POSITIVITY AND REPRESENTATIONS OF SURFACE GROUPS

OLIVIER GUICHARD, FRANÇOIS LABOURIE, AND ANNA WIENHARD

Abstract. In [18, 20] Guichard and Wienhard introduced the notion of \( \Theta \)-positivity, a generalization of Lusztig’s total positivity to real Lie groups that are not necessarily split.

Based on this notion, we introduce in this paper \( \Theta \)-positive representations of surface groups. We prove that \( \Theta \)-positive representations are \( \Theta \)-Anosov. This implies that \( \Theta \)-positive representations are discrete and faithful and that the set of \( \Theta \)-positive representations is open in the representation variety. We show that the set of \( \Theta \)-positive representations is closed within the set of representations that do not virtually factor through a parabolic subgroup. From this we deduce that for any simple Lie group \( G \) admitting a \( \Theta \)-positive structure there exist components consisting of \( \Theta \)-positive representations. More precisely we prove that the components parametrized using Higgs bundles methods in [5] consist of \( \Theta \)-positive representations.

1. Introduction

An important feature of Teichmüller space, seen as a connected component of the space of representations of the fundamental group of a closed connected orientable surface \( S \) of genus at least 2 in \( \text{PSL}_2(\mathbb{R}) \), is that it consists entirely of representations which are discrete and faithful. These representations are moreover quasi-isometries from \( \pi_1(S) \)
to $\text{PSL}_2(\mathbb{R})$. This situation does not extend to the case of any semisimple group, notably for simply connected complex ones, where the representation variety is irreducible as an algebraic variety [29].

However, this phenomenon was shown to happen for some groups of higher rank. Two families of (unions of) connected components of the variety of representations of the fundamental group of a closed connected orientable surface $S$ of genus at least 2, which consist entirely of discrete and faithful representations, have been singled out:

- Hitchin components, defined when $G$ is a real split group [23, 14, 11, 25, 17],
- spaces of maximal representations, which are defined when $G$ is Hermitian [13, 9, 7].

When $G$ is $\text{PSL}_2(\mathbb{R})$, the Hitchin component and the space of maximal representations both agree with Teichmüller space.

Both studies are closely related to the theory of Anosov representations as introduced in [25, 19]. For the above cases, it is known that these components consist entirely of Anosov representations [25, 7]. Being Anosov is a notion defined for any reductive Lie group with respect to a choice of parabolic subgroup. Every Anosov representations is in particular faithful, discrete and a quasi-isometric embedding [26, 19, 10].

Representations in the Hitchin components as well as maximal representations can be characterized in terms of equivariant curves from the boundary at infinity of $\pi_1(S)$ into an appropriate flag variety, which preserves some positivity [11, 25, 17, 9]. For Hitchin components this positivity is based on Lustztig’s total positivity [28], for maximal representations it is based on the maximality of the Maslov index and related to Lie semigroups in $G$.

In [20, 18], Guichard and Wienhard introduced the notion of $\Theta$-positivity. This notion extends Lusztig’s total positivity to generalized flag manifolds associated to a parabolic defined by a set $\Theta$ of simple roots.

They classified all possible simple Lie groups that admit a $\Theta$-positive structure. These include real split Lie groups, for which $\Theta$-positivity is Lusztig’s total positivity, Hermitian Lie groups of tube type, where $\Theta$-positivity is related to Lie-semigroups, but they also include two other families of Lie groups, namely the family of classical groups $\text{SO}(p, q)$ —with $p \neq q$— and an exceptional family consisting of the real rank 4 form of $F_4$, $E_6$, $E_7$, and $E_8$ respectively. They conjectured that $\Theta$-positivity provides the right underlying
algebraic structure for the existence of components made solely of discrete faithful representations. The goal of the present article is to fulfil that program.

As a corollary of our main result, we prove

**Theorem A.** If \( G \) admits a \( \Theta \)-positive structure, then there exists a connected component of the representation variety of the surface group in \( G \) that consists solely of discrete and faithful representations.

A \( \Theta \)-positive structure on \( G \) implies in particular the existence of a positive semigroup in the unipotent radical of the parabolic group \( P_{\Theta} \), which then leads to the notion of positive triples and positive quadruples in the flag variety \( F_{\Theta} \cong G/P_{\Theta} \). In the basic example of \( G = \text{PSL}_2(\mathbb{R}) \) and \( F_{\Theta} = \mathbb{P}^1(\mathbb{R}) \), a triple is positive if it consists of pairwise distinct points and a quadruple is positive if it is cyclically ordered.

Let us give a geometric picture of the positivity in the flag variety \( F_{\Theta} \). For this let \( a \) and \( b \) be two points in \( F_{\Theta} \) which are transverse to each other. Then \( \Theta \)-positivity provides the existence of preferred connected components of the set of all points in \( F_{\Theta} \) that are transverse to both \( a \) and \( b \). We call these preferred components the diamonds (with extremities \( a \) and \( b \)). There are exactly two diamonds with given extremities and they are disjoint. The semigroup property alluded to before translates into a nesting property of diamonds: if \( c \) is a point in a diamond \( V(a, b) \) with extremities \( a \) and \( b \), then there is exactly one diamond \( V(c, b) \) (with extremities \( c \) and \( b \)) included in \( V(a, b) \). These nesting properties of diamonds play an important role in our arguments.

If \( a \) and \( b \) are transverse, and \( c \) belongs to a diamond with extremities \( a \) and \( b \), we say the triple \( (a, b, c) \) is positive. Similarly if \( a \) and \( b \) are transverse, and \( c \) and \( d \) belong to the two distinct diamonds with extremities \( a \) and \( b \), then the quadruple \( (a, c, b, d) \) is called positive. We show in section 3 that being positive is invariant under all permutations for a triple, and invariant under the dihedral group for a quadruple.

We define a map \( \xi \) from of a cyclically ordered set \( A \) to \( F_{\Theta} \) to be positive if \( \xi \) maps triples of pairwise distinct points to positive triples and cyclically ordered quadruples to positive quadruples.

This allows us to define the notion of a \( \Theta \)-positive representation: A representation \( \rho: \pi_1(S) \to G \) is \( \Theta \)-positive if there exists a non-empty subset \( A \) of \( \partial_\infty \pi_1(S) \), invariant by \( \pi_1(S) \), and a \( \rho \)-equivariant positive boundary map from \( A \) to \( F_{\Theta} \).

We prove
Theorem B. Let $G$ be a simple Lie group that admits a $\Theta$-positive structure, and let $\rho : \pi_1(S) \to G$ be a $\Theta$-positive representation. Then $\rho$ is a $\Theta$-Anosov representation.

As a direct consequence we obtain that a $\Theta$-positive representation is faithful with discrete image, its orbit map into the symmetric space is a quasi-isometric embedding and the boundary map extends uniquely to a Hölder map [26, 19, 10, 6].

Theorem B provides a general proof of the Anosov property for all Hitchin representations and all maximal representations. This was proved for most Hitchin components in [25] and for all maximal representations which tightly embed into $\text{Sp}_{2n}(\mathbb{R})$ in [7]. However, the case of the Hitchin component of $\text{SO}(p, p)$ and of $F_4, E_6, E_7, E_8$ and the case of maximal representations into the exceptional Hermitian Lie group of tube type, which cannot be tightly embedded into $\text{Sp}_{2n}(\mathbb{R})$ [8, 22, 21], were not previously covered in the literature.

Using the openness of the set of $\Theta$-Anosov representation, a further consequence of Theorem B is the following

Corollary C. The set of $\Theta$-positive representations $\text{Hom}_{\Theta-\text{pos}}(\pi_1(S), G)$ is an open subset of the set of all homomorphism $\text{Hom}(\pi_1(S), G)$.

We now consider the set $\text{Hom}^*(\pi_1(S), G)$ of homomorphisms $\rho$ of $\pi_1(S)$ in $G$ that do not factor through a parabolic subgroup of $G$, not even when restricted to a finite index subgroup of $\pi_1(S)$. We show in Proposition 6.1 that $\text{Hom}_{\Theta-\text{pos}}(\Gamma, G)$ is a subset of $\text{Hom}^*(\pi_1(S), G)$. We show

Theorem D. The set of $\Theta$-positive representations $\text{Hom}_{\Theta-\text{pos}}(\pi_1(S), G)$ is a union of connected components of $\text{Hom}^*(\pi_1(S), G)$.

Note that special $\Theta$-positive representations arise from positive embeddings of $\text{SL}_2(\mathbb{R})$ into $G$. These positive embeddings of $\text{SL}_2(\mathbb{R})$ can be characterized and classified explicitly in terms of the embedding of the nilpotent generator of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. They have the property that the embedding induces a positive map from $\mathbb{P}^1(\mathbb{R})$ into $F_\Theta$. We call the image of such a map a positive circle. These circles play an important role in some of our arguments. Precomposing a positive embedding $\text{SL}_2(\mathbb{R}) \to G$ with a discrete embedding of $\pi_1(S)$ into $\text{SL}_2(\mathbb{R})$, we obtain a $\Theta$-positive representation. We call such representations $\Theta$-positive Fuchsian representations.
Recently, Bradlow, Collier, García-Prada, Gothen, and Oliveira [5] developed the theory of magical $\mathfrak{sl}_2$-triples, which is very closely related to the theory of $\Theta$-positivity. In fact a real simple Lie group is associated to a magical $\mathfrak{sl}_2$-triple if and only if it admits a $\Theta$-positive structure. Using methods from the theory of Higgs bundles, they then show that for any simple Lie group $G$, admitting a $\Theta$-positive structure; there exists a non-empty open and closed subset (i.e. a union of connected components) $\mathcal{P}_c(S,G)$ in $\text{Rep}^+(\pi_1(S)) = \text{Hom}^+(\pi_1(S),G)/G$, where $\text{Hom}^+(\pi_1(S),G)$ denotes the set of reductive representations, such that $\mathcal{P}_c(S,G)$ contains $\Theta$-positive representations. They furthermore show that these $\mathcal{P}_c(S,G)$ are contained in $\text{Rep}^+(\pi_1(S),G) = \text{Hom}^+(\pi_1(S),G)/G$ [5, Theorem A, Theorem E]. Using this, we deduce from Theorem D

**Theorem E.** Let $G$ be a simple Lie group that admits a $\Theta$-positive structure. Then the non-empty open and closed subsets $\mathcal{P}_c(S,G)$ in $\text{Rep}^+(\pi_1(S),G)$ consist entirely of $\Theta$-positive representations.

In particular, the union of connected components $\mathcal{P}_c(S,G)$ of $\text{Rep}^+(\pi_1(S),G)$ consists entirely of $\Theta$-Anosov representations, and hence of discrete and faithful representations, and we deduce Theorem A as a corollary.

Observe that the same phenomenon already occurred for Hitchin components: Hitchin introduced the Hitchin component [23] for real split groups using Higgs bundle methods. From his interpretation one gets that all representations in Hitchin components are in $\text{Hom}^+(\pi_1(S),G)$ [25, Lemma 10.1]. This last observation is then used in the proof by Labourie [25] as well as Fock and Goncharov [11] that these components consist entirely of discrete and faithful representations.

Note however that in our case Theorem E does not readily imply that the set $\mathcal{P}_c(S,G)$ agrees with the set of $\Theta$-positive representations, even though we expect this to be true [32, 5]. Theorem E also does not imply that the set of $\Theta$-positive representations is open and closed in $\text{Hom}^+(\pi_1(S),G)/G$, a property we conjectured in [32, Conjecture 19].

In the case when $G$ is locally isomorphic to $\text{SO}(p,q)$, $p < q$, Beyrer and Pozzetti [1] recently proved the closedness of the set of $\Theta$-positive Anosov representations in $\text{Hom}(\pi_1(S),G)$, thus by Theorem B also the closedness of the set of $\Theta$-positive representations. They derive this as a consequence of a family of collar lemmas and fine properties of the boundary map they establish. A similar approach should also work for the exceptional family of Lie groups admitting a $\Theta$-positive structure.
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Outline of the paper: In Section 2, we recall the necessary algebraic material from [20, 18] and introduce the main definitions: diamonds, positive configurations, positive circles and positive maps. In Section 3, we prove three propositions concerning combinatorial properties of configurations, proper inclusion of diamonds and extension of positive maps. In Section 4, we introduce the diamond metric on diamonds and establish its properties. With these preparations we prove Theorem B and Corollary C in Section 5, Theorem D in Section 6, and Theorem E in Section 7.

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2. Definitions

2.1. Lie algebra notations. Let $G$ be a simple group.

The roots of $G$ are the nonzero weights of the adjoint action of a Cartan subspace $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of $G$. They form a root system $\Sigma \subset \mathfrak{a}^*$ (nonreduced in some cases) and the choice of a vector space ordering on $\mathfrak{a}^*$ gives rise to the set $\Sigma^+$ of positive roots, and the set $\Delta$ of simple roots. The $\alpha$-weight space will be denoted by $\mathfrak{g}_\alpha \subset \mathfrak{g}$.

The parabolic subgroups of $G$ are the subgroups conjugated to one of the standard parabolic subgroups $P_\Theta$ (for $\Theta$ varying in the subsets of $\Delta$); namely $P_\Theta$ is the normalizer in $G$ of the Lie algebra $\mathfrak{u}_\Theta := \bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Delta \setminus \Theta)} \mathfrak{g}_\alpha$. A parabolic subgroup is its own normalizer so that the space $F_\Theta$ of parabolic subgroups conjugated to $P_\Theta$ is isomorphic to $G/P_\Theta$. The unipotent radical of $P_\Theta$ is the subgroup $U_\Theta = \exp(\mathfrak{u}_\Theta)$.

