On bound states in quantum field theory

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Abstract
The mechanism of formation of bound states in the relativistic quantum field theory is demonstrated by the Yukawa field model. It is shown that the weak coupling regime leads to the potential picture, i.e. it is equivalent to the nonrelativistic limit in the bound state problem. In the strong coupling regime the potential picture is not valid and the method of bosonisation of fermion currents (so-called $Z_2 = 0$ method) should be used. Essentially the nonlocal fermion currents are found to be responsible for the origin of bound states and ultraviolet convergence of fermion loops.
1 Introduction.

The greatest efforts are currently being made in order to describe physics of light and heavy bosons and baryons as bound states of light and heavy quarks from the first principles of QCD. The difficulty is that QCD like any local quantum field theory is well defined as perturbation expansion over an appropriate coupling constant, at the same time bound states arise as a result of nonperturbative interactions. Here I do not discuss the specific features of QCD such as confinement of quarks and gluons, asymptotic freedom and behaviour at large distances. The problem is more general: what is the mechanism of formation of bound states in quantum field theory? Great efforts were made to understand how bound states arise in the formalism of quantum field theory and to work out effective methods to calculate all characteristics of these bound states especially their masses and quantum numbers. Unfortunately, we can establish that there is no a well defined unique method, like the Schrödinger equation in the nonrelativistic quantum mechanics, which can be used for this aim.

On the other hand, there are two well-known approaches to investigate the bound state problem: the Potential picture (see, for example, [1]) including Bethe-Salpeter equation (see, for example, [2]) and the Bosonization of fermion currents or so-called $Z_2 = 0$ approach (see, for example, [3]). The potential or quasipotential picture in the standard approach can be obtained by taking into account the interaction of two particles-fermions due to one-boson or one-gluon exchange. This potential picture is formulated in the form of the Bethe-Salpeter or quasipotential equation which turns into the Schrödinger equation in the nonrelativistic limit.

Let us explain in a few words the idea of the method of the Bosonization of fermion currents or the $Z_2 = 0$ approach. Let a quantum field system of fermions and bosons like quark-gluons, electron-photon or nucleon-meson be given and the interaction be described by the Yukawa-type Lagrangian. If a bound state of a fermion-antifermion pair arises in one of these systems, this bound state, being a bosonic state, should be described by its own quantum bosonic field which is absent in the initial Lagrangian. We can introduce this additional quantum field into the initial Lagrangian adding the free Lagrangian of this field and the interaction Lagrangian which is the product of the boson field by the fermion current with appropriate quantum numbers. The fact that this bosonic field is the bound state of the fermion-antifermion
pair means that the constant of renormalization of the wave boson function is equal to zero, i.e. $Z_2 = 0$.

Effective practical application of this idea was always doubtful because the renormalization constants are divergent in the local quantum field theory where the fermion currents are usually chosen to be local. Moreover, the relation between the potential picture and the bosonization of fermion currents from the point of view of quantum field theory is not clarified up to now.

Naturally, the task which should be solved in the first place is to make clear the connection between these two approaches and to find the conditions which separate them. The aim of this paper is to answer this question by considering a relatively simple quantum field model. This answer will help us to understand more deeply what kinds of approximations are used in the standard well-known approaches.

An above-mentioned simple model under consideration is the Yukawa interaction of the so-called "scalar" one-component fermions $\Psi$ interacting with scalar bosons $\phi$. The Lagrangian density is

$$L(x) = (\Psi^+(x)(\Box - M^2)\Psi(x)) + \frac{1}{2} \phi(x)(\Box - m^2)\phi(x) + g(\Psi^+(x)\Psi(x))\phi(x).$$

The model makes it possible to retrace all details of arising of bound states in quantum field theory. Generalization to the case of the Dirac field has no difficulties of principle and leads to technical problems connected with the algebra of $\gamma$-matrices only. This model is superrenormalizable so that the renormalization procedure has the simplest form. The main aim of this paper will be to understand the general mechanism of arising of bound states in this quantum field model and to outline possible methods to investigate this problem.

The model contains three dimensionless parameters:

$$\lambda = \frac{1}{4\pi} \left( \frac{g}{M} \right)^2, \quad \xi = \left( \frac{m}{M} \right)^2, \quad b = \left( \frac{\mu}{2M} \right)^2,$$

where $\mu$ is the mass of a supposed bound state. The parameter $\xi$ is supposed to be smaller then 1. Our aim will be to find the condition under which the mass $\mu$ of the bound state belongs to the interval

$$0 < \mu < 2M \quad \text{or} \quad 0 < b = \left( \frac{\mu}{2M} \right)^2 < 1,$$
i.e. this bound state should be stable.

The value of the coupling constant $\lambda$ plays the crucial role to separate these two approaches: the potential picture and the bosonization of fermion currents. Namely

- the *Potential picture* takes place for

\[
\lambda = \frac{1}{4\pi} \left( \frac{g}{M} \right)^2 \ll 1 \quad (4)
\]

and $\mu = 2M - \epsilon$ where

\[
\epsilon \ll 2M \quad \text{or} \quad 1 - b = \left[ 1 - \left( \frac{\mu}{2M} \right)^2 \right] \ll 1, \]

i.e., the mass excess is very small;

- the *Bosonization of fermion currents* takes place for

\[
\lambda = \frac{1}{4\pi} \left( \frac{g}{M} \right)^2 \geq 1 \quad (5)
\]

and the mass excess can be comparable with $2M$, i.e.

\[
2M - \mu \sim M \quad \text{or} \quad b \sim \frac{1}{2}.
\]

The result of this paper is that in the relativistic quantum field theory the nonrelativistic potential picture takes place in the weak coupling regime and the bosonization of fermion currents takes place in the strong coupling regime.

In principle, this paper contains only the first rough scheme showing the way how to calculate masses and quantum numbers of possible bound states in quantum field systems. There are plenty of questions which have to be answered.

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2 The formulation of the problem

For simplicity we consider the interaction of scalar fermions described by the one-component field Ψ(\(x\)) and bosons described by the field \(\phi(x)\). All consideration is given in the Euclidean metrics. The total Lagrangian can be written in the form

\[
L[Ψ, φ] = \int dx L(x) = - (Ψ^+ D_M^{-1}Ψ) - \frac{1}{2}(ϕ D_M^{-1}ϕ) + g(Ψ^+Ψ)φ,
\]

\(D_M^{-1}D_M(x - y) = (-\Box + M^2)δ(x - y)\),

\(D_M(x - y) = \int \frac{dp}{(2π)^4} \frac{e^{ip(x-y)}}{M^2 + p^2} \).

The four-point Green function

\[
G(x_1, x_2, y_1, y_2) = \int DΨ DΨ^+ \int Dφ Ψ^+(x_1)Ψ(x_2)Ψ^+(y_1)Ψ(y_2) e^{L[Ψ, φ]},
\]

where the normalization should be introduced

\(DΨ DΨ^+ \rightarrow \frac{DΨ DΨ^+}{C}, \quad C = \int DΨ DΨ^+ \int Dφ e^{L[Ψ, φ]} \).

The four-point Green function contains all information about possible bound states in the two-fermion channels. The particles Ψ can be called constituent particles. We shall be interested in the bound states with definite quantum numbers which can be defined as

\[
Ψ^+(x_1)Ψ(x_2) \rightarrow J_Q(x) = (Ψ^+V_QΨ)_x = Ψ^+(x)V_Q(\vec{p}_x)Ψ(x),
\]

where the nonlocal vertex \(V_Q(\vec{p}_x)\) defines the quantum numbers \(Q\) of the current \(J_Q = (Ψ^+V_QΨ)\). It can be represented in the form

\[
V_Q(\vec{p}_x) = \int du \tilde{V}_Q(u)e^{i\vec{u}\vec{p}_x},
\]
The current $J_Q = (\Psi^+ V_Q \Psi)$ can be represented as

$$J_Q(x) = (\Psi^+ V_Q \Psi)_x = (\Psi^+(x)V_Q(\hat{p}_x)\Psi(x)) = \int du \; \Psi^+(x+u)V_Q(u)\Psi(x-u).$$

The Green function with quantum numbers $Q$ is defined as

$$G_Q(x-y) = \int D\Psi D\Psi^+ \int D\phi \; J_Q(x) J_Q(y) \; e^{L[\Psi,\phi]}. \quad (9)$$

This function has the following asymptotic behaviour

$$G_Q(x-y) \sim e^{-\mu_Q |x-y|} \quad \text{for} \quad |x-y| \to \infty,$$

The mass of the state $J_Q = (\Psi^+ V_Q \Psi)$ can be found as

$$\mu_Q = - \lim_{x \to \infty} \frac{1}{|x|} \ln G_Q(x). \quad (10)$$

The problem is to calculate the functional integral in the representation (9) and calculate the mass $\mu_Q$ according to (10).

