Bosonic Halperin fractional quantum Hall effect at filling factor $\nu = 2/5$

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Abstract

Quantum Hall effects with multicomponent internal degrees of freedom facilitate the playground of novel emergent topological orders. Here, we explore the correlated topological phases of two-component hardcore bosons at a total filling factor $\nu = 2/5$ in both lattice Chern band models and Landau level continuum model under the interplay of intracomponent and intercomponent repulsions. We show the theoretical discovery of the emergence of two competing distinct fractional quantum Hall states which have not been reported before: Halperin (441) fractional quantum Hall effect and Halperin (223) fractional quantum Hall effect. We elucidate their topological features including the degeneracy of the ground state and fractionally quantized topological Chern number matrix. Finally, we discuss scenarios related to phase transition between them when intercomponent nearest-neighbor coupling is tuned from weak to strong in lattice models.

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1 Introduction

Multicomponent systems provide an avenue for realizing a tremendous amount of physics that have no analogue in one-component systems, which also results in a question of fundamental interest: what kind of topological order can emerge in a microscopic interacting model. Two-component fractional quantum Hall (FQH) effect is such an example, which is conjectured to produce new incompressible ground states beyond Laughlin states. The earlier theoretical studies of an electron gas in spin-unpolarized and bilayer (or double quantum wells) systems such as AlGaAs, exemplified by integer quantum Hall state at $\nu = 1$ [1–3] and fractional quantum Hall state at $\nu = 1/2$ [4–7], can be described as Halperin’s two-component ($m|n$) wave functions with the $K = \begin{pmatrix} m & n \\ n & m \end{pmatrix}$ matrix [8]. Recently, experimental observations in graphene have greatly expanded our understanding of a widespread zoology of multicomponent FQH effects [9–12], and at the same time inspired the current study of multicomponent FQH effect in bosonic systems.

Compared to the fermionic systems, much less knowledge is learned from the study of two-component Bose gases. It was realized that there exists an intimate theoretical generalizations of the Halperin’s wave functions to clustered spin-singlet quantum Hall states [13, 14], and even a classification scheme for bosonic symmetry-protected topological (SPT) phases with no intrinsic topological order for multicomponent bosons [15], such as two-component bosonic integer quantum Hall liquid with the associated $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ matrix in the presence of symmetries [16, 17]. As the counterparts of fermionic states, the hunt for two-component bosonic fractional quantum Hall states has motivated several numerical proposals of new quantum Hall structures [18–21] based on strong synthetic gauge fields in cold atomic neutral systems that are not available in electronic FQH effect. Nevertheless, the vague outline of multicomponent bosonic systems discloses a large uncharted theoretical area of two-dimensional bosonic topological order [22] to be explored, aside from the experimental interest.

With these motivations, in this work, we are concerned with the internal correlation structures of two-component bosonic FQH effects at fixed filling factor $\nu = 2/5$ where a convincing theoretical evidence is still lacking. We will consider both the continuum Landau level and topological flat band models. While Landau level problem directly relates to the problems in the high magnetic field, topological flat band models dubbed “Chern insulators” are becoming an excitingly new platform [23, 24] for studying the quantum Hall effect at zero magnetic field, along with lots of experimental interests in topological Hofstadter-Harper [25, 26] and Haldane-honeycomb [27] bands for cold atom realization, topological moiré minibands for twisted multilayer graphene [30, 31] and other flat bands with nontrivial topology in geomet-
rically frustrated lattice \cite{33}. The emergence of FQH effect in topological flat bands (namely "fractional Chern insulators") requires a demanding understanding of the internal topological structure of interacting fractionalised phases, where an integer valued symmetric $K$ matrix was proposed to characterize different topological orders for Abelian multicomponent systems according to the Chern-Simons theory \cite{34,35}. Indeed, at partial fillings $\nu = 1/(kC + 1)$ (odd $k$ for hardcore bosons and even $k$ for spinless fermions) in topological flat bands with higher Chern numbers $C$, there fractionalised Abelian $C$-color-entangled states host a close relationship to $C$-component FQH states \cite{39,40}, where the general one-to-one correspondence is built up based on the classification of their $K$ matrices from the inverse of Chern number matrix for these gapped topological phases \cite{49,50}, where the quantized intercomponent drag Hall transport is identified as a primary evidence for the emergence of exotic correlated many-body topological states in multicomponent systems \cite{11,12}. Together with synthetic magnetic gauge fields in cold atomic neutral systems, these related progresses, thus enable new relevant prospects for the study of two-component bosonic FQH effects in both lattice and continuum models, which is the focus of our work.

