Multiscale analysis of the acoustic scattering by many scatterers of impedance type

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Abstract

We are concerned with the acoustic scattering problem, at a frequency $\kappa$, by many small obstacles of arbitrary shapes with impedance boundary condition. These scatterers are assumed to be included in a bounded domain $\Omega$ in $\mathbb{R}^3$ which is embedded in an acoustic background characterized by an eventually locally varying index of refraction. The collection of the scatterers $D_m$, $m = 1, ..., M$ is modeled by four parameters: their number $M$, their maximum radius $a$, their minimum distance $d$ and the surface impedances $\lambda_m$, $m = 1, ..., M$. We consider the parameters $M, d$ and $\lambda_m$’s having the following scaling properties: $M := M(a) = O(a^{3})$, $d := d(a) \approx a$ and $\lambda_m := \lambda_m(a) = \lambda_{m,0}a^{-\beta}$, as $a \to 0$, with non negative constants $s, t$ and $\beta$ and complex numbers $\lambda_{m,0}$’s with eventually negative imaginary parts.

We derive the asymptotic expansion of the farfields with explicit error estimate in terms of $a$, as $a \to 0$. The dominant term is the Foldy-Lax field corresponding to the scattering by the point-like scatterers located at the centers $z_m$’s of the scatterers $D_m$’s with $\lambda_m|_{\partial D_m}$ as the related scattering coefficients. This asymptotic expansion is justified under the following conditions

$$a \leq a_0, \ |\Re(\lambda_{m,0})| \geq \lambda_-, \ |\lambda_{m,0}| \leq \lambda_+ , \ \beta < 1 , \ 0 \leq s \leq 2 - \beta, \ \frac{s}{3} \leq t$$

and the error of the approximation is $C\ a^{3-2\beta-s}$, as $a \to 0$, where the positive constants $a_0, \lambda_-, \lambda_+$ and $C$ depend only on the apriori uniform bounds of the Lipschitz characters of the obstacles $D_m$’s and the ones of $M(a)a^{s}$ and $\frac{d(a)}{a}$. We do not assume the periodicity in distributing the small scatterers. In addition, the scatterers can be arbitrary close since $t$ can be arbitrary large, i.e. we can handle the mesoscale regime. Finally, for spherical scatterers, we can also allow the limit case $\beta = 1$ with a slightly better error of the approximation.

Keywords: Acoustic scattering, Small-scatterers, Multiple scattering, Foldy-Lax approximation.

1 Introduction and statement of the results

Let $B_1, B_2, ..., B_M$ be $M$ open, bounded and simply connected sets in $\mathbb{R}^3$ with Lipschitz boundaries containing the origin. We assume that the Lipschitz constants of $B_j$, $j = 1, ..., M$ are uniformly bounded. We set $D_m := \epsilon B_m + z_m$ to be the small bodies characterized by the parameter $\epsilon > 0$ and the locations $z_m \in \mathbb{R}^3$, $m = 1, ..., M$. Let $U^i$ be a solution of the Helmholtz equation $(\Delta + \kappa^2)U^i = 0$ in $\mathbb{R}^3$. We denote by $U^s$ the acoustic field scattered by the $M$ small bodies $D_m \subset \mathbb{R}^3$, due to the incident field $U^i$ (mainly the plane incident waves $U^i(x, \theta) := e^{i\kappa x \cdot \theta}$ with the incident direction $\theta \in S^2$, where $S^2$ being the unit sphere), with impedance boundary conditions. Hence the total field $U^t := U^i + U^s$ satisfies the following exterior impedance problem of the acoustic waves

$$(\Delta + \kappa^2)U^t = 0 \text{ in } \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M \bar{D}_m \right), \quad (1.1)$$

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\[
\frac{\partial U^t}{\partial \nu_m} + \lambda_m U^t \bigg|_{\partial D_m} = 0, \quad 1 \leq m \leq M, \quad (1.2)
\]
\[
\frac{\partial U^s}{\partial |x|} - i \kappa U^s = o \left( \frac{1}{|x|} \right), \quad |x| \to \infty, \quad \text{(S.R.C)} \quad (1.3)
\]
where \( \nu_m \) is the outward unit normal vector of \( \partial D_m \), \( \kappa > 0 \) is the wave number and S.R.C stands for the Sommerfeld radiation condition. The scattering problem (1.1,1.3) is well posed in the Hölder or Sobolev spaces, see [13, 14, 22, 24] for instance, in the case \( \exists \lambda_m > 0 \). We will see that this condition can be relaxed to allow \( \exists \lambda_m \) to be negative. Applying Green’s formula to \( U^s \), we can show that the scattered field \( U^s(x, \theta) \) has the following asymptotic expansion:
\[
U^s(x, \theta) = \frac{e^{i \kappa s|x|}}{4\pi|x|} U^\infty(\hat{x}, \theta) + O(|x|^{-2}), \quad |x| \to \infty, \quad (1.4)
\]
with \( \hat{x} := \frac{x}{|x|} \), where the function \( U^\infty(\hat{x}, \theta) \) for \( (\hat{x}, \theta) \in \mathbb{S}^2 \times \mathbb{S}^2 \) is called the far-field pattern. We recall that the fundamental solution, \( \Phi_\kappa(x, y) \), of the Helmholtz equation in \( \mathbb{R}^3 \) with the fixed wave number \( \kappa \) is given by
\[
\Phi_\kappa(x, y) := \frac{e^{i \kappa |x-y|}}{4\pi|x-y|}, \quad \text{for all } x, y \in \mathbb{R}^3.
\]

**Definition 1.1.** We define

1. \( a := \max_{1 \leq m \leq M} \text{diam}(D_m) \cdot \epsilon \max_{1 \leq m \leq M} \text{diam}(B_m) \),
2. \( d := \min_{1 \leq m \neq j \leq M} d_{mj}, \text{where } d_{mj} := \text{dist}(D_m, D_j) \).
3. \( \kappa_{\text{max}} \) as the upper bound of the used wave numbers, i.e. \( \kappa \in [0, \kappa_{\text{max}}] \).

The distribution of the scatterers is modeled as follows:

4. the number \( M := M(a) := O(a^{-s}) \leq M_{\text{max}} a^{-s} \) with a given positive constant \( M_{\text{max}} \)

5. the minimum distance \( d := d(a) \approx a^t \), i.e. \( a^t \leq d(a) \leq d_{\text{max}} a^t \), with given positive constants \( d_{\text{min}} \) and \( d_{\text{max}} \).

6. the surface impedance \( \lambda_m := \lambda_{m,0} a^{-\beta} \), where \( \lambda_{m,0} \neq 0 \) and might be a complex number.

Here the real numbers \( s, t \) and \( \beta \) are assumed to be non negative.

We call the upper bounds of the Lipschitz character of \( B_m \)'s, \( M_{\text{max}}, d_{\text{min}}, d_{\text{max}} \) and \( \kappa_{\text{max}} \) the set of the apriori bounds.

The goal of our work is to derive an asymptotic expansion of the scattered field by the collection of the small scatterers \( D_m, m = 1, \ldots, M \), taking into account these parameters. This is the object of the following theorem.

**Theorem 1.2.** There exist positive constants \( a_0, \lambda_- \) and \( \lambda_+ \) depending only on the set of the apriori bounds such that if
\[
a \leq a_0, |\lambda_{m,0}| \leq \lambda_+, |\Re(\lambda_{m,0})| \geq \lambda_-, \quad \beta < 1, \quad s \leq 2 - \beta, \quad \frac{s}{3} \leq t \quad (1.5)
\]
then the far-field pattern \( U^\infty(\hat{x}, \theta) \) has the following asymptotic expansion
\[
U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} e^{-i \kappa x \cdot z_m} Q_m + O \left( a^{3 - s - 2\beta} \right), \quad (1.6)
\]
uniformly in \( \hat{x} \) and \( \theta \) in \( \mathbb{S}^2 \). The constant appearing in the estimate \( O(\cdot) \) depends only on the set of the \( 3 \) apriori bounds, \( \lambda_- \) and \( \lambda_+ \). The coefficients \( Q_m, m = 1, \ldots, M \), are the solutions of the following linear algebraic system

\[
Q_m + \sum_{j=1}^{M} C_m \Phi_n(z_j, z_j) Q_j = -C_m U^s(z_m, \theta),
\]

for \( m = 1, \ldots, M \), with \( C_m := -\lambda_m |\partial D_m| \).

The algebraic system (1.7) is invertible under the conditions:

\[
|\Re(\lambda_{m,0})| \geq \lambda_-, \ s \leq 2 - \beta.
\]

Let us now assume that the background is not homogeneous but modeled by a locally variable index of refraction \( n \), i.e. there is a bounded set \( \Omega \) such that \( n(x) = 1, \ x \in \mathbb{R}^3 \setminus \Omega \) and bounded inside \( \Omega \), i.e. \( |n(x)| \leq n_{\text{max}}, \ x \in \Omega \). In this case we model our scattering problem as follows. The total field \( U^t := U^s + U^s \) satisfies the following exterior impedance problem of the acoustic waves

\[
(\Delta + \kappa^2 n^2(x)) U^t_0 = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} D_m \right),
\]

\[
\partial U^t_0 / \partial n_m + \lambda_m U^t_0 \bigg|_{\partial D_m} = 0, \ 1 \leq m \leq M,
\]

\[
\partial U^s / \partial |x| - i \kappa U^s = o \left( \frac{1}{|x|} \right), \ |x| \to \infty,
\]

Again, the scattering problem (1.9)-11 is well posed in the H"older or Sobolev spaces, see [13-14-22] in the case \( \Im \lambda_m > 0 \). As we said for (1.9-11), this last condition can be relaxed to allow \( \Im \lambda_m \) to be negative. Applying Green’s formula to \( U^t_0 \), we can show that the scattered field \( U^s(x, \theta) \) has the following asymptotic expansion:

\[
U^s_0(x, \theta) = \frac{e^{i|\hat{x}|}}{4\pi|x|} U^\infty_0(\hat{x}, \theta) + O(|x|^{-2}), \quad |x| \to \infty,
\]

where the function \( U^\infty_0(\hat{x}, \theta) \) for \( (\hat{x}, \theta) \in \mathbb{S}^2 \times \mathbb{S}^2 \) is the corresponding far-field pattern. As a corollary of Theorem 1.2 we derive the following result:

**Corollary 1.3.** There exist positive constants \( a_0, \lambda_-, \lambda_+ \) depending only on the set of the apriori bounds and on \( n_{\text{max}} \) such that if

\[
a \leq a_0, \ |\lambda_{m,0}| \leq \lambda_+, \ |\Re(\lambda_{m,0})| \geq \lambda_-, \ \beta < 1, \ s \leq 2 - \beta, \ \frac{s}{3} \leq t
\]

then the far-field pattern \( U^\infty_n(\hat{x}, \theta) \) has the following asymptotic expansion

\[
U^\infty_n(\hat{x}, \theta) = V^\infty_n(\hat{x}, \theta) + \sum_{m=1}^{M} V^t_n(z_m, -\hat{x}) Q_m + O \left( a_0^{s-\beta} \right),
\]

uniformly in \( \hat{x} \) and \( \theta \) in \( \mathbb{S}^2 \). The constant appearing in the estimate \( O(\cdot) \) depends only on the set of the apriori bounds, \( \lambda_- \), \( \lambda_+ \) and \( n_{\text{max}} \). The quantity \( V^t_n(z_m, -\hat{x}) \) is the total field, evaluated at the point \( z_m \) in the direction \( -\hat{x} \), corresponding to the scattering problem

\[
(\Delta + \kappa^2 n^2(x)) V^t_n = 0 \quad \text{in} \quad \mathbb{R}^3,
\]

\[1\] The following remarks apply for both Theorem 1.2 and Corollary 1.3. If \( s < 2 - \beta \), then the bounds \( \lambda_- \) and \( \lambda_+ \) can be arbitrary. If \( \Re \lambda_m \leq 0 \) then, for the invertibility of the systems (1.7) and (1.17), the condition \( s \leq 2 - \beta \) on the number of the small bodies can be replaced by the condition \( t \leq 2 - \beta \) on the minimum distance between them, see Remark 6.1.

\[2\] By the mixed reciprocity relations, we have \( V^\infty_n(z_m, -\hat{x}) = G^\infty_n(\hat{x}, z_m) \) where \( G^\infty_n(\hat{x}, z_m) \) is the farfield created by a point source \( G_n(x, z_m) \), located at the point \( z_m \), in the scattering model (1.15)-16.
\[ \frac{\partial V_n^s}{\partial |x|} - i\kappa V_n^s = o \left( \frac{1}{|x|} \right), \quad |x| \to \infty, \]  

(1.16)

The coefficients \( Q_m, m = 1, \ldots, M \), are the solutions of the following linear algebraic system

\[ Q_m + \sum_{j=1}^{M} C_m G_\kappa(z_m, z_j) Q_j = -C_m V^t(z_m, \theta), \]  

(1.17)

for \( m = 1, \ldots, M \), with

\[ C_m := -\lambda_m |\partial D_m|. \]  

(1.18)

Here \( G_\kappa(x, z) \) is the outgoing Green’s function corresponding to the scattering problem (1.15–1.16).

