Information Kernels

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Abstract

Given a set $X$ of finite strings, one interesting question to ask is whether there exists a member of $X$ which is simple conditional to all other members of $X$. Conditional simplicity is measured by low conditional Kolmogorov complexity. We prove the affirmative to this question for sets that have low mutual information with the halting sequence. There are two results with respect to this question. One is dependent on the maximum conditional complexity between two elements of $X$, the other is dependent on the maximum expected value of the conditional complexity of a member of $X$ relative to each member of $X$.

1 Introduction

In [Rom03], criteria for the amount of algorithmic information that can be extracted from a triplet of strings was established. In this paper, the notion of bunches was introduced. A $(k, l, n)$ bunch is a finite set of strings $X$ such that

1. $|X| = 2^k$,
2. $K(x_1|x_2) < l$ for all $x_1, x_2 \in X$,
3. $K(x) < n$ for all $x \in X$.

The term $K$ used above represents the conditional Kolmogorov complexity. In [Rom03], Theorem 5, it was shown that common information could be extracted from bunches.

**Theorem 5.** [Rom03] For $(k, l, n)$ bunch $X$, there exists a string $z$ such that $K(x|z) \leq l + O(|l - k| + \log n)$ and $K(z|x) = O(|l - k| + \log n)$ for any $x \in X$.

In our paper, we revisit bunches and show that every bunch that is not exotic has an element that is simple conditional to all other members. We show this over the class of non-exotic bunches, that is bunches whose encoding has low mutual information with the halting sequence. We also prove a similar result for a structure we call batches, which are defined in terms of expectation instead of max. In this paper, we use a slightly different definition of bunches (and batches), where there are no assumptions about the Kolmogorov complexity of its elements. We define a $(k, l)$ bunch $X$ to be a finite set of strings, where $k = \lceil \log |X| \rceil, l > k$, and for all $x, x' \in X$, $K(x|x') \leq l$. We define a $(k, l)$ batch $X$ to be a finite set of strings, where $k = \lceil \log |X| \rceil, l > k$, and for all $x \in X$, $E_{x' \in X}[K(x|x')] \leq l$. In this paper we prove the following two theorems.

**Theorem.** For $(k, l)$ batch $X$, $\min_{x \in X} E_{x' \in X}[K(x|x')] < \log l - k + I(X : \mathcal{H})$.

**Theorem.** For $(k, l)$ bunch $X$, $\min_{x \in X} \max_{x' \in X} K(x|x') < \log 2(l - k) + I(X : \mathcal{H})$. 

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The mutual information that a string (or any elementary object) $a$ has with the halting sequence, $\mathcal{H}$, is $I(a; \mathcal{H}) = K(a) - K(a|\mathcal{H})$. Both theorems show that for non-exotic sets, i.e., sets with low information with $\mathcal{H}$, there exist a string that is simple conditional to all the other strings. Due to information non-growth laws, there is no (randomized) algorithm which can create exotic sets. Therefore, there are no means to produce sets which don’t have elements that are simple relative to all other elements of the set.

An example exotic bunch is $R_n$, the set of all random strings of size $n$, where $x \in R_n$ iff $|x| = n$ and $K(x) >^+ n$. It is not hard to see that for all $x, x' \in R_n, K(x|x') < ^\log n$. So $R_n$ is a $(n - O(1), n + O(\log n))$ bunch. In addition, because $R_n$ contains all random strings of size $n$, $\min_{x \in X} \max_{x' \in X} K(x|x') > ^\log n$. Thus $R_n$ does not have such a conditionally simple element, and this implies it is exotic, because, due to the bunch theorem introduced above, $n < ^\log I(R_n: \mathcal{H})$. This bound is easily verifiable using the definition of $R_n$, since $K(R_n) >^+ n$ and $K(R_n|\mathcal{H}) < ^+ K(n)$, because given the halting sequence and $n$, there exists a simple program that can produce all random strings of size $n$.

Another example of a bunch is the set $S_{x,m}$, where $x$ is a string of arbitrary length, and $S_{x,m} = \{xy : y$ is a string of length $m\}$. This bunch is usually not exotic. It must be that for $\max_{x,x' \in S_{x,m}} K(y|x') < ^+ m + K(m)$ as all strings in $S_{x,m}$ differ by a substring of size $m$. Furthermore $\#S_{x,m} = m$. Therefore $S_{x,m}$ is a $(m, m + K(m) + O(1))$ bunch. Since $x$ and $m$ can be recovered from an encoding of the the set $S_{x,m}$, and of course $S_{x,m}$ can be created from $x$ and $m$, we have that $I(S_{x,m}; \mathcal{H}) = ^+ I(x,m; \mathcal{H}) < I(x; \mathcal{H}) + O(K(m))$. So by the above bunch theorem, $\min_{y \in S_{x,m}} \max_{x' \in S_{x,m}} K(y|x') < ^\log 2K(m) + I(S_{x,m}; \mathcal{H}) < ^\log I(x; \mathcal{H}) + O(K(m))$. Most $x$ has negligible information with the halting sequence, relative to its length. Furthermore it can be seen independently that $\min_{y \in S_{x,m}} \max_{x' \in S_{x,m}} K(y|x') < ^+ K(m)$, because for $y = x0^n \in S_{x,m}$, there is a program that given any member of $S_{x,m}$ and a program for $m$, can output $y$.

