On an equation involving fractional powers with one prime and one almost prime variables

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Abstract
In this paper we consider the equation \( \lfloor p^c \rfloor + \lfloor m^c \rfloor = N \), where \( N \) is a sufficiently large integer, and prove that if \( 1 < c < \frac{29}{28} \), then it has a solution in a prime \( p \) and an almost prime \( m \) with at most \( \left\lfloor \frac{52}{29-28c} \right\rfloor + 1 \) prime factors.

1 Introduction and statement of the result
A Piatetski–Shapiro sequence is a sequence of the form
\[
\{\lfloor t^c \rfloor \}_{n \in \mathbb{N}} \quad (c > 1, c \notin \mathbb{N}),
\] (1)
where \( \lfloor t \rfloor \) denotes the integer part of \( t \). In 1953 Piatetski–Shapiro [12] showed that if \( 1 < c < \frac{12}{11} \), then the sequence (1) contains infinitely many prime numbers. Since then, the upper bound for \( c \) has been improved many times and the strongest result is due to Rivat and Wu [13]. They proved that the sequence (1) contains infinitely many primes provided that \( 1 < c < \frac{243}{205} \).

For any natural number \( r \), let \( \mathcal{P}_r \) denote the set of \( r \)-almost primes, i.e. the set of natural numbers having at most \( r \) prime factors counted with multiplicity. There are many papers devoted to the study of problems involving Piatetsi–Shapiro primes and almost primes. In 2011 Cai and Wang [2], improving an earlier result of Peneva [11], showed that if \( 1 < c < \frac{30}{29} \), then there exist infinitely many primes \( p \) of the form \( \lfloor n^c \rfloor \) such that \( p + 2 \in \mathcal{P}_5 \). Later, in 2014, Baker, Banks, Guo, and Yeager [1] showed that if \( 1 < c < \frac{77}{76} \), then the sequence
\[
\{\lfloor n^c \rfloor \}_{n \in \mathcal{P}_8}
\]
contains infinitely many prime numbers.

Consider the equation
\[
\lfloor m_1^c \rfloor + \lfloor m_2^c \rfloor = N.
\] (2)
In 1973, Deshouillers [4] proved that if \( 1 < c < \frac{4}{3} \), then for every sufficiently large integer \( N \) the equation (2) has a solution with \( m_1 \) and \( m_2 \) integers. This result was improved by Gritsenko [6], and later by Konyagin [9]. In particular, the latter author showed that (2) has a solution in integers \( m_1, m_2 \) for \( 1 < c < \frac{3}{2} \) and \( N \) sufficiently large.

Kumchev [10] proved that if \( 1 < c < \frac{16}{15} \), then every sufficiently large integer \( N \) can be represented in the form (2), where \( m_1 \) is a prime and \( m_2 \) is an integer. On the other hand, the celebrated theorem of Chen [3] states that every sufficiently large even integer can be represented as a sum of a prime and an almost prime from \( \mathcal{P}_2 \). Having in mind this profound result, one can conjecture that there exists a constant \( c_0 > 1 \) such that if \( 1 < c < c_0 \), then the equation (2) has a solution with \( m_1 \) a prime and \( m_2 \in \mathcal{P}_2 \) provided that \( N \) is sufficiently large. In the present paper, we establish a result of this type and prove the following

**Theorem 1.** Suppose that \( 1 < c < \frac{29}{28} \). Then every sufficiently large integer \( N \) can be represented as

\[
[p^c] + [m^c] = N, \tag{3}
\]

where \( p \) is a prime and \( m \) is an almost prime with at most \( \left\lfloor \frac{52}{29-28c} \right\rfloor + 1 \) prime factors.

We note that the integer \( \left\lfloor \frac{52}{29-28c} \right\rfloor + 1 \) is equal to 53 if \( c \) is close to 1 and it is large if \( c \) is close to \( \frac{29}{28} \).

Our first step in the proof is to apply the linear sieve. After doing so, we could try to establish a relatively strong estimate for the exponential sum defined in (31) which is a rather difficult task since the function in the exponent depends on \( [p^c] \). Instead, we represent this sum as a linear combination of similar sums (see (60)) with a smooth function of \( p \) in the exponent. Then we use standard techniques to estimate these sums. We would like to mention that the sums in (60) are also studied by Kumchev [10]. However, we cannot use his work because we require stronger bounds for them.

### 2 Notation

We fix the following notation: \( \{t\} \) is the fractional part of \( t \), the function \( \rho(t) \) is defined by \( \rho(t) = \frac{1}{2} - \{t\} \) and \( e(t) = e^{2\pi it} \). We use Vinogradov’s notation \( A \ll B \), which is equivalent to \( A = O(B) \). If we have simultaneously \( A \ll B \) and \( B \ll A \), then we shall write \( A \asymp B \).

For us \( p \) will be reserved for prime numbers. By \( \varepsilon \) we denote an arbitrarily small positive number, which is not necessarily the same in the different formulae. As usual, \( \sum_{n \leq x} \) means \( \sum_{1 \leq n \leq x} \) and \( \mu(n) \), \( \Lambda(n) \) and \( \tau(n) \) are the Mobius function, von Mangold’s function and the number of positive divisors of \( n \), respectively.
3 Proof of the theorem

3.1 Beginning of the proof

Let \( N \) be a sufficiently large integer and let
\[
1 < c < \frac{29}{28}, \quad \gamma = \frac{1}{c}, \quad P = 10^{-9}N^\gamma. \tag{4}
\]

Suppose that \( \alpha > 0 \) is a constant, which will be specified later, and let
\[
z = N^\alpha, \quad B_z = \prod_{p < z} p. \tag{5}
\]

We consider the sum
\[
\Gamma = \sum_{P < p \leq 2P, m \in \mathbb{N}} \log p. \tag{6}
\]

If \( \Gamma > 0 \), then there is a prime number \( p \) and a natural number \( m \) satisfying the conditions imposed in the domain of summation of \( \Gamma \). From the condition \((m, B_z) = 1\) it follows that any prime factor of \( m \) is greater or equal to \( z \). Suppose that \( m \) has \( l \) prime factors, counted with the multiplicity. Then we have
\[
N^\gamma \geq m \geq z^l = N^{\alpha l}
\]
and hence \( l \leq \frac{\gamma}{\alpha} \). This implies that if \( \Gamma > 0 \) then (3) has a solution with \( p \) a prime and \( m \) an almost prime with at most \( \left\lfloor \frac{\gamma}{\alpha} \right\rfloor \) prime factors.

We denote
\[
D = N^\delta, \tag{7}
\]
where \( \delta > 0 \) is a constant which will be specified later. Let \( \lambda(d) \) be the lower bound Rosser weights of level \( D \), (see [5, Chapter 4]). Then we have
\[
\sum_{d \mid k} \mu(d) \geq \sum_{d \mid k} \lambda(d) \quad \text{for every} \quad k \in \mathbb{N}. \tag{8}
\]

Furthermore, we know that
\[
|\lambda(d)| \leq 1 \quad \text{for all} \quad d; \quad \lambda(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0. \tag{9}
\]

Finally, we have
\[
\sum_{d \mid B_z} \frac{\lambda(d)}{d} \geq \prod_{p < z} \left(1 - \frac{1}{p}\right) \left(f(s) + O((\log D)^{-\frac{3}{2}})\right), \tag{10}
\]
\[
\text{where} \quad s = \frac{\log D}{\log z} = \frac{\delta}{\alpha}. \tag{11}
\]
and where \( f(s) \) is the lower function of the linear sieve, for which we know that
\[
f(s) = \begin{cases} 
0 & \text{for } 0 < s < 2; \\
2e^Gs^{-1}\log(s-1) & \text{for } 2 < s < 3. 
\end{cases}
\] (12)
(Here \( G \) is the Euler constant).

