HYPERBOLICITY AND TYPES OF SHADOWING FOR $C^1$ GENERIC VECTOR FIELDS

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ABSTRACT. We study various types of shadowing properties and their implication for $C^1$ generic vector fields. We show that, generically, any of the following three hypotheses implies that an isolated set is topologically transitive and hyperbolic: (i) the set is chain transitive and satisfies the (classical) shadowing property, (ii) the set satisfies the limit shadowing property, or (iii) the set satisfies the (asymptotic) shadowing property with the additional hypothesis that stable and unstable manifolds of any pair of critical orbits intersect each other. In our proof we essentially rely on the property of chain transitivity and, in particular, show that it is implied by the limit shadowing property. We also apply our results to divergence-free vector fields.

1. INTRODUCTION

In studying dynamical systems one is often confronted with the necessity of approximation of individual orbits. For example, numerical studies require approximations due to the fact of the finite precision of a computer simulation. But it also appears in theoretical studies. There exist several concepts to formalize approximation. Let us start to explain some of them that will be studied in this paper.

Let us consider a closed $n$-dimensional Riemannian manifold $(M, d)$, $n \geq 3$. Denote by $\mathcal{X}^1(M)$ the set of all $C^1$ vector fields on $M$ endowed with the $C^1$ topology and consider the flow $(X_t)_{t \in \mathbb{R}}$ generated by $X \in \mathcal{X}^1(M)$.

Given $\delta > 0$, a (possibly) infinite sequence, $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$, of points $x_i \in M$ and positive integers $t_i$ is a $\delta$-pseudo orbit of $X$ if for all $i \in \mathbb{Z}$ we have

$$d(X_{t_i}(x_i), x_{i+1}) < \delta.$$ 

Let us now specify what we mean by approximation. Define the sequence $(s_i)_{i \in \mathbb{Z}}$ by

$$s_0 = 0, \quad s_n = \sum_{i=0}^{n-1} t_i, \quad \text{and} \quad s_{-n} = -\sum_{i=-n}^{-1} t_i, \quad n \in \mathbb{N}.$$ 

We say that a $\delta$-pseudo orbit, $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$, of $X$ is $\epsilon$-shadowed by the orbit through a point $x$, if there is an orientation preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ satisfying the following: For each $i \in \mathbb{Z}$ we have

$$d(X_{h(t)}(x), X_{t-s_i}(x_i)) < \epsilon \quad \text{for all} \quad s_i \leq t < s_{i+1}.$$ 

Observe that the homeomorphism $h$ does not alter trajectories, it only alters the speed of the displacement, that is, reparametrizes the trajectories.

It will be convenient for us to work with sets which are isolated in a certain sense. We call a invariant set $\Lambda \subset M$ isolated if there exist an open (isolating) neighborhood $U$ of $\Lambda$ such that $\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$.
Let $\Lambda$ be an isolated set of $X \in \mathcal{X}^1(M)$. The vector field $X$ has the shadowing property in $\Lambda$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that any $\delta$-pseudo orbit of $X$ in $\Lambda$ is $\epsilon$-shadowed by some orbit of $X$ in $\Lambda$. In the case that $\Lambda = M$ we simply say that $X$ has the shadowing property.

The shadowing property is central in hyperbolic dynamics. One of its consequences is, for example, the $C^1$ structural stability of uniformly hyperbolic dynamical systems (see, for example, [27] for more details and for the proof of this classical fact in the case of time-discrete and continuous systems).

Recall that a flow is called Anosov flow if the whole manifold $M$ is uniformly hyperbolic (see [12] for more information). It is well-known that every transitive Anosov flow has the shadowing property. More generally, if the Anosov flow is not transitive then its nonwandering set splits into a disjoint union of isolated transitive sets which have the shadowing property (see [13]). We know that the reverse implication is not true in general, that is, there are flows having the shadowing property which are not Anosov. For example, Morse Smale vector fields are structurally stable and therefore have the shadowing property, however they are not Anosov. In [33], Lewowicz constructed examples of transitive diffeomorphisms with the shadowing property which are not Anosov. So it is natural to ask when the shadowing property implies hyperbolicity. Recalling the above mentioned fact about spectral decomposition of non-transitivity sets, it is natural to restrict our attention to sets on which the flow is transitive. As we will be interested in pseudo-orbits, the appropriated concept to study will be chain transitivity.

Let $X \in \mathcal{X}^1(M)$ and $\Lambda \subset M$ isolated. We say that $X$ is chain transitive in $\Lambda$, or simply that $\Lambda$ is chain transitive, if for any points $x, y \in \Lambda$ and any $\delta > 0$, there exists a finite $\delta$-pseudo orbit $\{(x_i, t_i)\}_{0 \leq i \leq K}$ of $X$ contained in $\Lambda$, such that $x_0 = x$ and $x_K = y$. If $\Lambda = M$ we simply say that $X$ is chain transitive. Observe that transitivity implies chain transitivity and, generically, both concepts are equivalent (see below).

We believe that in general transitive isolated sets with the shadowing property are hyperbolic, although we are unable to prove this fact. Instead, we will prove this assertion for “most” systems, namely, for a residual set of systems. We recall that a subset $\mathcal{R} \subset \mathcal{X}^1(M)$ is called residual if it contains a countable intersection of open and dense subsets of $\mathcal{X}^1(M)$. Since $\mathcal{X}^1(M)$ is a Baire space, when equipped with the $C^1$-topology, any residual subset of $\mathcal{X}^1(M)$ is dense. A property is called $C^1$-generic if it holds in a residual subset of $\mathcal{X}^1(M)$. The expression $C^1$-generic vector field will refer to a vector field in a certain residual subset of $\mathcal{X}^1(M)$, which was previously displayed.

One of our motivations is the following result by Abdennur and Díaz [1]. They show that a locally maximal transitive set $\Lambda$ of a $C^1$-generic diffeomorphism $f$ is either hyperbolic, or there are a $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ and a neighborhood $\mathcal{V}$ of $\Lambda$ such that every $g \in \mathcal{U}(f)$ does not have the shadowing property on $\mathcal{V}$ (here they use the concept of shadowing appropriate for diffeomorphisms). We will generalize this result to vector fields for an isolated set. The following is our first main result.

**Theorem A.** Let $\Lambda$ be an isolated set. There exists a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that $X \in \mathcal{R}$ is chain transitive and has the shadowing property in $\Lambda$ if only if $\Lambda$ is a transitive hyperbolic set.

Observe that, as a consequence of Theorem A, if $X \in \mathcal{R}$ is chain transitive and has the shadowing property (on $M$) then $X_t$ is a transitive Anosov flow.

As a kind of generalization of the shadowing property, Blank [10] introduced the notion of the average shadowing property in the study of chaotic dynamical systems. Roughly speaking, it does allow large errors, however they must be compensated with small errors.
A sequence \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) is a \( \delta \)-average-pseudo orbit of \( X \), if \( t_i \geq 1 \) for every \( i \in \mathbb{Z} \) and there is a positive integer \( N \) such that for any \( n \geq N \) and \( k \in \mathbb{Z} \) we have
\[
\frac{1}{n} \sum_{i=1}^{n} d(X_{t_{i+k}}(x_{i+k}), x_{i+k+1}) < \delta.
\]

A \( \delta \)-average-pseudo orbit, \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \), of \( X \) is positively \( \epsilon \)-shadowed in average by the orbit of \( X \) through a point \( x \), if there is an orientation preserving homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) with \( h(0) = 0 \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{s_i}^{s_{i+1}} d(X_{h(t)}(x), X_{t-s_i}(x_i)) dt < \epsilon,
\]
where as before we put \( s_0 = 0 \) and \( s_n = \sum_{i=0}^{n-1} t_i, n \in \mathbb{N} \). We say that it is negatively \( \epsilon \)-shadowed in average by the orbit of \( X \) through \( x \), if there is an orientation preserving homeomorphism \( \tilde{h} : \mathbb{R} \to \mathbb{R} \) with \( \tilde{h}(0) = 0 \) for which (1) holds when replacing \( h \) by \( \tilde{h} \) and the limits of integration by \( -s_{-i} \) and \( -s_{-i+1} \) (in this case \( s_{-n} = \sum_{i=-n}^{-1} t_i \)).

Given an isolated set \( \Lambda \) of \( X \in \mathcal{X}^1(M) \), the vector field \( X \) has the average shadowing property in \( \Lambda \), if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that any \( \delta \)-average-pseudo orbit of \( X \) in \( \Lambda \) is both positively and negatively \( \epsilon \)-shadowed in average by some orbit of \( X \) in \( \Lambda \). We say that \( X \) has the average shadowing property when \( \Lambda \) is the whole \( M \).

We will study also one more shadowing-like concept which was introduced by Eirola, Nevalinna and Pilyugin [16]. They propose the notion of the limit-shadowing property. From the numerical point of view this property means that if we apply a numerical method of approximation to a vector field so that one-step errors tend to zero as time goes to infinity then the numerically obtained trajectories tend to real orbits to the flow. Such situations arise, for example, when one is not interested in the initial transient behavior of trajectories but instead wants to detect interesting asymptotic phenomena (for example certain behavior of an attractor) with good accuracy. To be more precise, we say that a sequence \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) is a limit-pseudo orbit of \( X \), if \( t_i \geq 1 \) for every \( i \in \mathbb{Z} \) and
\[
\lim_{|i| \to \infty} d(X_{t_i}(x_i), x_{i+1}) = 0.
\]

A limit-pseudo orbit, \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \), of \( X \) is positively shadowed in limit by an orbit of \( X \) through a point \( x \), if there is an orientation preserving homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) with \( h(0) = 0 \) such that
\[
\lim_{t \to \infty} \int_{s_i}^{s_{i+1}} d(X_{h(t)}(x), X_{t-s_i}(x_i)) dt = 0.
\]

Analogously, as we did before, we define when a limit-pseudo orbit is said to be negatively shadowed in limit by an orbit.

