CONSTRUCTION OF NEARLY PSEUDOCOMPACTIFICATIONS

BISWAJIT MITRA AND SANJIB DAS

Abstract. A space is nearly pseudocompact if and only if \( vX \setminus X \) is dense in \( \beta X \setminus X \). If we denote \( K = \text{cl}_{\beta X}(vX \setminus X) \), then \( \delta X = X \cup (\beta X \setminus K) \) is referred by Henriksen and Rayburn [3] as nearly pseudocompact extension of \( X \). Henriksen and Rayburn studied the nearly pseudocompact extension using different properties of \( \beta X \). In this paper our main motivation is to construct nearly pseudocompact extension of \( X \) independently and not using any kind of extension property of \( \beta X \). An alternative construction of \( \beta X \) is made by taking the family of all \( z \)-ultrafilters on \( X \) and then topologized in a suitable way. In this paper we also adopted the similar idea of constructing the \( \delta X \) from the scratch, taking the collection of all \( z \)-ultrafilters on \( X \) of some kind, called \( hz \)-ultrafilters, together with fixed \( z \)-ultrafilter and then be topologized in the similar way what we do in the construction of \( \beta X \). We have further shown that the extension \( \delta X \) is unique with respect to certain properties.

1. Introduction

Throughout the paper, by a space we shall always mean Tychonoff. For a space \( X \), let \( C(X) \) and \( C^*(X) \) be the rings of real-valued continuous and bounded continuous functions respectively. The Stone-\( \check{C} \)ech compactification of \( X \), usually denoted as \( \beta X \), is the unique compactification of \( X \) with respect to the extension property in the sense that every compact-valued continuous function can be uniquely extended to \( \beta X \). Similarly the Hewitt realcompactification \( vX \) of \( X \) is the unique realcompactification of \( X \) in the sense that every realcompact-valued continuous function can be continuously extended to \( vX \). In the year 1980, Henriksen and Rayburn [3] introduced nearly pseudocompact spaces. A space \( X \) is nearly pseudocompact if and only if \( vX \setminus X \) is dense in \( \beta X \setminus X \). If we denote \( K = \text{cl}_{\beta X}(vX \setminus X) \), then \( \delta X = X \cup (\beta X \setminus K) \) is referred by Henriksen and Rayburn [3] as nearly pseudocompact extension of \( X \). Henriksen and Rayburn studied the nearly pseudocompact extension using different properties of \( \beta X \) in [4]. In this paper our main motivation is to construct nearly pseudocompact extension of \( X \) independently and not using any kind of extension property of \( \beta X \). In chapter 6 of [2], the construction of \( \beta X \) was made by taking the family of all \( z \)-ultrafilters on \( X \) and then topologized in a suitable way. In this paper we also adopted the similar idea of constructing the \( \delta X \) by taking the collection of all \( hz \)-ultrafilters together with fixed \( z \)-ultrafilters on \( X \) and then topologized in the similar way what we do in the construction of \( \beta X \). We have further shown that the extension \( \delta X \) is unique with respect to certain properties.

In section 2, we discussed about few preliminary topics. In section 3, we initially started with an equivalent definitions of hard sets and nearly pseudocompact spaces where \( \beta X \) is not involved. Then we have provided alternative proofs of those theorems where properties of \( \beta X \) are involved in the original proof. We had to do this intending to avoid any kind of cyclic arguments. Finally we constructed \( \delta X \) as collection of all \( hz \)-ultrafilters and fixed \( z \)-ultrafilters and topologized it and proved that \( \delta X \) is indeed a nearly pseudocompactification of a space \( X \). In section 4, we tried to explore extension properties of certain kinds of maps such as \( h2pc \) map, hard map and have shown that \( \delta X \) is unique with respect to some properties.

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2. Preliminaries

In this paper, we used the most of preliminary concepts, notations and terminologies from the classic monograph of L.Gillman and M.Jerison, Rings of Continuous Functions [2]. However for ready references, we recall few notations, frequently used over here. For any $f \in C(X)$ or $C^*(X)$, $Z(f) = \{ x \in X : f(x) = 0 \}$, called zero set of $f$. Complement of zero set is called cozero set or cozero part of $f$, denoted as $cozf$. For any $f \in C(X)$ or $C^*(X)$, $cl_X(X \setminus Z(f))$ is called the support of $f$. A space is realcompact if and only if every $z$-ultrafilter with countable intersection property is fixed. In the year 1976, Rayburn [1] introduced hard set.

**Definition 1.** A subspace $H$ of $X$ is called hard in $X$ if $H$ is closed in $X \cup cl_{\beta X}(vX \setminus X)$.

It immediately follows that every hard set is closed in $X$, but the converse is obviously not true. In the same paper [Lemma 1.3, [1]], Rayburn proved the following characterization of hard subsets of $X$.

**Theorem 2.** A closed set $H$ is hard in $X$ if and only if there exists a compact set $K$ of $X$ such that for any open neighbourhood $G$ of $K$, $H \setminus G$ is completely separated from the complement of some realcompact set.

In particular, it follows that if a closed set is completely separated from complement of some realcompact space, then the set is hard in $X$. However the converse may not be true. The converse is also true if $X$ is locally realcompact, that is, for each point $x$, there is a realcompact neighbourhood of $x$ [Corollary 1.5, [1]].

In the year 1980, Henriksen and Rayburn [3] introduced nearly pseudocompact space. A space $X$ is nearly pseudocompact if $vX \setminus X$ is dense in $\beta X \setminus X$. It is immediate that every pseudocompact space is nearly pseudocompact. The converse is not true in general. In the same paper they have shown that every anti-locally realcompact space (i.e. the space with no points having realcompact neighbourhood) is nearly pseudocompact [Corollary 3.5, [3]] and produced a series of examples of anti-locally realcompact space which are not pseudocompact [Proposition 3.6, [3]]. However in [6], Example 4.14, the author gave an example of nearly pseudocompact but not pseudocompact, not even anti-locally realcompact.

Henriksen and Rayburn in their paper [[3] Theorem 3.2] furnished the following characterizations of nearly pseudocompact spaces.

**Theorem 3.** The followings are equivalent.

1. $X$ is nearly pseudocompact
2. Every hard set is compact.
3. Every regular hard set is compact.
4. Each decreasing sequence of non-empty regular hard sets has non-empty intersection.
5. $X$ can be expressed as $X_1 \cup X_2$, where $X_1$ is a regular closed almost locally compact pseudocompact subset and $X_2$ is regular closed anti-locally realcompact and $int_X(X_1 \cap X_2) = \emptyset$, where a subset of a space $T$ is almost locally compact if it is the $T$-closure of the set of all points in $T$ having compact neighbourhoods.

Mitra and Acharyya in [5] worked further on nearly pseudocompact spaces using the $C_H(X)$ and $H_\infty(X)$. $C_H(X)$ is the subring of $C(X)$ consisting of all those members of $C(X)$ which have hard support. $H_\infty(X)$ is the family of all real-valued continuous functions so that the set $\{ x \in X : |f(X)| \geq \frac{1}{n} \}$ is hard in $X$ for all $n \in \mathbb{N}$. $H_\infty(X)$ is also a subring of $C(X)$ and $C_H(X) \subseteq H_\infty(X)$. In [6], Mitra and Chowdhury showed that $H_\infty(X)$ is precisely $C_{RC}(X)$, the subring of $C(X)$ which is ideal of $C(X)$ too, consisting of all those continuous maps whose cozero part are realcompact. A result related to $C$-embeddedness has been used here. As every $C$-embedded subset is $C^*$-embedded, any zero set disjoint with a $C$-embedded subset can be completely separated [Chapter 1, [2]].