The algebraic system (1.17) is invertible under the conditions:

\[ |\Re(\lambda_{m,0})| \geq \lambda_-, \quad s \leq 2 - \beta. \]  

(1.19)

Before discussing these results comparing them to some of the literature, we add the following remark on the particular case when the scatterers are spherical.

**Remark 1.4.** If the scatterers have spherical shapes, then we can slightly improve the error estimates in (1.14) by slightly changing the form of the algebraic system (1.17). Precisely we have \( O(a^{3-s-\beta}) \) instead of \( O(a^{3-s-2\beta}) \). For these spherical shapes, we can also handle the case \( \beta = 1 \). The result reads as follows.

There exist positive constants \( a_0, \lambda_-, \lambda_+ \) depending only on the set of the apriori bounds and on \( n_{\text{max}} \) such that if

\[ a \leq a_0, \quad |\lambda_{m,0}| \leq \lambda_+, \quad |\Re(\lambda_{m,0})| \geq \lambda_-, \quad \beta \leq 1, \quad s \leq 2 - \beta, \quad \frac{s}{3} \leq t \]  

(1.20)

then the far-field pattern \( U^\infty_n(\hat{x}, \theta) \) has the following asymptotic expansion

\[ U^\infty_n(\hat{x}, \theta) = V^\infty_n(\hat{x}, \theta) + \sum_{m=1}^{M} V^t_n(z_m, -\hat{x}) Q_m + O(a^{3-s-\beta}), \]  

(1.21)

uniformly in \( \hat{x} \) and \( \theta \) in \( S^2 \). The coefficients \( Q_m, m = 1, \ldots, M \), are the solutions of the following linear algebraic system

\[ Q_m + \sum_{j=1}^{M} C_m G_\kappa(z_m, z_j) Q_j = -C_m V^t(z_m, \theta), \]  

(1.22)

for \( m = 1, \ldots, M \), where now we have a slight change in the constant \( C_m \) compared to (1.18), i.e.

\[ C_m := \frac{\lambda_m |\partial D_m|}{-1 + \lambda_m I_m}. \]  

(1.23)

where \( I_m := \int_{\partial D_m} \Phi_0(s_m, t)ds_m, \quad t \in \partial D_m, \) is a constant if \( D_m \) is a ball.\(^3\)

This algebraic system is invertible under the same conditions as (1.17).

These results say that the dominant term in the approximation (1.6), and similarly the one in (1.14), is the Foldy-Lax field corresponding to the scattering by the point-like scatterers located at the ‘centers’ \( z_m \)'s

\(^3\) The property that \( I_m \) is a constant only for balls is known as Gruber’s conjecture and it is solved for the 2D for Lipschitz regular domains and in 3D for convex domains, see [18]. This is the main restriction why we cannot obtain in Theorem 1.2 and Corollary 1.3 the same results as in Remark 1.4 for general shapes. Actually, if \( D_m \) is a ball then the value of \( I_m \) is nothing but the radius of \( D_m (= \epsilon \text{ radius}(B_m)) \).
of the scatterers $D_m$’s with $\lambda_m |\partial D_m|$ as the related scattering coefficients. This Foldy-Lax field describes the field that results in the multiple scattering between the different scatterers $D_m$’s, see [17] and the references therein for more information on this issue. The accuracy of the approximation of the scattered field by the Foldy-Lax field depends of course on the error estimates. In its generality, this issue is still largely open but there is an increase of interest to understand it, see for instance [7,9,11,19,21].

The formal asymptotic expansion (1.6) was already given in [25,26]. The results in Theorem 1.2 and in Corollary 1.3 provide the rigorous justification of the approximation of the scattered field taking into account all the involved parameters: the maximum diameter $a$, the number $M$, the minimum distance $d$ and the surface impedance $\lambda_m$’s including the regime defined by $M = M(a) := a^{-s}$, $d := d(a) := a^t$, $\lambda_m := \lambda_m(a) := \lambda_m,0 a^{-\beta}$. We characterized the set of parameters $s,t$ and $\beta$ where the approximation makes sense and we provided the approximation with explicit error estimates in terms of these parameters. These approximation formulas can be used for different purposes:

1. First, they can be used for imaging where the small anomalies are modeled by the small bodies. There is a large literature on the mathematical imaging of small anomalies, see for instance [3,4] and the references therein. Compared to those results, we can allow the small bodies to be dense and very close, since their number $M$ can be very large and the minimum distance $d$ can be as small as we want. This last property means that we deal with the mesoscale regime, where $\min_{j \neq m} \text{dist}(z_j, z_m) \sim 1$ and $\min_{j \neq m} \text{dist}(z_j, z_m)$ is the minimum distance between the centers of the small scatterers, compare with [19,20].

2. Second, these approximations can be used for the design of new indices of refraction. When the small scatterers are periodically distributed, this can be justified by homogenization, see [8,12,15,16]. Using the approximations in (1.6) and (1.14), we do not need such periodicity in the distribution of the small scatterers. This observation has been already made in [26] with quite formal computations. Let us observe that in our approximations in Theorem 1.2 and in Corollary 1.3, we allow the surface impedance to have negative imaginary parts. With this freedom of taking the surface impedance, we can generate a large class of indices of refraction with possible applications to the acoustic metamaterials. Details on this issue will be reported in a forthcoming work. Let us emphasize that we provide those results with explicit error estimates. This gives more credits to the feasibility of the design process, see our previous work [1] where this application is discussed.

3. Third, these approximations can be used to extract the values of the index of refraction $n(x), x \in \Omega$, from the farfields corresponding to few incident directions. The idea of adding small inclusions to deform the medium and then extract the field inside the support of the medium is already described and used in the literature, see for instance [2]. The extraction of the index of refraction (or other coefficients) from the internal measurements, related also to the hybrid imaging methods, has recently attracted the attention of many authors, see for instance [5,6] and the references therein. One of the main difficulties is to handle the possible zeros of the total field. What we propose is to deform the medium using multiple (and close) inclusions instead of only single ones. In this case, what we derive from the asymptotic expansions are the internal values of the Green’s function and not only the total fields. Finally, the values of the index of refraction can be extracted from the singularities of these Green’s function. Hence, we avoid the problems coming from the zeros of the internal fields. These arguments will be reported in a forthcoming work.

The rest of the paper is devoted to prove Theorem 1.2 and Corollary 1.3. The proof of Theorem 1.2 is done in sections 2 and 3 while the one of Corollary 1.3 is given in section 4. The justification of Remark 1.4 is discussed in section 5. In the appendix, we deal with invertibility of the algebraic system (1.7) (and similarly for (1.17)).
2 Proof of Theorem [1.2]

2.1 The representation via layer potential

We start with the following proposition on the solution of the problem (1.1-1.3) via the layer potential representation.

Proposition 2.1. Assume that the negative part of the imaginary part of \( \kappa \) are small enough. In addition, suppose that \( \kappa^2 \) is not a Dirichlet eigenvalue of the Laplacian in \( D_m \), for \( m = 1, \ldots, M \). Then for \( m = 1, 2, \ldots, M \), there exists \( \sigma_m \in L^2(\partial D_m) \) such that the problem (1.1-1.3) has one and a unique solution and it is of the form

\[
U^i(x) = U^i(x) + \sum_{m=1}^M \int_{\partial D_m} \Phi_\kappa(x, s) \sigma_m(s) \, ds, \quad x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M D_m \right),
\]

(2.1)

Proof of Proposition 2.1. We look for the solution of the problem (1.1-1.3) of the form (2.1), then from the impedance boundary condition (1.2), we obtain

\[
- \frac{\sigma_j(s_j)}{2} + \int_{\partial D_j} \frac{\partial \Phi_\kappa(s_j, s)}{\partial \nu_j(s)} \sigma_j(s) \, ds + \sum_{m=1}^M \int_{\partial D_m} \frac{\partial \Phi_\kappa(s_j, s)}{\partial \nu_j(s)} \sigma_m(s) \, ds
\]

(2.2)

One can write it in a compact form as \((-\frac{1}{2} I + DL^* + DK^* + \lambda(L + K))\sigma = -(\partial_\nu - \lambda)U^{in} \) with \( \partial_\nu := (\partial_{\nu m})_{m,j=1}^M, \lambda := (\lambda_{m,j})_{m,j=1}^M, DL^* := (DL^*_{m,j})_{m,j=1}^M, DK^* := (DK^*_{m,j})_{m,j=1}^M, L := (L_{m,j})_{m,j=1}^M \) and \( K := (K_{m,j})_{m,j=1}^M \), where

\[
\mathbf{I}_{mj} = \begin{cases} 
I, & m = j \\
0, & \text{else} 
\end{cases}, \quad DL^*_{mj} = \begin{cases} 
D^*_{mj}, & m = j \\
0, & \text{else} 
\end{cases}, \quad DK^*_{mj} = \begin{cases} 
D^*_{mj}, & m \neq j \\
0, & \text{else} 
\end{cases}
\]

(2.3)

\[
\partial_{\nu m} = \begin{cases} 
\partial_{\nu m}, & m = j \\
0, & \text{else} 
\end{cases}, \quad \lambda_{mj} = \begin{cases} 
\lambda_m, & m = j \\
0, & \text{else} 
\end{cases}, \quad L_{mj} = \begin{cases} 
S_{mj}, & m = j \\
0, & \text{else} 
\end{cases}, \quad K_{mj} = \begin{cases} 
S_{mj}, & m \neq j \\
0, & \text{else} 
\end{cases}
\]

(2.4)

\[
U^i = U^i(s_1, \ldots, s_M) := (U^i_1(s_1), \ldots, U^i_M(s_M))^T \quad \text{and} \quad \sigma = (\sigma_1(s_1), \ldots, \sigma_M(s_M))^T.
\]

Here, for the indices \( m \) and \( j \) fixed, \( S_{mj} \) is the integral operator acting as

\[
S_{mj}(\sigma_j)(t) := \int_{\partial D_j} \Phi_\kappa(t, s) \sigma_j(s) \, ds, \quad t \in \partial D_m,
\]

(2.5)

and \( D^*_{mj} \) is the adjoint of the integral operator defined by,

\[
D_{mj}(\sigma_j)(t) := \int_{\partial D_j} \frac{\partial \Phi_\kappa(t, s)}{\partial \nu_m(s)} \sigma_j(s) \, ds, \quad t \in \partial D_m.
\]

(2.6)

\footnote{This last condition is satisfied for every \( \kappa \) such that \( \kappa \leq \kappa_{\text{max}} \) and \( a < \frac{1}{\kappa_{\text{max}}} \frac{1}{1 + \sqrt{1 + \frac{4}{\kappa_{\text{max}}}}} \). Here, \( J_{1/2,1} \) is the 1st positive zero of the Bessel function \( J_{1/2,1} \).}

\footnote{The result of this proposition is valid regardless of the smallness of the obstacles \( D_m \)'s nor the conditions on \( M, d \) and \( \lambda_m \)'s. The emphasis is on the possibility to deal with surface impedance eventually having negative imaginary parts.}
Then the operator \( D_{mm}^* : L^2(\partial D_m) \to L^2(\partial D_m) \) is adjoint of the double layer operator \( D_{mm} \), defined by,

\[
D_{mm}(\sigma_m)(t) := \int_{\partial D_m} \frac{\partial \Phi_m(t,s)}{\partial n_m(s)} \sigma_j(s) ds, \quad t \in \partial D_m,
\]

and the operator \(-\frac{1}{2}I + D_{mm}^* : L^2(\partial D_m) \to L^2(\partial D_m)\) is isomorphic and hence Fredholm with zero index. For \( m \neq j \), \( D_{mj}^* : L^2(\partial D_j) \to L^2(\partial D_m) \) is compact, see [23, Theorem 4.1].

