LECTURE NOTES ON NON-COMMUTATIVE ALGEBRAIC GEOMETRY AND NONCOMMUTATIVE TORI

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INTRODUCTION

I would like to thank all the organizers, namely, M. Khalkhali, M. Marcolli, M. Shahshahani and M. M. Sheikh-Jabbari, of the International Workshop on Noncommutative Geometry, 2005 for giving me the opportunity to speak.

In section 1 we shall browse through some interesting definitions and constructions which will be referred to later on. In section 2 we shall discuss non-commutative projective geometry as initiated by Artin and Zhang in [AZ94]. This section is rather long and the readers can easily skip over some details. In section 3 we shall provide a brief overview of the algebraic aspects of noncommutative tori, which comprise the most widely studied class of noncommutative differentiable manifolds. These notes are not entirely self-contained and should be read in tandem with those of B. Noohi for background material and of J. Plazas for a better understanding of the topological and differential aspects of non-commutative tori.

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1. Some Preliminaries

In a paper entitled *Some Algebras Associated to Automorphisms of Elliptic Curves* Artin, Tate and Van den Bergh [TVdB90] gave a nice description of non-commutative algebras which should, in principle, be algebras of functions of some nonsingular “non-commutative schemes”. In the commutative case, nonsingularity is reflected in the regularity of the ring. However, this notion is insufficient for non-commutative purposes. So Artin and Schelter gave a stronger regularity condition which we call the *Artin-Schelter (AS)-regularity* condition. The main result of the above-mentioned paper says that AS-regular algebras of dimension 3 (global dimension) can be described neatly as some algebras associated to automorphisms of projective schemes, mainly elliptic curves. Also such algebras are both left and right Noetherian. This subsection is entirely based on the contents of [TVdB90].

To begin with, we fix an algebraically closed field $k$ of characteristic 0. We shall mostly be concerned with $\mathbb{N}$-graded $k$-algebras $A = \bigoplus_{i \geq 0} A_i$, that are finitely generated in degree 1, with $A_0$ finite dimensional as a $k$-vector space. Such algebras are called *finitely graded* for short, though the term could be a bit misleading at first sight. A finitely graded ring is called *connected graded* if $A_0 = k$. $A_+$ stands for the two-sided augmentation ideal $\bigoplus_{i > 0} A_i$.

**Definition 1.1. (AS-regular algebra)**

A connected graded ring $A$ is called Artin-Schelter (AS) regular of dimension $d$ if it satisfies the following conditions:

1. $A$ has global dimension $d$.
2. $\text{GKdim}(A) < \infty$.
3. $A$ is AS-Gorenstein.

It is worthwhile to say a few words about *Gelfand-Kirillov dimension* (GKdim) and the *AS-Gorenstein* condition of algebras.

Take any connected graded $k$-algebra $A$ and choose a finite dimensional $k$-vector space $V$ such that $k[V] = A$. Now set $F^n A = k + \sum_{i=1}^n V^i$ for $n \geq 1$. This defines a filtration of $A$. Then the GKdim($A$) is defined to be

$$\text{GKdim}(A) = \lim sup_n \frac{\ln(\dim_k F^n A)}{\ln(n)}.$$ 

Of course, one has to check that the definition does not depend on the choice of $V$.

**Remark 1.2.** Bergman [KL85] has shown that GKdim can be any real number $\alpha \geq 2$. However, if GKdim $\leq 2$, then it is either 0 or 1.

There are some equivalent formulations of the AS-Gorenstein condition available in literature. We shall be content by saying the following:
Definition 1.3. (AS-Gorenstein condition)

A connected graded \( k \)-algebra \( A \) of global dimension \( d < \infty \) is AS-Gorenstein if

\[
\text{Ext}_A^i(k, A) = 0 \quad \text{for} \quad i \neq d \quad \text{and} \quad \text{Ext}_A^d(k, A) \simeq k
\]

All regular commutative rings are AS-Gorenstein, which supports our conviction that the AS-Gorenstein hypothesis is desirable for non-commutative analogues of regular commutative rings.

Further, note that the usual Gorenstein condition (for commutative rings) requires that they be Noetherian of finite injective dimension as modules over themselves.

Now we take up the task of describing the minimal projective resolution (\( \ast \)) of an AS-regular algebra of dimension \( d = 3 \). As a fact, let us also mention that the global dimension of a graded algebra is equal to the projective dimension of the left module \( Ak \). Let

\[
0 \to P^d \to \ldots \to f_2 \to P^1 \xrightarrow{f_1} P^0 \to Ak \to 0
\]

be a minimal projective resolution of the left module \( Ak \). \( P^0 \) turns out to be \( A \); \( P^1 \) and \( P^2 \) need an investigation into the structure of \( A \) for their descriptions. Suppose \( A = T/I \), where \( T = k\{x_1, \ldots, x_n\} \) is a free associative algebra generated by homogeneous elements \( x_i \) with degrees \( l_{1j} \) (also assume that \( \{x_1, \ldots, x_n\} \) is a minimal set of generators). Then

\[
P^1 \simeq \bigoplus_{j=1}^{n} A(-l_{1j})
\]

The map \( P^1 \to P^0 \), denoted \( x \), is given by right multiplication with the column vector \( (x_1, \ldots, x_n)^t \).

Coming to \( P^2 \), let \( \{f_j\} \) be a minimal set of homogeneous generators for the ideal \( I \) such that \( \deg f_j = l_{2j} \). In \( T \), write each \( f_j \) as

\[
f_j = \sum_j m_{ij}x_j
\]

where \( m_{ij} \in T_{l_{2j} - l_{1j}} \). Let \( M \) be the image in \( A \) of the matrix \( (m_{ij}) \). Then

\[
P^2 \approx \bigoplus_{j} A(-l_{2j})
\]

and the map \( P^2 \to P^1 \), denoted \( M \), is just right multiplication by the matrix \( M \).

In general, it is not so easy to interpret all the terms of the resolution (1). However, for a regular algebra of dimension 3, the resolution looks like

\[
0 \to A(-s - 1) \xrightarrow{x^t} A(-s)^r \xrightarrow{M} A(-1)^r \xrightarrow{x} A \to Ak \to 0
\]

where \( (r, s) = (3, 2) \) or \( (2, 3) \). Thus such an algebra has \( r \) generators and \( r \) relations each of degree \( s \), \( r + s = 5 \). Set \( g = (x^t)M \); then

\[
g^t = ((x^t)M)^t = QMx = Qf
\]

for some \( Q \in GL_r(k) \).

Now, with some foresight, we introduce a new definition, that of a standard algebra, in which we extract all the essential properties of AS-regular algebras of dimension 3.
Definition 1.4. An algebra $A$ is called standard if it can be presented by $r$ generators $x_j$ of degree 1 and $r$ relations $f_i$ of degree $s$, such that, with $M$ defined by (3), $(r, s) = (2, 3)$ or $(3, 2)$ as above, and there is an element $Q \in GL_r(k)$ such that (6) holds.

Remark 1.5. For a standard algebra $A$, (5) is just a complex and if it is a resolution, then $A$ is a regular algebra of dimension 3.
Twisted Homogeneous Coordinate Rings

Here we sketch a very general recipe for manufacturing interesting non-commutative rings out of a completely “commutative geometric” piece of datum, called an abstract triple, which turns out to be an isomorphism invariant for “AS-regular algebras”.

**Definition 1.6.** An abstract triple $T = (X, \sigma, \mathcal{L})$ is a triple consisting of a projective scheme $X$, an automorphism $\sigma$ of $X$ and an invertible sheaf $\mathcal{L}$ on $X$.

It is time to construct the Twisted Homogeneous Coordinate Ring $B(T)$ out of an abstract triple. For each integer $n \geq 1$ set

$$(7) \quad \mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$$

where $\mathcal{L}^\sigma := \sigma^* \mathcal{L}$. The tensor products are taken over $\mathcal{O}_X$ and we set $\mathcal{L}_0 = \mathcal{O}_X$. As a graded vector space, $B(T)$ is defined as

$$B(T) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n)$$

For every pair of integers $m, n \geq 0$, there is a canonical isomorphism

$$\mathcal{L}_m \otimes_k \mathcal{L}_n \cong \mathcal{L}_{m+n}$$

and hence it defines a multiplication on $B(T)$

$$H^0(X, \mathcal{L}_m) \otimes_k H^0(X, \mathcal{L}_n) \rightarrow H^0(X, \mathcal{L}_{m+n}).$$

**Example 1.** Let us compute (more precisely, allude to the computation of) the twisted homogeneous coordinate ring in a very simple case. Let $T = (\mathbb{P}^1, \mathcal{O}(1), \sigma)$, where $\sigma(a_0, a_1) = (qa_0, a_1)$ for some $q \in k^*$. It is based on our understanding of Example 3.4 of [SvdB01].