2 Related Work

The study of Kolmogorov complexity originated from the work of [Kol65]. The canonical self-delimiting form of Kolmogorov complexity was introduced in [ZL70] and treated later in [Cha75]. The universal probability $m$ was introduced in [Sol64]. More information about the history of the concepts used in this paper can be found the textbook [LV08].

Information conservation laws were introduced and studied in [Lev74, Lev84]. Information asymmetry and the complexity of complexity were studied in [G75]. A history of the origin of the mutual information of a string with the halting sequence can be found in [VV04a].

The notion of the deficiency of randomness with respect to a measure follows from the work of [She83], and also studied in [KU87, VY87, She99]. At a Tallinn conference in 1973, Kolmogorov formulated notion of a two part code and introduced the structure function (see [VV04b] for more details). Related aspects involving stochastic objects were studied in [She83, She99, VY87, VY99].

The combination of complexity with distortion balls can be seen in [FLV06]. The work of Kolmogorov and the modelling of individual strings using a two-part code was expanded upon in [VV04b, GTV01]. These works introduced the notion of using the prefix of a “border” sequence to define a universal algorithmic sufficient statistic of strings. The generalization and synthesis of this work and the development of algorithmic rate distortion theory can be seen in the works of [VV04a, VV10]. More information on algorithmic statistics can be found in [VS17, VS15].

This paper uses theorems and lemmas found [Eps13] and [EL11]. An accessible game-theoretic proof to [EL11] can be found in [She12]. Bunches were first introduced by [Rom03], who used them to prove properties of common information of strings.
3 Conventions

We use \( \mathbb{N} \), \( \mathbb{Q} \), \( \mathbb{R} \), \( \Sigma \), \( \Sigma^* \), and \( \Sigma^\infty \) to represent natural numbers, rational numbers, reals, bits, finite strings, and infinite strings. Let \( X_{\geq 0} \) and \( X_{>0} \) be the sets of non-negative and of positive elements of \( X \). The length of a string \( x \in \Sigma^n \) is denoted by \( ||x|| = n \). The removal of the last bit of a string is denoted by \((y0^-) = (p1^-) = p \), for \( p \in \Sigma^* \). For the empty string \( \emptyset \), \((\emptyset^-) \) is undefined. We use \( \Sigma^\infty \) to denote \( \Sigma^* \cup \Sigma^\infty \), the set of finite and infinite strings. For \( x \in \Sigma^\infty \), \( y \in \Sigma^\infty \), we say \( x \sqcup y \) iff \( x = y \) or \( x \in \Sigma^* \) and \( y = xz \) for some \( z \in \Sigma^\infty \). The \( n \)-th bit of a string \( x \in \Sigma^\infty \) is denoted by \( x[i] \). The first \( n \) bits of a string \( x \in \Sigma^\infty \) is denoted by \( x[0..n] \). The indicator function of a mathematical statement \( A \) is denoted by \([A]\), where if \( A \) is true then \([A] = 1 \), otherwise \([A] = 0 \). The size of a finite set \( S \) is denoted to be \( |S| \) and also \#\( S = \lfloor \log |S| \rfloor \). For a finite set \( S \subset \Sigma^* \), and function \( f : \Sigma^* \rightarrow \mathbb{R} \), \( E_{x \in S}[f(x)] = \sum_{x \in S} f(x) \). As is typical of the field of algorithmic information theory, the theorems in this paper are relative to a fixed universal machine, and therefore their statements are only relative up to additive and logarithmic precision.

For positive real functions \( f \) the terms \( ^< f \), \( ^{+} f \), \( ^{-} f \) represent \( f + O(1) \), \( f - O(1) \), and \( f \pm O(1) \), respectively. In addition \( ^< f \), \( ^{+} f \), and \( ^{-} f \) denote \( f / O(1) \), \( f / O(1) \) and \( f \times O(1) \), respectively. For nonnegative real function \( f \), the terms \( ^{\log f} \), \( ^{\log f} \), \( ^{-} \log f \), \( ^{-} \log f \) represent the terms \( f + O(\log(f+1)) \), \( f - O(\log(f+1)) \), and \( f \pm O(\log(f+1)) \), respectively. A discrete measure is a nonnegative function \( Q : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \) over natural numbers. The support of a measure \( Q \) is the set of all elements \( a \in \mathbb{N} \) that have positive measure, with \( \text{Supp}(Q) = \{ a : Q(a) > 0 \} \). The measure is elementary if its support is finite and its range is a subset of \( \mathbb{Q} \). Elementary measures have an explicit finite encoding, in the natural way. The mean of a function \( f : \mathbb{N} \rightarrow \mathbb{R} \) by a measure \( Q \) is denoted by \( E_{a \sim Q}[f] = \sum_{a \in \mathbb{N} \leftarrow Q} f(a)Q(a) \). We say \( Q \) is a semimeasure iff \( E_{a \sim Q}[1] \leq 1 \). Furthermore, we say that \( Q \) is probability measure iff \( E_{a \sim Q}[1] = 1 \). For a set \( S \subset \mathbb{N} \), \( Q(S) = \sum_{x \in S} Q(x) \). For semimeasure \( Q \), we say that \( d : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \) is a \( Q \) test, if \( E_{a \sim Q}[2^{d(a)}] \leq 1 \).