Also notice that \( S_{mj} : L^2(\partial D_j) \to L^2(\partial D_m) \) is compact. So, \(-\frac{1}{2}I + DL^* + DK^* + \lambda(L + K)\) : \( \prod_{m=1}^M L^2(\partial D_m) \to \prod_{m=1}^M L^2(\partial D_m) \) is Fredholm with zero index. We induce the product of spaces by the maximum of the norms of the space. To show that \((\ref{eq:2.7})\) is invertible it is enough to show that it is injective. i.e. \((-\frac{1}{2}I + DL^* + DK^* + \lambda(L + K))\sigma = 0\) implies \(\sigma = 0\). We write

\[
\tilde{U}(x) = \sum_{m=1}^M \int_{\partial D_m} \Phi_m(x,s) \sigma_m(s) ds, \quad \text{in } \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M \bar{D}_m \right)
\]

and

\[
\tilde{\tilde{U}}(x) = \sum_{m=1}^M \int_{\partial D_m} \Phi_m(x,s) \sigma_m(s) ds, \quad \text{in } \bigcup_{m=1}^M D_m.
\]

Then \(\tilde{U}\) satisfies \(\Delta \tilde{U} + \kappa^2 \tilde{U} = 0\) for \(x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M \bar{D}_m \right)\), with S.R.C and \((\partial_{\nu_m} + \lambda_m)\tilde{U}(x) = 0\) on \(\bigcup_{m=1}^M \partial D_m\).

Let us assume for the moment that the exterior problem has a unique solution, then we deduce that \(\tilde{U} = 0\) on \(\bigcup_{m=1}^M \partial D_m\). In addition, we know that \(\Delta \tilde{\tilde{U}} + \kappa^2 \tilde{\tilde{U}} = 0\) for \(x \in \bigcup_{m=1}^M D_m\). From the condition on \(\kappa^2\), we deduce that \(\tilde{\tilde{U}} = 0\) in \(\bigcup_{m=1}^M D_m\).

By the jump relations, we have

\[
\frac{\partial \tilde{U}}{\partial \nu}(x) + \lambda_m \tilde{U}(x) = 0 \quad \implies (K^* \sigma_m)(x) - \frac{\sigma_m(x)}{2} + \sum_{j=1}^M (D_{mj}^* + \lambda_m S_{mj})(\sigma_j)(x) = 0 \quad (2.8)
\]

and

\[
\frac{\partial \tilde{\tilde{U}}}{\partial \nu}(x) + \lambda_m \tilde{\tilde{U}}(x) = 0 \quad \implies (K^* \sigma_m)(x) + \frac{\sigma_m(x)}{2} + \sum_{j=1}^M (D_{mj}^* + \lambda_m S_{mj})(\sigma_j)(x) = 0 \quad (2.9)
\]

for \(x \in \partial D_m\) and for \(m = 1, \ldots, M\). Here, \(K^*\) is the adjoint of the double layer operator \(K\),

\[
(K^* \sigma_m)(x) := \int_{\partial D_m} \frac{\partial}{\partial \nu_s} \Phi(x,s) \sigma_m(s) ds, \quad \text{for } m = 1, \ldots, M.
\]

Difference between (2.8) and (2.9) provides us, \(\sigma_m = 0\) for all \(m\).

We conclude then that \(-\frac{1}{2}I + DL^* + DK^* + \lambda(L + K) =: -\frac{1}{2}I + D^* + \lambda S : \prod_{m=1}^M L^2(\partial D_m) \to \prod_{m=1}^M L^2(\partial D_m)\) is invertible.

\(^6\)Observe that \(-\frac{1}{2}I + D_{mm}^*\) is the adjoint of \(-\frac{1}{2}I + D_{mm}\). Hence \(-\frac{1}{2}I + D_{mm}^*\) is Fredholm if we show that \(-\frac{1}{2}I + D_{mm}\) is Fredholm if we show that \(-\frac{1}{2}I + D_{mm}\) is invertible. In [23], this last property is proved for the case \(\kappa = 0\) and by a perturbation argument, we can obtain the same results for every \(\kappa\). Using the condition on \(\kappa^2\), we deduce that \(-\frac{1}{2}I + D_{mm}^*\) is an isomorphism.
We need now to show that the exterior problem $\Delta \hat{U} + \kappa^2 \hat{U} = 0$ for $x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \tilde{D}_m \right)$, with S.R.C. and $(\partial_{\nu} + \lambda_m)\hat{U}(x) = 0$ on $\bigcup_{m=1}^{M} \partial D_m$, has a unique solution. This result is known under the condition $\Im \lambda_m > 0$ (or $\Im \lambda_m = 0$) on $\bigcup_{m=1}^{M} \partial D_m$, see [13] for instance. To relax this positivity condition, we proceed as follows. Set $f := (f_1, ..., f_M)$ with $f_m := i(\Im \lambda)_- \hat{U}$ on $\partial D_m$, where $(\Im \lambda)_- := \max_{m=1}^{M} \partial D_m \{ -\Im \lambda_m, 0 \}$. Hence $\hat{U}$ satisfies $\Delta \hat{U} + \kappa^2 \hat{U} = 0$ for $x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \tilde{D}_m \right)$, with S.R.C and $(\partial_{\nu} + \lambda_m + i(\Im \lambda)_-)\hat{U}(x) = f_m$ on $\bigcup_{m=1}^{M} \partial D_m$. Since now $\Im (\lambda_m + i(\Im \lambda)_-) > 0$, then this last problem has a unique solution and it can be represented via single layer potentials $\sum_{m=1}^{M} S(\psi_j)$. Taking the normal trace on $\partial D_m$’s, we deduce that $\left( -\frac{1}{2} I + DL^* + DK^* + (\lambda + i(\Im \lambda)_-)(L + K) \right) \psi = f$, where $\psi := (\psi_1, ..., \psi_M)$. But from the form of $f$, we obviously have $f = i(\Im \lambda)_-(L + K)\psi$. Hence $\left( -\frac{1}{2} I + DL^* + DK^* + \lambda(L + K) \right) \psi = 0$. We know that the operator $-\frac{1}{2} I + DL^* + DK^* + (\lambda + i(\Im \lambda)_-)(L + K)$ is invertible. Hence if the negative part of the imaginary part of $\lambda$ is small so that

$$(\Im \lambda)_- \| L + K \| \| (-\frac{1}{2} I + DL^* + DK^* + (\lambda + i(\Im \lambda)_-)(L + K))^{-1} \| < 1,$$

then by the Neumann series expansion $-\frac{1}{2} I + DL^* + DK^* + \lambda(L + K)$ is also invertible and hence $\psi = 0$. $\blacksquare$

### 2.2 An appropriate estimate of the densities $\sigma_m$, $m = 1, \ldots, M$

From the above theorem, we have the following representation of $\sigma$:

$$\sigma = -(-\frac{1}{2} I + DL^* + DK^* + \lambda(L + K))^{-1}(\partial_{\nu} + \lambda)U^{I_n}$$

$$= -(-\frac{1}{2} I + DL^* + \lambda L)^{-1}(I + (-\frac{1}{2} I + DL^* + \lambda L)^{-1}(DK^* + \lambda K))^{-1}(\partial_{\nu} + \lambda)U^{I_n}$$

$$= -(-\frac{1}{2} I + DL^* + \lambda L)^{-1} \sum_{i=0}^{\infty} \left( (-\frac{1}{2} I + DL^* + \lambda L)^{-1}(DK^* + \lambda K) \right)^i (\partial_{\nu} + \lambda)U^{I_n},$$

if $\|(-\frac{1}{2} I + DL^* + \lambda L)^{-1}(DK^* + \lambda K)\| < 1$. By the assumption $a < \frac{1}{\kappa_{\max}} \sqrt{\frac{4\pi}{3}} J_{1/2,1}$, the operator $-\frac{1}{2} I + DL^*$ is invertible. We write $-\frac{1}{2} I + DL^* + \lambda L = (I + \lambda L(-\frac{1}{2} I + DL^*)^{-1})(-\frac{1}{2} I + DL^*)$. From the scaling of the operators $L$ and $(-\frac{1}{2} I + DL^*)$, we show that there exists a constant $\tilde{c}$, depending only on the Lipschitz character of the reference bodies $B_m$’s, such that if

$$\|\lambda\|a = \max_{m=1} \|\lambda_{m,0}\|a^{1-\beta} < \tilde{c},$$

we have $\|\lambda L(-\frac{1}{2} I + DL^*)^{-1}\| < 1$ and hence the operator $-\frac{1}{2} I + DL^* + \lambda L$ is invertible. The condition \[2.13\] is verified if $\beta < 1$ and $a$ small enough. For the convinience, we denote $\|\lambda\|$ by $|\lambda|$.

This implies that

$$\|\sigma\| \leq \frac{\|(-\frac{1}{2} I + DL^* + \lambda L)^{-1}\|}{1 - \|(-\frac{1}{2} I + DL^* + \lambda L)^{-1}\| \|DK^* + \lambda K\|} \|\partial_{\nu} + \lambda\|U^{I_n} \|. \tag{2.14}$$

Here we use the following notations:

$$\|DK^* + \lambda K\| := \|DK^* + \lambda K\|_{L^2 \left( \bigcup_{m=1}^{M} L^2(\partial D_m), \prod_{m=1}^{M} L^2(\partial D_m) \right)}$$
Let us consider a single obstacle 2.2.1 The case of a single obstacle

We divide the rest of the proof of Proposition 2.2 into two steps. In the first step, we assume we have a

we use the notations;

There exists \( \| \) and then estimate \( s \)

In the following proposition, we provide conditions under which \( 0 < \epsilon \leq 1 \) and \( s \leq 2 - \beta \), then we have the following estimate

\[
\| \sigma \|_{L^2(\partial D_m)} \leq c \epsilon^{1-\beta}
\]

where \( c \) is a positive constant depending only on the set of the apriori bounds.

Proof of Proposition 2.2

Suppose \( 0 < \epsilon \leq 1 \) and \( D_\epsilon := \epsilon B + z \subset \mathbb{R}^n \). For any functions \( f, g \) defined on \( \partial D_\epsilon \) and \( \partial B \) respectively, we use the notations:

\[
(f)^{\wedge}(\xi) := \hat{f}(\xi) := f(\epsilon \xi + z) \quad \text{and} \quad (g)^{\wedge}(x) := \hat{g}(x) := g\left(\frac{x - z}{\epsilon}\right). \tag{2.20}
\]

Then for each \( \psi \in L^2(\partial D_\epsilon) \), we have

\[
\| \psi \|_{L^2(\partial D_\epsilon)} = \epsilon^{\frac{n-1}{2}} \| \hat{\psi} \|_{L^2(\partial B)}. \tag{2.21}
\]

We divide the rest of the proof of Proposition 2.2 into two steps. In the first step, we assume we have a single obstacle and then in the second step we deal with the multiple obstacles case.

2.2.1 The case of a single obstacle

Let us consider a single obstacle \( D_\epsilon := \epsilon B + z \) with unit outword normal \( \nu \) to its boundary. Then define the operator \( D_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon) \) by

\[
(D_{D_\epsilon} \psi)(s) = \int_{\partial D_\epsilon} \frac{\Phi(s,t)}{\partial \nu(t)} \psi(t) dt. \tag{2.22}
\]
Following the arguments in the proof of Proposition 2.1, the integral operator $-(1/2)I + D_{D_2}^* : L^2(\partial D_2) \rightarrow L^2(\partial D_2)$ is invertible. If we consider the problem (1.1-1.3) in $\mathbb{R}^3 \setminus \overline{D}$, we obtain

$$\sigma = (-1/2 + D_{D_2}^* + \lambda S_{D_2})^{-1}(\partial \nu + \lambda)U,$$

and then

$$\|\sigma\|_{L^2(\partial D_2)} \leq \|(-1/2 + D_{D_2}^* + \lambda S_{D_2})^{-1}\|_{L^2(\partial D_2)}\|((\partial \nu + \lambda)U)\|_{L^2(\partial D_2)}.$$  

(2.23)

We have the following lemma, see [10] Lemma 2.4 and Lemma 2.15.