We may choose a parameter $u$ for $\mathbb{P}^1$, so that the standard affine open cover consisting of $U = \mathbb{P}^1 \setminus \{\infty\}$ and $V = \mathbb{P}^1 \setminus \{0\}$ has rings of regular functions $\mathcal{O}(U) = k[u]$ and $\mathcal{O}(V) = k[u^{-1}]$ respectively. Now we can identify $\mathcal{O}(1)$ with the sheaf of functions on $\mathbb{P}^1$ which have at most a simple pole at infinity; in other words, it is the subsheaf of $k(u) = k(\mathbb{P}^1)$ generated by $\{1, u\}$. It can be checked that $H^0(X, \mathcal{O}(n))$ is spanned by $\{1, u, \ldots, u^n\}$ and that, as a graded vector space $B(\mathbb{P}^1, \text{id}, \mathcal{O}(1)) = k\langle x, y \rangle$ (the free algebra over $k$ generated by $x$ and $y$ and not the usual polynomial ring), where $x = 1$ and $y = u$, thought of as elements of $B_1 = H^0(X, \mathcal{O}(1))$. It should be mentioned that $\sigma$ acts on the rational functions on the right as $f^\sigma(p) = f(\sigma(p))$ for any $f \in k(\mathbb{P}^1)$ and $p \in \mathbb{P}^1$. From the presentation of the algebra it is evident that $\mathcal{O}(1)^\sigma \cong \mathcal{O}(1)$. So as a graded vector space $B(T) \cong B(\mathbb{P}^1, \text{id}, \mathcal{O}(1))$. The multiplication is somewhat different though.

$$y \cdot x = y \otimes x^\sigma = u \otimes 1^\sigma = u \otimes 1 = u \in H^0(\mathbb{P}^1, \mathcal{O}(2)).$$

On the other hand,

$$x \cdot y = x \otimes y^\sigma = 1 \otimes u^\sigma = 1 \otimes qu = qu \in H^0(\mathbb{P}^1, \mathcal{O}(2)).$$

So we find a relation between $x$ and $y$, namely, $x \cdot y - qy \cdot x = 0$ and a little bit more work shows that this is the only relation. So the twisted homogeneous coordinate ring associated to $T$ is

$$B(T) = k\langle x, y \rangle/(x \cdot y - qy \cdot x).$$
A good reference for a better understanding of these rings is [AVdB90].
A cursory glance at Grothendieck Categories

For the convenience of the reader let me say a few things about a gener

ator of a category. An object \( G \) of a category \( C \) is called a generator if, given a pair of morphisms \( f, g : A \rightarrow B \) in \( C \) with \( f \neq g \), there exists an \( h : G \rightarrow A \) with \( fh \neq gh \) (more briefly, \( \text{Hom}(G, -) : C \rightarrow \text{Set} \) is a faithful functor). A family of objects \( \{G_i\}_{i \in I} \) is called a generating set if, given a pair of morphisms \( f, g : A \rightarrow B \) with \( f \neq g \), there exists an \( h_i : G_i \rightarrow A \) for some \( i \in I \) with \( fh_i \neq gh_i \).

Strictly speaking, this is a misnomer. In a cocomplete category (i.e., closed under all coproducts), a family of objects \( \{G_i\}_{i \in I} \) forms a generating set if and only if the coproduct of the family forms a generator.

Remark 1.7. Let \( C \) be a cocomplete abelian category. Then an object \( G \) is a generator if and only if, for any object \( A \in \text{Ob}(C) \), there exists an epimorphism

\[
G^\oplus I \rightarrow A
\]

for some indexing set \( I \).

Definition 1.8. Grothendieck Category

It is a (locally small) cocomplete abelian category with a generator and satisfying, for every family of short exact sequences indexed by a filtered category \( I \) [i.e., \( I \) is nonempty and, if \( i, j \in \text{Ob}(I) \) then \( \exists k \in \text{Ob}(I) \) and arrows \( i \rightarrow k \) and \( j \rightarrow k \), and for any two arrows \( i \rightarrow^u j \rightarrow k \) \( \in \text{Ob}(I) \) and an arrow \( w : j \rightarrow k \) such that \( wu = wv \) (think of a categorical formulation of a directed set)].

\[
0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0
\]

the following short sequence is also exact

\[
0 \rightarrow \colim_{i \in I} A_i \rightarrow \colim_{i \in I} B_i \rightarrow \colim_{i \in I} C_i \rightarrow 0
\]

i.e., passing on to filtered colimits preserves exactness. This is equivalent to the sup condition of the famous \textbf{AB5 Property}.

Remark 1.9. The original \textbf{AB5 Property} requires the so-called \textit{sup} condition, besides cocompleteness. An abelian category satisfies \textit{sup} if

for any ascending chain \( \Omega \) of subobjects of an object \( M \), the supremum of \( \Omega \) exists; and for any subobject \( N \) of \( M \), the canonical morphism

\[
\text{sup} \{L \cap N | L \in \Omega\} \rightarrow (\text{sup} \Omega) \cap N
\]

is an isomorphism. Hence, another definition of a Grothendieck category could be a cocomplete abelian category, having a generator and satisfying the sup condition, i.e., an \textbf{AB5} category with a generator.

Example 2. (Grothendieck Categories)

1. \( \text{Mod}(R) \), where \( R \) is an associative ring with unity.
2. The category of sheaves of \( R \)-modules on an arbitrary topological space.
3. In the same vein, the category of abelian pre-sheaves on a site \( T \). Actually this is just \( \text{Funct}(T^{\text{op}}, \text{Ab}) \).
4. QCoh(X), where X is a quasi-compact and quasi-separated scheme. (A morphism of schemes \( f : X \to Y \) is called quasi-compact if, for any open quasi-compact \( U \subseteq Y \), \( f^{-1}(U) \) is quasi-compact in X and it becomes quasi-separated if the canonical morphism \( \delta_f : X \to X \times_Y X \) is quasi-compact. A scheme X is called quasi-compact (resp. quasi-separated) if the canonical unique morphism \( X \to \text{Spec}(\mathbb{Z}) \), \( \text{Spec}(\mathbb{Z}) \) being the final object, is quasi-compact (resp. quasi-separated)).

Grothendieck categories have some remarkable properties which make them amenable to homological arguments.

1. Grothendieck categories are complete i.e., they are closed under products.
2. In a Grothendieck category every object has an injective envelope, in particular there are enough injectives.

The deepest result about Grothendieck categories is given by the following theorem.

**Theorem 1.10.** [Gabriel,Popescu] Let \( C \) be a Grothendieck category and let \( G \) be a generator of \( C \). Put \( S = \text{End}(G) \). Then the functor

\[
\text{Hom}(G, -) : C \to \text{Mod}(S^{op})
\]

is fully faithful (and has an exact left adjoint).

**Justification for bringing in Grothendieck Categories**

We begin by directly quoting Manin [Man88] - “...Grothendieck taught us, to do geometry you really don’t need a space, all you need is a category of sheaves on this would-be space.” This idea gets a boost from the following reconstruction theorem.

**Theorem 1.11.** [Gabriel,Rosenberg [Ros98]] Any scheme can be reconstructed uniquely up to isomorphism from the category of quasi-coherent sheaves on it.

**Remark 1.12.** When it is known in advance that the scheme to be reconstructed is an affine one, we can just take the centre of the category, which is the endomorphism ring of the identity functor of the category. More precisely, let \( X = \text{Spec} A \) be an affine scheme and let \( A \) be the category of quasi-coherent sheaves on \( X \), which is the same as \( \text{Mod}(A) \). Then the centre of \( A \), denoted \( \text{End}(\text{Id}_A) \), is canonically isomorphic to \( A \). [The centre of an abelian category is manifestly commutative and, in general, it gives us only the centre of the ring, that is, \( Z(A) \). But here we are talking about honest schemes and hence \( Z(A) = A \).

We can also get a derived analogue of the above result, which is, however, considerably weaker. Also it is claimed to be an easy consequence of the above theorem in [BO01].

**Theorem 1.13.** [Bondal,Orlov [BO01]] Let \( X \) be a smooth irreducible projective variety with ample canonical or anti-canonical sheaf. If \( D = \text{D}^b\text{Coh}(X) \) is equivalent as a graded category to \( \text{D}^b\text{Coh}(X') \) for some other smooth algebraic variety \( X' \), then \( X \) is isomorphic to \( X' \).

**Remark 1.14.** Notice that for elliptic curves or, in general, abelian varieties the theorem above is not applicable.

Finally, consider a pre-additive category with a single object, say \(*\). Then, being a pre-additive category, \( \text{Hom}(\ast, \ast) \) is endowed with an abelian group structure. If we define a product on it by composition, then it is easy to verify that the two operations satisfy the ring axioms. So \( \text{Hom}(\ast, \ast) \) or simply \( \text{End}(\ast) \) is a ring and that is all we need to know about the pre-additive category. Extrapolating this line of thought, we say that pre-additive categories generalise the concept of rings and since schemes are concocted from commutative rings, it is reasonable to believe that a pre-additive category with some geometric properties should give rise to a “non-commutative scheme”.
The geometric properties desirable in an abelian category were written down by Grothendieck as the famous **AB Properties** in [Gro57].
A brief discussion on construction of quotient categories

This discussion is based on Gabriel’s article [Gab62] (page 365) and curious readers are encouraged to go through the details from there.

Recall that we call a full subcategory \( \mathcal{C} \) of an abelian category \( \mathcal{A} \) thick if the following condition is satisfied:

for all short exact sequences in \( \mathcal{A} \) of the form \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \), we have

\[
M \in \mathcal{C} \iff \text{both } M', M'' \in \mathcal{C}
\]

Now we construct the quotient of \( \mathcal{A} \) by a thick subcategory \( \mathcal{C} \), denoted \( \mathcal{A} / \mathcal{C} \), as follows:

\[
\text{Ob}(\mathcal{A} / \mathcal{C}) = \text{objects of } \mathcal{A}.
\]

\[
\text{Hom}_{\mathcal{A} / \mathcal{C}}(M, N) = \varinjlim \text{ Hom}_{\mathcal{A}}(M', N / N').
\]

One needs to check that as \( M' \) and \( N' \) run through all subobjects of \( M \) and \( N \) respectively, such that \( M'/M' \) and \( N' \) are in \( \text{Ob}(\mathcal{C}) \) (take intersection and sum respectively), the abelian groups \( \text{Hom}_{\mathcal{C}}(M', N / N') \) form a directed system. It satisfies the obvious universal properties which the readers are invited to formulate. It comes equipped with a canonical quotient functor \( \pi : \mathcal{A} \rightarrow \mathcal{A} / \mathcal{C} \).

