Parameterized Model Checking of Token-Passing Systems

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\textbf{Abstract.} We revisit the parameterized model checking problem for token-passing systems and specifications in indexed CTL\textsuperscript{*}\textbackslash X. Emerson and Namjoshi (1995, 2003) have shown that parameterized model checking of indexed CTL\textbackslash X in uni-directional token rings can be reduced to checking rings up to some \textit{cutoff} size. Clarke et al. (2004) have shown a similar result for general topologies and indexed LTL\textbackslash X, provided processes cannot choose the directions for sending or receiving the token.

We unify and substantially extend these results by systematically exploring fragments of indexed CTL\textsuperscript{*}\textbackslash X with respect to general topologies. For each fragment we establish whether a cutoff exists, and for some concrete topologies, such as rings, cliques and stars, we infer small cutoffs. Finally, we show that the problem becomes undecidable, and thus no cutoffs exist, if processes are allowed to choose the directions in which they send or from which they receive the token.

\section{Introduction}

As executions of programs and protocols are increasingly distributed over multiple CPU cores or even physically separated computers, correctness of concurrent systems is one of the primary challenges of formal methods today. Many concurrent systems consist of an arbitrary number of identical processes running in parallel. The parameterized model checking problem (PMCP) for concurrent systems is to decide if a given temporal logic specification holds irrespective of the number of participating processes.

The PMCP is undecidable in many cases. For example, it is undecidable already for safety specifications and finite-state processes communicating by passing a binary-valued token around a uni-directional ring \textsuperscript{LSS}. However, decidability may be regained by restricting the communication primitives, the topologies under consideration (i.e., the underlying graph describing the communication paths between the processes), or the specification language. In particular, previous results have shown that parameterized model checking can sometimes be reduced to model checking a finite number of instances of the system, up to some \textit{cutoff} size.

\textsuperscript{*} This work was supported by the Austrian Science Fund through grant P23499-N23 and through the RiSE network (S11403, S11405, S11406, S11407-N23); ERC Starting Grant (279307: Graph Games); Vienna Science and Technology Fund (WWTF) grants PROSEED, ICT12-059, and VRG11-005.
For token-passing systems (TPSs) with uni-directional ring topologies, such
cutoffs are known for specifications in the prenex fragment of indexed CTL∗
without the next-time operator (CTL∗\X) [10]. For token-passing in general
topologies, cutoffs are known for the prenex fragment of indexed LTL\X, provided
that processes are not allowed to choose the direction to which the token is sent
or from which it is received [5]. In this paper we generalize these results and
elucidate what they have in common.

**Previous Results.** In their seminal paper [8], Emerson and Namjoshi consider
systems where the token does not carry messages, and specifications are in prenex
indexed temporal logic — i.e., quantifiers ∀ and ∃ over processes appear in a
block at the front of the formula. They use the important concept of a cutoff — a
number c such that the PMCP for a given class of systems and specifications can
be reduced to model checking systems with up to c processes. If model checking
is decidable, then existence of a cutoff implies that the PMCP is decidable. Conversely,
if the PMCP is undecidable, then there can be no cutoff for such systems.

For uni-directional rings, Emerson and Namjoshi provide cutoffs for formulas
with a small number k of quantified index variables, and state that their proof
method allows one to obtain cutoffs for other quantifier prefixes. In brief, cutoffs
exist for the branching-time specification language prenex indexed CTL∗\X and
the highly regular topology of uni-directional rings.

Clarke et al. [5] consider the PMCP for token-passing systems arranged in
general topologies. Their main result is that the PMCP for systems with arbi-
trary topologies and k-indexed LTL\X specifications (i.e., specifications with k
quantifiers over processes in the prenex of the formula) can be reduced to comb-
ing the results of model-checking finitely many topologies of size at most 2k
[5, Theorem 4]. Their proof implies that, for each k, the PMCP for linear-time
specifications in k-indexed LTL\X and general topologies has a cutoff.

**Questions.** Comparing these results, an obvious question is: are there cutoffs for
branching time temporal logics and arbitrary topologies (see Table 1)? Clarke et
al. already give a first answer [5, Corollary 3]. They prove that there is no cutoff
for token-passing systems with arbitrary topologies and specifications from 2-
indexed CTL\X. However, their proof makes use of formulas with unbounded
nesting-depth of path quantifiers. This lead us to the first question.

| Uni-Ring Topologies | Arbitrary Topologies |
|---------------------|---------------------|
| LTL\X               | [8]                 |
| indexed LTL\X       | this paper          |

**Table 1: Direction-Unaware TPSs.**

| Bi-Ring Topologies | Arbitrary Topologies |
|-------------------|---------------------|
| LTL\X             | [7]                 |
| indexed LTL\X     | this paper          |

**Table 2: Direction-Aware TPSs.**
**Question 1.** Is there a way to stratify $k$-indexed $\text{CTL}^*\setminus X$ such that for each level of the stratification there is a cutoff for systems with arbitrary topologies? In particular, does stratification by nesting-depth of path quantifiers do the trick?

Cutoffs for $k$-indexed temporal logic fragments immediately yield that for each $k$ there is an algorithm (depending on $k$) for deciding the PMCP for $k$-indexed temporal logic. However, this does not imply that there is an algorithm that can compute the cutoff for a given $k$. In particular, it does not imply that PMCP for full prenex indexed temporal logic is decidable.

**Question 2.** For which topologies (rings, cliques, all?) can one conclude that the PMCP for the full prenex indexed temporal logic is decidable?

Finally, an important implicit assumption in Clarke et al. [5] is that processes are not direction aware, i.e., they cannot sense or choose in which direction the token is sent, or from which direction it is received. In contrast, Emerson and Kahlon [7] show that cutoffs exist for certain direction-aware systems in bi-directional rings (see Section 8). We were thus motivated to understand to what extent existing results about cutoffs can be lifted to direction-aware systems, see Table 2.

**Question 3.** Do cutoffs exist for direction-aware systems on arbitrary topologies and $k$-indexed temporal logics (such as $\text{LTL} \setminus X$ and $\text{CTL} \setminus X$)?

**Our contributions.** In this paper, we answer the questions above, unifying and substantially extending the known cutoff results:

**Answer to Question 1.** Our main positive result (Theorem 7) states that for arbitrary parameterized topologies $G$ there is a cutoff for specifications in $k$-indexed $\text{CTL}^*\setminus X$ — the cutoff depends on $G$, the number $k$ of the process quantifiers, and the nesting depth $d$ of path quantifiers. In particular, indexed $\text{LTL} \setminus X$ is included in the case $d = 1$, and so our result generalizes the results of Clarke et al. [5].

**Answer to Question 2.** We prove (Theorem 14) that there exist topologies for which the PMCP is undecidable for specifications in prenex indexed $\text{CTL} \setminus X$ or $\text{LTL} \setminus X$. Note that this undecidability result does not contradict the existence of cutoffs (Theorem 7), since cutoffs may not be computable from $k, d$ (see the note on decidability in Section 2.4). However, for certain topologies our positive result is constructive and we can compute cutoffs given $k$ and $d$ (Theorem 15).

To illustrate, we show that rings have a cutoff of $2^k$, cliques of $k + 1$, and stars of $k + 1$ (independent of $d$). In particular, PMCP is decidable for these topologies and specifications in prenex indexed $\text{CTL}^*\setminus X$.

**Answer to Question 3.** The results just mentioned assume that processes are not direction-aware. Our main negative result (Theorem 17) states that if processes can control at least one of the directions (i.e., choose in which direction to send or from which direction to receive) then the PMCP for arbitrary topologies and $k$-indexed logic (even $\text{LTL} \setminus X$ and $\text{CTL} \setminus X$) is undecidable, and therefore does not have cutoffs. Moreover, if processes can control both in- and out-directions, then the PMCP is already undecidable for bi-directional rings and 1-indexed $\text{LTL} \setminus X$. 
Technical contributions relative to previous work. Our main positive result (Theorem 7) generalizes proof techniques and ideas from previous results [8,5]. We observe that in both of these papers the main idea is to abstract a TPS by simulating the quantified processes exactly and simulating the movement of the token between these processes. The relevant information about the movement of the token is this: whether there is a direct edge, or a path (through unquantified processes) from one quantified process to another. This abstraction does not work for $\text{CTL}_d \setminus \mathcal{X}$ and general topologies since the formula can express branching properties of the token movement. Our main observation is that the relevant information about the branching-possibilities of the token can be expressed in $\text{CTL}_d \setminus \mathcal{X}$ over the topology itself. We develop a composition-like theorem, stating that if two topologies (with $k$ distinguished vertices) are indistinguishable by $\text{CTL}_d \setminus \mathcal{X}$ formulas, then the TPSs based on these topologies and an arbitrary process template $P$ are indistinguishable by $\text{CTL}_d \setminus \mathcal{X}$ (Theorem 9). The machinery involves a generalization of stuttering trace-equivalence [17], a notion of $d$-contraction that serves the same purpose as the connection topologies of Clarke et al. [5, Proposition 1], and also the main simulation idea of Emerson and Namjoshi [8, Theorem 2].

