Generalized Hartree Method: A Novel Non-perturbative Approximation Scheme for Interacting Quantum systems

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Abstract

A self-consistent, non-perturbative scheme of approximation is proposed for arbitrary interacting quantum systems by generalization of the Hartree method. The scheme consists in approximating the original interaction term $\lambda H_I$ by a suitable 'potential' $\lambda V(\phi)$ which satisfies the following two requirements: (i) the 'Hartree Hamiltonian' $H_o$ generated by $V(\phi)$ is exactly solvable i.e., the eigen states $|n>$ and the eigenvalues $E_n$ are known and (ii) the 'quantum averages' of the two are equal, i.e. $<n|H_I|n> = <n|V(\phi)|n>$ for arbitrary $n'$. The leading-order results for $|n>$ and $E_n$, which are already accurate, can be systematically improved further by the development of a 'Hartree-improved perturbation theory' (HIPT) with $H_o$ as the unperturbed part and the modified interaction: $\lambda H' \equiv \lambda (H_I - V)$ as the perturbation. The HIPT is assured of rapid convergence because of the 'Hartree condition': $<n|H'|n> = 0$. This is in contrast to the naive perturbation theory developed with the original interaction term $\lambda H_I$ chosen as the perturbation, which diverges even for infinitesimal $\lambda$! The structure of the Hartree vacuum is shown to be highly non-trivial. Application of the method to the anharmonic-and double-well quartic-oscillators, anharmonic- sextic and octic- oscillators leads to very accurate results for the energy levels. In case of $\lambda\phi^4$ quantum field theory, the method reproduces, in the leading order, the results of Gaussian approximation, which can be improved further by the HIPT. We study the vacuum structure, renormalisation and stability of the theory in GHA.
1. Formulation

Consider a generic Hamiltonian describing an arbitrary interacting quantum system:

\[ H = H_s + \lambda H_I(\phi), \]  

(1)

where \( H_s \) is exactly solvable, \( \lambda H_I(\phi) \) is the interaction and \( \lambda \) is the strength of interaction. The generalized Hartree approximation (GHA) to the above Hamiltonian is:

\[ H_o \equiv H_s + \lambda V(\phi) \]  

(2)

where \( V(\phi) \equiv '\text{Hartree potential}' \) (HP) \([1]\) is the approximation to the original interaction \( H_I(\phi) \) and is required to satisfy the following two conditions: (i) \( H_o \) (\( \equiv '\text{Hartree Hamiltonian}' \) (HH)) is exactly solvable, i.e.,

\[ H_o |n> = E_n |n>, \quad <m|n> = \delta_{mn}, \]  

(3)

with the eigen-spectrum known and (ii) the 'quantum average' (QA) of \( V \) equals that of \( H_I \), \([1]\) i.e,

\[ <n|V(\phi)|n> = <n|H_I(\phi)|n> \]  

(4)

(The quantum average of any operator \( <\hat{A}(\phi)> \) is henceforth denoted by the notation : \( <\hat{A}> \equiv <n|\hat{A}|n> \)). Equations (2)-(4) defining the General Hartree Approximation (GHA) are hence forward referred to as "Hartree Conditions (HC)". The self-consistency of the procedure is implicit in eqs.(2-4): the states \( |n> \) which are obtained as the solution of \( H_o \), \([1]\) i.e,

\[ <n|V(\phi)|n> = <n|H_I(\phi)|n> \]  

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Note that, because of eq.(4), the QA of \( H' \) vanishes:

\[ <n|H'|n> = 0 \]  

(5)

where \( H' \equiv H_I - V \).

This naturally suggests a scheme of improved perturbation theory (\( \equiv '\text{Hartree-improved perturbation theory}' \) (HIPT)) in which \( H_0 \) is used as the unperturbed part and \( \lambda H' \) is considered as the perturbation (see, below).
2. Applications: (Quantum Mechanics)

2. (a) The quartic - oscillator

Consider the Hamiltonian for the (quartic) anharmonic oscillator (AHO) and the double-well oscillator (DWO):

\[ H = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2 + \lambda \phi^4, \]  

(6)

where \( g > 0 \) \((< 0)\) refers to the cases of the AHO(DWO) respectively. These systems have been widely studied in the literature owing to their theoretical importance as well as practical applications. The free-field \( \phi(t) \) and the conjugate momentum \( p(t) \) are parametrised in the standard manner:

\[ \phi(t) = \sigma + \frac{(b + b^\dagger)}{\sqrt{2\omega}}, \quad p(t) = i \sqrt{\frac{\omega}{2}} (b^\dagger - b), \]  

(7)

satisfying the usual commutation relation \([b, b^\dagger] = 1\). Here, \( \omega, \sigma \) are real constants with \( \omega > 0 \) and \( \sigma \) has the significance of the vacuum expectation value (VEV) of \( \phi \): \( \sigma \equiv \langle \phi \rangle \).

