On Distributed Dynamic Pricing of Multiscale Transportation Networks

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Abstract—We study transportation networks controlled by dynamic feedback tolls. We focus on a multiscale model whereby the dynamics of the traffic flows are intertwined with those of the drivers’ route choices. The latter are influenced by the current congestion through the network as well as decentralized congestion-dependent tolls set by the system planner. Our main result proves that a class of decentralized monotone congestion-dependent tolls allow for globally stabilising the transportation network around a generalized Wardrop equilibrium. In particular, our results imply that using decentralized marginal cost tolls, stability of the dynamic transportation network is guaranteed to be around the social optimum traffic assignment. This is particularly remarkable as such feedback tolls can be computed in a fully local way without the need for any global information about the network structure, its state, or the exogenous network loads. Through numerical simulations, we also compare the performance of such decentralized dynamic feedback tolls with constant off-line (and centrally) optimized tolls both in the asymptotic and in the transient regime and we investigate their robustness to delays.

Index Terms—Transportation networks, distributed control, user equilibrium, congestion pricing, dynamical flow networks.

I. INTRODUCTION

Over the past years there has been an increasing interest in the control systems analysis and synthesis of dynamical transportation networks. Such interest is especially motivated by the wide-spaying sensing, communication, information, and actuation technologies that are dramatically changing the transportation system dynamics and affecting the users’ decision making and behaviors. There is a growing awareness that the new opportunities and risks created by these technologies can be fully understood within a dynamical network framework.

Traditional traffic flow control strategies include mechanisms such as variable speed limits, ramp metering, or traffic signal control (see [1]–[4] and references therein). While such mechanisms do not consider neither the drivers’ perspective nor affect the total amount of vehicles, there has been also a significant research effort to understand the drivers’ answer to external communications from intelligent traveller information devices (see, e.g., [5]–[6]) and, in particular, studying the effect of such technologies on the drivers’ route choice behaviour and on the dynamical properties of the transportation network [7]. A traffic recommender which can announce potentially misleading travel time information and a new class of latency functions so as to influence the drivers’ behaviour was studied in [8] and [9], respectively. Moreover, it is known that if individual drivers make their own routing decisions to minimize their own experienced delays, overall network congestion can be considerably higher than if a central planner had the ability to explicitly direct traffic. Accordingly, to charge tolls for the purpose of influencing drivers to make routing choices that result in globally optimal routing was a central research focus (see [10–15]).

In this paper, we extend the model and results of [7] by introducing decentralized congestion-dependent tolls in order to influence the driver’s route choice behaviour. Specifically, we consider a multiscale dynamical model of the transportation network whereby the traffic dynamics describing the real time evolution of the local congestion level are coupled with those of the drivers’ path preferences. We assume that the latter evolve following a perturbed best response to global information about the congestion status of the whole network and to decentralized flow-dependent tolls.

Our main result shows that by using non-decreasing decentralized flow-dependent tolls and in the limit of a small update rate of the aggregate path preferences, the transportation network globally stabilises around the Wardrop equilibrium [16]. The latter is a configuration in which the perceived cost associated to any source-destination path chosen by a nonzero fraction of drivers does not exceed the perceived cost associated to any other path. As in [7], we assume that the drivers’ path preferences evolve at a slower time scale than the physical traffic flows and adopt a singular perturbation approach [17] to the stability analysis of the ensuing multiscale closed-loop traffic dynamics. Note that classic results of evolutionary game theory and population dynamics [18–19] cannot be applied to our framework since they suppose that the access to information take place at a single temporal and spatial scale and that the traffic dynamics are neglected by assuming that they are instantaneously equilibrated.

The introduction of tolls has long been studied as a way to influence the rational and selfish behaviour of drivers so that the associated Wardrop equilibrium can align with the system optimum network flow. A well-studied taxation mechanism that guarantees this alignment is marginal-cost pricing (see, e.g., [20] and [21]). Marginal-cost tolls do not require any global information about the network structure, user demands or state
and can be computed in a fully local way. Using marginal-cost tolls we prove that our transportation network stabilizes around the social optimum traffic assignment. It is worth observing that our results go well beyond the traditional setting [20] where only static frameworks are considered as well as [21] where only path preference dynamics are consider, neglecting the physical ones that are assumed equilibrated. In fact, our analysis is carried over in a fully dynamical flow network setting. In this respect, the global optimality guarantees that are obtained in this paper through decentralized feedback toll policies should be compared with other recent results on global performance and resilience results on robust distributed control of dynamical flow networks [22]–[25].

In the last part of the paper through numerical simulations we compare the performance both asymptotic and during the transient of the system by using distributed marginal cost tolls and constant marginal cost ones. The latter, know in the literature as “fixed” tolls (being the tolling function on each edge a constant function of edge flow) have been well studied, and it is known that they can be computed to enforce the social optimum equilibrium provided that the system planner has a complete knowledge of the network topology, user demand profile and delay functions. We show that not only is more convenient take into account the marginal cost tolls instead of the constant marginal cost ones. With the latter, the system remains stable and converges to a cycle-free, in this paper we allow for the presence of cycles. A node \( d \) is said to be reachable from another node \( o \) if there exists at least a path from \( o \) to \( d \). Observe that, in contrast to [7] where the transportation network was assumed to be cycle-free, in this paper we allow for the presence of cycles.

Throughout the paper, we will consider a given origin node \( o \) and a destination node \( d \neq o \) that is reachable from \( o \) and let \( \Gamma \) be the set of paths from \( o \) to \( d \) of any length \( l \geq 1 \). Denote the corresponding link-path incidence matrix by \( A \in \{0,1\}^{E \times \Gamma} \) with entries

\[
A_{\gamma} = \begin{cases} 
1 & \text{if } \gamma_s = i \text{ for some } 1 \leq s \leq l, \\
0 & \text{if } i \neq \gamma_1, \ldots, \gamma_l.
\end{cases}
\]

We shall refer to nonnegative vectors \( y \in \mathbb{R}_+^E \) generally as flow vectors. A flow vector \( y \) such that

\[
By = \lambda \left( \delta^{(o)} - \delta^{(d)} \right),
\]

for some \( \lambda \geq 0 \) will be referred to as an \( o \)-\( d \) equilibrium flow vector of throughput \( \lambda \). For \( \lambda \geq 0 \), let us consider the simplex

\[
S_{\lambda} = \left\{ z \in \mathbb{R}_+^E : 1^T z = \lambda \right\}.
\]

Observe that, for \( z \in S_{\lambda} \), one has \( B A z = \lambda (\delta^{(o)} - \delta^{(d)}) \), hence

\[
y^2 = A z
\]

is an \( o \)-\( d \) equilibrium flow vector of throughput \( \lambda \). Throughout the paper, we shall refer to any \( z \in S_{\lambda} \) as a path preference vector and to \( y^2 \) defined as in (3) as the associated equilibrium flow vector.

## II. Model description

### A. Transportation network

We model the topology of the transportation network as a directed multi-graph \( G = (V,E) \), where \( V \) is a finite set of nodes and \( E \) is a finite set of directed links. Each link \( i \in E \) is directed from its tail node \( \theta_i \) to its head node \( \kappa_i \neq \theta_i \). We shall allow for parallel links, i.e., links \( i \neq j \) such that \( \theta_i = \theta_j \) and \( \kappa_i = \kappa_j \), hence the prefix in multi-graphs. On the other hand, we shall assume that there are no self-loops, i.e., that \( \theta_i \neq \kappa_i \) for every \( i \in E \). We shall denote by \( B \in \{−1,0,1\}^{V \times E} \) the node-link incidence matrix of \( G \), whose entries are given by

\[
B_{vi} = \begin{cases} 
+1 & \text{if } v = \theta_i, \\
-1 & \text{if } v = \kappa_i, \\
0 & \text{if } v \neq \theta_i, \kappa_i.
\end{cases}
\]

A length-\( l \) path from a node \( v_0 \) to a node \( v_l \) is a string of links \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_l) \) such that the tail node of the first link is \( \theta_{\gamma_1} = v_0 \), the head node of the last link is \( \kappa_{\gamma_l} = v_l \), the tail node of the next link coincides with the head node of the previous link, i.e., \( v_s = \kappa_{\gamma_s} = \theta_{\gamma_{s+1}} \) for \( 1 \leq s \leq l - 1 \), and no node is visited twice, i.e., \( v_r \neq v_s \) for all \( 0 \leq r < s \leq l \), except possibly for \( v_0 = v_l \), in which case the path is referred to a cycle. A node \( d \) is said to be reachable from another node \( o \) if there exists at least a path from \( o \) to \( d \). Observe that, in contrast to [7] where the transportation network was assumed to be cycle-free, in this paper we allow for the presence of cycles.

