CRITICAL AND SUPER-CRITICAL ABSTRACT PARABOLIC EQUATIONS

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Abstract. Our purpose is to formulate an abstract result, motivated by the recent paper [8], allowing to treat the solutions of critical and super-critical equations as limits of solutions to their regularizations. In both cases we are improving the viscosity, making it stronger, solving the obtained regularizations with the use of Dan Henry’s technique, then passing to the limit in the improved viscosity term to get a solution of the limit problem. While in case of the critical problems we will just consider a 'bit higher’ fractional power of the viscosity term, for super-critical problems we need to use a version of the 'vanishing viscosity technique’ that comes back to the considerations of E. Hopf, O.A. Oleinik, P.D. Lax and J.-L. Lions from 1950th. In both cases, the key to that method are the uniform with respect to the parameter estimates of the approximating solutions. The abstract result is illustrated with the Navier-Stokes equation in space dimensions 2 to 4, and with the 2-D quasi-geostrophic equation. Various technical estimates related to that problems and their fractional generalizations are also presented in the paper.

1. Introduction. We are using an idea to approximate a 'critical' partial differential problem with a sequence of sub-critical pseudo-differential approximations that are easy to solve and study inside Dan Henry’s technique. The solutions of such approximations are easy to obtain and they have excellent regularity properties, just like the solutions of the heat equation. Next we will pass to the limit with the parameter involved in the approximation to obtain a kind of weak solution to the limiting critical, hard problem. Our task is to consider an abstract evolutionary semilinear Cauchy problem with sectorial positive operator $A$ in the main part

$$u_t + Au = F(u), \quad t > 0,\
\quad u(0) = u_0,$$

(1.1)
with the nonlinear term $F : D(A^a) \to D(A^b)$, where $0 \leq a - b \leq 1$, being Lipschitz continuous on bounded subsets of $D(A^a)$. When $|a - b| < 1$ such theory was developed in the famous monograph of Dan Henry \[16\], where problems (1.1) are studied using classical techniques of ordinary differential equations, modified however to cover equations with an unbounded operator in a Banach space. Our task in the present paper is to consider the case when such approach gives local in time solution under suitable choice of the fractional order space $X^b = D(A^b)$ we set the problem in, while the available a priori estimates are too weak to guarantee the global in time extendibility of the local solutions. Assume that an a priori estimate the problem in, while the available a priori estimates are too weak to guarantee the time solution under suitable choice of the fractional order space $X^b = D(A^b)$ we set the problem in, while the available a priori estimates are too weak to guarantee the global in time extendibility of the local solutions. Assume that an a priori estimate is available in a Banach space $Y \supset D(A^a)$, where $b \leq a$ with $a - b = 1$, together with the action of the sectorial positive operator $A$ we have a critical bound for nonlinearity taken on such arbitrary local solution $u(t)$:

$$\exists_{\text{nondecreasing } g: [0, \infty) \to [0, \infty)} \quad \|F(u(t))\|_{X^b} \leq g(\|u(t)\|_Y)(1 + \|u(t)\|_{X^{b+1}}^1),$$

(1.2)

for all $t \in (0, \tau_{u_0})$, where $\tau_{u_0}$ is the 'life time' of that solution. Note that the nonlinear term in that case is of the same order, or value, as the main part operator $A$. We will treat also, using a variant of the method known as vanishing viscosity technique, originated by E. Hopf, O.A. Oleinik, P.D. Lax in 1950th, the case when one need to replace the right hand side of the estimate above with $g(\|u(t)\|_Y)(1 + \|u(t)\|_{X^{b+1}}^\theta)$ where $\theta > 1$ is required, known as the super-critical case.

Recall that in the typical situation (compare \[4, 36\]), studying sub-critical problem (1.1), we assume stronger than (1.2) estimate:

$$\exists_{\text{nondecreasing } g: [0, \infty) \to [0, \infty)} \quad \|F(u(t))\|_{X^b} \leq g(\|u(t)\|_Y)(1 + \|u(t)\|_{X^a}^\theta),$$

(1.3)

for all $t \in (0, \tau_{u_0})$, where $|a - b| < 1$ and $\theta < 1$. The latter condition allows easily to bound the $X^a$ norm of the local solution, since the norm is bounded, such local solutions extends globally in time.

Our strategy, to treat the critical and super-critical problems, is the following. We will consider the regularization

$$u_t + Au + \epsilon A^\beta u = F(u), \quad t > 0,$n$$

$$u(0) = u_0,$n$$

(1.4)

of the critical/super-critical problem (1.1) improving the viscosity to the form $A^\beta$ ($\beta > 1$), where the value of $\beta$ is taken as large as needed for the nonlinearity to become sub-critical with respect to the previous a priori bound, we will additionally assume that such bound stays valid for the regularization (1.4), uniformly in $\epsilon > 0$ sufficiently small. Next, using the semigroup approach of Dan Henry, we will solve easily the sub-critical regularized problems to obtain a family (parameter $\epsilon$) of its solutions $u^\epsilon$. Using the assumed existing uniform in $\beta, \epsilon$, estimate (1.2) for $u^\epsilon$, we will finally pass to the limit in the regularized equation (1.4) in a suitable weak sense, to see that the limiting function is a weak form solution to the critical/super-critical problem. Depending on the available uniform a priori estimate, it has some suitable properties.

Recall here for completeness the definition of sectorial operator, due to Dan Henry. For $a \in \mathbb{R}$ and $\phi \in (0, \frac{\pi}{2})$, we introduce a sector of the complex plain (cf.\[16, Definition 1.3.1\]):

$$S_{a, \phi} := \{ \lambda \in \mathbb{C} : \phi \leq |\text{arg}(\lambda - a)| \leq \pi, \lambda \neq a \}.$$  

(1.5)

**Definition 1.** A linear, closed and densely defined operator $A : X \supset D(A) \to X$ acting in a Banach space $X$ is called sectorial operator in $X$, if and only if there
exist \( a \in \mathbb{R}, \phi \in (0, \frac{\pi}{2}) \) and \( M > 0 \) such that the resolvent set \( \rho(A) \) contains the sector \( S_{a,\phi} \), and

\[
\| (\lambda I - A)^{-1} \|_{\mathcal{L}(X,X)} \leq \frac{M}{|\lambda - a|}, \quad \text{for each } \lambda \in S_{a,\phi}.
\]

In most cases, to make the considerations simpler, we limit ourself here to the particular class of sectorial operators given by the self-adjoint and non-negative densely defined operators in a Hilbert space \( H \) (see [29, p.93]). They are sectorial in the usual sense, moreover their powers \( A^\beta \) with \( \beta > 1 \) stay sectorial, which is not the case for \( \beta \) powers of the general sectorial operators. It is worth mentioning that the ‘opening angel’ \( |\arg\lambda| < \pi - \beta \omega \) for \( (-A)^\beta \) with \( \beta > 1 \) is smaller than the sector \( |\arg\lambda| < \pi - \omega \) for \(-A\), as seen e.g. from [21, Theorem 10.3]. Consequently, sufficiently large power \((-A)^{1+\alpha}\) of sectorial operator need not be a sectorial operator because its sector containing resolvent set will become too small.

**Remark 2.** We shall pointed also, that the above idea of approximating ’critical’ problem with a sequence of its regularizations (with a bit higher viscosity exponent) allows us to recall an idea of the \( \epsilon \)-regular solution introduced by J. Arrieta and A.N. Carvalho [3]. Indeed, they were considering a fractional power scale of Banach spaces \( \{X^\alpha\}_{\alpha \geq 0} \) associated with a sectorial operator \( A \) in \( X \). An equation

\[
u_t + Au = f(u), \quad u(0) = u_0, \tag{1.7}
\]

was studied with nonlinear term \( f \) being locally Lipschitz between the spaces \( X^{1+\epsilon} \) and \( X^{\gamma(\epsilon)} \)

\[
\|f(u) - f(v)\|_{X^{\gamma(\epsilon)}} \leq c\|u - v\|_X (1 + \|u\|_{X^{1+\epsilon}}^{\rho-1} + \|v\|_{X^{1+\epsilon}}^{\rho-1}), \quad u, v \in X^{1+\epsilon}, \tag{1.8}
\]

where \( \rho > 1, \epsilon > 0 \) are such that \( \rho \epsilon \leq \gamma(\epsilon) < 1 \). Note that the latter assumption \( \rho \epsilon \leq \gamma(\epsilon) < 1 \) implies that \( \epsilon < \gamma(\epsilon) \), hence the ’distance on the scale’ between the phase space \( X^{1+\epsilon} \) and the base space \( X^{\gamma(\epsilon)} \) equals \( 1+\epsilon - \gamma(\epsilon) \) and is strictly less than one. Thus, for such a choice of the pair base/phase spaces, the standard approach (e.g. [4, 17]) to existence of solutions applies and the local in time solvability follows.

The motivation for our studies was the fractional generalization of the celebrated Navier-Stokes equation (N-S equation, for short) in space dimensions \( N = 2, 3, 4 \). We mean here the Dirichlet problem in bounded smooth domain \( \Omega \subset \mathbb{R}^N \), written in an abstract form of a differential equation in Banach space of divergence-free vector functions:

\[
u_t = -Au - \epsilon A^{\beta_N} u - P(u \cdot \nabla)u + Pf, \quad t > 0, \\
u(0) = u_0, \tag{1.9}
\]

where \( A \) is the Stokes operator, the exponents \( \beta_N \) fulfill

\[
\beta_2 > 1, \quad \beta_3 > \frac{5}{4}, \quad \beta_4 > \frac{3}{2}, \tag{1.10}
\]

(generally, \( \beta_N > \frac{N+2}{4} \), [26, Remarque 6.11]) and \( P \) is the projector onto the space of divergence-free functions. The nonlinear term \(-P(u \cdot \nabla)u\) is standard for the Navier-Stokes equation; see (1.14).
Remark 3. We will calculate that the exponents $\beta_N$ given above equal precisely the values for which the nonlinearity in the N-D N-S equation becomes critical with respect to the standard $L^\infty (0, T; [L^2 (\Omega)]^N)$ estimate. Considering the nonlinear term as a map between $D(A^{s-\frac{1}{2}})$ and $D(A^{-\frac{1}{2}})$ ($s > 0$ to be chosen), using the estimate borrowed from [13, p.19]
\[
\| A^{-\frac{1}{2}} P (u \cdot \nabla) v \|_{[L^r(\Omega)]^N} \leq M(r) \| u \|_{[L^r(\Omega)]^N} \| v \|_{[L^r(\Omega)]^N}, \tag{1.11}
\]
$1 < r < \infty$, with $r = 2$, and using the Nirenberg-Gagliardo inequality we obtain
\[
\| A^{-\frac{1}{2}} P (u \cdot \nabla) u \|_{[L^2(\Omega)]^N} \leq M(2) \| u \|_{[L^2(\Omega)]^N}^2 \leq c \| u \|_{[L^4(\Omega)]^N}^2
\]
\[
\leq c \| u \|_{[H^{2\alpha-1}(\Omega)]^N}^2 \| u \|_{[L^2(\Omega)]^N}^{2-\frac{2N}{\alpha}}. \tag{1.12}
\]
The nonlinear term becomes critical with respect to the $L^2(\Omega)$ estimate when the exponent on the $H^{2\alpha-1}(\Omega)$ norm equals 1, or equivalently, $s = \frac{1}{2} + \frac{N}{4}$, which is precisely the value of $\beta_N$ in [26].

Recall that the classical N-D Navier-Stokes equations has the form:
\[
\begin{align*}
&u_t = \nu \Delta u - \nabla p - (u \cdot \nabla) u + f, \quad \text{div} u = 0, \quad x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\
&u = 0, \quad x \in \partial \Omega, \quad t > 0, \\
&u(0, x) = u_0(x),
\end{align*}
\tag{1.13}
\]
where $\nu > 0$ is the viscosity coefficient, $u = (u_1(t, x), ..., u_N(t, x))^T$ denotes velocity, $p = p(t, x)$ pressure, and $f = (f_1(x), ..., f_N(x))$ external force. It is impossible to recall even the most important results devoted to that problem, since the corresponding literature is too large; see anyway [11, 12, 13, 24, 32, 33, 35, 37]. We prefer to have the nonlinear term written in an equivalent form (for divergence free functions):
\[
(u \cdot \nabla) u = \sum_{i=1}^{N} \frac{\partial (u_i u_j)}{\partial x_i}. \tag{1.14}
\]
Note that the right hand side is a vector with the $j$-row equal to $\sum_{i=1}^{N} \frac{\partial (u_i u_j)}{\partial x_i}$ ($j=1,...,N$).

The technique used in the present paper is similar to that in [8, 9]. We are following the idea proposed by J.-L. Lions in [26, Chapter 1, Remarque 6.11], to replace the $-\Delta$ operator for the 3-D N-S equation with $-\Delta + \kappa (-\Delta)^l$, $l \geq \frac{3}{4}$. Our ideas were inspired also by the vanishing viscosity technique originated in 1950th by E. Hopf, O.A. Oleinik, P.D. Lax, J.-L. Lions and many others.