It is to be noted that free-field operators defined by:

\[ \phi = \sigma + \frac{1}{\sqrt{2}}(a + a^\dagger), \quad p = \frac{i}{\sqrt{2}}(a^\dagger - a) \]  

(8)

satisfy identical commutation relation: \([a, a^\dagger] = 1\). Hence \((a, a^\dagger)\) and \((b, b^\dagger)\) must be related by Quantum canonical Transformation i.e.(Boguliobov-like Transformation). Denoting the eigen states of the number operator \(b^\dagger b\) as \(|n>\) we have the standard definition \(|n> = (b^\dagger)^n|vac>/\sqrt{n!}\) where \(b|vac>\equiv 0\); \(b^\dagger b|n> = n|n>\) and \(<m|n> = \delta_{mn}\).

It is to be noted that free field vacuum \(|0>\) is defined as: \(a|0>\equiv 0 \equiv <0|a^\dagger\). The QA are easily calculated: \(<\phi> = \sigma ; \quad <\phi^2> = \sigma^2 + \xi/\omega; \quad <\phi^3> = \sigma^3 + 3\sigma\xi/\omega\) and \(<\phi^4> = \sigma^4 + 6\sigma^2\xi/\omega + (3/(\omega)^2)((1 + 4\xi^2))\) where \(\xi \equiv n + 1/2\).

To set up the HH for the above system we propose the generic ansatz (valid for arbitrary anharmonicity):

\[ V(\phi) = A\phi^2 - B\phi + C \]  

(9)

which is amenable to immediate implementation of the HC. Self-consistency of the procedure is built in by imposing the constraint that the constants \(A,B,C\) are chosen as
suitable functions of \( < \phi^n > \). For the case of quartic anharmonicity, we determine them as follows:

\[
A = 6\sigma^2 + 3f(\xi)/\omega; \quad B = (1 + g) \sigma \omega^2/\lambda + 4 \omega^2 \sigma^3 + 12 \omega \sigma \xi;
\]
\[
C = < \phi^4 > - A < \phi^2 > + B < \phi >
\]

where \( f(\xi) = \xi + (1/4\xi) \). Note that, we have imposed the additional constraint of reproducing the variational "Gap-equation" and the "equation for the ground state" in the Gaussian-approximation [2], in determining the constants: A,B,C. With the choice above, the HH is reduced to the diagonalisable-structure, corresponding to a shifted-harmonic oscillator:

\[
H_0 = h_0 + \frac{1}{2} \Omega^2(\phi - \chi)^2 + \frac{1}{2} p^2,
\]

where \( \Omega^2 = 2\lambda A + g \), \( \chi = \lambda B/\Omega^2 \) and \( h_0 = \lambda C - \frac{1}{2} \Omega^2 \chi^2 \). However, demanding the consistency of the diagonalisation of \( H_0 \) to the defining ansatz eq.(8), requires that we make the identification:

\[
\omega^2 = \Omega^2; \quad \sigma^2 = \chi^2,
\]

which, in turn, imply the following constraints on \( \omega \) and \( \sigma \):

\[
\omega^3 - \omega (12\lambda \sigma^2 + g) - 6\lambda f(\xi) = 0
\]

\[
\sigma (4\lambda \sigma^2 + g + (12\lambda \xi/\omega)) = 0
\]

(Here inafter,eqs.(13 ) & (14 ) are referred to as the "gap-eqn (GE)" and the "equation for ground state" ( EGS ) respectively). (As stated earlier, \( g > 0 \) \( ( < 0) \) refers to the case of the AHO(DWO)). In the case of AHO the \( \sigma = 0 \), solution of eq(14 ) corresponds to the physical solution. This leads to the simplified GE, \( ( g = 1)\):

\[
\omega^3 - \omega - 6\lambda f(\xi) = 0.
\]

This gap-equation has been derived earlier by several authors [3 - 6 ] from a variety of different considerations. With the aid of eqs.(11) & (15), the energy-levels of \( H_0 \) are given by

\[
E_n = \frac{\xi}{4}(3\omega + \frac{1}{\omega}),
\]
where \( \omega' \) is the solution of eq.(15). The numerical results for \( E_n \) are given in Table – 1. The result is already accurate with error \( \sim 0.2 \% \) to \( 2 \% \) over the range of \( n' \) & \( \lambda' \) shown.

For the case of the DWO, there are two quantum phases: (i) \( 4\lambda\sigma^2 = -g - (12\lambda\xi/\omega) \), that leads to the spontaneously broken symmetry (SSB) phase and (ii) \( \sigma = 0 \), that leads to Symmetry-Restored phase (SR - phase) (see,eqn.( 14 ). The former exhibits the double-well structure of the ’effective-potential’(for which case, the VEV of the field is non-vanishing: \( \sigma^2 \neq 0 \) whereas the latter corresponds to the dynamical restoration of the single-well shape, with \( \sigma = 0 \).

