Quantum Delocalization of the Electric Charge

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Abstract
The classical Maxwell–Dirac and Maxwell–Klein–Gordon theories admit solutions of the field equations where the corresponding electric current vanishes in the causal complement of some bounded region of Minkowski space. This poses the interesting question of whether states with a similarly well localized charge density also exist in quantum electrodynamics. For a large family of charged states, the dominant quantum corrections at spacelike infinity to the expectation values of local observables are computed. It turns out that certain moments of the charge density decrease no faster than the Coulomb field in spacelike directions. In contrast to the classical theory, it is therefore impossible to define the electric charge support of these states in a meaningful way.

1 Introduction

Electrically charged systems are known to have poor localization properties with regard to measurements of the electric field, which extends to spacelike infinity according to Coulomb’s law. Whereas this delocalization is an inevitable consequence of Maxwell’s equations, the electron comes close to the idea of a point particle and one might infer that the charge density of such systems ought to have much better localization properties. Thus the interesting question arises of whether it is possible to assign a charge support in a consistent manner.
The idea of a clearcut distinction between charge and field support seems unproblematic in the context of classical physics. The possibility that it might be meaningful in quantum field theory too was first considered by Fröhlich [1] in a general discussion of the superselection structure of electrically charged states. More recently, a simple non–interacting model allowing a precise definition of the electric charge support of states was presented in [2], where it was also outlined how this notion could be used to analyze the statistics and symmetry properties of such models by generalizing methods developed in [3, 4] for analyzing strictly localized states. Thus a systematic investigation of the localization properties of the electric charge in more realistic (interacting) theories seems warranted.

It is the aim of the present article to carry out such an analysis in classical and quantum electrodynamics. In the classical Maxwell–Klein–Gordon and Maxwell–Dirac theories it turns out that a charge support can be sharply defined. More precisely, there exist finite energy solutions of the coupled field equations such that the corresponding electric current vanishes in the causal complement of some double cone in Minkowski space whereas the electric field is of Coulomb type there. These results fit perfectly with the heuristic picture of a point–like support of the electric charge.

The simple picture breaks down, however, if one takes quantum effects into account. Using perturbative methods, we shall determine, for a large family of charged physical states in quantum electrodynamics, the dominant quantum contributions to the matrix elements of local observables at spacelike infinity. These contributions have a simple form and thus can be computed in the cases of interest: whereas the expectation values of the charge density and their mean square fluctuations exhibit spatial decay properties which seem to corroborate the picture of a reasonably well localized charge distribution, the higher moments decay no faster than the Coulomb field. As a matter of fact, these moments can be used to determine the shape of the asymptotic electromagnetic field of the states. So the idea of discriminating the charge support from the field support fails in these examples.

The origin of this phenomenon will be traced back to vacuum polarization effects, namely the fact that observables which are related to the matter fields can generate states from the vacuum containing, with non–zero probability, only low energy photons. If the interaction is turned off, this effect disappears and the resulting states have a mass gap. This general mechanism is also at the root of a result by Swieca [5] who proved that the spatial integral of the charge density in electrically charged states exhibits an oscillatory behaviour in time, thereby leading to a Coulomb–like delocalization of the spatial components of the electric current. The present results show that the delocalizing effects of vacuum polarization also affect the charge density. Although we restrict attention here to a special family of charged states (corresponding to gauges of “Coulomb type”), our results provide evidence to the effect that this delocalization is generic.

Our paper is organized as follows. In Section 2 we use global existence theorems for the classical Maxwell–Klein–Gordon and Maxwell–Dirac theory to exhibit solutions of the field equations with sharp support properties of the electric
current. The quantum induced delocalization of electrically charged states is discussed in Section 3. By slightly modifying a perturbative method for constructing charged states, established by Steinmann [6], we exhibit a family of such states where the asymptotically leading quantum corrections to the matrix elements of local observables can conveniently be analyzed. The paper concludes with a brief discussion of the implications of our results for the general analysis of the superselection structure of theories involving charges of electric type.

2 Localizing the classical electric charge

We begin by discussing the localization properties of the electric charge at the classical level in the Maxwell–Klein–Gordon and Maxwell–Dirac theory. In the Maxwell–Klein–Gordon theory, the field equations are

\[ \partial^\nu F_{\mu\nu} = j_\mu, \quad D_\mu D^\mu \varphi = 0. \]  

(2.1)

Here \( F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \) is the electromagnetic field, \( A_\mu \) is the vector potential in the Coulomb gauge, \( \varphi \) is the charged scalar field and \( D_\mu = \partial_\mu - ieA_\mu \), where \( e \) is the unit of charge. The electric current \( j_\mu \) is given by

\[ j_\mu = e A_\mu \varphi^* \varphi - e \frac{i}{2} (\varphi^* \partial_\mu \varphi - \partial_\mu \varphi^* \varphi). \]  

(2.2)

By using the existence theorem of Klainermann and Machedon [7], we shall exhibit solutions of these equations where, in a given Lorentz system, both the current \( j_\mu \) and the magnetic field \( B_i \equiv \frac{1}{2} \varepsilon_{ijk} F_{jk} \) vanish in the spacelike complement of some compact region, whereas the electric field \( E_i \equiv F_{0i} \) is non-zero there, \( i = 1, 2, 3 \).

