The Broken Phase of the Topological Sigma model

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Abstract

The topological $\sigma$-model with the semi-infinite cigar-like target space (black hole geometry) is considered. The model is shown to possess unsuppressed instantons. The noncompactness of the moduli space of these instantons is responsible for an unusual physics. There is a stable vacuum state in which the vacuum energy is zero, correlation functions are numbers thus the model is in the topological phase. However, there are other vacuum states in which correlation functions show the coordinate dependence. The estimation of the vacuum energy indicates that it is nonzero. These states are interpreted as the ones with broken BRST-symmetry.

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1 Introduction

Topological Field Theories (TFT) received considerable attention during recent years. After the discovery of the topological Yang-Mills theory in four dimensions [1] and topological $\sigma$ models in two dimensions [2], a large class of 2D TFT have been studied, including conformal TFT and their deformations. Particularly interesting results have been obtained for topological $\sigma$-models formulated on Calabi-Yau spaces and for their relationship to N=2 superconformal theories in the Landau-Ginzburg formulation (see [3] and references therein).

The problem we are attempting to address in this paper is how TFT could serve physics. The puzzle stems from the very notion of TFT: they have no physical degrees of freedom. That means that they have no propagating particles; the configuration space in these theories is a finite dimensional one.

The most popular approach to TFT nowadays is the topological string theory. It was proven [4, 5] that the non-critical string theory at low dimensions (with central charge of matter $c < 1$) is equivalent to topological minimal matter [6] coupled to the topological 2D gravity [4]. However, it is not a surprising that low-dimensional string turns out to be a topological theory, because at $c < 1$ string has no physical degrees of freedom. The remaining open problem is whether TFT could be relevant to the string theory at $c \geq 1$. In particular, the case $c = 1$ is under intensive study [7, 8].

Another motivation to study TFT has been put forward by E. Witten in his original papers [1, 2]. In the framework of the field theory one can try to construct the theory of a physical gravity as a broken phase of the topological gravity. In this setup one could hope to arrive at the theory of gravity with good ultraviolet properties, at least renormalizability. The idea is similar to that in the electroweak theory: the spontaneous breakdown of the gauge symmetry does not affect ultraviolet properties of the model.

However, little is known about the possibility of the spontaneous breakdown of the BRST-symmetry in TFT. One known example is the topological quantum mechanics with double-well potential studied in ref.[27] (see [9] for a review). The mechanism of the BRST-symmetry breaking in that case is essentially the same as in the N=1 SUSY quantum mechanics considered by E. Witten long ago [10]. Namely, the supersymmetry is dynamically broken by instantons. The main reason for this phenomenon is the noncompactness
of the moduli space of instantons, which is a real line (rather than circle) in the case at hand. 

In TFT this breakdown manifest itself as a nonzero expectation value of BRST-exact operators. This is the case for the quantum mechanical example mentioned above, nevertheless such fundamental properties of TFT as coordinate and coupling constant independence of correlation functions of observables still remains untouched [27]. Therefore, one cannot speak about “liberating” any physical degree of freedom in that case. 

In this paper we consider the 2D topological \( \sigma \)-model [2] with the semi-infinite cigar-like target space. This geometry of the target space arises naturally [11] as a \( d=2 \) black hole solution in the string theory associated with \( SL(2,R)/U(1) \) gauged WZW model [12] (\( d \) is the dimension of the target space) 

In one of our previous papers [13] it is shown that the black hole model possesses world-sheet instantons, although the target space of the model is not compact. Here we discuss this point in some detail in the context of the topological \( \sigma \)-model with semi-infinite cigar-like target space. It follows from the standard topological reasoning that \( d=2 \) \( \sigma \)-models with compact target spaces have instantons, while \( \sigma \)-models with noncompact ones do not. We show that the case of the semi-infinite cigar-like \( \sigma \)-model is on the “borderline” between compact and noncompact cases and needs more careful regularization. Our result (which is one of the main results of this paper) is that the model at hand does have nonsuppressed instantons. The moduli space of these instantons turns out to be noncompact. This ensures the divergence of the integrals over the moduli space for the correlation functions of observables. These divergences provide the general reason for the BRST-symmetry breakdown in the model at hand. 

The physics in the model turns out to depend on the vacuum expectation value (VEV) of the \( \sigma \)-model boson field \( |v| \). We show that the true topological phase with zero vacuum energy and coordinate independent correlation functions is realized only for \( |v| \to \infty \). Instead, for finite \( |v| \) correlation functions of observables show the coordinate dependence. Moreover, we study anti-instantons in the model at hand and show that they

\[ \text{Note, however, that we do not consider the topological version of the } SL(2,R)/U(1) \text{ WZW theory here. The latter model would be a conformal TFT. The } \sigma \text{ model we study here is not a conformal one due to the absence of the dilaton term in the action.} \]
produce nonvanishing effects. Actually they generate a true interaction in the model which involves a coupling constant dependence. A preliminary study of the vacuum energy indicates that it is nonzero. Our interpretation of these results is that there are degenerative vacuum states at finite $|v|$ for which the BRST-symmetry is broken and some physical degrees of freedom are “liberated”. Although these states are unstable we present qualitative arguments that their lifetime could be infinitely large.

The plan of the paper is as follows. In Sec. 2 we review briefly general properties of $d = 2$ topological $\sigma$-models. In Sec. 3 we consider the $\sigma$-model with black hole metric and show that it has world-sheet instantons. In Sec. 4 we study correlation functions of observables and show their coordinate dependence for finite $|v|$. In Sec. 5 we develop the effective Lagrangian technique for anti-instanton effects and in Sec. 6 we estimate the vacuum energy. Sec. 7 contains our final discussion.

2 $d = 2$ topological $\sigma$-models.

Here we review some basic properties of $d = 2$ topological $\sigma$-models [2, 4]. The action of the model on the $d = 2$ Kahler target manifold reads

$$S = \frac{r^2}{\pi} \int d^2 x \left[ \frac{1}{2} g_{ij}(w) \partial_\mu w^i \partial_\mu w^j - \frac{1}{2} J_{ij}(w) \epsilon_{\mu\nu} \partial_\mu w^i \partial_\nu w^j \right] - g_{ij} \lambda^\mu i D_\mu \lambda^i - \frac{1}{8} R_{ij,kl} \lambda^i \lambda^j \lambda^\mu i \lambda_\mu^j. \quad (2.1)$$

Here $\mu, \nu = 1, 2$ are world-sheet indices, while $i, j = 1, 2$ are target space ones. For simplicity the world-sheet is considered to be a complex plane with Euclidean flat metric $h_{\mu,\nu} = \delta_{\mu,\nu}$ and complex structure $\epsilon_{1,2} = 1$, while $g_{ij}$ and $J_{ij}$ are the metric and the complex structure of the target space. $D_\mu$ is the covariant derivative with respect to the connection on the target manifold $\Gamma^i_{kl}$

$$D_\mu A^i = \partial_\mu A^i + \partial_\mu w^k \Gamma^i_{kl} A^l, \quad (2.2)$$

while $R^i_{j,kl}$ is the target space curvature tensor

$$R^i_{j,kl} = \partial_\kappa \Gamma^i_{lj} - \partial_\kappa \Gamma^i_{kj} + \Gamma^i_{ks} \Gamma^s_{lj} - \Gamma^i_{ls} \Gamma^s_{kj}. \quad (2.3)$$