The main findings of the present work is that, we characterize two robust bosonic FQH state at filling factor $\nu = 2/5$: (a) in the anisotropic limit $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} \ll 1$, the system is Halperin (441) FQH state with $K = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$, and (b) in the isotropic SU(2) limit $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} \simeq 1$, the system is Halperin (223) FQH state with $K = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$, as presented below ($V_{\sigma \sigma'}$ denotes the interactions between bosons of $\sigma$-component and those of $\sigma'$-component). Through the state-of-the-art methods, including ground state degeneracy, topological Chern number matrix on the torus manifold and fractional charge pumping, chiral edge excitations on the cylindrical geometry, we determine their topological nature.

The remainder of the paper is organized as follows. In Sec. 2, we give a description of the model Hamiltonian of interacting two-component bosons in both lattice and continuum models, with focuses on two typical topological lattice models, such as $\pi$-flux checkerboard and Haldane-honeycomb lattices. In Sec. 3 and Sec. 4, we study the many-body ground states of these two-component bosonic systems at $\nu = 2/5$ in both lattice and continuum models, and present detailed numerical demonstration of the $K$ matrix classification of competing quantum Hall states based on the ground state degeneracy, Chern number matrix. In Sec. 5, we discuss chiral edge physics. In Sec. 6, we qualitatively discuss the phase transition between competing quantum Hall states. Finally, we conclude in Sec. 7 with the prospect of investigating competing phases in two-component bosonic systems.

2 Model and Method

2.1 Topological lattice models

Here, we utilize both exact diagonalization (ED) and density-matrix renormalization group (DMRG) to study the low-energy properties of the Hamiltonian for two-component hardcore bosons with pseudospin degrees of freedom via intercomponent and intracomponent interactions at a total filling $\nu = 2/5$ in topological flat bands, and elucidate the physical mechanism of the competing Halperin FQH effects. The Hamiltonian built on topological $\pi$-flux checker-
board (CB) and Haldane-honeycomb (HC) lattices, is given by

\[
H_{CB} = \sum_{\sigma} \left[ -t \sum_{\langle r,r' \rangle} e^{i\varphi_{r,r'}} b_{r',\sigma}^\dagger b_{r,\sigma} - t' \sum_{\langle \langle r,r' \rangle \rangle} b_{r',\sigma}^\dagger b_{r,\sigma} - t'' \sum_{\langle \langle \langle r,r' \rangle \rangle \rangle} b_{r',\sigma}^\dagger b_{r,\sigma} + H.c. \right] + V_{\text{int}}, \tag{1}
\]

\[
H_{HC} = \sum_{\sigma} \left[ -t \sum_{\langle r,r' \rangle} b_{r',\sigma}^\dagger b_{r,\sigma} - t' \sum_{\langle \langle r,r' \rangle \rangle} e^{i\varphi_{r,r'}} b_{r',\sigma}^\dagger b_{r,\sigma} - t'' \sum_{\langle \langle \langle r,r' \rangle \rangle \rangle} b_{r',\sigma}^\dagger b_{r,\sigma} + H.c. \right] + V_{\text{int}}, \tag{2}
\]

where \(b_{r,\sigma}^\dagger\) is the hardcore bosonic creation operator of pseudospin \(\sigma = \uparrow, \downarrow\) at site \(r\). In what follows, we impose the tunnel couplings \(t' = 0.3t\), \(t'' = -0.2t\), \(\varphi = \pi/4\) for checkerboard lattice, while \(t' = 0.6t\), \(t'' = -0.58t\), \(\varphi = 2\pi/5\) for honeycomb lattice \([53,54]\), and choose the interaction strengths \(U = \infty\), \(V_{\uparrow\uparrow}/t = 100\) with variable \(V_{\uparrow\downarrow}\) ranging from anisotropic \(V_{\uparrow\uparrow}/V_{\uparrow\downarrow} = 0\) to isotropic \(V_{\uparrow\uparrow}/V_{\uparrow\downarrow} = 1\).

