Six-loop anomalous dimension of the $\phi^Q$ operator
in the $O(N)$ symmetric model

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Abstract

A technique of large-charge expansion provides a novel opportunity for calculation of critical dimensions of operators $\phi^Q$ with fixed charge $Q$. In the small-coupling regime the polynomial structure of the anomalous dimensions can be fixed from a number of direct perturbative calculations for a fixed $Q$. At the six-loop level one needs to include new diagrams that correspond to operators with five or more legs. The latter never appeared before in scalar-theory calculations. Here we show how to compute the anomalous dimension of the operator $\phi^{Q=5}$ at the six-loop order. In combination with results for operators with $Q < 5$, which are extracted from the six-loop beta-functions for general scalar theory, and with predictions from the large-charge expansion, our calculation allows us to derive the answer for general-$Q$ anomalous dimensions. At the critical point resummation in three dimensions enables us to compare the critical exponents with results of Monte-Carlo simulations and large-$N$ predictions.

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I. INTRODUCTION

The renormalization group (RG) method allows one to systematically improve the accuracy of calculations in perturbation theory. The key objects of the method are the renormalization group functions, which specify the response of various quantities to a scale variation.

Among many applications of the field-theoretical RG are studies of universal critical behaviour of different physical systems near second-order phase transitions.

The well-known examples are three-dimensional $O(N)$ universality classes. The corresponding critical indices can be derived by considering different operators within the $\phi^4$ model in $d = 4 - 2\varepsilon$ dimensions. In particular, the exponents are computed from the RG functions evaluated at a non-trivial Wilson-Fisher [1] (WF) fixed point, for which the quartic coupling $g = g^* = O(\varepsilon)$. Recent six- [2] and even seven-loop [3] results together with modern resummation techniques (see [2] and references therein) give very precise critical exponents, which can be compared both to experimental measurements and to values obtained by other methods.

While the expansion in small couplings seems natural and convenient, there exist several non-pertubative approaches to the calculation of critical indices. For example, in $O(N)$ theory one can consider the limit of large-$N$ and systematically obtain corrections for scaling exponents as series in $1/N$ [4, 5]. The latter are valid for any space dimensions and effectively resum infinite number of terms in the $\varepsilon$-expansion. The case of $\phi^Q$ operators was considered in Ref. [6].

Recently, a different method was proposed [7, 8] allowing one to study the anomalous dimensions of operators $\phi^Q$ with total charge $Q$ semi-classically via operator-state correspondence.

In Ref. [7] the $U(1) = O(2)$ model is considered and a new ‘t Hooft-like coupling is introduced $g^*Q$. The scaling dimension of $\phi^Q$-type operators are written as

$$\Delta_Q = \sum_{i=-1}^{\infty} (g^*)^i\Delta_i(g^*Q) = \frac{\Delta_{-1}(g^*Q)}{g^*} + \Delta_0(g^*Q) + \ldots$$ (1)

and the first two terms of the expansion in small $g^*$ for a fixed $g^*Q$ are computed. The approach was generalized to the case of $O(N)$ model [8] together with $U(N) \times U(N)$ [9] and $U(N) \times U(M)$ [10] theories.
Expanding $\Delta_{-1}$ and $\Delta_0$ in small $g^*Q$, one can predict leading and subleading terms in large $Q$ at arbitrary high loop. The latter can be compared with existing perturbative calculations (see, e.g., Ref. [11, 12]), which for the case of the $O(N)$ model are available (for arbitrary $Q$) up to five loops from recent paper [13].

In this work, we use the results of Ref. [8] as an input together with explicit computations for $Q = 1...5$, to deduce the six-loop terms in $\Delta_Q$ for general $Q$ in the $O(N)$ model. While the cases up to $Q = 4$ can easily be treated given our general formulae for beta-functions valid in arbitrary $\phi^4$ theory[14], the computation of the anomalous dimension for $Q = 5$ constitutes the main technical challenge of the current study.

The paper is organized as follows. In Section II we briefly review the $O(N)$ model and operators of our interest. In Section III we discuss the details of calculation. Section IV is devoted to our main results: six-loop expressions for the anomalous and scaling dimension of $\phi^Q$-type operators in $O(N)$ model. Discussion and conclusion and can be found in Section V.

II. FIXED-CHARGE OPERATORS IN $O(N)$ MODEL

The Euclidean Lagrangian that describes the $O(N)$-symmetric model is given by

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} \right) + \frac{g}{4!} \left( \vec{\phi} \cdot \vec{\phi} \right)^2, \tag{2}$$

where $\vec{\phi} = \{\phi_a\}, a = 1...N$ is a $N$-component scalar field.

Following Ref. [8], we consider lowest-lying $O(N)$ operators of fixed total charge $Q$, which can be represented as

$$\phi^Q = d_{i_1...i_Q} \phi_{i_1} \cdots \phi_{i_Q}, \quad d_{i_1...i_Q} = 0 \tag{3}$$

with $d_{i_1...i_Q}$ being symmetric traceless tensor.

To compute the anomalous dimension $\gamma_Q$ of (3) perturbatively one can consider the insertion of the $\phi^Q$ operator in the one-particle irreducible (1PI) Green function with $Q$ external legs (see., e.g., [12, 13] for four- and five-loop results). Using dimensional regularization [15] with $d = 4 - 2\varepsilon$ and modified minimal subtraction scheme ($\overline{\text{MS}}$), one extracts the renormalization constants $Z_{\phi^Q}$, which relate bare fields $\phi_B$ to the renormalized
operator $\phi^Q$:

$$
\phi_Q^B = Z_{\phi^Q} [\phi^Q].
$$

The anomalous dimension can be cast in the following general form:

$$
\gamma_Q(g) \equiv \frac{\partial \log Z_{\phi^Q}}{\partial g} (-2\epsilon g + \beta_g) = Q \sum_{l=1}^{\infty} g^l \gamma_{Q}^{(l)},
\gamma_Q^{(l)} = \sum_{r=0}^{l-r} \sum_{s=0}^{l} Q^r N^s \gamma_{r,s},
\gamma_{0,l}^{(l)} = 0.
$$

Here $\beta_g$ is 4d part of the well-known beta function of the self-coupling $g$ known up to six loops from Ref. [2]. In Eq. (5) we sum over $l$-loop contributions $\gamma_{Q}^{(l)}$. The latter are polynomials in $Q$ up to degree $l$, and for each monomial $Q^r$ the coefficient is a polynomial in $N$ up to degree $l - r$. The coefficients $\gamma_{r,s}^{(l)}$ are just numbers.

