Nearly Gorenstein rings arising from finite graphs

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Abstract

The classification of complete multipartite graphs whose edge rings are nearly
Gorenstein as well as that of finite perfect graphs whose stable set rings are nearly
Gorenstein is achieved.

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Gorenstein graded algebras associated to combinatorial objects like graphs or simplicial complexes have attracted a lot of interest. See, e.g., [5], [16], [2]. Recently several
extensions of the class of Gorenstein rings (inside the class of Cohen–Macaulay rings)
have been discussed in, e.g., [6], [7], hence it is natural to search for the combinatorial
counterpart.

According to [7], when $R$ is a Cohen–Macaulay graded $K$-algebra over the field $K$ with
canonical module $\omega_R$, it is called nearly Gorenstein if the canonical trace ideal $tr(\omega_R)$
contains the maximal graded ideal $m_R$ of $R$. Here $tr(\omega_R)$ is the ideal generated by the
image of $\omega_R$ through all homomorphism of $R$-modules into $R$. As $tr(\omega_R)$ describes the

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denote the maximal cardinality of cliques of graph.

Recall that a set \( C \) is Gorenstein if and only if all maximal cliques of \( G \) have the same cardinality \([16]\). A nonempty set \( W \) of vertices is called stable if there is no edge \( \{i, j\} \) in \( G \) with \( i, j \in W \). The stable set ring of \( G \) denoted \( \text{Stab}_G(G) \) is the \( \mathbb{K} \)-subalgebra of the polynomial ring \( \mathbb{K}[x_1, \ldots, x_d] \) generated by those monomials \( x_i x_j \) for all edges \( \{i, j\} \in E(G) \). When \( V(G) \) can be partitioned \( V(G) = \bigsqcup_{k=1}^n V_k \) with \( n \geq 2 \) and \( |V_k| = r_k \) for \( k = 1, \ldots, n \) such that \( E(G) \) consists of all the pairs \( \{i, j\} \) with \( i \in V_a \) and \( j \in V_b \) for \( 1 \leq a < b \leq n \), we say that \( G \) is a complete multipartite graph of type \( r_1, \ldots, r_n \), which is denoted \( K_{r_1, \ldots, r_n} \). Related algebraic properties for these graphs have been recently studied in \([10]\) and \([11]\). In Proposition 5 and in Theorem 6 we prove the following result.

**Theorem A.** Assume \( G = K_{r_1, \ldots, r_n} \). Set \( R = \mathbb{K}[G] \). Then

1. if \( n = 2 \) and \( 1 \leq r_1 \leq r_2 \), the ring \( R \) is nearly Gorenstein if and only if \( r_1 = 1 \), or \( r_2 \in \{r_1, r_1 + 1\} \).

2. if \( n \geq 3 \) the ring \( R \) is nearly Gorenstein if and only if \( R \) is Gorenstein.

Since Ohsugi and Hibi in \([14]\) have explicitly listed the complete multipartite graphs whose edge ring is Gorenstein (see Theorem 1 below), Theorem A offers a full description for the nearly Gorenstein property, as well.

The other class of algebras we consider deals with the stable sets in \( G \). A nonempty set \( W \) of vertices is called stable (or independent) if there is no edge \( \{i, j\} \) in \( G \) with \( i, j \in W \). The stable set ring of \( G \) denoted \( \text{Stab}_G(G) \) is the \( \mathbb{K} \)-subalgebra in the polynomial ring \( \mathbb{K}[x_1, \ldots, x_d, t] \) generated by those monomials \( (\prod_{i \in W} x_i) \cdot t \) with \( W \) any stable set in \( G \). When \( G \) is a perfect graph, it is known \([15]\) that \( \text{Stab}_G(G) \) is Cohen–Macaulay, and that it is Gorenstein if and only if all maximal cliques of \( G \) have the same cardinality \([16]\). Recall that a set \( C \subseteq V(G) \) is called a clique if the subgraph induced by \( C \) is a complete graph.