For finite system sizes with translational symmetry, \(\nu = \nu_{\uparrow} + \nu_{\downarrow} + \nu_{\uparrow\downarrow} = 2N_{\uparrow}/N_\sigma = \nu_{\downarrow} = 2N_{\downarrow}/N_\sigma = 1/5\), where \(N_\sigma\) is the total number of sites, and \(N_{\uparrow\downarrow}\) is the particle number of pseudospin. The energy states are labeled by the total momentum \(K = (K_x, K_y)\) in units of \((2\pi/N_x, 2\pi/N_y)\) in the Brillouin zone. For infinite system sizes, we exploit infinite DMRG on the cylindrical geometry with open boundary condition in the \(x\) direction and periodic boundary condition in the \(y\) direction, and the bond dimension of DMRG is kept up to \(M = 5000\) for well-convergent results we report here.

### 2.2 Landau continuum model

Meanwhile, as an unabridged physical counterpart, we also utilize ED to investigate the continuous model on periodic torus spanned by vectors \(\vec{L}_1 = L\hat{e}_x\) and \(\vec{L}_2 = L\hat{e}_y\). The single particle Hamiltonian, which can be realized in rapidly rotating bosonic gases \([55]\), is written as:

\[
H_0(A) = \frac{1}{2} \sum_a D_a(A)D_a(A). \tag{4}
\]

Where \(D_a(A) = p_a - qA_a = -i\hbar \nabla_a + |e|A_a\) is the canonical momentum in magnetic field and \(A = (-By, 0)\) is the vector potential. The total number of magnetic fluxes \(N_\sigma\) penetrating the torus is given by the Landau level degeneracy \(N_\sigma = \frac{\tilde{L}_1 \times \tilde{L}_2}{\pi e^2} = \frac{l^2}{2\pi e^2}\), where the magnetic length \(l = \sqrt{\hbar/|e|B}\) is taken as the unit of the length. The eigenstates of Eq. \(\ref{4}\) are called Landau levels. Here, we study the FQH state in the lowest Landau level with \(N_\sigma\)-fold degenerate orbits \([56]\):

\[
\psi_j = \frac{1}{\sqrt{\pi^2 L \ell}} e^{-\frac{y^2}{2\pi^2 L \ell}} \sum_{n \in \mathbb{Z}} \exp \left\{ -\pi N_\sigma (n + \frac{j}{N_\sigma})^2 + i2\pi N_\sigma (n + \frac{j}{N_\sigma}) \frac{z}{\ell} \right\} \quad (j = 0, 1, \cdots, N_\sigma - 1). \tag{5}
\]
Figure 1: (Color online) Numerical ED results for the low energy spectrum of two-component bosonic systems $\nu = 2/5$ with $U(1) \times U(1)$-symmetry for anisotropic interaction $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 0$ in different topological lattice models (a) checkerboard lattice and (b) honeycomb lattice. The red dotted box indicates the ground state degeneracy.

In the lowest Landau level, the two-body interactions between particles can be written as:

$$U_{\sigma_1\sigma_2} = \sum_{j_1,j_2,j_3,j_4} A_{j_1,j_2,j_3,j_4} b_{\sigma_1 j_1}^\dagger b_{\sigma_2 j_2}^\dagger b_{\sigma_3 j_3} b_{\sigma_4 j_4}$$

where $U(|q|) = \int \int dxdy U(|r_1 - r_2|) e^{-i\mathbf{q} \cdot \mathbf{r}}$ is the Fourier transformation of particle-particle interaction $U(|\mathbf{r}_1 - \mathbf{r}_2|)$ and $q = \frac{2\pi}{L} (n_x, n_y)$ is the reciprocal lattice vector. The symbol $\delta_{ij}^{mod N_s}$ means that the equivalence between $i$ and $j$ is defined by modulo $N_s$ and $\sigma$ represents the layer index (or pseudospin degree of freedom).

