (C) below occur during the first round of the game (so that in what follows, for any $n ≥ 1$, the “$n$-th round of the game” refers to the $n$-th round of the game after the given initial game configuration) and that $\gamma$ starts each round. That is, the inputs of Algorithms 4, 5 and 6 will be the initial game configurations when we prove their correctness. Furthermore, the phrase “between the $m$-th round of the game and the $n$-th round of the game” will always mean “between the $m$-th round of the game and the $n$-th round of the game inclusive” (unless explicitly stated otherwise). We will also assume that in the starting game configuration, there does not exist any face $i ∈ \{1, 2, 3, 4, 5\}$ such that $d_D(o_i, \lambda_j) > 196$ for all $j ∈ \{1, 2, 3\}$; otherwise, by Corollary 4, $\gamma$ can safely reach a centre in 196 rounds.

The reader familiar with the recent paper of Loh and Oh [13], which exhibited a strongly connected planar digraph with a classical cop number of at least 4, may notice some similarities between the robber’s general strategy in [13] and our strategy for $\gamma$. However, the details of how the robber’s high-level strategy will be implemented in the present proof are quite different due to the differences between the two models and the differences between the two classes of graphs.
Now suppose that it is currently \( \gamma \)'s turn and \( \lambda_1 \) is exactly 1 edge away from \( \gamma \), which lies at the centre \( o \) of \( U \). By symmetrical considerations, it suffices to assume that \( \lambda_1 \) is positioned at either vertex \( p_1 \), vertex \( p_2 \) or vertex \( p_3 \) as shown in Figure 2. If \( \lambda_1 \) moves away from \( o \) during the second turn of the game (so that \( \lambda_1 \) is 2 edges away from \( o \) at the end of the first round), then \( \gamma \) can simply return to \( o \) during the second round (see the proof of Lemma 8). Thus in our analysis of \( \gamma \)'s strategies in Cases (A), (B) and (C), it will be assumed that \( \lambda_1 \) does not move away from \( o \) during the first round of the game. Let \( u_1, u_2 \) and \( u_3 \) be the starting vertices occupied by \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) respectively. We will frequently use the following general subroutine in \( \gamma \)'s strategy (details depend on the individual cases considered; see Appendices A, B and C).

\textbf{Algorithm 3:} A strategy for \( \gamma \) when \( \gamma \) is at a corner

1. Suppose \( \gamma \) is at a corner \( v \). Let \( U, U' \) and \( U'' \) be the faces containing \( v \).
2. If there are two distinct faces \( U_i, U_j \) s.t. there is one cop (say \( \lambda_1 \)) s.t. \( d_D(\lambda_1, v) \geq 2 \), \( d_D(\lambda_2, U_i \cup U_j) \geq 2 \) and \( d_D(\lambda_3, U_i \cup U_j) \geq 2 \), then apply Lemma 9.
3. If there are at most two cops (say \( \lambda_1 \) and \( \lambda_2 \)) in \( U \cup U' \cup U'' \) s.t. \( d_D(\lambda_1, v) \geq 2 \) and \( d_D(\lambda_2, v) \geq 2 \), while the third cop \( \lambda_3 \) satisfies \( d_D(\lambda_3, U \cup U' \cup U'') \geq 2 \), then apply Lemma 10.
4. Else, move \( \gamma \) to some centre.

6.1 Case (A): \( U \) contains three cops when \( d_D(\lambda_1, o) = 1 \)

Note that there is at most one corner \( v' \) of \( L_{49} \) such that \( d_D(v_1, v') \leq 98 \). Let \( v_1, v_2, v_3, v_4, v_5 \) be the 5 corner vertices of \( L_{49} \), labelled clockwise, and \( m_1, m_2, m_3, m_4, m_5 \) be the 5 middle vertices of \( L_{49} \), also labelled clockwise. The vertex \( p \) is 1 edge away from \( m_4 \) and lies between \( m_4 \) and \( v_3 \), and the vertex \( q \) is 1 edge away from \( m_5 \) and lies between \( m_5 \) and \( v_4 \) (see Figure 3).

We summarise \( \gamma \)'s strategy in Algorithm 4. At each line of Algorithm 4 where a specific strategy is executed, the corresponding subcase in Appendix A is referenced. (A similar remark applies, mutatis mutandis, to Algorithms 5 and 6.)

\textbf{Lemma 12} For Case (A), Algorithm 4 correctly computes a strategy for \( \gamma \) such that \( \gamma \) succeeds in Step 3 of Algorithm 2.

As was mentioned earlier, every corner of \( D \) is a strategic location for \( \gamma \), and so \( \gamma \) will generally try to reach a corner if no cop is guarding it. To give an example of how Algorithm 4 works, suppose the starting configuration \( (D, p_1, m_1, m_3, o) \) (see Figures 2 and 3) is fed to Algorithm 4. By Line 11 of Algorithm 4, Lemma 9 will be applied. According to the strategy given in the proof of Lemma 8, \( \gamma \) will first move to \( m_4 \) in 98 rounds. If no cop is in \( U_4 \) at the end of the 98-th round, then \( \gamma \) can safely reach \( o_4 \) in another 98 rounds; otherwise, a straightforward
Algorithm 4: The Robber’s Strategy for Case (A)

**Input**: A game configuration \((D, u_1, u_2, u_3; o)\) such that \(o\) is the centre of some face \(U, \{u_1, u_2, u_3\} \subseteq V(U), u_1 \in \{p_1, p_2, p_3\}, d_D(u_2, o) \geq 2\) and \(d_D(u_3, o) \geq 2\)

**Output**: A strategy for \(\gamma\) if \(\exists\) a corner \(v'\) of \(L_{U,49}\) such that \(\lambda_2\) and \(\lambda_3\) are at least 99 edges away from \(v'\) and \(\lambda_1\) is at least 98 edges away from \(v'\) then

1. if \(d_D(v_1, u') \geq 99\) for all \(u' \in \{u_2, u_3\}\) and \(d_D(v_1, u_1) \geq 98\) then
2. if \(d_D(u_2, B_6) \leq 2\) and \(d_D(u_3, B_7) \leq 2\) then
3. apply Lemma 8 /* See Case (A.1.1) */
4. else
5. (w.l.o.g. assume that \(d_D(u_2, B_6) \geq 3\)) move \(\gamma\) from \(o\) to either \(o_6\) or \(o_1\) /* See Case (A.1.2) */
6. else
7. apply a strategy similar to that in Lines 2–6 /* See Case (A.1’) */
8. else
9. (w.l.o.g. assume that \(\lambda_3\) is at most 98 edges away from \(v_1\) and \(v_2\), \(\lambda_3\) is at most 98 edges away from \(v_2\) and \(v_3\), and \(\lambda_1\) is 97 edges away from \(v_4\))
10. apply Lemma 8 /* See Case (A.2) */

calculation shows that at the end of the 98-th round, \(\lambda_3\) cannot be in \(U_4\) while at most one of \(\{\lambda_1, \lambda_3\}\) is in \(U_4\). If either \(\lambda_1\) or \(\lambda_3\) is in \(U_4\) at the end of the 98-th round, then \(\gamma\) continues moving towards \(o_4\) until she reaches \(L_{U_4,r}\) for some \(r\) depending on the relative movements of the cops; at this point, she either moves safely to \(o_4\) or deviates from her original path towards \(o_4\) and moves to either \(q_4\) and then to \(o_9\) or to \(q_3\) and then to \(o_8\).

6.2 Case (B): \(U\) contains only \(\lambda_1\) when \(d_D(\lambda_1, o) = 1\)

We split \(\gamma\)'s strategy into two main subcases: either (i) there is a corner of \(L_{U,49}\) that \(\gamma\) can reach in 98 rounds without being caught, or (ii) for every corner \(v\) of \(L_{U,49}\), at least one of the following holds: (a) at least one of \(\{\lambda_2, \lambda_3\}\) is at a distance of at most 98 from \(v\), or (b) \(\lambda_1\) is at a distance of 97 from \(v\). Each subcase is further broken into cases depending on the relative initial positions of the cops. The specific strategies used by \(\gamma\) in each subcase are similar to those in Case (A) but the details are more tedious. \(\gamma\)'s strategy in the present case is summarised in Algorithm 5.

Lemma 13 For Case (B), Algorithm 4 correctly computes a strategy for \(\gamma\) such that \(\gamma\) succeeds in Step 3 of Algorithm 2.

6.3 Case (C): \(U\) contains exactly two cops when \(d_D(\lambda_1, o) = 1\)

Without loss of generality, assume that \(\lambda_3\) is in \(U\) and \(\lambda_2\) is not in \(U\). As in Case (A), we divide \(\gamma\)'s winning strategy into two subcases depending on whether or
Algorithm 5: The Robber’s Strategy for Case (B)

Input : A game configuration \( \langle D, u_1, u_2, u_3; o \rangle \) such that \( o \) is the centre of some face \( U, \{u_2, u_3\} \cap V(U) = \emptyset \) and \( u_1 \in \{p_1, p_2, p_3\} \)

Output : A strategy for \( \gamma \)

1. \( F \leftarrow U_{10} \cup U_6 \cup U_{1} \cup U_2 \cup U_7 \);
2. if \( \exists \) a corner \( v' \in \{v_1, v_2, v_3, v_5\} \) of \( L_U,49 \) such that every cop is at least 99 edges away from \( v' \) then
3. if \( d_D(v_1, u') \geq 99 \) for all \( u' \in \{u_1, u_2, u_3\} \) then
4. if \( \lambda_2 \) and \( \lambda_3 \) are in \( F \) then
5. depending on the cops’ positions, move \( \gamma \) from \( o \) to one of \( \{v_5, v_1, v_2, v_3, q_3\} \), then apply Algorithm 3 /* See Case (B.1.1) */
6. else if neither \( \lambda_2 \) nor \( \lambda_3 \) is in \( F \) then
7. depending on the cops’ positions, move \( \gamma \) from \( o \) to one of \( \{v_1, q_1, v_3, q_3\} \), then apply Algorithm 3
8. else
9. depending on the cops’ positions, move \( \gamma \) from \( o \) to one of \( \{v_5, v_1, v_2, v_3, q_5, q_3, q_2, q_4, t_9, q_1\} \), then apply Algorithm 3 /* See Case (B.1.2) */
10. else
11. apply a strategy similar to that in Lines 3,8 /* See Cases (B.1'), (B.1'') and (B.1''') */
12. else
13. (w.l.o.g. assume that \( \lambda_2 \) is in \( U_1 \) while \( \lambda_3 \) is in \( U_3 \))
14. if at least one of \( \lambda_2, \lambda_3 \) is at most 11 edges away from \( v_3 \) (w.l.o.g. assume that \( d_D(u_3, v_3) \leq 11 \)) then
15. depending on the cops’ positions, move \( \gamma \) from \( o \) to one of \( \{m_5, v_4, q_4, m_2\} \), then apply Algorithm 3 /* See Case (B.2.1) */
16. else
17. apply a strategy similar to that in Line 15 /* See Case (B.2.2) */
not \( \gamma \) can safely reach a corner of \( L_{U,49} \) in 98 rounds. \( \gamma \)'s winning strategy is outlined in Algorithm 6.