From (11) and (15) we find
\[
\Gamma = \sum_{P < p \leq 2P, \ m \in \mathbb{N}} (\log p) \sum_{d(m, B_x)} \mu(d) \geq \sum_{P < p \leq 2P, \ m \in \mathbb{N}} (\log p) \sum_{d(m, B_x)} \lambda(d).
\]

We change the order of summation to obtain
\[
\Gamma \geq \sum_{d \mid B_x} \lambda(d) G_d, \quad \text{where} \quad G_d = \sum_{P < p \leq 2P, \ m \in \mathbb{N}} \log p.
\] (13)

Now, we write the sum \( G_d \) in the form
\[
G_d = \sum_{P < p \leq 2P} (\log p) G'_{d, p}, \quad \text{where} \quad G'_{d, p} = \sum_{\substack{m \in \mathbb{N} \\
m \equiv 0 \pmod{d} \ \\
|p^{\gamma}| + |m^{\gamma}| = N}} 1.
\] (14)

We use the obvious identity
\[
\sum_{a \leq m < b} 1 = [-a] - [-b] = b - a - \rho(-b) + \rho(-a)
\]
to establish
\[
G'_{d, p} = \sum_{\substack{m \in \mathbb{N} \\
N - |p^{\gamma}| \leq m < N + 1 - |p^{\gamma}|}} 1 = \sum_{\frac{1}{d}(N - |p^{\gamma}|) \leq m < \frac{1}{d}(N + 1 - |p^{\gamma}|)} 1
\]
\[
= \frac{(N + 1 - |p^{\gamma}|)^\gamma - (N - |p^{\gamma}|)^\gamma}{d} + \rho \left( -\frac{1}{d}(N - |p^{\gamma}|)^\gamma \right) - \rho \left( -\frac{1}{d}(N + 1 - |p^{\gamma}|)^\gamma \right).
\]

We combine the above with (11) to obtain
\[
G_d = \frac{1}{d} A(N) + \sum_{P < p \leq 2P} (\log p) \left( \rho \left( -\frac{1}{d}(N - |p^{\gamma}|)^\gamma \right) - \rho \left( -\frac{1}{d}(N + 1 - |p^{\gamma}|)^\gamma \right) \right),
\] (15)
where
\[
A(N) = \sum_{P < p \leq 2P} (\log p)( (N - |p^{\gamma}|) + 1)^\gamma - (N - |p^{\gamma}|)^\gamma.
\]

From
\[
(N - |p^{\gamma}| + 1)^\gamma = (N - |p^{\gamma}|)^\gamma + \gamma(N - |p^{\gamma}|)^{\gamma-1} + O(N^{\gamma-2}),
\]
we deduce that
\[ A(N) = \gamma \sum_{P < p \leq 2P} (\log p) \left( (N - [p^\gamma])^{\gamma - 1} + O(N^{\gamma - 2}) \right), \]
and by Chebyshev’s prime number theorem and the definition of \(P\) in (4), we get
\[ A(N) \asymp N^{2\gamma - 1}. \quad (16) \]

From (13) and (15) we have
\[ \Gamma \geq \Gamma_0 + \Sigma_0 - \Sigma_1, \quad (17) \]
where
\[ \Gamma_0 = A(N) \sum_{d \mid B_z} \frac{\lambda(d)}{d}, \quad (18) \]
\[ \Sigma_j = \sum_{d \mid B_z} \lambda(d) \sum_{P < p \leq 2P} (\log p) \rho \left( -\frac{1}{d} (N + j - [p^\gamma])^{\gamma} \right), \quad j = 0, 1. \quad (19) \]

Consider \(\Gamma_0\). We use (5) and the Mertens formula to find
\[ \prod_{p < z} \left(1 - \frac{1}{p}\right) \asymp \frac{1}{\log z} \asymp \frac{1}{\log N}. \quad (20) \]

Assume that
\[ 2 < \frac{\delta}{\alpha} < 3. \quad (21) \]
Then, having in mind (11) and (12) we find that \(f(s) > \kappa\) for some constant \(\kappa > 0\) depending on \(\delta\) and \(\alpha\) only. Therefore, using (10) and (20) we get
\[ \sum_{d \mid B_z} \frac{\lambda(d)}{d} \gg \frac{1}{\log N}. \]
Thus, by (16) and (18) we obtain
\[ \Gamma_0 \gg \frac{N^{2\gamma - 1}}{\log N}. \quad (22) \]

We aim to establish the following bound for the sums \(\Sigma_j\) defined in (19):
\[ \Sigma_j \ll \frac{N^{2\gamma - 1}}{(\log N)^2}, \quad j = 0, 1. \quad (23) \]
This, together with (17) and (22) would imply
\[ \Gamma \gg \frac{N^{2\gamma - 1}}{\log N}, \]

\[ \text{5} \]
hence $\Gamma > 0$ for sufficiently large $N$. Then, as we already explained, the equation (3) would have a solution in a prime $p$ and an almost prime $m$ with no more than $\left\lfloor \frac{\gamma}{\alpha} \right\rfloor$ prime factors.

The remaining part of the paper will be devoted to the proof of the estimates (23) under the assumptions

$$\frac{28}{29} < \gamma < 1, \quad \delta < \frac{29\gamma - 28}{26}. \quad (24)$$

Then, it would remain to choose

$$\alpha = \frac{29\gamma - 28}{52} - \varepsilon_0$$

for some small $\varepsilon_0 > 0$ and to take

$$\delta \in \left(2\alpha, \frac{29\gamma - 28}{26}\right).$$

In this case, when $\varepsilon_0$ is small enough the condition (21) holds. With the choice (4) of $c$, it is easy to see that $\left\lfloor \frac{2}{\alpha} \right\rfloor \leq \left\lfloor \frac{52}{29\gamma - 28} \right\rfloor + 1$, which proves the theorem.

### 3.2 The estimation of the sums $\Sigma_1$ and $\Sigma_2$ — beginning

Consider the sum $\Sigma_j$ defined in (19). We apply Vaaler’s theorem [14], which states that for each $H \geq 2$ there are numbers $c_h (0 < |h| \leq H), d_h (|h| \leq H)$, such that

$$\rho(t) = \sum_{0 < |h| \leq H} c_h e(ht) + \Delta_H(t), \quad (25)$$

where

$$|\Delta_H(t)| \leq \sum_{|h| \leq H} \left| d_h e(ht) \right| \quad (26)$$

and

$$|c_h| \ll \frac{1}{|h|}, \quad |d_h| \ll \frac{1}{H}. \quad (27)$$

From (19) and (25) it follows that

$$\Sigma_j = \Sigma'_j + \Sigma''_j, \quad (28)$$

where

$$\Sigma'_j = \sum_{d \leq D} \lambda(d) \sum_{p < p' \leq 2p} (\log p) \sum_{0 < |h| \leq H} c_h e \left( \frac{h}{d} (N + j - [p^c])^\gamma \right), \quad (29)$$

$$\Sigma''_j = \sum_{d \leq D} \lambda(d) \sum_{p < p' \leq 2p} (\log p) \Delta_H \left( \frac{(N + j - [p^c])^\gamma}{d} \right). \quad (30)$$
Let
\[ W(v) = \sum_{P < p \leq 2P} (\log p) e(v(N + j - [p^c])^\gamma). \] (31)

We start with the sum \( \Sigma_j \). Changing the order of summation together with (9), (27) and (31) implies that
\[ \Sigma_j = \sum_{d \leq D} \lambda(d) \sum_{0 < |h| \leq H} c_h W \left( -\frac{h}{d} \right) \ll \sum_{d \leq D} \sum_{1 \leq h \leq H} \frac{1}{H} \left| W \left( \frac{h}{d} \right) \right|. \] (32)