Proposition 2.1 In the classical Maxwell–Klein–Gordon theory, for any double cone \( O \subset \mathbb{R}^4 \), there are finite energy solutions \( \varphi, A_\mu \) with non-zero charge,

\[ \int d^4x \ j_0(x, x_0) \neq 0, \]  

(2.3)

such that for any \( x = (x, x_0) \in O' \)

\[ j_\mu(x) = 0, \quad B(x) = 0 \]  

(2.4)

\[ E(x) = (4\pi)^{-1} \int d^3y \ (x - y) |x - y|^{-3} j_0(y, 0). \]  

(2.5)

Proof. By the main theorem of Klainermann and Machedon [7], the Maxwell-Klein–Gordon equations in the Coulomb gauge have unique global solutions with finite energy for all smooth initial data \( A_j, \partial_{0} A_j, \varphi, \partial_0 \varphi \) with support in the base of \( O \). Moreover, these solutions are smooth in all variables. We shall show that their support properties are as stated above.

To see this, we note that the field equation for \( \varphi \) can be read as a hyperbolic equation in the “external” field \( A_\mu \),

\[ \Box \varphi = -2ieA_\mu \partial^\mu \varphi + e^2 A_\mu A^\mu \varphi + ie \partial^0 A_0 \varphi \equiv J(\varphi, \partial_\mu \varphi). \]
Setting $\chi \equiv (\varphi, \partial_0 \varphi)$ gives a first order system satisfying the following integral equation with respect to the time variable

$$\chi(x_0) = G(x_0)\chi(0) + \int_0^{x_0} dy_0 \, G(x_0 - y_0) f(\chi(y_0)).$$

$$f(\chi(y_0)) \equiv (0, J(\varphi, \partial_\mu \varphi)(y_0)).$$

Here $G(x_0)$ is the Green’s function (propagator) of the equation for $e = 0$ and the spatial dependence has been suppressed. This integral equation is known to have a unique solution $\chi$, provided $f$ satisfies a suitable local Lipschitz condition, see e.g. [8]. In our case, this Lipschitz condition holds because $A_\mu$ is smooth, and therefore locally bounded.

Because of the hyperbolic character of the equation, the solution in a given double cone depends only on the initial data on the base of that double cone. Thus, for initial data $\varphi, \partial_0 \varphi$ of compact support contained in $O$, $\varphi$ and therefore $j_\mu$ vanish in the causal complement $O'$. Moreover, Maxwell’s equations give

$$\Box B = \text{curl } j,$$

and therefore $B$ vanishes in $O'$ if the initial data for $A$ have support in $O$. Finally, by Maxwell’s equations,

$$\partial_0 E = \text{curl } B - j = 0 \text{ in } O',$$

so $E$ is time independent in $O'$ and therefore given by equation (2.5). $\square$

Let us now turn to the Maxwell–Dirac theory with the field equations

$$\partial^\nu F_{\mu\nu} = j_\mu = e \bar{\psi} \gamma_\mu \psi,$$

$$(-i\gamma^\mu \partial_\mu + m) \psi = e \gamma^\mu A_\mu \psi, \quad (2.6)$$

where $\psi$ is the Dirac field and $\gamma^\mu$ are the gamma matrices. Here the results are slightly weaker than in the preceding case since the initial value problem is under control only for small initial data.

**Proposition 2.2** In the classical Maxwell–Dirac theory, given any double cone $O$, there are finite energy solutions $\psi, A_\mu$ with (small) non–zero charge $q$,

$$\int dx \, j_0(x, x_0) = q, \quad (2.7)$$

such that the corresponding current and the electromagnetic field have in $O'$ the properties (2.4), (2.5).

**Proof.** Theorem 2.5 of [9] (see also [10]) establishes the existence and uniqueness of finite energy solutions for sufficiently small smooth initial data. In particular, if the initial data for $\psi$ and $A$ are smooth, have compact support and are bounded by a sufficiently small constant and the initial data for $A_0$ are computed from the gauge condition

$$\partial_\mu A^\mu = 0$$
and the Gauss constraint

$$\Delta A_0 = -\partial^i \partial_0 A_i + e |\psi|^2,$$

a unique global solution exists and satisfies the preceding gauge conditions at all times. Moreover, this solution is smooth in the spatial variables and locally bounded in time. Thus, with the global Cauchy problem for the Maxwell–Dirac equations for sufficiently small smooth data (corresponding to small electric charge) under control, we can proceed as in the scalar case. The solution $\psi$ solves the integral equation

$$\psi(x_0) = S(x_0) \psi(0) + ie \int_0^{x_0} dy_0 S(x_0 - y_0) \gamma_0 \gamma^\mu A_\mu(y_0) \psi(y_0),$$

where $S(x_0)$ is the Green’s function (propagator) of the free Dirac equation which has the same hyperbolic properties as $G(x_0)$ in the scalar case and $A_\mu$ is regarded as an external field. Thus the non–linear term again satisfies a local Lipschitz condition since $A_\mu(x, x_0)$ is bounded in $x$ uniformly for $x_0$ in finite intervals. By the same argument as in the scalar case, therefore, $\psi$ vanishes in $O'$. This implies that $j_\mu$ vanishes in $O'$, and the results for $B$ and $E$ follow as before. □

Thus we find that, in classical field theory, the localization properties of the electric charge are not affected by the interaction between the electromagnetic field and the matter fields.