Let us mention here that $Q = 1$ case corresponds to the field anomalous dimension [2], while, due to tensor nature of the operator, $Q = 2$ is related to the crossover exponent $\nu [16]$.

Evaluating $\gamma_Q$ at the WF fixed point $g = g^* \approx \frac{6\epsilon}{N+8}$, we compute the scaling dimensions of the operators in the form of $\epsilon$-expansion:

$$
\Delta_Q = Q(1 - \epsilon) + \gamma_Q(g^*) = Q + Q \sum_{l=1}^{\infty} \sum_{r=0}^{l-r} (2\epsilon)^l \frac{Q^r}{(N+8)^p} P_{r,p},
P_{0,0}^{(l)} = 0, P_{0,2l}^{(l)} = 0.
$$

The ultimate goal of this paper is to compute 6-loop contributions $\gamma_{r,s}^{(6)}$ and $P_{r,p}^{(6)}$ to Eqs. (5) and (6), respectively. In what follows, we briefly review our approach to the calculation.

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1 And not to the anomalous dimension of the singlet $\phi^2 \equiv \phi^2$.  

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III. CALCULATION DETAILS

According to Eq. (5), six-loop contribution to the anomalous dimension is a polynomial in $Q$

$$\gamma^{(6)}_Q = Q^0 \left( \gamma^{(6)}_{0,0} + N \gamma^{(6)}_{0,1} + N^2 \gamma^{(6)}_{0,2} + N^3 \gamma^{(6)}_{0,3} + N^4 \gamma^{(6)}_{0,4} + N^5 \gamma^{(6)}_{0,5} \right)$$

$$+ Q^1 \left( \gamma^{(6)}_{1,0} + N \gamma^{(6)}_{1,1} + N^2 \gamma^{(6)}_{1,2} + N^3 \gamma^{(6)}_{1,3} + N^4 \gamma^{(6)}_{1,4} + N^5 \gamma^{(6)}_{1,5} \right)$$

$$+ Q^2 \left( \gamma^{(6)}_{2,0} + N \gamma^{(6)}_{2,1} + N^2 \gamma^{(6)}_{2,2} + N^3 \gamma^{(6)}_{2,3} + N^4 \gamma^{(6)}_{2,4} \right)$$

$$+ Q^3 \left( \gamma^{(6)}_{3,0} + N \gamma^{(6)}_{3,1} + N^2 \gamma^{(6)}_{3,2} + N^3 \gamma^{(6)}_{3,3} \right)$$

$$+ Q^4 \left( \gamma^{(6)}_{4,0} + N \gamma^{(6)}_{4,1} + N^2 \gamma^{(6)}_{4,2} \right)$$

$$+ Q^5 \left( \gamma^{(6)}_{5,0} + N \gamma^{(6)}_{5,1} \right)$$

$$+ Q^6 \left( \gamma^{(6)}_{6,0} \right),$$

which has seven $N$-dependent coefficients. The latter can, in principle, be determined from explicit perturbative results for seven anomalous dimensions corresponding, e.g., to $Q = 1 \ldots 7$. Frames of different color in (7) highlight contributions known from other methods and are discussed in detail in Section IV.

In this paper, we compute the anomalous dimensions for all operators up to $Q = 5$, giving five independent constraints on (7). The remaining two constraints are derived from the semi-classical result (1) of Ref. [8] expanded in $g^* Q$ up to relevant order$^2$:

$$6\text{-loop} : \left( -\frac{572}{243} Q + \frac{2}{729} [10191 - 64N - 2\zeta_3(1327 + 160N) - 2\zeta_5(1441 + 80N) $$

$$- 70\zeta_7(46 + N) - 21\zeta_9(126 + N)] \right) (g^* Q)^6.$$  

As it was noted in Ref. [12], the contribution (8) to the anomalous dimension $\gamma_Q(g^*)$ derived initially under the assumption $g = g^*$ is also valid away from the fixed point, so we can immediately read off the coefficients $\gamma^{(6)}_{6,0}, \gamma^{(6)}_{5,0},$ and $\gamma^{(6)}_{5,1}$ from Eq. (8).

It turns out that the calculation of the operators with charge up to $Q = 4$ is trivial. One can use our six-loop result [14] for the RG functions in the most general renormalizable $\phi^4$

$^2$ We fix a small misprint $2/279 \rightarrow 2/729$ in the published version of Ref. [8]
FIG. 1. Examples of non-factorizable 6-loop diagrams contributing to 1PI five-point function with an operator insertion (denoted by cross). Only integrals similar to the first one require special treatment. All the others are known from the $\phi^4$ renormalization [2].

theory. For example, to compute $\gamma_{Q=4}$ we introduce a coupling $\lambda_4$ for the $\phi^{Q=4}$ operator. To extract the corresponding anomalous dimension from the beta-function $\beta_{abcd}$ of general self-coupling $\lambda_{abcd}$, we substitute

$$\lambda_{abcd} \rightarrow \frac{g}{3}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \lambda_4d_{abcd} \quad (9)$$

and keep only terms that are linear in $\lambda_4$. The beta-function of $\lambda_4$ is related to $\gamma_{Q=4}$ defined in (5) via

$$\beta_{\lambda_4}(g) = \lambda_4\gamma_{Q=4}(g). \quad (10)$$

The cases $Q = 2$ and $Q = 3$ are treated in a similar fashion, i.e., we replace the mass parameter and the trilinear coupling by traceless symmetric tensors with two and three indices

$$m_{ab}^2 \rightarrow \lambda_2d_{ab}, \quad h_{abc} \rightarrow \lambda_3d_{abc}. \quad (11)$$

It is worth mentioning that we routinely utilize FORM [17] to implement traceless condition on $d_{i_1...i_Q}$ (3) and to contract dummy indices.

The case $Q = 5$ deserves special attention. We use DIANA [18] to generate five-point 1PI Green functions with a $\phi^{Q=5}$ insertion. The corresponding Feynman rule again involves the traceless symmetric tensor $d_{abcde}$. After carrying out $O(N)$ algebra and factoring $d_{abcde}$, we are left with scalar integrals multiplied by polynomials in $N$. It is obvious that some of the indices entering $d_{i_1...i_5}$ can be external (see, e.g., Fig. 1). In this case we effectively have a 1PI loop diagram with reduced number of external legs.

