Median bias reduction of maximum likelihood estimates

E. C. KENNE PAGUI, A. SALVAN and N. SARTORI
Department of Statistical Sciences, University of Padova
kenne@stat.unipd.it, salvan@stat.unipd.it, sartori@stat.unipd.it

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Abstract

For regular parametric problems, we show how median centering of the maximum likelihood estimate can be achieved by a simple modification of the score equation. For a scalar parameter of interest, the estimator is equivariant under interest respecting parameterizations and third-order median unbiased. With a vector parameter of interest, componentwise equivariance and third-order median centering are obtained. Like Firth’s (1993, Biometrika) implicit method for bias reduction, the new method does not require finiteness of the maximum likelihood estimate and is effective in preventing infinite estimates. Simulation results for continuous and discrete models, including binary and beta regression, confirm that the method succeeds in achieving componentwise median centering and in solving the infinite estimate problem, while keeping comparable dispersion and the same approximate distribution as its main competitors.

Some key words: Binary regression; Infinite estimate; Modified score; Parameterization invariance; Separation problem; Skew normal; Tensor.

1 Introduction

In regular parametric estimation problems, both the maximum likelihood estimator and the score estimating function have an asymptotic symmetric distribution centered at the true parameter value and at zero, respectively. However, the asymptotic behaviour may poorly reflect exact sampling distributions with small or moderate sample information, sparse data or complex models. Several proposals have been developed to correct the estimate or the estimating function.

Most available methods are aimed at approximate bias adjustment, either of the maximum likelihood estimator or of the profile score function when nuisance parameters are present. We refer to Kosmidis (2014) for a review of bias reduction for the maximum likelihood estimator and to McCullagh & Tibshirani (1990), Stern (1997) and subsequent literature for bias correction of the profile score.
In the absence of nuisance parameters, the score function is exactly unbiased and therefore no correction appears to be necessary. A change of parameterization does not affect this property and the solution of the score equation, namely the maximum likelihood estimator, behaves equivariantly under reparameterizations. On the other hand, bias correction of the maximum likelihood estimator is tied to a specific parameterization.

Lack of equivariance also affects the so-called implicit bias reduction methods (Kosmidis, 2014) that achieve first-order bias correction through a modification of the score equation, following Firth (1993). This lack of coherence is highlighted e.g. in Kosmidis (2014), but somehow overwhelmed by advantages in applications, possibly with a careful choice of the working parameterization (Kosmidis & Firth, 2010, § 4.2, Remark 3). Indeed, one major advantage of the approach in Firth (1993) and Kosmidis & Firth (2009) is that the modified estimating equation does not depend explicitly on the maximum likelihood estimate. The modified score equation has been found to overcome infinite estimate problems that may arise with positive probability mainly, but not only, in models for discrete or categorical data.

Considering first a scalar parameter of interest, we propose a new median modification of the score, or profile score, equation whose solution respects equivariance under monotone reparameterizations. Like Firth’s (1993) implicit method, this proposal does not rely on finiteness of the maximum likelihood estimate and is effective in preventing infinite estimates. The modification is obtained by considering the median, instead of the mean, as a centering index for the score and defining a new estimating function by subtracting from the score its approximate median.

Provided that the modified score equation has a unique solution, median centering of the score function implies median centering of the corresponding estimator. Therefore, the resulting estimator is approximately median unbiased (see e.g. Read, 1985), that is the true parameter value is approximately a median of the distribution of the estimator. In some instances exact median unbiased estimates can be obtained (see Hirji et al., 1989). Outside exactness cases, available approximations for median unbiased estimates are based on higher-order likelihood asymptotics. Approximations based on the modified signed likelihood ratio (Barndorff-Nielsen, 1986) have been developed in Pace & Salvan (1999), Giannone & Ventura (2002), Biehler et al. (2015). They rely, however, on finiteness of the maximum likelihood estimate. Third-order median unbiasedness of the new estimator is seen to hold in the continuous case and extensive numerical evidence indicates remarkable median centering also in the discrete case.

We show how the method can be extended to a vector parameter by simultaneously solving median bias corrected score equations for all parameter components. This leads to componentwise third-order median unbiasedness and parameterization equivariance.

Examples and simulation results in a number of models, including binary and beta regression, indicate that the new estimator provides a notable improvement
over the maximum likelihood estimator and solves the infinite estimates problem, both for a scalar and for a vector parameter.

2 Median modified score for a scalar parameter of interest

2.1 No nuisance parameters

For data $y$, consider a regular model with probability mass function $p_{y}(y; \theta)$, $\theta \in \Theta \subseteq \mathbb{R}$. Let $\ell(\theta)$ be the corresponding log likelihood and $U = U(\theta) = \partial \ell(\theta)/\partial \theta$, the score function. The maximum likelihood estimator $\hat{\theta}$ is solution of $U(\theta) = 0$. We assume that Fisher information, $i(\theta)$, and the third-order cumulant of $U(\theta)$ are finite and of order $O(n)$, where $n$ is the sample size or, more generally, an index of information in the data.

Using Cornish-Fisher expansion (see e.g. Pace & Salvan, 1997, § 10.6), the following asymptotic expansion holds for the median under $\theta$, $M_{\theta}(\cdot)$, of the score in the continuous case

$$
M_{\theta}\{U(\theta)\} = -\nu_{0,0,0}/\{6 \, i(\theta)\} + O(n^{-1}),
$$

with $\nu_{0,0,0} = \nu_{0,0,0}(\theta) = E_{\theta}\{U(\theta)^3\}$. A modified score equation can thus be defined by equating $U(\theta)$ to the leading term of its median. This suggests defining the median modified score

$$
\tilde{U}(\theta) = U(\theta) + \nu_{0,0,0}/\{6 \, i(\theta)\},
$$

(1)

where the modification term $\nu_{0,0,0}/\{6 \, i(\theta)\}$ is of order $O(1)$. Let $\tilde{\theta}$ be the estimator defined as solution of $\tilde{U}(\theta) = 0$.

For $\tilde{U}(\theta)$, we have $M_{\theta}\{\tilde{U}(\theta)\} = O(n^{-1})$ and it is shown in the Appendix that $\tilde{U}(\theta)$ is third-order median unbiased, i.e.

$$
P_{\theta}\{\tilde{U}(\theta) \leq 0\} = 1/2 + O(n^{-3/2}).$$

(2)

If $\tilde{\theta}$ is the unique solution of $\tilde{U}(\theta) = 0$, the events $\tilde{U}(\theta) \leq 0$ and $\tilde{\theta} \leq \theta$ are equivalent so that $\tilde{\theta}$ will be third-order median unbiased, i.e.

$$
P_{\theta}\{\tilde{\theta} \leq \theta\} = 1/2 + O(n^{-3/2}).$$

(3)

Like $\tilde{\theta}$, also $\tilde{\theta}$ is asymptotically $N\{\theta, i(\theta)^{-1}\}$, so that Wald-type confidence intervals only differ in location. Score-type confidence intervals can also be used, based on the asymptotic $N\{0, i(\theta)\}$ distribution of $\tilde{U}(\theta)$.

If $\omega(\theta)$ is a smooth reparameterization with inverse $\theta(\omega)$, ingredients of the modification term in (1) in the new parameterization are $\nu_{0,0,0}^{\omega} = \nu_{0,0,0}(\theta(\omega))\{\theta'(\omega)\}^3$ and $i^{\omega}(\omega) = i(\theta(\omega))\{\theta'(\omega)\}^2$, where $\theta'(\omega) = d\theta(\omega)/d\omega$. Hence, like $U(\theta)$, the modified score $\tilde{U}(\theta)$ transforms as a covariant tensor of order one, namely the
modified score in the \( \omega \) parameterization is \( \tilde{U}\{\theta(\omega)\} \theta'(\omega) \). Therefore, \( \tilde{\theta} \) behaves equivariantly as does \( \hat{\theta} \), and \( \tilde{\omega} = \omega(\tilde{\theta}) \) is also third-order median unbiased. [Firth's (1993) method gives an estimator \( \hat{\theta}^* \) with bias of order \( O(n^{-2}) \) in a chosen parameterization. For a scalar parameter, the corresponding modified score is

\[
U^*(\theta) = U(\theta) + (\nu_{0,0,0} + \nu_{0,00})/\{2i(\theta)\}, \tag{4}
\]

where \( \nu_{0,0,0} = E_{\theta}\{U(\theta)U_{\theta\theta\theta}(\theta)\} \), with \( U_{\theta\theta}(\theta) = \partial^2 \ell(\theta)/\partial \theta^2 \). As shown by Kosmidis & Firth (2010, § 3.4) in the vector parameter case, \( U^*(\theta) \) does not transform as a covariant tensor of order one under reparameterizations. This is because, while \( i(\theta) \) behaves tensorially, the same is not true for the term \( \nu_{0,0,0} + \nu_{0,00} \). Therefore, as is natural, first-order bias correction only operates in the reference parameterization. A suggestion in Kosmidis & Firth (2010, § 4.2, Remark 3) is to obtain the correction in a parameterization where the distribution of the maximum likelihood estimator is closer to normality, such as the logit for probability parameters, and then translate the result in the parameterization of interest.

The argument leading to (2) and (3) only holds in the continuous case. Indeed, in the discrete case, the Cornish-Fisher expansion involves also oscillatory terms (see e.g. Cai & Wang, 2009, formula (A.1)). These terms will be ignored in the following and the same adjustment will be employed both in the continuous and in the discrete case. Empirical results in the paper show a gain in median unbiasedness using (1) in place of \( U(\theta) \) also in the discrete case. The effect of omitting the oscillatory terms in a simple logistic regression is illustrated in detail in the Supplementary Material, showing that \( \tilde{\theta} \) is uniformly closer to the exact median unbiased estimator than \( \hat{\theta} \). Moreover, as the number of points in the support of the sufficient statistic increases, \( \tilde{\theta} \) gets much closer to the exact median unbiased estimator than \( \hat{\theta} \).