\( T_y(x) \) is the output of algorithm \( T \) (or \( \bot \) if it does not halt) on input \( x \in \Sigma^* \) and auxiliary input \( y \in \Sigma^\infty \). \( T \) is prefix-free if for all \( x, s \in \Sigma^* \) with \( s \neq \emptyset \), and \( y \in \Sigma^\infty \), either \( T_y(x) = \bot \) or \( T_y(xs) = \bot \). The complexity of \( x \in \Sigma^* \) with respect to \( T_y \) is \( K_T(x|y) = \inf \{ ||p|| : T_y(p) = x \} \).

There exist optimal for \( K \) prefix-free algorithm \( U \), meaning that for all prefix-free algorithms \( T \), there exists \( c_T \in \mathbb{N} \), where \( K_U(x|y) \leq K_T(x|y) + c_T \) for all \( x \in \Sigma^* \) and \( y \in \Sigma^\infty \). For example, one can take a universal prefix-free algorithm \( U \), where for each prefix-free algorithm \( T \), there exists \( t \in \Sigma^* \), with \( U_y(tx) = T_y(x) \) for all \( x \in \Sigma^* \) and \( y \in \Sigma^\infty \). \( K(x|y) \) is defined to be \( K_U(x|y) \) is the Kolmogorov complexity of \( x \in \Sigma^* \) relative to \( y \in \Sigma^\infty \). When we say that universal Turing machine is relativized to an object, this means that an encoding of the object is provided to the universal Turing machine on an auxiliary tape.

The complexity of a (partial) computable function \( f : \mathbb{N} \rightarrow \mathbb{N} \) is \( K(i) \) where \( D_f \) is the set of indices of functions equal to \( g \) in an enumeration of partial computable functions of the form \( \mathbb{N} \rightarrow \mathbb{N} \). A function \( f : \Sigma^* \rightarrow \mathbb{R}_{\geq 0} \) is lower semicomputable if the set \( \{ (x, q) : x \in \Sigma^* \land f(x) > q \in \mathbb{Q} \} \) is enumerable. The complexity of a lower semicomputable function \( f \) is \( \min_{i \in D_f} K \). The chain rule for Kolmogorov complexity is \( K(x, y) = K(x) + K(y|x, K(x)) \). The universal probability of a set \( D \subset \Sigma^* \) is \( m(D|y) = \sum_z |U(y,z) \in D|2^{-||z||} \). For strings \( x \in \Sigma^* \), we have \( m(x|y) = m(\{x\}|y) \). The coding theorem states \( -\log m(x|y) = K(x|y) \).

The halting sequence \( H \in \Sigma^\infty \) is the infinite string where \( H[i] = [U(i) \neq \bot] \) for all \( i \in \mathbb{N} \). We recall that the amount of mutual information that \( a \in \mathbb{N} \) has with \( H \) is denoted by \( I(a : H) = K(a) - K(a|H) \).
4 Left-Total Machines

The notion of total strings and the “left-total” universal algorithm is needed in the remaining sections of the paper. We say \( x \in \Sigma^* \) is total with respect to a machine if the machine halts on all sufficiently long extensions of \( x \). More formally, \( x \) is total with respect to \( T_y \) for some \( y \in \Sigma^{*\infty} \) iff there exists a finite prefix free set of strings \( Z \subset \Sigma^* \) where \( \sum_{z \in Z} 2^{-\|z\|} = 1 \) and \( T_y(xz) \neq \bot \) for all \( z \in Z \). We say (finite or infinite) string \( \alpha \in \Sigma^{*\infty} \) is to the “left” of \( \beta \in \Sigma^{*\infty} \), and use the notation \( \alpha \triangleright \beta \), if there exists a \( x \in \Sigma^* \) such that \( x0 \triangleright \alpha \) and \( x1 \triangleright \beta \). A machine \( T \) is left-total if for all auxiliary strings \( \alpha \in \Sigma^{*\infty} \) and for all \( x, y \in \Sigma^* \) with \( x \triangleright y \), one has that \( T_\alpha(y) \neq \bot \) implies that \( x \) is total with respect to \( T_\alpha \). An example can be seen in Figure 1.

![Figure 1](image)

Figure 1: The above diagram represents the domain of a left total machine \( T \) with the 0 bits branching to the left and the 1 bits branching to the right. For \( i \in \{1, \ldots, 5\} \), \( x_i \triangleright x_{i+1} \) and \( x_i \triangleright y \). Assuming \( T(y) \) halts, each \( x_i \) is total. This also implies each \( x_i^- \) is total as well.

For the remaining part of this paper, we can and will change the universal self delimiting machine \( U \) into a universal left-total machine \( U' \) by the following definition. The algorithm \( U' \) enumerates all strings \( p \in \Sigma^* \) in order of their convergence time of \( U(p) \) and successively assigns them consecutive intervals \( i_p \subset [0, 1] \) of width \( 2^{-\|p\|} \). Then \( U' \) outputs \( U(p) \) on input \( p' \) if the open interval corresponding to \( p' \) and not that of \( (p')^- \) is strictly contained in \( i_p \). The open interval in \([0,1]\) corresponding with \( p' \) is \( ((p')^-\|p'\|, ((p')^-\|p'\|+1)2^{-\|p'\|}) \) where \( (p) \) is the value of \( p \) in binary. For example, the value of both strings 011 and 0011 is 3. The value of 0100 is 4. The same definition applies for the machines \( U'_\alpha \) and \( U_\alpha \), over all \( \alpha \in \Sigma^{*\infty} \). We now set \( U \) to equal \( U' \).