The second example of critical problem discussed in the paper is the dissipative quasi-geostrophic equation with $\alpha = \frac{1}{2}$, which has the form:
\[
\begin{align*}
\theta_t + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta &= f, \quad x \in \mathbb{R}^2, \quad t > 0, \\
\theta(0, x) &= \theta_0(x),
\end{align*}
\tag{1.15}
\]
where $\theta$ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, $\alpha \in [\frac{1}{2}, 1]$ a fractional exponent, and $u = (u_1, u_2)$ is the velocity field determined by $\theta$ through the relation:
\[
u u = (-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}), \quad \text{where} \quad (-\Delta)^\frac{1}{2} \psi = -\theta, \tag{1.16}
\]
or, in a more explicit way,
\[
u u = (-R_2 \theta, R_1 \theta), \tag{1.17}
\]
where $R_i, i = 1, 2$ are the Riesz transforms. Critical case for equation (1.15) corresponds to the parameter $\alpha = \frac{1}{2}$, when both the main part operator of the equation and the nonlinear term contain first order derivatives of the solutions (or their equivalence).

1.1. Fractional power scale. Studying semilinear equation with sectorial operator (1.1), or its linear version, it is often convenient to consider it on a particular level of the fractional powers scale $X^\alpha = D(A^\alpha), \alpha \in \mathbb{R}$, connected with the sectorial positive operator $A$. As well known (e.g. [23, (3.9)]);

$$A^\mu : D(A^\alpha) \rightarrow D(A^{\alpha-\mu}), \alpha, \mu \in \mathbb{R},$$

as an isometry on the fractional powers scale.

Observe next, following [28, p.17], that when dealing with the Cauchy problem (1.1) in the base space $X^{\gamma}$, where $F : X^\gamma \rightarrow X^{\beta}$ is Lipschitz continuous on bounded sets of $X^\gamma$ and $0 \leq \gamma - \beta < 1$, we can set $v(t) := A^\beta u(t), t \geq 0$, and $v_0 = A^\beta u_0 \in X^{\gamma-\beta}$. Then, $v$ fulfills formally the equation in $X$

$$v_t + Av = F(v) = A^\beta \circ F(A^{-\beta}v),
\quad v(0) = A^\beta u_0 \in X^{\gamma-\beta},$$

(1.18)

which will be obtained from (1.1) applying to it the operator $A^\beta$. The above observation was exploited further in [2, Chapter V].

Notation. We are using standard notation of Sobolev spaces. Compare [34] or [4, Chapter 1] for properties of fractional order Sobolev spaces; see also [17] for Sobolev type embeddings. For $r \in \mathbb{R}$, let $r^-$ denotes a number strictly less than $r$ but close to it. Similarly, $r^+ > r$ and $r^+$ close to $r$. Throughout the text $c$ denotes an arbitrary positive constant, which may be different from line to line and even in the same line. When needed for clarity of the presentation, we mark the dependence of the solution $u$ of (1.4) on $\beta$, calling it $u_\beta$. Sometimes we are using the notation $A_2$ for the Stokes operator acting on divergence-free vector functions belonging to $[L^2(\Omega)]^N; A_2 \equiv A$ in that case.

2. Abstract result. Local solvability. We will shortly stated the local existence result borrowed from [4, Chapter 2]. Consider the Cauchy problem:

$$\begin{cases}
  u_t + Au = F(u), & t > 0, \\
  u(0) = u_0,
\end{cases}$$

(2.1)

where $A$ is sectorial in a Banach space $X$. Without lack of generality we may assume that $A$ is positive, i.e. its spectrum is contained in a positive half-line. Since the resolvent set of a closed linear operator is known to be open on the complex plane, then

$$\text{Re}(\sigma(A)) > a, \quad \text{for some positive } a,$$

so that the power spaces $X^\alpha$ with $\alpha > 0$ are defined in a usual way as the ranges of an appropriate power operators $A^{-\alpha}$. In the described situation we shall always take that for fixed $\alpha \in [0, 1)$ the nonlinear term $F$ in (2.1) is Lipschitz continuous on bounded subsets of $X^\alpha$. The latter means equivalently that there exists a non-decreasing function $L : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that the estimate

$$\|F(v) - F(w)\|_X \leq L(r)\|v - w\|_{X^\alpha},$$

holds for each $v, w \in B_{X^\alpha}(r)$, where $B_{X^\alpha}(r)$ denotes an open ball in $X^\alpha$ centered at zero with radius $r$. 
For shortage of further references we introduce the following assumption.

**Assumption.** Let $X$ be a Banach space, $A : D(A) \rightarrow X$ sectorial and positive operator in $X$ and for some $\alpha \in [0, 1)$, $F : X^\alpha \rightarrow X$ be Lipschitz continuous on bounded subsets of $X^\alpha$.

Then the notion of a local $X^\alpha$-solution of (2.1) is formulated.

**Definition 4.** Let $X$ be a Banach space, $\alpha \in [0, 1)$ and $u_0$ be an element of $X^\alpha$. If, for some real $\tau > 0$, a function $u \in C([0, \tau); X^\alpha)$ satisfies:

$$
\begin{align*}
&u(0) = u_0, \\
&u \in C^1((0, \tau); X), \\
&u(t) \text{ belongs to } D(A) \text{ for each } t \in (0, \tau), \\
&\text{the first equation in (2.1) holds in } X \text{ for all } t \in (0, \tau),
\end{align*}
$$

then $u$ is called a local $X^\alpha$-solution of (2.1).

We will recall further the following local existence result (cf. [4, Chapter 2], [16, Chapter 3]):

**Theorem 5.** Under the Assumption above, for each $u_0 \in X^\alpha$, there exists a unique $X^\alpha$-solution $u = u(t, u_0)$ of (2.1) defined on its maximal interval of existence $[0, \tau_{u_0})$ which means that either $\tau_{u_0} = +\infty$, or

$$
\text{if } \tau_{u_0} < +\infty \text{ then } \limsup_{t \rightarrow \tau_{u_0}} \|u(t, u_0)\|_{X^\alpha} = +\infty. \quad (2.2)
$$

As a consequence of the embeddings between fractional power spaces, if the function $F$ is Lipschitz continuous on bounded subsets of $X^\alpha$ ($\alpha \in [0, 1)$), then it possesses this property as a map from $X^\beta$ into $X$ for each $\beta \in [\alpha, 1)$. This allows for the extension of many results concerning the $X^\alpha$-solutions onto $X^\beta$ spaces (i.e. $X^\beta$-solutions) with arbitrary $\beta \in [\alpha, 1)$. In particular the conclusion below is valid.

**Corollary 6.** Under the Assumption, for each $\beta \in [\alpha, 1)$, $u_0 \in X^\beta$, there exists a unique $X^\beta$-solution $u = u(t, u_0)$ of (2.1) defined on its maximal interval of existence $[0, \tau_{u_0})$.

3. **Local in time solvability of the 2-D, 3-D and 4-D N-S and critical 2-D quasi-geostrophic equations.** For the local in time solvability, we will set the problem (1.13) in the base space $X^{-\frac{1}{2}}$ for space dimension $N = 2$, and in the base space $X^{-\frac{3}{4}}$ for the space dimension $N = 3$. The corresponding phase spaces will be $X^{\frac{3}{2}+} \subset [H^{1+}(\Omega)]^2$ in case $N = 2$, and $X^{\frac{3}{2}+} \subset [H^{\frac{3}{2}+}(\Omega)]^3$ in case $N = 3$ (e.g. [13, Proposition 1.4]). Note that, in both cases, the phase spaces are contained in the space $[L^\infty(\Omega)]^N$. For $N = 4$ we will leave the Hilbert spaces and set the 4-D N-S equation in the base space $[L^4(\Omega)]^4$.

We will formulate now the corresponding local existence results for $N = 2, 3, 4$.

3.1. **2-D N-S equation.** We will use a version of the estimate in [13, Lemma 2.2] (with $\delta = \frac{1}{4}, \theta = \rho = \frac{1}{2}$):

$$
\|A^{-\frac{1}{4}} P(u \cdot \nabla) v \|_{[L^2(\Omega)]^2} \leq M \|A^{\frac{1}{2}+} u\|_{[L^2(\Omega)]^2} \|A^{\frac{1}{2}+} v\|_{[L^2(\Omega)]^2}. \quad (3.1)
$$
Since the form above is bi-linear, we have also the following consequences of the last estimate:

\[
\|A^{\frac{1}{2}} P((u - v) \cdot \nabla) v\|_{L^2(\Omega)}^2 \leq M \|A^{\frac{1}{2}} (u - v)\|_{L^2(\Omega)} \|A^{\frac{1}{2}} v\|_{L^2(\Omega)}^2,
\]

\[
\|A^{\frac{1}{2}} P(u \cdot \nabla)(u - v)\|_{L^2(\Omega)}^2 \leq M \|A^{\frac{1}{2}} u\|_{L^2(\Omega)} \|A^{\frac{1}{2}} (u - v)\|_{L^2(\Omega)}^2.
\]

(3.2)

Therefore the nonlinear term \( F(u) = -P(u \cdot \nabla)u + Pf \) acts from \( D(A_2^{\frac{1}{2}+}) \subset [H^{1+}(\Omega)]^2 \) into \( D(A_2^{\frac{1}{2}+}) \) as a map Lipschitz continuous on bounded subsets of \( D(A_2^{\frac{1}{2}+}) \). According to [4, 16], this suffices to obtain a local in time mild solution of the 2-D equation (1.13), more precisely:

**Theorem 7.** When \( Pf \in D(A_2^{\frac{1}{2}+}), \; u_0 \in D(A_2^{\frac{1}{2}+}) \subset [H^{1+}(\Omega)]^2 \), then there exists a unique local in time mild solution \( u(t) \) to (1.13) in the phase space \( D(A_2^{\frac{1}{2}+}) \subset [H^{1+}(\Omega)]^2 \). Moreover,

\[
u \in C([0, \tau); D(A_2^{\frac{1}{2}+}) \cap C([0, \tau); D(A_2^3))], \quad u_t \in C([0, \tau); D(A_2^{\frac{1}{3}+})). \tag{3.3}\]

Here \( \tau > 0 \) is the 'life time' of that local in time solution. Moreover, the Cauchy formula is satisfied:

\[u(t) = e^{-A_2 t} u_0 + \int_0^t e^{-A_2 (t-s)} F(u(s)) ds, \; t \in [0, \tau),\]

where \( e^{-A_2 t} \) denotes the linear semigroup corresponding to the operator \( A_2 \).

3.2. **An extension.** Actually, in the 2-D case we can weaken the diffusion in the N-S equation and still obtain local and global existence, see e.g. [15]. More precisely, consider the following 2-D fractional extension of the N-S equation:

\[
u_t = -A^\alpha u - P(u \cdot \nabla) u + Pf, \; t > 0, \]

\[u(0) = u_0, \tag{3.4}\]

where the real exponent \( \alpha \in \left(\frac{1}{4}, 1\right] \). Such generalization possesses good properties of the 2-D N-S problem. We will formulate now the corresponding local existence result for it. It is based on the fact that \( H^s(\Omega) \subset L^\infty(\Omega), \; N = 2, \) when \( s > 1 \), which allows the following estimate of the nonlinear term in the N-S equation (written for divergence-free functions \( u, v \)):

\[
\|P(u \cdot \nabla)v\|_{L^2(\Omega)} \leq c \|u\|_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq c \|u\|_{D(A^\alpha)} \|v\|_{D(A^\alpha)}, \tag{3.5}\]

where we assume only that \( \alpha > \frac{1}{7} \), so the same will be true for \( \alpha^- \). Due to bilinearity of the nonlinear term of the N-S equation we have all the information required to formulate a local existence result.

