SQUARE FUNCTION AND MAXIMAL FUNCTION ESTIMATES FOR OPERATORS BEYOND DIVERGENCE FORM EQUATIONS

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Abstract. We prove square function estimates in $L^2$ for general operators of the form $B_1 D_1 + D_2 B_2$, where $D_i$ are partially elliptic constant coefficient homogeneous first order self-adjoint differential operators with orthogonal ranges, and $B_i$ are bounded accretive multiplication operators, extending earlier estimates from the Kato square root problem to a wider class of operators. The main novelty is that $B_1$ and $B_2$ are not assumed to be related in any way. We show how these operators appear naturally from exterior differential systems with boundary data in $L_2$. We also prove non-tangential maximal function estimates, where our proof needs only off-diagonal decay of resolvents in $L^2$, unlike earlier proofs which relied on interpolation and $L^p$ estimates.

1. Introduction

In this paper, we generalize the square function estimates from the Kato square root problem, to a wider class of operators on $L^2(\mathbb{R}^n; \mathbb{C}^N)$, $n, N \geq 1$. Previously, estimates were known for perturbations of a homogeneous first order constant coefficients self-adjoint partial differential operator $D$, of the form $DB$ or $BD$, with $B$ being a bounded multiplication operator which is accretive on the range of $D$. Let us first recall how such operators appear in connection with divergence form equations. The celebrated Kato square root estimate

$$\|\sqrt{-\text{div} A \nabla u}\|_2 \approx \|\nabla u\|_2$$

for divergence form operators with general bounded accretive coefficients $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$, were proved in one dimension by Coifman, McIntosh and Meyer [9] and in full generality by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [6]. It can be written $\|\sqrt{(BD)^2} [u, 0]\|_2 \approx \|BD[u, 0]\|_2$, with $D = \begin{bmatrix} 0 & \text{div} \\ -\nabla & 0 \end{bmatrix}$ and $B = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$, acting on vectors of dimension $N = 1 + n$. This estimate is in turn a consequence of square function estimates for the operator $BD$. This approach to the Kato square root estimate was developed in [7, 8].

Operators of the form $DB$ and $BD$ appear not only in connection with divergence form operators on $\mathbb{R}^n$, but also in connection with divergence form equations on the half-space $\mathbb{R}^{1+n}_+ := \{(t, x) ; t > 0, x \in \mathbb{R}^n\}$, with $L^2(\mathbb{R}^n)$ or $\dot{H}^1(\mathbb{R}^n)$ boundary data. We recall the following approach to boundary value problems from [5, 3]. Consider a divergence form equation

$$\text{div}_{t,x} A(t, x) \nabla_{t,x} u(t, x) = 0, \quad t > 0, x \in \mathbb{R}^n,$$
with coefficients $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, splitting $C^{1+n} = C \oplus C^n$. Write $e_0, e_1, \ldots, e_n$ for the standard basis in $\mathbb{R}^{1+n}$, with coordinates $x_0 = t, x_1, \ldots, x_n$, and $f_\bot := e_0 \cdot f$ for the normal component and $f_t := f - f_\bot e_0$ for the tangential part.

On the one hand, at the level of $H^1(\mathbb{R}^n)$ boundary data $u|_{\mathbb{R}^n}$, we consider the conormal gradient

$$f = \begin{bmatrix} f_\bot \\ f_t \end{bmatrix} := a\partial_t u + b\nabla_x u \quad \nabla_x u.$$

In terms of $f$, the divergence form equation is $\partial_t f_\bot + \text{div}_x (c(a^{-1}f_\bot - a^{-1}bf_\bot) + df_\parallel) = 0$. The conormal gradient $f$, with the inward conormal derivative as normal component $f_\bot$, is in one-to-one correspondence with the potential $u$, modulo curl-freeness and constants. Written in terms of $f$, the curl-free condition is $\partial_t f_\parallel = \nabla_x (a^{-1}f_\bot - a^{-1}bf_\bot)$, $\text{curl}_x f_\parallel = 0$. In vector notation this means that the divergence form equation for $u$ is equivalent to the vector valued ordinary differential equation

$$\partial_t f + DB f = 0$$

for $f$ under the constraint $f \in \overline{R(D)}$ for each $t > 0$, with $D := \begin{bmatrix} 0 & \text{div}_x \\ -\nabla_x & 0 \end{bmatrix}$ acting along $\mathbb{R}^n$ and $B := \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix}$ being a multiplication operator, which turns out to be accretive if and only if $A$ is so.

On the other hand, at the level of $L^2(\mathbb{R}^n)$ boundary data $u|_{\mathbb{R}^n}$, we can write

$$f_\bot = (A\nabla u)_\bot := \text{div}_x v_\parallel,$$

with a tangential vector field $v_\parallel$, for each fixed $t$, assuming appropriate decay of $u$ at infinity, since $\int_{\mathbb{R}^n} f_\bot dx = 0$ by the divergence theorem. Inserting this ansatz into the divergence form equation and commuting $\partial_t$ and $\text{div}_x$ yields $\text{div}_x (\partial_tv_\parallel + (A\nabla_t u)_\parallel) = 0$. Since $v_\parallel$ is only defined modulo tangent divergence free vector fields, we may choose it so that $\partial_tv_\parallel + (A\nabla_t u)_\parallel = 0$. In vector notation this means that the divergence form equation for $u$ is equivalent to the vector valued ordinary differential equation

$$\partial_t v + BD v = 0,$$

for the vector field

$$v = \begin{bmatrix} v_\parallel \\ v_t \end{bmatrix} := \begin{bmatrix} -u \\ \text{div}_x^{-1}(a\partial_t u + b\nabla_x u) \end{bmatrix}.$$

One can view both $\partial_t f + DB f = 0$ and $\partial_t v + BD v = 0$ as generalized Cauchy–Riemann systems. In particular, the $n$ components of $v_\parallel$ should be viewed as some generalized harmonic conjugate functions.

Estimates of operators of the form $DB$ or $BD$ are by now well understood, see [S] [3] [4]. The aim of this paper is to prove fundamental estimates for more general operators of the form

$$B_1D_1 + D_2B_2,$$

which appear for example when, similar to above, writing a more general exterior differential system as a vector valued ordinary differential equation in the variable transversal to the boundary. See Section [3] We assume that $D_1D_2 = 0$ but, unlike earlier results [4], not that $D_2B_2B_1D_1 = 0$. 
We next formulate our results in detail. Consider four operators $D_1, D_2, B_1$ and $B_2$ acting in the Hilbert space $L_2(R^n; C^N)$ with norm $\| \cdot \|_2$, where $n, N \geq 1$. We assume the following.

- The operators $D_1$ and $D_2$ are constant coefficient homogeneous first order differential operators which are self-adjoint and such that $R(D_i) \subset N(D_i)$. Assume the partial ellipticity estimates $\|D_if\|_2 \gtrsim \|f\|_{H^1(R^n)}$ for all $f \in R(D_1), i = 1, 2$.
- The operators $B_1$ and $B_2$ are bounded multiplication operators $B_i: f(x) \mapsto B_i(x)f(x), x \in R^n$, where $B_i(\cdot) \in L_\infty(R^n; L(C^N)), i = 1, 2$. Assume the partial accretivity estimates $\Re(B_if, f) \gtrsim \|f\|_2^2$ for all $f \in R(D_i), i = 1, 2$.

Denote by $\omega_i := \sup_{f \in R(D_i) \setminus \{0\}} |\arg(B_if, f)| < \pi/2$ the angle of accretivity for $B_i$ on $R(D_1), i = 1, 2$. For $0 \leq \alpha < \pi/2$, define the closed sectors $S_{\alpha+} := \{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \alpha \} \cup \{0\}$ and $S_{\alpha-} := -S_{\alpha+}$, and bisectors $S_{\alpha} := S_{\alpha+} \cup S_{\alpha-}$, as well as the corresponding open sectors/bisectors $S_{\alpha+}^o, S_{\alpha}^o$, being the interior of $S_{\alpha+}, S_{\alpha}$ respectively.

Denote by $R(\cdot), N(\cdot)$ and $D(\cdot)$ the range, null space and domain of an operator. Define the operator

$$T := B_1D_1 + D_2B_2, \quad D(T) := \{ f \in D(D_1) \mid B_2f \in D(D_2) \}.$$ 

The definition of operators $\psi(tT), \phi(tT)$ in the functional calculus of $T$, is found in Section [2]. For a function $h$ on $R_1^{+n}$, define the ($L_2$ Whitney averaged) nontangential maximal function

$$\tilde{N}_*h(x) := \sup_{t>0} \left( \int \int_{W(t,x)} |h(s,y)|^2 \frac{dsdy}{|W(t,x)|} \right)^{1/2}, \quad x \in R^n,$$

where $W(t,x)$ denotes a Whitney region around $(t, x)$, for example $W(t,x) = B(x,t) \times (t/2, 2t)$.

**Theorem 1.1.** Under the above hypothesis, $T$ is a closed and densely defined operator in $L_2(R^n; C^N)$, with spectrum $\sigma(T) \subset S_\omega$, where $\omega := \max(\omega_1, \omega_2)$, and resolvent estimates $\| (\lambda - T)^{-1} \| \lesssim 1/\text{dist}(\lambda, \sigma(T))$.

Moreover, the following estimates of holomorphic functions of the operator $T$ hold.

- We have square function estimates

$$\int_{R^n} \int_0^\infty |\psi(tT)f(x)|^2 \frac{dt dx}{t} \lesssim \int_{R^n} |f(x)|^2 dx, \quad f \in R(T),$$

for any holomorphic symbol $\psi : S^o_{\mu} \rightarrow C, \omega < \mu < \pi/2$, with estimates $|\psi(\lambda)| \lesssim \min(|\lambda|^s, |\lambda|^{-s})$ for some $s > 0$.

If furthermore $\psi|_{S_{\omega+}}$ and $\psi|_{S_{\omega-}}$ are not identically zero, then the reverse square function estimates $\gtrsim$ hold for all $f \in R(T)$.

