Noncommutative field theory
in formalism of first quantization.

A.Ya.Dymarsky†

Abstract

We present a first-quantized formulation of the quadratic non-commutative field theory in the background of abelian (gauge) field. Even in this simple case the Hamiltonian of a propagating particle depends non-trivially on the momentum (since external fields depend on location of the Landau orbit) so that one can not integrate out momentum to obtain a local theory in the second order formalism. The cases of scalar and spinning particles are considered. A representation for exact propagators is found and the result is applied to description of the Schwinger-type processes (pair-production in homogenous external field).

1 Introduction

The noncommutative (NC) fields theory have been extensively studied recently. However, most of consideration were based on the second-quantizes field theory. Here we are going to study the NC theory within the first quantization approach.

More concretely we claim that the unique difference between ordinary and NC theories in the formalism of first quantization is that the external fields in the NC case depend on “shifted” coordinates $q^\mu = x^\mu - \frac{1}{2} \theta^{\mu\nu} p_\nu$, which are the counterparts of coordinates of the Landau orbit for the charged particle in the magnetic field. Here $p_\mu$ are the momenta, $x^\mu$ are the coordinates of the particle and $\theta^{\mu\nu}$ is the parameter of noncommutativity (the constant antisymmetric tensor).

We consider the theories quadratic in quantum fields, since only such theories have first-quantized formulation.

This paper is organized as follows. We begin with verifying our claim in the case of scalar particle in the next section. In section 3 we apply our results to the pair-production process. Section 4 constructs the first-quantized theory of spinning particle. At last we end up with concluding remarks in section 5.

2 Scalar particle

In this section we consider correspondence between the first and second quantized description of the scalar particle in the example of the scalar QED with external currents. The main statement is that the first-quantized theory with the action (in the first order formalism)

$$S_1 = \int dt (p\dot{x} - (p - A(q))^2 - J(q) + m^2), \quad q^\mu = x^\mu - \frac{1}{2} \theta^{\mu\nu} p_\nu$$

(1)
corresponds to the second quantized theory with the action

$$S_2 = \int d^D x \left( D_\mu \phi^+ \phi^* D_\mu \phi - m^2 \phi^+ \phi \right) + J \phi^+ \phi, \quad D_\mu = \partial_\mu - iA_\mu.$$ 

(2)

† ITEP and MSU, Moscow, Russia; e-mail: dymarsky@gate.itep.ru
Here * is the Moyal product with the parameter of noncommutativity θμν. We omit the 1-dimensional metric along the trajectory from the action (1). It means that we already fixed the gauge: t is the natural parameter along the trajectory and t \in [0, T], where T is the proper time (the length of trajectory). Then, of the whole path integral over 1-dimensional metric, we are left with an ordinary integral over the proper time with the measure dT (for details see [3]).

We prove the correspondence between the theories identifying the exact propagators in the first and second quantized theories. The propagators will be obtained as perturbative series in both cases. The exact propagator for the first-quantized theory is described in [2]. The relativistic case and the Feynman rules for the field theory (2) from the particle theory (1). (The perturbative theory in the formalism of first quantization is described in [2]. The relativistic case and the correspondence to the field perturbation theory were discussed in [3].)

First we represent the path integral (3) as the limit of ordinary integrals

\[ G(x, y) = \int_0^\infty dT e^{im^2 T} G(x, y, T), \quad G(x, y, T) = \lim_{N \to \infty} \prod_{i=1}^{N-1} dx_i G(x_{i+1}, x_i) G(x_1, y) \frac{\partial^D p}{(2\pi)^D} \frac{e^{ip(x-y)}}{iS(x)} G(x, y, T). \]

G(x, y)_\Delta T is as usual defined up to the first oder in \Delta T only

\[ G(x, y)_\Delta T = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip(x-y)}}{iS(x)} \left( e^{-ip^2 \Delta T} + i(A_\mu(x') p_\mu + A_\mu(y') p_\mu - A_\mu(x') A_\mu(y') - J(x')) \Delta T, \right) \]

\[ x' = x - \frac{1}{2} \theta^{\mu\nu} p_\nu, \quad y' = y - \frac{1}{2} \theta^{\mu\nu} p_\nu. \] (5)

The points x’ and y’ are split in such a way that in the limit θμν → 0, the expression (5) becomes the ordinary one known in the first-quantized scalar QED.