Fermion fields $\chi^i$ and $\lambda^\mu i$ in (2.1) have integer spin zero and one respectively. Integer spins of fermions arises naturally after twisting [2] of the $N=2$ $\sigma$-model. On the other hand in the BRST approach to TFT [15] fermions $\chi^i$
and $\lambda^{i\mu}$ arise as ghosts and therefore have integer spins. Their role is to cancel out boson degrees of freedom in all correlation functions of observables. The field $\lambda^{i\mu}$ satisfies the constraint

$$\lambda^{i\mu} + \varepsilon^{\mu}_{\nu} f^i_j \lambda^{j\nu} = 0.$$  \hspace{1cm} (2.4)

The model (2.1) is invariant under scalar fermionic symmetry generated by the operator $Q$ ($Q^2 = 0$), which can be considered as a BRST operator \[\] [15, 9]. The transformation law for fields is

$$\{Q, w^i\} = \chi^i,$$  \hspace{1cm} (2.5)

$$\{Q, \chi^i\} = 0,$$

$$\{Q, \lambda^{i\mu}\} = 2i(\partial^\mu w^i - \varepsilon^{\mu\nu} f^i_j \partial^\nu w^j) + \lambda^{i\nu} \Gamma^i_{jk} \chi^k.$$  \hspace{1cm} (2.6)

Observables $O$ in TFT are elements of the $Q$-cohomology

$$\{Q, O\} = 0,$$  \hspace{1cm} (2.7)

$$\{Q, \tilde{O}\} \neq O.$$  \hspace{1cm} (2.8)

Conditions (2.6),(2.7) mean that we should consider only gauge invariant operators which are defined up to a gauge transformation.

Correlation functions we are interested in TFT are of the form

$$\langle O_1(x_1), \ldots O_n(x_n) \rangle,$$  \hspace{1cm} (2.9)

where $O_i$ are from $Q$-cohomology. Condition (2.7) means that if we substitute in (2.8) a $Q$-exact operator, then the correlation function (2.8) would be zero, provided other $O_i$ satisfy (2.6) and

$$Q \mid 0 \rangle = 0.$$  \hspace{1cm} (2.10)

The latter condition holds true if the $Q$-symmetry is not broken.

Topological $\sigma$-models (A-models, see the last ref. in \[\]) have two basic properties which could be viewed as a definition of a TFT (cohomological one), namely, energy-momentum tensor and the action are $Q$-exact

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\},$$  \hspace{1cm} (2.11)

$$S = \{Q, \Lambda\}.$$
In particular, (2.10) means that correlation functions (2.8) are world-sheet metric independent and (2.11) means that they are coupling constant \((1/r^2)\) independent. Indeed, the variation of (2.8) with respect to the world-sheet metric \(h_{\mu\nu}\) results in the insertion of the \(Q\)-exact operator in (2.8) as it follows from (2.10), and therefore, gives zero. In a similar way, the variation of (2.8) with respect to \(r^2\) gives zero because of (2.11).

The \(h_{\mu\nu}\) independence of (2.8) means, in particular, its independence of the world-sheet coordinates \(x_i\). The coupling constant independence makes it possible to consider the theory in the weak coupling limit \(r^2 \to \infty\). That means that the semiclassical approximation gives actually exact results. Thus correlation functions (2.8) are exactly given by their instanton contributions.

For the \(d = 2\) model (2.1) there are only two local observables in the cohomology (2.6),(2.7), namely

\[
1, \ A_{ij} \chi^i \chi^j(x),
\]

where \(A_{ij}\) is the antisymmetric tensor \(dA = 0, A \neq dB\). Tensor \(A_{ij}\) should have proper transformation law with respect to indices \(i, j\) to preserve the invariance under target space diffeomorphisms. Thus the natural choice for the only non-trivial local observable is

\[
O = i J_{ij}(w) \chi^i \chi^j(x).
\]

Let us now calculate the instanton contribution to the correlation function

\[
< O(x_1), \ldots O(x_n) >,
\]

where the operator \(O(x)\) is given by (2.13). We assume that the target space metric \(g_{ij}\) and the complex structure \(J_{ij}\) have the form

\[
g_{ij} = \delta_{ij} g(w),
\]

\[
J_{ij} = \varepsilon_{ij} g(w).
\]

Then the action (2.1) takes the form

\[
S = \frac{2r^2}{\pi} \int d^2 x \left\{ g(w) \partial \bar{\partial} \bar{w} \partial \bar{w} - \frac{i}{4} g(w) \left[ \lambda^{++} \bar{D} \chi \right. \right. \right.
\]

\[
+ \left. \left. \lambda^{-\bar{\nu}} \bar{D} \chi \right] + O(\lambda^{++}\lambda^{-\bar{\nu}}\bar{\chi} \chi) \right\}
\]
Eq.(2.16) is written down in complex coordinates both for the world-sheet and for the target space, for example, \( z = x_1 + i x_2, \partial = 1/2(\partial_1 - i \partial_2) \) and \( w = w^1 + i w^2, \bar{w} = \bar{w}^1 - i \bar{w}^2, \lambda^{++} = \lambda^{11} + i \lambda^{12} + i \lambda^{21} - \lambda^{22} \).

Note that the boson term in the action (2.16) comes as a sum of the kinetic term (the first term in (2.1))

\[
S_{\text{kin}} = \frac{r^2}{\pi} \int d^2 x g(w) \left( \partial w \partial \bar{w} + \bar{\partial} \bar{w} \partial w \right)
\] (2.17)

and the topological one (the second term in (2.1))

\[
S_{\text{top}} = \frac{r^2}{\pi} \int d^2 x g(w) \left( \bar{\partial} \bar{w} \partial w - \partial w \bar{\partial} \bar{w} \right)
\] (2.18)

The important feature of the topological \( \sigma \)-model (2.16) is that the topological term (2.18) appears with real coefficient in the action (the topological term with imaginary coefficient could be added to the action as well), that means that \( \Theta \)-angle is imaginary (see [1] for the discussion of unitarity and CPT-invariance).

The topological term (2.18) is proportional to the winding number of the map \( w(x) \) from the world sheet (which can be considered as \( S_2 \) with infinitely large radius) to the target space and thus is invariant under continuous deformations of the function \( w(x) \). Hence the functional space is divided into topological classes with a given winding number \( k \).

The minimum points of the action within each topological class are holomorphic maps, which obey \( \bar{\partial} w = 0 \), namely

\[
w(z) = v + \sum_{i=1}^{k} \frac{\rho_i}{z - z_i},
\] (2.19)

where \( v, \rho_1 \ldots \rho_k \) and \( z_1 \ldots z_k \) are \( 2k + 1 \) complex parameters, while \( k \) is the winding number of the instanton (2.19). Note, that instantons with singular behavior at infinity can by obtained from the one in (2.19) by coordinate inversion.

The action on the instanton (I) (2.19) is zero

\[
S_I = 0
\] (2.20)

because the contribution of the kinetic term (2.17) is exactly canceled by the one coming from the topological term (2.18).
Let us now restrict ourselves to the simplest case of the \( I \) with the winding number \( k = 1 \)

\[
I(z) = v + \frac{\rho}{z - z_0},
\]

(2.21)

Here \( x_0 \) is the center of the \( I \), \( \rho \) and \( \text{arg}(\rho) \) are its size and orientation.

\( I \) (2.21) has six real (three complex) boson zero modes (\( 2(2k + 1) \) in the general case)

\[
\begin{align*}
\nu_0 &= \frac{\partial w_I}{\partial v} = 1, \\
\nu_1 &= \rho \frac{\partial w_I}{\partial \rho} \frac{\rho}{z - z_0}, \\
\nu_2 &= \rho \frac{\partial w_I}{\partial z_0} \frac{\rho^2}{(z - z_0)^2}.
\end{align*}
\]

They satisfy the equation

\[
\bar{D} \nu = 0
\]

(2.22)

Consider now fermionic zero modes of the \( I \). They satisfy the equation

\[
\check{D} \chi = 0,
\]

(2.23)

which is the same as in (2.23). Thus we have six fermionic zero modes (\( 2(2k + 1) \) in the general case)

\[
\begin{align*}
\chi_0 &= \alpha_0, \\
\chi_1 &= \frac{\alpha_1}{z - z_0}, \\
\chi_2 &= \frac{\alpha_2 \rho}{(z - z_0)^2}.
\end{align*}
\]

(2.24)

Here \( \alpha_0, \alpha_1, \alpha_2 \) are Grassmanian variables with dimensions (0, -1, -1).