- We have non-tangential maximal function estimates

$$\int_{R^n} |\tilde{N}_*(\phi(tT)f)(x)|^2 dx \approx \int_{R^n} |f(x)|^2 dx, \quad f \in R(T),$$

for any holomorphic symbol $\phi : S^o_{\mu} \rightarrow C, \omega < \mu < \pi/2$, with estimates $|\phi'(\lambda)| \lesssim |\lambda|^s$ and $|\phi(\lambda)| \lesssim |\lambda|^{-s}$, for some $s > 0$. 

The estimates in Theorem 1.1 go back to the techniques from the solution of the Kato square root problem by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [6]. The connection between the Kato square root problem and square function estimates for first order differential operators was developed by Auscher, McIntosh and Nahmod [7]. More directly, both the square function and non-tangential maximal function estimates in Theorem 1.1 build on the author’s joint work [3] with Auscher and Hofmann.

So far, the Kato techniques have been applied to establish square function estimates for three main classes of first order differential operators.

1. Operators of the form $DB$ and $BD$, with $D$ being a self-adjoint constant coefficient homogeneous first order differential operator, and $B$ being a bounded multiplication operator which is accretive on $R(D)$.

2. Operators of the form $\Gamma + B_1\Gamma^*B_2$, with $\Gamma$ being a nilpotent (that is $\Gamma^2 = 0$) constant coefficient homogeneous first order differential operator, and $B_1, B_2$ being bounded multiplication operators such that $\Gamma^*B_2B_1\Gamma^* = 0 = \Gamma B_1B_2\Gamma$, which are accretive on $R(\Gamma^*)$ and $R(\Gamma)$ respectively.

3. Operators of the form $B_1D_1 + D_2B_2$, with $D_1, D_2$ being self-adjoint constant coefficient homogeneous first order differential operators with orthogonal ranges, and $B_1, B_2$ being bounded multiplication operators such that $D_2B_2B_1D_1 = 0$, which are accretive on $R(D_1)$ and $R(D_2)$ respectively.

Note that for all these classes of operators, there is essentially only one multiplication operator $B$ as we are considering a small generalization of the case $B_2 = B_1^{-1}$. This is in constrast to Theorem 1.1 where the two multiplication operators $B_1$ and $B_2$ are independent of each other.

The main example of operators of type (2) are Hodge–Dirac operators, with $\Gamma$ being the exterior derivative acting on differential forms, see Section 3. For operators of type (2), square function estimates were proved in [3, Thm. 2.7]. By a simple operator theoretic argument, square function estimates for operators of type (1) follow from such estimates of operators of type (2), as shown in [3, Thm. 3.1]. In fact this argument can be reversed. It was shown in [4, Sec. 8.1] that conversely square function estimates for operators of type (2) follow from such estimates for operators of type (1). Operators of type (3) are nothing but a direct sum of two operators of type (1), and hence square function estimates are immediate as shown in [4, Sec. 8.2]. The type (3) operators first appeared, in disguise, in the work by Auscher, Axelsson and Hofmann [3], where boundary value problems for Dirac equations of the form $(\Gamma + B_1\Gamma^*B_2)f = 0$, $\Gamma$ being the exterior derivative, were studied. Similarly to Section 3 here, it was shown in [4, Sec. 8.3], that solving for the $t$-derivatives, this equation can be written $(\partial_t + (BD_1 + D_2B^{-1}))Uf = 0$, under suitable similarity transformation $U$. It should be noted that in [3], $B_1$ and $B_2$ were related in exactly the way so that the associated operator $BD_1 + D_2B^{-1}$ is of type (3) and not of the more general form $T$ considered in Theorem 1.1 where the two multiplication operators are independent.

Coming to the non-tangential maximal function estimates in Theorem 1.1, these build on the estimates by Auscher, Axelsson and Hofmann in [3, Prop. 2.56]. Although set in the framework with Dirac equations, what was actually proved there
was non-tangential maximal function estimates for operators of type (1), in the special case when the differential operator is of the form $D = \begin{bmatrix} 0 & \text{div} \\ -\nabla & 0 \end{bmatrix}$. Even for operators of type (1) with more general $D$, the non-tangential maximal function estimate in Theorem [1.1] are new. Also the non-tangential maximal function estimates for general operators of type (2) and (3), which is a special case of Theorem [1.1] are new.

The key idea in the proof of the square function estimates in Theorem [1.1] is to use a splitting of $L_2$ adapted to the operators $B_1D_1$ and $D_2B_2$, in which the operator $B_1D_1 + D_2B_2$ is triangular due to the assumption $D_1D_2 = 0$. The proof of the non-tangential maximal function estimates in Theorem [1.1] is much inspired by the proof of [3, Prop. 2.56]. The main difference is that our proof here is a pure $L_2$ proof, in that it only requires $L_2$ off-diagonal decay of resolvents. In [3, Prop. 2.56], interpolation theory to prove $L_p$ off-diagonal decay, $p \approx 2$, was needed. Another novelty is a Caccioppoli type estimate, Lemma [5.2], for operators beyond divergence form equations.

The outline of this paper is as follows. In Section 3, we show how operators of the form $B_1D_1 + D_2B_2$ arise naturally in connection with exterior differential systems systems in $\mathbb{R}^{1+n}$ for differential forms. The special case of one-forms, that is vector fields, amounts to divergence form equations. In Section 2 we prove the resolvent estimates for the operator $T$, in Section 4 we prove the square function estimates for the operator $T$, and finally in Section 5 we prove the non-tangential maximal function estimates for the operator $T$. The (roadmap to the) proof of Theorem [1.1] is in Section 5.

2. Resolvent estimates

In this section, we establish the basic operator theoretical properties of the operator $T = B_1D_1 + D_2B_2$. A fundamental observation for the unperturbed operator $D_1 + D_2$ is the orthogonal splitting

$$L_2(\mathbb{R}^n; \mathbb{C}^N) = (N(D_1) \cap N(D_2)) \oplus \overline{\mathbb{R}(D_1)} \oplus \overline{\mathbb{R}(D_2)} =: \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2.$$  

The natural perturbation of this splitting which is adapted to the operator $T$ is

$$L_2(\mathbb{R}^n; \mathbb{C}^N) = (N(D_1) \cap N(D_2B_2)) \oplus \overline{\mathbb{R}(B_1D_1)} \oplus \overline{\mathbb{R}(D_2)}.$$  

Proposition 2.1. We have a topological (but in general not orthogonal) splitting

$$L_2(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{H}^{B_2}_{0} \oplus B_1\mathcal{H}_1 \oplus \mathcal{H}_2,$$

where $\mathcal{H}^{B_2}_{0} := (\mathcal{H}_0 \oplus \mathcal{H}_2) \cap B_2^{-1}(\mathcal{H}_0 \oplus \mathcal{H}_1)$, $B_2^{-1}V := \{ f \in L_2 \mid B_2f \in V \}$ and $B_1V := \{ B_1f \mid f \in V \}$.

Proof. We observe that we have two topological splittings

$$L_2 = B_1\mathcal{H}_1 \oplus (\mathcal{H}_0 \oplus \mathcal{H}_2)$$

and $L_2 = B_2^*\mathcal{H}_2 \oplus (\mathcal{H}_0 \oplus \mathcal{H}_1)$, since $B_1$ is accretive on $\mathcal{H}_1$ and $B_2$, and hence $B_2^*$, is accretive on $\mathcal{H}_2$. See for example [5, Prop. 3.3]. Taking orthogonal complements in the second splitting, we obtain a third topological splitting

$$L_2 = B_2^{-1}(\mathcal{H}_0 \oplus \mathcal{H}_1) \oplus \mathcal{H}_2,$$

since $B_2^{-1}(\mathcal{H}_0 \oplus \mathcal{H}_1) = (B_2^*\mathcal{H}_2)^\perp$ and $\mathcal{H}_2 = (\mathcal{H}_0 \oplus \mathcal{H}_1)^\perp$. 

The result is now a consequence of Lemma 2.2 below, with $X_1 = B_1 \mathcal{H}_1$, $X_2 = \mathcal{H}_0 \oplus \mathcal{H}_2$, $X_3 = B_2^{-1}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ and $X_4 = \mathcal{H}_2$. □

**Lemma 2.2.** Assume that a Banach space $X$ splits topologically in two ways

$$X = X_1 \oplus X_2 = X_3 \oplus X_4,$$

into closed subspaces such that $X_4 \subset X_2$. Then $X$ splits topologically into three closed subspaces

$$X = X_1 \oplus (X_2 \cap X_3) \oplus X_4.$$  

**Proof.** It is straightforward to verify that these three subspaces are closed and intersect pairwise only at 0. Also, given $x \in X$, we can write $x = x_1 + x_2$ and $x_2 = x_3 + x_4$ with $x_i \in X_i$, $i = 1, 2, 3, 4$. We have $x = x_1 + x_3 + x_4$, with $x_3 = x_2 - x_4 \in X_3 \cap X_2$, so the three subspaces span $X$. □

**Proposition 2.3.** For the operator $T$, the null space is $\mathcal{N}(T) = \mathcal{H}_0^{B_2}$, the range is $\mathcal{R}(T) = B_1 \mathcal{R}(D_1) + \mathcal{R}(D_2)$ and the domain is

$$D(T) = \{ u_0 + u_1 + u_2 \in \mathcal{H}_0^{B_2} \oplus B_1 \mathcal{H}_1 \oplus \mathcal{H}_2 : u_1 \in \mathcal{D}(D_1), u_1 + u_2 \in \mathcal{D}(D_2 B_2) \}.$$  

**Proof.** For the null space, we note that $f \in \mathcal{N}(T)$ if and only if $B_1 D_1 f = -D_2 B_2 f$. Since $B_1 \mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$, this is equivalent to $f \in \mathcal{N}(D_1) \cap B_2^{-1} \mathcal{N}(D_2)$.

Clearly $\mathcal{R}(T) \subset B_1 \mathcal{R}(D_1) + \mathcal{R}(D_2)$. For the converse implication, assume that $f_1 = B_1 D_1 u_1 \in B_1 \mathcal{R}(D_1)$ and $f_2 = D_2 B_2 u_2 \in \mathcal{R}(D_2) = \mathcal{R}(D_2 B_2)$. Write $u_1 = u_1^1 + u_1^2 \in \mathcal{H}_2 \oplus B_2^{-1}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ and $u_2 = u_2^1 + u_2^2 \in \mathcal{H}_2 \oplus B_2^{-1}(\mathcal{H}_0 \oplus \mathcal{H}_1)$. Then $(B_1 D_1 + D_2 B_2)(u_1^1 + u_2^1) = B_1 D_1 u_1 + D_2 B_2 u_2 = f_1 + f_2$, so $\mathcal{R}(T) = \mathcal{B}_1 \mathcal{R}(D_1) + \mathcal{R}(D_2)$.

