LIOUVILLE TYPE THEOREM FOR THE STATIONARY EQUATIONS OF MAGNETO-HYDRODYNAMICS

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Abstract We show that any smooth solution \((\mathbf{u}, \mathbf{H})\) to the stationary equations of magneto-hydrodynamics belonging to both spaces \(L^6(\mathbb{R}^3)\) and \(\text{BMO}^{-1}(\mathbb{R}^3)\) must be identically zero. This is an extension of previous results, all of which systematically required stronger integrability and the additional assumption \(\nabla \mathbf{u}, \nabla \mathbf{H} \in L^2(\mathbb{R}^3)\), i.e., finite Dirichlet integral.

Key words Liouville theorem; Caccioppoli inequality; Navier-Stokes equations; MHD

2010 MR Subject Classification 35B53; 35Q30; 76W05

1 Introduction

Liouville type theorems arise naturally when considering the regularity of solutions to the incompressible Navier-Stokes equations. Development in this direction has been led most notably by Chae, Nadirashvili, Seregin, and Šverák (cf. [2, 5, 7]). Intimately tied to the Navier-Stokes equations are the equations of magneto-hydrodynamics (MHD). The latter system models the motion of an incompressible fluid whose velocity field is affected by magnetic interactions, e.g., the movement of a magnetized plasma.

Liouville type theorems were known to hold for the MHD system, as demonstrated by the works [3, 9]. In [3], Chae proved that if a smooth solution of the stationary MHD equations is bounded in \(L^3(\mathbb{R}^3)\) and has finite Dirichlet integral, then it is identically zero. Later, in [9], Zhang-Yang-Qiu proved that if a smooth solution of the stationary MHD equations is bounded in \(L^2(\mathbb{R}^3)\) and has finite Dirichlet integral, then it is also identically zero. So far, no result exists without the finite Dirichlet integral assumption \(\nabla \mathbf{u}, \nabla \mathbf{H} \in L^2(\mathbb{R}^3)\).

The focus of this paper is to obtain a Liouville theorem for the equations of stationary MHD without the need for finite Dirichlet integral, and with only an \(L^6(\mathbb{R}^3)\) integrability criterion. To this end, we closely follow the scheme outlined by Seregin in [7]. In using this approach, we also reprove the original results in [3] and [9] without the requirement \(\nabla \mathbf{u}, \nabla \mathbf{H} \in L^2(\mathbb{R}^3)\). Although many of the estimates in this work are identical to those in [7], we go through them in detail for the sake of making this paper self-contained.

*Received February 6, 2018; revised October 15, 2018. The author is supported by the Engineering and Physical Sciences Research Council [EP/L015811/1].
2 Preliminaries

In what follows we employ the method of Seregin in [7], which first and foremost involves proving a Caccioppoli type inequality. Although Seregin’s paper is concerned with the stationary incompressible Navier-Stokes equations, his proof makes a similar Caccioppoli type inequality hold for the equations of magneto-hydrodynamics. In light of this, we structure our paper in the same way as was done in [7].

Below are the equations of stationary MHD. As per usual, $u$ is the velocity of the fluid and $H$ is the magnetic field,

$$
\begin{align*}
\text{div} u &= 0, \\
\mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p &= \mathbf{H} \cdot \nabla \mathbf{H}, \\
\text{div} \mathbf{H} &= 0, \\
\mathbf{u} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{u} &= \Delta \mathbf{H}.
\end{align*}
$$

(2.1)

**Definition 2.1** We say that $f \in \text{BMO}^{-1}(\mathbb{R}^3)$ if there exists a skew-symmetric tensor $d \in \text{BMO}(\mathbb{R}^3)$ such that $f = \text{div} \, d$ $\iff$ $f_i = d_{ij,j}$ for $i = 1, 2, 3$.

