On the generalized hypergeometric function, Sobolev orthogonal polynomials and biorthogonal rational functions.

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1 Introduction.

The theories of orthogonal polynomials on the real line and on the unit circle have a lot of contributions and applications [10], [3], [8], [9]. One of their possible generalizations, the theory of Sobolev orthogonal polynomials is now studied intensively by many mathematicians (see a survey in [5]). Important ingredients, which supported the importance of classical systems of polynomials \( \{ p_n(z) \}_{n=0}^{\infty} \), are the recurrence relation and the differential equations for \( p_n(z) \). Thus, it is natural to seek for such properties of Sobolev orthogonal polynomials. In this paper we shall provide a large class of polynomials \( g_n(z) \) which has both these properties. Moreover, polynomials \( g_n(z) \) are related to biorthogonal rational functions and Jacobi-type pencils.

Let \( p, q \) be some fixed non-negative integers. Denote

\[
g_n(z) = g_n(z; a_1, ..., a_p; b_1, ..., b_q) = \sum_{k=0}^{n} \frac{(a_1)_k \cdots (a_p)_k z^k}{(b_1)_k \cdots (b_q)_k k!}, \quad n = 0, 1, 2, ... \tag{1}
\]

where \( a_j, b_l \in \mathbb{C}\backslash\{0, -1, -2, ...\} \). Thus, \( g_n(z) \) is the \( n \)-th partial sum of the generalized hypergeometric series \( pFq(a_1, ..., a_p; b_1, ..., b_q; z) \), and \( \deg g_n = n \). As usual, \( p = 0 \) \( (q = 0) \) means that \( (a_j)_k \) \( (\text{respectively} \ (b_l)_k) \) are absent. By \( G_n(z) \) we denote the corresponding monic polynomials:

\[
G_n(z) = \frac{n!(b_1)_n \cdots (b_q)_n}{(a_1)_n \cdots (a_p)_n} g_n(z), \quad n \in \mathbb{Z}_+ \tag{2}
\]

Recall that a \( R_I \)-type continuos fraction is associated with a system of monic polynomials \( \{ P_n(z) \}_{n=0}^{\infty} \), generated by ([4], p. 5)

\[
P_n(z) = (z - c_n)P_{n-1}(z) - \lambda_n(z - a_n)P_{n-2}(z), \quad n = 1, 2, ..., \tag{3}
\]

where \( P_{-1}(z) := 0, P_0(z) := 1, \) and

\[
\lambda_{n+1} \neq 0, \quad P_n(a_n) \neq 0. \tag{4}
\]
Polynomials \( \{ P_n(z) \}_{n=0}^{\infty} \) are related to biorthogonal rational functions \cite{4} Theorem 2.1. The case \( a_n = 0, n \geq 2 \), is related to general \( T \)-fractions \cite{2}. It turns out that this is the case for the monic polynomials \( \{ G_n(z) \}_{n=0}^{\infty} \). On the other hand, recall the following definition from \cite{11}:

**Definition 1** A set \( \Theta = (J_3, J_5, \alpha, \beta) \), where \( \alpha > 0, \beta \in \mathbb{R}, J_3 \) is a Jacobi matrix and \( J_5 \) is a semi-infinite real symmetric five-diagonal matrix with positive numbers on the second subdiagonal, is said to be a **Jacobi-type pencil (of matrices)**.

Matrices \( J_3 \) and \( J_5 \) have the following form:

\[
J_3 = \begin{pmatrix}
  b_0 & a_0 & 0 & 0 & 0 & \cdots \\
  a_0 & b_1 & a_1 & 0 & 0 & \cdots \\
  0 & a_1 & b_2 & a_2 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad a_k > 0, \quad b_k \in \mathbb{R}, \quad k \in \mathbb{Z}_+; \quad (5)
\]

\[
J_5 = \begin{pmatrix}
  \alpha_0 & \beta_0 & \gamma_0 & 0 & 0 & 0 & \cdots \\
  \beta_0 & \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & \cdots \\
  \gamma_0 & \beta_1 & \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots \\
  0 & \gamma_1 & \beta_2 & \alpha_3 & \beta_3 & \gamma_3 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \alpha_n, \beta_n \in \mathbb{R}, \quad \gamma_n > 0, \quad n \in \mathbb{Z}_+.
\]

(6)