### 3 Quantum delocalization

We want to study now how quantum effects modify the localization properties of the electric charge. As a rigorous construction of quantum electrodynamics has not yet been accomplished, we have to rely on perturbative methods and results in this analysis.

Here, it is convenient to use the (indefinite metric) Gupta–Bleuler formalism of quantum electrodynamics based on the (unphysical) local Dirac field $\psi$ and the local vector potential $A_\mu$. The existence of the corresponding renormalized Green’s functions has been established to all orders in perturbation theory by various methods to a by now satisfactory degree of rigour [11–13].

The problem of constructing physical charged fields and states in the Gupta–Bleuler formalism, however, requires further analysis. As first pointed out by Dirac, such fields can be obtained by formally multiplying $\psi$ with non–local operators which restore the local gauge invariance,

$$\psi(x) \exp (ie A(g_x)), \quad (3.1)$$

where

$$A(g_x) \equiv \int dy A^\mu(y) g_{x\mu}(y) \quad \text{and} \quad \partial^\mu g_{x\mu}(y) = -\delta(y - x). \quad (3.2)$$

The rigorous treatment of these expressions requires control both of infrared and of ultraviolet problems. The infrared problems appear because of the slow decay
of the “gauge fixing functions” \( g_{x \mu} \) and the ultraviolet problems are due to the singular nature of the products of field operators involved in the definition of the exponential \( \exp (ieA(g_\mu)) \) (in the sense of formal power series) and of \( \psi(x) \). These problems have been extensively discussed by Steinmann \[3\], cf. also \[14\], who established the existence of physical charged fields for a large class of gauge fixing functions within the framework of perturbation theory. We will rely here on these results and manipulate formal expressions such as (3.1) freely, without going into the subtle details of their precise definition.

In our analysis we also make use of the following general properties of the Gupta–Bleuler formulation of quantum electrodynamics which have been established in perturbation theory. In order to keep the notation simple, we deal in the following with the unregularized fields \( \psi(x), A_\mu(y) \) etc. The subsequent statements are thus to be understood in the sense of distributions.

1) The Wightman functions (vacuum expectation values) of the renormalized fields \( \psi, A_\mu \) exist as tempered distributions satisfying locality, Poincaré covariance and the spectrum condition \[11, 15\].

2) The field \( \partial_\mu A^\mu \) is the generator of c–number gauge transformations in the sense that \[14, 16\]

\[
[\partial_\mu A^\mu(z), \psi(x)] = eD(z - x) \psi(x),
\]

\[
[\partial_\mu A^\mu(z), A_\nu(y)] = -i \partial_\nu D(z - y),
\]

where \( D \) is the Pauli–Jordan distribution. The latter equation implies

\[
[\partial_\mu A^\mu(z), \exp (ieA(g))] = e \int dy D(z - y) \partial^\mu g_\mu(y) \cdot \exp (ieA(g))
\]

for arbitrary \( g_\mu \).

3) Polynomials in the fields \( \psi(x), A_\mu(y) \) commuting with \( \partial_\mu A^\mu(z) \) are elements of the algebra of observables and have vacuum expectation values satisfying Wightman positivity in the sense of formal power series in \( e \), cf. \[17\]. As a consequence, these expectation values have the same cluster properties as in a positive metric Wightman field theory.

We begin our analysis by explaining how one can proceed from the charged fields (3.1) to unitary charge carrying operators. This step relies on the following two observations.

First, as shown in the appendix, for given \( R, T \) and \( x \) varying in the bounded region \( R = \{ y : |y| < R, |y_0| < T \} \) there exist gauge fixing functions \( g_{x \mu} \) decomposable as \( g_{x \mu} = g_{x \mu}^I + g_{x \mu}^{II} \), where \( g_{x \mu}^I, g_{x \mu}^{II} \) have the following specific properties. The functions \( y \to g_{x \mu}^I(y) \) have compact support in the region \( |y| < 4R, |y_0| < T \) and are, for \( |y| < R \), given by

\[
g^I_{x,0}(y) \equiv 0, \quad g^I_{x,i}(y) \equiv -(4\pi)^{-1} \delta(y_0 - x_0)(y_i - x_i)|y - x|^{-3}, \quad i = 1, 2, 3.
\]

Thus they coincide in the latter region with the gauge fixing functions considered by Steinmann \[3\]. The functions \( y \to g_{x \mu}^{II}(y) \), on the other hand, are given by

\[
g^{II}_0(y) \equiv 0, \quad g^{II}_i(y) \equiv -(4\pi)^{-1} y_i |y|^{-3} I(y) h(y_0), \quad i = 1, 2, 3,
\]

Further, as shown in the appendix, for given \( R, T \) and \( x \) varying in the bounded region \( R = \{ y : |y| < R, |y_0| < T \} \) there exist gauge fixing functions \( g_{x \mu} \) decomposable as \( g_{x \mu} = g_{x \mu}^I + g_{x \mu}^{II} \), where \( g_{x \mu}^I, g_{x \mu}^{II} \) have the following specific properties. The functions \( y \to g_{x \mu}^I(y) \) have compact support in the region \( |y| < 4R, |y_0| < T \) and are, for \( |y| < R \), given by

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g^{II}_0(y) \equiv 0, \quad g^{II}_i(y) \equiv -(4\pi)^{-1} y_i |y|^{-3} I(y) h(y_0), \quad i = 1, 2, 3,
\]
where \( h \) is a test function with support in \((-T, T)\), \( \int dy_0 h(y_0) = 1 \), and \( l \) is a smooth function which is equal to 0 for \(|y| < 3R\) and equal to 1 for \(|y| > 4R\). Thus the functions \( g^H_\mu \) do not depend on the choice of \( x \) within the above limitations.