If we consider perturbation expansion over the coupling constant $g$ for the four point Green function $G(x_1, x_2, y_1, y_2)$, we will get a series of the Feynman diagrams describing an interaction of two fermion particles $\Psi$. This series can be written in the form of the Bethe-Salpeter equation [2]. Bound states of two fermion particles in a channel $J_Q(x) = (\Psi^+ V_Q \Psi)_x$ can be found as solutions of this equation. Really, the Bethe-Salpeter equation can effectively be investigated only in the one-boson exchange approximation, i.e. in the weak coupling regime.

We proceed in another way, we shall consider the generating functional (9). Fortunately, it is possible to do the first integration over either the fermion field $\Psi(x)$ or the boson field $\phi(x)$. Thus, we can get two representations which are the starting points of two approaches: the Potential picture and the Bosonization of fermion currents.

### I. Potential picture

The integration in (7) over the fermion field $\Psi$ gives

$$\int D\Psi D\Psi^+ \; \Psi^+(x_1)\Psi(x_2)\Psi^+(y_1)\Psi(y_2)$$
\[
\exp \left\{ \int dx \Psi^+(x) \left( \Box - M^2 + g\phi(x) \right) \Psi(x) \right\} \\
= \left[ -S^+(x_1, y_2|\phi) S(x_2, y_1|\phi) + S(x_2, x_1|\phi) S(y_2, y_1|\phi) \right] \\
\cdot \exp \left\{ \text{tr} \ln [1 - g\phi D_M] \right\}.
\]

The Green function \( S(x, y|\phi) \) satisfies the equation

\[
[\Box - M^2 + g\phi(x)] S(x, y|\phi) = \delta(x-y)
\]

with

\[
S^+(x, y|\phi) = S(x, y|\phi).
\]

The Green function (9) can be written

\[
G_Q(x - y) = -G_Q^{(P)}(x - y) + G_Q^{(A)}(x - y),
\]

where

\[
L_P[\phi] = -\frac{1}{2} (\phi D_m^{-1} \phi) + \text{tr} \ln [1 - g\phi D_M].
\]

Here the functions \( G_Q^{(P)}(x) \) and \( G_Q^{(A)}(x) \) are said to be "potential" and "annihilation" Green functions, respectively. The approach based on the representation (13) shall be called the Potential picture.

II. Bosonization of fermion currents

The integration in (9) over the scalar field \( \phi(x) \) gives

\[
G_Q(x - y) = \int D\Psi D\Psi^+ e^{L_B[\Psi]} (\Psi^+ V_Q \Psi)_x (\Psi^+ V_Q \Psi)_y,
\]

where

\[
L_B[\Psi] = - (\Psi^+ D_M^{-1} \Psi) + \frac{g^2}{2} (\Psi^+ \Psi - \Psi D_m \Psi^+ \Psi).
\]
where
\[(\Psi^+ \Psi D_m \Psi^+ \Psi) = \int dx \int dy \Psi^+(x) \Psi(x) D_m(x - y) \Psi^+(y) \Psi(y), \quad (15)\]
and the following normalization should be introduced
\[D\Psi D\Psi^+ \rightarrow \frac{D\Psi D\Psi^+}{C}, \quad C = \int D\Psi D\Psi^+ e^{L_{\phi}[\Psi]}.
\]
The approach based on the representation (13) shall be called the **Bosonisation of fermion currents**.

## 3 The Potential picture.

The starting point of the Potential picture is the representation (13).

### 3.1 The Green function \( S(x, y|\phi) \)

The first step is that the fermion loops should be neglected, so that the Green functions \( G_P(x - y) \) and \( G_Q(x - y) \) are represented
\[
\begin{align*}
G_P(x - y) &= \int d\sigma w[\phi] \tilde{V}(u) S(x + u, y + v|\phi) \tilde{V}(v) S(x - u, y - v|\phi), \\
G_Q(x - y) &= \int d\sigma w[\phi] \tilde{V}(u) S(x + u, x - u|\phi) \cdot \tilde{V}(v) S(y + v, y - v|\phi) \\
d\sigma w[\phi] &= du dv D\phi \exp \left(-\frac{1}{2}(\phi(x)D^{-1}\phi)\right). \quad (16)
\end{align*}
\]

We would like to stress now that the neglect of fermion loops presupposes the dimensionless coupling constant \( \lambda \) (2) to be small enough.

The solution of (12) can be represented in the form of the following functional integral (see Appendix and, for example, [4]):
\[
S(x, y|\phi) = \frac{1}{\Box - M^2 + g\phi(x)} \cdot \delta(x - y) \quad (17)
\]
\[
= \int_0^\infty \frac{d\alpha}{8\pi^2 \alpha^2} e^{-\frac{1}{2}M^2} \int D\xi \exp \left\{-\int_0^\alpha d\tau \frac{\dot{\xi}^2(\tau)}{2} + \frac{g}{2} \int_0^\alpha d\tau \phi(\xi(\tau)) \right\},
\]
with the boundary conditions \( \xi(0) = y, \ \xi(\alpha) = x \) and the normalization
\[
\int D\xi \exp \left\{-\int_0^\alpha d\tau \frac{\dot{\xi}^2(\tau)}{2} \right\} = 1.
\]
3.2 The Green function $G_P(x)$

The function $G_P(x)$ after integration over $\phi$ has the form

$$G_P(x) = \left( \frac{M}{8\pi^2 x} \right)^2 \int d\Sigma_1 d\Sigma_2 \exp \{ W_{11} + 2W_{12} + W_{22} \}$$

$W_{ij} = \frac{g^2}{8} \int_0^{\alpha_i} d\tau_1 \int_0^{\alpha_j} d\tau_2 D_m (\xi_i(\tau_1) - \xi_j(\tau_2))$,

$$\int d\Sigma_j\{*\} = \int du_j \bar{V}(u_j) \int d\alpha_j e^{-\frac{x^2M^2}{2}} \int D\xi_j \exp \left\{ -\int_0^{\alpha_j} d\tau \frac{\dot{\xi}_j^2(\tau)}{2} \right\} \{*\},$$

$(j = 1, 2; \ u_1 = v, \ u_2 = v)$,

$$\xi_1(0) = v, \ \xi_1(\alpha_1) = x + u, \ \xi_2(0) = -v, \ \xi_2(\alpha_2) = x - u.$$  

where for simplicity we put $y = 0$.

Our task is to get the asymptotic behaviour of the functions $G_P(x)$ for asymptotically large $x = \sqrt{x^2} \to \infty$. To this end, let us introduce the following variables:

$$\alpha_j = \frac{x}{M s_j}, \quad \tau_j = \frac{\beta j}{M s_j},$$

$$\xi_1(\beta) = n\beta + \eta_1(\beta), \quad \xi_2(\beta) = n\beta + \eta_2(\beta), \quad n_\mu = \frac{x_\mu}{x}.$$  

Then, one can obtain

$$G_P(x) = \left( \frac{M}{8\pi^2 x} \right)^2 \int du \bar{V}(u) \int dv \bar{V}(v)$$

$$\cdot \int_0^\infty ds_1 ds_2 \exp \left\{ -\frac{xM}{2} \left( \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_2} \right) \right\} \cdot J_P(s_1, s_2; x),$$

$$J_P(s_1, s_2; x)$$

$$= \int \int D\eta_1 D\eta_2 \exp \left\{ -\int_0^x d\beta \left[ \frac{M s_1 \eta_1^2(\beta)}{2} + \frac{M s_2 \eta_2^2(\beta)}{2} \right] + W_x[\eta_1, \eta_2] \right\},$$

$$\eta_1(0) = v, \quad \eta_1(x) = u, \quad \eta_2(0) = -v, \quad \eta_2(x) = -u,$$
where
\[ W_x[\eta_1, \eta_2] = W_{11} + 2W_{12} + W_{22}, \]
\[ W_{ij} = \frac{g^2}{8M^2s_is_j} \int_0^x d\beta_1 \int_0^x d\beta_2 D_m (n(\beta_1 - \beta_2) + \eta_i(\beta_1) - \eta_j(\beta_2)), \]

The functional integral for \( J_x(s_1, s_2; x) \) looks like the Feynman path integral in the nonrelativistic statistic quantum mechanics for the four-dimensional motion of particles \( \eta_1(\beta) \) and \( \eta_2(\beta) \) with "masses" \( M_{S_1} \) and \( M_{S_2} \) where \( \beta \) plays the role of imaginary time or temperature. The interaction of these particles is defined by the nonlocal functional \( W_{11} + 2W_{12} + W_{22} \) which contains potential \( W_{12} \) and nonpotential \( W_{11} + W_{22} \) interactions.

