Tolman’s Luminosity-Distance, Poincaré’s Light-Distance and Cayley-Klein’s Hyperbolic Distance

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Abstract

(This is a paper of General Physics because it is entirely based on Lorentz Transformation (LT) but given that we introduce a new definition of light-distance in Einstein’s kinematics, it has cosmological implications)

We deduce Tolman’s formula of luminosity-distance in Cosmology from Poincaré’s definition of light-distance with LT. In Minkowski’s metric, if distance is proper time (as it is often argued) then light-distance must be also the shortest... like proper time (unlike Einstein’s longest length).

By introducing Poincaré’s proper light-distance in Einstein’s basic synchronization we deduce a $k$-dilated distance ($k = 1 + z$) with relativistic Doppler’s factor between the observer and the receding “mirror”.

Such a ”light-luminosity” distance corresponds not to an Euclidean distance (Einstein’s rigid rod) but to an Hyperbolic Distance (Cayley-Klein) with a Lobatchevskian Horizon.

From a basic proportionality ”hyperbolic distance- hyperbolic velocity”, we deduce the law of Hubble. With Hubble’s horizon $R_H$ we have not only the constant of Hubble $HR_H = c$, but also a minimal Milgrom’s acceleration $a_M R_H = c^2$. In basic Hyperbolic Rotation (active LT or Einstein’s boost), the former is an hyperbolic angular velocity and the latter a centrifugal (hyperbolic) proper acceleration.

The cosmological constant is the square $\Lambda = \rho_H^2$ of global Lobatchevskian Curvature $\rho_H = R_H^{-1}$.

By following Penrose’s Lobachevkian representation of LT, we transform SR into an Hyperbolic Cosmological Relativity (HCR) by using only the LT but the whole LT (Einstein’s active LT or Einstein’s boost).

In Tolman’s double reduction, given that photons are dispatching on a sphere for the source ($R = D$) and the observer ($R = (1 + z)D$) as well, the element of perpendicular area is transformed. We have to take into account LT of solid angle of aperture of emission light cone and therefore LT of spherical (isotropic) wavefront into ellipsoidal (anisotropic) wavefront (Poincaré 1908). The element of perpendicular area (longitudinal ”Lorentz contraction”) is therefore unchanged. Poincaré’s elongated light-ellipsoid becomes a direct explanation of Hubble’s expansion.

A non-transversal section in Minkowski’s cone is an ellipsoid which transforms pseudo-Euclidean $\pi - relativity$ (steradian is no longer a invariant), into Lobatchevskian $e - relativity$.

1 Luminosity-distance in Cosmology and invariance of solid angle

(Stationary Source) The standard method of estimating distances is given by the relation between observed and estimated luminosity. A stationary source $S$ having absolute luminosity $L_S$ placed at the distance $D$ will have apparent luminosity $l_{\text{stationary}}$ given by the inverse square law:

$$l_{\text{stationary}} = \frac{L_S}{4\pi D^2}$$  \hspace{1cm} D : \text{radius of emission sphere with center } S \tag{1}$$

The absolute luminosity $L_S$ is total luminous power (luminous energy per second) of the source and $l_{\text{stationary}}$ is luminous power per unit perpendicular area intercepted by unit of solid angle (steradian), received at a distance $D$ from the source. If $L_S$ can be estimated from a knowledge of the type of the source observed; then the stationary luminosity-distance $D$ can be calculated since $l_s$ is directly measurable. It is particularly important that the measurement of $l_s$ involves a small receptor area $dS$ (on the mirror) and therefore a small solid angle $d\Omega_{\text{stationary}} = \frac{dS}{D^2}$ (e for emitted sphere).

(Receding Source) At first pointed by Tolman if the source is receding in the line of sight then the luminosity actually observed will not be $l_{\text{stationary}}$ but a reduced value $l_{\text{recession}}$ owing to:

* The number effect: the reduction in the number of photons arriving because of the lengthening of the travel path of a receding source. this reduces incident radiation by a factor $1 + z$ (z is standard spectral redshift).
Tolman and Robertson suppose that the photons received by the observer $O$ are dispatching on a spherical wavefront (Einstein’s spheres).

* The energy effect: the energy of photons arriving is reduced because redshift lowers their frequency. the reduction of incident radiation is by a further factor $1 + z$. The reduced value of luminosity $l_{\text{recession}}$ on receptor is given by

$$ l_r = \frac{1}{(1+z)^2} l_s $$

(2)

where $l_r = l_{\text{recession}} = l_{\text{reception}}$ in system $K$ of receptor $O$ and $l_s = l_{\text{stationary}} = l_{\text{source}}$ in system $K'$ of source $S$. The factor $(1 + z)^2$ is called the “Tolman double reduction factor” [R.C. Tolman]. By introducing (1) in (2) we have

$$ l_r = \frac{L_S}{4\pi(1+z)^2 D^2} $$

(3)

With the invariance of the flux $l_s D^2 = l_r D'^2$

$$ l_r = \frac{L_S}{4\pi D'^2} \quad D_r = OS_{\text{LUMINOSITY}} : \text{radius of reception sphere with center } S $$

(4)

from which Tolman and also Robertson deduce the comoving distance $D$ (given by simultaneous events: position of the source at the time of observation) in function of receded luminosity-distance $D_r$ between the $S$ (source) and $O$ (Observer) ([E.P.Hubble & R.C. Tolman] & [M.P. Robertson])

$$ OS_{\text{LUMINOSITY}} = D_r = (1+z)D \quad D = \frac{D_r}{1+z} $$

(5)