With a Jacobi-type pencil of matrices \( \Theta \) one associates a system of polynomials \( \{ p_n(\lambda) \}_{n=0}^{\infty} \), such that

\[
 p_0(\lambda) = 1, \quad p_1(\lambda) = \alpha \lambda + \beta,
\]

and

\[
 (J_5 - \lambda J_3) \vec{p}(\lambda) = 0,
\]

where \( \vec{p}(\lambda) = (p_0(\lambda), p_1(\lambda), p_2(\lambda), \ldots)^T \). Here the superscript \( T \) means the transposition. Polynomials \( \{ p_n(\lambda) \}_{n=0}^{\infty} \) are said to be associated to the Jacobi-type pencil of matrices \( \Theta \). One can rewrite relation (8) in the scalar form:

\[
 \gamma_{n-2} p_{n-2}(\lambda) + (\beta_{n-1} - \lambda a_{n-1}) p_{n-1}(\lambda) + (\alpha_n - \lambda b_n) p_n(\lambda) + \\
 + (\beta_n - \lambda a_n) p_{n+1}(\lambda) + \gamma_n p_{n+2}(\lambda) = 0, \quad n \in \mathbb{Z}_+,
\]

where \( p_{-2}(\lambda) = p_{-1}(\lambda) = 0, \gamma_{-2} = \gamma_{-1} = a_{-1} = \beta_{-1} = 0 \).
In the case of positive parameters \(a_j, b_l\), the polynomials \(\{g_n(z)\}_{n=0}^\infty\) are connected with some Jacobi type pencils and their associated polynomials \(\{p_n(\lambda)\}_{n=0}^\infty\). This connection resembles the connection between orthogonal polynomials on the unit circle (OPUC) and orthogonal polynomials on \([-1, 1]\).

In Section 2, the announced recurrence relation, a differential equation for \(g_n(z)\), and Sobolev orthogonality relations for \(g_n(z)\) will be given in Theorem 1. Observe that the case \(p = q = 0\), leads to the exponential function. The corresponding partial sums appeared in [13], as a particular case with \(\rho = 1\).

Of course, polynomials \(g_n(z)\) have nice expressions for their coefficients. However, it is of interest to get integral representations for \(g_n(z)\), involving special functions. In particular, such integral representations would be useful for obtaining various estimates, as well as in the Fourier series analysis. We shall give two integral representations for \(g_n(z)\) in Theorem 2. Asymptotic properties of \(g_n(z)\) and location of their zeros are also described by this theorem. We shall discuss the partial sums of arbitrary power series with non-zero coefficients. They are also related to biorthogonal rational functions. Finally, we shall obtain a relation of polynomials \(g_n(z)\) to Jacobi-type pencils and their associated polynomials.

**Notations.** As usual, we denote by \(\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+\), the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. For \(k, l \in \mathbb{Z}\), we set \(\mathbb{Z}_{k,l} := \{j \in \mathbb{Z} : k \leq j \leq l\}\). Set \(\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}\), \(\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}\), \(\mathbb{D}_e := \{z \in \mathbb{C} : |z| > 1\}\). By \(\mathcal{B}(\mathbb{T})\) we mean the set of all Borel subsets of \(\mathbb{T}\). By \(\mathcal{P}\) we denote the set of all polynomials with complex coefficients. For a complex number \(c\) we denote \((c)_0 = 1, (c)_k = c(c+1)...(c+k-1), k \in \mathbb{N}\) (the shifted factorial or Pochhammer symbol). The generalized hypergeometric function is denoted by

\[
pFq(a_1, ..., a_p; b_1, ..., b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \frac{z^k}{k!},
\]

where \(p, q \in \mathbb{Z}_+, a_j, b_l \in \mathbb{C}\).

2 The partial sums of the hypergeometric series and different kinds of orthogonality.

Denote by \(\mu_0\) the (probability) normalized arc length measure on \(\mathbb{T}\), which may be identified with the Lebesgue measure on \([0, 2\pi]\). We shall use the
ideas from [12] to obtain the following theorem.

**Theorem 1** Let \( p, q \in \mathbb{Z}_+ \), be some fixed numbers, and \( a_1, ..., a_p; b_1, ..., b_q, \) be some parameters from \( \mathbb{C}\{0, -1, -2, ...\} \) (the case \( p = 0 \) (\( q = 0 \)) means that \( a_j \)'s (respectively \( b_l \)’s) are absent). The following statements hold:

(a) Polynomials \( g_n(z) \) from (7) satisfy the following recurrence relation:

\[
\frac{(n + 1)(b_1 + n)...(b_q + n)}{(a_1 + n)...(a_p + n)}(g_{n+1}(z) - g_n(z)) =
\]
\[
= z(g_n(z) - g_{n-1}(z)), \quad n \in \mathbb{Z}_+, \quad g_{-1}(z) := 0. \quad (10)
\]

