Theoretical Guarantees of Fictitious Discount Algorithms for Episodic Reinforcement Learning and Global Convergence of Policy Gradient Methods

Xin Guo *†¶ Anran Hu ‡ Junzi Zhang §¶

September 15, 2021

Abstract

When designing algorithms for finite-time-horizon episodic reinforcement learning problems, a common approach is to introduce a fictitious discount factor and use stationary policies for approximations. Empirically, it has been shown that the fictitious discount factor helps reduce variance, and stationary policies serve to save the per-iteration computational cost. Theoretically, however, there is no existing work on convergence analysis for algorithms with this fictitious discount recipe. This paper takes the first step towards analyzing these algorithms. It focuses on two vanilla policy gradient (VPG) variants: the first being a widely used variant with discounted advantage estimations (DAE), the second with an additional fictitious discount factor in the score functions of the policy gradient estimators. Non-asymptotic convergence guarantees are established for both algorithms, and the additional discount factor is shown to reduce the bias introduced in DAE and thus improve the algorithm convergence asymptotically. A key ingredient of our analysis is to connect three settings of Markov decision processes (MDPs): the finite-time-horizon, the average reward and the discounted settings. To our best knowledge, this is the first theoretical guarantee on fictitious discount algorithms for the episodic reinforcement learning of finite-time-horizon MDPs, which also leads to the (first) global convergence of policy gradient methods for finite-time-horizon episodic reinforcement learning.

1 Introduction

This paper studies episodic reinforcement learning with each episode consisting of a finite-time-horizon Markov decision process (MDP). For such finite-time-horizon episodic reinforcement learning problems, a popular heuristic approach is to introduce a fictitious discount factor and use

*University of California, Berkeley. Email: xinguo@berkeley.edu
†Amazon.com. Email: xnguo@amazon.com
‡University of California, Berkeley. Email: anran_hu@berkeley.edu
§Amazon.com. Email: junziz@amazon.com
¶Work done prior to joining or outside of Amazon.
stationary policies when designing algorithms; see for instance, the renowned DQN [44], DDPG [35], and recent works of [19, 66, 13, 27, 17, 61, 6].

Empirically, it has been shown that discount factors serve to reduce variance [62, 22], and stationary policies help save per-iteration computational costs. Theoretically, fictitious discount algorithms designed for *average reward* MDPs have been analyzed [40, 39] and the asymptotic local convergence has been established [11].

It remains open, however, to establish the non-asymptotic global convergence for this fictitious-discount-factor approach in the *finite-time-horizon* framework. The major challenges are to characterize the bias introduced by the discount factor, and to close the gap between the non-stationary optimal policies for finite-time-horizon MDPs and the stationary algorithm policies.

This paper takes the first steps towards rigorously analyzing the global and non-asymptotic convergence of fictitious discount algorithms for finite-time-horizon episodic reinforcement learning. It focuses on the convergence analysis of two concrete algorithms in the context of policy gradient methods. The first one is a widely used variant of the vanilla policy gradient (VPG) method with discounted advantage estimations (DAE). This variant was originally proposed for average reward problems [40, 9, 8, 39], later extended to episodic deep reinforcement learning setting [57] and implemented in popular solvers such as Spinning Up [2]. The second one is a new doubly discounted variant of VPG, with the introduction of an additional fictitious discount factor in the score functions of the policy gradient estimators. This additional discount factor is shown to help reduce the bias in DAE and thus improve asymptotically the algorithm convergence.

**Our approach.** There are three main ingredients in our analysis. The first is establishing quantitative connections among three settings of MDPs: the finite-time-horizon, the average award, and the discounted settings (cf. §2). These relations enable us to connect the finite-time-horizon sub-optimality gap with the average reward (cf. Theorem 14) and the discounted (cf. Theorem 18) ones. The second is utilizing the convergence property of value iteration algorithms to analyze the gap between the stationary policies of the average reward MDPs and the non-stationary optimal policies of the finite-time-horizon MDPs (cf. Lemma 6). The third one is deriving the gradient domination (cf. Lemma 8) and Lipschitz gradient (cf. Lemma 10) properties for average reward MDPs, which is critical to obtain the sub-optimality of algorithm policies for the average reward problem (cf. Theorem 13).

**Contributions.** The contributions of this paper are two-fold:

- It establishes the first (and non-asymptotic) connections between (a) the sub-optimality gap in finite-time-horizon MDPs and (b) the sub-optimality gaps in the average reward and the discounted reformulations (cf. Theorems 14 and 18).

- It obtains, for the first time, theoretical guarantees on fictitious discount algorithms for the episodic reinforcement learning of finite-time-horizon MDPs (cf. Theorems 15 and 19). The convergence is global, and not asymptotic. Moreover, it demonstrates explicit dependencies on both the time horizon and the fictitious discount factor. The analysis in this paper leads

---

1In this paper, “local convergence” indicates convergence to stationary points of value functions, and “global convergence” means convergence in terms of the value function sub-optimality gaps.
to the first global convergence of policy gradient methods for finite-time-horizon episodic reinforcement learning.

**Related work.** Since the seminal work of D. Blackwell [11], earlier works on the relationship among different settings of MDPs have been focusing on the discounted and average reward settings [28, 32, 29, 33, 38, 53]. In contrast, our focus is on the remaining two relations, namely (i) the connection between the finite-time-horizon and the discounted problems and (ii) the connection between the finite-time-horizon and the average reward problems.

Theoretical study on policy gradient methods started with the asymptotic local convergence [60, 31, 39]. Later, non-asymptotic rate of such local convergence has been established in a series of works [47, 65]. Recently, more attention has been shifted to the global convergence of policy gradient methods. However, the majority of these results have been on the discounted settings [69, 10, 4, 63, 58, 42, 14, 67]. Recent progress has been made on a particular class of finite-time-horizon MDPs, i.e., linear quadratic finite-time-horizon MDPs and their variants [23, 70, 24]. This paper, instead, studies global convergence of policy gradient methods for finite-time-horizon, finite-state-action MDPs with general dynamics and rewards.

**Outline.** §2 introduces three settings of MDPs and their mutual connections. §3 introduces DAE REINFORCE and establishes its global sub-optimality guarantee. A doubly discounted variant is then proposed in §4 with its global convergence analysis, showing the benefits of the additional discount factor. §5 concludes.

## 2 Problem setup and preliminaries

### 2.1 Problem Setup

Consider a Markov decision process $\mathcal{M}$ with a finite state space $\mathcal{S} = \{1, \ldots, S\}$, a finite action space $\mathcal{A} = \{1, \ldots, A\}$, a transition probability $p(s'|s,a)$ for the probability of transitioning from state $s$ to state $s'$ when taking action $a$, and a reward function $r(s,a)$ denoting the (deterministic) instantaneous reward for taking action $a$ in state $s$. Here, the initial state is assumed to follow a distribution $\rho \in \mathcal{P}(\mathcal{S})$, where $\mathcal{P}(\mathcal{S}) \subseteq \mathbb{R}^{\vert \mathcal{S} \vert}$ denotes the set of probability measures on over the set $\mathcal{S}$. Denote $R_{\text{max}}$ the maximum reward such that $R_{\text{max}} = \max_{s \in \mathcal{S}, a \in \mathcal{A}} \vert r(s,a) \vert$.

The focus of this paper is the finite-time-horizon MDP. Given a finite time horizon $H \geq 1$, decisions are made in the duration of timestamps from $h = 0$ to $h = H - 1$. This duration is also referred to as an “episode”. Such a horizon can either be naturally defined by the expiration time (e.g., the length of a video game) or manually specified by the decision maker (e.g., the length of affordable decision period). A (randomized) policy $\pi : \mathcal{S} \rightarrow \mathcal{P}(\mathcal{A})$ is a mapping from the state space to a distribution over the action space. For notational simplicity, we use $\pi(a|s)$ to denote the $a$-th entry of $\pi(s)$, i.e., the probability of taking action $a$ at state $s$ under a policy $\pi$. Then for any (randomized) policy sequence $\pi^H = \{\pi_h\}_{h=0}^{H-1}$, the performance metric $V^H(\pi^H)$ is the mean reward collected over the finite horizon episode of length $H$, i.e.,

$$V^H(\pi^H) = \frac{1}{H} \mathbb{E} \sum_{h=0}^{H-1} r(s_h, a_h), \quad (1)$$

3
where \( s_0 \sim \rho, a_h \sim \pi_h(s_h) \) and \( s_{h+1} \sim p(\cdot|s_h, a_h) \) for \( h = 0, \ldots, H - 2 \). The finite-time-horizon problem is the following optimization problem:

\[
\maximize_{\pi^h = \{\pi_0, \ldots, \pi_{H-1}\}} V^H(\pi^H).
\] (2)

Note that the optimal policy sequence \( \pi^{H,*} = \{\pi_h^{H,*}\}_{h=0}^{H-1} \) of problem (2) may be nonstationary, and we write \( V^{H,*} = V^H(\pi^{H,*}) \). When the policy sequence \( \pi^H = \{\pi\}_{h=0}^{H-1} \) is stationary, we will write it as \( \pi \) for notational simplicity. Here and below we use \( P_\pi \in \mathbb{R}^{S \times S} \) to denote the transition probability of the Markov chain induced by policy \( \pi \), i.e., \( P_\pi(s, s') = \sum_{a \in A} p(s'|s, a)\pi(a|s) \).

Throughout this paper, we make the following assumption as in [46]. Note that this assumption naturally holds when the transition probability \( p \) is component-wisely positive.

**Assumption 1.** For any deterministic stationary policy \( \pi \), the induced Markov chain with transition matrix \( P_\pi \) is irreducible and aperiodic.

With Assumption 1, we have the following proposition.

**Proposition 1.** Given Assumption 1, then there exist constants \( C_{p, S, A} > 1 \) and \( \alpha_{p, S, A} \in [0, 1) \) that depend only on the transition probability model \( p \), number of states \( S \) and number of actions \( A \) of the MDP \( \mathcal{M} \), such that for any policy \( \pi \) and \( h \geq 0 \),

\[
d_{TV}(\rho P_\pi^h, \mu_\pi) \leq C_{p, S, A} \alpha_{p, S, A}^h,
\] (3)

where \( \mu_\pi \) is the (unique) stationary distribution of the transition matrix \( P_\pi \).

The analysis of the above finite-time-horizon MDP will rely on two related MDPs: the average reward problem and the discounted one, both of which have stationary optimal policies under Assumption 1.

**Discounted problem.** It is to consider an infinite horizon and solve for

\[
\maximize_{\pi = \{\pi_h\}_{h=0}^{\infty}} V^\gamma(\pi)
\]

with

\[
V^\gamma(\pi) = (1 - \gamma) \mathbb{E} \sum_{h=0}^{\infty} \gamma^h r(s_h, a_h),
\]

where \( s_0 \sim \rho, a_h \sim \pi_h(s_h) \) and \( s_{h+1} \sim p(\cdot|s_h, a_h) \) for \( h \geq 0 \). Here \( \gamma \in [0, 1) \) is the discount factor, penalizing future rewards. It is well-known that for this discounted problem, there exists a stationary optimal policy sequence \( \pi^{\gamma,*} = \{\pi_h^{\gamma,*}\}_{h=0}^{\infty} \), where all \( \pi_h^{\gamma,*} = \pi^{\gamma,*} \) (\( h \geq 0 \)) are equal [50]. Similarly, we denote \( V^{\gamma,*} = V^\gamma(\pi^{\gamma,*}) \). Again, when the policy sequence \( \pi = \{\pi\}_{h=0}^{\infty} \) is stationary, we will write it as \( \pi \) for notational simplicity.

**Average reward problem.** The infinite horizon average reward of a (stationary) policy \( \pi \) is defined as

\[
\eta(\pi) = \lim_{H \to \infty} V^H(\pi) = \lim_{H \to \infty} \frac{1}{H} \mathbb{E} \sum_{h=0}^{H-1} r(s_h, a_h) = \sum_{s \in \mathcal{S}, a \in \mathcal{A}} \mu_\pi(s) \pi(a|s) r(s, a),
\] (4)

where \( \mu_\pi \) is defined in Proposition 1. The goal is to find \( \pi \) that maximizes \( \eta(\cdot) \). Note \( \eta(\pi) \) is well-defined as the limit in [4] is guaranteed to exist and be finite, and independent of the initial state distribution \( \rho \) under Assumption 1 [50]. Since \( |\eta(\pi)| \leq R_{\text{max}} \) and the set of all (stationary) policies (viewed as a subset \( \mathbb{R}^{SA} \)) is compact, the optimal (stationary) policy \( \pi^* \) (that maximizes \( \eta(\cdot) \)) exists and we denote the corresponding value function as \( \eta^* = \eta(\pi^*) \)
2.2 Connections of finite-time-horizon with discounted and average reward problems

Now we introduce our first set of main results, which characterize the connections within these three different MDP problems.

The first result bounds the error between $V^\gamma(\pi)$ (for the discounted problem) and $V^H(\pi)$ (for the finite-time-horizon problem) under an arbitrary stationary policy $\pi$.

**Lemma 2.** Given Assumption 1, then for any stationary policy $\pi$,

$$|V^\gamma(\pi) - V^H(\pi)| \leq 2R_{\text{max}}C_{p,S,A}(H(1-\gamma)\alpha_{p,S,A}^H + \frac{\alpha_{p,S,A} + |H(1-\gamma) - 1|}{1 - \alpha_{p,S,A}}),$$  \hspace{1cm} (5)

where $C_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0,1)$ are the constants in Proposition 1 and depend only on the transition probability model $p$, the number of states $S$ and the number of actions $A$ of $\mathcal{M}$, the underlying MDP.

The next lemma establishes a bound between $V^\gamma(\pi)$ (for the discounted problem) and $\eta(\pi)$ (for the average reward problem) under any stationary policy $\pi$.

**Lemma 3.** Given Assumption 1, then

$$|V^\gamma(\pi) - \eta(\pi)| \leq \frac{2(1-\gamma)R_{\text{max}}C_{p,S,A}}{1 - \alpha_{p,S,A}},$$  \hspace{1cm} (6)

where the constants $C_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0,1)$ are the same as in Lemma 2.

Maximizing over $\pi$, then immediately from Lemma 3, we have

**Corollary 4.** Given Assumption 1, then

$$|V^\gamma^* - \eta^*| \leq \frac{2(1-\gamma)R_{\text{max}}C_{p,S,A}}{1 - \alpha_{p,S,A}},$$  \hspace{1cm} (7)

where the constants $C_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0,1)$ are the same as in Lemma 2.

The following statement controls the gap between $V^H(\pi)$ (for the finite-time-horizon problem) and $\eta(\pi)$ (for the average reward problem) under any stationary policy $\pi$.

**Lemma 5.** Given Assumption 1, then

$$|V^H(\pi) - \eta(\pi)| \leq \frac{2R_{\text{max}}C_{p,S,A}}{H(1 - \alpha_{p,S,A})},$$  \hspace{1cm} (8)

where the constants $C_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0,1)$ are the same as in Lemma 2.

And finally, the bound of the gap between the optimal value functions $V^H^*$ (for the finite-time-horizon problem) and $\eta^*$ (for the average reward problem) is as follows.
Lemma 6. Given Assumption 7, then
\[ |V^{H,*} - \eta^*| \leq \frac{2 R_{\text{max}} D_{p,S,A}}{H}, \]  
(9)

where \( D_{p,S,A} > 1 \) is a constant that depends only on the transition probability model \( p \), the number of states \( S \) and the number of actions \( A \) of the underlying MDP \( \mathcal{M} \).

Remark 1. Lemma 6 cannot be directly implied by Lemma 5. The key issue is that the optimal policy for the average reward value function \( \eta(\cdot) \) is stationary, while the optimal policy for the finite-horizon value function \( V^H(\cdot) \) may be non-stationary. To bridge this gap between stationary and non-stationary policies, we need the convergence property of value iteration algorithms (cf. Appendix A.2).

These properties show that the three different settings are closely related for a large horizon \( H \), and are critical for the subsequent analyses.

2.3 Gradient properties

In this section, we review the basics of policy gradient methods and state some useful properties of policy gradients in the average reward and the discounted settings.

Policy gradient methods. Policy gradient methods start by parametrizing the policy with parameter \( \theta \in \Theta \), which we denote as \( \pi_\theta \). Here \( \Theta \) is the parameter space and the parametrization maps \( \theta \) to a randomized policy \( \pi_\theta : S \rightarrow \mathcal{P}(A) \). The (vanilla) policy gradient (VPG) methods then proceed by performing stochastic gradient ascent on a (regularized) value function in the parameter space, namely, for each iteration \( k \), \( \theta^k \) is updated to \( \theta^{k+1} \) with
\[ \theta^{k+1} = \theta^k + \alpha^k g_k. \]  
(10)
Here \( \theta^0 \) is the initial parameter, \( \alpha^k \) is the step-size, and \( g_k \) is a (possibly biased) stochastic gradient estimator of a regularized value function.

Throughout this paper, we will focus on the following regularized value function of the average reward problem:
\[ \bar{L}(\theta) = \eta(\pi_\theta) + \Omega(\theta), \]
and the regularized value function of the discounted problem:
\[ L^\gamma(\theta) = \frac{1}{1 - \gamma} V^\gamma(\pi_\theta) + \Omega(\theta). \]

Here \( \Omega : \Theta \rightarrow \mathbb{R} \) is a regularization term that serves to improve the convergence [71, 43, 26].

Below we specify additional assumptions about the problem setting. Note that the same set of assumptions have been made in [5, 68].

