Enumeration of finite inverse semigroups

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Abstract
We give an efficient algorithm for the enumeration up to isomorphism of the inverse semigroups of order \( n \), and we count the number \( S(n) \) of inverse semigroups of order \( n \leq 15 \). This improves considerably on the previous highest-known value \( S(9) \). We also give a related algorithm for the enumeration up to isomorphism of the finite inverse semigroups \( S \) with a given underlying semilattice of idempotents \( E \), a given restriction of Green’s \( D \)-relation on \( S \) to \( E \), and a given list of maximal subgroups of \( S \) associated to the elements of \( E \).

Keywords
Inverse semigroup · Enumeration · Semilattice · Maximal subgroup · Green’s relations

1 Introduction
The development of efficient algorithms for the enumeration of finite algebraic structures dates back at least to 1955, when the first successful computer-based enumeration of the semigroups of order \( n \) was accomplished, with the result that there are exactly 126 semigroups of order 4 [8]. The most recent result on the enumeration of finite semigroups is the 2012 result that there are exactly 12,418,001,077,381,302,684 semigroups of order 10 [3]. Currently, the number of semigroups of order \( n \) is known only for \( n \leq 10 \), and there is a database of the semigroups of order 1 through 8 [5].

This situation is in stark contrast to that for finite groups. The number of groups of order \( n \) is known for \( n \leq 2047 \), and the Small Groups Library contains all the groups of order 2000 or less (excluding 1024), a total of 423,164,062 groups [1].

Just as groups encode global symmetries, inverse semigroups encode partial symmetries [13]. Every group is an inverse semigroup, and every inverse semigroup is
a semigroup, but neither conversely. This paper marks the first attempt to enumerate finite inverse semigroups specifically. If we denote by $S(n)$ the number of inverse semigroups of order $n$ up to isomorphism, then the numbers $S(1), \ldots, S(9)$ are known from previous work on the enumeration of finite semigroups. $S(9)$ was computed by A. Distler in his 2010 Ph.D. thesis [4,6]. Previously $S(8)$ was computed by Satoh et al. [16]. The references in [16] contain information concerning the history of the computation of $S(n)$ for $n \leq 7$. In 2012 Distler et al. [3] found the number of semigroups of order 10 with the help of a parallelized computation which took approximately 130 CPU years. Although it may have been possible to compute $S(10)$ along the way during this computation, inverse semigroups are not discussed and $S(10)$ is not reported in [3]. At present there is no explicit formula for $S(n)$, and the only way to compute $S(n)$ is by a careful exhaustive search.

The approach to semigroup enumeration in [3,4,6] is based on the idea that any combinatorial enumeration problem can be written as a constraint satisfaction problem. To obtain their results in [3], the authors work out a collection of constraint satisfaction problems whose solutions comprise the semigroups of order $n$ which cannot be counted by any previously-known formula and feed those problems to the constraint satisfaction solver minion [9]. The constraint satisfaction approach can be adapted to count inverse semigroups by adding additional constraints to be satisfied. Using this idea and the code from [6] we were able to compute $S(10)$ in 11.5 hours of CPU time on a test machine with an Intel® Core™ i3 processor and 8 GB of RAM. This computation consumed about 7 GB of RAM. We were unable to go beyond $S(10)$ on our test machine with this approach.

In this paper we offer a specialized method for the enumeration of the inverse semigroups of order $n$ (Algorithm 3.4), which is both quicker and more memory-efficient than the generic approach of [3,4,6]. We have implemented our algorithm in Sage, an open-source computer algebra system [18]. For comparison, our implementation computes $S(10)$ on our test machine in 22 min of CPU time using only 256 MB of RAM. We have used our implementation to enumerate the inverse semigroups of order 1, \ldots, 15. Along the way we also counted commutative inverse semigroups, inverse monoids, and commutative inverse monoids. Our enumeration results are summarized in Table 1. More detailed counts and information regarding the efficiency of our algorithm are given in Sect. 6.

We have stored and made available the Cayley tables of the inverse semigroups of order 2, \ldots, 12, as well as the Sage code we used to count the inverse semigroups of order $\leq 15$.\(^1\) A key step of Algorithm 3.4 involves iterating over the unlabeled meet-semilattices of order 1, \ldots, $n$ up to isomorphism. For this step we used the algorithm of Heitzig and Reinhold [11] (see also [12]) for generating finite lattices. For $n \leq 15$ we have stored and made available the lattices of order $n$ as lists of cover relations.\(^2\)

Our approach to inverse semigroup enumeration is based on the Ehresmann–Schein–Nambooripad (ESN) theorem (see, e.g., [13, Ch. 4] and [7,15,17]), which essentially transfers the problem of enumerating inverse semigroups to the problem

\(^1\) Cayley tables and Sage code available at http://www.shsu.edu/mem037/ISGs.html.
\(^2\) Lattices available at http://www.shsu.edu/mem037/Lattices.html.
of enumerating a certain class of groupoids. Briefly, let \( r_1, \ldots, r_k \) be positive integers and let \( G_1, \ldots, G_k \) be finite groups. Let \( A \) denote the algebra

\[
A = \bigoplus_{i=1}^{k} M_{r_i}(C G_i).
\]

Let \( B \) be the collection of matrices in \( A \) which have a group element in exactly one position and 0 elsewhere. \( B \) is called the natural basis of \( A \). The set \( E(B) \) of idempotents of \( B \) is the set of matrices having a group identity on the diagonal. If \( \leq \) is a partial order on \( B \) for which \( (E(B), \leq) \) is a meet-semilattice and which satisfies some additional properties (see Theorem 3.1), then the ESN theorem may be used to construct from \( (B, \leq) \) an inverse semigroup \( S \) such that

\[
|S| = \sum_{i=1}^{k} r_i^2 |G_i|.
\]

Furthermore, to generate all inverse semigroups \( S \) of order \( n \), it suffices to apply the construction from the ESN theorem to all partial orders \( \leq \) on all natural bases \( B \) satisfying the hypotheses of Theorem 3.1, across all algebras \( A = \bigoplus_{i=1}^{k} M_{r_i}(C G_i) \) for which \( n = \sum_{i=1}^{k} r_i^2 |G_i| \).
Unfortunately, this process may create isomorphic inverse semigroups. To generate the inverse semigroups of order $n$ up to isomorphism, we generate inverse semigroups according to this process, testing each newly generated inverse semigroup for isomorphism against previously-generated inverse semigroups, and accepting only the inverse semigroups not isomorphic to any previously-generated inverse semigroup. To accomplish our isomorphism testing in an efficient manner we begin by separating generated inverse semigroups according to invariants. Any function $I$ such that whenever $S$ and $T$ are inverse semigroups with $S \cong T$ we have $I(S) = I(T)$ is called an invariant. Given an invariant $I$, a newly-generated inverse semigroup $S$ must be tested for isomorphism only against previously-generated inverse semigroups $T$ such that $I(S) = I(T)$.

The main computational challenges we face are the challenges of efficiently generating the partial orders on $B$, addressed in Sect. 4, giving an effective definition for $I$, addressed in Sect. 5.1, and quickly testing for isomorphism between pairs of inverse semigroups, addressed in Sect. 5.2. A complete description of our enumeration procedure is given as Algorithm 3.4 in Sect. 3.2. Our algorithm adopts a memory-efficient iteration order—only a relatively small percentage of the inverse semigroups of order $n$ need to be held in memory for isomorphism testing at any point during our algorithm’s execution.

We also give a modification of our algorithm that enumerates the inverse semigroups $S$ having a particular semilattice $E(S)$ of idempotents, a particular restriction of Green’s $D$-relation on $S$ to $E(S)$, and a particular collection of maximal subgroups of $S$ associated to the elements of $E(S)$. This modification is given as Algorithm 3.8 in Sect. 3.2.

## 2 Inverse semigroups

In this section we collect the basic definitions and facts about inverse semigroups we require to explain our algorithms and prove their correctness. Good references for the facts in this section include [2,10,13,19]. We recount the basic theory of inverse semigroups in Sect. 2.1, we discuss facts about inverse semigroup isomorphisms in Sect. 2.2, and we examine the restriction of Green’s $D$-relation on $S$ to $E(S)$ in Sect. 2.3. In this paper we write our semigroup operations multiplicatively.

### 2.1 Inverse semigroup basics

**Definition 2.1** An inverse semigroup is a nonempty semigroup $S$ where, for each $x \in S$, there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. We call $y$ the inverse of $x$, and we write $x^{-1} = y$.

There is a well-known alternate characterization of inverse semigroups.

**Theorem 2.2** A nonempty semigroup $S$ is an inverse semigroup if and only if $S$ is regular (meaning for each $x \in S$ there exists $y \in S$ such that $xyx = x$ and $yxy = y$) and the idempotents of $S$ commute.
Let $S$ be an inverse semigroup. We denote by $E(S)$ the set of idempotents of $S$. If $x \in S$, it is immediate that $xx^{-1}, x^{-1}x \in E(S)$, so $|E(S)| \geq 1$, and furthermore that if $e \in E(S)$, then $e = e^{-1}$. It is also clear from the alternate characterization that if $e, f \in E(S)$, then $ef \in E(S)$. It is also well known and easy to prove that $S$ is a group if and only if $|E(S)| = 1$.

**Definition 2.3** Let $S, T$ be semigroups. A map $\phi : S \to T$ is called a **homomorphism** if $\phi(st) = \phi(s)\phi(t)$ for all $s, t \in S$, and is called an **anti-homomorphism** if $\phi(st) = \phi(t)\phi(s)$ for all $s, t \in S$. An **isomorphism** is a bijective homomorphism and an **anti-isomorphism** is a bijective anti-homomorphism. $S$ and $T$ are **isomorphic** (resp., anti-isomorphic) if there is an isomorphism (resp., anti-isomorphism) from $S$ to $T$. If $S$ and $T$ are isomorphic, we write $S \cong T$.

Let $S$ be an inverse semigroup. We now recall the natural partial order on $S$.

**Definition 2.4** For $s, t \in S$, we define $s \leq t$ if and only if any of the following four equivalent conditions hold.

- $s = et$ for some $e \in E(S)$.
- $s = tf$ for some $f \in E(S)$.
- $s = ts^{-1}s$.
- $s = ss^{-1}t$.

Thus, the restriction of $\leq$ to $E(S)$ is given by the rule that, for $e, f \in E(S)$, $e \leq f$ if and only if $e = ef = fe$. Therefore $(E(S), \leq)$ is a meet-semilattice, where the meet $e \wedge f$ of $e, f \in E(S)$ is given by $e \wedge f = ef = fe$. Conversely, every meet-semilattice is an idempotent inverse semigroup under the meet operation.

As is common in inverse semigroup theory, for $s \in S$, let us write

$$\text{dom}(s) = s^{-1}s$$

and

$$\text{ran}(s) = ss^{-1}.$$

Next we recall Green’s $D$-relation, which takes on a particularly nice form for inverse semigroups.

**Definition 2.5** For $s, t \in S$, we say that $s$ is $D$-related to $t$ and we write $s D t$ if and only if any of the following equivalent conditions hold.

1. There exists $x \in S$ such that $\text{dom}(x) = \text{ran}(t)$ and $\text{ran}(x) = \text{ran}(s)$.
2. There exists $y \in S$ such that $\text{dom}(y) = \text{dom}(t)$ and $\text{ran}(y) = \text{dom}(s)$.
3. There exists $z \in S$ such that $\text{dom}(z) = \text{dom}(t)$ and $\text{ran}(z) = \text{ran}(s)$.

To see that these three conditions are equivalent, for (i) $\implies$ (ii), take $y = s^{-1}xt$, for (ii) $\implies$ (iii), take $z = sy$, and for (iii) $\implies$ (i), take $x = zt^{-1}$.

The relation $D$ is an equivalence relation on $S$, and the equivalence classes of $S$ under $D$ are called the $D$-classes of $S$. For finite inverse semigroups, an equivalent
characterization of $D$ is that, for $s, t \in S$, $s \mathcal{D} t$ if and only if $s$ and $t$ generate the same two-sided ideal in $S$.

Next we recall the notion of a maximal subgroup of an inverse semigroup.

**Definition 2.6** A subset $G$ of $S$ is a subgroup of $S$ if $G$ is a group with respect to the operation of $S$. A subgroup $G$ of $S$ is maximal if $G$ is not properly contained in any other subgroup of $S$.

Every idempotent $e$ of $S$ is the identity for a unique maximal subgroup of $S$, called the maximal subgroup of $S$ at $e$ and denoted $G_e$. For any subgroup $G$ of $S$ containing $e$, we have $G \subseteq G_e$. In fact

$$G_e = \{ s \in S : \text{dom}(s) = \text{ran}(s) = e \},$$

and if $e, f \in E(S)$ with $e \mathcal{D} f$, then $G_e \cong G_f$. Let $P$ denote the restriction of Green’s $\mathcal{D}$-relation on $S$ to $E(S)$. That is, let $P$ be the partition of $E(S)$ given by

$$P = \{ \{ e \in E(S) : e \mathcal{D} f \} : f \in E(S) \} = \{ \{ e \in E(S) : \exists x \in S \text{ such that } \text{dom}(x) = f \text{ and } \text{ran}(x) = e \} : f \in E(S) \}.$$

Let $X \in P$ and let $G$ be a group. If $G_e \cong G$ for some $e \in X$ (and hence for every $e \in X$), then we shall write $G_X \cong G$.

We now record several important basic properties.

**Theorem 2.7** Let $s, t, y, z \in S$ and let $e, f \in E(S)$.

1. If $s \leq e$, then $s \in E(S)$.
2. $s^{-1} \mathcal{D} s \mathcal{D} \text{dom}(s) \mathcal{D} \text{ran}(s)$.
3. If $s$ and $t$ are in the same maximal subgroup, then $s \mathcal{D} t$.
4. $ses^{-1}$ is idempotent.
5. If $s \leq t$, then $s^{-1} \leq t^{-1}$.
6. If $s \leq y$ and $t \leq z$, then $st \leq yz$.
7. If $e \leq \text{dom}(s)$, then $t = se$ is the unique element of $S$ such that $t \leq s$ and $\text{dom}(t) = e$.
8. If $e \leq \text{ran}(s)$, then $t = es$ is the unique element of $S$ such that $t \leq s$ and $\text{ran}(t) = e$.

**2.2 Facts about isomorphisms**

Results on the enumeration of semigroups are typically reported up to equivalence (meaning isomorphism or anti-isomorphism) [3,4,6,16], although some enumeration results have also been completed and reported up to isomorphism [4,6]. For inverse semigroups, these concepts agree.

**Theorem 2.8** Let $S$ and $T$ be inverse semigroups. Then $S$ and $T$ are isomorphic if and only if $S$ and $T$ are anti-isomorphic.
If \( s \) and \( t \) are lonely idempotents with \( s \neq t \), then \( st \) is not a lonely idempotent.

Proof: We begin by proving the contrapositive of part (i). Suppose \( st \) is a lonely idempotent. Write \( st = e \) for some \( e \in E(S) \). Then \( e = st = ss^{-1}st = ss^{-1}e \leq ss^{-1} \). Since \( e \) is not covered by any element of \( E(S) \), \( e \) is not covered by any element of \( S \). After all, if \( e < x \) for some \( x \in S \), then \( e < x^{-1} \), so \( e < xx^{-1} \) but \( xx^{-1} \in E(S) \). Thus \( e = ss^{-1} \). It follows that \( e \in D \) of \( S \), but since \( \{ e \} \) is a \( D \)-class, we have \( e = s \). Next, \( e = st = et \leq t \), so \( e = t \). Thus, \( s \) and \( t \) are both lonely idempotents.