The asymptotic form of the function \( J_P(s_1, s_2; x) \) looks like
\[ J_P(s_1, s_2; x) \sim \exp \{-xE_P(s_1, s_2)\}, \] 
where \( E_P(s_1, s_2) \) is the energy of the lowest bound state. The asymptotic behaviour of the functional \( G_P(x) \) as \( x \to \infty \) is determined by a saddle point of the integrals over \( s_1 \) and \( s_2 \) in the representation (18). Substituting expression (20) into (18), one can get
\[ \mu_P = -\lim_{x \to \infty} \frac{1}{|x|} \ln G_P(x), \]
\[ = \min_{(s_1, s_2)} \left[ M \left( \frac{1}{s_1} + \frac{1}{s_2} + s_1 + s_2 \right) + E_P(s_1, s_2) \right] \]
\[ = \min_{s} \left[ M \left( \frac{1}{s} + s \right) + E_P(s, s) \right]. \]

The main problem is to compute the functional integral (19). There is not much hope, at least in the near future, that the functional integral of the type (19) will be calculated exactly. We can only expect approximate methods to be worked out to compute integrals of this kind with acceptable accuracy especially for asymptotically large \( x \). We plan to calculate this functional integral applying the Gaussian equivalent representation method (see [4]), which was successfully used successfully for the polaron problem.

Besides one can see that this method is rough enough because it does not really feel the explicit form of the vertex \( \tilde{V}_Q(u) \) although it should extract a bound state with definite quantum numbers \( Q \). It means, in fact, that in
the general case, i.e., for any value of the coupling constant $g$, the modern analytical methods, applied to the functional integral (19), allow one to calculate with reasonable accuracy the energy of the lowest bound state only (see [4]).

3.3 The Green function $G_A(x - y)$

The function $G_A(x)$ after integration over $\phi$ has the form

$$G_A(x) = \left( \frac{M}{8\pi^2 x} \right)^2 \int d\Sigma_1 d\Sigma_2 \exp\{W_{11} + 2W_{12} + W_{22}\},$$

$$W_{jj} = \frac{g^2}{2} \int_0^{\alpha_j} d\tau_1 \int_0^{\alpha_j} d\tau_2 D_m (\xi_j(\tau_1) - \xi_j(\tau_2)), \quad (j = 1, 2),$$

$$W_{12} = \frac{g^2}{2} \int_0^{\alpha_1} d\tau_1 \int_0^{\alpha_2} d\tau_2 D_m (x - y + \xi_1(\tau_1) - \xi_2(\tau_2)).$$

Here

$$\xi_1(\tau) = x - u \left(1 - \frac{2\tau}{\alpha_1}\right) + \int_\tau^{\alpha_1} d\tau' \mu_1(\tau'),$$

$$\xi_2(\tau) = y - v \left(1 - \frac{2\tau}{\alpha_2}\right) + \int_\tau^{\alpha_2} d\tau' \mu_2(\tau'),$$

$$\xi_1(0) = -u, \quad \xi_1(\alpha_1) = u, \quad \xi_2(0) = -v, \quad \xi_2(\alpha_2) = v.$$ 

In the following we put $y = 0$ for simplicity. To study the asymptotic behaviour of the functions $G_j(x)$ as $x = \sqrt{x^2} \to \infty$, we introduce the variables

$$\alpha_j = \frac{x}{Ms_j}, \quad \tau_j = \frac{\beta_j}{Ms_j}.$$ 

One can obtain

$$G_A(x) = \left( \frac{M}{8\pi^2 x} \right)^2 \int du \tilde{V}(u) \int dv \tilde{V}(v)$$

$$\cdot \int_0^\infty ds_1 ds_2 \exp \left\{ -\frac{xM}{2} \left( \frac{1}{s_1} + s_1 + \frac{1}{s_2} + s_2 \right) \right\} \cdot J_A(s_1, s_2; x), \quad (22)$$

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\[ J_A(s_1, s_2; x) = \int \int D\eta_1 D\eta_2 \exp \left\{ - \int_0^x d\beta \left[ \frac{M s_1 \eta_1^2(\beta)}{2} + \frac{M s_2 \eta_2^2(\beta)}{2} \right] + W_x[\eta_1, \eta_2] \right\}, \tag{23} \]

where

\[ W_x[\zeta_1, \zeta_2] = W_{11} + 2W_{12} + W_{22}, \]

\[ W_{jj} = \frac{g^2}{8 M^2 s_j^2} \int_0^x \int_0^x d\beta_1 d\beta_2 D_m(\zeta_j(\beta_1) - \zeta_j(\beta_2)), \quad (j = 1, 2), \]

\[ W_{12} = \frac{g^2}{8 M^2 s_1 s_2} \int_0^x \int_0^x d\beta_1 d\beta_2 D_m(x + \zeta_1(\beta_1) - \zeta_2(\beta_2)), \]

\[ \zeta_1(0) = -u, \quad \zeta_1(x) = u, \quad \zeta_2(0) = -v, \quad \zeta_2(x) = v. \]

One can see that in the limit \( x \to \infty \)

\[ W_{12} \to 0 \]

and

\[ G_A(x) \to \left[ \frac{M}{8\pi^2 x} \int d\Sigma_1 e^{W_{11}} \right]^2, \]

so that no bound state arise in this case. Thus, the annihilation channel does not contain any bound states. In other words, intermediate pure boson states can not arange any bound state.

### 3.4 The Nonrelativistic Limit

In this section we obtain the nonrelativistic limit \( c \to \infty \) for the loop function \( G_P(x) \) in (18-19). To this end, let us restore the parameter \( c \) in our formulas:

\[ M \to Mc, \quad g \to \frac{g}{c}, \quad x_\mu = (x_4, \vec{x}) \to (ct, \vec{x}), \]

\[ D(x) \to cD(x) \]

\[ = c \int \frac{d^4k}{(2\pi)^4} \bar{D}(k^2)e^{ikx} = \int \frac{d^4\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{\infty} dv \bar{D} \left( \vec{k}^2 + \frac{v^2}{c^2} \right) e^{i(vt + \vec{k}\cdot\vec{x})} \]

\[ = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i(vt + \vec{k}\cdot\vec{x})} \int_{-\infty}^{\infty} dv \frac{e^{i(vt + \vec{k}\cdot\vec{x})}}{\vec{k}^2 + \frac{v^2}{c^2} + \kappa^2}. \]

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where $\kappa = mc$.

In the limit $c \to \infty$ we have

$$x = \sqrt{x^2} = \sqrt{c^2 t^2 + \vec{x}^2} \to ct, \quad n_\mu = (1, \vec{x}/ct) \to (1, 0),$$

Our task is to introduce the parameter $c$ in the explicit form into $G_P(x)$ (22) and then find the limit $c \to \infty$ in this expression for $G_P(x)$. Let us introduce in (22) new variables $\beta_j \to c\beta_j$. We obtain

$$G_P(t) = \left(\frac{M}{4\pi t}\right)^2 \int_0^\infty ds_1 ds_2 \exp \left\{ -\frac{M c^2}{2} \left( \frac{1}{s_1} + s_1 + \frac{1}{s_2} + s_2 \right) \right\}$$

$$\cdot J_P(s_1, s_2; t),$$

$$J_P(s_1, s_2; t) = \int D\eta_1 D\eta_2 \exp \left\{ -\int_0^t d\beta \left[ M s_1 \eta_1^2(\beta) + M s_1 \eta_2^2(\beta) \right] \right\}$$

$$\cdot \exp \{ W_i[\eta_1, \eta_2] \},$$

The functionals $W_{ij}$ become the form

$$W_{ij} = \frac{g^2}{8M^2 s_i s_j} \int_0^t \int_0^t d\beta_1 d\beta_2 \int \frac{d\vec{k}}{(2\pi)^3} \int_{-\infty}^\infty d\nu \tilde{D}_m \left( \vec{k}^2 + \frac{\nu^2}{c^2} \right)$$

$$\cdot \exp \left\{ i\nu \left( (\beta_1 - \beta_2) + \frac{1}{c} (\eta_4(\beta_1) - \eta_4(\beta_2)) \right) + i\vec{k} (\vec{\eta}_i(\beta_1) - \vec{\eta}_i(\beta_2)) \right\}.$$  \hspace{1cm} (25)

In the case of the standard scalar propagator one can get

$$\tilde{D}_m \left( \vec{k}^2 + \frac{\nu^2}{c^2} \right) \to \frac{1}{\vec{k}^2 + \kappa^2},$$

Here the parameter $\frac{1}{\kappa}$ defines the radius of the Yukawa potential; therefore, we will keep $\kappa = mc = \text{const}$ in the limit $c \to \infty$.