The size of the maximal cliques in \( G \) is also relevant to describe in Theorem 13 for which perfect graphs the algebra \( \text{Stab}_G(G) \) is nearly Gorenstein. We prove the following.

**Theorem B.** Let \( G \) be a perfect graph and \( G_1, \ldots, G_s \) its connected components. Let \( \delta_i \) denote the maximal cardinality of cliques of \( G_i \). Then \( \text{Stab}_G(G) \) is nearly Gorenstein if and only if for each \( G_i \) its maximal cliques have the same cardinality and \( |\delta_i - \delta_j| \leq 1 \) for \( 1 \leq i < j \leq s \).

To prove Theorems A and B we observe that the algebras \( R \) which occur are Cohen–Macaulay domains, so \( \omega_R \) can be identified with an ideal in \( R \). By \([7, \text{Lemma 1.1}]\), its trace can be computed as

\[
\text{tr}(\omega_R) = \omega_R \cdot \omega_R^{-1}, \text{ where }
\omega_R^{-1} = \{x \in Q(R) : x \cdot \omega_R \subseteq R\}
\]

is the anti-canonical ideal of \( R \) and \( Q(R) \) denotes the field of fractions of \( R \).

We refer the reader to \([1]\) and \([2]\) for the undefined graph or algebraic notions.
1 Edge rings

In this section unless stated otherwise $G = K_{r_1,...,r_n}$ is the complete multipartite graph on $[d]$ with vertices partitioned $V(G) = V_1 \cup \cdots \cup V_n$, $n \geq 2$, $|V_k| = r_k$ for all $k$. In this context $d = \sum_{k=1}^{n} r_k$ and without loss of generality, we will always assume that $1 \leq r_1 \leq \ldots \leq r_n$.

The graph $G$ satisfies the so called odd cycle condition, i.e. for any two odd cycles in $G$ which have no common vertex there is a bridge between them. Indeed, when $n = 2$ there is no odd cycle and anything to prove. Assume $n \geq 3$, and $C_1$ and $C_2$ be two disjoint odd cycles in $G$. Since $G$ is multipartite, each of these contains vertices from at least two of the components $V_1,\ldots,V_n$, so one finds $v \in C_1 \cap V_a$ and $w \in C_2 \cap V_b$ with $a \neq b$. Then $vw$ is an edge in $G$ and a bridge between $C_1$ and $C_2$. Consequently, by [13] the edge ring

$$R = \mathbb{K}[G] = \mathbb{K}[x_ix_j : i \in V_a, j \in V_b, 1 \leq a < b \leq n] \subset \mathbb{K}[x_1,\ldots,x_d]$$

is normal, hence a Cohen–Macaulay domain ([12]). Before we address the nearly Gorenstein property, we recall that Ohsugi and Hibi [14] classified the complete multipartite edge rings which are Gorenstein. With notation as above, their result is the following.

**Theorem 1.** (Ohsugi, Hibi [14, Remark 2.8]) The edge ring of the complete multipartite graph $K_{r_1,...,r_n}$ is Gorenstein if and only if

1. $n = 2$ and $(r_1,r_2) \in \{(1,m), (m,m) : m \geq 1\}$, or
2. $n = 3$ and $1 \leq r_1 \leq r_2 \leq r_2 \leq 2$, or
3. $n = 4$ and $r_1 = r_2 = r_3 = r_4 = 1$.

For some complete multipartite graphs the edge ring fits into classes of algebras for which the nearly Gorenstein property is already understood.

**Example 2.** When $r_1 = \cdots = r_n = 1$, the edge ring $R$ is the squarefree Veronese subalgebra of degree 2 in the polynomial ring $\mathbb{K}[x_1,\ldots,x_n]$, and according to [7, Theorem 4.14], $R$ is nearly Gorenstein if and only if it is Gorenstein. The latter property holds if and only if $n \leq 4$, by using work of De Negri and Hibi [5], or Bruns, Vasconcelos and Villarreal [3].