Assumption 2. (Setting)

- The policy is a soft-max policy parameterization, i.e., \( \pi_\theta(a|s) = \frac{\exp(\theta_{s,a})}{\sum_{a' \in A} \exp(\theta_{s,a'})} \), with the parameter space being \( \Theta = \mathbb{R}^{SA} \).
The regularization term is (with $\lambda > 0$)

$$\Omega(\theta) = \frac{\lambda}{SA} \sum_{s \in S, a \in A} \log \pi_\theta(a|s).$$

- The initial distribution is component-wisely positive, i.e., $\rho(s) > 0$ for any $s \in S$.
- The reward function $r(s, a) \in [0, 1]$, $\forall s \in S, a \in A$.

Some remarks on Assumption 2:
- The soft-max policy parametrization is simple yet forms the basis of the widely-used (neural network) energy based policies [22].
- The regularization term is a simplified version of the popular (relative) entropy regularization terms [48, 54], and has been demonstrated to be necessary to avoid exponential lower bounds when working with the soft-max policy parametrization in [34].
- The positivity assumption on the initial distribution is standard in the global convergence literature of policy gradient methods [5, 10, 42].
- The last assumption on the range of $r$ is merely for the simplicity of the subsequent discussions and can be easily relaxed to the general constant bound $r(s, a) \in [-R_{\text{max}}, R_{\text{max}}], \forall s \in S, a \in A$.

Properties of policy gradients. We are now ready to provide some useful properties regarding the gradients of the discounted and the average reward problems.

We first slightly tighten the gradient domination property established in [5, Theorem 5.2] for the discounted problems by utilizing the uniform ergodic property in Assumption 1.

Proposition 7. (Gradient domination for discounted problems) Given Assumptions 1 and 2, suppose that $\|\nabla_\theta L^\gamma(\theta)\|_2 \leq \lambda/(2SA)$. Then

$$V^\gamma,\pi - V^\gamma(\pi_\theta) \leq \lambda \min \left\{ \left\| \frac{d^{\gamma,\pi}_\rho}{\rho} \right\|_{\infty}, \frac{S\|d^{\gamma,\pi}_\rho\|_{\infty}}{1 - \alpha_{p,S,A}} \right\}.$$  

Here for any (randomized) policy $\pi : S \rightarrow \mathcal{P}(A)$,

$$d^{\gamma,\pi}_\rho(s) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \text{Prob}^{\pi}_\rho(s_t = s)$$

is the discounted state visitation distribution, where $\text{Prob}^{\pi}_\rho(s_t = s)$ is the probability of arriving at $s$ in step $t$ starting from $s_0 \sim \rho$ following policy $\pi$ in $\mathcal{M}$. In addition, the division in $d^{\gamma,\pi}_\rho / \rho$ is component-wise.

We next establish analogously the gradient domination property for the average reward problem.
Lemma 8. (Gradient domination for average reward problems) Given Assumptions 1 and 2, suppose that $\|\nabla_\theta \bar{L}(\theta)\|_2 \leq \lambda/(2SA)$. Then
\[
\eta^* - \eta(\pi_\theta) \leq \frac{\lambda S\|\mu_{\pi^*}\|_\infty}{1 - \alpha_{p,S,A}},
\]
where $\mu_{\pi^*}$ and $\alpha_{p,S,A}$ are defined as in Proposition 1.

The two statements above on gradient domination capture the sub-optimality results for policies satisfying certain gradient conditions.

Now recall the strongly smoothness property of the objectives for discounted problems [5].

Proposition 9. (Strongly smoothness for discounted problems [5, Lemma D.4]) Given Assumptions 1 and 2, $L^\gamma$ is strongly smooth with parameter $\beta_\lambda = \frac{8}{(1-\gamma)^3} + \frac{2\lambda}{S}$, i.e.,
\[
\|\nabla_\theta L^\gamma(\theta_1) - \nabla_\theta L^\gamma(\theta_2)\|_2 \leq \beta_\lambda \|\theta_1 - \theta_2\|_2
\]
for any $\theta_1, \theta_2 \in \Theta$.

We can establish analogously the strongly smoothness property for the average reward problem.

Lemma 10. (Strongly smoothness for average reward problems) Under Assumptions 1 and 2, $\bar{L}$ is strongly smooth with parameter $\bar{\beta}_\lambda = 22\sqrt{S}\left(\frac{2C_{p,S,A}}{1-\alpha_{p,S,A}} + 1\right)^3 + 2\lambda/S$, i.e.,
\[
\|\nabla_\theta \bar{L}(\theta_1) - \nabla_\theta \bar{L}(\theta_2)\|_2 \leq \bar{\beta}_\lambda \|\theta_1 - \theta_2\|_2
\]
for any $\theta_1, \theta_2 \in \Theta$. Here the constants $C_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0,1)$ are defined as in Proposition 1.

These two statements are critical for the subsequent analyses of the algorithms.

### 3 DAE REINFORCE algorithm

In this section, we first introduce a widely used vanilla policy gradient implementation [2], which we call the DAE REINFORCE algorithm (following its usage of DAE in [57]). In DAE REINFORCE, a stationary parametrized policy $\pi_\theta(a|s)$ is considered, and the parameter is updated by
\[
\theta^{k+1} = \theta^k + \alpha^k \hat{g}_k,
\]
where
\[
\hat{g}_k = \frac{1}{NH} \sum_{i=1}^N \sum_{h=0}^{H-1} \nabla_\theta \log \pi_{\theta^k}(a^i_h|s^i_h) \left( \sum_{h'=h}^{H-1} \gamma^{h'-h} r^i_{h'} - b(s^i_h) \right) + \nabla_\theta \Omega(\theta^k).
\]
Here $\gamma \in (0,1)$ is a fictitious discount factor, $N$ is the mini-batch size of the updates, $r^i_h = r(s^i_h, a^i_h)$, $\tau_i = (s^i_0, a^i_0, r^i_0, \ldots, s^i_{H-1}, a^i_{H-1}, r^i_{H-1})$ ($i = 1, \ldots, N$, $h = 0, \ldots, H-1$) are i.i.d. trajectories sampled under policy $\pi_{\theta^k}$, and $b$ is a baseline function that is independent of the trajectories. Throughout
the paper, we assume that the baseline $b$ is a.s. uniformly bounded, i.e., $\max_{s \in S} |b(s)| \leq B$ a.s. for some constant $B > 0$.

In the rest of the section, we establish the convergence of (a slightly modified version of) DAE REINFORCE, which we call Truncated DAE REINFORCE and summarize in Algorithm 1. Note that the estimator $\hat{g}_k$ is truncated in (12) (and for notational simplicity under the same symbol) with a truncation parameter $\beta \in (0, 1)$. The same truncation has been adopted for studying the standard REINFORCE algorithm (without DAE) in [68], where $\beta$ is introduced to ensure that the advantage function estimation is sufficiently accurate.

Algorithm 1 Truncated DAE REINFORCE

1: Input: Initialization $\theta^0$, step-sizes $\alpha_k$ for $k \geq 0$.
2: for $k = 0, 1, \ldots$ do
3: Sample $N$ i.i.d. trajectories $\{\tau_i\}_{i=1}^N$ under policy $\pi_{\theta_k}$.
4: Compute gradient estimator $\hat{g}_k$ as
5: $\hat{g}_k = \frac{1}{N[\beta H]} \sum_{i=1}^N \sum_{h=0}^{[\beta H]-1} \nabla_\theta \log \pi_{\theta_k}(a_{h}^i|s_{h}^i) \left( \sum_{h' = h}^{H-1} \gamma^{h'-h} r_{h'}^i - b(s_{h}^i) \right) + \nabla_\theta \Omega(\theta^k)$. (12)
6: Update $\theta^{k+1} = \theta^k + \alpha_k \hat{g}_k$.
7: end for

The main idea behind our convergence analysis is to use the average reward as a bridge to connect the original finite-time-horizon MDP and the DAE REINFORCE algorithm. The proof consists of two parts. The first part is to establish the sub-optimality of $\theta_k$, evaluated for the average reward problem. The second part is to establish the convergence of the algorithm for the finite-horizon problem by utilizing the connection between the average reward setting and the finite-horizon setting.

We begin the analysis by estimating the (upper) bound on the difference between the exact gradient and the sample gradient. Hereafter, we use $E_k$ to denote the conditional expectation given the $k$-th iteration $\theta^k$.

Lemma 11. Given Assumptions 1 and 2, then

$$
\|E_k[\hat{g}_k] - \nabla \tilde{L}(\theta^k)\|_2 \leq \frac{16C_{p,S,A}}{[\beta H](1 - \alpha_{p,S,A})} \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) \\
+ 8C_{p,S,A} \left( \frac{1 - \gamma}{(1 - \alpha_{p,S,A})^2} \right) + 4\gamma^{(1-\beta)H} \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right).$$

(13)

Here the constants $C_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0, 1)$ are defined in Proposition 1.

This lemma leads to the following bounds on the stochastic gradients, which are key to establishing the convergence of Algorithm 1.

In [4] we show that $\beta$ can be dropped if an additional discount factor is introduced in the gradient estimator.
Lemma 12. Given Assumptions 7 and 8, then
\[ \|g_k\|_2 \leq G^* + 2\lambda \quad \text{a.s.,} \]
\[ E_k \|g_k\|^2 \geq \|\nabla_{\theta} L(\theta^k)\|^2 - (G + 2\lambda)\Delta, \]
\[ E_k \|g_k\|^2 \leq 2\|\nabla_{\theta} L(\theta^k)\|^2 + \bar{M}. \]

Here \( G^* = \frac{2(1+(1-\gamma)B)}{1-\gamma} \), \( \bar{G} = 4 \left( 1 + \frac{C_{p,S,A}}{1-\alpha_{p,S,A}} \right) \), \( \bar{M} = 2\bar{\Delta}^2 + (G^* + 2\lambda)^2/N \), \( \bar{\Delta} \) is the right-hand side of (13), the constants \( C_{p,S,A} > 1 \) and \( \alpha_{p,S,A} \in [0,1) \) are defined in Proposition 7.

Remark 2. The second bound in Lemma 12 shows that \( \hat{g}_k \) is nearly unbiased, while the third bound shows that \( \hat{g}_k \) satisfies a bounded second-order moment growth condition. These conditions slightly generalize the standard ones used in analyzing stochastic gradient methods [12].

Now, we obtain first the sub-optimality behavior of \( \theta^k \) from the Truncated DAE REINFORCE algorithm (cf. Algorithm 1) in the average reward setting.

Theorem 13. Given Assumptions 7 and 8, let \( \tilde{\beta}_\lambda = 22\sqrt{3} \left( \frac{2C_{p,S,A}}{1-\alpha_{p,S,A}} + 1 \right)^3 + 2\lambda/S \). For a fixed \( \beta \in (0,1) \) and any \( \epsilon > 0, \delta \in (0,1) \), set \( \alpha^k = \frac{1}{2\beta_{\lambda} \sqrt{k+3 \log_2(k+3)}} \) and \( \lambda \) is the positive (larger) root of the following quadratic equation:
\[ 2(\bar{G} + 2\lambda)\bar{\Delta} = (\lambda - \epsilon)^2/(4S^2A^2), \]
where \( \bar{G} \) and \( \bar{\Delta} \) are defined as in Lemma 12. Then
\[ \min_{k=0,\ldots,K} \eta^* - \eta(\pi_{\theta_k}) \leq \frac{\|\mu_{\pi^*}\|_\infty}{1-\alpha_{p,S,A}} \left( S\epsilon + 8S^3A^2\Delta + 4S^2A\sqrt{\Delta\epsilon} + 4S^2A^2\Delta^2 + \bar{G}\bar{\Delta} \right) \]  
with probability at least \( 1 - \delta \), for any \( K \) such that
\[ K \geq O \left( \frac{S^4A^4\tilde{\beta}_{\lambda}^2(D + \sqrt{2C\log(2/\delta)})^2}{\epsilon^4} \log^2 \left( \frac{SA\tilde{\beta}_{\lambda}(D + \sqrt{2C\log(2/\delta)})}{\epsilon} \right) \right). \]  
Here the constants \( C_{p,S,A} > 1 \) and \( \alpha_{p,S,A} \in [0,1) \) are defined in Proposition 7 and the constants \( \bar{D} \) and \( \bar{C} \) are bounded by
\[ \bar{D} = O(\bar{M} + \lambda + 1), \]
\[ \bar{C} = O \left( \frac{(G^* + 2\lambda)^2}{S} \left( \frac{C_{p,S,A}^2}{(1-\alpha_{p,S,A})^2} + \lambda^2 + (G^* + 2\lambda)^2 \right) \right), \]  
where the constants hidden in the big-O notation may depend on \( \theta^0 \).

Next, by Lemma 5 and Lemma 8 we have the following theorem.

Theorem 14. Given Assumption 1, for any \( H \geq 1 \), if there exists a policy \( \tilde{\pi} \) such that \( |\eta^* - \eta(\tilde{\pi})| \leq \epsilon \) for some \( \epsilon > 0 \), then
\[ V_{H,\pi} - V_H(\tilde{\pi}) \leq \frac{2R_{\max}D_{p,S,A}}{H} + \epsilon + \frac{2R_{\max}C_{p,S,A}}{H(1-\alpha_{p,S,A})}. \]
Here the constants \( C_{p,S,A} > 1 \), \( D_{p,S,A} > 1 \) and \( \alpha_{p,S,A} \in [0,1) \) are the constants in Proposition 7 and Lemma 8, which depend only on the transition probability model \( p \), the number of states \( S \) and the number of actions \( A \) of the underlying MDP \( M \).
Combining Theorems 13 and 14 we can derive the convergence for Truncated DAE REINFORCE algorithm.

**Theorem 15.** Given Assumptions 7 and 14 let $\gamma = 1 - H^{-\sigma}$ for some $\sigma \in (0,1)$. For a fixed $\beta \in (0,1)$ and any $\epsilon > 0$, $\delta \in (0,1)$, set $\lambda, \beta$ and $\alpha^k$ to be the same as in Theorem 13. Then for any $K$ such that (15) is satisfied, with probability at least $1 - \delta$,

$$\min_{k=0,\ldots,K} V^H(\pi^{k}) - V^H(\pi^{\star}) \leq O \left( \frac{S}{1 - \alpha_{p,S,A}} \epsilon \right) + \text{bias}^\text{DAE}_H,$$

where

$$\text{bias}^\text{DAE}_H = O \left( \frac{S^2 AC^3_{p,S,A}}{(1 - \alpha_{p,S,A})^4} H^{-\sigma/2} + \frac{S^3 A^2 C^2_{p,S,A}}{(1 - \alpha_{p,S,A})^3} H^{-\sigma} + \left( D_{p,S,A} + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) H^{-1} \right).$$

Here $C_{p,S,A} > 1$, $D_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0,1)$ are constants in Proposition 7 and Lemma 6.

The choice of $\gamma$ is for ease of presentation. See also [36, 15].

4 Doubly Discounted REINFORCE algorithm

In Algorithm 1, a fictitious discount factor is introduced when computing advantage function estimates, while for the rest part it remains undiscounted. This introduces a bias term $\text{bias}^\text{DAE}_H$ as shown in Theorem 15 which remains nonzero for a fixed planning horizon $H$ even when the number of iterations $K$ goes to infinity and $\epsilon$ goes to 0. In this section, we propose the Doubly Discounted REINFORCE algorithm (cf. Algorithm 2) to reduce the bias introduced by DAE.

**Algorithm 2 Doubly Discounted REINFORCE**

1: Input: Initialization $\theta^0$, step-sizes $\alpha^k$ for $k \geq 0$.
2: for $k = 0, 1, \ldots$ do
3: Sample $N$ i.i.d. trajectories $\{\tau_i\}_{i=1}^N$ under policy $\pi^k$.
4: Compute gradient estimator $\tilde{g}_k$ as

$$\tilde{g}_k = \frac{1}{N} \sum_{i=1}^N \sum_{h=0}^{H-1} \gamma^h \nabla_\theta \log \pi^k(a^i_h|s^i_h) \left( \sum_{h'=h}^{H-1} \gamma^{h'-h} r^i_{h'} - b(s^i_h) \right) + \nabla_\theta \Omega(\theta^k).$$

5: Update $\theta^{k+1} = \theta^k + \alpha^k \tilde{g}_k$.
6: end for

Compared with Algorithm 1, Algorithm 2 introduces an additional discount factor when computing the score functions and gets rid of the artificial parameter $\beta \in (0,1)$ needed in Truncated DAE REINFORCE. As a result, the estimator (19) coincides with the vanilla policy gradient estimator for solving discounted problems [68] with a fixed-length trajectory truncation [37]. Note that a similar observation has been made for natural actor-critic methods in [62].

---

3See Appendix B.5 for more explicit bounds on the constants involved in (15).
Similar to the idea of §3, we first establish the sub-optimality of the Doubly Discounted REINFORCE algorithm, evaluated for the discounted problem. Parallel to Lemma 12, we have the following stochastic gradient bounds.