**Theorem 8.** For \( N = 2, \; \alpha \in \left(\frac{1}{2}, 1\right] \) and \( 0 < \epsilon < \alpha - \frac{1}{2} \), let \( Pf \in D(A_2^\epsilon) \) and \( u_0 \in D(A_2^{\alpha^-+}) \subset [H^{2(\alpha^-)}(\Omega)]^2 \). Then there exists a unique local in time mild solution to (3.4) in the phase space \( D(A_2^{\alpha^-+}) \). Moreover,

\[u \in C([0, \tau); D(A_2^{\alpha^-+}) \cap C([0, \tau); D(A_2^3)), \quad u_t \in C([0, \tau); D(A_2^{\frac{1}{3}+})), \]

and the Cauchy formula is satisfied.
We will formulate next the local existence result in a larger phase space. It is based on the following consequence of [13, Lemma 2.2], with \( r = 2 \), and \( \delta =: \epsilon + \frac{1}{2} \) which should also satisfy the restriction \( \epsilon < \frac{N}{4} \),

\[
\| A^{-\delta} P(u \cdot \nabla)v \|_{L^2(\Omega)^2} \leq c\| u \|_{L^r(\Omega)} \| v \|_{L^2(\Omega)^2} \leq c\| u \|_{L^{2r}(\Omega)} \| v \|_{L^{2r}(\Omega)^2}, \tag{3.6}
\]

where \( \frac{1}{r} = \frac{1}{2} + \frac{2N}{N} = \frac{1}{2} + \epsilon = \delta \) for \( N = 2 \). We will extend next the above estimate to the form

\[
\| P(u \cdot \nabla)v \|_{D(A^{-\delta}_2)} = \| A^{-\delta} P(u \cdot \nabla)v \|_{L^2(\Omega)^2} \leq c\| u \|_{L^{2r}(\Omega)} \| v \|_{L^{2r}(\Omega)^2} \leq c\| u \|_{D(A^{1-\epsilon}_2)} \| v \|_{D(A^{1-\epsilon}_2)}. \tag{3.7}
\]

Note that the ‘distance between the spaces’ \( D(A^{-\delta}_2) \) and \( D(A^{1-\epsilon}_2) \) equals \( \frac{1+\delta}{2} \) and it should be less than \( \alpha \) as required in Henry’s theory (since in equation (3.4) we are dealing with the sectorial operator \( A^\alpha \) rather than \( A \)). This requirement leads to the following restriction:

\[
\frac{1+\delta}{2} < \alpha \Rightarrow \frac{1}{2} + \epsilon < 2\alpha - 1 \Rightarrow \alpha > \frac{3}{4} + \frac{\epsilon}{2},
\]

or simply that \( \alpha > \frac{3}{2} \), since \( \epsilon \) is an arbitrary positive number. Therefore, due to bi-linearity of the nonlinear term of the N-S equation, we have all the information required to formulate a local existence result.

**Theorem 9.** Let \( N = 2 \), \( \alpha \in (\frac{3}{4}, 1) \), \( Pf \in D(A^{-\delta}_2) \) (base space) and \( u_0 \in D(A^{1-\epsilon}_2) \subset [H^{1-\delta}(\Omega)]^2 \). Then there exists a unique local in time mild solution to (3.4) in the phase space \( D(A^{1-\epsilon}_2) \). Moreover,

\[
u \in C([0, \tau); D(A^{1-\epsilon}_2)) \cap C((0, \tau); D(A^{(\alpha-\delta)}_2)), \quad u_t \in C((0, \tau); D(A^{(\alpha-\delta)^-})_2),
\]

and the Cauchy formula is satisfied.

### 3.3. 3-D N-S equation.

The main tool is the estimate taken from [13, Lemma 2.2] (with \( \delta = \frac{1}{2}, \theta = \rho = \frac{3}{2} \)):

\[
\| A^{-\frac{1}{2}} P(u \cdot \nabla)v \|_{L^2(\Omega)^3} \leq M \| A^{\frac{1}{2}} u \|_{L^2(\Omega)} \| A^{\frac{1}{2}} v \|_{L^2(\Omega)} \| A^{\frac{1}{2}} v \|_{L^2(\Omega)}. \tag{3.8}
\]

Since the form above is bi-linear, we have also the following consequences of the last estimate:

\[
\| A^{-\frac{1}{2}} P(u - v \cdot \nabla)v \|_{L^2(\Omega)} \leq M \| A^{\frac{1}{2}} (u - v) \|_{L^2(\Omega)} \| A^{\frac{1}{2}} v \|_{L^2(\Omega)} \| A^{\frac{1}{2}} v \|_{L^2(\Omega)},
\]

\[
\| A^{-\frac{1}{2}} P(u \cdot \nabla)(u - v) \|_{L^2(\Omega)} \leq M \| A^{\frac{1}{2}} u \|_{L^2(\Omega)} \| A^{\frac{1}{2}} (u - v) \|_{L^2(\Omega)} \| A^{\frac{1}{2}} v \|_{L^2(\Omega)}. \tag{3.9}
\]

Therefore the nonlinear term \( F(u) = -P(u \cdot \nabla) u + Pf \) acts from \( D(A^{\frac{3}{2}}_3) \subset [H^{\frac{3}{2}}(\Omega)]^3 \) into \( D(A^{\frac{3}{2}}_3) \) as a map, Lipschitz continuous on bounded subsets of \( D(A^{\frac{3}{2}}_3) \). According to [4, 16], this suffices to obtain a local in time solution of the 3-D equation (1.13), more precisely:

**Theorem 10.** When \( Pf \in D(A^{\frac{3}{2}}_3) \), \( u_0 \in D(A^{\frac{3}{2}}_3) \), then there exists a unique local in time mild solution \( u(t) \) to (1.13) in the phase space \( D(A^{\frac{3}{2}}_3) \subset [H^{\frac{3}{2}}(\Omega)]^3 \).

Moreover,

\[
u \in C([0, \tau); D(A^{\frac{3}{2}}_3)) \cap C((0, \tau); D(A^{\frac{3}{2}}_3)), \quad u_t \in C((0, \tau); D(A^{\frac{3}{2}}_3)). \tag{3.10}
\]
Here $\tau > 0$ is the 'life time' of that local in time solution. Moreover, the Cauchy formula is satisfied:

$$u(t) = e^{-A_2 t}u_0 + \int_0^t e^{-A_2 (t-s)}F(u(s))ds, \ t \in [0, \tau),$$

where $e^{-A_2 t}$ denotes the linear semigroup corresponding to the operator $A_2$.

3.4. 4-D N-S equation. In dimension 4 we will leave the Hilbert spaces and set the 4-D N-S equation in the base space $D(A^4_0)$ of divergence-free functions in $[L^4(\Omega)]^4$.

The Lipschitz condition on bounded sets of $D(A^3_0)$ (e.g. [13, p.271]) then reads

$$\|P(u \cdot \nabla)u - P(v \cdot \nabla)v\|_{[L^4(\Omega)]^4} \leq \|P((u - v) \cdot \nabla)u\|_{[L^4(\Omega)]^4} + \|P(v \cdot \nabla)(u - v)\|_{[L^4(\Omega)]^4}$$

$$\leq c\|u - v\|_{[L^8(\Omega)]^4} \|u\|_{[L^8(\Omega)]^4} + \|v\|_{[L^8(\Omega)]^4} \|\nabla (u - v)\|_{[L^4(\Omega)]^4}$$

$$\leq c\|u - v\|_{[W^{1,8}(\Omega)]^4} \|u\|_{[W^{1,8}(\Omega)]^4} + \|v\|_{[W^{1,8}(\Omega)]^4}$$

$$\leq c\|u - v\|_{[W^{4,8}(\Omega)]^4} \|u\|_{[W^{4,8}(\Omega)]^4} + \|v\|_{[W^{4,8}(\Omega)]^4}.$$

Consequently, we claim the following local existence result for the N-S equation in 4-D.

**Theorem 11.** When $Pf \in D(A^4_0)$ and $u_0 \in D(A^3_0)$, then there exists a unique local in time mild solution $u(t)$ to (1.13) in the phase space $D(A^3_0) \subset [W^{4,8}(\Omega)]^4$.

Moreover,

$$u \in C((0, \tau); D(A^3_0)) \cap C((0, \tau); D(A^1_0)), \ u_t \in C((0, \tau); D(A^1_0^-)).$$

Here $\tau > 0$ is the 'life time' of that local in time solution. Moreover, the corresponding Cauchy formula is satisfied.

3.5. 2-D critical quasi-geostrophic equation. The quasi-geostrophic equation was studied recently under various assumptions by many authors, see e.g. [6, 18, 20, 38].

Define $A_\alpha := \kappa[(-\Delta)^\alpha + I]$, $\alpha \in (\frac{1}{2}, 1)$, where $(-\Delta)^\alpha$ is the fractional Laplacian. Also, setting

$$F(\theta) = R_2\theta \frac{\partial \theta}{\partial x_1} - R_1\theta \frac{\partial \theta}{\partial x_2} + f + \kappa \theta,$$

the problem (1.15) will be written formally as

$$\theta_t + A_\alpha \theta = F(\theta), \ t \geq 0,$$

$$\theta(0) = \theta_0,$$

which is an abstract 'parabolic' equation with sectorial positive operator. As in [20], it is also possible to choose $H^s(\mathbb{R}^2)$ as a base space where we consider the equation (1.15). In that case the resulting phase space will be $H^{2\alpha + s}(\mathbb{R}^2)$ with $s > 1$. Using (1.17), for $\theta_1, \theta_2 \in B(r)$, we obtain

$$\|F(\theta_1) - F(\theta_2)\|_{H^s(\mathbb{R}^2)}$$

$$\leq \kappa \|\theta_1 - \theta_2\|_{H^s(\mathbb{R}^2)} + \|R_2(\theta_1 - \theta_2) \frac{\partial \theta_1}{\partial x_1} + R_2(\theta_2 \frac{\partial (\theta_1 - \theta_2)}{\partial x_1}\|_{H^s(\mathbb{R}^2)}$$

$$+ R_1(\theta_1 - \theta_2) \frac{\partial \theta_1}{\partial x_2} + R_1(\theta_2 \frac{\partial (\theta_1 - \theta_2)}{\partial x_2})\|_{H^s(\mathbb{R}^2)}.$$
Now we estimate the second term above using the property (e.g. [1, p. 115]), that when \( \Omega \) is a domain in \( \mathbb{R}^N \) having the cone property, then \( W^{m,r}(\Omega) \) is a Banach Algebra provided that \( m r > N \). In our case \( 2s > 2 \) since \( s > 1 \). Hence

\[
\| R_2(\theta_1 - \theta_2) \frac{\partial \theta_1}{\partial x_1} \|_{H^s(\mathbb{R}^2)} \leq c \| R_2(\theta_1 - \theta_2) \|_{H^s(\mathbb{R}^2)} \| \frac{\partial \theta_1}{\partial x_1} \|_{H^s(\mathbb{R}^2)} \\
\leq c \| u_1 - u_2 \|_{H^s(\mathbb{R}^2)} \| \theta_1 \|_{H^{2\alpha - s}}(\mathbb{R}^2),
\]

(3.16)

where \( u_i, i = 1, 2 \), correspond to \( \theta_i \) through relation (1.17).

Applying the property (5.3) to the first term in (3.16) we get

\[
\| R_2(\theta_1 - \theta_2) \frac{\partial \theta_1}{\partial x_1} \|_{H^s(\mathbb{R}^2)} \leq c \| \theta_1 - \theta_2 \|_{H^{2\alpha - s}}(\mathbb{R}^2) \| \theta_1 \|_{H^{2\alpha - s}}(\mathbb{R}^2).
\]

The other components in (3.15) are estimated analogously. Consequently we obtain that

\[
\| F(\theta_1) - F(\theta_2) \|_{H^s(\mathbb{R}^2)} \leq c (\| \theta_1 \|_{H^{2\alpha - s}}(\mathbb{R}^2) + \| \theta_2 \|_{H^{2\alpha - s}}(\mathbb{R}^2)) \| \theta_1 - \theta_2 \|_{H^{2\alpha - s}}(\mathbb{R}^2),
\]

which proves local solvability of (1.15) in the phase space \( H^{2\alpha - s}(\mathbb{R}^2) \). More precisely, following [4, 16], we formulate:

**Theorem 12.** Let \( s > 1 \) be fixed. Then, for \( f \in H^s(\mathbb{R}^2) \) and for arbitrary \( \theta_0 \in H^{2\alpha - s}(\mathbb{R}^2) \), there exists in the phase space \( H^{2\alpha - s}(\mathbb{R}^2) \) a unique local in time mild solution \( \theta(t) \) to the subcritical problem (1.15), \( \alpha \in (\frac{1}{2}, 1] \). Moreover,

\[
\theta \in C((0, \tau); H^{2\alpha - s}(\mathbb{R}^2)) \cap C([0, \tau); H^{2\alpha - s}(\mathbb{R}^2)), \quad \theta_t \in C((0, \tau); H^{2\gamma + s}(\mathbb{R}^2)),
\]

with arbitrary \( \gamma < \alpha \). Here \( \tau > 0 \) is the ‘life time’ of that local in time solution. Moreover, the Cauchy formula is satisfied:

\[
\theta(t) = e^{-A_\alpha t} \theta_0 + \int_0^t e^{-A_\alpha (t-s)} F(\theta(s)) ds, \quad t \in [0, \tau),
\]

where \( e^{-A_\alpha t} \) denotes the linear semigroup corresponding to the operator \( A_\alpha := \kappa (-\Delta)^\alpha + \mathbb{I} \) in \( H^s(\mathbb{R}^2) \), and \( F \) is given by formula (3.13).

4. Global in time solvability through the known natural a priori estimate.

4.1. Natural a priori estimate of the Navier-Stokes equation. We will recall now the natural a priori estimate known for solutions of the N-S equation, that we calculate here in 2-D case for the fractional extension

\[
u^\alpha := -A_\alpha u^\alpha - P(u^\alpha \cdot \nabla)u^\alpha + Pf,
\]

(4.1)

with the parameter \( \alpha \in (0, \infty) \). The estimates are uniform in the parameter \( \alpha \).