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**Algorithm 6:** The Robber’s Strategy for Case (C)

**Input:** A game configuration \( \langle D, u_1, u_2, u_3; o \rangle \) such that \( o \) is the centre of some face \( U \), \( u_3 \in V(U) \), \( u_2 \notin V(U) \) and \( u_1 \in \{ p_1, p_2, p_3 \} \)

**Output:** A strategy for \( \gamma \)

1. \( F \leftarrow U_{10} \cup U_6 \cup U_1 \cup U_2 \cup U_7 \);
2. if \( \exists \) a corner \( v' \) of \( L_{U,49} \) such that \( \lambda_2, \lambda_3 \) are at least 99 edges away from \( v' \) and \( \lambda_1 \) is at least 98 edges away from \( v' \) then
3. if \( d_D(v_1, u') \geq 99 \) for all \( u' \in \{ u_2, u_3 \} \) and \( d_D(v_1, u_1) \geq 98 \) then
4. depending on the cops’ positions, move \( \gamma \) from \( o \) to one of \( \{ v_2, v_1, q_1, v_3, q_3, v_5, v_2 \} \), then apply Algorithm 3 or apply strategy in Line 15 of Algorithm 5 /* See Case (C.1.1) */
5. else
6. move \( \gamma \) from \( o \) to one of \( \{ o_1, o_2, o_6 \} \) or apply a variant of Lemma 8 /* See Case (C.1.2) */
7. else
8. apply a strategy similar to that in Lines 3–7 /* See Cases (C.1'), (C.1''), (C.1'''), and (C.1''''') */
9. else
10. apply strategy in Line 15 of Algorithm 5 or apply strategy in Line 17 of Algorithm 5 or apply a variant of Lemma 8 /* See Case (C.2) */

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**Lemma 14** For Case (C), Algorithm 6 correctly computes a strategy for \( \gamma \) such that \( \gamma \) succeeds in Step 3 of Algorithm 2.

This completes the analysis, showing that at least 4 cops are necessary for capturing \( \gamma \) on \( D \).

7 Concluding Remarks

The present work established separation between the classical cops-and-robbers game and the one-cop-moves game on planar graphs by exhibiting a connected planar graph whose one-cop-moves cop number exceeds the largest possible classical cop number of connected planar graphs. We believe that this result represents an important first step towards understanding the behaviour of the one-cop-moves cop number of planar graphs. It is hoped, moreover, that some of the proof techniques used in this work could be applied more generally to the one-cop-moves game played on any planar graph.

This work did not prove any upper bound for the one-cop-moves cop number of \( D \); nonetheless, we conjecture that 4 cops are sufficient for catching the robber.
on $D$. It should also be noted that the Planar Separator Theorem of Lipton and Tarjan \[12\] may be applied to show that the one-cop-moves cop number of every connected planar graph with $n$ vertices is at most $O(\sqrt{n})$ (the proof is essentially the same as that in the case of planar directed graphs; see \[13\] Theorem 4.1). It may be asked whether or not the robber has a simpler winning strategy on $D$ than that presented in this paper. We have tried a number of different approaches to the problem, but all of them led to new difficulties. For example, one might suggest reducing Case (B) to Case (C) by allowing a single cop to chase the robber in a pentagonal face $U$ until a second cop arrives in $U$. However, such a strategy would generate new cases to consider since the relative positions of the robber and cop in $U$ just before a second cop reaches $U$ may vary quite widely. Again, in order to reduce the number of cases in our proof, we have chosen to let the robber wait until a cop is exactly one edge away from her; by symmetrical considerations, it would suffice to assume that when the robber starts moving away from her current position $o$, there is exactly one cop occupying one of only three possible vertices adjacent to $o$.

One reason it is not quite so easy to design a winning strategy for the robber on $D$ is that a key lemma of Aigner and Fromme in the classical cop-and-robbers game \[2\] – that a single cop can guard all the vertices of any shortest path $P$, in the sense that after a bounded number of rounds, if the robber ever moves onto a vertex of $P$, she will be captured by the cop – carries over to the one-cop-moves game.

The question of whether or not there exists a constant $k$ such that $c_1(G) \leq k$ for all connected planar graphs $G$ \[23\] remains open. It is tempting to conjecture that such an absolute constant does exist.

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A Proof of Lemma 12

Lemma 12. For Case (A), Algorithm 1 correctly computes a strategy for $\gamma$ such that Step 3 of Algorithm 2 succeeds.

Proof.

Case (1) There is at least one corner $v'$ of $L_{49}$ such that $d_D(v', u') \geq 99$ for all $u' \in \{u_2, u_3\}$ and $d_D(v', u_1) \geq 98$. First, assume that $d_D(v_1, u') \geq 99$ for all $u' \in \{u_2, u_3\}$; since $u_1 \in \{p_1, p_2, p_3\}$, it holds that $d_D(v_1, u_1) \geq 98$.

Case (1.1): $d_D(u_2, B_6) \leq 2$ and $d_D(u_3, B_7) \leq 2$. By Lemma 1 $d_D(u_2, m_4) \geq 99$ and $d_D(u_3, m_4) \geq 99$. The vertex $p$ of $L_{49}$ that is 1 edge away from $m_4$ and closer to $v_3$ than to $v_4$ also satisfies $d_D(u_2, v') \geq 99$ and $d_D(u_3, v') \geq 99$. It follows from Lemma 8 that $\gamma$ can escape to the centre of a pentagonal face.
Case (1.2): $d_D(u_2, B_6) \geq 3$ or $d_D(u_3, B_7) \geq 3$. It suffices to assume that $d_D(u_2, B_6) \geq 3$ and $d_D(u_3, B_7) \leq 2$. $\gamma$ then moves from $o$ to $v_1$. Suppose $\lambda_3$ remains stationary during more than 2 of the cops’ turns between the round when $\gamma$ is at $o$ and the round when $\gamma$ is at $v_1$. Now, by Lemma 4, a shortest path from $u_3$ to $q_1$ either (i) goes to $B_7$ (resp. $B_6$), passes along $B_7$ (resp. $B_6$) until $v_1$ is reached, then passes along $B_1$ until $q_1$ is reached, or (ii) goes to a point $v'$ in $B_7$ (resp. $B_6$), passes along the spoke connecting $v'$ to $o_2$ (resp. $o_3$), then goes along the spoke connecting $o_2$ (resp. $o_3$) to $q_1$. Since $d_D(u_3, v_1) \geq 99$, it follows that $d_D(u_3, q_1) \geq 196$. Also, $d_D(o, v_1) + d_D(v_1, q_1) = 198$. Consequently, after reaching $v_1$, $\gamma$ can successfully reach $o_6$ via the path $v_1 \xrightarrow{B_1} q_1 \xrightarrow{B_6} o_6$, where $B$ is the spoke connecting $q_1$ to $o_6$. Now suppose $\lambda_3$ remains stationary during at most 2 of the cops’ turns between the round when $\gamma$ is at $o$ and the round when $\gamma$ is at $v_1$. Then both $\lambda_1$ and $\lambda_2$ move a total of at most 2 steps as $\gamma$ is moving from $o$ to $v_1$. After reaching $v_1$, $\gamma$ moves to $o_1$ via the spoke connecting $v_1$ to $o_1$; since all the cops are at least 99 edges away from $o_1$ during the round when $\gamma$ reaches $v_1$, $\gamma$ can reach $o_1$ without being caught.

Case (1’): For some $v'' \in \{v_2, v_3, v_4, v_5\}$, $d_D(v'', u') \geq 99$ for all $u' \in \{u_2, u_3\}$ and $d_D(v'', u_1) \geq 98$. Notice that $\gamma$’s strategies in Cases (1.1) and (1.2) still apply (with an appropriate transformation of vertices; for example, if $v'' = v_2$, then we apply the mapping $v_2 \rightarrow v_1, v_1 \rightarrow v_5, v_5 \rightarrow v_4, v_4 \rightarrow v_3, v_3 \rightarrow v_2$, and extend this mapping so as to obtain an automorphism of $D$). To adapt Case (1.1) to the present case, $\gamma$ moves to the midpoint $m'$ of $L_{49}$ such that $d_D(m', v'') = 196$; note that a winning strategy similar to that in Case (1.1) still applies here because the distance between $u_2$ and $m'$ is at least 99, which may be even greater than the distance between $u_2$ and $m_4$ in Case (1.1). To adapt Case (1.2) to the present case, $\gamma$ first moves to $v''$ (which is still possible because $d_D(v'', u') \geq 99$ for all $u' \in \{u_2, u_3\}$ and $d_D(v'', u_1) \geq 98$).

Case (2): For every corner $v'$ of $L_{49}$, there is some $u' \in \{u_2, u_3\}$ such that $d_D(u', v') \leq 98$ or $d_D(u_1, v') \leq 97$ (or both inequalities hold). Without loss of generality, assume that $d_D(u_1, v_4) = 97, d_D(u_2, v_1) \leq 98, d_D(u_2, v_3) \leq 98, d_D(u_3, v_2) \leq
98 and \( d_P(u_3, v_3) \leq 98 \). Recall that \( q \) is the vertex of \( L_{49} \) that is one edge away from \( m_5 \) and \( p \) is the vertex of \( L_{49} \) that is one edge away from \( m_4 \) (as shown in Figure 3). Note that by Lemma 4 the condition imposed on the positions of \( \lambda_2 \) and \( \lambda_3 \), and the fact that neither \( \lambda_2 \) nor \( \lambda_3 \) is at \( a \), it holds that \( d_P(u_2, B_9) \geq 99 \) and \( d_P(u_3, B_{10}) \geq 99 \).

**Case (1.1.1):** Both \( \gamma \) and \( \lambda \) move safely to either \( U \) or \( \partial U \).

Note that since neither \( \lambda_2 \) nor \( \lambda_3 \) is at \( a \), it holds that \( d_P(u_2, B_9) \geq 99 \) and \( d_P(u_3, B_{10}) \geq 99 \).

**Case (1.1.2):** Either \( \gamma \) or \( \lambda \) moves to \( U \) or \( \partial U \).

98. As was observed earlier, \((\gamma_U, B_9) = 99 \). We show that \( d_P(u_3, B_9) \geq 50 \). Take any \( x \in V(B_9) \). If \( x \) lies between \( v_4 \) and \( m_4 \) inclusive, then \( d_P(u_3, x) \geq d_P(x, v_2) - d_P(u_3, v_2) \geq 150 - 98 = 52 \). If \( x \) lies between \( p \) and \( v_3 \) inclusive, then \( d_P(u_3, x) \geq d_P(p, u_3) - d_P(p, x) \geq 99 - 49 = 50 \). Since \( d_P(u_2, B_9) + d_P(u_3, B_9) \geq 149 \), Lemma 8 shows that \( \gamma \) can reach the centre of a pentagonal face.

**Case (2.2):** \( d_P(q, u_2) \geq 99 \). One can establish in a way similar to that used in Case (2.1) the inequality \( d_P(u_2, B_{10}) \geq 48 \), so that \( d_P(u_2, B_{10}) + d_P(u_3, B_{10}) \geq 148 \). An application of Lemma 8 then gives the required result.

**Case (2.3):** \( d_P(u_2, u_3) \leq 98 \) and \( d_P(q, u_2) \leq 98 \). Then \( d_P(u_2, m_2) \geq 99 \) and \( d_P(u_3, m_2) \geq 99 \). For any \( x \in V(B_7) \), \( d_P(u_2, x) \geq d_P(x, q) - d_P(q, u_2) \geq 151 - 98 = 53 \) and \( d_P(u_3, x) \geq d_P(x, p) - d_P(p, u_3) \geq 99 - 98 = 1 \). Thus \( d_P(u_2, B_7) + d_P(u_3, B_7) \geq 104 \), and so one may conclude from Lemma 8 that \( \gamma \) can move to \( m_2 \) and safely reach the centre of a pentagonal face.