**Proposition 1.15.** [Gabriel] Let \( \mathcal{C} \) be a thick subcategory of an abelian category \( \mathcal{A} \). Then the category \( \mathcal{A} / \mathcal{C} \) is abelian and the canonical functor \( \pi \) is exact.

The essence of this quotient construction is that the objects of \( \mathcal{C} \) become isomorphic to zero.

**Example 3.** Let \( \mathcal{A} = \text{Mod}(\mathbb{Z}) \) and \( \mathcal{C} = \text{Torsion groups} \). Then one can show that \( \mathcal{A} / \mathcal{C} \simeq \text{Mod}(\mathbb{Q}) \).

Let us define a functor from \( \mathcal{A} / \mathcal{C} \) to \( \text{Mod}(\mathbb{Q}) \) by tensoring with \( \mathbb{Q} \). We simplify the Hom sets of \( \mathcal{A} / \mathcal{C} \). Using the structure theorem, write every abelian group as a direct sum of its torsion part and torsion-free part. If one of the variables is torsion, it can be shown that in the limit Hom becomes 0. So we may assume that both variables are torsion-free and for simplicity let us consider both of them to be \( \mathbb{Z} \). Then,

\[
\text{Hom}_{\mathcal{A} / \mathcal{C}}(\mathbb{Z}, \mathbb{Z}) = \text{Hom}_{\mathcal{A}}(n\mathbb{Z}, \mathbb{Z})
\]

\[
\rightarrow n \cup \frac{1}{n} \mathbb{Z}
\]

\[
= \mathbb{Q} = \text{Hom}(\mathbb{Q}, \mathbb{Q})
\]

This says that the functor is full, and an easy verification shows that it is faithful and essentially surjective.

Now we are ready to discuss a model of non-commutative projective geometry after Artin and Zhang [AZ94]. We would also like to bring to the notice of readers the works of Verevkin (see [Ver92]). But before that let us go through one nice result in the affine case. Let \( X \) be an affine scheme and put \( A = \Gamma(X, \mathcal{O}_X) \). Then it is well-known that \( \text{QCoh}(X) \) is equivalent to \( \text{Mod}(A) \). This fact encourages us to ask: which Grothendieck categories can be written as \( \text{Mod}(A) \) for some possibly non-commutative ring \( A \)?

The answer to this question is given by the theorem below.
Theorem 1.16. [Ste75] Let $\mathcal{C}$ be a Grothendieck category with a projective generator $G$ and assume that $G$ is small [i.e., $\text{Hom}(G, -)$ commutes with all direct sums]. Then $\mathcal{C} \cong \text{Mod}(A^{\text{op}})$, for $A = \text{End}(G)$.

Note that the Gabriel-Popescu Theorem [1.10] gave just a fully faithful embedding with an exact left adjoint and not an equivalence.
2. Non-commutative Projective Geometry

Fix an algebraically closed field \( k \); then we shall mostly be dealing with categories which are \( k \)-linear abelian categories \([i.e., the bifunctor \text{Hom} ends up in \text{Mod}(k)]\). Since in commutative algebraic geometry one mostly deals with finitely generated \( k \)-algebras, which are Noetherian, here we assume that our \( k \)-algebras are at least right Noetherian. Later on we shall need to relax this Noetherian condition, but for now we stick to it. Let \( R \) be a graded algebra. Then we introduce some categories here:

\[
\begin{align*}
\text{QCoh}(X) &:= \text{category of quasi-coherent sheaves on a scheme } X. \\
\text{Coh}(X) &:= \text{category of coherent sheaves on } X. \\
\text{Mod}(A) &:= \text{category of right } A\text{-modules, where } A \text{ is a } k\text{-algebra.} \\
\text{Gr}(R) &:= \text{category of } \mathbb{Z}\text{-graded right } R\text{-modules, with degree 0 morphisms.} \\
\text{Tor}(R) &:= \text{full subcategory of } \text{Gr}(R) \text{ generated by torsion modules } \\
&\text{(i.e., } \forall x \in M, xR_{\geq s} = 0 \text{ for some } s), \text{ which is thick.} \\
\text{QGr}(R) &:= \text{the quotient category } \text{Gr}(R)/\text{Tor}(R) \text{ (refer to the quotient construction before).}
\end{align*}
\]

Notice that \( \text{QCoh}(X) \) is not obtained from \( \text{Coh}(X) \) by a quotient construction as \( \text{QGr}(R) \) is from \( \text{Gr}(R) \). In fact, when \( X \) is Noetherian, \( \text{Coh}(X) \) is the subcategory of \( \text{QCoh}(X) \) generated by all Noetherian objects in it.

Remark 2.1. Standard Convention. If \( XYuvw(\ldots) \) denotes an abelian category, then we shall denote by \( xYuvw(\ldots) \) the full subcategory consisting of Noetherian objects and if \( A, B, \ldots, M, N, \ldots \) denote objects in \( \text{Gr}(R) \) then we shall denote by \( A, B, \ldots, M, N, \ldots \) the corresponding objects in \( \text{QGr}(R) \).

Some people denote \( \text{QGr}(R) \) by \( \text{Tails}(R) \), but we shall stick to our notation. We denote the quotient functor \( \text{Gr}(R) \to \text{QGr}(R) \) by \( \pi \). It has a right adjoint functor \( \omega : \text{QGr}(R) \to \text{Gr}(R) \) and so, for all \( M \in \text{Gr}(R) \) and \( \mathcal{F} \in \text{QGr}(R) \) one obtains

\[
\text{Hom}_{\text{QGr}(R)}(\pi M, \mathcal{F}) \cong \text{Hom}_{\text{Gr}(R)}(N, \omega \mathcal{F}).
\]

The Hom’s of \( \text{QGr}(R) \) take a more intelligible form with the assumptions on \( R \). It turns out that for any \( N \in \text{gr}(R) \) and \( M \in \text{Gr}(R) \),

\[
\text{Hom}_{\text{QGr}(R)}(\pi N, \pi M) \cong \text{LimHom}_{\text{Gr}(R)}(N_{\geq n}, M)
\]

For any functor \( F \) from a \( k \)-linear category \( C \) equipped with an autoequivalence \( s \), we denote by \( F \) the graded analogue of \( F \) given by \( F(A) := \bigoplus_{n \in \mathbb{Z}} F(s^n A) \) for any \( A \in \text{Ob}(C) \). Further, to simplify notation we shall sometimes denote \( s^n A \) by \( A[n] \) when there is no chance of confusion.
Keeping in mind the notations introduced above we have

**Lemma 2.2.** $\omega \pi M \cong \lim_{R \geq n} \text{Hom}_R (R_{\geq n}, M)$

**Sketch of proof:**

\[
\omega \pi M = \text{Hom}_R (R, \omega \pi M) \quad \text{[since } R \in \text{gr}(R)]
\]

\[
= \bigoplus_{d \in \mathbb{Z}} \text{Hom}_{\text{Gr}(R)} (R, \omega \pi M [d]) \quad \text{[by adjointness of } \pi \text{ and } \omega]
\]

\[
= \bigoplus_{d \in \mathbb{Z}} \lim_{R \geq n} \text{Hom}_{\text{Gr}(R)} (R_{\geq n}, M [d]) \quad \text{[by (8)]}
\]

\[
= \lim_{R \geq n} \bigoplus_{d \in \mathbb{Z}} \text{Hom}_{\text{Gr}(R)} (R_{\geq n}, M [d])
\]

\[
= \lim_{R \geq n} \text{Hom}_R (R_{\geq n}, M)
\]

The upshot of this lemma is that there is a natural equivalence of functors $\omega \simeq \text{Hom}(\pi R, -)$.

**Proj $R$**

Let $X$ be a projective scheme with a line bundle $\mathcal{L}$. Then the homogeneous coordinate ring $B$ associated to $(X, \mathcal{L})$ is defined by the formula $B = \bigoplus_{n \in \mathbb{N}} \Gamma(X, \mathcal{L}^n)$ with the obvious multiplication. Similarly, if $\mathcal{M}$ is a quasi-coherent sheaf on $X$, $\Gamma_h(\mathcal{M}) = \bigoplus_{n \in \mathbb{N}} \Gamma(X, \mathcal{M} \otimes \mathcal{L}^n)$ defines a graded $B$-module. Thus, the composition of $\Gamma_h$ with the natural projection from $\text{Gr}(B)$ to $\text{QGr}(B)$ yields a functor $\tilde{\Gamma}_h : \text{QCoh}(X) \longrightarrow \text{QGr}(B)$. This functor works particularly well when $\mathcal{L}$ is ample, as is evident from the following fundamental result due to Serre.

**Theorem 2.3.** [Ser55] 1. Let $\mathcal{L}$ be an ample line bundle on a projective scheme $X$. Then the functor $\tilde{\Gamma}_h(\_)$ defines an equivalence of categories between $\text{QCoh}(X)$ and $\text{QGr}(B)$.

2. Conversely, if $R$ is a commutative connected graded $k$-algebra, that is, $R_0 = k$ and it is generated by $R_1$ as an $R_0$-algebra, then there exists a line bundle $\mathcal{L}$ over $X = \text{Proj}(R)$ such that $R = B(X, \mathcal{L})$, up to a finite dimensional vector space. Once again, $\text{QGr}(R) \simeq \text{QCoh}(X)$.

