What has been learnt from the analysis of the low-energy pion-nucleon data during the past three decades?

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Abstract

Over twenty-five years ago, two analyses of the pion-nucleon ($\pi N$) data at low energy (i.e., for pion laboratory kinetic energy $T \leq 100$ MeV) reported on the departure of the extracted scattering amplitudes, corresponding to the two elastic-scattering reactions $\pi^\pm p \to \pi^\pm p$ and to the $\pi^- p$ charge-exchange reaction $\pi^- p \to \pi^0 n$, from the triangle identity, which these amplitudes fulfil if the isospin invariance holds in the hadronic part of the $\pi N$ interaction. This discrepancy indicates that at least one of the following assumptions is not valid: first, that the absolute normalisation of the bulk of the low-energy $\pi N$ datasets is correct; second, that any residual contributions to the corrections, which aim at the removal of the effects of electromagnetic (EM) origin from the measurements, are not significant; and third, that the isospin invariance holds in the hadronic part of the $\pi N$ interaction. In view of the incompatibility of the results of the various schemes of removal of the so-called trivial EM effects at the $\pi N$ threshold ($T = 0$ MeV), the likelihood of residual effects of EM origin from the measurements, are not significant; and third, that the isospin invariance holds in the hadronic part of the $\pi N$ interaction. In view of the incompatibility of the results of the various schemes of removal of the so-called trivial EM effects at the $\pi N$ threshold ($T = 0$ MeV), the likelihood of residual effects of EM origin in the extracted $\pi N$ scattering amplitudes (from the data in the scattering region, i.e., above the $\pi N$ threshold) must be reassessed. This work emphasises the importance of the development of a unified scheme for the determination of reliable EM corrections, applicable at the $\pi N$ threshold and in the scattering region, in providing a resolution to the established discrepancy at low energy.

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1 Introduction

It was about mid November 2021 when Igor Strakovsky came up with the idea of a new book, summarising our knowledge of the properties of the pion, and kindly asked me to relate in that book my experience from the analysis
of the pion-nucleon ($\pi N$) experimental data at low energy. I found his idea appealing and, after a day or two, I made my decision to join the ‘celebration of the 75 years since the discovery of the pion’, and to attempt to explain to a broad, or - more accurately - to a broader than usual, audience of physicists why the interaction between pions and nucleons is interesting at all.

I ought to admit that my initial decision to take part in this effort was followed by a short interval of vacillation regarding the optimal organisation and presentation of the material. The trade-off between what I (as an author) expected of the average reader of Igor’s book versus what the average reader expected of me urged me to abandon mathematical rigour, to depart from my ‘obsession’ with technicalities, and to attempt to recount my experience in a more engaging manner, while retaining both accuracy and simplicity. Before commencing however, I recommend Ref. [1], as well as the works cited therein, to those of the readers who are keen on the mathematical details of this research programme, as well as on its development over time. As I started this project when I was a postdoctoral researcher at the Swiss Federal Institute of Technology in Zurich (Eidgenössische Technische Hochschule Zürich), I will refer to it as ‘the ETH $\pi N$ project’ henceforth.

I recently decided to make this work publicly available as a preprint. Unlikely as it might seem, my hope remains that the broader dissemination of this report (as opposed to its restricted function as a chapter in a book) will rekindle interest in the Pion-Physics domain.

Overall, Physics researchers are hardly amongst the most talented when it comes down to communicating efficiently the crux of their studies, even to colleagues as the case frequently is. In an effort to break this pattern, I will present the material in the form of a series of questions and answers in parts of this work; I have been asked these questions many times. This ‘didactical’ distraction averts a tedious monologue and brings the reader in plain view as an interlocutor.

The structure of this paper is as follows. After I discuss what makes the analysis of the $\pi N$ measurements at low energy interesting in Section 2, I provide some details about the experimental data. In Section 3, I describe the steps, which lead from the experimental data to the extraction of the hadronic part of the $\pi N$ interaction: these steps involve the modelling of that part, the inclusion of the electromagnetic (EM) effects, the selection of the input database (DB), and the selection of the optimisation procedure. In the subsequent section, I give a summary of what I have learnt from the various analyses of the low-energy $\pi N$ data thus far; that section establishes a disturbing discrepancy, called herein the ‘low-energy $\pi N$ enigma’. Section 5 is bound to be speculative, examining a few straightforward possibilities which could explain the observed discrepancies. The conclusions of this work, including a short
discussion about the steps towards a resolution to the low-energy $\pi N$ enigma, are set forth in the last section, Section 6.

In this work, all rest masses of particles and all 3-momenta will be expressed in energy units. All scattering lengths will be expressed in fm. Whenever two uncertainties accompany a physical result, the first one is statistical and the second systematic. The values of the physical constants have been fixed from the 2020 compilation of the Particle-Data Group [2].

2 Preliminaries

2.1 Why low energy?

Within the ETH $\pi N$ project, ‘low energy’ implies the restriction of the pion laboratory kinetic energy $T$ of the input experimental data to values up to 100 MeV. After 1994, all phase-shift analyses (PSAs) in this programme have been performed using data from $T = 0$ (known as ‘$\pi N$ threshold’, though technically one means the ‘$\pi^\pm p$ threshold’) to 100 MeV. A few words about this choice are in order. In retrospect, there are five reasons why the analyses are limited to the aforementioned energy region.

• Owing to the experiments conducted at the meson factories over three decades, the low-energy $\pi N$ DB became extensive enough to enable exclusive analyses.
• In its current form, the hadronic model of this programme (which will be referred to as ‘ETH model’ henceforth) is suitable for PSAs at low 4-momentum transfer $Q^2$. Although the description of the data even above the $\Delta(1232)$ resonance had been (successfully) attempted in the early 1990s, the sensitivity of the results (i.e., of the fitted values of the model parameters) to the inclusion (in the Feynman diagrams of the model) of hadronic form factors must be carefully addressed prior to the enhancement of the DB towards higher energy. At present, it can only be asserted that the impact of such effects on the results, obtained from the low-energy $\pi N$ DB, is insignificant.
• It is debatable whether the theoretical constraints, which are valid in the region of asymptotic freedom, also hold at low energy. To estimate the dispersion integrals, all dispersion-relation analyses rely (by and large) on high-energy data. The description of the low-energy data in an unbiased manner (i.e., without any high-energy influences) is not possible in such schemes. The exclusive analysis of the low-energy $\pi N$ DB, in terms of consistency and compliance with theoretical constraints, such as the isospin invariance in the hadronic part of the $\pi N$ interaction, is therefore interesting in its
within the ETH πN project, the removal of the effects of EM origin from the extracted πN phase shifts and partial-wave amplitudes rests upon the application of EM corrections, which are not available for $T > 100$ MeV (at least at present).

- Interest in the low-energy πN interaction was maintained for several decades by the prospect of obtaining estimates for the πN σ term using low-energy input exclusively (e.g., the πN phase shifts at low energy). It can be argued that the extrapolation of the πN scattering amplitudes into the unphysical region is more reliable when it rests upon ‘close-by’ data, thus avoiding any high-energy influences. This issue was first addressed (and realised) in Ref. [3].

### 2.2 Measurable quantities

As various definitions of the term ‘dataset’ have been in use, involving different choices of the experimental conditions which ought to remain stable/identical during the data acquisition, I feel that, before entering the description of the low-energy πN DB, I better explain what the term implies in this work. The properties of the incident beam and the (physical, geometrical) characteristics of the target were employed in the past in order to distinguish the results of experiments conducted at one place over a (short) period of time. However, datasets have appeared in experimental reports relevant to the πN interaction, which not only involved different beam energies, but also contained measurements of different reactions. In this work, the requisite for accepting a set of observations as comprising one dataset is that these observations share the same measurement of the absolute normalisation\(^1\) (and, consequently, identical normalisation uncertainty). All references to the experimental works may be found in Ref. [1] and in the works cited therein. Unless attention must be drawn to a specific dataset, no explicit reference will be given to the original experimental reports.

The composition of the low-energy πN DB in terms of number of entries (data points), arranged in datasets (following the definition of the previous paragraph), for the various low-energy πN measurable quantities (observables) is displayed in Table 1. The low-energy πN DB at finite (non-zero) $T$ comprises measurements of the differential cross section (DCS), analysing power (AP), partial-total cross section (PTCS), and total-nuclear cross section (TNCS) for the two elastic-scattering (ES) reactions ($\pi^\pm p \rightarrow \pi^\pm p$). One dataset of AP

\(^1\) Of course, this is a necessary, not a sufficient, condition. Additional requirements may apply after the examination of the original experimental reports, in particular regarding the off-line processing of the raw experimental data.
measurements contains data of both ES reactions, namely seven $\pi^+ p$ data-points and three $\pi^- p$ ES datapoints. In addition to the DCS and AP measurements, the DB of the $\pi^- p$ charge-exchange (CX) reaction ($\pi^- p \rightarrow \pi^0 n$) contains measurements of the total cross section (TCS). Furthermore, two experiments (conducted in the 1980s) measured the $\pi^- p$ CX DCS, but the experimental groups published the corresponding (fitted) values of the first three coefficients in the Legendre expansion (CLE) of their DCSs. At present, the two ES reactions and the $\pi^- p$ CX reaction are the only $\pi N$ processes which are experimentally accessible at low energy.

Table 1

The breakdown of the low-energy $\pi N$ DB into reactions and measured physical quantities. The entries represent the numbers of the datapoints and of the corresponding datasets in the DB. The data of this table have appeared in peer-reviewed Physics journals, in the sixteen issues of the $\pi N$ Newsletter, or in dissertations. Not included in this table (but contained in the SAID $\pi N$ DB) are 16 measurements (three datasets) of the $\pi^+ p$ DCS and 22 measurements (four datasets) of the $\pi^- p$ ES DCS, which had appeared in preprints, which were not submitted/accepted for publication, or had been privately communicated to the SAID group.

| Reaction | DCS | AP | PTCS | TNCS | TCS | CLE | $\epsilon_{1s}$ | $\Gamma_{1s}$ | Total |
|----------|-----|----|------|------|-----|-----|----------------|-------------|-------|
| $\pi^+ p$ | 682 | 31 | 24   | 6    | −   | −   | −              | −           | 743   |
| $\pi^- p$ ES | 525 | 85 | 3    | 6    | −   | −   | 2              | −           | 621   |
| $\pi^- p$ CX | 297 | 10 | −    | −    | 10  | 18  | −              | 2           | 337   |
| $\pi^+ p$ and $\pi^- p$ ES | − | 10 | −    | −    | −   | −   | −              | −           | 10    |
| Total   | 1504 | 136 | 27   | 12   | 10  | 18  | 2              | 2           | 1711  |

| Reaction | DCS | AP | PTCS | TNCS | TCS | CLE | $\epsilon_{1s}$ | $\Gamma_{1s}$ | Total |
|----------|-----|----|------|------|-----|-----|----------------|-------------|-------|
| $\pi^+ p$ | 53  | 5  | 19   | 6    | −   | −   | −              | −           | 83    |
| $\pi^- p$ ES | 40  | 9  | 3    | 6    | −   | −   | 2              | −           | 60    |
| $\pi^- p$ CX | 36  | 2  | −    | −    | 10  | 6   | −              | 2           | 56    |
| $\pi^+ p$ and $\pi^- p$ ES | − | 1  | −    | −    | −   | −   | −              | −           | 1     |
| Total   | 129 | 17 | 22   | 12   | 10  | 6   | 2              | 2           | 200   |

Unlike all other analyses of the $\pi N$ DB (which I am aware of), the studies performed within the ETH $\pi N$ project have made use (after the mid 1990s) of the $\pi^+ p$ PTCSs and TNCSs, as well as of the $\pi^- p$ CX TCSs. It would have
been controversial to include in the DB the nine measurements of the \( \pi^- p \) ES PTCSs and TNCSs (see Ref. [1] for a discussion).

The \( \pi N \) scattering amplitude \( F(\vec{q}', \vec{q}) \) at low energy may safely be confined to \( s \)- and \( p \)-wave contributions (e.g., see Ref. [4], pp. 17–18). Introducing the isospin of the pion as \( \vec{t} \) and that of the nucleon as \( \vec{\tau}/2 \), one may write (using natural units for the sake of brevity):

\[
F(\vec{q}', \vec{q}) = b_0 + b_1 \vec{\tau} \cdot \vec{t} + \left( c_0 + c_1 \vec{\tau} \cdot \vec{t} \right) \vec{q}' \cdot \vec{q} + i \left( d_0 + d_1 \vec{\tau} \cdot \vec{t} \right) \vec{\sigma} \cdot (\vec{q}' \times \vec{q}) , \tag{1}
\]

where \( \vec{\sigma} \) is (double) the spin of the nucleon; \( \vec{q}' \) and \( \vec{q} \) are the centre-of-mass (CM) 3-momenta of the incoming and outgoing pions, respectively.

Equation (1) defines the isoscalar (subscript 0) and isovector (subscript 1) \( s \)-wave scattering lengths (\( b_0 \) and \( b_1 \)) and \( p \)-wave scattering volumes (\( c_0, c_1, d_0, \) and \( d_1 \)), which may be projected onto the spin-isospin basis via standard transformations. The third term on the right-hand side of Eq. (1) is the no-spin-flip \( p \)-wave part of the \( \pi N \) scattering amplitude, whereas the fourth term is the spin-flip part.

Also included in the low-energy \( \pi N \) DB are the two \( \pi^- p \) scattering lengths \( a_{cc} \equiv b_0 - b_1 \) and \( a_{c0} \equiv \sqrt{2}b_1 \) (corresponding to the \( \pi^- p \) ES and CX reactions, respectively), obtained via the Deser formulae from measurements of the strong-interaction shift (henceforth, strong shift) \( \epsilon_{1s} \) [7,8] and of the total decay width \( \Gamma_{1s} \) [7,9] of the ground state in pionic hydrogen. The experiments were conducted at the Paul Scherrer Institut (PSI) during about one decade around the turn of the millennium. The quantities \( a_{cc} \) and \( a_{c0} \) represent the \( \pi^- p \) ES and CX scattering amplitudes at \( T = 0 \) MeV, respectively.

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2 These measurements do not enter the SAID \( \pi N \) DB.

3 I prefer this short form as reference to the two important relations developed by Deser, Goldberger, Baumann, and Thirring in 1954 [5], as well as by Trueman in 1961 [6], rather than Deser-Goldberger-Baumann-Thirring, Deser-Trueman, or Trueman-Deser relations.

4 The publication of the final result of the Pionic Hydrogen Collaboration (\( \Gamma_{1s} = 0.856(16)(22) \) eV [9]) for the total decay width of the ground state in pionic hydrogen in 2021 reinforces my claim that all former values, disseminated by the collaboration (i.e., \( \Gamma_{1s} \leq 850 \) meV in 2003, 765(56) meV in 2009, “from 750 to 900 meV with a central value at 850 meV” in 2015, etc.), were preliminary and, as such, they should never have entered any analyses, inasmuch as the final result (\( \Gamma_{1s} = 0.868(40)(38) \) eV) of the former experiment [7] was available since 1999. I see no reason why a preliminary result should be given priority over a final one, see also comments in Section 3.3 of Ref. [10].

5 The results of a predecessor experiment, which had yielded compatible estimates (but larger uncertainties) for \( \epsilon_{1s} \) and \( \Gamma_{1s} \), had already appeared in the mid 1990s [11,12]. The same collaboration also measured the strong shift and the total decay width of the ground state in pionic deuterium [13,14].
A few words about the precision, attained by the PSI experiments, are in order. After the raw results (measured distributions of the $3p \rightarrow 1s$ transition energy) of the two experiments are corrected for the various Quantum-Electrodynamical effects reported in Ref. [15] (e.g., vacuum-polarisation effects, Breit-Pauli interaction, finite-size effects, self-energy effects, and so on), the two results for the strong shift $\epsilon_{1s}$ [7,8] come out nearly identical\(^6\). However, the perfect agreement between the two results is not the only reason to rejoice. Judged by Hadronic-Physics standards, the precision, attained in these experiments, is unequalled: the (relative) statistical uncertainty of $\epsilon_{1s}$ in the former experiment is below the 2 per-mille level, whereas the systematic uncertainty does not exceed 0.5%; in case of the most recent experiment, both uncertainties are at the 1 per-mille level, thus making the strong shift $\epsilon_{1s}$ one of the best-known physical quantities in Hadronic Physics.

