List Message Passing Decoding of Non-binary Low-Density Parity-Check Codes

Emna Ben Yacoub
Institute for Communications Engineering
Technical University of Munich, Germany
Email: emna.ben-yacoub@tum.de

Abstract—A decoding algorithm for \( q \)-ary low-density parity-check codes over the \( q \)-ary symmetric channel is introduced. The exchanged messages are lists of symbols from \( \mathbb{F}_q \). A density evolution analysis for maximum list sizes 1 and 2 is developed. Thresholds for selected regular low-density parity-check code ensembles are computed showing gains with respect to a similar algorithm in the literature. Finite-length simulation results confirm the asymptotic analysis.

I. INTRODUCTION

Non-binary low-density parity-check (LDPC) codes have shown an outstanding error correction capability, outperforming their binary counterparts substantially at short block lengths [1]. Nevertheless, the complexity of the belief propagation (BP) decoder for these codes is high. Various works have reduced the decoding complexity of non-binary LDPC codes over the binary-input additive white Gaussian noise (biAWGN) channel [2]–[7] and the \( q \)-ary symmetric channel (QSC) [8]–[13]. In [3], an extension of the min-sum algorithm to non-binary fields was presented. The authors of [8] introduced a verification-based decoding algorithm for LDPC codes over large alphabets on the QSC. Similar approaches were considered in [10], [14]. It has been shown in [15] that LDPC codes on the QSC for large \( q \) can approach the Shannon limit. In [13], the authors introduced a decoding algorithm over the QSC (for any \( q \)) where the exchanged messages are sets of symbols from \( \mathbb{F}_q \) (including the empty set). The performance of this algorithm improves with the list size but, this comes at the cost of an increasing data flow within the decoder, with respect to the case where check and variable nodes exchange only hard decisions (i.e., symbols in \( \mathbb{F}_q \)).

Inspired by [13], we propose a message passing algorithm for \( q \)-ary LDPC codes over the QSC, referred to as scaled reliability list message passing (SRLMP), where the variable and check nodes exchange a set of symbols from \( \mathbb{F}_q \) of size at most \( \Gamma \). The difference between our algorithm and the one in [13] is the variable node (VN) update rule. In [13], the channel and the check node (CN) messages are mapped to binary vectors of length \( q \). The entry corresponding to the symbol \( a \in \mathbb{F}_q \) is one if the message contains the symbol \( a \) and it is zero otherwise. These vectors are summed at the VN decoder. Whereas, for SRLMP the channel and CN messages are converted to log-likelihood vectors at the VNs by modeling the extrinsic channel as a discrete memoryless channel (DMC) whose transition probabilities may be estimated via density evolution (DE). This technique is similar to the ternary message passing (TMP) decoder [16] for binary codes, and lays its foundation in the binary message passing (BMP) algorithm originally proposed in [17].

In this work, we describe the SRLMP decoder and develop the exact DE for \( \Gamma \in \{1, 2\} \). We provide the iterative decoding thresholds of some regular LDPC ensembles.

II. PRELIMINARIES

A. \( q \)-ary Symmetric Channel

Consider a QSC with error probability \( \epsilon \), input alphabet \( \mathcal{X} \) and output alphabet \( \mathcal{Y} \), with \( \mathcal{X} = \mathcal{Y} = \{0, \alpha^0, \ldots, \alpha^{q-2}\} \), where \( \alpha \) is a primitive element of \( \mathbb{F}_q \). Denote by \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \) the channel input and channel output, respectively. The transition probabilities of a QSC with error probability \( \epsilon \) are

\[
P(y|x) = \begin{cases} 1 - \epsilon & \text{if } y = x \\ \epsilon/(q-1) & \text{otherwise}. \end{cases}
\]

The capacity of the QSC, in symbols per channel use, is

\[
C = 1 + \epsilon \log_q \frac{\epsilon}{q-1} + (1-\epsilon) \log_q (1-\epsilon).
\]