In commutative algebraic geometry one defines the Proj of a graded ring to be the set of all homogeneous prime ideals which do not contain the augmentation ideal. This notion is not practicable over arbitrary algebras. However, Serre’s theorem filters out the essential ingredients to define the Proj of an arbitrary algebra. The equivalence is controlled by the category $\text{QCoh}(X)$, the structure sheaf $\mathcal{O}_X$ and the autoequivalence given by tensoring with $\mathcal{L}$, which depends on the polarization of $X$. Borrowing this idea we get to the definition of Proj. Actually one should have worked with a $\mathbb{Z}$-graded algebra $R$ and defined its Proj but it has been shown in [AZ94] that, with the definition to be provided below, Proj $R$ is the same as Proj $R_{\geq 0}$. Hence, we assume that $R$ is an $\mathbb{N}$-graded $k$-algebra. $\text{Gr}(R)$ has a shift operator $s$ such that $s(M) = M[1]$ and a special object, $R_R$. We can actually recover $R$ from the triple $(\text{Gr}(R), R_R, s)$ by

\[
R = \bigoplus_{i \in \mathbb{N}} \text{Hom}(R_R, s^i(R_R))
\]

and the composition is given as follows: $a \in R_i$ and $b \in R_j$, then $ab = s^j(a) \cdot b \in R_{i+j}$.

Let $\mathcal{R}$ denote the image of $R$ in $\text{QGr}(R)$ and we continue to denote by $s$ the autoequivalence induced by $s$ on $\text{QGr}(R)$. 

13
Definition 2.4. \((\text{Proj} \, R)\)

The triple \((\text{QGr}(R), \mathcal{R}, s)\) is called the projective scheme of \(R\) and is denoted \(\text{Proj} \, R\). Keeping in mind our convention we denote \((\text{agr}(R), \mathcal{R}, s)\) by proj\(R\). This is just as good because there is a way to switch back and forth between QGr\(R\) and agr\(R\).

**Characterization of Proj \(R\)**

We have simply transformed Serre’s theorem into a definition. It is time to address the most natural question: which triples \((\mathcal{C}, \mathcal{A}, s)\) are of the form Proj\((R)\) for some graded algebra \(R\)? This problem of characterization has been dealt with comprehensively by Artin and Zhang. We will be content with just taking a quick look at the important points. Let us acquaint ourselves with morphisms of such triples. A morphism between \((\mathcal{C}, \mathcal{A}, s)\) and \((\mathcal{C}', \mathcal{A}', s')\) is given by a triple \((f, \theta, \mu)\), where \(f : \mathcal{C} \rightarrow \mathcal{C}'\) is a \(k\)-linear functor, \(\theta : f(\mathcal{A}) \rightarrow \mathcal{A}'\) is an isomorphism in \(\mathcal{C}'\) and \(\mu\) is a natural isomorphism of functors \(f \circ s \rightarrow s' \circ f\). The question of characterization is easier to deal with when \(s\) is actually an automorphism of \(\mathcal{C}\). To circumvent this problem, an elegant construction has been provided in [AZ04], whereby one can pass to a different triple, where \(s\) becomes necessarily an automorphism. If \(s\) is an automorphism one can take negative powers of \(s\) as well and it becomes easier to define the graded analogues of all functors (refer to [2.1]). Sweeping that discussion under the carpet, henceforth, we tacitly assume that \(s\) is an automorphism of \(\mathcal{C}\) (even though we may write \(s\) as an autoequivalence).

The definition of Proj was conjured up from Serre’s theorem where the triple was \((\text{Q Coh}(X), \mathcal{O}_X, - \otimes \mathcal{L})\). Of course, one can easily associate a graded \(k\)-algebra to any \((\mathcal{C}, \mathcal{A}, s)\).

\[
\Gamma_h(\mathcal{C}, \mathcal{A}, s) = \bigoplus_{n \geq 0} \text{Hom}(\mathcal{A}, s^n\mathcal{A})
\]

with multiplication \(a \cdot b = s^n(a)b\) for \(a \in \text{Hom}(\mathcal{A}, s^m\mathcal{A})\) and \(b \in \text{Hom}(\mathcal{A}, s^n\mathcal{A})\).

**Remark 2.5.** Let \(X\) be a scheme, \(\sigma \in \text{Aut}(X)\) and \(\mathcal{L}\) be a line bundle on \(X\). Then one obtains the twisted homogeneous coordinate ring, as discussed in section 1, as a special case of the above construction applied to the triple \((\text{Q Coh}(X), \mathcal{O}_X, \sigma_*(- \otimes \mathcal{L}))\). [Hint: to verify this, use the projection formula for sheaves]

Notice that \(\mathcal{L}\) needs to be ample for Serre’s theorem to work. So we need a notion of ampleness in the categorical set-up.

**Definition 2.6. (Ampleness)**

Assume that \(\mathcal{C}\) is locally Noetherian. Let \(\mathcal{A} \in \text{Ob}(\mathcal{C})\) be a Noetherian object and let \(s\) be an autoequivalence of \(\mathcal{C}\). Then the pair \((\mathcal{A}, s)\) is called ample if the following conditions hold:

1. For every Noetherian object \(\mathcal{O} \in \text{Ob}(\mathcal{C})\) there are positive integers \(l_1, \ldots, l_p\) and an epimorphism from \(\bigoplus_{i=0}^p \mathcal{A}(-l_i)\) to \(\mathcal{O}\).

2. For every epimorphism between Noetherian objects \(\mathcal{P} \rightarrow \mathcal{Q}\) the induced map \(\text{Hom}(\mathcal{A}(-n), \mathcal{P}) \rightarrow \text{Hom}(\mathcal{A}(-n), \mathcal{Q})\) is surjective for \(n \gg 0\).

**Remark 2.7.** The first part of this definition corresponds to the standard definition of an ample sheaf and the second part to the homological one.

Now we are in good shape to state one part of the theorem of Artin and Zhang which generalises that of Serre.

**Theorem 2.8.** Let \((\mathcal{C}, \mathcal{A}, s)\) be a triple as above such that the following conditions hold:

(H1) \(\mathcal{A}\) is Noetherian,
Proposition 2.9. Let \( M \in \text{Gr}(B) \) and fix \( i \geq 0 \). There is a right \( B \)-module structure on \( \text{Ext}^n_B(B/B_+, M) \) coming from the right \( B \)-module structure of \( B/B_+ \). Then the following are equivalent:

1. for all \( j \leq i \), \( \text{Ext}^j_B(B/B_+, M) \) is a finite \( B \)-module;
2. for all \( j \leq i \), \( \text{Ext}^j_B(B/B_{\geq n}, M) \) is finite for all \( n \);
3. for all \( j \leq i \) and all \( n \in \text{Gr}(B) \), \( \text{Ext}^j_B(N/N_{\geq n}, M) \) has a right bound independent of \( n \);
4. for all \( j \leq i \) and all \( N \in \text{Gr}(B) \), \( \lim_{\rightarrow} \text{Ext}^j_B(N/N_{\geq n}, M) \) is right bounded.

The proof is a matter of unwinding the definitions of the terms suitably and then playing with them. We shall do something smarter instead - make a definition out of it.

Definition 2.10. (\( \chi \) conditions)

A graded algebra \( B \) satisfies \( \chi_n \) if, for any finitely generated graded \( B \)-module \( M \), one of the equivalent conditions of the above proposition is satisfied (after substituting \( i = n \) in them). Moreover, we say that \( B \) satisfies \( \chi \) if it satisfies \( \chi_n \) for every \( n \).

Remark 2.11. Since \( B/B_+ \) is a finitely generated \( B_0 \)-module \( (B_0 = k) \) we could have equally well required the finiteness of \( \text{Ext}^n_B(B/B_+, M) \) over \( B_0 = k \) i.e., \( \dim_k \text{Ext}^n_B(B/B_+, M) < \infty \) for \( \chi_n \).

Let \( B \) be an \( \mathbb{N} \)-graded right Noetherian algebra and \( \pi : \text{Gr}(B) \rightarrow \text{QGr}(B) \).

Theorem 2.12. If \( B \) satisfies \( \chi_1 \) as well, then \( (H1) \), \( (H2) \) and \( (H3) \) hold for the triple \( (\text{qgr}(B), \pi B, s) \). Moreover, if \( A = \Gamma_h(\text{QGr}(B), \pi B, s) \), then \( \text{Proj} \ B \) is isomorphic to \( \text{Proj} \ A \) via a canonical homomorphism \( B \rightarrow A \). [We have a canonical map \( B_n = \text{Hom}_B(B, B[n]) \rightarrow \text{Hom}(\pi B, \pi B[n]) = A_n \) given by the functor \( \pi \).

The proofs of the these theorems are once again quite long and involved. So they are omitted. What we need now is a good cohomology theory for studying such non-commutative projective schemes.

Cohomology of \( \text{Proj} \ R \)

The following rather edifying theorem due to Serre gives us some insight into the cohomology of projective (commutative) spaces.

Theorem 2.13. [Har77] Let \( X \) be a projective scheme over a Noetherian ring \( A \), and let \( \mathcal{O}_X(1) \) be a very ample invertible sheaf on \( X \) over \( \text{Spec} \ A \). Let \( \mathcal{F} \) be a coherent sheaf on \( X \). Then:

1. for each \( i \geq 0 \), \( H^i(X, \mathcal{F}) \) is a finitely generated \( A \)-module,
2. there is an integer \( n_0 \), depending on \( \mathcal{F} \), such that for each \( i > 0 \) and each \( n \geq n_0 \), \( H^i(X, \mathcal{F}(n)) = 0 \).

