New Bell Inequalities From No-Signalling Distributions

Thomas Cope† and Roger Colbeck‡

Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK
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A Bell inequality is a constraint on a set of correlations whose violation can be used to certify non-locality. They are instrumental for device-independent tasks such as key distribution or randomness expansion. In this work we consider bipartite Bell inequalities where two parties have \( m_A \) and \( m_B \) possible inputs and give \( n_A \) and \( n_B \) possible outputs, referring to this as the \((m_A, m_B, n_A, n_B)\) scenario. By exploiting knowledge of the set of extremal no-signalling distributions, we find all 175 Bell inequality classes in the \((4, 4, 2, 2)\) scenario, and all 7 classes in the \((3, 5, 2, 2)\) scenario, as well as providing a partial list of 18277 classes in the \((4, 5, 2, 2)\) scenario. We also use a probabilistic algorithm to obtain 5 classes of inequality in the \((2, 3, 3, 2)\) scenario, which we conjecture to be complete, and a partial list of 10143 classes in the \((3, 3, 3, 3)\) scenario. Our inequalities are given in supplementary files. Finally, we discuss the application of these to the detection loophole problem.

I. INTRODUCTION

Bell inequalities [1] can be thought of as constraints on the set of correlations realisable at spacelike separation by using shared classical randomness and freely chosen measurements. One of the most counterintuitive features of quantum theory is that by sharing quantum systems instead of classical randomness, these inequalities can be violated, a fact that has been subject to extensive experimental investigation [2–4]. This curious feature has since been used for cryptography [5] and shown to enable device-independent information processing tasks, such as quantum key distribution [8–11] and randomness expansion [12–15]. In essence, that shared classical randomness cannot explain random numbers can be distilled.

In spite of their usefulness, relatively little is known about the set of Bell inequalities in all but the simplest cases. In part, this is due to the complexity of finding them and the fact that the number of such inequalities grows rapidly as the number of inputs or outputs is increased. Bell inequalities can be thought of in a geometric way as the facets of the polytope of local (classically realisable) correlations. This insight means that Bell inequalities can be found by facet enumeration, a well-known problem in polytope theory that has been applied to find Bell inequalities in simple cases. However, as facet enumeration quickly becomes intractable, we turn to alternative algorithms. The algorithms we use are relatively easy to run, although they have the disadvantage that they don’t give a certificate when all the Bell inequalities have been found. In several cases we have been able to enumerate the complete list of Bell inequality classes [1], whilst in other cases we find lower bounds on the total number of classes of Bell inequality, which informs the complexity of doing facet enumeration in these cases. Our approach utilises knowledge of the set of extremal no-signalling distributions, as opposed to techniques based on enumerating the facets of simpler, related polytopes [16–18], or shelling techniques which travel along the polytope’s edges [19].

In particular, we provide a complete list of the 175 classes of Bell inequalities for the \((4, 4, 2, 2)\) scenario and the 7 classes for the \((3, 5, 2, 2)\) scenario, normalized in such a way to provide easy comparison. The number of inequalities for these scenario was already known [18] but a useable list was not provided, unlike for the \((2, 2, 2, 2)\) [19, 20], \((2, n, 2, m)\) [21], \((3, 3, 2, 2)\) [22], \((3, 4, 2, 2)\) [22] and \((2, 2, 3, 3)\) [23, 24] scenarios, which we summarize in Table 1.

In addition we investigate the \((2, 3, 3, 2)\) scenario, finding 5 classes of Bell inequality, which we conjecture to be complete. In the \((4, 5, 2, 2)\) scenario we find explicit representatives for 18277 classes of inequality, and in the \((3, 3, 3, 3)\) scenario we similarly give representatives for 10143 classes.

The structure of the paper is as follows. In Section II we overview of relevant polytope theory and the related topic of linear programming, before discussing Bell inequalities and detailing our representation of them. In Section III we discuss how to exploit knowledge of the set of extremal no-signalling distributions to obtain new Bell inequalities, as well as a technique for doing so without such knowledge. Section IV then gives the results. Finally, in Section V we apply these new inequalities to the problem of the detection loophole, presenting some numerical results and list the.

† thomas.cope@york.ac.uk
‡ roger.colbeck@york.ac.uk

‡ Two Bell inequalities are in the same class if they are related by a relabelling.
new inequalities which are the most promising candidates for lowering the detection threshold for small numbers of inputs and outputs.

II. PRELIMINARIES

A. Polytope Theory

A polytope is a convex set that can be described by the intersection of a finite number of half-spaces\(^2\). Given \(A \in \mathbb{R}^{r \times t}\) and \(c \in \mathbb{R}^r\) we can write

\[
P = \{x \in \mathbb{R}^t \mid Ax \geq c\} \tag{1}
\]

(each \(A_kx \geq c_k\) describes a half-space). This is called an \(H\)-representation of the polytope. Polytopes may also be described using a \(V\)-representation. If a polytope is bounded, then for some set \(\{x_k\}\) with \(x_k \in \mathbb{R}^t\) it can be written

\[
P = \left\{x = \sum_k \lambda_k x_k \mid \sum_k \lambda_k = 1, \ \lambda_k \geq 0\right\}. \tag{2}
\]

According to the Minkowski-Weyl theorem, every polytope admits both a \(V\)-representation and \(H\)-representation. We will always deal with \textit{minimal representations} (in which unnecessary half-spaces or points \(x_i\) are removed). In a minimal representation \(\{x_k\}\) are \textit{vertices}\(^3\). Given a polytope \(P \subset \mathbb{R}^t\) with dimension \(d \leq t\), the intersection of \(P\) with a bounding hyperplane \(A_kx = c_k\) is a \textit{facet} of the polytope if it has dimension \(d - 1\).

Converting from \(V\)-representation to \(H\)-representation is known as \textit{facet enumeration} whilst going from \(H\)-representation to \(V\)-representation is known as \textit{vertex enumeration}. For a polytope of dimension \(d\), given a \(V\)-representation with \(n\) vertices (or an \(H\)-representation with \(n\) half-spaces) there are algorithms that can perform this conversion in time \(O(ndr)\), with \(r\) the number of facets (vertices) enumerated\(^4\). When performing facet enumeration, \(r\) is generally not known in advance, and hence the worst case scenario is often used to provide an upper bound.

For a given dimension and number of vertices, the so called \textit{cyclic polytope}\(^2\) has the largest possible number of facets, \(\binom{n - \lfloor \frac{d}{n - d} \rfloor}{n - d} + \binom{n - \lfloor \frac{d}{n - d} \rfloor}{n - d - 1}\). Using this we obtain complexity \(O(n! \frac{d}{n})\)\(^2\). By contrast the simplest polytope is the simplex, with only \(d + 1\) facets.

In this work it is convenient to represent points using matrices rather than vectors. In this case, the \(H\)-representation of a polytope is based on a set \(\{B_i\}\) with \(B_i \in \mathbb{R}^{s \times t}\) and a set of real numbers \(c_i\) and can be expressed as

\[
\{\Pi \in \mathbb{R}^{s \times t} \mid \text{tr}(B_i^T \Pi) \geq c_i\}\). \tag{3}
\]

Likewise the \(V\)-representation is formed via a set \(\{\Pi_k\}\), \(\Pi_k \in \mathbb{R}^{s \times t}\) as

\[
\Pi = \sum_{k=1}^{n} \lambda_k \Pi_k \left\{\sum_{k=1}^{n} \lambda_k = 1, \ \lambda_k \geq 0\right\}. \tag{4}
\]

\(^2\) For a detailed summary of polytope theory, see \[28\].

\(^3\) For an unbounded polytope the \(V\)-representation will also have rays.

\(^4\) Some improvements on this have been achieved for specific classes of polytope.
B. Linear Programming

A linear programming problem involves the optimization of a linear objective function over a set of variables constrained by a finite number of linear equalities and/or inequalities \[28\]. The canonical form of a linear programming problem is as follows: given a fixed \(c \in \mathbb{R}^n, q \in \mathbb{R}^d\) and \(G \in \mathbb{R}^{d\times n}\),

\[
\max_{x} c^T x \quad \text{subject to} \quad Gx \leq q, \ x \in \mathbb{R}^n, \ x \geq 0.
\] (5)

For our later considerations it is convenient to rewrite this using \(A_i, Q \in \mathbb{R}^{s\times t}\) as

\[
\max_{\{x_i\}} \sum_i c_i x_i \quad \text{subject to} \quad \sum_i x_i A_i \leq Q, \ x_i \geq 0 \text{ for all } i.
\] (6)

We refer to this as the primal form. Any linear programming problem can be written in this way \[28\].

A linear programming problem is said to be infeasible, if there is no \(\{x_i\}\) satisfying the constraints. Otherwise we call the domain of \(\{x_i\}\) satisfying the constraints the feasible region. If the problem admits a finite solution, the problem is bounded. If the domain is bounded, then it forms a convex polytope, and the maximum principle \[29\] states that the optimum value is achieved at an extremal point of the polytope and is finite. Every linear program has an associated dual. For a problem written in the form (6), the dual problem is

\[
\min_M \text{tr}(M^T Q) \quad \text{subject to} \quad \text{tr}(M^T A_i) \geq c_i \text{ for all } i, \ M \in \mathbb{R}^{s\times t}, \ M \geq 0,
\] (7)

where the condition \(M \geq 0\) should be understood elementwise.

