Large $n$ Limit for the Product of Two Coupled Random Matrices

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Abstract: For a pair of coupled rectangular random matrices we consider the squared singular values of their product, which form a determinantal point process. We show that the limiting mean distribution of these squared singular values is described by the second component of the solution to a vector equilibrium problem. This vector equilibrium problem is defined for three measures with an upper constraint on the first measure and an external field on the second measure. We carry out the steepest descent analysis for a $4 \times 4$ matrix-valued Riemann–Hilbert problem, which characterizes the correlation kernel and is related to mixed type multiple orthogonal polynomials associated with the modified Bessel functions. A careful study of the vector equilibrium problem, combined with this asymptotic analysis, ultimately leads to the aforementioned convergence result for the limiting mean distribution, an explicit form of the associated spectral curve, as well as local Sine, Meijer-G and Airy universality results for the squared singular values considered.

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1. Introduction

The study of products of random matrices might be traced back to the work of Furstenberg and Kesten [34] in the context of random Schrödinger operators [18] and statistical physics relating to disordered and chaotic dynamical systems [21] in the 1960s. The emphasis at that time was put on the statistical behavior of individual entries as the number of factors in the product tends to infinity. The recent rapid developments, however, are focused on the eigenvalue or singular value distributions, at various scales, as the sizes of the matrices tend to infinity while the number of matrices in the product is kept fixed.

Among various progresses of this aspect, significant contributions are due to the works of Akemann, Ipsen, Kieburg and Wei [3,4], in which they showed that the squared singular values of products of independent complex Gaussian matrices (i.e., the matrices whose entries are independent with a complex Gaussian distribution, also known as Ginibre random matrices) form a determinantal point process over the positive real axis. The various local limits of the correlation kernel then reveal an interesting mathematical structure behind the products of independent random matrices, and various scaling limits can be predicted once one knows properties of the global distribution of the squared singular values [19,20]. On one hand, after proper centering and scaling, the correlation kernel tends to the sine kernel for points in the bulk, and to the Airy kernel for the right endpoint of the limiting spectrum [44], which obey the principle of universality in random matrix theory [36]. One the other hand, a new family of kernels, namely, the Meijer G-kernels, are found to describe the scaling limit of the correlation kernel near the origin [41]. The Meijer G-kernels generalize the classical Bessel kernel and represent a...
new universality class in random matrix theory as evidenced by their later appearances in many other random matrix models including Cauchy-chain matrix models [11,14], products of Ginibre matrices with inverse ones [32], Muttalib-Borodin ensembles [17, 39], a matrix model with Bures measure [33], among others. For more information about recent results for products of independent random matrices, we refer to the review article [2] and references therein.

In view of these interesting results obtained for products of independent complex Gaussian random matrices, a natural question to ask is how far such results remain valid, or yet if different ones arise, if some of the conditions on the models are relaxed. One attempt towards this direction is to drop the requirement of independence of the matrices in the product, as initiated by Akemann and Strahov [7] and further explored by them and Liu [5,6,43]. Following [7,43], let us consider a coupled two-matrix model defined by the probability distribution

\[ \frac{1}{\hat{Z}_n} \exp \left( -\beta \text{Tr}(X_1 X_1^* + X_2^* X_2) + \text{Tr}(\Omega X_1 X_2 + (\Omega X_1 X_2)^*) \right) \, dX_1 \, dX_2, \tag{1.1} \]

over pairs of rectangular complex matrices \((X_1, X_2)\) of sizes \(L \times M\) and \(M \times n\) respectively, where the superscript \(^*\) stands for the Hermitian transpose, \(dX_1\) and \(dX_2\) are the flat complex Lebesgue measures on the entries of \(X_1\) and \(X_2\), and \(\hat{Z}_n\) is a normalization constant. Here, \(\beta > 0\) and \(\Omega\) is a fixed \(n \times L\) complex matrix playing the role of coupling between \(X_1\) and \(X_2\), which should satisfy

\[ \Omega \Omega^* < \beta^2 I_n \tag{1.2} \]

to make sure the model is well defined, where \(I_n\) is the \(n \times n\) identity matrix. The interest lies in the singular values of the product matrix

\[ \hat{Y} := X_1 X_2, \tag{1.3} \]

where the matrices \(X_1\) and \(X_2\) are drawn from (1.1).

There are several motivations to study the product (1.3). First, if \(L = n\) and \(\Omega\) is a scalar matrix, the model (1.1) can be interpreted as the chiral two-matrix model [1,51], which was introduced in the context of quantum chromodynamics (QCD). In this case, an alternative formulation of the model is the following (see [7,43]). Let \(A\) and \(B\) be two independent matrices of size \(n \times M\) \((M \geq n)\) with independent and identically distributed standard complex Gaussian entries. Define two random matrices

\[ X_1 := \frac{1}{\sqrt{2}} (A - i \sqrt{\tau} B), \quad X_2 := \frac{1}{\sqrt{2}} (A^* - i \sqrt{\tau} B^*), \quad 0 < \tau < 1. \tag{1.4} \]

Then the pair \((X_1, X_2)\) is distributed according to (1.1) with

\[ L = n, \quad \beta = \frac{1 + \tau}{2\tau} \quad \text{and} \quad \Omega = \frac{1 - \tau}{2\tau} I_n, \tag{1.5} \]

and one can see \(\tau\) as an interpolation parameter between a model for singular values of the Ginibre matrix \(A\) (corresponding to \(\tau = 0\)) and a correlated product (for \(\tau > 0\)).

Also, in the context of QCD with a baryon chemical potential [51], the Dirac operator is realized as a block matrix whose diagonal entries are null matrices and the off-diagonal entries are matrices of the form (1.4). The singular values of \(\hat{Y}\) can be viewed as the correlations of complex eigenvalues of the QCD dirac operator.
In addition, as observed in [7], the product of $X_1$ and $X_2$ defined in (1.4) provides a new interpolating ensemble, in a sense extending (1.4)–(1.5) to a rectangular coupling matrix $\Omega$. It interpolates between the classical Laguerre ensemble [46] (for $\tau = 0$) and the product of two independent Ginibre random matrices (for $\tau = 1$).

A striking feature is that the squared singular values of $\hat{Y}$ are distributed according to a determinantal point process over the positive real axis [7,43]. This determinantal point process is a biorthogonal ensemble [17] with joint probability density function (see [43, Proposition 1.1])

$$\frac{1}{Z_n} \det \left[ I_{\kappa} (2\alpha_i \sqrt{x_j}) \right]_{i,j=1}^n \det \left[ x_j^{\nu+i-1} K_{\nu+\kappa+i-1} (2\beta \sqrt{x_j}) \right]_{i,j=1}^n,$$

with $I_\mu$ and $K_\nu$ being the modified Bessel functions of first kind and second kind, respectively, where

$$\kappa := L - n, \quad \nu := M - n,$$

$$\alpha_1, \ldots, \alpha_n$$ are the singular values of the coupling matrix $\Omega$ and $Z_n$ is a normalization constant explicitly known. The correlation kernel describing the point process (1.6) admits a double contour integral representation, which can be used to establish various limits near the origin if one further couples the $\alpha_i$'s and $\beta$ on one parameter; see [6,7,43] for details. In particular, the universal Meijer G-kernel also appears in one of these limits.

An interesting yet open question posed in [7] is to find the limiting mean distribution of the singular values for $\hat{Y}$ and the local limits of the correlation kernel beyond the origin. Due to the missing of independence of the matrices, the challenge we encounter is the fact that the approaches developed for the products of independent matrices are not applicable directly. The main contribution of this paper is to fully resolve this problem, and along the way obtain several other asymptotic results when, in contrast with the mentioned previous works, the parameters $\alpha$ and $\beta$ are not coupled together.

2. Statement of Results

2.1. The confluent case. We will focus on the confluent case that all the singular values of $\Omega$ are the same, i.e.,

$$\alpha_i \to \alpha > 0.$$  

In virtue of (1.2), we stress that

$$\alpha < \beta,$$

condition which is not a restriction but only ensures the model (1.1) is well defined. In addition, it is assumed that

$$M \geq L \geq n,$$

so that

$$\nu \geq \kappa \geq 0.$$ 

The condition $M, L \geq n$ assures us that, almost surely, $X_1$ and $X_2$ do not have 0 as a singular value, and the case $L < M$ can be handled by swapping the roles of $X_1$ and $X_2$. 
Under the condition (2.1), the vector space spanned by the functions \( x \mapsto I_\kappa(2\alpha_j \sqrt{x}) \), \( j = 1, \ldots, n \), becomes the linear space spanned by

\[
x \mapsto \partial_j^{-1} I_\kappa(2y \sqrt{x})|_{y=\alpha}, \quad j = 1, \ldots, n.
\]

Using the recurrence relations (see [50, Equation 10.29.1])

\[
I_{\mu-1}(z) - I_{\mu+1}(z) = \frac{2\mu}{z} I_\mu(z), \quad I_{\mu-1}(z) + I_{\mu+1}(z) = 2I'_\mu(z),
\]

satisfied by the modified Bessel functions of the first kind, it is readily seen that the resulting space is spanned by the functions \( x \mapsto x^{j-1/2} I_{\kappa+j-1/2}(2\alpha \sqrt{x}) |_{y=\alpha}, j = 1, \ldots, n \). Thus, a further algebraic calculation implies that the joint probability density function for the squared singular values of \( \widehat{Y} \) is given by

\[
\frac{1}{Z_n} \det \left[ x_{k}^{\kappa+j-1/2} I_{\kappa+j-1}(2\alpha \sqrt{x_{k}}) \right]_{j,k=1}^{n} \det \left[ x_{k}^{\nu-k+j-1/2} K_{\nu-k+j-1}(2\beta \sqrt{x_{k}}) \right]_{j,k=1}^{n},
\]

under the condition that the coupling matrix \( \Omega \) has a single singular value \( \alpha \). For the case \( \kappa = 0 \), this result was first obtained by Akemann and Strahov [7].

From general properties of biorthogonal ensembles [17], it is known that (2.3) is a determinantal point process with correlation kernel

\[
K_n(x, y) = \sum_{k=0}^{n-1} Q_k(x) P_k(y),
\]

where for each \( k = 0, 1, \ldots, \), \( Q_k \) belongs to the linear span of \( x^{\kappa+j} I_{\kappa+j}(2\alpha \sqrt{x}) \), \( j = 0, \ldots, k \), while \( P_k \) belongs to the linear span of \( x^{\nu-k+j} K_{\nu-k+j}(2\beta \sqrt{x}) \), \( j = 0, \ldots, k \), in such a way that

\[
\int_{0}^{\infty} Q_k(x) P_j(x) \, dx = \delta_{j,k},
\]

with \( \delta_{j,k} \) being the Kronecker delta.

This characterization of \( K_n \) will be the starting point of our work. To describe the large \( n \) limit of the correlation kernel \( K_n \), we introduce next a vector equilibrium problem.

2.2. A vector equilibrium problem. Given any two finite measures \( \mu \) and \( \nu \) on \( \mathbb{C} \), we denote by, as usual (cf. [54]),

\[
I(\mu, \nu) = \iint \frac{1}{|x - y|} \, d\mu(x) \, d\nu(y)
\]

their mutual logarithmic interaction, and by

\[
I(\mu) = I(\mu, \mu) = \iint \frac{1}{|x - y|} \, d\mu(x) \, d\mu(y)
\]

(2.5)
the logarithmic energy of the measure $\mu$.

The vector equilibrium problem relevant to the present work asks for minimizing the energy functional

$$E(v_1, v_2, v_3) = I(v_1) + I(v_2) + I(v_3) - I(v_1, v_2) - I(v_2, v_3) + 2(\beta - \alpha) \int \sqrt{x} \, dv_2(x),$$

(2.6)

over the set $M$ of admissible measures, which is defined to be the set of triples of measures $\nu = (v_1, v_2, v_3)$ satisfying the following conditions.

(E1) All three measures $v_1, v_2$ and $v_3$ have finite logarithmic energy.

(E2) $v_1$ is a measure on $\mathbb{R}^- := (-\infty, 0]$ with total mass $1/2$, i.e., $2|v_1| = 1$, and satisfies the upper constraint

$$v_1 \leq \sigma,$$

where $\sigma$ is the absolutely continuous measure on $\mathbb{R}^-$ with density

$$\frac{d\sigma}{dx}(x) = \frac{\alpha}{\pi \sqrt{|x|}}, \quad x < 0.$$

(2.7)

(E3) $v_2$ is a measure on $\mathbb{R}^+ := [0, \infty)$ with total mass $1$, i.e., $|v_2| = 1$.

(E4) $v_3$ is a measure on $\mathbb{R}^-$ with total mass $1/2$, i.e., $2|v_3| = 1$.

At first sight, the exact form of $E(\cdot)$ and the conditions on the measures might look mysterious. In the “Appendix A” we present the calculations that led us to this exact form. Similar vector equilibrium problems have appeared before in the literature [25, 29, 30, 37], and existence and uniqueness of solution are known under very mild conditions [9, 35] which include ours.

Our first result concerns the structure of the minimizer of the above equilibrium problem.

**Theorem 2.1.** There exists a unique vector of measures $\mu = (\mu_1, \mu_2, \mu_3) \in M$ that minimizes the energy functional (2.6) over $M$. In addition, the components $\mu_1, \mu_2$ and $\mu_3$ have the following properties.

(a) The support of $\mu_1$ is the negative real axis, and

$$\text{supp}(\sigma - \mu_1) = (-\infty, -q]$$

with

$$q = \frac{\alpha^6 + \beta^6 - 33 \alpha^4 \beta^2 - 33 \alpha^2 \beta^4 + \sqrt{(\alpha^4 + 14 \alpha^2 \beta^2 + \beta^4)^3}}{8 \alpha^2 \beta^2 (\beta^2 - \alpha^2)^2} > 0.$$  

(2.8)

Furthermore, $\mu_1$ is absolutely continuous with respect to the Lebesgue measure and satisfies

$$\frac{d\sigma}{dx}(x) - \frac{d\mu_1}{dx}(x) = c_1 (-q - x)^{\frac{1}{2}} (1 + o(1)), \quad x \to (-q)^-, \quad (2.9)$$

for some positive constant $c_1$. 
(b) The support of $\mu_2$ is
\[ \text{supp } \mu_2 = [0, p] \]
with
\[ p = \frac{-\alpha^6 - \beta^6 + 33\alpha^4\beta^2 + 33\alpha^2\beta^4 + \sqrt{(\alpha^4 + 14\alpha^2\beta^2 + \beta^4)^3}}{8\alpha^2\beta^2 (\beta^2 - \alpha^2)^2}. \]

Furthermore, $\mu_2$ is absolutely continuous with respect to the Lebesgue measure on $[0, p]$ and
\[ \frac{d\mu_2}{dx}(x) = \begin{cases} c_2 x^{-\frac{5}{2}} (1 + o(1)), & x \to 0^+, \\ \tilde{c}_2 (p - x)^{\frac{1}{2}} (1 + o(1)), & x \to p^-, \end{cases} \]
for some positive constants $c_2$ and $\tilde{c}_2$.

(c) The support of $\mu_3$ is the negative real axis and $\mu_3$ is absolutely continuous with respect to the Lebesgue measure with density
\[ \frac{d\mu_3}{dx}(x) = \frac{1}{2\pi \sqrt{|x|}} \int \sqrt{s} \frac{d\mu_2(s)}{s - x}, \]
\[ \frac{d\mu_3}{dx}(x) = c_3 |x|^{-\frac{5}{2}} (1 + o(1)), \quad x \to 0^-, \] for some positive constant $c_3$.

2.3. The spectral curve. One of the fundamental objects for a matrix model is its associated spectral curve that has been explored for various other matrix models [10, 12, 13, 40, 45, 49]. To describe the spectral curve for the model (1.3), denote by
\[ C^\mu(z) = \int \frac{d\mu(x)}{x - z}, \quad z \in \mathbb{C}\setminus \text{supp } \mu, \]
the Cauchy transform of a measure $\mu$, let $\mu = (\mu_1, \mu_2, \mu_3)$ be the unique minimizer given in Theorem 2.1 and set
\[ \xi_1(z) = C^{\mu_1}(z) + \frac{\alpha}{\sqrt{z}}, \quad z \in \mathbb{C}\setminus \mathbb{R}_-, \]
\[ \xi_2(z) = C^{\mu_2}(z) - C^{\mu_1}(z) - \frac{\alpha}{\sqrt{z}}, \quad z \in \mathbb{C}\setminus (-\infty, p], \]
\[ \xi_3(z) = C^{\mu_3}(z) - C^{\mu_2}(z) - \frac{\beta}{\sqrt{z}}, \quad z \in \mathbb{C}\setminus (-\infty, p], \]
\[ \xi_4(z) = -C^{\mu_3}(z) + \frac{\beta}{\sqrt{z}}, \quad z \in \mathbb{C}\setminus \mathbb{R}_-, \]
where the branch cut of the square root function $\sqrt{z}$ is taken along the negative real axis. The spectral curve for (1.3) takes the form of an algebraic equation and is given by our next theorem.
Theorem 2.2. The functions $\xi_1, \xi_2, \xi_3$ and $\xi_4$ are the four solutions to the algebraic equation

$$
\xi^4 - \frac{\alpha^2 + \beta^2}{z} \xi^2 + \frac{\alpha^2 - \beta^2}{z^2} \xi + \frac{\alpha^2 \beta^2}{z^2} = 0,
$$

(2.16)

and (2.16) is parametrized by

$$(z, \xi) = \left( \frac{t}{h(t)^2}, h(t) \right), \quad t \in \mathbb{C},$$

where

$$h(t) = \frac{t^2 - (\alpha^2 + \beta^2)t + \alpha^2 \beta^2}{\beta^2 - \alpha^2}.$$ 

(2.17)

Using the parametrization of (2.16), one can describe the densities of the components of $\mu$. For instance, the graph of the density of $\mu_2$ takes the form

$$
\left( x, \frac{d\mu_2}{dx}(x) \right) = \left( \frac{t}{h(t)^2}, \pi ih(t) \right), \quad t \in \gamma_2^-, 
$$

where $\gamma_2^-$ is a specific contour on $\mathbb{C}$ along which $h$ becomes purely imaginary. We refer the reader to Sect. 3.8 for details, in particular Fig. 2 where $\gamma_2^-$ is evaluated numerically.

2.4. Limiting mean distribution and hard edge scaling limit of the correlation kernel. Our main result is the following theorem relating the large $n$ limit of the correlation kernel $K_n(x, y)$ to the unique minimizer of the vector equilibrium problem introduced in Sect. 2.2.

Theorem 2.3. Let $K_n(x, y)$ be the correlation kernel defined in (2.4) for the squared singular values of $\hat{Y}$ (1.3) in the confluent case. With $\nu$ and $\kappa$ being fixed, we have

$$
\lim_{n \to \infty} nK_n \left( n^2x, n^2x \right) = \frac{d\mu_2}{dx}(x), \quad x > 0,
$$

(2.18)

where $\mu = (\mu_1, \mu_2, \mu_3) \in M$ is the unique minimizer of the energy functional (2.6) over $M$ stated in Theorem 2.1 and the limit above is uniform for $x$ in any compact subset of $[0, \infty)$.

According to (2.11), the density of $\mu_2$ blows up at $x = 0$, so it does not make sense to talk about the convergence (2.18) when $x = 0$. But, alternatively, the vector equilibrium problem stated in Sect. 2.2 is directly related to the matrix model (1.3) in the way we now explain, which then provides (2.18) also for $x = 0$ in a weaker sense.

Let us denote by $y_1, \ldots, y_n$ the squared singular values of the matrix $\hat{Y}$ in (1.3) and set

$$P_n(z) = \mathbb{E} \left( \prod_{j=1}^{n} (z - y_j) \right),$$

where $\mathbb{E}$ denotes the expectation.
where the expectation is over the $y_j$’s with respect to the density in (2.3). That is, $P_n$ is the average characteristic polynomial for $\hat{Y}^*\hat{Y}$. If we denote by $x_1, \ldots, x_n$ the zeros of $P_n$ and construct the sequence of zero counting measures

$$\mu(P_n) = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j/n^2},$$

with $\delta_a$ being the Dirac delta measure with mass at $a$, then the sequence $\{\mu(P_n)\}$ converges weakly to the second component $\mu_2$ of the minimizer given in Theorem 2.1. This claim follows from the uniform convergence above, or also from the Riemann–Hilbert (shortly RH) asymptotic analysis that we perform.

We next come to the hard edge scaling limit of the correlation kernel. As aforementioned, if the parameters $\alpha$ and $\beta$ are coupled in a specific way, it was shown in [7,43] that the hard edge scaling limit of $K_n$ is given by the universal Meijer G-kernel, which in a format appropriate for us takes the form [14,41]

$$K_{\nu_1,\nu_2}(x, y) = \int_0^1 G_{1,0}^{1,0}(0, -\nu_1, -\nu_2 \mid ux) G_{0,3}^{2,0}(v_1, v_2, 0 \mid uy) \, du, \quad (2.19)$$

where $G_{p,q}^{m,n}(a_1, \ldots, a_p \mid b_1, \ldots, b_q \mid z)$ is the Meijer G-function (see (10.16) below for the definition). We extend the results just mentioned to any fixed $\alpha$ and $\beta$.

**Theorem 2.4.** Let $K_n$ be the correlation kernel defined in (2.4) for the squared singular values of $\hat{Y}$ (1.3) in the confluent case. With $\nu$ and $\kappa$ being fixed, we have

$$\lim_{n \to \infty} \frac{1}{n(\beta^2 - \alpha^2)} K_n\left(\frac{x}{n(\beta^2 - \alpha^2)}, \frac{y}{n(\beta^2 - \alpha^2)}\right) = \left(\frac{y}{x}\right)^{\kappa/2} K_{\nu,\kappa}(y, x),$$

uniformly for $x, y$ in compact subsets of $(0, \infty)$, where the limiting kernel $K_{\nu,\kappa}$ is given in (2.19) and the parameters $\alpha, \beta$ satisfying (2.2) are fixed.

Our asymptotic analysis, leading to the proofs of the theorems above, also allows us to obtain the expected universality results for the local statistics of the squared singular values of $\hat{Y}$ beyond the origin. This means that the scaling limits of $K_n$ tend to the sine kernel when centered around a point $x_0 \in (0, p)$ (bulk universality), and to the Airy kernel for $x_0 = p$ (soft edge universality). All the ingredients for obtaining such results are presented, but we will not write the details down neither comment them any further; instead, we refer to [8,16,29] for a more detailed analysis in similar situations.

2.5. About the proofs and organization of the rest of the paper. The proofs of our asymptotic results rely on the fact that the biorthogonal functions $P_k$ and $Q_k$ in (2.4) can be interpreted as multiple orthogonal polynomials of mixed type [22], as it was first observed by the second-named author in [58]. This in particular implies a RH problem characterization [22] of the correlation kernel, which extends the classical results in [31,57] and, as of relevance for us here, takes the following form.

**RH Problem 2.5.** We look for a $4 \times 4$ matrix-valued function $Y: \mathbb{C}\setminus\mathbb{R}_+ \to \mathbb{C}^{4\times4}$ satisfying the following properties:

1. $Y$ is defined and analytic in $\mathbb{C}\setminus\mathbb{R}_+$. 
(2) \( Y \) has limiting values \( Y_{\pm} \) on \((0, \infty)\), where \( Y_+ (Y_-) \) denotes the limiting value from the upper (lower) half-plane, and

\[
Y_+(x) = Y_-(x) \begin{pmatrix} I_2 & W(x) \\ 0 & I_2 \end{pmatrix}, \quad x \in (0, +\infty),
\]

where \( W(x) \) is the rank-one matrix

\[
W(x) = \begin{pmatrix} \omega_{\kappa, \alpha}(x) & \rho_{\nu-\kappa, \beta}(x) & \rho_{\nu-\kappa+1, \beta}(x) \\ \omega_{\kappa+1, \alpha}(x) & \rho_{\nu-\kappa, \beta}(x) & \rho_{\nu-\kappa+1, \beta}(x) \end{pmatrix}
\]

with

\[
\omega_{\mu, a}(x) = x^\frac{\mu}{2} I_\mu(2an\sqrt{x}), \quad \mu > -1, \quad a > 0,
\]

and

\[
\rho_{\nu, b}(x) = x^\frac{\nu}{2} K_\nu(2bn\sqrt{x}), \quad \nu \geq 0, \quad b > 0.
\]

In (2.21), the parameters \( \kappa \) and \( \nu \) are given in (1.7).

(3) As \( z \to \infty \) and \( z \in \mathbb{C} \setminus \mathbb{R}_+ \), we have

\[
Y(z) = \left( I_4 + \frac{Y_1}{z} + O \left( \frac{1}{z^2} \right) \right) \text{diag} \left( z^{n_1}, z^{n_2}, z^{-n_1}, z^{-n_2} \right).
\]

with \( n_1 = \lfloor \frac{n-1}{2} \rfloor + 1 \) and \( n_2 = \lfloor \frac{n-2}{2} \rfloor + 1 \), where \( \lfloor x \rfloor = \max \{ n \in \mathbb{Z} : n \leq x \} \) stands for the integer part of \( x \).

(4) As \( z \to 0, z \in \mathbb{C} \setminus \mathbb{R}_+ \), the matrix \( Y(z) \) has the following behavior:

\[
Y(z) = \mathcal{O} \begin{pmatrix} 1 & 1 & h(z) & \tilde{h}(z) \\ 1 & 1 & h(z) & \tilde{h}(z) \\ 1 & 1 & h(z) & \tilde{h}(z) \\ 1 & 1 & h(z) & \tilde{h}(z) \end{pmatrix}, \quad Y^{-1}(z) = \mathcal{O} \begin{pmatrix} h(z) & h(z) & h(z) & h(z) \\ \tilde{h}(z) & \tilde{h}(z) & \tilde{h}(z) & \tilde{h}(z) \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},
\]

where

\[
h(z) = \begin{cases} 1, & \kappa > 0, \\ \log z, & \kappa = 0, \quad \nu > 0, \end{cases} \quad \tilde{h}(z) = \begin{cases} 1, & \kappa > 0, \\ \log z, & \kappa = 0, \quad \nu = 0, \end{cases}
\]

and the \( \mathcal{O} \) condition in (2.22) is understood in an entry-wise manner.

The above RH problem can be uniquely solved with the aid of mixed type multiple orthogonal polynomials associated with the modified Bessel functions; see [58]. Moreover, a general result in [22] shows that the correlation kernel (2.4) admits the following representation in terms of the solution of the RH problem 2.5 for \( Y \):
\[
\frac{n^2 K_n(n^2 x, n^2 y)}{2\pi i (x - y)} \left( \begin{array}{cc} 0 & 0 \\ \rho_{v-k,\beta}(y) & \rho_{v-k+1,\beta}(y) \end{array} \right) Y_+(y)^{-1} Y_+(x) \left( \begin{array}{c} \omega_{k,\alpha}(x) \\ \omega_{k+1,\alpha}(x) \\ 0 \\ 0 \end{array} \right).
\] (2.23)

We will then perform a Deift/Zhou steepest descent analysis [23,24] for the RH problem for \( Y \). The analysis consists of a series of explicit and invertible transformations

\[ Y \rightarrow X \rightarrow T \rightarrow S \rightarrow R, \] (2.24)

which leads to a RH problem for \( R \) tending to the identity matrix as \( n \rightarrow \infty \). Analyzing the effect of the transformations (2.24) gives us the large \( n \) limits of the correlation kernel in various regimes.

The rest of this paper is organized as follows. In Sect. 3, we analyze the equilibrium problem, along the way also extending some classical results from potential theory, introducing a four-sheeted Riemann surface built from the solution to the vector equilibrium problem and describing its uniformization in detail. Theorems 2.1 and 2.2 are finally established in Sects. 3.7 and 3.6, respectively.

Some auxiliary functions, constructed using objects from Sect. 3, are then introduced in Sect. 4 as a preparation for the asymptotic analysis.

Sections 5–11 are devoted to the steepest descent analysis of the RH Problem 2.5 for \( Y \) described above. In particular, we construct a local parametrix near the origin with the aid of the Meijer-G parametrix introduced by Bertola and Bothner in [11], using a recently introduced matching technique by Kuijlaars and Molag [38].

After the RH asymptotic analysis is finished, the conclusion of our main asymptotic results, i.e., Theorems 2.3 and 2.4, are presented in Sect. 12.

We conclude this paper with an Appendix to give some heuristic arguments on how to obtain the precise formulation of the vector equilibrium problem introduced in Sect. 2.2, which plays an important role in this paper.

Assumptions and notations Throughout this paper, it is assumed that \( n \) is an even number so that

\[ n_1 = n_2 = \frac{n}{2}. \]

This assumption is not essential and is only made to simplify the proof.

Since the asymptotic analysis of \( 4 \times 4 \) RH problems takes a substantial part of this work, it is notationally convenient to denote by \( E_{jk} \) the \( 4 \times 4 \) elementary matrix whose entries are all 0, except for the \((j, k)\)-entry, which is 1, that is,

\[ E_{jk} = \left( \delta_{l,j} \delta_{k,m} \right)_{l,m=1}^4. \] (2.25)

A fact of simple verification that comes in handy is the identity

\[ E_{jk} E_{lm} = \delta_{k,l} E_{jm}. \]

Finally, we adopt the notations

\[ \Delta_1 = (-\infty, -q), \quad \Delta_2 = (0, p), \quad \Delta_3 = (-\infty, 0), \] (2.26)
i.e., $\Delta_1$ is the interior of $\operatorname{supp}(\sigma - \mu_1)$, $\Delta_2$ is the interior of $\operatorname{supp} \mu_2$, $\Delta_3$ is the interior of $\operatorname{supp} \mu_3$, and also set
\[ c_\alpha = e^{i \alpha \pi}. \tag{2.27} \]
It is worthwhile to point out that for integer $\alpha$ the symmetry relation
\[ c_\alpha = c_{-\alpha} \]
takes place.

3. Analysis of the Vector Equilibrium Problem

The goal of this section is to analyze the equilibrium problem associated to the energy functional (2.6), which will ultimately lead to the proofs of Theorems 2.1, 2.2, and Proposition 3.5 about the relevant Euler-Lagrange variational conditions.

3.1. Preliminaries from potential theory. In this subsection, we will review some basic concepts and their properties from potential theory, which will be needed in what follows. For more details, we refer to the standard references [42,53,54,56].

Logarithmic potential and Cauchy transform of a measure Given a measure $\mu$ on $\mathbb{C}$, recall that its Cauchy transform $C^\mu$ was previously defined in (2.14). Closely connected is its logarithmic potential, which is defined by
\[ U^\mu(x) = \int \frac{\log |x - y|}{|x - y|} \, d\mu(y), \quad x \in \mathbb{C}, \]
whenever the integral makes sense as a finite real number.

By expanding the integrands into powers of $z$ around infinity, it immediately follows that, as $z \to \infty$,
\[ C^\mu(z) = - \frac{|\mu|}{z} (1 + o(1)), \quad U^\mu(z) = - |\mu| \log |z|(1 + o(1)). \tag{3.1} \]
If $\mu$ is compactly supported, the terms $o(1)$ in (3.1) can be replaced by $O(z^{-1})$. Furthermore, these functions are related through
\[ U^\mu(z) = \operatorname{Re} \int_\mathbb{C} C^\mu(s) \, ds + c, \]
where the constant $c$ is chosen so as to have the same asymptotic behavior as $z \to \infty$ on both sides of the identity above. This last relation implies that
\[ \frac{\partial U^\mu}{\partial z}(z) = \frac{1}{2} C^\mu(z), \quad z \in \mathbb{C} \setminus \operatorname{supp} \mu, \]
where $\frac{\partial}{\partial z} = \frac{1}{2} (\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$. This identity also extends to the $\pm$-boundary values on smooth arcs of $\operatorname{supp} \mu$. In this sense, for a measure $\mu$ on $\mathbb{R}$ with real-differentiable potential, we have
\[ \frac{dU^\mu}{dx}(x) = \operatorname{Re} C^\mu_{\pm}(x), \quad x \in \operatorname{supp} \mu, \tag{3.2} \]
so
\[
\frac{dU^\mu}{dx}(x) = \int \frac{d\mu(s)}{s-x} \begin{cases} > 0, & \text{if } x < \inf \text{supp } \mu, \\ < 0, & \text{if } x > \sup \text{supp } \mu. \end{cases} \tag{3.3}
\]

In addition, for \(z_0 \in \text{supp } \mu\) and \(\delta > 0\) for which \(\text{supp } \mu \cap \{|z-z_0| < \delta\} = \gamma\) is an analytic arc with complex line element \(ds\), the Sokhotski-Plemelj relations
\[
C^\mu_+(z) - C^\mu_-(z) = 2\pi i \frac{d\mu}{ds}(z), \quad C^\mu_+(z) + C^\mu_-(z) = 2 \text{PV} \int \frac{d\mu(x)}{x-z}, \tag{3.4}
\]
hold for \(z \in \gamma\), where \(\text{PV}\) denotes the Cauchy principal value.

