Subtracted Geometry from Harrison Transformations: II

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Abstract

We extend our previous study (arXiv:1203.5088) to the case of five-dimensional multi-charge black holes, thus showing that these configurations and their subtracted geometries also lie in a 3d duality orbit. In order to explore the 3d duality orbit, we do a timelike reduction from 5d to 4d and a spacelike reduction from 4d to 3d. We present our analysis in the notation of Euclidean N=2 supergravity and its c-map. We also relate our analysis to that of Cvetič, Guica, and Saleem.

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1 Introduction and Summary

String theory has come a long way in addressing the question of entropy for extremal black holes. However, very little is known about microscopic origins of thermodynamic properties of general non-extreme black holes. An obvious obstacle one faces in making progress on this problem is that the specific heats of non-extreme black holes in asymptotically flat settings are typically negative, whereas those of unitary field theories are typically positive. To get around some of these difficulties Cvetič and Larsen [1, 2] have put forward the proposal of subtracted geometry.

The notion of subtracted geometry offers several attractive features. Firstly, it makes the idea of hidden conformal symmetry of Castro, Maloney, and Strominger [3] geometrical. Secondly, it provides us with a classical geometry that can be regarded, in some respect, as the “near-horizon” analog for non-extreme black holes. Thirdly, perhaps the most attractive feature of subtracted geometries is that they lift to AdS$_3$ times a sphere in one higher dimension. These features have attracted the attention of several researchers [4, 5, 6, 7]. However, it remains to be seen how these ideas pan out in paving a path towards a detailed microscopic understanding of general non-extreme black holes in string theory. A summary of the current literature on this subject is as follows.

Since the beginning an important aspect in the discussion of subtracted geometry has been how to obtain these geometries systematically starting from a black hole solution. The initial papers [1, 2] focused on the STU model where the subtraction procedure was implemented as an adjustment of the warp factor of multi-charge asymptotically flat black holes. It was then not clear how to implement such a procedure for other settings. Around that time, Bertini, Cacciatori, and Klemm [8] realized that SU(2,1) Harrison transformations$^1$ of Einstein Maxwell theory, which map 4d Schwarzschild solution to AdS$_2 \times$ S$^2$, also map Schwarzschild hidden conformal symmetry generators to isometries of AdS$_2$. Significant progress happened when Cvetič and Gibbons [5] showed that the subtracted geometry can be obtained by a scaling limit of asymptotically flat multi-charge black holes. They also conjectured that these geometries lie in a three-dimensional duality orbit of the original black hole, and hence should be obtainable using a particular Harrison transformation on the original black holes. This conjecture was confirmed in our previous work [9]. For earlier related work see [10, 11]. Motivated by these developments Baggio, de Boer, Jottar, and Mayerson [6] argued that at least in the 4d static case the adjustment of the warp factor can be implemented dynamically by means of an interpolating flow. They also constructed an interpolating solution. From their point of view a dual description of the 5d uplift of the 4d multi-charge asymptotically flat black hole can be obtained by turning on certain specific irrelevant deformations in the Maldacena-Strominger-Witten (MSW) CFT. In another line of investigation $^1$These transformations are also sometimes known as Kinnersley-Chitre transformations.
Chakraborty, Jana, and Krishnan [7] have studied various aspects of interpolating solutions in the extremal limit and their relation to the attractor mechanism. They have also proposed a more general notion of subtracted geometries. Recently Cvetič, Guica, and Saleem [12] have constructed interpolating solution for both 4d and 5d rotating black holes. For other related developments see also [13, 14, 15].

In this paper we extend our previous study [9] to the case of 5d multi-charge black holes. Our study provides a somewhat different perspective on the five-dimensional analysis of [12]. We show that 5d multi-charge, rotating, non-extreme black hole configurations and their subtracted geometries lie in a 3d duality orbit. The duality orbit we explore is of a rather special type: we first do a timelike reduction (for details on timelike reduction see [16, 17]) from 5d to 4d and then a spacelike reduction from 4d to 3d. Since we do timelike reduction first, we get a Euclidean theory in 4d. This theory is nothing but the Euclidean version of the N=2 STU supergravity. We make connection with the well developed formalism of Euclidean N=2 supergravity [18, 19, 20, 21].

The rest of the paper is organized as follows. In section 2 we perform the dimensional reduction. In section 3 we present the main result of our investigation, first for static black holes and then for rotating black holes. In this section we also relate our analysis to that of Cvetič, Guica, and Saleem [12].

## 2 Dimensional Reduction

In this section we present an appropriate dimensional reduction of five-dimensional U(1)$^3$ supergravity to three dimensions.

### 2.1 Timelike Reduction: 5d to 4d

Our starting point is the Lagrangian of five-dimensional U(1)$^3$ supergravity

$$L_5 = R_5 \ast_5 1 - \frac{1}{2} G_{I,J} \ast_5 dh^I \wedge dh^J - \frac{1}{2} G_{I,J} \ast_5 F^I_{[2]} \wedge F^J_{[2]} + \frac{1}{6} C_{IJK} F^I_{[2]} \wedge F^J_{[2]} \wedge A^K_{[1]}, \quad (2.1)$$

where $C_{IJK} = |\epsilon_{IJK}|$, with $I = 1, 2, 3$, and $G_{I,J}$ is diagonal with entries $G_{II} = (h^I)^{-2}$. The scalars $h^I$’s satisfy the constraint $h^1 h^2 h^3 = 1$ that must be solved before computing variations of the action in order to equations of motion for various fields.

To perform KK reduction we parameterize our 5d spacetime as

$$ds^2 = \epsilon_1 f^2 (dt + \tilde{A}_{[1]}^0)^2 + f^{-1} ds_4^2, \quad (2.2)$$

and 5d vectors as

$$A_{[1]}^I = \chi^I (dt + \tilde{A}_{[1]}^0) + \tilde{A}_{[1]}^I, \quad (2.3)$$
where we use $\epsilon_1$ to keep track of minus signs. When $\epsilon_1$ is $+1$ we are performing the standard spacelike reduction that, for example, is presented in detail in [9]. The case of interest for the present discussion is $\epsilon_1 = -1$.

Upon KK reduction the 4d graviphoton $\tilde{A}^0_{[1]}$ and the 4d vectors $\tilde{A}^I_{[1]}$ together form a symplectic vector $\tilde{A}^\Lambda_{[1]}$ with $\Lambda = 0, 1, 2, 3$. We define the field strength for the symplectic vector to be simply $F^\Lambda_{[2]} = d\tilde{A}^\Lambda_{[1]}$.

