RANDOM 3-NONCROSSING PARTITIONS

JING QIN AND CHRISTIAN M. REIDYS

Abstract. In this paper, we introduce polynomial time algorithms that generate random 3-noncrossing partitions and 2-regular, 3-noncrossing partitions with uniform probability. A 3-noncrossing partition does not contain any three mutually crossing arcs in its canonical representation and is 2-regular if the latter does not contain arcs of the form \((i, i + 1)\). Using a bijection of Chen et al. [2, 4], we interpret 3-noncrossing partitions and 2-regular, 3-noncrossing partitions as restricted generalized vacillating tableaux. Furthermore, we interpret the tableaux as sampling paths of Markov-processes over shapes and derive their transition probabilities.

1. Introduction

Recently, a paper written by Chen et. al. [3] attracts our attention. According to the bijection between the \(k\)-noncrossing matchings and the oscillating tableaux [2], they identify the latter as stochastic processes over Young tableaux of less than \(k\) rows in order to uniformly generate a \(k\)-noncrossing matching. Furthermore, since the generating function of the corresponding oscillating lattice walks in \(\mathbb{Z}^{k-1}\) that remains in the interior of the dominant Weyl chamber has been given by Grabiner and Magyar [7], the key quantities, the transition probabilities of the specific stochastic processes can be derived with linear time complexity.

The objective to enumerate \(k\)-noncrossing partitions is much more difficult since the corresponding lattice walks are not reflectable [6] in \(\mathbb{Z}^{k-1}\) in case of \(k \geq 3\). Only the case for \(k = 3\) has been solved by Bousquet-Mélou and Xin in [1] via the celebrated kernel method. Also in their paper, they conjecture that \(k\)-noncrossing partitions are not \(P\)-recursive for \(k \geq 4\). For \(k = 3\), what Bousquet-Mélou and Xin need in order to enumerate 3-noncrossing partition is the number of corresponding

\textit{Key words and phrases.} 3-noncrossing partition, 2-regular 3-noncrossing partition, uniform generation, kernel method.
lattice walks starting and ending at \((1,0) \in \mathbb{Z}^2\). However, our main idea is to interpret the corresponding vacillating tableaux as sampling paths of Markov-processes over shapes and derive their transition probabilities, see Fig. 1. In order to derive the transition probabilities, what we need is the number of the corresponding lattice walks ending at arbitrary \((i,j) \in \mathbb{Z}^2\) such that \(i > j \geq 0\).

This paper is organized as follows. Section 2 describes the basic facts of 3-noncrossing partitions and 2-regular, 3-noncrossing partitions. Section 3 shows the reader how we explore more information from the kernel equations. In the meantime, we generate a 3-noncrossing braid since we will prove there exists a bijection between the set of \(k\)-noncrossing partitions over \([N] = \{1,2,\ldots,N\}\) and the set of \(k\)-noncrossing braids over \([N−1]\). Furthermore, in the Section 4 we will show the reader how to arrive at the transition probabilities of corresponding lattice walks for 2-regular, 3-noncrossing partitions from the fact that there exists a bijection between the set of 2-regular, \(k\)-noncrossing partitions over \([N]\) and the set of \(k\)-noncrossing braids without loops over \([N−1]\).

![Diagram](image)

**Figure 1.** The idea behind the uniform generation: we consider a stochastic process over shapes (A) extract a specific sampling path (B) and (C) display the corresponding 3-noncrossing partition structure induced by this path. The probabilities given in (B) are the conditional probabilities with respect to their last step. The transition probabilities for given \(k = 3\) and \(N\) are computed in Theorem 3.3 as a preprocessing step in polynomial time.
2. Some basic facts

A set partition \( P \) of \([N]\) is a collection of nonempty and mutually disjoint subsets of \([N]\), called blocks, whose union is \([N]\). A \( k \)-noncrossing partition is called \( m \)-regular, \( m \geq 1 \), if for any two distinct elements \( x, y \) in the same block, we have \(|x - y| \geq m \). A partial matching and a matching is a particular type of partition having block size at most two and exactly two, respectively. Their standard representation is a unique graph on the vertex set \([N]\) whose edge set consists of arcs connecting the elements of each block in numerical order, see Fig. 2.

![Figure 2](image)

**Figure 2.** Standard representation and \( k \)-crossings: we display the partition \( \{2\}, \{8\}, \{11\}, \{1\}, \{5\}, \{6\}, \{12\}, \{3\}, \{7\}, \{10\}, \{9\}, \{15\} \) of \([15]\). Elements within blocks are connected in numerical order. The arcs \( \{(1, 5)\}, \{(3, 7)\}, \{(3, 7), (6, 12)\}, \{(7, 10)\}, \{(9, 15)\} \) and \( \{(6, 12), (9, 15)\} \) are all 2-crossings.

Given a (set) partition \( P \), a \( k \)-crossing is a set of \( k \) edges \( \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\} \) such that \( i_1 < i_2 < \ldots < i_k < j_1 < j_2 < \ldots < j_k \), see Fig. 2. A \( k \)-noncrossing partition is a partition without any \( k \)-crossings. We denote the sets of \( k \)-noncrossing partitions and \( m \)-regular, \( k \)-noncrossing partitions by \( P_k(N) \) and \( P_{k,m}(N) \), respectively. For instance, the set of 2-regular, 3-noncrossing partitions are denoted by \( P_{3,2}(N) \).

A (generalized) vacillating tableau \( V^N_\lambda \) of shape \( \lambda \) and length \( 2N \) is a sequence \( \lambda^0, \lambda^1, \ldots, \lambda^{2N} \) of shapes such that (1) \( \lambda^0 = \emptyset \), \( \lambda^{2N} = \lambda \) and (2) for \( 1 \leq i \leq N \), \( \lambda^{2i-1}, \lambda^{2i} \) are derived from \( \lambda^{2i-2} \) by elementary moves (EM) defined as follows: \((\emptyset, \emptyset)\): do nothing twice; \((-\Box, \emptyset)\): first remove a square then do nothing; \((\emptyset, +\Box)\): first do nothing then add a square; \((\pm\Box, \pm\Box)\): add/remove a square at the odd and even steps, respectively. We use the following notation: if \( \lambda_{i+1} \) is obtained from \( \lambda_i \) by adding, removing a square from the \( j \)-th row, or doing nothing we write \( \lambda_i \vdash_{+\Box} \ lambda_{i+1} \), \( \lambda_i \vdash_{-\Box} \ lambda_{i+1} \) or \( \lambda_i \vdash_{\emptyset} \ lambda_{i+1} \), respectively, see Fig. 3.