**Lemma 16.** Given Assumptions 1 and 2, then
\[
\|\bar{g}_k\|^2 \leq G + 2\lambda \quad \text{a.s.,}
\]
\[
E_k\bar{g}_k^T \nabla_{\theta} L^\gamma(\theta_k) \geq \|
abla_{\theta} L^\gamma(\theta_k)\|^2 - (G + 2\lambda)\Delta,
\]
\[
E_k\|\bar{g}_k\|^2 \leq 2\|
abla_{\theta} L^\gamma(\theta_k)\|^2 + M.
\]

Here \(G = \frac{2(1 + B(1 - \gamma))}{(1 - \gamma)^2}\), and the constants \(\Delta\) and \(M\) are defined by
\[
\Delta = 2\gamma \frac{H}{1 - \gamma} - \gamma \left(\frac{H + 1}{1 - \gamma}\right), \quad M = 2\Delta^2 + (G + 2\lambda)^2/N.
\]

Based on the above conditions, we now establish the sub-optimality of \(\theta_k\) from the Doubly Discounted REINFORCE algorithm for the discounted problem.

**Theorem 17.** Given Assumptions 1 and 2, let \(\beta_\lambda = \frac{8}{(1 - \gamma)^3} + \frac{2\lambda}{S}\). For any \(\epsilon > 0\) and \(\delta \in (0, 1)\), set \(\alpha_k = \frac{1}{2\beta_\lambda} \frac{1}{\sqrt{k+3\log_2(k+3)}}\) and \(\lambda\) to be the positive (larger) root of the following quadratic equation:
\[
2(G + 2\lambda)\Delta = \left(\frac{\lambda - \epsilon}{4S^2A^2}\right).
\]

Then
\[
\min_{k = 0, \ldots, K} V^{\gamma*} - V^{\gamma}(\pi_{\theta_k}) \leq \min\left\{\frac{d_\rho,\pi^{\gamma*}}{\rho}, \frac{S\|d_\rho,\pi^{\gamma*}\|_\infty}{1 - \alpha_{p,S,A}}\right\}
\times (\epsilon + 8S^2A^2\Delta + 4SA\sqrt{\Delta\epsilon + 4S^2A^2\Delta^2 + G\Delta})
\]

with probability at least \(1 - \delta\), for any \(K\) such that
\[
K \geq O\left(\frac{S^4A^4\beta_\lambda^2(D + \sqrt{2C\log(2/\delta)})^2}{\epsilon^4} \frac{\log^2\left(\frac{SA\beta_\lambda(D + \sqrt{2C\log(2/\delta)})}{\epsilon}\right)}{1 - \alpha_{p,S,A}}\right).
\]

Here the constant \(\alpha_{p,S,A} \in (0, 1)\) is defined in Proposition 11 and the constants \(D\) and \(C\) are bounded by
\[
D = O(M + 1/(1 - \gamma) + \lambda),
\]
\[
C = O((G + 2\lambda)^2(1/(1 - \gamma)^4 + \lambda^2 + (G + 2\lambda)^2)),
\]
where the constants hidden in the big-O notation may depend on \(\theta^0\).

The next result is parallel to Theorem 14 and is based on Lemma 2, Corollary 4, and Lemma 6.
Theorem 18. Given Assumption 1, if there exists a policy \( \hat{\pi} \) such that \( V^{\gamma,*} - V^\gamma(\hat{\pi}) \leq \epsilon \) for some \( \epsilon > 0 \), then for any \( H \geq 1 \),

\[
V^{H,*} - V^H(\hat{\pi}) \leq 2R_{\text{max}}C_{p,S,A}(1-H^{1-\gamma})H^\gamma + \epsilon \\
+ \frac{2R_{\text{max}}}{H} \left( \frac{C_{p,S,A}(1-H^{1-\gamma}) + o_{p,S,A} + |H(1-\gamma) - 1|}{1 - \alpha_{p,S,A}} + D_{p,S,A} \right),
\]

(23)

where \( C_{p,S,A} > 1 \), \( D_{p,S,A} > 1 \) and \( \alpha_{p,S,A} \in [0,1) \) are the constants in Proposition 1 and Lemma 6, which depend only on the transition probability model \( p \), the number of states \( S \) and the number of actions \( A \) of the underlying MDP \( \mathcal{M} \).

Combining Theorems 17 and 18 we obtain the final convergence result for the Doubly Discounted REINFORCE algorithm (in parallel to Theorem 15).

Theorem 19. Given Assumptions 1 and 2, let \( \gamma = 1 - H^{-\sigma} \) for some \( \sigma \in (0,1) \). For any \( \epsilon > 0 \), \( \delta \in (0,1) \), set \( \lambda, \beta, \alpha \) and \( \alpha_k \) to be the same as in Theorem 17. Then for any \( K \) such that (21) is satisfied, \( 4 \) with probability at least \( 1 - \delta \),

\[
\min_{k=0,\ldots,K} V^{H,*} - V^H(\pi_{gk}) \leq O \left( \epsilon \min \left\{ \left\| \frac{1}{\rho} \right\|_\infty, \frac{S}{1 - \alpha_{p,S,A}} \right\} \right) + \text{bias}^{DD}_H,
\]

(24)

where

\[
\text{bias}^{DD}_H = O \left( \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}}H^{-\sigma} + C_{p,S,A}H^{-1} + \frac{S^3A^2}{1 - \alpha_{p,S,A}}H^{1+3\sigma}e^{-H^{1-\sigma}/2} + C_{p,S,A} \right).
\]

Here \( C_{p,S,A} > 1 \), \( D_{p,S,A} > 1 \) and \( \alpha_{p,S,A} \in [0,1) \) are constants in Proposition 1 and Lemma 6.

Comparison with DAE REINFORCE. Here we compare the convergence of (truncated) DAE REINFORCE (cf. Algorithm 1) and Doubly Discounted REINFORCE (cf. Algorithm 2). Note that in both (18) and (24), the global sub-optimality bounds consist of two parts: a vanishing \( \epsilon \) term that goes to zero as the number of iterations \( K \) goes to infinity and a remaining bias term (bias\(^{\text{DAE}}_H\) and bias\(^{\text{DD}}_H\), respectively) resulting from the fictitious discount factor. Below we focus on comparing the bias terms with the same fictitious discount factor \( \gamma = 1 - H^{-\sigma} \), with \( \sigma \in (0,1) \). Recall that

\[
\text{bias}^{\text{DAE}}_H = O \left( \frac{S^2AC^3_{p,S,A}}{(1 - \alpha_{p,S,A})^3}H^{-\frac{\sigma}{2}} \right) + \text{lower order terms in } H,
\]

\[
\text{bias}^{\text{DD}}_H = O \left( \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}}H^{-\sigma} \right) + \text{lower order terms in } H.
\]

Comparing the above two bounds, we see the power of the additional discounting. Indeed, with further discounting, Doubly Discounted REINFORCE improves over DAE REINFORCE, especially in terms of \( H \) (from \( H^{-\sigma/2} \) to \( H^{-\sigma} \)) as it grows. More precisely, the constant before the \( H^{-\sigma} \) term is improved from \( O(S^2A^2C^3_{p,S,A}/(1 - \alpha_{p,S,A})^3) \) to \( O(C_{p,S,A}/(1 - \alpha_{p,S,A})) \), the constant before the \( H^{-1} \) term is improved from \( O(D_{p,S,A} + C_{p,S,A}/(1 - \alpha_{p,S,A})) \) to \( O(D_{p,S,A}) \), while the \( H^{-\sigma/2} \) term is improved to be exponentially decaying as \( H \) grows.

\(^4\)See Appendix C.3 for more explicit bounds on the constants involved in (21).
5 Conclusion and extensions

This paper focuses on two concrete fictitious discount algorithms in the context of policy gradient methods, namely DAE REINFORCE and Doubly Discounted REINFORCE. Rigorous convergence analyses are established for the two algorithms, which, for the first time, shed light on the non-asymptotic global convergence of fictitious discount algorithms.

Given recent development in (global) convergence analysis of algorithms in the discounted setting \[5, 63, 58\] and in the average reward framework \[45, 1\], it is natural to extend our study for natural policy gradient \[30\], natural actor-critic \[49\], TRPO \[55\], PPO \[56\], as well as deep learning based algorithms such as DQN \[44\] and DDPG \[35\].

Meanwhile, it remains to see if one can generalize our work to the general weakly communicating MDPs \[7\] or MDPs with more general state and action spaces, and to remove the need for an exploratory initial distribution (i.e., $\rho > 0$ component-wisely) (e.g., by combining with the policy cover approach in \[3\]).

References

[1] Y. Abbasi-Yadkori, P. Bartlett, K. Bhatia, N. Lazic, C. Szepesvari, and G. Weisz. Politex: Regret bounds for policy iteration using expert prediction. In International Conference on Machine Learning, pages 3692–3702, 2019.

[2] J. Achiam. OpenAI Spinning Up: Vanilla policy gradient, 2018.

[3] A. Agarwal, M. Henaff, S. Kakade, and W. Sun. PC-PG: Policy cover directed exploration for provable policy gradient learning. arXiv preprint arXiv:2007.08459, 2020.

[4] A. Agarwal, N. Jiang, and S. Kakade. Reinforcement Learning: Theory and Algorithms. Technical report, Department of Computer Science, University of Washington, 2019.

[5] Alekh Agarwal, Sham M. Kakade, Jason D Lee, and Gaurav Mahajan. On the theory of policy gradient methods: Optimality, approximation, and distribution shift. arXiv preprint arXiv:1908.00261, 2019.

[6] Ron Amit, Ron Meir, and Kamil Ciosek. Discount factor as a regularizer in reinforcement learning. In International conference on machine learning, pages 269–278. PMLR, 2020.

[7] P. Bartlett and A. Tewari. REGAL: A regularization based algorithm for reinforcement learning in weakly communicating mdps. arXiv preprint arXiv:1205.2661, 2012.

[8] J. Baxter and P. Bartlett. Infinite-horizon policy-gradient estimation. Journal of Artificial Intelligence Research, 15:319–350, 2001.

[9] Jonathan Baxter and Peter L. Bartlett. Direct gradient-based reinforcement learning: I. gradient estimation algorithms. Technical report, Citeseer, 1999.

[10] J. Bhandari and D. Russo. Global optimality guarantees for policy gradient methods. arXiv preprint arXiv:1906.01786, 2019.
[11] David Blackwell. Discrete dynamic programming. *The Annals of Mathematical Statistics*, pages 719–726, 1962.

[12] L. Bottou, F. Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. *SIAM Review*, 60(2):223–311, 2018.

[13] Yuri Burda, Harrison Edwards, Amos Storkey, and Oleg Klimov. Exploration by random network distillation. *arXiv preprint arXiv:1810.12894*, 2018.

[14] S. Cen, C. Cheng, Y. Chen, Y. Wei, and Y. Chi. Fast global convergence of natural policy gradient methods with entropy regularization. *arXiv preprint arXiv:2007.06558*, 2020.

[15] Shi Dong, Benjamin Van Roy, and Zhengyuan Zhou. Simple agent, complex environment: Efficient reinforcement learning with agent state. *arXiv preprint arXiv:2102.05261*, 2021.

[16] Eyal Even-Dar, Sham M. Kakade, and Yishay Mansour. Online markov decision processes. *Mathematics of Operations Research*, 34(3):726–736, 2009.

[17] William Fedus, Carles Gelada, Yoshua Bengio, Marc G Bellemare, and Hugo Larochelle. Hyperbolic discounting and learning over multiple horizons. *arXiv preprint arXiv:1902.06865*, 2019.

[18] Eugene A Feinberg and Adam Shwartz. Constrained discounted dynamic programming. *Mathematics of Operations Research*, 21(4):922–945, 1996.

[19] Vincent François-Lavet, Raphael Fonteneau, and Damien Ernst. How to discount deep reinforcement learning: Towards new dynamic strategies. *arXiv preprint arXiv:1512.02011*, 2015.

[20] Bolin Gao and Lacra Pavel. On the properties of the softmax function with application in game theory and reinforcement learning. *arXiv preprint arXiv:1704.00805*, 2017.

[21] András György Gergely Neu, Csaba Szepesvári, and András Antos. Online markov decision processes under bandit feedback. In *Proceedings of the Twenty-Fourth Annual Conference on Neural Information Processing Systems*, 2010.

[22] Tuomas Haarnoja, Haoran Tang, Pieter Abbeel, and Sergey Levine. Reinforcement learning with deep energy-based policies. In *International Conference on Machine Learning*, pages 1352–1361. PMLR, 2017.

[23] Ben Hambly, Renyuan Xu, and Huining Yang. Policy gradient methods for the noisy linear quadratic regulator over a finite horizon. *Available at SSRN*, 2020.

[24] Ben Hambly, Renyuan Xu, and Huining Yang. Policy gradient methods find the Nash equilibrium in n-player general-sum linear-quadratic games. *Available at SSRN 3894471*, 2021.

[25] Moshe Haviv and Ludo Van der Heyden. Perturbation bounds for the stationary probabilities of a finite markov chain. *Advances in Applied Probability*, pages 804–818, 1984.

[26] Florian Henkel. *A Regularization Study for Policy Gradient Methods/submitted by Florian Henkel*. PhD thesis, Universität Linz, 2018.
[27] Matteo Hessel, Joseph Modayil, Hado Van Hasselt, Tom Schaul, Georg Ostrovski, Will Dabe
ney, Dan Horgan, Bilal Piot, Mohammad Azar, and David Silver. Rainbow: Combining improve
ments in deep reinforcement learning. In Thirty-second AAAI conference on artificial intelligen
tce, 2018.

[28] Arie Hordijk and Alexander A. Yushkevich. Blackwell optimality. In Handbook of Markov de
cision processes, pages 231–267. Springer, 2002.

[29] Sham Kakade. Optimizing average reward using discounted rewards. In International Confer
cence on Computational Learning Theory, pages 605–615. Springer, 2001.

[30] Sham M. Kakade. A natural policy gradient. Advances in neural information processing
systems, 14, 2001.

[31] V. Konda and J. Tsitsiklis. On actor-critic algorithms. SIAM journal on Control and Opti
mization, 42(4):1143–1166, 2003.

[32] JB Lasserre. Conditions for existence of average and blackwell optimal stationary policies in
denumerable markov decision processes. Journal of mathematical analysis and applications,
136(2):479–489, 1988.

[33] Mark E. Lewis and Martin L. Puterman. Bias optimality. In Handbook of Markov decision
processes, pages 89–111. Springer, 2002.

[34] Gen Li, Yuting Wei, Yuejie Chi, Yuantao Gu, and Yuxin Chen. Softmax policy gradient
methods can take exponential time to converge. arXiv preprint arXiv:2102.11270, 2021.

[35] Timothy P. Lillicrap, Jonathan J. Hunt, Alexander Pritzel, Nicolas Heess, Tom Erez, Yuval
Tassa, David Silver, and Daan Wierstra. Continuous control with deep reinforcement learning.
arXiv preprint arXiv:1509.02971, 2015.

[36] Shuang Liu and Hao Su. γ-regret for non-episodic reinforcement learning. arXiv e-prints,
pages arXiv–2002, 2020.

[37] Yanli Liu, Kaiqing Zhang, Tamer Basar, and Wotao Yin. An improved analysis of (varian
treduced) policy gradient and natural policy gradient methods. Advances in Neural Information
Processing Systems, 33, 2020.

[38] Sridhar Mahadevan. Sensitive discount optimality: Unifying discounted and average reward
reinforcement learning. In ICML, pages 328–336. Citeseer, 1996.

[39] P. Marbach and J. Tsitsiklis. Simulation-based optimization of Markov reward processes. IEEE
Transactions on Automatic Control, 46(2):191–209, 2001.

[40] Peter Marbach. Simulation-based optimization of Markov decision processes. PhD thesis, Mas
sachusetts Institute of Technology, 1998.

[41] Peter Marbach and John N Tsitsiklis. Approximate gradient methods in policy-space optimization
of markov reward processes. Discrete Event Dynamic Systems, 13(1):111–148, 2003.
[42] J. Mei, C. Xiao, C. Szepesvari, and D. Schuurmans. On the global convergence rates of softmax policy gradient methods. *arXiv preprint arXiv:2005.06392*, 2020.

[43] Volodymyr Mnih, Adria Puigdomenech Badia, Mehdi Mirza, Alex Graves, Timothy Lillicrap, Tim Harley, David Silver, and Koray Kavukcuoglu. Asynchronous methods for deep reinforcement learning. In *International conference on machine learning*, pages 1928–1937. PMLR, 2016.

[44] Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Andrei A Rusu, Joel Veness, Marc G Bellemare, Alex Graves, Martin Riedmiller, Andreas K Fidjeland, Georg Ostrovski, et al. Human-level control through deep reinforcement learning. *Nature*, 518(7540):529–533, 2015.

[45] G. Neu, A. Jonsson, and V. Gómez. A unified view of entropy-regularized Markov decision processes. *arXiv preprint arXiv:1705.07798*, 2017.

[46] Ronald Ortner. Regret bounds for reinforcement learning via Markov chain concentration. *Journal of Artificial Intelligence Research*, 67:115–128, 2020.

[47] M. Papini, D. Binaghi, G. Canonaco, M. Pirotta, and M. Restelli. Stochastic variance-reduced policy gradient. *arXiv preprint arXiv:1806.05618*, 2018.

[48] J. Peters, K. Mülling, and Y. Altun. Relative entropy policy search. In *AAAI*, volume 10, pages 1607–1612. Atlanta, 2010.

[49] Jan Peters and Stefan Schaal. Natural actor-critic. *Neurocomputing*, 71(7-9):1180–1190, 2008.

[50] Martin L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, 2014.

[51] Jeffrey S. Rosenthal. Convergence rates for Markov chains. *Siam Review*, 37(3):387–405, 1995.

[52] Ernest K. Ryu and Stephen Boyd. Primer on monotone operator methods. *Appl. Comput. Math.*, 15(1):3–43, 2016.