From the Lagrangian (2.1) using the ansatzes (2.2) and (2.3) we obtain,

$$L_4 = R \star_4 1 - \frac{1}{2} G_{IJ} \star_4 dh^I \wedge dh^J - \frac{3}{2f^2} \star_4 df \wedge df - \epsilon_1 \frac{f^3}{2} \star_4 \tilde{F}^0_{[2]} \wedge \tilde{F}^0_{[2]}$$

$$- \epsilon_1 \frac{1}{2f^2} G_{IJ} \star_4 d\chi^I \wedge d\chi^J - \frac{f}{2} G_{IJ} \star_4 (\tilde{F}^I_{[2]} + \chi^I \tilde{F}^0_{[2]}) \wedge (\tilde{F}^J_{[2]} + \chi^J \tilde{F}^0_{[2]})$$

$$+ \frac{1}{2} C_{IJK} \chi^I \tilde{F}^J_{[2]} \wedge \tilde{F}^K_{[2]} + \frac{1}{2} C_{IJK} \chi^I \chi^J \tilde{F}^0_{[2]} \wedge \tilde{F}^K_{[2]} + \frac{1}{6} C_{IJK} \chi^I \chi^J \chi^K \tilde{F}^0_{[2]} \wedge \tilde{F}^0_{[2]}.$$ (2.4)

Note that there are two signs in the above expression, namely the kinetic term for the graviphoton $\tilde{F}^0_{[2]}$ and the kinetic terms for the scalars $\chi^I$'s, that depend on the sign $\epsilon_1$.

### 2.2 Euclidean Supergravity

The reduced Lagrangian (2.4) can also be obtained using prepotential formalism [22] of Euclidean $N = 2$ supergravity [18, 19, 20, 21]. We briefly summarize this formalism. It uses the so-called split complex numbers. Recall that split complex numbers [23] satisfy the standard conjugation relation but the imaginary unit $e$ squares to $+1$ instead of $-1$,

$$\bar{e} = -e, \quad e^2 = +1.$$ (2.5)

As in the Lorentzian case, the action of Euclidean $N = 2$ supergravity coupled to $n$ vector multiplets is also governed by a prepotential function $F$, which is now a function of $n + 1$ split complex variables $X^\Lambda$ ($0 \leq \Lambda \leq n$). The gauge invariant bosonic degrees of freedom are the Euclidean metric $g_{\mu\nu}$, split complex scalars $X^\Lambda$, and $n + 1$ vectors $\tilde{A}^\Sigma$. The Lagrangian [22] is given by [18, 19, 20, 21]

$$L_4 = R \star_4 1 - 2g_{IJ} \star_4 dX^I \wedge dX^J + \frac{1}{2} \bar{F}^\Lambda_{[2]} \wedge \tilde{G}^\Lambda_{[2]}.$$ (2.6)

where $\bar{F}^\Lambda_{[2]} = d\tilde{A}^\Lambda_{[1]}$. The indices $I, J$ run from 1 to $n$, and $g_{IJ} = \partial_I \partial_J K$ with the potential

$$K = -\log \left[-e(\bar{X}^\Lambda F_\Lambda - \tilde{F}_\Lambda X^\Lambda)\right],$$ (2.7)

and where $F_\Lambda = \partial_\Lambda F$. The two form $\tilde{G}^\Lambda_{[2]}$ is defined as

$$\tilde{G}^\Lambda_{[2]} = (\text{Re}N)_{\Lambda\Sigma} \bar{F}^\Sigma_{[2]} + (\text{Im}N)_{\Lambda\Sigma} \star_4 \tilde{F}^\Sigma_{[2]}.$$ (2.8)
where the split complex symmetric matrix $N_{\Lambda\Sigma}$ is constructed from the prepotential as

$$N_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2e^{\frac{(\text{Im}F \cdot X)_{\Lambda}(\text{Im}F \cdot X)_{\Sigma}}{X \cdot \text{Im}F \cdot X}},$$

(2.9)

and where $F_{\Lambda\Sigma} = \partial_{\Lambda} \partial_{\Sigma} F$. Most of the above formulas are similar to the ones used in our previous work [9]. However, note that there are some sign changes with respect to [9], and most importantly the imaginary unit $i$ of the standard complex numbers is replaced with the imaginary unit $e$ of the split complex numbers at all places.

We are interested in the Euclidean STU model. For this model $n = 3$ and the prepotential function takes the form

$$F(X) = -\frac{X^1X^2X^3}{X^0}.$$  

(2.10)

Using the gauge fixing $X^0 = 1$ and $X^I = -\chi^I + efh^I$, the resulting Lagrangian can be shown to be identical to (2.4) with $\epsilon_1 = -1$.

2.3 Spacelike Reduction: 4d to 3d

Now we perform a spacelike reduction from four to three dimensions to obtain an $\text{SO}(4,4)/\left(\text{SO}(2,2) \times \text{SO}(2,2)\right)$ coset model. The denominator subgroup is a different $\text{SO}(2,2) \times \text{SO}(2,2)$ compared to the one used in [9] where the reduction was first done over a spacelike direction and then over a timelike direction.

For this KK reduction of the Lagrangian (2.4) we parameterize our four-dimensional Euclidean space as

$$ds^2_4 = e^{2U}(dz + \omega_3)^2 + e^{-2U}ds^2_3,$$

(2.11)

and the 4d one-forms as

$$A^A_{[i]} = \zeta^A(dz + \omega_3) + A^A_3,$$

(2.12)

where $A^A_3$ and $\omega_3$ are three-dimensional one-forms. Since now the reduction is done differently, the dualization equations are different compared to [9]. To find the correct dualization equations we proceed as in the lecture notes by Pope [24].

