T–Duality and Spinning Solutions in 2 + 1 Gravity

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Abstract
Starting with the 2 + 1 Einstein–Maxwell–Dilaton system with a cosmological constant and assuming two commuting Killing symmetries we derive the corresponding 1 + 0 σ–model. It is shown that, for general values of the coupling parameters, the T–duality group is SL(2, R), which coincides with the group of linear coordinate transformations along the Killing orbits. This duality, along with a suitable parameter choice, is applied to obtain some new spinning solutions in the alternative gravity theories: Einstein–Maxwell, Brans–Dicke and Einstein–Maxwell–Dilaton.

Key words: T-duality, spinning solutions

1 Introduction
Gravity in 2 + 1 dimensions has attracted much attention since Bañados, Teitelboim and Zanelli [1,2] discovered a black hole solution in the (2 + 1)–dimensional Einstein theory with a negative cosmological constant. The BTZ black hole is a surprisingly simple solution of 2 + 1 gravity exhibiting almost all usual features of a black hole in spite of the fact that the spacetime curvature is constant. Soon after, various spinning electrically charged solutions were found [3] in the Einstein–Maxwell theory. Moreover, from an (anti-) self-duality assumption for the Maxwell field a horizonless regular particle-like “dyon” solution was obtained [4]. It has been pointed out that the angular momentum and the mass of the (anti-) self dual solutions diverge at spatial infinity [5].

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Generally in three dimensions (unlike the four-dimensional case) there is no duality between electric and magnetic fields except for the static and rotationally symmetric configurations [6]. Thus we should consider the electric and magnetic solutions separately. The static magnetically charged solution in the Einstein–Maxwell theory was suggested in [7] and interpreted as a magnetic monopole; it has no event horizon and is particle-like. Finally, using a coordinate transformation, Clément found general spinning electrically charged BTZ black holes [8]. All charged solutions appear to have a logarithmic divergence of mass and angular momentum densities at spatial infinity. In order to regularize divergence a topological Chern-Simons term was added directly [8] or through the boundary conditions [9].

The extended theories including a dilaton field, such as Brans–Dicke and Einstein–Maxwell–Dilaton (EMD) theories, were studied by many authors. For Brans–Dicke theory static and stationary black hole solutions were analyzed in [10,11] using the string frame. For EMD theory, Chan and Mann [12] have found static electrically charged solutions and later spinning uncharged solutions [13]. Similarly to the situation in Einstein–Maxwell theory, magnetic EMD solutions were also obtained in [14]. Most of the solutions were found by solving the field equations directly using different assumptions and technical tricks in order to simplify them. But no general spinning charged solutions have been found in this way since the spin necessitates both electric and magnetic fields being present thus increasing the complexity of the field equations.

Here we suggest to use $T$–duality for generating new solutions. Assuming that the $2 + 1$ spacetime permits two Killing vector fields, we construct an $1 + 0$ equivalent $\sigma$–model by a dimensional reduction of the metric and a vector field. The resulting $\sigma$–model possesses $T$–duality symmetry which can be directly applied to known solutions to construct some new ones. In this paper we consider the action including the dilaton and the Maxwell field with arbitrary coupling constants. In the general case, the $T$–duality locally is just a general coordinate transformation and part of it was used to get stationary solutions previously [8,11]. Here we want to emphasize that globally the coordinate transformations used to generate spinning solutions from static ones have more appropriate interpretation as $T$–duality of the reduced $2 + 1$ theory. Moreover, for particular set of coupling constants the symmetry is enhanced and the system becomes fully integrable.

Throughout this paper, the $T$–duality as well as a suitable choice of parameters are used to generate spinning solutions. In the Einstein–Maxwell theory we found a spinning electric solution which is equivalent to Clément’s solution [8] and a new spinning magnetic solution. Moreover a dyon solution can be obtained either from electric or magnetic one by identifying the spin parameter with the charge parameter. This is a generalization of the (anti-) self dual
solutions [4].

For the Brans–Dicke case we had found a set of static and stationary solutions which are similar to [10,11] but are obtained in the Einstein frame. For the EMD case we found two new classes of spinning solutions: electric and magnetic. The spinning charged solution has some special features distinguishing it from all previous ones. First, it necessarily contains the region where the angle $\theta$ becomes time–like and hence there exists closed time–like curves. Second, the solution possesses an inner horizon. Unfortunately, we did not find an explicit formula for the radius of the outer horizon. Hence the general discussion of the black hole thermodynamics is still lacking.