**Example 3.** According to Higashitani and Matsushita [10, Proposition 2.2], when $n = 2$, or when $n = 3$ and $r_1 = 1$, the corresponding edge ring is isomorphic to a Hibi ring, and for the latter the nearly Gorenstein property is described in [7]. We refer to [9] for background on Hibi rings.

**Theorem 4** ([7, Theorem 5.4]; [9]). Let $P$ be a finite poset. Then the Hibi ring $R$ of the distributive lattice of the order ideals in $P$ is nearly Gorenstein if and only if $P$ is the disjoint union of pure connected posets $P_1,\ldots,P_q$ such that $|\text{rank}(P_i) - \text{rank}(P_j)| \leq 1$ for $1 \leq i < j \leq q$.

In particular, $R$ is a Gorenstein ring if and only if $P$ is pure.
Based on that, when \( G \) is a complete bipartite graph we obtain the following classification.

**Proposition 5.** Let \( G = K_{r_1, r_2} \) be the complete bipartite graph with \( 1 \leq r_1 \leq r_2 \). Then the edge ring \( \mathbb{K}[G] \) is nearly Gorenstein if and only if \( r_1 = 1 \), or \( r_1 \geq 2 \) and \( r_2 \in \{r_1, r_1 + 1\} \).

When \( 2 \leq r_1 = r_2 - 1 \), the ring \( \mathbb{K}[G] \) is nearly Gorenstein and not Gorenstein.

**Proof.** By [10, Proposition 2.2], \( \mathbb{K}[G] \) is isomorphic to the Hibi ring associated to the distributive lattice of order ideals in the poset \( P \) which consists of two disjoint chains with \( r_1 - 1 \) and \( r_2 - 1 \) elements, respectively. By Theorem 4, \( \mathbb{K}[G] \) is nearly Gorenstein if and only if \( r_1 = 1 \), or \( r_1 \geq 2 \) and \( r_2 \in \{r_1, r_1 + 1\} \).

For non-bipartite graphs we prove the following result.

**Theorem 6.** Let \( R \) be the edge ring of a complete multipartite graph \( K_{r_1, \ldots, r_n} \) with \( n \geq 3 \). The following statements are equivalent:

(i) \( R \) is a Gorenstein ring;

(ii) \( R \) is a nearly Gorenstein ring.

**Proof.** Clearly, (i) \( \Rightarrow \) (ii). We’ll prove the converse.

When \( n = 3 \) and \( r_1 = 1 \leq r_2 \leq r_3 \), by [10, Proposition 2.2] the ring \( R \) is isomorphic to the Hibi ring associated to the distributive lattice of order ideals in a poset \( Q \) with maximal chains \( q_1 < \cdots < q_{r_1}, q_{r_1+1} < \cdots < q_{r_1+r_2} \) and \( q_1 < q_{r_1+r_2} \). The poset \( Q \) is connected, hence \( R \) is nearly Gorenstein if and only if it is Gorenstein, i.e. \( 1 = r_1 \leq r_2 \leq r_3 \leq 2 \).

We now consider the remaining cases: either \( n = 3 \) and \( r_1 \geq 2 \), or \( n \geq 4 \). Assume, by contradiction that \( R \) is nearly Gorenstein and not Gorenstein, i.e.

\[ \text{tr}(\omega_R) = m_R. \] (1)

The monomials in \( R \) and \( \omega_R \) have a nice combinatorial description as feasible integer solutions to some systems of inequalities. This can be described as follows. We denote \( H = \sum_{(i,j) \in E(G)} \mathbb{N}(e_i + e_j) \subset \mathbb{N}^d \) the affine semigroup generated by the columns of the vertex-edge incidence matrix for \( G \), and \( C = \mathbb{R}_+ H \) the rational cone over \( H \).