**Lemma 2.3.** Let $\phi, \psi \in L^2(\partial D_2)$. Then,

$$S_{D_2}\psi = \epsilon (S_B^* \hat{\psi})^\vee,$$

(2.24)

$$\|S_{D_2}\|_{L^2(\partial D_2)} = \epsilon \|S_B^*\|_{L^2(\partial B)},$$

(2.25)

$$D_{D_2}^*\psi = (D_B^* \hat{\psi})^\vee,$$

$$(-1/2 + D_{D_2}^*) \psi = \left((-1/2 + D_B^*) \hat{\psi}\right)^\vee,$$

(2.27)

$$\left(-1/2 + D_{D_2}^*\right)^{-1} \phi = \left((-1/2 + D_B^*)^{-1} \hat{\phi}\right)^\vee,$$

(2.28)

$$\left(-1/2 + D_{D_2}^*\right)^{-1} = \left(-1/2 + D_B^*\right)^{-1},$$

(2.29)

and hence

$$\left(-1/2 + D_{D_2}^* + \lambda S_{D_2}\right) \psi = \left((-1/2 + D_B^* + \lambda \epsilon S_B) \hat{\psi}\right)^\vee,$$

(2.30)

$$\left(-1/2 + D_{D_2}^* + \lambda S_{D_2}\right)^{-1} \phi = \left((-1/2 + D_B^* + \lambda \epsilon S_B)^{-1} \hat{\phi}\right)^\vee,$$

(2.31)

$$\left(-1/2 + D_{D_2}^* + \lambda S_{D_2}\right)^{-1} = \left(-1/2 + D_B^* + \lambda \epsilon S_B\right)^{-1},$$

(2.32)

with $S_B^* \hat{\psi}(\xi) := \int_{\partial B} \Phi(\xi, \eta) \hat{\psi}(\eta) d\eta$, $D_B^* \hat{\psi}(\xi) := \int_{\partial B} \frac{\partial \Phi^*(\xi, \eta)}{\partial \nu} \hat{\psi}(\eta) d\eta$ and $\Phi^*(\xi, \eta) := \frac{e^{i\pi(\xi - \eta)}}{4\pi|\xi - \eta|}$.

Let us estimate the norm of $\|S_B^*\|_{L^2(\partial B), L^2(\partial B)}$.

**Lemma 2.4.** The operator norm of the compact operator $S_{D_2} : L^2(\partial D_2) \rightarrow L^2(\partial D_2)$, defined in (2.22), is estimated by $\epsilon$, i.e.

$$\|S_{D_2}\|_{L^2(\partial D_2), L^2(\partial D_2)} \leq \epsilon \left(\|S_B^*\|_{L^2(\partial B), L^2(\partial B)} + \frac{1}{2\pi \epsilon^2 |\partial B|}\right),$$

(2.33)
Proof of Lemma 2.4. To estimate the operator norm of $S_{D_{\epsilon}}$, we decompose $S_{D_{\epsilon}} = S_{D_{\epsilon}}^{i_{\kappa}} + S_{D_{\epsilon}}^{d_{\kappa}}$ into two parts $S_{D_{\epsilon}}^{i_{\kappa}}$ (independent of $\kappa$) and $S_{D_{\epsilon}}^{d_{\kappa}}$ (dependent of $\kappa$) given by

$$S_{D_{\epsilon}}^{i_{\kappa}} \psi(x) := \int_{\partial D_{\epsilon}} \frac{1}{4\pi|x-y|} \psi(y) dy, \quad (2.34)$$

$$S_{D_{\epsilon}}^{d_{\kappa}} \psi(x) := \int_{\partial D_{\epsilon}} e^{ix|x-y|} \frac{1}{4\pi|x-y|} \psi(y) dy. \quad (2.35)$$

With this definition, $S_{D_{\epsilon}}^{i_{\kappa}} : L^2(\partial D_{\epsilon}) \rightarrow L^2(\partial D_{\epsilon})$ and $S_{D_{\epsilon}}^{d_{\kappa}} : L^2(\partial D_{\epsilon}) \rightarrow L^2(\partial D_{\epsilon})$ are compact. From (2.25), it can be observed that

$$\|S_{D_{\epsilon}}^{i_{\kappa}}\|_{L^2(\partial D_{\epsilon}),L^2(\partial D_{\epsilon})} = \epsilon \|S_{B}^{i}\|_{L^2(\partial B),L^2(\partial B)} = \epsilon \|S_{B}^{i}\|_{L^2(\partial B),L^2(\partial B)} \quad (2.36)$$

On the other hand, as mentioned in (2.30) of [10, Lemma 2.5], the following estimate can be obtained,

$$\|S_{D_{\epsilon}}^{d_{\kappa}}\|_{L^2(\partial D_{\epsilon}),L^2(\partial D_{\epsilon})} \leq \frac{1}{2\pi} \kappa e^{3|\partial B|} \quad \text{for } \kappa_{\text{max}} \leq 1. \quad (2.37)$$

Hence the result follows.\[\Box\]

Before proceeding further we make the following observation. For $m = 1, \ldots, M$ fixed, we distinguish between the obstacles $D_j$, $j \neq m$ by keeping them into different layers based on their distance from $D_m$. Let $\Omega_m$, $1 \leq m \leq M$ be the cubes of center $z_m$ such that each side is of size $(\frac{a}{2} + d^a)$ with $0 \leq \alpha \leq 1$ and it contains only $D_m$. Let us suppose that these cubes are arranged in a cuboid, for example unit rubics cube, see Fig 1, in different layers such that the total cubes upto the $n^{th}$ layer consists $(2^n + 1)^3$ cubes for $n = 0, \ldots, [d^a]$, and $\Omega_m$ is located on the center. Hence the number of obstacles located in the $n^{th}$ layer will be $[(2^n + 1)^3 - (2^{n-1})^3]$ and their distance from $D_m$ is more than $nd^a$.

![Figure 1: Rubics cube consisting of two layers](image)

2.2.2 The multiple obstacle case

Lemma 2.5. For $m, j = 1, 2, \ldots, M$, the operator $S_{mj} : L^2(\partial D_j) \rightarrow L^2(\partial D_m)$ defined in Proposition 2.1, see (2.5), satisfies the following estimates,

- For $j = m$,

$$\|S_{m m}\|_{L^2(\partial D_m),L^2(\partial D_m)} \leq \epsilon \left( \|S_{B}^{i}\|_{L^2(\partial B),L^2(\partial B)} + \frac{1}{2\pi} \kappa e^{3|\partial B|} \right), \quad (2.38)$$

for $\kappa_{\text{max}} \leq 1$. 
For \( j \neq m, \) such that \( D_j \in N_m^\alpha, n = 1, \ldots, [d-\alpha] \)
\[
\|S_{mj}\|_{L(L^2(\partial D_j), L^2(\partial D_m))} \leq \frac{1}{4\pi nd^\alpha} \frac{1}{|\partial B|} \epsilon^2,
\]
(2.39)

where \( |\partial B| := \max_m |\partial B_m| \) and \( \|S_{\partial B}^j\| := \max_m \|S_{\partial B}^j\|_{L(L^2(\partial B_m), L^2(\partial B_m))}. \)

Proof of Lemma 2.5 The estimate (2.38) is nothing else but (2.33) of Lemma 2.4 replacing \( B \) by \( B_m, \)
\( z \) by \( z_m \) and \( D_i \) by \( D_m \) respectively. The proof of the estimate (2.39) is a straightforward consequences of
(2.37) in [10, Lemma 2.6].

Proposition 2.6. For \( m, j = 1, 2, \ldots, M, \) the operator \( D^*_m: L^2(\partial D_j) \rightarrow L^2(\partial D_m) \) defined in Proposition
2.1 see (2.6), satisfies the following estimates,

- For \( j = m, \)
\[
\left\| \left( \frac{-1}{2}I + D^*_{mm} \right)^{-1} \right\|_{L(L^2(\partial D_m), L^2(\partial D_m))} \leq \hat{C}_{6m},
\]
(2.40)

where \( \hat{C}_{6m} := \frac{2\pi \| \left( \frac{-1}{2}I + D^*_{mm} \right)^{-1} \|_{L(L^2(\partial D_m), L^2(\partial D_m))}}{2\pi - \kappa^2d^2 |\partial B_m| \| \left( \frac{-1}{2}I + D^*_{mm} \right) \|_{L(L^2(\partial D_m), L^2(\partial D_m))}}. \)

- For \( j \neq m, \) such that \( D_j \in N_m^\alpha, n = 1, \ldots, [d-\alpha] \)
\[
\| D^*_m \|_{L(L^2(\partial D_j), L^2(\partial D_m))} \leq \frac{1}{4\pi} \left( \frac{\kappa}{nd^\alpha} + \frac{1}{n^2d^2\alpha} \right) |\partial B| \epsilon^2,
\]
(2.41)

where \( |\partial B| := \max_m |\partial B_m|. \)

In addition, as a consequence of (2.31), we can also prove that

- For \( j = m, \)
\[
\left\| \left( \frac{-1}{2}I + D^*_{mm} + \lambda_m S_{mm} \right)^{-1} \right\|_{L(L^2(\partial D_m), L^2(\partial D_m))} \leq C_{6m},
\]
(2.42)

where \( C_{6m} := \frac{2\pi \| \left( \frac{-1}{2}I + D^*_{mm} + \lambda_m S_{mm} \right)^{-1} \|_{L(L^2(\partial D_m), L^2(\partial D_m))}}{2\pi - (\lambda_m + \kappa)n\epsilon \|\partial B_m\| \| \left( \frac{-1}{2}I + D^*_{mm} + \lambda_m S_{mm} \right)^{-1} \|_{L(L^2(\partial D_m), L^2(\partial D_m))}}. \)

Proof of Proposition 2.6 This result can be proved in the similar lines of the proof of [10, Proposition
2.17].

End of the proof of Proposition 2.2 By substituting (2.39) and (2.41) in (2.15), (2.42) in (2.16), we obtain
\[
\|DK^* + \lambda K\| \equiv \max_{1 \leq m \leq M} \sum_{j=1}^M \|D^*_m + \lambda_m S_{mj}\|_{L(L^2(\partial D_j), L^2(\partial D_m))}
\leq \sum_{n=1}^{[d-\alpha]} \left[ (2n+1)^3 - (2n-1)^3 \right] \frac{1}{4\pi} \left( \frac{\kappa}{nd^\alpha} + \frac{1}{n^2d^2\alpha} \right) |\partial B| \epsilon^2
\]
where
\[
|\|D^\alpha|\|_{L^2(D_m)} + 1 \cdot \left( |\|\lambda| + \kappa \right) \sum_{n=1}^{[d-\alpha]} \left( 12 + \frac{1}{n} \right) \left[ \left( 6 \right)^{d-3\alpha} + 7d^{-2\alpha} \right] |\|\partial B|\| \epsilon^2
\]
and
\[
\left\| \left( -\frac{1}{2} I + DL^* + \lambda L \right)^{-1} \right\|_{L^2(\partial D_m)} \leq C_{6m}. \tag{2.44}
\]
Hence, (2.43)(2.44) provides
\[
\left\| \left( -\frac{1}{2} I + DL^* + \lambda L \right)^{-1} \right\|_{L^2(\partial D_m)} \leq \left( \max_{1 \leq m \leq M} C_{6m} \right) |\|\partial B|\| \frac{1}{2\pi} \left[ \left( |\|\lambda| + \kappa \right)(6 + 7d^\alpha) + 13 \right] d^{-3\alpha} \epsilon^2. \tag{2.45}
\]
By imposing the condition \(\left\| \left( -\frac{1}{2} I + DL^* + \lambda L \right)^{-1} \right\|_{L^2(\partial D_m)} \leq 1\), we have from (2.14) and (2.17-2.19):
\[
\left\| \sigma_m \right\|_{L^2(\partial D_m)} \leq |\|\sigma|\| \leq \left[ \left( -\frac{1}{2} I + DL^* + \lambda L \right)^{-1} \right] \left\| DK^* + \lambda K \right\| \left\| \left( -\frac{1}{2} I + DL^* + \lambda L \right)^{-1} \right\| \left\| DK^* + \lambda K \right\| \left\| \partial \nu + \lambda \right\| U^i \|_{L^2(D_m)} \left( C_p \geq \frac{1}{1 - C_s} \right)
\]
for all \(m \in \{1, 2, \ldots, M\}\). But,
\[
\left\| \left( \partial \nu + \lambda \right) U^i \right\|_{L^2(\partial D_m)} \leq |\|\lambda|\| U^i \|_{L^2(\partial D_m)} + |\|\partial \nu U^i \|_{L^2(\partial D_m)} = |\|\lambda|\| \left| \partial B_m \right|^{1/2} + k \left| \partial B_m \right|^{1/2} \quad (\text{Since } U^i(x, \theta) = e^{i\kappa x \cdot \theta})
\]
\[
\leq (|\|\lambda| + \kappa) |\|\partial B|\|^{1/2}, \forall m = 1, 2, \ldots, M. \tag{2.47}
\]
Now by substituting (2.47) in (2.46), for each \(m = 1, \ldots, M\), we obtain
\[
\left\| \sigma_m \right\|_{L^2(\partial D_m)} \leq C(\kappa) \epsilon, \tag{2.48}
\]
where \(C(\kappa) := C |\|\partial B|\|^{1/2} (|\|\lambda| + \kappa)\).