Given a function \(\omega(x)\) on \(K, K = \mathbb{R}_-\) or \(K = \mathbb{R}_+\), with
\[
\omega(x) = c|x|^a(1 + o(1)), \quad x \to 0 \text{ along } K, \quad a > -1,
\]
its Cauchy transform
\[
C^\omega(z) := C^\mu \omega(z), \quad d\mu_\omega(x) := \omega(x) \, dx
\]
satisfies \([48, \text{Section 29}]\)
\[
C^\omega(z) = \begin{cases} O(1), & a > 0, \\ O(\log z), & a = 0, \quad \text{as } z \to 0, \\ C_K z^a(1 + o(1)), & -1 < a < 0, \end{cases}
\] \(\tag{3.6}\)
where, for \(-1 < a < 0\), the branch of \(z^a\) is chosen so that
\[
\lim_{\delta \to 0^+} (x + i\delta)^a = |x|^a, \quad x \in K,
\]
and
\[
C_K = \begin{cases} \frac{c\pi e^{a\pi i}}{\sin(a\pi)}, & \text{if } K = \mathbb{R}_-, \\ -\frac{c\pi e^{-a\pi i}}{\sin(a\pi)}, & \text{if } K = \mathbb{R}_+. \end{cases}
\]

Obviously, the behavior near the origin in (3.5) and (3.6) could be replaced by \(x-x_0 \to 0\) for any finite point \(x_0 \in \mathbb{R}\).

**Logarithmic capacity** The *logarithmic capacity* \(\text{cap } K\) of a compact set \(K \subset \mathbb{C}\) is defined by
\[
\text{cap } K = \sup_{\|\mu\|=1 \atop \text{supp } \mu \subset K} \text{exp} \left( -\inf_{\|\mu\|=1 \atop \text{supp } \mu \subset K} I(\mu) \right),
\]
where we emphasize that the inf/sup is taken over probability measures supported on \(K\) and \(I(\mu)\) is the logarithmic energy of \(\mu\) previously defined in (2.5). In particular, if \(\text{cap } K = 0\), then there is no probability measure on \(K\) with finite logarithmic energy.

If \(G \subset \mathbb{C}\) is an arbitrary Borel set, its capacity is defined by
\[
\text{cap } G = \sup \{\text{cap } K \mid K \subset G, \ K \text{ compact}\}.
\]
A property is said to hold \textit{quasi-everywhere} (shortly \textit{q.e.}), if it holds everywhere except on a set of capacity zero. For a general treatise on capacity and its relation to complex analysis, we refer the reader to [52,53].

\textbf{Balayage measure} Given a closed set $K \subset \mathbb{C}$ with positive capacity and a finite measure $\mu$ on $\mathbb{C}$, the balayage measure of $\mu$ associated with $K$ is the unique measure $\hat{\mu}$ such that $|\mu| = |\hat{\mu}|$ and

$$U^{\hat{\mu}}(z) = U^{\mu}(z) + c, \quad \text{q.e. } z \in K,$$  \hspace{1cm} (3.7)

where $c$ is a constant. In particular, if $K$ has an unbounded connected component, then comparing the behavior of both sides of (3.7) as $z \to \infty$ tells us that $c = 0$. When needed, we write

$$\hat{\mu} = \text{bal}(\mu, K)$$

to emphasize the underlying set $K$. A direct relation between the measures $\mu$ and $\text{bal}(\mu, K)$ is given by the formula

$$\text{bal}(\mu, K) = \int \text{bal}(\delta_z, K) \, d\mu(z),$$  \hspace{1cm} (3.8)

where $\delta_z$ denotes the Dirac measure at the point $z$.

A choice of our particular interest is

$$K = K_c = (-\infty, -c], \quad c \geq 0.$$  

In this case, if $z > -c$, then $\text{bal}(\delta_z, K_c)$ is absolutely continuous with respect to the Lebesgue measure, and

$$\frac{d \text{bal}(\delta_z, K_c)}{dx}(x) = \frac{1}{\pi} \frac{\sqrt{z+c}}{\sqrt{|x+c|}(z-x)}, \quad x \in K_c.$$  \hspace{1cm} (3.9)

For a measure $\mu$ with $\text{supp } \mu \subset [-c, +\infty)$, for simplicity we denote

$$\hat{\mu}_c = \text{bal}(\mu, K_c).$$

Assuming that $\mu([-c]) = 0$, it is easily seen from (3.8) and (3.9) that

$$\frac{d\hat{\mu}_c}{dx}(x) = \frac{1}{\pi \sqrt{|x+c|}} \int \frac{\sqrt{z+c}}{z-x} \, d\mu(z), \quad x \in K_c.$$  \hspace{1cm} (3.10)

As an application of (3.10), we have the following two simple lemmas which will be essential in establishing the characterization of $\text{supp } \mu_1$ given by Theorem 2.1.

\textbf{Lemma 3.1}. If $\mu$ is a finite measure on $[-c, +\infty)$ with $\mu([-c]) = 0$, then the function

$$x \mapsto \sqrt{|x|} \frac{d\hat{\mu}_c}{dx}(x)$$

is increasing on $K_c$. 

Proof. By (3.10), it follows that
\[
\sqrt{|x|} \frac{d\hat{\mu}_c}{dx}(x) = \frac{1}{\pi} \sqrt{\frac{|x|}{|x + c|}} \int \frac{\sqrt{z + c}}{z - x} d\mu(z), \quad x \in K_c.
\]
Since both \(\sqrt{\frac{|x|}{|x + c|}}\) and the integrand on the right-hand side of the above formula are increasing functions of \(x\) on \(K_c\), the lemma follows immediately. \(\square\)

With the measure \(\sigma\) introduced in (2.7), we have

**Lemma 3.2.** If \(\mu\) is an absolutely continuous finite measure on \(K_c\) for which \(\sqrt{|x|} \frac{d\mu}{dx}(x)\) is increasing on \(K_c\), then the positive part \((\mu - \sigma)^+\) of the signed measure \(\mu - \sigma\) is either zero or satisfies
\[
\text{supp}((\mu - \sigma)^+) = [-\tilde{c}, -c],
\]
for some \(\tilde{c} > c\).

**Proof.** Because \(\mu\) is finite but \(\sigma\) is not, we are sure that \(\frac{d\mu}{dx} - \frac{d\sigma}{dx}\) is negative for \(x\) large. By (2.7), we can write
\[
\frac{d\mu}{dx}(x) = \frac{\pi}{\alpha} \left( \sqrt{|x|} \frac{d\mu}{dx}(x) \right).
\]
Thus, the previous Lemma tells us that the quotient on the left-hand side above is strictly increasing, so there exists at most one point in which this quotient changes from smaller to bigger than 1. That is, there is at most one point for which the difference \(\frac{d\mu}{dx} - \frac{d\sigma}{dx}\) changes from negative to positive, and the result follows. \(\square\)

**3.2. An extension of the Lower Envelope Theorem.** In this subsection, we will extend the so-called Lower Envelope Theorem. The results presented here are well-known under the stronger assumption that the underlying measures are supported in a fixed compact set of \(\mathbb{C}\), but later we will need these results for measures with unbounded support.

**Proposition 3.3.** Let \(\{\mu_n\}\) be a sequence of probability measures on \(\mathbb{C}\) that converges weakly to a probability measure \(\mu\) on \(\mathbb{C}\) and satisfies the following conditions:

(i) The quantities
\[
\int \log(1 + |z|^2) d\mu_n(z)
\]
are finite and uniformly bounded in \(n\).

(ii) As \(R \to \infty\), the quantities
\[
\int_{|z| \geq R} \log(1 + |z|^2) d\mu_n(z)
\]
converge to zero uniformly in \(n\).
Then, we have
\[ U^\mu(z) \leq \liminf_{n \to \infty} U^{\mu_n}(z), \quad z \in \mathbb{C}, \]
and
\[ \liminf_{n \to \infty} U^{\mu_n}(z) = U^\mu(z) \]
for quasi-every \( z \in \mathbb{C} \).

Proof. We follow an idea in [35] and map the Riemann sphere \( \mathbb{C} \) to the sphere \( S \subset \mathbb{R}^3 \) centered at \((0, 0, 1/2)\) with radius \( 1/2 \) through the stereographic projection
\[
T(z) = \begin{cases} 
\left( \frac{\text{Re}(z)}{1+|z|^2}, \frac{\text{Im}(z)}{1+|z|^2}, \frac{|z|^2}{1+|z|^2} \right), & z \in \mathbb{C}, \\
(0, 0, 1), & z = \infty.
\end{cases}
\]
It is straightforward to check that the mapping \( T \) satisfies
\[
\|T(z) - T(w)\| = \frac{|z - w|}{\sqrt{1+|z|^2} \sqrt{1+|w|^2}}, \quad z, w \in \mathbb{C},
\]
where \( \| \cdot \| \) stands for the standard Euclidean norm in \( \mathbb{R}^3 \).

For a measure \( \nu \) on \( \mathbb{C} \), denote by \( \nu^T \) its pushforward measure induced by \( T \). That is, \( \nu^T \) is a measure on \( S \) determined by the condition that
\[
\int f(x) \, d\nu^T(x) = \int f(T(z)) \, d\nu(z).
\]
With
\[
V^\lambda(x) = \int \log \frac{1}{\|x - y\|} \, d\lambda(y)
\]
denoted by the potential of a measure \( \lambda \) on \( S \), it follows from (3.11) that if a measure \( \nu \) on \( \mathbb{C} \) satisfies
\[
\int \log(1+|z|^2) \, d\nu(z) < \infty
\]
and \( U^\nu \) is finite at \( z \), then \( V^{\nu^T}(z) \) is finite at \( T(z) \) and
\[
V^{\nu^T}(T(z)) = V^\nu(z) + \frac{1}{2} \int \log(1+|w|^2) \, d\nu(w) + \frac{1}{2} |\nu| \log(1+|z|^2).
\]

Since \( \mu_n \Rightarrow \mu \), it then follows from (3.12) that \( \mu_n^T \Rightarrow \mu^T \). Thus, by replacing the measure \( \nu \) in (3.13) by \( \mu_n \), it is readily seen that our proposition follows if we can show that
(a) The following limit holds:
\[
\lim_{n \to \infty} \int \log(1+|w|^2) \, d\mu_n(w) = \int \log(1+|w|^2) \, d\mu(w).
\]
(b) Whenever \( \{ \nu_n \} \) is a sequence of probability measures on \( S \) converging weakly to \( \nu \), then
\[
V^\nu(x) \leq \liminf_{n \to \infty} V^{\nu_n}(x), \tag{3.15}
\]
for every \( x \in S \).

(c) For the measures \( \{ \nu_n \} \) and \( \nu \) as in (b), there exists a log-polar set \( E \subset S \) such that
\[
V^\nu(x) = \liminf_{n \to \infty} V^{\nu_n}(x), \tag{3.16}
\]
for \( x \in S \setminus E \).

In part (c), by a log-polar set we mean that
\[
\int V^\nu(x) \, d\nu(x) = +\infty,
\]
for any probability measure \( \nu \) supported on \( E \). We note that \( E \) is log-polar if, and only if, \( T^{-1}(E) \) has zero capacity in \( \mathbb{C} \).

The proof of (3.15) follows immediately from the weak convergence of \( \nu_n \) to \( \nu \), and the fact that the function
\[
y \mapsto \log \frac{1}{\|x - y\|}
\]
is lower semi-continuous on any compact subset of \( \mathbb{R}^3 \), whereas the proof of (3.16) follows in the same steps as its analogue for measures supported in a fixed compact set of the plane \([54, \text{Theorem I.6.9}]\).

We finally provide a proof of (3.14). Since the non-negative function \( \log(1 + |z|^2) \) is lower semi-continuous on \( \mathbb{C} \), the weak convergence \( \mu_n \rightharpoonup \mu \) immediately implies that
\[
0 \leq \int \log(1 + |z|^2) \, d\mu(z) \leq \liminf_{n \to \infty} \int \log(1 + |z|^2) \, d\mu_n(z). \tag{3.17}
\]
By the condition (i), the right-hand side of the above inequality is finite.

Let \( \{ \lambda_n \} \) be a sequence of probability measures on \( \mathbb{C} \) defined by
\[
d\lambda_n(z) = \frac{1}{c_n} \log(1 + |z|^2) \, d\mu_n(z),
\]
where
\[
c_n = \int \log(1 + |z|^2) \, d\mu_n(z).
\]
If \( \limsup_{n \to \infty} c_n = 0 \), then the proof is over. Hence, we may assume that, without loss of generality,
\[
c_n \to \limsup_{n \to \infty} c_n := c > 0.
\]
From the condition (i), the limsup above is finite and thus \( \{ \lambda_n \} \) is a well-defined sequence of probability measures on \( \mathbb{C} \). Furthermore, from the condition (ii), we see that this
sequence is tight. By Prohorov’s theorem, we can assume that, after extracting a subsequence, it converges weakly to a probability measure \( \lambda \) on \( C \). Thus, if \( f \) is any bounded continuous function on \( C \), we have

\[
\lim_{n \to \infty} \int f(z) \, d\lambda_n(z) = \int f(z) \, d\lambda(z).
\]

If, in addition, the function \( f \) has compact support, then function \( f(z) \log(1 + |z|^2) \) is continuous and bounded on \( C \). The weak convergence \( \mu_n \to \mu \) then implies that

\[
\int f(z) \log(1 + |z|^2) \, d\mu_n(z) \to \int f(z) \log(1 + |z|^2) \, d\mu(z),
\]

and consequently

\[
\int f(z) \log(1 + |z|^2) \, d\mu(z) = c \int f(z) \, d\lambda(z),
\]

for every compactly supported continuous function \( f \). Considering a sequence \( \{f_m\} \) of such functions with the extra conditions that \( f_m \geq 0 \) and \( f_m \not\to 1 \) pointwise, it follows from the Monotone Convergence Theorem that

\[
\int \log(1 + |z|^2) \, d\mu(z) = c \int d\lambda(z) = c = \limsup_{n \to \infty} \int \log(1 + |z|^2) \, d\mu_n(z).
\]

This, together with (3.17), gives us (3.14).

This completes the proof of Proposition 3.3. \( \square \)

3.3. A scalar constrained equilibrium problem. Let \( \rho \) be a probability measure on \( \mathbb{R}_+ \). The so-called \( \sigma \)-constrained equilibrium measure \( \mu^\sigma_\rho \) of \( \mathbb{R}_- \), if it exists, is the measure that minimizes the functional

\[
I(\mu) - 2 \int U^\rho(z) \, d\mu(z)
\]

over all probability measures \( \mu \) on \( \mathbb{R}_- \) subject to the condition \( \mu \leq \sigma \), where \( \sigma \) is a given measure on \( \mathbb{R}_- \).

The characterization of the measure \( \mu_1 \) in Theorem 2.1 that we are looking for will follow from the following proposition.

**Proposition 3.4.** With the measure \( \sigma \) given in (2.7), the \( \sigma \)-constrained measure \( \mu^\sigma_\rho \) exists uniquely. Furthermore, there exists a constant \( c \geq 0 \) such that

\[
supp(\sigma - \mu^\sigma_\rho) = K_c = (-\infty, -c], \quad (3.18)
\]

and the following Euler-Lagrange variational conditions hold:

\[
U^{\mu^\sigma_\rho}(z) - U^\rho(z) = 0, \quad z \in supp(\sigma - \mu^\sigma_\rho), \quad (3.19)
\]

\[
U^{\mu^\sigma_\rho}(z) - U^\rho(z) \leq 0, \quad z \in \mathbb{R}_-. \quad (3.20)
\]
Proof. Existence, uniqueness and characterization through the variational conditions of the minimizer, with possibly a nonzero constant \( \ell \) on the right-hand side of (3.19) and (3.20), follow from the standard theory, we refer the reader to [28] for details. To see that \( \ell = 0 \) is the correct constant, we first observe that
\[
\sigma((−\infty, −a]) = +\infty \text{ for any } a > 0.
\]
This, together with the fact that \( \mu_\rho^\sigma \) is a probability measure, implies that \( \text{supp}(\sigma - \mu_\rho^\sigma) \) is unbounded. Thus, we can take the limit \( z \to -\infty \) in (3.19) and use the behavior of \( U^{\mu_\rho^\sigma}(z) - U^\rho(z) \) near \( \infty \) (see (3.1)) to conclude that \( \ell = 0 \).

To show (3.18), we follow the ideas similar to the ones in [27,29,30], which are based on the iterative balayage algorithm introduced by Dragnev [26]. To proceed, we set
\[
\nu_1 = \text{bal}(\rho, \mathbb{R}_-). \tag{3.21}
\]
The measure \( \nu_1 \) then has the following properties:

(a) For \( z \in \mathbb{R}_- \), we have
\[
U^{\nu_1}(z) = U^\rho(z),
\]
that is, \( \nu_1 \) is the unconstrained equilibrium measure of \( \mathbb{R}_- \) with the external field \(-2U^\rho\). This property follows from the definition of the balayage measure.

(b) From Lemmas 3.1 and 3.2, we have that
\[
\text{supp}((\nu_1 - \sigma)^+) = [-c_1, 0],
\]
for some \( c_1 \geq 0 \).

(c) For \( c_1 \) as above, we have
\[
\mu_\rho^\sigma|_{[-c_1, 0]} = \sigma|_{[-c_1, 0]}.
\]
This follows from property (a) and the Saturation Principle [28, Theorem 2.6].

We now define inductively
\[
\nu_{k+1} = \nu_k|_{K_{c_k}} + \sigma|_{[-c_k, 0]} + \tilde{\nu}_k, \quad k \geq 1, \tag{3.22}
\]
with
\[
\tilde{\nu}_k = \text{bal}((\nu_k - \sigma)^+, K_{c_k}). \tag{3.23}
\]
In (3.23), if \( k \geq 2 \), the constant \( c_k \geq 0 \) is, as we will show in a moment, uniquely defined through the condition
\[
\begin{cases}
  c_k = c_{k-1}, & \text{if } (\nu_k - \sigma)^+ = 0, \\
  \text{supp}((\nu_k - \sigma)^+) = [-c_k, -c_{k-1}], & \text{if } (\nu_k - \sigma)^+ \neq 0.
\end{cases} \tag{3.24}
\]
In words, we swap out the part of \( \nu_k \) that saturates \( \sigma \) to the set \( K_{c_k} \). From (3.22)–(3.24), we also observe that
\[
\nu_k|_{[-c_{k-1}, 0]} = \sigma|_{[-c_{k-1}, 0]} \tag{3.25}
\]
and that $\nu_k$ has no mass points. This particularly implies that

$$|\nu_{k+1}| = \nu_k(K_{c_k}) + \sigma([-c_k, 0]) + |(\nu_k - \sigma)^+|$$

$$= \nu_k(K_{c_k}) + \sigma([-c_k, -c_{k-1}]) + \sigma([-c_{k-1}, 0]) + \nu_k([-c_k, -c_{k-1}]) - \sigma([-c_k, -c_{k-1}])$$

$$= \nu_k(K_{c_k}) + \nu_k([-c_k, -c_{k-1}]) + \nu_k([-c_{k-1}, 0]) + \nu_k([-c_k, -c_{k-1}]) - \sigma([-c_k, -c_{k-1}])$$

$$= |\nu_k|,$$

and because $|\nu_1| = 1$ we get that $|\nu_k| = 1$ for every $k$.

To see that (3.24) indeed uniquely defines $c_k$, we will proceed inductively. We start with the observation that the function

$$K_{c_k} \ni x \mapsto \sqrt{|x|} \frac{d\nu_{k+1}}{dx}(x)$$

(3.26)

is increasing, once $\nu_1, \ldots, \nu_{k+1}$ are all well defined. In fact, because $\nu_1$ is absolutely continuous and $\nu_{k+1}$ is obtained from $\nu_k$ and $\sigma$ by sums, balayages and restrictions, which are operations that preserve the absolutely continuity, it follows that $\nu_{k+1}$ is always absolutely continuous. By (3.22), it is readily seen that for $x \in K_{c_k},$

$$\sqrt{|x|} \frac{d\nu_{k+1}}{dx}(x) = \sqrt{|x|} \frac{d\nu_k}{dx}(x) + \sqrt{|x|} \frac{d\nu_k}{dx}(x).$$

Because of (3.23), Lemma 3.1 tells us that the second term in the sum on the right-hand side above is increasing. Under induction hypothesis for (3.26), the first term in this sum is increasing as well. Hence, by induction it follows that (3.26) is always increasing.

Thus, once we know that $c_k$ as in (3.24) exists, the corresponding measure $\nu_{k+1}$ in (3.22) is well defined. Since $\sqrt{|x|} \frac{d\nu_{k+1}}{dx}(x)$ is increasing on $K_{c_k}$, we conclude (3.24) for $k + 1$ with the aid of Lemma 3.2, showing that the recursions (3.22)–(3.24) are well defined.

We also remark that, for $x \in K_{c_k},$

$$U^1_{\nu_k+1} (x) = U^1_{\nu_k | K_{c_k}} (x) + U^\sigma|[-c_k, 0]| (x) + U^\tilde{\nu_k} (x)$$

$$= U^1_{\nu_k | K_{c_k}} (x) + U^\sigma|[-c_k, 0]| (x) + U^\nu_k|[-c_k, -c_{k-1}]| (x) - U^\sigma|[-c_k, -c_{k-1}]| (x)$$

$$= U^1_{\nu_k | K_{c_k}} (x) + U^\sigma|[-c_{k-1}, 0]| (x) + U^\nu_k|[-c_k, -c_{k-1}]| (x)$$

$$= U^1_{\nu_k} (x),$$

(3.27)

where the first equality simply follows from the definition of $\nu_k$ in (3.22), the second equality is a consequence of the definition (3.23) of $\tilde{\nu}_k$ as a balayage measure and the assumption that $x \in K_{c_k}$, and for the final equality we have made use of (3.25). Furthermore, from the Principle of Domination [54], we also know that

$$U^\tilde{\nu_k} (x) \leq U^1_{\nu_k}|[-c_k, -c_{k-1}]| (x) - U^\sigma|[-c_k, -c_{k-1}]| (x), \quad x \in \mathbb{C}.$$  

Thus, by performing similar calculations as in (3.27) but replacing the second equality by an inequality, we conclude that

$$U^1_{\nu_k+1} (x) \leq U^1_{\nu_k} (x), \quad x \in \mathbb{C} \setminus K_{c_k}.$$  

(3.28)
We claim that the sequence \( \{c_k\} \) is convergent. Indeed, from its construction, it is readily seen that \( c_k \geq c_{k-1} \), so this sequence is increasing. It is also bounded, because by (3.25), we have
\[
\sigma([-c_k, 0]) = \nu_k([-c_k, 0]) \leq 1,
\]
but \( \sigma([x, 0]) \to +\infty \) when \( x \to -\infty \). Hence,
\[
\lim_{k \to \infty} c_k = c \tag{3.29}
\]
for some \( c \geq 0 \).

Our next goal is to show that the measures \( \{\nu_k\} \) has a weakly convergent subsequence. To see this, we observe from (3.10), (3.21) and (3.22) that
\[
\frac{d\nu_k}{dx}(x) = O(|x|^{-3/2}), \quad x \to -\infty, \tag{3.30}
\]
where the bound is uniform in \( k \). Thus, given any \( \varepsilon > 0 \), we can find \( M = M(\varepsilon) > c \) such that
\[
\nu_k((\infty, -M]) < \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, this shows that \( \{\nu_k\} \) is a tight sequence of probability measures. By Prokhorov’s theorem, there is a subsequence \( \{\nu_{kj}\} \) converging weakly to a probability measure \( \nu \) on \( \mathbb{R}^- \).

Let \( G \subset \mathbb{R}^- \) be any bounded open subset. From the weak convergence, we have
\[
\nu(G) \leq \liminf_{j \to \infty} \nu_{kj}(G).
\]
If \( G \subset [-c, 0] \), by (3.25), it is easily seen that \( \nu(G) = \sigma(G) \). If \( G \subset K_c \), note that \( \frac{d\nu_k/dx}{d\sigma/dx} \) is strictly increasing on \( K_{ck} \), it then follows from (3.24) and (3.29) that
\[
\nu(G) < \sigma(G).
\]

Moreover, the bound (3.30) implies that the requirements, and thus, the conclusions of Proposition 3.3 are applicable to the sequence \( \{\nu_{kj}\} \). This, together with (3.27), tells us that
\[
U^\nu(c) \leq \liminf_{j \to \infty} U^{\nu_{kj}}(c) = U^{\nu_1}(c) = U^\rho(c) < +\infty,
\]
hence, \( \nu \) cannot have a point mass at \( z = c \). A combination of all these results then shows that
\[
\nu \leq \sigma, \quad \text{on} \ \mathbb{R}^-,
\]
and
\[
\text{supp}(\sigma - \nu) = K_c.
\]

Finally, using Proposition 3.3 and equations (3.27)–(3.28), we have that \( \nu \) also satisfies the two conditions in (3.19). Hence, by uniqueness of the minimizer, it follows that \( \nu = \mu^\rho_\sigma \) and \( \text{supp}(\sigma - \mu^\rho_\sigma) = \text{supp}(\sigma - \nu) = K_c \).

This completes the proof of Proposition 3.4. \( \square \)
3.4. Qualitative properties for the vector equilibrium measure. To obtain qualitative properties for the vector of measures $\mu = (\mu_1, \mu_2, \mu_3) \in \mathcal{M}$ that minimizes (2.6), we recall the Euler-Lagrange conditions of the problem, which here take the form of the following set of equalities and inequalities:

$$2U^{\mu_1}(x) - U^{\mu_2}(x) = \ell_1,$$  \quad \text{q.e. } x \in \text{supp}(\sigma - \mu_1), \quad (3.31)

$$2U^{\mu_1}(x) - U^{\mu_2}(x) \leq \ell_1,$$  \quad x \in \mathbb{R}_- \setminus \text{supp}(\sigma - \mu_1), \quad (3.32)

$$2U^{\mu_2}(x) - U^{\mu_1}(x) - U^{\mu_3}(x) + 2(\beta - \alpha)\sqrt{x} = \ell_2,$$  \quad \text{q.e. } x \in \text{supp} \mu_2, \quad (3.33)

$$2U^{\mu_2}(x) - U^{\mu_1}(x) - U^{\mu_3}(x) + 2(\beta - \alpha)\sqrt{x} \geq \ell_2,$$  \quad x \in \mathbb{R}_+ \setminus \text{supp} \mu_2, \quad (3.34)

$$2U^{\mu_3}(x) - U^{\mu_2}(x) = \ell_3,$$  \quad \text{q.e. } x \in \text{supp} \mu_3, \quad (3.35)

$$2U^{\mu_3}(x) - U^{\mu_2}(x) \geq \ell_3,$$  \quad x \in \mathbb{R}_- \setminus \text{supp} \mu_3, \quad (3.36)

where $\ell_1$, $\ell_2$ and $\ell_3$ are three constants. These equations actually follow from the Euler-Lagrange conditions of the problem, which here take the form of the following set of equalities and inequalities:

$$2U^{\mu_1}(x) - U^{\mu_2}(x) = \ell_1,$$  \quad \text{q.e. } x \in \text{supp}(\sigma - \mu_1), \quad (3.31)

$$2U^{\mu_1}(x) - U^{\mu_2}(x) \leq \ell_1,$$  \quad x \in \mathbb{R}_- \setminus \text{supp}(\sigma - \mu_1), \quad (3.32)

$$2U^{\mu_2}(x) - U^{\mu_1}(x) - U^{\mu_3}(x) + 2(\beta - \alpha)\sqrt{x} = \ell_2,$$  \quad \text{q.e. } x \in \text{supp} \mu_2, \quad (3.33)

$$2U^{\mu_2}(x) - U^{\mu_1}(x) - U^{\mu_3}(x) + 2(\beta - \alpha)\sqrt{x} \geq \ell_2,$$  \quad x \in \mathbb{R}_+ \setminus \text{supp} \mu_2, \quad (3.34)

$$2U^{\mu_3}(x) - U^{\mu_2}(x) = \ell_3,$$  \quad \text{q.e. } x \in \text{supp} \mu_3, \quad (3.35)

$$2U^{\mu_3}(x) - U^{\mu_2}(x) \geq \ell_3,$$  \quad x \in \mathbb{R}_- \setminus \text{supp} \mu_3, \quad (3.36)

where $\ell_1$, $\ell_2$ and $\ell_3$ are three constants. These equations actually follow from the Euler-Lagrange conditions of the problem, which here take the form of the following set of equalities and inequalities:

$$I(\nu) + \int Q_2(x) \, d\nu(x)$$

among all the probability measures $\nu$ on $\mathbb{R}_+$, where

$$Q_2(x) = -U^{\mu_1}(x) - U^{\mu_3}(x) + 2(\beta - \alpha)\sqrt{x}, \quad x \in \mathbb{R}_+,$$  \quad (3.37)

is interpreted as the external field. Thus, given $\mu_1$ and $\mu_3$, the component $\mu_2$ is characterized by equations (3.33) and (3.34). The other variational conditions can be derived in a similar manner.

In our setup, (3.31)–(3.36) are improved with the next result, which also provides some of the statements claimed in Theorem 2.1.

**Proposition 3.5.** There exists a unique minimizer $\mu = (\mu_1, \mu_2, \mu_3) \in \mathcal{M}$ of the energy functional (2.6) over $\mathcal{M}$ stated in Theorem 2.1. Moreover, supp $\mu_1 = \mathbb{R}_-$, supp $\mu_3 = \mathbb{R}_+$, and for some positive numbers $p$ and $q$,

$$\text{supp} \sigma - \mu_1 = (-\infty, -q], \quad \text{supp} \mu_2 = [0, p].$$  \quad (3.38)

In addition, the three measures $\mu_1$, $\mu_2$ and $\mu_3$ are absolutely continuous with respect to the Lebesgue measure, and their densities are bounded except possibly at the origin.

Furthermore, there exists a constant $\ell \in \mathbb{R}$ such that

$$2U^{\mu_2}(x) - U^{\mu_1}(x) - U^{\mu_3}(x) + 2(\beta - \alpha)\sqrt{x} = \ell, \quad x \in \text{supp} \mu_2 = [0, p],$$  \quad (3.39)

$$2U^{\mu_2}(x) - U^{\mu_1}(x) - U^{\mu_3}(x) + 2(\beta - \alpha)\sqrt{x} > \ell, \quad x \in (p, +\infty).$$  \quad (3.40)

Finally, we have

$$2U^{\mu_1}(x) - U^{\mu_2}(x) = 0, \quad x \in \text{supp} \sigma - \mu_1 = (-\infty, -q],$$  \quad (3.41)

$$2U^{\mu_1}(x) - U^{\mu_2}(x) < 0, \quad x \in (-q, 0],$$  \quad (3.42)

and

$$2U^{\mu_3}(x) - U^{\mu_2}(x) = 0, \quad x \in \text{supp} \mu_3 = \mathbb{R}_-. \quad (3.43)$$
Proof. The existence and uniqueness of the minimizer $\mu = (\mu_1, \mu_2, \mu_3)$ claimed by Theorem 2.1 follows from the standard theory, we refer the reader to [35] for details, and also [25,29,30] where similar equilibrium problems appeared.

Also, observe that once two among the measures $\mu_1$, $\mu_2$ and $\mu_3$ are fixed, the total potential acting on the third measure is real analytic on the set supporting it, except possibly at the origin. This immediately implies that the three measures are absolutely continuous, and also that their densities are bounded except possibly at the origin. For the same reason, the q.e. conditions on (3.31), (3.33) and (3.35) are actually valid everywhere on the corresponding supports, so yielding (3.39), (3.41) and (3.43), where for the latter two the fact that the variational constants $\ell_1$ and $\ell_3$ are zero will follow from the unboundedness of the supports of $\sigma - \mu_1$ and $\mu_3$, to be shown in a moment.

We first show the properties of $\mu_1$. Since $\sigma((−∞, −x)) = +∞$ for any $x ≥ 0$ and $\mu_1$ is finite, we get that $\text{supp}(\sigma - \mu_1)$ is unbounded. In addition, it is readily seen from (3.31) and (3.32) that

$$\mu_1 = \frac{1}{2} \mu_\rho^{\sigma} \quad \text{with} \quad \rho = \mu_2,$$

where the measure $\mu_\rho^{\sigma}$ is defined in Proposition 3.4. Hence, it follows that $\text{supp}(\sigma - \mu_1) = (−\infty, −q]$ for some $q ≥ 0$, as well as (3.42) with possibly weak inequality and also (3.43). To see that the inequality is indeed strict, we start with the functions $\xi_1$ and $\xi_2$ in (2.15) that are at this point already defined off the real axis, and compute from (3.2) that

$$\Re \int_{-q}^x (\xi_{1,+}(s) - \xi_{2,+}(s)) \, ds = 2U^{\mu_1}(x) - U^{\mu_2}(x), \quad x \in (-q, 0). \quad (3.44)$$

This, together with Remark 3.8 below\(^1\), implies that the inequality (3.32) is strict. All the conditions on $\text{supp}\mu_1$ are thus proven.