Let us now calculate correlation functions (2.14), where the operator \( O \) from (2.13) becomes

\[
O = g(w) \bar{\chi} \chi.
\]

(2.25)

The instanton measure has the form

\[
d\mu_I = -d^2v d^2x_0 d^2 \rho d^2 \alpha_0 d^2 \alpha_1 d^2 \alpha_2
\]

(2.26)
Note, that the factor $\exp -const r^2$ usual in instanton calculus is absent in (2.27) due to the cancellation of the instanton action in (2.20). This is in accordance with the general rule which requires the $r^2$-independence of (2.14) (more precisely, if we introduce a new coefficient $\gamma$ different from $r^2$ in front of the topological term in the action (2.1), then (2.14) could depend only on the difference $r^2 - \gamma$ [4]). Note also the cancellation of the determinants coming from boson and fermion degrees of freedom in (2.27).

Now counting the number of fermion zero modes we conclude that only the three point correlation function in (2.14) is nonzero (in general case $n = 2k + 1$). This is a consequence of the $U(1)$ charge anomaly relation. We have

$$< O(x_1), O(x_2), O(x_3) > = - \int d^2v d^2x_0 d^2\rho d^2\alpha_0 d^2\alpha_1 d^2\alpha_2$$

$$g(w)\bar{\chi}(x_1)g(w)\bar{\chi}(x_2)g(w)\bar{\chi}(x_3)$$

(2.28)

Here fields $\chi, \bar{\chi}$ should be understood as a sum over zero modes (2.25). Now instead of integration over $v, \rho, x_0$ in (2.28) let us proceed to new variables defined as follows. Fix points $x_1, x_2, x_3$, and consider $w(x_1), w(x_2), w(x_3)$, as functions of $v, \rho, x_0$ (see (2.21)). Then using equations (2.25), (2.22) it is easy to see that functions $\chi$ and $\bar{\chi}$ in (2.29) represent the jacobian needed to pass from $v, \rho, x_0$ to $w_1 = w(x_1), w_2 = w(x_2), w_3 = w(x_3)$. We get finally

$$< O(x_1), O(x_2), O(x_3) > = \int d^2w_1 g(w_1) d^2w_2 g(w_2) d^2w_3 g(w_3).$$

(2.29)

The coordinate and coupling constant independence of the correlation function becomes evident in (2.29) (provided the integrals are convergent). The moduli space of instanton is just the product of three copies of the target manifold and the correlation function (2.29) is equal to the $A^3$, where $A$ is the area of the target space. In general if $k \neq 1$ the correlation function (2.14) for $n = 2k + 1$ is given by the equation similar to the (2.29) one, the moduli space of I being the product of $2k + 1$ copies of the target manifold.

In the next two sections we re-examine the above outlined procedure for the calculation of correlation functions of observables (2.14) for the special case of the semi-infinite cigar-like target space.
3 Cigar-like target space.

Let us consider the target space metric
\[
g(w) = \frac{1}{1 + |w|^2}.
\] (3.1)

Its difference from, say, the metric of the sphere for the \(O(3)\) \(\sigma\)-model
\[
g(w)^{\text{sphere}} = \frac{1}{(1 + |w|^2)^2}
\] (3.2)
is in its slow fall-off at large \(|w| \to \infty\).

To see that (3.1) represents the black hole metric let us perform the change of variables
\[
w = \sinh r e^{-i\phi}
\]
\[
\bar{w} = \sinh r e^{i\phi}
\] (3.3)

Under this transformation the kinetic term for bosons (2.17) becomes
\[
S_{\text{kin}} = \frac{r^2}{2\pi} \int d^2 x \left\{ \left( \partial_\mu r \right)^2 + \tanh r^2 (\partial_\mu \phi)^2 \right\}.
\] (3.4)

The target manifold defined in (3.4) has the form of the semi-infinite cigar. Its metric has been studied in [11]. In fact in ref. [11] the gauged coset WZW model \(SL(2, R)/U(1)\) has been considered. The integration over gauge field leads to the theory of two real fields \(r\) and \(\phi\) with the target space geometry given by (3.4). It has been interpreted as a Euclidean version of the string theory for the two dimensional black hole [11]. Note, that the black hole model in [11] contains also a dilaton term added to the one in (3.4) which is responsible for its conformal invariance. The model we consider here (defined by (2.16), (3.1)) has no dilaton term and therefore is not a topological version of the \(SL(2,R)/U(1)\) coset (the latter is the special case of the Kazama-Suzuki model [14], see also [8]).

Let us now address a question [13]: do holomorphic instantons (2.19) still exist in the \(\sigma\)-model with metric (3.1). The main problem is that the topological term
\[
S_{\text{top}} \sim \frac{r^2}{\pi} \int \frac{dw \wedge d\bar{w}}{1 + |w|^2}
\] (3.5)
becomes logarithmically divergent at large $w$. However, the coefficient in front of the logarithm does not depend on the regularization scheme and is still proportional to the winding number $k$. Indeed, substituting (2.19) into (2.18) and using (3.1) for the metric we get

$$S_{\text{top}} = -2r^2k \left[ \log \frac{1}{a} + \text{const} \right], \quad (3.6)$$

where $1/a$ is the UV cutoff. Thus, we still have the configuration space divided into topological classes and holomorphic instantons (2.19) are still minimum points of the kinetic term (2.17) in a given topological class.

Above arguments led us to the conclusion that $I$ (2.19) is a solution to equations of motion of the model (2.16), (3.1) and we can expand around them using the standard saddle-point method. Now let us check this directly. Equations of motion for $\sigma$-model (2.16) read

$$\frac{\delta S}{\delta \bar{w}} = gD\bar{w} = 0 \quad (3.7)$$

However, as a matter of fact the r.h.s. is actually never zero on the holomorphic instanton (2.19) even in $\sigma$-model with a compact target space. For example, for O(3) $\sigma$-model with metric (3.2) one gets

$$\frac{\delta S}{\delta \bar{w}} = gD\bar{w}_I \sim \sum_l (x - x_l)^2 \delta^{(2)}(x - x_l). \quad (3.8)$$

To see what is going on let us expand the action around such a “solution” to equations of motion

$$S(w_I + \nu) = S(w_I) + \int d^2x \nu^i(x) \frac{\delta S}{\delta \bar{w}_I} \quad (3.9)$$

$$+ c.c. + \frac{1}{2} \int d^2x d^2y \nu^i(x) \frac{\delta^2 S}{\delta w_I^i(x) \delta w_I^j(y)} \nu^j(y) + \cdots,$$

where $w_I$ is the instanton solution and $\nu$ is a quantum fluctuation. In order to use the standard saddle-point solution what one really needs is the absence of the linear in $\nu$ term in (3.9). Substituting (3.8) into (3.9) we indeed get zero linear term, provided $\nu(x)$ is non-singular in the instanton center. Quantitative universal $\nu$ is supposed to be non-singular, otherwise it should be treated as another instanton, rather than the quantum fluctuation.
In this sense instanton is indeed a solution to equations of motion in $O(3)$ \(\sigma\)-model.