Now we are able to go to the limit $c \to \infty$. We can put all terms with $\frac{1}{c}$ equal to zero. The $\delta$-function $\delta(\beta_1 - \beta_2)$ arises because the fourth components $\eta_{j4}(\beta)$ disappear in the integrand. Thus, we get

$$W_{ij} \to \frac{g^2}{8M^2 s_i s_j} \int_0^t \int_0^t d\beta_1 d\beta_2 \delta(\beta_1 - \beta_2) \int \frac{d\vec{k}}{(2\pi)^3} \tilde{D}_m (\vec{k}^2) e^{i\vec{k} \cdot (\vec{\eta}_i(\beta_1) - \vec{\eta}_i(\beta_2))}$$

$$= \frac{1}{s_i s_j} \int_0^t d\beta U (\vec{\eta}_i(\beta) - \vec{\eta}_j(\beta)).$$

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where
\[ U(\vec{r}) = \frac{g^2}{8M^2} \int \frac{d\vec{k}}{(2\pi)^3} \tilde{D}_m \left( \vec{k}^2 \right) e^{i\vec{k}\vec{r}} = \frac{\lambda}{8} \cdot \frac{e^{-\kappa r}}{r} \]
is the nonrelativistic potential.

We shall consider only the term \( W_{12} \), because the terms \( W_{11} \) and \( W_{22} \) are equal to
\[ W_{jj} = \frac{t}{s_j^2} U(0) = \frac{g^2 t}{8M^2 s_j^2} \int \frac{d\vec{k}}{(2\pi)^3} \tilde{D}_m \left( \vec{k}^2 \right) \]
and contribute to the mass renormalization of particles as well as have purely relativistic origin.

In the limit \( c \to \infty \) in the functional integral for \( J_P(s_1, s_2; t) \) we can integrate over the fourth components of the 4-vectors \( \eta_1 \) and \( \eta_2 \):
\[
\int D\eta_1 \exp \left\{ -\int_0^t d\tau M \frac{s_1^2}{2} \frac{\dot{\eta}_2^2(\tau)}{2} \right\} = \exp \left\{ -\frac{Ms}{2} \cdot \frac{(v_4 - u_4)^2}{t} \right\} \]
where \( \eta_4(0) = u_4, \quad \eta_4(t) = v_4. \)

Now we can integrate over the fourth components of the 4-vectors \( u \) and \( v \):
\[
\int du_4 \tilde{V}(u) = \int du_4 \tilde{V}(u_4, \bar{u}) = \psi(\bar{u}). \quad (26)
\]

The next point is that the saddle points of the integrals over \( s_1 \) and \( s_2 \), which determine the asymptotic behaviour of the functional \( G_P(x) \) as \( x \to \infty \) in the limit \( c \to \infty \), are \( s_1 = s_2 = 1 \). Thus, the functional integral for \( J_P(t) = J_P(1, 1; t) \) becomes
\[
J_P(t) = e^{-12Mc^2} \int d\bar{u} \psi(\bar{u}) \int d\bar{v} \psi(\bar{v}) K(\bar{v}, t; \bar{u}, 0), \quad (27)
\]
\[
K(\bar{v}, t; \bar{u}, 0) = \int D\bar{\eta}_1 D\bar{\eta}_2 \cdot \exp \left\{ -\int_0^t d\beta \left[ \frac{M\bar{\eta}_1^2(\beta)}{2} + \frac{M\bar{\eta}_2^2(\beta)}{2} - U(\bar{\eta}_1(\beta) - \bar{\eta}_2(\beta)) \right] \right\},
\]
\[
\bar{\eta}_1(0) = \bar{u}, \quad \bar{\eta}_2(0) = -\bar{u}, \quad \bar{\eta}_1(t) = \bar{v}, \quad \bar{\eta}_2(t) = -\bar{v}.
\]
One can see that this representation for the function $K(\vec{v}, t; \vec{u}, 0)$ coincides with the Feynman path integral in the quantum mechanics [6] for the situation where there are two particles $\vec{\eta}_1$ and $\vec{\eta}_2$ with masses $M$ and the interaction between these particles is described by the potential $U(\vec{\eta}_1 - \vec{\eta}_2)$.

Let us introduce the standard variables

$$\vec{\xi}_1(\beta) = \vec{R}(\beta) + \frac{1}{2} \vec{r}(\beta), \quad \vec{\xi}_2(\beta) = \vec{R}(\beta) - \frac{1}{2} \vec{r}(\beta),$$

$$\vec{R}(0) = \vec{R}(t) = 0, \quad \vec{r}(0) = 2\vec{u}, \quad \vec{r}(t) = 2\vec{v}$$

and integrate over $\vec{R}(\beta)$. We obtain

$$K(\vec{v}, t; \vec{u}, 0) = \int D\vec{r} \exp \left\{ -\int_0^t d\beta \left[ \frac{M_r}{2} \dot{\vec{r}}^2(\beta) - V(\vec{r}(\beta)) \right] \right\},$$

$$= \sum_N \psi_N(\vec{v}) e^{-tE_N} \psi_N(\vec{u})$$

(28)

where the boundary conditions are $\vec{r}(0) = 2\vec{u}$, $\vec{r}(t) = 2\vec{v}$ and $M_r = M/2$. Here $\psi_N(\vec{r})$ and $E_N$ are eigenfunctions and eigenvalues for the quantum number $N$ connected with the space $R^3$ of the Schrödinger equation

$$\left[ \frac{p^2}{2M_r} - U(\vec{r}) \right] \psi_N(\vec{r}) = E_N \psi_N(\vec{r})$$

(29)

where $U(\vec{r}) > 0$, i.e., it is the attractive potential.

As a result, the function $J_P(t)$ for $t \to \infty$ behaves like

$$J_P(t) = \sum_N e^{-tE_N} \left[ \int d\vec{u} \psi(\vec{u}) \psi_N(\vec{u}) \right]^2.$$

If we choose

$$\psi(\vec{u}) = \psi_{N_0}(\vec{u})$$

then

$$\int d\vec{u} \psi_N(\vec{u}) \psi_{N_0}(\vec{u}) = \delta_{NN_0}$$

and, finally, for large $t$ we have

$$J_P(t) \to e^{-tE_{N_0}}.$$
where $E_{N_0}$ is the energy of the bound state of two nonrelativistic particles in the quantum state $N_0$ arising due to the potential $U(\vec{r})$. The mass of the bound state in the nonrelativistic approach is

$$
\mu_{N_0} = \frac{1}{c^2} \left[ Mc^2 + E_{N_0} + O \left( \frac{1}{c} \right) \right] = 2M + \frac{E_{N_0}}{c^2} + O \left( \frac{1}{c^3} \right). \tag{30}
$$

Thus, in the nonrelativistic limit the mass of the bound state of two scalar particles is the sum of their masses plus the bound state energy which is defined by the nonrelativistic potential interaction.

### 3.5 The Nonrelativistic Limit. Dimensional analysis.

Here we show that the nonrelativistic limit is completely equivalent to the weak coupling regime when $\lambda \to 0$.