Throughout the paper, we will consider a given origin node \( o \) and a destination node \( d \neq o \) that is reachable from \( o \) and let \( \Gamma \) be the set of paths from \( o \) to \( d \) of any length \( l \geq 1 \). Denote the corresponding link-path incidence matrix by \( A \in \{0,1\}^{E \times \Gamma} \) with entries

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A_{\gamma} = \begin{cases} 
1 & \text{if } \gamma_s = i \text{ for some } 1 \leq s \leq l, \\
0 & \text{if } i \neq \gamma_1, \ldots, \gamma_l.
\end{cases}
\]
Moreover, its first derivative is given by
\[ y_i = \varphi_i(x_i), \quad i \in \mathcal{E}, \]  

satisfying the following property\footnote{Note that in classical transportation theory the flow-density function are typically not strictly increasing, but here our assumption is valid as long as we confine ourselves to the free-flow region, as is done in [29].}

**Assumption 1.** For every link \( i \in \mathcal{E} \) the flow function \( \varphi_i : \mathbb{R}_+ \to \mathbb{R}_+ \) is twice continuously differentiable, strictly increasing, strictly concave, and such that \( \varphi_i(0) = 0, \quad \varphi_i'(0) < \infty \).

For every link \( i \in \mathcal{E} \), let
\[ C_i := \sup\{\varphi_i(x_i) : x_i \geq 0\} \]
be its maximum flow capacity. Denote by \( \tau_i : \mathbb{R} \to [0, +\infty] \),
\[ \tau_i(y_i) := \begin{cases} +\infty & \text{if } y_i \geq C_i \\ \varphi_i^{-1}(y_i)/y_i & \text{if } 0 < y_i < C_i \\ 1/\varphi_i'(0) & \text{if } y_i = 0, \end{cases} \]
the cell’s latency function, returning the delay incurred by drivers traversing link \( i \in \mathcal{E} \), when the current flow out of it is \( y_i \). The following simple but useful result is proven in Appendix \footnote{Note that in classical transportation theory the flow-density function are typically not strictly increasing, but here our assumption is valid as long as we confine ourselves to the free-flow region, as is done in [29].}

**Lemma 1.** Let \( \varphi_i : \mathbb{R}_+ \to \mathbb{R}_+ \) be a flow function satisfying Assumption 1. Then, the corresponding latency function \( \tau_i \) defined in (5) is twice continuously differentiable, strictly increasing on the interval \([0, C_i]\), and such that \( \tau_i(0) > 0 \). Moreover, its first derivative is given by
\[ \tau_i'(y_i) = \frac{y - x\varphi'_i(x)}{\varphi_i(x) y^2}, \quad x = \varphi_i^{-1}(y), \]
and the function \( y \mapsto y \tau_i(y) \) is strictly convex on \([0, C_i]\).

Let us now define the set of feasible flow vectors as
\[ \mathcal{F} = \{y \in \mathbb{R}_+^E : y_i < C_i, \quad i \in \mathcal{E}\} \]
and the set of feasible path preferences as
\[ \mathcal{Z} := \{z \in S_\lambda : y^z \in \mathcal{F}\}. \]

Moreover, let the total latency associated to a flow vector \( y \in \mathbb{R}_+^E \) be
\[ L(y) = \sum_{i \in \mathcal{E}} y_i \tau_i(y_i). \]

Observe that the total latency \( L(y) \) is finite if and only if the flow vector \( y \) is feasible. In fact, as a consequence of Lemma \footnote{Note that in classical transportation theory the flow-density function are typically not strictly increasing, but here our assumption is valid as long as we confine ourselves to the free-flow region, as is done in [29].} we have that the total latency function \( L(y) \) is a strictly convex function of \( y \in \mathcal{F} \). Notice that, by the max-flow min-cut theorem (see [29], Thm. 4.1), the set of feasible flows \( \mathcal{F} \)

Figure 1. Example of network with cycle.

contains equilibrium \( o-d \) flows if and only if the throughput \( \lambda < C_{\text{min cut}}^{o,d} \), where
\[ C_{\text{min cut}}^{o,d} = \min_{\Phi \subseteq \mathcal{V}} \sum_{i \in \mathcal{E}} C_i \quad \min_{\nu \in \mathcal{U}} \sum_{i \in \mathcal{E}} C_i \]
is the min-cut capacity. It then follows that, for every \( \lambda \in [0, C_{\text{min cut}}^{o,d}] \), the total latency \( L(y) \) admits a unique minimizer \( y^*(\lambda) \) in the set of feasible equilibrium \( o-d \) flows of throughput \( \lambda \). We shall refer to such unique minimizer
\[ y^*(\lambda) := \arg\min_{y \in \mathcal{F}} L(y) \]
as the social optimum equilibrium flow.

**Example 1.** Consider the network in Figure 1 with node set \( \mathcal{N} = \{o,a,b,d\} \) and link set \( \mathcal{L} = \{e_1, e_2, e_3, e_4, e_5, e_6\} \). It contains four distinct paths from \( o \) to \( d \). In fact, we may write \( \Gamma = \{\gamma(1), \gamma(2), \gamma(3), \gamma(4)\} \), where \( \gamma(1) = (e_1, e_5) \), \( \gamma(2) = (e_2, e_6) \), \( \gamma(3) = (e_1, e_3, e_6) \), and \( \gamma(4) = (e_2, e_4, e_5) \). Note that there is a cycle \( \gamma(o) = (e_3, e_4) \). For each link \( i \in \mathcal{E} \), let the flow functions be given by
\[ \varphi_i(x_i) = C_i(1 - e^{-x_i}), \quad i \in \mathcal{E}, \]
where \( C_i > 0 \) is link \( i \)'s capacity. Then, the corresponding latency functions are given by
\[ \tau_i(y_i) = \begin{cases} +\infty & \text{if } y_i \geq C_i \\ 1/y_i \log \left( \frac{C_i}{C_i - y_i} \right) & \text{if } y_i \in (0, C_i), \\ 1/C_i & \text{if } y_i = 0. \end{cases} \]

Plots of the flow function \( \Phi \) and of the latency function \( \tau \) are reported in Figure 2. In the special case when the link capacities are
\[ C_{e_1} = 3, \quad C_{e_2} = 1, \quad C_{e_3} = 1, \quad C_{e_4} = 1, \quad C_{e_5} = 1, \quad C_{e_6} = 3, \]
the min-cut capacity is \( C_{\text{min cut}}^{o,d} = 3 \) and the minimum total latency and corresponding social optimum flow are plotted in Figure 3 as a function of the throughput \( \lambda \in [0, C_{\text{min cut}}^{o,d}] \).

**B. Multi-scale model of network traffic flow dynamics**

We shall consider a physical traffic flow consisting of indistinguishable homogeneous drivers entering the network from the origin node \( o \) at a constant unit rate, traveling through it on the different paths and finally exiting the network from
the destination node \(d\). Conservation of mass implies that the traffic volume on link \(i\) at time \(t\) evolves as

\[
\dot{x}_i(t) = \lambda S_{\theta_i} + \sum_{j \in E} R_{ij}(t)y_j(t) - y_i(t),
\]

where

\[
y_i(t) = \varphi_i(x_i(t))
\]

is the total outflow from link \(i\), the term \(R_{ij}(t)\) stands for the fraction of outflow from link \(i\) that moves directly towards link \(j\), and the term \(\lambda S_{\theta_i}\) accounts for the constant exogenous inflow in the origin node \(o\). Topological constraints and mass conservation imply that \(R_{ij}(t) = 0\) whenever \(\theta_j \neq \theta_i\), i.e., whenever link \(j\) is not immediately downstream of link \(i\), and that \(\sum_{j \in E} R_{ij}(t) = 1\) for every \(i \in E\) such that \(\theta_i \neq d\). The matrix \(R(t) = (R_{ij}(t))_{i,j \in E}\) will be referred to as the routing matrix.

