NISNEVICH DESCENT FOR DELIGNE MUMFORD STACKS

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Abstract. We prove the excision property for the $K$-theory of perfect complexes on Deligne-Mumford stacks. We define the Nisnevich site on the category of such stacks which restricts to the usual Nisnevich site on schemes. Using the refinements of the localization sequence of [17] and [32], we then show that the $K$-theory of perfect complexes satisfies the Nisnevich descent on the category of tame Deligne-Mumford stacks. This is done by showing that the above Nisnevich site is given by a cd-structure which is complete, regular and bounded.

1. Introduction

It is by now well known that the localization and the Mayer-Vietoris are one of the fundamental desired properties for the algebraic $K$-theory of schemes. It was shown by Brown and Gersten [4] that the above two properties yield the Zariski descent for the $K$-theory of coherent sheaves on schemes. In this paper, we concentrate on the study of such properties for the $K$-theory of perfect complexes which is a much more difficult task than the $K$-theory of coherent sheaves.

One of the main features of the descent property is that it essentially reduces the problem of computing the $K$-group of a scheme in terms of the $K$-groups of local schemes which is expectedly an easier task. For the reason of reducing the study of $K$-theory to the local calculations on better behaved schemes and stacks, it is only desirable that one has a descent property for the $K$-theory with respect to the Zariski and possibly finer topologies. The Zariski and the Nisnevich descent for the $K$-theory of perfect complexes on schemes was established by Thomason and Trobaugh [30]. They proved it by showing that this $K$-theory has the localization and the excision properties, both of which are difficult results in their own way.

The $K$-theory of quotient stacks that result from the action of a given group on schemes was developed by Thomason [27]. The algebraic $K$-theory of stacks were later studied in [12] and [31], where the localization sequence for the $K$-theory of coherent sheaves was established. The general form of the localization sequence for the $K$-theory of perfect complexes on stacks is still not known. However, for the Deligne-Mumford stacks which are tame, have coarse moduli schemes and have the resolution property, such a localization sequence was first shown to exist in [17]. A more general and categorical form of such a localization sequence has been recently obtained by Toën [32].

The first form of descent theorem for the $K$-theory of stacks was proven by Thomason in [29], where he showed that the mod-$n$ $K$-theory of perfect complexes on smooth quotient stacks arising from the action of a fixed smooth and affine group scheme on schemes, satisfies the descent with respect to the isovariant étale topology if one inverts the Bott element. Our goal in this paper is to prove the localization, excision and descent theorems for the $K$-theory of perfect complexes on Deligne-Mumford stacks in as general form as possible. We briefly describe these results in the following paragraphs.

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Let $S$ be a fixed noetherian affine scheme. We denote the category of separated Deligne-Mumford stacks of finite type over $S$ by $\mathcal{DM}_S$. Using the characterization of perfect complexes in terms of the compact objects in the unbounded derived category of complexes of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves and its version with support, and then combining this with the results of [32], we first establish the appropriate localization for the $K$-theory of perfect complexes on separated and tame Deligne-Mumford stacks in Theorem 4.5. Our next main result of this paper is Theorem 5.5, where we establish an excision theorem for the above class of stacks. This is the first result of its kind for stacks and generalizes the corresponding result of [30] in the world of stacks. As a consequence of the above results, we show in Corollary 5.6 that the $K$-theory of perfect complexes satisfies the Mayer-Vietoris property in the category of separated and tame Deligne-Mumford stacks with coarse moduli schemes. This plays a very crucial role in our proof of the Nisnevich descent for the $K$-theory.

In Section 6, we introduce the Nisnevich site on the category $\mathcal{DM}_S$. It turns out that this is a Grothendieck site and its restriction to the full subcategory of schemes is the usual Nisnevich site for schemes. We also introduce a Nisnevich $cd$-structure on $\mathcal{DM}_S$ in the sense of [34]. In Theorem 7.6 we show that the Nisnevich site of $\mathcal{DM}_S$ is in fact given by the above $cd$-structure. It is further shown in Section 8 that this $cd$-structure is complete, regular and bounded. Our final main result is Theorem 9.5 where we show that the $K$-theory of perfect complexes satisfies the Nisnevich descent for representable maps. We also draw several consequences, one of which is to show that the Nisnevich (and Zariski) cohomological dimension of a stack in $\mathcal{DM}_S$ is bounded by the dimension of the stack. This is first result of its kind for stack. We also deduce a Brown-Gersten spectral sequence for the $K$-theory of such stacks.

The computation of the equivariant algebraic $K$-theory and equivariant higher Chow groups of schemes with group action has been a very active area of research ever since Thomason developed this theory. It often turns out that the equivariant $K$-theory is easier to compute using the tools of representation theory and one can often use them to compute the non-equivariant $K$-theory via spectral sequences. We refer to [18] for some results in this direction and to [21] for applications of these techniques. It is often difficult to prove various expected results about the equivariant $K$-theory which are otherwise known in the non-equivariant case. Our perspective is to look at these equivariant $K$-groups as special cases of the $K$-groups of stacks, which constitute a much larger class of geometric objects and where one can develop the general theory and prove those results for stacks which are hitherto known only for schemes, using the categorical point of view.

We end the introduction with some remarks about our ongoing and future work of which the present paper is the first part. Our main motivation for writing this paper was to undertake a deeper study of the $K$-theory of perfect complexes on stacks. In the upcoming sequel [6] to this paper, we study the homotopy $K$-theory of Deligne-Mumford stacks, introduce the $cdh$-site of such stacks and the final goal is to show that the homotopy $K$-theory of stacks satisfies the $cdh$-descent. We hope to deduce several important consequences of these results for the $K$-theory of stacks which have been proven for schemes in recent years.

Finally, as the reader will observe, our focus in this paper is to study certain fundamental properties of the $K$-theory of Deligne-Mumford stacks. There is a vast majority of stacks occurring in geometry which are of this kind. A very interesting class of such stacks are the moduli stacks of stable maps from $n$-pointed stable curves of genus $g$ to a projective variety $X$, denoted by $\mathcal{M}_{g,n}(X,\beta)$. There has been a lot of interest in the $K$-theory and quantum $K$-theory of these moduli stacks from the point of view of Gromov-Witten theory. One should notice that these moduli stacks are mostly singular even if $X$ is smooth. Nonetheless, it would be interesting to know if the results of this paper can be proven for more general
Artin stacks. As the techniques used in this paper suggest, such a generalization is very much expected and we hope to take this up in a future project.

2. Preliminaries

In this section, we recall the definition and some basic properties of Deligne-Mumford stacks which we shall use frequently. We refer the reader to [19] for more details. All the stacks in this article will be defined over a fixed noetherian and base scheme $S$. In some cases, we shall take $S$ to be the spectrum of a field in this paper.

**Definition 2.1.** A stack $X$ is called a Deligne-Mumford stack if

1. The diagonal $\Delta_X : X \to X \times_S X$ is representable, quasi-compact and separated.
2. There is an $S$-scheme $U$ and an étale surjective morphism $U \to X$.

The scheme $U$ is called an atlas of $X$. Note that the representability of the diagonal implies that any atlas $U \to X$ such as in the second condition above is automatically representable. The stack $X$ is called separated if the diagonal is proper. It is called a noetherian stack if $U$ can be chosen to be a noetherian scheme. An algebraic space over $S$ is a Deligne-Mumford stack where the diagonal is an embedding. It is separated if the diagonal is a closed embedding. We say that $X$ is of finite type over $S$ if it has an atlas which is of finite type over $S$. The following well known theorem provides a large class of examples of Deligne-Mumford stacks.

**Theorem 2.2** (Deligne-Mumford, [3]). Let $X/S$ be a noetherian scheme of finite type and let $G/S$ be a smooth affine group scheme (of finite type over $S$) acting on $X$ such that the stabilizers of geometric points are finite and reduced. Then the quotient stack $[X/G]$ (cf. [3] 7.17) is a noetherian Deligne-Mumford stack. If the stabilizers are trivial, then $[X/G]$ is an algebraic space. Furthermore, the stack is separated if and only if the action is proper.

**Remark 2.3.** If the action $\Psi : G \times X \to X \times X$ is proper, then the fiber of the action map over a geometric point $x$ of $X$ is a closed group subscheme of $G$ which is proper over Spec($k(x)$). Now since $G$ is affine over $S$, this fiber must be finite. Thus proper actions have finite stabilizers. In particular, if $S$ is defined over a field of characteristic zero, then the quotient stack $[X/G]$ for a proper action is a Deligne-Mumford stack. However, in positive characteristic, this need not be true since the stabilizers of geometric points can be non-reduced.

A stack in this paper will always mean a noetherian and separated Deligne-Mumford stack over a noetherian base scheme $S$, unless mentioned otherwise. For a noetherian and affine base scheme $S$, we shall denote the category of noetherian and separated Deligne-Mumford stacks over $S$ by $\mathcal{DM}_S$. Let $\text{Aff}/S$ denote the category of morphisms $T \to S$, where $T$ is an affine scheme. The category $\mathcal{DM}_S$ is clearly closed under fiber products. To say more about this category, recall that a stack $X$ is quasi-compact if it has an étale atlas $U \to X$ such that $U$ is quasi-compact. In particular, all noetherian stacks are quasi-compact. We also recall that a morphism $f : X \to Y$ of stacks is quasi-compact if for all $T \in \text{Aff}/S$ and all morphisms $T \to Y$, the stack $X \times_Y T$ is quasi-compact. A morphism $f : X \to Y$ of stacks is called separated if the diagonal $\Delta_X : X \to X \times_Y X$ is proper, i.e., it is universally closed. The morphism $f$ is said to be representable if for every morphism $Z \to Y$, where $Z$ is an algebraic space, the stack $X \times_Y Z$ is also an algebraic space. We say that $f$ is strongly representable if $X \times_Y Z$ is a scheme whenever $Z$ is a scheme. It follows from [19] Lemme 4.2 that any morphism $f : X \to Y$, where $X$ is a scheme, is strongly representable. This fact will be used in this paper repeatedly without any further mention of the above reference.

**Proposition 2.4.** Every morphism $f : X \to Y$ in $\mathcal{DM}_S$ is quasi-compact and separated.
\textbf{Proof.} We first show that \( f \) is quasi-compact. Let \( U \xrightarrow{u} Y \) be a noetherian affine atlas and let \( f' : Z = X \times_Y U \to U \) be the base change. We first claim that the quasi-compactness of \( f' \) implies the same for \( f \). To prove this claim, let \( T \xrightarrow{g} Y \) be a morphism with \( T \in \text{Aff}/S \) and let \( X' = X \times_Y T \). Consider the following commutative diagram, where \( W = U \times_Y T \).

\[
\begin{array}{ccc}
V \to T \times_Y Z & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
W & \xrightarrow{f'} & T \\
\downarrow & & \downarrow \\
U & \xrightarrow{u} & Y
\end{array}
\]

Since \( Y \) is separated and \( U, T \) are affine, we see that \( W \) is affine. Now the quasi-compactness of \( f' \) implies that \( T \times_Y Z \) is quasi-compact and hence there is an atlas \( V \to T \times_Y Z \) which is quasi-compact. On the other hand, as \( u \) is étale and surjective, we see that \( T \times_Y Z \to X' \) is also étale and surjective. In particular, \( V \to X' \) is an atlas, which shows that \( X' \) is quasi-compact. This proves the claim.

Using the claim, we can assume that \( Y \) is a noetherian affine scheme. Let \( T \to U \) be a morphism with \( T \in \text{Aff}/S \) and put \( X' = T \times_Y X \). Let \( V \xrightarrow{v} X \) be a noetherian atlas of \( X \). Then it is easy to see that \( V \times_T W = W \to X' \) is an étale atlas of \( X' \). On the other hand, \( V \to Y \) is a morphism of noetherian schemes and hence quasi-compact, which means that \( W \) is a quasi-compact scheme. In particular, \( f \) is quasi-compact.

We now prove the separatedness of \( f \). Since \( X \) and \( Y \) are separated Deligne-Mumford stacks over \( S \), it suffices to show that if \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are morphisms of stacks such that \( h = g \circ f \) is separated, then \( f \) is also separated. This is easily proved using the valuative criteria of separatedness. So let \( V = \text{Spec}(V) \) be the spectrum of a valuation ring with the valuation field \( K \). Let \( U = \text{Spec}(K) \) and let \( t : V \to Y \). Put \( t' = g \circ t \). Let \( x_1, x_2 \in \text{Ob}(X_V) \) and let \( \beta : f \circ x_1 \xrightarrow{\sim} f \circ x_2 \) and \( \alpha : (x_1)_U \xrightarrow{\sim} (x_2)_U \) be given such that \( f(\alpha) = (\beta)_U \). Put \( \beta' = g(\beta) : h \circ x_1 \xrightarrow{\sim} h \circ x_2 \). This implies that \( h(\alpha) = (\beta')_U \). Hence the separatedness of \( h \) implies that there exists \( \tilde{\alpha} : x_1 \xrightarrow{\sim} x_2 \) extending \( \alpha \). In particular, \( f \) is separated.

Let \( P \) be a property of morphisms of schemes which is of local nature for the étale topology. Examples of such properties are a morphism being flat, smooth, étale, unramified, etc.

\textbf{Definition 2.5.} A morphism \( f : X \to Y \) of stacks is said to have property \( P \) if there are étale atlases \( U \to X, V \to Y \) and a compatible morphism \( U \to V \) with the property \( P \).

\textbf{Definition 2.6.} Let \( P \) be a property of schemes which is of local nature in the étale topology. We say that the stack \( X \) has property \( P \) if there is an (étale) atlas \( U \to X \) such that \( U \) has property \( P \). The stack \( X \) is said to have dimension \( n \) if it has an atlas \( U \to X \) of Krull dimension \( n \).

A stack \( X \) is called a \textit{tame} stack if for any geometric point \( s : \text{Spec}(\Omega) \to X \), the group \( \text{Aut}_\Omega(s) \) has order prime to the characteristic of the field \( \Omega \). In particular, if \( G \) is an algebraic group acting on a scheme \( X \) of finite type over a field \( k \) such
that the stabilizers of geometric points are finite and reduced, then the quotient stack \([X/G]\) is tame if and only if the orders of stabilizer groups are prime to the characteristic of \(k\). Thus, all stacks over a field of characteristic zero are tame.

The following lemma about the separated Deligne-Mumford stacks is well-known.

**Lemma 2.7** ([1], Lemma 2.2.3). Let \(X\) be a noetherian and separated Deligne-Mumford stack. Then there is an étale cover \(\{X_\alpha \to X\}_{\alpha \in I}\) such that each \(X_\alpha\) is of the form \(X_\alpha = [U_\alpha/\Gamma_\alpha]\), where \(U_\alpha\) is an affine noetherian scheme and \(\Gamma_\alpha\) is a finite group acting on \(U_\alpha\).

**Proof.** (Sketch) It follows from [23, Theorem 6.12] that \(X\) has a coarse moduli space \(q : X \to Y\) which is a noetherian and separated algebraic space. Choose a geometric point \(y_\alpha\) of \(Y\) and let \(Y_\alpha^{sh}\) denote the spectrum of the strict henselization of \(Y\) at the point \(y_\alpha\). In particular, \(Y_\alpha^{sh}\) is a noetherian affine scheme. It is then shown in *loc. cit.* that there is a scheme \(U_\alpha\) which is finite over \(Y^{sh}\) and a finite group \(\Gamma_\alpha\) acting on \(U_\alpha\) such that \(q^{-1}(Y_\alpha^{sh}) = [U_\alpha/\Gamma_\alpha]\) and \(Y_\alpha^{sh} = U_\alpha/\Gamma_\alpha\). Since \(U_\alpha \to Y_\alpha^{sh}\) is finite, we see that \(U_\alpha\) is a noetherian affine scheme. Since \(Y_\alpha^{sh}\) is an inverse limit of affine étale covers, we see that there is an étale cover \(\{Y_\alpha \to Y\}\) by noetherian affine schemes such that for each \(\alpha \in I\), one has \(q^{-1}(Y_\alpha) = [U_\alpha/\Gamma_\alpha]\) with \(U_\alpha\) noetherian and affine. \(\square\)

**2.1. Sheaves on Deligne-Mumford stacks.** We recall from [19, 12.1] that the (small) étale site \(\text{ét}(X)\) of a Deligne-Mumford stack is the category whose objects are the representable étale morphisms \(u : U \to X\) with \(U\) a scheme (denoted by \((U, u)\)) and morphisms between \((U, u)\) and \((V, v)\) are 1-morphisms of schemes \(f : U \to V\) such that there is a 2-isomorphism \(\alpha : v \circ f \to u\). The Grothendieck topology on \(\text{ét}(X)\) is generated by the coverings \(\text{Cov}(U, u)\) whose objects are families of morphisms \(\alpha_i : (U_i, u_i) \to (U, u)\) such that the morphism

\[
\coprod_i u_i : \coprod_i U_i \to U
\]

of schemes is surjective.