B. Log-Likelihood Vector

For a given channel output \( y \) of a DMC with input alphabet \( \mathcal{X} = \mathbb{F}_q \), we introduce the normalized log-likelihood vector, also referred to as \( L \)-vector,

\[
L(y) = [L_0(y), L_1(y), \ldots, L_{q-2}(y)]
\]

whose elements are defined as

\[
L_a(y) = \log (P(y|u)) \quad \forall u \in \mathbb{F}_q.
\]

The \( L \)-vector will be fundamental to the development of a decoding algorithm for non-binary LDPC codes over the QSC. In particular, we will focus on a message passing decoding algorithm where the exchanged messages are list of symbols from \( \mathbb{F}_q \) of size at most \( \Gamma \). In this case, a message sent from a CN to a VN can be modeled as the observation of the random variable (RV) \( X \) after transmission over a \( q \)-ary input \( |M_\Gamma| \)-ary output discrete memoryless extrinsic channel [18] Fig. 3], where \( M_\Gamma \) is the message alphabet. In the decoding algorithm, we will use the \( L \)-vector \( L(y) \) of the communication channel observation and the \( L \)-vectors of the CN messages \( L(m), \forall m \in M_\Gamma \). While the transition probabilities of the
communication channel, which is a QSC with error probability \( \epsilon \), are given in \([1]\), the transition probabilities of the extrinsic channel are in general unknown but accurate estimates can be obtained via DE analysis, as suggested in \([17]\).

C. Non-binary LDPC Codes

Consider next non-binary LDPC codes defined by an \( m \times n \) sparse parity-check matrix \( \mathbf{H} \) with coefficients in \( \mathbb{F}_q \). The parity-check matrix can be represented by a Tanner graph with \( n \) VNs corresponding to codeword symbols and \( m \) CNs corresponding to parity checks. The edge label associated to the edge connecting \( v \) and \( c \) is denoted by \( h_{v,c} \), with \( h_{v,c} \in \mathbb{F}_q \setminus \{0\} \). The sets \( \mathcal{N}(v) \) and \( \mathcal{N}(c) \) denote the neighbors of VN \( v \) and CN \( c \), respectively. The degree of a VN \( v \) is the cardinality of the set \( \mathcal{N}(v) \). Similarly, the degree of a CN \( c \) is the cardinality of the set \( \mathcal{N}(c) \). The VN edge-oriented degree distribution polynomial of an LDPC code graph is given by \( \lambda(x) = \sum_i \lambda_i x^{i-1} \) where \( \lambda_i \) corresponds to the fraction of edges incident to VNs with degree \( i \). Similarly, the CN edge-oriented degree distribution polynomial is given by \( \rho(x) = \sum_i \rho_i x^{i-1} \) where \( \rho_i \) corresponds to the fraction of edges incident to CNs with degree \( i \). An unstructured irregular LDPC code ensemble \( \mathcal{E}_{q,n}^c \) is the set of all \( q \)-ary LDPC codes with block length \( n \) and degree distributions \( \lambda(x) \) and \( \rho(x) \) and edge labels uniformly chosen in \( \mathbb{F}_q \setminus \{0\} \).

III. SRLMP Decoding

This section introduces an extension of the symbol message passing (SMP) algorithm \([12]\) for transmission over a QSC. An exchanged message between a check and a variable node is a list of symbols from \( \mathbb{F}_q \) of size between 0 (empty set) and \( \Gamma \), i.e., the message alphabet is \( \mathcal{M}_\Gamma \) which contains all possible sets of symbols in \( \mathbb{F}_q \) of size less than or equal to \( \Gamma \) (including the empty set). The cardinality of the message alphabet is \( |\mathcal{M}_\Gamma| = \sum_{i=0}^{\Gamma} \binom{\ell}{i} \).

i. Initialization

Each VN sends its channel observation \( y \) to its neighboring CNs:

\[
m^{(0)}_{v\rightarrow c} = y.
\]

ii. Check to variable update

Consider a CN \( c \) and a VN \( v \) connected to it. If all of the incoming messages to \( c \) from the other neighboring VNs are not empty, \( c \) computes the set of all symbols that satisfy the parity check equation given the received VN messages. Formally, it computes

\[
\mathcal{U}_{v}^{(c)} = -h_{v,c}^{-1} \sum_{v' \in \mathcal{N}(c) \setminus v} h_{v',c} m^{(c-1)}_{v'\rightarrow c}.
\]