A braid over \([N]\) can be represented via introducing loops \((i, i)\) and drawing arcs \((i, j)\) and \((j, \ell)\) with \( i < j < \ell \) as crossing, see Fig. 4(B1). A \( k \)-noncrossing braid is a braid without any \( k \)-crossings. We denote the set of \( k \)-noncrossing braids over \([N]\) with and without isolated points by \( B_k(N) \) and \( B^*_k(N) \), respectively. Chen et al. \cite{2} have shown that each \( k \)-noncrossing partition corresponds
A1 (A2)

(B1)  

(C1)  

(C2)  

Figure 3. Vacillating tableaux and elementary moves: (A1) shows a general vacillating tableau with EM (A2). (B1) displays another vacillating tableau and its set of EM, \{(-\emptyset, \emptyset), (\emptyset, +\emptyset), (\emptyset, \emptyset), (-\emptyset, +\emptyset)\} shown in (B2). In (C1) we present a vacillating tableau with the EM \{(-\emptyset, \emptyset), (\emptyset, +\emptyset), (\emptyset, \emptyset), (+\emptyset, -\emptyset)\} displayed in (C2).

uniquely to a vacillating tableau of empty shape, having at most \(k-1\) rows, obtained via the EM \{(-\emptyset, \emptyset), (\emptyset, +\emptyset), (\emptyset, \emptyset), (-\emptyset, +\emptyset)\}, see Fig. (A2). In \[4\], Chen et al. proceed by proving that vacillating tableaux of empty shape, having at most \((k-1)\)-rows which are obtained by the EMs \{(-\emptyset, \emptyset), (\emptyset, +\emptyset), (\emptyset, \emptyset), (+\emptyset, -\emptyset)\} correspond uniquely to \(k\)-noncrossing braids.

Figure 4. Vacillating tableaux, partitions and braids: in (A1) we show a 3-noncrossing partition and in (A2) the associated vacillating tableau. (B1) shows a 3-noncrossing braid and (B2) its vacillating tableau.

In the following, we consider 3-noncrossing partition or a 2-regular, 3-noncrossing partition of \(N\) vertices, where \(N \geq 1\). To this end, we introduce two \(\mathbb{Z}^2\) domains \(Q_2 = \{(a_1, a_2) \in \mathbb{Z}_2 \mid a_1, a_2 \geq 0\}\) and \(W_2 = \{(a_1, a_2) \in \mathbb{Z}_2 \mid a_1 > a_2 \geq 0\}\). Let \(D \in \{Q_2, W_2\}\), \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\). A
$\mathcal{P}_D$-walk is a lattice walk in $D$ starting at $(1,0)$ having steps $\pm \mathbf{e}_1, \pm \mathbf{e}_2$, $0 = (0,0)$, such that even steps are $\pm \mathbf{e}_1, \pm \mathbf{e}_2$, or $0$ and odd steps are either $-\mathbf{e}_1, -\mathbf{e}_2$, or $0$. Analogously, a $\mathcal{B}_D$-walk is a lattice walk in $D$ starting at $(1,0)$ whose even steps are either $-\mathbf{e}_1, -\mathbf{e}_2$, or $0$ and whose odd steps are $+\mathbf{e}_1, +\mathbf{e}_2$, or $0$. By abuse of language, we will omit the subscript $D$.

Interpreting the number of squares in the rows of the shapes as coordinates of lattice points, we immediately obtain

**Theorem 2.1.** [4]

(1) The number of 3-noncrossing partitions over $[N]$ equals the number of $P_{W_2}$-walks from $(1,0)$ to itself of length $2N$.

(2) The number of 3-noncrossing braids over $[N]$ equals the number of $B_{W_2}$-walks from $(1,0)$ to itself of length $2N$.

### 3. Random 3-noncrossing partitions

In the following, set $\bar{x} = 1/x$ and $\bar{y} = 1/y$. We work in the ring of power series in $t$ whose coefficients are Laurent polynomials in $x$ and $y$, denoted by $Q[x, \bar{x}, y, \bar{y}][t]$. We first review some calculations of [1]. Let $a_{i,j}^s$ denote the number of $P_{Q_2}$-walks of length $s$ ending at $(i,j) \in Q_2$ and

$$F_e(x, y; t) = \sum_{i,j,\ell} a_{i,j}^{i,j} x^i y^j t^{2\ell}$$

$$F_o(x, y; t) = \sum_{i,j,\ell} a_{i,j}^{i,j+1} x^i y^j t^{2\ell+1}$$

are the generating functions of $P_{Q_2}$-walks of even and odd length and ending at $(i,j)$, respectively. In particular, let $H_e(x; t) = F_e(x, 0; t)$ and $V_e(y; t) = F_e(0, y; t)$ denote the generating functions of even $P_{Q_2}$-walks ending on the $x$-axis and $y$-axis.