The condition \(\left\| \left( -\frac{1}{2} I + DL^* + \lambda L \right)^{-1} \right\|_{L^2(\partial D_m)} \leq 1\) is satisfied if
\[
C_s = \left( \max_{1 \leq m \leq M} C_{6m} \right) |\|\partial B|\| \frac{1}{2\pi} \left[ \left( |\|\lambda| + \kappa \right) \left( 6 + 7d^\alpha \right) + 13 \right] d^{-3\alpha} \epsilon^2 < 1. \tag{2.49}
\]
Since \( \lambda_m = \lambda_0 a^{-\beta} \) and \( d \approx a^t \), in particular \( d \geq d_{\text{min}} a^t \), then (2.49) reads as \( a^{-3\alpha t + 2 - \beta} < c \), where we set\(^7\)

\[
\hat{c} := \left( \left[ (\lambda_+ + \kappa_{\text{max}} a^\beta)(6 + 7|\text{max}|^\alpha) + 13a^\beta \right] \frac{1}{2} \left[ \text{max}_{1 \leq m \leq M} \frac{|\partial B|}{d_{\text{min}} a^t \text{diam}(B_m)^2} \right] \text{max}_{1 \leq m \leq M} C_{6m} d_{\text{min}}^{-\alpha} \right)^{-1} > 1 \quad (2.50)
\]

with \( \lambda_+ := \text{max}_{1 \leq m \leq M} |\lambda_0| \) and with the rewritten form of \( C_{6m} \) mentioned in Proposition 2.6 as

\[
C_{6m} := \frac{2\pi \left( -\frac{1}{2} \mathbb{1} + D_{B_m}^{s^*} + \frac{\lambda_+ e^{-\beta} |\partial B_m|}{\text{max}_{1 \leq m \leq M} \text{diam}(B_m)^2} \right)^{-1} \left( \frac{\lambda_+ e^{-\beta} |\partial B_m|}{\text{max}_{1 \leq m \leq M} \text{diam}(B_m)^2} \right)^{-1} \right)}{L(2(\partial B_m), L^2(\partial B_m))}.
\]

Observe that \( s = 3\alpha t \). Hence (2.49) makes sense if \( s \leq 2 - \beta \) and \( \lambda_+ \) satisfies (2.50). Again, since \( \lambda_m = \lambda_0 a^{-\beta} \), (2.48) can be rewritten as

\[
\|\sigma_m\|_{L^2(\partial D_m)} \leq \hat{C}(\kappa) e^{1 - \beta},
\]

with\(^8\)

\[
\hat{C}(\kappa) := C |\partial B|^2 \left( \frac{\lambda_+}{\text{max}_{1 \leq m \leq M} \text{diam}(B_m)^2} + \kappa e^\beta \right) \left( < C |\partial B|^2 \left( \frac{\lambda_+}{\text{max}_{1 \leq m \leq M} \text{diam}(B_m)^2} + \kappa_\text{max} \right) \right).
\]

\( \blacksquare \)

**Definition 2.7.** We call the \( \sigma_m \)'s used in (2.1), the solution of the problem (1.1-1.3), the surface charge distributions. Using these surface charge distributions, we define the total charge on each surface \( \partial D_m \) denoted by \( Q_m \) as

\[
Q_m := \int_{\partial D_m} \sigma_m(s) ds. \quad (2.52)
\]

### 2.2.3 Estimates on the total charges \( Q_m \), \( m = 1, \ldots, M \)

In the following proposition, we provide an approximate of the far-fields in terms of the total charges \( Q_m \).

**Proposition 2.8.** The far-field pattern \( U^\infty \) of the scattered solution of the problem (1.1-1.3) has the following asymptotic expansion

\[
U^\infty(x) = \sum_{m=1}^M e^{-i\kappa z_m} Q_m + O(\kappa a^{3-\beta}), \quad (2.53)
\]

with \( Q_m \) given by (2.52), if \( \kappa_{\text{max}} a < 1 \) where \( O(\kappa a^{3-\beta}) \leq C \kappa a^{3-\beta} \) and \( C := \frac{|\partial B|C(\lambda_+ + \kappa_{\text{max}})}{\text{max}_{1 \leq m \leq M} \text{diam}(B_m)^2} \).

\( \blacksquare \)

\( \text{It is important to remark that, if we do not distinguish the near by and far obstacles, as it discussed immediately after} \quad (2.37) \quad \text{and by following the way as it done in} \quad [10, 11] \quad \text{we can get the estimate} \quad ||DR^* + \lambda K|| \leq M^{-1} \left( \frac{|\lambda_+ + \kappa_\text{max}}{2} + \frac{a}{\pi} \right) |\partial B| e^2 \quad \text{in place of} \quad (2.49) \quad \text{and hence the condition} \quad (2.49) \quad \text{will be replaced by} \quad (M - 1)\xi^2_{\text{max}} c_0 \quad \text{for some suitable constant} \quad c_0. \) However, this condition is too strong to enable us to apply our asymptotic expansion to the effective medium theory where we need to choose \( M \sim a^{-\xi} \) with \( s = 2 - \beta \) and \( d \sim a^t \) with \( t \geq \frac{4}{3} \).
Proof of Proposition 2.8. From (2.1), we have

\[ U^s(x) = \sum_{m=1}^{M} \int_{\partial D_m} \Phi_m(x,s) \sigma_m(s) ds, \text{ for } x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \bar{D}_m \right). \]

Hence

\[
U^\infty(\hat{x}) = \sum_{m=1}^{M} \int_{\partial D_m} e^{-i\kappa \hat{x} \cdot s} \sigma_m(s) ds
= \sum_{m=1}^{M} \left( \int_{\partial D_m} e^{-i\kappa \hat{x} \cdot z_m} \sigma_m(s) ds + \int_{\partial D_m} [e^{-i\kappa \hat{x} \cdot s} - e^{-i\kappa \hat{x} \cdot z_m}] \sigma_m(s) ds \right)
= \sum_{m=1}^{M} \left( e^{-i\kappa \hat{x} \cdot z_m} Q_m + \int_{\partial D_m} [e^{-i\kappa \hat{x} \cdot s} - e^{-i\kappa \hat{x} \cdot z_m}] \sigma_m(s) ds \right). \tag{2.54}
\]

For every \( m = 1, 2, \ldots, M \), we have from Proposition 2.2

\[
\left| \int_{\partial D_m} |\sigma_m(s)| ds \right| \leq \|1\|_{L^2(\partial D_m)} \|\sigma_m\|_{L^2(\partial D_m)} \leq \tag{2.51}
\leq \|1\|_{L^2(\partial D_m)} C |\partial B| \frac{1}{2} (\lambda_+ + \kappa_{\max}) \epsilon^{1-\beta}
\leq |\partial B| C \left( \frac{\lambda_+}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \right)^{\beta} + \kappa_{\max} \epsilon^{2-\beta} \tag{2.55}
\]

with

\[
C := \frac{|\partial B| C (\lambda_+ + \kappa_{\max} \max_{1 \leq m \leq M} \text{diam}(B_m))^{\beta}}{\left( \max_{1 \leq m \leq M} \text{diam}(B_m) \right)^2}. \tag{2.56}
\]

It gives us the following estimate:

\[
\left| \int_{\partial D_m} [e^{-i\kappa \hat{x} \cdot s} - e^{-i\kappa \hat{x} \cdot z_m}] \sigma_m(s) ds \right| \leq \int_{\partial D_m} |e^{-i\kappa \hat{x} \cdot s} - e^{-i\kappa \hat{x} \cdot z_m}| \sigma_m(s) ds
\leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa^l |s - z_m|^l |\sigma_m(s)| ds
\leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa^l \left( \frac{a}{2} \right)^l |\sigma_m(s)| ds
\leq C a^{2-\beta} \sum_{l=1}^{\infty} \kappa^l \left( \frac{a}{2} \right)^l
= \frac{1}{2} C \kappa^{a^{3-\beta}} \frac{1}{1 - \kappa \frac{a}{2}}, \text{ if } a < \frac{2}{\kappa_{\max}} \left( \leq \frac{2}{\kappa} \right) \tag{2.57}
\]

which means

\[
\int_{\partial D_m} [e^{-i\kappa \hat{x} \cdot s} - e^{-i\kappa \hat{x} \cdot z_m}] \sigma_m(s) ds \leq C \kappa a^{3-\beta}, \text{ for } a \leq \frac{1}{\kappa_{\max}}. \tag{2.58}
\]

Now substitution of (2.58) in (2.54) gives the required result (2.53).
Lemma 2.9. For $m = 1, 2, \ldots, M$, we have

$$|Q_m| \leq \tilde{c} \epsilon^{2-\beta},$$

(2.59)

where $\tilde{c} := |\partial B|C(\frac{\lambda_+}{\max_{1 \leq m \leq M} \text{diam}(B_m)^\beta} + \kappa_{\max})$ with $\partial B$ and $C$ are defined in (2.39) and (2.46) respectively.

Proof of Lemma 2.9. The proof follows as below;

$$|Q_m| = \left| \int_{\partial D_m} \sigma_m(s) ds \right| \leq \left[ 1 \right] L^2(\partial D_m) \| \sigma_m \| L^2(\partial D_m)

\leq \left[ \frac{\lambda_+}{\max_{1 \leq m \leq M} \text{diam}(B_m)^\beta} + \kappa_{\max} \right] \epsilon^{1-\beta}

\leq |\partial B|C(\frac{\lambda_+}{\max_{1 \leq m \leq M} \text{diam}(B_m)^\beta} + \kappa_{\max}) \epsilon^{2-\beta}.$$

Proposition 2.10. For $m = 1, 2, \ldots, M$, the total charge $Q_m$ on each surface $\partial D_m$ of the small scatterer $D_m$ can be calculated from the algebraic system

$$\frac{Q_m}{C_m} = -U_i(z_m) - \sum_{j=1}^M C_j \Phi_\kappa(z_m, z_j) \frac{Q_j}{C_j} + \text{Err}$$

(2.60)

where $\text{Err} := O(\alpha^{1-\beta} + \frac{\epsilon^{2-\beta}}{\epsilon^3})$ and $C_m := -\lambda_m |\partial D_m|.$

Proof of Proposition 2.10. For $s_m \in \partial D_m$, using the impedance boundary condition (1.2), we have

$$0 = \frac{\partial U}{\partial \nu_m}(s_m) + \lambda_m U_i(s_m) = -\frac{\sigma_m(s_m)}{2} + \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s) \sigma_m(s) ds + \sum_{j=1}^M \int_{\partial D_j} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s) \sigma_j(s) ds

+ \lambda_m \sum_{j=1}^M \int_{\partial D_j} \Phi_\kappa(s_m, s) \sigma_m(s) ds + \frac{\partial U}{\partial \nu_m}(s_m) + \lambda_m U_i(s_m)$$

(2.61)

Integrating the above on $\partial D_m$, we can write it as

$$-\frac{1}{2} \int_{\partial D_m} \sigma_m(s_m) ds_m + \int_{\partial D_m} \left( \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s) ds_m \right) \sigma_m(s) ds + \sum_{j=1}^M \int_{\partial D_j} \left( \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s) ds_m \right) \sigma_j(s) ds