A subset of $X$ is called $z$-embedded in a space $Y$ if every zero set in $X$ is the trace of some zero set in $Y$ on $X$. It is indeed a generalized notion of both $C$-embedding and $C^*$-embedding. Every cozero set in $X$ is $z$-embedded in $X$. Countable union of $z$-embedded real compact subsets
is again realcompact. All the literature of $z$-embeddedness are available in the book, Hewitt-Nachbin Spaces by M. Weir [Page 108-120, [8]].

We also recall few facts about $z$-filter, prime $z$-filter and $z$-ultrafilter. A $z$-filter $F$ is called fixed if $\beta F \neq \emptyset$. Otherwise it is called free $z$-filter. Lemma 4.10 of [2] tells that a zero set $Z$ is compact if and only if $Z$ is not a member of any free $z$-filter. $z$-ultrafilters are maximal $z$-filters and prime $z$-filters are those $z$-filters where if union of two zero sets is a member of the $z$-filter then any one of them must be a member of that $z$-filter. A prime $z$-filter $F$ converges to a point $p$ if and only if it clusters at the point $p$ if and only if $\cap F = \{p\}$. Every $z$-ultrafilter is a prime $z$-filter. However the converse may not be true. Fixed $z$-ultrafilters on $X$ are precisely of the form $\{A_x : x \in X\}$, where $A_x := \{Z \in \mathcal{Z}[X] : x \in Z\}$. From theorem 5.7 of [2] it follows that a $z$-ultrafilter is real if and only if for any $f \in C(X)$ there exists a natural number $n$ such that $Z_n(f) \notin U$ where $Z_n(f) := \{x \in X : |f(x)| \geq n\}$. Let $\tau : X \rightarrow Y$ be a continuous map. If $F$ be a $z$-filter on $X$ then $\tau^*(F) = \{Z \in \mathcal{Z}(X) : \tau^{-1}(Z) \in F\}$ is a $z$-filter on $Y$. If $F$ is prime then $\tau^*(F)$ is prime. However analogous version of $z$-ultrafilter does not hold true. If $S \subseteq X$ and $F$ be a $z$-ultrafilter on $X$ then $i^*(F) \cap S = \{Z \cap S : Z \in i^*(F)\}$.

### 3. Construction of nearly pseudocompactifications

In the year 1987, Henriksen and Rayburn in their paper [4] discussed about nearly pseudocompact extension. They call $X \cup (\beta X \setminus K)$ as nearly pseudocompact extension of $X$, where $K = cl_{\beta X}(\nu X \setminus X)$, denoted by $\delta X$ and studied different properties of this extension. For instance a set $H$ is hard in $X$ if and only if there exists a compact set $K$ in $\delta X$ so that $K \cap X = H$. But all these proofs were done by crucially using the various properties, specially the extension property of $\delta X$. In this section we shall try to explore a direct construction of nearly pseudocompactification $\delta X$. Likewise the construction of $\beta X$ in chapter 6 of [2], we shall almost follow the same lines of arguments. In this paper we may only use $\beta X$ as a collection of all $z$-ultrafilters on $X$ which is Tychonoff and nothing more. In consequence, we hereby recall the construction of $\beta X$. If $Z[X]$ denotes the family of all zero sets in $X$, then $\{Z : Z \in Z[X]\}$ forms a base for closed sets for some topology on $\beta X$ which is compact and Hausdorff and hence Tychonoff in which $X$ is densely embedded, where $\overline{\mathcal{Z}} = \{p \in \beta X : Z \in p\}$. In order to avoid junk of notations, we shall denote fixed $z$-ultrafilters by the corresponding points of $X$. So when $x \in X$, we shall treat it as per requirement either as an element of $X$ or a fixed $z$-ultrafilter $A_x := \{Z \in \mathcal{Z}(X) : x \in Z\}$ that converges to the point $x$. In this frame work, $x \in Z \iff Z \in x$ is no more self-contradictory statement and we denote this statement by ($*$) for future references. Likewise we shall construct nearly pseudocompactification of $X$ as a collection of certain type of $z$-ultrafilters. But prior to that we redefine some relevant notions and alternate proofs of few results to be used here without involving $\beta X$ as a Stone-Čech compactification of $X$.

Let $RX$ denotes the family of all free real $z$-ultrafilters on $X$ and $HRX$ denotes the family of all hyper-real $z$-ultrafilters. Then $\beta X \setminus X$ is the union of $RX$ and $HRX$.

We already know that a space $X$ is pseudocompact if every real-valued continuous function on $X$ is bounded. However we hired the following definition of realcompact space from Mandelker [7], Theorem 5.1, who used this definition to study some likewise properties of realcompactness in non-Tychonoff set up.

**Definition 4.** A space $X$ is called realcompact if every stable family of closed sets with finite intersection property is fixed.

It immediately follows [Theorem 5.2,[7]] that every compact set is realcompact, every closed subset of a realcompact space is realcompact and a space is compact if and only if it is pseudocompact and realcompact. The following theorem is given in chapter 5 and chapter 8 of [2]. But since our definition of realcompact is different from the definition given in [2], we give here an alternate proof of the following theorem.

**Theorem 5.** The followings are equivalent.

1. A space is realcompact.
(2) Any prime \( z \)-filter with countable intersection property is fixed.
(3) Any \( z \)-ultrafilter with countable intersection property is fixed.

Proof. (1) \( \Rightarrow \) (2): Let \( F \) be a prime \( z \)-filter on \( X \) with countable intersection property. Let for each \( n \), \( Z_n(f) = \{ x \in X : |f(x)| \geq n \} \) and \( Z_n(f) = \{ x \in X : |f(x)| \leq n \} \). Then for each \( f \in C(X) \) there exists \( n \) such that \( Z_n(f) \notin F \). Then \( Z_n(f) \in F \) as \( F \) is prime. So \( F \) turns out to be stable family of closed subsets \( X \). Hence it is fixed.

(2) \( \Rightarrow \) (3): Trivial.

(3) \( \Rightarrow \) (1): Let \( S \) be a stable family of closed subsets of \( X \). \( F = \{ Z \in Z[X] : Z \supseteq F, \text{for some } F \in S \} \). Then \( F \) is a family of zero sets having finite intersection property. Hence it can be extended to a \( z \)-ultrafilter \( U \). Then for each \( f \in C(X) \), there exists a \( D \in S \), such that \( f \) is bounded on \( D \). So there exists an \( n \) such that \( Z_n(f) \supseteq D \) and hence \( Z_n(f) \in F \). So \( Z_n(f) \notin U \). So \( U \) is a real \( z \)-ultrafilter. Hence, \( \cap U \neq \emptyset \). As \( X \) is Tychonoff, the family of zero sets form a base for closed sets and therefore \( \cap U \subseteq \cap F = \cap S \). So \( \cap S \neq \emptyset \). Hence \( X \) is realcompact. \( \square \)

The following lemma is very useful in this paper and can be easily derived from the proof of Lemma 8.12 of [2].

**Lemma 6.** Any prime \( z \)-filter with countable intersection property is contained in unique real-\( z \)-ultrafilter.

The following theorem was proved in [2], corollary 8.14 involving intricately the extension property of \( \beta X \) and \( vX \). Here we have proved this results without using any kind of extension property of \( \beta X \).