Note that the parameter of noncommutativity θμν is contained only in terms, describing the interaction with external fields. Therefore, the free propagator obtained from (3) with action (1) coincides with the ordinary one. This is why in the NC theory the internal lines remain ordinary. This fact is in coincidence with the field theory result.

Now we begin to consider the vertex part of Feynman diagrams. To this end, we need the following key formula

\[ \langle x| f * g |y \rangle = \int \frac{d^D p}{(2\pi)^D} e^{ip(x-y)} f(x^\mu + \alpha(y-x)^\mu - \frac{1}{2} \theta^{\mu\nu} p_\nu), \] (6)

for arbitrary α ∈ [0, 1]. Note that the dependence on α vanishes in (6) after the integration over p. The crucial point here is that θμν is constant and (6) has the very simple form. Yet another useful formula, which follows from the previous one, is

\[ \langle x|(f * g) * |y \rangle = \int \frac{d^D p}{(2\pi)^D} e^{ip(x-y)} f(x^\mu - \frac{1}{2} \theta^{\mu\nu} p_\nu) g(y^\mu - \frac{1}{2} \theta^{\mu\nu} p_\nu). \] (7)

It is evident now that the vertex term (formula (5) without the term \( e^{-ip^2 \Delta T} \) which describes free propagation) corresponds to the NC theory

\[ \int \frac{d^D p}{(2\pi)^D} e^{ip(x-y)} iA_\mu(x') p_\mu + A_\mu(y') p_\mu - A_\mu(x') A_\mu(y') - J(x') \Delta T = \]

\[ = -i < x| 2i(A_\mu *) \partial_\mu + i(\partial_\mu A_\mu) * + (A_\mu * A_\mu) * + J * |y \rangle. \] (8)
It is also obvious now how to construct for arbitrary scalar theory the first-quantized formulation from the second-quantized one.

3 Schwinger type processes

One can not calculate the exact propagator for arbitrary external field. However when the external field is homogenous it is possible. In this section, we calculate the probability for the pair production (the imaginary part of the propagator) in this the background in the formalism of first quantization and compare our result with the field theory calculation.

Let us consider a particle on the noncommutative plane in the classical external field $A_{\mu}(q) = \frac{1}{2}B_{\sigma\mu}q^\sigma$ with homogenous field’s strength $B_{\mu\nu} = \text{const}$ and $J = 0$. In this case, one can easy calculate the probability of pair production since the action (1) is quadratic in all variables. It is convenient to change variables from $p, q$ to the canonical $\pi, y$

$$\pi_{\mu} = p_{\mu}K_{\nu}^{\mu}, \quad y^{\mu} = x^{\nu}(K^{-1})_{\nu}^{\mu}, \quad K_{\mu}^{\nu} = \delta_{\nu}^{\mu} + \frac{1}{4}B_{\sigma\mu}\theta^{\sigma\nu}, \quad (9)$$

where $x^{\mu} = q^{\mu} + \frac{1}{2}\theta^{\mu\nu} p_{\nu}$. In the new variables, the action can be rewritten as

$$S = \int dt(\pi_{\mu} \dot{y}^{\mu} - (\pi_{\mu} - \tilde{A}_{\mu}(y))^2 + m^2), \quad \tilde{A}_{\mu}(y) = A_{\mu}(Ky) = \frac{1}{2}B_{\sigma\mu}K_{\nu}^{\sigma}y^{\nu}. \quad (10)$$

It is remarkable that this action corresponds to an ordinary particle in external field with the strength

$$\tilde{F}_{\mu\nu} = \frac{\partial\tilde{A}_{\nu}}{\partial y^{\mu}} - \frac{\partial\tilde{A}_{\mu}}{\partial y^{\nu}}. \quad (11)$$

Now one can calculate the probability of pair creation in the usual way. The result (gauge invariant quantity) will depend only on $\tilde{F}_{\mu\nu}$. At the same time, $\tilde{F}_{\mu\nu}$ changes under gauge transformations

$$A_{\mu} \rightarrow g * A_{\mu} * g^+ - i\partial_{\mu}g * g^+ + g^+ * g = 1. \quad (12)$$