The gap equations in respective phases are easily obtained : for the SSB phase ( see eq.(13) )

\[
\omega_a^3 + 2g \omega_a + 6\lambda p(\xi) = 0
\]

where \( p(\xi) \equiv (5\xi - 1/4\xi) \) with the physical solution given by :

\[
\omega_a = 2\sqrt{-\frac{2g}{3}} \cos \left[ \frac{\pi}{6} + \left( \frac{1}{3} \right) \sin^{-1} \left( \frac{\lambda}{\lambda_c} \right) \right]
\]

\[
\lambda_c(\xi) = (-2g/3)^{3/2}/3p(\xi)
\]

where \( \lambda_c = ” \text{critical coupling” } \). For the ground state ( \( n = 0, g= -1, \xi = 1/2 \) ) : \( \lambda_c(1/2) \approx 0.09007 \). This shows that the SSB-phase is realized for \( \lambda < \lambda_c \) and the SR phase is favoured for \( \lambda > \lambda_c \). The energy levels in SSB -phase are given by

\[
E_n^{SSB} = \frac{\xi}{4}(3\omega_a + \frac{2}{\omega_a}) - \frac{g^2}{16\lambda}
\]

For the SR phase:

\[
\omega_s^3 - g \omega_s - 6\lambda f(\xi) = 0
\]

The corresponding expression for the energy-levels, are given by :

\[
E_n^{SR} = (\xi/4)[3\omega_s + g/\omega_s]
\]

where \( \omega_s \) is the solution of eqn.(21) and \( \lambda > \lambda_c \) (see,Table-2).
2. (b) Cases of higher harmonicity

Other cases of anharmonicity are studied in analogous manner. For the anharmonic sextic-oscillator, the Hamiltonian is given by

\[ H = \frac{1}{2}p^2 + \frac{1}{2} g \phi^2 + \lambda \phi^6 \]  

(23)

where \( g > 0 \) (< 0) refers to the case of AHO (DWO) respectively. The Hartree-Hamiltonian for this case is chosen as:

\[ H_0 = \frac{1}{2}p^2 + \frac{1}{2} g \phi^2 + \lambda V(\phi) \]  

(24)

with the condition,

\[ < H > = < H_0 > \]  

(25)

Identical ansatz, eq.(9), for \( V(\phi) \) is assumed: \( V(\phi) = A \phi^2 - B\phi + C \). The constants are calculated in the analogous manner and given as:

\[ A = 15 \sigma^4 + 45\sigma^2(4\xi^2 + 1)/4\xi\omega + (15/8\omega^2)(4\xi^2 + 5); \]

\[ B = \sigma [(1 + g)\omega^2/\lambda + 6\omega^2\sigma^4 + 60\sigma^2\xi\omega + (45/4)(4\xi^2 + 1)]; \]

\[ C = < \phi^6 > - A < \phi^2 > + B < \phi > \]  

(26)

With the above choice the Hatree-Hamiltonian, \( H_0 \) is again reduced to the diagonalisable form with a shifted field \( \tilde{\phi} = \phi - \sigma \). The structure of HH is identical to eq.(11); together with eq.(12):

\[ H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2(\phi - \sigma)^2 + h_0 \]  

(27)

where \( \omega^2 = 2\lambda A + g, \sigma = \lambda B/\omega^2 \) and \( h_0 = \lambda C - \frac{1}{2}\omega^2\sigma^2 \). The 'gap -eqn' in this case is given by

\[ \omega^4 - \omega^2(g + 30\lambda\sigma^4) - 45\lambda(\sigma^2\omega/2\xi)(4\xi^2 + 1) - (15\lambda/4)(4\xi^2 + 5) = 0 \]  

(28)

whereas, the ground-state-configuration is governed by the following equation:

\[ \sigma [g + 6\lambda(\sigma^4 + 10\xi\sigma^2/\omega + 15(4\xi^2 + 1)/8\omega^2)] = 0. \]  

(29)
In the case of AHO, the $\sigma = 0$ solution of eqn. (29) corresponds to the physical solution. This leads to the simplified "gap-eqn."

$$\omega^4 - g\omega^2 - (15\lambda/4)(4\xi^2 + 5) = 0$$

By using eqs.(27) and(30) the energy levels of $H_0$ are given by

$$E_n = \frac{\xi}{3}(2\omega + \frac{g}{\omega})$$

where 'ω' is the solution of eq(30). The numerical results for $E_n$ are given in Table-3.