This is (5) Tolman-Robertson’s formula of expanding spherical wavefront in $K$. There is however a very subtle (§3.5) hypothesis in this standard accepted reasoning. Given that area units are on emitted sphere ($e$) $S_e = D^2$ and on received light sphere ($r$) $S_r = D'^2$, the element of perpendicular area obviously cannot be an invariant $dS_e \neq dS_r$ if we have... spheres (center $S$). So implicit non-relativistic Tolman’s hypothesis is the invariance of element of solid angle of aperture of light cone in both $K'$(source) and $K$(observer) ([J. F. Barrett (Luminosity]) & [J. F. Barrett (Lobatchevski)]):

$$ d\Omega_e = \frac{dS_e}{D^2} = d\Omega_r = \frac{dS_r}{D'^2} $$

(6)

The element of perpendicular area can be an invariant if the sphere of emission is transformed into... an observed ellipsoid. Let us suppose that the spherical wavefront of emission (set of simultaneous events) from the moving source $S$ of $K'$ be not transformed into a spherical wavefront (set of simultaneous events) but into an (elongated) ellipsoidal wavefront (the events are no longer simultaneous) for the observer in $K$ (Poincaré 1908). We will show that, with Lorentz Transformation ($LT$, 8)

$$ dS_r = D'^2 d\Omega_r = dS_e = D^2 d\Omega $$

(7)

And given that relativistic transformation is $d\Omega = d\Omega' \frac{1}{(1+v^2)}$, we will deduce, immediately from (completed) SR, Tolman-Robertson’s law (5). In many papers we showed that Poincaré’s ellipsoid ($x, y, z$) is directly inscribed in $LT$ (Relativity of Simultaneity). Given that it is an ellipsoid of Revolution, we showed at two dimensions that Poincaré’s elongated ellipse ($x, y$) involves an original definition of light-distance (that is not Einstein’s one) ([Y. Piereaux 2006, Y. Piereaux 2004, Y. Piereaux 2007]). We prefer here to take as point of departure basic Minkowski’s diagram (and basic Einstein’s synchronization) at only one space dimension $x$. In this case Poincaré’s ellipse is reduced to only one point but we will see that one point $M$ (Fig3-4) is sufficient for initiating the Revolution (of ellipse around Ox).

2 New symmetry duration-distance from Minkowski’s calibration hyperbolas

Let us consider fundamental hyperbolas along $Ot$ and $Ox$ in Minkowski’s space-time with the axis, $x, t$ of system $K$ (observer) and with light velocity $c = 1$ (Fig1). The symmetrical scale (or calibration) hyperbolas determine the space-time units of measure ($x^2 - t^2 = \pm 1$) with the invariance by $LT$ (8) of timelike interval $t^2 - x^2 = t'^2 = T^2$ or spacelike interval $x^2 - t^2 = x'^2 = D^2$ ($\gamma = (1 - \beta^2)^{-\frac{1}{2}}$) ([H. Minkowski 1908]):
\[ x' = \gamma(x - \beta t) \quad t' = \gamma(t - \beta x) \quad x = \gamma(x' + \beta t') \quad t = \gamma(t' + \beta x') \] (8)

The light asymptotes and the standard hyperbolic rotation (HR: axis \( x', t' \) "in scissors") of system \( K' \) are represented on FigA or Fig1 (Y. Pierseaux (2009), http://arxiv.org/abs/0904.3332). Minkowski’s proper duration \( T \) (a timelike interval) is determined by the duration between two events-at-the-same-place \( x' = 0 \) at \( O' \) in \( K' \), we define proper distance (a spacelike interval) \( D = O'P' \) by the distance between two "two events-at-the-same-time" \( t' = 0 \) in \( K' \) (simultaneous events, see Einstein’s synchronization, §3). Observed in \( K \), Einstein’s dilated duration \( t_\gamma \) is determined by "two events-non-at-the-same-place" (8), \( (0, 1) \xrightarrow{LT} \gamma \). We focus the attention on light-point \( M' \), \( L,T \rightarrow M(k, k) \) with \( k = \gamma(1 + \beta) \) (8-2, \( M \) is a point of Poincaré’s ellipse, note 2, §3-5)

\[ t_k = kT \quad x_k = kD \quad \Rightarrow \quad \frac{D}{T} = c = 1 = \frac{x_k}{t_k} \] (8.1)

These basic proportions (\( \gamma - \text{dilation} \) 8-1 & \( k - \text{dilation} \) 8-2) are not possible with Einstein’s standard contraction of distance (10). We note that unlike \( \gamma - \text{dilation}, k - \text{dilation} \) involves \( O'M' \xrightarrow{LT} \text{OM} \) and therefore a transformation of proper distance \( D \) with respect to \( O' \) into a "true" distance \( x_k \) with respect to \( O \). Until now our definition 8-2 however is a purely geometrical definition without physical meaning.

### 3 Einstein’s synchronization and Poincaré’s proper light-distance

Let us now examine in details Einstein’s physical procedure of synchronization [A. Einstein (1905)] in order to define physically the new distance (8-2).