For both $A^A_3$ and $\omega_3$ we define the field strengths simply as $F^A_3 := dA^A_3$ and $F_3 := d\omega_3$. The procedure of dualization interchanges the role of Bianchi identities and the field equations. The easiest way to achieve dualization is to treat $F^A_3$ and $F_3$ as fundamental fields in their own right. We impose the Bianchi identities by adding the following Lagrange multiplier terms to the 3d Lagrangian

$$-\bar{\zeta}_A F^A_3 - \frac{1}{2}(\sigma + \zeta^A \bar{\zeta}_A) F_3.$$  

(2.13)
Thus the total three dimensional Lagrangian we consider is

\[
\mathcal{L}_3 = R \star_3 1 - 2 \star_3 dU \wedge dU - \frac{1}{2} e^{4U} \star_3 F_3 \wedge F_3 - 2 g_{1, j} \star_3 dX^I \wedge dX^J \\
+ \frac{1}{2} e^{2U} (\text{Im } N)_{\Lambda \Sigma} \star_3 (F^\Lambda_3 + \zeta^\Lambda F_3) \wedge (F^\Sigma_3 + \zeta^\Sigma F_3) + \frac{1}{2} e^{-2U} (\text{Im } N)_{\Lambda \Sigma} \star_3 d\zeta^\Lambda \wedge d\zeta^\Sigma \\
+(\text{Re } N)_{\Lambda \Sigma} \, d\zeta^\Lambda \wedge (F^\Sigma_3 + \zeta^\Sigma F_3) - \tilde{\zeta}_\Lambda dF_3^\Lambda - \frac{1}{2} (\sigma + \zeta^\Lambda \tilde{\zeta}_\Lambda) dF_3.
\] (2.14)

Clearly, variations of this Lagrangian with respect to \( \sigma \) and \( \tilde{\zeta}_\Lambda \) give the required Bianchi identities. Equation for \( F^\Sigma_3 \) and \( F_3 \) are purely algebraic. These equations allow us to do the dualizations of the one-forms. We find

\[- d\tilde{\zeta}_\Lambda = e^{2U} (\text{Im } N)_{\Lambda \Sigma} \star_3 (F^\Sigma_3 + \zeta^\Sigma F_3) + (\text{Re } N)_{\Lambda \Sigma} d\zeta^\Sigma, \]

and

\[- d\sigma = -2 e^{4U} \star_3 F_3 + \tilde{\zeta}_\Lambda d\zeta^\Lambda - \zeta^\Lambda d\tilde{\zeta}_\Lambda. \]

Substituting these back into Lagrangian (2.14) we find \( \mathcal{L}_3 \) takes the form (from now on we drop the prime)

\[
\mathcal{L}_3 = R \star_3 1 - \frac{1}{2} G_{ab} \partial \varphi^a \partial \varphi^b,
\] (2.17)

where the target space is a Lorentzian manifold (as expected). It is parameterized by 16 scalars \( \varphi^a \) and is of signature (8, 8). The metric in our conventions is

\[
G_{ab} \, d\varphi^a d\varphi^b = 4 dU^2 + 4 g_{1, j} dz^I d\bar{z}^J - \frac{1}{4} e^{-4U} \left( d\sigma + \tilde{\zeta}_\Lambda d\zeta^\Lambda - \zeta^\Lambda d\tilde{\zeta}_\Lambda \right)^2 \\
+ e^{-2U} \left[ -(\text{Im } N)_{\Lambda \Sigma} d\zeta^\Lambda d\zeta^\Sigma + ((\text{Im } N)^{-1})_{\Lambda \Sigma} \left( d\tilde{\zeta}_\Lambda + (\text{Re } N)_{\Lambda \Sigma} d\zeta^\Sigma \right) \left( d\tilde{\zeta}_\Sigma + (\text{Re } N)_{\Sigma \Gamma} d\zeta^\Gamma \right) \right].
\] (2.18)

It is a different analytic continuation of the c-map of Ferrara and Sabharwal \cite{25} compared to the one used in \cite{9}. The analytic continuation is as follows\(^3\)

\[
\chi^I \rightarrow i \chi^I \\
\sigma \rightarrow i \sigma \\
\zeta^0 \rightarrow -i \zeta^0 \\
\tilde{\zeta}_0 \rightarrow -\tilde{\zeta}_0 \\
\zeta^I \rightarrow \zeta^I \\
\tilde{\zeta}_I \rightarrow i \tilde{\zeta}_I.
\] (2.19-2.24)

---

\(^2\)Extra care must be exercised in checking this analytic continuation. Recall that in all the expressions above, real and imaginary parts refer to the split complex numbers, whereas in Ferrara and Sabharwal \cite{25} or in \cite{26} the real and imaginary parts refer to the standard complex numbers. Perhaps the easiest way to check this analytic continuation is to express the three-dimensional Lagrangian explicitly in terms of various scalars and then compare it with the analytically continued expressions of Ferrara and Sabharwal.

\(^3\)In these equations \( i \) refers to the imaginary unit of the standard complex numbers.
Note that only 8 scalars pick up a factor of $i$, so that the signature of the target space is $(8,8)$.

The symmetric space (2.18) can be parameterized in the Iwasawa gauge by the coset element

$$
\mathcal{V} = e^{-U H_0} \cdot \prod_{I=1,2,3} \left( e^{-\frac{1}{2} \left[ \log(f^I) \right] H_I} \cdot e^{\lambda_I E_I} \right) \cdot e^{-\zeta_\Lambda E_{0\Lambda}} \cdot e^{-\frac{1}{2} \sigma E_0}.
$$

(2.25)

For the Lie algebra generators we use the same notation as [9] (see also appendix A). The metric (2.18) is obtained through the Maurer-Cartan one-form $\theta = d\mathcal{V} \cdot \mathcal{V}^{-1}$ as follows (for details see for example [24, 27])

$$
G_{ab} d\phi^a d\phi^b = \text{Tr}(P_a P_a),
$$

(2.26)

$$
P_a = \frac{1}{2} (\theta - \tilde{\tau}(\theta)),
$$

(2.27)

where the involution $\tilde{\tau}$ that defines the coset is:

$$
\tilde{\tau}(H_0) = -H_0, \quad \tilde{\tau}(H_I) = -H_I, \quad \tilde{\tau}(E_0) = +F_0, \quad \tilde{\tau}(E_I) = +F_I, \quad \tilde{\tau}(E_{0\rho}) = +F_{0\rho}, \quad \tilde{\tau}(E_{qI}) = -F_{qI}, \quad \tilde{\tau}(E_{p0}) = -F_{p0}, \quad \tilde{\tau}(E_{pI}) = +F_{pI}.
$$

(2.28) - (2.31)

3 5d Charged Black Holes and their Subtracted Geometry

In this section we make use of the formalism presented in the previous section and show that subtracted geometry of five-dimensional three-charge black hole lies in a three-dimensional duality orbit of the black hole itself. Cvetič, Guica, and Saleem have recently obtained [12] a closely related result. They have shown that in the five-dimensional case subtracted geometry can be obtained from STU transformations of the Euclidean STU supergravity. We also obtain this result, however, our presentation and analysis is complementary to theirs. We show this by establishing that the three-dimensional duality transformations that we apply to obtain subtracted geometry are in fact part of the four-dimensional duality group. We establish this using the analysis of Bossard, Nicolai, and Stelle [28].

