Comment on ‘The tan \( \theta \) theorem with relaxed conditions’, by Y. Nakatsukasa

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ABSTRACT. We show that in case of the spectral norm, one of the main results of the recent paper The tan \( \theta \) theorem with relaxed conditions, by Yuji Nakatsukasa, published in Linear Algebra and its Applications is a corollary of the tan \( \theta \) theorem proven in [V. Kostrykin, K. A. Makarov, and A. K. Motovilov, On the existence of solutions to the operator Riccati equation and the tan \( \theta \) theorem, IEOT 51 (2005), 121 – 140]. We also give an alternative finite-dimensional matrix formulation of another tan \( \theta \) theorem proven in [S. Albeverio and A. K. Motovilov, The a priori tan \( \theta \) theorem for spectral subspaces, IEOT 73 (2012), 413 - 430].

In a recent paper [7] published in Linear Algebra and its Applications, Y. Nakatsukasa obtains two bounds on the tangent of the canonical angles between an approximate and an exact spectral subspace of a Hermitian matrix. These bounds (see [7, Theorems 1 and 2]) extend respectively the tan \( \theta \) theorem and the generalized tan \( \theta \) theorem proven by C. Davis and W. M. Kahan in their celebrated paper [2]. Actually, an extension of the tan \( \theta \) theorem similar to [7, Theorem 1] has already been given in [4], in the wider context of the perturbation theory for self-adjoint operators on a Hilbert space.

In our discussion below we restrict ourselves to the spectral norms of the matrices involved, that is, by \( \|S\| \) we always understand the maximal singular value of a matrix \( S \). If \( \mathfrak{A} \) and \( \mathfrak{L} \) are subspaces of \( \mathbb{C}^n \), the notation \( \angle(\mathfrak{A}, \mathfrak{L}) \) is used for the largest principal angle between \( \mathfrak{A} \) and \( \mathfrak{L} \).

We begin with presenting a relevant finite-dimensional version of the tan \( \theta \) theorem from [4] (see [4, Theorem 2]).

**Proposition 1.** Assume that a Hermitian matrix \( L \in \mathbb{C}^{n \times n} \) is block partitioned in the form

\[
L = \begin{bmatrix}
A_1 & B \\
B^H & A_2
\end{bmatrix}
\]

(1)

with \( A_1 \in \mathbb{C}^{k \times k}, 1 < k < n \). Let the spectrum of \( A_1 \) lie in \((-\infty, \alpha - \delta] \cup [\beta + \delta, \infty)\), where \( \alpha \leq \beta \) and \( \delta > 0 \). Suppose that \( \mathfrak{L}_1 \) and \( \mathfrak{L}_2 \) are complementary orthogonal reducing subspaces of \( L \) such that \( \dim(\mathfrak{L}_1) = k \) and the spectrum of the restriction \( L|_{\mathfrak{L}_2} \) of (the operator) \( L \) on the reducing subspace \( \mathfrak{L}_2 \) is confined in \([\alpha, \beta]\). Also, let \( \mathfrak{A}_1 \) be the subspace of \( \mathbb{C}^n \) spanned by the first \( k \) columns of the identity matrix \( I_n \). Then

\[
\tan \angle(\mathfrak{A}_1, \mathfrak{L}_1) \leq \frac{\|B\|}{\delta}.
\]

(2)

**Remark 2.** Actually, [4, Theorem 2] (combined with [4, Theorem 2.3]) suggests the equivalent bound \( \delta \tan ||\Theta|| \leq ||B|| \) for the operator angle \( \Theta \) between the orthogonal complements \( \mathfrak{A}_2 \) and \( \mathfrak{L}_2 \) of the subspaces \( \mathfrak{A}_1 \) and \( \mathfrak{L}_1 \), respectively, provided that \( \mathfrak{L}_2 \) is the graph of an operator from \( \mathfrak{A}_2 \) to \( \mathfrak{A}_1 \). But the latter, in the finite-dimensional case under consideration, holds true automatically. This is seen from the following lemma.
Lemma 3. Assume the hypothesis of Proposition 1. Then $\mathfrak{L}_2 \cap \mathfrak{L}_1 = \mathfrak{A}_1 \cap \mathfrak{L}_2 = \{0\}$ and, hence, the reducing subspace $\mathfrak{L}_2$ is the graph of an operator from $\mathfrak{L}_2$ to $\mathfrak{A}_1$.