To extract the renormalization constant $Z_5$ from the ultraviolet (UV) divergences, we apply $\mathcal{KR}'$-operation to each logarithmically-divergent diagram $G_i$:

$$Z_5^5Z_5 = 1 - \sum_i \mathcal{KR}'G_i, \quad (12)$$
where $Z_\phi$ is a field renormalization constant $\phi_B = Z_\phi \phi$.

The $\mathcal{K}\mathcal{R}'$-operation can be written recursively as

$$
\mathcal{K}\mathcal{R}' G = \mathcal{K} G + \sum_{\gamma} \mathcal{K} \left[ \prod_{\gamma_i \in \gamma} (-\mathcal{K}\mathcal{R} \gamma_i) \ast G / \gamma \right],
$$

(13)

where $G$ is the original diagram, and $\mathcal{K} G$ extracts its singular $O(1/\epsilon)$ part. The sum goes over all sets of disjoint UV-divergent 1PI subgraphs $\{\gamma\} = \bigcup_i \gamma_i$ (with $G$ itself excluded), and $G / \gamma$ is a co-graph obtained from $G$ after shrinking all $\gamma_i$ belonging to $\gamma$. To implement (13) at the six-loop order, one needs to compute all lower-loop counterterms $\mathcal{K}\mathcal{R}' \gamma_i$. Some of the diagrams contain cut-vertices (see, e.g., Fig. 2). In this case $\mathcal{K}\mathcal{R}'$ factorizes. In spite of the fact that these diagrams do not contribute to the anomalous dimension of the operator, we keep them for further crosschecks (see below).

Drastic simplification comes from the application of the infrared (IR) rearrangement trick [19] to the logarithmically divergent diagrams. One can set all but one of the external momenta to zero and re-route the momentum flow in a way to avoid (as much as possible) the appearance of spurious IR divergences. The choice of the IR-safe routing together with the UV-subgraph identification was automated by means of the private computer code.

It turns out that all integrals but one entering $\mathcal{K} G$ and $G / \gamma$ can be calculated with IR-safe non-exceptional external momentum routing in terms of graphical functions [3, 20, 21] implemented in HyperlogProcedures package\(^3\).

There remains a single diagram for which the IR-safe routing leads to an integral not calculable with this approach. Due to this, the external momentum routing was changed at the price of introduction of IR divergent subgraphs. The latter were treated manually via infrared $\mathcal{K}\mathcal{R}'$ operation [22–25]. Further details are provided in Appendix A.

Given $Z_5$, the required anomalous dimension is derived via

$$
\gamma_{Q5} = -\frac{\partial \log Z_5}{\partial \log \mu} = (2\epsilon g - \beta_3) \frac{\partial \log Z_5}{\partial g}.
$$

(14)

As usual only single poles in $\epsilon$ contribute to $\gamma_{Q5}$. However, the crucial crosscheck of the obtained expression is the cancellations of $\epsilon$ poles in the final formula for (14).

\(^3\) Available for download from https://www.math.fau.de/person/oliver-schnetz/
IV. RESULTS

Given large-charge prediction (1) and explicit results for the anomalous dimensions of the first five operators $\phi^Q$, we can fix all the coefficients in $\gamma^{(6)}_Q$:

$$\gamma^{(6)}_Q = \sum_{r=0}^{6} \sum_{s=0}^{6-r} Q^r N^s \gamma^{(6)}_{r,s}, \quad \gamma^{(6)}_{0,0} = 0,$$

as

$$\gamma^{(6)}_{6,0} = \frac{-572}{243},$$

$$\gamma^{(6)}_{5,1} = \frac{-640\zeta_3}{729} - \frac{320\zeta_5}{729} - \frac{140\zeta_7}{729} - \frac{14\zeta_9}{243} - \frac{128}{729},$$

$$\gamma^{(6)}_{5,0} = \frac{-5308\zeta_3}{729} - \frac{5764\zeta_5}{729} - \frac{6440\zeta_7}{729} - \frac{196\zeta_9}{243} + \frac{6794}{243},$$

$$\gamma^{(6)}_{4,2} = \frac{-70\zeta_3^2}{729} + \zeta_3 \left( -\frac{46\zeta_5}{729} - \frac{14}{81} - \frac{100\zeta_5}{729} - \frac{49\zeta_7}{729} + \frac{14}{243} + \frac{7\pi^4}{7290} + \frac{10\pi^6}{137781} + \frac{7\pi^8}{1968300} \right),$$

$$\gamma^{(6)}_{4,1} = \frac{-236\zeta^2_3}{243} - \frac{176\zeta^3_3}{729} + \zeta_3 \left( \frac{1564}{243} - \frac{808\zeta_5}{729} \right) - \frac{490\zeta_5}{243} - \frac{11935\zeta_7}{1458},$$

$$\gamma^{(6)}_{4,0} = \frac{-2368\zeta^2_3}{729} - \frac{2720\zeta^3_3}{729} + \zeta_3 \left( \frac{39340}{729} - \frac{9208\zeta_5}{729} \right) + \frac{26564\zeta_5}{729} + \frac{10451\zeta_7}{729},$$