For the case of the octic-anharmonic oscillator the Hamiltonian and the HH of the system are given respectively by

$$H = \frac{1}{2}p^2 + \frac{1}{2}g\phi^2 + \lambda\phi^8 \tag{32}$$

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}g\phi^2 + \lambda V(\phi) \tag{33}$$

with identical ansatz for $V(\phi)$:

$$V(\phi) = A\phi^2 - B\phi + C.$$  The constants A,B,C are analogously determined:

$$A = 28\sigma^6 + 105\sigma^4(4\xi^2 + 1)/2\xi\omega + (105/2\omega^2)\sigma^2(4\xi^2 + 5) + 35h(\xi)/2\omega^3;$$

$$B = \sigma [(1 + g)\omega^2/\lambda + 8\omega^2\sigma^6 + 168\sigma^4\xi\omega + 105\sigma^2(4\xi^2 + 1) + 35\xi(4\xi^2 + 5)/\omega ];$$

$$C = <\phi^8> - A <\phi^2> + B <\phi> \tag{34}$$

where $h(\xi) = \xi^3 + (7\xi/2) + (9/16\xi)$.

The "gap-eqn" and eqn. for "ground-state" are as follows:

$$\omega^5 - \omega^3(g + 56\lambda\sigma^6) - 105\omega^2(\lambda\sigma^4/\xi)(4\xi^2 + 1) - 105\omega\lambda\sigma^2(4\xi^2 + 5) - 35\lambda h(\xi) = 0 \tag{35}$$

$$\sigma [g + \lambda(8\sigma^6 + 168\sigma^4(\xi/\omega) + 105\sigma^2(4\xi^2 + 1)/\omega^2 + 35\xi(4\xi^2 + 5)/\omega^3) ] \tag{36}$$

For the case of anharmonic-octic oscillator the solution $\sigma = 0$ is the physical solution. In this case the "gap-eqn" is simplified to

$$\omega^5 - g\omega^3 - 35\lambda h(\xi) = 0 \tag{37}$$
which leads to the energy-levels of $H_0$

$$E_n = \left(\frac{\xi}{8}\right)(5\omega + \frac{3g}{\omega})$$

(38)

with $\omega$ as the solution of eqn(37). The numerical results are given in Table-4, compared with earlier results [7].

The generality of the above method is thus apparent from the above examples of increasing anharmonicity. In each case, the interacting system has been effectively reduced to an exactly solvable system while preserving the inherent non-linearity (through the gap-eqn.& the Hartree-condition) of the interacting theory. The resulting energy-levels are uniformly accurate to within a few percent of the ‘exactly’ computed values even in the zeroth-order, which is chosen to reproduce the results of the gaussian-approximation [2]. Further physical significance of the GHA is obtained by studying the structure and stability of the Hartree-vacuum as discussed below.

3. Structure and significance of the Hartree-vacuum

Starting from the alternative expansion of the field in terms of the ‘free’-field operators,

$$\phi = \sigma + (a + a^\dagger) / \sqrt{2}; \quad p = (i / \sqrt{2})(a^\dagger - a), \quad [a, a^\dagger] = 1,$$

(39)

it follows (from eqs.(7)) that the two sets of operators must be related by a Boguliobov-type quantum canonical transformation:

$$b = a \cosh(\alpha) - a^\dagger \sinh(\alpha)$$
$$b^\dagger = a^\dagger \cosh(\alpha) - a \sinh(\alpha),$$

(40)

with the two vacua related by the transformation

$$|vac> = \exp[(1/2) \tanh(\alpha) (a^\dagger a^\dagger - aa)]|0>;$$
$$\alpha \equiv (1/2) \ln(1/\omega)$$

(41)

The set of equations: (40,41) imply a highly non-trivial structure of the Hartree-vacuum analogous to the case of the super-fluid ground state [8] and the hard-sphere-Bose gas [9]. In particular, the free-particle number-density in the Hartree-vacuum is non-zero
and depends strongly on the strength of interaction:

\[ n_0 \equiv \langle \text{vac}| a^\dagger a |\text{vac} \rangle = \sinh^2(\alpha) = (1/4)[\omega + (1/\omega) - 2] \] (42)

such that \( n_0 \sim \lambda^{1/3} \) for \( \lambda >> 1 \) whereas \( n_0 \to 0 \) for \( \lambda \to 0 \) (as expected). The above result also implies an alternative interpretation of \( \omega' \), i.e; \( u \equiv (1 - \omega)/(1 + \omega) \) measures the non-trivial structure \([10]\) of the Hartree-vacuum. Moreover, it can be shown \([10]\) that the free-field ground state, \( |0> \) is highly unstable compared to the Hartree-vacuum, \( |\text{vac}> \) signifying that the true vacuum is much better approximated by the latter.

4. Perturbative Improvement

An improved perturbation theory (HIPT) can be developed by treating the Hartree Hamiltonian \( H_0 \) as the unperturbed part and \( \lambda H' \) as the perturbation. The HIPT is, by construction (because of eq.(4)), guaranteed to be convergent, in contrast to the ordinary perturbation theory (wherein the entire interaction \( \lambda H_I \) is treated as the perturbation) which is divergent \([11-13]\) even for infinitesimal \( \lambda \). The first-order correction in HIPT vanishes and the 2\(^{nd}\) order -correction is given by the standard formula :

\[ \Delta E^{(2)}_n = \sum_{m \neq n} |\langle n|\lambda H'|n \rangle|^2/(E_n - E_m) \] (43)

Only a finite number of matrix-elements contribute to eq.(43) corresponding to \( m = n \pm 2, n \pm 4 \) for the case of the quartic AHO (DWO). Inclusion of this correction significantly improves the zeroth-order results, both in case of the AHO (Table-1) and the DWO (Table-2), over the full range of investigation: \( 0.1 \leq \lambda \leq 100 \) and \( 0 \leq n \leq 40 \).