The solution of this puzzle is that the NC strength

$$F_{\mu\nu} = \frac{\partial A_{\nu}}{\partial q^{\mu}} - \frac{\partial A_{\mu}}{\partial q^{\nu}} - i[A_{\mu}, A_{\nu}], \quad (13)$$

is still gauge invariant for homogenous fields. Moreover, (a little surprise) $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are equal to each other

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} = B_{\mu\nu} + \frac{1}{4}\theta^{\sigma\rho}B_{\sigma\mu}B_{\rho\nu}. \quad (14)$$

Therefore, the probability is given by the standard formula (see \cite{4}) but with noncommutative field strength $\tilde{F}_{\mu\nu}$. For example, in the 4-dimensional space, when the magnetic field vanishes ($\tilde{F}_{i,j} = 0$, $i, j = 1, 3$), the probability is

$$w = \frac{e^2|\tilde{E}|^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp\left(-\frac{\pi nm^2}{|\epsilon\tilde{E}|}\right), \quad \epsilon\tilde{E}_i = \tilde{F}_{0i}. \quad (15)$$

This result certainly coincides with the field theory calculation (see \cite{5, 6}).

4 Spinning particle

Now we are going to construct the exact propagator for the spinning particle in the NC case similar to what we did below.

First, we rederive the analogue of formula (3) for the spinning particle and calculate the first correction in external fields to the exact propagator in the ordinary (commutative)
In this section, we work in the Euclidean space for simplicity. We start from another representation for the propagator of scalar particle (see [1])

\[
G(p_2, p_1) = \int_0^\infty \frac{dTe^{-mT}}{(2\pi)^2D} \int Dx(t)e^{ip_2x(T)}\delta(\dot{x}^2 - 1)e^{\int A_x e^{-ip_1x(0)}}.
\]  

(16)

In further consideration we mainly follow the approach of spin quantization developed in [7, 8, 9] (see also [10]). In accordance with their approach, in order to quantize the spin one needs to add a term \( e^{iS_{spin}} \) to (16). Note that the delta-function \( \delta(\dot{x}^2 - 1) \) keeps the vector \( \dot{x}^\mu \) in the sphere. Thus, in fact, in (16) one sums over the trajectories lying on the sphere (this fact is important). The term \( S_{spin} \) is the integral of a special external 1-form connected with the \( SO(D) \) group, along such a trajectory. Note that this sphere is the configuration space for the particle’s velocity \( \dot{n}^\mu = \dot{x}^\mu \) (or the phase space for the spin), not for the particle. In order to distinguish these spaces, we denote the space-time trajectory via \( x(t) \), and “spin” space trajectory via \( \eta(t) \). We also use the convenient variables \( n^\mu = i\eta^\mu \), \( Dn = D\eta \delta(\dot{n}^2 - 1) \). Then the exact propagator for spinning particle has the form

\[
G(p_2, p_1) = \int_0^\infty \frac{dTe^{-mT}}{(2\pi)^2D} \int D\eta(t)e^{ip_2\eta(T)}\delta(\dot{\eta}^2 - 1)e^{iS_{spin}e}\int A_\mu \eta^\mu e^{-ip_1\eta(0)} = \\
= \int_0^\infty \frac{dTe^{-mT}}{(2\pi)^2D} \int D\eta(t)e^{ip_2} \int_0^\infty e^{\int A_\mu \eta^\mu \delta(p_2 - p_1)}. 
\]  

(17)

In this case, the perturbative theory is more complicated than in the scalar case. This is why we first demonstrate in detail how the vertex terms emerge from (17). The first correction in \( A_\mu \) is

\[
G^1(p_2, p_1) = \lim_{\Delta t \to 0} \int_\Gamma \frac{dTdTe^{-mT}}{(2\pi)^2D} \int D\eta(t)e^{ip_2\eta T} A_\mu (\eta_T)(\eta'_T - \eta_T)^\mu \delta(\dot{\eta}^2 - 1)e^{iS_{spin}e},
\]

\[
t' = t + \Delta t, \quad \Gamma : T \in [0, \infty], t \in [0, T].
\]  

(18)

In order to transform it to the standard form, we use the following important trick: we multiply (18) by unity

\[
1 = \int \frac{dn_\mu'dk_2}{(2\pi)^D} \frac{dn_\mu'dk_1}{(2\pi)^D} e^{ik_2(\eta_T - \eta'_T)}e^{ik_1(\eta_T - \eta'_T)},
\]  