A sheaf (resp. presheaf) on \(X\) is a sheaf (resp. presheaf) on the étale site \(\text{ét}(X)\) in the sense of [2]. In particular, a sheaf of \(\mathcal{O}_X\)-modules (resp. quasi-coherent sheaf) \(\mathcal{F}\) on \(X\) is the following data.

(i) For each atlas \(u : U \to X\), an \(\mathcal{O}_U\)-module (resp. quasi-coherent sheaf) \(\mathcal{F}_U\) on \(U\).

(ii) For any morphism of atlases \(f : (U, u) \to (V, v)\), an isomorphism \(\mathcal{F}_U \xrightarrow{\sim} f^*(\mathcal{F}_V)\). The above isomorphisms are required to satisfy the usual cocycle conditions.

**Remark 2.8.** We shall often consider the bigger étale site \(\mathcal{E}(X)\), whose objects are all morphisms \([X' \xrightarrow{f} X]\), where \(X'\) is a Deligne-Mumford stack and \(f\) is an étale morphism (not necessarily representable). The morphisms in this site are defined as in \(\text{ét}(X)\). Note that every covering in the Grothendieck site \(\mathcal{E}(X)\) has refinements which are from \(\text{ét}(X)\). It is then easy to see that the sites \(\text{ét}(X)\) and \(\mathcal{E}(X)\) have the same topos. We shall denote this common étale topos by \(\text{ét}(X)\).

A coherent sheaf is a quasi-coherent sheaf \(\mathcal{F}\) such that \(\mathcal{F}_U\) is coherent for each atlas \(U \to X\). A vector bundle is a coherent sheaf \(\mathcal{F}\) such that \(\mathcal{F}_U\) is locally free for each atlas \(U \to X\). Let \(\text{Sh}(X)\) (resp. \(\text{Mod}(X)\)) (resp. \(\text{QC}(X)\)) denote the abelian category of sheaves of abelian groups (resp. sheaves of \(\mathcal{O}_X\)-modules) (resp. quasi-coherent sheaves) on \(X\).
For any sheaf of abelian groups $\mathcal{F}$, we define the global section functor $\Gamma(X, \mathcal{F})$ as $\text{Hom}(\mathbb{Z}_X, \mathcal{F})$. This is a left exact functor and its right derived functors are denoted by $H^i(X, \mathcal{F})$ for $i \geq 0$. It is easy to check that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $\Gamma(X, \mathcal{F}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \Gamma(S, \pi_*(\mathcal{F}))$, where $\pi : X \to S$ is the structure morphism. Moreover, $H^i(X, -)$ is same as the derived functors of $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, -)$ computed in the category $\text{Mod}(X)$. The following result was proven for the special case of strongly representable maps in \cite[Lemma 1.6]{17}.

**Lemma 2.9.** Let $X$ be a stack and let $j : Y \to X$ be an étale morphism of stacks. Then there is a functor $j_* : \text{Sh}(Y) \to \text{Sh}(X)$ which is exact and is a left adjoint to the restriction functor $j^*$. Furthermore, $j_*$ takes $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules. In particular, $j^*$ preserves limits and injective sheaves of $\mathcal{O}_X$-modules.

**Proof.** Except for the last assertion, this result is already shown in \cite[Lemma 1.6]{17}. The last assertion is also shown there for the strongly representable maps. We just have to show that the general case can be reduced from the case of strongly representable maps. So let $(U, u) \in \text{Ét}(X)$ and let $\mathcal{F} \in \text{Mod}(Y)$. It is enough to show that $u^* \circ j_!(\mathcal{F})$ is an $\mathcal{O}_U$-module. Consider the Cartesian diagram

$$
\begin{array}{ccc}
V & \xrightarrow{j'} & U \\
\downarrow{u'} & & \downarrow{u} \\
Y & \xrightarrow{j} & X
\end{array}
$$

We now note from loc. cit. that

\begin{equation}
(2.1)
\quad j_!(\mathcal{F})(((U, u)) = \bigoplus_{u' : U \to Y | j_0 u' = u} \mathcal{F}((U, u')),
\end{equation}

from which it is easy to see that $u^* \circ j_!(\mathcal{F}) = j'_! \circ u'^* (\mathcal{F})$. This reduces the problem to the case when $X$ is a scheme. We now let $V \to Y$ an étale atlas. There is a natural map $v_!(v^* (\mathcal{F})) \to \mathcal{F}$ and $v_!(v^* (\mathcal{F}))$ is a sheaf of $\mathcal{O}_Y$-modules since $v$ is strongly representable. Since $v^* \circ v_!$ is identity, as follows from (2.1), we see that this map is an isomorphism when restricted to $V$. Since $V \to Y$ is an atlas, this map must be an isomorphism. Thus we can assume that $\mathcal{F} = v_!(\mathcal{G})$, where $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules. We have then $j_!(\mathcal{F}) = j_* \circ v_!(\mathcal{G}) = (j \circ v)_!(\mathcal{G})$ and this reduces to the case when $X$ is a scheme and $Y$ is an étale open subset of $X$. In this case, the definition in (2.1) agrees with the definition of $j_!$ in \cite[Remark 3.18]{20} for schemes, which is known to preserve the sheaves of $\mathcal{O}_X$-modules. \hfill $\square$

**Lemma 2.10.** (\cite[Lemma 1.7]{17}) Let $X$ be a stack. Then $\text{QC}(X)$ and $\text{Mod}(X)$ are Grothendieck categories and hence have enough injectives and all limits.

**Proof.** Since there is a minor error in the proof of \cite[Lemma 1.7]{17}, we reproduce the complete proof here. Recall that an abelian category is a Grothendieck category if it has inductive limits, a family of generators and if the filtered inductive limits are exact. The first and the last conditions are easily verified for the given categories. So we only need to show that both the categories have families of generators. Since every quasi-coherent sheaf on $X$ is a direct limit of a family of its coherent subsheaves (cf. \cite[Proposition 15.4]{19}), the family of the isomorphism classes of coherent sheaves clearly forms a generating family for the category $\text{QC}(X)$ (cf. \cite[B.3]{30}).

To give a generating family for $\text{Mod}(X)$, let $u : U \to X$ be an atlas. We claim that $\{(u \circ j)_!(\mathcal{O}_Y)| V \to U \text{ open} \}$ is the required family. As before, let $\mathcal{F} \hookrightarrow \mathcal{G}$ be
an inclusion in $\text{Mod}(X)$ such that $G/F$ is not zero. Then there is an open subset $V \to U$ such that this condition remains true when we restrict these sheaves to $V$. Now we can further restrict $V$ to ensure that there is map $O_V \to (u \circ j)^*(G)$ such that the composite $O_V \to (u \circ j)^*(G/F)$ is not zero. Now we use Lemma 2.9 to get a map $(u \circ j)^*(O_V) \to G$ such that the composite $(u \circ j)^*(O_V) \to G/F$ is not zero. This proves the claim.

Finally, it is known that a Grothendieck category has enough injectives and the Gabriel-Popescu embedding theorem implies that such categories have all limits.

\[\square\]

**Remark 2.11.** It is well known (and easy to check) that the category $\text{Sh}(X)$ is also a Grothendieck category (cf. [2], Theorem 1.6).

### 3. Perfect complexes on stacks

Let $X$ be a stack. Let $C(qc/X)$ and $C_{qc}(X)$ respectively denote the categories of (possibly unbounded) cochain complexes of quasi-coherent sheaves and cochain complexes of $O_X$-modules with quasi-coherent cohomology. Let $D(qc/X)$ and $D_{qc}(X)$ denote the corresponding derived categories. Note that $D_{qc}(X)$ is a full subcategory of the derived category $D(X)$ of all $O_X$-modules. Let $C(X)$ denote the category of unbounded cochain complexes of $O_X$-modules. If $K$ is a complex of $O_X$-modules, we denote its cohomology sheaves by $H^i(K)$. Recall that a stack $X$ is said to have the resolution property if every coherent sheaf on $X$ is a quotient of a locally free sheaf. It follows from the results of Thomason [28] that the quotient stack $[X/G]$ for a linear action of an algebraic group $G$ on a quasi-projective variety $X$ over a field has the resolution property. In particular, this property is satisfied for practically all geometric stacks.

For a Grothendieck category $\mathcal{A}$, let $C(\mathcal{A})$ denote the abelian category of all (possibly unbounded) complexes of objects in $\mathcal{A}$ and let $D(\mathcal{A})$ denote its derived category. Spaltenstein [26] defined a complex $I$ over $\mathcal{A}$ to be $K$-injective if for every acyclic complex $J$, the complex of abelian groups $\text{Hom}^\bullet(J, I)$ is acyclic. This is equivalent to saying that in the homotopy category, there are no nonzero morphisms from an acyclic complex to $I$. Serpe [21] has shown that every unbounded complex over $\mathcal{A}$ has a $K$-injective resolution. On the other hand, since $\mathcal{A}$ has enough injectives, a complex over $\mathcal{A}$ also has a Cartan-Eilenberg resolution (cf. [15] Appendix A). It is known [26] that a Cartan-Eilenberg resolution of an unbounded complex over $\mathcal{A}$ need not give a $K$-injective resolution. In other words, the Cartan-Eilenberg hypercohomology may not coincide with the derived functor cohomology for complexes in a Grothendieck category. The following result was proven in [17], Proposition 2.2, Corollary 2.6] whose scheme version was shown by Keller in [15] Appendix A).

**Theorem 3.1.** Let $X$ be a stack of dimension $n$ and let $K \in C_{qc}(X)$. Let $K \to I^{••}$ be a Cartan-Eilenberg resolution of $K$ in $C(X)$. Then $K \to \text{Tot}I$ is a $K$-injective resolution of $K$, where $\text{Tot}I$ is the product total complex of $I^{••}$. If $X$ is a tame stack with the resolution property, then every Cartan-Eilenberg resolution of a complex $K \in C(qc/X)$ is a $K$-injective resolution.

Let $X$ be a stack and let $E$ be a cochain complex of $O_X$-modules with quasi-coherent cohomology.

**Definition 3.2** ([13]). We say that $E$ is pseudo-coherent if all cohomology sheaves $H^i(E)$ are coherent, and $E$ is cohomologically bounded above, i.e., $H^i(E) = 0$ for $i \gg 0$. 

Definition 3.3. We say that $E$ is strictly perfect if it is a bounded complex of vector bundles on $X$. We say that $E$ is perfect if there is an atlas $U \rightarrow X$ such that $u^*(E)$ is isomorphic to a strictly perfect complex in $D(U)$.

The following properties of perfect complexes were proven in [17, Proposition 4.3]. We also refer the reader to [14, Proposition 2.9] for some other properties of perfect complexes on more general stacks.

Proposition 3.4. Let $X$ be as above and let $E \in C_{qc}(X)$.

(i) If $E$ is perfect, then it is pseudo-coherent and cohomologically bounded.

(ii) $E$ is perfect if and only if there is an atlas $V \rightarrow X$ and a strictly perfect complex $F$ on $V$ with a morphism $F \rightarrow v^*(E)$ which is a quasi-isomorphism.

(iii) If $X$ is a scheme, then $E$ is perfect if and only if it is perfect in the sense of [30].

(iv) If $E \cong F$ in $D_{qc}(X)$, then $E$ is perfect if and only if $F$ is so.

(v) If $E = E_1 \oplus E_2$, then $E$ is perfect if and only if $E_1$ and $E_2$ are so.

Our aim in this and the next section is to characterize the perfect complexes on stacks in terms of the full subcategory of compact objects in certain triangulated category. We begin with a brief recall of the relevant notions. Recall that for a triangulated category $\mathcal{C}$ admitting arbitrary direct sums, an object $E$ in $\mathcal{C}$ is compact if $\text{Hom}_{\mathcal{C}}(E, -)$ commutes with direct sums. Let $\mathcal{C}^c$ be the full subcategory of $\mathcal{C}$ consisting of compact objects. Then $\mathcal{C}^c$ is a triangulated category and $\mathcal{C}$ is called compactly generated if $\mathcal{C}$ is generated by $\mathcal{C}^c$. If $\mathcal{E} = (A_\lambda)_{\lambda \in I}$ is a set of objects in $\mathcal{C}$, then we say that $\mathcal{E}$ classically generates $\mathcal{C}$ if the smallest thick triangulated subcategory of $\mathcal{C}$ containing $\mathcal{E}$ is equal to $\mathcal{C}$ itself. We say that $\mathcal{C}$ is finitely generated if it is classically generated by one object. We shall need the following results in the next section.

Lemma 3.5. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor between Grothendieck categories which has an exact left adjoint. Then $F : C(\mathcal{A}) \rightarrow C(\mathcal{B})$ preserves the $K$-injective complexes.

Proof. Let $G$ be the exact left adjoint of $F$. Let $A$ be a $K$-injective complex in $C(\mathcal{A})$ and let $B$ be an acyclic complex in $C(\mathcal{B})$. Then the exactness of $G$ implies that $G(B)$ is acyclic. Now the adjointness of the pair $(G, F)$ implies that the complex $\text{Hom}_\mathcal{A}^*(B, F(A))$ is canonically isomorphic to the complex $\text{Hom}_\mathcal{A}^*(G(B), A)$, and this latter complex is acyclic because $G(B)$ is acyclic and $A$ is $K$-injective. □

Corollary 3.6. Consider the Cartesian diagram of stacks

\[
\begin{array}{ccc}
W & \xrightarrow{t} & Y \\
g \downarrow & & \downarrow f \\
Z & \xrightarrow{s} & X,
\end{array}
\]

where $s$ is étale. For any $A \in D(Y)$, the natural map $s^* Rf_*(A) \rightarrow Rg_* t^*(A)$ is an isomorphism in $D(Z)$.

Proof. Since $s$ is étale, we see that $t$ is also étale and hence $s^*$ and $t^*$ are exact functors on $\text{Mod}(\mathcal{X})$ and $\text{Mod}(\mathcal{Y})$ respectively. It follows from Lemmas 2.9, 2.10 and 3.5 that $s^*$ and $t^*$ preserve $K$-injective complexes. To prove the corollary, we can assume that $A$ is $K$-injective. Then we have $s^* Rf_*(A) = s^* f_*(A)$ and $Rg_* t^*(A) = g_* t^*(A)$. Thus we only need to show that for a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules, the natural map $s^* f_*(\mathcal{F}) \rightarrow g_* t^*(\mathcal{F})$ is an isomorphism. But this is immediate from

\[\text{Definition 3.3.} \quad \text{We say that } E \text{ is strictly perfect if it is a bounded complex of vector bundles on } X. \ \text{We say that } E \text{ is perfect if there is an atlas } U \rightarrow X \text{ such that } u^*(E) \text{ is isomorphic to a strictly perfect complex in } D(U).\]

\[\text{The following properties of perfect complexes were proven in [17, Proposition 4.3]. We also refer the reader to [14, Proposition 2.9] for some other properties of perfect complexes on more general stacks.}\]

\[\text{Proposition 3.4. Let } X \text{ be as above and let } E \in C_{qc}(X).\]

\[\text{(i) If } E \text{ is perfect, then it is pseudo-coherent and cohomologically bounded.}\]

\[\text{(ii) } E \text{ is perfect if and only if there is an atlas } V \rightarrow X \text{ and a strictly perfect complex } F \text{ on } V \text{ with a morphism } F \rightarrow v^*(E) \text{ which is a quasi-isomorphism.}\]

\[\text{(iii) If } X \text{ is a scheme, then } E \text{ is perfect if and only if it is perfect in the sense of [30].}\]

\[\text{(iv) If } E \cong F \text{ in } D_{qc}(X), \text{ then } E \text{ is perfect if and only if } F \text{ is so.}\]

\[\text{(v) If } E = E_1 \oplus E_2, \text{ then } E \text{ is perfect if and only if } E_1 \text{ and } E_2 \text{ are so.}\]

\[\text{Our aim in this and the next section is to characterize the perfect complexes on stacks in terms of the full subcategory of compact objects in certain triangulated category. We begin with a brief recall of the relevant notions. Recall that for a triangulated category } \mathcal{C} \text{ admitting arbitrary direct sums, an object } E \text{ in } \mathcal{C} \text{ is compact if } \text{Hom}_{\mathcal{C}}(E, -) \text{ commutes with direct sums. Let } \mathcal{C}^c \text{ be the full subcategory of } \mathcal{C} \text{ consisting of compact objects. Then } \mathcal{C}^c \text{ is a triangulated category and } \mathcal{C} \text{ is called compactly generated if } \mathcal{C} \text{ is generated by } \mathcal{C}^c. \text{ If } \mathcal{E} = (A_\lambda)_{\lambda \in I} \text{ is a set of objects in } \mathcal{C}, \text{ then we say that } \mathcal{E} \text{ classically generates } \mathcal{C} \text{ if the smallest thick triangulated subcategory of } \mathcal{C} \text{ containing } \mathcal{E} \text{ is equal to } \mathcal{C} \text{ itself. We say that } \mathcal{C} \text{ is finitely generated if it is classically generated by one object. We shall need the following results in the next section.}\]

