Scale Invariance, Mass and Cosmology

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Abstract

The possibility of mass in the context of scale-invariant, generally covariant theories, is discussed. The realizations of scale invariance which are considered, are in the context of a gravitational theory where the action, in the first order formalism, is of the form

\[ S = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x \]

where \( \Phi \) is a density built out of degrees of freedom independent of gravity, which we call the "measure fields". For global scale invariance, a "dilaton" \( \phi \) has to be introduced, with non-trivial potentials \( V(\phi) = f_1 e^{\alpha \phi} \) in \( L_1 \) and \( U(\phi) = f_2 e^{2\alpha \phi} \) in \( L_2 \). This leads to non-trivial mass generation and potential for \( \phi \). Mass terms for an arbitrary matter field can appear in a scale invariant form both in \( L_1 \) and in \( L_2 \) where they are coupled to different exponentials of the field \( \phi \). Implications of these results for cosmology, having in mind in particular inflationary scenarios, models of the late universe and modified gravitational theories are discussed.

1 Introduction

The concept of scale invariance appears as an attractive possibility for a fundamental symmetry of nature. In its most naive realizations, such a symmetry is not a viable symmetry, however, since nature seems to have chosen some typical scales.
Here we will find that scale invariance can nevertheless be incorporated into realistic, generally covariant field theories. However, scale invariance has to be discussed in a more general framework than that of standard generally relativistic theories, where we must allow in the action, in addition to the ordinary measure of integration $\sqrt{-g}d^4x$, another one, $\Phi d^4x$, where $\Phi$ is a density built out of degrees of freedom independent of that of $g_{\mu\nu}$. To achieve global scale invariance, also a “dilaton” $\phi$ has to be introduced.

As will be discussed, a potential consistent with scale invariance can appear for the $\phi$ field. Such a potential has a shape which makes it suitable for the satisfactory realization of an inflationary scenario$^3$ of the improved type$^4$. Alternatively, it can be of use in a slowly rolling $\Lambda$—scenario for the late universe$^5$.

Finally, we also discuss how scale invariant mass terms, which lead to phenomenologically acceptable dynamics, can be introduced into the theory. We discuss some properties of such types of mass terms and their implication for the early universe inflationary cosmology, for the cosmology of a late universe filled with matter and for the possibility of obtaining modified gravitational dynamics.

2 The Non Gravitating Vacuum Energy (NGVE) Theory. Strong and Weak Formulations.

When formulating generally covariant Lagrangian formulations of gravitational theories, we usually consider the form

$$S_1 = \int L\sqrt{-g}d^4x, g = det g_{\mu\nu}$$

As it is well known, $d^4x$ is not a scalar but the combination $\sqrt{-g}d^4x$ is a scalar. Inserting $\sqrt{-g}$, which has the transformation properties of a density, produces a scalar action (1), provided $L$ is a scalar.

One could use nevertheless other objects instead of $\sqrt{-g}$, provided they have the same transformation properties and achieve in this way a different generally covariant formulation.

For example, given 4-scalars $\varphi_a$ ($a = 1, 2, 3, 4$), one can construct the density

$$\Phi = \varepsilon^{\mu\nu\rho\beta} \varepsilon_{abcd} \partial_\mu \varphi_a \partial_\nu \varphi_b \partial_\rho \varphi_c \partial_\beta \varphi_d$$

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and consider instead of (1) the action

\[ S_2 = \int L \Phi d^4x. \]  

(3)

\( L \) is again some scalar, which contains the curvature (i.e. the gravitational contribution) and a matter contribution, as it is standard also in (1).

In the action (3) the measure carries degrees of freedom independent of that of the metric and that of the matter fields. The most natural and successful formulation of the theory is achieved when the connection coefficients are also treated as an independent degrees of freedom. This is what is usually referred to as the first order formalism.

One can notice that \( \Phi \) is the total derivative of something, for example, one can write

\[ \Phi = \partial_\mu (\varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} \phi_a \partial_\nu \phi_b \partial_\alpha \phi_c \partial_\beta \phi_d). \]  

(4)

This means that a shift of the form

\[ L \rightarrow L + \text{constant} \]  

(5)

just adds the integral of a total divergence to the action (3) and it does not affect therefore the equations of motion of the theory. The same shift, acting on (1) produces an additional term which gives rise to a cosmological constant. Since the constant part of \( L \) does not affect the equations of motion resulting from the action (3), this theory is called the Non Gravitating Vacuum Energy (NGVE) Theory\(^1\).