#### 3.1 Synchronized clocks, rigid system and rigid rod (Einstein 1905)

**(Stationary Mirror)** Let us first consider Einstein’s rigid rod \( x_M = O'M' = L \) at rest in system \( K' \), with a light source at \( O' \) and a mirror at \( M' \). A light signal is emitted from \( O' \) at \( t' = 0 \), it is reflected in \( t' = T \) at \( M' \) and returns to \( O' \) at \( t' = 2T = 2L/c \) \( (c = 1) \), the "time out" \( T \) being equal to the "back time" \( T \). Einstein’s clock synchronization uses three successive physical events 1, 2, 3:

\[ O'(0, 0)_1 \quad M'(L, T)_2 \quad O'(0, 2T)_3 \quad O'M' = M'O' \] (9)

Einstein synchronizes the two clocks at the ends, \( O' \) & \( M' \), of the rigid rod by defining the simultaneity of two events "at a distance". These two simultaneous events \( (0, T)_2 \) & \( (L, T)_2 \) are however not explicitly written in Einstein’s 1905 paper. Given that each end of rigid rod \( L \) is defined in \( K' \) for any time \( t' \), the length in \( K \) is defined by Einstein with simultaneous \( (t = 0) \) positions of the ends in \( K \) and therefore by first LT (8)

\[ x' = \gamma(x - \beta t) \Rightarrow x_M' = \gamma x_M \Rightarrow x_M = \gamma^{-1}x_M' = \gamma^{-1}L \] (10)

This is Einstein’s kinematical interpretation of Lorentz contraction \( \gamma^{-1} \).

#### 3.2 Synchronous distance, abstract system and light-distance (Poincaré 1908)

**(Stationary Mirror)** Consider now the same situation but without Einstein’s rigid rod (given a priori) and with a single clock in \( O' \). The mirror \( M' \) is at rest in \( K' \) (a distant reflecting object) at an unknown distance. A light signal is emitted at \( O' \) at \( t' = 0 \), reflected in \( M' \) in \( T \) and returns to \( O' \) at \( t' = 2T \):

\[ O'M' = D(\text{radar}) = T(\text{Minkowski}) \quad \text{(round trip)} \] (11)
The "synchronous" distance $D$ may be measured by a single clock in $O'$ with a $\frac{1}{2}$"round trip" $2T$ ("two-ways") signal. Such a distance (Bondi’s radar method, [H. Bondi]). Until now nothing is changed because we can replace Einstein’s rigid rod $L$ (ν′t′) by "one half light travel time distance". Suppose now that "synchronous" distance $D$ be a proper distance basically defined by the difference of space coordinates $\Delta x' = D$ between two simultaneous events in $K'(t' = T)$:

\[(0, T)_{2} \quad \text{and} \quad (D, T)_{2} \quad \text{FigA or Fig3-4:} \quad O' (0, 1)_{2} \quad \text{and} \quad M'(1, 1)_{2} \quad (12)\]

This definition $t' = T$ in $K'$ is not compatible with Einstein’s definition $t = 0$ in $K$ (10) because the simultaneity is relative: if both ends of such a distance (12) are given at the same time $T$ in $K'$, they cannot be determined at the same time in $K$ (10). We add here a new element because the reciprocal (Poincaré) interpretation of Einstein’s synchronisation involves that exactly as "simultaneity at a distance" cannot be defined without the velocity of light, the distance itself cannot be defined without this velocity. This proper light-distance $D$ then becomes an "invariant" of LT ($x^2 - t^2 = D^2 = s^2$) in the same sense as the proper duration $T$ is an "invariant" of LT ($t^2 - x^2 = T^2 = s^2$). In summary Poincaré’s proper distance (hyperbola along $Ox$) is the exact symmetric of Minkowski’s proper time (hyperbola along (Ot)). Poincaré’s interpretation of Lorentz contraction involves that proper distance $D$, like proper time, is the shortest in $K'$ (in Einstein’s (10), it is the longest in $K$). "This Lorentz hypothesis is the immediate translation of Michelson’s experiment, if the lengths are defined by the time that light takes to travel through them"([H. Poincaré 1908]). As a last analysis and from a historical viewpoint, Einstein’s work is based on the direct theorem (the $O't'$ axis), while Poincaré’s opened the way to the reciprocal (the $O'x'$ axis).

3.3 Poincaré’s $k-$dilated round-trip light-distance ”Observer-(receding)Mirror”

(Receding Mirror). Let us examine now the light-distance from system $K$ where the mirror is receding from $O$. Travel-duration in $K$ is given by the difference of time coordinates $\Delta t = 2\gamma T$ between two not-at-the-same-place events,

\[O'(0, 0)_{1} \quad \& \quad O'(0, 2T)_{3} \xrightarrow{LT} (0, 0)_{1} \quad \& \quad (2\gamma\beta T, 2\gamma\beta T)_{3} \quad \Delta t = 2\gamma T = 2t_\gamma \quad \text{travel-duration} \quad (13)\]

Automatically (11) involves that light-"distance" $O'M'$ in $K$ is given by $\frac{1}{2}\Delta t = x_\gamma$ (8-1). Obviously the one-way $O'M'$ in $K$ is also given by the difference of $x$ coordinates $\Delta x = \gamma T$ between two not-at-the-same-time events

\[O'(0, 0)_{2} \quad \& \quad M'(D, 0)_{2} \xrightarrow{LT} (0, 0)_{2} \quad \& \quad (\gamma D, \gamma\beta D)_{2} \quad \Delta x = \gamma D = x_\gamma \quad \text{light-"distance"} \quad (14)\]

If distance is really time (see Penrose, conclusion), the so formed distance $D$ must be LTed (8) into $\gamma D$ like the duration. Unfortunately such a $\gamma = \text{dilated "distance"}$ (14) is not a physical distance because until now we have only considered $O'M'$ but not the distance between the receding mirror and the observer $O$. We have only deduced round-trip light "distance" $O'M'O'$ in $K'$ (11) and in $K$ (14) but not round-trip light distance $OMO$ in $K$. With our new symmetry duration-distance we can calculate $OM$. We know that the signal is in $O'$ in $2\gamma T$ (13) and that at this $K - time$ the $K - distance$ $OO'$ is $2\beta\gamma T$ (13). We deduce the total travel distance $2\gamma T + 2\beta\gamma T$. Automatically (11) involves that light-distance $OM$ in $K'$

\[OM = \sqrt{\frac{1 + \beta}{1 - \beta}} D \quad (15)\]

In the case of stationary mirror (11) and receding mirror (15) the light-distance is given by half total travel duration as well. This $k = \text{dilated light distance}$ is Tolman’s luminosity distance except that Tolman’s distance is a one-way distance $OS$ (see FigA or Fig3-4, point $M(x_k, t_k)$).