Multiplying (4.1) by \( u^\alpha \), noting that the nonlinear term vanishes for divergence free functions, we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u^\alpha|^2 dx \leq - \int_\Omega |A_\alpha^{\frac{\alpha}{2}} u^\alpha|^2 dx + \int_\Omega Pf u^\alpha dx.
\]

(4.2)

Next note an extension of the Poincaré inequality valid for fractional powers of the Stokes operator (following e.g. from [22, Chapter I, §7])

\[
\int_\Omega |A_\alpha^{\frac{\alpha}{2}} \phi|^2 dx \geq (\nu \lambda_1)^{\alpha} \int_\Omega \phi^2 dx,
\]

(4.3)
where \( \lambda_1 > 0 \) is the first eigenvalue of the \(-\Delta\) in \( \Omega \) with zero Dirichlet boundary condition. Using the last estimate in the differential inequality we get

\[
\frac{d}{dt} \| u^\alpha \|_{L^2(\Omega)}^2 \leq - \int_\Omega |A^\frac{\alpha}{2} u^\alpha|^2 \, dx + c \| Pf \|_{L^2(\Omega)}^2 \\
\leq - (\nu \lambda_1)^\alpha \| u^\alpha \|_{L^2(\Omega)}^2 + c \| Pf \|_{L^2(\Omega)}^2,
\]

where \( c \) can be taken independent on \( \alpha \). Consequently, we obtain an estimate

\[
\| u^\alpha(t) \|_{L^2(\Omega)}^2 \leq \| u^\alpha(0) \|_{L^2(\Omega)}^2 + \frac{c \| Pf \|_{L^2(\Omega)}^2}{(\nu \lambda_1)^\alpha}.
\]  

(4.5)

Returning to the differential inequality (4.4) we find that

\[
\int_0^T \int_\Omega |A^\frac{\alpha}{2} u^\alpha|^2 \, dx \, dt \leq \| u^\alpha(0) \|_{L^2(\Omega)}^2 + c \int_0^T \| Pf \|_{L^2(\Omega)}^2 \, dt,
\]

which is the second required bound. Consequently,

\[
u \in L^\infty(0, T; [L^2(\Omega)]^2) \cap L^2(0, T; D(A^\frac{\alpha}{2})).
\]  

(4.7)

It is clear that the estimates can be made uniformly in \( \alpha \in (0, \infty) \).

**A better norm estimate.** We will show next that a better norm a priori estimate is valid for solution \( u^\alpha \) of (4.1) when \( \alpha > 1 \). Multiplying equation by \( A^\alpha u^\alpha \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |A^\frac{\alpha}{2} u^\alpha|^2 \, dx = - \int_\Omega |A^\alpha u^\alpha|^2 \, dx - \int_\Omega A^\alpha u^\alpha P(u^\alpha \cdot \nabla) u^\alpha \, dx + \int_\Omega A^\alpha u^\alpha Pf \, dx.
\]  

(4.8)

For the nonlinear term we use an estimate

\[
| \int_\Omega A^\alpha u^\alpha P(u^\alpha \cdot \nabla) u^\alpha \, dx | \leq c \| A^\alpha u^\alpha \|_{[L^2(\Omega)]^2} \| u^\alpha \|_{L^6(\Omega)} \| \nabla u^\alpha \|_{[L^2(\Omega)]^2}
\]

\[
\leq c \| A^\alpha u^\alpha \|_{[L^2(\Omega)]^2}^{1+\frac{\alpha}{2}} \| u^\alpha \|_{[L^2(\Omega)]^2}^{2-\frac{\alpha}{2}},
\]  

(4.9)

following from the Nirenberg-Gagliardo inequality. Consequently, when \( \alpha > 1 \), by Young’s inequality we get an estimate

\[
\frac{d}{dt} \int_\Omega |A^\frac{\alpha}{2} u^\alpha|^2 \, dx \leq - \int_\Omega |A^\alpha u^\alpha|^2 \, dx + c \| u^\alpha \|_{[L^2(\Omega)]^2}^{2(\alpha-1)} + c \| Pf \|_{L^2(\Omega)}^2,
\]  

(4.10)

leading, together with the natural a priori estimate (4.7), to the bound

\[
u \in L^\infty(0, T; D(A^\frac{\alpha}{2})) \cap L^2(0, T; D(A^\frac{\alpha}{2})).
\]  

(4.11)

Note that the value \( \alpha = 1 \) is critical for validity of the calculations above. The last estimate will be crucial for extending the local solutions globally in time.

**Global in time solutions of the 4-D N-S equation with small data.** We will analyze such estimate in more detail in a particular case \( N = 4 \) for the original N-S equation. Such type estimates are used to show global in time extendibility of the local solutions corresponding to small data.

Multiplying the original N-S equation by \( Au \), using standard estimates, we obtain a differential inequality

\[
\frac{1}{2} \frac{d}{dt} \| u \|_{H^1(\Omega)}^2 \leq - \alpha \| u \|_{H^2(\Omega)}^2 + c \| u \|_{W^{1,4}(\Omega)} \| u \|_{L^4(\Omega)} \| u \|_{H^2(\Omega)}^4 + \| Pf \|_{[L^2(\Omega)]^2} \| u \|_{[H^2(\Omega)]^4}.
\]  

(4.12)
In the 4-D N-S equation the following two Sobolev type embeddings are valid

\[ \| \phi \|_{W^{1,4}(\Omega)} \leq c \| \phi \|_{H^2(\Omega)}, \quad \| \phi \|_{L^4(\Omega)} \leq c \| \phi \|_{H^1(\Omega)}, \]  

allowing to extend the estimate of nonlinear term to the form

\[ \| u \|_{W^{1,4}(\Omega)}^4 \| u \|_{L^4(\Omega)}^4 \| u \|_{H^2(\Omega)}^4 \leq c \| u \|_{H^2(\Omega)}^7 \| u \|_{H^1(\Omega)}^4. \]

Inserting the last inequality to (4.12), using Cauchy’s inequality to the final term, we get a differential inequality for \( y(t) = \| u(t, \cdot) \|_{H^1(\Omega)}^4 \)

\[ \frac{1}{2} \frac{d}{dt} y(t) \leq \left( -c v + \epsilon + c \| u \|_{H^1(\Omega)}^4 \right) \| u \|_{H^2(\Omega)}^2 + c \| Pf \|_{L^2(\Omega)}^4. \]  

In case the bracket above is negative, the latter can be extended with the use of the embedding \( H^2(\Omega) \subset H^1(\Omega) \) to the form

\[ \frac{1}{2} \frac{d}{dt} y(t) \leq c \left( -c v + \epsilon + c g^\frac{1}{2}(t) \right) y(t) + c \| Pf \|_{L^2(\Omega)}^4 =: g(y). \]

A nice while simple analysis of the right hand side \( g(y) \) shows that since \( g(0) = c \| Pf \|_{L^2(\Omega)}^4 > 0 \) and \( g'(0) = c(-cv + \epsilon) < 0 \), then there is a first positive zero \( z_1 \) of \( g \), provided \( \| Pf \|_{L^2(\Omega)}^4 \) is sufficiently small. More precisely, we need to assume that the minimum of \( g \) attained in \( y_{\text{min}} > 0 \) is negative, that is

\[ \frac{1}{3} y_{\text{min}} = \frac{2}{3} \frac{c v - \epsilon}{c}, \]

\[ g(y_{\text{min}}) = -c(c v - \epsilon) \left( \frac{2}{3} \frac{c v - \epsilon}{c} \right)^2 + c^2 \left( \frac{2}{3} \frac{c v - \epsilon}{c} \right)^3 + c \| Pf \|_{L^2(\Omega)}^4 < 0. \]  

The last condition provides a smallness restriction on \( Pf \)

\[ c^2 \| Pf \|_{L^2(\Omega)}^4 < \frac{4}{27} (c v - \epsilon)^3. \]

Note that considering sufficiently small initial data \( \| u_0 \|_{H^1(\Omega)}^4 \leq y_{\text{min}}^{\frac{1}{2}} = \frac{2}{3} \frac{c v - \epsilon}{c} \), we justify also (4.16). This is the required smallness restriction for \( u_0 \). The corresponding to such restricted \( u_0 \) and \( Pf \) local solution \( u(t) \) will be bounded in \( H^1(\Omega) \) for all \( t \geq 0 \), hence also in \( L^4(\Omega) \) since \( N = 4 \). Such a priori estimate is sufficient to extend \( u(t) \) globally in time; compare Remark 21.

An alternative way of getting the smallness restrictions goes through direct estimates on integral equation. In case of the 4-D Navier-Stokes equation (1.13) in \( D(A_4^\frac{3}{4}) \) we will find conditions guaranteeing that its \( \| u(t, u_0) \|_{D(A_4^\frac{3}{4})} \) norm does not blow up in a finite time. That is,

\[ \forall T > 0 \limsup_{t \to T^-} \| u(t) \|_{D(A_4^\frac{3}{4})} < \infty. \]
Applying Lemma 2.2 in [13] and the usual estimates for analytic semigroups (e.g. [9]), we get
\[
\|u(t)\|_{D(A^3_4)} \leq \|e^{-A^3_4 t}u_0\|_{D(A^3_4)} + \int_0^t \|e^{-A^3_4 (t-s)}(P(u \cdot \nabla)u + Pf)\|_{D(A^3_4)} ds
\]
\[
\leq c\|u_0\|_{D(A^3_4)} + c\int_0^t e^{-a(t-s)}(t-s)^{-\frac{7}{4}} \|P(u \cdot \nabla)u + Pf\|_{D(A^4_4)} ds
\]
\[
\leq c\|u_0\|_{D(A^3_4)} + c\int_0^t e^{-a(t-s)}(t-s)^{-\frac{7}{4}} \left(\|A^{-\frac{7}{4}} P(u \cdot \nabla)u\|_{L^4(\Omega)}^4 + \|Pf\|_{L^4(\Omega)}^4\right) ds
\]
\[
\leq c\|u_0\|_{D(A^3_4)} + c\int_0^t e^{-a(t-s)}(t-s)^{-\frac{7}{4}} \|Pf\|_{L^4(\Omega)}^4 + c\int_0^t e^{-a(t-s)}(t-s)^{-\frac{7}{4}} \|u\|_{D(A^3_4)}^2 ds,
\]
where \(a > 0\) is such that \(Re \sigma(A) > a\).

Choose \(u_0\) and \(f\) sufficiently small such that
\[
m_0 := c^2 a^{-\frac{7}{4}} \Gamma \left(\frac{1}{8}\right) \left(\|u_0\|_{D(A^3_4)} + a^{-\frac{7}{4}} \Gamma \left(\frac{1}{8}\right) \|Pf\|_{L^4(\Omega)}^4\right) < 1. \tag{4.19}
\]

Then it follows from the Gronwall type inequality in Lemma 22 that
\[
\|u(t)\|_{D(A^3_4)} \leq c \left(\|u_0\|_{D(A^3_4)} + a^{-\frac{7}{4}} \Gamma \left(\frac{1}{8}\right) \|Pf\|_{L^4(\Omega)}^4\right)^{\frac{1}{1-m_0}}.
\]

**Theorem 13.** If \(Pf \in D(A^3_4)\) and \(u_0 \in D(A^3_4)\) fulfill the smallness restriction (4.19), then the \(D(A^3_4)\) norm of the solution \(u\) is bounded uniformly in time \(t \geq 0\). Consequently the solution \(u\) will be extended globally in time.