Without loss of generality, the complexity terms of this paper are defined with respect to the universal left total machine \( U \). The infinite border sequence \( B \in \Sigma^{\infty} \) represents the unique infinite sequence such that all its finite prefixes have total and non-total extensions. The term “border” is used because for any string \( x \in \Sigma^* \), \( x \triangleright B \) implies that \( x \) total with respect to \( U \) and \( B \triangleright x \) implies that \( U \) will never halt when given \( x \) as an initial input. Figure 2 shows the domain of \( U \) with respect to \( B \).

For total string \( b \), let \( btime(b) \), be the slowest running time of a program that extends \( b \) or is to the left of \( b \). With respect to the universal Turing machine \( U \) defined above, \( btime(b) \) would be the running time of the rightmost extension of \( b \) that halts. For total string \( b \in \Sigma^* \), and \( x, y \in \Sigma^* \),
5 Stochasticity

In algorithmic statistics, a string is stochastic if it is typical of a simple probability measure. Properties of stochastic (and non-stochastic) strings can be found in the survey [VS17]. The deficiency of randomness of $x$ with respect to elementary probability measure $Q$ and $v \in \Sigma^*$ is $d(x|Q,v) = [-\log Q(x)] - K(x|v)$. The function $d(\cdot|Q,v)$ is a $Q$-test (up to an additive constant). It is also universal, in that for any lower semicomputable $Q$-test $d$, and $v \in \Sigma^*$, for all $x \in \Sigma^*$, $d(x) < d(x|Q,v) + K(d|v) + K(Q|v)$, as shown in [G13].

For some $j, k \in \mathbb{N}$, we say that $x \in \mathbb{N}$ is $(j,k)$-stochastic if there exists $v \in \Sigma^j$, with $U(v) = Q$, $Q$ being an elementary probability measure, and $d(x|Q,v) \leq k$. The stochasticity of $x \in \mathbb{N}$, is measured by $\Lambda(x) = \min\{j + 3k : x \text{ is } (j,k) \text{-stochastic}\}$. The conditional stochasticity form$^1$ is represented by $\Lambda(x|\alpha)$, for $\alpha \in \Sigma^{\mathbb{N}}$.

Stochasticity follows non-growth laws; a total computable function cannot increase the stochasticity of a string by more than a constant factor dependent on its complexity. Lemma 1 illustrates this point. Another variant of the same idea can be found in Proposition 5 in [VS17].

$^1$This is formally represented as $\Lambda(x|\alpha) = \min\{j + 3k : \exists v \in \{0, 1\}^j, U_\alpha(v) = (Q), d(x|Q,v,\alpha) \leq k \in \mathbb{N}\}$. 

Figure 2: The above diagram represents the domain of the universal left-total algorithm $U$, with the 0 bits branching to the left and the 1 bits branching to the right. The strings in the above diagram, 0v0 and 0v1, are halting inputs to $U$ with $U(0v0) \neq \perp$ and $U(0v1) \neq \perp$. So 0v is a total string. The infinite border sequence $B \in \Sigma^\infty$ represents the unique infinite sequence such that all its finite prefixes have total and non-total extensions. All finite strings branching to the right of $B$ will cause $U$ to diverge.

let $m_b(x|y)$ be the algorithmic weight of $x$ from programs conditioned on $y$ in time $\texttt{bbtime}(b)$. More formally, $m_b(x|y) = \sum\{2^{-\|p\|} : U_y(p) = x \text{ in time } \texttt{bbtime}(b)\}$. The term $m_b(x|y)$ is 0 if $b$ is not total. Let $K_b(x|y) = \lceil -\log m_b(x|y) \rceil$, and $K_b(x|y) = \infty$ if $m_b(x|y)$ is 0.
Lemma 1  Given total computable function $f : \Sigma^* \to \Sigma^*$, $\Lambda(f(x)) < \Lambda(x) + O(K(f))$.

Proof. Let $v \in \Sigma^*$ realize $\Lambda(x)$, with $U(v) = Q$, $\|v\| + 3 \max \{d(x|Q,v), 1\} = \Lambda(x)$. Let $f(Q)$ be the image distribution of $Q$ with respect to $f$. Thus $f(Q)(a) = \sum_{b : f(b) = a} Q(b)$. The function $d(f(\cdot)|f(Q), v)$ is a Q-test (relative to $v$ and up to an additive constant), because

$$\sum_a 2^{d(f(a)|f(Q), v)} Q(a) = \sum_b 2^{d(b|f(Q), v)} f(Q(b)) < O(1).$$

Also $d(f(\cdot)|f(Q), v)$ is lower semi-computable given $v$, with $K(d(f(\cdot)|f(Q), v)|v) <^+ K(f|v)$. So due to the universality of $d$, $d(f(x)|f(Q), v) <^+ d(x|Q, v) + K(f|v) <^+ d(x|Q, v) + K(f)$. Let $v' = v_0 v_f \in \Sigma^*$ compute $f(Q)$, where $v_0$ is helper code of size $O(1)$ and $v_f$ is a shortest program that computes $f$, with $\|v_f\| = K(f)$. So $v' <^+ \|v\| + K(f)$. Since $K(x|v) <^+ K(x|v') + K(v'|v) <^+ K(x|v') + K(f)$, we have that $d(f(x)|f(Q), v') <^+ d(x|Q, v) + O(K(f))$. So

$$\Lambda(f(x)) \leq \|v'\| + 3 \max \{d(f(x)|f(Q), v'), 1\}$$

$$<^+ \|v\| + 3 \max \{d(f(x)|f(Q), v'), 1\} + K(f)$$

$$< \|v\| + 3 \max \{d(x|Q, v), 1\} + O(K(f))$$

$$\leq \Lambda(x) + O(K(f)).$$

\[\Box\]

The following lemma is taken from [EL11]. It states that the stochasticity measure of a string lower bounds its information with the halting sequence. Another version of the lemma can be found in [Eps13]. Even though the stochasticity measure $\Lambda$ is larger in this paper than in [Eps13], changing from $3 \log d$ to $3d$, the arguments in the proof still hold.