There is an analogue of the above result and we zero in on that. We have already come across the \( \chi \) conditions, which have many desirable consequences. Actually the categorical notion of
ampleness doesn’t quite suffice. For the desired result to go through, we need the algebra to satisfy \( \chi \) too. Without inundating our minds with all the details of \( \chi \) we propose to get to the point i.e., cohomology. Set \( \pi R = \mathcal{R} \). On a projective (commutative) scheme \( X \) one can define the sheaf cohomology of \( \mathcal{F} \in \text{Coh}(X) \) as the right derived functor of the global sections functor, i.e., \( \Gamma \). But \( \Gamma(X, \mathcal{F}) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \). Buoyed by this fact, the proposed definition of the cohomology for every \( \mathcal{M} \in \text{qgr}(R) \) is

\[
H^n(\mathcal{M}) := \text{Ext}^n_{\mathcal{R}}(\mathcal{R}, \mathcal{M})
\]

However, taking into consideration the graded nature of our objects we also define the following:

\[
\overline{H}^n(\text{Proj } R, \mathcal{M}) := \text{Ext}^n_{\mathcal{QGr}(R)}(\mathcal{R}, \mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} \text{Ext}^n_{\mathcal{Gr}(R)}(\mathcal{R}_{\geq n}, \mathcal{M}) \quad \text{[by (8)]}
\]

The category \( \mathcal{QGr}(R) \) has enough injectives and one can choose a nice “minimal” injective resolution of \( \mathcal{M} \) to compute its cohomologies, the details of which are available in chapter 7 of [AZ94].

Let \( M \in \text{Gr}(R) \) and write \( M = \pi M \). Then one should observe that

\[
\overline{H}^n(\mathcal{M}) = \text{Ext}^n_{\mathcal{QGr}(R)}(\mathcal{R}, \mathcal{M}) \\
\cong \lim_{\rightarrow} \text{Ext}^n_{\mathcal{Gr}(R)}(\mathcal{R}_{\geq n}, M) \quad \text{[by (8)]}
\]

As \( R \)-modules we have the following exact sequence,

\[
0 \rightarrow \mathcal{R}_{\geq n} \rightarrow \mathcal{R} \rightarrow \mathcal{R}/\mathcal{R}_{\geq n} \rightarrow 0
\]

For any \( M \in \text{Gr}(R) \), the associated \( \text{Ext} \) sequence in \( \text{Gr}(R) \) looks like

\[
\cdots \rightarrow \text{Ext}^i(\mathcal{R}/\mathcal{R}_{\geq n}, M) \rightarrow \text{Ext}^i(\mathcal{R}, M) \rightarrow \text{Ext}^i(\mathcal{R}_{\geq n}, M) \rightarrow \cdots
\]

Since \( R \) is projective as an \( R \) module, \( \text{Ext}^j(\mathcal{R}, M) = 0 \) for every \( j \geq 1 \). Thus, we get the following exact sequence:

\[
0 \rightarrow \text{Hom}(\mathcal{R}/\mathcal{R}_{\geq n}, M) \rightarrow M \rightarrow \text{Hom}(\mathcal{R}_{\geq n}, M) \rightarrow \text{Ext}^1(\mathcal{R}/\mathcal{R}_{\geq n}, M) \rightarrow 0
\]

and, for every \( j \geq 1 \), an isomorphism

\[
\text{Ext}^j(\mathcal{R}_{\geq n}, M) \cong \text{Ext}^{j+1}(\mathcal{R}/\mathcal{R}_{\geq n}, M)
\]

The following theorem is an apt culmination of all our efforts.

**Theorem 2.14. (Serre’s finiteness theorem)**

Let \( R \) be a right Noetherian \( \mathbb{N} \)-graded algebra satisfying \( \chi \), and let \( \mathcal{F} \in \text{qgr}(R) \). Then,

(\( H_4 \)) for every \( j \geq 0 \), \( H^j(\mathcal{F}) \) is a finite right \( R_0 \)-module, and

(\( H_5 \)) for every \( j \geq 1 \), \( H^j(\mathcal{F}) \) is right bounded; i.e., for \( d \gg 0 \), \( H^j(\mathcal{F}[d]) = 0 \).

**Sketch of proof:**

Write \( \mathcal{F} = \pi M \) for some \( M \in \text{gr}(R) \). Suppose that \( j = 0 \). Since \( \chi_1(M) \) holds, \( \text{Ext}^i_{\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, M) \) is a finite \( R \)-module for each \( i = 1, 2 \) and together with (11) it implies that \( \omega \mathcal{F} \cong H^0(\mathcal{F}) \) is finite (recall \( \omega \) from the paragraph after [2.1]). Now taking the 0-graded part on both sides we get (\( H_4 \)) for \( j = 0 \).

Suppose that \( j \geq 1 \). Since \( R \) satisfies \( \chi_{j+1} \), invoking proposition [2.9] we get

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16
\[
\lim_{n \to \infty} \text{Ext}_R^{j+1}(R/R_{\geq n}, M)
\]
is right bounded. Combining (10) and (12) this equals \(H^j(F)\). This immediately proves \((H5)\) as \(H^j(F) = H^j(F[d])\). We now need left boundedness and local finiteness of \(H^j(F)\) to finish the proof of \((H4)\) for \(j \geq 1\). These we have already observed (at least tacitly) but one can verify them by writing down a resolution of \(R/R \geq n\) involving finite sums of shifts of \(R\), and then realising the cohomologies as subquotients of a complex of modules of the form \(\text{Hom}_R(\oplus R[l_i], M)\).

Our discussion does not quite look complete unless we investigate the question of the “dimension” of the objects that we have defined.

**Dimension of Proj \(R\)**

The **cohomological dimension** of Proj \(R\), denoted by \(cd(\text{Proj} \ R)\), is defined to be

\[
cd(\text{Proj} \ R) := \begin{cases} 
\sup \{i \mid H^i(M) \neq 0 \text{ for some } M \in \text{qgr}(R)\} & \text{if it is finite}, \\
\infty & \text{otherwise}.
\end{cases}
\]

**Remark 2.15.** As \(H^i\) commutes with direct limits one could have used QGr(\(R\)) in the definition of cohomological dimension.

The following proposition gives us what we expect from a Proj construction regarding dimension and also provides a useful way of calculating it.

**Proposition 2.16.**

1. If \(cd(\text{Proj} \ R)\) is finite, then it is equal to \(\sup \{i \mid H^i(R) \neq 0\}\).

2. If the left global dimension of \(R\) is \(d < \infty\), then \(cd(\text{Proj} \ R) \leq d - 1\).

**Sketch of proof:**

1. Let \(d\) be the cohomological dimension of Proj \(R\). It is obvious that \(\sup \{i \mid H^i(R) \neq 0\} \leq d\). We need to prove the other inequality. So we choose an object for which the supremum is attained, i.e., \(M \in \text{qgr}(R)\) such that \(H^d(M) \neq 0\) and, hence, \(H^d(M) \neq 0\). By the ampleness condition we may write down the following exact sequence:

\[
0 \to N \to \bigoplus_{i=0}^p R[-l_i] \to M \to 0
\]

for some \(N \in \text{qgr}(R)\). By the long exact sequence of derived functors \(H^i\) we have

\[
\ldots \to \bigoplus_{i=0}^p H^d(R[-l_i]) \to H^d(M) \to H^{d+1}(N) = 0
\]

This says that \(H^d(R[-l_i]) \neq 0\) for some \(i\) and hence, \(H^d(R) \neq 0\).

2. It has already been observed that \(H^i(M) \cong \lim_{n \to \infty} \text{Ext}_R^i(R_{\geq n}, M)\) for all \(i \geq 0\). Now, if the left global dimension of \(R\) is \(d\), then \(\text{Ext}_R^j(N, M) = 0\) for all \(j > d\) and all \(N, M \in \text{Gr}(R)\). Putting \(N = R/R_{\geq n}\) and using (12) we get \(H^d(M) = 0\) for all \(M \in \text{Gr}(R)\). Therefore, \(cd(\text{Proj} \ R) \leq d - 1\).

**Remark 2.17.** If \(R\) is a Noetherian AS-regular graded algebra, then the Gorenstein condition can be used to prove that \(cd(\text{Proj} \ R)\) is actually equal to \(d - 1\).

**Some Examples (mostly borrowed from [AZ94])**
Example 4. (Twisted graded rings)

Let σ be an automorphism of a graded algebra \( A \). Then define a new multiplication \(*\) on the underlying graded \( k\)-module \( A = \oplus A_n \) by

\[ a * b = a\sigma^n(b) \]

where \( a \) and \( b \) are homogeneous elements in \( A \) and \( \deg(a) = n \). Then algebra is called the twist of \( A \) by \( \sigma \) and it is denoted by \( A^\sigma \). By \cite{TVdB91} and \cite{Zha96} \( \text{gr}(A) \simeq \text{gr}(A^\sigma) \) and hence, \( \text{proj}(A) \simeq \text{proj}(A^\sigma) \).

For example, if \( A = k[x,y] \) where \( \deg(x) = \deg(y) = 1 \), then any linear operator on the space \( A_1 \) defines an automorphism, and hence a twist of \( A \). If \( k \) is an algebraically closed field then, after a suitable linear change of variables, a twist can be brought into one of the forms \( k_q[x,y] := k\{x,y\}/(xy - qxy) \) for some \( q \in k \), or \( k_j[x,y] := k\{x,y\}/(x^2 + xy - yx) \). Hence, \( \text{proj} k[x,y] \simeq \text{proj} k_q[x,y] \simeq \text{proj} k_j[x,y] \). The projective scheme associated to any one of these algebras is the projective line \( \mathbb{P}^1 \).

Example 5. (Changing the structure sheaf)

Though the structure sheaf is a part of the definition of Proj, one might ask, given a \( k\)-linear abelian category \( C \), which objects \( A \) could serve the purpose of the structure sheaf. In other words, for which \( A \) do the conditions (H1), (H2) and (H3) of Theorem \cite{ZS} hold? Since (H3) involves both the structure sheaf and the polarization \( s \), the answer may depend on \( s \). We propose to illustrate the possibilities by the simple example in which \( C = \text{Mod}(R) \) when \( R = k_1 \oplus k_2 \), where \( k_i = k \) for \( i = 1, 2 \) and where \( s \) is the automorphism which interchanges the two factors. The objects of \( C \) have the form \( V \simeq k_1^{n_1} \oplus k_2^{n_2} \), and the only requirement for (H1), (H2) and (H3) is that both \( r_1 \) and \( r_2 \) not be zero simultaneously.