Linear programming problems are strongly dual: an optimum solution \(\{x^*_i\}\) for the problem (6), and \(M^*\) for the problem (7) satisfy \(\sum_i c_i x^*_i = \text{tr}((M^*)^T Q)\). If the primal is unbounded, then the dual is infeasible and vice versa. Solutions \(\{x^*_i\}, M^*\) also satisfy the complimentary slackness conditions \[28\], which in our notation are

\[
\text{tr} \left( (M^*)^T \left( Q - \sum_i x^*_i A_i \right) \right) = 0,
\] (8)

\[
\sum_i (\text{tr}((M^*)^T A_i) - c_i)x^*_i = 0.
\] (9)

Two common approaches for solving linear programming problems are simplex algorithms \[30\] and interior point methods \[31\]. Simplex algorithms exploit the fact that the optimum of a linear program is always achieved at a vertex. Such algorithms move between vertices by following edges that improve the value of the objective function until no further improvement is possible. Interior point methods make successive steps towards the optimal solution while remaining in the interior of the feasible region. Since we are interested in finding extremal Bell inequalities, simplex algorithms will be most useful for us.

C. Representing Probability Distributions and Bell Inequalities.

In this work we focus on the bipartite case. We consider two spacelike separate measurements, modelling the inputs using random variables \(X\) and \(Y\), and the respective outputs \(A\) and \(B\). We label the possible values of \(X\) by \(\{1, \ldots, m_A\}\). Likewise, \(Y\) takes values from \(\{1, \ldots, m_B\}\), \(A\) takes values from \(\{1, \ldots, n_A\}\) and \(B\) takes values from \(\{1, \ldots, n_B\}\)\(^5\). If the measurements \(X = x\) and \(Y = y\) are performed\(^6\), the joint distribution over the outputs is \(P_{AB|xy}\), i.e., for all \(x, y, a, b\) we have \(0 \leq P_{AB|xy}(ab) \leq 1\), and for all \(x, y\) we have \(\sum_{a,b} P_{AB|xy}(ab) = 1\). A distribution is said to be no-signalling if it satisfies

\[
\sum_b P_{AB|xy}(ab) = \sum_b P_{AB|xy'}(ab) \quad \forall a, x, y, y', \quad \text{and} \quad \sum_a P_{AB|xy}(ab) = \sum_a P_{AB|x'y}(ab) \quad \forall b, y, x, x'.
\]

\(^5\) In this work we do not consider the more general case where different inputs may have different numbers of outputs, although our techniques readily extend to such cases.

\(^6\) We use upper case to denote random variables and lower case for particular values of these.
Since we consider measurements made at spacelike separation, all distributions will be no-signalling.

Using notation from Tsirelson [32], we express the conditional distribution \( P_{AB|X} \) using an \( m_A n_A \times m_B n_B \) matrix:

\[
\Pi = \begin{pmatrix}
P_{AB|1}(1,1) & \cdots & P_{AB|1}(1,n_B) & \cdots & \cdots & P_{AB|m_B}(1,1) & \cdots & P_{AB|m_B}(1,n_B) \\
P_{AB|1}(n_A,1) & P_{AB|1}(n_A,n_B) & \cdots & \cdots & \cdots & P_{AB|m_B}(n_A,1) & P_{AB|m_B}(n_A,n_B) \\
\vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \ddots & \ddots \\
P_{AB|m_A}(1,1) & \cdots & P_{AB|m_A}(1,n_B) & \cdots & \cdots & P_{AB|m_A}(m_B,1) & \cdots & P_{AB|m_A}(m_B,n_B) \\
P_{AB|m_A}(n_A,1) & \cdots & P_{AB|m_A}(n_A,n_B) & \cdots & \cdots & \vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
\]

(10)

This notation makes it convenient to check the no-signalling conditions. However, it has some redundancy: the no-signalling conditions and the normalization condition mean that the true dimension of the space is \((m_A(n_A - 1) + 1)(m_B(n_B - 1) + 1) - 1\) [32].

A local deterministic distribution is one for which \( P_{AB|X} = P_{A|X}P_{B|Y} \) and for which \( P_{A|X}(a) \in \{0,1\} \) for all \( a, x \) and \( P_{B|Y}(b) \in \{0,1\} \) for all \( b, y \). There are \((n_A)^m_A(n_B)^m_B\) such distributions, and we use \( P_{AB|X}^{i,i} \) to denote the \( i \)-th local distribution for \( i = 1, \ldots, (n_A)^m_A(n_B)^m_B \). A local distribution is then one that can be written as a convex combination of local deterministic distributions, i.e., \( P_{AB|X} = \sum_i \lambda_i P_{AB|X}^{i,i} \), where \( \lambda_i \geq 0 \) and \( \sum_i \lambda_i = 1 \). We use \( L_{(m_A,m_B,n_A,n_B)} \) for the set of local distributions in each scenario. For all \((m_A,m_B,n_A,n_B)\), these form a polytope (the local polytope) with the local deterministic distributions as its vertices.

A Bell inequality is a linear inequality that is satisfied if and only if a distribution is local. The most important class of these are the facet Bell inequalities, which represent the facets of the local polytope. In principle, these can be found by facet enumeration starting from the local deterministic distributions. We can represent every Bell inequality \( P \) as a linear inequality that is satisfied if and only if a distribution is local. The most important Bell inequality is a no-signalling type and \( P \) is no-signalling type and \( P \) is the Bell inequality

\[
\sum_{i=1}^m A_i \prod_{j=1}^n B_j \geq 0
\]

where \( \sum_{i=1}^m A_i \prod_{j=1}^n B_j \geq 0 \) and \( \sum_{i=1}^m A_i = 0 \). We say that a matrix \( B \) is no-signalling type and \( B \) is identity type if we have \( \sum_{i=1}^m A_i \prod_{j=1}^n B_j \geq 0 \) and \( \sum_{i=1}^m A_i = 0 \). Although \( \sum_{i=1}^m A_i \prod_{j=1}^n B_j \geq 0 \), both \( B \) and \( B \) are representations of the same Bell inequality. For a Bell inequality of the form \( \sum_{i=1}^m A_i \prod_{j=1}^n B_j \geq 0 \), any local distribution, \( B \), for which \( \sum_{i=1}^m A_i \prod_{j=1}^n B_j = 0 \) is said to saturate the inequality.

Given a Bell inequality, we can construct others in the same scenario by relabelling inputs and outputs. In addition, if \( n_A = n_B \) and \( m_A = m_B \) then we can also swap parties to construct others. Two Bell inequalities related by such labellings are said to be in the same class. If \( n_A = n_B \) and \( m_A = m_B \) then there are \( 2(n_A)^m_A(n_B)^m_B \) ways to relabel and half as many otherwise, although some relabellings may be equivalent to others. Because labellings do not change the essential features of a Bell inequality, we focus on acquiring a representative of each Bell inequality class, rather than the full list of inequalities.

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7 For example, when \((m_A,m_B,n_A,n_B) = (2,2,2,2)\), there are 32 relabellings, but applying each of these to a CHSH inequality generates only 8 unique inequalities.
Another important property of Bell inequalities is that they may be “lifted” to apply to scenarios with more inputs and/or outputs. We can add an input by setting the coefficients corresponding to the new input to 0. (This means that the new Bell inequality ignores the new input.) To increase the number of outputs, the lifting involves copying the coefficients for one of the existing outputs\(^8\). This corresponds to treating the new output in the same way as one of the existing outputs. This copying is needed to ensure that the new Bell inequality has the same bound. Note that this method of lifting has the property that the lifting of a facet Bell inequality always gives a facet Bell inequality [33].

To illustrate the concept of lifting, we present three Bell inequalities: The first (\(B_{CHSH}\)) is the CHSH inequality for the \((2, 2, 2, 2)\) scenario. The second (\(B_I\)) lifts this to become a \((2, 3, 2, 2)\) inequality, whilst the third (\(B_O\)) is a lifting of \(B_C\) to a \((2, 2, 2, 3)\) inequality. All three are facet Bell inequalities with the form \(\text{tr}(B^T\Pi) \geq 1\).

\[
B_{CHSH} = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}, \quad
B_I = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0
\end{pmatrix}, \quad
B_O = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

**D. The sets of no-signalling and quantum distributions**

For a given scenario, the set of all no-signalling distributions, \(\mathcal{NS}_{(m_A, m_B, n_A, n_B)}\), forms a polytope whose facets correspond to the positivity of probabilities. The extremal no-signalling distributions can in principle be found by vertex enumeration on these facets. In the general case, we do not know how to express all of these, but local deterministic distributions are always vertices of \(\mathcal{NS}_{(m_A, m_B, n_A, n_B)}\). Furthermore, if both parties have only two outputs per measurement, i.e., \(n_A = n_B = 2\), it is known that (up to relabelling) all non-local extremal no-signalling distributions have the form [34]

\[
\begin{pmatrix}
S & S & \ldots & S & L & \ldots & L \\
S & A & S/A & \ldots & S/A & L & \ldots & L \\
S & S/A & S/A & \ldots & S/A & L & \ldots & L \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
K & K & \ldots & K & M & \ldots & M \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
K & K & \ldots & K & M & \ldots & M
\end{pmatrix}
\]

where \(g \in \{0, 1, \ldots, m_B - 2\}, h \in \{0, 1, \ldots, m_A - 2\}\) and with the following \(2 \times 2\) blocks:

\[
S = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}, \quad
A = \begin{pmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0
\end{pmatrix}, \quad
K = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}, \quad
L = \begin{pmatrix}
\frac{1}{2} & 0 \\
\frac{1}{2} & 0
\end{pmatrix}, \quad
M = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