Two parabolic subgroups $P$ and $P'$ are called transverse or opposite if their intersection $P \cap P'$ is a reductive subgroup (i.e. its unipotent radical is trivial); this is equivalent to having $\text{UniRad}(P) \cap P' = \{1\}$. In that case, there exists $\Theta \subset \Delta$ such that the pair $(P, P')$ is conjugated to $(P_\Theta, P_\Theta^{\text{opp}})$ where $P_\Theta^{\text{opp}}$ is the normalizer of $\bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Delta \setminus \Theta)} \mathfrak{g}_{-\alpha}$. The intersection $L_\Theta := P_\Theta \cap P_\Theta^{\text{opp}}$ is a Levi factor of $P_\Theta$ (and of $P_\Theta^{\text{opp}}$).

We will always work with a parabolic subgroup $P \simeq P_\Theta$ such that $P_\Theta$ is conjugated to its opposite; in this situation it makes sense to look at transverse elements in $F_\Theta \simeq G/P_\Theta$. In particular we will use the following notation, for $x$ in $F_\Theta$,

$$P_x := \text{Stab}(x),$$
$$U_x := \text{UniRad}(P_x),$$
$$\Omega_x := \{ y \in F_\Theta \mid y \text{ is transverse to } x \},$$
$$S_x := F_\Theta \setminus \Omega_x .$$

We will sometimes use that, if $a$ and $b$ are transverse points, then $L_{a,b} := P_a \cap P_b$ is a Levi factor of $P_a$ and $P_b$. Recall that $\Omega_x$ is an open transitive orbit of $U_x$, and that $S_x$ is a proper algebraic subvariety of $F_\Theta$.

Given a point $a$ in $F_\Theta$, a unipotent pinning, or $U$-pinning at $a$, is an identification $s$ of $U$ with $U_a$. Observe that there are finitely many up to the action of $L_x$ and that the associated Weyl group acts on those $U$-pinnings.

2.2. Cones and semigroup.

**Definition 2.1.** [20] A positive structure with respect to $F_\Theta$ (or a $\Theta$-positive structure) is a nonempty open semigroup $N$ of $U = U_\Theta$ that is
invariant by conjugation by the connected component \( L^o := L^o_\Theta \) of \( L_\Theta \) and such that, for any \( h, k \in \overline{N} \), if \( hk = 1 \), then \( h = k = 1 \) (i.e. the only invertible element in \( \overline{N} \) is the identity element).

We shall see that given \( a \) and \( b \) transverse in \( F_\Theta \) and an identification of \( U \) with \( U_a \) which sends \( N \) to a subgroup \( N_a \) of \( U_a \), then \( N_a \cdot b \) is a connected component of \( \Omega_a \cap \Omega_b \).

In [20], Guichard and Wienhard have classified all \( \Theta \)-positive structures. In particular, they have shown that this new notion of positivity encompasses Lusztig’s total positivity for real split groups, and positivity in the context of Shilov boundaries of Hermitian symmetric subspaces. Moreover they have shown that new classes of groups and parabolics appear; also, up to natural symmetries, the semigroup \( N \) in the definition is unique.

We first present some conclusion of their construction that we are going to use in this paper, then concentrate on the notions of diamonds and positive configurations that play a crucial role in this paper.

2.2.1. The parameterization of the positive semigroup. The papers [20, Theorem 4.5] and [18] give a precise description of the possible parameterizations of the semigroup \( N \). We recall here the material necessary for our purpose.

Fact 2.2. There exist \( N \geq 1 \), and \( C \) a \( L^o \)-invariant cone in \( (u_\Theta)^N \) such that the map

\[
(u_\Theta)^N \rightarrow U \\
(x_1, \ldots, x_N) \mapsto \exp(x_1) \cdots \exp(x_N)
\]

induces by restriction a \( L^o \)-equivariant diffeomorphism

\[
\Psi : C \rightarrow N .
\]

Furthermore the stabilizer in \( L^o \) of any point \( h \) in \( C \) is a compact subgroup of \( L^o \).

The closure \( \overline{C} \) is also \( L^o \)-invariant and the definition 2.1 implies that the cone \( \overline{C} \) is salient, i.e. the intersection of \( \overline{C} \) and \( -\overline{C} \) is reduced to \( \{0\} \).

Remark 2.3. A little more precisely, we have that \( N \) is the length of the longest element in a finite Coxeter group associated with \( \Theta \) and, for every \( i = 1, \ldots, N \), there is a \( L^o \)-invariant cone \( C_i \) contained in an \( L \)-irreducible factor of \( u_\Theta \) (and \( C_i \) is open in that factor) and such that \( C = C_1 \times C_2 \times \cdots \times C_N \).
2.3. **Diamonds.** Let $a$ and $b$ be two transverse points in $\mathbf{F}_\Theta$.

**Definition 2.4.** A *diamond* with extremities $a$ and $b$, associated to a U-pinning $s_a$ at $a$, is the subset

$$s_a(N) \cdot b .$$

The terminology *diamond* was coined in [27] in the context of $G = \text{SO}(2, n)$. To give an idea, in that context $\mathbf{F}_\Theta$ is covered by charts which are identified with the Minkowski space $\mathbb{R}^{1,n-1}$. Then, in a suitable chart, a diamond is the intersection of the future time cone $F^+$ of $a$, with the past time cone $F^-$ of $b$.

By definition, we observe that a diamond makes sense for a real closed field and is a semi-algebraic set.

We list some first properties of diamonds that are proved in [20].

**Proposition 2.5.**

1. A diamond with extremities $a$ and $b$ is a connected component of $\Omega_a \cap \Omega_b$.

2. Given a diamond $s_a(N) \cdot b$, there exists a U-pinning $s_b$ so that

$$s_a(N) \cdot b = s_b(N) \cdot a .$$

3. Given a diamond $V_{s_b(N)} \cdot a$ with extremities $a$ and $b$, the opposite diamond is $V^* = s_b(N^{-1}) \cdot a$. A diamond and its opposite are disjoint. Finally any point in $V$ is transverse from any point in $V^*$.

As a consequence of the proposition, if $c$ is an element in a diamond with extremities $a$ and $b$, will denote by

1. $V_c(a,b)$ the unique diamond containing $c$ with extremities $a$ and $b$,

2. $V^c(a,b) = V_c(a,b)$ the diamond opposite to the diamond containing $c$,

**Proof of Proposition 2.5.** The first item is a consequence of [20, Theorem 4.7]. The second item is a consequence of [18, section 6].

The third item comes from [20, Remark 4.9]. In particular, if $x \in V$, then $x = s_b(n) \cdot a$ with $n \in \mathbb{N}$, while if $y \in V^*$, then $y = s_b(m^{-1}) \cdot a$ with $m \in \mathbb{N}$. Thus

$$x = s_b(nm) \cdot y .$$

Since $\mathbb{N}$ is a semigroup, this means that $x$ belongs to a diamond with extremities $y$ and $b$. By the first point $x$ is transverse to $y$. □

As an immediate consequence of the semigroup property we obtain the following result that we shall use freely:
Lemma 2.6 (Nesting property). Let $c$ be a point in a diamond with extremities $(a, b)$. Then there exist unique diamonds $V(a, c)$ and $V(c, b)$ with extremities $(a, c)$ and $(c, b)$ respectively so that

$$V(a, c) \subset V_c(a, b), \ V(c, b) \subset V_c(a, b).$$

![Figure 1. The nesting of $V(c, b)$ in $V(a, b)$](image)

Note that, as a consequence of the third item of Proposition 2.5 and of the inclusion properties of diamonds, we have

$$V(a, c) \cap V(c, b) = \emptyset.$$ 

Proof. Let us write $V(a, b) = N_b \cdot a = N_a \cdot b$ and consider the diamonds

$$V(c, b) = N_b \cdot c, \ V(a, c) = N_a \cdot c.$$ 

By construction $c = n_b \cdot a = n_a \cdot b$ with $n_b \in N_b$ and $n_c \in N_b$. By the semigroup property

$$N_b \cdot n_b \subset N_b, \ N_a \cdot n_a \subset N_a,$$

which leads to the inclusions

$$V(a, c) \subset V_c(a, b), \ V(c, b) \subset V_c(a, b).$$

Using the fact that there are exactly two diamonds with extremities $a$ and $c$, and that these two diamonds are disjoint, the uniqueness property follows easily. □
2.4. Positive configurations. The following definition plays a central role in this article:

**Definition 2.7 (Positive configuration).** Let us equip \( \{1, \ldots, p\} \) with the usual cyclic order, with \( p \geq 3 \). We say the configuration \((a_1, \ldots, a_p)\) is **positive**, if there exists diamonds \( V_{i,j} \) with extremities \((a_i, a_j)\) for all \( i \neq j \) so that

1. \( V_{i,j} = V_{j,i}^* \)
2. \( a_i \) belongs to \( V_{j,k} \), if \((j, i, k)\) is oriented,
3. we have \( V_{i,j} \subset V_{k,m} \), if \((k, i, j, m)\) is oriented.

![Figure 2. A positive 5-configuration and some diamonds](image)

Proposition 3.1 will give easier criteria to understand positive triples and quadruples and show that the definition is equivalent to the definition given in the introduction.

Observe that the choice of \( V_{i,j} \) is forced once the cyclic order is chosen.

By construction, every subconfiguration of a positive configuration is positive. On the real projective line, a configuration is positive exactly if it is cyclically oriented.

Moreover
Proposition 2.8. Positivity of a configuration is invariant under cyclic and order reversing permutations. In particular

1. to be positive for a triple is invariant under all permutations,
2. to be positive for a quadruple is invariant under the dihedral group.

Proof. The definition is invariant under cyclic transformation. If \( \sigma_0 \) is the reverse ordering, we choose the new diamonds \( V_{\sigma_0(i),\sigma_0(j)} = V_{i,j} \). \( \square \)

Finally, since the Definition 2.7 involves each time to check at most 4 indices, to check that a configuration is positive, it is enough to check that subquadruples are positive. In the next section we will give more properties of positive configurations.

2.5. Positive circles and \( \text{PSL}_2(\mathbb{R}) \). Let \( H \) be a subgroup in \( G \) isomorphic to \( \text{PSL}_2(\mathbb{R}) \). An \( H \)-circle in \( F_\Theta \) is a closed \( H \)-orbit, it can be parameterized by a circle map which is a \( \text{PSL}_2(\mathbb{R}) \)-equivariant from \( P^1(\mathbb{R}) \) to \( F_\Theta \). The group \( H \) is proximal if it contains a proximal element in \( F_\Theta \).

Proposition 2.9 (Positive circle). Given a positive structure, there exists \( \mathcal{H} \), a \( G \)-orbit of pairs \((H, C)\) so that \( H \) is a subgroup of \( G \) isomorphic to \( \text{PSL}_2(\mathbb{R}) \), \( C \) is an \( H \)-circle, satisfying the following properties

1. \( H \) has a compact centralizer,
2. \( H \) is proximal and \( C \) is the set of attractive fixed points of hyperbolic elements of \( H \),
3. given transverse points \( a \) and \( b \) in \( F_\Theta \), there exists \((H, C)\) in \( \mathcal{H} \) passing through \( a \) and \( b \), if \( c \) is a point in \( C \) different from \( a \) and \( b \), then \( V_c(a,b) \cap C \) is an arc with extremities \( a \) and \( b \).

Proof. It is enough to construct one such group isomorphic to \( \text{PSL}_2(\mathbb{R}) \) and to use the \( G \)-action that is transitive on the pairs of transverse points in \( F_\Theta \). For this, one just picks the Lie subgroup associated with a \( \text{sl}_2 \)-triple given by the Jacobson–Morozov theorem applied to the nilpotent element \( n = \log(h) \) for \( h \) a fixed element in the semigroup \( N \). \( \square \)

We fix once and for all such a class \( \mathcal{H} \) and call for \((H, C)\) in \( \mathcal{H} \), \( H \) a positive \( \text{PSL}_2(\mathbb{R}) \) and \( C \) a positive circle. As an important example of positive configuration, we have

Proposition 2.10. Any cyclically ordered configuration of points in a positive circle is positive.
Proof. It is enough to prove the results for triples and quadruples.

Let first \((a_0, a_1, a_2)\) be an oriented triple on a circle \(C\). By the last property of circles, \(a_{i+1}\) belongs to a diamond with extremities \((a_i, a_{i+2})\).

Let us define (where indices are taken modulo 3)
\[
V_{i,i+2} := V_{a_{i+1}}(a_i, a_{i+2}), V_{i,i+1} := V^*_{i+1,i}.
\]

Then property (2) of Definition 2.7 is obviously satisfied and the triple is positive. Let now consider \((a_0, a_1, a_2, a_3)\) an oriented quadruple on a circle. Observe that from the last property of circles we have
\[
V_{a_{i+2}}(a_i, a_{i+3}) = V_{a_{i+1}}(a_i, a_{i+3}).
\]

We now define
\[
W_{i,i+2} := V_{a_{i+1}}(a_i, a_i + 2),
W_{i,i+3} := V_{a_{i+2}}(a_i, a_{i+3}) = V_{a_{i+1}}(a_i, a_{i+3}),
W_{i,i+1} := W^*_{i+1,i}.
\]

The uniqueness part in Lemma 2.6 proves the positivity of \((a_0, a_1, a_2, a_3)\). □

2.6. Positive maps.

Definition 2.11 (Positive map). Let \(S\) be a cyclically ordered set containing at least three points. Then a map \(f\) from \(S\) to \(\Theta\) is positive if the image of every ordered quadruple is a positive quadruple, and the image of every ordered triple is a positive triple.¹

It follows in particular that the image of every cyclically ordered configuration by a positive map is a positive configuration.