For a one parameter exponential family with canonical parameter \( \theta \), i.e. with

\[
p_Y(y; \theta) = \exp\{\theta t(y) - K(\theta)\}h(y), \tag{5}
\]

the median modified score function has the form

\[
\tilde{U}(\theta) = U(\theta) + K_{\theta\theta\theta}/\{6 K_{\theta\theta}\},
\]

where \( K_{\theta\theta\theta} = \partial^3 K(\theta)/\partial \theta^3 \) and \( K_{\theta\theta} = \partial^2 K(\theta)/\partial \theta^2 = i(\theta) \). In this parameterization, \( \tilde{U}(\theta) \) can be seen as the score of the penalized log likelihood \( \tilde{\ell}(\theta) = \ell(\theta) + \{\log i(\theta)\}/6 \). On the other hand, Firth’s (1993) modified score takes the form

\[
U^*(\theta) = U(\theta) + K_{\theta\theta\theta}/\{2 K_{\theta\theta}\}. \tag{6}
\]

The effect of the median modification is thus to penalize the likelihood by \( i(\theta)^{1/6} \), while (6) implies a Jeffreys prior penalization.
Under model (5), $U(\theta) = t(y) - E_\theta(t(Y))$, hence, if $K(\theta) = O(n)$, the estimating equation $\hat{U}(\theta) = 0$ provides, in the continuous case, an approximate version of the optimal median unbiased estimator for monotone likelihood ratio families, calculated as the value $\hat{\theta}^*$ of $\theta$ such that $P_\theta(T \leq t) = 1/2$ (Lehmann & Romano, 2005, §3.5). Use of $\hat{U}(\theta) = 0$ amounts to replace the exact $P_\theta(T < t)$ with its Edgeworth expansion up to terms of order $O(n^{-1})$. It is straightforward to see that $\hat{\theta} - \hat{\theta}^* = O_p(n^{-2})$.

In general, a regular model has locally a monotone likelihood ratio with respect to the score function (Cox & Hinkley, 1974, §4.8.i). As a consequence, optimality of $\hat{\theta}$ as defined e.g. in Pace & Salvan (1997, formula (3.58)) will hold locally in a neighbourhood of $\theta_0$.

Example 1. Normal distribution with known mean. Let $y_1, \ldots, y_n$ be a random sample from $N(\mu, \psi)$, with known $\mu$. Quantities for computing (1) and (4) are given in Firth (1992, §4.2). In particular, the adjustment in (4) is equal to zero, so that $\hat{\psi} = \hat{\psi}^* = s(\mu)/n$, with $s(\mu) = \sum_{i=1}^n (y_i - \mu)^2$, is exactly unbiased. The median modified score (1) is equal to $-\left((n-2/3)/(2\sigma^2}\right)+s(\mu)/(2\sigma^2)^2$, giving $\hat{\psi} = s(\mu)/(n-2/3)$, equal to the optimal median unbiased estimator $s(\mu)/\chi^2_{n,0.5}$ plus an error of order $O(n^{-2})$. Consider now the parameterization with the standard deviation $\omega = \psi^{1/2}$. By equivariance, $\hat{\omega} = \hat{\psi}^{1/2}$ and $\hat{\omega} = \psi^{1/2}$, while the bias reduced estimator calculated in the new parameterization is $\hat{\omega}^* = \{s(\mu)/(n - 1/2)\}^{1/2}$.

Example 2. Skew normal shape parameter. Let $y_1, \ldots, y_n$ be $n$ independent realizations of a skew normal distribution with shape parameter $\theta \in \mathbb{R}$ and density $p(y; \theta) = 2\phi(y)\Phi(\theta y)$, where $\phi$ and $\Phi$ denote the standard normal density and distribution functions, respectively, and $y \in \mathbb{R}$. The log likelihood is $\ell(\theta) = \sum_{i=1}^n \zeta_0(\theta y_i)$, where $\zeta_0(x) = \log\{2\Phi(x)\}$. With $\zeta_m(x) = \partial^m \zeta_0(x)/\partial x^m$, $m = 1, 2, \ldots$, the score function is $U(\theta) = \sum_{i=1}^n \zeta_1(\theta y_i)y_i$. Let $a_{kh}(\theta) = E_\theta(Y^k \zeta_1(\theta Y)h)$. The expected quantities needed to compute the median modified score (1) are $i(\theta) = n a_{22}(\theta)$ and $a_{0\theta} = n a_{33}(\theta)$, giving $\hat{U}(\theta) = U(\theta) + a_{33}(\theta)/\{6a_{22}(\theta)\}$. The modified score (4) (see Sartori, 2006) is $U^*(\theta) = U(\theta) - \theta a_{22}(\theta)/\{2a_{22}(\theta)\}$.

The performance of $\theta$, $\hat{\theta}^*$ and $\hat{\theta}$ has been investigated by Monte Carlo simulations with 5,000 replications. Results are displayed in Table 1. Estimators are compared in terms of empirical probability of underestimation, median absolute error, bias, root mean squared error and coverage of 95% Wald-type and score-type confidence intervals. The empirical probability of underestimation is the summary of primary interest for $\hat{\theta}$, as the estimator is designed to satisfy (3).

A natural associated measure of dispersion is the median absolute error. Estimated bias and root mean squared error are also reported to enable a fair comparison with $\hat{\theta}^*$. While $\hat{\theta}^*$ and $\hat{\theta}$ are always finite, in some samples the maximum likelihood estimate is infinite. The simulation frequency of finite maximum likelihood estimates, $\%(\hat{\theta} < +\infty)$, is reported in the table. As in Kosmidis & Firth (2009, §6.2), estimated bias, root mean squared error and coverage probability of confidence intervals for $\hat{\theta}$ are conditional upon its finiteness. Although this favours $\hat{\theta}$, both $\hat{\theta}$
Table 1: Simulation results for estimates of the skew normal shape parameter. For \( \hat{\theta}, \), B, RMSE and coverage are conditional upon finiteness of the estimates

| \( \theta \) | \( n \) | PU | MAE | B  | RMSE | Wald | Score | \( \%((\hat{\theta} < +\infty) \) |
|-----|-----|-----|-----|----|------|------|-------|-----------------|
| 5   | 20  | \( \hat{\theta} \) | 36.2 | 2.31 | 1.90 | 8.44 | 94.5 | 94.7 | 72.2 |
|     |     | \( \hat{\theta}^* \) | 92.8 | 1.91 | -1.70 | 2.01 | 68.4 | 87.0 |
|     |     | \( \tilde{\theta} \) | 53.8 | 1.73 | 0.94 | 4.02 | 91.1 | 92.5 |
| 50  | 20  | \( \hat{\theta} \) | 41.0 | 1.31 | 1.93 | 8.67 | 96.5 | 95.0 | 96.0 |
|     |     | \( \hat{\theta}^* \) | 67.7 | 1.20 | -0.28 | 1.79 | 86.2 | 90.3 |
|     |     | \( \tilde{\theta} \) | 50.3 | 1.21 | 1.25 | 4.82 | 93.9 | 93.4 |
| 100 | 20  | \( \hat{\theta} \) | 42.7 | 0.86 | 0.82 | 3.64 | 96.9 | 96.1 | 99.9 |
|     |     | \( \hat{\theta}^* \) | 60.9 | 0.84 | 0.00 | 1.50 | 91.9 | 92.8 |
|     |     | \( \tilde{\theta} \) | 49.8 | 0.84 | 0.49 | 2.20 | 95.5 | 95.3 |
| 10  | 20  | \( \hat{\theta} \) | 29.7 | +\( \infty \) | 2.12 | 20.11 | 90.6 | 89.5 | 49.2 |
|     |     | \( \hat{\theta}^* \) | 99.8 | 6.16 | -5.94 | 6.06 | 20.4 | 83.7 |
|     |     | \( \tilde{\theta} \) | 73.0 | 3.57 | -1.36 | 5.35 | 83.2 | 91.7 |
| 50  | 20  | \( \hat{\theta} \) | 36.9 | 3.73 | 5.11 | 30.11 | 95.5 | 95.2 | 80.2 |
|     |     | \( \hat{\theta}^* \) | 87.2 | 3.25 | -2.59 | 3.56 | 73.5 | 88.4 |
|     |     | \( \tilde{\theta} \) | 52.6 | 3.10 | 2.30 | 8.67 | 92.0 | 93.4 |
| 100 | 20  | \( \hat{\theta} \) | 40.1 | 2.50 | 3.92 | 15.95 | 96.1 | 95.4 | 96.0 |
|     |     | \( \hat{\theta}^* \) | 68.0 | 2.28 | -0.52 | 3.53 | 86.9 | 90.8 |
|     |     | \( \tilde{\theta} \) | 49.6 | 2.32 | 2.57 | 10.01 | 93.9 | 93.9 |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald and Score, percentage coverage of 95% Wald-type and score-type confidence intervals.

and \( \hat{\theta}^* \) are uniformly better. Median centering improvement attained by \( \tilde{\theta} \), as measured by empirical probability of underestimation, is remarkable, both for small and moderate sample sizes. On the other hand, the estimated root mean squared error is much smaller for \( \hat{\theta}^* \) than for \( \hat{\theta} \). In this case, values of \( \tilde{\theta} \) are intermediate between those of \( \hat{\theta} \) and \( \hat{\theta}^* \). This effect is illustrated, for the same sample as in Example 1 of Sartori (2006), in Fig. 1. Score-type confidence intervals have overall better coverage than Wald-type intervals, although this effect is substantial only for \( \hat{\theta}^* \). Indeed, the penalization implied by \( \hat{\theta}^* \) is excessive, leading to poor coverage of Wald-type confidence intervals. Coverage probabilities for maximum likelihood should be judged with caution since samples with infinite estimates are excluded.

2.2 Presence of nuisance parameters

With \( \theta = (\theta_1, \ldots, \theta_p) \), we denote by \( U_r = \partial \ell(\theta)/\partial \theta^r \), \( r = 1, \ldots, p \), the elements of the score vector \( U(\theta) \). Let \( i_{rs} \) be a generic entry of Fisher information, \( i(\theta) \), and \( i^{rs} \) an entry of its inverse, \( r, s, \ldots = 1, \ldots, p \). Let \( U_{rs} \) and \( U_{rst} \) be higher order partial derivatives of \( \ell(\theta) \) with respect to elements of \( \theta \) with indices \( r, s, t \). More-
over, expected values of log likelihood derivatives are denoted as $\nu_{rs} = E_\theta(U_{rs}) = -i_{rs}$, $\nu_{rst} = E_\theta(U_{rst})$, $\nu_{r,s,t} = E_\theta(U_r U_s U_t)$ and $\nu_{r,s,t} = E_\theta(U_r U_s U_t)$.