The reason we work with three-dimensional duality orbits is manifold. (i) Once we realized that from 5d to 4d timelike reduction is required to perform our analysis we found it natural to relate our analysis to the well developed Euclidean N=2 supergravity formalism [18, 19, 20, 21] including its c-map. (ii) The original construction of five-dimensional three-charge black hole by Cvetič and Youm [29] was done using 3d duality transformations. We found it useful to relate our analysis to theirs at certain intermediate steps. (iii) Lastly, four-dimensional Euclidean STU
transformations are part of the three-dimensional duality transformations. Thus, our analysis offers a different perspective on the results of Cvetič, Guica, and Saleem.

3.1 Three-dimensional Duality Orbit

To explore 3d duality orbit we proceed as in [9, 27]. Having constructed the coset representative \( V \) we define the generalized transposition \( \sharp(x) = -\tilde{\tau}(x), \forall x \in \mathfrak{so}(4, 4) \). Next we encode all 16 scalars of the \( \text{SO}(4, 4) / \text{SO}(2, 2) \times \text{SO}(2, 2) \) coset in a matrix \( M \) defined as \( M = (V^\sharp V) \). Under \( \text{SO}(4, 4) \) group action the matrix \( M \) transforms as \( M \rightarrow M' = g^\sharp M g \). The transformed solution is constructed using the new matrix \( M' \). Since the involution \( \tilde{\tau} \) is different in the present case compared to the one used in [9], a separate Mathematica implementation is required for extracting scalars from the matrix \( M \). Similarly when dualizing appropriate scalars to one-forms additional care is required in the present case since the real and imaginary parts of the matrix \( N \) in equation (2.15) refer to the real and imaginary parts with respect to split-complex numbers.

3.2 5d Subtracted Geometry from Harrison Transformations

As in our previous work [9] to obtain subtracted geometry of three-charge five-dimensional black hole we act on the black hole with a series a transformations. This investigation was initiated in [8, 5]. The key insight that our previous work [9] brought to this analysis was that certain negative roots of Lie algebra \( \mathfrak{so}(4, 4) \) are required to perform the appropriate Harrison boosts. For the four-charge black hole in four-dimensions three negative roots were used. Each negative root brings down a power of \( r \) in the warp factor \( \Delta \). As a result, after the application of these transformations \( \Delta \) grows linearly with \( r \), whereas for asymptotically flat black holes it grows as \( r^4 \) [2].

For the five-dimensional analysis very similar discussion applies. Having constructed the three-charge black hole by the action of [29]

\[
\begin{align*}
g_{\text{charging}} & = \exp[\beta_1(E_1 + F_1)] \cdot \exp[\beta_2(E_2 + F_2)] \cdot \exp[\beta_3(E_3 + F_3)] \\
\end{align*}
\]

on the five-dimensional Schwarzschild black hole, we act with

\[
\begin{align*}
g_{\text{subtraction}} & = \exp[F_1 + F_2].
\end{align*}
\]

Note that in this step we exponentiate only two negative roots of the \( \mathfrak{so}(4, 4) \) Lie algebra. Each negative root brings down the power of \( r \) by two in the five-dimensional warp factor and as a result \( \Delta \) grows as \( r^2 \) after the action of these generators. In the asymptotically flat space it grows as \( r^6 \) [1].
As in the 4d case [9] one also needs to perform certain scaling transformations to get the subtracted geometry in the form of [5]. These transformations are as follows

\[ g_{\text{scaling}} = \exp[-c_1 H_1 - c_2 H_2 - c_3 H_3]. \]  

(3.3)

As the next step, we change variables following the suggestion of [5] and choose \( c_1, c_2, c_3 \) in some specific way. The choice

\[ \beta_1 = c_1 = \frac{1}{4} \ln \left[ (\Pi_c^2 - \Pi_s^2) \gamma_1 \right], \quad \beta_2 = c_2 = \frac{1}{4} \ln \left[ (\Pi_c^2 - \Pi_s^2) \gamma_2 \right] \]  

(3.4)

and

\[ \beta_3 = \sinh^{-1} \left[ \frac{\Pi_s}{\sqrt{\Pi_c^2 - \Pi_s^2}} \right], \quad c_3 = -\frac{1}{2} \ln \left[ (\Pi_c^2 - \Pi_s^2) \right] \]  

(3.5)

leads to subtracted geometry in exactly the form as given in the appendix of [5]. The resulting geometry still has \( \gamma_1 \) and \( \gamma_2 \) are parameters, but they appear exclusively as constant terms in five-dimensional vector fields and hence can be gauged away.

For simplicity we first present various fields and further details for the static case. After the action of \( g_{\text{charging}} \) on 5d Schwarzschild black hole various five-dimensional fields take the following form. The metric is

\[ ds^2_5 = -\Delta^{-\frac{4}{3}} r^2 (r^2 - 2m) dt^2 + \Delta^{\frac{4}{3}} ds^2_4 \]  

(3.6)

where

\[ ds^2_4 = \frac{1}{r^2 - 2m} dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2, \]  

(3.7)

and where the warp factor \( \Delta \) is

\[ \Delta = r^6 H_1 H_2 H_3, \]  

(3.8)

with the Harmonic functions

\[ H_I = 1 + \frac{2m \sinh^2 \beta_I}{r^2}. \]  

(3.9)

The five-dimensional vectors and scalars respectively take the form

\[ A_I = \frac{m \sinh 2 \beta_I}{r^2 H_I}, \quad h_I = H_I^{-1} (H_1 H_2 H_3)^{\frac{1}{3}}. \]  

(3.10)

On this solution\(^4\) after the action of \( g_{\text{subtraction}} \) and \( g_{\text{scaling}} \) we get

\[ \Delta = 4m^2 \left( r^2 (\Pi_c^2 - \Pi_s^2) + 2m \Pi_s^2 \right), \]  

(3.11)

\(^4\)The KK reductions are done over first \( t \) and and then over \( \psi \).
in the metric (3.6) and $d\delta^2$ remains unchanged. The new 5d vectors are scalars respectively

\begin{align*}
A^1 &= -r^2 + m + m \sqrt{\gamma_1 (\Pi_c^2 - \Pi_s^2)} \frac{dt}{2m} \equiv -\frac{r^2}{2m} dt, \\
A^2 &= -r^2 + m + m \sqrt{\gamma_2 (\Pi_c^2 - \Pi_s^2)} \frac{dt}{2m} \equiv -\frac{r^2}{2m} dt, \\
A^3 &= \frac{(2m)^3 \Pi_c \Pi_s}{(\Pi_c^2 - \Pi_s^2)} \Delta dt,
\end{align*}

and

\begin{align*}
h_1 &= \frac{\Delta^\frac{1}{2}}{2m}, \quad h_2 = \frac{\Delta^\frac{1}{2}}{2m}, \quad h_3 = 4m^2 \Delta^{-\frac{3}{2}}.
\end{align*}

These are exactly the fields given in reference [5].

