The linear-non-linear substitution 2-monad

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We introduce a general construction on 2-monads. We develop background on maps of 2-monads, their left semi-algebras, and colimits in 2-category. Then we introduce the construction of a colimit induced by a map of 2-monads, show that we obtain the structure of a 2-monad and give a characterisation of its algebras. Finally, we apply the construction to the map of 2-monads between free symmetric monoidal and the free cartesian 2-monads and combine them into a Linear Non-Linear 2-monad.

This paper is concerned with a particular general construction on 2-monads in the sense of Cat-enriched monad theory [11]. Prima facie, the construction is not a universal one in a standard 2-category of 2-monads. All the same we are able precisely to characterise the 2-category of algebras for the 2-monad which we construct. This is a first step and further work will involve 2-dimensional monad theory in the sense of [5]. Specifically, in the future, we shall address the question of extending our constructed 2-monad on the 2-category Cat of small categories to the corresponding bicategory Prof of profunctors or distributeurs [1, 2, 9]. We shall then use a resulting Kleisli bicategory [16] as the setting for an analysis of the foundations of the differential calculus as it appears in the differential λ-calculus [8, 12, 14]. This will involve an extension of the approach of variable binding and substitution in abstract syntax [13, 15, 19, 21, 25].

For readers not familiar with substitution and variable binding, we recall that it is based on the use of suitable 2-monads \( \mathcal{T} \) on Cat which extend to pseudo-monads on Prof [16]. In the corresponding Kleisli bicategory, we can consider monads \( M : A \to \mathcal{T}A \) and these can be identified as generalised multicategories. For example, in case \( \mathcal{T} \) is the 2-monad for symmetric monoidal categories, these are exactly what are called many coloured operads or symmetric multicategories [10]. Similarly, in case \( \mathcal{T} \) is the 2-monad for categories with products, the monads in the Kleisli bicategory are essentially many sorted algebraic theories [20]. In this paper, we show how to construct a 2-monad \( Q \) which would give rise to a Linear Non-Linear multicategory.

Our project is based on 2-monads on a 2-category \( \mathcal{K} \) in the setting of the pioneering paper [5]. Here, for a 2-monad \( \mathcal{T} \) on \( \mathcal{K} \), we follow the practice of that paper in writing \( \mathcal{T} \)-Alg_s for the 2-category of strict \( \mathcal{T} \)-algebras, strict \( \mathcal{T} \)-algebra maps and \( \mathcal{T} \)-algebra 2-cells. We shall use more detailed information from [5] in further papers.

In (enriched) categories of algebras for a monad, limits are easy and it is colimits which are generally of more interest. We assume throughout that our ambient 2-category \( \mathcal{K} \) is cocomplete, that our 2-monads \( \mathcal{T} \) are such that the 2-categories \( \mathcal{T} \)-Alg_s are also cocomplete. In fact, we shall only need rather innocent looking colimits in \( \mathcal{T} \)-Alg_s, specifically the co-lax colimit of an arrow. However, even that requires an infinite construction [22]. So it does not seem worth worrying about minimal conditions for our results: we assume that we are in a situation where all our 2-categories are cocomplete. That happens for example if our basic 2-category is locally finitely presentable and our monads are finitary [23].

Content

In Section 1, we first describe the background on maps of 2-monads (Subsection 1.1), left-semi algebras (Subsection 1.2) and colimits (Subsection 1.3), needed in our main Section 2. We first define the colimits obtained from a map of monads (Subsection 2.1) and exhibit their properties.
(Subsection 2.2). Inspired by these properties, we define what we simply call the structure 2-category (Subsection 2.3). We finally use (Subsection 2.4) the properties of the structure 2-category to prove, in Theorem 2.9 that the colimit is a monad; and finally we prove our main Theorem 2.12 which states that the structure 2-category is isomorphic to the 2-category of strict algebras over the colimit monad. We end by spelling out the construction for two examples, the first one generates the left-semi algebra 2-category (Proposition 2.13) and the second, what we call the Linear-Non-Linear monad (Section 3) which was the original intention for developing this theory.

1 BACKGROUND

1.1 Maps of 2-monads

The construction which we introduce here takes for its input a map \( \lambda : \mathcal{L} \to \mathcal{M} \) of 2-monads on \( \mathcal{K} \). For clarity we stress that the usual diagrams commute on the nose. We rehearse some folklore related to this situation.

First, it is elementary categorical algebra that the monad map \( \lambda : \mathcal{L} \to \mathcal{M} \) induces a 2-functor \( \lambda^* : \mathcal{M}\text{-Alg} \to \mathcal{L}\text{-Alg} \). On objects \( \lambda^* \) takes an \( \mathcal{M} \)-algebra \( \mathcal{M} X \to X \) to an \( \mathcal{L} \)-algebra \( \mathcal{L} X \xrightarrow{\lambda} \mathcal{M} X \to X \). It is equally evident that \( \lambda : \mathcal{L} \to \mathcal{M} \) induces a 2-functor \( \lambda_! : \text{kl}(\mathcal{L}) \to \text{kl}(\mathcal{M}) \) between the corresponding Kleisli 2-categories. We have the standard locally full and faithful comparisons: \( \text{kl}(\mathcal{L}) \to \mathcal{L}\text{-Alg} \) and \( \text{kl}(\mathcal{M}) \to \mathcal{M}\text{-Alg} \).

Suppose we interpret \( \lambda_! \) as acting on the free \( \mathcal{L} \)-algebra \( \mathcal{L}^2 A \xrightarrow{\lambda^*} \mathcal{L} A \to \text{free} \mathcal{M} \text{-algebra} \mathcal{M}^2 A \xrightarrow{\mu^M} \mathcal{M} A \). Then we can see \( \lambda_! \) as a restricted left adjoint to \( \lambda^* \) in the following sense. Given the free \( \mathcal{L} \)-algebra \( \mathcal{L}^2 A \xrightarrow{\lambda^*} \mathcal{L} A \) on \( A \) and \( \mathcal{M} B \xrightarrow{b} B \) an arbitrary \( \mathcal{M} \)-algebra, we have \( \mathcal{L}\text{-Alg}_s(\mathcal{L} A, \lambda^* B) \equiv \mathcal{M}\text{-Alg}_s(\lambda_! \mathcal{L} A, B) \). For \( \lambda_!(\mathcal{L}^2 A \xrightarrow{\lambda^*} \mathcal{L} A) \equiv \mathcal{M}^2 A \xrightarrow{\mu^M} \mathcal{M} A \) and so both sides are isomorphic to \( \mathcal{K}(A, B) \).