For the sum \( \Sigma_j'' \) we use (9), (26), (27) and (31) to get
\[ \Sigma_j'' \ll \sum_{d \leq D} \sum_{P < p \leq 2P} (\log p) \sum_{|h| \leq H} d_h e \left( -\frac{h}{d}(N + j - [p^c])^\gamma \right) \]
\[ = \sum_{d \leq D} \sum_{|h| \leq H} d_h W \left( -\frac{h}{d} \right) \ll \sum_{d \leq D} \sum_{|h| \leq H} \frac{1}{H} \left| W \left( \frac{h}{d} \right) \right|. \]

From (4), (31) and Chebyshev’s prime number theorem we find that \( W(0) \approx N^\gamma \) and hence
\[ \Sigma_j'' \ll \sum_{d \leq D} \frac{N^\gamma}{H} + \sum_{d \leq D} \sum_{1 \leq h \leq H} \frac{1}{H} \left| W \left( \frac{h}{d} \right) \right|. \] (33)

We let
\[ H = dN^{1-\gamma}(\log N)^3. \] (34)

Now, using (7), (28) and (32) – (34) we obtain
\[ \Sigma_j \ll \frac{N^{2\gamma-1}}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \left| W \left( \frac{h}{d} \right) \right|, \quad j = 0, 1. \] (35)

### 3.3 Consideration of the sum \( W(v) \)

As we mentioned earlier, it is hard to estimate directly the exponential sum \( W(v) \), defined by (31). Instead, we can write it as a linear combination of similar sums which are easier to be dealt with.

Let \( Z \geq 2 \) be an integer, which we shall specify later. We apply the well-known Vinogradov’s “little cups” lemma (see [3], Chapter 1, Lemma A) with parameters
\[ \alpha = -\frac{1}{4Z}, \quad \beta = \frac{1}{4Z}, \quad \Delta = \frac{1}{2Z}, \quad r = \lfloor \log N \rfloor \]
and construct a function \( g(t) \) which is periodic with period 1 and has the following properties:
\[ g(0) = 1; \quad 0 < g(t) < 1 \text{ for } 0 < |t| < \frac{1}{2Z}; \quad g(t) = 0 \text{ for } \frac{1}{2Z} \leq |t| \leq \frac{1}{2}. \] (36)
Furthermore, the Fourier series of \( g(t) \) is given by
\[
g(t) = \frac{1}{2Z} + \sum_{n \in \mathbb{Z}} \beta_n \, e(nt), \quad \text{with} \quad |\beta_n| \leq \min \left( \frac{1}{2Z}, \frac{1}{|n|} \right) \left( \frac{2Z \log N}{\pi |n|} \right)^{\log N}.
\]

From the above estimate of \( |\beta_n| \) one easily obtains
\[
\sum_{|n| > Z(\log N)^4} |\beta_n| \ll N^{-\log \log N}
\]
with an absolute constant in the \( \ll \)-symbol. Hence we have
\[
g(t) = \sum_{|n| \leq Z(\log N)^4} \beta_n \, e(nt) + O \left( N^{-\log \log N} \right),
\]
where the implied constant is absolute and
\[
|\beta_n| \leq \frac{1}{2Z}.
\]

Finally, one can easily see that the function \( g(t) \), constructed in the proof of [8, Chapter 1, Lemma A], is even and also satisfies
\[
g(t) + g \left( t - \frac{1}{2Z} \right) = 1 \quad \text{for} \quad 0 \leq t \leq \frac{1}{2Z}.
\]

Let
\[
g_z(t) = g \left( t - \frac{z}{2Z} \right) \quad \text{for} \quad z = 0, 1, 2, \ldots, 2Z - 1.
\]

Obviously, each \( g_z(t) \) is a periodic function with period 1. From (36) we find that
\[
0 < g_z(t) \leq 1 \quad \text{if} \quad \left| t - \frac{z}{2Z} \right| < \frac{1}{2Z};
\]
\[
g_z(t) = 0 \quad \text{if} \quad \frac{1}{2Z} \leq \left| t - \frac{z}{2Z} \right| \leq \frac{1}{2}.
\]

From (40) it follows that if \( \beta_n^{(z)} \) is the \( n \)-th Fourier coefficient of the function \( g_z(t) \), then
\[
\beta_n^{(z)} = \beta_n e \left( -\frac{zn}{2Z} \right) \quad \text{and hence} \quad |\beta_n^{(z)}| = |\beta_n|.
\]

From this observation and (37), as well as the estimate for \( |\beta_n| \) given above, we find that for \( z = 0, 1, \ldots, 2Z - 1 \) we have
\[
g_z(t) = \sum_{|n| \leq Z(\log N)^4} \beta_n^{(z)} \, e(nt) + O \left( N^{-\log \log N} \right),
\]
where the constant in the \( O \)-symbol is absolute and
\[
|\beta_n^{(z)}| \leq \frac{1}{2Z}.
\]
Finally, from (39), (40) and (42) we find
\[ 2Z^{-1} \sum_{z=0}^{2Z-1} g_z(t) = 1 \quad \text{for all} \quad t \in \mathbb{R}. \]  
(45)

(We leave the easy verification to the reader).

Now, we consider the sum \( W(v) \) which was defined in (31). From (45) it follows that
\[ W(v) = \sum_{P < p \leq 2P} (\log p) g_z(p^c) e(v(N + j - [p^c])^\gamma). \]  
(46)

where
\[ W_z(v) = \sum_{P < p \leq 2P} (\log p) g_z(p^c) e(v(N + j - [p^c])^\gamma). \]  
(47)

It is clear that
\[ W_0(v) \ll \sum_{P < p \leq 2P} (\log p) g(p^c) \ll \log N \sum_{P < k \leq 2P} g(k^c). \]  
We apply (4), (37) and (38) to get
\[ W_0(v) \ll (\log N) \frac{P}{Z} + (\log N) \left| \sum_{P < k \leq 2P} \sum_{0 < |n| \leq Z/(\log N)^4} \beta_n e(nk^c) \right| + 1 \]
\[ \ll (\log N) \frac{N^\gamma}{Z} + \log N \sum_{n \leq Z/(\log N)^4} |\mathcal{H}_n| + 1, \]  
(48)

where
\[ \mathcal{H}_n = \sum_{P < k \leq 2P} e(nk^c). \]

If \( \theta(x) = nx^c \), then \( \theta''(x) = c(c-1)nx^{c-2} \sim nP^{c-2} \) uniformly for \( x \in [P, 2P] \). Hence, we can apply Van der Corput’s theorem (see [8], Chapter 1, Theorem 5) to get
\[ \mathcal{H}_n \ll P \left( nP^{c-2} \right)^{\frac{1}{2}} + \left( nP^{c-2} \right)^{-\frac{1}{2}} \ll P^\frac{c}{2} n^\frac{1}{2}. \]  
(49)

Henceforth we assume that
\[ Z \ll N^{\frac{1}{3}} (\log N)^{-4}. \]  
(50)

Then using (4) and (48) – (50) we obtain
\[ W_0(v) \ll (\log N) \left( \frac{N^\gamma}{Z} + N^\frac{c}{2} \frac{Z^{\frac{1}{2}} \log^6 N}{Z} \right) \ll (\log N) \frac{N^\gamma}{Z}. \]  
(51)
Now, we restrict our attention to the sums $W_z(v)$ for $1 \leq z \leq 2Z - 1$. Using (12) we see that $g_z(p^r)$ vanishes unless $\{p^r\} \in \left[\frac{z}{2Z}, \frac{z+1}{2Z}\right]$. Hence, the only summands in the sum (17) are those for which $\{p^r\} = \frac{z}{2Z} + O\left(\frac{1}{Z}\right)$. In this case we have

$$v(N + j - [p^r])^\gamma = v \left( N + j - p^r + \frac{z}{2Z} \right)^\gamma + O\left(\frac{v N^{\gamma - 1}}{Z}\right)$$

and respectively

$$e \left( v(N + j - [p^r])^\gamma \right) = e \left( v \left( N + j - p^r + \frac{z}{2Z} \right)^\gamma \right) + O\left(\frac{v N^{\gamma - 1}}{Z}\right).$$