Given an isolated set \( \Lambda \) of \( X \in \mathcal{X}^1(M) \), the vector field \( X \) has the limit shadowing property in \( \Lambda \), if every limit-pseudo orbit in \( \Lambda \) is both positively and negatively shadowed in limit by an orbit of \( X \) in \( \Lambda \). We say that \( X \) has the limit shadowing property when \( \Lambda \) is the whole \( M \).

Finally, Gu [23] introduced the notion of the asymptotic average shadowing property for flows which is particularly well adapted to random dynamical systems. A sequence
\{ (x_i, t_i) \}_{i \in \mathbb{Z}} \) is an asymptotic average-pseudo orbit of \( X \), if \( t_i \geq 1 \) for every \( i \in \mathbb{Z} \) and

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=-n}^{n} d(X_{t_i}(x_i), x_{i+1}) = 0.
\]

An asymptotic average-pseudo orbit, \( \{ (x_i, t_i) \}_{i \in \mathbb{Z}} \), of \( X \) is positively asymptotically shadowed in average by an orbit of \( X \) through \( x \), if there exists an orientation preserving homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} d(X_{h(t)}(x), X_{t-s}(x_i)) dt = 0.
\]

Again where \( s_0 = 0 \) and \( s_n = \sum_{i=0}^{n-1} t_i \), \( n \in \mathbb{N} \). Similarly an asymptotic average-pseudo orbit is negatively asymptotically shadowed in average by an orbit of \( X \) through \( x \), if there exists an orientation preserving homeomorphism \( \tilde{h} : \mathbb{R} \to \mathbb{R} \) with \( \tilde{h}(0) = 0 \) for which the limit (3) is true when replacing \( h \) by \( \tilde{h} \) and the limits of integration by \( -s_{-1} \) and \( -s_{-1} \) (in this case \( s_{-1} = \sum_{i=-1}^{-n} t_i \)).

Given an isolated set \( \Lambda \) of \( X \in \mathcal{X}^1(M) \). The vector field \( X \) has the asymptotic average shadowing property in \( \Lambda \), if every asymptotic average-pseudo orbit in \( \Lambda \) is both positively and negatively asymptotically shadowed in average by an orbit of \( X \) in \( \Lambda \). We say that \( X \) has the asymptotic average shadowing property when \( \Lambda \) is the whole \( M \).

Before formulating our second main result, let us briefly discuss the relation between the above shadowing concepts. Note that they are not equivalent. Recall that Morse Smale vector fields admit sinks and sources. As we will see below (see Propositions 13 and 20) the average shadowing, the asymptotic average shadowing, and the limit shadowing properties each imply that there are neither sinks nor source in the system. Thus, a Morse Smale vector field is an example of a vector field which has the shadowing property but does not have any of the other shadowing concepts. Examples of systems which has the asymptotic average shadowing property or the limit shadowing property, but do not have the shadowing property can found in [22] and [34], respectively.

We will show that any of the above shadowing concepts relate to hyperbolicity. Here we will include in our studies flow that in general can have singularities. Posteriori we will derive that the latter four shadowing properties in fact imply the absence of singularities. We refer to Komuro [28] for a discussion of shadowing in the Lorenz flow containing one singularity. Let us denote by \( \text{Crit}(X) \) the set of critical orbits of \( X \), that is, the set formed by all periodic orbits and all singularities of \( X \). By the Stable Manifold Theorem [26], if \( O \) is a hyperbolic critical orbit of \( X \) with splitting

\[
T_O M = E^u_O \oplus E^s_O \oplus E^n_O
\]

then its unstable set \( W^u(O) \) is an immersed submanifold tangent to the subbundle \( E^u_O \oplus E^s_O \), and its stable set \( W^s(O) \) is an immersed submanifold tangent to the subbundle \( E^u_O \oplus E^s_O \). In this case \( W^s(O) \) and \( W^u(O) \) are called the stable and the unstable manifold of \( O \), respectively.

Finally, let us state our second main result that relates hyperbolicity to the various concepts of shadowing defined above.

**Theorem B.** Let \( \Lambda \) be an isolated set. There exists a residual subset \( \mathcal{R} \) of \( \mathcal{X}^1(M) \) such that if \( X \in \mathcal{R} \) satisfies any one of the following properties in \( \Lambda \):

1. \( X \) has the limit shadowing property in \( \Lambda \),
(2) \( X \) has the average shadowing property and \( W^s(O) \cap W^u(O') \neq \emptyset \) for any pair of critical orbits \( O, O' \) of \( X \) in \( \Lambda \).

(3) \( X \) has the asymptotic average shadowing property and \( W^s(O) \cap W^u(O') \neq \emptyset \) for any pair of critical orbits \( O, O' \) of \( X \) in \( \Lambda \).

then \( \Lambda \) is a transitive hyperbolic set.

As before, we can observe that in the case that \( X \in \mathcal{R} \) satisfy one of the properties of Theorem B (on \( M \)) then \( X_t \) is a transitive Anosov flow.

Let us discuss some results related to the one above. First, in \([30]\) the author shows that a \( C^1 \)-generic diffeomorphism in a locally maximal homoclinic class (in a set which \textit{a priori} contains saddle-type periodic points) has the limit shadowing property if, and only if, the homoclinic class is hyperbolic. Here we will prove that a \( C^1 \)-generic vector field which is chain transitive in an isolated set admits periodic points in the set, and prove also that such vector fields admits neither sinks nor sources, so in fact \textit{a posteriori} we show that there are saddle-type periodic points in the set.

Second, Theorem A is the versions for vector fields of the corresponding results \([31]\) for diffeomorphisms. The analysis of shadowing for flows is certainly more complicated than for maps due to presence of reparametrization of the trajectories and the (possible) presence of singularities. The results in \([30]\) and \([31]\) can now be obtained from the above results by the consideration of suspension flows. Complementing these results with the other types of shadowing which we consider here, we have the following.

**Corollary 1.** Let \( \Lambda \) be an isolated set. There exists a residual subset \( \mathcal{R} \) in the space of \( C^1 \)-diffeomorphisms such that, if \( f \in \mathcal{R} \) satisfies one of the properties that are stated in Theorems A or B, then \( \Lambda \) is transitive and uniformly hyperbolic\(^1\). In particular, a \( C^1 \)-generic diffeomorphism satisfying one of the shadowing properties of Theorem A or B (on \( M \)) is a transitive Anosov.

Now we observe that the main results are verified for divergence-free vector fields in manifolds of the dimensions greater than or equal 3. We denote by \( \mu \) the Lebesgue measure induced by the Riemannian volume form on \( M \). We say that a vector field \( X \) is \textit{divergence-free} if its divergence is equal to zero or equivalently if the measure \( \mu \) is invariant for the associated flow, \( X_t, t \in \mathbb{R} \). In this case we say that the flow is \textit{conservative} or volume-preserving. We denote by \( \mathcal{X}_\mu^1(M) \) the space of \( C^1 \)-divergence-free vector fields of \( M \).

We assume that \( \mathcal{X}_\mu^1(M) \) is endowed with the \( C^1 \) Whitney (or strong) vector field topology which turn these space completed in the sense of Baire.

We recall that an isolated set \( \Lambda \) is \textit{topologically mixing} if for all open sets \( U \) and \( V \) of \( \Lambda \) there is \( N > 0 \) such that \( U \cap X_t(V) \neq \emptyset, \forall t \geq N \). In the case that \( \Lambda = M \) we simply say that \( X \) is \textit{topologically mixing}. Clearly, the topologically mixing vector fields are transitive (so, chain transitive).

**Theorem 2.** Let \( \Lambda \) be an isolated set. There exists a residual subset \( \mathcal{R} \) of \( \mathcal{X}_\mu^1(M) \) such that if \( X \in \mathcal{R} \) satisfies any one of the following properties in \( \Lambda \):

1. \( X \) has the shadowing property,
2. \( X \) has the limit shadowing property,
3. \( X \) has the average shadowing property and \( W^s(O) \cap W^u(O') \neq \emptyset \) for any pair of critical orbits \( O, O' \) of \( X \) in \( \Lambda \).

\(^1\)The definitions of chain transitivity, hyperbolicity and shadowing for the case of diffeomorphisms are analogous to the ones in the case of flows. See \([27, 37]\) for definitions.
(4) \( X \) has the asymptotic average shadowing property and \( W^s(O) \cap W^u(O') \neq \emptyset \) for any pair of critical orbits \( O, O' \) of \( X \) in \( \Lambda \),

then \( \Lambda \) is a topologically mixing hyperbolic set. In particular, if \( X \) satisfy one of the
properties above in the manifold \( M \), then \( X_t \) is a topologically mixing Anosov flow.

Let us briefly sketch the proof of Theorem A and B. We first prove a more general
result (Theorem 2) that says that for a \( C^1 \)-generic vector field a chain transitive isolated
set \( \Lambda \) whose invariants manifolds of the critical orbits in \( \Lambda \) have non-empty intersection,
transitive uniformly hyperbolic. Then we will construct explicitly a residual set, and show
that the vector field satisfying the hypotheses of Theorem A (Theorem B) meets also the
hypothesis of Theorem 9.