2 1 + 0 $\sigma$–Model and T–Duality

Consider a $(2 + 1)$–dimensional gravity theory including the Maxwell and dilaton fields with general coupling parameters. In the Einstein frame the action reads

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} \left\{ R - 4\gamma(\partial\phi)^2 - e^{-4\alpha\phi} F^2 - 2e^{\beta\phi} \Lambda \right\},$$

where $R$ is the three–dimensional scalar curvature, $\Lambda$ is a cosmological constant ($\Lambda < 0$ corresponds to anti–de–Sitter), $F$ is the Maxwell two–form field and $\phi$ is the dilaton which couples to $F^2$ and $\Lambda$ with strengths $\alpha$ and $\beta$ respectively. As particular cases this model includes Einstein–Maxwell ($\alpha = \beta = \gamma = 0$), Brans–Dicke ($\beta = 4$ and $F = 0$), and low energy string theory ($\alpha = 1$, $\beta = 4$ and $\gamma = 1$).

Field equations corresponding to this action are

$$R_{\mu\nu} = 4\gamma \partial_\mu \phi \partial_\nu \phi + e^{-4\alpha\phi} \left( 2F_{\mu\lambda} F^\lambda_{\nu} - g_{\mu\nu} F^2 \right) + 2g_{\mu\nu} e^{\beta\phi} \Lambda,$$

$$4\gamma \nabla^2 \phi + 2\alpha e^{-4\alpha\phi} F^2 - \beta e^{\beta\phi} \Lambda = 0,$$

$$\nabla_\mu \left( e^{-4\alpha\phi} F^{\mu\nu} \right) = 0,$$

while the Bianchi identity for the Maxwell field is

$$\epsilon^{\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0.$$

Now we assume the existence of two commuting Killing vectors, so that in the adapted coordinates all quantities depend only on one variable (assumed to
be spacelike), which is commonly denoted as \( r \):

\[
ds^2 = h_{ij}(r)dx^i dx^j + e^\varphi f^{-2}(r)dr^2, \quad A = A_i(r)dx^i,
\]

where \( \varphi = \ln |\det h| \). The ‘radial’ component of the 3–potential \( A_r \) was set zero as a pure gauge quantity. Substituting this ansatz into the action (1) and dividing it by the two–dimensional volume of the Killing orbits one arrives at the 0 + 1 action

\[
S = \frac{1}{2\kappa^2} \int dr \left\{ f \left( \frac{1}{4} \partial_r \varphi \partial_r \varphi + \frac{1}{4} \partial_i h_{ij} \partial_i \varphi_f - 4\gamma \partial_r \phi \partial_r \phi 
- 2e^{-4\alpha \phi} h_{ij} \partial_i A_r \partial_j A_r \right) - 2f^{-1}e^{\beta \phi + \varphi} \Lambda \right\}.
\]

Clearly, the function \( f \) is pure gauge, and it enters into the action as a Lagrange multiplier.

Now it is convenient to fully decouple the scale degree of freedom normalizing the moduli \( h_{ij} \) as

\[
H_{ij} = e^{-\varphi/2} h_{ij}, \quad H_{ij}^{-1} = e^{\varphi/2} h_{ij}.
\]

Then, in matrix notation, the action will read:

\[
S = \frac{1}{2\kappa^2} \int dr \left\{ f \left( \frac{1}{8} \varphi^2 + \frac{1}{4} \text{Tr}(H' H'^{-1}) - 4\gamma \phi^2
- 2e^{-4\alpha \phi - \varphi/2} \text{Tr}(A'^T H'^{-1} A') \right) - 2f^{-1}e^{\beta \phi + \varphi} \Lambda \right\},
\]

where \( A \) is a column of covariant components \( A_i \).

It is easy to see that the \( \sigma \)–model action (9) is invariant under the \( T \)–duality transformation:

\[
H \rightarrow \Omega^T H \Omega, \quad A \rightarrow \Omega^T A, \quad \text{where} \quad \Omega \in SL(2, R).
\]

This transformation holds for any values of \( \alpha, \beta \) and \( \gamma \) therefore it can be applied to all theories included in the action (1). It is not a surprise because, locally, this \( T \)–duality is just a coordinate transformation.