Any \( \mathcal{L} \)-algebra \( \mathcal{L} A \xrightarrow{a} A \) lies in a coequalizer diagram in \( \mathcal{L}\text{-Alg}_s : \mathcal{L}^2 A \xrightarrow{\lambda^*} \mathcal{L} A \xrightarrow{a} A \). So to extend \( \lambda_! \) to a full left adjoint \( \lambda_! : \mathcal{L}\text{-Alg}_s \Rightarrow \mathcal{M}\text{-Alg}_s \) one has only to take the coequalizer of the corresponding pair in \( \mathcal{M}\text{-Alg}_s : \mathcal{M} \mathcal{L} A \xrightarrow{\mu^M_M \lambda_!} \mathcal{M} A \). As it happens, we do not need the full left adjoint, but we shall need the unit of the adjunction given by the \( \mathcal{L} \)-algebra map \( \lambda_A \) from \( \mathcal{L}^2 A \xrightarrow{\lambda^*} \mathcal{L} A \) to \( \lambda^\ast \lambda_A : (\mathcal{L}^2 A \xrightarrow{\mu^M} \mathcal{L} A) = \mathcal{L} \mathcal{M} A \xrightarrow{\lambda M} \mathcal{M}^2 A \xrightarrow{\mu^M} \mathcal{M} A \).

If \( \mathcal{L} A \xleftarrow{g} \mathcal{L} B \) is an \( \mathcal{L} \)-algebra 2-cell then the corresponding 2-cell \( \lambda^* \lambda_A g \Rightarrow \lambda^* \lambda_A g' \) is given by the composite \( \mathcal{M} A \xrightarrow{\lambda_! \mathcal{M} g} \mathcal{M} \mathcal{L} A \xrightarrow{\mu^M_M} \mathcal{M} \mathcal{L} B \xrightarrow{\lambda_! \mu^M} \mathcal{M} B \xrightarrow{\mu^M} \mathcal{M} B \) so that

\[
\begin{align*}
\mathcal{L} A & \xleftarrow{g} \mathcal{L} B \\
\mathcal{L} A & = \mathcal{L} A \\
\mathcal{M} A & \xrightarrow{\lambda^* \lambda_A g} \mathcal{M} B
\end{align*}
\]

\( (1) \)
1.2 Left-semi Algebras

In this section we present a theory of a generalization of the notion of \( T \)-algebra for a 2-monad \( T \). In effect, it is a mere glimpse of an extensive theory of semi-algebra structure, in the sense of structure "up to a retraction", a terminology well-established in computer science. We do not need to have this background in place for the results which we give in this paper: we give only what is required to make the paper comprehensible. However, some impression of what is involved can be obtained by looking at [18] which gives some theory in the 1-dimensional context.

**Definition 1.1.** Let \( T \) be a 2-monad on a 2-category \( K \). A left-semi \( T \)-algebra structure on an object \( Z \) of \( K \) consists of a 1-cell \( T \overset{\mu}{\rightarrow} Z \) and a 2-cell \( \eta : z \cdot \Rightarrow 1_Z \) satisfying the following 1-cell and 2-cell equalities:

\[
\begin{aligned}
T^2 Z & \xrightarrow{Tz} T Z \\
\downarrow \mu & \downarrow z \\
TZ & \xrightarrow{z} Z
\end{aligned}
\quad (2)
\begin{aligned}
TZ & \xrightarrow{\eta} T Z \\
\downarrow \epsilon & \downarrow z \\
Z & \xrightarrow{z} Z
\end{aligned}
\quad (3)

**Remark 1.** (1) The diagrams demonstrate that Condition (2) implies that the boundaries of the 2-cells in Condition (3) do match.

(2) Condition (2) is the standard composition for a strict \( T \)-algebra, while Condition (3) is the unit condition for a colax \( T \)-algebra.

**Definition 1.2.** Suppose that \( T \overset{\xi}{\rightarrow} Z, \epsilon : z \cdot \Rightarrow 1_Z \) and \( T \overset{\eta}{\rightarrow} W, \epsilon : w \cdot \Rightarrow 1_W \) are left-semi \( T \)-algebras. A strict map from the first to the second consists of \( p : Z \rightarrow W \) satisfying the following 1-cell and 2-cell equalities:

\[
\begin{aligned}
TZ & \xrightarrow{Tp} TW \\
\downarrow z & \downarrow w \\
Z & \xrightarrow{p} W \\
\end{aligned}
\quad (4)
\begin{aligned}
Z & \xrightarrow{p} W \\
\downarrow \eta & \downarrow w \\
W & \xrightarrow{\eta} TZ
\end{aligned}
\quad (5)

**Remark 2.** (1) The Condition (4) with the naturality of \( \eta \) imply that the boundaries of the 2-cells in (3) do match.

(2) The definition is the restriction to left-semi algebras of the evident notion of strict map of colax \( T \)-algebras.

(3) If \( T \overset{\xi}{\rightarrow} Z, \epsilon : z \cdot \Rightarrow 1_Z \) is a left-semi algebra, then \( T \overset{\xi}{\rightarrow} Z \) is a strict map to it from the free algebra \( T^2 Z \overset{\mu}{\rightarrow} T Z \).

**Proposition 1.3.** Suppose that \( T \overset{\xi}{\rightarrow} Z, \epsilon : z \cdot \Rightarrow 1_Z \) is a left-semi algebra. Then the composite \( f : Z \overset{\eta}{\rightarrow} T Z \overset{\xi}{\rightarrow} Z \) is a strict endomap of the left-semi algebra.

Finally, we consider 2-cells between maps of left-semi algebras.
Definition 1.4. Suppose that \( p, q : Z \to W \) are strict maps of left-semi algebras from \( \mathcal{T}Z \xrightarrow{z} Z, \varepsilon : z.\eta \Rightarrow 1_Z \) to \( \mathcal{T}W \xrightarrow{w} W, \varepsilon : w.\eta \Rightarrow 1_W \). A 2-cell from \( p \) to \( q \) consists of a 2-cell \( \gamma : p \Rightarrow q \) such that the equality \( \mathcal{T}Z \xrightarrow{z} Z \xrightarrow{p} W = \mathcal{T}Z \xrightarrow{\varphi} \mathcal{T}W \xrightarrow{w} W \) holds.

Remark 3. Again, this is simply the restriction to the world of left-semi algebras of the definition of 2-cells for colax \( \mathcal{T} \)-algebras.

Proposition 1.5. Suppose that \( \mathcal{T}Z \xrightarrow{z} Z, \varepsilon : z.\eta \Rightarrow 1_Z \) is a left-semi \( \mathcal{T} \)-algebra, so that both \( z.\eta \) and \( 1_Z \) are strict endomaps. Then \( \varepsilon : z.\eta \Rightarrow 1_Z \) is a left-semi \( \mathcal{T} \)-algebra 2-cell.