5. \( \lambda \phi^4 \) - Quantum Field Theory

The above formalism is easily extended to \( \lambda \phi^4 \) quantum field theory described by the Lagrangian: \( \mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)(\partial^{\mu}\phi) - \frac{1}{2}m^2 \phi^2 - \lambda \phi^4 \) which leads to the Hamiltonian:

\[ \mathcal{H} = \frac{1}{2} \left[ (\partial \phi/\partial t)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right] + \lambda \phi^4 \] (44)

The field \( \phi \) is Fourier-expanded in terms of the 'interacting' Fock-space operators in the standard manner:

\[ \phi = \sigma + \int \frac{d^3k}{\Omega_k(M)} \left[ b(k) \exp(-ikx) + b^\dagger(k) \exp(ikx) \right] \] (45)
where, $\Omega_k(M) \equiv (2\pi)^3 2\sqrt{k^2 + M^2} = (2\pi)^3 2\omega_k(M)$, $[b(k), b^\dagger(q)] = \Omega_k(M) \delta^3(k - q)$, 

$M$ = bare mass of the physical particle and $|vac>$ is defined by $b(k)|vac> = 0$.

Similarly, the physical one-particle state is defined by $|k> \equiv b^\dagger(k)|vac>$. The Hartree-Hamiltonian is constructed in an analogous manner:

$$H_0 = \frac{1}{2} \left[ (\partial \phi / \partial t)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right] + \lambda V(\phi)$$

with $V(\phi)$ parameterised as before:

$$V(\phi) = A \phi^2 - B \phi + C$$

The various QA’s are now given by:

$< \phi^2 > = \sigma^2 + I_0, \quad < \phi^4 > = \sigma^4 + 6\sigma^2 I_0 + 3I_0^2, \quad < (\partial \phi / \partial t)^2 > = I_1, \quad < (\nabla \phi)^2 > = I_1 - M^2 I_0$, where $< \hat{A} > \equiv < vac|\hat{A}|vac>$ and

$$I_n = I_n(M^2) \equiv \int \frac{d^3k}{\Omega_k(M)} [\omega^2_k(M)]^n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

are the Stevenson-Integrals [14]. Applying the Hartree-condition, eq.(4), and ensuring equivalence to the GEP [14] leads, as before, to the complete determination of the parameters in $V(\phi)$:

$$A = 6 < \phi^2 >; \quad B = 8 \sigma^3 \quad \text{and} \quad C = 3\sigma^4 - 6\sigma^2 I_0 - 3I_0^2$$

By analogous procedure, the Hartree - Hamiltonian is then reduced to diagonal form (corresponding to an effectively free Klein-Gordon theory) by a shift of the field and energy:

$$H_0 = \frac{1}{2} \left[ (\partial \varphi / \partial t)^2 + (\nabla \varphi)^2 + M^2 \varphi^2 \right] + h_0$$

where $\varphi \equiv \phi - \sigma$ and $h_0 = -(1/2)M^2 \sigma^2 + \lambda C$, with ‘C’ given in eq.(49).

The ‘gap-equation’ is given by

$$M^2 = m^2 + 12\lambda \sigma^2 + 12\lambda I_0(M^2)$$

and the VEV now satisfies the consistency condition:

$$\sigma[M^2 - 8 \lambda \sigma^2] = 0$$
The physical solution of this equation for the vacuum-configuration is at $\sigma = 0$, as intuitively expected for the symmetric $\lambda \phi^4$ theory (i.e. with $m^2 > 0$) considered here. This is further verified by the computation of the effective potential and the renormalized parameters as shown below.

Employing the standard definition of the effective potential $U(\sigma)$: $\langle \text{vac} | H | \text{vac} \rangle = \langle \text{vac} | H_0 | \text{vac} \rangle \equiv U(\sigma)$, we have,

$$U(\sigma) = I_1 - 3\lambda I_0^2 + (1/2)m^2\sigma^2 + \lambda \sigma^4$$

(53)

It is important to note that, eqs. (51 - 53) coincide with the results based on the Gaussian Effective Potential (GEP) [14]. It is then straightforward to carry out the renormalisation program as has been done in ref. [14]. The renormalized parameters are given by the following expressions:

$$m_R^2 \equiv d^2U/d\sigma^2|_{\sigma=0} = m^2 + 12\lambda I_0(\bar{M}^2); \bar{M}^2 \equiv M^2(\sigma = 0)$$

(54)

$$\lambda_R \equiv (1/4!)d^4U/d\sigma^4|_{\sigma=0} = \lambda \left[ \frac{1 - 12\lambda I_{-1}}{1 + 6\lambda I_{-1}} \right].$$

(55)

It has been shown [14] in case of the GEP that a non-trivial version of $\lambda \phi^4$ theory emerges which cannot be realized in lattice theories (for any finite lattice-spacing). Identical conclusion holds for the GHA in the leading order, because of the equivalence established here.