$$\gamma^{(6)}_{3,3} = \pi^4 \left( \frac{13\zeta_3}{87480} + \frac{1}{4860} \right) + \frac{2\zeta_3}{243} + \frac{65\zeta^2_3}{2916} - \frac{59\zeta_5}{2916} - \frac{25\zeta_7}{2916} - \frac{1}{243} + \frac{25\pi^6}{1102248}.$$
\[
\begin{align*}
\gamma^{(6)}_{3,2} &= \pi^4 \left( \frac{131\zeta_3}{43740} - \frac{133}{87480} \right) + \frac{607\zeta_2^2}{1458} - \frac{8\zeta_3^3}{81} + \zeta_3 \left( \frac{2537}{2916} - \frac{218\zeta_5}{729} \right) - \frac{37\zeta_5}{54} \\
&\quad - \frac{20143\zeta_7}{5832} - \frac{1063\zeta_9}{729} - \frac{335}{729} + \frac{160\pi^6}{137781} + \frac{5417\pi^8}{22049600} \\
\gamma^{(6)}_{3,1} &= \pi^4 \left( \frac{296\zeta_3}{10935} - \frac{49}{729} \right) - \frac{788\zeta_2^2}{729} - \frac{128\zeta_3^3}{81} + \zeta_3 \left( -\frac{12296\zeta_5}{729} - \frac{35633}{1458} \right) \\
&\quad + \frac{512\zeta_5}{243} + \frac{11\zeta_7}{2} - \frac{5738\zeta_9}{729} - \frac{2(7191\zeta_{5,3} + 3275)}{1215} \\
&\quad + \frac{860\pi^6}{137781} + \frac{1213171\pi^8}{275562000} \\
\gamma^{(6)}_{3,0} &= \pi^4 \left( \frac{368\zeta_3}{3645} - \frac{7151}{21870} \right) - \frac{84580\zeta_2^2}{729} + \frac{5152\zeta_3^3}{729} + \zeta_3 \left( \frac{59144\zeta_5}{729} - \frac{44186}{243} \right) \\
&\quad - \frac{59306\zeta_5}{729} - \frac{12529\zeta_7}{46256\zeta_9} + \frac{10768\zeta_{5,3}}{10768\} \\
&\quad + \frac{91750}{243} - \frac{2585\pi^6}{137781} + \frac{727847\pi^8}{34445250} \\
\gamma^{(6)}_{2,4} &= \frac{\zeta_3}{972} + \frac{\zeta_2^2}{729} - \frac{\zeta_5}{486} + \frac{1}{5832} - \frac{\pi^4}{174960} + \frac{\pi^6}{1102240} \\
\gamma^{(6)}_{2,3} &= \pi^4 \left( \frac{53\zeta_3}{43740} - \frac{1}{9720} \right) + \frac{7\zeta_2^2}{36} + \zeta_3 \left( \frac{5\zeta_5}{729} - \frac{631}{5832} \right) - \frac{35\zeta_5}{1944} \\
&\quad - \frac{7735\zeta_7}{11664} - \frac{\zeta_{5,3}}{45} + \frac{1619}{46656} + \frac{263\pi^6}{1102248} + \frac{2063\pi^8}{61236000} \\
\gamma^{(6)}_{2,2} &= \pi^4 \left( \frac{641}{43740} - \frac{49\zeta_3}{21870} \right) - \frac{163\zeta_2^2}{729} + \frac{32\zeta_3^3}{729} - \frac{283\zeta_5}{162} + \zeta_3 \left( \frac{448\zeta_5}{243} - \frac{3535}{972} \right) \\
&\quad + \frac{553\zeta_7}{5832} - \frac{2296\zeta_9}{6561} + \frac{4\zeta_{5,3}}{45} + \frac{3541}{3888} + \frac{241\pi^6}{137781} + \frac{29453\pi^8}{137781000} \\
\gamma^{(6)}_{2,1} &= \pi^4 \left( \frac{5599}{21870} - \frac{1426\zeta_3}{10935} \right) + \frac{12799\zeta_2^2}{729} - \frac{2992\zeta_3^3}{729} - \frac{43871\zeta_5}{1458} + \zeta_3 \left( \frac{42052\zeta_5}{729} + \frac{14243}{486} \right) \\
&\quad - \frac{46361\zeta_7}{972} - \frac{142852\zeta_9}{6561} + \frac{14428\zeta_{5,3}}{14428} + \frac{41047}{5832} + \frac{185\pi^6}{275562} - \frac{1274101\pi^8}{137781000} \\
\gamma^{(6)}_{2,0} &= \pi^4 \left( \frac{3449}{3645} - \frac{3824\zeta_3}{10935} \right) + \frac{380672\zeta_2^2}{729} + \frac{32\zeta_3^3}{729} + \frac{5050\zeta_5}{81} + \zeta_3 \left( \frac{100720\zeta_5}{243} + \frac{22307}{81} \right) \\
&\quad - \frac{320719\zeta_7}{1458} - \frac{596264\zeta_9}{6561} + \frac{26944\zeta_{5,3}}{3367853} + \frac{8146\pi^6}{137781} - \frac{299533\pi^8}{3444525} \\
\gamma^{(6)}_{1,5} &= \frac{-\zeta_3}{11664} + \frac{\zeta_5}{3888} - \frac{1}{23328} - \frac{\pi^4}{699840} \\
\gamma^{(6)}_{1,4} &= \frac{169\zeta_3}{29160} - \frac{\zeta_2^2}{243} + \frac{13\zeta_5}{1458} - \frac{299}{103680} - \frac{37\pi^4}{874800} - \frac{\pi^6}{367416} \\
\gamma^{(6)}_{1,3} &= \pi^4 \left( \frac{1091\zeta_3}{437400} - \frac{493}{437400} \right) - \frac{659\zeta_2^2}{1620} + \zeta_3 \left( \frac{12001}{58320} - \frac{5\zeta_5}{243} + \frac{3389\zeta_5}{14580} \right)
\end{align*}
\]
\[\gamma^{(6)}_{1,2} = \pi^4 \left( \frac{7837\zeta_3}{218700} - \frac{539}{9720} \right) - \frac{4834\zeta_3^2}{3645} + \frac{136\zeta_3^3}{243} + \zeta_3 \left( \frac{96511}{29160} - \frac{340\zeta_5}{729} \right) \]
\[+ \frac{48371\zeta_5}{3645} + \frac{171947\zeta_7}{10935} - \frac{45287\zeta_9}{521\zeta_5,3} + \frac{5321\pi^6}{2187} + \frac{10499}{2160} - \frac{7693807\pi^8}{2755620000} \]
\[= \frac{1636\zeta_7}{729} + \frac{\zeta_5,3}{15} - \frac{10403}{155520} - \frac{137\pi^6}{122472} - \frac{2063\pi^8}{204120000} \]  
\[\gamma^{(6)}_{1,1} = \pi^4 \left( \frac{8158\zeta_3}{54675} - \frac{6853}{10935} \right) - \frac{135304\zeta_3^2}{3645} + \frac{64\zeta_3^3}{27} + \zeta_3 \left( \frac{-8956\zeta_5}{729} - \frac{1256211}{14580} \right) \]
\[+ \frac{424234\zeta_5}{3645} + \frac{14111\zeta_7}{729} - \frac{159772\zeta_9}{290408\zeta_5,3} + \frac{675}{6561} - \frac{1401281\pi^6}{1640\zeta_5,3} \]
\[= \frac{3484\zeta_3}{18225} - \frac{97517}{54675} \right) - \frac{3006466\zeta_3^2}{3645} + \frac{4640\zeta_3^3}{729} + \zeta_3 \left( \frac{-302656\zeta_5}{729} - \frac{1567481}{7290} \right) \]
\[+ \frac{314518\zeta_5}{729} + \frac{696211\zeta_7}{7290} + \frac{4163188\zeta_9}{290408\zeta_5,3} - \frac{675}{6561} \]
\[\gamma^{(6)}_{1,0} = \pi^4 \left( \zeta_3 - \frac{\zeta_5}{15552} - \frac{1}{3888} + \frac{466560}{248832} \right) \]
\[\gamma^{(6)}_{0,5} = \pi^4 \left( \frac{2207}{1749600} - \frac{313\zeta_3}{729} \right) + \frac{3401\zeta_3^2}{14580} + \zeta_3 \left( \frac{10\zeta_5}{729} - \frac{1667}{19440} \right) - \frac{61\zeta_5}{270} \]
\[\gamma^{(6)}_{0,4} = \frac{18341\zeta_7}{11664} - \frac{2\zeta_5,3}{45} + \frac{69623}{933120} + \frac{965\pi^6}{1102248} + \frac{2063\pi^8}{306180000} \]
\[\gamma^{(6)}_{0,3} = \frac{197}{4374} - \frac{667\zeta_3}{36450} + \frac{4334\zeta_3^2}{3645} + \frac{3688\zeta_3^3}{729} + \zeta_3 \left( \frac{-740\zeta_5}{729} - \frac{2297}{9720} \right) - \frac{4519\zeta_5}{405} \]
\[\gamma^{(6)}_{0,2} = \frac{16885\zeta_7}{648} - \frac{103330\zeta_9}{6561} + \frac{100471}{1225} + \frac{11617\pi^6}{4592700000} \]
\[\gamma^{(6)}_{0,1} = \frac{38221}{87480} - \frac{836\zeta_3}{18225} + \frac{78049\zeta_3^2}{3645} - \frac{3392\zeta_3^3}{729} + \zeta_3 \left( \frac{112751}{14580} - \frac{664\zeta_5}{243} \right) \]
\[\gamma^{(6)}_{0,0} = \pi^4 \left( \frac{3148\zeta_3}{54675} + \frac{242993}{218700} \right) + \frac{504412\zeta_3^2}{1215} - \frac{2368\zeta_3^3}{243} + \frac{232190\zeta_5}{729} \]
\[= \frac{68848\zeta_5}{729} + \frac{550921}{7290} - \frac{1736897\zeta_7}{2430} + \frac{1161368\zeta_9}{2187} + \frac{359144\zeta_5,3}{2025} \]
Evaluating the anomalous dimension at the fixed point, we obtain the $\epsilon$-expansion of the scaling dimension $\Delta_Q$ (6). We reproduce the five-loop results [13]. The new six-loop coefficients are given by