Our main negative result, undecidability of PMCP for direction-aware systems (Theorem 17), is proven by a reduction from the non-halting problem for 2-counter machines (as is typical in this area [8,11]). Due to the lack of space, full proofs are omitted, and can be found in the full version [1].

2 Definitions and Existing Results

Let $\mathbb{N}$ denote the set of positive integers. Let $[k]$ for $k \in \mathbb{N}$ denote the set $\{1, \ldots, k\}$. The concatenation of strings $u$ and $w$ is written $uw$ or $u \cdot w$.

Let $\text{AP}$ denote a countably infinite set of atomic propositions or atoms. A labeled transition system (LTS) over $\text{AP}$ is a tuple $(Q, Q_0, \Sigma, \delta, \lambda)$ where $Q$ is the set of states, $Q_0 \subseteq Q$ are the initial states, $\Sigma$ is the set of transition labels (also called action labels), $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation, and $\lambda : Q \to 2^\text{AP}$ is the state-labeling and satisfies that $\lambda(q)$ is finite (for every $q \in Q$). Transitions $(q, \sigma, q') \in \delta$ may be written $q \xrightarrow{\sigma} q'$.

A state-action path of an LTS $(Q, Q_0, \Sigma, \delta, \lambda)$ is a finite sequence of the form $q_0 \sigma_0 q_1 \sigma_1 \ldots q_n \in (Q \Sigma)^*Q$ or an infinite sequence $q_0 \sigma_0 q_1 \sigma_1 \ldots \in (Q \Sigma)^\omega$ such that $(q_i, \sigma_i, q_{i+1}) \in \delta$ (for all $i$). A path of an LTS is the projection $q_0 q_1 \ldots$ of a state-action path onto states $Q$. An action-labeled path of an LTS is the projection $\sigma_0 \sigma_1 \ldots$ of a state-action path onto transition labels $\Sigma$.

2.1 System Model (Direction-Unaware)

In this section we define the LTS $P^G$ — it consists of replicated copies of a process $P$ placed on the vertices of a graph $G$. Transitions in $P^G$ are either internal (in which exactly one process moves) or synchronized (in which one
process sends the token to another along an edge of $G$). The token starts with the process that is at the initial vertex of $G$.

Fix a countably infinite set of (local) atomic propositions $\mathsf{AP}_{pr}$ (to be used by the states of the individual processes).

**Process Template $P$.** Let $\Sigma_{\text{int}}$ denote a finite non-empty set of internal-transition labels. Define $\Sigma_{pr}$ as the disjoint union $\Sigma_{\text{int}} \cup \{\text{rcv}, \text{snd}\}$ where $\text{rcv}$ and $\text{snd}$ are new symbols.

A process template $P$ is a LTS $(Q, Q_0, \Sigma_{pr}, \delta, \lambda)$ over $\mathsf{AP}_{pr}$ such that:

i) The state set $Q$ is finite and can be partitioned into two non-empty sets, say $T \cup \mathcal{N}$. States in $T$ are said to have the token.

ii) The initial state set is $Q_0 = \{q_0, q_n\}$ for some $q_0 \in T, q_n \in \mathcal{N}$.

iii) Every transition $q \xrightarrow{\text{snd}} q'$ satisfies that $q$ has the token and $q'$ does not.

iv) Every transition $q \xrightarrow{\text{rcv}} q'$ satisfies that $q'$ has the token and $q$ does not.

v) Every transition $q \xrightarrow{a} q'$ with $a \in \Sigma_{\text{int}}$ satisfies that $q$ has the token if and only if $q'$ has the token.

vi) The transition relation $\delta$ is total in the first coordinate: for every $q \in Q$ there exists $\sigma \in \Sigma_{pr}$, $q' \in Q$ such that $(q, \sigma, q') \in \delta$ (i.e., the process $P$ is non-terminating).

vii) Every infinite action-labeled path $a_0a_1\ldots$ is in the set $(\Sigma_{\text{int}} \text{ snd} \Sigma_{\text{int}} \text{ rcv})^\omega \cup (\Sigma_{\text{int}} \text{ rcv} \Sigma_{\text{int}} \text{ snd})^\omega$ (i.e., $\text{snd}$ and $\text{rcv}$ actions alternate continually along every infinite action-labeled path of $P$).\footnote{This restriction was introduced by Emerson and Namjoshi in \cite{Emerson87}. Our positive results that cutoffs exist also hold for a more liberal restriction (see Section\ref{sec:positive}).}

The elements of $Q$ are called local states and the transitions in $\delta$ are called local transitions (of $P$). A local state $q$ such that the only transitions are of the form $q \xrightarrow{\text{snd}} q'$ (for some $q'$) is said to be send-only. A local state $q$ such that the only transitions are of the form $q \xrightarrow{\text{rcv}} q'$ (for some $q'$) is said to be receive-only.

**Topology $G$.** A topology is a directed graph $G = (V, E, x)$ where $V = [k]$ for some $k \in \mathbb{N}$, vertex $x \in V$ is the initial vertex, $E \subseteq V \times V$, and $(v, v) \notin E$ for every $v \in V$. Vertices are called process indices.

We may also write $G = (V_G, E_G, x_G)$ if we need to disambiguate.

**Token-Passing System $P^G$.** Let $\mathsf{AP}_{sys} := \mathsf{AP}_{pr} \times \mathbb{N}$ be the indexed atomic propositions. For $(p, i) \in \mathsf{AP}_{sys}$ we may also write $p_i$. Given a process template $P = (Q, Q_0, \Sigma_{pr}, \delta, \lambda)$ over $\mathsf{AP}_{pr}$ and a topology $G = (V, E, x)$, define the token-passing system (TPS) $P^G$ as the finite LTS $(S, S_0, \Sigma_{\text{int}} \cup \{\text{tok}\}, \Delta, \Lambda)$ over atomic propositions $\mathsf{AP}_{sys} := \mathsf{AP}_{pr} \times \mathbb{N}$, where:

- The set $S$ of global states is $Q^V$, i.e., all functions from $V$ to $Q$. If $s \in Q^V$ is a global state then $s(i)$ denotes the local state of the process with index $i$.
- The set of global initial states $S_0$ consists of the unique global state $s_0 \in Q^V_0$ such that only $s_0(x)$ has the token (here $x$ is the initial vertex of $G$).
- The labeling $\Lambda(s) \subset \mathsf{AP}_{sys}$ for $s \in S$ is defined as follows: $p_i \in \Lambda(s)$ if and only if $p \in \lambda(s(i))$, for $p \in \mathsf{AP}_{pr}$ and $i \in V$.
- The global transition relation $\Delta$ is defined to consist of the set of all internal transitions and synchronous transitions:

$$\Delta = \sigma^\omega \cup \rho^\omega \cup \pi^\omega$$
• An internal transition is an element \((s, a, s')\) of \(S \times \Sigma_{\text{int}} \times S\) for which there exists a process index \(v \in V\) such that
  i) \(s(v) \xrightarrow{d} s'(v)\) is a local transition of \(P\), and
  ii) for all \(w \in V \setminus \{v\}\), \(s(w) = s'(w)\).

• A token-passing transition is an element \((s, \text{tok}, s')\) of \(S \times \{\text{tok}\} \times S\) for which there exist process indices \(v, w \in V\) such that \((v, w) \in E\) and
  i) \(s(v) \xrightarrow{\text{snd}} s'(v)\) is a local transition of \(P\),
  ii) \(s(w) \xrightarrow{\text{rcv}} s'(w)\) is a local transition of \(P\), and
  iii) for every \(u \in V \setminus \{v, w\}\), \(s'(u) = s(u)\).

In words, the system \(P^G\) can be thought of as the asynchronous parallel composition of \(P\) over topology \(G\). The token starts with process \(x\). At each time step either exactly one process makes an internal transition, or exactly two processes synchronize when one process sends the token to another along an edge of \(G\).

### 2.2 System Model (Direction-Aware)

Inspired by direction-awareness in the work of Emerson and Kahlon [7], we extend the definition of TPS to include additional labels on edges, called directions. The idea is that processes can restrict which directions are used when they send or receive the token.

Fix finite non-empty disjoint sets \(\text{Dir}_{\text{snd}}\) of sending directions and \(\text{Dir}_{\text{rcv}}\) of receiving directions. A direction-aware token-passing system is a TPS with the following modifications.