One can generalize this structure and allow both geometrical objects to enter the theory and consider

\[ S_3 = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x \]  

(6)

Now instead of (5), the shift symmetry can be applied only on \( L_1 \) (\( L_1 \rightarrow L_1 + \text{constant} \)). Since the structure has been generalized, we call this formulation the weak version of the NGVE - theory. Here \( L_1 \) and \( L_2 \) are \( \varphi_a \) independent.

There is a good reason not to consider mixing of \( \Phi \) and \( \sqrt{-g} \), like for example using

\[ \frac{\Phi^2}{\sqrt{-g}} \]  

(7)
this is because (6) is invariant (up to the integral of a total divergence) under the infinite dimensional symmetry

\[ \varphi_a \rightarrow \varphi_a + f_a(L_1) \]  

where \( f_a(L_1) \) is an arbitrary function of \( L_1 \) if \( L_1 \) and \( L_2 \) are \( \varphi_a \) independent. Such symmetry (up to the integral of a total divergence) is absent if mixed terms (like (7)) are present. Therefore (6) is considered for the case when no dependence on the measure fields (MF) appears in \( L_1 \) or \( L_2 \).

In this paper we will see that the existence of two independent measures of integrations as in (6) allows new realizations of global scale invariance with most interesting consequences when the results are viewed from the point of view of cosmology.

### 3 The Action Principle for a Scalar Field in the Weak NGVE - Theory

We will study now the dynamics of a scalar field \( \phi \) interacting with gravity as given by the following action

\[ S_\phi = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x \]  

\[ L_1 = -\frac{1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \]  

\[ L_2 = U(\phi) \]  

\[ R(\Gamma, g) = g^{\mu\nu} R_{\mu\nu}(\Gamma), R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\nu\lambda} \]  

\[ R^\lambda_{\mu\nu\sigma}(\Gamma) = \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu} + \Gamma^\lambda_{\alpha\sigma} \Gamma^\alpha_{\mu\nu} - \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\sigma}. \]

In the variational principle \( \Gamma^\lambda_{\mu\nu}, g_{\mu\nu} \), the measure fields scalars \( \varphi_a \) and the scalar field \( \phi \) are all to be treated as independent variables although the variational principle may result in equations that allow us to solve some of these variables in terms of others.
4 Global Scale Invariance

If we perform the global scale transformation \((\theta = \text{constant})\)
\[ g_{\mu\nu} \rightarrow e^\theta g_{\mu\nu} \]  
then (9) is invariant provided \(V(\phi)\) and \(U(\phi)\) are of the form
\[ V(\phi) = f_1 e^{\alpha \phi}, U(\phi) = f_2 e^{2\alpha \phi} \]
and \(\varphi_a\) is transformed according to
\[ \varphi_a \rightarrow \lambda_a \varphi_a \]
(no sum on a) which means
\[ \Phi \rightarrow \left( \prod_a \lambda_a \right) \Phi \equiv \lambda \Phi \]
such that
\[ \lambda = e^\theta \]
and
\[ \phi \rightarrow \phi - \frac{\theta}{\alpha}. \]

In this case we call the scalar field \(\phi\) needed to implement scale invariance "dilaton".

5 The Equations of Motion

We will now work out the equations of motion for arbitrary choice of \(V(\phi)\) and \(U(\phi)\). We study afterwards the choice (15) which allows us to obtain the results for the scale invariant case and also to see what differentiates this from the choice of arbitrary \(U(\phi)\) and \(V(\phi)\) in a very special way.