Beyond proving the equivalence of the GHA to GEP in the leading order, the new results derived are:

(i) the vacuum-structure:

$$|\text{vac}>= \exp\{\frac{1}{2} \int \frac{d^3k}{\Omega_k(m)} \beta(k) [a^\dagger(k)a^\dagger(-k) - a(k)a(-k)]\}|0>$$

(56)

where $\beta(k)$ is analogous to $\alpha(k)$ (see, eqn.(40) with $\alpha \rightarrow \alpha(k)$). The "vacuum-structure-function" is denoted by $u(k)$ given by

$$u(k) = \omega_k(m)/\omega_k(M) = \left[ (k^2 + m^2)/(k^2 + M^2) \right]^{1/2}$$

(57)

The free-particle number-density in the Hartree-vacuum is given by ($V$ = spatial volume)

$$n(k) \equiv \langle \text{vac} | a^\dagger(k)a(k) | \text{vac} \rangle / V \Omega_k(m)$$

(58)
One then calculates $\rho(k)$ as defined below

$$\lim_{\Lambda \to \infty} n(k)/n(0) \equiv \rho(k) = [1 + (k^2/m^2_R)]^{-1/2}$$

(59)

where $n(0) = (m/m_R)/(32\pi^3)$ is the maximum value of $n(k)$. Eqs. (57-58) show the condensate-structure of the physical vacuum consisting of correlated off-shell particle pairs (with momenta $k$ and $-k$) with a non-trivial dependence on $|k|$. This condensate-structure is expected to play significant role in the thermodynamic-properties of the system, if the behaviour persists to non-zero value of temperature, $T$.

The static-potential $U(r)$ of the $\lambda\varphi^4$ theory is easily calculable from the 2-particle correlation function $U(x - y)$ defined as:

$$U(x - y) \equiv \langle \text{vac} | \varphi(x,0)\varphi(y,0)|\text{vac} \rangle / \langle \text{vac} | \text{vac} \rangle$$

(60)

which leads to

$$U(r) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \exp(ik.r)(k^2 + m^2_R)^{-1/2}$$

(61)

where $r \equiv |x - y|$. The above integral can be related to the modified Bessel-function $K_1(m_Rr)$ and one obtains:

$$U(r) = m_RK_1(m_Rr)/4\pi^2r,$$

(62)

where, $r \equiv |r|$. Note that $U(r)$ diverges in the limit $r \to 0$, as it should (in any local quantum field-theory) and behaves asymptotically as

$$\lim_{r \to \infty} U(r) \sim r^{-3/2}\exp(-m_Rr)$$

(63)

which shows a fall-off faster than the Yukawa-potential! To our knowledge, the important information (contained in Eqs. (60 - 63)) regarding the static-potential of the symmetric $\lambda\varphi^4$ theory are new results.

6. Summary and Conclusion

The generalised Hartree approximation is, in principle, applicable to arbitrary quantum-systems with interaction. The method consists in mapping the interacting system to an exactly solvable system while preserving the essential non-linearity and the major effects
of the interaction of the original system through the self-consistent feed-back mechanism. The basic approximation, which is non-perturbative in character, lends itself to be systematically improved by modified perturbation theory developed about the Hartree-vacuum and the Hartree-Hamiltonian chosen as the unperturbed part and the modified interaction term $\lambda H'$ taken as the perturbation. This modified perturbation theory can be shown to be rapidly convergent in contrast to the divergence of the naive perturbation theory developed about the free-field vacuum with the entire interaction term $\lambda H_1$ chosen as perturbation. We conjecture that the instability of the free-field vacuum and the divergence of perturbation theory developed about this vacuum may be intimately related. The structure of the Hartree-vacuum emerges to be highly non-trivial with a strong dependence on $\lambda$ analogous to the case of the ground states of superfluid Helium, hard-sphere Bose-gas etc.

The method and its perturbative improvement applied to the case of the quartic, sextic and the octic anharmonic - oscillator and the quartic double-well oscillator leads to excellent results over the entire allowed range of the interaction strength $\lambda$ and excitation energy level $'n'$. The generalisation to the case of $\lambda \phi^4$ field theory reproduces, in the leading order, the results derived from the Gaussian effective potential [14]. Going beyond the leading order through the HIPT, the results of the Gaussian approximation can be systematically improved. The structure of the Hartree-vacuum again emerges quite non-trivial and interesting, being characterized by the condensation of off-shell, correlated particle-pairs and leading to the definition of a 'vacuum - structure function'. It has also been established elsewhere [10] that the effective potential based upon the free-field ('perturbative') vacuum leads either to instability or to triviality. We also derive the inter-particle (static) potential by evaluating the two-particle correlation-function in the Hartree vacuum and show that the potential falls faster than the Yukawa potential! The generalization to finite temperature appears straight-forward.