Then using (47) we find

$$W_z(v) = V_z(v) + O\left(\frac{v N^{\gamma - 1}}{Z} \sum_{P < p \leq 2P} (\log p) g_m(p^r)\right),$$

(52)

where

$$V_z(v) = \sum_{P < p \leq 2P} (\log p) g_z(p^r) e \left( v \left( N + j - p^r + \frac{z}{2Z} \right)^\gamma \right).$$

(53)

Therefore, from (46), (51) and (52) we obtain

$$W(v) = \sum_{z=1}^{2Z-1} W_z(v) + W_0(v) = \sum_{z=1}^{2Z-1} V_z(v) + O(\Xi) + O\left(\left(\log N\right)\frac{N^{\gamma}}{Z}\right),$$

where

$$\Xi = \frac{v N^{\gamma - 1}}{Z^2} \sum_{P < p \leq 2P} (\log p) \sum_{z=1}^{2Z-1} g_z(p^r).$$

Now we use (11), (15) and Chebyshev’s prime number theorem to find that

$$\Xi \ll \frac{v N^{2\gamma - 1}}{Z}$$

and therefore

$$W(v) = \sum_{z=1}^{2Z-1} V_z(v) + O\left(\frac{v N^{2\gamma - 1}}{Z}\right) + O\left(\left(\log N\right)\frac{N^{\gamma}}{Z}\right).$$

(54)

From this point onwards we assume that

$$v = \frac{h}{d}, \quad \text{where} \quad 1 \leq d \leq D, \quad 1 \leq h \leq H.$$  

(55)
Then using (34) we see that 
\[ vN^{2\gamma - 1} \ll N^\gamma (\log N)^3, \]

hence formula (54) can be written as
\[ W(v) = \sum_{z=1}^{2Z-1} V_z(v) + O\left( (\log N)^3 \frac{N^\gamma}{Z} \right). \] (56)

From (34), (35) and (56) we find
\[ \sum_j \ll N^{2\gamma - 1} \frac{1}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{z=1}^{2Z-1} |V_z(v)| + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \frac{N^\gamma}{Z} (\log N)^3. \] (57)

We choose \( Z \) such that
\[ Z \approx dN^{1-\gamma} (\log N)^7. \] (58)

From (7) and (24) it follows that the condition (50) holds. Consequently from (57) and (58) we find
\[ \sum_j \ll N^{2\gamma - 1} \frac{1}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{z=1}^{2Z-1} |V_z(v)|. \] (59)

Now, we consider the sum \( V_z(v) \) defined in (53), where \( v \) satisfies (55). By (43) we find that
\[
V_z(v) = \sum_{p < p \leq 2P} (\log p) \left( \sum_{|r| \leq Z(\log N)^4} \beta_r^{(z)} e(rp) \right) e \left( v \left( N + j - p + \frac{z}{2Z} \right) \right) + O(N^{-10})
\]
\[
= \sum_{|r| \leq Z(\log N)^4} \beta_r^{(z)} U \left( N + j + \frac{z}{2Z}, r, v \right) + O(N^{-10}),
\]

where
\[
U = U(T, r, v) = \sum_{p < p \leq 2P} (\log p) e(rp + v(T - p)^\gamma). \] (60)

We would like to point out that when \( 0 \leq j \leq 1 \) and \( 1 \leq z \leq 2Z - 1 \) then
\[ N + j + \frac{z}{2Z} \in [N, N+2]. \]

Furthermore, when we also take into account (43) and (58) we obtain
\[ V_z(v) \ll N^{-10} + \frac{1}{Z} \sum_{|r| \leq R} \sup_{T \in [N,N+2]} |U(T, r, v)|. \]

where
\[ R = dN^{1-\gamma} (\log N)^{12} \] (61)

When we substitute the above expression for \( V_z(v) \) in formula (59) we find that
\[ |\Sigma_1| + |\Sigma_2| \ll \frac{N^{2\gamma - 1}}{(\log N)^2} + \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{|r| \leq R} \sup_{T \in [N,N+2]} |U(T, r, v)|. \] (62)
3.4 Application of Vaughan’s identity

Let us introduce the notations

\[ \phi(t) = rt^c + v(T - t^c)^\gamma, \]

\[ f(m, l) = \phi(ml) = r(ml)^c + v(T - (ml)^c)^\gamma. \]

For the sum \( U \), defined in (60), we have

\[ U = \sum_{P < n \leq 2P} \Lambda(n) e(\phi(n)) + O\left(P^{1/2}\right). \]

Now we apply Vaughan’s identity (see [15]) and find that

\[ U = U_1 - U_2 - U_3 - U_4 + O\left(P^{1/2}\right), \]

where

\[ U_1 = \sum_{m \leq P^{1/3}} \mu(m) \sum_{\frac{P^c}{m} < l \leq \frac{2P^c}{m}} (\log l) e(f(m, l)), \]

\[ U_2 = \sum_{m \leq P^{1/3}} c(m) \sum_{\frac{P^c}{m} < l \leq \frac{2P^c}{m}} e(f(m, l)), \]

\[ U_3 = \sum_{P^{1/3} < m \leq P^{1/2}} c(m) \sum_{\frac{P^c}{m} < l \leq \frac{2P^c}{m}} e(f(m, l)), \]

\[ U_4 = \sum_{P < ml \leq 2P} \sum_{m > P^{1/3}, l > P^{1/3}} a(m) \Lambda(l) e(f(m, l)), \]

and where

\[ |c(m)| \leq \log m \quad \text{and} \quad |a(m)| \leq \tau(m). \]

Hence from (7), (24), (34), (61), (62) and (65) we have

\[ |\Sigma_1| + |\Sigma_2| \ll \frac{N^{2\gamma-1}}{(\log N)^2} + \sum_{i=1}^{4} \Omega_i, \]

where

\[ \Omega_i = \sum_{d \leq D} \sum_{h \leq H} \frac{1}{k} \sum_{|r| \leq R} \sup_{T \in [N, N+2]} |U_i|. \]

By (71) and (72), in order to prove that (23) is satisfied, it suffices to show that

\[ \Omega_i \ll \frac{N^{2\gamma-1}}{(\log N)^2} \quad \text{for} \quad i = 1, 2, 3, 4. \]
3.5 The estimation of the sums $\Omega_1$ and $\Omega_2$

We begin with the study of $\Omega_2$. From (64) we get
\[ f''_ll(m, l) = \pi_1 - \pi_2, \]  
(74)

where
\[ \pi_1 = m^2rc(c - 1)(ml)^{c-2}, \quad \pi_2 = m^2v(c - 1)T(ml)^{c-2}(T - (ml)^c)^{\gamma-2}. \]  
(75)

From (4), (55) and the conditions
\[ P < ml \leq 2P, \quad N \leq T \leq N + 2 \]  
(76)
we have
\[ |\pi_1| \asymp |r|m^2N^{1-2\gamma} \quad \text{and} \quad \pi_2 \asymp vm^2N^{-\gamma}. \]  
(77)

It follows from (74) and (77) that there exists sufficiently small constant $\alpha_0 > 0$ such that if $|r| \leq \alpha_0vN^{\gamma-1}$, then $|f''_ll| \asymp |r|m^2N^{1-2\gamma}$.

Similarly, from (74) and (77) we conclude that there exists sufficiently large constant $A_0 > 0$ such that if $|r| \geq A_0vN^{\gamma-1}$, then $|f''_ll| \asymp |r|m^2N^{1-2\gamma}$.