Let us begin by considering the equations which are obtained from the variation of the fields that appear in the measure, i.e. the \(\varphi_a\) fields. We obtain then
\[ A^a_\mu \partial_\mu L_1 = 0 \]
where $A_\mu^a = \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d$. Since it is easy to check that $A_\mu^a \partial_\mu \varphi_a' = \frac{\delta a'}{4} \Phi$, it follows that $\det (A_\mu^a) = \frac{4^{-1}}{4!} \Phi^3 \neq 0$ if $\Phi \neq 0$. Therefore if $\Phi \neq 0$ we obtain that $\partial_\mu L_1 = 0$, or that

$$L_1 = -\frac{1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V = M$$

where $M$ is constant.

Let us study now the equations obtained from the variation of the connection $\Gamma^\lambda_{\mu\nu}$. We obtain then

$$-\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\beta\mu} g^{\beta\lambda} g_{\alpha\nu} + \delta^\lambda_{\beta\mu} g^\alpha_{\nu} - g_{\alpha\nu} \partial_\mu \varphi^\alpha_{\lambda} + \delta^\lambda_{\beta\mu} g_{\alpha\nu} \partial_\beta \varphi^\alpha_{\gamma} - \delta^\lambda_{\beta\mu} \frac{\Phi_{\mu\nu}}{\Phi} + \delta^\lambda_{\beta\mu} \frac{\Phi_{\nu\mu}}{\Phi} = 0$$

(22)

If we define $\Sigma^\lambda_{\mu\nu}$ as $\Sigma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \{\lambda_{\mu\nu}\}$ where $\{\lambda_{\mu\nu}\}$ is the Christoffel symbol, we obtain for $\Sigma^\lambda_{\mu\nu}$ the equation

$$-\sigma_{\lambda} g_{\mu\nu} + \sigma_{\mu} g_{\nu\lambda} - g_{\mu\lambda} \Sigma^\alpha_{\nu\lambda} + g_{\mu\nu} \Sigma^\alpha_{\mu\lambda} + g_{\nu\lambda} g_{\mu\nu} \Sigma^\alpha_{\beta\gamma} \Sigma^\alpha_{\beta\gamma} = 0$$

(23)

where $\sigma = \ln \chi, \chi \equiv \frac{\Phi}{\sqrt{-g}}$.

The general solution of (23) is

$$\Sigma^\alpha_{\mu\nu} = \delta^\alpha_{\mu} \lambda_{\nu} + \frac{1}{2} (\sigma_{\mu} \delta^\alpha_{\nu} - \sigma_{\nu} \delta^\alpha_{\mu} g^{\alpha\beta})$$

(24)

where $\lambda$ is an arbitrary function due to the $\lambda$ - symmetry of the curvature $R^\lambda_{\mu\nu\alpha}(\Gamma)$,

$$\Gamma^\lambda_{\mu\nu} \rightarrow \Gamma^\lambda_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \delta^\lambda_{\mu} Z_{\nu}$$

(25)

$Z$ being any scalar (which means $\lambda \rightarrow \lambda + Z$).

If we choose the gauge $\lambda = \frac{z}{2}$, we obtain

$$\Sigma^\alpha_{\mu\nu}(\sigma) = \frac{1}{2} (\delta^\alpha_{\mu} \sigma_{\nu} + \delta^\alpha_{\nu} \sigma_{\mu} - \sigma_{\beta} g_{\mu\nu} g^{\alpha\beta}).$$

(26)

Considering now the variation with respect to $g^{\mu\nu}$, we obtain

$$\Phi \left( -\frac{1}{\kappa} R_{\mu\nu}(\Gamma) + \frac{1}{2} g_{\mu\nu} \Phi_{\nu} \right) - \frac{1}{2} \sqrt{-g} U(\varphi) g_{\mu\nu} = 0$$

(27)

Solving for $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$ and introducing in (21), we obtain a constraint,

$$M + V(\varphi) - \frac{2 U(\varphi)}{\chi} = 0$$

(28)
that allows us to solve for $\chi$,

$$\chi = \frac{2U(\phi)}{M + V(\phi)}.$$  \hspace{1cm} (29)