**B Proof of Lemma 13**

**Lemma 13.** For Case (B), Algorithm 5 correctly computes a strategy for \( \gamma \) such that Step 3 of Algorithm 2 succeeds.

**Proof.**

**Case (1):** There is at least one corner \( v_i \in \{ v_1, v_2, v_3, v_5 \} \) of \( L_{49} \) such that \( d_P(v_i, u') \geq 99 \) for all \( u' \in \{ v_1, u_2, u_3 \} \). We first consider the case \( i = 1 \). Define \( F := U_{10} \cup U_6 \cup U_1 \cup U_2 \cup U_7 \).

**Case (1.1):** Both \( \lambda_2 \) and \( \lambda_3 \) are currently in \( F \).

**Case (1.1.1):** \( d_P(u_2, U_4 \cup U_3) \leq 1 \) or \( d_P(u_3, U_5 \cup U_4) \leq 1 \); without loss of generality, assume that \( \lambda_2 \) is currently in \( U_7 \). If \( \lambda_3 \) is currently lying on \( B_5 \), then \( \gamma \) moves to \( v_5 \) in 98 rounds. If \( \lambda_1 \) reaches \( v_3 \) within the 98-th round, then one can apply Lemma 9 to \( U \cup U_5 \) (note that both \( \lambda_2 \) and \( \lambda_3 \) are at least 2 edges away from \( U \cup U_5 \) one turn after \( \lambda_1 \) reaches \( v_3 \)). If \( \lambda_1 \) does not reach \( v_3 \) within the 98-th round, then \( \gamma \) moves safely to \( v_5 \) in another 98 rounds. Now suppose that \( \lambda_3 \) is currently not lying on \( B_1 \). If \( \gamma \) moves to \( v_1 \) in 98 rounds. Suppose that \( \lambda_3 \) does not move on any turn as \( \gamma \) moves to \( v_1 \). Note that since neither \( \lambda_1 \) nor \( \lambda_2 \) can reach \( U_1 \cup U_2 \) in 98 rounds, \( \gamma \) will be able to move safely to either \( v_1 \) or \( v_2 \) in another 98 rounds after reaching \( v_1 \). On the other hand, suppose that \( \lambda_3 \) moves during at least one turn as \( \gamma \) moves to \( v_1 \). Note that both \( \lambda_1 \) and \( \lambda_2 \) must be at least 2 edges away from \( U_1 \cup U_2 \) one turn after \( \gamma \) reaches \( v_1 \), so that one may apply Lemma 9 to \( U_1 \cup U_2 \).
Case (1.1.2): $d_D(u_2, U_8 \cup U_9) \geq 2$ and $d_D(u_3, U_8 \cup U_9) \geq 2$.

Case (1.1.2.1): At most one of the following holds: $d_D(u_2, U_3 \cup U_8) \leq 101$, $d_D(u_3, U_3 \cup U_8) \leq 101$. $\gamma$ first moves to $v_3$ in 98 rounds. If $\lambda_1$ does not reach $B_9$ during the turn after $\gamma$ reaches $v_3$, then $\gamma$ can safely reach $o_4$ in another 98 rounds. If $\lambda_1$ does reach $B_9$ during the turn after $\gamma$ reaches $v_3$, then $\gamma$ continues moving along $B_9$ until she reaches $q_{3}$ in another 100 rounds (note that $\gamma$ can reach $q_{3}$ without being caught due to the fact that $d_D(u_2, U_8 \cup U_9) \geq 2$ and $d_D(u_3, U_3 \cup U_9) \geq 2$). Since at most one of $\lambda_2$ and $\lambda_3$ is less than 102 edges away $U_3 \cup U_8$ when $\gamma$ is at $o$, either $\gamma$ can safely move to $o_8$ using an additional 98 turns or Lemma 10 may be applied to $U_3 \cup U_4 \cup U_8$.

Case (1.1.2.2): $d_D(u_2, U_3 \cup U_8) \leq 101$ and $d_D(u_3, U_3 \cup U_8) \leq 101$. $\gamma$ moves to $v_5$ in 98 rounds; Lemma 9 can then be applied to $U_1 \cup U_5$.

Case (1.2): Exactly one of $\lambda_2$ and $\lambda_3$ is currently in $F$. Suppose that $\lambda_2$ is currently in $F$ and $\lambda_3$ is currently not in $F$.

Case (1.2.1): $d_D(u_3, U_1 \cup U_2) \leq 99$. First, assume that $\lambda_3$ is currently in $U_3$. Suppose that $d_D(u_2, U_1) \geq 3$. Then $\gamma$ can safely reach $v_5$ in 98 rounds. If $\lambda_1$ reaches a vertex that is at most 1 edge away from $U_5$ when $\gamma$ reaches $v_5$, then $\lambda_3$ is still at least 1 edge away from $U_1$ when $\gamma$ is at $v_5$, and therefore $\gamma$ can safely reach either $o_1$ or $o_5$. If $\lambda_1$ moves at most 95 steps as $\gamma$ is moving towards $v_5$, then $\lambda_1$ and $\lambda_3$ are both still at least 2 edges away from $U_1 \cup U_5$ when $\gamma$ is at $v_5$, and so Lemma 9 can be applied to $U_1 \cup U_5$. For the rest of the present case, it will be assumed that $d_D(u_2, U_1) \leq 2$.

Case (1.2.1.1): $d_D(u_2, U_6) \leq 39$. Then $\gamma$ can safely reach $v_5$ in 98 rounds. Suppose $\lambda_2$ is not at $q_5$ during the turn after $\gamma$ reaches $v_5$. If $\lambda_1$ does not reach $v_4$ during the turn after $\gamma$ reaches $v_5$ or if $\lambda_2$ is in $U_5$ but not in $U_1$, then $\gamma$ can simply move to either $o_5$ or $o_1$ in another 98 rounds without being caught. Suppose $\lambda_1$ does reach $v_4$ and $\lambda_2$ is in $U_1$ during the turn after $\gamma$ reaches $v_5$. Notice that in this case, $\lambda_2$ can move at most 1 step as $\gamma$ moves from $o$ to $v_5$. $\gamma$ then starts moving towards $o_5$ until she reaches $L_{U_5-2}$ (see Figure 9). If, during the turn when $\gamma$ reaches $L_{U_5-2}$, $\lambda_1$ is at least 2 vertices farther away from $o_5$ than $\gamma$ is, then $\gamma$ can reach $o_5$ without being caught in another 4 rounds. Otherwise, $\gamma$ moves towards $z_3$ via the path highlighted in Figure 9 in 100 rounds.

Suppose $\gamma$ is now at $z_3$. Moreover, suppose that as $\gamma$ moves from $t_1$ to $z_3$ (see Figure 9), $\lambda_2$ moves $\ell$ steps towards $o_{10}$, so that at the end of the round when $\gamma$ reaches $z_3$, $\lambda_2$ is at most $98 - \ell$ edges away from $o_{10}$. Now consider the following case distinction.

First, suppose that $\ell \geq 43$. If $\lambda_2$ can reach a neighbour of $v'$ of $q_5$ in another 47 rounds, then, since $\lambda_2$ started at a vertex that is at least 60 edges away from $q_5$ when $\gamma$ started moving from $t_1$ to $z_3$, $\lambda_2$ must have moved a total of at least $43 + 12 = 55$ steps between the round when $\gamma$ was at $t_1$ and the round when $\gamma$ reached $z_3$. This means that $\lambda_1$ could have moved at most 45 steps between the same two rounds. Notice that $d_D(t_2, U_5) = 100$. Therefore $\lambda_1$ must be at least 55 edges away from $U_5$ during the turn after $\gamma$ reaches $z_3$. $\gamma$ can thus safely move
to \(z_2\) in 53 rounds, and thence to \(o_9\) in another 98 rounds. Suppose now that \(\lambda_2\) cannot reach any neighbour of \(q_5\) in 47 rounds. If \(\lambda_1\) can reach a neighbour of \(q_5\) in 47 rounds, then \(\lambda_1\) is currently at least 93 edges away from \(U_9\). \(\gamma\) can then move to \(z_2\) in 53 rounds, and thence to \(o_9\) in another 98 rounds. Suppose, on the other hand, that \(\lambda_1\) cannot reach any neighbour of \(q_5\) in 47 rounds. \(\gamma\) may then safely reach \(q_5\) in 47 rounds. An application of Lemma 10 shows that \(\gamma\) can thus either (i) safely reach \(o_{10}, o_1\) or \(o_5\), or (ii) oscillate infinitely often between \(q_5\) and an adjacent vertex in \(U_1 \cup U_5 \cup U_{10}\).

Second, suppose \(\ell \leq 42\). Suppose \(\lambda_2\) moves at least 12 steps towards \(q_5\) during the sequence of turns when \(\gamma\) moves from \(t_1\) to \(z_3\), so that \(d_{D}(u_2, q_5) \leq 47\). Notice that in this case, as \(\gamma\) moves from \(t_1\) to \(z_3\), \(\lambda_1\) could have moved at most \(100 - \ell - 12 = 88 - \ell\) steps. \(\gamma\) first moves to \(m_6\) in 3 rounds, and then she starts moving towards \(o_{10}\). If \(\lambda_2\) skips more than \(\ell + 3\) turns or stops moving towards \(o_{10}\) as \(\gamma\) is moving from \(m_6\) to \(o_{10}\), then \(\gamma\) can reach \(o_{10}\) without being caught. Suppose that for some \(j \leq \ell + 3\), \(\lambda_2\) moves towards \(o_{10}\), skipping \(j\) turns as \(\gamma\) is moving from \(z_3\) to \(m_6\) and then to \(o_{10}\). \(\gamma\) starts moving from \(m_6\) to \(o_{10}\) until she reaches \(L_{U_{10}, 7 + \ell - j}\). (In other words, for each subsequent turn that \(\lambda_2\) skips, \(\gamma\) moves another 2 steps towards \(o_{10}\) after reaching \(L_{U_{10}, 7 + \ell - j}\).) After reaching \(L_{U_{10}, 7 + \ell - j}\), \(\gamma\) continues along the path shown in Figure 10 until she reaches \(z_1\). The total distance covered by \(\gamma\) in moving from \(t_3\) to \(z_1\) via the path highlighted in Figure 10 is \(108 - j + \ell\), and \(\gamma\) can safely reach \(z_1\) due to the following inequalities:

\[
108 - j + \ell < 111 + \ell - j
\]
\[
108 - j + \ell < 109 + \ell - j;
\]

note that the minimum number of turns required by \(\lambda_1\) to reach \(z_1\) after \(\gamma\) starts moving from \(t_3\) to \(z_1\) is bounded below by \(111 + \ell - j\), while the minimum number of turns required by \(\lambda_2\) to reach \(z_1\) after \(\gamma\) starts moving from \(t_3\) to \(z_1\) is bounded below by \(109 + \ell - j\) (given the present case constraints). Observe that the total distance covered by \(\gamma\) in moving from \(t_1\) to \(z_1\) is \(297 - \ell + j\), while the total number of turns needed by \(\lambda_1\) and \(\lambda_2\) to reach \(U_9\) and \(U_{11}\) respectively (under the present case constraints) after \(\gamma\) starts moving away from \(t_1\) is 308. \(\gamma\) may thus safely reach either \(o_{11}\) or \(o_9\) in another 98 rounds.