Next we handle the conditions on $\text{supp}\mu_2$. Observe that for $j = 1, 3$, and $x > 0$,

$$\frac{d}{dx}(x(U^{\mu_j})'(x)) = \int \frac{x}{(s-x)^2} \, d\mu_j(s) + \int \frac{1}{s-x} \, d\mu_j(s)$$

$$= \int \left( \frac{x}{(s-x)^2} + \frac{s-x}{(s-x)^2} \right) \, d\mu_j(s) = \int \frac{s}{(s-x)^2} \, d\mu_j(s) < 0,$$

where we have made use of the fact that $\mu_j$ is a positive measure supported on $\mathbb{R}_-$. Furthermore, a simple calculation also shows that

$$(x(\sqrt{x}))' > 0, \quad x > 0.$$

Hence, on account of (3.37), (2.2) and the above two inequalities, we conclude that

$$(xQ'_2(x))' > 0, \quad x > 0.$$

By [54, Theorem IV.1.10 - (c)], this implies that

$$\text{supp}\mu_2 = [\bar{p}, p].$$

\(^1\) In fact, in (3.44) the +-boundary value can be omitted. Our proof that the left-hand side of (3.44) does not vanish relies on the equalities (3.39), (3.41) and (3.43) in an implicit manner, but obviously not on their corresponding inequalities.
for some \( p > \tilde{p} \geq 0 \) and also that the inequality (3.34) is strict. To see that \( \tilde{p} = 0 \), we note that the equality (3.39), already proven, gives us that

\[
2U^{\mu_2}(\tilde{p}) + Q_2(\tilde{p}) = \ell.
\]  
(3.45)

If \( \tilde{p} > 0 \), it then follows from (3.3) that the function \( Q_2 + 2U^{\mu_2} \) is strictly increasing on \((0, \tilde{p})\). This, together with (3.45), implies that

\[
2U^{\mu_2}(x) + Q_2(x) < \ell, \quad x \in (0, \tilde{p}),
\]
contradicting the inequality (3.34). Hence, we have to have that \( \tilde{p} = 0 \), which concludes (3.38).

As for \( \mu_3 \), it is a consequence of (3.7) that the measure \( \frac{1}{2} \text{bal}(\mu_2, \mathbb{R}_-) \) is fully supported on \( \mathbb{R}_- \) and satisfies the equality (3.43) everywhere on its support. Hence, we must have

\[
\mu_3 = \frac{1}{2} \text{bal}(\mu_2, \mathbb{R}_-), \quad \ell_3 = 0,
\]
and (2.12) follows immediately from (3.10).

This completes the proof of Proposition 3.5. \( \square \)

The arguments above give us the qualitative properties claimed by Theorem 2.1. The proofs of the quantitative claims of Theorem 2.1, namely formulas (2.8), (2.9), (2.10), (2.11) and (2.13), will be given in Sect. 3.7.

3.5. A four-sheeted Riemann surface \( \mathcal{R} \). To prove Theorem 2.2, we need a Riemann surface consisting of four sheets \( \mathcal{R}_j, j = 1, 2, 3, 4 \), given by

\[
\begin{align*}
\mathcal{R}_1 &= \mathbb{C} \setminus (-\infty, -q], \\
\mathcal{R}_2 &= \mathbb{C} \setminus ((-\infty, -q] \cup [0, p]), \\
\mathcal{R}_3 &= \mathbb{C} \setminus (-\infty, p], \\
\mathcal{R}_4 &= \mathbb{C} \setminus (-\infty, 0],
\end{align*}
\]  
(3.46)

where the constants \( p, q \) are given in (2.10) and (2.8), respectively.

The sheet \( \mathcal{R}_1 \) is connected to the sheet \( \mathcal{R}_2 \) through \((-\infty, -q]\), \( \mathcal{R}_2 \) is connected to \( \mathcal{R}_3 \) through \([0, p]\) and \( \mathcal{R}_3 \) is connected to \( \mathcal{R}_4 \) through \((-\infty, 0]\). All these gluings are performed in the usual crosswise manner; see Fig. 1. We then compactify the resulting surface by adding a common point at \( \infty \) to the sheets \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), and a common point at \( \infty \) to the sheets \( \mathcal{R}_3 \) and \( \mathcal{R}_4 \). We denote this compact Riemann surface by \( \mathcal{R} \).

The surface \( \mathcal{R} \) has the following branch points:

- Common branch points to \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) at \( \infty \) and \( z = -q \).
- A common branch point to \( \mathcal{R}_2 \) and \( \mathcal{R}_3 \) at \( z = p \).
- A common branch point to \( \mathcal{R}_3 \) and \( \mathcal{R}_4 \) at \( \infty \).
- A common branch point to \( \mathcal{R}_2, \mathcal{R}_3 \) and \( \mathcal{R}_4 \) at \( z = 0 \).

The last branch point enlisted above has ramification index 3, whereas the others have ramification index 2. Consequently, it follows from the Riemann Hurwitz formula (cf. [47]) that \( \mathcal{R} \) has genus 0.
**Proposition 3.6.** The function $\xi_j$ defined in (2.15) is analytic on $R_j$.

**Proof.** From the description of the supports of $\mu_1, \mu_2$ and $\mu_3$ in Proposition 3.5, it follows that the $\xi_j$’s are analytic in their domains of definition as in (2.15). A comparison of these domains with (3.46) then shows that we only need to show that $\xi_1$ and $\xi_2$ are analytic across $(-q, 0)$, and in addition that $\xi_1$ does not have a singularity at $z = 0$.

With the constraint measure $\sigma$ given in (2.7), a simple residue calculation shows that

$$C^\sigma(z) = -\frac{\alpha}{\sqrt{z}}, \quad z \in \mathbb{C} \setminus R_-.$$  \hfill (3.47)

By (2.15), it is then readily seen that

$$\xi_1(z) = C^{\mu_1}(z) - C^\sigma(z), \quad z \in \mathbb{C} \setminus R_-.$$  \hfill (3.48)

On account of the fact that $\mu_1 = \sigma$ in $(-q, 0)$ and the first equation in (3.4), we obtain that

$$\xi_{1,+}(x) - \xi_{1,-}(x) = 0, \quad x \in (-q, 0),$$

thus concluding that $\xi_1$ is indeed analytic across $(-q, 0)$, and also that $z = 0$ is an isolated singularity. However, because $\mu_1$ is equal to $\sigma$ near $z = 0$, it follows from (3.5) and (3.6) that, as $z \to 0$,

$$C^{\mu_1}(z) = \mathcal{O}(|z|^{-1/2}).$$

Hence, $\xi_1(z) = \mathcal{O}(|z|^{-1/2})$ as well, which implies that $z = 0$ is in fact a removable singularity of $\xi_1$.

The proof for $\xi_2$ follows from the fact that

$$\xi_2(z) = -\xi_1(z) + C^{\mu_2}(z),$$

and that supp $\mu_2$ does not intersect $(-q, 0)$.

This completes the proof of Proposition 3.6. $\square$
With the functions $\xi_j$, $j = 1, 2, 3, 4$, defined in (2.15), set

$$
\xi : \bigcup_{j=1}^{4} \mathcal{R}_j \to \mathbb{C}, \quad \xi|_{\mathcal{R}_j} = \xi_j.
$$

(3.49)

From the previous Proposition, $\xi$ is a well-defined meromorphic function on each of the sheets. It turns out that, in fact, it extends meromorphically to the whole surface $\mathcal{R}$, as claimed by our next result.

**Proposition 3.7.** The function $\xi$ defined in (3.49) extends to a meromorphic function on the Riemann surface $\mathcal{R}$, and its unique pole is the branch point at $z = 0$.

**Proof.** We need to show that the analytic continuation of $\xi_j$ to $\mathcal{R}_{j+1}$ is $\xi_{j+1}$, $j = 1, 2, 3$. For the sake of brevity, we will only consider the case when $j = 1$, while the other cases can be proved similarly.

To show that the analytic continuation of $\xi_1$ to $\mathcal{R}_2$ is $\xi_2$, we note from (3.2)–(3.4) that for $x < -q$

$$
2 \frac{d}{dx} U^{\mu_1}(x) = 2 \text{PV} \int \frac{d\mu_1(s)}{s-x} = C_+^{\mu_1}(x) + C_-^{\mu_1}(x)
$$

and

$$
\frac{d}{dx} U^{\mu_2}(x) = C_+^{\mu_2}(x) = C_-^{\mu_2}(x).
$$

Thus, by taking derivatives with respect to $x$ on both sides of (3.41), it follows that

$$
0 = C_+^{\mu_1}(x) + C_-^{\mu_1}(x) - C_+^{\mu_2}(x) = C_+^{\mu_1}(x) + \frac{\alpha}{(\sqrt{x})_+} + C_+^{\mu_1}(x) - C_+^{\mu_2}(x) + \frac{\alpha}{(\sqrt{x})_-}
$$

$$
= \xi_1,+(x) - \xi_2,+(x), \quad x \in (-\infty, -q),
$$

as required, where we have made use of the fact that $(\sqrt{x})_+ = -(\sqrt{x})_-$ for $x < 0$ in the second equality.

Thus, the only possible poles of $\xi$ have to be at the branch points. Proposition 3.5 already tells us that the densities of $\mu_1$, $\mu_2$ and $\mu_3$ remain bounded except possibly at the origin. At this stage, we already know that $\xi$ - and hence each $\mu_j$ - is algebraic, so for each of $\mu_1$, $\mu_2$ and $\mu_3$ the behavior (3.5)–(3.6) has to take place as $x \to p, -q$, for some $a > 0$, giving us that $\xi$ cannot blow up at these points.

This way, we have shown that the only possible poles of $\xi$ are $z = 0, \infty$. However, the large $z$ asymptotics of $\xi_j$, $j = 1, 2, 3, 4$, (which are immediate from (2.15) but for convenience also given in (3.58) below) show that the function $\xi$ is analytic at $\infty$ and non-constant, so the point $z = 0$ common to the last three sheets has indeed to be a pole of $\xi$.

This completes the proof of Proposition 3.7. □
3.6. Proof of Theorem 2.2. By Proposition 3.7, we have that the functions \( \xi_j, j = 1, 2, 3, 4, \) are the four distinct solutions to the following algebraic equation of order four:

\[
0 = \prod_{i=1}^{4} (\xi - \xi_i) = \xi^4 + R_3(z)\xi^3 + R_2(z)\xi^2 + R_1(z)\xi + R_0(z),
\]

where the functions \( R_j(z), j = 0, 1, 2, 3, \) are rational functions whose set of poles coincide with the set of poles for \( \xi, \) so they can have poles only at \( z = 0. \) In view of (2.15), it is easily seen that

\[
R_3(z) = -\xi_1 - \xi_2 - \xi_3 - \xi_4 = 0.
\]

To show that \( R_1, R_2 \) and \( R_3 \) are indeed given by the ones in (2.16), we need to know the local behavior of each \( \xi_j, j = 2, 3, 4, \) near the origin.

Because \( R \) has a branch point of ramification index 3 at \( z = 0, \) we have that, as \( z \to 0, \)

\[
\xi_j(z) = \tilde{c}_j z^{\hat{\delta}} (1 + o(z)), \quad j = 2, 3, 4, \tag{3.50}
\]

for some nonzero integer \( \hat{\delta} \) and some nonzero constants \( \tilde{c}_2, \tilde{c}_3, \tilde{c}_4. \) Thus, in virtue of the Sokhotski-Plemelj relations (3.4) and (3.6), it follows that the densities of the three measures \( \mu_1, \mu_2 \) and \( \mu_3 \) behave algebraically near the origin as well, that is, as \( z \to 0, \)

\[
\frac{d\mu_j}{dx}(x) = c_j z^{q_j} (1 + o(1)), \quad j = 1, 2, 3, \tag{3.51}
\]

for some nonzero constants \( c_1, c_2, c_3 \) and some rational numbers \( q_1, q_2 \) and \( q_3 \) with \( q_j > -1. \) We note that the latter condition holds because the measures \( \mu_j \)'s are finite. Also, we see from (3.43) and (3.2) that

\[
2 \text{ Re } C^\mu_3(x) = \text{ Re } C^\mu_2(x), \quad x < 0.
\]

This, together with (3.4) and (3.6), implies that either \( q_2, q_3 \geq 0 \) or \(-1 < q_2 = q_3 < 0.\) Hence, we further obtain from (3.50) and the definition of \( \xi_3 \) in (2.15) that

\[
\frac{\hat{\delta}}{3} = \begin{cases} 
-\frac{1}{2}, & \text{if } q_2, q_3 \geq 0, \\
\min\{q_3, -\frac{1}{2}\}, & \text{if } -1 < q_2 = q_3 < 0.
\end{cases}
\]

Because \( \hat{\delta} \) is an integer, we learn from the above formula that the only possibility left is

\[
\hat{\delta} = -2, \tag{3.52}
\]

or equivalently,

\[
q_2 = q_3 = -\frac{2}{3}. \tag{3.53}
\]

In view of the Vieta relations, (3.50), (3.52) and the fact that \( \xi_1 \) is analytic near \( z = 0, \) we obtain that, as \( z \to 0, \)

\[
R_0(z) = \xi_1 \xi_2 \xi_3 \xi_4 = \mathcal{O}(z^{-2}), \tag{3.54}
\]

\[
R_1(z) = -\xi_1 \xi_2 \xi_3 - \xi_1 \xi_2 \xi_4 - \xi_1 \xi_3 \xi_4 - \xi_2 \xi_3 \xi_4 = \mathcal{O}(z^{-2}). \tag{3.55}
\]
\[ R_2(z) = \sum_{j\neq k, 1 \leq j, k \leq 4} \xi_j \xi_k = O(z^{-4/3}). \]  

(3.56)

Since \( R_j, j = 0, 1, 2, \) are rational functions with possible finite poles only at \( z = 0, \) we conclude that

\[ R_j(z) = \frac{P_j(z)}{z^2}, \quad j = 0, 1, \text{ and } R_2(z) = \frac{P_2(z)}{z}, \]  

(3.57)

for some polynomials \( P_0, P_1, \) and \( P_2. \)

On the other hand, as \( z \to \infty, \) it follows from (3.1) and the local coordinates on \( \mathcal{R} \) around the branch points at \( \infty \) that

\[ \xi_1(z) = \frac{\alpha}{\sqrt{z}} - \frac{1}{2z} - \frac{c_1}{z^{3/2}} + O(z^{-2}), \quad \xi_2(z) = -\frac{\alpha}{\sqrt{z}} - \frac{1}{2z} + \frac{c_1}{z^{3/2}} + O(z^{-2}), \]  

\[ \xi_3(z) = -\frac{\beta}{\sqrt{z}} + \frac{1}{2z} - \frac{c_3}{z^{3/2}} + O(z^{-2}), \quad \xi_4(z) = \frac{\beta}{\sqrt{z}} + \frac{1}{2z} + \frac{c_3}{z^{3/2}} + O(z^{-2}), \]  

(3.58)

for some constants \( c_1 \) and \( c_3. \)

Looking at the polynomial part of (3.58), and expanding as in (3.54)–(3.56) but near \( z = \infty, \) we see from (3.57) that the coefficients \( R_j, j = 0, 1, 2, \) reduce to the ones given in (2.16).

This completes the first part of the proof of Theorem 2.2.

To obtain the rational parametrization for (2.16), which is known to exist because \( \mathcal{R} \) has genus 0, we first remark that the point \( (\xi, z) = (0, \infty) \) is the only point of high order branching of the curve, as all the other points are either simple branch points or regular points. As a consequence, the line

\[ z = \frac{t}{\xi^2}, \quad t \in \mathbb{C}, \]  

(3.59)

should intersect the point \( (0, \infty) \) with high multiplicity. Substituting (3.59) into (2.16), we arrive at

\[ t^2 - (\alpha^2 + \beta^2)t + (\alpha^2 - \beta^2)\xi + \alpha^2 \beta^2 = 0, \]  

from which it follows that \( \xi = h(t) \) with \( h \) given in (2.17). Thus, the map

\[ (\xi, z) = H(t) := \left(h(t), \frac{t}{h(t)^2}\right), \quad t \in \mathbb{C}, \]  

(3.60)

is a rational parametrization of the Riemann surface \( \mathcal{R}. \) Counting its degree, we see that this parametrization is maximal [55, Theorem 4.21].

This completes the proof of Theorem 2.2. \( \square \)
3.7. Proof of Theorem 2.1. As we observed at the end of Sect. 3.4, Proposition 3.5 already provides most of the claims in Theorem 2.1, and it only remains to prove (2.8), (2.9), (2.10), (2.11) and (2.13).

The local behavior of the density functions near the origin for the measures $\mu_2$ and $\mu_3$ claimed in (2.11) and (2.13) was already obtained; see (3.51) and (3.53). To verify the other formulas, we need an analysis of the spectral curve (2.16).

From the construction of the Riemann surface $\mathcal{R}$, its only finite branch points are $p$, $-q$ and 0. The discriminant of (2.16) with respect to $\xi$, as computed with Mathematica, is

$$\frac{1}{z^8} \left( \alpha^2 - \beta^2 \right)^2 D_1(z)$$

with

$$D_1(z) = -27 \left( \alpha^2 - \beta^2 \right)^2 + 4z \left( \alpha^2 + \beta^2 \right) \left( \alpha^4 - 34\alpha^2\beta^2 + \beta^4 \right) + 16z^2\alpha^2\beta^2 \left( \alpha^2 - \beta^2 \right)^2$$

being a quadratic polynomial. The leading coefficient of $D_1$ is positive and

$$D_1(0) = -27 \left( \alpha^2 - \beta^2 \right)^2 < 0,$$

so we have that the discriminant of the spectral curve has two simple zeros with distinct signs. Hence, these two real roots have to be the nonzero branch points of $\mathcal{R}$, namely $p$ and $-q$, and the formulas (2.8) and (2.10) are obtained by solving the quadratic equation $D_1(z) = 0$.

Finally, from the relation (3.48), the definition of $\xi_2$ in (2.15), and the fact that $p$ and $-q$ are two simple zeros of (3.61), we conclude (2.9) and the local behavior of $\mu_2$ near $z = p$ as stated in (2.11).

This completes the proof of Theorem 2.1. \(\square\)

Remark 3.8. We note that the arguments above also imply that

$$\xi_j(x) - \xi_k(x) \neq 0, \quad j \neq k,$$

for $x \in (-q, 0) \cup (p, +\infty)$, because the discriminant (3.61) does not vanish on these two intervals.

3.8. The uniformization of the Riemann surface $\mathcal{R}$ in detail. For later purpose, it is convenient to give a geometric description of the opens sets $\mathcal{D}_k$ that are uniquely determined by

$$\mathcal{D}_k = H^{-1}(\mathcal{R}_k), \quad k = 1, 2, 3, 4,$$

where $H$ is given in (3.60). To obtain these sets, we first analyze the images of the branch points of $\mathcal{R}$ on the $t$-sphere.

The finite branch points of $\mathcal{R}$ where $\xi$ remains bounded, that is, the branch points $z = p$ and $z = -q$, are determined as the values of $t$ for which the equation

$$z = z(t) = \frac{t}{h(t)^2} = \frac{t(\beta^2 - \alpha^2)^2}{(t - \alpha^2)(t - \beta^2)^2}$$

(3.63)
has multiple solutions. Since
\[ z'(t) = -\frac{(\beta^2 - \alpha^2)^2}{(t - \alpha^2)^3(t - \beta^2)^3} \hat{h}(t), \]
where
\[ \hat{h}(t) := 3t^2 - t(\alpha^2 + \beta^2) - \alpha^2 \beta^2, \]
these points are the roots of \( \hat{h}(t) \), i.e.,
\[ t_\pm = \frac{1}{6}(\beta^2 + \alpha^2 \pm \sqrt{\alpha^4 + 14\alpha^2\beta^2 + \beta^4}) \]
with \( t_- < 0 < t_+ \). As a consequence,
\[ z(t_+) = \frac{t_+}{h(t_+)^2} > 0, \quad z(t_-) = \frac{t_-}{h(t_-)^2} < 0, \]
so actually
\[ z(t_+) = p, \quad z(t_-) = -q, \]
which is also consistent with (2.10) and (2.8).

To find the \( t \)-points corresponding to \( \infty^{(1)} = \infty^{(2)} \) and \( \infty^{(3)} = \infty^{(4)} \), we must find the values of \( t \) for which \( z(t) \) blows up. These are thus given by the zeros of \( h(t) \), that is,
\[ t = \alpha^2 \quad \text{or} \quad t = \beta^2. \]

To identify the images, we see from (3.58) that
\[ \sqrt{z} \xi_{1,2}(z) = \pm \alpha + o(1), \quad \sqrt{z} \xi_{3,4}(z) = \mp \beta + o(1), \quad z \to \infty, \]
whereas using the rational parametrization \( H \),
\[ |\sqrt{z} \xi(z)| = \left| \sqrt{\frac{t}{h(t)^2}} h(t) \right| = \begin{cases} \alpha + o(1), & t \to \alpha^2, \\ \beta + o(1), & t \to \beta^2, \end{cases} \]
hence,
\[ z(\alpha^2) = \infty^{(1)} = \infty^{(2)}, \quad z(\beta^2) = \infty^{(3)} = \infty^{(4)}. \]
Moreover, since
\[ \hat{h}(\alpha^2) = -2\alpha^2(\beta^2 - \alpha^2) < 0, \quad \hat{h}(\beta^2) = 2\beta^2(\beta^2 - \alpha^2) > 0, \]
we have the ordering
\[ t_- < 0 < \alpha^2 < t_+ < \beta^2. \]

The remaining branch point of \( R \) is the one at \( z = 0 \) connecting \( R_2, R_2 \) and \( R_3 \). According to Proposition 3.7, this branch point corresponds to the only \( t \)-point for which \( h(t) = \xi \) blows up, so it is \( t = \infty \).

In summary, we have the following proposition regarding the mapping properties of the rational parametrization \( H \) defined in (3.60).
Table 1. The $z \leftrightarrow t$ correspondence for the branch points of $\mathcal{R}$

| Branch points on $\mathcal{R}$ | Points on $t$-sphere (in increasing order of magnitude) |
|--------------------------------|--------------------------------------------------------|
| $\infty(1) = \infty(2)$      | $t_-$                                                  |
| $p$                           | $t_+$.                                                 |
| $\infty(3) = \infty(4)$      | $\beta^2$.                                             |
| $0$                           | $\infty$.                                              |

Proposition 3.9. The $z \leftrightarrow t$ correspondence for the branch points of the Riemann surface $\mathcal{R}$ under the rational parametrization $H$ is listed in Table 1. Furthermore, the local coordinate $z(t)$ admits the following behavior near each of its critical points.

\[
\begin{align*}
    z(t) &= \frac{\alpha^2}{(\beta^2 - \alpha^2)^2} \frac{1}{(t - \alpha^2)^2} (1 + \mathcal{O}(t - \alpha^2)), & t &\to \alpha^2, \\
    z(t) &= \frac{\beta^2}{(\beta^2 - \alpha^2)^2} \frac{1}{(t - \beta^2)^2} (1 + \mathcal{O}(t - \beta^2)), & t &\to \beta^2, \\
    z(t) &= -q + \mathcal{O}((t - t_-)^2), & t &\to t_- , \\
    z(t) &= p + \mathcal{O}((t - t_+)^2), & t &\to t_+, \\
    z(t) &= \frac{(\beta^2 - \alpha^2)^2}{t^3} (1 + \mathcal{O}(t^{-1})), & t &\to \infty.
\end{align*}
\]

Proof. We have already proved the images of the branch points of $\mathcal{R}$ in the $t$-sphere, while the local behavior of $z$ near each of its critical points follows directly from (3.63).

\[\square\]

The inverse map $H^{-1}$ maps the branch cuts $\Delta_k$ of $\mathcal{R}$ to simple analytic arcs $\gamma_k^\pm \subset \mathbb{C}$, $k = 1, 2, 3$ that can only intersect at the points of the $t$-sphere enlisted in Table 1. Due to the symmetry, $\gamma_k^-$ is the complex conjugate of $\gamma_k^+$, and the $+$-sign indicates that $\gamma_k^+$ is on the upper half plane. The index of each of these arcs is determined by the following rules.

- $\gamma_1^\pm$ is the arc that connects $t_-$ and $\alpha^2$.
- $\gamma_2^\pm$ is the arc that connects $t_+$ and $\beta^2$.
- $\gamma_3^\pm$ is the arc that connects $\beta^2$ and $\infty$.

A basic geometric analysis of the conformal map $H$ then shows the following.

- The contour $H(\gamma_1^+)$ ($H(\gamma_1^-)$) is the upper (lower) part of the interval $\Delta_1$ on $\mathcal{R}_1$, which is the same as the lower (upper) part of this interval on $\mathcal{R}_2$.
- The contour $H(\gamma_2^-)$ ($H(\gamma_2^+)$) is the upper (lower) part of the interval $\Delta_2$ on $\mathcal{R}_2$, which is the same as the lower (upper) part of this interval on $\mathcal{R}_3$.
- The contour $H(\gamma_3^-)$ ($H(\gamma_3^+)$) is the upper (lower) part of the interval $\Delta_3$ on $\mathcal{R}_3$, which is the same as the lower (upper) part of this interval on $\mathcal{R}_4$.

This also means that each of the arcs $\gamma_k := \gamma_k^+ \cup \gamma_k^-$ is an analytic closed contour on $\overline{\mathbb{C}}$, which is the common boundary component of $D_k$ and $D_{k+1}$, $k = 1, 2, 3$, where $D_k$ is defined in (3.62). The above correspondence is illustrated in Fig. 2.

Finally, we observe that $H$ maps the intervals $(t_-, 0)$ and $(-\infty, t_-)$ to the interval $(-q, 0)$ on the sheets $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. This is an immediate consequence of the description above, combined with real symmetry.
4. Auxiliary Functions

In this section, we introduce some auxiliary functions for later use.

4.1. The $\lambda$-functions. The $\lambda$-functions are defined as the anti-derivative of the $\xi$-functions (2.15):

\[
\lambda_1(z) = \int_{-q}^{z} \xi_1(s) \, ds + \int_{-q}^{p} \xi_{2,-}(s) \, ds, \quad z \in \mathbb{C}\backslash(-\infty, -q], \quad (4.1)
\]

\[
\lambda_2(z) = \int_{p}^{z} \xi_2(s) \, ds, \quad z \in \mathbb{C}\backslash(-\infty, p], \quad (4.2)
\]

\[
\lambda_3(z) = \int_{p}^{z} \xi_3(s) \, ds, \quad z \in \mathbb{C}\backslash(-\infty, p], \quad (4.3)
\]

\[
\lambda_4(z) = \int_{0}^{z} \xi_4(s) \, ds - \int_{0}^{p} \xi_{2,+}(s) \, ds, \quad z \in \mathbb{C}\backslash(-\infty, 0]. \quad (4.4)
\]

We have the following asymptotic behaviors of the $\lambda$-functions for large $z$.

**Proposition 4.1.** As $z \to \infty$, we have

\[
\lambda_1(z) = 2\alpha \sqrt{z} - \frac{1}{2} \log z + \theta_1 + \frac{2c_1}{\sqrt{z}} + O(z^{-1}), \quad (4.5)
\]

\[
\lambda_2(z) = -2\alpha \sqrt{z} - \frac{1}{2} \log z + \theta_1 - \pi i - \frac{2c_1}{\sqrt{z}} + O(z^{-1}), \quad (4.6)
\]

\[
\lambda_3(z) = -2\beta \sqrt{z} + \frac{1}{2} \log z + \theta_3 + \frac{2c_3}{\sqrt{z}} + O(z^{-1}), \quad (4.7)
\]

\[
\lambda_4(z) = 2\beta \sqrt{z} + \frac{1}{2} \log z + \theta_3 - \pi i - \frac{2c_3}{\sqrt{z}} + O(z^{-1}), \quad (4.8)
\]

for some constants $\theta_1$ and $\theta_3$, where $c_1$ and $c_3$ are the same as in (3.58).
Proof. In virtue of (3.58), it is readily seen that, as $z \to \infty$,
\[
\lambda_1(z) = 2\alpha \sqrt{z} - \frac{1}{2} \log z + \theta_1 + \frac{2c_1}{\sqrt{z}} + O(z^{-1}),
\]
\[
\lambda_2(z) = -2\alpha \sqrt{z} - \frac{1}{2} \log z + \theta_2 - \frac{2c_1}{\sqrt{z}} + O(z^{-1}),
\]
\[
\lambda_3(z) = -2\beta \sqrt{z} + \frac{1}{2} \log z + \theta_3 + \frac{2c_3}{\sqrt{z}} + O(z^{-1}),
\]
\[
\lambda_4(z) = 2\beta \sqrt{z} + \frac{1}{2} \log z + \theta_4 - \frac{2c_3}{\sqrt{z}} + O(z^{-1}),
\]
for some constants $\theta_1, \theta_2, \theta_3, \theta_4$. To show that $\theta_2 = \theta_1 - \pi i$, we note from (4.1) and (4.2) that, if $x < -q$,
\[
\lambda_{1,+}(x) - \lambda_{2,-}(x) = \int_{-q}^{x} \xi_{1,+}(s) \, ds + \int_{\pi}^{x} \xi_{2,-}(s) \, ds - \int_{\pi}^{x} \xi_{2,-}(s) \, ds
\]
\[
= \int_{-q}^{x} (\xi_{1,+}(s) - \xi_{2,-}(s)) \, ds = 0,
\]
since $\xi_{1,+}(s) = \xi_{2,-}(s)$ for $s < -q$. Inserting (4.5) and (4.6) into the above equality yields that $\theta_2 = \theta_1 - \pi i$.

In a similar manner, it is easily seen that
\[
\lambda_{3,-}(x) - \lambda_{4,+}(x) = 0, \quad x < 0.
\]
This, together with (4.7) and (4.8), implies that $\theta_4 = \theta_3 - \pi i$, as required.

This completes the proof of Proposition 4.1. □

4.2. The $\phi$-functions. For the sake of clarity, we also define the following $\phi$-functions:
\[
\phi_1(z) = \int_{-q}^{z} (\xi_1(s) - \xi_2(s)) \, ds, \quad z \in \mathbb{C} \setminus ((-\infty, -q] \cup \mathbb{R}_+), \quad (4.9)
\]
\[
\phi_2(z) = \int_{p}^{z} (\xi_2(s) - \xi_3(s)) \, ds, \quad z \in \mathbb{C} \setminus (-\infty, p], \quad (4.10)
\]
\[
\phi_3(z) = \int_{0}^{z} (\xi_3(s) - \xi_4(s)) \, ds, \quad z \in \mathbb{C} \setminus (-\infty, p], \quad (4.11)
\]
where the path of integration in $\phi_3$ emerges from $z = 0$ in the upper half plane. Note that each of the $\lambda$-functions and the $\phi$-functions is analytic in its domain of definition.

Some properties of these auxiliary functions are collected in the following proposition.

Proposition 4.2. Let $x \in \mathbb{R}$, with $\Delta_i, i = 1, 2, 3$, defined in (2.26), we have
\[
\lambda_{1,+}(x) - \lambda_{1,-}(x) = \phi_{1,+}(x), \quad x \in \Delta_1,
\]
\[
\lambda_{2,+}(x) - \lambda_{2,-}(x) = \begin{cases} 
\phi_{2,+}(x), & x \in \Delta_2, \\
-2\pi i, & x \in \Delta_3 \setminus \Delta_1, \\
-2\pi i + \phi_{1,-}(x), & x \in \Delta_1,
\end{cases}
\]
\[ \lambda_{3,+}(z) - \lambda_{3,-}(z) = \begin{cases} \phi_{2,-}(x), & x \in \Delta_2, \\ 2\pi i + \phi_{3,+}(x), & x \in \Delta_3, \end{cases} \]
\[ \lambda_{4,+}(x) - \lambda_{4,-}(x) = 2\pi i + \phi_{3,-}(x), & x \in \Delta_3, \]

and

\[ \begin{align*}
\lambda_{1,+}(x) - \lambda_{2,-}(x) &= 0, & x \in \Delta_1, \\
\lambda_{1,-}(x) - \lambda_{2,+}(x) &= 2\pi i, & x \in \Delta_1, \\
\lambda_{2,\pm}(x) - \lambda_{3,\mp}(x) &= 0, & x \in \Delta_2, \\
\phi_1(x), & x \in \Delta_3 \setminus \Delta_1, \\
\phi_2(x), & x \in \Delta_2 \setminus \Delta_1, \\
\phi_3(x), & x \in \Delta_3 \setminus \Delta_1, \\
\end{align*} \]

Furthermore, we have

\[ \begin{align*}
\phi_{1,+}(x) + \phi_{1,-}(x) &= 0, & x \in \Delta_1, \\
\phi_{2,+}(x) + \phi_{2,-}(x) &= 0, & x \in \Delta_2, \\
\phi_{3,+}(x) + \phi_{3,-}(x) &= -2\pi i, & x \in \Delta_3. \\
\end{align*} \]

**Proof.** These formulas follow directly from the definitions of the \( \lambda \)-functions and the \( \phi \)-functions given in (4.1)–(4.4) and (4.9)–(4.11), as well as Proposition 3.7. We omit the details here. \( \Box \)

Finally, we present some inequalities satisfied by the \( \phi \)-functions in the neighborhoods of their branch cuts. These inequalities will be essential in our further asymptotic analysis.