Lemma 2  For $x \in \Sigma^*$, $\Lambda(x) < \log I(x; H)$.

The following lemma is also from [Eps13]. It shows is that if a prefix of the border sequence is simple relative to a string $x$, then it will be the common information between $x$ and the halting sequence $H$. Note that if a string $b$ is total and $b^-$ is not, then $b^- \sqsubset B$, due to the fact that $b^-$ has total and non-total extensions.

Lemma 3  If $b \in \Sigma^*$ is total and $b^-$ is not, and $x \in \Sigma^*$, then $K(b) + I(x; H|b) < \log I(x; H) + K(b|x, \|b\|)$.

The following theorem is from [EL11]. It states that sets that are not exotic, i.e. sets with low mutual information with the halting sequence, have simple members that contain a large portion of the algorithmic weight of the sets. It is compatible with this paper’s stochasticity definition because the term $\Lambda$ used here is larger than the stochasticity measure used in [EL11].

Theorem 1  For finite set $D \subset \Sigma^*$, $\min_{x \in D} K(x) <^+ [-\log m(D)] + 2K([-\log m(D)]) + \Lambda(D)$. 

6 Batches

We recall that a \((k, l)\) batch \(X\) is a finite set of strings, where \(k = \#X\), \(l > k\), and for all \(x \in X\), \(E_{x' \in X}[K(x|x')] \leq l\). The following theorem says that for non-exotic batches, there is an element of \(X\) that is simple, on average, conditional to all other members of \(X\).

**Theorem 2** For \((k, l)\) batch \(X\), \(\min_{x \in X} E_{x' \in X}[K(x|x')] < \log l - k + I(X : H)\).

**Proof.** We can assume that \(k > 2\), otherwise the theorem is trivially proven. Let \(b\) be the shortest total string where \(\max_{y \in X} E_{x' \in X}[K_b(y|x')] < l + 2\), dubbed property \(A\). Thus \(K(b|X, \|b\|) < K((l - k))\), as \(l\) can be constructed from \((l - k)\) and \(X\). In addition there exists a program that can enumerate all total programs of length \(\|b\|\) and select the first one with property \(A\). The first one selected will be \(b\), otherwise there exists a \(b' \neq b\), \(\|b'\| = \|b\|\), with property \(A\). This implies there exists a total \(b'' \subset b\), \(b'' \subset b\), and so \(K_{b''} \leq K_b\) and \(K_{b''} \leq K_b\) and thus property \(A\) holds for \(b''\), contradicting the minimal length of \(b\). This also implies \(b''\) is not total.

Let \(S = \text{Supp}(m_b)\) be the support of \(m_b\), which is finite. Let \(G\) be the infinite set of all functions \(g : S \to \mathbb{N}\). Since \(S\) is finite, each \(g \in G\) can be encoded in an explicit finite string. Let \(\kappa : G \to \mathbb{R}_{\geq 0}\) be a probability measure where \(\kappa(g) = \prod_{a \in S} 2^{-g(a)}\). So for all \(a \in S\), it must be that \(\kappa(\{g : g(a) = n\}) = 2^{-n}\) and \(\kappa(\{g : g(a) \geq n\}) = 2^{-n+1}\).

For any finite set \(H \subseteq \Sigma^*, \#H > 2\), let \(\mathcal{G}_H^H\) be the set of functions \(g \in G\), there exists \(x_g \in H\) with \(g(x_g) = \#H - 2\). Using the fact that \((1 - m)e^m \leq 1\) for \(m \in [0, 1]\), we have that

\[
\kappa(\mathcal{G}_1^H) \leq \prod_{a \in H} \left(1 - 2^{-\#H + 2}\right) \leq \left(1 - 2^{-\#H + 2}\right)^{2\#H - 1} \leq e^{-2^{-\#H + 2}2\#H - 1} = e^{-2} < 0.25.
\]

So \(\kappa(\mathcal{G}_1^H) > 0.75\). We use measures \(P_g(y|x') : \Sigma^* \to \mathbb{R}_{\geq 0}\), indexed by \(g \in G\) and \(x' \in S\). The measure \(P'\) is defined as \(P_g(y|x') = [\delta_g(y, x') \geq 2]2^{-\delta_g(y, x')}\delta_g(y, x') - 2 + \delta_g(y, x') < 2\), where \(\delta_g(y, x') = K_b(y|x') - g(y)\). Noting the definition of measures, for a set \(B \subseteq \Sigma^*\), we have that \(P_g(B|x') = \sum_{a \in B} P_g(a|x')\). We define a second set of functions \(\mathcal{G}_2^H = \{g : E_{x' \in H}[P_g(S|x')] \leq 10, g \in G\}\). So