\[\text{Lemma 3.5. Let } F : \mathcal{A} \rightarrow \mathcal{B} \text{ be a functor between Grothendieck categories which has an exact left adjoint. Then } F : C(\mathcal{A}) \rightarrow C(\mathcal{B}) \text{ preserves the } K\text{-injective complexes.}\]

\[\text{Proof. Let } G \text{ be the exact left adjoint of } F. \text{ Let } A \text{ be a } K\text{-injective complex in } C(\mathcal{A}) \text{ and let } B \text{ be an acyclic complex in } C(\mathcal{B}). \text{ Then the exactness of } G \text{ implies that } G(B) \text{ is acyclic. Now the adjointness of the pair } (G, F) \text{ implies that the complex } \text{Hom}_{\mathcal{A}}^*(B, F(A)) \text{ is canonically isomorphic to the complex } \text{Hom}_{\mathcal{A}}^*(G(B), A), \text{ and this latter complex is acyclic because } G(B) \text{ is acyclic and } A \text{ is } K\text{-injective.}\]

\[\text{Corollary 3.6. Consider the Cartesian diagram of stacks}\]

\[
\begin{array}{ccc}
W & \xrightarrow{t} & Y \\
g \downarrow & & \downarrow f \\
Z & \xrightarrow{s} & X,
\end{array}
\]

\[\text{where } s \text{ is étale. For any } A \in D(Y), \text{ the natural map } s^* Rf_*(A) \rightarrow Rg_* t^*(A) \text{ is an isomorphism in } D(Z).\]

\[\text{Proof. Since } s \text{ is étale, we see that } t \text{ is also étale and hence } s^* \text{ and } t^* \text{ are exact functors on } \text{Mod}(\mathcal{X}) \text{ and } \text{Mod}(\mathcal{Y}) \text{ respectively. It follows from Lemmas 2.9, 2.10 and 3.5 that } s^* \text{ and } t^* \text{ preserve } K\text{-injective complexes. To prove the corollary, we can assume that } A \text{ is } K\text{-injective. Then we have } s^* Rf_*(A) = s^* f_*(A) \text{ and } Rg_* t^*(A) = g_* t^*(A). \text{ Thus we only need to show that for a sheaf } \mathcal{F} \text{ of } \mathcal{O}_Y\text{-modules, the natural map } s^* f_*(\mathcal{F}) \rightarrow g_* t^*(\mathcal{F}) \text{ is an isomorphism. But this is immediate from}\]

the definition of the push-forward and the pull-back maps on the étale topoi of the underlying stacks.

4. CHARACTERIZATION OF PERFECT COMPLEXES ON STACKS

In this section, we prove our main result on the characterization of the category of perfect complexes on a separated Deligne-Mumford stack \( X \) in terms of the full subcategory of the compact objects in \( D_{qc}(X) \). We begin with the following result.

**Proposition 4.1.** Let \( f : Y \to X \) be a strongly representable morphism in \( DM_\mathcal{S} \). Then \( Rf_* \) maps \( D_{qc}(Y) \) into \( D_{qc}(X) \) and it commutes with arbitrary direct sums on \( D_{qc}(Y) \).

**Proof.** Let \( U \to X \) be a noetherian atlas for \( X \) and consider the Cartesian diagram

\[
\begin{array}{ccc}
V & \to & Y \\
\downarrow g & & \downarrow f \\
U & \to & X
\end{array}
\]

Since \( f \) is strongly representable, we see that \( g \) is a morphism of noetherian schemes. To show that \( Rf_* \) maps \( A \in D_{qc}(X) \) into \( D_{qc}(X) \) and it commutes with arbitrary direct sums on \( D_{qc}(Y) \), it suffices to show that \( u^* f^*(A) \) has quasi-coherent cohomology sheaves. However, it follows from Corollary 3.6 that \( u^* f^*(A) \xrightarrow{\sim} g_* v^*(A) \) on \( U \) of \( X \).

To show the second part, let \( (A_\lambda) \) be a family of objects in \( D_{qc}(Y) \) and let \( A = \bigoplus A_\lambda \). It suffices again to show that the natural map \( u^* \left( \bigoplus \frac{Rf_* A_\lambda}{\lambda} \right) \to u^* Rf_* (A) \) is an isomorphism on any étale atlas \( U \) of \( X \). However, we have

\[
u^* Rf_* (A) \cong Rg_* v^*(A) \qquad \text{(by Corollary 3.6)}
\]

\[
\cong Rg_* \left( \bigoplus \lambda v^* A_\lambda \right)
\]

\[
\cong \bigoplus \lambda Rg_* v^* A_\lambda
\]

\[
\cong \bigoplus \lambda u^* Rf_* A_\lambda \qquad \text{(by Corollary 3.6)}
\]

\[
\cong u^* \left( \bigoplus \lambda Rf_* A_\lambda \right),
\]

where the second and the last isomorphisms follow from the fact that \( V \in \acute{E}t(Y) \) and \( U \in \acute{E}t(X) \) and the third isomorphism follows from [3, Corollary 3.3.4] as \( g \) is a morphism of noetherian schemes and hence quasi-compact and separated by Proposition 2.4. This completes the proof. \( \square \)

**Corollary 4.2.** Let \( X \) be a stack and let \( U \to X \) be an open substack with complement \( Z \to X \). Let \( p : V \to X \) be a representable and étale morphism of stacks and let \( T = p^{-1}(Z) \). Then \( p^* : D_{qc}(X) \to D_{qc}(V) \) on \( Z \to V \) preserves perfect and compact objects.

**Proof.** Since \( p \) is étale and representable, it is in fact strongly representable by Proposition 2.4 and [16, Corollary II.6.17]. Since it is immediate from the definitions that an étale map preserves perfect complexes, we only need to show that \( p \)
preserves the compact objects. Suppose that \( A \in D_{qc}(X \text{ on } Z) \) is compact. Let \( W = p^{-1}(U) \) and consider the Cartesian diagram

\[
\begin{array}{ccc}
W & \xrightarrow{j'} & V \\
p' & & p \\
U & \xrightarrow{j} & X.
\end{array}
\]

Let \((A_\lambda)\) be a family of objects in \(D_{qc}(V \text{ on } T)\). We have then

\[
\text{Hom}_{D_{qc}(V \text{ on } T)}(p^*A, \bigoplus \lambda A_\lambda) = \text{Hom}_{D_{qc}(X)}(A, Rp_* (\bigoplus \lambda A_\lambda)) = \text{Hom}_{D_{qc}(X)}(A, \bigoplus \lambda Rp_* A_\lambda) \quad \text{(by Proposition 4.1)}
\]

\[
= \text{Hom}_{D_{qc}(X \text{ on } Z)}(A, \bigoplus \lambda Rp_* A_\lambda) = \bigoplus \lambda \text{Hom}_{D_{qc}(X \text{ on } Z)}(A, Rp_* A_\lambda) = \bigoplus \lambda \text{Hom}_{D_{qc}(X)}(A, Rp_* A_\lambda) = \bigoplus \lambda \text{Hom}_{D_{qc}(V \text{ on } T)}(p^*A, A_\lambda).
\]

Here, the second and the one before the last equality follow from Lemma 2.10 and [17, Lemma 3.3] since \((p^*, p_*)\) is a pair of adjoint functors with \(p_*\) exact. The fourth equality follows from Corollary 3.6 since \(j\) is representable and étale and the fifth equality holds because \(A\) is compact in \(D_{qc}(X \text{ on } Z)\). This shows the desired compactness of \(p^*A\).

The following result is a generalization of [3, Theorem 3.1.1].

**Corollary 4.3.** Let \(X\) be a noetherian and separated scheme and let \(U \xhookrightarrow{i} X\) be an open subscheme with complement \(Z \xhookrightarrow{j} X\). Let \(D_{qc}(X \text{ on } Z)\) denote the full subcategory of \(D_{qc}(X)\) consisting of complexes which are acyclic over \(U\). Then an object of \(D_{qc}(X \text{ on } Z)\) is compact if and only if it a perfect complex on \(X\) which is acyclic over \(U\).

**Proof.** If \(A \in D_{qc}(X \text{ on } Z)\) is perfect, then it is also perfect as an object of \(D_{qc}(X)\) and hence compact by [3, Theorem 3.1.1]. But an object of \(D_{qc}(X \text{ on } Z)\) which is compact in \(D_{qc}(X)\) is also compact as an object of \(D_{qc}(X \text{ on } Z)\).

Suppose now that \(A \in D_{qc}(X \text{ on } Z)\) is compact. It follows from Corollary 4.2 that the restriction of \(A\) to any given affine open subscheme \(V \xhookrightarrow{i} X\) is compact in \(D_{qc}(V \text{ on } V \cap Z)\). Since \(V\) is noetherian and affine, it has the resolution property (in fact has an ample line bundle) and hence \(A\) is perfect on \(V\) by [30, Proposition 5.4.2] (see also [?]). Hence \(X\) has a finite affine open cover \(\{X_\alpha \xhookrightarrow{i} X\}\) such that the restriction of \(A\) to each \(D_{qc}(X_\alpha \text{ on } X_\alpha \cap Z)\) is perfect. We conclude that \(A\) must be perfect on \(X\) which is acyclic over \(U\).

The following are our main results of this section.
Theorem 4.4. Let \( X \) be stack and let \( U \to X \) be an open substack with complement \( Z \to X \). Then every compact object in \( D_{qc}(X \text{ on } Z) \) is perfect. If \( X \) is a tame stack with a coarse moduli scheme, then every perfect complex in \( D_{qc}(X \text{ on } Z) \) is compact.

Proof. We first assume that \( A \) is compact in \( D_{qc}(X \text{ on } Z) \) and show that it is perfect. Let \( U \to X \) be a noetherian atlas of \( X \). We have seen before that \( u \) is strongly representable. We can then apply Corollary 4.2 to conclude that \( u^*A \) is a compact object of \( D_{qc}(U \text{ on } u^{-1}(Z)) \). It subsequently follows from Corollary 4.3 that \( u^*A \) is a perfect complex on the scheme \( U \). We now apply Proposition 3.4 \((iii)\) to get an étale cover \( V \to U \) such that \( v^*u^*A \) is quasi-isomorphic to a strictly perfect complex on \( V \). Since the composite \( V \to X \) is an étale atlas of \( X \), we conclude that \( A \) is a perfect complex which is acyclic over \( U \).

To prove that a perfect complex \( A \) in \( D_{qc}(X \text{ on } Z) \) is a compact object of \( D_{qc}(X \text{ on } Z) \), we note that \( A \) is then a perfect complex on \( X \). Hence it is enough to prove the case when \( Z = \emptyset \). So we assume now that \( X \) is a tame stack and has a coarse moduli scheme \( q: X \to Y \). Then \( Y \) is a noetherian and separated scheme by [23, Theorem 6.12]. It follows from Lemma 2.7 that \( Y \) has a finite affine cover \( \{Y_i \to Y\}_{1 \leq i \leq n} \) such that for each \( i \), the diagram

\[
\begin{array}{ccc}
[U_i/\Gamma_i] & \xrightarrow{j'_i} & q^{-1}(Y_i) \\
\downarrow{p_i} & & \downarrow{q_i} \\
V_i & \xrightarrow{j_i} & Y_i
\end{array}
\]

is Cartesian, \( U_i \) and \( V_i \) are affine noetherian schemes and the horizontal maps are étale and surjective and \([U_i/\Gamma_i]\) is a tame stack. It follows that each \( p_i \) is a coarse moduli space.

To show the compactness of an object, we first observe from the above definition of the compactness that an object \( A \in D_{qc}(X) \) is compact if and only if given any family \( (A_\lambda) \), any morphism \( A \to \bigoplus A_\lambda \) factors through the morphism into a finite direct sum of \( A_\lambda \)'s. From this criterion, it is easy to see that \( A \) is compact if there is a finite Zariski open cover \( X_i \to X \) such that the restriction of \( A \) to each open substack \( X_i \) is compact.

To prove now the second part of the theorem when \( Z = \emptyset \), let \( A \) be a perfect complex on \( X \). From the above observation, it suffices to show that the restriction of \( A \) to each \( X_i = q^{-1}(Y_i) \) is compact, where \( \{Y_i \to Y\}_{1 \leq i \leq n} \) is an affine open cover as in \((4.4)\). Since the perfectness is clearly preserved under the restriction to an open substack, we see that \( A|_{X_i} \) is perfect for each \( i \). Thus, we are reduced to proving the theorem when \( X \) a tame stack with a coarse moduli scheme \( q: X \to Y \) such that \( Y \) is noetherian and affine and there is a Cartesian diagram

\[
\begin{array}{ccc}
[U/\Gamma] & \xrightarrow{j'} & X \\
\downarrow{p} & & \downarrow{q} \\
V & \xrightarrow{j} & Y
\end{array}
\]
where the horizontal morphisms are étale covers, $U$ and $V$ are noetherian and
affine schemes and $\Gamma$ is a finite group acting on $U$ such that $X' = [U/\Gamma]$ is a tame
stack. The map $X' \xrightarrow{\eta} V$ is then a coarse moduli scheme.

We now note that as a locally presentable dg-category in the sense of [32, Section 3], the category $C_{qc}(X)$ is simply the unit dg-category $1$ over $X$. Let $\alpha$ be the locally presentable dg-category on $Y$ defined by

$$(W \to Y) \mapsto C_{qc}(g^{-1}(W)).$$

It is then clear that in the notations of [32], $C_{qc}(X) = L_1(X) = L_\alpha(Y)$. Thus, it suffices to show that $A$ is a compact object in $L_\alpha(Y)$. Since $V \xrightarrow{j} Y$ is an étale and surjective map between affine schemes, it suffices to show using [32, Lemma 3.4] that $j^*(A)$ is compact in $L_\alpha(V)$. However, we see as above that $L_\alpha(V) = L_{j^*(1)}(X') = L_1(X') = C_{qc}(X')$. In particular, it suffices to show that $j^*(A)$ is compact in $D_{qc}(X')$.

Since $j'$ is étale, we see that $A' = j'^*(A)$ is a perfect complex on $X'$. Since $U$ is an affine noetherian scheme, it clearly has the resolution property. Since $\Gamma$ is a finite group acting on $U$, it follows from [28, Lemma 2.14] that $X'$ also has the resolution property. The tameness of $[U/\Gamma]$ and [17, Lemma 3.5] imply that there is an equivalence of categories $D(qc/X') \xrightarrow{\cong} D_{qc}(X')$. It follows now from [17, Theorem 4.12] that $A'$ is compact in $D_{qc}(X')$. This completes the proof of the theorem.

**Theorem 4.5 (Localization).** Let $X$ be a tame stack which admits a coarse moduli scheme. Let $U \hookrightarrow X$ be an open substack with the complement $Z \hookrightarrow X$. Then there is a homotopy fibration sequence of non-connective spectra

$$K(X \text{ on } Z) \to K(X) \to K(U).$$

In particular, one has a long exact sequence

$$\cdots \to K_i(X \text{ on } Z) \to K_i(X) \to K_i(U) \to K_{i-1}(X \text{ on } Z) \to \cdots$$

for all $i \in \mathbb{Z}$.

**Proof.** This follows immediately from Theorem 4.4 and [32, Corollary 4.3].

5. Excision for $K$-theory of stacks

In this section, we study the excision property of the $K$-theory of perfect complexes on stacks. For a stack $X$ and a closed substack $Z \hookrightarrow X$, let $C_{Z}(Perf/X)$ denote the dg-category of perfect complexes on $X$ which are acyclic on the complement of $Z$. Let $D_{Z}(Perf/X)$ denote the corresponding derived category. Then $D_{Z}(Perf/X)$ is a full triangulated subcategory of $D_{qc}(X)$ on $Z$. We write $D(Perf/X)$ for the full triangulated subcategory of perfect complexes on $X$. Let $K(X \text{ on } Z)$ denote the non-connective $K$-theory spectrum of the dg-category $C_{Z}(Perf/X)$. Let $K(X)$ denote the non-connective $K$-theory spectrum of the perfect complexes on $X$. We refer the reader to [5] for details about the non-connective $K$-theory of dg-categories.