For practical applications we can parameterize \( \Omega \) as

\[
\Omega = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \xi & \xi \nu \\ \xi \mu & \xi^{-1} + \xi \mu \nu \end{pmatrix},
\]
\begin{align}
\Omega^{-1} = \begin{pmatrix}
\xi^{-1} + \xi \mu \nu & -\xi \nu \\
-\xi \mu & \xi
\end{pmatrix}.
\end{align}

Assuming the seed solution to have the form
\begin{align}
  h^0_{ij} = \begin{pmatrix}
  -H^2_0(r) & 0 \\
  0 & H^2_1(r)
\end{pmatrix},
  A^0_i = \begin{pmatrix}
  -A_0(r) \\
  A_1(r)
\end{pmatrix},
\end{align}

after $T$–duality transformation we get a new solution
\begin{align}
  h_{ij} = \begin{pmatrix}
  -\xi^2 H^2_0 + \xi^2 \mu^2 H^2_1 & -\xi^2 \nu H^2_0 + \mu (1 + \xi^2 \mu \nu) H^2_1 \\
  -\xi^2 \nu H^2_0 + \mu (1 + \xi^2 \mu \nu) H^2_1 & -\xi^2 \nu^2 H^2_0 + \xi^{-2} (1 + \xi^2 \mu \nu)^2 H^2_1
\end{pmatrix},
  A_i = \begin{pmatrix}
  -\xi A_0 + \xi \mu A_1 \\
  -\xi \nu A_0 + \xi^{-1} (1 + \xi^2 \mu \nu) A_1
\end{pmatrix}.
\end{align}

In the following sections we will illustrate how to apply this $T$–duality to generate the spin parameter.

### 3 Einstein-Maxwell Theory

#### 3.1 Spinning Electric Solution

We start applications by generating a spinning solution from the EM system ($\alpha = \beta = \gamma = 0$) in order to demonstrate the procedure. For a general 2+1 EMD system with arbitrary coupling constants the $T$–duality is identical with a linear coordinate transformation and does not include a Harris on–type transformation. Hence the charge parameter can not be generated by (13). Thus, as the seed, one should take a charged solution.

Consider the static electrically charged BTZ solution \[1,2\] which is characterized by three parameters $m, q, r_0$:
\begin{align}
ds^2 = -V^2 dt^2 + r^2 d\theta^2 + V^{-2} dr^2, \quad A_t = -q \ln \left( \frac{r}{r_0} \right),
\end{align}

where
\begin{align}
V^2 = -\Lambda r^2 - m - 2q^2 \ln \left( \frac{r}{r_0} \right).
\end{align}
After the transformation (13) we get the nonvanishing metric components

\[ h_{tt} = \xi^2 (\Lambda + \mu^2) r^2 + \xi^2 \left[ m + 2q^2 \ln \left( \frac{r}{r_0} \right) \right], \quad (16) \]

\[ h_{t\theta} = \left[ \xi^2 \nu (\Lambda + \mu(1 + \xi^2 \mu \nu)) \right] r^2 + \xi^2 \nu \left[ m + 2q^2 \ln \left( \frac{r}{r_0} \right) \right], \quad (17) \]

\[ h_{\theta\theta} = \left[ \xi^2 \nu^2 \Lambda + \xi^{-2}(1 + \xi^2 \mu \nu)^2 \right] r^2 + \xi^2 \nu^2 \left[ m + 2q^2 \ln \left( \frac{r}{r_0} \right) \right]. \quad (18) \]

To get the spin one should make a choice of parameters such that the \( r^2 \) term in \( h_{t\theta} \) vanish and the coefficient of the \( r^2 \) term in \( h_{\theta\theta} \) is equal to one. Under this choice the three parameters \( \xi, \mu \) and \( \nu \) of the \( SL(2, R) \) \( T \)-duality are related by

\[ \xi^2 = \frac{\Lambda}{\Lambda + \mu^2}, \quad \nu = -\frac{\mu}{\Lambda}. \quad (19) \]

The remaining one is just the desired spin parameter. After this choice the metric \( h_{ij} \) becomes to

\[ h_{tt} = \Lambda r^2 + \frac{\Lambda}{\Lambda + \mu^2} \left[ m + 2q^2 \ln \left( \frac{r}{r_0} \right) \right], \quad (20) \]

\[ h_{t\theta} = -\frac{\mu}{\Lambda + \mu^2} \left[ m + 2q^2 \ln \left( \frac{r}{r_0} \right) \right], \quad (21) \]

\[ h_{\theta\theta} = r^2 + \frac{\mu^2}{\Lambda(\Lambda + \mu^2)} + \left[ m + 2q^2 \ln \left( \frac{r}{r_0} \right) \right]. \quad (22) \]

In order to have a better presentation of this solution we change the parameters as following

\[ \omega := \frac{\mu}{\Lambda}, \quad M := \frac{\Lambda}{\Lambda + \mu^2} m, \quad Q^2 := \frac{\Lambda}{\Lambda + \mu^2} q^2. \quad (23) \]

then metric of the spinning electric solution has the usual form

\[ ds^2 = - \left[ -\Lambda r^2 - M - 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] dt^2 - 2\omega \left[ M + 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] dt d\theta \\
+ \left\{ r^2 + \omega^2 \left[ M + 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] \right\} d\theta^2 \\
+ \left\{ -\Lambda r^2 - (1 + \omega^2 \Lambda) \left[ M + 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] \right\}^{-1} dr^2, \quad (24) \]
and the potential of the Maxwell field is
\[ A = Q \ln \left( \frac{r}{r_0} \right) (-dt + \omega d\theta). \] (25)