Let us now turn to the \(\sigma\)-model with black hole metric (3.1). One has

\[
\frac{\delta S}{\delta \bar{w}} = g D \bar{\partial} w_I \sim \sum_l (x - x_l) \delta^{(2)}(x - x_l).
\]  

(3.10)

Plugging (3.10) into (3.9) one finds that the linear term is also zero in this case. As we expected above the instanton (2.19) is a solution to equations of motion for the \(\sigma\)-model with black hole metric and one can expand around it using the saddle-point method.

What about the instanton action, which is the sum of (2.17) and (2.18). Note, that both the kinetic term (2.17) and the topological one (2.18) are logarithmically divergent on I. Due to the singularities introduced by \(\delta\)-functions the instanton action turns out to be nonzero in the contrast to the case of the compact target space (2.20). Let us calculate it. One has

\[
S_I = \frac{2r^2}{\pi} \int d^2 x \frac{\bar{\partial} w_I \partial \bar{w}_I}{1 + |w_I|^2}.
\]  

(3.11)

Substituting \(\bar{\partial} w_I = \pi \sum_l \rho_l \delta^{(2)}(x - x_l)\) and using some regularization for the \(\delta\)-function, say,

\[
\delta^{(2)}(x) = \frac{1}{\pi} \frac{a^2}{(x^2 + a^2)^2},
\]  

(3.12)

one gets

\[
S_I = k \frac{r^2}{3}.
\]  

(3.13)

We see that the logarithmic divergences are canceled out in (3.13), the instanton action is nonzero but finite. Eq. (3.13) means that instanton contribution to correlation functions are weighed with the factor \(g^I\), where

\[
g^I = e^{-\frac{r^2}{3}}.
\]  

(3.14)

3 The author is indebted to A. Losev for this remark.

4 Note that the exact coefficient in front of \(r^2\) in (3.13) and (3.14) could depend on the regularization scheme. Therefore, in what follows we consider \(g_I\) as a new independent coupling constant. It turns out to be the true coupling constant of the model, since there is no direct dependence of observables on \(r^2\).
The result (3.13) means that instantons are not suppressed in the topological $\sigma$-model with the black hole metric (3.1).

We would like to stress now that the target space with metric (3.1) is the limiting case of a noncompact manifold for which unsuppressed instantons exist. Indeed, consider the metric

$$g_q(w) = \frac{1}{(1 + |w|^2)^q}$$

At $q > 1$ the target space is compact and we have conventional instantons with zero action in the topological version of the $\sigma$-model. At $q = 1$ we have instanton solutions (as it follows from (3.10)) with finite action (3.13). At $1/2 < q < 1$ we still have instanton solutions to equations of motion as it follows from the considerations similar to the above ones (see eqs. (3.8)-(3.10)). However, the instanton action diverges as $1/a^{2(1-q)}$, hence contributions of these instantons are suppressed in all the correlation functions. At $q < 1/2$ we have no instantons as holomorphic curve (2.19) fails to be a solution to equations of motion.

We see that $q = 1$ case is on the borderline between the compact and noncompact target space models. Moreover, the black hole metric, perhaps, should be thought of as being “compact” because there are nonsuppressed instantons in the topological version of $\sigma$-model with this metric.

The existence of unsuppressed instantons in the topological $\sigma$-model with the black hole metric is one of the main results of the present paper. These instantons have unusual physical properties. In particular, as we show below the moduli space of these instantons is noncompact. Therefore, new logarithmic divergences emerge in the theory. We study some of these effects in the remainder of the paper.

4 Correlation functions

Now let us calculate correlation functions (2.14) of the observable (2.26) for the $\sigma$-model with black hole metric (3.1). Again for the sake of simplicity let us restrict ourselves to the case of the instanton (2.21) with the winding number $k = 1$. Zero modes of $I$ is given essentially by eqs. (2.22), (2.25). The only exception is that we are not going now to integrate over parameter $v$ in the instanton measure. The reason is that $\nu_0$, $\chi_0$ zero modes in (2.22),
have the quadratically divergent norm on the world-sheet taken to be a complex plane. Moreover, \( v \) is a boundary condition for field \( w \) at infinity. In fact it has a meaning of vacuum expectation value (VEV) \( \langle w \rangle \).

Usually we do not integrate over VEV in QFT. That would mean summing up all the different vacuums of the theory. Instead, we minimize the vacuum energy with respect to VEV to find the true vacuum. Of course, if physics do not depend on VEV (as is the case, say, for the phase of the scalar field in the Gauge-Higgs model) then one can safely integrate over it; nothing would change except some numerical factor in front of the partition function. However, once we suspect that physics could be \(|v|\) dependent, in the case at hand, it is better to keep \( v \) fixed.

Thus the instanton measure now is

\[
d\mu_I = -g_I d^2x_0 d^2\rho d^2\alpha_1 d^2\alpha_2,
\]

instead of \( (2.27) \). Here \( g_I \) is given by \( (3.14) \). Eq. \( (1.1) \) leads to another selection rule for correlation functions, namely \( (2.14) \) is nonzero if \( n = 2k \). Thus for \( k = 1 \) we have \( n = 2 \). Repeating the same calculation, which leads us from \( (2.28) \) to \( (2.29) \) one gets in the case at hand

\[
\langle O(x_1), O(x_2) \rangle = -\int d^2w_1 g(w_1) d^2w_2 g(w_2).
\]

Note that deriving this equation we have not actually used the concrete form of the black hole metric \( (3.1) \). Therefore, \( (4.2) \) holds true also for topological \( \sigma \)-model with compact target space (with coupling constant \( g_I = 1 \)). We see, that our prescription not to integrate over constant modes defines another formulation of the theory based on the same Lagrangian \( (2.16) \). It is of course also a topological one (for the \( \sigma \)-model with compact target space) as it is clear already from \( (4.2) \). The new selection rule \( n = 2k \) means, in particular that the partition function \( Z = const \) instead of being zero (which is more usual in conventional QFT), while \( \langle O(x) \rangle = 0 \). In what follows we study topological \( \sigma \)-model in this formulation. The physical motivation for this as discussed above, is based essentially on the observation that it is not very useful to integrate over VEV of the field \( w \) if physics depend on it.

Now let us specialize for the case of the black hole metric \( (3.1) \). Substituting \( (3.1) \) into \( (4.2) \) we see that integral over moduli space of \( I \) becomes logarithmically divergent (like ones for topological and kinetic terms, see \( (3.6) \))
and needs regularization. We regularize the integral in (4.2) in the same way as before, introducing UV and IR cutoffs on the world-sheet, \( \frac{1}{a} \) and \( \frac{1}{L} \) respectively. Roughly speaking, (4.2) gives the answer \( \sim \log^2 |w_{\text{max}}| \). To see what is \( |w_{\text{max}}| \) in terms of cutoff parameters on the world-sheet one recalls that

\[
\begin{align*}
  w_1 &= v + \frac{\rho}{z_1 - z_0}, \\
  w_2 &= v + \frac{\rho}{z_2 - z_0}.
\end{align*}
\]

Taking into account (4.3) one gets with the double logarithm accuracy at finite \( |v| \) (\( |v| \ll \frac{L}{a} \))

\[
< O(x_1)O(x_2) > = -2(2\pi)^2 g_I \log \left| \frac{x_{12}}{a} \right| \log \frac{L}{|x_{12}|} + O(\log),
\]

where \( x_{12} = x_1 - x_2 \). The \( x_{12}\)-dependence here comes as follows. To get the double logarithm one has to consider either region \( x_0 \to x_1 \) \( (a \ll |x_0 - x_1| \ll |x_{12}|) \) or region \( x_0 \to x_2 \) \( (a \ll |x_0 - x_2| \ll |x_{12}|) \). Suppose \( x_0 \to x_1 \), then the integral over \( |w_1| \) goes from \( \rho/|x_{12}| \) to \( \rho/a \) and gives \( \log |x_{12}|/a \). Further, at \( x_0 \to x_1 \mid w_{\text{max}} \mid = L/|x_{12}| \). This gives \( \log \frac{L}{|x_{12}|} \). Factor 2 arises when one takes onto account the second option \( x_0 \to x_2 \). These arguments are just to explain qualitatively the origin of the answer (4.4). Of course, the integral in (4.2) can be calculated in a more rigorous way.

