Image Deblurring Using a Projective Inertial Parallel Subgradient Extragradient-Line Algorithm of Variational Inequality Problems

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Abstract. In this paper, we introduce projective inertial parallel subgradient extragradient-line algorithm for solving variational inequalities of L-Lipschitz continuous and monotone mappings which L is unknown. We prove a strong convergence result under some mild conditions in Hilbert space. We also present some numerical examples in Euclidean space \( \mathbb{R}^3 \) compared with Parallel-Viscosity-Type Subgradient Extragradient-Line Method. Finally, we deblur the Grey and RGB images from common types of blur matrices Gaussian blur, Out of focus blur and Motion blur using our proposed algorithm and show the better efficiency when the number of types of blur matrices is large.

1. Introduction and Definitions

We consider the classical variational inequality problem (VIP) in a real Hilbert space \( H \) which is to find a point \( x^* \in C \) such that

\[
(Bx^*, x - x^*) \geq 0, \quad \forall x \in C,
\]

where \( C \) is a nonempty closed and convex subset of \( H \) and \( B : H \to H \) is a mapping. Let us denote the solution set of VIP (1) by \( \text{VI}(C,B) \). It is well known that \( x^* \) solves the VIP (1) if and only if \( x^* \) solves the fixed point equation

\[
x^* = P_C(x^* - \lambda Bx^*), \quad \lambda > 0,
\]

where \( \lambda \) is any positive real number. Many problems in the real-world can formulated in the form of the VIP (1), such as economics, engineering mechanics, signal processing, image recovery, transportation, and others (see, for example [3, 6, 9, 13, 14]). Several numerical methods have been constructed for solving variational inequalities and related optimization problems, in [1, 2, 15, 20, 22, 25–29] and the references cited therein.

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One of the important methods for solving VIP is the extragradient method (EGM) introduced by Korpelevich [10] in 1976 for solving the saddle point problems and then extended to solve the VIPs. The extragradient method was stated as follows:

\[
\begin{align*}
&x_1 \in C, \\
y_k = P_C(x_k - \lambda Bx_k), \\
x_{k+1} = P_C(x_k - \lambda By_k),
\end{align*}
\]

where \( \lambda \in (0, \frac{1}{L}) \) and \( P_C \) denotes the metric projection from \( H \) onto \( C \). This method converges if \( B \) is \( L \)-Lipschitz continuous and monotone operator.

In 2011, Censor et al. [27] improved the extragradient method (2) by introducing the subgradient extragradient method (SEGM), in which the second projection onto \( C \) is replaced by a projection onto a specific constructible half-space. Their algorithm is of the form

\[
\begin{align*}
&x_1 \in C, \\
y_k = P_C(x_k - \lambda Bx_k), \\
T_k = \{w \in H : \langle x_k - \lambda Bx_k - y_k, w - y_k \rangle \leq 0\}, \\
x_{k+1} = P_{T_k}(x_k - \lambda By_k),
\end{align*}
\]

where \( \lambda \in (0, \frac{1}{L}) \). The subgradient extragradient method for solving VIP (1) has received great attention by many authors (see, e.g., [7, 18] and the references therein).

Motivated and inspired by the results of Alvarez and Attouch in [8], and of Censor et al. in [27], used the inertial technique with the SEGM. The method have been called the inertial subgradient extragradient method, that is, for any initial \( x_0, x_1 \in H \), the sequence \( \{x_k\} \) is generated by

\[
\begin{align*}
&w_k = x_k + \alpha_k(x_k - x_{k-1}), \\
y_k = P_C(w_k - \lambda Bw_k), \\
T_k = \{x \in H : \langle w_k - \lambda Bw_k - y_k, x - y_k \rangle \leq 0\}, \\
x_{k+1} = P_{T_k}(w_k - \lambda By_k),
\end{align*}
\]

where \( \lambda > 0, \alpha_k \geq 0 \). Under suitable conditions, they proved the weak convergence of \( \{x_k\} \) to an element of \( \text{VI}(C,B) \).

