Collinear Divergences at One-loop Order
for External Particles in a Heat-bath.

Saumen Datta\textsuperscript{1}, Sourendu Gupta\textsuperscript{2}, V. Ravindran\textsuperscript{3}

Abstract

In hard interactions between external particles incident on a heat-bath, we show that large logarithms are generated when a radiated or absorbed gauge boson is collinear with the initial fermion momentum. These logarithms can be absorbed into process independent splitting/absorption probabilities. Unlike the zero-temperature case, however, they depend explicitly on the temperature and the scale of the interaction.

\textsuperscript{1}Theory Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India, Email— saumen@theory.tifr.res.in
\textsuperscript{2}Theory Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India, Email— sgupta@theory.tifr.res.in
\textsuperscript{3}Theory Group, Tata Institute of Fundamental Research, Navrangpura, Ahmedabad 380009, India, Email— ravi@prl.ernet.in
In recent years the problem of calculating scattering cross sections or rates for external particles travelling in a thermal medium has become phenomenologically interesting \cite{1}. In a previous paper we have shown that to one-loop order there are no infrared divergences when two external particles collide in a heatbath kept at a temperature $T$ \cite{2}. We were able to resum the one-loop result and calculate the distribution of the mismatch in initial and final momenta due to soft radiation. The result was finite and contained some large logarithms of the type which usually arise from collinear emissions. In this paper we study this collinear behaviour in detail. We find that divergences exist, and are signalled by the familiar “large terms” of the order $\log(Q^2/m^2)$ ($Q^2$ is the scale of the process and $m$ is a regulating mass). We show that the collinear part gives rise to a certain universal splitting/absorption probability which explicitly depends on $Q$ and $T$.

There has been recent interest in collinear singularities associated with soft particles \cite{3}, because they might spoil the hard thermal loop resummation \cite{4}. Since we study a hard particle in a heat-bath, we have nothing further to say about this.

Consider the simplest scattering process in which the collinear singularities make their appearance. This is the inclusive cross section for the scattering of a charged electron with a space-like photon, $\gamma^*$, in a QED heat-bath, which we consider in a perturbation theory of the gauge coupling. The initial electron momentum is denoted by $p$, the $\gamma^*$ momentum by $q$ and the final electron momentum by $p'$. The 4-velocity of the heat-bath in any frame, $u$ ($u^2 = 1$), is a new vector in the problem \cite{5}. Compared to the $T = 0$ case, there are extra scalars $p \cdot u$ and $q \cdot u$ which have to be taken into account. For the rest, we use the standard notation

\begin{equation}
q^2 = -Q^2 \quad \text{and} \quad p \cdot q = \frac{Q^2}{2x},
\end{equation}

and work in the limit $Q^2 \gg m^2$, where $p^2 = p'^2 = m^2$. We also take $Q^2 \gg T^2$ where $T$ is the temperature of the heat-bath.

The inclusive cross section can be written as a perturbation expansion in the form

\begin{equation}
\sigma = \alpha \left( \sigma_0 + \alpha \sigma_1 + \alpha^2 \sigma_2 + \cdots \right),
\end{equation}

where $\alpha$ is the gauge coupling. At the lowest order in perturbation theory, one has only the interaction between $e$ and $\gamma^*$, and hence $\sigma_0$ is independent
of $T$. At higher orders, interactions with thermal photons have to be taken into account, and $\sigma_i \ (i > 0)$ may change from its $T = 0$ value.

It is useful to write the cross section in the form

$$\sigma = \frac{\rho^{\mu\nu} W_{\mu\nu}}{4\sqrt{m^2 Q^2 + (p \cdot q)^2}}$$

where

$$\rho_{\mu\nu} = \sum_{\lambda} \epsilon_{\mu}^{*\lambda}(q) \epsilon_{\nu}^{\lambda}(q).$$

(3)

$\epsilon_{\mu}^{\lambda}(q)$ is the polarisation vector of the off-shell photon in the polarisation state $\lambda$. The tensor $\rho$ is a density matrix for the polarisation states of $\gamma^*$ and is symmetric in its indices. The quantity to be computed in perturbation theory is the symmetric part of the rank-2 tensor $W_{\mu\nu}$, which is the vacuum expectation value of the product of the electromagnetic current coupling to $\gamma^*$.