The result for the domain follows from the facts that $\mathcal{H}_0^{B_2} \subset \mathcal{D}(D_1) \cap \mathcal{D}(D_2 B_2)$ and $\mathcal{H}_2 \subset \mathcal{D}(D_1)$. □

We now express the resolvents of $T$ in terms of the resolvents

$$R_1^1 := (I + it B_1 D_1)^{-1} \quad \text{and} \quad R_2^1 := (I + it D_2 B_2)^{-1}$$

of $B_1 D_1$ and $D_2 B_2$. It is known that $\sigma(B_1 D_1) \subset S_{\omega_1} \cup \{0\}$ with resolvent estimates $\|R_1^1\| \lesssim 1/(|t| \text{dist}(i/t, S_{\omega_1}))$, and that $\sigma(D_2 B_2) \subset S_{\omega_2} \cup \{0\}$ with resolvent estimates $\|R_2^1\| \lesssim 1/(|t| \text{dist}(i/t, S_{\omega_2}))$. See for example [5, Prop. 3.3].

**Proposition 2.4.** The operator $T$ is closed and densely defined in $L_2(\mathbb{R}^n; C^N)$. The spectrum is contained in the bisector $S_\omega \cup \{0\}$, $\omega = \max(\omega_1, \omega_2)$, and in the splitting $L_2(\mathbb{R}^n; C^N) = \mathcal{H}_0^{B_2} \oplus B_1 \mathcal{H}_1 \oplus \mathcal{H}_2$, the resolvent has the expression

$$(I + it)^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & R_1^1 & 0 \\ 0 & (R_2^1 - I)R_1^1 & R_2^1 \end{bmatrix}, \quad i/t \notin S_{\omega} \cup \{0\},$$

with estimates $\|(I + it)^{-1}\| \lesssim 1/(|t| \text{dist}(i/t, S_{\omega}))$.

**Proof.** Consider $(I + it)u = f$ with $u \in \mathcal{D}(T)$, and write $u = u_0 + u_1 + u_2$ and $f = f_0 + f_1 + f_2$ in the splitting from Proposition 2.1. Then

$$u_0 = f_0, \quad u_1 + it B_1 D_1 u_1 = f_1,$$

$$u_2 + it D_2 B_2 (u_1 + u_2) = f_2.$$
Solving for $u$, we equivalently have
\[
    u_0 = f_0, \quad u_1 = (I + itB_1D_1)^{-1}f_1, \quad u_2 = (I + itD_2B_2)^{-1}f_2 - itD_2B_2(I + itD_2B_2)^{-1}(I + itB_1D_1)^{-1}f_1.
\]
This shows that $I + iT$ is injective with the stated resolvent expression and estimate.

To show surjectivity, given $f \in L_2(\mathbb{R}^n; \mathbb{C}^N)$, define $u := f_0 + (I + itD_2B_2)^{-1}f_2 + (I + itD_2B_2)^{-1}(I + itB_1D_1)^{-1}f_1$. Then reversing the above calculation, shows that $u \in \mathcal{D}(T)$ and $(I + iT)u = f$. It follows that $I + iT$ is surjective and that $T$ is a closed operator. That $T$ is densely defined, follows from the fact that $\mathcal{D}(T)$ contains the dense subspace $\mathcal{D}(D_2B_2) \cap R(D_2B_2)$. \(\square\)

**Proposition 2.5.** The adjoint of $T$ is $T^* = B_2^*D_2 + D_1B_1^*$, with domain $\mathcal{D}(T^*) := \{ f \in \mathcal{D}(D_2) ; B_1^*f \in \mathcal{D}(D_1) \}$.

**Proof.** It suffices to show that if
\[
    (1) \quad \mathcal{D}(T) \to \mathbb{C} : u \mapsto (Tu, v)
\]
is $L_2$ continuous, then $v \in \mathcal{D}(D_2) \cap \mathcal{D}(D_1B_1^*)$. The splitting for $v$ analogous to Proposition 2.7 for $u$, is
\[
    v = v_0 + v_1 + v_2 \in \left( (\mathcal{H}_0 \oplus \mathcal{H}_1) \cap (B_1^*)^{-1}(\mathcal{H}_0 \oplus \mathcal{H}_2) \right) \oplus B_2^*\mathcal{H}_2 \oplus \mathcal{H}_1.
\]
We need to show $v_1 \in \mathcal{D}(D_2)$ and $v_1 + v_2 \in \mathcal{D}(D_1B_1^*)$. To this end, let $u = u_2 \in \mathcal{H}_2 \cap \mathcal{D}(D_2B_2) \subset \mathcal{D}(T)$ in (1). Then
\[
    |(D_2B_2u_2, v_1)| = |(Tu_2, v)| \lesssim \|u_2\|_2.
\]
It follows that $v_1 \in \mathcal{D}(D_2)$, since $(D_2B_2)^* = B_2^*D_2$. Therefore, for general $u \in \mathcal{D}(T)$, we have
\[
    (Tu, v) = (B_1D_1u_1, v) + (u, B_2^*D_2v_1),
\]
so $|(B_1D_1u_1, v_1 + v_2)| \lesssim \|u\|_2$. Since $(B_1D_1)^* = D_1B_1^*$, it follows that $v_1 + v_2 \in \mathcal{D}(D_1B_1^*)$. \(\square\)

We end this section with a short discussion of the definition of the functional calculus of bisectorial operators. For further details see [1], where the corresponding theory for sectorial operators is readily adapted to bisectorial operators.

Given a bisectorial operator $T$ in a Hilbert space $\mathcal{H}$, that is a closed and densely defined operator $T$ with $\sigma(T) \subset S_{\omega}$ for some $\omega < \pi/2$ and resolvent bounds
\[
    \|(\lambda I - T)^{-1}\| \lesssim 1/\text{dist} (\lambda, S_{\omega}),
\]
there is a natural definition of $\phi(T)$ for any rational function $\phi(\lambda)$ which is bounded and without poles in $S_{\omega}$. Useful such symbols in this paper are for example $1/(1 + t^2\lambda^2)$ and $t\lambda/(1 + t^2\lambda^2)$, with scale parameter $t > 0$.

If $T$ is not injective, then there is a topological splitting $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, with $\mathcal{H}_0 = N(T)$ and $\mathcal{H}_1 = \overline{R(T)}$. Indeed,
\[
    I = (I + iT)^{-1} + iT(I + iT)^{-1},
\]
where the two terms converge strongly to the projections onto $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively as $t \to \infty$. Thus $T = 0 \oplus T_1$ and $(\lambda I - T)^{-1} = \lambda^{-1}I \oplus (\lambda I - T_1)^{-1}$, where $T_1 := T|_{\mathcal{H}_1}$ is an injective bisectorial operator in $\mathcal{H}_1$. 

\section*{Estimates for Operators Beyond Divergence Form Equations}
With the Dunford integral
\[ \psi(T) := \psi(0)I_{\mathcal{H}_0} + \frac{1}{2\pi i} \int_{\gamma} \psi(\lambda)(\lambda I_{\mathcal{H}_1} - T_1)^{-1}d\lambda, \]
the functional calculus is extended to all symbols \( \psi : S^0_\mu \cup \{0\} \rightarrow \mathbb{C} \) which are holomorphic on an open bisector \( S^0_\mu \supset S_\omega \setminus \{0\} \), with estimates \( |\psi(\lambda)| \lesssim \min(|\lambda|^s,|\lambda|^{-s}) \) for some \( s > 0 \), so that the integral is convergent in the operator norm on \( \mathcal{H}_1 \). Here the curve \( \gamma = \{te^{\pm\theta} : t \in \mathbb{R}\}, \omega < \theta < \mu \), is oriented counter clockwise around \( S_\omega \).

To obtain \( \phi(T) \) as bounded operators on \( \mathcal{H} \) for general bounded holomorphic symbols \( \phi \), without decay at 0 and \( \infty \), square function estimates
\[ \int_0^\infty \|\psi(tT)f\|^2_{\mathcal{H}_1} dt \approx \|f\|^2_{\mathcal{H}_1}, \quad f \in \mathcal{H}_1, \]
are required. Usually, it suffices to show estimates \( \lesssim \), as estimates \( \gtrsim \) for \( T \) follow from estimates \( \lesssim \) for \( T^* \). Another basic result concerning square function estimates, is that if such hold for one symbol \( \psi \), then they hold for all \( \tilde{\psi} \) such that \( |\tilde{\psi}(\lambda)| \lesssim \min(|\lambda|^s,|\lambda|^{-s}) \) for some \( s > 0 \), and \( \tilde{\psi} \big|_{S_\mu} \neq 0 \).

Given such square function estimates, it follows that \( \psi_n(T) \) are uniformly bounded and converges strongly in \( L(\mathcal{H}) \) whenever \( \sup_{\mu \in \mathbb{Z_+}, \lambda \in S^0_\mu} |\psi_n(\lambda)| < \infty \) and \( \psi_n(\lambda) \) converges for each \( \lambda \in S^0_\mu \cup \{0\} \). Through such a limiting argument, we construct a bounded homomorphism
\[ \phi \mapsto \phi(T), \]
taking bounded symbols \( \phi : S^0_\mu \cup \{0\} \rightarrow \mathbb{C} \) which are holomorphic on \( S^0_\mu \) to bounded linear operators \( \phi(T) \) on \( \mathcal{H} \).

3. APPLICATIONS TO EXTERIOR DIFFERENTIAL SYSTEMS

In this section, we show how operators of the form \( B_1D_1 + D_2B_2 \) appear in connection with exterior differential systems for differential form. We first fix notation. Instead of writing \( \{dx_0, dx_1, \ldots, dx_n\} \) for the basis one-forms, we shall keep the notation \( \{e_0, e_1, \ldots, e_n\} \) from Section 1 for the basis vectors, and we use the terminology \( k \)-vector field instead of \( k \)-form, in the euclidean space \( \mathbb{R}^{1+n} \).