(b) The corresponding monic polynomials \( G_n(z) \) from (2) satisfy the following recurrence relation:

\[
G_n(z) = (z + \delta_n)G_{n-1}(z) - \delta_{n-1}zG_{n-2}(z), \quad n = 1, 2, ..., \quad (11)
\]

where \( G_{-1}(z) := 0, \) and \( \delta_0 := 0, \)

\[
\delta_k := \frac{k(b_1 + k - 1)...(b_q + k - 1)}{(a_1 + k - 1)...(a_p + k - 1)}, \quad k \in \mathbb{N}. \quad (12)
\]

Therefore \( G_n(z) \) are related to general \( T \)-fractions and biorthogonal rational functions.

(c) Polynomials \( g_n(z) \) obey the following differential equation:

\[
\theta R g_n(z) - nR g_n(z) = 0, \quad n \in \mathbb{Z}_+, \quad (13)
\]

where

\[
R = \frac{d}{dz}\prod_{j=1}^{q}(\theta + b_j - 1) - \prod_{j=1}^{p}(a_j + \theta), \quad \theta := z \frac{d}{dz}. \quad (14)
\]

(d) Polynomials \( g_n(z) \) are Sobolev orthogonal polynomials on the unit circle:

\[
\int_{T} \left(g_n(z), g'_n(z), ..., g^{(\rho)}_n(z)\right) M \begin{pmatrix} g_m(z) \\ g'_m(z) \\ \vdots \\ g^{(\rho)}_m(z) \end{pmatrix} d\mu_0 = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+, \quad (15)
\]
where
\[ M = (c_0(z), c_1(z), ..., c_p(z))^T (c_0(z), c_1(z), ..., c_p(z)), \]
and \( c_j(z) \in \mathbb{P} \), are the coefficients of the differential operator:
\[ - \frac{n! (b_1)_n ... (b_q)_n}{(a_1)_{n+1} ... (a_p)_{n+1}} R = \sum_{l=0}^{\rho} c_l(z) \frac{d^l}{dz^l}, \]
with \( \rho = \max(p, q + 1) \).

Note that the case \( p = 0 \) (\( q = 0 \)) means that all \( (a_j)_k \), \( (a_j + k) \) (respectively \( (b_l)_k \), \( (b_l + k) \)) in the above formulas are replaced by 1. The same takes place with the products \( \prod_{j=1}^{p} \) and \( \prod_{j=1}^{q} \).

Proof.
(a): Observe that
\[ g_n(z) - g_{n-1}(z) = \frac{(a_1)_{n} ... (a_p)_{n}}{(b_1)_n ... (b_q)_n} z^n \frac{z^n}{n!}, \quad n \in \mathbb{Z}_+, \]
where \( g_{-1} = 0 \). Using this relation with
\[ z^{n+1} = z z^n, \quad n \in \mathbb{Z}_+, \]
we immediately obtain the required recurrence relation \(10\).
(b): It follows directly from (a).
(c): Using the known idea of proof for the differential equation of \( p F_q \) \(17\) we may write:
\[ \theta \prod_{j=1}^{q} (\theta + b_j - 1) g_n(z) = \sum_{k=0}^{n} \frac{(a_1)_k ... (a_p)_k}{(b_1)_k ... (b_q)_k} \frac{1}{k!} k! \prod_{j=1}^{q} (k + b_j - 1) z^k = \]
\[ = z \sum_{k=1}^{n} \frac{(a_1)_k ... (a_p)_k}{(b_1)_k ... (b_q)_k} \frac{z^{k-1}}{(k-1)!} = z \sum_{l=0}^{n-1} \frac{(a_1)_{l+1} ... (a_p)_{l+1}}{(b_1)_{l+1} ... (b_q)_{l+1}} \frac{z^l}{l!} = \]
\[ = z \sum_{l=0}^{n-1} \frac{(a_1)_l ... (a_p)_l}{(b_1)_l ... (b_q)_l} \frac{1}{l!} \prod_{j=1}^{p} (a_j + l) z^l = z \prod_{j=1}^{p} (a_j + \theta) \sum_{l=0}^{n-1} \frac{(a_1)_l ... (a_p)_l}{(b_1)_l ... (b_q)_l} \frac{z^l}{l!} = \]
\[ = z \prod_{j=1}^{p} (a_j + \theta) \left( g_n(z) - \frac{(a_1)_{n} ... (a_p)_{n}}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!} \right), \quad n \in \mathbb{Z}_+. \]
\[
-\frac{n!(b_1)_{n}...(b_q)_{n}}{(a_1)_{n+1}...(a_p)_{n+1}}Rg_n(z) = z^n, \quad n \in \mathbb{Z}_+,
\]

(19)

where \( R \) is defined by (14). Here we assumed that \( p, q \in \mathbb{N} \), while the other cases are similar and lead to the same formula. Using

\[
\theta z^n = nz^n, \quad n \in \mathbb{Z}_+,
\]

and relation (19) we obtain the differential equation (13).

