A simplified proof of the Johansson-Molloy Theorem using the Rosenfeld counting method

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Abstract

We show that any triangle-free graph with maximum degree $\Delta$ has chromatic number at most $(1 + o(1))\Delta / \log \Delta$.

1 Introduction

The Johansson-Molloy Theorem states that any triangle-free graph $G$ with maximum degree $\Delta$ has chromatic number at most $(1 + o(1))\Delta / \ln \Delta$ as $\Delta \to \infty$. In an interesting new development on this problem, a recent article by Hurley and Pirot [1] present an elementary proof based on the Rosenfeld counting method [2]. In fact, their result is quite a bit stronger in that it works even if the underlying graph contains a moderate amount of triangles, it extends to list colorings, and give explicit error terms. In this note, we will present a simplified version of their proof in the case of triangle-free graphs. We prove the following.

Theorem 1.1. For any $\varepsilon > 0$ there exists a constant $\Delta_0 = \Delta_0(\varepsilon)$ such that any triangle-free graph with maximum degree at most $\Delta \geq \Delta_0$ is $\lceil (1 + \varepsilon)\Delta / \ln \Delta \rceil$-colorable.

Let us start by fixing some notation. Let $k = \lceil (1 + \varepsilon)\Delta / \ln \Delta \rceil$ denote the number of colors, and for any graph $G$, let $\mathcal{C}(G)$ the set of proper $k$-colorings of $G$.

We will show this theorem following the format of Rosenfelds counting method, by proving by induction that the number of proper colorings of a triangle-free graph grows by a large factor whenever a vertex is added. To this end, let $\ell = \ell(\Delta)$ be any function such that $\ln \Delta \ll \ell = \Delta \circ(1)$, for instance $\ell(\Delta) = \ln^2 \Delta$.

Claim 1.2. Let $\Delta$ be sufficiently large and let $G$ be any triangle-free graph with maximum degree at most $\Delta$. Then $|\mathcal{C}(G)|/|\mathcal{C}(G - v)| \geq \ell$ for any $v \in G$.

Note that this immediately implies Theorem 1.1. In fact, the number of proper $k$-colorings of $G$ grows as $\ell^{|V(G)|} > 0$.

We will prove Claim 1.2 by induction of the number of vertices in $G$. Note that the statement is trivially true when $G$ consists of a single vertex. So we may assume below that $G$ has at least two vertices and that $v \in G$ denotes an arbitrary but fixed vertex.

For any partial proper coloring $c$ of $G$, let $L_c(u)$ denote the set of available colors for $u$. That is, the set of colors not present in the neighborhood of $u$. In other words, $L_c(u)$ is the set of

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colors we can (re-)assign to \( u \) while keeping the coloring proper. Note that this definition holds regardless of whether \( u \) is already assigned a color.

The core of our argument is to study the behavior of a uniformly chosen proper coloring \( c \) of the graph \( G - v \). That is, \( c \) is taken uniformly at random from \( \mathcal{C}(G - v) \). We first note that the conclusion of the theorem can be expressed in terms of the expectation of the number \( |L_c(v)| \) of available colors for \( v \) in \( c \). Observe that each proper coloring of \( G \) can be formed by taking a proper coloring of \( G - v \) and extending it by coloring \( v \) in one of its available colors. Hence, we have the equality

\[
\mathbb{E}[|L_c(v)|] = \sum_{c \in \mathcal{C}(G-v)} |L_c(v)| \cdot |\mathcal{C}(G)| / |\mathcal{C}(G-v)| = |\mathcal{C}(G)| / |\mathcal{C}(G-v)|,
\]

which means that the claim follows if we can prove that, for sufficiently large \( \Delta \), \( \mathbb{E}|L_c(v)| \geq \ell \).

In fact, we will show the stronger statement that, whp as \( \Delta \to \infty \), we have \( |L_c(v)| \gg \ell \).

The proof of this is based on two observations about the distribution of \( c \). Let \( u \) be any neighbor of \( v \) in \( G \), and let \( t \geq 0 \) be a number to be chosen later. By the induction hypothesis, we know (in fact, for any vertex) that \( |\mathcal{C}(G-v)| \geq \ell \cdot |\mathcal{C}(G-v-u)| \). We will use this to show that it is unlikely for \( u \) to have only few available colors in \( c \). Note that any proper coloring of \( G - v \) in which \( v \) has at most \( t \) available colors can be formed by taking some proper coloring of \( G - v - u \) and coloring \( u \) by picking one out of at most \( t \) options. Hence, there are at most \( t \cdot \mathbb{E}|\mathcal{C}(G-v-u)| \) such colorings. Plugging in the induction hypothesis as above, it follows that

\[
\Pr(|L_c(u)| \leq t) \leq \frac{t \cdot \mathbb{E}|\mathcal{C}(G-v-u)|}{\mathbb{E}|\mathcal{C}(G-v)|} \leq \frac{t}{\ell}.
\]

In particular, this means that the expected number of neighbors of \( v \) with at most \( t \) available colors in \( c \) is at most \( t \Delta / \ell \). So if we let \( 1 \ll t \ll \ell / \ln \Delta \), it follows by Markov’s inequality that, with high probability, all but at most \( o(k) = o(\Delta / \ln \Delta) \) neighbors of \( v \) have more than \( t = o(1) \) available colors. Crucially, then only a small fraction of colors will be unavailable for \( v \) due to these vertices. Thus if suffices to show that it is unlikely for the remaining neighbors of \( v \) to block too many of the other \( k - o(k) \) colors.