We divide the sum $\Omega_2$ into four sums according to the value of $r$ as follows:
\[ \Omega_2 = \Omega_{2,1} + \Omega_{2,2} + \Omega_{2,3} + \Omega_{2,4}, \]  
(78)

where
\[ \text{in } \Omega_{2,1} : \quad |r| \leq \alpha_0vN^{\gamma-1}, \]  
(79)
\[ \text{in } \Omega_{2,2} : \quad -A_0vN^{\gamma-1} < r < -\alpha_0vN^{\gamma-1}, \]  
(80)
\[ \text{in } \Omega_{2,3} : \quad \alpha_0vN^{\gamma-1} < r < A_0vN^{\gamma-1}, \]  
(81)
\[ \text{in } \Omega_{2,4} : \quad A_0vN^{\gamma-1} \leq |r| \leq R. \]  
(82)

We note that from (34), (55) and (61) it follows that
\[ vN^{\gamma-1} \ll (\log N)^3 \ll \frac{R}{\log N}. \]  
(83)

Let us consider $\Omega_{2,4}$ first. We have
\[ \Omega_{2,4} = \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{A_0vN^{\gamma-1} \leq |r| \leq R} \sup_{T \in [N, N+2]} |U_2|. \]  
(84)
We shall estimate the sum $U_2$, defined by (67), provided that the condition (82) holds. We recall that the constant $A_0$ is chosen in such a way, that if $|r| \geq A_0N^{\gamma-1}$, then uniformly for $l \in \left(\frac{P}{m}, \frac{2P}{m}\right]$ we have $|f''_l(m,l)| \approx |r|m^2N^{1-2\gamma}$. Hence we are in a position to use again Van der Corput’s theorem (see [8, Chapter 1, Theorem 5]) and having also in mind (4) we obtain

$$e(f(m,l)) \ll \frac{P}{m} \left(|r|m^2N^{1-2\gamma}\right)^{\frac{1}{2}} + \left(|r|m^2N^{1-2\gamma}\right)^{-\frac{1}{2}} \ll |r|^{\frac{3}{2}}N^{\frac{1}{2}}.$$  

Then from (67) and (70) we find

$$U_2 \ll |r|^{\frac{3}{2}}N^{\frac{1}{2}+\frac{\gamma}{2}}(\log N). \quad (85)$$

We substitute this expression for $U_2$ in (84) and use (34), (55) and (61) to get

$$\Omega_{2,4} \ll D_{\gamma} \frac{N^{2-\frac{2\gamma}{3}+\varepsilon}}{N^\gamma}.$$  

Hence from (7) and (24) we obtain

$$\Omega_{2,4} \ll \frac{N^\gamma \log N}{(\log N)^2}. \quad (86)$$

We carry on with $\Omega_{2,3}$. We have to study the sum $U_2$ defined by (67) provided that the condition (81) holds. To do so we use (64) and compute

$$f''_l(m, l) = m^3rc(c-1)(c-2)(ml)^{c-3}$$

$$+ m^3v(c-1)T(T-(ml)^c)^{\gamma-3}(ml)^{c-3}((2-c)T-(c+1)(ml)^c). \quad (87)$$

From (74), (75) and (87) we find

$$(c-2)f''_l(m, l) - l f''_l(m, l) = v(c-1)(2c-1)Tm^{2c}l^{2e-2}(T-(ml)^c)^{\gamma-3}.$$  

The above, together with (1) and (76) implies

$$|(c-2)f''_l(m, l) - l f''_l(m, l)| \approx vm^2N^{-\gamma}.$$  

Therefore, there exists $\kappa_0 > 0$, depending only on the constant $c$, such that for every $l \in \left(\frac{P}{m}, \frac{2P}{m}\right]$ at least one of the following inequalities holds:

$$|f''_l(m, l)| \geq \kappa_0vm^2N^{-\gamma}, \quad (88)$$

or

$$|f''_l(m, l)| \geq \kappa_0vm^3N^{-2\gamma}. \quad (89)$$

We are going to show that the interval $\left(\frac{P}{m}, \frac{2P}{m}\right]$ can be divided into at most 7 intervals such that if $J$ is one of them, then at least one of the following statements holds:

We have (88) for all $l \in J$.  

(90)
We have (89) for all \( l \in J \).

To establish this it is enough to show that the equation
\[
|f''(m, l)| = \kappa_0 vx^{2}N^{-\gamma}
\]
has at most 6 solutions in real numbers \( l \in \left( \frac{P}{m}, \frac{2P}{m} \right) \). Hence, it is enough to show that if \( C \) does not depend on \( l \) then the equation \( f''(m, l) = C \) has at most 3 solutions in real numbers \( l \in \left( \frac{P}{m}, \frac{2P}{m} \right) \). According to Rolle’s theorem, between any two solutions of the last equation there is a solution (in real numbers \( l \)) of the equation \( f'''(m, l) = 0 \). So, we use (87) to conclude that it is enough to show that
\[
v T (T - (ml)^\gamma)^{-3} ((2 - c)T - (c + 1)ml)^c) = rc(2 - c)
\]
has at most 2 solutions in \( l \in \left( \frac{P}{m}, \frac{2P}{m} \right) \), which is equivalent to the assertion that the equation
\[
(T - X)^{\gamma - 3}((2 - c)T - (c + 1)X) = \frac{rc(2 - c)}{vT}
\]
has at most 2 solutions in \( X \in (P, (2P)^c) \). Alternatively, instead of the last equation one can look at
\[
(\gamma - 3) \log (T - X) + \log ((2 - c)T - (c + 1)X) = \log \frac{rc(2 - c)}{vT}.
\]
Let \( H(X) \) denote the function on the left side of (92). From Rolle’s theorem we know that between any two solutions of (92) there is a solution of \( H'(X) = 0 \). Since
\[
H'(X) = \frac{3 - \gamma}{T - X} - \frac{c + 1}{(2 - c)T - (c + 1)X}
\]
it is easy to see that \( H'(X) \) vanishes for at most 1 value of \( X \). Therefore, (92) has at most 2 solutions in \( X \) and our assertion is proved.