Throughout, we shall assume that the routing matrix is determined by the drivers’ preferences that are continuously updated in response to available congestion information and dynamic tolls. Formally, the relative appeal of the different paths to the drivers is modelled by a time-varying nonnegative vector \(z(t)\) in the simplex \(S_{\lambda_{\theta_d}}\), to be referred to as the current aggregate path preference. We shall assume that such aggregate path preferences determine the routing matrix as

\[
R_{ij}(t) = \begin{cases} 
G_j(z(t)) & \text{if } \theta_j = \theta_i, \\
0 & \text{if } \theta_j \neq \theta_i,
\end{cases}
\]

for \(i, j \in E\) and \(t \geq 0\), where \(G : \mathcal{Z} \to \mathbb{R}_+^E\) is given by

\[
G_j(z) = \begin{cases} 
\sum_{i \in E : \theta_i = \theta_j} y_i^+ & \text{if } \sum_{i \in E : \theta_i = \theta_j} y_i^+ > 0 \\
\left\{ i \in E : \theta_i = \theta_j \right\} & \text{if } \sum_{i \in E : \theta_i = \theta_j} y_i^+ = 0,
\end{cases}
\]

for each cell \(j \in E\) and it is continuously differentiable. Equations (14) and (15) state that at every junction, represented by a node \(v \in \mathcal{V}\), the outflow from every incoming cell \(i\) such that \(\kappa_i = v\) gets split among the cells \(j\) immediately downstream (i.e., such that \(\theta_j = v\)) according to the proportion associated to the equilibrium flow vector \(y^*\) corresponding to the path preference \(z\), provided that \(y^*\) is such there is flow passing through node \(v\), and otherwise the split is uniform among the immediately downstream cells.

In our dynamical network traffic model, the aggregate path preference vector \(z(t)\) is continuously updated as drivers access global information about the current congestion status of the whole network embodied by the vector

\[
l(t) = (l_i(t))_{i \in E}, \quad l_i(t) = \tau_i(y_i(t)),
\]

of current latencies on the different links. The aggregate path preference vector is also influenced by a vector \(w(t) = (w_i(t))_{i \in E}\) of dynamic tolls, that are to be determined by the transportation system operator. Specifically, let the cost perceived by each user, crossing a link \(i \in E\), be given by the sum of the latency \(l_i(t)\) and the toll \(w_i(t)\) so that the perceived total cost that a driver expects to incur on a path \(\gamma \in \Gamma\) assuming that the congestion levels on that path won’t change during the journey is \(\sum_i A_{\gamma i}(l_i(t) + w_i(t))\). We shall then assume that the path preferences are updated at some rate \(\eta > 0\), according to a noisy best response dynamics

\[
\dot{z}(t) = \eta F(\beta)(l(t), w(t)) - z(t)),
\]

where for every fixed noise parameter \(\beta > 0\) the function \(F(\beta) : \mathbb{R}_+^E \times \mathbb{R}_+^E \to \mathcal{Z}\) is the perturbed best response defined as follows:

\[
F(\beta)(l, w) = \frac{\lambda \exp(-\beta(A(l + w))}{\text{TV} \exp(-\beta(A(l + w)))}.
\]

We shall rewrite the coupled dynamics of the physical flow and the path preferences defined in (12)-(13) in the compact notation

\[
\begin{cases} 
\dot{x}(t) = H(y(t), z(t)), \quad y(t) = \varphi(x(t)) \\
\dot{z}(t) = \eta F(\beta)(l(t), w(t)) - z(t)
\end{cases}
\]

where \(H : 
\mathcal{F} \to \mathbb{R}_+^E\) is defined as

\[
H_i(y, z) := G_i(z)\left(\lambda \theta_i + \sum_{j : \kappa_j = \theta_i} y_j\right) - y_i, \quad i \in E.
\]

III. PROBLEM STATEMENT AND MAIN RESULTS

The goal of this paper is to design robust scalable feedback pricing policies

\[
\omega : \mathcal{F} \to \mathbb{R}_+^E
\]
determining in real time the dynamic tolls
\[ w(t) = \omega(y(t)) \] (22)
with the objectives of guaranteeing stability and achieving social optimality for the closed-loop network traffic flow dynamics (19)–(22).

Observe that, for any given fixed inflow \( \lambda \) and constant tolls \( w \), stability and convergence to the corresponding Wardrop equilibrium—as defined later in this section—follow from the results in [7]. In fact, given full knowledge of the exogenous inflow \( \lambda \) and of the whole transportation network characteristics, one could use classical results in order to pre-compute static tolls that would align such Wardrop equilibrium with the social optimum. However, such an approach would result in an inadequate solution as it would lack robustness with respect to the value of the exogenous input \( \lambda \), as well as to changes of the network characteristics in response, e.g., to accidents and other disruptions.

In contrast, we seek to design feedback pricing policies that are universal with respect to values of the exogenous inflow and robustly adapt in real time to changes of the network characteristics. We shall particularly focus on the class of decentralized monotone feedback pricing policies, as defined below.

**Definition 1.** In a transportation network with topology \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), a feedback pricing policy \( \omega : \mathcal{F} \to \mathbb{R}_+^\mathcal{E} \) is said to be:

(i) monotone if \( \omega(y) \geq \omega(y') \) for every \( y, y' \in \mathcal{F} \) such that \( y \geq y' \), where inequalities are meant to hold true entrywise;

(ii) decentralized if, for every \( i \in \mathcal{E} \), the toll \( w_i = \omega_i(y) \) is a function of the flow \( y_i \) on link \( i \) only.

Throughout the rest of the paper, we shall emphasize the local structure of decentralized pricing policies by writing \( w_i = \omega_i(y_i) \), with a slight abuse of notation. As shown in the following, such robustly local feedback pricing policies can be designed with global guarantees on stability and optimality. Before stating our main results, we introduce the notion of generalized Wardrop equilibrium with feedback pricing.

**Definition 2.** (Generalized Wardrop equilibrium with feedback pricing). For a transportation network with topology \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) and latency functions \( \tau_i \), let \( o \) and \( d \) in \( \mathcal{V} \), with \( d \neq o \) reachable from \( o \), be an origin and a destination, respectively, and let \( \Gamma \) the set of \( o-d \) paths and \( A \) the induced matrix. Then, for a feedback pricing policy \( \omega : \mathcal{F} \to \mathbb{R}_+^\mathcal{E} \), an \( o-d \) equilibrium flow vector \( y \in \mathcal{F} \) of throughput \( \lambda \) is a generalized Wardrop equilibrium if \( y = Az \) for some path preference vector \( z \in S_\lambda \) such that for all \( \gamma \in \Gamma \), if \( z_\gamma > 0 \) then
\[
(A' (\tau(y) + \omega(y)))_\gamma \leq (A' (\tau(y) + \omega(y)))_\bar{\gamma} \quad \forall \gamma, \bar{\gamma} \in \Gamma. \] (23)

Equation (23) states that the sum of the total delay and the total toll associated to an \( o-d \) path \( \gamma \) at the equilibrium flow \( y \) are less than or equal to the sum of the total delay and the total toll associated to any other \( o-d \) path \( \bar{\gamma} \). Hence, a generalized Wardrop equilibrium with feedback pricing is characterized as being the flow associated to a path preference vector supported on the subset of paths with minimal sum of total latency plus total toll. In the special case with no tolls, i.e., when the feedback pricing policy \( \omega(y) \equiv 0 \), this reduces to the classical notion of Wardrop equilibrium [16]. More in general, for constant tolls \( \omega(y) \equiv w \) we get the standard notion of Wardrop equilibrium with tolls. For general decentralized monotone feedback pricing policies, existence and uniqueness of a generalized Wardrop equilibrium are guaranteed by the following result proven in Appendix B.

**Proposition 1.** Consider a transportation network with topology \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) and strictly increasing latency functions. Let \( o \) and \( d \) in \( \mathcal{V} \), with \( d \neq o \) reachable from \( o \), be an origin and a destination, respectively. Then, for every throughput \( \lambda \in [0, \lambda_{\text{min cut}}] \) and every decentralized monotone feedback pricing policy \( \omega : \mathcal{F} \to \mathbb{R}_+^\mathcal{E} \), there exists a unique generalized Wardrop equilibrium \( y(\omega) \) and it can be characterized as the solution of the convex optimization problem
\[
y(\omega) = \arg\min \sum_{i \in \mathcal{E}} D_i(y_i), \tag{24}
\]
where, for each \( i \in \mathcal{E} \),
\[
D_i(y_i) = \int_0^{y_i} (\tau_i(s) + \omega_i(s)) \, ds \tag{25}
\]
is the primitive of the perceived cost \( \tau_i(f_i) + \omega_i(f_i) \).

In the following, we shall prove that for small \( \eta \) and large \( \beta \), the long-time behaviour of the system (19) is approximately at Wardrop equilibrium which, under proper distributed feedback pricing policies, coincides with the social optimum equilibrium. The following is the main result of the paper. It will be proved in the next section using a singular perturbation approach.

