GLOBAL WEAK SOLUTION TO 3D COMPRESSIBLE FLOWS
WITH DENSITY-DEPENDENT VISCOSITY AND
FREE BOUNDARY

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Abstract. In this paper, we obtain the global weak solution to the 3D spherically symmetric compressible isentropic Navier-Stokes equations with arbitrarily large, vacuum data and free boundary when the shear viscosity $\mu$ is a positive constant and the bulk viscosity $\lambda(\rho) = \rho^\beta$ with $\beta > 0$. The analysis of the upper and lower bound of the density is based on some well-chosen functionals. In addition, the free boundary can be shown to expand outward at an algebraic rate in time.

1. Introduction. The compressible isentropic Navier-Stokes equation in $\mathbb{R}^3$ is described by

$$\begin{cases}
\rho_t + \text{div}(\rho U) = 0, \\
(\rho U)_t + \text{div}(\rho U \otimes U) + \nabla P = \mu \Delta U + \nabla((\mu + \lambda(\rho))\text{div}U),
\end{cases} \tag{1}$$

where $\rho$, $U$ and $P(\rho)$ are the density, the velocity and the pressure respectively. The pressure is given by

$$P(\rho) = \rho^\gamma, \quad (\gamma > 1)$$

Here, it is assumed that

$$\mu = \text{const.} > 0, \quad \lambda(\rho) = \rho^\beta, \quad \beta \geq 0. \tag{2}$$

There are a large number of literatures on the well-posedness theories of the compressible Navier-Stokes equation (1) when both viscosities are positive constants. It is well-known that the global well-posedness theory for the one-dimension case is rather satisfactory, see [12, 29, 25, 26] and the references therein. On multidimensional case, the local well-posedness theory of classical solutions was established in the absence of vacuum (see [33, 17, 38, 31]) and the global well-posedness theory of classical solutions was obtained for initial data close to a non-vacuum

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steady state (see [31, 11, 4, 3] and references therein). While, the global well-posedness of classical solutions to the 3D isentropic compressible Navier-Stokes equations with small energy was proved by Huang-Li-Xin [16]. The global existence of weak solutions with large initial data permitting vacuums was proposed in [28, 5, 18]. The analysis allow that the initial data are arbitrarily large and the with vacuums, then the solution will also have vacuums and the global well-posedness could not be expected, see [39, 36, 40] for blow-up results of classical solutions.

The case that both the shear and bulk viscosities depend on the density has also received a lot of attention recently, see [1, 2, 4, 6, 18, 19, 20, 21, 24, 27, 30, 32, 37] and the references therein. In geophysical flow, the viscous Saint-Venant system for the shallow water corresponds exactly to a kind of compressible Navier-Stokes equations with density-dependent viscosities. However, except for the one-dimensional problems, few results are available for the multi-dimensional problems and even the short time well-posedness of classical solutions in the presence vacuum remains open. The first successful example is attributed to Guo, Jiu and Xin in [6] with spherically symmetric initial data and fixed boundary conditions and later Guo, Li, and Xin extended it to the free boundary conditions with discontinuously symmetric initial data [7].

The system (1)-(2) was first proposed by Vaigant-Kazhikhov in [26] where they showed that the solution to the 2D periodic problem will not develop vacuum states in any finite time provided the initial density is uniformly away from vacuum. Then the global existence and large time behavior of weak solutions was studied by Perepelitsa in [35] in the special case $\beta = \gamma > 3$. If the initial values may contain vacuum states, Jiu-Wang-Xin [23] first proved the global well-posedness of the classical solution to the 2D periodic problem with large data provided $\beta > 3$ and some compatibility conditions are satisfied. Then Huang-Li improved the index $\beta$ to be $\beta > \frac{4}{3}$ in [14]. For the 2D Cauchy problems with vacuum states at far fields, Jiu-Wang-Xin [22] and Huang-Li [15] independently considered the global well-posedness of classical solution in quite different weighted spaces. However, similar result for the 3D case is completely open due to the complicated nonlinear structures.

In the present paper, we prove the global existence of weak solutions of the Vaigant-Kazhikhov model to the Navier-Stokes equations of compressible isentropic flow in three dimensional space with large and spherically symmetric initial data for free boundary. Our work is motivated by the paper of Guo-Wang-Wang [9] and Guo-Li [8]. In [9], it has the global weak solution of the Vaigant-Kazhikhov model with large data in the whole space and in [8] they studied the spherically symmetric weak solution with large data and free boundary for the constant viscosity, and established the global existence of weak solutions to the compressible nonbarotropic Navier-Stokes equations in the “fluid region”. Our results generalize the one by Guo-Li [8] to the Kazhikhov-Vaigant Model and the Guo-Wang-Wang [9] to the free boundary case. In this paper, the key point in the proof for the upper and lower bound of the density given by some well-chosen functionals. we can show that the solution to 3D spherically symmetric problem will not develop vacuum states in any finite time away from the origin of the symmetry provided the initial density is away from the vacuum states.

The subsequent contents of the paper are organized as follows. In section 2 we will present the main result of this paper. In section 3 we construct an approximate solution sequence and derive a priori estimates for the approximate solutions.
key uniform estimates away from the symmetry center are established in section 4. In this section, these estimates do not depend on $\epsilon$. Based on these, in section 5, we take the limits to obtain the global existence of weak solutions of the original system.

**Notation.** Let $\Omega$ be a domain in $\mathbb{R}^3$ and $m$ be an integer $1 \leq p \leq \infty$. By $W^{m,p}(\Omega)$ ($W^{m,p}_{0}(\Omega)$) we denote the usual Sobolev space defined over $\Omega$. $W^{m,2}(\Omega) \equiv H^m(\Omega)(W^{m,2}_{0}(\Omega) \equiv H^m_0(\Omega))$.

\[ \mathcal{L}^p(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \int_{\Omega} |f(r)|^{p r^2} dr < \infty \right\}, \]

with norm \[ \|f\|_{\mathcal{L}^p(\Omega)} := \int_{\Omega} |f(r)|^{p r^2} dr. \]

For simplicity we also use the following abbreviations:

\[ \|f\|_{L^p(\Omega)} \equiv \|f\|_{\mathcal{L}^p(\Omega)}. \]

The same letter $C$ (sometimes used as $C(\mathfrak{X})$ to emphasize the dependence of $C$ on $\mathfrak{X}$) will denote various positive constants.

2. **Main result.** More precisely, we are mainly concerned with the following spherically symmetric solution to the problem (1), that is,

\[ r = |x|, \quad \rho(x,t) = \rho(r,t), \quad U(x,t) = u(r,t) \frac{x}{r}. \]

Thus, the system (1) becomes

\[ \begin{cases} \rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_r + \frac{2\rho u^2}{r} = \left( (\lambda(\rho) + 2\mu)(u_r + \frac{2u}{r}) \right)_r, \end{cases} \tag{3} \]

where $(r,t) \in \Omega_T$ with

\[ \Omega_T = \{(r,t)|0 \leq r \leq a(t), \quad 0 \leq t \leq T\} \]

and the initial values are given by

\[ (\rho,\rho u)(r,0) = (\rho_0, m_0)(r) =: (\rho_0, \rho_0 u_0)(r), \quad r \in (0, a_0), \tag{4} \]

the free boundary condition is imposed by

\[ \left( P(\rho) - (\lambda(\rho) + 2\mu) \left( u_r + \frac{2u}{r} \right) \right)(a(t), t) = 0, \quad t \geq 0. \tag{5} \]

At the center of symmetry we impose the Dirichlet boundary condition

\[ u(0, t) = 0, \quad t \geq 0, \tag{6} \]

where $a(t)$ is the free boundary defined by

\[ \begin{cases} \frac{da(t)}{dt} = u(a(t), t), \quad t \geq 0, \\ a(0) = a_0, \end{cases} \tag{7} \]

which is the interface separating the gas from the vacuum. Before giving the main result, we introduce first the definition of the global weak solution to (3)-(6).
Definition 2.1 (Weak solution). \((\rho(t, r), u(t, r), a(t))\) with \(\rho \geq 0\) a.e. is said to be a weak solution to the free surface problem (3)-(6) on \(\Omega_T \times [0, T]\), provided that it holds that

1) \(\rho \in L^\infty(0, T; \mathcal{L}^1(\Omega_t) \cap \mathcal{L}^\gamma(\Omega_t)), \quad (\rho^{-\frac{1}{2}})_r \in L^\infty(0, T; \mathcal{L}^2(\Omega_t)), \)
\[\sqrt{\rho} u \in L^\infty(0, T; \mathcal{L}^2(\Omega_t)), \quad u_r \in \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega_t)), \]
\[\rho(a(t), t) > 0, \quad t \in [0, T], \quad a(t) \in H^1([0, T]) \cap C^0([0, T]), \quad (8)\]

2) For any \(t_2 \geq t_1 \geq 0\) and \(\phi \in C^1(\Omega_t \times [0, t])\) there holds
\[\int_{\Omega_t} \rho|\phi_t|^2 - \int_{t_1}^{t_2} \int_{\Omega_t} (\rho \phi_t + \rho u \phi_r) r^2 dr dt = 0; \quad (9)\]

3) For \(\psi \in C^1(\Omega_t \times [0, t])\) satisfying \(\psi(r, t) = 0\) on \(\partial \Omega_t\), there holds
\[\int_{\Omega_t} \rho u \psi r^2 dr|t_1^t_2 - \int_{t_1}^{t_2} \int_{\Omega_t} \left\{\rho u \psi_t + \rho u^2 \psi_r\right\} r^2 dr dt \]
\[= \int_{t_1}^{t_2} \int_{\Omega_t} [P - (2\mu + \lambda(\rho))(u_r + \frac{2u}{r}) + 2\psi_r + \frac{2\psi}{r}] r^2 dr dt. \quad (10)\]

The free boundary condition (5) is satisfied in the sense of trace.