The set of quantum distributions is a subset of \(\mathcal{NS}\). It is convex, but not a polytope. A distribution \(P_{AB|XY}\) is quantum if there exist POVMs \(\{E_a^x\}_a\) and \(\{F_b^y\}_b\) and a bipartite quantum state \(\rho_{AB}\) such that \(P_{AB|xy}(a,b) = \text{tr}((E_a^x \otimes F_b^y) \rho_{AB})\) for all \(a, b, x, y\). We use \(Q_{(m_A, m_B, n_A, n_B)}\) to denote the set of quantum distributions in each scenario. Given a distribution \(P_{AB|XY}\) it is difficult to decide whether it is quantum. To cope with this a series of outer approximations to the quantum set were introduced [33]. For integer \(k\), we use \(Q_{(m_A, m_B, n_A, n_B)}^k\) to denote the set of correlations at the \(k\)th level. These levels form a hierarchy, in that

\[
Q_{(m_A, m_B, n_A, n_B)} \subseteq Q_{(m_A, m_B, n_A, n_B)}^k \subseteq Q_{(m_A, m_B, n_A, n_B)}^l
\]

for any integers \(k > l\). The advantage of using these sets is that testing for membership of \(Q_{(m_A, m_B, n_A, n_B)}^k\) is a semidefinite program, which is in practice tractable for small enough \(k, m_A, m_B, n_A, n_B\).

Given a distribution \(\Pi\), we use the following measure of its non-locality:

\[^8\] The choice of output to copy may vary with each input.
Definition 1 ([36]). The local weight of a distribution $\Pi$ is the solution to the problem:

$$
\begin{aligned}
\max_{\{x_i\}} & \quad \sum_i x_i \\
\text{subject to} & \quad \sum_i x_i P_{L,i} \leq \Pi \\
& \quad x_i \geq 0 \text{ for all } i,
\end{aligned}
$$

(12)

where $\sum_i x_i P_{L,i} \leq \Pi$ is interpreted component-wise.$^9$

If $\Pi$ is local then the local weight is 1, while if $\Pi$ is a non-local extremal no-signalling distribution, its local weight is 0. Note that computing the local weight is a linear program. Its dual can be written

$$
\begin{aligned}
\min_M & \quad \text{tr}(M^T \Pi) \\
\text{subject to} & \quad \text{tr}(M^T P_{L,i}) \geq 1 \text{ for all } i \\
& \quad M \geq 0,
\end{aligned}
$$

(13)

where again $M \geq 0$ is treated component-wise, and $i$ runs over all local deterministic distributions. Note that $\text{tr}(M^T P_{L,i}) \geq 1$ for all $i$ implies that $\text{tr}(M^T P_{L,i}) \geq 1$ is a Bell inequality. If $\Pi$ is non-local, the matrix $M^*$ that achieves the minimum is the Bell inequality that $\Pi$ violates the most.

Definition 2. Let $\{x^*_i\}$ be the argument that achieves the optimum in the definition of the local weight of a distribution. The local part of a distribution $\Pi$ is $\sum_i x^*_i P_{L,i}$ and the non-local part is $\Pi - \sum_i x^*_i P_{L,i}$.

III. GENERATING FACET BELL INEQUALITIES

We can use the insight of the previous section to find Bell inequalities by solving the dual problem (13) for non-local distributions. Note that the Bell inequalities that emerge as solutions have the form where all entries are positive and have local bound 1. It turns out that all Bell inequalities can be represented in such a form (see Lemma 1 in Appendix A). Furthermore, for every non-trivial facet Bell inequality of this form there exists a non-local extremal no-signalling distribution which gives value 0 for this Bell expression (see Lemma 3 in Appendix A). In addition, there is a non-local extremal no-signalling distribution achieving this and that takes the form (11) with $g = h = 0$ (see Theorem 4 in Appendix A).

This suggests that we could find all the Bell inequalities by running the dual program for all non-local extremal no-signalling distributions (in cases where these are known). However, there is a hidden subtlety: although 0 is the minimum possible value for any Bell expression of this form (hence no other Bell inequality can have a larger violation) there can be several Bell expressions all of which have value 0 at the same extremal no-signalling distribution. This means that the output Bell inequality may not be a facet inequality, and that some facet inequalities may be missed. To mitigate this problem, we can reduce the degeneracy by mixing extremal no-signalling distributions with local distributions prior to using them as the dual problem’s objective function. This is the idea behind our algorithm. In principle one can choose enough local distributions to break this degeneracy completely (see Appendix C); in practice though we only mix with two local distributions to reduce the degeneracy whilst keeping a reasonable runtime.

A. A Linear Programming Algorithm for Bell Inequalities

Our algorithm needs a few sub-algorithms.

1. This is a way to decide whether a Bell inequality $B$ is a facet. To do so, we find the set of local deterministic distributions for which $\text{tr}(B^T P_{L,i}) = 1$ (i.e., those that achieve equality in the Bell inequality). Call the values of $i$ giving equality $a(1), a(2), \ldots, a(t)$. These all lie on a face, which is a facet if its dimension is one less than that of the entire space. In other words we have to check how many dimensions are spanned by $\{P_{L,a(i)} - P_{L,a(1)}\}_{i=2}$.

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$^9$ This condition ensures that $\Pi - \sum_i x_i P_{L,i}$ is equal to $(1 - \sum_i x_i) \tilde{P}$ for $(1 - \sum_i x_i) \geq 0$ and with $\tilde{P}$ as a valid distribution.
If this is \((m_A(n_A - 1) + 1)(m_B(n_B - 1) + 1) - 2\) then \(B\) represents a facet Bell inequality. (This dimensionality can be found by computing the rank of a matrix whose rows comprise the elements of \(P^L,a(\xi) - P^L,a(1)\) for \(i = 2, \ldots, t\).)

2. This checks whether two matrices \(B\) and \(\tilde{B}\) are representations of the same inequality. To do so we compute two vectors, \(v\) and \(\tilde{v}\) with components \(v_i = \text{tr}(B^T P^L,i)\) and \(\tilde{v}_i = \text{tr}(B^T P^L,i)\). By construction, the smallest element of each of these vectors is 1. Let the second smallest value of \(v\) be \(s > 1\). We perform an affine transformation that maps 1 to 1 and \(s\) to 2, (i.e., the function \(x \mapsto \frac{1}{s-1}x + \frac{s-1}{s-1}\)) to each component of \(v\) forming \(v'\). A similar procedure is performed on \(\tilde{v}\) forming \(\tilde{v}'\). The matrices \(B\) and \(\tilde{B}\) represent the same inequality if and only if \(v' = v'\).

3. Because we are interested in classes of Bell inequality, rather than the inequalities themselves, we also need to check whether two matrices are equivalent up to relabellings. This algorithm checks whether \(v' = T_m(\tilde{v}')\) where \(m\) runs over all the relabellings, and \(T_m\) is the permutation on the entries of \(\tilde{v}'\) corresponding to the \(m\)th relabelling (a list of such permutations can be computed once before commencing the main algorithm to speed up this check, although for larger cases a lot of memory is required to store them all). Note that if \(v'\) and \(\tilde{v}'\) do not have the same numbers of each type of entry (the same tally) then there cannot be such a permutation. We hence first check for this before running over all the permutations corresponding to relabellings.

Algorithm 1

This algorithm generates new facet inequalities for cases where \(n_A = n_B = 2\). It takes input \(\epsilon \in (0, 2/3)\), and a list \(W\) of known facet inequalities (which could be empty).

1. Set \(j = 1\).
2. Set \(\Pi\) to be the \(j\)th extremal no-signalling distribution of the form \((1,1)\) with \(g = h = 0\).
3. Solve the dual problem \(\text{(13)}\) for \(\Pi\) using a simplex algorithm, giving the matrix \(M\) that minimizes \(\text{tr}(M^T \Pi)\).
4. Generate a list of values of \(i\) such that \(\text{tr}(M^T P^L,i) = 1\). Call these \(a(1), a(2), \ldots, a(t)\).
5. Check whether \(M\) defines a facet (using subalgorithm \(\text{(1)}\)) and whether it or a Bell inequality in the same class is in the list \(W\) (using subalgorithms \(\text{(2)}\) and \(\text{(3)}\)). If not, add \(M\) to \(W\).
6. Choose 2 distinct elements \(k, l\) from \(\{1, 2, \ldots, t\}\) and form \(\Pi' = (1 - \frac{3\epsilon}{2})\Pi + \epsilon P^L,a(k) + \frac{\epsilon}{2} P^L,a(l)\).
7. Solve the dual problem \(\text{(13)}\) for \(\Pi'\) using a simplex algorithm, giving the matrix \(M'\) that minimizes \(\text{tr}((M')^T \Pi')\).
8. Check whether \(M'\) defines a facet (using subalgorithm \(\text{(1)}\)) and whether it or a Bell inequality in the same class is in the list \(W\) (using subalgorithms \(\text{(2)}\) and \(\text{(3)}\)). If not, add \(M'\) to \(W\).
9. Repeat steps \(\text{(1)}\) \text{ running over all distinct pairs } k, l \text{ from } \{1, 2, \ldots, t\}.
10. If \(j < 2^{(m_A-1)(m_B-1)\frac{1}{2}}\) set \(j = j + 1\) and return to step \(\text{(2)}\) otherwise end the algorithm, outputting \(W\).