**Theorem 1.** Consider a transportation network with topology \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) and flow functions satisfying Assumption [7]. Let \( \omega : \mathcal{F} \to \mathbb{R}_+^\mathcal{E} \) a Lipschitz-continuous monotone decentralized feedback pricing policy. Then, there exists a perturbed equilibrium flow \( y(\omega, \beta) \in \mathcal{F} \) such that, for every initial condition \((z(0), x(0)) \in \mathcal{Z} \times \mathbb{R}_+^\mathcal{E} \), the solution of the closed-loop network traffic flow dynamics (19)–(22) satisfies
\[
\limsup_{t \to \infty} \|y(t) - y(\omega, \beta)\| \leq \delta(\eta), \quad \eta > 0, \tag{26}
\]
where \( \delta(\eta) \) is a nonnegative real-valued, nondecreasing function such that \( \lim_{\eta \to 0} \delta(\eta) = 0 \). Moreover,
\[
\lim_{\beta \to \infty} y(\omega, \beta) = y(\omega). \tag{27}
\]

Theorem [7] states that the system planner globally stabilizes the transportation network around the Wardrop equilibrium using non-decreasing decentralized congestion-dependent tolls.

**Remark 1.** Note that Theorem [7] does not follow from Theorem 2.5 in [2]. Indeed, although the functions \( \tau \) and \( \omega \) both depend on the flow \( y \), it is not possible consider an auxiliary function \( \bar{\tau} = \tau + \omega \) and directly apply the result from [2] due to the specific structure imposed on \( \tau \) in [5]. Moreover, to better understand the evolution process of the multiscale transportation networks, see Figure [7].
Now, we choose as decentralized feedback tolls the marginal cost tolls, namely,
\[ w_i(t) = w_i(y_i(t)) = y_i(t) \tau_i'(y_i(t)), \quad i \in \mathcal{E}. \quad (28) \]

Due the properties of the delay function \( \tau_i \), the above tolls \( w_i \) are increasing, then the Theorem 1 continue to hold. Moreover the following holds

**Corollary 1.** Consider (28) as the decentralized congestion-dependent tolls that influence the system (19). Then the latter globally stabilises the transportation network around the social optimum traffic assignment \( y^*(\lambda) \), i.e., one has
\[ \lim_{\beta \to \infty} y^{(\omega, \beta)} = y^*(\lambda), \]
for every \( \lambda > 0 \) and this is possible without knowing arrival rates or the network structure.

**Proof.** First notice that with feedback marginal cost tolls \( \omega_i(y_i) = y_i \tau_i'(y_i) \), the perceived cost \( \tau_i(y_i) + \omega_i(y_i) \) on each link \( i \in \mathcal{E} \) has primitive
\[ D_i(y_i) = \int_0^{y_i} \left( \tau_i(s) + s \tau_i'(s) \right) ds = y_i \tau_i(y_i), \]
so that
\[ \sum_{i \in \mathcal{E}} D_i(y_i) = L(y) \]
coincides with the total latency. It then follows from the characterization [24] of Proposition 1 that
\[ y^{(\omega)} = \arg \min_{y \in \mathbb{R}_+^s} \sum_{i \in \mathcal{E}} D_i(y_i) = \arg \min_{y \in \mathbb{R}_+^s, B_y = 0} L(y) = y^*(\lambda). \]

The claim then follows as a direct application of Theorem 1.

**Remark 2.** It can be easily seen that Corollary 1 holds true also if the dynamic feedback marginal cost tolls (28) are replaced by the constant marginal cost tolls
\[ w_i^* = y_i^* (\lambda) \tau_i'(y_i^*(\lambda)), \quad i \in \mathcal{E}. \quad (30) \]

However, in contrast to the dynamic feedback marginal cost tolls (28), such constant marginal cost tolls (30) require knowledge both of the social optimum flow and the exogenous inflow \( \lambda \) and lack robustness with respect to changes of the value of \( \lambda \), as well as to changes of the network.

**Remark 3.** In order to implement the dynamic feedback marginal cost tolls (28), each local controller is required to compute the product \( y_i \tau_i'(y_i) \) of the link’s current flow times the link’s latency function’s derivative. Notice that, using (4), we get
\[ \omega_i(y_i) = \frac{1}{\varphi_i'(x_i)} - \frac{x_i}{y_i}, \]

Hence, it is possible to carry the problem of the knowledge of \( \tau_i \) to the computation of the derivative of the flow function \( \varphi_i \).

**IV. PROOF OF THEOREM 1**

In this section, we prove Theorem 1. First of all, notice that since the functions \( F^{(\beta)}, G, \) and \( \varphi \) are differentiable, standard results imply the existence and uniqueness of a solution of the initial value problem associated to (19). In order to prove the stability result, we shall adopt a singular perturbation approach. Our strategy consists in thinking of the path preference vector \( z \) as quasi-static when we analyse the fast-scale dynamics (12), and considering the flow vector \( y \) almost equilibrated (i.e., close to \( y^* \)) when study the slow-scale dynamics (17). Then we will give a series of intermediate results that will turn out to be useful to complete the proof of Theorem 1.

Before proceeding, we introduce some notation to be used throughout the section. Similar to (16) and (22) let
\[ l^\omega(t) = (l_i^\omega(t))_{i \in \mathcal{E}}, \quad l_i^\omega(t) = \tau_i(y_i^\omega(t)) \]
and
\[ w^\omega(t) = (w_i^\omega(t))_{i \in \mathcal{E}}, \quad w_i^\omega(t) = \omega_i(y_i^\omega(t)) \]
be respectively, the vector of current latencies and the one of dynamic tolls both corresponding to the flow \( y^\omega \) associated to the path preference \( z \).

Furthermore, observe that the perturbed best response function (18) satisfies
\[ F^{(\beta)}(l, w) := \arg \min_{\alpha \in \mathcal{Z}_h} \left\{ \alpha' A'(l + w) + h(\alpha) \right\}, \]
where \( h : \mathcal{Z} \to \mathbb{R} \) is the negative entropy function defined as
\[ h(z) = \beta^{-1} \sum_{\gamma \in \mathcal{F}} z_{\gamma} \log z_{\gamma}, \]
using the standard convention that \( 0 \log 0 = 0 \). In fact, all our analysis and results apply to a more general setting where the perturbed best response function is defined as
\[ F^{(h)}(l, w) := \arg \min_{\alpha \in \mathcal{Z}_h} \left\{ \alpha' A'(l + w) + h(\alpha) \right\}, \]
for some admissible perturbation \( h : \mathcal{Z}_h \to \mathbb{R} \) such that \( \mathcal{Z}_h \subseteq \mathcal{Z} \) is a closed convex set, \( h(\cdot) \) is strictly convex, twice differentiable in int(\( \mathcal{Z}_h \)), and \( \lim_{x \to \partial \mathcal{Z}_h} \| \nabla h(z) \| = \infty \). These conditions on \( h \) imply that \( F^{(h)}(l, w) \in \text{int}(\mathcal{Z}_h) \) and that it is continuously differentiable on \( \mathbb{R}_+^s \times \mathbb{R}_+^s \). Notice that clearly the negative entropy function (32) is an admissible perturbation as defined above. We shall then proceed to proving Theorem 1 in this more general setting.
Now, let
\[ x_i^\gamma := \varphi_i^{-1}(y_i^\gamma), \quad \sigma_i := \text{sgn}(x_i - x_i^\gamma) = \text{sgn}(y_i - y_i^\gamma) \]
denote, respectively, the density corresponding to the flow associated to the path preference \( z \) and the sign of the difference between it and the actual density \( x_i \). Then, we define the functions
\[ V(y, z) = \|y - y^\gamma\|_1, \quad \text{and} \quad W(x, z) = \|x - x^\gamma\|_1. \quad (34) \]
The technical Lemmas aim at showing that (34) are Lyapunov functions for the fast-scale dynamics \([12]\) with stationary path preference \( z \).

**Lemma 2.** Let \( \bar{E} \subseteq E \) be a nonempty set of cells. Then,
\[ \max_{j \in \bar{E}} \left(1 - \sum_{i \in \bar{E}} G_i(z)\right) \geq \frac{1}{|V|} \quad (35) \]

*Proof.* Let \( \bar{V} = \{v \in V : v = \kappa_i, i \in \bar{E}\} \). Observe that
\[ \sum_{i \in \bar{E}} G_i(z) = 0, \]
so that, if \( d \in \bar{V} \) then
\[ \max_{j \in \bar{E}} \left(1 - \sum_{i \in \bar{E}} G_i(z)\right) = 1, \]
and the claim follows immediately.