**Direction-aware Topology.** A direction-aware topology is a topology \(G = (V, E, x)\) with labeling functions \(\text{dir}_{\text{rcv}} : E \rightarrow \text{Dir}_{\text{rcv}}, \text{dir}_{\text{snd}} : E \rightarrow \text{Dir}_{\text{snd}}\).

**Direction-aware Process Template.** For process templates of direction-aware systems, transition labels are taken from \(\Sigma_{\text{pr}} := \Sigma_{\text{int}} \cup \text{Dir}_{\text{snd}} \cup \text{Dir}_{\text{rcv}}\). The definition of a direction-aware process template is like that in Section 2.1 except that in item iii) \(\text{snd}\) is replaced by \(d \in \text{Dir}_{\text{snd}}\), in iv) \(\text{rcv}\) is replaced by \(d \in \text{Dir}_{\text{rcv}}\), and in vii) \(\text{snd}\) is replaced by \(\text{Dir}_{\text{snd}}\) and \(\text{rcv}\) by \(\text{Dir}_{\text{rcv}}\).

**Direction-aware Token Passing System.** Fix \(\text{Dir}_{\text{snd}}\) and \(\text{Dir}_{\text{rcv}}\), let \(G\) be a direction-aware topology and \(P\) a direction-aware process template. Define the direction-aware token-passing system \(P^G\) as in Section 2.1 except that token-passing transitions are now direction-aware: direction-aware token-passing transitions are elements \((s, \text{tok}, s')\) of \(S \times \{\text{tok}\} \times S\) for which there exist process indices \(v, w \in V\) with \((v, w) \in E\), \(\text{dir}_{\text{snd}}(v, w) = d\), and \(\text{dir}_{\text{rcv}}(v, w) = e\), such that:
  i) \(s(v) \xrightarrow{d} s'(v)\) is a local transition of \(P\).
  ii) \(s(w) \xrightarrow{e} s'(w)\) is a local transition of \(P\).
  iii) For every \(u \in V \setminus \{v, w\}\), \(s'(u) = s(u)\).

**Notations** \(\mathcal{P}, \mathcal{P}_{\text{snd}}, \mathcal{P}_{\text{rcv}}, \mathcal{P}_{\text{rcv}}\). Let \(\mathcal{P}\) denote the set of all process templates for which \(|\text{Dir}_{\text{snd}}| = |\text{Dir}_{\text{rcv}}| = 1\). In this case \(P^G\) degenerates to a direction-unaware TPS as defined in Section 2.1. If we require \(|\text{Dir}_{\text{rcv}}| = 1\), then processes cannot choose from which directions to receive the token, but possibly in which
direction to send it. Denote the set of all such process templates by \( P_{\text{snd}} \). Similarly define \( P_{\text{rcv}} \) to be all process templates where \(|\text{Dir}_{\text{snd}}| = 1\) — processes cannot choose where to send the token, but possibly from which direction to receive it. Finally, let \( P_{\text{sndrcv}} \) be the set of all direction-aware process templates.

**Examples.** Figure 1(a) shows a bi-directional ring with directions \( \text{cw} \) (clockwise) and \( \text{ccw} \) (counterclockwise). Every edge \( e \) is labeled with an outgoing direction \( \text{dir}_{\text{snd}}(e) \) and an incoming direction \( \text{dir}_{\text{rcv}}(e) \). Using these directions, a process that has the token can choose whether he wants to send it in direction \( \text{cw} \) or \( \text{ccw} \). Depending on its local state, a process waiting for the token can also choose to receive it only from direction \( \text{cw} \) or \( \text{ccw} \).

Figure 1(b) depicts a topology in which process 1 can choose between two outgoing directions. If it sends the token in direction \( \text{snd}_1 \), it may be received by either process 2 or 3. If however process 2 blocks receiving from direction \( \text{rcv}_1 \), the token can only be received by process 3. If 3 additionally blocks receiving from \( \text{rcv}_2 \), then this token-passing transition is disabled.

### 2.3 Indexed Temporal Logics

*Indexed temporal logics (ITL)* were introduced in [4,9,8] to model specifications of certain distributed systems. Subsequently one finds a number of variations of indexed temporal-logics in the literature (linear vs. branching, restrictions on the quantification). Thus we introduce Indexed-CTL* which has these variations (and those in [5]) as syntactic fragments.

**Syntactic Fragments of CTL*, and \( \equivTL \).** We assume the reader is familiar with the syntax and semantics of CTL*, for a reminder see [2]. For \( d \in \mathbb{N} \) let \( \text{CTL}_d^* \setminus \text{X} \) denote the syntactic fragment of CTL*\( \setminus \text{X} \) in which the nesting-depth of path quantifiers is at most \( d \) (for a formal definition see [17] Section 4).

Let \( \text{TL} \) denote a temporal logic (in this paper these are fragments of CTL*\( \setminus \text{X} \)). For temporal logic \( \text{TL} \) and LTSs \( M \) and \( N \), write \( M \equivTL N \) to mean that for every formula \( \phi \in \text{TL} \), \( M \models \phi \) if and only if \( N \models \phi \).

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\( ^2 \) For notational simplicity, we denote both outgoing direction \( \text{snd}_{\text{cw}} \) and incoming direction \( \text{rcv}_{\text{cw}} \) by \( \text{cw} \), and similarly for \( \text{ccw} \).
Indexed-CTL*. Fix an infinite set \( \text{Vars} = \{x, y, z, \ldots \} \) of index variables, i.e., variables with values from \( \mathbb{N} \). These variables refer to vertices in the topology.

**Syntax.** The Indexed-CTL* formulas over variable set \( \text{Vars} \) and atomic propositions \( \text{AP} \) are formed by adding the following rules to the syntax of CTL* over atomic propositions \( \text{AP} \times \text{Vars} \). We write \( p_x \) instead of \((p, x) \in \text{AP} \times \text{Vars}\).

If \( \phi \) is an indexed-CTL* state (resp. path) formula and \( x, y \in \text{Vars}, Y \subseteq \text{Vars} \), then the following are also indexed-CTL* state (resp. path) formulas:

- \( \forall x. \phi \) (i.e., for all/some vertices in the topology, \( \phi \) should hold),
- \( \forall x. x \in Y \rightarrow \phi \) and \( \exists x. x \in Y \land \phi \) (for all/some vertices that are designated by variables in \( Y \)),
- \( \forall x. x \in E(y) \rightarrow \phi \) and \( \exists x. x \in E(y) \land \phi \) (for all/some vertices to which there is an edge from the vertex designated by the variable \( y \)).

**Index Quantifiers.** We use the usual shorthands (e.g., \( \forall x \in Y. \phi \) is shorthand for \( \forall x. x \in Y \rightarrow \phi \)). The quantifiers introduced above are called called index quantifiers, denoted \( Qx \).

**Semantics.** Indexed temporal logic is interpreted over a system instance \( P^G \) (with \( P \) a process template and \( G \) a topology). The formal semantics are in the full version of the paper [1]. Here we give some examples. The formula \( \forall i. EF p_i \) states that for every process there exists a path such that, eventually, that process is in a state that satisfies atom \( p \). The formula \( EF\forall i.p_i \) states that there is a path such that eventually all processes satisfy atom \( p \) simultaneously. We now define the central fragment that includes the former example and not the latter.

**Prenex indexed TL and \( \{\forall, \exists\}^k\text{-TL} \).** Prenex indexed temporal-logic is a syntactic fragment of indexed temporal-logic in which all quantifiers are at the front of the formula, e.g., prenex indexed LTL\( \setminus X \) consists of formulas of the form \((Q_1 x_1) \ldots (Q_k x_k) \varphi \) where \( \varphi \) is an LTL\( \setminus X \) formula over atoms \( \text{AP} \times \{x_1, \ldots, x_k\} \), and the \( Q_i x_i \)s are index quantifiers. Such formulas with \( k \) quantifiers will be referred to as \( k\text{-indexed} \), collectively written \( \{\forall, \exists\}^k\text{-TL} \). The union of \( \{\forall, \exists\}^k\text{-TL} \) for \( k \in \mathbb{N} \) is written \( \{\forall, \exists\}^*\text{-TL} \) and called (full) prenex indexed TL.

### 2.4 Parameterized Model Checking Problem, Cutoffs, Decidability

A parameterized topology \( G \) is a countable set of topologies. E.g., the set of unidirectional rings with all possible initial vertices is a parameterized topology.

**PMCP**\(_G\)(\(-, -\)). The parameterized model checking problem (PMCP) for parameterized topology \( G \), processes from \( \mathcal{P} \), and parameterized specifications from \( \mathcal{F} \), written \( \text{PMCP}_G(\mathcal{P}, \mathcal{F}) \), is the set of pairs \((\varphi, P) \in \mathcal{F} \times \mathcal{P} \) such that for all \( G \in G \), \( P^G \models \varphi \). A solution to \( \text{PMCP}_G(\mathcal{P}, \mathcal{F}) \) is an algorithm that, given a formula \( \varphi \in \mathcal{F} \) and a process template \( P \in \mathcal{P} \) as input, outputs ‘Yes’ if for all \( G \in G \), \( P^G \models \varphi \), and ‘No’ otherwise.

