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Abstract

The chaos bound in the near-horizon regions has been studied through the expansions of the metric functions on the horizon. In this paper, we investigate the chaos bound in the near-horizon region and at a certain distance from the horizon of a charged Kiselev black hole. The value of the Lyapunov exponent is accurately calculated by a Jacobian matrix. The angular momentum of a charged particle around the black hole affects not only the exponent, but also the position of the equilibrium orbit. This position gradually moves away from the horizon with the increase of the angular momentum. We find that the bound is violated at a certain distance from the horizon and there is no violation in the near-horizon region when the charge mass ratio of the particle is fixed. The small value of the normalization factor is more likely to cause the violation.
I. INTRODUCTION

Chaos is a kind of unpredictable and random motion in deterministically and nonlinearly dynamic systems due to the sensitivity to initial conditions. In quantum many-body systems, an important probe of chaotic behaviors is out-of-time ordered correlators (OTOCs). Through the diagnosis of the chaos, people found that it increases exponentially with time, $C(t) \approx \exp(\lambda t)$, where $\lambda$ is a Lyapunov exponent to characterize the sensitive dependence on the initial conditions [1–5].

Recently, Maldacena, Shenker and Stanford researched the chaos in the thermal quantum system with a large number of degrees of freedom by using an OTOC. Based on the factorization assumption and the assumption that there is a large hierarchy between the dissipation time and the scrambling time, they put forward a conjecture that there is a universal upper bound for the Lyapunov exponent,

$$\lambda \leq \frac{2\pi T}{\hbar},$$

where $T$ is the temperature of the system [6]. After this seminal conjecture was put forward, it immediately attracted much attention. The chaos in various models and gravitational theories was researched [7–17, 19–44]. These researches play an important role in black hole physics and quantum information. When a particle is subjected to sufficiently strong electromagnetic or scalar forces, the particle can be very close to a black hole without falling into it. At this time, the chaotic behavior of the particle was discussed in [16]. The value of
the Lyapunov exponent was gotten and equal to the surface gravity of the black hole. From the relation between the surface gravity and temperature, this result supports the conjecture of Maldacena et al.

In the recent work, some violations of the chaos bound were found [45–49]. The static equilibrium of a charged probe particle around a black hole can be provided by the Lorentz force. Adjusting the charge mass ratio of the particle make it infinitely close to the event horizon. Taking into account the contributions of the sub-leading terms in the near-horizon expansion, Zhao et al studied the chaos bound in the near-horizon regions by using the effective potentials [45]. They found that the bound was satisfied by the Reissner-Nordström and Reissner-Nordström-AdS black holes and violated by a large number of charged black holes. In the derivation, they only considered the contribution of the radial directions. In fact, the angular momentum of the particle has an influence on the Lyapunov exponent. This is because the angular momentum affects the effective potential and increases the magnitude of the chaotic behavior of the particle. When this influence is considered, the chaos bound in the near-horizon regions of the Reissner-Nordström and Reissner-Nordström-AdS black holes was studied again [49]. It was found that the bound is violated in the near-horizon regions when the charge of the black holes and the angular momentum of the particle are large. In the rotating charged black holes, the violation of the bound was also found through the calculation of the effective potentials [46, 47].

In this paper, we investigate the influence of the angular momentum of a charged particle on the chaos bound through circular motions of the particle around a charged Kiselev black hole. This black hole describes a space-time surrounded by an anisotropic fluid. It was first gotten by Kiselev [50] and its properties were studied in [51, 52]. Based on the work of Kiselev, some exact black hole solutions were gotten in [53, 55] and their properties were discussed in [56, 63]. Here the value of the Lyapunov exponent is accurately derived by a Jacobian matrix. A position of a equilibrium orbit of the particle is affected by its charge mass ratio and angular momentum. By fixing the charge mass ratio and changing the value of the angular momentum, we obtain positions of different equilibrium orbits. The bound is numerically discussed in the near-horizon region and at a certain distance from the horizon.

The rest is organized as follows. In the next section, taking into account a motion of a charged particle in the equatorial plane of a spherically symmetric black hole, we obtain a general expression of Lyapunov exponent by calculating the eigenvalue of the Jacobian
matrix. In Section 3, we investigate the influence of the angular momentum of the particle on the chaos bound in the near-horizon region and at a certain distance from the horizon of the charged Kiselev black hole. Section 4 is devoted to our conclusions.