We have \( s^n(V) = k_1^{n_1} \oplus k_2^{n_1} \) if \( n \) is odd and \( s^n(V) = V \) otherwise. Thus, if we set \( A = V \) and \( A = \Gamma_h(C, A, s) \), then \( A_n \simeq k_1^{r_1 \times r_1} \oplus k_2^{r_2 \times r_2} \) if \( n \) is even, and \( A_n \simeq k_1^{r_1 \times r_2} \oplus k_2^{r_2 \times r_1} \) otherwise. For example, if \( r_1 = 1 \) and \( r_2 = 0 \), then \( A \simeq k[y] \), where \( y \) is an element of degree 2. Both of the integers \( r_i \) would need to be positive if \( s \) were the identity functor.

Example 6. (Commutative Noetherian algebras satisfy \( \chi \)-condition)

Let \( A \) be a commutative Noetherian \( k\)-algebra. Then the module structure on \( \text{Ext}^0_n(A/A_+, M) \) can be obtained both from the right \( A \)-module structure of \( A/A_+ \) and that of \( M \). Choose a free resolution of \( A/A_+ \), consisting of finitely generated free modules. The cohomology of this complex of finitely generated \( A \)-modules is given by the \( \text{Ext}^i \)'s, whence they are finite.

Example 7. (Noetherian \( AS\)-regular algebras satisfy \( \chi \)-condition)

If \( A \) is a Noetherian connected \( \mathbb{N}\)-graded algebra having global dimension 1, then \( A \) is isomorphic to \( k[x] \), where \( \deg(x) = n \) for some \( n > 0 \), which satisfies the condition \( \chi \) by virtue of the previous example. In higher dimensions we have the following proposition.

Proposition 2.18. Let \( A \) be a Noetherian \( AS\)-regular graded algebra of dimension \( d \geq 2 \) over a field \( k \). Then \( A \) satisfies the condition \( \chi \).

Sketch of proof:

\( A \) is Noetherian and locally finite (due to finite \( \text{GKdim} \)). For such an \( A \) it is easy to check that \( \text{Ext}^i(N, M) \) is a locally finite \( k\)-module whenever \( N, M \) are finite. \( A_0 \) is finite and hence, \( \text{Ext}^i(A_0, M) \) is locally finite for every finite \( M \) and every \( j \). Since \( A \) is connected graded, \( A_0 = k \).

For any \( n \) and any finite \( A \)-module \( M \) we first show that \( \text{Ext}^n(A_0, M) = \text{Ext}^n(k, M) \) is bounded using induction on the projective dimension of \( M \). If \( \text{pd}(M) = 0 \), then \( M = \bigoplus_{i=0}^p A[-l_i] \). By the
Gorenstein condition (see definition [1.3]) of an AS-regular algebra $A$, $\text{Ext}^n(k, A[-l_i])$ is bounded for each $i$. Therefore, so is $\text{Ext}^n(k, M)$. If $\text{pd}(M) > 0$, we choose an exact sequence

$$0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$$

where $P$ is projective. Then $\text{pd}(N) = \text{pd}(M) - 1$. By induction, $\text{Ext}^n(k, N)$ and $\text{Ext}^n(k, P)$ are bounded, hence, so is $\text{Ext}^n(k, M)$. Now $A/A_+$ is finite and we have just shown $\text{Ext}^n(k, M)$ is finite (since bounded together with locally finite implies finite); then $\text{Hom}(A/A_+, \text{Ext}^n(k, M)) \cong \text{Ext}^n(A/A_+, M)$ is locally finite and clearly bounded. Therefore, $\text{Ext}^n(A/A_+, M)$ is finite for every $n$ and every finite $M$.

Most of these examples are taken directly from the original article by Artin and Zhang [AZ94]. There is a host of other examples on algebras satisfying $\chi$ up to varying degrees, for which we refer the interested readers to [AZ94], [Rog02] and [SZ94]. Also one should take a look at [TvdB90] where these ideas, in some sense, germinated. Finally, a comprehensive survey article by J. T. Stafford and M. van den Bergh [SvdB01] should be consulted for further curiosities in the current state of affairs in non-commutative algebraic geometry.
3. Algebraic aspects of non-commutative tori

The section is mostly based on the article *Noncommutative two-tori with real multiplication as noncommutative projective spaces* by A. Polishchuk [Pol04b]. Noncommutative two-tori with real multiplication will be explained by Jorge Plazas and by now we know what we mean by non-commutative projective varieties. In the following paragraph the gist of the article has been provided (objects within quotes will be explained either by Jorge Plazas or by B. Noohi or the reader is expected to look it up for himself/herself). Interested readers are also encouraged to take a look at the 4th chapter entitled *Fractional dimensions in homological algebra* of [Man06], where Manin gives a very insightful overview of this work. We merely fill in some details here for pedagogical reasons.

One considers the category of “holomorphic vector bundles” on a noncommutative torus $\mathbb{T}_\theta$, $\theta$ being a real parameter, whose algebra of smooth functions is denoted $A_\theta$. We always assume $\theta$ to be irrational. In keeping with the general philosophy, $\mathbb{T}_\theta$ and $A_\theta$ will be used interchangeably for a non-commutative torus. There is a fully faithful functor [PS03] from the “derived category of holomorphic bundles on a noncommutative torus $A_\theta$” to the derived category of coherent sheaves on a complex elliptic curve $X$, denoted by $D^b(X)$ (refer to the remark below (3.2)), sending holomorphic bundles to the “heart” of a “$t$-structure” depending on $\theta$, denoted by $\mathcal{C}^\theta$. The elliptic curve $X$ is determined by the choice of a “complex structure” on $A_\theta$, depending on a complex parameter $\tau$ in the lower half plane i.e., $X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. It follows from [Pol04a] that the category of holomorphic vector bundles on $A_\theta$ is actually equivalent to the heart $\mathcal{C}^\theta$ and the “standard bundles” end up being the so-called “stable” objects of $D^b(X)$. We also know that the heart has “cohomological dimension” 1 and is derived equivalent to $D^b(X)$. The “real multiplication” of $A_\theta$ gives rise to an auto-equivalence, say $F$, of $D^b(X)$, which preserves the heart up to a shift. One knows when it actually preserves the heart, viz., when the matrix inducing the real multiplication has positive real eigenvalues. Now by choosing a “stable” object, say $\mathcal{G}$, in $D^b(X)$, one can construct graded algebras from the triple $\mathcal{C}^\theta$, $\mathcal{G}$ and $F$ as described before (see (9)). Some criteria for the graded algebras to be generated in degree 1, quadratic and “Koszul” are also known. For the details one may refer to e.g., [Pla].

Remark 3.1. It is known that two noncommutative tori, say $A_\theta$ and $A_{\theta'}$, are Morita equivalent if $\theta' = g\theta$ for some $g \in SL(2, \mathbb{Z})$ [Rie81].

The equivalence defined in [PS03] between the “derived category of holomorphic bundles on $A_\theta$” and $D^b(X)$ actually sends the holomorphic bundles on $A_\theta$ to $\mathcal{C}^{−\theta−1}$ up to some shift and the real multiplication on $A_\theta$ descends to an element $F \in Aut(D^b(X))$, which preserves $\mathcal{C}^{−\theta−1}$ (up to some shift). This is not too bad, as \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \theta = -\theta^{-1}
\]
(action by fractional linear transformation), which says that $A_{−\theta−1}$ is Morita equivalent to $A_\theta$. The generators of $SL(2, \mathbb{Z})$ are $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The matrix $g$ acts by translation by 1 and so up to Morita equivalence $\theta$ may be brought within the interval $[0, 1]$ and $\theta$, being irrational, $\theta \in (0, 1)$.

The image of this interval under $x \mapsto -x^{-1}$ is $(-\infty, -1)$. We label the noncommutative torus by $A_{−\theta−1}$, $-\theta^{-1} \in (-\infty, -1)$, so that when we pass on to the heart $\mathcal{C}^\theta$, $\theta \in (0, 1)$.

1Usually $A_\theta$ is used to denote the algebra of continuous functions and $A_0$ is used to denote the smooth ones. To ease LaTeX-ing, the algebra of smooth functions has been consistently denoted by $A_\theta$. 

20
Remark 3.2. By the bounded derived category of coherent sheaves one should actually understand $D^b(Coh(X))$. However, we may not be able to find injective resolutions in Coh(X). So the precise category we want is $D^b(X) := D^b_{Coh(X)}(Q Coh(X))$, i.e., the bounded derived category of complexes of quasi-coherent sheaves on $X$ with the cohomology objects in Coh(X). According to Lemma 2.3 of [ST01] one knows that the two categories under consideration are equivalent. It is known that for a smooth curve $X$, every object of $D^b(X)$ (which is a complex) is quasi-isomorphic to the direct sum of its cohomologies. This is not true in higher dimensions.