**Cutoff.** A cutoff for \( \text{PMCP}_G(\mathcal{P}, \mathcal{F}) \) is a natural number \( c \) such that for every \( P \in \mathcal{P} \) and \( \varphi \in \mathcal{F} \), the following are equivalent:

- \( P^G \models \varphi \) for all \( G \in G \) with \( |V_G| \leq c \);
- \( P^G \models \varphi \) for all \( G \in G \).
Thus \( PMCP_G(P, F) \) does not have a cutoff iff for every \( c \in \mathbb{N} \) there exists \( P \in P \) and \( \varphi \in F \) such that \( P^G \models \varphi \) for all \( G \in G \) with \( |V_G| \leq c \), and there exists \( G \in G \) such that \( P^G \not\models \varphi \).

**Observation 1.** If \( PMCP_G(P, F) \) has a cutoff, then \( PMCP_G(P, F) \) is decidable.

Indeed: if \( c \) is a cutoff, let \( G_1, \ldots, G_n \) be all topologies \( G \) in \( G \) such that \( |V_G| \leq c \). The algorithm that solves \( PMCP \) takes \( P, \varphi \) as input and checks whether or not \( P^G_i \models \varphi \) for all \( 1 \leq i \leq n \).

**Note About Decidability.** The following statements are not, a priori, equivalent (for given parameterized topology \( G \) and process templates \( P \)):
- For every \( k \in \mathbb{N} \), \( PMCP_G(P, \{\forall, \exists\}^k-\mathcal{LTL}) \) is decidable.
- \( PMCP_G(P, \{\forall, \exists\}^*-\mathcal{LTL}) \) is decidable.

The first item says that for every \( k \) there exists an algorithm \( A_k \) that solves the \( PMCP \) for \( k \)-indexed \( \mathcal{LTL} \). This does not imply the second item, which says that there exists an algorithm that solves the \( PMCP \) for \( \bigcup_{k \in \mathbb{N}} \{\forall, \exists\}^k-\mathcal{LTL} \). If the function \( k \mapsto A_k \) is also computable (e.g., Theorem 10), then indeed the second item follows: given \( P, \varphi \), extract the size \( k \) of the prenex block of \( \varphi \), compute a description of \( A_k \), and run \( A_k \) on \( P, \varphi \).

For instance, the result of Clarke et al. — that there are cutoffs for \( k \)-index \( \mathcal{LTL}/X \) and arbitrary topologies — does not imply that the \( PMCP \) for \( \{\forall, \exists\}^*-\mathcal{LTL}/X \) and arbitrary topologies is decidable. Aware of this fact, the authors state (after Theorem 4) “Note that the theorem does not provide us with an effective means to find the reduction [i.e. algorithm]...”.

In fact, we prove (Theorem 14) that there is some parameterized topology such that \( PMCP \) is undecidable for prenex indexed \( \mathcal{LTL}/X \).

**Existing Results.** We restate the known results using our terminology.

A uni-directional ring \( G = (V, E, x) \) is a topology with \( V = [n] \) for some \( n \in \mathbb{N} \), there are edges \((i, i + 1)\) for \( 1 \leq i \leq n \) (arithmetic is modulo \( n \)), and \( x \in V \). Let \( R \) be the parameterized topology consisting of all uni-directional rings.

**Theorem 2 (Implicit in [5]).** For every \( k \in \mathbb{N} \), there is a cutoff for the problem \( PMCP_R(P_u, \{\forall, \exists\}^k-\mathcal{LTL}/X) \).

Although Clarke et al. [5] do not explicitly state the following theorem, it follows from their proof technique, which we generalize in Section 8.

**Theorem 3 (Implicit in [5]).** For every parameterized topology \( G \), and every \( k \in \mathbb{N} \), the problem \( PMCP_G(P_u, \{\forall, \exists\}^k-\mathcal{LTL}/X) \) has a cutoff.

**Theorem 4 ([5, Corollary 3]).** There exists a parameterized topology \( G \) and process \( P \in P_u \) such that the problem \( PMCP_G(\{P\}, \{\exists\}^2-\mathcal{CTL}/X) \) does not have a cutoff.

---

3 The paper explicitly contains the result that 4 is a cutoff for \( \forall\mathcal{CTL}^*/X \) on rings. However the proof ideas apply to get the stated theorem.

4 Khalimov et al. [14, Corollary 2] state that 2k is a cutoff if \( G \) is taken to be \( R \). However this is an error: 2k is a cutoff only for formulas with no quantifier alternations. See Remark 16 in Section 5.
The proof of this theorem defines $G$, process $P$, and for every $c \in \mathbb{N}$ a formula $\varphi_c$, such that if $G \in G$ then $P^G \models \varphi_c$ if and only if $|V_G| \leq c$. The formula $\varphi_c$ is in 2-indexed $\text{CTL}\backslash X$ and has nesting depth of path quantifiers equal to $c$.

3 Method for Proving Existence of Cutoffs

We give a method for proving cutoffs for direction-unaware TPSs that will be used to prove Theorem 5.

In a $k$-indexed TL formula $Q_1x_1 \ldots Q_kx_k \varphi$, every valuation of the variables $x_1, \ldots, x_k$ designates $k$ nodes of the underlying topology $G$, say $\bar{g} = g_1, \ldots, g_k$. The formula $\varphi$ can only talk about (the processes in) $\bar{g}$. In order to prove that the PMCP has a cutoff, it is sufficient (as the proof of Theorem 5 will demonstrate) to find conditions on two topologies $G, G'$ and $\bar{g}, \bar{g}'$ that allow one to conclude that $P^G$ and $P^{G'}$ are indistinguishable with respect to $\varphi$.

We define two abstractions for a given TPS $P^G$. The first abstraction simulates $P^G$, keeping track only of the local states of processes indexed by $\bar{g}$. We call it the projection of $P^G$ onto $\bar{g}$. The second abstraction only simulates the movement of the token in $G$, restricted to $\bar{g}$. We call it the graph LTS of $G$ and $\bar{g}$.

Notation. Let $\bar{g}$ denote a tuple $(g_1, \ldots, g_k)$ of distinct elements of $V_G$, and $\bar{g}'$ a $k$-tuple of distinct elements of $V_{G'}$. Write $v \in \bar{g}$ if $v = g_i$ for some $i$.

The projection $P^G|\bar{g}$. Informally, the projection of $P^G$ onto a tuple of process indices $\bar{g}$ is the LTS $P^G$ and a new labeling that, for every $g_i \in \bar{g}$, replaces the indexed atom $p_{g_i}$ by the atom $p@i$; all other atoms are removed. Thus $p@i$ means that the atom $p \in \text{AP}_{pr}$ holds in the process with index $g_i$. In other words, process indices are replaced by their positions in $\bar{g}$.

Formally, fix process $P$, topology $G$, and $k$-tuple $\bar{g}$ over $V_G$. Define the projection of $P^G = (S, S_0, \Sigma_{\text{int}} \cup \{\text{tok}\}, \Delta, A)$ onto $\bar{g}$, written $P^G|\bar{g}$, as the LTS $(S, S_0, \Sigma_{\text{int}} \cup \{\text{tok}\}, \Delta, L)$ over atomic propositions $\{p@i : p \in \text{AP}_{pr}, i \in [k]\}$, where for all $s \in S$ the labeling $L(s)$ is defined as follows: $L(s) := \{p@i : p_{g_i} \in \Lambda(s), i \in [k]\}$.

The graph LTS $G|\bar{g}$. Informally, $G|\bar{g}$ is an LTS where states are nodes of the graph $G$, and transitions are edges of $G$. The restriction to $\bar{g}$ is modeled by labeling a state with the position of the corresponding node in $\bar{g}$.

Let $G = (V, E, x)$ be a topology, and let $\bar{g}$ be a $k$-tuple over $V_G$. Define the graph LTS $G|\bar{g}$ as the LTS $(Q, Q_0, \Sigma, \Delta, \Lambda)$ over atomic propositions $\{1, \ldots, k\}$, with state set $Q := V$, initial state set $Q_0 := \{x\}$, action set $\Sigma = \{a\}$, transition relation with $(v, a, w) \in \Delta$ iff $(v, w) \in E$, and labeling $\Lambda(v) := \{i\}$ if $v = g_i$ for some $1 \leq i \leq k$, and $\emptyset$ otherwise.\footnote{Emerson and Namjoshi \cite{EmersonNamjoshi} Section 2.1} define the related notion “LTS projection”.\footnote{Atomic propositions that have to be true in exactly one state of a structure are called nominals in \cite{EmersonNamjoshi}.}
Fix a non-indexed temporal logic \( \mathcal{TL} \), such as \( \mathcal{CTL}^X \). We now define what it means for \( \mathcal{TL} \) to have the reduction property and the finiteness property. Informally, the reduction property says that if \( G \) and \( G' \) have the same connectivity (with respect to \( \mathcal{TL} \) and only viewing \( k \)-tuples \( \bar{g}, \bar{g}' \)) then \( P^G \) and \( P^{G'} \) are indistinguishable (with respect to \( \mathcal{TL} \)-formulas over process indices in \( \bar{g}, \bar{g}' \)).