The space of \( k \)-vectors in \( \mathbb{R}^{1+n} \) we define to be the \( \binom{1+n}{k} \) dimensional complex linear space
\[ \wedge^k \mathbb{R}^{1+n} := \text{span}_\mathbb{C}\{e_{s_1} \wedge \ldots \wedge e_{s_k} ; 0 \leq s_1 < s_2 < \ldots s_k \leq n\}, \quad k = 2, \ldots, n+1, \]
and we let \( \wedge^0 \mathbb{R}^{1+n} := \mathbb{C}, \wedge^1 \mathbb{R}^{1+n} := \mathbb{C}^{1+n} \) and \( \wedge^k \mathbb{R}^{1+n} := \{0\} \) if \( k \notin \{0,1,\ldots,n+1\} \).

Given a vector \( v = \sum_{j=0}^n v_j e_j \in \wedge^1 \mathbb{R}^{1+n} \) and a \( k \)-vector \( w = \sum_{0 \leq s_1 < \ldots < s_k \leq n} w_{s_1} \ldots w_{s_k} e_{s_1} \ldots e_{s_k} \in \wedge^k \mathbb{R}^{1+n} \), writing \( s = \{s_1, \ldots, s_k\} \) and \( e_s := e_{s_1} \wedge \ldots \wedge e_{s_k} \), we have in particular the exterior product \( v \wedge w \in \wedge^{k+1} \mathbb{R}^{1+n} \) and the (left) interior product \( v \iota w \in \wedge^{k-1} \mathbb{R}^{1+n} \) defined bilinearly using
\[ e_j \wedge e_s := \begin{cases} \epsilon(j,s) e_{\{j\}\cup s} , & j \notin s , \\ 0 , & j \in s , \end{cases} \quad e_j \iota e_s := \begin{cases} 0 , & j \notin s , \\ \epsilon(j,s \setminus \{j\}) e_{s\setminus\{j\}} , & j \in s , \end{cases} \]
where the permutation sign is \( \epsilon(j,s) := (-1)^{|\{s_i ; j > s_i\}|} \). Defining inner products on \( \wedge^k \mathbb{R}^{1+n}, k = 0,1,\ldots,n+1 \), so that the standard bases above are ON-bases, we have that
\[ v \wedge (\cdot) : \wedge^k \mathbb{R}^{1+n} \rightarrow \wedge^{k+1} \mathbb{R}^{1+n} \quad \text{and} \quad v \iota (\cdot) : \wedge^{k+1} \mathbb{R}^{1+n} \rightarrow \wedge^k \mathbb{R}^{1+n}, \]
are adjoint multiplication operators if \( v \) is a vector, that is \( v \in \wedge^1 \mathbb{R}^{1+n} \), with real coefficients. The corresponding differential operators are the exterior derivative operator

\[
\nabla_{t,x} \wedge f := \sum_{j=0}^{n} e_j \wedge \partial_j f,
\]

mapping \( k \)-vector fields \( f : \mathbb{R}^{1+n} \to \wedge^k \mathbb{R}^{1+n} \) to \( k+1 \)-vector fields, and the interior derivative operator

\[
\nabla_{t,x} \lrcorner g := \sum_{j=0}^{n} e_j \lrcorner \partial_j g,
\]

mapping \( k+1 \)-vector fields \( g : \mathbb{R}^{1+n} \to \wedge^{k+1} \mathbb{R}^{1+n} \) to \( k \)-vector fields. As special cases of these operators, we have the gradient and curl, being the exterior derivative acting on scalars and vectors \((k = 0 \text{ and } k = 1 \text{ respectively})\), and the divergence being the interior derivative acting on vectors. We also note the duality \( \int \int (\nabla_{t,x} \wedge f, g) dt dx = -\int \int (f, \nabla_{t,x} \lrcorner g) dt dx \) for compactly supported fields.

The basic exterior differential system in \( \mathbb{R}^{1+n} \) that we want to consider is

\[
\begin{aligned}
\nabla_{t,x} \lrcorner \hat{f}_{k+1} &= \nabla_{t,x} \wedge \hat{f}_{k-1}, \\
\nabla_{t,x} \wedge \hat{f}_{k+1} &= 0 \quad \nabla_{t,x} \lrcorner \hat{f}_{k-1},
\end{aligned}
\]

for a \( k + 1 \)-vector field \( \hat{f}_{k+1} \) and a \( k - 1 \)-vector field \( \hat{f}_{k-1} \). Two important special cases are the following. If \( k = 0 \), then the system reads \( \text{div}_{t,x} \hat{f}_1 = 0 = \text{curl}_{t,x} \hat{f}_1 \), since \( \hat{f}_{-1} = 0 \). This is nothing but the Laplace equation, written for the gradient as in Section 3. If \( k = 1 \), then the system reads \( \nabla_{t,x} \lrcorner \hat{f}_2 = \nabla_{t,x} \hat{f}_0, \nabla_{t,x} \wedge \hat{f}_2 = 0 \). This equation is the Stokes’ system of linearized hydrostatics, written for the vorticity \( \hat{f}_2 \) and the pressure \( \hat{f}_0 \).

Consider next a bilipschitz map \( \rho : \mathbb{R}^{1+n}_+ \to \Omega \subset \mathbb{R}^{1+n} \), and the system (2) in \( \Omega \). We want to pull back this system of equations to \( \mathbb{R}^{1+n}_+ \), and recall therefore the following facts from differential geometry. At a fixed point in \( \mathbb{R}^{1+n}_+ \), denote by \( \rho \) the Jacobian matrix of all partial derivatives of \( \rho \). Extend this linear map as a \( \Lambda^\ast \)-homomorphism to \( \Lambda^k \mathbb{R}^{1+n} \), letting

\[
\rho(e_{s_1} \wedge \ldots \wedge e_{s_k}) := \left(\rho e_{s_1}\right) \wedge \ldots \wedge \left(\rho e_{s_k}\right).
\]

Given a \( k \)-vector field \( f : \Omega \to \wedge^k \mathbb{R}^{1+n} \), we define the pullback of \( f \) by \( \rho \) to be the \( k \)-vector field

\[
\rho^* f(t,x) := \rho_{(t,x)}^* (f(\rho(t,x)))
\]

in \( \mathbb{R}^{1+n}_+ \), where \( \rho_{(t,x)}^* \) is the adjoint of the Jacobian matrix at \((t,x)\). A fundamental well known result is that

\[
\nabla_{t,x} \wedge (\rho^* f) = \rho^* (\nabla_{t,x} \wedge f).
\]

Less commonly used is the equivalent dual result that

\[
\nabla_{t,x} \lrcorner (J_\rho \rho_{(t,x)}^{-1} g) = J_\rho \rho_{(t,x)}^{-1} (\nabla_{t,x} \lrcorner g),
\]

where \( J_\rho \) is the Jacobian determinant of \( \rho \) and

\[
\rho_{(t,x)}^{-1} g(t,x) := \rho_{(t,x)} (g(\rho(t,x)))
\]
is the push forward of $g$ by $\rho^{-1}$. Applying (3) and (4), we find that (2) in $\Omega$ is equivalent to

\[
\begin{align*}
\nabla_{t,x} \lceil (A_{k+1}(t,x)f_{k+1}) &= A_k(t,x)(\nabla_{t,x} \wedge f_{k-1}), \\
\nabla_{t,x} \wedge f_{k+1} &= 0 = \nabla_{t,x} \lceil (A_{k-1}(t,x)f_{k-1})
\end{align*}
\]

in $\mathbb{R}_+^{1+n}$, where $f_j := \rho^* \tilde{f}_j$ and $A_j := J_\rho(\rho^* \rho_\ast)^{-1}$ is the Jacobian determinant times the inverse of the metric tensor $G = \rho^* \rho_\ast$, extended as a $\wedge$-homomorphism to $\wedge^j \mathbb{R}_+^{1+n}$.

We now show, analogous to the case $k = 0$ in the introduction, how (5) is equivalent to a vector valued ordinary differential equation $\partial_t f_i + T f_i = 0$, for general bounded measurable and accretive coefficients $A_j(t,x) \in \mathcal{L}(\wedge^j \mathbb{R}_+^{1+n})$, $j = k - 1, k, k + 1$, with an infinitesimal generator $T$ of the form $T = B_1 D_1 + D_2 B_2$. We use the natural identifications

\[
\begin{align*}
\wedge^k \mathbb{R}_+^{n} \oplus \wedge^{k+1} \mathbb{R}_+^{n} &= \wedge^{k+1} \mathbb{R}_+^{1+n} : \tilde{f}_k \oplus \tilde{f}_{k+1} \approx e_0 \wedge \tilde{f}_k + \tilde{f}_{k+1} = f_{k+1}, \\
\wedge^{k-2} \mathbb{R}_+^{n} \oplus \wedge^{k-1} \mathbb{R}_+^{n} &= \wedge^{k-1} \mathbb{R}_+^{1+n} : \tilde{f}_{k-2} \oplus \tilde{f}_{k-1} \approx e_0 \wedge \tilde{f}_{k-2} + \tilde{f}_{k-1} = f_{k-1},
\end{align*}
\]

with corresponding splittings of the coefficient matrices so that

\[
\begin{align*}
A_{k+1}f_k &= e_0 \wedge (a_{k+1} \tilde{f}_k + b_{k+1} \tilde{f}_{k+1}) + (c_{k+1} \tilde{f}_k + d_{k+1} \tilde{f}_{k+1}), \\
A_{k-1}f_k &= e_0 \wedge (a_{k-1} \tilde{f}_{k-2} + b_{k-1} \tilde{f}_{k-1}) + (c_{k-1} \tilde{f}_{k-2} + d_{k-1} \tilde{f}_{k-1}),
\end{align*}
\]

and similarly for $A_k$.

