K3 SURFACES WITH NON-SYMPLECTIC AUTOMORPHISMS OF ORDER THREE AND CALABI-YAU MANIFOLDS

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Abstract. Let $S$ be a K3 surface that admits a non-symplectic automorphism $\rho$ of order 3. We divide $S \times \mathbb{P}^1$ by $\rho \times \psi$ where $\psi$ is the multiplication by a third root of unity. There exists a threefold ramified cover of the crepant resolution of the quotient that is a Calabi-Yau manifold. We compute the Euler characteristic of our examples and obtain values ranging from $-96$ to $336$.

1. Introduction

The motivation for this article are various constructions of Calabi-Yau manifolds that involve non-symplectic automorphisms of K3 surfaces. The oldest example is the construction of Borcea [4] and Voisin [21]. Their idea is to divide the product of a K3 surface $S$ and an elliptic curve $E$ by the product $\rho \times \psi$ of two involutions. $S$ shall act non-trivially on $H^{2,0}(S)$. Such an involution is called non-symplectic. The quotient $(S \times E)/(\rho \times \psi)$ is an orbifold whose singularities are the product of ordinary double points and the curves in the fixed locus of $\rho$. These singularities can be blown up and the authors obtain smooth Calabi-Yau threefolds.

Kovalev and Lee [14] replace $E$ by the Riemann sphere $\mathbb{P}^1$ and $\psi$ by the map $\psi(z) := -z$. The resulting threefold $X$ is not a Calabi-Yau manifold. Nevertheless, one can remove an anti-canonical K3 divisor $D$ and there exists an asymptotically cylindrical Ricci-flat Kähler metric on $X \setminus D$. In other words, $X \setminus D$ is a non-compact Calabi-Yau manifold. The manifolds that are constructed by this method can be used as building blocks for compact $G_2$-manifolds.

It is a natural question if it is possible to construct Calabi-Yau manifolds with help of non-symplectic automorphisms of higher order. These automorphisms are classified in [1, 20] for order 3 and for arbitrary prime order in [2]. Rohde [18,19] and Dillies [11] divide a product $S \times E$ by the cyclic group that is generated by $\rho \times \psi$ where $\rho$ and $\psi$ are of order 3. The crepant resolutions of those quotients are smooth Calabi-Yau manifolds. Dillies [11] also considers quotients of the product of two K3 surfaces by a cyclic group.
that is generated by the product of two non-symplectic automorphisms of the same order. In those cases where a crepant resolution exists, he obtains Calabi-Yau manifolds of dimension four. In fact, $S$ and $E$ can be replaced by arbitrary Calabi-Yau manifolds that admit a non-symplectic automorphism of order 3 [9]. Finally, Garbagnati [12] obtains Calabi-Yau threefolds by dividing a suitable product of a K3 surface and an elliptic curve by a non-symplectic automorphism of order 4.

If we divide $S \times \mathbb{P}^1$ by $\rho \times \psi$ where $\psi(z) := \exp \left( -\frac{2\pi i}{3} \right) z$ and $\rho$ is of order $p$, the K3 divisor is not anti-canonical anymore and we cannot apply the method of [14]. In the case $p = 3$, we are nevertheless able to construct a threefold ramified cover that is a compact Calabi-Yau manifold. A similar method is applied in an article of Cynk [8] where Calabi-Yau manifolds that are threefold ramified covers of Fano varieties are constructed. This method was generalized by Casnati [5] to almost-Fano threefolds.

The article is organized as follows. In the second section, we introduce the most important facts on non-symplectic automorphisms of order 3 and their classification. After that, we describe our construction and prove that it yields indeed a Calabi-Yau manifold. In the fourth section, we compute the Euler characteristic of our examples.