[53] Manuel Schneckenreither. Average reward adjusted discounted reinforcement learning: Near-blackwell-optimal policies for real-world applications. *arXiv preprint arXiv:2004.00857*, 2020.

[54] J. Schulman, X. Chen, and P. Abbeel. Equivalence between policy gradients and soft Q-learning. *arXiv preprint arXiv:1704.06440*, 2017.

[55] J. Schulman, S. Levine, P. Abbeel, M. Jordan, and P. Moritz. Trust region policy optimization. In *International conference on machine learning*, pages 1889–1897, 2015.

[56] J. Schulman, F. Wolski, P. Dhariwal, A. Radford, and O. Klimov. Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*, 2017.

[57] John Schulman, Philipp Moritz, Sergey Levine, Michael Jordan, and Pieter Abbeel. High-dimensional continuous control using generalized advantage estimation. *arXiv preprint arXiv:1506.02438*, 2015.
[58] L. Shani, Y. Efroni, and S. Mannor. Adaptive trust region policy optimization: Global convergence and faster rates for regularized MDPs. *arXiv preprint arXiv:1909.02769*, 2019.

[59] G. Stewart. Matrix perturbation theory. 1990.

[60] R. Sutton, D. McAllester, S. Singh, and Y. Mansour. Policy gradient methods for reinforcement learning with function approximation. In *Advances in neural information processing systems*, pages 1057–1063, 2000.

[61] Chen Tessler and Shie Mannor. Reward tweaking: Maximizing the total reward while planning for short horizons. *arXiv preprint arXiv:2002.03327*, 2020.

[62] Philip Thomas. Bias in natural actor-critic algorithms. In *International conference on machine learning*, pages 441–448. PMLR, 2014.

[63] Lingxiao Wang, Qi Cai, Zhuoran Yang, and Zhaoran Wang. Neural policy gradient methods: Global optimality and rates of convergence. *arXiv preprint arXiv:1909.01150*, 2019.

[64] Neng-Yi Wang, Liming Wu, et al. Convergence rate and concentration inequalities for Gibbs sampling in high dimension. *Bernoulli*, 20(4):1698–1716, 2014.

[65] Pan Xu, Felicia Gao, and Quanquan Gu. Sample efficient policy gradient methods with recursive variance reduction. *arXiv preprint arXiv:1909.08610*, 2019.

[66] Zhongwen Xu, Hado van Hasselt, and David Silver. Meta-gradient reinforcement learning. *arXiv preprint arXiv:1805.09801*, 2018.

[67] Junyu Zhang, Alec Koppel, Amrit Singh Bedi, Csaba Szepesvari, and Mengdi Wang. Variational policy gradient method for reinforcement learning with general utilities. *arXiv preprint arXiv:2007.02151*, 2020.

[68] Junzi Zhang, Jongho Kim, Brendan O’Donoghue, and Stephen Boyd. Sample efficient reinforcement learning with REINFORCE. *arXiv preprint arXiv:2010.11364*, 2020.

[69] K. Zhang, A. Koppel, H. Zhu, and T. Başar. Global convergence of policy gradient methods to (almost) locally optimal policies. *arXiv preprint arXiv:1906.08383*, 2019.

[70] Kaiqing Zhang, Xiangyuan Zhang, Bin Hu, and Tamer Başar. Derivative-free policy optimization for risk-sensitive and robust control design: Implicit regularization and sample complexity. *arXiv preprint arXiv:2101.01041*, 2021.

[71] Tingting Zhao, Gang Niu, Ning Xie, Jucheng Yang, and Masashi Sugiyama. Regularized policy gradients: direct variance reduction in policy gradient estimation. In *Asian Conference on Machine Learning*, pages 333–348. PMLR, 2016.
Appendix

A Preliminary facts

In this section, we show the proofs of results in §2: Propositions 1 and 7, Lemmas 2, 3, 5, 6, 8 and 10. For ease of notation, we define \( c = H(1 - \gamma) \) so that \( \gamma = 1 - c/H \) and \( c \in (0, H] \).

Notation and terminology. Here and below we use \( P_\pi \in \mathbb{R}^{S \times S} \) to denote the transition probability of the Markov chain induced by policy \( \pi \), i.e., \( P_\pi(s, s') = \sum_{a \in A} p(s'|s, a) \pi(a|s) \). In general, a matrix \( P \in \mathbb{R}^{S \times S} \) is called a stochastic matrix if \( P(s, s') \geq 0 \) for any \( s, s' \in S \) and \( \sum_{s' \in S} P(s, s') = 1 \) for any \( s \in S \). If in addition we also have \( P(s, s') > 0 \) for any \( s, s' \in S \), then we say that \( P \) is a positive stochastic matrix. We say that a policy \( \pi \) is deterministic if for any \( s \in S \), \( \pi(a_s|s) = 1 \) for some \( a_s \in A \). Unless otherwise stated, all state distributions (e.g., \( \rho \)) are row vectors.

We also introduce the following notation to be used in the proof. The first three quantities are defined for the discounted setting, while the last three quantities are defined for the average reward setting. In all cases, \( \tau = (s_0, a_0, r_0, \ldots, s_h, a_h, r_h, \ldots) \) is a trajectory sampled under policy \( \pi \).

- discounted value function:
  \[
  V_{\gamma, \pi}(s) = (1 - \gamma) \mathbb{E} \left[ \sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| s_0 = s, \pi \right].
  \]

- discounted action-value function:
  \[
  Q_{\gamma, \pi}(s, a) = (1 - \gamma) \mathbb{E} \left[ \sum_{h=0}^{\infty} \gamma^h r(s_h, a_h) \middle| s_0 = s, a_0 = a, \pi \right].
  \]

- discounted advantage function: \( A_{\gamma, \pi}(s, a) = Q_{\gamma, \pi}(s, a) - V_{\gamma, \pi}(s) \).

- average reward bias value function:
  \[
  \bar{V}_\pi(s) = \lim_{H \to \infty} \mathbb{E} \left[ \sum_{h=0}^{H-1} r(s_h, a_h) - \eta(\pi) \middle| s_0 = s, \pi \right].
  \]

- average reward action-value function:
  \[
  \bar{Q}_\pi(s, a) = \lim_{H \to \infty} \mathbb{E} \left[ \sum_{h=0}^{H-1} r(s_h, a_h) - \eta(\pi) \middle| s_0 = s, a_0 = a, \pi \right].
  \]

- average reward advantage function: \( \bar{A}_\pi(s, a) = \bar{Q}_\pi(s, a) - \bar{V}_\pi(s) \).

Note that we have \( V_{\gamma}(\pi) = \sum_{s \in S} \rho(s)V_{\gamma, \pi}(s) \).
A.1 Proof of Proposition 1

We first show two propositions which will be used in the proof of Proposition 1. The following well-known fact about the convergence rate of finite and ergodic Markov chains (also known as Dobrushin’s inequality) is central for our proofs.

**Proposition 20.** Let \( P \in \mathbb{R}^{S \times S} \) be a positive stochastic matrix. Then for any distribution \( \rho \in \mathcal{P}(S) \) (viewed as a row vector of length \( S \)), we have for any \( h \geq 0 \),
\[
    d_{TV}(\rho P^h, \mu_P) \leq \alpha_P^h,
\]
where \( d_{TV} \) is the total variation distance between two measures, \( \mu_P \) is the (unique) stationary distribution of the transition matrix \( P \), and \( \alpha_P = 1 - S \min_{s, s' \in S} P(s, s') \in [0, 1) \).

A proof of the above proposition can be found in standard textbooks [51, 64].

Another important property is the following proposition for representing an arbitrary (randomized) policy as a convex combination of finitely many deterministic policies.

**Proposition 21.** Suppose that \( |S| = S < \infty \) and \( |A| = A < \infty \). Let \( \pi : S \rightarrow \mathcal{P}(A) \) be an arbitrary policy. Then there exist \( n_{S,A} = S(A-1) + 1 \) deterministic policies \( \pi_1, \ldots, \pi_{n_{S,A}} \) and nonnegative constants \( c_1, \ldots, c_{S_A} \), such that
\[
    \pi(a|s) = \sum_{i=1}^{n_{S,A}} c_i \pi_i(a|s), \quad \forall s \in S, a \in A,
\]
where \( \sum_{i=1}^{n_{S,A}} c_i = 1 \) and \( c_i \geq 0 \) (i = 1, \ldots, \( n_{S,A} \)).

The above proposition is implied by the proof of [18, Theorem 5.1]. For self-containedness, we also provide a simple proof by induction below.

**Proof.** Define the index of an arbitrary (randomized) policy \( \pi \) as
\[
    \sum_{s \in S} (|\{a \in A | \pi(a|s) > 0\}| - 1),
\]
i.e., the difference between the total number of non-zero entries in \( \pi \) (when viewed as a vector of length \( SA \) or a matrix of size \( S \times A \)) and the total number of non-zero entries in a deterministic policy (i.e., the number of states). By definition, the index of a policy is at most \( S(A-1) \). Below we prove the following claim, which immediately implies the desired result of Proposition 21 by taking index equal to \( S(A-1) \):

**Claim 1.** For a policy with index \( m \), there exist \( m+1 \) deterministic policies \( \pi_1, \ldots, \pi_{m+1} \) and nonnegative constants \( c_1, \ldots, c_{m+1} \), such that
\[
    \pi(a|s) = \sum_{i=1}^{m+1} c_i \pi_i(a|s), \quad \forall s \in S, a \in A,
\]
where \( \sum_{i=1}^{m+1} c_i = 1 \) and \( c_i \geq 0 \) (i = 1, \ldots, \( m+1 \)).

We prove this claim by induction on the index of \( \pi \).
Base step. When the index of \( \pi \) is 0, the policy \( \pi \) is deterministic, and hence we can simply take \( c_1 = 1 \) and \( \pi_1 = \pi \).

Induction step. Suppose that Claim 1 holds for index \( m - 1 \) \((m \geq 1)\). Then for a policy with index \( m \), let

\[
(s_{\min}, a_{\min}) \in \arg\min_{s \in S, a \in A} \pi(a|s) > 0 \pi(a|s),
\]

and \( \pi_{\min} = \pi(a_{\min}|s_{\min}) \in (0, 1) \). Note that \( \pi_{\min} < 1 \) since otherwise the index would be 0, which contradicts the assumption that \( m \geq 1 \).

Now define \( \pi_{m+1} \) as a deterministic policy such that \( \pi(a_{\min}|s_{\min}) = 1 \) and that for any \( s \neq s_{\min} \), \( \pi_{m+1}(a_s|s) = 1 \) for some (arbitrary) \( a_s \) with \( \pi(a_s|s) > 0 \). Note that such a policy exists by the trivial fact that for any \( s \in S \), \( \pi(a|s) > 0 \) for some \( a \in A \). By taking \( c_{m+1} = \pi_{\min} \in (0, 1) \), we can define a policy \( \pi' \) with

\[
\pi'(a|s) = (\pi(a|s) - c_{m+1}\pi_{m+1})/(1 - c_{m+1}), \quad \forall s \in S, a \in A.
\]

It’s easy to see that \( \pi' \) is indeed a policy \( i.e., \pi'(a|s) \geq 0 \) for any \( s \in S \), \( a \in A \) and \( \sum_{a \in A} \pi'(a|s) = 1 \). In addition, by definition of \( c_{m+1} \) and \( \pi_{m+1} \), we also have

\[
\{a \in A \mid \pi'(a|s_{\min}) > 0\} \subseteq \{a \in A \mid \pi(a|s_{\min}) > 0\}\setminus\{a_{\min}\}
\]

and

\[
\{a \in A \mid \pi'(a|s) > 0\} \subseteq \{a \in A \mid \pi(a|s) > 0\}, \quad \forall s \in S, s \neq s_{\min},
\]

and hence the index of \( \pi' \) is at most \( m - 1 \). By the induction hypothesis, there exist \( m \) deterministic policies \( \pi_1, \ldots, \pi_m \) and nonnegative constants \( c_1', \ldots, c_m' \), such that

\[
\pi'(a|s) = \sum_{i=1}^{m} c_i'\pi_i(a|s), \quad \forall s \in S, a \in A,
\]

\[
\sum_{i=1}^{m} c_i' = 1 \text{ and } c_i' \geq 0 \ (i = 1, \ldots, m), \text{ which immediately implies that}
\]

\[
\pi(a|s) = \sum_{i=1}^{m+1} c_i\pi_i(a|s), \quad \forall s \in S, a \in A,
\]

with \( c_i = (1 - c_{m+1})c_i' \ (i = 1, \ldots, m) \). Since \( c_i \geq 0 \ (i = 1, \ldots, m+1) \) and \( \sum_{i=1}^{m+1} c_i = 1 \) by definition, we have proved the claim for index \( m \). By induction, this completes the proof. \( \square \)

Proof of Proposition 7. Let \( \Pi_{\text{det}} \) be the set of all deterministic policies \( \pi \). By the finiteness of the state and action spaces, \( \Pi_{\text{det}} \) is also a finite set. For any \( \pi \in \Pi_{\text{det}} \), since \( P_{\pi} \) is irreducible and aperiodic, there exists a positive integer \( m_{\pi} \) such that \( P_{\pi}^m \) is componentwisely positive for any \( m \geq m_{\pi} \). Now by the finiteness of \( \Pi_{\text{det}} \), we can define \( m_p = \max_{\pi \in \Pi_{\text{det}}} m_{\pi} < \infty \), and then \( P_{\pi}^m \) is componentwise positive for any \( \pi \in \Pi_{\text{det}} \) and \( m \geq m_p \). Accordingly, we also define

\[
p_{\min} = \min_{\pi \in \Pi_{\text{det}}, s, s' \in S} P_{\pi}^{m_p}(s, s') > 0.
\]

By Proposition 21 for any (randomized) policy \( \pi \), there exist \( n_{S,A} = S(A - 1) + 1 \) policies \( \pi_1, \ldots, \pi_{n_{S,A}} \in \Pi_{\text{det}} \) and nonnegative constants \( c_1, \ldots, c_{n_{S,A}} \), such that

\[
\pi(a|s) = \sum_{i=1}^{n_{S,A}} c_i\pi_i(a|s), \quad \forall s \in S, a \in A,
\]
\[ \sum_{i=1}^{n_{S,A}} c_i = 1 \] and \[ c_i \geq 0 \] \((i = 1, \ldots, n_{S,A})\). By the linearity of \[ P_\pi \] in \[ \pi \], we have
\[ P_\pi = \sum_{i=1}^{n_{S,A}} c_{\pi_i} P_{\pi_i}. \]

This implies that for any \( s, s' \in S \), we have
\[ P^m_\pi(s, s') \geq \sum_{i=1}^{n_{S,A}} c_{\pi_i} P^m_\pi(s, s') \geq p_{\min} \sum_{i=1}^{n_{S,A}} c_{\pi_i} \]
(by convexity of \( x^m \) for \( x \geq 0 \) \( \geq p_{\min} n_{S,A} (1/n_{S,A})^m = p_{\min}/n_{S,A}^{m-1} \).

Accordingly, by Proposition 20, for any (randomized) policy \( \pi : S \rightarrow \mathcal{P}(A) \), there exists a constant
\[ \alpha_\pi = 1 - S \min_{s, a' \in S} P^m_\pi(s, a'), \]
such that for any \( r \geq 0 \),
\[ d_{TV}(\rho(P^m_\pi)^r, \mu_\pi) \leq \alpha_\pi^r. \]

By (26), we have \( \alpha_\pi \in [0, 1 - S p_{\min}/n_{S,A}^{m-1}] \subseteq [0, 1) \), which implies that
\[ d_{TV}(\rho(P^m_\pi)^r, \mu_\pi) \leq \tilde{\alpha}_r, \]
where \( \tilde{\alpha}_p = \alpha_\pi 1/m \) and \( C_{p,S,A} = 1/\tilde{\alpha}_p \). This completes the proof.

A.2 Proofs of Lemmas 2, 3, 5 and 6

Proof of Lemma 2. By reorganization of the summations, we have
\[ V^\gamma(\pi) = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \sum_{s \in S, a \in A} \text{Prob}_\pi^\gamma(s_h = s, a_h = a|s_0 \sim \rho)r(s, a) \]
\[ = (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \sum_{s \in S, a \in A} [\rho P^h_\pi]_s \pi(a|s)r(s, a) \]
\[ = \sum_{s \in S} w^{\gamma}(s; \pi) \sum_{a \in A} \pi(a|s)r(s, a), \]
where
\[ w^{\gamma}(s; \pi) = (1 - \gamma) \sum_{h=0}^{\infty} [\rho(\gamma P_\pi)^h]_s = (1 - \gamma) [\rho(I - \gamma P_\pi)^{-1}]_s, \]
\(\text{Prob}^\pi(s_h = s, a_h = a|s_0 \sim \rho)\) is the probability of arriving at state \(s\) and action \(a\) in step \(h\) starting from \(s_0 \sim \rho\) following policy \(\pi\), and for a vector \(x \in \mathbb{R}^S\), we use \(x_s\) or \([x]_s\) alternatively to denote its \(s\)-th element. Note that here \(w(\gamma; \pi) = d(\gamma, \pi, \rho)\), and we use them alternatively throughout the appendix. In fact, for most of the time in the appendix, we use the former for simplicity and clarity (as \(\rho\) is always fixed in our paper, while \(\gamma\) may change) except for the final statements.