\[
E_{g \sim \kappa} E_{x' \in H}[P_g(S|x')] = \left|\mathcal{G}_1^H\right|^{-1} \sum_{x' \in H} \sum_{y \in S} m_b(y|x') \\
= \left|\mathcal{G}_1^H\right|^{-1} \sum_{x' \in H} \sum_{y \in S} \left(\sum_{c = 1}^{K_b(y|x') - 2} 2^{c - K_b(y|x')(K_b(y|x') - c)^2} \kappa(\{g : g \in G, g(y) = c\}) + \kappa(\{g : g \in G, g(y) \geq K_b(y|x') - 1\})\right) \\
\leq \left|\mathcal{G}_1^H\right|^{-1} \sum_{x' \in H} \sum_{y \in S} \left(m_b(y|x') \sum_{c = 1}^{K_b(y|x') - 2} (K_b(y|x') - c)^{-2} + 2^{K_b(y|x') + 2}\right) \\
\leq \left|\mathcal{G}_1^H\right|^{-1} \sum_{x' \in H} m_b(S|x') < 5.
\]

So by the Markov inequality, \(\kappa(\mathcal{G}_2^H) \geq 0.5\). So for all finite \(H \subseteq \Sigma^*, \#H > 2\), \(\kappa(\mathcal{G}_1^H \cap \mathcal{G}_2^H) > 0.25\). We use the following probability measure \(P_g(y|x')\), indexed by \(g \in G\) and \(x' \in S\), defined as
\[ P_g(y|x') = \{ y \in S | P_g'(y|x')/P_g'(S|x') \}. \] Thus \( P_g(\Sigma^*|x') = 1 \) for all \( x' \in S, \ g \in \mathcal{G} \). So for any \( g \in \mathcal{G}_1 \cap \mathcal{G}_2^H \), there exists \( x_g \in H \) where \( g(x_g) = \#H - 2 \) and also
\[
\mathbf{E}_{x' \in H}[- \log P_g(x_g|x')]
= \mathbf{E}_{x' \in H}[- \log P_g'(x_g|x') + \log P_g'(S|x')]
= \mathbf{E}_{x' \in H}[- \log P_g'(x_g|x') + \log P_g'(S|x')]
\leq \mathbf{E}_{x' \in H}[- \log P_g(x_g|x') + \log \mathbf{E}_{x' \in H}[P_g'(S|x')]]
<^+ \mathbf{E}_{x' \in H}[- \log P_g(x_g|x')]
=^+ \mathbf{E}_{x' \in H}[- \log P_g(x_g|x') \geq \log \mathbf{E}_{x' \in H}[\delta_g(x_g, x') + 2 \log \delta_g(x_g, x'), O(1)]
<^+ \mathbf{E}_{x' \in H}[\delta_g(x_g, x') + 2 \log \mathbf{E}_{x' \in H}[\delta_g(x_g, x')], O(1)]
<^+ \log \max \{\mathbf{E}_{x' \in H}[K_b(x_g|x')] - \#H, O(1)\}. \tag{1}
\]

Let \( \{G_i\} \) be a computable enumeration of all finite subsets of \( \mathcal{G} \). Let \( f \) be a function that when given a set \( H \subset \Sigma^*, \#H > 2 \), outputs an encoding of the first finite subset \( W \subset \mathcal{G} \) in the list \( \{G_i\} \) such that \( W \subset \mathcal{G}_1^H \cap \mathcal{G}_2^H \) and \( \kappa(W) > 0.25 \). On all other inputs which are not an encoded finite set \( H \subset \Sigma^* \) with \( \#H > 2 \), \( T \) outputs the empty string. The function \( f \) is total computable relative to \( b \), with \( K(f|b) = O(1) \), because given \( H \) and \( b \), it is computable to determine whether a given function \( g \in \mathcal{G} \) is in \( \mathcal{G}_1^H \cap \mathcal{G}_2^H \).

Let \( D = f(X) \). Invoking Theorem 1, conditional to \( b \), gives \( g \in D \), where \( K(g|b) <^+ [- \log m(D|b)] + 2K([- \log m(D|b)]) + \Lambda(D|b) \). Since \([- \log m(D|b)] <^+ - \log \kappa(D) + K(\kappa|b) < O(1) \), we have that \( K(g|b) <^+ \Lambda(D|b) \). Lemma 1, relativized to \( b \), using total computable function \( f \), gives \( K(g|b) <^+ \Lambda(X|b) \). Lemma 2, gives
\[
K(g|b) < I(X : H|b) + O(\log I(X : H|b)). \tag{2}
\]

Since \( g \in D \subset \mathcal{G}_1^X \cap \mathcal{G}_2^X \), there exists \( x_g \in X \) where, due to Equation 1,
\[
\mathbf{E}_{x' \in X}[- \log P_g'(x_g|x')] <^+ \log \max \{\mathbf{E}_{x' \in X}[K_b(x_g|x')] - \#X, O(1)\} <^+ \log l - k. \tag{3}
\]

So we have that
\[
\mathbf{E}_{x' \in X}[K(x_g|b, x')] <^+ \mathbf{E}_{x' \in X}[K(x_g|b, g, x') + K(g|b)]
=^+ \mathbf{E}_{x' \in X}[K(x_g|b, g, x')] + K(g|b)
< \mathbf{E}_{x' \in X}[- \log P_g'(x_g|x')] + I(X : H|b) + O(\log I(X : H|b)) \tag{4}
< l - k + I(X : H|b) + O(\log I(X : H|b) + \log(l - k)) \tag{5}
\]
\[
\mathbf{E}_{x' \in X}[K(x_g|x') - K(b)] < l - k + I(X : H|b) + O(\log I(X : H|b) + \log(l - k))
\mathbf{E}_{x' \in X}[K(x_g|x')] < l - k + K(b) + I(X : H|b) + O(\log (I(X : H|b) + K(b)) + \log(l - k)) \tag{6}
\mathbf{E}_{x' \in X}[K(x_g|x')] < \log l - k + I(X : H) + K(b | X, |b|) \tag{7}
\]
\[
\mathbf{E}_{x' \in X}[K(x_g|x')] < \log l - k + I(X : H). \tag{7}
\]