2. NON-SYMPLECTIC AUTOMORPHISMS OF ORDER 3

A smooth compact complex surface $S$ with trivial canonical bundle and $\pi_1(S) = \{0\}$ is called a K3 surface. The cohomology group $H^2(S, \mathbb{Z})$ together with the intersection form is a lattice that is isomorphic to $L := H^3 \oplus (-E_8)^2$ where $H$ is the hyperbolic plane lattice and $-E_8$ is the root lattice with negative signature that corresponds to the root system $E_8$. We call $L$ the K3 lattice. A K3 surface together with a lattice isometry $\phi : H^2(S, \mathbb{Z}) \to L$ is called a marked K3 surface.

An automorphism $\rho : S \to S$ is called of order 3 if $\rho \neq \text{Id}_S$ but $\rho^3 = \text{Id}_S$. Moreover, it is called non-symplectic if $\rho^*$ acts on the one-dimensional complex space $H^{2,0}(S)$ as multiplication by $\zeta_3 := \exp \frac{2\pi i}{3}$. The non-symplectic automorphisms of order 3 are classified in [11 20].

In order to explain this classification we have to introduce some further concepts. $\phi \circ \rho^* \circ \phi^{-1}$ is an isometry of $L$ that we abbreviate by $\rho^*$. Let

$$L^\rho := \{ x \in L | \rho^*(x) = x \}$$

be the fixed lattice. The discriminant group of $L^\rho$ is defined as $L^{\rho^*}/L^\rho$ where $L^{\rho^*} = \text{Hom}(L^\rho, \mathbb{Z})$. It is a finite group that is isomorphic to $\mathbb{Z}_3^a$ for a $a \in \mathbb{N}$. Lattices whose discriminant group is of that kind are called $3$-elementary.
The rank of $L^\rho$ is even and we define $m := \frac{1}{2}L^\rho \perp$. In \cite{1, 20} the following result is proven:

**Theorem 2.1.** Let $L^\rho$ be the invariant lattice of a K3 surface with a non-symplectic automorphism of order 3. Then the isomorphism type of $L^\rho$ is determined by the invariants $(m,a)$ that are defined above.

Moreover, we have:

**Theorem 2.2.** Let $(m,a)$ be a pair of natural numbers from the following list:

| (1,1) | (5,3) | (7,5) |
| (2,0) | (5,5) | (7,7) |
| (2,2) | (6,0) | (8,2) |
| (3,1) | (6,2) | (8,4) |
| (3,3) | (6,4) | (9,1) |
| (4,2) | (6,6) | (9,3) |
| (4,4) | (7,1) | (10,0) |
| (5,1) | (7,3) | (10,2) |

Then there exists a K3 surface with an automorphism of order 3 such that its fixed lattice has invariants $(m,a)$.

The authors of \cite{1, 20} show that the invariants $(m,a)$ determine the topological type of the fixed locus $S^\rho := \{ p \in S | \rho(p) = p \}$

**Theorem 2.3.** Let $S$ be a K3 surface with a non-symplectic automorphism $\rho$ of order 3. The fixed locus of $\rho$ is not empty and consists of

- three isolated points if $m = a = 7$,
- $n$ isolated points, $k$ smooth rational curves, and one smooth curve of genus $g$ where
  - $n = 10 - m$,
  - $k = 6 - \frac{m+a}{2}$,
  - $g = \frac{m-a}{2}$

For our construction and the computation of the Euler characteristic we are mainly interested in the shape of the fixed locus. We refer the reader for information on further topics such as the moduli space of all K3 surfaces that admit a non-symplectic automorphism of order 3 to the literature \cite{1, 2, 20}.

### 3. The construction of the Calabi-Yau manifolds

The aim of this section is to prove the following theorem.
Theorem 3.1. Let $S$ be a K3 surface that admits a non-symplectic automorphism $\rho$ of order 3. Moreover, let $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be defined by $\psi(z) := \zeta_3^2 z$. Then we can construct a Calabi-Yau threefold $X$ by the following three steps.

1. We divide $S \times \mathbb{P}^1$ by the cyclic group that is generated by $\rho \times \psi$. The quotient shall be denoted by $Z$ and the projection map by $p : S \times \mathbb{P}^1 \rightarrow Z$.
2. Let $\phi : X_0 \rightarrow Z$ be a crepant resolution. Moreover, let $D_0 := \phi^{-1}(p(S \times \{z\}))$, where $z \in \mathbb{P}^1 \setminus \{0, \infty\}$.
3. Finally, let $\pi : X \rightarrow X_0$ be a suitable 3-fold cover that is ramified along $D_0$.