Algorithm 2

In cases where the complete set of extremal no-signalling vertices is not known, we use another algorithm to find facet Bell inequalities. This works by picking random quantum-realisable distributions instead of extremal no-signalling distributions. This algorithm takes as input a number of iterations, \(j_{\text{max}}\), and a list \(W\) of known facet inequalities (which could be empty).

1. Set \(j = 1\).
2. Randomly choose a pure quantum state of dimension \((\max(n_A, n_B))^2\) and \(m_A\) random projective measurements of dimension \(\max(n_A, n_B)\) with \(n_A\) outcomes and \(m_B\) random projective measurements of dimension \(\max(n_A, n_B)\) with \(n_B\) outcomes.
max(n_A, n_B) with n_B outcomes. Form the (quantum) distribution Π by computing the distribution that would be observed for this state and measurements.

3. Solve the dual problem \[13\] for Π using a simplex algorithm, giving the matrix \( M \) that minimizes \( \text{tr}(M^T\Pi) \).

4. Check whether \( M \) defines a facet (using subalgorithm \[1\]) and whether it or a Bell inequality in the same class is in the list \( W \) (using subalgorithms \[2\] and \[3\]). If not, add \( M \) to \( W \).

5. If \( j < j_{\text{max}} \) set \( j = j + 1 \) and return to step \[2\] otherwise end the algorithm, outputting \( W \).

B. Comments on the algorithms

We use the simplex algorithm as implemented by Mathematica 11.3.0.0. Unfortunately, full details of this specific implementation are not publicly available (to our knowledge). In particular, for our problem, in spite of the steps taken to break some of the degeneracy, for a given \( \Pi \) there remain many \( M' \) that achieve the optimum for the dual. Which one is given out by the simplex algorithm depends on the details of how it decides which edge to travel along when faced with several possibilities. Mathematica’s implementation is deterministic, but for fixed \( k \) and \( l \), small changes in \( \epsilon \) can lead to a different solution based on \( \Pi' \). It is hence useful to rerun the algorithm for several values of \( \epsilon \).

As mentioned above, one disadvantage of the above algorithm is that in many cases the output of the dual program does not correspond to a facet inequality. It is possible to alter the problem such that the solution space is the polar analogue of Lemma \[3\] does not hold). This is further discussed in Appendix \[10\].

Running through all permutations to check whether two Bell matrices are in the same class can take time, so Algorithm 1 can also be run with a modified subalgorithm \[3\] in which \( M \) is added to \( W \) if the vectors \( v' \) and \( \tilde{v}' \) have different tallies. Running the algorithm in this way can generate many classes of Bell inequality, but without also running through all permutations, some classes may be missed. For instance, in the case \((m_A, m_B, n_A, n_B) = (4, 4, 2, 2)\), for \( i = 1, 2 \), the Bell inequalities \( \text{tr}(B_i^T\Pi) \geq 1 \) where

\[
B_1 = \begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and \( B_2 = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0
\end{pmatrix} \)

are in different classes, but the corresponding vectors \( v' \) and \( \tilde{v}' \) have the same tallies.

Another disadvantage of our algorithm is that was do not have a criteria for deciding when the list of classes found is complete. In cases where we are sure we have found all Bell inequalities, we know this only because the total number had already been found by other means.

Note that because Algorithm \[2\] is based on quantum distributions, it is unable to find Bell inequalities for which there is no quantum violation \[37\]. Furthermore, the chosen quantum distribution may be local, in which case the dual program will not output a Bell inequality. To circumvent these disadvantages, other ways to pick non-local distributions can be used. For instance for the (2, 3, 3, 2) scenario we have used a second method in which we generate a quantum distribution, \( \Pi \), and, if it is non-local, we remove its local part and renormalize the remaining distribution \((\Pi - \sum x_i^*P_{L,i})\). If there were an extension of the work of \[34\] to cases with \( n_A > 2 \) or \( n_B > 2 \), we could use Algorithm 1 in these cases. We expect that this would be a quicker way to generate new Bell inequalities.

\[10\] For more details on how we do this, see Appendix \[15\]
| Size of Class | Number of Classes | Notable cases |
|--------------|------------------|--------------|
| 64           | 1                | Positivity   |
| 288          | 1                | CHSH [19]    |
| 9216         | 2                | $I_{3322}$ [22] |
| 18432        | 4                | $I_{4422}$ [22] |
| 24576        | 1                |              |
| 36864        | 4                |              |
| 49152        | 2                |              |
| 73728        | 8                |              |
| 98304        | 2                |              |
| 147456       | 61               |              |
| 294912       | 89               |              |
| 36391264     | 175              |              |

TABLE I: The size of each facet class for the (4,4,2,2) local polytope. 294912 is the size of the relabelling symmetry group.

IV. RESULTS USING THE ALGORITHM

In this section we summarise the results we have obtained using the above algorithm. Due to the large number of inequality classes found we present them in separate files available at [38], along with a file explaining how to import and use them in both Mathematica and Matlab. We give both the version the algorithm found (the “raw” version) and a second version after an affine transformation analogous to that mentioned in subalgorithm 2 has been applied (the “affine” version). These are presented after relabellings that make obvious input/output liftings.

A. (4,4,2,2) scenario

For this scenario, we have enumerated all 175 inequivalent classes of facet Bell inequality (including the trivial positivity inequality). Whilst the number of classes was already known [18], a complete list was not provided. A partial list of 129 non-trivial inequalities was given in [17]. Our generation of these inequalities was performed on a standard desktop computer and took a few days.

In order to give an idea of the symmetries of this polytope, Table I gives the number of members of each class.

B. (3,5,2,2) scenario

As in the previous case, the number of facet classes (7) was given in [18]. Six of these are also facet classes of the (3,4,2,2) scenario, given in [22]. The new Bell inequality was found using Algorithm 1 to be $\text{tr}(I_{3522}^T \Pi) \geq 1$, where

$$I_{3522} = \begin{pmatrix}
0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3}
\end{pmatrix}.$$  (14)

C. (2,3,3,2) scenario

The extremal no-signalling distributions are not known for this scenario, but using the modification of Algorithm 2 mentioned at the end of Section III B (in which we sample quantum distributions and then remove their local part), we found five classes of Bell inequality: the positivity condition, a lifting of CHSH and three new inequality classes.
\[ \text{tr}((I_{2332}^1)^T \Pi) \geq 1 \text{ and } \text{tr}((I_{2332}^2)^T \Pi) \geq 1 \text{ and } \text{tr}((I_{2332}^3)^T \Pi) \geq 1 \] with representative matrices

\[
I_{2332}^1 = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
I_{2332}^2 = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad
I_{2332}^3 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(Here, for clarity, we have added dividing lines to the matrices to indicate the different measurements.) Given that these five were found within the first 802 distributions sampled and no more classes were found for the next 567928 distributions, we conjecture this is the full list of classes.

\section*{D. (4, 5, 2, 2) scenario}

There is no known total of inequivalent classes for this scenario. Using Algorithm 1 we have established the existence of at least 18277 classes and have a representative of each. We expect the total number of classes to be significantly larger.\textsuperscript{11}

\section*{E. (3, 3, 3, 3) scenario}

Again for this case we do not know all the extremal no-signalling distributions, so we use Algorithm 2 to find facet inequalities. In this case we have found a lower bound of 10143 classes of inequality and have a representative of each. Again, we expect the total number of classes to be significantly larger (see Footnote 11).

\section*{V. APPLICATION TO THE DETECTION LOOPHOLE}

In the remainder of this article we investigate whether these inequalities allow us to lower the efficiency required for closing the detection loophole.

To perform a bipartite Bell experiment, entangled photons are sent to two detectors where they are measured. After repeating many times using different randomly chosen measurements we can build up an estimate of the distribution \( \Pi \), from which we can see whether a particular Bell inequality is violated or not. One of the issues with such an experiment is that real detectors sometimes fail to detect. The question is then how to certify that the setup is non-local in the presence of such no-click events. In particular, we would like to know the minimal detection efficiency at which we can still certify the presence of non-locality. Given that certifying the presence of non-locality is necessary for device-independent tasks, it is important to be able to treat cases with imperfect detectors.

To model this, we assume that each detector has an efficiency, representing the probability that it will click when it should. For simplicity, we consider the case where the efficiencies of each detector are the same and call this parameter \( \eta \). No-click can be treated as an additional outcome for each measurement that occurs with probability \( 1 - \eta \) (independently for each measurement). Given a probability distribution \( P_{AB|XY} \), we use \( P_{AB|XY}^\eta \) to denote the inefficient detector version with efficiency \( \eta \), which is formed by adding the possible outcome “\( N \)” to each measurement and taking

\[
\begin{align*}
P_{AB|xy}^\eta(a, b) &= \eta^2 P_{AB|xy}(a, b), \\
P_{AB|xy}^\eta(N, b) &= \eta(1 - \eta) P_{B|y}(b), \\
P_{AB|xy}^\eta(a, N) &= \eta(1 - \eta) P_{A|x}(a), \\
P_{AB|xy}^\eta(N, N) &= (1 - \eta)^2,
\end{align*}
\]

\textsuperscript{11} We stopped the algorithm after roughly 1 week but new inequality classes were being found regularly.
for all \( a \in \{1, 2, \ldots, n_A\}, \) \( b \in \{1, 2, \ldots, n_B\}, \) \( x \in \{1, 2, \ldots, m_A\} \) and \( y \in \{1, 2, \ldots, m_B\}. \) We use \( \Pi^\eta \) to denote the matrix representation of \( P_{AB;XY}^\eta. \)

It is worth noting that for a given \( \Pi \in \mathcal{NS}(m_A, m_B, n_A, n_B) \) we have that \( \Pi^\eta \notin \mathcal{L}(m_A, m_B, n_A, n_B+1) \) if \( \Pi \notin \mathcal{L}(m_A, m_B, X, n_A, n_B). \) and that for \( \eta_1 \geq \eta_2, \) \( \Pi^{\eta_2} \notin \mathcal{L}(m_A, m_B, n_A, n_B, n_B+1) \) implies \( \Pi^{\eta_1} \notin \mathcal{L}(m_A, m_B, n_A, n_B+1). \)

We now define the detection threshold for a given Bell inequality, and for a given scenario.

