A Convex Maximization Problem: Discrete Case

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Abstract. We study a specific convex maximization problem in \( n \)-dimensional space. The conjectured solution is proved to be a vertex of the polyhedral feasible region, but only a partial proof of local maximality is known. Integer sequences with interesting patterns arise in the analysis, owing to the number theoretic origin of the problem.

Dedicated in memory of Jill Spurr Titus, with love

1. Problem

For each positive integer \( n \), maximize the convex function

\[
\sum_{i=1}^{n} \frac{1}{x_i}
\]

over the polyhedron in \( n \)-dimensional real space \( \mathbb{R}^n \) defined by

\[
(j + 1)x_j + x_i \geq (j + 1)i + \varepsilon_{ij} \quad \text{for} \quad 1 \leq j \leq i \leq n
\]

where \( \varepsilon_{ij} = 1 \) if \( i = j = 1 \) and \( \varepsilon_{ij} = 0 \) otherwise. Prove that:

(i) a global maximum \((a_1, a_2, ..., a_n)\) exists and is unique

(ii) the components \( a_i \) of the global maximum satisfy

\[
a_1 = 1, \quad a_2 = 2, \quad a_3 = 4
\]

and, when \( i \geq 4 \),

\[
a_i = (j + 1)(i - a_j)
\]

for any \( j \) with \((j + 1)a_j - ja_{j-1} \leq i < (j + 2)a_{j+1} - (j + 1)a_j\).

Remark. A solution of this problem will imply the truth of a certain number theoretic conjecture due to Levine and O’Sullivan \[\footnote{3}\].
2. Partial Solution

For fixed $n$, let $\xi = (x_1, x_2, ..., x_n)$,

$$f(\xi) = \sum_{i=1}^{n} \frac{1}{x_i}$$

and

$$P = \{\xi : (j + 1)x_j + x_i \geq (j + 1)i + \varepsilon_{ij}, 1 \leq j \leq i \leq n\}$$

If $\xi \in P$, then clearly $x_i \geq 1$ for $1 \leq i \leq n$. As a consequence, $P$ contains no lines and $f$ is bounded above on $P$; therefore, the supremum of $f$ over $P$ is attained at one or more vertices of $P$ [5]. This proves the existence part of (i). While we do not know how to prove the remainder of (i) or (ii), we show here that the conjectured global maximum $\alpha = (a_1, a_2, ..., a_n)$ is:

(a) well-defined
(b) feasible (that is, $\alpha \in P$)
(c) a vertex of $P$

and, if certain key inequalities hold,

(d) a local maximum of $f(\xi)$ subject to $\xi \in P$.

2.1. Proof of (a). The well-definition issue arises because of the conceivable non-uniqueness (or even non-existence) of $j$ when determining $a_i$ for $i \geq 4$. Let

$$c_j = (j + 2)a_{j+1} - (j + 1)a_j$$

for $j \geq 1$. Expressed using $c_1, c_2, ...$, the definition of $a_4, a_5...$ is

$$a_i = \begin{cases} 
3(i - 2) & \text{for } 4 = c_1 \leq i < c_2 = 10 \\
4(i - 4) & \text{for } 10 = c_2 \leq i < c_3 = 14 \\
5(i - 6) & \text{for } 14 = c_3 \leq i < c_4 = 24 \\
& \vdots
\end{cases}$$

We prove that both sequences $a_1, a_2, ...$ and $c_1, c_2, ...$ are strictly increasing. Hypothesize inductively that $c_{i-1} > c_{i-2} > ... > c_2 > c_1 = 4$, where $i \geq 4$ is fixed. Since $c_{i-1} > i$, there exists uniquely $j < i$ with $c_{j-1} \leq i < c_j$. If $i + 1 < c_j$, then $c_{j-1} \leq i + 1 < c_j$ and hence

$$a_{i+1} - a_i = (j + 1)(i + 1 - a_j) - (j + 1)(i - a_j) = j + 1 > 0$$
If \( i + 1 = c_j \), then \( c_j \leq i + 1 < c_{j+1} \) and hence
\[
a_{i+1} - a_i = (j + 2)(i + 1 - a_{j+1}) - (j + 1)(i - a_j) \\
= [(j + 2)(i + 1) - (j + 1)i] - [(j + 2)a_{j+1} - (j + 1)a_j] \\
= [(j + 2)(i + 1) - (j + 1)i] - c_j = [(j + 2)(i + 1) - (j + 1)i] - (i + 1) = j + 1 > 0
\]