Similarly, we also have

\[
V^H(\pi) = \frac{1}{H} \sum_{h=0}^{H-1} \sum_{s \in S, a \in A} [\rho P^h \pi]_s \pi(a|s)r(s, a) = \sum_{s \in S} w^H(s; \pi) \sum_{a \in A} \pi(a|s)r(s, a),
\]

where

\[
w^H(s; \pi) = \frac{1}{H} \sum_{h=0}^{H-1} [\rho P^h \pi]_s .
\]

Hence we have

\[
\left| V^\gamma(\pi) - V^H(\pi) \right| \leq \sum_{s \in S} |w^\gamma(s; \pi) - w^H(s; \pi)| \sum_{a \in A} |r(s, a)| \pi(a|s)
\]

\[
\leq R_{\max} \sum_{s \in S} \left| (1 - \gamma) [\rho(I - \gamma P^\pi)^{-1}]_s - \frac{1}{H} \sum_{h=0}^{H-1} [\rho P^h \pi]_s \sum_{a \in A} \pi(a|s) \right|
\]

\[
= R_{\max} \left\| \rho \left( (1 - \gamma) I - \frac{1}{H} \sum_{h=0}^{H-1} P^h \pi (I - \gamma P^\pi) \right) (I - \gamma P^\pi)^{-1} \right\|_1.
\]

By Proposition \([\text{Prop}]\), we have

\[
d_{TV}(\rho P^h \pi, \mu^\pi) \leq C_{p, S, A}^h \alpha_{p, S, A}^h,
\]

for some constant \(\alpha_{p, S, A} \in [0, 1]\) that depends only on \(p, S, A\).

Noticing that \(1 - \gamma = c/H\), we have

\[
\rho \left( (1 - \gamma) I - \frac{1}{H} \sum_{h=0}^{H-1} P^h \pi (I - \gamma P^\pi) \right) = \left( 1 - \gamma - \frac{1}{H} \right) \rho - \frac{1}{H} \sum_{h=1}^{H-1} (1 - \gamma) \rho P^h \pi + \frac{1}{H} \gamma \rho P^H \pi
\]

\[
= \frac{1}{H} \left( 1 - \frac{c}{H} \right) \rho P^H \pi - \frac{c}{H} \sum_{h=1}^{H-1} \rho P^h \pi + \frac{c - 1}{H} \rho
\]

\[
= J_1 + J_2,
\]

where

\[
J_1 = \frac{1}{H} \left( \frac{H - c}{H} (\rho P^H \pi - \mu^\pi) - \frac{c}{H} \sum_{h=1}^{H-1} (\rho P^h \pi - \mu^\pi) \right), \quad J_2 = \frac{c - 1}{H} (\rho - \mu^\pi).
\]
Now using $\|\mu - \nu\|_1 = 2d_{TV}(\mu, \nu)$ for any $\mu, \nu \in \mathcal{P}(S)$, we have
\[
\|J_1\|_1 \leq \frac{1}{H} \left( \frac{H-c}{H} \|\rho P^H - \mu\|_1 + \frac{c}{H} \sum_{h=1}^{H-1} \|\rho P^h - \mu\|_1 \right) \\
\leq \frac{2}{H} \left( \frac{H-c}{H} d_{TV}(\rho P^H, \mu) + \frac{c}{H} \sum_{h=1}^{H-1} d_{TV}(\rho P^h, \mu) \right) \\
\leq \frac{2C_{p,S,A}}{H} \left( \frac{H-c}{H} \alpha_{p,S,A}^H + \frac{c}{H} \sum_{h=1}^{H-1} \alpha_{p,S,A}^h \right) \\
= \frac{2C_{p,S,A}(H-c)}{H^2} \alpha_{p,S,A}^H + \frac{2cC_{p,S,A}\alpha_{p,S,A}}{(1 - \alpha_{p,S,A})H^2}.
\]

Similarly, we have
\[
\|J_2(I - \gamma P^H)^{-1}\|_1 = \frac{|c-1|}{H} \left[ \rho \sum_{h=0}^{\infty} \gamma^h P^h - \mu \sum_{h=0}^{\infty} \gamma^h P^H \right]_1 \\
= \frac{|c-1|}{H} \left[ \sum_{h=0}^{\infty} \gamma^h P^h - \sum_{h=0}^{\infty} \gamma^h \mu \right]_1 \leq \frac{|c-1|}{H} \sum_{h=0}^{\infty} \gamma^h \|P^h - \mu\|_1 \\
= \frac{2|c-1|}{H} \sum_{h=0}^{\infty} \gamma^h d_{TV}(\rho P^h, \mu) \leq \frac{2|c-1|C_{p,S,A}}{(1 - \gamma \alpha_{p,S,A})H} \leq \frac{2|c-1|C_{p,S,A}}{(1 - \alpha_{p,S,A})H}.
\]

Finally, since we have
\[
\|(I - \gamma P^H)^{-1}\|_\infty = \left\| \sum_{h=0}^{\infty} (\gamma P^H)^h \right\|_\infty \leq \sum_{h=0}^{\infty} \gamma^h = \frac{1}{1 - \gamma} = H/c,
\]
we conclude that
\[
|V^\gamma(\pi) - V^H(\pi)| \leq R_{\max} \|J_1\|_1 \|(I - \gamma P^H)^{-1}\|_\infty + R_{\max} \|J_2(I - \gamma P^H)^{-1}\|_1 \\
\leq 2R_{\max} C_{p,S,A} \left( \frac{H-c}{cH} \alpha_{p,S,A}^H + \alpha_{p,S,A} + \frac{|c-1|}{(1 - \alpha_{p,S,A})H} \right)
\]

Here we use the fact that for any row vector $x \in \mathbb{R}^S$ and matrix $A \in \mathbb{R}^{S \times S}$,
\[
\|xA\|_1 = \|A^T x\|_1 \leq \|A^T\|_1 \|x\|_1 = \|A\|_\infty \|x\|_1.
\]

This completes the proof. \qed
Proof of Lemma 3. The proof is similar to that of Lemma 2. In fact, by (28) and (4), we have
\[ |V^\gamma(\pi) - \eta(\pi)| \leq R_{\text{max}} \| (1 - \gamma) [\rho(I - \gamma P)^{-1}] - \mu_\pi \|_1 \]
\[ = R_{\text{max}} \| (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h \rho P_\pi^h - \mu_\pi \|_1 = R_{\text{max}} \| (1 - \gamma) \sum_{h=0}^{\infty} \gamma^h (\rho P_\pi^h - \mu_\pi) \|_1 \]
\[ \leq \frac{2cR_{\text{max}}}{H} \sum_{h=0}^{\infty} \gamma^h d_{TV}(\rho P_\pi^h, \mu_\pi) \]
(by Proposition 1) \[ \leq \frac{2cR_{\text{max}} C_{p,S,A}}{H} \sum_{h=0}^{\infty} \gamma^h \alpha_{p,S,A} = \frac{2cR_{\text{max}} C_{p,S,A}}{H} \frac{1}{1 - \gamma \alpha_{p,S,A} (1 - \alpha_{p,S,A} H)}. \]

This completes the proof.

Proof of Lemma 5. The key is to notice that we have
\[ \sum_{s \in S} |w^H(s; \pi) - \mu_\pi(s)| = \left\| \frac{1}{H} \sum_{h=0}^{H-1} [\rho P_\pi^h] - \mu_\pi \right\|_1 \leq \frac{1}{H} \sum_{h=0}^{H-1} 2C_{p,S,A} \alpha_{p,S,A} \leq \frac{2C_{p,S,A}}{H(1 - \alpha_{p,S,A})}. \]
Hence by (4) and (29), we have
\[ |V^H(\pi) - \eta(\pi)| = \left| \sum_{s \in S} w^H(s; \pi) \sum_{a \in A} \pi(a|s) r(s,a) - \sum_{s \in S, a \in A} \mu_\pi(s) \pi(a|s) r(s,a) \right| \]
\[ \leq \sum_{s \in S, a \in A} |w^H(s; \pi) - \mu_\pi(s)| \pi(a|s) r_{\text{max}} \leq \frac{2R_{\text{max}} C_{p,S,A}}{H(1 - \alpha_{p,S,A})}. \]

This completes the proof.

Proof of Lemma 6. Let \( L : \mathbb{R}^S \rightarrow \mathbb{R}^S \) be the Bellman operator, with
\[ [LJ]_s = \max_{a \in A} \left( r(s,a) + \sum_{s' \in S} p(s'|s,a) J(s') \right) \]
for any \( J \in \mathbb{R}^S \). Then by the well-known dynamic programming principle [50], we have \( V^{H,*} = \frac{1}{H} \sum_{s \in S} p_s [J^{H,*}]_s \), where
\[ J^{H,*} = L^{H-1} r_{\text{max}} \]
where \( r_{\text{max}} \in \mathbb{R}^S \) is defined by \( r_{\text{max}}(s) = \max_{a \in A} r(s,a) \).

On the other hand, by the convergence property of value iteration algorithm for the infinite horizon average reward setting [50 Prop. 8.5.1, Theorem 8.5.2], we have that for any nonnegative integers \( r \) and \( k \),
\[ \text{sp}(L^{r\rho_p+k+1}r_{\text{max}} - L^{r\rho_p+k}r_{\text{max}}) \leq \beta_{p,S,A}^{r} \text{sp}(L^{k+1}r_{\text{max}} - L^{k}r_{\text{max}}), \]
where \( \text{sp}(J) \) is the span function defined as
\[ \text{sp}(J) = \max_{s \in S} J(s) - \min_{s \in S} J(s) \]
25
for any $J \in \mathbb{R}^S$, $m_p$ is the positive integer defined in the proof of Proposition \[\text{I}\] and $\tilde{\beta}_{p,S,A} \in [0,1)$ is defined by $\tilde{\beta}_{p,S,A} = 1 - S_{p_{\min}}$, where $p_{\min} > 0$ is again defined as in the proof of Proposition \[\text{I}\]. Hence if we write $H - 1$ as $H - 1 = rm_p + k$ for some nonnegative integers $r$ and $s$ with $0 \leq k \leq m_p - 1$, then we have

$$\text{sp}(L J^{H,*} - J^{H,*}) \leq \tilde{\beta}_{p,S,A}^r \text{sp}(L^{k+1}r_{\max}^r - L^k r_{\max}^r) \leq 4R_{\max} E_{p,S,A} m_p^p \beta_{H-1}^{p,S,A},$$

where $\beta_{p,S,A} = \tilde{\beta}_{p,S,A}^{1/m_p}$, $E_{p,S,A} = 1/\tilde{\beta}_{p,S,A}$, and we use the fact that for any $s \in S$,

$$|[L^k r_{\max}^r(s)| \leq (k + 1) R_{\max}.\]

Finally, by \[\text{I}\] Theorem 8.5.5], we have that for any $J \in \mathbb{R}^S$,

$$\min_{s \in S} [LJ - J]_s \leq \eta^* \leq \max_{s \in S} [LJ - J]_s,$$

which immediately implies that

$$|\eta^* - \sum_{s \in S} \rho_s [LJ - J]_s| \leq \text{sp}(LJ - J).$$

By plugging in $J = J^{H,*}$ and noticing that

$$\sum_{s \in S} \rho_s L J^{H,*} = \sum_{s \in S} \rho_s [L^H r_{\max}^s]_s = (H + 1) V^{H+1,*},$$

we have

$$|\eta^* - ((H + 1)V^{H+1,*} - H V^{H,*})| \leq 4R_{\max} E_{p,S,A} m_p^p \beta_{H-1}^{p,S,A},$$

which implies that

$$|(H + 1)(V^{H+1,*} - \eta^*)| - |H(V^{H,*} - \eta^*)| \leq |(H + 1)(V^{H+1,*} - \eta^*) - H(V^{H,*} - \eta^*)| \leq 4R_{\max} E_{p,S,A} m_p^p \beta_{H-1}^{p,S,A}.$$

By telescoping the inequality from 1 to $H - 1$, we obtain that

$$H|V^{H,*} - \eta^*| \leq |V^{1,*} - \eta^*| + 4R_{\max} E_{p,S,A} m_p^{H} \sum_{h=1}^{H-1} \beta_{p,S,A}^{h-1} \leq 2R_{\max} + 4R_{\max} E_{p,S,A} m_p \beta_{p,S,A} \frac{1 - \beta_{p,S,A}}{1 - \beta_{p,S,A}},$$

which shows that for any $H \geq 1$,

$$|V^{H,*} - \eta^*| \leq \frac{2R_{\max} D_{p,S,A}}{H},$$

where $D_{p,S,A} = 1 + 2E_{p,S,A} m_p \beta_{p,S,A}/(1 - \beta_{p,S,A})$. This completes the proof. \[\square\]
## A.3 Proofs of Proposition 7, Lemma 8 and Lemma 10

The proof of Proposition 7 follows the same steps as [5, Theorem 5.2], with some modifications leading to a slightly tightened bound. For completeness, we provide a self-contained proof below.

**Proof of Proposition 7.** By [5, Lemma C.1], the policy gradient of $V^\gamma$ has the following form:

$$\frac{\partial V^\gamma(\pi_\theta)}{\partial \theta_{s,a}} = d^\gamma_{\rho,\pi_\theta}(s) \pi_\theta(a|s) A^\gamma_{\pi_\theta}(s, a), \quad (30)$$

and the gradient of the regularization term $\Omega$ has the form

$$\frac{\partial \Omega(\theta)}{\partial \theta_{s,a}} = \frac{\lambda}{SA} - \frac{\lambda}{S} \pi_\theta(a|s). \quad (31)$$

Now since $\nabla L^\gamma(\theta) \parallel_2 \leq \lambda/(2SA)$, we have for any $s \in S$ and $a \in A$,

$$\frac{\partial L^\gamma(\theta)}{\partial \theta_{s,a}} = \frac{1}{1-\gamma} d^\gamma_{\rho,\pi_\theta}(s) \pi_\theta(a|s) A^\gamma_{\pi_\theta}(s, a) + \frac{\lambda}{SA} - \frac{\lambda}{S} \pi_\theta(a|s) \leq \lambda/(2SA),$$

from which we see that

$$A^\gamma_{\pi_\theta}(s, a) \leq \frac{\lambda(1-\gamma)}{d^\gamma_{\rho,\pi_\theta}(s)} \left( \frac{1}{S} - \frac{1}{2SA \pi_\theta(a|s)} \right) \leq \frac{\lambda(1-\gamma)}{S d^\gamma_{\rho,\pi_\theta}(s)}.$$

Now notice that for any stationary policy $\pi$ and any state $s \in S$,

$$d^\gamma_{\rho}(s) = w^\gamma(s; \pi) = (1-\gamma) \sum_{h=0}^{\infty} \gamma^h \rho^{P^h}[\pi]_s \geq (1-\gamma) \sum_{k=0}^{\infty} \gamma^{km_p} \rho^{P^{km_p}}[\pi]_s \geq (1-\gamma) \sum_{k=0}^{\infty} \gamma^{km_p} p_{\text{min}}/n_{S,A}^{m_p-1},$$

where the quantities $m_p$, $p_{\text{min}}$ and $n_{S,A}$ are defined in Proposition 1. In addition, by definition, we also have $d^\gamma_{\rho}(s) \geq (1-\gamma) \rho(s)$ for any stationary policy $\pi$ and state $s \in S$.

Finally, by the performance difference lemma in the discounted setting [5, Lemma 3.2], we have

$$V^{\gamma,*} - V^\gamma(\pi_\theta) = \sum_{s \in S} d^\gamma_{\rho,\pi_\theta}(s) \sum_{a \in A} \pi^{\gamma,*}(a|s) A^\gamma_{\pi_\theta}(s, a)$$

$$\leq \lambda \frac{(1-\gamma)}{S} \sum_{s \in S} \pi_{\text{min}}/n_{S,A}^{m_p-1} \left\{ \frac{d^\gamma_{\rho,\pi_\theta}(s)}{(1-\gamma)\rho(s)}, \frac{d^\gamma_{\rho,\pi_\theta}(s)n_{S,A}^{m_p-1}(1-\gamma)^{m_p}}{(1-\gamma)p_{\text{min}}} \right\}$$

$$\leq \lambda \left\{ \left\| \frac{d^\gamma_{\rho,\pi_\theta}}{\rho} \right\|_\infty, S \left\| d^\gamma_{\rho,\pi_\theta} \right\|_\infty \right\} \frac{1}{1 - \alpha_{p,S,A}},$$

where the last step uses the fact that

$$\alpha_{p,S,A} = \frac{1}{m_p} \geq \tilde{\alpha}_{p,S,A} = 1 - Sp_{\text{min}}/n_{S,A}^{m_p-1},$$

which comes from the proof of Proposition 1. This completes the proof. \(\square\)
Proof of Lemma 8 relies on the following lemma.