**REDUCTION property for \( \mathcal{TL} \)**

For every \( k \in \mathbb{N} \), process \( P \in \mathcal{P}_u \), topologies \( G, G' \), \( k \)-tuples \( \bar{g}, \bar{g}' \), if \( P|\bar{g} \equiv_{\mathcal{TL}} G'|\bar{g}' \) then \( P^G|\bar{g} \equiv_{\mathcal{TL}} P^{G'}|\bar{g}' \).

**FINITENESS property for \( \mathcal{TL} \)**

For every \( k \in \mathbb{N} \), there are finitely many equivalence classes \( [G|\bar{g}]_{\equiv_{\mathcal{TL}}} \) where \( G \) is an arbitrary topology, and \( \bar{g} \) is a \( k \)-tuple over \( V_G \).

**Theorem 5 (REDUCTION & FINITENESS \( \implies \) Cutoffs for \( \{\forall, \exists\}^k \)-\( \mathcal{TL} \)).** If \( \mathcal{TL} \) satisfies the REDUCTION and the FINITENESS property, then for every \( k, G \), \( \text{PMCP}_G(\mathcal{P}_u, \{\forall, \exists\}^k \)-\( \mathcal{TL}\) has a cutoff.

**Proof.** Fix quantifier prefix \( Q_kx_1 \ldots Q_kx_k \). We prove that there exist finitely many topologies \( G_1, \ldots, G_N \in \mathcal{G} \) such that for every \( G \in \mathcal{G} \) there is an \( i \leq N \) such that for all \( P \in \mathcal{P}_u \), and all \( \mathcal{TL} \)-formulas \( \varphi \) over atoms \( \mathcal{AP}_r \) \( \times \{x_1, \ldots, x_k\} \)

\[ P^G \models Q_1x_1 \ldots Q_kx_k \varphi \iff P^{G_i} \models Q_1x_1 \ldots Q_kx_k \varphi \]

In particular, \( \max\{|V_{G_i}| : 1 \leq i \leq N\} \) is a cutoff for \( \text{PMCP}_G(\mathcal{P}_u, \{\forall, \exists\}^k \)-\( \mathcal{TL}\).

Suppose for simplicity of exposition that \( Q_i x_i \) is a quantifier that also expresses that the value of \( x_i \) is different from the values of \( x_j \in \{x_1, \ldots, x_{i-1}\} \). Fix representatives of \( [G|\bar{g}]_{\equiv_{\mathcal{TL}}} \) and define a function \( \tau \) that maps \( G|\bar{g} \) to the representative of \( [G|\bar{g}]_{\equiv_{\mathcal{TL}}} \). Define a function \( \text{rep} \) that maps \( P^G|\bar{g} \) to \( P^H|\bar{h} \), where \( \tau(G|\bar{g}) = H|\bar{h} \).

For each \( P^G|\bar{g} \) satisfying \( \varphi \), introduce a new Boolean proposition \( \text{rep}(G|\bar{g}) \). By the FINITENESS property of \( \mathcal{TL} \) there are finitely many such Boolean propositions, say \( n \).

Define a valuation \( e_\varphi \) (that depends on \( \varphi \)) of these new atoms by

\[
e_\varphi(\text{rep}(G|\bar{g})) := \begin{cases} \top & \text{if } P^H|\bar{h} \models \varphi[p_{xj} \mapsto p_{xj}] \\ \bot & \text{otherwise.} \end{cases}
\]

For every \( G \in \mathcal{G} \) define Boolean formula \( B_G := (Q_1g_1 \in V_G) \ldots (Q_kg_k \in V_G) q_{\tau(G|\bar{g})} \), where \( Q \) is the Boolean operation corresponding to \( Q \), e.g., \( \exists i \in V_G \) is interpreted as \( \bigvee_{g \in V_G \setminus \{g_1, \ldots, g_{k-1}\}} \).

\[ \text{Properties of this type are sometimes named \( \text{composition} \) instead of \( \text{reduction} \), see for instance [15].} \]

\[ \text{All the types of quantifiers defined in Section 2.3 such as } \exists x \in E(y) \text{, can be dealt with similarly at the cost of notational overhead.} \]

\[ \text{Note that in the Boolean propositions } G \text{ is fixed while } \bar{g} = (g_1, \ldots, g_n) \text{ ranges over } (V_G)^k \text{ and is determined by the quantification.} \]
Then (for all $P, G$ and $\varphi$)
\[ P^G \models Q_1x_1 \ldots Q_kx_k \cdot \varphi \]
\[ \iff Q_1g_1 \in V_G \ldots Q_kg_k \in V_G : P^G \models \varphi[p_{x_j} \mapsto p_{g_j}] \]
\[ \iff Q_1g_1 \in V_G \ldots Q_kg_k \in V_G : P^G[g] \models \varphi[p_{x_j} \mapsto p_{g_j}] \]
\[ \iff Q_1g_1 \in V_G \ldots Q_kg_k \in V_G : \text{rep}(P^G[g]) \models \varphi[p_{x_j} \mapsto p_{g_j}] \]
\[ \iff e_{\varphi}(B_G) = \top \]

Here $\varphi[p_{x_j} \mapsto p_{g_j}]$ is the formula resulting from replacing every atom in $\varphi$ of the form $p_{x_j}$ by the atom $p_{g_j}$, for $p \in AP_{pr}$ and $1 \leq j \leq k$. Similarly $\varphi[p_{x_j} \mapsto \text{rep}(p_{g_j})]$ is defined as the formula resulting from replacing (for all $p \in AP_{pr}, j \leq k$) every atom in $\varphi$ of the form $p_{x_j}$ by the atom $\text{rep}(p_{g_j})$. The first equivalence is by the definition of semantics of indexed temporal logic; the second is by the definition of $P^G[g]$; the third is by the REDUCTION property of TL; the fourth is by the definition of $e_{\varphi}$ and $\text{rep}$.

Fix $B_{G_1}, \ldots, B_{G_N}$ (with $G_i \in \mathbb{G}$) such that every $B_G$ ($G \in \mathbb{G}$) is logically equivalent to some $B_{G_i}$. Such a finite set of formulas exists since there are $2^{2^n}$ Boolean formulas (up to logical equivalence) over $n$ Boolean propositions, and thus at most $2^{2^n}$ amongst the $B_G$ for $G \in \mathbb{G}$.

By the equivalences above conclude that for every $G \in \mathbb{G}$ there exists $i \leq N$ such that $P^G \models Q_1x_1 \ldots Q_kx_k \cdot \varphi$ if and only if $P^{G_i} \models Q_1x_1 \ldots Q_kx_k \cdot \varphi$. Thus $\forall G \in \mathbb{G}, P^G \models \varphi$ is equivalent to $\bigwedge_{i \leq N} e_{\varphi}(B_{G_i})$ and so the integer $c := \max\{|V_{G_i}| : 1 \leq i \leq N\}$ is a cutoff for $\text{PMCP}_G(P, \forall, \exists)^{k-\text{TL}}$. $\square$

**Remark 6.** The theorem implies that for every $k, G$, $\text{PMCP}_G(P, \forall, \exists)^{k-\text{TL}}$ is decidable. Further, fix $G$ and suppose that given $k$ one could compute the finite set $G_1, \ldots, G_N$. Then by the last sentence in the proof one can compute the cutoff $c$. In this case, $\text{PMCP}_G(P, \forall, \exists)^{k-\text{TL}}$ is decidable.

### 4 Existence of Cutoffs for $k$-indexed $\text{CTL}^*_d \setminus \mathcal{X}$

The following theorem answers Question 1 from the introduction.

**Theorem 7.** Let $G$ be a parameterized topology. Then for all $k, d \in \mathbb{N}$, the problem $\text{PMCP}_G(P, \forall, \exists)^{k-\text{CTL}^*_d \setminus \mathcal{X}}$ has a cutoff.

**Corollary 8.** Let $G$ be a parameterized topology. Then for all $k, d \in \mathbb{N}$, the problem $\text{PMCP}_G(P, \forall, \exists)^{k-\text{CTL}^*_d \setminus \mathcal{X}}$ is decidable.