Before we prove the above theorem, we describe the second and the third step in more detail. Since we are in complex dimension 3, a crepant resolution always exists. Moreover, we can show that it is unique. The singular locus of $Z$ is a disjoint union of complex curves and isolated points. Along the curves we have ADE-singularities of type $A_2$ that can be blown up. The singularities at the isolated points are of type $\mathbb{C}^3/\mathbb{Z}_3$ where $\mathbb{Z}_3$ is the cyclic group that is generated by scalar multiplication with $\zeta_3$. In general, there exists more than one crepant resolution of a given singularity. If the singularity is of type $\mathbb{C}^3/G$ where $G$ is an abelian subgroup of $SL(3, \mathbb{C})$, there is an algorithm that counts the crepant resolutions of $\mathbb{C}^3/G$. We sketch how this algorithm works in our case. For a more comprehensive description and the proof that the algorithm yields the correct result, we refer the reader to [7, 10] and references therein. Since $G$ is abelian, its elements can be diagonalized simultaneously. Any $g \in G$ can be written as

\[
\begin{pmatrix}
\exp\left(\frac{2\pi i}{r}a_1\right) & 0 & 0 \\
0 & \exp\left(\frac{2\pi i}{r}a_2\right) & 0 \\
0 & 0 & \exp\left(\frac{2\pi i}{r}a_3\right)
\end{pmatrix},
\]

where $r := |G|$ and $a_1, a_2, a_3 \in \{0, \ldots, r-1\}$. Let $L \subset \mathbb{R}^3$ be the lattice that is generated by $\mathbb{Z}^3$ and all triples $\frac{1}{r}(a_1, a_2, a_3)$. We define $\triangle$ as the triangle with vertices

\[
(1, 0, 0), \quad (0, 1, 0) \quad \text{and} \quad (0, 0, 1).
\]

The intersection $L \cap \triangle$ can be visualized as a triangle with a finite number of interior points. A triangulation of this set is called basic if it cannot be triangulated further. There is a one-to-one correspondence between the crepant resolutions of $\mathbb{C}^3/G$ and the basic triangulations of $L \cap \triangle$. In our situation, the only interior point is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and we obtain the following picture:
Since there exists only the above basic triangulation, the resolution of $\mathbb{C}^3/\mathbb{Z}^3$ is unique. An explicit description of that resolution can be found in [16, 17]. Since the blow-up is the unique crepant resolution of an $A_2$-singularity, there exists only one possibility for $\phi : X_0 \to Z$ and we cannot obtain further Calabi-Yau threefolds by choosing another $\phi$.

For the third step of our construction we need the following lemma that is proven in Chapter I.17 of [3].

**Lemma 3.2.** Let $M_0$ be a connected complex manifold and $B$ be an effective divisor of $M_0$. We assume that there exists a line bundle $L$ on $M_0$ such that

\[(4) \quad L^n = [B] \]

where $[B]$ is the line bundle that is associated to $B$. Then there exists an $n$-fold ramified covering $\pi : M \to M_0$ with branch-locus $B$.

Since we need the explicit construction of $M$, we include a proof of the above lemma in our article.

**Proof.** There exists a global holomorphic section $s$ of $L^n$ that has a zero of order 1 at $B$ and is non-zero outside of $B$. Let $\alpha : \mathcal{L} \to \mathcal{L}^n$ be the map with $\alpha(v) = v^\otimes n$. $M$ can be defined as $\alpha^{-1}(s(M_0)) \subseteq \mathcal{L}$ and $\pi : M \to M_0$ is the restriction of the projection map $p : \mathcal{L} \to M_0$. \qed

The ramified cover from the above lemma is not necessarily unique since there may exist several line bundles satisfying (4). In order to simplify our proof, we choose a particular bundle for our construction of $X$. We are now able to prove Theorem 3.1.