For the sake of a future use let us calculate (4.2) also in the limit of \( |v| \to \infty \), namely \( |v| \gg \frac{L}{a} \). Then \( |w_1| \sim |w_2| \sim |v| \) and one gets

\[
< O(x_1)O(x_2) > = -\pi^2 \frac{L}{|x_{12}|^2} \frac{1}{a^2} \frac{1}{|v|^4} \to 0.
\]

This goes to zero as \( |v| \to \infty \).

The correlation function in (4.4) shows \( x_{12}\)-dependence. This is a signal for the BRST-symmetry breaking. Indeed, the derivative of an element of the Q-cohomology is always Q-exact \( \mathbb{H} \). Explicitly

\[
\partial_\mu O(x) = \{Q, iJ_{ij} \partial_\mu w^i \chi^j \}.
\]

Hence, nonzero value for \( < \partial_\mu O(x_1)O(x_2) > \) means that

\[
Q \mid 0 \neq 0,
\]

15
and the BRST-symmetry is broken.

Now let us observe that we run into a problem. Indeed, the topological $\sigma$-model with $d=2$ has the value of the Witten index $I_W = 2 \neq 0$ (it is equal to the number of the physical states, which is the same as the number of elements of the Q-cohomology, see for example $[9]$). Hence, supersymmetry cannot be broken in the model at hand. So, what is going on?

One suggestion which can save the day is that the BRST-symmetry is in fact not broken for the true stable vacuum state of the theory which corresponds to $|v| \to \infty$. Indeed, (4.5) shows that at $|v| \to \infty < O(x_1) O(x_2) > \to 0$ and thus shows no $x_{12}$-dependence in this limit.

We show in the remainder of this paper that this assumption is correct. In fact, in Sect. 6 we show that for finite $|v|$ the model possess an infinite number of degenerative vacuum states with nonzero vacuum energy $E_{vac}$. However, as $|v| \to \infty E_{vac} \to 0$. Thus, the theory has a stable vacuum state at $|v| \to \infty$ for which BRST-symmetry is not broken. This resolves the puzzle with the nonzero value of the Witten index. However, for the unstable vacuum states at finite $|v|$ we have $x_{12}$-dependent correlation function (4.4) and the BRST-symmetry broken down.

The similar phenomenon of “vacuum running to infinity” is known to occur in the 4D SUSY QCD $[21]$. Instantons generate the superpotential which removes the degeneracy of the vacuum states associated with different values of the VEV of the scalar field. As a result the true vacuum state with zero vacuum energy corresponds only to VEV going to infinity. The theory also has nonzero Witten index.

5 Anti-instantons

Our aim now is to show that the vacuum energy in the $\sigma$-model with black hole metric is nonzero for finite $|v|$. However, it is clear that the perturbative theory gives zero for $E_{vac}$ since our model is a topological one at the perturbative level. Instantons by themselves do not contribute to the $E_{vac}$ because of the anomalous selection rule $n=2k$ (instantons have fermion zero modes). What we need to get the nonzero vacuum energy are the instanton-anti-instanton pairs ($I\bar{I}$). In order to calculate the contribution of $I\bar{I}$ into $E_{vac}$ we study anti-instantons ($\bar{I}$) in the model at hand in this section and derive the effective Lagrangian which takes them into account.
$I$ is an anti-holomorphic map from the world-sheet to the target

$$w = v + \sum_{l=1}^{p} \frac{\bar{\rho}_{al}}{\bar{z} - \bar{z}_a}. \quad (5.1)$$

Here $-p$ is the winding number, $x_{al}$, $\bar{\rho}_{al}$ are new complex parameters. Of course, (5.1) satisfies equations of motion in the same sense as $I$ (2.19) do (see (3.10))

$$\bar{D}\partial w \sim 0 \quad (5.2)$$

Again we restrict ourselves to $\bar{I}$ with $p=1$:

$$w = v + \frac{\bar{\rho}_a}{\bar{z} - \bar{z}_a}. \quad (5.3)$$

where $x_a$ and $\bar{\rho}_a$ has the interpretation of the position and the orientation-size vectors.

In nontopological versions of $\sigma$-models $I$'s and $\bar{I}$'s come on the same ground. Instead, in topological $\sigma$-models with compact target space $I$'s come with the zero action (2.20) while

$$S_{\bar{I}} = 2r^2A. \quad (5.4)$$

Indeed, the topological term (2.18) cancels the contribution of the kinetic one in (2.16) for $I$ and doubles it for $\bar{I}$. Now it is clear that $\bar{I}$'s cannot contribute to correlation functions in the topological $\sigma$-model with unbroken BRST-symmetry. This is a consequence of the general statement of $r^2$-independence of observables. Indeed, $\bar{I}$ contribution would involve the factor $\exp{-pS_{\bar{I}}}$ in the contradiction with the general rule.

Now we are going to study $\bar{I}$'s in the topological $\sigma$-model with metric (3.1) and show that they do contribute to observables of the model at finite $|v|$.

Zero modes of $\bar{I}$ (5.3) can be studied in a way similar to that for $I$ (2.21). We have four boson zero modes which are associated with variations of (5.3) with respect to $x_a$ and $\rho_a$ like in (2.22) (note, that we keep $v$ fixed). Furthermore, we have four fermion zero modes. They are

$$\lambda_1^{++} = \frac{\beta_1}{\bar{z} - \bar{z}_a}, \quad (5.5)$$

$$\lambda_2^{++} = \frac{\bar{\rho}_a\beta_2}{(\bar{z} - \bar{z}_a)^2}.$$
They satisfy the equation
\[ D_\mu \lambda^\mu = 0, \]  
(5.6)
which reduces to \( \partial \lambda^{++} = 0 \).

The action on \( \bar{I} \) (5.3) is logarithmically divergent as it is clear from the discussion above. It reads
\[ S_{\bar{I}} = 4r^2 \log \frac{|\rho_a|}{a}. \]  
(5.7)
Eq. (5.7) means that large size \( \bar{I} \) ’s are suppressed in the path integral. However, small size \( \bar{I} \) ’s (with size \( |\rho_a| \sim a \)) can induce a new point-like interaction (see [13]).

In general, the effect of instantons in any theory in which they are present can be described by means of the effective Lagrangian [18, 13]. In \( d = 2 \) \( \sigma \)-model this \( \bar{I} \)-induced vertex is (cf. [13])
\[ V_{\bar{I}} = - \int d\mu \bar{I} e^{-S_{\bar{I}} \left( \frac{ir^2}{2} \right)^4 g^4(v) \bar{\beta}_1 \chi \bar{\beta}_1 \bar{\chi}} \]
\[ \rho_a \bar{\beta}_2 \partial \chi \bar{\rho}_a \beta_2 \partial \bar{\chi} \exp \{ 2r^2 g(v) [\rho \partial w + \bar{\rho} \bar{\partial} \bar{w}] \}, \]  
(5.8)
where \( d\mu \bar{I} \) is the \( \bar{I} \) measure to be discussed below. This vertex should be added to the action (2.16) to get the effective action which mimics the effects of \( \bar{I} \)’s at the perturbative level. To check this let us calculate the following correlation function
\[ < w(x_1) \ldots w(x_n) \lambda^{++}(x) \lambda^{++}(x') >_{\bar{I}} \]  
(5.9)
in the one-\( \bar{I} \) background.