**Theorem 7.** Every cozero subset of a realcompact space is realcompact.

Proof. Let \( S \) be a cozero subset of \( X \). Suppose \( X \setminus S = Z(f) \) for some \( f \in C(X) \). Let \( F \) be a real-\( z \)-ultrafilter on \( S \). Then \( i^\#(F) \) is a prime-\( z \)-filter on \( X \) and \( i^\#(F) \cap S = F \) as \( S \) is \( z \)-embedded in \( X \). It is contained in the unique real \( z \)-ultrafilter \( U \) on \( X \). If \( Z(f) \notin U \), then \( \{ x \in X : |f(x)| \leq \frac{1}{n} \} \) is a member of \( U \), \( \forall n \) and \( \{ x \in X : |f(x)| \geq \frac{1}{n} \} \notin U \), for all \( n \). As \( i^\#(F) \) is a prime \( z \)-filter, \( \{ x \in X : |f(x)| \leq \frac{1}{n} \} \in i^\#(F) \), for all \( n \). Hence \( \{ x \in X : |f(x)| \leq \frac{1}{n} \} \cap S \in F \). As \( F \) is real \( z \)-ultrafilter on \( S \), \( \cap \{ x \in X : |f(x)| \leq \frac{1}{n} \} \cap S \neq \emptyset \). So there exists a point \( Z(f) \leq \frac{1}{n} \) for all \( n \). Hence \( f(z) = 0 \), which is a contradiction. So \( Z(f) \notin U \). Then \( \cap U \cap Z(f) = \emptyset \), otherwise \( Z(f) \) would intersect every member of \( U \). This would imply that \( Z(f) \notin U \), a contradiction. As \( U \) is real and \( X \) is realcompact, \( \cap U \neq \emptyset \). Now \( \cap U \subseteq (\cap U \cap S) \subseteq (\cap i^\#(F)) \cap S = \cap (i^\#(F) \cap S) = \cap F \). This follows that \( F \) is fixed. Hence \( S \) is realcompact. \( \square \)

Now we shall give definition of a hard set. As the original definition of it involves \( \beta X \), we therefore adopted the intrinsic characterization of hard set given by Rayburn[1], menioned in Theorem 2 of this paper as a definition of hard set.

**Definition 8.** A subset \( H \) of a space \( X \) is called hard in \( X \) if it is closed in \( X \) and there exists a compact set \( K \) such that for any open neighbourhood \( V \) of \( K \) there exists a realcompact subset \( P \) such that \( H \setminus V \) is completely separated from the complement of \( P \). Henceforth we shall refer \( K \) as the compact set satisfying the property of hardness.

**Remark 9.** It immediately follows that a closed subset of \( X \) which is completely separated from a complement of realcompact subset containing the closed set, is hard in \( X \) (Simply take \( K = \emptyset \)). Hence any zero set contained in a realcompact cozero set is hard. Moreover every closed subset of hard set is hard as if \( L \) is a closed subset of a hard set \( H \) and \( K \) is the compact set satisfying the proprty of hardness of \( H \), then same \( K \) will satisfy the property of hardness of \( L \).

The following theorem is a little modification of the definitin of hard set but equally important in our paper.
\textbf{Theorem 10.} A closed set \( H \) is hard in \( X \) if and only if there exists compact subset \( K \) of \( H \) such that for any open neighbourhood \( G \) of \( K \), \( H \setminus G \) is completely separated from the complement of some realcompact set.

\textit{Proof.} Suppose \( H \) be a closed set in \( X \) which is hard in \( X \).

We have to show that there exists a compact set \( K \) in \( H \) which satisfies the property of hardness. Since \( H \) is hard, there exists a compact subset \( L \) of \( X \) which satisfies the property of hardness. \( X \) being Tychonoff, \( L \) is closed and so is \( H \cap L \). Again \( H \cap L \subset L \) which implies \( H \cap L \) is compact subset of \( H \), we call it as \( K \). Now, our claim is, \( K \) satisfies the property of hardness.

Let \( V \) be an open neighbourhood of \( K \). Then \( L \setminus V \) is compact subset of \( X \) and \( H \) is closed in \( X \) with \( H \cap (L \setminus V) = \emptyset \). Then \( H \) and \( L \setminus V \) can be strongly separated. So, there exists an open set \( U \) in \( X \) containing \( L \setminus V \) and \( H \cap U = \emptyset \). Now \( V \cup U \) is an open set in \( X \) such that \( L \subset V \cup U \) i.e. \( V \cup U \) is an open neighbourhood of \( L \). Therefore by the property of hardness of \( L \), \( H \setminus (V \cup U) \) is completely separated from the complement of a realcompact subset of \( X \). But \( H \setminus (V \cup U) = (H \setminus V) \cap (H \setminus U) = H \setminus V \), as \( H \cap U = \emptyset \). So \( H \setminus V \) is completely separated from the complement of a realcompact subset of \( X \). Hence we have a compact subset \( K \) of \( H \) which satisfies the property of hardness.

Conversely, let \( H \) be a closed subset of \( X \) and there exists a compact subset \( K \) of \( H \) such that for any open neighbourhood \( G \) of \( K \), \( H \setminus G \) is completely separated from the complement of some realcompact subset of \( X \).

Since \( K \) is compact in \( H \), it is compact in \( X \) also. Hence we have a compact set \( K \) in \( X \) such that the given condition holds. Therefore \( H \) is hard in \( X \).

\( \blacksquare \)

The following trivial but important observations are indeed direct consequences of the Theorem 10.

\textbf{Theorem 11.} Let \( H \) be a closed subset of \( X \). Then the followings are equivalent.

1. \( H \) is hard in \( X \).
2. There exists a compact subset \( K \) of \( H \) such that for any open neighbourhood \( V \) of \( K \), there exist a zero set \( Z_1 \) and a realcompact cozero set \( X \setminus Z_2 \) such that \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \).
3. There exists a compact subset \( K \) of \( H \) such that for any open neighbourhood \( V \) of \( K \), there exist a regular hard zero set \( Z_1 \) and a realcompact cozero set \( Z_2 \) with \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \).

\textit{Proof.} (1) \( \Rightarrow \) (2) : Let \( H \) is hard in \( X \). So there exists a compact subset \( K \) of \( H \) such that for any open neighbourhood \( V \) of \( K \), \( H \setminus V \) is completely separated from the complement of a realcompact subspace \( P \) of \( X \). i.e. \( H \setminus V \) and \( X \setminus P \) are completely separated. Then there exist two disjoint zero sets \( Z_1 \) and \( Z_2 \) such that \( H \setminus V \subset Z_1 \) and \( X \setminus P \subset Z_2 \) with \( Z_1 \cap Z_2 = \emptyset \). So \( Z_1 \subset X \setminus Z_2 \) and \( X \setminus P \subset Z_2 \Rightarrow X \setminus Z_2 \subset P \). As \( P \) is realcompact and \( X \setminus Z_2 \) is a cozero subset of \( P \), this implies \( X \setminus Z_2 \) is a realcompact cozero set in \( X \) and \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \).

(2) \( \Rightarrow \) (1) : Let there exists a compact set \( K \) of \( H \) such that for any open neighbourhood \( V \) of \( K \), there exist a zero set \( Z_1 \) and a realcompact cozero set \( X \setminus Z_2 \) such that \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \). Therefore \( Z_1 \cap Z_2 = \emptyset \), then \( H \setminus V \) and \( Z_2 \) are completely separated where \( Z_2 \) is the complement of a realcompact subset \( X \setminus Z_2 \).

(1) \( \Rightarrow \) (3) : Let \( H \) is hard in \( X \). Then as (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (1), so there exists a compact set \( K \) in \( H \) such that for any open neighbourhood \( V \) of \( K \), there exist a zero set \( Z_1 \) and a real compact cozero set \( X \setminus Z_2 \) such that \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \). i.e. \( Z_1 \cap Z_2 = \emptyset \), so \( Z_1 \) and \( Z_2 \) are completely separated. Then there exists \( g \in C(X) \) such that \( g(Z_1^c) = \{0\} \) and \( g(Z_2^c) = \{1\} \).

Let \( Z_1 = \{x \in X : g(x) \leq \frac{1}{2}\} \) and \( Z_2 = \{x \in X : g(x) \geq \frac{1}{2}\} \). Therefore \( Z_1 \subset Z_2 \), \( Z_2 \subset Z_2 \) and \( Z_1 \cap Z_2 = \emptyset \). Also \( \text{cl}_{X \setminus Z_1} Z_1 = Z_1 \Rightarrow Z_1 \) is a regular closed zero set and \( Z_1 \subset X \setminus Z_2 \) where \( X \setminus Z_2 \) is a realcompact cozero set in \( X \). Hence \( Z_1 \) is a regular hard zero set and \( X \setminus Z_2 \) is a realcompact cozero set such that \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \).