When \( M = 0 \), the solution (24) is equivalent to the solution of Clément [8]. For the uncharged case, \( Q = 0 \), it reduces to the BTZ solution [1,2] in the standard form
\[
ds^2 = -\left(-\Lambda R^2 - M' + \frac{J^2}{4R^2}\right) dt^2 + R^2 \left(d\theta - \frac{J}{2R^2} dt\right)^2 \\
+ \left(-\Lambda R^2 - M' + \frac{J^2}{4R^2}\right)^{-1} dr^2, \tag{26}
\]
where
\[
R^2 := r^2 + \omega^2 M, \quad M' := (1 - \omega^2 \Lambda) M, \quad J := 2\omega M, \quad \Lambda := -l^{-2}. \tag{27}
\]

It is worth noting that the quasilocal mass and angular momentum are divergent (except for the uncharged BTZ solution) because of the presence of a logarithmic term. This problem can be solved by introducing a Chern-Simons term [8] or by a regularization [9].

### 3.2 Spinning Magnetic Solution

An electric-magnetic duality does not exist in three–dimensions. Hence the magnetic solution should be considered separately. In the EM theory the static magnetic monopole solution was found by Hirschmann and Welch [7] \[7\]
\[
ds^2 = -r^2 dt^2 + \left[-\Lambda^{-1} r^2 + m + 2p^2 \ln \left( \frac{r}{r_0} \right) \right] d\theta^2 \\
+ \left\{-\Lambda r^2 + \Lambda \left[m + 2p^2 \ln \left( \frac{r}{r_0} \right) \right] \right\}^{-1} dr^2, \\
A_\theta = p \ln \left( \frac{r}{r_0} \right), \tag{28}
\]
Here we take this solution as a seed to find the corresponding spinning solution in the same way as in the previous subsection. After a simple calculation and a redefinition of parameters we get the spinning magnetic solution as
\[\text{Our form of the magnetic solution is slightly different from [7]. The original form can be achieved by the following transformation: } r^2 \to \frac{r^2}{l^2} - M, m \to (1 + l^2)M, \Lambda \to -\frac{1}{l^2}, p \to Q_m \text{ and } r_0 \to 1.\]
\[ ds^2 = - \left\{ r^2 - \omega^2 \left[ M + 2P^2 \ln \left( \frac{r}{r_0} \right) \right] \right\} dt^2 - 2\omega \left[ M + 2P^2 \ln \left( \frac{r}{r_0} \right) \right] dt d\theta \\
+ \left[ \Lambda^{-1} r^2 + M + 2P^2 \ln \left( \frac{r}{r_0} \right) \right] d\theta^2 \\
+ \left\{ -\Lambda r^2 + \Lambda (\Lambda + \omega^2) \left[ M + 2P^2 \ln \left( \frac{r}{r_0} \right) \right] \right\}^{-1} dr^2, \] 
\tag{29}

and

\[ A = P \ln \left( \frac{r}{r_0} \right) (-\omega dt + d\theta). \] 
\tag{30}

### 3.3 Static Dyon

Furthermore, both spinning electric or magnetic solutions can be treated as a static dyon by setting \( \omega = P/Q \) in the electric or \( \omega = Q/P \) in the magnetic solutions. The static dyon is

\[ ds^2 = - \left[ -\Lambda r^2 - M - 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] dt^2 - \frac{2P}{Q} \left[ M + 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] dt d\theta \\
+ \left\{ r^2 + \frac{P^2}{Q^2} \left[ M + 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] \right\} d\theta^2 \\
+ \left\{ -\Lambda r^2 - (1 + \frac{P^2}{Q^2} \Lambda) \left[ M + 2Q^2 \ln \left( \frac{r}{r_0} \right) \right] \right\}^{-1} dr^2, \] 
\tag{31}

with

\[ A = \ln \left( \frac{r}{r_0} \right) (-Q dt + P d\theta). \] 
\tag{32}

This is a generalization of the (anti-) self dual solutions [4] which reduces to them if \( Q = \pm P \).