For \( u = (u_1, \ldots, u_d) \in \mathbb{N}^d \), it follows from [13] and [18, Proposition 3.4] that \( u \in H \) (equivalently, \( x^u \in R \)) if and only if

\[ \sum_{i=1}^{d} u_i \equiv 0 \mod 2, \] (2)

\[ u_1, \ldots, u_d \geq 0, \quad \text{and} \quad \sum_{i \not\in V_k} u_i \geq \sum_{j \in V_k} u_j \text{ for all } k = 1, \ldots, n. \]

The latter inequalities are equivalent to

\[ \sum_{i=1}^{d} u_i \geq 2 \sum_{j \in V_k} u_j \text{ for } k = 1, \ldots, n. \] (3)
Since $R$ is normal, by [4], [17] (see also [2, Theorem 6.3.5(b)]), a $\mathbb{K}$-basis for $\omega_R$ is given by the monomials $x^u$ where $u = (u_1, \ldots, u_d) \in \mathbb{Z}^d$ satisfies
\[
\sum_{i=1}^d u_i \equiv 0 \mod 2, \quad (4)
\]

$u_1, \ldots, u_d \geq 1$, and
\[
\sum_{i=1}^d u_i \geq 2 + 2 \sum_{j \in V_k} u_j, \text{ for } k = 1, \ldots, n. \quad (5)
\]

From the equations above it is easy to see that if the monomial $x^u$ is in $R$ or in $\omega_R$, we can permute the exponents $x_i$ and $x_j$ whenever $i, j \in V_k$ for some $k$, and we obtain another monomial in $R$, or in $\omega_R$, respectively.

In what follows $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$.

For a monomial $x^u \in \omega_R$ and $1 \leq k \leq n$ we say that $V_k$ (or simply, $k$) is a heavy component in $u$ if
\[
d \sum_{i=1}^d u_i = 2 + 2 \sum_{j \in V_k} u_j. \quad (6)
\]

**Claim 7.** For any $x^u \in \omega_R$ there exist at most two heavy components in $u$. In particular, there is at least one non-heavy component in $u$.

**Proof.** Indeed, if $k_1 < k_2 < k_3$ are heavy components in $u$, then by adding the equations (7) for these indices we get
\[
3 \sum_{i=1}^d u_i = 6 + \sum_{j \in V_{k_1} \cup V_{k_2} \cup V_{k_3}} 2u_j,
\]
If $n = 3$, then $\sum_{i=1}^d u_i = 6$. Since $u_i \geq r_i \geq 2$ for all $i$, we infer that $r_1 = r_2 = r_3 = 2$, and $\mathbb{K}[G]$ is a Gorenstein ring (by Theorem 1), which is not the case.

If $n \geq 4$, then $\sum_{i=1}^d u_i < 6$. As $\sum_{i=1}^d u_i$ is even, we get that $n = 4$ and $r_1 = r_2 = r_3 = r_4 = 1$. Example 2 implies that $R$ is a Gorenstein ring, which is false.

**Claim 8.** For any $1 \leq i \leq d$ there exists a monomial $x^u \in \omega_R$ such that $u_i = 1$.

**Proof.** We fix $i$ and we denote $a_i = \min\{u_i : \prod x_i^{u_i} \in \omega_R\}$. By (5), $a_i \geq 1$. Assume $a_i \geq 2$, and say $i \in V_k$.

If $r_k > 1$, we may pick $j \in V_k$, $j \neq i$. Then it is easy to check that the monomial $m = \prod x_i^{a_i} x_j \in \omega_R$ and $\deg_{x_i}(m) = a_i - 1$, a contradiction.

When $r_k = 1$, then $n \geq 4$ and by the previous claim there is at least one non-heavy component $V_{k_1}$ in $u$ which is different from $V_k$. We pick $j \in V_{k_1}$ and since the monomial $m = \prod x_i^{a_i} x_j \in \omega_R$ and $\deg_{x_i}(m) = a_i - 1$ we obtain a contradiction.
It follows at once that
\[
gcd(x^u : x^u \in \omega_R) = \prod_{i=1}^{d} x_i,
\]
where the greatest common divisor is computed in the polynomial ring \(S = \mathbb{K}[x_1, \ldots, x_d]\).