We deduce that \((a_{i+1} - a_i) - (a_i - a_{i-1}) \geq 0\) and thus
\[
c_i - c_{i-1} = (i + 2)a_{i+1} - 2(i + 1)a_i + i a_{i-1} = 2a_{i+1} + i(a_{i+1} + a_{i-1}) - 2(i + 1)a_i \\
\geq 2a_{i+1} + 2i a_i - 2(i + 1)a_i = 2(a_{i+1} - a_i) > 0
\]

This completes the inductive proof, from which well-definition follows immediately. As a consequence, we may define
\[
b_i = \begin{cases} 
1 & \text{if } 1 \leq i \leq 3 \\
j & \text{if } 4 \leq i \leq n \text{ and } c_{j-1} \leq i < c_j 
\end{cases}
\]
without ambiguity.

2.2. Proof of (b). This is trivial if \( 1 \leq i \leq 3 \). If \( i \geq 4 \) and \( j = b_i \), it follows that
\[
(j + 1)(i - a_j) - (j + k + 1)(i - a_{j+k}) = \sum_{m=0}^{k-1} (c_{j+m} - i) > 0
\]
for \( 1 \leq k \leq n - j \) and
\[
(j + 1)(i - a_j) - (j - k + 1)(i - a_{j-k}) = \sum_{m=1}^{k} (i - c_{j-m}) \geq 0
\]
for \( 0 \leq k \leq j - 1 \). Both series are telescoping and the inequalities are consequences of part (a). We deduce that
\[
a_i = (j + 1)(i - a_j) \geq (p + 1)(i - a_p)
\]
for any \( 1 \leq p \leq n \), from which feasibility of \( \alpha \) follows immediately.

2.3. Proof of (c). This is true since \( \alpha \) lies at the intersection of the \( n \) hyperplanes
\[
(b_i + 1)x_{b_i} + x_i = (b_i + 1)i + \varepsilon_i b_i \quad 1 \leq i \leq n
\]
2.4. Key Inequalities. Before discussing part (d), we need to state certain key inequalities which, although unproven, appear to be true for all \( n \leq 10000 \) via computer check.

**Definition.** Fix integers \( i \) and \( j \) with \( 1 \leq i < j \). Let

\[
\begin{align*}
k_1 &= j - 1 \\
k_2 &= b_{k_1} \\
k_3 &= b_{k_2} \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Reader is not supported
and $v$ is the 24-vector with $i^{th}$ element $1/a_i^2$. The inverse, $M^{-1}$, of $M$ is given by

$$
\begin{pmatrix}
1 & -2 & -2 & 6 & 6 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 8 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -30 & -36 \\
1 & 0 & -3 & -3 & -3 & -3 & -3 & 0 & 0 & 0 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 18 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & -5 & -5 & -5 & -5 & -5 & -5 & -5 & -5 & -5 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & & & & & & & & & & & & & & & & & & & & .
\end{pmatrix}
$$

and the entries $x_i^*$ of $\xi^* = M^{-1}v$ are prescribed by the above summation formula. In the case $n = 24$, we compute

$$
x_1^* = \frac{123587941503427}{187646731272000} > 0 $$

$$
x_2^* = \frac{3536905093973}{27799515744000} > 0 $$

$$
x_3^* = \frac{44159}{1016064} > 0 $$

$$
x_4^* = \frac{9439261073843}{750586925088000} > 0 $$

$$
x_5^* = \frac{47}{4050} > 0 $$

and these positivity results are consistent with the Conjecture. Of course, $x_i^* > 0$ for $i > b_n$ immediately.

2.5. Partial proof of (d). It suffices to solve the following (primal) linear programming problem:

Minimize $g(\xi) = (-\xi) \cdot \nabla f(\alpha) = \sum_{i=1}^{n} \frac{x_i}{a_i^2}$ subject to $\xi \epsilon Q$

where $Q$ is the polyhedron

$$Q = \{ \xi : (b_i + 1)x_{b_i} + x_1 \geq (b_i + 1)i + \varepsilon_i b_i, 1 \leq i \leq n \}$$
Note that $Q$ contains $P$ and possesses a unique vertex, $\alpha$. Note also that, by the Conjecture, the dual linear programming problem has nonempty feasible region

\[
R = \left\{ \xi : \begin{aligned}
x_j + (j + 1) \sum_{i=c_j-1}^{\min\{c_j-1,n\}} x_i &= \frac{1}{a_j^2} \quad \text{for } 1 \leq j \leq b_n \\
x_j &= \frac{1}{a_j^2} \quad \text{for } b_n < j \leq n \\
x_j &\geq 0 \quad \text{for all } j
\end{aligned} \right\} = \{ M^{-1}v \} = \{ \xi^* \}
\]

hence $g$ is bounded below on $Q$. Therefore $\alpha$ is the global minimum of $g(\xi)$ subject to $\xi \in Q$, which implies that $\alpha$ is a local maximum of $f(\xi)$ subject to $\xi \in P$.