(3) \( \Rightarrow \) (1) : Let the given condition holds. i.e. \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \). Then \( Z_1 \cap Z_2 = \emptyset \). So \( H \setminus V \) and \( Z_2 \) are completely separated where \( Z_2 \) is the complement of a realcompact subset of \( X \). Hence \( H \) is hard in \( X \).
Likewise the definition of hard set, we now give definition of nearly pseudocompact space. Again its original definition involves $\beta X$. We here adopted the intrinsic characterization of nearly pseudocompact space given by Henriksen and Rayburn in their paper [3], mentioned in Theorem 3 (5) in this paper.

**Definition 12.** A space $X$ is nearly pseudocompact if $X$ can be expressed as $X_1 \cup X_2$, where $X_1$ is a regular closed almost locally compact pseudocompact subset and $X_2$ is regular closed anti-locally realcompact and $\text{int}_X(X_1 \cap X_2) = \emptyset$.

The following facts are immediate from the definition of the nearly pseudocompact space. If $X$ is almost locally compact nearly pseudocompact space then $X = X_1$. Hence $X$ is pseudocompact. On the other side if $X$ is anti-locally realcompact space then we may take $X_1 = \phi$, which follows from the definition that $X$ is nearly pseudocompact. We have the following theorem already done by Henriksen and Rayburn in [3]. But they have proved the theorem involving $\beta X$. Here we have given an alternate proof without using $\beta X$.

**Theorem 13.** The followings are equivalent.

1. A space $X$ is nearly pseudocompact.
2. Every hard set is compact.
3. Every regular hard set is compact.

**Proof.** (1) $\Rightarrow$ (2) : Suppose $X$ is nearly pseudocompact. $X = X_1 \cup X_2$, where $X_1$ is a regular closed almost locally compact pseudocompact subset and $X_2$ is regular closed anti-locally realcompact and $\text{int}_X(X_1 \cap X_2) = \emptyset$. Let $H$ be a hard set. There exists a compact subset $K \subseteq H$ satisfying the property of hardness. Let $U = \{U_\alpha : \alpha \in \lambda\}$ be an open cover of $H$. There exist finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $K \subseteq \bigcup_{i=1}^n U_{\alpha_i} = V$ (say). Then there exists a regular zero set $Z$ such that $H \setminus V \subseteq \text{int}_X Z \subseteq Z$ and $Z$ is realcompact. That means $\text{int}_X Z \cap X_2 = \emptyset$, otherwise $X_2$ would contain a point having realcompact neighbourhood, a contradiction. So $\text{int}_X Z \subseteq X_1$. Then $\text{cl}_X \text{int}_X Z$ being a regular closed subset of the pseudocompact space $X_1$ and $Z$, is both pseudocompact and realcompact. Hence it is compact. Now $H \setminus V$ being a closed subset of $\text{cl}_X \text{int}_X Z$, is also compact and to cover it we need another finite sub-collection of $U$ and hence as a conclusion there exists atmost finitely many members of $U$, to cover $H$. Hence $H$ is compact.

(2) $\Rightarrow$ (3) : Trivial

(3) $\Rightarrow$ (1) : Suppose $X$ is not nearly pseudocompact. Then $X \neq X_2$, otherwise $X$ would be anti-locally realcompact and hence nearly pseudocompact. Then $X_1 \neq \emptyset$ and also $X_1$ is not pseudocompact and hence $\text{int}_X X_1$ is not relatively pseudocompact as the closure of open relatively pseudocompact subset is pseudocompact. There exists $f \in C(X)$ such that $f \geq 1$ and is unbounded on $\text{int}_X X_1$ and hence contains a copy of $\mathbb{N}$ along which $f$ tends to infinity. Now for each $n \in \mathbb{N}$, there exists a cozero set $R_n$, such that $\text{cl}_X R_n$ is compact and is contained in $X_1$. So $N \subseteq \cup_n R_n \subseteq \cup_n \text{cl}_X R_n$, which is a $\sigma$-compact subset of $X$ and hence realcompact. As countable union of cozero set is again a cozero set, $\cup_n R_n = P$ (say), a cozero set contained in the realcompact set $\cup_n \text{cl}_X R_n$. Hence $P$ is a realcompact cozero set containing $N$. As $N$ is $C$-embedded in $X$, $N$ and $X \setminus P$ can be completely separated. So there exists a regular zero set $W$ such that $N \subseteq W \subseteq P$. Hence $W$ is a regular hard zero set in $X$. As $N$ is $C$-embedded in $X$, $N$ is closed in $X$ and hence in $W$. This follows that $W$ is not compact, a contradiction. So $X$ is nearly pseudocompact.

The following theorem is already proved by Mitra and Acharyya [Theorem 2.5, [5]] but again using $\beta X$. Here we give an alternate proof not involving $\beta X$.

**Theorem 14.** $X$ is nearly pseudocompact if and only if $C_H(X) \subseteq C^*(X)$, where $C_H(X) = \{f \in C(X) : \text{support of } f \text{ is hard in } X\}$.

**Proof.** If $X$ is nearly pseudocompact, every hard set is compact. So $C_H(X) = C_K(X) \subseteq C^*(X)$. Conversely, if $X$ is not nearly pseudocompact, it follows from the proof of the previous theorem that there exists $N$, a copy of $\mathbb{N}$, $C$-embedded in $X$, contained in a realcompact cozero set $P$. There exists a continuous function $f$ satisfying $f(n) = n$, for all $n \in N$. Choose $g \in C(X)$ such
that \( g = 1 \) on \( N \) and equals to 0 on a neighbourhood of \( X \setminus P \). Then \( \cl_X(X \setminus Z(fg)) \subseteq P \) is hard in \( X \). So \( fg \in \cl_H(X) \) but \( fg \) is unbounded on \( N \). Hence \( fg \notin C^*(X) \).

We begin our discussion with the following definitions of \( hz \)-filter and \( hz \)-ultrafilter.

**Definition 15.** A \( z \)-filter is said to be \( hz \)-filter if it has a base consisting of hard zero sets in \( X \). Maximal \( hz \)-filter is called \( hz \)-ultrafilter.

Likewise we can define prime \( hz \)-filter. However it is not that much relevant in this paper. It is further to be noted that the maximal \( hz \)-filter and maximal \( z \)-filter containing an \( hz \)-filter are essentially same as follows from the following proposition.

**Proposition 16.** Any \( z \)-filter containing a hard zero set is \( hz \)-filter.

*Proof.* If \( B \) is a base for a \( z \)-filter \( A \) and \( H \) be a hard zero set in \( A \), then \( B \cap H = \{H \cap B | B \in B\} \) is a base of \( A \) consisting of hard zero sets. So this fact immediately tells us that maximal \( hz \)-filter is the maximal \( z \)-filter containing \( hz \)-filter.

We have the following characterization of nearly pseudocompact spaces using the ideas of \( hz \)-filter and \( hz \)-ultrafilters.

**Theorem 17.** The followings are equivalent.

1. \( X \) is nearly pseudocompact
2. Every \( hz \)-filter is fixed
3. Every \( hz \)-ultrafilter is fixed.

*Proof.* (1) \( \Rightarrow \) (2) As \( X \) is nearly pseudocompact, every hard set is compact by theorem 13, every \( hz \)-filter is fixed.

(2) \( \Rightarrow \) (3) Trivial.

(3) \( \Rightarrow \) (1) Suppose \( X \) is not nearly pseudocompact. By theorem 14, there exists \( f \in C_H(X) \) which is unbounded. Then \( Z_n = \{x \in X : |f(x)| \geq n\} \) being closed subset of the hard set \( \cl_X(X \setminus Z(f)) \) is hard zero sets in \( X \) and hence can be extended to a \( hz \)-ultrafilter which is not fixed. So we arrive at a contradiction.