Second, letting \( G_0 := G - v - N(v) \), we consider the distribution of \( c \) conditioned on \( c_{G_0} = c_0 \) for some \( c_0 \in \mathcal{C}(G_0) \). Observe that, as \( G \) is triangle-free, there cannot be any edges between two vertices in \( N(v) \). Thus, for any \( u \in N(v) \), we have that \( L_c(u) \) is completely determined by \( c_0 \) according to \( L_{c_0}(u) \). Moreover, any \( c \in \mathcal{C}(G-v) \) such that \( c_{G_0} = c_0 \) can be constructed from \( c_0 \) by, for each \( v \in N(v) \), color \( u \) by an arbitrary color in \( L_{c_0}(u) \). As \( c \) is uniformly distributed, any such extension of \( c_0 \) is equally likely. Hence, conditioned on \( c_{G_0} = c_0 \), we get that the colors of the neighbors of \( v \) are conditionally independent and uniformly chosen from the respective sets \( L_{c_0}(u) \). It remains to show that when \( L_c(u) \) is large for most \( u \in N(v) \), the set of colors present in \( N(v) \) behaves like a coupon collector process.

We have now reduced the problem to the following elementary statement: We are given sets \( L_1, L_2, \ldots, L_d \subseteq [k] \) for some \( d \leq \Delta \) where all but \( o(k) \) sets satisfies \( |L_i| > t = o(1) \), and random variables \( X_1 \in L_1, \ldots, X_d \in L_d \) chosen independently and uniformly from the respective sets. We wish to show that, whp, \( X := [k] \setminus \{X_1, X_2, \ldots, X_d\} \) contains \( \gg \ell \) elements.

Note first that the events \( j \in X \) for \( j \in [k] \) are negatively correlated. This is intuitively clear as if some number \( j \) never appears among \( X_1, \ldots, X_d \), then any \( X_i \) such that \( j \in L_i \) is consequently more likely to hit any value in \( L_i \setminus \{j\} \). It follows that \( \Var(|X|) \leq \mathbb{E}|X| \), so by Chebyshev’s inequality it suffices to show that \( \mathbb{E}|X| \gg \ell \).

Condition on the values of \( X_i \) for each \( i \) such that \( |L_i| \leq t \), and let \( B \) be the set of colors assigned to these variables. Then \( |B| = o(k) \). By linearity of expectation, the conditional
expected value of $|X|$ given these values is

$$\sum_{j \in [k] \setminus B} \prod_{L_i \ni j, |L_i| > t} \left(1 - \frac{1}{|L_i|}\right).$$

It only remains to find a natural lower bound to this expression. By applying the AM-GM inequality to the right-hand side sum, we can lower bound the previous expression by

$$(k - |B|) \left(\prod_{j \in [k] \setminus B} \prod_{L_i \ni j, |L_i| > t} \left(1 - \frac{1}{|L_i|}\right)\right)^{1/(k - |B|)}.$$

Now, by reordering the products, we get

$$\prod_{i; |L_i| > t} \prod_{j \in L_i \setminus B} \left(1 - 1/|L_i|\right) \geq \left((1 - 1/t)^\Delta\right)^\Delta = e^{-\left(1 + O(1/t)^\Delta\right)},$$

meaning we can further lower bound the above expression by

$$(k - |B|) \exp\left(-\frac{(1 + o(1)) \cdot \Delta}{k - |B|}\right) = \Theta\left(\frac{\Delta}{\ln \Delta}\right) \cdot \exp\left(-\frac{1 + o(1)}{1 + \varepsilon} \ln \Delta\right) \geq \Delta^{\varepsilon/(1 + \varepsilon) - o(1)},$$

which is clearly $\gg \ell$, as desired.

We end this note with two short remarks. First, by carefully choosing parameters $\ell$ and $t$ in the argument, it is possible to let $\varepsilon$ depend on $\Delta$ according to $\varepsilon = \Omega(\log \log \Delta / \log \Delta)$. Second, it is strictly speaking not needed to argue for concentration of $|X|$ in the proof. As one only wishes to conclude that $|L_c(v)|$ is large in expectation, estimating $E|X|$ suffices on its own. Nevertheless, we have chosen to write the proof with concentration as we find the statement that $|L_c(v)|$ is large whp intuitively simpler than that $E\left[|L_c(v)|\mid c \in G_0\right]$ is large whp.

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**References**

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[2] Matthieu Rosenfeld, *Another Approach to Non-Repetitive Colorings of Graphs of Bounded Degree*, Electronic Journal of Combinatorics 27(3), 2020.