Let us now comment on the interpolating solution [6, 12] and the generalized notion of subtracted geometry [7]. From the point of view of the Harrison transformations if instead of (3.2) we act with

\begin{equation}
\exp[d_1 F_1 + d_2 F_2],
\end{equation}

then we obtain an interpolating geometry, which in the limit $d_1, d_2 \to 0$ gives the original black hole and in the limit $d_1, d_2 \to 1$ gives its subtracted geometry. If instead we exponentiate not just two negative roots but say three

\begin{equation}
\exp[d_1 F_1 + d_2 F_2 + d_3 F_3],
\end{equation}

then we obtain a bigger class of interpolating geometries, which in the limit, say $d_1 \to 1, d_2 = 0, d_3 = 0$ realizes one of the generalized subtracted geometry introduced in [7] (upon taking the extremal limit). Another one of these geometries can be realized as $d_1, d_2, d_3 \to 1$.

We end this section with a discussion of rotating five-dimensional black holes. Exactly the same transformations are applicable as in the static case. Only the expressions are more cumbersome. The 5d Myers-Perry black hole metric in our convention is (we use $x = \cos \theta$, but otherwise our
conventions are exactly the same as Cvetic-Youm [29]):

\[
g_{tt} = -\frac{\Sigma - 2m}{2m}, \tag{3.18}
\]

\[
g_{t\psi} = -\frac{2ml_2x^2}{\Sigma}, \tag{3.19}
\]

\[
g_{t\phi} = -\frac{2ml_1(1 - x^2)}{\Sigma}, \tag{3.20}
\]

\[
g_{\psi\psi} = -\frac{x^2}{\Sigma} \left( (r^2 + l_2^2)\Sigma + 2ml_2^2x^2 \right), \tag{3.21}
\]

\[
g_{\phi\phi} = \frac{1 - x^2}{\Sigma} \left( (r^2 + l_1^2)\Sigma + 2ml_1^2(1 - x^2) \right), \tag{3.22}
\]

\[
g_{\psi\phi} = \frac{2ml_1l_2x^2(1 - x^2)}{\Sigma}, \tag{3.23}
\]

\[
g_{rr} = \frac{\Sigma r^2}{(r^2 + l_1^2)(r^2 + l_2^2) - 2mr^2}, \tag{3.24}
\]

\[
g_{xx} = \frac{\Sigma}{1 - x^2}, \tag{3.25}
\]

where

\[
\Sigma = r^2 + l_1^2x^2 + l_2^2(1 - x^2). \tag{3.26}
\]

We do KK reduction over \( t \) and \( \psi \). The resulting three-dimensional metric is

\[
ds_3^2 = \left[ \frac{(r^2 + l_1^2)(\Sigma - 2m) + 2ml_2^2x^2}{r^2(r^2 + l_1^2 + l_2^2 - 2m) + l_1^2l_2^2} \right] r^2x^2dt^2
\]

\[
+ \left[ (r^2 + l_2^2)(\Sigma - 2m) + 2ml_2^2x^2 \right] \frac{x^2}{1 - x^2}dx^2
\]

\[
+ \left[ r^2(r^2 + l_1^2 + l_2^2 - 2m) + l_1^2l_2^2 \right] x^2(1 - x^2)d\phi^2, \tag{3.27}
\]

and the non-zero scalars are (we also use the notation \( y^I = fh^I \)):

\[
U_{\text{MP}} = \frac{1}{2} \ln \left[ \frac{(r^2 + l_2^2)(\Sigma - 2m) + 2ml_2^2x^2}{\sqrt{\Sigma(\Sigma - 2m)}} \right]^2, \tag{3.28}
\]

\[
\sigma_{\text{MP}} = 4l_1l_2mx^4 \frac{(\Sigma - m)}{\sqrt{\Sigma(\Sigma - 2m)}}, \tag{3.29}
\]

\[
\zeta^0_{\text{MP}} = \frac{2ml_2x^2}{\Sigma - 2m}, \tag{3.30}
\]

\[
\bar{\zeta}_{0\text{MP}} = -\frac{2ml_1x^2}{\Sigma}, \tag{3.31}
\]

\[
y_{\text{MP}} := y^1 = y^2 = y^3 = \sqrt{\frac{\Sigma - 2m}{\Sigma}}, \tag{3.32}
\]

After the action\(^5\) of \( g_{\text{charging}} \), \( g_{\text{subtraction}} \), and \( g_{\text{scaling}} \) the resulting scalars in terms of the MP scalars

\(^5\)We do not list the resulting fields after the action of \( g_{\text{charging}} \) because the five-dimensional multi-charge black hole solution is written in detail in several references, e.g., [29, 30, 31, 32, 5].
are

\[ U = U_{\text{MP}}, \quad \text{(as expected)} \]  \hspace{1cm} (3.33)

\[ \sigma = \sigma_{\text{MP}}, \quad \text{(as expected)} \]  \hspace{1cm} (3.34)

\[ y^1 = \frac{y_{\text{MP}}}{1 - y_{\text{MP}}^2}, \]  \hspace{1cm} (3.35)

\[ y^2 = \frac{y_{\text{MP}}}{1 - y_{\text{MP}}^2}, \]  \hspace{1cm} (3.36)

\[ y^3 = \frac{\Pi - \Pi y_{\text{MP}}^2}{\Pi - \Pi y_{\text{MP}}^2}, \]  \hspace{1cm} (3.37)

\[ \chi^1 = -\frac{1}{2(1 - y_{\text{MP}}^2)} \left( 1 + y_{\text{MP}}^2 - \sqrt{\gamma \Pi^2 - \Pi^2} \right), \]  \hspace{1cm} (3.38)

\[ \chi^2 = -\frac{1}{2(1 - y_{\text{MP}}^2)} \left( 1 + y_{\text{MP}}^2 - \sqrt{\gamma \Pi^2 - \Pi^2} \right), \]  \hspace{1cm} (3.39)

\[ \chi^3 = \frac{1}{(1 - y_{\text{MP}}^2)\Pi c \Pi s}, \]  \hspace{1cm} (3.40)

\[ \zeta^0 = \zeta_{\text{MP}}^{\Pi c} + \tilde{\zeta}_{\text{MP}}^{\Pi s}, \]  \hspace{1cm} (3.41)