We consider the double-layer system with all bosons in each layer limited to the lowest Landau level, and the interaction Hamiltonian between bosons contain three parts:

$$H = U_{\uparrow\uparrow} + U_{\uparrow\downarrow} + U_{\downarrow\downarrow},$$

where $U_{\sigma_1\sigma_2}$ is defined by Eq. 6 and $U_{\uparrow\uparrow}, U_{\downarrow\downarrow}$ is the intra-layer interaction in the first and second layer while $U_{\uparrow\downarrow}$ denotes the inter-layer interactions between different layers.

3 Halperin (441) Fractional Quantum Hall Effect

In this section, we first delve into the topological ground state degeneracies for anisotropic interaction $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 0$, where the SU(2)-symmetric repulsion is broken down to $U(1) \times U(1)$-symmetry, and discuss its stability. Next we give a detailed demonstration of the $K$ matrix classification from the inverse of the Chern number matrix $C$ based on ED calculation of topologically invariant Chern number and DMRG simulation of drag charge pumping in the periodic parameter plane.
3-fold degenerate ground states in a single irreducible Brillouin zone and a total of $3 \times 5 = 15$-fold degeneracy for the whole Brillouin zone. The inset depicts the finite size scaling of energy gap which persists in the thermodynamic limit.

### 3.1 Ground state degeneracy

The key property of the existence of topological fractionalised ordered phases lies in their ground state degeneracies. For Halperin (mmn) quantum Hall state, the ground state degeneracy is given by the determinant of the $K$ matrix. Thus, first we demonstrate the ground state degeneracy on periodic lattice in different interacting regimes. As shown in Figs. 1(a) and 1(b) for different topological systems in the anisotropic limit $V_{↑↓}/V_{↑↑} = 0$, there exists a low-energy manifold with fifteen-fold degenerate ground states, which are separated from higher energy levels by a robust gap.

Similarly, we also study the ground state degeneracy in the continuum Landau level model as an intuitive understanding. The energy states are labeled by the pseudo-momentum $K = (K_x, K_y)$ in units of $(2\pi/L, 2\pi/L)$ in the irreducible Brillouin zone $[57]$. Since the translational symmetry of center of mass, there are five equivalent irreducible Brillouin zones. Thus, for FQH states in $\nu = 2/5$, we have at least five-fold degenerated ground states. When we choose Haldane pseudopotentials $v_0 = v_2 = 1.0$ for $U_{↑↑}(U_{↓↓})$ and $v_0 = 1.0$ for $U_{↑↓}$, the ground states with exact zero energy host three-fold degeneracy in each irreducible Brillouin zone (namely 15-fold for five equivalent irreducible Brillouin zones when the center-of-mass translation invariance is taken into account) as shown in Fig. 2, consistent with the theoretical prediction of Halperin (441) state. In the continuum model, we can access three different system sizes. For all of these system sizes, the ground state degeneracy is robust, and the energy gap separating the ground state manifold and the excitations is finite which is nonzero in the finite-size scaling (see the inset of Fig. 2(a)).

Further, in order to demonstrate the topological equivalence of these ground states, we calculate the low energy spectra flux under the insertion of flux quanta, by utilizing twisted boundary conditions $\psi(r_x + N_\alpha) = \psi(r_x) \exp(i\theta_\sigma^\alpha)$ where $\theta_\sigma^\alpha$ is the twisted angle for pseudospin $\sigma$ particles in the $\alpha$ direction. For $V_{↑↑}/V_{↑↑} = 0$, as shown in Fig. 3(a), these fifteen-fold ground states evolve into each other without mixing with the higher levels, and the system returns back to itself upon the insertion of five flux quanta for both $\theta_↑^α = \theta_↓^α = \theta$ and $\theta_↑^α = \theta, \theta_↓^α = 0$. The robustness of degenerate ground states reveal the existence of universal internal structures with the behavior of fractional quantization.
In essence, on both lattice model and continuum models, we have identified the robustness of fifteen-fold ground state degeneracy on the torus geometry at $\nu = 2/5$. This Halperin state survives in both lattice Chern band models and Landau level continuum model, which demonstrate its robustness for future detecting.

