An explicit sum-product estimate in $\mathbb{F}_p$

M. Z. Garaev
Instituto de Matemáticas, UNAM
Campus Morelia, Apartado Postal 61-3 (Xangari)
C.P. 58089, Morelia, Michoacán, México
garaev@matmor.unam.mx

Abstract

Let $\mathbb{F}_p$ be the field of residue classes modulo a prime number $p$ and let $A$ be a non-empty subset of $\mathbb{F}_p$. In this paper we give an explicit version of the sum-product estimate of Bourgain, Katz, Tao and Bourgain, Glibichuk, Konyagin on the size of $\max\{|A+A|, |AA|\}$. In particular, our result implies that if $1 < |A| \leq p^{7/13}(\log p)^{-4/13}$, then

$$\max\{|A+A|, |AA|\} \gg \frac{|A|^{15/14}}{(\log |A|)^2/7}.$$ 

2000 Mathematics Subject Classification: 11B75, 11T23

Key words: sum set, product set, sum-product estimates

1 Introduction

Let $\mathbb{F}_p$ be the field of residue classes modulo a prime number $p$ and let $A$ be a non-empty subset of $\mathbb{F}_p$. Consider the sum set

$A+A = \{a+b : a \in A, b \in A\}$

and the product set

$AA = \{ab : a \in A, b \in A\}$. 

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Bourgain, Katz, Tao \cite{5} and Bourgain, Glibichuk, Konyagin \cite{4} have shown that if $|A| < p^{1-\delta}$, where $\delta > 0$, then one has the sum-product estimate

$$\max\{|A + A|, |AA|\} \gg |A|^{1+\varepsilon}$$  \hspace{1cm} (1)$$

for some $\varepsilon = \varepsilon(\delta) > 0$. This result has found a number of spectacular applications in combinatorial problems and exponential sum estimates, see \cite{1}–\cite{5}.

Bound (1) does not yield an explicit relationship between $\varepsilon$ and $\delta$. Hart, Iosevich and Solymosi \cite{9}, by using Kloosterman sums, could obtain a concrete value of $\varepsilon$ in certain ranges of $|A|$. More precisely, they proved that

$$\max\{|A + A|, |AA|\} \gg \begin{cases} |A|^{3/2}p^{-1/4}, & \text{if } p^{1/2} < |A| < p^{7/10}, \\ |A|^{2/3}p^{1/3}, & \text{if } p^{7/10} < |A| \leq p. \end{cases}$$

(2)

The aim of the present paper is to obtain an explicit sum-product estimate for any range of $|A|$.

\textbf{Theorem 1.} Let $|A| > 1$. Then

$$\max\{|A + A|, |AA|\} \gg \min \left\{ \frac{|A|^{15/14} \max\left\{ 1, |A|^{1/7}p^{-1/14} \right\}}{(\log |A|)^{2/7}}, \frac{|A|^{11/12}p^{1/12}}{(\log |A|)^{1/3}} \right\}.$$  \hspace{1cm} (3)

In particular, if $1 < |A| < p^{7/13}(\log p)^{-4/13}$, then we have

$$\max\{|A + A|, |AA|\} \gg \frac{|A|^{15/14}}{(\log |A|)^{2/7}}.$$  \hspace{1cm} (4)

The proof of Theorem 1 uses results and tools from arithmetical combinatorics. When $|A|$ is larger than $p^{5/9}$, combinatorial arguments and trigonometric sums can be used together to get a better estimate.

\textbf{Theorem 2.} Let $|A| > 1$. Then

$$\max\{|A + A|, |AA|\} \gg \min\left\{ |A|^{5/3}p^{-1/3}(\log |A|)^{-1/3}, |A|^{2/3}p^{1/3}(\log |A|)^{-1/3} \right\}.$$  \hspace{1cm} (5)

In particular, if $1 < |A| < p^{2/3}$, then we have

$$\max\{|A + A|, |AA|\} \gg |A|^{5/3}p^{-1/3}(\log |A|)^{-1/3}.$$
When \( p^{7/10}(\log p)^{-1/3} < |A| < p \), the inequality (2) is preferable. We remark that (2) can also be proved using the method described in the present paper.

In the corresponding problem for integers (i.e., if the field \( \mathbb{F}_p \) is replaced by the set of integers) the conjecture of Erdős and Szemerédi \[6\] is that
\[
\max\{|A + A|, |AA|\} \gg c(\varepsilon)|A|^{2-\varepsilon}
\]
for any \( \varepsilon > 0 \). At present the best known bound in the integer problem is
\[
\max\{|A + A|, |AA|\} \gg |A|^{14/11}(\log |A|)^{-3/11}
\]
due to Solymosi \[13\].