The computation is simplified by decomposing the tensor $W_{\mu\nu}$ into scalar functions multiplying all symmetric tensors which can be built out of the vectors in the problem. Furthermore, the gauge invariance of the current implies that only those tensors orthogonal to $q$ are relevant. There are four such tensors—

$$T^1_{\mu\nu} = g_{\mu\nu} + \frac{1}{Q^2} q_{\mu} q_{\nu} \quad T^2_{\mu\nu} = P_{\mu} P_{\nu}$$

$$T^3_{\mu\nu} = U_{\mu} U_{\nu} \quad T^4_{\mu\nu} = U_{\mu} P_{\nu} + P_{\mu} U_{\nu}.$$  

(4)

We have used a shorthand notation for the components of $p$ and $u$ orthogonal to $q$—

$$P_{\mu} = p_{\mu} + \frac{p \cdot q}{Q^2} q_{\mu}, \quad U_{\mu} = u_{\mu} + \frac{u \cdot q}{Q^2} q_{\mu}.$$  

(5)

As a result,

$$W_{\mu\nu} = \sum_{i=1}^{4} W_i(x, Q^2, p \cdot u, q \cdot u) T^i_{\mu\nu},$$

(6)

and hence there are four “structure functions” in this problem. At $T = 0$ only the two structure functions $W_1$ and $W_2$ appear. Even for $T > 0$, at the leading order of perturbation theory $W_3 = W_4 = 0$, since $\sigma_0$ does not contain any terms involving $u$. Hence the Callan-Gross relation [6] is also valid to this order, with corrections generated at higher orders, through the usual vacuum ($T = 0$) processes, as well as by additional interactions with real thermal gauge bosons.
Figure 1: The real photon emission and absorption corrections. Absorbed thermal photons come in from the left, emitted photons exit to the right. In the planar gauge only the diagrams with photons attached to the initial leg contribute to the collinear limit.
Reviews of the real-time thermal field theory techniques we use can be found in [7]. Using these we can generate the processes contributing to \( \sigma_1 \) and the rules for their evaluation. The relevant diagrams with real thermal photon emission and absorption are shown in Figure (1). For one photon emission (absorption) the two-body phase space measure can be taken to be

\[
d\Gamma_\pm = \frac{1}{(2\pi)^4} d^4k 2\pi \delta^\pm(k^2) 2\pi \delta^+((p + q - k)^2) B(k \cdot u),
\]  

where \( k \) is the four-momentum of the thermal photon, \( \delta^\pm(x^2) = \delta(x^2)\theta(\pm x_0) \) and \( B(x) \) is the Bose distribution \( 1/[\exp(|x|/T) - 1] \). The positive sign in eq. (7) corresponds to the emission process and negative to absorption.

Note that for the diagrams in Figure (1), the vector \( u \) appears only in the measure. This leads to a singularity as \( k \cdot u \to 0 \). However, this is not a collinear singularity but the previously analysed soft singularity [8]. Since \( u^2 = 1 \), a boost to the rest frame of \( u \) can always be done. This gives us the correct interpretation of the divergence. The only collinear singularities then arise from the matrix elements. These can be identified as divergences in the limit \( k \cdot p \to 0 \) when \( m \to 0 \), and have the same origin as those occurring at \( T = 0 \).

We choose to work in a planar gauge [8], which is a ghost-free gauge specified by the gauge fixing part of the Lagrangian—

\[
\mathcal{L}_{gf} = -\frac{1}{2v^2}(v_\mu A^\mu)\partial^2(v_\nu A^\nu).
\]  

The vector

\[
v_\mu = C_1 p'_\mu + C_2 p
\]

defines the gauge choice. The coefficients \( C_1 \) and \( C_2 \) are chosen such that \( v^2 \neq 0 \) and the sum over polarisations of the real photons becomes

\[
d_{\alpha\beta} \equiv \sum_\lambda \epsilon_\alpha^\lambda(k) \epsilon_\beta^{\star \lambda}(k) = -g_{\alpha\beta} + \frac{k_\alpha v_\beta + k_\beta v_\alpha}{k \cdot v}.
\]  

where \( \epsilon_\mu^\lambda(k) \) is the polarisation vector of a photon with polarisation \( \lambda \) and momentum \( k \).