To get the physical content of the theory, it is convenient to go to the Einstein conformal frame where

$$\bar{g}_{\mu\nu} = \chi g_{\mu\nu}$$  \hspace{1cm} (30)

and $\chi$ given by (29). In terms of $\bar{g}_{\mu\nu}$ the non Riemannian contribution $\Sigma_{\mu\nu}^a$ disappears from the equations, which can be written then in the Einstein form ($R_{\mu\nu}(\bar{g}_{\alpha\beta})$ = usual Ricci tensor)

$$R_{\mu\nu}(\bar{g}_{\alpha\beta}) - \frac{1}{2} \bar{g}_{\mu\nu} R(\bar{g}_{\alpha\beta}) = \frac{\kappa}{2} T_{\mu\nu}^{\text{eff}}(\phi)$$  \hspace{1cm} (31)

where

$$T_{\mu\nu}^{\text{eff}}(\phi) = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \phi_{,\alpha} \phi_{,\beta} \bar{g}^{\alpha\beta} + \bar{g}_{\mu\nu} V_{\text{eff}}(\phi)$$  \hspace{1cm} (32)

and

$$V_{\text{eff}}(\phi) = \frac{1}{4U(\phi)}(V + M)^2.$$  \hspace{1cm} (33)

In terms of the metric $\bar{g}^{\alpha\beta}$, the equation of motion of the Scalar field $\phi$ takes the standard General - Relativity form

$$\frac{1}{\sqrt{-\bar{g}}} \partial_{\mu}(\bar{g}^{\mu\nu} \sqrt{-\bar{g}} \partial_{\nu} \phi) + V'_{\text{eff}}(\phi) = 0.$$  \hspace{1cm} (34)

Notice that if $V + M = 0$, $V_{\text{eff}} = 0$ and $V'_{\text{eff}} = 0$ also, provided $V'$ is finite and $U \neq 0$ and regular there. This means the zero cosmological constant state is achieved without any sort of fine tuning. This is the basic feature that characterizes the NGVE - theory and allows it to solve the cosmological constant problem$^1$. It should be noticed that the equations of motion in terms of $\bar{g}_{\mu\nu}$ are perfectly regular at $V + M = 0$ although the transformation (30) is singular at this point. In terms of the original metric $g_{\mu\nu}$ the equations do have a singularity at $V + M = 0$. The existence of the singular behavior in the original frame implies the vanishing of the vacuum energy for the true vacuum state in the bar frame, but without any singularities there.
In what follows we will study (33) for the special case of global scale invariance, which as we will see displays additional very special features which makes it attractive in the context of cosmology.

Notice that in terms of the variables $\phi, g_{\mu\nu}$, the "scale" transformation becomes only a shift in the scalar field $\phi$, since $g_{\mu\nu}$ is invariant (since $\chi \rightarrow \lambda^{-1}\chi$ and $g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}$)

$$g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}, \phi \rightarrow \phi - \frac{\theta}{\alpha}.$$ \hspace{1cm} (35)

6 Analysis of the Scale - Invariant Dynamics

If $V(\phi) = f_1 e^{\alpha\phi}$ and $U(\phi) = f_2 e^{2\alpha\phi}$ as required by scale invariance (14), (16), (17), (18), (19), we obtain from (33)

$$V_{eff} = \frac{1}{4f_2} (f_1 + Me^{-\alpha\phi})^2$$ \hspace{1cm} (36)

Since we can always perform the transformation $\phi \rightarrow -\phi$ we can choose by convention $\alpha > O$. We then see that as $\phi \rightarrow \infty, V_{eff} \rightarrow \frac{f_1^2}{4f_2} = const.$ providing an infinite flat region. Also a minimum is achieved at zero cosmological constant for the case $\frac{f_1}{M} < O$ at the point

$$\phi_{min} = \frac{-1}{\alpha} \ln |\frac{f_1}{M}|.$$ \hspace{1cm} (37)

Finally, the second derivative of the potential $V_{eff}$ at the minimum is

$$V''_{eff} = \frac{\alpha^2}{2f_2} |f_1|^2 > O$$ \hspace{1cm} (38)

if $f_2 > O$, there are many interesting issues that one can raise here. The first one is of course the fact that a realistic scalar field potential, with massive excitations when considering the true vacuum state, is achieved in a way consistent with the idea (although somewhat generalized) of scale invariance.

The second point to be raised is that there is an infinite region of flat potential for $\phi \rightarrow \infty$, which makes this theory an attractive realization of the improved inflationary model$^4$. 