\[ \zeta^1 = \frac{1}{2} \left( \zeta_{\text{MP}}^{\Pi c} \left[ 1 - \sqrt{\gamma \Pi^2 - \Pi^2} \right] - \tilde{\zeta}_{\text{MP}}^{\Pi s} \left[ 1 + \sqrt{\gamma \Pi^2 - \Pi^2} \right] \right), \]  \hspace{1cm} (3.42)

\[ \zeta^2 = \frac{1}{2} \left( \zeta_{\text{MP}}^{\Pi c} \left[ 1 - \sqrt{\gamma \Pi^2 - \Pi^2} \right] - \tilde{\zeta}_{\text{MP}}^{\Pi s} \left[ 1 + \sqrt{\gamma \Pi^2 - \Pi^2} \right] \right), \]  \hspace{1cm} (3.43)

\[ \zeta^3 = -\frac{1}{2} \left( \zeta_{\text{MP}}^{\Pi c} \left[ 1 + \sqrt{\gamma \Pi^2 - \Pi^2} \right] \right), \]  \hspace{1cm} (3.44)

\[ \tilde{\zeta}_0 = \frac{1}{4(\Pi^2 - \Pi^2)} \left\{ \zeta_{\text{MP}}^{\Pi c} \left[ 1 - \sqrt{\gamma \Pi^2 - \Pi^2} \right] \left[ 1 - \sqrt{\gamma \Pi^2 - \Pi^2} \right] + \tilde{\zeta}_{\text{MP}}^{\Pi c} \left[ 1 + \sqrt{\gamma \Pi^2 - \Pi^2} \right] \right\}, \]  \hspace{1cm} (3.45)

\[ \tilde{\zeta}_1 = \frac{1}{2(\Pi^2 - \Pi^2)} \left\{ \zeta_{\text{MP}}^{\Pi c} \left[ 1 + \sqrt{\gamma \Pi^2 - \Pi^2} \right] - \zeta_{\text{MP}}^{\Pi s} \left[ 1 - \sqrt{\gamma \Pi^2 - \Pi^2} \right] \right\}, \]  \hspace{1cm} (3.46)

\[ \tilde{\zeta}_2 = \frac{1}{2(\Pi^2 - \Pi^2)} \left\{ \zeta_{\text{MP}}^{\Pi c} \left[ 1 + \sqrt{\gamma \Pi^2 - \Pi^2} \right] - \zeta_{\text{MP}}^{\Pi s} \left[ 1 - \sqrt{\gamma \Pi^2 - \Pi^2} \right] \right\}, \]  \hspace{1cm} (3.47)

\[ \tilde{\zeta}_3 = \frac{1}{4} \left\{ \zeta_{\text{MP}}^{\Pi s} \left[ 1 + \sqrt{\gamma \Pi^2 - \Pi^2} \right] \right\}, \]  \hspace{1cm} (3.48)

The five-dimensional fields constructed from the above scalars precisely match the expressions in the appendix of [5] (provided certain typos in the field $A^3$ are fixed in [5]). Explicitly, the final geometry is

\[ ds^2 = -\Delta^{-\frac{2}{3}}(\Sigma - 2m)(dt + A)^2 + \Delta^{\frac{1}{3}}d\Sigma^2 \]  \hspace{1cm} (3.49)

where

\[ \Delta = (2m)^2 \Sigma^2 - \Pi^2 + (2m)^3 \Pi^2, \]  \hspace{1cm} (3.50)
\[ A = 2m(1 - x^2) \left( \frac{\Pi_c}{\Sigma - 2m} l_1 - \frac{\Pi_s l_2}{\Sigma} \right) d\phi + 2mx^2 \left( \frac{\Pi_c}{\Sigma - 2m} l_2 - \frac{\Pi_s l_1}{\Sigma} \right) d\phi, \] (3.51)

and

\[ ds_4^2 = \frac{dx^2}{1 - x^2} + \frac{r^2 dr^2}{(r^2 + l_1^2)(r^2 + l_2^2) - 2m r^2} + \frac{x^2}{\Sigma} \left[ r^2 + l_1^2 + \frac{2ml_2 x^2 (1 - x^2)}{\Sigma - 2m} \right] d\psi^2 + \frac{1 - x^2}{\Sigma} \left[ r^2 + l_2^2 + \frac{2ml_1 x^2 (1 - x^2)}{\Sigma - 2m} \right] d\phi d\psi. \] (3.52)

The scalars are

\[ h^1 = h^2 = (h^3)^{-\frac{1}{2}} = \frac{\Delta^\frac{1}{2}}{2m}, \] (3.53)

and the vectors are (where we have fixed minor typos in \( A^3 \) compared to [5]),

\[ A^1 = A^2 = -\frac{\Sigma}{2m} dt + x^2 (l_1 \Pi_s - l_2 \Pi_c) d\psi + (1 - x^2)(l_2 \Pi_s - l_1 \Pi_c) d\phi \] (3.54)

\[ A^3 = \frac{(2m)^3\Pi_s \Pi_c}{(\Pi_c^2 - \Pi_s^2) \Delta} dt + \frac{(2m)^3(l_1 \Pi_c - l_2 \Pi_s)}{\Delta} x^2 d\psi + \frac{(2m)^3(l_2 \Pi_c - l_1 \Pi_s)}{\Delta} (1 - x^2) d\phi. \] (3.55)

### 3.3 Relation to Cveti\v{c}-Guica-Saleem Analysis

Cveti\v{c}, Guica, and Saleem [12] showed that in the five-dimensional case subtracted geometry can be obtained using STU transformations of the Euclidean STU supergravity. One reason this computation works is that both the five-dimensional subtracted geometry and the five-dimensional black hole have the same 4d Euclidean base space metric. We have also obtained the same result, however, our presentation and analysis looks very different from that of [12]. To relate the two discussions we must figure out the embedding of four-dimensional STU transformations in the three-dimensional duality group. Having obtained this embedding if we show that the transformations we have used to obtain subtracted geometry are from the 4d STU subgroup of the 3d duality group, then we have at least qualitatively related our analysis to that of [12]. In fact, quantitatively too the corresponding expressions can be readily compared.