On the one hand (5.9) can be calculated (in the leading order in \( 1/r^2 \)) substituting classical expressions (5.3) and (5.5) for fields \( w \) and \( \lambda \) into (5.9). The result is
\[ \prod_{i=1}^n \left( v + \frac{\bar{\rho}_a}{z_i - z_a} \right) \frac{\bar{\beta}_1}{\bar{z} - z_a} \frac{\bar{\rho}_a \beta_2}{(\bar{z}' - \bar{z}_a)^2}. \]  
(5.10)
On the other hand the same result can be reproduced in the purely perturbative manner, inserting (5.8) into the action (2.16) of the model. Indeed, taking in the expansion of \( \exp -V_{\bar{I}} \) the only first power of \( V_{\bar{I}} \) (this corresponds to the one-\( \bar{I} \) contribution) one gets the same answer as in (5.10)
contracting fields in (5.9) with fields in (5.8) and taking into account the propagation functions of the model

\[ < w(x), w(0) > = \frac{1}{g(v)r^2} \log \frac{L}{x} + v^2; \]  
(5.11)

\[ < \lambda^{++}(x), \chi(o) > = - \frac{2i}{g(v)r^2} \frac{1}{z}, \]

\[ < \lambda^{--}(x), \chi(o) > = - \frac{2i}{g(v)r^2} \frac{1}{z}. \]

Note that \( v \) is the VEV of the field \( w \) one expands around in the perturbation theory.

We have derived the vertex (5.8) above only in the semiclassical approximation \( r^2 \gg 1 \). To get also quantum corrections one can use the following trick: substitute \( w \) instead of \( v \) in factors \( g(v) \) which appear in (5.8). It is clear that the effective Lagrangian should depend on the field \( w \) rather than its VEV if there is no explicit symmetry breaking. One gets

\[ V_{\bar{I}} = - \frac{r^8}{16} \int d\mu_{\bar{I}} e^{-S_{\bar{I}}} \left| \rho_a \right|^2 \frac{g^2(w)}{r^2} \tilde{\beta}_1 \beta_1 \beta_2 g(w) \bar{\chi} \chi \]

\[ g(w) \bar{D} \chi D \chi \exp \{ 2r^2 g(v) \left[ \rho \partial w + \bar{\rho} \bar{\partial} \bar{w} \right] + O(r^2 \rho^2 \partial^2 w) \} \]

(5.12)

Here \( O(r^2 \rho^2 \partial^2 w) \) denotes the possible contributions of higher derivative terms \( \rho^n \partial^n \) which is beyond our control here. Note, that the trick of the substitution \( v \to w \) gives us nontrivial quantum corrections in (5.12). To check these explicitly one has to go through two-loop (and higher loop) calculation in the \( \bar{I} \) background. Note, also the obvious generalization \( \partial \chi \to D \chi, \bar{\partial} \bar{\chi} \to \bar{D} \bar{\chi} \) which also accounts for quantum corrections.

Now let us discuss what \( \bar{I} \) measure \( d\mu_{\bar{I}} \) in (5.12) is. The following part of this section is rather technical, therefore the reader who is more interested in physics than in technical details could go straight to our final result for \( V_{\bar{I}} \) in eqs.(5.26), (5.28). According to general rules the \( \bar{I} \) measure yields (we assume \( r^2 >> 1 \)) here to justify the one loop calculation.

\[ d\mu_{\bar{I}} = \frac{d^2 x d^2 \rho a d^2 \beta_1 d^2 \beta_2}{a^4} \left( \frac{\det' \Delta_{F}^{b}}{\det \Delta_{F}^{b}} \right)^{-1/2} \left( \frac{\det' \Delta_{F}^{b}}{\det \Delta_{F}^{b}} \right)^{1/2} \]

(5.13)

\[ ^5 \text{Cf.} \quad \text{[13]} \quad \text{where these corrections remain undetermined because the VEV v of the field w have been set to zero} \]
Here $\Box_B^I_i$ is the operator of the quadratic fluctuations in the $I$ background

$$\Box_B^I_i = -[D^p_{\mu\nu}D^{p}_{\mu\nu} + R^i_{j\ell \mu^\nu}(w)\partial^\mu w^\ell \partial^\nu],$$ (5.14)

where $R^i_{j\ell \mu^\nu}$ is the curvature tensor for the target space (2.3) while $\Box_B^0$ is the same operator for the trivial background field $w = v$. In an analogous way $\Box_F^i$ is the fermion operator

$$\Box_F^i = -[D^p_{\mu\nu}D^{p}_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu}J^i_{k\ell \mu^\nu}(w)\partial^\mu w^n \partial^\nu w^m]$$ (5.15)

Prime over det’s in (5.13) means the product of nonzero eigenvalues. It is a rather complicated problem to compute determinants in (5.13) exactly. However, we need not do that. All we need is the dependence of $d\mu_i$ on $\rho_a$ and $v$, which fix $d\mu_i$ up to a numerical constant. The dependence on $\rho_a$ is easy to calculate since $\rho_a$ can enter determinants in (5.13) only as a ratio $|\rho_a|/a$. The latter is completely fixed by the one loop metric-complex structure $\beta$-functions (note that the model at hand is not a conformal one).

So let us first discuss $\beta$-functions. As is well known the first coefficient of the metric $\beta$-function is proportional to the Ricci tensor of the target space [23]. Explicitly, expanding $\det \Box_B^I$ in powers of any given external field $w$ and taking into account the term with $\log 1/a$ one gets

$$-\frac{1}{2} \log \left( \frac{\det \Box_B^I}{\det \Box_B^0} \right) \bigg|_{\log 1/a} = \log \left| \frac{\rho_a}{a} \right| \left\{ \frac{1}{4\pi} \int d^2 x R^i_{j\ell \mu^\nu} \partial^\mu w^\ell \partial^\nu w^j - 4 \right\}$$ (5.16)

$$= \log \left| \frac{\rho_a}{a} \right| \left\{ -\frac{1}{\pi} \int d^2 x (\partial w \partial \bar{w} + \partial \bar{w} \partial w) \left[ \frac{g'}{g} + \left( \frac{g''}{g} - \frac{g'^2}{g^2} \right) |w|^2 \right] - 4 \right\}.

Here $R^i_{j\ell \mu^\nu} = R^k_{i,kj}$ and $g$ is considered as a function of $|w|^2$, $g'$ is its derivative with respect to $|w|^2$. Term $-4 \log |\rho_a|/a$ accounts for the subtraction of zero modes. It just cancels the factor $1/a^4$ in (5.13) as the a-dependence of $d\mu_i$ can come only from coupling constant renormalization (and from a-dependence of $S_I$ in (5.7) for the particular case of the black hole metric). The size of $\bar{I} |\rho_a|$ is put in (5.16) just as a IR cutoff. In a similar way one can calculate the fermion contribution to one-loop $\beta$-function[23]. One gets the complex structure renormalization

$$\frac{1}{2} \log \left( \frac{\det \Box_F^i}{\det \Box_F^0} \right) \bigg|_{\log 1/a} = \log \left| \frac{\rho_a}{a} \right| \left\{ \frac{1}{8\pi} \int d^2 x \epsilon_{\mu\nu} J^i_{k\ell \mu^\nu} \partial^\mu w^n \partial^\nu w^m \right\}$$

20
We observe now that for $I$ background ($\partial w = 0$) both factors (5.16) and (5.17) cancels each other. This, of course, should be expected: as there is no coupling constant dependence (more generally, no target space metric $g$ dependence) of $I$ contributions at the classical level, coupling constant (more generally metric) renormalization can not arise as well. Instead, for $I$ background ($\partial w = 0$) the fermion $\beta$-function in (5.17) doubles the boson one in (5.16). Thus the measure $d\mu_I$ is proportional to

$$
\left( \frac{\rho_a}{a} \right)^{2b},
$$

(5.18)

where $b$ is

$$
b = -\frac{1}{\pi} \int d^2x \partial \bar{w}_I \partial w_I \left[ \frac{g'}{g} + \left( \frac{g''}{g} - \frac{g'^2}{g^2} \right) |w_I|^2 \right].
$$

(5.19)

Say, for $O(3)$ $\sigma$-model $b_{\text{sphere}} = 2$, moreover, renormalization (5.16), (5.17) does not change the form (3.2) of the metric and complex structure. For black hole metric (3.1)

$$
b_{BH} = 1,
$$

(5.20)

however, the form of the metric and complex structure in (3.1) is not stable under renormalization (5.16), (5.17).