Suppose that the spherically symmetric initial values \((\rho_0, u_0)(r)\) satisfy
\[0 \leq \rho_0 \leq C, \quad u_0 \in \mathcal{H}^1(0, a_0), \quad \rho_0(r), \rho_0(r) \in \mathcal{L}^2(0, a_0). \quad (11)\]
where \(C\) are positive constants. The main results of the present paper can be stated in the following

Theorem 2.2. Assume that (11) holds, the initial data and boundary values are consistent in the sense
\[\left(\rho^\gamma_0 - (\lambda(\rho_0) + 2\mu)\left(u_{0r} + \frac{2u_0}{a_0}\right)\right)(a_0) = 0, \quad \rho_0(a_0) \geq 0. \quad (12)\]

Then, the FBVP (3)-(6) has a global spherically symmetric weak solution \((\rho(t, r), u(t, r), a(t))\) for which the support of \(\rho\) is bounded on the left by a curve \(r(t)\), satisfying the following:

(a) \(c \leq a(t) \leq C_T, \quad a(t) \in C^{1/2}([0, T]), \quad a(t) \in H^1([0, T]). \)

(b) The function \(r : [0, \infty) \rightarrow [0, \infty)\) is a semicontinuous curve, so that if \(F\) is the set
\[F := \{(t, r) | t \geq 0, \quad \text{and} \quad r(t) < r(t) < a(t)\}, \]
then \(F \cap \{t > 0\} \cap \{r < a(t)\}\) is open.

(c) The density \(\rho \in L^\infty_F(F), u\) is locally Hölder continuous in \(F \cap \{t > 0\}\), and the Navier-Stokes equations (3) hold in \(\mathcal{D}'(F \cap \{t > 0\} \cap \{r < a(t)\}).\)

(d) The density \(\rho \in C([0, \infty); W^{1, \infty}[0, a(t)])^*\). Also, \(\rho(\cdot, t) \equiv 0\) in \(F^c\), and if \(pu\) is taken to be zero in \(F^c\), then the weak form of the mass equation holds for test function \(\phi \in C^1(\Omega_t \times [0, t]):\)
\[\int_{\Omega_t} \rho \phi_r r^2 dr|t_1^t_2 - \int_{t_1}^{t_2} \int_{\Omega_t} (\rho \phi_t + \rho u \phi_r) r^2 dr dt = 0, \quad (13)\]
For \( \psi \in C^2(\Omega_t \times [0,T]) \) satisfying \( \psi(r,t) = 0 \) on \( \partial \Omega_t \), there holds
\[
\int_{\Omega_t} \rho u \psi^2 dr dt|_{t=0} - \int_{t_1}^{t_2} \int_{\Omega_t} \left\{ (\rho u \psi_t + \rho u^2 \psi_r) \right\}^{2} dr dt \\
+ \int_{t_1}^{t_2} \int_{\Omega_t} [P - (2\mu + \lambda(\rho))(u_r + \frac{2u}{r})] (\psi_r + \frac{2\psi}{r}) r^2 dr dt.
\]
(14)

The main ideas of the proof of Theorem 2.2 are motivated by the paper of Hoff-Jenssen [10] where they studied the spherically and cylindrically symmetric nonbarotropic flows with large data and forces, and established the global existence of weak solutions to the compressible nonbarotropic Navier-Stokes equations in the “fluid region”.

3. Global existence of approximate FBVP. Consider a modified FBVP problem for Eq. (3) with the following initial data and boundary conditions for any fixed small enough \( \epsilon > 0 \):
\[
(\rho^\epsilon, u^\epsilon)|_{t=0} = (\rho_0 + \epsilon, u_0)(r), \quad \epsilon \leq r \leq a_0,
\]
(15)
\[
u(\epsilon, t) = 0, \quad (P(\rho^\epsilon) - (\lambda(\rho) + 2\mu)(u^\epsilon_r + \frac{2u^\epsilon}{r}))(a^\epsilon(t), t) = 0, \quad t \geq 0,
\]
(16)
where
\[
\left\{ \begin{array}{l}
\frac{da^\epsilon(t)}{dt} = u^\epsilon(a^\epsilon(t), t), \quad t \geq 0, \\
a(0) = a_0.
\end{array} \right.
\]
(17)

For simplicity of notation, we denote \((\rho^\epsilon, u^\epsilon)\) as \((\rho, u)\) in the following if without confusions.

Proposition 1. Let \( T > 0 \), and \( \epsilon > 0 \) be fixed, assume the initial data \((\rho_0, u_0)\) satisfies
\[
\epsilon \leq \rho_0 \leq C, \quad (\rho_0, u_0) \in H^1([\epsilon, a_0]).
\]
(18)

Then, there exists a unique global smooth solution \((\rho, u, a)\) of the FBVP (3) and (15)-(16) which satisfies
\[
c \leq a(t) \leq C_T, \quad c_T \leq \rho \leq C_{\epsilon, T},
\]
(19)
\[
\| \rho, u \|_{H^1([\epsilon, a(t)])} + \int_0^T \| \rho_{rr}, u_r, u_{rr} \|_{L^2([\epsilon, a(t)])}^2 dt \\
+ \int_0^T \| \rho^\gamma - (2\mu + \lambda(\rho))(u_r + \frac{2u}{r}) \|_{L^2([\epsilon, a(t)])}^2 dt + \int_0^T |(a, a')(t)|^2 dt \\
\leq C(\epsilon, T).
\]
(20)

Furthermore, if \( u_t(r, 0) \in L^2([\epsilon, a_0]) \), then
\[
\| u \|_{H^2([\epsilon, a(t)])} + \| u_t \|_{L^2([\epsilon, a(t)])} + \| \rho^\gamma - (2\mu + \lambda(\rho))(u_r + \frac{2u}{r}) \|_{H^1([\epsilon, a(t)])} \\
+ \int_0^T |u_t|_{L^2([\epsilon, a(t)])}^2 + |a''(t)|^2 dt \leq C(\epsilon, T).
\]
(21)

In this section, we establish the a-priori estimates for any approximate solution \((\rho, u, a)\) with \( \rho > 0 \) to FBVP (3) and (15)-(16). We start with a basic energy estimate.
Lemma 3.1. Let $\gamma > 1$, $T > 0$, and $(\rho, u, a)$ with $\rho > 0$ be any regular solution to the FBVP (3) and (15)-(16) for $t \in [0, T]$ under the assumptions of Proposition 1. Then

$$\int_0^{a(t)} \rho r^2 dr = \int_0^{a_0} \rho_0 r^2 dr = M_0, \quad (22)$$

$$\int_0^{a(t)} \left( \frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho \gamma \right) r^2 dr + \int_0^t \int_0^{a(s)} \lambda(\rho) + 2\mu \left( u_r + \frac{2u}{r} \right)^2 r^2 drds = E_0, \quad (23)$$
or

$$\int_0^{a(t)} \left( \frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho \gamma \right) r^2 dr + \int_0^t \int_0^{a(s)} \lambda(\rho) \left( u_r + \frac{2u}{r} \right)^2 r^2 drds$$

$$+ 2\mu \int_0^t \int_0^{a(s)} \left( (u_r)^2 + \frac{2u^2}{r^2} \right) r^2 drds + 2\mu \int_0^t u^2(a(s), s) a(s) ds = E_0. \quad (24)$$

Proof. Multiplying (3) by $r^2 u$, using (3), (17) and integration by parts, one gets

$$\int_0^{a(t)} \left( \frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho \gamma \right) r^2 dr + \int_0^t \int_0^{a(s)} \lambda(\rho) \left( u_r + \frac{2u}{r} \right)^2 r^2 drds = E_0.$$ 