The multiplication in \((4)\) is performed element-wise over \( \mathbb{F}_q \) and the sum is over sets of symbols. The sum over two sets \( \mathcal{A} \) and \( \mathcal{B} \) is defined as the Minkowski sum, i.e.,

\[
\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}.
\]

Algorithm 1 VN Update Rule.

1: Initialize the set \( \mathcal{T} = \emptyset \)
2: Find one symbol \( a \in \mathbb{F}_q \) with \( L_{\text{ex},a}^{(t)} = \max_{u \in \mathbb{F}_q \setminus \mathcal{T}} L_{\text{ex},u}^{(t)} \)
3: Update the set \( \mathcal{T} = \mathcal{T} \cup \{a\} \)
4: if \( L_{\text{ex},c}^{(t)} > L_{\text{ex},u}^{(t)} + \Delta^{(t)} \forall u \in \mathcal{T} \) and \( \forall u \in \mathbb{F}_q \setminus \mathcal{T} \) then
5: \( m^{(t)}_{c\rightarrow v} = \mathcal{T} \)
6: else
7: if \( |\mathcal{T}| < \Gamma \) then
8: return to 2
9: \( m^{(t)}_{c\rightarrow v} = \emptyset \)
10: end if
11: end if

If the size of \( \mathcal{U}_{c}^{(t)} \) is larger than \( \Gamma \) or \( c \) receives at least one empty set from its neighboring VNs, then \( c \) sends an empty set to \( v \), otherwise it sends the set \( \mathcal{U}_{c}^{(t)} \). Formally, we write

\[
m^{(t)}_{c\rightarrow v} = \begin{cases} \mathcal{U}_{c}^{(t)} & \text{if } m_{c\rightarrow v}^{(t-1)} \neq \emptyset \forall u' \in \mathcal{N}(c) \setminus v \\ \emptyset & \text{and } |\mathcal{U}_{c}^{(t)}| \leq \Gamma \\ \emptyset & \text{otherwise.} \end{cases}
\]

iii. Variable to check update

First, each VN computes

\[
L_{\text{ex}}^{(t)} = \left[ L_{\text{ex},0}^{(t)}, L_{\text{ex},1}^{(t)}, \ldots, L_{\text{ex},\alpha-2}^{(t)} \right] = L(y) + \sum_{c \in \mathcal{N}(v) \setminus c} L_m_{c\rightarrow v}^{(t)}.
\]

Then, the outgoing VN message is obtained by applying Algorithm \([1]\) For \( \Gamma = 1 \), Algorithm \([1]\) simplifies to

\[
m_{v\rightarrow c}^{(t)} = \begin{cases} \{a\} & \text{if } \exists a \in \mathbb{F}_q : L_{\text{ex},a}^{(t)} > L_{\text{ex},u}^{(t)} + \Delta^{(t)} \\ \emptyset & \text{otherwise} \end{cases}
\]

and for \( \Gamma = 2 \)

\[
m_{v\rightarrow c}^{(t)} = \begin{cases} \{a\} & \text{if } \exists a \in \mathbb{F}_q : L_{\text{ex},a}^{(t)} > L_{\text{ex},u}^{(t)} + \Delta^{(t)} \\ \emptyset & \text{otherwise} \end{cases}
\]

In \((5)\), the \( L \)-vector \( L(y) \) corresponding to the channel observation is obtained from \((3)\) using the transition probabilities of the QSC communication channel given in \([1]\). Further, we model each CN to VN message as an observation of the symbol \( X \) (associated to \( v \)) at the output of an extrinsic channel with input alphabet \( \mathcal{X} = \mathbb{F}_q \) and output alphabet \( \mathcal{Z} = \mathcal{M}_\Gamma \). The transition probabilities of the extrinsic channel can be estimated via DE and are used to compute the \( L \)-vectors of the CN.
messages as shown in [2] and [3]. The parameters $\Delta^{(t)}$ are chosen to maximize the iterative decoding threshold and can be chosen for each iteration individually.

