We propose a theoretical description of the superconducting state of under- to overdoped cuprates, based on the short coherence length of these materials and the associated strong pairing fluctuations. The calculated $T_c$ and the zero temperature excitation gap $\Delta(0)$, as a function of hole concentration $x$, are in semi-quantitative agreement with experiment. Although the ratio $T_c/\Delta(0)$ has a strong $x$ dependence, different from the universal BCS value, and $\Delta(T)$ deviates significantly from the BCS prediction, we obtain, quite remarkably, quasi-universal behavior, for the normalized superfluid density $\rho_s(T)/\rho_s(0)$ and the Josephson critical current $I_J(T)/I_J(0)$, as a function of $T/T_c$. While experiments on $\rho_s(T)$ are consistent with these results, future measurements on $I_J(T)$ are needed to test this prediction.

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Pseudogap phenomena in the cuprates are of interest not only because of the associated unusual normal-state properties, but more importantly because of the constraints which these phenomena impose on the nature of the superconductivity and its associated high $T_c$. Moreover, this superconducting state presents an interesting challenge to theory: while the normal state is highly unconventional, the superconducting phase exhibits some features of traditional BCS superconductivity along with others which are strikingly different.

Thus far, there is no consensus on a theory of cuprate superconductivity. Scenarios which address the pseudogap state are needed to test this prediction.
As a consequence, in evaluating the associated contribution to the self-energy, the main contribution to the $Q$ sum comes from this small $Q$ divergent region so that $\Sigma_{pg}(K) \approx -G_o(-K)\Delta_{pg}^2\phi_k^2$, where we have defined the pseudogap parameter $\Gamma$.

$$\Delta_{pg}^2 = -\sum_Q t_{pg}(Q) = -\sum_Q \frac{g}{1 + g\chi(Q)}.$$  

(2a)

Thus, both $\Sigma_{pg}$ and the total self-energy $\Sigma$ can be well approximated by a BCS-like form, i.e., $\Sigma(K) \approx \Delta^2 \varphi_k^2 / (i\omega_n + \epsilon_k)$, where $\Delta \equiv \sqrt{\Delta_{sc}^2 + \Delta_{pg}^2}$ is the magnitude of the total excitation gap, with the $k$ dependence given by the $d$-wave function $\varphi_k$. Within the above approximations, the gap and number equations reduce to

$$1 + g \sum_k \frac{1 - 2f(E_k)}{2E_k} \varphi_k^2 = 0,$$  

(2b)

$$\sum_k \left[ 1 - \frac{\epsilon_k}{E_k} + \frac{2\epsilon_k}{E_k} f(E_k) \right] = n,$$  

(2c)

where the quasiparticle energy dispersion $E_k = (\epsilon_k^2 + \Delta_k^2)^{1/2}$ contains the full excitation gap $\Delta$.

The superfluid density can be written in terms of the London penetration depth as $\rho_{s,ab}(T)/\rho_{s,ab}(0) = \lambda_{ab}(0)/\lambda_{ab}(T)$, where

$$\lambda_{ab} = \frac{4\pi e^2}{c^2} \sum_k \frac{\varphi_k}{E_k} \left[ 1 - \frac{2f(E_k)}{2E_k} \right] + f'(E_k).$$  

(3)

During the calculation special attention should be paid to lattice effects [12] and to the vertex correction (associated with the pseudogap self-energy) which enforces gauge invariance via the generalized Ward identity. This identity insures that $\rho_s \propto \Delta_{sc}^2$ and it vanishes identically at and above $T_c$. The prefactor $\Delta_{sc}^2 = \Delta^2 - \Delta_{pg}^2$, in Eq. (3) indicates that pairs (in addition to quasiparticles) serve to reduce the superfluid density.

In a related fashion, we address c-axis Josephson tunneling between two identical high $T_c$ superconductors. This situation is relevant to both break junction experiments [13] and to intrinsic Josephson tunneling [14] as well. An expression for the Josephson critical current [15] can be derived under the presumption that the tunneling matrix element

$$|T_{kp}|^2 = |T_0|^2 \delta_{kp} + |T_1|^2,$$

where only the first (coherent) term contributes for a $d$-wave order parameter,

$$I_c = 2e|T_0|^2 \sum_{kp} \delta_{kp} \frac{\varphi_k \varphi_p}{E_k E_p} \left[ 1 - f(E_k) - f(E_p) \right].$$  

(4)

Equation (4), like Eq. (3), differs from the usual BCS form (as well as that assumed by Lee and Wen [16]) in that the prefactor $\Delta_{sc}^2$ is no longer the total excitation gap $\Delta$.