On the other hand, from (74), (77) and (87) we see that under the condition on \( r \) imposed in (81) we have
\[
f''(m, l) \ll v x^{2}N^{-\gamma} \quad \text{and} \quad f'''(m, l) \ll v x^{3}N^{-2\gamma}.
\]
Hence, the interval \( \left( \frac{P}{m}, \frac{2P}{m} \right) \) can be divided into at most 7 intervals such that if \( J \) is one of them, then at least one of the following assertions holds:
\[
|f''(m, l)| \asymp v x^{2}N^{-\gamma} \quad \text{uniformly for} \quad l \in J,
\]
\[
|f'''(m, l)| \asymp v x^{3}N^{-2\gamma} \quad \text{uniformly for} \quad l \in J.
\]
If (93) is fulfilled, then we use Van der Corput’s theorem (see [8, Chapter 1, Theorem 5]) for the second derivative and find
\[
\sum_{l \in J} e(f(m, l)) \ll \frac{P}{m} \left( v x^{2}N^{-\gamma} \right)^{\frac{1}{2}} + \left( v x^{2}N^{-\gamma} \right)^{-\frac{1}{2}} \ll v^{\frac{3}{2}}N^{\frac{3}{2}} + v^{-\frac{1}{2}}m^{-1}N^{\frac{3}{2}}.
\]
In the case (94) we apply Van der Corput’s theorem for the third derivative to get
\[ \sum_{l \in J} e(f(m, l)) \ll \frac{P}{m} \left( v m^3 N^{-2\gamma} \right)^{\frac{1}{m}} + \left( \frac{P}{m} \right)^{\frac{1}{2}} \left( v m^3 N^{-2\gamma} \right)^{-\frac{1}{m}} \ll v^\frac{1}{2} m^{-\frac{1}{2}} N^{\frac{2\gamma}{3}} + v^{-\frac{1}{2}} m^{-1} N^{\frac{2\gamma}{3}}. \]
Hence, in either case \( \sum_{l \in J} e(f(m, l)) \) can be estimated by the sum of the expressions on the right sides of the inequalities above. Therefore,
\[ \sum_{\frac{P}{m} < l \leq 2P} e(f(m, l)) \ll v^\frac{1}{2} N^{\frac{2\gamma}{3}} + v^{-\frac{1}{2}} m^{-1} N^{\frac{2\gamma}{3}} + v^\frac{1}{2} N^{\frac{2\gamma}{3}} + v^{-\frac{1}{2}} m^{-1} N^{\frac{2\gamma}{3}}. \] (96)
Then from (4), (67) and (70) we find that
\[ U_2 \ll (\log N)^2 \left( v^\frac{1}{2} N^{\frac{2\gamma}{3}} + v^{-\frac{1}{2}} N^{\frac{2\gamma}{3}} + v^\frac{1}{2} N^{\frac{2\gamma}{3}} + v^{-\frac{1}{2}} N^{\frac{2\gamma}{3}} \right). \] (97)
We use (34), (55), (81), (83) and (97) to get
\[ \Omega_{2,3} = \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{\alpha_0 \neq \nu_1, \nu_2 < \rho < A_0 \nu_1, \nu_2} \sup_{T \in [N, N+2]} |U_2| \]
\[ \ll N^\varsigma \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \left( h^\frac{1}{2} N^{\frac{2\gamma}{3}} + h^{-\frac{1}{2}} N^{\frac{2\gamma}{3}} + h^\frac{1}{2} N^{\frac{2\gamma}{3}} + h^{-\frac{1}{2}} N^{\frac{2\gamma}{3}} \right) \]
\[ \ll N^\varsigma \left( D N^{\frac{1}{2} + \frac{\gamma}{3}} + D^2 N^{\frac{2\gamma}{3}} + D N^{\frac{1}{2} + \frac{\gamma}{3}} + D^2 N^{\frac{2\gamma}{3}} \right) \] (98)
and from (7) and (24) we deduce that
\[ \Omega_{2,3} \ll \frac{N^{2\gamma - 1}}{(\log N)^2}. \] (99)
Let us consider \( \Omega_{2,1} \). We have chosen the constant \( \alpha_0 \) in such a way, that from (76) and from the condition on \( r \) imposed in (79) it follows that \( |f''_m(m, l)| \propto v m^3 N^{-\gamma} \) uniformly for \( l \in \left( \frac{P}{m}, \frac{2P}{m} \right] \). Hence the sum \( \sum_{\frac{P}{m} < l \leq 2P} e(f(m, l)) \) can be estimated by the expression on the right side of (95) and certainly the estimate (96) holds again. From this observation we see that \( \Omega_{2,1} \) can be estimated in the same way as \( \Omega_{2,3} \), i.e.
\[ \Omega_{2,1} \ll \frac{N^{2\gamma - 1}}{(\log N)^2}. \] (100)
The sum \( \Omega_{2,2} \) can be studied in the same way. From (74) – (76) and (80) it follows that \( |f''_m(m, l)| \propto v m^3 N^{-\gamma} \) and hence the estimate (96) is correct again. Therefore
\[ \Omega_{2,2} \ll \frac{N^{2\gamma - 1}}{(\log N)^2}. \] (101)
From (78), (86) and (99) – (101) we conclude that
\[ \Omega_2 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \] (102)

Consider now \( \Omega_1 \). For \( U_1 \) defined by (66), we use Abel’s transformation to get rid of the factor \( \log l \) in the inner sum. Then we proceed as in the estimation of \( \Omega_2 \) to obtain
\[ \Omega_1 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \] (103)

### 3.6 The estimation of the sums \( \Omega_3 \) and \( \Omega_4 \) and the end of the proof

Consider the sum \( \Omega_4 \), defined in (72). We divide the sum \( U_4 \) given by (69) into \( O(\log N) \) sums of the form
\[ W_{M,L} = \sum_{L < l \leq 2L} b(l) \sum_{M < m \leq 2M} \sum_{P^l < m \leq 2P^l} a(m) e(f(m, l)), \] (104)
where
\[ a(m) \ll N^\varepsilon, \quad b(l) \ll N^\varepsilon, \quad P^\frac{1}{3} \leq M \leq P^\frac{2}{3} \ll L \ll P^\frac{2}{3}, \quad ML \sim P. \] (105)

From (104), (105) and Cauchy’s inequality we find that
\[ |W_{M,L}|^2 \ll N^\varepsilon L \sum_{L < l \leq 2L} \left| \sum_{M_1 < m \leq M_2} a(m) e(f(m, l)) \right|^2, \] (106)
where
\[ M_1 = \max \left( M, \frac{P}{l} \right), \quad M_2 = \min \left( 2M, \frac{2P}{l} \right). \] (107)

Now we apply the well-known inequality
\[ \left| \sum_{a < m \leq b} \xi(m) \right|^2 \leq \frac{b - a + Q}{Q} \sum_{|q| < Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{m \in [a,b]} \sum_{m \in [a-q,b-q]} \xi(m + q) \xi(m), \] (108)
where \( Q \in \mathbb{N}, a, b \in \mathbb{R}, 1 \leq b - a \) and \( \xi(m) \) is any complex function. (A proof can be found in [7, Lemma 8.17]). In our setting \( \xi(m) = a(m) e(f(m, l)) \), \( a = M_1, b = M_2 \). The exact value of \( Q \) will be chosen later. For now we only require that
\[ Q \leq M. \] (109)
Then we find

$$|W_{M,L}|^2 \ll N^\varepsilon L \sum_{L < l \leq 2L} \frac{M}{Q} \left| \sum_{|q| \leq Q} \left( 1 - \frac{|q|}{Q} \right) \right| \times \sum_{M_1 < m \leq M_2} a(m + q) a(m) \left| e(f(m + q, l) - f(m, l)) \right|.$$ 

We estimate the contribution coming from the terms with $q = 0$, then we change the order of summation and using (105) and (107) we find

$$|W_{M,L}|^2 \ll N^\varepsilon (LM)^2 \frac{Q}{Q} + N^\varepsilon LM \sum_{0 < |q| \leq Q} \sum_{M < m \leq 2M} \sum_{M_1 < m + q \leq M_2} e \left( Y_{m,q}(l) \right),$$

where

$$L_1 = \max \left( L, \frac{P}{m}, \frac{P}{m + q} \right), \quad L_2 = \min \left( 2L, \frac{2P}{m}, \frac{2P}{m + q} \right)$$

and

$$Y(l) = Y_{m,q}(l) = f(m + q, l) - f(m, l).$$

It is now easy to see that the sum over negative $q$ in formula (110) is equal to the sum over positive $q$, hence we obtain

$$|W_{M,L}|^2 \ll N^\varepsilon (LM)^2 \frac{Q}{Q} + N^\varepsilon LM \sum_{1 \leq |q| \leq Q} \sum_{M < m \leq 2M - q} \sum_{L_1 < l \leq L_2} e \left( Y_{m,q}(l) \right).$$

Consider the function $Y(l)$. Using (63), (64) and (112) we find that

$$Y(l) = \int_m^{m+q} f'(t, l) dt = \int_m^{m+q} l \phi'(tl) dt$$

and therefore

$$Y''(l) = \int_m^{m+q} (2t \phi''(tl) + lt \phi'''(tl)) dt, \quad Y'''(l) = \int_m^{m+q} (3t^2 \phi'''(tl) + lt^2 \phi^{(4)}(tl)) dt.$$ 