**Remark 2.2** Observe that the requirement that a vector be the divergence of a skew-symmetric tensor is “equivalent” to this vector being equal to a curl. Formally, we have

$$
f = \text{div} \, d \text{ for } d = (d^3)_{i,j=1}^{3} \text{ skew-symmetric } \iff f = \nabla \times g \text{ for } g = \begin{pmatrix} d_{23} \\ -d_{13} \\ d_{12} \end{pmatrix} \iff \text{div} \, f = 0.
$$

**Remark 2.3** If $d \in \text{BMO}(\mathbb{R}^3)$, then

$$
\Gamma(s) := \sup_{x_0 \in \mathbb{R}^3, r > 0} \left( \int_{B(x_0, r)} |d - [d]_{x_0, r}|^s \, dx \right)^{\frac{1}{s}} < \infty
$$

for each $1 \leq s < \infty$. Here, $[d]_{x_0, r}$ denotes the mean value of $d$ in the ball $B(x_0, r)$. We recurrently use the finiteness of this quantity in our later estimates.

We begin by showing the following theorem.

**Theorem 2.4** Let $(u, H)$ be a smooth solution of system (2.1) with $u, H \in \text{BMO}^{-1}(\mathbb{R}^3)$. If we additionally require that $u, H \in L^q(\mathbb{R}^3)$ for $q \in (2, 6)$, then $u \equiv 0$ and $H \equiv 0$.

Note that the above covers the cases explored by Chae in [3] and Zhang-Yang-Qiu in [9]. However, unlike them, we do not additionally require $\nabla u, \nabla H \in L^2(\mathbb{R}^3)$. A supplementary argument then yield the result claimed in the abstract, which is contained in the theorem underneath.

**Theorem 2.5** Let $(u, H)$ be a smooth solution of system (2.1) with $u, H \in \text{BMO}^{-1}(\mathbb{R}^3)$. If we additionally require that $u, H \in L^6(\mathbb{R}^3)$, then $u \equiv 0$ and $H \equiv 0$. 

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3 Proof of the Main Results

3.1 Caccioppoli Type Inequality

Much like in [7], we have at the heart of our proof a Caccioppoli type inequality, which we develop in this portion of the paper. We state this inequality below.

**Lemma 3.1** Let \((u, H)\) be a smooth solution to system (2.1) with \(u, H \in \text{BMO}^{-1}(\mathbb{R}^3)\), and let \(v := u + H\). Then the Caccioppoli type inequality

\[
\int_{B(x_0, R/2)} |\nabla v|^2 \, dx \leq c R^{1-6/s} \left( \int_{B(x_0, R)} |v - v_0|^s \, dx \right)^{\frac{2}{s}}
\]

(3.1)

holds for any ball \(B(x_0, R) \subset \mathbb{R}^3\), any constant \(v_0 \in \mathbb{R}^3\), and any \(s > 2\).

**Proof** Begin by adding the two evolution equations together to obtain

\[
\begin{cases}
\div v = 0, \\
(u - H) \cdot \nabla v - \Delta v = -\nabla p.
\end{cases}
\]

(3.2)

Note that, since both \(u\) and \(H\) are in \(\text{BMO}^{-1}(\mathbb{R}^3)\), we know that their difference \(u - H\) and \(v\) are also \(\text{BMO}^{-1}\) vector fields. In particular, we know that there exists a skew-symmetric tensor \(d \in \text{BMO}(\mathbb{R}^3)\) such that \(u - H = \div d\).

Take an arbitrary ball \(B(x_0, R) \subset \mathbb{R}^3\) and a non-negative cut-off function \(\varphi \in C_c^\infty(B(x_0, R))\) with the properties: \(\varphi(x) = 1\) in \(B(x_0, \rho)\), \(\varphi(x) = 0\) outside of \(B(x_0, r)\), and \(|\nabla \varphi(x)| \leq c/(r - \rho)\) for any \(R/2 \leq \rho < r \leq R\). We let \(d = d - [d]_{x_0, R}\), where \([d]_{x_0, R}\) is the mean value of \(d\) on the ball \(B(x_0, R)\). From here on, we write \(\bar{v} = v - v_0\), where \(v_0\) is any constant in \(\mathbb{R}^3\).