Second, it follows from the properties of \( g_{x\mu} \) and the commutation relations (3.3), (3.5) that the slightly modified charged fields

\[
\psi(x) \exp(ieA(g^I_x)) \exp(ieA(g^H)), \quad x \in \mathcal{R}, \tag{3.8}
\]

commute with \( \partial_\mu A^\mu(z) \) and consequently create charged physical states from the vacuum. The advantage of these fields is that the non–local effects, needed to describe the asymptotic Coulomb field, are clearly separated from the local effects encoded in \( \psi \).

We proceed from the fields (3.8) to (unbounded) operators by integrating them with test functions \( f \) with support in \( \mathcal{R} \). As \( g^H_\mu \) does not depend on \( x \), this integration affects only the first two factors in the product (3.8). Moreover, since the functions \( g^I_{x\mu} \) are well–behaved extensions of the gauge fixing functions (3.5) considered by Steinmann [6], the ultraviolet problems involved in defining the product could be controlled by similar methods. We therefore anticipate that the expression

\[
\chi(f) \equiv \int dx f(x) \psi(x) \exp(ieA(g^I_x)) \tag{3.9}
\]

is meaningful and defines a closable operator which is localized in some bounded space–time region fixed by the support properties of \( f \) and \( g^I_{x\mu} \).

Making use of the commutation relations (3.3), (3.5) and the specific properties of \( g_{x\mu} = g^I_{x\mu} + g^H_\mu \), one finds that the local operators \( \chi(f)^* \chi(f) \) and \( \chi(f) \chi(f)^* \) commute with \( \partial_\mu A^\mu(z) \) and therefore are observables. Skipping some technical details, it follows that the partial isometry \( V_f \) appearing in the polar decomposition \( \chi(f) = V_f |\chi(f)| \) is a local operator transforming under gauge transformations in the same way as \( \chi(f) \). Moreover, multiplying \( \chi(f) \) from the left and right with suitable elements of the algebra of local observables, one can always arrange that \( V_f \) be unitary [18].

The preceding arguments thus provide evidence to the effect that in quantum electrodynamics unitary operators of the form

\[
U_f \equiv V_f \exp(ieA(g^H)) \tag{3.10}
\]

exist which are invariant under local gauge transformations and carry electric charge. These operators have the interesting feature that the \( f \)–dependent contributions of the Dirac field \( \psi \) are completely absorbed in the local operators \( V_f \), whereas the properties of the corresponding asymptotic electromagnetic field are encoded in the \( f \)–independent operator \( \exp(ieA(g^H)) \). As we shall see, this clearcut separation of the local and asymptotic features of the charge carrying operators greatly simplifies the analysis of the corresponding charged states.

Keeping \( f \) fixed in the following, we define with the help of the unitaries \( U_f \) the maps

\[
C \to \rho_f(C) \equiv U_f^{-1} C U_f, \tag{3.11}
\]
where \( C \) are arbitrary local observables. If \( C \) is localized in the region \( O_r + \mathbf{x} \), where \( O_r \) denotes the double cone of radius \( r \) centred at 0 and \( \mathbf{x} \) is a sufficiently large spatial translation, then \( C \) and \( V_f \) commute by locality. Thus in this case

\[
\rho_f(C) = \exp(-i e A(g^{II})) C \exp(i e A(g^{II}))
\]

\[
= \sum_{k=0}^{\infty} \frac{(-i e)^k}{k!} \left( \text{Ad} A(g^{II}) \right)^k(C)
\]

(3.12)

holds in the sense of formal power series. In the following lemma we analyze the action of the approximants \( \rho^{(n)} \), given by the first \( n \) terms in this series, on observables \( C \) localized in spacelike asymptotic regions. As the fields are unbounded, our results are to be understood in the sense of sesquilinear forms on \( D \times D \), where \( D \) is the linear span of vectors obtained by applying locally regularized gauge–invariant polynomials \( C' \) in the fields to the vacuum vector \( \Omega \).

**Lemma 3.1** Let \( C \) be any local observable, let \( n \in \mathbb{N} \), and let \( t > 0 \). Then, for \( |x_0| < t \) and large \( |x| \),

\[
\rho^{(n)}(C(x)) = C(x) + i e a_C(x) \cdot 1 + R_C(x)
\]

(3.13)
on \( D \times D \). Here

\[
a_C(x) \equiv (4\pi)^{-1} \int dz |z|^{-3} h(z_0) (\Omega, [A^i(z), C(x)], \Omega)
\]

(3.14)

with \( h \) as in (3.7), and the matrix elements of the remainder \( R_C(x) \) decrease at least like \( |x|^{-4} \), uniformly in \( |x_0| < t \).