At this point, it is straightforward to check that left-semi \( \mathcal{T} \)-algebras, strict maps and 2-cells form a 2-category that we denote as \( \text{ls-} \mathcal{T} \text{-Alg} \).

Looking more closely at what we showed above we see that if we set \( f = z.\eta \), then we have \( f = f^2 \) and \( \varepsilon.\varepsilon = \text{id}_f = f.\varepsilon \). So, in fact, we have the following.

Proposition 1.6. Suppose that \( \mathcal{T}Z \xrightarrow{z} Z, \varepsilon : z.\eta \Rightarrow 1_Z \) is a left-semi \( \mathcal{T} \)-algebra. Then, in the 2-category \( \text{ls-} \mathcal{T} \text{-Alg} \), the 1-cell \( f \) and the 2-cell \( \varepsilon : f \Rightarrow 1_Z \) equips the left-semi \( \mathcal{T} \)-algebra with the structure of a strictly idempotent comonad.

Applying the evident forgetful 2-functor we get that \( f = f^2 \) and \( \varepsilon : f \Rightarrow 1_Z \) equips \( Z \) with the structure of a strictly idempotent comonad in the underlying 2-category \( \mathcal{K} \).

Proposition 1.7. Suppose that \( \mathcal{T}X \xrightarrow{X} X \) is a \( \mathcal{T} \)-algebra and \( f = f^2 : X \to X \) and \( \varepsilon : f \Rightarrow 1_X \) equips \( X \) with the structure of a strictly idempotent comonad natural in \( \mathcal{T} \text{-Alg} \). Then \( \mathcal{T}X \xrightarrow{X} X \xrightarrow{f} X, \varepsilon : f.\eta \Rightarrow 1_X \) is a left-semi \( \mathcal{T} \)-algebra.

Proof Sketch. The 1-cell part is routine and the 2-cell uses that \( \varepsilon \) is a 2-cell in \( \mathcal{T} \text{-Alg} \). \( \square \)

Definition 1.8. Suppose that \( S \) and \( \mathcal{T} \) are 2-monads. A left-semi monad map from the first to the second consists of \( \lambda : S \to \mathcal{T} \) satisfying the following equalities

\[
\begin{array}{c}
1 \\ \eta \\
\downarrow \varepsilon \\
\downarrow \lambda \\
S \to \mathcal{T}
\end{array} = \begin{array}{c}
\begin{array}{c}
S \\
\eta \\
\downarrow S \eta \\
S^2 \to S \mathcal{T}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\lambda \\
\eta \\
\downarrow \lambda \eta \\
\mathcal{T} \to S \mathcal{T}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\lambda \\
\eta \\
\downarrow \lambda \eta \\
\mathcal{T} \to S \mathcal{T}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Proposition 1.9. Suppose that \( \mathcal{T}Z \xrightarrow{z} Z, \varepsilon : z.\eta \Rightarrow 1_Z \) is a left-semi \( \mathcal{T} \)-algebra and \( S \xrightarrow{\lambda} \mathcal{T}, \gamma : \lambda.\eta \Rightarrow \eta \) is a left-semi monad map. Then \( S \mathcal{T} \xrightarrow{\lambda \gamma} \mathcal{T}Z \xrightarrow{z} Z, \varepsilon.\gamma : z.\lambda.\eta \Rightarrow 1_Z \) is a left-semi \( S \)-algebra.
Proof sketch. The 1-cell part is routine and the 2-cell parts use the naturality of $\lambda$ to separate the two 2-cells $\gamma$ and $\epsilon$. □

1.3 Colax colimits induced by a map in 2-category

In this section we review the notion of colax colimits in a cocomplete 2-category specialised to our context [4, 24].

In the 2-category $\mathcal{K}$, suppose that $\alpha$ is a colax cocone $(k, \ell, \alpha)$ under the arrow $\lambda$ (see Figure 1, left). Then, for every $D$, composition with $\alpha$ induces an isomorphism of categories between $\mathcal{K}(C, D)$ and the category of colax cocones under the arrow $\lambda$ with objects $(f, g, \phi)$ (see Figure 1, center) and 1-cells $(f, g, \phi) \rightarrow (f', g', \phi')$ given by 2-cells $f \Rightarrow f'$ and $g \Rightarrow g'$ such that $\rho \star \phi = \phi' \star \sigma \lambda$ (see Figure 1, right).

This isomorphism of categories has two universal aspects, the first is 1-dimensional and the second is 2-dimensional:

- for any $A \xrightarrow{\lambda} B$ there is a unique $r$ such that $A \xrightarrow{\lambda} B \xrightarrow{\alpha} C \xrightarrow{r} D = \phi$

- for any $A \xrightarrow{f^\prime} B$ there is a unique $r$ such that $A \xrightarrow{\lambda} B \xrightarrow{g} D = \phi$

$$A \xrightarrow{\phi} D = A \xrightarrow{k} C \xrightarrow{r} D \quad \text{and} \quad B \xrightarrow{\sigma} D = B \xrightarrow{\ell} C \xrightarrow{r} D$$ (9)

Although we will compute colax colimits in the 2-category of $\mathcal{L}$-Alg, where what happens is more subtle, we illustrate this definition by computing colax colimits in the 2-category Cat.

Example 1.10. In Cat, $A \xrightarrow{\lambda} B$ is a functor between categories. The colax colimit under $\lambda$ is a category $C$ which consists of separate copies of $A$ and $B$ together with, for every object $a \in A$, new maps $\lambda(a) \xrightarrow{a} a$, composition of such and evident identifications. Precisely, maps from $b \in B$ to $a \in A$ are given by $b \xrightarrow{\ell} \lambda(a) \xrightarrow{\alpha a} a$ and $C(b, a) \simeq B(b, \lambda(a))$.

2 THE COLIMIT 2-MONAD INDUCED BY A MAP OF 2-MONADS

From now on, we assume that $\mathcal{L}$ is a finitary 2-monad, so that $\mathcal{L}$-Alg is cocomplete [23].
2.1 Definition of the colimit and its 2-naturality

Definition 2.1. Suppose that \( \lambda : \mathcal{L} \to \mathcal{M} \) is a map of 2-monads. Then the colax colimit \((QX, u)\) under the induced \( \lambda_X : (\mathcal{L}X, \mu^L) \to (\mathcal{M}X, \mu^M) \) in \( \mathcal{L}\text{-Alg} \) satisfies

\[
\begin{array}{c}
\mathcal{L}X \\
\downarrow \lambda \\
\mathcal{M}X \\
\alpha \\
\downarrow \ell \\
QX
\end{array}
\]

Proposition 2.2. The colax colimit \((QX, u)\) is natural in \((\mathcal{L}X, \mu^L)\).