The dimension of our variables and parameters is

$$[M] = [g] = \left[ \frac{1}{x} \right] = \left[ \frac{1}{\beta_j} \right] = [\eta_j].$$

Let us consider three functions in formula (18) for $G_P(x)$

$$A = \frac{xM}{2} \left( \frac{1}{s_1} + s_1 + \frac{1}{s_2} + s_2 \right),$$

$$K = \int_0^x d\beta \left[ \frac{Ms_1\eta_1^2(\beta)}{2} + \frac{Ms_2\eta_2^2(\beta)}{2} \right],$$

$$W_{ij} = \frac{g^2}{8M^2s_is_j} \int_0^x \int_0^x d\beta_1 d\beta_2 D \left( n(\beta_1 - \beta_2) + \eta_i(\beta_1) - \eta_j(\beta_2) \right)$$

where $n = (1, \vec{0})$ and

$$D(x) = \int \frac{dk}{(2\pi)^4} \tilde{D}(k^2) e^{i k x} = \int \frac{d\vec{k}}{(2\pi)^3} \int \frac{d\vec{k}_4}{2\pi} \tilde{D}(\vec{k}^2 + k_4^2) e^{i(\vec{k}_4 \cdot \vec{x} + \vec{k}_4 \cdot \vec{x})}.$$
\[ \bar{k} = g\bar{q}, \quad k_4 = \frac{g^2}{Mv}. \]

One obtains
\[ A = X \left( \frac{M}{g} \right)^2 \left( \frac{1}{s_1} + s_1 + \frac{1}{s_2} + s_2 \right), \]
\[ K = \int_{\tau_1}^{X} d\tau \left[ \frac{s_1 \dot{\zeta}_1^2(\tau)}{2} + \frac{s_2 \dot{\zeta}_2^2(\tau)}{2} \right], \]
\[ W_{ij} = \frac{g}{M} \int_{\tau_1}^{X} d\tau d\tau_2 \int \frac{d\bar{q}}{(2\pi)^3} \int \frac{dv}{2\pi} g^2 D \left( g^2 \left[ \bar{q}^2 + \frac{g^2}{M^2}v^2 \right] \right) \]
\[ \cdot \exp \left\{ iv \left[ (\tau_1 - \tau_2) + \frac{g}{M} (\zeta_4(\tau_1) - \zeta_4(\tau_2)) \right] + i\bar{q}(\zeta_4(\tau_1) - \zeta_4(\tau_2)) \right\}. \]

In particular for
\[ \bar{D}(k^2) = \frac{1}{k^2 + m^2} \]
we get
\[ g^2 \bar{D} \left( g^2 \left[ \bar{q}^2 + \frac{g^2}{M^2}v^2 \right] \right) = \frac{1}{\bar{q}^2 + \frac{g^2}{M^2}v^2 + \frac{m^2}{g^2}}. \]

In the weak coupling regime \( \lambda \ll 1 \) we have
\[ \int \frac{dv}{2\pi} g^2 \bar{D} \left( g^2 \left[ \bar{q}^2 + \frac{g^2}{M^2}v^2 \right] \right) e^{iv(\tau_1 - \tau_2) + \frac{g}{M} (\zeta_4(\tau_1) - \zeta_4(\tau_2))} \]
\[ \rightarrow \int \frac{dv}{2\pi} g^2 \bar{D} \left( g^2 \bar{q}^2 \right) e^{iv(\tau_1 - \tau_2)} = g^2 \bar{D} \left( g^2 \bar{q}^2 \right) \delta(\tau_1 - \tau_2) \]
and
\[ W_{ij} \rightarrow \frac{1}{M 8s_is_j} \int_{\tau_1}^{X} d\tau \int \frac{d\bar{q}}{(2\pi)^3} g^2 \bar{D} \left( g^2 \bar{q}^2 \right) e^{i\bar{q}(\zeta_i(\tau) - \zeta_j(\tau))} \]
\[ = \frac{1}{s_is_j} \int_{\tau_1}^{X} d\tau U(\zeta_i(\tau) - \zeta_j(\tau)). \]
where

\[ U(\zeta) = \frac{g}{8M} \int \frac{d\vec{q}}{(2\pi)^3} q^2 \tilde{D} \left( g^2 q^2 \right) e^{i\vec{q}\cdot\vec{\zeta}} \]

\[ = \frac{g}{8M} \int \frac{d\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{\zeta}}}{q^2 + \frac{m^2}{g^2}} = g \frac{1}{M} \frac{e^{-\frac{\mu}{g^2}}}{32\pi} \]

\[ = M \frac{\lambda e^{-\mu\eta}}{8\eta} \]

where \( \zeta = |\vec{\zeta}| \) and \( \eta = |\vec{\eta}| \).

Thus, nonrelativistic limit is realized in the weak coupling regime.

### 3.6 Relativistic incompleteness of quantum mechanics of two particles.

Here we would like to pay attention to the Schrödinger equation describing two nonrelativistic particles

\[ H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - U(\vec{r}_1 - \vec{r}_2), \]

where the potential is attractive.

Let us pass to the inertial system in the standard way

\[ \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2. \]

The Hamiltonian takes the form

\[ H = \frac{p^2}{2M} + \frac{p_r^2}{2\mu} - U(r) \]

\[ M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \]

The solution of the Schrödinger equation

\[ H\Psi = E\Psi \]

can be written as

\[ \Psi(\vec{R}, \vec{r}) = e^{i\vec{p}\cdot\vec{R}} \psi(r) \]
where $\vec{p}$ is the momentum of the total system and $\psi(r)$ is the eigenfunction of the equation

$$\left[ \frac{\vec{p}^2}{2\mu} - U(r) \right] \psi = -\varepsilon \psi$$

and $-\varepsilon$ ($\varepsilon > 0$) is an eigenvalue of a bound state. Then, the eigenvalue or the energy of the state $\Psi(\vec{R}, \vec{r})$ in the case $\vec{p} \neq 0$ is

$$E = \frac{\vec{p}^2}{2M} - \varepsilon.$$

From the physical point of view this energy has no reasonable sense. Indeed, we should get

$$E = \frac{p^2}{2M} - \varepsilon \rightarrow \frac{p^2}{2M_{phys}},$$

$$M_{phys} = m_1 + m_2 - \Delta, \quad \Delta = \frac{\varepsilon}{c^2}$$

i.e., the interaction between two particles should give the mass excess.

On the other hand, the last formula can be obtained from the pure energy in the nonrelativistic limit

$$E = \sqrt{M_{phys}^2c^4 + p^2c^2} = M_{phys}c^2 + \frac{p^2}{2M_{phys}} + O(p^2)$$

$$= (m_1 + m_2 - \Delta)c^2 + \frac{p^2}{(m_1 + m_2 - \Delta)} + O(p^2)$$

$$= (m_1 + m_2)c^2 + \frac{p^2}{2(m_1 + m_2)} - \Delta c^2 + O\left( \frac{\Delta}{M} \right) + O(p^2),$$

and the mass of the bound state

$$M_{phys} = M c^2 - \frac{\varepsilon}{c^2} + O\left( \frac{1}{c^2} \right).$$

Thus, we can consider the nonrelativistic Schrödinger equation describing two nonrelativistic particles as a fragment of an unknown relativistic equation describing the relativistic interaction of two particles.
4 Bosonization of fermion currents.

The starting point of the Bosonization of fermion currents is the representation (14).

4.1 Fierz transformation.

Let us consider the four fermion term (15) and perform the ”Fierz transformation”:

\[
\begin{align*}
(\Psi^+ \Psi D_m \Psi^+ \Psi) &= \int \int dy_1 dy_2 \Psi^+(y_1) \Psi(y_1) D_m(y_1 - y_2) \Psi^+(y_2) \Psi(y_2) \\
&= - \int \int dy_1 dy_2 \Psi^+(y_1) \Psi(y_2) D_m(y_1 - y_2) \Psi^+(y_2) \Psi(y_1).
\end{align*}
\]

(32)

Let us introduce the new variables

\[
y_1 = x + \frac{u}{2}, \quad y_2 = x - \frac{u}{2}
\]

Then the four fermion term looks like

\[
(\Psi^+ \Psi D_m \Psi^+ \Psi) = - \int dx \int du D_m(u) J(x, u) J^+(x, u)
\]

where

\[
J(x, u) = \left( \Psi^+ \left( x + \frac{u}{2} \right) \Psi \left( x - \frac{u}{2} \right) \right) = \Psi^+ (x) e^{\bar{\Psi} \hat{D} \Psi} \Psi(x),
\]

(34)

\[
J^+(x, u) = J(x, -u).
\]

4.2 Orthonormal system.

The main point is that in the representation (33) the boson Green function \(D_m(u)\) can be considered as a weight function inducing uniquely the system of orthogonal polynomials in the space \(R^4\). Thus, the full orthonormal system of real functions

\[
\{ f_{\{n\}}^{\{\mu \}}(u) \} = \{ f_Q(u) \} = \{ Q \}, \quad Q = n l \{ \mu \},
\]

which are symmetric for all permutations \( \mu_i \leftrightarrow \mu_j \) and

\[
f_{\{\nu \mu \}}^{\{n\}}(u) \equiv 0,
\]

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can be chosen in the form
\[ f_Q(u) = \sqrt{\rho(u)} P_Q(u), \quad \text{(35)} \]
\[ \rho(u) = D_m(u) = \int \frac{dk}{(2\pi)^4} \frac{e^{iku}}{m^2 + k^2} = \frac{m^2}{(2\pi)^2} \cdot \frac{1}{mu} K_1(mu), \quad u = \sqrt{u^2} \]
where \( P_Q(u) \) are real polynomials, satisfying
\[ P_Q(-u) = (-1)^l P_Q(u). \]