For part (ii), suppose \( s \) and \( t \) are lonely idempotents with \( s \neq t \). Since \( s \) and \( t \) cover \( m \) and \( st = s \wedge t \), the meet of \( s \) and \( t \) in \( E(S) \), we have \( st = m \).

Theorem 2.12 Suppose \( \phi : S \to T \) is an isomorphism, the set of lonely idempotents of \( S \) is \( L(S) = \{ e_1, \ldots, e_r \} \), and the set of lonely idempotents of \( T \) is \( \{ f_1, \ldots, f_r \} \). Then \( r_1 = r_2 \), and if we define \( \gamma : S \to T \) by
\[ \gamma(e_i) = f_i \quad \text{for } e_i \in L(S); \]
\[ \gamma(x) = \phi(x) \quad \text{for } x \notin L(S), \]

then \( \gamma \) is an isomorphism.

**Proof** Let \( \phi : S \to T \) be an isomorphism.

Suppose \( e \in S \) is a lonely idempotent. Then \( \{\phi(e)\} \) is a \( \mathcal{D} \)-class of \( T \) (since \( \phi \) maps \( \mathcal{D} \)-classes to \( \mathcal{D} \)-classes), \( G_\phi(e) \cong \mathbb{Z}_1 \) (since \( \phi \) restricts to a bijection between maximal subgroups), \( \phi(e) \) covers the minimal element of \( E(T) \) (since \( \phi \) is a poset isomorphism), and \( \phi(e) \) is not covered by any element of \( E(T) \) (since \( \phi \) is a poset isomorphism). That is, \( \phi(e) \) is a lonely idempotent. Since \( \phi \) is an isomorphism, the same argument applied to \( \phi^{-1} \) shows that \( e \in S \) is a lonely idempotent if and only if \( \phi(e) \in T \) is a lonely idempotent and hence \( r_1 = r_2 \).

Let \( r = r_1 = r_2 \) and let \( \sigma \) be the permutation of \( \{1, \ldots, r\} \) such that \( \phi(e_i) = f_\sigma(i) \) for all \( i \in \{1, \ldots, r\} \). It is clear that \( \gamma \) is a bijection. Let \( s, t \in S \). We must show \( \gamma(st) = \gamma(s)\gamma(t) \).

Case 1: Suppose neither \( s \) nor \( t \) is a lonely idempotent. Then \( st \) is not a lonely idempotent, and \( \gamma(st) = \phi(st) = \phi(s)\phi(t) = \gamma(s)\gamma(t) \).

Case 2: Suppose exactly one of \( s, t \) is a lonely idempotent. Then \( st \) is not a lonely idempotent. Let \( m \) denote the minimal element of \( E(T) \).

Suppose \( x \in T \) is any lonely idempotent and \( y \in T \) is not a lonely idempotent. Then \( xy^{-1} \) is not a lonely idempotent, so we have \( m = xy^{-1} \), so \( my = xy \). Similarly we have \( ym = yx \).

If \( s \) is a lonely idempotent, then so is \( \gamma(s) \), while \( \gamma(t) \) is not. Say \( s = e_i \).

We have \( \gamma(st) = \phi(st) = \phi(s)\phi(t) = f_\sigma(i)\gamma(t) = m\gamma(t) = f_i\gamma(t) = \gamma(s)\gamma(t) \).

On the other hand, if \( t \) is a lonely idempotent, then so is \( \gamma(t) \), while \( \gamma(s) \) is not. Say \( t = e_i \). Then \( \gamma(st) = \phi(st) = \phi(s)\phi(t) = \gamma(s)f_\sigma(i) = \gamma(s)m = \gamma(s)f_i = \gamma(s)\gamma(t) \).

Case 3: Suppose \( s \) and \( t \) are both lonely idempotents. Then so are \( \phi(s), \phi(t), \gamma(s), \) and \( \gamma(t) \). Let \( m_S \) and \( m_T \) denote the minimal elements of \( S \) and \( T \), respectively.

Say \( s = e_i \) and \( t = e_j \). If \( s = t \), then \( \gamma(st) = \gamma(s) = \gamma(e_i) = f_i = \phi(e_{\sigma^{-1}(i)}) = \phi(e_{\sigma^{-1}(i)}e_{\sigma^{-1}(i)}) = \phi(e_{\sigma^{-1}(i)}\phi(e_{\sigma^{-1}(i)}) = \gamma(e_i)\gamma(e_i) = \gamma(s)\gamma(t) \).

On the other hand, if \( s \neq t \), then \( st = ms \), and \( \gamma(s) \neq \gamma(t) \), so \( \gamma(s)\gamma(t) = m_T \). Therefore \( \gamma(st) = \gamma(m_S) = \phi(m_S) = m_T = \gamma(s)\gamma(t) \).

\( \square \)

### 2.3 The restriction of Green’s \( \mathcal{D} \)-relation to \( E(S) \)

Let \( E \) be a meet-semilattice and \( P \) a set partition of \( E \). Write \( \sim \) for the equivalence relation on \( E \) induced by \( P \). That is, for \( e, f \in E \), \( e \sim f \) if and only if \( e \) and \( f \) are in the same part of \( P \).

**Definition 2.13** We say \( P \) is a \( \mathcal{D} \)-partition of \( E \) if, whenever \( e_1, e_2, f \in E \) with \( e_1 \sim e_2 \) and \( f \leq e_1 \), we have

\[ |h \in E : h \leq e_1 \text{ and } h \sim f| = |h \in E : h \leq e_2 \text{ and } h \sim f|. \]
The main result of this section is Theorem 2.15, which states that any set partition $P$ of $E$ for which there exists an inverse semigroup $S$ such that $E(S) = E$ and such that the restriction of $D$ from $S$ to $E(S)$ is $P$ is a $D$-partition of $E$. In fact we have the following more general result. Let $S$ be an inverse semigroup.

**Proposition 2.14** Let $s \in S$, $e \in E(S)$, and $s \not\in e$. Let $f \in S$ and $f \leq e$. Then

$$|\{t \in S : t \leq s \text{ and } t \not\in f\}| = |\{g \in S : g \leq e \text{ and } g \not\in f\}|.$$

**Proof** Let $s \in S$, $e \in E(S)$, and $s \not\in e$. Let $f \in S$ with $f \leq e$, so $f$ is idempotent. Let $x \in S$ such that $xx^{-1} = e$ and $x^{-1}x = ss^{-1}$. Define

$$\psi : \{g \in S : g \leq e \text{ and } g \not\in f\} \to \{t \in S : t \leq s \text{ and } t \not\in f\}$$

by $\psi(g) = x^{-1}gx$. We claim that $\psi$ is a bijection.

First, to show that the codomain of $\psi$ really is $\{t \in S : t \leq s \text{ and } t \not\in f\}$, let $g \in S$ with $g \leq e$ (so $g$ is idempotent and $g = eg = ge$) and $g \not\in f$. Note that $x^{-1}gx$ is idempotent (by part (iv) of Theorem 2.7), so $\psi(g) \leq s$. We need to show that $\psi(g) \not\in f$. We have that

$$\text{ran}(\psi(g)) = \psi(g)\psi(g)^{-1} = x^{-1}gx(ss^{-1})x^{-1}gx = x^{-1}gx(x^{-1}x)x^{-1}gx = x^{-1}g(xx^{-1})(xx^{-1})gx = x^{-1}(xx^{-1})(xx^{-1})ggx = x^{-1}gx,$$

so we need to show that there exists $z \in S$ such that $zz^{-1} = x^{-1}gx$ and $z^{-1}z = f$. Since $g \not\in f$, let $y \in S$ such that $yy^{-1} = g$ and $y^{-1}y = f$. Let $z = x^{-1}y$. Then $zz^{-1} = x^{-1}yy^{-1}x = x^{-1}gx$, as desired. Note that $y = yy^{-1}y = gy$, so we also have

$$z^{-1}z = y^{-1}xx^{-1}y = y^{-1}ey = y^{-1}egy = y^{-1}gy = y^{-1}y = f,$$

so $z^{-1}z = f$, as desired. Thus $\psi(g) \not\in f$ and the codomain of $\psi$ is as claimed.

Next, to show $\psi$ is injective, suppose $g_1, g_2 \leq e$ (so $eg_1 = g_1$ and $eg_2 = g_2$) and $\psi(g_1) = \psi(g_2)$. Then

$$x^{-1}g_1xs = x^{-1}g_2xs \implies x^{-1}g_1xs^{-1} = x^{-1}g_2xs^{-1} \implies x^{-1}g_1xx^{-1}x = x^{-1}g_2xx^{-1}x \implies x^{-1}g_1x = x^{-1}g_2x \implies xx^{-1}g_1xx^{-1} = xx^{-1}g_2xx^{-1}.$$
Finally, to show that \( \psi \) is surjective, let \( t \leq s \) and \( t \mathcal{D} f \). Then \( t = gs \) for some \( g \in E(S) \), and there exists \( y \in S \) such that \( yy^{-1} = tt^{-1} \) and \( y^{-1}y = f \). We claim that \( xgx^{-1} \leq e, xgx^{-1} \mathcal{D} f \), and \( \psi(xgx^{-1}) = t \). To see why, first note that \( xgx^{-1} \) is idempotent and \( xgx^{-1} = xgx^{-1}xx^{-1} = xgx^{-1}e \), so \( xgx^{-1} \leq e \). Next, we show that \( xgx^{-1} \mathcal{D} f \) by showing that there exists \( z \in S \) such that \( zz^{-1} = xgx^{-1} \) and \( z^{-1}z = f \). If we take \( z = xy \), then we have

\[
zz^{-1} = xyy^{-1}x^{-1} = xtt^{-1}x^{-1} = xgs^{-1}gx^{-1} = xgx^{-1}xx^{-1} = xgx^{-1}.
\]

Note that \( tt^{-1} = gss^{-1}g = gx^{-1}xg = gx^{-1}x \), which implies that

\[
y = yy^{-1}y = tt^{-1}y = gx^{-1}xy.
\]

Note also that \( y^{-1} = y^{-1}yy^{-1} = fy^{-1} \). Therefore we have

\[
z^{-1}z = y^{-1}x^{-1}xy = fy^{-1}x^{-1}xy = fy^{-1}x^{-1}xgx^{-1}xy = fy^{-1}gx^{-1}xx^{-1}xy = fy^{-1}gx^{-1}xy = fy^{-1}y = ff = f,
\]

so \( xgx^{-1} \mathcal{D} f \). Finally we show that \( \psi(xgx^{-1}) = t \). We have \( \psi(xgx^{-1}) = x^{-1}xgx^{-1}xs = gx^{-1}xx^{-1}xs = gss^{-1}ss^{-1}s = gs = t \). Thus \( \psi \) is surjective, completing the proof. \( \square \)

Combining Proposition 2.14 and part (i) of Theorem 2.7, we obtain the following result.
Theorem 2.15 Let $e_1, e_2, \in E(S)$ with $e_1 \mathcal{D} e_2$. Suppose $f \in E(S)$ with $f \leq e_1$. Then
\[ |h \in E(S) : h \leq e_1 \text{ and } h \mathcal{D} f| = |h \in E(S) : h \leq e_2 \text{ and } h \mathcal{D} f|. \]

3 Enumeration via the Ehresmann–Schein–Nambooripad theorem

3.1 The ESN theorem

The Ehresmann–Schein–Nambooripad (ESN) theorem provides an isomorphism between the category of inverse semigroups and homomorphisms and the category of inductive groupoids and inductive functors. See, e.g., [13, Ch. 4] and [7,15,17]. In this section we review the portion of the ESN theorem we need through the lens of B. Steinberg’s construction of an isomorphism between the algebra of a finite inverse semigroup and a direct sum of matrix algebras over group algebras [19].

Let $r_1, \ldots, r_k$ be positive integers and let $G_1, \ldots, G_k$ be finite groups. Let $A$ denote the algebra
\[ A = \bigoplus_{i=1}^{k} M_{r_i}(\mathbb{C}G_i). \]

The natural basis $B$ of $A$ is the set of matrices in $A$ having a single group element in one position and 0 elsewhere. Let $E(B)$ denote the set of idempotents of $B$. It is easy to see that $E(B)$ consists precisely of the elements of $B$ having a group identity on the diagonal. Even though $B$ is not generally an inverse semigroup, for $b \in B$, let $b^{-1}$ denote the matrix obtained by taking the transpose of $b$ and replacing the group element in $b$ with its (group) inverse. It is easy to see that $bb^{-1}b = b$ and $b^{-1}bb^{-1} = b^{-1}$, so $bb^{-1}$ and $b^{-1}b$ are idempotent. We continue to use the notation
\[ \text{dom}(b) = b^{-1}b \]
and
\[ \text{ran}(b) = bb^{-1}, \]
so for $a, b \in B$, $ab$ is nonzero if and only if $\text{dom}(a) = \text{ran}(b)$.

First we review how the ESN theorem allows us to construct inverse semigroups from $B$.

Theorem 3.1 (ESN theorem pt. 1) Let $B$ be the natural basis of the algebra $\bigoplus_{i=1}^{k} M_{r_i}(\mathbb{C}G_i)$ and let 0 denote the zero matrix. Let $\leq$ be any partial order on $B$ satisfying the following properties.

(i) $\leq$ restricted to $E(B)$ forms a meet-semilattice.
(ii) $\forall s, t \in B$, if $s \leq t$ then $s^{-1} \leq t^{-1}$. 
(iii) \( \forall s, t, y, z \in B, \text{if } s \leq y, t \leq z, st \neq 0, \text{and } yz \neq 0, \text{then } st \leq yz. \)

(iv) \( \forall e, s \in B, \text{if } e \leq \text{dom}(s), \text{then } \exists! t \in B \text{ such that } t \leq s \text{ and } \text{dom}(t) = e. \)

(v) \( \forall e, s \in B, \text{if } e \leq \text{ran}(s), \text{then } \exists! t \in B \text{ such that } t \leq s \text{ and } \text{ran}(t) = e. \)

For \( b \in B, \) let

\[
\overline{b} = \sum_{a \in B: a \leq b} a.
\]

Then \( \overline{B} = \{ \overline{b} : b \in B \} \) is an inverse semigroup under matrix multiplication, with

\[
|\overline{B}| = \sum_{i=1}^{k} r_i^2 |G_i|.
\]

Furthermore \( E(\overline{B}) = \{ \overline{e} : e \in E(B) \} \) and the map \( E(B) \to E(\overline{B}) \) given by \( e \mapsto \overline{e} \) is an isomorphism of posets. If \( B_i \) denotes the natural basis of \( M_{r_i}(CG_i) \), then the \( \mathcal{D} \)-classes of \( \overline{B} \) are \( D_1, \ldots, D_k \), with \( D_i = \{ \overline{b} : b \in B_i \} \). If \( e \in E(B) \cap B_i \), we have \( G_{\overline{e}} \cong G_i \). Finally, the natural partial order \( \leq \) on \( \overline{B} \) is given by \( \overline{s} \leq \overline{t} \iff s \leq t. \)

The ESN theorem also asserts that the construction in Theorem 3.1 is sufficient to construct any finite inverse semigroup \( S \) up to isomorphism. In particular, we have the following.