The remainder of this paper is directed towards understanding three experimental characteristics of the cuprates: (i) the phase diagram, (ii) the superfluid density and (iii) the Josephson critical current.

(i) In order to generate physically realistic values of the various energy scales, we make two assumptions: (1) We take $g$ as doping-independent (which is not unreasonable in the absence of any more detailed information about the pairing mechanism) and (2) incorporate the effect of the Mott transition at half filling, by introducing an $x$-dependence into the in-plane hopping matrix elements $t_{\parallel}$, as would be expected in the limit of strong on-site Coulomb interactions in a Hubbard model [17]. Thus the hopping matrix element is renormalized as $t_{\parallel}(x) \approx t_0 (1 - n) = t_0 x$, where $t_0$ is the matrix element in the absence of Coulomb effects. This $x$ dependent energy scale is consistent with the requirement that the plasma frequency vanish at $x = 0$. These assumptions leave us with one free parameter $-g/4t_0$, for which we assign the value 0.15 to optimize the overall fit of the phase diagram to experiment. We take $t_{\perp}/t_{\parallel} \approx 0.01$ [18], and $t_0 \approx 0.6$ eV, which is reasonably consistent with experimentally based estimates [19].

The results for $T_c$, obtained from Eqs. (3), as a function of $x$ are plotted in Fig. 1. Also indicated is the corresponding zero temperature excitation gap $\Delta(0)$ as well as the pseudogap $\Delta_{pg}$ at $T_c$. These three quantities provide us, for use in...
of clear indication of a "quasi-universal" behavior with respect to the highly overdoped to highly underdoped regimes. These plots context of the present formalism.

shown in the inset) as phenomenological inputs, within the lower inset: energy gaps as a function of $T/T_c$, for $x = 0.125$. Upper inset: (A) the slope given by the low temperature expansion assuming a constant $\Delta_{sc}(T) = \Delta(0)$; (B) the ratio $\frac{\Delta^2_{sc}(T)}{\Delta^2_{sc}(0)}$ at $T/T_c = 0.2$; and (A+B) the sum of two contributions.

subsequent calculations, with energy scales which are in reasonable agreement with the data of Ref. [3], shown in the inset. The temperature dependences of the energy gaps in Fig. 1 are shown as the lower inset to Fig. 2, for a slightly underdoped case with $x = 0.125$. The relative size of $\Delta_{pg}(T_c)$, compared to $\Delta(0)$, increases with decreasing $x$. In the highly overdoped limit this ratio approaches zero, and the BCS limit is recovered. This inset illustrates the general behavior as a function of $T/T_c$: the excitation gap $\Delta$ is, generally, finite at $T_c$, the superconducting gap $\Delta_{sc}$ is established at and below $T_c$, while the pseudogap $\Delta_{pg}$ decreases to zero as $T$ is reduced from $T_c$ to 0. This last observation is consistent with general expectations for $\Delta_{pg} \approx \left(1 - \frac{T}{T_c}\right)^2 \Delta_{sc}(0)$ [11].

It is important to stress, that our subsequent results for the superfluid density and Josephson current, need not be viewed as contingent on the detailed $x$-dependence used to derive the phase diagram. One can approach the calculations of these quantities by taking $T_c(x)$ and the various gap parameters (shown in the inset) as phenomenological inputs, within the context of the present formalism.

(ii) The superfluid density (normalized to its $T = 0$ value), given by Eq. (3), is plotted in Fig. 2 as a function of $T/T_c$ for several representative values of $x$, ranging from the highly over- to highly underdoped regimes. These plots clearly indicate a "quasi-universal" behavior with respect to $x$: $\rho_s(T)/\rho_s(0)$ vs. $T/T_c$ depends only slightly on $x$. Moreover, the shape of these curves follows closely that of the weak-coupling BCS theory. The, albeit, small variation with $x$ is systematic, with the lowest value of $x$ corresponding to the top curve. Recent experiments provide some preliminary evidence for this universal behavior [20,21]. However, a firm confirmation requires further experiments on a wider range of hole concentrations, from extreme under- to overdoped samples [22].