**Proof.** We have to prove that $X$ is a complex Kähler manifold with trivial canonical bundle $K_X$. We will also prove the condition $h^{1,0}(X) = 0$ that is sometimes included in the definition of a Calabi-Yau manifold. $X$ as well
as the intermediate spaces $Z$ and $X_0$ are equipped with canonical complex structures such that $p$, $\phi$, and $\pi$ become holomorphic maps.

Our next step is to show that $K_X$ is trivial. Let $z \in \mathbb{P}^1 \setminus \{0, \infty\}$ be arbitrary. $\{\{z\}\}$ is the hyperplane line bundle $H$. The bundle $\bigwedge^{1,0} T^* \mathbb{P}^1$ is isomorphic to $H^{-2}$. By extending the transition functions of $H$ canonically to $S \times \mathbb{P}^1$ we obtain a line bundle $H'$ on $S \times \mathbb{P}^1$. Since $S$ has vanishing first Chern class, the canonical bundle of $S \times \mathbb{P}^1$ is $H'^{-2} = [S \times \{z\}]^{-2}$.

There exists a $(3,0)$-form $\eta$ on $S \times \mathbb{P}^1$ with a pole of second order along $S \times \{z\}$. $\eta \otimes (\rho \times \psi)^* \eta \otimes (\rho \times \psi)^* \eta$ is a $\rho \times \psi$-invariant section of $\left(\bigwedge^{3,0} T^* (S \times \mathbb{P}^1)\right)^3$ with poles of second order along $S \times \{z\} \cup (S \times \{\zeta_3 \cdot z\}) \cup (S \times \{\zeta_3^2 \cdot z\})$. Therefore, there exists a section $\alpha$ of $\left(\bigwedge^{3,0} T^* Z\right)^3$ with $p^* \alpha = \eta \otimes (\rho \times \psi)^* \eta \otimes (\rho \times \psi)^* \eta$. $\alpha$ has a pole of second order along $p(S \times \{z\})$ and we thus have

\[
(5) \quad K_Z^3 \cong [p(S \times \{z\})]^{-2}.
\]

Since $\phi$ is a crepant resolution, we also have

\[
(6) \quad K_{X_0}^3 \cong [D_0]^{-2}.
\]

We define a line bundle $L := [D_0] \otimes K_{X_0}$. We have $L^3 \cong [D_0]$. The conditions of Lemma 3.2 are satisfied and we can take $L$ to construct a threefold ramified cover $\pi : X \to X_0$ with branching locus $D_0$. We denote $\pi^{-1}(D_0)$ by $D$. In the situation of Lemma 3.2, we have (see Chapter I.17 of [3])

\[
(7) \quad K_M \cong \pi^*(K_{M_0} \otimes L^{n-1}).
\]

We conclude that

\[
(8) \quad K_X \cong \pi^*(K_{X_0} \otimes L^2) \cong \pi^*(K_{X_0}^3 \otimes [D_0]^2) \cong \mathbb{C} \times X.
\]

Next, we have to prove that $X$ is Kähler. Let $g$ be a Kähler metric on $S \times \mathbb{P}^1$. $g + (\rho \times \psi)^* g + (\rho \times \psi)^* g$ is a $\rho \times \psi$-invariant Kähler metric on $S \times \mathbb{P}^1$. $Z$ therefore carries an orbifold Kähler metric. Among the crepant resolutions of a Kähler orbifold there exists at least one that admits a Kähler metric [13, p. 137]. Since $\phi$ is unique, $X_0$ is Kähler.