**Proposition 4.3.** For each \( i = 1, 2, 3 \), there exists an open neighborhood \( G_i \) of the interval \( \Delta_i \), such that the following inequalities hold:

\[ \text{Re} \phi_1(z) > 0, & z \in G_1 \setminus \Delta_1, \\
\text{Re} \phi_2(z) < 0, & z \in G_2 \setminus \Delta_2, \\
\text{Re} \phi_3(z) < 0, & z \in G_3 \setminus \Delta_3. \]

Furthermore, we also have that

\[ \phi_2(x) > 0, & x > p, \\
\phi_1(x) < 0, & x \in \Delta_3 \setminus \Delta_1. \]

**Proof.** We will only prove the existence of \( G_1 \), since the existence of \( G_2 \) and \( G_3 \) follow in a similar manner.

If \( x \in \Delta_1 = (-\infty, -q) \), note that \( \xi_{1,\pm}(x) = \xi_{2,\mp}(x) \), it is readily seen from (2.15), (3.4), (3.47) and (4.9) that
\[ \phi_{1, \pm}(x) = \pm \int_{-q}^{x} (\xi_{1,+}(s) - \xi_{1,-}(s)) \, ds = \pm \int_{-q}^{x} \left( C^{\mu_1}_+(s) - C^{\mu_1}_ -(s) + \frac{\alpha}{\sqrt{s_+}} - \frac{\alpha}{\sqrt{s_-}} \right) \, ds \]

\[ = \pm \int_{-q}^{x} \left( C^{\mu_1-\sigma}_+(s) - C^{\mu_1-\sigma}_-(s) \right) \, ds = \pm 2\pi i (\sigma - \mu_1)((x, -q)). \]

Thus \( \phi_{1, \pm}(x) \) is purely imaginary along \( \Delta_1 \), and the functions

\[ x \mapsto \text{Im} \phi_{1, +}(x), \quad x \mapsto \text{Im} \phi_{1, -}(x), \]

are strictly decreasing and increasing, respectively. By the Cauchy-Riemann equations, we then get immediately that \( \text{Re} \, \phi_1(z) \) is strictly positive above and below the interval \((-\infty, -q)\), assuring the existence of \( \mathcal{G}_1 \).

To conclude the first inequality in (4.15), we start with the identity

\[ \text{Re} \, \phi_2(z) = \text{Re} \int_{p}^{z} (\xi_2(s) - \xi_3(s)) \, ds \]

\[ = \text{Re} \int_{p}^{z} \left( 2C^{\mu_2}(s) - C^{\mu_1}(s) - C^{\mu_3}(s) + \frac{\beta - \alpha}{\sqrt{s}} \right) \, ds \]

\[ = 2U^{\mu_2}(z) - U^{\mu_1}(z) - U^{\mu_3}(z) + 2(\beta - \alpha) \text{Re} \, \sqrt{z} - c, \quad z \in \mathbb{C} \setminus (-\infty, p], \]

for some constant \( c \). This identity extends to \( \mathbb{C} \) by continuity, and in virtue of the equality (3.39), we get

\[ 0 = \text{Re} \, \phi_2(p) = \ell - c, \]

so \( c = \ell \). The inequality then follows directly from (3.40).

In a similar fashion, the second inequality in (4.15) follows from (3.44) and (3.42). This completes the proof of Proposition 4.3. \( \Box \)

We are now ready to carry out asymptotic analysis of the RH problem 2.5 for \( Y \).

5. First Transformation \( Y \to X \)

The aim of this transformation to simplify the block matrix \( W(x) \) appearing in the jump condition (2.20) for \( Y \). The cost we have to pay is to create a new jump on the negative real axis. Following [25,37], the main idea is to use the special properties of modified Bessel functions.

We start by setting

\[ y_{1,a}(z) = z^{(a+1)/2} I_{a+1}(2\sqrt{z}), \quad y_{2,a}(z) = z^{(a+1)/2} K_{a+1}(2\sqrt{z}), \quad (5.1) \]

where \( a > -1 \) is a real parameter. In general, we have that both \( y_{1,a} \) and \( y_{2,a} \) are analytic in the complex plane with a cut along the negative real axis. Some properties of \( y_{i,a} \) are collected in what follows for later use.

- Connection formulas (see [50, Formulas 10.34.1 and 10.34.2]): if \( x < 0 \),

\[ \begin{aligned}
(y_{1,a})_+ (x) &= e^{2a \pi i} (y_{1,a})_- (x), \\
(y_{2,a})_+ (x) &= (y_{2,a})_- (x) + i \pi e^{a \pi i} (y_{1,a})_- (x),
\end{aligned} \quad (5.2) \]

where the orientation of \( \mathbb{R}_- \) is taken from the left to the right.
Derivatives (see [50, Formulas 10.29.2 and 10.29.5]):

\[ y'_{1,a}(z) = z^{a/2} I_a(2\sqrt{z}) = y_{1,a-1}(z), \quad y'_{2,a}(z) = -z^{a/2} K_a(2\sqrt{z}) = -y_{2,a-1}(z). \]  

(5.3)

The Wronskian relation (see [50, Formula 10.28.2]):

\[ y_{1,a}(z)y'_{2,a}(z) - y'_{1,a}(z)y_{2,a}(z) = -z^a/2, \quad z \in \mathbb{C}\setminus\mathbb{R}_-. \]  

(5.4)

By (5.1) and (5.3), it is readily seen that the matrix \( W(x) \) in (2.21) can be rewritten as

\[ W(x) = w_1(x)^T w_2(x), \]  

(5.5)

where

\[ w_1(x) := (\omega_{\kappa,\alpha}(x) \omega_{\kappa+1,\alpha}(x)) = \left( \tau_1^{-\kappa} y_{1,\kappa}'(\tau_1^2 x) \quad \tau_1^{-\kappa-1} y_{1,\kappa}(\tau_1^2 x) \right), \]  

(5.6)

\[ w_2(x) := (\rho_{\nu-\kappa,\beta}(x) \rho_{\nu-\kappa+1,\beta}(x)) = \left( -\tau_2^{\nu-\kappa} y_{2,\nu-\kappa}'(\tau_2^2 x) \quad \tau_2^{\nu-\kappa-1} y_{2,\nu-\kappa}(\tau_2^2 x) \right). \]  

(5.7)

with

\[ \tau_1 := \alpha n, \quad \tau_2 := \beta n. \]

With the help of functions \( y_{i,a}(z) \) given in (5.1), we further define two \( 2 \times 2 \) matrices

\[ A_1(z) = \tau_1^{-\kappa} z^{-\frac{\kappa}{2}} \left( \begin{array}{cc} -\frac{1}{\pi i} y_{2,\kappa}'(\tau_1^2 z) & y_{1,\kappa}'(\tau_1^2 z) \\ -\frac{1}{\pi i} y_{2,\kappa}'(\tau_1^2 z) & \tau_1^{-1} y_{1,\kappa}(\tau_1^2 z) \end{array} \right), \]  

(5.8)

and

\[ A_2(z) = 2\tau_2^{\nu-\kappa} z^{-\frac{\kappa+\nu}{2}} \left( \begin{array}{cc} y_{1,\nu-\kappa}(\tau_2^2 z) & -\frac{1}{\pi i} y_{2,\nu-\kappa}(\tau_2^2 z) \\ \tau_2 y_{1,\nu-\kappa}'(\tau_2^2 z) & -\frac{1}{\pi i} y_{2,\nu-\kappa}'(\tau_2^2 z) \end{array} \right). \]  

(5.9)

In view of (5.4), it is easily seen that

\[ \det A_1(z) = \frac{1}{2\pi i \tau_1} \quad \text{and} \quad \det A_2(z) = \frac{2\tau_2}{\pi i}. \]  

(5.10)

Our first transformation is then defined by

\[ X(z) = C_X Y(z) \text{ diag}(A_1(z), A_2(z)) \text{ diag} \left( z^{\frac{\kappa}{2} \sigma_3}, z^{\frac{\kappa+\nu}{2} \sigma_3} \right), \]  

(5.11)

where \( \sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) is the third Pauli matrix and

\[ C_X = \text{ diag} \left( \sqrt{2\pi \tau_1} \begin{pmatrix} i & 0 \\ \frac{i}{4(\kappa+1)^2-1} & 1 \end{pmatrix}, \sqrt{\frac{\pi}{2\tau_2}} \begin{pmatrix} 1 & 0 \\ \frac{4(v-\kappa+1)^2-1}{16\tau_2} & i \end{pmatrix} \right). \]

By (5.10), it is easily seen that \( \det X = 1 \). We further have that \( X \) satisfies the following RH problem.
**Lemma 5.1.** The function $X$ defined in (5.11) has the following properties:

1. $X$ is defined and analytic in $\mathbb{C}\setminus\mathbb{R}$.
2. For $x \in \mathbb{R}$, $X(z)$ satisfies the jump conditions

$$X_+(x) = X_-(x) \begin{cases} I_4 + x^\kappa E_{23}, & \text{if } x > 0, \\ I_4 - |x|^\kappa E_{21} - |x|^{\nu-\kappa} E_{34}, & \text{if } x < 0, \end{cases}$$

where the $4 \times 4$ matrix $E_{ij}$ is defined in (2.25).

3. As $z \to \infty$, we have

$$X(z) = (I_4 + \mathcal{O}(z^{-1})) B(z) \begin{pmatrix} z^n e^{-2\tau_1 z^{\frac{1}{2}}} , z^n e^{2\tau_2 z^{\frac{1}{2}}} , z^{-n} e^{2\tau_2 z^{\frac{1}{2}}} , z^{-n} e^{-2\tau_2 z^{\frac{1}{2}}} \end{pmatrix},$$

where

$$B(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ z^{-\frac{1}{2}} & 1 & z^{\frac{1}{2}} & 1 \\ z^{-\frac{1}{2}} & z^{\frac{1}{2}} & 1 & 1 \\ z^{-\frac{1}{2}} & z^{\frac{1}{2}} & z^{\frac{1}{2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & i & 1 & i \\ i & 1 & i & 1 \\ 1 & i & 1 & i \\ i & 1 & i & 1 \end{pmatrix} \begin{pmatrix} z^{\kappa \sigma_3} & z^{\kappa-v \sigma_3} \\ z^{\kappa-\sigma_3} & z^{\kappa-v \sigma_3} \\ z^{\kappa-\sigma_3} & z^{\kappa-\sigma_3} & z^{\kappa-v \sigma_3} \\ z^{\kappa-\sigma_3} & z^{\kappa-\sigma_3} & z^{\kappa-\sigma_3} & z^{\kappa-v \sigma_3} \end{pmatrix}.$$ 

4. $X$ has the following local behaviors near the origin.

- For $\kappa > 0$, $\nu > 0$, $\nu \neq \kappa$

  $$X(z) = \mathcal{O}(1), \quad z \to 0.$$ 

- For $\kappa = \nu > 0$

  $$X(z) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \mathcal{O}(1), \quad z \to 0.$$ 

- For $\kappa = 0$, $\nu > 0$

  $$X(z) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \mathcal{O}(1), \quad z \to 0.$$ 

- For $\kappa = \nu = 0$

  $$X(z) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \mathcal{O}(1), \quad z \to 0.$$ 

**Proof.** To show the jump condition as stated in item (2), we see from (5.11) and (2.20) that

$$X_-(x) X_+(x) = \begin{cases} \begin{pmatrix} I_2 x^{-\frac{\kappa}{2} \sigma_3} A_{1,-}^{-1}(x) W(x) A_2(x) x^{\frac{\kappa-v}{2} \sigma_3} \\ 0 \end{pmatrix}, & \text{if } x > 0, \\ \begin{pmatrix} x^{-\frac{\kappa}{2} \sigma_3} A_{1,+}^{-1}(x) A_{1,+}(x) x^{\frac{\kappa}{2} \sigma_3} \\ 0 \end{pmatrix}^{-1 \times \frac{\kappa-v}{2} \sigma_3} A_{2,-}^{-1}(x) A_2(x) x^{\frac{\kappa-v}{2} \sigma_3}, & \text{if } x < 0. \end{cases}$$

(5.15)
By (5.5)–(5.9), it follows from (5.4) and a straightforward calculation that

\[ A_1^{-1}(x)w_1(x)^T = \left( \begin{array}{c} 0 \\ \frac{z}{x^2} \end{array} \right), \quad w_2(x)A_2(x) = \left( x^{\frac{v-\kappa}{2}} 0 \right), \quad (5.16) \]

so

\[ x^{-\frac{v}{2}} A_1^{-1}(x)W(x)A_2(x)x^{\frac{v-\kappa}{2}} = x^{-\frac{v}{2}} A_1^{-1}(x)w_1(x)^T w_2(x)A_2(x)x^{\frac{v-\kappa}{2}} \]

\[ = x^{-\frac{v}{2}} \left( \begin{array}{cc} 0 & 0 \\ \frac{z}{x^2} & 0 \end{array} \right) x^{\frac{v-\kappa}{2}} = \left( \begin{array}{cc} 0 & 0 \\ \frac{z}{x^2} & 0 \end{array} \right). \quad (5.17) \]

Similarly, by making use of (5.2) and (5.4), one can check that if \( x < 0 \),

\[ x^{-\frac{v}{2}} A_1^{-1}(x)A_1^+(x)x^{\frac{v}{2}} = x^{-\frac{v}{2}} \left( \begin{array}{cc} e^{-\kappa \pi i} & 0 \\ -1 & e^{\kappa \pi i} \end{array} \right) x^{\frac{v}{2}} = \left( \begin{array}{cc} 0 & 0 \\ -|x|^\kappa & 1 \end{array} \right) \]

and

\[ x^{-\frac{v}{2}} A_2^{-1}(x)A_2^+(x)x^{\frac{v}{2}} = x^{-\frac{v}{2}} \left( \begin{array}{cc} e^{(\nu-\kappa)\pi i} & -1 \\ 0 & e^{(\kappa-\nu)\pi i} \end{array} \right) x^{\frac{v}{2}} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right). \quad (5.18) \]

Inserting (5.17)–(5.18) into (5.15) gives us (5.12).

To establish the large \( z \) behavior of \( X \), it suffices to derive the asymptotics of \( A_1 \) and \( A_2 \). We follow closely [37] and start with known asymptotic formulas for the Bessel functions [50, Formulas (10.40.1) and (10.40.2)] to obtain

\[ y_{1,a}(\tau^2 z) = \frac{1}{2\sqrt{\pi}} \tau^{a+\frac{1}{2}} \frac{2\tau^2}{z^2} e^{2\tau^2 \frac{1}{2}} \times \left( 1 - \frac{4(a + 1)^2 - 1}{16\tau^2} + \frac{(4(a + 1)^2 - 1)(4(a + 1)^2 - 9)}{512\tau^2 z} + O(z^{-\frac{3}{2}}) \right), \]

\[ y_{2,a}(\tau^2 z) = \sqrt{\pi} \tau^{a+\frac{1}{2}} \frac{2\tau^2}{z^2} e^{-2\tau^2 \frac{1}{2}} \times \left( 1 + \frac{4(a + 1)^2 - 1}{16\tau^2} + \frac{(4(a + 1)^2 - 1)(4(a + 1)^2 - 9)}{512\tau^2 z} + O(z^{-\frac{3}{2}}) \right), \]

for \( z \to \infty \) with \( |\text{arg} \, z| < \pi \) and \( \tau > 0 \). This, together with (5.8) and (5.3), implies that, as \( z \to \infty \),

\[ A_1(z) = -\frac{i}{2\sqrt{\pi \tau_1}} z^{\frac{\sigma_1}{2}} \times \left[ \left( \begin{array}{cc} 1 & i \\ -1 & -i \end{array} \right) + \frac{D_1}{z^2} \left( \begin{array}{cc} 1 & -i \\ -1 & -1 \end{array} \right) + \frac{D_2}{z} \left( \begin{array}{cc} 1 & i \\ -1 & -1 \end{array} \right) + O(z^{-\frac{3}{2}}) \right] e^{-2\tau_1 z^{\frac{1}{2}} \frac{\sigma_3}{2}}, \]
where
\[
D_1 = \frac{1}{16\tau_1} \begin{pmatrix} 4\kappa^2 - 1 & 0 \\ 0 & 4(\kappa + 1)^2 - 1 \end{pmatrix},
\]
\[
D_2 = \frac{1}{512\tau_1^2} \begin{pmatrix} (4\kappa^2 - 1)(4\kappa^2 - 9) & 0 \\ 0 & (4(\kappa + 1)^2 - 1)(4(\kappa + 1)^2 - 9) \end{pmatrix}.
\]

Using the identity
\[
\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix},
\]
we further simplify the previous formula to
\[
A_1(z) = -\frac{i}{2\sqrt{\pi \tau_1}} z^{-\frac{\tau_1}{4}} \left[ I_2 + \frac{D_1}{z^2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + O\left(z^{-\frac{3}{2}}\right) \right] \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} i \right) e^{-2\tau_1 z \frac{1}{2}\sigma_3}.
\]
On account of the fact that
\[
\frac{z^{-\frac{\tau_1}{4}}}{z^{-\frac{1}{2}}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -z^{-1} \\ -1 & 0 \end{pmatrix} z^{-\frac{\tau_1}{4}},
\]
we finally arrive at
\[
A_1(z) = -\frac{i}{2\sqrt{\pi \tau_1}} \left[ I_2 + D_1 \begin{pmatrix} 0 & -z^{-1} \\ -1 & 0 \end{pmatrix} + O\left(z^{-1}\right) \right] z^{-\frac{\tau_1}{4}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} i e^{-2\tau_1 z \frac{1}{2}\sigma_3}
\]
\[
= -\frac{i}{2\sqrt{\pi \tau_1}} \left[ \begin{pmatrix} 1 \\ -\frac{4(\kappa + 1)^2 - 1}{16\tau_1} \end{pmatrix} + O\left(z^{-1}\right) \right] z^{-\frac{\tau_1}{4}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} i e^{-2\tau_1 z \frac{1}{2}\sigma_3}
\]
\[
= \frac{1}{2\sqrt{\pi \tau_1}} \begin{pmatrix} -i \\ \frac{4(\kappa + 1)^2 - 1}{16} \end{pmatrix} + O\left(z^{-1}\right) \right] z^{-\frac{\tau_1}{4}} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}} e^{-2\tau_1 z \frac{1}{2}\sigma_3}, \tag{5.19}
\]
which is valid for $z \to \infty$ along $\mathbb{C} \setminus \mathbb{R}_-$.

In a similar way, we also obtain that if $z \to \infty$ along $\mathbb{C} \setminus \mathbb{R}_-$,
\[
A_2(z) = -\frac{\sqrt{\tau_2}}{\pi} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{4(\nu - \kappa + 1)^2 - 1}{16\tau_2} \end{pmatrix} + O\left(z^{-1}\right) \right] z^{\frac{\tau_2}{4}} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} e^{2\tau_2 z \frac{1}{2}\sigma_3}
\]
\[
= \frac{\sqrt{2\tau_2}}{\pi} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{4(\nu - \kappa + 1)^2 - 1}{16} \end{pmatrix} + O\left(z^{-1}\right) \right] z^{\frac{\tau_2}{4}} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} e^{2\tau_2 z \frac{1}{2}\sigma_3}. \tag{5.20}
\]

A combination of (5.11), (5.19) and (5.20) then gives us (5.13).

Finally, it follows from the known behavior of the modified Bessel functions near the origin (cf. [50, Formulas 10.30.1–10.30.3]) that, as $z \to 0$,
\[
y_1(z) \sim \frac{1}{\Gamma(a + 2)} z^{a+1}, \quad y'_1(z) \sim \frac{1}{\Gamma(a + 1)} z^a,
\]
\[
y_2(z) \sim \frac{1}{2} \Gamma(a + 1), \quad y'_2(z) \sim \begin{cases} -\frac{1}{2} \Gamma(a), & a > 0, \\
\frac{1}{2} \log(z), & a = 0, \\
-\frac{1}{2} \Gamma(-a) z^a, & a < 0. \end{cases}
\]

The behavior of $X$ near the origin in item (4) then follows from a straightforward calculation.

This completes the proof of Lemma 5.1. \qed
6. Second Transformation $X \to T$

With the $\lambda$-functions given in (4.1)–(4.4), we define the second transformation $X \to T$ by

$$T(z) = C_T X(z) \text{diag} \left( e^{n\lambda_1(z)}, e^{n\lambda_2(z)}, e^{n\lambda_3(z)}, e^{n\lambda_4(z)} \right), \quad (6.1)$$

where

$$C_T = (I_4 - 2nc_1 E_{21} + 2nc_3 i E_{34}) \text{diag} \left( e^{-n\theta_1}, e^{-n\theta_1}, e^{-n\theta_3}, e^{-n\theta_3} \right)$$

with the constants $c_1, c_3, \theta_1, \theta_3$ as in Proposition 4.1. Then, $T$ satisfies the following RH problem.

**Lemma 6.1.** The function $T$ defined in (6.1) has the following properties:

1. $T$ is defined and analytic in $\mathbb{C} \setminus \mathbb{R}$.
2. For $x \in \mathbb{R}$, $T$ satisfies the jump condition

$$T_+(x) = T_-(x) J_T(x), \quad (6.2)$$

where

$$J_T(x) = \begin{cases} 
I_4 + x^\kappa e^{-n\phi_2(x)} E_{23}, & x > p, \\
\text{diag} \left( 1, e^{n\phi_2(x)}, e^{n\phi_2(-x)}, 1 \right) + x^\kappa E_{23}, & x \in \Delta_2, \\
\text{diag} \left( 1, 1, e^{n\phi_3(x)}, e^{n\phi_3(-x)} \right) - |x|^\kappa e^{n\phi_1(x)} E_{21} - |x|^{\nu-\kappa} E_{34}, & x \in \Delta_3 \setminus \Delta_1, \\
\text{diag} \left( e^{-n\phi_1(-x)}, e^{-n\phi_1(x)}, e^{n\phi_3(x)}, e^{n\phi_3(-x)} \right) - |x|^\kappa E_{21} - |x|^{\nu-\kappa} E_{34}, & x \in \Delta_1, 
\end{cases} \quad (6.3)$$

and where the $\phi$-functions are defined in (4.9)–(4.11).

3. As $z \to \infty$, we have

$$T(z) = (I_4 + O(z^{-1})) B(z), \quad (6.4)$$

where the function $B$ is given in (5.14).

4. The matrix $T$ has the same behavior as $X$ as $z \to 0$.

**Proof.** To show the jump condition (6.2), it is readily seen from (6.1) and (5.12) that

$$J_T(x) = \begin{cases} 
\text{diag} \left( e^{n(\lambda_j + (x) - \lambda_j - (x))} \right)_{j \leq 4} + x^\kappa e^{n(\lambda_3 + (x) - \lambda_2 - (x))} E_{23}, & \text{if } x > 0, \\
\text{diag} \left( e^{n(\lambda_j + (x) - \lambda_j - (x))} \right)_{j \leq 4} - |x|^\kappa E_{21} e^{n(\lambda_1 + (x) - \lambda_2 - (x))} - |x|^{\nu-\kappa} e^{n(\lambda_4 + (x) - \lambda_3 - (x))} E_{34}, & \text{if } x < 0. 
\end{cases} \quad (6.3)$$

This formula simplifies further to (6.3) with the aid of Proposition 4.2.

For the asymptotic behavior of $T$ near infinity, we observe from (5.13) and Proposition 4.1 that, as $z \to \infty$, the $\lambda$-functions given in (4.1)–(4.4), we define the second transformation $X \to T$ by
In a similar spirit, we use (4.14) to see that for

\[ X(z) \text{ diag} \left( e^{n\lambda_1(z)} , e^{n\lambda_2(z)} , e^{n\lambda_3(z)} , e^{n\lambda_4(z)} \right) \]

\[ = \left( \text{diag} \left( e^{n\theta_1(z)} , e^{n\theta_2(z)} , e^{n\theta_3(z)} , e^{n\theta_4(z)} \right) + O(z^{-1}) \right) B(z) \]

\[ \times \left( \text{diag} \left( 1 + \frac{2nc_1}{\sqrt{\varepsilon}} , 1 - \frac{2nc_1}{\sqrt{\varepsilon}} , 1 + \frac{2nc_3}{\sqrt{\varepsilon}} , 1 - \frac{2nc_3}{\sqrt{\varepsilon}} \right) + O(z^{-1}) I_4 \right). \]

By moving the last diagonal matrix in the above formula to the left, it follows that

\[ X(z) \text{ diag} \left( e^{n\lambda_1(z)} , e^{n\lambda_2(z)} , e^{n\lambda_3(z)} , e^{n\lambda_4(z)} \right) = C_T^{-1} (I_4 + O(z^{-1})) B(z). \]

This, together with (6.1), implies (6.4).

Finally, since each of the \( \lambda \)-functions is bounded near the origin, it is clear that the matrix \( T \) has the same behavior as \( X \) as \( z \to 0 \).

This completes the proof of Lemma 6.1. \( \square \)

7. Third Transformation \( T \to S \)

The third transformation involves the so-called lens opening. The goal of this step is to convert the highly oscillatory jumps into a more convenient form on the original contours while creating extra jumps tending to the identity matrices exponentially fast on the new contours. This transformation is based on the following classical factorizations:

\[ \begin{pmatrix} e^{-u} & v \\ 0 & e^u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & v \\ -v^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^{-1} & 1 \end{pmatrix} - u \]

and

\[ \begin{pmatrix} e^{-u} & 0 \\ v & e^u \end{pmatrix} = \begin{pmatrix} 1 & v^{-1} e^{-u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -v^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & v^{-1} e^u \\ 0 & 1 \end{pmatrix}. \]

Note that the jump matrices in (6.3) can be viewed as \( 2 \times 2 \) block matrices, the factorizations above can be easily applied. For instance, if \( x \in \Delta_2 \), it follows from (4.13) that

\[ J_T(x) = \text{diag} \left( 1, \begin{pmatrix} e^{n\phi_2,+(x)} & x^k \\ 0 & e^{n\phi_2,-(x)} \end{pmatrix}, 1 \right) \]

\[ = \text{diag} \left( 1, \begin{pmatrix} 1 & 0 \\ -x^{-\kappa} e^{n\phi_2,-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & x^k \\ 0 & 1 \end{pmatrix}, 1 \right) \]

\[ = (I_4 + x^{-\kappa} e^{n\phi_2,-(x)} E_{32}) \text{ diag} \left( 1, \begin{pmatrix} 0 & x^k \\ -x^{-\kappa} & 0 \end{pmatrix}, 1 \right) (I_4 + x^{-\kappa} e^{n\phi_2,+(x)} E_{32}). \]

(7.1)

In a similar spirit, we use (4.14) to see that for \( x \in \Delta_3 \setminus \Delta_1 \),

\[ J_T(x) = \text{diag} \left( I_2, \begin{pmatrix} e^{n\phi_3,+(x)} & -|x|^\nu x^{-\kappa} e^{n\phi_3,-(x)} \\ 0 & e^{n\phi_3,-(x)} \end{pmatrix} \right) - |x|^\nu e^{n\phi_1(x)} E_{21} \]

\[ = (I_4 - c_{\kappa-\nu} x^{-\nu} e^{n\phi_3,-(x)} E_{43}) \]
\[
\times \left[ \text{diag}\left( I_2, \begin{pmatrix} 0 & -|x|^{\nu-k} \\ |x|^{\kappa} & 0 \end{pmatrix} \right) - |x|^\kappa e^{n\phi_1(x)} E_{21} \right]
\times (I_4 - c_{v-k} x_+^{\kappa-v} e^{n\phi_3,x}(x) E_{43}), \tag{7.2}
\]
where \( c_{\alpha} \) is defined in (2.27), and, finally, using (4.12), we obtain that for \( x \in \Delta_1 \),
\[
J_T(x) = \text{diag}\left( \begin{pmatrix} e^{-n\phi_1,-(x)} & 0 \\ -|x|^{\kappa} & e^{-n\phi_1,+(x)} \end{pmatrix}, \begin{pmatrix} e^{n\phi_3,-(x)} & -|x|^{\nu-k} \\ 0 & 0 \end{pmatrix} \right)
= (I_4 - c_{v-k} x_-^{\kappa-v} e^{n\phi_3,-(x)} E_{43})(I_4 - c_{\kappa} x_-^{\kappa-v} e^{-n\phi_1,-(x)} E_{12})
\times \text{diag}\left( \begin{pmatrix} 0 & |x|^{-\kappa} \\ -|x|^{\kappa} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -|x|^{\nu-k} \\ |x|^{\kappa} & 0 \end{pmatrix} \right)
\times (I_4 - c_{\kappa} x_+^{\kappa-v} e^{-n\phi_1,+(x)} E_{12})(I_4 - c_{v-k} x_+^{\kappa-v} e^{n\phi_3,+(x)} E_{43}). \tag{7.3}
\]

For each \( k = 1, 2, 3 \), we set simply connected domains \( \mathcal{L}_k^\pm \) (the lenses) on the \( \pm \)-side of \( \Delta_k \), with oriented boundaries \( \partial \mathcal{L}_k^\pm \cup \Delta_k \) as shown in Fig. 3. Moreover, it is required that
\[
\partial \mathcal{L}_1^\pm \subset \mathcal{L}_3^\pm \quad \text{and} \quad \overline{\mathcal{L}_k^\pm} \setminus \Delta_k \subset G_k, \tag{7.4}
\]
where the open neighborhood \( G_k \) of the interval \( \Delta_k \) is given in Proposition 4.3.

Based on the decompositions of \( J_T \) given in (7.1)–(7.3) and also on the lenses just defined, the third transformation reads
\[
S(z) = T(z) \begin{cases} 
(I_4 \mp z^{-\kappa} e^{n\phi_2(z) E_{32}}), & z \in \mathcal{L}_2^\pm, \\
(I_4 \pm c_{v-k} z^{\kappa-v} e^{n\phi_1(z) E_{43}}), & z \in \mathcal{L}_3^\pm \setminus \mathcal{L}_1^{\pm}, \\
(I_4 \pm c_{v-k} z^{\kappa-v} e^{n\phi_3(z) E_{43}})(I_4 \pm c_{\kappa} z^{-\kappa} e^{-n\phi_1(z) E_{12}}), & z \text{ outside the lenses.} \\
I_4, & z \notin \mathcal{L}_1^\pm.
\end{cases} \tag{7.5}
\]
Since both \( \kappa \) and \( \nu \) are integers, it is easily seen that
\[c_{v-k} = c_{\kappa-v}, \quad c_{\kappa} = c_{-\kappa}.\]
Also note that the factors of the form \((I_4 + (\ast) E_{12})\) and \((I_4 + (\ast) E_{43})\) appearing above commute, it is then straightforward to check that the matrix \( S \) satisfies the following RH problem.
**RH Problem 7.1.** The function $S$ defined in (7.5) has the following properties:

1. $S$ is defined and analytic in $\mathbb{C} \setminus \Gamma_S$, where

   $$\Gamma_S := \mathbb{R} \cup \left( \bigcup_{j=1}^{3} \partial L_j^\pm \right).$$

   (7.6)

2. For $z \in \Gamma_S$, $S$ satisfies the jump condition

   $$S^+(z) = S^-(z) J_S(z),$$

   where

   $$J_S(z) = \begin{cases} 
   J_T(z) = I_4 + x^4 e^{-n\phi_2(z)} E_{23}, & z \in (p, +\infty), \\
   I_4 + z^{-\kappa} e^{n\phi_2(z)} E_{32}, & z \in \partial L_2^\pm, \\
   I_4 - c_{\nu-\kappa} z^{\nu-\kappa} e^{n\phi_3(z)} E_{43}, & z \in \partial L_3^\pm, \\
   I_4 - c_{\kappa} z^{\kappa} e^{-n\phi_1(z)} E_{12}, & z \in \partial L_1^\pm, \\
   \begin{pmatrix} 1 & 0 \\
   -z^{-\kappa} & 1 \end{pmatrix}, & z \in \Delta_2, \\
   \begin{pmatrix} I_2 & 0 \\
   |z|^{\nu-\kappa} & 0 \end{pmatrix} - |z|^{\nu-\kappa} e^{n\phi_1(z)} E_{21}, & z \in \Delta_3 \setminus \Delta_1, \\
   \begin{pmatrix} 0 & |z|^{\nu-\kappa} \\
   -|z|^{\nu-\kappa} & 0 \end{pmatrix}, & z \in \Delta_1. 
   \end{cases}
   \quad (7.7)

3. As $z \to \infty$, we have

   $$S(z) = (I_4 + o(z^{-1})) B(z),$$

   where the function $B$ is given in (5.14).