Since \(\omega_R\) is generated by monomials, one gets that \(\omega_{R-1}\) is also generated by monomials in \(\mathbb{K}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]\). If \(f = x^u/x^v \in \omega_{R-1}\) with \(x^u\) and \(x^v\) coprime monomials in \(S\), then \(x^v\) divides the greatest common divisor of the monomials in \(\omega_R\). Hence, in order to determine a system of generators for the \(R\)-module \(\omega_{R-1}\) it is enough to scan among the (non-reduced) fractions \(f = x^u/(x_1 \ldots x_d)\), where \(x^u\) is in the set
\[
B = \left\{ x^u \in S : \sum_{i=1}^{d} u_i \equiv 0 \text{ mod } 2, \quad x^u \cdot \omega_R \subseteq x_1 \ldots x_d R \right\}.
\]
A monomial \(x^u\) is in \(B\) if and only if \(\sum_{i=1}^{d} u_i \equiv 0 \text{ mod } 2\) and
\[
x_1^{u_1+v_1-1} \cdots x_d^{u_d+v_d-1} \in R
\]
for all \(x_1^{v_1} \cdots x_d^{v_d}\) in \(\omega_R\). That is equivalent, via (2), (4), (3), to the fact that
\[
\sum_{i=1}^{d} u_i \equiv d \text{ mod } 2, \quad \text{and} \quad \sum_{i=1}^{d} u_i + \sum_{i=1}^{d} v_i \geq d - r_k + 2 \sum_{j \in V_k} u_j + 2 \sum_{j \in V_k} v_j, \tag{9}
\]
for \(k = 1, \ldots, d\), and any \(x^v \in \omega_R\).

For \(k = 1, \ldots, n\) we set
\[
E_k = \min \left\{ \sum_{i=1}^{d} v_i - 2 \sum_{j \in V_k} v_k : x^v \in \omega_R \right\}.
\]
Therefore, (9) is equivalent to
\[
\sum_{i=1}^{d} u_i \geq d - r_k - E_k + 2 \sum_{j \in V_k} u_j \quad \text{for } k = 1, \ldots, n. \tag{10}
\]
Before computing \(E_k\) we make a simple observation regarding \(d\) and the \(r_i\)'s.

**Claim 9.** \(2r_i + 2 \leq d\) for all \(i = 1, \ldots, n - 1\).

**Proof.** Indeed, if that were not the case, then \(2r_n + 2 \geq 2r_{n-1} + 2 > d\), hence \(2r_n \geq 2r_{n-1} \geq d - 1\). This implies \(r_n + r_{n-1} \geq d - 1\), equivalently that \(1 = \sum_{i=1}^{n-2} r_i\), which is not possible in our setup. \(\square\)
Next we show that $E_k$ does not depend on $k$.

**Claim 10.** $E_k = 2$ for $k = 1, \ldots, n$.

**Proof.** We fix $1 \leq k \leq n$. Clearly, $E_k \geq 2$, by (6). Then $E_k = 2$ once we find $x \in \omega_R$ such that

$$\sum_{i=1}^{d} v_i = 2 + 2 \sum_{j \in V_k} v_j. \quad (11)$$

Using Eqs. (4), (5), (6), and translating $v_i = r_i + s_i$ for $i = 1, \ldots, n$, we observe that finding $v$ as in (11) is equivalent to finding integers $s_1, \ldots, s_n$ such that

$$s_1, \ldots, s_n \geq 0, \quad (12)$$

$$\sum_{i=1}^{n} s_i \geq 2s_{\ell} + 2r_{\ell} + 2 - d, \text{ for } 1 \leq \ell \leq n, \ell \neq k, \text{ and } \quad (13)$$

$$\sum_{i=1}^{n} s_i = 2s_k + 2 + 2r_k - d. \quad (14)$$

The $s_{\ell}$ represents the sum of the components of $v$ from $V_{\ell}$, each decreased by one. Note that (14) already implies that $\sum_{i=1}^{n} s_i \equiv d \mod 2$.

We have two cases to consider.