Since

$$\int_0^t \int_0^{a(s)} 2\mu \left( u_r + \frac{2u}{r} \right)^2 r^2 drds$$

$$= 2\mu \int_0^t \int_0^{a(s)} ((u_r)^2 r^2 + 4u^2 + 2(u^2 r)) drds$$

$$= 2\mu \int_0^t \int_0^{a(s)} ((u_r)^2 r^2 + 2u^2) drds + 2\mu \int_0^t 2u^2(a(s), s) a(s) ds,$$

thus we obtain (24). \hfill \Box

Lemma 3.2. Under the same assumptions as Lemma 3.1. Then

$$c_0 \leq E_0^{-\frac{1}{\gamma - 1}} M_0^{\frac{1}{\gamma - 1}} \leq a(t) \leq C(1 + t)^{1/3}. \quad (25)$$

Lemma 3.3. Under the same assumptions as Lemma 3.1. Then

$$\rho(a(t), t) \leq C(1 + t)^{-\frac{t}{\gamma}}.$$

Proof. From (16) and Eq. (3), we get

$$\frac{d}{dt} \rho(a(t), t) = -\frac{1}{\lambda(\rho) + 2\mu \rho^{\gamma + 1}(a(t), t)},$$

thus, we have

$$\rho(a(t), t) \leq \rho_0(a_0)(1 + \frac{\gamma}{2\mu} \rho_0^2(a_0) t)^{-\frac{1}{\gamma}} = \left( \frac{\gamma}{2\mu} \rho_0^2(a_0) t \right)^{-\frac{1}{\gamma}} \left( \frac{2\mu}{\gamma} \rho_0^{-\gamma}(a_0) + t \right)^{-\frac{1}{\gamma}},$$

which gives Lemma 3.3 and $\rho(a(t), t) \leq \rho_0(a_0)$. \hfill \Box

In order to estimate $\rho$, we first establish the following technical result.
Lemma 3.4. Under the same assumptions as Lemma 3.1. Define
\[ \varphi(r,t) = (\lambda(\rho) + 2\mu)(u_r + \frac{2u}{r}) - \rho u^2 - \rho \gamma - \int_t^s \frac{2\rho u^2}{y} dy, \] (26)
and
\[ \psi(r,t) = \int_0^t \varphi(r,s)ds + \int_t^s \rho_0 u_0 dy. \] (27)
Then there exists a constant \( C(\epsilon,T), \) such that
\[ |\psi(r,t)| \leq C(\epsilon,T) \quad \text{for} \quad r \in [\epsilon,a(t)], \quad t \in [0,T]. \]
Proof. We observe that
\[ (\rho u)_t - \varphi(r,t)_r = 0, \]
and
\[ \psi_t = \varphi, \quad \psi_r = \rho u. \]
Thus, we have complete the proof of Lemma 3.4.

Lemma 3.5. Under the same assumptions as Lemma 3.1, Then
\[ c(\epsilon) \leq \rho \leq C(\epsilon), \quad (r,\tau) \in [\epsilon,a(t)] \times [0,T]. \] (28)
Proof. Let \( F(\psi,\rho) \) be a function of \( \psi, \rho \) to be determined. We can compute that
\[ (\rho F(\psi,\rho))_t + u(\rho F(\psi,\rho))_r \]
\[ = \rho_1 F + \rho F \psi \psi_t + \rho F \rho_t + u \rho_r F + u F \psi \psi_r + u F \rho \rho_r \]
\[ = - \left( \rho u \right)_r F + \rho F \psi \varphi - \rho F \left( \rho u \right)_r + \frac{2\rho u}{r} \right) + u \rho_r F + u F \psi \psi_r + u F \rho \rho_r \]
\[ = \left( \rho u + \frac{2\rho u}{r} \right) (-F + (2\mu + \lambda(\rho)) F_\psi - \rho F_\rho) - \rho P(\rho) F_\psi - \rho \int_t^s \frac{2\rho u^2}{s} ds F_\psi. \]
Choose \( F \) satisfying the following equation
\[ - F - \rho F_\rho + (2\mu + \lambda(\rho)) F_\psi = 0. \] (29)
Now we solve the above first-order hyperbolic partial differential equation. Set \( G(\rho,\psi) = \rho F(\rho,\psi). \) Then \( G \) satisfies the equation
\[ G_\rho - \frac{2\mu + \lambda(\rho)}{\rho} G_\psi = 0. \]
By the characteristic method, we can solve that
\[ G(\rho, \psi) = f(\psi + 2\mu \ln \rho + \frac{\rho^\beta - 1}{\beta}), \]
with \( f(\cdot) \) being any smooth function. Here, we choose \( f(z) = e^{\frac{1}{2} z} \), and thus
\[ G(\rho, \psi) = e^{\frac{1}{2} (\psi + 2\mu \ln \rho + \frac{\rho^\beta - 1}{\beta})} = \rho e^{\frac{1}{2} (\psi + \frac{\rho^\beta - 1}{\beta})}. \]
Then we can get from (29) that
\[ D_t G(\rho, \psi) = D_t (\rho F(\psi, \rho)) = -\left( P(\rho) + \int_\varepsilon^r 2\rho u^2 s ds \right) G_\psi \leq 0. \]
Integrating the above ordinary differential equation over \([0, t]\) with respect to \( \tau \) along the particle path \( X(\tau; t, x) \) defined by
\[ \begin{align*}
  \frac{dX(\tau; r, t)}{d\tau} &= u(X(\tau; r, t), \tau), \\
  X(0; r, t) &= a_0,
\end{align*} \]
it holds that
\[ \rho(r, t) e^{\frac{1}{2} \left( \psi(r, t) + \frac{\rho^\beta(r, t) - 1}{\beta} \right)} \leq \rho(X(0; r, t), 0) e^{\frac{1}{2} \left( \psi(X(0; r, t), 0) + \frac{\rho^\beta(X(0; r, t)) - 1}{\beta} \right)}, \]
Thus, one has
\[ \rho(r, t) \leq \rho_0(X(0; r, t)) e^{\frac{1}{2} \left( \psi(X(0; r, t), 0) + \frac{\rho^\beta(X(0; r, t)) - 1}{\beta} \right)} e^{-\frac{1}{2} \left( \psi(r, t) + \frac{\rho^\beta(r, t) - 1}{\beta} \right)}. \]
For this, let \( F_1(\psi, \rho) \) be a function of \( \psi, \rho \) to be determined later. We can compute that
\[ \begin{align*}
  \left( \frac{F_1(\psi, \rho)}{\rho} \right)_t + u \left( \frac{F_1(\psi, \rho)}{\rho} \right)_r \\
  = (u_r + 2\mu \varepsilon) \left[ -F_1 + \frac{F_1}{\rho} + 2\mu + \lambda(\rho) F_1 \right] - F_1 \left[ P(\rho) + \int_\varepsilon^r 2\rho u^2 s ds \right].
\end{align*} \]
Choose \( F_1(\psi, \rho) \) satisfying the following equation
\[ -F_1 + \frac{F_1}{\rho} + 2\mu + \lambda(\rho) F_1 = 0, \]
then we can solve that
\[ \frac{F_1}{\rho} = g\left( \psi + 2\mu \ln \rho + \frac{\rho^\beta - 1}{\beta} \right), \]
where \( g(\cdot) \) is any smooth function. At the same time, by (32), it holds that
\[ D_t \left( \frac{F_1}{\rho} \right) = \left( \frac{F_1}{\rho} \right)_r \left( P(\rho) + \int_\varepsilon^r 2\rho u^2 s ds \right). \]
Therefore, one has
\[ D_t \left( \psi + 2\mu \ln \rho + \frac{\rho^\beta - 1}{\beta} \right) = -\left( P(\rho) + \int_\varepsilon^r 2\rho u^2 s ds \right). \]
Integrating the above ordinary differential equation over \([0,t]\) with respect to \(\tau\) along the particle path \(X(\tau; t, x)\), it holds that
\[
\left( \psi + 2\mu \ln \rho + \frac{\rho^\beta - 1}{\beta} \right) (r, t) - \left( \psi + 2\mu \ln \rho + \frac{\rho^\beta - 1}{\beta} \right) (X(0; r, t), 0)
= - \int_0^t \left[ P(\rho)(X(\tau; r, t), \tau) + \int_{\epsilon}^{X(\tau; r, t)} \frac{2\rho u^2(s, \tau)}{s} ds \right] d\tau,
\]
which implies that
\[
2\mu \ln \rho(r, t) \leq \left| \left( \psi + 2\mu \ln \rho + \frac{\rho^\beta}{\beta} \right) (X(0; r, t), 0) \right| + \left| \psi(r, t) \right| + \left| \frac{\rho^\beta}{\beta} (r, t) \right|
+ \left| \int_0^t \left[ P(\rho)(X(\tau; r, t), \tau) + \int_{\epsilon}^{X(\tau; r, t)} \frac{2\rho u^2(s, \tau)}{s} ds \right] d\tau \right|
\leq C(\epsilon, T).
\]
Thus, there exist positive constant \(c(\epsilon, T)\) such that
\[
\rho(r, t) \geq c(\epsilon, T), \quad \forall r \in [\epsilon, a(t)], \quad \forall t \in [0, T].
\]
Therefore, we completed the proof of Lemma 3.5. \(\square\)