**Theorem 3.2** (ESN theorem pt. 2) Suppose \( S \) is a finite inverse semigroup and the partition of \( E(S) \) obtained by restricting Green’s \( \mathcal{D} \)-relation on \( S \) to \( E(S) \) is \( \{ X_1, \ldots, X_k \} \). Suppose \( |X_i| = r_i \) and \( G_{X_i} \cong G_i \) for all \( i \in \{ 1, \ldots, k \} \). Let \( \sqsubseteq \) be a partial order on \( E(B) \) for which \( (E(S), \sqsubseteq) \cong (E(B), \sqsubseteq) \). Then \( S \) may be obtained, up to isomorphism, from the construction of Theorem 3.1 for some partial order on \( B \) which restricts to \( \sqsubseteq \) on \( E(B) \).

Our statement of Theorem 3.2 is perhaps nonstandard. We include a proof in Sect. 7.1.

### 3.2 Our enumeration algorithms

We take \( \mathbb{N} = \{ 1, 2, \ldots \} \). Recall that a composition of \( m \in \mathbb{N} \) is a list of positive integers whose sum is \( m \). An integer partition (or just partition) of \( m \in \mathbb{N} \) is a composition of \( m \) whose elements are in weakly decreasing order. If \( \lambda \) is a partition or composition, denote the number of entries of \( \lambda \) (also called the length of \( \lambda \)) by \( |\lambda| \) and denote the \( i \)-th entry of \( \lambda \) by \( \lambda_i \). Any set partition \( P = \{ X_1, \ldots, X_k \} \) of a finite set \( X \) gives rise to a partition \( \lambda \) of \( |X| \) called the shape of \( P \). Specifically, by relabeling if necessary we may assume \( |X_1| \geq \cdots \geq |X_k| \). The shape of \( P \) is then the partition \((|X_1|, \ldots, |X_k|)\).

**Definition 3.3** If \( m, n \in \mathbb{N}, \lambda \) is a partition of \( m \), and \( C \) is a composition with \( |C| = |\lambda| \), we say \( C \) is an admissible composition for \((n, \lambda)\) if

\[
n = \sum_{i=1}^{\lambda_1} \lambda_i^2 C_i.
\]
For instance, the admissible compositions for \((10, (2, 1))\) are \((2, 2)\) and \((1, 6)\), and the only admissible composition for \(\binom{n}{(1, \ldots, 1)}\) is \((1, \ldots, 1)\).

We shall write the pseudocode for our algorithms using Python-esque syntax. In particular \([\ ]\) denotes a new empty list, \(\text{dict}([\ ])\) denotes a new empty dictionary, \([x]\) denotes a new list containing \(x\), and if \(x\) is a key in a dictionary \(d\), then \(d[x]\) denotes the associated value. A single equals sign denotes variable assignment, and nonempty lists are indexed beginning at 0. We now give the specifications for the functions we shall use in our enumeration algorithms.

- **Groups** \((n)\) accepts \(n \in \mathbb{N}\), chooses a representative from each isomorphism class of the groups of order not exceeding \(n\), and returns this set of representatives. This function can be computed easily with standard computational mathematical suites such as Sage [18] or GAP [20].
- **Partitions** \((m)\) accepts \(m \in \mathbb{N}\) and returns the set of partitions of \(m\).
- **AdmissibleComposition** \((n, \lambda)\) accepts \(n \in \mathbb{N}\) and a partition \(\lambda\) of a positive integer not exceeding \(n\), and returns the set of admissible compositions for \((n, \lambda)\).
- **MeetSemilattices** \((m)\) accepts \(m \in \mathbb{N}\) and returns the set of meet-semilattices of order \(m\) up to isomorphism. The algorithm of Heitzig and Reinhold for enumerating finite lattices up to isomorphism [11] may be used to implement this function by returning the lattices of order \(m + 1\) up to isomorphism with their maximal elements removed.
- **DPartitions** \((E, \lambda)\) accepts a finite meet-semilattice \(E\) and a partition \(\lambda\) of \(|E|\), and returns the \(\mathcal{D}\)-partitions of \(E\) of shape \(\lambda\) as ordered tuples. Specifically, if \(P = \{X_1, \ldots, X_k\}\) is a \(\mathcal{D}\)-partition of \(E\) of shape \(\lambda\), then by relabeling if necessary we may assume \(|X_1| \geq \cdots \geq |X_k|\). This function outputs \((X_1, \ldots, X_k)\) for \(P\).
- **GroupMaps** \((P, C, L)\) accepts a composition \(C\), a tuple \(P = (X_1, \ldots, X_{|C|})\) of length \(|C|\), and a set \(L\) of groups, and returns the set of all functions \(f : \{X_1, \ldots, X_{|C|}\} \to L\) such that \(|f(X_i)| = C_i\) for all \(i \in \{1, \ldots, |C|\}\).
- **Write** () for the identity of any group. **EGroupoid** \((E, P, f)\) accepts a finite meet-semilattice \(E\), a \(\mathcal{D}\)-partition \(P = (X_1, \ldots, X_k)\) of \(E\) (with \(|X_1| \geq \cdots \geq |X_k|\)), and a function \(f\) for which \(f(X_i)\) is a finite group for all \(i \in \{1, \ldots, k\}\), and returns \((B, \leq_{E(B)})\), where \(B\) is the natural basis of the algebra \(\bigoplus_{i=1}^{k} M_{|X_i|}(\mathbb{C} f(X_i))\) (where the rows and columns of the \(i\)th block are indexed by the elements of \(X_i\)), and \(\leq_{E(B)}\) is the partial order on \(E(B)\) given by, for \(a, b \in E\),

\[
\big(\big(0\big)_{a,a} \leq_{E(B)} \big(0\big)_{b,b} \iff a \leq b.
\]

- **GPosets** \((G)\) accepts an output \((B, \leq_{E(B)})\) of **EGroupoid**, and returns the set of partial orders \(\leq\) on \(B\) which restrict to \(\leq_{E(B)}\) on \(E(B)\) and meet the hypotheses of Theorem 3.1. An implementation of this function is given in Sect. 4.
- **ESN** \((G, \leq)\) accepts an output \(G\) of **EGroupoid** and an output \(\leq\) of **GPosets** \((G)\), and returns the inverse semigroup \(S\) obtained from the construction of Theorem 3.1 applied to \((G, \leq)\), after renaming \((0)_{e,e}\) as \(e\) for all \(e \in E\).
- **Invariants**$(S)$ accepts a finite inverse semigroup $S$ and returns a tuple $I(S)$ having the property that, for finite inverse semigroups $S$, $T$, if $E(S) = E(T)$ and $S \cong T$, then $I(S) = I(T)$. An implementation of this function is given in Sect. 5.1.
- **IsNew**$(S, I, \text{isgs})$ accepts a finite inverse semigroup $S$, $I \equiv \text{Invariants}(S)$, and a dictionary $\text{isgs}$. If the key $I$ is not present in $\text{isgs}$, this function returns True. On the other hand, if the key $I$ is present in $\text{isgs}$, then this function returns False if $S$ is isomorphic to some inverse semigroup in the list $\text{isgs}[I]$, and returns True otherwise. An implementation of this function, incorporating our implementation of Invariants, is given in Sect. 5.2.
- **Output**$(\text{isgs})$ accepts a dictionary $\text{isgs}$ and outputs to file (or console) every element of $\text{isgs}[I]$, for every key $I$ in $\text{isgs}$.

**Algorithm 3.4** Algorithm for enumerating the inverse semigroups of order $n$ up to isomorphism.

```plaintext
1 Input: n
2 \text{Gn} = \text{Groups}(n)
3 for m in \{1, ..., n\}:
4     for \lambda in \text{Partitions}(m):
5         \text{S}_\lambda = \text{AdmissibleCompositions}(n, \lambda)
6         for E in \text{MeetSemilattices}(m):
7             for \lambda in \text{Partitions}(m):
8                 \text{isgs} = \text{dict}([], \text{isgs})
9                 for C in S_\lambda:
10                    for P in \text{DPartitions}(E, \lambda):
11                        for f in \text{GroupMaps}(P, C, \text{Gn}):
12                            G = \text{EGroupoid}(E, P, f)
13                            for \leq in \text{GPosets}(G):
14                                S = \text{ESN}(G, \leq)
15                                I = \text{Invariants}(S)
16                                if \text{IsNew}(S, I, \text{isgs}):
17                                    if I in \text{isgs}:
18                                        \text{isgs}[I].append(S)
19                                    else:
20                                        \text{isgs}[I] = [S]
21 Output(\text{isgs})
```

We note that Algorithm 3.4 is easily parallelized at line 9 by starting a new task for each meet-semilattice $E$. We also note that the renaming in the specification of $\text{ESN}$ guarantees that if $S$ and $T$ are inverse semigroups generated by this algorithm for the same meet-semilattice $E$ (that is, within the same pass of the loop beginning at line 9), then $E(S) = E(T) = E$.

**Theorem 3.5** (Correctness of Algorithm 3.4) The output of Algorithm 3.4 is precisely the collection of inverse semigroups of order $n$ up to isomorphism.

**Proof** Let $\text{G}_n = \text{Groups}(n)$. By Theorem 3.2, we may obtain the inverse semigroups of order $n$ up to isomorphism by iterating over all possible combinations of meet-semilattices $E$ of order $1, \ldots, n$ up to isomorphism, set partitions $P = \{X_1, \ldots, X_k\}$
of $E$ (for $k \in \{1, \ldots, |E|\}$), functions $f : P \to \mathbb{G}_n$, and partial orders $\leq$ on the natural basis $B$ of $A = \bigoplus_{i=1}^k M_{|X_i|}([f(X_i)])$ (where the rows and columns of the $i$th block of $A$ are indexed by the elements of $X_i$) for which

- $a \leq b \in E \iff (a,a) \leq (b,b) \in E(B)$,
- the hypotheses of Theorem 3.1 are satisfied, and
- $n = \sum_{i=1}^k |X_i|^2 |f(X_i)|$,

and outputting each inverse semigroup $S$ afforded by the construction of Theorem 3.1, provided $S$ is not isomorphic to any previously-output inverse semigroup. We show that Algorithm 3.4 is an implementation of this procedure. In particular, we must justify lines 12, 13, 18–23, and the placement of line 11 of Algorithm 3.4.

If $S$ and $T$ are isomorphic inverse semigroups then it is straightforward to verify that $E(S) \cong E(T)$, both as posets and as inverse semigroups under the meet operation. Therefore we only need to test newly-generated inverse semigroups $S$ against previously-generated inverse semigroups $T$ for which $E(S) = E(T)$. In particular, line 11 may be placed below line 9. Furthermore, if $S$ and $T$ are isomorphic finite inverse semigroups with $E(S) = E(T) = E$, and $P_1$ and $P_2$ are the set partitions of $E$ induced by the restrictions of $D$ on $S$ and $T$, respectively, to $E$, then by parts (v) and (vi) of Theorem 2.9, $P_1$ and $P_2$ have the same shape. We therefore also only need to test newly-generated inverse semigroups $S$ for isomorphism against previously-generated inverse semigroups $T$ for which the shape of $D$ restricted to $E(S)$ is equal to the shape of $D$ restricted to $E(T)$. In particular, the placement of line 11 is correct.

By Theorem 2.15 we only need to consider $D$-partitions of each meet-semilattice $E$, so line 13 is correct.

Let $S$ be a semigroup generated by the procedure in the first paragraph of the proof, generated from the parameters $P$ and $f$. By relabeling if necessary, suppose $|X_1| \geq \cdots \geq |X_k|$. Then the composition $([f(X_1)], \ldots, [f(X_k)])$ is an admissible composition for $(n, ([|X_1|], \ldots, [|X_k|]))$. Therefore line 12 of the algorithm is correct.

Lines 18–23 of the algorithm sort the generated inverse semigroups according to their invariants for isomorphism testing, and therefore serve only to make the algorithm more efficient. In particular, since the placement of line 11 is correct, in lines 18–23 any newly-generated inverse semigroup $S$ is tested for isomorphism against every previously-generated inverse semigroup $T$ to which $S$ could possibly be isomorphic.

\begin{remark}
If one desires to enumerate only inverse monoids, a simple modification of Algorithm 3.4 can be used to do so. To enumerate the inverse monoids of order $n$ instead of the inverse semigroups of order $n$, iterate over the lattices of order 1, \ldots, $n$ instead of the meet-semilattices of order 1, \ldots, $n$ on line 9. (A finite inverse semigroup $S$ is a monoid if and only if $E(S)$ is a lattice.)
\end{remark}

\begin{remark}
We comment on the idea behind the use of Admissible Compositions and line 12 in Algorithm 3.4. Given a finite meet-semilattice $E$, to iterate over the $D$-partitions of $E$ it would suffice to iterate over all set partitions of $E$ and check which ones are $D$-partitions. However for the purpose of generating the inverse semigroups of order $n$ this is highly inefficient, as the number of set partitions of $E$ grows rapidly as $|E|$ grows, and for a typical meet-semilattice $E$ of order $m \leq n$, \[\mathbb{G}_n \cong \bigoplus_{i=1}^k M_{|X_i|}([f(X_i)])\]
the vast majority of set partitions $P$ of $E$ cannot serve to generate an inverse semigroup of order $n$ simply because of (1). In particular, if $P = \{X_1, \ldots, X_k\}$ is a $D$-partition of $E$ for which there exists an inverse semigroup of order $n$ such that $E(S) = E$ and the restriction of $D$ on $S$ to $E$ is $P$, then

$$|S| = n = \sum_{i=1}^{k} |X_i|^2 \lambda_i$$

for some $\lambda_1, \ldots, \lambda_k \in \mathbb{N}$ (namely, $\lambda_i = |G|$ for any group $G$ such that $G_{X_i} \cong G$). Indeed, if $m \in \{n, n - 1\}$, then the only partition $P$ of $E$ we need to consider is the finest partition of $E$ (which is automatically a $D$-partition). If $m \in \{n - 2, n - 3\}$ then we only need to consider partitions of $E$ of shape $(1, \ldots, 1)$ and $(2, 1, \ldots, 1)$, and so on. In general, since we only need to consider set partitions of $E$ whose shapes $\lambda$ have an admissible composition for $(n, \lambda)$, this reduces the amount of work (done by DPartitions) required to generate all possible $D$-partitions of $E$ needed by the rest of the algorithm.

Now, let $E$ be a finite meet-semilattice, $P$ a $D$-partition of $E$, and $f$ a function from $P$ to Groups$(n)$. Algorithm 3.4 is easily modified to enumerate the inverse semigroups $S$ for which $E(S) = E$, $D$ restricted to $E$ is equal to $P$, and $\forall e \in E \forall X \in P, e \in X \implies G_e \cong f(X)$. In particular, we have the following.