3.4 Poincaré’s one-way $k-$dilated light-distance, Tolman’s luminosity-distance and relativistic ”Doppler” formula

Let us consider in details the LT of Einstein’s first two (1 and 2) successive events $O'M'$

\[O'(0, 0)_{1} \quad \& \quad M'(D, T)_{2} \xrightarrow{LT} (0, 0)_{1} \quad \& \quad (\gamma(D + \beta T), \gamma(T + \beta D))_{2} \quad (16)\]
The one-way forth light-distance $\Delta x$ is given by difference (final and initial) of space coordinate $\Delta x = x_f - x_i$ or time coordinate $\Delta t = t_f - t_i$ as well (light-distance=travel-duration)

$$\Delta x_{\text{forth}}(O' \rightarrow M') = \gamma(1 + \beta)T = \sqrt{\frac{1 + \beta}{1 - \beta}}D \quad (17)$$

This is the point (Fig A or FigB-4) $M'(1,1) \xrightarrow{LT} M(k,k)$. With (2 and 3) successive events $M'O'$ the one-way back light-“distance” $\Delta x$ is given by difference (final and initial)

$$M'(D,T)_2 \& \ O'(0,2T)_3 \xrightarrow{LT} (\gamma(D + \beta T), \gamma(T + \beta D))_2 \& \ (2\gamma\beta T, 2\gamma T)_3 \quad (18)$$

of space coordinate $\Delta x = x_f - x_i$ or time coordinate $\Delta t = t_f - t_i$ as well

$$\Delta x_{\text{back}}(M' \rightarrow O') = \gamma(1 - \beta)T = \sqrt{\frac{1 - \beta}{1 + \beta}}(-D) \quad (19)$$

We see that if $\Delta t$ is always positive, that is not the case for algebraic $\Delta x$ : we have $T = D$ if the travel light is in positive $Ox$ sense and $T = -D$ in negative sense (note 3). We rediscover $(\sqrt{\frac{1 + \beta}{1 - \beta}}O'M') + (\sqrt{\frac{1 - \beta}{1 + \beta}}M'O) = 2\gamma$ and given that $\Delta x_{\text{oneway}}(O' \rightarrow M') \neq \Delta x_{\text{oneway}}(M' \rightarrow O')$, the round-trip definition of distance seems to be in physics the only one possible\(^1\). However until now we have only considered one-way $O'M'$ and $M'O'$ in $K'$ and in $K$ but not the observer $O$. According to observer $O$, the receding mirror is $M$. Let us consider now one-way light distance $OM$ and $MO$. In the first case (16) $OM$ is given by (17) the light travel duration from the emission in $O$ and the reception in $M$ and therefore $OM = \Delta x_{\text{oneway}}(O' \rightarrow M')$. In the second case $MO$ is not given by (19), $MO \neq \Delta x_{\text{oneway}}(M' \rightarrow O')$ but by light travel duration from the reflection in $M$ and the reception in $O$ (and not $O',19$). Given that light signal is in $O'$ at $\gamma(1 + \beta)T + \gamma(1 - \beta)T = 2\gamma T$ (18) and that at this $K$ – time the $K$ – distance $O'O'$ is $2\beta\gamma T$(18) we have a consistent new definition of $k$–dilation light-distance with $\gamma(1 - \beta)T + 2\beta\gamma T$, 15).

$$OM = MO = \sqrt{\frac{1 + \beta}{1 - \beta}}D = kT \quad (20)$$

In summary, given that source and mirror are at rest in $K'$ and that $O'$ and $O$ coincide in $t = t' = 0$, we have Einstein’s equality of one way travel time $O'M' = M'O'$ and Poincaré’s equality $OM = MO$ as well\(^2\) This is a physical new definition of distance if we reverse the situation ”source-mirror” in $K'$; the remote source $S$ is now at proper distance $D$ in $K'$ at $t = t' = 0$ when $O$ and $O'$ coincide and the mirror of the telescope is in $O$. What is the cosmological light-distance $OS$ or $MS$? . Two events, ”coincidence” $O \equiv O' (0,0)$ and ”emission”, $(D,0)$ are simultaneous\(^3\) in $K'$ but not in $K(0,0)$ and $(\gamma D, \gamma\beta D)$. Then the time of emission $t_i = t_e$ is not the same in $K'$ and $K$. Given that the signal is in $O'$ in $\gamma T$ and the distance $OO' = \gamma\beta T$, the total duration until $O$ gives $\gamma T + \gamma\beta T = \gamma\tau T(1 + \beta) = \sqrt{\frac{1 + \beta}{1 - \beta}}T$. We obtain now the identity between Poincaré’s light-distance and Tolman’s luminosity-distance $D_r$(5).

$$OS_{\text{LIGHT}} = \sqrt{\frac{1 + \beta}{1 - \beta}}D = D(1 + z) = OS_{\text{LUMINOUS}} \quad (21)$$

where $D$ is a proper or a comoving distance. Unlike $\gamma$ – diluted distance (14), $k$ – diluted distance is a physical distance. We can also obtain this basic result with $t_e - t_i > 0$ by taking into account the negative sense of travel light. $D = -T$ ; $S(-T,0)_i$ \& $O'(0,0)_f \xrightarrow{LT} (-\gamma T,-\gamma\beta T)_e$ \& $(\gamma\beta D, \gamma T)_f$ and therefore (21, with the same method we transform 19 in 20).