It is easy to see that the normal bundle of $D_0$ and of $D$ is trivial. We can therefore choose a tubular neighborhood $T$ of $D$ that can be identified with
As before, we identify suitable tubular neighborhoods of $S \times B_{\epsilon}(0)$, where $B_{\epsilon}(0) := \{ z \in \mathbb{C} | |z| < \epsilon \}$. Let $u^1, u^2$ be local complex coordinates on $S$ and $u^3$ be the standard coordinate on $B_{\epsilon}(0)$. Moreover, let $g_0$ be a Kähler metric with Kähler form $\omega_0$ on $X_0$. The form $\pi^* \omega_0$ on $X$ is not Kähler anymore. The reason is that $d\pi_p(\frac{\partial}{\partial u^3}) = d\pi_p(\frac{\partial}{\partial u^3}) = 0$ for any $p \in D$ since $\pi$ behaves near $D$ as $\pi(u^1, u^2, u^3) = (u^1, u^2, (u^3)^3)$. The coefficient functions $g_{33}^0, g_{33}^0 \in \mathbb{R}$ of $g^0 := \pi^* g_0$ have order of vanishing $4$ at $D$ and the coefficients $g_{3l}$ and $g_{3l}$ with $l \in \{1, 2, 3\}$ have order of vanishing $2$.

Let $h_0 : [0, \epsilon^2] \rightarrow \mathbb{R}_{\geq 0}$ be a smooth function that vanishes outside of $[0, \frac{1}{2}\epsilon^2]$. We define a function $h$ on $T$ by $h(u^1, u^2, u^3) := h_0(|u^3|^2)$ and a closed $2$-form $\omega := \pi^* \omega_0 + \frac{1}{2} h du^3 \wedge d\overline{u}^3$. $h$ can be chosen such that $\omega$ determines a Riemannian metric and we thus have proven that $X$ is Kähler.

Finally, we prove that $h^{1,0}(X) = 0$ or equivalently that $\pi_1(X)$ is finite. As before, we identify suitable tubular neighborhoods of $D_0$ and $D$ with $S \times B_{\epsilon}(0)$ and obtain open covers of type

$$X_0 = (X_0 \setminus D_0) \cup (S \times B_{\epsilon}(0))$$

$$X = (X \setminus D) \cup (S \times B_{\epsilon}(0))$$

The intersection of both open sets is diffeomorphic to $S \times B_{\epsilon}(0) \setminus \{0\}$. There are the following Mayer-Vietoris sequences:

$$\ldots \rightarrow \{0\} \rightarrow H^1(X_0, \mathbb{R}) \rightarrow H^1(X_0 \setminus D_0, \mathbb{R}) \oplus H^1(S \times B_{\epsilon}(0), \mathbb{R})$$

$$\rightarrow H^1(S \times B_{\epsilon}(0) \setminus \{0\}, \mathbb{R}) \xrightarrow{d} H^2(X_0, \mathbb{R}) \rightarrow \ldots$$

$$\ldots \rightarrow \{0\} \rightarrow H^1(X, \mathbb{R}) \rightarrow H^1(X \setminus D, \mathbb{R}) \oplus H^1(S \times B_{\epsilon}(0), \mathbb{R})$$

$$\rightarrow H^1(S \times B_{\epsilon}(0) \setminus \{0\}, \mathbb{R}) \xrightarrow{d} H^2(X, \mathbb{R}) \rightarrow \ldots$$