### 3.2 Chern number matrix

Following the above discussion, we continue to analyze the fractionally topological quantization, characterized by the Chern number matrix $C = \begin{pmatrix} C_{\uparrow\uparrow} & C_{\uparrow\downarrow} \\ C_{\downarrow\uparrow} & C_{\downarrow\downarrow} \end{pmatrix}$ of the many-body ground state wavefunction $\psi$ for interacting systems \cite{Halperin1984, Sticlet2017}. In the parameter plane $(\theta^x, \theta^y)$, the matrix elements are defined by $C_{\sigma\sigma'} = \int d\theta^x d\theta^y F_{\sigma\sigma'}(\theta^x, \theta^y)/2\pi$ with the Berry curvature

$$ F_{\sigma\sigma'}(\theta^x, \theta^y) = \text{Im} \left( \frac{\partial \psi}{\partial \theta^x} \frac{\partial \psi}{\partial \theta^y} - \frac{\partial \psi}{\partial \theta^y} \frac{\partial \psi}{\partial \theta^x} \right). $$

By numerically calculating the Berry curvatures using $m \times m$ mesh Wilson loop plaquette in the boundary phase space with $m \geq 10$, one can obtain the quantized topological invariant $C_{\sigma\sigma'}$ of the gapped ground states at momentum $K$, and $C_{\uparrow\uparrow} = C_{\downarrow\downarrow}, C_{\uparrow\downarrow} = C_{\downarrow\uparrow}$ under the prescribed symmetry $b_{\uparrow\uparrow} \leftrightarrow b_{\downarrow\downarrow}$. In the ED study of finite system size, we find that for $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 0$, the three ground states at $K = (0,0)$ host the total Chern numbers $\sum_{i=1}^{3} C_{\uparrow\uparrow}^i = 4/5$ and $\sum_{i=1}^{3} C_{\downarrow\uparrow}^i = -1/5$, as indicated in Figs. 4(a) and 4(b) respectively.
Figure 4: (Color online) Numerical ED results for Berry curvatures $F^{xy}_{\sigma} \Delta \theta^x_{\sigma} \Delta \theta^y_{\sigma}/2\pi$ of the three ground states at $K = (0,0)$ of two-component bosonic systems $N_\uparrow = N_\downarrow = 5, N_s = 2 \times 3 \times 5$ for anisotropic interaction $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 0$ in topological checkerboard lattice in different parameter planes. (a) $(\theta^x_\uparrow, \theta^y_\uparrow)$ and (b) $(\theta^x_\uparrow, \theta^y_\downarrow)$.