Suppose now that the proper light-distance $D$ of a very remote monochromatic source $S$ is unknown and only the length-wave $\lambda_S$ and the intensity of source $L_S$ (absolute luminosity) are known. We have then a basic proportionality distance-lengthwave with a new redshift Light-Luminosity Distance

\(^1\)But imagine a physical situation where the round trip is by definition impossible (in Cosmology).

\(^2\)M is the only (right) point of Poincaré’s elongated ellipse in $K$ with the observer $O$ at the (left) focus and the source at the center $O'$. (Y. Pierseaux 2006)

\(^3\)According to a non-relativistic calculation (absolute simultaneity) the light signal catch up with $O$ at the distance $\frac{D}{1 + \beta}$. In relativistic point of view we have rigorously proved (15 and 20, the principle of inverse return of the light) that we have $\sqrt{\frac{1 + \beta}{1 - \beta}}D$. 

\(^3\)
\[
\frac{D_r}{D} = \sqrt{\frac{1 + \beta}{1 - \beta}} = \frac{\lambda_{\text{Obs}}}{\lambda_{\text{Source}}} = k = 1 + z
\]  

(22)

This is obvious because Poincaré’s length is basically a travel length by a wave. So we deduce Tolman-Robertson’s law (5), coupled with relativistic” Doppler4 redshift law (22), directly from LT and new definition of distance (inscribed in Minkowski’s diagram, FigA or Fig1-3). This is the reason why we suggest to call the new SR with Cosmological Relativity (CR). The \( k \)-dilation (21 or 22) is a law of expanding universe. We note however that Tolman-Robertson’s law (5) is based on spherical waves and thus on an area on the mirror of the telescope: we have now to prove that this law is immediately deductible from LT at 3 dimensions.

3.5 Tolman’s double reduction, Double Reduction of solid angle of emission and Poincaré’s space-time light ellipsoid

Light-distance by travel-duration can be generalized at 3 space dimensions; \( r(x, y, z) \) & \( r'(x, y, z) \), with the norms \( r = t \) and \( r' = t'(c = 1) \) and with a source emitting a spherical wavefront in \( K' \). Given that the azimutal angle is Lorentz invariant (ellipsoid of revolution, see §1), we consider only the angle \( \theta \) and \( \theta' \) (respectively in \( x, y \) and \( x', y' \) planes) and solid angle of the light cone of emission \( \Omega = 2\pi(1 - \cos \theta) \) LTed into \( \Omega' = 2\pi(1 - \cos \theta') \).

\[
\begin{align*}
    x &= \gamma (x' + \beta t') \\
    r \cos \theta &= \gamma (r' \cos \theta' + \beta r') \\
    y &= y' \\
    r \sin \theta &= r' \sin \theta' \\
    t &= \gamma (t' + \beta x') \\
    r &= \gamma r'(1 + \beta \cos \theta')
\end{align*}
\]

(23)

and we rediscover Einstein’s aberration formula \( \cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'} \), Penrose’s formula \( \tan \frac{1}{2} \theta = k^{-1} \tan \frac{1}{2} \theta' \) and polar equation of Poincaré’s ellipse \( r = t = \frac{\rho}{\gamma (1 - \beta \cos \theta)} \) where \( \rho \) corresponds to \( D (M \in \text{ellipse, FigA or Fig3-4, note 2, [Y. Pierseaux 2006]}) \). The isotropic (spherical) emission of a moving source (\( S \) in \( K' \)) is anisotropic (ellipsoidal5) observed from \( K \) with relativistic transformation of the solid angle ([H. Poincaré 1908])

\[
\Omega = \Omega' \frac{1}{\gamma^2 (1 + \beta \cos \theta')^2}
\]

(25)

This is reduction of the angle of aperture of the cone of emission of a moving source. For small angle \( \theta' \) we have no aberration (Robertson, the motion is along the line of sight) but a headlight effect (Lorentz reduction of solid angle given that we have necessarily a small area on the mirror of telescope):

\[
d\Omega = d\Omega' \frac{1 - \beta}{1 + \beta} = d\Omega' k^{-2} = d\Omega' \frac{1}{(1 + z)^2} \\
r = r' \sqrt{\frac{1 + \beta}{1 - \beta}} = k \rho = (1 + z) \rho \\
(d\theta = k^{-1} d\theta')
\]

(26)

Where \( r \) corresponds to \( D_r \) (20 or 21). And so we deduce directly the law of Tolman-Robertson from new fundamental relativistic invariant in CR the element of “perpendicular” area must be an invariant (purely longitudinal6 elongation of Poincaré’s ellipse, [Y. Pierseaux 2004])

\[
dS = r^2 d\Omega = dS' = r'^2 d\Omega' \\
d\Omega = \frac{d\Omega}{dS'} = \frac{l_{\text{recession}}}{l_{\text{stationary}}}
\]

(27)

Poincaré’s double reduction of angle of aperture of emission cone (26) is exactly Tolman’s double reduction of luminosity (2). Poincaré’s space-time elongated light ellipsoid7 is therefore a direct explanation of Hubble’s expansion (Observer is at the focus of the meridian section of ellipsoid and therefore the geometrical measure of solid angle, steradian, is not a Lorentz invariant). We must prove now that Poincaré’s light-distance necessarily involves an Lobatchevskian distance with an Horizon.