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Table 1: Sample results for the zeroth-order (GHA) compared with results of earlier calculations from ref.[3] (shown in parentheses), over a wide range of 'λ' and 'n'. Also shown are the results (in square brackets) of the Hartree-improved perturbation theory (HIPT) up to second order (see text).

| λ   | $E_0$  | $E_1$  | $E_2$  | $E_4$  | $E_{10}$ | $E_{40}$ |
|-----|--------|--------|--------|--------|----------|----------|
| 0.1 | 0.56031| 1.7734 | 3.1382 | 6.2052 | 17.2267  | 94.84    |
|     | (0.55915) | (1.7695) | (3.1386) | (6.2203) | (17.352) | (90.56)  |
|     | [0.55911] | [1.7694] | [3.1391] | [6.2239] | [17.374] | [95.766] |
| 1.0 | 0.81250| 2.7599 | 5.1724 | 10.902 | 32.663   | 192.79   |
|     | (0.80377) | (2.7379) | (5.1792) | (10.902) | (32.963) | (194.60) |
|     | [0.80321] | [2.7367] | [5.1824] | [10.982] | [33.013] | [195.15] |
| 10.0| 1.5313 | 5.3821 | 10.3240| 22.248 | 68.177   | 409.89   |
|     | (1.5050) | (5.3216) | (10.3471) | (22.409) | (68.804) | (413.94) |
|     | [1.5030] | [5.3177] | [10.356] | [22.457] | [68.996] | [415.18] |
| 100.0| 3.1924 | 11.325 | 21.853 | 47.349 | 145.843  | 880.55   |
|     | (3.1314) | (11.187) | (21.907) | (47.707) | (147.231) | (889.32) |
|     | [3.1266] | [11.178] | [21.927] | [47.817] | [147.652] | [892.03] |
| 1000.0| 6.8280 | 24.272 | 46.902 | 101.742 | 313.720  | 1895.90  |
|     | (6.6942) | (23.972) | (47.017) | (102.516) | (−−) | (−−) |
|     | [6.6836] | [23.952] | [47.062] | [102.75] | [317.65] | [1920.70] |
**Table- 2:** The computed energy levels of the quartic - DWO in the zeroth order of GHA compared with the earlier calculations including twenty - orders of perturbation theory [3] shown for sample values of $'\lambda'$ and $'n'$. Also shown are the results after inclusion of second order correction in HIPT, denoted as $E_n^{(2)}$.

| $\lambda$ | $n$ | $E_n^{(0)}$ | $E_n^{(2)}$ | Ref [3] |
|-----------|-----|-------------|-------------|--------|
| 0.1       | 0   | 0.5496      | 0.4606      | 0.4702 |
|           | 1   | 0.8430      | 0.7553      | 0.7703 |
|           | 2   | 1.5636      | 1.6547      | 1.6300 |
|           | 4   | 3.5805      | 3.7232      | 3.6802 |
|           | 10  | 12.192      | 12.517      | 12.400 |
| 1.0       | 0   | 0.5989      | 0.5752      | 0.5800 |
|           | 1   | 2.1250      | 2.0800      | 2.1800 |
|           | 2   | 4.2324      | 4.2600      | 4.2500 |
|           | 4   | 9.4680      | 9.5950      | 9.5600 |
|           | 10  | 30.530      | 30.650      | 30.420 |
| 10.0      | 0   | 1.4098      | 1.3752      | 1.3800 |
|           | 1   | 5.0650      | 4.9910      | 5.0900 |
|           | 2   | 9.8660      | 9.9050      | 9.8900 |
|           | 4   | 21.561      | 21.791      | 21.700 |
|           | 10  | 66.950      | 67.820      | 67.620 |
| 100.0     | 0   | 3.1340      | 3.0650      | 3.0700 |
|           | 1   | 11.175      | 11.024      | 11.002 |
|           | 2   | 21.638      | 21.715      | 21.700 |
|           | 4   | 47.023      | 47.505      | 47.200 |
|           | 10  | 145.27      | 147.10      | 146.70 |
Table- 3: Sample results for the zeroth - order (GHA) for the sextic - AHO compared with results of earlier calculations from ref.[ 7 ] (shown in parentheses), over a wide range of 'λ' and 'n'. Percentage of error is shown in square bracket.