We do not know what the optimal lower bound for \( \max\{|A + A|, |AA|\} \) in terms of \( |A| \) and \( p \) should be. It is known that the analogy of the Erdős and Szemerédi conjecture in the form
\[
\max\{|A + A|, |AA|\} \geq c(\varepsilon) \min\{|A|^{2-\varepsilon}, p^{1-\varepsilon}\}
\]
for all subsets \( A \) of \( \mathbb{F}_p \) does not hold.

In what follows, all the sets under consideration are assumed to be non-empty. For a set \( X \subset \mathbb{F}_p \) and for an element \( a \in \mathbb{F}_p \) we use the notation
\[
a \ast X = \{ax : x \in X\}.
\]

2 Lemmas

The following lemma follows from the work of Glibichuk and Konyagin \[8\].

**Lemma 1.** Let \( A_1 \subset \mathbb{F}_p \) with \( 1 < |A_1| < p^{1/2} \). Then there exist elements \( a_1, a_2, b_1, b_2 \in A_1 \) such that \( a_1 \neq a_2 \) and
\[
|(a_1 - a_2) \ast A_1 + (a_1 - a_2) \ast A_1 + (b_1 - b_2) \ast A_1| \geq 0.5|A_1|^2.
\]

When \( |A_1| > p^{1/2} \), we will use the following statement (see \[4\] or \[7\]).

**Lemma 2.** Let \( A_1 \subset \mathbb{F}_p \) with \( |A_1| > p^{1/2} \). Then there exist \( a_1, a_2, b_1, b_2 \in A_1 \) such that
\[
|(a_1 - a_2) \ast A_1 + (b_1 - b_2) \ast A_1| \geq 0.5p.
\]

The following lemmas are due to Ruzsa (see \[10\], \[11\], \[12\], \[14\]). They hold for subsets of any abelian group, but here we state them only for subsets of \( \mathbb{F}_p \). We will repeatedly use these lemmas to prove our result. Lemma \[3\] is called Ruzsa’s triangle inequality.
Lemma 3. For any subsets $X, Y, Z$ of $\mathbb{F}_p$ we have

$$|X - Z| \leq \frac{|X - Y||Y - Z|}{|Y|}.$$ 

Lemma 4. Let $X, B_1, \ldots, B_k$ be any subsets of $\mathbb{F}_p$ with

$$|X| = n, \ |X + B_i| = \alpha_i n, \ (i = 1, \ldots, k).$$

Then there is an $X_1 \subset X$ such that

$$|X_1 + B_1 + \ldots + B_k| \leq \alpha_1 \ldots \alpha_k |X_1|$$

An important corollary of Lemma 4 is the inequality

$$|B_1 + \ldots + B_k| \leq \frac{|X + B_1| \ldots |X + B_k|}{|X|^{k-1}}.$$ 

Below, when we refer to Lemma 4 we will always understand this inequality. The case $k = 2$ illustrates another version of Ruzsa’s triangle inequality.

3 Proof of Theorem 1

We use an idea of the proof of Katz-Tao lemma presented in [14, Section 2.8]. That proof, as it was mentioned in [14], used Bourgain’s idea from [2].

We may assume that $|A|^2 \geq 100|AA|$ and that $0 \notin A$. Let $J$ denote the number of solutions of the equation

$$ax = by, \ a, b, x, y \in A.$$ 

From the well-known relationship between the cardinality of a set and the number of solutions of the corresponding equation, we have that

$$J \geq \frac{|A|^4}{|AA|}.$$ 

Since

$$J = \sum_{a \in A} \sum_{b \in A} |a \ast A \cap b \ast A|,$$
there exists a fixed element $b = b_0 \in A$ such that

$$\sum_{a \in A} |a \ast A \cap b_0 \ast A| \geq \frac{|A|^3}{|AA|}.$$ 

For a given positive integer $j \leq \log |A|/\log 2 + 1$, let $D_j$ be the set of all $a \in A$ for which

$$2^{j-1} \leq |a \ast A \cap b_0 \ast A| < 2^j.$$ 

Then,

$$\sum_j \sum_{a \in D_j} 2^j \geq \frac{|A|^3}{|AA|}.$$ 

Let the quantity $\sum_{a \in D_j} 2^j$ takes its biggest value (or one of its biggest values if there are several such ones) when $j = j_0$. Denote

$$N = 2^{j_0-1}, \quad A_1 = D_{j_0} \subset A.$$ 

Then, for any $a \in A_1$, we have

$$1 \leq N \leq |a \ast A \cap b_0 \ast A| < 2N, \quad N|A_1| \gg \frac{|A|^3}{|AA| \log |A|}. \quad (3)$$ 

In particular, since $N \leq |A|$ and $|A_1| \leq |A|$, we get

$$|N| \gg \frac{|A|^2}{|AA| \log |A|}, \quad |A_1| \gg \frac{|A|^2}{|AA| \log |A|}. \quad (4)$$ 