Now, under the assumption that \(\ell \leq 42\), suppose that the distance between \(\lambda_2\) and \(q_5\) is more than \(47\) after \(\gamma\) reaches \(z_3\). Suppose \(\lambda_1\) can reach a neighbour of \(q_5\) in another 47 rounds. A direct calculation gives that \(\lambda_1\) must currently be at least 55 edges away from \(z_2\). As in the previous case, \(\gamma\) moves from \(z_3\) to \(m_6\) in 3 rounds and starts moving towards \(o_{10}\). she can then either safely reach \(o_{10}\), or reach \(t_3\), a vertex in \(L_{U_{10}, 7 + \ell - 1}\), and then move from \(t_3\) to \(z_1\) via the path highlighted in Figure 10. Notice that in the case where \(\gamma\) moves to \(z_1\), \(\gamma\) can avoid being caught by \(\lambda_1\) due to the inequality

\[
108 - j + \ell \leq 150 - j < 152 - j,
\]

where the minimum number of turns required by \(\lambda_1\) to reach \(z_1\) after \(\gamma\) starts moving from \(t_3\) to \(z_1\) is bounded below by \(152 - j\) (given the present case constraints).
constraints). After reaching $z_1$, one can argue as before that $\gamma$ can safely move to either $o_{11}$ or $o_9$ in another 98 rounds. Next, suppose $\lambda_1$ cannot reach any neighbour of $q_5$ in 47 rounds after $\gamma$ reaches $z_3$. $\gamma$ then moves to $q_5$ in another 47 rounds, and Lemma 10 may be applied to $U_{10} \cup U_1 \cup U_5$.

Fig. 9. The first escape path of $\gamma$ in Case (1.2.1.1). $m_6$ is the midpoint of the path connecting $z_2$ and $q_5$.

Fig. 10. The second escape path of $\gamma$ in Case (1.2.1.1). $m_6$ is the midpoint of the path connecting $z_2$ and $q_5$.

Case (1.2.1.2): $d_D(u_2, U_6) \geq 40$. Let $\ell' = d_D(u_2, U_5)$ and $d' = d_D(u_3, v_3)$.

Case (1.2.1.2.1): $\ell' + d' \leq 97$. $\gamma$ first moves to $v_1$ in 98 rounds. Note that $d_D(u_3, U_2) \geq 100 - d'$ and $d_D(u_2, q_1) \geq 196 - \ell'$. Since the total distance traversed by $\gamma$ in moving from $o$ to $q_1$ is 198 but $d_D(u_3, U_2) + d_D(u_2, q_1) \geq 100 - d' + 196 - \ell' \geq 296 - 97 = 199$, $\gamma$ can either safely reach $o_2$ in another 98 rounds, or reach $q_1$
in another 100 rounds without being caught. Suppose \( \gamma \) can reach \( q_1 \) without being caught. Since 
\[
d_P(u_2, U_6) + d_P(u_3, U_2) + (d_P(u_1, U_1 \cup U_2 \cup U_6) - 1) \geq 100 - d' + 100 - \ell' + 98 \geq 298 - 97 = 201,
\]
which is greater than the total distance traversed by \( \gamma \) in moving from \( o \) to \( q_1 \), at least one of the following must hold after \( \gamma \) reaches \( q_1 \): (a) \( \gamma \) can safely reach \( o_6 \) in another 98 rounds; (b) \( \gamma \) can safely reach \( o_2 \) in another 98 rounds; (c) there are at most 2 cops in \( U_1 \cup U_2 \cup U_6 \), each of which is at least 3 edges away from \( \gamma \), and the third cop is at least 2 edges away from \( U_1 \cup U_2 \cup U_6 \). In Case (c), one may apply Lemma 10 to \( U_1 \cup U_2 \cup U_6 \).

**Case (1.2.1.2.2):** \( \ell' + d' \geq 98 \). 
First, suppose \( d' \geq 12 \). Suppose \( d_P(u_1, v_4) > 97 \). \( \gamma \) first moves to \( v_4 \) in 98 rounds, and then moves to \( q_4 \) in another 100 rounds. Note that 
\[
d_P(u_2, U_9) + d_P(u_3, U_4) + (d_P(u_1, U_5 \cup U_4 \cup U_9) - 1) \geq \ell' + 100 + d' + 96 \geq 294,
\]
and the latter quantity is greater than 198, the distance traversed by \( \gamma \) in moving from \( o \) to \( v_4 \) and then to \( q_4 \). Thus after \( \gamma \) reaches \( q_4 \), \( \gamma \) can either safely reach \( o_6 \) in another 98 rounds, or Lemma 10 may be applied to \( U_5 \cup U_9 \cup U_4 \).

Next, suppose \( d_P(u_1, v_4) > 97 \). \( \gamma \) starts by moving towards \( m_4 \). Suppose \( \lambda_1 \) moves at least 48 steps as \( \gamma \) is moving towards \( m_4 \). This means that \( \lambda_2 \) and \( \lambda_3 \) can move a total of at most 50 steps as \( \gamma \) is moving towards \( m_4 \), and therefore \( \gamma \) can safely reach \( m_4 \). Suppose \( \lambda_3 \) is the first cop to reach \( U_4 \), and that \( \lambda_3 \) is \( \ell \) vertices closer to \( o_4 \) than \( \gamma \) is after \( \gamma \) reaches \( m_4 \). Note that \( \ell \leq 38 \). \( \gamma \) then starts moving towards \( o_4 \). Suppose that as \( \gamma \) approaches \( o_4 \), \( \lambda_2 \) skips \( j \) turns. If \( j > \ell \), then \( \gamma \) can reach \( o_4 \) without being caught. Now assume that \( j \leq \ell \). \( \gamma \) moves towards \( o_4 \) until she reaches \( L_{U_4,4+\ell-j} \). She then continues moving along the path highlighted in Figure 11 until she reaches \( t_8 \). Furthermore, \( \gamma \) can safely reach \( t_8 \) due to the following inequalities:

\[
101 + (4 + \ell - j) < \ell - j + 100 + 49 - (4 + \ell - j) \\
3(4 + \ell - j) + 3 < 2(4 + \ell - j) - \ell + j + 2(4 + \ell - j);
\]

note that \( 101 + (4 + \ell - j) \) is the distance traversed by \( \gamma \) in moving from \( t_5 \) to \( t_8 \), \( \ell - j + 100 + 49 - (4 + \ell - j) \) is a lower bound on the distance between \( \lambda_2 \) and \( t_8 \) after \( \gamma \) reaches \( t_5 \), \( 3(4 + \ell - j) + 3 \) is the distance traversed by \( \gamma \) in moving from \( t_5 \) to \( t_7 \), and \( 2(4 + \ell - j) - \ell + j + 2(4 + \ell - j) \) is a lower bound on the distance between \( \lambda_3 \) and \( t_7 \) after \( \gamma \) reaches \( t_5 \). In addition, note that the total distance traversed by \( \gamma \) in moving from \( o \) to \( t_8 \) is \( 98 + 196 - 4(4 + \ell - j) + 3(4 + \ell - j) + 3 + 51 + (4 + \ell - j), \) which is smaller than the sum of \( d_P(u_2, U_9), d_P(u_3, U_4) \) and \( d_P(u_1, U_4) - 1 \) — this means that after reaching \( t_8 \), \( \gamma \) can either safely reach \( o_9 \) in another 98 rounds, or Lemma 10 may be applied to \( U_4 \cup U_9 \cup U_5 \).

Now suppose that as \( \gamma \) is approaching \( m_4 \), \( \lambda_1 \) moves \( z \) steps for some \( z \leq 47 \). We distinguish two cases: (i) \( d_P(u_2, v_5) \geq 3 \) and (ii) \( d_P(u_2, v_5) \leq 2 \).
(i) \( \gamma \) then moves to \( L_{U,z+2} \), continuing along the side path of \( L_{U,z+2} \) parallel \( B_9 \) until she reaches the corner of \( L_{U,z+2} \) that is \( 98 - 2(z + 2) \) edges away from \( v_4 \). \( \gamma \) now moves to \( v_4 \) (observe that because \( d_P(u_2, v_5) \geq 3 \), \( \lambda_2 \) cannot reach \( v_4 \) during the turn after \( \gamma \) reaches \( v_4 \) and then to \( q_4 \) in another 198 \(-2(z+2) \) rounds. One can argue as in the preceding case that either \( \gamma \) can safely reach \( o_9 \) in another 98 rounds or Lemma 10 may be applied to \( U_5 \cup U_4 \cup U_9 \).
(ii) Since $\ell' + d' \geq 98$, one has that $d' \geq 96$. $\gamma$ first moves to $m_4$ in 98 rounds. Note that either $\lambda_1$ or $\lambda_3$ must reach $U_4$ just after $\gamma$ reaches $m_4$; otherwise, $\gamma$ can safely reach $o_4$ in another 98 rounds.

If $\lambda_1$ reaches $U_4$ first, then $\lambda_3$ and $\lambda_2$ could have moved a total of at most 1 step. $\gamma$ continues moving towards $o_4$ until she reaches $L_{U_4,20}$. $\gamma$ then moves along the path highlighted in Figure 12. After $\gamma$ reaches $t_{15}$, she can either safely reach $o_8$ in another 98 rounds, or she can continue moving along the side path of $U_4$ containing $t_{15}$ and $t_9$ until she reaches $t_9$. In the latter case, Lemma 10 may be applied to $U_4 \cup U_9 \cup U_8$.

If $\lambda_3$ reaches $U_4$ first, then $\lambda_1$ and $\lambda_2$ could have moved a total of at most 3 steps. Thus $\gamma$ can move to $v_4$ in 50 rounds, then to $q_4$ in another 100 rounds, and finally move safely to $o_9$ using an additional 98 turns.

Fig. 11. An escape path of $\gamma$ in Case (1.2.1.2).

Fig. 12. An escape path of $\gamma$ in Case (1.2.1.2).
Second, suppose $d' \leq 11$. We distinguish two subcases: (a) $d_D(u_2, q_1) \geq 98$ and (b) $d_D(u_2, q_1) \leq 97$.

(a): $\gamma$ first moves to $v_1$ in 98 rounds. $\gamma$ then approaches $o_2$ until she reaches $L_{U_{2,21}}$, and continues along the path highlighted in Figure 13. Note that $\lambda_3$ may skip at most 9 rounds as $\gamma$ is approaching $o_2$; otherwise, $\gamma$ can safely reach $o_2$. The total distance traversed by $\gamma$ in moving from $t_{10}$ to $t_{12}$ is 172; the minimum distance between $\lambda_3$ and $m_7$ when $\gamma$ is at $t_{10}$ is at least 139, while the distance between $\lambda_3$ and $U_7$ when $\gamma$ is at $t_{10}$ is at least 89, and the distance between $\lambda_2$ and $U_6$ when $\gamma$ is at $o$ is at least 40 (under the present case constraints). One can now verify that at least one of the following must hold after $\gamma$ reaches $t_{12}$: (i) $\gamma$ can safely reach $o_6$ or $o_7$, or (ii) Lemma [9] may be applied to $U_6 \cup U_7$.