This paper is organized as follows. In Section 2 we exhibit some properties of chain
transitive sets, that will be needed for the proofs of the main results. In Section 3 we prove
a general result that will be the key to prove Theorems A and B. In Section 4 we prove
that \( C^1 \)-generic vector fields with the limit shadowing property are chain transitive, which
in turn is a necessary condition to prove one of the items of Theorem B. Finally we prove
Theorems A and B. The Section 5 is devoted to a brief discussion of the arguments for
divergence-free vector fields that are used in the proof of Theorem 2.

2. Properties of Chain Transitive Sets

In the sequel we will always consider a vector field \( X \in \mathfrak{X}^1(M) \), where \( M \) is a closed
Riemannian manifold of dimension \( n \geq 3 \) and \( \Lambda \) is an isolated set of \( M \) which is not
simply a periodic orbit or a singularity. In this section we will exhibit some properties of
chain transitive sets that will be needed below. First we remark that chain transitivity
implies non-existence of sinks and sources in \( \Lambda \). More general, an invariant set is chain
transitive if, and only if, it does not contain proper attractors. Let us recall this notions.

A compact invariant set \( \Lambda \) is attracting if \( \Lambda = \bigcap_{t \geq 0} X_t(U) \) for some neighborhood \( U \)
of \( \Lambda \) satisfying, \( X_t(U) \subset U \) for all \( t > 0 \). An attractor of \( X \) is a transitive attracting set
of \( X \) and a repeller is an attractor for \( -X \). We say that \( \Lambda \) is a proper attractor or repeller
if \( \emptyset \neq \Lambda \neq M \). A sink (source) of \( X \) is a attracting (repelling) critical orbit of \( X \).

We recall that a point \( x \in M \) is a chain recurrent point if for any \( \epsilon > 0 \) there is an
\( \epsilon \)-pseudo orbit of \( X \) from \( x \) to \( x \). A subset \( A \subset M \) is chain recurrent set if any \( x \in A \) is a
chain recurrent point.

The next result say that the chain transitivity implies the absence of proper attractor and
the inverse implication is true.

**Proposition 3.** A vector field \( X \) is chain transitive in an isolated set \( \Lambda \) if, and only if, \( \Lambda \)
has no proper attractor for \( X \).

**Proof.** Necessity. Assume there is a proper attractor \( \Lambda \) in \( \Lambda \). Then \( \Lambda \neq \emptyset \) and \( \Lambda \setminus \Lambda \neq \emptyset \).
Since \( \Lambda \) is an attractor, there is an \( \epsilon > 0 \) such that \( \Lambda \) attracts the open \( \epsilon \)-neighborhood \( U \)
of \( \Lambda \), hence there exists \( \lambda < 1 \) such that if \( d(x, \Lambda) < \epsilon \) then \( d(X_t(x), \Lambda) < \lambda d(x, \Lambda) \)
for any \( t \geq 1 \). Choose \( y \in \Lambda \setminus U \), \( x \in \Lambda \) and \( \eta > 0 \). Let \( \{(x_i, t_i)\}_{i=0}^{m} \) \((m \geq 1)\) be a \( \eta \)-pseudo
orbit of \( X \) in \( \Lambda \) connecting \( x = x_0 \) to \( y = x_m \). If \( \eta \) is sufficiently small, we obtain that
\[
d(x_{i+1}, x_0) \leq \eta \sum_{k=0}^{i} \lambda^k \leq \frac{\eta}{1-\lambda}.
\]
Therefore, we cannot have \( x_m = y \). This contradicts the chain transitivity.

Sufficiency. Given \( a, b \in \Lambda \) and \( \epsilon > 0 \), is sufficient to prove that there exists an \( \epsilon \)-pseudo
orbit from \( a \) to \( b \). Since \( \Lambda \) has no proper attractor, we have that [14 p. 37] implies that \( \Lambda \)
is chain recurrent set. Let $V$ be a set formed by all points $x \in \Lambda$ for which there is an $\epsilon$-pseudo orbit connecting $a$ to $x$; this set contains $a$.

Let $z \in V$, then there is an $\epsilon$-pseudo orbit, $\{(z_i, t_i)\}_{i=0}^{k}$, with $z_0 = a$, $z_k = z$. Since

$$\lim_{x \to z} d(X_{t_{k-1}}(z_{k-1}), x) = d(X_{t_{k-1}}(z_{k-1}), z),$$

there is an open neighborhood $U$ of $z$ in $\Lambda$ such that for any $x \in U$, $d(X_{t_{k-1}}(z_{k-1}), x) < \epsilon$. Then the sequence $\{(z_i, t_i)\}_{i=0}^{k}$ defined above, with $z_k = x$ instead of $z_k = z$, is an $\epsilon$-pseudo orbit in $\Lambda$, connecting $a$ to $x$. So, $U \subset V$ and $V$ is an open set in $\Lambda$.

Now, we assert that $X_t(V) \subset V$ for all $t > 0$. Indeed, for any $z \in V$, by continuity of $X_t$ at $z$, we can choose a $y \in V$ such that $d(X_t(y), X_t(z)) < \epsilon$ for all $0 < t \leq 1$. Let $\{(y_i, t_i)\}_{i=0}^{m}$ an $\epsilon$-pseudo orbit in $\Lambda$ connecting $a$ to $y$, $(y_0 = a$ and $y_m = y)$. Then the sequence $\{(\tilde{y}_i, \tilde{t}_i)\}_{i=0}^{m+1}$ defined as follows: $\tilde{y}_i = y_i$, $0 \leq i \leq m$, and $\tilde{y}_{m+1} = X_1(z)$ where $\tilde{t}_i = t_i$, $0 \leq i < m$ and $\tilde{t}_m = 1$, is an $\epsilon$-pseudo orbit connecting $a$ to $X_1(z)$. Therefore, $X_1(z) \in V$. Proceeding inductively, with this same reasoning we obtain the conclusion of the statement.

By compactness of $\Lambda$ and [25] Lemma 3.1.1 applied to $X$ in $\Lambda$, follows that $\omega(V)$ is nonempty, compact, invariant and

$$\omega(V) = \bigcap_{t \geq 0} X_t(V).$$

Since $X_t(V) \subset V$ for all $t > 0$, we have $\omega(V) \subset V$ and hence $\omega(V) = X_t(\omega(V)) \subset V$. Therefore, $\omega(V)$ is an attractor in $\Lambda$. As $\Lambda$ no has proper attractor we have that $\omega(V) = \Lambda$. Now, $b \in \Lambda = V$, then by the definition of $V$, there is an $\epsilon$-pseudo orbit in $\Lambda$ connecting $a$ to $b$. Therefore, we conclude that $X$ is chain transitive in $\Lambda$.

Here we will quote a result by Bonatti and Crovisier. For a $C^1$-generic vector field the non-wandering set coincides with the chain recurrent set (see [11] for the case of diffeomorphisms and [7] for the case of conservative flows). The proof for the general case is verbatim to [7]. As a direct consequence of [11 Theorem 1.1] we have the following result.

**Theorem 4.** There exists a $C^1$-residual $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that if $X \in \mathcal{R}$ is chain transitive then $X$ is transitive.

We also recall that the Hausdorff distance between two compact subsets $A$ and $B$ of $M$ is given by:

$$d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

**Theorem 5.** [15] Theorem 4) There exists a residual set $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that for any vector field $X \in \mathcal{R}$, a compact invariant set $\Lambda$ is the limit (for the Hausdorff distance) of a sequence of periodic orbits if and only if $X$ is chain transitive in $\Lambda$.

As a consequence we have the next result for isolated sets.

**Lemma 6.** Let $\Lambda$ be an isolated set and $X \in \mathcal{X}^1(M)$ be a $C^1$-generic vector field which is chain transitive in $\Lambda$, then $X$ has a periodic orbit in $\Lambda$.

**Proof.** Let $X \in \mathcal{X}^1(M)$ be a vector field in the residual of Theorem 5 and $U$ be a neighborhood isolating of $\Lambda$. Then there exists a sequence of hyperbolic periodic orbits $O_n$ of $X$ such that

$$d_H(O_n, \Lambda) < \frac{1}{n}. $$
For $n$ sufficiently large the periodic orbits are contained in $U$, $O_n \subset U$. Therefore, $\Lambda$ contains a hyperbolic periodic orbit of $X$. □

Let us recall the notion of Morse index. By a closed orbit we mean a periodic orbit or a singularity. A closed orbit $O$ is hyperbolic if it does as a compact invariant set. In such a case we define its Morse index $I(O) = \dim (E^s_O)$, where $\dim(.)$ stands for the dimension operation. If $O$ reduces to a singularity $\sigma$, then we write $I(\sigma)$ instead of $I(\{\sigma\})$.

A vector field $X \in \mathcal{X}^1(M)$ is Kupka-Smale if its critical orbits are all hyperbolic and moreover their stable and unstable manifolds intersect transversally. The Kupka-Smale vector fields form a residual subset in $\mathcal{X}^1(M)$, (see [38] for diffeomorphisms).

Lemma 7. Let $\Lambda$ be an isolated set and $X \in \mathcal{X}^1(M)$ a $C^1$-generic vector field which is chain transitive in $\Lambda$. If $W^s(O) \cap W^u(O') \neq \emptyset$ for any hyperbolic critical orbits $O, O'$ of $X$ in $\Lambda$, then $X$ has no singularities in $\Lambda$.

Proof. Let $X$ be a Kupka Smale vector field in the residual set given by Theorem 5 and $\gamma$ a hyperbolic periodic orbit of $X$ in $\Lambda$ with index $i$. Suppose that $X$ has a hyperbolic singularity $\sigma \in \Lambda$ with index $i$.