To this end we proceed along the lines of section 2 of Bossard, Nicolai, and Stelle [28]. Recall that we are considering U(1)\(^3\) theory in five-dimensions, which upon dimensional reduction over a timelike direction gives rise to a Euclidean STU Einstein Maxwell theory with duality group \( \mathfrak{G}_4 = \text{SL}(2, \mathbb{R})^3 \). The Maxwell degrees of freedom transform under some representation \( t_4 \) of \( \mathfrak{G}_4 \) (in the present case \( \mathbf{8} \) of \( \text{SL}(2, \mathbb{R})^3 \)). Since we are interested in axisymmetric configurations only, we consider them as solutions of 3d Euclidean theory. This dimensional reduction yields one dilatonic scalar from the metric (scalar \( U \) in our notation) and one scalar each from the Maxwell field each (scalars \( \zeta^A \) in our notation) together with the scalars of the four-dimensional theory. In addition there is a vector field from the metric (\( \omega_3 \)) and one vector field (\( A_3^\Sigma \)) each from the four-dimensional
Maxwell field. The Maxwell vectors become scalars after dualization in three-dimensions (scalars $\tilde{\zeta}_\Lambda$ in our notation).

The axisymmetric Euclidean solutions of vacuum gravity admit the so-called Ehlers symmetry $\text{SL}(2,\mathbb{R})/\text{SO}(2)$. This symmetry together with the duality symmetry in four-dimensions $\mathfrak{G}_4$ results in

$$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g}_4$$

as a set of symmetry generators. In addition the ‘electric’ scalars $\zeta^A$ admit shift symmetry. After dimensional reduction the ‘magnetic’ scalars $\tilde{\zeta}_\Lambda$ also admit this shift symmetry. Since Maxwell vectors transform in the representation $l_4$ of $\mathfrak{g}_4$, the shift symmetries also transform in $l_4$ of $\mathfrak{g}_4$.

The commutator of Ehlers $\mathfrak{sl}(2,\mathbb{R})$ generators on these shift symmetries give rise to new generators that also belong to the $l_4$ representation of $\mathfrak{g}_4$ [33, 28]. These new generators are also non-linearly realized on the 3d fields. Altogether, the whole three-dimensional duality group becomes a simple Lie group (in the present case $\text{SO}(4,4)$), for which the Lie algebra admits a five-grading

$$\mathfrak{g} \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g}_4 \oplus (2 \otimes l_4) \simeq 1^{(-2)} \oplus l_4^{(-1)} \oplus (1 \oplus \mathfrak{g}_4)^{(0)} \oplus l_4^{(1)} \oplus 1^{(2)}. \quad (3.57)$$

The key point to note is that all of $\mathfrak{g}_4$ has grading level 0 in this decomposition. We now show that all the generators we use to obtain the subtracted geometry are at level 0 in this five-grading.

In our case Ehlers $\mathfrak{sl}(2,\mathbb{R})$ generators are $^6 (H_0, E_0, F_0)$, where $E_0$ and $F_0$ respectively have grading +2 and −2: $[H_0, E_0] = 2E_0, [H_0, F_0] = -2F_0$. The generators $E_q^A$ and $E_p^\Lambda$ have level +1 and $F_q^\Lambda$ and $F_p^A$ have level −1. In the construction of the subtracted geometry starting from the charged black hole we do not use any of these generators. The three sets of generators $(H_I, E_I, F_I)$ and the generator $H_0$ have level 0. The $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ generated by $(H_I, E_I, F_I)$ is thus the four-dimensional duality group, and notice that these are the only generators that we have used in the construction of the subtracted geometry starting from the charged black hole. Thus, although our computations are differently organized compared to [12], the duality symmetries that our analysis uses and the duality symmetry that their analysis uses are exactly the same. Our presentation and analysis has the advantage that it uses the more widely used notation of N=2 supergravity. It can be thought of as a direct continuation of our previous work [9] and can be naturally generalized to other supergravities. It also offers a different and a useful perspective on the analysis of [12]. It can be an interesting exercise to understand in detail the relation between our Harrison transformations and the spectral flows of [34, 12]. Such an analysis is beyond the scope of the present considerations, but perhaps it can be used to shed some light on the question of interpreting the timelike Melvin twists of [34, 12].

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$^6$Recall that $U$ and $\sigma$ are with $H_0$ and $E_0$ respectively in equation (2.25).
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A \( \text{SO}(4,4) \) basis

Since the generators of \( \mathfrak{so}(4,4) \) Lie algebra feature prominently in our work and we make reference to the explicit basis we use at various places, here we list all 28 generators in the fundamental representation \( \mathbf{8} \). The basis we use is identical to the one used in [9] and also in reference [26]. The symbol \( E_{ij} \) denotes a \( 8 \times 8 \) matrix with 1 in the \( i \)-th row and \( j \)-th column and 0 elsewhere.

\[
\begin{align*}
H_0 &= E_{33} + E_{44} - E_{77} - E_{88} & H_1 &= E_{33} - E_{44} - E_{77} + E_{88} \\
H_2 &= E_{11} + E_{22} - E_{55} - E_{66} & H_3 &= E_{11} - E_{22} - E_{55} + E_{66}
\end{align*}
\]

(A.1)

\[
\begin{align*}
E_0 &= E_{47} - E_{38} & E_1 &= E_{87} - E_{34} \\
E_2 &= E_{25} - E_{16} & E_3 &= E_{65} - E_{12} \\
F_0 &= E_{74} - E_{83} & F_1 &= E_{78} - E_{43} \\
F_2 &= E_{52} - E_{61} & F_3 &= E_{56} - E_{21}
\end{align*}
\]

(A.2)

\[
\begin{align*}
E_{q_0} &= E_{41} - E_{58} & E_{q_1} &= E_{57} - E_{31} \\
E_{q_2} &= E_{46} - E_{28} & E_{q_3} &= E_{42} - E_{68} \\
F_{q_0} &= E_{14} - E_{85} & F_{q_1} &= E_{75} - E_{13} \\
F_{q_2} &= E_{64} - E_{82} & F_{q_3} &= E_{24} - E_{86}
\end{align*}
\]

(A.3)

\[
\begin{align*}
E_{p_0} &= E_{17} - E_{35} & E_{p_1} &= E_{18} - E_{45} \\
E_{p_2} &= E_{67} - E_{32} & E_{p_3} &= E_{27} - E_{36} \\
F_{p_0} &= E_{71} - E_{53} & F_{p_1} &= E_{81} - E_{54} \\
F_{p_2} &= E_{76} - E_{23} & F_{p_3} &= E_{72} - E_{63}.
\end{align*}
\]

(A.4)

References

[1] M. Cvetič and F. Larsen, “Conformal Symmetry for General Black Holes,” JHEP 1202, 122 (2012) [arXiv:1106.3341 [hep-th]].
[2] M. Cvetič and F. Larsen, “Conformal Symmetry for Black Holes in Four Dimensions,” arXiv:1112.4846 [hep-th].