\[ P^{(6)}_{6,6} = -1716, \]
\[ P^{(6)}_{5,7} = -38400, \]
\[ P^{(6)}_{5,6} = -2(94\zeta_3 + 1602\zeta_5 + 2660\zeta_7 + 2478\zeta_9 - 16463), \]
\[ P^{(6)}_{5,5} = -2(320\zeta_3 + 160\zeta_5 + 70\zeta_7 + 21\zeta_9 + 64), \]
\[ P^{(6)}_{4,8} = -529200, \]
\[ P^{(6)}_{4,7} = -24(2212\zeta_3 + 3500\zeta_5 + 4725\zeta_7 - 27264), \]
\[ P^{(6)}_{4,6} = 3552\zeta_3^2 - 1312\zeta_3^3 + 51124\zeta_5 - 4\zeta_3(1422\zeta_5 + 935) \]
\[ + 86975\zeta_7 + \frac{257848\zeta_9}{9} - 265526, \]
\[ P^{(6)}_{4,5} = 412\zeta_3^2 - 176\zeta_3^3 + 1930\zeta_5 - 24\zeta_3(3\zeta_5 - 437) \]
\[ - \frac{9107\zeta_7}{2} - \frac{42412\zeta_9}{9} + 1898 + \frac{14\pi^4}{15} + \frac{20\pi^6}{27} + \frac{7\pi^8}{50}, \]
\[ P^{(6)}_{4,4} = -70\zeta_3^2 - 100\zeta_5 - 2\zeta_3(23\zeta_5 + 63) - 49\zeta_7 + 42 + \frac{7\pi^4}{10} + \frac{10\pi^6}{189} + \frac{7\pi^8}{2700}, \]
\[ P^{(6)}_{3,9} = -6048000, \]
\[ P^{(6)}_{3,8} = -5040(301\zeta_3 + 275\zeta_5 - 1602), \]
\[ P^{(6)}_{3,7} = -79776\zeta_3^2 + \zeta_3(951996 - 95040\zeta_5) + 1306920\zeta_5 + 704025\zeta_7 - 4540356, \]
\[ P^{(6)}_{3,6} = -33870\zeta_3^2 + 9760\zeta_3^3 - 425010\zeta_5 + 24\zeta_3(1973\zeta_5 + 856) - \frac{1029567\zeta_7}{2} - \frac{152896\zeta_9}{9}, \]
\[ - \frac{23328\zeta_5}{5} + 1148614 + \frac{386\pi^4}{5} + \frac{1100\pi^6}{63} + \frac{1044\pi^8}{875}, \]
\[ P^{(6)}_{3,5} = -7984\zeta_3^2 + \frac{1}{60}\pi^4(336\zeta_3 - 493) + 17146\zeta_5 - \frac{3}{2}\zeta_3(5072\zeta_5 + 44331) \]
\[ + \frac{286097\zeta_7}{4} + 11270\zeta_9 - \frac{23706\zeta_5}{5} - 10492 - \frac{2150\pi^6}{189} + \frac{14419\pi^8}{42000}, \]
\[ P^{(6)}_{3,4} = 986\zeta_3^2 - 72\zeta_3^3 - \frac{1}{24}\pi^4(10\zeta_3 + 221) + \zeta_3 \left( \frac{5993}{4} - 218\zeta_5 \right) \]
\[ + \frac{999\zeta_5}{2} - \frac{18943\zeta_7}{8} - 1063\zeta_9 - 243\zeta_5 - \frac{939}{2} + \frac{115\pi^6}{756} + \frac{5417\pi^8}{30240}. \]
\[ P_{3,3}^{(6)} = \frac{1}{120} \pi^4 (13 \zeta_3 + 18) + \frac{1}{4} (24 \zeta_3 - 65 \zeta_3^2 - 59 \zeta_5 - 25 \zeta_7 - 12) + \frac{25 \pi^6}{1512}, \]  
\[ P_{2,10}^{(6)} = -68040000, \]  
\[ P_{2,9}^{(6)} = -6048000(51 \zeta_3 - 142), \]  
\[ P_{2,8}^{(6)} = -252 \left( -114928 \zeta_3 + 9072 \zeta_3^2 - 18200 \zeta_5 + 195133 \right), \]  
\[ P_{2,7}^{(6)} = 6 \left( 219240 \zeta_3^2 + 21 \zeta_3 (3360 \zeta_5 - 72017) + 351 \pi^4 \right) - 30(187650 \zeta_5 + 33957 \zeta_7 - 538628), \]  
\[ P_{2,6}^{(6)} = 74436 \zeta_3^2 - 21856 \zeta_3^3 + \frac{1}{5} \pi^4 (648 \zeta_3 - 6619) + \zeta_3 (447674 - 176400 \zeta_5) - \frac{760 \pi^6}{9} \]  
\[ + 2091276 \zeta_3 - \frac{1646491 \zeta_7}{2} + \frac{923144 \zeta_9}{9} + \frac{235872 \zeta_{5,3}}{5} - \frac{2793137}{1508 \pi^8}, \]  
\[ P_{2,5}^{(6)} = 27831 \zeta_3^2 + 2480 \zeta_3^3 - \frac{2}{15} \pi^4 (297 \zeta_3 - 844) - \frac{476097 \zeta_5}{2} + \zeta_3 \left( 46708 \zeta_5 + \frac{361793}{2} \right) - \frac{790853 \zeta_7}{4} + \frac{224572 \zeta_9}{9} + \frac{109116 \zeta_{5,3}}{5} + \frac{52581}{2} + \frac{24205 \pi^6}{378} - \frac{40673 \pi^8}{9000}, \]  
\[ P_{2,4}^{(6)} = -4162 \zeta_3^2 + 32 \zeta_3^3 + \frac{7}{60} \pi^4 (2 \zeta_3 + 351) + 6263 \zeta_5 + \zeta_3 \left( 1704 \zeta_5 - \frac{11473}{2} \right) + \frac{133063 \zeta_7}{8} - \frac{22964 \zeta_9}{9} + \frac{2268 \zeta_{5,3}}{5} + \frac{14669}{8} - \frac{385 \pi^6}{54} - \frac{11707 \pi^8}{27000}, \]  
\[ P_{2,3}^{(6)} = \frac{1}{80} (8780 \zeta_3^2 + 6390 \zeta_5 + 100 \zeta_3 (4 \zeta_5 - 63) - 38675 \zeta_7 - 1296 \zeta_{5,3} + 2175) - \frac{1}{120} \pi^4 (106 \zeta_3 + 161) + \frac{11 \pi^6}{72} + \frac{2063 \pi^8}{84000}, \]  
\[ P_{2,2}^{(6)} = \frac{3 \zeta_3}{4} + \zeta_3^2 - \frac{3 \zeta_5}{2} + \frac{1}{8} - \frac{\pi^4}{240} + \frac{\pi^6}{1512}, \]  
\[ P_{1,11}^{(6)} = -1020600000, \]  
\[ P_{1,10}^{(6)} = -272160000(18 \zeta_3 - 41), \]  
\[ P_{1,9}^{(6)} = -7560 \left( -77856 \zeta_3 + 5184 \zeta_3^2 + 16800 \zeta_5 + 71741 \right), \]  
\[ P_{1,8}^{(6)} = 126 \left( 324144 \zeta_3^2 + 610860 \zeta_5 + 1250399 + 270 \pi^4 \right) - 126 (30 \zeta_3 (2016 \zeta_3 + 73393) + 66150 \zeta_7), \]  
\[ P_{1,7}^{(6)} = -13852800 \zeta_3^2 + 50688 \zeta_3^3 + \frac{1296}{5} \pi^4 (9 \zeta_3 - 133) - 6666360 \zeta_5 + 24 \zeta_3 (110880 \zeta_3 + 2632463) - 2469033 \zeta_7 + 966080 \zeta_9 + \frac{497664 \zeta_{5,3}}{5} - \frac{121888917}{4} + 1200 \pi^6 - \frac{22272 \pi^8}{875}, \]  
\[ P_{1,6}^{(6)} = \pi^4 \left( \frac{24247}{2} - \frac{8172 \zeta_3}{5} \right) + 1369356 \zeta_3^2 - 41696 \zeta_3^3 - 3917542 \zeta_5. \]
\[-\frac{3}{2}\zeta_3(155136\zeta_5 + 4083031) + \frac{5677959\zeta_7}{2} - \frac{8721100\zeta_9}{9} - \frac{1313064\zeta_{5,3}}{5} + \frac{28571673}{8} - \frac{24965\pi^6}{63} + \frac{782339\pi^8}{10500},\]