**Case $k = n$:**

We let $s_{\ell} = \lfloor d/2 \rfloor - r_{\ell} - 1$ for $\ell = 1, \ldots, n - 1$. For (14) to hold, we must let

$$s_n = \sum_{i=1}^{n-1} s_i - 2 - 2r_n + d = (n-1)\lfloor d/2 \rfloor - d + r_n - (n-1) - 2 - 2r_n + d$$

$$= (n-1)(\lfloor d/2 \rfloor - 1) - r_n - 1 \geq 2(\lfloor d/2 \rfloor - 1) - r_n - 1 \geq d - r_n - 2 \geq 0.$$ 

For $\ell < n$, one has $s_{\ell} \geq 0$ by the previous Claim. Also, $2s_{\ell} + 2r_{\ell} - d$ is either 0 or 1, depending on $d$ being even or odd. Therefore, (13) and (12) are all verified.

**Case $1 \leq k \leq n - 1$:**

We let $s_n = 0$ and $s_{\ell} = \lfloor d/2 \rfloor - r_{\ell} - 1$ for $\ell = 1, \ldots, n - 1$ where $\ell \neq k$. For (14) to hold, we must let

$$s_k = \left( \sum_{1 \leq i \leq n-1, i \neq k} s_i \right) + s_n - 2 - 2r_k + d. \quad (15)$$

Clearly, $s_k \geq 0$ since $d \geq 2r_k + 2$. Arguing as in the other case, for $k \neq \ell < n$ one has $s_{\ell} \geq 0$ and (13) holds. We are left to verify that

$$\sum_{i=1}^{n} s_i \geq 2s_n + 2r_n + 2 - d. \quad (16)$$
Substituting (14) into the left hand side term above, (16) is equivalent to
\[ s_k + r_k \geq s_n + r_n. \]

Using (15) we get that
\[
s_k + r_k = \left( \sum_{1 \leq i \leq n-1, i \neq k} s_i \right) + s_n + d - r_k - 2
\]
\[ = \left( \sum_{1 \leq i \leq n-1, i \neq k} s_i \right) + s_n + r_n + (d - r_k - r_n - 2) \geq s_n + r_n,
\]
where for the latter inequality we used the observation that \( d \geq r_k + r_n + 2 \) in our setup. Consequently, \( s_1, \ldots, s_n \) fulfil (12), (13), (14), and \( E_k = 2 \).

We can now finish the proof of Theorem 6.

Let \( m = x_1^{a_1} \cdots x_d^{a_d} \) be a monomial generator for \( \omega_R \). Then \( \deg m = \sum_{i=1}^d a_i \geq 2 + 2 \sum_{j \in V_k} a_j \) for all \( k = 1, \ldots, n \). In particular, \( \deg m \geq 2r_n + 2 \).

Let \( f = x^u / (x_1 \cdots x_d) \) be a monomial in \( \omega_R^{-1} \), with \( x^u \in B \). By (10),
\[
\deg x^u = \sum_{i=1}^d u_i \geq d - r_k - 2 + 2 \sum_{j \in V_k} u_j \text{ for all } k = 1, \ldots, n.
\]

Since \( d > r_n + 2 \) in our setup, we find a component \( k_1 \) such that \( \sum_{j \in V_{k_1}} u_j > 0 \).

The product \( m \cdot f \) is a monomial in \( R \) of degree at least
\[
(2r_n + 2) + (d - r_{k_1} - 2 + 2 \sum_{j \in V_{k_1}} u_j) - d \geq 2r_n - r_{k_1} + 2 \geq 3.
\]
Consequently, \( \text{tr}(\omega_R) = \omega_R \cdot \omega_R^{-1} \subsetneq m_R \), a contradiction with (1). \( \square \)

2 Stable set rings

In this section we consider an algebra generated by the monomials coming from the stable sets of a graph.