iv. **Final decision.**

Each variable node computes

$$L_{\text{app}}^{(t)} = L_{\text{app}}^{(t),0} = (L_{\text{app}}^{(t),1}, \ldots, L_{\text{app}}^{(t),q+1-2})$$

$$= L(y) + \sum_{e \in \mathcal{X}} L \left( m_{e\rightarrow v}^{(t)} \right).$$

$$\hat{x}^{(t)} = \arg \max_{a \in \mathcal{F}_q} L_{\text{app},a}^{(t)}.$$  \hspace{1cm} \text{(6)}

In [4], if multiple maximizing arguments exist we choose one of them uniformly at random.

Note that for $\Gamma = 1$, the SRLMP is similar to the SMP [12] but SRLM includes an additional empty set.

IV. **Density Evolution Analysis for SRLMP with \( \Gamma = 1 \)**

This section provides a DE analysis for SRLMP with maximum list size $\Gamma = 1$ for non-binary irregular LDPC code ensembles. For $\Gamma = 1$, the cardinality of the message alphabet is $|\mathcal{M}_1| = q + 1$. In the DE, the probabilities of VN to CN and CN to VN messages are tracked as iterations progress and we consider the limit as $n \to \infty$. Due to symmetry and under the all-zero codeword assumption, we can partition the messages in the same set have the same probability. We have

$$\mathcal{I}_0 = \{\emptyset\}$$

$$\mathcal{I}_1 = \{\{0\}\}$$

$$\mathcal{I}_2 = \{\{a\} : a \in \mathcal{F}_q \setminus \{0\}\}. \hspace{1cm} \text{(7)}$$

Note that $|\mathcal{I}_0| = |\mathcal{I}_1| = 1$, $|\mathcal{I}_2| = q - 1$. Let $P_{I_k}^{(t)}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the $\ell$-th iteration, i.e., a VN to CN message takes the value $a \in \mathcal{I}_k$ with probability $P_{\mathcal{I}_k}^{(t)}/|\mathcal{I}_k|$. Similarly, $S_{I_k}^{(t)}$ is the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0, 1, 2\}$.

Initially, we have

$$P_{I_0}^{(0)} = 0$$

$$P_{I_0}^{(1)} = 1 - \epsilon$$

$$P_{I_0}^{(1)} = \epsilon.$$

At the $\ell$-th iteration, we have

$$S_{I_0}^{(\ell)} = 1 - P_{I_0}^{(\ell)}(1 - P_{I_0}^{(\ell - 1)}$$

$$S_{I_1}^{(\ell)} = \frac{1}{q} \left[ \rho \left( P_{I_1}^{(\ell - 1)} + P_{I_2}^{(\ell - 1)} \right) 
\right.$$  \hspace{1cm} \text{(8)}

$$\left. + (q - 1) \rho \left( P_{I_1}^{(\ell - 1)} - \frac{P_{I_2}^{(\ell - 1)}}{q - 1} \right) \right].$$

The extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = \mathcal{M}_1$ and transition probabilities

$$P(z|u) = \begin{cases} 
S_{\mathcal{I}_0}^{(t)} & \text{if } z = \emptyset \\
S_{\mathcal{I}_1}^{(t)} & \text{if } z = \{u\} \\
S_{\mathcal{I}_2}^{(t)} & \text{if } z = \{e\} \
\end{cases} \hspace{1cm} e \in \mathcal{F}_q \setminus \{u\}. \hspace{1cm} \text{(10)}$$