I find it unfortunate that the high precision at which $\epsilon_{1s}$ is experimentally known does not translate into solid knowledge of the $\pi^- p$ ES length $a_{sc}$. Additional EM effects must be removed, and, as I will demonstrate in Section 5.2.1, the overall status of these corrections (at present) is below the expectations.

### 3 Extraction of the hadronic quantities from the data

What does one need in order to attempt the description of the $\pi N$ data? The general approach comprises four steps. One needs to:

a) model the hadronic part of the $\pi N$ interaction,
b) include the EM effects,
c) select an input DB, and
d) select/establish an optimisation procedure.

A number of studies omit step (b) above, by fitting to the $s$- and $p$-wave $\pi N$ phase shifts of popular partial-wave analyses (PWAs), usually of those performed by the SAID group [16]. One such solution in the recent past was their WI08 solution [17], whereas their ‘current’ solution has been named XP15 [18]. The differences between these two solutions are small. Two comments are in order.

- Such a practice frequently gives rise to a misleading complacency: many authors and readers tend to forget that the problem of the EM corrections

\(^6\) The corrected $\epsilon_{1s}$ result of Ref. [7] was given in Ref. [15] as: $-7.085(13)(34)$ eV, whereas the corresponding result of Ref. [8] (retaining the sign convention of Ref. [7]) was: $-7.0858(71)(64)$ eV. Both values ought to be corrected by $+0.0055$ eV, due to changes in some physical constants, namely in the charged-pion and proton rest masses, as well as in the charge radii of both particles.
has been bypassed, not resolved. Fitting to $\pi N$ phase shifts does not fix by itself any problems regarding the inclusion of the EM interaction; this issue cannot be relegated by simply changing the type of the input to a study.

- Owing to the overconstrained fits, dispersion-relation analyses cannot provide meaningful estimates for the uncertainties of the $\pi N$ phase shifts; only their single-energy solutions are accompanied by such uncertainties. This implies that working uncertainties must be assigned in the studies which use the $\pi N$ phase shifts of the SAID solutions as input. On the contrary, all results obtained within the ETH $\pi N$ project (e.g., for the low-energy constants of the $\pi N$ interaction, for the $\pi N$ phase shifts, etc.) are accompanied by (1σ) uncertainties which reflect the statistical and systematic fluctuations of the fitted data.

### 3.1 Modelling of the hadronic part of the $\pi N$ interaction

The low-energy region corresponds to the non-perturbative domain of Quantum Chromodynamics (QCD), i.e., of the gauge field theory which describes the strong interaction between quarks and gluons. The largeness of the strong coupling constant $\alpha_S$ in this domain (which is also known as ‘confinement region’) renders the framework of perturbation theory ineffective as far as the evaluation is concerned of the important physical characteristics of the various hadronic processes (i.e., of the scattering amplitudes and of the measurable quantities emanating thereof). To enable the modelling of the strong interaction between pions and nucleons, a number of approaches have been developed.

- Several approaches are based on dispersion relations. Information on the various forms of such schemes (fixed-$t$, forward, hyperbolic dispersion relations, etc.) may be found in Section 6 of Höhler’s celebrated work [19]. This category also includes the recent attempts to account for the data on the basis of the Roy-Steiner equations [20,21].
- Several effective-field models have emerged, inspired by the Chiral-Perturbation Theory ($\chi$PT) [22] in its various forms, e.g., Heavy Baryon $\chi$PT (HB$\chi$PT) [23,24], Covariant Baryon $\chi$PT (CB$\chi$PT) [25,26,27], etc. A Lorentz-invariant formulation of the Baryon $\chi$PT, featuring the method of infrared regularisation as the means to preserve the power-counting features of $\chi$PT at low energy, surfaced in Ref. [28]. Additional works are listed in Ref. [29] (as well as in a variety of other studies).
- Several models, based on hadronic exchanges, were put forward during the last half century or so. Most of these efforts were critically examined in

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7 The reader must bear in mind that providing a complete list of references on the topic of this section lies well beyond the scope of this work. This subject is covered in dedicated review articles.
Several phenomenological/empirical models have appeared, which are based on hadronic potentials [32], on the polynomial parameterisation of the s- and p-wave $K$-matrix elements [33,34], on the parameterisation of the s- and p-wave phase shifts [35,36], etc. All aforementioned methods model the hadronic part of the $\pi N$ interaction on the basis of some parameters. (Some $\chi$PT-based approaches also model the EM part of the $\pi N$ interaction.) The methods of the first two categories are more constrained, in that they fulfil the theoretical constraints of unitarity (see Section 5 of Ref. [19]), analyticity (see Section 7 of Ref. [19]), crossing symmetry (see Section 6.1 of Ref. [19]), and isospin invariance (see Section 2.2 of Ref. [19], as well as the historical perspective of Ref. [37] on the concept of isospin). The partial-wave amplitudes of the hadronic model of Ref. [1] fulfill unitarity, crossing symmetry, and isospin invariance. On the other hand, the phenomenological/empirical models are compatible only with the unitarity constraint. This remark, however, must not be taken as indicating a disadvantage; being devoid of the theoretical constraints of crossing symmetry and isospin invariance, these data-driven approaches

- grant maximal freedom to the fitted data and
- are indispensable tools in the tests of these theoretical constraints at low energy.

Given its relevance in this work, a few words about the isospin invariance are in order. Assuming that the isospin invariance holds in the hadronic part of the $\pi N$ interaction, only two (complex) scattering amplitudes enter the description of the three low-energy $\pi N$ reactions (e.g., see Appendix 1 in Ref. [38]): the isospin $I = 3/2$ amplitude ($f_3$) and the $I = 1/2$ amplitude ($f_1$). Fulfilment of the isospin invariance in the hadronic part of the $\pi N$ interaction implies that the $\pi^+p$ reaction is described by $f_3$, the $\pi^-p$ ES reaction by the linear combination $(f_3 + 2f_1)/3$, and the $\pi^-p$ CX reaction by $\sqrt{2}(f_3 - f_1)/3$. From these relations, the following expression (known as ‘triangle identity’) links together the amplitudes $f_{\pi^+p}$, $f_{\pi^-p}$, and $f_{CX}$:

$$f_{\pi^+p} - f_{\pi^-p} = \sqrt{2}f_{CX}. \tag{2}$$

In the following, ‘isospin invariance in the $\pi N$ interaction’ will be used as the short form of ‘isospin invariance in the hadronic part of the $\pi N$ interaction’; it

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8 The scattering amplitudes of the two ES reactions are linked via the interchange $s \leftrightarrow u$ in the two invariant amplitudes $A_\pm(s,t,u)$ and $B_\pm(s,t,u)$, where $s$, $t$, and $u$ are the standard Mandelstam variables.
is known that the isospin invariance is broken in the EM part of the interaction. One of the proposed indicators of the violation of the isospin invariance is the symmetrised relative difference $R_2$:

$$R_2 := \frac{2 \Re \left[ f_{CX} - f_{\text{extr}}^{CX} \right]}{\Re \left[ f_{CX} + f_{\text{extr}}^{CX} \right]} = \frac{2 \Re \left[ f_{\pi^+ p} - f_{\pi^- p} - \sqrt{2} f_{\text{extr}}^{CX} \right]}{\Re \left[ f_{\pi^+ p} - f_{\pi^- p} + \sqrt{2} f_{\text{extr}}^{CX} \right]},$$

where the operator $\Re$ returns the real part of a complex number and the $f_{\text{extr}}^{CX}$ is extracted from the data (as opposed to the reconstructed amplitude $f_{CX}$, obtained via Eq. (2)).

Of interest in this work are those of the approaches which can be used as tools to test the theoretical constraint of the isospin invariance in the $\pi N$ interaction. This personal predilection is the product of the experience I gained from the analysis of the low-energy $\pi N$ DB during the last three decades. In retrospect, the separate analysis of the measurements of the three low-energy $\pi N$ reactions (or the joint analyses of the measurements after leaving out one of these reactions), has given rise to persistent discrepancies. In this respect, of interest in this work are the models of Refs. [1,32,33,34,38]. It is interesting to examine the similarities of and the differences between the aforementioned models in the chronological order in which the first analyses, involving these models, appeared.

- In their pioneering work [32], Gibbs and collaborators extracted the $\pi N$ scattering amplitudes $f_{\pi^+ p}$, $f_{\pi^- p}$, and $f_{\text{extr}}^{CX}$ from the (sparse, at that time) low-energy $\pi N$ DB using hadronic potentials of different shapes. Subsequently, the authors evaluated the difference between the extracted $f_{\text{extr}}^{CX}$ and the reconstructed $f_{CX}$ scattering amplitudes, and established a significant discrepancy in the $s$-wave part of the interaction (see their Fig. 1); a smaller discrepancy was reported in the no-spin-flip $p$-wave part of the interaction (see their Fig. 2). These discrepancies were found to be independent (on the whole) of the type of the hadronic potential used in the data description (see their Table 1). Another interesting result in that paper is frequently overlooked: the zero-crossings of the extracted $f_{\text{extr}}^{CX}$ and reconstructed $f_{CX}$ scattering amplitudes (a phenomenon which is due to the $s$- and $p$-wave destructive interference at forward CM scattering angles $\theta$) occur at different energies (see their Fig. 3). Subsequent studies [1,38], using sizeably larger (in comparison with Ref. [32]) DBs, corroborated these results [9]. This is more remarkable than it sounds because, for the sake of argument, the $\pi^- p$ CX DB has been increased from 46 to 337 datapoints

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9 The only outcome of the analyses, performed by Gibbs and collaborators, which cannot be corroborated by the results obtained within the ETH $\pi N$ project, is the equality (within the uncertainties) of the two $p$-wave phase shifts $\delta_{3/2}^{1-} (P_{31})$ and $\delta_{1/2}^{1+} (P_{13})$ [35].
since the time the analysis of Ref. [32] was performed.

- When Fettes and I sought in the mid 1990s the description of the low-energy $\pi^+p$ data on the basis of simple polynomial forms for the $s$- and $p$-wave $K$-matrix elements [33], I could not foresee the pivotal role, which this novel model-independent way of analysis would assume within the ETH $\pi N$ project in the years to follow. This is due to two reasons: first, the method has been firmly established as the means of identification of the outliers which the low-energy $\pi N$ DB contains; second, it provides a model-independent way of analysing the measurements, one which is devoid of theoretical constraints. There are two issues in analyses using similar parameterisation:

1. the number of terms which one retains from the power series (in a suitable variable, e.g., in the pion CM kinetic energy $\epsilon$ or, more frequently, in the square of the CM 3-momentum in the initial state [34]), and
2. the forms which one uses in the modelling of the resonant contributions.

Experience has shown that the polynomial parameterisation of the $s$- and $p$-wave $K$-matrix elements successfully captures the dynamics of the $\pi N$ interaction at low energy. A number of tests have been performed, demonstrating beyond doubt that the outliers are flanked by measurements which can successfully be accounted for. Therefore, the identification of datapoints as outliers in the fits cannot be attributed to the inadequacy of the employed parametric forms to account for the energy dependence of the $\pi N$ phase shifts; consequently, the outliers are suggestive of experimental shortcomings. In the data analysis, terms up to $O(\epsilon^2)$ (or up to $O(q^4)$ in the parameterisation of the $K$-matrix elements in powers of $q^2$, where $q := |\vec{q}|$) are retained [34]. Owing to the current uncertainties of the measurements, the coefficients of higher orders cannot be determined reliably from the data.

- The backbone of the ETH $\pi N$ project is the isospin-invariant hadronic model whose Feynman diagrams are shown in Fig. 1. The question, of course, arises as to how an isospin-invariant model can be relevant in the tests of the isospin invariance; in fact, there are two ways by which it can. One approach rests upon the extraction of a phase-shift solution from the ES DBs and the subsequent evaluation of the scattering amplitude, corresponding to the $\pi^-p$ CX reaction, using Eq. (2) [38]. Having reconstructed the amplitude $f_{CX}$, one may generate predictions for the various observables, corresponding to the kinematical quantities ($T, \theta$) of the measurements in the $\pi^-p$ CX DB. The comparison between these predictions and the experimental data enables the assessment of the amount by which the triangle identity (equivalently, the isospin invariance) is violated; this method does not require the extraction of the amplitude $f_{CX}^{extr}$ from the data. A second approach was more recently established [1]. Two kinds of fits may be pursued: one uses as input DB the data of two ES reactions, whereas the second uses the data of the $\pi^+p$ and $\pi^-p$ CX reactions. In both cases, the isospin $I = 3/2$ partial-wave amplitudes are fixed (predominantly) from the $\pi^+p$ ES reaction, whereas the two $\pi^-p$ reactions establish the $I = 1/2$ partial-wave
Fig. 1. The main Feynman diagrams of the ETH model: scalar-isoscalar ($I = J = 0$) and vector-isovector ($I = J = 1$) $t$-channel graphs (upper part), and $N$ and $\Delta(1232)$ $s$- and $u$-channel graphs (lower part). Not shown in this figure, but also analytically included in the model, are the small contributions from the well-established (four-star) $s$ and $p$ higher baryon resonances with masses below 2 GeV and known branching fractions to $\pi N$ decay modes, as well as those from the $t$-channel exchanges of four (three scalar-isoscalar and one vector-isovector) mesons with masses below 2 GeV and known branching fractions to $\pi \pi$ decay modes, see Ref. [1] for details.

It is true that those of the approaches, which fix their parameters from fits to popular phase-shift solutions, could (in principle) test the isospin invariance in the $\pi N$ interaction, e.g., by using as input the $\pi N$ phase shifts obtained from specific reactions or combinations of reactions (e.g., two such sets of $\pi N$ phase shifts are generated within the ETH $\pi N$ project). However, this has

\[ \chi^2 \text{ value corresponding to the optimal description of the input DB.} \]

\[ ^{10} \text{When an isospin-invariant model is used in the description of data which contain isospin-breaking effects (such as the measurements of the } \pi^- p \text{ CX reaction presumably do), the isospin-breaking effects are absorbed in the fitted values of the model parameters, which thus become effective. Such changes have been observed in the joint fits of the ETH model to the } \pi^+ p \text{ and } \pi^- p \text{ CX data: affected are the Fermi-like coupling associated with the } t \text{-channel } \rho \text{-meson-exchange Feynman diagram (model parameter } G_{\rho} \text{) and (to a lesser degree) the } \pi N \text{ coupling constant (model parameter } g_{\pi NN} \text{)} [1]. \text{ The result of the presence of effects in the data, which have no counterpart in the modelling of the hadronic part of the } \pi N \text{ interaction, is the inevitable increase in the } \chi^2 \text{ value corresponding to the optimal description of the input DB.} \]
never been done; all such works choose to make use of $\pi N$ phase shifts which have been obtained by global fits to the $\pi N$ data (see end of Section 3.3).

3.2 Inclusion of the EM effects

The EM effects come in two forms:

- as direct contributions to the $\pi N$ scattering amplitude (Coulomb amplitude) for the two ES reactions and
- as distortion effects on the $\pi N$ phase shifts and partial-wave amplitudes, originating from the interference of the EM and strong interactions.

The first systematic attempt to determine the EM corrections, applicable to the data in the scattering region ($T > 0$ MeV), was performed in the 1970s, and led to three papers by the NORDITA group [39,40,41]. Their results are listed in tabular form (for several CM momenta) spanning a broad energy domain. The NORDITA programme did not develop a procedure for correcting any measurements at the $\pi N$ threshold, though (not known to some) the issue of the EM corrections at the $\pi N$ threshold had already been addressed [42,43,44,45] a few years prior to the publication of the NORDITA papers. In Ref. [45], Rasche and Woolcock developed the methodology for the inclusion of the effects of the $\pi^- p \rightarrow \gamma n$ channel (simply $\gamma n$ channel henceforth) at the $\pi N$ threshold, whereas the same authors presented in a subsequent paper [46] a method for the extraction of the (corrected) strong shift and total decay width in pionic atoms. The numerical results in Section 3 of Ref. [46] were tailored to the $2p \rightarrow 1s$ transition in pionic hydrogen, which was considered (on the basis of the yield) the most promising transition at that time. Comments on the corrections of Ref. [46] may be found in Section 4 of Ref. [47]. The corrections of Ref. [46] were superseded by those of Ref. [48].

