The stability issues in problems of mathematical modeling

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Abstract. In the paper it is briefly considered various aspects of stability concepts, which are used in physics, mathematics and numerical methods of solution. The interrelation between these concepts is described, the questions of preliminary stability research before the numerical solution of the problem and the correctness of the mathematical statement of the physical problem are discussed. Examples of concrete mathematical statements of individual physical problems are given: a nonlocal problem for the heat equation, the Korteweg-de Fries equation with boundary conditions at infinity, the sine-Gordon equation, the problem of propagation of femtosecond light pulses in a area with a cubic nonlinearity.

1. Introduction

The stability concept is multifaceted. The concept of physical stability differs from the stability definition in mathematics. But physicists and mathematicians try to solve one problem together - the problem of adequate mathematical modeling. To obtain a correct solution, it is very important to study in detail the stability at different stages of solving the mathematical modeling problem: in a physical experiment, in a mathematical formulation of the problem, in a difference scheme, i.e. in a numerical method for the problem solving.

The interpretation of the stability concept in physics is given in S.I. Ozhegov's dictionary [1]: "Stable." Standing, holding firmly, without hesitation, not falling. "Stable prop." Stable equilibrium (restored after a slight deviation.)"

Thus, from the physical point of view, we can talk about stability, if, with the introduction of small deviations in the system, they will have little effect on the behavior dynamics. See, for example, the problem of oscillations [2]. Sometimes physical stability is essentially technological stability. That is, the physical process has a property: with a random effect of some perturbation on this process, it spontaneously returns to its original state. The stability problems of technological processes are singled out in a separate series. A vivid example of technological stability is the problem of the MHD stability of an aluminum cell [3]. Stable process of electrolysis has technological stability.

Usually, the mathematical formulation of the problem of a physical experiment modeling is a system of ODEs (ordinary differential equations) or a system of non-stationary partial differential equations. Here the question arises of the correspondence between the numerical solution and the result of the physical experiment. Moreover, it is quite often in the construction of a mathematical model, some data that are considered insignificant can be ignored.

There are two questions: the correspondence of the obtained numerical solution to the unknown exact solution and the correspondence of the unknown exact solution to the observed result of the physical experiment. In the 20th century, many theorems on the stability of various problems were developed, which were based on the stability theory of A.M. Lyapunov [4]. This theory was developed
to study the dynamics of the solution of ODE systems in the second half of the 19th century. For example, there is the Arnold-Moser theorem on the stability of the equilibrium positions of a Hamiltonian system with two degrees of freedom in the general elliptic case [5, 6].

The basis of A.M. Lyapunov’s theory is precisely definition of physical stability. This the stability of a differential problem definition is closely connected with the notion of the problem correctness [7]:

1) the solution exists;
2) the solution is unique;
3) for small perturbations of the boundary conditions the solution also varies insignificantly.

With the existence and uniqueness of the differential problem solution, we can speak about its correctness and obtain a numerical solution. That is, we can build a difference scheme (set a difference problem). As a rule, mathematical statement of spatial homogeneous models reduces to a numerical solution of systems of differential equations, since an analytical solution can be written out only for an extremely narrow class of problems. If the difference scheme is stable, we obtain convergence to the solution of the differential problem. If difference scheme is unstable, there is no convergence: it is necessary to choose another difference scheme. We note that a class of ill-posed problems [8] exists. The theory for such problems solution is currently developing rapidly.

The basis of the numerical solution of the problem lies in the transition from continuous data to discrete data. We introduce a difference scheme with step it in time and hover space in the case of a spatially-inhomogeneous problem. It should be noted that the numerical differentiation is incorrect from the machine point of view. The reason is the limitation of the computer mantissa: when we calculate the difference derivative by the formula \( \frac{\Delta y}{\Delta t} \) at \( \Delta t \to 0 \), we obtain division by zero and the result is infinity.

Nevertheless, in practice, conditions are possible with a finite \( \Delta t \) and numerical solution that coincides with the specified accuracy with the solution of the differential problem. But it is necessary, as mentioned above, to consider preliminary the questions of the correctness of the problem mathematical formulation, and also the stability of the chosen numerical method for its solution.

There are different approaches to the problem of finding the solution numerically.

2. Analytical solution

Of course, solving non-stationary equations and finding a solution as a function of time, we can investigate its behavior with increasing \( t \).

But first, it is often not possible to obtain an analytical solution in general form. And secondly, it is rather difficult to analyze the already written out solution. Therefore, such approach is extremely rare.