**Proof.** The leading term \( C(x) \) in the asymptotic expansion of \( \rho^{(n)}(C(x)) \) corresponds to the \( k = 0 \) contribution to the series (3.12). The next term is given by the vacuum expectation value of the \( k = 1 \) contribution which has the form

\[
-ie \int dz g^H_\mu(z) (\Omega, [A^\mu(z), C(x)] \Omega).
\]

Plugging into the integral the expression given in (3.7) and taking locality into account, one obtains, for sufficiently large translations \( \mathbf{x} \), the function \( i e a_C \) appearing in the statement.

The proof that the remainder

\[
R_C(x) = \rho^{(n)}(C(x)) - C(x) - i e a_C(x) \cdot 1
\]

has the stated decay properties requires more work. We begin by noting that if \( C \) is localized in the double cone \( O_r \), the multiple commutators \( (\text{Ad} A(g^{II}))^k(C(x)) \) contributing to \( \rho^{(n)}(C(x)) \) are localized in \( O_{r+kT+|x_0|} + \mathbf{x} \), as a consequence of the support properties of \( g^H_\mu \) and locality. So, in view of the spacelike commutativity of local gauge–invariant polynomials in the fields, it suffices to establish the
asymptotic decay properties of matrix elements of $R_C(x)$ between vectors of the form $C'\Omega$ and the vacuum $\Omega$. Next, we introduce the notation

$$A_h^j(z) \equiv \int dz_0 h(z_0) A(z, z_0)$$

and recall that, as a consequence of temperedness and the spectrum condition, it suffices to regularize the fields in the time variable in order to obtain operators depending smoothly on the spatial variables on their natural domain of definition $[19]$. Now for large $x$ as above, the contribution arising from the $k = 1$ term in $R_C(x)$ has the form

$$(C_0', \Omega, [A(g'H), C(x)] \Omega)$$

$$= (4\pi)^{-1} \int dz_0 \frac{dz}{|z|^{-3}} (C_0', [A_h^j(z), C(x)] \Omega)$$

$$= (4\pi)^{-1} \int d\mathbf{z} (z_j + x_j)|z + x|^{-3} (C_0', [A_h^j(z + x), C(x)] \Omega),$$

apart from a factor $ie$. Here $C'$ has been replaced by $C'_0 \equiv C' - (\Omega, C'\Omega) \cdot 1$ since the vacuum expectation value of the commutator has been subtracted in $R_C(x)$. The second equality is obtained by substituting $z \rightarrow z + x$, which is legitimate in the present setting since the matrix element under the integral is continuous in all variables.

Because of locality, the latter integral extends over a bounded region $\mathcal{K} \subset \mathbb{R}^3$ which can be held fixed for $|x_0| < t$ and $\mathbf{x} \in \mathbb{R}^3$. Moreover, for $|x_0| < t$ and $\mathbf{z} \in \mathbb{R}^3$ the operators $[A_h^j(z), C(x_0)]$ are localized in the fixed double cone $\mathcal{O}_{r + T + t}$ and are gauge invariant, like $C''_0$. So we can apply the Araki–Hepp–Ruelle cluster theorem [20] to their vacuum expectation values, cf. the properties of the Gupta–Bleuler formalism stated above. Thus

$$|(C_0', [A_h^j(z + x), C(x)] \Omega)| < c_j |x|^2,$$

uniformly in $|x_0| < t$ and $\mathbf{z} \in \mathbb{R}^3$. Combining this estimate with the preceding information, we obtain the bound

$$|(C_0', [A(g'H), C(x)] \Omega)| \leq c_j' \int_{\mathcal{K}} d\mathbf{z} |z_j + x_j| |z + x|^{-3} |x|^{-2} \leq c'' |x|^{-4}.$$

The higher order terms ($k \geq 2$) can be treated similarly. In fact, for $|x_0| < t$, $\mathbf{x} \in \mathbb{R}^3$ we have

$$|(C', [A(g'H), \ldots A(g'H), C(x)] \ldots \Omega)|$$

$$\leq (4\pi)^{-k} \int_{\mathcal{K}} d\mathbf{z} |z_j + x_j| |z + x|^{-3} \ldots \int_{\mathcal{K}} d\mathbf{z} |z_k + x_k| |z_k + x|^{-3} \times$$

$$\times |(C', [A_h^{j_1}(z_1 + x), \ldots, A_h^{j_k}(z_k + x), C(x)] \ldots \Omega)|,$$
where $\mathcal{K} \subset \mathbb{R}^3$ is some fixed compact set. The multiple commutator function is bounded in $z_1, \ldots, z_k \in \mathbb{R}^3$, uniformly in $|x_0| < t$ and $x \in \mathbb{R}^3$. Thus the integral is bounded by $c \cdot |x|^{-2k}$, completing the proof of the statement. □

It is important here that the form of the leading terms of the asymptotic expansion given in the preceding lemma does not depend on the order $n > 1$ of the approximants $\rho^{(n)}$ of the map $\rho_f$. This fact allows us to establish the following statement on the spacelike asymptotic properties of the charged states $U_f \Omega$.

**Proposition 3.2** Let $C$ be any local observable and let $t > 0$. Then for large $|x|$ in any order of perturbation theory,

\[
(U_f \Omega, C(x) U_f \Omega) = (\Omega, \rho_f(C(x)) \Omega) + i(e/4\pi) x_j |x|^{-3} \int dz \, h(z_0)(\Omega, [A_j(z), C(x)] \Omega)
\]

apart from terms which decrease at least like $|x|^{-3}$, uniformly in $|x_0| < t$.