For a stack $X$, let $D_{qc}^{-}(X)$ and $D_{qc}^{+}(X)$ denote the full triangulated subcategories of $D_{qc}(X)$ consisting of the bounded above and bounded below complexes respectively. We define $D_{qc}^{-}(X)$ on $Z$ and $D_{qc}^{+}(X)$ on $Z$ in an obvious way. For a complex $E \in D(X)$, let

$$\tau^{\leq n}(E) = (\to E^{i} \to \cdots \to E^{n-1} \to Z^{n}E \to 0)$$

and
\[ \tau^{\geq n}(E) = (0 \rightarrow E^n/B^n E \rightarrow E^{n+1} \rightarrow \cdots) \]
denote the good truncations, where \( B^n E \) and \( Z^n E \) denote the boundaries and the kernels of the differential of \( E \). There is a natural isomorphism

\begin{equation}
\lim_n \tau^{\leq n}(E) \xrightarrow{\sim} E.
\end{equation}

\begin{equation}
E \xrightarrow{\sim} \lim_n \tau^{\geq n}(E).
\end{equation}

**Lemma 5.1.** Let \( f : Y \rightarrow X \) be a strongly representable morphism of stacks. Then \( Rf_* : D_{qc}(Y) \rightarrow D_{qc}(X) \) maps \( D_{qc}^{-}(Y) \) into \( D_{qc}^{-}(X) \) and \( D_{qc}^{+}(Y) \) into \( D_{qc}^{+}(X) \).

**Proof.** We first assume that \( X \) is a scheme. The strong representability then implies that \( Y \) is also a scheme. The first assertion then follows from [3, Theorem 3.3.3]. To prove the second assertion, we can use Corollary [3, 4] to assume that \( X \) is affine. If \( \{Y_i \rightarrow Y\}_{1 \leq i \leq n} \) is a finite affine cover of \( Y \), let \( U = Y_2 \cup \cdots Y_n \) and \( V = U \cap Y_1 \).

Then the separatedness of \( Y \) implies that \( U \) and \( V \) can be covered by at most \( n-1 \) affine open sets. Let \( U \xrightarrow{j} Y, Y_1 \xrightarrow{j} Y \) and \( V \xrightarrow{j'} Y \) denote the inclusion maps. Then the exact triangle

\[ Rf_*E \rightarrow Rf_*(j^*E) \oplus Rf_*(j^*A) \rightarrow Rf_*(j'^*A) \rightarrow \]

and an induction on the number of open sets in an affine cover of \( Y \) show that it suffices to prove the case when \( Y \) is also affine. But in this case, there is an equivalence \( D^{+}(qc/X) \xrightarrow{\sim} D^{+}_{qc}(X) \) by [30, B.16] and similarly for \( Y \). The lemma now follows in the scheme case since \( f_* \) is exact on the category of quasi-coherent sheaves for maps between affine schemes. In general, we can choose an atlas \( U \rightarrow X \) and use the strong representability of \( f \) and Corollary [3, 6] to reduce to the case of schemes. \( \square \)

**Proposition 5.2.** Let \( f : Y \rightarrow X \) be a representable and étale morphism of stacks.

Let \( Z \xrightarrow{i} X \) be a closed substack such that the restriction \( f : Z \times_X Y \rightarrow Z \) is an isomorphism. Then for any \( E \in D^{+}_{qc}(X) \) and \( F \in D^{+}_{qc}(Y) \), the natural maps \( E \rightarrow Rf_* f^*(E) \) and \( F \rightarrow Rf_* f^*(F) \) are isomorphisms.

**Proof.** We first prove the case when \( X \) and \( Y \) are schemes. Put \( W = Z \times_X Y \) and let \( E \in D^{+}_{qc}(X) \). For a Cartan-Eilenberg resolution \( E \xrightarrow{\sim} \mathcal{I}^{\bullet} \) of \( E \), we let \( \tau^{\leq n}(\mathcal{I}^{\bullet}) \) be the double complex obtained by taking the good truncation of \( \mathcal{I}^{\bullet} \) in each row. Put \( \mathcal{I} = \text{Tot}(\mathcal{I}^{\bullet}) \) and \( \mathcal{I}^{[\leq n]} = \text{Tot}(\tau^{\leq n}(\mathcal{I}^{\bullet})) \) for \( n \in \mathbb{Z} \). Then we get for any \( i \in \mathbb{Z} \),

\[ \mathcal{I}^{i} = \bigoplus_{\alpha + \beta = i} (\mathcal{I}^{\alpha, \beta}) = \mathcal{I}^{[\leq n]} \]

for all large \( n \). In particular, we get

\begin{equation}
\lim_n \mathcal{I}^{[\leq n]} \xrightarrow{\sim} \mathcal{I}.
\end{equation}

Lemma [2, 9] implies that \( f^*(E) \xrightarrow{f^*(i)} f^*(\mathcal{I}^{\bullet}) \) is a Cartan-Eilenberg resolution of \( f^*(E) \). Since \( E \) is bounded below, the product total complex of \( f^*(\mathcal{I}^{\bullet}) \) coincides with the corresponding direct sum total complex. In particular, we get that \( \text{Tot}(f^*(\mathcal{I}^{\bullet})) \cong f^*(\mathcal{I}) \). Similarly, \( f^*(\tau^{\leq n}(E)) \rightarrow f^*(\mathcal{I}^{[\leq n]}) \) is a Cartan-Eilenberg resolution and \( \text{Tot}(f^*(\tau^{\leq n}(\mathcal{I}^{\bullet}))) \cong f^*(\text{Tot}(\tau^{\leq n}(\mathcal{I}^{\bullet}))) \). We conclude
from this and Theorem 3.1 that the following are the $K$-injective resolutions of the complexes on the left.

$$
\begin{align*}
E \to \mathcal{I}, & \quad \tau^{\leq n}(E) \to \mathcal{I}^{\leq n}, \\
f^*(E) \to f^*(\mathcal{I}), & \quad f^*(\tau^{\leq n}(E)) \to f^*(\mathcal{I}^{\leq n}).
\end{align*}
$$

This implies in particular that $Rf_* \circ f^*(E) \cong f_* \circ f^*(\mathcal{I})$ and $Rf_* \circ f^*(\tau^{\leq n}(E)) \cong f_* \circ f^*(\mathcal{I}^{\leq n})$. Combining this with 5.1 and 5.3, we get

$$
Rf_* \circ f^*(E) \cong f_* \circ f^*(\mathcal{I}) \cong \lim_{\to} f_* \circ f^*(\mathcal{I}^{\leq n}),
$$

(5.4)

where the third isomorphism follows, for example, by the exact triangle

$$
\oplus_{n} \mathcal{I}^{\leq n} \to \oplus_{n} \mathcal{I}^{\leq n} \to \lim_{\to} \mathcal{I}^{\leq n}
$$

(5.5)

and Proposition 4.1. Thus, we get a commutative diagram

$$
\begin{array}{ccc}
\lim_{\to} \tau^{\leq n}(E) & \to & \lim_{\to} Rf_* \circ f^*(\tau^{\leq n}(E)) \\
\downarrow & & \downarrow \\
E & \to & Rf_* \circ f^*(E),
\end{array}
$$

where the left vertical map is isomorphism by (5.1) and the right vertical map is isomorphism by (5.6). Since each $\tau^{\leq n}(E)$ is a good truncation of $E$, it is in $D^+_q(X \text{ on } Z)$. Moreover, as $\tau^{\leq n}(E)$ is a bounded complex, the top horizontal map above is an isomorphism by [30, Theorem 2.6.3]. We now conclude that

$$
E \cong Rf_* \circ f^*(E).
$$

Next we prove the second assertion of the proposition for schemes. So let $F \in D^+_q(Y \text{ on } W)$. Choosing a Cartan-Eilenberg resolution $F \xrightarrow{\sim} \mathcal{I}^{**}$, we have as in the above case, the commutative diagram of isomorphisms

$$
\begin{array}{ccc}
\lim_{\to} \tau^{\leq n}(F) & \xrightarrow{\sim} & F \\
\downarrow & & \downarrow \\
\lim_{\to} \mathcal{I}^{\leq n} & \xrightarrow{\sim} & \mathcal{I},
\end{array}
$$

where the complexes on the bottom are the $K$-injective resolutions of the corresponding complexes on the top. In particular, we get

$$
Rf_*(F) \cong f_* (\mathcal{I}) \cong \lim_{\to} f_* (\mathcal{I}^{\leq n}) \cong \lim_{\to} Rf_* (\tau^{\leq n}(F)),
$$

(5.6)
where the second isomorphism follows as before by (5.5). Applying \( f^* \) to above, we get

\[
\lim_n f^* \circ Rf_* (\tau^{\leq n}(F)) \xrightarrow{\sim} f^* \left( \lim_n Rf_* (\tau^{\leq n}(F)) \right) \xrightarrow{\sim} f^* \circ Rf_*(F).
\]

Thus we get a commutative diagram

\[
\begin{array}{ccc}
\lim_n f^* \circ Rf_* (\tau^{\leq n}(F)) & \xrightarrow{\sim} & \lim_n \tau^{\leq n}(F) \\
\downarrow & & \downarrow \\
\lim_n f^* \circ Rf_*(F) & \cong & F,
\end{array}
\]

where the left vertical arrow is an isomorphism by (5.7) and the right vertical arrow is an isomorphism by (5.1). Since each \( \tau^{\leq n}(F) \) is a bounded complex in \( D^+_{qc}(Y \text{ on } W) \), the top horizontal arrow is an isomorphism by [30, Theorem 2.6.3]. We conclude that

\[
f^* \circ Rf_*(F) \xrightarrow{\sim} F.
\]

This proves the proposition for noetherian schemes.

To complete the proof of the proposition, let \( X \) be a stack and let \( u : U \to X \) be an étale atlas for \( X \) and consider the Cartesian diagram 5.1. Put \( T = u^{-1}(Z) \) and \( F = v^{-1}(W) \). Since we have already seen that \( f \) is strongly representable and étale, we see that \( g \) is an étale morphism of noetherian and separated schemes which is an isomorphism over \( T \). It suffices to show that the natural maps

\[
u^* (E) \to u^* (Rf_* \circ f^*(E)) \text{ and } v^* (f^* \circ Rf_*(F)) \to v^*(F)
\]

are isomorphisms for \( E \in D^+_{qc}(X \text{ on } Z) \) and \( F \in D^+_{qc}(Y \text{ on } W) \). However, Corollary 3.6 implies that \( u^* (Rf_\circ f^*(E)) \cong Rg_* \circ v^* \circ f^*(E) \cong Rg_* \circ g^* \circ u^*(E) \), which in turn is isomorphic to \( u^*(E) \) from the case of schemes proved above since \( U \) is a noetherian scheme and \( v^*(E) \in D^+_{qc}(U \text{ on } T) \). Similarly, we have

\[
v^* (f^* \circ Rf_*(F)) \cong g^* \circ u^* \circ Rf_*(F) \cong g^* \circ Rg_* \circ v^*(F),
\]

again by Corollary 3.6. The last term is isomorphic to \( v^*(F) \), again from the case of schemes since \( V \) is a noetherian scheme and \( v^*(F) \in D^+_{qc}(V \text{ on } P) \).

\[\square\]

**Lemma 5.3.** Let \( f : Y \to X \) be a representable and étale morphism of stacks. Let \( Z \hookrightarrow X \) be a closed substack such that the restriction \( f : Z \times_X Y \to Z \) is an isomorphism. Then for any \( E \in D^+_{qc}(X \text{ on } Z) \), the natural map \( E \to Rf_* \circ f^*(E) \) is an isomorphism.

**Proof.** The isomorphism in (5.2) and Lemma 2.9 imply that there are isomorphisms

\[
E \xrightarrow{\sim} \lim_n \tau^{\geq n} E, \quad f^*(E) \xrightarrow{\sim} \lim_n f^* (\tau^{\geq n} E).
\]

Let \( f^*(E) \to \mathcal{I}\mathcal{I}^\bullet \) be a Cartan-Eilenberg resolution. It follows from Theorem 3.1 that \( f^*(E) \to \mathcal{I} = \text{Tot} (\mathcal{I}\mathcal{I}^\bullet) \) is a \( K \)-injective resolution of \( f^*(E) \). Furthermore, \( f^* (\tau^{\geq n} E) \to \tau^{\geq n} (\mathcal{I}\mathcal{I}^\bullet) \) is a Cartan-Eilenberg resolution, and hence \( f^* (\tau^{\geq n} E) \to \)
\[ \mathcal{I}^{[\geq n]} = \text{Tot} (\tau^{\geq n} (\mathcal{I}^{\bullet})) \] is a \( K \)-injective resolution for each \( n \). One also checks that \( \mathcal{I} \Rightarrow \varprojlim_n \mathcal{I}^{[\geq n]} \). In particular, we get

\[ (5.9) \quad Rf_* \circ f^*(E) \cong f_*(\mathcal{I}) \cong f_*(\varprojlim_n \mathcal{I}^{[\geq n]}) . \]

We claim that the natural map \( f_* \left( \varprojlim_n \mathcal{I}^{[\geq n]} \right) \rightarrow \varprojlim_n f_* (\mathcal{I}^{[\geq n]}) \) is an isomorphism. To see this, we note first from Lemma 2.10 that \( \text{Tot}(X) \) and \( \text{Tot}(Y) \) are Grothendieck categories. Hence we can use [17, Lemma 2.3] to get

\[ \left( f_* \left( \varprojlim_n \mathcal{I}^{[\geq n]} \right) \right)^i = f_* \left( \left( \varprojlim_n \mathcal{I}^{[\geq n]} \right)^i \right) = f_* \left( \varprojlim_n \mathcal{I}^{[\geq n]_i} \right) . \]

On the other hand, since \( f_* \) has a left adjoint, we have

\[ f_* \left( \varprojlim_n \mathcal{I}^{[\geq n]_i} \right) = \varprojlim_n f_* \left( (\mathcal{I}^{[\geq n]_i})^i \right) = \varprojlim_n \left( f_* (\mathcal{I}^{[\geq n]}) \right)^i = \left( \varprojlim_n f_* (\mathcal{I}^{[\geq n]}) \right)^i , \]

where the last isomorphism follows again from [17, Lemma 2.3]. This proves the claim.

Using this claim in (5.9), we obtain

\[ (5.10) \quad Rf_* \circ f^*(E) \cong \varprojlim_n f_* (\mathcal{I}^{[\geq n]}) \cong \varprojlim_n Rf_* \circ f^* (\tau^{\geq n}(E)) . \]

Finally, we consider the following commutative diagram.

\[
\begin{array}{ccc}
E & \longrightarrow & Rf_* \circ f^*(E) \\
\downarrow & & \downarrow \\
\varprojlim_n \tau^{\geq n}(E) & \longrightarrow & \varprojlim_n Rf_* \circ f^* (\tau^{\geq n}(E))
\end{array}
\]

The left vertical arrow is an isomorphism by (5.8) and the right vertical arrow is an isomorphism by (5.10). Since \( \tau^{\geq n}(E) \) is a good truncation of \( E \), we see that \( \tau^{\geq n}(E) \in D^+_{qc}(X \text{ on } Z) \) for each \( n \) and hence the bottom horizontal arrow is an isomorphism by Proposition 5.2. We now conclude that \( E \rightarrow Rf_* \circ f^*(E) \) is an isomorphism.

**Theorem 5.4.** Let \( f : Y \rightarrow X \) be a representable and étale morphism of stacks. Let \( Z \hookrightarrow X \) be a closed substack such that the restriction \( f : Z \times_X Y \rightarrow Z \) is an isomorphism. Then the pullback \( f^* : D_{qc}(X \text{ on } Z) \rightarrow D_{qc}(Y \text{ on } Z \times_X Y) \) is an equivalence of triangulated categories.

**Proof.** Before we begin the proof, one should observe that \( f^* \) maps \( D_{qc}(X \text{ on } Z) \) into \( D_{qc}(Y \text{ on } Z \times_X Y) \) because \( f \) is étale. Put \( W = Z \times_X Y \). We first show that \( f^* \) induces a full embedding of \( D_{qc}(X \text{ on } Z) \) into \( D_{qc}(Y \text{ on } W) \). In fact, we will show that the former is a reflexive full subcategory of the latter.
To show this, let $E, E' \in D_{qc}(X \text{ on } Z)$. We have then

$$\text{Hom}_{D_{qc}(X \text{ on } Z)}(E, E') \cong \text{Hom}_{D_{qc}(X)}(E, Rf_*(f^*(E'))) \quad \text{(by Lemma 5.3)}$$

$$\cong \text{Hom}_{D_{qc}(X)}(E, Rf_*(f^*(E')))$$

$$\cong \text{Hom}_{D_{qc}(Y)}(f^*(E), f^*(E'))$$

$$\cong \text{Hom}_{D_{qc}(Y \text{ on } W)}(f^*(E), f^*(E')) ,$$

where the third isomorphism follows from [17, Lemma 3.3]. Hence $f^*$ is full and faithful. Lemma 5.3 now implies that $D_{qc}(X \text{ on } Z)$ is a reflexive full triangulated subcategory of $D_{qc}(Y \text{ on } W)$. To prove the theorem, it suffices now to show that $f^*$ is surjective on objects.