Let $\mathcal{H}_j^i$ denote the closure of the range of $\nabla_x \wedge (\cdot) : L_2(\mathbb{R}_+^{n}; \wedge^{j-1} \mathbb{R}_+^{n}) \to L_2(\mathbb{R}_+^{n}; \wedge^j \mathbb{R}_+^{n})$, and let $\mathcal{H}_j$ be the closure of the range of $\nabla_x \wedge (\cdot) : L_2(\mathbb{R}_+^{n}; \wedge^{j+1} \mathbb{R}_+^{n}) \to L_2(\mathbb{R}_+^{n}; \wedge^j \mathbb{R}_+^{n})$. Fundamental results are that $\mathcal{H}_j^i$ is the null space of $\nabla_x \wedge (\cdot) : L_2(\mathbb{R}_+^{n}; \wedge^{j-1} \mathbb{R}_+^{n}) \to L_2(\mathbb{R}_+^{n}; \wedge^{j+1} \mathbb{R}_+^{n})$, $\mathcal{H}_j$ is the nullspace of $\nabla_x \wedge (\cdot) : L_2(\mathbb{R}_+^{n}; \wedge^j \mathbb{R}_+^{n}) \to L_2(\mathbb{R}_+^{n}; \wedge^{j-1} \mathbb{R}_+^{n})$, and we have an orthogonal Hodge splitting

\[L_2(\mathbb{R}_+^{n}; \wedge^j \mathbb{R}_+^{n}) = \mathcal{H}_j^i \oplus \mathcal{H}_j^j.
\]

**Proposition 3.1.** Assume that $A_j \in L_\infty(\mathbb{R}_+^{n}; \mathcal{L}(\wedge^{j-1} \mathbb{R}_+^{n} \oplus \wedge^j \mathbb{R}_+^{n}))$ are $t$-independent and accretive on $\mathcal{H}_j^{j-1} \oplus \mathcal{H}_j^j$, $j = k - 1, k, k + 1$. Define the $\wedge^{k-2} \mathbb{R}_+^{n} \oplus \wedge^{k-1} \mathbb{R}_+^{n} \oplus \wedge^k \mathbb{R}_+^{n} \oplus \wedge^{k+1} \mathbb{R}_+^{n}$ valued function $\tilde{f}$ by

\[
\tilde{f} = \begin{bmatrix} \tilde{f}_{k-2} \\ \tilde{f}_{k-1} \\ \tilde{f}_k \\ \tilde{f}_{k+1} \end{bmatrix} = \begin{bmatrix} a_{k-1} \tilde{f}_{k-2} + b_{k-1} \tilde{f}_{k-1} \\ a_{k+1} \tilde{f}_k + b_{k+1} \tilde{f}_{k+1} \end{bmatrix}.
\]

Then the exterior differential system (5) for the $\wedge^{k-1} \mathbb{R}_+^{1+n} \oplus \wedge^{k+1} \mathbb{R}_+^{1+n}$ valued function $f_{k-1} \oplus f_{k+1}$ is equivalent to the vector valued ordinary differential equation

\[
\partial_t \hat{f} + (B_1 D_1 + D_2 B_2) \hat{f} = 0,
\]
together with the constraint \( \hat{f} \in \overline{R(B_1D_1 + D_2B_2)} \) for each \( t \), where

\[
B_1 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & a_k^{-1} & -a_k^{-1}b_k & 0 \\
0 & c_k a_k^{-1} & d_k - c_k a_k^{-1}b_k & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
D_1 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \nabla_x \langle \cdot \rangle & 0 \\
0 & -\nabla_x \wedge \langle \cdot \rangle & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
D_2 := \begin{bmatrix}
a_{k-1}^{-1} & -a_{k-1}^{-1}b_{k-1} & 0 & 0 \\
c_{k-1} a_{k-1}^{-1} & d_{k-1} - c_{k-1} a_{k-1}^{-1}b_{k-1} & 0 & 0 \\
0 & 0 & a_{k+1}^{-1} & -a_{k+1}^{-1}b_{k+1} \\
0 & 0 & c_{k+1} a_{k+1}^{-1} & d_{k+1} - c_{k+1} a_{k+1}^{-1}b_{k+1}
\end{bmatrix},
\]

\[
B_2 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \nabla_x \langle \cdot \rangle & 0 & 0 \\
0 & 0 & -\nabla_x \wedge \langle \cdot \rangle & 0 \\
0 & 0 & 0 & \nabla_x \langle \cdot \rangle
\end{bmatrix}
\]

satisfy the hypothesis in Theorem 1.1.

**Proof.** The equation \( \nabla_t x \wedge f_{k+1} = 0 \) is equivalent to

\[
\begin{cases}
\partial_t \hat{f}_{k+1} - \nabla_x \wedge \hat{f}_k = 0, \\
\nabla_x \wedge \hat{f}_{k+1} = 0.
\end{cases}
\]

The equation \( \nabla_t x \wedge (A_{k-1} f_{k-1}) = 0 \) is equivalent to

\[
\begin{cases}
\nabla_x \wedge (a_{k-1} \hat{f}_{k-2} + b_{k-1} \hat{f}_{k-1}) = 0, \\
\partial_t (a_{k-1} \hat{f}_{k-2} + b_{k-1} \hat{f}_{k-1}) + \nabla_x \wedge (c_{k-1} \hat{f}_{k-2} + d_{k-1} \hat{f}_{k-1}) = 0.
\end{cases}
\]

The equation \( \nabla_t x \wedge (A_{k+1} f_{k+1}) = A_k (\nabla_t x \wedge f_{k+1}) \) is equivalent to

\[
\begin{cases}
-\nabla_x \wedge (a_{k+1} \hat{f}_k + b_{k+1} \hat{f}_{k+1}) = a_k (\partial_t \hat{f}_{k-1} - \nabla_x \wedge \hat{f}_{k-2}) + b_k (\nabla_x \wedge \hat{f}_{k-1}), \\
\partial_t (a_{k+1} \hat{f}_k + b_{k+1} \hat{f}_{k+1}) + \nabla_x \wedge (c_{k+1} \hat{f}_k + d_{k+1} \hat{f}_{k+1}) \\
= c_k (\partial_t \hat{f}_{k-1} - \nabla_x \wedge \hat{f}_{k-2}) + d_k (\nabla_x \wedge \hat{f}_{k-1}).
\end{cases}
\]

Written in terms of \( \hat{f} \), the four evolution equations are

\[
\begin{cases}
\partial_t \hat{f}_{k-2} + \nabla_x \wedge (c_{k-1} \hat{f}_{k-2} + d_{k-1} \hat{f}_{k-1}) = 0, \\
\partial_t \hat{f}_{k-1} - \nabla_x \wedge \hat{f}_{k-2} + a_k^{-1} b_k (\nabla_x \wedge \hat{f}_{k-1}) + a_k^{-1} \nabla_x \wedge (a_{k+1} \hat{f}_k + b_{k+1} \hat{f}_{k+1}) = 0, \\
\partial_t \hat{f}_k + \nabla_x \wedge (c_{k+1} \hat{f}_k + d_{k+1} \hat{f}_{k+1}) - c_k (\partial_t \hat{f}_{k-1} - \nabla_x \wedge \hat{f}_{k-2}) - d_k (\nabla_x \wedge \hat{f}_{k-1}) = 0, \\
\partial_t \hat{f}_{k+1} - \nabla_x \wedge \hat{f}_k = 0,
\end{cases}
\]

and the remaining two equations give the constraints \( \nabla_x \wedge \hat{f}_{k+1} = 0 = \nabla_x \wedge \hat{f}_{k-2} \). We next write the evolution equations in terms of \( \hat{f} \), using \( \hat{f}_{k-2} = a_{k-1}^{-1} (\hat{f}_{k-2} - b_{k-1} \hat{f}_{k-1}) \) and \( \hat{f}_k = a_{k+1}^{-1} (\hat{f}_k - b_{k+1} \hat{f}_{k+1}) \). The tangential derivatives in the evolution equations
that appear with coefficients to the right are
\[
\begin{bmatrix}
\nabla_s \cdot (c_{k-1}a^{-1}_{k-1}(\dot{f}_{k-2} - b_{k-1}\dot{f}_{k-1}) + d_{k-1}\dot{f}_{k-1})
- \nabla_s \cdot (a^{-1}_{k-1}(\dot{f}_{k-2} - b_{k-1}\dot{f}_{k-1}))
\nabla_s \cdot (c_{k+1}a^{-1}_{k+1}(\dot{f}_k - b_{k+1}\dot{f}_{k+1}) + d_{k+1}\dot{f}_{k+1})
- \nabla_s \cdot (a^{-1}_{k+1}(\dot{f}_k - b_{k+1}\dot{f}_{k+1}))
\end{bmatrix}
= D_2B_2\dot{f}.
\]

The tangential derivatives in the evolution equations that appear with coefficients to the left are
\[
\begin{bmatrix}
0
a^{-1}_k b_k \nabla_s \cdot \dot{f}_{k-1} + a^{-1}_k \nabla_s \cdot \dot{f}_k
\c_k(a^{-1}_k b_k \nabla_s \cdot \dot{f}_{k-1} + a^{-1}_k \nabla_s \cdot \dot{f}_k) - d_0 \nabla_s \cdot \dot{f}_{k-1}
0
\end{bmatrix}
= B_1D_1\dot{f}.
\]

This shows that the evolution equation for \( \dot{f} \) is \( \partial_t \dot{f} + (B_1D_1 + D_2B_2)\dot{f} = 0 \). To show that the constraint \( \nabla_s \cdot \dot{f}_{k+1} = 0 = \nabla_s \cdot \dot{f}_{k-2} \) is equivalent to \( f \in R(B_1D_1 + D_2B_2) \), we note that
\[
R(B_1D_1 + D_2B_2) = B_1R(D_1) + R(D_2)
\]

by Proposition 3.1 and a \( L_2 \) Hodge splitting of \( L_2(R^n; \wedge^{k-1}R^n + \wedge^kR^n) \) adapted to \( B_1 \).