The orthonormality conditions look like
\[ (f_Q f_{Q'}) = \int d^4u f_Q(u) f_{Q'}(u) = \int d^4u \rho(u) P_Q(u) P_{Q'}(u) = \delta_{QQ'}, \]
\[ \rho(u) \sum_Q P_Q(u) P_Q(u') = \delta(u - u') \]

The symbol \( \delta_{\mu\nu\{\mu'\}} \) is defined as
\[ \sum_{\mu'} \delta_{\mu\nu\mu'} f_{\{\mu\}}^{(nl)}(u) = f_{\{\mu\}}^{(nl)}(u). \]

The orthonormal condition can be rewritten in the form
\[ (f_Q f_{Q'}) = P_Q \left( i \frac{\partial}{\partial k} \right) P_{Q'} \left( i \frac{\partial}{\partial k} \right) \frac{1}{m^2 + k^2} \bigg|_{k=0} = \delta_{QQ'}. \quad \text{(36)} \]

For the lowest states one can get
\[ P_{00} = m, \quad P_{10}(u) = \frac{m}{\sqrt{2}} \left( 1 - \frac{m^2 u^2}{8} \right), \]
\[ P_{01\{\mu\}}(u) = \frac{m^2}{\sqrt{2}} u_\mu, \quad P_{11\{\mu\}}(u) = \frac{m^2}{\sqrt{2}} \left( 1 - \frac{m^2 u^2}{24} \right) u_\mu, \]
\[ P_{02\{\mu\nu\}}(u) = \frac{m^3}{4} \left( u_\mu u_\nu - \frac{1}{4} \delta_{\mu\nu} u^2 \right), \quad \text{and so on.} \]

The normalization conditions for these states look like
\[ (f_{00} f_{00}) = (f_{10} f_{10}) = 1, \]
\[ (f_{01\{\mu\}} f_{01\{\nu\}}) = (f_{11\{\mu\}} f_{11\{\nu\}}) = \delta_{\mu\nu}, \]
\[ (f_{02\{\mu\nu\}} f_{02\{\alpha\beta\}}) = \delta_{\{\mu\nu\}\{\alpha\beta\}} = \frac{1}{2} \left[ \delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha} - \frac{1}{2} \delta_{\mu\nu} \delta_{\alpha\beta} \right]. \]
Let us introduce the vertex
\[ V_Q(p) = (-i)^l \int du f_Q(u) \sqrt{\rho(u)} e^{iup} = (-i)^l \int du P_Q(u) \rho(u) e^{iup} \]
\[ = (-i)^l P_Q \left( -i \frac{\partial}{\partial p} \right) \frac{1}{m^2 + p^2}, \]
\[ V_Q^+(p) = V_Q(p), \quad V_Q^+(-p) = (-1)^i V_Q(p). \tag{37} \]

In particular
\[ V_{00}(p) = \frac{m}{m^2 + p^2}, \quad V_{01(\mu)}(p) = \frac{m^2}{\sqrt{2}} \cdot \frac{2p_\mu}{(m^2 + p^2)^2}, \]
\[ V_{10}(p) = \frac{m}{\sqrt{2}} \cdot \frac{p^2(2m^2 + p^2)}{(m^2 + p^2)^3}. \]

Then, the representation is valid
\[ \sqrt{\rho(u)} e^{iup} = \sum_Q i f_Q(u) V_Q \left( \frac{p}{2} \right). \]

### 4.3 Bilocal and nonlocal currents.

The bilocal current \( J(x, u) \) can be represented as
\[ \sqrt{D_m(u)} J(x, u) = \sqrt{\rho(u)} (\Psi^+(x) e^{\frac{i}{\sqrt{2}} \hat{D}_x} \Psi(x)) = \sum_Q i f_Q(u) \cdot I_Q(x), \]
\[ I_Q(x) = \left( \Psi^+(x) V_Q \left( \frac{\hat{p}_x}{2} \right) \Psi(x) \right) = \left( \Psi^+ V_Q \Psi \right), \]
\[ I_Q^2(x) = I_Q(x). \]

Thus, the bilocal currents \( J(x, u) \) are represented in the form of a sum of the nonlocal hermitian currents \( I_Q(x) \). As a result, the representation is valid
\[ (\Psi^+ \Psi D_m \Psi^+ \Psi) = - \sum_Q \int dx [I_Q(x)]^2. \tag{38} \]

The next step is to use the Gaussian representation
\[ \exp \left\{ \frac{g^2}{2} (\Psi^+ D_m \Psi \Psi^+) \right\} = \exp \left\{ - \frac{g^2}{2} \sum_Q \int dx I_Q^2(x) \right\} \]
\[ = \int D\Phi \exp \left\{ -\frac{1}{2}(\Phi \Phi) + ig(\Phi I) \right\}. \]
where

\[ D\Phi = \prod_Q D\Phi_Q, \quad (\Phi \Phi) = \sum_Q \int dx \Phi_Q^2(x), \]

\[ (\Phi I) = \sum_Q \int dx \Phi_Q(x) I_Q(x) = \sum_Q (\Psi^+ [\Phi_Q V_Q]\Psi) = (\Psi^+ [\Phi V]\Psi). \]

The hermitian field variables

\[ \Phi_Q = \Phi_{\{\mu_1...\mu_l\}}^{(nl)}(u) \]

are symmetric for all permutations \( \mu_i \leftrightarrow \mu_j \) and

\[ \Phi_{\{\nu_\mu \mu_3...\mu_l\}}^{(nl)}(u) \equiv 0. \]

Let us suppose that \( V = g V_{Q_0} \). For \( x \neq y \) we have

\[
G_{Q_0}(x - y) = \int D\Psi D\Phi^+ (\Psi^+ V_{Q_0}\Psi)_x (\Psi^+ V_{Q_0}\Psi)_y \exp \left\{ (\Psi^+ D^{-1}_M \Psi) \right\} \cdot \\
\cdot \int D\Phi \exp \left\{ -\frac{1}{2}(\Phi \Phi) + ig(\Psi I) \right\} \\
= - \int D\Phi \exp \left\{ -\frac{1}{2}(\Phi \Phi) \right\} \frac{\delta^2}{\delta \Phi_{Q_0}(x) \delta \Phi_{Q_0}(y)} \cdot \\
\cdot \int D\Psi D\Phi^+ \exp \left\{ (\Psi^+ D^{-1}_M \Psi) + ig \int dx (\Psi^+ [\Phi V]\Psi) \right\} \\
= - \int D\Phi \exp \left\{ -\frac{1}{2}(\Phi \Phi) \right\} \\
\cdot \frac{\delta^2}{\delta \Phi_{Q_0}(x) \delta \Phi_{Q_0}(y)} \exp \left\{ \text{tr} \ln [1 + ig(\Phi V) D_M] \right\}. \\
\]

Integrating by parts we finally get

\[
G_{Q_0}(x - y) = - \int D\Phi \ \Phi_{Q_0}(x) \Phi_{Q_0}(y) e^{L_{eff}[\Phi]}, \quad (39)\]

where

\[
L_{eff}[\Phi] = -\frac{1}{2}(\Phi \Phi) + \text{tr} \ln [1 + ig(\Phi V) D_M]. \quad (40)\]

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First of all we would like to stress that the representation (39) is completely equivalent to the initial representation (14). The Green function $G_{Q_0}(x - y)$ can be considered as the Green function of the field $\Phi_{Q_0}(x)$ and these fields $\{\Phi_{Q_0}\}$ are described by the effective Lagrangian $L_{\text{eff}}[\Phi]$. In addition, we want to pay attention to the sign “minus” in front of the expression (39) and the nonhermitian interaction Lagrangian in (39) and (40).