Let us suppose now that the parameter is partitioned as $\theta = (\psi, \lambda)$, with $\psi$ a scalar parameter of interest. When exact elimination of $\lambda$ by conditioning or by marginalization is feasible, arguments in the previous subsection may be applied to the conditional or marginal score for $\psi$. See e.g. Hirji et al. (1989) for exact conditional median unbiased estimators in logistic regression. In more general situations, or when an expression for the exact solution is not available, we propose a modification of the profile score. Let us denote by $\ell_p(\psi) = \ell(\psi, \hat{\lambda}_\psi)$ the profile log likelihood for $\psi$, where $\hat{\lambda}_\psi$ is the maximum likelihood estimate of $\lambda$ for a given value of $\psi$. The profile score is $U_p(\psi) = \partial \ell_p(\psi) / \partial \psi$. Let us use subscript $\psi$ when referring to $\psi$ and indices $a,b,c, \ldots$ to refer to components of $\lambda$, so that elements of $U(\theta)$ are $U_\psi = U_\psi(\psi, \lambda) = \partial \ell(\psi, \lambda) / \partial \psi$ and $U_a = U_a(\psi, \lambda) = \partial \ell(\psi, \lambda) / \partial \lambda_a$, $a = 1, \ldots, p - 1$. As is well known, $U_p(\psi) = U_\psi(\psi, \hat{\lambda}_\psi)$ and approximate expressions for the first three cumulants of $U_p(\psi)$ are

$$\begin{align*}
\kappa_{1\psi} &= -\frac{1}{2} \nu^{ab} \{ (\nu_{\psi,ab} - \gamma_{\psi,c} \nu^{c,ab}) + (\nu_{\psi,a,b} - \gamma_{\psi,c} \nu^{a,b,c}) \} \\
\kappa_{2\psi} &= \nu_{\psi,\psi} - \gamma_{\psi,a} \nu_{\psi,a} \\
\kappa_{3\psi} &= \nu_{\psi,\psi,\psi} - 3 \gamma_{\psi,a} \nu_{\psi,a} + 3 \gamma_{\psi,a} \gamma_{\psi,b} \nu_{\psi,a,b} - \gamma_{\psi,a} \gamma_{\psi,b} \gamma_{\psi,c} \nu_{a,b,c},
\end{align*}$$

where the error term is of order $O(n^{-1})$ in $\kappa_{1\psi}$ and of order $O(1)$ in $\kappa_{2\psi}$ and $\kappa_{3\psi}$. In (7), Einstein summation convention is used, i.e. summation over repeated indices $a,b,\ldots$ is understood. The quantity $\nu^{ab}$ is an element of the inverse of the square matrix of order $p - 1$ with entries $\nu_{a,b}$, and $\gamma_{\psi,a} = \nu^{ab} \nu_{a,b}$. $\nu_{a,b}$ is a regression coefficient of $U_\psi$ on the vector with elements $U_a$, $a = 1, \ldots, p - 1$. The above expression for $\kappa_{1\psi}$ was obtained in McCullagh & Tibshirani (1990). Approximations

Figure 1: The left panel shows $U(\theta)$ (solid), $U^*(\theta)$ (dot-dashed) and $\tilde{U}(\theta)$ (dashed) with data from Sartori (2006, Example 1) with corresponding estimates 5.40, 2.84 and 4.10, respectively. The right panel is relative to the same data with a change of sign of the only negative observation, with estimates $+\infty$, 3.92 and 10.82.
\(\kappa_{2\psi}\) and \(\kappa_{3\psi}\) are the second and third cumulants of the efficient score for \(\psi\), namely \(\hat{U}_\psi = U_\psi - \gamma_\psi(a)\), which is the leading term of the expansion of \(U_\psi(\psi)\). They are obtained from formulae (7.15) and (7.16) in Barndorff-Nielsen & Cox (1989) for cumulants of residuals.

In a continuous model, using a Cornish-Fisher expansion, the median of the standardized profile score \((U_\psi(\psi) - \kappa_{1\psi})/\sqrt{\kappa_{2\psi}}\) is equal to \(-\kappa_{3\psi}/(6\kappa_{2\psi}^{3/2}) + O(n^{-3/2})\). Therefore, the median modified profile score is

\[
\hat{U}_\psi(\psi) = U_\psi(\psi) - \kappa_{1\psi} + \kappa_{3\psi}/(6\kappa_{2\psi})
\]

and has median zero with error of order \(O(n^{-1})\). The same argument as in the proof of (3) shows that \(P_\theta(\hat{U}_\psi(\psi) \leq 0) = 1/2 + O(n^{-3/2})\). Let \(\hat{\psi}_p\) be the estimator defined as solution of \(\hat{U}_\psi(\psi) = 0\) with \(\lambda\) replaced by \(\hat{\lambda}_\psi\). If the resulting estimating equation has a unique solution, third-order median unbiasedness of \(\hat{\psi}_p\) follows. Although this argument only holds in the continuous case, empirical results for binary regression in Examples 4 and 7 show a gain in median unbiasedness using (8) in place of \(U_\psi(\theta)\) also in the discrete case. See the Supplementary Material for a numerical comparison of \(\psi\) with the exact conditional median unbiased estimator. The asymptotic distribution of \(\hat{\psi}_p\) is the same as that of \(\psi\), that is \(N(\psi, \kappa_{2\psi}^{-1})\). This can be used to construct Wald-type confidence intervals. Score-type confidence intervals can also be used, based on the asymptotic \(N(0, \kappa_{2\psi})\) distribution of \(\hat{U}_\psi(\psi)\).

Substituting \(\hat{\lambda}_\psi\) for \(\lambda\) has the drawback of requiring the solution of \(U_a = 0\) for fixed \(\psi\), \(a = 1, \ldots, p - 1\). Although infinite values of the constrained estimate of \(\lambda\) may not be a problem in (8), joint estimation as described in § 3 is often preferable.

Parameterization equivariance of \(\hat{\psi}_p\) holds under interest respecting reparameterizations. In detail, let \(\omega = (\varphi, \chi)\) be a smooth reparameterization with \(\varphi = \varphi(\psi)\) and \(\chi = \chi(\psi, \lambda)\) and \(\varphi = \varphi(\psi)\) a one-to-one function of \(\psi\) with inverse \(\psi(\varphi)\). Then, the modified score for \(\varphi\) in the new parameterization is \(\hat{U}_\psi(\varphi(\psi))\varphi'(\varphi)\), so that \(\hat{\varphi}_p = \varphi(\hat{\psi}_p)\). This tensorial behaviour of the modified profile score follows from the tensorial behaviour of the profile score and of its first-order expectation (Pace & Salvan, 1997, § 9.5.3). In addition, the efficient score \(U_\psi\) also transforms tensorially and therefore so does the ratio \(\kappa_{3\psi}/\kappa_{2\psi}\).

If \(p_\psi(y; \theta)\) is an exponential family of order \(p\) with canonical parameter \((\psi, \lambda)\), i.e.

\[
p_\psi(y; \psi, \lambda) = \exp\{\psi t(y) + \lambda^T s(y) - K(\psi, \lambda)\} h(y),
\]

quantities (7) are simply obtained from derivatives of \(K(\psi, \lambda)\). In particular, \(\nu^{ab}\) is a generic element of \((\partial^2 K(\psi, \lambda)/\partial \lambda \partial \lambda^T)^{-1}\). \(\nu_{\psi,ab} = \nu_{c,ab} = 0\), and all other \(\nu\) quantities are the derivatives of \(K(\psi, \lambda)\) with respect to components of \((\psi, \lambda)\) appearing as subscripts. Here, \(U_\psi(\psi) - \kappa_{1\psi}\) is an approximation with error of order \(O(n^{-1})\) of the score for \(\psi\) in the conditional model given \(s(y)\) (see e.g. Pace & Salvan, 1997, § 10.10.2). In the continuous case, the estimator from (8) is an approximation of the optimal conditional median unbiased estimator (Lehmann & Romano, 2005, § 5.4), solution with respect to \(\psi\) of \(P_\psi(T \leq t\mid S =
the relationship between the histology of the endometrium of 79 patients and three risk factors: neovasculation, pulsatility index of arteria uterina and endometrium height. Logistic regression has been fitted with parameter \( \beta \). The corresponding estimate is \( \hat{\beta} \), with both \( \beta \) and \( \psi \) unknown and let \( \psi \) be of interest. The maximum likelihood estimator is \( \hat{\psi} = s(\bar{y})/n \), with \( \bar{y} = \sum_{i=1}^{n} y_i/n \). \( \hat{\psi} \) is equal to the optimal median unbiased estimator \( s(\bar{y})/(n-1) \), which coincides with the usual unbiased estimator. Formula (8) gives \( \hat{U}_p(\psi) = -(n-1-2/3)/(2\psi) + s(\bar{y})/(2\psi^2) \), so that \( \hat{\psi}_p = s(\bar{y})/(n-1-2/3) \), that is equal to the optimal median unbiased estimator \( s(\bar{y})/\chi^2_{n-1;0.5} \) plus an error of order \( O(n^{-2}) \). In the \((\mu, \omega)\) parameterization, with \( \omega = \psi^{1/2} \), the bias reduced estimator is \( \hat{\omega}^* = \{s(\bar{y})/(n-3/2)\}^{1/2} \).

Example 3. Normal distribution (cont.). Consider again the setting of Example 1 with both \( \mu \) and \( \psi \) unknown and let \( \psi \) be of interest. The maximum likelihood estimator is \( \hat{\psi} = s(\bar{y})/n \), with \( \bar{y} = \sum_{i=1}^{n} y_i/n \). \( \hat{\psi} \) is equal to the optimal median unbiased estimator \( s(\bar{y})/(n-1) \), which coincides with the usual unbiased estimator. Formula (8) gives \( \hat{U}_p(\psi) = -(n-1-2/3)/(2\psi) + s(\bar{y})/(2\psi^2) \), so that \( \hat{\psi}_p = s(\bar{y})/(n-1-2/3) \), that is equal to the optimal median unbiased estimator \( s(\bar{y})/\chi^2_{n-1;0.5} \) plus an error of order \( O(n^{-2}) \). In the \((\mu, \omega)\) parameterization, with \( \omega = \psi^{1/2} \), the bias reduced estimator is \( \hat{\omega}^* = \{s(\bar{y})/(n-3/2)\}^{1/2} \).