Given Theorem 1.1 and Proposition 3.1, we can proceed as in [5, Thm. 2.3], where the case \( k = 0 \) was treated, to represent solutions to the exterior differential system [5] with functional calculus as outlined in Section 2. To this end, define symbols
\[
e^{-\lambda t}\chi^+(\lambda) := \begin{cases} e^{-\lambda t}, & \text{Re } \lambda > 0, \\ 0, & \text{Re } \lambda \leq 0, \end{cases}, \quad t \geq 0,
\]
\[
e^{-\lambda t}\chi^-(\lambda) := \begin{cases} 0, & \text{Re } \lambda \geq 0, \\ e^{-\lambda t}, & \text{Re } \lambda < 0, \end{cases}, \quad t \leq 0.
\]

For \( t = 0 \), we obtain bounded spectral projections \( \chi^+(T) \), with \( \chi^+(T) + \chi^-(T) \) being the projection onto \( R(T) \) along \( N(T) \). The following result roughly states that the spectral subspace \( \chi^+(T)L_2 := R(\chi^+(T)) \) is a Hardy type subspace containing traces of solutions to [5] in \( R^{1+n}_+ \), whereas the spectral subspace \( \chi^-(T)L_2 := R(\chi^-(T)) \) is a Hardy type subspace containing traces of solutions to [5] in \( R^{1+n}_- \), and the operators \( e^{-\lambda t}\chi^\pm(T) \) are Cauchy integral type operators, giving the value of the function at \( (t, \cdot) \) from the boundary trace.

**Theorem 3.2.** Consider the exterior differential system [5], with bounded, \( t \)-independent, accretive coefficients \( A_{k-1}, A_k, A_{k+1} \), and the associated operator \( T = B_1D_1 + D_2B_2 \) as in Proposition 3.1. Given \( \hat{f}_0^+ \in \chi^+(T)L_2 \), the function \( f \approx \hat{f} \) defined by
\[
\hat{f}^+(t, x) := (e^{-\lambda t}\chi^+(T))\hat{f}_0^+(x), \quad (t, x) \in R^{1+n}_+,
\]
is a solution to [5], with limits \( \lim_{t \to 0^+} \|\hat{f}_t^+\|_2 = 0 \) and \( \lim_{t \to \pm \infty} \|\hat{f}_t^+\|_2 = 0 \). Conversely, any solution \( f^+ \approx \hat{f}^+ \) to [5] in \( R^{1+n}_+ \) with estimates \( \sup_{t>0} \int_{t<s<2t} \|f_s\|^2ds < \)
is of this form, and in particular has the stated limits at \( t = 0 \), for some \( f_0^\pm \in \chi^\pm(T)L_2 \), and \( t = \infty \).

These solutions have square function, non-tangential maximal function and \( L^1_\infty L^2 \) estimates

\[
\int_{t>0} \|\partial_t \hat{f}^\pm\|_2^2 dt \approx \|\tilde{N}_s(\hat{f}^\pm)\|_2^2 \approx \sup_{t>0} \|\hat{f}^\pm\|_2^2 \approx \|\hat{f}_0^\pm\|_2^2.
\]

The idea of proof is found in [5, Thm. 3.2] and [2, Thm. 8.2]. In particular, the estimates follow from Theorem 1.1 using the symbol \( \psi(\lambda) = \lambda e^{-\lambda^2} \chi^\pm(\lambda) \) for the square function estimates, and the symbol \( \phi(\lambda) = e^{-\lambda^2} \chi^\pm(\lambda) \) for the non-tangential maximal function estimates and the \( L^1_\infty L^2 \) estimates. We omit the details.

4. SQUARE FUNCTION ESTIMATES

In this section, we prove the square function estimates for the operator \( T \) in Theorem 1.1. We start by simplifying the problem with Lemma 4.1, and we use the following operators.

\[
P^1_t := (I + t^2(B_1 D_1))^2 - 1 = \frac{1}{2}(R^1_t + R^1_{-t}),
Q^1_t := tB_1 D_1 (I + t^2(B_1 D_1))^2 - 1 = \frac{1}{2}(R^1_t - R^1_{-t}),
P^2_t := (I + t^2(D_2 B_2))^2 - 1 = \frac{1}{2}(R^2_t + R^2_{-t}),
Q^2_t := tD_2 B_2 (I + t^2(D_2 B_2))^2 - 1 = \frac{1}{2}(R^2_t - R^2_{-t}).
\]

It is known that these operators are uniformly bounded for \( t > 0 \) and that square function estimates

\[
\int_0^\infty \|Q^1_t f\|_2^2 dt + \int_0^\infty \|Q^2_t f\|_2^2 dt \lesssim \|f\|_2^2, \quad f \in L^2_2(\mathbb{R}^n; C^N),
\]

hold. See for example [5, Thm. 3.4].

Lemma 4.1. Let

\[
\Theta_t := Q^2_t R^1_t B_1.
\]

Assume that we have square function estimates

\[
\int_0^\infty \|\Theta_t f\|_2^2 dt \lesssim \|f\|_2^2, \quad \text{for all } f \in \mathcal{H}_1.
\]

Then we have square function estimates \( \int_0^\infty \|\psi(tT)f\|_2^2 dt \lesssim \|f\|_2^2, f \in L^2_2(\mathbb{R}^n; C^N), \) for \( \psi \) as in Theorem 1.1.

Proof. It is known, see [1], that it suffices to prove the square function estimate for \( \psi(\lambda) = \lambda/(1 + \lambda^2) \). For this \( \psi \), we see from Proposition 2.4 that

\[
\psi(tT) = \frac{1}{2}((I + itT)^{-1} - (I - itT)^{-1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q^1_t & 0 \\ 0 & (R^2_t - I)Q^1_t + Q^2_t R^1_t & Q^2_t \end{bmatrix}, \quad t > 0,
\]

by writing

\[
\frac{1}{2} \left( \frac{tD_2 B_2}{I + itD_2 B_2} \frac{1}{I + itB_1 D_1} + \frac{tD_2 B_2}{I - itD_2 B_2} \frac{1}{I - itB_1 D_1} \right)
= \frac{itD_2 B_2}{I - itD_2 B_2} \frac{tB_1 D_1}{I + t^2(B_1 D_1)^2} + \frac{itD_2 B_2}{I + t^2(D_2 B_2)^2} \frac{1}{I + itB_1 D_1}.
\]
Since $R^2_{t}$ are uniformly bounded and since we have square function estimates for $Q^1_t$ and $Q^2_t$, it suffices to prove square function estimates for $Q^2_t R^1_t$ on $B_tH_1$ as claimed.

We now prove the square function estimates for $\Theta_t$ using techniques from the proof of the Kato square root estimate, following [3, Sec. 4].

**Definition 4.2.** Let $D = \bigcup_{j \in \mathbb{Z}} D_{2^{-j}}$ denote the dyadic cubes in $\mathbb{R}^n$, with

$$D_j := \{2^{-j}(0,1)^n + 2^{-j}k \mid k \in \mathbb{Z}^n\}, \quad 2^{-j-1} < t \leq 2^{-j}, j \in \mathbb{Z}.$$ Write $\ell(Q)$ for side length and $|Q|$ for measure of a cube $Q$. Given $Q \in D$, write

$$A_0(Q) := Q, \quad A_k(Q) := (2^k Q) \setminus (2^{k-1} Q), \quad k \geq 1,$$

for the dyadic annuli around $Q$, where $aQ$ denote the cube with same center as $Q$ but with $\ell(aQ) = a\ell(Q)$.

The key tool in the proof of the square function estimates, as well as for the non-tangential maximal function estimates, are the following $L_2$ off-diagonal estimates.

**Proposition 4.3.** For any $m < \infty$, there exists $C_m < \infty$ such that

$$\|\Theta_t f\|_{L_2(F)} \lesssim C_m \left(\frac{t}{\text{dist}(E,F)}\right)^m \|f\|_2,$$

for all $f \in L_2(\mathbb{R}^n; C^N)$ with supp $f \subset E$, and any closed subsets $E,F \subset \mathbb{R}^n$ such that $\text{dist}(E,F) := \inf\{|x-y| \mid x \in E, y \in F\} > 0$.

**Proof.** These estimates are known to hold for $R^2_t$, and therefore for $R^2_t$, $P^1_t$ and $Q^1_t$, $j = 1, 2$, see for example [4, Sec. 5]. From this, the estimates for $\Theta_t := Q^2_t(I - R^1_t)B_1$ follow as in [3, Lem. 2.26].

These $L_2$ off-diagonal estimates enable us to approximate the family of operators $\{\Theta_t\}_{t>0}$ by a family of multiplication operators $\{\gamma_t\}_{t>0}$, where formally $\gamma_t = \Theta_t 1$. More precisely, we let

$$\gamma_t(x)v := \sum_{k=0}^{\infty} \Theta_t(v \chi_{A_k(Q)})(x), \quad x \in D_t, v \in C^N,$$

where $\chi_{A_k(Q)}$ denotes the characteristic function of the dyadic annulus $A_k(Q)$. From Proposition 4.3 we have the estimate

$$\|\gamma_t\|_{L_2(Q)} \lesssim \sum_{k=0}^{\infty} 2^{-km} 2^{kn/2} \leq C$$

uniformly for all $Q \in D_t, t > 0$, by choosing $m > n/2$.

We also need the following Sobolev–Poincaré inequality, see [10, Sec. 7.8].

**Lemma 4.4.** Let $1 < r < n$, or $r = n = 1$, and $1/r^* = 1/r - 1/n$. Assume that $1 \leq q \leq r^*$ and $r \leq p \leq \infty$. Then there exists $C < \infty$ such that for all $u \in H^1_{loc}(\mathbb{R}^n)$ and $0 < r \leq R < \infty$, we have the estimate

$$\|u - u_S\|_{L_p(Q)} \leq C |\Omega|^{1/q - 1/p + 1/n} R^n S^{-1} \|\nabla u\|_{L_p(S)},$$

for any convex set $\Omega$ with diameter $R$ and measure $|\Omega|$ and any measurable subset $S \subset \Omega$ with measure $|S|$.
Proposition 4.5. We have the estimate
\[ \int_0^\infty \| \Theta_tf - \gamma_tE_tf \|^2_2 \frac{dt}{t} \lesssim \|f\|^2_2, \]
for all \( f \in \mathcal{H}_1 \), where \( E_t \) denotes the dyadic averaging operator
\[ E_tf(x) = E_Qf := \frac{1}{|Q|} \int_Q f(y)dy, \quad x \in Q \in \mathcal{D}_t. \]

Proof. Let \( P_t, Q_t \) denote the unperturbed operators \( P_t^1, Q_t^1 \), that is
\[ P_t := (I + t^2D_t^2)^{-1}, \quad Q_t := tD_t(I + t^2D_t^2)^{-1}. \]

Write
\[ \Theta_t f - \gamma_tE_tf = \Theta_t(I - P_t)f + (\Theta_t - \gamma_tE_t)P_t f + (\gamma_tE_t)E_t(P_t - I)f =: I + II + III. \]

For the first term we have
\[ \|I\|_2 = \|Q_t^2(I - R_t^1)Q_t f\|_2 \lesssim \|Q_t f\|_2, \]

since \((I + itB_1D_1)^{-1}B_1(t^2D_1^2(I + t^2D_1^2)) = (I + itB_1D_1)^{-1}(tB_1D_1)Q_t\), and square function estimates for \( Q_t \) give the desired estimate.