It is convenient to deal with the FBVP (3) in the Lagrangian coordinates. For simplicity we assume that \(\int_0^{a(t)} \rho_0 r^2 dr = 1\), which implies
\[
\int_0^{a(t)} \rho r^2 dr = \int_0^{a_0} \rho_0 r^2 dr = 1.
\]
For \(r \in [\epsilon, a(t)], t \in [0, T]\), define the Lagrangian coordinates transformation
\[
x(r, t) = \int_{\epsilon}^{r} \rho(y, t) y^2 dy, \quad \tau = t.
\]
which translates the domain \([0, T] \times [\epsilon, a(t)]\) into \([0, T] \times [0, 1]\) and satisfies
\[
\frac{\partial x}{\partial \tau} = \rho r^2, \quad \frac{\partial x}{\partial t} = -\rho \tau r^2, \quad \frac{\partial \tau}{\partial \tau} = 0, \quad \frac{\partial \tau}{\partial t} = 1 \text{ and } \frac{\partial r}{\partial \tau} = u.
\]
The free boundary value problem (3) and (15)-(16) are changed to
\[
\begin{cases}
\rho \tau + \rho^2 (u^2)_x = 0, \\
u \tau + P(\rho) = r^2((2\mu + \lambda(\rho))\rho (u^2)_x)_x,
\end{cases}
\tag{33}
\]
for \((x, \tau) \in [0, 1] \times [0, T]\), with the initial data and boundary conditions given by
\[
(u, \rho)|_{t=0} = (\rho_0, u_0)(x), \quad x \in [0, 1],
\tag{34}
\]
\[
u(0, \tau) = 0, \quad \left( P(\rho) - (2\mu + \lambda(\rho))\rho (u^2)_x \right)(1, \tau) = 0, \quad \tau \in [0, T],
\tag{35}
\]
where \(r = r(x, \tau)\) is defined by
\[
\frac{dr(x, \tau)}{d\tau} = u(x, \tau), \quad (x, \tau) \in [0, 1] \times [0, T],
\tag{36}
\]
and the fixed boundary \(x = 1\) corresponds to the free boundary \(a(\tau) = r(1, \tau)\) in Eulerian form determined by
\[
\frac{da(\tau)}{d\tau} = u(1, \tau), \quad \tau \in [0, T], \quad a(0) = a_0.
\tag{37}
\]
Energy estimation of Lemma 3.1 has follow form in Lagrangian coordinates.
Lemma 3.6 (Energy estimation).

\[
\int_0^1 \left[ \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \rho^{\gamma - 1} \right] dx + \int_0^t \int_0^1 (2 \mu + \lambda(\rho)) \rho [(ur^2)_x]^2 dx ds = E_0, \tag{38}
\]

or

\[
\int_0^1 \left[ \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \rho^{\gamma - 1} \right] dx + \int_0^t \int_0^1 \lambda(\rho) \rho [(ur^2)_x]^2 dx ds
\]
\[+ \int_0^t \int_0^1 2 \mu \rho (ur^2)^2 dx ds + 4 \mu \int_0^t \int_0^1 \frac{1}{\rho} \left( \frac{\mu}{\gamma} \right)^2 dx ds + \int_0^t 2 \mu u^2 (1, s) ds = E_0. \tag{39}
\]

Lemma 3.7 (Entropy estimation). Under the same assumptions as Lemma 3.1, then

\[
\int_0^1 \left[ u + r^2 (2 \mu \ln \rho + \frac{\rho^\beta}{\beta})_x \right]^2 dx + \frac{8 \mu}{\gamma} \int_0^t \int_0^1 \left( (\rho^{\gamma/2})_x r^2 \right)^2 dx ds
\]
\[+ \frac{4 \gamma}{(\beta + \gamma)^2} \int_0^t \int_0^1 \left( (\rho^{\frac{\beta + \gamma}{2}}) r^2 \right)^2 dx ds \leq C.
\]

Proof. From (33), one has

\[
\left[ u + r^2 \left( 2 \mu \ln \rho + \frac{\rho^\beta}{\beta} \right)_x \right] - 2ru (2 \mu \ln \rho + \frac{\rho^\beta}{\beta})_x + P_x r^2 = 0,
\]

Multiplying above equation by \( u + r^2 \left( 2 \mu \ln \rho + \frac{\rho^\beta}{\beta} \right)_x \), integrating the result with respect to \( x \) over \([0, 1]\), we get

\[
\frac{d}{dt} \int_0^1 \left[ u + r^2 \left( 2 \mu \ln \rho + \frac{\rho^\beta}{\beta} \right)_x \right]^2 dx
\]
\[= - \int_0^1 P_x r^2 \left[ u + r^2 \left( 2 \mu \ln \rho + \frac{\rho^\beta}{\beta} \right)_x \right] dx
\]
\[+ \int_0^1 2ru \left( 2 \mu \ln \rho + \frac{\rho^\beta}{\beta} \right)_x \left[ u + r^2 \left( 2 \mu \ln \rho + \frac{\rho^\beta}{\beta} \right)_x \right] dx \tag{40}
\]
\[=: I_1 + I_2.
\]

In the sequel, we derive bounds for each term on the right-hand side of (40) as follows:

By the upper bound of the density, one has

\[
P(\rho)_x = \frac{\gamma \rho^\gamma}{2\mu + \rho^\beta} (2 \mu \ln \rho + \frac{\rho^\beta}{\beta})_x \leq \frac{\gamma \rho^\gamma}{2\mu} (2 \mu \ln \rho + \frac{\rho^\beta}{\beta})_x \leq C(\epsilon, T) (2 \mu \ln \rho + \frac{\rho^\beta}{\beta})_x
\]
and then

\[ I_1 = - \int_0^1 P_x u r^2 \, dx - \frac{8\mu}{\gamma} \int_0^1 \left( (\rho^{\gamma/2})_x r^2 \right)^2 \, dx - \frac{4\gamma}{(\beta + \gamma)^2} \int_0^1 \left( (\rho^{(\beta+\gamma)/2})_x r^2 \right)^2 \, dx \]

\[ \leq C(\epsilon, T) \int_0^1 \left[ u - u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] u \, dx - \frac{8\mu}{\gamma} \int_0^1 \left( (\rho^{\gamma/2})_x r^2 \right)^2 \, dx \]

\[ \leq C(\epsilon, T) \int_0^1 \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] u \, dx + C(\epsilon, T) - \frac{8\mu}{\gamma} \int_0^1 \left( (\rho^{\gamma/2})_x r^2 \right)^2 \, dx \]

\[ \leq C(\epsilon, T) \int_0^1 \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] u \, dx + C(\epsilon, T) \int_0^1 \left( n^2 \right) \, dx \]  \hspace{1cm} (41)

and

\[ I_2 = \int_0^1 \frac{2u}{r} r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] \, dx \]

\[ = \int_0^1 \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] - u \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] \, dx \]

\[ \leq C(\|u\|_{L^\infty}) \int_0^1 \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] ^2 \, dx + C(\|u\|_{L^\infty}) \int_0^1 u^2 \, dx. \]  \hspace{1cm} (42)

Substituting (41) and (42) into (40), we get

\[ \int_0^1 \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] ^2 \, dx + \frac{8\mu}{\gamma} \int_0^1 \left( (\rho^{\gamma/2})_x r^2 \right)^2 \, dx \]

\[ + \frac{4\gamma}{(\beta + \gamma)^2} \int_0^1 \left( (\rho^{(\beta+\gamma)/2})_x r^2 \right)^2 \, dx \]

\[ \leq E_1 + C(h, T) \int_0^1 \left( \|u\|_{L^\infty} + 1 \right) \int_0^1 \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^3}{\beta} \right)_x \right] ^2 \, dx \]

\[ + C(\epsilon) \int_0^T \|u\|_{L^\infty} \, ds, \]  \hspace{1cm} (43)
where \( E_1 = \int_0^1 \left[ u_0 + r^2 \left( 2\mu \ln \rho_0 + \frac{\rho_0^\beta}{\beta} \right) \right]^2 dx \). Then, from Gronwall’s inequality one has
\[
\int_0^1 \left[ u + r^2 \left( 2\mu \ln \rho + \frac{\rho^\beta}{\beta} \right) \right]^2 dx + \frac{4\gamma}{(\beta + \gamma)^2} \int_0^T \int_0^1 \left( (\rho^{(\beta+\gamma)/2})^2 r^2 \right)^2 dx ds
\leq C(\epsilon, T) e^{C(\epsilon, T) f_0^T (\|u\|_{L^\infty} + 1) dt} \left( E_1 + \int_0^T \|u\|_{L^\infty} ds \right).
\]
(44)

Using Lemma 3.2 and Lemma 3.5, we get
\[
\int_0^T \|u(x, t)\|_{L^\infty}^2 dt \leq \int_0^T \|u(x, t)\|_{L^2}^2 \|u_x(x, t)\|_{L^2}^2 dt
\leq \int_0^T \frac{1}{pr^2} \|r^2 u_x(x, t)\|_{L^2}^2 dt \leq C(\epsilon, T).
\]
(45)

Thus one has
\[
\int_0^T \|u(x, t)\|_{L^\infty} dt \leq C(\epsilon, T),
\]
which together with (44) gives the Lemma.