Observe that by Proposition 2.10, the circle map of a positive circle is positive.

3. Properties of positivity

We prove in this section, three main propositions concerning positivity.

- The first one, Proposition 3.1, gives various combinatorial properties of positive triples, quadruples and configurations.
- The second one, Proposition 3.8, gives information about the limit of diamonds included in a given diamond.
- The last one, Proposition 3.13, shows that positive maps share the property of monotone maps: they coincide on a dense subset with a left-continuous positive map.

¹When \(S\) has more than three points, the second requirement follows from the first
3.1. **Combinatorics of positivity.** The following items of the next proposition are fundamental properties of positive triples and quadruples.

- The first one gives an easy criterion for positivity of triples, while the second and third for quadruples. In particular, this shows that the definition of positivity given in the introduction is equivalent to Definition 2.7.
- The fourth one gives a recursive way to build positive configuration and, formally, the previous property can be deduced from it.
- The fifth and sixth gives “exclusion” properties that are not used in this paper.

We are going to prove this proposition and its corollary in the context of a group defined over \( \mathbb{R} \), although by Tarski Theorem, the statements will be true over every real closed field.

**Proposition 3.1 (Combinatorial properties).**

1. Assume \((a, b)\) are transverse and \(c\) belongs to a diamond with extremities \(a\) and \(b\), then \((a, b, c)\) is positive.
2. Assuming \((a, c, b)\) is positive and \(d\) belongs to \(V^a(c, b)\), then \((a, c, d, b)\) is positive.
3. Assuming \((a, x_0, b)\) and \((a, y_0, b)\) are positive then \((a, x_0, b, y_0)\) is positive if and only if \(V_{x_0}(a, b) = V_{y_0}(a, b)\).
4. Assume that \((x_0, x_1, \ldots, x_p)\) is a positive configuration and that \(y \in V^{x_2(x_0, x_1)}\) then
   \[(x_0, y, x_1, \ldots, x_p)\]
   is a positive configuration.
5. [Exclusion for triples] Assume \((a, b, c, d)\) is positive, then \((a, c, b, d)\) is not positive.
6. [Exclusion for quadruples] Let \(x_0, x_1,\) and \(x_2\) be three points so that \((a, x_i, b)\) is positive, then the three quadruples \((a, x_0, b, x_1)\), \((a, x_1, b, x_2)\), and \((a, x_2, b, x_0)\) cannot be all positive.

The proof of this proposition and of the next Corollary 3.2 will be given in Section 3.1.3. It is important to remark that all these properties are true for configurations on \(\mathbb{P}^1(\mathbb{R})\).

Although the last three properties are not used in the sequel they are important in the study of positivity. Finally we have,

**Corollary 3.2 (Necklace property).** Let \((a, b, c)\) be a positive triple. Let \(\alpha, \beta\) and \(\gamma\) be elements of \(V_a(b, c), V_b(a, c)\) and \(V_c(a, b)\) respectively. Then \((\alpha, \beta, \gamma)\) is a positive triple.
3.1.1. Triples and quadruples.

**Lemma 3.3.** A triple \((a_0,a_1,a_2)\) is positive if and only if \(a_0, a_1, a_2\) all belongs to diamonds with extremities \((a_1,a_2), (a_2,a_0)\) and \((a_0,a_1)\) respectively.

**Proof.** We just need to prove the “if” part. Let then
\[
V_{i,i+1} := V_{a_{i+2}}(a_i, a_{i+1}) \quad V_{i,i+2} := V_{a_{i+1}}(a_i, a_{i+2}).
\]
Observe that
\[
V_{i,i+1} = V_{a_{i+2}}(a_i, a_{i+1}) = V_{a_{i+2}}(a_i) = V_{i+1,i}^*.
\]
Then Lemma 2.6 provides all the necessary inclusions needed to prove the triple is positive. □

The following lemma gives a way to go from positive triple to positive quadruple.

**Lemma 3.4.** Let \((a_0,a_1,a_2,a_3)\) be a quadruple. Assume that all subtriples are positive. Then the quadruple \((a_0,a_1,a_2,a_3)\) is positive, if and only if
\[
a_i \in V_{a_{i+2}}(a_{i+1}, a_{i+3}) \quad (1)
\]
\[
a_{i+2} \in V_{a_{i+1}}(a_i, a_{i+3}) \quad (2).
\]

**Proof.** The “only if” part follows from the definition. It remains to prove the “if” part. Let (indices are taken modulo 4)
\[
V_{i,i+1} := V_{a_{i+2}}(a_i, a_{i+1}) \quad V_{i,i+2} := V_{a_{i+1}}(a_i, a_{i+2}) = V_{a_{i+2}}(a_i, a_{i+2}) \quad V_{i,i+3} := V_{a_{i+1}}(a_i, a_{i+3}) = V_{a_{i+2}}(a_i, a_{i+3}),
\]
where in the second line we used the hypothesis (1), while in the last we used the hypothesis (2). Hence by definition
\[
V_{i,i+1} = V_{i+1,i}^* \quad V_{i,i+2} = V_{i+2,i}^*.
\]
It thus follows that for all \(i\) and \(j\),
\[
V_{i,j} = V_{i,j}^* \quad (3).
\]
From the positivity of all subtriples \((a_i,a_{i+1},a_{i+2})\) and the previous lemma, we get the inclusions
\[
V_{i,i+1} \subset V_{i,i+2} \quad V_{i,i+1} \subset V_{i-1,i+1} \quad (4).
\]

From the positivity of the triple \((a_i,a_{i+1},a_{i+3})\) we get the inclusions
\[
V_{i,i+1} = V_{a_{i+3}}(a_i, a_{i+1}) \subset V_{a_{i+1}}(a_i, a_{i+3}) = V_{i,i+3} \quad (5)
\]
\[
V_{i+1,i+3} = V_{a_{i+1}}(a_{i+1}, a_{i+3}) \subset V_{a_{i+1}}(a_{i}, a_{i+3}) = V_{i,i+3} \quad (6).
\]
Similarly the positivity of the triple \((a_i, a_{i+2}, a_{i+3})\) yields
\[
V_{i+2,i+3} = V^a(a_{i+2}, a_{i+3}) \subset V_{a_{i+2}}(a_i, a_{i+3}) = V_{i+3}, \tag{7}
\]
\[
V_{i,i+2} = V^a_i(a_i, a_{i+2}) \subset V_{a_i}(a_i, a_{i+3}) = V_{i+3}. \tag{8}
\]
All together the equation (3) as well as the inclusions (4), (5), (6), (7), and (8), prove that \((a_0, a_1, a_2, a_3)\) is a positive quadruple. □

3.1.2. Deformation lemmas. We need to prove some deformation lemmas.

**Lemma 3.5 (Deforming triples).** Let \(a(t), b(t), \) and \(c(t)\) be continuous arcs from \([0, 1]\) to \(F_\Theta\) so that

(1) for all \(t\) in \([0, 1]\), the points in the triple \((a(t), b(t), c(t))\) are all pairwise transverse,

(2) the triple \((a(0), b(0), c(0))\) is positive.

Then, for all \(t\), \((a(t), b(t), c(t))\) is a positive triple.

**Proof.** The hypothesis tells us that
\[
c(t) \in \Omega_{a(t)} \cap \Omega_{b(t)}, \quad a(t) \in \Omega_{c(t)} \cap \Omega_{b(t)}, \quad b(t) \in \Omega_{a(t)} \cap \Omega_{c(t)}.\n\]
By hypothesis, denoting \(V(e, d)\) a diamond with extremities \(e\) and \(d\).
\[
c(0) \in V(a(0), b(0)), \quad a(0) \in V(c(0), b(0)), \quad b(0) \in V(c(0), a(0)).
\]
We now use the fact that a diamond with extremities \(c\) and \(d\) is a connected component of \(\Omega_c \cap \Omega_d\) by Proposition 2.5. Then for all \(t\) by continuity
\[
c(t) \in V(a(t), b(t)), \quad a(t) \in V(c(t), b(t)), \quad b(t) \in V(c(t), a(t)).
\]
Thus the result follows from the definition. □

Similarly

**Lemma 3.6 (Deforming quadruples).** Let \(\gamma\) and \(\eta\) be continuous arcs from \([0, 1]\) to \(F_\Theta\) so that there exists \(a\) and \(b\) in \(F_\Theta\) satisfying

(1) for all \(t\) in \([0, 1]\), the points in the quadruple \((a, \gamma(t), b, \eta(t))\) are all pairwise transverse,

(2) the quadruple \((a, \gamma(0), b, \eta(0))\) is positive.

Then, for all \(t\), \((a, \gamma(t), b, \eta(t))\) is a positive quadruple.
Proof. By applying Lemma 3.5, we obtain that all the subtriples of \((a, \gamma(t), b, \eta(t))\) are positive. By Lemma 3.4, we only need to check that
\[
a \notin V_b(\gamma(t), \eta(t)) , \quad b \notin V_a(\gamma(t), \eta(t)) ,
\]
\[
\gamma(t) \notin V_{\gamma(0)}(a, b) , \quad \eta(t) \notin V_{\eta(0)}(a, b) ,
\]
\[
\gamma(t) \in V_a(\eta(t), b) , \quad b \in V_{\gamma(t)}(a, \eta(t)) ,
\]
\[
\eta(t) \in V_b(\gamma(t), b) , \quad a \in V_{\eta(t)}(b, \gamma(t)) .
\]
We use again the fact that a diamond with extremities \(c\) and \(d\) is a connected component of \(\Omega_c \cap \Omega_d\) by Proposition 2.5. Then, the statement follows by continuity. □

Finally we also have as an immediate consequence of the connectedness of the positive cone:

Lemma 3.7 (Connectedness). Let \(a\) and \(b\) two transverse points. Let \(C\) be any positive circle through \(a\) and \(b\).

1. Assume \(c\) is so that \((a, c, b)\) is positive. Then there is an arc \(t \mapsto c(t)\) from \([0, 1]\) to \(V_c(a, b)\) connecting \(c = c(0)\) to \(c(1)\) so that \((a, c(1), b)\) is a positive triple on \(C\).

2. Assume furthermore that \(d\) belongs to \(V^a(c, b)\) then there are a path \(t \mapsto c(t)\) as in the previous item and a path \(t \mapsto d(t)\) from \([0, 1]\) to \(V_c(a, b)\), so that \(d(t) \in V^a(c(t), b)\) and \((a, c(1), d(1), b)\) are on \(C\).

Proof. Using the U-pinning \(s_b\) at \(b\), we identify \(N\) with a positive semigroup \(N_b\) in \(U_b\). Recall that we have \(V_c(a, b) = N_b \cdot a\). The first point just follows from the connectedness of the positive cone \(N_b\). For use in the second point we take a path which is constant for \(t > 1/2\).

Recall that \(d = m_0 \cdot c\), with \(m_0 \in N_b\). Let then for \(t \in [0, 1/2]\),
\[
d(t) = m_0 \cdot c(t) ,
\]
then we have by the semigroup property \(d(t) \in V^a(c(t), b)\). Observe also that \(d(0) = d\). Then for \(t \in [1/2, 1]\), we have \(c(t) = c(1/2)\) and we choose, using the first part, a path \(t \mapsto d(t)\) with \(d(t) \in V^a(c(1/2), d)\), and so that \(d(1)\) belongs to \(C\). □

3.1.3. Proof of the combinatorial properties.

Proof of item (1) of Proposition 3.1. Assume \((a, b, c)\) satisfies the hypothesis. Let \(c(t)\) be path connecting \(c\) to a point \(d\) in \(V_c(a, b)\) obtained by Lemma 3.7. On a circle, a triple is positive if and only if the three points are distinct, the result thus follows from Lemma 3.5. □

Proof of item (2) of Proposition 3.1. From the connectedness Lemma 3.7 we obtain deformation \(c(t), d(t)\) so that \((a, c(t), d(t), b)\) are pairwise
transverse, \( c(0) = c, d(0) = d, (a, c(1), d(1), b) \) on a circle and \( d(1) \in V_a(c(1), b) \). In particular \( (a, c(1), d(1), b) \) is positive and thus by the deforming Lemma 3.6, \( (a, c, d, b) \) is positive. \( \square \)

Proof of item (3) of Proposition 3.1. The “only if” part follows from the definition. Then for the “if” part we find by the first part of Lemma 3.7, paths \( x(t) \) and \( y(t) \) in \( V_{x_0}(a, b) \) and \( V_{y_0}(a, b) \) respectively, so that \( (x(0), y(0)) = (x_0, y_0) \) and \( x(1), y(1) \) are on a circle passing through \( a \) an \( b \). Then \( (a, x(1), b, y(1)) \) is positive and so is \( (a, x_0, b, y_0) \) by Lemma 3.6, since \( x(t) \) and \( y(t) \) are transverse thanks to Proposition 2.5. \( \square \)