**Proposition 3.1.** [1] Let $V(y; t) = V_e(y; t^2)$, $H(x; t) = H_e(x; t^2)$, then

$$F_e(x, y; t) = x + F_o(x, y; t)(1 + x + y)t,$$

$$K(x, y; t) F(x, y; t) = xy + x^2 y + x^2 - xH(x, t) - yV(y, t),$$

where $F(x, y; t) = \sum_{i,j,\ell} f_{i,j}^{i,j} x^i y^j t^{\ell}$ is given by $F_o(x, y; t) = tF(x, y; t^2)$ and

$$K(x, y; t) := xy - t(1 + x + y)(x + y + xy)$$

is called the kernel of eq. (3.3).
Eq. (3.22) follows from $a_{0}^{1,0} = 1$ and $a_{2t}^{i,j} = a_{2t-1}^{i-1,j} + a_{2t-1}^{i,j-1} + a_{2t-1}^{i-1,j-1}$ for $1 \leq \ell \leq N$. In view of Proposition 3.1, $F(x, y; t)$ is the key for enumerating $\mathcal{P}_{Q_{1}}$-walks. We observe that

$$Y_0 = \frac{1 - (\bar{x} + 3 + x)t - \sqrt{(1 - (1 + x + \bar{x}t)^2 - 4t}}{2(1 + \bar{x})t}$$

is the unique power-series solution in $t$ of the kernel equation $K(x, y; t) = 0$, whence

$$xY_0 = t(1 + x + Y_0)(x + Y_0 + xY_0).$$

Since $K(x, y; t)$ is quadratic in both $x$ and $y$, $(x, Y_0)$ can be regarded as a pair of Laurent series in $t$ with coefficients satisfying $K(x, Y_0; t) = 0$. As $(\bar{x}Y_0, Y_0)$ and $(\bar{x}Y_0, \bar{x})$ also solve $K(x, y; t) = 0$, we arrive at

$$xH(x; t) + Y_0V(Y_0; t) = xY_0 + x^2Y_0 + x^2,$$

$$\bar{x}Y_0H(\bar{x}Y_0; t) + Y_0V(Y_0; t) = \bar{x}Y_0^2 + \bar{x}^2Y_0^3 + \bar{x}^2Y_0^2,$$

$$\bar{x}Y_0H(\bar{x}Y_0; t) + \bar{x}V(\bar{x}; t) = \bar{x}^2Y_0 + \bar{x}^3Y_0^2 + \bar{x}^2Y_0^2.$$

Indeed, we can conclude from eqs. (3.6)-(3.8)

$$xH(x; t) + \bar{x}V(\bar{x}) = x^2 + (\bar{x}^2 + x + x^2)Y_0 + (\bar{x}^3 - \bar{x})Y_0^2 - \bar{x}^2Y_0^3.$$

Since $H$ and $V$ are generating functions, $xH(x; t)$ contains only positive powers of $x$. Similarly, $\bar{x}V(\bar{x}; t)$ contains only negative powers of $x$. Therefore, setting $NT_x$ and $PT_x$ to be the operators that extract the positive and negative powers of $x$ from a power series in $Q[x, \bar{x}][[t]]$, we arrive at

$$xH(x; t) = \text{PT}_x[x^2 + (\bar{x}^2 + x + x^2)Y_0 + (\bar{x}^3 - \bar{x})Y_0^2 - \bar{x}^2Y_0^3]$$

$$\bar{x}V(\bar{x}; t) = \text{NT}_x[x^2 + (\bar{x}^2 + x + x^2)Y_0 + (\bar{x}^3 - \bar{x})Y_0^2 - \bar{x}^2Y_0^3].$$

Via Lagrange inversion [5], we derive the formula

$$[x^m \bar{x}^s]x^mY_0^m = \sum_{j} \frac{m}{s} \binom{s}{j} \binom{s}{j + m} \binom{2j + m}{j - \ell}.$$  

The enumeration of $\mathcal{P}_{W_2}$-walks follows from the reflection principle [5]. Let $\omega_{t}^{i,j}$ denote the number of $\mathcal{P}_{W_2}$-walks ending at $(i, j)$ of length $\ell$. The reflection-principle implies

$$\omega_{t}^{i,j} = a_{t}^{i,j} - a_{t}^{j,i}.$$

In the following lemma, we express $\omega_{t}^{i,j}$ via the coefficients of $F(x, y; t)$ and establish their corresponding recurrence relations.
Lemma 3.2. (a) Suppose \( 1 \leq \ell \leq N - 1 \) and \((i, j) \in \mathbb{Z}^2\), then \( \omega_{s}^{i,j} \) is given by

\[
\omega_{s}^{i,j} = \begin{cases} 
  f_{\ell}^{i,j} - f_{\ell}^{i,0} & \text{for } s = 2\ell + 1, \\
  f_{\ell}^{i,j} + f_{\ell}^{i-1,j} + f_{\ell}^{i,j-1} - f_{\ell}^{i-1,i} - f_{\ell}^{i,j-1} & \text{for } s = 2\ell + 2,
\end{cases}
\]

where (1) \( \omega_{0}^{i,j} = 1 \), for \( i = 1, j = 0 \) and \( \omega_{0}^{i,j} = 0 \), otherwise; (2) \( f_{\ell}^{i,j} = 0 \) for \( i \not\in \{0, 1, \ldots, \ell\} \) or \( j \not\in \{0, 1, \ldots, \ell\} \).

(b) \( f_{\ell}^{i,j} \) satisfies the recursion

\[
f_{\ell}^{i,j} = \begin{cases} 
  [y^{j}t^{j+1}]V(y; t) - f_{\ell}^{0,j-1} & \text{for } i = 0, j \neq 0 \\
  [x^{i}t^{i+1}]H(x; t) - f_{\ell}^{i-1,0} & \text{for } i \neq 0, j = 0 \\
  f_{\ell-1}^{i+1,j+1} - f_{\ell-1}^{i+1,j+1} - f_{\ell-1}^{i,j+1} + 3f_{\ell-1}^{i,j} - f_{\ell-1}^{i,j} - f_{\ell-1}^{i,j} & \text{otherwise.}
\end{cases}
\]

Proof. We first prove assertion (a). Indeed, according to Proposition 3.1 we have \( F_{o}(x, y; t) = tF(x, y; t^{2}) \) and \( F_{e}(x, y; t) = x + F_{o}(x, y; t)(1 + x + y)t \) or equivalently

\[
a_{2\ell+1}^{i,j} = f_{\ell}^{i,j} \text{ and } a_{2\ell}^{i,j} = f_{\ell-1}^{i,j} + f_{\ell-1}^{i-1,j} + f_{\ell-1}^{i,j-1},
\]

whence eq. \( 3.14 \) follows from \( \omega_{s}^{i,j} = a_{s}^{i,j} - a_{s}^{i,j} \).