Recently, Censor, Gibali and Reich [23, 24] introduced the common solutions to variational inequality problem (CSVIP), which consists of finding common solutions to unrelated variational inequality. The general form of the CSVIP is the following: Let \( C \) be a nonempty closed and convex subset of \( H \). Let \( B_i : H \to H, i = 1, 2, ..., N \) be mappings. The CSVIP is to find \( x^* \in C \) such that

\[
\langle B_i x^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, i = 1, 2, ..., N.
\]

If \( N = 1 \), CSVIP (5) becomes VIP (1).

Very recently, using a modified viscosity-type subgradient extragradient-line method, Suantai et al. [19] introduced the parallel viscosity-type subgradient extragradient-line method (PVSEGM) for solving the VIP. The strong convergence theorem was proved when each of the operator \( A_i \) is Lipschitz continuous monotone mapping that the Lipschitz constant is unknow. They introduced the following algorithm:

\[
\begin{align*}
&x_1 \in H, \\
y_1' = P_C(x_1 - \lambda_1' B_1 x_1), \\
T_1 = \{z \in H : \langle x_1 - \lambda_1' B_1 x_1 - y_1', z - y_1' \rangle \leq 0\}, \\
x_{1+1} = \alpha_{1,1}^0 f(x_1) + \sum_{i=1}^{N} \hat{\alpha}_i^0 z_{1,i}, \quad k \geq 1,
\end{align*}
\]
where $\rho, \mu \in (0, 1), \{a_k\}_{k=1}^\infty \subseteq (0, 1), \lim_{k \to \infty} a_k^0 = 0, \sum_{k=1}^\infty a_k^0 = \infty$, and they proved that the sequence $\{x_k\}_{k=1}^\infty$ generated by (6) converges strongly to $x^* \in \text{VI}(C, B)$. The benefit of the PVSEGM was presented to solve the problem of multiblur effects in an image restoration. As a result, the resulting image quality can be improved sharper by using the PVSEGM in the resolution of common resolution VIP problems.

Motivated and inspired by the works in the literature, we study strong convergence of the algorithm for solving common solution of variational inequality problem (5). The algorithm is generated by the hybrid inertial techniques and a parallel subgradient extragradient-line method. Several numerical experiments are implemented to support the theoretical results. Our numerical results have illustrated the better convergence of the new algorithms over the PVSEGM method of Suantai et al. [19]. Finally, we present a solution to the problem of multiblur effects in an image is solved by applying our algorithm.

2. Main result

In this section we present a new algorithm for solving the CSVIP (5). Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$. Let $B_i : H \to H$ be monotone and Lipschitz continuous on $H$ with the constant $L_i$ but $L_i$ is unknown for all $i = 1, 2, ..., N$. Moreover, we denote $\Psi := N \bigcap_{i=1}^N \text{VI}(C, B_i) \neq \emptyset$. Suppose $\{x_k\}_{k=1}^\infty$ is generated in the following algorithm:

Algorithm 2.1. Initialization: Given $\gamma > 0, \mu \in (0, 1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Compute $x_{k+1}$ as follows:

Step 1. Set $w_k = x_k + a_k(x_k - x_{k-1})$ and compute

$$y_k' = P_C(w_k - \tau_k^i B_i w_k), \quad \forall k \geq 1,$$

where $\tau_k^i = \gamma \ell_k^i$ and $\ell_k^i$ is the smallest nonnegative integer such that

$$\tau_k^i \| B_i w_k - B_i y_k' \| \leq \mu \| w_k - y_k' \| . \quad (7)$$

Step 2. Compute

$$z_k' = P_{T_k^i}(w_k - \tau_k^i B_i y_k'),$$

where $T_k^i := \{x \in H | \langle w_k - \tau_k^i B_i w_k - y_k', x - y_k' \rangle \leq 0\}$.