**Proof.** As $(U_f \Omega, C(x) U_f \Omega) = (\Omega, \rho_f(C(x)) \Omega)$ and $\rho_f$ can be replaced in any given order of perturbation theory by $\rho^{(n)}$ for sufficiently large $n$, the statement follows from the preceding lemma by extracting the asymptotically leading contribution in $|x|$ from the function $a_C$ appearing there. □

Thus in the charged states, besides the leading vacuum contribution of the observable $C$, a term appears behaving asymptotically like the Coulomb field, whenever the corresponding integral does not vanish. In view of the commutator appearing in this expression, we call this sub–leading contribution the asymptotic quantum correction, for short.

It is instructive to study the form of this contribution for specific observables. Making use of locality, Lorentz covariance and the spectrum condition, it follows that the commutator function of the vector potential has the form

\[
(\Omega, [A_\mu(u), A_\nu(v)] \Omega) = -ig_{\mu\nu}K(u - v) - i\partial_\mu \partial_\nu L(u - v),
\]

where $K$, $L$ are causal, Lorentz invariant distributions whose Fourier transforms have supports in the forward and backward light cones. Hence the expectation value of the electric field in the charged states has the asymptotic form

\[
(U_f \Omega, E(x) U_f \Omega) = -(e/4\pi) x |x|^{-3} \int dz \, h(z_0 + x_0) \partial_0 K(z),
\]

in agreement with the expected Coulomb behaviour. But, in contrast to the situation in classical field theory, the Coulomb field is modulated by an additional time dependent factor: only if $K$ is equal to the massless Pauli–Jordan commutator function, i.e. in zeroth order perturbation theory, does this factor equal 1. Higher order (loop) corrections induce an additional oscillatory behaviour which may be attributed to vacuum polarization (i.e. quantum) effects which interfere with
the asymptotic Coulomb–like contributions of the states $U_f\Omega$. We emphasize that these oscillations are not in conflict with the fact that the charge of the underlying state is equal to $e$. Determining the total charge of a state from the expectation values of the charge density requires in general not only an integration over all of space but also a suitable mean over time [21]. Such a procedure also works for electrically charged states [22]. It is in fact easily checked that, as a consequence of Eqs. (3.4) and (3.16), the mean of the time dependent factor in relation (3.17) is equal to 1, in agreement with the charge content of the underlying state.

If $C$ is the sum of spatial derivatives of local operators, the integral appearing in the asymptotic quantum correction vanishes. As $j_0 = \text{div}E$ and $B = \text{curl}A$, we conclude that the matrix elements of the charge density and of the magnetic field decrease at least like $|x|^{-3}$. As a matter of fact, a more refined analysis shows that they behave like $|x|^{-6}$. Moreover, since the vacuum expectation values of triple products of the electromagnetic field vanish, as a consequence of the charge conjugation symmetry, the mean square fluctuations of the charge density in the charged states coincide asymptotically with those in the vacuum, up to contributions which decrease like $|x|^{-4}$. So, in this sense, the charged states have localization properties with respect to these observables coming close to those in the classical theory.

A first clear deviation from the classical situation appears in the case of the spatial components of the current. Whereas classically these components have the same support properties as the charge density, one gets in the quantum case

$$(U_f\Omega, j(x) U_f\Omega) = (e/4\pi) x |x|^{-3} \int dz h(z_0 + x_0) \partial^2_0 K(z)$$

(3.18)
as the leading contribution. So these expectation values decrease asymptotically no faster than the Coulomb field. This result is related to the temporal oscillations of the electric field, mentioned above, and shows that quantum effects lead to a substantial asymptotic delocalization.

It is a priori not clear whether quantum corrections and a corresponding asymptotic Coulomb like behaviour can also appear for higher moments of the better behaved observables, such as the charge density and the magnetic field. As a matter of fact, in spite of the restriction on the energy–momentum transfer of the vector potential by the spacetime integration in the asymptotic expansion given in Proposition 3.2, these expressions are not controlled by general low energy theorems [23]. The question can be decided, however, by perturbative calculations where one finds that, at one loop level, the asymptotic quantum corrections do not vanish for the triple product $C = j_0(x_1)j_0(x_2)j_0(x_3)$, even after averaging over time. We are indebted to O. Steinmann and O. Tarasov for communicating to us the results of these perturbative calculations.

Recalling that for full information on a quantum observable all of its moments are needed, these results show that the charged states have no better localization properties with respect to measurements of the charge density than the Coulomb field (although the amplitudes of the delocalizing terms are suppressed by powers of the fine structure constant). Thus a meaningful separation between the charge...
and field support of these states is impossible, in contrast to the simple model considered in [2].

Instead of reproducing here the preceding statements about the asymptotic quantum corrections of higher moments of certain specific observables by explicit computations, we sketch a quite general related result. Namely, given any subalgebra $C$ of local observables, stable under translations and irreducible in the vacuum sector, we shall show that the quantum corrections cannot vanish for all elements of $C$. In particular, the algebra generated by the charge density and the magnetic field can be shown to satisfy these conditions. So there must be polynomials in these observables giving rise to non-trivial quantum corrections, in accordance with the computational results.