In the following section, we will prove that $H^1(X_0, \mathbb{R}) = 0$. Moreover, $H^1(S \times B_{\epsilon}(0) \setminus \{0\}, \mathbb{R})$ is generated by $d\theta$ where $\theta$ is the argument and $H^1(S \times B_{\epsilon}(0), \mathbb{R})$ is trivial. The only possibility to write $d\theta$ as a difference between forms on $X_0 \setminus D_0$ or $X \setminus D$ and $S \times B_{\epsilon}(0)$ is therefore $d\theta = (d\theta + \alpha) - \alpha$ where $\alpha$ is exact. The maps "$d$" in the above sequences vanish consequently. With help of this information, we can conclude that $H^1(X_0 \setminus D_0, \mathbb{R}) = \mathbb{R}$. We therefore have $H_1(X_0 \setminus D_0, \mathbb{Z}) = \mathbb{Z} \oplus \text{Tor}$, where $\text{Tor}$ is the torsion group. The restriction of $\pi$ to $X \setminus D$ is an ordinary threefold cover. $\pi_1(X \setminus D)$ is either isomorphic to $\pi_1(X_0 \setminus D_0)$ or to $\pi_1(X_0 \setminus D_0)/\mathbb{Z}_3$. Since $H_1(X \setminus D, \mathbb{Z})$ is the abelianization of $\pi_1(X \setminus D)$, we again have $H_1(X \setminus D, \mathbb{Z}) = \mathbb{Z} \oplus A$ for a finite group $A$ and therefore $b_1(X \setminus D) = 1$. We conclude with help of the second exact sequence that $b^1(X) = h^{1,0}(X) = 0$. \qed
Remark 3.3. If we had divided $S \times \mathbb{P}^1$ by an automorphism of order $p$ and defined $\mathcal{L}$ as $\lbrack D_0 \rbrack \otimes K_{X_0}$, we would have $\mathcal{L}^p = \lbrack D_0 \rbrack^{p-2}$. Since we need $\mathcal{L}^p = \lbrack D_0 \rbrack$ to construct a $p$-fold ramified cover, the condition $p = 3$ in our theorem is necessary.

4. THE EULER CHARACTERISTIC

We go through each step of our construction and determine the hodge numbers of the intermediate threefolds. In the end, we obtain the Euler characteristic of our Calabi-Yau manifolds. Since the Hodge numbers of a K3 surface are well-known, the K"unneth theorem yields

$$h^{*,*}(S \times \mathbb{P}^1) = \begin{array}{ccc} \ 1 \ & \ 0 \ & \ 0 \ \\ 0 & 1 & 2 \ & 1 & 0 \ & 0 \ & 0 \ \\ 1 & 2 \ & 1 \ & 0 & 0 \ & 1 \ & \ 1 \ \end{array}$$

(11)

The Hodge diamond of $Z$ can be calculated with help of the Chen-Ruan orbifold cohomology that was introduced in [6]. We shortly describe how to determine that cohomology in our situation. Let $M$ be a complex manifold of dimension $n$. Moreover, let $G$ be a finite group that acts holomorphically on $M$. For any $x \in M$ let $G_x$ be the stabilizer group of the $G$ at $x$. Any $g \in G_x$ can be diagonalized to a matrix

$$\text{diag}(e^{2\pi i m_1}, \ldots, e^{2\pi i m_n}) \quad \text{with} \quad m_i \in [0, 1) \cap \mathbb{Q}.$$  

We define the age $\text{age}(g, x)$ of $g$ at $x$ as

$$\text{age}(g, x) = m_1 + \ldots + m_n.$$  

(13)

In our situation, we have $\text{age}(g, x) \in \mathbb{Z}$. The age of $g$ is constant along any connected component $\Sigma$ of the fixed point set $M^g$ of $g \in G$. We therefore denote it by $\text{age}(g, \Sigma)$. If $M$ is K"ahler and $G$ is a cyclic group of prime order $p$, the orbifold cohomology of $M/G$ is

$$H_{\text{orb}}^{k,l}(M/G) = H^{k,l}(M)^G \oplus \bigoplus_{i=1, \ldots, p-1, \Sigma \subset M^g} H^{k-\text{age}(\gamma^i, \Sigma), l-\text{age}(\gamma^i, \Sigma)}(\Sigma).$$

(14)
In the above formula, $H^{k,l}(M)^G$ is the subspace of $H^{k,l}(M)$ that acts trivially on and $\gamma$ is a generator of $G$. We refer the reader to [4] for the general definition of $H^{*,*}_{\text{orb}}$ that can be applied to any complex orbifold. The main reason why we are interested in the orbifold cohomology is the following theorem of Yasuda [22]:

**Theorem 4.1.** Let $X$ and $X'$ be complete varieties with Gorenstein quotient singularities. Moreover, let $\pi : Z \to X$ and $\pi' : Z \to X'$ be proper birational morphisms such that the pull-backs $\pi^*K_X$, $\pi'^*K_{X'}$ of the canonical bundles coincide. Then the orbifold cohomology groups of $X$ and $X'$ have the same Hodge structure.