Now let $a$ be an arbitrary element of $A_1$. From Lemma 3 and the inequality (3), we have

$$\frac{|a \ast A - b_0 \ast A|}{N} \leq \frac{|a \ast A + (a \ast A \cap b_0 \ast A)| - (a \ast A \cap b_0 \ast A) - b_0 \ast A}{N} \leq \frac{|a \ast A + a \ast A||b_0 \ast A + b_0 \ast A|}{N} = \frac{|A + A|^2}{N}.$$ 

Furthermore, using Lemma 4 with

$$k = 2, \quad X = a \ast A \cap b_0 \ast A, \quad B_1 = a \ast A, \quad B_2 = b_0 \ast A,$$
we obtain
\[
|a \cdot A + b_0 \cdot A| \leq \frac{|a \cdot A + (a \cdot A \cap b_0 \cdot A)|}{|b_0 \cdot A|} |b_0 \cdot A + (a \cdot A \cap b_0 \cdot A)| \leq \frac{|a \cdot A \cap b_0 \cdot A|}{|b_0 \cdot A|}.
\]

Thus, the bound
\[
|a \cdot A \pm b_0 \cdot A| \ll \frac{|A + A|^2}{N} \quad (5)
\]
holds for any \(a \in A_1\) and for any choice of the sign “±”.

There are two cases to consider.

**Case 1:** \(|A_1| < p^{1/2}\).

First of all, in this case besides of (4), we also have
\[
N p^{1/2} > N|A_1| \geq \frac{|A|^3}{|A| \log |A|}.
\]

Thus, together with (4), we have
\[
N > \max \left\{ \frac{|A|^2}{|A| \log |A|}, \frac{|A|^3}{p^{1/2}|A| \log |A|} \right\}. \quad (6)
\]

Since \(A_1 \subset A\), according to Lemma 1 there exist \(a_1, a_2, b_1, b_2 \in A_1\) such that \(a_1 \neq a_2\) and
\[
0.5|A_1|^2 \leq |(a_1 - a_2) \cdot A + (a_1 - a_2) \cdot A + (b_1 - b_2) \cdot A|.
\]

We apply Lemma 4 with \(k = 3\) and
\[
X = B_1 = B_2 = (a_1 - a_2) \cdot A, \quad B_3 = (b_1 - b_2) \cdot A.
\]

Then we get
\[
0.5|A_1|^2 \leq \frac{|A + A|^2 |(a_1 - a_2) \cdot A + (b_1 - b_2) \cdot A|}{|A|^2}. \quad (7)
\]

Next, we apply Lemma 4 with \(k = 4\) and
\[
X = b_0 \cdot A, \quad B_1 = a_1 \cdot A, \quad B_2 = -a_2 \cdot A, \quad B_3 = b_1 \cdot A, \quad B_4 = -b_2 \cdot A.
\]
Then,

\[
|(a_1 - a_2) \ast A + (b_1 - b_2) \ast A| \leq \frac{|b_0 \ast A + a_1 \ast A||b_0 \ast A - a_2 \ast A||b_0 \ast A + b_1 \ast A||b_0 \ast A - b_2 \ast A|}{|A|^3}.
\]

(8)

Applying the inequality (5) to the right hand side of (8) and incorporating the resulting estimate to (7), we get

\[
|A + A|^{10} \gg |A_1|^2 |A|^5 N^4.
\]

Taking into account (3) to substitute $N|A_1|$ and then (5) to substitute $N$, we conclude that

\[
|A + A|^{10} |A A|^4 (\log |A|)^4 \gg |A|^{15} \max \{1, \frac{|A|^2}{p}\}.
\]

This proves Theorem 1 in Case 1.

**Case 2:** $|A_1| > p^{1/2}$.

In this case, according to Lemma 2 there exist elements $a_1, a_2, b_1, b_2 \in A_1$ such that

\[
0.5p \leq |(a_1 - a_2) \ast A + (b_1 - b_2) \ast A|.
\]

Applying Lemma 4 with $k = 4$ and

\[
X = b_0 \ast A, \quad B_1 = a_1 \ast A, \quad B_2 = -a_2 \ast A, \quad B_3 = b_1 \ast A, \quad B_4 = -b_2 \ast A,
\]

we get

\[
0.5p \leq \frac{|b_0 \ast A + a_1 \ast A||b_0 \ast A - a_2 \ast A||b_0 \ast A + b_1 \ast A||b_0 \ast A - b_2 \ast A|}{|A|^3}.
\]

Taking into account the inequality (5), we obtain that

\[
p \ll \frac{|A + A|^8}{|A|^3 N^4}.
\]