It can be verified that in this gauge all collinear singularities come from the squares of the diagrams with the real photon attached to the initial
fermion leg. For the first emission diagram, we obtain

\[ |M|^2 = -2e^4 \frac{1}{(p-k)^2} \]

\[ \text{Tr} \left[ \gamma_\nu \not{p}^\nu \{ \not{q} \left( \frac{2p \cdot v}{k \cdot v} - 1 \right) + 2k \cdot \left( 1 - \frac{p \cdot v}{k \cdot v} \right) + \frac{k \cdot p}{k \cdot v} \} \right]. \quad (11) \]

We are interested in extracting the leading terms in the collinear limit, \( p \cdot k \to 0 \). The most transparent way of doing this is to use the Sudakov parametrisation \[ k = (1 - \rho)p + \beta(q + xp) + k_T, \text{ where } p \cdot k_T = q \cdot k_T = 0. \quad (12) \]

It is clear that this is a Lorentz invariant decomposition. The integration variables are changed to \( \rho, \beta \) and the two independent components of \( k_T \). The Jacobian is simply

\[ \frac{d^3k}{d\rho d\beta d^2k_T} = \frac{Q^2}{2x}. \quad (13) \]

The variables \( \rho \) and \( \beta \) are fixed by the \( \delta \)-function constraints in the measure \( d\Gamma_+ \). In the collinear limit, the solution which leads to \( p \cdot k \to 0 \) as \( k_T^2 \to 0 \) is

\[ \rho = x + \mathcal{O}(k_T^2), \quad \beta = \frac{xk_T^2}{Q^2(1-x)} + \mathcal{O}(k_T^4). \quad (14) \]

Requiring \( \rho \) and \( \beta \) to be real, in the limit \( m \to 0 \) we find

\[ 0 \leq k_T^2 \leq \frac{Q^2}{4x}(1-x). \quad (15) \]

The \( \theta \)-functions place no further restrictions, and may be dropped to give the phase space measure

\[ d\Gamma_+ = \frac{xd\rho d\beta d^2k_T}{2Q^2(1-x)(2\pi)^2} \delta \left( \beta - \frac{xz}{1-\rho} \right) \delta(\rho - x). \quad (16) \]

Absorption is handled by making the change of variables \( k_\mu \to -k_\mu \), and then writing a Sudakov parametrisation as before. \( d\Gamma_- \) differs from \( d\Gamma_+ \) only in the change

\[ \delta(\rho - x) \to \delta(\rho - 2 + x). \quad (17) \]
In the collinear limit, the denominator on the right of eq. (11) becomes zero, and hence some care is required in taking this limit. We retain the fermion mass, $m$, as a regulator in the denominator and write
\[ \frac{1}{(p-k)^2} = \frac{1}{m^2(1-2\rho) - Q^2\beta/x}. \] (18)

The most singular terms in the collinear limit can then be found by evaluating the trace in eq. (11) for $m=0$ and neglecting terms in $k^2_{\perp}$ (and hence in $\beta$). After some straightforward manipulations, we find that the most singular contribution to the tensor $W_{\mu\nu}$ is
\[ -8\epsilon^4 \left[ 2xT^2_{\mu\nu} - \frac{Q^2}{2x} T^1_{\mu\nu} \right] \int d\Gamma + B[(1-\rho)p \cdot u] \left[ \frac{1 + \rho^2}{1-\rho} \right] \frac{1}{m^2(1-2\rho) - Q^2\beta/x}. \] (19)

The contribution from the corresponding absorption diagram, obtained by changing $k \rightarrow -k$ in the matrix element and using the phase space measure $d\Gamma_-$, is simply obtained by setting $\rho$ to $2-\rho$ is the above expression.

In the chosen gauge, the squares of the other two diagrams do not have singular denominators and hence can be neglected in the collinear limit. The cross terms do have a singular denominator. However, the trace contains
\[ \frac{\text{Lt}}{k \rightarrow (1-\rho)p} d_{\alpha\beta}(k)p^\alpha = 0. \] (20)

Hence this term can also be neglected.

Two facts about eq. (19) and its analogue for photon absorption are worth pointing out. First, both terms contain only the tensors $T^1$ and $T^2$. Hence there are no collinear contributions to the thermal structure functions $W_3$ and $W_4$ at this order. Second, the integrand has a probability interpretation. The part $(1 + \rho^2)/(1-\rho)$ can be interpreted as the probability that an incoming electron radiates a photon carrying a fraction $1-\rho$ of the initial momentum. The Bose distribution factor is the probability that the radiated photon is indistinguishable from a thermal photon. The integrand has support on $0 \leq \rho \leq 1$, which is consistent with this interpretation. Similarly, the absorption process can also be interpreted as the product of two probabilities. The Bose distribution is the probability of finding a photon in the heat-bath carrying a fraction $\rho - 1$ of the incoming electron’s momentum and the factor
\(\frac{(1 + (2 - \rho)^2)}{(1 - (2 - \rho))}\) is the probability of absorption. This integral has support on \(1 \leq \rho \leq 2\).