Equation 4 is due to Equation 2. Equation 5 is due to Equation 3. Equation 6 is due to the invocation of Lemma 3. Equation 7 is due to the fact that \( K(b|X, |b|) <^+ K((l - k)) \). □
7 Bunches

We recall that a \((k, l)\) bunch \(X\) is a finite set of strings, where \(k = \#X, l > k\), and for all \(x, x' \in X\), \(K(x|x') \leq l\). The following theorem says that for non-exotic bunches, there is an element of \(X\) that is simple conditional to all other members of \(X\).

**Theorem 3** For \((k, l)\) bunch \(X\), \(\min_{x \in X} \max_{x' \in X} K(x|x') < \log 2(l - k) + I(X : H)\).

**Proof.** Let \(z = l - k\). Let \(b\) be the shortest total string where \(\max_{x, x' \in X} K_b(x|x') < l + 2\), which we call satisfying property A. Thus \(K(b|X, \|b\|) < K(z)\). This is because there exists a program that can enumerate all total strings of length \(\|b\|\) and return the first \(b\) that satisfies property A. This \(b\) is unique, otherwise there exists \(b' \neq b\), \(\|b'\| = \|b\|\) that satisfies property A. This implies the existence of \(b'' \subset b'\), \(b'' \subset b\) that also satisfies property A, contradicting the definition of \(b\). This also implies \(b^-\) is not total. Let \(s = \langle b, z\rangle\).

Let \(v' \in \Sigma^*\), elementary probability measure \(Q'\) minimize \(\Lambda(X|s)\), where \(U_s(v') = Q'\). Let \(Q\) be an elementary probability measure equal to \(Q'\) conditioned on the largest set of encoded sets \(G\) such that for all \(x, x' \in G\), \(K_b(x|x') < \#G + z + 2\), which we call satisfying property \(B\). Thus more formally \(Q(G) = [G \in T]\frac{Q(G)}{Q'(T)}\), where \(T \subset \Sigma^*\), the support of \(Q\), is defined as \(T = \{\langle G\rangle : G \in \text{Supp}(Q')\}\). Let \(v = v_0v'\) be a program such that \(U_s(v) = \langle Q\rangle\), where \(v_0 \in \Sigma^*\) is helper code. So \(\|v\| < + \|v'\|\).

\[
\|v\| + 3d < + \|v'\| + 3d
\]
\[
= + \|v'\| + 3(\max\{-\log Q(X) - K(X|v, s), 1\})
\]
\[
< + \|v'\| + 3(\max\{-\log Q'(X) - K(X|v, s), 1\})
\]
\[
< + \|v'\| + 3(\max\{-\log Q'(X) - K(X|v', s) + K(v|v', s), 1\})
\]
\[
< + \|v'\| + 3(\max\{-\log Q'(X) - K(X|v', s), 1\})
\]
\[
\|v\| + 3d < + \Lambda(X|s).
\] (8)

Let \(S = \bigcup\{Y : \langle Y\rangle \in \text{Supp}(Q)\}\) be the union of all sets encoded in the support of \(Q\). Since \(Q\) is elementary, \(|S| < \infty\). Let \(\mathcal{G}\) be the set of all functions \(g : S \rightarrow \mathbb{N}\). Since \(S\) is finite, each \(g \in \mathcal{G}\) can be encoded with an explicit finite string. Let \(\kappa : \mathcal{G} \rightarrow \mathbb{R}_{\geq 0}\) be a probability measure over \(\mathcal{G}\), where \(\kappa(g) = \prod_{a \in S} 2^{-g(a)}\). So for all \(a \in S\), \(\kappa(\{g : g(a) \geq n\}) = 2^{-n+1}\). We define the following function \(t\) over \(\text{Supp}(Q)\), parameterized by \(g \in \mathcal{G}\). Let \(c \in \mathbb{N}\) be a constant solely dependent on \(U\) to be determined later. We assume that \(|Y| > 16(c + d)\). Otherwise, \(k < + \log d\), and then \(\min_{x \in X} \max_{x' \in X} K(x|x', s) \leq l < + z + \log d < + 2z + \Lambda(X|s)\). From this point, the reasoning starting at Equation 9 can be used to prove the theorem. With this assumption, let \(t_g(Y) = e^{2(d+c)-1}\) if \(Y \cap \{a : g(a) \geq \log(|Y|/(c + d))\} = \emptyset\), and \(t_g(Y) = 0\), otherwise. So, using the
For all functions $g$, furthermore, we extend the domain $I_{\text{Supp}}(Y)$ formally, let $t \sim E$ to the Markov inequality, semi-measure for at least half of the elements $y$. Thus $x, y$ for each $g$, parameterized by $\kappa$, we define the following function, with $E \sim \kappa \cdot \gamma + 1$. We define the function $t' : \gamma + 1$ parameterized by $g \in G$, which will give a set $Y$ a zero score iff $P_g(\gamma + 1)$ is a semi-measure for at least half of the elements $y \in Y$. Otherwise $t'_g(Y)$ gives $Y$ a high score. More formally, let $t'_g(Y) = 0$ if $I_g(Y) < 0.5|Y|$ and $t'_g(Y) = 2^{(d+c)-7}$, otherwise. Thus we have that, due to the Markov inequality,