The inclusion of the EM effects in the scattering region for $T < 300$ MeV was also addressed by Bugg in 1973 [49]: these ‘Coulomb-barrier’ corrections, determined by solving a relativised Schrödinger equation, were used in many analyses of the $\pi N$ data at former times.

To my knowledge, the only programme which addressed the issue of the EM corrections throughout the low-energy region (also including, in a consistent manner, the corrections which ought to apply to the measurements of the $\epsilon_{1s}$ and $\Gamma_{1s}$ in pionic hydrogen) was carried out at the University of Zurich in the late 1990s and the early 2000s. The programme resulted in the publication of the EM corrections for $\pi^+ p$ scattering [50], for $\pi^- p$ scattering [51], as well as for the strong shift $\epsilon_{1s}$ and the total decay width $\Gamma_{1s}$ [48]. The programme rested upon the determination of the hadronic potentials which optimally account for the low-energy $\pi N$ DB in the scattering region, and made use of
the same potentials to determine the corrections at the $\pi N$ threshold using a three-channel ($\pi^- p \rightarrow \pi^- p, \pi^0 n, \gamma n$) calculation. Unfortunately, this method is applicable only at low energy.

Some of the available schemes [35,39,40,50,51] for the determination of the EM corrections for $\pi N$ scattering experiments are compared in Ref. [36]. Two conclusions may be drawn from that work: first, that the effects of the EM corrections on the low-energy $\pi N$ DCs cannot be neglected; second, that the differences among the available schemes are generally small. The second realisation is encouraging for analysts of the low-energy $\pi N$ data, in that the selection of the EM-correction scheme seems to be of lesser importance than applying no EM corrections to the $\pi N$ phase shifts and partial-wave amplitudes, obtained from the modelling of the hadronic part of the $\pi N$ interaction.

To summarise, no consistent approach exists for the treatment of the EM corrections in the entirety of the energy range of the available data, i.e., from the $\pi N$ threshold to the few-GeV region.

### 3.3 Selection of an analysis method/optimisation scheme

In Particle Physics, the commonest (by far) choice in optimisation procedures is the minimisation of $\chi^2$-based functions. Of interest herein is the $\chi^2$-based minimisation with rescaling of the input datasets [52], though also $\chi^2$-based methods without rescaling [33], as well as more robust methods [38], have been used in the analysis of the low-energy $\pi N$ data. Some in-between methods, listed in Section 4.1 of Ref. [10], could be tested in the future. In any case, by variation of the parameters of the various hadronic models, one achieves the optimisation of the description of the input data.

The PSAs of the $\pi N$ data by the SAID group, as well as those performed within the ETH $\pi N$ project during the last two decades, make use of the minimisation function which Arndt and Roper introduced half a century ago [52]. The contribution of the $j$-th dataset to the overall $\chi^2$ reads as:

$$\chi^2_j = \sum_{i=1}^{N_j} \left( \frac{y_{ij}^{\exp} - z_j y_{ij}^{\th}}{\delta y_{ij}^{\exp}} \right)^2 + \left( \frac{z_j - 1}{\delta z_j} \right)^2,$$

where $y_{ij}^{\exp}$ denotes the $i$-th datapoint of the $j$-th dataset, $y_{ij}^{\th}$ the corresponding fitted ('theoretical') value, $\delta y_{ij}^{\exp}$ the statistical uncertainty of $y_{ij}^{\exp}$, $z_j$ a scale factor (which applies to the entire dataset), $\delta z_j$ the normalisation uncertainty (reported or assigned), and $N_j$ the number of the accepted datapoints in the dataset (i.e., of the datapoints which remain in the dataset after the removal of any outliers). At each optimisation step, the fitted values $y_{ij}^{\th}$ are supplied by the modelling of the hadronic part of the $\pi N$ interaction and the inclusion of
the EM effects. As the scale factor \( z_j \) appears only in \( \chi_j^2 \), the minimisation of the overall \( \chi^2 := \sum_{j=1}^{N} \chi_j^2 \) (where \( N \) denotes the number of the datasets in the fit) implies the fixation of each scale factor \( z_j \) from the condition \( \partial \chi_j^2 / \partial z_j = 0 \). The unique solution

\[
z_j = \frac{\sum_{i=1}^{N_j} y_{ij}^{\text{exp}} y_{ij}^{\text{th}} / (\delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}}
\]

leads to

\[
(\chi_j^2)_{\text{min}} = \sum_{i=1}^{N_j} \frac{(y_{ij}^{\text{exp}} - y_{ij}^{\text{th}})^2}{(\delta y_{ij}^{\text{exp}})^2} - \frac{\left( \sum_{i=1}^{N_j} (y_{ij}^{\text{exp}} - y_{ij}^{\text{th}}) y_{ij}^{\text{th}} / (\delta y_{ij}^{\text{exp}})^2 \right)^2}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}}.
\]

The sum of the contributions \( \sum_{j=1}^{N} (\chi_j^2)_{\text{min}} \) is a function of the parameters entering the modelling of the hadronic part of the \( \pi N \) interaction. By variation of these parameters, the overall \( \chi^2 \) is minimised, yielding \( \chi^2_{\text{min}} \).

The numerical minimisation may be achieved by using many commercial and non-commercial software libraries. Within the ETH \( \pi N \) project, the MINUIT package [53] of the CERN library (FORTRAN version) has exclusively been used. Each optimisation is achieved by following the sequence: SIMPLEX, MINIMIZE, MIGRAD, and MINOS. The calls to the last two methods involve the high-level strategy in the numerical minimisation.

- SIMPLEX uses the simplex method of Nelder and Mead [54].
- MINIMIZE calls MIGRAD, but reverts to SIMPLEX in case that MIGRAD fails to converge.
- MIGRAD, undoubtedly the workhorse of the MINUIT software package, is a variable-metric method, based on the Davidon-Fletcher-Powell algorithm. The method checks for the positive-definiteness of the Hessian matrix.
- MINOS performs a detailed error analysis, separately for each model parameter, taking into account the non-linearities in the problem, as well as the correlations among the parameters.

All aforementioned methods admit an optional argument, fixing the maximal number of calls to each method (separately). If this limit is reached, the corresponding method is terminated (by MINUIT, internally) regardless of whether or not that method converged. Inspection of the (copious) MINUIT output may easily ascertain whether or not the application terminated successfully and its methods converged.

Before entering the details of the analysis of the \( \pi N \) data, I will outline the types of fits which one may pursue.

- In the global fits, all measurements of the three experimentally-accessible \( \pi N \) reactions at low energy comprise the input DB, whose description ought
to be optimised. Global fits to the $\pi N$ data are routinely performed by the SAID group. Not known to some, global fits to the low-energy $\pi N$ data are possible within the ETH $\pi N$ project since the mid 1990s. Such fits are carried out (for the sake of completeness), though (due to reasons which will become clear later on) the results have never been reported/used.

- In the \textit{joint} fits, measurements of two $\pi N$ reactions comprise the input DB. Such fits are performed within the ETH $\pi N$ project, using either the ETH model or the polynomial parameterisation of the $s$- and $p$-wave $K$-matrix elements. At present, two such fits are carried out: the first uses as input DB the ES data, whereas the second imports the data of the $\pi^+ p$ and $\pi^- p$ CX reactions. The third combination of reactions, i.e., the selection as input of the measurements contained in the DBs of the two $\pi^- p$ reactions (ES and CX), remains a possibility; such fits have not been carried out yet.

- In the \textit{separate} fits, measurements of only one $\pi N$ reaction comprise the input DB. Such fits may be performed using the phenomenological/empirical models of Section 3.1 [32,33,34,35,36]. Due to the sizeable correlations among the model parameters, separate fits of the ETH model to the $\pi N$ data have not been attempted for a long time.

4 So, what has been learnt?

The studies, which do not pursue global fits to the low-energy $\pi N$ data (and can therefore test the isospin invariance in the $\pi N$ interaction), are those performed by Gibbs and collaborators [32], as well as the ones carried out within the ETH $\pi N$ project since the mid 1990s (see cited works in Ref. [1]). In spite of their substantial differences in the modelling of the hadronic part of the $\pi N$ interaction, in the inclusion of the EM effects, in the input DB (e.g., the low-energy $\pi N$ DB grew substantially after Ref. [32] appeared), and in the choice of the optimisation procedure, the results of these works strongly suggest that, regarding the $\pi N$ interaction at low energy, not all pieces fall into place. Reference [32] demonstrated that the discrepancy in the $s$ wave between the extracted $f_{\text{extr}}^{\pi N}$ and the reconstructed $f_{\text{CX}}$ scattering amplitudes is statistically significant between $T = 30$ and $50$ MeV, e.g., see Fig. 1 therein. Given the growing importance of the $p$-wave part of the $\pi N$ scattering amplitude with increasing energy, Ref. [32] essentially predicts (though not explicitly mentioned in the paper) an energy-dependent isospin-breaking effect in the DCSs of the $\pi^- p$ CX reaction.

Within the ETH $\pi N$ project, such an effect has been established [1]. Integrated between 0 and 100 MeV, the discrepancy between measured (or extracted) and predicted (or reconstructed) cross sections would be equivalent to an effect at the level of $(12.8 \pm 1.3) \cdot 10^{-2}$ or, naively converted into a relative difference between the two $\pi^- p$ CX scattering amplitudes, of $r_2 = 6.19(63)$ %. The
The aforementioned discrepancy between measured and predicted cross sections indicates that

\[ \left\langle \frac{|f_{\text{extr}}^{|CX|^2}}{|f_{\text{CX}}|^2} \right\rangle > 1 \]

where \( f_{\text{CX}} \) is obtained from \( f_{\pi+p} \) and \( f_{\pi-p} \) via Eq. (2). As a result,

\[ \left\langle \frac{|f_{\text{extr}}^{|CX|^2}|^2}{|f_{\text{CX}}|^2} \right\rangle = (1 + r_2)^2 \]

for some \( r_2 \in \mathbb{R}_{>0} \). To obtain a working (though, given the pronounced energy dependence of the effect, not mathematically meaningful) \( R_2 \) result, one could assume the constancy of the ratio on the left-hand side of the previous equation, in which case

\[ \frac{|f_{\text{extr}}^{|CX}|}{|f_{\text{CX}}|} = \sqrt{2} \frac{|f_{\text{extr}}^{|CX}|}{|f_{\pi+p} - f_{\pi-p}|} = (1 + r_2) . \]  

(6)

At low energy, the imaginary parts of the scattering amplitudes may be omitted, and one obtains

\[ \frac{\Re[f_{\text{extr}}^{|CX}|]}{\Re[f_{\pi+p} - f_{\pi-p}]} \approx \frac{1 + r_2}{\sqrt{2}} . \]

Using Eq. (3), one ends up with the following relation for \( r_2 \ll 1 \).

\[ R_2 \approx -\frac{2r_2}{2 + r_2} = -r_2 + \frac{r_2^2}{2} - \frac{r_2^3}{4} + \ldots \]

Therefore, \( R_2 \approx -r_2 \) to a first approximation (the real part of the scattering amplitude \( f_{\text{CX}} \), which is obtained from the two ES scattering amplitudes via the triangle identity, is less negative than \( f_{\text{extr}}^{|CX|} \) at low energy); retaining all higher-order contributions, \( R_2 \approx -6.01(59) \) %. Although these percentages should not be taken too seriously, they agree on the whole with the findings of Ref. [32]. Of course, the indicator \( R_2 \) of Eq. (3) has been obtained in other analyses at several energies, separately for the \( s \) and \( p \) waves, i.e., not in the form of an overall relative difference (which is how the discrepancy is currently presented within the ETH \( \pi N \) project). In future reports relating to the ETH \( \pi N \) project, the indicator \( R_2 \) will be evaluated at a few energies, separately for the \( s \) and \( p \) waves.

A few words on the stability of the results of Refs. [1,32,38] are worth including in this preprint. To my knowledge, the results of Ref. [32] have never been updated. A relative deviation from the triangle identity (in the \( s \) and \( p \) waves, average value over three energies) had been obtained in Ref. [38]: \( 6.4\pm1.4 \) %. Between 1997 and 2019, only the \( \chi^2 \) value (and the resulting p-value) of the reproduction of the low-energy \( \pi^-p \) CX data on the basis of the solution, obtained from the fits of the ETH model to the ES data, was used as a measure of the departure from the triangle identity. The quantity \( r_2 \) of Eq. (6) was first
evaluated in 2019 (6.17(64) %), and then updated in 2020 (6.19(63) %). Owing to the enhancement of the low-energy $\pi^- p$ CX DB over the past twenty-five years, the statistical significance of the departure from the triangle identity improved over time, though the amount of the departure itself has remained nearly unchanged.

The outcome of the optimisation within the ETH $\pi N$ project is routinely subjected to further analysis. As aforementioned, two kinds of fits are pursued: one uses as input DB the data of two ES reactions, the other the data of the $\pi^+ p$ and $\pi^- p$ CX reactions. Given that the Arndt-Roper formula [52], see Eq. (4), is used in the optimisation, the expectation is that the datasets which are scaled ‘upwards’ ($z < 1$) balance (on average) those which are scaled ‘downwards’ ($z > 1$) in both cases. Furthermore, the energy dependence of the scale factors $z_j$ of Eq. (5) must not be significant. If these prerequisites are not fulfilled, the description of the input DB cannot be considered satisfactory.

To cut a long story short, the fitted values of the scale factor $z$ in the former case (fit to the ES data) do not show any significant departure from the statistical expectation (i.e., they are centred on 1 and exhibit no significant energy dependence). On the other hand, significant systematic effects have been established in the latter case (fit to the data of the $\pi^+ p$ and $\pi^- p$ CX reactions) throughout the time this analysis has been performed: the modelling of the hadronic part of the $\pi N$ interaction at low energy generates (on average) overestimated DCS values for the $\pi^+ p$ reaction and underestimated ones for the $\pi^- p$ CX reaction.

To summarise, Refs. [1,32,38] essentially demonstrate that the $\pi N$ scattering amplitudes $f_{\pi^+ p}$, $f_{\pi^- p}$, and $f_{\text{extr}}$, extracted from the low-energy $\pi N$ DB, fail to satisfy the triangle identity of Eq. (2). Henceforth, I will refer to this failure as the ‘low-energy $\pi N$ enigma’. This discrepancy suggests that at least one of the following assumptions is not valid.

a) Any systematic effects (e.g., systematic underestimation/overestimation) in the determination of the absolute normalisation of the datasets of the low-energy $\pi N$ DB are not significant. The absolute normalisation of each dataset is subject only to statistical fluctuation, regulated by the (reported or assigned) normalisation uncertainty of that dataset.

b) Any residual contributions to the EM corrections of Refs. [35,39,40,41,48,50,51] are not significant.

c) The isospin invariance, i.e., Eq. (2), holds in the hadronic part of the $\pi N$ interaction.

The three possibilities, arising from the non-fulfilment of each of these assumptions, will be examined in Section 5. Before that, I ought to address two important questions.
4.1 Can global fits to all low-energy $\pi N$ data be performed?

The answer to this question is: without doubt. A more meaningful question, however, would have been: are such fits satisfactory? Had I been asked that question, I would have replied without hesitation: hardly so. The low-energy $\pi N$ enigma cannot be swept under the carpet by submitting the available low-energy data to a global fit; one way or another, the discrepancy is bound to find its way to the output. In this section, I will examine one way by which this comes about, using the current solution of the SAID website. I ought to emphasise that the only input (the fitted values of the scale factor $z$ are available to two decimal places online [16]) in this part of the study comes from the SAID website (copied on March 31, 2022); there is no input from any other source. Their $T_j$ and $\delta z_j$ values will be used.