**Definition 3.** Given a \( m_A(n_A + 1) \times m_B(n_B + 1) \) matrix \( B \) such that \( \text{tr}(B^T \Pi) \geq c \) for all \( \Pi \in \mathcal{L}(m_A, m_B, n_A, n_B+1) \), the detection threshold for \( B \), is defined by

\[
\eta_B := \inf \{ \eta \in [0, 1] : \exists \Pi \in \mathcal{Q}(m_A, m_B, n_A, n_B) : \text{tr}(B^T \Pi) < c \}. \tag{15}
\]

This is the smallest detection efficiency such that \( B \) can certify non-locality using quantum states and measurements for all higher efficiencies. Note that \( B \) is a Bell inequality for the \((m_A, m_B, n_A + 1, n_B + 1)\) scenario. Some Bell inequalities of this type can be formed from those for the \((m_A, m_B, n_A, n_B)\) scenario by lifting, as discussed in Section III C.

**Definition 4.** The detection threshold for the \((m_A, m_B, n_A, n_B)\) scenario is defined by

\[
\eta(m_A, m_B, n_A, n_B) := \inf \{ \eta \in [0, 1] : \exists \Pi \in \mathcal{Q}(m_A, m_B, n_A, n_B) : \Pi^\eta \notin \mathcal{L}(m_A, m_B, n_A, n_B+1) \}. \tag{16}
\]

We can also define a detection threshold for a set \( S(m_A, m_B, n_A, n_B) \subseteq \mathcal{NS}(m_A, m_B, n_A, n_B) \).

**Definition 5.** The detection threshold for \( S(m_A, m_B, n_A, n_B) \) is defined by

\[
\eta^S(m_A, m_B, n_A, n_B) := \inf \{ \eta \in [0, 1] : \exists \Pi \in S(m_A, m_B, n_A, n_B) : \Pi^\eta \notin \mathcal{L}(m_A, m_B, n_A, n_B+1) \}. \tag{17}
\]

We will use the fact that if \( Q \) is a subset of \( S \) then the detection threshold for \( S \) is lower than the quantum one, i.e., \( Q(m_A, m_B, n_A, n_B) \subseteq S(m_A, m_B, n_A, n_B) \) implies \( \eta^S(m_A, m_B, n_A, n_B) \leq \eta(m_A, m_B, n_A, n_B). \)

It is known that there exist states and measurements for which this threshold tends to 0 as \( d \to \infty \). However, to achieve this, the number of measurements required scales as \( O(2^d) \). For practical reasons we are interested in cases with small numbers of inputs and outputs.

In the \((2, 2, 2, 2)\) scenario, taking \( \Pi \) to be quantum correlations that achieve Tsirelson’s bound \([40]\) for the CHSH inequality, \( \text{tr}(B^T \Pi) = 2 - \sqrt{2} \) it has been shown that \( \Pi^\eta \) is non-local if and only if \( \eta > 2(2\sqrt{2} - 1) = 2.88% \). A lower value was found by Eberhard \([42]\), who showed that one can use a two-qubit state of the form \( \cos \theta |00\rangle + \sin \theta |11\rangle \) and appropriate 2-outcome measurements to give rise to a distribution \( \Pi_\theta \) such that for any \( \eta > 2/3 \) there exists \( \theta \) such that \( \Pi_\theta^\eta \) is non-local, while for \( \eta = 2/3 \) the distribution \( \Pi_\theta^\eta \) is local for all \( \theta \).

Somewhat counterintuitively, the state demonstrating non-locality has \( \theta \to 0 \) as \( \eta \to 2/3 \). In \([13]\), a \((4, 4, 2, 2)\) inequality (which we refer to as \( I_{4422} \)) was considered and a state and measurements on a four dimensional Hilbert space were given demonstrating that \( \eta_{I_{4422}} \leq (\sqrt{5} - 1)/2 \approx 61.8\% \). The state used has the form \( \sqrt{(1 - \epsilon^2)/3} |00\rangle + |11\rangle + |22\rangle + \epsilon |33\rangle \), with the value \( (\sqrt{5} - 1)/2 \) achieved in the limit \( \epsilon \to 0 \).

In \([44]\), the problem was abstracted away from specific Bell inequalities, instead giving an explicit local-hidden variable construction which can replicate any inefficient no-signalling distribution, provided the detection efficiency is below \((m_A + m_B - 2)/(m_A m_B - 1) \). This hence corresponds to a lower bound on \( \eta(m_A, m_B, n_A, n_B) \). In the next subsection we improve on these lower bounds in cases where \( n_A = n_B = 2 \).

## A. A Fundamental Lower Bound on the Detection Threshold

In this section we show how to bound \( \eta(m_A, m_B, n_A, n_B) \) using knowledge of the set of no-signalling distributions. As discussed above, since \( \mathcal{Q}(m_A, m_B, n_A, n_B) \subseteq \mathcal{NS}(m_A, m_B, n_A, n_B) \), a lower bound for the detection threshold for all no-signalling distributions will apply to the quantum case too.

In cases where we have a complete set of extremal no-signalling distributions, we can obtain an arbitrarily good estimate \( \eta^{NS}(m_A, m_B, n_A, n_B) \) using the following algorithm.

**Algorithm 3**

The algorithm takes input \( \delta \in (0, 1) \), the tolerance we look for in the solutions.

1. Set \( j = 1, \) \( \eta_c = 1 \) and \( j_{\text{max}} \) to be the total number of extremal no-signalling distributions.
TABLE II: The maximum detection efficiency such that $\Pi^\eta$ can be generated classically for any $\Pi \in NS(m_A,m_B,2,2)$. The $(5,6,2,2)$ case was not evaluated due to the high number of non-local extremal no-signalling distributions. The final column is the lower bound of [44]. The main part of the table is populated with rational numbers, even though the algorithm outputs real numbers. Strictly we should say that the value is consistent with this rational number to 7 decimal places.

| $m_A$ | 2 | 3 | 4 | 5 | 6 | $2/(m_B + 1)$ |
|-------|---|---|---|---|---|---------------|
| $m_B$ | 2/3 | 2/3 | 2/3 | 2/3 | 2/3 | $2/3$ |
| 3     | 4/7 | 5/9 | 5/9 | 5/9 | 1/2 | $1/2$ |
| 4     | 1/2 | 1/2 | 1/2 | 1/2 | 2/5 | $1/2$ |
| 5     | 4/9 | 2/5 | 5/9 | 5/9 | 2/3 | $1/3$ |

2. Set $\Pi$ to be the $j^{th}$ non-local extremal no-signalling distribution.
3. Set $\eta_{\text{min}} = 0$ and $\eta_{\text{max}} = 1$.
4. Set $\eta' = (\eta_{\text{min}} + \eta_{\text{max}})/2$ and generate $\Pi_{\eta'}$.
5. Find the local weight of $\Pi_{\eta'}$ by solving the linear program (12), setting this to $w$.
6. If $w = 1$, set $\eta_{\text{min}} = \eta'$, otherwise, set $\eta_{\text{max}} = \eta'$.
7. If $\eta_{\text{max}} - \eta_{\text{min}} > \delta$, go to step 4.
8. If $(\eta_{\text{min}} + \eta_{\text{max}})/2 < \eta_c$, set $\eta_c = (\eta_{\text{min}} + \eta_{\text{max}})/2$.
9. If $j < j_{\text{max}}$, set $j = j + 1$ and return to step 2; otherwise the algorithm ends, outputting $\eta_c$.

This algorithm runs through all the non-local extremal no-signalling distributions, and computes at what $\eta$ they become local. Then by taking the minimum over all such distributions we obtain the no-signalling detection threshold for this scenario.

We have applied this algorithm to the $(m_A,m_B,2,2)$ scenario for various $m_A$ and $m_B$. The results are shown in Table II, where we have also compared with the lower bound from [44] in order to highlight the improvement we obtain.

B. Bounding Detection Thresholds using the Semidefinite Hierarchy

We can think of these no-signalling values as lower bounds on the quantum detection thresholds. When interpreted this way, we do not expect these lower bounds to be tight, although, somewhat surprisingly, they are in the case where $m_B = 2$ [42]. In order to give better bounds, we can use other supersets of the quantum set, for instance, those based on the semidefinite hierarchy [35]. In other words, we can try to find $\eta^Q_k(m_A,m_B,n_A,n_B)$ for some level $k \in \mathbb{N}$ of the hierarchy.