### 4 Brans-Dicke Theory

In the string frame, a set of static and stationary solutions of Brans–Dicke theory, \( \beta = 4 \) and \( F = 0 \), was found in [10,11]. Here we consider similar solutions in the Einstein frame. The static solutions can be easy obtained by solving the field equations under Schwarzschild gauge, i.e. \( -g_{tt} = g_{rr}^{-1} \). Then the corresponding stationary solution can be found by \( T \)-duality.
According to the coupling constant $\gamma$, the static solutions can be divided into four cases. For one of them, $\gamma = 0$, there exist only asymptotically flat solutions, i.e. $\Lambda$ should be zero. So it is not considered here.

4.1 Case $\gamma \neq -1, 0, \frac{1}{2}$

The static solution for the case $\gamma \neq -1, 0, \frac{1}{2}$, by solving field equations, is

$$ds^2 = -W_1(r)dt^2 + c^2r^{\frac{2\gamma}{\gamma+1}}d\theta^2 + W_1^{-1}(r)dr^2,$$

$$\phi = -\frac{1}{2(\gamma+1)}\ln\left(\frac{r}{r_0}\right),$$

where

$$W_1(r) = -ar^{\frac{2\gamma}{\gamma+1}} - mr^{\frac{1}{\gamma+1}}, \quad a := \frac{2(\gamma + 1)^2\Lambda r_0^{\frac{1}{\gamma+1}}}{\gamma(2\gamma - 1)}.$$  (34)

This solution is related to the solution of [10]. Using it as a seed we apply the T–duality and get

$$h_{tt} = \xi^2(a + \mu^2c^2)r^{\frac{2\gamma}{\gamma+1}} + \xi^2mr^{\frac{1}{\gamma+1}},$$

$$h_{t\theta} = \left[\xi^2\nu a + \mu(1 + \xi^2\mu\nu)c^2\right]r^{\frac{2\gamma}{\gamma+1}} + \xi^2\nu mr^{\frac{1}{\gamma+1}},$$

$$h_{\theta\theta} = \left[\xi^2\nu^2 a + \xi^{-2}(1 + \xi^2\mu\nu)^2c^2\right]r^{\frac{2\gamma}{\gamma+1}} + \xi^2\nu^2mr^{\frac{1}{\gamma+1}}.$$  (37)

In this case, in order to generate the spin we should choose the parameters in such a way that the $r^2$ term in $h_{t\theta}$ vanishes and the coefficient of $r^2$ term in $h_{\theta\theta}$ is equal to $c^2$. This corresponds to the following relations

$$\xi^2 = \frac{a}{a + \mu^2c^2}, \quad \nu = -\frac{\mu c^2}{a},$$  (38)

and the metric $h_{ij}$ reduces to

$$h_{tt} = ar^{\frac{2\gamma}{\gamma+1}} + \frac{a}{a + \mu^2c^2}mr^{\frac{1}{\gamma+1}},$$  (39)

$$h_{t\theta} = -\frac{\mu c^2}{a + \mu^2c^2}mr^{\frac{1}{\gamma+1}},$$  (40)

$\gamma \rightarrow \omega + 2, r \rightarrow r^{\frac{\omega+3}{\omega+1}}, r_0 \rightarrow a r^{\frac{\omega+3}{\omega+1}}, c^2 \rightarrow r^{\frac{\omega+3}{\omega+1}}, \Lambda \rightarrow -2\lambda^2$. 

\[ \]
\[ h_{\theta\theta} = c^2 r^{\frac{2\gamma}{1+\gamma}} + \frac{\mu^2 \gamma}{a(a + \mu^2 c^2) m} r^{\frac{1}{1+\gamma}}. \] \tag{41}

Redefining the parameters as

\[ \omega := \frac{\mu c^2}{a}, \quad M := \frac{a}{a + \mu^2 c^2} m, \] \tag{42}

one obtains the metric of new solution as

\[ rs^2 = -\left( -ar^{\frac{2\gamma}{1+\gamma}} - Mr^{\frac{1}{1+\gamma}} \right) dt^2 - 2\omega Mr^{\frac{1}{1+\gamma}} dtd\theta \]
\[ + \left( c^2 r^{\frac{2\gamma}{1+\gamma}} + \omega^2 Mr^{\frac{1}{1+\gamma}} \right) d\theta^2 \]
\[ + \left[ -ar^{\frac{2\gamma}{1+\gamma}} - (1 + \frac{\omega^2}{c^2 a}) Mr^{\frac{1}{1+\gamma}} \right]^{-1} dr^2. \] \tag{43}

The dilaton is invariant under the transformation.