Example 4. Binary regression. Let \( y_i, i = 1, \ldots, n \), be independent realizations of binary random variables with probability \( \pi_i = F(\eta_i) \), where \( \eta_i = x_i\beta \), \( x_i = (x_{i1}, \ldots, x_{i4}) \) is a row vector of covariates and \( F \) is a known cumulative distribution function. We assume that a generic scalar component of \( \beta \) is of interest and treat the remaining components as nuisance parameters. Quantities needed for (8) are given in the Supplementary Material.

As an example, we consider the endometrial cancer grade dataset analyzed, among others, in Agresti (2013, § 5.7.1). The goal of the study was to evaluate the relationship between the histology of the endometrium of 79 patients and three risk factors: neovasculation, pulsatility index of arteria uterina and endometrium height. Logistic regression has been fitted with parameter \( \beta = (\beta_1, \beta_2, \beta_3, \beta_4)^\top \), where \( \beta_1 \) is an intercept and the remaining parameters correspond to neovasculation, pulsatility index of arteria uterina, and endometrium height, respectively. Maximum likelihood leads to infinite maximum likelihood estimate of \( \beta_2 \) due to quasi-complete separation. Let us consider \( \beta_2 \) as the parameter of interest while the remaining regression coefficients are treated as nuisance parameters. Both \( \beta_2^* \) and \( \tilde{\beta}_2 \) are finite with \( \beta_2^* = 2.929 \) and \( \tilde{\beta}_2 = 3.883 \). The corresponding standard errors are 1.551 and 2.407, respectively.

To assess the properties of estimators of \( \beta_2 \), we performed a simulation study with sample size and covariates as in the endometrial dataset and with \( \beta = (1.5, 2, 0, -2)^\top \). The results are presented in Table 2. We found 684 samples out of 10,000 with data separation. Empirical probability of underestimation indicates that \( \tilde{\beta}_2 \) has a remarkable performance in terms median centering. On the other hand, as expected, \( \tilde{\beta}_2^* \) has estimated bias close to zero. Coverages of Wald-type confidence intervals based on \( \tilde{\beta}_2^* \) and on \( \tilde{\beta}_2 \) are comparable, while those based on \( \tilde{\beta}_2^* \) are favored by being computed only using samples with finite estimates. Score-type intervals based on \( \tilde{U}_p, (\beta_2) \) perform slightly better than Wald-type ones, while score-type confidence intervals for scalar components of the parameter are not available when
Table 2: Simulation results for endometrial cancer study. For $\hat{\beta}_2$, B, RMSE and coverage are conditional upon finiteness of the estimates

|       | PU   | MAE | B     | RMSE | Wald | Score |
|-------|------|-----|-------|------|------|-------|
| $\hat{\beta}_2$ | 43.0 | 0.66 | 0.12  | 0.90 | 97.5 | 98.9  |
| $\hat{\beta}_2^*$ | 53.1 | 0.56 | 0.02  | 0.90 | 97.4 | –     |
| $\tilde{\beta}_2$ | 49.7 | 0.60 | 0.16  | 1.09 | 97.7 | 95.7  |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald and Score, percentage coverage of 95% Wald-type and score-type confidence intervals.

using bias reduction.

The modified profile score (8) can also be seen as a median modification of a first order bias corrected profile score $U_p(\psi) - \kappa_1\psi$, with $\kappa_1\psi$ evaluated at $(\psi, \hat{\lambda}_\psi)$ (McCullagh & Tibshirani, 1990). This is equivalent to the score of an adjusted profile likelihood, such as the modified profile likelihood (Barndorff-Nielsen, 1983). Many available adjustments of the profile likelihood share indeed the common feature of reducing the score bias to $O(n^{-1})$ (DiCiccio et al., 1996). In the presence of many nuisance parameters, typically the term $\kappa_1\psi$ dominates $\kappa_3\psi/\kappa_2\psi$. For instance, in a stratified setting with independent $y_{aj}$, $a = 1, \ldots, q$ and $j = 1, \ldots, m$, having marginal distribution depending on $(\psi, \lambda_a)$ with both $q$ and $m$ diverging as in Sartori (2003), the term $\kappa_1\psi$ in (8) is of order $O(q)$, while $\kappa_3\psi/\kappa_2\psi$ is of order $O(1)$. Therefore, the difference between $\tilde{\psi}_P$ and $\hat{\psi}_M$, the maximizer of the modified profile likelihood, is of order $O\left\{1/(qm)\right\}$ and both estimators have the standard asymptotic behaviour provided that $q = o(m^3)$, as opposed to the stronger condition $q = o(m)$ for the maximum likelihood estimator.

Example 5. Gamma samples with common shape parameter. Let $y_{aj}$, $a = 1, \ldots, q$ and $j = 1, \ldots, m$, be realizations of independent gamma random variables with shape parameter $\psi$ and scale parameter $1/\lambda_a$. The needed quantities in (8) are

$$U_p(\psi) = t +qm \log m\psi - m\Psi^{(0)}(\psi), \quad \nu_{\psi,\psi} = mq\Psi^{(1)}(\psi), \quad \nu_{\psi,a} = -m/\lambda_a,$$

$$\nu_{a,a} = (m\psi)/\lambda_a^2, \quad \nu_{a,\psi,\psi} = 0, \quad \nu_{a,b} = \nu_{a,b,\psi} = \nu_{a,b,c}, \quad a \neq b,$$

$$\nu_{\psi,\psi,\psi} = mq\Psi^{(2)}(\psi), \quad \nu_{a,a,\psi} = m/\lambda_a^2, \quad \nu_{a,a,a} = -(2m\psi)/\lambda_a^3,$$

where $t = \sum_{a=1}^q \sum_{j=1}^m \log y_{aj}$ and $\Psi^{(k)}(\psi) = d^{k+1} \log \Psi(\psi)/d\psi^{k+1}$ is the poly-gamma function of order $k$. Here, the conditional maximum likelihood estimator, $\hat{\psi}_C$, based on the distribution of $t$ given the stratum sums is also available and is asymptotically equivalent to both $\hat{\psi}_P$ and $\hat{\psi}_M$, provided that $q = o(m^3)$ (Sartori, 2003, Example 2).

Simulation results with 10,000 replications are shown in Table 3 for $q = 1, 50$, $m = 5, 10$, $\psi = \exp(1)$. We compared $\hat{\psi}$, $\hat{\psi}_M$, $\hat{\psi}_C$, $\hat{\psi}_P$, the bias reduced estimator
\( \hat{\psi}^* \) in the \((\psi, \lambda)\) parameterization and the estimator \( \hat{\psi}^{**} = \exp(\hat{\varphi}^*) \), where \( \hat{\varphi}^* \) is the bias reduced estimator of \( \varphi \) in the parameterization \((\varphi, \chi)\), with \( \varphi = \log \psi, \chi = \log \lambda \). Median centering of \( \hat{\psi}_p \) is considerable, even in the most extreme setting with \( q = 50 \). Median bias reduction shows coverage of Wald-type confidence intervals closer to nominal values than bias reduction. Score-type intervals based on \( \hat{U}_p(\psi) \) are slightly more accurate than Wald-type ones. As expected, the \( \varphi \) parameterization is more favourable than the \( \psi \) parameterization for bias reduction.

**Example 6.** Common odds ratio in \( 2 \times 2 \) tables. Consider \( q \) independent pairs of observations \((y_{a1}, y_{a2})\), realizations of independent binomial variables \(Bi(1, p_{a1})\) and \(Bi(m, p_{a2})\). Let

\[
\begin{align*}
    p_{a1} &= \exp(\lambda_a + \psi)/(1 + \exp(\lambda_a + \psi)) \\
    p_{a2} &= \exp(\lambda_a)/(1 + \exp(\lambda_a))
\end{align*}
\]

This model may arise in case-control studies, with 1 case and \( m \) controls in each table, and where interest is about \( \psi \), representing the influence of some risk factor. As in Breslow (1981), we consider sparse settings with large \( q \) and small \( m \), where improvements over the maximum likelihood estimator are particularly needed. This is also an instance where invariance is important, since results are often reported in terms of odds ratio \( \rho = \exp(\psi) \). The median modified profile score is a special case of that in Example 4. The conditional maximum likelihood estimator \( \hat{\rho}_C \) is available, based on the conditional distribution of \( t = \sum_{a=1}^q y_{a1} \) given \( s_a = y_{a1} + y_{a2}, a = 1, \ldots, q \).

The aim is to compare the various methods with conditional maximum likelihood, which gives consistency also for fixed \( m \) and can be considered as a gold standard. The comparison is made on the odds ratio scale, using \( \hat{\rho}^* = \exp(\hat{\psi}^*) \) for bias reduction and equivariance for the other estimators. As in Sartori (2003, Example 3), we focus on particular instances with odd values of \( m \) and with \( s_a = (m+1)/2, a = 1, \ldots, q \), so that all tables have the same number of successes and failures. In this case, \( \hat{\rho}_C = (t/q)/(1-t/q) \) and, for a given \( m \), both \( \hat{\rho} \) and \( \hat{\rho}_M \) are functions of \( t/q \) only. Although \( \hat{\rho}^* \) and \( \hat{\rho}_p \) depend also on \( q \), numerical evidence indicates that such dependence vanishes as \( q \) increases. For \( q = 300 \) and various values of \( m \), estimates of the odds ratio are plotted versus \( \hat{\rho}_C \) in Figure 2. Median modified estimates are almost indistinguishable from those based on modified profile likelihood, as expected, and both are the closest to \( \hat{\rho}_C \). On the contrary, \( \hat{\rho}^* \) markedly departs from \( \hat{\rho}_C \), especially for small \( m \) and as \( \hat{\rho}_C \) increases, while \( \hat{\rho}^* \) overcorrects, in particular for large values of \( \hat{\rho}_C \). Other values of \( q \) give the same results in terms of estimates, while accuracy of inference is affected since standard errors decrease as \( q \) increases.