Let \( G \) be a finite simple graph on \( [n] \) and \( E(G) \) is the set of edges of \( G \). A subset \( C \subset [n] \) is a clique of \( G \) if \( \{i, j\} \in E(G) \) for all \( i, j \in C \) with \( i \neq j \). A subset \( W \subset [n] \) is stable in \( G \) if \( \{i, j\} \not\in E(G) \) for all \( i, j \in W \) with \( i \neq j \). In particular, the empty set as well as each \( \{i\} \subset [n] \) is both a clique of \( G \) and a stable subset of \( G \). Let \( \Delta(G) \) denote the clique complex of \( G \) which is the simplicial complex on \( [n] \) consisting of all cliques of \( G \). Let \( \delta \) denote the maximal cardinality of cliques of \( G \). Thus \( \dim \Delta(G) = \delta - 1 \). We say that \( G \) is pure if \( \Delta(G) \) is a pure simplicial complex, i.e. the cardinality of each maximal clique of \( G \) is \( \delta \). The chromatic number of a graph is the smallest number of colors that can be used for its vertices such that no adjacent vertices have the same color. The graph
$G$ is called **perfect** if for all induced subgraphs $H$ of $G$, including $G$ itself, the chromatic number is equal to the maximal cardinality of cliques contained in $H$, see [1, p. 165].

Let $\mathbb{K}[x_1, \ldots, x_n, t]$ denote the polynomial ring in $n+1$ variables over the field $\mathbb{K}$. If, in general, $W \subset [n]$, then $x^W t$ stands for the squarefree monomial

$$x^W t = \left( \prod_{i \in W} x_i \right) \cdot t \in \mathbb{K}[x_1, \ldots, x_n, t].$$

Let $\text{Stab}_G(G)$ denote the subalgebra of $\mathbb{K}[x_1, \ldots, x_n]$ which is generated by those $x^W t$ for which $W$ is a stable set of $G$. Letting $\text{deg}(x^W t) = 1$ for any stable set $W$, the algebra $\text{Stab}_G(G)$ becomes standard graded. We call $\text{Stab}_G(G)$ the **stable set ring** of $G$.

It is known [15, Example 1.3 (c)] that $\text{Stab}_G(G)$ is normal if $G$ is perfect. It follows that, when $G$ is perfect, $\text{Stab}_G(G)$ is spanned over $\mathbb{K}$ by those monomials $(\prod_{i=1}^n x_i^{a_i}) t^q$ with $\sum_{i \in C} a_i \leq q$ for each maximal clique $C$ of $G$. Furthermore, the canonical module $\omega_{\text{Stab}_G(G)}$ of $\text{Stab}_G(G)$ is spanned over $\mathbb{K}$ by those monomials $(\prod_{i=1}^n x_i^{a_i}) t^q$ with each $a_i > 0$ and with $\sum_{i \in C} a_i < q$ for each maximal clique $C$ of $G$. Thus [16, Theorem 2.1 (b)] implies that $\text{Stab}_G(G)$ is Gorenstein if and only if $G$ is pure.

The following lemma captures a sufficient combinatorial condition for $\text{Stab}_G(G)$ to be nearly Gorenstein.

**Lemma 11.** Let $G$ be a finite simple perfect graph such that $\text{Stab}_G(G)$ is nearly Gorenstein. Then every connected component of $G$ is pure.

**Proof.** Assume $V(G) = [n]$. Denote $R = \text{Stab}_G(G)$. Since each $x_i t$ as well as $t$ belongs to $R$, the quotient field of $R$ is the rational function field $\mathbb{K}(x_1, \ldots, x_n, t)$ over $\mathbb{K}$.

Suppose $G_1$ is a connected component of $G$ which is not pure. Let $\delta$ and $\delta_1$ denote the maximal cardinality of cliques of $G$ and of $G_1$, respectively. Then there is an edge $\{i_0, j_0\} \in E(G_1)$ for which $i_0$ belongs to a clique $C$ of $G$ with $|C| = \delta_1$ and for which $j_0$ belongs to no clique $C$ of $G$ with $|C| = \delta_1$.