We can then focus on the case when \( d \notin \bar{V} \). Let
\[ \alpha = \sum_{i : \kappa_i \notin \bar{V}} y_i^\gamma + \lambda \delta_i^{(o)} \quad (36) \]
be the total inflow in \( \bar{V} \) which is also equal to the total outflow from \( \bar{V} \). Indeed \( \alpha \) in (36) can also be written as
\[ \alpha = \sum_{i : \kappa_i \notin \bar{V}} y_i^\gamma = \sum_{v \in \bar{V}} \sum_{i : \kappa_i \notin \bar{V} \atop \theta_i = v} y_i^\gamma \leq \sum_{v \in \bar{V}} \sum_{i : \kappa_i \notin \bar{V} \atop \theta_i = v} y_i^\gamma \quad (37) \]
Now, let \( \gamma_v = \sum_{i : \kappa_i = v} y_i^\gamma \) be outflow from a single node \( v \) and observe that \( \gamma_v \leq \alpha \) for every node \( v \). Using this and (37) we get
\[ \alpha \leq \sum_{v \in \bar{V}} \sum_{i : \kappa_i = v \atop \theta_i = v} y_i^\gamma = \sum_{v \in \bar{V}} \gamma_v \sum_{i : \kappa_i = v \atop \theta_i = v} G_i(z) \leq \alpha \sum_{v \in \bar{V}} \sum_{i : \kappa_i = v \atop \theta_i = v} G_i(z). \quad (38) \]

Hence,
\[ \frac{1}{|\bar{V}|} \leq \frac{1}{|\bar{V}|} \leq \frac{1}{|\bar{V}|} \sum_{v \in \bar{V}} \sum_{i : \kappa_i = v \atop \theta_i = v} G_i(z) \leq \max_{v \in \bar{V}} \sum_{i : \kappa_i = v \atop \theta_i = v} G_i(z), \quad (39) \]
so that
\[ \max_{j \in \bar{E}} \left(1 - \sum_{i \in \bar{E}} G_i(z)\right) = \max_{v \in \bar{V}} \sum_{i : \kappa_i = v \atop \theta_i = v} G_i(z) \geq \frac{1}{|\bar{V}|}, \]
hence proving the claim.

**Lemma 3.** For every \( y = \varphi(x) \in \mathcal{F} \) and \( z \in \mathcal{Z} \)
\[ \nabla_x W(x, z)'H(y, z) = -\zeta V(y, z), \]
where \( \zeta = 1/|\bar{V}| \).

*Proof.* Observe that by (15) we get
\[ y_i^\gamma = G_i(z) \left(\lambda \delta_i^{(o)} + \sum_{j : \kappa_j = \kappa_i} y_j^\gamma\right). \]
We will use the above in the second equality of the computation below. Indeed we have
\[ \nabla_x W(x, z)'H(y, z) = \sum_{i \in \bar{E}} \sigma_i \left(G_i(z) \left(\lambda \delta_i^{(o)} + \sum_{j : \kappa_j = \kappa_i} y_j\right) - y_i\right) = \sum_{i \in \bar{E}} \sigma_i \left(G_i(z) \left(\lambda \delta_i^{(o)} + \sum_{j : \kappa_j = \kappa_i} y_j\right) - G_i(z) \left(\lambda \delta_i^{(o)} + \sum_{j : \kappa_j = \kappa_i} y_j^\gamma\right)\right) + \sum_{i \in \bar{E}} \sigma_i (y_i^\gamma - y_i).
\]
Now, define
\[ \bar{E} = \{i \in \mathcal{E} : \sigma_i = 0\} \]
and put
\[ \delta_i = |y_i - y_i^\gamma|, \quad i \in \bar{E}. \]
We have that
\[ \delta_i \geq \max_{k \in \bar{E}} \delta_k, \leq \frac{\|\delta\|_1}{|\bar{E}|}, \quad \forall \ i \in \bar{E}. \]
Then by (40)
\[ \sum_{i \in \bar{E}} \sigma_i \left(G_i(z) \sum_{j : \kappa_j = \kappa_i} (y_j - y_j^\gamma)\right) - \sum_{i \in \bar{E}} \sigma_i (y_i^\gamma - y_i) \leq \sum_{i \in \bar{E}} \sum_{j : \kappa_j = \kappa_i} G_i(z) \delta_j - \sum_{i \in \bar{E}} \delta_i \]
\[ = - \sum_{j \in \bar{E}} \delta_j \left(1 - \sum_{i \in \bar{E} \atop \kappa_i = \kappa_j} G_i(z)\right) \leq - \frac{\|\delta\|_1}{|\bar{E}|} \max_{j \in \bar{E}} \left(1 - \sum_{i \in \bar{E} \atop \kappa_i = \kappa_j} G_i(z)\right) \leq - \frac{\|\delta\|_1}{|\bar{E}|} \frac{\zeta}{|\bar{E}|} \]
by using Lemma 2.

The following two results show that both \( y_i^\gamma(t) \) and \( y_i(t) \) stay bounded away from the maximum flow capacity \( C_i \).

**Lemma 4.** Given the admissible perturbation \([12]\), there exists \( t_0 \in \mathbb{R}_+ \) and, for every link \( i \in \mathcal{E} \), a finite positive constant \( \bar{C}_i \), dependent on \( h \), but not on \( y \), such that for every initial condition \((z(0), x(0)) \in \mathcal{Z} \times [0, +\infty)^E\),
\[ y_i^\gamma(t) \leq \bar{C}_i < C_i \quad \forall t \geq t_0, \forall i \in \mathcal{E}. \]

*Proof.* The fact that \( y_i^\gamma(t) \leq \lambda \) for all \( i \in \mathcal{E} \) follows from the fact that the arrival rate at the origin is unitary. Hence, for
all \( i \in E \) with \( C_i > \lambda \) (and therefore also for \( C_i = \lambda \)) the claim follow with \( C_i = \lambda \) and \( t_0 = 0 \). We now consider the case when \( C_i < \lambda \) for all \( i \in E \). Recall that by the definition of admissible perturbation, the domain of (32) is a closed set \( \mathcal{Z}_\beta \subseteq \text{int}(Z) \). This implies that
\[
\xi_i := C_i - \sup \{(A\alpha)_i : \alpha \in \mathcal{Z}_\beta \} > 0.
\]
It follows from (18) that
\[
C_i - \xi_i = \sup \{(A\alpha)_i : \alpha \in \mathcal{Z}_\beta \} \geq \sup \{(AF^{(b)}(l, w))_i \}.
\]
Hence, one gets
\[
\frac{d}{dt} y_i^z(t) = \eta(A(F^{(b)}(l, w))_i - z(t)_i) \leq \eta(C_i - \xi_i - y_i^z).
\]

This implies that
\[
y_i^z(t) - C_i + \xi_i \leq (y_i^z(0) - C_i + \xi_i)e^{-\eta t} \leq \lambda e^{-\eta t}, \quad t \geq 0, \quad (42)
\]
where the last inequality comes from the fact that \( y_i^z(0) \leq \lambda \) and \( C_i \geq \xi_i \). The lemma for \( i \in E \) with \( C_i < \lambda \) now follows from (42) by choosing, for example, \( C_i := \frac{C_i - \xi_i}{2} \) with \( \xi_i := \min \{\xi_i : i \in E \text{ s.t. } C_i < \lambda \} \) and \( t_0 := -\frac{1}{\eta} \log(\xi<i\lambda>)/2\lambda) \).

**Lemma 5.** Given the admissible perturbation (32), there exists \( \eta^* > 0 \) and a finite positive constant \( C_i \) for every \( i \in E \), dependent on \( h \), but not on \( \eta \), such that for every \( \eta < \eta^* \) and every initial condition \( (z(0), x(0)) \in \mathcal{Z} \times [0, +\infty)^E \),
\[
y_i(t) \leq C_i < C_i, \quad \forall t \geq 0, \quad \forall i \in E.
\]

**Proof.** For \( t \geq 0 \), let us define
\[
\zeta(t) := W(x(t), z(t)), \quad \chi(t) := V(y(t), z(t)),
\]
where \( V \) and \( W \) are defined in (34). By the Lemma 4 there exists \( t_0 \geq 0 \) and a positive constant \( C_i \) for every \( i \in E \), such that for every \( t \geq t_0 \) and applying the inverse of the function \( \varphi_i \) we get
\[
x_i^z(t) \leq x_i^*, \quad x_i^* := \varphi_i^{-1}(C_i) \quad \forall i \in E.
\]

Since \( x_i^z(t) \geq 0 \), (43) implies that if \( |x_i^z(t) - x_i^z| \geq 2x_i^* \) for some \( t \geq t_0 \), then \( x_i^z(t) \geq 2x_i^* \) for \( t \geq t_0 \). Hence \( y_i(t) - y_i^z(t) \geq x_i^* \) for all \( t \geq t_0 \), where \( x_i^* = \varphi_i(2x_i^*) - C_i \). Being \( \varphi_i \) strictly increasing, then one has
\[
\chi_i = \varphi_i(2x_i^*) - C_i > \varphi_i(x_i^*) - C_i = 0.
\]

Now, let
\[
\zeta^* := 2|E| \max \{x_i^* : i \in E \}, \quad \chi^* := \min \{\chi_i^* : i \in E \},
\]
and observe that
\[
W(x, z) \leq |E| \max \{|x_i - x_i^*| : i \in E \}, \quad V(y, z) \geq |y_i - y_i^*| \quad \forall i\in E.
\]

Therefore, it follows that for any \( t \geq t_0 \), if \( \zeta(t) \geq \zeta^* \), then for some \( i' \in E \) we have that \( |x_i(t) - x_i^z(t)| \geq 2x_i^* \) for \( t \geq t_0 \). This in turn involves that \( \chi(t) \geq \chi^* > 0 \forall t \geq t_0 \).