### 3 Median modified score for a vector parameter

For estimation of the full vector parameter \( \theta \), with \( p > 1 \), a direct extension of the rationale leading to (1) does not seem to be practicable due to lack of a manageable definition of multivariate median. Actually, a number of definitions have been proposed (Oja 2013), but none seems suitable for developing a median modification.
Table 3: Simulation results for estimates of the common gamma shape parameter

| q | PU | MAE | B  | RMSE | Wald | Score |
|---|---|-----|----|------|------|-------|
| 1 | 5 | \( \hat{\psi} \) | 29.9 | 1.41 | 3.48 | 8.36 | 97.5 | 97.5 |
|   |   | \( \hat{\psi}_M \) | 40.9 | 1.22 | 2.31 | 6.51 | 95.2 | 95.1 |
|   |   | \( \hat{\psi}_C \) | 41.0 | 1.22 | 2.30 | 6.51 | 95.1 | 95.0 |
|   |   | \( \hat{\psi}^* \) | 73.4 | 1.27 | 0.04 | 2.99 | 76.1 | – |
|   |   | \( \hat{\psi}^{**} \) | 56.5 | 1.22 | 1.06 | 4.68 | 84.3 | – |
|   | 10| \( \hat{\psi} \) | 50.1 | 1.19 | 1.51 | 5.29 | 89.2 | 94.9 |
|   |   | \( \hat{\psi}_M \) | 35.7 | 0.85 | 1.03 | 2.36 | 97.1 | 97.1 |
|   |   | \( \hat{\psi}_C \) | 44.3 | 0.80 | 0.68 | 2.03 | 95.6 | 95.5 |
|   |   | \( \hat{\psi}^* \) | 64.5 | 0.85 | 0.03 | 1.48 | 85.6 | – |
|   |   | \( \hat{\psi}^{**} \) | 54.7 | 0.80 | 0.31 | 1.73 | 91.2 | – |
|   |   | \( \hat{\psi}_P \) | 50.5 | 0.79 | 0.45 | 1.83 | 92.8 | 95.3 |
| 50| 5 | \( \hat{\psi} \) | 1.2 | 0.62 | 0.64 | 0.72 | 40.5 | 40.5 |
|   |   | \( \hat{\psi}_M \) | 47.0 | 0.17 | 0.03 | 0.26 | 95.1 | 95.1 |
|   |   | \( \hat{\psi}_C \) | 48.3 | 0.17 | 0.03 | 0.26 | 95.0 | 95.0 |
|   |   | \( \hat{\psi}^* \) | 58.0 | 0.18 | -0.04 | 0.26 | 90.2 | – |
|   |   | \( \hat{\psi}^{**} \) | 51.2 | 0.06 | 0.00 | 0.09 | 92.2 | – |
|   |   | \( \hat{\psi}_P \) | 48.4 | 0.17 | 0.03 | 0.26 | 92.5 | 97.1 |
| 10| 5 | \( \hat{\psi} \) | 6.2 | 0.27 | 0.28 | 0.34 | 67.8 | 67.8 |
|   |   | \( \hat{\psi}_M \) | 48.6 | 0.11 | 0.01 | 0.17 | 95.1 | 95.1 |
|   |   | \( \hat{\psi}_C \) | 49.0 | 0.11 | 0.01 | 0.17 | 95.1 | 95.1 |
|   |   | \( \hat{\psi}^* \) | 53.6 | 0.12 | -0.01 | 0.17 | 93.4 | – |
|   |   | \( \hat{\psi}^{**} \) | 50.5 | 0.04 | 0.00 | 0.06 | 93.9 | – |
|   |   | \( \hat{\psi}_P \) | 49.5 | 0.12 | 0.01 | 0.17 | 94.0 | 96.0 |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald and Score, percentage coverage of 95% Wald-type and score-type confidence intervals.
Figure 2: Estimates of odds ratio as functions of the conditional maximum likelihood estimate for $q = 300$ and $m = 1, 3, 5, 7$ (clockwise from top left): $\hat{\rho}_C$ (solid), $\hat{\rho}$ (dashed), $\hat{\rho}^*$ (long-dashed), $\hat{\rho}_M$ (dotted), $\hat{\rho}_P$ (dot-dashed).
of the score vector. For instance, with the simplest definition, i.e. taking the vector of approximate marginal medians as an approximate median of the score vector, dependence among score components is ignored. Other available definitions of multivariate median would involve the joint distribution of the score vector in a rather complex way and do not seem to provide feasible proposals.

Instead, the approach we follow is to set up a system of estimating equations giving, for each \( \theta_r, r = 1, \ldots, p \), the same estimate as \( \tilde{J} \), up to terms of order \( O_p(n^{-1}) \) included. This is obtained by defining the median modified score vector \( \tilde{U}(\theta) \) with components

\[
\tilde{U}_r = U_r - \gamma r \gamma U_a + M_r, \quad r = 1, \ldots, p,
\]

(9)

where \( M_r = -\kappa_{1r} + \kappa_{3r}/(6\kappa_{2r}) \), and \( \kappa_{jr}, j = 1, 2, 3 \), are as in \( \psi = \theta_r \). In \( \tilde{U} \), and in related formulae \( \tilde{J} \), indices \( a, b \ldots \) take values in \( \{1, \ldots, p\} \setminus \{r\} \), and are summed when repeated. Moreover, all quantities involved are evaluated at \( \theta \), so that no constrained estimates are involved. Then, the joint estimate \( \hat{\theta} \) is defined as solution of \( \tilde{U}(\theta) = 0 \).

For each \( r = 1, \ldots, p \), \( \tilde{U}_r \) behaves tensorially under interest respecting reparameterizations of \( \theta_r \). As a consequence, \( \hat{\theta} \) is equivariant under joint reparameterizations that transform each component of \( \theta \) separately.

Denoting by \( \tilde{U}(\theta) \) the vector with components given by the efficient scores \( \tilde{U}_r = U_r - \gamma r \gamma U_a \), we can write \( \tilde{U}(\theta) = A(\theta)U(\theta) \), with \( A(\theta) \) a nonsingular and nonstochastic matrix of order \( p \). As shown in (16), \( H(\theta) = E\theta \{ -\partial \tilde{U}(\theta)/\partial \theta^T \} = \{ \text{diag}(i(\theta)^{-1}) \}^{-1} \). Moreover, \( H(\theta) = A(\theta)i(\theta) \), so that \( A(\theta) = H(\theta)i(\theta)^{-1} \). Hence, solving \( \tilde{U}(\theta) = 0 \) is equivalent to solving

\[
U(\theta) + i(\theta)M_1(\theta) = 0,
\]

(10)

with \( M_1(\theta) \) having elements \( M_{1r} = M_r/\kappa_{2r} \). There is no general guarantee that (10) has a solution. However, \( i(\theta)M_1(\theta) \) is of order \( O(1) \), so that, asymptotically, existence of \( \hat{\theta} \) is guaranteed whenever \( \theta \) exists. Moreover, \( \hat{\theta} - \tilde{\theta} = O_p(n^{-1}) \) and the asymptotic distribution of \( \hat{\theta} \) is the same as that of \( \tilde{\theta} \).

Let \( \hat{\theta}_r \) be the \( r \)-th component of \( \hat{\theta} \) and \( \hat{\theta}_{r_p} \) the solution of \( \hat{U}_p(\theta_r) = 0 \), with \( \hat{U}_p(\cdot) \) given by \( \tilde{J} \). In a regular model, we show that

\[
\hat{\theta}_r - \hat{\theta}_{r_p} = O_p(n^{-3/2}), \quad r = 1, \ldots, p.
\]

(11)

A proof of (11) is given in the Appendix. A key property for the result is that \( H(\theta) \) is a diagonal matrix, so that \( \hat{U}(\theta) \) satisfies

\[
E\theta(\partial \hat{U}_r/\partial \theta_s) = O(1), \quad r, s = 1, \ldots, p, \quad s \neq r.
\]

(12)

Following Jorgensen & Knudsen (2004), we call \( \hat{U}_r \) first-order insensitive to \( \theta \) components other than \( \theta_r, r = 1, \ldots, p \). Due to (12), terms up to order \( O_p(n^{-1}) \) in the expansion of \( \hat{\theta}_r - \theta_r \) are not affected by terms of order \( O(1) \) in \( \hat{U}_s, s \neq r \).
Table 4: Endometrial cancer study. Estimates (s.e.) for logistic regression (top rows) and probit regression (bottom rows)

| $\hat{\beta}$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ |
|---------------|-----------|-----------|-----------|-----------|
| $\hat{\beta}$ | 4.305 (1.637) | $+\infty$ ($+\infty$) | -0.042 (0.044) | -2.903 (0.846) |
| $\hat{\beta}^*$ | 3.775 (1.489) | 2.929 (1.551) | -0.035 (0.040) | -2.604 (0.776) |
| $\hat{\beta}$ | 3.969 (1.552) | 3.869 (2.298) | -0.039 (0.042) | -2.708 (0.803) |

Using delta method arguments as in Hall (1992, § 2.7), it follows from (11) that, in the continuous case, $P_{\theta_r}(\tilde{\theta}_r \leq \theta_r) = P_{\tilde{\theta}_r}(\tilde{\theta}_r \leq \tilde{\theta}_r) + o(n^{-3/2})$, so that componentwise median unbiasedness of $\tilde{\theta}$ with error of order $O(n^{-3/2})$ follows from the analogous property of $\tilde{\theta}_r$.

Equation (10) has the same structure as the estimating equation for bias reduction. Hence, some of the ideas in Kosmidis & Firth (2009, 2010) for the implementation of bias reduction can be adapted for median bias reduction. For instance, a modified Fisher scoring iteration can be written as

$$\tilde{\theta}^{(k+1)} = \tilde{\theta}^{(k)} + M_1(\tilde{\theta}^{(k)})^{-1} U(\tilde{\theta}^{(k)}),$$

which differs from the analogue for $\hat{\theta}$ only by the addition of the term $M_1$. When available, $\tilde{\theta}$ is a convenient starting value. As happens for bias reduction (Kosmidis & Firth, 2010), convergence or otherwise of (13) depends on the properties of the specific assumed model. Nonetheless, assuming convergence of (13), it will be to a solution of (10).

Example 7. Binary regression (continued). Quantities needed for (9) in binary regression are the same as those in Example 4. Moreover, (13) simplifies to a modified iterative reweighted least squares procedure. Details are provided in the Supplementary Material and an implementation is given in the R package mbrglm (Kenne Pagui et al., 2017).

For the endometrial cancer grade dataset, estimates of the model parameters using (9) for logistic and probit regression are given in Table 4. The estimate $\hat{\beta}_2$ is very close to $\tilde{\beta}_2$, obtained in Example 4 as a solution of (8).