### 4.2. Natural a priori estimate of the quasi-geostrophic equation with linear damping term

We consider the following quasi-geostrophic equation with linear damping term,
\[
\theta_t + \lambda \theta + u \cdot \nabla \theta + \kappa (-\Delta)^{\alpha} \theta = f, \quad x \in \mathbb{R}^2, \ t > 0,
\]
\[
\theta(0, x) = \theta_0(x), \tag{4.20}
\]
where \(\lambda > 0\), \(\kappa > 0\) and \(\alpha \in \left(\frac{1}{2}, 1\right)\). Note that for smooth solutions after multiplying the nonlinear term \(u \cdot \nabla \theta\) by \(|\theta|^{q-1} \text{sgn} \theta\) with \(q \geq 2\) and integrating the result over \(\mathbb{R}^2\) the resulting term will vanish:
\[
\int_{\mathbb{R}^2} \left(\frac{\partial \theta}{\partial x_1} \frac{\partial}{\partial x_2} [-(-\Delta)^{-\frac{1}{2}} \theta] - \frac{\partial \theta}{\partial x_2} \frac{\partial}{\partial x_1} [-(-\Delta)^{-\frac{1}{2}} \theta]\right) |\theta|^{q-1} \text{sgn} \theta \, dx
\]
\[
= \frac{1}{q} \int_{\mathbb{R}^2} \left(\frac{\partial |\theta|^{q-1}}{\partial x_1} \frac{\partial}{\partial x_2} [-(-\Delta)^{-\frac{1}{2}} \theta] - \frac{\partial |\theta|^{q-1}}{\partial x_2} \frac{\partial}{\partial x_1} [-(-\Delta)^{-\frac{1}{2}} \theta]\right) \, dx = 0,
\]
thanks to integration by parts. Consequently, multiplying (1.15) by \(|\theta|^{q-1} \text{sgn} \theta\), we obtain
\[
\int_{\mathbb{R}^2} \theta_1 |\theta|^{q-1} \text{sgn} \theta \, dx + \lambda \int_{\mathbb{R}^2} \theta |\theta|^{q-1} \text{sgn} \theta \, dx + \kappa \int_{\mathbb{R}^2} (-\Delta)^{\alpha} \theta \cdot |\theta|^{q-1} \text{sgn} \theta \, dx
\]
\[
= \int_{\mathbb{R}^2} f |\theta|^{q-1} \text{sgn} \theta \, dx.
\]
Since for all the values \(1 \leq q < \infty\) and \(0 \leq \alpha \leq 1\), as was shown in [6] and [7, Lemma 2.5],

\[
\int_{\mathbb{R}^2} (-\Delta)\alpha \theta |\theta|^{q-1} \text{sgn}(\theta) dx \geq 0,
\]

using Hölder’s and Young’s inequalities, we get

\[
\frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^2} |\theta|^{q} dx + \lambda \int_{\mathbb{R}^2} |\theta|^{q} dx \leq \int_{\mathbb{R}^2} f|\theta|^{q-1} \text{sgn}(\theta) dx
\]

\[
\leq \frac{1}{q\lambda \gamma - 1} \|f\|_{L^q(\mathbb{R}^2)}^{q} + \frac{\lambda (q - 1)}{q} \|\theta\|_{L^q(\mathbb{R}^2)}^{q}.
\]

Solving the above differential inequality we get

\[
\|\theta(t, \cdot)\|_{L^q(\mathbb{R}^2)}^{q} \leq \|\theta_0\|_{L^q(\mathbb{R}^2)}^{q} e^{-\lambda t} + \frac{1}{\lambda \gamma} \|f\|_{L^q(\mathbb{R}^2)}^{q} (1 - e^{-\lambda t}). \tag{4.21}
\]

**Further a priori estimate.** Let \(s > 1\) be fixed and \(f \in H^s(\mathbb{R}^2)\). Multiplying the equation by \((-\Delta)^{\alpha+s} \theta\) with \(\alpha \in [\frac{1}{2}, 1]\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} [(-\Delta)^{\frac{\alpha+s}{2}} \theta]^{2} dx + \lambda \int_{\mathbb{R}^2} [(-\Delta)^{\frac{\alpha+s}{2}} \theta]^{2} dx + \kappa \int_{\mathbb{R}^2} [(-\Delta)^{\frac{2\alpha+s}{2}} \theta]^{2} dx
\]

\[
= \int_{\mathbb{R}^2} (-\Delta)^{\frac{s}{2}} f(-\Delta)^{\frac{2\alpha+s}{2}} \theta dx - \int_{\mathbb{R}^2} u \cdot \nabla \theta (-\Delta)^{\alpha+s} \theta dx. \tag{4.22}
\]

Noticing that \(\nabla \cdot u = 0\), we have

\[
\int_{\mathbb{R}^2} u \cdot \nabla (-\Delta)^{\frac{s}{2}} \theta (-\Delta)^{\alpha+s} \theta dx = 0.
\]

Since \(\nabla\) and \((-\Delta)^{\frac{s}{2}}\) are commutable ([18] or Remark 23), the nonlinear term is transformed as follows:

\[
| - \int_{\mathbb{R}^2} u \cdot \nabla \theta (-\Delta)^{\alpha+s} \theta dx | = | - \int_{\mathbb{R}^2} (-\Delta)^{\frac{s}{2}} (u \cdot \nabla \theta) (-\Delta)^{\alpha+s} \theta dx |
\]

\[
= \int_{\mathbb{R}^2} (-\Delta)^{\frac{s}{2}} (u \cdot \nabla \theta) (-\Delta)^{\alpha+s} \theta - u \cdot \nabla (-\Delta)^{\frac{s}{2}} \theta (-\Delta)^{\alpha+s} \theta dx \tag{4.23}
\]

\[
\leq \|(-\Delta)^{\frac{s}{2}} (u \cdot \nabla \theta) - u \cdot (-\Delta)^{\frac{s}{2}} \nabla \theta\|_{L^{p_2}(\mathbb{R}^2)} \|(-\Delta)^{\alpha+s} \theta\|_{L^{p_2}(\mathbb{R}^2)},
\]

where

\[
\frac{1}{p_1} + \frac{1}{p_2} = 1, \quad p_2 \in (1, 2), \quad p_2 \in (2, \infty).
\]

Taking \(p_1 = 4\alpha + 2s\) and \(p_2 = \frac{4\alpha + 2s}{\alpha + s}\), then by Lemma 24 and (5.2) we obtain

\[
\|(-\Delta)^{\frac{s}{2}} (u \cdot \nabla \theta) - u \cdot (-\Delta)^{\frac{s}{2}} \nabla \theta\|_{L^{p_2}(\mathbb{R}^2)} \leq c \|\nabla u\|_{L^{p_1}(\mathbb{R}^2)} \|(-\Delta)^{\frac{\alpha+s}{2}} \theta\|_{L^{p_2}(\mathbb{R}^2)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^{p_2}(\mathbb{R}^2)} \|\nabla \theta\|_{L^{p_1}(\mathbb{R}^2)}.
\]

Using the Nirenberg-Gagliardo inequality, we have

\[
\|\nabla \theta\|_{W^{1, p_1}(\mathbb{R}^2)} \leq c \|\theta\|_{H^{2\alpha+s}(\mathbb{R}^2)} \|\theta\|_{L^{\infty}(\mathbb{R}^2)}^{1 - \frac{1}{(\alpha+s)p_1}},
\]

\[
\|\theta\|_{W^{\alpha+s, p_2}(\mathbb{R}^2)} \leq c \|\theta\|_{H^{\alpha+s}(\mathbb{R}^2)} \|\theta\|_{L^{\infty}(\mathbb{R}^2)}^{1 - \frac{\alpha+s}{\alpha+s+p_2}}.
\]

Thus, estimate (4.23) extends to

\[
| - \int_{\mathbb{R}^2} u \cdot \nabla \theta (-\Delta)^{\alpha+s} \theta dx | \leq c \|\theta\|_{H^{2\alpha+s}(\mathbb{R}^2)} \|\theta\|_{L^{\infty}(\mathbb{R}^2)}^{3 - \frac{2\alpha+2s+1}{\alpha+s}}. \tag{4.24}
\]
Note that \( \theta_0 \in H^{2\alpha+1}(\mathbb{R}^2) \), we deduce that \( \theta_0 \in L^\infty(\mathbb{R}^2) \). Then it follows from [1, Theorem 2.8] and (4.21) that
\[
\|\theta\|_{L^\infty(\mathbb{R}^2)} \leq \|\theta_0\|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\lambda} \|f\|_{L^\infty(\mathbb{R}^2)} < \infty. \quad (4.25)
\]
In the following, we will deal with (4.24). Since \( \alpha > \frac{1}{2} \) implies \( \frac{2\alpha+2+1}{2\alpha+\beta} < 2 \), then by Young’s inequality, we have
\[
| \int_{\mathbb{R}^2} u \cdot \nabla \theta (-\Delta)^{\alpha+\beta} \theta \, dx | \leq \frac{\kappa}{4} \|\theta\|_{H^{2\alpha+\beta}(\mathbb{R}^2)}^2 + \text{const}(\|\theta\|_{L^\infty(\mathbb{R}^2)}),
\]
where \( \text{const} \) depends in a non-decreasing way on its argument.

Consequently, from (4.22) and the above estimate we obtain a differential inequality:
\[
\frac{d}{dt} \int_{\mathbb{R}^2} [(-\Delta)^{\frac{\alpha+\beta}{2}} \theta]^2 \, dx \leq 2\lambda \int_{\mathbb{R}^2} [(-\Delta)^{\frac{\alpha+\beta}{2}} \theta]^2 \, dx + \kappa \int_{\mathbb{R}^2} [(-\Delta)^{\frac{2\alpha+\beta}{2}} \theta]^2 \, dx \leq \text{const}(\|\theta\|_{L^\infty(\mathbb{R}^2)}),
\]
providing us, together with (4.25), the bound for \( \|\theta\|_{H^{\alpha+\beta}(\mathbb{R}^2)} \).

4.3. **Global in time weak solutions.** Our next task is to extend the constructed above local solution globally in time (that means, onto arbitrary large time interval \([0,T], T > 0 \)). Consider the family of regularizations to (2.1)
\[
\begin{align*}
    u_t + A^{1+\alpha} u &= F(u), \\
    u(0) &= u_0,
\end{align*}
\]
with parameter \( \alpha \in (0, \epsilon), \epsilon > 0 \).

For \( \beta \in [0, 1+\alpha) \), assume that the nonlinear term \( F : X^\beta \to X \) is Lipschitz continuous on bounded sets \( B \subset X^\beta \), that is
\[
\|F(u) - F(v)\|_X \leq L_B \|u - v\|_{X^\beta} \quad \text{whenever } u, v \in B \subset X^\beta. \quad (4.28)
\]
Thanks to the abstract theory we have local in time solvability of (2.1) in the \( X^\beta \) phase space.

Observe further that when \( \beta + \epsilon < 1+\alpha, \epsilon > 0 \), extending (4.28), for any bounded set \( \tilde{B} \subset X^{\beta+\epsilon} \)
\[
\|F(u) - F(v)\|_X \leq \tilde{L}_B \|u - v\|_{X^{\beta+\epsilon}} \quad \text{whenever } u, v \in \tilde{B}. \quad (4.29)
\]
However, if \( \beta + \epsilon \geq 1 \) the introduced earlier local existence theory will not be applied to the problem (2.1) in the \( X^{\beta+\epsilon} \) phase space.

Assume further that for each local \( X^\beta \) solutions of (2.1), we have an *a priori estimate*: \( \|u(t)\|_Y \leq c \) in a certain auxiliary space \( Y \), such that \( D(A^\beta) \subset Y \subset D(A^\nu) \) for certain \( 0 \leq \eta \leq \beta \), and
\[
\forall \text{nondecreasing } g : [0, \infty) \to [0, \infty), \quad \exists \theta \in (0,1) \quad \|F(u(t))\|_X \leq g(\|u(t)\|_Y)(1 + \|u(t)\|_{X^\beta}),
\]
(4.30)
that means, the problem (2.1) is *critical with respect to such a priori estimate*.

From that moment, together with the original problem (2.1), we will consider its regularization (4.27) with small \( \alpha > 0 \) fixed. We assume that the operator \( A^{1+\alpha} \) remains sectorial and positive. It is then easy to note that for such regularization, as a consequence of the Lipschitz continuity (4.28), local in time solvability holds. Denote the corresponding local solution by \( u^\alpha \). We need also to note, that the *fractional scale parameter* connected with \( A^{1+\alpha} \) will change (the original parameter corresponding to \( A \) should be divided by \( 1+\alpha \)). Denote the modified scale by \( Z^\alpha \), so
that $Z^{\frac{k}{1+k}} = X^k$ and $X:=X^0 = Z$. For that new scale and the local solutions $u^\alpha$ of the regularized problem (4.27), using the Nirenberg-Gagliardo inequality, condition (4.30) will take the form
\[
\exists_{\text{nondecreasing } g:[0,\infty)\to [0,\infty]} \|F(u^\alpha(t))\|_Z \leq g(\|u^\alpha(t)\|_Y)(1 + \|u^\alpha(t)\|_Z^{\frac{1+k}{1+k}}) \\
\leq g(\|u^\alpha(t)\|_Y)(1 + \|u^\alpha(t)\|^\delta_1 \|u^\alpha(t)\|^{\frac{1-\delta_1}{\delta_1}}),
\]
for $\delta \in (0, \alpha)$ with $\theta(\delta) < 1$, so that the local solution $u^\alpha$ will be extended globally in time.

Our task now is to pass to the limit over a sequence $\alpha_n \to 0^+$ extracting a subsequence of approximating solutions $u^{\alpha_n}(t)$ convergent for $t \in [0,\infty)$ to certain weak solution of the limit critical problem (2.1). A uniform in $\alpha > 0$ estimate of the solutions $\tilde{u}^\alpha$ of (4.27) in $Y$ will be the main information used in that procedure.

We will describe next the way of passing to the limit in (4.27). Thanks to the uniform in $\alpha \in (0, \epsilon)$ boundedness of $u^\alpha$, by weak compactness, we can extract a subsequence $u^{\alpha_n}$ convergent weakly to $u$ in the corresponding space. Note that, by [27, Theorems 3.1.6 and 7.1.1], $A^{-1}\alpha \phi \to A^{-1}\phi$ when $\alpha \to 0^+$. As for the nonlinear term, we will use the approach of J.-L. Lions ([26, Theorem 5.1]).