(b): If $d_D(u_2, q_1) \leq 97$, then $d_D(u_2, v_5) \geq 99$ and $d_D(u_2, U_5) \geq 99$. $\gamma$ can then employ the winning strategy used in Case (1.2.1.1).

Fig. 13. An escape path of $\gamma$ in Case (1.2.1.2). $m_7$ is the midpoint of the path in $U_2$ connecting $q_1$ and the vertex intersecting $U_6, U_2$ and $U_7$.

To conclude the proof for Case (1.2.1), observe that if $\lambda_3$ is in $U_5$ when $\gamma$ is at $o$, then $\gamma$ may adopt a winning strategy symmetrical to that for the case when $\lambda_3$’s starting position is in $U_3$; the main difference is that instead of moving to $v_5$ and $v_4$, $\gamma$ moves to $v_2$ and $v_3$ respectively.

**Case (1.2.2):** $d_D(u_3, U_1 \cup U_2) \geq 100$. First, suppose that $\lambda_2$ is currently not lying on $B_1$. Then $\gamma$ moves to $v_1$ in 98 rounds. If $\lambda_2$ moves during some turn as $\gamma$ is moving towards $v_1$, then both $\lambda_1$ and $\lambda_3$ must be at least 2 edges away from $U_1 \cup U_2$ when $\gamma$ reaches $v_1$, and therefore Lemma [9] may be applied to $U_1 \cup U_2$. If $\lambda_2$ does not move during any turn as $\gamma$ is moving towards $v_1$, then $\gamma$ can safely reach either $o_1$ or $o_2$ in another 98 rounds after reaching $v_1$.

Second, suppose that $\lambda_2$ is currently lying on $B_1$. If $\lambda_3$ is lying in $U_{11}$, then $\gamma$ can safely reach $v_3$ in 98 rounds. $\gamma$ can then move to either $o_4$ or $o_3$ without being caught.
Now suppose that $\lambda_3$ is lying in $U_5 \cup U_9 \cup U_4 \cup U_8 \cup U_3$. Assume that $d_D(u_3, U_2 \cup U_3 \cup U_7) \geq 50$ and $d_D(u_3, U_7) \geq 196$. $\gamma$ now moves to $v_2$ in 98 rounds. If neither $\lambda_1$ nor $\lambda_3$ is in $U_3$ during the turn after $\gamma$ reaches $v_2$, then $\gamma$ can safely move along $B_2$ until she reaches $q_2$ in 100 rounds. Note that at most one of $\lambda_1$, $\lambda_3$ can reach $U_3$ during the turn after $\gamma$ reaches $v_2$. Furthermore, owing to the assumption that $d_D(u_3, U_7) \geq 196$, $\gamma$ can safely move along $B_2$ until she reaches $q_2$ in 100 rounds. Note that when $\gamma$ is at $o$, the cop nearest to $U_7$, $\lambda_2$, is 100 edges away from $U_7$; $\lambda_1$ is 99 edges away from $U_2 \cup U_3 \cup U_7$, while $\lambda_3$ needs at least 50 rounds to reach $U_2 \cup U_3 \cup U_7$. Hence after $\gamma$ reaches $q_2$, either $\gamma$ can safely reach $o_7$ using an additional 98 turns, or Lemma 10 may be applied to $U_2 \cup U_3 \cup U_7$.

Now if $d_D(u_3, U_1 \cup U_5 \cup U_{10}) \geq 50$ and $d_D(u_3, U_{10}) \geq 196$, then an argument parallel to that in the preceding paragraph applies. In this case, $\gamma$ should first move to $v_5$; she can then safely reach either $o_5$ or $q_5$.

**Case (1.3):** Neither $\lambda_2$ nor $\lambda_3$ is currently in $F$. We distinguish two subcases:

(a) $d_D(u_2, U_1 \cup U_2) \geq 2 \lor d_D(u_3, U_1 \cup U_2) \geq 2$ and (b) $d_D(u_2, U_1 \cup U_2) \leq 1 \land d_D(u_3, U_1 \cup U_2) \leq 1$.

(a): Suppose $d_D(u_2, U_1 \cup U_2) \geq 2$. $\gamma$ first moves to $v_1$ in 98 rounds. After $\gamma$ reaches $v_1$, the assumption $d_D(u_2, U_1 \cup U_2) \geq 2$ implies that at least one of the following must hold: (i) $\gamma$ can safely reach $q_1$ in another 100 rounds, or (ii) $\gamma$ can safely reach either $o_1$ or $o_2$ in another 98 rounds. If (i) holds, then Lemma 10 may be applied to $U_6 \cup U_1 \cup U_2$.

(b): Notice that in this case, $d_D(v_3, u') \geq 99$ for all $u' \in \{u_1, u_2, u_3\}$. $\lambda_1$ first moves to $v_3$ in 98 rounds. If $\lambda_1$ fails to reach $U_4$ during the turn after $\gamma$ reaches $v_3$, then $\gamma$ can safely reach $o_1$ in another 98 rounds. If $\lambda_1$ does reach $v_4$ during the turn after $\gamma$ reaches $v_3$, then $\gamma$ can continue moving along $B_3$ until she reaches $q_3$ in another 100 rounds. Now at least one of the following must hold: (i) $\gamma$ can safely reach $o_8$ in another 98 rounds, or (ii) Lemma 10 may be applied to $U_4 \cup U_3 \cup U_8$.

Second, suppose that both $\lambda_2$ and $\lambda_3$ are in $U_3$. $\gamma$ can first move to $v_1$ in 98 rounds, and then safely reach $o_1$ in another 98 rounds.

To finish the proof for Case (1), we describe how to extend $\gamma$’s winning strategy to handle the cases $i = 2$, $i = 3$ and $i = 5$.

**Case (1'):** $d_D(u_2, u') \geq 99$ for all $u' \in \{u_1, u_2, u_3\}$. Notice that this case is entirely symmetrical to the case that $d_D(u_1, u') \geq 99$ for all $u' \in \{u_1, u_2, u_3\}$. $\lambda_1$ is 1 edge away from $o$, and either (i) $d_D(u_1, m_5) \leq 97$, or (ii) $d_D(u_1, m') \leq 97$, where $m'$ is the vertex on $L_{196, 49}$ that is 1 edge away from $m_5$ and between $m_5$ and $v_4$, or (iii) $d_D(u_1, v_4) \leq 97$. Furthermore, observe that $\gamma$’s strategies in Cases (1.1), (1.2) and (1.3) apply to the latter case as well (that is, even if $\lambda_1$ can reach either $m_3$, $m'$ or $v_4$ in 97 rounds and $d_D(v_1, u') \geq 99$ for all $u' \in \{u_1, u_2, u_3\}$.)

**Case (1''):** $d_D(v_3, u') \geq 99$ for all $u' \in \{u_1, u_2, u_3\}$, and that there are $u', v' \in \{u_1, u_2, u_3\}$ such that $d_D(u', v_1) \leq 98$ and $d_D(v', v_2) \leq 98$. Define $F := U_1 \cup U_2 \cup U_{10} \cup U_6 \cup U_7$ as before. We shall follow the proof for the case $i = 1$, dividing the strategy into the subcases (1.1') both $\lambda_2$ and $\lambda_3$ are currently in $F$ and (1.2') exactly one of $\lambda_2$ and $\lambda_3$ is currently in $F$. Note that since there are
If \( d_\gamma \) is currently either in \( U_5 \) or in \( U_3 \).

First, suppose that \( \lambda_3 \) is currently in \( U_5 \). Since there are \( u', v' \in \{u_1, u_2, u_3\} \) such that \( d_P(u', v_1) \leq 98 \) and \( d_P(v', v_2) \leq 98 \), at least one of the cops \( \lambda_2, \lambda_3 \) must currently be in \( F \). Since the proofs that \( \gamma \)’s strategies in Cases (1.1’’') and (1.2’’’) succeed are so similar to those of cases (1.1) and (1.2), many proof details will be omitted.

**Case (1.1’’):** Both \( \lambda_2 \) and \( \lambda_3 \) are currently in \( F \).

**Case (1.1.1’’):** Exactly one of \( \lambda_2 \) and \( \lambda_3 \), say \( \lambda_2 \), is at most 1 edge away from \( U_8 \cup U_9 \).

If \( \lambda_2 \) is currently in \( U_7 \), then \( \lambda_3 \) must be in \( U_2 \) (so that \( d_P(u_3, v_1), d_P(u_3, v_2) \leq 98 \)). \( \gamma \) first moves to \( v_5 \) in 98 rounds. If \( \lambda_1 \) reaches \( v_4 \) during the turn after \( \gamma \) reaches \( v_5 \), then \( \gamma \) continues moving along \( B_3 \) until she reaches \( q_5 \) using a further 100 rounds. \( \gamma \) can then safely reach \( o_5 \) in another 98 rounds. If \( \lambda_1 \) does not reach \( v_4 \) during the turn after \( \gamma \) reaches \( v_5 \), then \( \gamma \) can safely move to \( o_5 \) in another 98 rounds.

If \( \lambda_2 \) is currently in \( U_{10} \), then \( \lambda_3 \) must again be in \( U_2 \). \( \gamma \) first moves to \( v_3 \) in 98 rounds. If \( \lambda_1 \) does not reach \( B_9 \) just after \( \gamma \) reaches \( v_3 \), then \( \gamma \) can safely move to \( o_4 \) using another 98 turns. If \( \lambda_1 \) does reach \( B_9 \) just after \( \gamma \) reaches \( v_3 \), then Lemma \( \square \) may be applied to \( U \cup U_3 \cup U_4 \).

**Case (1.1.2’’):** Both \( \lambda_2 \) and \( \lambda_3 \) are at least 2 edges away from \( U_8 \cup U_9 \). \( \gamma \) employs the strategies in Cases (1.1.2.1) and (1.1.2.2) (these strategies are valid in the present case because they do not involve \( \gamma \) moving to \( v_1 \)).

**Case (1.2’’):** Exactly one of \( \lambda_2 \) and \( \lambda_3 \) is currently in \( F \). Suppose that \( \lambda_2 \) is currently in \( F \) and \( \lambda_3 \) is currently not in \( F \).

**Case (1.2.1’’):** \( d_P(u_3, U_1 \cup U_2) \leq 99 \). Then \( \lambda_3 \) is currently either in \( U_5 \) or in \( U_3 \).

First, suppose that \( \lambda_3 \) is currently in \( U_5 \). Since there are \( u', v' \in \{u_1, u_2, u_3\} \) such that \( d_P(u', v_1) \leq 98 \) and \( d_P(v', v_2) \leq 98 \), \( \lambda_2 \) must currently be in \( U_2 \). \( \gamma \) first moves to \( v_3 \) in 98 rounds. If \( \lambda_2 \) does not reach \( U_3 \) (using up at least 4 turns in the process) before or just after \( \gamma \) reaches \( v_3 \), then \( \gamma \) can safely reach \( o_3 \) in another 98 rounds. If \( \lambda_2 \) does use up at least 4 turns to move to \( U_3 \) as \( \gamma \) is approaching \( v_3 \), then \( \gamma \) can move unobstructed along \( B_3 \) until she reaches \( q_3 \) using another 100 turns. After \( \gamma \) reaches \( q_3 \), either \( \gamma \) can safely move to \( o_5 \) in another 98 rounds, or Lemma \( \square \) may be applied to \( U_1 \cup U_3 \cup U_8 \).