[3] A. Castro, A. Maloney and A. Strominger, “Hidden Conformal Symmetry of the Kerr Black Hole,” Phys. Rev. D 82, 024008 (2010) [arXiv:1004.0996 [hep-th]].

[4] J. de Boer, M. Johnstone, M. M. Sheikh-Jabbari and J. Simon, “Emergent IR Dual 2d CFTs in Charged AdS5 Black Holes,” Phys. Rev. D 85, 084039 (2012) [arXiv:1112.4664 [hep-th]].

[5] M. Cvetič and G. W. Gibbons, “Conformal Symmetry of a Black Hole as a Scaling Limit: A Black Hole in an Asymptotically Conical Box,” JHEP 1207, 014 (2012) [arXiv:1201.0601 [hep-th]].

[6] M. Baggio, J. de Boer, J. I. Jottar and D. R. Mayerson, “Conformal Symmetry for Black Holes in Four Dimensions and Irrelevant Deformations,” JHEP 1304, 084 (2013) [arXiv:1210.7695 [hep-th]].

[7] A. Chakraborty and C. Krishnan, “Subtractor in CFT,” arXiv:1212.1875 [hep-th].

A. Chakraborty and C. Krishnan, “Attraction, with Boundaries,” arXiv:1212.6919 [hep-th].

S. Jana and C. Krishnan, “A Kaluza-Klein Subtractor,” arXiv:1303.3097 [hep-th].

[8] S. Bertini, S. L. Cacciatori and D. Klemm, “Conformal structure of the Schwarzschild black hole,” Phys. Rev. D 85, 064018 (2012) [arXiv:1106.0999 [hep-th]].

[9] A. Virmani, “Subtracted Geometry From Harrison Transformations,” JHEP 1207, 086 (2012) [arXiv:1203.5088 [hep-th]].

[10] H. J. Boonstra, B. Peeters and K. Skenderis, “Duality and asymptotic geometries,” Phys. Lett. B 411, 59 (1997) [hep-th/9706192].

[11] K. Sfetsos and K. Skenderis, “Microscopic derivation of the Bekenstein-Hawking entropy formula for nonextremal black holes,” Nucl. Phys. B 517, 170 (1998) [hep-th/9711138].

[12] M. Cvetič, M. Guica and Z. H. Saleem, “General black holes, untwisted,” arXiv:1302.7032 [hep-th].

[13] C. Keeler and F. Larsen, “Separability of Black Holes in String Theory,” JHEP 1210, 152 (2012) [arXiv:1207.5928 [hep-th]].

[14] E. Malek, “Timelike U-dualities in Generalised Geometry,” arXiv:1301.0543 [hep-th].
[15] L. Andrianopoli, R. D’Auria, A. Gallerati and M. Trigiante, “Extremal Limits of Rotating Black Holes,” arXiv:1303.1756 [hep-th].

[16] C. M. Hull and B. Julia, “Duality and moduli spaces for timelike reductions,” Nucl. Phys. B 534, 250 (1998) [hep-th/9803239].

[17] E. Cremmer, I. V. Lavrinenko, H. Lu, C. N. Pope, K. S. Stelle and T. A. Tran, “Euclidean signature supergravities, dualities and instantons,” Nucl. Phys. B 534, 40 (1998) [hep-th/9803259].

[18] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, “Special geometry of Euclidean supersymmetry. 1. Vector multiplets,” JHEP 0403, 028 (2004) [hep-th/0312001].

[19] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, “Special geometry of euclidean supersymmetry. II. Hypermultiplets and the c-map,” JHEP 0506, 025 (2005) [hep-th/0503094].

[20] V. Cortes and T. Mohaupt, “Special Geometry of Euclidean Supersymmetry III: The Local r-map, instantons and black holes,” JHEP 0907, 066 (2009) [arXiv:0905.2844 [hep-th]].

[21] J. B. Gutowski and W. A. Sabra, “Euclidean N=2 Supergravity,” Phys. Lett. B 718, 610 (2012) [arXiv:1209.2029 [hep-th]].

[22] A. Ceresole, R. D’Auria, S. Ferrara and A. Van Proeyen, “Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity,” Nucl. Phys. B 444, 92 (1995) [hep-th/9502072].

[23] http://en.wikipedia.org/wiki/Split-complex_number, accessed on May 10 2013.

[24] Christopher Pope, “Kaluza-Klein Theory,” http://faculty.physics.tamu.edu/pope/ihplec.pdf.

[25] S. Ferrara and S. Sabharwal, “Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces,” Nucl. Phys. B 332, 317 (1990).

[26] G. Bossard, Y. Michel and B. Pioline, “Extremal black holes, nilpotent orbits and the true fake superpotential,” JHEP 1001, 038 (2010) [arXiv:0908.1742 [hep-th]].

[27] G. Comper`e, S. de Buyl, E. Jamsin and A. Virmani, “G2 Dualities in D=5 Supergravity and Black Strings,” Class. Quant. Grav. 26, 125016 (2009) [arXiv:0903.1645 [hep-th]].

[28] G. Bossard, H. Nicolai and K. S. Stelle, “Universal BPS structure of stationary supergravity solutions,” JHEP 0907, 003 (2009) [arXiv:0902.4438 [hep-th]].

[29] M. Cveti´c and D. Youm, “General rotating five-dimensional black holes of toroidally compactified heterotic string,” Nucl. Phys. B 476, 118 (1996) [hep-th/9603100].
[30] S. Giusto, S. D. Mathur and A. Saxena, “Dual geometries for a set of 3-charge microstates,” Nucl. Phys. B 701, 357 (2004) [hep-th/0405017].

[31] P. Figueras, E. Jamsin, J. V. Rocha and A. Virmani, “Integrability of Five Dimensional Minimal Supergravity and Charged Rotating Black Holes,” Class. Quant. Grav. 27, 135011 (2010) [arXiv:0912.3199 [hep-th]].

[32] J. L. Hornlund and A. Virmani, “Extremal limits of the Cvetič-Youm black hole and nilpotent orbits of G2(2),” JHEP 1011, 062 (2010) [arXiv:1008.3329 [hep-th]].

[33] P. Breitenlohner, D. Maison and G. W. Gibbons, “Four-Dimensional Black Holes from Kaluza-Klein Theories,” Commun. Math. Phys. 120, 295 (1988).

[34] I. Bena, M. Guica and W. Song, “Un-twisting the NHEK with spectral flows,” JHEP 1303, 028 (2013) [arXiv:1203.4227 [hep-th]].