Moreover by (43) follows that there exist some \( \mu > 0 \) such that
\[
\sum_{i \in E} \frac{1}{\varphi_i'(x_i^z(t))} \leq \mu \quad \forall t \geq t_0.
\]

By combining the above with Lemma 5 one finds that for any \( u, t \geq t_0 \),
\[
\begin{align*}
\zeta(t) - \zeta(u) &= \int_u^t \sum_{i \in E} \sigma_i \left( \frac{d}{ds} x_i - \frac{d}{ds} x_i^z \right) ds \\
&\leq \int_u^t \nabla_x W(x, z)' \hat{H}(y, z) ds \\
&+ \int_u^t \sum_{i \in E} \frac{\eta}{\varphi_i'(x_i^z(t))} |(AF^{(b)}(l, w))_i - (Az)_i| ds \\
&\leq \int_u^t \left( -\chi(t) + 2\lambda \eta \mu \right) ds.
\end{align*}
\]

Now, by contradiction, let us assume that \( \limsup_{t \to \infty} y_i(t) \geq C_i \) for some \( i \in E \). Since \( y_i(t) = \varphi_i(x_i(t)) < C_i \) for every \( t \geq 0 \) then \( \limsup_{t \to \infty} x_i(t) = \infty \). From this follows that the limsup_{t \to \infty} \zeta(t) = \infty. Then, in particular, the set \( T := \{t > 0 : \zeta(t) > \zeta(s) \forall s < t\} \) is an unbounded union of open intervals with \( \lim_{t \to T} \zeta(t) = \infty \). This and (43) imply that there exist a nonnegative constant \( t^* \geq t_0 \) such that
\[
\chi(t) \geq \chi^* \quad \forall t \in T \cap [t^*, \infty).
\]

Now defining \( \eta^* := \chi^*/2\lambda \mu \), for every \( \eta < \eta^* \), (45) and (46) give
\[
\begin{align*}
\zeta(t) - \zeta(u) &\leq \int_u^t \left( -\chi(t) + 2\lambda \eta \mu \right) ds \\
&\leq \int_u^t \left( -\chi^* + 2\lambda \eta \mu \right) ds < 0
\end{align*}
\]
for any \( t > u > t^* \) such that \( t \) and \( u \) belong to the same connected component of \( T \). But this contradicts the definition of the set \( T \). Hence, if \( \eta < \eta^* \) then \( \limsup_{t \to \infty} y_i(t) < C_i \) for any \( i \in E \). Since on every compact time interval \( T \subseteq \mathbb{R}_+ \), one has \( \sup_{t \in T} y_i(t) = y_i(T) < C_i \) for some \( T \in T \), the previous implies the claim.

**Lemma 6.** There exists \( K > 0 \) and \( t_1 > 0 \) such that for every initial condition \( (z(0), x(0)) \in \mathcal{Z} \times [0, +\infty)^E \), \( \|\nabla_z h(z(t))\| \leq K \) for all \( t \geq t_1 \).

**Proof.** By the Lemma 5 there exist \( T^*, \eta^* > 0 \) such that \( |l(t)| \leq T^* \) and \( \|w(t)\| \leq \eta^* \) for all \( t > 0 \). This fact together with the definition of \( F^{(b)}(l, w) \) (18) imply that \( F^{(b)}(l(t), w(t)) \in \text{int}(\mathcal{Z}_\beta) \) and \( \nabla_z h(F^{(b)}(l(t), w(t))) = -\Phi A'(l(t) + w(t)) \). Hence \( \|\nabla_z h(F^{(b)}(l(t), w(t)))\| \leq \|\Phi\|\|A'\|S^*, \) with \( S^* = T^* + \eta^* \). This implies the existence of a convex compact \( K \subset \text{int}(\mathcal{Z}_\beta) \) such that \( F^{(b)}(l(t), w(t)) \in K \) for all \( t \geq 0 \). Define
\[
\Delta(t) := \frac{\eta}{1 - e^{-\eta t}} \int_0^t e^{-\eta(t-s)} F^{(b)}(l(s), w(s)) ds.
\]
Since \( \Delta(t) \) is an average of elements of the convex set \( K \), then \( \Delta(t) \in K \forall t \geq 0 \). Moreover, \( z(t) = e^{-\eta t} z(0) + (1 - e^{-\eta t}) \Delta(t) \) approaches \( K \), which implies that for large enough \( t, z(t) \in K_1 \), where \( K_1 \) is a closed subset of \( \text{int}(\mathcal{Z}_\beta) \) that contains \( K \). Hence, after large enough \( t \), say, \( t_1, \nabla_z h(z(t)) \) stays bounded.

**Lemma 7.** There exist \( \ell > 0 \) and \( t_0 > 0 \) such that for every initial condition \( (z(0), x(0)) \in \mathcal{Z} \times [0, +\infty)^E \),
\[
\nabla_z W(x(t), z(t))'(F^{(b)}(l(t), w(t)) - z(t)) \leq 2\lambda \|E\| \quad \forall t \geq t_0.
\]
Proof. Observe that thanks to Lemma 3 there exist $t_0 \geq 0$ such that $t_i := \sup \{1/\varphi_i'(x_i^z(t)) : t \geq t_0\} < +\infty$. Put $\ell := \max \{t_i : i \in E\}$. Then, for every path $\gamma \in \Gamma$ and for every $t \geq t_0$, one has

$$\left| \frac{\partial W(x, z)}{\partial z_\gamma} \right| = \left| -\sum_{i \in E} \sigma_i \frac{\partial}{\partial z_\gamma} x_i^z \right| = \left| \sum_{i \in E} \sigma_i \frac{\partial}{\partial z_\gamma} \varphi_i^{-1} \left( \sum_{\gamma} A_{i \gamma} z_\gamma \right) \right| \leq \sum_{i \in E} A_{i \gamma} \frac{1}{\varphi_i'(x_i^z)} \leq \sum_{i \in E} A_{i \gamma} t_i \leq \ell \|E\|.$$ 

Therefore,

$$2\lambda \ell \|E\| \geq \sum_{\gamma} F_{\beta}^y (l, w) \left| \frac{\partial W(x, z)}{\partial z_\gamma} \right| + \sum_{\gamma} z_\gamma \left| \frac{\partial W(x, z)}{\partial z_\gamma} \right| \geq \sum_{\gamma} F_{\beta}^y (l, w) \frac{\partial W(x, z)}{\partial z_\gamma} - \sum_{\gamma} z_\gamma \frac{\partial W(x, z)}{\partial z_\gamma} = \nabla_z W(x, z)'(F(\beta)(l, w) - z).$$

A. Proof of Theorem 7

Now we are able to prove Theorem 7. Let us consider the function

$$\Theta : Z \to \mathbb{R}_+, \quad \Theta (z) := \int_{0}^{y_f} \left( \tau_i (s) + \omega_i (s) \right) ds \quad (47)$$

and observe that

$$\check{\nabla} \Theta (z) = \Phi A'(l^2 + w^2) \quad \forall z \in \operatorname{int} (Z). \quad (48)$$

Note that since $\tau_i (y_f) + \omega_i (y_f)$ is strictly increasing, then each term of $\int_{0}^{y_f} \left( \tau_i (s) + \omega_i (s) \right) dy_i$ is convex in $y_f$. Hence, the composition with the linear map $z \mapsto y_f = \sum_{\gamma} A_{i \gamma} z_\gamma$ is convex in $z$, which in turn implies convexity of $\check{\nabla} \Theta$ over $Z$. Then using the perturbation (42) we obtain the strict convexity of $\Theta (z) + h(z)$ on $Z_\beta$. Moreover, being $Z_\beta$ a compact and convex set, there exists a unique minimizer

$$z^\beta := \arg \min \{ \Theta (z) + h(z) : z \in Z_\beta \}. \quad (49)$$

Let now $y^{(\omega, \beta)} := z^\beta$. Then the following hold.