4.2 Case \( \gamma = \frac{1}{2} \)

The static solution for the case \( \gamma = \frac{1}{2} \) is

\[ ds^2 = -W_2(r) dt^2 + c^2 r^2 d\theta^2 + W_2^{-1}(r) dr^2, \]
\[ \phi = -\frac{1}{3} \ln \left( \frac{r}{r_0} \right), \] \tag{44}

where

\[ W_2(r) = -\frac{3}{2} \Lambda r^{\frac{4}{3}} r^\frac{2}{3} \ln(br). \] \tag{45}

After \( T \)-duality the metric becomes

\[ h_{tt} = -\xi^2 W_2(r) + \xi^2 \mu^2 c^2 r^\frac{2}{3}, \] \tag{46}
\[ h_{t\theta} = -\xi^2 \nu W_2(r) + \mu(1 + \xi^2 \mu \nu) c^2 r^\frac{2}{3}, \] \tag{47}
\[ h_{\theta\theta} = -\xi^2 \nu^2 W_2(r) + \xi^{-2}(1 + \xi^2 \mu \nu)^2 c^2 r^\frac{2}{3}. \] \tag{48}

There are two possibilities of stationary solutions depending on the choice of parameters. One choice is

\[ \xi^2 = 1, \quad \mu = 0, \quad \nu = \omega, \] \tag{49}
and the stationary solution takes the form

\[ ds^2 = -W_2(r)dt^2 - 2\omega W_2(r)dtd\theta + \left[ c^2r^2 - \omega^2 W_2(r) \right] d\theta^2 + W_2^{-1}(r)dr^2. \]  
(50)

The other choice is

\[ \xi^2 = 1, \quad \mu = -\omega, \quad \nu = 0, \]  
(51)

then the stationary solution becomes

\[ ds^2 = \left[ -W_2(r) + \omega^2 c^2 r^2 \right] dt^2 - 2\omega c^2 r^2 dtd\theta + c^2 r^2 d\theta^2 + W_2^{-1}(r)dr^2. \]  
(52)

4.3 Case \( \gamma = -1 \)

For the case \( \gamma = -1 \) the Schwarzschild gauge does not hold. However, the static solution is not difficult to obtain

\[ ds^2 = -b^2 r^2 dt^2 + c^2 r^2 d\theta^2 + \left( -\frac{2}{3} \Lambda r_0^{-2} r^4 \right)^{-1} dr^2, \]
\[ \phi = -\frac{1}{2} \ln \left( \frac{r}{r_0} \right), \]  
(53)

After \( T \)-duality the metric becomes

\[ h_{tt} = -\xi^2 b^2 r^2 + \xi^2 \mu^2 c^2 r^2, \]  
(54)
\[ h_{t\theta} = -\xi^2 \nu b^2 r^2 + \mu (1 + \xi^2 \mu \nu) c^2 r^2, \]  
(55)
\[ h_{\theta\theta} = -\xi^2 \nu^2 b^2 r^2 + \xi^{-2} (1 + \xi^2 \mu \nu)^2 c^2 r^2. \]  
(56)

We make the following choice of the parameters

\[ \xi^2 = 1, \quad \mu = 0, \quad \nu = \omega. \]  
(57)

The stationary solution is

\[ ds^2 = -b^2 r^2 dt^2 - 2\omega b^2 r^2 dtd\theta + \left( c^2 r^2 - \omega^2 b^2 r^2 \right) d\theta^2 + \left( -\frac{2}{3} \Lambda r_0^{-2} r^4 \right)^{-1} dr^2. \]  
(58)
5 Einstein-Maxwell-Dilaton Theory

5.1 Spinning Electric Solution

A general static electric charged solution in the EMD theory was found by Chan and Mann [12]. It is characterized by four constants $c, m, q, r_0$

\[
\begin{align*}
  ds^2 &= -U(r)dt^2 + c^2 r^N d\theta^2 + U^{-1}(r)dr^2, \\
  \phi &= k \ln \left( \frac{r}{r_0} \right), \\
  A_t &= -qr^\frac{N}{2} - 1,
\end{align*}
\]  

(59)

where

\[
U(r) = -\frac{8\Lambda r_0^{2-N}}{N(3N-2)} r^N - mr^{1-\frac{N}{2}} - 2(N-2)r_0^{N-2}q^2,
\]

\[
k = \pm \frac{1}{4} \sqrt{\frac{N(2-N)}{\gamma}},
\]

\begin{align*}
  4\alpha k &= \beta k = N - 2, \quad 4\alpha = \beta, \quad 0 < N < 2. 
\end{align*}

(60)

Taking this solution as a seed, the corresponding $H_0$ and $H_1$ in (12) are

\begin{align*}
  H_0^2 &= -Br^N - mr^{1-\frac{N}{2}} - C, \\
  H_1^2 &= c^2 r^N.
\end{align*}