The same simulated samples as in Example 4 allow to evaluate the properties of estimators of the vector $\beta$. Table 4 shows that the new method is remarkably accurate in achieving median centering for all the parameter components. It should be recalled that 684 samples out of 10,000 produced infinite maximum likelihood estimates, so that results for $\hat{\beta}$ should be judged accordingly. The three approaches are comparable in terms of coverage of Wald-type confidence intervals, while profile score-type intervals show some improvement. Similar results have been found with a probit model and are reported in the Supplementary Material.
Table 5: Simulation results for endometrial cancer study. For maximum likelihood, B, RMSE and coverage are conditional upon finiteness of the estimates

|   | PU | MAE | B  | RMSE | Wald | Score |
|---|----|-----|----|------|------|-------|
| $\hat{\beta}$ | 45.1 | 0.97 | 0.29 | 1.60 | 95.8 | 94.8 |
|     | 43.0 | 0.66 | 0.12 | 0.90 | 97.4 | 95.2 |
|     | 51.0 | 0.03 | 0.00 | 0.04 | 95.0 | 94.2 |
|     | 56.0 | 0.57 | -0.26 | 1.02 | 96.0 | 94.9 |
| $\hat{\beta}^*$ | 52.6 | 0.86 | 0.00 | 1.38 | 96.6 | –    |
|     | 53.0 | 0.56 | 0.02 | 0.90 | 97.4 | –    |
|     | 49.6 | 0.02 | 0.00 | 0.04 | 96.3 | –    |
|     | 44.4 | 0.52 | 0.01 | 0.83 | 94.8 | –    |
| $\tilde{\beta}$ | 50.1 | 0.90 | 0.09 | 1.46 | 96.4 | 95.0 |
|     | 49.7 | 0.59 | 0.15 | 1.07 | 97.5 | 95.3 |
|     | 50.7 | 0.02 | 0.00 | 0.04 | 96.1 | 94.3 |
|     | 49.6 | 0.52 | -0.10 | 0.89 | 95.8 | 94.7 |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald and Score, percentage coverage of 95% Wald-type and score-type confidence intervals.

Table 6: Food expenditure. Estimates (s.e.) for beta regression

|   | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\phi$ |
|---|-----------|-----------|-----------|-------|
| $\hat{\theta}$ | -0.623 (0.224) | -0.012 (0.003) | 0.118 (0.035) | 35.610 (8.080) |
| $\hat{\theta}^*$ | -0.621 (0.239) | -0.012 (0.003) | 0.118 (0.038) | 30.922 (7.005) |
| $\tilde{\theta}$ | -0.621 (0.235) | -0.012 (0.003) | 0.118 (0.037) | 32.160 (7.289) |

Example 8. Beta regression. Let $y_i$, $i = 1, \ldots, n$, be independent realizations of beta random variables with parameters $\phi \mu_i$ and $\phi (1 - \mu_i)$, i.e. with expected value $\mu_i$ and precision parameter $\phi$. We assume a regression structure for the expected value $\mu_i = g^{-1}(\eta_i)$, where $\eta_i = x_i \beta$, $x_i = (x_{i1}, \ldots, x_{ip})$ is a vector of covariates and $g(\cdot)$ is a given link function, such as the logit. The needed quantities for (9) with $\theta = (\beta_1, \ldots, \beta_p, \phi)^T$ are the same as those required for bias reduction (Kosmidis & Firth, 2010). Details are given in the Supplementary Material. An R implementation of (13) is given in function mbrbetareg, available on GitHub.

As an application, we consider data in Griffiths et al. (1993, Table 15.4) on food expenditure for a random sample of 38 households in a large U.S. city, also available in the R package betareg. The objective is to model the proportion of income spent on food ($y$) as a function of income ($x_2$) and number of persons ($x_3$). Estimates of $\theta = (\beta_1, \beta_2, \beta_3, \phi)^T$, where $\beta_1$ is an intercept, and with the logit link are given in Table 6. Values for the regression coefficients and corresponding standard errors are essentially the same for all methods, while differences are observed for the dispersion parameter.
Table 7: Simulation results for food expenditure

|        | PU   | MAE  | B     | RMSE | Wald | Score |
|--------|------|------|-------|------|------|-------|
| \( \hat{\phi} \) | 32.7 | 6.25 | 5.46  | 11.79| 95.1 | 95.1  |
| \( \hat{\phi}^* \) | 56.5 | 5.74 | 0.06  | 9.07 | 91.8 | –     |
| \( \tilde{\phi} \) | 49.8 | 5.69 | 1.49  | 9.56 | 93.7 | 95.8  |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald and Score, percentage coverage of 95% Wald-type and score-type confidence intervals.

We performed a simulation study with the same sample size and covariates as in the food expenditure data and with parameter fixed at \( \hat{\theta} \). Results obtained from 100,000 simulated samples show identically accurate behaviour for estimators of regression parameters. Hence, only results for estimators of \( \phi \) are displayed in Table 10, in line with those of previous examples. The complete table is reported in the Supplementary Material, together with an additional example with a smaller ratio \( n/p \) leading to larger differences among estimators of \( \phi \). This also implies different confidence intervals for the regression coefficients and corresponding coverages.

Appendix

Proof of (2). Let \( \rho_3 = \nu_0,\theta,\theta,\theta / i(\theta)^{3/2} \) be the third standardized cumulant of \( U(\theta) \), of order \( O(n^{-1/2}) \). Then, with a standard Edgeworth expansion,

\[
\Phi(\frac{\tilde{\theta}(\theta) - \theta}{\sqrt{i(\theta)}/6}) = \Phi(\frac{\hat{\theta}(\theta) - \theta}{\sqrt{i(\theta)}/6}) + \rho_3(\theta) + O(n^{-3/2}),
\]

where the error is of order \( O(n^{-3/2}) \) because the \( O(n^{-1}) \) term in the Edgeworth expansion is a linear combination with coefficients of order \( O(n^{-1}) \) of odd Hermite polynomials evaluated at \( -\rho_3/6 \). The result in (2) follows using the expansions \( \Phi(-\rho_3/6) = 1/2 - \rho_3(0)/6 + O(n^{-3/2}) \) and \( \Phi(-\rho_3/6) = \phi(0) + O(n^{-1}) \).

Proof of (14). First, an expansion of \( \bar{\theta}_{rr} - \theta_r \) is readily obtained from an expansion for \( \theta_r - \theta_e \) (see e.g. [199]), taking into account the effect of the modification to the score \( U_r(\theta_r) \) given in (8). In detail, being \( -\partial U_r(\theta_r)/\partial \theta_r = \kappa_{2r} + O_p(n^{1/2}) \), we get

\[
\bar{\theta}_{rr} - \theta_r = \hat{\theta}_r - \theta_r - \kappa_{1r}/\kappa_{2r} + \kappa_{3r}/\{6\kappa_{2r}^2\} + O_p(n^{-3/2}).
\]  

Second, an expansion for \( \hat{\theta}_r - \theta_r \) from (2) is obtained using standard asymptotic expansions for estimating equations. Let \( g = g(\theta) = g(\theta; y) \) be an estimating function with generic component \( g_r \). We assume that \( g \) is of order \( O_p(n^{1/2}) \) with
expected value $O(1)$. Let $g_{r/s} = \partial g_r/\partial \theta_s$, $g_{r/st} = \partial^2 g_r/(\partial \theta_s \partial t)$ and let $\xi_r = E_\theta(g_r) = O(1)$, $\xi_{r/s} = E_\theta(g_{r/s})$, $\xi_{r/st} = E_\theta(g_{r/st})$, the latter two quantities being typically of order $O(n)$. Moreover, let $D_{r/s} = g_{r/s} - \xi_{r/s}$, $D_{r/st} = g_{r/st} - \xi_{r/st}$. Let $\tau^r$s be a generic entry of the inverse of the matrix with entries $-\xi_{r/s}$. An asymptotic expansion for $g(\bar{\theta}) = 0$ gives

$$
\bar{\theta}_r - \theta_r = \tau^r r g_s + \tau^r t u D_{s/t} g_u + \frac{1}{2} \tau^r u \tau^w \xi_\sigma_g + O_p(n^{-3/2}). \tag{15}
$$

When $g_r = U_r$, we obtain $\bar{\theta} = \bar{\theta}$, so that expansion (15) gives the usual expansion for $\bar{\theta}_r - \theta_r$. The same is true if $g_r = U_r$, being $U_r$ a linear transformation of $U_r$. However, in the latter case, $\tau^r = 0$ if $r \neq s$, while $\tau^r = \bar{\tau} = \kappa_2^{-1}$. Indeed, $\xi_{r/s} = E_\theta(U_{r/s} - \gamma_{r/s}) = -(i_{r/s} - \gamma_{r/s})$. \tag{16}

Since, when $s \neq r$, we have $r_{ab} i_{ab} = \delta^b_s$, the indicator of $b = s$, it follows that $\xi_{r/s} = -(i_{r/s} - i_r \delta^b_s) = -(i_{r/s} - i_r \delta^b_s) = 0$ if $s \neq r$. On the other hand, $\xi_{r/s} = -\kappa_2 r$.

When (15) is applied to (9), we have $\tau^r = \bar{\tau} + O(n^{-2})$ and $\tau^r = O(n^{-2})$ if $r \neq s$. Therefore, terms up to order $O_p(n^{-1})$ in the expansion for $\bar{\theta}_r - \theta_r$ do not involve modification terms of order $O(1)$ of $\bar{U}_r$ with $s \neq r$. The desired expansion for $\bar{\theta}_r - \theta_r$ is thus equivalently obtained from the system

$$
\bar{U}_r = 0, \quad \bar{U}_s = 0, s \neq r.
$$

This is the same as the expansion from $\bar{U}_r = 0, r = 1, \ldots, p$, plus a $O(n^{-1})$ term given by the modification term in (9) divided by $\kappa_2 r$. Therefore, the resulting expansion coincides with (14).

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Supplementary material

Supplementary material includes some discussion on the discrete case and details and quantities for the implementation of the method, together with additional simulation results for Examples 4, 7 and 8.

A Comparison with exact median unbiased estimator in simple binomial regression models

Consider first a simple binomial regression model with \( y_1, \ldots, y_5 \) realizations of independent \( Bi(m, \pi_i) \) random variables, with \( \log\{\pi_i/(1 - \pi_i)\} = \theta x_i, \ i = 1, \ldots, 5 \), with covariate values \( (x_1, \ldots, x_5) = (-0.560, -0.230, 0.071, 0.129, 1.559) \), generated from a standard normal distribution. The sufficient statistic is \( t = \sum_{i=1}^{5} y_i x_i \) and takes \( (m + 1)^5 \) distinct values.