**Lemma 22** ([16, 21]. Average reward performance difference lemma). Suppose that Assumption 1 holds. Then we have

\[
\eta(\pi) - \eta(\pi') = \sum_{s \in S} \mu_\pi(s) \sum_{a \in A} \pi(a|s) \bar{A}'(s, a).
\] (32)

**Proof of Lemma 8**. By the well-known policy gradient theorem [60] and some simplification, we have

\[
\frac{\partial \eta(\pi_\theta)}{\partial \theta_{s,a}} = \mu_{\pi_\theta}(s) \pi_\theta(a|s) \bar{A}^\pi_\theta(s, a).
\] (33)

Now since \(\|\nabla_\theta \bar{L}(\theta)\|_2 \leq \lambda/(2SA)\), recalling the form of \(\nabla_\theta \Omega(\theta)\) in the proof of Proposition 7, we have that for any \(s \in S\) and \(a \in A\),

\[
\frac{\partial \bar{L}(\theta)}{\partial \theta_{s,a}} = \mu_{\pi_\theta}(s) \pi_\theta(a|s) \bar{A}^\pi_\theta(s, a) + \frac{\lambda}{SA} - \frac{\lambda}{S} \pi_\theta(a|s) \leq \lambda/(2SA).
\] (34)

Hence we have

\[
\bar{A}^\pi_\theta(s, a) \leq \frac{1}{\mu_{\pi_\theta}(s)} \left( \frac{\lambda}{S} - \frac{\lambda}{2SA \pi_\theta(a|s)} \right) \leq \frac{\lambda}{\mu_{\pi_\theta}(s) S}.
\]

Now notice that for any stationary policy \(\pi\), \(\mu_\pi P = \mu_\pi\) and hence \(\mu_\pi(s) = \sum_{s' \in S} \mu_\pi(s') P^m_\pi(s', s) \geq p_{\min}/n_{S,A}^{m_p-1}\), where the quantities \(\mu_\pi, m_p, p_{\min}\) and \(n_{S,A}\) are defined in Proposition 1.

Finally, we have

\[
\eta^* - \eta(\pi_\theta) = \sum_{s \in S} \mu_\pi^*(s) \sum_{a \in A} \pi^*(a|s) \bar{A}^\pi_\theta(s, a)
\]

\[
\leq \lambda S \sum_{s \in S} \mu_{\pi_\theta}(s) \leq \lambda \|\mu_\pi^*\|\infty n_{S,A}^{m_p-1} p_{\min} \leq \lambda S \|\mu_\pi^*\|\infty 1/1 - \alpha_{p,S,A},
\]

where again we use the fact that

\[
\alpha_{p,S,A} = \frac{1}{m_p} \geq \tilde{\alpha}_{p,S,A} = 1 - Sp_{\min}/n_{S,A}^{m_p-1}.
\]

This completes the proof. \(\square\)

Now we are ready to show the proof of Lemma 10.

**Proof of Lemma 10**. We prove a slightly generalized version of the claimed results assuming only that \(|r(s, a)| \leq R_{\max}\) instead of \(r(s, a) \in [0, 1]\) as in Assumption 2.

Firstly, we show that

\[
\|\mu_{\pi_{\theta_1}} - \mu_{\pi_{\theta_2}}\|_1 \leq \frac{2\sqrt{SC_{p,S,A}}}{1 - \alpha_{p,S,A}} \|\theta_1 - \theta_2\|_2.
\] (35)

To see this, first notice that

\[
\mu_{\pi_{\theta_1}} - \mu_{\pi_{\theta_2}} = \mu_{\pi_{\theta_1}}(P_1 - P_2)Y_2,
\] (36)
where $P_i = P_{\pi_i}$ and $Y_i = \sum_{h=0}^{\infty} (P_i^h - P_i^\infty)$, with $1 \in \mathbb{R}^S$ being the all-one vector and $P_i^\infty = 1\mu_{\pi_i} = \lim_{h \to \infty} P_i^h$ ($i = 1, 2$).

By Proposition 3, we have that for any policy $\pi$,
\[
\|e_j P_i^h - \mu_p\|_1 \leq 2C_{p,S,A} \alpha_{p,S,A},
\]
where $e_j$ is the coordinate vector with 1 in the $j$-th coordinate and 0 elsewhere. Hence
\[
\|P_i^h - P_i^\infty\|_\infty \leq \max_{j=1,\ldots,S} \|e_j P_i^h - \mu_i\|_1 \leq 2C_{p,S,A} \alpha_{p,S,A},
\]
where $P_i^\infty = 1\mu_i$, which implies that for $i = 1, 2$,
\[
\|Y_i\|_\infty \leq 2C_{p,S,A} \sum_{h=0}^{\infty} \alpha_{p,S,A} \leq \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}}.
\]

Hence we have
\[
\|\mu_{\pi_1} - \mu_{\pi_2}\|_1 \leq \|P_1 - P_2\|_\infty \|Y_2\|_\infty \|\mu_{\pi_1}\|_1 \leq \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} \|P_1 - P_2\|_\infty.
\]

By noticing that
\[
\|P_1 - P_2\|_\infty = \max_{s \in S} \sum_{s' \in S} \sum_{a \in A} p(s'|s,a)(\pi_{\theta_1}(a|s) - \pi_{\theta_2}(a|s))
\leq \max_{s \in S} \sum_{a \in A} \sum_{s' \in S} p(s'|s,a)\pi_{\theta_1}(a|s) - \pi_{\theta_2}(a|s)
\leq \max_{s \in S} \|\pi_{\theta_1}(\cdot|s) - \pi_{\theta_2}(\cdot|s)\|_1 \leq \sqrt{S}\|\theta_1 - \theta_2\|_2,
\]
we obtain (35). Here the last step uses [20] Proposition 4 (soft-max function is 1-Lipschitz in $\ell_2$-norm) and the fact that $\|x\|_1 \leq \sqrt{S}\|x\|_2$ for any $x \in \mathbb{R}^S$.

Secondly, we show that for any $s \in S$ and $a \in A$, we have
\[
|\bar{A}_{\pi_1}(s,a) - \bar{A}_{\pi_2}(s,a)| \leq 2R_{\max}\sqrt{S}\|\theta_1 - \theta_2\|_2 \left(\left(\frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1\right)^3 + \frac{7C_{p,S,A}}{1 - \alpha_{p,S,A}} + 2\right).
\]

To see this, first notice that for $i = 1, 2$,
\[
Q^{\pi_i}(s,a) = r(s,a) - \eta(\pi_i)
+ \sum_{h=1}^{\infty} \left( \sum_{s \in S, a \in A} [\rho_1 P_i^{h-1}]_s \pi_{\theta_i}(a|s) r(s,a) - \sum_{s \in S, a \in A} \mu_{\pi_i}(s) \pi_{\theta_i}(a|s) r(s,a) \right)
= r(s,a) - \eta(\pi_i) + \sum_{s \in S, a \in A} \pi_{\theta_i}(a|s) r(s,a) \sum_{h=0}^{\infty} ([\rho_1 P_i^h]_s - \mu_{\pi_i}(s))
= r(s,a) - \eta(\pi_i) + \sum_{s \in S, a \in A} \pi_{\theta_i}(a|s) r(s,a) [\rho_1 Y_i]_s,
\]

29
where $\rho_1(s') = p(s'|s, a)$ for any $s' \in S$. This implies that

$$\left| \bar{Q}^{\pi_{\theta_1}}(s, a) - \bar{Q}^{\pi_{\theta_2}}(s, a) \right| \leq |\eta(\pi_{\theta_1}) - \eta(\pi_{\theta_2})|$$

$$+ \sum_{s \in S, a \in A} |r(s, a)| |\pi_{\theta_1}(a|s)[\rho_1 Y_1]_s - \pi_{\theta_2}(a|s)[\rho_1 Y_2]_s|$$

$$\leq |\eta(\pi_{\theta_1}) - \eta(\pi_{\theta_2})| + R_{\max} \sum_{s \in S, a \in A} \pi_{\theta_1}(a|s)[\rho_1 Y_1]_s - [\rho_1 Y_2]_s$$

$$+ R_{\max} \sum_{s \in S, a \in A} |\pi_{\theta_1}(a|s) - \pi_{\theta_2}(a|s)||[\rho_1 Y_1]_s|$$

$$\leq |\eta(\pi_{\theta_1}) - \eta(\pi_{\theta_2})| + R_{\max} \|\rho_1 Y_1 - \rho_1 Y_2\|_1$$

$$+ R_{\max} \sum_{s \in S} ||[\rho_1 Y_2]_s|||\pi_{\theta_1}(\cdot|s) - \pi_{\theta_2}(\cdot|s)||_1$$

$$\leq \eta(\pi_{\theta_1}) - \eta(\pi_{\theta_2}) + R_{\max} \|Y_1 - Y_2\|_\infty + R_{\max} \sqrt{S} \|\theta_1 - \theta_2\|_2 \|Y_2\|_\infty.$$  

We now bound each of the three terms on the right-hand side. Firstly, by (4), we have

$$|\eta(\pi_{\theta_1}) - \eta(\pi_{\theta_2})| \leq \sum_{s \in S, a \in A} |r(s, a)| \mu_{\pi_{\theta_1}}(s) \pi_{\theta_1}(a|s) - \mu_{\pi_{\theta_2}}(s) \pi_{\theta_2}(a|s)|$$

$$\leq R_{\max} \sum_{s \in S, a \in A} (\pi_{\theta_1}(a|s) \mu_{\pi_{\theta_1}}(s) - \mu_{\pi_{\theta_2}}(s)| + |\pi_{\theta_1}(a|s) - \pi_{\theta_2}(a|s)| \mu_{\pi_{\theta_2}}(s)$$

$$\leq R_{\max} \|\mu_{\pi_{\theta_1}} - \mu_{\pi_{\theta_2}}\|_1 + R_{\max} \sqrt{S} \|\theta_1 - \theta_2\|_2$$

$$\leq R_{\max} \sqrt{S} \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right) \|\theta_1 - \theta_2\|_2.$$  

Next, notice that for $i = 1, 2$, we have (25)

$$Y_i = (I - P_i + P_i^\infty)^{-1} - P_i^\infty.$$  

Hence we have

$$\|Y_i\|_\infty + \|P_i^\infty\|_\infty \leq \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1.$$  

Now by Banach perturbation lemma [53] III.2.2, Theorem 2.5], we have

$$\|Y_1 - Y_2\|_\infty \leq \|Y_1\|_\infty \|Y_2\|_\infty + \|P_1 - P_2\|_\infty + \|P_1^\infty - P_2^\infty\|_\infty$$

$$+ \|P_1 - P_2\|_\infty \|Y_2\|_\infty$$

$$\leq \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^2 \|P_1 - P_2\|_\infty + \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^2 \|P_1^\infty - P_2^\infty\|_\infty$$

$$\leq \sqrt{S} \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^2 \|\theta_1 - \theta_2\|_2 + \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^2 \|\mu_{\pi_{\theta_1}} - \mu_{\pi_{\theta_2}}\|_1$$

$$\leq \sqrt{S} \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^3 \|\theta_1 - \theta_2\|_2 + \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} \|\theta_1 - \theta_2\|_2.$$  

30
Putting these together, we obtain that

\[
|\bar{Q}^{\pi_1}(s, a) - \bar{Q}^{\pi_2}(s, a)| \leq R_{\text{max}} \sqrt{S}\|\theta_1 - \theta_2\|_2 \left( \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^3 + \frac{6C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right).
\]

By the fact that \( \bar{V}^\pi(s) = E_{a \sim \pi_\pi(s)} Q^\pi(s, a) \), we also have for any \( s \in S \),

\[
|\bar{V}^{\pi_1}(s) - \bar{V}^{\pi_2}(s)| \leq \sum_{a \in A} |\pi_{\theta_1}(a|s)\bar{Q}^{\pi_1}(s, a) - \pi_{\theta_2}(a|s)\bar{Q}^{\pi_2}(s, a)|
\]

\[\leq \sum_{a \in A} \pi_{\theta_1}(a|s)|\bar{Q}^{\pi_1}(s, a) - \bar{Q}^{\pi_2}(s, a)|
\]

\[+ \sum_{a \in A} |\pi_{\theta_1}(a|s) - \pi_{\theta_2}(a|s)||\bar{Q}^{\pi_2}(s, a)|
\]

\[\leq R_{\text{max}} \sqrt{S}\|\theta_1 - \theta_2\|_2 \left( \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^3 + \frac{8C_{p,S,A}}{1 - \alpha_{p,S,A}} + 3 \right),
\]

where the last step uses the fact that for \( i = 1, 2 \) and any \( s \in S \) and \( a \in A \),

\[
|\bar{Q}^{\pi_i}(s, a)| \leq |r(s, a)| + |\eta(\pi_{\theta_i})| + \sum_{s \in S, a \in A} \pi_{\theta_i}(a|s)|r(s, a)||[\rho_1 Y_i]|s
\]

\[
\leq 2R_{\text{max}} + R_{\text{max}}\|\rho_1 Y_i\|_1 \leq 2R_{\text{max}}(1 + C_{p,S,A}/(1 - \alpha_{p,S,A})).
\]

These immediately imply (37). Note that again by \( \bar{V}^\pi(s) = E_{a \sim \pi_\pi(s)} \bar{Q}^\pi(s, a) \), (38) also holds when \( \bar{Q}^{\pi_i}(s, a) \) is replaced with \( \bar{V}^{\pi_i}(s) \).

Finally, combining (33), (35), (37), (38) and the fact that the soft-max function is 1-Lipschitz in \( \ell_2 \)-norm, we have

\[
\|\nabla_\theta(\pi_{\theta_1}) - \nabla_\theta(\pi_{\theta_2})\|_2 \leq \|\nabla_\theta(\pi_{\theta_1}) - \nabla_\theta(\pi_{\theta_2})\|_1 = \sum_{s \in S, a \in A} \left| \frac{\partial \eta(\pi_{\theta_1})}{\partial \theta_{s,a}} - \frac{\partial \eta(\pi_{\theta_2})}{\partial \theta_{s,a}} \right|
\]

\[
\leq \sum_{s \in S, a \in A} \left( \mu_{\pi_{\theta_1}}(s)\pi_{\theta_1}(a|s)|\bar{A}^{\pi_1}(s, a) - \bar{A}^{\pi_2}(s, a)|
\right.
\]

\[+ \mu_{\pi_{\theta_2}}(s)|\pi_{\theta_1}(a|s) - \pi_{\theta_2}(a|s)|\bar{A}^{\pi_2}(s, a)|
\]

\[+ \left( \mu_{\pi_{\theta_1}}(s) - \mu_{\pi_{\theta_2}}(s) \right)|\pi_{\theta_2}(a|s)|\bar{A}^{\pi_2}(s, a)|
\]

\[\leq 2R_{\text{max}} \sqrt{S}\|\theta_1 - \theta_2\|_2 \left( \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^3 + \frac{7C_{p,S,A}}{1 - \alpha_{p,S,A}} + 2 \right)
\]

\[+ 4R_{\text{max}} \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) \sqrt{S}\|\theta_1 - \theta_2\|_2
\]

\[+ 4R_{\text{max}} \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) \frac{2\sqrt{S}C_{p,S,A}}{1 - \alpha_{p,S,A}}\|\theta_1 - \theta_2\|_2
\]

\[= 2R_{\text{max}} \sqrt{S}\|\theta_1 - \theta_2\|_2 \left( \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^3 + 4 \left( \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right)^2 + 13C_{p,S,A} + 4 \right)
\]

\[\leq 22R_{\text{max}} \sqrt{S} \left( \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + 1 \right)^3 \|\theta_1 - \theta_2\|_2.
\]
where the last step uses the fact that $2C_{p,S,A}/(1 - \alpha_{p,S,A}) > 1$.

Finally, noticing that $\Omega(\theta)$ is $\frac{2\lambda}{\beta}$-strongly smooth \cite[Lemma D.4]{5}, the proof is finished. \qed

\section{Proofs for DAE REINFORCE algorithm}

In this section we provide the proofs for the convergence result of DAE REINFORCE algorithm.

\subsection{Proof of Lemma \ref{lemma:11}}

In this section, we prove a slightly generalized version of Lemma \ref{lemma:11} which may be useful for future research.

\begin{lemma} (Slight generalization of Lemma \ref{lemma:11}) \label{lemma:23}
Suppose that Assumption \ref{assumption:1} holds. In addition, suppose that the policy parametrization is such that $\|\nabla_\theta \log \pi_\theta(a|s)\|_2 \leq \tilde{C}$ for any $\theta \in \Theta$, $s \in S$ and $a \in A$ and $\Omega(\theta)$ is differentiable. Then we have the following gradient estimation error:

\begin{equation}
\|E_k[\hat{g}_k] - \nabla \tilde{L}(\theta^k)\|_2 \leq \frac{8\tilde{C}R_{\text{max}}C_{p,S,A}}{[\beta H](1 - \alpha_{p,S,A})} \left(1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}}\right)
+ 4\tilde{C}R_{\text{max}}C_{p,S,A} \frac{1 - \gamma}{(1 - \alpha_{p,S,A})^2}
+ 2\tilde{C}R_{\text{max}}^\gamma(1 - \beta)H \left(1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}}\right).
\end{equation}

Here the constants $C_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0,1)$ are defined in Proposition \ref{proposition:1}. In particular, when Assumptions \ref{assumption:1} and \ref{assumption:2} hold, $\Omega(\theta)$ is obviously differentiable and we have $\tilde{C} = 2$ and $R_{\text{max}} = 1$.