4.4 One-loop approach.

The next problem is to give the standard particle interpretation to the effective Lagrangian (40). For this aim, let us extract the quadratic term

$$\text{tr} \ln [1 + ig(\Phi V)D_M] = \frac{g^2}{2}(\Phi \Pi \Phi) + \text{int}[, \Phi].$$

$$g^2(\Phi \Pi \Phi) = \sum_{QQ'}(\Phi_Qg^2[V_QD_MV_{Q'}D_M]\Phi_{Q'})$$

$$= \int dx \int dy \sum_{QQ'}\Phi_Q(x)g^2\Pi_{QQ'}(x - y)\Phi_{Q'}(y)$$

$$= \int dp \sum_{QQ'}\tilde{\Phi}_Q(p)g^2\tilde{\Pi}_{QQ'}(p)\tilde{\Phi}_{Q'}(p)$$

where the polarization operator is

$$g^2\Pi_{QQ'}(x - y)$$

$$= g^2\left[V_Q\left(\frac{\vec{p}_x - \vec{p}_x}{2}\right)D_M(x - y)V_{Q'}\left(\frac{\vec{p}_y - \vec{p}_y}{2}\right)D_M(y - x)\right],$$

$$= g^2\int \frac{dk}{(2\pi)^4}\tilde{D}_M(k)\int \frac{dk'}{(2\pi)^4}\tilde{D}_M(k')e^{i(x - y)(k - k')},$$

$$\cdot \left[V_Q\left(\frac{k + k'}{2}\right)V_{Q'}\left(\frac{k + k'}{2}\right)\right],$$

and in the momentum space

$$g^2\tilde{\Pi}_{QQ'}(p) = \int dx e^{i px}g^2\Pi_{QQ'}(x) = g^2\tilde{\Pi}_{QQ'}(p).$$

$$g^2\tilde{\Pi}_{QQ'}(p) = g^2\int \frac{dk}{(2\pi)^4}\frac{V_Q(k)V_{Q'}(k)}{\left(M^2 + \left(k + \frac{p}{2}\right)^2\right)\left(M^2 + \left(k - \frac{p}{2}\right)^2\right)}.$$
The index structure of the polarization operator looks like

$$\tilde{\Pi}_{QQ}(p) = \tilde{\Pi}_{\{\mu_{1}\} \{\mu_{2}\}}^{(nl)}(p) = \tilde{\Pi}_{(nl)}(p^2)\delta_{\{\mu_{1}\} \{\mu_{2}\}} + \sum_{j} g^{2}\tilde{\Pi}_{j}^{(nl)}(p^2)t_{\{\mu_{1}\} \{\mu_{2}\}}^{j}(p)$$

where the tensors $t_{\{\mu_{1}\} \{\mu_{2}\}}^{j}(p)$ contain the vectors $p_{\mu}$.

The Green function $G_{Q0}$ takes the form

$$G_{Q0}(x - y) = -\int D\Phi \Phi_{Q0}(x)\Phi_{Q0}(y) \exp \left\{ -\frac{1}{2}(\Phi[1 - g^{2}\Pi]\Phi) + I_{int}[\Phi] \right\}.$$ 

(42)

where

$$I_{int}[\Phi] = \text{tr} \ln [1 + ig(\Phi V)D_{M}] - g^{2}/2(\Phi\Pi\Phi).$$

The diagonal part of the quadratic form of (12) gives the equation of motion on the field $\Phi_{(nl)}^{\{\nu_{\mu_{2}}...\mu_{l}\}}(x)$

$$\left[\delta_{QQ'} - g^{2}\tilde{\Pi}_{QQ'}(x) \left( \frac{\partial}{i\partial x} \right) \right] \Phi_{Q'}(x) = 0,$$

or

$$\left[\delta_{QQ'} - g^{2}\tilde{\Pi}_{QQ'}(p) \right] \tilde{\Phi}_{Q'}(p) = 0.$$ 

(43)

The requirement that this equation on the mass shell should be the Klein-Gordon equation gives the constrain

$$\frac{\partial}{\partial x_{\nu}}\Phi_{\nu_{\mu_{2}}...\mu_{l}}^{(nl)}(x) = 0$$

or

$$p_{\nu}\Phi_{\nu_{\mu_{2}}...\mu_{l}}^{(nl)}(p) = 0$$

on the mass shell. Then, the function $\tilde{\Phi}_{\{\mu\}}^{(nl)}(p)$ satisfies the equation

$$\left[1 - \tilde{\Pi}^{(nl)}(p^2) \right] \tilde{\Phi}_{\{\mu_{1}...\mu_{l}\}}^{(nl)}(p) = 0.$$ 

(44)

The mass of the state with quantum numbers $Q = (nl)$ is defined by the equation

$$1 - g^{2}\tilde{\Pi}^{(nl)}(-\mu_{(nl)}^{2}) = 0.$$ 

(45)
Let us represent

\[ 1 - g^2 \tilde{\Pi}^{(nl)}(p^2) = [1 - g^2 \tilde{\Pi}^{(nl)}(-\mu_{(nl)}^2)] - g^2 \tilde{\Pi}'_{(nl)}(-\mu_{(nl)}^2)(p^2 + \mu_{(nl)}^2) - g^2 \tilde{\Pi}_{reg}^{(nl)}(p^2, \mu_{(nl)}^2) = -Z_{(nl)}(p^2 + \mu_{(nl)}^2) - g^2 \tilde{\Pi}_{reg}^{(nl)}(p^2, \mu_{(nl)}^2) \]

where the constant

\[ Z_{(nl)} = g^2 \tilde{\Pi}'_{(nl)}(-\mu_{(nl)}^2) \]

is positive.

The crucial point is that in the representation (39) the sign of the measure of the functional integral is defined by

\[ e^{-\frac{1}{2}(\Phi \Phi)} \]

but in the representation (42) by

\[ e^{-\frac{1}{2}(\Phi[1 - g^2 \Pi]\Phi)} \]

and according to (46) by

\[ e^{\frac{1}{2}(\Phi Z_{(nl)}[p^2 + \mu_{(nl)}^2]\Phi)}. \]

We can see that the sign of the quadratic form becomes positive. It requires to do the rotation

\[ \Phi \rightarrow -i\phi \]

to have the decreasing measure. Then, let us introduce new field variables

\[ \Phi_Q(x) = \frac{-i}{\sqrt{Z_{(nl)}}} \varphi_Q(x). \]

Finally we get

\[ G_{Q_0}(x - y) = \frac{1}{Z_{Q_0}} G_{Q_0}(x - y), \]

\[ G_{Q_0}(x - y) = \int D\varphi \varphi_{Q_0}(x)\varphi_{Q_0}(y) e^{L_{eff}[\varphi]} = D_{\mu_{(nl)}}(x - y) + O(h^2). \]
The normalization constant in (47) should be implied

$$\varphi \to \frac{D\varphi}{C}, \quad C = \int D\varphi \ e^{L_{eff}[\varphi]}.$$  

The effective Lagrangian looks like

$$L_{eff}[\varphi] = \frac{1}{2}(\varphi \left[ -\Box +\mu^2 \right] \varphi) + I_{int}[\varphi] \tag{48}$$

Here

$$(\varphi \left[ -\Box +\mu^2 \right] \varphi) = \int dx \sum_Q \varphi_Q(x) \left[ -\Box +\mu_{(nl)}^2 \right] \varphi_Q(x)$$

$$= \int dp \sum_Q \tilde{\varphi}_Q^+(p) \left[ p^2 +\mu_{(nl)}^2 \right] \tilde{\varphi}_Q(p),$$

$$I_{int}[\varphi] = \text{tr} \ln \left[ 1 + h\varphi V D_M \right] + (\varphi \Pi \varphi), \quad (\varphi \Pi \varphi) = \sum_{Q \neq Q'} h_Q h_{Q'} (\varphi_Q \Pi_{QQ'} \varphi_{Q'}) + \sum_Q h_Q^2 (\varphi_Q \Pi_{QQ}^{reg} \varphi_Q), \quad h_{\varphi V} = \sum_{Q} h_Q \varphi_Q V_Q, \quad I_{int}^+[\varphi] = I_{int}[\varphi].$$

The effective coupling constants are defined as

$$h_Q = h_{(nl)} = \frac{g}{\sqrt{Z_{(nl)}}} = \frac{1}{\sqrt{W_{(nl)}^\prime \left( -\mu_{(nl)}^2 \right)}}. \tag{50}$$

Now we want to repeat and to stress that eq. (45) and the positivity of $Z_{(nl)}$ require the rotation $\Phi \to -i\dot{\varphi}$, so that the sign $(-i)^2 = -1$ arises in front of the Green function $G_{Q0}(x - y)$ and compensates the sign $(-1)$ in (39). Besides, the interaction Lagrangian becomes hermitian. Thus, we get the representation (47) which has the standard physical sense with the positive metrics and the hermitian effective Lagrangian (48).

As a result, the final representation (47) can be interpreted as a generating functional of the quantum field system of the bosonic fields $\{\phi_{Q}\}$ which is described by the effective Lagrangian (48).
4.5 The functions $\tilde{\Pi}_{(nl)}(p^2)$.