To prove this, let $F \in D_{qc}(Y \text{ on } W)$. We first consider the case when $F$ is bounded above. If $X$ is a scheme, then we have seen before that $Y$ is also a scheme and the assertion then follows directly from the stronger result in [30, Theorem 2.6.3] that the map $f^* Rf_*(F) \to F$ is an isomorphism. If $X$ is a stack, we can choose an atlas $U \to X$ and note that $U \times_X Y$ is a scheme since $f$ is representable and étale and hence strongly representable. Now we argue as in the proof of Proposition 5.2 to show that $f^* Rf_*(F) \to F$ is an isomorphism. It follows moreover from Corollary 3.6 that $Rf_*(F) \in D_{qc}(X \text{ on } Z)$.

To complete the proof in general, we write

$$f^* \left( \lim_{\to n} Rf_* \left( \tau^{\leq n}(F) \right) \right) \cong \lim_{\to n} f^* \circ Rf_* \left( \tau^{\leq n}(F) \right) \to \lim_{\to n} \tau^{\leq n}(F) \to F,$$

where the first isomorphism follows from (5.5). Since each $\tau^{\leq n}(F) \in D_{qc}(Y \text{ on } W)$, we have just shown that the second map is also an isomorphism, and the last map is an isomorphism by (5.11).

Putting $E = \lim_{\to n} Rf_* \left( \tau^{\leq n}(F) \right)$, we see that $f^*(E) \cong F$. Thus, it suffices to prove that $E \in D_{qc}(X \text{ on } Z)$ to finish the proof of the theorem.

Let $T = X - Z$ and let $j : T \hookrightarrow X$ be the inclusion morphism. It suffices to show that $j^* (E) = 0$. To show this, we have for any $i \in \mathbb{Z}$,

$$H^i (j^* (E)) \cong j^* \left( H^i (E) \right)$$

$$\cong j^* \left( \lim_{\to n} Rf_* \left( \tau^{\leq n}(F) \right) \right)$$

$$\cong \lim_{\to n} j^* \circ H^i \left( Rf_* \left( \tau^{\leq n}(F) \right) \right)$$

$$\cong \lim_{\to n} H^i \left( j^* \circ Rf_* \left( \tau^{\leq n}(F) \right) \right)$$

$$\cong 0,$$

where the last term is zero since $\tau^{\leq n}(F) \in D_{qc}(Y \text{ on } W)$ and hence $Rf_* \left( \tau^{\leq n}(F) \right) \in D_{qc}(X \text{ on } Z)$ as shown above. This completes the proof of the theorem. \[\square\]

**Theorem 5.5 (Excision).** Let $f : Y \to X$ be a representable and étale morphism of tame stacks which admit coarse moduli schemes. Let $Z \hookrightarrow X$ be a closed substack


such that the restriction \( f : Z \times_X Y \to Z \) is an isomorphism. Then the map \( f^* : C(X) \to C(Y) \) induces an equivalence of spectra \( K(X \text{ on } Z) \xrightarrow{\cong} K(Y \text{ on } Z \times_X Y) \).

Proof. If we let \( W = Z \times_X Y \), then \( f^* \) clearly induces the morphism of dg-categories \( f^* : C_Z(Perf/X) \to C_W(Perf/Y) \). To show that the induced morphism of the \( K \)-theory spectra is a homotopy equivalence, it suffices to show that the map \( f^* : D_Z(Perf/X) \to D_W(Perf/Y) \) is an equivalence of the triangulated categories. Suppose we show that

\[
(5.12) \quad f^* : D_{qc}(X \text{ on } Z) \to D_{qc}(Y \text{ on } W)
\]

is an equivalence. Since \( f \) is étale, \( f^* \) takes perfect complexes which are acyclic on the complement of \( Z \) to perfect complexes on \( Y \) which are acyclic on the complement of \( W \). It follows from Theorem 4.4 that \( f^* \) takes compact objects of \( D_{qc}(X \text{ on } Z) \) into the compact objects of \( D_{qc}(Y \text{ on } W) \). The equivalence in (5.12) will then restrict to an equivalence of the corresponding full subcategories of compact objects. Another application of Theorem 4.4 will show that \( f^* \) induces an equivalence between the derived categories of perfect complexes. Thus, we only need to show (5.12). But this is shown in Theorem 5.4. \( \square \)

Corollary 5.6 (Mayer-Vietoris). Let \( f : Y \to X \) be a representable and étale morphism of tame stacks which admit coarse moduli schemes. Let

\[
(5.13) \quad \begin{array}{ccc}
V & \to & Y \\
\downarrow^g & & \downarrow^f \\
U & \to & X
\end{array}
\]

be a Cartesian square such that \( U \hookrightarrow X \) is an open substack and \( f \) is an isomorphism over the complement of \( U \). Then, this induces a homotopy Cartesian square of \( K \)-theory spectra

\[
(5.14) \quad \begin{array}{ccc}
K(X) & \xrightarrow{f^*} & K(U) \\
\downarrow & & \downarrow^g \\
K(Y) & \xrightarrow{g^*} & K(V)
\end{array}
\]

Proof. Let \( Z \hookrightarrow X \) be the complement of \( U \) and let \( W = Z \times_X Y \). By Theorem 4.3, there is a commutative diagram of morphism of spectra

\[
(5.15) \quad \begin{array}{ccc}
K(X \text{ on } Z) & \xrightarrow{f^*} & K(X) \to K(U) \\
\downarrow & & \downarrow^g \\
K(Y \text{ on } W) & \xrightarrow{g^*} & K(Y) \to K(V)
\end{array}
\]

such that each row is a homotopy fibration. Moreover, the left vertical morphism is a homotopy equivalence by Theorem 5.3. This proves the corollary. \( \square \)

6. Nisnevich site of stacks

In this section, we define the Nisnevich site of a stack and study its basic properties. It will turn out in the subsequent sections that the resulting Grothendieck topology is given by a certain \( cd \)-structure in the sense of [34] on the big étale site of \( X \). This fact will be subsequently used in this paper to prove the Nisnevich descent for the \( K \)-theory of perfect complexes on stacks. The equivariant Nisnevich
site in the category $Sm^f_k$ of smooth schemes over a field with action of a finite group, has also appeared in [9] in the study of equivariant stable homotopy theory.

Let $S$ be a noetherian affine scheme. From now on, we shall assume that all the objects in the category $\mathcal{D}\mathcal{M}_S$ are of finite type over $S$. Recall that a Deligne-Mumford stack $X$ is of finite type over $S$ if there is an étale atlas $U \rightarrow X$ such that $U$ is a scheme of finite type over $S$. Recall from Remark 2.8 that $E(X)$ is the category whose objects are morphisms $[X' \rightarrow X]$ in $\mathcal{D}\mathcal{M}_S$, where $f$ is étale but not necessarily representable.

6.1. Nisnevich topology. Let $X$ be a stack. We recall from [19, Chapter 5] that a point of the stack $X$ is an equivalence class of morphisms $\text{Spec}(K) \rightarrow X$, where $K$ is a $S$-field and where the two points $(x, K)$ and $(y, L)$ are equivalent if there is a common field extension $E$ of $K$ and $L$ such that the composite maps $\text{Spec}(E) \rightarrow \text{Spec}(K) \rightarrow X$ and $\text{Spec}(E) \rightarrow \text{Spec}(L) \rightarrow X$ are isomorphic. If $X$ is a scheme, this gives the usual notion of points on schemes. We often denote a point $(x, K)$ of $X$ simply by $x$ keeping the above meaning in mind. Every point $x : \text{Spec}(K) \rightarrow X$ has a unique factorization

$$
\text{Spec}(K) \rightarrow \eta_x \hookrightarrow X
$$

as fppf stacks, and it turns out that $\eta_x$ is in fact a fppf stack and hence a Deligne-Mumford stack. Moreover, there is a uniquely defined field $k(\eta_x)$ with a unique map $\eta_x \rightarrow \text{Spec}(k(\eta_x))$ which is a gerbe. The field $k(\eta_x)$ is called the residue field of the residual gerbe $\eta_x$ at the point $x$. Moreover, one can always choose a representative $(x, K)$ of $x$ such that $k(\eta_x) \hookrightarrow K$ is a finite extension and there is a Cartesian diagram of stacks

$$
\begin{array}{ccc}
\text{Spec}(K) & \rightarrow & \eta_x \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \rightarrow & \text{Spec}(k(\eta_x))
\end{array}
$$

(6.1)

where $G_\eta$ is the isotropy group scheme at the point $x$. The residual gerbe $\eta_x$ is uniquely defined by the equivalence class of $x$ and $G_\eta$ is a group scheme over $\eta_x$. Note that $G_\eta$ is a finite étale group scheme since $X$ is a Deligne-Mumford stack. We shall often denote the residual gerbe $\eta_x$ at the point $x$ simply by $\eta$. The morphism $\eta_x \rightarrow X$ is a monomorphism and representable and the map $\eta_x \rightarrow \text{Spec}(k(\eta_x))$ is of finite type. It follows in particular that this map is a coarse moduli space map. Every stack is a union of its finitely many irreducible components and every irreducible component has a unique generic point. The residual gerbe at the generic point of an irreducible component will be called a generic residual gerbe of the stack. The following elementary results about the residual gerbes on Deligne-Mumford stacks are well known.

**Lemma 6.1.** Let $X$ be a stack with a coarse moduli space $X$. Let $x$ be a point of $X$ and let $\text{Spec}(k) = \overline{x} \in X$ be its image in the coarse moduli space. Then $k$ is the residue field of the residual gerbe $\eta$ at $x$ and the diagram

$$
\begin{array}{ccc}
\eta & \rightarrow & X \\
\downarrow & & \downarrow \\
\overline{x} & \rightarrow & X
\end{array}
$$

(6.2)

is Cartesian.
Proof. (Sketch) The composite map \( \text{Spec}(K) \to X \to \underline{X} \) gives a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \to & \eta \\
\downarrow q & & \downarrow p \\
\underline{x} & \to & \underline{X}
\end{array}
\]

where the square is Cartesian. Since \( \underline{x} \) is the image of a point on \( X \), the map \( p^{-1}(\underline{x}) \to \underline{x} \) is a coarse moduli space map. Since \( x \) is the spectrum of a field, it follows from [19, Theorem 11.5]) that \( q \) is a gerbe. It follows now from [loc. cit. Lemme 3.17] that \( \eta \to p^{-1}(\underline{x}) \) is an epimorphism. Since \( \eta \to X \) is a monomorphism, we see that \( \eta \to p^{-1}(\underline{x}) \) is an isomorphism. \( \square \)

**Lemma 6.2.** Let \( f : X \to Y \) be a morphism of stacks. Let \( x \) be a point of \( X \) and let \( y \) be its image in \( Y \). Then the induced map of the residual gerbes \( \eta_x \to \eta_y \) is an isomorphism if and only if the induced maps on the corresponding isotropy groups and the residue fields are isomorphisms.

**Proof.** Since the spectrum of the residue field of the residual gerbe is its coarse moduli space and since the residual gerbe has only one point, we see easily that the isomorphism of the residual gerbes implies the isomorphism of the residue fields and the isotropy groups. To prove the converse, let \( k \) be the residue field of \( \eta_x \) and \( \eta_y \). Then, there is a finite extension \( l \) of \( k \) such that the base extension of \( \eta_x \) and \( \eta_y \) over \( l \) are the neutral gerbes of the form \([\text{Spec}(l)/G]\) and \([\text{Spec}(l)/G]\), where \( G \) is the common isotropy group. Thus, the natural map \( \eta_x \to \eta_y \) becomes an isomorphism after an étale cover. Hence the map must be an isomorphism. \( \square \)

**Definition 6.3.** Let \( X \) be a stack over \( S \). A family of morphisms \( \{X_i \to X\} \) in the étale site \( \mathcal{E}(X) \) (cf. Remark 2.5) is called a Nisnevich cover of \( X \) if for every point \( x \in X \), there is a member \( X_i \) and a point \( x_i \in X_i \) such that \( f_i(x_i) = x \) and the induced map of the residual gerbes \( \eta_{x_i} \to \eta_x \) is an isomorphism.

**Lemma 6.4.** The category \( \mathcal{D}M_S \) with covering maps \( \{X_i \to X\} \) given by the Nisnevich covers, defines a Grothendieck topology on \( \mathcal{D}M_S \).

**Proof.** All the conditions for a Grothendieck topology are immediate except possibly the condition that a Nisnevich cover has the base change property. To see this, let \( Y \to X \) be a morphism in \( \mathcal{D}M_S \) and let \( \{X_i \to X\} \) be a Nisnevich cover. It is clear that \( X_i \times_X Y \to Y \) is étale. Now let \( y \in Y \) be a point with \( f(y) = x \) and let \( x_i \in X_i \) be such that \( f_i(x_i) = x \) and \( \eta_{x_i} \cong \eta_x \). This implies that \( \eta_{x_i} \times_{\eta_y} \eta_y \cong \eta_y \). Thus we see that \( \eta_{x_i} \times_{\eta_y} \eta_y \) defines a point \( y_i \in X_i \times_X Y \) which maps to \( y \) such that \( \eta_{y_i} \cong \eta_y \). \( \square \)

We shall call the Grothendieck topology defined by the Nisnevich coverings, the Nisnevich topology on \( \mathcal{D}M_S \). The corresponding site will be denoted by \((\mathcal{D}M_S)_{Nis}\). Note that since all stacks in this category are noetherian and of finite type over \( S \), every Nisnevich cover has a refinement by a finite cover. In other words, \((\mathcal{D}M_S)_{Nis}\) is a noetherian site.

Let \((\mathcal{D}M_S)_{Nis}^f\) be the topology on \( \mathcal{D}M_S \) generated by those Nisnevich coverings \( \{X_i \to X\} \) where each \( f_i : X_i \to X \) is representable. Since the representable morphisms have the base change property, we see that \((\mathcal{D}M_S)_{Nis}^f\) is also a Grothendieck site on \( \mathcal{D}M_S \).
6.2. **Stacks with moduli schemes.** To prove the main results of this paper, we mostly focus on those stacks which have coarse moduli schemes. We need the following results in order to show that the Nisnevich topology restricts to a similar topology on the category of those stacks which have the coarse moduli schemes. We first quote the following result from [23].

**Theorem 6.5.** Let \( X \) be a stack over \( S \). Then \( X \) has the coarse moduli space \( X \rightarrow X_p \), where \( X \) is an algebraic space. The map \( p \) is separated, proper and quasi-finite. Moreover, \( X \) is separated, noetherian and of finite type over \( S \). In particular, \( X \in \mathcal{DM}_S \).

**Proof.** Cf. [23, Theorem 6.12]. \( \square \)

**Lemma 6.6.** Let
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p \downarrow & & \downarrow q \\
X_p & \xrightarrow{g} & Y
\end{array}
\]
be a commutative diagram in \( \mathcal{DM}_S \) (cf. Theorem 6.5) where the bottom arrow is the map of coarse moduli spaces of the top arrow. Then,

(i) \( g \) is separated, quasi-compact and of finite type.

(ii) \( f \) is quasi-finite \( \Rightarrow \) \( g \) is quasi-finite.

(iii) \( f \) is closed \( \Rightarrow \) \( g \) is closed.

(iv) \( f \) is proper and \( Y \) tame \( \Rightarrow \) \( g \) is proper.

(v) \( f \) quasi-finite and \( Y \) is a scheme \( \Rightarrow \) \( X \) is a scheme.

**Proof.** Since the moduli spaces are quasi-compact (in fact noetherian), separated and of finite type over \( S \) by Theorem 6.5, we conclude from Proposition 2.4 that \( g \) is quasi-compact and separated and of finite type. If \( f \) is quasi-finite, then Theorem 6.5 implies that \( g \circ p \) is quasi-finite. But, \( g \) must then be quasi-finite.

Suppose now that \( f \) is closed and let \( Z \subset X \) be a closed subspace. Then \( Z = p^{-1}(Z) = \overline{Z \times_X Y} \) is a closed substack of \( X \) and hence \( g(Z) = q \circ p(Z) = q \circ f(Z) \) is closed in \( Y \) since \( g \) is proper by Theorem 6.5 and hence closed.

Assume now that \( Y \) is a tame stack and \( f \) is proper. We need to show that \( g \) is universally closed. So let \( Z \xrightarrow{h} Y \) be a morphism of algebraic spaces. Since \( Y \) is tame, it follows from [11, Lemma 2.3.3] that \( \overline{Z \times_Y Y} \rightarrow Z \) is a coarse moduli space. We need to show that this map is closed. Since the base change of a proper map is proper and hence closed, the above proof now shows that the map \( \overline{Z \times_Y Y} \rightarrow Z \) is closed.

Finally, assume that \( Y \) is a scheme and \( f \) is quasi-finite. Then we see from the other assertions of the lemma that \( g \) is a quasi-finite map of noetherian algebraic spaces which is of finite type and separated with \( Y \) a scheme. Hence, it follows from [16, Corollary II.6.16] that \( X \) is also a scheme. \( \square \)

**Proposition 6.7.** Let \( S \) be the spectrum of a field and let \( \widetilde{\mathcal{DM}}_S \) be the full subcategory of \( \mathcal{DM}_S \) consisting of those stacks which have coarse moduli schemes. Then \( \widetilde{\mathcal{DM}}_S \) is closed under fiber products.