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A peculiar feature of the potential (36), is that the integration constant $M$, provided it has the correct sign, i.e. that $f_1/M < 0$, does not affect the physics of the problem. This is because if we perform a shift
\[
\phi \rightarrow \phi + \Delta
\]  
(39)
in the potential (36), this is equivalent to the change in the integration constant $M$
\[
M \rightarrow M e^{-\alpha \Delta}.
\]  
(40)

We see therefore that if we change $M$ in any way, without changing the sign of $M$, the only effect this has is to shift the whole potential. The physics of the potential remains unchanged, however. This is reminiscent of the dilatation invariance of the theory, which involves only a shift in $\phi$ if $\bar{g}_{\mu\nu}$ is used (see eq. (35) ).

This is very different from the situation for two generic functions $U(\phi)$ and $V(\phi)$ in (33 ). There, $M$ appears in $V_{\text{eff}}$ as a true new parameter that generically changes the shape of the potential $V_{\text{eff}}$, i.e. it is impossible then to compensate the effect of $M$ with just a shift. For example $M$ will appear in the value of the second derivative of the potential at the minimum, unlike what we see in eq. (38), where we see that $V''_{\text{eff}}(\text{min})$ is $M$ independent.

In conclusion, the scale invariance of the original theory is responsible for the non appearance (in the physics) of a certain scale, that associated to $M$. However, masses do appear, since the coupling to two different measures of $L_1$ and $L_2$ allow us to introduce two independent couplings $f_1$ and $f_2$, a situation which is unlike the standard formulation of globally scale invariant theories, where usually no stable vacuum state exists.

Notice that we have not considered all possible terms consistent with global scale invariance. Additional terms in $L_2$ of the form $e^{\alpha \phi} R$ and $e^{\alpha \phi} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi$ are indeed consistent with the global scale invariance (14), (16), (17), (18), (19) but they give rise to a much more complicated theory, which will be studied in a separate publication. There it will be shown that for slow rolling and for $\phi \rightarrow \infty$ the basic features of the theory are the same as what has been studied here. Let us finish this section by comparing the appearance of the potential $V_{\text{eff}}(\phi)$, which has privileged some point depending on $M$ (for example the minimum of the potential will have to be at some specific point), although the theory has the "translation invariance" (35), to the physics of solitons.
In fact, this very much resembles the appearance of solitons in a spacetranslation invariant theory: The soliton solution has to be centered at some
point, which of course is not determined by the theory. The soliton of course
breaks the space translation invariance spontaneously, just as the existence of
the non trivial potential $V_{eff}(\phi)$ breaks here spontaneously the translations
in $\phi$ space, since $V_{eff}(\phi)$ is not a constant.

Notice however, that the existence for $\phi \to \infty$, of a flat region for $V_{eff}(\phi)$
can be nicely described as a region where the symmetry under translations
(35) is restored.

7 Cosmological Applications of the Model

Since we have an infinite region in which $V_{eff}$ as given by (36) is flat ($\phi \to \infty$),
we expect a slow rolling (new inflationary) scenario to be viable, provided
the universe is started at a sufficiently large value of the scalar field $\phi$.

One should point out that the model discussed here gives a potential with
two physically relevant parameters $\frac{f_1^2}{4f_2}$, which represents the value of $V_{eff}$
as $\phi \to \infty$, i.e. the strength of the false vacuum at the flat region and $\frac{\alpha^2 f_1^2}{2f_2}$,
representing the mass of the excitations around the true vacuum with zero
cosmological constant (achieved here without fine tuning).

When a realistic model of reheating is considered, one has to give the
strength of the coupling of the $\phi$ field to other fields. It remains to be seen
what region of parameter space provides us with a realistic cosmological
model.

Furthermore, one can consider this model as suitable for the very late
universe rather than for the early universe, after we suitably reinterpret the
meaning of the scalar field $\phi$.

This can provide a long lived almost constant vacuum energy for a long
period of time, which can be small if $f_1^2/4f_2$ is small. Such small energy
density will eventually disappear when the universe achieves its true vacuum
state. For a more detailed scenario which includes the effect of matter other
than the dilaton $\phi$ see next section.