Next, we prove (b). We distinguish the following three cases. First, suppose \( i \neq 0 \) and \( j \neq 0 \). Equating the coefficients of \( x^{i}y^{j}t^{\ell} \) in eq. \( 3.3 \), we derive

\[
f_{\ell}^{i,j} = f_{\ell-1}^{i+1,j+1} - f_{\ell-1}^{i+1,j+1} - f_{\ell-1}^{i,j+1} + 3f_{\ell-1}^{i,j} - f_{\ell-1}^{i,j} - f_{\ell-1}^{i,j}.
\]

In case of \( i = 0 \) or \( j = 0 \), let \( [x^{i}t^{j}]xH(x; t) \) and \( [y^{j}t^{i}]yV(y; t) \) denote the coefficients of \( x^{i}t^{j} \) in \( xH(x; t) \) and the coefficient of \( y^{j}t^{i} \) in \( yV(y; t) \), respectively. Then we have

\[
\begin{align*}
  f_{\ell-1}^{0,j-1} + f_{\ell-1}^{0,j-2} &= [y^{j}t^{i}]yV(y; t), \\
  f_{\ell-1}^{i-1,0} + f_{\ell-1}^{i-2,0} &= [x^{i}t^{j}]xH(x; t).
\end{align*}
\]

According to eq. \( 3.10 \) and eq. \( 3.11 \), the coefficients \( [x^{i}t^{j}]xH(x; t) \) and \( [y^{j}t^{i}]yV(y; t) \) are given as follows:

\[
\begin{align*}
  [x^{i}t^{j}]H(x, t) &= [x^{0}t^{i}]x^{i-1}(x^{2} + (\bar{x}^{2} + x + x^{2})Y_{0} + (\bar{x}^{3} - \bar{x}^{2})Y_{0}^{2} - \bar{x}^{2}Y_{0}^{3}), \\
  [y^{j}t^{i}]V(y, t) &= [x^{0}t^{i}]x^{i+1}(x^{2} + (\bar{x}^{2} + x + x^{2})Y_{0} + (\bar{x}^{3} - \bar{x}^{2})Y_{0}^{2} - \bar{x}^{2}Y_{0}^{3}).
\end{align*}
\]

whence (b) and the proof of the lemma is complete.
Lemma 3.2 allows us to compute $\omega_{i,j}^\ell$ for all $i, j \in \mathbb{Z}^2$ and $1 \leq \ell \leq 2N$. In the following, we consider a vacillating tableaux as the sampling path of a Markov-process, whose transition probabilities can be calculated via the terms $\omega_{i,j}^\ell$.

**Theorem 3.3.** Algorithm 1 generates a random 3-noncrossing partition after a pre-processing step having $O(N^4)$ time and $O(N^3)$ space complexity with uniform probability in linear time and space.

**Algorithm 1** Uniform generation of 3-noncrossing partitions

1: $i = 1$
2: Tableaux (Initialize the sequence of shapes, $\{\lambda^i\}_{i=0}^{2N}$)
3: $\lambda^0 = \emptyset$, $\lambda^{2N} = \emptyset$
4: while $i < 2N$ do
5: if $i$ is even then
6:   $X[0] \leftarrow V_3(\lambda^i+1\emptyset, 2N - (i + 1))$
7:   $X[1] \leftarrow V_3(\lambda^i+1\square_1, 2N - (i + 1))$
8:   $X[2] \leftarrow V_3(\lambda^i+1\square_2, 2N - (i + 1))$
9: end if
10: if $i$ is odd then
11:   $X[0] \leftarrow V_3(\lambda^i+1\emptyset, 2N - (i + 1))$
12:   $X[1] \leftarrow V_3(\lambda^i+1\square_1, 2N - (i + 1))$
13:   $X[2] \leftarrow V_3(\lambda^i+1\square_2, 2N - (i + 1))$
14: end if
15: $sum \leftarrow X[0] + X[1] + X[2]$
16: Shape $\leftarrow$ Random($sum$) (Random generates the random shape $\lambda^{i+1}_{\square_1,\square_2}$ with probability $X[j]/sum$ or $\lambda^{i+1}_{\emptyset}$ with probability $X[0]/sum$)
17: $i \leftarrow i + 1$
18: Insert Shape into Tableau (the sequence of shapes).
19: end while
20: Map(Tableau) (maps Tableau into its corresponding 3-noncrossing partition)

**Proof.** The main idea is to interpret tableaux of 3-noncrossing partitions as sampling paths of a stochastic process. We label the $(i+1)$-th shape, $\lambda^{i+1}_\alpha$ by $\alpha = \lambda^{i+1}_i \setminus \lambda^i \in \{+\square_1, +\square_2, -\square_1, -\square_2, \emptyset\}$, where the labeling specifies the transition from $\lambda^i$ to $\lambda^{i+1}$.

Let $V(\lambda^{i+1}_\alpha, 2N - (i + 1))$ denote the number of vacillating tableaux of length $(i + 1)$ such that $\lambda^{i+1}_i \setminus \lambda^i = \alpha$. We remark here that if $\lambda^{i+1}$ has $(a - 1)$ in the first row and $b$ boxes in the second
row, then $V(\lambda^{i+1}, 2N - (i + 1)) = \omega^{a,b}_{2N - (i + 1)}$. Let $(X^i)_{i=0}^{2N}$ be given as follows:

- $X^0 = X^{2N} = \emptyset$ and $X^i$ is a shape having at most 2 rows;
- for $1 \leq i \leq N - 1$, we have $X_{2i+1} \setminus X_{2i} \in \{\emptyset, -\Box_1, -\Box_2\}$ and $X_{2i+2} \setminus X_{2i+1} \in \{\emptyset, +\Box_1, +\Box_2\}$.
- for $1 \leq i \leq 2N - 1$, we have

$$P(X^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{V(\lambda^{i+1}, 2N - i - 1)}{V(\lambda, 2N - i)}.$$

In view of eq. (3.22), we immediately observe

$$\prod_{i=0}^{2N-1} P(X^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{V(\lambda^{2N} = \emptyset, 0)}{V(\lambda^0 = \emptyset, N)} = \frac{1}{V(\emptyset, 2N)}.$$

Consequently, the process $(X^i)_{i=0}^{2N}$ generates random 3-noncrossing partitions with uniform probability in $O(N)$ time and space.