The integrals can be performed completely. The thermal leading log part is

\[
W_{\mu\nu} = 4\pi\alpha^2 P(x, Q/T) \log \left(\frac{Q^2}{m^2}\right) \frac{x}{Q^2} \left[2xT_{\mu\nu}^2 - \frac{Q^2}{2x} T_{\mu\nu}^1 \right].
\]

We have introduced the finite temperature part of the “splitting function”

\[
P(x, Q/T) = \frac{2}{\exp[(1-x)p \cdot u/T] - 1} \left(\frac{1 + x^2}{1 - x}\right).\]

This can be given an interpretation as the probability of an external electron splitting off a thermal photon. Unlike the case at \(T = 0\), this factor has an explicit dependence on \(Q/T\). Note also that \(P(x, Q/T)\) has a singularity as \(x \to 1\). This is the region of phase space where the thermal photon is soft. We have shown in [2] that the cross section is finite in this limit provided virtual corrections are taken into account. Using this result, we can simply write down a regulated version of eq. (22) as the splitting probability for the external electron.

At \(T = 0\) the leading soft divergence in the real diagrams is logarithmic, and shows up in the splitting functions as a divergence of the form \(1/(1 - x)\), in the limit \(x \to 1\). It is cured by taking into account the virtual diagrams. The regulated form of the splitting functions is then given by the familiar prescription

\[
\int dx P_{\mu\nu}(x)f(x) = \int dx P(x) [f(x) - f(1)], \quad (T = 0),
\]

where \(f(x)\) is a test function. Note that the first moment of any distribution vanishes when convoluted with the splitting function so regularised.

For \(T > 0\) the leading soft divergence is quadratic. This is signalled by a divergence of the form \(1/(1 - x)^2\) \((x \to 1)\) in the splitting functions. Its cancellation against virtual contributions has been shown in [2]. The sub-leading logarithmic divergence has also been shown to cancel against virtual corrections [9]. Consequently, we just write down an appropriately regularised version of the \(T > 0\) part of the splitting function—

\[
\int dx P_{\mu\nu}(x)f(x) = \int dx P(x) [f(x) - f(1) - (x - 1)f'(1)].
\]
Note that the first two moments of \( P_+ \) vanish with this regularisation. The vanishing second moment implies that finite temperature effects do not change the expectation value of the parton’s momentum. This is expected \(^2\) and is a consequence of detailed balance.

The universality of this additional finite temperature term in the splitting functions derived here can be easily checked. The calculation for Fermion pair-production is very similar to the computation presented in this paper, and yields precisely the same collinear term derived here.

In QED, since the electron mass is non-zero, our results are complete. However, in a real experiment we shall have to deal with a QCD heatbath. Our main results, eq. (21), along with eqs. (22) and (24), can be carried over to this case with the simple replacement \( \alpha^2 \rightarrow C_F \alpha \alpha_s \). The crucial change is that \( m = 0 \) for quarks and hence the results are singular.

At \( T = 0 \), these collinear singularities are handled by factoring them into universal quark distributions inside hadrons. A similar procedure will have to be developed for external hadrons or jets impinging on a plasma. In order to complete this program, a suitable definition of the QCD running coupling at finite temperature \(^1\) must be provided. At \( T = 0 \), this is sufficient information to sum the one-loop iterated ladder diagrams into the DGLAP equations \(^1\).

For \( T > 0 \) the situation is more complicated. This is clear from the fact that the analogue of the splitting function contains the dimensionless variable \( Q/T \) in addition to \( x \). At finite temperature and arbitrary scale \( Q^2 \), we are forced to consider two scales in the renormalisation group \(^2\). In various domains these simplify. For example, when \( Q \gg T \), one expects to be able to use a single scale. In this limit, the regularised version of eq (22) vanishes, and the evolution in \( Q^2 \) is the same as at \( T = 0 \). However, the parton distributions at each \( T \) must then be separately measured. Only by keeping two scales can information at \( T = 0 \) be evolved to \( T > 0 \). This work is left to the future.

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