Let us evaluate the values of $\lambda$ for some typical parameters $\frac{m}{M}$ and possible masses $\frac{\mu^2}{2M}$ of the lowest bound states with quantum number (00), (10), (01). We have

$$
g^2\tilde{\Pi}_{(n0)}(p^2) = g^2 \int \frac{dk}{(2\pi)^4} \cdot \frac{[V_{n0}(k^2)]^2}{\left( M^2 + (k + \frac{p}{2})^2 \right) \left( M^2 + (k - \frac{p}{2})^2 \right)} = \lambda \cdot \frac{\xi}{8\pi b} \cdot \int \frac{dt w_{(n)}(\xi, t)}{1 - bt} \cdot \left( \sqrt{(1 - bt)^2 + 4bt(1 - t)} \right) - 1
$$

$$
w_{(0)}(\xi, t) = \frac{1}{(1 - (1 - \xi)t)^2},
$$

$$
w_{(1)}(\xi, t) = \frac{1}{2} \left[ \frac{(1 - t)(1 - (1 - 2\xi)t)}{(1 - (1 - \xi)t)^3} \right]^2,
$$

where

$$
\xi = \left( \frac{m}{M} \right)^2, \quad b = -\frac{p^2}{4M^2} = \frac{\mu^2}{4M^2}.
$$

For the state (01) in the integrand in (42) we have

$$
k_{\mu}k_{\nu} \rightarrow \frac{\delta_{\mu\nu}}{3} \left[ k^2 - \frac{(kp)^2}{p^2} \right] + \frac{p_{\mu}p_{\nu}}{p^2} \frac{1}{3} \left[ -k^2 + 4 \frac{(kp)^2}{p^2} \right]
$$

and for the diagonal part of the polarization operator $\tilde{\Pi}_{(01)}(p^2)$ one can get

$$
g^2\tilde{\Pi}_{(01)}(p^2) = \frac{g^2}{3} \int \frac{dk}{(2\pi)^4} \cdot \frac{2m^2}{(m^2 + k^2)^4} \cdot \frac{k^2 - \frac{(kp)^2}{p^2}}{\left( M^2 + (k + \frac{p}{2})^2 \right) \left( M^2 + (k - \frac{p}{2})^2 \right)} = \lambda \cdot \frac{\xi^2}{48\pi b^2} \cdot \int \frac{dt}{1 - (1 - \xi)t} \cdot \left\{ \frac{[(1 - bt)^2 + 4bt(1 - t)]^{3/2}}{1 - bt} - (1 - bt)^2 - 6bt(1 - t) \right\}
$$
It is convenient to introduce the notion

\[ g^2 \Pi_{\Pi_{(nl)}}(p^2) = \lambda J_{\Pi_{(nl)}}(\xi, b). \]  

(53)

The dimensionless coupling constant can be defined as

\[ h_{(nl)} = \frac{1}{4\pi} \frac{\xi}{[J_{\Pi_{(nl)}}(\xi, b)]_b^r}. \]  

(54)

The numerical results are given in the table.

| \( m/M \) | \( \lambda \) | \( \mu/M \) | \( h_{(00)} \) | \( \mu/M \) | \( h_{(10)} \) | \( \mu/M \) | \( h_{(01)} \) |
|---|---|---|---|---|---|---|---|
| .1 | 449 | .2 | .22 | 1.30 | .15 | 1.60 | .044 |
| .1 | 412 | .5 | .20 | 1.36 | .12 | 1.61 | .039 |
| .1 | 292 | 1.0 | .12 | 1.52 | .07 | 1.65 | .026 |
| .3 | 129 | .2 | .71 | 1.55 | .43 | 1.53 | .140 |
| .3 | 120 | .5 | .64 | 1.59 | .38 | 1.54 | .130 |
| .3 | 92 | 1.0 | .41 | 1.72 | .24 | 1.59 | .085 |
| .5 | 90 | .2 | 1.64 | 1.78 | .83 | 1.51 | .28 |
| .5 | 85 | .5 | 1.49 | 1.81 | .72 | 1.52 | .25 |
| .5 | 67 | 1.0 | 1.01 | 1.90 | .44 | 1.56 | .18 |

4.6 Conclusion

In conclusion the following points should be emphasized:

- the effective Lagrangian \( \mathcal{L}_{eff} \) is Hermitian,
- the effective interaction Lagrangian \( \mathcal{L}_{int} \) does not depend on the input coupling constant \( g \),
- the input coupling constant \( g \) defines the spectrum of bound states only,
• the dimensionless coupling constant $\lambda = \frac{1}{4\pi} \left( \frac{g}{M} \right)^2$ should be large,

• the effective coupling constants $h_{(nl)}$ depend on the mass of the corresponding bound state with the quantum number $(nl)$,

• the effective coupling constants $h_{(nl)}$ are small, so that the effective interaction Lagrangian $L_{\text{eff}}$ can be computed by the perturbation method,

• in principle, the quadratic part of the total Lagrangian should be diagonalized, i.e., the matrix $U_{QQ'}(p)$ should be found for which

$$\left[ \delta_{QQ'} - g^2 \Pi_{QQ'}(p) \right] = \sum_{Q_1} U_{QQ_1}(p) \Sigma_{Q_1}(p) U_{Q_1Q'}^T(p).$$

4.7 Appendix

The solution of (12) can be represented in the form of the following functional integral (see Appendix and, for example, [4]):

$$S(x, y|\phi) = \frac{1}{\Box - M^2 + g\phi(x)} \cdot \delta(x - y)$$

$$= \frac{1}{2} \int_0^\infty d\alpha e^{-\frac{\alpha}{2} M^2} T_\tau \exp \left\{ \frac{1}{2} \int_0^\alpha d\tau \left( \frac{\partial}{\partial x(\tau)} \right)^2 + \frac{g}{2} \int_0^\alpha d\tau \phi(x(\tau)) \right\} \delta(x - y)$$

$$= \frac{1}{2} \int_0^\infty d\alpha e^{-\frac{\alpha}{2} M^2} \int D\nu \exp \left\{ -\frac{1}{2} \int_0^\alpha d\tau \nu^2(\tau) \right\}$$

$$\cdot T_\tau \left\{ \int_0^\alpha d\tau \left( \nu(\tau) \frac{\partial}{\partial x(\tau)} \right) + \frac{g}{2} \int_0^\alpha d\tau \phi(x(\tau)) \right\} \delta(x - y)$$

$$= \frac{1}{2} \int_0^\infty d\alpha e^{-\frac{\alpha}{2} M^2} \cdot \delta \left( x - y + \int_0^\alpha d\tau \nu(\tau) \right)$$

$$\cdot \int D\nu \exp \left\{ -\frac{1}{2} \int_0^\alpha d\tau \nu^2(\tau) + \frac{g}{2} \int_0^\alpha d\tau \phi \left( x + \int_0^\tau d\tau' \nu(\tau') \right) \right\}.$$

The function

$$D(x, \alpha; y, 0|\phi)$$

30
\[
= T_\tau \exp \left\{ \frac{1}{2} \int_0^\alpha d\tau \left( \frac{\partial}{\partial x(\tau)} \right)^2 - \frac{M^2\alpha}{2} + \frac{g}{2} \int_0^\alpha d\tau\phi(x(\tau)) \right\} \delta(x - y)
\]

is the Green function of the equation

\[
- \frac{\partial}{\partial \alpha} Y(x, \alpha) = \left[ -\frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 + \frac{M^2}{2} - \frac{g}{2} \phi(x) \right] Y(x, \alpha), \quad (55)
\]
i.e.
\[
Y(x, \alpha) = \int dy D(x, \alpha; y, 0|\phi) Y(y, 0).
\]
This equation can be considered as the Schrödinger equation where \(\alpha\) plays the role of the imaginary time or "temperature", \(x \in \mathbb{R}^4\) and \(\phi(x)\) is the Gaussian random potential with the correlation function

\[
\langle \phi(x)\phi(y) \rangle_\phi = D_m(x - y).
\]

The Green function \(S(x, y|\phi)\) can be written in a more suitable form. Let us introduce the variables

\[
\nu(\tau) = \nu_0 + \mu(\tau), \quad \int_0^\alpha d\tau\mu(\tau) = 0
\]
and integrate over \(\nu_0\) using the \(\delta\)-function in the representation (55). Introducing the variable

\[
\xi(\tau) = y \left(1 - \frac{\tau}{\alpha} \right) + x\frac{\tau}{\alpha} + \int_\tau^\alpha d\tau'\mu(\tau')
\]
one can get after some transformations

\[
S(x, y|\phi) = \int d\Sigma \exp \left\{ \frac{g}{2} \int_0^\alpha d\tau \phi(\xi(\tau)) \right\},
\]

\[
d\Sigma = \frac{d\alpha}{8\pi^2\alpha^2} e^{-\alpha M^2} D\xi \exp \left\{ -\frac{1}{2} \int_0^\alpha \frac{\dot{x}^2(\tau)}{} \right\},
\]

\[
\xi(0) = y, \quad \xi(\alpha) = x.
\]
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