2.6. Partial proof of the Conjecture. The key inequalities are provably true when $i$ is sufficiently large relative to $n$. More precisely, if

\[
b_{b_{b_n}} < i \leq n
\]

then

\[
x_i^* \geq \frac{1}{a_i^2} - (i + 1) \sum_{j=c_i-1}^{c_i-1} \frac{1}{a_j^2} > 0
\]

To see this, we prove two lemmas.

**Lemma One.** $d_i, i+1 = -(i + 1)$ for all $i \geq 1$ and $d_i, j \geq 0$ if $i > b_{b_n}$ and $i + 1 < j \leq b_n + 1$.

**Proof of Lemma One.** The first part is trivial. The second part is proved by noting that $m > 1$ since $k_1 = j - 1 > i$, so either $m = 2$ (which implies that $d_i, j \geq 0$) or $m = 3$ since

\[
k_3 = b_{b_{i-1}} \leq b_{b_n} < i
\]

(which, in turn, implies that $d_i, j = 0$). QED.

**Lemma Two.** $\frac{1}{a_j^2} - (j + 1) \sum_{i=c_j-1}^{c_j-1} \frac{1}{a_i^2} > 0$ for all $j \geq 1$.

**Proof of Lemma Two.** Direct computation proves the inequality for $j = 1, 2, 3, 4, 10$ and 14. For all other values of $j$, we will show that

\[
\frac{1}{a_j^2} - \frac{c_j - c_{j-1}}{j + 1} \frac{1}{(c_{j-1} - a_j)^2} > 0
\]
that is, 
\[ e_j \equiv (j + 1)c_{j-1}(c_{j-1} - 2a_j) + [(j + 1) - (c_j - c_{j-1})]a_j^2 > 0 \]
which implies the truth of the Lemma. Observe that, if \( c_{k-1} < j \leq c_k \), then
\[ c_{j-1} - 2a_j = (k + 1)a_k > 0 \]
and
\[ (j + 1) - (c_j - c_{j-1}) = \begin{cases} j - 2k - 1 \geq 0 & \text{if } c_{k-1} < j < c_k \\ -2k - 3 < 0 & \text{if } j = c_k \end{cases} \]
These inequalities yield \( e_j > 0 \) when \( j \neq c_k \) for any \( k \). In the event \( j = c_k \) for some \( k \geq 4 \), the argument is only slightly more complicated:
\[ e_j = (k + 1)^2 \left[ (c_k + 1)(2c_k - a_k)a_k - (2k + 3)(c_k - a_k)^2 \right] \geq (k + 1)^2 \left[ (c_k - a_k)(2c_k - 2a_k)a_k - (2k + 3)(c_k - a_k)^2 \right] = (k + 1)^2(c_k - a_k)^2 [2a_k - (2k + 3)] > 0 \]
for all \( k \geq 4 \). QED.

3. Closing Words

Techniques for numerical convex maximization abound [4]. A vertex enumeration scheme has led to verification that \( \alpha \) is the global maximum of \( f(\xi) \) subject to \( \xi \epsilon P \) for small \( n \) only. Keith Briggs has used the general-purpose optimization programs AMPL and LANCELOT to confirm the global maximum claim up to \( n = 24 \), and CFSQP to do likewise up to \( n = 121 \).

The continuous analog of this problem (with summations replaced by integrals) is discussed in a companion paper.

An outcome of Levine and O’Sullivan’s work [3] is that, for any \( n \), there is a global maximum \( \alpha \) that satisfies \( a_1 = 1, a_2 = 2, a_3 = 4 \) and either \( a_4 = 6 \) or \( a_4 \geq 28 \), where \( \xi \) is restricted to integer points in \( P \) (that is, to \( \xi \in P \cap \mathbb{Z}^n \)). Their proof unfortunately does not extend to the real case.

Do there exist other functions \( f \) and polyhedra \( P \) for which the maximizing vertex \( \alpha \) is ”self-generating” as the dimension \( n \) increases? A simple characterization of such pairs \( (f, P) \) may lead to the insight necessary to solve this problem.
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