From (63) and (115) we get

$$Y''(l) = \int_m^{m+q} (\Phi_1(t) - \Phi_2(t)) dt,$$
where
\begin{align}
\Phi_1(t) &= r e^2 (c - 1) t^{e-1} e^{-2}, \\
\Phi_2(t) &= v (c - 1) T t^{e-1} e^{-2} (T - (tl)^c)^{\gamma - 3} (cT + (c - 1)(tl)^c).
\end{align}

If \( t \in [m, m + q] \) then \( tl \approx P \). Thus, by (11) and the condition
\[ N \leq T \leq N + 2 \]
we find that uniformly for \( t \in [m, m + q] \) we have
\[ |\Phi_1(t)| \approx |r| m N^{1-2\gamma} \quad \text{and} \quad \Phi_2(t) \approx v m N^{-\gamma}. \]

From (116) and (120) we see that there exists a sufficiently small constant \( \alpha_1 > 0 \) such that if \( |r| \leq \alpha_1 v N^{\gamma - 1} \), then \( |Y''(l)| \approx q v m N^{-\gamma} \). Similarly, we conclude that there exists a sufficiently large constant \( A_1 > 0 \) such that if \( |r| \geq A_1 v N^{\gamma - 1} \), then \( |Y''(l)| \approx |r| q m N^{1-2\gamma} \). Hence, it makes sense to divide the sum \( \Omega_4 \) into four sums according to the value of \( r \) as follows:
\[ \Omega_4 = \Omega_{4,1} + \Omega_{4,2} + \Omega_{4,3} + \Omega_{4,4}, \]

where
\begin{align}
\text{in } \Omega_{4,1} : & \quad |r| \leq \alpha_1 v N^{\gamma - 1}, \\
\text{in } \Omega_{4,2} : & \quad -A_1 v N^{\gamma - 1} < r < -\alpha_1 v N^{\gamma - 1}, \\
\text{in } \Omega_{4,3} : & \quad \alpha_1 v N^{\gamma - 1} < r < A_1 v N^{\gamma - 1}, \\
\text{in } \Omega_{4,4} : & \quad A_1 v N^{\gamma - 1} \leq |r| \leq R.
\end{align}

Let us consider \( \Omega_{4,4} \) first. From (72) and (125) we have
\[ \Omega_{4,4} \ll (\log N) \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{A_1 v N^{\gamma - 1} \leq |r| \leq R} \sup_{M, L \in [N, N+2]} |W_{M,L}|. \]

(The supremum is taken over \( T \in [N, N + 2] \) and \( M, L \) satisfying the conditions imposed in (105)).

Consider the sum \( W_{M,L} \). We already mentioned that if \( |r| \geq A_1 v N^{\gamma - 1} \), then uniformly for \( l \in (L_1, L_2] \) we have \( Y''(l) \approx |r| q m N^{1-2\gamma} \). Hence we can use Van der Corput’s theorem (see [8 Chapter 1, Theorem 5]) for the second derivative and by (3), (105) and (111) we obtain
\[ \sum_{L_1 \leq l \leq L_2} e(Y(l)) \ll L(|r| q m N^{1-2\gamma})^{\frac{1}{2}} + (|r| q m N^{1-2\gamma})^{-\frac{1}{2}} \ll |r|^{\frac{1}{2}} q^{\frac{1}{2}} M^{-\frac{1}{2}} N^{\frac{1}{2}}. \]
Then from (1), (105) and (113) we find

\[ W_{M,L} \ll N^\varepsilon \left( N^{-\frac{1}{2}} + |r|^\frac{1}{2} Q^{\frac{5}{4}} N^{\frac{1}{4} + \frac{5}{8}} \right) . \]  

(127)

From (34), (61) and (126) we have

\[ \Omega_{4,4} \ll N^\varepsilon \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{|r| \leq d N^{1-\gamma} (\log N)^{12}} \left( N^{-\frac{1}{2}} + |r|^\frac{1}{2} Q^{\frac{5}{4}} N^{\frac{1}{4} + \frac{5}{8}} \right) \]

\[ \ll N^\varepsilon \left( D^2 N^{-\frac{1}{2}} + Q^{\frac{5}{4}} D^2 N^\frac{3}{4} - \frac{5}{8} \right) . \]

We choose

\[ Q = \left[ D^{-\frac{1}{2}} N^{\frac{5}{8} - \frac{3}{8}} \right] . \]  

(128)

It is now easy to verify that the condition (109) holds. Hence, from (1) and [24] we obtain

\[ \Omega_{4,4} \ll \frac{N^{2\gamma - 1}}{(\log N)^2} . \]  

(129)

Let us now consider \( \Omega_{4,3} \). From (72) and (124) we have

\[ \Omega_{4,3} \ll (\log N) \sum_{d \leq D} \sum_{h \leq H} \frac{1}{h} \sum_{\alpha_1 \in N^{\gamma - 1}} \sum_{\alpha_1 < r < A_1 N^{\gamma - 1}} \sup_{T \in [N, N + 2]} |W_{M,L}| . \]  

(130)

Consider the sum \( W_{M,L} \) from the expression in the above formula. Using (116) – (118) we find

\[ Y''''(l) = \int \frac{m+q}{m} (\Psi_1(t) + \Psi_2(t)) dt, \]  

(131)

where

\[ \Psi_1(t) = r c^2 (c - 1) (c - 2) t^{c-1} t^{-3}, \]  

(132)

\[ \Psi_2(t) = v (c - 1) T t^{c-1} t^{-3} (T - (t l)^c)^{c-4} \times \left( c(2 - c) T^2 + (-4c^2 + 3c - 2) T (tl)^c + (1 - c^2) (tl)^{2c} \right) . \]  

(133)

From (116) – (118) and (131) – (133) we obtain

\[ lY''''(l) + (2 - c)Y''(l) = - \int \frac{m+q}{m} \Theta(t) dt, \]  

where

\[ \Theta(t) = v(c - 1)Tl^{2c-1}l^{2c-2} (T - (tl)^c)^{c-4} \left( 2c(2c - 1)T + (2c^2 - 3c + 1)(tl)^c \right) . \]  

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Using (4), (105) and (119) we find that uniformly for \( t \in [m, m + q] \) we have

\[
\Theta(t) \approx vmN^{-\gamma}.
\]

Hence

\[
|(2 - c)Y''(l) + lY'''(l)| \approx qmvN^{-\gamma}
\]

uniformly for \( l \in (L_1, L_2) \). Therefore, there exists \( \kappa_1 > 0 \) which depends only on \( c \) and such that, at least one of the following inequalities holds for every \( l \in (L_1, L_2) \):

\[
|Y''(l)| \geq \kappa_1 vqmN^{-\gamma}, \quad (134)
\]

\[
|Y'''(l)| \geq \kappa_1 vqm^2N^{-2\gamma}. \quad (135)
\]

The next step is to show that the interval \( (L_1, L_2) \) can be divided into at most 13 intervals such that if \( J \) is one of them, then at least one of the following assertions holds:

We have (134) for all \( l \in J \). \quad (136)

We have (135) for all \( l \in J \). \quad (137)

To establish this, it suffices to show that the equation \( |Y''(l)| = \kappa_1 vqmN^{-\gamma} \) has at most 12 solutions in real numbers \( l \in (L_1, L_2) \). Hence, it is enough to show that if \( C \) does not depend on \( l \), then the equation \( Y''(l) = C \) has at most 6 solutions in real numbers \( l \in (L_1, L_2) \). According to Rolle’s theorem, between any two solutions of the last equation there is a solution of the equation

\[
Y'''(l) = 0. \quad (138)
\]

Hence, it is enough to show that \( (138) \) has at most 5 solutions in real numbers \( l \in (L_1, L_2) \).