Step 3. Compute $z_k$, i.e.

$$z_k = \text{argmax} \{\|z_k' - x_k\| : i = 1, 2, ..., N\}. \quad (8)$$

Step 4. Compute

$$x_{k+1} = P_{C_{k+1}} x_k,$$

where $C_{k+1} := \{z \in C_k : \|z - z_k\| \leq \|w_k - z_k\|\}$.

Again set $k := k + 1$ and go to Step 1.

Theorem 2.2. Assume that condition $\sum_{k=1}^\infty a_k \| x_k - x_{k-1} \| < \infty$ holds. Then the sequences $\{x_k\}$ generated by Algorithm 2.1 converge strongly to $z \in P_{\Psi} x_1$. 

Proof **Claim 1.** We prove that the sequence \( \{x_k\} \) is well defined and \( \lim \|x_k - x_1\| \) exists. Since \( C = C_1, C_1 \) is closed and convex. Assume that \( C_k \) is closed and convex. Let \( x^* \in \Psi \) and \( h^i_k = w_k - \tau_k^i B_i y^i_k, \forall k \geq 1, i = 1, 2, ..., N, \) we have

\[
\|z_k - x^*\|^2 = \|P_{T_k^i}(h^i_k) - h^i_k\|^2 + 2\langle P_{T_k^i}(h^i_k) - h^i_k, h^i_k - x^* \rangle + \|h^i_k - x^*\|^2.
\]

Since \( x^* \in \Psi \subseteq C \subseteq T_k^i \) and by the characterization of the metric projection \( P_{T_k^i} \), we obtain

\[
2\|h^i_k - P_{T_k^i}(h^i_k)\|^2 + 2\langle P_{T_k^i}(h^i_k) - h^i_k, h^i_k - x^* \rangle = 2\langle h^i_k - P_{T_k^i}(h^i_k), x^* - P_{T_k^i}(h^i_k) \rangle \leq 0.
\]

This implies that

\[
\|h^i_k - P_{T_k^i}(h^i_k)\|^2 + 2\langle P_{T_k^i}(h^i_k) - h^i_k, h^i_k - x^* \rangle \leq -\|h^i_k - P_{T_k^i}(h^i_k)\|^2.
\]

By the inequalities (9), (10) and the definition of Algorithm 2.1, we obtain

\[
\|z_k - x^*\|^2 \leq \|h^i_k - x^*\|^2 - \|h^i_k - z_k\|^2 = \|\|w_k - x^*\|^2 - \|w_k - z_k\|^2 - \|w_k - z_k\|^2 - 2\tau_k^i \langle B_i y^i_k, y^i_k - z_k \rangle = \|w_k - x^*\|^2 - \|w_k - z_k\|^2 - 2\tau_k^i \langle x^* - z_k, B_i y^i_k \rangle.
\]

By the monotonicity of the operator \( B_i, \) we have

\[
0 \leq \langle B_i y^i_k - B_i x^*, y^i_k - x^* \rangle = \langle B_i y^i_k, y^i_k - x^* \rangle - \langle B_i x^*, y^i_k - x^* \rangle \leq \langle B_i y^i_k, y^i_k - x^* \rangle = \langle B_i y^i_k, y^i_k - z_k \rangle + \langle B_i y^i_k, z_k - x^* \rangle.
\]

Thus

\[
\langle x^* - z_k, B_i y^i_k \rangle \leq \langle B_i y^i_k, y^i_k - z_k \rangle.
\]

Using (13) in (12), we obtain

\[
\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \|w_k - z_k\|^2 - 2\tau_k^i \langle B_i y^i_k, y^i_k - z_k \rangle = \|w_k - x^*\|^2 - \|w_k - y^i_k\|^2 - \|y^i_k - z_k\|^2 - 2\tau_k^i \|B_i y^i_k - w_k\| \|w_k - y^i_k\| \|y^i_k - z_k\|^2.
\]

Observe that

\[
\langle w_k - \tau_k^i B_i y^i_k, y^i_k - z_k \rangle = \langle w_k - \tau_k^i B_i y^i_k, y^i_k - z_k \rangle + \langle \tau_k^i B_i y^i_k, z_k - y^i_k \rangle \leq \langle \tau_k^i B_i y^i_k, z_k - y^i_k \rangle.
\]

