Heavy quark masses from QCD sum rules

C. A. Dominguez

Institute of Theoretical Physics and Astrophysics, University of Cape Town, Rondebosch 7700, South Africa

Abstract

I review recent determinations of the (on-shell) charm- and beauty-quark masses in the framework of relativistic and non-relativistic ratios of Laplace transform QCD moment sum rules. The validity of the non-relativistic version of QCD sum rules in this particular application is discussed.
With the advent of the Heavy Quark Effective Theory (HQET) there has been a revived interest in applications of QCD sum rules to the heavy quark sector. The purpose being the update of old determinations, as well as the performance of new calculations to extract accurate values of various dynamical quantities entering the HQET. Chief of these quantities is the ubiquitous heavy-quark mass. I review here recent determinations of the (on-shell) charm- and beauty-quark masses [1] performed by confronting very accurate experimental data on the charmonium and the upsilon systems [2] with ratios of relativistic and non-relativistic Laplace transform QCD moments. The latter theoretical framework, suggested by Bertlmann [3], offers several advantages, e.g. radiative and non-perturbative corrections are well under control, and the non-relativistic limit follows quite naturally from quantum mechanical analogues [4]. This version of QCD sum rules leads to an expansion in powers of the inverse of the heavy quark mass which allows one to test the range of validity of the non-relativistic limit, and more generally, to assess the role of mass corrections. This might be of interest for calculations based on the simplifying assumption $\Lambda_{QCD}/m_Q \ll 1$. Also, a comparison of the results from the relativistic and the non-relativistic determinations provides an estimate of the systematic uncertainties affecting this technique.

I begin by considering the two-point function

$$\Pi_{\mu\nu}(q) = i \int d^4 x \exp(i qx) \langle 0|T(V_{\mu}(x)V_{\nu}^+(0))|0 \rangle = (-g_{\mu\nu}q^2 + q_{\mu}q_{\nu})\Pi(q^2) ,$$

(1)

with $V_{\mu}(x) = \bar{Q}(x)\gamma_{\mu}Q(x)$. The function $\Pi(q^2)$ has been calculated in perturbative QCD at the two-loop level [5], with its imaginary part given by

$$\frac{1}{\pi} \text{Im} \Pi(s)|_{QCD} = \frac{1}{8\pi^2} v(3 - v^2) \left\{ 1 + \frac{4\alpha_s}{3} \left[ \frac{\pi}{2v} - \frac{(v + 3)(\pi - 3)}{4(4\pi)} \right] \right\} \theta(s - 4m^2_Q) ,$$

(2)

where $v = \frac{\sqrt{1 - 4m^2_Q/s}}{2}$, and $m_Q$ is the charm- or the beauty-quark on-shell mass: $m_Q = m_Q(Q^2 = m^2_Q)$. The leading non-perturbative term in the operator product expansion of $\Pi(q^2)$ involves the gluon condensate, i.e.

$$\Pi(s)|_{NP} = \frac{1}{48} \times \left[ \frac{3(v^2 + 1)(1 - v^2)^2}{2v^5} \ln \frac{1 + v - 3v^4 - 2v^2 + 3}{v^2} \right] \frac{\alpha_s G^2}{\pi} ,$$

(3)

The function $\Pi(q^2)$ satisfies a once-subtracted dispersion relation, and the subtraction constant can be disposed of e.g. by taking the Laplace transform

$$\Pi(\sigma) = \int_0^\infty ds \exp(-\sigma s) \text{Im} \Pi(s) .$$

(4)

The quantity of interest to us here is the ratio of the first two Laplace moments, which can be expressed as

$$R(\sigma) = -\frac{d}{d\sigma} \ln \Pi(\sigma) .$$

(5)
Substituting Eq.(2) into Eq.(4) one can carry out the integration analytically. The result for the ratio Eq.(5), with $\omega = 4m_Q^2\sigma$, is

$$\mathcal{R}(\omega) = 4m_Q^2 \left[ 1 - \frac{A'(\omega)}{A(\omega)} - \frac{a'(\omega)\alpha_s + b'(\omega)\phi}{1 + a(\omega)\alpha_s + b(\omega)\phi} \right],$$

where

$$\alpha_s(Q^2) = \frac{-2\pi}{\beta_1 \ln Q^2/m_Q^2},$$

$$\phi = \frac{\pi}{36} < \frac{\alpha_s G^2}{m^4_Q},$$

$$G(b, c, \omega) = \frac{\omega^{-b}}{\Gamma(c)} \int_0^\infty dt \, t^{c-1} e^{-t/(1 + t/\omega)} - b.$$