\end{lemma}

\begin{proof}[Proof of Lemma \ref{lemma:11}]
By the well-known policy gradient theorem \cite{60}, we have

$$
\nabla_\theta \eta(\pi_\theta) = \sum_{s \in S} \mu_{\pi_\theta}(s) \sum_{a \in A} \nabla_\theta \pi_\theta(a|s) \tilde{A}^{\pi_\theta}(s,a).
$$

In addition, by \cite[Lemma 4.10]{4} we have that for any function $f : S \to \mathbb{R}$ independent of the trajectories $\tau_i$ ($i = 1, \ldots, N$),

$$
E_k[\nabla_\theta \log \pi_{\theta_k}(a_i^h | s_h^i) f(s_h^i)] = 0.
$$

Hence by first taking $f = b$ and then $f = V^{\gamma,\pi_{\theta_k}}$, we have (for an arbitrary $i = 1, \ldots, N$)

$$
E_k[\hat{g}_k] - \nabla_\theta \Omega(\theta_k) = J_1 - J_2,
$$

where

$$
J_1 = E_k \left[ \frac{1}{[\beta H]} \sum_{h=0}^{[\beta H]-1} \nabla_\theta \log \pi_{\theta_k}(a_h^i | s_h^i) \left( \sum_{h'=h}^\infty \gamma^{h'-h} r_h^i | s_h^i, a_h^i, \pi_{\theta_k} \right) - \frac{1}{1 - \gamma} V^{\gamma,\pi_{\theta_k}}(s_h^i) \right],
$$

$$
J_2 = \sum_{s \in S} w^{:\beta H}(s; \pi_{\theta_k}) \sum_{a \in A} \pi_{\theta_k}(a|s) \nabla_\theta \log \pi_{\theta_k}(a|s) \left( \frac{1}{1 - \gamma} Q^{\gamma,\pi_{\theta_k}}(s, a) - \frac{1}{1 - \gamma} V^{\gamma,\pi_{\theta_k}}(s) \right),
$$

\end{proof}
and

\[ J_2 = E_k \left[ \frac{1}{|\beta H|} \sum_{h=0}^{|\beta H|-1} \nabla_\theta \log \pi_{\theta^k} (a_h^i | s_h^i) E_k \left[ \sum_{h'=H}^\infty \gamma^{h'-h} r_{h'}^i | s_h^i, a_h^i, \pi_{\theta^k} \right] \right]. \]

Let’s first consider \( J_1 \). Notice that we have

\[
\frac{1}{1 - \gamma} Q^{\gamma, \pi_{\theta^k}} (s, a) - \frac{1}{1 - \gamma} V^{\gamma, \pi_{\theta^k}} (s) = E_k \left[ \sum_{h'=h}^\infty \gamma^{h'-h} (r_{h'}^i - \eta(\pi_{\theta^k})) | s_h^i = s, a_h^i = a, \pi_{\theta^k} \right]
- E_k \left[ \sum_{h'=h}^\infty \gamma^{h'-h} (r_{h'}^i - \eta(\pi_{\theta^k})) | s_h^i = s, \pi_{\theta^k} \right]
= I_1(s, a) - I_2(s),
\]

where

\[
I_1(s, a) = \sum_{h'=h}^\infty \gamma^{h'-h} \left( \sum_{s' \in S, a' \in A} \text{Prob}^{\pi_{\theta^k}} (s_h^i = s', a_h^i = a', s_h^i = s, a_h^i = a) \rho(s, a) - \eta(\pi_{\theta^k}) \right)
\]

\[
I_2(s) = \sum_{h'=h}^\infty \gamma^{h'-h} \left( \sum_{s' \in S, a' \in A} \text{Prob}^{\pi_{\theta^k}} (s_h^i = s', a_h^i = a', s_h^i = s) \rho(s, a) - \eta(\pi_{\theta^k}) \right)
\]

By writing out the conditional expectations explicitly, we have

\[
I_1(s, a) = \rho(s, a) - \eta(\pi_{\theta^k}) + \gamma \sum_{h'=h+1}^\infty \gamma^{h'-h-1} \left( \sum_{s' \in S, a' \in A} \rho_{h+1,i} P^{h'-h-1}_{\pi_{\theta^k}} (a' | s') \rho(s', a') - \sum_{s' \in S, a' \in A} \mu_{\pi_{\theta^k}} (s') \rho(s', a') \right)
\]

where \( \rho_{h+1,i} (s'') = \text{Prob}^{\pi_{\theta^k}} (s_{h+1}^i = s'' | s_h^i = s, a_h^i = a) = p(s'' | s, a) \) for any \( s'' \in S \).

Hence by Proposition 11, we have

\[
|I_1(s, a)| \leq 2R_{\max} + 2C_{p,S,A} R_{\max} \frac{\gamma}{1 - \alpha_{p,S,A} \gamma} \leq 2R_{\max} + \frac{2C_{p,S,A} R_{\max}}{1 - \alpha_{p,S,A}}.
\]

In addition, noticing that

\[
Q^{\pi_{\theta^k}} (s, a) = E_k \left[ \sum_{h'=h}^\infty \rho(s_{h'}^i, a_{h'}^i) - \eta(\pi_{\theta^k}) | s_h^i = s, a_h^i = a, \pi_{\theta^k} \right],
\]

33
we also have

\[
|I_1(s, a) - \bar{Q}^{\pi_k}(s, a)| \leq \sum_{h'=h+1}^{\infty} (1 - \gamma^{h' - h}) \sum_{s', a' \in A} \left[ \beta_{h+1,i} P_{\pi_k}^{h' - h - 1}(s') - \mu_{\pi_k}(s') \right] \pi_k(a'|s') r(s', a')
\]

\[
\leq 2C_{p, S, A} R_{\max} \sum_{h'=h+1}^{\infty} (1 - \gamma^{h' - h}) \alpha^{h' - h - 1}_{p, S, A}
\]

\[
\leq 2C_{p, S, A} R_{\max} \frac{\gamma}{1 - \gamma \alpha_{p, S, A}} - \frac{1}{1 - \alpha_{p, S, A}}
\]

\[
\leq 2C_{p, S, A} R_{\max} \frac{1 - \gamma}{(1 - \alpha_{p, S, A})^2}
\]

Noticing that \(I_2(s) = E_{a \sim \pi_k(\cdot|s)}[I_1(s, a)|\theta^k]\) and \(\bar{V}^\pi = E_{a \sim \pi_k(\cdot|s)}[\bar{Q}^{\pi}(s, a)|\theta^k]\), we see that the same bounds above for \(I_1\) hold for \(I_2\). More precisely, we have

\[
\|I_2(s)\|_2 \leq 2R_{\max} + \frac{2C_{p, S, A} R_{\max}}{1 - \alpha_{p, S, A}}, \quad \|I_2(s) - \bar{V}^{\pi_k}(s)\|_2 \leq 2C_{p, S, A} R_{\max} \frac{1 - \gamma}{(1 - \alpha_{p, S, A})^2}
\]

Hence we conclude that

\[
\|J_1 - \nabla \theta \eta(\pi_{\theta_k})\|_2 \leq \sum_{s \in S} |w^{[\beta H]}(s; \pi_{\theta_k}) - \mu_{\pi_{\theta_k}}(s)| \sum_{a \in A} \pi_{\theta_k}(a|s) \|\nabla \theta \log \pi_{\theta_k}(a|s)\|_2 |I_1(s, a) - I_2(s)|
\]

\[
+ \sum_{s \in S} \mu_{\pi_{\theta_k}}(s) \sum_{a \in A} \pi_{\theta_k}(a|s) \|\nabla \theta \log \pi_{\theta_k}(a|s)\|_2 |I_1(s, a) - I_2(s) - \bar{A}^{\pi_k}(s, a)|_2
\]

\[
\leq \frac{8C_{\max} C_{p, S, A}}{[\beta H](1 - \alpha_{p, S, A})} \left(1 + \frac{C_{p, S, A}}{1 - \alpha_{p, S, A}}\right) + 4\tilde{C} R_{\max} C_{p, S, A} \frac{1 - \gamma}{(1 - \alpha_{p, S, A})^2}
\]

Similarly, for \(J_2\), following the same analysis as above, we have that

\[
\|J_2\|_2 \leq \sum_{s \in S} |w^{[\beta H]}(s, \pi_{\theta_k})| \sum_{a \in A} \pi_{\theta_k}(a|s) \|\nabla \theta \log \pi_{\theta_k}(a|s)\|_2 \times \frac{\gamma^{(1-\beta)H}}{1 - \gamma} \|Q^{\gamma, \pi_k}(s, a)\|
\]

\[
\leq 2\tilde{C} R_{\max} \gamma^{(1-\beta)H} (1 + C_{p, S, A}/(1 - \alpha_{p, S, A})).
\]

Here we use the fact that \(\frac{1}{1 - \gamma} Q^{\gamma, \pi_k}(s, a) = I_1(s, a)\).

Finally, combining the above bounds of \(\|J_1 - \nabla \theta \eta(\pi_{\theta_k})\|_2\) and \(\|J_2\|_2\), we obtain the desired result. \(\square\)

### B.2 Proof of Lemma 12

**Proof.** We prove a slightly generalized version of the claimed results assuming only that \(|r(s, a)| \leq R_{\max}\) instead of \(r(s, a) \in [0, 1]\) as in Assumption 2.

Firstly, by definition, we have

\[
\|\dot{g}_k\|_2 \leq \frac{2(R_{\max} + (1 - \gamma)B)}{1 - \gamma} + 2\lambda,
\]

where we use the fact that for the soft-max policy parametrization, \(\|\nabla \theta \log \pi_{\theta}(a|s)\|_2 \leq 2\) for any \(\theta \in \Theta, s \in S\) and \(a \in A\) (cf. the proof of [68 Lemma 2]).
Then by Lemma 11, we have
\[ E_k \hat{g}_k^T \nabla \bar{L}(\theta^k) = (E_k \hat{g}_k - \nabla \bar{L}(\theta^k))^T \nabla \bar{L}(\theta^k) + \| \nabla \bar{L}(\theta^k) \|^2 \]
\[ \geq \| \nabla \bar{L}(\theta^k) \|^2 - (\bar{G} + 2\lambda) \bar{\Delta}. \]
Here \( \bar{\Delta} \) is the right-hand side of (13), \( \bar{G} = 4R_{\max} \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) \), and we use the fact that
\[ \| \nabla \bar{L}(\theta^k) \|_2 \leq \| \nabla \bar{L}(\theta^k) \|_1 \leq \sum_{s \in S, a \in A} \mu_{\pi_{\theta_k}}(s) \pi_{\theta_k}(a|s) |\bar{A}^\pi_{\theta_k}(s, a)| + 2\lambda \]
\[ \leq \max_{s \in S, a \in A} |\bar{A}^\pi_{\theta_k}(s, a)| + 2\lambda \leq 4R_{\max} \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) + 2\lambda. \] (40)
Finally, by Lemma 11 we have
\[ \| E_k \hat{g}_k \|_2 \leq \| \nabla \bar{L}(\theta^k) \|_2 + \bar{\Delta}, \]
and hence
\[ E_k \| \hat{g}_k \|_2^2 = \| E_k \hat{g}_k \|_2^2 + \text{Var}_k \hat{g}_k \]
\[ = \| E_k \hat{g}_k \|_2^2 + \text{Var}_k \hat{g}_k^1 \]
\[ \leq 2 \| \nabla \bar{L}(\theta^k) \|_2^2 + 2\bar{\Delta}^2 + \text{Var}_k \hat{g}_k \]
\[ \leq 2 \| \nabla \bar{L}(\theta^k) \|_2^2 + 2\bar{\Delta}^2 + (G^\gamma + 2\lambda)^2 \]
where \( \hat{g}_k^1 \) denotes the special case of \( \hat{g}_k \) with \( N = 1 \), \( G^\gamma = \frac{2(R_{\max}+(1-\gamma)B)}{1-\gamma} \), and for a vector \( X \in \mathbb{R}^n \), \( \text{Var}_k X := \sum_{i=1}^n \text{var}_k X_i \) and \( \text{var}_k \) is the standard conditional variance given the \( k \)-th iteration \( \theta_k \). Here we use the fact that \( \text{Var}_k X \leq \sum_{i=1}^n E_k X_i^2 = E_k \| X \|_2^2 \). This completes the proof. \( \square \)

B.3 Proof of Theorem 13

Proof. By Lemma 10 and an equivalent definition of strongly smoothness (cf. [52, Appendix]), we have
\[ -\bar{L}(\theta^{k+1}) - (-\bar{L}(\theta^k)) \leq -\nabla \bar{L}(\theta^k)^T (\theta^{k+1} - \theta^k) + \frac{\bar{\beta}_{\lambda}}{2} \| \theta^{k+1} - \theta^k \|^2 \]
\[ = -\alpha^k - \nabla \bar{L}(\theta^k)^T \hat{g}_k + \frac{\bar{\beta}_{\lambda} (\alpha^k)^2}{2} \| \hat{g}_k \|^2. \]
Let \( Z_k = Y_k - E_k[Y_k] \). Then the above inequality implies that
\[ \bar{L}(\theta^k) - \bar{L}(\theta^{k+1}) \]
\[ \leq -\alpha^k \nabla \bar{L}(\theta^k)^T E_k \hat{g}_k + \frac{\bar{\beta}_{\lambda} (\alpha^k)^2}{2} \| E_k \hat{g}_k \|^2 + Z_k \]
\[ \leq -\alpha^k \left( \| \nabla \bar{L}(\theta^k) \|^2_2 - (G + 2\lambda) \bar{\Delta} \right) + \frac{\bar{\beta}_{\lambda} (\alpha^k)^2}{2} (\bar{M} + 2 \| \nabla \bar{L}(\theta^k) \|^2_2) + Z_k \]
\[ = -\alpha^k (1 - \bar{\beta}_{\lambda} \alpha^k) \| \nabla \bar{L}(\theta^k) \|^2_2 + \alpha^k (\bar{G} + 2\lambda) \bar{\Delta} + \frac{\bar{\beta}_{\lambda} \bar{M} (\alpha^k)^2}{2} + Z_k \] (41)
\[ \leq -\alpha^k \| \nabla \bar{L}(\theta^k) \|^2_2 + \alpha^k (G + 2\lambda) \bar{\Delta} + \frac{\bar{\beta}_{\lambda} \bar{M} (\alpha^k)^2}{2} + Z_k. \]

Here we use the fact that
\[ \bar{\beta}_\lambda \alpha^k \leq \bar{\beta}_\lambda / (2\bar{\beta}_\lambda) = 1/2. \]
Now define \( X_K = \sum_{k=0}^{K-1} Z_k \) (with \( X_{t,0} = 0 \)), then
\[
E(X_{K+1}|F_K) = \sum_{k=0}^{K-1} Z_k + E(Y_K - E_K Y_K|F_K) = X_K.
\] (42)
Here \( F_K \) is the filtration up to episode \( K \), i.e., the \( \sigma \)-algebra generated by all iterations \( \{\theta^0, \ldots, \theta^K\} \) up to the \( K \)-th one. Notice that the second equality makes use of the fact that given the current policy, the correspondingly sampled trajectory is conditionally independent of all previous policies and trajectories (as is always implicitly assumed in the literature of episodic reinforcement learning (e.g., cf. [39]).

In addition, for any \( K \geq 1 \),
\[
|X_K - X_{K-1}| = |Z_{K-1}| \leq \alpha^{K-1} \|
\nabla_\theta \bar{L}(\theta^{K-1})\|_2 \|E_{K-1} \bar{g}_{K-1} - \bar{g}_{K-1}\|_2
\]
\[+ \frac{\bar{\beta}_\lambda (\alpha^{K-1})^2}{2} |E_{K-1}\| \bar{g}_{K-1}\|_2 - \bar{g}_{K-1}\|_2| \]
\[\leq 2(G^\gamma + 2\lambda) \left( 4 \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) + 2\lambda \right) \alpha^{K-1} + \frac{\bar{\beta}_\lambda}{2} (G^\gamma + 2\lambda)^2 (\alpha^{K-1})^2. \]

Here we use the fact that
\[
\|
\nabla_\theta \bar{L}(\theta^k)\|_2 \leq 4 \left( 1 + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) + 2\lambda,
\]
which follows from [10] in the proof of Lemma [12] with \( R_{\text{max}} = 1 \). The above inequality on \( |X_K - X_{K-1}| \) also implies that \( E|X_K| < \infty \), which, together with (42), implies that \( X_K \) is a martingale.

Now by the definition of \( \alpha^k \), it’s easy to see that \( \sum_{K=1}^{\infty} c_K^2 \leq C < \infty \), where
\[
C = \frac{8(G^\gamma + 2\lambda)^2}{\bar{\beta}_\lambda^2} \left( 2 + \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + \lambda \right)^2 + \frac{(G^\gamma + 2\lambda)^4}{32\bar{\beta}_\lambda^2}. \] (43)

Hence by Azuma-Hoeffding inequality, for any \( c > 0 \) and \( K \geq 0 \),
\[
\text{Prob}(|X_K| \geq c) \leq 2e^{-c^2/(2C)}. \] (44)

Then by summing up the inequalities (41) from \( k = 0 \) to \( K \), we obtain that
\[
\frac{1}{2} \sum_{k=0}^{K} \alpha^k \|
\nabla_\theta \bar{L}(\theta^k)\|_2^2 \leq \sum_{k=0}^{K} \alpha^k (\bar{G} + 2\lambda) \Delta + \frac{\bar{\beta}_\lambda \bar{M} \sum_{k=0}^{\infty} (\alpha^k)^2}{2} + \sum_{k=0}^{K} Z_k + \sup_{\theta \in \Theta} \bar{L}(\theta) - \bar{L}(\theta^0)
\]
\[\leq \sum_{k=0}^{K} \alpha^k (\bar{G} + 2\lambda) \Delta + \frac{\bar{\beta}_\lambda \bar{M}}{2} \sum_{k=0}^{\infty} (\alpha^k)^2 + X_{K+1} + \eta^* - \bar{L}(\theta^0)
\]
\[\leq \frac{\bar{M}}{8\bar{\beta}_\lambda} + \eta^* - \bar{L}(\theta^0) + X_{K+1} + (\bar{G} + 2\lambda) \Delta \sum_{k=0}^{K} \alpha^k,
\]
\[\text{D} \]
36
where we use the fact that the regularization term \( \Omega(\theta) \leq 0 \) for all \( \theta \in \Theta \).