Notice that a small value of $\frac{f_1^2}{f_2}$ can be achieved if we let $f_2 >> f_1$. In this
case $\frac{f_1^2}{f_2} << f_1$, i.e. a very small scale for the energy density of the universe is
obtained by the existence of a very high scale (that of $f_2$) the same way as a
small fermion mass is obtained in the see-saw mechanism\(^7\) from the existence also of a large mass scale.

## 8 Introducing Scale Invariant Mass Terms for Additional Matter Fields and Cosmological Implications

So far we have studied a theory which contains the metric tensor \(g_{\mu\nu}\), the measure fields \(\varphi_a\) (\(a=1,2,3,4\)) and the "dilaton" \(\phi\), which makes global scale invariance possible in a non-trivial way. All of the above fields have some kind of geometrical significance, but if we are to describe the real world, the list of fields and/or particles to be introduced has to be enlarged.

To see how scale invariant mass terms are possible, let us start with the simplest possible example, i.e. the case of a point particle.

A point particle can be discussed by a contribution to \(L_1\) and \(L_2\) in (9) of the form

\[
L_{1p} = m_1 \int e^{-\alpha\phi/2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \delta(4)(x - x(\lambda)) \sqrt{-g} d\lambda \tag{41}
\]

and

\[
L_{2p} = m_2 \int e^{\alpha\phi/2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \delta(4)(x - x(\lambda)) \sqrt{-g} d\lambda. \tag{42}
\]

In this case, the contribution of \(L_{1p}\) to the first term of (9) and of \(L_{2p}\) to the second term of (9) give rise to scale invariant contributions under the transformations (14), (16), (17), (18) and (19).

Now, going through the same steps that lead us to the constraint (28), we get now instead

\[
M + V(\phi) - \frac{2U(\phi)}{\chi} + \frac{1}{\chi} (L_{1p} - \frac{1}{\chi} L_{2p}) = 0 \tag{43}
\]

If we are not located exactly on the particle, \(\chi\) is given by the old answer, i.e. \(\chi = 2U(\phi)/(M + V(\phi))\). If, however, we are located exactly at the point particle, the first three terms in (43) can be ignored, since they are
non-singular and we must then have $L_{1p} - \frac{1}{\chi}L_{2p} = 0$, which means
\[ \chi = \frac{m_2}{m_1} e^{\alpha \phi} . \] (44)

If an extended particle description of matter is taken, then (44) is obtained in the region of high density of matter while $\chi = 2U(\phi)/(M + V(\phi))$ is obtained for the low density of matter.

If (44) is inserted into (41), (42) and then both contributions of $L_{1p}$ and $L_{2p}$ are inserted in (9), we obtain for the particle contribution to the action
\[ S_p = 2\sqrt{m_1 m_2} \int \sqrt{g_{\mu\nu}} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda \] (45)
where a transformation to the Einstein Frame $\bar{g}_{\mu\nu} = \chi g_{\mu\nu}$ has been made. We see than that no time dependent masses are obtained, since in the Einstein Frame the $\phi$ dependence of the particle action totally disappears.

Similar results are obtained if instead of a point particle, we use an extended distribution of matter or a field, provided we use a high density approximation.

Taking, for example, the case of a fermion $\psi$, where the kinetic term of the fermion is chosen to be part of $L_1$
\[ S_{fk} = \int L_{fk} \Phi d^4x \] (46)

\[ L_{fk} = \frac{i}{2} \bar{\psi} \left[ \gamma^a V^\mu_a \left( \partial_\mu + \frac{1}{2} \omega^c_{\mu \sigma} \sigma_{cd} \right) - \left( \partial_\mu + \frac{1}{2} \omega^c_{\mu \sigma} \sigma_{cd} \right) \gamma^a V^\mu_a \right] \psi \] (47)
there $V^\mu_a$ is the vierbein, $\sigma_{cd} = \frac{1}{2} [\gamma_c, \gamma_d]$, the spin connection $\omega^c_{\mu \sigma}$ is determined by variation with respect to $\omega^c_{\mu \sigma}$ and, for self-consistency, the curvature scalar is taken to be (if we want to deal with $\omega^a_{\mu \nu}$ instead of $\Gamma_{\mu \nu}^\lambda$ everywhere)
\[ R = V^{a \mu} V^{b\nu} R_{\mu\nu ab}(\omega), R_{\mu\nu ab}(\omega) = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + (\omega^c_{\mu a} \omega_{\nu bc} - \omega^c_{\nu a} \omega_{\mu bc}) . \] (48)