4. As $z \to 0$, $S$ has the same behavior as $T$ provided $z \to 0$ outside the lenses that end in 0.

**8. Global Parametrix**

By (7.4), (7.7) and Proposition 4.3, it is easily seen that, as $n \to \infty$,

$$J_S(z) = I_4 + o(1), \quad z \in \bigcup_{j=1}^{3} \partial L_j^\pm \cup (p, +\infty),$$

uniformly valid for $z$ bounded away from the endpoints of the sets $\Delta_k$, $k = 1, 2, 3$. This, together with the second inequality in (4.15), leads us to the following model RH problem, also called *global parametrix* RH problem.
RH Problem 8.1. We look for a $4 \times 4$ matrix-valued function $G$ satisfying the following properties:

1. $G$ is defined and analytic in $\mathbb{C} \setminus (-\infty, p]$.
2. $G$ satisfies the jump condition
   \[ G_+(x) = G_-(x)J_G(x), \]
   where
   \[ J_G(x) = \begin{cases} 
   \text{diag} \left( \begin{pmatrix} 0 & |x|^{-\kappa} \\
   -|x|^\kappa & 0 \end{pmatrix}, \begin{pmatrix} 0 & -|x|^{\nu-\kappa} \\
   |x|^{\kappa-\nu} & 0 \end{pmatrix} \right), & x \in \Delta_1, \\
   \text{diag} \left( \begin{pmatrix} 1 & 0 \\
   -x^{-\kappa} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\
   x^{\kappa} & 1 \end{pmatrix} \right), & x \in \Delta_2, \\
   \text{diag} \left( \begin{pmatrix} I_2 & 0 \\
   |x|^{\kappa-\nu} & 0 \end{pmatrix} \right), & x \in \Delta_3 \setminus \overline{\Delta_1}. \]

3. As $z \to \infty$ away from $\mathbb{R}_-$, we have
   \[ G(z) = (I_4 + \mathcal{O}(z^{-1}))B(z), \]
   where $B$ is as in (5.14).

Note that we are not imposing any endpoint behaviors for $G$, so the solution to RH problem 8.1 might not be unique. Nevertheless, we will construct some $G$ explicitly that will be enough to finish the further asymptotic analysis. The construction relies on the uniformization map of the Riemann surface described in Sect. 3.8.

8.1. Construction of the global parametrix for $\kappa = \nu = 0$. In this section, we will solve the model RH problem 8.1 with $\kappa = \nu = 0$, whose solution will be denoted by $G_0$. The basic idea is to lift the original RH problem to the Riemann surface $\mathcal{R}$, and then transform the matrix-valued RH problem into several scalar RH problems on the $t$-plane with the aid of the rational parametrization (3.60).

To proceed, let $t_k = t_k(z), k = 1, 2, 3, 4$, be the inverse of the map $z = z(t)$ in (3.63) restricted to $\mathcal{R}_k$, i.e.,

\[ t_k : \mathcal{R}_k \to \mathbb{C}. \]

We then have the following proposition.

Proposition 8.2. A solution of the model RH problem 8.1 with $\kappa = \nu = 0$ is given by

\[ G_0(z) = \left(G_k(t_j(z))\right)^4_{k,j=1}, \]

where

\[ G_1(t) = \varepsilon_1 \left( \frac{t - \alpha^2}{t - t_-} \right)^{\frac{1}{2}} \left( \frac{t - \beta^2}{t - t_+} \right)^{\frac{3}{2}}, \]
\[ G_2(t) = \varepsilon_2 \frac{t \left( t - \beta^2 \right)^{\frac{1}{2}} \left( t - \alpha^2 \right) \left( t - t_- \right)^{-\frac{1}{2}}}{\left( t - t_+ \right)^{\frac{3}{2}}}. \]
\[ G_3(t) = \epsilon_3 \left( t - \alpha^2 \right) \left( \frac{t - \alpha^2}{t - t_-} \right)^{\frac{1}{2}} (t - t_+)^{-\frac{1}{2}} (t - \beta^2)^{-\frac{1}{2}}, \quad (8.6) \]

\[ G_4(t) = \epsilon_4 \left( t - \alpha^2 \right) \left( \frac{t - \alpha^2}{t - t_-} \right)^{\frac{1}{2}} (t - t_+)^{-\frac{1}{2}} (t - \beta^2)^{\frac{1}{2}}. \quad (8.7) \]

Here, \( t_\pm \) is given in (3.64), the branch cut for the root of \( (t - \alpha^2)^{\frac{1}{2}} (t - t_-)^{\frac{1}{2}} \) is taken along \( \gamma_1^+ \), the branch cuts of \( (t - t_+)^{\frac{1}{2}} \) and \( (t - \beta^2)^{\frac{1}{2}} \) are taken along \( \gamma_2^- \) and \( \gamma_3^- \), respectively, and \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) are explicitly computable non-zero constants.

**Proof.** Suppose that

\[ G_0(z) = \left( g_{k,j}(z) \right)_{k,j=1}^4 \]

solves the model RH problem 8.1 with \( \kappa = \nu = 0 \). We lift the RH problem to the Riemann surface \( \mathcal{R} \) by treating each entry \( g_{k,j}(z) \) of the \( k \)-th row of \( G_0 \) as defined on the sheet \( \mathcal{R}_j \) of \( \mathcal{R} \) and define

\[ g_k : \mathcal{R} \to \mathbb{C}, \quad g_k|_{\mathcal{R}_j} = g_{k,j}, \quad j, k = 1, 2, 3, 4. \quad (8.8) \]

It is then easily seen that the RH problem for \( G_0 \) is equivalent to the following RH problem on \( \mathcal{R} \).

**RH Problem 8.3.** For \( k = 1, 2, 3, 4 \), the function \( g_k \) defined in (8.8) has the following properties:

1. \( g_k \) is analytic in \( \mathcal{R} \setminus \Gamma_g \), where
   \[ \Gamma_g := H(\gamma_1^+) \cup H(\gamma_2^-) \cup H(\gamma_3^-) \]
   with \( H \) being the rational parametrization (3.60). Here, each of the contours \( H(\gamma_1^+), H(\gamma_2^-) \) and \( H(\gamma_3^-) \) is a real interval on \( \mathcal{R} \) with the orientation taken from the left to the right.
2. \( g_k \) satisfies the jump condition
   \[ g_{k,+}(z) = -g_{k,-}(z), \quad z \in \Gamma_g. \]
3. \( g_k \) has the following large \( z \) asymptotic behaviors.
   - As \( z \to \infty \) along \( \mathcal{R}_1 \),
     \[ g_1(z) = \frac{z^{-\frac{1}{4}}}{\sqrt{2}} (1 + \mathcal{O}(z^{-\frac{1}{2}}), \quad g_2(z) = \frac{iz^{\frac{1}{4}}}{\sqrt{2}} (1 + \mathcal{O}(z^{-1})), \]
     \[ g_3(z) = g_4(z) = \mathcal{O}(z^{-\frac{3}{4}}). \]
   - As \( z \to \infty \) along \( \mathcal{R}_2 \),
     \[ g_1(z) = \frac{iz^{-\frac{1}{4}}}{\sqrt{2}} (1 + \mathcal{O}(z^{-\frac{1}{2}}), \quad g_2(z) = \frac{z^{\frac{1}{4}}}{\sqrt{2}} (1 + \mathcal{O}(z^{-1})), \]
     \[ g_3(z) = g_4(z) = \mathcal{O}(z^{-\frac{3}{4}}). \]
As $z \to \infty$ along $R_3$,

$$g_1(z) = g_2(z) = O(z^{-\frac{3}{4}}), \quad g_3(z) = \frac{z^{\frac{1}{4}}}{\sqrt{2}} (1 + O(z^{-1})),
$$

$$g_4(z) = \frac{iz^{\frac{1}{4}}}{\sqrt{2}} (1 + O(z^{-\frac{1}{2}})).$$

As $z \to \infty$ along $R_4$,

$$g_1(z) = g_2(z) = O(z^{-\frac{3}{4}}), \quad g_3(z) = \frac{iz^{\frac{1}{4}}}{\sqrt{2}} (1 + O(z^{-1})),
$$

$$g_4(z) = \frac{z^{\frac{1}{4}}}{\sqrt{2}} (1 + O(z^{-\frac{1}{2}})).$$

Using the rational parametrization (3.60), we further transfer the above RH problem for $g_k$ to a scalar RH problem on the $t$-complex plane by setting

$$G_k(t) = g_k(z(t)). \quad (8.9)$$

The RH problem for $g_k$ is then equivalent to the following RH problem for $G_k$.

**RH Problem 8.4.** For $k = 1, 2, 3, 4$, the function $G_k$ defined in (8.9) has the following properties:

1. $G_k$ is analytic in $\mathbb{C}\setminus \Gamma_G$, where

   $$\Gamma_G = H^{-1}(\Gamma_g) = \gamma_1^+ \cup \gamma_2^- \cup \gamma_3^-.$$

2. $G_k$ satisfies the jump condition

   $$G_{k,+}(t) = -G_{k,-}(t), \quad t \in \Gamma_G.$$

3. As $t \to \alpha^2$, we have

   $$G_1(t) = \frac{1}{\sqrt{2}} \left( \frac{\alpha}{\beta^2 - \alpha^2} \right)^{-\frac{1}{2}} (\alpha^2 - t)^{\frac{1}{4}} (1 + O(t - \alpha^2)),
   $$

   $$G_2(t) = \frac{i}{\sqrt{2}} \left( \frac{\alpha}{\beta^2 - \alpha^2} \right)^{\frac{1}{2}} (\alpha^2 - t)^{-\frac{1}{4}} (1 + O(t - \alpha^2)),
   $$

   $$G_3(t) = O((t - \alpha^2)^{\frac{3}{2}}), \quad k = 3, 4.
   $$

4. As $t \to \beta^2$, we have

   $$G_1(t) = (O(t - \beta^2)^{\frac{3}{2}}), \quad k = 1, 2,
   $$

   $$G_3(t) = \frac{1}{\sqrt{2}} \left( \frac{\alpha}{\beta^2 - \alpha^2} \right)^{\frac{1}{2}} (\beta^2 - t)^{-\frac{1}{2}} (1 + O(t - \beta^2)),
   $$

   $$G_4(t) = \frac{i}{\sqrt{2}} \left( \frac{\alpha}{\beta^2 - \alpha^2} \right)^{-\frac{1}{2}} (\beta^2 - t)^{\frac{1}{4}} (1 + O(t - \beta^2)).
It is straightforward to check that the function $G_k$ defined in (8.4)–(8.7) with specified branch cuts satisfies the RH problem 8.4. In particular, the constants $e_\bullet$ are determined by the explicit leading coefficients given in items (3) and (4) of the above RH problem. This completes the proof of Proposition 8.2. □

From (8.4)–(8.7), it follows that

$$G_k(t) = \begin{cases} e_k t (1 + O(t^{-1})), & t \to \infty, \\ O((t - t_+)^{-1}), & t \to t_+, \\ O((t - t_-)^{-1}), & t \to t_- \end{cases},$$

This, together with (8.3) and Proposition 3.9, implies that the following rough estimate of $G_0$ near the endpoints of the jump contours:

$$G_0(z) = \begin{cases} O(z^{-\frac{1}{4}}), & z \to 0, \\ O((z - p)^{-\frac{1}{4}}), & z \to p, \\ O((z + q)^{-\frac{1}{4}}), & z \to -q. \end{cases} \tag{8.10}$$

8.2. Construction of the global parametrix for general $\kappa$ and $\nu$. With the aid of $G_0$ in (8.3), we could construct the global parametrix for general parameters $\kappa$ and $\nu$. To state the result, let us define

$$\log(t - \alpha^2) : \mathbb{C}\setminus(\gamma_1^+ \cup (-\infty, t_-]) \to \mathbb{C}, \quad \log(t - \beta^2) : \mathbb{C}\setminus\gamma_3^- \to \mathbb{C}, \tag{8.11}$$

where both the branches are chosen to be purely real for large positive values of $t$, and further set

$$F_k(z) = e^{-\kappa \log(t_k(z) - \alpha^2) - (\kappa - \nu) \log(t_k(z) - \beta^2)}, \quad z \in \mathcal{R}_k, \quad k = 1, 2, 3, 4. \tag{8.12}$$

**Proposition 8.5.** A solution of the model RH problem 8.1 is given by

$$G(z) = \text{diag}(f_1, f_1, f_3, f_3)G_0(z) \times \text{diag}\left( e_\kappa F_1(z), F_2(z)e^{-\kappa \log z}, F_3(z), e_{\kappa - \nu} F_4(z)e^{(\nu - \kappa) \log z} \right), \tag{8.13}$$

where $G_0$ given in (8.3) solves the RH problem 8.1 with $\kappa = \nu = 0$, the function $F_k$, $k = 1, 2, 3, 4$, is defined in (8.12), the branch cut of $\log z$ is taken along the negative real axis, and $f_1, f_3$ are explicitly computable non-zero constants.

**Proof.** By the definition (8.11), it is easily seen that the maps

$$\mathcal{R}_k \ni z \mapsto \log(t_k(z) - \alpha^2), \ \log(t_k(z) - \beta^2),$$

satisfy the following boundary relations:

- if $x \in \Delta_1$,

$$\begin{align*}
(\log(t_1(z) - \alpha^2))_+ - (\log(t_2(z) - \alpha^2))_- &= -2\pi i, \\
(\log(t_1(z) - \alpha^2))_- - (\log(t_2(z) - \alpha^2))_+ &= 0,
\end{align*}$$
• if \( x \in \Delta_3 \),

\[
\log(t_3(z) - \beta^2)_+ - \log(t_4(z) - \beta^2)_- = 0,
\]

\[
\log(t_3(z) - \beta^2)_- - \log(t_4(z) - \beta^2)_+ = 2\pi i,
\]

• if \( x \in \Delta_3 \setminus \bar{\Delta}_1 \),

\[
\log(t_2(z) - \alpha^2)_+ - \log(t_2(z) - \alpha^2)_- = -2\pi i,
\]

and are otherwise analytic in their domains of definition. As a consequence, the function

\[ F : \mathbb{R} \to \mathbb{C}, \quad F|_{\mathbb{R}_k} = F_k, \quad k = 1, 2, 3, 4, \]

with \( F_k \) given in (8.12) extends to a meromorphic function on \( \mathbb{R} \), and it is easy to check that the function \( G \) defined in (8.13) satisfies the jump condition (8.1).

Finally, in virtue of the expansions in (3.65), we have that, as \( z \to \infty \),

\[
F_1(z) = \frac{\epsilon_{x_K}}{f_1} z^{\frac{x}{3}}(1 + \mathcal{O}(z^{-1/2})), \quad F_2(z) = \frac{1}{f_1} z^{\frac{x}{3}}(1 + \mathcal{O}(z^{-1/2})),
\]

\[
F_3(z) = \frac{1}{f_3} z^{\frac{x}{3}}(1 + \mathcal{O}(z^{-1/2})), \quad F_4(z) = \frac{\epsilon_{x_K}}{f_3} z^{\frac{x}{3}}(1 + \mathcal{O}(z^{-1/2})),
\]

for some non-zero constants \( f_1, f_3 \), which implies the large \( z \) asymptotics stated in (8.2).

This completes the proof of Proposition 8.5. \( \square \)

By Proposition 3.9, it is also readily seen that

\[
G(z) = \begin{cases} \mathcal{O}((z - p)^{-\frac{1}{2}}), & z \to p, \\ \mathcal{O}((z + q)^{-\frac{1}{2}}), & z \to -q. \end{cases}
\]

The local behavior of \( G \) near the origin, however, is crucial in our further analysis. By setting

\[ \mathcal{U}^+ = \hat{\mathcal{U}} \text{ diag}(\omega^{\frac{x}{3}} \sigma_3, 1), \quad \mathcal{U}^- = \mathcal{U}^+ \text{ diag} \left( \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, 1 \right), \quad (8.14) \]

where

\[
\hat{\mathcal{U}} = \begin{pmatrix} \omega^- & \omega^+ & 1 \\ -1 & -1 & -1 \\ \omega^+ & \omega^- & 1 \end{pmatrix} \quad (8.15)
\]

with \( \omega = e^{2\pi i/3}, \omega^\pm = \omega^{\pm 1} \), we have the following proposition regarding the asymptotics of \( G \) near the origin.

**Proposition 8.6.** The matrix

\[ \hat{G}(z) := G(z) \text{ diag} \left( 1, z^{\frac{A}{2}} (\mathcal{U}^\pm)^{-1} z^{\frac{B}{2}} \right), \quad \pm \text{Im} z > 0, \quad (8.16) \]

is analytic in a neighborhood of \( z = 0 \), and has an analytic inverse as well, where

\[ A = A(v, \kappa) = \text{ diag}(v + \kappa, v - 2\kappa, \kappa - 2v), \quad (8.17) \]

\[ B = \text{ diag}(1, 0, -1), \quad (8.18) \]

and the matrices \( \mathcal{U}^\pm \) are defined through (8.14)–(8.15).
Proof. It is clear that $\hat{G}$ defined in (8.16) is analytic in the upper and lower half planes. We now compute its jumps across the real axis in a neighborhood of the origin. For $0 < x < p$, it follows from (8.1) and (8.14) that

$$
(\hat{G}_-(x))^{-1} \hat{G}_+(x) = \text{diag} \left( 1, x^{-\frac{B}{2}} \mathcal{U}^{-1} x^{\frac{A}{2}} \right) \text{diag} \left( 1, x^{\frac{A}{2}} (\mathcal{U}^+)^{-1} x^{\frac{B}{2}} \right)
$$

$$
= \text{diag} \left( 1, x^{-\frac{B}{2}} \mathcal{U}^{-1} \right) \text{diag} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) (\mathcal{U}^+)^{-1} x^{\frac{B}{2}}
$$

$$
= \text{diag} \left( 1, x^{-\frac{B}{2}} \mathcal{U} (\mathcal{U}^+)^{-1} x^{\frac{B}{2}} \right) = I_4.
$$

Similarly, if $-q < x < 0$, we use again (8.1) and compute

$$
(\hat{G}_-(x))^{-1} \hat{G}_+(x) = \text{diag} \left( 1, x^{-\frac{B}{2}} \mathcal{U}^{-1} x^{\frac{A}{2}} \right) \text{diag} \left( 1, x^{\frac{A}{2}} (\mathcal{U}^+)^{-1} x^{\frac{B}{2}} \right)
$$

$$
= \text{diag} \left( 1, x^{-\frac{B}{2}} \mathcal{U}^{-1} \right) \text{diag} \left( c_{2(x+y)}, \begin{pmatrix} 0 & -c_{\frac{x+y}{2}} \\ c_{\frac{x+y}{2}} & 0 \end{pmatrix} \right) (\mathcal{U}^+)^{-1} x^{\frac{B}{2}}
$$

$$
= \text{diag} \left( 1, x^{-\frac{B}{2}} \hat{\mathcal{U}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \hat{\mathcal{U}}^{-1} x^{\frac{B}{2}} \right).
$$

Note that

$$
\hat{\mathcal{U}}^{-1} = \frac{1}{3} \begin{pmatrix} \omega^+ & -1 & \omega^- \\ \omega^- & -1 & \omega^+ \\ 1 & -1 & 1 \end{pmatrix},
$$

a straightforward calculation shows that

$$
(\hat{G}_-(x))^{-1} \hat{G}_+(x) = \text{diag} \left( 1, x^{-\frac{B}{2}} \right) \text{diag} \left( \omega^-, 1, \omega^+ \right) x^{\frac{B}{2}} = I_4, \quad -q < x < 0.
$$

Hence, we can conclude that $G$ is analytic in a neighborhood of the origin with $z = 0$ being an isolated singularity.

We next show that $z = 0$ is a removable singularity. Note that, as $z \to 0$,

$$
F_1(z) = \mathcal{O}(1), \quad F_{2,3,4}(z) = \mathcal{O}(z^{\frac{2x-y}{3}}).
$$

Thus,

$$
\text{diag} \left( c_k F_1(z), F_2(z) e^{-\kappa \log z}, F_3(z), c_{k-y} F_4(z) e^{(y-\kappa) \log z} \right) = \hat{F}(z) \text{diag} \left( 1, z^{-\frac{A}{3}} \right),
$$

where $\hat{F}$ is a diagonal matrix satisfying

$$
\hat{F}(z) = F_0 + \mathcal{O}(z^{\frac{1}{3}}), \quad z \to 0,
$$

for some non-singular constant matrix $F_0$. This, together with (8.13) and (8.10), implies that

$$
G(z) \text{diag} \left( 1, z^{\frac{A}{3}} \right) = \text{diag}(f_1, f_1, f_3) G_0(z) \hat{F}(z) = \mathcal{O}(z^{\frac{1}{3}}), \quad z \to 0.
$$
By (8.16), we further get that
\[ \hat{G}(z) = \mathcal{O}(z^{-\frac{2}{3}}), \quad z \to 0, \]
so \( z = 0 \) must be a removable singularity, as claimed.

Finally, the existence of the analytic inverse follows immediately because the determinants of \( G, \mathcal{U}^\pm, z^{\frac{4}{3}} \) and \( z^{\frac{8}{3}} \) are all constant and non-zero, so the same is true for \( \det \hat{G} \).

This completes the proof of Proposition 8.6. \( \square \)

9. Local Parametrices Near \( p \) and \( -q \)

From our definition of \( \phi \)-functions given in (4.9)–(4.11), it is readily seen that
\[
\begin{align*}
\phi_2(z) &= C_2(z - p)^{\frac{2}{3}}(1 + \mathcal{O}(z - p)), \quad z \to p, \\
\phi_1(z) &= -C_1(z + q)^{\frac{3}{2}}(1 + \mathcal{O}(z + p)), \quad z \to -q,
\end{align*}
\]
for some positive constants \( C_1 \) and \( C_2 \). Hence, by setting \( D_p(\delta) \) and \( D_{-q}(\delta) \) with \( \delta > 0 \) sufficiently small to be two small disks around \( p \) and \( -q \), we could construct local parametrices \( L_p \) and \( L_{-q} \) in each of the disk with the aid of the standard \( 2 \times 2 \) Airy parametrix [24]. Since this construction is very well-known, we omit the details but mention that as one of the outcomes we get the matching
\[ L_j(z) = (I_4 + \mathcal{O}(n^{-1}))G(z), \quad n \to \infty, \] (9.1)
uniformly for \( z \in \partial D_j(\delta), j = p, -q \).

10. Local Parametrix Near the Origin

In this section, we will construct the local parametrix near the origin, which is somewhat involved and performed in several steps. The main difficulty lies in the fact one cannot expect a nice matching like (9.1) immediately in this case, and this phenomenon is quite common in the asymptotic analysis of higher order RH problem; cf. [11,15,38]. Here, we follow a novel technique recently developed by Kuijlaars and Molag [38], which requires to construct a matching condition on two circles.

Let \( D(\delta) \) and \( D(r) \) be disks centered at the origin with radii \( \delta > r > 0 \). We will take \( D(\delta) \) to be small but fixed and \( D(r) = D(r_n) \) to be shrinking with \( n \). A more precise requirement on \( r \) will be given later.

On account of the second inequality in (4.15) and the fact that \( \kappa \geq 0 \), we could simply ignore the \((2,1)\)-entry of \( J_S \) on \((-\delta,0)\) for large \( n \) and this leads us to consider the following RH problem.

RH Problem 10.1. We look for a \( 4 \times 4 \) matrix-valued function \( L_0 \) with the following properties:

1. \( L_0 \) is defined and analytic in \( D(\delta) \setminus ((\Gamma_S \cap D(r)) \cup (-\delta, \delta)) \), where the contour \( \Gamma_S \) is defined in (7.6).
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(2) \( L_0 \) satisfies the jump condition

\[
L_{0,+}(z) = L_{0,-}(z) \begin{cases} J_G(z), & z \in (-\delta, \delta) \setminus [-r, r], \\ J_{L_0}(z), & z \in D(r) \cap \Gamma_S, \end{cases}
\]

where \( J_G \) is defined in (8.1), and

\[
J_{L_0}(z) = \begin{cases} J_S(z), & z \in (D(r) \cap \Gamma_S) \setminus (-r, 0), \\ \text{diag} \left( I_2, \begin{pmatrix} 0 & -|z|^{\nu-\kappa} \\ |z|^{\kappa-\nu} & 0 \end{pmatrix} \right), & z \in (-r, 0), \end{cases}
\]

with \( J_S \) given in (7.7).

(3) As \( z \to 0 \), \( L_0 \) has at worse a power log singularity.

(4) As \( n \to \infty \), we have the matching conditions

\[
L_0(z) = (I_4 + \mathcal{O}(n^{-1}))G(z), \quad z \in \partial D(\delta) \setminus (-\delta, \delta),
\]

where \( G \) is the global parametrix (8.13), and

\[
L_{0,+}(z) = (I_4 + \mathcal{O}(n^{-1}))L_{0,-}(z), \quad z \in \partial D(r) \setminus \Gamma_S,
\]

where the orientation of the circle is taken in a counter-clockwise manner and the error terms in (10.1) and (10.2) are uniform in \( z \).

In previous works in the literature, only the matching (10.1) is present, with possibly a shrinking radius \( \delta = \delta_n \). In these scenarios, one often has to make several post-corrections to the matching, as the initial error term is not of the appropriate order. As mentioned earlier, in [38] Kuijlaars and Molag explored the introduction of this new matching condition (10.2), which allows to keep \( \delta \) fixed but make \( r = r_n \) shrinking. It turns out that this double-matching (10.1)–(10.2) makes the coming calculations more systematic, and this will be the approach we follow. We next present some preliminary work before doing that.

As the first step to solve the RH Problem 10.1, we remove all the \( \phi \)-functions from the jumps of \( L_0 \) by defining

\[
P(z) = L_0(z) \text{diag} \left( 1, e^{-n(\lambda_2(z)+c)}, e^{-n(\lambda_3(z)+c)}, e^{-n(\lambda_4(z)+c)} \right), \quad z \in D(r) \setminus \Gamma_S,
\]

where

\[
c = \int_0^p \xi_{2,+}(s) \, ds = 2\pi i + \int_0^p \xi_{2,-}(s) \, ds = \int_0^p \xi_{3,+}(s) \, ds = 2\pi i + \int_0^p \xi_{3,-}(s) \, ds,
\]

and the \( \lambda \)-functions are defined in (4.2)–(4.4).

An appropriate, but straightforward, combination of (4.2)–(4.4), (4.10)–(4.11) and Proposition 4.2 leads us to consider the following RH Problem that \( P \) must satisfy.
RH Problem 10.2. The function $P$ defined in (10.3) has the following properties:

1. $P$ is defined and analytic on $D(r) \setminus \Gamma_S$.
2. $P$ satisfies the jump condition

$$P_+(z) = P_-(z) J_P(z), \quad z \in \Gamma_S \cap D(r),$$

where

$$J_P(z) = \begin{cases}
    I_4 + z^{-k} E_{32}, & z \in D(r) \cap \partial L_2^\pm, \\
    I_4 - c_{v-k} z^{-v} E_{43}, & z \in D(r) \cap \partial L_3^\pm, \\
    \text{diag} \left( 1, \left( \begin{array}{cc} 0 & z^k \\ -z^{-k} & 0 \end{array} \right), 1 \right), & z \in (0, r), \\
    \text{diag} \left( I_2, \left( \begin{array}{cc} 0 & -|z|^{v-k} \\ |z|^{k-v} & 0 \end{array} \right) \right), & z \in (-r, 0).
\end{cases}$$

3. As $z \to 0$, $P$ has at worse a power log singularity.

Note that we do not pose any asymptotic behavior of $P$ on $\partial D(r)$. We will give an explicit solution to the above RH problem, and, after some further manipulations, modify $P$ in such a way that, at the end of the day, the corresponding matrix $L_0$ solves the RH problem 10.1.

For later use, we introduce the functions $\hat{\lambda}_k^\pm(z)$ defined by

$$\hat{\lambda}_k^\pm(z) = \int_0^z \xi_k(s) \, ds, \quad \pm \text{Im } z > 0, \quad k = 2, 3, 4, \quad (10.5)$$

where the path for $\hat{\lambda}_k^+(z)$ ($\hat{\lambda}_k^-(z)$) is contained in the upper (lower) half plane. It is easily seen from (4.2)–(4.4), (10.4) and (10.5) that

$$e^{-(\lambda_k(z)+c)} = e^{-n\hat{\lambda}_k^\pm(z)}, \quad \pm \text{Im } z > 0, \quad k = 2, 3, 4. \quad (10.6)$$

The asymptotic behaviors of $\hat{\lambda}_k^\pm$ near the origin are collected in the following proposition.

Proposition 10.3. There exist analytic functions $f_4$, $g_4$ and $h_4$ in a neighborhood of $z = 0$ so that for $\pm \text{Im } z > 0$

$$\hat{\lambda}_2^\pm(z) = \omega^\pm z^{1/3} f_4(z) + \omega^\pm z^{2/3} g_4(z) + zh_4(z), \quad (10.7)$$

$$\hat{\lambda}_3^\pm(z) = \omega^\pm z^{1/3} f_4(z) + \omega^\pm z^{2/3} g_4(z) + zh_4(z), \quad (10.8)$$

$$\hat{\lambda}_4^\pm(z) = z^{1/3} f_4(z) + z^{2/3} g_4(z) + zh_4(z). \quad (10.9)$$

Furthermore, we have

$$f_4(0) = 3(\beta^2 - \alpha^2)^{1/3} > 0. \quad (10.10)$$

Proof. From the local behavior of the $\xi$-functions near $z = 0$ (which can be derived from (3.50), (3.52) and the spectral curve (2.16)), it is readily seen that, as $z \to 0$,

$$\hat{\lambda}_k^\pm(z) = z^{1/3} f_k^\pm(z) + z^{2/3} g_k^\pm(z) + zh_k^\pm(z), \quad \pm \text{Im } z > 0, \quad k = 2, 3, 4, \quad (10.11)$$
where \( f_k^\pm \), \( g_k^\pm \) and \( h_k^\pm \) are analytic in a neighborhood of \( z = 0 \) and all the roots are taken the principal branches with cuts along the negative axis. The jump relations for the \( \xi \)-functions across the positive axis imply in particular that
\[
\begin{align*}
 f_2^\pm(z) &= f_3^\mp(z), & g_2^\pm(z) &= g_3^\mp(z), & h_2^\pm(z) &= h_3^\mp(z), \\
 f_4^\pm(z) &= f_4^\pm := f_4(z), & g_4^\pm &= g_4 := g_4(z), & h_4^\pm(z) &= h_4(z),
\end{align*}
\]
while the jump conditions across the negative axis give that
\[
\begin{align*}
 f_2^\mp(z) &= \omega f_2^+(z), & g_2^\mp(z) &= \omega^- g_2^+(z), & h_2^\mp(z) &= h_2^+(z), \\
 f_3^\pm(z) &= \omega^\mp f_4(z), & g_3^\pm(z) &= \omega^\pm g_4(z), & h_3^\pm(z) &= h_4(z).
\end{align*}
\]
As a consequence, we obtain the relations
\[
f_2^\pm(z) = f_3^\mp(z) = \omega^\pm f_4(z), \quad g_2^\pm(z) = g_3^\mp(z) = \omega^\pm g_4(z), \quad h_2^\pm(z) = h_3^\pm(z) = h_4(z).
\]
Inserting the above formulas into (10.11) gives us (10.7)–(10.9).

Finally, the fact that \( \xi_4 > 0 \) on the positive axis (see (2.15)) allows us to conclude from (2.16) that
\[
\xi_4(z) \sim 3(\beta^2 - \alpha^2)^{1/3}z^{-2/3}, \quad z \to 0,
\]
which in turn implies (10.10).