We compare the maximum likelihood estimator, \( \hat{\theta} \), which amounts to considering only the leading term of the Cornish-Fisher expansion for the median of \( U(\theta) \),
and the median bias reduced estimator, $\tilde{\theta}$, with the exact median unbiased estimator, $\hat{\theta}^e$, for increasing values of $m$. All three estimators vary monotonically with $t$ and the latter estimator (see, for instance, Hirji et al., 1989) is defined as $\hat{\theta}^e = (\theta_e + \theta_{e*})/2$, where $\theta_e$ and $\theta_{e*}$ are such that

$$P_{\theta_e}(T \leq t) \geq 1/2, \quad P_{\theta_{e*}}(T \geq t) \geq 1/2.$$ 

When $t$ is equal to either the maximum or the minimum of its possible values, then only one of $\theta_e$ or $\theta_{e*}$ is defined. In such case, $\hat{\theta}^e$ is taken to be $\theta_e$ or $\theta_{e*}$, whichever exists. This estimator satisfies

$$P_{\theta}({\tilde{\theta}^e} \leq \theta) \geq 1/2, \quad P_{\theta}({\tilde{\theta}^e} \geq \theta) \geq 1/2.$$ 

For $m = 1, 2, 3$, $t$ takes respectively 32, 243 and 1024 distinct values. Figure 3 shows the differences $\hat{\theta} - \hat{\theta}^e$ and $\hat{\theta} - \tilde{\theta}^e$ as functions of $\hat{\theta}^e$ in the three situations. We note that the two points corresponding to the minimum and maximum values of $t$ are not reported since $\hat{\theta}$ is respectively $-\infty$ or $+\infty$. The proposed estimator $\tilde{\theta}$ is closer to $\hat{\theta}^e$ than $\hat{\theta}$ in all three situations, with relative differences getting smaller as the number of points in the sample space increases.

As an example with $p > 1$, consider the hypothetical clinical trial data in Hirji et al. (1989, Table 2) with $n = 30$ patients belonging to two age groups (age less or equal than 30 years, and age greater than 30 years) of size 20 and 10, respectively. Each group is randomized to receive one of two treatments, with 9 and 6 patients receiving the first treatment in the first and second age group, respectively. Let $y_i$ be the binary disease outcome ($y_i = 1$ for a positive outcome, $y_i = 0$ otherwise). Moreover, let $x_{i2}$ be a binary age indicator ($x_{i2} = 1$ if age is less or equal to 30 years, $x_{i2} = 0$ if age is greater than 30 years) and $x_{i3}$ be a binary treatment indicator ($x_{i3} = 1$ for the first treatment, $x_{i3} = 0$ for the second treatment), $i = 1, \ldots, 30$. Then, with $\pi_i$ the probability of a positive outcome, a logistic model relating the response of the $i$-th patient to treatment and age can be written as

$$\log\{\pi_i/(1 - \pi_i)\} = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3}, \quad i = 1, \ldots, 30.$$ 

Here $\beta_3$ is the relative log odds of response for treatment 1 versus treatment 2, and can be considered as the parameter of interest. The exact conditional median unbiased estimator $\hat{\beta}_3$ (Hirji et al., 1989) can be obtained using the definition above, applied to the conditional distribution of $t = \sum_{i=1}^{30} x_{i3} y_i$ given $s = (\sum_{i=1}^{30} y_i, \sum_{i=1}^{30} x_{i2} y_i)$. As in Hirji et al. (1989, Table 2), we compare in Table 3 $\hat{\beta}_3$, $\tilde{\beta}_3$ and $\hat{\beta}_3^e$, for all possible values of $t$ in the conditional distribution of $T$ given $s = (16, 12)$. Estimate $\hat{\beta}_3$ is the third component of the joint bias reduced estimate $\hat{\beta}$, as in Example 7, and it is third order equivalent to $\hat{\beta}_3$, from (8). The bias reduced estimator is uniformly closer to the exact conditional median unbiased estimator than the maximum likelihood estimate.
Figure 3: Simple binomial regression. Differences $\hat{\theta} - \tilde{\theta}^e$ (solid) and $\tilde{\theta} - \tilde{\theta}^e$ (dashed) as a function of $\theta^e$ when $m$ is equal to 1 (top), 2 (middle) and 3 (bottom). Ticks on the horizontal axes represent values of $\tilde{\theta}^e$. 
We give details on the computation of the needed quantities for (8) and (9), used in Examples 4 and 7 of the paper, respectively. We assume \( y_i, i = 1, \ldots, n \), as independent realizations of binary random variables with probability \( \pi_i = F(\eta_i) \), where \( \eta_i = x_i \beta, x_i = (x_{i1}, \ldots, x_{ip}) \) is a row vector of covariates, \( \beta = (\beta_1, \ldots, \beta_p)^T \) and \( F \) is a known cumulative distribution function. Below, indices \( r, s, \) and \( t \) refer to the components of \( \beta \). We have

\[
\begin{align*}
U_r &= \sum_{i=1}^{n} x_{ir} A(\eta_i) \{ y_i - F(\eta_i) \}, \\
i_{rs} &= \sum_{i=1}^{n} x_{ir} x_{is} A(\eta_i) F'(\eta_i), \\
\nu_{rs,t} &= \sum_{i=1}^{n} x_{ir} x_{is} x_{it} B(\eta_i) F'(\eta_i), \\
\nu_{r,s,t} &= \sum_{i=1}^{n} x_{ir} x_{is} x_{it} A(\eta_i)^3 F(\eta_i) \{1 - F(\eta_i)\} \{1 - 2F(\eta_i)\},
\end{align*}
\]

with

\[
\begin{align*}
A(\eta_i) &= \frac{F'(\eta_i)}{F(\eta_i) \{1 - F(\eta_i)\}}, \\
B(\eta_i) &= \frac{F''(\eta_i)}{F(\eta_i) \{1 - F(\eta_i)\} + \frac{F'(\eta_i)^2}{F(\eta_i)^2 \{1 - F(\eta_i)\}^2}},
\end{align*}
\]

where \( F'(\cdot) \) and \( F''(\cdot) \) are first and second derivatives of \( F(\cdot) \). If \( F(\cdot) \) is the logistic cumulative distribution function, \( A(\cdot) = 1 \) and \( B(\cdot) = 0 \).

Ingredients of Fisher scoring equation (13) may be written in matrix form as

\[
U(\beta) = X^T W(\beta) v(\beta) \quad \text{and} \quad i(\beta) = X^T W(\beta) X,
\]

where \( X \) is the design matrix.
Table 9: Simulation results for endometrial cancer study with probit link. For maximum likelihood, B, RMSE and coverage are conditional upon finiteness of the estimates

|        | PU | MAE | B   | RMSE | Wald | Score |
|--------|----|-----|-----|------|------|-------|
| $\hat{\beta}$ | 43.1 | 0.57 | 0.21 | 0.96 | 95.3 | 95.3  |
|        | 43.4 | 0.38 | 0.44 | 1.55 | 97.1 | 95.0  |
|        | 50.5 | 0.01 | -0.00 | 0.02 | 94.1 | 94.2  |
|        | 58.1 | 0.33 | -0.18 | 0.61 | 95.5 | 95.4  |
| $\hat{\beta}$* | 52.8 | 0.51 | -0.01 | 0.80 | 95.9 | –     |
|        | 51.7 | 0.33 | 0.01 | 0.52 | 97.1 | –     |
|        | 49.2 | 0.01 | -0.00 | 0.02 | 96.4 | –     |
|        | 45.0 | 0.30 | 0.01 | 0.48 | 94.5 | –     |
| $\tilde{\beta}$ | 50.5 | 0.53 | 0.05 | 0.85 | 96.1 | 95.2  |
|        | 49.3 | 0.34 | 0.06 | 0.58 | 97.0 | 94.9  |
|        | 50.1 | 0.01 | -0.00 | 0.02 | 96.0 | 94.2  |
|        | 49.8 | 0.31 | -0.06 | 0.51 | 95.6 | 95.0  |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald and Score, percentage coverage of 95% Wald-type and Score-type confidence intervals.

with entries $x_{ir}$, $W(\beta)$ is a diagonal matrix with diagonal elements $\{F'(\eta_i)\}^2/[F(\eta_i)\{1 - F(\eta_i)\}]$, and $v(\beta)$ is a vector with elements $v_i(\beta) = \{y_i - F(\eta_i)/F'(\eta_i)\}$.

We denote by $\theta = (\beta_1, \ldots, \beta_p, \phi)^T$ the full vector of parameters. The log-likelihood has the form

$$\ell(\theta) = \sum_{i=1}^{n} \{\mu_i \phi t_i + (1 - \mu_i) \phi z_i + \log \Gamma(\phi) - \log \Gamma(\mu_i \phi) - \log \Gamma\{(1 - \mu_i) \phi\}\},$$

where $t_i = \log(y_i)$ and $z_i = \log(1 - y_i)$.
Let $\bar{t}_i = t_i - \Psi^{(0)}(\phi \mu_i) + \Psi^{(0)}(\phi)$ and $\bar{z}_i = z_i - \Psi^{(0)}(\phi(1 - \mu_i)) + \Psi^{(0)}(\phi)$, with $\Psi^{(r)}(k) = d^{r+1} \log \Gamma(k)/dk^{r+1}$ the polygamma function of order $r$, $r = 0, 1, 2, \ldots$. The needed quantities for (9) are

$$U_{\beta'} = \phi \sum_{i=1}^{n} x_{ir}(\bar{t}_i - \bar{z}_i)(d\mu_i/d\eta_i), \quad U_{\phi} = \sum_{i=1}^{n} \mu_i(\bar{t}_i - \bar{z}_i) + \bar{z}_i,$$

$$i_{\beta',\beta'} = \phi^2 \sum_{i=1}^{n} x_{ir}x_{is}[\Psi^{(1)}(\phi \mu_i) - \Psi^{(1)}\{\phi(1 - \mu_i)\}](d\mu_i/d\eta_i)^2,$$

$$i_{\beta',\phi} = \phi \sum_{i=1}^{n} x_{ir}(\mu_i[\Psi^{(1)}(\phi \mu_i) + \Psi^{(1)}\{\phi(1 - \mu_i)\}] - \Psi^{(1)}\{\phi(1 - \mu_i)\}](d\mu_i/d\eta_i),$$