If $j < i$, then
$$\dim W^u(\sigma) + \dim W^s(\gamma) \leq \dim M.$$  
As $X$ is a Kupka-Smale vector field we have $\dim W^u(\sigma) + \dim W^s(\gamma) = \dim M$. By hypothesis, we can consider $x \in W^u(\sigma) \cap W^s(\gamma)$. Then $O(x) \subset W^u(\sigma) \cap W^s(\gamma)$ and we can split
$$T_x(W^u(\sigma)) = T_x(O(x)) \oplus E^1 \quad \text{and} \quad T_x(W^s(\gamma)) = T_x(O(x)) \oplus E^2.$$  
So, $\dim(T_x(W^u(\sigma)) + T_x(W^s(\gamma))) < \dim W^u(\sigma) + \dim W^s(\gamma) = \dim M$. This is a contradiction, because $X$ is a Kupka-Smale vector field.

If $j \geq i$, then $\dim W^u(\sigma) + \dim W^s(\gamma) \leq \dim M$ and by the same arguments we have a contradiction. Thus $X$ has no singularities in $\Lambda$. □

As a consequence of Propositions 5 and Lemma 7 we have that the hyperbolic periodic orbits of $X$ in $\Lambda$ have constant indices.

Lemma 8. Let $\Lambda$ be an isolated set and $X \in \mathcal{X}^1(M)$ be a $C^1$-generic vector field which is chain transitive in $\Lambda$. If $W^s(O) \cap W^u(O') \neq \emptyset$ for any critical orbits $O, O'$ of $X$ in $\Lambda$, then the hyperbolic periodic orbits of $X$ in $\Lambda$ have constant index.

Proof. Let $\gamma_1$ and $\gamma_2$ be hyperbolic periodic orbits saddle-type of $X$ in $\Lambda$, since by Proposition 5 $X$ not admits sink or source. Suppose that $\gamma_1$ has index $i$ and $\gamma_2$ has index $j$, with $i \neq j$.

If $j < i$ (the other case is similar) then
$$\dim W^u(\gamma_1) + \dim W^s(\gamma_2) \leq \dim M.$$  
Take $X$ a Kupka-Smale vector field, then $\dim W^u(\gamma_1) + \dim W^s(\gamma_2) = \dim M$. Proceeding the analyses as in the proof of Lemma 7 we obtain a contradiction. □

3. Chain Transitivity and Hyperbolicity

This section is devoted to prove the following more general result that will be the key to prove Theorems A and B.

Theorem 9. Let $\Lambda$ be an isolated set. There exists a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that if $X \in \mathcal{R}$ is chain transitive in $\Lambda$ and $W^s(O) \cap W^u(O') \neq \emptyset$ for any pair of critical orbits $O, O'$ of $X$ in $\Lambda$, then $\Lambda$ is a transitive hyperbolic set.
Let us begin by setting our context and recalling some standard definitions:

**Definition 10.** A compact invariant set $\Lambda$ of $X$ is partially hyperbolic if there is a continuous invariant splitting

$$T\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$$

such that the following properties hold for some positive constants $K, \lambda$:

1. $E^s_\Lambda$ is contracting, that is,
   $$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t}$$
   for all $x \in \Lambda$ and $t \geq 0$;
2. $E^c_\Lambda$ dominates $E^s_\Lambda$, that is,
   $$\frac{\|DX_t(x)/E_x^c\|}{\|DX_t(x)/E_x^s\|} \leq Ke^{-\lambda t}$$
   for all $x \in \Lambda$ and $t \geq 0$.

We say that $\Lambda$ has contracting dimension $d$ if $\dim(E^s_x) = d$ for all $x \in \Lambda$. Moreover, we say that the central subbundle $E^c_\Lambda$ is sectional expanding if $\dim(E^c_x) \geq 2$ and

$$\|Det(DX_t(x)/L_x)\| \geq K^{-1}e^{\lambda t},$$

for all $x \in \Lambda$ and $t \geq 0$ and all two-dimensional subspace $L_x$ of $E^c_x$. Here $Det(.)$ denotes the jacobian.

**Definition 11.** A seccional hyperbolic set is a partially hyperbolic set with only hyperbolic singularities and the central subbundle is seccional expanding.

**Remark 12.** We remark that by hyperbolicity, if $O_X$ is a hyperbolic critical orbit of a vector field $X$ then there exists a neighborhood $U$ of $O_X$ in $M$ and a $C^1$-neighborhood $U$ of $X$ in $\mathcal{X}^1(M)$ such that if $Y \in U$, then $Y$ has a hyperbolic critical orbit $O_Y \subset U$ with $\text{index}(O_Y) = \text{index}(O_X)$. Such an orbit $O_Y$ is called the continuation of $O_X$.

**Definition 13.** We say that $\Lambda$ has an index, $0 \leq \text{Ind}(\Lambda) \leq n-1$, if there are a neighborhood $U$ of $X$ in $\mathcal{X}^1(M)$ and a neighborhood $U$ of $\Lambda$ in $M$ such that $I(O) = \text{Ind}(\Lambda)$ for every hyperbolic periodic orbit $O \subset U$ of every vector field $Y \in U$. In such a case we say that $\Lambda$ is strongly homogeneous (of index $\text{Ind}(\Lambda)$).

**Lemma 14.** There is a residual set $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that every $X \in \mathcal{R}$ satisfies the following property: For any closed invariant set $\Lambda$ of $X$, if there are a sequence of vector fields $X_n$ converging to $X$ and a sequence of hyperbolic periodic orbits $O_n$ of $X_n$ with index $k$ verifying

$$d_H(O_n, \Lambda) < \frac{1}{n},$$

then there is a sequence of hyperbolic periodic orbits $Q_n$ of $X$ with index $k$ such that $\Lambda$ is the Hausdorff limit of $Q_n$.

**Proof.** Let $\mathcal{K}(M)$ be the space of all nonempty compact subsets of $M$ equipped with the Hausdorff metric. Note that $\mathcal{K}(M)$ it is a compact metric space. Consider a countable basis $\{V_n\}_{n \in \mathbb{N}}$ of the space $\mathcal{K}(M)$.

For each pair $(n,k)$ with $n \geq 1$ and $k \geq 0$, we define the set $A_{n,k}$ as the set of vector fields such that exist a $C^1$-neighborhood $U$ in $\mathcal{X}^1(M)$ of $X$ such that for every $Y \in U$,
there is a hyperbolic periodic orbit $Q \in \mathcal{V}_n$ of $Y$ with index $k$. Observe that $A_{n,k}$ is an open set. Define $B_{n,k} = \mathcal{X}^1(M) \setminus A_{n,k}$. Thus the set
\[
\mathcal{R} = \bigcap_{n \in \mathbb{Z}^+; k = 0, \ldots, \dim(M)} (A_{n,k} \cup B_{n,k})
\]
is residual of $\mathcal{X}^1(M)$. If $X$ belongs to $\mathcal{R}$ satisfies the hypothesis of the lemma in a closed invariant subset $\Lambda$ of $X$, then there exist a sequence of vector fields $X_n$ converging to $X$ and a sequence of periodic orbits $O_n$ of $X_n$ with index $k$ such that $\Lambda$ is the Hausdorff limit of $O_n$.

Since $\{\mathcal{V}_n\}_{n \in \mathbb{N}}$ is a basis for $\mathcal{K}(M)$, there exist $l \in \mathbb{N}$ such that $\Lambda \in \mathcal{V}_l$. So, $X \in \mathcal{A}_{l,k}$. As $X \in \mathcal{R}$, we have $X \in \mathcal{A}_{l,k}$ and therefore, $X$ has hyperbolic periodic orbit, $Q_n$, in $\mathcal{V}_l$ with index $k$. This finishes the proof. □

**Lemma 15.** Let $X \in \mathcal{X}^1(M)$ a $C^1$-generic vector field and $\Lambda$ be an isolated set of $X$. If the hyperbolic periodic orbits of $X$ in $\Lambda$ have constant index then $\Lambda$ is strongly homogeneous.

**Proof.** Let $X$ be a vector field in the residual set of Lemma 14 and assume that the hyperbolic periodic orbits of $X$ in $\Lambda$ has constant index $i$. Consider the neighborhoods $U_n$ of $X$ in $\mathcal{X}^1(M)$ and $V_n$ of $\Lambda$ such that $U_n$ has radius $1/n$ and
\[
d_H(V_n, \Lambda) < \frac{1}{n}.
\]
Suppose that $\Lambda$ is not strongly homogeneous. Then, for each $n \in \mathbb{N}$, there exists a vector fields $Y_n$ in $U_n$ and hyperbolic periodic orbits $O_n$ and $Q_n$ of $Y_n$ in $V_n$ with different indices $i$ and $j$, respectively. Taking a subsequence if necessary, we can assume that the sequences $O_n$ and $Q_n$ has constant indices $i$ and $j$, respectively.

The sequence of vector fields $Y_n$ with the hyperbolic periodic orbits $O_n$, $Q_n$ satisfies the conditions of the hypothesis of Lemma 14. Then there are sequences of hyperbolic periodic orbits $\tilde{O}_n$ and $\tilde{Q}_n$ of $X$, with constant index $i$ and $j$, respectively, converging to $\Lambda$ in the Hausdorff distance. Since $\Lambda$ is an isolated set, $X$ has hyperbolic periodic orbits $O_1$ and $O_2$ in $\Lambda$ with index $i$ and $j$, respectively. This is a contradiction, and thus we conclude the proof. □

As a consequence of Lemmas 8 and 15 we have the following result.