Let $X$ be an elliptic curve over the complex numbers. For $F \in Coh(X)$, let $\text{rk}(F)$ stand for the generic rank of $F$ and $\chi(F)$ for the Euler characteristic of $F$. Since $X$ has genus 1, by Riemann-Roch the degree of $F$ is the same as the Euler characteristic of $F$, i.e., $\text{deg}(F) = \chi(F) := \dim_{\mathbb{C}} \text{Hom}(\mathcal{O}_X, F) - \dim_{\mathbb{C}} \text{Ext}^1(\mathcal{O}_X, F)$. So the slope of a coherent sheaf $F$, denoted by $\mu(F)$, which is just the rational number $\frac{\text{deg}(F)}{\text{rk}(F)}$, also equals $\frac{\chi(F)}{\text{rk}(F)}$. The latter fraction is more suitable for our purposes and, hence, we take that as the definition of slope. We define the rank (resp. the Euler characteristic) of a complex in the derived category of Coh(X) as the alternating sum of the ranks (resp. the Euler characteristics) of the individual terms of the corresponding cohomology complex. Then the same definition as above extends the notion of slope to the objects of the derived category. A coherent sheaf $F$ is called semistable (resp. stable) if for any nontrivial exact sequence $0 \to F' \to F \to F'' \to 0$ one has $\mu(F') \leq \mu(F)$ (resp. $\mu(F') < \mu(F)$) or equivalently $\mu(F) \leq \mu(F'')$ (resp. $\mu(F) < \mu(F'')$).

It is well-known that every coherent sheaf on $X$ splits as a direct sum of its torsion and torsion-free parts. Since $X$ is smooth, projective and of dimension 1, every torsion-free coherent sheaf is locally free and for any $F \in Coh(X)$ there exists a unique filtration $[\text{HN}75]$:

\begin{equation}
F = F_0 \supset F_1 \supset \cdots \supset F_n \supset F_{n+1} = 0
\end{equation}

such that

- $F_i/F_{i+1}$ for $0 \leq i \leq n$ are semistable and
- $\mu(F_0/F_1) < \mu(F_1/F_2) < \cdots < \mu(F_n)$.

The filtration above is called the Harder-Narasimhan filtration of $F$ and the graded quotients $F_i/F_{i+1}$ are called the semistable factors of $F$. We set $\mu_{\text{min}}(F) = \mu(F_0/F_1)$ and $\mu_{\text{max}}(F) = \mu(F_n)$.

One calls an object $F \in D^b(X)$ stable if $F = V[n]$, where $V$ is either a stable vector bundle (stable as above) or a coherent sheaf supported at a point (the stalk is the residue field).

3.1. t-structures on $D^b(X)$ depending on $\theta$. One way to obtain t-structures is via “torsion theories”. So let us define a torsion pair $(\text{Coh}_{>\theta}, \text{Coh}_{\leq \theta})$ in Coh(X).

\[
\text{Coh}_{>\theta} := \{F \in Coh(X) : \mu_{\text{min}}(F) > \theta\}
\]
\[
\text{Coh}_{\leq \theta} := \{F \in Coh(X) : \mu_{\text{max}}(F) \leq \theta\}
\]

We consider the full subcategories generated by these objects inside Coh(X). Notice that torsion sheaves, having slope $= \infty$, belong to $\text{Coh}_{>\theta}$.

To show that this is indeed a torsion pair we need to verify two conditions.

---

2By induction, it is enough to show for complexes of length 2. Let $F^* \in D^b(X)$.

$F^* = \cdots \supseteq 0 \to F_{-1} \to F_0 \to 0 \to \cdots$

Consider the triangle: $\ker f[1] \to F^* \to \text{cone} \theta \to \ker f[2]$. Check that in $D^b(X)$ cone $\theta$ is $\text{coker } f$. Now $\xi \in \text{Hom}(\text{coker } f, \ker f[2]) \equiv \text{Hom}^2(\text{coker } f, \ker f) = 0$. The last equality is due to Fact 2 (it appears later on). So $F^*[1] = \text{coker } f[1] \oplus \ker f[2]$. Note that coker $f$ and ker $f$ are the cohomologies of $F^*$. 

---
1. \( \text{Hom}(T, F) = 0 \) for all \( T \in \text{Coh}_{>\theta} \) and \( F \in \text{Coh}_{<\theta} \)

**Lemma 3.3.** Let \( F \) and \( F' \) be a pair of semistable bundles. Then \( \mu(F) > \mu(F') \) implies that \( \text{Hom}(F, F') = 0 \).

**Proof:** Suppose \( f : F \to F' \) is a nonzero morphism. Let \( G \) be the image of \( f \). Then \( G \) is a quotient of \( F \) and so one has \( \mu(G) \geq \mu(F) \). On the other hand, \( G \) is a torsion-free subsheaf of a vector bundle on a smooth curve and so it is locally free. Thus, one has \( \mu(G) \leq \mu(F') \), which implies \( \mu(F) \leq \mu(G) \leq \mu(F') \). Take the contrapositive to obtain the desired result.

Let \( T \in \text{Coh}_{>\theta} \) and \( F \in \text{Coh}_{<\theta} \). Further, suppose \( \sigma \in \text{Hom}(T, F) \). Let us write down the Harder-Narasimhan filtrations of \( T \) and \( F \) respectively:

\[
0 \to T_0 \to \cdots \to T_{m-1} \to T_m = T
\]

\[
0 \to F_0 \to \cdots \to F_{n-1} \to F_n = F
\]

Restrict \( \sigma : T \to F \) to \( T_0 \) and compose it with the canonical projection onto \( F_n/F_{n-1} \). Now \( T_0 \) is a semistable factor of \( T \) and \( F_n/F_{n-1} \) that of \( F \). Since \( T \in \text{Coh}_{>\theta} \) and \( F \in \text{Coh}_{<\theta} \), by the lemma above this map is 0. So the image lies in \( F_{n-1} \). Apply the same argument after replacing \( F_n/F_{n-1} \) by \( F_{n-1}/F_{n-2} \) to conclude that the image lies in \( F_{n-2} \). Iterating this process we may conclude that \( \sigma \) restricted to \( T_0 \) is 0. So \( \sigma \) factors through \( T/T_0 \). The Harder-Narasimhan filtration of \( T/T_0 \) is

\[
0 \to T_1/T_0 \to \cdots \to T_{m-1}/T_0 \to T_m/T_0 = T/T_0
\]

This filtration has the same semistable factors as that of \( T \) and so they satisfy the conditions of the Harder-Narasimhan filtration. So by the uniqueness of the Harder-Narasimhan filtration this is that of \( T/T_0 \). Iterate the argument above after replacing \( T \) by \( T/T_0 \) and taking the induced map of \( \sigma \) between \( T/T_0 \) and \( F \) to conclude that \( \sigma \) vanishes on \( T_1/T_0 \). But \( \sigma \) also vanishes on \( T_0 \). So it must vanish on \( T_1 \). Repeating this argument finitely many times one may show that \( \sigma \) vanishes on the whole of \( T \).

2. For every \( F \in \text{Coh}(X) \) there should be an exact sequence (necessarily unique up to isomorphism)

\[
0 \to t(F) \to F \to F/t(F) \to 0
\]

such that \( t(F) \in \text{Coh}_{>\theta} \) and \( F/t(F) \in \text{Coh}_{<\theta} \).

**Proof:** Let \( 0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F \) be the Harder-Narasimhan filtration of \( F \). Let \( i \) be the unique integer such that \( \mu(F_i/F_{i-1}) > \theta \) and \( \mu(F_{i+1}/F_i) \leq \theta \). Then set \( t(F) = F_i \). It is easy to see that \( F_i \in \text{Coh}_{>\theta} \) and \( F/F_i \in \text{Coh}_{<\theta} \). By the way, if no such \( i \) exists, then \( F \) is already either an element of \( \text{Coh}_{>\theta} \) or \( \text{Coh}_{<\theta} \).

**Fact 1.** (see for instance [HRS96]) Let \( (\mathcal{T}, \mathcal{F}) \) be a torsion pair on an abelian category \( \mathcal{A} \). Let \( \mathcal{C} \) be the heart of the associated t-structure. Then \( \mathcal{C} \) is an abelian category, equipped with a torsion pair \((\mathcal{F}[1], \mathcal{T})\).
Recall that the **cohomological dimension** (perhaps, global dimension is a more appropriate term) of an abelian category $\mathcal{A}$ is the minimum integer $n$ such that $\text{Ext}^i(A, B) = 0$ for all $A, B \in \mathcal{A}$ and for all $i > n$ and $\infty$ if no such $n$ exists.

**Fact 2.** [Ser55] If $X$ is a smooth projective curve (i.e., $\dim X = 1$), then the cohomological dimension of $\text{Coh}(X)$ is 1.

Now we shall associate a $t$-structure to this torsion pair (see for instance [HRS96]) as follows:

$D^{\theta, \leq 0} := \{ K \in D^b(X) : H^0(K) = 0, H^0(K) \in \text{Coh}_{\leq \theta} \}$

$D^{\theta, \geq 1} := \{ K \in D^b(X) : H^0(K) = 0, H^0(K) \in \text{Coh}_{\geq \theta} \}$

It is customary to denote $D^{\theta, \leq 0}[-n]$ by $D^{\theta, \leq n}$ and $D^{\theta, \geq 0}[-n]$ by $D^{\theta, \geq n}$. Let $\mathcal{C}^{\theta} := D^{\theta, \leq 0} \cap D^{\theta, \geq 0}$ be the heart of the $t$-structure, which is known to be an abelian category. An interesting thing is that $(\text{Coh}_{\leq \theta}[1], \text{Coh}_{\geq \theta})$ defines a torsion pair on $\mathcal{C}^{\theta}$ (refer to Fact 1 above). As a matter of convention, the family of $t$-structures is extended to $\theta = \infty$ by putting the standard $t$-structure on it, whose heart is just $\text{Coh}(X)$.

Our next aim is to show that $\mathcal{C}^{\theta}$ has cohomological dimension 1.

**ASIDE on Serre Duality:** Let $X$ be a smooth projective scheme of dimension $n$. Then there is a dualizing sheaf $\omega$ such that one has natural isomorphisms

$$H^i(X, F) \cong \text{Ext}^{n-i}(F, \omega)^*$$

where $F$ is any coherent sheaf on $X$.

**Remark 3.4.** The definition of a dualizing sheaf exists for all proper schemes. For nonsingular projective varieties it is known that the dualizing sheaf is isomorphic to the canonical sheaf.