+ \lambda_m \sum_{j=1}^M \int_{\partial D_j} \left( \int_{\partial D_m} \Phi_\kappa(s_m, s) ds_m \right) \sigma_m(s) ds = -\int_{\partial D_m} \frac{\partial U}{\partial \nu_m}(s_m) ds_m - \int_{\partial D_m} \lambda_m U_i(s_m) ds_m.$$
It can be rewritten as

\[-\frac{1}{2}Q_m + \frac{1}{2} \sum_{j \neq m} \int_{\partial D_m \cap \partial D_j} \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) d\nu \sigma_j(s) ds + \sum_{j \neq m} \int_{\partial D_m \cap \partial D_j} \left( \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) d\nu \right) \sigma_j(s) ds \]

\[=: A \]

\[+ \lambda_m \int_{\partial D_m} \left( \int_{\partial D_m} \Phi_\kappa (s, s_m) d\nu \right) \sigma_j(s) ds + \lambda_m \sum_{j \neq m} \int_{\partial D_m} \left( \int_{\partial D_m} \Phi_\kappa (s, z_j) d\nu \right) \sigma_j(s) ds \]

\[=: B \]

\[= - \int_{\partial D_m} \frac{\partial U^i}{\partial \nu_m} (s, s_m) ds - \int_{\partial D_m} \lambda_m U^i (s, s_m) + A' + \lambda_m D', \]

with

\[A' := \int_{\partial D_m} \left( \int_{\partial D_m} \left[ \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) - \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) \right] ds \right) \sigma_m(s) ds \quad (2.62)\]

\[D' := \sum_{j \neq m} \int_{\partial D_j} \left( \int_{\partial D_m} \left[ \Phi_\kappa (s, s_m) - \Phi_\kappa (s, z_j) \right] ds \right) \sigma_j(s) ds. \quad (2.63)\]

- We can approximate \(A\) and \(A'\) as follows:

\[A = \int_{\partial D_m} \left( \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) ds \right) \sigma_m(s) ds \]

\[= \int_{\partial D_m} \left[ K_{D_m}^{\kappa} (1) \right] (s) \sigma_m(s) ds \]

\[= -\frac{1}{2} \int_{\partial D_m} \sigma_m(s) ds \]

\[= -\frac{1}{2} Q_m. \quad (2.64)\]

Here \(K_{D_m}^{\kappa}\) is the double layer operator defined as in (2.10) but with zero frequency and on the boundary of \(D_m\). Observe that,

\[|A'| = \left| \int_{\partial D_m} \left( \int_{\partial D_m} \left[ \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) - \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) \right] ds \right) \sigma_m(s) ds \right| \]

\[= \int_{\partial D_m} \left( \int_{\partial D_m} \frac{ik(s_m - s) \cdot \nu_m(s_m)}{4\pi |s_m - s|^2} \sum_{l=1}^{\infty} (ik |s_m - s|)^l \left( \frac{1}{l!} - \frac{1}{(l+1)!} \right) ds \right) \sigma_m(s) ds \]

\[\leq \int_{\partial D_m} \left( \int_{\partial D_m} \left[ \frac{\kappa^2}{4\pi} \sum_{l=0}^{\infty} \frac{(\kappa a)^l}{l!} \right] ds \right) \sigma_m(s) ds \]

\[= O(\kappa^2 a^{4-\beta}). \quad (2.65)\]

- Now, we can approximate \(B\)

\[B = \sum_{j \neq m} \int_{\partial D_j} \left( \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m} (s, s_m) ds \right) \sigma_j(s) ds \]

\[= \sum_{j \neq m} \int_{\partial D_j} \left( \int_{D_m} \Delta \Phi_\kappa (y, s) dy \right) \sigma_j(s) ds \]
\[
\sum_{j \neq m} \int_{\partial D_j} \left( \int_{D_m} \left[ -\kappa^2 \Phi_\kappa(y_m, s) \right] dy_m \right) \sigma_j(s) ds \\
= \sum_{j \neq m} \int_{\partial D_j} \left[ O \left( \kappa^2 \frac{a^3}{d_{m_j}} \right) \right] \sigma_j(s) ds \\
= \sum_{n=1}^{[d^{-\alpha}]} [(2n+1)^3 - (2n-1)^3] \left[ O \left( \kappa^2 \frac{a^{5-\beta}}{nd^{\alpha}} \right) \right] \\
= O \left( 2\kappa^2 \frac{a^{5-\beta}}{d^{2\alpha}} \sum_{n=1}^{[d^{-\alpha}]} \left[ 12n + \frac{1}{n} \right] \right) \\
= O \left( 2\kappa^2 \frac{a^{5-\beta}}{d^{2\alpha}} \left[ 6 + 7 \right] \right) . \tag{2.66}
\]

\bullet \text{ Since } \Phi_\kappa(s_m, s) - \Phi_\kappa(s_m, z_j) = O \left( \frac{s_m}{d_{m_j}} + \frac{s_m}{d_{m_j}} \right) \text{ for } s \in \partial D_j, j \neq m, \text{ we can approximate } D \text{ and } D' \text{ as follows;}

\[
D = \sum_{j \neq m} \int_{\partial D_j} \left( \int_{\partial D_m} \Phi_\kappa(s_m, z_j) ds_m \right) \sigma_j(s) ds \\
= \sum_{j \neq m} \left[ \int_{\partial D_j} \left( \int_{\partial D_m} \Phi_\kappa(z_m, z_j) ds_m \right) \sigma_j(s) ds \right. \\
+ \left. \int_{\partial D_j} \left( \int_{\partial D_m} \left[ \Phi_\kappa(s_m, z_j) - \Phi_\kappa(z_m, z_j) \right] ds_m \right) \sigma_j(s) ds \right] \\
= \sum_{j \neq m} \left[ \Phi_\kappa(z_m, z_j) Q_j |\partial D_m| + O \left( \frac{a}{d_{m_j}} \left( \kappa + \frac{1}{d_{m_j}} \right) |\partial D_m| \left( \int_{\partial D_j} \sigma_j(s) ds \right) \right) \right] \\
= \sum_{j \neq m} \left[ \Phi_\kappa(z_m, z_j) Q_j |\partial D_m| + O \left( \frac{a^{5-\beta}}{d_{m_j}} \left( \kappa + \frac{1}{d_{m_j}} \right) \right) \right] \\
= \sum_{j \neq m} \Phi_\kappa(z_m, z_j) Q_j |\partial D_m| + \sum_{n=1}^{[d^{-\alpha}]} [(2n+1)^3 - (2n-1)^3] O \left( \frac{a^{5-\beta}}{nd^{\alpha}} \left( \kappa + \frac{1}{nd^{\alpha}} \right) \right) \\
= \sum_{j \neq m} \Phi_\kappa(z_m, z_j) Q_j |\partial D_m| + O \left( 2\frac{a^{5-\beta}}{d^{2\alpha}} \sum_{n=1}^{[d^{-\alpha}]} \left[ 12n^2 + \frac{1}{n} \right] \left( \kappa + \frac{1}{nd^{\alpha}} \right) \right) \\
= \sum_{j \neq m} \Phi_\kappa(z_m, z_j) Q_j |\partial D_m| + O \left( 2\frac{a^{5-\beta}}{d^{2\alpha}} \left[ 7\kappa + 6\kappa + 13 \right] \right) \tag{2.67}
\]

and

\[
|D'| = \sum_{j \neq m} \int_{\partial D_j} \left( \int_{\partial D_m} \left[ \Phi_\kappa(s_m, s) - \Phi_\kappa(s_m, z_j) \right] ds_m \right) \sigma_j(s) ds \\
= \sum_{j \neq m} O \left( \frac{a^{5-\beta}}{d_{m_j}} \left( \kappa + \frac{1}{d_{m_j}} \right) \right)
\]
\[
\sum_{n=1}^{[d-\alpha]} \left( (2n+1)^3 - (2n-1)^3 \right) O \left( \frac{a^{5-\beta}}{nd^\alpha} \left( \kappa + \frac{1}{nd^\alpha} \right) \right) \\
= O \left( \frac{a^{5-\beta}}{d^\alpha} \left[ 12n^2 + 1 \right] \left( \kappa + \frac{1}{nd^\alpha} \right) \right) \\
= O \left( \frac{a^{5-\beta}}{d^\alpha} \left[ 7\kappa + \frac{6\kappa + 13}{d^\alpha} \right] \right). \tag{2.68}
\]

Now, let us approximate \( C \). Since \( |\int_{\partial D_m} \Phi_\kappa(s_m, s) ds_m| \leq \frac{1}{4\pi} \int_{\partial D_m} \frac{1}{|s_m-s|} ds_m = O(a) \), then

\[
|C| = \left| \int_{\partial D_m} \left( \int_{\partial D_m} \Phi_\kappa(s_m, s) ds_m \right) \sigma_m(s) ds \right| \\
= O(a^{3-\beta}). \tag{2.69}
\]

Hence, from (2.62, 2.69), we obtain the approximation below

\[
-Q_m + \sum_{j \neq m}^{M} \Phi_\kappa(z_m, z_j) \lambda_m |\partial D_m| Q_j \\
= -\lambda_m |\partial D_m| e^{i\kappa \theta} z_m + O \left( |\lambda_m| + \kappa a^3 \right) + \lambda_m O \left( a^{3-\beta} \right), \\
+ O \left( \kappa^2 a^{4-\beta} \right) + O \left( 2\kappa^2 a^{5-\beta} \left[ \frac{6}{d^\alpha} + 7 \right] \right) + \lambda_m O \left( \frac{2a^{5-\beta}}{d^\alpha} \left[ 7\kappa + \frac{6\kappa + 13}{d^\alpha} \right] \right). \tag{2.70}
\]

Indeed, \( \int_{\partial D_m} \lambda_m (U^i(s_m) - U^i(z_m)) ds_m = O(|\lambda| \kappa a^3) \) and \( \int_{\partial D_m} \frac{\partial U^i}{\partial z_m} (s_m) ds_m = O(\kappa^2 a^3) \). In addition, since \( \beta < 1 \), then \( \lambda_m O(a^{3-\beta}) = O(a^{3-2\beta}) = o(a^{2-\beta}) = o(\lambda |\partial D_m| e^{i\kappa \theta} z_m) \).

We can rewrite the above system as

\[
-\frac{1}{\lambda_m |\partial D_m|} Q_m = -e^{i\kappa \theta} z_m - \sum_{j \neq m}^{M} \Phi_\kappa(z_m, z_j) Q_j + Err \tag{2.71}
\]

with

\[
Err := \frac{1}{\lambda_m |\partial D_m|} \left[ O \left( (\lambda_+ + \kappa \epsilon^3) \kappa a^{3-\beta} \right) + O \left( \kappa^2 a^{4-\beta} \right) + O \left( 2\kappa^2 a^{5-\beta} \left[ \frac{6}{d^\alpha} + 7 \right] \right) \right] \\
+ O \left( \frac{a^{5-2\beta}}{d^\alpha} \left[ 7\kappa + \frac{6\kappa + 13}{d^\alpha} \right] \right) + O \left( a^{3-2\beta} \right) \\
= \frac{1}{\lambda_m |\partial D_m|} O \left( a^{3-2\beta} + \frac{a^{5-2\beta}}{d^\alpha} \right) \\
= O \left( a^{1-\beta} + \frac{a^{3-\beta}}{d^\alpha} \right). \tag{2.72}
\]

The last two lines in the above approximation are due to \( \kappa \leq \kappa_{\text{max}} \) and \( d \leq d_{\text{max}} \).\]
3 The algebraic system

Define the algebraic system

\[
\frac{\bar{Q}_m}{C_m} := -U^i(z_m) - \sum_{j \neq m} \bar{C}_j \Phi_k(z_m, z_j) \frac{\bar{Q}_j}{C_j}
\]  
(3.1)

for all \( m = 1, 2, \ldots, M \). It can be written in a compact form as

\[
B\bar{Q} = U^i,
\]  
(3.2)

where \( \bar{Q}, U^i \in \mathbb{C}^{M \times 1} \) and \( B \in \mathbb{C}^{M \times M} \) are defined as

\[
B := \begin{pmatrix}
- \frac{1}{C_1} & -\Phi_k(z_1, z_2) & -\Phi_k(z_1, z_3) & \cdots & -\Phi_k(z_1, z_M) \\
-\Phi_k(z_2, z_1) & - \frac{1}{C_2} & -\Phi_k(z_2, z_3) & \cdots & -\Phi_k(z_2, z_M) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\Phi_k(z_M, z_1) & -\Phi_k(z_M, z_2) & \cdots & \cdots & - \frac{1}{C_M}
\end{pmatrix},
\]  
(3.3)

\[
\bar{Q} := ( \bar{Q}_1 \; \bar{Q}_2 \; \ldots \; \bar{Q}_M )^\top \text{ and } U^i := ( U^i(z_1) \; U^i(z_2) \; \ldots \; U^i(z_M) )^\top.
\]  
(3.4)

The above linear algebraic system is solvable for \( \bar{Q}_j, 1 \leq j \leq M \), when the matrix \( B \) is invertible. Now we give sufficient conditions for the invertibility of system (3.2). We recall that \( \lambda_m := \lambda_m a^{-\beta}, \beta \geq 0 \).