As for the derivation of the transition probabilities, suppose we are given a shape $\lambda^h$ having exactly $a$ and $b$ squares in the first and second row, respectively. According to Lemma 3.2, we obtain

$$\omega^{a,b}_{2N-h} = \begin{cases} f^{a,b}_\ell - f^{b,a}_\ell & \text{for } 2N - h = 2\ell + 1, \\
 f^{a,b}_\ell + f^{a,b-1}_\ell + f^{b,a}_\ell - f^{b-1,a}_\ell - f^{b,a-1}_\ell & \text{for } 2N - h = 2\ell + 2, \end{cases}$$

which reduces the problem to the calculation of at most six coefficients $f^{i,j}_\ell$ for fixed $i$, $j$ and $\ell$.

Claim. The coefficients $f^{i,j}_\ell$ for all $0 \leq i, j \leq \ell$ and $1 \leq \ell \leq N - 1$, can be computed in $O(N^4)$ time and $O(N^3)$ space complexity.

Step 1. For fixed $i$, $j$ and $\ell$, $f^{1,0}_\ell$ and $f^{0,j}_\ell$ can be computed in $O(N^2)$ time. Indeed, according to eq. (3.15), we have

$$f^{1,0}_\ell = \sum_{i_1} (-1)^{i_1} [x^{i_1}\ell^{i_1+1}]H(x; t),$$
whence \( f^{i,0}_\ell \) can be calculated via a \textbf{For}-loop summing over the terms \((-1)^{q-i} x^{q+i+\ell+1} H(x;t)\). In view of eq. (3.21), the \([x^{q+i+\ell+1}] H(x;t)\) terms in turn are expressed via \([x^{q+i+\ell+1}] x^{q}Y_{0}^{m}\), for \(q \in \{-i-4,-i-3,-i-1,-i, -i+1\}\) and \(m \in \{0,1,2\}\). According to eq. (3.12), the quantities \([x^{q+i+\ell+1}] x^{q}Y_{0}^{m}\) can be calculated via a \textbf{For}-loop summing over the terms \(\frac{m}{\ell+1} (\ell+1) (\ell+1) (\ell+1) \lambda_{j}^{\ell+m} \) for \(1 \leq j \leq \ell + 1\). Consequently, for fixed \(i, \ell\) the coefficient \(f^{i,0}_\ell\) can be derived in \(O(N^2)\) time. Using the same arguments we obtain for fixed \(j, \ell\) the coefficient \(f^{0,j}_\ell\) in \(O(N^2)\) time.

**Step 2.** We compute \(f^{i,0}_\ell\) and \(f^{0,j}_\ell\) for all \(i, j\) and \(\ell\) via two nested \textbf{For}-loops in \(O(N^2)\) time.

**Step 3.** Once the coefficients \(f^{i,0}_\ell\) and \(f^{0,j}_\ell\) are calculated for all \(i, j, \ell\), we compute \(f^{i,j}_\ell\) for arbitrary \(i, j, \ell\) employing three nested \textbf{For}-loops and the recurrence of eq. (3.15) since there exists boundary conditions as follows:

\[
\begin{align*}
\text{(3.24)} & \quad f^{i,j}_\ell = 0 \quad i > \ell \text{ or } j > \ell, \\
\text{(3.25)} & \quad f^{i,j}_0 = \begin{cases} 
1 & i = 1, \ j = 0; \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Therefore, we compute \(f^{i,j}_\ell\) for arbitrary \(i, j, \ell\) in \(O(N^4)\) time and \(O(N^3)\) space and the claim follows.

Given the coefficients \(f^{i,j}_\ell\) for all \(i, j, \ell\), we can derive the transition probabilities in \(O(1)\) time. Accordingly, we obtain all transition probabilities \(V(\lambda^{i}, 2N - i)\) in \(O(N^4)\) time and \(O(N^3)\) space complexity.

\[ \square \]

4. Random 2-regular, 3-noncrossing partitions

In this section, we generate random 2-regular, 3-noncrossing partitions employing a bijection between the set of 2-regular, 3-noncrossing partitions over \([N]\) and the set of \(k\)-noncrossing braids without loops over \([N - 1]\).

**Lemma 4.1.** Set \(k \in \mathbb{N}, \ k \geq 3\). Suppose \((\lambda^{i})_{i=0}^{2\ell+1}\) in which \(\lambda^{0} = \emptyset\) is a sequence of shapes such that \(\lambda^{2j}_h \backslash \lambda^{2j-1}_h \in \{+\square_h^{k-1}_h, \emptyset\}\) and \(\lambda^{2j-1}_h \backslash \lambda^{2j-2}_h \in \{-\square_h^{k-1}_h, \emptyset\}\). Then \((\lambda^{i})_{i=0}^{2\ell+1}\) induces a unique sequence of shapes \((\mu^{i})_{i=0}^{2\ell}\) with the following properties

\[
\begin{align*}
\text{(4.1)} & \quad \mu^{2j+1}_h \backslash \mu^{2j}_h \in \{+\square_h^{k-1}_h, \emptyset\} \text{ and } \mu^{2j+2}_h \backslash \mu^{2j+1}_h \in \{-\square_h^{k-1}_h, \emptyset\}, \\
\text{(4.2)} & \quad \mu^{2j} = \lambda^{2j+1} \\
\text{(4.3)} & \quad \mu^{2j+1} \neq \lambda^{2j+2} \implies \mu^{2j+1} \in \{\lambda^{2j+1}, \lambda^{2j+3}\}.
\end{align*}
\]
Proof. Since $\lambda^1 = \emptyset$, $(\lambda_j^{2i+1})_{j=1}^{\ell}$ corresponds to a sequence of pairs $((x_i, y_i))_{i=1}^{\ell}$ given by $x_i = \lambda^{2i} \setminus \lambda^{2i-1}$ and $y_i = \lambda^{2i+1} \setminus \lambda^{2i}$ such that

$$\forall 1 \leq i \leq \ell; \quad (x_i, y_i) \in \{(\emptyset, \emptyset), (+\square_h, \emptyset), (\emptyset, -\square_h), (+\square_h, -\square_j)\}.$$  