Hence we have

\[
\min_{k=0, \ldots, K} \| \nabla_{\theta} \bar{L}(\theta^k) \|_2^2 \leq \frac{\sum_{k=0}^K \alpha^k \| \nabla_{\theta} \bar{L}(\theta^k) \|_2^2}{\sum_{k=0}^K \alpha^k} \leq \frac{2(\bar{D} + |X_{K+1}|)}{\sum_{k=0}^K \alpha^k} + 2(\bar{G} + 2\lambda)\bar{\Delta} \leq 6\beta_{\lambda} \frac{\bar{D} + |X_{K+1}|}{\sqrt{K + 3}} \log_2(K + 3) + 2(\bar{G} + 2\lambda)\bar{\Delta},
\]

where we use the fact that \( \bar{D} \geq 0 \).

Finally, by combining with the tail bound of (44), we conclude that for any \( \epsilon > 0 \) and \( \delta \in (0, 1) \), for any

\[
K \geq O \left( \frac{S^4 A^4 \beta_{\lambda}^2 (D + \sqrt{2C_p \log(2/\delta)})^2}{\epsilon^4} \log^2 \left( \frac{SA \beta_{\lambda} (D + \sqrt{2C_p \log(2/\delta)})}{\epsilon} \right) \right),
\]

we have that with probability at least \( 1 - \delta \),

\[
\min_{k=0, \ldots, K} \| \nabla_{\theta} \bar{L}(\theta^k) \|_2 \leq \frac{\epsilon}{2SA} + \sqrt{2(\bar{G} + 2\lambda)\bar{\Delta}} = \frac{\lambda}{2SA}
\]

and hence (20) is satisfied as desired. Here the last equality comes from noticing that our choice of \( \lambda \) is a root of the following quadratic equation:

\[2(\bar{G} + 2\lambda)\bar{\Delta} = \frac{(\lambda - \epsilon)^2}{4S^2 A^2}.
\]

Here since \( \beta_{\lambda} \geq 8 \), \( \eta(\pi) \in [0, 1] \), we have

\[
D = O(M + \lambda + 1), \quad C = O \left( \frac{(G^2 + 2\lambda)^2}{S} \left( \frac{C_{p,S,A}^2}{(1 - \alpha_{p,S,A})^2} + \lambda^2 + (G^2 + 2\lambda)^2 \right) \right)
\]

where the constants hidden in the big-O notation may depend on \( \theta^0 \) (and the constants \( B \) and \( \beta \)).

\[\Box\]

**B.4 Proof of Theorem 14**

*Proof of Theorem 14* By Lemmas 5 and 6 and the fact that \( \eta^* - \eta(\hat{\pi}) \leq \epsilon \), we have

\[
V^{H,*} - V^H(\hat{\pi}) \leq |V^{H,*} - \eta^*| + |\eta^* - \eta(\hat{\pi})| + |\eta(\hat{\pi}) - V^H(\hat{\pi})| \\
\leq 2R_{\max} \frac{D_{p,S,A}}{H} + \epsilon + \frac{2R_{\max} C_{p,S,A}}{H(1 - \alpha_{p,S,A})}.
\]

This completes our proof. \[\Box\]
B.5 A more detailed statement of Theorem 15

In this section, we provide a more detailed statement of Theorem 15 which displays the dependencies of the constants on the problem and algorithm parameters in a more explicit manner and provides a slightly tighter sub-optimality bound in terms of the (non-dominating) constants.

**Theorem 24.** Given Assumptions 1 and 2, let \( \gamma = 1 - \beta H \) for some \( \sigma \in (0, 1) \). For any \( \epsilon > 0 \) and \( \delta \in (0, 1) \), set \( \lambda, \beta, \) and \( \alpha^k \) to be the same as in Theorem 13. Then for any \( K \) such that (15) is satisfied with

\[
\tilde{C} = \frac{8(G^\gamma + 2\lambda)^2}{\beta^3} \left( 2 + \frac{2C_{p,S,A}}{1 - \alpha_{p,S,A}} + \lambda \right)^2 + \frac{(G^\gamma + 2\lambda)^4}{32\beta^2} \left( H^{4\sigma} + \tilde{E}^2 H^{2\sigma} + S^2 A^2 (S^2 A^2 + \tilde{E}) \left( \frac{E^4}{\beta H^2} + \frac{E^4}{H^2} \right) + S^4 A^4 (S^4 A^4 + \tilde{E}^2) \left( \frac{E^4}{\beta^2 H^2} + \frac{E^4}{H^2} \right) \right),
\]

where \( \tilde{E} = C_{p,S,A}/(1 - \alpha_{p,S,A}) \), with probability at least \( 1 - \delta \), we have

\[
\min_{k=0,\ldots,K} V^{H,\ast} - V^H (\pi_{\theta^k}) \leq O \left( \frac{S}{1 - \alpha_{p,S,A}} \left( \epsilon + \frac{S^2 A^2 C_{p,S,A}^2}{(1 - \alpha_{p,S,A})^2} ((\beta H)^{-1} + H^{-\sigma}) 
+ \frac{S A C_{p,S,A}^3}{(1 - \alpha_{p,S,A})^3} ((\beta H)^{-1/2} + H^{-\sigma/2}) 
+ \left( D_{p,S,A} + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} \right) H^{-1} \right) \right).
\]

Here \( G^\gamma \) and \( \tilde{M} \) are the constants defined in Lemma 12, while \( C_{p,S,A} > 1, D_{p,S,A} > 1 \) and \( \alpha_{p,S,A} \in [0, 1) \) are the constants in Proposition 12 and Lemma 6.

**Proof.** The key is to notice that we have

\[
\tilde{\Delta} = O \left( \frac{C_{p,S,A}^2}{\beta (1 - \alpha_{p,S,A})^2 H} + \frac{C_{p,S,A}}{(1 - \alpha_{p,S,A})^2} H^{-\sigma} + \frac{C_{p,S,A}}{1 - \alpha_{p,S,A}} e^{-(1-\beta)H^{1-\sigma}} \right) = O \left( \frac{C_{p,S,A}^2}{(1 - \alpha_{p,S,A})^2 H} + \frac{1}{\beta H} + H^{-\sigma} \right).
\]

The proof then follows by plugging in the constants and elementary simplifications, and is hence omitted.

C Proofs for Doubly Discounted REINFORCE algorithm

In this section we provide the proofs for the convergence result of Doubly Discounted REINFORCE algorithm.
Proof of Lemma 16 is a direct implication of [68, Lemmas 2 and 12] and [37, Lemma B.1]. We omit the details here. Below we provide the proof of Theorem 17 and a more detailed statement of Theorem 19.

C.1 Proof of Theorem 17

The proof of Theorem 17 follows similar steps as in Theorem 13. But for self-containedness, we still include the complete proof below.

Proof. By Proposition 9 and an equivalent definition of strongly smoothness (cf. [52, Appendix]), we have

\[ -L^\gamma(\theta^{k+1}) - (-L^\gamma(\theta^k)) \leq -\nabla_\theta L^\gamma(\theta^k)^T(\theta^{k+1} - \theta^k) + \frac{\beta_\lambda}{2}\|\theta^{k+1} - \theta^k\|^2 \]

\[ = -\alpha^k\nabla_\theta L^\gamma(\theta^k)^T\tilde{g}_k + \frac{\beta_\lambda(\alpha^k)^2}{2}\|\tilde{g}_k\|^2. \]

Let \( Z_k = Y_k - E_k|Y_k \). Then the above inequality implies that

\[ L^\gamma(\theta^k) - L^\gamma(\theta^{k+1}) \leq -\alpha^k\nabla_\theta L^\gamma(\theta^k)^TE_k\tilde{g}_k + \frac{\beta_\lambda(\alpha^k)^2}{2}E_k\|\tilde{g}_k\|^2 + Z_k \]

\[ \leq -\alpha^k\left(\|\nabla_\theta L^\gamma(\theta^k)\|^2_2 - (G + 2\lambda)\Delta\right) + \frac{\beta_\lambda(\alpha^k)^2}{2}(M + 2\|\nabla_\theta L^\gamma(\theta^k)\|^2_2) + Z_k \]

\[ = -\alpha^k(1 - \beta_\lambda\alpha^k)\|\nabla_\theta L^\gamma(\theta^k)\|^2_2 + \alpha^k(G + 2\lambda)\Delta + \frac{\beta_\lambda M(\alpha^k)^2}{2} + Z_k \]

Here we use the fact that \( \beta_\lambda\alpha^k \leq \beta_\lambda/(2\beta_\lambda) = 1/2 \).

Now define \( X_K = \sum_{k=0}^{K-1} Z_k \) (with \( X_{t,0} = 0 \)), then

\[ E(X_{K+1}|\mathcal{F}_K) = \sum_{k=0}^{K-1} Z_k + E(Y_K - E_K Y_K|\mathcal{F}_K) = X_K. \]

Here \( \mathcal{F}_K \) is the filtration up to episode \( K \), i.e., the \( \sigma \)-algebra generated by all iterations \( \{\theta^0, \ldots, \theta^K\} \) up to the \( K \)-th one. Notice that the second equality makes use of the fact that given the current policy, the correspondingly sampled trajectory is conditionally independent of all previous policies and trajectories.

In addition, for any \( K \geq 1 \),

\[
|X_K - X_{K-1}| = |Z_{K-1}| \leq \alpha^{K-1}\|\nabla_\theta L^\gamma(\theta^{K-1})\|_2\|E_{K-1}\tilde{g}_{K-1} - \tilde{g}_{K-1}\|_2 \\
+ \frac{\beta_\lambda(\alpha^{K-1})^2}{2}\|E_{K-1}\|\tilde{g}_{K-1}\|_2 - \|\tilde{g}_{K-1}\|_2^2 \]

\[
\leq 2(G + 2\lambda)\left(\frac{2}{(1 - \gamma)^2} + 2\lambda\right)\alpha^{K-1} + \frac{\beta_\lambda}{2}(G + 2\lambda)^2(\alpha^{K-1})^2.
\]

39
Here we use the fact that
\[ \|\nabla_\theta L^\gamma(\theta^{K-1})\|_2 \leq 2/(1 - \gamma)^2 + 2\lambda, \]
which follows from (30) and (31) similarly as in (40). The above inequality on \(|X_K - X_{K-1}|\) also implies that \(E|X_K| < \infty\), which, together with (19), implies that \(X_K\) is a martingale.

Now by the definition of \(\alpha^k\), it's easy to see that \(\sum_{k=0}^{K} \alpha^k = \alpha^k(G + 2\lambda)\Delta + \frac{\beta M \sum_{k=0}^{\infty} (\alpha^k)^2}{2} + \sum_{k=0}^{K} Z_k + \sup_{\theta \in \Theta} L^\gamma(\theta) - L^\gamma(\theta^0)\)
\[ \leq \sum_{k=0}^{K} \alpha^k(G + 2\lambda)\Delta + \frac{\beta M}{2} \sum_{k=0}^{\infty} (\alpha^k)^2 + X_{K+1} + V^{\gamma,*}/(1 - \gamma) - L^\gamma(\theta^0) \]
\[ \leq \frac{M}{8\beta_\lambda} + V^{\gamma,*}/(1 - \gamma) - L^\gamma(\theta^0) + X_{K+1} + (G + 2\lambda)\Delta \sum_{k=0}^{K} \alpha^k, \]
where we use the fact that the regularization term \(\Omega(\theta) \leq 0\) for all \(\theta \in \Theta\).

Hence we have
\[ \min_{k=0,\ldots,K} \|\nabla_\theta L^\gamma(\theta^k)\|_2 \leq \frac{\sum_{k=0}^{K} \alpha^k \|\nabla_\theta L^\gamma(\theta^k)\|_2}{\sum_{k=0}^{K} \alpha^k} \leq \frac{2(D + |X_{K+1}|) + 2(G + 2\lambda)\Delta}{\sum_{k=0}^{K} \alpha^k} \leq 6\beta_\lambda \frac{D + |X_{K+1}|}{\sqrt{K + 3}} \log_2(K + 3) + 2(G + 2\lambda)\Delta, \]
where we use the fact that \(D \geq 0\).

Finally, by combining with the tail bound of (51), we conclude that for any \(\epsilon > 0\) and \(\delta \in (0, 1)\), for any
\[ K \geq \frac{O \left( \frac{S^4 A^4 \beta_\lambda^2 (D + \sqrt{2C \log(2/\delta)})^2}{\epsilon^4} \log^2 \left( \frac{SA\beta_\lambda (D + \sqrt{2C \log(2/\delta)})}{\epsilon} \right) \right)}{2SA}, \]
we have that with probability at least \(1 - \delta\),
\[ \min_{k=0,\ldots,K} \|\nabla_\theta L^\gamma(\theta^k)\|_2 \leq \frac{\epsilon}{2SA} + \sqrt{2(G + 2\lambda)\Delta} = \frac{\lambda}{2SA} \]
and hence (20) is satisfied as desired. Here the last equality comes from noticing that our choice of \(\lambda\) is a root of the following quadratic equation:
\[ 2(G + 2\lambda)\Delta = \frac{(\lambda - \epsilon)^2}{4S^2 \lambda^2}. \]
Here since $\beta_\lambda \geq 8$, $V^\gamma(\pi) \in [0, 1]$, we have
\[
D = O(M + 1/(1 - \gamma) + \lambda), \quad C = O((G + 2\lambda)^2(1/(1 - \gamma)^4 + \lambda^2 + (G + 2\lambda)^2)).
\]
where the constants hidden in the big-$O$ notation may depend on $\theta^0$ (and the constant $B$).

C.2 Proof of Theorem 18

Proof of Theorem 18 Notice that we have
\[
V^{H,*} - V^H(\hat{\pi}) \leq |V^{H,*} - \eta^*| + |\eta^* - V^{\gamma,*}| + |V^{\gamma,*} - V^\gamma(\hat{\pi})| + |V^\gamma(\hat{\pi}) - V^H(\hat{\pi})|.
\]
Hence combining Lemma 2, Corollary 4 and Lemma 6 and by the fact that $V^{\gamma,*} - V^\gamma(\hat{\pi}) \leq \epsilon$, the proof is complete.

C.3 A more detailed statement of Theorem 19

Similarly, in this section, we provide a more detailed statement of Theorem 19 which displays the dependencies of the constants on the problem and algorithm parameters in a more explicit manner and provides a slightly tighter sub-optimality bound in terms of the (non-dominating) constants.

Theorem 25. Given Assumptions 1 and 2, let $\gamma = 1 - H^{-\sigma}$ for some $\sigma \in (0, 1)$. For any $\epsilon > 0$, $\delta \in (0, 1)$, set $\lambda, \beta_\lambda$ and $\alpha^k$ to be the same as in Theorem 17. Then for any $K$ such that (21) is satisfied with
\[
C = \frac{8(G + 2\lambda)^2}{\beta_\lambda^2} \left( \frac{1}{(1 - \gamma)^2} + \lambda \right)^2 + \frac{(G + 2\lambda)^4}{32\beta_\lambda^2} = O((H^{4\sigma} + S^4A^4H^{1+3\sigma}e^{-H^{1-\sigma}})^2),
\]
\[
D = \frac{M}{8\beta_\lambda} + V^{\gamma,*}/(1 - \gamma) - L^\gamma(\theta^0)
= O \left( S^2A^2H^{\frac{1+3\sigma}{2}}e^{-\frac{H^{1-\sigma}}{2}} + H^{\sigma} + \frac{S^4A^4H^{1+3\sigma}e^{-H^{1-\sigma}} + H^{4\sigma}}{N} \right),
\]
with probability at least $1 - \delta$,
\[
\min_{k=0,...,K} V^{H,*} - V^H(\pi_{\theta^k}) \leq O \left( \min \left\{ \left\| \frac{S}{\rho} \right\|_\infty, \frac{S}{1 - \alpha_{p,S,A}} \right\} \left( \epsilon + S^2A^2H^{\frac{1+3\sigma}{2}}e^{-H^{1-\sigma}/2} \right) + C_{p,S,A} \left( \frac{1}{H^{1-\sigma}} - \frac{1}{H} \right) \alpha_{p,S,A}^H + \frac{1}{H} \left( \frac{C_{p,S,A}(H^{1-\sigma} + \alpha_{p,S,A})}{1 - \alpha_{p,S,A}} + D_{p,S,A} \right) \right).
\]
(54)

Here $G$ and $M$ are constants defined in Lemma 16, while $C_{p,S,A} > 1$, $D_{p,S,A} > 1$ and $\alpha_{p,S,A} \in [0, 1)$ are the constants in Proposition 7 and Lemma 6.

Proof. The key is to notice that we have
\[
\Delta = 2H^\sigma(1 - H^{-\sigma})^H(H + H^{\sigma}) \leq 4H^{1+\sigma}((1 - 1/H^{\sigma})H^{\sigma})^{H^{1-\sigma}} \leq 4H^{1+\sigma}e^{-H^{1-\sigma}}
\]
The proof then follows by plugging in the constants and elementary simplifications, and is hence omitted.
Remark 3. Note that from the slightly more refined bound above, we see that compared with the bias term in DAE REINFORCE, additional constant improvements in the exponential term can be achieved when \( \min_{s \in S} \rho(s) \) is relatively large (e.g., when it is lower bounded by \( \frac{1 - \alpha_{p,S,A}}{S} \)).