**Algorithm 3.8** Algorithm for enumerating the inverse semigroups having a specified semilattice of idempotents $E$, restriction of $D$ to $E$, and collection of maximal subgroups.

```python
1 Input: E, P, f
2 isgs = dict([])
3 G = EGroupoid(E, P, f)
4 for ≤ in GPosets(G):
5     S = ESN(G, ≤)
6     I = Invariants(S)
7     if IsNotNew(S, I, isgs):
8         if I in isgs:
9             isgs[I].append(S)
10        else:
11            isgs[I] = [S]
12 Output(isgs)
```

Note that Algorithm 3.8 is not strictly a subroutine of Algorithm 3.4, as in Algorithm 3.4 a wider amount of isomorphism testing is necessary. In particular, in Algorithm 3.4 any newly generated inverse semigroup must be tested for isomorphism against any previously generated inverse semigroup having the same underlying partition for the restriction of $D$ to its semilattice of idempotents.
4 GPosets

In this section we give an implementation of the GPosets function for Algorithms 3.4 and 3.8.

**Definition 4.1** Let \((Y, \leq)\) be a finite poset. The *down-levels* (or just *levels*) of \(Y\) are defined inductively. Let \(Y_1 = Y\). For \(i \in \mathbb{N}\), let \(L_i\) consist of the maximal elements of \(Y_i\), and let \(Y_{i+1}\) be the poset obtained by removing \(L_i\) from \(Y_i\). Let \(d\) be the first value of \(i\) for which \(L_{d+1} = \emptyset\). We call \(\{L_1, \ldots, L_d\}\) the set of down-levels of \(Y\). For \(i \in \{1, \ldots, d\}\), the set \(L_i\) is called the *ith down-level* of \(Y\).

By definition, the levels of \(Y\) form a partition of \(Y\). Let \((E, \leq)\) be a finite meet-semilattice and let \(P = \{X_1, \ldots, X_k\}\) be a \(D\)-partition of \(E\). Let \(G_1, \ldots, G_k\) be finite groups. For \(i \in \{1, \ldots, k\}\), index the rows and columns of \(M_{|X_i|}(\mathbb{C}G_i)\) by the elements of \(X_i\), and denote the natural basis of \(M_{|X_i|}(\mathbb{C}G_i)\) by \(B_i\). Let \(B = \bigcup_{i=1}^k B_i\). If \(y \in B_i\), \(z \in B_j\), and \(i \neq j\), we declare \(yz = 0\). Let \(E(B)\) denote the set of idempotents of \(B\). Write \(\()\) for the identity of any group. Let \(\leq_{E(B)}\) be the partial order on \(B\) inherited from \((E, \leq)\)—for \(a, b \in E\),

\[
\()a, a \leq_{E(B)} ()b, b \iff a \leq b.
\]

Denote the levels of \(E\) by \(L_1, \ldots, L_d\). If there exists an idempotent \(\()r, r\in B_i\) such that \(r \in L_j\), indicate this by writing \(B_i \cap L_j \neq \emptyset\). It is possible to have \(B_i \cap L_{j_1} \neq \emptyset\) and \(B_i \cap L_{j_2} \neq \emptyset\) for distinct \(L_{j_1}, L_{j_2}\). Let \(I(B_i) = \max\{j : B_i \cap L_j \neq \emptyset\}\). Define a linear order \(\prec\) on the \(B_i\) by setting

\[
\{B_i : I(B_i) = d\} \prec \{B_i : I(B_i) = d - 1\} \prec \cdots \prec \{B_i : I(B_i) = 1\},
\]

and then ordering the \(B_i\) within each set in (2) arbitrarily. By relabeling if necessary, we may assume that \(B_1 \prec B_2 \prec \cdots \prec B_k\). If there exist idempotents \(f \in B_i\) and \(e \in B_j\) such that \(f\) covers \(e\), let us say that \(B_i\) covers \(B_j\).

With this ordering of the \(B_i\) we will build the partial orders on \(B\) that we seek by finding all extensions of the partial order \(\leq_{E(B)}\) on \(E(B)\) to \(E(B) \cup B_1\), and then finding all extensions of such extensions to \(E(B) \cup B_1 \cup B_2\), and so on. Let

\[
\hat{B}_i = E(B) \cup B_1 \cup \cdots \cup B_i.
\]

Our implementation of GPosets((\(B, \leq_{E(B)}\))) may be described as a depth-first search in the following search tree. Let \(\leq_1 = \leq_{E(B)}\). The root of our search tree is \((\hat{B}_1, \leq_1, 1)\). Nodes of our search tree are of the form \((\hat{B}_i, \leq_i, i)\) (where \(\leq_i\) is a partial order on \(\hat{B}_i\)), for \(i \in \{1, \ldots, k\}\), and for a node \(N = (\hat{B}_i, \leq_i, i)\) with \(i < k\), the children of \(N\) are given by the following algorithm, with sub-functions as specified following the algorithm.

**Algorithm 4.2** Children((\(\hat{B}_i, \leq_i, i)\))

\[
\leq_i = \leq_{E(B)}.
\]
In this section we discuss our implementation of the Invariants and IsNew functions for Algorithms 3.4 and 3.8. Our implementation of IsNew makes use of our implementation of Invariants, which we discuss first.
5.1 Invariants

For the purposes of our algorithms, a tuple of invariants for a finite inverse semigroup $S$ is a tuple $I(S)$ having the property that for finite inverse semigroups $S, T$, if $E(S) = E(T)$ and $S \cong T$, then $I(S) = I(T)$. In Algorithms 3.4 and 3.8 we use invariants to sort generated inverse semigroups for isomorphism testing — when a new inverse semigroup $S$ is generated, it must be tested for isomorphism only against previously-generated inverse semigroups $T$ for which $I(S) = I(T)$. In this section we give the invariants we used in our implementation of Algorithm 3.4.

Remark 5.1 The invariants specified in this section are remarkably efficient at separating inverse semigroups in Algorithm 3.4. It is not uncommon for Algorithm 3.4 to be able to accept a newly generated inverse semigroup $S$ that is isomorphic to some previously-generated inverse semigroup $T$, and would be useful in an implementation of Algorithm 3.8 if a more refined tuple of invariants could be given.

It is straightforward to verify the following theorem.

Theorem 5.2 Suppose $X$ and $Y$ are isomorphic finite posets, with $\phi : X \to Y$ an isomorphism. If the levels of $X$ are $X_1, \ldots, X_m$ and the levels of $Y$ are $Y_1, \ldots, Y_{m'}$, then $m = m'$ and $\phi$ restricts to a bijection between $X_i$ and $Y_i$ for all $i$.

Let $S$ and $T$ be finite inverse semigroups. Recall that by part (iv) of Theorem 2.9 we have that if $S \cong T$, then as posets $(S, \leq) \cong (T, \leq)$. Let $S_1, \ldots, S_m$ denote the levels of $(S, \leq)$. Let $\text{Lev}(S) = (|S_1|, \ldots, |S_m|)$. By Theorem 5.2, if $S \cong T$ then $\text{Lev}(S) = \text{Lev}(T)$.

Definition 5.3 Let $(X, \leq)$ be a finite poset. The up-levels of $X$ are defined by replacing maximal with minimal in Definition 4.1.

Definition 5.4 Let $E$ be a finite meet-semilattice and let $L^U$ and $L^D$ denote the up-levels and the down-levels of $E$, respectively. The up-down levels of $E$ is the meet $L^U \wedge L^D$ in the lattice of set partitions of $E$. That is, the up-down levels of $E$ is given by the collection of nonempty pairwise intersections between the elements of $L^U$ and $L^D$.

Let $S$ be a finite inverse semigroup with $E = E(S)$. Let $D_{S,E}$ be the restriction of Green’s $\mathcal{D}$-relation on $S$ to $E(S)$, so $D_{S,E}$ is a $\mathcal{D}$-partition of $E$. For $e \in E$, let $D_{S,E}(e)$ denote the element of $D_{S,E}$ containing $e$, and let $G_S(e)$ be the isomorphism class of the maximal subgroup of $S$ at $e$. For each element $L$ of the up-down levels $U$ of $E$, let $X_S(L)$ denote the multiset of ordered pairs $X_S(L) = \{(|D_{S,E}(e)|, G_S(e)) : e \in L\}$.

Theorem 5.5 Let $S$ and $T$ be finite inverse semigroups with $E(S) = E(T) = E$. Let $U$ denote the up-down levels of $E$. If $S \cong T$, then $X_S(L) = X_T(L)$ for all $L \in U$.
Proof Suppose $\phi : S \to T$ is an isomorphism. Then $\phi$ restricted to $E(S) = E$ is an automorphism of posets $\phi|_E : E \to E(T) = E$. Any automorphism of a finite poset must preserve the up-levels and the down-levels of that poset, and hence must preserve the up-down levels of that poset as well. Thus, $\phi$ preserves the up-down levels of $E$. Note that $|D_{S,E}(e)|$ is simply the number of idempotents in the $D$-class of $e$. Since $\phi$ also maps $D$-classes to $D$-classes and restricts to isomorphisms of maximal subgroups, if $L \in U$ and $e \in L$, we have $\phi(e) \in L$, $G_S(e) = G_T(\phi(e))$, and $|D_{S,E}(e)| = |D_{T,E}(\phi(e))|$. Thus, as multisets we have $X_S(L) = X_T(L)$. \qed

For the purposes of Algorithm 3.4, for an inverse semigroup $S$ generated by Algorithm 3.4 and $S \cong T$ among the bijections from $S$ to $T$ or certify that among these bijections no homomorphism exists. Fortunately, it is only necessary to check a small number of these bijections to find an isomorphism. Our strategy for computing IsIsoc($S$, $T$) is to find a homomorphism from $S$ to $T$ among the bijections from $S$ to $T$ or certify that among these bijections no homomorphism exists. Fortunately, frequently it is only necessary to check a small number of these bijections to find an isomorphism or certify non-isomorphism.

Let $D_{S,E}$ be the partition of $E = E(S)$ obtained by restricting Green’s $D$-relation on $S$ to $E$. We specify the following functions.

- DRestriction($S$) takes $S$ and returns the partition $D_{S,E}$. The data structure for $D_{S,E}$ must be implemented in such a way that the iteration order of $D_{S,E}$ is fixed.

5.2 IsNew

In this section all inverse semigroups are assumed to have been created by the function ESN in Algorithm 3.4 or 3.8. In particular, given a finite inverse semigroup $S$, we have a finite meet-semilattice $E$ such that $E = E(S)$, and every non-idempotent element of $S$ is of the form $\overline{g_{a,b}}$ for some group element $g$ and some idempotents $a, b \in E$. To simplify the notation in this section we write the idempotents of $S$ in the same way, so we write $\overline{1}_{e,e}$ for $e \in E$ (where $\overline{1}$ denotes the identity of any group). Further, let $\mathbb{G}_n$ be the set constructed by Groups($n$) on line 3 of Algorithm 3.4. For $e \in E$ we have $G_e = \{\overline{g_{e,e}} : g \in G\}$ for some $G \in \mathbb{G}_n$, so there is an obvious isomorphism between $G_e$ and $G$.

The function IsNew($S$, $I$, isgs) accepts an inverse semigroup $S$, $I = \text{Invariants}(S)$, and a dictionary isgs such that isgs[$I$] is a list of inverse semigroups $T$ for which $\text{Invariants}(S) = \text{Invariants}(T) = I$, $E(S) = E(T) = E$, and the shape of Green’s $D$-relations on $S$ and $T$, restricted to $E$, are equal. The function IsNew($S$, $I$, isgs) returns False if $S \cong T$ for some $T \in \text{isgs}[I]$ and returns True otherwise. We implement this by iterating over the inverse semigroups $T \in \text{isgs}[I]$ and for each such $T$, determining whether or not $S$ and $T$ are isomorphic. Let IsIsoc($S$, $T$) be True if $S \cong T$ and False otherwise. We now describe our implementation of IsIsoc.
• \( \text{EColoring}(S, D_{S, E}) \) returns a copy of \( E = E(S) \), with nodes colored in the following manner. Let \( e \in E(S) \) and fix an ordering \( Sl_1, \ldots, Sl_j \) of the lonely idempotents of \( S \).

  - If \( (e, e) \in S \) is not a lonely idempotent, \( e \) is colored by \( (G, |DS, E(e)|) \), where \( G \in \mathbb{G}_n \) is such that \( G_e \cong G \).
  - If \( (e, e) \in S \) is a lonely idempotent, say \( Sl_i \), then \( e \) is colored by \( i \).

• \( \text{ColoredIsoms}(E_S, E_T) \) accepts two outputs \( E_S \) and \( E_T \) of \( \text{EColoring} \) and returns the set of color-preserving (poset) isomorphisms from \( E_S \) to \( E_T \). This function can be computed with standard graph-theoretic software such as \( \text{nauty} \) [14].

• \( \text{MaxSubgp}(X, S) \) takes an element \( X \in DS, E = \text{DRestriction}(S) \) and returns the element \( G \) of \( \mathbb{G}_n \) for which \( G_e \cong G \) for any \( e \in X \).

• \( \text{Bijections}(A, B) \) takes two equal-sized sets or lists \( A \) and \( B \), and returns the set of bijections from \( A \) to \( B \).

• \( \text{IsISGHomomorphism}(d) \) takes a dictionary \( d \) whose keys are elements of an inverse semigroup \( S \) and whose values are elements of an inverse semigroup \( T \) (that is, \( d \) is a map from \( S \) to \( T \)), and returns \( \text{True} \) if \( d \) is a homomorphism and \( \text{False} \) otherwise.

\[ \text{Algorithm 5.6 Implementation of } \text{IsIsoc}(S, T) \]

1. \( \text{Input: } S, T \)
2. \( D_{S, E} = \text{DRestriction}(S) \)
3. \( D_{T, E} = \text{DRestriction}(T) \)
4. \( E_S = \text{EColoring}(S, D_{S, E}) \)
5. \( E_T = \text{EColoring}(T, D_{T, E}) \)
6. \( \text{for } p \in \text{ColoredIsoms}(E_S, E_T): \)
7. \( \quad \text{if } p(D_{S, E}) == D_{T, E}: \)
8. \( \quad \quad \text{ToCp} = [ ] \)
9. \( \quad \quad \text{for } X \in D_{S, E}: \)
10. \( \quad \quad \quad G = \text{MaxSubgp}(X, S) \)
11. \( \quad \quad \quad \text{for } j \in X: \)
12. \( \quad \quad \quad \quad \text{for } k \in X: \)
13. \( \quad \quad \quad \quad \quad \text{if } j == k: \)
14. \( \quad \quad \quad \quad \quad \quad \text{ToCp}.\text{append}(G.\text{automorphisms}()) \)
15. \( \quad \quad \quad \quad \quad \text{else:} \)
16. \( \quad \quad \quad \quad \quad \quad \text{ToCp}.\text{append}(\text{Bijections}(G, G)) \)
17. \( \quad \quad \quad \text{for } T \in \text{CartesianProduct}(\text{ToCp}): \)
18. \( \quad \quad \quad \quad d = \text{dict}([ ]) \)
19. \( \quad \quad \quad \quad i = 0 \)
20. \( \quad \quad \quad \text{for } X \in D_{S, E}: \)
21. \( \quad \quad \quad \quad \text{for } j \in X: \)
22. \( \quad \quad \quad \quad \quad \text{for } g \in T[i]: \)
23. \( \quad \quad \quad \quad \quad \quad d[g, j, k] = \frac{(T[i])(g)_{p(j), p(k)}}{i = i+1} \)
24. \( \quad \quad \quad \quad \text{if } \text{IsISGHomomorphism}(d): \)
25. \( \quad \quad \quad \quad \quad \text{return True} \)
26. \( \quad \quad \quad \text{return False} \)
Proposition 5.7 (Correctness of Algorithm 5.6) For (finite) inverse semigroups $S$ and $T$ generated by the ESN function in Algorithm 3.4 or 3.8, if $E = E(S) = E(T)$, $I(S) = \text{Invariants}(S) = \text{Invariants}(T) = I(T)$, and the shape of Green’s $D$-relations on $S$ and $T$ restricted to $E$ are equal, then this implementation of $\text{IsIsoc}(S,T)$ returns True if $S \cong T$ and returns False otherwise.