Consider now the VN to CN messages. We define the random vector $F^{(t)} = (F_{\{0\}}^{(t)}, \ldots, F_{\{q-2\}}^{(t)}, F_{\{0\}}^{(t)})$, where $F_{\{a\}}^{(t)}$ denotes the RV associated to the number of incoming CN messages to a degree $d$ VN that take value $a \in \mathcal{M}_1$ at the $\ell$-th iteration, and $F_{\{a\}}^{(t)}$ is its realization. The entries of $L \left( m_{e\rightarrow v}^{(t)} \right)$ in [5] are

$$L_u \left( m_{e\rightarrow v}^{(\ell)} \right) \left( 1 \right) = D_1^{(t)}(f_u^{(\ell)}) + D_2 \delta_{uy} + K_1$$

where

$$K_1 = \log(\epsilon/(q - 1)) + f_0^{(t)} \log(s_{I_0}^{(t)}) + (d - 1) f_0^{(t)} \log(s_{I_2}^{(t)}/(q - 1))$$

$$\text{D}_2 \delta_{uy} = \log(1 - \epsilon) - \log(\epsilon/(q - 1))$$

$$\text{D}_1^{(t)} = \log(s_{I_1}^{(t)}) - \log(s_{I_2}^{(t)}/(q - 1))$$

and $\delta_{uy}$ is the Kronecker delta function. Note that $K_1$ in [11] is independent of $u$. Thus, it can be ignored when computing the extrinsic $L$-vector.

We obtain

$$P_{I_0}^{(t)} = \sum_{d} \lambda_d \sum_{y \in \mathcal{F}_q} \text{Pr}\{Y = y\} \sum_{f^{(t)}} \text{Pr}\{F^{(t)} = f^{(t)}\} \times$$

$$\left[ 1 - \sum_{a \in \mathcal{F}_q} \prod_{u \in \mathcal{F}_q \setminus \{a\}} \mathbb{I}(L_{\text{ex},a}^{(t)} > L_{\text{ex},u}^{(t)} + \Delta^{(t)}) \right]$$

$$P_{I_1}^{(t)} = \sum_{d} \lambda_d \sum_{y \in \mathcal{F}_q} \text{Pr}\{Y = y\} \sum_{f^{(t)}} \text{Pr}\{F^{(t)} = f^{(t)}\} \times$$

$$\prod_{u \in \mathcal{F}_q \setminus \{0\}} \mathbb{I}(L_{\text{ex},0}^{(t)} > L_{\text{ex},u}^{(t)} + \Delta^{(t)})$$

$$P_{I_2}^{(t)} = 1 - P_{I_0}^{(t)} - P_{I_1}^{(t)}.$$

where $\mathbb{I}(A)$ is an indicator function and the inner sum is over all length $q + 1$ non-negative integer vectors $f^{(t)}$ whose entries sum to $d - 1$ and

$$\text{Pr}\{F^{(t)} = f^{(t)}\} = \left( f^{(t)}_{\{0\}}, \ldots, f^{(t)}_{\{0\}} \right) \prod_{k=0}^{d-1} \left( \frac{S_{I_k}^{(t)}}{|I_k|} \right) f_{I_k}^{(t)}.$$
The iterative decoding threshold $\epsilon^*$ is the maximum QSC error probability such that $p_{\epsilon^*}^{(\ell)} \to 1$ as $\ell \to \infty$.

V. DENSITY EVOLUTION ANALYSIS FOR SRLMP WITH $\Gamma = 2$

This section gives a DE analysis for SRLMP with maximum list size $\Gamma = 2$. For $\Gamma = 2$, the cardinality of the message alphabet is $|M_2| = 1+q+\binom{q}{2}$. Due to symmetry and under the all-zero codeword assumption, we can partition the message alphabet $M_2$ into 5 disjoint sets $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ such that the messages in the same set have the same probability. We have $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ as defined in (7) and

\[ \mathcal{I}_3 = \{0, a\} : a \in \mathbb{F}_q \setminus \{0\} \]
\[ \mathcal{I}_4 = \{a, e : a, e \in \mathbb{F}_q \setminus \{0\} \text{ and } a \neq e\} \]

Note that $|\mathcal{I}_0| = |\mathcal{I}_1| = 1, |\mathcal{I}_2| = |\mathcal{I}_3| = q - 1$ and $|\mathcal{I}_4| = (q-1)^2$. Let $p_{\epsilon^*}^{(\ell)}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the $\ell$-th iteration. Similarly $s_{\mathcal{I}_k}^{(\ell)}$ is the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0, 1, 2, 3, 4\}$.