Proof sketch. Assume \( \mathcal{L}A \xrightarrow{g} \mathcal{L}B \) is an \( \mathcal{L} \)-algebra 2-cell. For each 1-cell we get by 2-cell naturality a cocone and so we get a unique maps \( \hat{g} \) mapping \( QA \) to \( QB \) arising from 1-cell universality. We then have

\[
\begin{array}{c}
\mathcal{L}A \xrightarrow{g} \mathcal{L}B \\
\downarrow \alpha \\
\mathcal{M}B \\
\downarrow \ell \\
QB
\end{array}
= \begin{array}{c}
\mathcal{L}A \\
\downarrow \alpha \\
\mathcal{M}A \\
\downarrow \ell \\
QA \xrightarrow{\hat{g}} QB
\end{array}
\]

and similarly for \( g' \) and \( \hat{g}' \). By 2-cell universality (9), we then get:

\[
\begin{array}{c}
\mathcal{L}A \xrightarrow{g'} \mathcal{L}B \\
\downarrow \alpha \\
\mathcal{M}B \\
\downarrow \ell \\
QB = \mathcal{L}A \xrightarrow{k} QA \xrightarrow{\hat{g}} QB
\end{array}
\]

\[
\begin{array}{c}
\mathcal{M}A \xleftarrow{\lambda' \cdot g} \mathcal{M}B \\
\downarrow \lambda' \\
\mathcal{M}A \\
\downarrow \ell \\
QA \xrightarrow{\hat{g}} QB
\end{array}
= \begin{array}{c}
\mathcal{M}A \\
\downarrow \lambda' \\
\mathcal{M}A \\
\downarrow \ell \\
QA \xrightarrow{\hat{g}} QB
\end{array}
\]

\( \square \)

2.2 A left semi-algebra

We explore the properties of \( QX \) by considering 1 and 2 dimensional aspects of trivial cocones under \( \lambda \). From the identity cocone under \( \lambda \), a unique \( \mathcal{L} \)-algebra map \( h \) arises by 1-dimensional universality.

\[
\begin{array}{c}
\mathcal{L}X \\
\downarrow \lambda \\
\mathcal{M}X \\
\downarrow \ell \\
QX \\
\downarrow h \\
\mathcal{M}X
\end{array}
= \begin{array}{c}
\mathcal{L}X \\
\downarrow \lambda \\
\mathcal{M}X \\
\downarrow \ell \\
QX \\
\downarrow h \\
\mathcal{M}X
\end{array}
\quad\text{and}\quad \begin{array}{l}
h \cdot k = \lambda_X \\
h \ell = 1_{\mathcal{M}X} \\
h \alpha = \text{id}_{\mathcal{L}A}
\end{array}
\]

(11)

If \( \mathcal{L}A \xrightarrow{g} \mathcal{L}B \) is an \( \mathcal{L} \)-algebra 2-cell, then by 2-dimensional universality, so \( h \) is natural

\[
\begin{array}{c}
QA \xrightarrow{h} \mathcal{M}A \\
\uparrow \lambda' \cdot g' \\
\mathcal{M}B = \begin{array}{c}
\mathcal{M}A \\
\uparrow \lambda' \cdot g
\end{array} \xrightarrow{\hat{g}'} \begin{array}{c}
QB \\
\uparrow \hat{g}
\end{array} \xrightarrow{h} \mathcal{M}B.
\end{array}
\]
From the 2-cells \(id_\ell : \ell = \ell\) and \(\alpha : \ell \lambda \Rightarrow k\), arises a unique \(\mathcal{L}\text{-Alg}_s\) 2-cell \(\beta : \ell h \Rightarrow 1_{QX}\) s.t.

\[
\begin{array}{c}
\mathcal{L}X \\
\downarrow^k
\end{array}
\xrightarrow{1_{QX}}
\begin{array}{c}
QX \\
\downarrow^h
\end{array}
\xrightarrow{\beta} 
\begin{array}{c}
MX \\
\downarrow^\ell
\end{array}
\xrightarrow{\ell h} 
\begin{array}{c}
QX \\
\downarrow^\lambda
\end{array}
\xrightarrow{\alpha} 
\begin{array}{c}
QX
\end{array}
\]

\[
\begin{array}{c}
\mathcal{L}X \\
\downarrow^k
\end{array}
\xrightarrow{1_{QX}}
\begin{array}{c}
QX \\
\downarrow^h
\end{array}
\xrightarrow{\beta} 
\begin{array}{c}
MX \\
\downarrow^\ell
\end{array}
\xrightarrow{\ell h} 
\begin{array}{c}
QX \\
\downarrow^\lambda
\end{array}
\xrightarrow{\alpha} 
\begin{array}{c}
QX
\end{array}
\]

\[
\begin{array}{c}
MX \\
\downarrow^k
\end{array}
\xrightarrow{1_{QX}}
\begin{array}{c}
QX \\
\downarrow^h
\end{array}
\xrightarrow{\beta} 
\begin{array}{c}
MX \\
\downarrow^\ell
\end{array}
\xrightarrow{\ell h} 
\begin{array}{c}
QX \\
\downarrow^\lambda
\end{array}
\xrightarrow{\alpha} 
\begin{array}{c}
QX
\end{array}
\]

Denote \(f = \ell h\). Then \(QX\) is a \(\mathcal{L}\)-algebra and \(f = f^2 : QX \to QX\) and \(\beta : f \Rightarrow 1_{QX}\) equips \(QX\) with the structure of a strictly idempotent comonad natural in \(\mathcal{L}\text{-Alg}_s\) as \(\beta, \ell = \text{id}_\ell, \beta, \lambda = \alpha\), and thus \(h, \beta = \text{id}_h\). We apply Proposition 1.7 and get

**Proposition 2.3.** \(\mathcal{L}QX \xrightarrow{\mu} QX \xrightarrow{\ell} MX \xrightarrow{\beta} QX\) with \(\beta : \ell h \eta^\ell = \ell h \Rightarrow 1_{QX}\) is a left-semi \(\mathcal{L}\)-algebra.

**Proposition 2.4.** Assume \(z\) denotes the map \(\mathcal{M}QX \xrightarrow{\mu h^M} \mathcal{M}^2X \xrightarrow{\mu^M} \mathcal{M}X \xrightarrow{\ell} QX\). Then \(QX\) together with \(z\) and \(\eta^\mathcal{M} = \ell h \beta \Rightarrow 1_{QX}\) is a left-semi \(\mathcal{M}\)-algebra.