Figure 5: (Color online) Quantized charge transfers for two-component bosonic systems $\nu = 2/5$ with $U(1) \times U(1)$-symmetry on the infinite $N_y = 5$ cylinder for anisotropic interaction $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 0$ under the insertion of flux quantum $\theta^x_\uparrow = \theta, \theta^y_\downarrow = 0$ in one component for (a) checkerboard lattice and (b) honeycomb lattice.
As topological Chern number is related to the Hall transport response \[60\], we further calculate the charge pumping induced by the Berry curvature once the flux quantum is adiabatically inserted on infinite cylinder systems using DMRG \[61\]. Numerically we cut the infinite cylinder into left-half and right-half parts along the \(x\) direction, and obtain the net charge transfer of the pseudospin \(\sigma\) from the right side to the left side by calculating the evolution of \(N_{\sigma}(\theta_{\sigma'}) = \text{tr}[\hat{\rho}_L(\theta_{\sigma'}) \hat{N}_{\sigma}]\) as a function of \(\theta_{\sigma'}\), (here \(\hat{\rho}_L\) the reduced density matrix of the left part, classified by the quantum numbers \(\Delta Q^{\uparrow}, \Delta Q^{\downarrow}\)). For \(V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 0\), as shown in Fig. 5(a) and 5(b), we get fractionally quantized charge transfer in different topological lattice models with pumping values \(\Delta N^{\uparrow} = N^{\uparrow}(2\pi) - N^{\uparrow}(0) \simeq C^{\uparrow\uparrow} = 4/15, \Delta N^{\downarrow} = N^{\downarrow}(2\pi) - N^{\downarrow}(0) \simeq C^{\downarrow\uparrow} = -1/15\) upon threading one flux quantum \(\theta^{\uparrow}_{\sigma'}\) of the pseudospin \(\uparrow\) with \(\theta^{\downarrow}_{\sigma'} = 0\) for two-component bosons.

In view of quantized topological invariants, our ED and DMRG studies establish the essential diagnosis of distinct topological orders, benefitting from the merit of the well-defined Chern number matrix of the gapped ground state, namely

\[
C = \frac{1}{15} \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}, \quad K = C^{-1}
\]

for \(V_{\uparrow\downarrow}/V_{\uparrow\uparrow} \ll 1\) where the system falls into Halperin’s (441) FQH state.

4 Halperin (223) Fractional Quantum Hall Effect

In this section, we turn to analyze the topological ground state properties for isotropic interactions \(V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 1\), where the SU(2)-symmetric repulsion is restored. Similarly, we would present numerical proofs of the \(K\) matrix classification from the inverse of the Chern number matrix \(C\) based on ED calculation of topologically invariant Chern number and DMRG simulation of drag charge pumping in the periodic parameter plane.

Figure 6: (Color online) Numerical ED results for the low energy spectrum of two-component bosonic systems \(\nu = 2/5\) with \(U(1) \times U(1)\)-symmetry for isotropic SU(2) interaction \(V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 1\) in different topological lattice models (a) checkerboard lattice and (b) honeycomb lattice. The red dotted box indicates the ground state degeneracy.
4.1 Ground state degeneracy

In contrast to the limit $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} \ll 1$, we find that in the isotropic SU(2) limit $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 1$, there exists a low-energy manifold with five-fold degenerate ground states, which are separated from higher energy levels by a robust gap, as shown in Figs. 6(a) and 6(b). Further, in order to demonstrate their topological equivalence, we calculate the low energy spectra flux under the insertion of flux quanta, by utilizing twisted boundary conditions $\psi(r_\sigma + N_\alpha) = \psi(r_\sigma) \exp(i\theta_\sigma^\alpha)$ where $\theta_\sigma^\alpha$ is the twisted angle for pseudospin $\sigma$ particles in the $\alpha$ direction. For $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 1$, as shown in Fig. 7, these five-fold ground states evolve into each other without mixing with the higher levels, and the system returns back to itself upon the insertion of five flux quanta for both $\theta_\uparrow^\sigma = \theta_\downarrow^\sigma = 0$ and $\theta_\uparrow^\sigma = \theta, \theta_\downarrow^\sigma = 0$. The robustness of inherent five-fold degenerate ground states in the isotropic SU(2) limit $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 1$, implies the existence of novel internal topological structures, in distinction from the behavior of Halperin (441) FQH states in the anisotropic limit $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 0$.

Figure 7: (Color online) Numerical ED results for the low energy spectral flow of two-component bosonic systems $\nu = 2/5, N_s = 2 \times 3 \times 5$ with $U(1) \times U(1)$-symmetry for isotropic SU(2) interaction $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 1$ in topological checkerboard lattice under the insertion of two types of flux quanta $\theta_\uparrow^\sigma = \theta, \theta_\downarrow^\sigma = 0$ (upper panel) and $\theta_\uparrow^\sigma = \theta_\downarrow^\sigma = \theta$ (lower panel).