Proof of item (4) of Proposition 3.1. This is an immediate consequence of item (3) and the fact that in order to check the positivity of a configuration one only needs to check the positivity of subtriples and subquadruples. \( \square \)

Proof of item (5) of Proposition 3.1. If \( (a, b, c, d) \) is positive and \( (a, c, b, d) \) is positive we have strict inclusions \( V^a(c, d) \subset V^a(b, d) \) and \( V^a(b, d) \subset V^a(c, d) \), hence a contradiction. \( \square \)

Proof of item (6) of Proposition 3.1. Assume that \( V(a, x_0, b, x_1) \) is positive. Let \( V := V_{x_0}(a, b) \) and \( V^* = V_{x_1}(a, b) \) the opposite to \( V \). If both \( (a, x_1, b, x_2) \) and \( (a, x_0, b, x_2) \) are positive then we get that \( x_2 \) belongs to both \( V \) and \( V^* \), which is a contradiction. \( \square \)

Proof of the necklace property (Corollary 3.2). Let us first remark that from item (4) of Proposition 3.1, applied three times, the configuration

\[
(a, \gamma, b, \alpha, c, \beta),
\]

is positive. Thus \( (\gamma, \alpha, \beta) \) is positive. \( \square \)

3.2. Inclusion of diamonds.

Proposition 3.8 (Bounded property). Let \( (a, b, d) \) be a positive triple and let \( c \in V_b(a, d) \). Assume that there exists sequences \( \{b_m\}_{m \in \mathbb{N}} \) and \( \{c_m\}_{m \in \mathbb{N}} \), converging respectively to \( b \) and \( c \) and such that, for all \( m \), \( (a, b_m, c_m, d) \) is a positive quadruple. then

\[
\lim_{m \to \infty} \left( V^d(b_m, c_m) \right) \subset V_c(a, d).
\]

In particular,

Corollary 3.9 (Inclusion). Let \( (a, b, c, d) \) be a positive quadruple points in \( F_{\Theta} \). Then

\[
\overline{V^d(b, c)} \subset V_b(a, d).
\]

Proposition 3.8 will be proved in Section 3.2.2.
3.2.1. Preliminaries on circles. Let then \( V(a,d) \) be a diamond and \( C \) a positive circle passing through \( a \) and \( d \) so that \( C \) is an orbit of a positive \( H \), isomorphic to \( \text{PSL}_2(\mathbb{R}) \).

- Let \( \delta = \{ \delta_t \mid t \in \mathbb{R} \} \) be the 1-parameter group in \( H \) for which \( d \) is the attractive fixed point and \( a \) is the repulsive fixed point.
- Let \( \gamma = C \setminus \{a,d\} \).
- Let \( \delta \) be the 1-parameter group in \( H \) for which \( d \) is the attractive fixed point and \( a \) is the repulsive fixed point.

The result of this section is

**Proposition 3.10.** For any \( e \) in \( \gamma \), we have \( \overline{V^a(e,d)} \subset \Omega_a \).

This proposition implies Corollary 3.9:

**Proof of the Corollary 3.9.** Applying Proposition 3.10, we get that

\[
\overline{V^a(b,c)} \subset \overline{V^a(b,d)} \cap \overline{V^d(c,a)} \subset \Omega_d \cap \Omega_a .
\]

Since \( V_c(a,d) \) is a connected component of \( \Omega_d \cap \Omega_a \) containing \( V^a(b,c) \) it follows that

\[
\overline{V^a(b,c)} \subset V_c(a,d) .
\]

In order to prove to prove Proposition 3.10, we introduce the following sets

\[
J(e) := \overline{V^a(e,d)} \cap S_a , \quad F(e) := J(e) \cap F , \quad O^+ := \{ x \in F_\emptyset \mid \lim_{t \to \infty} (\delta_t(x) = d) \} , \quad O^- := \{ x \in F_\emptyset \mid \lim_{t \to -\infty} (\delta_t(x) = a) \} .
\]

We will first prove that the sets \( J(e) \) and \( F(e) \) are empty. We first prove the following lemma

**Lemma 3.11.** For any \( e \) in \( \gamma \),

1. \( J(e) \) is invariant by the semigroup \( \delta^+ := \{ \delta_t \mid t > 0 \} \), i.e. \( \delta_t(J(e)) \subset J(e) \) for all \( t > 0 \),
2. \( F(e) \) is independent of the choice of \( e \),
3. if \( J(e) \) is not empty, so is \( F(e) \),
4. for all \( c \) in \( O^- \) and \( b \) in \( F_\emptyset \) so that \( (a,b,c,d) \) is a positive quadruple then

\[
F(e) \subset S_c \cap S_b .
\]

**Proof.** We prove the first point. By Proposition 2.10 for \( t > 0 \)

\[
V^a(\delta_t(e),d) \subset V^a(e,d) .
\]

This implies that \( V^a(e,d) \) is invariant by \( \delta^+ \) and so is \( J(e) \).
The second point is a consequence that $F$ is pointwise fixed by $\delta_t$:

$$J(\delta_t(e)) \cap F = \delta_t(J(e)) \cap F = J(e) \cap F.$$ 

The third point is a consequence of the compactness of $F_\Theta$ which implies that any non-empty closed set in $F_\Theta$ invariant by the semigroup $\delta^+$ has a non-empty intersection with $F$.

Let us prove the last point now. First given $b$ and $c$ as in the proposition, then

$$V^a(c, d) \cap \gamma \neq \emptyset.$$ 

Indeed, by the definition of $O^-$ we know that $\lim_{t \to -\infty} V^a(\delta_{-t}(c), d)$ contains a diamond with extremities $a$ and $d$. Thus for $t$ large enough

$$V^a(\delta_{-t}(c), d) \cap \gamma \neq \emptyset,$$

and the result holds since $C$ is invariant by $\delta_t$. Let thus

$$e \in V^a(c, d) \cap \gamma.$$ 

Remark that by Lemma 2.6,

$$V^a(e, d) \subset V^a(c, d).$$ 

and thus

$$F(e) \subset J(e) = \left(V^a(e, d) \cap S_a\right) \subset \left(V^a(c, d) \cap S_a\right). \quad (9)$$ 

Now, we remark that by positivity of $(a, b, c, d)$,

$$V^a(c, d) \subset V_c(a, b) \subset \Omega_a.$$ 

Thus

$$\overline{V^a(c, d)} \cap S_a = \left(\overline{V^a(c, d)} \cap V^a(c, d)\right) \cap S_a. \quad (10)$$ 

But since $V^a(c, d)$ is a connected component of the open set

$$\Omega_c \cap \Omega_d = F_\Theta \setminus (S_c \cup S_d),$$

we get

$$\left(\overline{V^a(c, d)} \setminus V^a(c, d)\right) \subset (S_c \cup S_d). \quad (11)$$

Combining inclusions (9), (10) and (11), we get that

$$F(e) \subset (S_c \cup S_d). \quad \square$$

We can now prove Proposition 3.10, in other words that $J(e)$ is empty. By item (4) of Lemma 3.11, it suffices to show that $F(e)$ is empty. The fact that $F(e)$ is empty follows from item (3) of Lemma 3.11 and the following result.
Lemma 3.12. Let $Q$ be a subset of $F_\Theta$. Assume that there exists nonempty open sets $U$ and $V$ so that for all $c$ in $U$, and all $d$ in $V$,

$$Q \subset S_c \cup S_d,$$

then $Q$ is empty.

Proof. Let $q$ be in $Q$ and set $Z = S_q$.

Then $Z$ is a proper closed Zariski subset of $F_\Theta$. Observe that if $u < Z$, then $q < S_u$.

On the other hand we can find $c$ in the nonempty set $U \setminus Z$ and $d$ in the nonempty set $V \setminus Z$, and by hypothesis $q \in (S_u \cup S_v)$. This shows that $q \neq q$

and concludes the proof. □

3.2.2. Proof of the boundedness Proposition 3.8.

Proof. We use the notation of the previous paragraph. Let $C$ be a circle though $a$ and $d$. Let $\gamma := V_c(a,d) \cap C$. Since being positive is an open condition for quadruples, we can find $e$ and $f$ in $\gamma$ so that $(e, b_m, c_m, f)$ is positive for $m$ large enough as well as $(a, e, f, d)$. Thus

$$V^a(b_m, c_m) \subset V^a(e, f)$$

Applying Proposition 3.10, we get that

$$\overline{V^a(b_m, c_m)} \subset \overline{V^a(e, f)} \subset V_c(a,d),$$

which easily implies the result. □

3.3. Left and right limits of positive maps. Our main result is

Proposition 3.13 (Existence of left and right limits). Let $S$ be a totally ordered set and $\phi$ be a positive map from $S$ to $F_\Theta$.

Let $\{b_n\}_{n \in \mathbb{N}}$ be a sequence of points in $S$ so that $a < b_n \leq b_{n+1} \leq b < c$, for $a$, $b$, and $c$ in $S$.

Then $\{\phi(b_n)\}_{n \in \mathbb{N}}$ converges to a point $y$ in $V_b(a,c)$. Symmetrically, let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of points so that $c < a \leq a_{n+1} \leq a_n < b$. Then $\{\phi(a_n)\}_{n \in \mathbb{N}}$ converges to a point $y$ in $V_a(c,b)$.

As an immediate corollary, we show that positive maps defined on dense subsets extend to positive maps. More precisely:

Corollary 3.14 (Extension of positive maps). Let $A$ be dense subset in $[0, 1]$. Assume that we have a positive map $\xi$ from $A$ to $F_\Theta$. Then there exists

• a unique left-continuous positive map $\xi^+$ from $[0, 1]$ to $F_\Theta$ so that $\xi$ coincide with $\xi^+$ on a dense subset of $A$. 
• a unique right-continuous positive map $\xi$ from $[0, 1]$ to $F_\Theta$ so that $\xi$ coincide with $\xi_-$ on a dense subset of $A$.

Moreover,

• for any triple of pairwise distinct points $(x, y, z)$ in $[0, 1]$

\[
(\xi_a(x), \xi_b(y), \xi_c(z)) ,
\]

is a positive triple for any choice of $a, b,$ and $c$ in $\{+,-\}$,

• if $\{x_m\}_{m \in \mathbb{N}}, \{z_m\}_{m \in \mathbb{N}}$ are sequences in $[0, 1]$ converging to $y$, so that $x_m < y < z_m$, then

\[
\lim_{m \to \infty} \xi_{a_m}(x_m) = \xi_-(y), \quad \lim_{m \to \infty} \xi_{b_m}(z_m) = \xi_+(y),
\]

for any sequences $\{a_m\}_{m \in \mathbb{N}}$ and $\{b_m\}_{m \in \mathbb{N}}$ in $\{+,-\}$.

Proof of Proposition 3.13. Let us define $x_n = \phi(b_n)$. We can write $x_n = m_n \cdot x_{n-1}$, with $m_n \in N_{\phi(c)}$. Thus by induction we have

\[
x_n = m_n \cdots m_1 \cdot x_0 .
\]

But we know that $V^{\phi(c)}(x_0, \phi(b))$ is a relatively compact region of $\Omega_{\phi(c)}$ by Proposition 3.8. Thus $\{x_n\}_{n \in \mathbb{N}}$ is a relatively compact region of $\Omega_{\phi(c)}$. It follows that

\[
\pi_n = m_n \cdots m_1,
\]

is a bounded sequence in $N_{\phi(c)}$. We now prove that this sequence converges. Assume that we have subsequences that converge to different limits $u$ and $v$. After extracting further subsequence, we may find subsequences

\[
q_i = \pi_{n_i}, \quad p_i = \pi_{m_i}, \quad \text{with } n_i \leq m_i
\]

such that $\{q_i\}_{i \in \mathbb{N}}$ converges to $u$ and $\{p_i\}_{i \in \mathbb{N}}$ converges to $v$. It follows that $u = w_1 \cdot v$ with $w_1 \in \overline{N}_{\phi(c)}$. Symmetrically, $v = w_0 \cdot u$ with $w_0 \in \overline{N}_{\phi(c)}$. It follows that $w_0 \cdot w_1 = 1$, thus $w_0$ and $w_1$ are invertible in the closed semigroup $\overline{N}_{\phi(c)}$, hence equal to the identity. In particular $u = v$ and $\{x_n\}_{n \in \mathbb{N}}$ converges.

The proof for the sequence $\{a_n\}_{n \in \mathbb{N}}$ is symmetric. \qed

3.3.1. Positivity and continuity. In some cases, it suffices to show that the image of every triple is positive

Proposition 3.15 (Triples suffice). Let $\phi$ be a continuous map from an interval $I$ to $F_\Theta$ so that the image of every ordered triple is positive, then $\phi$ is positive.
Proof. Let \((x, y)\) be a pair of distinct points. Let \(z\) such that \(x < z < y\). Let \(W_z(x, y) = V_{\phi(z)}(\phi(x), \phi(y))\). Since \([x, y]\) is connected \(W_z(x, y)\) does not depend on \(z\) and we denote it \(W(x, y)\). It follows by construction that if \(a < b < c < d\), then \(W(b, d) \subset W(a, d)\). In particular \(\phi(a)\) does not belong to \(W(b, d)\). Thus \(\phi(c) \in W(b, d) = V_{\phi(a)}(\phi(b), \phi(d))\). This means by item (2) of Proposition 3.1 that \((\phi(a), \phi(b), \phi(c), \phi(d))\) is positive, which is what we wanted to prove. \(\square\)

4. Triples, Tripods and metrics

In this section, we construct for every positive triple \((a, b, c)\) a complete metric on the diamond \(V_c(a, b)\) in Definition 4.9. We also show that this metric satisfy contraction properties (Propositions 4.11 and 4.12).