If $Y$ is a crepant resolution of $M/G$, we thus have

$$h^{k,l}(Y) = h^{k,l}_{\text{orb}}(M/G),$$

since the orbifold cohomology coincides with the Dolbeault cohomology for nonsingular varieties. In particular, $h^{k,l}(Y)$ is independent of the choice of the crepant resolution. The action of $\rho \times \psi$ on the tangent space of a fixed point is

$$\begin{pmatrix}
\zeta_3 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \zeta_3^2
\end{pmatrix}$$

if the point is on a fixed curve and the age is 1. If the fixed point is isolated, the action is given by

$$\begin{pmatrix}
\zeta_3^2 & 0 & 0 \\
0 & \zeta_3^2 & 0 \\
0 & 0 & \zeta_3^2
\end{pmatrix}$$

and the age is 2. The action of $(\rho \times \psi)^2$ at an isolated or non-isolated fixed point is 1. All in all, we have:

$$h^{*,*}(X_0) = h^{*,*}(S \times \mathbb{P}^1)^{\rho \times \psi} + 2 \cdot h^{*-1,*-1}(C) + h^{*-2,*-2}(P),$$

where $P$ denotes the union of all isolated points and $C$ the union of all curves in the fixed locus. We calculate the part of $H^{*,*}(S \times \mathbb{P}^1)$ that is invariant under $\rho \times \psi$. Any harmonic form on $\mathbb{P}^1$ is $\psi$-invariant. By assumption, the $(2,0)$-form on $S$ is not fixed by $\rho$ and $\dim H^{1,1}(S)^{\rho} = 22 - 2m$. Therefore, we have
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\begin{equation}
\begin{pmatrix}
1 \\
0 & 0 \\
0 & 23 - 2m & 0 \\
0 & 23 - 2m & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1
\end{pmatrix}
\end{equation}

Since we do not consider the case \( m = a = 7 \), \( C \) is non-empty and it consists of two copies of \( C_g \cup E_1 \cup \ldots E_k \). We obtain

\begin{equation}
\begin{pmatrix}
2 + 2k \\
2 + 2k
\end{pmatrix}
\end{equation}

We conclude that the Hodge diamond of \( X_0 \) is

\begin{equation}
\begin{pmatrix}
1 \\
0 & 0 \\
0 & 71 - 6m - 2a & 0 \\
0 & 71 - 6m - 2a & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1
\end{pmatrix}
\end{equation}

In the exceptional case \( m = a = 7 \), we obtain the same Hodge diamond. This means that \( h^{1,1}(X_0) = 15 \) and \( h^{1,2}(X_0) = 0 \). Our final step is to calculate the Euler characteristic of our Calabi-Yau manifold \( X \). It is well-known that the Euler characteristic of a quotient of a manifold \( M \) by a discrete group \( G \) is given by

\begin{equation}
\chi(M/G) = \frac{1}{|G|} \sum_{g \in G} \chi(M^g).
\end{equation}

Thus, we obtain

\begin{equation}
\chi(X) = 3\chi(X_0) - 2\chi(D) = 384 - 48m.
\end{equation}

We go through the table from \([1]\) and obtain the following numerical values:
| \((m, a)\) | \(\chi\) |
|-----------|--------|
| (1,1)     | 336    |
| (2,0), (2,2) | 288   |
| (3,1), (3,3) | 240   |
| (4,2), (4,4) | 192   |
| (5,1), (5,3), (5,5) | 144   |
| (6,0), (6,2), (6,4), (6,6) | 96    |
| (7,1), (7,3), (7,5), (7,7) | 48    |
| (8,2), (8,4) | 0     |
| (9,1), (9,3) | -48   |
| (10,0), (10,2) | -96   |

**Remark 4.2.** It is possible to calculate \(\chi\) more directly, without determining the Hodge diamond of the intermediate manifolds. We have nevertheless chosen this approach since we obtain \(h^{1,0}(X_0) = 0\) as a by-product.

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