Proof Since $I(S) = I(T)$, $S$ and $T$ have the same number of lonely idempotents. Suppose the lonely idempotents of $S$ are $Sl_1, \ldots, Sl_j$ and the lonely idempotents of $T$ are $Tl_1, \ldots, Tl_j$. Lines 3–6 set up the loop beginning at line 7 to iterate over the automorphisms $p : E \to E$ such that

- for every $e \in E$, $G_e \cong G_{p(e)}$,
- for every $e \in E$, $|D_{S,E}(e)| = |D_{T,E}(p(e))|$, and
- $p(Sl_i) = Tl_i$ for all $i \in \{1, \ldots, j\}$.

By Theorem 2.12 and parts (i), (ii), (iii), and (vi) of Theorem 2.9, $S \cong T$ if and only if there exists an isomorphism $d : S \to T$ extending some such $p$. Furthermore, by parts (i) and (v) of Theorem 2.9, we only need to consider extensions of $p$ for which we have, as sets, $p(D_{S,E}) = D_{T,E}$. The loop on lines 18–28 searches for an isomorphism $d$ extending such a $p$. In particular, this loop checks every extension of such a $p$ to a bijection $d : S \to T$ such that

- for each $e \in E$, $d|_{G_e \subseteq S} : G_e \to G_{d(e)} \subseteq T$ is an isomorphism of groups,
- for each $s \in S$, $d(\text{ran}(s)) = p(\text{ran}(s)) = \text{ran}(d(s))$, and
- for each $s \in S$, $d(\text{dom}(s)) = p(\text{dom}(s)) = \text{dom}(d(s))$.

By parts (ii) and (vii) of Theorem 2.9, $S \cong T$ if and only if for some $p$, some such $d$ is a homomorphism. \hfill \Box

6 The inverse semigroups of order $\leq 15$

If $E$ is a meet-semilattice of order $n$, then up to isomorphism the only inverse semigroup $S$ of order $n$ such that $E(S) = E$ is $S = E$ itself. Therefore to count the number of inverse semigroups of order $n$ it suffices to iterate over $m \in \{1, \ldots, n - 1\}$ on line 5 of Algorithm 3.4, count the number of inverse semigroups output by the algorithm, and add the result to the number of meet-semilattices of order $n$.

We have implemented Algorithm 3.4 and have used our implementation to count the inverse semigroups of order 1 through 15. Along the way we also counted the number of commutative inverse semigroups, inverse monoids, and commutative inverse monoids. These counts were given in Table 1 in Sect. 1.

Tables 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15 contain more detailed information. In these tables “ISGs” stands for inverse semigroups, “IMs” stands for inverse monoids, and “Comm.” stands for commutative. We report in these tables the number of inverse semigroups, commutative inverse semigroups, inverse monoids, and commutative inverse monoids $S$ of order 2 through 15, broken down by number $|E(S)|$ of idempotents and the shape of the set partition $D_{S,E}$ of $E(S)$ given by restricting Green’s $D$-relation on $S$ to $E(S)$. In particular, given $n$, a number $m$ of idempotents, and a partition $\lambda$ of $m$, an entry of the form $X//Y$ in the ISGs//Semilattices column...
Table 2  The inverse semigroups of order 2

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|---------------------|---------------------|--------------------------|---------------|---------------------|
| 1           | (1)                 | 1/1                 | 1/1                      | 1/1           | 1/1                 |
| 2           | (12)                | 1/1                 | 1/1                      | 1/1           | 1/1                 |
| Semigroup totals | 2                   | 2                   | 2                        | 2             | 2                   |

Table 3  The inverse semigroups of order 3

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|---------------------|---------------------|--------------------------|---------------|---------------------|
| 1           | (1)                 | 1/1                 | 1/1                      | 1/1           | 1/1                 |
| 2           | (12)                | 2/1                 | 2/1                      | 2/1           | 2/1                 |
| 3           | (13)                | 2/2                 | 2/2                      | 1/1           | 1/1                 |
| Semigroup totals | 5                   | 5                   | 4                        | 4             | 4                   |

Table 4  The inverse semigroups of order 4

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|---------------------|---------------------|--------------------------|---------------|---------------------|
| 1           | (1)                 | 2/1                 | 2/1                      | 2/1           | 2/1                 |
| 2           | (12)                | 4/1                 | 4/1                      | 4/1           | 4/1                 |
| 3           | (13)                | 5/2                 | 5/2                      | 3/1           | 3/1                 |
| 4           | (14)                | 5/5                 | 5/5                      | 2/2           | 2/2                 |
| Semigroup totals | 16                  | 16                  | 11                       | 11            | 11                  |

of Table $n$ indicates that there are $Y$ meet-semilattices $E$ of order $m$ for which there exists an inverse semigroup $S$ of order $n$ such that $E(S) = E$ and the restriction of $D$ on $S$ to $E$ has shape $\lambda$, and that there are $X$ such inverse semigroups. The pairs of numbers in the other columns have analogous interpretations. Cells are left blank if their entries are $0//0$.

If $E$ is a meet-semilattice of order $m \leq n$, then there is always at least one inverse semigroup $S$ of order $n$ such that $E(S) = E$ and such that the shape of the restriction of $D$ on $S$ to $E$ is the all-ones partition, so the value $Y$ in the $X//Y$ pair in the all-ones partition portion of the $m$th row and the ISGs//semilattices (resp. IMs//lattices) column is just the number of meet-semilattices (resp. lattices) of order $m$.

To avoid writing repeated ones in our partitions, we use the notation $1_j$ to indicate $j$ repeated ones. So, for instance, we write $(3, 3, 2, 1, 1, 1, 1) = (3, 3, 2, 14)$ and $(1, 1, 1, 1, 1) = (15)$.

We parallelized our implementation of Algorithm 3.4 at line 9 by spawning a new thread to carry out the computations for each meet-semilattice $E$. One benefit of our approach is that the parallelized threads do not need to communicate with one another,
**Table 5** The inverse semigroups of order 5

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|---------------------|---------------------|--------------------------|---------------|---------------------|
| 1           | (1)                 | 1//1                | 1//1                      | 1//1          | 1//1                |
| 2           | (12)                | 6//1                | 6//1                      | 6//1          | 6//1                |
| 3           | (2, 1)              | 1//1                |                          |               |                     |
|             | (13)                | 13//2               | 13//2                     | 8//1          | 8//1                |
| 4           | (14)                | 16//5               | 16//5                     | 7//2          | 7//2                |
| 5           | (15)                | 15//15              | 15//15                    | 5//5          | 5//5                |
| Semigroup totals |             | 52                  | 51                       | 27           | 27                  |

**Table 6** The inverse semigroups of order 6

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|---------------------|---------------------|--------------------------|---------------|---------------------|
| 1           | (1)                 | 2//1                | 1//1                      | 2//1          | 1//1                |
| 2           | (12)                | 12//1               | 12//1                     | 12//1         | 12//1               |
| 3           | (2, 1)              | 2//1                |                          |               |                     |
|             | (13)                | 26//2               | 26//2                     | 16//1         | 16//1               |
| 4           | (2, 12)             | 4//4                | 4//4                      | 1//1          |                     |
| 5           | (14)                | 49//5               | 49//5                     | 22//2         | 22//2               |
| 6           | (15)                | 60//15              | 60//15                    | 21//5         | 21//5               |
| 7           | (16)                | 53//53              | 53//53                    | 15//15        | 15//15              |
| Semigroup totals |             | 208                 | 201                      | 89           | 87                  |

**Table 7** The inverse semigroups of order 7

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|---------------------|---------------------|--------------------------|---------------|---------------------|
| 1           | (1)                 | 1//1                | 1//1                      | 1//1          | 1//1                |
| 2           | (12)                | 10//1               | 8//1                      | 10//1         | 8//1                |
| 3           | (2, 1)              | 2//1                |                          |               |                     |
|             | (13)                | 51//2               | 51//2                     | 33//1         | 33//1               |
| 4           | (2, 12)             | 13//4               | 4//1                      |               |                     |
|             | (14)                | 118//5              | 118//5                    | 54//2         | 54//2               |
| 5           | (2, 13)             | 17//14              |                          | 4//4          |                     |
|             | (15)                | 215//15             | 215//15                   | 76//5         | 76//5               |
| 6           | (16)                | 262//53             | 262//53                   | 75//15        | 75//15              |
| 7           | (17)                | 222//222            | 222//222                  | 53//53        | 53//53              |
| Semigroup totals |             | 911                 | 877                      | 310          | 300                 |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|---------------------|---------------------|--------------------------|--------------|---------------------|
| 1          | (1)                 | 5//1                | 3//1                      | 5//1          | 3//1                |
| 2          | (12)                | 22//1               | 18//1                     | 22//1         | 18//1               |
| 3          | (2, 1)              | 5//1                |                          |              |                     |
| (13)       |                     | 85//2               | 80//2                     | 54//1         | 51//1               |
| 4          | (2, 12)             | 26//4               |                          |              |                     |
| (14)       |                     | 269//5              | 269//5                    | 124//2        | 124//2              |
| 5          | (2, 13)             | 70//14              |                          |              |                     |
| (15)       |                     | 601//15             | 601//15                   | 215//5        | 215//5              |
| 6          | (2, 14)             | 82//52              |                          |              |                     |
| (16)       |                     | 1079//53            | 1079//53                  | 311//15       | 311//15             |
| 7          | (17)                | 1315//222           | 1315//222                 | 315//53       | 315//53             |
| 8          | (18)                | 1078//1078          | 1078//1078                | 222//222      | 222//222            |
| Semigroup totals |                | 4637                | 4443                      | 1311          | 1259                |
| Idempotents | Shape of $D_{S,E}$ | ISGs/semilattices | Comm. ISGs/semilattices | IMs/lattices | Comm. IMs/lattices |
|------------|-------------------|-------------------|------------------------|---------------|---------------------|
| 1          | (1)               | 2//1              | 2//1                   | 2//1          | 2//1                |
| 2          | (12)              | 23//1             | 16//1                  | 23//1         | 16//1               |
| 3          | (2, 1)            | 3//1              |                        |               |                     |
|            | (13)              | 126//2            | 111//2                 | 82//1         | 72//1               |
| 4          | (2, 12)           | 47//4             |                        |               |                     |
|            | (14)              | 520//5            | 504//5                 | 245//2        | 238//2              |
| 5          | (2, 2, 1)         | 3//3              |                        |               |                     |
|            | (2, 13)           | 192//14           |                        | 53//4         |                     |
|            | (15)              | 1555//15          | 1555//15               | 562//5        | 562//5              |
| 6          | (2, 14)           | 410//52           |                        | 92//14        |                     |
|            | (16)              | 3460//53          | 3460//53               | 1003//15      | 1003//15            |
| 7          | (2, 15)           | 445//221          |                        | 82//52        |                     |
|            | (17)              | 6137//222         | 6137//222              | 1480//53      | 1480//53            |
| 8          | (18)              | 7505//1078        | 7505//1078             | 1537//222     | 1537//222           |
| 9          | (19)              | 5994//5994        | 5994//5994             | 1078//1078    | 1078//1078          |
| Semigroup totals | 26,422       | 25,284             | 6253                  | 5988         |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|--------------------|-------------------|--------------------------|--------------|-------------------|
| 1          | (1)                | 2/1               | 1/1                      | 2/1          | 1/1               |
| 2          | (12)               | 48/1              | 30/1                     | 48/1         | 30/1              |
| 3          | (2, 1)             | 10/1              |                          |              |                   |
|            | (13)               | 235/2             | 193/2                    | 151/1        | 125/1             |
| 4          | (3, 1)             | 1/1               |                          |              |                   |
|            | (2, 12)            | 92/4              |                          | 23/1         |                   |
|            | (14)               | 981/5             | 918/5                    | 462/2        | 433/2             |
| 5          | (2, 2, 1)          | 7/3               |                          |              |                   |
|            | (2, 13)            | 424/14            |                          | 118/4        |                   |
|            | (15)               | 3499/15           | 3439/15                  | 1273/5       | 1252/5            |
| 6          | (2, 2, 12)         | 27/24             |                          | 3/3          |                   |
|            | (2, 14)            | 1387/52           |                          | 321/14       |                   |
|            | (16)               | 10,016/53         | 10,016/53                | 2928/15      | 2928/15           |
| 7          | (2, 15)            | 2629/221          |                          | 508/52       |                   |
|            | (17)               | 22,254/222        | 22,254/222               | 5389/53      | 5389/53           |
| 8          | (2, 16)            | 2704/1077         |                          | 445/221      |                   |
|            | (18)               | 39,164/1078       | 39,164/1078              | 8077/222     | 8077/222          |
| 9          | (19)               | 48,061/5994       | 48,061/5994              | 8583/1078    | 8583/1078         |
| 10         | (110)              | 37,622/37,622     | 37,622/37,622            | 5994/5994    | 5994/5994         |
| Semigroup totals |                  | 169,163        | 161,698                   | 34,325       | 32,812            |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|-------------------|--------------------|-------------------------|--------------|-------------------|
| 1          | (1)               | 1//1               | 1//1                    | 1//1         | 1//1              |
| 2          | (1_2)             | 26//1              | 18//1                   | 26//1        | 18//1             |
| 3          | (2, 1)            | 4//1               |                         |              |                   |
|            | (1_3)             | 301//2             | 215//2                  | 198//1       | 141//1            |
| 4          | (3, 1)            | 2//1               |                         |              |                   |
|            | (2, 1_2)          | 113//4             |                         | 34//1        |                   |
|            | (1_4)             | 1707//5            | 1495//5                 | 808//2       | 710//2            |
| 5          | (3, 1_2)          | 5//5               |                         | 1//1         |                   |
|            | (2, 2, 1)         | 7/3                |                         |              |                   |
|            | (2, 1_3)          | 904//14            |                         |              |                   |
|            | (1_5)             | 7407//15           | 7108//15                | 2723//5      | 2616//5           |
| 6          | (2, 2, 1_2)       | 105//24            |                         | 14//3        |                   |
|            | (2, 1_4)          | 3660//52           |                         | 866//14      |                   |
|            | (1_6)             | 25,503//53         | 25,241//53              | 7507//15     | 7432//15          |
| 7          | (2, 2, 1_3)       | 216//149           |                         | 27//24       |                   |
|            | (2, 1_5)          | 10,518//221        |                         | 2085//52     |                   |
|            | (1_7)             | 71,439//222        | 71,439//222             | 17,439//53   | 17,439//53        |
| 8          | (2, 1_6)          | 18,510//1077       |                         | 3134//221    |                   |
|            | (1_8)             | 158,478//1078      | 158,478//1078           | 32,845//222  | 32,845//222       |
| 9          | (2, 1_7)          | 18,232//5993       |                         | 2704//1077   |                   |
|            | (1_9)             | 277,347//5994      | 277,347//5994           | 49,905//1078 | 49,905//1078      |
| 10         | (1_10)            | 341,390//37,622    | 341,390//37,622         | 54,055//5994 | 54,055//5994      |
| 11         | (1_11)            | 262,776//262,776   | 262,776//262,776        | 37,622//37,622 | 37,622//37,622 |
| Semigroup totals |               | 1,198,651         | 1,145,508               | 212,247     | 202,784          |
### Table 12: The inverse semigroups of order 12