**Corollary 16.** Let $\Lambda$ be an isolated set and $X \in \mathcal{X}^1(M)$ be a $C^1$-generic vector field which is chain transitive in $\Lambda$. If $W^s(O) \cap W^u(O') \neq \emptyset$ for any critical orbits $O, O'$ of $X$ in $\Lambda$, then $\Lambda$ is strongly homogeneous.

Next, we state the following result verified in [4].

**Proposition 17.** A nontrivial transitive set $\Lambda$ with singularities (all hyperbolic) which is strongly homogeneous satisfying
\[
\text{Ind}(\sigma) > \text{Ind}(\Lambda)(\text{Ind}(\sigma) < \text{Ind}(\Lambda)) \quad \forall \sigma \in \text{Sing}(X) \cap \Lambda,
\]
is sectional hyperbolic for $X$ (resp.-$X$).
3.1. **Proof of Theorem 9**. Let \( \mathcal{R} \subset \mathcal{X}(M) \) be the residual composed by the intersection of the four residuals: the residual formed by Kupka-Smale vector fields and those given by Theorem 4, Theorem 5, and Lemma 14.

Let \( X \in \mathcal{R} \) be a chain transitive vector field. By Theorem 4, Lemma 7, and Corollary 16, the set \( \Lambda \) satisfies the conditions of Proposition 17. Once any sectional hyperbolic set without singularities is hyperbolic (Hyperbolic lemma of [6]), we concluded the proof of Theorem 9.

### 4. Proof of Theorems A and B

Let \( \mathcal{R} \) be the residual of Theorem 9. In this section we prove that vector fields in \( \mathcal{R} \) satisfying the hypotheses of Theorems A and B meet also the hypothesis of Theorem 9. To this end, we begin by proving that the limit shadowing property implies chain transitivity.

**Proposition 18.** Let \( \Lambda \) an isolated set of \( X \). If \( X \) has the limit shadowing property in \( \Lambda \), then \( \Lambda \) has no proper attractor.

**Proof.** Assume there is a proper attractor \( A \subset \Lambda \). Then \( A \neq \emptyset \) and \( \Lambda \setminus A \neq \emptyset \). Since \( A \) is an attractor, there is an \( \epsilon_0 > 0 \) such that \( A \) attracts the open \( \epsilon_0 \)-neighborhood \( U \) of \( A \) in \( \Lambda \). Choose \( b \in \Lambda \setminus U \) and \( a \in A \). Consider the sequence:

\[
\begin{align*}
  x_i &= X_t(a), \quad t_i = 1, \quad i \leq 0 \\
  x_i &= X_t(b), \quad t_i = 1, \quad i > 0,
\end{align*}
\]

with \( i \in \mathbb{Z} \). Clearly the sequence \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) is a limit-pseudo orbit of \( X \). By the limit shadowing property there are an orientation preserving homeomorphism \( h \) and a point \( z \in \Lambda \) which shadow positively and negatively the limit-pseudo orbit. Then, there exists \( N > 0 \) sufficiently large such that \( X_t(h(-N))(z) \in U \). Thus, \( X_t(X_h(-N))(z) \in U \) for all \( t > 0 \). Taking \( -t_k = h(-N) \) we have that \( z = X_{t_k}(X_{-t_k}(z)) \in U \). Therefore, by definition of \( U \), we have that

\[
d(X_t(z), X_t(b)) > \epsilon_0 \quad \forall \ t > 0,
\]

which contradicts (3) and proves the desired. \( \square \)

As a consequence direct of this result, together with Proposition 3 we obtain chain transitivity for vectors fields with the limit shadowing property.

**Corollary 19.** If \( X \) has the limit shadowing property in \( \Lambda \) then \( X \) is chain transitive in \( \Lambda \).

**Proposition 20.** (Theorem 2.1 of [23] and Theorem 3.1 of [24]) If \( X \) has the (asymptotic) average shadowing property in \( \Lambda \) then \( X \) is chain transitive in \( \Lambda \).

Now, for the vector fields of Theorems A and B satisfying the hypotheses of Theorem 9 it suffices to prove that the invariant manifolds of its critical orbits has nonempty intersection.

Let us recall the notions on local stable and unstable manifolds. Given \( \epsilon > 0 \), the local stable and the local unstable manifold of a hyperbolic critical orbit \( \mathcal{O} \) of \( X \) is defined by

\[
W^s_\epsilon(\mathcal{O}) = \{ y \in M; d(X_t(y), \mathcal{O}) < \epsilon, \quad \forall \ t \geq 0 \},
\]

and by

\[
W^u_\epsilon(\mathcal{O}) = \{ y \in M; d(X_t(y), \mathcal{O}) < \epsilon, \quad \forall \ t \leq 0 \},
\]

respectively.

**Remark 21.** If \( V \) is a neighborhood of \( \mathcal{O} \) such that \( X_t(q) \in V \) for all \( t \geq 0 \), then \( q \in W^s(\mathcal{O}) \). Analogously, if \( X_t(q) \in V \) for all \( t \leq 0 \), then \( q \in W^u(\mathcal{O}) \).
Lemma 22. If $X$ has the shadowing property and is chain transitive in $\Lambda$, then $W^s(O) \cap W^u(O') \neq \emptyset$ for any pair of hyperbolic critical orbits $O, O' \in \text{Crit}(X) \cap \Lambda$.

Proof. Let $p, q \in \Lambda$ two critical points, such that the hyperbolic orbits $O(p)$ and $O(q)$ are disjoint. We analyze different cases:

Case 1.
Suppose that $p, q \in \Lambda$ are two hyperbolic periodic points. Given $\epsilon > 0$, by chain transitivity of $X$ in $\Lambda$, there is a $(\delta/2)$-pseudo orbit, $\{(z_i, t_i)\}_{0 \leq i \leq K}$, joining $p$ to $q$, where $\delta > 0$ is the constant of the shadowing property.

Consider the following sequence of points and times:

\[
\begin{align*}
    x_i &= X_i(p), & t_i &= 1, & i < 0, \\
    x_i &= z_i, & t_i &= \bar{t}_i, & 0 \leq i < K, \\
    x_i &= X_{i-K}(q), & t_i &= 1, & i \geq K.
\end{align*}
\]

The sequence $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$ is a $\delta$-pseudo orbit. Indeed, we have that

\[
\begin{align*}
    d(X_i(x_i), x_{i+1}) &= 0, \quad \forall \; i < 0 \text{ and } i \geq K \quad \text{and} \\
    d(X_i(x_i), x_{i+1}) &< \frac{\delta}{2}, \quad \forall \; 0 \leq i < K.
\end{align*}
\]

Thus, for all $i \in \mathbb{Z}$ it holds

\[
d(X_i(x_i), x_{i+1}) < \frac{\delta}{2}.
\]

Since $X$ has the shadowing property, there is a point $w \in \Lambda$ and an orientation preserving homeomorphism, $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$, such that

\[
\begin{align*}
    d(X_{h(t)}(w), X_{t-s}(x_i)) &< \epsilon, \quad \text{for } i \leq t < i+1, \; i \geq 0, \quad \text{and} \\
    d(X_{h(t)}(w), X_{t+i}(x_i)) &< \epsilon, \quad \text{for } -i \leq t < -i+1, \; i \geq 0.
\end{align*}
\]

Thus, we obtain the inequalities,

\[
\begin{align*}
    d(X_{h(t)}(w), X_t(q)) &< \epsilon, \quad \forall \; t > K \quad \text{and} \\
    d(X_{h(t)}(w), X_t(p)) &< \epsilon, \quad \forall \; t < 0.
\end{align*}
\]

Therefore, $w \in W^s_p \subset W^u(O(p))$ and $w \in W^s_q \subset W^s(O(q))$. Similarly, we have $W^s(O(p)) \cap W^u(O(q)) \neq \emptyset$.

Case 2.
Suppose that $p$ is a hyperbolic periodic point and $\sigma$ is a hyperbolic singularity. Let $u \in W^u(\sigma)$. By chain transitivity of $X$ there is $(\delta/2)$-pseudo-orbit, $\{(y_i, \tilde{t}_i)\}_{0 \leq i \leq T}$, joining $u$ to $p$. Consider the sequence:

\[
\begin{align*}
    x_i &= X_i(u), & t_i &= 1, & i < 0, \\
    x_i &= y_i, & t_i &= \tilde{t}_i, & 0 \leq i < T, \\
    x_i &= X_{1-T}(p), & t_i &= 1, & i \geq T.
\end{align*}
\]

Note that the sequence $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$ is a $\delta$-pseudo orbit. Indeed,

\[
\begin{align*}
    d(X_{i-1}(x_{-1}), x_0) &= d(X_1(X_{-1}(u)), y_0) = d(u, u) = 0, \\
    d(X_{1-T-1}(x_{T-1}), x_T) &= d(X_{1-T-1}(y_{T-1}), X_0(p)) \\
    &= d(X_{1-T-1}(y_{T-1}), y_T) < \frac{\delta}{2}.
\end{align*}
\]
By the shadowing property, there is a point \( \bar{w} \in \Lambda \) and an orientation preserving homeomorphisms, \( \alpha : \mathbb{R} \to \mathbb{R} \) with \( \alpha(0) = 0 \), such that \( 4 \) and is satisfied with \( \bar{w} \) and \( \alpha \) instead of \( w \) and \( h \), respectively. As in \( 5 \) we have

\[
\begin{align*}
    d(X_{\alpha(t)}(\bar{w}), X_t(u)) &< \epsilon, \forall \ t < 0 \text{ and} \\
    d(X_{\alpha(t)}(\bar{w}), X_t(p)) &< \epsilon, \forall \ t > T.
\end{align*}
\]

Therefore, \( \bar{w} \in W^u(\sigma) \subset W^u(\bar{\sigma}) \) and \( \bar{w} \in W^s(O(p)) \subset W^s(O(p)) \). We concluded that \( \bar{w} \in W^u(\sigma) \cap W^s(\sigma) \), as desired. Similarly, we have \( W^u(\sigma) \cap W^s(\sigma) \neq \emptyset \).