**Lemma 3.1.** We distinguish the following two cases:

- Let \( \Re(\lambda_m) < 0 \) \[\text{and assume that } \min_{1 \leq m \leq M} \frac{\Re(\bar{C}_m)}{\pi d_t^2} > \frac{\sqrt{2M_{\max}}}{\pi d_t} \] then the matrix \( B \) is invertible and the solution vector \( \bar{Q} \) of (3.2) satisfies the estimate

\[
\sum_{m=1}^M |\bar{Q}_m|^2 \leq 4 \left( \frac{\min_{1 \leq m \leq M} \Re(\bar{C}_m)}{\max_{1 \leq m \leq M} |C_m|^2} - \frac{\sqrt{2M_{\max}}}{\pi d_t} \right)^{-2} \sum_{m=1}^M |U^i(z_m)|^2.
\]  
(3.5)

- Let \( \Re(\lambda_m) > 0 \) \[\text{and assume that } \max_{1 \leq m \leq M} \frac{\Re(\bar{C}_m)}{|C_m|^2} > \frac{\sqrt{2M_{\max}}}{\pi d_t} \] then the matrix \( B \) is invertible and the solution vector \( \bar{Q} \) of (3.2) satisfies the estimate

\[
\sum_{m=1}^M |\bar{Q}_m|^2 \leq 4 \left( \frac{\min_{1 \leq m \leq M} \Re(\bar{C}_m)}{\max_{1 \leq m \leq M} |C_m|^2} - \frac{\sqrt{2M_{\max}}}{\pi d_t} \right)^{-2} \sum_{m=1}^M |U^i(z_m)|^2.
\]  
(3.6)

where \( M_{\max} := M a^s \), recalling that \( M := O(a^{-s}) \), as \( a \to 0 \).

Since \( \bar{C}_m := -\lambda_m |\partial D_m| \) and \( d \geq d_{\min} a^t \), the condition \( \min_{1 \leq m \leq M} \frac{\Re(\bar{C}_m)}{\max_{1 \leq m \leq M} |C_m|^2} > \frac{\sqrt{2M_{\max}}}{\pi d_t} \) is satisfied if

\[
\frac{4\lambda_+ a^{-(2-\beta)}}{\pi \lambda_+^2} > \frac{\sqrt{2M_{\max}}}{\pi d_t^2}.
\]

This is possible if \( s < 2 - \beta \) and \( a \) is small enough or \( s = 2 - \beta \) and \( \lambda_- \) and \( \lambda_+ \) satisfy

\[
\frac{4\lambda_+ a^{-(2-\beta)}}{\pi \lambda_+^2} > \frac{\sqrt{2M_{\max}}}{\pi d_{\min}^2}.
\]

Recall that \( t \geq \frac{s}{3} \), hence \( t \leq 3 \). If we assume \( d_{\min} \leq 1 \), then the last condition is satisfied if \( \frac{4\lambda_+}{\pi \lambda_+^2} > \frac{\sqrt{2M_{\max}}}{\pi d_{\min}} \) and if \( d_{\min} \geq 1 \) then we take \( \frac{4\lambda_+}{\pi \lambda_+^2} > \frac{\sqrt{2M_{\max}}}{\pi} \).

\[\text{In this case, we can actually relax the condition } s \leq 2 - \beta \text{ on the number of small scatterers } s, \text{ see Remark 6.1.}\]
3.1 End of the proof of Theorem 1.2

We can rewrite the inequalities (3.5) and (3.6) using norm inequalities as

\[
\sum_{m=1}^{M} |\bar{Q}_m| \leq 2 \left( \frac{\min_{1 \leq m \leq M} \Re C_m}{\max_{1 \leq m \leq M} |C_m|} \right) - \frac{\sqrt{2M \max_{1 \leq m \leq M} |C_m|}}{\pi \max_{1 \leq m \leq M} |C_m|} \left( M \max_{1 \leq m \leq M} |C_m| \right) \left( \max_{1 \leq m \leq M} |\bar{C}_m| \right) - \frac{1}{M \max_{1 \leq m \leq M} |\bar{C}_m|} \left( \max_{1 \leq m \leq M} |\bar{C}_m| \right) \left( \max_{1 \leq m \leq M} |U_i(z_m)| \right) \]  

(3.7)

which holds for the cases \( \Re \lambda_m \leq 0 \) and \( \Re \lambda_m \geq 0 \) respectively. The difference between (3.1) and (2.60) produce the following

\[
\frac{Q_m - \bar{Q}_m}{C_m} = - \sum_{j=1, j \neq m}^{M} \Phi_\kappa(z_m, z_j) (Q_j - \bar{Q}_j) + Err, 
\]

(3.9)

for \( m = 1, \ldots, M \). Comparing the above system of equations (3.9) with (3.1) and by making use of the estimates (3.7) and (3.8), we obtain

\[
\sum_{m=1}^{M} (Q_m - \bar{Q}_m) = O \left( Ma^{2-\beta} Err \right). 
\]

(3.10)

We can evaluate the \( \bar{Q}_m \)'s from the algebraic system (3.1). Hence, by using (3.10) in (2.53) we can represent the far-field pattern in terms of \( \bar{Q}_m \) as below:

\[
U^\infty(\tilde{x}) = \sum_{m=1}^{M} \left[ e^{-i\kappa \cdot \tilde{x}} z_m Q_m + O(\kappa a^{3-\beta}) \right] = \sum_{m=1}^{M} \left[ e^{-i\kappa \cdot \tilde{x}} z_m (Q_m - \bar{Q}_m) + O(\kappa a^{3-\beta}) \right] = \sum_{m=1}^{M} e^{-i\kappa \cdot \tilde{x}} z_m Q_m + O(\kappa a^{2-\beta} [Err + (\kappa a)]) \\
\]  

(2.72)

\[
= \sum_{m=1}^{M} e^{-i\kappa \cdot \tilde{x}} z_m \bar{Q}_m + O \left( Ma^{2-\beta} \left( a^{1-\beta} + \frac{a^{3-\beta}}{a^{2\alpha}} \right) \right) = \sum_{m=1}^{M} e^{-i\kappa \cdot \tilde{x}} z_m \bar{Q}_m + O \left( a^{3-s-2\beta} \right) 
\]

(3.11)

since, as \( s = 3t\alpha \leq 2 - \beta \), we have \( \frac{a^{3-\beta}}{a^{2\alpha}} \sim a^{3-\beta - 3t\alpha} = O(a) = o(a^{1-\beta}) \) if \( \beta < 1 \). Hence Theorem 1.2 is proved with the replacement of \( C_m \) in the statement by \( \bar{C}_m \).

\[\square\]

4 Proof of Corollary 1.3

The steps of the proof of Corollary 1.3 are the same as for the proof of Theorem 1.2. Here we only explain the main changes that are needed to derive it.
We recall that, for an obstacle $D_\epsilon$ of radius $\epsilon$, $S(\phi)(s) := \int_{\partial D_\epsilon} \Phi_\kappa(s,t)\phi(t)dt$ and $D(\phi)(s) := \int_{\partial D_\epsilon} \frac{\partial \Phi_\kappa(s,t)}{\partial n(t)}\phi(t)dt$. Similarly, we set $S_G(\phi)(s) := \int_{\partial D_\epsilon} G_\kappa(s,t)\phi(t)dt$ and $D_G(\phi)(s) := \int_{\partial D_\epsilon} \frac{\partial G_\kappa(s,t)}{\partial n(t)}\phi(t)dt$.

We see that $W_\kappa(x,z) := G_\kappa(x,z) - \Phi_\kappa(x,z)$ satisfies

$$(\Delta + \kappa^2 n^2)W_\kappa = \kappa^2(1 - n^2)\Phi_\kappa, \quad \text{in } \mathbb{R}^3$$

with the Sommerfeld radiation conditions. Since $\Phi_\kappa(\cdot,z), z \in \mathbb{R}^3$ is bounded in $L^p(\Omega)$, for $p < 3$, by interior estimates, we deduce that $W(\cdot,z), z \in \mathbb{R}^3$ is bounded in $W^{2,p}(\Omega)$, for $p < 3$, and hence, in particular, the normal traces are bounded in $L^2(\partial D_\epsilon)$. Then we can show that the norms of the operators

$$(S_G - S : L^2(\partial D_\epsilon) \to H^1(\partial D_\epsilon))$$

and

$$(D_G - D : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon))$$

are of the order $O(\epsilon)$ at least. The representation (2.1), in Proposition 2.1, needs to be replaced by

$$U_n^\epsilon(x) = V_n^\epsilon(x) + \sum_{m=1}^{M} \int_{\partial D_m} H_\kappa(x,s)\sigma_m(s)ds, \quad x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \tilde{D}_m \right),$$

where $V_n^\epsilon$ is the total field corresponding to the background modeled by the index of refraction $n$, see (1.15). We use the single layer potentials defined by the Greens’ function $G_\kappa$ instead of the fundamental function $\Phi_\kappa$. The main tools used in justifying Proposition 2.1 are the invertibility properties of the corresponding integral operators (i.e. the Fredholm property) and the jumps of the double layer potentials defined by $\Phi_\kappa$. These two tools are satisfied also when we use $G_\kappa$ instead of $\Phi_\kappa$ due to the error bounds in (4.2) and (4.3).

The apriori estimates on the densities $\sigma_m$’s derived in section 2.2 are quantitative versions of the result in Proposition 2.1. Due to error bounds in (4.2) and (4.3), those estimates can then be translated to the densities used in the representation (4.4). Of course, in Lemma 2.3 one needs to replace the equalities by inequalities to estimate the properties of the operators defined by $G_\kappa$, on the scaled obstacles $D_\epsilon$, in terms of the properties of the operators defined by $\Phi_\kappa$ on the original obstacles $B$.

Finally, to do the same analysis as in section 2.2.3, one needs to split $G_\kappa$ as $G_\kappa = \Phi_\kappa + (G_\kappa - \Phi_\kappa)$ and use the results derived in section 2.2.3 and again error bounds in (4.2) and (4.3).

5 Justification of Remark 1.4

We show only the main changes needed in the proof of Theorem 1.2. This change occurs in the proof of Proposition 2.10 and precisely in evaluating the term $C$, i.e. (2.69). We rewrite $C$ as

$$C = \int_{\partial D_m} \left( \int_{\partial D_m} \Phi_0(s_m,s)ds \right) \sigma_m(s)ds + \int_{\partial D_m} \left( \int_{\partial D_m} \Phi_\kappa(s_m,s) - \Phi_0(s_m,s)ds \right) \sigma_m(s)ds.$$

Since

$$\left| \int_{\partial D_m} \left( \int_{\partial D_m} [\Phi_\kappa(s_m,s) - \Phi_0(s_m,s)]ds \right) \sigma_m(s)ds \right| < \frac{1}{2} \sum_{l=1}^{\infty} \frac{\kappa^l a^{l-1}}{l!} \left| \partial D_m \right| \int_{\partial D_m} |\sigma_m(s)|ds$$

$$= O(\kappa a^{4-\beta})$$

then we can write $C$ as follows:

$$C = \int_{\partial D_m} \left( \int_{\partial D_m} \Phi_0(s_m,s)ds \right) \sigma_m(s)ds + O(\kappa a^{4-\beta})$$

$$= I_m Q_m + O(\kappa a^{4-\beta}),$$

(5.1)
where \( I_m = \int_{\partial D_m} \Phi_0(s_m, t) ds_m \), is a constant for each \( t \in \partial D_m \) if \( D_m \) is a ball. Indeed, for the single layer potential \( (S^\varepsilon_{D_m} \psi)(s) = \int_{\partial D_m} \Phi_0(s, t) \psi(t) dt \), we have the jump condition as \( \frac{\partial}{\partial n} (S^\varepsilon_{D_m} \psi)(s)|_+ = \left( \frac{1}{2} I + K^\varepsilon_{D_m} \right) \psi(s) \) recalling that \( K^\varepsilon_{D_m} \) is the adjoint of the double layer operator. Since \( K^\varepsilon_{D_m}[1] = K^\varepsilon_{D_m}[1] = -1/2 \), then we have \( \frac{\partial}{\partial n} S^\varepsilon_{D_m}[1] = 0 \) on \( \partial D_m \). Hence \( S^\varepsilon_{D_m}[1] \) is a constant, namely \( I_m \), as it satisfies \( \Delta S^\varepsilon_{D_m}[1] = 0 \) in \( D_m \) and has zero normal derivative on \( \partial D_m \).