\[P_{1,5}^{(6)} = -122928\zeta_3^2 + 5696\zeta_3^3 + \frac{4}{15}\pi^4(951\zeta_3 - 5767) + \zeta_3\left(\frac{249699}{4} - 44604\zeta_5\right) + 878664\zeta_5 - \frac{1581769\zeta_9}{4} + \frac{997516\zeta_9}{9} - \frac{24138\zeta_{5,3}}{5} - \frac{59181}{2} - \frac{1660\pi^6}{21} + \frac{68489\pi^8}{14000},\]

\[P_{1,4}^{(6)} = \pi^4\left(\frac{61}{4} - \frac{35\zeta_3}{12}\right) + \frac{21937\zeta_3^2}{2} + 664\zeta_3^3 + \zeta_3\left(\frac{831}{8} - 2188\zeta_5\right) - \frac{101553\zeta_5}{2} - \frac{31659\zeta_7}{8} + \frac{194069\zeta_9}{9} + \frac{6237\zeta_{5,3}}{5} - \frac{77953}{32} + \frac{9467\pi^6}{378} - \frac{14767\pi^8}{108000},\]

\[P_{1,3}^{(6)} = -\frac{1415\zeta_3^2}{4} + \frac{1}{120}\pi^4(247\zeta_3 - 5) + \zeta_3\left(\frac{6225}{16} - 15\zeta_5\right) + \frac{2645\zeta_5}{4} + 1724\zeta_7 + \frac{243\zeta_{5,3}}{5} - \frac{10273}{128} - \frac{1733\pi^6}{1512} - \frac{2063\pi^8}{28000},\]

\[P_{1,2}^{(6)} = -4\zeta_3 - 3\zeta_3^2 - \zeta_5 - \frac{383}{256} + \frac{13\pi^4}{120} - \frac{\pi^6}{504},\]

\[P_{1,1}^{(6)} = \frac{1}{960}\left(-30(2\zeta_3 - 6\zeta_5 + 1) - \pi^4\right),\]

\[P_{0,11}^{(6)} = 1020600000,\]

\[P_{0,10}^{(6)} = 3402000(144\zeta_3 - 323),\]