Let $\varphi$ be given by

$$\varphi((x_i, y_i)) = \begin{cases} 
(x_i, y_i) & \text{for } (x_i, y_i) = (+\square_h, -\square_j) \\
(y_i, x_i) & \text{otherwise},
\end{cases}$$

and set $((\varphi(x_i, y_i))_{i=1}^{\ell} = (a_i, b_i))_{i=1}^{\ell}$. Note that $(a_i, b_i) \in \{(-\square_h, \emptyset), (\emptyset, +\square_h), (\emptyset, \emptyset), (+\square_h, -\square_j)\}$, where $1 \leq h, j \leq k - 1$. Let $(\mu_i^{2i})_{i=1}^{\ell}$ be the sequence of shapes induced by $(a_i, b_i)_{i=1}^{\ell}$ according to $a_i = \mu^{2i} \setminus \mu^{2i-1}$ and $b_i = \mu^{2i-1} \setminus \mu^{2i-2}$ initialized with $\mu_0 = \emptyset$. Now eq. (4.1) is implied by eq. (4.4) and eq. (4.2) follows by construction. Suppose $\mu^{2i+1} \neq \lambda^{2j+2}$ for some $0 \leq j \leq \ell - 1$. By definition of $\varphi$, only pairs containing "$\emptyset$" in at least one coordinate are transposed from which we
can conclude \( \mu^{2j+1} = \mu^{2j} \) or \( \mu^{2j+1} = \mu^{2j+2} \), whence eq. (4.3). I.e. we have the following situation

\[
\begin{align*}
\lambda^{2j+1} & \rightarrow \lambda^{2j+2} \\
\mu^{2j} & \rightarrow \mu^{2j+1} \rightarrow \mu^{2j+2},
\end{align*}
\]

and the lemma follows. \( \Box \)

Lemma 4.1 establishes a bijection between \( \mathcal{P}_{W_{k-1}} \)-walks of length \( 2\ell + 1 \) and \( \mathcal{B}_{W_{k-1}} \)-walks of length \( 2\ell \), where \( W_{k-1} = \{(a_1, a_2, \ldots, a_{k-1}) \mid a_1 > a_2 > \cdots > a_{k-1}\} \).

Corollary 4.2. Let \( \mathcal{P}_k(N) \) and \( \mathcal{B}_k(N-1) \) denote the set of \( k \)-noncrossing partitions on \([N]\) and \( k \)-noncrossing braids on \([N-1]\). Then

(a) we have a bijection

\[
\vartheta: \mathcal{P}_k(N) \longrightarrow \mathcal{B}_k(N-1).
\]

(b) \( \vartheta \) induces by restriction a bijection between 2-regular, \( k \)-noncrossing partitions on \([N]\) and \( k \)-noncrossing braids without loops on \([N-1]\).

Proof. Assertion (a) follows from Lemma 4.1 since a partition is completely determined by its induced \( \mathcal{P}_{W_{k-1}} \)-walk of shape \( \lambda^{2n-1} = \emptyset \). (b) follows immediately from the fact that, according to the definition of \( \varphi \) given in Lemma 4.1 any pair of consecutive EMs \((\emptyset, \square_1), (-\square_1, \emptyset)\) induces an EM \((+\square_1, -\square_1)\). Therefore, \( \vartheta \) maps 2-regular, \( k \)-noncrossing partitions into \( k \)-noncrossing braids without loops. \( \Box \)

According to Corollary 4.2 each 2-regular, 3-noncrossing partition on \([N]\) corresponds to a 3-noncrossing braid without loops over \([N-1]\). Each such braid uniquely corresponds to a \( \mathcal{B}_{W_{2}} \)-walk of length \( 2N - 2 \) in which there does not exist any odd \( +e_1 \)-step followed by an even \( -e_1 \)-step. We call such a walk a \( \mathcal{B}_{W_{2}}^* \)-walk.

Lemma 4.3. The number of \( \mathcal{B}_{W_{2}}^* \)-walks ending at \((i, j)\) of length \( 2\ell \), is given by

\[
\sigma_{2\ell}^{i,j*} = \sum_h (-1)^h \binom{\ell}{h} \omega_{2(\ell-h)+1}^{i,j}.
\]
Furthermore, we obtain the recurrence of $\sigma_{2\ell+1}^{i,j;*}$ given by

$$\sigma_{2\ell+1}^{i,j;*} = \sigma_{2\ell}^{i-1,j;*} + \sigma_{2\ell}^{i,j-1;*} + \sigma_{2\ell}^{i,j;*}.$$  

Proof. According to Lemma 4.1, any $B_{W_2}$-walk of length $2\ell + 1$ corresponds to a unique $B_{W_2}$-walk of length $2\ell$, whence $\sigma_{2\ell}^{i,j} = \omega_{2\ell+1}^{i,j}$. Let $A_{2\ell}(h)$ denote the set of $B_{W_2}$-walks of length $2\ell$ in which there exist at least $h$ pairs of shapes $(\mu_{2\ell-1}, \mu_{2\ell})$ induced by the EM $(+e_1, -e_1)$. Since the removal of $h$ pairs of $(+e_1, -e_1)$-EMs from such a $B_{W_2}$-walk, results in a $B_{W_2}$-walk of length $2(\ell - h)$, we derive $A_{2\ell}(h) = \binom{\ell}{h} \sigma_{2\ell-2h}^{i,j}$. Using the inclusion-exclusion principle, we arrive at

$$\sigma_{2\ell}^{i,j;*} = \sum_{h} (-1)^{h} \binom{\ell}{h} \sigma_{2(\ell-h)}^{i,j;*} = \sum_{h} (-1)^{h} \binom{\ell}{h} \omega_{2(\ell-h)+1}^{i,j;*}.$$  