Second, suppose that \( \lambda_3 \) is currently in \( U_3 \). Since \( d_P(v_3, u') \geq 99 \) for all \( u' \in \{u_1, u_2, u_3\} \), \( \gamma \)’s winning strategy in Case (1.2.1.2.2) when \( d' \geq 12 \) may be applied to the present case (notice that \( \gamma \)’s winning strategy in Case (1.2.1.2.2) when \( d' \geq 12 \) does not involve \( \gamma \) moving to \( v_1 \)).

**Case (1.2.2’’):** \( d_P(u_3, U_1 \cup U_2) \geq 100 \). Since \( d_P(u_3, v_2) \geq 99 \), \( \lambda_2 \) must lie in \( U_2 \) (so that \( d_P(u_2, v_1), d_P(u_2, v_2) \leq 98 \)). Moreover, note that \( d_P(u_2, U_1) \geq 4 \) and \( d_P(u_2, U_3) \geq 4 \).

First, suppose that \( d_P(u_3, q_5) \geq 196 \) and \( d_P(u_3, U_{10}) \geq 99 \). \( \gamma \) first moves to \( v_5 \) in 98 rounds. If, during the turn after \( \gamma \) reaches \( v_5 \), \( \lambda_2 \) does not reach \( U_1 \), then \( \gamma \) can safely reach \( o_1 \) in another 98 rounds. Suppose that \( \lambda_2 \) uses at least 4 turns to reach \( U_1 \). \( \gamma \) then continues moving along \( B_5 \) until she reaches \( q_5 \) in another 100
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195, then $d_\gamma$ used up at least 4 turns as $\gamma$ moved from $o$ to $v_5$. After $\gamma$ reaches $q_5$, one can directly verify that either $\gamma$ can safely reach $o_8$ in another 98 rounds, or Lemma 10 may be applied to $U_1 \cup U_{10} \cup U_5$.

Second, suppose that either $d_\gamma(u_3, q_5) \leq 195$ or $d_\gamma(u_3, U_{10}) \leq 98$. If $d_\gamma(u_3, q_5) \leq 195$, then $d_\gamma(u_3, U_8) \geq 99$. If $d_\gamma(u_3, U_{10}) \leq 98$, then either $d_\gamma(u_3, U_4) \geq 98$ or $d_\gamma(u_3, U_8) \geq 98$. Thus in the present case, either $d_\gamma(u_3, U_4) \geq 98$ or $d_\gamma(u_3, U_8) \geq 98$ holds. Furthermore, $\lambda_3$ cannot lie in $U_4$ when $\gamma$ is at $o$. $\gamma$ first moves to $v_3$ in 98 rounds.

Suppose $d_\gamma(u_3, U_4) \geq 98$. If $\lambda_2$ does not reach $U_3$ during the turn after $\gamma$ reaches $v_3$, then $\gamma$ can safely reach $o_3$ using another 98 turns. Suppose $\lambda_2$ does use up at least 4 turns to reach $U_3$ before or just after $\gamma$ reaches $v_3$. Then neither $\lambda_1$ nor $\lambda_3$ can reach $U_4$ during the turn after $\gamma$ reaches $v_3$, and therefore $\gamma$ can safely move to $o_4$ in another 98 rounds.

Suppose $d_\gamma(u_3, U_8) \geq 98$. If $\lambda_2$ does not reach $U_3$ during the turn after $\gamma$ reaches $v_3$, then $\gamma$ can safely reach $o_8$ using another 98 turns. Suppose $\lambda_2$ does use up at least 4 turns to reach $U_3$ before or just after $\gamma$ reaches $v_3$. $\gamma$ moves along $B_3$ until she reaches $q_3$ in another 100 rounds. Note that since $\lambda_3$ is not in $U_4$ when $\gamma$ is at $o$ and $d_\gamma(u_3, U_8) \geq 98$, $\lambda_3$ cannot catch $\gamma$ before or just after $\gamma$ reaches $q_3$. After $\gamma$ reaches $q_3$, one can verify that either $\gamma$ can safely reach $o_8$ using another 98 turns, or Lemma 10 may be applied to $U_4 \cup U_3 \cup U_8$.

**Case (1′):** $d_\gamma(v_5, u') \geq 99$ for all $u' \in \{u_1, u_2, u_3\}$ while there are $u', v', w' \in \{u_1, u_2, u_3\}$ such that $d_\gamma(u', v_1), d_\gamma(w', v_2), d_\gamma(w', v_3) \leq 98$.

Notice that this case is entirely symmetrical to the case that $d_\gamma(v_3, u') \geq 99$ for all $u' \in \{u_1, u_2, u_3\}$, there are $u', v', w' \in \{u_1, u_2, u_3\}$ such that $d_\gamma(u', v_1), d_\gamma(u', v_2), d_\gamma(w', v_5) \leq 98$, $\lambda_1$ is 1 edge away from $o$, and either (i) $d_\gamma(u_1, m_5) \leq 97$, or (ii) $d_\gamma(u_1, m') \leq 98$, where $m'$ is the vertex that is 1 edge away from $m_5$ and between $m_5$ and $v_1$ in 98 rounds, or (iii) $d_\gamma(u_1, v_4) \leq 98$. Furthermore, it may be directly verified that $\gamma's strategies in Cases (1.1′) and (1.1′) may be applied to the latter case.

**Case (2):** For every corner $v' \in \{v_1, v_2, v_3, v_5\}$ of $L_{49}$, there is some $u' \in \{u_2, u_3\}$ such that $d_\gamma(v', u') \leq 98$. Then there is exactly one cop, say $\lambda_2$, that is in $U_1$, while exactly one other cop, say $\lambda_3$, is in $U_3$.

**Case (2.1):** $d_\gamma(u_3, v_3) \leq 11$. The following two cases are distinguished.

**Case (2.1.1):** $d_\gamma(u_2, v_5) \geq 12$. $\gamma$ begins moving towards $m_5$. We further distinguish two cases.

**Case (2.1.1.1):** $\lambda_1$ moves at least 47 steps as $\gamma$ is moving towards $m_5$. Suppose that as $\gamma$ is approaching $m_5$, $\lambda_1$ moves $z$ steps for some $z \geq 47$. $\gamma$ then continues moving until she reaches $m_5$ in 98 rounds. Note that $\lambda_2$ and $\lambda_3$ can move a total of at most 51 steps between the turn $\gamma$ moves away from $o$ and the turn after $\gamma$ reaches $m_5$. So $\lambda_3$ is at most 39 vertices closer to $o_5$ than $\gamma$ is after $\gamma$ reaches $m_5$. We may assume that at least one of $\lambda_1, \lambda_2$ reaches $U_5$ just after $\gamma$ reaches $m_5$ (otherwise, $\gamma$ can safely reach $o_5$ in another 98 rounds).
Case (2.1.1.1.1): $\lambda_1$ reaches $U_5$ before $\lambda_2$. Note that $\lambda_2$ is still at least 11 edges away from $U_5$ just after $\gamma$ reaches $m_5$. $\gamma$ starts by moving towards $o_5$ until she reaches $L_{U_5,4}$; $\gamma$ then moves along the path highlighted in Figure 14. An argument very similar to those used in earlier cases shows that either $\gamma$ can move to $o_10$ without being caught after reaching $t_{19}$, or $\gamma$ can continue moving until she safely reaches $z_2$, at which point Lemma 10 may be applied to $U_{10} \cup U_5 \cup U_9$.

Case (2.1.1.1.2): $\lambda_2$ reaches $U_5$ before $\lambda_1$. Suppose that $\lambda_2$ is $\ell$ vertices closer to $o_5$ than $\gamma$ is during the turn after $\gamma$ reaches $m_5$. Note that $\ell \leq 39$. $\gamma$ starts by moving towards $o_5$. Suppose that as $\gamma$ is approaching $o_5$, $\lambda_2$ skips $j$ turns. If $j > \ell$ then $\gamma$ can safely reach $o_5$. So assume that $j \leq \ell$. $\gamma$ continues moving towards $o_5$ until she reaches $L_{U_5,\ell+j}$. She then moves along the path highlighted in Figure 15. One can verify that after reaching $q_4$, either $\gamma$ can safely reach $o_9$ in another 98 rounds, or Lemma 10 may be applied to $U_5 \cup U_4 \cup U_9$ (for a very similar argument, see Case (1.2.1.1)).

Case (2.1.1.2): $\lambda_1$ moves at most 46 steps as $\gamma$ is moving towards $m_5$. Suppose that $\lambda_1$ moves $\ell$ steps towards $v_4$, where $\ell \leq 46$. $\gamma$ first moves to $L_{U,\ell+3}$ parallel $B_{10}$ until she reaches the corner of $L_{U,\ell+3}$ that is $92 - 2\ell$ edges away from $v_4$. $\gamma$ then moves to $v_4$ in $92 - 2\ell$ rounds; note that $\lambda_3$ cannot catch $\gamma$ just after $\gamma$ reaches $v_4$ because he is at least 4 edges away from $U_4$. Since $d_D(u_2,v_5) \geq 12$, $\gamma$ can safely reach either $o_5$ or $q_4$ after reaching $v_4$. Note that if $\gamma$ moves to $q_4$ using the preceding winning strategy, then she requires a total of $202 + \ell$ turns (starting at the turn when she moves away from $o$). On the other hand, the cops need at least 196 turns to reach $U_9$, $\lambda_2$ needs at least 12 rounds to reach $U_5$, and $\lambda_1$ needs at least 96 rounds to reach a neighbour of $U_5 \cup U_4 \cup U_9$. Thus if $\gamma$ can safely reach $q_4$ in another 100 rounds, then Lemma 10 may be applied to $U_5 \cup U_4 \cup U_9$. 

![Fig. 14. An escape path of $\gamma$ in Case (2.1.1.1).](image-url)
Fig. 15. An escape path of $\gamma$ in Case (2.1.1.2).

**Case (2.1.2):** $d_D(u_2, v_5) \leq 11$. Both $d_D(u_2, v_5) \leq 11$ and $d_D(u_3, v_3) \leq 11$ hold. $\gamma$ starts moving towards $m_2$. Note that at most one of $\lambda_2$ and $\lambda_3$ can reach $U_2$ before or just after $\gamma$ reaches $m_2$. We may assume that either $\lambda_2$ or $\lambda_3$ reaches $U_2$ before or just after $\gamma$ reaches $m_2$.

Suppose that $\lambda_3$ reaches $U_2$ before $\lambda_2$. Suppose $\lambda_3$ is $\ell$ vertices closer to $o_2$ than $\gamma$ is just after $\gamma$ reaches $m_2$. Note that $\ell \leq 9$. $\gamma$ starts moving towards $o_2$. Suppose $\lambda_3$ skips $j$ turns as $\gamma$ is approaching $o_2$. If $j > \ell$, then $\gamma$ can safely reach $o_2$. Assume now that $j \leq \ell$. $\gamma$ moves towards $o_2$ until she reaches $U_{4+\ell-j}$, continuing along the path highlighted in Figure 16 until she reaches $q_1$. One may directly verify (in a way that is similar to earlier cases) that either $\gamma$ can safely reach $o_6$, or Lemma 10 may be applied to $U_6 \cup U_1 \cup U_2$. The case that $\lambda_2$ reaches $U_2$ before $\lambda_3$ may be handled similarly; in this case $\gamma$ should move from $t_{22}$ to $q_2$ instead.