Using (4), we get

\[
|A + A|^8 |A A|^4 (\log |A|)^4 \gg p |A|^{11}.
\]

Theorem 1 is proved.
4 Proof of Theorem 2

We remark that the proof of Lemma 2, as well as the proof of Lemma 1, uses the fact that for any sets \( X, Y, G \subset \mathbb{F}_p \) there exists \( \xi \in G \) such that

\[
|X + \xi * Y| \geq \frac{|X||Y||G|}{|X||Y| + |G|}.
\]

This estimate is nontrivial when \( |G| \) is larger than \( |X| \) and \( |Y| \). In order to prove Theorem 2, we would like to have a similar estimate which in certain cases would be nontrivial when the cardinality of \( G \) is smaller than those of \( X \) and \( Y \). The following lemma provides with such an estimate.

**Lemma 5.** Let \( X, Y, G \subset \mathbb{F}_p \). Then there exists \( \xi \in G \) such that

\[
|X + \xi * Y| \geq \frac{p|X||Y||G|}{|X||Y||G| + p^2}.
\]

It is easy to see that the bounds (3), (4), (5) and Lemma 5 imply Theorem 2. Indeed, apply Lemma 5 with \( X = -b_0 * A, Y = A, G = A_1 \).

Then for some \( a \in A_1 \), in view of (5), we have

\[
\frac{|A + A|^2}{N} \gg |a * A - b_0 * A| \gg \min\left\{ p, \frac{|A|^2|A_1|}{p} \right\}.
\]

Thus,

\[
|A + A|^2 \gg \min\left\{ pN, \frac{|A|^2N|A_1|}{p} \right\} \gg \min\left\{ p|A|^2|AA|^{-1}, \frac{|A|^2N|A_1|}{p} \right\}.
\]

Recalling (3), (4), we get that either

\[
|A + A|^2|AA| \log |A| \geq p|A|^2
\]

or

\[
|A + A|^2|AA| \log |A| \gg p|A|^5.
\]

In particular,

\[
\max\{|A + A|, |AA|\} \gg \min\left\{ |A|^{5/3}p^{-1/3}(\log |A|)^{-1/3}, |A|^{2/3}p^{1/3}(\log |A|)^{-1/3} \right\}.
\]
Thus, if we prove Lemma 5 then we are done. To this end, let $I$ denote the number of solutions of the equation

$$x + gy = x_1 + gy_1, \quad g \in G, \ x, x_1 \in X, \ y, y_1 \in Y.$$

We express $I$ via trigonometric sums:

$$I = \frac{1}{p} \sum_{n=0}^{p-1} \sum_{g \in G} \left| \sum_{x \in X} e^{2\pi i n x / p} \right|^2 \left| \sum_{y \in Y} e^{2\pi i g y / p} \right|^2.$$

Picking up the term corresponding to $n = 0$ and then extending the summation over $g$ to the whole range $0 \leq g \leq p - 1$, we obtain

$$I \leq \frac{|X|^2 |Y|^2 |G|}{p} + \frac{1}{p} \sum_{n=1}^{p-1} \left| \sum_{g \in G} \sum_{x \in X} e^{2\pi i n x / p} \right|^2 \left| \sum_{y \in Y} e^{2\pi i g y / p} \right|^2 \leq$$

$$\frac{|X|^2 |Y|^2 |G|}{p} + \frac{1}{p} \left( \sum_{n=0}^{p-1} \left| \sum_{x \in X} e^{2\pi i n x / p} \right|^2 \right) \left( \sum_{g=0}^{p-1} \left| \sum_{y \in Y} e^{2\pi i g y / p} \right|^2 \right) =$$

$$\frac{|X|^2 |Y|^2 |G|}{p} + p |X|| |Y|.$$

Therefore, there exists a fixed element $\xi \in G$ such that the number $I_0$ of solutions of the equation

$$x + \xi y = x_1 + \xi y_1, \quad x, x_1 \in X, \ y, y_1 \in Y,$$

satisfies

$$I_0 \leq \frac{|X|^2 |Y|^2}{p} + \frac{p |X|| |Y|}{|G|}.$$

From the relationship between the cardinality of a set and the number of solutions of the corresponding equation, we derive that

$$|X + \xi * Y| \geq \frac{|X|^2 |Y|^2}{I_0} \geq \frac{|X|^2 |Y|^2}{\frac{|X|^2 |Y|^2}{p} + \frac{p |X|| |Y|}{|G|}} = \frac{p |X|| |Y|| |G|}{|X|| |Y|| |G| + p^2}.$$

This proves Lemma 5 and thus Theorem 2 is also proved.

Acknowledgement. The author is grateful to S. V. Konyagin and I. Z. Ruzsa for very useful remarks, comments and references. This work was supported by the Project PAPIIT IN 100307 from UNAM.
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