Lemma 9. The perturbed equilibrium flow $y^{(\omega, \beta)} \in F$ is such that

$$\lim_{\beta \to \infty} y^{(\omega, \beta)} = y^{(w)}.$$

Proof. Since $\{Az^\beta\} \subseteq AZ$, and $AZ$ is compact, there exists a converging subsequence $\{Az^{\beta_k}\} : k \in \mathbb{N}$. Let us denote by $\hat{y} := \lim_k \hat{y}^{Az^{\beta_k}} \in AZ$ its limit and choose some $\hat{z} \in Z$ such that $\hat{y} = \hat{A} \hat{z}^\beta$.

Notice that since $\sup \{\tau_i (y_f) + \omega_i (y_f) : z \in Z_\beta\} < +\infty$ for all $z \in Z$, the differentiability of $h$ in $\operatorname{int} (Z_\beta)$ implies that the minimizer in (49) has to be in the interior of $Z_\beta$. As a consequence, one finds that necessarily

$$\nabla_z h(z^{\beta_k}) = -\Phi A'(\tau (Az^{\beta_k}) + \omega (Az^{\beta_k})), \quad (49)$$

which successively implies that $F^{\beta_k} (\tau (Az^{\beta_k}) + \omega (Az^{\beta_k})) = z^{\beta_k}$. Then, using the general definition of perturbed best response (33) one finds that

$$\left( Az^{\beta_k} \right)^* (\tau (Az^{\beta_k}) + \omega (Az^{\beta_k})) \leq (Az^{\beta_k} \prime (Az^{\beta_k}) + \omega (Az^{\beta_k} + \omega (Az^{\beta_k})), \quad (50)$$

for all $\alpha \in Z_{\beta_k}$. Now, fix any $z \in Z$. Since $Z_\beta \to \overline{Z}$ then there exists a sequence $\{z_k\}$ such that $z_k \in Z_{\beta_k}$ for all $k$ and $\lim_k z^{\beta_k} = z$. Hence, taking $\alpha = z_k$ in (50) and passing to the limit as $k$ grows large, one finds that

$$z' A'(\tau (\hat{y}) + \omega (\hat{y})) \leq z' A'(\tau (\hat{y}) + \omega (\hat{y})) \quad \forall z, \hat{y}, \hat{z}. \quad (50)$$

In turn, the above can be easily shown to be equivalent to the condition (23) characterizing Wardrop equilibria. From the uniqueness of the Wardrop equilibrium, it follows that necessarily $y = y^{(w)}$. Then the claim follows from the arbitrariness of the accumulation point $\hat{y}$, hence $y^{(\omega, \beta)} \to y^{(w)}$. $\square$

The convergence $\lim_{\beta} Z_\beta = \overline{Z}$ holds with respect to the Hausdorff metric and $\overline{Z}$ is the closure of $Z$. 

We now combine Lemmas 3 and 7 in order to estimate the behaviour in time of $W(x(t), z(t))$.

Lemma 8. There exist $L, \eta, \eta^* > 0$ and $t_0 \geq 0$ such that for every initial condition $z(0) \in Z$, $x(0) \in [0, +\infty)^E$,

$$W(x(t), z(t)) \leq \frac{2\lambda \ell \eta \|E\|}{L} + e^{-\varsigma (t-t_0)/L} \left( W(x(t_0), z(t_0)) - \frac{2\lambda \ell \eta \|E\|}{L} \right),$$

for $t \geq t_0$ and $\eta < \eta^*$. 

Proof. Define $\zeta (t) := W(x(t), z(t))$. Note that thanks to Lemmas 4 and 5 there exist $L > 0$, $\eta^* > 0$ and $t_0 \geq 0$ such that for any $\eta < \eta^*$,

$$|x_i(t) - x_i^z(t)| \leq L |y_i(t) - y_i^z(t)| \quad \forall i \in E, t \geq t_0.$$ 

This involves that

$$V(y(t), z(t)) \geq \frac{1}{L} W(x(t), z(t)) = \frac{1}{L} \zeta (t) \quad \forall \eta < \eta^*, t \geq t_0.$$ 

Moreover $W(x, z)$ is a Lipschitz function of $x$ and $z$, while both $x(t)$ and $z(t)$ are Lipschitz on every compact time interval. Therefore $\zeta (t)$ is Lipschitz on every compact time interval and hence absolutely continuous. Thus $d\zeta (t)/dt$ exists for almost every $t \geq 0$, and, thanks to Lemmas 3 and 7, it satisfies

$$\frac{d\zeta (t)}{dt} = \frac{dW(x(t), z(t))}{dt} = \nabla_z W(x, z)'(F\beta (l, w) - z) \leq -\varsigma V(y, z) + 2\lambda \ell \|E\| \leq -\frac{\varsigma \zeta (t)}{L} + 2\lambda \ell \|E\|.$$ 

Then, integrating both sides we get the claim. $\square$
We now estimate the time derivative of $\Theta(z) + h(z)$ along trajectories of our dynamical system. Hence define

$$
\Psi(t) := \Theta(z(t)) + h(z(t)),
$$
$$
\psi(t) := \Phi A'(l(t^2) + w(t^2)) + \nabla_z h(z(t)).
$$

Then, using (48), we get

$$
\tilde{\psi}(t) = \left( \nabla z \Theta + \nabla h(z) \right) \dot{z} = \eta \psi(t) \left( (F(\beta))(l(t), w(t)) - z(t) \right)
$$
$$
= \eta \psi(t) \left( (F(\beta))(l(t^2), w(t^2)) - z(t) \right)
$$
$$+ \eta \psi(t) \left( (F(\beta))(l(t), w(t)) - (F(\beta))(l(t^2), w(t^2)) \right).
$$

(51)

By Lemma 8 there exist $t_2 > 0$, $\eta^* > 0$ and $M_1 > 0$ such that for any $\eta < \eta^*$, $W(x, z(t)) \leq \eta M_1$ for all $t \geq t_2$. By the definition of $W$ follows that $W(x, z) \geq \|x - x^2\|_1/|E|$ for all $x, z$. Moreover, by the properties of $\varphi$, follows that $\|y - y_\eta^2\|_1 \leq \mathcal{L}\|x - x^2\|_1$ for all $y, z$, and $\mathcal{L} := \max|\varphi'(0)| : i \in E$. Combining all these relationships we get that there exists a $M > 0$ such that for every $\eta < \eta^*$,

$$
\|y(t) - y^2(t)\| \leq \eta M \quad \forall t \geq t_2,
$$

(52)

where $M = |E|M_1\mathcal{L}$. Thanks to the differentiability of $F(\beta)$ on $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ and the boundedness of both $y^2(t)$ and $y(t)$ one gets

$$
\|F(\beta)(l(t), w(t)) - F(\beta)(l^2(t), w^2(t))\| \leq K_1 \eta
$$

for some positive constant $K_1$, $\eta < \eta^*$ and $t$ large enough. Since by Lemmas 3 and 6 follows that $l^2(t), w^2(t))$ and $\nabla_z h(z(t))$ are eventually bounded, then there exists a positive constant $K_2$ such that $\|\psi(t)\| \leq K_2$ for $t$ large enough. This implies that the second addend in the last line of (51) can be bounded as

$$
\eta \psi(t) \left( (F(\beta))(l(t), w(t)) - (F(\beta))(l^2(t), w^2(t)) \right) \leq K_2 \eta^2
$$

(53)

$\forall \eta < \eta^*$, $\forall t \geq t_3$ where $K = K_1K_2$ and for some sufficiently large but finite value of $t_3$. Now, observe that for every $z \in Z$

$$
\Phi A'(l^2(t) + w^2(t))) = -\nabla_z h(F(\beta)(l^2(t), w^2(t)))
$$

so that the first addend in the last line of (51) may be rewritten as

$$
\psi(t) \left( (F(\beta))(l^2(t), w^2(t)) - z(t) \right) = -Y(z(t)),
$$

(54)

where

$$
Y(z(t)) = \left( \nabla_z h(F(\beta)(l^2(t), w^2(t))) - \nabla_z h(z(t)) \right)'.
$$

(55)

From the strict convexity of $h(z)$ on $Z_\beta$, $Y(z(t)) \geq 0$ for every $z$, with equality if and only if $z = z^\beta$. Now, put

$$
\delta(r) = \begin{cases}
\sup\{\|y^\beta - y(\omega, \beta)\| : Y(z) \leq Kr \} + Kr & \text{if } 0 \leq r < \eta^*, \\
\tilde{C}/|E| & \text{if } r \geq \eta^*,
\end{cases}
$$

where $\tilde{C} := \max\{1, \tilde{C}_i : i \in E\}$, with $\tilde{C}_i$ as defined in Lemma 5. It can be proved that $\delta(r)$ is nondecreasing, right-continuous, and such that $\lim_{r \to 0} \delta(\eta) = \delta(0) = 0$. Then, (52) and (55) imply that for $\eta < \eta^*$,

$$
\lim sup_{t \to \infty} \|y(t) - y(\omega, \beta)\| \leq \delta(\eta).
$$

(56)

Note that since $y(t) \in [0, \tilde{C}]^E$ and $y(\omega, \beta) \in AZ \subset [0, 1]^E$ then $\|y(t) - y(\omega, \beta)\| \leq \max\{\tilde{C}_i, 1\} \leq \tilde{C}$ for all $i \in E$ and hence $\|y(t) - y(\omega, \beta)\|^2 \leq |E|\tilde{C}^2$. Then (55) also holds for $\eta \geq \eta^*$, since in that range $\delta(r) = \tilde{C}/|E|$. This together with Lemma 9 conclude the proof.