Let \( HX \) be the family of all \( hz \)-ultrafilters on \( X \) and \( \eta X \) be the family of all non-\( hz \)-ultrafilters on \( X \). Then \( \beta X = HX \cup \eta X \). We may recall here that \( \beta X \) is the family of all \( z \)-ultrafilters on \( X \).

The following theorem is very important to us.

**Theorem 18.** Every real \( hz \)-ultrafilter is fixed. That is, \( RX \subseteq \eta X \).

*Proof.* Let \( U \) be a real \( hz \)-ultrafilter. Let \( H \in U \) be a hard zero set in \( X \). Then \( H \cap U = \{H \cap Z : Z \in U\} \) is a prime \( z \)-filter on \( H \). Moreover as \( U \) has countable intersection property, \( H \cap U \) has countable intersection property. So \( \bigcap (H \cap U) \) is non empty as \( H \) is realcompact and hence every prime \( z \)-filter in \( H \) with countable intersection property is fixed. So \( U \) is fixed.

**Theorem 19.** If \( q \) is a free \( hz \)-ultrafilter on \( X \). Then \( q \) contains a regular hard zero set in \( X \).

*Proof.* Let \( q \) be a free \( hz \)-ultrafilter on \( X \). Let \( H \) be a hard zero set in \( q \). Then there exists a compact set \( K \) which satisfies the property of hardness. Now every member of \( q \) does not meet with \( K \). If it would be so, \( q \) turns out to be fixed \( z \)-ultrafilter. So there exists a zero set \( Z \in q \) such that \( Z \cap K = \emptyset \). \( X \setminus Z \) is an open neighbourhood of \( K \). Hence \( Z \cap H = H \setminus (X \setminus Z) \) is completely separated by complement of a realcompact subset. So there exists a regular zero set \( Z_0 \) containing \( Z \cap H \) and is contained in a realcompact cozero set in \( X \). Then \( Z_0 \) is a regular hard zero set containing \( Z \cap H \). Hence \( Z_0 \) is a member of \( q \) as \( Z \cap H \) is a member of \( q \).

We shall now develop a construction of nearly pseudocompactification of a space \( X \) in a formal way. For each space \( X \), let \( \delta X \) be the collection of all \( hz \)-ultrafilters along with every fixed \( z \)-ultrafilter on \( X \). So \( \delta X = X \cup HX \). It is clear that \( \delta X \) is a subset of \( \beta X \). Likewise we define \( \overline{Z} = \{p \in \delta X : Z \in p\} \) for each \( Z \in Z(X) \). Then \( \{\overline{Z} : Z \in Z(X)\} \) is a base for
closed sets of some topology on $\delta X$. With respect to this topology $\overline{Z}$ turns out to be $cl_{\delta X}Z$. In fact this topology is indeed the subspace topology under $\beta X$, as already mentioned at the beginning of this section. However treating $\delta X$ as only a subspace of $\beta X$ does no way impede to the progress of our motivation. We have later shown avoiding any properties of $\beta X$, that $\delta X$ is indeed a nearly pseudocompact space in which $X$ is densely embedded. 

We already know that the following theorem is true for $\beta X$. Without any modification it can be easily shown that the following theorem is true in $\delta X$ also.

**Theorem 20.** Any two disjoint zero sets in $X$ has disjoint closure in $\delta X$ and for any two zero sets $Z_1, Z_2$ in $X$, $cl_{\delta X}Z_1 \cap cl_{\delta X}Z_2 = \overline{cl_{\delta X}(Z_1 \cap Z_2)}$.

*Proof.* Let $Z_1$ and $Z_2$ be two disjoint zero sets in $X$. Suppose $p \in cl_{\delta X}Z_1 \cap cl_{\delta X}Z_2$. Then $Z_1 \cap p$ and $Z_2 \cap p$, but $Z_1 \cap Z_2 = \emptyset$, a contradiction. Hence $cl_{\delta X}Z_1 \cap cl_{\delta X}Z_2 = \emptyset$.

Let $p \in cl_{\delta X}Z_1 \cap cl_{\delta X}Z_2$. So $Z_1, Z_2 \in p$. Hence $Z_1 \cap Z_2 \in p$. Thus $p \in \overline{Z_1 \cap Z_2}$. Other side follows trivially. 

**Theorem 21.** If $Z$ is a hard zero set, $cl_{\delta X}Z$ is compact subset of $\delta X$ and also if $H$ be a hard set in $X$ then $cl_{\delta X}H$ is compact.

*Proof.* Let $Z$ be a hard zero set in $X$. Let $\{Z_\alpha : \alpha \in \lambda\}$ be a family of zero sets in $X$ such that $\{\overline{Z_\alpha} \cap Z : \alpha \in \lambda\}$ is a family of basic closed sets in $\overline{Z}$ having finite intersection property. Hence $\{Z_\alpha \cap \overline{Z} : \alpha \in \lambda\}$ is a family of hard zero sets in $X$ satisfying finite intersection property and hence can be extended to a $hz$-ultrafilter $q \in \delta X$. As $Z_\alpha, Z \in q$ for all $\alpha, q \in Z, \overline{Z_\alpha}$ for all $\alpha$. So $q \in \bigcap \{Z_\alpha \cap Z\}$. Hence $\overline{Z}$ is compact.

Second part: let $H$ be a hard set in $X$. There exists a compact set $K$ in $H$ satisfying the property of hardness. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $cl_{\delta X}H$ in $\delta X$. There exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $K \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha = W$. Then there exists a zero set $Z$ and a realcompact cozero set $P$ such that $H \setminus W \subseteq Z \subseteq P$. It is clear that $Z$ is a hard zero set in $X$. As $W$ is open in $\delta X$, $cl_{\delta X}H \setminus \overline{W} \subseteq cl_{\delta X}Z$. By the first part of the theorem, $cl_{\delta X}Z$ is compact. Hence $cl_{\delta X}H \setminus \overline{W}$ is also compact. Another set of finitely many members from the above cover are required to cover $\overline{cl_{\delta X}H \setminus \overline{W}}$. So $cl_{\delta X}H$ can be covered by finitely many members of $\{V_\alpha : \alpha \in \Lambda\}$. Therefore $cl_{\delta X}H$ is compact.

**Theorem 22.** Let $K$ be a compact set in $\delta X$, then $K \cap X$ is hard in $X$. If $H$ be a closed set in $X$ such that $cl_{\delta X}H$ is compact in $\delta X$, then $H$ is hard in $X$.

*Proof.* Let $H = K \cap X$ is a closed subset of $X$ and let $T = K \cap cl_{\delta X}(\eta X \cap X)$ be a compact subset of $H$. Let $V$ be an open neighbourhood of $T$ in $\beta X$. Then $K \setminus V$ is a compact set not intersecting $cl_{\beta X}\eta X$. So there exist zero set neighbourhoods $Z_1$ and $Z_2$ of $K \setminus V$ and $cl_{\beta X}\eta X$ respectively in $\beta X$ with $Z_1 \cap Z_2 = \emptyset$. Then $cl_{\beta X}(Z_2 \cap X) \subseteq \eta X$. Then complement of $Z_2 \cap X$ in $X$ is realcompact containing $Z_1 \cap X$, which again contains $H \setminus (V \cap X)$. Hence $H$ is hard as any neighbourhood of $T$ in $X$ is indeed of the form $V \cap X$ where $V$ is open in $\beta X$.

Second part is trivial as $cl_{\delta X}H \cap X = H$.

**Theorem 23.** A zero set is hard in $X$ if and only if it is not contained in any z-filter which is not hz.

*Proof.* If $H$ is hard zero set in $X$, as discussed above, any z-filter containing $H$ is $hz$-filter. So it can not be member of any non-$hz$-filter.