Now let us discuss the $g(v)$ dependence of the measure in (5.10). It is very important for us because we are going to study the $|v|$ dependence of $V_I$. To calculate it we use the method of ref. [16] which allows us to compute the variation of determinants in (5.13) with respect to given parameters (now we are interested in variations with respect to $v$) in terms of low eigenvalue and high eigenvalue limits of certain exponential operators. One has, say, for $\det \square_B$

$$
-\frac{1}{2} \delta_v \left( \log \frac{\det' \square_B}{\det \square_B} \right) = -\frac{1}{2} \text{Tr} \left\{ g^{-1} \delta_v g \left( e^{-t\square_B} - e^{-t\triangle_B} \right) \right\} |_{t=\infty} \quad (5.21)
$$

$$
+ \frac{1}{2} \text{Tr} \left\{ g^{-1} \delta_v g \left( e^{-t\square_B} - e^{-t\triangle_B} \right) \right\} |_{t=0}.
$$
Here the trace is understood as both the trace of operator and the one over target space induces, while

$$\Delta^i_{Bj} = - \left( D^i_{\mu p} D^p_{\mu j} - R^i_{i,jk} \partial_{\mu} w^k \partial_{\mu} w^j \right) . \quad (5.22)$$

In fact one should be more careful considering the high eigenvalue term \((t = 0)\) in \((5.21)\) \([16] \). Namely, one should introduce the world-sheet metric \(h_{\mu\nu}(x)\) because this term gives

$$\frac{d}{8\pi} \int d^2 x \delta \nu [\log g] R^{(2)}, \quad (5.23)$$

where \(R^{(2)}\) is the world sheet curvature and \(d = 2\). This is nothing other than the dilaton \(\beta\) -function related to the Polyakov anomaly \([17] \). The high eigenvalue term of the fermion determinant gives the same result as in \((5.23)\) but with the inverse sign, thus both factors cancels each other. This should be expected since the total central charge is zero in the TFT.

The low eigenvalue term in \((5.21)\) is the contribution of zero modes. It gives \(\delta \nu \left(S_I g(v) \log L/a\right)^{-1}\). For the \(I\) background the fermion low eigenvalue term gives the same result, thus one gets the factor

$$\frac{1}{g^2(v) S^2_I \log^2 L/a} \quad (5.24)$$

Now, taking into account \((5.18)\) and \((5.24)\) one obtain

$$d\mu_I = \text{const} \frac{d^2 x a d^2 \rho_a}{|\rho_a|^4 g^2(v) S^2_I} \frac{1}{d^2 \beta_1 d^2 \beta_2} \left( \frac{|\rho_a|}{a} \right)^{2b} \quad (5.25)$$

Making the substitution \(v \to w\) in the argument of metric here and inserting the result in \((5.12)\) one gets finally (after integration over \(\beta_1, \beta_2\))

$$V_I = - g_I \int d^2 x \frac{d^2 \rho_a}{|\rho_a|^2 S^2_I} \left( \frac{|\rho_a|}{a} \right)^{2b} e^{-S_I g(w) \bar{\chi} \chi \partial^2_\mu (g(w) \bar{\chi} \chi)} \quad (5.26)$$

$$\exp \left\{ 2r^2 g(w) \left[ \rho_a \partial w + \bar{\rho}_a \bar{\partial} \bar{w} \right] + O(r^2 \rho^2_\alpha \partial^2 w) \right\} ,$$

\(\text{\footnotesize 6}\) It is curious to note, that this term had been calculated in \([13] \) before the Polyakov’s original paper \([13] \) appeared however, was not interpreted in terms of anomaly.
where we introduced a constant $g_I$, to be treated as a new independent coupling constant of the model. Operators $\chi, \bar{\chi}$ here could give nonzero contributions only if we consider them in the external instanton field. Therefore, one can replace $g\bar{D}\chi D\chi$ in (5.12) by $\partial^2 \mu (g\bar{\chi}\chi)$ using the fact that $\bar{D}\chi = 0, D\chi = 0$ for I field. The effective vertex (5.26) holds true for any $d = 2$ topological $\sigma$-model. The only factors which depend on the particular metric are $S_I$ and $b$.

It is easy to see that $V_I$ in (5.26) is a Q-invariant one. Moreover, it is Q-exact. This can be checked using eq. (4.6), which shows that operator $\partial^2 \mu O(x)$ in (5.26) is Q-exact. Therefore, indeed, if the BRST-symmetry is not broken then $I'$s plays no role, in accordance with the general rule.

Of course, the similar effective vertex can be obtained for instantons. Let us write it down for the sake of completeness (we are not going to use it in this paper).

$$V_I = -g_I \int d^2 x d^2 \rho \left[ \rho \bar{\rho} g^2(w)g(w)\lambda^{--}\lambda^{--} D\lambda^{--} \bar{D}\lambda^{++} \right]$$

Expanding $\exp - V$ in powers of $V_I, \bar{V}_I$ and taking into account nonlinear terms in $V$ one can study effects of $I\bar{I}$ interactions in the instanton medium [18].

Effective vertices (5.26) (5.27) reproduce effects of $I'$s and $\bar{I}'$s in the framework of the perturbation theory. They have to be added to the original action (2.16) of the model which afterwards should be treated perturbatively. In particular, expanding $\exp - V$ in powers of $V_I, \bar{V}_I$ and taking into account nonlinear terms in $V$ one can study effects of $I\bar{I}$ interactions in the instanton medium [18].

Now let us concentrate on the case of the black hole metric (3.1). The $\bar{I}$ action has the form (5.7). Therefore, the integral over $\rho_a$ in (5.26) is UV divergent at $r^2 >> 1$ and could be performed in the closed form. Cutting it off at $|\rho_a| \sim a$ one gets

$$V_I = -g_I \int d^2 x g(w)\bar{\chi}\chi \partial^2 \mu (g(w)\bar{\chi}\chi). \quad (5.28)$$

To conclude this section, let us observe that the vertex (5.28) is invariant under global target space diffeomorphisms. Indeed, it can be rewritten as

$$V_I = -g_I \int d^2 x iJ_{ij}(w)\chi^i\chi^j \partial^2 \mu \left( iJ_{kl}(w)\chi^k\chi^l \right), \quad (5.29)$$
which is manifestly covariant. This provides a check for our computation of the $v$ dependence of the $\bar{I}$ measure in (5.26). The vertex in (5.28) is also obviously $Q$-invariant as it is constructed with the help of operator (2.13) which is the one from the $Q$-cohomology of the $\sigma$-model.