$$i_{\phi \phi} = \sum_{i=1}^{n}[\Psi^{(1)}(\phi \mu_i)\mu_i^2 + \Psi^{(1)}\{\phi(1 - \mu_i)\}(1 - \mu_i)^2] - n\Psi^{(1)}(\phi),$$

$$\nu_{\beta',\beta',\beta'} = \phi^3 \sum_{i=1}^{n} x_{ir}x_{is}x_{it}[\Psi^{(2)}(\phi \mu_i) - \Psi^{(2)}\{\phi(1 - \mu_i)\}](d\mu_i/d\eta_i)^3,$$

$$\nu_{\beta',\beta',\phi} = \phi^2 \sum_{i=1}^{n} x_{ir}x_{is}[\Psi^{(2)}(\phi \mu_i)\mu_i + \Psi^{(2)}\{\phi(1 - \mu_i)\}(1 - \mu_i)(d\mu_i/d\eta_i)^2,$$

$$\nu_{\beta',\phi,\phi} = \phi \sum_{i=1}^{n} x_{ir}[\Psi^{(2)}(\phi \mu_i)\mu_i^2 - \Psi^{(2)}\{\phi(1 - \mu_i)\}(\mu_i - 1)^2](d\mu_i/d\eta_i),$$

$$\nu_{\phi,\phi,\phi} = \sum_{i=1}^{n}[\Psi^{(2)}(\phi \mu_i)\mu_i^3 - \Psi^{(2)}\{\phi(1 - \mu_i)\}(\mu_i - 1)^3] - n\Psi^{(2)}(\phi),$$

$$\nu_{\beta',\beta',\beta'} = \phi^2 \sum_{i=1}^{n} x_{ir}x_{is}x_{it}[\Psi^{(1)}(\phi \mu_i) - \Psi^{(1)}\{\phi(1 - \mu_i)\}](d\mu_i/d\eta_i)(d^2\mu_i/d\eta_i^2),$$

$$\nu_{\beta',\beta',\phi} = \phi \sum_{i=1}^{n} x_{ir}x_{is}[\Psi^{(1)}(\phi \mu_i) + \Psi^{(1)}\{\phi(1 - \mu_i)\}](d\mu_i/d\eta_i)^2,$$

$$\nu_{\beta',\phi,\phi} = 0,$$

$$\nu_{\phi,\beta',\beta'} = \phi \sum_{i=1}^{n} x_{ir}(\mu_i[\Psi^{(1)}(\phi \mu_i) + \Psi^{(1)}\{\phi(1 - \mu_i)\}] - \Psi^{(1)}\{\phi(1 - \mu_i)\})(d\mu_i/d\eta_i)^2,$$

$$\nu_{\phi,\phi,\phi} = 0.$$

An R implementation of the method, using formula (13) of the paper, is given in function mbrbetareg available at https://github.com/eulogepagui/mbrbetareg.
Table 10: Simulation results for food expenditure

| PU  | MAE | B   | RMSE | Wald | Score |
|-----|-----|-----|------|------|-------|
| $\hat{\theta}$ | 50.0 | 0.15 | 0.00 | 0.22 | 93.1  | 94.3  |
|      | 50.6 | 0.00 | 0.00 | 0.00 | 93.5  | 94.8  |
|      | 50.5 | 0.02 | 0.00 | 0.04 | 93.2  | 94.5  |
|      | 32.7 | 6.25 | 5.46 | 11.79| 95.1  | 95.1  |
| $\hat{\theta}^*$ | 49.8 | 0.15 | 0.00 | 0.22 | 94.7  | –     |
|      | 50.2 | 0.00 | 0.00 | 0.00 | 95.2  | –     |
|      | 50.9 | 0.02 | 0.00 | 0.04 | 94.9  | –     |
|      | 56.5 | 5.74 | 0.06 | 9.07 | 91.8  | –     |
| $\tilde{\theta}$ | 49.9 | 0.15 | 0.00 | 0.22 | 94.3  | 94.4  |
|      | 50.3 | 0.00 | 0.00 | 0.00 | 94.8  | 94.8  |
|      | 50.8 | 0.02 | 0.00 | 0.04 | 94.5  | 94.5  |
|      | 49.8 | 5.69 | 1.49 | 9.56 | 93.7  | 95.8  |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald and Score, percentage coverage of 95% Wald-type and score-type confidence intervals.

Complete simulation results for Example 8 are in Table 10.

As a further example, we consider the gasoline yield data as in Kosmidis & Firth (2010, Section 4.3). Here $n = 32$ and the response variable is the proportion of crude oil converted to gasoline after distillation and fractionation. Covariates are 9 indicators representing the 10 distinct experimental settings in the data and the temperature in degrees Fahrenheit at which all gasoline has vaporized. Estimates of $\theta = (\beta_1, \ldots, \beta_{11}, \phi)^T$, where $\beta_1$ is an intercept, $\beta_2, \ldots, \beta_{10}$ are the coefficients of the 9 indicators, $\beta_{11}$ is the coefficient of the temperature, and with the logit link are given in Table 11. Values for the regression coefficients are essentially the same for all methods, while notable differences are observed for the dispersion parameter. These in turn influence estimates of standard errors for the regression coefficients.

We performed a simulation study with the same sample size and covariates as in the gasoline yield data and with parameter fixed at $\hat{\theta}$. Results obtained from 100,000 simulated samples in Table 12 show marked differences among estimators of $\phi$, with $\tilde{\phi}$ achieving median centering. These differences also imply different coverages of confidence intervals for the regression coefficients. In this rather extreme case score-type confidence intervals are numerically very unstable and only results for Wald-type intervals are reported.
Table 11: Gasoline data. Regression estimates (s.e.).

|   | $\hat{\beta}$     | $\hat{\beta}^*$ | $\hat{\beta}$     |
|---|------------------|------------------|------------------|
| $\beta_1$ | -6.160 (0.182) | -6.142 (0.236) | -6.144 (0.228) |
| $\beta_2$ | 1.728 (0.101) | 1.723 (0.131) | 1.724 (0.127) |
| $\beta_3$ | 1.323 (0.118) | 1.319 (0.153) | 1.319 (0.148) |
| $\beta_4$ | 1.572 (0.116) | 1.567 (0.150) | 1.568 (0.145) |
| $\beta_5$ | 1.060 (0.102) | 1.057 (0.132) | 1.058 (0.128) |
| $\beta_6$ | 1.134 (0.104) | 1.130 (0.134) | 1.131 (0.130) |
| $\beta_7$ | 1.040 (0.106) | 1.037 (0.137) | 1.038 (0.133) |
| $\beta_8$ | 0.544 (0.109) | 0.542 (0.141) | 0.543 (0.137) |
| $\beta_9$ | 0.496 (0.109) | 0.494 (0.141) | 0.495 (0.136) |
| $\beta_{10}$ | 0.386 (0.119) | 0.385 (0.154) | 0.385 (0.148) |
| $\beta_{11}$ | 0.011 (0.000) | 0.011 (0.001) | 0.011 (0.001) |
| $\phi$ | 440.278 (110.026) | 261.038 (65.216) | 279.409 (69.809) |
Table 12: Simulation results for gasoline data

| PU   | MAE  | B    | RMSE | Wald |
|------|------|------|------|------|
| \( \hat{\theta} \) | 52.63 | 0.124 | -0.015 | 0.183 | 86.82 |
|      | 48.67 | 0.068 | 0.004 | 0.102 | 86.86 |
|      | 48.14 | 0.081 | 0.005 | 0.119 | 86.95 |
|      | 48.75 | 0.078 | 0.004 | 0.116 | 87.19 |
|      | 49.18 | 0.070 | 0.002 | 0.103 | 87.12 |
|      | 49.55 | 0.070 | 0.002 | 0.103 | 87.70 |
|      | 49.10 | 0.071 | 0.003 | 0.107 | 87.20 |
|      | 49.84 | 0.073 | 0.001 | 0.109 | 87.29 |
|      | 49.69 | 0.074 | 0.002 | 0.110 | 86.57 |
|      | 49.60 | 0.081 | 0.002 | 0.118 | 87.23 |
|      | 47.51 | 0.000 | 0.000 | 0.000 | 87.14 |
|      | 5.65  | 254.489 | 302.277 | 395.193 | 74.92 |
| \( \hat{\theta}^* \) | 50.00 | 0.125 | -0.002 | 0.182 | 94.76 |
|      | 49.95 | 0.069 | 0.001 | 0.102 | 94.69 |
|      | 49.16 | 0.081 | 0.002 | 0.119 | 94.32 |
|      | 49.81 | 0.078 | 0.000 | 0.116 | 94.86 |
|      | 50.06 | 0.069 | 0.000 | 0.102 | 94.73 |
|      | 50.44 | 0.070 | 0.000 | 0.103 | 94.95 |
|      | 49.82 | 0.071 | 0.001 | 0.106 | 94.42 |
|      | 50.02 | 0.073 | 0.000 | 0.109 | 94.45 |
|      | 49.96 | 0.074 | 0.001 | 0.110 | 94.57 |
|      | 49.81 | 0.081 | 0.001 | 0.118 | 94.97 |
|      | 49.82 | 0.000 | 0.000 | 0.000 | 94.67 |
|      | 58.12 | 93.669 | 0.209 | 151.152 | 84.83 |
| \( \tilde{\theta} \) | 50.30 | 0.125 | -0.004 | 0.182 | 93.83 |
|      | 49.67 | 0.069 | 0.002 | 0.102 | 93.97 |
|      | 49.08 | 0.081 | 0.002 | 0.119 | 93.60 |
|      | 49.67 | 0.078 | 0.001 | 0.116 | 94.04 |
|      | 49.83 | 0.069 | 0.001 | 0.102 | 93.98 |
|      | 50.25 | 0.070 | 0.000 | 0.103 | 94.24 |
|      | 49.64 | 0.071 | 0.001 | 0.106 | 93.67 |
|      | 49.95 | 0.073 | 0.001 | 0.109 | 93.68 |
|      | 49.88 | 0.074 | 0.001 | 0.110 | 93.90 |
|      | 49.81 | 0.081 | 0.001 | 0.118 | 94.29 |
|      | 49.51 | 0.000 | 0.000 | 0.000 | 94.02 |
|      | 49.76 | 92.138 | 31.176 | 164.736 | 88.72 |

PU, percentage of underestimation; MAE, median absolute error; B, bias; RMSE, root mean squared error; Wald, percentage coverage of 95% Wald-type confidence intervals.