V. ASYMPTOTIC AND TRANSIENT PERFORMANCES

In this section, we will present a numerical study comparing both the asymptotic and the transient performance of multi-scale transportation networks controlled by dynamic feedback marginal cost tolls (28) and constant marginal cost tolls (30).

We performed several numerical simulations with different graph topologies and for several values of the parameter $\eta$. In all of our simulations we found that dynamic feedback marginal cost tolls outperform constant marginal ones. Indeed:

- concerning the transient convergence, it appears that the time needed to reach the perturbed equilibrium associated to the dynamic feedback marginal cost tolls is lower than the time to reach the perturbed equilibrium associated to the constant marginal ones.
- as the noise parameter $\beta$ of the route choice goes to infinity the perturbed equilibrium associated to dynamic feedback marginal cost tolls asymptotically converges to the social optimum flow faster than the one associated to the constant marginal cost tolls.

We illustrate these findings in the following simple case:

- network topology $G$ as in Fig. 5.
- flow-density function as in (9) and corresponding delay function as in (10), with capacity $C_i = 2$ for every $i \in E$;
- $F(\beta)$ as in (18), $\eta = 0.1$, $G$ as in (13) and $\lambda = 1$;
- initial conditions: $z_i(0) = 1/2$, $z_i(2) = 1/6$, $z_i(3) = 1/3$ $x_i(0) = 4$, $x_i(2) = 2$, $x_i(3) = 3$, $x_i(4) = 1$, $x_i(5) = 5$.

Having settled a time horizon $T = 350$, Fig. 6 displays the 1-norm distance and the latency loss of $y(\omega, \beta)(T)$ from the system optimum $y^\beta$, for different values of the noise parameter $1/\beta$. This is done both considering (28) and the constant marginal tolls (30). Note that while our theoretical results guarantee that $y(\omega, \beta)(T)$ converges to $y^\beta$ only in the double limit of large $T$ (asymptotically in time) and large $\beta$ (vanishing noise), in our

![Figure 5. The graph topology used for the simulations.](image-url)
numerical examples convergence is practically observed already for relatively small values of $\beta$. Our simulations also suggest

that convergence of $y^{(\omega,\beta)}(T)$ to the system optimum is faster for the feedback marginal cost tolls (28) than for the fixed marginal cost (30). Hence, in addition to variations of network’s parameters and exogenous loads, feedback marginal cost tolls appear to be more robust than their constant counterparts also when it comes to noise.

A. Effect of delays

In this section we study the effects of delays in the global information of the slow scale dynamics (17) on the system (19). Considering at first the case of marginal cost tolls, we take a time-delay $\phi$ and $l_i(t-\phi) + w_i(t-\phi)$ as the cost perceived by each user crossing a link $i \in E$. Fixing the noise parameter $\beta$ and varying $\phi$, we observe how changes the time-evolution of the density $x$ and how the correspondent flow $y$ approximates the social optimum flow $y^*(\lambda)$ with $\lambda = 1$. For that, we use the graph topology as in Fig. 5 and the same parameters as before. Then, fixing $\beta = 5$, we perform the trajectory $x$ for different value of $\phi$ as show in the Fig. 7. In Figure 7(a) and 7(b) we can note that the density converges to the equilibrium. By numerical simulations one gets that $\phi = 9$ is the bigger value for which one has convergence (see Fig. 7(b)), because for $\phi > 9$ one witness a phase transition of the system, namely one observes an oscillatory behaviour. We can also note in Figure 7(c) and 7(d) that the bigger $\phi$ is the greater the oscillation amplitude and phase. A similar situation can be seen in the plot of the 1-norm distance of $y$ from $y^*$ in Fig. 8, for the same value of $\phi$ used in Figure 7.

Consider now the case of constant marginal cost tolls (30). Let $\phi$ be the time delay as before and $\tau_i(y_i(t-\phi)) + w_i^*$ the cost perceived by each user crossing a link $i \in E$. Still using the graph topology as in Fig. 5 and fixing $\beta = 5$ we perform the trajectory $x$ and the 1-norm distance of the correspond flow $y$ from $y^*$ with the different values of $\phi$ used before. From Figure 9 we can note that for all considered values of $\phi$ the trajectory $x$ converges to the equilibrium. This differs from what happens using the marginal cost tolls (see Figure 7) and highlights how time-delays affect marginal cost tolls more than their constant counterpart. The plot of the 1-norm, Fig. 10 confirms the same trend, indeed after some initial oscillations, the 1-norm is the same for the different values of $\phi$. 

![Figure 6](image_url)

Figure 6. Plot of $\|y^{(\omega,\beta)}(T) - y^*\|_1$ and $\mathcal{L}(y^{(\omega,\beta)}(T)) - \mathcal{L}(y^*)$ for decentralised marginal and constant marginal tolls.

![Figure 7](image_url)

Figure 7. The trajectory $x(t)$ for different values of the delay $\phi$.

![Figure 8](image_url)

Figure 8. Plot of $\|y(t) - y^*\|_1$ for different values of the delay $\phi$.

![Figure 9](image_url)

Figure 9. Trajectories with constant marginal tolls, for different values of the delay $\phi$. 

![Figure 10](image_url)

Figure 10. Plot of the 1-norm, Fig. 10 confirms the same trend, indeed after some initial oscillations, the 1-norm is the same for the different values of $\phi$. 

![Figure 11](image_url)

Figure 11. Plot of $\|y^{(\omega,\beta)}(T) - y^*\|_1$ and $\mathcal{L}(y^{(\omega,\beta)}(T)) - \mathcal{L}(y^*)$ for decentralised marginal and constant marginal tolls.
VI. CONCLUSION

In this paper, we studied the stability of Wardrop equilibria of multi-scale transportation networks with distributed dynamic tolls. In particular, we prove that if the frequency of updates of path preferences is sufficiently small and considering positive, non-decreasing decentralized flow-dependent tolls, then the state of the network ultimately approaches a neighborhood of the Wardrop equilibrium. Then, using a particular class of tolls, i.e., the marginal cost ones, we observe that the stability is around the social optimum equilibrium and, thanks to numerical experiments, the performances both asymptotic and during the transient of the system is better than the one obtained considering the constant marginal tolls. Moreover, assuming the existence of a time delay in the global information of the slow-scale dynamics, we highlight, through numerical simulations, as this delay influences the stability and convergence of the dynamical system, both in the case in which one considers marginal cost tolls and constant marginal cost ones. In future research, inspired by the numerical results we will provide analytic estimates about the different convergence rates. Moreover, we also plan to define a more general class of tolls that does not require the knowledge neither of the derivatives of the delay functions and at the same time guarantees the convergence to the social optimum.

APPENDIX A

PROOF OF LEMMA 1

The fact that $\tau_i$ is twice continuously differentiable on $[0, C_i]$, strictly increasing and such that $\tau_i(0) > 0$ directly follows from Assumption 1. We prove now that $y \rightarrow y\tau_i(y)$ is strictly convex computing its second derivative. For a given $y \in [0, C_i)$, let $x = \varphi_i^{-1}(y)$, $a = \varphi_i'(x)$, and $b = \varphi_i''(x)$. Then,

\[ (y\tau_i(y))'' = 2\tau_i''(y) + y\tau_i''(y) = \frac{2(y-xa)}{ay^2} + \frac{-y^4b/a - 2y^3a + 2y^2xa^2}{y^4a^2} = -\frac{b}{a^2}. \]

Now, observe that Assumption 1 guarantees that $a > 0$ and $b > 0$. Hence, $(y\tau_i(y))'' > 0$ and therefore $y\tau_i(y)$ is strictly convex, thus completing the proof.

APPENDIX B

PROOF OF PROPOSITION 1

From Assumption 1 and the fact that the toll on a link is a non-decreasing function of the flow on that link only, it follows that the perceived cost function $\tau_i(y_i) + \omega_i(y_i)$ on link $i$ is continuous, strictly increasing, and grater than zero when $y_i = 0$. The claim then follows as a direct application of Theorems 2.4 and 2.5 in [30].

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