By (87) and (112) one can easily see that \( (138) \) is equivalent to the equation

\[
(m + q)^c(T - (m + q)cT)^{\gamma-3}((2 - c)T - (c + 1)(m + q)c^T)
\]

\[
- m^c(T - m^cT)^{\gamma-3}((2 - c)T - (c + 1)m^cT) = \frac{rc(2 - c)((m + q)^c - m^c)}{vT}.
\]

Let \( X = t^c \). Define

\[
\mathcal{F}(X) = (m + q)^c(T - (m + q)cX)^{\gamma-3}((2 - c)T - (c + 1)(m + q)cX)
\]

\[
- m^c(T - m^cX)^{\gamma-3}((2 - c)T - (c + 1)m^cX).
\]

It would be enough to show that if \( B \) does not depend on \( X \), then the equation \( \mathcal{F}(X) = B \) has at most 5 solutions with \( X \in (L_1^c, L_2^c) \).
Once more, we refer to Rolle’s theorem to justify that it is enough to prove that the equation $F'(X) = 0$ has no more than 4 solutions with $X \in (L_1^c, L_2^c)$. One could write $F'(X) = 0$ as

$$(m + q)^{2c}(T - (m + q)^c X)^{\gamma - 4}((4c + 2\gamma - 6)T + (2c - \gamma + 1)(m + q)^c X),$$

which, in turn, is equivalent to

$$G(X) = \log m^c - \log (m + q)^c,$$

where

$$G(X) = (\gamma - 4) \log (T - (m + q)^c X) + \log ((4c + 2\gamma - 6)T + (2c + 1 - \gamma)(m + q)^c X)$$

$$- (\gamma - 4) \log (T - m^c X) - \log ((4c + 2\gamma - 6)T + (2c + 1 - \gamma)m^c X).$$

By the same argument as before it is enough to establish that the equation

$$G'(X) = 0$$

has at most 3 solutions with $X \in (L_1^c, L_2^c)$. This can easily be shown because

$$G'(X) = \frac{(4 - \gamma)(m + q)^c}{T - (m + q)^c X} + \frac{(2c + 1 - \gamma)(m + q)^c}{(4c + 2\gamma - 6)T + (2c + 1 - \gamma)(m + q)^c X}$$

$$- \frac{(4 - \gamma)m^c}{T - m^c X} - \frac{(2c + 1 - \gamma)m^c}{(4c + 2\gamma - 6)T + (2c + 1 - \gamma)m^c X}.$$

Therefore, the number of solutions of (139) does not exceed the number of roots of non-zero polynomial of degree at most 3.

On the other hand, from (116), (120), (124) and (131) – (133) we have

$$Y''(l) \ll vqmN^{-\gamma} \quad \text{and} \quad Y'''(l) \ll vqm^2N^{-2\gamma}.$$ (140)

Hence, we conclude that the interval $(L_1, L_2)$ can be divided into at most 13 intervals such that if $J$ is one of them, then at least one of the following assertions holds:

$$|Y''(l)| \asymp vqmN^{-\gamma} \quad \text{uniformly for} \quad l \in J,$$ (141)

$$|Y'''(l)| \asymp vqm^2N^{-2\gamma} \quad \text{uniformly for} \quad l \in J.$$ (142)

If (131) holds, then we use (4), (105) and Van der Corput’s theorem (see [8, Chapter 1, Theorem 5]) for the second derivative to get

$$\sum_{l \in J} e(Y(l)) \ll L(qvmN^{-\gamma})^{\frac{1}{2}} + (qvmN^{-\gamma})^{-\frac{1}{2}}$$

$$\ll q^{\frac{1}{2}}v^{\frac{1}{2}}LM^{\frac{1}{2}}N^{-\frac{1}{2}} + q^{-\frac{1}{2}}v^{-\frac{1}{2}}M^{-\frac{1}{2}}N^{\frac{1}{2}}.$$ (143)
In the case when \((142)\) is satisfied we apply \((1), (105)\) and Van der Corput’s theorem for the third derivative to get

\[
\sum_{l \in J} e(Y(l)) \ll L(qvm^2N^{-2\gamma})^{1/4} + L^{1/4}(qvm^2N^{-2\gamma})^{1/4} \\
\ll q^{1/4}v^{1/4}LM^{1/4}N^{-1/8} + q^{-1/4}v^{-1/4}M^{1/4}N^{-1/8} + q^{1/4}v^{1/4}LM^{1/4}N^{-1/8} + q^{-1/4}v^{-1/4}L^{1/4}M^{-1/4}N^{1/8}.
\]

(144)

Hence, in each case, \(\sum_{l \in J} e(Y(l))\) can be estimated by the sum of the expressions on the right sides of the inequalities \((143)\) and \((144)\). Therefore, we obtain

\[
\sum_{L_1 < l \leq L_2} e(Y(l)) \ll q^{1/4}v^{1/4}LM^{1/4}N^{-1/8} + q^{-1/4}v^{-1/4}M^{1/4}N^{-1/8} + q^{1/4}v^{1/4}LM^{1/4}N^{-1/8} + q^{-1/4}v^{-1/4}L^{1/4}M^{-1/4}N^{1/8}.
\]

We use \((1), (105), (111)\) and \((113)\) and find that

\[
W_{M,L} \ll N^\varepsilon \left( N^{7/4}Q^{-1/4} + v^{1/4}Q^{1/4}N^{7/8} + v^{-1/4}Q^{-1/4}N^{7/8} + v^{1/4}Q^{1/4}N^{11/12} + v^{-1/4}Q^{-1/4}N^{11/12} \right).
\]

We apply the above estimate for \(W_{M,L}\) in \((130)\). Then, by \((34)\) and \((55)\) we obtain

\[
\Omega_{4,3} \ll N^\varepsilon \sum_{d \leq D} \sum_{h \leq H} \sum_{r < A_1 \log^3 N} \left( N^{7/4}Q^{-1/4} + \left( \frac{h}{d} \right)^{1/4} Q^{1/4}N^{7/8} \right)
\]

\[
+ \left( \frac{h}{d} \right)^{1/4} Q^{-1/4}N^{7/8} + \left( \frac{h}{d} \right)^{1/4} Q^{1/4}N^{11/12} + \left( \frac{h}{d} \right)^{-1/4} Q^{-1/4}N^{23/24}
\]

\[
\ll N^\varepsilon \left( DN^{7/4}Q^{-1/4} + DQ^{1/4}N^{4/5 + 23/24} + D^{1/4}Q^{-1/4}N^{7/8} + DQ^{1/4}N^{11/12 + 23/24} + D^{1/4}Q^{-1/4}N^{23/24} \right).
\]

With the choice of \(Q\) which we made in \((128)\) it is now clear that

\[
\Omega_{4,3} \ll \frac{N^{2\gamma-1}}{(\log N)^2}.
\]

(145)

Now, let us carry on with the study of \(\Omega_{4,1}\). We have chosen the constant \(\alpha_1\) in such a way, that from \((70)\) and \((122)\) it follows that \(Y''(l) \approx qvmN^{-\gamma}\) uniformly for \(l \in (L_1, L_2)\). Then the sum \(\sum_{m < l < m + 2M} e(Y(l))\) can be bounded by the expression on the right side of \((143)\). This observation illustrates that \(\Omega_{4,1}\) is bounded by the same quantity as \(\Omega_{4,3}\), i.e.

\[
\Omega_{4,1} \ll \frac{N^{2\gamma-1}}{(\log N)^2}.
\]

(146)

In a very similar manner one can show that

\[
\Omega_{4,2} \ll \frac{N^{2\gamma-1}}{(\log N)^2}.
\]

(147)
Then, from (121), (129) and (145) – (147) we get

$$\Omega_4 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (148)$$

It remains to find a bound for $\Omega_3$. The same argument as the one for $\Omega_4$ can be applied here once more to show that

$$\Omega_3 \ll \frac{N^{2\gamma-1}}{(\log N)^2}. \quad (149)$$

From (102), (103), (148) and (149) we conclude that (73) is satisfied and the theorem is proved.

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