Using the inequality (15) in (14) and the existence of the step size \( \tau_k^i \) of Lemma 3.1 in [30], we have

\[
\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \|w_k - y^i_k\|^2 - \|y^i_k - z_k\|^2 - 2\tau_k^i \|B_i y^i_k - w_k\| \|w_k - y^i_k\| \|y^i_k - z_k\|^2.
\]

This implies that \( \|z_k - x^*\| \leq \|w_k - x^*\|, \) so \( x^* \in C_k, \forall k \in N. \) This shows that \( \{x_k\} \) is well-defined. From \( x_k = P_{C_k} x_1 \) and \( x_{k+1} \in C_k, \) for all \( k \geq 1, \) we get

\[
\|x_k - x_1\| \leq \|x_{k+1} - x_1\|, \quad \forall k \geq 1.
\]
On the other hand, as $\Psi \subset C_r$, we obtain
$$
\|x_k - x_i\| \leq \|x^r - x_i\|, \ \forall k \geq 1.
$$
(18)

From (17) and (18) that the sequence $\{x_k\}$ is bounded and nondecreasing. Therefore
$$
\lim_{k \to \infty} \|x_k - x_i\| \text{ exists.}
$$

Claim 2. Show that $x_k \to v \in C$ as $k \to \infty$. For $m > r$, by the definition of $C_r$, since $x_m = P_{C_m} x_1 \in C_m \subset C_r$, so by the property of the metric projection $P_{C_r}$, [11], we have
$$
\|x_m - x_i\|^2 \leq \|x_m - x_i\|^2 - \|x_r - x_i\|^2.
$$

Since $\lim_{k \to \infty} \|x_k - x_i\|$ exists, we have $\|x_m - x_i\| \to 0$, as $m, r \to \infty$ this means that $\{x_k\}$ is a Cauchy sequence. Hence, there exists $v \in C$ such that $x_k \to v$ as $k \to \infty$. Therefore,
$$
\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0.
$$
(19)

Claim 3. Show that $\lim_{k \to \infty} \|z_k^i - x_k\| = \lim_{k \to \infty} \|x_k - y_k^i\| = \lim_{k \to \infty} \|y_k^i - z_k^i\| = 0, \forall i = 1, 2, ..., N$. From the definition of $C_k$ and $x_{k+1} \in C_{k+1} \subset C_k$, we have
$$
\|z_k - x_{k+1}\| \leq \|x_{k+1} - \bar{w}_k\| \leq \|x_{k+1} - x_k\| + \|x_k - \bar{w}_k\| = \|x_{k+1} - x_k\| + \alpha_k \|x_k - x_{k-1}\|.
$$

From (19) and condition in Theorem 2.2, we obtain
$$
\lim_{k \to \infty} \|z_k - x_{k+1}\| = 0.
$$

This the triangle inequality $\|z_k - x_k\| \leq \|z_k - x_{k+1}\| + \|x_{k+1} - x_k\|$ implies that
$$
\lim_{k \to \infty} \|z_k - x_k\| = 0.
$$
(20)

From the definition of $z_k^i$ and (20), we get
$$
\lim_{k \to \infty} \|z_k^i - x_k\| = 0, \ \forall i = 1, 2, ..., N.
$$
(21)