**Proof.** Let \( X \xrightarrow{f} Z \) and \( Y \xrightarrow{g} Z \) be the morphisms in \( \mathcal{DM}_S \) and let \( W = X \times_Z Y \). We need to show that \( W \) has a coarse moduli scheme if \( X, Y \) and \( Z \) have so. We
first notice that $W$ has a coarse moduli space by Theorem 6.5. We next observe that the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{h} & X \times_S Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\Delta_Z} & Z \times_S Z
\end{array}
$$

is a Cartesian diagram. Since $Z$ is a Deligne-Mumford stack, $\Delta_Z$ is separated and quasi-finite by [19, Lemme 4.2] and hence so is the map $h$. We see from Lemma 6.6 that the coarse moduli space of $W$ is a scheme if the same holds for the coarse moduli space of $X \times_S Y$.

Let us now assume that the coarse moduli spaces $X$ and $Y$ of $X$ and $Y$ respectively are schemes. Since $S$ is the spectrum of a field, the map $X \times_S Y \to Y$ is flat and hence the map $X \times_S Y \to X \times_S Y$ is a coarse moduli space by [11, Lemma 2.2.2]. On the other hand, the map $X \times_S Y \to X \times_S Y$ is of finite-type, separated, and quasi-finite map of noetherian stacks by Theorem 6.5. Hence the map $T \to X \times_S Y$ is of finite type, separated and quasi-finite map of noetherian algebraic spaces by Lemma 6.6 where $T$ is the coarse moduli space of $X \times_S Y$. It follows again from Lemma 6.6 that $T$ is a scheme. \qed

**Remark 6.8.** The reader would notice in the above proof that the only reason to assume $S$ to the spectrum of a field was to ensure that the moduli space has the flat base change. However, this base change also holds if all the stacks considered are tame by [11, Lemma 2.3.3]. Thus we see that if we are working in the category of tame stacks, then Proposition 6.7 holds over any noetherian and affine base $S$.

**Corollary 6.9.** Let $(\overline{\mathcal{D}\mathcal{M}}_S)_{\text{Nis}}$ and $(\overline{\mathcal{D}\mathcal{M}}_S)^r_{\text{Nis}}$ denote the restrictions of $(\mathcal{D}\mathcal{M}_S)_{\text{Nis}}$ and $(\mathcal{D}\mathcal{M}_S)^r_{\text{Nis}}$ respectively to the full subcategory of those stacks which have coarse moduli schemes. Then $(\overline{\mathcal{D}\mathcal{M}}_S)_{\text{Nis}}$ and $(\overline{\mathcal{D}\mathcal{M}}_S)^r_{\text{Nis}}$ are Grothendieck sites.

**Proof.** Follows immediately from Proposition 6.7. \qed

**Remark 6.10.** Let $G$ be an affine and smooth group scheme over $S$ and let $\text{Sch}_S^G$ denote the category of noetherian and separated schemes of finite type over $S$ with $G$-action and $G$-equivariant morphisms. For a $G$-scheme $X$, let $[X/G]$ denote the associated quotient stack. It follows essentially from the definitions that the atlas $X \to [X/G]$ is strongly representable. Moreover, for a $G$-equivariant map $X \xrightarrow{f} Y$, the associated map of stacks $[X/G] \to [Y/G]$ is always representable. In particular, it is strongly representable if $f$ is étale. We conclude that if we consider the $G$-schemes with proper action, then $\text{Sch}_S^G$ is canonically equivalent to a subcategory (not full) $\overline{\text{Sch}}_S^G$ of $\mathcal{D}\mathcal{M}_S$ whose Nisnevich topology is in fact the restriction of the Nisnevich site $(\mathcal{D}\mathcal{M}_S)^r_{\text{Nis}}$.

**Example 6.11.** Let $G$ be a smooth affine group scheme over $S$ and let $\text{Sch}_S^G$ be as in Remark 6.10. The following example shows that for a $G$-equivariant map $X \xrightarrow{f} Y$ which is a Nisnevich cover of $Y$ in the category $\text{Sch}/S$ of schemes over $S$, the associated map of quotient stacks need not be a Nisnevich cover.

Let $k$ be a field with $\text{char}(k) \neq 2$. Let $G = \mathbb{Z}/2$ acting trivially on $Y = \text{Spec}(k)$. Let $X = \text{Spec}(k) \coprod \text{Spec}(k)$ where $G$ acts by switching the components. The natural morphism $f : X \to Y$ which is identity on both components, is clearly $G$-equivariant. Moreover, $f$ is obviously a Nisnevich covering map of schemes if we forget the group action (cf. [25, Remark 4.1]). However, the associated map
$\mathcal{f} : [X/G] \rightarrow [Y/G]$ in the category $\mathcal{DM}_k$ is étale but not a Nisnevich cover. This is because the isotropy group of every point of $[X/G]$ is trivial while the isotropy group of the only point of $[Y/G]$ is all of $G$. Hence by Lemma 6.2, $\mathcal{f}$ cannot be a Nisnevich cover. In fact, it is not very difficult to see that a morphism $g : W \rightarrow [k/G]$ from a stack is a Nisnevich cover if and only if this map is étale and has a section.

7. Nisnevich topology via cd-structure

In [34], Voevodsky introduced the notion of cd-structures on a category $\mathcal{C}$. It turns out that such a cd-structure naturally defines a Grothendieck topology on $\mathcal{C}$. He also showed that a Grothendieck topology which comes from a cd-structure, has several nice properties. We refer the reader to [34] and [35] for applications in the homotopy theory of simplicial sheaves on schemes. It was also shown by Voevodsky that the Nisnevich topology on schemes is induced by a cd-structure. In this section, we use these ideas of Voevodsky to define a cd-structure on $\mathcal{DM}_S$ and then show that the resulting Grothendieck topology coincides with the Nisnevich topology as defined in Section 6. This will be later used to prove the Nisnevich descent for the $K$-theory of perfect complexes on stacks.

Let $\mathcal{C}$ be a category with an initial object. Recall from [34] that a cd-structure on $\mathcal{C}$ is a collection $P$ of commutative squares of the form

\begin{equation}
\begin{array}{c}
B \rightarrow Y \\
\downarrow \\
A \rightarrow X
\end{array}
\end{equation}

such that if $Q \in P$ and $Q'$ is isomorphic to $Q$, then $Q' \in P$. The squares of the collection $P$ are called the distinguished squares of the cd-structure. One can define different cd-structures and/or restrict them to subcategories considering only the squares which lie in the corresponding subcategory. In particular, for an object $X$ of $\mathcal{C}$, the cd-structure of $\mathcal{C}$ defines a cd-structure on the category $\mathcal{C}/X$.

We define the topology $t_P$ associated with a cd-structure $P$ as the smallest Grothendieck topology such that for a distinguished square of the form (7.1), the morphisms $\{Y \rightarrow X, A \rightarrow X\}$ form a covering and such that the empty sieve is a covering of the initial object. The class of simple coverings in this topology is the smallest class of families which contains all isomorphisms and satisfies the condition that for a distinguished square of the form (7.1) and families $\{p_i : Y_i \rightarrow Y\}$, $\{q_j : A_j \rightarrow A\}$ in this class, the family $\{p \circ p_i, e \circ e_j\}$ is also a covering in this family. In other words, a simple covering is an iteration of the coverings coming from distinguished squares.

We now define a cd-structure on $\mathcal{DM}_S$. Note that this category has an initial object in terms of the empty scheme.

**Definition 7.1.** A square in $\mathcal{DM}_S$ is called distinguished if it is a Cartesian diagram of the form (7.1), where $e$ is an open immersion of stacks with complement $Z$ and $p$ is an étale morphism (not necessarily representable) of stacks such that the induced morphism $Z \times_X Y \rightarrow Z$ is an isomorphism of the associated reduced closed substacks.

It is clear from the definition that these distinguished squares define a cd-structure on $\mathcal{DM}_S$. Let $t_{Nis}$ denote the Grothendieck topology on $\mathcal{DM}_S$ defined by this cd-structures. We shall call this as the Nisnevich cd-topology on $\mathcal{DM}_S$.

Let $t'_{Nis}$ denote the Grothendieck topology on $\mathcal{DM}_S$ defined by the class of those distinguished squares of the form (7.1), where the map $p$ is also assumed to be representable. We shall often call the corresponding cd-structure as the
representable Nisnevich cd-structure. We denote the restriction of $t^\ast_{Nis}$ on the full subcategory $\tilde{DM}_S$ by $\tilde{t}^\ast_{Nis}$.

It is clear from the above that every covering in the Nisnevich cd-topology is also a covering in the Nisnevich topology on $\tilde{DM}_S$. Our aim now is to show that both define the same topology on $\tilde{DM}_S$. We follow the argument of Voevodsky [35, Proposition 2.17] in order to prove this.

We recall that a morphism $f : Y \to X$ of stacks is called split if there is a sequence of closed embeddings

$$\emptyset = Z_{n+1} \to Z_n \to \cdots \to Z_1 \to Z_0 = X$$

such that for each $i = 0, \ldots, n$, the morphism $(Z_i - Z_{i+1}) \times_X Y \to (Z_i - Z_{i+1})$ has a section.

Lemma 7.2. Let $X$ be an irreducible stack with the generic residual gerbe $\eta_X$ and let $Y$ be an irreducible and noetherian $S$-scheme with the generic point $\eta_Y$. Let $t : \eta_Y \to \eta_X$ be a morphism. Then, there is an affine open subset $U \subset Y$ and morphism $\tilde{t} : U \to X$ which restricts to $t$ on $\eta_Y$.

Proof. We can assume that $Y = \text{Spec}(A)$ is affine. Let $(x, K)$ be a representative of the generic point of $X$ with the residual gerbe $\eta_X$. Then by [19, Théorème 6.2], there is an irreducible affine and noetherian scheme $W$ with an action of a finite group $G$ and a Cartesian diagram

$$\begin{array}{ccc}
\text{Spec}(K) & \to & [W/G] \\
\downarrow{id} & & \downarrow{\phi} \\
\text{Spec}(K) & \longrightarrow & X,
\end{array}$$

where $\phi$ is representable, étale and separated. Thus, we can assume that our stack $X$ is of the form $[W/G]$, where $G$ is a finite group acting on an irreducible, affine and noetherian scheme $W$.

Since a map $Z \to X = [W/G]$ is equivalent to a diagram

$$\begin{array}{ccc}
Z' & \xrightarrow{f} & W \\
\downarrow{\pi} & & \downarrow{p} \\
Z & \to & X,
\end{array}$$

where $\pi$ is a principal $G$-bundle and $f$ is a $G$-equivariant map, we see that the morphism $t : \eta_Y \to \eta_X \hookrightarrow X$ is equivalent to a principal $G$-bundle $T \xrightarrow{\pi} \eta_Y$ and a $G$-equivariant map $T \xrightarrow{\eta_Y} W$. Since $G$ is a finite group, the principal $G$-bundle $\pi$ extends to a principal $G$-bundle $V' \xrightarrow{\tilde{\pi}} V$, where $V \subset Y$ is an affine open subset. Thus, we can replace $Y$ by the appropriate open set and get a diagram of the form

$$\begin{array}{ccc}
Y' & \xleftarrow{T} & W \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
Y & \leftarrow & \eta_Y \xrightarrow{t} X,
\end{array}$$

where all the vertical maps are $G$-bundles and the top horizontal map is $G$-equivariant. Since $T$ is the generic fiber of $\tilde{\pi}$, it is the set of generic points of
\( Y' \) and hence is a finite set. In particular, there is an open subset \( V' \subset Y' \) containing \( T \) and a morphism (not necessarily \( G \)-equivariant) \( f' : V' \to W \) which extends the map \( f \). Put
\[
\tilde{T} = \bigcap_{g \in G} gV' \subset V' \quad \text{and let} \quad \tilde{f} = f'|\tilde{T}.
\]
Since \( V' \) contains all the generic points of \( Y' \), it is dense and hence \( gV' \) is dense open in \( Y' \) for all \( g \in G \). This implies that \( \tilde{T} \) is open dense in \( Y' \) which is now \( G \)-invariant. Letting \( U = \tilde{T}/G \), we see that the map \( t \) extends to a map \( \tilde{t} : U \to X \) where \( U \) is an open subscheme of \( Y \). By shrinking, we can assume further that \( U \) is affine.

**Lemma 7.3.** Let \( f : Y \to X \) be an étale morphism of irreducible stacks which is an isomorphism at the generic residual gerbes. Then there are dense open substacks \( V \leftrightarrow \tilde{Y} \) and \( U \leftrightarrow X \) such that \( g = f|_V : V \to U \) is an isomorphism.

**Proof.** Since \( X \) is a separated Deligne-Mumford stack which is irreducible, we can use [19, Théorème 6.1] and assume that \( X = [W/G] \), where \( G \) is a finite group acting on an irreducible affine and noetherian scheme \( W \). Thus we have a commutative diagram
\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{p} & Y \\
g \downarrow & & \downarrow f \\
W & \xrightarrow{q} & X
\end{array}
\]
where \( \eta_X \) and \( \eta_Y \) are the generic residual gerbes of \( X \) and \( Y \) respectively and the square on the left is Cartesian. In particular, \( \tilde{Y} \) is a noetherian and separated Deligne-Mumford stack with an action of \( G \) such that \( g \) is \( G \)-equivariant and the horizontal maps on the left square are principal \( G \)-bundles. Let \( \zeta \) denote the generic point of \( W \) and let \( \tilde{Y}_\zeta \) be the generic fiber of \( g \). Since \( \zeta \) is the only point which maps onto \( \eta_X \), the isomorphism of the right vertical vertical map implies that the map \( \tilde{Y}_\zeta \to \zeta \) is an isomorphism. This in turn implies that \( \tilde{Y} \) has only one generic point with the residual gerbe \( \theta = \tilde{Y}_\zeta \).

By Lemma 7.2 there is an affine open subset \( \tilde{V} \subset W \) and a map \( h' : \tilde{V} \to \tilde{Y} \) such that the composite \( \tilde{V} \to g^{-1}(V) \to V \) is identity as the composite is identity on the generic points. Putting \( \tilde{U} = \bigcap_{g \in G} g\tilde{V} \), we see that \( \tilde{U} \subset W \) is \( G \)-invariant and there is a \( G \)-equivariant map \( h : \tilde{U} \to \tilde{Y} \) such that the composite \( \tilde{U} \to g^{-1}(\tilde{U}) \to \tilde{V} \) is identity. Since \( g \) is étale and the composite \( g \circ h \) is identity, we see that \( \tilde{U} \) maps isomorphically onto an open substack \( \tilde{V}' \) of \( \tilde{Y} \) which is \( G \)-invariant. Putting \( V = [\tilde{V}'/G] \) and \( U = [\tilde{U}/G] \), we see that \( V \) is an open dense substack of \( Y \) which maps isomorphically to an open dense substack \( U \) of \( X \). \( \Box \)

**Proposition 7.4.** Let \( f : Y \to X \) be an étale morphism of stacks. Assume that for every generic residual gerbe \( \eta \) of \( X \), there is a generic residual gerbe \( \zeta \) of \( Y \) which maps isomorphically onto \( \eta \). Then there is an open substack \( V \) of \( Y \) containing a generic residual gerbe which maps isomorphically onto an open substack \( U \) of \( X \) containing a generic residual gerbe.

**Proof.** We fix a generic gerbe \( \eta \) of \( X \) and let \( Z \leftrightarrow X \) be the irreducible component of \( X \) such that \( \eta \) is the residual gerbe of the generic point of \( Z \). Let \( Z' \leftrightarrow X \) be the union of all other irreducible components of \( X \) and put \( U = X - Z' \). Then \( U \) is
an open substack of $X$ which contains the generic residual gerbe $\eta$. We choose an irreducible component $W$ of $Y$ such that the residual gerbe $\zeta$ at the generic point of $W$ maps isomorphically onto $\eta$ and let $W'$ be the union of all other irreducible components of $Y$. Put $V = U \times_X (Y - W')$.

We now see that $V$ is an open and irreducible substack of $Y$ containing the generic residual gerbe $\zeta$ such that $V$ maps to an open and irreducible substack $U$ of $X$ containing the generic residual gerbe $\eta$ of $X$. Hence by Lemma 7.3, there are open substacks $V' \hookrightarrow V$ and $U' \hookrightarrow U$ containing the generic residual gerbes $\zeta$ and $\eta$ respectively such that $V'$ maps isomorphically onto $U'$. Since $V'$ and $U'$ are also open substacks of $Y$ and $X$ respectively, we get the desired assertion. \hfill \square

**Theorem 7.5.** Let $X$ be a stack and let $\{X_i \to X\}$ be a finite Nisnevich covering of $X$. Then the morphism $f = \coprod f_i : Y = \coprod X_i \to X$ is split.