3. Numerical methods

The method should also be stable. The application of the method stability to physical stability is carried out. The physical stability of such problems is determined experimentally. That is, when it is impossible to obtain an analytical solution, the operator (linear, nonlinear, self–conjugate, non–self–adjoint) is discretized, the solution is solved numerically (an experiment is carried out) – and the solution behavior is examined on physical experiment. The solution of the problem by a numerical method must ensure convergence to the solution of the differential problem at the grid nodes.

4. Special methods that move away from the classical, according to K.I. Babenko

Methods K.I. Babenko – methods without saturation [9], i.e. the smoothness of the solution is used. Ordinarily, the smoothness of the function is not used. However, one differential equation can be taken of different orders for the difference scheme. By the methods of K.I. Babenko it is recognized about the smoothness of the solution experimentally, and these data are used to construct the scheme.

5. Method of separation of variables with subsequent study of the spectrum
The method of separation of space-time variables is one of the most popular methods of solving (linear problems). As a result of which we obtain an eigenvalue problem

\[ X(x,t) = \lambda(x) \phi(x) e^{\gamma t}. \]

The analysis of the operator spectrum more easily and visually characterizes the dynamics of the process. Depending on the properties of the spectrum, we can conclude that the initial stationary problem is stable or not. In this case, the instability of the physical process is characterized by an unlimited increase in the characteristics of the given process, for example, the amplitude of the pendulum oscillations. This is clearly seen in physical experiments.

There is a whole direction of stability research using the properties of the spatial operator spectrum. Based on the theory of A.A. Samarskii for operator problems, the spectral theory of stability research was developed by A.V. Gulin [11]. It is based on the spectral properties of the basic operator of the difference scheme.

According to the theory of A.V. Gulin a difference problem must be stable too. Its stability, as well as the differential analog, is also characterized by the properties of the spectrum, the solution of the discrete eigenvalue problem. Moreover, all the properties of the difference problem, including the spectral ones, must coincide with an accuracy of the approximation order for the corresponding differential problem.

The application of the spectral theory is shown on the example of a nonlocal problem for the heat equation in [12]. For a differential problem with a nonlocal boundary condition

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,
\]

\[ u(x,0) = u_0(x), \quad 0 < x < 1, \]

\[ u(0,t) = 0, \quad \gamma \frac{\partial u}{\partial x} (0,t) = \frac{\partial u}{\partial x} (1,t), \]

where \( \gamma \) is given numerical parameter, the stability of the difference scheme with weights

\[
\frac{\gamma_{i}^{n+1} - \gamma_{i}^{n}}{\tau} = \alpha \gamma_{x,i}^{n+1} + (1 - \sigma) \gamma_{x,i}^{n}, \quad i = 1,2,...,N - 1,
\]

\[
\frac{\gamma_{N}^{n+1} - \gamma_{N}^{n}}{\tau} = 2 \left[ \sigma \left( \gamma_{x,0}^{n+1} - \gamma_{x,1}^{n+1} \right) + (1 - \sigma) \left( \gamma_{y,0}^{n} - \gamma_{y,1}^{n} \right) \right],
\]

\[ \gamma_{0}^{0} = u_0(x_i), \quad i = 0,1,...,N, \quad \gamma_{0}^{n+1} = 0, \quad n = 0,1,...
\]

is studied on the basis of the spectral properties of second-order non-self-adjoint difference operators with a parameter \( \gamma \), appearing in nonlocal boundary conditions. Stability criteria are obtained in the space with a special energy norm that does not increase on the solution of the difference problem and is equivalent to the standard mean-square norm.

The differential and difference problems can be linear and nonlinear by the properties of the operator. The spectral theory works well for linear eigenvalue problems, whose solution can form a discrete spectrum, a continuous spectrum, or a piecewise-continuous spectrum. But we have to study the stability of nonlinear schemes. Then we obtain a nonlinear nonstationary eigenvalue problem. The study of nonlinear problems has been carried out for a long time and is really relevant at the present time.