From the Lemma 3.5 and Lemma 3.7, we get

**Corollary 1.** Under the same assumptions as Lemma 3.1, Then
\[
\int_0^1 r^4 (\rho_x)^2 dx \leq C(\epsilon, T).
\]

We now introduce a functional of higher-order derivatives for \( u_x \),
\[
A(T) := \sup_{0 \leq t \leq T} \int_0^1 (2\mu + \lambda(\rho)) \rho (u^2)_x \right)^2 dx + \int_0^T \int_0^1 (u_x^2) dx dt.
\]

**Lemma 3.8 (Higher-order boundedness).** Let \( h > 0 \) and \( T > 0 \) be given. Then there is a constant \( C = C(T, \epsilon) \) such that
\[
A(T) \leq C(T, \epsilon).
\]

**Proof.** Multiplying (33) by \( u_x \) and integrating, we have
\[
\int_0^1 (u_x^2) dx - \int_0^1 P(u_x r^2)_x dx
= - \int_0^1 (2\mu + \lambda(\rho)) \rho (u^2)_x (u_x r^2)_x dx
= - \int_0^1 (2\mu + \lambda(\rho)) \rho (u^2)_x ((u^2)_x - 2ru^2)_x dx
= - \frac{d}{dt} \frac{1}{2} \int_0^1 (2\mu + \lambda(\rho)) \rho ((u^2)_x)^2 dx + \frac{1}{2} \int_0^1 (\rho (2\mu + \lambda(\rho)))_x (u^2)_x^2 dx
+ \int_0^1 (2\mu + \lambda(\rho)) \rho (u^2)_x (2ru^2)_x dx
\]
(46)
This leads to
\[
\frac{1}{2} \int_0^1 (2\mu + \lambda(\rho)) \rho ((ur^2)_x)^2 \, dx + \int_0^T \int_0^1 (u_r)^2 \, dx \, dt
\]
\[
= E_2 + \int_0^T \int_0^1 P(u_r r^2)_x \, dx \, dt + \frac{1}{2} \int_0^1 (\rho(2\mu + \lambda(\rho)))_x ((ur^2)_x)^2 \, dx + \int_0^1 (2\mu + \lambda(\rho)) \rho ((ur^2)_x)^2 \, dx
\]
\[
= E_2 + J_1 + J_2 + J_3,
\]
where \( E_2 = \frac{1}{2} \int_0^1 (2\mu + \lambda(\rho)) \rho ((ur^2)_x)^2 \, dx \).

We first estimate \( J_1 \). (11) and Lemma 3.1 implies
\[
J_1 = \int_0^T \int_0^1 P_{ux} r^2 dx \, dt + \int_0^T \int_0^1 P_{ur} \frac{2}{r^2} \, dx \, dt
\]
\[
\leq \int_0^1 P_{ux} r^2 \, dx - \int_0^1 P_{ur} r^2 (x,0) \, dx - \int_0^T \int_0^1 u_x (P_r r^2 + 2ruP) \, dx \, dt
\]
\[
+ \int_0^T \int_0^1 \rho^{-1} u_x^2 \frac{2}{r^2} \, dx \, dt
\]
\[
\leq \int_0^1 P_{ux} r^2 \, dx + C(\epsilon, T) \int_0^T \int_0^1 \rho ((ur^2)_x)^2 \, dx \, dt + \delta \int_0^T \int_0^1 u_x^2 \, dx \, dt + C(\epsilon, T)
\]
\[
\leq \delta \int_0^1 (2\mu + \lambda(\rho)) \rho ((ur^2)_x)^2 \, dx + C(\epsilon, T) \int_0^T \int_0^1 (2\mu + \lambda(\rho)) \rho ((ur^2)_x)^2 \, dx \, dx
\]
\[
+ \delta \int_0^T \int_0^1 u_x^2 \, dx \, dt + C(\epsilon, T).
\]

The equation (33) implies
\[
J_2 \leq C(\epsilon, T) \int_0^T \int_0^1 |(ur^2)_x|^3 \, dx \, dt.
\]

Young’s inequality and (45) give
\[
J_3 = \int_0^T \int_0^1 (2\mu + \lambda(\rho)) \rho (ur^2)_x (2ruu_x) \, dx \, dt
\]
\[
\leq C(\epsilon, T) \int_0^T \|u\|_{L^\infty} \times \left\{ \int_0^1 (2\mu + \lambda(\rho)) \rho ((ur^2)_x)^2 \, dx + \int_0^1 \frac{4}{\rho} (\frac{u}{\rho^2})^2 \, dx + \int_0^1 \rho^2 u_x^2 \, dx \right\} \, dt
\]
\[
\leq C(\epsilon, T) \int_0^T \|u\|_{L^\infty} \int_0^1 (2\mu + \lambda(\rho)) \rho ((ur^2)_x)^2 \, dx \, dt + C(\epsilon, T).
\]
Substituting (48) – (50) into (47), we obtain
\[
\int_0^1 (2\mu + \lambda(\rho))\rho ((ur^2)_x)^2 dx + \int_0^T \int_0^1 (u_t)^2 dx d\tau \\
\leq C(\epsilon, T) \int_0^T (\|u\|_{L^\infty} + 1) \int_0^1 (2\mu + \lambda(\rho))\rho ((ur^2)_x)^2 dx d\tau \\
+ C(\epsilon, T) \left( \int_0^T \int_0^1 \|(ur^2)_x^3\| dx d\tau + 1 \right)
\]
(51)

the first term in (51) can estimated as follows
\[
\int_0^T \int_0^1 (r^2u)_x^3 dx d\tau \\
\leq \sup_{0 \leq t \leq T} \left( \int_0^1 (r^2u)_x^2 dx \right)^{1/2} 
\times \left\{ \int_0^T \int_0^1 (r^2u)_x^2 dx d\tau \right. \\
\left. + \left( \int_0^T \int_0^1 (r^2u)_x^2 dx d\tau \right)^{1/4} \left( \int_0^T \int_0^1 (r^2u)_x^2 dx d\tau \right)^{3/4} \right\}
\]
(52)
\[
\leq C(\epsilon, h)A(T)^{1/2} \left( 1 + \int_0^T \int_0^1 \|(r^2u)_x^2\| dx d\tau \right)^{1/4},
\]

Equation (33) can re-write as
\[
(r^2u)_x = \frac{1}{(2\mu + \lambda(\rho))\rho}(u_t - [(2\mu + \lambda(\rho))\rho]_x (r^2u)_x).
\]
Which and the fact \(\|u\|_{L^\infty} \leq \|u\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2}\) give
\[
\int_0^T \int_0^1 (r^2u)_x^2 dx d\tau \\
\leq C(T, \epsilon) \left( \int_0^T \int_0^1 u_t^2 dx dt + \int_0^T \int_0^1 |P_x|^2 dx dt + \int_0^T (r^2u)_x^2 dx d\tau \right)^{1/2} \left( 1 + \|\rho_x^2\|_{L^2} dt \right)
\]
\[
\leq C(T, \epsilon) \left( \int_0^T \int_0^1 u_t^2 dx dt + \int_0^T \|(r^2u)_x\|_{L^2}^2 dt \right) + \delta \int_0^T \|(r^2u)_x\|_{L^2}^2 dt.
\]
Consequently
\[
\int_0^T \int_0^1 (r^2u)_x^2 dx d\tau \leq C(T, \epsilon) \left( \int_0^T \int_0^1 u_t^2 dx dt + \int_0^T \|(r^2u)_x\|_{L^2}^2 dt \right) \\
\leq C(T, \epsilon) (A(T) + 1),
\]
thus, by (52) we have
\[
\int_0^T \int_0^1 (r^2u)_x^3 dx d\tau \leq C(T, \epsilon)A(T)^{1/2} (1 + A(T)^{1/4}) \leq C(T, \epsilon) \left( 1 + A(T)^{3/4} \right).
\]