(61)

Here we introduce following combinations of the parameters

\begin{align*}
  B &:= \frac{8\Lambda r_0^{2-N}}{N(3N-2)}, \\
  C &:= \frac{2(N-2)r_0^{N-2}}{N} q^2,
\end{align*}

(62)

to simplify the notation. After the $T$–duality transformation we have

\begin{align*}
  h_{tt} &= \xi^2 (B + \mu^2 c^2) r^N + \xi^2 \left( mr^{1-\frac{N}{2}} + C \right), \\
  h_{t\theta} &= \left[ \xi^2 \nu B + \mu(1 + \xi^2 \mu \nu)c^2 \right] r^N + \xi^2 \nu \left( mr^{1-\frac{N}{2}} + C \right), \\
  h_{\theta\theta} &= \left[ \xi^2 \nu^2 B + \xi^{-2}(1 + \xi^2 \mu \nu)c^2 \right] r^N + \xi^2 \nu^2 \left( mr^{1-\frac{N}{2}} + C \right).
\end{align*}

(63) \quad (64) \quad (65)

We should make a choice of parameters such that the $r^2$ term in $h_{t\theta}$ vanishes and the coefficient of $r^2$ term in $h_{\theta\theta}$ is equal to $c^2$, which means

\[
\xi^2 = \frac{B}{B + \mu^2 c^2}, \quad \nu = -\frac{\mu c^2}{B}.
\]  

(66)
Then the metric $h_{ij}$ becomes

\begin{align*}
h_{tt} & = Br^N + \frac{B}{B + \mu^2 c^2} \left( mr^{1-\frac{N}{2}} + C \right), \\
h_{t\theta} & = -\frac{\mu c^2}{B + \mu^2 c^2} \left( mr^{1-\frac{N}{2}} + C \right), \\
h_{\theta\theta} & = c^2 r^N + \frac{\mu^2 c^4}{B(B + \mu^2 c^2)} \left( mr^{1-\frac{N}{2}} + C \right).
\end{align*}

(67)

It is convenient to present the spinning charged solution in terms of the following variables

\begin{align*}
\omega & := \frac{\mu c^2}{B}, \\
M & := \frac{B}{B + \mu^2 c^2} m, \\
Q^2 & := \frac{B}{B + \mu^2 c^2} q^2, \\
C' & := \frac{2(N-2)r_0^{N-2}}{N} Q^2.
\end{align*}

(70)

Then the metric takes the form

\begin{align*}
\text{d}s^2 & = - \left( -Br^N - Mr^{1-\frac{N}{2}} - C' \right) \text{d}t^2 - 2\omega \left( Mr^{1-\frac{N}{2}} + C' \right) \text{d}t \, \text{d}\theta \\
& + \left[ c^2 r^N + \omega^2 \left( Mr^{1-\frac{N}{2}} + C' \right) \right] \text{d}\theta^2 \\
& + \left[ -Br^N - (1 + \frac{\omega^2}{c^2} B)(Mr^{1-\frac{N}{2}} + C') \right]^{-1} \text{d}r^2.
\end{align*}

(71)

The dilaton and the 1-form are

\begin{align*}
\phi & = k \ln \left( \frac{r}{r_0} \right), \\
A & = Q r^{\frac{N}{2}-1} (-\text{d}t + \omega \text{d}\theta).
\end{align*}

(72)

For the uncharged case, $q = 0$, our solution reduces to the spinning black hole solution in [13] with the parameter normalization $r_0^{2-N} = c^2$.

The horizon radius is defined by the equation

\begin{align*}
Br_H^N + (1 + \frac{\omega^2}{c^2} B)(Mr_H^{1-\frac{N}{2}} + C') = 0.
\end{align*}

(73)

Unfortunately it can not be solved explicitly. But it is easy to check that the charged spinning black holes could have an inner horizon. Moreover the horizon in the extremal case can be found in general

\begin{align*}
r_H^{(\text{ext})} = \left[ \frac{(2 - N)(c^2 + \omega^2 B)M}{2c^2 NB} \right]^{\frac{1}{2N-2}},
\end{align*}

(74)
and the extremality condition is

\[ B^{\frac{N-3}{2N}} \left[ (1 + \frac{\omega^2 B}{c^2}) M \right]^{\frac{2}{3N-2}} \left\{ \left( \frac{2 - N}{2N} \right)^{\frac{2N}{3N-2}} - \left( \frac{2 - N}{2N} \right)^{\frac{2N}{3N-2}} \right\} + \left( 1 + \frac{\omega^2 B}{c^2} \right) C' = 0. \] (75)

Besides this, let us consider the \( g_{\theta \theta} \) component of the metric

\[ g_{\theta \theta} = c^2 r^N + \omega^2 \left( Mr^{1 - \frac{N}{2}} + C' \right). \] (76)

In general we should assume this term to be positive otherwise the angle \( \theta \) will becomes a time–like coordinate which will produce a closed time–like curve. This requirement always holds for the static solution and can be easy achieved by assuming \( M > 0 \) for spinning uncharged solution. But due to the term \( C' \) it could be negative so that we can not preclude the existence of a closed time–like curve for the spinning charged solution.