From (16) and $\bar{w}_k = x_k + \alpha_k (x_k - x_{k-1})$, we have
$$
\|z_k^i - x^r\|^2 \leq \|x_k + \alpha_k (x_k - x_{k-1}) - x^r\|^2 - (1 - \mu) \|x_k + \alpha_k (x_k - x_{k-1}) - y_k^i\|^2 + \|y_k^i - z_k^i\|^2
$$
$$
= \|(x_k - x^r) + \alpha_k (x_k - x_{k-1})\|^2 - (1 - \mu) \|(x_k - y_k^i) + \alpha_k (x_k - x_{k-1})\|^2 + \|y_k^i - z_k^i\|^2
$$
$$
\leq \|(x_k - x^r\|^2 + 2\alpha_k (x_k - x_{k-1}, \bar{w}_k - x^r)
$$
$$
- \|x_k - x^r\|^2 - (1 - \mu) \|(x_k - y_k^i\|^2 + \|y_k^i - z_k^i\|^2) + 2\alpha_k (x_k - x_{k-1}, \bar{w}_k - x^r)
$$
$$
2\alpha_k (1 - \mu) (x_k - x_{k-1}, \bar{w}_k - y_k^i).
$$
(22)

From (22), for each $c \in \Psi$, we obtain
$$
(1 - \mu) \|x_k - y_k^i\|^2 \leq \|x_k - c\|^2 - \|z_k^i - c\|^2 + 2\alpha_k (x_k - x_{k-1}, \bar{w}_k - c)
$$
$$
- 2\alpha_k (1 - \mu) (x_k - x_{k-1}, \bar{w}_k - y_k^i).
$$
(23)

From (21), (23) and the boundedness of $\{\bar{w}_k\}, \{x_k\}, \{y_k^i\}, \{z_k^i\}$ and condition in Theorem 2.2, we have
$$
\lim_{k \to \infty} \|x_k - y_k^i\| = 0, \ \forall i = 1, 2, ..., N.
$$
(24)
Using the equation (21) and (24), we obtain
\[ \lim_{k \to \infty} \|y_k^i - z_k^i\| = 0, \quad \forall i = 1, 2, \ldots, N. \]  
(25)

**Claim 4.** Show that \( v \in \Psi \) and \( v = P_C x_1 \). Now, \( x_k - y_k^i \to 0 \) implies that \( y_k^i \to v \) and since \( y_k^i \in C \), we then obtain \( v \in C \). For all \( x \in C \) and using the property of the projection \( P_C \), we have (Since \( B_i \) is monotone)

\[
0 \leq \langle y_k^i - w_k + \tau_k^i B_i w_k, x - y_k^i \rangle \\
= \langle y_k^i - w_k, x - y_k^i \rangle + \langle \tau_k^i B_i w_k, x - x_k \rangle + \langle \tau_k^i B_i w_k, x_k^i - y_k^i \rangle \\
\leq \langle y_k^i - x_k, x - x_k^i \rangle + \tau_k^i \langle B_i x_k, x - x_k \rangle + \tau_k^i \langle B_i x_k, x_k^i - y_k^i \rangle \\
+ \langle \alpha_k (x_k - x_k - 1), x - y_k^i \rangle + \tau_k^i \langle B_i \alpha_k (x_k - x_k - 1), x_k^i \rangle \\
+ \tau_k^i \langle B_i \alpha_k (x_k - x_k - 1), x_k^i - y_k^i \rangle.
\]  
(26)

Taking the limit as \( k \to \infty \) in (26), we obtain (recall that \( \inf_{k \geq 1} \tau_k > 0 \) by Remark 3.2 in [21])

\[ 0 \leq \langle B_i v, x - v \rangle, \quad \forall x \in C. \]

This implies that \( v \in V(I[C, B_i]) \) for all \( i = 1, 2, \ldots, N \). It follows from (18) that, for \( x' \in \Psi \|v - x_1\| \leq \|x' - x_1\| \). This implies that \( v = P_C x_1 \). The proof is completed.

We give the following numerical example to illustrate Theorem 2.2.