Which together with (51) give
\[
A(T) \leq C(T, \epsilon) \left( 1 + A(T)^{3/4} + \int_0^T \|\psi\|_{L^\infty} A(\tau) d\tau \right),
\]
Applying Cauchy’s inequality and Gronwall’s inequality we finally obtain
\[ \mathcal{A}(T) \leq C(T, \epsilon). \]

From the Lemma 3.5 and Lemma 3.8, we get

**Corollary 2.** Under the same assumptions as Lemma 3.1, Then
\[
\int_0^1 (u_x)^2 \, dx + \int_0^T \int_0^1 (u_x^2)^2 \, dx \, d\tau \leq C(T, \epsilon).
\]

**Lemma 3.9.** Under the same assumptions as Lemma 3.1, Then
\[
\int_0^T \int_0^1 (u_{xx})^2 + ((P - (2\mu + \lambda(\rho))\rho(\nabla u^2)_x)^2 \, dx \, d\tau
+ \int_0^T ((a(\tau))^2 + (a'(\tau))^2) \, d\tau \leq C(T, \epsilon),
\]
and
\[
\int_0^1 ((u_\tau)^2 + (u_{xx})^2) \, dx + \int_0^1 ((P - (2\mu + \lambda(\rho))\rho(\nabla u^2)_x)^2 \, dx
+ \int_0^T \int_0^1 (u_{xx})^2 \, dx \, d\tau \leq C(\epsilon, T).
\]

**Proof.** From Equation (33)2 can we have
\[
\int_0^T \int_0^1 (u_{xx})^2 \, dx \, d\tau
\leq \int_0^T \int_0^1 ((u_\tau)^2 + (\rho_x)^2 + (\rho_x u_x)^2 + (u_x)^2 + (u \rho_x)^2 + u^2) \, dx \, d\tau
\leq \int_0^T \int_0^1 ((\rho_x u_x)^2 + (u \rho_x)^2) \, dx \, d\tau + C(T, \epsilon)
\leq \int_0^1 (\|u_x\|_{L^\infty}^2 + \|u\|_{L^\infty}^2) \int_0^1 (\rho_x)^2 \, dx \, d\tau + C(T, \epsilon)
\leq \delta \int_0^T \int_0^1 (u_{xx})^2 \, dx \, d\tau + C(T, \epsilon),
\]
Thus
\[
\int_0^T \int_0^1 (u_{xx})^2 \, dx \, d\tau \leq C(T, \epsilon).
\]

By (24) and (25)
\[
\int_0^T ((a(\tau))^2 + (a'(\tau))^2) \, d\tau \leq C(T, \epsilon),
\]
and
\[
\int_0^T \int_0^1 ((P - (2\mu + \lambda(\rho))\rho(\nabla u^2)_x)^2 \, dx \, d\tau
\leq \int_0^T \int_0^1 (r^{-2} u_{xx})^2 \, dx \, d\tau \leq C(T, \epsilon).
\]

Differentiating (33)2 with respect to \( \tau \) gives
\[ r^{-2} u_{\tau \tau} - 2r^{-3} u_{\tau \rho} + P_{\tau xx} = ((2\mu + \lambda(\rho))\rho(\nabla u^2)_x)_{xx}. \]
Taking the inner product of (53) with $u_\tau$ over $[0, 1]$
\[
\frac{d}{dt} \int_0^1 \frac{1}{2} r^{-2}(u_\tau)^2 dx + \int_0^1 \left( (2\mu + \lambda(\rho))\rho (u_{xx})^2 r^2 dx + ((2\mu + \lambda(\rho)) \frac{u_\tau^2}{r} (1, \tau) \right)
\]
\[
= \int_0^1 r^{-3} u(u_\tau)^2 dx + \int_0^1 P u_{xx} dx - \int_0^1 ((2\mu + \lambda(\rho)\rho) (ur^2)_x u_{xx} dx
\]
\[
- \int_0^1 2((2\mu + \lambda(\rho)) \rho uu^2 u_{xx} dr - \int_0^1 ((2\mu + \lambda(\rho)) \frac{u_\tau^2}{r} dx + \int_0^1 \frac{1}{2} \frac{2u_\tau u_{xx}}{r} dx + \int_0^1 ((2\mu + \lambda(\rho)) \frac{2u_\tau^2 u_{xx}}{r} dx)
\]
\[
\leq C \int_0^1 r^{-2}(u_\tau)^2 dx + \delta \int_0^1 \rho (u_{xx})^2 r^2 dx + C \int_0^1 (\rho (u_\tau)^2 dx
\]
\[
- \int_0^1 ((2\mu + \lambda(\rho)) \rho r (ur^2)_x u_{xx} dx + C \int_0^1 (u_\tau)^2 dx + C \int_0^1 u^2 dx
\]
\[
\leq \int_0^1 r^{-2}(u_\tau)^2 dx + \delta \int_0^1 \rho (u_{xx})^2 r^2 dx + \int_0^1 (u_\tau)^2 dx,
\]
where we have used the following estimate
\[
- \int_0^1 ((2\mu + \lambda(\rho)) \rho r (ur^2)_x u_{xx} dx
\]
\[
= \int_0^1 ((2\mu + \lambda(\rho)) \rho^2 (ur^2)_x u_{xx} dx
\]
\[
\leq C(\epsilon, T) \frac{\|u_{xx}\|_{L^\infty}}{\int_0^1 (\rho (u_{xx})^2 dx) \left( \int_0^1 (\rho (u_{xx})^2 dx) \right)^{1/2}} \left( \int_0^1 (\rho (u_{xx})^2 dx) \right)^{1/2}
\]
\[
\leq C(\epsilon, T) \frac{\|u_{xx}\|_{L^\infty}}{\int_0^1 (\rho (u_{xx})^2 dx) \left( \int_0^1 (\rho (u_{xx})^2 dx) \right)^{1/2}} \left( \int_0^1 (\rho (u_{xx})^2 dx) \right)^{1/2}
\]
\[
\leq C(\epsilon, T) \frac{\|u_{xx}\|_{L^\infty}^2 + \delta \int_0^1 (\rho (u_{xx})^2 r^2 dx
\]
\[
\leq C(\epsilon, T) \frac{\|u_{xx}\|_{L^\infty}^2 + \delta \int_0^1 (\rho (u_{xx})^2 r^2 dx
\]
\[
\leq C(\epsilon, T) \frac{\|u_{xx}\|_{L^\infty}^2 + \|u_x\|_{L^2}^2 + \|u\|_{L^2} + \|u_x\|_{L^2} + \|u\|_{L^2}}{\delta \int_0^1 (\rho (u_{xx})^2 r^2 dx
\]
\[
\leq C(\epsilon, T) \frac{\|u_{xx}\|_{L^\infty}^2 + \|u_x\|_{L^2} + \|u\|_{L^2}}{\delta \int_0^1 (\rho (u_{xx})^2 r^2 dx
\]

Using Gronwall’s inequality to (53), we obtain
\[
\int_0^1 r^{-2}(u_\tau)^2 dx + \int_0^T \int_0^1 ((2\mu + \lambda(\rho)) \rho (u_{xx})^2 r^2 dx d\tau
\]
\[
+ \int_0^T ((2\mu + \lambda(\rho)) \frac{u_\tau^2}{r} (1, \tau) d\tau \leq C(\epsilon, T).
\]
Consequently by (33) and (37), we have
\[
\int_0^1 ((P - ((2\mu + \lambda(\rho)) \rho (ur^2)_x)) \right)^2 dx \leq \int_0^1 (r^{-2} u_{xx})^2 dx \leq C(T, \epsilon),
\]
\[
\int_0^T (a''(\tau))^2 d\tau \leq C(T, \epsilon),
\]
and
\[
\int_0^1 (u_{xx})^2 \, dx \leq \int_0^1 ((u_x)^2 + (u_T)^2 + (\rho_x)^2) \, dx \leq C(\epsilon, T).
\]

The proof of proposition 1. First, we can change Proposition 1 in the Lagrangian coordinates. Next, with the help of Lemmas 3.1-3.9 for \((\rho, u, a)\) and continuity argument, the global strong solution to the FBVP (3) and (15)-(16) under the assumptions of Proposition 1 can be shown by standard argument as in [34].