The function $G(b, c, \omega)$ is related to the Whittaker function $W_{\lambda, \mu}(\omega)$ through

$$G(b, c, \omega) = \omega^{-b/2} e^{\omega/2} W_{\lambda, \mu}(\omega).$$
with \( \mu = (c - b)/2 \), and \( \lambda = (1 - c - b)/2 \).

The above expressions involve no approximations, other than the two-loop perturbative expansion, and the truncation of the operator product expansion beyond the leading non-perturbative term. We shall refer to Eq.(6) as the fully relativistic Laplace ratio. In the non-relativistic (heavy quark-mass) limit, the Laplace transform Eq.(4) becomes

\[
\Pi(\tau) = \int_0^\infty dE \exp(-\tau E) \text{Im} \Pi(E),
\]

where \( \tau = 4m_Q\sigma \), and \( s = (2m_Q + E)^2 \) so that \( E \geq 0 \). The Laplace ratio Eq.(5) is now given by

\[
\mathcal{R}(\tau) = 2m_Q - \frac{d}{d\tau} \ln \Pi(\tau).
\]

After expanding the functions \( G(b, c, \omega) \) entering Eq.(6), one obtains the non-relativistic ratio

\[
\mathcal{R}(\tau) = 2m_Q \left\{ 1 + \frac{3}{4} \frac{1}{m_Q\tau} \left[ 1 - \left( \frac{2}{3} + \frac{3}{8\pi^2} \right) \frac{1}{m_Q\tau} + \frac{1}{32} \left( 107 + \frac{51}{\pi^2} \right) \frac{1}{m_Q^2\tau^2} \right] \right. \\
- \frac{\sqrt{\pi}}{3} \frac{\alpha_s}{\sqrt{m_Q^2\tau}} \left[ 1 - \left( \frac{2}{3} + \frac{3}{8\pi^2} \right) \frac{1}{m_Q\tau} + \frac{1}{32} \left( 107 + \frac{51}{\pi^2} \right) \frac{1}{m_Q^2\tau^2} \right] \\
+ \frac{\pi}{48} \frac{\tau^2}{m_Q^2} \langle \alpha_s G^2 \rangle \left[ 1 + \frac{4}{3} \frac{1}{m_Q\tau} - \frac{5}{12} \frac{1}{m_Q^2\tau^2} \right] \right\},
\]

where the expansion has been truncated at the next-to-next to leading order in \( 1/m_Q \).

Notice that the appearance of \( \sqrt{m_Q} \) above is only an artifact of the change of variables; written in terms of \( \sigma \), Eq.(20) contains no such term. The theoretical ratios of the first two Laplace moments Eqs.(6) and (20) must now be confronted with a corresponding ratio involving the experimental data on the \( J/\psi \) and the \( \Upsilon \) systems. In the case of the \( J/\psi \) one parametrizes the data by a sum of two narrow resonances below \( D\bar{D} \) threshold, followed by a hadronic continuum modelled by perturbative QCD. In the case of the \( \Upsilon \), three narrow resonances below the \( B\bar{B} \) threshold are required, at least in principle. One obtains

\[
\Pi(\sigma)_{\text{EXP}} = \frac{3}{4\pi} \frac{1}{e_Q^2 a_{EM}^2} \sum_V \Gamma_V^{ee} M_V \exp(-\sigma M_V^2) + \frac{1}{\pi} \int_{s_0}^\infty ds \exp(-\sigma s) \text{Im} \Pi(s)|_{\text{QCD}}.
\]

The experimental ratio is then calculated using Eq.(21) in Eq.(5). The continuum threshold \( s_0 \) is chosen at or below the \( D\bar{D} \) (\( B\bar{B} \)) threshold. Reasonable changes in the value of \( s_0 \) have essentially no impact on the results, as \( \Pi(\sigma) \) is saturated almost entirely by the first two \( J/\psi \) narrow resonances in the case of charm, and the first \( \Upsilon \) state in the case of beauty. In the theoretical ratios the following current values of the QCD parameters
have been used: \( \Lambda = 200 - 300 \text{ MeV} \), for four flavours, and \( \Lambda = 100 - 200 \text{ MeV} \), for five
flavours \[2\], and \( < \alpha_s G^2 > = 0.063 - 0.19 \text{ GeV}^4 \[3\].