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|---------------------|--------------------|-------------------------|---------------|---------------------|
| 1           | (1)                 | 5//1               | 2//1                    | 5//1          | 2//1                |
| 2           | (1,2)               | 93//1              | 56//1                   | 93//1         | 56//1               |
| 3           | (2, 1)              | 26//1              |                         |               |                     |
|             | (1,3)               | 544//2             | 367//2                  | 349//1        | 236//1              |
| 4           | (3, 1)              | 3//1               |                         |               |                     |
|             | (2, 1, 2)           | 227//4             |                         | 59//1         |                     |
|             | (1,4)               | 3081//5            | 2535//5                 | 1473//2       | 1220//2             |
| 5           | (3, 1, 2)           | 19//5              |                         | 4//1          |                     |
|             | (2, 2, 1)           | 20//3              |                         |               |                     |
|             | (2, 1, 3)           | 1650//14           |                         | 466//4        |                     |
|             | (1,5)               | 14,725//15         | 13,552//15              | 5430//5       | 5010//5             |
| 6           | (3, 1, 3)           | 26//23             |                         | 5//5          |                     |
|             | (2, 2, 1, 2)        | 209//24            |                         | 25//3         |                     |
|             | (2, 1, 4)           | 8865//52           |                         | 2118//14      |                     |
|             | (1,6)               | 60,352//53         | 58,761//53              | 17,905//15    | 17,444//15          |
| 7           | (2, 2, 1, 3)        | 1078//149          |                         | 151//24       |                     |
|             | (2, 1, 5)           | 32,320//221        |                         | 6546//52      |                     |
|             | (1,7)               | 202,397//222       | 201,082//222            | 49,742//53    | 49,427//53          |
| 8           | (2, 2, 1, 4)        | 1780//883          |                         | 216//149      |                     |
|             | (2, 1, 6)           | 85,146//1077       |                         | 14,739//221   |                     |
|             | (1,8)               | 559,264//1078      | 559,264//1078           | 116,857//222  | 116,857//222        |
| 9           | (2, 1, 7)           | 142,296//5993      |                         | 21,476//1077  |                     |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|-------------------|-------------------|------------------------|--------------|-------------------|
| 10         | (19)              | 1,237,965//5994   | 1,237,965//5994         | 224,095//1078 | 224,095//1078     |
|            | (2, 18)           | 135,249//37,621   |                        | 18,232//5993  |                   |
| 11         | (110)             | 2,157,481//37,622 | 2,157,481//37,622       | 344,406//5994 | 344,406//5994     |
|            | (111)             | 2,660,921//262,776| 2,660,921//262,776      | 379,012//37,622| 379,012//37,622   |
| 12         | (112)             | 2,018,305//2,018,305 | 2,018,305//2,018,305   | 262,776//262,776| 262,776//262,776  |
| Semigroup totals |         | 9,324,047         | 8,910,291               | 1,466,180    | 1,400,541         |
Table 13: The inverse semigroups of order 13

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|-------------------|--------------------|--------------------------|--------------|-------------------|
| 1           | (1)               | 1/1                | 1/1                      | 1/1          | 1/1               |
| 2           | (1,2)             | 38/1               | 24/1                     | 38/1         | 24/1              |
| 3           | (2, 1)            | 8/1                |                          |              |                   |
|             | (1,3)             | 634/2              | 412/2                    | 419/1        | 272/1             |
| 4           | (3, 1)            | 7/1                |                          |              |                   |
|             | (2, 1,2)          | 295/4              |                          |              |                   |
|             | (1,4)             | 4717/5             | 3479/5                   | 2246/2       | 1660/2            |
| 5           | (3, 1,2)          | 44/5               |                          |              |                   |
|             | (2, 2, 1)         | 14/3               |                          |              |                   |
|             | (2, 1,3)          | 2777/14            |                          |              |                   |
|             | (1,5)             | 28,025/15          | 24,490/15                | 10,385/5     | 9106/5            |
| 6           | (3, 1,3)          | 134/23             |                          |              |                   |
|             | (2, 2, 1,2)       | 428/24             |                          |              |                   |
|             | (2, 1,4)          | 18,873/52          |                          |              |                   |
|             | (1,6)             | 132,846/53         | 125,672/53               | 39,675/15    | 37,592/15         |
| 7           | (3, 1,4)          | 153/117            |                          |              |                   |
|             | (2, 2, 2, 1)      | 13/12              |                          |              |                   |
|             | (2, 2, 1,3)       | 3063/149           |                          |              |                   |
|             | (2, 1,5)          | 88,364/221         |                          |              |                   |
|             | (1,7)             | 528,405/222        | 518,948/222              | 130,955/53   | 128,668/53        |
| 8           | (2, 2, 1,4)       | 10,719/883         |                          |              |                   |
Table 13 continued

| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|-------------|--------------------|--------------------|--------------------------|--------------|----------------------|
| (2, 1)      |                    | 298,708/1077       | 1,734,284/1078           | 52,723/221   | 366,740/222          |
| (18)        |                    | 1,741,789/1078     |                          | 365,203/222  |                      |
| 9           |                    | 15,456/5435        |                          | 1780/883     |                      |
| (2, 15)     |                    | 737,996/5993       |                          | 113,535/1077 |                      |
| (10)        |                    | 4,764,281/5994     |                          | 869,969/1078 |                      |
| 10          |                    | 1,187,056/37,621   |                          | 161,843/5993 |                      |
| (18)        |                    | 10,518,061/37,622  |                          | 1,691,090/5994 |                |
| 11          |                    | 1,093,871/262,775  |                          | 135,249/37,621 |                |
| (11)        |                    | 18,265,468/262,776 |                          | 2,623,757/37,622 |          |
| 12          |                    | 22,545,079/2,018,305 |                        | 2,923,697/262,776 |          |
| (12)        |                    | 16,873,364/16,873,364 |                        | 2,018,305/2,018,305 |          |
| 13          |                    | 78,860,687         |                          | 11,167,987   | 10,669,344           |
| Semigroup totals |                | 75,373,563         |                          |              |                      |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|------------------|---------------------|-------------------------|--------------|------------------|
| 1          | (1)              | 2/1                 | 1/1                     | 2/1          | 1/1              |
| 2          | (1_2)            | 95/1                | 40/1                    | 95/1         | 40/1             |
| 3          | (2, 1)           | 20/1                | 706/2                   | 801/1        | 465/1            |
|            | (1_3)            | 1225/2              |                         |              |                  |
| 4          | (3, 1)           | 3/1                 |                         |              |                  |
|            | (2, 1_2)         | 629/4               |                         |              |                  |
|            | (1_4)            | 8460/5              | 5977/5                  | 4071/2       | 2899/2           |
| 5          | (3, 1_2)         | 91/5                |                         |              |                  |
|            | (2, 2, 1)        | 51/3                |                         |              |                  |
|            | (2, 1_3)         | 5309/14             |                         |              |                  |
|            | (1_5)            | 51,551/15           | 42,161/15               | 19,261/5     | 15,848/5         |
| 6          | (3, 2, 1)        | 7/6                 |                         |              |                  |
|            | (3, 1_3)         | 418/23              |                         |              |                  |
|            | (2, 2, 1_2)      | 816/24              |                         |              |                  |
|            | (2, 1_4)         | 37,344/52           |                         |              |                  |
|            | (1_6)            | 278,911/53          | 254,127/53              | 83,827/15    | 76,586/15        |
| 7          | (3, 1_4)         | 976/117             |                         | 170/23       |                  |
|            | (2, 2, 2, 1)     | 32/12               |                         |              |                  |
|            | (2, 2, 1_3)      | 7216/149            |                         |              |                  |
|            | (2, 1_5)         | 213,876/221         |                         |              |                  |
|            | (1_7)            | 1,279,242/222       | 1,230,949/222           | 319,361/53   | 307,637/53       |
| 8          | (3, 1_5)         | 999/653             |                         | 153/117      |                  |
|            | (2, 2, 2, 1_2)   | 240/191             |                         |              |                  |
|            | (2, 2, 1_4)      | 38,341/883          |                         |              |                  |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|---------------------|---------------------|--------------------------|--------------|---------------------|
|            |                     |                     |                          |              |                     |
| (2, 16)    |                     | 912,857//1077        |                          | 163,837//221 |                     |
| (18)       |                     | 4,967,113//1078      | 4,904,704//1078          | 1,055,099//222 | 1,042,206//222     |
| 9          | (2, 2, 15)          | 108,619//5435        |                          | 12,998//883  |                     |
|            | (2, 17)             | 2,908,054//5993      |                          | 455,377//1077 |                     |
|            | (19)                | 16,156,724//5994     | 16,108,663//5994         | 2,975,421//1078 | 2,966,838//1078   |
| 10         | (2, 2, 16)          | 142,385//35,893      |                          | 15,456//5435 |                     |
|            | (2, 18)             | 6,838,144//37,621    |                          | 948,366//5993 |                     |
|            | (110)               | 43,822,653//37,622   | 43,822,653//37,622       | 7,111,831//5994 | 7,111,831//5994   |
| 11         | (2, 19)             | 10,673,677//262,775  |                          | 1,329,810//37,621 |                     |
|            | (111)               | 96,447,794//262,776  | 96,447,794//262,776      | 13,966,574//37,622 | 13,966,574//37,622 |
| 12         | (2, 110)            | 9,569,171//2,018,304 |                          | 1,093,871//262,775 |                     |
|            | (112)               | 166,932,647//2,018,305 | 166,932,647//2,018,305 | 21,834,255//262,776 | 21,834,255//262,776 |
| 13         | (113)               | 205,966,795//16,873,364 | 205,966,795//16,873,364 | 24,563,384//2,018,305 | 24,563,384//2,018,305 |
| 14         | (114)               | 152,233,518//152,233,518 | 152,233,518//152,233,518 | 16,873,364//16,873,364 | 16,873,364//16,873,364 |
| Semigroup totals |             | 719,606,005          | 687,950,735              | 92,889,294  | 88,761,928         |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|--------------------|--------------------|--------------------------|---------------|-------------------|
| 1          | (1)                | 1/1                | 1/1                      | 1/1           | 1/1               |
| 2          | (1,2)              | 59/1               | 34/1                     | 59/1          | 34/1              |
| 3          | (2,1)              | 7/1                | 1017/2                   | 580/2         | 672/1             |
|            | (1,3)              |                    |                          |               |                   |
| 4          | (3,1)              | 12/1               |                          |               |                   |
|            | (2,1,2)            | 445/4              |                          |               |                   |
|            | (1,4)              | 11963/5            | 7588/5                   | 5734/2        | 3650/2            |
| 5          | (3,1,2)            | 164/5              |                          |               |                   |
|            | (2,2,1)            | 22/3               |                          |               |                   |
|            | (2,1,3)            | 8202/14            |                          |               |                   |
|            | (1,5)              | 89,791/15          | 68,092/15                | 33,434/5      | 25,468/5          |
| 6          | (3,2,1)            | 18/6               |                          |               |                   |
|            | (3,1,3)            | 1020/23            |                          |               |                   |
|            | (2,2,1,2)          | 1266/24            |                          |               |                   |
|            | (2,1,4)            | 71,562/52          |                          |               |                   |
|            | (1,6)              | 557,310/53         | 482,754/53               | 168,321/15    | 146,380/15        |
| 7          | (3,2,1,2)          | 86/66              |                          | 7/6           |                   |
|            | (3,1,4)            | 3741/117           |                          |               |                   |
|            | (2,2,2,1)          | 32/12              |                          |               |                   |
|            | (2,2,1,3)          | 15,957/149         |                          |               |                   |
|            | (2,1,5)            | 476,897/221        |                          |               |                   |
|            | (1,7)              | 2,928,371/222      | 2,739,985/222            | 736,701/53    | 690,700/53        |
| 8          | (3,1,4)            | 7561/653           | 2,739,985/222            | 736,701/53    | 690,700/53        |
|            | (2,2,2,1,2)        | 1020/192           |                          |               |                   |
| Idempotents | Shape of $D_{S,E}$ | ISGs//semilattices | Comm. ISGs//semilattices | IMs//lattices | Comm. IMs//lattices |
|------------|-------------------|-------------------|------------------------|--------------|------------------|
| (2, 2, 14) | 107,023//883       | 14,722//149       |                        |              |                  |
| (2, 16)    | 2,467,528//1077    | 449,724//221      |                        |              |                  |
| (18)       | 13,101,797//1078   | 12,745,673//1078  | 2,806,507//222         | 2,732,450//222 |                  |
| (3, 16)    | 7225//4049         | 999//653          |                        |              |                  |
| (2, 2, 13) | 3381//2062         | 240//191          |                        |              |                  |
| (2, 2, 15) | 462,898//5435      | 57,163//883       |                        |              |                  |
| (2, 17)    | 9,837,747//5993    | 1,566,722//1077   |                        |              |                  |
| (19)       | 49,918,237//5994   | 9,280,078//1078   | 9,198,343//1078        |              |                  |
| 9          |                   |                   |                        |              |                  |
| (2, 2, 16) | 1,142,433//3589    | 127,151//5435     |                        |              |                  |
| (2, 18)    | 29,864,170//37,621 | 4,210,461//5993   |                        |              |                  |
| (110)      | 160,561,088//37,622 | 26,299,900//5994 | 26,245,845//5994      |              |                  |
| 10         |                   |                   |                        |              |                  |
| (2, 2, 17) | 1,390,467//257,001 | 142,385//35,893   |                        |              |                  |
| (2, 19)    | 67,509,604//262,775 | 8,541,243//37,621 |                        |              |                  |
| (111)      | 432,247,509//262,776 | 63,212,608//37,622 | 63,212,608//37,622    |              |                  |
| 11         |                   |                   |                        |              |                  |
| (2, 2, 10) | 102,805,707//2,018,304 | 11,815,609//262,775 |                        |              |                  |
| (112)      | 948,037,628//2,018,305 | 125,084,221//262,776 | 125,084,221//262,776  |              |                  |
| 12         |                   |                   |                        |              |                  |
| (2, 110)   | 89,902,414//16,873,363 | 9,569,171//2,018,304 |                        |              |                  |
| (113)      | 1,635,389,858//16,873,364 | 196,698,551//2,018,305 | 196,698,551//2,018,305 |              |                  |
| 13         |                   |                   |                        |              |                  |
| (114)      | 2,014,968,017//152,233,518 | 222,840,159//16,873,364 | 222,840,159//16,873,364 |              |                  |
| 14         |                   |                   |                        |              |                  |
| (115)      | 1,471,613,387//1,471,613,387 | 152,233,518//152,233,518 | 152,233,518//152,233,518 |              |                  |
| 15         |                   |                   |                        |              |                  |
| Semigroup totals | 7,035,514,642 | 6,727,985,390 | 836,021,796 | 799,112,310 |                  |
so the work required by the algorithm is easily distributable across several CPUs and/or computer systems if necessary.