4CR is based not on plane wavefront but on spherical wavefront ([Y. Pierseaux 2007])

5Poincaré’s ellipsoid of revolution is the direct kinematical explanation of the very physical “headlight effect” (in synchrotron radiation, bremsstrahlung...) The LTed ellipsoidal wavefront is an equiphase surface (a non-transverse section in Minkowski’s cone)

6According to Poincaré, the ellipsoid is elongated because units, meters and steradians are reduced [H. Poincaré 1908]. Of course the right angle of perpendicular area is LTed.

7Let us remark that Robertson, in order to prove Tolman’s formula, uses Einstein’s 1905 formula of LTed volumic density of energy in Complex of Light. But Einstein’s spherical Complex is also LTed into an ellipsoid ([A. Einstein (1905), 8]).
4 Hyperbolic velocity, hyperbolic distance and hyperbolic Hubble’s law

In Friedman-Lemaître’s model (in Robertson-Walker’s metric), the theoretical Hubble law is defined by an apparent velocity \( V_{\text{exp}} \) of expansion of geometrical space itself \( R(t) \), the scale factor, that is not limited by the velocity of light
\[
V_{\text{exp}} = \dot{R}(t) = H(t)R(t)
\]
(28)
In standard model \( RW \) the ”constant” \( H(t) \) of Hubble is defined by its present value \( H \) and the law of expansion is not connected with relativistic Doppler’s formula in SR (scale factor \( k \)). However, the experimental measurements are made not on the space itself but on the moving bodies. So the empirical form of Hubble’s law is a relation between spectral redshift \( z \) and distance \( \rho \) (or \( D \)) of remote objects; deduced from Tolman’s law (5) generally written
\[
v = cz = H \rho \quad \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}} = 1 + \frac{v}{c} = 1 + z
\]
(28bis)
with non-relativistic Doppler law ([E.P.Hubble & R.C. Tolman]). The ”constant” \(^8\) of Hubble, that is defined by this empirical law is directly confirmed when the redshift shift is small compared with unity \( v << c \). When this is not the case, for example quasar 3C9 a wave length ratio of 3.01 \((z = 2.01)\) for which \( v > c \) a correction with Einstein’s Doppler law is necessary \( 1 + z = k \Rightarrow v/c = 0.8 \). Thanks to this correction on velocity \( v \), we have \( \rho < \text{present} \) \( R(t) \) with \( \text{present} \) \( R(t) = \frac{c}{H(t)} \). So we have the paradox in standard \( RW \) that (28) has nothing to do with SR (non-Minkowskian metric) whilst its experimental form (28bis) is directly connected with SR but only with spectral lengthwave \( \lambda \) (not for length ”itself” \( \rho \)). And we showed that in CR lengthwave and length are LTed in the same way (22).

How can we deduce rigorously a basic law of Hubble, i.e. a basic proportionality between \( \beta \) and \( \rho \) in CR (with LT)? The equation \( r = \rho \sqrt{\frac{1 + z}{1 - z}} \) (26) or (21) suggests an hyperbolic definition of Poincaré’s light distance in the meaning of Cayley and Klein. In Beltrami’s model of hyperbolic geometry : a circle of radius \( R_H \) is regarded as an horizon (a circle ”at infinity”) and a straight line is interpreted as a line segment within this circle. Cayley and Klein define an hyperbolic distance by the cross-ratio formula. Consider the hyperbolic radial distance \( r_H \) from origin of the circle to a point \( P \) with Cartesian distance \( \rho \):
\[
r_H = \frac{R_H}{2} \ln \frac{R(R + \rho)}{R(R + \rho)} \Rightarrow r_H = R_H \arctanh \frac{\rho}{R_H} = R_H \ln \sqrt{1 + \frac{\rho^2}{R_H^2}}
\]
(29)
By taking the Neperian logarithm (number \( e \)) of the fundamental formula (22) it turns out \((c = 1)\):
\[
\beta_H = \ln \sqrt{1 + \frac{1 + \beta}{1 - \beta}} = \ln \frac{\lambda_{\text{obs}}}{\lambda_{\text{source}}} = \ln k = \ln(1 + z) = Z
\]
(30)
where \( \beta_H \) is the hyperbolic velocity in \( SR \). So if we compare (29) and 30, we have a fundamental Hyperbolic proportionality between \( \beta_H \) and \( r_H \) if and only if \( \frac{R}{H_R} = \beta \) (\( \Rightarrow k\beta = Hr \)). We note \( Z \) as the logarithmic spectral redshift\(^9\) ([J. F. Barrett (Luminosity)]):
\[
\beta = H \rho \quad \beta_H = Hr_H = Z \quad (c\beta = Hr) \quad (c\beta_H = Hr_H)
\]
(31)
hyperbolic or non-hyperbolic expression of Hubble’s law which becomes a basic law of CR completely defined by LT (8) and therefore by Hyperbolic Rotation \( HR \) (\( \beta_H \) is hyperbolic angle of rotation). Given that in any Rotation motion, there is an acceleration, Hubble’s constant appears as a basic Hyperbolic Angular Velocity (Euclidean angular velocity \( v = \omega R \)). Let us examine if Hubble constant correspond to a basic Hyperbolic Acceleration \((Hc = \alpha_M = \frac{c^2}{R_H})\) ([Y. Pierseaux (2009)]). Suppose Einstein’s basic boost where \( K’ \) (\( d\tau \) element of proper time) is uniformly accelerated (from 0 to \( \beta_H \)) with respect to \( K \)
\[
Hc = \alpha_M = \frac{c^2}{R_H} \quad \text{then} \quad \frac{d\beta_H}{dt} = \gamma^2 \frac{d\beta}{dt} \quad \frac{d\beta_H}{d\tau} = \gamma^2 a = \beta_H = \alpha_M \quad (c = 1)
\]
(32)
\(^8\)The cosmologists introduced a magnitude without dimension \( h_0 \) a fraction of \( 100 \text{km/sec/Megaparsec} \) (\( h_0 = 0.5’\)).
\(^9\)Given that \( Z \) is a strictly increasing function of the wavelength ratio which is zero when \( \frac{\lambda_{\text{observer}}}{\lambda_{\text{source}}} = 1 \), \( Z \approx z \) when \( z \) is small and for infinitesimal wavelength shift we have \( \delta Z = \delta z = \frac{\Delta \lambda}{\lambda} \) (logarithmic derivation).
$\alpha_M$ being a minimal non-null norm of spacelike 4-vector of acceleration (a can be as small $\gamma^{-3}$ as we wish). In standard $SR$, given that $HR$ is not a motion, we have for active $LT$ or $HR: \frac{d^2\beta}{d\tau^2} = 0$. Suppose now that $HR$ (8) be a physical motion (Born-Rindler’s hyperbolic "rigid" motion, (Fig2, Y. Pierseaux (2009))) we have