For the proof of the above statement, we consider the maps $\delta_j, j = 1, 2, 3$, from $C$ into the algebra of all local observables, given by

$$\delta_j(C) \equiv i \int dz h(z_0) [E_j(z), C], \quad C \in C. \quad (3.19)$$

Because of locality, these maps are well defined. Assuming that the quantum corrections of all elements of $C$ vanish yields (since $C$ is invariant under translations)

$$0 = \int dz h(z_0) (\Omega, [A_j(z), \partial_0 C] \Omega) = - \int dz h(z_0) (\Omega, [\partial_0 A_j(z), C] \Omega)$$
$$= - \int dz h(z_0) (\Omega, [E_j(z), C] \Omega), \quad (3.20)$$

where the third equality is a consequence of the fact that $\partial_j A_0(z)$ does not contribute to the integral because of locality. Thus

$$(\Omega, \delta_j(C) \Omega) = 0, \quad C \in C, \quad (3.21)$$

so one can consistently define Hermitian operators $Q_j$ in the vacuum sector, setting

$$Q_j C\Omega \equiv \delta_j(C) \Omega, \quad C \in C. \quad (3.22)$$

Moreover, it follows from the generalization of a famous result of Coleman, cf. [24], that these operators are (combinations of) constants of motion. Thus we conclude that the components $E_j$ of the electric field are subject to a conservation law. This is indeed so for the free electromagnetic field, where $\partial^\nu F_{\nu\mu} = 0$, but clearly not so in quantum electrodynamics. So non-trivial asymptotic quantum corrections inevitably appear in this case for some elements of $C$.

We conclude this section by noting that the existence of some subalgebra $C$ of observables where all quantum corrections vanish would be a prerequisite for applying the methods outlined in [2]. So an analysis of the superselection structure and statistics of the electrically charged states considered here cannot be carried out along these lines.

4 Conclusions

The quantum delocalization of the electric charge, established in the present investigation, is due to a combination of quantum effects and the influences of
interaction. The subtle interplay between these ingredients causes a Coulomb-like spreading of (higher moments of) the charge density, present neither in the interacting classical theory nor in quantum field theory if the interaction is turned off. In fact, the amplitudes of these long range contributions are suppressed by powers of the fine structure constant and consequently are extremely small. It may therefore be difficult, if not impossible to establish their existence experimentally.

On the theoretical side, however, this delocalization of the charge means that the notion of charge support is fraught with conceptual difficulties. For the qualitative picture of a well-localized charge distribution does not have a clearcut mathematical counterpart. This fact gives rise to complications in the discussion of the statistics and fusion structure of electrically charged states, where localization properties matter. In particular, the general methods outlined in [2] cannot be applied in this case.

Although it seems impossible to discriminate the electric charge and field support, the coarser notion of causal support is still meaningful. We recall that a state is causally supported in a region if it can be generated from the vacuum by some physical isometric operation localized there. The causal support of states carrying electric charge is clearly non-compact, but it can be confined to an arbitrary spatial cone. For there are charged physical fields with gauge fixing functions having support in such cones [1, 23] and one can proceed from them to corresponding charged isometries by a process of polar decomposition, described in Section 3.

As pointed out in [23], such cone-like localized operators could be the starting point for a systematic analysis of the statistics and fusion structure of the superselection sectors in theories with electromagnetic forces. This would require, however, a better understanding of the relation between operators localized in different cones. In particular, it would be necessary to show that these operators are related by suitable limits of local observables which merely describe different configurations of low energy photons, in analogy to the situation discussed in [26]. It should be possible to provide evidence to this effect by perturbative methods similar to those used in the present investigation.

**Appendix**

In this appendix we establish the existence of certain specific (generalized) solutions $y \rightarrow g_{\mu}(y)$ of the equation

$$\partial^\mu g_{\mu}(y) = -\delta(y - x),$$  \hspace{1cm} (A.1)

where $x = (x, x_0)$ lies in the bounded region $|x| < R, |x_0| < T$. To avoid overburdening the notation, we will not indicate the dependence of these functions on $x$ in the following.

The desired solutions can be decomposed into $g_{\mu} = g^{I}_{\mu} + g^{II}_{\mu}$, where $g^{I}_{\mu}, g^{II}_{\mu}$ have the following specific properties. The function $g^{I}_{\mu}$ has compact support in the region $|y| < 4R, |y_0| < T$, for $x$ varying within the above limitations, and is
given by

\[ g_0^I(y) = 0, \quad g_i^I(y) = -(4\pi)^{-1} (y_i - x_i) |y - x|^{-3} \delta(y_0 - x_0), \quad i = 1, 2, 3 \] (A.2)

for \( y \in \mathcal{R}_I \equiv \{ z : |z| < R \} \). The function \( g^H_\mu \) does not depend on the choice of \( x \), vanishes for \( |y| < 3R \), and is, for \( y \in \mathcal{R}_H \equiv \{ z : |z| > 4R \} \), given by

\[ g_0^H(y) = 0, \quad g_i^H(y) = -(4\pi)^{-1} y_i |y|^{-3} h(y_0), \quad i = 1, 2, 3, \] (A.3)

where \( h \) is a test function with support in \( I_T \equiv (-T, T) \) and \( \int dy_0 h(y_0) = 1 \). Note that \( g^I_\mu \), \( g^H_\mu \) are local solutions of equation (A.1) in the regions \( \mathcal{R}_I \) and \( \mathcal{R}_H \), respectively.