Initially, we have

\[ p_{\epsilon^*}^{(0)} = 1 - \epsilon \]
\[ p_{\epsilon^*}^{(0)} = \epsilon \]
\[ p_{\epsilon^*}^{(0)} = p_{\epsilon^*}^{(0)} = p_{\epsilon^*}^{(0)} = 0 . \]

For the CN to VN messages, $s_{\mathcal{I}_0}, s_{\mathcal{I}_1}, s_{\mathcal{I}_2}, s_{\mathcal{I}_3}, s_{\mathcal{I}_4}$ are given in (8), (9), (14) and (15), respectively and

\[ s_{\mathcal{I}_0}^{(\ell)} = 1 - s_{\mathcal{I}_1}^{(\ell)} - s_{\mathcal{I}_2}^{(\ell)} - s_{\mathcal{I}_3}^{(\ell)} - s_{\mathcal{I}_4}^{(\ell)} . \]

In this case, the extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = M_2$ and transition probabilities

\[ P(z|u) = \begin{cases} s_{\mathcal{I}_0}^{(\ell)} & \text{if } z = \{u\} \\ s_{\mathcal{I}_1}^{(\ell)} & \text{if } z = \{e\} \\ s_{\mathcal{I}_2}^{(\ell)} & \text{if } z = \{u, e\} \\ s_{\mathcal{I}_3}^{(\ell)} & \text{if } z = \{a, e\} \\ s_{\mathcal{I}_4}^{(\ell)} & \text{if } z = \emptyset. \end{cases} \]

Consider now the VN to CN messages. We extend the random vector $F^{(\ell)}$ to

\[ F^{(\ell)} = \left(F^{(\ell)}_{\{0\}}, \ldots, F^{(\ell)}_{\{a\}}, F^{(\ell)}_{\{1\}}, \ldots, F^{(\ell)}_{\{a\}}, F^{(\ell)}_{\emptyset}\right), \]

where $F^{(\ell)}_{\{a\}}$ denotes the RV associated to the number of incoming CN messages to a degree $d$ VN that take value $a \in M_2$ at the $\ell$-th iteration. The entries of $L\left(m_{c'}^{(\ell)}\right)$ in (5) are given by

\[ L_u\left(m_{c'}^{(\ell)}\right) = \log\left(P(m_{c'}^{(\ell)}|u)\right) \]

where $m_{c'}^{(\ell)} \in M_2, u \in \mathbb{F}_q$ and $P(z|u)$ is given in (16) for $\forall z \in M_2$. Hence, the entries $L_{ex,u}^{(\ell)}$ of the aggregated extrinsic $L$-vector in (5) are related to $f_{u}^{(\ell)}$ and the channel observation $y$ by

\[ L_{ex,u}^{(\ell)} = D_1^{(\ell)} f_{u}^{(\ell)} + D_2^{(\ell)} \sum_{a \in \mathbb{F}_q} f_{a}^{(\ell)} + D_{ch} \delta_{yu} + K_2 \]

where $D_1^{(\ell)}$ and $D_2^{(\ell)}$ are given in (12) and (13) and we have

\[ D_1^{(\ell)} = \log\left(s_{\mathcal{I}_0}^{(\ell)} / |\mathcal{I}_0|\right) - \log\left(s_{\mathcal{I}_0}^{(\ell)} / |\mathcal{I}_0|\right) \]
\[ K_2 = \log(\epsilon/(q-1)) + \sum_{a \in \mathbb{F}_q} f_{a}^{(\ell)} \log\left(s_{\mathcal{I}_2}^{(\ell)} / |\mathcal{I}_2|\right) \]
\[ \sum_{a,e \in \mathbb{F}_q} f_{\{a,e\}}^{(\ell)} \log\left(s_{\mathcal{I}_4}^{(\ell)} / |\mathcal{I}_4|\right) + f_{\emptyset}^{(\ell)} \log(s_{\mathcal{I}_4}^{(\ell)}). \]

Note that $K_2$ in (17) can be ignored in the VN update rule since it is independent of $u$.