For the converse, suppose $H$ is a zero set not containing in any non-$hz$-filter on $X$. Let $\{\overline{Z_\alpha} : \alpha \in cl_{\delta X}H\}$ be a family of basic closed sets in $cl_{\delta X}H$ having finite intersection property. Then $\{Z_\alpha\} \cup \{H\}$ satisfies finite intersection property and hence can be extended to a $z$-ultrafilter $q$. For all $\alpha$, $Z_\alpha \in q$ and $H \in q$. By assumption $q$ is an $hz$-ultrafilter. So $q \in \delta X$. Hence $q \in \bigcap \{cl_{\delta X}H \cap Z_\alpha\}$. Hence $cl_{\delta X}H$ is compact. By theorem 22 $H$ is hard in $X$.

**Theorem 24.** $f \in C_{RC}(X)$ if and only if $Z(f) \in q$, for all $q \in RX$, that is $RX \subseteq cl_{\beta X}Z(f)$.
Proof. Let \( f \in C_{RC}(X) \). Suppose \( Z(f) \notin p \) for some \( p \in RX \). There exists \( Z \in p \) so that \( Z \cap Z(f) = \emptyset \). Then \( Z \subseteq X \setminus Z(f) \) which is realcompact. So \( Z \) is hard in \( X \). Then \( p \notin RX \), a contradiction as every real \( hz \)-ultrafilter is fixed.

Conversely, suppose \( f \in C(X) \) such that \( Z(f) \notin q \) for all \( q \in RX \). Let \( U \) be a \( z \)-ultrafilter in \( X \setminus Z(f) = D \) (say) with countable intersection property. Then \( i^\#(U) \) is a prime \( z \)-filter with countable intersection property and hence is contained in a real \( z \)-ultrafilter (say) \( p \) on \( X \). If \( Z(f) \in p \), then \( \{ x \in X : |f(x)| > \frac{1}{n} \} \) is a member of \( p \), for all \( n \). Hence \( \{ x \in X : |f(x)| \leq \frac{1}{n} \} \in i^\#(U) \), for all \( n \). As \( i^\#(U) \) is a prime \( z \)-filter, \( \{ x \in X : |f(x)| \leq \frac{1}{n} \} \in i^\#(U) \), for all \( n \). Hence \( \{ x \in X : |f(x)| \leq \frac{1}{n} \} \cap D \neq \emptyset \). So there exists a point \( z \in D \), such that \( |f(z)| \leq \frac{1}{n} \), for all \( n \). Hence \( f(z) = 0 \), which is a contradiction. So \( Z(f) \notin p \). So \( p \) can not be member of \( RX \). So \( p \) must be fixed. As \( Z(f) \notin p \), by our convention (\( * \)), \( p \notin Z(f) \). So \( p \in D \). As \( i^\#(U) \in p \), \( p \in \cap (i^\#(U)) \). As \( p \in D \) and \( D \) being \( z \)-embedded, \( p \in \cap (i^\#(U)) \cap D = \cap U \). So \( \cap U \) is fixed. Hence \( D \) is realcompact and therefore \( f \in C_{RC}(X) \).

\[ \square \]

**Theorem 25.** If \( p \in \delta X \setminus X \), then \( p \) has a compact neighbourhood in \( \delta X \).

Proof. : Let \( p \in \delta X \setminus X \). There exists a hard zero set \( H \) in \( p \). Let \( K \) be the compact subset of \( H \) satisfying the property of hardness. Let \( U \) and \( V \) be two \( \delta X \)-open sets such that \( p \in U \) and \( K \subseteq V \) with \( U \cap V = \emptyset \), so \( p \notin cl_X V \). Now \( H \setminus V = H \setminus (U \cap X) \) is contained in a zero set \( Z_1 \) and is contained in a realcompact cozero set \( X \setminus Z_2 \), where \( Z_2 \) is a zero set in \( X \). Then \( p \notin cl_X Z_2 \) as \( p \in cl_X (H \setminus V) \subseteq Z_1 \) and \( Z_1 \cap Z_2 = \emptyset \). Then there exists a zero set neighbourhood \( W \) of \( p \) in \( \delta X \) such that \( W \cap Z_2 = \emptyset \). Then \( W \cap X \subseteq X \setminus Z_2 \). Hence \( W \cap X \) is hard in \( X \) and \( p \in W \subseteq cl_X (W \cap X) \) which is compact. So \( p \) has a compact neighbourhood in \( \delta X \).

\[ \square \]

**Theorem 26.** \( cl_X (\eta X \cap X) \) is closed in \( \delta X \) and is precisely the set of points having no hard neighbourhood or equivalently any realcompact neighbourhood.

Proof. It is clear that \( cl_X (\eta X \cap X) \) is a closed in \( X \). It is enough to show that no point of \( \delta X \setminus X \) is a limit point of \( cl_X (\eta X \cap X) \). Let \( p \in \delta X \setminus X \). By previous theorem there exists a compact neighbourhood \( K \) of \( p \) in \( \delta X \) such that \( p \in int_{\delta X} K \subseteq K \). Then \( int_{\delta X} K \cap (\eta X \cap X) = \emptyset \). Otherwise suppose \( x \in int_{\delta X} K \cap (\eta X \cap X) \neq \emptyset \), there exists a zero set \( Z \) in \( X \) containing \( x \) and contained in \( int_{\delta X} K \cap (\eta X \cap X) \). Then \( cl_X Z \) being a closed subset of \( K \), is compact. So by theorem 21 \( Z \) is hard in \( X \). As \( x \in Z \), by our convention (\( * \)), \( Z \subseteq X \). So \( x \) is a fixed \( hz \)-ultrafilter and hence does not belongs to \( \eta X \), a contradiction. Therefore \( int_{\delta X} K \cap (\eta X \cap X) = \emptyset \) which further implies \( int_{\delta X} K \cap cl_X (\eta X \cap X) = \emptyset \). Thus \( cl_X (\eta X \cap X) \) is closed in \( \delta X \).

Second part: it is clear that the set of points having hard neighbourhood is an open subset of \( X \) and does not intersect \( (\eta X \cap X) \). Hence it does not intersect \( cl_X (\eta X \cap X) \). So no point of \( cl_X (\eta X \cap X) \) has hard (equivalently realcompact) neighbourhood in \( X \). Conversely suppose a point \( p \) in \( X \) does not belong to \( cl_X (\eta X \cap X) \), then \( p \) does not belong to \( cl_{\beta X} \eta X \). Let \( W \) and \( L \) be zero set neighbourhoods of \( cl_{\beta X} \eta X \) and \( p \) respectively such that \( W \cap L = \emptyset \) as \( \beta X \) is Tychonoff. Let \( W \cap X = Z(f) \) and \( L \cap X = Z(g) \). Then \( \eta X \) is contained in \( cl_{\beta X} Z(f) \) and \( Z(g) \) is a neighbourhood of \( p \) in \( X \). Then \( f \in C_{RC}(X) \). Since \( Z(g) \) is contained in \( X \setminus Z(f) \) which is realcompact, \( Z(g) \) is a hard neighbourhood of \( p \). So \( cl_X (\eta X \cap X) \) is precisely the set of points having no hard neighbourhood or equivalently any realcompact neighbourhood.

\[ \square \]

**Theorem 27.** Union of hard and compact set is hard.

Proof. As \( H \) is hard, we have a compact set \( T \) satisfying the property of hardness. Let \( K \) be a compact set. Then \( W = K \cup T \) is also compact. Then it is straightforward to check that any open neighbourhood \( G \) of \( W \). \( (H \cup K) \setminus G \) can be completely separated from the complement of realcompact cozero set. Hence \( H \cup K \) is hard in \( X \).

\[ \square \]

**Theorem 28.** For any \( p \in \eta X \), any \( \beta X \)-open neighbourhood of \( p \) contains a free real \( z \)-ultrafilter.