**Example 2.3.** Let \( B_1, B_2, B_3 : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by \( B_1 x = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} x \), \( B_2 x = \begin{pmatrix} 6 & -3 & 3 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{pmatrix} x \) and \( B_3 x = \begin{pmatrix} 10 & -5 & 5 \\ -5 & 10 & -5 \\ 5 & -5 & 10 \end{pmatrix} x \) for all \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Let \( C = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 4 \} \). The stopping criterion is defined by \( \|x_k - x_k - 1\| < 10^{-5} \). We choose parameters

\[
\alpha_k = \begin{cases} \frac{0.2}{k \|x_k - x_k - 1\|} & \text{if } x_k \neq x_k - 1 \text{ and } k \leq 1000 \\
10^{-12} & \text{if } x_k \neq x_k - 1 \text{ and } k > 1000 \\
0 & \text{Otherwise}, \end{cases}
\]

\[ \gamma = 0.45 \] and \( \mu = 0.35 \) for our algorithm, and choose \( \alpha_1 = 1 - \frac{3k}{\mu \rho}, \alpha_2 = \frac{k}{\mu \rho}, \alpha_3 = \frac{k}{\mu \rho}, \alpha_4 = \frac{k}{\mu \rho}, \alpha_5 = 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 \). \( \rho = 0.2 \) and \( \mu = 0.1 \) for PVSEGM.

Table 1: Comparison of the number of iterations in Theorem 2.2 and Theorem 1 [19] of Example 2.3 by choosing \( x_0 = (-2.15, -4.35, 1.12) \) and \( x_1 = (6.13, -5.24, -1.19) \).
From Table 1 and Figure 1, we see that the common solution of two or more inputting $B_i$ gives the number of iterations smaller than inputting one and the comparison between our Algorithm and PVSEGM and the inertial term ($\alpha_k \neq 0$) can speed up the convergence of the algorithm. We see that our Algorithm get the good CPU Time and number of iterations more than PVSEGM.

3. Application to image restoration problems

Image restoration is the process of recovering a degraded image that is blurred and noisy image. The image restoration problem can be formulated in the linear equation system as follows:

$$b = Ax + \omega,$$

(27)

where $x \in \mathbb{R}^{n \times 1}$ is an original image, $b \in \mathbb{R}^{m \times 1}$ is the unknown image, $\omega$ is additive noise and $A \in \mathbb{R}^{m \times n}$ is the blurring matrix. For solving the problem of image recovery (27) is an approximation of the original image $x$. In some case, finding $x = A^{-1}(b - \omega)$ maybe a hard task, thus finding the solution $x$ by mean of convex minimization can overbear such hard, which is called a least squares (LS) problem as follows:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|b - Ax\|_2^2,$$

(28)

where $\|\cdot\|$ is $l_2$-norm defined by $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$. The solution of (28) can be approximated by many well known iteration methods [5, 12, 16, 17].

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|A_1 x - b_1\|_2^2, \min_{x \in \mathbb{R}^n} \frac{1}{2} \|A_2 x - b_2\|_2^2, ... , \min_{x \in \mathbb{R}^n} \frac{1}{2} \|A_N x - b_N\|_2^2,$$

(29)

where $x$ is the original true image, $A_i$ is the blurred matrix, $b_i$ is the blurred image by the blurred matrix $A_i$ for all $i = 1, 2, ..., N$. For Algorithm 2.1 can apply to solve the problem (29), we know that $A_i^T(A_i x - b_i)$ is Lipschitz continuous for each $i = 1, 2, ..., N$. This algorithm is generated as follows:
Algorithm 3.1. Initialization: Given $\gamma > 0$, $\mu \in (0, 1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate $x_{k+1}$ as follows:

Step 1. Set $w_k = x_k + \alpha_k (x_k - x_{k-1})$ and compute

$$y^i_k = P_C(w_k - \tau^i_k A^T_i (A_i w_k - b_i)), \quad \forall k \geq 1,$$

where $\tau^i_k = \gamma \ell^i_k$ and $\ell^i_k$ is the smallest nonnegative integer such that

$$\tau^i_k \| A^T_i (A_i w_k - b_i) - A^T_i (A_i y^i_k - b_i) \| \leq \mu \| w_k - y^i_k \|.

(30)$$

Step 2. Compute

$$z^i_k = P_{T_k}(w_k - \tau^i_k A^T_i (A_i y^i_k - b_i)),$$

where $T_k := \{x \in H | \langle A_i w_k - \tau^i_k A_i w_k - y^i_k, x - y^i_k \rangle \leq 0 \}$.