Fig. 16. An escape path of $\gamma$ in Case (2.1.2).
Case (2.2): \(d_D(u_1, v_3) \geq 12\). Observe that this case is almost symmetrical to Case (2.1.1) and a parallel argument may be applied. More precisely, note that if one maps the set of corner vertices of \(U\) to itself as follows: \(v_4 \to v_6, v_5 \to v_3, v_3 \to v_5, v_1 \to v_2, v_2 \to v_1\), and extend this mapping so as to obtain an automorphism of \(D\), then one may apply \(\gamma\)'s winning strategy in Case (2.1.1) to \(D'\) (with the appropriate transformed vertices).

C Proof of Lemma 14

Lemma 14. For Case (C), Algorithm 6 correctly computes a strategy for \(\gamma\) such that Step 3 of Algorithm 2 succeeds.

Proof. Without loss of generality, assume that \(\lambda_2\) is currently not in \(U\) while both \(\lambda_1\) and \(\lambda_3\) are currently in \(U\). As the proof techniques in the present case are so similar to those in Cases (1) and (2), we will omit many proof details and refer to strategies for \(\gamma\) in previous cases.

Case (1): There is at least one corner \(v_i\) of \(L_{U,49}\) such that \(d_D(v_i, u') \geq 99\) for all \(u' \in \{u_2, u_3\}\) and \(d_D(v_i, u_1) \geq 98\). We first assume that \(i = 1\). As in Case (B), define \(F := U_{10} \cup U_6 \cup U_1 \cup U_2 \cup U_7\).

Case (1.1): \(\lambda_2\) is currently in \(F\). First, suppose that \(d_D(v_5, u_2) \leq 98\). (This implies that \(\lambda_2\) is currently in \(U_1\).) If it also holds that \(d_D(v_5, u_3) \leq 98\), then \(\gamma\) moves to \(v_2\) in 98 rounds; Lemma 8 may then be applied to \(U_2 \cup U_3\). Now suppose that \(d_D(v_5, u_3) \geq 99\). If \(d_D(v_3, u_3) \geq 12\), then \(\gamma\) may apply the winning strategy in Case (B.2.1.1). Now suppose that \(d_D(v_2, u_3) \leq 11\). If \(d_D(u_3, U_7) \leq 50\), then \(\gamma\) may apply a slight variant of the winning strategy in Lemma 8 first moving to \(p\) in 98 rounds. Now suppose \(d_D(u_3, U_7) \geq 51\). \(\gamma\) first moves to \(v_1\) in 98 rounds. Note that \(d_D(u_2, q_1) \geq 185\). If \(\lambda_3\) is not in \(U_2\) just after the round when \(\gamma\) reaches \(v_1\), then \(\gamma\) can safely reach \(o_2\) in another 98 rounds. If \(\lambda_3\) is in \(U_2\) just after the round when \(\gamma\) reaches \(v_1\), then \(\lambda_3\) could have moved at most 47 steps between the 1-st and the 98-th round. Thus \(\lambda_2\) is at least 149 edges away from \(q_1\) just after the 98-th round. \(\gamma\) can now safely move to \(q_1\) in 100 rounds, and then to \(o_6\) in another 98 vertices.

Second, suppose that \(d_D(u_2, v_3) \geq 99\). We distinguish the following cases.

Case (1.1.1): \(d_D(u_3, v_3) \leq 100\) and \(d_D(u_3, v_4) \leq 100\). First, suppose that \(d_D(u_2, v_2) \leq 98\). If \(d_D(u_3, U_4) + d_D(u_2, U_3) \geq 5\), then \(\gamma\) first moves to \(v_3\) in 98 rounds. If neither \(\lambda_1\) nor \(\lambda_3\) reaches \(U_4\) at the end of the 98-th round, then \(\gamma\) can move to \(o_4\) without being caught in another 98 rounds. If either \(\lambda_1\) or \(\lambda_3\) reaches \(U_4\) at the end of the 98-th round (or if both \(\lambda_1\) and \(\lambda_3\) reach \(U_4\) at the end of the 98-th round), then \(\gamma\) can safely reach \(q_3\) in another 100 rounds. After reaching \(q_3\), either \(\gamma\) can safely reach \(o_6\) in another 98 rounds, or Lemma 10 may be applied to \(U_4 \cup U_3 \cup U_8\).

If \(d_D(u_3, U_4) + d_D(u_2, v_2) \leq 4\), then \(\gamma\) first moves to \(v_1\) in 98 rounds. If \(\lambda_3\) does not reach \(U_1\) at the end of the 98-th round, then \(\gamma\) can safely reach \(o_1\).
in another 98 rounds. If $\lambda_3$ reaches $U_1$ at the end of the 98-th round, then, since $d_D(u_2, q_1) + d_D(u_3, U_1) \geq 100 - d_D(u_3, U_4) + 196 - d_D(u_2, U_3) \geq 288$, $\gamma$ can move unmolested towards $q_1$ in another 100 rounds, and then safely reach $o_6$ using an additional 98 rounds.

**Case (1.1.2):** $d_D(u_3, v_3) \leq 100$ and $d_D(u_3, v_4) \leq 100$. Note that $\lambda_3$ and $\lambda_1$ are each at least 99 edges away from $U_1 \cup U_2$. First, suppose that $\lambda_2$ is currently not at $q_1$. $\gamma$ moves to $v_3$ in 98 rounds. If $\lambda_2$ does not move between the 1-st and the 98-th round, then some $U_i \in \{U_1, U_2\}$ does not contain any cop at the end of the 98-th round. $\gamma$ may then safely reach $o_5$ in another 98 rounds. If $\lambda_2$ moves at least one step between the 1-st and the 98-th round, then both $\lambda_3$ and $\lambda_1$ are each at least 2 edges away from $U_2 \cup U_3$ at the end of the 98-th round. One may then apply Lemma 9 to $U_1 \cup U_2$.

Second, suppose that $\lambda_2$ is currently at $q_1$. If $d_D(\lambda_3, B_{10}) \geq 50$, then $\gamma$ moves to $v_5$ in 98 rounds. At the end of the 98-th round, $\gamma$ can either safely reach $o_5$ in another 98 rounds, or move towards $q_5$ and then to $o_{10}$ in another 198 rounds. If $d_D(\lambda_3, B_8) \geq 50$, then $\gamma$ moves to $v_2$ in 98 rounds. An argument similar to that in the preceding case (that is, when $d_D(\lambda_3, B_{10}) \geq 50$) shows that $\gamma$ can either safely reach $o_3$ in another 98 rounds or safely reach $o_7$ in another 198 rounds.

**Case (1.1.3):** $d_D(u_3, v_2) \leq 100$ and $d_D(u_3, v_3) \leq 100$. $\gamma$ moves to $v_5$ in 98 rounds. An argument very similar to that in the last paragraph of Case (1.1.2) shows that $\gamma$ can either safely reach $o_5$ in another 98 rounds, or safely reach $o_{10}$ in another 198 rounds.

**Case (1.1.4):** $d_D(u_3, v_1) \leq 100$ and $d_D(u_3, v_2) \leq 100$. $\gamma$ pursues the same winning strategy as that in Case (1.1.3).

**Case (1.1.5):** $d_D(u_3, v_1) \leq 100$ and $d_D(u_3, v_5) \leq 100$. First, suppose that $d_D(u_2, v_2) \leq 98$. $\gamma$ moves to $v_3$ in 98 rounds. If neither $\lambda_1$ nor $\lambda_3$ is in $U_4$ at the end of the 98-th round, then $\gamma$ may safely reach $o_4$ in another 98 rounds. If either $\lambda_1$ or $\lambda_3$ is in $U_4$ at the end of the 98-th round, then $\gamma$ must still be at least 195 edges away from $q_3$ at the end of the 98-th round. Thus $\gamma$ may safely move to $q_3$ in 100 rounds, and then to $o_8$ in another 98 rounds.

Second, suppose that $d_D(u_2, v_2) \geq 99$. If $\lambda_2$ is currently at $q_2$, then $\gamma$ employs the winning strategy in the preceding case (that is, the case when $d_D(u_2, v_2) \leq 98$). If $\lambda_2$ is currently not at $q_2$, then $\gamma$ moves to $v_2$ in 98 rounds. If $\lambda_2$ moves at least one step between the 1-st and the 98-th round, then $\lambda_1$ and $\lambda_3$ are each at least 2 edges away from $U_2 \cup U_3$ at the end of the 98-th round, and therefore Lemma 9 may be applied to $U_2 \cup U_3$. If $\lambda_2$ does not move between the 1-st and the 98-th round, then there is some $U_i \in \{U_2, U_3\}$ such that $U_i$ does not contain any cop at the end of the 98-th round. Thus $\gamma$ may safely reach $o_i$ in another 98 rounds.

**Case (1.2):** $\lambda_2$ is currently not in $F$. First, suppose that $d_D(u_2, U_1 \cup U_2) \geq 3$ or $d_D(u_3, U_1 \cup U_2) \geq 3$. $\gamma$ first moves to $v_1$ in 98 rounds. If some $U_i \in \{U_1, U_2\}$
does not contain any cop at the end of the 98-th round, then $\gamma$ moves to $o_i$ in another 98 rounds. If both $U_1$ and $U_2$ contain at least one cop at the end of the 98-th round, then $\gamma$ continues moving towards $q_1$. Since each cop requires at least 196 rounds (from his starting position) to reach $q_1$ but at least 2 cops need more than 2 rounds to reach $U_1 \cup U_2$ (and no cop can reach $v_1$ in 98 rounds), $\gamma$ can safely get from $v_1$ to $q_1$ in 100 rounds, and then move from $q_1$ to $o_5$ in another 98 rounds.

Second, suppose that $d_D(u_2, U_1 \cup U_2) \leq 2$ and $d_D(u_3, U_1 \cup U_2) \leq 2$. $\gamma$ then moves to $p$ in 98 rounds. One may then apply a slight variant of Lemma 13 to obtain a winning strategy for $\gamma$.

**Case (1'):** $d_D(u', v_2) \geq 99$ for all $u' \in \{u_2, u_3\}$ and $d_D(u_1, v_2) \geq 98$. Notice that this case is almost symmetrical to the case $i = 1$, but with the initial position of $\lambda_1$ slightly different. In particular, we will assume that $d_D(v_1, u') \geq 99$ for all $u' \in \{u_2, u_3\}$, $d_D(u_1, o) = 1$, and either $d_D(v_4, u_1) = 97$, $d_D(m_5, u_1) = 97$ or $d_D(p', u_1) = 97$, where $p'$ is the vertex between $m_5$ and $v_4$ such that $d_D(p', m_5) = 1$. $\gamma$'s strategies in Cases (1.1) and (1.2) apply in the present case if the winning strategy does not involve $\gamma$ first moving to $m_5$. The only scenario in Cases (1.1) and (1.2) where $\gamma$ may move to $m_5$ occurs when $12 \leq d_D(u_2, v_5) \leq 98$ and $d_D(u_3, v_5) \geq 99$.