**Proof Sketch.** The 2-cell property relies on \(\beta, \ell = \text{id}_\ell\) and \(h, \beta = \text{id}_h\). \(\square\)

As \(\lambda\) is a map of 2-monads, it is a left-semi monad map. We apply Proposition 1.9 and get

**Proposition 2.5.** \(\mathcal{L}QX \xrightarrow{\lambda Q} \mathcal{M}QX \xrightarrow{\mu^M} \mathcal{M}^2X \xrightarrow{\mu^M} \mathcal{M}X \xrightarrow{\ell} QX\) together with the 2-cell \(\beta : z(\lambda Q) \eta^\mathcal{M} = \ell h \Rightarrow 1_{QX}\) is a left-semi \(\mathcal{L}\)-algebra.

The following is an immediate consequence of the definitions.

**Proposition 2.6.** The left-semi \(\mathcal{L}\)-algebras of Proposition 2.3 and 2.5 are equal.

Let us recap the properties of \(QX\). It is equipped with an \(\mathcal{L}\)-algebra structure \(u\) and a left-semi \(\mathcal{M}\)-algebra structure \(z\) whose 2-cell \(\beta\) lies in \(\mathcal{L}\text{-Alg}_s\) and such that the two resulting left-semi \(\mathcal{L}\)-algebra structures coincide.

In order to prove that \(Q\) is a 2-monad (Theorem 2.9) and that these properties characterise \(Q\)-algebras (Theorem 2.12), we introduce an eccentric lemma. Given this structure on a general object \(X\), we can build a map \(QX \to X\) in a sufficiently functorial way that both theorems follow. What we need is the 1-cell and 2-cell aspects associated to these properties.

### 2.3 The Structure category

Let us define the Structure category \(\mathcal{Q}\)

- an object of \(\mathcal{Q}\) consists of an object \(X\) of \(\mathcal{K}\) equipped with
  - the structure \(\mathcal{L}X \xrightarrow{\omega} X\) of an \(\mathcal{L}\)-algebra
  - the structure \(\mathcal{M}X \xrightarrow{\zeta} X\) of a left-semi \(\mathcal{M}\)-algebra
  - such that \(f\) is an endomap of the \(\mathcal{L}\)-algebra \(\mathcal{L}X \xrightarrow{\omega} X\) and \(\varepsilon\) is an \(\mathcal{L}\)-algebra 2-cell
  - the two induced left-semi \(\mathcal{L}\)-algebra structures, with structure maps \(\mathcal{L}X \xrightarrow{\omega} X \xrightarrow{f} X\)
  - and \(\mathcal{L}X \xrightarrow{\lambda} \mathcal{M}X \xrightarrow{\zeta} X\) are equal
- a map in \(\mathcal{Q}\) between objects \(X\) and \(X'\) equipped as above is a map \(p : X \to X'\) in \(\mathcal{K}\) which is both an \(\mathcal{L}\)-algebra map and a left-semi \(\mathcal{M}\)-algebra map
- a 2-cell between two such maps \(p\) and \(q\) is a 2-cell \(p \Rightarrow p'\) which is both an \(\mathcal{L}\)-algebra and a left-semi \(\mathcal{M}\)-algebra 2-cell.
Remark 4. (1) In the definition, the condition regarding the left-semi $L$-algebra structures amounts to the claim that $f \cdot w = z \cdot \lambda$. The equality of the 2-cells is then automatic.

(2) It is a consequence of the definition that $z : MX \to X$ is a map of $L$-algebras. Indeed, if we consider the three following conditions, any two of them implies the third.
- $f$ is an endomap of $L$-algebras,
- $f \cdot w = \lambda \cdot z$
- $z$ is a map of $L$-algebras

**Proposition 2.7.** $QX$ together with $u$, $z$ and $\alpha$ is an object in $\mathbb{Q}$.

Assume $X$ together with $w$, $z$, and $\epsilon$ is an object in $\mathbb{Q}$. Then we define $QX \xrightarrow{x} X$ to be the unique $L$-$\text{Alg}$ map arising from the colax cocone

$$
\begin{array}{ccc}
L \times X & \xrightarrow{w} & X \\
\downarrow \quad \downarrow f \quad \downarrow \xi & & \downarrow \quad \downarrow k \\
MX & \xrightarrow{\lambda z} X \\
\end{array} = 
\begin{array}{ccc}
L \times X & \xrightarrow{k} & QX \\
\downarrow \quad \downarrow \alpha & & \downarrow \quad \downarrow x \\
MX & \xrightarrow{\ell} QX & \xrightarrow{x} X \\
\end{array}
$$

(12)

**Proposition 2.8.** Assume $X$ together with $w$, $z$, and $\epsilon$ is an object in $\mathbb{Q}$ and $x$ denotes the associated map. Then $x : QX \to X$ is a map in $\mathbb{Q}$ which is natural in $X$.

**Sketch Proof.** Assume $X'$ together with $w'$, $z'$, $\epsilon'$ in $\mathbb{Q}$ associated with $x'$ and $p \xrightarrow{\rho} q$ a 2-cell in $\mathbb{Q}$. Then

$$
\begin{array}{ccc}
QX' \xrightarrow{Q\epsilon} QX & \xrightarrow{Qx} X = QX' \xrightarrow{x'} X' & \xrightarrow{\mu^Q} X
\end{array}
$$

by 2-cell universality. □

### 2.4 The colimit is a monad

As $QX$ is an object in $\mathbb{Q}$ (Proposition 2.7), the induced map $Q^2X \xrightarrow{\mu^Q} QX$ is a map in $\mathbb{Q}$ (Proposition 2.8).

Assume $(X, w, z, \epsilon)$ in $\mathbb{Q}$. Then the induced map $QX \xrightarrow{x} QX$ is a map in $\mathbb{Q}$. We apply the 1-cell part of the naturality (Proposition 2.8) with $p = x$ and $x' = \mu^Q$ and get

$$
\begin{array}{ccc}
Q^2X & \xrightarrow{\mu^Q} & QX \\
QX & \xrightarrow{x} & X
\end{array}
$$

in particular, setting $x = \mu^Q$.

$$
\begin{array}{ccc}
Q^3X & \xrightarrow{\mu^Q} & QX \\
Q^2X & \xrightarrow{\mu^Q} & QX
\end{array}
$$

Theorem 2.9. $Q$ is a 2-monad with multiplication $\mu^Q$ and unit $X \xrightarrow{\eta^Q} LX \xrightarrow{k} QX$.