Global scale invariance (14), (16), (17), (18) and (19) is obtained provided $\psi$ is also allowed to transform, as in
\[ \psi \rightarrow \lambda^{-\frac{1}{4}} \psi \] (49)
In this scale invariant case mass terms are of the form

\[ S_{fm} = m_1 \int \bar{\psi} \psi e^{\alpha \phi / 2} \phi d^4 x + m_2 \int \bar{\psi} \psi e^{3 \alpha \phi / 2} \sqrt{-g} d^4 x. \] (50)

If we once again consider the situation where \( m_1 e^{\alpha \phi / 2} \bar{\psi} \psi \) or \( m_2 e^{3 \alpha \phi / 2} \bar{\psi} \psi \)
are much bigger than \( V(\phi) + M \), i.e. a high density approximation, we obtain that instead of the constraint (28), the following holds,

\[ (3m_2 e^{3 \alpha \phi / 2} + m_1 e^{\alpha \phi / 2} \chi) \bar{\psi} \psi = 0 \] (51)

which means

\[ \chi = -\frac{3m_2}{m_1} e^{\alpha \phi}. \] (52)

Inserting (52) into (50), we obtain the \( \phi \) independent mass term after going to the conformal Einstein frame, which involves, when fermions are present, also a transformation of the fermion fields, necessary so as to achieve simultaneously the standard Einstein-Cartan form for both the gravitational and fermion equations. These transformations are, \( \bar{g}_{\mu \nu} = \chi g_{\mu \nu} \) (or \( \bar{V}_\mu = \chi^{\frac{3}{2}} V_\mu \)) and \( \psi' = \chi^{-\frac{1}{4}} \psi \) and they lead to a mass term,

\[ S_{fm} = -2m_2 \left( \frac{|m_1|}{3|m_2|} \right)^{3/2} \int \sqrt{-\bar{g}} \psi' \psi' d^4 x \] (53)

Once again, as in the case of the point particle, the \( \phi \) dependence of the mass term has disappeared (in this case, however, under the approximation of high density of the fermion fields). The situation for low density can be much more complicated and in fact could lead to a non-conventional type of dynamics in this limit. Connections to proposals for deviations from Newtonian dynamics appear possible, if one can correlate low densities, where deviations are expected here, to low accelerations, as is the case in Ref[8].

There is one situation where the low density of matter can also give results which are similar to those obtained in the high density approximation, in that the coupling of the \( \phi \) field disappears and that the mass term becomes of a conventional form in the Einstein conformal frame. For the point particle model this can never be the case, since the point particle always produces an infinite energy density at the point where it is located, but such a discussion can be made in a meaningful way in the case of an extended distribution, like that of a Dirac particle.
This is the case, when we study the theory for the limit $\phi \to \infty$. Then $U(\phi) \to \infty$ and $V(\phi) \to \infty$. In this case, taking $m_1 e^{3\alpha/2 \overline{\psi} \psi}$ and $m_2 e^{3\alpha/2 \overline{\psi} \psi}$ much smaller than $V(\phi)$ or $U(\phi)$ respectively and since also $M$ can be ignored in the constraint in this limit, we get then,

$$\chi = \frac{2f_2}{f_1} e^{\alpha \phi}.$$  \hspace{1cm} (54)

If (54) is inserted in (50), we get

$$S_{fm} = m \int \sqrt{-g} \psi' d^4x$$  \hspace{1cm} (55)

where

$$m = m_1 (\frac{f_1}{2f_2})^{1/2} + m_2 (\frac{f_1}{2f_2})^{3/2}$$  \hspace{1cm} (56)

Comparing (55)-(56) and (53) and taking for example $m_1$ and $m_2$ of the same order of magnitude, we see that the mass of the Dirac particle is much smaller in the region $\phi \to \infty$, for which (55), (56) are valid, than it is in the region of high density of the Dirac particle relative to $V(\phi) + M$, as displayed in eq. (53), if the assumption $\frac{f_2}{f_1} << 1$, which was motivated in section 8, is made.