We ran our implementation on a computational server hosted at Sam Houston State University, which has four AMD Opteron\textsuperscript{TM} 6272 processors (a total of 64 cores), with each core running at 2.1 GHz, and 256 GB of RAM. The following running times are given in terms of the computational power of one core of our server. Including time spent testing for commutativity and counting monoids and inverse monoids along the way, our algorithm took a total of 11.1 CPU years to count the inverse semigroups of order 1, \ldots, 15. Approximately 20\% of this time was spent on isomorphism tests. 92\% of this time was spent on $n = 15$. We estimate that it would take approximately 100 CPU years for our implementation to count the inverse semigroups of order 16.

Thanks to our implementation of Invariants in Sect. 5.1, of the 6,201,659,106 inverse semigroups of order 1, \ldots, 15 generated by our algorithm, 4,317,895,179 of them (69.62\%) were accepted as new inverse semigroups immediately (with no isomorphism testing), and 2,824,933,733 of those (65.42\%) were never involved in an isomorphism test. A total of 5,491,416,345 isomorphism tests were performed, for an overall rate of 0.885 isomorphism tests per generated inverse semigroup. Statistics regarding the effectiveness of our implementation of Invariants, broken down by $n$, are given in Table 16. These statistics show that while our implementation of Invariants becomes less effective as $n$ grows, it remains highly effective for all $n \leq 15$.

Although it is impossible to certify that our implementation of our algorithm (which consists of thousands of lines of \texttt{Sage} code) is bug-free and ran without error, all of the evidence we have points in this direction. First, our implementation correctly computed the number of inverse semigroups, commutative inverse semigroups, inverse monoids,

### Table 16 Effectiveness of Invariants

| $n$ | \% of generated ISGs accepted immediately | \% of these never involved in isomorphism test | \# isomorphism tests done/\# generated ISGs |
|-----|----------------------------------------|---------------------------------------------|------------------------------------------|
| 2   | 100                                    | 100                                         | 0.0                                      |
| 3   | 100                                    | 100                                         | 0.0                                      |
| 4   | 100                                    | 90.9                                        | 0.091                                    |
| 5   | 100                                    | 83.8                                        | 0.189                                    |
| 6   | 97.4                                   | 75.5                                        | 0.316                                    |
| 7   | 94.3                                   | 72.3                                        | 0.409                                    |
| 8   | 90.0                                   | 69.9                                        | 0.500                                    |
| 9   | 85.9                                   | 68.7                                        | 0.559                                    |
| 10  | 81.9                                   | 67.8                                        | 0.618                                    |
| 11  | 78.5                                   | 67.2                                        | 0.668                                    |
| 12  | 75.6                                   | 66.7                                        | 0.722                                    |
| 13  | 73.2                                   | 66.3                                        | 0.776                                    |
| 14  | 71.1                                   | 65.8                                        | 0.833                                    |
| 15  | 69.4                                   | 65.4                                        | 0.892                                    |
and commutative inverse monoids of order \( n \) for all previously-known values of \( n (n = 1, \ldots, 9) \), and agrees with the output of Distler’s code [6] for \( n = 10 \). Second, to guard against system errors unrelated to our implementation that could nevertheless affect its output, we ran our program multiple times on multiple systems, including our server, for \( n \leq 13 \). We ran our program on our server multiple times for \( n = 14 \) and twice for \( n = 15 \), and we obtained the same results every time. Finally, in search of greater speed and memory efficiency, over the course of this project we recoded, entirely from scratch, several key steps of our algorithm in a number of different ways, and we obtained the same results regardless of which of our implementations of these key steps we used.

7 Additional proofs

In this section we prove Theorem 3.2 and we prove the correctness of our implementation of GPosets in Sect. 4.

7.1 Proof of Theorem 3.2

In this section we prove Theorem 3.2. Our proof is essentially an elaboration of the main idea in Section 4 of [19].

Proof of Theorem 3.2 Let \( S \) be a finite inverse semigroup. Recall that the semigroup algebra \( \mathbb{C} S \) is a \( \mathbb{C} \)-vector space with basis \( \{ s \}_{s \in S} \), where multiplication is given by the extension of the multiplication in \( S \) via the distributive law. B. Steinberg defines another basis (called the groupoid basis) \( \{ \lfloor s \rfloor \}_{s \in S} \) of \( \mathbb{C} S \) as follows [19]. For \( s \in S \), let

\[
[ s ] = \sum_{t \in S: t \leq s} \mu(t, s)t,
\]

where \( \mu \) is the Möbius function of the natural partial order \( \leq \) on \( S \).

The groupoid basis multiplies in the following manner. For \( s, t \in S \),

\[
[ s ][ t ] = \begin{cases} 
[ st ] & \text{if } \text{dom}(s) = \text{ran}(t); \\
0 & \text{otherwise.}
\end{cases}
\]

The groupoid basis is thus a basis of \( \mathbb{C} S \), whose elements multiply as in a groupoid (where we interpret 0 as undefined). Of course, the natural basis of \( \mathbb{C} S \) can be recovered by Möbius inversion. Specifically, for \( s \in S \), in \( \mathbb{C} S \) we have

\[
s = \sum_{t \in S: t \leq s} [ t ].
\]
The natural partial order of $S$ gives rise to a partial order on the groupoid basis: For $s, t \in S$, let

$$[s] \leq [t] \iff s \leq t.$$ 

With this notation we can write $s$ in terms of the groupoid basis and its partial order:

$$s = \sum_{i : S : [i] \leq [s]} [i]. \quad (4)$$

Note that the semilattice $E(S)$ is isomorphic to the semilattice $([e] : e \in E(S)), \leq$ by $e \mapsto [e]$.

Now suppose the partition of $E(S)$ obtained by restricting Green’s $D$-relation on $S$ to $E(S)$ is $\{X_1, \ldots, X_k\}$. Suppose $|X_i| = r_i$ and $G_{X_i} \cong G_i$ for all $i \in \{1, \ldots, k\}$. Denote the $D$-class of $S$ containing $X_i$ by $D_i$, and denote the $\mathbb{C}$-span of $H_i = \{[s] : s \in D_i\}$ by $\mathbb{C}H_i$. It is clear from (3) that, as an algebra, $\mathbb{C}S = \bigoplus_{i=1}^k \mathbb{C}H_i$. For each $D$-class $D_i$, fix an idempotent $e_i$, so $G_{e_i} \cong G_i$ for all $i$.

Steinberg gives the following explicit algebra isomorphism from $\mathbb{C}H_i$ to $M_{r_i}(\mathbb{C}G_{e_i})$. For each $e \in X_i$, fix an element $p_e \in S$ such that $\text{dom}(p_e) = e_i$ and $\text{ran}(p_e) = e$, taking $p_{e_i} = e_i$. Note that $p_e \in D_i$, so $p_e^{-1} \in D_i$ as well. Viewing $r_i \times r_i$ matrices as being indexed by pairs of elements of $X_i$, define a map $\Phi_i : H_i \rightarrow M_{r_i}(\mathbb{C}G_{e_i})$ by

$$\Phi_i([s]) = p_{\text{ran}(s)^{-1}}^{-1} p_{\text{dom}(s)} E_{\text{ran}(s), \text{dom}(s)},$$

where $E_{\text{ran}(s), \text{dom}(s)}$ is the standard $r_i \times r_i$ matrix with a 1 in the $\text{ran}(s)$, $\text{dom}(s)$ position and 0 elsewhere. The linear extension of $\Phi_i$ to $\mathbb{C}H_i$ is Steinberg’s isomorphism, with inverse induced by, for $s \in G_{e_i}$,

$$sE_{e, f} \mapsto [p_{e} s p_{f}^{-1}].$$

Note that $p_{\text{ran}(s)^{-1}}^{-1} p_{\text{dom}(s)} \in G_{e_i}$ by construction, so $\Phi_i$ is a bijection between $H_i$ and the natural basis $\{sE_{e, f} : s \in G_{e_i} \text{ and } e, f \in X_i\}$ of $M_{r_i}(\mathbb{C}G_{e_i})$. Note further that if $e \in X_i$, then $\Phi_i([e]) = e_i E_{e, e}$. That is, $\Phi_i$ maps $[e]$ to the matrix which contains the identity of $G_{e_i}$ in the $e$, $e$ position, and hence $\Phi_i$ restricts to a bijection between $\{[e] : e \in X_i\}$ and the set of idempotents of the natural basis of $M_{r_i}(\mathbb{C}G_{e_i})$.

Since $\mathbb{C}S = \bigoplus_{i=1}^k \mathbb{C}H_i$, we may glue the $\Phi_i$ together to obtain an isomorphism

$$\Phi : \mathbb{C}S \rightarrow \bigoplus_{i=1}^k M_{r_i}(\mathbb{C}G_{e_i}).$$

By hypothesis we have $G_{e_i} \cong G_i$, so let $\omega_i : G_{e_i} \rightarrow G_i$ be an isomorphism. Extend $\omega_i$ to an isomorphism $\omega_i : M_{r_i}(\mathbb{C}G_{e_i}) \rightarrow M_{r_i}(\mathbb{C}G_i)$ by declaring $\omega_i(g E_{e, f}) = \omega_i(g) E_{e, f}$ and extending linearly. Glue the $\omega_i$ together to obtain an isomorphism

$$\Omega : \bigoplus_{i=1}^k M_{r_i}(\mathbb{C}G_{e_i}) \rightarrow \bigoplus_{i=1}^k M_{r_i}(\mathbb{C}G_i).$$
Then
\[ \Omega \circ \Phi : \mathbb{C}S \to \bigoplus_{i=1}^{k} M_{r_i}(\mathbb{C}G_i) \]
is an isomorphism.

Let \( B \) and \( C \) denote the natural bases of \( \bigoplus_{i=1}^{k} M_{r_i}(\mathbb{C}G_i) \) and \( \bigoplus_{i=1}^{k} M_{r_i}(\mathbb{C}G_e) \), respectively. \( \Phi \) restricts to a bijection between the groupoid basis of \( \mathbb{C}S \) and \( C \) and \( \Omega \) restricts to a bijection between \( C \) and \( B \), so we may use \( \Omega \circ \Phi \) to define a partial order \( \leq_{\Omega \circ \Phi} \) on \( B \): for \( b_1, b_2 \in B \), let
\[ b_1 \leq_{\Omega \circ \Phi} b_2 \iff (\Omega \circ \Phi)^{-1}(b_1) \leq (\Omega \circ \Phi)^{-1}(b_2). \]

We now show that \( \leq_{\Omega \circ \Phi} \) is a partial order on \( B \) satisfying the hypotheses of Theorem 3.1 and that \( S \) is recoverable up to isomorphism from the construction of Theorem 3.1 applied to \((B, \leq_{\Omega \circ \Phi})\).

Let \( E(B) \) and \( E(C) \) denote the set of idempotents of \( B \) and \( C \), respectively. \( \Phi \) restricts to a bijection between \( \{[e] : e \in E(S)\} \) and \( E(C) \), and \( \Omega \) restricts to a bijection between \( E(C) \) and \( E(B) \). From the definition of \( \Omega \) it follows that the semilattice \( ([e] : e \in E(S)), \leq \) is isomorphic to \( (E(B), \leq_{\Omega \circ \Phi}) \) by \([e] \mapsto \Omega \circ \Phi([e])\). In particular, \((E(B), \leq_{\Omega \circ \Phi})\) is a meet-semilattice.

Note that for \( b \in B \) and \( s \in S \), if \( b = \Omega \circ \Phi([s]) \) then \( b^{-1} = \Omega \circ \Phi([s^{-1}]) \). From this and parts (v)–(viii) of Theorem 2.7 it is straightforward to check that \( \leq_{\Omega \circ \Phi} \) satisfies hypotheses (ii)–(v) of Theorem 3.1.

By (4), we can recover \( S \) up to isomorphism from \( \leq_{\Omega \circ \Phi} \) and the multiplication of \( B \). In particular, for \( b \in B \), if we let
\[ \bar{b} = \sum_{a \in B : a \leq_{\Omega \circ \Phi} b} a, \]
then \( \{\bar{b} : b \in B\} \) is an inverse semigroup isomorphic to \( S \).

Finally, let \( \sqsubseteq \) be any partial order on \( E(B) \) for which \((E(S), \leq) \cong (E(B), \sqsubseteq)\). Write \( () \) for the identity of any group. Let \( \phi : E(S) \to E(B) \) be the function for which we have, for \( e \in E(S) \),
\[ e \mapsto ()_{\phi(e),\phi(e)} \]
in this isomorphism. Define \( \gamma : B \to B \) by
\[ \gamma(ga,b) = g_{\phi(a),\phi(b)} \]
and define \( \sqsubseteq' \) on \( B \) by
\[ g_{\phi(a),\phi(b)} \sqsubseteq' h_{\phi(c),\phi(d)} \iff ga,b \leq_{\Omega \circ \Phi} hc,d. \]
It is then straightforward to verify that \( \gamma \) is a bijective operation-preserving map, that \((E(B), \sqsubseteq') = (E(B), \sqsubseteq)\), and that \((B, \sqsubseteq')\) is a poset isomorphic to \((B, \leq_{\Omega \circ \Phi})\).
It follows that \( \sqsubseteq' \) is a partial order on \( B \) which restricts to \( \sqsubseteq \) on \( E(B) \), meets the hypotheses of Theorem 3.1, and yields an inverse semigroup isomorphic to \( S \) from the construction of Theorem 3.1.

\[\square\]

### 7.2 GPosets

In this section we prove the correctness of the implementation of \texttt{GPosets} described in Sect. 4. We require a sequence of lemmas.

**Lemma 7.1** If \( S \) is a finite inverse semigroup and \( s, t \in S \) with \( s \mathcal{D} t \) and \( s \leq t \), then \( s = t \).

**Proof** Suppose \( S \) is a finite inverse semigroup. Let \( s, t \in S \) with \( s \mathcal{D} t \) and \( s \leq t \). Then \( s^{-1} \leq t^{-1} \), so \( \text{dom}(s) = s^{-1} s \leq \text{dom}(t) = t^{-1} t \). Since \( s \mathcal{D} t \), by part (ii) of Theorem 2.7 we have \( \text{dom}(s) \mathcal{D} \text{dom}(t) \). Thus by Theorem 2.15 we have

\[|\{ x \in S : x \leq \text{dom}(s) \text{ and } x \mathcal{D} \text{dom}(s) \}| = |\{ x \in S : x \leq \text{dom}(t) \text{ and } x \mathcal{D} \text{dom}(s) \}|.\]

If we were to have \( \text{dom}(s) < \text{dom}(t) \), then by transitivity the quantity on the right would be strictly larger than the quantity on the left. Therefore we must have \( \text{dom}(s) = \text{dom}(t) \).

We emphasize that the condition that \( S \) be finite is necessary for Lemma 7.1, as there exist infinite inverse semigroups for which the natural partial order does not reduce to equality on \( \mathcal{D} \)-classes.