$$E_{g \sim \kappa}E_{Y \sim Q}[t'_g(Y)] \leq \sum_Y Q(Y)\kappa(\{g : I_g(Y) \geq 0.5|Y|\})2^{c+d-7}$$

$$= \sum_Y Q(Y)\gamma(\{g : I_g(Y) \geq 0.5|Y|\})2^{c+d-7}$$

$$\leq \sum_Y Q(Y)2^{-(c+d)+5}2^{c+d-7}$$

$$= 0.25.$$
By probabilistic arguments, there exists $g \in \mathcal{G}$, such that $\mathbb{E}_{Y \sim Q}[t_g(Y)] \leq 1$ and $\mathbb{E}_{Y \sim Q}[t'_g(Y)] \leq 1$. So both $t_g(\cdot)Q(\cdot)$ and $t'_g(\cdot)Q(\cdot)$ are semi-measures. Furthermore, $\mathbf{K}(g|c,d,v,s) = O(1)$. It must be that $t_g(X) = 0$. Otherwise, for proper choice of $c$ solely dependent on $U$,

$$
\text{d}(X|Q,v,s) = [\log Q(X) - \mathbf{K}(X|v,s)] - O(1)
> -\log Q(X) - (\log t_g(X)Q(X) + \mathbf{K}(t_g(\cdot)Q(\cdot)|v,s)) - O(1)
> -\log Q(X) - (\log t_g(X)Q(X) + \mathbf{K}(g,Q|v,s)) - O(1)
> 2(c + d)(\log e) - \mathbf{K}(c,d) - O(1)
> d,
$$
causing a contradiction. The same reasoning can be used to show that $t'_g(X) = 0$. We roll $c$ into the additive constants of the theorem and remove it from consideration for the rest of the proof. Therefore, since $t_g(X) \neq 0$, there exists $a \in X$ where for all $y \in X$, using the fact that $|Y| > 16(c + d)$,

$$
g(a) \geq |\log(|Y|/(d + c)| \geq |\log |Y| - \log(d + c)|
\geq k - 1 - |\log(d + c)|
\geq \max\{\mathbf{K}_b(a|y) - z - |\log(d + c)| - 3, 1\}.
$$

Furthermore, since $t'_g(X) \neq 0$, there is a subset $X' \subseteq X$, $|X'| > 2^{k-2}$, where for all $y \in X'$, $P_g(\cdot|y)$ is a semimeasure. For such $y$, $\mathbf{K}(a|y, s) < + \log P_g(a|y) + \mathbf{K}(g|d, v, s) + \mathbf{K}(d, v|s) < + z + 3d + \|v\| < + z + \Lambda(X|s)$, using Equation 8. Therefore for all $y' \in X \setminus X'$,

$$
\mathbf{K}(a|y', s) < + \log \sum_{y \in X'} 2^{-\mathbf{K}(a|y, s) - \mathbf{K}(y|y', s)}
< + \log \sum_{y \in X'} 2^{l - z - \Lambda(X|s)}
< + 2z + \Lambda(X|s).
$$

So for all $x \in X$,

$$
\mathbf{K}(a|x, s) < + 2z + \Lambda(X|s) \quad (9)
\mathbf{K}(a|x) < + 2z + \mathbf{K}(s) + \Lambda(X|s)
< 2z + \mathbf{K}(b) + \Lambda(X|s) + O(\log z)
< 2z + \mathbf{K}(b) + \mathbf{I}(X : \mathcal{H}|s) + O(\log z + \log \mathbf{I}(X : \mathcal{H}|s)) \quad (10)
< 2z + \mathbf{K}(b) + \mathbf{I}(X : \mathcal{H}|b) + O(\log z + \log(\mathbf{I}(X : \mathcal{H}|b) + \mathbf{K}(b)))
< \log 2z + \mathbf{I}(X : \mathcal{H}) + \mathbf{K}(b|X, \|b\|) \quad (11)
< \log 2z + \mathbf{I}(X : \mathcal{H}). \quad (12)
$$

Equation 10 is due to the application of Lemma 2. Equation 11 is due to the application of Lemma 3. Equation 12 is due to the fact that $\mathbf{K}(b|X, \|b\|) < + \mathbf{K}(z)$. \qed
8 Discussion

There exists a generalization for batches to elementary probability measures, where for batch $X$, $l \geq \max_{x \in X} E_{x'} \sim X [K(x|x')]$ and $k = \# \text{Supp}(X)$. For bunches, there is a way to achieve a comparable result to Theorem 3, for enumerative sets $X$, where instead of the bounds being in terms of $I(X : \mathcal{H})$, the bounds are in terms of $I(p : \mathcal{H})$, where $p$ is a program that enumerates $X$. In both cases, we leave the details of the proofs to the reader.

The stochasticity method has been proven fruitful in characterizing elementary objects that have low mutual information with the halting sequence. Further work involves publishing results regarding stochasticity and the $M$ measure of prefix free sets, where $M$ is the universal lower computable continuous semi-measure. This work has application in the minimal complexity of completions of partial binary predicates. Other work involves looking at stochasticity and combinatorial objects, such as graphs or matroids.

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