**Case 3.**

Let \( \sigma_1 \) and \( \sigma_2 \) be two hyperbolic singularities saddle-type. Take \( v_1 \in W^u(\sigma_1) \) and \( v_2 \in W^s(\sigma_2) \) and let \( \{(\bar{y}_i, \bar{t}_i)\}_{0 \leq i \leq N} \) be the \( \delta/2 \)-pseudo orbit joining \( v_1 \) to \( v_2 \). Consider the sequences of points and times:

\[
\begin{align*}
    x_i &= X_i(v_1), \quad t_i = 1, \quad i < 0, \\
    x_i &= \bar{y}_i, \quad t_i = \bar{t}_i, \quad 0 \leq i < N, \\
    x_i &= X_i(v_2), \quad t_i = 1, \quad i \geq N.
\end{align*}
\]

Note that the sequence \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) is a \( \delta \) pseudo-orbit. Indeed,

\[
\begin{align*}
    d(X_{t_i-1}(x_{i-1}), x_0) &= d(X_1(X_{t_i-1}(v_1)), \bar{y}_0) = d(v_1, \bar{y}_0) = 0, \\
    d(X_{t_{N-1}}(x_{N-1}), x_N) &= d(X_{t_{N-1}}(\bar{y}_{N-1}), X_0(v_2)) \\
    &= d(X_{t_{N-1}}(\bar{y}_{N-1}), \bar{y}_N) < \frac{\delta}{2}.
\end{align*}
\]

By the shadowing property, there is a point \( \tilde{z} \in \Lambda \) and an orientation preserving homeomorphisms, \( \tilde{\alpha} : \mathbb{R} \to \mathbb{R} \) with \( \tilde{\alpha}(0) = 0 \), such that \( 4 \) is satisfied with \( \tilde{z} \) and \( \tilde{\alpha} \) instead of \( w \) and \( h \). So,

\[
\begin{align*}
    d(X_{\tilde{\alpha}(t)}(\tilde{z}), X_t(v_1)) &< \epsilon, \forall \ t < 0 \text{ and} \\
    d(X_{\tilde{\alpha}(t)}(\tilde{z}), X_t(v_2)) &< \epsilon, \forall \ t > N.
\end{align*}
\]

Therefore, \( \tilde{z} \in W^u(\sigma_1) \subset W^u(\sigma_2) \) and \( \tilde{z} \in W^s(\sigma_2) \subset W^s(\sigma_1) \). We concluded that \( \tilde{z} \in W^u(\sigma_1) \cap W^s(\sigma_2) \), as desired. Similarly, we prove that \( W^s(\sigma_1) \cap W^s(\sigma_2) \neq \emptyset \).

In addition, in all the cases above, if \( X \) is \( C^1 \) generic, we can take \( X \) a Kupka-Smale vector field and, therefore, the intersections are transversal. \( \square \)

### 4.1. Proof of Theorem A

Let \( \mathcal{R} \subset X^1(\Lambda) \) be the residual of the Teorema \( 9 \). If \( X \in \mathcal{R} \) satisfies the hypotheses of Theorem A then, by Lemma \( 22 \), \( X \) satisfy the conditions of Theorem \( 9 \). Therefore, \( \Lambda \) is a transitive hyperbolic set. \( \square \)

**Lemma 23.** If \( X \) has the limit shadowing property in \( \Lambda \) then \( W^s(O) \cap W^u(O') \neq \emptyset \) for any pair of hyperbolic critical orbits \( O, O' \in \text{Crit}(X) \cap \Lambda \).

Recall that any \( O \in \text{Crit}(X) \cap \Lambda \) is a saddle-type critical orbit.

**Proof.** Let \( p \) and \( q \) be hyperbolic critical points of \( X \) in \( \Lambda \) such that the orbits \( O(p) \) and \( O(q) \) are disjoint. By compactness of \( M \) there exist \( K \in \mathbb{N} \) such that \( d(p, q) \leq K \).

**Case 1.**
Suppose that $p$ and $q$ are hyperbolic periodic points. Consider the limit pseudo orbit $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$:

\[
\begin{align*}
x_i &= X_i(p), \quad t_i = 1, \quad i < 0 \\
x_i &= X_i(q), \quad t_i = 1, \quad i \geq 0.
\end{align*}
\]

It is easy to see that the sequence $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$ is a limit pseudo orbit in $\Lambda$. Indeed,

\[
\begin{align*}
d(X_1(x_i), x_{i+1}) &= 0, \quad \forall i \in \mathbb{Z}, \quad i < -1 \text{ and } i \geq 0, \\
d(X_1(x_{-1}), x_0) &= d(X_1(X_{-1}(p)), X_0(q)) = d(p, q) \leq K.
\end{align*}
\]

So it can be positively and negatively shadowed in limit by the orbit of $X$ through some point $z \in \Lambda$, that is, there is an orientation preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that

\[
\begin{align*}
\lim_{i \to -\infty} \int_{-i}^{-i+1} d(X_{h(t)}(z), X_{t+i}(x_{-i}))dt &= 0 \quad \text{and} \\
\lim_{i \to \infty} \int_{i}^{i+1} d(X_{h(t)}(z), X_{t-i}(x_i))dt &= 0.
\end{align*}
\]

**Claim 1.** There is $n_0 \in \mathbb{N}$ such that for all $i > n_0$,

\[
d(X_{h(t)}(z), X_{t-i}(x_i)) < \frac{1}{i} \quad \forall t \in [i, i+1].
\]

We prove the claim.

**Proof.** Given $\eta > 0$. Suppose that for all $n \in \mathbb{N}$ there is an integer $i_n \geq n$ such that

\[
d(X_{h(t_{i_n})}(z), X_{t_{i_n}-i_n}(x_{i_n})) > \eta,
\]

for some $t_{i_n} \in [i_n, i_n+1]$. Thus,

\[
\int_{i_n}^{i_n+1} d(X_{h(t)}(z), X_{t-i_n}(x_{i_n}))dt > \eta.
\]

Therefore we obtain a subsequence $\{(x_{i_n}, t_{i_n})\}_{i_n \in \mathbb{N}}$ of $\{(x_i, t_i)\}_{i \in \mathbb{N}}$, satisfying the inequality (7), and consequently (8). Then,

\[
\lim_{n \to \infty} \int_{i_n}^{i_n+1} d(X_{h(t)}(z), X_{t-i_n}(x_{i_n}))dt \geq \eta.
\]

This contradicts (6) and proves the Claim [1].

Now, using that the limit-pseudo orbit $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$ is shadowed negatively in limit by the orbit of $X$ through the point $z \in \Lambda$, and proceeding as above, we prove the following claim.

**Claim 2.** There is $n_1 \in \mathbb{N}$ such that for all $i > n_1$,

\[
d(X_{h(t)}(z), X_{t+i}(x_{-i})) < \frac{1}{i} \quad \forall t \in [-i, -i+1].
\]

Returning to the proof of Theorem 23 let $N = \max \{n_0, n_1\}$. By the Claims [1] and [2] above, there is $N \in \mathbb{N}$ big enough, such that for every $i > N$ it holds:

\[
\begin{align*}
d(X_{h(t)}(z), X_{t+i}(x_{-i})) &< \frac{1}{i}, \quad \forall t \in [-i, -i+1] \quad \text{and} \\
d(X_{h(t)}(z), X_{t-i}(x_i)) &< \frac{1}{i}, \quad \forall t \in [i, i+1].
\end{align*}
\]
Thus, we obtain the inequalities:
\[
d(\tilde{X}_{h(t)}(z), X_t(p)) < \frac{-1}{t+1}, \quad \forall \ t < -N \quad \text{and} \\
d(\tilde{X}_{h(t)}(z), X_t(q)) < \frac{1}{t-1}, \quad \forall \ t > N.
\]

Then,
\[
\lim_{t \to -\infty} d(\tilde{X}_{h(t)}(z), O(p)) = 0 \quad \text{and} \quad \lim_{t \to \infty} d(\tilde{X}_{h(t)}(z), O(q)) = 0.
\]

As the reparametrization does not change the trajectory, only the speed, by Remark 21 we concluded that \( z \in W^u(O(p)) \) and \( z \in W^s(O(q)) \).

Thus, \( w \in W^u(O(p)) \cap W^s(O(q)) \). Similarly, we have \( W^s(O(q)) \cap W^u(O(p)) \neq \emptyset \).

**Case 2.**

Suppose that \( p \) is a hyperbolic periodic point and \( \sigma \) is a hyperbolic singularities saddle-type. Let \( u \in W^u(\sigma) \). Consider the sequence:
\[
\begin{align*}
x_i &= X_i(u) & t_i &= 1 & i < 0, \\
x_i &= X_i(p) & t_i &= 1 & i \geq 0.
\end{align*}
\]

Note that the sequence \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) is a limit-pseudo orbit. Indeed,
\[
\begin{align*}
d(X_t(x_i), x_{i+1}) &= 0 \quad \forall \ i \in \mathbb{Z}, \ i < -1 \text{ and } i \geq 0, \\
d(X_{t-1}(x-1), x_0) &= d(X_1(x-1(u)), x_0) = d(u, p) \leq K, \text{ for some } K \in \mathbb{N}.
\end{align*}
\]

By the limit shadowing property, there is a point \( w \in \Lambda \) and an orientation preserving homeomorphisms, \( \tilde{h} : \mathbb{R} \to \mathbb{R} \) with \( \tilde{h}(0) = 0 \), such that (6) is satisfied with \( w \) and \( \tilde{h} \), instead of \( z \) and \( h \), respectively.