Following an analysis of the energy dependence of the scale factors $z_j$, a systematic bias was established in Ref. [55] in the output of the SAID solution WI08 [17]. Herein, I will re-examine the issue using the more recent SAID solution XP15 [18]. There is no difference between the two sets of fitted values of the scale factor $z$ for the ES datasets, whereas small changes can be seen in some $\pi^-p$ CX datasets in the region of the $s$- and $p$-wave destructive interference, namely in the scale factors $z_j$ of (some of) the FITZGERALD86 datasets\(^\text{11}\) [56].

I will now report on a simple analysis of the scale factors $z_j$ in the XP15 solution for $T \leq 100$ MeV, starting from the results representing all available data of the three low-energy $\pi N$ reactions (see Fig. 2). There is no doubt that, in spite of the large fluctuation which is present in the plot, the fitted values of the scale factor $z$ show no significant bias: their linear fit results in an intercept which is compatible with 1 and a slope which is compatible with 0, see Table 2. Therefore, the conclusion at first glance is that their fit yields results which are compatible with the statistical expectation (i.e., with the unbiased outcome of an optimisation based on the Arndt-Roper formula [52]).

However, the SAID $\pi N$ DB comprises three distinct parts, i.e., corresponding to the three low-energy $\pi N$ reactions. Had their fit been truly satisfactory, a similar behaviour of the scale factors $z_j$ (in terms of their energy dependence),

\(^{11}\)The FITZGERALD86 measurements of the $\pi^-p$ CX DCS in the kinematical region of the $s$- and $p$-wave interference minimum are arranged into seven three-point datasets. A fourth value was obtained in Ref. [56] via the extrapolation to $\theta = 0^\circ$ of a quadratic form fitted to the three datapoints of each dataset. The SAID $\pi N$ DB includes these additional datapoints, though (technically) they do not constitute independent observations. I do not insinuate that the addition of these seven datapoints is of any significance in practice.
Fig. 2. Plot of the scale factors $z_j$ of Eq. (5) corresponding to the XP15 solution [18]. $T_j$ denotes the pion laboratory kinetic energy of the $j$-th dataset. The data shown correspond to DCSs only (upward triangles: $\pi^+p$, filled circles: $\pi^-p$ ES, downward triangles: $\pi^-p$ CX). The datasets which were practically floated (the SAID group assign a normalisation uncertainty of 100% to datasets of suspicious absolute normalisation) have not been used. The dashed straight line represents the optimal, weighted linear fit to the data shown (see Table 2), whereas the shaded band represents $1\sigma$ uncertainties. The red line is the ideal outcome of the optimisation.

which is seen in Fig. 2, should also have been observed in any arbitrary subset of their DB, complying with the basic principles of the Sampling Theory (adequate population, representative sampling). The fitted values of the scale factor $z$, relating to the description of the SAID low-energy $\pi N$ DBs with the XP15 solution, are shown (separately for the three reactions) in Figs. 3-5. Had the three conditions (a)-(c), which are mentioned at the end of the previous section, been fulfilled in the SAID analysis at all energies, the fitted values of the scale factor $z$ of Figs. 3-5 should have come out independent of the beam energy and should have been centred on 1 (as the case was for the results of the global fit of Fig. 2). However, the bulk of the data for $T \leq 100$ MeV (represented by the shaded bands in Figs. 3-5) seem to be either underestimated by the XP15 solution (i.e., the measurements of the $\pi^-p$ CX reaction) or over-
Table 2

The fitted values of the parameters of the weighted linear fit to the data shown in Figs. 2-5, as well as the fitted uncertainties, corrected with the application of the Birge factor \( \sqrt{\chi^2/NDF} \), which takes account of the goodness of each fit [57]. Also quoted is the reduced \( \chi^2 \) value of each linear fit; NDF stands for the number of degrees of freedom.

| Reaction        | Intercept | Slope \( (10^{-3} \text{ MeV}^{-1}) \) | \( \chi^2/NDF \) |
|-----------------|-----------|---------------------------------------|------------------|
| All three \( \pi N \) reactions | 0.989 ± 0.014 | 0.05 ± 0.19 | 355.45/108 ≈ 3.29 |
| \( \pi^+p \)     | 0.936 ± 0.028 | 0.70 ± 0.34 | 167.53/39 ≈ 4.30 |
| \( \pi^-p \) ES  | 0.972 ± 0.021 | 0.16 ± 0.27 | 65.10/28 ≈ 2.33 |
| \( \pi^-p \) CX  | 1.051 ± 0.022 | −0.49 ± 0.41 | 69.88/37 ≈ 1.89 |

estimated by it (i.e., the ES measurements, the effects for the \( \pi^+p \) reaction being more pronounced), see also Table 2. One notices that the difference to the statistical expectation decreases with increasing beam energy, vanishing in the vicinity of \( T = 100 \text{ MeV} \). Such behaviour is in general agreement with the conclusions of Refs. [1,32,38] for an energy-dependent isospin-breaking effect.

From Figs. 3-5 and from Table 2, one may conclude that the XP15 solution does not describe sufficiently well the bulk of the low-energy measurements. Evidently, the XP15 solution at low energy represents a fictitious, average \( \pi N \) process, one which does not adequately capture the dynamics of the three low-energy \( \pi N \) reactions in sufficient detail.

In my judgment, only the ES data can jointly be submitted to an optimisation. There is no indication of violation of the isospin invariance in these two reactions. One of the possibilities in which models, based on hadronic exchanges, could accommodate a departure from the isospin invariance would have to involve \( \rho^0 - \omega \) mixing Feynman diagrams; however, the impact of this mechanism is suppressed at low energy, see Ref. [58] and the works cited therein. Experience shows that any optimisation, involving the \( \pi^-p \) CX DB, departs from the expected statistical norms, resulting in a sizeable increase in \( \chi^2_{\text{min}} \) and in a pronounced energy dependence of the scale factors \( z_j \). It ought to be reminded that the \( \rho^0 - \eta \) mixing mechanism was proposed over four decades ago [59] as a potential source of isospin-breaking effects in the \( \pi^-p \) CX reaction. It may be argued that, as the mechanism affects nearly all Feynman diagrams of the ETH model (see Fig. 6), significant bias in the results might be expected whenever the data of the \( \pi^-p \) CX reaction are included in the input DB. Although the coupling of the \( \eta \) meson (compared with that of the pion) to the nucleon could be stronger, the range of the published \( g_{\eta NN} \) values is fairly large. An average value \( g_{\eta NN} = 3.71(71) \) may be obtained from the entries of Table 2.
Fig. 3. Plot of the scale factors $z_j$ of Eq. (5) corresponding to the XP15 solution [18]. $T_j$ denotes the pion laboratory kinetic energy of the $j$-th dataset. The data correspond to $\pi^+p$ DCSs only. The datasets which were practically floated (the SAID group assign a normalisation uncertainty of 100 \% to datasets of suspicious absolute normalisation) have not been used. The dashed straight line represents the optimal, weighted linear fit to the data shown (see Table 2), whereas the shaded band represents 1\$ uncertainties. The red line is the ideal outcome of the optimisation.

of Ref. [60], translating into the pseudovector coupling $f_{\eta NN}^2 = 0.093(36)$ or $f_{\eta NN} = 0.306(58)$; in comparison, $f_c^2 := f_{\pi^\pm pn}^2 = 0.07629^{+0.00065}_{-0.00062}$ [10]. It thus seems that the $\pi^0 - \eta$ transition could have a significant impact on the $\pi N$ dynamics, at least at low energy.

The conclusion seems to be inevitable. Even in analyses, which assume that the isospin invariance holds and pursue global fits to the data (as the case is for the SAID solutions), the isospin-breaking effects, albeit somewhat hidden, manifest themselves as a systematic bias in the output of the optimisation, and can be uncovered after the results of the optimisation for the three low-energy $\pi N$ reactions are separately analysed.
4.2 Can the low-energy $\pi N$ enigma be explained in terms of inadequate modelling of the hadronic part of the $\pi N$ interaction?

On account of several reasons, inadequate modelling of the hadronic part of the $\pi N$ interaction is the most unlikely of the explanations, which could be put forward in order to resolve the low-energy $\pi N$ enigma. To start with, the approach of Ref. [32] is general and transparent, and the results, obtained on the basis of the hadronic potentials in that study, seem to be consistent among themselves. The approach of Refs. [33,34] is devoid of theoretical constraints, save for the expected behaviour of the $s$- and $p$-wave $K$-matrix elements at low energy. The optimal values of the coefficients of the polynomials, used in the parameterisation of these quantities, are obtained from optimisations which terminate successfully, posing no convergence problems whatsoever.

The ETH model contains Feynman diagrams of all well-established hadrons as intermediate states. Although the completeness of such a model can never be ascertained (in fact, one might also argue that all such models are bound to be ‘incomplete’), any residual contributions (e.g., from higher baryon resonances)
Fig. 5. The equivalent of Fig. 3 for the SAID $\pi^-p$ CX DB.

Fig. 6. Feynman diagrams involving the $\pi^0 - \eta$ admixture, a mechanism for the violation of the isospin invariance in the hadronic part of the $\pi N$ interaction in case of the $\pi^-p$ CX reaction [59].
cannot be significant (due to the largeness of their rest masses). In addition, the results obtained with the ETH model may be (and, in most cases, have been) verified by similar analyses performed using the approach of Refs. [33,34] (which, in fact, was the main reason that that approach had been put forward in the first place over two and a half decades ago).

The modelling of the hadronic part of the $\pi N$ interaction is different in Refs. [1,32,38]: it is based on hadronic potentials in the work of Ref. [32], whereas the two other works featured the ETH model. The results obtained with the ETH model are routinely compared with those extracted when using the polynomial parameterisation of the s- and p-wave $K$-matrix elements [33,34]. The inclusion of the EM effects in Ref. [32] was based on algorithms developed by Gibbs and collaborators, whereas Refs. [33,38] had used the NORDITA corrections. All analyses performed within the ETH $\pi N$ project after 2000 have used the EM corrections developed at the University of Zurich [48,50,51]. The low-energy $\pi N$ DBs, which all these efforts rested upon, are different, enhanced as time went by. As a result, it appears to be remarkable that the analyses of Refs. [32,38] had come up with the same statistical significance of the discrepancy in the $\pi N$ interaction at low energy, equivalent to a $4\sigma$ effect in the normal distribution; the statistical significance of the effect improved over the years, though the level of the discrepancy itself remained virtually constant [1].

Regarding the studies [32,38], the occasionally-voiced criticism (mainly by $\chi$PT-inspired researchers) is that such models are inconsistent in their treatment of the EM and hadronic effects. It is true that within the ETH $\pi N$ project, the EM corrections to the $\pi N$ phase shifts and partial-wave amplitudes must be imported from external sources; I am not sure that the criticism is entirely justified in case of the work of Gibbs and collaborators [32]. In any case, the same argument can be advanced against the plethora of works on the $\pi N$ interaction, which had been carried out within the $\chi$PT framework, but had used the SAID $\pi N$ phase shifts as input. In addition, though $\chi$PT is habitually presented in studies in ways which overstress its qualities, it nonetheless suffers from the same problems which plague all other effective-field methods. The evaluations of physical quantities are complete at one order, if they contain the contributions from all relevant physical effects at that order; in this sense, $\chi$PT provides a systematic basis for the treatment of the various physical effects. However, until the calculations are carried out at the next order, it remains unknown what changes they might bring. Furthermore, the convergence cannot easily be ascertained; it is usually assumed when two evaluations (of a physical quantity) at successive orders differ by an amount which is smaller than a user-defined threshold, though there is little assurance that the results of such evaluations should necessarily form a monotonic sequence. Even if a monotonic behaviour is expected in a problem (e.g., due to the smallness of the expansion parameter and the expected behaviour of the
contributions from the physical effects entering that problem) and the convergence criterion has been met, the importance of the residual effects, i.e., those corresponding to all higher orders (above the one at which the evaluation was last carried out), can hardly be assessed.

5 How could the low-energy $\pi N$ enigma be explained?

5.1 Experimental shortcomings

The first attempt to provide an explanation for the low-energy $\pi N$ enigma involves an obvious effect, namely the faulty assessment of the absolute normalisation of the low-energy $\pi N$ datasets. Is such a possibility tenable? After all, there have been problems with $\pi N$ datasets at low energy over the years, involving bizarre angular distributions of the DCS, but also erroneous absolute normalisation. However, what is of relevance in this section is a systematic misapplication of corrections to the raw experimental data\textsuperscript{12}, i.e., effects which impact on the bulk of the measurements, not sporadic experimental shortcomings. Given that the experiments were conducted at different places and at times spanning nearly four decades, and that they involved various experimental groups, beamlines, targets, and detectors, the possibility of systematic experimental flaws, affecting the bulk of the measurements, appears to be doubtful. Having said that, let me elaborate further on two common (and related) concerns regarding the outcome of the $\pi N$ experimentation at low energy.

One concern is about the absolute normalisation of some low-energy experiments. One feels being at a loss to come up with an explanation for the significant effects in the rescaling of some of the datasets, which the PSAs of the $\pi N$ data, performed within both the SAID and the ETH $\pi N$ projects, occasionally reveal. (Those who wish to object should take a better look at the fluctuation of the plotted data in Fig. 2.) The significant departure of the fitted values of the scale factor $z$ from 1 for some of the datasets may be due to at least one of the following reasons.

- The energy of the incoming beam had not been what the experimental group anticipated\textsuperscript{13}.

\textsuperscript{12}For the sake of example, such a situation could arise if a part of the final-state pions evaded detection or if the flux of the incident beam were overestimated in the $\pi N$ experiments at low energy. In both cases, the DCS would be systematically underestimated.

\textsuperscript{13}For the sake of example, corrections were applied in the early 1990s to the energy calibration of the M11 channel at TRIUMF. I am not aware of formal corrections
• The effects of the contamination of the incoming beam had been underestimated in the experiment.
• The normalisation uncertainty in the experiment had been underestimated.
• The determination of the absolute normalisation in the experiment had been erroneous.

Another concern relates to the smallness of the normalisation uncertainty reported in some experiments. Arguments have been presented [61], to substantiate the point of view that it is very likely that the published uncertainties in the \(\pi N\) experiments at low energy had been underestimated \textit{on average}. For instance, for nearly half of the DCS datasets (with known normalisation uncertainty) in the ES DB, normalisation uncertainties below 3\% had been reported. The smallest normalisation uncertainty in that DB is equal to a mere 1.2\%, which (based on the experience gained after three decades of relevant experimentation) borders on the impossible (at present). While pondering over these issues, one cannot help thinking that it would make sense to disregard all claims of such exaggerated precision, and to assign to all relevant datasets a reasonable normalisation uncertainty, say, 3\% (though this value might be optimistic too). However, such an approach would seem to be arbitrary and would undoubtedly (and, to an extent, justifiably) provide ground for criticism. Such revisions ought to be instigated by the experimental groups which had been responsible for the measurements, not by analysts. Regrettably, it is unrealistic to expect that such effects could be re-examined, in particular with a critical eye, given the time which elapsed since most of the \(\pi N\) experiments at low energy were conducted, the unfortunate loss of relevant information, and the shifting interests of the experimental groups towards other research domains as Pion Physics gradually phased out during the recent years.