$$\frac{d\beta_H}{d\tau} = H = \frac{1}{k} \frac{\delta z}{1 + z}$$

(33)

In standard static metric we have obviously $k = const$. Recent observations indicate a variation $\delta z$ in our proper time $\delta \tau$. In Cosmology we measures always $z$ and $D_r$ but never $R(t)$. This ad hoc scale factor in RW’s metric with an absolute time $t$ can be eliminated with Occam’s razor in HCR (Hyperbolic Cosmological Relativity). By integration with standard initial conditions of basic Lorentz boost ($O \equiv O', t = t' = 0$)

$$d\beta_H = H d\tau \implies \beta_H = H \tau \implies r_H = \tau$$

(34)

We deduce the physical relativistic meaning of $r_H$: hyperbolic distance is proper time $\tau c$ (see Penrose, conclusion) in a basic Einstein’s boost or basic HR (where $H$ is the angular velocity). We have used only the LT and the whole LT (HR: passive-LT but also "active-LT or Einstein boost").

5 Conclusion: From Penrose’s Lobatchevskian SR to Cosmological Hyperbolic Relativity

The question now is: how can we modify or adapt standard SR to the new symmetrical hyperbolic light-distance? Nothing is changed about proper time (and therefore about standard relativistic dynamics). Penrose writes about SR and Hyperbolic geometry:

I said that I like hyperbolic, Lobatchevskian geometry the best. One of the reason is that the group of symmetries is exactly the same as.. the Lorentz group, the group of SR. (...) Distance in Minkowskian geometry is time, the proper time that is physically measured by moving clocks. It turns out that the intrinsic geometry of the "sphere" (in Minkowskian space-time) is Lobatchevskian hyperbolic geometry [R. Penrose].

It could be argued that with a rigorous definition of light-distance by proper time, nothing is changed with standard Einstein’s asymmetrical contraction of distance (“Gedanken experiment” never experimentally observed). But if distance is proper time ($§3$), it must be dilated, in rest frame like... proper time. We showed that Einstein’s (contracted) rigid rod is no longer valid for Cosmological distances (in light-years). If distance is proper time (from O’ in K’), light-distance (from O in K) must be $k – dilated$ with travel duration (21). Such a distance determines not an Euclidean rigid rod by an Hyperbolic distance. We showed finally that Hyperbolic Distance $r_H$ is directly proportional to Hyperbolic Velocity $\beta_H$ (law of Hubble, 31) but also that $r_H$ is proper time $\tau$ (34) in elastic motion (Born-Rindler "rigid" motion without Einstein’s rigid rod).

Unlike Galilean invariant with Euclidean distance defined by plus (+) signs $r^2 = x^2 + y^2$, Lorentz invariant involves one minus (−) sign. So a standard objection could be that SR is already hyperbolic because (at one dimension for example, FigA) we have a minus sign in particular for scale hyperbola along Ox $x^2 - t^2 = x^2 (t = 0)$.

Standard SR (signature: $(1,−1)$ is not completely Lobatchevskian because Lobatchevskian geometry involves necessarily an Horizon $x' = R_H$ (and a curvature $\varrho_H = \frac{1}{R_H}$), unlike Euclidean geometry which involves $x' \rightarrow \infty$ (without horizon and flat $\varrho_E = 0$). So when all physicists, during more than one century, write

$$x^2 - t^2 = \varrho (t = 0)$$

(LEFT Member: Hyperbolic) $x^2 - t^2 \rightarrow \infty$ (RIGHT Member: Euclidean, Flat) (35)

they introduce an Euclidean definition of infinity in the second member of Minkowski’s basic invariant: this Euclidean flatness is a "stranger" in an hyperbolic interval $(1,−1)$. So they obtain the standard flat pseudo-Euclidean geometry. What does it mean physically? Minkowski claimed, that "space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality". But an "infinite" interval$^{10}$ ($x^2 - t^2 = \infty, \forall t$) should mean that independent space is given for any $t$ and therefore the Return of the Shadow (Absolute Space Ox, $\forall t$). So the pseudo-Euclidean flatness promotes

$^{10}$If we delete scale hyperbolas we have a non-relativistic infinite interval and a flat space-time
the Return of the Phantom (Remake II: the dark energy\textsuperscript{11}?). It is therefore logically essential to put an horizon in (35).

\[ x^2 - t^2 = R_H^2 = \varrho_H^{-2} \]

Horizon \( R_H \), Global negative Curvature \( \varrho_H \) (36)

Only (36) is consistent. This recalibration is also a renormalization of Minkowski’s metric because we eliminate Euclidean inappropriate infinity. In this way, we transform pseudo-Euclidean SR into Hyperbolic Cosmological Relativity (HCR).