Since $Q^k(m_A,m_B,n_A,n_B)$ is not a polytope, we cannot directly use the method of Section V A. Instead, for each value $\eta$ we perform a semidefinite optimisation over $Q^k(m_A,m_B,n_A,n_B)$, minimising $\text{tr}(B^T\Pi^\eta)$ over $\Pi \in Q^k(m_A,m_B,n_A,n_B)$ for a specific Bell inequality $B$ for the $(m_A, m_B, n_A + 1, n_B + 1)$ scenario. If the minimum is at least 1, we can conclude that there is no $\Pi \in Q(m_A,m_B,n_A,n_B)$ for which $\Pi^\eta$ violates $\text{tr}(B^T\Pi^\eta) \geq 1$. Performing the computation for higher levels $k$ of the hierarchy gives successively tighter bounds.

C. Results

All of the results in this section were obtained using the convex optimisation interface CVX [45] for Matlab, with MOSEK [46] as the solver. Unless stated otherwise, tests were run at the default CVX precision. Note also that in the code, a value of $\text{tr}(B^T\Pi^\eta) \geq 1 - \epsilon$ was considered local for $\epsilon \approx 1.49 \times 10^{-8}$. This mitigates against the possibility of concluding a distribution is non-locality because of the solver precision, when in fact it is not. However, it can mean that we sometimes incorrectly conclude a distribution is local. Thus, the values we obtain will be upper bounds of the threshold over the considered set.
In the supplementary files these results are given in the following format: first the inequality considered is given, followed by the lifting choices of Alice, then the lifting choices of Bob, followed by the threshold found.

1. \((4,4,2,2)\) Scenario

For this scenario an explicit quantum construction is given in [43] achieving \((\sqrt{5} - 1)/2 \approx 0.6180\) (61.80\%) requiring shared entangled states with local dimension 4. We were able to test every inequality in this scenario at level 2, with no inequality beating \(I_{4422}\), which was the inequality used in [43]. We were only able to obtain the value 61.83\% for the \(I_{4422}\) construction, implying we cannot rely on the results beyond 3 s.f., and an alternative lifting of the same inequality was able to achieve 61.8\% also. Repeating the analysis for these two with the CVX precision variable set to “high”, gave a value of 61.82\% for both liftings, which also means we cannot improve the number of significant figures this way.

2. \((3,5,2,2)\) Scenario

For the single new inequality \((3,5,2,2)\), we tested it with the CVX precision set to high. Note that we are considering \((3,5,3,3)\) probability distributions, so lifting the inequality by adding an extra output for each input can be done in \(2^{3+5} = 256\) ways. For the first level, \(Q^1\), we obtained an optimal threshold of 64.6\%, and for \(Q^2, Q^3\) we obtained a threshold of 66.7\%. This, together with the fact that restricting to \(m_A \in \{1,3\}\) and \(m_B \in \{3,5\}\) (with reference to the form of \(I_{3522}\) in (14)), reduces to a CHSH inequality for which the threshold is known to be \(2/3\), suggests that \(\eta_{I_{3522}} = 2/3\).

The supplementary file for this case omits the inequality as only one was considered.

3. \((3,3,3,3)\) Scenario

For each new inequality there are \(3^{3+3} = 729\) possible liftings. Rather than finding the detection threshold for each, we looked for inequalities with low thresholds in the following way. We first tested each lifting at \(\eta = 0.65\), discarding it if no non-local distributions \(\Pi^{0.65}\) could be found for \(\Pi \in Q^2\). 207 inequality/lifting combinations achieved a threshold below this value; 8 of them obtained value 61.8\% — due to the precision of 3 s.f. we are unable to compare them definitively with \(I_{4422}\). Testing these 8 at level 3 of the hierarchy we find two of them maintain the value 61.8\%. We denote these \(I^A_3\) and \(I^B_3\), presented below along with arrows indicating the input that is copied to generate the specific lifting.

\[
I^A_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix} \\
I^B_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

These two inequalities match the threshold \((\sqrt{5} - 1)/2\) up to 3 s.f. but we have not found an explicit quantum construction achieving this value. If it turns out that the thresholds for \(I_{4422}\) and either of these two are the same, this cannot be explained by a set of correlations common to both, unlike in the case of \(I_{3522}\) and CHSH, whose thresholds match due to \(I_{3522}\) having a CHSH submatrix (cf. Section V C 2).

Added note

These results formed part of TC’s Ph.D. thesis submitted to the University of York in September 2018. Since then the work [47] appeared which independently found all the Bell inequalities in the \((4,4,2,2)\) scenario. Some analysis
of these was performed in \cite{18}.

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dual program (13) for \( \Pi \), the optimum is obtained when \( M \) gives value 0 for the Bell inequality. If \( \Pi \) have \( \Pi \) non-negative such that \( \Pi \) is a Bell inequality. There is a matrix \( \tilde{B} \) whose entries are all non-negative such that \( \tilde{B}^T \Pi \geq 1 \) represents the same Bell inequality.

Proof. Let \( B \) be the matrix with every entry 1. The matrix \( B' = B + \frac{1-\alpha}{\alpha m_B} \mathbf{1} \) is such that \( (B')^T \Pi \geq 1 \) is a Bell inequality. If \( B' \) has no negative entries we are done. Otherwise, suppose the minimum entry of \( B' \) is \( -\alpha \). If we choose \( \tilde{B} = (B' + \alpha \mathbf{1})/(1 + \alpha m_B) \), so that, by construction, it has no negative entries, then for any local \( \Pi \) we have

\[
\text{tr}(\tilde{B}^T \Pi) = \frac{1}{1 + \alpha m_B} \text{tr}(B'^T \Pi) + \frac{\alpha}{1 + \alpha m_B} \text{tr}(\Pi) \geq \frac{1}{1 + \alpha m_B} + \frac{\alpha m_B}{1 + \alpha m_B} = 1.
\]

\[\square\]

Corollary 2. Let \( B \) be a matrix such that \( \text{tr}(B^T \Pi) \leq c \) is a Bell inequality. There is a matrix \( \tilde{B} \) whose entries are all non-negative such that \( \tilde{B}^T \Pi \geq 1 \) represents the same Bell inequality.

Proof. The original Bell inequality is equivalent to \( \text{tr}((-B)^T \Pi) \geq c \) from which we can apply Lemma 1.

\[\square\]

Lemma 3. Let \( B \) be a matrix with no negative entries and such that \( \text{tr}(B^T \Pi) \geq 1 \) is a facet Bell inequality. There exists an extremal no-signalling distribution \( \Pi^{NS} \) such that \( \text{tr}(B^T \Pi^{NS}) = 0 \).

Proof. Since \( B \) represents a violatable Bell inequality, there exists a no-signalling point \( \Pi \) such that \( \text{tr}(B^T \Pi) < 1 \) but \( \text{tr}(B^T \Pi) \geq 1 \) for all other matrices \( \tilde{B} \) that represent facet Bell inequalities with local bound 1. Thus, if we run the dual program (13) for \( \Pi \), the optimum is obtained when \( M^* \) corresponds to the same Bell inequality as \( B \) (it may be that \( M^* \neq B \), but it must represent the same Bell inequality). Using the complementary slackness condition (8), we have

\[
\text{tr} \left( B^T \left( \Pi - \sum_i x_i^* P_{L,i} \right) \right) = 0,
\]

where \( \{x_i^*\} \) achieve the optimum in the primal problem (12). This implies that the non-local part, \( \Pi - \sum_i x_i^* P_{L,i} \), of \( \Pi \) gives value 0 for the Bell inequality \( B \). This non-local part is a convex combination of extremal non-local no-signalling distributions, \( \{\Pi^{NS}_j\} \), each satisfying \( \text{tr}(B^T \Pi^{NS}_j) = 0 \), as required.

\[\square\]
Remark 1. For the problem in the previous lemma, the second complementary slackness condition \( \sum_i (\text{tr}(B^T P^{L,i}) - 1) x_i^* = 0 \).

\[ (A2) \]

Hence, for all \( i \) either \( x_i^* = 0 \) or \( (\text{tr}(B^T P^{L,i}) - 1) = 0 \) (both values are non-negative). As \( x_i^* \) is non-zero if and only if the local distribution \( P^{L,i} \) is in the local part of \( \Pi \), we can conclude that \( \text{tr}(B^T P^{L,i}) = 1 \) for these local distributions, and so the local part of \( \Pi \) satisfies the Bell inequality with equality.

**Theorem 4.** Consider the \((m_A, m_B, 2, 2)\) scenario and let \( B \) be a matrix with no negative entries and such that \( \text{tr}(B^T \Pi) \geq 1 \) is a facet Bell inequality. There exists an extremal no-signalling distribution \( \Pi \) that, up to relabellings, takes the form of \((\Pi)\) with \( g = h = 0 \) such that \( \text{tr}(B^T \Pi) = 0 \).

**Proof.** By Lemma 3 there exists an extremal no-signalling distribution \( \Pi \) such that \( \text{tr}(B^T \Pi) = 0 \). Suppose that \( \Pi \) has the form \((\Pi)\) with either \( g \neq 0, h \neq 0 \), or both. Our aim is to show that in these cases there is always another extremal no-signalling distribution \( \tilde{\Pi} \) of the form \((\Pi)\) with \( g = h = 0 \) such that \( \text{tr}(B^T \tilde{\Pi}) = 0 \).