The fluctuation, which is present in Fig. 2, attests to the seriousness of these concerns; the result of the linear fit \(\chi^2 \approx 355.45\) for 108 degrees of freedom indicates a poor description of the data, suggestive of erroneous absolute normalisation of and/or underestimated normalisation uncertainty in many low-energy \(\pi N\) datasets. On the other hand, the plot also indicates that some of the problematic datasets have overestimated absolute normalisation, whereas a similar number have underestimated one. On the whole, the two categories balance one another (which, in fact, is favoured due to the use of the Arndt-Roper minimisation function, as opposed to more robust procedures, in the optimisation), and the distribution of the fitted values of the scale factor \(z\) comes out nearly centred on 1. Let me next examine the distribution of the normalised residuals of the scale factor \(z\), defined as: \(\zeta_j := (z_j - 1)/\delta z_j\). Ideally, \(\zeta\) should be normally distributed, i.e., it should follow the \(N(\mu = 0, \sigma^2 = 1)\) distribution. The average \(\zeta\) value, obtained from the 110 low-energy \(\pi N\) datasets in the SAID DB, comes out equal to \(\hat{\mu} = -0.19(17)\), suggesting no signifi-
cant departure from a symmetrical distribution (about $0$). On the contrary, for the square root of the unbiased variance, one obtains $\hat{\sigma} \approx 1.82$, suggesting a sizeable departure of the $\zeta$ distribution from the $N(0,1)$ distribution, see Fig. 7. From these results, one can hypothesise that the large reduced $\chi^2$ value for the description of the data in Fig. 2 is due to smaller $\delta z_j$ values having been used in the definition of the normalised residuals $\zeta_j$. Absorbing the fluctuation in Fig. 2 into a redefinition of the normalisation uncertainty $\delta z$ favours the scenario that that quantity had been underestimated on average in the low-energy $\pi N$ experimentation, by possibly as much as 45 % (i.e., by a factor of nearly 2). Last but not least, the three low-energy $\pi N$ reactions are not equally affected by the systematic underestimation of the normalisation uncertainty of the experiments; Table 2 leaves no doubt that the imprint of the effect on the $\pi^+ p$ experiments is deeper.

To summarise, a systematic bias in the absolute normalisation of the low-energy $\pi N$ data, one which would impact on the bulk of the available measurements, however unlikely, cannot be altogether removed from consideration. The sizeable fluctuation in Fig. 2 (or, equivalently, the presence of substantial tails in the distribution of the normalised residuals $\zeta$ in Fig. 7) indicates that it is very likely that the normalisation uncertainty had been underestimated on average in the $\pi N$ experiments at low energy. Be that as it may, ‘outsiders’ cannot resolve such issues; any experimental shortcomings ought to be addressed (and resolved) by the members of the original experimental groups, a prospect which seems to be remote at present.

5.2 Residual EM contributions

The completeness of the EM corrections in the $\pi N$ interaction is an important issue which must be addressed with the seriousness it deserves. The way I understand this matter in the context of a Universe $\mathcal{A}$, in which only hadronic and EM phenomena are of relevance, may be summarised in one sentence. If complete, an EM correction to a value of a physical quantity, obtained in Universe $\mathcal{A}$, will translate it into the corresponding result in Universe $\mathcal{B}$, which is devoid of all EM effects. If my understanding/interpretation is correct, then one may pose an inevitable question: what happens to the particle ‘proton’ itself when the EM interaction is switched off? Without doubt, the rest mass of the proton receives EM contributions, which ought to be deducted in Universe $\mathcal{B}$. Therefore, the particle ‘proton’ of Universe $\mathcal{A}$ will have another mass in Universe $\mathcal{B}$ (and, of course, will be neutral). The same applies to all other (charged or composite) particles, namely to the ‘neutron’ and to the ‘pions’. If the strong interaction treats the members of isospin multiplets on an equal footing, then the hadronic masses of protons and neutrons must be identical. The same applies to the charged and neutral pions, which must share one
hadronic mass. Therefore, the four rest masses of Universe \( \mathcal{A} \) (proton, neutron, charged pion, and neutral pion) would reduce to two hadronic masses in Universe \( \mathcal{B} \), namely the hadronic mass of the nucleon and that of the pion. The former ought to be smaller than the rest mass of the proton, whereas the latter ought to be smaller than (or equal to?) the rest mass of the neutral pion. Some suggest that the hadronic mass of the pion should be taken to be the rest mass of the neutral pion. That would be a breakthrough (as one hadronic mass in the \( \pi N \) interaction would be fixed), but I believe that one can argue further, against this possibility, in that a neutral pion consists of \( q\bar{q} \) pairs. Evidently, as we descended one level into the structure of matter, another question is posed, while the original one remains unanswered: what are the EM contributions to the ‘physical’ masses of the quarks?
In Ref. [61], a clear distinction is made between stage-1 and stage-2 EM corrections. The stage-1 EM corrections should provide estimates for the so-called trivial EM effects (direct Coulomb amplitude, effects of the extended charge distributions of the interacting hadrons, vacuum polarisation, and all relevant interferences) and, in the case of $\pi^- p$ scattering, for the external mass differences and the effects of the $\gamma n$ channel. The three studies on the EM corrections, which were carried out at the University of Zurich in the late 1990s and the early 2000s, aimed at the removal of the stage-1 EM effects in low-energy scattering [50,51], as well as at the $\pi N$ threshold [48], in a consistent manner. On the other hand, the stage-2 EM corrections should go one step further: in addition to Feynman diagrams with loops and internal photon lines, they should also take care of the effects relating to the use of the physical rest masses of the particles in the stage-1 EM corrections, instead of the hadronic ones.

Needless to emphasise that the available EM-correction schemes in the scattering region [35,39,40,41,50,51] aim at the removal of the trivial effects relating to the EM interaction of the particles involved, but assume that no change is induced on the particles themselves as the result of the ‘switching-off’ of the EM interaction. The underlying assumption in all aforementioned studies is that the hadronic masses of the various particles are the corresponding physical ones. At this point, one basic question calls for a satisfactory answer: are the EM corrections supposed to also remove the EM contributions to the rest masses? If the answer to this question is ‘yes’, then all known EM-correction schemes are incomplete, and all analyses of the $\pi N$ data had been pleasant assignments, largely reminiscent of Tinguely’s contraptions. If the answer to the aforementioned question is ‘yes’, then the hadronic part of the $\pi N$ interaction is not sufficiently well known at present.

I believe that knowledge about the hadronic part of the $\pi N$ interaction will not be furthered by new experiments (even if they could be conducted somewhere), while the question of the EM corrections remains unresolved. Even for perfect data and no discrepancies whatsoever in the DB, one would still need to address the extraction of the important (hadronic) information from those perfect measurements. Important issues, which ought to be addressed in the reassessment of the EM effects in the $\pi N$ interaction, include the following.

- Clear definitions of what is meant by ‘EM part’ and by ‘hadronic part’ of the $\pi N$ interaction.
- Suitable methodology to deal with the hadrons after they (or their constituents, i.e., the quarks and antiquarks) have been deprived of their EM features.

\footnote{14 Jean Tinguely (1925-1991) was a Swiss sculptor, whose creations I always considered highly innovative and totally useless.}
Suitable methodology to deal with the altered kinematics, after the interacting hadrons have been deprived of the EM contributions to their rest masses.

To summarise, an advancement of knowledge in Pion Physics could only be instigated by a theoretical breakthrough, in particular in relation to the reliable removal of the EM effects.

In the subsequent section, I will demonstrate that, due to the sizeably different results of the various EM-correction schemes (at present), even the extraction of the values of the $\pi N$ scattering lengths from the accurate PSI measurements at the $\pi N$ threshold is not as precise as it should have been. And, if the evaluation of the EM effects at the $\pi N$ threshold is controversial to the extent that the hadronic $\pi N$ scattering lengths are not sufficiently well known, there can be no doubt that the $\pi N$ phase shifts are even less 'well-known'. In my opinion, prior to the development of advanced methods to extrapolate the $\pi N$ scattering amplitudes to the unphysical region, the theoretical innovation should be directed to the physical one.

5.2.1 How does the possibility of residual EM effects gain momentum in the light of the reported corrections at the $\pi N$ threshold?

The first of the Deser formulae relates the strong shift $\epsilon_{1s}$ in pionic hydrogen with the 'untreated' (i.e., containing effects of EM origin) scattering length $a_{cc}$. Following the sign convention of Ref. [7],

$$\epsilon_{1s} := E_{np \rightarrow 1s}^{\text{EM}} - E_{np \rightarrow 1s}^{\text{measured}} = -4 \left| \frac{E_{1s}}{r_B} \right| a_{cc} = -\frac{2\alpha^3\mu^2}{\hbar c} a_{cc}, \quad (7)$$

where $E_{1s} = -\alpha^2 \mu/2$ is the (point-Coulomb) EM binding energy of the $1s$ level and $r_B = \hbar c/(\alpha \mu)$ is the Bohr radius; $\alpha$ denotes the fine-structure constant and $\mu$ stands for the reduced mass of the $\pi^- p$ system.

The second of the Deser formulae, put into its current form by Trueman [6], enables the extraction of the scattering length $a_{c0}$ from the total decay width $\Gamma_{1s}$ in pionic hydrogen.

$$\Gamma_{1s} = 8 q_0 \frac{|E_{1s}|}{r_B \hbar c} \left( 1 + P^{-1} \right) a_{c0}^2 = 4 q_0 \frac{\alpha^3 \mu^2}{(\hbar c)^2} \left( 1 + P^{-1} \right) a_{c0}^2, \quad (8)$$

where $q_0$ denotes the magnitude of the CM 3-momentum of the outgoing $\pi^0$ (or neutron) at the $\pi^- p$ threshold and $P = 1.546(9)$ [62] is the Panofsky ratio, the ratio between the CX and the radiative-capture partial decay widths:

$$P := \Gamma_{1s}(\pi^- p \to \pi^0 n)/\Gamma_{1s}(\pi^- p \to \gamma n).$$

Corrections must be applied, to rid $a_{cc}$ and $a_{c0}$ of all effects of EM origin,
and lead to estimates for the hadronic scattering lengths $\tilde{a}_{cc}$ and $\tilde{a}_{c0}$. The EM corrections are usually expressed in the form of two quantities, $\delta_\epsilon$ for $a_{cc}$ and $\delta_\Gamma$ for $a_{c0}$. The two hadronic scattering lengths are obtained from $a_{cc}$ and $a_{c0}$ according to the following definitions.

\begin{align}
\tilde{a}_{cc} &= a_{cc}/(1 + \delta_\epsilon) \quad (9) \\
\tilde{a}_{c0} &= a_{c0}/(1 + \delta_\Gamma) \quad (10)
\end{align}

It is understood that the scattering lengths $a_{cc}$ and $a_{c0}$ in Eqs. (9,10) are associated with the original Deser formulae (7,8). These formulae represent leading-order (LO) evaluations of $\epsilon_{1s}$ and $\Gamma_{1s}$, namely determinations at $O(\alpha^3)$. In several works, the quantities $a_{cc}$ and $a_{c0}$ of Eqs. (7,8) are therefore denoted as $a_{cc}^{LO}$ and $a_{c0}^{LO}$, as opposed to the quantities entering the upgraded forms of the Deser formulae, i.e., the expressions obtained at higher orders of $\alpha$. At present, only the next-to-leading-order (NLO) evaluations of $\epsilon_{1s}$ and $\Gamma_{1s}$ are available, i.e., the evaluations at $O(\alpha^4)$. Some authors denote the scattering lengths, entering the NLO evaluations of $\epsilon_{1s}$ and $\Gamma_{1s}$, as $a_{cc}^{NLO}$ and $a_{c0}^{NLO}$. In this work, $a_{cc}$ and $a_{c0}$ will represent $a_{cc}^{LO}$ and $a_{c0}^{LO}$, respectively. I will introduce $A_{cc}$ and $A_{c0}$ later on, when referring to $a_{cc}^{NLO}$ and $a_{c0}^{NLO}$, respectively.

Several schemes of removal of the EM effects from the measurements were developed after the first experimental results at the $\pi N$ threshold became available; some of these schemes aim at the removal of the trivial EM effects, i.e., of those comprising stage-1 EM corrections. The models of Sections 5.2.1.1 and 5.2.1.2 are supposed to remove only these effects. On the contrary, works carried out within the $\chi$PT framework also attempt the removal of effects which are associated with the mass difference of the two light quarks ($u$ and $d$). The models of Sections 5.2.1.3-5.2.1.6 belong to this category. When comparing the results of the various schemes, one ought to bear in mind the distinction between these two categories of corrections.

5.2.1.1 Potential models for the removal of the EM effects

In their assessment of the EM effects at the $\pi N$ threshold, Refs. [47,48] made use of suitable hadronic potentials \cite{15}. An estimate for $\delta_\epsilon$ was obtained in Ref. [47] by means of a two-channel calculation, along with the phenomenological inclusion of the effects of the $\gamma n$ channel: $\delta_\epsilon = -2.1(5) \cdot 10^{-2}$. Using a genuine three-channel calculation, Ref. [48] obtained an incompatible result: $\delta_\epsilon = 0.67(67) \cdot 10^{-2}$. The difference in the $\delta_\epsilon$ results between the two studies is due to the different treatment of the effects of the $\gamma n$ channel.

\cite{15} Regrettably, I failed to mention the 1987 paper of Kaufmann and Gibbs [63] in the first version of this preprint. Comments on that work may be found in Refs. [47,48].
In Ref. [47], the estimate for $\delta_\Gamma$ of $-1.3(5) \cdot 10^{-2}$ had been extracted. On the other hand, Ref. [48] obtained: $\delta_\Gamma = -1.66(33) \cdot 10^{-2}$. Therefore, the two $\Gamma_1$s corrections [47,48] agree within the uncertainties; evidently, the correction $\delta_\Gamma$ is less sensitive to the way by which the effects of the $\gamma n$ channel are included in the calculation.

5.2.1.2 The model of Ericson and collaborators [64]

In 2004, Ericson and collaborators [64] followed a non-relativistic approach using Coulomb wavefunctions, with a short-range strong interaction and extended charge distributions, and treated four sources of EM corrections. The first two take account of effects relating to the vacuum polarisation and the extended charge distributions of the interacting particles. The remaining two corrections relate to renormalisation and gauge effects: the former takes account of the continuity and smoothness of the wavefunction on the spherical boundary which delimits the application of the strong interaction, whereas the latter correction is due to the adjustment of the energy level in such a way that it corresponds to the scattering off an extended charge Coulomb potential close to the origin.

The estimates of Ref. [64] for the corrections $\delta_\epsilon$ and $\delta_\Gamma$ may be found in their Table 1: $\delta_\epsilon = -0.62(29) \cdot 10^{-2}$ and $\delta_\Gamma = 1.02(23) \cdot 10^{-2}$. The discrepancy in $\delta_\Gamma$ between the results, obtained with the potential models of Section 5.2.1.1 and with the model of this part, is noticeable. The correction $\delta_\epsilon$ of Ref. [64] lies in-between the results of the two potential models of the previous section, slightly closer to the estimate of Ref. [48].

The subject of the corrections $\delta_\epsilon$ and $\delta_\Gamma$ within the model of Ericson and collaborators [64] was recently revisited [65], using the results of fits to a subset of the currently-available low-energy $\pi N$ measurements, rather than importing information from the outdated Karlsruhe analyses [19] (which Ref. [64] had actually done). As Ref. [65] has not been subjected to the peer-review process, I decided to quote the original $\delta_\epsilon$ and $\delta_\Gamma$ estimates [64] in Table 3 and to use the same estimates for the purposes of Figs. 8-11.

Comments on the approach of Ref. [64] may be found in Section 4 of Ref. [48]; there is no point in repeating them here. It must be borne in mind that Ericson and collaborators [64] had also expressed criticism about the models of Section 5.2.1.1, on the basis of the inconsistency of the coupled-channel formalism with the low-energy expansion of the $K$-matrix elements, see also Ref. [66].

5.2.1.3 The Lyubovitskij-Rusetsky correction to $a_{cc}$ [67]

I consider the 2000 paper of Lyubovitskij and Rusetsky [67] important for two reasons.
The authors presented a calculation of the strong shift $\epsilon_{1s}$ at $\mathcal{O}(\alpha^4)$; this is an essential upgrade of Eq. (7). Part of the effects, which ought to be taken care of by the EM corrections in case that Eq. (7) is used, are contained in the upgraded expression.

Their work constituted the first attempt to determine the isospin-breaking corrections, which are due to the mass difference of the two light quarks, within the $\chi$PT framework.