We first do it for special triples that we call tripods.

4.1. Tripods and metrics.

**Definition 4.1.** A tripod is a triple of pairwise distinct triple of points on a positive circle. A tripod is always positive. If \(\tau = (x, t, y)\) is a tripod, we write

\[
\tau^- = x, \quad \tau^0 = t, \quad \tau^+ = y, \quad \tau := (y, t, x), \quad V_\tau := V_{\tau^0}(\tau^-, \tau^+).
\]

Let \(\mathcal{T}_0\) be the set of tripods. Observe that the stabilizer of any tripods is compact and that \(G\) acts transitively on the space of tripods. Let then \(d\) be a \(G\)-invariant metric on \(\mathcal{T}_0\).

4.1.1. Tripods and the parameterization. Let us consider as in Paragraph 2.2.1, the \(L^0\)-equivariant parameterization \(\Psi : C \to N\) where \(C\) is a convex cone in \(u_N^\Theta\) for some \(N\) (and \(\Psi\) is constructed from the exponential map). Note that \(\Psi\) extends continuously to a map \(\overline{C} \to U\) that is also \(L^0\)-equivariant.

Let \(h\) be the element of \(G\) corresponding to the unipotent associated to the preferred \(\text{SL}_2(\mathbb{R})\) — see Proposition 2.10. Let \(K_h\) be the stabilizer of \(h\) in \(L^0 = L^0_\Theta\). Since the stabilizer of a positive triple is compact, it follows that \(K_h\) is compact.

If now \(x\) and \(y\) are transverse points in \(F_\Theta\) and \(\sigma\) is an isomorphism of \(U\) with \(U_y\) (i.e. \(\sigma\) is a \(U\)-pinning at \(y\)), then the map

\[
\Psi^{\sigma} : C \mapsto F_\Theta, \quad u \mapsto \sigma \circ \Psi(u) \cdot x,
\]

is a parameterization of the diamond \(V_c(x, y)\) with \(t := \Psi^{\sigma}(h)\). We then define

**Definition 4.2.** Given a tripod \(\tau = (x, t, y)\) a parameterization of the diamond \(V_\tau\), is a map \(\Psi_\tau\) of the form \(\Psi^\sigma\) so that \(\Psi^\sigma(h) = t\).
From the definition follows

**Proposition 4.3.** Given a tripod \( \tau \), a parameterization of the diamond exists and is unique up to post-composition by the stabilizer of \( \tau \), or up to precomposition by \( K_h \).

The next proposition is crucial; it insures that a sequence of parameterizations of diamonds associated with tripods converges to the constant map as soon as one sequence in the image converges, precisely

**Proposition 4.4** *(Contraction in corners).* Let \( \{ \tau_m \}_{m \in \mathbb{N}} \) be a sequence of tripods, with \( \tau_m = (x_m, t_m, y) \).

Assume that \( \{x_m\}_{m \in \mathbb{N}} \) converges to a point \( x_0 \) transverse to \( y \). Assume that there exists a converging sequence \( \{k_m\}_{m \in \mathbb{N}} \) in the cone \( C \), so that

\[
\lim_{m \to \infty} \Psi_{\tau_m}(k_m) = x_0 .
\]

Then for any convergent sequence \( \{k'_m\}_{m \in \mathbb{N}} \) in \( C \),

\[
\lim_{m \to \infty} \Psi_{\tau_m}(k'_m) = x_0 .
\]

**Proof.** Let \( \{\tau_m\}_{m \in \mathbb{N}}, \{x_m\}_{m \in \mathbb{N}}, \) and \( \{t_m\}_{m \in \mathbb{N}} \) as in the statement. Since \( x_0 \) is transverse to \( u \), by replacing \( \tau_m \) by \( \mu_m \tau_m \) where \( \{\mu_m\}_{m \in \mathbb{N}} \) is a converging sequence in \( U_y \), we may as well assume that \( \{x_m\}_{m \in \mathbb{N}} \) is constant and equal to \( x \).

Using the fact that \( G \) acts transitively on tripods, let us write \( t_m = g_m \cdot t_0 \), with \( g_m \) in the connected component of the identity of \( L_{x,y} = P_x \cap P_y \). Thus

\[
\Psi_{\tau_m}(h) = g_m \cdot t_0 = g_m \cdot \Psi_{\tau_0}(h) = g_m \sigma(\Psi(h)) \cdot x .
\]

Note that the U-pinning \( \sigma : U \to U_y \) extends to an isomorphism \( \sigma \) of \( P \) with \( P_y \). We denote by \( g^0_m \) the element of \( L^0 \) such that \( \sigma(g^0_m) = g_m \). Up to maybe precomposing by an element of \( K_h \), we may assume that \( \Psi_{\tau_m} \) is the map \( k \mapsto \sigma(\Psi(g^0_m \cdot k)) \cdot x \).

Therefore we have, for any \( \{\ell_m\}_{m \in \mathbb{N}} \) in \( \overline{C} \), that \( \{\Psi_{\tau_m}(\ell_m)\}_{m \in \mathbb{N}} \) converges to \( x = x_0 \) if and only if the sequence \( \{g^0_m \cdot \ell_m\}_{m \in \mathbb{N}} \) converges to 0 in \( \overline{C} \).

For any \( y \) in \( \overline{C} \), let

\[
K(y) := \overline{C} \cap (y - \overline{C}) ,
\]

where \( u - A := \{u - x \mid x \in A\} \).

Since \( \overline{C} \) is salient, \( K(y) \) is compact for any \( y \).

From the previous discussion, we get that the sequence \( \{c_m = g^0_m \cdot k_m\}_{m \in \mathbb{N}} \) converges to 0. Thus, using again the fact that \( \overline{C} \) is salient,
for every positive real $R$, the sequence of compact sets $\{K(R\cdot c_m)\}_{m\in\mathbb{N}}$ converges to $\{0\}$.

Let now $\{k'_m\}_{m\in\mathbb{N}}$ be a sequence converging in $\mathbb{C}$. Since by hypothesis $\{k_m\}_{m\in\mathbb{N}}$ converges in $\mathbb{C}$, there exists a positive real $R$ such that, for all $m$, $R\cdot k_m - k'_m$ belongs to $\overline{C}$. In other words: $k'_m$ belongs to $R\cdot k_m - \overline{C}$. Thus, for all $m$, $g_0^m \cdot k'_m$ belongs to $K(R\cdot c_m)$. Hence the sequence $\{g_0^m \cdot k'_m\}_{m\in\mathbb{N}}$ converges to 0. This means that the sequence $\{\Psi_{\tau_m}(k'_m)\}_{m\in\mathbb{N}}$ converges to $x = x_0$ as wanted. □

4.1.2. Tripod metrics. We choose once and for all a Euclidean distance $d_0$, associated to the Riemannian $g_0$, on $\mathbb{C}$ which is invariant by $K_0$. Note that this distance extends to $\mathbb{C}$.

**Definition 4.5.** Given a tripod $\tau = (x, t, y)$, let $\Psi_{\tau}$ be a parameterization of $V_{\tau}$, let

$$g^+_{\tau} := \Psi_{\tau}^* g_0, \quad g^-_{\tau} := \Psi_{\tau}^* g_0, \quad g^\tau := g^+_{\tau} + g^-_{\tau},$$

as well as $d^+_{\tau}, d^-_{\tau}$, and $d_{\tau}$ the associated distance so that

$$d^\tau_{\tau} \leq d_{\tau} \leq d^+_{\tau} + d^-_{\tau}.$$

The metric $g^\tau_{\tau}$ is the diamond metric (for tripods) on $V_{\tau}$, while $d_{\tau}$ is the diamond distance.

We then have

**Proposition 4.6 (Completeness).** The diamond metric is independent of the choice of the parameterization and only depends on $\tau$. Moreover, $d_{\tau}$ is complete on $V_{\tau}$.

**Proof of Proposition 4.6.** The independence on the parameterization is a consequence of Proposition 4.3 and the fact that $d_0$ itself is invariant under the group $K_0$.

Let us now prove the completeness. Let $\{u_m\}_{m\in\mathbb{N}}$ be a Cauchy sequence for $d_{\tau}$, then it is a Cauchy sequence for both $d^+_{\tau}$ and $d^-_{\tau}$. It follows that $\{v_m\}_{m\in\mathbb{N}}$ and $\{w_m\}_{m\in\mathbb{N}}$ are both Cauchy sequences, where

$$v_m = \Psi_{\tau}^{-1}(u_m), \quad w_m = \Psi_{\tau}^{-1}(u_m).$$

Since $\overline{C}$ is complete with respect to the metric $d_0$, there exist $v$ and $w$ in $\overline{C}$ so that

$$\lim_{m\to\infty} v_m = v, \quad \lim_{m\to\infty} w_m = w.$$

Since by construction $\Psi_{\tau}$ and $\Psi_{\tau}$ extend continuously to the closure of $\mathbb{C}$,

$$\Psi_{\tau}(v) = \Psi_{\tau}(w) = \lim_{m\to\infty} u_m =: u.$$
Obviously $u$ belongs to $V_z$. Thus by construction $u = ny \cdot x$ and $u = nx \cdot y$ where $ny$ belongs to $U_y$ and $nx$ belongs to $U_x$. Since $x$ and $y$ are transverse, it follows that $u$ is transverse to both $y$ and $x$. Thus $u$ belongs to $V_\tau$. Hence $d_\tau$ is complete. $\Box$

**Proposition 4.7 (Contraction for tripods).** Let $\{\tau_m\}_{m \in \mathbb{N}}$ be a sequence of tripods. Assume that $V_{\tau_{m+1}} \subset V_{\tau_m}$ and that

\[
\bigcap_{m \in \mathbb{N}} V_{\tau_m} = \{z\} .
\]

(14)

For any positive $R$, let $V_{\tau_m}(R)$ be the ball of radius $R$ and center $\tau_m^0$ with respect to $d_{\tau_m}$. Then on $V_{\tau_m}(R)$, we have

\[
g_{\tau_0} \leq k_m g_{\tau_m},
\]

with $\{k_m\}_{m \in \mathbb{N}}$ converging to zero.

**Proof.** Since $G$ acts transitively on the space of tripods $T_0$, it follows that $\tau_m = h_m \tau_0$, for some $h_m$ in $G$. Since our construction is $G$-equivariant, we observe that $g_{\tau_m} = h_m g_{\tau_0}$. Then, we take

\[
k_m = \sup \left\{ \frac{g_{\tau_0}(w, w)}{g_{\tau_m}(w, w)} \mid w \in TV_{\tau_m}(R) \right\}
\]

\[
= \sup \left\{ \frac{g_{\tau_0}(w, w)}{g_{\tau_0}(Th_m^{-1} w, Th_m^{-1} w)} \mid w \in TV_{\tau_m}(R) \right\}
\]

\[
= \sup \left\{ \frac{g_{\tau_0}(Th_m(v), Th_m(v))}{g_{\tau_0}(v, v)} \mid v \in TV_{\tau_0}(R) \right\} .
\]

(15)

The hypothesis (14) says that $h_m$ converges uniformly on every compact set of $V_{\tau_0}$ to the constant map. It follows that $\{h_m\}_{m \in \mathbb{N}}$ also converges $C^1$ to zero on any compact set in $V_{\tau_0}$. Thus, equality (15) shows that $\{k_m\}_{m \in \mathbb{N}}$ converges to zero. $\Box$

### 4.2. Positive triples, tripods and metric.

Our goal is to construct a complete metric on the diamond associated to a positive triple and to prove a generalization of the contraction property (Proposition 4.4).