(d): It follows from the orthonormality relations for \( \{z^n\}_{n=0}^\infty \), and relations (19), (17). \( \square \)

Integral representations for polynomials \( g_n(z) \) and some other their basic properties are described in the next theorem.

**Theorem 2** In conditions of Theorem 1, the following statements hold:

(i) If \( p \leq q \), then polynomials \( g_n \) admit the following integral representation:

\[
g_n(e^{i\tau}) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1 - e^{i(n+1)(\tau-t)}}{1 - e^{i(\tau-t)}} \right) {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; e^{it}) dt,
\]

\( \tau \in [0,2\pi), \quad n \in \mathbb{Z}_+. \)

(20)

(ii) Polynomials \( g_n \) have the following integral representation:

\[
g_n(x) = -(n+1)x^{n+1} \int_{-\infty}^x t^{-n-2} {}_{p+1}F_{q+1}(-n, a_1, ..., a_p; -n-1, b_1, ..., b_q; t) dt,
\]

\( x < 0, \quad n \in \mathbb{Z}_+. \)

(21)

(iii) Polynomials \( g_n \) have simple roots. If \( p \leq q \), and

\[
0 < a_j \leq b_j, \quad j \in \mathbb{Z}_{1,p};
\]

\[
b_k \geq 1, \quad k \in \mathbb{Z}_{p+1,q},
\]

then all roots of \( g_n \) are located in \( \mathbb{D}_e \setminus (1, \infty) \).

(iv) Polynomials \( g_n(z) \) tend to \( {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z) \), as \( n \to \infty \), in \( \mathbb{T} \)

(in \( \mathbb{C} \)), if \( p = q + 1 \) (respectively \( p \leq q \)).
Proof.

(i): If \( p \leq q \), then the function \( g(z) := pF_q(a_1, ..., a_p; b_1, ..., b_q; z) \), is analytic in the whole plane. In particular, we have

\[
g(e^{i\tau}) = \sum_{k=0}^{\infty} \xi_k e^{ik\tau} = \lim_{n \to \infty} g_n(e^{i\tau}), \quad \tau \in [0, 2\pi),
\]

where

\[
\xi_k := \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{1}{k!}.
\]

Let us check that

\[
\frac{1}{2\pi} \int_{0}^{2\pi} g(e^{i\tau}) e^{-ij\tau} d\tau = \xi_j, \quad j \in \mathbb{Z}_+.
\]

Observe that

\[
|g_n(e^{i\tau}) e^{-ij\tau}| \leq \sum_{k=0}^{n} |\xi_k| \leq \sum_{k=0}^{\infty} |\xi_k| =: C < \infty.
\]

By (24), (27) and the dominated convergence theorem it follows that

\[
\frac{1}{2\pi} \int_{0}^{2\pi} g_n(e^{i\tau}) e^{-ij\tau} d\tau \to_{n \to \infty} \frac{1}{2\pi} \int_{0}^{2\pi} g(e^{i\tau}) e^{-ij\tau} d\tau.
\]

The left-hand side of (28) is equal to \( \xi_j \), when \( n \geq j \). Therefore relation (26) holds true. We may write

\[
g_n(e^{i\tau}) = \sum_{k=0}^{n} \xi_k e^{ik\tau} = \frac{1}{2\pi} \int_{0}^{2\pi} g(e^{i\tau}) \left( \sum_{k=0}^{n} e^{ik(\tau-t)} \right) dt, \quad \tau \in [0, 2\pi).
\]

Notice that the following relation:

\[
d_n = d_n(u) := \sum_{k=0}^{n} u^k = \frac{1 - u^{n+1}}{1 - u}, \quad n \in \mathbb{Z}_+, \quad u \in D.
\]

holds for all complex \( u \) (not only for \( u \in D \)). It follows from the recurrence relation: \( d_{n+1} - ud_n = 1 \), with \( d_0 = 1 \). By (29) and (30) we obtain relation (20).