For the proof that a corresponding global interpolating solution exists, we make an ansatz of the form

\[ g_i(y, y_0) = -f_i(y) k(y, y_0), \quad i = 1, 2, 3. \] (A.4)

With this ansatz, Eq. (A.1) is clearly satisfied if

\[ \partial^j f_i(y) = \delta(y - x), \] (A.5)

\[ \delta(y - x) k(y) = \delta(y - x) \] (A.6)

and \( g_0 \) is a solution of

\[ \partial^0 g_0(y) = f_i(y) \partial^j k(y, y_0). \] (A.7)

If \( k \) is chosen to have compact support in \( I_T \) with respect to \( y_0 \) and

\[ \int dy_0 k(y, y_0) = 1 \quad \text{for all } y, \] (A.8)

the solutions \( g_\mu \) have compact support as well. In fact, integrating the right hand side of Eq. (A.7) with respect to \( y_0 \) yields 0 for all \( y \), hence Eq. (A.7) has a (unique) solution \( g_0 \) which has compact support in \( I_T \) with respect to \( y_0 \).

The given form of the functions \( g^I_\mu \), \( g^H_\mu \) in the regions \( \mathcal{R}_I \) and \( \mathcal{R}_H \), respectively, is consistent with the above ansatz. More concretely, \( g^I_\mu \) corresponds to the choice

\[ f_i^I(y) = (4\pi)^{-1} (y_i - x_i) |y - x|^{-3}, \quad k^I(y) = \delta(y_0 - x_0) \] (A.9)

in Eqs. (A.4), (A.7) and similarly \( g^H_\mu \) is fixed by

\[ f_i^H(y) = (4\pi)^{-1} y_i |y|^{-3}, \quad k^H(y) = h(y_0). \] (A.10)

To interpolate these local solutions and solve Eq. (A.1) on all of \( \mathbb{R}^4 \), we first construct functions \( f_i \) which satisfy Eq. (A.3) and coincide with \( f_i^I \) for \( |y| < R \) as well as with \( f_i^H \) for \( |y| > 3R \), \( i = 1, 2, 3 \). These functions are obtained as components of the electric field corresponding to the retarded solution of Maxwell’s equations for a suitable current. The corresponding charge density is given by

\[ j_0(z, z_0) \equiv \delta(z - x(z_0)), \] (A.11)
where \( z_0 \rightarrow x(z_0) \) is assumed to be a smooth function which is equal to 0 for \( z_0 < -3R \), equal to \( x \) for \( z_0 > -2R \), and which satisfies \( |dx(z_0)/dz_0| < 1 \). As \( |x| < R \), such a function exists. The spatial components of the current are given by

\[
j_i(z, z_0) \equiv \delta(z - x(z_0)) dx_i(z_0)/dz_0, \quad i = 1, 2, 3,
\]

so the resulting current \( j_\mu \) is conserved. The corresponding retarded potential is

\[
a_\mu(y) = \int dz \frac{|z - y|}{|z - y|}^{-1} j_\mu(z, y_0 - |z - y|).
\]

Taking into account the spacetime features of the current, a moment’s reflection shows that the associated electric field

\[
f_i(y) \equiv \partial_0 a_i(y, 0) - \partial_i a_0(y, 0), \quad i = 1, 2, 3,
\]

satisfies Eq. (A.3) and coincides with \( f_i^I \) and \( f_i^{II} \) in the regions \( |y| < R \) and \( |y| > 3R \), respectively.

It remains to show that there is a function \( y \rightarrow k(y) \) with properties specified above interpolating between \( k^I \) and \( k^{II} \). To this end we pick a smooth function \( y \rightarrow l(y) \) which is equal to 0 for \( |y| < 3R \) and equal to 1 for \( |y| > 4R \) and put

\[
k(y) \equiv (1 - l(y)) \delta(y_0 - x_0) + l(y) h(y_0)
\]

with \( h \) as above. Taking into account the restrictions on \( x \), it follows that \( k \) has support in \( I_T \) with respect to \( y_0 \) and satisfies Eqs. (A.6) and (A.8).

By making use of Eqs. (A.4) and (A.7) with \( f_i \) and \( l \) defined as above, we arrive at solutions \( g_\mu \) of (A.1). The corresponding functions \( g_\mu^I, g_\mu^{II} \) are given by

\[
g_0^I(y) \equiv -f_i(y) \partial^i l(y) (H(y_0) - \theta(y_0 - x_0)),
g_i(y) \equiv -f_i(y) (1 - l(y)) \delta(y_0 - x_0), \quad i = 1, 2, 3,
\]

where \( H \) is the primitive of \( h \) vanishing for \( y_0 < -T \), and

\[
g_0^{II}(y) \equiv 0, \quad g_i^{II}(y) \equiv -(4\pi)^{-1} y_i |y|^{-3} l(y) h(y_0), \quad i = 1, 2, 3.
\]

Bearing in mind the properties of the \( f_i \) and \( l \), it follows that these functions have all the desired features stated above.

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