4.2 Chern number matrix

Similar to Sec. 3, to further clarify the distinctive topological phase in the isotropic SU(2) limit $V_{\uparrow\downarrow}/V_{\uparrow\uparrow} = 1$, we calculate the Chern number matrix $C = \begin{pmatrix} C_{\uparrow\uparrow} & C_{\uparrow\downarrow} \\ C_{\downarrow\uparrow} & C_{\downarrow\downarrow} \end{pmatrix}$ of the many-body ground state wavefunction $\psi$ for interacting systems in the parameter plane $(\theta_\sigma^\uparrow, \theta_\sigma^\downarrow)$. In the ED study of finite system size, for single ground state at $K = (0,0)$, we plot the corresponding
Berry curvatures \( F_{\uparrow\uparrow}, F_{\uparrow\downarrow} \) which vary smoothly, as indicated in Figs. 8(a) and 8(b) respectively. We obtain the fractionally quantized topological invariants \( C_{\uparrow\uparrow} = -\frac{2}{5}, C_{\uparrow\downarrow} = \frac{3}{5} \).

![Figure 8](image_url)

Figure 8: (Color online) Numerical ED results for Berry curvatures \( F_{xy} \Delta \theta_x \Delta \theta_y / 2\pi \) of the single ground state at \( \theta = (0, 0) \) of two-component bosonic systems \( N_{\uparrow} = N_{\downarrow} = 5, N_s = 2 \times 3 \times 5 \) for isotropic SU(2) interaction \( V_{\uparrow\downarrow} / V_{\uparrow\uparrow} = 1 \) in topological checkerboard lattice in different parameter planes. (a) \((\theta_1^x, \theta_2^x)\) and (b) \((\theta_1^x, \theta_2^y)\).

For larger system sizes, we continue to calculate the charge pumping on infinite cylinder systems utilizing the same DMRG simulation of adiabatic flux insertion in Sec. 3. For \( V_{\uparrow\downarrow} / V_{\uparrow\uparrow} = 1 \), as shown in the left panel of Fig. 9, we get fractionally quantized charge transfer in topological checkerboard lattice model, encoded by \( \Delta N_{\uparrow} = N_{\uparrow}(2\pi) - N_{\uparrow}(0) \simeq C_{\uparrow\uparrow} = -\frac{2}{5}, \Delta N_{\downarrow} = N_{\downarrow}(2\pi) - N_{\downarrow}(0) \simeq C_{\uparrow\downarrow} = \frac{3}{5} \) upon threading one flux quantum \( \theta_1^y \) of the pseudospin \( \uparrow \) with \( \theta_2^y = 0 \) for two-component bosons.

Therefore, combined with the results of ground state degeneracy, we can establish the \( K \) matrix classification of Halperin (223) FQH state, derived from the inverse of the Chern number matrix of the gapped ground states, namely

\[
C = \frac{1}{5} \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}, K = C^{-1}
\]

for \( V_{\uparrow\downarrow} / V_{\uparrow\uparrow} \simeq 1 \).

## 5 Chirality of Edge Physics

Moreover, we analyze the edge physics of these topological ordered phases according to the bulk-edge correspondence, especially the chiral Luttinger liquid character which can be used to characterize the topological orders in the different FQH states \([62, 63]\). The edge chirality is determined by the signs of the eigenvalues \( m \pm n \) of the \( K = \begin{pmatrix} m & n \\ n & m \end{pmatrix} \) matrix.