The relation at $\mathcal{O}(\alpha^4)$ between $\epsilon_{1s}$ and the (untreated) $\pi^- p$ ES length (denoted as $A$ in Ref. [67], $A_{cc}$ in this work) reads as:

$$\epsilon_{1s} = -\frac{2\alpha^3\mu^2}{hc}A_{cc}\left(1 + 2\alpha(1 - \ln \alpha)\frac{\mu A_{cc}}{hc}\right),$$

(11)

which, after appending the effects of the vacuum-polarisation correction to the ground-state wavefunction $\varphi \approx 0.483 \cdot 10^{-2}$ of Ref. [68] (these effects had not been included in Ref. [67]), may be rewritten as

$$\epsilon_{1s} = -\frac{2\alpha^3\mu^2}{hc}A_{cc}\left(1 + \varphi + 2\alpha(1 - \ln \alpha)\frac{\mu A_{cc}}{hc}\right).$$

(12)

Lyubovitskij and Rusetsky did not identify $A_{cc}$ with $\tilde{a}_{cc}$. Additional corrections (denoted as $\epsilon$ in Ref. [67], $\Delta A_{cc}$ in this work), to be understood as contributions originating from residual EM effects and from the mass difference of the two light quarks, were evaluated in Ref. [67] at $\mathcal{O}(p^2)$ in $\chi$PT. The relation between the quantities $A_{cc}$ and $\tilde{a}_{cc}$ was given in Ref. [67] as:

$$\tilde{a}_{cc} = A_{cc} - \Delta A_{cc}.$$

The correction $\Delta A_{cc}$ depends on three low-energy constants (LECs) $c_1$, $f_1$, and $f_2$, one of which ($f_1$) is poorly known. According to Ref. [67]:

$$\Delta A_{cc} = \frac{m_p h c}{2(m_p + m_c)} \left(\frac{2(m_c^2 - m_0^2)}{\pi F_\pi^2} c_1 - \alpha (4f_1 + f_2)\right),$$

(13)

where $m_0$, $m_c$, and $m_p$ are the rest masses of the neutral pion, of the charged pion, and of the proton, respectively; $F_\pi = 92.07(85)$ MeV is the pion-decay constant, see Eqs. (71.13,71.14) in the chapter ‘Leptonic Decays of Charged Pseudoscalar Mesons’ of Ref. [2] (their $f_\pi$ is equal to $\sqrt{2}F_\pi$).

For $c_1$, Lyubovitskij and Rusetsky used a result from the Karlsruhe programme of the mid 1980s [19], privately communicated to the authors: that value was equal to $-0.925 \text{ GeV}^{-1}$ (no uncertainty was quoted in Ref. [67]). Also using input from the same source, Gasser and collaborators [69] came up (in 2002) with $c_1 = -0.93(7) \text{ GeV}^{-1}$. In the same year, Lyubovitskij and collaborators [70] imported (from a work of 2001) a different $c_1$ value, namely $c_1 = -1.2(1) \text{ GeV}^{-1}$. More recent works [71,72] recommend: $c_1 = -1.0(3) \text{ GeV}^{-1}$. 

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Regarding the LEC $f_2$, Ref. [67] used $f_2 = -0.97(38) \text{ GeV}^{-1}$, which is the recommended value in Ref. [69].

As aforementioned, the LEC $f_1$ is poorly known. To obtain an estimate for the correction $\delta_\epsilon$, Lyubovitskij and Rusetsky assumed in Ref. [67] that $|f_1| \leq |f_2|$. However, Lyubovitskij and collaborators [70] arrived in 2002 at a mismatching result for the ratio $f_1/f_2$, namely $2.24(26)$. The authors favoured $f_1 = -2.29(19) \text{ GeV}^{-1}$, which does not seem to be in line with the other ‘expectations’ for this LEC. Gasser and collaborators [69] mention their “order of magnitude” estimate for $|f_1|$ at about $1.4 \text{ GeV}^{-1}$.

Be that as it may, Ref. [67] reported a ‘large’ negative correction: $\delta_\epsilon = (-4.8 \pm 2.0) \cdot 10^{-2}$. I set out to re-evaluate the correction $\delta_\epsilon$ in the Lyubovitskij-Rusetsky scheme, using Eq. (12), rather than Eq. (11) which the authors had used. As Ref. [67] mentions no uncertainty in the LEC $c_1$, I first assumed that $c_1$ was not varied in their analysis. However, the resulting uncertainty of $\delta_\epsilon$ turned out to be nearly a factor of 2 smaller than the one quoted in Ref. [67]. Therefore, I concluded that also $c_1$ was varied in Ref. [67] and proceeded by changing the assigned $c_1$ uncertainty, until the final result for $\delta_\epsilon$ matched the reported $\delta_\epsilon$ uncertainty of Ref. [67]. My conclusion is that Lyubovitskij and Rusetsky had most likely used a $\delta c_1$ value between 0.2 and 0.3 GeV$^{-1}$ in their work. In any case, the $\delta c_1$ value of 0.3 GeV$^{-1}$, also recommended in Refs. [71,72], appears to be reasonable and conservative. For the needs of Table 3, I obtain the correction $\delta_\epsilon$ using Eq. (12) with the $\varphi$ value of Ref. [68] and $\delta c_1 = 0.3 \text{ GeV}^{-1}$. The other two LECs are varied according to Ref. [67].

As the models of Sections 5.2.1.1 and 5.2.1.2 do not contain any stage-2 EM corrections, the comparison of their $\delta_\epsilon$ values with the result of this part makes little sense. On the other hand, one could pose the question whether a comparison could be meaningful if $\tilde{a}_{cc}$ were identified with the scattering length $A_{cc}$, obtained from Eq. (12). There is no doubt that some of the effects, which are treated by the models of Sections 5.2.1.1 and 5.2.1.2, are contained in $\Delta A_{cc}$ of Eq. (13). Unfortunately, it is not clear to me how to disentangle the EM contributions and those relating to the $m_u \neq m_d$ effects in Eq. (13). Therefore, it seems that there is no assurance that a comparison of the results of this part with those obtained with the models of Sections 5.2.1.1 and 5.2.1.2 is meaningful. Nevertheless, I will also obtain a $\delta_\epsilon$ value corresponding to the case that $A_{cc}$ of Eq. (12) is identified with $\tilde{a}_{cc}$. This intermediate result could be helpful later on, in assessing the importance of the isospin-breaking effects at $O(p^2)$.

In 2002, Lyubovitskij and collaborators [70] provided an update of $\delta_\epsilon$, claiming improved knowledge of $f_1$ after deploying their “perturbative chiral quark model;” the new value was $\delta_\epsilon = -2.8 \cdot 10^{-2}$, quoted in Ref. [70] without an uncertainty. Evidently, the updated value of Ref. [70] is not incompatible with the 1996 result extracted with the potential model of Ref. [47].
5.2.1.4 Isospin-breaking corrections evaluated at $O(p^3)$ in $\chi PT$ [69]

An even larger (and more uncertain) correction $\delta_\epsilon$ was extracted in 2002 [69] within a calculation at NLO ($O(p^3)$) in isospin breaking and in the low-energy expansion: $(-7.2 \pm 2.9) \cdot 10^{-2}$.

5.2.1.5 Leading-order correction $\delta_\Gamma$ in $\chi PT$ [73]

The LO correction to $a_{c0}$, obtained in 2004 within the $\chi PT$ framework [73], was found small: $\delta_\Gamma = 0.6(2) \cdot 10^{-2}$, see Eq. (5.26) therein. One notices that, within the $\chi PT$ framework, the correction $\delta_\Gamma$ is more precisely known than $\delta_\epsilon$. This is due to the fact that the LECs $c_1$ and $f_1$ do not enter the determination of $\delta_\Gamma$, e.g., see Eq. (16).

5.2.1.6 The corrections developed by the Bonn-Jülich group

Another correction scheme for the pionic-hydrogen measurements, along the lines of those detailed in the last three sections, was developed between 2005 and 2011, see Refs. [71,72] and the relevant papers therein. In addition, corrections for the strong shift of the $1s$ state in pionic deuterium were advanced.

Regarding the strong shift $\epsilon_{1s}$ in pionic hydrogen, Ref. [71] uses Eq. (12) to extract $A_{cc}$, which the authors call $a_{\pi^-p}$ in their paper. They subsequently associate $A_{cc}$ with the difference $b_0 - \tilde{b}_1$.

$$b_0 - \tilde{b}_1 = A_{cc} - \Delta a_{cc} \hbar c ,$$

where the isoscalar scattering length is to be thought of as untreated \(^{16}\) (as the lack of the tilde over it indicates) and $\Delta a_{cc} = (-2.0 \pm 1.3) \cdot 10^{-3} m^{-1}$.

For the relation between $\Gamma_{1s}$ of pionic hydrogen and the corresponding scattering length $A_{c0}$, the authors used the expression:

$$\Gamma_{1s} = 4q_0^2 \frac{\alpha^3 \mu^2}{(\hbar c)^2} \left(1 + P^{-1}\right) A_{c0}^2 \left(1 + \varphi + 4\alpha (1 - \ln \alpha) \frac{\mu A_{cc}}{\hbar c} + 2(m_p + m_c - m_n - m_0) \frac{\mu_0^2}{(\hbar c)^2} \right) ,$$

where $m_n$ is the mass of the neutron and $A_{c0} = \sqrt{2}b_1 + \Delta a_{c0} \hbar c$, with $\Delta a_{c0} = 0.4(9) \cdot 10^{-3} m^{-1}$. Equations (14,15) contain two unknowns: $b_0$ and $\tilde{b}_1$. The

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\(^{16}\) The untreated isoscalar scattering length $b_0$ also enters the strong shift $\epsilon_{1s}$ in pionic deuterium. A combined analysis of the $\epsilon_{1s}$ values in pionic hydrogen and deuterium, and of $\Gamma_{1s}$ in pionic hydrogen enables a more accurate determination of the two scattering lengths (in comparison with the use of the information extracted only from pionic hydrogen), see Ref. [7,11,12].
quantity $\tilde{b}_1$ may be evaluated by use of a simple recurrence relation; the convergence is fast. The quantity $b_0$ is subsequently obtained via Eq. (14). Evident from Refs. [71,72] is that the isospin-breaking effects have a larger impact on the isoscalar part of the $\pi N$ interaction at the $\pi N$ threshold. For this correction, the authors give the expression:

$$\tilde{b}_0 = b_0 - \frac{m_p \hbar c}{m_p + m_c} \left( \frac{m_c^2 - m_0^2}{\pi F_\pi^2} c_1 - 2\alpha f_1 \right),$$

where the values and uncertainties of the LECs $c_1$ and $f_1$, used in Refs. [71,72], have already been given in Section 5.2.1.3. Comparison with Eq. (13) implies that the correction to $b_1$ reads as

$$b_1 - \tilde{b}_1 = \frac{m_p \hbar c}{m_p + m_c} \frac{\alpha f_2}{2},$$

and comes out equal to $-0.43(17) \cdot 10^{-3} m_c^{-1}$. Presumably, this correction is contained in $\Delta a_{cc}$ of Eq. (14). The corrected $\tilde{a}_{cc}$ may then be obtained as the difference $\tilde{b}_0 - \tilde{b}_1$, whereas $\tilde{a}_{c0} = \sqrt{2} \tilde{b}_1$.

### 5.2.2 Comparison of results and last remarks on the corrections at the $\pi N$ threshold

The important results of the application of the aforementioned correction schemes to $\epsilon_{1s}$ and $\Gamma_{1s}$, so that the two hadronic scattering lengths be extracted, are listed in Table 3. To enable a comparison with the results obtained with the models of Sections 5.2.1.1 and 5.2.1.2, and provide an impression of the largeness of the $O(p^2)$ corrections in Ref. [67], a $\delta_\epsilon$ result was extracted after identifying $A_{cc}$ with $\tilde{a}_{cc}$ or, equivalently, after ignoring the correction $\Delta A_{cc}$ of Eq. (13). The difference between the corrections $\delta_\epsilon$ of Ref. [67] and the corresponding value of Table 3 is accounted for by the use of Eq. (12) herein, as opposed to Eq. (11) in Ref. [67].

The values of the two scattering lengths $\tilde{a}_{cc}$ and $\tilde{a}_{c0}$, corrected for the EM effects using the methods of Sections 5.2.1.1 and 5.2.1.2, and for the EM effects and effects of hadronic origin of Sections 5.2.1.3-5.2.1.6 are given in Figs. 8 and 9, respectively. Shown in Figs. 10 and 11 are the corresponding results for the isoscalar $\tilde{b}_0$ and the isovector $\tilde{b}_1$ scattering lengths, respectively. Inspection of these plots provides an impression as to the uncertainties in the original experimental data and the ones which the various correction schemes introduce: the uncertainties of the quantities $\alpha$ and $\mu$ in the two Deser formulae of Eqs. (7,8) are tiny, whereas the relative uncertainty of $P$ in the second of these equations is close to 6 per-mille. In spite of the large uncertainty, noticeable is the change in the isoscalar part of the $\pi N$ interaction when
Table 3

The corrections to the measurements of the strong shift $\epsilon_{1s}$ and the total decay width $\Gamma_{1s}$ in pionic hydrogen. The input, common in all cases, comprises the average results for $\epsilon_{1s}$ and $\Gamma_{1s}$ of the two PSI experiments, with no uncertainties (statistical or systematic). All corrections are expressed in the form of $\delta_{\epsilon}$ and $\delta_{\Gamma}$, see Eqs. (9,10). A $\delta_{\epsilon}$ result was also obtained (last row in the upper part of the table) after identifying $A_{cc}$ of Eq. (12) with $\tilde{a}_{cc}$.

| Source | $\delta_{\epsilon}$ ($10^{-2}$) | $\delta_{\Gamma}$ ($10^{-2}$) |
|--------|-----------------|-----------------|
| Methods aiming at the removal of the trivial EM effects |
| [47]   | $-2.1(5)$       | $-1.3(5)$       |
| [48]   | $0.67(67)$      | $-1.66(33)$     |
| [64]   | $-0.62(29)$     | $1.02(23)$      |
| [67], $\tilde{a}_{cc} \equiv A_{cc}$ | 1.12            |
| Methods aiming at the removal of EM effects, as well as of effects due to $m_u \neq m_d$ |
| [67]   | $-4.3 \pm 2.2$  | $-$             |
| [69]   | $-7.2 \pm 2.9$  | $-$             |
| [73]   | $-$             | $0.6(2)$        |
| [71,72] | $-7.2 \pm 2.6$  | $0.56(72)$      |

applying to the pionic-hydrogen measurements the results of the method of Section 5.2.1.6.

Inspection of Table 3 leads to the following conclusions.

- A consistent picture for the corrections $\delta_{\epsilon}$ and $\delta_{\Gamma}$ does not emerge from the upper part of this table (Methods aiming at the removal of the trivial EM effects).
- One notices that the uncertainty of the $\delta_{\epsilon}$ correction of Ref. [64] is sizeably smaller than those obtained with all other methods; this is due to the fact that only their gauge term is accompanied by an appreciable uncertainty. To obtain an estimate for that contribution, Ref. [64] had imported information from the Karlsruhe programme of the 1980s [19], which is - to a great extent - outdated by today’s standards; current information suggests that the values of the two range parameters, entering the contributions of the gauge term to $\delta_{\epsilon}$ and $\delta_{\Gamma}$, are (first) different and (second) better known [65], in comparison with the values used in Ref. [64]. As a result, the updated $\delta_{\epsilon}$ and $\delta_{\Gamma}$ values come out different now and are (even) better known; in my judgment, the uncertainties of Ref. [64], as well as those of the updated corrections [65],...
Fig. 8. The scattering length $\tilde{a}_{cc}$, obtained after the application of the correction methods of Sections 5.2.1.1-5.2.1.4 and 5.2.1.6 to the experimental results of the two PSI experiments. The last three correction schemes on the right take account of effects which go beyond the scope of the other three correction methods. Filled circles: Ref. [7]; open circles: Ref. [8]. To err on the side of caution, the statistical and systematic uncertainties of the experimental results were linearly combined for both experiments.

are model-dependent and overly optimistic.