By construction, an odd step in a $B_{W_2}$-walk is either $-e_1$, $-e_2$ or $0$, whence

$$\sigma_{2\ell+1}^{i,j;*} = \sigma_{2\ell}^{i-1,j;*} + \sigma_{2\ell}^{i,j-1;*} + \sigma_{2\ell}^{i,j;*}.$$

□
Via Lemma 4.3 we have explicit knowledge about the numbers of $B_{W_2}^*$-walks, i.e. $\sigma^{i,j,*}_\ell$ for all $i,j \in \mathbb{Z}^2$ and $1 \leq \ell \leq 2N$. Accordingly, we are now in position to generate 2-regular, 3-noncrossing partitions with uniform probability via 3-noncrossing, loop-free braids.

**Theorem 4.4.** A random 2-regular, 3-noncrossing partition can be generated, in $O(N^4)$ preprocessing time and $O(N^3)$ space complexity, with uniform probability in linear time. Each 2-regular, 3-noncrossing partition is generated with $O(N)$ space and time complexity.

**Proof.** We interpret $B_{W_2}^*$-walks as sampling paths of a stochastic process. To this end, we again label the $(i+1)$-th shape, $\lambda_{i+1}^j$ by $\alpha = \lambda_{i+1}^j \lambda_i^j \in \{+\Box_1, +\Box_2, -\Box_1, -\Box_2, \emptyset \}$, where the labeling specifies the transition from $\lambda_i^j$ to $\lambda_{i+1}^j$. In the following we distinguish even and odd labeled shape.

For $i = 2s$, we let $W^*(\lambda_{2s}^2, 2N - 2s)$ denote the number of $B_{W_2}^*$-walks such that $\alpha = \lambda_{2s}^2 \setminus \lambda_{2s-1}^2$. By construction, assume $\lambda_{2s}^2$ has $(a-1)$ and $b$ boxes in its first and second row, respectively, then we have

$$W^*(\lambda_{2s}^2, 2N - 2s) = \sigma^{a,b,*}_{2N-2s}$$

i.e. $W^*(\lambda_{2s}^2, 2N - 2s)$ is independent of $\alpha$ and we write $W^*(\lambda_{2s}^2, 2N - 2s) = W^*(\lambda_{2s}^2, 2N - 2s)$. For $i = 2s + 1$, let $V^*(\lambda_{2s+1}^2, 2N - (2s + 1))$ denote the number of $B_{W_2}^*$-walks of shape $\lambda_{2s+1}^2$ of length $2N - (2s + 1)$ where $\lambda_{2s+1}^2 \setminus \lambda_{2s}^2 = \alpha$. Then we have setting $u = 2N - 2s - 2$

$$V^*(\lambda_{2s+1}^2, u + 1) = \begin{cases} W^*(\lambda_{2s+2}^2, u) & \text{for } \alpha = +\Box_1 \\ W^*(\lambda_{2s+2}^2, u) + W^*(\lambda_{2s+2}^-\Box_2, u) & \text{for } \alpha = +\Box_2 \\ W^*(\lambda_{2s+2}^2, u) + W^*(\lambda_{2s+2}^{-}\Box_1, u) + W^*(\lambda_{2s+2}^-, u) & \text{for } \alpha = \emptyset \\ W^*(\lambda_{2s+2}^2, u) & \text{for } \alpha = -\Box_1, -\Box_2. \end{cases}$$

We are now in position to specify the process $(X^i)_{i=0}^{2N}$:

- $X^0 = X_{2N} = \emptyset$, $X^i$ is a shape with at most 2 rows.
- for $1 \leq i \leq N - 1$, $(X^{2i+1} \setminus X^{2i}, X^{2i+2} \setminus X^{2i}) \in \{(-\Box, \emptyset), (\emptyset, +\Box), (\emptyset, \emptyset), (+\Box, -\Box) \}$.
- there does not exist any subsequence $(X^{2i}, X^{2i+1}, X^{2i+2})$ such that $(X^{2i+1} \setminus X^{2i}, X^{2i+2} \setminus X^{2i}) = (+\Box_1, -\Box_1)$.
- the transition probabilities are given as follows:

(1) for $i = 2\ell$, we obtain

$$\mathbb{P}(X^{i+1} = \lambda_{i+1}^+ | X^i = \lambda^i) = \frac{V^*(\lambda_{i+1}^+, 2n - i - 1)}{W^*(\lambda_{i+1}^+, 2N - i)}.$$
Algorithm 2 Uniform generation of 2-regular, 3-noncrossing partitions.