Now, consider the following Dirichlet problem

\[
\begin{cases}
\div w = \div (\varphi \bar{v}) & \text{in } B(x_0, r), \\
w = 0 & \text{on } \partial B(x_0, r).
\end{cases}
\]

Since the right-hand side of the equation integrates to zero (by the divergence theorem) and is locally integrable, we deduce from Theorem 3.6 in Chapter 1 of [6] (or from [1]) that there exists \(w \in W_0^{1,s}(B(x_0, r))\) solving the above, and for which the following inequality holds for \(1 < s < \infty\),

\[
\int_{B(x_0, r)} |\nabla w|^s \, dx \leq c \int_{B(x_0, r)} |\div (\varphi \bar{v})|^s \, dx
\]

\[
= c \int_{B(x_0, r)} |\nabla \varphi \cdot \bar{v}|^s \, dx
\]

\[
\leq \frac{c}{(r - \rho)^s} \int_{B(x_0, r)} |\bar{v}|^s,
\]

here \(c = c(s)\) and is independent of \(x_0\) and \(R\).

Next, we follow the bounds as in [7], i.e., we test the second equation in (3.2) against \(\varphi \bar{v} - w\), to get

\[
\int_{B(x_0, r)} \varphi |\nabla v|^2 \, dx = -\int_{B(x_0, r)} \nabla v : (\nabla \varphi \otimes \bar{v}) \, dx + \int_{B(x_0, r)} \nabla w : \nabla v \, dx
\]

\[
- \int_{B(x_0, r)} (\div \bar{d} \cdot \nabla v) \cdot \varphi \bar{v} \, dx + \int_{B(x_0, r)} (\div \bar{d} \cdot \nabla v) \cdot w \, dx.
\]

We denote the previous integrals by \(I_1, \cdots, I_4\).
Remark 3.2 The term involving $\nabla p$ has vanished, since $\varphi \tilde{v} - \mathbf{w}$ is divergence-free and

$$\int_{B(x_0,r)} (\varphi \tilde{v} - \mathbf{w}) \cdot \nabla p \, dx = - \int_{B(x_0,r)} \text{div} (\varphi \tilde{v} - \mathbf{w}) p \, dx = 0,$$

where the boundary term has vanished due to the compact support of our test function.

Now we bound the numbered integrals $I_1, \ldots, I_4$,

$$|I_1| \leq \int_{B(x_0,r)} |\nabla v||\nabla \varphi||\tilde{v}| \, dx \leq \frac{c}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{c}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^s \, dx \right)^{\frac{1}{2}} \leq \frac{c}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^s \, dx \right)^{\frac{1}{2}} \leq \frac{c R^3 (\frac{s-2}{2})}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^s \, dx \right)^{\frac{1}{2}}.$$

Similarly,

$$|I_2| \leq \int_{B(x_0,r)} |\nabla v||\nabla w| \, dx \leq \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\nabla w|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\nabla w|^s \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^s \, dx \right)^{\frac{1}{2}} \leq \frac{c R^3 (\frac{s-2}{2})}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\nabla w|^s \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^s \, dx \right)^{\frac{1}{2}}.$$

Note that we have implicitly assumed that $s > 2$.

For $I_3$ and $I_4$ we need to use the skew-symmetry of $d$.

$$|I_3| = \left| \int_{B(x_0,r)} \tilde{d}_{jm,m} v_{i,j} \varphi \tilde{v}_i \, dx \right| = \left| \int_{B(x_0,r)} \tilde{d}_{jm} v_{i,j} \varphi m \tilde{v}_i \, dx \right| \leq \frac{c}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{d}|^2 |\tilde{v}|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{c}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^s \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{d}|^{\frac{2s}{s-2}} \, dx \right)^{\frac{s-2}{2s}} \leq \frac{c R^3 (\frac{s-2}{2})}{r - \rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\tilde{v}|^s \, dx \right)^{\frac{1}{2}}.$$

For the fourth integral

$$|I_4| = \left| \int_{B(x_0,r)} \tilde{d}_{jm,m} v_{i,j} w_i \, dx \right| = \left| \int_{B(x_0,r)} \tilde{d}_{jm} v_{i,j} w_{i,m} \, dx \right|.$$
\[
\begin{align*}
&\leq \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |d|^2 |\nabla w|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\nabla w|^4 \, dx \right)^{\frac{1}{4}} \left( \int_{B(x_0,r)} |d|^{\frac{2m}{m-2}} \, dx \right)^{\frac{m-2}{2m}} \\
&\leq cR^3 \left( \frac{\frac{\varepsilon}{2}}{r-\rho} \right) \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\bar{v}|^s \, dx \right)^{\frac{1}{2}}.
\end{align*}
\]