**Definition 3.5.** ([BO01] Defn. 1.2., Prop. 1.3. and Prop. 1.4.) Let $\mathcal{D}$ be a $k$-linear triangulated category with finite dimensional Hom’s. An auto-equivalence $S : \mathcal{D} \to \mathcal{D}$ is called a **Serre functor** if there are bi-functorial isomorphisms

$$\text{Hom}_\mathcal{D}(A, B) \cong \text{Hom}_\mathcal{D}(B, SA)^*$$

which are natural for all $A, B \in \mathcal{D}$.

Bondal and Kapranov have shown that in a reasonable manner Serre Duality of a smooth projective scheme $X$ can be reinterpreted as the existence of a Serre functor [if it exists it is unique up to a graded natural isomorphism] on $D^b(X)$ [BK89].

It is also known that for a smooth projective variety of dimension $n$ the Serre functor is $- \otimes \omega_X[n]$. For an elliptic curve $X$ the Serre functor will be just the translation functor $[1]$ (since $\dim X = 1$ and the canonical sheaf $\omega_X$ of an elliptic curve is trivial).

**Lemma 3.6.** Any $F \in \text{Coh}(X)$ is isomorphic to the direct sum of its semistable factors.

**Proof:**

The proof is by induction on the length of the Harder-Narasimhan filtration and for simplicity we treat only the case of length two. The category $\text{Coh}(X)$ has the so-called Calabi-Yau property, which says that $\text{Ext}^1(F, G) \cong \text{Hom}(G, F)^*$ for all $F, G \in \text{Coh}(X)$ (this follows from Serre Duality as discussed above). Let $F \in \text{Coh}(X)$ and let
be its Harder-Narasimhan filtration.

Then its semistable factors are $F_1$ and $F/F_1 =: G$. Thus we obtain an exact sequence

$$0 \to F_1 \to F \to G \to 0$$

where $F_1$ and $G$ are semistable. By the properties of the Harder-Narasimhan filtration we have $\mu(G) < \mu(F_1)$. Due to the Calabi-Yau property we know that $\Ext^1(G, F_1) \cong \Hom(F_1, G)^*$. From Lemma 3.3 it follows that $\Hom(F_1, G) = 0$ and hence $\Ext^1(G, F_1) = 0$. Therefore the short exact sequence above splits and we obtain $F \cong F_1 \oplus G$.

**Proposition 3.7.** $C^\theta$ has cohomological dimension 1.

**Proof:**
First of all, observe that it is enough to show $\Hom^{i+1}_{D^b(X)}(A, B) = 0$ for all $A, B$ belonging to $\Coh_{>\theta}$ and $\Coh_{<\theta}$ only. Now $\Coh(X)$ has cohomological dimension 1 (Fact 2) and $\Coh_{>\theta}$ and $\Coh_{<\theta}$ are full subcategories of $\Coh(X)$. So if $A, B$ were both either in $\Coh_{>\theta}$ or in $\Coh_{<\theta}[1]$ then there would have been nothing to prove. Let $A \in \Coh_{>\theta}$ and $B \in \Coh_{<\theta}[1]$.

Then $\Hom^i_{D^b(X)}(A, B) = \Hom^{i+1}_{D^b(X)}(A, B[-1])$. But $B \in \Coh_{<\theta}$, which is a subcategory of $\Coh(X)$. So $\Hom^{i+1}_{D^b(X)}(A, B[-1]) = 0$ for all $i \geq 1$. On the other hand,

$$\Hom^i_{D^b(X)}(B, A) = \Hom^{i-1}_{D^b(X)}(B[-1], A)$$

$$= \Hom_{D^b(X)}(B[-1], A[1-i])$$

$$\cong \Hom_{D^b(X)}(A[1-i], B)^*$$

(use Serre functor $= [1]$ as explained before)

$$= \Hom_{D^b(X)}(A[2-i], B[-1])^*$$

$$= \Hom^{2-i}_{D^b(X)}(A, B[-1])^*$$

So, for $i > 2$, $\Hom^i_{D^b(X)}(B, A)$ is evidently 0. Due to the first axiom of a torsion pair, $\Hom_{\Coh(X)}(A, B[-1]) = 0$ when $i = 2$.

**Proposition 3.8.** The categories $C^\theta$ and $\Coh(X)$ are derived equivalent, i.e., $D^b(C^\theta) \cong D^b(X)$.

**Proof:** It is known that if a torsion pair $(T, F)$ in an abelian category $\mathcal{A}$ is cotilting, i.e., every object of $\mathcal{A}$ is a quotient of an object in $\mathcal{F}$, then the heart of the t-structure induced by the torsion pair is derived equivalent to $\mathcal{A}$ (see for instance Proposition 5.4.3. and the remark thereafter in [BvdB03]). Thus, it is enough to check that the torsion pair $(\Coh_{>\theta}, \Coh_{<\theta})$ is cotilting.

Given any $F \in \Coh(X)$ we need to produce an object in $\Coh_{<\theta}$ which surjects onto $F$. Let $L$ be an ample line bundle on $X$, i.e., $\deg(L) > 0$. By Serre’s theorem one may twist $F$ by a large enough power of $L$ such that it becomes generated by global sections, i.e., the quotient of a free sheaf. In other words, there exists $N > 0$ large enough such that for all $n > N$ there is an epimorphism $\oplus_{i \in I} \mathcal{O}_X \to F \otimes L^n$, $I$ finite. One may twist it back to obtain an epimorphism $\oplus_{i \in I} \tilde{L}^n \to F$, where $\tilde{L}$ is the dual line bundle. This shows that there exists an epimorphism onto $F$ from a finite direct sum of copies of $\tilde{L}^n$. Since $\deg(\tilde{L}^n) = -n \cdot \deg(L) < 0$ it is possible to make the slope of $\tilde{L}^n$, 

$$0 \subset F_1 \subset F_2 = F$$
which is equal to \( \text{deg}(\mathcal{L}^n) \), less than \( \theta \) by choosing a large enough \( n \). Being a line bundle \( \mathcal{L}^n \) is clearly semistable and we observe that the direct sum of copies of \( \mathcal{L}^n \) lies in \( \text{Coh}_{\leq \theta} \).

\[ \sqrt{\mathcal{L}} \]

Remark 3.9. In fact, all “bounded” t-structures on \( D^b(X) \) come from some cotilting torsion pair in \( \text{Coh}(X) \). All such cotilting torsion pairs (up to an action of \( \text{Aut}(D^b(X)) \)) have been listed in [GKR04]. I am especially thankful to S. A. Kuleshov for explaining to me the above argument.

Since \( \mathcal{C}^\theta \) has cohomological dimension 1, if it were equivalent to \( \text{Coh}(Y) \) for some \( Y \), then \( Y \) had better be a smooth curve (c.f., Fact 2). The problem of dealing with categories of holomorphic bundles on \( A^\theta \) has been reduced to studying t-structures on \( D^b(X) \).

We have already seen some technical conditions involving a categorical incarnation of “ampleness” \[ \] to verify when a given \( k \)-linear (\( k = \mathbb{C} \) now) abelian category is of the form \( \text{Proj} \mathcal{R} \) for some graded \( k \)-algebra \( \mathcal{R} \). One of the requirements of \[ \] is that the category be locally Noetherian (i.e., the category has a Noetherian set of generators). Unfortunately, this condition fails to be true in our situation.

Proposition 3.10. \( \theta \) irrational implies that every nonzero object in \( \mathcal{C}^\theta \) is not Noetherian.

We would still like to say that what we have seen so far was not entirely useless. Recall that in our discussion of \( \text{Proj} \mathcal{R} \) after Artin and Zhang we had assumed our graded algebra to be right Noetherian. Polishchuk has shown that even if one dispenses with the Noetherian assumption there is a way to recover Serre’s theorem [2.8]. He gives an analogue of an “ample sequence of objects” and proves that if a \( k \)-linear abelian category has an ample sequence of objects then it is equivalent to “cohproj \( \mathcal{R} \)”, where \( \mathcal{R} \) is a “coherent” \( \mathbb{Z} \)-algebra. Unfortunately the words in quotes in the previous sentence will not be explained anymore. Interested readers are encouraged to look them up from [Pol05]. Finally, as an apt culmination of all our efforts we have the following theorem [Pol04b].

Theorem 3.11 (Polishchuk). For every quadratic irrationality \( \theta \in \mathbb{R} \) there exists an auto-equivalence \( F : D^b(X) \rightarrow D^b(X) \) preserving \( \mathcal{C}^\theta \) and a stable object \( \mathcal{G} \in \mathcal{C}^\theta \) such that the sequence \( \langle F^n\mathcal{G}, n \in \mathbb{Z} \rangle \) is ample (in the modified sense of Polishchuk). Hence, the corresponding algebra \( A_{F,\mathcal{G}} := \Gamma_h(\mathcal{C}^\theta, \mathcal{G}, F) \) is right coherent and \( \mathcal{C}^\theta \simeq \text{cohproj} A_{F,\mathcal{G}} \).

Remark 3.12. There is some anomaly in the choice of the algebra \( \mathcal{R} \), whose cohproj \( \mathcal{R} \) should be equivalent to \( \mathcal{C}^\theta \). However, even in the commutative case one can show that if \( S \) and \( S' \) are two graded commutative rings, such that \( S_n \cong S'_n \) for all \( n >> 0 \), then \( \text{Proj} S \simeq \text{Proj} S' \) (commutative Proj construction).

Final Remark: “Another perspective for the future work is to try to connect our results with Manin’s program in [Man06] to use noncommutative two-tori with real multiplication for the explicit construction of the maximal Abelian extensions of real quadratic fields”. ——— A. Polishchuk.

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