This completes the proof of Proposition 10.3. \( \square \)

The explicit construction of \( P \) is based on the bare Meijer-G parametrix which is described in the next section.

### 10.1. The Meijer-G parametrix of Bertola-Bothner.

The model RH problem we need to solve for \( P \) was introduced by Bertola and Bothner in the context of a model of several coupled positive-definite matrices [11], which is called ‘bare Meijer-G parametrix for \( p \)-chain’, \( p = 2, 3, \ldots \), therein. The one that is relevant to the present work corresponds to the case \( p = 2 \) and reads as follows:\(^2\)

**RH Problem 10.4.** The function \( \Psi \) is a \( 3 \times 3 \) matrix-valued function satisfying the following properties:

\(^2\) For convenience, the correspondence between our notations and those used in [11] is listed below:
\[
\begin{align*}
 a_1 &= \kappa, \quad a_2 = \nu - \kappa, \quad A_1 = -\nu - \kappa, \quad A_2 = 2\kappa - \nu, \quad A_3 = 2\nu - \kappa,
\end{align*}
\]
and
\[
(a_{j,k})_{j,k=1,2} = \begin{pmatrix} \kappa & \nu \\ 0 & \nu - \kappa \end{pmatrix}, \quad \Omega_\pm = \text{diag}(\omega^\pm, \omega^\mp, 1).
\]

Moreover, in [11], \( \omega = \omega_{BB} = e^{\pi i/3} \), so \( \omega^\pm_{BB} = \omega \) and the contours are
\[
\begin{align*}
 t_0 &= \Gamma_0, & t_1 &= \Gamma_1, & t_2 &= \Gamma_5, & t_3 &= \Gamma_2, & t_4 &= \Gamma_4, & t_5 &= \Gamma_3,
\end{align*}
\]
with all the \( t_k \)'s oriented from the origin towards \( \infty \).
(1) $\Psi$ is defined and analytic in $\mathbb{C}\setminus\Gamma_\Psi$, where

$$
\Gamma_\Psi := \bigcup_{k=0}^5 \Gamma_k, \quad \Gamma_k = e^{k\pi i/3}[0, +\infty), \quad k = 0, \ldots, 5,
$$

with the orientations as shown in Fig. 4.

(2) $\Psi$ satisfies the jump condition

$$
\Psi_+(z) = \Psi_-(z) J_\Psi(z), \quad z \in \Gamma_\Psi,
$$

where

$$
J_\Psi(z) = \begin{cases}
\begin{pmatrix}
0 & z^k & 0 \\
-z^{-k} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, & z \in \Gamma_0, \\
\begin{pmatrix}
1 & 0 & 0 \\
z^{-k} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, & z \in \Gamma_1 \cup \Gamma_5, \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -c_{\nu-k} z^{\nu-k} & 1
\end{pmatrix}, & z \in \Gamma_2 \cup \Gamma_4, \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -|z|^\nu \\
|z|^k & 0 & 0
\end{pmatrix}, & z \in \Gamma_3.
\end{cases}
$$

(10.12)

(3) As $z \to 0$, $\Psi$ has at worse a power-log singularity. In particular, we have, as $z \to 0$,

$$
\Psi(z) \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T = 0(1).
$$

(10.13)
(4) As $z \to \infty$ with $\pm \operatorname{Im} z > 0$, we have
\[
\Psi(z) = z^{-\frac{n}{2}} U^\pm K(z) z^{-\frac{d}{4}} \operatorname{diag}(e^{-3z^{1/3} \omega^\pm}, e^{-3z^{1/3} \omega^\mp}, e^{-3z^{1/3}}),
\] (10.14)
where the diagonal matrices $A$ and $B$ are as in (8.17) and (8.18), $U^\pm$ are given in (8.14) and $K$ admits an asymptotic expansion of the form
\[
K(z) \sim I_3 + \sum_{j=1}^{\infty} K_j z^{-j}, \quad z \to \infty,
\] (10.15)
where the coefficient $K_j$, $j = 1, 2, 3, \ldots$, possibly depends on the sector $\Theta_j$, $j = 0, \ldots, 5$, along which $z \to \infty$. Here, $\Theta_k$, $k = 0, 1, \ldots, 5$, denotes the region between the contours $\Gamma_k$ and $\Gamma_{k+1}$; see Fig. 4 for an illustration.

The precise asymptotic behavior of $\Psi$ as $z \to 0$ depends on whether the values $\kappa$ and $\nu$ are zero or equal to one another. This behavior is indicated in [11] as the behavior of certain iterated Cauchy transforms (see the RH Problem 4.22 and also equations (2.6) and (2.7) therein), but for our purposes the behavior as in item (3) above will suffice.

In the construction carried out by Bertola and Bothner [11, Theorem 4.23], they actually only show that $K(z) = I + O(z^{-1/3})$. Nevertheless, the existence of the full asymptotic expansion as in (10.15) follows from the existence of the asymptotic expansion for the entries of $\Psi$ which, as we now explain, are given by Meijer G-functions - hence this parametrix bears the name the Meijer-G parametrix.

To describe it, recall that the Meijer G-function is given by the following contour integral in the complex plane:
\[
G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| \xi \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + u) \prod_{j=1}^{n} \Gamma(1 - a_j - u) \prod_{j=m+1}^{q} \Gamma(1 - b_j - u) \prod_{j=n+1}^{p} \Gamma(a_j + u)}{(1 + \xi - s) \Gamma(1 + \kappa - s)} \left( \frac{\kappa}{s} \right)^{n-s} ds.
\] (10.16)

Here, it is assumed that
\begin{itemize}
  \item $0 \leq m \leq q$ and $0 \leq n \leq p$, where $m, n, p$ and $q$ are integer numbers;
  \item The real or complex parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ satisfy the conditions
    
    \[ a_k - b_j \neq 1, 2, 3, \ldots, \quad \text{for} \quad k = 1, 2, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots, m, \]

    i.e., none of the poles of $\Gamma(b_j + u)$, $j = 1, 2, \ldots, m$, coincides with any poles of $\Gamma(1 - a_k - u)$, $k = 1, 2, \ldots, n$.
\end{itemize}

The contour $L$ is chosen in such a way that all the poles of $\Gamma(b_j + u)$, $j = 1, \ldots, m$, are on the left of the path, while all the poles of $\Gamma(1 - a_k - u)$, $k = 1, \ldots, n$, are on the right, which is usually taken to go from $-i \infty$ to $i \infty$. We now set
\[
g_1(z) = \frac{c_1}{2\pi i} \int_L \frac{\Gamma(s)}{\Gamma(1 + \kappa - s) \Gamma(1 + \nu - s)} z^{-s} ds = c_1 G_{0,3}^{1,0} \left( \begin{array}{c} - \\ 0, -\kappa, -\nu \end{array} \bigg| z \right),
\]
\[
g_2^{(\pm)}(z) = \frac{c_2}{2\pi i} \int_L \frac{\Gamma(s) \Gamma(\kappa) e^{\pm \pi i s}}{\Gamma(1 + \nu - \kappa - s)} e^{-z s} ds = c_2 G_{0,3}^{2,0} \left( \begin{array}{c} - \\ 0, \kappa, -\nu \end{array} \bigg| e^{\pm \pi i z} \right),
\]
\[
g_3(z) = \frac{c_3}{2\pi i} \int_L \frac{\Gamma(s + \nu) \Gamma(s + \nu - \kappa) \Gamma(s)}{\Gamma(s + \nu - \kappa - s)} z^{-s} ds = c_3 G_{0,3}^{3,0} \left( \begin{array}{c} - \\ 0, \nu - \kappa, \nu \end{array} \bigg| z \right),
\]
where

\[ c_k = (2\pi i)^{3-k} \frac{\sqrt{3}}{2\pi}, \quad k = 1, 2, 3, \]

and define the auxiliary function

\[
\tilde{\Psi}(z) = \begin{pmatrix}
g_1(z) & g_2(\pm)(z) & g_3(z) \\
\frac{d g_1}{dz}(z) & (\frac{d}{dz} - \kappa) g_2(\pm)(z) & (\frac{d}{dz} - \nu) g_3(z) \\
\frac{d}{dz} g_1(z) & 2 \frac{d}{dz} g_2(\pm)(z) & 2 \frac{d}{dz} g_3(z)
\end{pmatrix}, \quad \pm \text{Im } z > 0.
\]

Then, the solution to the model RH problem 10.4 for \( \Psi \) is given by

\[
\Psi(z) = \begin{cases}
\tilde{\Psi}(z), & z \in \Theta_1 \cup \Theta_4, \\
\tilde{\Psi}(z) \text{ diag} \left( \begin{pmatrix} 1 \\ \zeta_{\nu-k}^{-} z_{\kappa-\nu}^{+} \\ 1 \end{pmatrix} \right), & z \in \Theta_2, \\
\tilde{\Psi}(z) \text{ diag} \left( \begin{pmatrix} 1 \\ -z_{-\kappa}^{-} 1 \\ 0 \end{pmatrix} \right), & z \in \Theta_0, \\
\tilde{\Psi}(z) \text{ diag} \left( \begin{pmatrix} 1 \\ -\zeta_{\nu-k}^{-} z_{\kappa-\nu}^{+} \\ 1 \end{pmatrix} \right), & z \in \Theta_3, \\
\tilde{\Psi}(z) \text{ diag} \left( \begin{pmatrix} 1 \\ 0 \\ z_{-\kappa}^{-} 1 \end{pmatrix} \right), & z \in \Theta_5,
\end{cases}
\]

recall that the region \( \Theta_k, k = 0, 1, \ldots, 5 \), is shown in Fig. 4.

We conclude this section with some auxiliary results for later purposes.

**Lemma 10.5.** The matrix-valued function

\[ Q_1(z) = z^{-\frac{B}{3}} \mathcal{U}^\pm \text{ diag}(e^{-3z^{1/3} \omega^\pm}, e^{-3z^{1/3} \omega^\mp}, e^{-3z^{1/3} \omega^\mp})(\mathcal{U}^\pm)^{-1} z^{\frac{B}{3}}, \quad \pm \text{Im } z > 0, \]

(10.18)

is entire, where the matrices \( \mathcal{B} \) and \( \mathcal{U}^\pm \) are defined in (8.18) and (8.14). Similarly, for any function \( \vartheta \) analytic near the origin, the matrix-valued function

\[ Q_2(z) = z^{-\frac{B}{3}} \mathcal{U}^\pm \text{ diag}(e^{\frac{2}{3} \omega^\varphi \vartheta(z)}, e^{\frac{2}{3} \omega^\pm \vartheta(z)}, e^{\frac{2}{3} \vartheta(z)})(\mathcal{U}^\pm)^{-1} z^{\frac{B}{3}}, \quad \pm \text{Im } z > 0, \]

is analytic near the origin as well.

**Proof.** Both \( Q_1 \) and \( Q_2 \) take the form

\[ Q(z) = z^{-\frac{B}{3}} \mathcal{U}^\pm \text{ diag}(e^{\vartheta_1(z)}, e^{\vartheta_2(z)}, e^{\vartheta_3(z)})(\mathcal{U}^\pm)^{-1} z^{\frac{B}{3}}, \]

where \( \vartheta_1, \vartheta_2 \) and \( \vartheta_3 \) are analytic functions on \( \mathcal{V} \setminus \mathbb{R} \) (\( \mathcal{V} = \mathbb{C} \) for \( Q_1 \), and \( \mathcal{V} \) is a neighborhood of the origin for \( Q_2 \)), with jumps across \( \mathbb{R} \) related through

\[
\begin{align*}
\vartheta_{1,+}(x) - \vartheta_{1,-}(x) &= \vartheta_{2,+}(x) - \vartheta_{1,-}(x) = \vartheta_{3,+}(x) - \vartheta_{3,-}(x) = 0, \quad x > 0, \\
\vartheta_{1,+}(x) - \vartheta_{1,-}(x) &= \vartheta_{2,+}(x) - \vartheta_{3,-}(x) = \vartheta_{3,+}(x) - \vartheta_{2,-}(x) = 0, \quad x < 0.
\end{align*}
\]
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After a cumbersome but straightforward calculation, these relations combined assure us that $Q$ has no jumps across the real axis, so $z = 0$ is an isolated singularity of $Q$. Furthermore, it is clear that

$$Q(z) = O(z^{-2/3}),$$

so $z = 0$ is actually a removable singularity of $Q$, as required.

This completes the proof of Lemma 10.5.

10.2. Construction of the local parametrix $P$. We now construct the parametrix $P$ that solves the RH problem 10.2. To do so, recall the function $f_4(z)$ given in Proposition 10.3 and set

$$\varphi(z) = \frac{1}{27}z(f_4(z))^3.$$

From Proposition 10.3, it follows that the function $\varphi$ is conformal in a neighborhood of $z = 0$. Even more so, we actually have

$$\varphi(z) = z(\beta^2 - \alpha^2)(1 + O(z)), \quad z \to 0. \quad (10.19)$$

Furthermore, by deforming the lenses if needed, we can assume that $\Gamma_S \cap D(r)$ is mapped by $z \mapsto n^3 \varphi(z)$ to the union of the contours $\Gamma_0 \cup \cdots \cup \Gamma_5$ and, in virtue of (10.19), also that $\varphi((0, r)) \subset \Gamma_0$. We then define

$$P(z) = \text{diag} \left( 1, \left( \frac{n}{3} f_4(z) \right)^B \Psi(n^3 \varphi(z)) \left( \frac{n}{3} f_4(z) \right)^A \right), \quad z \in D(r) \setminus \Gamma_S, \quad (10.20)$$

where $\Psi$ is the Meijer-G parametrix (10.17), and the matrices $A$, $B$ are given in (8.17) and (8.18), respectively.

**Proposition 10.6.** The matrix-valued function $P(z)$ defined in (10.20) solves the RH problem 10.2.

**Proof.** It is easily seen that $P$ is analytic on $D(r) \setminus \Gamma_S$. To show $P$ satisfies other items of the RH problem 10.2, we start with checking the jump condition. If $z \in (0, r)$, with $J_\Psi$ defined in (10.12), we have

$$J_P(z) = P^-(z)^{-1} P^+(z)$$

$$= \text{diag} \left( 1, \left( \frac{n}{3} f_4(z) \right)^{-A} J_\Psi(n^3 \varphi(z)) \left( \frac{n}{3} f_4(z) \right)^A \right)$$

$$= I_4 + \left( \frac{n}{3} f_4(z) \right)^{-v-\kappa} \left( n^3 \varphi(z) \right)^{\kappa} \left( \frac{n}{3} f_4(z) \right)^{v-2\kappa} E_{23}$$

$$- \left( \frac{n}{3} f_4(z) \right)^{-v+2\kappa} \left( n^3 \varphi(z) \right)^{-\kappa} \left( \frac{n}{3} f_4(z) \right)^{v+\kappa} E_{32}$$

$$= I_4 + \left( \frac{n}{3} f_4(z) \right)^{-3\kappa} \left( n \frac{n}{3} f_4(z) \right)^{3\kappa} E_{23} - \left( \frac{n}{3} f_4(z) \right)^{3\kappa} \left( \frac{n}{3} \left( \frac{n}{3} f_4(z) \right)^{-3\kappa} E_{32} \right.$$
as expected. The jump matrix of $P$ on other parts of $D(r) \cap \Gamma_S$ can be computed similarly, we omit the details here. Finally, the behavior of $P$ near the origin follows from the behavior of $\Psi$ given in item (3) of the RH problem 10.4 and the fact that all the other terms in (10.20) remain bounded as $z \to 0$.

This completes the proof of Proposition 10.6. $\square$

We further set
\[
\hat{P}(z) := P(z) \text{ diag } \left( 1, \text{ diag}(e^{n\hat{\lambda}^\pm_k(z)})_{k=1}^3 \right),
\] (10.21)
where the functions $\hat{\lambda}^\pm_k(z)$, $k = 2, 3, 4$, are defined in (10.5). On account of (10.3) and (10.6), it is easily seen that $\hat{P}(z)$ satisfies the same jump condition as $L_0$ for $z \in \Gamma_S \cap D(r)$ and item (3) of the RH problem 10.1 for $L_0$. As shown later, we will solve the RH problem 10.1 with the aid of $\hat{P}$. For that purpose, we next explore the asymptotics of $\hat{P}$ on the boundary of the disk.

From now on, we assume, as mentioned before, that $\delta > 0$ is sufficiently small and fixed but make $r = r_n$ shrink with $n$, namely,
\[
r = r_n = n^{-\frac{3}{2}}.
\] (10.22)

Since
\[
n^3 \varphi(z) \to \infty, \quad z \in \partial D(r_n), \quad n \to \infty,
\] under the scaling (10.22), we can use (10.21), (10.20) and (10.14) to compute
\[
\hat{P}(z) = \text{ diag } \left( 1, \left( \frac{n}{3} f_4(z) \right)^B \Psi(n^3 \varphi(z)) \left( \frac{n}{3} f_4(z) \right)^A \text{ diag}(e^{n\hat{\lambda}^\pm_k(z)})_{k=1}^3 \right),
\]
\[
= \text{ diag } \left( 1, z^{-\frac{B}{3}} K(z) U^\pm z^{-\frac{4}{3}} D_n(z) \right), \quad z \in \partial D(r_n), \quad n \to \infty,
\] (10.23)
where $K = K_n$ is an error matrix explicitly given by
\[
K(z) := U^\pm K(n^3 \varphi(z))(U^\pm)^{-1}, \quad \pm \text{Im } z > 0,
\] (10.24)
with $K$ being given in the asymptotic formula (10.14), and
\[
D_n(z) = \text{ diag } \left( e^{n(\hat{\lambda}_2^\pm(z)-3\omega^\pm \varphi(z)^{1/3})}, e^{n(\hat{\lambda}_3^\pm(z)-3\omega^\mp \varphi(z)^{1/3})}, e^{n(\hat{\lambda}_4^\pm(z)-3\varphi(z)^{1/3})} \right)
\]
\[
= e^{nz h_4(z)} \text{ diag } \left( e^{n\omega^\mp z^{2/3} g_4(z)}, e^{n\omega^\pm z^{2/3} g_4(z)}, e^{nz^{2/3} g_4(z)} \right), \quad \pm \text{Im } z > 0.
\] (10.25)

In the second equality of (10.25), we have made use of Proposition 10.3, which also implies that $D_n(z)$ remains bounded for $z \in \overline{D(r_n)}$ under the scaling (10.22). By defining
\[
\hat{D}_n(z) := \text{ diag } \left( 1, (n^{3/2} z)^{-\frac{B}{3}} U^\pm D_n(z)(U^\pm)^{-1}(n^{3/2} z)^{\frac{4}{3}} \right), \quad \pm \text{Im } z > 0,
\] (10.26)
it then follows from (8.16) that

$$\text{diag} \left( 1, z^{-\frac{n}{3}} U^{\pm} z^{-\frac{n}{3}} D_n(z) \right) = n^{\frac{B}{2}} \hat{D}_n(z) n^{-\frac{B}{2}} \text{diag} \left( 1, z^{-\frac{n}{3}} U^{\pm} z^{-\frac{n}{3}} \right)$$

$$= n^{\frac{B}{2}} \hat{D}_n(z) n^{-\frac{B}{2}} \hat{G}(z)^{-1} G(z),$$

where

$$\hat{B} := \text{diag}(0, B) = \text{diag}(0, 1, 0, -1). \quad (10.27)$$

Thus, we could rewrite the asymptotics in (10.23) as

$$\hat{P}(z) = \text{diag} \left( 1, z^{-\frac{n}{3}} K(z) z^{\frac{n}{3}} \right) n^{\frac{B}{2}} \hat{D}_n(z) n^{-\frac{B}{2}} \hat{G}(z)^{-1} G(z), \quad (10.28)$$

for \(z \in \partial D(r_n)\) and \(n \to \infty\).

We will need some auxiliary results on the matrices \(K\) and \(\hat{D}_n\) in the above formula that we discuss next.

**Lemma 10.7.** With the function \(K(z)\) defined in (10.24), we have that for \(z \in \partial D(r_n)\) and large \(n\), \(K(z)\) admits a formal asymptotic expansion of the form

$$K(z) \sim I_3 + \sum_{j=1}^{\infty} \frac{K_j}{n^j z^j}, \quad (10.29)$$

where the matrix coefficients \(K_j\) are independent of \(z\) and \(n\), and take the following structures:

$$K_j = \begin{cases} 
\begin{pmatrix} * & 0 & 0 \\
0 & * & 0 \\
0 & 0 & * \end{pmatrix}, & j \equiv 0 \mod 3, \\
\begin{pmatrix} * & 0 & 0 \\
0 & 0 & * \\
0 & * & 0 \end{pmatrix}, & j \equiv 1 \mod 3, \\
\begin{pmatrix} 0 & * & 0 \\
0 & * & 0 \\
0 & 0 & * \end{pmatrix}, & j \equiv 2 \mod 3.
\end{cases} \quad (10.30)$$

**Proof.** If \(z \in \partial D(r_n)\), we have that \(z = O(n^{-\frac{2}{3}})\) and \(n^3 \varphi(z) = O(n^3 z) = O(n^{3 \frac{2}{3}})\), so the existence of the expansion (10.29) with coefficients independent of \(n\), but possibly depending on the sector along which \(z \to \infty\), follows from the asymptotic expansion of \(K\) given in (10.15).

Let \(u \in \mathbb{C}\) be a dummy variable. It follows from a calculation similar to that carried out in the proof of Proposition 8.6 that \(\Psi(u) u^{\frac{A}{2}} (U^{\pm})^{-1} u^{\frac{B}{2}} \) has no jump on \(\mathbb{R} \setminus \{0\}\). Furthermore, from (10.14), we find that

$$\Psi(u) u^{\frac{A}{2}} (U^{\pm})^{-1} u^{\frac{B}{2}} = u^{-\frac{B}{2}} U^{\pm} K(u) (U^{\pm})^{-1} u^{\frac{B}{2}} Q_1(u), \quad u \to \infty, \quad \pm \text{Im} u > 0,$$
where $Q_1$ defined in (10.18) is an entire function. Setting $u = n^3 \varphi(z)$, this yields

$$z^{-\frac{B}{3}} K(z) z^{\frac{B}{3}} = \left( \frac{n}{3} f_4(z) \right)^B \Psi(u) u^{4/3} (U^{\pm})^{-1} u^{\frac{B}{3}} Q_1(u)^{-1} \left( \frac{n}{3} f_4(z) \right)^{-B},$$

(10.31)

which should be understood in the scaling (10.22) and $n$ sufficiently large but fixed. Now, the functions in $u$ appearing on the right-hand side of (10.31) do not have jumps on the real axis, and neither do the functions in $z$ because they are entire. Thus, the right-hand side admits an asymptotic expansion in integer powers of $z$ (recall that $u = u(z)$ is conformal), with $n$-dependent coefficients.

On the other hand, the left-hand side admits an asymptotic expansion in inverse powers of $z^{1/3}$, but possibly with different coefficients in different sectors of the plane. A comparison of the asymptotic expansions on both sides then yields that the expansion

$$z^{-\frac{B}{3}} K(z) z^{\frac{B}{3}} \sim I_3 + \sum_{j=1}^{\infty} z^{-\frac{B}{3}} K_j z^{\frac{B}{3}} \frac{1}{n^j z^{\frac{1}{3}}},$$

must involve only inverse integer powers, and furthermore the coefficients should not depend on the sector along which $z \to \infty$. Further noticing the identity

$$z^{-\frac{B}{3}} K_j z^{\frac{B}{3}} = O \left( \begin{array}{ccc} 1 & z^{-\frac{1}{3}} & z^{-\frac{2}{3}} \\ z^{\frac{1}{3}} & 1 & z^{-\frac{1}{3}} \\ z^{\frac{2}{3}} & z^{\frac{1}{3}} & 1 \end{array} \right),$$

we then conclude the structure (10.30).

This completes the proof of Lemma 10.7. ⊓⊔

For any $a, b, c \in \mathbb{C}$, it is straightforward to verify the following commutation relations:

$$z^{-\frac{B}{3}} \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix} z^{\frac{B}{3}} = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z^{\frac{1}{3}} + \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z^{-\frac{2}{3}},$$

$$z^{-\frac{B}{3}} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} z^{\frac{B}{3}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c & 0 & 0 \end{pmatrix} z^{\frac{1}{3}} + \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} z^{-\frac{1}{3}}.$$

(10.32)

Thus, we obtain from (10.32), Lemma 10.7 and a rearrangement of terms that for $z \in \partial D(r_n)$ and large $n$,

$$\text{diag} \left( 1, z^{-\frac{B}{3}} K(z) z^{\frac{B}{3}} \right) = n^{\frac{B}{3}} \left( I_4 + T_0 + E_n^{(1)}(z) \right) n^{-\frac{B}{3}},$$

(10.33)

where the matrix $T_0$ is a strictly lower triangular constant matrix with first column zero, and the error term $E_n^{(1)}$ admits an asymptotic expansion of the form

$$E_n^{(1)}(z) \sim \sum_{j=1}^{\infty} \frac{(I_4 + T_0) A_j^{(1)}}{n^3 j z^j}, \quad z \in \partial D(r_n), \quad n \to \infty,$$

(10.34)

with the coefficients $A_j^{(1)}$ being independent of $n$. In (10.34), the factor $I_4 + T_0$ is added just for later convenience, to avoid some coming cumbersome notations.

To explore the properties of $\hat{D}_n$, we need the following basic fact.
Lemma 10.8. Suppose that \( \{M_n(z)\} \) is a sequence of matrix-valued functions, analytic and uniformly bounded in a neighborhood \( D(2\varepsilon) \) of the origin with \( \varepsilon = \varepsilon_n \to 0 \) as \( n \to \infty \). Then, we have

\[
M_n(z) - M_n(w) = O(\varepsilon^{-1}(z - w)),
\]

(10.35)

and

\[
M_n(z) = M_n(0) + O(\varepsilon^{-1}z),
\]

(10.36)

uniformly for \( z, w \in D(\varepsilon) \) as \( n \to \infty \).

Proof. The estimate (10.36) follows immediately from (10.35). To show (10.35), we fix \( z, w \in D(\varepsilon) \) and use Cauchy’s Theorem to write

\[
M_n(z) - M_n(w) = \frac{1}{2\pi i} \left( \int_{|t| = 2\varepsilon} \frac{M_n(t)}{t - z} \, dt - \int_{|t| = 2\varepsilon} \frac{M_n(t)}{t - w} \, dt \right)
\]

\[
= \frac{z - w}{2\pi i} \int_{|t| = 2\varepsilon} \frac{M_n(t)}{t - z} \frac{dt}{t - w}.
\]

Because \( \{M_n(z)\} \) is uniformly bounded, the identity above immediately implies (10.35).

This completes the proof of Lemma 10.8. \( \square \)

Proposition 10.9. The matrix-valued function \( \hat{D}_n(z) \) defined in (10.26) is invertible. Furthermore, the matrices \( \hat{D}_n(z)^{\pm 1} \) are analytic near the origin and uniformly bounded for \( z \in \bar{D}(r_n) \) as \( n \to \infty \) with the estimates

\[
\hat{D}_n(z)^{\pm 1} - \hat{D}_n(w)^{\pm 1} = O(n^{3/2}(z - w)),
\]

(10.37)

and

\[
\hat{D}_n(z)^{\pm 1} = \hat{D}_n(0)^{\pm 1} + O(n^{3/2}z),
\]

(10.38)

all valid uniformly for \( z, w \in \bar{D}(r_n) \) as \( n \to \infty \).

Proof. The invertibility of \( \hat{D}_n \) follows immediately from its definition (10.26). For ease of notation, we will focus on \( \hat{D}_n \) in what follows, since the arguments for \( \hat{D}_n^{-1} \) are essentially the same.

Recalling the definition of \( D_n \) given in (10.25), the fact that \( \hat{D}_n \) is analytic near the origin follows from a direct application of the second part of Lemma 10.5. Moreover, under the scaling (10.22), the function \( D_n(z) \) as well as \( (n^{3/2}z)^{\pm 1/3} \) remain uniformly bounded for \( z \in \partial D(r_n) \) as \( n \to \infty \). This implies that \( \hat{D}_n \) is uniformly bounded for \( z \in \partial D(r_n) \) and, as a consequence of the maximum principle, also on the whole set \( \bar{D}(r_n) \). The estimates (10.37) and (10.38) are then immediate from Lemma 10.8. This completes the proof of Proposition 10.9. \( \square \)

10.3. Construction of the local parametrix \( L_0 \). With the above preparations, we are finally ready to build a solution to the RH problem 10.1 for \( L_0 \), construction which is carried out in five steps as explained next.
**Initial step** As the initial step, we define

\[
L_0^{(1)}(z) = \begin{cases} 
\hat{G}(z)n^{\frac{\hat{B}}{z}}\hat{D}_n(z)^{-1}n^{-\frac{\hat{B}}{z}}\hat{P}(z), & z \in \mathcal{D}(r_n)\setminus \Gamma_S, \\
G(z), & z \in \mathcal{D}(\delta)\setminus \left(\mathcal{D}(r_n) \cup (-\delta, \delta)\right), 
\end{cases}
\]

(10.39)

where the matrices \(\hat{G}\) and \(\hat{P}\) are given in (8.16) and (10.21), respectively. We then have the following proposition.

**Proposition 10.10.** The matrix-valued function \(L_0^{(1)}(z)\) defined in (10.39) satisfies items (1)–(3) and the matching condition (10.1) of the RH problem 10.1 for \(L_0\). Moreover, we have, as \(n \to \infty\),

\[
L_0^{(1)}(z) = (I_4 + \mathcal{O}(n^{1/2}))L_{0,-}^{(1)}(z) - \hat{B}z\hat{P}(z)\hat{G}(z) - \hat{B}z\hat{G}(z),
\]

(10.40)

**Proof.** Note that \(\hat{P}\) is analytic in \(\mathcal{D}(r_n)\setminus \Gamma_S\) and the global parametrix \(G\) is analytic in \(\mathbb{C}\setminus (-\infty, p]\). Thus, the analyticity properties of \(L_0\) claimed in item (1) follows from the fact that both \(\hat{G}(z)\) and \(\hat{D}_n(z)^{-1}\) are analytic everywhere near \(z = 0\). The jumps claimed in item (2) follow from the jumps of \(\hat{P}\) and \(G\) and again by the analyticity of \(\hat{G}(z)\) and \(\hat{D}_n(z)^{-1}\). The local behavior of \(L_0^{(1)}\) near \(z = 0\) can be seen from the behavior of \(\hat{P}\) near \(z = 0\) and it is also easily seen that the matching condition (10.1) is actually exact.

To show (10.40), we obtain from (10.39), (10.28) and (10.33) that, for \(z \in \partial \mathcal{D}(r_n)\setminus \Gamma_S\) and \(n \to \infty\),

\[
L_0^{(1)}(z) = \hat{G}(z)n^{\frac{\hat{B}}{z}}\hat{D}_n(z)^{-1}n^{-\frac{\hat{B}}{z}}\hat{P}(z)\hat{G}(z) - \hat{B}z\hat{P}(z)\hat{G}(z) - \hat{B}z\hat{G}(z).
\]

(10.41)

Since \(T_0\) is a strictly lower triangular constant matrix with first column zero, we have

\[
\hat{B}zT_0n^{-\frac{\hat{B}}{z}} = \mathcal{O}(n^{-1/2}),
\]

and by (10.34),

\[
n^{\frac{\hat{B}}{z}}E_n^{(1)}(z)n^{-\frac{\hat{B}}{z}} = \mathcal{O}(n^{-1/2}), \quad z \in \partial \mathcal{D}(r_n)\setminus \Gamma_S.
\]

Inserting the above two estimates into (10.41), we arrive at (10.40) on account of the analyticity and boundedness in \(n\) of both \(\hat{G}(z)\) and \(\hat{D}_n(z)^{-1}\) near the origin.

This completes the proof of Proposition 10.10. ☐

In view of (10.40), it follows that the matching condition (10.2) is not satisfied. The next few steps are then devoted to refine the error term in (10.40).
Second step towards the matching. In the second step, we eliminate the term $T_0$ in (10.41) by defining

$$L_0^{(2)}(z) = \begin{cases} \hat{G}(z)n^{\frac{b}{2}}\hat{D}_n(z)^{-1}n^{rac{b}{2}} & \\ \times (I_4 + T_0)^{-1}n^{-\frac{b}{2}}\hat{D}_n(z)n^{-\frac{b}{2}}\hat{G}(z)^{-1}L_0^{(1)}(z), & \ z \in D(r_n)\setminus \Gamma_S, \\
L_0^{(1)}(z) = G(z), & \ z \in D(\delta)\setminus \left(D(r_n) \cup (-\delta, \delta)\right). \end{cases}$$

(10.42)

On account of the triangularity structure of $T_0$, we have that $T_0^3 = 0$, which implies

$$(I_4 + T_0)^{-1} = I_4 - T_0 + T_0^2.$$ \hspace{1cm} (10.43)

Thus, $L_0^{(2)}$ is well defined, and it has the following properties.