- One may argue that the genuine three-channel calculation of Ref. [48] constitutes an improvement over the approach of Ref. [47], and thus proceed to compare the $\delta_\epsilon$ and $\delta_\Gamma$ results of Ref. [48] with those obtained with the only other approach which does not deploy $\chi$PT, namely Ref. [64]. Obviously, there is no matching; the signs are opposite in both corrections $\delta_\epsilon$ and $\delta_\Gamma$. Moreover, the discrepancy between the two corrections $\delta_\Gamma$ is disturbing.

- The correction $\delta_\epsilon$ of Ref. [48] appears to be compatible with the result obtained from the upgraded form of the Deser formula for $\epsilon_{1s}$ (see Eq. (12)), whereas the corresponding result of Ref. [64] is not. Nevertheless, it is not clear that such a comparison is meaningful. Part of the EM corrections of Refs. [48,64] are contained in the upgraded form of the Deser formula for $\epsilon_{1s}$; another part is contained in the correction $\Delta A_{cc}$; a third part is not
Fig. 9. The scattering length $\tilde{a}_{c0}$, obtained after the application of the correction methods of Sections 5.2.1.1, 5.2.1.2, 5.2.1.5, and 5.2.1.6 to the experimental results of the two PSI experiments. The last two correction schemes on the right take account of effects which go beyond the scope of the other three correction methods. Filled circles: Ref. [7]; open circles: Ref. [9]. In both cases, the statistical and systematic uncertainties of the experimental results were linearly combined.

contained in the correction $\Delta A_{cc}$. Therefore, the compatibility between the correction $\delta e$ of Ref. [48] with the result obtained from the upgraded form of the Deser formula for $\epsilon_{1s}$ may be coincidental.

- The correction $\bar{\delta}_T$ extracted in Ref. [64] is not incompatible with the two estimates obtained within the $\chi$PT framework in Refs. [71,72,73]; the same goes for the updated result of Ref. [65], which (in fact) is in perfect agreement with the estimates of Refs. [71,72,73]. It has been suggested that potential models have a tendency to yield negative corrections $\delta_T$. Given the outcome of Refs. [47,48], there might be some truth in this supposition.

- It is time I discussed the corrections $\tilde{\delta}e$ obtained within the $\chi$PT framework. The corrections $\tilde{\delta}e$ of Refs. [67,69,71,72] are large and, even worse, poorly known. The sizeable uncertainties are mostly attributable to the poor knowledge of the LEC $f_1$. The difference between Refs. [67,69] is that, in the former work, the additional isospin-breaking effects are treated at
Fig. 10. The isoscalar scattering length $\tilde{b}_0 \equiv \tilde{a}_{cc} + \tilde{a}_{c0}/\sqrt{2}$, obtained after the application of the correction methods of Sections 5.2.1.1, 5.2.1.2, and 5.2.1.6 to the experimental results of the two PSI experiments. The last correction scheme on the right take account of effects which go beyond the scope of the other correction methods. Filled circles: Ref. [7]; open circles: Refs. [8,9]. In both cases, the statistical and systematic uncertainties of the experimental results were linearly combined.

$\mathcal{O}(p^2)$; in Ref. [69], they are treated at $\mathcal{O}(p^3)$. If, as the result of the application of the correction $\Delta A_{cc}$ of Eq. (13), $\delta_\epsilon$ changes by as much as $-5.4\%$ (i.e., from +1.1% to −4.3%) and the result of the correction at the next order brings another −2.9%, then I do wonder what the calculation at $\mathcal{O}(p^4)$ might bring. I see no signs of convergence.

• The evaluation of the corrections $\delta_\epsilon$ within the $\chi$PT framework (Sections 5.2.1.3, 5.2.1.4, and 5.2.1.6) also involves the mass difference of the two light quarks. Apart from the effects, which in the language of Refs. [47,48] constitute ‘mass differences of the particles in the initial and final states’, such procedures remove contributions, which should be categorised herein under Section 5.3. Therefore, it is not surprising that the corrections $\delta_\epsilon$ of Refs. [67,69,71,72] come out sizeably larger than those of Refs. [47,48,64]. One remark is in order. The deduction of the EM contributions from the light-quark masses is not addressed in Refs. [67,69,71,72].
Fig. 11. The isovector scattering length $\tilde{b}_1 \equiv \tilde{a}_{c0}/\sqrt{2}$, obtained after the application of the correction methods of Sections 5.2.1.1, 5.2.1.2, 5.2.1.5, and 5.2.1.6 to the experimental results of the two PSI experiments. The last two correction schemes on the right take account of effects which go beyond the scope of the other three correction methods. Filled circles: Ref. [7]; open circles: Refs. [8,9]. In both cases, the statistical and systematic uncertainties of the experimental results were linearly combined.

- The only positive conclusion from the inspection of Table 3 is the overall agreement of Refs. [64,71,72,73] regarding the magnitude of the correction $\delta_1$: they all suggest that this correction does not exceed $\approx 1 \%$. This implies that the isovector part of the $\pi N$ scattering amplitude is better known (in comparison with the isoscalar part).

At this point, I cannot resist one comment. Between 1990 and 1995, I had heard at least four prominent theorists lamenting the lack of precise experimental information at the $\pi N$ threshold. The ETHZ-Neuchâtel-PSI Collaboration delivered $\epsilon_{1\sigma}$ with an uncertainty well below $1 \%$, whereas both statistical and systematic uncertainties, reported by the Pionic-Hydrogen Collaboration, were at the level of $0.1 \%$. Such accuracy is unprecedented in Pion Physics. After this precise information became available, the theorists discovered that
no competitive, up-to-date EM-correction scheme had been developed\(^{17}\) to enable the extraction of the important (hadronic) information from the experimental results. If the best Theory can do is to provide corrections at the \(\pi N\) threshold which are one order of magnitude less precise than the experimental results, then my opinion remains that Theory must find a way of ‘narrowing the gap’.

To summarise, the agreement of the results of the available schemes of removal of the EM effects at the \(\pi N\) threshold is poor at present.

- The overall incompatibility of the results of the various schemes of removal of the trivial EM effects at the \(\pi N\) threshold (upper part of Table 3), ought to be explained.
- Regarding the corrections, which have been developed within the \(\chi PT\) framework, a twofold improvement would be welcomed: to assess the convergence, the isospin-breaking corrections ought to be obtained at order \(\mathcal{O}(p^3)\); they must also become more precise (after proposing ways, which are within the bounds of possibility, to narrow down the uncertainty of the LEC \(f_1\)). In this context, it ought to be understood that the determination of accurate corrections at the \(\pi N\) threshold is undoubtedly a good cause, but not the only one; the data analysis calls for the development of corresponding corrections in the scattering region.

In one phrase: a unified scheme for the development of reliable EM corrections, \textit{applicable at the \(\pi N\) threshold and in the scattering region}, seems (to me) to be the only step forwards. Without doubt, we must first decide what effects should be removed from the experimental data (or from quantities emerging thereof).

\(5.3\) Violation of the isospin invariance

The discrepancies, which the analyses of the low-energy \(\pi N\) data have established [1,32,38], may be taken to suggest that the isospin invariance is broken in the \(\pi N\) interaction at a level exceeding the \(\chi PT\) expectations [74]. I left this option for the end as, admittedly, it is the most compelling one in Physics terms. Although the conclusions of the studies [32,38] had not been received with boundless enthusiasm, one is tempted to raise the question: ‘Why should the isospin invariance hold in the first place?’ After all, the bare masses of the \(u\) and \(d\) quarks \textit{are} different. It seems, therefore, that the right question to ask is not whether the isospin invariance is broken in the \(\pi N\) interaction, but at which level it is.

\(^{17}\)In fact, the lack of such a scheme was the motivation for Sigg and collaborators to set out to examine the EM effects in pionic hydrogen in Ref. [47].
Piekarewicz has suggested [75] that the isospin breaking in the NN interaction can “originate from: (i) isovector-isoscalar mixing in the meson propagator - such as the $\rho^0 - \omega$ mixing; (ii) isospin-breaking in the nucleon wavefunction - through the neutron-proton mass difference; and (iii) isospin-breaking in the meson-nucleon and photon-nucleon vertices - as in electromagnetic scattering.” Piekarewicz added that the aforementioned isospin-breaking mechanisms (ii) and (iii) “also operate in the $\pi N$ system.” A few remarks are in order. As Piekarewicz suggested, the $\rho^0 - \omega$ mixing does not seem to play a role in $\pi N$ ES [58], yet another isovector-isoscalar mixing, the one involving the QM admixture of the $\pi^0$ and the $\eta$ meson, a state with quantum properties $I^G(J^{PC}) = 0^+(0^{-+})$ and a rest mass of 547.862(17) MeV [2], has long been known as potentially affecting the $\pi^-p$ CX reaction [59]. In the language of the models based on hadronic exchanges, one would categorise the sources of isospin-breaking effects in the $\pi N$ interaction as follows (in order of increasing importance, in my opinion):

- mass differences in Feynman diagrams involving different members of the isospin multiplets;
- splitting effects in the various coupling constants and, presumably, vertex factors; and
- Feynman diagrams containing a $\pi^0 - \eta$ transition vertex (see Fig. 6). Regarding the $\pi^0 - \eta$ mixing mechanism, it would be interesting to also have an estimate for the potential difference between the coupling constants $g_{\eta pp}$ and $g_{\eta nn}$ (though this might take some time).

Some authors call the effects of the first type ‘static’ or (less frequently) ‘kinematical’, whereas those of the second are usually described as ‘dynamical’. The effects due to the $\pi^0 - \eta$ transition should be categorised as ‘dynamical’.

Already in 1995, Piekarewicz [75] put forward an explanation for the surprising result of Ref. [32]; using a non-relativistic constituent-quark model, he evaluated the changes in the $\pi N$ coupling constant due to the mass difference of the $u$ and $d$ quarks. Regarding the discrepancy $D = \Re [f_{\text{extr}}^{CX} - f_{\text{CX}}]$ for the $s$ wave, which had come out equal to $-0.012(3)$ fm in Ref. [32], Piekarewicz obtained in his paper the relation:

$$D = -\sqrt{2} \hbar c \frac{g_{\pi NN}^2}{4\pi} \frac{g_{\pi}^\pi}{M} \left(1 + \frac{m_c}{M}\right)^{-1} \left(1 - \frac{m_c^2}{4M^2}\right)^{-1}, \quad (17)$$

where $g_{\pi NN}$ stands for the isospin-conserving coupling constant (for which $g_{\pi NN}^2/(4\pi) = 14.21$ was used in Ref. [75]) and $M$ is the nucleon mass (average of the proton and neutron masses); $g_{\pi}^\pi$ denotes the isospin-violating component in the $\pi^0 NN$ coupling constant, for which Piekarewicz obtained the relation

$$g_{\pi}^\pi = 0.3 \Delta m \approx 0.004,$$
where $m$ and $\Delta m$ denote the average of the constituent masses of the two light quarks ($m = M/3$) and their difference, respectively. In his evaluation, he chose $\Delta m = 4.1$ MeV, which actually coincides with the central value appearing in a peculiar inequality $m_d - m_u > (4.1 \pm 0.3)$ MeV [76], and obtained $D \approx -0.0145$ fm, i.e., a value compatible with the result of Ref. [32]. Fixing $\Delta m$ to 4 MeV from Ref. [77], see Table 4.4 therein (p. 135), and using the average $f_2^c$ value of Ref. [10], one extracts from Eq. (17) about the same discrepancy which Piekarewicz had obtained in 1995 ($D \approx -0.0138$ fm).

The splitting effects in the $\pi N$ coupling constant were also studied two years later [78]: the authors suggested that $g_{\pi^0 \rho \pi}$ should be equal to the average of the two $g_{\pi^0 NN}$ values and provided an estimate for the splitting $(g_{\pi^0 pp} - g_{\pi^0 nn})/g_{\pi^0 \rho \pi}$. A short review on the status of the determination of the various $\pi N$ coupling constants may be found in Ref. [10]. The result of Ref. [78] implies that $f_p > f_n$ or, equivalently, that $f_p^2 > f_n^2$; the odds in favour of the opposite inequality come out to be about $2 : 1$ from Ref. [10] and about $3.5 : 1$ from Ref. [79]. In 2017, two studies [80,81] also reported sizeable (however contradicting) splitting effects in the $\pi N$ coupling constant.

It is interesting to note that the starting point in both studies [75,78] is the difference between the masses of the two light quarks (Ref. [75] makes use of the constituent masses, Ref. [78] of the bare ones). The authors then proceed to evaluate the splitting effect which that difference would imply for the $\pi N$ coupling constant. Provided that I interpret the quantity $g_{\pi^0}^\rho$ of Ref. [75] correctly, the resulting effects differ sizeably between the two studies: the former work came up with a splitting effect of the order of 0.4 %, whereas the latter one reported effects between 1.2 and 3.7 %. The former work concluded that the reported discrepancy $D$ of Ref. [32] could be accounted for, whereas the latter study did not examine that issue. I find it surprising that Ref. [75] could account for the discrepancy reported in Ref. [32] on the basis of a small splitting effect in the $\pi N$ coupling constant and without needing to invoke the $\pi^0 - \eta$ mixing mechanism.

Let me finally provide a synopsis of the four (thus far) $\chi$PT attempts to identify the source of the discrepancy reported in Refs. [32,38]: in none of these four papers did the authors refer to the previous work of Ref. [75], which sought an explanation for the same discrepancy.

The first attempt by HB$\chi$PT to predict the level of the violation of the isospin invariance in the $\pi N$ interaction at low energy, on the basis of the mass difference of the two light quarks and the dominant virtual-photon effects, is detailed in Ref. [82]. Eight indicators $R_{1,8}$ were used therein, to establish the type and quantify the level of the violation of the isospin invariance. One of these quantities, the indicator $R_2$, see Eq. (3) of this work, quantifies the amount of the departure from the triangle identity of Eq. (2). The results of
the fits, appearing in their Table 1, suggest that their $R_2$ prediction is small, at the level of about +1 %, i.e., not only of smaller magnitude in comparison with the results of Refs. [32,38], but also opposite in sign. The authors commented that additional effort is needed, to include in the evaluation all virtual-photon effects, and pointed “towards the necessity of a fourth-order calculation.”

Reference [83] provides an extension of Ref. [82] in terms of energy. Their Figs. 2 suggest that the indicator $R_2$ for the s wave comes out about $-2.5 \%$, nearly constant in the low-energy region. Sizeable effects are seen in the p waves, see their Figs. 3-6. The effect in the s wave, albeit of the same sign now, still remains about three to four times smaller than the effects reported in Refs. [32,38].

Reference [84], the last work by HB$\chi$PT for several years to follow, included all electromagnetic effects to third order. Regarding the violation of the isospin invariance, the authors wrote: “The precise description of the scattering process also allows us to address the question of isospin violation in the strong interaction. For the usually employed triangle relation, we find an isospin breaking effect of $-0.7 \%$ in the s wave, whereas the p waves show effects of $-1.5 \%$ and $-4$ to $-2.5 \%$, respectively, for pion laboratory momenta between 25 and 100 MeV.” Evidently, there seems to be another change in their $R_2$ prediction for the s wave, leading to a less negative value in comparison with the results obtained in Ref. [83] by the same authors, similar in magnitude (but of opposite sign) to their first result [82].

To my knowledge, the last work, addressing the issue of the violation of the isospin invariance in the $\pi N$ interaction within the $\chi$PT framework, may be found in Ref. [74]. Using CB$\chi$PT and including all effects due to the mass difference of the two light quarks, as well as to real and virtual photons, to third order in the chiral expansion, the authors find substantial differences to the results of all earlier studies [82,83,84], in both the s and p waves, see Section 3.3 therein. If their Fig. 18 is correct, then their upgraded corrections bring the indicator $R_2$ (for the s wave) further away from the results of Refs. [1,32,38]. In my judgment, it is counter-intuitive that $R_2$ (for the s wave) increases (in magnitude) with increasing energy. At present, I do not see a coherent picture, emerging from these four attempts.

The subject of the isospin breaking in the $\pi N$ interaction may be considered in the light of what has long been known for the $NN$ system. Let me next touch upon the ways by which the isospin breaking has been established in the $NN$ interaction (see Ref. [85] for details).