6 Vacuum energy.

Now let us use the effective $\bar{I}$ vertex in (5.28) to calculate the $I\bar{I}$ contribution to the vacuum energy. Observe first that this calculation is essentially the same as the one for the correlation function (4.2) in the one $I$ background we already performed in section 4. Indeed, (5.28) gives

$$E_{\text{vac}}^{I\bar{I}} = \langle V_{\bar{I}} >_I = -g_I \delta_{x_1}^2 < O(x_1)O(x_2) > |_{x_2 \to x_1}$$

(6.1)

The eq. (6.1) shows that if there were no $x_{12}$ - dependence of the correlation function (4.2) the vacuum energy would be zero. More generally speaking $V_I$ in (5.28) is obviously $Q$-exact. Thus, if the BRST-symmetry is not broken, then $E_{\text{vac}} = 0$ in (6.1) as should be the case on the general ground. However, once we get $x_{12}$-dependence of $< O(x_1)O(x_2) >$ for finite $|v|$ in eq.(4.4), the vacuum energy is nonzero. Indeed, plugging (4.4) into (6.1) one arrives at finite $v$ at

$$E_{\text{vac}} = 16\pi^2 g_I g_{\bar{I}} \frac{[\log L/a + O(1)]}{a^2} V^{(2)}$$

(6.2)

where $V^{(2)}$ is the volume of the world sheet. We use, here the regularization of the $\delta$ function (3.12) which gives $\delta^{(2)}(0) = \frac{1}{\pi a^2}$. Instead, if $|v| \to \infty$, namely $|v| >> L/a$ one uses eq.(4.5) for the correlation function in (6.1). One gets the vacuum energy

$$E_{\text{vac}} = \frac{4\pi^2 g_I g_{\bar{I}}}{a^2} \frac{L^4}{|v|^4 a^4} V^{(2)}$$

(6.3)

which goes to zero at $|v| \to \infty$.

Results (6.2), (6.3) show that at finite $|v|$ we have an infinite number of unstable degenerative vacuum states (each of which corresponds to a given $|v|$) with nonzero vacuum energy (6.2). Hence the BRST-symmetry is broken for these states. Moreover, the correlation function (4.4) shows the coordinate dependence. Thus, the theory at finite $|v|$ can be interpreted as a broken phase of the TFT.
Instead, for $|v| \to \infty$ we have $E_{\text{vac}} \to 0$. On the other hand the correlation function in (4.5) looses its $x_{12}$-dependent correlation function $\langle O(x_1)O(x_2) \rangle$. The BRST symmetry is not broken in that stable vacuum state in accordance with the nonzero value of the Witten index. This state is interpreted as a topological phase of the model at hand.

\section{Discussion}

In this paper we study the topological $\sigma$ model with the black hole target space metric (3.1). We show that there are peculiar instantons in that model with the noncompact moduli space. As a result of divergences of integrals over the moduli space of instantons the unbroken topological phase is realized only at the VEV of the field $w$ going to infinity. Instead, at finite $|v|$ the theory possess an infinite number of unstable degenerative vacuum states with nonzero vacuum energy (6.2) and $x$-dependent correlation function (4.4). These vacuum states correspond to broken phase of the TFT.

The breakdown of the BRST-symmetry for these states can be considered as a spontaneous one. Indeed, effective $I$-induced vertex (5.28) is $Q$-invariant, so the BRST-symmetry is broken by the choice of the vacuum state at finite $|v|$. However, the appearance of UV cut-off parameter in our results for correlation function (4.4) and vacuum energy (6.2) signals that the non-perturbative conformal invariance breaking occurs (in addition to the perturbative one, associated with the non-zero $\beta$-function). The latter is perhaps related to some new anomaly associated with the non-compactness of the target space. The situation could be similar to that with chiral anomaly in 4D Gauge theories. In the latter case instantons do not produce the chiral symmetry breaking by themselves. They just saturate the chiral anomaly relation.

Now let us discuss the question of the lifetime of states with finite $|v|$. We give a qualitative arguments below that it is infinite. The decay rate of an unstable vacuum state is proportional to

$$\Gamma \sim e^{-S_b}, \quad (7.1)$$

where $S_b$ is the action on the bubble of the true vacuum inside the false one. In order to estimate $\Gamma$, one can try to pick up some bubble configuration $w_b$.
which goes to infinity at some point \( x_0 \), while at large \( |x - x_0| \) approaches finite VEV \( v \).

Observe now, that instanton (2.19) by itself plays a role of such a bubble. However, it is a stable solution of equations of motion and has no negative modes. Therefore it can not give a contribution to the decay rate of the false vacuum. Furthermore, consider the \( I \bar{I} \) pair. Sometimes in nontopological theories \( I \bar{I} \)-induced vacuum energy could have an imaginary part which reflects the non-borel summable behavior of the perturbation theory \([22]\). However, we got no imaginary part in (6.2) (at least to the leading order in \( \log L/a \)). That means that \( \Gamma \) in (7.1) is suppressed at least as \( 1/\log L/a \). Thus we can live forever in a broken phase of TFT in the toy model we consider in this paper and never realize that it is unstable.

Let us note, however that our calculation of the \( I \bar{I} \) vacuum energy in Sect.6 could be viewed only as a preliminary estimate. The point is that the answer in (6.2) contains the IR logarithmic divergence. It actually arises from the integral over the size of the instanton (the size of \( \bar{I} \) is small \( |\rho_a| \sim a \). Similar infrared divergences (related to the integration of the \( I \) size) is known to occur in \( O(3) \) \( \sigma \)-model \([16]\). In \( O(3) \) \( \sigma \)-model each \( I \) can be represented as a dipole of some “charge” and “anticharge” \([16]\). The infinitely growing size of \( I \) then means the phase transition (of the Kosterlitz-Thouless type) from the dipole phase of these “charges” to the plasma one. Something similar could happen in the model at hand. The issue needs a future study. If the above mentioned phenomenon happened in the black hole topological \( \sigma \)-model, then the scalar potential and the vacuum energy could be different from the one in (6.2) (still nonzero, of course).

Another question which remains unsolved in this paper is: what kind of physics emerge in the broken phase of the topological \( \sigma \) model at hand. The result in (4.4) shows that some physical degrees of freedom are “liberated”. However, it is unclear if (4.4) could be interpreted as an exchange of one or two boson particles or not. To clarify this question one has to study multipoint correlation functions (2.14) which receive contributions of instantons with \( k > 1 \). It is also clear, that the \( \bar{I} \) vertex in (5.28) gives rise to a nontrivial interactions, which would be interesting to study in more details.

Another intriguing direction of thinking is the possible string theory applications of the breakdown of the BRST-symmetry we find in the present paper. In particularly, in ref.\([8]\) the \( c = 1 \) string is shown to be equivalent to topological Kazama-Suzuki \( SL(2,R)/U(1) \) coset. On the top of this it
is interesting to consider the possible instanton solutions in $SL(2, R)/U(1)$ gauged $WZW$ theory. It differs from the model we study in this paper by the presence of the dilaton term. As dilaton term is a quantum correction one could expect that its presence cannot change the fact of the existence of instantons in the model. Then the emergence of instantons could produce dramatic consequences for the string theory. Of course, if the conformal invariance of 2D theory is broken it cannot serve as a string vacuum state any longer. However, if we think of quantum string theory, we might have to consider these states as well. This point of view has been recently taken up in refs. [23]. In latter papers the renormalization group flow (which could occur in certain black hole $\sigma$-models if instantons are taken into account) is interpreted as a decay of the false string vacuum and related to the black hole information loss paradox [24]. If the phenomenon we observe in this paper is indeed related to some kind of anomaly (as we conjecture above) than it could appear to be an unavoidable feature of the string in the black hole background.

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