**Proposition 10.11.** The matrix-valued function $L_0^{(2)}(z)$ defined in (10.42) satisfies items (1)–(3) and the matching condition (10.1) of the RH problem 10.1 for $L_0$. Moreover, we have, as $n \to \infty$,

$$L_{0,0}(z) = (I_4 + O(n^{1/2}))L_{0,-}(z), \quad z \in \partial D(r_n)\setminus \Gamma_S.$$ \hspace{1cm} (10.44)

To prove the above proposition and for later convenience, we need the following lemma, which is a version of the key observation [38, Proposition 5.15] adapted to our setting.

**Lemma 10.12.** With $A_k^{(1)}$, $k = 1, 2, \ldots$, being the constant matrix in (10.34), define

$$A_k^{(2)}(z) := \hat{D}_n(z)^{-1}n^\frac{b}{2}A_k^{(1)}n^{-\frac{b}{2}}\hat{D}_n(z).$$ \hspace{1cm} (10.45)

Then, $A_k^{(2)}(z)$ is analytic near the origin, and we have, as $n \to \infty$, for any indices $k_1, k_2, \ldots, k_m$,

$$A_{k_1}^{(2)}(z)A_{k_2}^{(2)}(z)\cdots A_{k_m}^{(2)}(z) = O(n),$$ \hspace{1cm} (10.46)

uniformly for $z \in \overline{D(r_n)}$, and

$$A_{k_1}^{(2)}(z_1)A_{k_2}^{(2)}(z_2)\cdots A_{k_m}^{(2)}(z_m) = O(n^{m-l}),$$ \hspace{1cm} (10.47)

uniformly for $z_1, \ldots, z_m \in \overline{D(r_n)}$, where

$$l := \#\{j|1 \leq j \leq m, z_j = z_{j+1}\}.$$ 

**Proof.** The analyticity of $A_k^{(2)}(z)$ near the origin follows directly from its definition and the analyticity of $\hat{D}_n(z)^{\pm 1}$.

By (10.45), it is readily seen that

$$A_{k_1}^{(2)}(z)\cdots A_{k_m}^{(2)}(z) = \hat{D}_n(z)^{-1}n^\frac{b}{2}A_{k_1}^{(1)}\cdots A_{k_m}^{(1)}n^{-\frac{b}{2}}\hat{D}_n(z).$$

This, together with the fact that $\hat{D}_n(z)$ remains bounded for $z \in \overline{D(r_n)}$ as $n \to \infty$ (see Proposition 10.9) gives us (10.46). The proof of (10.47) is similar to that of (10.46), we omit the details here.

This completes the proof of Lemma 10.12.  

\boxed{\begin{align*}
\end{align*}}
Proof of Proposition 10.11. In (10.42), the factor multiplying $L^{(1)}_0$ to the left is analytic on $D(\delta)$, which then gives that $L^{(2)}_0$ still satisfies items (1)–(3) of the RH problem 10.1, and the matching condition (10.1) is obvious as well.

To show (10.44), we see from (10.42), (10.41) and (10.34) that

$$L^{(2)}_{0,+}(z) L^{(2)}_{0,-}(z)^{-1}$$

$$= \widehat{G}(z) n^{\frac{\hat{b}}{2}} \widehat{D}_n(z)^{-1} n^{-\frac{\hat{b}}{2}} (I_4 + T_0)^{-1} n^{-\frac{\hat{b}}{2}} \widehat{D}_n(z) n^{-\frac{\hat{b}}{2}} \widehat{G}(z)^{-1} L^{(1)}_0(z) G(z)^{-1}$$

$$= \widehat{G}(z)^n \frac{\hat{b}}{2} \widehat{D}_n(z)^{-1} n^{-\frac{\hat{b}}{2}} (I_4 + T_0)^{-1} (I_4 + T_0 + E^{(1)}_n(z)) n^{-\frac{\hat{b}}{2}} \widehat{D}_n(z) n^{-\frac{\hat{b}}{2}} \widehat{G}(z)^{-1}$$

$$= \widehat{G}(z) \left( I_4 + n^{-\frac{\hat{b}}{2}} \widehat{D}_n(z)^{-1} n^{-\frac{\hat{b}}{2}} (I_4 + T_0)^{-1} E^{(1)}_n(z) n^{-\frac{\hat{b}}{2}} \widehat{D}_n(z) n^{-\frac{\hat{b}}{2}} \right) \widehat{G}(z)^{-1}$$

$$= \widehat{G}(z) \left( I_4 + \frac{n^{-\frac{\hat{b}}{2}} A_k^{(2)}(z) n^{-\frac{\hat{b}}{2}}}{n^3 z} + E^{(2)}_n(z) \right) \widehat{G}(z)^{-1}, \quad (10.48)$$

uniformly for $z \in \partial D(r_n)$ as $n \to \infty$, where

$$E^{(2)}_n(z) \sim \sum_{k=2}^\infty \frac{n^{\frac{\hat{b}}{2}} A_k^{(2)}(z) n^{-\frac{\hat{b}}{2}}}{n^3 z^k}, \quad z \in \partial D(r_n), \quad n \to \infty, \quad (10.49)$$

and $A_k^{(2)}(z)$ is defined in (10.45).

By (10.46) and (10.22), it follows that

$$\frac{n^{\frac{\hat{b}}{2}} A_k^{(2)}(z) n^{-\frac{\hat{b}}{2}}}{n^3 z} = O(n^{1/2}), \quad z \in \partial D(r_n), \quad n \to \infty.$$

Similarly, we obtain from (10.49), (10.46) and (10.22) that

$$E^{(2)}_n(z) = O(n^{-1}), \quad z \in \partial D(r_n), \quad n \to \infty.$$

A combination of the above two estimates, (10.48) and Proposition 8.6 then gives us (10.44).

This completes the proof of Proposition 10.11. □

Third step towards the matching In the third step, we eliminate the growing term in (10.48) by defining

$$L^{(3)}_0(z) =$$

$$\left\{ \begin{array}{ll}
\widehat{G}(z) n^{\frac{\hat{b}}{2}} \left( I_4 - \frac{A_k^{(2)}(z) - A_k^{(2)}(0)}{n^3 z} \right) n^{-\frac{\hat{b}}{2}} \widehat{G}(z)^{-1} L^{(2)}_0(z), & z \in D(r_n) \setminus \Gamma_S, \\
\widehat{G}(z) n^{\frac{\hat{b}}{2}} \left( I_4 - \frac{A_k^{(2)}(0)}{n^3 z} \right)^{-1} n^{-\frac{\hat{b}}{2}} \widehat{G}(z)^{-1} L^{(2)}_0(z), & z \in D(\delta) \setminus \left( D(r_n) \cup (-\delta, \delta) \right). 
\end{array} \right. \quad (10.50)$$

Here, we observe from (10.46) and (10.22) that, as $n \to \infty$,

$$\frac{A_k^{(2)}(0)}{n^3 z} = \left\{ \begin{array}{ll}
O(n^{-1/2}), & z \in \partial D(r_n), \\
O(n^{-2}), & z \in \partial D(\delta), 
\end{array} \right. \quad (10.51)$$
which implies that the inverse of $I_4 - \frac{A_1^{(2)}(0)}{n^3z}$ is well-defined in the definition of $L_0^{(3)}$. We then have the following proposition.

**Proposition 10.13.** The matrix-valued function $L_0^{(3)}(z)$ defined in (10.50) satisfies items (1)–(3) and the matching condition (10.1) of the RH problem 10.1 for $L_0$. Moreover, we have, as $n \to \infty$,

$$L_{0,+}^{(3)}(z) = (I_4 + O(1))L_{0,-}^{(3)}(z), \quad z \in \partial D(r_n) \setminus \Gamma_5. \quad (10.52)$$

**Proof.** As before, the fact that the prefactors multiplying $L_0^{(2)}$ to the left are analytic makes sure that $L_0^{(3)}$, too, satisfies items (1)–(3) of the RH problem 10.1. Since $L_0^{(2)}$ already satisfies (10.1), it is then easily seen from (10.51) that $L_0^{(3)}$ satisfies (10.1) as well.

To show (10.52), we begin with some elementary estimates. From Lemma 10.12 and (10.49), it is easily seen that for $z_1, z_2 \in \partial D(r_n)$ and $n \to \infty$,

$$n^{-\frac{6}{7}} E_n^{(2)}(z_1)n^{-\frac{6}{7}} = O(n^{-2}), \quad \frac{A_1^{(2)}(z_1)A_1^{(2)}(z_2)}{n^6z_2^2} = \begin{cases} O(n^{-1}), & z_1 \neq z_2, \\ O(n^{-2}), & z_1 = z_2. \end{cases} \quad (10.53)$$

Thus, it follows from (10.51) and the above formula that

$$\left( I_4 - \frac{A_1^{(2)}(z) - A_1^{(2)}(0)}{n^3z} \right) \left( I_4 + \frac{A_1^{(2)}(z)}{n^3z} + n^{-\frac{6}{7}} E_n^{(2)}(z)n^{-\frac{6}{7}} \right)$$

$$= I_4 + \frac{A_1^{(2)}(0)}{n^3z} + \frac{A_1^{(2)}(0)A_1^{(2)}(z)}{n^6z^2} + O(n^{-2}),$$

uniformly on $\partial D(r_n)$ as $n \to \infty$. Combining this with (10.48), (10.50), (10.51) and (10.53), we get that, as $n \to \infty$,

$$L_{0,+}^{(3)}(z)L_{0,-}^{(3)}(z)^{-1}$$

$$= \hat{G}(z)n^{-\frac{6}{7}} \left( I_4 - \frac{A_1^{(2)}(z) - A_1^{(2)}(0)}{n^3z} \right) n^{-\frac{6}{7}} \hat{G}(z)^{-1}L_{0,+}^{(2)}(z)L_{0,-}^{(2)}(z)^{-1}$$

$$\times \hat{G}(z)n^{-\frac{6}{7}} \left( I_4 - \frac{A_1^{(2)}(0)}{n^3z} \right) n^{-\frac{6}{7}} \hat{G}(z)^{-1}$$

$$= \hat{G}(z)n^{-\frac{6}{7}} \left( I_4 - \frac{A_1^{(2)}(z) - A_1^{(2)}(0)}{n^3z} \right) \left( I_4 + \frac{A_1^{(2)}(z)}{n^3z} + n^{-\frac{6}{7}} E_n^{(2)}(z)n^{-\frac{6}{7}} \right)$$

$$\times \left( I_4 - \frac{A_1^{(2)}(0)}{n^3z} \right) n^{-\frac{6}{7}} \hat{G}(z)^{-1}.$$
\[
\hat{G}(z) n^{\frac{\hat{b}}{2}} \left( I_4 + \frac{A_1^{(2)}(0)}{n^3 z} + \frac{A_1^{(2)}(0)A_1^{(2)}(z)}{n^6 z^2} + \mathcal{O}(n^{-2}) \right) \left( I_4 - \frac{A_1^{(2)}(0)}{n^3 z} \right) n^{-\frac{\hat{b}}{2}} \hat{G}(z)^{-1}
\]
\[
= \hat{G}(z) n^{\frac{\hat{b}}{2}} \left( I_4 + \frac{A_1^{(2)}(0)A_1^{(2)}(z)}{n^6 z^2} - \frac{A_1^{(2)}(0)A_1^{(2)}(z)A_1^{(2)}(0)}{n^9 z^3} + \mathcal{O}(n^{-2}) \right) n^{-\frac{\hat{b}}{2}} \hat{G}(z)^{-1},
\]

(10.54)

uniformly for \( z \in \partial D(r_n) \). Again, by Lemma 10.12, we see that
\[
n^{\frac{\hat{b}}{2}} A_1^{(2)}(0)A_1^{(2)}(z) n^{-\frac{\hat{b}}{2}} = \mathcal{O}(1), \quad n^{\frac{\hat{b}}{2}} A_1^{(2)}(0)A_1^{(2)}(z)A_1^{(2)}(0) n^{-\frac{\hat{b}}{2}} = \mathcal{O}(n^{-1/2}),
\]

uniformly on \( \partial D(r_n) \) as \( n \to \infty \). Inserting the above estimates into (10.54) gives us (10.52).

This completes the proof of Proposition 10.13. \( \square \)

As a preparation for the next step, we now introduce some new functions to rewrite (10.54) in a convenient form. With \( A_1^{(2)}(z) \) defined in (10.45), write
\[
A_1^{(2)}(0)A_1^{(2)}(z) = zA_1^{(3)}(z) + A_1^{(2)}(0)^2,
\]

(10.55)

where
\[
A_1^{(3)}(z) := \frac{1}{z} A_1^{(2)}(0)(A_1^{(2)}(z) - A_1^{(2)}(0))
\]

(10.56)
is analytic on \( D(r_n) \). In view of Lemma 10.12, we have
\[
A_1^{(3)}(z) = \mathcal{O}(n^{7/2}),
\]

(10.57)

and by applying Lemma 10.8 to the bounded analytic function \( A_1^{(3)}(z)/n^{7/2} \), it follows that
\[
A_1^{(3)}(z) = A_1^{(3)}(0) + \mathcal{O}(n^5 z),
\]

(10.58)

both of which being valid uniformly for \( z \in \overline{D(r_n)} \) as \( n \to \infty \). Combining (10.55) with (10.53) then gives us
\[
\frac{A_1^{(2)}(0)A_1^{(2)}(z)}{n^6 z^2} = \frac{A_1^{(3)}(z)}{n^6 z} + \mathcal{O}(n^{-2}).
\]

(10.59)

In a way similar to (10.55), we also rewrite the other fraction in (10.54) into the form
\[
- \frac{A_1^{(2)}(0)A_1^{(2)}(z)A_1^{(2)}(0)}{n^9 z^3} = \frac{A_2^{(3)}(z)}{n^9 z} + \frac{A_1^{(2)}(z)^3 - A_1^{(2)}(z)^2 A_1^{(2)}(0) - A_1^{(2)}(0) A_1^{(2)}(z)^2}{n^9 z^3},
\]

(10.60)

where
\[
A_2^{(3)}(z) = \frac{1}{z^2} (A_1^{(2)}(z) - A_1^{(2)}(0)) A_1^{(2)}(z)(A_1^{(2)}(0) - A_1^{(2)}(z))
\]

(10.61)
is an analytic function near the origin. Applying the same arguments as in (10.57) and (10.58), it is readily seen that

\[ A_2^{(3)}(z) = \mathcal{O}(n^6), \quad A_2^{(3)}(z) = A_2^{(3)}(0) + \mathcal{O}(n^{15/2}z), \]  

which is valid, as always, uniformly for \( z \in \overline{D(r_n)} \) as \( n \to \infty \). Due to the decomposition (10.60), we again obtain from Lemma 10.12 that

\[ -\frac{A_1^{(2)}(0)A_1^{(2)}(z)A_1^{(2)}(0)}{n^{9}z^{3}} = \frac{A_2^{(3)}(z)}{n^{9}z} + \mathcal{O}(n^{-5/2}), \]  

Inserting the estimates (10.59) and (10.63) into (10.54) we obtain

\[ L_{0,+}^{(3)}(z)L_{0,-}^{(3)}(z)^{-1} = \hat{G}(z)n^{\frac{6}{5}} \left( I_4 + \frac{A_1^{(3)}(z)}{n^{6}z} + \frac{A_2^{(3)}(z)}{n^{9}z} + \mathcal{O}(n^{-2}) \right) n^{-\frac{6}{5}} \hat{G}(z)^{-1}, \]  

uniformly valid for \( z \in \partial D(r_n) \) as \( n \to \infty \).

**Fourth step towards the matching**  
In a format already familiar to the reader, we define in the fourth step the following transformation:

\[ L_0^{(4)}(z) = \left\{ \begin{array}{ll}
\hat{G}(z)n^{\frac{6}{5}} \left( I_4 - \frac{A_1^{(3)}(z) - A_1^{(3)}(0)}{n^{6}z} \right) n^{-\frac{6}{5}} \hat{G}(z)^{-1}L_{0}^{(3)}(z), & z \in D(r_n) \setminus \Gamma_S, \\
\hat{G}(z)n^{\frac{6}{5}} \left( I_4 - \frac{A_1^{(3)}(0)}{n^{6}z} \right)^{-1} n^{-\frac{6}{5}} \hat{G}(z)^{-1}L_{0}^{(3)}(z), & z \in D(\delta) \setminus \left( \overline{D(r_n)} \cup (-\delta, \delta) \right), 
\end{array} \right. \]  

where \( A_1^{(3)}(z) \) is given in (10.56). In view of (10.57), it follows that, as \( n \to \infty \),

\[ \frac{A_1^{(3)}(0)}{n^{6}z} = \begin{cases} 
\mathcal{O}(n^{-1}), & z \in \partial D(r_n), \\
\mathcal{O}(n^{-5/2}), & z \in \partial D(\delta), 
\end{cases} \]

which implies that the inverse of \( I_4 - \frac{A_1^{(3)}(0)}{n^{6}z} \) is well defined, and thus is \( L_0^{(4)} \). Furthermore, we have the following proposition.

**Proposition 10.14.** The matrix-valued function \( L_0^{(4)}(z) \) defined in (10.65) satisfies items (1)–(3) and the matching condition (10.1) of the RH problem 10.1 for \( L_0 \). Moreover, we have, as \( n \to \infty \),

\[ L_{0,+}^{(4)}(z) = (I_4 + \mathcal{O}(n^{-1/2}))L_{0,-}^{(4)}(z), \quad z \in \partial D(r_n) \setminus \Gamma_S. \]  

(10.66)
Proof. It suffices to show (10.66), while the other claims can be verified directly. For $z \in \partial D(r_n) \setminus \Gamma_S$ and $n \to \infty$, it is readily seen from (10.65) and (10.64) that

$$\begin{align*}
L^{(4)}_{0,+}(z)L^{(4)}_{0,-}(z)^{-1} & = \hat{G}(z)n^{\frac{\hat{b}}{2}} \left( I_4 - \frac{A^{(3)}_1(z) - A^{(3)}_1(0)}{n^6 z} \right) \left( I_4 + \frac{A^{(3)}_1(z)}{n^6 z} + \frac{A^{(3)}_2(z)}{n^9 z} + O(n^{-2}) \right) \\
& \quad \times \left( I_4 - \frac{A^{(3)}_1(0)}{n^6 z} \right) n^{-\frac{\hat{b}}{2}} \hat{G}(z)^{-1} \\
& = \hat{G}(z)n^{\frac{\hat{b}}{2}} \left( I_4 + \frac{A^{(3)}_1(0)}{n^6 z} + \frac{A^{(3)}_2(z)}{n^9 z} + O(n^{-2}) \right) \left( I_4 - \frac{A^{(3)}_1(0)}{n^6 z} \right) n^{-\frac{\hat{b}}{2}} \hat{G}(z)^{-1} \\
& = \hat{G}(z)n^{\frac{\hat{b}}{2}} \left( I_4 + \frac{A^{(3)}_2(z)}{n^9 z} + O(n^{-2}) \right) n^{-\frac{\hat{b}}{2}} \hat{G}(z)^{-1}, \quad (10.67)
\end{align*}$$

where for the second and third equality we have made use of the estimates (10.57), (10.58) and (10.62) to suppress the error terms. By (10.62), we further have

$$\frac{A^{(3)}_2(z)}{n^9 z} = O(n^{-3/2}), \quad z \in \partial D(r_n), \quad n \to \infty,$$

which, together with (10.67), yields (10.66).

This completes the proof of Proposition 10.14. \( \square \)

Last step towards the matching As the fifth and last step, we modify $L^{(4)}_0$ to

$$\begin{align*}
L_0(z) = L^{(5)}_0(z) & = \left\{ \begin{array}{ll}
\hat{G}(z)n^{\frac{\hat{b}}{2}} \left( I_4 - \frac{A^{(3)}_1(z) - A^{(3)}_1(0)}{n^6 z} \right) n^{-\frac{\hat{b}}{2}} \hat{G}(z)^{-1}L^{(4)}_0(z), & z \in D(r_n) \setminus \Gamma_S, \\
\hat{G}(z)n^{\frac{\hat{b}}{2}} \left( I_4 - \frac{A^{(3)}_2(z)}{n^9 z} \right)^{-1} n^{-\frac{\hat{b}}{2}} \hat{G}(z)^{-1}L^{(4)}_0(z), & z \in D(\delta) \setminus \left( D(r_n) \cup (-\delta, \delta) \right),
\end{array} \right.
\end{align*} \quad (10.68)$$

where $A^{(3)}_2(z)$ is given in (10.61). In view of (10.62), it follows that, as $n \to \infty$,

$$\frac{A^{(3)}_2(0)}{n^9 z} = \begin{cases} O(n^{-3/2}), & z \in \partial D(r_n), \\ O(n^{-3}), & z \in \partial D(\delta), \end{cases}$$

which implies that the inverse of $I_4 - \frac{A^{(3)}_2(0)}{n^9 z}$ is well defined, thus so is $L^{(5)}_0$. Following the same arguments as in the proof of Proposition 10.14, it is straightforward to conclude that

**Proposition 10.15.** The matrix-valued function $L^{(5)}_0(z)$ defined in (10.68) solves the RH problem 10.1 for $L_0$.

This completes the construction of the local parametrix near the origin.
11. Final Transformation $S \rightarrow R$

With the global parametrix $G$ given in Proposition 8.5, the local parametrices $L_p$ and $L_{-q}$ near $p$ and $-q$ briefly discussed in Sect. 9 and the local parametrix $L_0$ near the origin constructed in (10.68), the final transformation is defined by

$$
R(z) = \begin{cases} 
S(z)L_0(z)^{-1}, & z \in D(\delta), \\
S(z)L_p(z)^{-1}, & z \in D_p(\delta), \\
S(z)L_{-q}(z)^{-1}, & z \in D_{-q}(\delta), \\
S(z)G(z)^{-1}, & \mathbb{C}\setminus D_R,
\end{cases}
$$

where

$$
D_R := D(\delta) \cup D_p(\delta) \cup D_{-q}(\delta).
$$

It is then straightforward to check that $R$ satisfies the following RH problem.

**RH Problem 11.1.** The function $R$ defined in (11.1) has the following properties:

1. $R$ is defined and analytic in $\mathbb{C}\setminus \Gamma_R$, where

$$
\Gamma_R := \partial D_R \cup \partial D(\delta) \cup (-\delta, 0) \cup \partial C \setminus \gamma_R \\
\cup \left( \bigcup_{j=1}^{3} \partial \mathcal{L}^{\pm}_j \right) \setminus \left( D(\delta) \cup D_p(\delta) \cup D_{-q}(\delta) \right)
$$

with the orientations as illustrated in Fig. 5.

2. For $z \in \Gamma_R$, $R$ satisfies the jump condition

$$
R_+(z) = R_-(z)J_R(z),
$$

where

$$
J_R(z) = \begin{cases} 
G(z)J_S(z)G(z)^{-1}, & z \in \Gamma_R \setminus (D_R \cup (-\delta, 0)), \\
L_0(z)J_S(z)L_0(z)^{-1}, & z \in \bigcup_{j=1}^{3} \left( \partial \mathcal{L}^{\pm}_j \cap D(\delta) \right) \setminus \partial D(\delta), \\
I_4 - |z|^\kappa e^{n\phi_1(z)}G_-(z)E_2G_-(z)^{-1}, & z \in (-\delta, 0), \\
I_4 - |z|^\kappa e^{n\phi_1(z)}L_{0,-}(z)E_2L_{0,-}(z)^{-1}, & z \in (-\delta, 0), \\
G(z)L_p(z)^{-1}, & z \in \partial D_p(\delta), \\
G(z)L_{-q}(z)^{-1}, & z \in \partial D_{-q}(\delta), \\
G(z)L_0(z)^{-1}, & z \in \partial D(\delta), \\
L_{0,-}(z)L_{0,+}(z)^{-1}, & z \in \partial D(\delta),
\end{cases}
$$

(11.2)
and where $J_5(z)$ is given in (7.7).

(3) As $z \to \infty$, we have

$$ R(z) = I_4 + O(z^{-1}). $$

It comes out that the jump matrix of $R$ on each jump contour tends to the identity matrix for large $n$ with the convergence rate given in the next lemma.

**Lemma 11.2.** Let $J_R(z)$ be defined in (11.2). There exists two positive constants $c_1, c_2$ such that, as $n \to \infty$,

$$ J_R(z) = \begin{cases} 
I_4 + O(n^{-1}), & z \in \partial D_R \cup \partial D(r_n), \\
I_4 + O(e^{-c_1n^{1/2}}), & z \in \bigcup_{j=2,3} \left( \partial \mathcal{L}_j^\pm \cap D(\delta) \setminus D(r_n) \right), \\
I_4 + O(e^{-c_2n}), & \text{elsewhere on } \Gamma_R,
\end{cases} \tag{11.3} $$

uniformly for $z$ on the indicated contours.

**Proof.** By (11.2), the first estimate of $J_R(z)$ in (11.3), that is, on the boundaries of the four disks $\partial D_R \cup \partial D(r_n)$, follows directly from (9.1), (10.1) and (10.2).

For the estimate of $J_R(z)$ on $\bigcup_{j=2,3} \left( \partial \mathcal{L}_j^\pm \cap D(\delta) \setminus D(r_n) \right)$, we first focus on the case $j = 2$. From (11.2) and (7.7), it follows that

$$ J_R(z) = I_4 + z^{-\kappa} e^{n\phi_2(z)} L_0(z) E_{32} L_0(z)^{-1}, \quad z \in \partial \mathcal{L}_2^\pm \cap D(\delta) \setminus D(r_n). \tag{11.4} $$

Since $L_0(z)^{\pm 1}$ has at most power log singularities near the origin, the estimate of $J_R(z)$ then essentially relies on the behavior of $e^{n\phi_2(z)}$ near $z = 0$. In view of (4.2), (4.3), (4.10), (10.6) and Proposition 10.3, we have, for $z \in \partial \mathcal{L}_2^\pm \cap D(\delta) \setminus D(r_n),$

$$ |e^{n\phi_2(z)}| = |e^{n(\lambda_2(z) - \hat{\lambda}_3(z))}| = |e^{n(\hat{\lambda}_2^+(z) - \hat{\lambda}_3^-(z))}| \\
= |e^{n((\omega_2^+ - \omega_3^-) f_4(0) e^{1/3} + O(z^{2/3}))}| \leq e^{-cn|z|^{1/3}}, $$

for some $c > 0$. Note that $|z| > n^{-3/2}$ on the annulus $D(\delta) \setminus D(r_n)$, which together with the above estimate and (11.4) gives us

$$ J_R(z) = I_4 + O(e^{-cn^{1/2}}), \quad z \in \partial \mathcal{L}_2^\pm \cap D(\delta) \setminus D(r_n), $$

for large $n$. If $z \in \partial \mathcal{L}_3^\pm \cap D(\delta) \setminus D(r_n)$, the estimate of $J_R(z)$ can be derived in a similar manner, where one needs to explore the behavior of $e^{n\phi_3(z)}$ near $z = 0$. We omit the details.

Finally, for $z$ belonging to other parts of $\Gamma_R$, we note from (11.2) and (7.7) that, if $z \in (p + \delta, +\infty),$ 

$$ J_R(z) = I_4 + z^\kappa e^{-n\phi_2(z)} G(z) E_{23} G(z)^{-1}. $$

Since $G(z)^{\pm 1}$ grows at most in a power law for large $z$ (see (8.2)) and $\phi_2(z) > c$ for some $c > 0$ on $(p + \delta, +\infty)$ (see (4.15)), it is immediate to conclude from the above formula that

$$ J_R(z) = I_4 + O(e^{-cn}), \quad z \in (p + \delta, +\infty), $$
for large \( n \). The estimate of \( J_R(z) \) on \((-q + \delta, 0) \cup \left( \frac{3}{\sqrt{j = 1}} \partial \mathcal{L}_j^+ \right) \setminus D_R \) can be obtained by applying similar arguments.

A little extra effort is needed to handle the case \( z \in (-\delta, 0) \). Similarly as above, from (11.2) and (4.15), it suffices to show that \( L_0(z) E_{21} L_0(z)^{-1} \) has power growth in \( n \). To see this, from the definition of \( L_0 \) given in (10.68), and tracing back the transformations \( L_5 \to L_0 \to \ldots \to L_1 \), it is readily seen that

\[
L_0(z) = L_0^{(5)}(z) = \mathcal{A}_n(z) \hat{P}(z), \quad |z| < r_n,
\]

where the prefactor \( \mathcal{A}_n \) is analytic and invertible near the origin, with \( \mathcal{A}_n \) and \( \mathcal{A}_n^{-1} \) having at worse power growth as \( n \to \infty \), and \( \hat{P} \), defined in (10.21), contains the Meijer-G parametrix. From the structure of \( \hat{P} \) (see the first identity in (10.23)) and from (10.13) we see that, as \( z \to 0 \),

\[
\hat{P}(z) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T = O(1), \quad (1, 0, 0, 0) \hat{P}(z)^{-1} = O(1).
\]

Thus,

\[
L_0(z) E_{21} L_0(z)^{-1} = \mathcal{A}_n(z) \hat{P}(z) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T (1, 0, 0, 0) \hat{P}(z)^{-1} \mathcal{A}_n(z)^{-1}
\]

has at worse power growth as \( n \to \infty \).

This completes the proof of Lemma 11.2. \( \square \)

As a consequence of the above lemma, we conclude from the standard arguments in the RH analysis (cf. [23] and [15, Appendix A]) that

\[
R(z) = I_4 + O(n^{-1}), \quad n \to \infty, \quad (11.5)
\]

uniformly for \( z \in \mathbb{C} \setminus \Gamma_R \).

12. Proofs of Asymptotic Results

In this section, we will prove Theorems 2.3 and 2.4 by inverting the transformations (2.24).

12.1. Proof of Theorem 2.3. Let \( x, y \in \Delta_2 = (0, p) \) be fixed. In view of the representation of \( K_n \) given in (2.23), and having in mind (5.6)–(5.7) and the calculation (5.16), we obtain from (5.11) and a straightforward calculation that

\[
n^2 K_n \left( n^2 x, n^2 y \right) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & 0 & w_2(y) \end{pmatrix} \text{diag} \left( A_1(y)^{\frac{\nu}{\sigma_3}} A_2(y)^{\frac{\nu - \kappa}{\sigma_3}} \right) \times X_+(y)^{-1} X_+(x) \text{diag} \left( x^{\frac{\nu}{\sigma_3}} A_1(x)^{-1} x^{\frac{\nu - \kappa}{\sigma_3}} A_2(x)^{-1} \right) \begin{pmatrix} w_1(x) & 0 & 0 \end{pmatrix}^T \]

\[
= \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} X_+(y)^{-1} X_+(x) \begin{pmatrix} 0 & x & 0 \end{pmatrix}^T.
\]
From (6.1), this becomes
\[ n^2 K_n \left( n^2 x, n^2 y \right) = \frac{1}{2\pi i (x - y)} \begin{pmatrix} 0 & 0 \\ e^{n\lambda_3,+(y)} & 0 \end{pmatrix} T_+(y)^{-1} T_+(x) \begin{pmatrix} 0 \\ e^{-n\lambda_2,+}(x) \\ 0 \\ 0 \end{pmatrix}. \]

A further appeal to (7.5) and Proposition 4.2 yields
\[ n^2 K_n \left( n^2 x, n^2 y \right) = \frac{1}{2\pi i (x - y)} \left( 0 - y^{-\kappa} e^{n(\phi_2,+)(y)+\lambda_3,+(y)} e^{n\lambda_3,+(y)} 0 \right) S_+(y)^{-1} \]
\[ \times S_+(x) \left( 0 x^\kappa e^{-n\lambda_2,+}(x) e^{n(\phi_2,+)(x) - \lambda_2,+}(x)} 0 \right)^T \]
\[ = \frac{1}{2\pi i (x - y)} \left( 0 - y^{-\kappa} e^{n\lambda_2,+}(y) e^{n\lambda_3,+(y)} 0 \right) S_+(y)^{-1} \]
\[ \times S_+(x) \left( 0 x^\kappa e^{-n\lambda_2,+}(x) e^{-n\lambda_3,+(x)} 0 \right)^T. \tag{12.1} \]

Since both \( x \) and \( y \) are fixed, we may assume that \( \delta \) is chosen so as that \( x \) and \( y \) are outside the discs around the edges 0 and \( p \). From (11.5) and the analyticity of \( G_+ \) away from 0 and \( p \), we obtain that
\[ S_+(y)^{-1} S_+(x) = I_4 + O(x - y), \quad x \to y, \quad \tag{12.2} \]
uniformly for \( x, y \in [\delta, p - \delta] \) as \( n \to \infty \). Next, noticing that \( \lambda_2,\pm(x) = \lambda_3,\pm(x) \) for \( x \in \Delta_2 \) (see Proposition 4.2), by taking \( y \to x \), it then follows from (12.1), (12.2), L’Hôpital’s rule, (4.3) and (2.15) that
\[ n^2 K_n \left( n^2 x, n^2 x \right) = -\frac{n}{2\pi i} (\xi_{3,+}(x) - \xi_{3,-}(x)) + O(1) \]
\[ = \frac{n}{2\pi i} \left( C_{+\pm}^\mu(x) - C_{-\pm}^\mu(x) \right) + O(1) \]
\[ = n \frac{d\mu_\pm}{dx}(x) + O(1), \]
which implies that
\[ n K_n \left( n^2 x, n^2 x \right) = \frac{d\mu_\pm}{dx}(x) \left( 1 + O(n^{-1}) \right), \]
uniformly for \( x \in (\delta, p - \delta) \) as \( n \to \infty \). Similarly, it can be shown that
\[ \lim_{n \to \infty} n K_n \left( n^2 x, n^2 x \right) = 0, \quad x > p, \]
as desired.