For the term \( III \) we note from (6) that \( \|\gamma_tE_t\|_{L^2 \to L^2} \leq C \). Square function estimates for \( E_t(P_t - I) \) can be proved as in [8, Prop. 5.7], replacing \( \Pi \) there by the operator \( D_1 \).

The term \( II \) we write
\[ (\Theta_t - \gamma_tE_t)P_t f = \sum_{k \geq 0} \Theta_t((P_t f - E_Q(P_t f))\chi_{A_k(Q)}), \quad \text{on } Q \in \mathcal{D}_t. \]

Proposition 4.4 and Poincaré’s inequality in Lemma 4.3 yields
\[ \|II\|^2_2 \lesssim \sum_{Q \in \mathcal{D}_t} \left( \sum_{k=0}^\infty 2^{-km} \|P_t f - E_Q(P_t f)\|_{L^2(A_k(Q))} \right)^2 \]
\[ \lesssim \sum_{Q \in \mathcal{D}_t} \sum_{k=0}^\infty 2^{-km} \|P_t f - E_Q(P_t f)\|^2_{L^2(A_k(Q))} \]
\[ \lesssim \sum_{Q \in \mathcal{D}_t} \sum_{k=0}^\infty 2^{-km+2kn+1}\|t\nabla P_t f\|^2_{L^2(A_k(Q))} \approx \|t\nabla P_t f\|_2, \]
if we choose \( m \) sufficiently large. Since \( D_1 \) is elliptic on \( R(D_1) \), the square function estimate for \( II \) follows from that for \( Q_t \). \( \square \)

To prove square function estimates for the remaining paraproduct term \( \gamma_tE_tf \), we use the following test functions. For a small fixed parameter \( \epsilon > 0 \), we define for all dyadic cubes \( Q \in \mathcal{D} \) and unit vectors \( v \in \mathcal{O}^N \), the test function
\[ f_Q^v := (I + (\epsilon\ell(Q)D_1B_1)^2)^{-1}(\eta_Q v), \]

where \( \eta_Q = 1 \) on \( 2Q \) and \( \text{supp} \eta_Q \subset (3Q) \), \( \|\nabla \eta_Q\|_\infty \lesssim 1/\ell(Q) \). The parameter \( \epsilon \) is chosen so that the accretivity condition
\[ \text{Re} \left( v, \int_Q f_Q^v dx \right) \geq |Q|/2 \]
holds. This is possible since it is known that we have the estimate
\[ \left| \frac{1}{|Q|} \int_Q \tilde{f}_Q^n dx - v \right| \lesssim \sqrt{\epsilon}. \]

This can be proved by applying [8, Lem. 5.6] with the operator \( D_1 \), similar to [8, Lem. 5.10].

**Proposition 4.6.** We have the estimate
\[ \int_0^\infty \int_{\mathbb{R}^n} |E_t f(x)|^2 |\gamma_t(x)|^2 \frac{dxdt}{t} \lesssim \| f \|^2, \quad \text{for all } f \in L_2({\mathbb{R}^n}; C^N). \]

**Proof.** By Carleson’s embedding theorem, it suffices to show that
\[ \int_0^{\ell(Q)} \int_Q |\gamma_t(x)|^2 \frac{dxdt}{t} \lesssim |Q|, \quad \text{for all } Q \in \mathcal{D}. \]

Following the proof of the Kato square root problem, see for example [8, Sec. 5.3], we now do (1) a sufficiently fine sectorial decomposition of \( \mathcal{L}(C^N) \), run (2) a stopping time argument to construct an large sawtooth sub region of the Carleson box \( Q \times (0, \ell(Q)) \) where the test function \( f^n \) is paraaccretive, and make (3) a John–Nirenberg bootstrapping argument for Carleson measure, to show that it suffices to prove the estimate
\[ \int_0^{\ell(Q)} \int_Q |\gamma_t(x) E_t f^n_Q| \frac{dxdt}{t} \lesssim |Q| \]
for all \( Q \in \mathcal{D} \) and unit vectors \( v \in C^N \).

We first note that
\[ \| \Theta_t f^n_Q \|_2 = |Q|^2 (B_1 f^n_Q) - i t Q^2 R^1_t D_1 B_1 (I + (\epsilon(Q) D_1 B_1)^{-1}) (\eta_Q v) \|_2 \lesssim |Q|^2 (B_1 f^n_Q) + t/\epsilon(Q) \sqrt{|Q|}, \]
since \( Q^2_t \) and \( R^1_t \) and \( \epsilon(Q) D_1 B_1 (I + (\epsilon(Q) D_1 B_1)^{-1}) \) are uniformly bounded. Therefore
\[ \int_0^{\ell(Q)} \| \Theta_t f^n_Q \|^2 \frac{dt}{t} \lesssim \| B_1 f^n_Q \|^2 + (Q^2) \lesssim |Q|, \]
so it suffices to prove
\[ \int_0^{\ell(Q)} \int_Q |(\Theta_t - \gamma_t E_t) f^n_Q| \frac{dxdt}{t} \lesssim |Q|. \]

Although \( f^n_Q \notin \mathcal{H}_1 \) in general, we have that \( f^n_Q - \eta_Q v = \eta(Q) D_1 B_1 f^n_Q \in \mathcal{R}(D_1) \). Thus Proposition 4.3 applies to \( f = f^n_Q - \eta_Q v \), and it remains to show
\[ \int_0^{\ell(Q)} \int_Q |(\Theta_t - \gamma_t E_t)(\eta_Q v)| \frac{dxdt}{t} \lesssim |Q|. \]

We have on \( Q \)
\[ (\Theta_t - \gamma_t E_t)(\eta_Q v) = \sum_{k \geq 0} \Theta_t (v(\eta_Q - 1) \chi_{A_k(Q)}) = \sum_{k \geq 2} \Theta_t (v(\eta_Q - 1) \chi_{A_k(Q)}), \]
and Proposition 4.3 gives
\[ \| (\Theta_t - \gamma_t E_t)(\eta_Q v) \|_{L^2(Q)} \lesssim \sum_{k=2}^\infty (t/2^k \ell(Q))^m (2^{nk}|Q|)^{1/2} \lesssim (t/\ell(Q))^m |Q|^{1/2}, \]
if $m$ is chosen large enough. We obtain
\[
\int_0^{\ell(Q)} \int_Q |(\Theta_t - \gamma_t E_t)(\eta Qv)|^2 \frac{dxdt}{t} \lesssim \int_0^{\ell(Q)} (t/\ell(Q))^{2m} |Q| \frac{dt}{t} \approx |Q|
\]
and the proof is complete. \hfill \Box

5. Non-tangential maximal function estimates

In this section, we prove the non-tangential maximal function estimates for the operator $T$ in Theorem 1.1. We first consider the operator $D_2 B_2$, that is the operator $T$ in the special case when $D_1 = 0$. We prove the following result.

**Theorem 5.1.** We have non-tangential maximal function estimates
\[
\|\tilde{N}_*(\phi(tD_2B_2)f)\|_2 \lesssim \|f\|_2, \quad f \in \mathcal{H}_2,
\]
for $\phi$ as in Theorem 1.1.

This result was proved in [3, Prop. 2.56] for operators $DB$ with $D$ of the form $D = \begin{bmatrix} 0 & \text{div} \\ -\nabla & 0 \end{bmatrix}$, which appear in connection with boundary value problems for divergence form elliptic system. Below we give a simplified proof for general operators of the form $DB$. Before doing so, we complete the proof of Theorem 1.1 using Theorem 5.1.

**Proof of Theorem 1.1.** Proposition 2.4 proves that $\tilde{T}$ is closed, densely defined and has the stated estimates of spectrum and resolvents. The square function estimates for $\psi(tT)$ follows from Propositions 4.5 and 4.6 using Lemma 4.1. Using Proposition 2.8, the reverse square function estimates follow by duality.

To prove the non-tangential maximal function estimates, we first note that estimates $\gtrsim$ hold since $\|\phi(tT)f - f\|_2 \to 0$ as $t \to 0$ and
\[
\sup_{t>0} t^{-1} \int_t^{2t} \|\phi(sT)f\|_2^2 ds \lesssim \|\tilde{N}_*(\phi(tT)f)\|_2,
\]
as proved in [2, Lem. 5.3]. For the estimate $\lesssim$, it suffices to consider the resolvents $(I + itT)^{-1}$, since
\[
\|\tilde{N}_*(\phi(tT)f - (I + itT)^{-1}f)\|_2^2 \lesssim \int_0^{\infty} \|\phi(tT)f - (I + itT)^{-1}f\|_2^2 \frac{dt}{t} \lesssim \|f\|_2^2,
\]
where a proof of the first estimate can be found in [2, Lem. 5.3] and the second estimate follows from square function estimates with $\psi(\lambda) := \phi(\lambda) - (1 + i\lambda)^{-1}$. By Proposition 2.4, it remains to prove
\[
(7) \quad \|\tilde{N}_*(R^2_t f)\|_2 \lesssim \|f\|_2, \quad f \in \mathcal{H}_2,
\]
\[
(8) \quad \|\tilde{N}_*(R^1_t f)\|_2 \lesssim \|f\|_2, \quad f \in B^1 \mathcal{H}_1,
\]
\[
(9) \quad \|\tilde{N}_*(R^2_t R^1_t f)\|_2 \lesssim \|f\|_2, \quad f \in B^1 \mathcal{H}_1.
\]