We obtain

\[ p_{\mathcal{I}_2}^{(0)} = \sum_d \lambda_d \sum_{y \in \mathbb{F}_q} \Pr\{Y = y\} \sum_{f^{(\ell)}} \Pr\{F^{(\ell)} = f^{(\ell)}\} \times \prod_{u \in \mathbb{F}_q \setminus \{0\}} \left(1 - L_{ex,u}^{(\ell)} + \Delta^{(\ell)}\right) \]
\[ p_{\mathcal{I}_2}^{(0)} = \sum_d \lambda_d \sum_{a \in \mathbb{F}_q \setminus \{0\}} \Pr\{Y = y\} \sum_{f^{(\ell)}} \Pr\{F^{(\ell)} = f^{(\ell)}\} \times \prod_{u \in \mathbb{F}_q \setminus \{a\}} \left(1 - L_{ex,u}^{(\ell)} + \Delta^{(\ell)}\right) \]
\[ p_{\mathcal{I}_2}^{(0)} = \sum_d \lambda_d \sum_{a \in \mathbb{F}_q \setminus \{0\}} \Pr\{Y = y\} \sum_{f^{(\ell)}} \Pr\{F^{(\ell)} = f^{(\ell)}\} \times \sum_{a,e \in \mathbb{F}_q \setminus \{0,a\}} \left(1 - L_{ex,a}^{(\ell)} - L_{ex,e}^{(\ell)} \right) \times \prod_{u \in \mathbb{F}_q \setminus \{a,e\}} \left(1 - L_{ex,u}^{(\ell)} + \Delta^{(\ell)}\right) \]
\[ p_{\mathcal{I}_2}^{(0)} = \sum_d \lambda_d \sum_{y \in \mathbb{F}_q} \Pr\{Y = y\} \sum_{f^{(\ell)}} \Pr\{F^{(\ell)} = f^{(\ell)}\} \times \prod_{a,e \in \mathbb{F}_q \setminus \{0,a\}} \left(1 - L_{ex,a}^{(\ell)} - L_{ex,e}^{(\ell)} \right) \times \prod_{u \in \mathbb{F}_q \setminus \{a,e\}} \left(1 - L_{ex,u}^{(\ell)} + \Delta^{(\ell)}\right) \]
\[ p_{\mathcal{I}_2}^{(0)} = \sum_d \lambda_d \sum_{y \in \mathbb{F}_q} \Pr\{Y = y\} \sum_{f^{(\ell)}} \Pr\{F^{(\ell)} = f^{(\ell)}\} \times \prod_{a \in \mathbb{F}_q \setminus \{0,a\}} \left(1 - L_{ex,a}^{(\ell)} - L_{ex,e}^{(\ell)} \right) \times \prod_{u \in \mathbb{F}_q \setminus \{a\}} \left(1 - L_{ex,u}^{(\ell)} + \Delta^{(\ell)}\right) \]
\[ p_{\mathcal{I}_2}^{(0)} = 1 - p_{\mathcal{I}_1}^{(0)} - p_{\mathcal{I}_2}^{(0)} - p_{\mathcal{I}_3}^{(0)} - p_{\mathcal{I}_4}^{(0)} \]

where the inner sum is over all length $|M_2|$ non-negative integer vectors $f^{(\ell)}$ whose entries sum to $d - 1$.
The iterative decoding threshold $\epsilon^*$ is defined as the maximum channel error probability such that $p_{\ell}^{(\ell)} \rightarrow 1$ as $\ell \rightarrow \infty$.