**Proof.** We choose a generic residual gerbe $\eta$ of $X$. By the definition of the Nisnevich covering, there a member $X_i$ of the family and a point $x_i$ with the residual gerbe $\zeta$ such that $\zeta$ maps isomorphically onto $\eta$. Since $f$ is étale, $x_i$ must be a generic point of $X_i$ and hence of $Y$. By Proposition 7.4, there is an open substack $V$ of $Y$ containing $\zeta$ which maps isomorphically onto an open substack $U$ of $X$ containing $\eta$. In particular, the map $U \times_X Y \to U$ has a section. Put $Z = X - U$. Since $Z$ is a proper closed substack of $X$ and since $Z \times_X Y \to Z$ is a Nisnevich cover, we conclude from the noetherian induction that the map $Z \times_X Y \to Z$ is split. Hence the map $f$ is split. \hfill \square

The following is our main result comparing the Nisnevich and Nisnevich cd-sites on stacks.

**Theorem 7.6.** Let $S$ be a noetherian and affine scheme and let $X \in DM_S$. Then a family $\{X_i \to X\}$ is a covering in the Nisnevich topology if and only if it is covering in the Nisnevich cd-topology. In particular, the Grothendieck sites $t_{Nis}^n$ and $(DM_S)_Nis$ are equivalent on $DM_S$. The Grothendieck sites $t_{Nis}^n$ and $(DM_S)^r_{Nis}$ are also equivalent.

**Proof.** We prove the equivalence of the sites $t_{Nis}^n$ and $(DM_S)_Nis$. The equivalence between $t_{Nis}^n$ and $(DM_S)^r_{Nis}$ follows exactly in the same without any change once we note that the splitting for a representable Nisnevich cover is given by the representable maps as is clear from the proof of Proposition 7.4.

We have already seen that a covering in the Grothendieck topology given by the Nisnevich cd-structure is also a Nisnevich covering. To prove the converse, let $\{X_i \to X\}$ be a Nisnevich covering. Since our stacks are noetherian, we can assume that this a finite covering. In view of Theorem 7.5, it suffices now to show that an étale map $f : \tilde{X} \to X$ which is split, is a covering in the Nisnevich cd-topology. We prove by the induction on the length of the splitting sequence. We shall construct a distinguished square of the form (7.1) based on $\tilde{X}$ such that the pull-back of $f$ to $Y$ has a section and the pull-back of $f$ to $A$ has a splitting sequence of length less than the length of the splitting sequence for $f$. The result will then follow by induction.

Let the map $f$ have the splitting sequence of the form (7.2). We take $A = X - Z_n$. To define $Y$, consider the section $s$ of $Z_n \times_X \tilde{X} \to Z_n$ which exists by the definition of the splitting sequence. Since $f$ is étale, it is étale over $Z_n$ and hence $s$ maps isomorphically onto an open substack $\tilde{Z}_n$ of $Z_n \times_X \tilde{X}$. Let $W$ be its complement. Then $W$ is a closed substack of $\tilde{X}$ and we take $Y = \tilde{X} - W$. It is immediate from the construction that the pull-back square defined by $\{A \to X, Y \to X\}$ is a distinguished square for the Nisnevich cd-structure. Moreover, the pull-back of $f$
to $Y$ has a section and the pull-back of $f$ to $A$ has a splitting sequence of length less than $n$. □

**Corollary 7.7.** Let $S$ be the spectrum of a field. Then the Grothendieck sites $\tilde{\mathcal{M}}_{Nis}$ and $(\tilde{\mathcal{M}})_{Nis}^r$ on $\tilde{\mathcal{M}}_S$ are equivalent. In other words, the site $(\tilde{\mathcal{M}})_{Nis}^r$ is given by a cd-structure.

**Proof.** This follows directly from Proposition 6.7 and Theorem 7.6. □

## 8. Some properties of Nisnevich cd-structure

In order to prove the Nisnevich descent for the $K$-theory of perfect complexes on stacks, we show in this section that the Nisnevich cd-structure on $\mathcal{DM}_S$ is complete, regular and bounded in the sense of [34]. We broadly follow the idea of [35] where this was shown for schemes. We need the following results about the étale and unramified morphisms of stacks which are well known for schemes. Recall that a morphism $f : X \rightarrow Y$ of stacks is étale (resp. unramified) if there are atlases $V \xrightarrow{v} X$ and $U \xrightarrow{u} Y$ and a diagram

\begin{equation}
V \xrightarrow{g} U \\
\downarrow{v} \quad \downarrow{u} \\
X \xrightarrow{f} Y
\end{equation}

such that $g$ is an étale (resp. unramified) map of $S$-schemes.

**Lemma 8.1.** A morphism $f : X \rightarrow Y$ in $\mathcal{DM}_S$ is unramified if and only if the diagonal morphism $\Delta_X : X \rightarrow X \times_Y X$ is étale. If $f$ is representable, then it is unramified if and only if $\Delta_X$ is an open immersion. The map $f$ is étale if and only if it is smooth and unramified.

**Proof.** Cf. [22, Appendix B.1, B.2]. □

**Proposition 8.2.** Let $S$ be a noetherian affine scheme. The Nisnevich cd-structure on $\mathcal{DM}_S$ is complete and regular. The same holds for the representable Nisnevich cd-structure on $\mathcal{DM}_S$.

**Proof.** Since it is clear from the definition of a Nisnevich distinguished square that the pull-back of any distinguished square of the form (7.1) is also a distinguished square, it follows from [34, Lemma 2.5] that the Nisnevich cd-structure on $\mathcal{DM}_S$ is complete.

We now show that the Nisnevich cd-structure on $\mathcal{DM}_S$ is regular. Let us consider a Nisnevich distinguished square of the form (7.1) based on a stack $X$. By [34, Lemma 2.11], it suffices to show that this is a pull-back square, $e$ is a monomorphism and the square

\begin{equation}
\begin{array}{ccc}
B & \xrightarrow{e'} & Y \\
\downarrow{\Delta_B} & & \downarrow{\Delta_Y} \\
B \times_A B & \xrightarrow{\sim} & Y \times_X Y
\end{array}
\end{equation}

is a Nisnevich distinguished square.

The first and the second assertions are obvious from the definitions since a Nisnevich distinguished square is a pull-back square and an open immersion of stacks is a monomorphism. So we only need to show (8.2).
Note first that as \( e \) is an open immersion, so is \( e' \) and hence (8.2) is a pull-back square. For the same reason, \( B \times_A B \to Y \times_Y Y \) is an open immersion. Let \( Z \) be the complement of \( A \) in \( X \) and let \( W = \overline{Z} \times_X Y \) with reduced structures. Then \( W \) is the complement of \( B \) in \( Y \) and we have \( W \cong Z \) via \( p \). Let \( Z' \) be the complement of \( B \times_A B \) in \( Y \times_X Y \). Then we have \( Z' = (Y \times_X W) \cup (W \times_X Y) \).

Since the map \( W \xrightarrow{e} Z \) is an isomorphism, we see that \( Z' \cong W \times_X W \cong W \times_Z W \). Since \( W = \Delta^{-1}_Y(W \times_Z W) \to W \times_Z W \) is an isomorphism, we see that \( \Delta^{-1}_Y(Z') \to Z' \) is an isomorphism. Furthermore, as \( p \) is étale, it follows from Lemma 8.1 that it is unramified and hence \( \Delta_Y \) is étale. This shows that (8.2) is a Nisnevich distinguished square. The same proof also works for the completeness and boundedness of the representable Nisnevich \( cd \)-structure. We only need to observe that the diagonal maps in (8.2) are representable. But this follows from [19, Lemme 7.7].

8.1. Boundedness of the Nisnevich \( cd \)-structures. To show that the Nisnevich \( cd \)-structure on \( DM_S \) is bounded, we first recall the topological space associated to a stack from [19, Chapter 5]. Let \( X \) be a stack \( DM_S \). Let \( |X| \) denote the set of the equivalence classes of points \( (x, K) \) on \( X \). One defines the Zariski topology on \( |X| \) by declaring a subset \( U' \subset |X| \) to be open if and only if it is of the form \( |U| \), where \( U \) is an open substack of \( X \). In particular, there is one-to-one correspondence between the open (resp. reduced and closed) substacks of \( X \) and open (resp. closed) subsets of \( |X| \). Moreover, this correspondence preserves the inclusion (and strict inclusion) relation between open (resp. reduced and closed) substacks of \( X \) (cf. [19, Corollaire 5.6.3]). It also follows from [loc. cit., Corollaire 5.7.2] that \( |X| \) is a finite union of its uniquely defined irreducible components and every irreducible closed subset \( T \subset |X| \) has a unique generic point \( t \) such that \( T = \{ \overline{\{ t \}} \} \) is of the form \( |Z| \) where \( Z \hookrightarrow X \) is an irreducible and reduced closed substack with generic point \( t \). It is also easy to check that \( |X| \) is a noetherian topological space of dimension equal to the dimension of the stack \( X \).

**Definition 8.3.** We define a density structure on \( DM_S \) by assigning to any \( X \in DM_S \), a sequence \( D_0(X), D_1(X), \ldots \) of families of morphisms in \( DM_S \) as follows. For \( i \geq 0 \), let \( D_i(X) \) be the family of inclusion of open substacks \( U \hookrightarrow X \) such that for every irreducible and reduced closed substack \( Z \hookrightarrow (X - U) \), there is a chain of inclusions of irreducible and reduced closed substacks \( Z = Z_0 \subset Z_1 \subset \cdots \subset Z_i \) in \( X \). A density structure such as above will be often denoted by \( D_*(-) \). Note here that all the maps in the definition of \( D_*(-) \) are closed and open immersions and hence representable.

It is easy to check that the above indeed defines a density structure on the category \( DM_S \) in the sense of [34, Definition 2.20]. We define a density structure on the category \( NTop \) of noetherian topological spaces to be the standard one (cf. [34, Example 2.23]). Thus, \( D_i(X) \) is the family of open subsets \( U \subset X \) such that for every point \( x \in (X - U) \), there is sequence of points \( x = x_0, x_1, \ldots, x_i \) in \( X \) such that \( x_j \neq x_{j+1} \) and \( x_j \in \{ x_{j+1} \} \) for each \( j \). Such a sequence is called an increasing sequence of length \( i \). For \( (U \hookrightarrow X) \in D_i(X) \), we shall often write \( U \in D_i(X) \). The following lemma is now elementary whose proof follows easily from the above, noting that the correspondence between the reduced and irreducible closed substacks of \( X \) and the irreducible closed subsets of \( |X| \) preserves the strict inclusion. We leave the detail as an easy exercise.

**Lemma 8.4.** Let \( X \) be a stack and let \( U \hookrightarrow X \) be an open substack. Then \( U \in D_i(X) \) if and only if \( |U| \in D_i(|X|) \).
Lemma 8.5. Let $f : X \to Y$ be an étale morphism of Deligne-Mumford stacks. Then the fibers of $f$ and $\overline{f} : |X| \to |Y|$ are zero-dimensional.

Proof. This is also elementary and known to experts. We only give a sketch. Let $\text{Spec}(K) \xrightarrow{\eta} Y$ be a point and let $W = \text{Spec}(K) \times_Y X$. Then $\tilde{f} : W \to \text{Spec}(K)$ is also étale. Let $U \xrightarrow{\eta} W$ be an étale atlas. Then the composite $g \circ u : U \to \text{Spec}(K)$ is an étale map of schemes and hence $U$ is zero-dimensional and so is $W$.

Now, let $y$ be the point of $|Y|$ defined by the point $\text{Spec}(K) \xrightarrow{\eta} Y$ above and let $T = (\tilde{f})^{-1}(y)$. Then there are natural maps $|W| \to T \to \{y\}$ such that the first map is surjective by \cite{19} Proposition 5.4. Since we have shown that $|W|$ is zero-dimensional, we see that $T$ must also be so. \hfill \Box

Lemma 8.6. Let $f : X \to Y$ be an étale and surjective morphism of stacks in $\mathcal{DMS}$ and let $U \in D_i(X)$. Then $U$ defines an unique open substack $V \hookrightarrow Y$ which is in $D_i(Y)$.

Proof. Since $f$ is étale, it is an open morphism. In particular, the image of $U$ in $Y$ defines a unique open substack $V \hookrightarrow Y$. We have to show that $V \in D_i(Y)$. By Lemma 8.3 it suffices to show that $[V] \in D_i([Y])$.

Let $y_0 \in (|Y| - |V|)$. Since $f$ is surjective, the map $\overline{f} : |X| \to |Y|$ is clearly surjective. In particular, there is a point $x_0 \in |X|$ with $\overline{f}(x_0) = y_0$. As $x_0 \notin |U|$ and since $|U| \in D_i(|X|)$ by Lemma 8.4, there is an increasing sequence $x_0, x_1, \cdots, x_i$ of length $i$ in $|X|$. Put $y_j = \overline{f}(x_j)$. We claim that $y_0, y_1, \cdots, y_i$ is an increasing sequence of length $i$ in $|Y|$.

To show that $y_j \in \{y_{j+1}\}$, let $V'$ be an open subset of $|Y|$ containing $y_j$ and let $U' = (\overline{f})^{-1}(V')$. Then $U'$ is an open subset of $|X|$ containing $x_j$ and hence also contains $x_{j+1}$, which in turn implies that $y_{j+1} \in V'$, that is, $y_j \in \{y_{j+1}\}$. Finally, suppose $y_j = y_{j+1} = y$, say, and put $T = (\overline{f})^{-1}(y)$. Let $T_j$ be the closure of $\{x_j\}$ in $T$. Then, the fact that $x_j \in \{x_{j+1}\}$ implies that $W_j \subset W_{j+1}$. Moreover, since $W_j$ is the closure of a point in $T$, it is easy to see that it is irreducible, that is, we get a chain $W_j \subset W_{j+1}$ of irreducible closed subsets in $T$ which means that the dimension of $T$ is at least one. But this is a contradiction from Lemma 8.5 since $f$ is étale. This completes the proof of the lemma. \hfill \Box

Lemma 8.7. Let

\begin{equation}
\begin{array}{ccc}
B & \xrightarrow{e} & Y \\
q \downarrow & & \downarrow p \\
A & \xrightarrow{\varphi} & X
\end{array}
\end{equation}

be a commutative square in $\mathcal{DMS}$ such that $e$ is an open immersion. Then this square is Cartesian if and only if the square of the associated topological spaces

\begin{equation}
\begin{array}{ccc}
|B| & \xrightarrow{\overline{e}} & |Y| \\
\overline{q} \downarrow & & \downarrow \overline{p} \\
|A| & \xrightarrow{\overline{\varphi}} & |X|
\end{array}
\end{equation}

is Cartesian.
Proof. We only need to prove the ‘only if’ part. The other one follows easily from this in the same way. So suppose the square (8.3) is Cartesian and let $T = |Y| \times_{|X|} |A|$. This gives a natural map $|B| \to T$ which factors $\overline{e_Y}$ and $\overline{q}$ in the square (8.4). Moreover, the map $b$ is surjective by [19, Proposition 5.4]. Since $e$ is an open immersion, we see that the maps $e_Y, e, e_Y$ and $T \to |Y|$ are all open immersions. Thus we get the maps $|B| \to T \to |Y|$ such that the second map and the composite map are both open immersions and the first map is surjective. The first map must then be an isomorphism. □

Proposition 8.8. The Nisnevich and the representable Nisnevich $cd$-structures on $\mathcal{DM}_S$ are bounded with respect to the density structure $D^*_\ast(\mathcal{S})$ defined above.

Proof. We show the boundedness of the Nisnevich $cd$-structure as the same proof also works verbatim for the representable Nisnevich $cd$-structure. Following the proof of [35] in the scheme case, we show that any Nisnevich distinguished square of stacks is reducing with respect to our density structure. So let

\[
\begin{array}{ccc}
B & \overset{e_Y}{\to} & Y \\
\downarrow & & \downarrow^p \\
A & \overset{e}{\to} & X
\end{array}
\]

be a Nisnevich distinguished square. Let $B_0 \in D_{i-1}(B)$, $A_0 \in D_i(A)$ and $Y_0 \in D_i(Y)$ be given. Then we have to find another Nisnevich distinguished square

\[
\begin{array}{ccc}
B' & \overset{e_Y'}{\to} & Y' \\
\downarrow & & \downarrow^{p'} \\
A' & \overset{e}{\to} & X'
\end{array}
\]

and a morphism of distinguished squares

\[
\begin{array}{ccc}
B' & \overset{e_Y'}{\to} & Y' \\
\downarrow & & \downarrow^{p'} \\
A' & \overset{e}{\to} & X' \\
\downarrow & & \downarrow \\
A & \overset{e}{\to} & X
\end{array}
\]

such that $B' \to B$ factors through $B' \to B_0$, $Y' \to Y$ factors through $Y' \to Y_0$, $A' \to A$ factors through $A' \to A_0$ and $X' \in D_i(X)$.