To prove the Theorem it is enough, by Theorem 7 to show that the logic $\{\forall, \exists\}^{k-\text{CTL}^*_d \setminus \mathcal{X}}$ has the REDUCTION property and the FINITENESS property.

**Theorem 9 (Reduction).** For all $d, k \in \mathbb{N}$, topologies $G, G'$, processes $P \in P_u$, $k$-tuples $g$ over $V_G$ and $k$-tuples $g'$ over $V_{G'}$:

If $G[g] \equiv \text{CTL}^*_d \setminus \mathcal{X} G'[g']$ then $P^G[g] \equiv \text{CTL}^*_d \setminus \mathcal{X} P^{G'}[g']$. 

The idea behind the proof is to show that paths in $P^G$ can be simulated by paths in $P^{G'}$ (and vice versa). Given a path $\pi$ in $P^G$, first project it onto $G$ to get a path $\rho$ that records the movement of the token, then take an equivalent path $\rho'$ in $G'$ which exists since $G[\bar{g}] \equiv_{\text{CTL}^* \backslash X} G'[\bar{g}'$, and then lift $\rho'$ up to get a path $\pi'$ in $P^{G'}$ that is equivalent to $\pi$. This lifting step uses the assumption that process $P$ is in $P_u$, i.e., $P$ cannot control where it sends the token, or from where it receives it. The proof can be found in the full version of the paper [1].

Remark 10. As immediate corollaries we get that the REDUCTION property holds with $\text{TL} = \text{LTL} \backslash X$ (take $d = 1$), $\text{CTL}^* \backslash X$ (since the assumption holds with $\text{TL} = \text{CTL}^* \backslash X$ then the conclusion holds with $\text{TL} = \text{CTL}^* \backslash X$ for all $d \in \mathbb{N}$, and thus also for $\text{TL} = \text{CTL}^* \backslash X$) and, if $P$ is finite, also for $\text{TL} = \text{CTL} \backslash X$ (since $\text{CTL} \backslash X$ and $\text{CTL}^* \backslash X$ agree on finite structures).

**Finiteness Theorem.** Theorem 4 ([1, Corollary 3]) states that there exists $G$ such that the problem PMCP$_G(P_u, \exists \text{-CTL}^* \backslash X)$ does not have a cutoff. We observed that the formulas from their result have unbounded nesting depth of path quantifiers. This leads to the idea of stratifying $\text{CTL}^* \backslash X$ by nesting depth.

Recall from Section 2.3 that i) $\text{CTL}^*_d \backslash X$ denotes the syntactic fragment of $\text{CTL}^* \backslash X$ in which formulas have path-quantifier nesting depth at most $d$; ii) $M \equiv_{\text{CTL}^*_d \backslash X} N$ iff $M$ and $N$ agree on all $\text{CTL}^*_d \backslash X$ formulas. Write $[M]_{\text{CTL}^*_d \backslash X}$ for the set of all LTSS $N$ such that $M \equiv_{\text{CTL}^*_d \backslash X} N$.

Following the method of Section 3 we prove that the following finiteness property holds (where $k$ represents the number of process-index quantifiers in the prenex indexed temporal logic formula).

Remark 11. For ease of exposition we sketch a proof under the assumption that path quantifiers in formulas ignore runs in which the token does not visit every path quantifiers. This is an explicit restriction in [3] and implicit in [8]. In the full version [1] we remove this restriction. For the purpose of this paper this restriction only affects the explicit cutoffs in Theorem 11.

**Theorem 12 (Finiteness).** For all positive integers $k$ and $d$, there are finitely many equivalence classes $[G[\bar{g}]]_{\text{CTL}^*_d \backslash X}$ where $G$ is an arbitrary topology, and $\bar{g}$ is a $k$-tuple over $V_G$.

**Proof Idea.** We provide an algorithm that given positive integers $k, d$, topology $G$, $k$-tuple $\bar{g}$ over $V_G$, returns a LTS $\text{CON}_d G[\bar{g}]$ such that $G[\bar{g}] \equiv_{\text{CTL}^*_d \backslash X} \text{CON}_d G[\bar{g}]$. Moreover, we prove that for fixed $k, d$ the range of $\text{CON}_d \_\_$ is finite.

Recursively define a marking function $\mu_d$ that associates with each $v \in V_G$ a finite set (of finite strings over alphabet $\mu_{d-1}(V_G)$). For the base case define $\mu_0(v) := \Lambda(v)$, the labeling of $G[\bar{g}]$. The marking $\mu_d(v)$ stores (representatives) of all strings of $\mu_{d-1}$-labels of paths that start in $v$ and reach some element in $\bar{g}$. The idea is that $\mu_d(v)$ determines the set of $\text{CTL}^*_d \backslash X$ formulas that hold in $G[\bar{g}]$ with initial vertex $v$, as follows: stitch together these strings, using elements of $\bar{g}$ as stitching points, to get the $\text{CTL}^*_d \backslash X$ types of the infinite paths starting in
v. This allows us to define a topology, called the \( d \)-contraction \( \text{CON}_d G[\bar{g}] \), whose vertices are the \( \mu_d \)-markings of vertices in \( G \). In the full version [1] we prove that \( G[\bar{g}] \) is \( \text{CTL}^*_d \backslash X \)-equivalent to its \( d \)-contraction, and that the number of different \( d \)-contractions is finite, and depends on \( k \) and \( d \).

**Definition of \( d \)-contraction \( \text{CON}_d G[\bar{g}] \).** Next we define \( d \)-contractions.

**Marking \( \mu_d \).** Fix \( k, d \in \mathbb{N} \), topology \( G \), and \( k \)-tuple \( \bar{g} \) over \( V_G \). Let \( A \) be the labeling-function of \( G[\bar{g}] \), i.e., \( A(v) = \{ i \} \) if \( v = g_i \), and \( A(v) = \emptyset \) for \( v \not\in \bar{g} \). For every vertex \( v \in V_G \) define a set \( X(v) \) of paths of \( G \) as follows: a path \( \pi = \pi_1 \ldots \pi_t \), say of length \( t \), is in \( X(v) \) if \( \pi \) starts in \( v \), none of \( \pi_1, \ldots, \pi_{t-1} \) is in \( \bar{g} \), and \( \pi_t \in \bar{g} \). Note that \( X(g_i) = \{ g_i \} \).

Define the marking \( \mu_d \) inductively:

\[
\mu_d(v) := \begin{cases} 
  A(v) & \text{if } d = 0 \\
  \{ \text{destutter}(\mu_{d-1}(\pi_1) \ldots \mu_{d-1}(\pi_t)) : \pi_1 \ldots \pi_t \in X(v), t \in \mathbb{N} \} & \text{if } d > 0,
\end{cases}
\]

where destutter\( (w) \) is the maximal substring \( s \) of \( w \) such that for every two consecutive letters \( s_i \) and \( s_{i+1} \) we have that \( s_i \neq s_{i+1} \). Informally, remove identical consecutive letters of \( w \) to get the ‘destuttering’ destutter\( (w) \).

The elements of \( \mu_d(v) \) \((d > 0)\) are finite strings over the alphabet \( \mu_{d-1}(V_G) \). For instance, strings in \( \mu_1(v) \) are over the alphabet \( \{ \{1\}, \{2\}, \ldots, \{k\}, \emptyset \} \).

**Equivalence relation \( \sim_d \).** Vertices \( v, u \in V_G \) are \( d \)-equivalent, written \( u \sim_d v \), if \( \mu_d(v) = \mu_d(u) \). We say that \( \sim_d \) refines \( \sim_j \) if \( u \sim_d v \) implies \( u \sim_j v \).

**Lemma 13.** If \( 0 \leq j < d \), then \( \sim_d \) refines \( \sim_j \).

Indeed, observe that for all nodes \( v \), all strings in \( \mu_d(v) \) start with the letter \( \mu_{d-1}(v) \). Thus \( \mu_d(v) = \mu_d(u) \) implies that \( \mu_{d-1}(v) = \mu_{d-1}(u) \). In other words, if \( u \sim_d v \) then \( u \sim_{d-1} v \), and thus also \( u \sim_j v \) for \( 0 \leq j < d \).

**\( d \)-contraction \( \text{CON}_d G[\bar{g}] \).** Define an LTS \( \text{CON}_d G[\bar{g}] \) called the \( d \)-contraction of \( G[\bar{g}] \) as follows. The nodes of the contraction are the \( \sim_d \)-equivalence classes. Put an edge between \( [u]_{\sim_d} \) and \( [v]_{\sim_d} \) if there exists \( u' \in [u]_{\sim_d}, v' \in [v]_{\sim_d} \) and an edge in \( G \) from \( u' \) to \( v' \). The initial state is \( [x]_{\sim_d} \) where \( x \) is the initial vertex of \( G \). The label of \( [u]_{\sim_d} \) is defined to be \( A(u) \) — this is well-defined because, by Lemma 13, \( \sim_d \) refines \( \sim_0 \).