In total, we have
\[
\int_{B(x_0,\rho)} |\nabla v|^2 \, dx \leq \frac{cR^3 \left( \frac{\varepsilon}{2} \right)}{r-\rho} \left( \int_{B(x_0,r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)} |\bar{v}|^s \, dx \right)^{\frac{1}{2}}.
\]

Applying a weighted Cauchy-Schwarz inequality, we obtain
\[
\int_{B(x_0,\rho)} |\nabla v|^2 \, dx \leq \frac{1}{4} \int_{B(x_0,r)} |\nabla v|^2 \, dx + cR^3 \left( \frac{\varepsilon}{2} \right) \left( \frac{r-\rho}{2} \right)^2 \left( \int_{B(x_0,r)} |\bar{v}|^s \, dx \right)^{\frac{1}{2}}.
\]

Suitable iterations then give the following Caccioppoli type inequality
\[
\int_{B(x_0,R/2)} |\nabla v|^2 \, dx \leq cR^3 \left( \frac{s}{2} \right)^{-2} \left( \int_{B(x_0,R)} |\bar{v}|^s \, dx \right)^{\frac{1}{2}}
\]
as required. \[\square\]

**Remark 3.3** The positive constant \( c \) is independent of \( x_0 \) and \( R \), and depends only on \( s \).

### 3.2 The Proof of Theorem 2.4

The proof of Theorem 2.4 rests entirely on the observation that we can make the exponent \( 1 - 6/s \) negative in the Caccioppoli type inequality (3.1). In view of this, we present our proof.

**Proof of Theorem 2.4** Suppose \( u, H \in L^q(\mathbb{R}^3) \) for \( 2 < q < 6 \), and let \( \varepsilon := 6/q - 1 \). Observe that \( \varepsilon > 0 \), so by choosing \( v_0 = 0 \) the Caccioppoli type inequality (3.1) now reads
\[
\int_{B(x_0,R/2)} |\nabla v|^2 \, dx \leq cR^{-\varepsilon} |v|_L^2(\mathbb{R}^3).
\]

By taking the limit as \( R \to \infty \) we recover \( \nabla v \equiv 0 \). This implies that \( v \) is constant, but since \( v \in L^q(\mathbb{R}^3) \), we know that this constant must be zero. Hence \( u \equiv -H \).

Using this relation, we know from the first evolution equation for \( u \) in (2.1) that
\[
\begin{align*}
&\text{div } u = 0, \\
&\Delta u = \nabla p. \\
\end{align*}
\]
(3.3)

As before, we can find a \( w \in W_0^{1,q}(B(x_0,r)) \) such that \( \text{div } w = \text{div } (\varphi \hat{u}) \), where \( \hat{u} = u - u_0 \) for some arbitrary constant \( u_0 \) in \( \mathbb{R}^3 \). Here, \( \varphi \) is the same cut-off function that we used in the proof of the Caccioppoli type inequality. Testing (3.3) against \( \varphi \hat{u} - w \) we obtain
\[
\int_{B(x_0,r)} \varphi |\nabla u|^2 \, dx = -\int_{B(x_0,r)} \nabla u : (\nabla \varphi \otimes \hat{u}) \, dx + \int_{B(x_0,r)} \nabla w : \nabla u \, dx.
\]

Once again, we obtain
\[
\int_{B(x_0,R/2)} |\nabla u|^2 \, dx \leq cR^{1-6/q} \left( \int_{B(x_0,R)} |\hat{u}|^q \, dx \right)^{\frac{2}{q}},
\]
so choosing $u_0 = 0$ we get

$$\int_{B(x_0,R/2)} |\nabla u|^2 \, dx \leq cR^{-\varepsilon}||u||^2_{L^6(\mathbb{R}^3)}.$$

Taking the limit as $R \to \infty$ we recover $u \equiv 0$, which concludes the proof of the theorem. \qed