This completes the proof of Theorem 2.3 away from the endpoints \( x = p \) and \( x = 0 \). The case for \( x = p \) can be handled similarly, with the Airy parametrix appearing instead of the global parametrix \( G \), and with a worse error term. \( \square \)
12.2. Proof of Theorem 2.4. To prove Theorem 2.4, let us start with \( u \) and \( v \) in the shrinking interval \((0, r_n)\) and trace back all the transformations (2.24). The transformations \( Y \mapsto X \) and \( X \mapsto T \), given in (5.11) and (6.1) respectively, are defined globally, whereas the transformation \( T \mapsto S \), defined in (7.5), is the same on the plus side of \( \Delta_2 \). Thus, even for \( u, v \) in the shrinking interval \((0, r_n)\), it holds

\[
n^2 K_n(n^2 u, n^2 v) = \frac{1}{2\pi i} \left( (0, -v^{-\kappa} e^{\eta \lambda_2,+(v)} e^{\eta \lambda_3,+(v)}) \right) \times S_+(v)^{-1} S_+(u) \left( (0, u^{-\kappa} e^{-\eta \lambda_2,+(u)} e^{-\eta \lambda_3,+(u)}) \right)^T ;
\]

see (12.1). Using now the transformation (11.1) on \( D(r_n) \subset D(\delta) \), we obtain

\[
S_+(v)^{-1} S_+(u) = L_0,+(v)^{-1} R(v)^{-1} R(u) L_0,+(u). \tag{12.4}
\]

We now scale \( u = u_n = \frac{x}{n^3(\beta^2 - \alpha^2)} \) and \( v = v_n = \frac{y}{n^3(\beta^2 - \alpha^2)} \), where \( x, y \) are in fixed compact subsets of \((0, \infty)\). Note that with this scaling the points \( u \) and \( v \) fall inside \((0, r_n)\) and the calculations above are bona fide. To estimate (12.4) for large \( n \), we will need the following lemma, which is a refined version of Lemma 10.8.

**Lemma 12.1.** Suppose that \( \{M_n\} \) is a sequence of matrix-valued functions satisfying the conditions of Lemma 10.8 and for which there exists a bounded sequence of constant matrices \( \{\tilde{M}_n\} \) for which

\[
M_n(z) - \tilde{M}_n = O(\delta_n), \quad n \to \infty, \tag{12.5}
\]

uniformly for \( z \in \partial D(2\varepsilon_n) \), where \( \{\delta_n\} \) is a sequence of bounded positive numbers (possibly with \( \delta_n \to 0 \) but not necessarily). Then, (10.35) can be improved to

\[
M_n(z) - M_n(w) = O\left( \frac{\delta_n}{\varepsilon_n} (z - w) \right), \quad n \to \infty,
\]

uniformly for \( z, w \in \overline{D(\varepsilon_n)} \).

**Proof.** Similarly as in the proof of Lemma 10.8, we write

\[
M_n(z) - M_n(w) = M_n(z) - \tilde{M}_n - (M_n(w) - \tilde{M}_n) = \frac{z - w}{2\pi i} \int_{|t| = 2\varepsilon} \frac{M_n(t) - \tilde{M}_n}{t - z} \frac{dw}{t - w}.
\]

It remains to estimate the numerator using (12.5), and the lemma follows. \( \square \)

We start estimating \( R \). The following lemma also appears in [38, Lemma 6.5], although the proof has to be slightly modified to account for the jump of \( R \) along \((-r_n, 0)\) that appears here but not in the mentioned work.

**Lemma 12.2.** The matrix \( R \) satisfies

\[
R(v_n)^{-1} R(u_n) = I_4 + O(n^{-5/2}(x - y))
\]

uniformly for \( x, y \) in compact subsets of \((0, \infty)\).
Proof. For γ being any contour for which R is analytic in its interior and encircling \( u_n \) and \( v_n \) counter-clockwise, we write with the help of Cauchy’s integral formula

\[
R(u_n) - R(v_n) = R(u_n) - I_4 - (R(v_n) - I_4) = \frac{u_n - v_n}{2\pi i} \oint_{\gamma} \frac{R(s) - I_4}{(s - u_n)(s - v_n)} \, ds.
\]

We apply this to \( \gamma \) being the boundary of the slit disk \( D(r_n/2) \backslash (-r_n/2, 0] \) and obtain

\[
R(u_n) - R(v_n) = \frac{u_n - v_n}{2\pi i} \left( \oint_{|s|=\frac{r_n}{2}} \frac{R(s) - I_4}{(s - u_n)(s - v_n)} \, ds + \int_{-\frac{r_n}{2}}^{0} \frac{R_+(s) - R_-(s)}{(s - u_n)(s - v_n)} \, ds \right).\tag{12.6}
\]

For \( s \in (-r_n/2, 0] \), it is readily seen from Lemma 11.2 that

\[
R_+(s) - R_-(s) = R_- (s)(J_R(s) - I_4) = O(e^{-c_2 n}),
\]

where we have also made use of the fact that \( R \) remains uniformly bounded near 0. Moreover, since \(|s - u_n|, |s - v_n|\) decay with \( O(n^{-3}) \) along the interval \((-r_n/2, 0]\), it follows that

\[
\frac{u_n - v_n}{2\pi i} \int_{-\frac{r_n}{2}}^{0} \frac{R_+(s) - R_-(s)}{(s - u_n)(s - v_n)} \, ds = O((x - y)e^{-c_n}),
\]

for some constant \( c > 0 \). The first integral in (12.6) can be estimated from (11.5) and using the same approach in the proof of Lemma 12.1, allowing us to conclude that

\[
R(u_n) - R(v_n) = O((x - y)n^{-5/2}),
\]

uniformly for \( x, y \) in compact subsets of \((0, \infty)\). To conclude the lemma, simply write

\[
R(v_n)^{-1}R(u_n) = I_4 + R(v_n)^{-1}(R(u_n) - R(v_n))
\]

and use that \( R \) remains bounded near the origin.

This completes the proof of Lemma 12.2. \( \Box \)

Next, we need to estimate \( L_0 \), which is more cumbersome. We start by spelling it out after unraveling the transformations \( L_0 = L_0^{(5)} \leftrightarrow L_0^{(1)} \), which are given in Sect. 10.3, giving us that

\[
L_0(z) = \hat{L}_0(z) \hat{P}(z),
\]

\[
\hat{L}_0(z) := \hat{G}(z)n^\frac{2}{5}A^{(3)}(z)A^{(2)}(z)A^{(1)}(z)\hat{D}_n(z)^{-1}n^\frac{2}{5}(I_4 + T_0)^{-1}n^{-\hat{B}},\tag{12.7}
\]

with

\[
A^{(j)}(z) = I_4 - \frac{A_1^{(j+1)}(z) - A_1^{(j+1)}(0)}{n^{3j}z}, \quad j = 1, 2,
\]

\[
A^{(3)}(z) = I_4 - \frac{A_2^{(3)}(z) - A_2^{(3)}(0)}{n^9z}.
\]
Lemma 12.3. The matrix-valued function \( \hat{L}_0(z) \) defined in (12.7) satisfies
\[
\hat{L}_0(v_n)^{-1}\hat{L}_0(u_n) = n^{\frac{\hat{\beta}}{2}} \left( I_4 + \mathcal{O}(n^{-3/2}(x - y)) \right) n^{-\hat{\beta}}, \quad n \to \infty,
\]
uniformly for \( x, y \) in compact subsets of \( (0, \infty) \).

**Proof.** The analyticity of \( \hat{G} \) and its inverse near the origin (recall Proposition 8.6) and the fact that they do not depend on \( n \) gives
\[
\hat{G}(v_n)^{-1}\hat{G}(u_n) = I_4 + \hat{G}(v_n)^{-1}(\hat{G}(u_n) - \hat{G}(v_n)) = I_4 + \mathcal{O}(u_n - v_n) = I_4 + \mathcal{O}((x - y)n^{-3}), \quad n \to \infty,
\]
(12.8)
uniformly for \( x, y \in (0, \infty) \).

The function \( \mathcal{A}^{(3)} \) is analytic in a neighborhood of \( \overline{D(r_n)} \), and in virtue of (10.62),
\[
\mathcal{A}^{(3)}(z) - I_4 = \mathcal{O}(n^{-3/2}),
\]
so from Lemma 12.1 with \( \delta_n = n^{-3/2} = \varepsilon_n \) and \( \hat{M}_n = I_4 \), it follows that
\[
\mathcal{A}^{(3)}(u_n) - \mathcal{A}^{(3)}(v_n) = \mathcal{O}(n^{-3}(x - y)),
\]
and consequently as in (12.8)
\[
\mathcal{A}^{(3)}(v_n)^{-1}\mathcal{A}^{(3)}(u_n) = I_4 + \mathcal{O}(n^{-3}(x - y)). \quad (12.9)
\]
Similarly, using (10.58), we find that
\[
\mathcal{A}^{(2)}(v_n)^{-1}\mathcal{A}^{(2)}(u_n) = I_4 + \mathcal{O}(n^{-5/2}(x - y)). \quad (12.10)
\]
Finally, from (10.45), it is readily seen that
\[
A_1^{(2)}(z)\hat{D}_n(z)^{-1} = \hat{D}_n(z)^{-1}n^{\frac{\hat{\alpha}}{2}}A_1^{(1)}n^{-\frac{\hat{\beta}}{2}}.
\]
This, together with (10.38), implies that
\[
\mathcal{A}^{(1)}(z)\hat{D}_n(z)^{-1} = \hat{D}_n(z)^{-1} - \left( \hat{D}_n(z)^{-1} - \hat{D}_n(0)^{-1} \right) n^{\frac{\hat{\alpha}}{2}}A_1^{(1)}n^{-\frac{\hat{\beta}}{2}} + \hat{D}_n(0)^{-1}n^{\frac{\hat{\alpha}}{2}}A_1^{(1)}n^{-\frac{\hat{\beta}}{2}} \mathcal{O}(n^{3/2}z)
\]
\[
= \hat{D}_n(z)^{-1} + \mathcal{O}(n^{-1/2}).
\]
According to Proposition 10.9, the right-hand side above is bounded, so from Lemma 10.8, we obtain
\[
\hat{D}_n(v_n)A^{(1)}(v_n)^{-1}A^{(1)}(u_n)\hat{D}_n(u_n)^{-1} = I_4 + \mathcal{O}(n^{3/2}(u_n - v_n)) = I_4 + \mathcal{O}(n^{-3/2}(x - y)). \quad (12.11)
\]
Moving towards the end of the proof, let us combine all the equations (12.8)–(12.11) into the definition (12.7) of \( \hat{L}_0 \) to obtain
\[ \mathcal{L}_0(v_n)^{-1} \mathcal{L}_0(u_n) = n^{\frac{\tilde{B}}{2}} (I_4 + T_0) n^{-\frac{\tilde{B}}{2}} \left( I_4 + \mathcal{O}(n^{-3/2}(x - y)) \right) n^{\frac{\tilde{B}}{2}} (I_4 + T_0)^{-1} n^{-\frac{\tilde{B}}{2}}. \] (12.12)

Now, having in mind (10.43), it follows that
\[ n^{\frac{\tilde{B}}{2}} (I + T_0) n^{-\frac{\tilde{B}}{2}} = I_4 + n^{\frac{\tilde{B}}{2}} T_0 n^{-\frac{\tilde{B}}{2}} \quad \text{with} \quad n^{\frac{\tilde{B}}{2}} T_0^k n^{-\frac{\tilde{B}}{2}} = \mathcal{O}(n^{-1/2}), \quad k = 1, 2, \]
and also
\[ n^{\frac{\tilde{B}}{2}} (I_4 + T_0)^{-1} n^{-\frac{\tilde{B}}{2}} = I_4 - n^{\frac{\tilde{B}}{2}} T_0 n^{-\frac{\tilde{B}}{2}} + n^{\frac{\tilde{B}}{2}} T_0^2 n^{-\frac{\tilde{B}}{2}}. \]

Plugging these last two identities into (12.12) concludes the proof. □

Using Lemmas 12.2 and 12.3 in (12.4), we see that
\[ S_+(v_n)^{-1} S_+(u_n) = \hat{P}_+(v_n)^{-1} n^{\frac{\tilde{B}}{2}} \left( I_4 + \mathcal{O}(n^{-3/2}(x - y)) \right) n^{-\frac{\tilde{B}}{2}} \hat{P}_+(u_n), \quad n \to \infty, \]
uniformly for \( x, y \) in compact subsets of \((0, \infty)\). Thus, it is readily seen from (12.3) that
\[
n^2 K_n \left( \frac{x}{n(\beta^2 - \alpha^2)}, \frac{y}{n(\beta^2 - \alpha^2)} \right) = n^2 K_n(n^2 u_n, n^2 v_n) \\
= n^3(\beta^2 - \alpha^2) \frac{1}{2\pi i (x - y)} \left(0 - v_n^{-\kappa} e^{n\lambda_{2,+}(v_n)} e^{n\lambda_{3,+}(v_n)} 0\right) \hat{P}_+(v_n)^{-1} n^{\frac{\tilde{B}}{2}} \\
\times \left( I_4 + \mathcal{O}(n^{-3/2}(x - y)) \right) n^{-\frac{\tilde{B}}{2}} \hat{P}_+(u_n) \left(0 u_n^\kappa e^{-n\lambda_{2,+}(u_n)} e^{-n\lambda_{3,+}(u_n)} 0\right)^T,
\]
and then using (10.21),
\[
\frac{1}{n(\beta^2 - \alpha^2)} K_n \left( \frac{x}{n(\beta^2 - \alpha^2)}, \frac{y}{n(\beta^2 - \alpha^2)} \right) \\
= \frac{1}{2\pi i (x - y)} \left(0 - v_n^{-\kappa} 1 0\right) \\
\times P_+(v_n)^{-1} n^{\frac{\tilde{B}}{2}} \left( I_4 + \mathcal{O}(n^{-3/2}(x - y)) \right) n^{-\frac{\tilde{B}}{2}} P_+(u_n) \left(0 u_n^\kappa 1 0\right)^T.
\] (12.13)

To simplify it further, we use (10.20), Proposition 10.3 and the definitions of \( A \) and \( \hat{B} \) in (8.17) and (10.27), respectively, to get
\[
(0 - v_n^{-\kappa} 1 0) P_+(v_n)^{-1} \\
= n^{-\nu+2\kappa} \left(0 - y^{-\kappa}(\beta^2 - \alpha^2)^\kappa 1 0\right) \\
\times \text{diag} \left(1, \left(\frac{f_4(u_n)}{3} \right)^{-A} \Psi_+(n^3 \varphi(v_n))^{-1} \left(\frac{f_4(v_n)}{3} \right)^{-B} \right) n^{-\hat{B}}
\]
and

\[ P_+(u_n) \left( \begin{array}{c} 0 \\ u_n^\kappa \\ 1 \\ 0 \end{array} \right)^T = n^{v-2\kappa} n^{\hat{B}} \times \text{diag} \left( 1, \left( \frac{f_4(u_n)}{3} \right)^B \Psi_+(n^3 \varphi(u_n)) \left( \frac{f_4(u_n)}{3} \right)^A \right) \left( \begin{array}{c} 0 \\ x^\kappa (\beta^2 - \alpha^2)^{-\kappa} \\ 1 \\ 0 \end{array} \right)^T. \]

Moving forward, we now use Proposition 10.3 to obtain

\[ \frac{f_4(u_n)}{3} = (\beta^2 - \alpha^2)^{\frac{1}{3}} + O(n^{-3}) = \frac{f_4(v_n)}{3}, \quad n \to \infty, \]

and from (10.19)

\[ n^3 \varphi(u_n) = x(1 + O(n^{-3})), \quad n^3 \varphi(v_n) = y(1 + O(n^{-3})), \quad n \to \infty, \]

where the error terms above are uniform for \( x, y \) in compact subsets of \((0, +\infty)\). Combining with the analyticity of \( \Psi_+ \), we thus conclude

\[ \left( \begin{array}{c} 0 \\ -v_n^\kappa \\ 1 \\ 0 \end{array} \right) P_+(v_n)^{-1} = n^{-v+2\kappa} (\beta^2 - \alpha^2)^{-\frac{2\kappa-v}{3}} \left( \begin{array}{c} 0 \\ -y^\kappa \\ 1 \\ 0 \end{array} \right) \times \text{diag} \left( 1, \Psi_+(y)^{-1} \right) (\beta^2 - \alpha^2)^{-\frac{2}{3}} (I_4 + \mathcal{E}_n(v_n))^{-1} n^{-\hat{B}}, \]

where \( \{\mathcal{E}_n\} \) is a sequence of \( 4 \times 4 \) matrix-valued analytic functions on \( D(r_n) \) with

\[ \mathcal{E}_n(z) = O(n^{-3}) \quad \text{uniformly for} \quad z \in D(r_n) \quad \text{as} \quad n \to \infty, \quad (12.14) \]

and

\[ P_+(u_n) \left( \begin{array}{c} 0 \\ u_n^\kappa \\ 1 \\ 0 \end{array} \right)^T = n^{v-2\kappa} (\beta^2 - \alpha^2)^{-\frac{v-2\kappa}{3}} n^{\hat{B}} (I_4 + \mathcal{E}_n(u_n)) \times (\beta^2 - \alpha^2)^{\frac{2}{3}} \text{diag} \left( 1, \Psi_+(x) \right) \left( \begin{array}{c} 0 \\ x^\kappa \\ 1 \\ 0 \end{array} \right)^T \]

with the same error function \( \mathcal{E}_n \).

Inserting these last two identities into (12.13), we have

\[ \frac{1}{n(\beta^2 - \alpha^2)^2} K_n \left( \frac{x}{n(\beta^2 - \alpha^2)}, \frac{y}{n(\beta^2 - \alpha^2)} \right) = \frac{1}{2\pi i(x - y)} \left( \begin{array}{c} 0 \\ -y^\kappa \\ 1 \\ 0 \end{array} \right) \text{diag} \left( 1, \Psi_+(y)^{-1} \right) (\beta^2 - \alpha^2)^{-\frac{2}{3}} (I_4 + \mathcal{E}_n(v_n))^{-1} n^{-\hat{B}} \]

\[ \times \left( I_4 + O(n^{-\frac{3}{2}}(x - y)) \right) n^{\frac{2\hat{B}}{3}} (I_4 + \mathcal{E}_n(u_n)) (\beta^2 - \alpha^2)^{\frac{2}{3}} \text{diag} \left( 1, \Psi_+(x) \right) \left( \begin{array}{c} 0 \\ x^\kappa \\ 1 \\ 0 \end{array} \right) \]

\[ = \frac{1}{2\pi i(x - y)} \left( \begin{array}{c} 0 \\ -y^\kappa \\ 1 \\ 0 \end{array} \right) \text{diag} \left( 1, \Psi_+(y)^{-1} \right) (\beta^2 - \alpha^2)^{-\frac{2}{3}} \]

\[ \times \left( (I_4 + \mathcal{E}_n(v_n))^{-1} (I_4 + \mathcal{E}_n(u_n)) + O(n^{-\frac{1}{2}}(x - y)) \right) (\beta^2 - \alpha^2)^{\frac{2}{3}} \text{diag} \left( 1, \Psi_+(x) \right) \left( \begin{array}{c} 0 \\ x^\kappa \\ 1 \\ 0 \end{array} \right). \]
In virtue of (12.14), we can once more apply Lemma 10.8 to get that

\[(I_4 + \mathcal{E}_n(v_n))^{-1}(I_4 + \mathcal{E}_n(u_n)) = I_4 + \mathcal{O}(n^{-3}(x - y)), \quad \text{as } n \to \infty,\]

and we finally arrive at

\[
\frac{1}{n(\beta^2 - \alpha^2)} K_n \left( \frac{x}{n(\beta^2 - \alpha^2)}, \frac{y}{n(\beta^2 - \alpha^2)} \right) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & -y^{-\kappa} & 1 & 0 \end{pmatrix} \Psi_+(y)^{-1} \left( I_4 + \mathcal{O}(n^{-\frac{1}{2}}(x - y)) \right) \times \Psi_+(x) \begin{pmatrix} 0 & x^\kappa & 1 & 0 \end{pmatrix}^T + \mathcal{O}(n^{-1/2}),
\]

where, as always, the error term is uniform for \(x, y\) in compact subsets of \((0, \infty)\). Hence, we obtain that

\[
\lim_{n \to \infty} \frac{1}{n(\beta^2 - \alpha^2)} K_n \left( \frac{x}{n(\beta^2 - \alpha^2)}, \frac{y}{n(\beta^2 - \alpha^2)} \right) = K_\infty(x, y)
\]

uniformly for \(x, y\) in compact subsets of \((0, \infty)\), where

\[
K_\infty(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & -y^{-\kappa} & 1 & 0 \end{pmatrix} \Psi_+(y)^{-1} \Psi_+(x) \begin{pmatrix} x^\kappa & 1 \\ 0 \end{pmatrix}.
\]

To conclude the proof of Theorem 2.4, it remains to relate \(K_\infty\) with \(K_{\nu,\kappa}\) as in (2.19). To do so, first observe that \(\Psi\) - and hence \(K_\infty\) - does not depend on \(\alpha\) and \(\beta\), as can be seen from the RH problem 10.4 whose conditions do not depend on \(\alpha\) and \(\beta\). Thus, it is enough to relate \(K_\infty\) with \(K_{\nu,\kappa}\) for one specific choice of \(\alpha\) and \(\beta\), which we take to be matching those in (1.5), that is,

\[
\beta = \frac{1}{2\tau} + \frac{1}{2}, \quad \alpha = \frac{1}{2\tau} - \frac{1}{2}, \quad \text{so} \quad \beta^2 - \alpha^2 = \frac{1}{\tau},
\]

where \(0 < \tau < 1\) is any fixed number. For this specific coupling, our model (1.1) coincides with the model considered by Liu [43], so comparing\(^3\) with [43, Theorem 1.3(i) and Equation (5.20)] we arrive at

\[
K_\infty(x, y) = \left( \frac{y}{x} \right)^{\kappa/2} K_{\nu,\kappa}(y, x).
\]

Alternatively, the above relation can be seen from the RH characterization of the Meijer G-kernel commented in [11, Section 4.2.5].

This completes the proof of Theorem 2.4. \(\Box\)

\(^3\) The correspondence between our parameters \(\alpha = \alpha_{SZ}, \beta = \beta_{SZ}\) and \(\tau = \tau_{SZ}\) and Liu’s parameters \(\delta_L, \alpha_L\) and \(\mu_L\) is \(\beta_{SZ} = \alpha_L, \alpha_{SZ} = \delta_L\) and \(\tau_{SZ} = \beta_L\).
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Appendix A: Heuristics on the Vector Equilibrium Problem

In this section, we give some heuristic arguments on how to formulate the vector equilibrium problem introduced in Sect. 2.2, which is closely related to the asymptotic analysis of the RH problem for \( Y \).

Recall that the goal of the second transformation \( X \rightarrow T \) is to ‘normalize’ the large \( z \) asymptotics of \( X \) and to prepare for the opening of lenses. We assume that, at this moment, it takes the following form:

\[
T(z) = \hat{C} X(z) \text{diag}(e^{n\lambda_1(z)}, e^{n\lambda_2(z)}, e^{n\lambda_3(z)}, e^{n\lambda_4(z)}),
\]

(A.1)

where \( \hat{C} \) is a constant matrix and the \( \lambda \)-functions are of the form

\[
\begin{align*}
\lambda_1(z) &= \int_{-\infty}^{z} C^{\mu_1}(s) \, ds + V_1(z), \\
\lambda_2(z) &= \int_{-\infty}^{z} C^{\mu_2}(s) \, ds - \int_{-\infty}^{z} C^{\mu_1}(s) \, ds + V_2(z), \\
\lambda_3(z) &= \int_{-\infty}^{z} C^{\mu_3}(s) \, ds - \int_{-\infty}^{z} C^{\mu_2}(s) \, ds + V_3(z), \\
\lambda_4(z) &= -\int_{-\infty}^{z} C^{\mu_3}(s) \, ds + V_4(z).
\end{align*}
\]

(A.2)

In the above formulas, \( C^{\mu}(z) \) is the Cauchy transform of a measure \( \mu \) given in (2.14), \( \mu_1, \mu_2 \) and \( \mu_3 \) are three measures satisfying

\[
\begin{align*}
\text{supp } \mu_1 &\subset \mathbb{R}_-, \quad \text{supp } \mu_2 \subset \mathbb{R}_+, \quad \text{supp } \mu_3 \subset \mathbb{R}_-, \\
2|\mu_1| &= |\mu_2| = |\mu_3| = 1,
\end{align*}
\]

(A.3)

and \( V_1, V_2, V_3, V_4 \) are four functions to be determined.

As \( z \rightarrow \infty \), it is readily seen from (A.1) and (5.13) that,

\[
T(z) = (I_4 + \mathcal{O}(z^{-1})) B(z) \times \text{diag} \left( z^{-\frac{n}{2}} e^{n(\lambda_1 - 2\alpha z^\frac{1}{2})}, z^{\frac{n}{2}} e^{n(\lambda_2 + 2\alpha z^\frac{1}{2})}, z^{-\frac{n}{2}} e^{n(\lambda_3 + 2\beta z^\frac{1}{2})}, z^{\frac{n}{2}} e^{n(\lambda_4 - 2\beta z^\frac{1}{2})} \right).
\]

The normalization requirement then invokes us to expect that, as \( z \rightarrow \infty \),

\[
\begin{align*}
\lambda_1(z) - 2\alpha z^\frac{1}{2} + \frac{1}{2} \log z &= o(1), \quad &\lambda_2(z) + 2\alpha z^\frac{1}{2} + \frac{1}{2} \log z &= o(1), \\
\lambda_3(z) + 2\beta z^\frac{1}{2} - \frac{1}{2} \log z &= o(1), \quad &\lambda_4(z) - 2\beta z^\frac{1}{2} - \frac{1}{2} \log z &= o(1).
\end{align*}
\]

(A.5)
On the other hand, in view of (3.1), it follows that, as $z \to \infty$, 
\[
\lambda_1(z) = V_1(z) - \frac{1}{2} \log z + \mathcal{O}(z^{-1}), \quad \lambda_2(z) = V_2(z) - \frac{1}{2} \log z + \mathcal{O}(z^{-1}), \\
\lambda_3(z) = V_3(z) + \frac{1}{2} \log z + \mathcal{O}(z^{-1}), \quad \lambda_4(z) = V_4(z) + \frac{1}{2} \log z + \mathcal{O}(z^{-1}).
\]

Comparing these asymptotics with (A.5), it is easily seen that we should have 
\[
V_1(z) = 2\alpha z^\frac{1}{2}, \quad V_2(z) = -2\alpha z^\frac{1}{2}, \\
V_3(z) = -2\beta z^\frac{1}{2}, \quad V_4(z) = 2\beta z^\frac{1}{2}, \quad (A.6)
\]

We next come to the jump condition satisfied by $T$. Taking into account (A.1), (A.3) and (5.12), it is readily seen that 
\[
T_+(x) = T_-(x) J_T(x), \quad x \in \mathbb{R},
\]
where 
\[
J_T(x) = \text{diag} \left( 1, e^{\ell(x)} \right),
\]
and 
\[
J_T(x) = \Lambda \text{diag} \left( e^{\ell(x)}, e^{\mu(x)} \right) \left( E_{21} - e^{\ell(x)} E_{34} \right), \quad x \in \mathbb{R}_-,
\]
with 
\[
\Lambda := \text{diag} (e^{-\pi i \sigma_3}, e^{\pi i (\nu - \kappa) \sigma_3}).
\]

We now look at the non-diagonal entries of the jump matrix $J_T$. It is expected that these entries to be constant on the supports of the measures. Taking their real part, we arrive at the following conditions.

- (2, 3)-entry on $\mathbb{R}_+$: 
  \[
  2U^{\mu_2}(x) - U^{\mu_1}(x) - U^{\mu_3}(x) + \text{Re} \left( V_2(x) - V_3(x) \right) = \ell_2;
  \]

- (2, 1)-entry on $\mathbb{R}_-$: 
  \[
  2U^{\mu_1}(x) - U^{\mu_2}(x) + \text{Re} \left( V_{1,+}(x) - V_{2,-}(x) \right) = \ell_1;
  \]

- (3, 4)-entry on $\mathbb{R}_-$: 
  \[
  2U^{\mu_3}(x) - U^{\mu_2}(x) + \text{Re} \left( V_{3,-}(x) - V_{4,+}(x) \right) = \ell_3.
  \]
where $\ell_j$, $j = 1, 2, 3$, is constant. From (A.6), we thus find that the potentials $Q_1$, $Q_2$ and $Q_3$ acting on the measures $\mu_1$, $\mu_2$ and $\mu_3$ should be

\[
\begin{align*}
Q_1(x) &= \Re (V_{1,+}(x) - V_{2,-}(x)) = 2\alpha(\sqrt{x})_+ + 2\alpha(\sqrt{x})_- = 0, \\
Q_2(x) &= \Re (V_2(x) - V_3(x)) = 2(\beta - \alpha)\sqrt{x}, \\
Q_3(x) &= \Re (V_{3,-}(x) - V_{4,+}(x)) = -2\beta(\sqrt{x})_- - 2\beta(\sqrt{x})_+ = 0,
\end{align*}
\]

as shown in (2.6).

Finally, we explain the upper constraint. The fact that there is an upper constraint for $\mu_1$ but not for $\mu_2$, $\mu_3$ is connected to the form of the jumps: the equilibrium conditions for $\mu_1$ play a role in a lower triangular block of the jump matrix, whereas for the remaining measures the corresponding equilibrium conditions appear in an upper triangular block. In virtue of the direction of the variational inequalities for the equilibrium problem, we thus expect that associated to $\mu_1$ there should be an upper constraint, but no upper constraint should appear on the remaining measures.

To find the explicit form of the constraint, again some ansatz is needed. We expect that the functions $\lambda_1'$, $\lambda_2'$, $\lambda_3'$ and $\lambda_4'$ should all be solutions to the same algebraic equation (a.k.a. spectral curve). From the sheet structure for the associated Riemann surface, we also expect that $\lambda_1'$ is analytic across the places where $\sigma$ is active, that is, $\lambda_1'$ should be analytic across $\mathbb{R}_- \setminus \text{supp}(\sigma - \mu_1)$. Hence,

\[
\lambda_{1,+}'(x) - \lambda_{1,-}'(x) = 0, \quad x \in \mathbb{R}_- \setminus \text{supp}(\sigma - \mu_1).
\]

Using the explicit expression for $\lambda_1$ (see (A.2) and (A.6)) and Plemelj’s formula (3.4), we can rewrite the identity above as

\[
\frac{1}{2\pi i} \left( C_{1,1}'(x) - C_{1,1}'(x) + V_{1,+}'(x) - V_{1,-}'(x) \right) = \frac{d\mu_1}{dx}(x) + \frac{\alpha}{2\pi i} \left( (x^{-\frac{1}{2}})_+ - (x^{-\frac{1}{2}})_- \right) = 0, \quad x \in \mathbb{R}_- \setminus \text{supp}(\sigma - \mu_1).
\]

Taking into account that $\mu_1 = \sigma$ on $\mathbb{R}_- \setminus \text{supp}(\sigma - \mu_1)$, the identity above gives us

\[
\frac{d\sigma}{dx}(x) = -\frac{\alpha}{\pi \sqrt{|x|}}, \quad x \in \mathbb{R}_- \setminus \text{supp}(\sigma - \mu_1),
\]

which is (2.7).

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