Therefore if space is populated by these diluted Dirac particles of this type, the mass of these particles will grow substantially if we go to the true vacuum state valid in the absence of matter, i.e. $V = M = 0$, as dictated by $V_{\text{eff}}$ given by eq. (36).

The presence of matter pushes therefore the minimum of energy to a state where $V + M > 0$. The real vacuum in the presence of matter should not be located in the region $\phi \to \infty$, which minimizes the matter energy, but maximizes the potential energy $V_{\text{eff}}$ and not at $V = M = 0$, which minimizes $V_{\text{eff}}$, and where particle masses are big, but somewhere in a balanced intermediate stage. Clearly how much above $V = M = 0$ such true vacuum is located must be correlated to how much particle density is there in the Universe.

The situation described by eq. (55)-(56) represents the situation of low energy density of particles as compared to for example the false vacuum energy density, having there a very small mass. This mass can then grow a lot when the dilaton field approaches the minimum of its potential. This
situation resembles very much the scenario developed by Felder, Kofman and Linde\textsuperscript{9}, where, in the context of an inflationary scenario, particles, initially created with a very low mass, increase their mass considerably through the evolution of the inflaton field. This ”fattening” of the particle masses can play a role in making the transfer of energy from the inflaton field (in our case the dilaton field $\phi$) to matter very efficient.

Finally, let us mention that vector particles, even if massive, can be incorporated in a very simple way into the dilatation invariant theory described above. A scale invariant action, including mass is given by

$$S_{\text{vector}} = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4 x + \frac{m^2}{2} \int A_\mu A^\mu \Phi d^4 x$$

(57)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(58)

Notice that if done in this way, mass for the vector field $A_\mu$ is consistent with dilatation invariance, without need of a coupling of such a field to the dilaton $\phi$.

9 Discussion and Conclusions

In this paper we have seen that realistic realizations of scale invariance can be obtained when in addition to the standard measure of integration $\sqrt{-g}$, we also consider a measure of integration which is given in terms of degrees of freedom which are independent of the metric. A dilaton field $\phi$ has to be introduced in order to make global scale invariance possible. Masses and potentials are possible in a way consistent with scale invariance. This is achieved generically by allowing couplings of fields to both possible measures. Then a non trivial dilaton potential appears which has attractive features from the point of view of cosmology, like an infinite region of flat potential which is desirable in new inflation.

Masses for other fields different from the dilaton can be obtained also by coupling the mass terms to the two different measures. The coupling to two different measures can be done even if we do not require scale invariance and in this sense such idea can be exploited even outside the context of scale invariant theories and this is of interest by itself. For the scale invariant case additional surprises appear. In the first place, for the high density
approximation, particles behave like regular particles and the coupling to the dilaton totally disappears when we analyze the theory in the CEF. In some cases the low density approximation of the fields can give also a normal propagation, i.e. standard equations for the particles. This, as we have seen, happens in the infinite flat region of the dilaton potential. In this region the mass is different to that obtained in the high density approximation, more reasonable for the region near the true vacuum. Particles whose mass can naturally change and the cosmological application of this in Felder, Kofman and Linde type scenarios in connection with the effective transfer of energy from the dilaton field to matter have been discussed.

Also the fact that particle masses grow as we approach the state with zero cosmological constant can lead to an effect where the true vacuum in the presence of matter is not the zero cosmological constant state (found to be the true vacuum in the absence of matter), but a state where $V_{\text{eff}} > 0$. How much above zero is $V_{\text{eff}}$, depends on the amount of particles present.

From the point of view of particle physics, this theory could give a new approach to the family problem, since here we have a situation where the same particle in different states can have a different, although well defined mass.

Finally the theory when applied to the study of particles which are both low density and not in the infinite flat region of the potential could lead to a non conventional type of dynamics, which could be related to that discussed by Bekenstein and Milgrom, for example, if one can correlate low densities (needed here to get non conventional behavior) to low accelerations (as in the non conventional behavior of the models by Bekenstein and Milgrom).

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