(19)

and rewrite it as

\[
G^1(p_2, p_1) = \lim_{\Delta t \to 0} \int_\Gamma \frac{dTdTe^{-mT}}{(2\pi)^2D} \frac{dk_2dk_1}{(2\pi)^2D} \int D\eta e^{i\int (p_2\dot{\eta} + p_2k_2 + k_2\eta_T) A_\mu (\eta_T)(\eta'_T - \eta_T)^\mu e^{i(-k_1\eta'_T + \int k_1\dot{\eta} + \eta_0(k_1 - p_1))} \delta(\dot{\eta}^2 - 1)e^{iS_{spin}}. 
\]  

(20)

We have to remove \( e^{ik_2(\eta_T - \eta'_T)} \) from formula (20), analogously we remove \( e^{-ip^2\Delta T} \) from (8) before taking the continuum limit \( \Delta t \to 0 \). Only after this, the first correction acquires the form corresponding to QED

\[
G^1(p_2, p_1) = \int dk_1dk_2d\eta G^0(p_2, k_2)e^{ik_2\eta}A_\mu(\eta)\gamma^\mu e^{-ik_1\eta}G^0(1, p_1).
\]  

(21)

The important moment is that we remove here the integral \( \int D\eta \delta(\dot{\eta}^2 - 1)^iS_{spin} \) and change all \( i\eta^\mu \) for the gamma-matrices \( \gamma^\mu \) (see the works [5] and [11]).
Now we demonstrate how to deform the propagator (17) in the NC case. The main difficulty is that in formula (17) we sum over the trajectories in the special “spin” space, when the particle propagates straightforwardly ($p = \text{const}$). However, the terms which depend on $q^\mu$ need to be integrated over the trajectories in the “space-time” phase space. This is why if we want to change the argument of external fields in (17), similarly to (1), we have to add in (17) the sum over trajectories in such a space. We can do this in the following way: represent the path integral as the continuum limit of ordinary integrals similarly we did with (3). After that, multiply (17) by unity
\[ 1 = \int \frac{dx dp}{(2\pi)^D} e^{ip(\eta - x)} \] (22)
for all $\eta(t)$. It is evident that the dependencies of external fields on $\eta$ or on $x$ are equivalent. Then, we integrate out the delta-function and pass from integrating over $\eta$ to integrating over $n$. (Note that in a similar way one can obtain (3) with the commutative ($\theta^{\mu\nu} = 0$) action (1) from (16), by adding new degrees of freedom.) After that, we change the arguments of all external fields from $x^\mu$ to $q^\mu$. Since we work with propagator in the momentum representation, the boundary conditions allow us to change the integration over variables $x^\mu$ to the integration over variables $q^\mu$. Finally, one obtains
\[ G = \int_0^\infty dT e^{-mT} \int Dq(t) Dp(t) Dn(t) e^{i \int_0^T (p dq + \frac{1}{2} p^2 dp)} e^{i \int_0^T p n dt} e^{i \int_0^T A_\mu(q) n^\mu dt} e^{i S_{\text{spin}}[n(t)]}. \] (23)

The perturbation series for this formula corresponds to the NCQED. For example, the first correction to the propagator has the form
\[ G^1(p_2, p_1) = \int dk_1 dk_2 dx \ G^0(p_2, k_2) e^{ik_2 q} A_\mu(q) \gamma^\mu e^{i(-k_1 q + k_1 \theta(k_2 - k_1))} G^0(k_1, p_1) = \]
\[ = \int dk_1 dk_2 dq \ G^0(p_2, k_2) e^{ik_2 q} A_\mu(q) \gamma^\mu e^{-ik_1 q} G^0(k_1, p_1). \] (24)

It is obvious that we have to add $e^{\int_0^T J(q) dt}$ to (23) in order to obtain the first-quantized formulation of NCQED with external current
\[ S_2 = \int d^D x (\bar{\psi} (D_\mu \gamma^\mu + J) * \psi). \] (25)

5 Concluding remarks

We demonstrated that quadratic noncommutative field theory can be described in terms of particles similarly to the ordinary case. However, in this case the particle action depends on the momentum nontrivially and the theory is non-local. We also constructed the exact propagators in the case of scalar and spinning particle in the background of classical abelian gauge field and current. The result is applied to the pair-production processes.

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