Now let notation be as in Sect. 4.

**Lemma 7.2** Write \( \leq_f \) for \( \leq_{E(B)} \). If there exist idempotents \( e \in B_i \) and \( f \in B_j \) such that \( f \leq e \), then \( B_j \sqsubseteq B_i \).

**Proof** Suppose there are idempotents \( e \in B_i \), \( f \in B_j \) with \( f \leq e \). If \( e = f \) then by definition of \( B_i \) and \( B_j \) we have \( B_i = B_j \). So suppose \( f < e \) and for the sake of contradiction suppose \( B_j \not\sqsubseteq B_i \). Then \( B_i \prec B_j \), so we have

\[I(B_i) = \max(r \in \mathbb{Z} : B_i \cap L_r) \geq \max(r \in \mathbb{Z} : B_j \cap L_r) = I(B_j).\]

Let \( r_i = I(B_i) \) and let \( e' \in B_i \) with \( e' = \emptyset_{a,a} \) for some \( a \in L_{r_i} \). Then, since \( P \) is a \( \mathcal{D} \)-partition of \( E \), we have

\[|\{ h \in B_j : h < e' \}| = |\{ h \in B_j : h < e \}|.
\]

Since \( f < e \) the quantity on the right is positive, so there exists \( h' \in B_j \) with \( h' < e' \). Therefore

\[\max(r \in \mathbb{Z} : B_j \cap L_r) > r_i = I(B_i),\]

a contradiction.\[\square\]
Parts (i)–(iv) of the following lemma concern the membership of elements in $\hat{B}_i$ in restrictions to the $\hat{B}_j$ of the partial orders on $B$ we seek. Part (v) is a technical result that will be used in the proof of Lemma 7.4.

Lemma 7.3 Suppose $\leq$ is a partial order on $B$ satisfying the hypotheses of Theorem 3.1 and $1 \leq i \leq k$. Then:

(i) $\forall t \in \hat{B}_i$, if $s \leq t$ then $s^{-1} \leq t^{-1}$ and $s, s^{-1} \in \hat{B}_i$.

(ii) $\forall y, z \in \hat{B}_i$, if $s \leq y, t \leq z$, $st \neq 0$, and $yz \neq 0$, then $st \leq yz$ and $s, t, st, yz \in \hat{B}_i$.

(iii) $\forall s \in \hat{B}_i$, if $e \leq \text{dom}(s)$, then $\exists! t \in B$ such that $t \leq s$ and $\text{dom}(t) = e$. We also have $t \in \hat{B}_i$.

(iv) $\forall s \in \hat{B}_i$, if $e \leq \text{ran}(s)$, then $\exists! t \in B$ such that $t \leq s$ and $\text{ran}(t) = e$. We also have $t \in \hat{B}_i$.

(v) If $h, h' \leq i$, $b \in B_h$, $b' \in B_{h'}$, and $b$ covers $b'$, then there exist idempotents $e \in B_h$, $f \in B_{h'}$, such that $e$ covers $f$. Furthermore $h' < h$.

Proof To show (i)–(iv) we only need to establish the claimed membership in $\hat{B}_i$. It follows from hypotheses (ii) and (iii) of Theorem 3.1 that if $s, t \in B$ with $s \leq t$, then $\text{ran}(s) \leq \text{ran}(t)$. The claimed membership in $\hat{B}_i$ in (i)–(iv) follows from Lemma 7.2.

To show (v), first suppose to the contrary that there exists $b \in B_h$ which covers $b' \in B_{h'}$, and for all idempotents $e \in B_h, f \in B_{h'}$, $e$ does not cover $f$. Then $\text{ran}(b)$ does not cover any idempotents in $B_{h'}$. From the hypotheses of Theorem 3.1 it follows that $\text{ran}(b') < \text{ran}(b)$. Since $\text{ran}(b)$ does not cover $\text{ran}(b')$, there exists an idempotent $f' \in B_r$ for some $r \leq i$ for which $\text{ran}(b') < f' < \text{ran}(b)$. By (iv), then, there exists $x \in B_r$ such that $x < b$ and $\text{ran}(x) = f'$. Then, since $\text{ran}(b') \in B_{h'}$ and $\text{ran}(b') < f'$, there exists $y < x$ such that $y \in B_{h'}$ and $\text{ran}(y) = \text{ran}(b')$. In addition, there is a unique element $u \in B_{h'}$ such that $u < b$ and $\text{ran}(u) = \text{ran}(b')$. Since $b', y \in B_{h'}$, $b' < b$, $y < b$, and $\text{ran}(y) = \text{ran}(b')$, we have that $b' = y$. But then we have $b' = y < x < b$, contradicting the assumption that $b$ covers $b'$. The final statement of (v) follows from Lemmas 7.1 and 7.2.

Lemma 7.4 Let $N = (\hat{B}_i, \preceq, i)$ be a node of the search tree for GPosets with $1 \leq i < k$. Then the children of $N$ (produced by Algorithm 4.2) consist precisely of all possible nodes $(\hat{B}_{i+1}, \preceq_{i+1}, i + 1)$ such that

(C1) if $a, b \in \hat{B}_i$, then $a \preceq_i b$ if and only if $a \preceq_{i+1} b$,

(C2) $\forall s, t \in \hat{B}_{i+1}$, if $s \preceq_{i+1} t$ then $s^{-1} \preceq_{i+1} t^{-1}$,

(C3) $\forall s, t, y, z \in \hat{B}_{i+1}$, if $s \preceq_{i+1} y, t \preceq_{i+1} z, st \neq 0$, and $yz \neq 0$, then $st \preceq_{i+1} yz$,

(C4) $\forall e, s \in \hat{B}_{i+1}$, if $e \preceq_{i+1} \text{dom}(s)$, then $\exists! t \in \hat{B}_{i+1}$ such that $t \preceq_{i+1} s$ and $\text{dom}(t) = e$, and

(C5) $\forall e, s \in \hat{B}_{i+1}$, if $e \preceq_{i+1} \text{ran}(s)$, then $\exists! t \in \hat{B}_{i+1}$ such that $t \preceq_{i+1} s$ and $\text{ran}(t) = e$.

Proof Suppose $\preceq_{i+1}$ is a partial order on $\hat{B}_{i+1}$ satisfying (C1)–(C5). By part (v) of Lemma 7.3, if $b$ covers $b'$ in $\preceq_{i+1}$ and $b \in B_{i+1}$, then $b' \in B_j$ for some $j \leq i$ and for which $B_{i+1}$ covers $B_j$. Furthermore, the construction of Theorem 3.1 applied to $(\hat{B}_{i+1}, \preceq_{i+1})$ would produce a finite inverse semigroup, so by Lemma 7.1 we have

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that \( \leq i+1 \) restricts to equality on \( B_h, \forall h \leq i+1 \). Finally, by Proposition 2.14, if \( h \leq i \), then for all \( s, t \in B_{i+1}, [|s' \in B_h : s' \leq i+1 s|] = [|t' \in B_h : t' \leq i+1 t|] \). Therefore every partial order \( \leq i+1 \) on \( \hat{B}_{i+1} \) satisfying (C1)–(C5) appears among the children of \( N \).

We now show conversely that every child of \( N \) satisfies (C1)–(C5). (C1) is satisfied by construction (in particular, by the specification of the function \text{PosetPossibilities}). Let \( (\hat{B}_{i+1}, \leq i+1, i+1) \) be a child of \( N \), so \( \leq i+1 \) is a partial order on \( \hat{B}_{i+1} \) such that

(i) \( \forall s, t \in \hat{B}_{i+1}, \text{if } s \leq i+1 t \text{ then } s^{-1} \leq i+1 t^{-1} \),

(ii) \( \forall s, t, y, z \in \hat{B}_{i+1}, \text{if } s \leq i+1 y, t \leq i+1 z, \text{ and } yz \neq 0, \text{ and } st \leq i+1 yz, \)

(iii) \( \forall e, s \in \hat{B}_{i+1}, \text{if } e \leq i+1 \text{ dom}(s), \text{then } \exists! t \in \hat{B}_{i+1} \text{ such that } t \leq i+1 s \text{ and } \text{dom}(t) = e, \)

(iv) \( \forall e, s \in \hat{B}_{i+1}, \text{if } e \leq i+1 \text{ ran}(s), \text{then } \exists! t \in \hat{B}_{i+1} \text{ such that } t \leq i+1 s \text{ and } \text{ran}(t) = e, \)

and

(v) \( \forall s \in \hat{B}_{i+1}, \text{then } \forall h \in \{1, \ldots, i\}, [|t \in B_h : t \leq i+1 s|] = [|e \in B_h : e \leq i+1 \text{ ran}(s)|]. \)

Furthermore, if \( B_{i+1} \) covers \( B_j \), then

(vi) \( \forall t \in B_{i+1}, \text{if } s \in B_j \text{ with } s \leq i+1 t \text{ then } s^{-1} \leq i+1 t^{-1}, \)

(vii) \( \forall y, z \in B_{i+1}, \text{if } s \leq i+1 y \text{ with } s, t \in B_j, \text{ and } st \neq 0, \text{ and } yz \neq 0, \text{ then } st 

\leq i+1 yz, \)

(viii) \( \forall s \in B_{i+1}, \text{if } e \leq i+1 \text{ dom}(s) \text{ with } e \in B_j, \text{ then } \exists! t \in B_j \text{ such that } t \leq i+1 s \text{ and } \text{dom}(t) = e, \)

and

(ix) \( \forall s \in B_{i+1}, \text{if } e \leq i+1 \text{ ran}(s) \text{ with } e \in B_j, \text{ then } \exists! t \in B_j \text{ such that } t \leq i+1 s \text{ and } \text{ran}(t) = e. \)

In order, we explain why \( (\hat{B}_{i+1}, \leq i+1, i+1) \) satisfies (C2), (C4), (C5), and (C3). For the remainder of the proof, write \( \leq \) for \( \leq i+1 \).

(C2) Suppose \( s, t \in \hat{B}_{i+1} \) and \( s \leq t \). If \( t \in \hat{B}_i \) or \( t = s \) we are done, so suppose \( t \in \hat{B}_{i+1} \) and \( s < t \). Then \( s \in B_h \) for some \( h \leq i \). If \( B_{i+1} \) covers \( B_h \) we are done, so suppose \( B_{i+1} \) does not cover \( B_h \). Let \( j \leq i \) and \( t' \in B_j \) such that \( B_{i+1} \) covers \( B_j \) and \( s \leq t' \leq t \). Then \( t'^{-1} \leq t^{-1} \) (by (vi)) and \( s^{-1} \leq t'^{-1} \) (by (i)), so \( s^{-1} \leq t^{-1} \).

(C4) Let \( e, s \in \hat{B}_{i+1} \) and \( e \leq \text{dom}(s) \). If \( s \in \hat{B}_i \) or \( e \in B_{i+1} \) we are done, so suppose \( s \in B_{i+1} \) and \( e \in B_h \) for some \( h \leq i \). If \( B_{i+1} \) covers \( B_h \) we are done, so suppose \( B_{i+1} \) does not cover \( B_h \). Let \( j \leq i \) such that \( B_{i+1} \) covers \( B_j \) and there exists \( f \in B_j \) such that \( e \leq f \leq \text{dom}(s) \). Then \( \exists s' \in B_j \text{ such that } s' \leq s \text{ and } \text{dom}(s') = f \) (by (viii)) and \( \exists \hat{t} \in B_h \text{ such that } t \leq s' \text{ and } \text{dom}(\hat{t}) = e \) (by (iii)), so \( \exists \hat{t} \in B_h \text{ such that } t \leq s \text{ and } \text{dom}(\hat{t}) = e \). The uniqueness of \( t \) follows from (v)—in particular, if \( \{e \in B_h : e \leq \text{dom}(s)\} = \{e_1, \ldots, e_p\} \), then for \( 1 \leq q \leq p \), let \( \phi(e_q) = \{t \in B_h : t \leq s, \text{dom}(t) = e_q\} \). By the preceding argument, \( |\phi(e_q)| \geq 1 \) for all \( 1 \leq q \leq p \). Since the \( e_q \) are distinct, the \( \phi(e_q) \) do not overlap. Furthermore \( \cup_{q=1}^p \phi(e_q) = \{t \in B_h : t \leq s, \text{dom}(t) \leq \text{dom}(s)\} \), so \( |\{t \in B_h : t \leq s, \text{dom}(t) \leq \text{dom}(s)\}| \geq p \). But by (v), \( |\{t \in B_h : t \leq s, \text{dom}(t) \leq \text{dom}(s)\}| = p \) and \( |\phi(e_q)| = 1 \) for all \( 1 \leq q \leq p \).

(C5) Similar to (C4).
C3) Let $s, t, y, z \in \hat{B}_{i+1}$, $s \leq y$, $t \leq z$, $st \neq 0$, and $yz \neq 0$. We need to show $st \leq yz$. Since $yz \neq 0$, we have $y, z \in B_x$ for some $x \leq i + 1$. If $x \leq i$ we are done so suppose $y, z \in B_{i+1}$. Similarly, we have $s, t \in B_h$ for some $h \leq i + 1$. If $h = i + 1$ or $B_{i+1}$ covers $B_h$ we are done, so suppose $h \leq i$ and $B_{i+1}$ does not cover $B_h$. By hypothesis we have $\text{dom}(s) = \text{ran}(t)$ and $\text{dom}(y) = \text{ran}(z)$.

Let $j \leq i$ be such that $B_{i+1}$ covers $B_j$ and there exists $y' \in B_j$ such that $s \leq y' \leq y$. It follows from (vi) and (vii) that $\text{dom}(y') \leq \text{dom}(y)$, and from (i) and (ii) that $\text{dom}(s) \leq \text{dom}(y')$. Since $\text{dom}(y') \leq \text{dom}(y) = \text{ran}(z)$, by (ix) there is a unique $z' \in B_j$ such that $z' \leq z$ and $\text{ran}(z') = \text{dom}(y')$. By (vii) we have $y'z' \leq yz$. We claim that $t \leq z'$, for if not there would exist $t' \in B_h$ such that $t \neq t'$, $t' \leq z'$, and $\text{ran}(t') = \text{dom}(s)$; but then we would have distinct elements $t, t' \in B_h$ with $t, t' \leq z$ and $\text{ran}(t) = \text{ran}(t') = \text{dom}(s)$, contradicting (C5). So $t \leq z'$. By (ii), then, $st \leq y'z'$, so by transitivity we have $st \leq yz$. □

**Proposition 7.5** (Correctness of implementation of $\text{GPosets}$) The set of partial orders in the leaves of the search tree for $\text{GPosets}$ is precisely the set of partial orders on $B = \hat{B}_k$ satisfying the hypotheses of Theorem 3.1.

**Proof** Suppose $\leq$ is a partial order on $B$ satisfying the hypotheses of Theorem 3.1. By part (v) of Lemma 7.3, the restriction of $\leq$ to $\hat{B}_1$ is $\leq_{E(B)}$. Lemmas 7.3 and 7.4 and induction establish that $\leq$ may be found as one of the leaves of the search tree.

Conversely, Lemma 7.4 establishes that the partial order $\leq_k$ of every leaf ($\hat{B}_k$, $\leq_k$, $k$) of the search tree satisfies the hypotheses of Theorem 3.1. □

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