Here we harness the low-lying momentum-resolved entanglement spectrum to identify the topological nature on the infinite cylinder \([64]\): (i) for \( m > n \), two propagating chiral modes in the same direction are obtained \([49]\), (ii) for \( m < n \), two chiral modes propagate in the opposite directions, i.e. bosonic integer quantum Hall phase \([65]\), (iii) for \( m = n \), only one chiral mode is left, i.e. Halperin (111) exciton phase \([66]\). For \( V_{\uparrow\downarrow} / V_{\uparrow\uparrow} = 1 \), as shown in the right panel of Fig. 9, we observe two oppositely moving branches of low-lying excitations,
Figure 9: (Color online) Left panel: Quantized charge transfers for two-component bosonic systems $\nu = 2/5$ with $U(1) \times U(1)$-symmetry on the infinite $N_y = 5$ cylinder for isotropic SU(2) interaction $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} = 1$ under the insertion of flux quantum $\theta_y^x = \theta, \theta_y^y = 0$. Right panel: Non-chiral edge nature for two-component bosonic systems $\nu = 2/5$ with $U(1) \times U(1)$-symmetry for isotropic SU(2) interaction $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} = 1$ on the infinite $N_y = 5$ cylinder in the typical charge sector $\Delta Q_{\uparrow} = 0, \Delta Q_{\downarrow} = -1$ for topological checkerboard lattice.

consistent with the non-chiral nature of Halperin (223) FQH state. However, for $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} = 0$, Figures 10(a) and 10(b) depict two parallel forward-moving branches of low-lying excitations, matching with the level counting $1, 2, 5, \cdots$ of WZW conformal field description for two gapless free bosonic edge theories of Halperin (441) FQH state.

Figure 10: (Color online) Chiral edge mode identified from the momentum-resolved entanglement spectrum for two-component bosonic systems $\nu = 2/5$ with $U(1) \times U(1)$-symmetry for anisotropic interaction $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} = 0$ on the infinite $N_y = 6$ cylinder in the typical charge sectors (a) $\Delta Q_{\uparrow} = \Delta Q_{\downarrow} = 0$ and (b) $\Delta Q_{\uparrow} = 0, \Delta Q_{\downarrow} = -1$. The horizontal axis shows the relative momentum $\Delta K = K_y - K_y^0$ (in units of $2\pi/N_y$). The numbers below the red dashed line label the level counting: $1, 2, 5, \cdots$. 
6 Phase Transition

Finally, since two different competing topological orders emerge in the same clean system, one fundamental question is about the possible phase transition nature between them with varying $V_{\uparrow \downarrow}/V_{\uparrow \uparrow}$. When tuning $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} = 0$ continuously to $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} = 1$, we plot the evolution of the low energy spectrum in Fig. 11. Even a moderate intercomponent interaction $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} \sim 0.01$ tends to destroy the topological degenerate manifold of Halperin (441) FQH states by diminishing the protecting energy gap. For finite system sizes, our ED study gives a continuous crossover among these degenerate ground states, with a possible level repulsion at the intermediate regime $V_{\uparrow \downarrow}/V_{\uparrow \uparrow} \sim 0.04$. However, a much detailed study (taking account of finite size effects) is particularly demanded to make conclusive statements about the nature of the transition. It is still possible that in the thermodynamic limit the system can then undergo a simple level crossing, resulting in a first-order transition due to the mismatched nature of topological order [67]. A more refined numerical simulation of the transition nature based on DMRG is required to describe the critical theory, in analogy to emergent multi-flavor QED$_3$ at the critical transition between different fractional Chern insulators [68], which is deferred to future study.

7 Conclusion and Outlook

So far we have introduced both microscopic lattice model and Landau continuum model that realize two-component bosonic FQH effects at the filling $\nu = 2/5$, including Halperin (441) quantum Hall states in the extremely anisotropic interacting limit and Halperin (223) quantum Hall states in the nearly isotropic SU(2) interacting limit. Using ED and DMRG simulations, we find that the ground state shows several characteristic topological properties in connection to the $\mathbf{K}$ matrix: (i) the ground state degeneracy, (ii) fractional topological
Chern number in relation to Hall conductance and (iii) chiral edge modes. A tiny interaction perturbation trying to restore the SU(2) symmetry would quickly destroy the stability of this Halperin (441) FQH effect. Our study thus offers a unique perspective of competing quantum Hall physics in the clean system with multiple components, and might furthermore serve as a promising paradigm for engineering this mechanism just by tuning intercomponent and intracomponent interactions using artificial topological bands in future experiments on cold atomic gases [69] or rapidly rotating bosonic gases.

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