#### 4.2.1. Approximating triples: the tripod norm.

We will first approximate in a rough sense positive triples by tripods. For any positive triple $t = (x, z, y)$, let

\[
K(t) := \inf \{d_\tau(z, \tau^0) \mid \tau \in T_0, (\tau^+, \tau^-) = (x, y)\} .
\]
We call $K(t)$ the tripod norm. Observe that $K(t)$ depends continuously on $t$, and that the tripod norm vanishes for tripods. Let also

$$
D(t, K_0) := \{ \tau \in T_0 \mid (\tau^+, \tau^-) = (x, y), d_\tau(z, \tau_0) \leq K(t) \},
$$

$$
D(t) := D(t, K(t)).
$$

**Proposition 4.8.**

1. Given $K_0 \geq K(t)$, the set $D(t, K_0)$ is compact and non-empty.
2. $K(t) = 0$ if and only if $t$ is a tripod,
3. For any $K_0$ there exists a constant $A$, such that if $t$ is a tripod with $K(t) \leq K_0$, then for $\tau_0$ and $\tau_1$ in $D(t)$, then

$$
g_{\tau_0} \leq A g_{\tau_1}.
$$

**Proof.** Let $t = (a, b, c)$ be a positive triple. Let $\{\tau_m\}_{m \in \mathbb{N}}$ be a sequence of tripods such that $(\tau_m^+, \tau_m^-) = (a, c)$ and

$$
\{d_\tau(b, \tau_m^0)\}_{m \in \mathbb{N}},
$$

is bounded. Let $\{g_m\}_{m \in \mathbb{N}}$ be a sequence of elements in $L_{a,c} = P_a \cap P_c$ so that $\{g_m^{-1}(\tau_m^0)\}_{m \in \mathbb{N}}$ is constant and equal to $\tau^0$. Let $\tau := (a, \tau^0, c)$. It follows that

$$
\{d_\tau(g_m(c), \tau^0)\}_{m \in \mathbb{N}},
$$

is bounded. Since $d_\tau$ is a complete metric, the sequence $\{g_m(c)\}_{m \in \mathbb{N}}$ —after extracting a subsequence— converges. Thus $\{\tau_m\}_{m \in \mathbb{N}}$ converges to a tripod $\tau_\infty$, with $\tau_\infty^0$ in $V_t$. Since $D(t, K_0)$ is non-empty for $K_0 > K(t)$, it follows that $D(t, K_0)$ is compact. The result for $K_0 = K(t)$ follows from the fact that

$$
D(t) = \cap_{K_0 > K(t)} D(t, K_0).
$$

The second assertion is an immediate consequence of the first. The third follows from the first as a consequence of the $G$-equivariance of the assignment $\tau \mapsto d_\tau$. □

4.2.2. **The diamond metric for triples.** The following definition is one of the goals of this section

**Definition 4.9.** Let $t$ be a positive triple – which is not a tripod. The **diamond metric** (for triples) $g_t$ is the Riemannian metric on $V_t$ defined by

$$
g_t := \frac{1}{\text{Vol}(D(t))} \int_{D(t)} g_\tau \, d\text{Vol}(\tau),
$$

where $\text{Vol}$ is a $G$-invariant volume form on $T_0$. The associated distance is the **diamond metric** $d_t$.

As an immediate corollary of Proposition 4.8 and Proposition 4.6, we have
Corollary 4.10. The diamond metric takes finite values and is complete. Moreover if the sequence of positive triples \( \{t_m\}_{m \in \mathbb{N}} \) converges to a tripod \( \tau \), then \( \{g_{t_m}\}_{m \in \mathbb{N}} \) converges to \( g_{\tau} \) on every compact of the diamond \( V_{\tau} \).

The following are two contractions properties of the diamond metric that we shall use in the sequel,

**Proposition 4.11 (Contraction).** Let \( \{t_m\}_{m \in \mathbb{N}} \) be a sequence of positive triples, with \( t_m = (a_m, b_m, c_m) \). Assume that the sequence \( \{K(t_m)\}_{m \in \mathbb{N}} \) of tripod norms is bounded. Assume that \( V_{t_{m+1}} \subset V_{t_m} \) and that

\[
\bigcap_{m \in \mathbb{N}} V_{t_m} = \{z\}.
\]

For any positive \( R \), let \( V_{t_m}(R) \) be the ball of radius \( R \) and center \( a_m \) with respect to \( d_{t_m} \). Then on \( V_{t_m}(R) \), we have

\[
g_{t_0} \leq k_m \cdot g_{t_m},
\]

with \( \{k_m\}_{m \in \mathbb{N}} \) converges to zero.

**Proposition 4.12 (Contraction in corners).** Let \( \{t_m\}_{m \in \mathbb{N}} \) be a sequence of positive triples, where \( t_m = (a_m, b_m, c_m) \). Assume that

1. the sequence \( \{K(t_m)\}_{m \in \mathbb{N}} \) of tripod norms is bounded,
2. \( \{a_m\}_{m \in \mathbb{N}} \) and \( \{c_m\}_{m \in \mathbb{N}} \) converge to transverse points \( a \) and \( c \),
3. \( \{u_m\}_{m \in \mathbb{N}} \) is a sequence of elements of \( F_\Theta \), so that \( u_m \) belongs to \( V_{t_m} \), the sequence \( \{d_{t_m}(b_m, u_m)\}_{m \in \mathbb{N}} \) uniformly bounded, and \( \lim_{m \to \infty} u_m = a \).

Then \( \lim_{m \to \infty} (b_m) = a \).

**Proof of Proposition 4.11.** By the Definition 4.9 of the diamond metric for triples, and Proposition 4.8 it follows that we can find a constant \( A \), such that for all \( m \), we can find a tripod \( \tau_m \), with

\[
d_{\tau_m}(\tau_m^0, b_m) \leq A, \quad \frac{1}{A}g_{\tau_m} \leq g_{t_m} \leq Ag_{\tau_m}.
\]

The result now follows from the corresponding proposition for tripods: Proposition 4.7.

**Proof of Proposition 4.12.** Let \( \{t_m\}_{m \in \mathbb{N}} \) be a sequence of positive triples satisfying the hypothesis of the proposition where \( t_m = (t_m^-, t_m^0, t_m^+) \). By the first hypothesis and Proposition 4.8, we can find a constant \( A \), a sequence of tripods \( \{\tau_m\}_{m \in \mathbb{N}} \) with \( \tau_m^\pm = t_m^\pm \) and so that

\[
d_{\tau_m} \leq A d_{t_m}.
\]

In particular, we have that \( \{d_{\tau_m}(t_m, \tau_m^0)\}_{m \in \mathbb{N}} \) and \( \{d_{\tau_m}(u_m, \tau_m^0)\}_{m \in \mathbb{N}} \) are uniformly bounded. The result now follows from a double application of Proposition 4.4. Indeed, since \( \{d_{\tau_m}(t_m, \tau_m^0)\}_{m \in \mathbb{N}} \) is uniformly
bounded, it follows that $t_m = \Psi_{\tau_m}(k_m)$ with $\{k_m\}_{m \in \mathbb{N}}$ bounded. Hence by Proposition 4.4, with $k_m' = h$, yields that
\[
\lim_{m \to \infty} (\tau_m) = a.
\]
Applying again Proposition 4.4 to $\{k_m'\}_{m \in \mathbb{N}}$ with $\Psi_{\tau_m}(k_m') = u_m$ yields that
\[
\lim_{m \to \infty} (u_m) = a.
\]
This concludes the proof. □

5. Positive representations are Anosov

In this section we introduce the notion of positive representations of a surface group. We then show that any $\Theta$-positive representation is $\Theta$-Anosov, establishing Theorem B and Corollary C from the introduction.

**Definition 5.1 (Positive representations).** Let $G$ be a simple Lie group admitting a $\Theta$-positive structure. A representation $\rho: \pi_1(S) \to G$ is $\Theta$-positive if there exists a non-empty $\pi_1(S)$-invariant subset $A$ of $\partial_\infty \pi_1(S)$ and a positive $\rho$-equivariant map $\xi$ from $A$ to $F_\Theta$. The map $\xi$ is called the positive boundary map of $\rho$ and the image of $\xi$ is the positive limit curve of $\rho$.

We also say that a representation is positive if it is $\Theta$-positive for some $\Theta$.

To establish the Anosov property we first extend the positive boundary map to a left- and a right-continuous boundary map using Corollary 3.14. We prove that these extensions are continuous, and then deduce the Anosov property using the contraction property of the diamond metric (Proposition 4.11).

5.1. **Properness.** The following definition will be used several times in the sequel: an application $f$ defined on a subset $A$ of a topological set $X$, with value in some topological set $Y$ is bounded if for every compact set $K$ in $X$, $f(A \cap K)$ is empty or relatively compact.

**Lemma 5.2.** Let $A$ be a dense set in the circle. Let $\phi$ be a positive map from $A$ to $F_\Theta$. Let $A_3^3$ be the intersection of $A^3$ with the set $T_{S_3}$ of pairwise distinct triple in $S^1$. Then $\phi^3$ is bounded as a map from $A_3^3$ in $T$.

**Proof.** Let $\chi = (x_1, x_2, y_1, y_2, z_1, z_2)$ be an oriented sextuplet in $S_1$. Let $I_\chi$ be the subset of $(S^1)^3$ given by
\[
I_\chi = \{(X, Y, Z) \mid x_1 < X < x_2 < y_1 < Y < y_2 < z_1 < Z < z_2\}.
\]
Let $\phi$ be a positive map and

$$K := \phi^3 (I_x \cap A).$$

It is enough to show that $\overline{K} \subset T$, where the closure is taken in $F^3_\Theta$.

Thus by the necklace Corollary 3.2, if

$$(x_0, y_0, z_0) \in F^3_\Theta,$$

we get

$$(x_1, y_1, z_1, x_2, y_2, z_2) \in \overline{K}.$$ 

Thus if

$$(x, y, z) \in \overline{K},$$

then

$$x \in V^{a_i}(c_2, a_0), \quad y \in V^{b_i}(a_2, b_0), \quad z \in V^{c_i}(b_2, c_0).$$

Then

$$a \in V^{a_i}(c_2, a_0), \quad b \in V^{b_i}(a_2, b_0), \quad c \in V^{c_i}(b_2, c_0).$$

Thus using the inclusion Corollary 3.9, we get

$$a \in V^{b_i}(a_1, a_1), \quad b \in V^{c_i}(a_1, b_1), \quad c \in V^{c_i}(b_1, c_1).$$

Thus by the necklace Corollary 3.2, $(a, b, c)$ is a positive triple and thus belongs to $T$. This concludes the proof. $\square$

Let $\rho$ be a $\Theta$-positive representation, $A$ a non-empty $\pi_1(S)$-invariant subset of $\partial_\infty \pi_1(S)$ and $\xi : A \to F_\Theta$ the positive $\rho$-equivariant boundary map. Then, by Corollary 3.14, there exists a unique right-continuous $\rho$-equivariant boundary map $\xi^+ : \partial_\infty \pi_1(S) \to F_\Theta$ and a unique left-continuous $\rho$-equivariant boundary map $\xi^- : \partial_\infty \pi_1(S) \to F_\Theta$, coinciding with the map $\xi$ on some dense subset of $A$.

**Proposition 5.3.** Let $\rho : \pi_1(S) \to G$. Let $\xi^+$ be a positive $\pi_1(S)$-invariant map from $\partial_\infty \pi_1(S)$ to $F_\Theta$.

Let $T_{\pi_1(S)}$ be the set of pairwise distinct triple in $\partial_\infty \pi_1(S)$, $T$ the set of positive triples in $F_\Theta$. Let $\Xi^+$ be the map from $T_{\pi_1(S)}$ to $T / G$, defined by

$$\Xi^+(x, y, z) := [\xi^+(x), \xi^+(y), \xi^+(z)].$$

Then the image of $\Xi^+$ is relatively compact.

**Proof.** The map $\Xi^+$ is invariant by the diagonal action of $\pi_1(S)$. The result follows then from Lemma 5.2 using the fact that $\pi_1(S)$ acts cocompactly on $T_{\pi_1(S)}$. $\square$
5.2. **An a priori bound on the tripod norm.** For any positive triple \( t \), let \( K(t) \) be the real constant and \( D(t) \) the set of tripods, both defined in paragraph 4.2. Then Proposition 5.3 implies an a priori bound on the tripod norm.

**Proposition 5.4.** Let \( \rho: \pi_1(S) \to G \). Let \( \xi \) be a \( \rho \)-equivariant positive map from a \( \pi_1(S) \)-invariant dense subset of \( \partial_{\infty}\pi_1(S) \) to \( F_{\Theta} \). Then there exists a constant \( A \) so that for all triple of pairwise distinct points \( t \) in the closure of \( \xi(\partial_{\infty}\pi_1(S)) \), we have

\[
K(t) \leq A.
\]

**Proof.** This is an immediate consequence of Proposition 5.3 and the fact that \( K \) is a continuous function on \( T^+ \).

5.3. **Continuity of equivariant positive maps.** Let \( \xi^+: \partial_{\infty}\pi_1(S) \to F_{\Theta} \) be the right-continuous \( \rho \)-equivariant boundary map, and \( \xi^-: \partial_{\infty}\pi_1(S) \to F_{\Theta} \) the left-continuous \( \rho \)-equivariant boundary agreeing on a dense subset of \( A \).

Let \( T_{\pi_1(S)} \) be the set of triples of pairwise distinct points of \( \partial_{\infty}\pi_1(S) \). For \( t = (x, y, z) \) in \( T_{\pi_1(S)} \), let us define

\[
\tau(t) = (\xi^+(x), \xi_+(y), \xi_+(z)).
\]

**Lemma 5.5.** The \( \pi_1(S) \)-invariant function \( f \) defined by

\[
f(x, y, z) = d_{\tau(t)}(\xi_+(y), \xi_-(y))
\]

is bounded.

**Proof.** The function \( f \) is \( \pi_1(S) \)-invariant, it thus suffices to check that \( f \) is bounded on a fundamental domain \( U \) for the action of \( \pi_1(S) \) on \( \partial_{\infty}\pi_1(S)^3 \).

We know that \( \xi^+_3 \) is bounded. Thus there exists a compact \( K \) in \( T \) which contains \( \xi^+_3(U) \) and thus \( \xi^+_3(U) \) and \( \xi^-_3(U) \).

It follows that for any complete distance \( d_0 \) on \( F_{\Theta} \) there exists \( \epsilon > 0 \) such that for any \( t = (x, z, y) \) in \( U \)

\[
d_0(\partial V_{\tau(t)}, \xi_-(z)) \geq \epsilon,
\]

\[
d_0(\partial V_{\tau(t)}, \xi_+(z)) \geq \epsilon.
\]

Recall now that \( d_{\tau(t)} \) is complete on \( V_y(x, z) \). Since \( K \) is compact, and \( \tau(t) \) belongs to \( K \), it follows that all the metrics \( d_{\tau(t)} \) are equivalent on compact sets of the diamond. This implies that for all \( t \) in \( U \),

\[
d_{\tau(t)}(\xi_+(y), \xi_-(y)) < A,
\]

where \( A \) depends on \( K \) and \( \epsilon_0 \). In other words, \( f \) is bounded by \( A \).