Proof. Let \( p \in \eta X \). So \( p \) is a non-\( hz \)-ultrafilter on \( X \). For \( Z \in Z[X] \), let \( p \in \beta X \setminus Z \). Then \( p \) is not a member of \( Z \), so \( Z \notin p \). There exists a member \( W \) of \( p \) such that \( W \cap Z = \emptyset \). Suppose
Z belongs to all free real z-ultrafilter. Then $X \setminus Z$ is realcompact. So $W$ is hard in $X$. This contradicts $p$ to be non-hz-ultrafilter. So there exists a free real z-ultrafilter $q$ such that $q \notin \mathbb{Z}$. Hence $q \in \beta X \setminus \mathbb{Z}$.

The above theorem shows that $\text{cl}_{\beta X} \eta X$ is precisely the $\text{cl}_{\beta X} RX$.

**Theorem 29.** $\delta X$ is nearly pseudocompact.

**Proof.** Let $\delta X_{lc}$ be the set of points in $\delta X$ having compact neighbourhood. Then $\delta X \setminus \delta X_{lc}$ is an open subset of $\delta X$, $\delta X_{lc} \cap X \neq \emptyset$. And $\text{cl}_{\delta X} (\delta X_{lc} \cap X) = \delta X_{lc}$. Hence $\text{cl}_{\delta X} \delta X_{lc} = \text{cl}_{\delta X} (\delta X_{lc} \cap X)$. Let $\delta X_1 = \text{cl}_{\delta X} \delta X_{lc}$. Let $f \in C(\delta X)$ such that $f$ is unbounded on $\delta X_1$. Then $f$ is unbounded on $\delta X_{lc} \cap X$. Hence there exists a copy $N$ of $\mathbb{N}$, C-embedded in $\delta X$. Then each point of $N$ is contained in realcompact cozero set. As countable union of realcompact cozero sets (in fact z-embedded sets) is realcompact cozero sets, $N$ is contained in a realcompact cozero set in $\delta X$. As $N$ is C-embedded in $\delta X$, $N$ can be completely separated from the complement of the realcompact cozero set. Hence $N$ is hard in $\delta X$. Hence $N$ is compact which is absurd. So $\delta X_{lc}$ is relatively pseudocompact and hence $\delta X_1$ is pseudocompact. Let $\delta X_2 = \delta X \setminus \delta X_{lc}$ is a regular closed subset of $\delta X$ which is anti-locally realcompact. Clearly $\text{int}_{\delta X} (\delta X_1 \cap \delta X_2) = \emptyset$. So $\delta X$ is nearly pseudocompact. \hfill \Box

4. **Uniqueness of $\delta X$ with respect to extension property**

In this section we shall show that the nearly pseudocompactification, $\delta X$, of $X$ is unique, with respect to some properties. Prior to that, we hereby introduce a subring $S(X)$ of $C(X)$ containing $C^*(X)$.

**Definition 30.** Let $S(X) = \{f \in C(X) : f$ is bounded on every hard set in $X\}$. We call such $f$ as hard-bounded continuous map.

Then clearly $C^*(X) \subseteq S(X) \subseteq C(X)$. Moreover as closed subset of hard set is hard in $X$ and finite union of hard sets is again a hard set, it follows that $S(X)$ is indeed a subring of $C(X)$.

We call a space $X$ to be S-embedded in $Y$ if every hard-bounded continuous map on $X$ can be continuously extended to $Y$.

**Definition 31.** A continuous map $f : X \to Y$ is called h2pc map if $f$ takes a hard set to a pre-compact set in $Y$. i.e. if $H$ is hard in $X$ then $\text{cl}_Y f(H)$ is compact.

**Example 32.** (i) Any hard map [1](that takes hard set to a hard set ) from a space to a nearly pseudocompact space is h2pc.

(ii) Every hard-bounded continuous map is h2pc: Infact, if $H$ is hard in $X$, $f$ is a hard-bounded continuous map, then $f(H)$ is bounded subset of $\mathbb{R} \Rightarrow \text{cl}_R f(H)$ is compact.

(iii) Then the inclusion map $i : X \to \delta X$ is also h2pc map.

(iv) Any continuous map from a nearly pseudocompact space is also h2pc map: Infact if $H$ is hard in nearly pseudocompact space $X$, $H$ is compact. Let $f : X \to Y$ be a continuous map. Then $f(H)$ is compact $\Rightarrow \text{cl}_Y f(H) = f(H)$ is also compact. So $f$ is h2pc.

**Theorem 33.** Composition of two h2pc map is also h2pc.

**Proof.** Let $f : X \to Y$ and $g : Y \to Z$ be h2pc maps. Let $H$ be a hard set in $X$. Then $\text{cl}_Y f(H)$ is compact in $Y$. By continuity, $g(\text{cl}_Y f(H)) \subseteq \text{cl}_Z g(f(H))$. Now we have $g(f(H)) \subseteq g(\text{cl}_Y f(H)) \subseteq \text{cl}_Z g(f(H))$. But $g(\text{cl}_Y f(H))$ is compact. Hence $g(\text{cl}_Y f(H)) = \text{cl}_Z g(f(H)) \Rightarrow g f$ is h2pc map. \hfill \Box

**Theorem 34.** Let $f : X \to Y$ be a hard map and $g : Y \to Z$ be an h2pc map. Then $g \circ f : X \to Z$ is an h2pc map.

Though the proof of the above theorem is trivial, its importance is reflected in the last paragraph of this section.

**Theorem 35.** Let $X$ be dense in $T$ and each point of $T \setminus X$ is in $T$-closure of a hard set in $X$. Then each point of $T \setminus X$ is in the $T$-closure of hard zero set (equivalently regular hard zero set) in $X$. 
Proof. Let \( p \in T \setminus X \) and \( H \) be a hard set in \( X \) such that \( p \in \text{cl} H \). Let \( K \) be the compact subset of \( H \) satisfying the property of hardness. As \( p \notin K \), \( \exists \) open sets \( U, V \) in \( T \) such that \( p \in U \) and \( K \subset V \) with \( U \cap V = \emptyset \). Since \( K \) satisfies the property of hardness, there exist a zero set \( Z_1 \) and a cozero set \( X \setminus Z_2 \) in \( X \) where \( Z_2 \), a zero set, such that \( H \setminus V \subset Z_1 \subset X \setminus Z_2 \) and \( X \setminus Z_1 \) is realcompact. This implies that \( Z_1 \) is a hard zero set in \( X \). (Infact, here we may even choose \( Z_1 \) to be regular zero set. Then \( Z_1 \) becomes a regular hard zero set.) Now \( p \notin \text{cl}_T (V \cap X) \) as \( U \cap V = \emptyset \), but \( p \in \text{cl} H \). So \( p \in \text{cl}_T (H \setminus V) \). Hence \( p \in \text{cl}_T Z_1 \). \( \square \)

Theorem 36. Suppose \( X \) is dense in \( T \) such that every point \( p \in T \setminus X \) is in the \( T \)-closure of some hard set in \( X \). Then the following are equivalent.

1. If \( \tau \) is a \( h2pc \) map from a space \( X \) to \( Y \) then \( \tau \) can be continuously extended upto \( T \).
2. \( X \) is \( S \)-embedded in \( T \).
3. For any two zero sets \( Z_1, Z_2 \) in \( X \), \( Z_1 \cap Z_2 = \emptyset \Rightarrow \text{cl}_T Z_1 \cap \text{cl}_T Z_2 = \emptyset \).
4. For any two zero sets \( Z_1, Z_2 \) in \( X \), \( \text{cl}_T (Z_1 \cap Z_2) = \text{cl}_T Z_1 \cap \text{cl}_T Z_2 \).
5. Each point of \( T \) is the limit of either a fixed \( z \)-ultrafilter or a \( hz \)-ultrafilter on \( X \).