\[P_{0,9}^{(6)} = 15120\left(-37968\zeta_3 + 2592\zeta_3^2 + 8400\zeta_5 + 33763\right),\]

\[P_{0,8}^{(6)} = -126\left(309888\zeta_3^2 + 661460\zeta_5 + 1065945 + 270\pi^4\right) + 126(2\zeta_3(30240\zeta_5 + 1055743) + 66150\zeta_7),\]

\[P_{0,7}^{(6)} = 12961872\zeta_3^3 - 50688\zeta_3 - \frac{486\pi^4}{5}(24\zeta_3 - 343) + 12986880\zeta_5 - 18\zeta_3(166080\zeta_3 + 3400493) + 2778048\zeta_7 - 966080\zeta_9 - \frac{497664\zeta_{5,3}}{5} + \frac{87424317}{4} - 1200\pi^6 + \frac{22272\pi^8}{875},\]

\[P_{0,6}^{(6)} = -1493358\zeta_3^2 + 55104\zeta_3^3 + \frac{1}{10}\pi^4(15048\zeta_3 - 116131) + 1861686\zeta_5 + \zeta_3(367440\zeta_5 + 6850142) - \frac{6511113\zeta_7}{2} + \frac{3194632\zeta_9}{3} + 220104\zeta_{5,3} - \frac{32994733}{16} + \frac{30445\pi^6}{63} - \frac{133639\pi^8}{2100},\]

\[P_{0,5}^{(6)} = 109509\zeta_3 - 8000\zeta_3^2 - \frac{1}{30}\pi^4(6588\zeta_3 - 48821) - 655038\zeta_5 + \zeta_3\left(5576\zeta_5 - \frac{608795}{2}\right) + \frac{2210257\zeta_7}{4} - \frac{1280728\zeta_9}{9} - \frac{61272\zeta_{5,3}}{5} + \frac{892443}{32} + \frac{7055\pi^6}{378} - \frac{26969\pi^8}{31500}.,\]
\[ p_{0,4}^{(6)} = -\frac{15819\zeta_3^2}{2} + \frac{624\zeta_3^3}{8} + \frac{1}{80}\pi^4(248\zeta_3 - 5521) + \frac{95231\zeta_5}{2} + \zeta_3 \left(748\zeta_5 + \frac{80693}{8}\right) \]
\[ -\frac{106765\zeta_7}{8} - \frac{53846\zeta_9}{3} - 1458\zeta_{5,3} + \frac{34005}{32} - \frac{4493\pi^6}{252} + \frac{8161\pi^8}{5040}, \]  
\[ p_{0,3}^{(6)} = -\frac{1073e^2}{4} + \frac{7}{240}\pi^4(44\zeta_3 - 73) - 917\zeta_5 + \zeta_3 \left(10\zeta_5 - \frac{3807}{8}\right) - \frac{19749\zeta_7}{16} - \frac{162\zeta_{5,3}}{5} \]
\[ + \frac{1693}{32} + \frac{173\pi^6}{168} + \frac{2063\pi^8}{42000}, \]  
\[ p_{0,2}^{(6)} = -\frac{111\zeta_3}{16} + 2\zeta_3^3 + \frac{5\zeta_5}{2} + \frac{343}{512} - \frac{59\pi^4}{480} + \frac{\pi^6}{756}, \]  
\[ p_{0,1}^{(6)} = -\frac{240\zeta_3 - 960\zeta_5 + 15 + 8\pi^4}{5120}. \]  

Provided results for coefficients \( \gamma_{r,s}^{(l)} \) and \( P_{r,p}^{(l)} \) in addition to the Riemann zeta functions \( \zeta_n = \sum_{i=1}^{\infty} 1/i^n \) contain multiple zeta value \( \zeta_{5,3} = \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} 1/(i^5 j^3) \approx 0.0377077 \).

To verify our expressions we consider various limits of Eq. (5). For example, defining \( J = Q/N \) we get an expansion in \( J \). Neglecting terms further suppressed by large \( N \), we obtain (for \( g = g^* \))

\[ \frac{A_Q}{Q} \approx 1 - \varepsilon + \sum_{l=1}^{\infty} (6\varepsilon)^l \sum_{k=1}^{1} j^k \gamma_{k,l-k}^{(l)} + O \left( \frac{1}{N} \right) \]
\[ = 1 - \varepsilon + \sum_{l=1}^{\infty} (2\varepsilon)^l \sum_{k=1}^{l} j^k P_{k,k}^{(l)} + O \left( \frac{1}{N} \right) \]
\[ (90) \]

with

\[ P_{k,k}^{(l)} = \gamma_{k,l-k}^{(l)}. \]  
\[ (91) \]

It turns out that the expansion of \( \Delta_Q \) in small \( J \) for large \( N \) can be found for a general dimension \( d \) [26]:

\[ \frac{\Delta Q}{Q} = \frac{d}{2} - 1 + h_2(d)j + h_3(d)j^2 + \ldots, \]
\[ (92) \]

where, e.g.,

\[ h_2(d) = -\frac{2^{d-3}d \sin \left( \frac{\pi d}{2} \right) \Gamma \left( \frac{d-1}{2} \right)}{\pi^{3/2} \Gamma \left( \frac{d}{2} + 1 \right)}. \]
\[ (93) \]

We follow [26] to compute the \( \varepsilon \)-expansion of \( h_i(d) \), for \( i = 2...7 \), and find perfect agreement with our perturbative result (90). In this way, we check \( \gamma_{r,b-r}^{(6)} \) for \( r = 1...6 \). 

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In addition, we also compare our result with the first two non-trivial orders of large-$N$ expansion[6], which begins as
\[
\frac{\Delta Q}{Q} = \frac{d}{2} - 1 + \frac{h_2(d)}{N} \left( Q - 2 + \frac{4}{d} \right) + \ldots,
\] (94)

At six loops only five coefficients of Eq. (5) contribute to large-$N$ expansion of our result (6) up to $O(1/N^2)$. Two of them $\gamma^{(6)}_{1,5}$ and $\gamma^{(6)}_{2,4}$ also enter (92), while the comparison with $\varepsilon$-expansion of (94) provides additional checks for $\gamma^{(6)}_{0,6}, \gamma^{(6)}_{1,4}, \gamma^{(6)}_{0,5}$.

V. DISCUSSION AND CONCLUSION

In this paper we derived the six-loop anomalous dimension of the charged $\phi^Q$ operator in the $O(N)$ model. Our computation was based on the combination of semi-classical results and explicit diagram calculations. Our expression up to five loops coincides with that obtained recently in Refs. [13, 27].