**Lemma 5.6.** The map \( \xi_+ \) is left-continuous.
Proof. Let \( \{x_m\}_{m \in \mathbb{N}} \) be a sequence of points of \( \partial_\infty \pi_1(S) \) so that \((x_m, y, z)\) is oriented with respect to the orientation on \( \partial_\infty \pi_1(S) \), and that \( \{x_m\}_{m \in \mathbb{N}} \) converges to \( z \). Let \( t_m = (\xi_+(x_m), \xi_+(x), \xi_+(y)) \).

Recall that by Corollary 3.14, \( \{\xi_+(x_m)\}_{m \in \mathbb{N}} \) converges to \( \xi_-(x) \).

We now apply Proposition 4.12 to the following setting:

\[
a_m = \xi_+(x_m), \quad b = \xi_+(x), \quad u_m = \xi_-(x), \quad c_m = \xi_+(y).
\]

Since \( \{dt_m(b_m, u_m)\}_{m \in \mathbb{N}} = \{dt_m(\xi_+(x), \xi_-(x))\}_{m \in \mathbb{N}} \), is bounded by Lemma 5.5 and \( \{K(t_m)\}_{m \in \mathbb{N}} \) is bounded by Proposition 5.4, we get that

\[
\lim_{m \to \infty} \xi_+(x_m) = \xi_+(x).
\]

This proves that \( \xi_+ \) is left-continuous. \( \square \)

As a consequence we obtain

**Proposition 5.7.** Let \( \rho : \pi_1(S) \to G \) be a positive representation and \( \xi \) the positive \( \rho \)-invariant boundary map from a \( \pi_1(S) \)-invariant dense subset of \( \partial_\infty \pi_1(S) \) to \( F_\Theta \). Then \( \xi \) extends to a \( \rho \)-equivariant positive continuous map.

5.4. The Anosov property. We are now in the position that we can prove Theorem B in the introduction. More precisely we show

**Proposition 5.8.** Let \( G \) admit a \( \Theta \)-positive structure. Let \( \rho : \pi_1(S) \to G \) be a positive representation and \( \xi : \partial_\infty \pi_1(S) \to F_\Theta \) the \( \rho \)-equivariant continuous positive boundary map. Then \( \rho \) is \( \Theta \)-Anosov and its boundary map is \( \xi \).

**Proof.** Let us choose an hyperbolization of \( S \) which defines a \( \pi_1(S) \)-invariant cross ratio on \( \partial_\infty \pi_1(S) \). Let us also define an orientation on \( \partial_\infty \pi_1(S) \). For any oriented positive triple \( t = (x, y, z) \), let us consider the harmonic (with respect to the cross ratio) quadruple \( q(t) = (x, y, z, w(t)) \). Let then

\[
Y_t := V_{\xi(z)}(\xi(y), \xi(w(t))).
\]

By construction \( Y_t \) is an open neighborhood of \( \xi(z) \). Moreover if \((x, t_1, t_0, y)\) is an oriented positive quadruple,

\[
Y_{t_0} \subset Y_{t_1}.
\]  \( \quad (17) \)

Finally, since \( \xi \) is continuous,

\[
\lim_{y \to z} Y_{x,y,z} = \{\xi(z)\}.
\]  \( \quad (18) \)

This is precisely the Anosov property.

Let us spell out the details. Recall that we have chosen a uniformisation of the surface. Let us now identify the space of triples in the boundary at infinity with the unit tangent bundle of the universal
Let \( \phi_s \) be the geodesic flow on \( \mathbb{UH}^2 \). Let \( F_\Theta \times \mathbb{UH}^2 \) be the trivial bundle. The action of \( \pi_1(S) \) on \( \mathbb{UH}^2 \) and \( F_\Theta \) through \( \rho \) gives rise to an action of \( \pi_1(S) \) on \( F \).

Let \( U \) be the subbundle with open fibers given by

\[
U = \{ (x, v) \in F \mid v \in \mathbb{UH}^2, \ x \in Y_v \}.
\]

The bundle \( U \) is invariant by the \( \pi_1(S) \)-action, moreover it has a canonical section \( \sigma_0 \) given by

\[
\sigma_0(x, y, z) = \xi(z).
\]

Let us lift the flow \( \{\phi_s\}_{s \in \mathbb{R}} \) to a flow \( \{\Phi_s\}_{s \in \mathbb{R}} \) on \( F \) acting trivially on the first factor. By assertion (17), for all positive \( s \)

\[
\Phi_{-s}(U) \subset U.
\]

Moreover the section \( \sigma_0 \) is invariant by \( \{\Phi_s\}_{s \in \mathbb{R}} \).

The diamond metric \( g_t \) and the diamond distance \( d_t \) on each \( Y_t \) gives a metric on each fiber of \( U \) which depends continuously on the base and is equivariant under the action of \( \pi_1(S) \).

For any \( R \), let \( U(R) \) the neighborhood of the image of the section \( \sigma_0 \), given by

\[
U(R) = \{ (x, v) \in F \mid v \in \mathbb{UH}^2, \ d_v(x, \sigma_0(v)) \leq R \}.
\]

It now follows for assertion (18) and Proposition 4.11, that for every \( u \) in \( \mathbb{UH}^2/\pi_1(S) \), there is \( s_u \) so that, for all \( (x, u) \) in \( U(R) \)

\[
\text{for all } s \geq s_u, \ g_{\Phi_s(x,u)} \circ T_{(x,u)} \Phi_{-s} \leq \frac{1}{2} g_{(x,u)}.
\]

Let now \( s \) be the real valued function on \( \mathbb{UH}^2 \) defined by

\[
s(u) = \inf \{ s_u \mid s_u \text{ satisfies assertion (19) on } U(R) \}.
\]

The function \( u \mapsto s(u) \) is upper semicontinuous and invariant under the action of \( \pi_1(S) \). Thus by compactness of \( \mathbb{UH}^2/\pi_1(S) \) the function has an upper bound \( s_0 \). Then, for all \( s \) greater than \( s_0 \), for all \( w \) in \( U(R) \)

\[
\Phi_{-s}^w g \leq \frac{1}{2} g,
\]

on \( U(R) \). In other words, the action of \( \{\Phi_{-s}\}_{s \in \mathbb{R}} \) is contracting on \( U \) and \( \sigma_0 \) is an invariant section.

Thus \( \rho \) is \( F_\Theta \)-Anosov and \( z \mapsto \xi(z) \) is the limit curve. \( \square \)

Now Corollary C in the introduction follows now directly from the openness of the set of \( \Theta \)-Anosov representation.
Remark 5.9. The definition of $\Theta$-positive representations can be made in more generality for non-elementary word hyperbolic group $\Gamma$ whose boundary admits a cyclic ordering. This holds if $\Gamma$ is a surface group, but also if $\Gamma$ is virtually free. In this case we define a $\Theta$-positive representation to be a homomorphism $\rho: \Gamma \to G$ which admits a $\rho$-equivariant positive boundary map $\xi: \partial_\infty \Gamma \to F_\Theta$. Then an appropriate extension of the arguments in this section show that $\Theta$-positive representations of $\Gamma$ are $\Theta$-Anosov.

6. Closedness

In this section we consider the space $\text{Hom}^*(\pi_1(S), G)$ of homomorphisms from $\pi_1(S)$ to $G$, which (even when restricted to a finite index subgroup) do not factor through a proper parabolic subgroup of $G$. We show that the set of $\Theta$-positive representations $\text{Hom}_{\Theta-\text{pos}}(\pi_1(S), G)$ is an open and closed subset, hence a union of connected components of $\text{Hom}^*(\pi_1(S), G)$.

Proposition 6.1. The set of $\Theta$-positive representations is a subset of $\text{Hom}^*(\pi_1(S), G)$.

Proof. Let us first note that since the centralizer of a positive triple is compact, the centralizer of a positive representation is compact as well. Let $\rho: \pi_1(S) \to G$ be a positive representation with $\rho$-equivariant positive boundary map $\xi: \partial_\infty \pi_1(S) \to F_\Theta$. Then $\rho$ is $\Theta$-Anosov with boundary map $\xi$. This remains true when restricting the representation to a finite index subgroup. To argue by contradiction we can thus assume that without loss of generality $\rho(\pi_1(S))$ is contained in a proper parabolic subgroup of $G$. We consider the semi-simplification $\rho^{ss}$ of $\rho$, which is its projection to a Levi factor of the parabolic subgroup. Then by [16, Proposition 2.39] the semi-simplification is $\Theta$-Anosov with $\rho^{ss}$-equivariant boundary map $\xi^{ss}$. Since the semisimplification is the $G$-orbit of $\rho^{ss}$ in $\text{Hom}(\pi_1(S), G)$ is the unique closed orbit in the closure of the $G$-orbit of $\rho$. There exists a path $g_t \in G$ such that $\rho^{ss}$ is the limit of $g_t \rho g_t^{-1}$. Since the boundary map $\xi^{ss}$ is transverse, Lemma 3.5 implies that the boundary map $\xi^{ss}$ is also positive as well. But this is a contradiction because the centralizer of $\rho^{ss}$ in $G$ contains the center of the Levi factor of the parabolic subgroup which is non-compact. □

Since the set of $\Theta$-positive representations is open in $\text{Hom}(\pi_1(S), G)$ by Corollary C, it is also open in $\text{Hom}^*(\pi_1(S), G)$.

We will now show

Theorem 6.2. The set of $\Theta$-positive homomorphisms is closed in the set $\text{Hom}^*(\pi_1(S), G)$. 

This implies Theorem D in the introduction.

We consider a sequence $\{\rho_m\}_{m \in \mathbb{N}}$ of $\Theta$-positive representations converging to a representation $\rho_\infty$ in $\text{Hom}^*(\pi_1(S), G)$. Let $\{\xi_m\}_{m \in \mathbb{N}}$ be the corresponding sequence of positive limit maps.

We fix a countable set $A$ in $\partial_\infty \pi_1(S)$, invariant by $\pi_1(S)$. We may now assume after extracting a subsequence, by Cantor diagonal argument, that $\{\xi_m\}_{m \in \mathbb{N}}$ converges point-wise for all elements of $A$ to a map $\xi_\infty$ from $A$ to $F_\Theta$.

We now consider the two cases,

1. there exists $x, y \in A$ such that $\xi_\infty(x)$ is transverse to $\xi_\infty(y)$,
2. for all $x, y \in A$ the flags $\xi_\infty(x)$ and $\xi_\infty(y)$ are not transverse.

If we are in case (2), Theorem A.1 applied to $H = \rho_\infty(\pi_1(S))$ implies that $\rho_\infty(\pi_1(S))$ is contained in a proper parabolic subgroup of $G$. This contradicts that $\rho_\infty$ lies in $\text{Hom}^*(\pi_1(S), G)$.

Thus we can assume that we are in case (1) and have

**Lemma 6.3.** There exists $x$ and $y$ in $A$ so that $\xi_\infty(x)$ and $\xi_\infty(y)$ are transverse.

### 6.1. Limiting diamonds

For any pairs of distinct points $x, y$ in $\partial_\infty \pi_1(S)$, let $c$ so that $(c, x, y)$ are pairwise distinct and

$$W_\infty(x, y) := \lim_{n \to \infty} V_{\xi_n(c)}(\xi_n(x), \xi_n(y))$$

Observe that $W_\infty(x, y)$ only depends on $x, y$ and the choice of a connected component of $\partial_\infty \pi_1(S) \setminus \{x, y\}$.

**Lemma 6.4.** Assume that $\xi_\infty(x)$ and $\xi_\infty(y)$ are transverse then $W_\infty(x, y)$ is a closure of a diamond with vertices $x$ and $y$ and is Zariski dense.

**Proof.** Since $\xi_\infty(x)$ and $\xi_\infty(y)$ are transverse, $W_\infty(x, y)$ is the closure of a diamond (see Proposition 3.8). It thus contains an open set, and in particular is Zariski dense. □

**Lemma 6.5.** For every pairs of distinct points $(x, y)$ and $(z, t)$ in $\partial_\infty \pi_1(S)$, after possibly extracting a subsequence.

$$W_\infty(x, y)^Z = W_\infty(z, t)^Z$$

where $M^Z$ denotes the Zariski closure of a set $M$.

**Proof.** We shall use freely the following fact. If $\gamma$ is an algebraic automorphism of a variety $V$, if $B$ is a Zariski closed subset so that $\gamma(B) \subset B$ then $\gamma(B) = B$.