4. Uniform estimates away from symmetry center. In order to obtain the global weak solutions containing the symmetric center \(r = 0\), we need further the uniform-in-\(\epsilon\) estimates to pass the limit \(\epsilon \to 0^+\). For this, the idea is originated by Hoff [13] and Hoff-Jessen [10]. Roughly speaking, for any given \(h > 0\), define the particle path \(r^\epsilon_h(t)\) by
\[
h = \int_0^{r^\epsilon_h(t)} \rho^\epsilon(r, t) r^2 \, dr.
\]
\[
\frac{\partial r^\epsilon_h}{\partial t} = u^\epsilon(r^\epsilon_h, t). \tag{55}
\]
Thus \(r^\epsilon_h(t)\) is the position at time \(t\) of a fixed fluid particle. So, given \(h > 0\) there is a positive constant \(C\) independent of \(\epsilon\) and \(T\), such that
\[
Ch^{\frac{\gamma-1}{\gamma}} \leq r^\epsilon_h(t).
\]
Then we consider the problem on the region \(r \in [r^\epsilon_h(t), a(t)]\). Due to the fact that \(r^\epsilon_h(t) \geq c(h)\) with the positive constant \(c(h)\) independent of \(\epsilon\), we can get some uniformly interior estimates in \(\epsilon\) for the solution on the region \(r \in [r^\epsilon_h(t), a(t)]\). Therefore, we can pass the limit \(\epsilon \to 0\) for any fixed \(h > 0\) and then take \(h \to 0\) to get a weak solution. However, due to the bulk viscosity here depends on the density, the analysis in [10] can not be applied directly since we do not have
\[
| \int_{t_1}^{t_2} u^\epsilon_{\tau, h}(r_h(t), t) \, dt | \leq C(h, T), \quad \forall t_1, t_2 \in [0, T] \tag{56}
\]
by the elementary energy estimates. Thus we can not get the uniform estimates for the cut-off function along the particle path as in [10]. In order to overcome this difficulty, we first use the Lagrangian transformation to get the uniform interior estimates for the solution. Then we can change back to the Eulerian coordinates to pass to the limit \(\epsilon \to 0\) and then \(h \to 0\) to get the global weak solution.

Thus the Lagrangian coordinates transformation translates the domain \([0, T] \times [r^\epsilon_h(t), a(t)]\) into \([0, T] \times [h, 1]\).

In this section, we derive some desired uniform estimates for \((\rho^\epsilon, u^\epsilon, a^\epsilon)\) to the modified FBVP (3) and (15)-(16). To simplify the presentation, we drop the superscript \(\epsilon\).

**Lemma 4.1.** The upper bound of the density:
\[
c(h) \leq \rho \leq C(h), \quad (r, \tau) \in [r^\epsilon_h(t), a(t)] \times [0, T].
\]

**Proof.** Thanks to the fact \(Ch^{\frac{\gamma}{\gamma-1}} \leq r^\epsilon_h(t)\), this lemma can be proved similarly as Lemma 3.5. So the details are omitted here. \(\square\)

Similar to the proof of Lemma 3.7 we have the following Entropy estimation.
Lemma 4.2 (Entropy estimation).

\[
\int_0^1 (u + r^2(2\mu \ln \rho + \frac{\rho \beta}{\beta})_x)^2 dr + \frac{8\mu}{\gamma} \int_0^1 \int_0^1 \left( (\rho^{\gamma/2})_x r^2 \right)^2 dxds \\
+ \frac{4\gamma}{(\beta + \gamma)^2} \int_0^1 \int_0^1 \left( (\rho^{\beta + \gamma})_x r^2 \right)^2 dxds \leq C(h).
\]

We shall make repeated use of a cut-off function which is convected with the flow and which vanishes near the origin. So we construct a cut-off function \( \phi(x) \) satisfies

\[
\phi(x) = \begin{cases} 
0 & x \in [0, h], \\
\frac{1}{h} (x - h) & x \in (h, 2h), \\
1 & x \in [2h, 1].
\end{cases}
\]

Now we introduce the higher-order functional for a given solution: let \( \sigma(t) = \min\{1, t\} \) and define

\[
\mathcal{B}(T) = \sup_{0 \leq t \leq T} \sigma(t) \int_0^1 \phi^2 [(r^2 u)_x]^2 dx + \int_0^T \int_0^1 \phi^2 u_x^2 dxdT.
\] (57)

For which corresponding to the following form in Euler coordinates:

\[
\mathcal{B}(T) := \sup_{0 \leq t \leq T} \sigma(t) \int_{r_h(t)}^{a(t)} \phi^2 u_x^2 r^2 dr + \int_0^T \int_{r_h(t)}^{a(t)} \sigma \phi^2 \dot{u}_x^2 r^2 drdt.
\] (58)

where

\[
\frac{\partial}{\partial \xi} = \frac{\partial}{\partial r} + \frac{2}{r}
\]

and

\[
\dot{u} = u_t + uu_r.
\]

Lemma 4.3. Let \( h > 0 \) and \( T > 0 \) be given. Then there exists a constant \( C = C(h, T) \) such that, if \( \mathcal{B} \) is as defined above in (57), then

\[
\mathcal{B}(t) \leq C(h, T).
\]

Proof. Multiplying (33) by \( \sigma \phi^2 u_r \), one has

\[
\int_0^1 \left\{ (2\mu + \rho \beta) \rho \frac{[(r^2 u)_x]^2}{2} \sigma \phi^2 \right\} (x, t) dx + \int_0^T \int_0^1 \sigma \phi^2 u_x^2 dxdt \\
= -\int_0^T \int_0^1 \sigma \phi^2 u_t r^2 P(\rho)_x dxdt - \int_0^T \int_0^1 \frac{[(r^2 u)_x]^3}{2} \sigma \phi^2 [2\mu + (\beta + 1)\rho^\beta] \rho^2 dxdt \\
+ \int_0^T \int_0^1 \frac{[(r^2 u)_x]^2}{2} \sigma_i \phi^2 (2\mu + \rho \beta) \rho dxdt \\
- \int_0^T \int_0^1 2 \sigma \phi \rho x^2 u_t (2\mu + \rho \beta) \rho (r^2 u)_x dxdt \\
+ \int_0^T \int_0^1 \sigma \phi^2 (2\mu + \rho^3) \rho (r^2 u)_x dxdt \\
=: \sum_{i=1}^5 K_i.
\] (59)
In the sequel, we derive bounds for $K_j$ ($1 \leq j \leq 3$) on the right-hand side of (59).

\[
K_1 = - \int_0^T \int_0^1 \sigma \phi^2 u_t r^2 P(\rho) dx dt
\leq \frac{1}{8} \int_0^T \int_0^1 \sigma \phi^2 u_t^2 dx dt + C \int_0^T \int_0^1 \sigma \phi^2 r^4 P(\rho) dx dt
\leq \frac{1}{8} \int_0^T \int_0^1 \sigma \phi^2 u_t^2 dx dt + C(h, T).
\]

\[
K_2 = \int_0^T \int_0^1 \frac{[(r^2u)_x]^2}{2} \sigma \phi^2 (2\mu + \rho^3) \rho^2 dx dt \leq C.
\]

\[
K_3 = - \int_0^T \int_0^1 \frac{[(r^2u)_x]^3}{2} \sigma \phi^2 [2\mu + (\beta + 1)\rho^3] \rho^2 dx dt \leq C \int_0^T \int_0^1 \sigma \phi^2 [(r^2u)_x]^3 dx dt.
\]

\[
K_4 = - \int_0^T \int_0^1 2\sigma \phi_\tau r^2 u_t (2\mu + \rho^3) \rho (r^2u)_x dx dt
= - \int_0^T \int_0^1 2\sigma \phi_\tau r^2 u_t (2\mu + \rho^3) \rho (r^2u)_x dx dt
\leq \frac{1}{8} \int_0^T \int_0^1 \sigma \phi^2 r^2 u_t^2 dx dt + C \int_0^T \int_0^1 \sigma [(2\mu + \rho^3) \rho^2 [(r^2u)_x]^2 dx dt
\leq \frac{1}{8} \int_0^T \int_0^1 \sigma \phi^2 r^2 u_t^2 dx dt + C(h, T).
\]