In the full version [1] we prove that \( G[\bar{g}] \) is \( \text{CTL}^*_d \backslash X \)-equivalent to its \( d \)-contraction, and that the number of different \( d \)-contractions is finite, and depends on \( k \) and \( d \). \( \square \)

5 **Cutoffs for \( k \)-index \( \text{CTL}^* \backslash X \) and Concrete Topologies**

The following two theorems answer Question 2 from the introduction, regarding the PMCP for specifications from \( \{ \forall, \exists \}^* \text{-CTL}^* \backslash X \).

First, the PMCP is undecidable for certain (pathological) parameterized topologies \( G \) and specifications from \( \{ \forall, \exists \}^* \text{-CTL}^* \backslash X \).
Theorem 14. There exists a process $P \in \mathcal{P}_w$, and parameterized topologies $G$, $H$, such that the following PMCPs are undecidable:

1. $\text{PMCP}_G(\{P\}, \{\forall, \exists\}^*\text{-CTL}\setminus X)$.
2. $\text{PMCP}_H(\{P\}, \{\forall, \exists\}^2\text{-CTL}\setminus X)$.

Moreover, $G$ and $H$ can be chosen to be computable sets of topologies.

Second, PMCP is decidable for certain (regular) parameterized topologies and specifications from $\{\forall\}^*\text{-CTL}\setminus X$. This generalizes results from Emerson and Namjoshi [8] who show this result for $\{\forall\}^k\text{-CTL}\setminus X$ with $k = 1, 2$ and uni-directional ring topologies. By Remark 11 these cutoffs apply under the assumption that we ignore runs that do not visit every process infinitely often.

Theorem 15. If $G$ is as stated, then $\text{PMCP}_G(\mathcal{P}_w, \{\forall\}^k\text{-CTL}\setminus X)$ has the stated cutoff:

1. If $G$ is the set of uni-directional rings, then $2k$ is a cutoff.
2. If $G$ is the set of bi-directional rings, then $2k$ is a cutoff.
3. If $G$ is the set of cliques, then $k + 1$ is a cutoff.
4. If $G$ is the set of stars, then $k + 1$ is a cutoff.

Consequently, for each $G$ listed, $\text{PMCP}_G(\mathcal{P}_w, \{\forall\}^*\text{-CTL}\setminus X)$ is decidable.

This theorem is proved following Remark 6 given $k, d$, we compute a set $G_1, \ldots, G_N \in G$ such that every $B_G$ for $G \in G$ is logically equivalent to some $B_{G_i}$, where the Boolean formula $B_G$ is defined as $\bigwedge_{g} g_{\text{con}_d(G)[g]}$. By doing this, note that $B_G$ is logically equivalent to $B_H$ if and only if $\{\text{con}_dG[g] : g \in V_G\} = \{\text{con}_dH[h] : h \in V_H\}$ (this is where we use that there is no quantifier alternation). So it is sufficient to prove that, if $c$ is the stated cutoff,

$$|V_G|, |V_H| \geq c \implies \{\text{con}_dG[g] : g \in V_G\} = \{\text{con}_dH[h] : h \in V_H\}$$

To illustrate how to do this, we analyze the case of uni-directional rings and cliques (the other cases are similar).

Uni-directional rings. Suppose $G$ are the uni-directional rings and let $G \in G$. Fix a $k$-tuple of distinct elements of $V_G$, say $(g_1, g_2, \ldots, g_k)$. Define a function $f : V_G \to \{g_1, \ldots, g_k\}$ that maps $v$ to the first element of $G$ on the path $v, v + 1, v + 2, \ldots$ (addition is mod $|V_G|$). In particular $f(g_i) = g_i$ for $i \in [k]$. In the terminology of the proof of Theorem 12 $X(v)$ consists of the simple path $v, v + 1, \ldots, f(v)$.

We now describe $\mu_d$. Clearly $\mu_d(g_i) = \{\mu_{d-1}(g_i)\}$. By induction on $d$ one can prove that if $v \notin g$ with $f(v) = g_j$ then $\mu_d(v) = \{\mu_{d-1}(v) \cdot \mu_{d-1}(g_j)\}$. So for every $d > 1$, the equivalences $\sim_d$ and $\sim_1$ coincide.

| $d$ | 0 | 1 | 2 | \ldots |
|-----|---|---|---|-------|
| $\mu_d(v)$ for $v = g_i$ | $\{\{i\}\}$ | $\{\{i\}\}$ | \ldots |
| $\mu_d(v)$ if $v \notin g$ and $f(v) = g_j$ | $\emptyset \cdot \{\{j\}\}$ | $\{\emptyset \cdot \{j\}\}$ | \ldots |

Thus for every $k \in \mathbb{N}$, the $d$-contraction $\text{con}_dG[g]$ is a ring of size at most $2k$ (in particular, it is independent of $d$). In words, the $d$-contraction of $G$ is the ring
resulting by identifying adjacent elements not in $\bar{g}$. It is not hard to see that if $G, H$ are rings such that $|V_G|, |V_H| \geq 2k$ then for every $\bar{g}$ there exists $\bar{h}$ such that $\text{CON}_d G[\bar{g}] = \text{CON}_d H[\bar{h}]$.

**Cliquets.** Fix $n \in \mathbb{N}$. Let $G$ be a clique of size $n$. That is: $V_G = [n]$ and $(i, j) \in E_G$ for $1 \leq i \neq j \leq n$. Fix a $k$-tuple of distinct elements of $V_G$, say $(g_1, g_2, \ldots, g_k)$. We now describe $\mu_d(v)$. Clearly $\mu_d(g_i) = \{\mu_{d-1}(g_i)\}$ and for $v \notin \bar{g}$ we have $\mu_d(v) = \{\mu_{d-1}(v) : j \in [k]\}$. So for every $d > 1$, the equivalences $\sim_d$ and $\sim_1$ coincide, and the $d$-contraction $\text{CON}_d G[\bar{g}]$ is the clique of size $k + 1$. In words, the $d$-contraction of $G$ results from $G$ by identifying all vertices not in $\bar{g}$. It is not hard to see that if $G, H$ are cliques such that $|V_G|, |V_H| \geq k + 1$ then for every $\bar{g}$ (of size $k$) there exists $\bar{h}$ such that $\text{CON}_d G[\bar{g}] = \text{CON}_d H[\bar{h}]$.

**Remark 16.** For cliques and stars, $k + 1$ is also a cutoff for $\{\forall, \exists\}^k$-$\text{CTL}^* \setminus X$. Also, $2k$ is not a cutoff for uni-rings and $\{\forall, \exists\}^k$-$\text{LTL} \setminus X$ as stated in [13] Corollary 2. To see this, let $tok_i$ express that the process with index $i$ has the token, and $\text{adj}(k, i) := tok_k \rightarrow tok_i \lor tok_i \rightarrow tok_k \lor tok_i$. Then the formula $\exists j \forall k. \text{adj}(k, i) \lor \text{adj}(k, j)$, holds in the ring of size 6, but not 7.

### 6 There are No Cutoffs for Direction-Aware Systems

In the following, we consider systems where processes can choose which directions are used to send or receive the token, i.e., process templates are from $\mathcal{P}_{\text{snd}}, \mathcal{P}_{\text{rcv}}$, or $\mathcal{P}_{\text{sndrcv}}$. Let $B$ be the parameterized topology of all bi-directional rings, with directions $\text{cw}$ (clockwise) and $\text{ccw}$ (counter-clockwise). The following theorem answers Question 3 from the introduction.

**Theorem 17.**
1. $\text{PMCP}_B(\mathcal{P}_{\text{sndrcv}}, \forall$-$\text{LTL} \setminus X)$ is undecidable.
2. For $\mathcal{F}$ equal to $\{\forall\}^3$-$\text{LTL} \setminus X$ or $\{\exists\}^3$-$\text{CTL} \setminus X$, and $\mathcal{P} \in \{\mathcal{P}_{\text{snd}}, \mathcal{P}_{\text{rcv}}\}$, there exists a parameterized topology $G$ such that $\text{PMCP}_G(\mathcal{P}, \mathcal{F})$ is undecidable.

**Proof Idea.** We reduce the non-halting problem of two-counter machines (2CMs) to the PMCP. The idea is that one process, the controller, simulates the finite-state control of the 2CM. The other processes, arranged in a chain or a ring, are memory processes, collectively storing the counter values with a fixed memory per process. This allows a given system to simulate a 2CM with bounded counters. Since a 2CM terminates if and only if it terminates for some bound on the counter values, we can reduce the non-halting problem of 2CMs to the PMCP. The main work is to show that the controller can issue commands, such as ‘increment counter 1’ and ‘test counter 1 for zero’. We give a detailed proof sketch for part 1 of the theorem, and then outline a proof for part 2.