We first prove that if $[u, v] \subset [w, s]$, then we have

$$W_\infty(u, v)^Z = W_\infty(w, s)^Z.$$  \hspace{1cm} (20)
Let us prove this fact. We can always find an element \( \gamma \) of \( \pi_1(S) \) such that 
\[
\gamma[w, s] \subset [u, v].
\]
Thus 
\[
\rho_\infty(\gamma)\left(\overline{W_\infty(w, s)}\right) \subset \overline{W_\infty(u, v)} \subset \overline{W_\infty(w, s)}.
\]
Thus by the initial observation we get that 
\[
\overline{W_\infty(w, s)} \subset \overline{W_\infty(u, v)} \subset \overline{W_\infty(w, s)},
\]
and thus the assertion (20) follows. Let now \( \gamma \) in \( \pi_1(S) \) so that 
\[
\gamma[x, y] \subset [x, y], \gamma[x, y] \cup [z, t] \neq \partial_\infty\pi_1(S).
\]
We can then find distinct points \( u \) and \( v \) so that 
\[
(\gamma[x, y] \cup [z, t]) \subset [u, v].
\]
Thus, applying thrice assertion (20), we have 
\[
\overline{W_\infty(x, y)} = \overline{W_\infty(\gamma \cdot x, \cdot y)} = \overline{W_\infty(u, v)} = \overline{W_\infty(z, t)}.
\]
We are now in the position to show that \( \rho_\infty \) is \( \Theta \)-positive. This follows from the following proposition:

**Proposition 6.6.** For any triple of points \( (x, y, z) \), the triple 
\[
(\xi_\infty(x), \xi_\infty(y), \xi_\infty(z)),
\]
consists of pairwise transverse points. For any positive quadruple of points \( (x, y, z, w) \), the quadruple 
\[
(\xi_\infty(x), \xi_\infty(y), \xi_\infty(z), \xi_\infty(w)),
\]
is positive.

**Proof.** Let \( x, y, \) and \( z \) be a triple of pairwise distinct points in \( \partial_\infty\pi_1(S) \). Let us denote for simplification \( x_n = \xi_n(x), y_n = \xi_n(y) \) and \( z_n = \xi_n(z) \) for \( n \) in \( \mathbb{N} \cup \{\infty\} \). We choose the diamonds by letting 
\[
V_{n}^{0} = V^{\geq n}(x_n, y_n), \quad V_{n}^{1} = V^{y_n}(x_n, z_n) \quad V_{n}^{2} = V^{y_n}(z_n, y_n).
\]
Since \( W_\infty(x, y) \), \( W_\infty(x, y) \) and \( W_\infty(x, y) \) are Zariski dense, we can pick three points \( a, b, \) and \( c \) so that 
(1) \( a \in W_\infty(x, y), b \in W_\infty(y, z), c \in W_\infty(z, x), \)
(2) \( a, b, c \) are pairwise transverse, 
(3) any point in \( \{a, b, c\} \) is transverse to any point in \( \{x_\infty, y_\infty, z_\infty\} \).
Let now pick sequences \( \{a_m\}_{m \in \mathbb{N}}, \{b_m\}_{m \in \mathbb{N}}, \text{ and } \{c_m\}_{m \in \mathbb{N}} \) in \( V_n^0, V_n^1, \) and \( V_n^2 \) respectively and converging to \( a, b, \) and \( c \) respectively.

We will now apply the necklace property several times. By Proposition 3.2, \((a_n, b_n, c_n)\) is a positive triple and since \((a, b, c)\) are pairwise transverse it follows that \((a, b, c)\) is a positive triple.

Then, since \( x_n \) belongs to \( V^b(a_n, c_n) \), it follows that \( x_\infty \) belongs to \( V^a(a, c) \). Since \( x_\infty \) is transverse to both \( a \) and \( c \), \( x_\infty \) belongs to \( V^a(a, c) \). Symmetrically \( y_\infty \) belongs to \( V^c(a, b) \), \( z_\infty \) belongs to \( V^c(c, b) \). Applying Proposition 3.2 again, \((x_\infty, y_\infty, z_\infty, w_\infty)\) is a positive triple.

The fact that the quadruple \((x_\infty, y_\infty, z_\infty, w_\infty)\) is positive, now follows from Proposition 3.1.

\[ \square \]

7. Connected Components

In this section, we prove that Theorem D implies the existence of components consisting solely of discrete faithful representations for any simple Lie group \( G \) admitting a \( \Theta \)-positive structure. More precisely, we show that the unions of the connected components \( \mathcal{P}_c(\Sigma, G) \) of \( \text{Rep}^+(\pi_1(S), G) \), (where \( \Sigma \) is a Riemann surface modelled on \( S \)) introduced in [5] using methods from the theory of Higgs bundles consist entirely of \( \Theta \)-positive representations. In particular they consist solely of discrete faithful representations.

Let us recall that for real split Lie groups \( G \), the Hitchin component was originally defined by Hitchin as the image of the Hitchin section \( \Phi \) which assigns to a tuple of holomorphic differentials on a Riemann surface \( \Sigma \) a \( G \)-Higgs bundle on \( \Sigma \). Let us denote the image of \( \Phi \) by \( \mathcal{P}(\Sigma, G) \). Through the non-abelian Hodge correspondence the set \( \mathcal{P}(\Sigma, G) \) corresponds to a subset of the representation variety \( \text{Rep}^+(\pi_1(S), G) \), which we denote by the same symbol. Hitchin showed that \( \mathcal{P}(\Sigma, G) \) is open and closed and the map \( \Phi \) gives an explicit parameterization of the components of \( \mathcal{P}(\Sigma, G) \). In the case of maximal representations a similar but more complicated parameterization of the space of maximal representations was obtained in several works by Bradlow, García-Prada and Gothen. For simple Lie groups admitting a \( \Theta \)-positive structure, the authors of [2] (see also [5] for classical groups and [12] for \( \text{Sp}(2n) \)) in a similar way define subsets \( \mathcal{P}_c(\Sigma, G) \) of the moduli space of \( G \)-Higgs bundles by giving precise parameterizations of the set. They prove that the sets \( \mathcal{P}_c(\Sigma, G) \) are open and closed in \( \text{Rep}^+(\pi_1(S), G) \). They further prove that all representations in \( \mathcal{P}_c(\Sigma, G) \) have compact centralizer and do not factor through a proper parabolic subgroup. This implies that \( \mathcal{P}_c(\Sigma, G) \) is contained in \( \text{Rep}^+(\pi_1(S), G) \).
They further show that these sets $\mathcal{P}_e(\Sigma, G)$ contain an open set of $\Theta$-positive representations.

Thus Theorem D implies

**Theorem 7.1.** The open and closed subsets $\mathcal{P}_e(\Sigma, G)$ in $\text{Hom}^+\left(\pi_1(S), G\right)/G$ consist entirely of $\Theta$-positive representations.

**Proof.** It is an immediate consequence of Theorem D and the fact that $\mathcal{P}_e(\Sigma, G) \subset \text{Rep}^+\left(\pi_1(S), G\right)$ that if a connected component of $\mathcal{P}_e(\Sigma, G) \subset \text{Rep}^+\left(\pi_1(S), G\right)$ contains a $\Theta$-positive representation, then the entire component of $\mathcal{P}_e(\Sigma, G) \subset \text{Rep}^+\left(\pi_1(S), G\right)$ consists of $\Theta$-positive representations. Here is one way to construct a $\Theta$-positive representations. Consider an embedding of $\text{SL}_2(\mathbb{R})$, such that the induced map from $\text{PSL}_2(\mathbb{R})$ to $F_\Theta$ is a positive circle, then the corresponding Fuchsian representation is positive. These Fuchsian representations can now in addition be twisted by a representation of $\pi_1(S)$ into the centralizer of this $\text{SL}_2(\mathbb{R})$ in $G$. This is called a twisted positive Fuchsian representation. We call a component of $\mathcal{P}_e(\Sigma, G)$ standard if it contains a twisted positive Fuchsian representation.

When $G$ is not locally isomorphic to $\text{Sp}_4(\mathbb{R})$, or $\text{SO}(p, p + 1)$, every component of $\mathcal{P}_e(\Sigma, G)$ is standard [5]. For $\text{Sp}_4(\mathbb{R})$ there exist exceptional connected components which do not contain any twisted positive Fuchsian representation [15], but embedding $\text{Sp}_4(\mathbb{R})$ diagonally into $\text{Sp}_8(\mathbb{R})$ these exceptional components are sent into a standard component for $\text{Sp}_8(\mathbb{R})$. Thus the exceptional components for $\text{Sp}_4(\mathbb{R})$ consist entirely $\Theta$-positive representations. For $\text{SO}(p, p + 1)$ there also exist exceptional connected components which do not contain any twisted positive Fuchsian representation, but embedding $\text{SO}(p, p + 1) \to \text{SO}(p, p + 2)$, these components are sent to standard components for $\text{SO}(p, p + 2)$ and thus they consist also entirely of $\Theta$-positive representations [5].

□

As a corollary we obtain

**Corollary 7.2.** The subsets $\mathcal{P}_e(\Sigma, G)$ of $\text{Hom}^+\left(\pi_1(S), G\right)/G$ only contains discrete faithful representations.

We expect that the sets $\mathcal{P}_e(\Sigma, G)$ coincide precisely with the set of $\Theta$-positive representations. We also wonder whether positivity is the only phenomenon for representation varieties of surface groups that leads to entire components made of discrete faithful representations.

**Appendix A. Transversality in the space of flags**

Our goal is to prove the following result of independent interest:
**Theorem A.1.** Let $P$ be a parabolic and $P^*$ an opposite parabolic. Let $H$ be a subgroup of $G$. Assume that no element in a given orbit of $H$ in $G/P$ is transverse to $P^*$. Then there is a finite index subgroup of $H$ which is included in some parabolic.

This result is also sketched in [11, Lemma 7.9], but we prefer to give an independent and simple proof based on Tits systems.

A.0.1. *Bruhat cell decomposition.* We use freely the following standard facts on Bruhat decompositions, see for example [24, 31].

Let $G$ be a semisimple Lie group over $\mathbb{R}$. Let $B$ be a minimal parabolic subgroup, $A$ a maximal split real torus, $W$ the Weyl group, $\Delta$ a choice of simple roots, that we identify with the corresponding reflections on the Lie algebra of $A$. Then we have the *Bruhat cell decomposition*

$$G = \bigsqcup_{w \in W} BwB.$$ 

Moreover, let us denote by $\ell$ the word metric on $W$ coming from the generators $\Delta$, calling $\ell(w)$ the *length* of $w$. Let $C(w)$ the Zariski closure of $BwB$ then

$$C(w) = \bigsqcup_{h \in W(w)} BwB,$$

where $W(w)$ is a subset of $W$ containing $w$ so that for any $h$ in $W(w)$ different from $w$,

$$\ell(h) < \ell(w).$$

There exists a unique element $w_0$ of longest length in $W$ and the cell $Bw_0B$ is precisely the union of all subgroups $B'$ conjugate to $B$ and transverse to it.

If $J$ is a proper subset of $\Delta$, let $W_J$ be the subgroup of $W$ generated by the reflections in $J$, then

$$P_J := \bigsqcup_{w \in W_J} BwB,$$

is a *standard parabolic subgroup* of $G$.

We shall need the following lemma which is true for general Coxeter groups by [30]. For Weyl groups, it follows from [4, Chap. VI, §1, Exercise 17-b].

**Lemma A.2.** Every involution in $W$ different from $w_0$, is conjugated to an involution of $W_J$ for some proper subset $J$ of $\Delta$. 
Proof of Theorem A.1. After conjugation we may assume that the parabolic $P$ contains the minimal parabolic subgroup $B$. Let us first notice that if $\pi$ be the projection from $G/B$ to $G/P$ and $\pi^*$ be the projection on $G/P^*$ where $P^*$ is the opposite parabolic. Then if $x$ and $y$ in $G/B$ are so that $\pi(x)$ and $\pi^*(y)$ are not transverse, then $x$ and $y$ are not transverse. Thus it is enough to prove the proposition when $P = B$.

Let $Z$ be the Zariski closure of $H$ and $Z_0$ the irreducible component (as an algebraic variety) of $Z$ containing the identity. Recall that $Z_0$ is a finite index subgroup of $Z$ [3, Paragraph 1.5]. It is thus enough to prove the result for $H$ an irreducible (as an algebraic variety) algebraic group of $G$.

We will use the algebraic irreducibility for the following fact, which follows at once from the definition of irreducibility as an algebraic variety:

(*) If $H$ is included in a finite union of $C(w)$ for $w$ in a subset $A$ of $W$ then $H$ is included in $C(a)$ for some $a$ in $A$.

The hypothesis now translates into the fact that $H$ does not intersect $Bw_0B$, where $w_0$ is the longest element in $W$. Thus by (*) and the Bruhat cell decomposition $H$ is included in $C(w)$ for $w$ in $W$ different from $w_0$. Let $s$ be the element in $W$ with the shortest length amongst the $w$ so that $H \subset C(w)$. Since $H$ is invariant under taking the inverse, $H$ is also included in $C(s^{-1})$. We now show by contradiction that $s$ is an involution. If $s$ is different from $s^{-1}$, then

$$H \subset C(s) \cap C(s^{-1}) = \bigcup_{a \in A_s} C(a),$$

where for all $a$ in $A_s$, $\ell(a) < \ell(s)$. Using (*) again, equation (21) implies that $H \subset C(a)$ for some $a$ with $\ell(a) < \ell(s)$, a contradiction.

Thus $s$ is an involution different from $w_0$. Then Lemma A.2 implies that $C(s)$—hence $H$—is included in a parabolic. The result is proved. □

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**Université de Strasbourg et CNRS, IRMA, UMR 7501, 7 rue Descartes, F-67000 Strasbourg**

*Email address*: olivier.guichard@math.unistra.fr

**Université Côte d’Azur, CNRS, LJAD, France**

*Email address*: francois.labourie@univ-cotedazur.fr

**Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Germany**

*Email address*: wienhard@uni-heidelberg.de