1: Tableaux (Initialize the sequence of shapes to be a list \( \{\lambda^i\}_{i=0}^{2N} \))
2: \( \lambda^0 = \emptyset, \lambda^{2N} = \emptyset, i=1 \)
3: while \( i < 2N \) do
4:   flag0 ← 1, flag1 ← 1, flag2 ← 1, flag3 ← 1
5:   if \( i \) is even then
6:     \( X[0] ← V^*(\lambda^i+1, 2n - (i + 1)) \)
7:     \( X[1] ← V^*(\lambda^i+1, 2N - (i + 1)), X[2] ← V^*(\lambda^i+1, 2n - (i + 1)) \)
8:     \( X[3] ← V^*(\lambda^i+1, 2N - (i + 1)), X[4] ← V^*(\lambda^i+1, 2N - (i + 1)) \)
9:   end if
10:  if \( i \) is odd then
11:     \( X[0] ← W^*(\lambda^i+1, 2n - (i + 1)) \)
12:     \( X[1] ← W^*(\lambda^i+1, 2N - (i + 1)), X[2] ← W^*(\lambda^i+1, 2n - (i + 1)) \)
13:     \( X[3] ← W^*(\lambda^i+1, 2N - (i + 1)), X[4] ← W^*(\lambda^i+1, 2N - (i + 1)) \)
14:   end if
15:  if flag0 =0 then
16:     \( X[0] ← 0, X[1] ← 0, X[2] ← 0, X[3] ← 0 \)
17:  end if
18:  if flag1 =0 then
19:     \( X[1] ← 0, X[2] ← 0, X[0] ← 0, X[3] ← 0 \)
20:  end if
21:  if flag2 =0 then
22:     \( X[1] ← 0, X[2] ← 0, X[3] ← 0, X[4] ← 0 \)
23:  end if
24:  if flag3 =0 then
25:     \( X[1] ← 0, X[2] ← 0, X[0] ← 0 \)
26:  end if
27:  sum ← \( X[0] + X[1] + X[2] + X[3] + X[4] \)
28:  if \( i \) is even and Shape=\( \lambda^i+1 \) then
29:     \( flag1 ← 0 \)
30:  end if
31:  if \( i \) is even and Shape=\( \lambda^i+1 \) then
32:     \( flag2 ← 0 \)
33:  end if
34:  if \( i \) is even and Shape=\( \lambda^i+1 \) then
35:     \( flag3 ← 0 \)
36:  end if
37:  if \( i \) is even and Shape=\( \lambda^i+1 \) then
38:     \( flag0 ← 0 \)
39:  end if
40:  Insert Shape into Tableaux (the sequence of shapes)
41:  \( i ← i + 1 \)
for $i = 2\ell + 1$, we have
\[
\mathbb{P}(X_i^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{W^*(\lambda^{i+1}, 2n - i - 1)}{W^*(\lambda^{i+1}, 2N - i)}.
\]
By construction,
\[
\prod_{i=0}^{2N-1} \mathbb{P}(X_i^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{W^*(\lambda^{2N} = \emptyset, 0)}{W^*(\lambda^0 = \emptyset, n)} = \frac{1}{W^*(\emptyset, 2N)},
\]
whence $(X_i^{i})_{i=0}^{2N}$ generates random 2-regular, 3-noncrossing partitions with uniform probability in $O(N)$ time and space, see Figure 8.

As for the derivation of the transition probabilities, supposed that the terms $\omega^{i,j}_\ell$ for $0 \leq i, j \leq \ell$ and $1 \leq \ell \leq 2N$ can be calculated in $O(N^4)$ time and $O(N^3)$ space according to Theorem 3.3. We claim for fix the indices $i_1, j_1$ and $s$ such that $0 \leq i_1, j_1 \leq s$ and $1 \leq s \leq 2N$, $\sigma^{i_1,j_1 s}_s$ can be computed in $O(N)$ time. Consider the parity of $s$, there are two cases. First, in case of $s = 2\ell_1$, via using a For-loop summing over the terms $(-1)^h (\ell_1 - h) \omega^{j_1,j_1}_2(\ell_1 - h + 1)$, we obtain $\sigma^{i_1,j_1 s}_s$. Otherwise, in case of $s = 2\ell_1 + 1$, we first calculate $\sigma^{i_1-1,j_1 s}_s$, $\sigma^{i_1,j_1-1 s}_s$, and $\sigma^{i_1,j_1 s}_s$ via using a For-loop according to eq. (4.10), then $\sigma^{i_1,j_1 s}_s$ follows from $\sigma^{i_1,j_1 s}_s = \sigma^{i_1-1,j_1 s}_s + \sigma^{i_1,j_1-1 s}_s + \sigma^{i_1,j_1 s}_s$.

Furthermore, via using three nested For loops for $s, i_1$ and $j_1$ from outside to inside, we derive $\sigma^{i,j s}_s$ for arbitrary $i_1, j_1$ and $s$. Consequently, we compute $\sigma^{i,j s}_s$ for all $i_1, j_1, s$ with $O(N^4) + O(N^3 \times N) = O(N^4)$ time and $O(N^3)$ space complexity. Obviously, once the terms $\sigma^{i,j s}_s$ for $i_1, j_1$ and $s$ are calculated, we can compute the transition probabilities in $O(1)$ time. Therefore we obtain the transition probabilities $W^*(\lambda^{i+1}, 2N - i)$ in $O(N^4)$ time and $O(N^3)$ space complexity.

\[\square\]

References

[1] M. Bousquet-Mélou and G. Xin. On partitions avoiding 3-crossings. Séminaire Lotharingien de Combinatoire, 54(B54e), 2006.
[2] W.Y.C. Chen, E.Y.P. Deng, R.R.X. Du, R.P. Stanley, and C.H. Yan. Crossing and nesting of matchings and partitions. Trans. Amer. Math. Soc., 359:1555–1575, 2007.
[3] W.Y.C. Chen, H.S.W Han, and C.M. Reidys. Random $k$-noncrossing RNA structures. 2009.
[4] W.Y.C. Chen, J. Qin, and C.M. Reidys. Crossings and nestings in tangled diagrams. Electron. J. Comb., 15:R86, 2008.
[5] I. M. Gessel and D. Zeilberger. Random walk in a Weyl chamber. Proc. Am. Math. Soc., 115:27–31, 1992.
[6] I.M. Gessel and D. Zeilberger. Random walk in a weyl chamber. Proc. Amer. Math. Soc., 115:27–31, 1992.
[7] D.J. Grabiner and P. Magyar. Random walks in Weyl chambers and the decomposition of tensor powers. J. Algebr. Comb., 2:239–260, 1993.
[8] R.P. Stanley. Enumerative Combinatorics, volume 1. Cambridge University Press, Cambridge, 1996.
Figure 8. Uniform generation of 2-regular, 3-noncrossing partitions: the stochastic process over shapes (A), extract a specific sampling path (B) which we transform in (C) into the corresponding 2-regular, 3-noncrossing partition. The probabilities specified in (B) are the transition probabilities.

E-mail address: reidys@santafe.edu