**Case 1:** Suppose \( g = 0, h \neq 0 \). Since the coefficients of \( B \) are non-negative, \( B \) must have zero entries whenever \( \Pi \) has non-zero entries. Hence, \( B(0b|xy) = 0 \) for \( b \in \{0,1\}, x \in \{m_A - h + 1, \ldots, m_A\} \) and \( y \in \{1, \ldots, m_B\} \).

Suppose there exists a local deterministic distribution, \( P^1 \), such that \( \text{tr}(B^T P^1) = 1 \) and for which \( P^1_{A|x'}(1) = 1 \) for some \( x' \in \{m_A - h + 1, \ldots, m_A\} \). Consider now another local deterministic distribution \( P^2 \) that is identical to \( P^1 \) except that \( P^2_{A|x'}(0) = 1 \) (i.e., \( P^2 \) is formed by exchanging the row of \( P^1 \) corresponding to \( P^1_{A|x'}(0) \) with the row corresponding to \( P^2_{A|x'}(1) \)). It follows that \( 1 \leq \text{tr}(B^T P^2) \leq \text{tr}(B^T P^1) = 1 \), i.e., \( P^2 \) saturates the Bell inequality if \( P^1 \) does. It also follows that the \( 2 \times 2 \) blocks of \( B \) corresponding to a measurement of \( X = x' \) and any \( Y \in \{1, \ldots, m_B\} \) have the form \[
\begin{pmatrix}
0 & 0 \\
0 & \gamma
\end{pmatrix}
\] (depending on whether \( P^1_{B|y}(0) = 1 \) or \( P^1_{B|y}(1) = 1 \)), where \( \gamma \) is an arbitrary non-negative value.

We can therefore replace the \( 2 \times 2 \) blocks of \( \Pi \) corresponding to \( X = x', Y \in \{1, \ldots, m_B\} \) by \( A \) or \( S \) without affecting the value of \( \text{tr}(B^T \Pi) \). In other words, if \( \text{tr}(B^T P^1) = 1 \) and \( P^1_{A|x'}(1) = 1 \) for some \( x \in \{m_A - h + 1, \ldots, m_A\} \), there exists another extremal no-signalling distribution, \( \Pi' \), with a smaller value of \( h \) that also has \( \text{tr}(B^T \Pi') = 0 \).

If we can reduce in this way until \( h = 0 \), we are done. Alternatively, we reduce to the case where \( \text{tr}(B^T P) = 1 \) implies \( P^1_{A|x'}(0) = 1 \) for all \( x \in \{m_A - h + 1, \ldots, m_A\} \). The affine span of the local deterministic distributions satisfying \( \text{tr}(B^T P) = 1 \) is hence the same as the affine span of those satisfying \( \text{tr}(B^T \tilde{P}) = 1 \), where \( \tilde{B} \) and \( \tilde{P} \) comprise the first \( (m_A - h) \) rows of \( B \) and \( P \) respectively. This is at most \((m_A - h)(m_A - 1) + 1)(m_B(m_B - 1) + 1) - 2\), and hence contradicts the assumption that \( B \) is a facet inequality.

**Case 2:** If \( g \neq 0, h = 0 \), we can run an analogous argument.

**Case 3:** Suppose that both \( g \neq 0 \) and \( h \neq 0 \), and consider some \( x' \in \{m_A - h + 1, \ldots, m_A\} \) and \( y' \in \{m_B - g + 1, \ldots, m_B\} \). We must have \( B(0b|x'y') = 0 \) for \( b \in \{0,1\} \) and \( y \in \{1, \ldots, m_B - g\} \), \( B(a0|xy') = 0 \) for \( a \in \{0,1\} \) and \( x \in \{(1, \ldots, m_A - h)\} \), and \( B(00|x'y') = 0 \).

Suppose there exists a local deterministic distribution, \( P^1 \), such that \( \text{tr}(B^T P^1) = 1 \) and for which \( P^1_{A|x'}(1) = 1 \). Let \( P^2 \) be another local deterministic distribution that is identical to \( P^1 \) except that \( P^2_{A|x'}(0) = 1 \) and \( P^2_{B|y}(0) = 1 \) for all \( y \in \{m_B - g + 1, \ldots, m_B\} \). It follows that \( 1 \leq \text{tr}(B^T P^2) \leq \text{tr}(B^T P^1) = 1 \), i.e., \( P^2 \) saturates the Bell inequality if \( P^1 \) does.

It also follows that

- For \( X = x' \) and any \( Y \in \{1, \ldots, m_B - g\} \) the corresponding \( 2 \times 2 \) blocks of \( B \) have the form \[
\begin{pmatrix}
0 & 0 \\
0 & \gamma
\end{pmatrix}
\] (depending on whether \( P^1_{B|y}(0) = 1 \) or \( P^1_{B|y}(1) = 1 \)).

- For \( X = x' \) and \( Y \in \{m_B - g + 1, \ldots, m_B\} \) the analogous blocks of \( B \) have the form \[
\begin{pmatrix}
0 & \gamma_1 \\
\gamma_2 & 0
\end{pmatrix}
\] or \[
\begin{pmatrix}
0 & \gamma_1 \\
0 & \gamma_2
\end{pmatrix}
\].

We can therefore replace the \( 2 \times 2 \) blocks of \( \Pi \) corresponding to \( X = x' \) by either \( A, S \) or \( L \) (whichever matches the zeros of \( B \)) without changing the value of \( \text{tr}(B^T \Pi) \). In other words, if there exists a local deterministic distribution, \( P^1 \), such that \( \text{tr}(B^T P^1) = 1 \) and for which \( P^1_{A|x'}(1) = 1 \), then there exists an extremal no-signalling distribution, \( \Pi' \), with a smaller value of \( h \) that also satisfies \( \text{tr}(B^T \Pi') = 0 \).
Case 2. Alternatively, we reduce to the case where $\text{tr}(P) = 1$ implies $\hat{P}_A(0) = 1$ for all $x \in \{m_A - h + 1, \ldots, m_A\}$, and $\hat{P}_B(y)(0) = 1$ for all $y \in \{m_B - g + 1, \ldots, m_B\}$. The affine span of the local deterministic distributions satisfying $\text{tr}(P) = 1$ is hence the same as the affine span of those satisfying $\text{tr}(\hat{P}) = 1$, where $\hat{B}$ and $\hat{P}$ comprise the first $2(m_A - h)$ rows and $2(m_B - g)$ columns of $B$ and $P$ respectively. This is at most $(m_A - h)(n_A - 1 + (m_B - g)(n_B - 1) + 1) - 2$, and hence contradicts the assumption that $\hat{B}$ is a facet inequality.

To get the idea of the proof, let us consider an example. Take $\Pi = \begin{pmatrix} S & S & L \\ S & A & L \\ K & K & M \end{pmatrix}$. A Bell inequality with $\text{tr}(B\Pi) = 0$ must have the form

$$B = \begin{pmatrix} 0 & v_1 & 0 & v_2 & 0 & v_3 \\ v_4 & 0 & v_5 & 0 & 0 & v_6 \\ 0 & v_7 & v_8 & 0 & 0 & v_9 \\ v_{10} & 0 & 0 & v_{11} & 0 & v_{12} \\ 0 & 0 & 0 & 0 & v_{13} \\ v_{14} & v_{15} & v_{16} & v_{17} & v_{18} & v_{19} \end{pmatrix},$$

where $v_i$ denotes an arbitrary non-negative entry.

Consider now the local deterministic distributions

$$P^1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{P}^1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

If $\text{tr}(B^TP^1) = 1$ then we must have $\text{tr}(B^TP^2) = 1$, and hence $v_{15} = v_{17} = v_{18} = 0$. In this case the distribution

$$\Pi' = \begin{pmatrix} S & S & L \\ S & A & L \\ S & S & L \end{pmatrix}$$

will also satisfy $\text{tr}(B\Pi') = 0$. We can then apply the argument of case 2 to this distribution.

Similarly, if $\text{tr}(B^T\hat{P}^1) = 1$ then we must have $\text{tr}(B^T\hat{P}^2) = 1$, and hence $v_3 = v_9 = v_{15} = v_{17} = v_{19} = 0$. In this case the distribution

$$\Pi' = \begin{pmatrix} S & S & A \\ S & A & A \\ S & S & S \end{pmatrix}$$

will also satisfy $\text{tr}(B\Pi') = 0$. Note that by relabelling the outputs for $y = 3$ corresponds to exchanging the $A$ entries for $S$s in the final column of $\Pi'$, bringing us into a form matching (11).

By arguments of this kind it follows that either we can reduce to a case with a lower value of $g$ or $h$, or all the local deterministic distributions with $\text{tr}(B^TP) = 1$ have zeros in the final row and column ($P_{X|a=3}(0) = 1$ and $P_{Y|b=3}(0) = 1$). In the latter case the dimension of the plane containing the saturating local deterministic distributions is insufficient for $B$ to be a facet inequality.