By claims 1 and 2 of the Case 1, there is \( M \in \mathbb{N} \) large enough such that:
\[
\begin{align*}
d(\tilde{X}_{\tilde{h}(t)}(w), X_t(u)) &< \frac{-1}{t+1}, \quad \forall \ t < -M \quad \text{and} \\
d(\tilde{X}_{\tilde{h}(t)}(w), X_t(p)) &< \frac{1}{t-1}, \quad \forall \ t > M.
\end{align*}
\]

Since,
\[
d(\tilde{X}_{\tilde{h}(t)}(w), \sigma) < d(\tilde{X}_{\tilde{h}(t)}(w), X_t(u)) + d(X_t(u), \sigma) < \frac{-1}{t+1} + d(X_t(u), \sigma), \quad \forall \ t < -M.
\]

we obtain that
\[
\lim_{t \to -\infty} d(\tilde{X}_{\tilde{h}(t)}(w), O(p)) = 0 \quad \text{and} \quad \lim_{t \to -\infty} d(\tilde{X}_{\tilde{h}(t)}(w), \sigma) = 0.
\]

Thus \( w \in W^u(\sigma) \cap W^s(O(p)) \). Similarly, we obtain \( W^s(\sigma) \cap W^u(O(p)) \neq \emptyset \).

**Case 3.**

Let \( \sigma_1 \) and \( \sigma_2 \) be two hyperbolic singularities saddle-type and take \( v_1 \in W^u(\sigma_1) \) and \( v_2 \in W^s(\sigma_2) \). Consider the sequences of points and times:
\[
\begin{align*}
x_i &= X_i(v_1), & t_i &= 1, & i < 0, \\
x_i &= X_i(v_2), & t_i &= 1, & i \geq 0.
\end{align*}
\]
Note that the sequence \( \{ (x_i, t_i) \}_{i \in \mathbb{Z}} \) above is a limit pseudo-orbit. Indeed,
\[
\begin{align*}
    d(X_{t_i}(x_i), x_{i+1}) &= 0, \quad \forall \ i < -1 \text{ and } \forall \ i > 0, \\
    d(X_{t_{i-1}}(x_{i-1}), x_0) &= d(X_1(X_{-1}(v_1)), v_2) \\
    &= d(v_1, v_2) \leq T, \text{ for some } T \in \mathbb{N}.
\end{align*}
\]

By the limit shadowing property, there is a point \( \bar{z} \in \Lambda \) and an orientation preserving homeomorphism, \( \bar{h} : \mathbb{R} \to \mathbb{R} \) with \( \bar{h}(0) = 0 \), such that (6) is satisfied with \( \bar{z} \) and \( \bar{h} \) instead of \( z \) and \( h \), respectively.

By claims 1 and 2 of Case 1, there is a residual subset \( \Lambda \) of \( \mathbb{R} \) for \( X \) verifies
\[
\begin{align*}
    d(X_{\bar{h}(t)}(\bar{z}), X_t(v_1)) &< \frac{-1}{t + 1}, \quad \forall \ t < -M, \text{ and} \\
    d(X_{\bar{h}(t)}(\bar{z}), X_t(v_2)) &< \frac{1}{t - 1}, \quad \forall \ t > M.
\end{align*}
\]

Since,
\[
\begin{align*}
    d(X_{\bar{h}(t)}(\bar{z}), \sigma_1) &< d(X_{\bar{h}(t)}(\bar{z}), X_t(v_1)) + d(X_t(v_1), \sigma_1) \quad \text{and} \\
    d(X_{\bar{h}(t)}(\bar{z}), \sigma_2) &< d(X_{\bar{h}(t)}(\bar{z}), X_t(v_2)) + d(X_t(v_2), \sigma_2).
\end{align*}
\]

we obtain that
\[
\lim_{t \to -\infty} d(X_{\bar{h}(t)}(\bar{z}), \sigma_1) = 0 \quad \text{and} \quad \lim_{t \to \infty} d(X_{\bar{h}(t)}(\bar{z}), \sigma_2) = 0.
\]

Thus, \( \bar{z} \in W^u(\sigma_1) \cap W^s(\sigma_2) \). Similarly, we have \( W^s(\sigma_1) \cap W^u(\sigma_2) \neq \emptyset \).

In addition, if the vector field at the cases above is Kupka-Smale the intersections are transversal.

We do not know how to prove the result above for a vector field \( X \) with the (asymptotic) average shadowing (not even generically). But if this condition is satisfied we obtain the hyperbolicity of the isolated set for fields \( C^1 \) generic.

4.2. **Proof of Theorem B.** Let \( \mathcal{R} \subset \mathcal{X}^1(M) \) be the residual of the Theorem B. If \( X \in \mathcal{R} \) satisfies the conditions of Theorem B -1 then, by Corollary 19 and Lemma 23, \( X \) verifies the hypothesis of Theorem B. If \( X \) satisfies Theorem B -(2) or (3) then, by Proposition 20, \( X \) verifies the conditions of Theorem B. In any case we conclude that \( \Lambda \) is a transitive hyperbolic set.

\[
\square
\]

5. **Theorem**

In the sequel we will consider a vector field \( X \in \mathcal{X}^1(M) \), where \( M \) is a closed Riemannian manifold of dimension \( n \geq 3 \) and \( \Lambda \) is an isolated set of \( M \) which is not simply a periodic orbit or a singularity. In this section we will mention the arguments for divergence-free vector fields that permit us proceed as before and obtain the same conclusions of Theorems A and B for such fields.

We point out that Theorems A and B are consequence of Theorem (Section 3). Here, for \( X \in \mathcal{X}^1(M) \), the key to prove Theorem is a version of Theorem for divergence-free vector fields, that we announce next.

**Theorem 24.** Let \( \Lambda \) be an isolated set. There exists a residual subset \( \mathcal{R} \) of \( \mathcal{X}^1(M) \) such that if \( X \in \mathcal{R} \) is transitive in \( \Lambda \) and \( W^s(O) \cap W^u(O') \neq \emptyset \) for any pair of critical orbits \( O, O' \) of \( X \) in \( \Lambda \), then \( \Lambda \) is a hyperbolic set and \( X \) is topologically mixing in \( \Lambda \).
Note that Theorem 9 implies that $\Lambda$ is a transitive hyperbolic set, while Theorem 24 implies that $\Lambda$ is a topological mixing hyperbolic set. This is due to [7, Theorem 1.1]. The proof of Theorem 24 is analogous to the proof of Theorem 9 and so we will not give the proof in details. Instead, we will point out the necessary steps to adapt the proof of Theorem 9 in this context.

We start recalling that in [36], the author proved that the set of Kupka Smale vector fields in a manifold with dimension greater or equal than 3 form a $C^1$-residual in the $X^1_\mu(M)$. So, all the critical orbits of $C^1$-generic divergence-free vector fields in manifolds with dimension greater than 3 are hyperbolic.

Now we enunciate a stronger result than Theorem 4. It says that $C^1$-generic divergence free vector fields are topological mixing, and therefore, transitive.

**Theorem 25.** [7, Theorem 1.1] There exists a $C^1$-residual subset $R \subset X^1_\mu(M)$ such that if $X \subset R$ then $X$ is a topological mixing vector field.

Next we will use [11, Theorem 3] to obtain periodic orbits from this transitivity. For this, first recall that an invariant set $\Lambda$ is weakly transitive if for any two non-empty open sets $U$ and $V$ that intersect $\Lambda$ and any neighborhood $W$ of $\Lambda$ there exists a segment of orbit $X_{[0,T]}(x) = \{X_t(x); 0 \leq t \leq T\} \subset W$, where $x \in U$, $X_T(x) \in V$ and $T \geq 1$. Observe that every transitive vector field is also weakly transitive.

Before enunciate [11, Theorem 3], we let us explain why it is true for divergence-free vector fields. Indeed, its proof follows from a technical perturbation result [11, Proposition 8], whose proof, by its turn, is based on a connecting lemma, that holds in this setting (conservative flows), see [39, 35]. Moreover, the author has mentioned in his work [11, Subsection 2.5] that the perturbation result used holds for conservative systems. Then [11, Theorem 3] is true for divergence-free vector fields and can be state as follows.

**Theorem 26.** Let $X \subset X^1_\mu(M)$ be a Kupka-Smale vector field, $U$ be a neighborhood of $X$, and $\Lambda$ be a compact weakly transitive set. Then, for every $\epsilon > 0$ there exist $Y \in U$ and a periodic orbit $O(p)$ of $Y$ such that $d_H(O(p), \Lambda) < \epsilon$.

Let $X \in X^1_\mu(M)$ be a Kupka-Smale vector field in the residual $R$ of Theorem 25. Then, Lemmas [6, 7] stated at Section 8 hold for divergence-free vector fields in $R$ and their proofs follow analogously.

Now we explain why the results used in the proof of Theorem 9 holds in the context of divergence-free vector fields. So, we have to check if Lemmas [14] and [15] and Proposition [17] are true for divergence-free vector fields. Since Lemmas [14] and [15] are true for divergence-free vector fields, it remains to verify that Proposition [17] holds in this context.