Estimate (7) follows from Theorem 5.1. For (8), we estimate
\[
\|\tilde{N}_*(R^1_t B_1 v)\|_2 = \|\tilde{N}_*(B_1 (I + itD_1 B_1)^{-1}v)\|_2 \lesssim \|v\|_2 \lesssim \|B_1 v\|_2, \quad v \in \mathcal{H}_1,
\]
using that $B_1$ is a bounded multiplication operator and Theorem 5.1 with $D_2 B_2$ replaced by $D_1 B_1$, for the first estimate, and that $B_1$ is accretive on $\mathcal{H}_1$ for the second estimate.
To prove (9), we introduce the auxiliary non-tangential maximal functions
\[
\tilde{N}_s^k u(x) := \sup_{t > 0} \left(2^{-kn}t^{-(1+n)} \int_{B(x,2^k t) \times (t/2,t)} |u(s,y)|^2 ds dy\right)^{1/2}, \quad k \geq 0.
\]
From $L_2$ off-diagonal estimates for $R_t^2$, as in Proposition 4.3, we get that
\[
\tilde{N}_s^0 (R_t^2 R_t^1 f)(x) \lesssim \sum_{k=0}^{\infty} 2^{-kn} \tilde{N}_s^k (R_t^1 f)(x).
\]
We claim that
\[
\| \tilde{N}_s^k (R_t^1 f) \|_2 \lesssim 2^{kn/2} \| \tilde{N}_s^0 (R_t^1 f) \|_2.
\]
Thus, choosing $m$ large, we obtain the desired estimates
\[
\| \tilde{N}_s^0 (R_t^2 R_t^1 f) \|_2 \lesssim \| \tilde{N}_s^0 (R_t^1 f) \|_2 \lesssim \| f \|_2.
\]
To prove the claim, assume that $\tilde{N}_s^k g(x) > \lambda$. It follows that there exists $x_1 \in B(x_0,2^k t)$ such that $t^{-(1+n)} \int_{B(x_1,t/2) \times (t/2,t)} |g|^2 \gtrsim \lambda^2$. We conclude that for all $y \in B(x_1, t/2)$ we have $\tilde{N}_s^k g(y) \gtrsim \lambda$, and therefore
\[
| \{ x ; \tilde{N}_s^k g(x) > \lambda \} | \lesssim | \{ x ; M(\chi_{\{y ; \tilde{N}_s^0 g(y) > \lambda\}})(x) \gtrsim 2^{-kn} \} | \lesssim 2^{kn} | \{ y ; \tilde{N}_s^0 g(y) \gtrsim \lambda \} |,
\]
using the weak $L_1$ boundedness of the Hardy–Littlewood maximal function $M$, from which the claim follows. \hfill \Box

We now turn to the proof of Theorem 3, where we write $D := D_2$ and $B := B_2$ to simplify notation. We use the following version of the Caccioppoli estimate for equations of the form $\partial_t v + BDv = 0$.

**Lemma 5.2.** Let $\eta \in C_0^\infty (\mathbb{R}^{1+n})$ and consider a function $v$ solving $\partial_t v + BDv = 0$ on a neighbourhood of $\text{supp } \eta$. Then
\[
\iint |Dv(t,x)|^2 |\eta(t,x)|^2 dt dx \lesssim \iint |v(t,x)|^2 |\nabla \eta(t,x)|^2 dt dx.
\]

**Proof.** Integrating by part twice, using anti-symmetry of $\partial_t$ and symmetry of $D$, we obtain
\[
\iint (BDv,Dv) \eta^2 = - \iint (\partial_t v,Dv) \eta^2
\]
\[
= \iint (v, \partial_t Dv) \eta^2 + 2 \iint (v,Dv) \eta \partial_t \eta = - \iint (v, BDv) \eta^2 + 2 \iint (v,Dv) \eta \partial_t \eta
\]
\[
= - \iint (D(\eta^2 v), BDv) + 2 \iint (v,Dv) \eta \partial_t \eta.
\]
We get the estimate
\[
\text{Re} \iint (BDv,Dv) \eta^2 \lesssim \iint |v||Dv||\eta||\nabla \eta|.
\]
For each $t \in \mathbb{R}$, we have by the accretivity of $B$ that
\[
\int |D(\eta v)|^2 dx \lesssim \text{Re} \int (BD(\eta v), D(\eta v)) dx.
\]
Integrating both sides with respect to $t$ and using the product rule for the derivatives, we get
\[
\iint |Dv|^2 \eta^2 dtdx \leq \iint (|Dv|\eta)(|v||\nabla \eta|) dtdx + \iint |v|^2|\nabla \eta|^2 dtdx.
\]
Using the absorption inequality, we obtain the stated estimate. \hfill \square

**Proof of Theorem 5.1.** To prove the non-tangential maximal function estimates, it suffices to estimate the semigroups $e^{-tDB} \chi^+(DB)$ and $e^{tDB} \chi^-(DB)$, since
\[
\|\tilde{N}_s(\phi(tDB)f - e^{-tDB} \chi^+(DB)f - e^{tDB} \chi^-(DB)f)\|^2_2 \\
\leq \int_0^\infty \|\phi(tT)f - e^{-tDB} \chi^+(DB)f - e^{tDB} \chi^-(DB)f\|^2_2 dt \lesssim \|f\|^2,
\]
where the second estimate follows from square function estimates for $DB$ with $\psi(\lambda) := \phi(\lambda) - e^{-|\lambda|}$. Moreover, by a limiting argument, we may assume that $f \in R(D)$.

Consider first $e^{-tDB} \chi^+(DB)f$ and a Whitney region $W = B(x_0, t_0) \times (t_0/2, t_0)$. Write $\chi^+(DB)f = Dv$ with $v \in \chi^+(BD)L_2 \cap \mathcal{D}(D)$. Let $P$ be the orthogonal projection onto $\overline{R(D)}$, and denote by $[Pv] := |B(x_0, t_0)|^{-1} \int_{B(x_0, t_0)} P\psi(x) dx$ the average of $P\psi$ in $\overline{R(D)}$. Note that $Dv = D(Pv)$ since $P$ projects along the null space $N(D)$.

Define the function
\[
w(t, x) := t\psi_1(tBD)BDv(x) + \phi_1(tBD)(Pv - [Pv])(x), \quad (t, x) \in \mathbb{R}^{1+n}_+,
\]
where
\[
\psi_1(\lambda) := \lambda^{-1}(e^{-\lambda} - (1 - \lambda)/(1 + \lambda^2)) \chi^+(\lambda),
\]
\[
\phi_1(\lambda) := (1 - \lambda)/(1 + \lambda^2).
\]
In the second term, $[Pv]$ is regarded as a constant function. Since $\phi_1(tBD)$ is a linear combination of resolvents, it extends to a bounded operator $L_\infty(\mathbb{R}^n) \to \mathcal{L}_2(B(x_0, t_0))$ as in the estimate \([4]\). For the first term, the decay of $\psi_1$ at $\lambda = 0$ and $\infty$ shows that we have square function estimates for $\psi_1(tBD)$. We calculate for $t > 0$ that
\[
Dw = De^{-tBD}v = e^{-tDB} \chi^+(DB)f,
\]
\[
\partial_t w = -BD e^{-tBD}v,
\]
using $D\phi_1(tBD)[Pv] = \partial_t \phi_1(tBD)[Pv] = 0$. Since $\partial_t w + BDw = 0$, we get from Lemma \([5,2]\) that
\[
\iint_W |e^{-tDB} \chi^+(DB)f|^2 dtdx = \iint_W |Dw|^2 dtdx \lesssim \iint_{\tilde{W}} |t^{-1}w|^2 dtdx,
\]
with a slightly enlarged Whitney region $\tilde{W} = B(x_0, 2t_0) \times (t_0/4, 2t_0)$. Using square function estimates, we obtain
\[
\|\tilde{N}_s(\psi_1(tBD)BDv)\|^2_2 \lesssim \int_0^\infty \|\psi_1(tBD)BDv\|^2_2 dt \lesssim \|BDv\|^2_2 \lesssim \|f\|^2_2.
\]
so it remains to estimate \( \phi_1(t BD)(Pv - [Pv]) \). To this end, for fixed \( t/4 < t < 2t_0 \), use the \( L_2 \) off-diagonal estimates for the resolvents of \( BD \) to get

\[
\| \phi_1(t BD)(Pv - [Pv]) \|^2_{L_2(B(x_0, 2t_0))} \lesssim \sum_{k=0}^\infty 2^{-km} \| P^* - [Pv] \|^2_{L_2(B(x_0, 2^k t_0))} \lesssim \sum_{k=0}^\infty 2^{k(-m+3n-2n/p+2)} t_0^{-2n/p+2} \| \nabla v \|^2_{L^p(B(x_0, 2^k t_0))},
\]

using the Sobolev–Poincaré inequality from Lemma 4.4 with suitable \( 1 \leq p < 2 = q \), \( \Omega = B(x_0, 2^k t_0) \) and \( S = B(x_0, 2t_0) \). Integrating with respect to \( t \), we get

\[
t_0^{1-n} \int_W \| t^{-1} \phi_1(t BD)(Pv - [Pv]) \|^2 \, dt \, dx \lesssim \sum_{k=0}^\infty 2^{k(-m+3n+2)} M(\| \nabla Pv \|_p)(x_0)^{2/p} \lesssim M(\| \nabla Pv \|_p)(x_0)^{2/p},
\]

choosing the parameter \( m \) large. Therefore, boundedness of the Hardy–Littlewood maximal function \( M \) on \( L_{2/p} \) gives

\[
\| \tilde{N}_*(e^{-tBD} \chi^+(DB)f) \|^2 \lesssim \| f \|^2 + \| \nabla Pv \|^2 \lesssim \| f \|^2 + \| DPv \|^2 \lesssim \| f \|^2,
\]

by the ellipticity of \( D \) on \( R(D) \).

The estimate of the semigroup \( e^{tDB} \chi^-(DB) \) follows from the above estimate upon replacing \( D \) by \( -D \), since \( e^{-(-DB)} \chi^+(DB) = e^{tDB} \chi^-(DB) \). This proves the non-tangential maximal function estimate for \( \phi(tDB) \). \( \square \)

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