With this correction at hand the estimate (2.70) becomes

\[
(-1 + \lambda_m I_m)Q_m + \sum_{j \neq m}^{M} \Phi_\kappa(z_m, z_j) \lambda_m |\partial D_m| Q_j = -e^{i\kappa \theta} z_m + O \left( (|\lambda_m| + \kappa) \kappa \alpha^3 \right) + \lambda_m O \left( a^{4-\beta} \right),
\]

\[
+ O \left( \kappa^2 a^{4-\beta} \right) + O \left( 2\kappa^2 a^{5-\beta} - \frac{6}{d^2} + 7 \right) + \lambda_m O \left( 2\kappa a^{5-\beta} \left[ 7\kappa + \frac{6\kappa + 13}{d^3} \right] \right)
\]

and the system (2.71) as

\[
-\frac{1}{\lambda_m} \sum_{j \neq m}^{M} \Phi_\kappa(z_m, z_j) Q_j + \text{Err} = -e^{i\kappa \theta} z_m - \sum_{j \neq m}^{M} \Phi_\kappa(z_m, z_j) Q_j + \text{Err}
\]

with

\[
\text{Err} := \frac{1}{\lambda_m |\partial D_m|} \left[ O \left( (|\lambda_m + \kappa \alpha^3 \beta) \kappa \alpha^3 \right) + O \left( \kappa^2 a^{4-2\beta} \right) + O \left( 2\kappa^2 a^{5-\beta} - \frac{6}{d^2} + 7 \right) + O \left( \kappa a^{4-\beta} \right) \right]
\]

\[
= \frac{1}{\lambda_m |\partial D_m|} \left[ O \left( a^{3-\beta} + \frac{a^{5-2\beta}}{d^{2\alpha}} \right) \right] = \frac{1}{\lambda_m |\partial D_m|} \left[ O \left( a^{3-\beta} + \frac{a^{5-2\beta}}{d^{2\alpha}} \right) \right]
\]

Finally, the estimate (3.11) becomes

\[
U^\infty(\hat{x}) = \sum_{m=1}^{M} \left[ e^{-i\kappa \hat{x} \cdot z_m} Q_m + O(\kappa a^{3-\beta}) \right]
\]

\[
= \sum_{m=1}^{M} \left[ e^{-i\kappa \hat{x} \cdot z_m} [Q_m + (Q_m - \bar{Q}_m)] + O(\kappa a^{3-\beta}) \right]
\]

\[
= \sum_{m=1}^{M} e^{-i\kappa \hat{x} \cdot z_m} Q_m + O(M a^{2-\beta} [\text{Err} + (\kappa a)])
\]

\[
= \sum_{m=1}^{M} e^{-i\kappa \hat{x} \cdot z_m} Q_m + O(M a^{2-\beta} \left( a + \frac{a^{3-\beta}}{d^{3\alpha}} \right))
\]

\[
= \sum_{m=1}^{M} e^{-i\kappa \hat{x} \cdot z_m} Q_m + O(a^{3-\beta})
\]

since, as \( s = 3t \alpha \leq 2 - \beta \), we have \( \frac{a^{3-\beta}}{d^{3\alpha}} = O(a^{3-\beta-3t \alpha}) = O(a) \).
6 Appendix: Proof of Lemma 3.1

We start by factorizing \( B \) as \( B = -(C^{-1} + B_n) \) where \( C := \text{Diag}(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_M) \in \mathbb{R}^{M \times M} \), \( I \) is the identity matrix and \( B_n := -C^{-1} - B \). We have \( B : \mathbb{C}^M \rightarrow \mathbb{C}^M \), so it is enough to prove the injectivity in order to prove its invertibility. For this purpose, let \( X, Y \) are vectors in \( \mathbb{C}^M \) and consider the system

\[
(C^{-1} + B_n)X = Y. \tag{6.1}
\]

Let \((\cdot)^{\text{real}}\) and \((\cdot)^{\text{img}}\) denotes the real and the imaginary parts of the corresponding complex number/vector/matrix. For convenience, let us denote \( C^{-1} \) by \( C_I \). Now, the following can be written from (6.1):

\[
(C_I^{\text{real}} + B_n^{\text{real}})X^{\text{real}} - (C_I^{\text{img}} + B_n^{\text{img}})X^{\text{img}} = Y^{\text{real}},
\]

\[
(C_I^{\text{real}} + B_n^{\text{real}})X^{\text{real}} + (C_I^{\text{img}} + B_n^{\text{img}})X^{\text{real}} = Y^{\text{img}},
\]

which leads to

\[
\begin{align*}
\langle (C_I^{\text{real}} + B_n^{\text{real}})X^{\text{real}}, X^{\text{real}} \rangle - \langle (C_I^{\text{img}} + B_n^{\text{img}})X^{\text{img}}, X^{\text{real}} \rangle &= \langle Y^{\text{real}}, X^{\text{real}} \rangle, \tag{6.4} \\
\langle (C_I^{\text{real}} + B_n^{\text{real}})X^{\text{img}}, X^{\text{img}} \rangle + \langle (C_I^{\text{img}} + B_n^{\text{img}})X^{\text{real}}, X^{\text{img}} \rangle &= \langle Y^{\text{img}}, X^{\text{img}} \rangle. \tag{6.5}
\end{align*}
\]

By summing up (6.4) and (6.5) will give

\[
\begin{align*}
\langle C_I^{\text{real}}X^{\text{real}}, X^{\text{real}} \rangle + \langle B_n^{\text{real}}X^{\text{real}}, X^{\text{real}} \rangle + \langle C_I^{\text{img}}X^{\text{img}}, X^{\text{img}} \rangle + \langle B_n^{\text{img}}X^{\text{img}}, X^{\text{img}} \rangle \\
= \langle Y^{\text{real}}, X^{\text{real}} \rangle + \langle Y^{\text{img}}, X^{\text{img}} \rangle.
\end{align*}
\]

We can observe that, the right-hand side in (6.6) does not exceed

\[
\begin{align*}
\langle X^{\text{real}}, X^{\text{real}} \rangle^{1/2}\langle Y^{\text{real}}, Y^{\text{real}} \rangle^{1/2} + \langle X^{\text{img}}, X^{\text{img}} \rangle^{1/2}\langle Y^{\text{img}}, Y^{\text{img}} \rangle^{1/2} \\
\leq 2\langle X^{||}, X^{||} \rangle^{1/2}\langle Y^{||}, Y^{||} \rangle^{1/2}.
\end{align*}
\]

At this stage, we divide the proof the in to two cases.

1. If \( \Re(\lambda_{m,0}) < 0 \) for each \( m \). In this case \( \Re \tilde{C}_m > 0 \). We know that:

\[
|\langle B_n^{\text{real}}X^{\text{real}}, X^{\text{real}} \rangle| \leq \|B_n^{\text{real}}\|_2\|X^{\text{real}}\|_2 \tag{6.8}
\]

where \( \|B_n^{\text{real}}\|_2 := \sum_{i,j=1}^{M} (B_n^{\text{real}})_{i,j}^{2} \) and \( (B_n^{\text{real}})_{i,j} := \Re \Phi(z_i, z_j) \) if \( i \neq j \) and \( (B_n^{\text{real}})_{i,i} := 0 \) for \( i, j = 1, \ldots, M \). Hence \( |(B_n^{\text{real}})_{i,j}| \leq \frac{1}{4\pi |z_i - z_j|}, i \neq j \). Arguing as in (2.43), we see that

\[
\sum_{i,j=1}^{M} (B_n^{\text{real}})_{i,j}^{2} \leq \frac{d^{-\alpha}}{n^2 d^{2\alpha}} \leq \frac{M \cdot 2d^{-2\alpha}}{\pi^2} \sum_{n=1}^{d^{-\alpha}} 1 = \frac{2Md^{-3\alpha}}{\pi^2}. \tag{6.9}
\]

Observing that \( M = O(a^{-s}) \leq M_{\max}a^{-s} \) and that \( s = 3t\alpha \), we obtain:

\[
\|B_n^{\text{real}}\|_2 \leq \frac{\sqrt{2M_{\max}}}{\pi a^{-\hat{t}}}. \tag{6.10}
\]

From (6.6) and (6.7), we deduce that

\[
\left( \frac{\min_{1 \leq m \leq M} \Re \tilde{C}_m}{\max_{1 \leq m \leq M} |\tilde{C}_m|^2} \right) \sum_{m=1}^{M} |X_m|^2 \leq 2 \left( \sum_{m=1}^{M} |X_m|^2 \right)^{1/2} \left( \sum_{m=1}^{M} |Y_m|^2 \right)^{1/2}, \tag{6.11}
\]

From (6.6) and (6.7), we deduce that
which yields
\[ \sum_{m=1}^{M} |x_m|^2 \leq 4 \left( \min_{1 \leq m \leq M} \Re(\tilde{C}_m) \right) \left( \frac{2M_{\text{max}}}{\pi d \tau} \right)^2 \sum_{m=1}^{M} |y_m|^2. \] (6.12)

Thus, if \( \min_{1 \leq m \leq M} \Re(\tilde{C}_m) > \frac{2M_{\text{max}}}{\pi d \tau} \), then the matrix B in algebraic system (3.2) is invertible.

2. If \( \Re(\lambda_{m,0}) > 0 \) for each \( m \). In this case \( \Re(\tilde{C}_m) < 0 \) and so to prove the invertibility of B in algebraic system (3.2), consider \((-C^{-1} - B_n)X = Y \) in place of (6.1) and proceed in the way as it done for the case \( \Re(\lambda_{m}) > 0 \) for each \( m \). Then we can get the following estimate
\[ \sum_{m=1}^{M} |x_m|^2 \leq 4 \left( \min_{1 \leq m \leq M} \Re(-\tilde{C}_m) \right) \left( \frac{2M_{\text{max}}}{\pi d \tau} \right)^2 \sum_{m=1}^{M} |y_m|^2. \] (6.13)

and the invertibility of the matrix B, under the assumption that \( \min_{1 \leq m \leq M} \Re(-\tilde{C}_m) > \frac{2M_{\text{max}}}{\pi d \tau} \).

3. We are left with the case where \( \Re(\lambda_{m,0}) > 0 \) for few \( m \)’s. In this case we multiply every line of the system (6.1) corresponding to \( \Re(\lambda_{m,0}) > 0 \) by \(-1 \). Hence, the system (6.1) becomes
\[ (\tilde{C}^{-1} + \tilde{B}_n)X = \tilde{Y}. \] (6.14)

where now every component of the diagonal matrix \( \tilde{C}^{-1} \) is positive and \( \|\tilde{B}_n\|_2 = \|B_n\|_2 \). Then we are in the case (1).

\[ \square \]

**Remark 6.1.** Assume that \( \Re(\lambda_{m,0}) < 0 \). Following the computations in [10, Lemma 2.22] and assuming that \( \frac{5\pi}{4} \left( \min_{1 \leq m \leq M} \Re(C_m) \right) > \frac{\pi}{2} \) and \( \gamma := \min_{j \neq m, 1 \leq j, m \leq M} \cos(\kappa|z_m - z_j|) \geq 0 \), we can prove that
\[ \left( \min_{1 \leq m \leq M} \Re(\tilde{C}_m) \right) \left( \max_{1 \leq m \leq M} |C_m|^2 \right) - \frac{3\gamma}{5\pi d} \sum_{m=1}^{M} |x_m|^2 \leq 2 \left( \sum_{m=1}^{M} |x_m|^2 \right)^{1/2} \left( \sum_{m=1}^{M} |y_m|^2 \right)^{1/2}, \]

which yields
\[ \sum_{m=1}^{M} |x_m|^2 \leq 4 \left( \min_{1 \leq m \leq M} \Re(\tilde{C}_m) \right) \left( \max_{1 \leq m \leq M} |C_m|^2 \right) - \frac{3\gamma}{5\pi d} \sum_{m=1}^{M} |y_m|^2. \]

and thus the invertibility of the matrix B in the algebraic system (3.2). Observe that the condition \( \min_{1 \leq m \leq M} \Re(\tilde{C}_m) > \frac{2M_{\text{max}}}{\pi d \tau} \) is satisfied if \( t < 2 - \beta \).

We can see that if the number of the scatterers is \( O(a^{-s}) \) such that \( s \leq 2 - \beta \), then we do not need upper bound on \( t \), which allow as to have very close scatterers. However, if we want to have larger number of scatterers, i.e. \( s \) limited only by the bound \( s \leq 3 \), then we need a condition on \( t \), i.e. \( t \leq 2 - \beta \), which makes more restrictions on the minimum distance between the scatterers.
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