4.4. 2-D N-S equation. We consider the following approximating equation for the 2-D Navier-Stokes equation:
\[
u = -A^{1+\alpha} u - P(u \cdot \nabla)u + Pf, \quad t > 0,
\]
with $\alpha \in (0, \frac{1}{2})$, and denote its solution by $u^\alpha$. Using the similar estimates in (4.7), we obtain $u^\alpha \in L^\infty(0, T; [L^2(\Omega)]^2) \cap L^2(0, T; D(A^{\frac{1+\alpha}{2}}))$. Thanks to uniform boundedness (in $\alpha$) of $u^\alpha$ in $L^\infty(0, T; [L^2(\Omega)]^2)$ and $L^2(0, T; [H^1_0(\Omega)]^2)$, we can find a subsequence (which we shall relabel $u^\alpha$) such that $u^\alpha \rightharpoonup u$ ($\alpha \to 0^+$)-weakly in $L^\infty(0, T; [L^2(\Omega)]^2)$ and weakly in $L^2(0, T; [H^1_0(\Omega)]^2)$ with $u \in L^\infty(0, T; [L^2(\Omega)]^2) \cap L^2(0, T; [H^1_0(\Omega)]^2)$. To obtain similar convergence of $A^{1+\alpha} u^\alpha$ to $Au$ we will use Theorem 3.1.6 in [27], or more precisely [9, Lemma 7.2] and the expression for $A^\psi$ using the 'special basis' as in [26, Section 6.4]
\[
A^\psi = \sum_{j=1}^\infty g_j(t)\lambda_j w_j,
\]
to see that $A^{1+\alpha} \psi - A \psi = (A^\alpha - I)(A \psi) \to 0$, whenever $\psi \in D(A_2)$. Consequently the equality holds
\[
\int_0^T < A^{1+\alpha} u^\alpha, \psi > - < Au, \psi > dt \\
= \int_0^T < u^\alpha - u, A^{1+\alpha} \psi > dt - \int_0^T < u, A^{1+\alpha} \psi - A \psi > dt \to 0,
\]
as $\alpha \to 0$, valid for any $\psi \in L^2(0, T; D(A_2))$, where $< \cdot, \cdot >$ denotes the duality product between $[H^1_0(\Omega)]^2$ and $[H^{-1}(\Omega)]^2$.

Note that
\[
\|A^{\frac{1}{2}} P(u^\alpha \cdot \nabla)u^\alpha\|_{L^2(\Omega)^2} \leq c \|u^\alpha\|_{L^4(\Omega)^2} \leq c' \|u^\alpha\|_{L^4(\Omega)^2} \|u^\alpha\|_{[H^1(\Omega)]^2}.
\]
Furthermore, we have
\[
A^{-\frac{1+\alpha}{2}} u^\alpha = -A^{-\frac{1+\alpha}{2}} u^\alpha - A^{-\frac{1+\alpha}{2}} P(u^\alpha \cdot \nabla)u^\alpha + A^{-\frac{1+\alpha}{2}} Pf \in L^2(0, T; [L^2(\Omega)]^2),
\]
so \(\{u^\alpha\}\) is uniformly bounded in \(L^2(0, T; D(A_2^{-\frac{3}{4}}))\) (even in \(L^2(0, T; D(A_2^{-\frac{5}{4}} + a_0))\) with arbitrary small positive \(a_0\)), therefore \(u^\alpha_{t} \rightharpoonup u_t\) (the distributional time derivative) weakly in \(L^2(0, T; \mathcal{H}^{-\frac{3}{4}}_2))\) as \(\alpha \to 0\), and the Lions compactness lemma [26, Theorem 5.1] will guarantee that there is a subsequence \(\{u^\alpha\}\) (after relabeling) converging to \(u\) strongly in \(L^2(0, T; [H^1(\Omega)]^2)\) and almost everywhere in \([0, T] \times \Omega\). Then for any \(w \in C_c^\infty(0, T; [H^1_0(\Omega)]^2)\),

\[
\left| \int_0^T \int_{\Omega} P(u^\alpha \cdot \nabla)u^\alpha w - P(u \cdot \nabla)uw \, dx dt \right| \\
\leq \left| \int_0^T \int_{\Omega} ((u^\alpha - u) \cdot \nabla)wdx dt \right| + \left| \int_0^T \int_{\Omega} (u \cdot \nabla)(u^\alpha - u)wdx dt \right| \\
\leq \int_0^T \|u^\alpha - u\|_{L^2(\Omega)} \left( \|u^\alpha\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right) \|\nabla w\|_{L^2(\Omega)} \, dt \\
\leq \|\nabla w\|_{L^\infty(0,T;L^2(\Omega))} \left( \|u^\alpha\|_{L^2(0,T;H^{-\frac{3}{4}}_1(\Omega))} + \|u\|_{L^2(0,T;H^{-\frac{3}{4}}_1(\Omega))} \right) \times \|u^\alpha - u\|_{L^2(0,T;H^{-\frac{3}{4}}_1(\Omega))} \to 0 \quad \text{as} \quad \alpha \to 0.
\]

Since \(C_c^\infty(0, T; [H^1_0(\Omega)]^2)\) is dense in \(L^2(0, T; [H^1_0(\Omega)]^2)\), (4.34) implies that \(P(u^\alpha \cdot \nabla)u^\alpha \rightharpoonup P(u \cdot \nabla)u\) weakly in \(L^2(0, T; [H^{-\frac{3}{4}}_1(\Omega)]^2)\). Letting \(\alpha \to 0\) in (4.32), we see that the equality

\[
u_t = -Au - P(u \cdot \nabla)u + Pf
\]

is fulfilled in \(L^2(0, T; D(A_2^{-\frac{3}{4}}))\).

Note further that since \(u \in L^\infty(0, T; \{L^2(\Omega)\}^2), u_t \in L^2(0, T; D(A^{-\frac{3}{4}}_2))\), then by [26, Lemma 1.2, Chapter I] also \(u \in C^0([0, T]; D(A^{-\frac{3}{4}}_2))\). Consequently \(u \in C^0([0, T]; D(A^{-\frac{3}{4}}_2))\), and since \(u \in L^\infty(0, T; \{L^2(\Omega)\}^2)\), then by [31, Theorem 2.1] \(u \in C_w([0, T]; \{L^2(\Omega)\}^2)\). This allows to define properly the initial value \(u_0\).

Concluding the above considerations, we formulate:

**Theorem 14.** Let \(Pf \in D(A_2^{-\frac{3}{4}})\). Then if \(u_0 \in D(A_2^{\frac{1}{4}}) \subset [H^{1+}(\Omega)]^2\), there exists a weak solution \(u(t)\) to (1.13) in 2-D, that for any \(T > 0\),

\[
u \in L^\infty(0, T; \{L^2(\Omega)\}^2) \cap L^2(0, T; [H^1_0(\Omega)]^2),
\]

and (4.35) holds as equality in \(L^2(0, T; D(A_2^{-\frac{3}{4}}))\). Moreover \(u \in C_w([0, T]; \{L^2(\Omega)\}^2)\).

**Remark 15.** As well known (e.g. [26, Theorem 6.2]), the solution of the 2-D N-S equation with \(u \in L^\infty(0, T; \{L^2(\Omega)\}^2) \cap L^2(0, T; D(A_2^{\frac{1}{4}}))\) and \(u_t \in L^2(0, T; D(A_2^{-\frac{3}{4}}))\) is unique. Indeed, if \(u_1, u_2\) are two such (weak) solutions, applying projector \(P\), taking the difference of the equations and multiplying the result by \(w = u_1 - u_2\), we get:

\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2(\Omega)}^2 = -\|A^\frac{1}{2}_2 w\|_{L^2(\Omega)}^2 - <P(w \cdot \nabla)u_1, w>_{[L^2(\Omega)]^2} - <P(u_2 \cdot \nabla)w, w>_{[L^2(\Omega)]^2}.
\]

(4.36)

The last term vanishes for divergence-free functions. The earlier term, using an equivalent form of the nonlinearity in (1.14) and the Nirenberg-Gagliardo inequality, is estimated as follows:

\[
|<P(w \cdot \nabla)u_1, w>_{[L^2(\Omega)]^2}| \leq c\|w\|_{L^4(\Omega)}^2 \|A^\frac{1}{2}_2 u_1\|_{[L^2(\Omega)]^2}^2 \\
\leq c\|w\|_{L^2(\Omega)} \|A^\frac{1}{2}_2 w\|_{L^2(\Omega)} \|A^\frac{1}{2}_2 u_1\|_{[L^2(\Omega)]^2}.
\]

(4.37)
Inserting the last estimate into (4.36), using Young’s inequality, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2(\Omega)}^2 \leq -\|A^{\frac{1}{2}} w\|_{L^2(\Omega)}^2 + c\|w\|_{L^2(\Omega)} \|A^{\frac{1}{2}} w\|_{L^2(\Omega)} \|A^{\frac{1}{2}} u_1\|_{L^2(\Omega)}^2 \\
\leq c\|w\|_{L^2(\Omega)}^2 \|A^{\frac{1}{2}} u_1\|_{L^2(\Omega)}^2.
\]
Since \(\|w(0)\|_{L^2(\Omega)}^2 = 0\) then \(\|w(t)\|_{L^2(\Omega)}^2 = 0\) for all \(t \in [0, T]\), due to the classical Gronwall lemma. Note that \(w \in C^0([0, T]; [L^2(\Omega)^2])\) because \(w \in L^2(0, T; D(A^{\frac{1}{2}}))\) with \(w_t \in L^2(0, T; D(A^{\frac{1}{2}}))\) (e.g. [32, Lemma 1.2, Chap. III]).

4.5. 3-D N-S equation. We want now to extend the above observation to the case of the 3-D N-S equation. We write the corresponding approximating equation:
\[
u_s \frac{d}{dt} \|w(t)\|_{L^2(\Omega)}^2 \leq -\|A^{\frac{1}{2}} w\|_{L^2(\Omega)}^2 + c\|w\|_{L^2(\Omega)} \|A^{\frac{1}{2}} w\|_{L^2(\Omega)} \|A^{\frac{1}{2}} u_1\|_{L^2(\Omega)}^2 \\
\leq c\|w\|_{L^2(\Omega)}^2 \|A^{\frac{1}{2}} u_1\|_{L^2(\Omega)}^2.
\]

with \(s \in (\frac{5}{4}, \frac{3}{2})\) (but close to \(\frac{5}{4}\)), and denote its solution by \(w'\). Arguing as in (4.7), we obtain that \(w' \in L^\infty(0, T; [\mathcal{L}^2(\Omega)^3] \cap L^2(0, T; D(A^{\frac{s}{2}})), and \sqrt{\nu_s} A^{\frac{s}{2}} w' \in L^2(0, T; [\mathcal{L}^2(\Omega)^3])\). Since \(w'\) are bounded, uniformly in \(\epsilon\), in \(L^\infty(0, T; [\mathcal{L}^2(\Omega)^3])\) and in \(L^2(0, T; D(A^{\frac{s}{2}}))\), then there exists a function \(u \in L^\infty(0, T; [\mathcal{L}^2(\Omega)^3] \cap L^2(0, T; D(A^{\frac{s}{2}}))^2\) such that \(w' \rightharpoonup u^*\)-weakly in \(L^\infty(0, T; [\mathcal{L}^2(\Omega)^3])\) and weakly in \(L^2(0, T; D(A^{\frac{s}{2}}))\) as \(\epsilon \to 0\). On the other hand, for any \(v \in L^2(0, T; D(A^{\frac{s}{2}}))\), we have
\[
\nu_s \int_0^T \int_\Omega A^{\frac{s}{2}} w' \cdot v dxdt \leq \frac{\nu_s}{\gamma_s} \|\sqrt{\nu_s} A^{\frac{s}{2}} w'\|_{L^2(0, T; [\mathcal{L}^2(\Omega)^3])} \|A^{\frac{s}{2}} v\|_{L^2(0, T; [\mathcal{L}^2(\Omega)^3])} \to 0
\]
as \(\epsilon \to 0\), hence \(A^{\frac{s}{2}} w' \to 0\) as \(\epsilon \to 0\), weakly in \(L^2(0, T; D(A^{\frac{s}{2}}))\), as \(\epsilon \to 0\).