- First, the hadronic part of the low-energy $NN$ interaction is characterised by three scattering lengths, corresponding to the $^1S_0$ states $pp$, $nn$, and $np$. 

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If the charge independence (which is used in the $NN$ domain as a synonym for the isospin invariance) held, then these three scattering lengths would have been equal. In reality, after the removal of the EM effects, their values are [86]:

$$a_{pp} = -17.3(4) \text{ fm}, \quad a_{nn} = -18.8(3) \text{ fm}, \quad a_{np} = -23.77(9) \text{ fm}.$$  
(In Ref. [86], these scattering lengths carry the superscript ‘N’, indicating that they are nuclear ones, obtained after the EM corrections had been applied.) Evidently, these values violate charge independence and, to a lesser extent, charge symmetry, as

$$\Delta a_{CD} = (a_{pp} + a_{nn})/2 - a_{np} = 5.7(3) \text{ fm}$$

and

$$\Delta a_{CSD} = a_{pp} - a_{nn} = 1.5(5) \text{ fm}$$

are significantly non-zero. These values correspond to the violation of the charge independence in the low-energy $s$-wave part of the $NN$ scattering amplitude by about 27% and of charge symmetry by about 8%.

- Second, there is a discrepancy between the theoretical expectations and the experimental results regarding the binding-energy differences of mirror nuclei. The effect was established in the 1960s [87,88] and is known as the ‘Okamoto-Nolen-Schiffer anomaly’.

- Third, measurements of the spin-dependent left-right asymmetries in ES of polarised neutrons off polarised protons in the late 1980s and the early 1990s established significant differences between the APs of the two nucleons [89,90,91,92].

The level of the charge-independence breaking in the $NN$ interaction exceeds by far the magnitude of the effects which were reported in Refs. [1,32,38] for the low-energy $\pi N$ interaction. If the $\pi N$ interaction is viewed as the basis for the description of the $NN$ interaction (as the case is in meson-exchange models of the $NN$ interaction), it is a plausible expectation that part of the (large) isospin-breaking effects, observed in the $NN$ system, could originate from the $\pi N$ interaction. In the mid 1990s, Gardner and collaborators had examined the impact of the splitting in the meson-baryon coupling constants, arising from the mass difference of the two light quarks, on the observables in the $NN$ interaction [93,94], and found out that the aforementioned discrepancy between the APs of the two nucleons could be accounted for.

At present, the source of the isospin-breaking effect in the $\pi N$ interaction at low energy, which was reported over twenty-five years ago [32,38], remains unknown (or even “mysterious” [83]). The attempts to attribute the effect to the difference between the bare masses of the two light quarks, which have been carried out within the $\chi$PT framework, have not brought fruit yet, in that none of their predictions comes anywhere close to the level of the reported effects.
of course, it may turn out in the end that the origin of the effect involves more ‘mundane’ phenomena than the mass difference of the two light quarks. Although the constituent-quark model of Ref. [75] has demonstrated that the effect may be understood as originating from the difference between the constituent masses of the $u$ and $d$ quarks, it would be desirable to obtain a corroboration from another - perhaps less phenomenological - approach, so that the source of the discrepancy be unequivocally identified.

6 Conclusions

Over twenty-five years ago, two analyses [32,38] of the pion-nucleon ($\pi N$) data at low energy (i.e., for pion laboratory kinetic energy $T \leq 100$ MeV) reported on the departure of the extracted $\pi N$ scattering amplitudes, corresponding to the three reactions which are experimentally accessible at low energy (namely the two elastic-scattering (ES) reactions $\pi^\pm p \rightarrow \pi^\pm p$ and the $\pi^- p$ charge-exchange (CX) reaction $\pi^- p \rightarrow \pi^0 n$), from the ‘triangle identity’ of Eq. (2), a relation which is fulfilled, if the isospin invariance holds in the $\pi N$ interaction. The aforementioned studies had used different ways of modelling the hadronic part of the $\pi N$ interaction, different schemes of application of the electromagnetic (EM) corrections, different (though overlapping) databases (DBs), and different methodologies. In spite of these differences, the two studies agreed well on the level of the departure from the triangle identity, thus giving rise to what I call herein the ‘low-energy $\pi N$ enigma’. This result has been corroborated several times by subsequent phase-shift analyses of the low-energy $\pi N$ data; as newer measurements emerged and were included in the DB, the statistical significance of the effect improved over time [1]. Point No. 1: The analysis of the low-energy $\pi N$ data gives rise to a significant departure of the extracted $\pi N$ scattering amplitudes from the triangle identity of Eq. (2). The existence of this effect can hardly be disputed; its explanation, however, is open to speculation.

The first question arises: can global fits to the entirety of the $\pi N$ data be performed? Section 4.1 starts with a positive answer to this question. The question however, whether or not such fits are satisfactory, admits a negative answer. For the sake of example, examined herein was one such popular solution, the ‘current’ (XP15) solution of the SAID website [18]. When the Arndt-Roper formula [52] is used in the optimisation (see Eq. (4)), the statistical expectation is that the datasets which are scaled ‘upwards’ balance (on average) those which are scaled ‘downwards’. Furthermore, the energy dependence of the scale factors $z_j$ of Eq. (5) must not be significant. The fitted values of the scale factor $z$ in the XP15 solution (for $T \leq 100$ MeV) show no significant bias, see Fig. 2: the linear fit results in an intercept which is compatible with 1 and a slope value which is compatible with 0, see Table 2.
Therefore, the output of their global fit passes the first test of compatibility with the statistical expectation (i.e., with an unbiased outcome of optimisations resting upon the Arndt-Roper formula [52]). However, the SAID $\pi N$ DB comprises three distinct parts, i.e., the sets of observations in the three low-energy $\pi N$ reactions. Had their global fit been truly satisfactory, a similar behaviour of the scale factors $z_j$ (in terms of their energy dependence), which is seen in Fig. 2, should also have been observed in any arbitrary subset of their DB, complying with the basic principles of the Sampling Theory (adequate population, representative sampling). The fitted values of the scale factor $z$, relating to the description of the SAID low-energy $\pi N$ DBs with the XP15 solution, are shown (separately for the three reactions) in Figs. 3-5. If the global fit were truly satisfactory, then the scale factors $z_j$ of Figs. 3-5 would have come out independent of the beam energy and would have been centred on 1 (as the case was for the results of the analysis of all scale factors $z_j$, shown in Fig. 2). However, the bulk of the data for $T \leq 100$ MeV (represented by the shaded bands in Figs. 3-5) seem to be either underestimated by the XP15 solution (i.e., in case of the $\pi^-p$ CX reaction) or overestimated by it (i.e., in case of the two ES reactions, the effects for the $\pi^+p$ reaction being more pronounced), see also Table 2. One notices that the mismatches decrease with increasing beam energy, converging to $z = 1$ in the vicinity of $T = 100$ MeV. Such behaviour is in general agreement with the conclusions of Refs. [1,32,38] for an energy-dependent isospin-breaking effect. From Figs. 3-5 and Table 2, one may conclude that the XP15 solution does not describe sufficiently well the bulk of the low-energy measurements. Nearly identical results in case of the former solution of the SAID group, the WI08 solution [17], were reported in Ref. [55]. Point No. 2: In global fits of isospin-invariant analyses using the Arndt-Roper formula [52], the low-energy $\pi N$ enigma gives rise to systematic energy-dependent effects in the scale factors $z_j$ of Eq. (5), which (as expected) are different for the three low-energy $\pi N$ reactions.

Further analysis of the low-energy $\pi N$ data demonstrates that the joint optimisation of the ES measurements yields unbiased results, in accordance with the statistical expectation, see Ref. [1] and the works cited therein. Therefore, there is no indication at present of violation of the isospin invariance in the two ES reactions. The departure from the statistical expectation occurs when one attempts to also include in the fit the $\pi^-p$ CX data (thus pursuing a global fit of the data of all three reactions). For that reason, it seems that the low-energy $\pi N$ enigma is due to the inability to accommodate the measurements of the $\pi^-p$ CX reaction in a global fit.

The first (perhaps naive) attempt to provide an explanation for the low-energy $\pi N$ enigma involves an obvious effect, namely the systematic inaccuracy of the absolute normalisation of the low-energy $\pi N$ datasets. However, this issue cannot be addressed by analysts, but should rather be resolved by the experimental groups which had been responsible for the measurements. At present,
such a re-examination, even of a small fraction of the available datasets (e.g.,
of those datasets which look suspicious either because of their absolute nor-
malisation or because of the smallness of their published normalisation un-
certainty), seems to be a remote prospect. Point No. 3: It is unlikely that the
reassessment of the correctness of the absolute normalisation of the datasets
of the low-energy $\pi N$ DB will (or, in all probability, even could) take place.
Given that this possibility seems to be barred, I see no other option (for the
time being) but to accept the correctness of the absolute normalisation of the
bulk of the experimental data at low energy. On the other hand, there is strong
indication that the normalisation uncertainties had been seriously underesti-
mated on average in the $\pi N$ experiments at low energy, in particular in the
$\pi^+ p$ experiments, see Table 2.

The second attempt to provide an explanation for the low-energy $\pi N$ enigma
involves significant residual contributions in the results of the various avail-
able EM-correction schemes in the scattering region $[35,39,40,41,50,51]$. Such
schemes aim at the removal of the EM effects from the $\pi N$ phase shifts
and partial-wave amplitudes, and lead to the extraction of the (important)
hadronic quantities, e.g., of the parameters of the hadronic potentials in Ref. [32],
of the parameters of the hadronic model of Ref. [38], etc. The results of the
aforementioned EM-correction schemes were compared in Ref. [36]; two con-
clusions were drawn therein:

- the differences among the results of the tested schemes are small and
- the low-energy $\pi N$ enigma seems to be several times more pronounced than
  the impact which the EM corrections have on the analyses at low energy.

Consequently, one could draw the tentative conclusion that an explanation for
the low-energy $\pi N$ enigma in terms of significant contributions missing from
the established schemes of removal of the EM effects from the scattering data
$[35,39,40,41,50,51]$ does not seem to be credible.

Can the situation at the $\pi N$ threshold provide some insight into the matter?
This issue is addressed in detail in Section 5.2.1. Very precise measurements
of the strong-interaction shift $\epsilon_{1s} [7,8]$ and less-precise ones of the total decay
width $\Gamma_{1s} [7,9]$ of the ground state in pionic hydrogen were obtained at the Paul
Scherrer Institut (PSI) between the mid 1990s and the early 2000. To extract
the $\pi N$ scattering lengths, corresponding to the $\pi^- p$ ES and CX reactions,
one must remove the effects of EM origin. To this end, three schemes were de-
developed between 1996 and 2007: two potential-model approaches $[47,48]$ (the
latter study may be thought of as an extension of the former) and one method
relying on the use of Coulomb wavefunctions, with a short-range strong inter-
action and extended charge distributions $[64]$. Although these schemes aim at
the removal of the same effects (i.e., of the so-called trivial EM effects), their
results mismatch.
In addition to the removal of the EM effects, corrections have surfaced, which also attempt to remove from the pionic-hydrogen measurements effects of hadronic origin (i.e., effects emanating from the mass difference of the two light quarks); such studies have been carried out within the framework of the Chiral-Perturbation Theory ($\chi$PT) [67,69,71,72,73]. The corrections to the $\pi^-p$ ES length in Refs. [67,69,71,72] are larger (in magnitude) than those obtained from potential models [47,48] or from the model of Ericson and collaborators [64], and (furthermore) they are accompanied by sizeably larger uncertainties, see Table 3. Evidently, there is no consensus at present even on what effects should be removed from the accurate PSI measurements.

In view of the incompatibility of the results of the various schemes of removal of the trivial EM effects at the $\pi N$ threshold (upper part of Table 3), it cannot be excluded that the established schemes of application of the EM corrections to the scattering data fail to remove a sizeable part of effects of EM origin. This is why Section 5.2.1 concludes with a recommendation: to establish a unified scheme for the development of reliable EM corrections, applicable at the $\pi N$ threshold and in the scattering region. Point No. 4: Are significant contributions missing from the established schemes of removal of the EM effects from the scattering data [35,39,40,41,50,51]?

The third attempt to provide an explanation for the low-energy $\pi N$ enigma is via the violation of the isospin invariance in the $\pi N$ interaction at low energy. The question of the isospin breaking in the $\pi N$ interaction may be considered in the light of the large breaking effects, which have long been established in the $NN$ system [86]. The argument resting upon the inevitable question ‘if the $NN$ system is affected, then why should the $\pi N$ system not be?’ might be a convincing one, but even more convincing would have been the elimination of the possibility of a systematic bias in the absolute normalisation of the low-energy $\pi N$ datasets and the application of reliable EM corrections to the entirety of the $\pi N$ data, from the $\pi N$ threshold to the few-GeV region.

I will finalise this study with a few comments on issues which have troubled me over the years, listed below in order of decreasing importance.

The most troubling question is why the analysis of the DCSs of the CHAOS collaboration [95] is not possible within the context of the ETH $\pi N$ project [96,97]. As the beam energy in the CHAOS DCSs is low, an investigation of the description of these data within the $\chi$PT framework (e.g., using the method of Ref. [25]) or following the method of Ref. [21] should be possible. Given the availability of the data for over one and a half decades, I find it surprising that this issue has not been pursued yet.

The process of the identification of the outliers in the DB has been perfected in the analyses carried out within the ETH $\pi N$ project [1]. To reduce the model
dependence of the process, the polynomial parameterisation of the s- and p-wave $K$-matrix elements (i.e., the ETH parameterisation in the language of Ref. [34]) is exclusively used in that task. On the whole, I am fond of the use of robust methods in fits to data containing outliers, in particular of methods which apply soft weights to all datapoints, different at each step of the optimisation, depending on their distance (at that step) to the bulk of the fitted values. Such a dynamical optimisation process enables one to retain the initial input DB, allowing all datapoints to be present at all steps of the optimisation, with suitable weights regulating their contribution to the minimisation function. To come up with a reliable robust version of the Arndt-Roper formula has been a task which I have not been able to accomplish yet.

As the $K$-matrix approach is followed by both methods used in the modelling of the hadronic part of the $\pi N$ interaction within the ETH project, the unitarity of the $\pi N$ partial-wave amplitudes is fulfilled. Although the ETH model is occasionally categorised as a ‘tree-level model’, the association of the $K$-matrix elements and the partial-wave amplitudes via the unitarisation scheme does take account of higher-order effects; one way of understanding this is by considering the low-energy expansion of the $\pi N$ partial-wave amplitudes:

$$\mathcal{F}_{IJ} = K_{IJ} (1 - iqK_{IJ})^{-1} = K_{IJ} (1 + iqK_{IJ} - \mathcal{O}(q^2)).$$

The first term within the brackets would correspond to the tree-level approximation, the second to Feynman diagrams with one loop, and so on. The continuation of the unitarisation scheme into the unphysical region has been one question to which I have found no satisfactory solution yet.

At present, only the s- and p-wave $K$-matrix elements of the ETH model are used in the analyses; the (small) d- and f-wave contributions are imported from the SAID analyses (their current solution), which (given that they make use of high-energy data) are expected to be reliable. The contribution of the Feynman diagrams of the ETH model to the d- and f-wave amplitudes are known since 1993, yet the contributions from the six well-established d and f higher baryon resonances with masses below 2 GeV and known branching fractions to $\pi N$ decay modes introduce additional parameters, which neither can be treated as free in the optimisation nor can they be fixed from external sources. These states are:

- $N(1520) \ (3/2)^- \ (D_{13}),$
- $N(1675) \ (5/2)^- \ (D_{15}),$
- $N(1680) \ (5/2)^+ \ (F_{15}),$
- $\Delta(1700) \ (3/2)^- \ (D_{33}),$
- $\Delta(1905) \ (5/2)^+ \ (F_{35})$, and
- $\Delta(1950) \ (7/2)^+ \ (F_{37}).$

Although the issue is of no practical significance, the appropriate inclusion of these effects would reduce the amount of information which needs to be
imported into the analyses, featuring the ETH model, from external sources.

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