**Appendix B: Generating Quantum Distributions**

As stated in the main body, one may use our linear programming algorithm using quantum distributions rather than extremal no-signalling ones. To generate these, one must first fix the dimension of the state $d$. One then creates a normalised real vector $\lambda$ of $d$ non-zero real elements. We take these as the Schmidt coefficients of the pure entangled state $|\phi\rangle = \sum_{i=1}^{d} \lambda_i |i\rangle \otimes |i\rangle$. We then generate $m_A + m_B$ random unitaries $\{U^A_1, U^A_2, \ldots, U^A_{m_A}, U^B_1, U^B_2, \ldots, U^B_{m_B}\}$, each unitary corresponding to a measurement. Since the columns of each $U^i$ are orthonormal, we can define projection operators $P_k^i := \sum_{k \in S_k^i} |U^i_k\rangle \langle U^i_k|$, where $\{S_k^i\}$ is a partition of $[n_A]$ or $[n_B]$ as appropriate. These projectors satisfy

$$\sum_k P_k^i = (U^i)\dagger U_i = I_d.$$ 

Thus, we obtain the probability distribution:

$$p(ab|xy) = \langle \phi | (P_a^{A^*} \otimes P_b^{B^*}) | \phi \rangle.$$  

(B1)
Appendix C: Guaranteeing All Inequality Classes

Although for the scenarios in which the number of inequality classes was known, Algorithm 1 was able to generate a representative of every class, due to the degeneracy of the optimal solutions it does not guarantee that this generally will be the case. In this appendix we discuss an alteration to the algorithm that can, in principle, provide this. However, the run time of such an algorithm is prohibitive. We nevertheless state the method here, because it gives the idea behind Algorithm 1.

Suppose we have a specific Bell inequality \( B \) we wish to find as a solution to \([13]\). Let \( \Pi \) be an extremal no-signalling distribution \( \Pi \) such that \( \text{tr}(B^T \Pi) = 0 \) and let \( \{P_j\}_{j=1}^{d} \) be a set of linearly independent local deterministic distributions saturating \( B \) (where \( d \) is the dimension of the local polytope). If we define \( \Pi' = (1 - \delta) \Pi + \delta \sum_{j=1}^{d} P_j/d \), then \( \text{tr}(B^T \Pi') = \delta \).

Furthermore, the matrix \( B \) represents the unique Bell inequality that achieves the minimum solution to the dual problem \([13]\) with input \( \Pi' \). To see this, note that no matrix \( \hat{M} \) for which \( \text{tr}(M^T \Pi) \geq 1 \) is a Bell inequality can give a lower value because \( \text{tr}(M^T \Pi) \) is bounded below by 0 and \( \text{tr}(M^T P_i) \geq 1 \). In addition, no other matrix \( \hat{M} \) with non-negative entries for which \( \text{tr}(M^T \hat{P}) \geq 1 \) is a different Bell inequality will achieve this value because \( \text{tr}(M^T \Pi') \geq \delta \sum_{j=1}^{d} \text{tr}(M^T P_j)/d \). It cannot be that \( \text{tr}(M^T P_j) = 1 \) for all \( j \in \{1, \ldots, d\} \) because this would mean \( \{P_j\}_{j=1}^{d} \) also all lie on the facet formed by \( \hat{M} \), which is only possible if the facets are identical.

It follows that a Bell inequality of every class can be generated by considering all possible objective functions of the form \( (1 - \delta) \Pi + \delta \sum_{j=1}^{d} P_j/d \) where \( \Pi \) is an extremal no-signalling point of the form in Eq. \([11]\) and \( \{P_j\}_{j=1}^{d} \) are linearly independent deterministic local distributions. However, this algorithm is impractical for all but the smallest cases. In the case \( n_A = n_B = 2 \), we wish to choose \( d = (m_A + 1)(m_B + 1) - 1 \) local deterministic distributions from from \( 2^{m_A+m_B} \) possible choices and the number of ways to do this scales roughly as \( 2^{m_A m_B (m_A+m_B)} \), which is prohibitively large. This is why in Algorithm 1 we mix each no-signalling point with only two local deterministic distributions, chosen from a strict subset of the local deterministic distributions. For small values of \( m_A \) and \( m_B \) the algorithm can be run in reasonable time.

Appendix D: Using the Polar Dual

One disadvantage of our algorithm is that not all solutions of the linear programming problem are facet inequalities. This increases the calculation time because we have to check the affine dimension of every solution and discard many non-facet cases. In this appendix we consider an alternative approach which guarantees facet outputs, by taking advantage of the polar dual of the local polytope.

Definition 6. Given a polytope \( \mathcal{P} \subset \mathbb{R}^{n \times t} \), its polar dual\(^{12}\) is the set of points:

\[
\mathcal{P}^* := \left\{ B \in \mathbb{R}^{n \times t} \mid \text{tr}(B^T \Pi) \leq 1 \text{ } \forall \text{ } \Pi \in \mathcal{P} \right\}.
\] (D1)

If the co-ordinate origin is interior to \( \mathcal{P} \) (i.e., \( \Pi = 0 \) is a non-boundary element of \( \mathcal{P} \)), then the polar dual satisfies \((\mathcal{P}^*)^* = \mathcal{P}\), and the two are linked by the following \([22]\).

| H-representation | V-representation |
|------------------|------------------|
| \( \mathcal{P} = \left\{ \Pi \in \mathbb{R}^{n \times t} \mid \text{tr}(B_j^T \Pi) \leq 1 \text{ } \forall j \right\} \) | \( \mathcal{P} = \left\{ \Pi = \sum_i \lambda_i \Pi_i \mid \sum_i \lambda_i = 1, \lambda_i \geq 0 \right\} \) (D2) |
| \( \mathcal{P}^* = \left\{ B \in \mathbb{R}^{n \times t} \mid \text{tr}(B^T \Pi_i) \leq 1 \text{ } \forall i \right\} \) | \( \mathcal{P}^* = \left\{ B = \sum_j \lambda_j B_j \mid \sum_j \lambda_j = 1, \lambda_j \geq 0 \right\} \), (D3) |

where \( \Pi_i, B_j \in \mathbb{R}^{n \times t} \). Hence, there is a one-to-one correspondence between the vertices of the primal and the facets of the dual, and vice versa.

\(^{12}\) In the study of convex bodies, there are other types of a dual, which we do not use here.
The motivation for considering the polar dual is that by optimizing with the polar dual of the local polytope as the solution space, a simplex algorithm will always give a solution corresponding to a facet of the local polytope.

In order to use this form of the polar dual we require the origin to lie in the interior. We hence perform a translation of coordinates. A natural choice of the new origin is the distribution $\Pi^u$ whose entries are all $1/n_A n_B$. For example, in the $(2, 2, 2, 2)$ scenario this would map the extremal no-signalling distribution

$$
\begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}
to
\begin{pmatrix}
\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4}
\end{pmatrix},
$$

Note that this representation of a distribution can have negative entries. The linear programming problem we wish to solve is then

$$
\begin{aligned}
\max_{M} & \quad \text{tr}(M^T \overrightarrow{\Pi}) \\
\text{subject to} & \quad \text{tr}(M^T \overrightarrow{B^L,i}) \leq 1 \text{ for all } i.
\end{aligned}
$$

(D4)

where $\overrightarrow{\Pi}$ refers to a distribution after shifting origin. The canonical form of a linear program requires positive entries. To take care of this, we can write $M = M_+ - M_-$ where $M_+, M_- \geq 0$ component-wise, making the problem

$$
\begin{aligned}
\max_{M_+, M_-} & \quad \text{tr}(M_+^T \overrightarrow{\Pi}) - \text{tr}(M_-^T \overrightarrow{\Pi}) \\
\text{subject to} & \quad \text{tr}(M_+^T \overrightarrow{B^L,i}) - \text{tr}(M_-^T \overrightarrow{B^L,i}) \leq 1 \text{ for all } i, \quad M_+, M_- \geq 0.
\end{aligned}
$$

(D5)

Although the solution space is technically unbounded due to rays of the form $(M_+)_ij = (M_-)_ij$, these cannot contribute to the objective function, and the problem (D5) is bounded. However, our conversion of the problem into the form (D4) means we no longer have a known bound on the objective function\(^{13}\). To illustrate the problem with this, we perform the optimisation (D5), for all extremal non-local no-signalling distributions. For the $(4, 4, 2, 2)$ scenario, the largest value obtained is $12/5$, and the smallest is $2$. (Unlike in problem (E3), the optimal value varies depending on the extremal no-signalling point used.)

We now consider a particular $(4, 4, 2, 2)$ Bell inequality of the form:

$$
\overrightarrow{B^L}_{\text{ex}} = \begin{pmatrix}
0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} \\
0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 \\
-\frac{1}{5} & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & -\frac{1}{5} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0
\end{pmatrix}, \quad \text{tr}(\overrightarrow{B^L}_{\text{ex}} \overrightarrow{\Pi}) \leq 1,
$$

(D6)

and the corresponding problem of maximizing $\text{tr}(\overrightarrow{B^L}_{\text{ex}} \overrightarrow{\Pi})$ over all $\overrightarrow{\Pi} \in N^5_{(m_A, m_B, n_A, n_B)}$. Performing this optimisation we find the maximal value to be $9/5 < 2$. We can therefore conclude there is no extremal no-signalling point that, when used as the objective function for the problem (D4), will give the solution $M = \overrightarrow{B^L}_{\text{ex}}$. By using the polar dual, if $\epsilon$ is too small there are certain facets that Algorithm 1 (with this new linear programming problem) will never output, and it is not clear whether $\epsilon$ can be chosen in such a way that all facets could in principle be output. However, this restriction does not apply to Algorithm 2, for which this translation may be useful.

\(^{13}\) In the original form, we had that $\text{tr}(M^T \Pi) \geq 0$. 