For the solution \(w'\) to (4.38), applying to the corresponding equation the operator \(A^{-s+\frac{1}{2}}\), we find
\[
A^{-s+\frac{1}{2}} u'_t = -A^{-s+\frac{1}{2}} u' - \frac{\nu_s}{\gamma_s} A^{\frac{s}{2}} u' - A^{-s+\frac{1}{2}} P(u' \cdot \nabla) u' + A^{-s+\frac{1}{2}} Pf.
\]
Using the nonlinear term the estimate of [13, Lemma 2.2] with \(d = s - \frac{1}{2}, \epsilon = s - 1\), we get
\[
\|A^{-d} P(u' \cdot \nabla) u'\|_{L^2(\Omega)^3} \leq c\|u'\|_{L^2(\Omega)^3} \leq c\|u'\|_{L^2(\Omega)^3}^2 \\
\leq c\|u'\|_{H^s(\Omega)^3} \|u'\|_{L^2(\Omega)^3}^{2s-\frac{3}{2}},
\]
where \(\frac{1}{2} = \frac{1}{2} + \frac{3}{4}\), consequently \(z = \frac{6}{3+4\cdot(\frac{3}{2})} < \frac{3}{2}\) (but close). Due to the standard \(L^2\) estimate, all the right hand side components in (4.39) belong to \(L^2(0, T; [\mathcal{L}^2(\Omega)^3])\), i.e.,
\[
A^{-s+\frac{1}{2}} u'_t \in L^2(0, T; [\mathcal{L}^2(\Omega)^3]), \text{ or } u'_t \in L^2(0, T; D(A^{-s+\frac{1}{2}})).
\]

Thus the Lions compactness lemma will be used as before to conclude that there exists a subsequence \(\{u'_t\}\) (after relabeling) convergent in \(L^2(0, T; [\mathcal{H}^s_0(\Omega)^3])\) and almost everywhere in \([0, T] \times \Omega\), and \(u'_t \rightharpoonup u_t\) (\(u_t\) is the distributional time derivative) weakly in \(L^2(0, T; D(A^{-s+\frac{1}{2}}))\) as \(\epsilon \to 0\). As in (4.34), we get \(P(u' \cdot \nabla) u' \rightharpoonup P(u \cdot \nabla) u\) (\(\epsilon \to 0\)) weakly in \(L^2(0, T; [H^1(\Omega)^3])\). Passing to the limit in (4.38), we obtain the equality
\[
u_s \frac{d}{dt} \|w(t)\|_{L^2(\Omega)}^2 \leq -\|A^{\frac{1}{2}} w\|_{L^2(\Omega)}^2 + c\|w\|_{L^2(\Omega)} \|A^{\frac{1}{2}} w\|_{L^2(\Omega)} \|A^{\frac{1}{2}} u_1\|_{L^2(\Omega)}^2 \\
\leq c\|w\|_{L^2(\Omega)}^2 \|A^{\frac{1}{2}} u_1\|_{L^2(\Omega)}^2.
\]
in $L^2(0, T; D(A_2^{-1}))$.

Note also that since $u \in L^\infty(0, T; [L^2(\Omega)]^3)$, $u_t \in L^2(0, T; D(A_2^{-1}))$, by [26, Lemma 1.2, Chapter I] also $u \in C^0(0, T; D(A_2^{-1}))$. Consequently $u \in C_w([0, T]; D(A_2^{-1}))$, and since $u \in L^\infty(0, T; [L^2(\Omega)]^3)$, then by [31, Theorem 2.1] $u \in C_w([0, T]; [L^2(\Omega)]^3)$.

**Theorem 16.** Let $P f \in D(A_2^{\frac{\alpha}{2}})$ and $u_0 \in D(A_2^{\frac{\alpha}{2}})$, then there exists a weak solution $u(t)$ to (1.13) in 3-D, such that for any $T > 0$,

$$u \in L^\infty(0, T; [L^2(\Omega)]^3) \cap L^2(0, T; D(A_2^{\frac{\alpha}{2}})),$$

and (4.41) holds as an equality in $L^2(0, T; D(A_2^{-1}))$. Moreover,

$$u \in C_w([0, T]; [L^2(\Omega)]^3).$$

Note that the weak solution obtained above is global in time, while eventually not unique, since it depends on the chosen subsequence. However, for regular initial data $u_0 \in D(A_2^{\frac{\alpha}{2}})$, it must coincide for small times with the unique local strong solution described in Theorem 10, since the last exists on a certain time interval $t \in [0, \tau)$ and fulfills (4.41) for $T < \tau$.

4.6. **Quasi-geostrophic equation.** We consider the following equation:

$$\theta_t + \lambda \theta + u \cdot \nabla \theta + A_\alpha \theta = f + \kappa \theta, \quad x \in \mathbb{R}^2, \quad t > 0,$$

$$\theta^\alpha(0, x) = \theta_0(x),$$

(4.42)

where $\lambda$, $\alpha$, and $\kappa$ are like in (4.20), and $A_\alpha = \kappa(-\Delta)^\alpha + I$. Analogous to the arguments of Theorem 12, we get a unique solution $\theta^\alpha$ to the problem (4.42). We now investigate the limiting behavior of $\theta^\alpha$ as $\alpha \to \frac{1}{2}^+$.

Choose the smooth (at least $C^2$, but we prefer $\eta \in C^\infty$) cut-off function $\eta : \mathbb{R}^2 \to [0, 1],$

$$\eta(x) = \begin{cases} 1, & |x| \geq 2, \\ 0, & |x| \leq 1. \end{cases}$$

**Lemma 17.** For any $\alpha \in (0, 1)$, there exists a constant $M = M(\alpha, \eta) > 0$ such that

$$|(-\Delta)^\alpha \eta(x)| \leq M < \infty \quad \text{for all} \ x \in \mathbb{R}^2.$$

Let $\eta_k(\cdot) = \eta(\frac{x}{k})$, $k = 1, 2, \ldots$. Then the following identity is obvious.

**Lemma 18.** For any $s \in (0, 2),$

$$(-\Delta)^s \eta_k(x) = \frac{1}{k^s}(-\Delta)^s \eta(z)|_{z = \frac{x}{k}}.$$

Consequently, we obtain the so called tail estimates for solution $\theta$ of (4.42) in $L^2(\mathbb{R}^2)$ as introduced in [35].

**Lemma 19.** For each $\epsilon > 0$ and arbitrary $\theta_0 \in H^{2\alpha-\epsilon + s}(\mathbb{R}^2)$, there exists a constant $k = k(\epsilon; \|\theta_0\|_{L^2(\mathbb{R}^2)}; \|\theta_0\|_{L^4(\mathbb{R}^2)})$ such that the solution $\theta(t)$ of (4.42) corresponding to $\theta_0$ satisfies

$$\int_{\mathcal{O}_k} |\theta(t)|^2 dx \leq \epsilon \quad \text{for any} \ t \geq 0,$$

where $\mathcal{O}_k = \{x \in \mathbb{R}^2 : |x| \geq k\}$ and $\epsilon$ is independent on $k$. 

Proof. Taking the scalar product of (4.42) with $\theta \eta$, we obtain
\[
\int_{\mathbb{R}^2} \theta_t \theta \eta dx + \lambda \int_{\mathbb{R}^2} \theta^2 \eta dx = -\int_{\mathbb{R}^2} u \cdot \nabla \theta \eta dx - \kappa \int_{\mathbb{R}^2} (-\Delta)^a \theta \eta dx + \int_{\mathbb{R}^2} f \theta \eta dx.
\]
We will transform the components one by one. We have
\[
\int_{\mathbb{R}^2} \theta_t \theta \eta dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \theta^2 \eta dx.
\]
Integrating by parts, due to (5.2), we get
\[
-\int_{\mathbb{R}^2} u \cdot \nabla \theta \eta \eta dx = -\frac{1}{2} \int_{\mathbb{R}^2} u \cdot \nabla \theta^2 \eta dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial x_1} \theta^2 \frac{\partial}{\partial x_2} \psi \eta - \frac{\partial}{\partial x_2} \theta^2 \frac{\partial}{\partial x_1} \psi \eta \right) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} \left( -\theta^2 \frac{\partial}{\partial x_1} \theta \frac{\partial}{\partial x_2} \psi \eta - \theta^2 \frac{\partial}{\partial x_2} \theta \frac{\partial}{\partial x_1} \psi \eta + \theta^2 \frac{\partial}{\partial x_2} \theta \frac{\partial}{\partial x_1} \psi \eta \right) dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^2} \theta^2 |u| |\nabla \eta| dx \leq \frac{c}{k} \left( \int_{\mathbb{R}^2} \theta^4 dx + \int_{\mathbb{R}^2} \theta^2 dx \right).
\]

Thanks to the pointwise estimate from [7] which states that
\[
\forall \alpha \in [0,1] \forall \phi \in C_0^2(\mathbb{R}^2) \quad 2 \phi (\Delta)^a \phi \geq (-\Delta)^a (\phi^2),
\]
we have
\[
-\kappa \int_{\mathbb{R}^2} (-\Delta)^a \theta \eta dx \leq -\frac{\kappa}{2} \int_{\mathbb{R}^2} \theta^2 (-\Delta)^a (\eta) dx \leq \frac{c}{k^{2a}} \int_{\mathbb{R}^2} \theta^2 dx.
\]
The last component is transformed as follows
\[
\int_{\mathbb{R}^2} f \theta dx \leq \frac{1}{2\lambda} \int_{\mathbb{R}^2} f^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} \theta^2 dx \leq \frac{1}{2\lambda} \int_{\mathcal{O}_k} f^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} \theta^2 dx.
\]
Consequently,
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \theta^2 \eta dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} \theta^2 \eta dx
\]
\[
\leq \frac{c}{k^{2a}} \int_{\mathbb{R}^2} \theta^2 dx + \frac{c}{k} \left( \int_{\mathbb{R}^2} \theta^4 dx + \int_{\mathbb{R}^2} \theta^2 dx \right) + \frac{1}{2\lambda} \int_{\mathcal{O}_k} f^2 dx.
\]
Solving the above inequality, we deduce that
\[
\int_{\mathcal{O}_k} |\theta(t)|^2 dx \leq e^{-\lambda t} \int_{\mathcal{O}_k} |\theta_0|^2 dx + \left( \frac{c}{k^{2a}} + \frac{c}{k} \right) \|\theta\|^2_{L^2(\mathbb{R}^2)}
\]
\[
+ \frac{c}{k} \|\theta\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{2\lambda^2} \int_{\mathcal{O}_k} f^2 dx.
\]
(4.43)

Note that, since $\theta_0 \in H^{2a^-+s}(\mathbb{R}^2)$ and $f \in H^s(\mathbb{R}^2)$, by the Sobolev type embeddings, we obtain that $\theta_0$ and $f$ belong to $L^2(\mathbb{R}^2)$, then
\[
\int_{\mathcal{O}_k} |\theta_0|^2 dx \to 0 \quad \text{and} \quad \int_{\mathcal{O}_k} f^2 dx \to 0 \quad \text{as} \quad k \to \infty.
\]
(4.44)

Combining (4.43) together with (4.44), and $L^q(\mathbb{R}^2)$ uniform in time estimate of solution $\theta$, $q \in [2, \infty)$, we obtain the thesis. \qed
It follows from (4.26) that $\theta^\alpha \in L^\infty(0,T;H^{\alpha+\gamma}(\mathbb{R}^2)) \cap L^2(0,T;H^{\alpha+2\gamma}(\mathbb{R}^2))$. Indeed, $\theta^\alpha$ is uniformly bounded in $L^\infty(0,T;H^{\alpha+\frac{1}{2}}(\mathbb{R}^2)) \cap L^2(0,T;H^{\alpha+1}(\mathbb{R}^2))$, so by extracting a subsequence (which we shall relabel $\theta^\alpha$) we can ensure that when $\alpha \to \frac{1}{2}^+$, $\theta^\alpha \to \theta$ *weakly* in $L^\infty(0,T;H^{\alpha+\frac{1}{2}}(\mathbb{R}^2))$ and weakly in $L^2(0,T;H^{\alpha+1}(\mathbb{R}^2))$ with $\theta \in L^\infty(0,T;H^{\alpha+\frac{1}{2}}(\mathbb{R}^2)) \cap L^2(0,T;H^{\alpha+1}(\mathbb{R}^2))$. In particular, arguing as in (4.33), it follows from Theorem 3.1.6 in [27] that $A_\alpha \theta^\alpha \to A_\frac{1}{2} \theta$ ($\alpha \to \frac{1}{2}^+$) weakly in $L^2(0,T;H^\gamma(\mathbb{R}^2))$.