The utilized approach relies on diagram-by-diagram computation of Feynman graphs with an $\phi^5$ insertion, and, thus, facilitates further six-loop studies of the charged operators in models with other symmetries once semi-classical results are available.

Given our result, one can apply various resummation techniques to the $\varepsilon$-expansion of $\Delta_Q$ and obtain numerical values of scaling dimension in a bunch of three-dimensional $O(N)$ models.

| $N$ \ $Q$ | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-----------|------|------|------|------|------|------|------|------|------|------|
| 2         | 0.5187(5) | 1.2351(6) | 2.1085(7) | 3.112(2) | 4.230(3) | 5.45(1) | 6.75(2) | 8.15(2) | 9.62(3) | 11.17(3) |
| 4         | 0.5181(5) | 1.1885(3) | 1.985(14) | 2.892(3) | 3.896(7) | 4.99(2) | 6.16(3) | 7.40(3) | 8.72(4) | 10.10(4) |
| 6         | 0.5163(5) | 1.1537(13) | 1.895(3) | 2.728(5) | 3.643(11) | 4.63(2) | 5.70(3) | 6.82(3) | 8.01(4) | 9.26(4) |
| 8         | 0.5146(4) | 1.127(2) | 1.827(5) | 2.604(8) | 3.45(2) | 4.36(2) | 5.34(3) | 6.37(3) | 7.46(4) | 8.60(4) |

TABLE I. Scaling dimensions of $\phi^Q$ ($d = 3$) obtained by resumming six-loop result for $N = 2, 4, 6, 8$.

Based on the fact that we are working at the same order of perturbation theory and in the same theory, we use in straightforward manner the advanced technique of Ref. [2] to compute numerical values and uncertainties of $\Delta_Q$ for $Q = 1 \ldots 10$ and $N = 2, 4, 6, 8$ (see Table I). The latter can be compared to the Monte-Carlo results [28–30].
It is interesting to study how our computation matches the large $Q$ expansion \[31\]
\[
\Delta_Q = c_{3/2}Q^{3/2} + c_{1/2}Q^{1/2} + c_0 + \mathcal{O}(Q^{-1/2})
\] (95)

Here $c_{3/2}, c_{1/2}$ are $N$-dependent constants, while $c_0 \approx -0.0937254$ originates from Casimir energy and is universal. In Ref. \[32\] the large-$N$ limit was considered and the following predictions were obtained
\[
c_{3/2} = \frac{2}{3}N^{-1/2}, \quad c_{1/2} = \frac{1}{6}N^{1/2}
\] (96)

which are valid for $1 \ll Q \ll N \ll Q^2$. We follow \[28, 30\] and fit our numerical data to

| $N$ | $c_{3/2}$ | $c_{1/2}$ | $c_{-1/2}$ |
|-----|-----------|-----------|------------|
| 2   | 0.3324(6) | 0.271(3)  | 0.009(3)   |
| 4   | 0.2925(4) | 0.3241(13)| -0.0048(11)|
| 6   | 0.2612(6) | 0.371(3)  | -0.022(2)  |
| 8   | 0.2371(9) | 0.408(3)  | -0.036(3)  |

TABLE II. Fit from resummation results using $Q = 1 \ldots 6$ expressions with $c_0 = -0.0937$ fixed and additional free parameter $c_{-1/2}$ in $c_{-1/2}Q^{-1/2}$ term appended to the ansatz (95).

![FIG. 3. Leading coefficients in large-charge expansion $c_{3/2}$ and $c_{1/2}$ as functions of $N$. We also add results of Monte-Carlo simulation for $N = 2, 4, 6, 8$ from Ref. \[30\] (MC-1), for $N = 2$ from Ref. \[28\] (MC-2) and for $N = 4$ from Ref. \[29\] (MC-3).]
The extracted coefficient for a fixed $c_0$ are given in Table II. One can see satisfactory agreement between our values and Monte-Carlo results [28–30] (see Fig. 3). While our fits for $c_{3/2}$ are close to the Monte-Carlo values and approaches large-$N$ limit (96), we see a rather big discrepancies in $c_{1/2}$. We expect that our error estimates according to Ref. [2] may be too optimistic and require further studies. Nevertheless, we see that according to Eq. (96) the coefficient $c_{3/2}$ decreases with $N$, while $c_{1/2}$ increases. We believe our result contributes important information to CFT data (see, e.g.,[27] for review). While we rely on non-perturbative large $Q$ expressions to fix part of the coefficients in the anzats (7), it would be interesting to reproduce them independently, e.g., by considering Green functions with $\phi^{Q=6}$ and $\phi^{Q=7}$ insertions. We postpone this verification for the future.

ACKNOWLEDGMENTS

We thank O. Antipin, J. Henriksson, S. Kosvou, M.Kompaniets, A. Kudlis, and N.Lebedev for fruitful discussions. Furthermore, we are grateful to the Joint Institute for Nuclear Research for using their supercomputer “Govorun.”

Appendix A: Details of calculation with $\mathcal{KR}^*$ operation

Here, we present the expression used for manual computation of a single diagram by means of $\mathcal{KR}^*$ operation. We closely follow the notation of Refs. [24, 25]. One can see that there is only one non-trivial IR-divergent counterterm

$$\gamma_{\text{IR}} = \left( \gamma_i \right) = \frac{1}{\epsilon},$$

which cancels spurious IR divergences appearing in $\Gamma_i = G/\gamma_i$ for a UV subgraph $\gamma_i$. The graph $\tilde{\Gamma}_i = \Gamma_i \setminus \gamma_{\text{IR}}$ is obtained from $\Gamma_i$ by deleting lines and internal vertices (denoted by filled dots) of $\gamma_{\text{IR}}$. One can see that even if $\Gamma_i$ can be zero in dimensional regularization the corresponding IR counterterm can give a non-trivial contribution to the final result. All $\mathcal{KR}^*(\gamma_i)$ are known from lower-loop calculations, while $G$, $\tilde{G}$, $\Gamma_i$, and $\tilde{\Gamma}_i$ calculated with
HyperlogProcedures.

\[ KR' = K \left\{ \begin{array}{c} G \\ \hat{G} \end{array} \right\} + \left\{ \begin{array}{c} \gamma_1 \\ \gamma_1 \\ \gamma_2 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_5 \\ \gamma_6 \\ \gamma_6 \\ \gamma_7 \\ \gamma_7 \end{array} \right\} \cdot \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_7 \end{array} \right) \]
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