In the case of charm, one finds that theoretical and experimental ratios match in the
wide sum rule window: \( \sigma \simeq 0.8 - 1.5 \text{ GeV}^{-2} \), for \( m_c = 1.39 - 1.46 \text{ GeV} \) in the fully
relativistic case, and \( \sigma \simeq 0.6 - 0.8 \text{ GeV}^{-2} \), \( m_c = 1.40 - 1.53 \text{ GeV} \) in the non-relativistic
case. For values of \( \sigma \) inside the sum rule window, the hierarchy of the various terms in
the non-relativistic Laplace ratio (20) guarantees a fast convergence. In fact, the leading
correction in \( 1/m_c \) is at the 15-20\% level, the radiative correction and the non-perturbative
contribution amount both to less than 10\% . At the same time, the next, and next-to-
next to leading (in \( 1/m_c \)) terms everywhere in Eq.(20) are safely small, as it can be easily
verified from Eq.(20) noticing that if \( \sigma \simeq 1/2 \text{ GeV}^{-2} \), then \( \tau \simeq 2m_c \). Clearly, the complete
analysis at the level of accuracy of these next-to-leading mass corrections would require
the evaluation of the perturbative \( O(\alpha_s^2) \) terms. Combining the results from both versions
of the Laplace ratios, leads to the result

\[
m_c(Q^2 = m_c^2) = 1.46 \pm 0.07 \text{ GeV} .
\] (22)

In the case of beauty, at small and intermediate values of \( \sigma \) the \( \Upsilon(1S) \) provides the
bulk of the hadronic contribution, i.e. the \( \Upsilon(2S) \), \( \Upsilon(3S) \), and the continuum represent
a small correction, below the spread in the theoretical ratio due to variations in \( \Lambda \) and
in \( < \alpha_s G^2 > \). Theoretical and experimental ratios match inside the wide regions: \( \sigma \simeq
0.4 - 0.8 \text{ GeV}^{-2} \), for \( m_b = 4.63 - 4.67 \text{ GeV} \) in the fully relativistic case, and \( \sigma \simeq 0.20 -
0.35 \text{ GeV}^{-2} \), for \( m_b = 4.69 - 4.77 \text{ GeV} \) in the non-relativistic one. In the latter case
all correction terms in Eq.(20) are at the safe level of a few percent. The subleading
quark mass corrections, though small, are important. For instance, the term of order
\( O(\alpha_s/m_b \sqrt{m_b}) \) in Eq.(20) is of the same size and sign as the non-perturbative term.
Hence, it is not fully justified to keep the latter and ignore the former. After combining
the results from both methods one predicts

\[
m_b(Q^2 = m_b^2) = 4.70 \pm 0.07 \text{ GeV} .
\] (23)

When comparing the results reported here, Eqs.(22)-(23), with previous determinations
based on various versions of QCD sum rules \[3\], \[8\] - \[9\], it is important to know which
values of \( \Lambda \) and \( < \alpha_s G^2 > \) have been used, as well as which renormalization point has been
chosen, e.g. some authors determine \( m_Q(Q^2 = -m_Q^2) \), which is related to the on-shell
mass \( m_Q(m_Q^2) \) through

\[
m_Q^2(m_Q^2) = m_Q^2(-m_Q^2)(1 + \frac{4 \ln 2}{\pi} \alpha_s) .
\] (24)

The results from the present method are in very good agreement with those of \[3\] and \[8\].
Comparison with other analyses \[9\] is often made difficult by the lack of information on
the specific values used for \( \Lambda \) and the gluon condensate.
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