**Proposition 2.10.** $L \xrightarrow{k} Q$ is a map of monads.

**Proof Sketch.** The unit aspect is by definition of $\eta^Q$. As $k$ is a map of $L$-algebra and $\mu^Q k = u$ by cocone equality (12), we get the multiplication diagram. □

**Proposition 2.11.** $M \xrightarrow{\ell} Q$ is a left-semi map of monads.

**Proof Sketch.** Recall that $h \ell = 1$ and that $\mu^Q (\ell Q) = z$ by cocone equality (12). Then, the multiplication diagram (8) follows since $\mu^Q (\ell Q) (L \ell) = z (L \ell) = \ell \mu^M (Mh) (M \ell) = \ell \mu^M$. 

8
We define the unit 2-cell $\gamma : \ell \eta^M \Rightarrow \eta^Q$ in (6) as

$$
\begin{array}{c}
X \\ \eta^M \\
\downarrow \eta^L \\
\mathcal{L}X \\
\downarrow \alpha \\
\mathcal{M}X \\
\downarrow \ell \\
QX
\end{array}
$$

We prove Equalities (7). Recall that $\alpha = \beta.k$ and $\beta.\ell = \text{id}_\ell$. As $\mu^Q(\ell Q) = z = \ell \mu^M(Mh)$ and $h.\alpha = h.\beta.k = \text{id}_\ell.k$

$$
\begin{array}{c}
\mathcal{M}X \\
\downarrow M\eta^M \\
\mathcal{M}^2X \\
\downarrow M\ell \\
\mathcal{M}QX \\
\downarrow M\ell Q \\
\mathcal{Q}^2X \\
\downarrow \ell \\
QX
\end{array}
$$

As $\mu^Q.\alpha = \beta.u$ (see Equality (12) with $x = \mu^Q$), and as $u$ is an $\mathcal{L}$-algebra $u(\eta^LQ) = 1_{QX}$ so the second 2-cell equality follows: $\mu^Q.\alpha.(\eta^LQ) \ell = \beta.u(\eta^LQ) \ell = \beta.\ell = \text{id}_\ell$. □

**Theorem 2.12.** The 2-category $\mathcal{Q Alg}_s$ of algebras of the 2-monad $\mathcal{Q}$ is isomorphic to the Structure category.

**Proof Sketch.** It remains to prove the direct implication. Assume $QX \xrightarrow{X} X$ is a $\mathcal{Q}$-algebra.

- Since $k : \mathcal{L} \rightarrow \mathcal{Q}$ is a monad map, $w : \mathcal{L}X \xrightarrow{k} QX \xrightarrow{x} X$ is an $\mathcal{L}$-algebra.
- By Propositions 1.9, since $\ell : \mathcal{M} \rightarrow \mathcal{Q}$ is a left-semi monad map, $z : \mathcal{M}X \xrightarrow{\ell} QX \xrightarrow{X} X$ is a left-semi $\mathcal{M}$-algebra with 2-cell $\alpha$ where we denote $f_x = z \eta^M$.

$$
\begin{array}{c}
X \\
\downarrow f_x \\
\mathcal{L}X \\
\downarrow \psi_x \\
X
\end{array}
= 
\begin{array}{c}
X \\
\downarrow \eta^M \\
\mathcal{M}X \\
\downarrow \ell \\
\mathcal{Q}X \\
\downarrow \psi_x \\
X
\end{array}
$$

(13)

- We know that $h.\ell = \lambda$ and $h.\ell = 1_{QX}$ and $z = x.\ell$ is a left-semi $\mathcal{M}$-algebra. We deduce $\mathcal{L}X \xrightarrow{w} X \xrightarrow{f} X = \mathcal{L}X \xrightarrow{\lambda} \mathcal{M}X \xrightarrow{z} X$ using the following.
We prove that \( \epsilon \) is in \( \mathcal{L} \)-Alg. We first remark that \( x.\beta = \epsilon.x \). Indeed, by naturality of \( \eta^M \) and of \( \alpha \), we have \( \alpha.\eta^L.x = x.(Qx).\alpha.\eta^L \). Because \( x \) is a \( Q \)-algebra, \( x.\alpha.\eta^L = \mu^Q.\alpha.\eta^L \) and we conclude as \( \mu^Q.\alpha.\eta^L = \beta \).

Then, as \( \beta \) is an \( \mathcal{L} \)-algebra 2-cell by construction and \( x \) is a \( \mathcal{L} \)-algebra, so that \( \epsilon.x \) is an \( \mathcal{L} \)-algebra 2-cell. This can be represented by the lhs 2-cell equality which results in the rhs equality by precomposition by \( \mathcal{L}\eta^Q \). This proves that \( \epsilon \) is an \( \mathcal{L} \)-algebra 2-cell.

Our analysis of the 2-monad \( Q \) involved consideration of left-semi \( M \)-algebras. We can immediately say something about them. Suppose that \( M^+ \) is the result of applying our construction to the map \( \eta : I \to M \) of monads given by the unit. By Theorem 2.12, we deduce the following.

**Proposition 2.13.** \( M^+ \)-Alg\( s \) is isomorphic to \( \text{ls}\-M\text{-Alg}\( s \)

So the 2-category of left-semi \( M \)-algebras is in fact monadic over the base \( \mathcal{K} \).

### 3 The Linear Non-linear 2-Monad

In this section, we show how our theory applies in the case of most immediate interest to us. We take for \( \mathcal{L} \) the 2-monad for symmetric strict monoidal categories: we give a concrete presentation in Subsection 3.1. We take for \( \mathcal{M} \) the 2-monad for categories with strict finite products: we give a concrete presentation in Subsection 3.2. There is an evident map of monads \( \mathcal{L} \to \mathcal{M} \) and in Subsection 3.3, we describe the 2-monad \( Q \) obtained by our construction.

In further work we shall develop general theory to show that this \( Q \) in particular extends from \( \text{CAT} \) to profunctors. This gives a notion of algebraic theory in the sense of Hyland [20] and we shall use that to handle the linear and non-linear substitutions appearing in differential lambda-calculus [12].

#### 3.1 The 2-monad for symmetric strict monoidal categories

For a category \( A \), let \( \mathcal{L}A \) be the following category. The objects are finite sequences \( \langle a_i \rangle_{i \in [n]} \) with \( n \in \mathbb{N} \) and \( a_i \in A \). The morphisms

\[
\langle a_i \rangle_{i \in [n]} \to \langle a'_j \rangle_{j \in [m]}
\]

consist of a bijection \( \sigma : [n] \to [m] \) (so \( n \) and \( m \) are equal) and for each \( j \in [m] \) a map \( a_{\sigma(j)} \to a'_j \) in \( A \). The identity and composition are evident.