Step 3. Compute $z_k$, i.e.

$$z_k = \arg\max ||z^i_k - x_k|| : i = 1, 2, ..., N.

(31)$$

Step 4. Compute

$$x_{k+1} = P_{C_{k+1}} x_k,$$

where $C_{k+1} := \{z \in C_k : ||z_k - z|| \leq ||w_k - z||\}$.

Again set $k := k + 1$ and go to Step 1.

We will present the advantages of our Algorithm 3.1 in images corrupted by the following three blur types:

(I) Gaussian blur of filter size $9 \times 9$ with standard deviation $\sigma = 4$ (blur matrix $A_1$).

(II) Out of focus blur (Disk) with radius $r = 6$ (blur matrix $A_2$).

(III) Motion blur specifying with motion length of 21 pixels ($\text{len} = 21$) and motion orientation $11^\circ$ ($\theta = 11$) (blur matrix $A_3$).

We will show the original RGB and grey images in the following figure 2-3.

![Figure 2-3: The matrix size of RGB and grey images are 277 x 370 x 3 and 277 x 370, respectively. Three different types of blurred RGB and grey images degraded by the blurring matrices $A_1$, $A_2$ and $A_3$ are shown in figures 4-9.](image-url)
We apply the PVSEG and our Algorithm 3.1 to obtain the solution of the deblurring problem (VIP) with the three blurring matrices $A_1$, $A_2$ and $A_3$. The results of the PVSEG and our Algorithm 3.1 are considered in following seven cases:
Case I: Inputting $A_1$ on the PVSEG and Algorithm 3.1,
Case II: Inputting $A_2$ on the PVSEG and Algorithm 3.1,
Case III: Inputting $A_3$ on the PVSEG and Algorithm 3.1,
Case IV: Inputting $A_1$ and $A_2$ on the PVSEG and Algorithm 3.1,
Case V: Inputting $A_1$ and $A_3$ on the PVSEG and Algorithm 3.1,
Case VI: Inputting $A_2$ and $A_3$ on the PVSEG and Algorithm 3.1,
Case VII: Inputting $A_1$, $A_2$ and $A_3$ on the PVSEG and Algorithm 3.1.

The following parameters are used for our algorithm:

$$
\alpha_k = \begin{cases} 
0.2 & \text{if } x_k \neq x_{k-1} \text{ and } k \leq 10,000 \\
\frac{1}{\|x_{k} - x_{k-1}\|} & \text{if } x_k \neq x_{k-1} \text{ and } k > 10,000 \\
0 & \text{Otherwise,}
\end{cases}
$$

$\gamma = 0.2$ and $\mu = 0.3$. We choose $\mu = 0.95$, $\rho = 0.5$, $\alpha_0^0 = 1 - \frac{2\gamma}{\|x_{k}\|}$, $\alpha_1^1 = \frac{1}{\|x_{k}\|}$, $\alpha_2^2 = \frac{1}{\|x_{k}\|}$, $\alpha_3^3 = 1 - \alpha_0^0 - \alpha_1^1 - \alpha_2^2$ for PVSEG.

Figures 10-15 show the reconstructed RGB and grey images with 10000 iterations. It comprises RGB and gray image quality restored, and PSNR.
Case I
PSNR = 35.95345

Case II
PSNR = 37.92625

Case III
PSNR = 42.21131

Figure 10-15: The reconstructed RGB and grey images with their PSNR for Case I - Case III using the proposed our Algorithm 3.1 presented on 10000th iterations, respectively.

Later in figures 16-21, we can see that the image quality restored using our algorithms for solving common problem resolution (VIP) problems with (N = 2) has been improved when compare with the previous results in Figures 10-15.

Case IV
PSNR = 57.88788

Case V
PSNR = 63.35813

Case VI
PSNR = 57.72298

Case V
PSNR = 46.08089

Case IV
PSNR = 42.12156

Case VI
PSNR = 47.87718

Figure 16-21: The reconstructed RGB and grey images with their PSNR for Case IV - Case VI using the proposed our Algorithm 3.1 presented on 10000th iterations, respectively.