If $d_D(u_1, v_4) = 97$, then the starting game configuration is precisely one of the starting configurations handled in Case (1.1). Now suppose $d_D(u_1, v_4) \geq 99$. If $d_D(u_3, v_3) > 100$ holds, then $\gamma$ can safely reach $v_5$ in 98 rounds, and then move to either $o_4$ or $o_3$ in another 98 rounds. If $d_D(u_3, v_4) > 100$ holds, then $\gamma$ can safely reach $v_4$ in 98 rounds, and then move to either $o_5$ or $o_4$ in another 98 rounds, or move to $o_6$ in another 198 rounds. Now suppose that $d_D(u_3, v_4) \leq 100$ and $d_D(u_3, v_5) \leq 100$ both hold. If $d_D(u_3, B_8) \geq 2$, then $\gamma$ first moves to $v_2$ in 98 rounds. If $\lambda_3$ reaches $U_3$ and $\lambda_2$ reaches $U_2$ at the end of the 98-th round, then $\gamma$ can continue moving towards $q_2$, eventually reaching $o_7$ in another 198 rounds. If $\lambda_3$ does not reach $U_3$ at the end of the 98-th round, then $\gamma$ can safely reach $o_3$ in another 98 rounds; if $\lambda_2$ does not reach $U_2$ at the end of the 98-th round, then $\gamma$ can safely reach $o_2$ in another 98 rounds. If $d_D(u_3, B_8) \leq 1$, then $\gamma$ first moves to $v_3$ in 98 rounds, then to $q_4$ in another 100 rounds; $\gamma$ may then safely get from $q_4$ to $o_9$ using an additional 98 rounds.

**Case (1''):** $d_D(u', v_3) \geq 99$ for all $u' \in \{u_2, u_3\}$ and $d_D(u_1, v_3) \geq 98$. Assume that there are $u', v' \in \{u_2, u_3\}$ such that $d_D(u', v_1) \leq 98$ and $d_D(v', v_2) \leq 98$. Observe that this case is almost symmetrical to the case $i = 1$, except that $d_D(u_1, m_1) = 97$ or $d_D(u_1, v_5) = 97$ or $d_D(u_1, p'') = 97$, where $p''$ is the vertex between $v_5$ and $m_1$ that is 1 edge away from $m_1$. In addition, one may assume that there exist $u'', v'' \in \{u_2, u_3\}$ such that $d_D(u'', v_2) \leq 98$ and $d_D(v'', v_3) \leq 98$. (This implies that if $\lambda_2$ is currently in $F$, then either $d_D(u_3, v_1) > 100$ or $d_D(u_3, v_2) > 100$; furthermore, either $d_D(u_3, v_1) > 100$ or $d_D(u_3, v_5) > 100$.) If $\lambda_2$ is currently not in $F$, then $\gamma$ may apply the winning strategy in Case (1.2).

Now suppose that $\lambda_2$ is currently in $F$. First, suppose that $d_D(u_3, v_4) \leq 100$ and $d_D(u_3, v_3) \leq 100$. Since at least one cop must be at most 98 edges away from $v_2$,
it must hold that \( d_3(u_2, v_2) \leq 98 \), which implies that \( \lambda_2 \) is currently in \( U_2 \) and \( d_3(u_2, U_6) \geq 99 \). In addition, since \( d_3(u_2, v_1) \geq 99 \), it holds that \( d_3(u_2, U_1) \geq 99 \). \( \gamma \) starts by moving to \( v_1 \) in 98 rounds. Note that only \( \lambda_1 \) can reach \( U_1 \) at the end of the 98-th round; if he does not reach at the end of the 98-th round, then \( \gamma \) can safely reach \( o_1 \) in another 98 rounds. Suppose that \( \lambda_1 \) does reach \( U_1 \) by the end of the 98-th round, using up 97 turns in the process. \( \gamma \) then continues moving towards \( o_1 \) until she reaches \( L_{U_1, U_6} \); she then moves along the path highlighted in Figure 17. After reaching \( t_{28} \), \( \gamma \) moves to \( o_6 \) in another 98 rounds if \( \lambda_2 \) is not in \( U_6 \); otherwise, \( \gamma \) moves from \( t_{28} \) to \( u_3 \) in another 100 rounds, and then safely reaches \( o_{10} \) using an additional 98 rounds.

Second, suppose that \( d_3(u_3, v_4) \leq 100 \) and \( d_3(u_3, v_5) \leq 100 \). Suppose \( d_3(u_3, U_1) \geq 98 \). \( \gamma \) may pursue the winning strategy given in the case \( d_3(u_3, v_4) \leq 100 \) and \( d_3(u_3, v_5) \leq 100 \). Now suppose \( d_3(u_3, U_1) \leq 97 \). \( \gamma \) first moves to \( v_3 \) in 98 rounds. If \( \lambda_3 \) is not in \( U_4 \) at the end of the 98-th round, then \( \gamma \) can safely reach \( o_4 \) in another 98 rounds. If \( \lambda_3 \) reaches \( U_4 \) by the end of the 98-th round, using up at least 3 turns in the process, then \( \gamma \) moves to \( q_3 \) in another 100 rounds. After reaching \( q_3 \), either \( \gamma \) may safely reach \( o_8 \) in another 98 rounds, or \( \gamma \) may apply the winning strategy in Lemma 10 to \( U_3 \cup U_4 \cup U_8 \).

Third, suppose that \( d_3(u_3, v_2) \leq 100 \) and \( d_3(u_3, v_3) \leq 100 \). Suppose first that at least one of the inequalities \( d_3(u_2, U_3) \leq 97 \) and \( d_3(u_3, U_2) \geq 3 \) holds. \( \gamma \) moves to \( v_1 \) in 98 rounds. If there is some \( U_4 \in \{ U_1, U_2 \} \) that does not contain any cop at the end of the 98-th round, then \( \gamma \) can simply move to \( o_1 \) in 98 rounds. If both \( U_1 \) and \( U_2 \) contain at least one cop at the end of the 98-th round, then \( \gamma \) can move to \( q_1 \) in another 100 rounds, and finally safely reach \( o_6 \) using an additional 98 rounds. Next, suppose that \( d_3(u_2, U_6) \geq 98 \) and \( d_3(u_3, U_2) \geq 2 \). \( \gamma \) moves to \( v_4 \) in 98 rounds. If \( \lambda_3 \) is not in \( U_4 \) at the end of the 98-th round, then \( \gamma \) can safely reach \( o_4 \) in another 98 rounds. If \( \lambda_3 \) does reach \( U_4 \) by the end of the 98-th round, using up at least 98 turns in the process, then \( \gamma \) can move unmolested to \( q_4 \) in another 100 rounds. \( \gamma \) may then either safely reach \( o_8 \) using another 98 rounds, or apply the winning strategy in Lemma 10 to \( U_4 \cup U_5 \cup U_9 \).

Case (1''): \( d_3(u', v_3) \geq 99 \) for all \( u' \in \{ u_2, u_3 \} \) and \( d_3(u_1, v_5) \geq 98 \). Observe that this case is symmetrical to the case that \( d_3(u', v_1) \geq 99 \) for all \( u' \in \{ u_2, u_3 \} \) and \( \lambda_1 \) is 97 edges away from either \( v_5 \) or \( m_5 \) or the vertex \( v'' \) between \( m_5 \) and \( v_5 \) that is 1 edge away from \( m_5 \). Thus \( \gamma \) may employ a winning strategy analogous to that in the previous case (that is, \( d_3(u', v_3) \geq 99 \) for all \( u' \in \{ u_2, u_3 \} \)).

Case (1''): \( d_3(u', v_4) \geq 99 \) for all \( u' \in \{ u_2, u_3 \} \) and \( d_3(u_1, v_4) \geq 98 \), and for each \( v_i \in \{ v_1, v_2, v_3, v_5 \} \), there exists some \( v' \in \{ u_2, u_3 \} \) such that \( d_3(v', v_i) \leq 98 \). This implies that one of the following holds: (i) \( d_3(u_2, v_1) \leq 98, d_3(u_2, v_5) \leq 98, d_3(u_3, v_2) \leq 98, d_3(u_3, v_3) \leq 98 \), or (ii) \( d_3(u_3, v_1) \leq 98, d_3(u_3, v_5) \leq 98, d_3(u_2, v_2) \leq 98, d_3(u_2, v_3) \leq 98 \). Suppose (i) holds. \( \gamma \) starts by moving to \( v_4 \) in 98 rounds. If \( \lambda_2 \) does not reach \( U_5 \) by the end of the 98-th round, then \( \gamma \) can safely reach \( o_5 \) in another 98 rounds. If \( \lambda_2 \) is in \( U_5 \) at the end of the 98-th round (using up at least 2 turns in the process), then \( \gamma \) can continue moving along \( B_4 \) until she reaches \( q_4 \) in another 100 rounds. \( \gamma \) may then either safely reach \( o_9 \) in
another 98 rounds, or apply the winning strategy in Lemma 10 to \(U_4 \cup U_5 \cup U_9\). 
\(\gamma\) may apply an analogous winning strategy in Case (ii) (interchanging \(\lambda_2\) and \(\lambda_3\) throughout the winning strategy).

**Case (2):** For each corner \(v'\) of \(L_{49}\), it holds that \(d_D(v', u') \leq 98\) for some \(u' \in \{u_2, u_3\}\) or \(d_D(v', u_1) \leq 97\) (or both inequalities hold).

**Case (2.1):** \(d_D(u_2, v_2) \leq 98, d_D(u_2, v_3) \leq 98, d_D(u_3, v_1) \leq 98, d_D(u_4, v_5) \leq 98\) and \(d_D(u_1, v_4) = 97\). Suppose that \(d_D(u_2, v_3) \geq 12\). \(\gamma\) may then apply the winning strategy in Case (B.2.2). Now suppose that \(d_D(u_2, v_3) \leq 11\). Then \(d_D(u_2, U_2) \geq 89\) and \(d_D(u_2, m_2) \geq 139\). Consider the following case distinction: (i) \(d_D(u_3, m_2) \leq 98\) and (ii) \(d_D(u_3, m_2) \geq 99\).

(i) Notice that in this case, \(d_D(u_3, m_5) \geq 99\) and \(d_D(u_3, v_5) \geq 49\). \(\gamma\) may then apply the winning strategy in Case (B.2.1.1).

(ii) It follows from the inequalities \(d_D(u_3, m_2) \geq 99\) and \(d_D(u_3, v_5) \leq 98\) that \(d_D(u_3, v_1) \geq 49\). \(\gamma\) may then apply the winning strategy in Lemma 8 first moving to \(m_2\) in 98 rounds (note that the winning strategy applies in this case even though \(\lambda_2\) is not in \(U\)).

**Case (2.2):** \(d_D(u_2, v_1), d_D(u_2, v_5) \leq 98, d_D(u_3, v_2), d_D(u_3, v_3) \leq 98\) and \(d_D(u_1, v_4) = 97\). Similar to Case (3.2.1), we first suppose that \(d_D(u_2, v_5) \geq 12\). \(\gamma\) may then employ the winning strategy in Case (B.2.1.1). Now suppose that \(d_D(u_2, v_5) \leq 11\). Then \(d_D(u_2, m_2) \geq 139\) and \(d_D(u_2, v_1) \geq 89\).

(i) \(d_D(u_3, m_2) \geq 99\). Then \(d_D(u_3, v_2) \geq 49\). \(\gamma\) moves to \(m_2\) in 98 rounds, employing the winning strategy in Lemma 8.

(ii) \(d_D(u_3, m_2) \leq 98\). Then \(d_D(u_3, m_4) \geq 99\). \(\gamma\) moves to \(m_4\) in 98 rounds, employing the winning strategy in Case (B.2.2).