Proof. By the hypothesis, the dimensions of the subspaces $\mathfrak{L}_1$ and $\mathfrak{A}_1$ coincide, $\dim(\mathfrak{L}_1) = \dim(\mathfrak{A}_1) = k$. Then by using the canonical orthogonal decomposition of $\mathbb{C}^n$ with respect to the orthogonal projections onto $\mathfrak{A}_1$ and $\mathfrak{L}_1$ (see, e.g. [3, Theorem 2.2]) one verifies that

$$\dim(\mathfrak{L}_2 \cap \mathfrak{L}_1) = \dim(\mathfrak{A}_1 \cap \mathfrak{L}_2).$$

Suppose that $\mathfrak{A}_1 \cap \mathfrak{L}_2 \neq \{0\}$. In such a case, there is a vector $y \in \mathfrak{L}_2$ of the form $y = \begin{bmatrix} x \\ 0_{n-k} \end{bmatrix}$, where the lower subcolumn $0_{n-k}$ consists of exactly $n-k$ zeros and the upper subcolumn $x$ contains at least one nonzero element. For $c = (\alpha + \beta)/2$ one then obtains

$$\|(L - c \lambda y\| = \|(A_1 - c \lambda I_k)x\| + \|Bx\| \geq \|(A_1 - c \lambda I_k)x\|^2 \geq \left(\frac{1}{2}(\beta - \alpha) + \delta\right)^2 \|y\|^2,$$

since $\|y\| = \|x\|$ and the spectrum of $A_1$ belongs to $(-\infty, \alpha - \delta] \cup [\beta + \delta, \infty)$. On the other hand, for $y \in \mathfrak{L}_2$ we should have $\|(L - c \lambda y\| \leq \frac{1}{2}(\beta - \alpha) \|y\|$ since the spectrum of the restriction $L|_{\mathfrak{L}_2}$ lies in $[\alpha, \beta]$. Hence, $y = 0$, a contradiction, which yields $\mathfrak{A}_1 \cap \mathfrak{L}_2 = \{0\}$. Applying [3, Theorem 3.2] completes the proof. □

Now we show that for the spectral norm the tan $\theta$ theorem proven in [7] is a corollary of Proposition 1. We reproduce the corresponding statement from [7] in the following form (see [7, Theorem 1]).

Proposition 4 ([7]). Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Let $X = [X_1 \ X_2]$ be a unitary eigenvector matrix of $A$ with $X_1 \in \mathbb{C}^{n \times k}$, $1 < k < n$, so that $X^HAX = \text{diag}(\Lambda_1, \Lambda_2)$ is diagonal and $\Lambda_1$ has $k$ columns. Assume that the columns of a matrix $Q_1 \in \mathbb{C}^{n \times k}$ are orthonormal and let $R = AQ_1 - Q_1A_1$, where $A_1 = Q_1^H AQ_1$. Furthermore, assume that for some $\alpha \leq \beta$ and $\delta > 0$ the spectrum of $A_1$ lies in $(-\infty, \alpha - \delta] \cup [\beta + \delta, \infty)$ and the spectrum of $\Lambda_2$ belongs to $[\alpha, \beta]$. Then

$$\tan \angle(\Omega_1, X_1) \leq \frac{\|R\|}{\delta},$$

where $\Omega_1$ and $X_1$ are the subspaces spanned by the columns of $Q_1$ and $X_1$, respectively.

Proof. Assume that $Q_1$ is a submatrix of a unitary $n \times n$ matrix $Q = [Q_1 \ Q_2]$ and let $L = Q^H AQ$. The matrix $L$ has the form (1) with $A_1 = Q_1^H AQ_1$, $A_2 = Q_2^H AQ_2$, and $B = Q_2^H AQ_1$. Since $A$ is unitarily equivalent to the diagonal matrix $\Lambda = \text{diag}(\Lambda_1, \Lambda_2)$, the same is true for $L$. Moreover, the $k$-dimensional subspace $\mathfrak{L}_1 = Q^H X_1$ and its orthogonal complement $\mathfrak{L}_2 = \mathbb{C}^n \ominus \mathfrak{L}_1$ are reducing subspaces of $L$. The spectrum of the restriction $L|_{\mathfrak{L}_2}$ coincides with the spectrum of $\Lambda_2$ and, hence, it lies in $[\alpha, \beta]$. If the subspace $\mathfrak{A}_1$ is as in Proposition 1, then, just by this proposition, the largest principal angle between $\mathfrak{A}_1$ and $\mathfrak{L}_1$ satisfies the bound (2). Meanwhile, the subspaces $\Omega_1$ and $X_1$ are obtained from $\mathfrak{A}_1$ and $\mathfrak{L}_1$ by the same unitary transformation: $\Omega_1 = Q \mathfrak{A}_1$ and $X_1 = Q \mathfrak{L}_1$. Hence, $\angle(\Omega_1, X_1) = \angle(\mathfrak{A}_1, \mathfrak{L}_1)$. Observing that $B = Q_2^H (AQ_1 - Q_1A_1) = Q_2^H R$, one infers $\|B\| = \|R\|$ and then (2) implies (4). □