Finally, the common solution of the deblurring problem (VIP) with (N= 3) using the proposed algorithm was also tested (Inputting $A_1$, $A_2$ and $A_3$ in the proposed algorithm).
Figure 22-23: The reconstructed RGB and grey images with their PSNR for Case VII using the proposed our Algorithm 3.1 presented on 10000th iterations, respectively.

Figure 22 and 23 show the reconstructed RGB and grey images with thousand iteration. The quality of the recovered RGB and grey images obtained by this algorithm were the highest compared to the previous two algorithms.

Figures 24-37 show the reconstructed RGB and grey images using the proposed algorithm in obtaining the common solution of the following problem with the same PSNR.

Figure 24-30: The reconstructed RGB images of all cases being used our Algorithm 3.1 with PSNR = 48.
From Table 2 we will compared our algorithm with PVSEGM, when PSNR of 10000th and number of iterations 43 PSNR of RGB images. Moreover, the Cauchy error, the Image error and the peak signal-to-noise ratio (PSNR) for recovering processes of the degraded RGB images by using the proposed method within the first 10000th iterations are shown in figures 38-40.
Figure 38-40: Cauchy error, Figure error and PSNR quality plots of the proposed iteration in all cases of RGB images.

Table 3: Comparison of the number of iterations in grey images.

| Inputting | PSNR of 10000th | Number of Iterations 34 PSNR |
|-----------|-----------------|-------------------------------|
|           | Our Algorithm   | PVSEGEM                       | Our Algorithm | PVSEGEM |
| A<sub>1</sub> | 35.95345        | 34.42142                      | 2184<sup>th</sup> | 7088<sup>th</sup> |
| A<sub>2</sub> | 37.92625        | 35.31389                      | 1110<sup>th</sup> | 5213<sup>th</sup> |
| A<sub>3</sub> | 42.21131        | 37.58995                      | 132<sup>th</sup>  | 1901<sup>th</sup> |
| A<sub>1</sub>,A<sub>2</sub> | 42.12156        | 39.05422                      | 461<sup>th</sup>  | 1108<sup>th</sup> |
| A<sub>1</sub>,A<sub>3</sub> | 46.08089        | 42.89091                      | 138<sup>th</sup>  | 542<sup>th</sup>   |
| A<sub>2</sub>,A<sub>3</sub> | 47.87718        | 42.95463                      | 212<sup>th</sup>  | 699<sup>th</sup>   |
| A<sub>1</sub>,A<sub>2</sub>,A<sub>3</sub> | 50.73579        | 45.43944                      | 158<sup>th</sup>  | 482<sup>th</sup>   |

From Table 3 we will compared our algorithm with PVSEGEM, when PSNR of 10000<sup>th</sup> and number of iterations 34 PSNR of grey images. Moreover, the Cauchy error, the Image error and the peak signal-to-noise ratio (PSNR) for recovering processes of the degraded grey images by using the proposed method within the first 10000<sup>th</sup> iterations are shown in figures 41-43.
4. Conclusions

In this paper, we propose projective inertial parallel subgradient extragradient-line algorithm for solving variational inequalities. Under some suitable conditions imposed on parameters, we have proved the strong convergence of the algorithm. A numerical example illustrating the proposed algorithm performance in comparison with PVSEGM, see Table 1 and Figure 1. Our algorithm can solve image recovery under unknown situation of blur matrix type, to demonstrate the computational performance see in Figures 10-23 and Figures 24-37. Finally, we apply our proposed algorithm to recover RGB image, when PSNR of $10000^{th}$ and number of iterations $34$ PSNR and grey image, when PSNR of $10000^{th}$ and number of iterations $34$ PSNR compared to PVSEGM, see in Figures 38-43. Our algorithm is more efficient than PVSEGM see in Table 2 and 3.

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