### 5.2 Spinning Magnetic Solution

Here we consider separately the generation of a spinning magnetic charged solution in the EMD theory. It is not difficult, by solving field equations like those in [12], to find the following static magnetic solution

\[ ds^2 = -c^2 r^N dt^2 + \left( -Br^N + mr^{1 + \frac{N}{2}} + C \right) d\theta^2 + \left( -Br^N + mr^{1 + \frac{N}{2}} + C \right)^{-1} dr^2, \]
\[ \phi = k \ln \left( \frac{r}{r_0} \right), \quad A_\theta = pr^{\frac{N}{2}-1}. \] (77)

Using this solution as a seed, after the \( SL(2, R) \) transformation (13) and a suitable choice of parameters, we get a spinning magnetic solution. The metric of this new solution is

\[ ds^2 = - \left[ cr^2 - \omega^2 \left( Mr^{1 - \frac{N}{2}} + C' \right) \right] dt^2 - 2\omega \left( Mr^{1 - \frac{N}{2}} + C' \right) dtd\theta + \left( -Br^N + Mr^{1 - \frac{N}{2}} + C' \right) d\theta^2 + \left[ -Br^N + \left( 1 + \frac{\omega^2}{c^2} B \right) \left( Mr^{1 - \frac{N}{2}} + C' \right) \right]^{-1} dr^2, \] (78)
and the dilaton and the form fields are
\[ \phi = k \ln \left( \frac{r}{r_0} \right), \quad A = Pr^{\frac{N}{2} - 1}(-\omega dt + d\theta). \] (79)

5.3 Static Dyon Solution

Both spinning electric or magnetic solutions can be treated as a dyon solution by setting \( \omega = P/Q \) in the electric or \( \omega = Q/P \) in the magnetic solutions. The dyon solution is

\[
\begin{align*}
\text{ds}^2 &= - \left( -Br^N - Mr^{1-\frac{N}{2}} - C' \right) dt^2 - 2\frac{P}{Q} \left( Mr^{1-\frac{N}{2}} + C' \right) dt d\theta \\
&\quad + \left[ c^2 r^N + \frac{P^2}{Q^2} \left( Mr^{1-\frac{N}{2}} + C' \right) \right] d\theta^2 \\
&\quad + \left[ -Br^N - (1 + \frac{P^2}{c^2 Q^2} B)(Mr^{1-\frac{N}{2}} + C') \right]^{-1} dr^2,
\end{align*}
\] (80)

and

\[ \phi = k \ln \left( \frac{r}{r_0} \right), \quad A = r^{\frac{N}{2} - 1}(-Qdt + Pd\theta). \] (81)

6 Conclusion

Using the \( T \)-duality symmetry in 2 + 1-dimensional gravity theory we can generate the spin parameter. It enables us to find spinning charged solutions from static ones without solving the complicated field equations. Applying this symmetry we recover the spinning electric charged solution [8] and obtain a new spinning magnetically charged solution and a static dyon in the Einstein-Maxwell theory with a cosmological constant.

For the Brans-Dicke theory a set of static and stationary solutions was obtained in the Einstein frame. Most of them are equivalent the solution of [10,11]. For the EMD theory we find new spinning electric and magnetic charged solutions. The existence of a charge provides the possibility of an inner horizon. The static dyon solution is also presented. All the new spinning charged solutions have quite different properties from the earlier static charged or spinning uncharged solutions. This point requires further investigation.
The solutions presented here do not seem to be the most general ones with or without a dilaton field. A general $SL(2,R)$ transformation includes three parameters. But by a choice of parameters we had restricted them to produce only one spin parameter. It seems that this condition can be relaxed. One of the other two parameters corresponds to rescaling of the radial coordinate, but the other which produces an $r^2$ term in $N^\theta$ can lead to a globally different structure. It is equivalent to applying a coordinate transformation $\theta \rightarrow \theta + Tt$ to the solutions derived here.

The BTZ solution can be interpreted as an exact three-dimensional black string solution [15]. In three dimensions the 3–form field is equivalent to a cosmological constant. So the solutions presented here should have the corresponding charged string counterparts.

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