(ii): Formula (21) can be checked by the direct integration of the polynomial under the integral sign, and some algebraic simplifications.
(iii): The simplicity of zeros of \( g_n \) follows from the three-term recurrence relation (10) (notice that \( g_n(0) = 1 \)).

Let \( p \leq q \), and conditions (22),(22) be satisfied. Since coefficients of \( g_n \) are positive, it takes positive values on \((0, +\infty)\). Conditions (22),(23) imply that the coefficients of \( g_n \) form a monotone sequence. Thus, one can apply the Eneström–Kakeya Theorem ([6, p. 136]) to conclude that the roots of \( g_n \) are located in \( \mathbb{D}_e \).

(iv): It is a known property of \( pF_q \). □

**Generalizations.** Consider a power series

\[
    f(z) = \sum_{k=0}^{\infty} d_k z^k, \quad d_k \in \mathbb{C}\{0\}.
\]

Define

\[
    f_n(z) = \sum_{k=0}^{n} d_k z^k, \quad n \in \mathbb{Z}_+,
\]

\[
    F_n(z) = \frac{1}{d_n} f_n(z), \quad n \in \mathbb{Z}_+.
\]

We have

\[
    \frac{1}{d_n} (f_n(z) - f_{n-1}(z)) = z^n, \quad n \in \mathbb{Z}_+,
\]

where \( f_{-1} := 0 \). By (18),(34) we obtain a recurrence relation for \( f_n \). For the monic polynomials \( F_n \) it takes the following form:

\[
    F_n(z) = \left( z + \frac{d_{n-1}}{d_n} \right) F_{n-1}(z) - \frac{d_{n-2}}{d_{n-1}} z F_{n-2}(z), \quad n = 1, 2, ..., \quad (35)
\]

where \( F_{-1}(z) := 0 \), and \( d_{-1} := 1 \). Consequently, polynomials \( F_n(z) \) are also related to biorthogonal rational functions. In order to obtain some equations for \( f_n(z) \) with respect to \( z \) (e.g., differential, difference, \( q \)-difference equations), it looks promising to consider the power series \( f(z) \), corresponding to the basic hypergeometric function, elliptic hypergeometric functions and other special functions.

Suppose now that all coefficients \( d_k \) are positive. We may write:

\[
    \text{Re} f_n(e^{i\tau}) = \sum_{k=0}^{n} d_k T_k(x), \quad (36)
\]

\[
    \text{Im} f_{n+1}(e^{i\tau}) = \sin \tau \sum_{k=1}^{n+1} d_k U_{k-1}(x) = \sin \tau \sum_{j=0}^{n} d_{j+1} U_j(x),
\]

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\[ x = \cos \tau, \quad \tau \in (0, \pi), \quad n \in \mathbb{Z}_+, \quad (37) \]

where \( T_k(x) = \cos(k \arccos x) \), \( U_k(x) = \frac{\sin((k+1) \arccos x)}{\sqrt{1-x^2}} \), are Chebyshev polynomials of the first and the second kinds. Therefore the following systems of polynomials:

\[
\left\{ \text{Re } f_n(e^{i \arccos x}) \right\}_{n=0}^{\infty}, \quad \left\{ \frac{1}{\sin \arccos x} \text{Im } f_{n+1}(e^{i \arccos x}) \right\}_{n=0}^{\infty},
\]

are modified kernel polynomials, see [4] formula (5). They are associated to Jacobi-type pencils and possess special orthogonality relations.

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On the generalized hypergeometric function, Sobolev orthogonal polynomials and biorthogonal rational functions.

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It turned out that the partial sums \( g_n(z) = \sum_{k=0}^{n} \frac{(a_1)_{k} \cdots (a_p)_{k}}{(b_1)_{k} \cdots (b_q)_{k}} \frac{z^k}{k!} \), of the generalized hypergeometric series \( _pF_q(a_1, ..., a_p; b_1, ..., b_q; z) \), with parameters \( a_j, b_l \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \), are Sobolev orthogonal polynomials. The corresponding monic polynomials \( G_n(z) \) are polynomials of \( R_I \) type, and therefore they are related to biorthogonal rational functions. Polynomials \( g_n \) possess a differential equation (in \( z \)), and a recurrence relation (in \( n \)). We study integral representations for \( g_n \), and some other their basic properties. Partial sums of arbitrary power series with non-zero coefficients are shown to be also related to biorthogonal rational functions. We obtain a relation of polynomials \( g_n(z) \) to Jacobi-type pencils and their associated polynomials.

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