\( \mathcal{L} \) extends readily to a \( 2 \)-functor on \( \text{CAT} \) and it has the structure of a 2-monad where \( \eta^\mathcal{L} : A \to \mathcal{L}A \) takes \( a \) to the singleton \( \langle a \rangle \) and \( \mu^\mathcal{L} : \mathcal{L}^2A \to \mathcal{L}A \) acts on objects by concatenation of sequences.

Each \( \mathcal{L}A \) has the structure of a symmetric empty sequence and tensor product is given by concatenation. One can check directly that \( \eta^\mathcal{L} : A \to \mathcal{L}A \) makes \( \mathcal{L}A \) the free symmetric strict monoidal category on \( A \). Moreover to equip \( A \) with the structure of a symmetric strict monoidal category is to give \( A \) an \( \mathcal{L} \)-algebra structure. Maps and 2-cells are as expected so we identify...
\( L, \text{-Alg} \) as the 2-category of strict monoidal categories, strict monoidal functors and monoidal 2-cells.

### 3.2 The 2-monad for categories with products

For a category \( A \), let \( M A \) be the following category. The objects are finite sequences \( \langle a_i \rangle_{i \in [n]} \) with \( n \in \mathbb{N} \) and \( a_i \in A \). The morphisms

\[
\langle a_i \rangle_{i \in [n]} \to \langle a'_j \rangle_{j \in [m]}
\]

consist of a map \( \phi : [m] \to [n] \) and for each \( j \in [m] \) a map \( a_{\phi(j)} \to a'_j \) in \( A \). The identity and composition are evident.

\( M \) extends readily to a 2-functor on CAT and it has the structure of a 2-monad where \( \eta^M : A \to MA \) takes \( a \) to the singleton \( \langle a \rangle \) and \( \mu^M : M^2 A \to MA \) acts on objects by concatenation of sequences.

Each \( MA \) has the structure of a category with strict products: the terminal object is the empty sequence and product is given by concatenation. Again, one can check directly that \( \eta^M : A \to MA \) makes \( MA \) the free category with strict products on \( A \). Again, to equip \( A \) with the structure of a category with strict products is to give \( A \) a \( M \)-algebra structure. Maps and 2-cells are as expected so we identify \( M, \text{-Alg}_\approx \) as the 2-category of categories with strict products, functors preserving these strictly and appropriate 2-cells.

### 3.3 The 2-monad for linear-non-linear substitution

There is a map \( \lambda : \mathcal{L} \to M \) which on objects takes \( \langle a_i \rangle_{i \in [n]} \in \mathcal{L}A \) to \( \langle a_i \rangle_{i \in [n]} \in MA \) and includes the maps in \( LA \) into those in \( MA \) in the obvious way. It accounts for the evident fact that every category with strict product is a symmetric strict monoidal category. We describe the 2-monad \( Q \) obtained from \( \lambda \) by our colimit construction.

For a category \( A \), \( QA \) is the following category. The objects are \( \langle a^{e_i} \rangle_{i \in [n]} \) with \( n \in [n] \), \( a_i \in A \) and the indices \( e_i \) chosen from the set \( \{ L, M \} \) (\( L \) indicates linear and \( M \) non-linear). For \( a = \langle a^{e_i} \rangle_{i \in [n]} \), write \( L_a \) for \( \{ i \mid e_i = L \} \). Then a morphism

\[
\langle a_i \rangle_{i \in [n]} \to \langle a'_j \rangle_{j \in [m]}
\]

is given by first a map \( \phi : [m] \to [n] \) satisfying the condition

\[
\phi^{-1}(L_a) \subseteq L_{a'} \quad \text{and} \quad \phi|_{\phi^{-1}(L_a)} : \phi^{-1}(L_a) \to L_a \text{ is a bijection};
\]

and secondly by for each \( j \in [m] \), a map \( a_{\phi(j)} \to a'_j \) in \( A \).

\( Q \) extends readily to a 2-functor on CAT and it has the structure of a 2-monad as follows. The unit \( \eta^Q : A \to QA \) takes \( a \in A \) to \( \langle a^L \rangle \). The multiplication \( \mu^Q : A \to QA \) acts by concatenating the objects and with the following behaviour on indices: objects of \( Q^2 A \) have shape

\[
\langle \langle \ldots \rangle \langle \ldots \rangle a^e \ldots \rangle \eta \rangle \langle \ldots \rangle
\]

so that each \( a \in A \) has two indices; in the concatenated string in \( QA \) a has index \( L \) just when both \( e \) and \( \eta \) are \( L \).

One can now readily see the structure on \( QA \) involved in its definition.

- \( QA \) is clearly an \( L \)-algebra and \( k : \mathcal{L}A \to QA \) sends \( \langle a_1, \ldots, a_n \rangle \) to \( \langle a^L_1, a^L_n \rangle \)
- \( \ell : MA \to QA \) sends \( \langle a_1, \ldots, a_n \rangle \) to \( \langle a^M_1, \ldots, a^M_n \rangle \) given by the identity on \([n]\) and is evidently an \( L \)-algebra map.
Now, we can use our Theorem 2.12 to give a description of what a $Q$-algebra is in this case. It is an object of our structure category described in Subsection 2.3. That means it is a symmetric monoidal category $X$ equipped with a strictly idempotent comonad $f : X \to X$ with $\epsilon : f \Rightarrow 1_X$ and with that structure in the 2-category of symmetric monoidal categories and strict maps; it is such that the full subcategory of fixpoints of $f$ is equipped with the structure of a category with products; moreover the effect of tensoring objects of $X$ and then applying $f$ is equal to that of first applying $f$ and then taking the product.

3.4 Next steps

Starting from the observation that the 2-monad $L$ for strict monoidal categories and the 2-monad $M$ for categories with strict products can be combined into a 2-monad $Q$ mixing the two related structures, we have introduced a new notion for combining 2-monads as the colimit of a map of monads. We have proved that our construction gives rise to a 2-monad in Theorem 2.9 and characterised its algebras in Theorem 2.12.

Our next step will be to give conditions under which $Q$ admits an extension to a pseudomonad on $\textbf{Prof}$ [16]. We draw attention to the following issue which we need to address. It is clear from [16] that the 2-monad $L$ for symmetric strict monoidal categories and $M$ for categories with strict products admit extensions to pseudomonads on $\textbf{Prof}$. However, we cannot use our colimit construction at this level as we only have access to bicolimits. All the same, the characterisation of Theorem 2.12 will be useful to describe pseudo $Q$-algebras. Then one can show that the presheaf construction has a lifting to pseudo $Q$-algebras and so deduce by [16] the wanted extension of $Q$ to $\textbf{Prof}$.