Proof. (1) \( \Rightarrow \) (2) : Trivial as every hard-bounded continuous map is \( h2pc \) map.

(2) \( \Rightarrow \) (3) : Trivially follows from Theorem 6.4 [3] as \( C^* (X) \subset S (X) \).

(3) \( \Rightarrow \) (4) : Trivially follows from Theorem 6.4 [3].

(4) \( \Rightarrow \) (5) : Let \( p \in T \). If \( p \in X \), then \( p \) is the limit of unique fixed \( z \)-ultrafilter \( A_p = \{ Z \in Z [X] : p \in Z \} \). So suppose now, \( p \in T \setminus X \). There exists a hard zero set \( H \) in \( X \) such that \( p \in \text{cl} H \). Let \( \mathcal{N}_p \) be the family of all zero set neighbourhoods of \( p \). Then \( \{ Z \cap X : Z \in \mathcal{N}_p \} \cup \{ H \} \) satisfies finite intersection property. Hence can be extended to a \( z \)-ultrafilter \( \mathcal{U} \) on \( X \). As \( H \in \mathcal{U} \), \( \mathcal{U} \) is then a \( hz \)-ultrafilter. That \( \mathcal{U} \) is unique and converges to \( p \), follows the same line of arguments as given in Theorem 6.4 [3].

(5) \( \Rightarrow \) (1) : Let \( \tau : T \to Y \) be a \( h2pc \) map. Let \( p \in T \). Then \( \mathcal{U} \to p \) where \( \mathcal{U} \) is either a fixed \( z \)-ultrafilter on \( X \) or \( \mathcal{U} \) is a \( hz \)-ultrafilter on \( X \) by (5). Now \( \tau^* (\mathcal{U}) \) is a prime \( z \)-filter on \( Y \). If \( \tau^* (\mathcal{U}) \) is fixed then there exists \( y_p \in Y \) such that \( \bigcap \tau^* (\mathcal{U}) = \{ y_p \} \). Then we shall define \( \tau^* (p) = y_p \). Now if \( \mathcal{U} \) is fixed then \( p \in X \) and \( \mathcal{U} = A_p \). \( Z \in \tau^* (\mathcal{U}) \Leftrightarrow \tau^* (Z) \subset A_p \Leftrightarrow p \in \tau^* (Z) \subset Z \). So \( \tau (p) \in \bigcap \tau^* (\mathcal{U}) \Rightarrow \tau^* (p) = \tau (p) \). This also shows that \( \tau^* \) is an extension of \( \tau \). Now suppose \( p \in T \setminus X \), then there exists unique \( hz \)-ultrafilter \( \mathcal{U} \) on \( X \) which converges to \( p \). Let \( Z \in \tau^* (\mathcal{U}) \Rightarrow \tau^* (Z) \subset \mathcal{U} \Rightarrow \exists \) a hard zero set \( H \) in \( U \) such that \( H \subset \tau^* (Z) \Rightarrow \tau (H \subset Z) \Rightarrow \text{cl}_Y (\tau (H \subset Z)) \subset Z \). Now \( \text{cl}_Y (\tau (H \subset Z)) \) is compact and \( \text{cl}_Y (\tau (H \subset Z)) \subset \tau^* (\mathcal{U}) \) satisfies finite intersection property. Hence \( \bigcap \tau^* (\mathcal{U}) \neq \emptyset \). Clearly \( \bigcap \tau^* (\mathcal{U}) \neq \emptyset \). Let \( y_p \in \tau^* (\mathcal{U}) \Rightarrow \tau (p) = \{ y_p \} \). So we have defined \( \tau^* : \delta X \to Y \) which is an extension of \( \tau \). That \( \tau^* \) is continuous, follows again the same line of arguments as given in Theorem 6.4 [3]. \( \square \)

By construction of \( \delta X \), it is clear that \( \delta X \) is precisely the collection of limits of all fixed \( z \)-ultrafilter and \( hz \)-ultrafilter on \( X \). Further if \( p \in \delta X \), there exists a compact neighbourhood \( K \) of \( p \) in \( \delta X \). Then \( K \cap X \) is hard in \( X \) and \( p \in \text{cl}_\delta X (K \cap X) \). So from the previous theorem we conclude the following theorem.

Theorem 37. For each space \( X \), there exists a nearly pseudocompactification \( \delta X \) of \( X \) which satisfies the following equivalent properties.

1. Any \( h2pc \) map from a space \( X \) to \( Y \) has continuous extension upto \( \delta X \).
2. \( X \) is \( S \)-embedded in \( \delta X \).
3. For any two zero sets \( Z_1, Z_2 \) in \( X \), \( Z_1 \cap Z_2 = \emptyset \Rightarrow \text{cl}_\delta X Z_1 \cap \text{cl}_\delta X Z_2 = \emptyset \).
4. For any two zero sets \( Z_1, Z_2 \) in \( X \), \( \text{cl}_\delta X (Z_1 \cap Z_2) = \text{cl}_\delta X Z_1 \cap \text{cl}_\delta X Z_2 \).
5. Every fixed \( z \)-ultrafilter and \( hz \)-ultrafilter converges in \( \delta X \).

Next theorem shows that \( \delta X \) is unique with respect to certain properties.

Corollary 38. Let \( X \) be dense in \( Y \) such that any \( h2pc \) map on \( X \) has continuous extension to \( Y \) and \( Y \)-closure of each hard set in \( X \) is compact in \( Y \), then \( Y \) is homeomorphic with \( \delta X \).
Proof. We note that the inclusion map $i_{\delta X} : X \to \delta X$ is $h2pc$ map and $\delta X$-closure of hard set in $X$ is compact. As per the conditions, the inclusion map $i_Y : X \to Y$ is also a $h2pc$ map. Now $\exists f : \delta X \to Y$ and $g : Y \to \delta X$ such that $f|_X = I_X$ and $g|_X = I_X$. Then $f \circ g = I_Y$ and $g \circ f = I_{\delta X} \Rightarrow f$ is a homeomorphism. So $Y$ is homeomorphic with $\delta X$. □

Corollary 39. If $f : X \to Y$ be a hard map from a space $X$ to a nearly pseudocompact space $Y$, then $f$ can be uniquely extended to a continuous map $f^* : \delta X \to Y$.

Proof. It trivially follows from the theorem 37, as any hard map from a space to a nearly pseudocompact space is $h2pc$ map. □

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References

[1] M.C.Rayburn - On hard sets, General Topology and its Applications 6(1976),21-26.
[2] L. Gillman and M. Jerison, Rings of Continuous Functions, University Series in Higher Math, Van Nostrand, Princeton, New Jersey, 1960. 1, 2, 2, 3, 3, 3, 3, 3
[3] M. Henriksen and M. Rayburn - On nearly pseudocompact spaces, Top. Appl. 11 (1980), 161-172. (document), 1, 2, 3, 3, 3
[4] Melvin Henriksen and Marlon C. Rayburn - Nearly pseudocompact extensions, Math Japonica 32, No. 4 (1987), 569-582.
[5] B. Mitra and S.K. Acharyya, Characterizations of Nearly Pseudocompact spaces and Related spaces (With S.K. Acharyya), Topology Proceedings, Vol 29, No. 2, 2005, 577 - 594.
[6] B. Mitra and D. Chowdhury, A characterization of C-type subrings of C(X) of some kind (With Debojjoti Chowdhury), Positivity, volume 24, (2020) 1181–1192.
[7] M. Mandelker - Supports of continuous functions, Trans. Amer. Math. Soc 156 (1971), 73-83.
[8] R.C. Walker - The Stone-$\check{C}$ech Compactification, North-Holland Publishing Company, Amsterdam, 1975.

Department of Mathematics, The University of Burdwan, Burdwan 713104, West Bengal, India

Email address: bmitra@math.buruniv.ac.in

Email address: ruin.sanjibdas893@gmail.com