VI. Numerical Results

We investigate the asymptotic performance of SRLMP with maximum list size 1 and 2 obtained by DE. Tables I and II show the iterative decoding thresholds of SRLMP for (3, 4) and (3, 5) regular ensembles and various values of $q$. For the sake of comparison, we provide the belief propagation thresholds $\epsilon_{\text{BP}}$, the Shannon limit $\epsilon_{\text{Sh}}$ and the thresholds of the SMP decoder [12], which is similar to the SRLMP with $\Gamma = 1$ (excluding the empty set). By comparing the thresholds for $\Gamma = 1$ with the SMP ones, we see that significant gains are obtained if empty sets are allowed in the decoding algorithm. Increasing $\Gamma$ improves the threshold but this comes at the cost of an increasing complexity. We believe that increasing $\Gamma$ further will significantly increase the decoding complexity and will not achieve significant gains compared to the case of $\Gamma = 2$. Note that the SRLMP outperforms the decoding algorithm in [13] for the same maximum list size.

Since the CN update rule of both decoders is the same, the gain is probably due to the VN update rule which is more complex for the case of the SRLMP decoder. In fact, the VNs in [13] compute the sum of binary vectors, whereas, here the incoming messages are converted to $L$-vectors before summation. To check the finite-length performance under SRLMP, we consider the performance of a regular (3, 5) code where we set the maximum number of iterations $\ell_{\text{max}} = 50$. The code has a block length $n = 60000$ and its Tanner graph is obtained via the progressive edge-growth (PEG) algorithm [19]. Finite-length simulation results for $\Gamma = 1$ and $\Gamma = 2$ are shown in Fig. 1 in terms of symbol error rate (SER) versus the QSC error probability $\epsilon$. We keep $\Delta^{(t)}$ constant over the iterations and use $\Delta^{(t)} = 1$ for $\Gamma = 1$ and $\Delta^{(t)} = 1.25$ for $\Gamma = 2$. As a reference, we provide the simulation results under the SMP decoder [12] and under the decoding algorithm in [13] for $\Gamma = 1$.

VII. Conclusions

A decoding algorithm for $q$-ary LDPC codes on the QSC was introduced and analyzed. We presented a DE analysis for maximum list size 1 and 2. The DE yields iterative decoding thresholds and provides the reliabilities of the extrinsic channel needed for the VN update rule. Our algorithm outperforms competing algorithms with similar complexity.

\[
s^{(t)}_{I_3} = q - 1 \left[ -2\rho \left( p_{I_3}^{(t-1)} + p_{I_2}^{(t-1)} \right) + 2\rho \left( p_{I_3}^{(t-1)} + p_{I_2}^{(t-1)} + \frac{p_{I_2}^{(t-1)}}{q-1} + \frac{p_{I_3}^{(t-1)}}{q-1} \right) \right.
\]
\[
+ (q-2)\rho \left( p_{I_3}^{(t-1)} - \frac{p_{I_2}^{(t-1)}}{q-1} + \frac{p_{I_3}^{(t-1)}}{q-1} - \frac{2p_{I_3}^{(t-1)}}{(q-1)(q-2)} \right) - (q-2)\rho \left( p_{I_3}^{(t-1)} - \frac{p_{I_2}^{(t-1)}}{q-1} \right) \right]
\]

\[
s^{(t)}_{I_4} = \frac{(q-1)(q-2)}{q} \left[ \rho \left( p_{I_4}^{(t-1)} + p_{I_2}^{(t-1)} + \frac{p_{I_3}^{(t-1)}}{q-1} + \frac{p_{I_2}^{(t-1)}}{q-1} \right) - \rho \left( p_{I_4}^{(t-1)} + p_{I_2}^{(t-1)} \right) \right.
\]
\[
- \rho \left( p_{I_4}^{(t-1)} - \frac{p_{I_2}^{(t-1)}}{q-1} + \frac{p_{I_2}^{(t-1)}}{q-1} - \frac{2p_{I_3}^{(t-1)}}{(q-1)(q-2)} \right) + \rho \left( p_{I_4}^{(t-1)} - \frac{p_{I_2}^{(t-1)}}{q-1} \right) \right]
\]

![Graph showing SER versus channel error probability $\epsilon$ for 4-ary regular (3, 5) LDPC code with $n = 60000$.](image-url)
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