Since the map $(e \bigsqcup p) : A \bigsqcup Y \to X$ is étale and surjective, and since $(A_0 \bigsqcup Y_0) \in D_i(A \bigsqcup Y)$, we see from Lemma 8.6 that $X_0 = (e \bigsqcup p)(A_0 \bigsqcup Y_0) \in D_i(X)$. Since $D_i(X)$ is closed under finite intersections, by base changing to $X_0 \to X$, we can assume that $A_0 = A$ and $Y_0 = Y$.

We take $B' = B_0$ and $A' = A$ and let $Z = (X - A)$ and $T = B - B_0$. Following the notations of Lemma 8.7, we put $V' = |Y| - \left(\overline{e_Y}(|B - B_0|)\right)$ and $W' = |X| - \left(\overline{e_X}(|B - B_0|)\right)$. Since
\[(\{X - A\} \cap (p \circ e_Y)(B - B_0))\]. Let \(Z_Y\) and \(Z_X\) be the unique reduced closed substacks of \(Y\) and \(X\) respectively such that \(|Z_Y| = e_Y(B - B_0)|\) and \(|Z_X| = (p \circ e_Y)(B - B_0)|\). Then we have \(W' = |X - (Z \cap Z_X)|\) and \(V' = |Y - Z_Y|\). Put \(X' = X - (Z \cap Z_X)\) and \(Y' = Y - Z_Y\). We then see that \(A' = A \hookrightarrow X'\) and the map \(p\) then induces a map \(Y' \to X'\), which we write as \(p'\). Moreover, as \(Y'\) is an open substack of \(Y\), the map \(p'\) is also étale.

By Lemma \(8.7\), the square \((8.4)\) is Cartesian, \(\overline{e}\) is an open immersion and the map \(\overline{p}: (\overline{p})^{-1}(|Z|) \to |Z|\) is an isomorphism. From this, one easily checks that the square

\[
\begin{array}{ccc}
|B'| & \xrightarrow{|Y'|} & |Y'| \\
\overline{p} & & \overline{p} \downarrow \\
|A'| & \xrightarrow{|X'|} & |X'|
\end{array}
\]

is Cartesian, \(\overline{e}\) is an open immersion and \((\overline{p})^{-1}(|X' - A'|) \to |X' - A'|\) is an isomorphism and \(|X'| \in D_i(|X|)\) (cf. \[35, Proposition 2.10\]). Thus, we have a commutative diagram

\[
\begin{array}{ccc}
B' & \xrightarrow{e_Y} & Y' \\
\downarrow & & \downarrow \overline{p} \\
A' & \xrightarrow{e_X} & X',
\end{array}
\]

where \(e' = e\) and \(e_Y'\) is the restriction of \(e_Y\) on the open substack \(B'\) such that \((8.8)\) is the diagram of the associated topological spaces. Since this square is Cartesian and since \(e'\) is an open immersion, it follows from Lemma \(8.7\) that \((8.9)\) is a Cartesian square and the map \(p'^{-1}(X' - A') \to (X' - A')\) is an isomorphism of the associated reduced closed substacks. In particular, \((8.9)\) is a Nisnevich distinguished square which clearly maps to \((8.5)\). To complete the proof of the proposition, we only need to show that \(X' \in D_i(X)\). But this follows immediately from Lemma \(8.4\) since \(|X'| \in D_i(|X|)\) as shown above. \(\square\)

9. Nisnevich descent and other consequences

In this section, we prove the Nisnevich descent for the \(K\)-theory of perfect complexes on stacks using the results of the previous sections and we also draw some other important consequences for the cohomology of sheaves on stacks.

Recall that \((\mathcal{DM}_S)_{Nis}\) is a Grothendieck site on the category \(\mathcal{DM}_S\) which is noetherian. Let \(*\) be a chosen final object in the category of sets. For \(X \in \mathcal{DM}_S\), let \(Z_X\) denote the sheaf on the restricted Grothendieck site \(\mathcal{DM}_S/X\) of all objects lying over \(X\), which is the sheaf given by the free abelian group on the presheaf which takes every object to \(*\). For a sheaf \(\mathcal{F}\) on \((\mathcal{DM}_S)_{Nis}\), let \(H^i_{Nis}(X, \mathcal{F})\) denote the cohomology groups \(\text{Ext}^i_{Nis}(\mathbb{Z}_X, \mathcal{F})\).

**Corollary 9.1.** Let \(\mathcal{F}\) be a presheaf on \((\mathcal{DM}_S)_{Nis}\). Then \(\mathcal{F}\) is a sheaf if and only if \(\mathcal{F}(\emptyset) = *\) and for every Nisnevich distinguished square of the form \((7.1)\), the
is a pull-back.

**Proof.** It follows immediately from Theorem 7.6, Propositions 8.2, 8.8 and Lemma 2.9, Corollary 2.17.

**Corollary 9.2.** Let $X$ be a stack of dimension $n$ and let $F$ be a sheaf of abelian groups on $\mathcal{DM}_S/X$. Then

$$H^i_{\text{Nis}}(X, F) = 0 \text{ for } i > n.$$  

**Proof.** It follows immediately from Theorem 7.6, Propositions 8.2, 8.8 and Theorem 2.27.

**Remark 9.3.** We remark that the same proof also shows that the Zariski cohomological dimension of a stack is bounded by its Krull dimension.

In the special case when $X$ is a tame stack and $F$ is a quasi-coherent sheaf, the above result was proven in [17, Theorem 1.10].

### 9.1. Nisnevich descent.

Let $\mathcal{C}$ be a Grothendieck site and let $\text{Pres}(\mathcal{C})$ denote the category of presheaves of spectra on $\mathcal{C}$. Recall from [7, Section 3] that a morphism $f : \mathcal{E} \to \mathcal{E}'$ of presheaves of spectra is called a **global weak equivalence** if $\mathcal{E}(X) \to \mathcal{E}'(X)$ is a weak equivalence of spectra for every object $X$ in $\mathcal{C}$. It is called a **local weak equivalence** if it induces an isomorphism on the sheaves of stable homotopy groups of the presheaves of spectra. Recall from [10] (see also [11]) that there is model structure, called the **injective model structure** on $\text{Pres}(\mathcal{C})$ for which the weak equivalence is the local weak equivalence. Moreover, a morphism $f : \mathcal{E} \to \mathcal{E}'$ of presheaves of spectra is a cofibration if each $\mathcal{E}(X) \to \mathcal{E}'(X)$ is a cofibration, and the fibrations are defined by the right lifting property with respect to trivial cofibrations. Recall also that in the above model structure, a **fibrant replacement** of $\mathcal{E}$ is a trivial cofibration $\mathcal{E} \to \mathcal{E}'$ such that $\mathcal{E}'$ is fibrant. In particular, $\mathcal{E} \to \mathcal{E}'$ is a local weak equivalence. We also recall here the following from [7].

**Definition 9.4.** A presheaf of spectra $\mathcal{E}$ on $\mathcal{C}$ is said to satisfy the **descent** in the Grothendieck topology of the site $\mathcal{C}$ if the fibrant replacement map $\mathcal{E} \to \mathcal{E}'$ is a global weak equivalence.

For the presheaf of non-connective spectra $\mathcal{K}$ on $\mathcal{DM}_S$ which associates to a stack $X$, the $K$-theory spectrum of perfect complexes on $X$, let $\mathcal{K}^{\text{Nis}}$ denote its fibrant replacement in the Grothendieck topology $(\mathcal{DM}_S)_{\text{Nis}}$. One says that $K$-theory satisfies the Nisnevich descent if the presheaf of spectra $\mathcal{K}$ satisfies descent in the Nisnevich topology $(\mathcal{DM})_{\text{Nis}}$ on stacks. For any $q \in \mathbb{Z}$, let $\mathcal{K}_q$ denote the sheaf of $q$th stable homotopy groups of $\mathcal{K}$ on $\mathcal{DM}_S$. Thus, $\mathcal{K}_q$ is the sheafification of the presheaf $X \mapsto K_q(X)$. The following is our main result about the Nisnevich descent.

**Theorem 9.5.** Let $S$ be the spectrum of a field $k$ and let $\mathcal{DM}S$ denote the full subcategory of $\mathcal{DM}_S$ consisting of tame stacks. Then, the $K$-theory of perfect
complexes satisfies the Nisnevich descent on the Grothendieck site \( \widetilde{\mathcal{DM}}_S \). In particular, if \( \text{char}(k) = 0 \), then \( K \)-theory satisfies the Nisnevich descent on \( \widetilde{\mathcal{DM}}_S \).

**Proof.** It is an elementary fact that the fiber product of representable maps of tame stacks is also tame. The theorem now follows immediately from Corollaries 5.6, 7.7, Propositions 8.2, 8.8 and [7, Theorem 3.4]. \( \square \)

**Corollary 9.6.** Let \( S \) be the spectrum of a field and let \( X \) be in \( \widetilde{TDM}_S \). Then there is a strongly convergent spectral sequence

\[
E^{p,q}_2 = H^p_{\text{Nis}}(X, K_q) \Rightarrow K_{q-p}(X).
\]

**Proof.** This is an immediate consequence of Theorem 9.5 and Corollary 9.2. \( \square \)

The following is also an immediate consequence of Theorem 9.5.

**Corollary 9.7.** Let \( S \) be the spectrum of a field and let \( X \) be in \( \widetilde{TDM}_S \). Then for any representable Nisnevich cover \( U \to X \), the natural map

\[
K_*(X) \to H^*(U, K)
\]

is an isomorphism. \( \square \)

**Remark 9.8.** If \( S = \text{Spec}(k) \) with \( \text{char}(k) = 0 \) and \( G \) is an affine smooth group scheme over \( S \), then it follows from the Remark 6.10 and Theorem 9.5 that the equivariant \( K \)-theory satisfies the Nisnevich descent on the category \( Sch^G_S \) of schemes of finite type over \( S \) with proper action of \( G \).

**Example 9.9.** Let \( S \) be a noetherian scheme and let \( G \) be a smooth and affine group scheme over \( S \). Let \( Sch^G_S \) denote the category of schemes of finite type over \( S \) with proper action of \( G \). We have shown in Remark 6.10 that there is a naturally defined Nisnevich site on \( Sch^G_S \) where a \( G \)-equivariant map \( f : X \to Y \) is a Nisnevich cover if the associated map of quotient stacks \([X/G] \to [Y/G]\) is a Nisnevich cover in the category \( DM_S \). We have also shown that this site on \( Sch^G_S \) is given by a complete, regular and bounded cd-structure. In [25], an isovariant Nisnevich site on \( Sch^G_S \) is studied, where a \( G \)-equivariant map \( f : X \to Y \) is defined to be a Nisnevich cover if \( f \) is a Nisnevich cover of schemes, forgetting the \( G \)-action and it preserves the isotropy groups. We show in the following example that this site is not a reasonable site to consider and may not be given by a cd-structure.

So let \( k = \mathbb{R} \) and let \( G = \mathbb{Z}/2 \). We set \( X = \text{Spec}(\mathbb{C}) \) and \( Y = \text{Spec}(\mathbb{R}) \) with the obvious map \( f : X \to Y \). Note that there is a \( G \)-action on \( X \) induced by the conjugation on \( \mathbb{C} \) and \( f \) is the quotient map. We consider the commutative diagram

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\phi} & X \\
\downarrow{p} & & \downarrow{f} \\
X & \xrightarrow{f} & Y,
\end{array}
\]

where \( \phi \) is the action map which is \( G \)-equivariant if we let \( G \) act diagonally on \( G \times X \) for the trivial action on \( X \) and the left multiplication action on \( G \). The map \( p \) is the projection. Since \( f \) is a principal bundle quotient, the above diagram is Cartesian. Since \( G \) acts freely on itself, it acts freely on \( G \times X \cong X \coprod X \), and the map \( \phi \) takes each component isomorphically onto \( X \) as schemes. Thus, we find that \( \phi \) is a \( G \)-equivariant map which is Nisnevich and isovariant. However,
the map $f$ of quotients is not a Nisnevich map. This shows in particular that the isovariant Nisnevich site (cf. [25]) of $X$ is not equivalent to the usual Nisnevich site of $X/G$. This also shows that the isovariant Nisnevich site on $\text{Sch}^G_{G}$ may not be given by a cd-structure.

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References

[1] D. Abramovich, A. Vistoli, Compactifying the space of stable maps, J. Amer. Math. Soc., 15, no. 1, (2001), 27-75.
[2] M. Artin, Grothendieck Topologies, Harvard University Seminar, Spring, (1962).
[3] A. Bondal, M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J., 3, (2003), no. 1, 1-36.
[4] K. Brown, S. Gersten, Algebraic $K$-theory as generalized sheaf cohomology, Lecture Notes in Math., 341, Springer-Verlag, New York, 1973, 85-147.
[5] D. Cisinski, G. Tabuada, Non-connective $K$-theory via universal invariants, math.AG/0903.3717, (2009).
[6] D. Cisinski, A. Krishna, P. Østvaer, Descent theorems for homotopy invariant $K$-theory of stacks, In preparation, (2010).
[7] G. Cortinas, C. Haesemeyer, M. Schlichting, C. Weibel Cyclic homology, cdh-cohomology and negative $K$-theory, Ann. of Math., (2) 167, (2008), no. 2, 549-573.
[8] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, Inst. Hautes Etudes Sci. Publ. Math., 36, (1969), 75-109.
[9] P. Hu, I. Kriz, K. Ormsby, Equivariant and real motivic stable homotopy theory, $K$-theory Preprint arxiv., (2010).
[10] J. Jardine, Simplicial presheaves, J. Pure Appl. Algebra, 47, (1987), no. 1, 35-87.
[11] J. Jardine, Generalized étale cohomology theories, Progress in Mathematics, 146. Birkhauser Verlag, Basel, 1997.
[12] R. Joshua, Higher intersection theory on algebraic stacks. II, $K$-Theory, 27, (2002), no. 3, 197-244.
[13] R. Joshua, $K$-theory and absolute cohomology of algebraic stacks, Preprint, (2005).
[14] R. Joshua, Bredon-style homology, cohomology and Riemann-Roch for algebraic stacks, Adv. Math., 209 (2007), no. 1, 1-68.
[15] B. Keller, On the cyclic homology of ringed spaces and schemes, Doc. Math., 3, (1998), 231-259.
[16] D. Knutson, Algebraic spaces, Lecture Notes in Mathematics, 203, Springer-Verlag, Berlin, 1971.
[17] A. Krishna, Perfect complexes on Deligne-Mumford stacks and applications, J. K-Theory, 4, (2009), no. 3, 559-603.
[18] A. Krishna, Equivariant $K$-theory and Higher Chow Groups, math.AG/0906.3109, (2009).
[19] G. Laumon, L. Moret-Baily Champs algébriques, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 39, Springer-Verlag, Berlin, (2000).
[20] J. Milne, Étale cohomology, Princeton University Press, Princeton, (1980).
[21] A. Merkurjev, Equivariant $K$-theory, Handbook of K-theory, Springer-Verlag, 1, 2, (2005), 925-954.
[22] D. Rydh, The canonical embedding of an unramified morphism in an étale morphism, Math. Z., to appear, (2010).
[23] D. Rydhl, Existence of quotients by finite groups and coarse moduli spaces, math.AG/0708.3333, (2009).
[24] C. Serpe, Resolution of unbounded complexes in Grothendieck categories, J. Pure Appl. Algebra, 177 (2003), no. 1, 103-112.
[25] C. Serpe, Descent properties of equivariant $K$-theory, math.AG/1002.2565, (2010).
[26] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math., 65, (1988), 121-154.

[27] R. Thomason, *Algebraic K-theory of group scheme actions*, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), 539-563, Ann. of Math. Stud., 113, Princeton Univ. Press, Princeton, NJ, 1987.

[28] R. Thomason, *Equivariant resolution, linearization, and Hilbert’s fourteenth problem over arbitrary base schemes*, Adv. in Math., 65, (1988), no. 1, 16-34.

[29] R. Thomason, *Equivariant algebraic vs. topological K-homology Atiyah-Segal-style*, Duke Math. J., 56, (1988), no. 3, 589-636.

[30] R. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 247–435, Progr. Math., 88, Birkhauser Boston, Boston, MA, 1990.

[31] B. Töen, *Théorème de Riemann-Roch pour les champs de Deligne-Mumford*, K-Theory, 18, (1999), 33-76.

[32] B. Töen, *Derived Azumaya’s algebras and generators for twisted derived categories*, math.AG/1002.2599, (2010).

[33] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math., 97, (1989), 613-670.

[34] V. Voevodsky, *Homotopy theory of simplicial sheaves in completely decomposed topology*, J. Pure Appl. Algebra, 214 (2010), 1384-1398.

[35] V. Voevodsky, *Unstable motivic categories in Nisnevich and cdh-topology*, J. Pure Appl. Algebra, 214 (2010), 1399-1406.

[36] C. Weibel, *Cyclic homology for schemes*, Proc. Amer. Math. Soc., 124, (1996), no. 6, 1655-1662.

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