| $\beta$ | $E_0$ | $E_1$ | $E_2$ | $E_4$ | $E_6$ | $E_{10}$ | $E_{14}$ | $E_{17}$ |
|---------|-------|-------|-------|-------|-------|----------|----------|---------|
| 0.2     | 1.193 | 3.966 | 7.420 | 16.15 | 26.88 | 53.24    | 85.01    | 111.9   |
|         | (1.174) | (3.901) | (7.382) | (16.30) | (27.29) | (54.31)  | (86.78)  | (114.0) |
| 2.0     | 1.676 | 5.931 | 11.61 | 26.48 | 45.08 | 91.17    | 147.0    | 194.4   |
|         | (1.610) | (5.749) | (11.54) | (26.83) | (45.94) | (93.26)  | (150.4)  | (198.3) |
| 10.0    | 2.323 | 8.420 | 16.74 | 38.73 | 66.36 | 135.0    | 218.3    | 289.0   |
|         | (2.206) | (8.115) | (16.64) | (39.29) | (67.70) | (138.2)  | (223.4)  | (294.9) |
| 100.0   | 3.947 | 14.52 | 29.16 | 68.01 | 117.0 | 238.7    | 386.6    | 512.1   |
|         | (3.717) | (13.95) | (28.98) | (69.05) | (119.4) | (244.5)  | (395.7)  | (559.1) |
| 400.0   | 5.521 | 20.39 | 41.03 | 95.90 | 165.1 | 337.1    | 546.2    | 723.7   |
|         | (5.188) | (19.56) | (40.78) | (97.38) | (168.5) | (345.3)  | (559.1)  | (835.6) |
| 2000.0  | 8.206 | 30.37 | 61.18 | 143.2 | 246.5 | 503.8    | 816.3    | 1082.0  |
|         | (7.702) | (29.12) | (60.81) | (145.4) | (251.7) | (516.1)  | (835.6)  | (1104.0) |
Table- 4:  Sample results for the octic- AHO in the zeroth - order (GHA) compared with results of earlier calculations from ref.[ 7 ] (shown in parentheses), over a wide range of 'λ' and 'n'.

| $\lambda$ | 0.1 | 1.0 | 5.0 | 50.0 | 200.0 |
|-----------|-----|-----|-----|------|------|
| $E_0$     |     |     |     |      |      |
|           | 1.3005 | 1.7794 | 2.3290 | 3.5565 | 4.6425 |
|           | (1.2410) | (1.6413) | (2.1145) | (3.1886) | (4.1461) |
| $E_1$     |     |     |     |      |      |
|           | 4.4717 | 6.3946 | 8.5167 | 13.172 | 17.259 |
|           | (4.2754) | (5.9996) | (7.9296) | (12.1950) | (15.9519) |
| $E_2$     |     |     |     |      |      |
|           | 8.6264 | 12.717 | 17.126 | 26.698 | 35.062 |
|           | (8.4530) | (12.421) | (16.711) | (26.033) | (34.183) |
| $E_4$     |     |     |     |      |      |
|           | 19.763 | 30.026 | 40.863 | 64.165 | 84.444 |
|           | (19.9930) | (30.4605) | (41.4947) | (65.20180) | (85.8251) |
| $E_6$     |     |     |     |      |      |
|           | 34.217 | 52.669 | 72.044 | 113.48 | 149.47 |
|           | (35.0560) | (54.1403) | (74.0830) | (116.7629) | (153.8278) |
| $E_8$     |     |     |     |      |      |
|           | 51.570 | 80.013 | 109.65 | 172.99 | 227.97 |
|           | (53.145590) | (82.6496) | (113.3486) | (178.9215) | (235.8193) |
| $E_9$     |     |     |     |      |      |
|           | 61.239 | 95.255 | 130.64 | 206.23 | 271.81 |
|           | (63.2253) | (98.5529) | (135.2598) | (213.6157) | (281.5864) |
| $E_{10}$  |     |     |     |      |      |
|           | 71.532 | 111.49 | 153.01 | 241.64 | 318.52 |
|           | (73.9545) | (115.4899) | (158.5991) | (250.5751) | (330.3433) |
| $E_{11}$  |     |     |     |      |      |
|           | 824.24 | 128.68 | 176.69 | 279.14 | 368.00 |
|           | (85.3079) | (133.4201) | (183.3103) | (289.7106) | (381.9720) |
| $E_{12}$  |     |     |     |      |      |
|           | 93.893 | 146.79 | 201.65 | 318.67 | 420.14 |
|           | (97.2636) | (152.3080) | (209.3443) | (330.9440) | (436.3695) |
| $E_{13}$  |     |     |     |      |      |
|           | 105.92 | 165.79 | 227.84 | 360.14 | 474.85 |
|           | (109.7967) | (172.1125) | (236.6436) | (374.1834) | (493.4143) |
| $E_{14}$  |     |     |     |      |      |
|           | 118.49 | 185.65 | 255.21 | 403.50 | 532.06 |
|           | (122.8909) | (192.8082) | (265.1732) | (419.3737) | (553.0335) |