We now make a further estimate on $\theta^\alpha$. Along with (5.2), we can apply the Nirenberg-Gagliardo inequality to deduce that

$$ ||u^\alpha \cdot \nabla \theta^\alpha||_{L^2(\mathbb{R}^2)} \leq ||\theta^\alpha||_{L^1(\mathbb{R}^2)} ||\nabla \theta^\alpha||_{L^1(\mathbb{R}^2)} \leq c ||\theta^\alpha||_{L^2(\mathbb{R}^2)}^{\frac{4}{s}} ||\theta^\alpha||_{H^{s+1}(\mathbb{R}^2)}^{\frac{2}{s}}, $$

thus we get $\theta^\alpha_t = -\lambda \theta - u^\alpha \cdot \nabla \theta^\alpha - A_\alpha \theta^\alpha + f + \kappa \theta^\alpha \in L^2(0,T;L^2(\mathbb{R}^2))$. By the uniform boundedness (in $\alpha$) of $\theta^\alpha_t$ in $L^2(0,T;L^2(\mathbb{R}^2))$, it follows that $\theta^\alpha_t \to \theta_t$ ($\theta_t$ is the distributional time derivative) weakly in $L^2(0,T;L^2(\mathbb{R}^2))$ as $\alpha \to \frac{1}{2}^+$. Note that the tail estimates in $H^{\alpha+\frac{1}{2}}$ norm follows directly from Lemma 19, uniform boundedness of $\theta^\alpha$ in $L^2(0,T;H^{\alpha+1}(\mathbb{R}^2))$ and interpolation inequality

$$ ||\theta^\alpha(t)||_{H^{\alpha+\frac{1}{2}}(\mathcal{O}_k)} \leq c ||\theta^\alpha(t)||_{H^{s+1}(\mathbb{R}^2)}^{\frac{s+\frac{1}{2}}{s}} ||\theta^\alpha(t)||_{L^2(\mathbb{R}^2)}^{\frac{1}{s}}. $$

Thus, for any $\epsilon > 0$ and $\theta^\alpha \in L^2(0,T;H^{\alpha+1}(\mathbb{R}^2))$, we can choose $k > 0$ sufficiently large such that

$$ \int_0^T ||\theta^\alpha(t)||_{H^{\alpha+\frac{1}{2}}(\mathcal{O}_k)}^2 dt < \frac{\epsilon}{8}. \quad (4.45) $$

On the other hand, using the Lions compactness lemma, we can find a subsequence $\{\theta^\alpha\}$ (after relabeling) that converges to $\theta$ strongly in $L^2(0,T;H^{\alpha+\frac{1}{2}}(B_k))$, where $B_k$ denotes a closed ball in $\mathbb{R}^2$ centered at zero with radius $k$ (note that $\mathbb{R}^2 = B_k \cup \mathcal{O}_k$), which implies that $\theta^\alpha$ is a Cauchy sequence in $L^2(0,T;H^{\alpha+\frac{1}{2}}(B_k))$, i.e., for any $\epsilon > 0$ and $\theta^\alpha \in L^2(0,T;L^2(\mathbb{R}^2))$, there exists a positive constant $N = N(\epsilon)$ such that

$$ \int_0^T ||\theta^{\alpha_n} - \theta^{\alpha_m}||_{H^{\alpha+\frac{1}{2}}(B_k)}^2 dt < \frac{\epsilon}{2}, \quad \text{for } n, m > N. \quad (4.46) $$

Combining (4.45) and (4.46), for any $\epsilon > 0$ and $\theta^\alpha \in L^2(0,T;L^2(\mathbb{R}^2))$, we have

$$ \int_0^T ||\theta^{\alpha_n} - \theta^{\alpha_m}||_{H^{\alpha+\frac{1}{2}}(\mathbb{R}^2)}^2 dt $$

$$ \leq \int_0^T ||\theta^{\alpha_n} - \theta^{\alpha_m}||_{H^{\alpha+\frac{1}{2}}(B_k)}^2 dt + \int_0^T ||\theta^{\alpha_n} - \theta^{\alpha_m}||_{H^{\alpha+\frac{1}{2}}(\mathcal{O}_k)}^2 dt $$

$$ \leq \int_0^T ||\theta^{\alpha_n} - \theta^{\alpha_m}||_{H^{\alpha+\frac{1}{2}}(B_k)}^2 dt + 2 \int_0^T ||\theta^{\alpha_n}||_{H^{\alpha+\frac{1}{2}}(\mathcal{O}_k)}^2 + ||\theta^{\alpha_m}||_{H^{\alpha+\frac{1}{2}}(\mathcal{O}_k)}^2 dt $$

$$ < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon, \quad \text{for } n, m > N, $$

which implies that $\theta^\alpha$ is also a Cauchy sequence in $L^2(0,T;H^{\alpha+\frac{1}{2}}(\mathbb{R}^2))$, therefore, $\{\theta^\alpha\}$ converges to $\theta$ strongly in $L^2(0,T;H^{\alpha+\frac{1}{2}}(\mathbb{R}^2))$. Using boundedness of the Riesz
Lemma 22. Let the function $x, a, b$ and $k$ be continuous and nonnegative on $J = [\alpha, \beta]$, and $n$ be a positive integer ($n \geq 2$) and $\frac{n}{2}$ be a nondecreasing function. If

$$x(t) \leq a(t) + b(t) \int_{\alpha}^{t} k(s)x^{n}(s)ds, \quad t \in J,$$

operator, for $w \in C^{\infty}_{0}(0, T; L^{2}(\mathbb{R}^{2}))$, we get

$$\begin{align*}
| \int_{0}^{T} \int_{\mathbb{R}^{2}} u^{\alpha} \cdot \nabla \theta^{\alpha} w - u \cdot \nabla \theta w dx dt |
\leq | \int_{0}^{T} \int_{\mathbb{R}^{2}} (u^{\alpha} - u) \cdot \nabla \theta^{\alpha} w + u \cdot \nabla (\theta^{\alpha} - \theta) w dx dt |
\leq \int_{0}^{T} \left( \| u^{\alpha} - u \|_{L^{2}(\mathbb{R}^{2})} \| \theta^{\alpha} \|_{W^{1,4}(\mathbb{R}^{2})} + \| u \|_{L^{4}(\mathbb{R}^{2})} \| \theta^{\alpha} - \theta \|_{W^{1,4}(\mathbb{R}^{2})} \| w \|_{L^{2}(\mathbb{R}^{2})} dt 
\leq c \int_{0}^{T} \left( \| \theta^{\alpha} \|_{H^{s+\frac{1}{2}}(\mathbb{R}^{2})} + \| \theta \|_{H^{s+\frac{1}{2}}(\mathbb{R}^{2})} \right) \| \theta^{\alpha} - \theta \|_{H^{s+\frac{1}{2}}(\mathbb{R}^{2})} \| w \|_{L^{2}(\mathbb{R}^{2})} dt
\leq c \| w \|_{L^{\infty}(0, T; L^{2}(\mathbb{R}^{2}))} \left( \| \theta^{\alpha} \|_{L^{2}(0, T; H^{s+\frac{1}{2}}(\mathbb{R}^{2}))} + \| \theta \|_{L^{2}(0, T; H^{s+\frac{1}{2}}(\mathbb{R}^{2}))} \right)
\times \| \theta^{\alpha} - \theta \|_{L^{2}(0, T; H^{s+\frac{1}{2}}(\mathbb{R}^{2}))} \rightarrow 0 \quad \text{as} \quad \alpha \rightarrow \frac{1}{2},
\end{align*}$$

this shows that $u^{\alpha} \cdot \nabla \theta^{\alpha} \to u \cdot \nabla \theta$ weakly in $L^{2}(0, T; L^{2}(\mathbb{R}^{2}))$ as $\alpha \to \frac{1}{2}$.

Passing to the limit $\alpha \to \frac{1}{2}$ in (4.42), we obtain the equation

$$\theta_{t} + \lambda \theta + u \cdot \nabla \theta + A_{\frac{1}{2}} \theta = f + \kappa \theta, \quad x \in \mathbb{R}^{2}, \quad t > 0,
\theta(0, x) = \theta_{0}(x), \quad (4.47)$$

in $L^{2}(0, T; L^{2}(\mathbb{R}^{2}))$.

Concluding, we have shown the following result.

**Theorem 20.** Let $s > 1$ be fixed and $f \in H^{s}(\mathbb{R}^{2})$. Then for arbitrary $\theta_{0} \in H^{2s+1}(\mathbb{R}^{2})$, there exists a weak solution $\theta(t)$ of the problem (4.42) such that, for any $T > 0$,

$$\theta \in L^{\infty}(0, T; H^{s+\frac{1}{2}}(\mathbb{R}^{2})) \cap L^{2}(0, T; H^{s+1}(\mathbb{R}^{2})), \quad \text{and the equation } (4.47) \quad \text{holds as an equality in } L^{2}(0, T; L^{2}(\mathbb{R}^{2})).$$

5. Closing remarks.

**Remark 21.** It is a well known fact (e.g. T. Kato [19]), that an a priori estimate in $L^{N}$ is sufficient to make the N-D Navier-Stokes equation critical with respect to such estimate. The corresponding estimate takes the form

$$\begin{align*}
\| P(u \cdot \nabla) u \|_{X^{-\frac{1}{2}}} &= \| A^{-\frac{1}{2}} P(u \cdot \nabla) u \|_{L^{2}(\Omega)^{N}} \\
&\leq c \| u \|_{L^{2}(\Omega)^{N}} \leq c \| u \|_{L^{2^{\frac{N}{2-n}}}(\Omega)^{N}} \| u \|_{L^{\infty}(\Omega)^{N}} \quad (5.1)
\end{align*}$$

for $N \geq 3$. Thus, if one can get a $L^{N^{+}}$ estimate for the N-D Navier-Stokes equation, the problem will automatically be sub-critical with respect to such estimate, and all the difficulties connected nowadays with the 3-D case will disappear.

Next we recall a Gronwall type inequality taken from [10, Theorem 25].

**Lemma 22.** Let the function $x, a, b$ and $k$ be continuous and nonnegative on $J = [\alpha, \beta]$, and $n$ be a positive integer ($n \geq 2$) and $\frac{n}{2}$ be a nondecreasing function. If

$$x(t) \leq a(t) + b(t) \int_{\alpha}^{t} k(s)x^{n}(s)ds, \quad t \in J,$$
then
\[ x(t) \leq a(t) \left\{ 1 - (n - 1) \int_{\alpha}^{t} k(s)b(s)a^{n-1}(s)ds \right\} \frac{\alpha^n}{n}, \quad \alpha \leq t \leq \beta_n, \]

where
\[ \beta_n = \sup \left\{ t \in J : (n - 1) \int_{\alpha}^{t} k(s)b(s)a^{n-1}(s)ds < 1 \right\}. \]

Further, we quote the observation (e.g. [27, p. 299]), that the Riesz transforms \( R_j \) are bounded operators from \( L^q(\mathbb{R}^N) \) into \( L^r(\mathbb{R}^N) \), \( 1 < q < \infty \):
\[ \exists C > 0 \forall \psi \in L^q(\mathbb{R}^N) \| R_j(\psi) \|_{L^r(\mathbb{R}^N)} \leq C \| \psi \|_{L^q(\mathbb{R}^N)}, \quad j = 1, 2, ..., N. \tag{5.2} \]
Moreover ([38, p.12]),
\[ \| D^j\theta(t, \cdot) \|_{L^q(\mathbb{R}^2)} \leq \| D^j\theta(t, \cdot) \|_{L^q(\mathbb{R}^2)}, \quad q \in (1, \infty), |j| \leq k. \tag{5.3} \]

Following [27, pp. 299, 300], we formulate next an observation we were unable to locate explicitly in the literature:

**Remark 23.** In case of \( R^N \) the operators \( \frac{\partial}{\partial x_i} \) and \( (-\Delta)^{\alpha}, \alpha > 0 \), are commutable on \( D((-\Delta)^{\alpha+1/2}) \).

Note first, that for \( f \in D((-\Delta)^{\alpha}) \)
\[ -R_j((-\Delta)^{1/2}f) = \frac{\partial f}{\partial x_j}, \quad j = 1, ..., N. \]

Observe next ([27, (12.24)]), that for \( f \in D((-\Delta)^{\beta}), \beta > 0 \)
\[ (-\Delta)^{\beta}R_j f = R_j((-\Delta)^{\beta})f. \]

Finally, for \( f \in D((-\Delta)^{\alpha+1/2}) \), we conclude that
\[ \frac{\partial}{\partial x_j}(-\Delta)^{\alpha}f = -R_j((-\Delta)^{1/2})(-\Delta)^{\alpha}f = -(-\Delta)^{\alpha}R_j(-\Delta)^{1/2}f = (-\Delta)^{\alpha} \frac{\partial}{\partial x_j} f. \]

The following commutator estimates are known in the literature [14, 18].

**Lemma 24.** Suppose that \( s > 0 \) and \( p \in (1, \infty) \). If \( f, g \in S \), the Schwartz class, then
\[ \| (-\Delta)^{\bar{s}}(fg) - f(-\Delta)^{\bar{s}}g \|_{L^p(\mathbb{R}^2)} \]
\[ \leq c(\| \nabla f \|_{L^{p_1}(\mathbb{R}^2)}^{\frac{1}{p_1}} \| (-\Delta)^{\frac{s}{2}} g \|_{L^{p_2}(\mathbb{R}^2)} + \| (-\Delta)^{\frac{s}{2}} f \|_{L^{p_3}(\mathbb{R}^2)} \| g \|_{L^{p_4}(\mathbb{R}^2)}) \]
and
\[ \| (-\Delta)^{\bar{s}}(fg) \|_{L^p(\mathbb{R}^2)} \leq c(\| f \|_{L^{p_1}(\mathbb{R}^2)} \| (-\Delta)^{\frac{s}{2}} g \|_{L^{p_2}(\mathbb{R}^2)} + \| (-\Delta)^{\frac{s}{2}} f \|_{L^{p_3}(\mathbb{R}^2)} \| g \|_{L^{p_4}(\mathbb{R}^2)}), \]
with \( p_2, p_3 \in (1, \infty) \) such that
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]
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