The extension of $Q$ to $\textbf{Prof}$ will give a notion of Linear Non-Linear multicategory which will serve as a basis for describing the substitution structure at play in differential $\lambda$-calculus [17]. In parallel, we shall compare our approach to existing approaches to the combination of linearity and non-linearity which arises from Linear Logic [3]. We hope to show that starting from a Linear Non-Linear category, we can obtain a $Q$-algebra (or at least a $Q$-multicategory) which accounts for the usual practice of modelling Linear Non-Linear calculi. This is not a straightforward issue and indications are given in the Appendix A.

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A COMPARISON WITH EXISTING APPROACHES

Readers could certainly be forgiven for thinking that this paper presents foundations for an approach to differentiation essentially equivalent to existing ones [6, 7, 12] which are based on ideas coming from Linear Logic. So here we explain in rough outline why that seems not to be the case. The issues are particularly clear if one considers the approach to Linear Logic via Linear-Non-linear Lambda Calculus (LNL) due originally to Benton [3]. What we shall do here is compare what one naturally derives from that with a pseudo version of the structure 2-category of this paper. The latter is central to our next step (see Paragraph 3.4) of extending the 2-monad $Q$ to a pseudomonad on $Prof$ and we shall give full details in a further paper.

The pseudo version of the structure category is essentially obtained by making the following modifications to the definition of the Structure category in Section 2.3.

- Replace the the $L$-algebra $w : LX \to X$ by a pseudo $L$-algebra. This involves a couple of isomorphism 2-cells satisfying standard coherence conditions.
- Replace the left-semi $M$-algebra $z : MX \to X$ and $e$ by a pseudo such. That introduces a further isomorphism 2-cell and further coherence. Inter alia this 2-cell gives rise to an evident 2-cell $\delta : f \Rightarrow f^2$.
- Replace the $L$-algebra conditions on $f$ and $e$ by the conditions that $f$ is a pseudo map, which introduces a further isomorphism 2-cell and some coherence, together with the condition that $e$ and $\delta$ are $L$-algebra 2-cells.
- There are then two natural pseudo left-semi $L$-algebra structures available. Replace the equality condition with the structure of an isomorphism between the two. That is a 2-cell satisfying yet further evident conditions.
It is not completely trivial that what we have described makes sense but for the purposes of this paper all one really needs to appreciate is the overall shape of the definition. We shall see that a categorical model for LNL naturally gives rise to a structure of a similar shape but with clear differences. It will help in appreciating that to have in mind the fact that though the 2-cells introduced above are isomorphisms, they have a natural direction which is that corresponding to co-lax algebras and co-lax maps.

Now we give a brief description of the basic structure of the standard categorical models for LNL. We omit the closed structure as it plays no role here. In a notation adapted to this paper, a categorical model for LNL has at its basis a symmetric monoidal category \( X \), a category \( Y \) equipped with products, the two being equipped with a monoidal adjunction \( r : X \to Y \), \( s : Y \to X \) with \( s \dashv r \). Let us take \( \mathcal{L} \) and \( \mathcal{M} \) the 2-monads for symmetric monoidal categories and categories with products and consider what we naturally get from a model for LNL.

1. First of all \( X \) is a symmetric monoidal category so we have an \( \mathcal{L} \)-algebra structure \( w : \mathcal{L}X \to X \). We may as well take this to be a strict algebra though nothing stand or falls by that.
2. Second \( Y \) is a category with products so that we have an \( \mathcal{M} \)-algebra structure \( y : \mathcal{M}Y \to Y \). We can try to transfer that from \( Y \) to \( X \) by considering \( z = s\mu r : MX \to \mathcal{M}Y \to Y \to X \). Using the unit and counit of the adjunction we find that we get a co-lax \( \mathcal{M} \)-algebra structure. That should give us pause but maybe we can shrug that off for the moment. If we assume that the unit of the adjunction is an isomorphism so that \( s : Y \to X \) is (essentially) a coreflective subcategory then the multiplication square is an isomorphism. (And we might be encouraged by the fact that at this point the 2-cells are appearing in the right direction!)
3. Now we look at the endomap \( f \). Here it is equal at the functor level to the composite \( sr : X \to Y \to X \), which lies in a paragraph of this form:

```
\[
\begin{array}{c}
\mathcal{L}X \\
\downarrow w
\end{array} \xrightarrow{r} \begin{array}{c}
\mathcal{L}Y \\
\downarrow \lambda
\end{array} \xrightarrow{s} \begin{array}{c}
\mathcal{L}X
\end{array}
\]
\[
\begin{array}{c}
X \\
\downarrow r
\end{array} \xrightarrow{s} \begin{array}{c}
Y \\
\downarrow \mu
\end{array} \xrightarrow{Y} \begin{array}{c}
X
\end{array}
\]
```

Now, by an old observation of Brian Day, the left adjoint \( s \) is a strong map of symmetric monoidal categories so the 2-cell \( \overline{s} \) is an isomorphism. But there is no way the 2-cell \( \overline{s} \) is an isomorphism. So what we have is that \( f \) is naturally a lax map of \( \mathcal{L} \)-algebras and nothing more. That is a distinguishing feature and there is no getting round it. We also have from the previous point the 2-cells \( \epsilon : f \to 1 \) and \( \delta : f \to f^2 \) and it follows immediately from the structure of a monoidal adjunction that these are 2-cell maps.

4. Finally we can look for two comparable structures. On the one hand we can derive from \( z \) and its associated data a co-lax \( \mathcal{L} \)-algebra structure with structure map \( z\lambda : \mathcal{L}X \to MX \to X \). That is fine. On the other hand we can try to find a structure for the map \( fw : \mathcal{L}X \to X \to X \). Again we get a co-lax \( \mathcal{L} \)-algebra structure. Both the structures come from the unit and counit of the adjunction \( s \dashv r \). So one might hope they were isomorphic but the natural 2-cell from \( z\lambda \) to \( fw \) which is a map of structures is sadly not invertible: it makes use of the 2-cell \( \overline{f} \) above. So again at this point we do not quite get what we have in our structure 2-category.
The crucial difference between our setting (see Subsection 3.3) and the Linear Logic setting is that our idempotent comonad is a strict monoidal functor. In Linear Logic, the ! whether or not idempotent is only lax monoidal. We put off explaining what we think is going on to a further paper but very simply our view is this. When using the traditional approach to Linear Non-Linear calculi to axiomatise differentiation the essential work uses a hidden structure of $Q$-multicategory in the sense implied by this paper. But evidently - if we are right - it cannot be obtained in the direct way we have just sketched.