When studying the stability of a nonlinear problem, the concept of orbital stability is often used. This concept of stability was first introduced by A. Poincaré [13], based on the theory of A.M. Lyapunov. We consider not a concrete solution, but a phase space, the points of which characterize the state of the dynamic or physical model as a whole, the change of this state with the passage of time. Orbital stability describes the behavior of a closed trajectory (orbit) under the action of small external perturbations.
A bright example of a nonlinear problem is the problem of studying waves and oscillations. Waves of small amplitude are described by linear differential equations, which are studied. But if the amplitude is not small, the differential equation becomes nonlinear. For example, the Korteweg-de Fries equation (KDF) [14]

\[
\frac{du}{dt} + 6u \frac{du}{dx} + d^3u = 0,
\]

which describes waves in shallow water. In the form of a traveling wave \( u(x,t) = f(x-ct) \) the KDF equation can be solved analytically. If \( u(x,t) \xrightarrow{t \to \infty} 0 \), then one of the exact solutions has the form

\[
u_1(x,t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x-ct+\sigma) \right),
\]

where \( \sigma \) is an integration constant. The solution \( u(x,t) \) is a solitary wave localized in a small region. KDF has an infinite number of exact solutions \( u_n(x,t) \), that contain two arbitrary parameters \( c_i \) and \( \sigma_j \), and also behave as a superposition of independent isolated waves of the form \( u_i(x,t) \) for \( t << 0 \) or \( t >> 0 \). These waves can catch up with each other or collide with each other for a finite period of time, but then they always return to their original state (although a phase shift is possible). They do not annihilate, they behave like particles. The solution \( u_n(x,t) \), that consists of \( n \) isolated waves is called an \( n \)-soliton solution.

If the differential equation is linear, then the superposition principle holds for it, that is, if \( u_i(x,t), i = 1, ..., N \) is a particular solution, then \( \sum_{i=1}^{N} \alpha_i u_i \) is also a solution. For nonlinear equations this principle is not valid, but the existence of solutions \( u_n(x,t) \) is an analogue of the superposition principle. KDF can be reduced to a system that has infinitely many degrees of freedom, that is, it is an absolutely integrable system and has infinitely many first integrals (invariants).

In addition to KDF, the Kadomtsev-Petviashvili (KP) equations, Toda equations and other soliton equations have the property of the superposition principle. We emphasize that these equations also have ordinary nonlinear solutions, apart from the isolated soliton solutions. Soliton solutions are stable by definition.

There are methods for finding soliton solutions: the inverse scattering method [15], the Hirota method (that is, the bilinear method), the theory of quasi-periodic solutions, etc., and also the specially developed methods M1 and M2 [16] and the method for constructing difference schemes for nonstationary equations [17], which are based on finding the invariants of equations systems.

Numerical solutions of methods must converge to soliton solutions of different modes, and the convergence depends little on the form of the initial function. For example, in the paper [18] we can see the application of the iterative method M2 for finding soliton solutions of various nonlinear problems:

for the KDF equation with boundary conditions at infinity

\[
\frac{\partial u(x,t)}{\partial t} + 6u(x,t) \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \quad x \in R, \quad t > 0,
\]

\[
u(\pm \infty, t) = 0, \quad \nu'(\pm \infty, t) = 0, \quad \nu''(\pm \infty, t) = 0, \quad t > 0,
\]

\[
u(x,0) = u_0(x), \quad x \in R,
\]

for the sine–Gordon equation of the form
\[
\frac{\partial^2 u(x,t)}{\partial t^2} = a \frac{\partial^2 u(x,t)}{\partial x^2} + b \sin(cu(x,t)),
\]
where \( a, b, c \) are given;
for the problem of femtosecond light pulses propagating in a cubic nonlinear medium [19]
\[
\frac{\partial u}{\partial z} + i \frac{\partial^2 u}{\partial t^2} + i \alpha \frac{\partial}{\partial t} \left( |u|^2 u \right) = 0, \quad 0 < t < L, \quad z > 0,
\]
\[
u(z,0) = u(z;L) = 0,\]
\[
u(0,t) = u_0(t),
\]
where \( u(z,t) \) is the complex amplitude of a pulse propagating along the \( z \) axis (measured in units of the dispersion spreading length) normalized to the maximum value, \( t \) is the time in the accompanying coordinate system normalized to the duration of the main pulse,
\( L \) is the dimensionless time interval, \( \alpha \) is the ratio of the initial pulse power to the characteristic self-action power, \( \gamma \) is the coefficient characterizing the rate of change of the nonlinear polarization.

Thus, the study of the problem stability at all stages of its solution ensures the correctness of the solution, i.e. the correlation of the solution of the differential problem with the solution of the difference problem, and also the obtaining of a model whose solution corresponds to the results of a physical experiment. All of the above provides a qualitative correspondence of the conclusion a difference problem, and also the obtaining of a model whose solution corresponds to the results of a physical experiment.

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