Indeed, Proposition [17] verified in [4], is a generalization of following result proved in [20, 21, 32]: A $C^1$ robustly transitive set $\Lambda$ with singularities (all hyperbolic) which is strongly homogeneous satisfying $I(\sigma) > Ind(\Lambda)$ is sectional hyperbolic for $X$. The authors, in [4], concluded that is not necessary the condition of robustness of transitivity to obtain the result above. For this is used the notion of preperiodic set [40], instead of the continuation of a robustly transitive set, as in [20]. More specifically, the authors related the notion of local star flow [4] with nontrivial transitive strongly homogeneous sets and preperiodic sets. For this proof is also used the Pugh’s closing lemma, to approximate such sets by periodic orbits. See [20] for more details. Once made these modifications, the proof of Proposition [17] follows similarly to that in [20, 21, 32].

---

2A vector field $X$ is called star vector field in $U$, where $U$ is an isolating neighborhood, if $X$ has a $C^1$ neighborhood $U \in X^1(M)$ such that, for every $Y \in U$, every critical orbits of $Y$ contained in $U$ is hyperbolic.
Next we note that the results mentioned above ([20, 21, 32]), hold to divergence-free vector fields. We start by the perturbations lemmas used in the proofs. The $C^1$-closing lemma for conservative flows, proved by Pugh and Robinson (see [35]) and its improved version by Arnaud [5]. Finally, a conservative version of the Ergodic Closing Lemma [5].

Along the proofs, it is used a kind of Frank’s lemma (see [18]) for conservative flows [9, Lemma 3.2]. Its proof is obtained using techniques of Pasting Lemma [3] that is also true in the conservative context for $X \in \mathcal{X}^C_{\mu}(M)$, instead of $X \in \mathcal{X}^C_\mu(M)$, because of the improved Smooth $C^1$-pasting lemma proved in [8, Lemma 5.2]. Finally, the results of Gan, Wen. [19] Theorems A and A', were also proved for conservative flows (see [2, 17]).

So, we concluded that Proposition 17 is true for divergence-free vector fields. And now we are ready to sketch the proof of Theorems 24 and 2.

Proof. (of Theorem 24) Let $R \subset \mathcal{X}^C_{\mu}(M)$ be the residual composed by intersection of the four residuais: the formed by Kupka Smale vector fields and those given by Theorem 25, Theorem 26 and by Lemma 14. By Lemma 7 and Corollary 16 for all vector fields in $R$ the set $\Lambda$ satisfies the condition of Proposition 17 (for divergence-free vector fields). Therefore, $\Lambda$ is a hyperbolic set and $X$ is topologically mixing in $\Lambda$.

Now, we can see that Lemmas 22 and 23 are true for divergence-free vector fields. So, the conclusion of Theorem 2 follows. More precisely,

Proof. (of Theorem 2) Let $R \subset \mathcal{X}^C_{\mu}(M)$ be the residual of Theorem 24. If $X \in R$ satisfies Theorem 2 (1) then, by Lemma 22 $X$ satisfies the conditions of Theorem 24. If $X$ satisfies Theorem 2 (2) then, by Lemma 23 $X$ verifies the hypothesis of Theorem 24. If $X$ satisfies Theorem 2 (3) or 2-(4) then, directly $X$ verifies the conditions of Theorem 24. In any case we conclude that $\Lambda$ is hyperbolic and $X$ is topologically mixing in $\Lambda$.

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REFERENCES

[1] Abdenur, F.; Díaz, L. J. Pseudo-orbit shadowing in the $C^1$ topology. Discrete Contin. Dyn. Syst. 17 (2007), no. 2, 223-245.
[2] Arbieto, A.; Catalan, T. Hyperbolicity in the volume preserving scenario. Preprint arXiv:1004.1664.
[3] Arbieto, A.; Matheus, C. A pasting lemma and some applications for conservative systems. Ergodic Theory Dynam. Systems 27 (2007), 1399-1417.
[4] Arbieto, A.; Morales, C. A dichotomy for higher-dimensional flows. Preprint arXiv:1110.3720.
[5] Arnaud, M C. Le “closing lemma” en topologie $C^1$. Mem. Soc. Math. Fr., Nouv. Série (1998), no. 74, 1-120.
[6] Bautista, S.; Morales. C. Lectures on Sectional Anosov Flows. Preprint IMPA, available at 2011.
[7] Bessa, M. A generic incompressible flow is topological mixing. C.R. Math. Acad. Sci. Paris 346 (2008), no. 21-22, 1169-1174.
[8] Bessa, M.; Rocha, J. Contributions to the geometric and ergodic theory of conservative flows. Ergod. Th. & Dynam. Sys. (at press).
[9] Bessa, M.; Rocha, J. On $C^1$-robust transitivity of volume-preserving flows. Journal of Differential Equations 245 (2008), no. 11, 3127-3143.
[10] Blank, M. L. Metric properties of minimal solutions of discrete periodical variational problems. Nonlinearity 2 (1989), no.1, 1-22.
[11] Bonatti, C.; Crovisier, S. Récurrence et généricité. Invent. Math. 158 (2004), no. 1, 33-104.
[12] Bowen, R. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Mathematics, Vol. 470. Springer-Verlag, Berlin-New York, 1975.
[13] Bowen, R. *On Axiom A diffeomorphisms*. Regional Conference Series in Mathematics, no 35, Amer. Math. Soc., Providence, R.I., 1978.

[14] Conley, C. *Isolated Invariant sets and the Morse Index*. CBMS Regional Conference Series in Mathematics, 38, American Mathematical Society, Providence, R.I., 1978.

[15] Crovisier, S. *Periodic orbits and chain-transitive sets of $C^1$-diffeomorphisms*. Publ. Math. Inst. Hautes Études Sci. (2006), no. 104, 87-141.

[16] Eirola, T.; Nevalinna, O.; Pilyugin, S. *Limit shadowing property*. Numer. Funct. Anal. Optim. 18 (1997), no. 1-2, 75-92.

[17] Ferreira, C. *Stability properties of divergence-free vector fields*. Dyn. Syst. 27 (2012), no. 2, 223-238.

[18] Franks, J. *Necessary conditions for the stability of diffeomorphisms*. Trans. Amer. Math. Soc. 158 (1971) 301-308.

[19] Gan, S.; Wen, L. *Nonsingular star flows satisfy Axiom A and the no-cycle condition*. Invent. Math. 164, (2006), 279-315.

[20] Gan, S.; Li, M.; Wen, L. *Robustly transitive singular sets via approach of an extended linear Poincaré flow*. Discrete Contin. Dyn. Syst. 13 (2005), no. 2, 239-269.

[21] Gan, S.; Wen, L.; Zhu, S. *Indices of singularities of robustly transitive sets*. Discrete Contin. Dyn. Syst. 21 (2008), no. 3, 945-957.

[22] Gu, R. *The asymptotic average shadowing property and transitivity*. Nonlinear Anal. 67 (2007), no. 6, 1680-1689.

[23] Gu, R. *The asymptotic average-shadowing property and transitivity for flows*. Chaos Solitons Fractals 41 (2009), no. 5, 2234-2240.

[24] Gu, R.; Sheng, Y.; Xia, Z. *The average-shadowing property and transitivity for continuous flows*. Chaos Solitons Fractals 23 (2005), no. 3, 989-995.

[25] Hale, J. K. *Asymptotic Behaviour of Dissipative Systems*. Math. Surveys and Monographs 25, Amer. Math. Soc., Providence, RI, 1988.

[26] Hirsch, M.; Pugh, C.; Shub, M. *Invariant Manifolds*. Lecture Notes in Math. 583 1997, Springer-Verlag.

[27] Katok, Anatole; Hasselblatt, Boris. *Introduction to the Modern Theory of Dynamical Systems*. Encyclopedia of Mathematics and its Applications, 54, Cambridge University Press, Cambridge, 1995.

[28] Komuro, M. *Lorenz attractors do not have the pseudo-orbit tracing property*. J. Math. Soc. Japan 37 (1985), no. 3, 489-514.

[29] Kupka, I. *Contribution à la théorie des champs génériques*. Contributions to Differential Equations 2 (1963), 457-484.

[30] Lee, M. *Usual limit shadowable homoclinic classes of generic diffeomorphisms*. Adv. Difference Equ. (2012).

[31] Lee, K.; Wen, X. *Shadowable chain transitive sets of $C^1$-generic diffeomorphisms*. Bull. Korean Math. Soc. 49 (2012), no. 2, 263-270.

[32] Metzger, R.; Morales, M. *Sectional-hyperbolic systems*. Ergodic Theory Dynam. Systems 28 (2008), no. 5, 1587-1597.

[33] Lewowicz, J. *Lyapunov functions and topological stability*. J. Differential Equations 38 (1980), no. 2, 192-209.

[34] Pilyugin, S. Y. *Shadowing in Dynamical Systems*, Lecture Notes in Mathematics 1706, Springer-Verlag, Berlin, 1999.

[35] Pugh, C.; Robinson, C. *The $C^1$ closing lemma, including Hamiltonians*, Ergodic Theory Dynam. Systems 3 (1983), no. 2, 261-313.

[36] Robinson, C. *Generic properties of conservative systems*. Amer. J. Math. 92 (1970), 562-603.

[37] Shub, M. *Global Stability of Dynamical Systems*, Springer-Verlag, New York, 1987.

[38] Smale, S. *Stable manifolds for differential equations and diffeomorphisms*. Ann. Sc. Norm. Super. Pisa. III. Ser. 17, (1963) 97-116.

[39] Wen, L.; Xia, Z. *$C^1$-connecting lemmas*. Trans. Amer. Math. Soc. 352 (2000), no. 11, 5213-5230.

[40] Wen, L. *On the preperiodic set*. Discrete Contin. Dyn. Systems 6 (2000), no. 1, 237-241.

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