This universal behavior appears surprising at first sight [14] because of the strong $x$ dependence in the ratio $T_c/\Delta(0)$ (see Fig. 1). It should be noted that universality would not persist if $\rho_s(T)/\rho_s(0)$ were plotted in terms of $\rho_s(0)/\Delta(0)$. The nontrivial origin of this effect has a simple explanation within the present theory. At low to intermediate temperatures, it follows from Eq. (3) that $\lambda_{ab}^2(0)/\lambda_{ab}^2(T) = (\Delta_{sc}^2(T)/\Delta(0)) \left(1 - \frac{Tc}{Tc} + \mathcal{O}\left(T/Tc\right)^2\right) \approx 1 - \left[A + B(T)\right] \left(T/Tc\right)^2$, where temperatures of order $\left(T/Tc\right)^2$ and higher have been neglected. Here $A = 32\sqrt{2}\ln 2 \left(e^2\lambda_{ab}^2(0)/\varepsilon\right) T_{c}^{1/2} \times (T_c/\Delta(0))$ represents the standard contribution to the linear $T$ dependence of $\rho_s(T)$. The new term $B(T) = (T/T_c) \Delta_{pg}^2(T)/\Delta^2(0)$ derives from the pseudogap contribution and has a weaker than linear $T$ dependence (as can be inferred from the lower inset in Fig. 3) [23]. For the purposes of illustration, these two terms are plotted in the upper inset of Fig. 2 at $T/T_c = 0.2$. Note that the effective (negative) "slope" $A + B$ is relatively $x$ independent over the physical range of hole concentrations. Physically, the terms $A$ and $B$ are associated with two compensating contributions, arising from the quasiparticle and pair excitations, respectively, so that quasi-universal behavior results at low $T$. It can be shown that the same compensating effect obtains all the way to $T_c$, as is exhibited in Fig. 2. Thus, the destruction of the superconducting state comes predominantly from pair excitations at low $x$, and quasiparticle excitations at high $x$.

(iii) Finally, as plotted in Fig. 3, we obtain from Eq. (3), similarly, unexpected quasi-universal behavior for the normalized $c$-axis Josephson critical current for the same wide range of $x$ as in Fig. 2. This behavior is in contrast to the strongly $x$ dependent quasiparticle tunneling characteristics which can be inferred from the temperature dependent excitation gap plotted in the upper inset of Fig. 3. The origin of this univer-
sality is essentially the same as that for $\rho_s$, deriving from two compensating contributions. At this time, there do not appear to be detailed studies of $I_c(T)$ as a function of $x$, although future measurements will, ultimately, be able to determine this quantity. In these future experiments the quasiparticle tunneling characteristics should be simultaneously measured, along with $I_c(T)$, so that direct comparison can be made to the excitation gap; in this way, the predictions indicated in Fig. 3 and its upper inset can be tested. Indications, thus far [13,24], are that this tunneling excitation gap coincides rather well with values obtained from photoemission data (see Fig. 1).

In summary, in this paper we have proposed a scenario for the superconducting state of the cuprates. This state evolves continuously with hole doping $x$, exhibiting unusual features at low $x$ (associated with a large excitation gap at $T_c$) and manifesting the more conventional features of BCS theory at high $x$. In this scenario the pseudogap state is associated with pair excitations, which act in concert with the usual quasiparticle tunneling gap; in this way, the predictions indicated in Fig. 3 and its upper inset can be tested. Indications, thus far [13,24], are that this tunneling excitation gap coincides rather well with values obtained from photoemission data (see Fig. 1).

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This expression for $C_2$ defines the $T$-matrix $t(Q)$. The combination of bare and fully dressed Green’s functions enter $A(K; Q)$; for $t(Q) = t_{sc}(Q) + t_{pg}(Q)$, recalling the definition of the pair susceptibility $\chi(Q)$, and using the expression for the gap equation, one has $\langle \tilde{\Delta}^2 \rangle = -\sum_Q t(Q) \langle g x(Q) \rangle^2 = |\langle \Delta_{sc} \rangle|^2 + \left(-\sum_Q t_{pg}(Q) \langle g x(Q) \rangle^2 \right) \approx |\Delta_{sc}|^2 + \Delta_{pg}^2$, where we have exploited once again the fact that, for $T \leq T_c$, $t_{pg}(Q)$ is highly peaked around $Q = 0$.

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