1. $\forall$-$\text{LTL} \setminus X$ and $\mathcal{P}_{\text{sndrcv}}$ in bidirectional rings.
   The process starting with the token becomes the controller, all others are memory, each storing one bit for each counter of the 2CM. The current value of a counter $c$ is the total number of corresponding bits ($c$-bits) set to 1. Thus, a system with $n$ processes can store counter values up to $n - 1$.  

Fix a numbering of 2CM-commands, say 0 ↦ ‘increment counter 1’, 1 ↦ ‘decrement counter 1’, 2 ↦ ‘test counter 1 for zero’, etc. Every process has a command variable that represents the command to be executed when it receives the token from direction ccw.

If the controller sends the token in direction cw, the memory processes will increment (mod 6) the command variable, allowing the controller to encode which command should be executed. Every process just continues to pass the token in direction cw, until it reaches the controller again.

If the controller sends the token in direction ccw, then the memory processes try to execute the command currently stored. If it is an 'increment counter c' or 'decrement counter c' command, the memory process tries to execute it (by incrementing/decrementing its c-bit). If the process cannot execute the command (because the c-bit is already 1 for an increment, or 0 for a decrement), then it passes the token along direction ccw and remembers that a command is being tried. If the token reaches a memory process which can execute the command, then it does so and passes the token back in direction cw. The processes that remembered that a command is being tried will receive the token from direction cw, and know that the command has been successfully executed, and so will the controller. If the controller gets the token from ccw, the command failed. In this case, the controller enters a loop in which it just passes the token in direction cw (and no more commands are executed).

If the command stored in the memory processes is a ‘test for zero counter c’, then the processes check if their c-bit is 0. If this is the case, it (remembers that a command is being tried and) passes the token to the next process in direction ccw. If the token reaches a process for which the c-bit is 1, then this process sends the token back in direction cw. Other memory processes receiving it from cw (and remembering that the command is being tried), pass it on in direction cw. In this case, the controller will receive the token from cw and know that counter c is not zero. On the other hand, if all memory processes store 0 in their c-bit, then they all send the token in direction ccw. Thus, the controller will receive it from ccw and knows that counter c currently is zero. To terminate the command, it sends the token in direction cw, and all processes (which remembered that a command is being tried), know that execution of this command is finished.

With the description above, a system with \( n - 1 \) memory processes can simulate a 2CM as long as counter values are less than \( n \). Let \( HALT \) be an atomic proposition that holds only in the controller’s halting states. Then solving the PMCP for \( \forall i G \neg HALT_i \) amounts to solving the non-halting problem of the 2CM.

2. \{∀\}^9\text{LTL} \setminus X and \( P_{\text{end}} \).

We give a proof outline. In this case there are \( 2n \) memory processes, \( n \) for each counter \( c \in \{1, 2\} \). The remaining 9 processes are special and called ‘controller’, ‘counter c is zero’, ‘counter c is not zero’, ‘counter c was incremented’, and ‘counter c was decremented’. When the controller wants to increment or decrement counter c, it sends the token non-deterministically to some memory process for counter c. When the controller wants to test counter c for zero, it sends the token to the first memory process. When a memory process receives the token
it does not know who sent it, and in particular does not know the intended command. Thus, it non-deterministically takes the appropriate action for one of the possible commands. If its bit is set to 0 then it either i) increments its bit and sends the token to a special process ‘counter c was incremented’, or ii) it sends the token to the next memory node in the chain, or to the special process ‘counter c is zero’ if it is the last in the chain. If its bit is set to 1 then it either i) decrements its bit and sends the token to a special process ‘counter c was decremented’, or ii) sends the token to a special process ‘counter c is not zero’.

Even though incoming directions are not available to the processes, we can write the specification such that, out of all the possible non-deterministic runs, we only consider those in which the controller receives the token from the expected special node (the formula requires one quantified index variable for each of the special nodes). So, if the controller wanted to increment counter c it needs to receive the token from process ‘counter c was incremented’. If the controller receives the token from a different node, it means that a command was issued but not executed correctly, and the formula disregards this run. Otherwise, the system of size $2n+1$ correctly simulates the 2CM until one of the counter values exceeds $n$.

7 Extensions

There are a number of extensions of direction-unaware TPSs for which the theorems that state existence of cutoffs (Theorems 7 and 15) still hold. We describe these in order to highlight assumptions that make the proofs work:

1. Processes can be infinite-state.
2. The EN-restriction on the process template $P$ can be relaxed: replace item $vii$) in Definition [2.1] by “For every state $q$ that has the token there is a finite path $q \ldots q'$ such that $q'$ does not have the token, and for every $q$ that does not have the token there is a finite path $q \ldots q'$ such that $q'$ has the token”.
3. One can further allow direction-sensing TPSs, which is a direction-aware TPS with an additional restriction on the process template: “If $q \overset{d}{\rightarrow} q' \in \delta$ for some direction $d \in \text{Dir}_{\text{snd}}$, then for every $d \in \text{Dir}_{\text{snd}}$ there exists a transition $q \overset{d}{\rightarrow} q'' \in \delta''$; and a similar statement for $\text{Dir}_{\text{rcv}}$. Informally: we can allow processes to change state according to the direction that the token is (non-deterministically) sent to or received, but the processes are not allowed to block any particular direction.
4. One can further allow the token to carry a value but with the strong restriction that from every state that has the token and every value $v$ there is a path of internal actions in $P$ which eventually sends the token with value $v$, and the same for receiving.

These conditions on $P$ all have the same flavor: they ensure that a process can not choose what information to send/receive, whether that information is a value on the token or a direction for the token.
8 Related work

Besides the results that this paper is directly based on \cite{18,8,5}, there are several other relevant papers. Emerson and Kahlon \cite{7} consider token-passing in uni- and bi-directional rings, where processes are direction-aware and tokens carry messages (but can only be changed a bounded number of times). However, the provided cutoff theorems only hold for specifications that talk about two processes (in a uni-directional ring) or one process (in a bi-directional ring), process templates need to be deterministic, and no cutoffs depend on the size of the process implementation.

German and Sistla \cite{12} provide cutoffs for the PMCP for systems with pairwise synchronization. Although pairwise synchronization can simulate token-passing, their cutoff results are restricted to cliques and 1-indexed LTL. Moreover, their proof uses vector-addition systems with states and their cutoff depends on the process template and the specification formula.

Delzanno et al. \cite{6} study a model of broadcast protocols on arbitrary topologies, in which a process can synchronize with all of its available neighbors ‘at once’ by broadcasting a message (from a finite set of messages). They prove undecidability of PMCP for systems with arbitrary topologies and 1-indexed safety properties, and that the problem becomes decidable if one restricts the topologies to ‘graphs with bounded paths’ (such as stars). Their proof uses the machinery of well-structured transitions systems, and no cutoffs are provided. They also show undecidability of the PMCP in the case of non-prenex indexed properties of the form $G(\exists i.s(i) \in B)$ on general and the restricted topologies.

Rabinovich \cite{15, Section 4} proves, using the composition method, that if monadic second-order theory of the set of topologies in \(G\) is decidable, then the PMCP is decidable for propositional modal logic. The systems considered are defined by a very general notion of product of systems (which includes our token-passing systems as a subcase).

The PMCP for various fragments of non-prenex indexed LTL is undecidable, see German and Sistla \cite{12, Section 6} for systems with pairwise synchronization, and John et al. \cite{13, Appendix A} for systems with no synchronization at all.

9 Summary

The goal of this work was to find out under what conditions there are cutoffs for temporal logics and token-passing systems on general topologies. We found that stratifying prenex indexed CTL*\(\setminus X\) by nesting-depth of path quantifiers allowed us to recover the existence of cutoffs; but that there are no cutoffs if the processes are allowed to choose the direction of the token. In all the considered cases where there is no cutoff we show that the PMCP problem is actually undecidable.

Our positive results are provided by a construction that generalizes and unifies the known positive results, and clearly decomposes the problem into two aspects: tracking the movement of the token through the underlying topology, and simulating the internal states of the processes that the specification formula
can see. The construction yields small cutoffs for common topologies (such as rings, stars, and cliques) and specifications from prenex indexed $\text{CTL}^* \setminus X$.

Acknowledgments. We thank Roderick Bloem for detailed comments on numerous drafts and Krishnendu Chatterjee for important comments regarding the structure of the paper. We thank Roderick Bloem, Igor Konnov, Helmut Veith, and Josef Widder for discussions at an early stage of this work that, in particular, pointed out the relevance of direction-unawareness in [5].

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