Remark 5. In its turn, Proposition 1 may be viewed as a particular version of Proposition 4 for the case where $[Q_1 \ Q_2]$ is taken equal to the identity matrix $I_n$. Thus, in fact these two propositions are equivalent to each other.

We next note that there is another sharp tan $\theta$ bound established in [1, Theorem 1] (see also [6, Theorem 2] for an earlier result). The following assertion represents a finite-dimensional version of [1, Theorem 1] reformulated in the style of Proposition 4.
Proposition 6. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix and $Q = [Q_1 \, Q_2]$ a unitary matrix with $Q_1 \in \mathbb{C}^{n \times k}$, $1 < k < n$. Assume that for some $a \leq b$ and $d > 0$ the spectrum of $A_1 = Q_1^H A Q_1$ lies in $(-\infty, a - d] \cup [b + d, \infty)$ and that the spectrum of $A_2 = Q_2^H A Q_2$ belongs to $[a, b]$. Let $R = A Q_1 - Q_1 A_1$ and suppose that $\|R\| < \sqrt{2d}$. Then $n$ orthonormal eigenvectors of $A$ may be numbered in such an order that the corresponding unitary eigenvector matrix $X = [X_1 \, X_2]$ with $X_1 \in \mathbb{C}^{n \times k}$ reduces $A$ to the diagonal form $X^H AX = \text{diag}(\Lambda_1, \Lambda_2)$ with $\Lambda_1 \in \mathbb{C}^{k \times k}$ having its spectrum in $(-\infty, a - d] \cup [b + d, \infty)$, and with $\Lambda_2$ having all its eigenvalues in $[a - \delta_R, b + \delta_R]$, where $\delta_R = \|R\| \tan \left( \frac{1}{2} \arctan \frac{2\|R\|}{d} \right) < d$. Moreover,

$$\tan \angle(\Omega_1, \mathcal{X}_1) \leq \frac{\|R\|}{d},$$

where $\Omega_1$ and $\mathcal{X}_1$ are the subspaces spanned by the columns of $Q_1$ and $X_1$, respectively.

Proof. The matrix $L = Q^H A Q$ has the form (1) with $A_1$ and $A_2$ defined in the hypothesis, and $B = Q_1^H A Q_1$. As in the proof of Proposition 4 we have $\|B\| = \|R\|$. Hence $\|B\| < \sqrt{2d}$ and then the statement on the eigenvalue matrix $\Lambda$ and, in particular, on the spectral inclusions for $\Lambda_1$ and $\Lambda_2$, is an immediate corollary of [5, Theorem 2]. Furthermore, for the case under consideration, the bound from [1, estimate (1.3) in Theorem 1] may be equivalently written as $d \tan \angle(\Omega_1, \mathcal{X}_1) \leq \|R\|$, where $\Omega_1$ is as in Proposition 1 and $\mathcal{X}_1$ is the spectral subspace of $L$ associated with the set $(-\infty, a - d] \cup [b + d, \infty)$. By the unitarity argument we already used in the proof of Proposition 4, the bound $d \tan \angle(\Omega_1, \mathcal{X}_1) \leq \|R\|$ implies the bound (5). \hfill \Box

Remark 7. In general, condition $\|R\| < \sqrt{2d}$ cannot be removed. If this condition is violated, the matrix $A$ may not have eigenvalues in the interval $(a - d, b + d)$ at all (see [5, Example 1.6]).

If we estimate $\angle(\Omega_1, \mathcal{X}_1)$ by using inequality (5), no knowledge on the exact eigenvalues of $A$ is required. Unlike the bound (4), the estimate (5) involves the separation distance $d$ between the respective eigenvalue sets of the matrices $A_1$ and $A_2$. In applications, these sets are usually treated as an approximate spectrum of $A$ and their separation distance is assumed to be known prior to further calculations. Following [6] and [1], it is appropriate thus to call the bound (5) the a priori $\tan \theta$ theorem. Similarly, the bound (4) may be called the (semi-)a posteriori $\tan \theta$ theorem since it involves the separation distance between one approximate and one exact spectral sets.

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