Let $z = \prod_{i=1}^n x_i^{a_i} t^q \in \omega_R^{-1}$. Set $v_1 = x_1 \cdots x_n t^{\delta_1+1}$. It is easy to check that $v_1 \in \omega_R$ and that each monomial belonging to $\omega_R$ is divisible (in $\mathbb{K}[x_1, \ldots, x_n, t]$) by $v_1$. Hence $a_i \geq -1$ for all $i$. Clearly, $x_j v_1 \in \omega_R$ and $1 \neq x_j v_1 z \in R$, hence $q' \geq -\delta$.

Since $G$ is not pure, $R$ is not a Gorenstein ring and thus

$$\text{tr} (\omega_R) = \omega_R \cdot \omega_R^{-1} = m_R.$$  

Let $w' = \prod_{i=1}^n x_i^{a_i} t^q \in \omega_R$ and $w = \prod_{i=1}^n x_i^{a_i} t^q \in \omega_R$ with $w' w = x_{i_0} t$. Since $q' \geq -\delta$ and $q \geq \delta + 1$, one has $q' = -\delta$ and $q = \delta + 1$.

Let $v = x_1 x_2 \cdots x_n t^{\delta_1+1} \cdot x_i^{-\delta_1}$. One has $v \in \omega_R$ and $x_j v \in \omega_R$. We claim that $w' \cdot x_j v \in m_R$ is divisible by $x_{i_0} x_{j_0} t$, but it is not divisible by $t^\delta$. This is clear when $\delta > \delta_1$. In case $\delta = \delta_1$, since $i_0$ belongs to a clique $C$ of $G$ with $|C| = \delta$, one has $a_{i_0} = 1$. Thus $a_{i_0}' = 0$ and the claim is verified.

Thus $w' \cdot x_j v$ must be of the form $x^W t$, where $W$ is a stable set of $G$, which contradicts $\{i_0, j_0\} \in E(G)$. Hence $m_R \not\subseteq \text{tr}(\omega_R)$, as desired. 

\qed

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Recall that the $a$-invariant of any graded algebra $R$ with canonical module $\omega_R$ is defined as $a(R) = -\min\{i : (\omega_R)_i \neq 0\}$.

**Corollary 12.** If $G$ is a perfect graph then $a(\text{Stab}_K(G)) = -\dim \Delta(G) - 2$.

**Proof.** Let $\delta$ be the maximal size of a clique in $G$. From the proof of the Lemma 11, $v = x_1 \cdots x_n^{\delta+1}$ is in $(\omega_{\text{Stab}_K(G)})_{\delta+1}$ and $v$ divides every monomial in $\omega_{\text{Stab}_K(G)}$. This gives the conclusion.

**Theorem 13.** Let $G$ be a finite simple graph with $G_1, \ldots, G_s$ its connected components and suppose that $G$ is perfect. Let $\delta_i$ denote the maximal cardinality of cliques of $G_i$. Then $\text{Stab}_K(G)$ is nearly Gorenstein if and only if each $G_i$ is pure and $|\delta_i - \delta_j| \leq 1$ for $1 \leq i < j \leq s$.

**Proof.** Suppose that $\text{Stab}_K(G)$ is nearly Gorenstein. It follows from Lemma 11 that each $G_i$ is pure and each $\text{Stab}_K(G_i)$ is Gorenstein. Since $\text{Stab}_K(G)$ is the Segre product of $\text{Stab}_K(G_1), \ldots, \text{Stab}_K(G_s)$, it follows from [7, Corollary 4.16] and [8, Corollary 2.8] that

$$|a(\text{Stab}_K(G_i)) - a(\text{Stab}_K(G_j))| \leq 1 \text{ for all } i, j.$$

Corollary 12 yields $|\delta_i - \delta_j| \leq 1$ for $1 \leq i < j \leq s$. Furthermore, the “If” part also follows from [7, Corollary 4.16] and [8, Corollary 2.8].

**Corollary 14.** Let $G$ be a finite simple graph which is perfect and connected. Then the ring $\text{Stab}_K(G)$ is nearly Gorenstein if and only if it is Gorenstein.

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