### 3.3 The Proof of Theorem 2.5

In the case where $s = 6$ we cannot argue as we did previously. Putting $s = 6$ and $v_0 = 0$ in (3.1) yields

$$\int_{B(x_0,R/2)} |\nabla v|^2 \, dx \leq c||v||^2_{L^6(\mathbb{R}^3)}.$$

Hence, passing to the limit $R \to \infty$ gives the reverse Sobolev inequality

$$||\nabla v||_{L^2(\mathbb{R}^3)} \leq c||v||_{L^6(\mathbb{R}^3)},$$

(3.4)

This is not particularly useful in itself, and does not readily produce a reverse Sobolev inequality for the individual vector fields $u$ and $H$. Instead, one can pick $s = 3$ and $v_0 = [v]_{x_0,R}$ with the aim of constructing an inequality between maximal functions. This is precisely how the proof of Theorem 2.5 runs, which we elaborate on in the next few paragraphs.

**Proof of Theorem 2.5** Firstly recall the Gagliardo-Nirenberg type inequality

$$||\bar{v}||_{L^3(B(x_0,R))} \leq c||\nabla \bar{v}||_{L^{\frac{3}{2}}(B(x_0,R))},$$

(3.5)

Now choose $s = 3$ and $v_0 = [v]_{x_0,R}$ in the Caccioppoli type inequality (3.1), and couple this with (3.5) to obtain the reverse Hölder inequality

$$\int_{B(x_0,R/2)} |\nabla v|^2 \, dx \leq c\left(\int_{B(x_0,R)} |\nabla v|^\frac{3}{2} \, dx\right)^{\frac{2}{3}},$$

(3.6)

where $c$ is independent of $x_0$ and $R$, as per usual.

Define the function $h := |\nabla v|^\frac{1}{2} \in L^\frac{3}{2}(\mathbb{R}^3)$ and let

$$M_h(x_0) = \sup_{R>0} \int_{B(x_0,R)} h(x) \, dx$$

be its Hardy-Littlewood maximal function. Now the reverse Hölder inequality (3.6) reads

$$M_h^\frac{2}{3}(x_0) \leq cM_h^\frac{3}{2}(x_0), \quad \forall x_0 \in \mathbb{R}^3.$$

From the maximal function inequality in $L^p(\mathbb{R}^3)$ for $p > 1$ (c.f. [8]), we know that there exists a universal constant $c_0 > 0$ such that

$$\int_{\mathbb{R}^3} M_h^\frac{2}{3}(x) \, dx \leq c_0 \int_{\mathbb{R}^3} h^\frac{3}{2}(x) \, dx$$

$$= c_0 \int_{\mathbb{R}^3} |\nabla v|^2 \, dx$$

$$\leq c||v||^2_{L^6(\mathbb{R}^3)},$$

where the last inequality is exactly (3.4). Thus we have shown that both $h^\frac{3}{2}$ and its maximal function $M_h^\frac{3}{2}$ are $L^1(\mathbb{R}^3)$ functions, which is only possible if $h \equiv 0$ (c.f. [8]). This implies that $v$ is constant, thus once again we arrive at $u \equiv -H$. \qed

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We now show that we must have \( u \equiv 0 \). Making use of the relation \( u \equiv -H \) as we did in the proof of Theorem 2.4, we recover (3.3) and the Caccioppoli type inequality
\[
\int_{B(x_0, R/2)} |\nabla u|^2 \, dx \leq c R^{1-6/q} \left( \int_{B(x_0, R)} |\bar{u}|^q \, dx \right)^{\frac{2}{q}}.
\]
Picking \( q = 6 \) and \( u_0 = 0 \) we recover
\[
||\nabla u||_{L^2(\mathbb{R}^3)} \leq c ||u||_{L^6(\mathbb{R}^3)},
\]
as expected. Selecting \( q = 3 \) and \( u_0 = [u]_{x_0, R} \) and using the same strategy as before, we arrive at the maximal function inequality
\[
M_{\tilde{h}}^4(x_0) \leq c M_{\tilde{h}}^\frac{2}{3}(x_0) \quad \forall x_0 \in \mathbb{R}^3,
\]
where \( \tilde{h} = |\nabla u|^\frac{2}{3} \). The same argument as before then yields \( u \equiv 0 \), as required. \( \square \)

Acknowledgements The author wishes to thank Gui-Qiang Chen and Gregory Seregin for useful discussions.

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