We rediscover our equation (2) of our \( \Lambda \)-paper ([Y. Pierseaux (2009), equation 2]) because Cosmological Constant \( \Lambda = \varrho_H^2 \) is the square of Lobatchevskian basic global (negative) Curvature. Given that LT is an Hyperbolic Rotation (Fig A) and that in any Rotation, there is an acceleration (Einstein’s boost) we showed, from Born-Rindler’s accelerated ”rigid” (?) motion ([W. Rindler (SR)]), that Hyperbolic Rotation Motion is an elastic\textsuperscript{12} motion involving a centrifugal\textsuperscript{13} acceleration (coupled with expanding distance). With the horizon of Hubble, we deduce a basic Hyperbolic Acceleration in basic Einstein’s boost that is Milgrom’s minimal proper acceleration \( \alpha_M = \frac{x^2}{R_H} \) (MOND\textsuperscript{14}, [M. Milgrom MOND]). Hubble constant is a basic Hyperbolic Angular Velocity (\( H_C = \alpha_M \)). We rediscover our equation (4) in \( \Lambda \)-paper

\[ \frac{1}{R_H} = \varrho_H = \Lambda = \frac{\alpha_M^2}{c^4} = \frac{H^2}{c^2} \quad (c = 1) \]  (4-\( \Lambda \))

In HCR true constants \( R_H, H, T_H \) are directly connected to constant \( \Lambda (c = 3 \times 10^{10} \text{cm/s}, H \approx 10^{-18} \text{s}, R_H \approx 3.10^{-58} \text{cm}, \alpha_M \approx 3.10^{-54} \text{cm/s}^2 \Lambda \approx 10^{-57} \text{cm}^{-2}) \). Can we expect with hyperbolic distance a small adaptation or a very large adaptation of SR? In one sense it is a small adaptation because Minkowski’s signature is unchanged. The ”renormalization” (recalibration) of standard flat element of Minkowski’s metric is

\[ dt^2 - dx^2 = d\tau^2 = \alpha_M^{-2} d\beta_H^2 = \varrho_H^{-2} d\beta_H^2 \]  (34-\( \Lambda \))

But in another sense it is a very big adaptation because we do not need in Cosmology the standard factor of scale \( R(t) \) (28) in \( RW \)’s metric (we underline that hyperbolic metric for \( \kappa < 0 \) in \( RW \) obviously is not Minkowskian).

So the standard hypothesis (Lemaître-Gamow-Paccelli) \( R(t) = 0 \) can be replaced by a well-tempered (in proper time \( \tau \)) ”steady state” infinite (Hyperbolic) space-time (Hoyle-Bondi).

On an historical point of view there are two curious ideas in Poincaré’s work with respect to Einstein’s standard SR: the light ellipsoid (kinematics) and the negative gravitational pressure of classical vacuum (dynamics). Poincaré unfortunately has had no enough time to develop both ideas that are today completely forgotten. HCR is a kinematical synthesis between Einstein’s kinematics (5) and Poincaré’s light distance (8). In our next paper we will develop the corresponding dynamics and more precisely ”ELECTRO-dynamics”. We will prove that in CEMB (Cosmological ElectroDynamics of Moving Bodies), Poincaré’s 1908 (negative) pressure of classical vacuum ([H. Poincaré 1905]) and Einstein’s 1917 cosmological constant are directly connected.

The irony of history is that the complete hyperbolization of SR (30) is induced by a non-transversal section in Minkowski’s cone an therefore by an... ellipsoid which transforms pseudo-euclidean \( \pi - \text{relativity} \), given that (ste)radian is no longer an invariant, into Lobatchevskian \( e - \text{relativity} \). (Penrose). According to Poincaré, in french dans le texte ”les cercles divisés dont nous nous servons pour mesurer les angles sont déformés par la translation, ils deviennent des ellipses” ([H. Poincaré 1908]).

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\textsuperscript{11}It is a remake of Absolute physics because dark matter is not dark energy! (\( E = mc^2 \), Einstein 1905)

\textsuperscript{12}We will show in CEMB that the elasticity constant is \( G^{-1} \) (the inverse of gravitational constant in units \( c = 1 : g/cm \)).

\textsuperscript{13}Euclidean Rotation Motion involves a centripetal acceleration

\textsuperscript{14}In HCR we have therefore a basic proportionality with the inverse of distance and not with the inverse-square of distance.
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Minkowski’s hyperbolic interval: with or without Lobatchevskian Horizon?

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**FigA** New Symmetry in Minkowski’s scale hyperbolas $(\gamma, \gamma)$: Proper duration $T = 1$ in $K'$ with dilated duration $t = \gamma \approx 1.16$ in $K$. Proper distance $D = O'P' = 1$ in $K'$ with dilated “distance” $x = \gamma \approx 1.16$ in $K$. The new distance will be given (§3) by $(1, 1) \xrightarrow{LT} (k, k)$ with $k = \sqrt{\frac{1+\beta}{1-\beta}} \approx 1.7$. With $D = RH$ (the scale of Hubble) we have naturally a Lobatchevskian Horizon in Hyperbolic interval. $x^2 - t^2$