\[
K_5 = \int_0^T \int_0^1 \sigma \phi^2 (2ru^2)_x (2\mu + \rho^3) \rho (r^2u)_x dx dt
= \int_0^T \int_0^1 \sigma \phi^2 \left( \frac{2u^2}{\rho r^2} + 4ruu_x \right) (2\mu + \rho^3) \rho (r^2u)_x dx dt
\leq C(h, T) \int_0^T \|u\|_{L^\infty([h, 1])} \left[ \int_0^1 \sigma \phi^2 u^2 (2\mu + \rho^3) \rho dx 
+ \int_0^1 \sigma \phi^2 (2\mu + \rho^3) \rho [(r^2u)_x]^2 dx \right] dt
+ \int_0^T \int_0^1 \sigma \phi^2 r^2 uu_x (2\mu + \rho^3) \rho (r^2u)_x dx dt
\leq C(h, T) \int_0^T \|u\|_{L^\infty([h, 1])} \int_0^1 \sigma \phi^2 (2\mu + \rho^3) \rho [(r^2u)_x]^2 dx dt + C(h, T)
+ \int_0^T \|u\|_{L^\infty([h, 1])} \int_0^1 \sigma \phi^2 [(r^2u)_x] \rho - \frac{2u}{\rho r^4} (2\mu + \rho^3) \rho (r^2u)_x dx dt
\leq C(h, T) \int_0^T \|u\|_{L^\infty([h, 1])} \int_0^1 \sigma \phi^2 (2\mu + \rho^3) \rho [(r^2u)_x]^2 dx dt + C(h, T).
\]
Substituting (60) – (64) into (59), we obtain
\[
\begin{align*}
& \frac{1}{2} \int_{0}^{1} \sigma \phi^2 (2\mu + \rho^3) \rho [(r^2 u)_x]^2 \, dx + \int_{0}^{T} \int_{0}^{1} \sigma \phi^2 u^2_x \, dx \, d\tau \\
& \leq C(h, T) \int_{0}^{T} \int_{0}^{1} \sigma \phi^2 [(r^2 u)_x]^3 \, dx \, d\tau \\
& \quad + C(h, T) \int_{0}^{T} \|u\|_{L^\infty([h, 1])} \int_{0}^{1} \sigma \phi^2 \rho [(ur^2)_x]^2 \, dx \, d\tau + C.
\end{align*}
\]

Lemma 4.4. Let \( J(x, \tau) \) be a smooth function defined on \( \{(x, \tau) | h \leq x < 1, \tau \in [0, T]\} \). Then
\[
\int_{0}^{T} \int_{0}^{1} \sigma \phi^2 |J|^3 \, dx \, d\tau 
\leq \sup_{0 \leq t \leq T} \left( \sigma(t) \int_{h}^{1} \phi^2 J^2 \, dx \right) \frac{1}{2} \times \left( \int_{0}^{T} \int_{0}^{1} \sigma \phi^2 J^2_x \, dx \, d\tau + \left( \int_{0}^{T} \int_{0}^{1} \sigma \phi^2 J^2_x \, dx \, d\tau \right) \frac{1}{2} \left( \int_{0}^{T} \int_{0}^{1} J^2 \, dx \, d\tau \right)^{\frac{1}{2}} \right).
\]

Taking \( J = (r^2 u)_x \), we have
\[
\int_{0}^{T} \int_{0}^{1} \sigma \phi^2 |(r^2 u)_x|^3 \, dx \, d\tau \leq C(h, T) \sup_{0 \leq t \leq T} \left( \sigma(t) \int_{h}^{1} \phi^2 [(r^2 u)_x]^2 \, dx \right)^{\frac{1}{2}} \times \left\{ 1 + \left( \int_{0}^{T} \int_{0}^{1} \sigma \phi^2 [(r^2 u)_x]^2 \, dx \, d\tau \right)^{\frac{1}{2}} \right\},
\]
by the equation (33), we have
\[
(r^2 u)_{xx} = r^{-2} u_t + P(\rho)_x - [(2\mu + \rho^3) \rho](r^2 u)_x.
\]
Thus,
\[
\int \int \sigma \phi^2 |(r^2 u)_{xx}|^3 \, dx \, dt 
\leq \int \int \sigma \phi^2 r^{-4} u^4_x \, dx \, dt + \int \int \sigma \phi^2 \rho_x^4 \, dx \, dt + \int \int \sigma \phi^2 \rho^3_x [(r^2 u)_x]^2 \, dx \, dt \\
\leq C(h, T) \left[ \int \int \sigma \phi^2 u^4_x \, dx \, dt + 1 + \int_{0}^{T} \|\sqrt{\sigma} \phi (r^2 u)_x\|_{L^\infty} \int \int r^4 \rho^3_x \, dx \, dt \right] \\
\leq C(h, T) \left[ \int \int \sigma \phi^2 u^4_x \, dx \, dt + 1 + \int_{0}^{T} \|\sqrt{\sigma} \phi (r^2 u)_x\|_{L^2} \|\sqrt{\sigma} \phi (r^2 u)_x\|_{L^2} + \|\sqrt{\sigma} \phi (r^2 u)_{xx}\|_{L^2} \right] \\
\leq C(h, T) \left[ \int \int \sigma \phi^2 u^4_x \, dx \, dt + 1 + \int_{0}^{T} \|\sqrt{\sigma} \phi (r^2 u)_x\|_{L^2} \|\sqrt{\sigma} \phi (r^2 u)_{xx}\|_{L^2} \, dt + 1 \right] \\
\leq \frac{1}{2} \int \int \sigma \phi^2 [(r^2 u)_{xx}]^3 \, dx \, dt + C(h, T) \left[ \int \int \sigma \phi^2 u^4_x \, dx \, dt + 1 \right],
\]
\[
\int_{0}^{T} \int_{0}^{1} \sigma \phi^2 [(r^2 u)_x]^3 \, dx \, d\tau \leq C(h, T) B^4 \times \left\{ 1 + \mathcal{B}^4 \right\},
\]
which together with (65) yield

$$B(T) \leq C(h, T)B^\frac{1}{2} \left[1 + B(T)^{\frac{1}{2}}\right] + \int_0^T \left\|u\right\|_{L^\infty([h, 1])}^2 + 1)B(T)d\tau$$

$$\leq C(h, T)\left[1 + B(T)^{\frac{1}{2}}\right] + \int_0^T \left\|u\right\|_{L^\infty([h, 1])}B(\tau)d\tau.$$  

By Gronwall’s inequality, we get

$$B(T) \leq C(h, T).$$

Thus, we have complete the proof of Lemma 4.3.

5. Proof of Theorem 2.2. By virtue of the a priori estimates derived in Sections 4, we are now able to prove our main theorem by taking appropriate limits in a manner analogous to that in [10].

To begin with, we let \((\rho_0, u_0)\) be initial data satisfying the hypotheses of Theorem 2.2. Let \(\psi(x) \in C_0^\infty(\mathbb{R})\) satisfy \(\psi(x) = 1\) when \(|x| < \frac{a_0}{2}\) and \(\psi(x) = 0\) when \(|x| \geq a_0\), and define \(\psi_\delta(x) = \psi(\frac{x}{\delta})\). We denote by \(H_\delta\) a standard mollifier (in \(r\)) of width \(\delta\). For simplicity we still denote by \((\rho_0^\epsilon, u_0^\epsilon)\) the extension of \((\rho_0^\epsilon, u_0^\epsilon)\) in \(\mathbb{R}\), i.e.

$$\rho_0^\epsilon(r) \doteq \begin{cases} 
\rho_0(\epsilon) + \epsilon, & r \in [0, \epsilon] \\
\rho_0(r) + \epsilon, & r \in [\epsilon, a_0] \\
\rho_0(a_0) + \epsilon, & r \in [a_0, \infty),
\end{cases}$$

$$u_0^\epsilon(r) \doteq \begin{cases} 
u_0(r), & r \in (2\delta + \epsilon, a_0], \\
0, & \text{otherwise}.
\end{cases}$$

For \(\epsilon < \min\{a_0(1 - \delta), \delta\}\), we define the smooth approximate initial data \((\rho_0^{\epsilon, \delta}, u_0^{\epsilon, \delta})\) as follows:

$$\rho_0^{\epsilon, \delta}(r) \doteq (\rho_0^\epsilon * H_\delta)(r),$$

$$u_0^{\epsilon, \delta}(r) \doteq \left\{ \begin{array}{ll}
 \frac{1}{2\mu + \lambda(\rho_0^\epsilon (a_0))} - \frac{2u_0^\epsilon (a_0)}{a_0} \\
 \int_\epsilon^r \psi_\delta(a_0 - x)dx
\end{array} \right. \doteq (u_0^\epsilon * H_\delta)(r) \left\{1 - \psi_\delta(a_0 - r)\right\} + (u_0^\epsilon * H_\delta)(a_0)\psi_\delta(a_0 - r)$$

The resulting data \((\rho_0^{\epsilon, \delta}, u_0^{\epsilon, \delta})\) then satisfy the hypotheses of Theorem 2.2 with the constants independent of \(\epsilon\) and \(\delta\). Thus, there is a global-in-time smooth solution \((\rho^{\epsilon, \delta}, u^{\epsilon, \delta})\) of the system (3)-(6)). This is a result of Okada’s work [34] in the annular domain \([\epsilon, a(t)]\). Next, we can pass the limit to get a global weak solution. It is referred to Guo-Li[8] for more details and the proof of Theorem 2.2 is finished.

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