II. LYAPUNOV EXPONENT IN CHARGED BLACK HOLES

We consider a circular motion of a charged particle in the equatorial plane of a spherically symmetric black hole to obtain the Lyapunov exponent. The black hole is given by

$$ds^2 = -F(r)dt^2 + \frac{1}{N(r)}dr^2 + C(r)d\theta^2 + D(r)d\phi^2,$$

with an electromagnetic potential $A_\mu = A_t dt$. When the particle with charge $q$ moves around the black hole, its Lagrangian is

$$L = \frac{1}{2} \left( -F \dot{t}^2 + \frac{\dot{r}^2}{N} + D \dot{\phi}^2 \right) - qA_t \dot{t},$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ and $\tau$ is a proper time. Using the definition of the generalized momentum $\pi_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$, we get

$$\pi_t = -F \dot{t} - qA_t = -E, \quad \pi_r = \frac{\dot{r}}{N}, \quad \pi_\phi = D \dot{\phi} = L.$$

In the above equation, $E$ and $L$ denote the energy and angular momentum of the particle, respectively. Thus the Hamiltonian of the particle is

$$H = \frac{-(\pi_t + qA_t)^2 + \pi_r^2 FN + \pi_\phi^2 D^{-1}F}{2F}.$$

From the Hamiltonian, the equations of motion are gotten, which are

\[
\begin{align*}
\dot{i} &= \frac{\partial H}{\partial \pi_t} = -\frac{\pi_t + qA_t}{F}, \quad \dot{\pi}_t = -\frac{\partial H}{\partial t} = 0, \quad \dot{r} = \frac{\partial H}{\partial \pi_r} = \pi_r N, \\
\dot{\pi}_r &= -\frac{\partial H}{\partial r} = -\frac{1}{2} \left[ \pi_r^2 N' - \frac{2qA'_t(\pi_t + qA_t)}{F} + \frac{(\pi_t + qA_t)^2 F'}{F'^2} - \pi_\phi^2 D^{-2}D' \right], \\
\dot{\phi} &= \frac{\partial H}{\partial \pi_\phi} = \frac{\pi_\phi}{D}, \quad \dot{\pi}_\phi = -\frac{\partial H}{\partial \phi} = 0.
\end{align*}
\]
In the above equations, "$r'$" represents the derivative of $r$. Using the equations of motion, we get the relations between the radial coordinate and time and between the radial momentum and time

$$\frac{dr}{dt} = \frac{\dot{r}}{t} = -\frac{\pi_r FN}{\pi_t + qA_t},$$

$$\frac{d\pi_r}{dt} = \frac{\dot{\pi}_r}{t} = -qA_t' + \frac{1}{2} \left[ \frac{\pi_r^2 FN'}{\pi_t + qA_t} + \frac{(\pi_t + qA_t)F'}{F} - \frac{\pi_\phi^2 D^{-2} D' F'}{\pi_t + qA_t} \right].$$  \(7\)

For convenience, we define $F_1 = \frac{dr}{dt}$ and $F_2 = \frac{d\pi_r}{dt}$. The normalization of the four-velocity of a particle is given by $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \eta$, where $\eta = 0$ describes the case of a photon, and $\eta = -1$ corresponds to the case of a massive particle. Here the particle is charged. Using the normalization and the metric (2) yields a constrain condition

$$\pi_t + qA_t = -\sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}. \quad (8)$$

We use this constrain and rewrite Eq. (7) as

$$F_1 = \frac{\pi_r FN}{\sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}},$$

$$F_2 = -qA_t' - \frac{\pi_r^2 (NF)' + F'}{2 \sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}} - \frac{\pi_\phi^2 (D^{-1} F)'}{2 \sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}}. \quad (9)$$

The effective potential of the particle plays an important role in the acquisition of the Lyapunov exponent \[45\]–\[47\]. Here the Lyapunov exponent is derived by the eigenvalue of a Jacobian matrix in the phase space $(r, \pi_r)$. In this phase space, the Jacobian matrix is defined by $K_{ij}$ and the matrix elements are
\[ K_{11} = \frac{\partial F_1}{\partial r} = \frac{\pi_r (NF)'}{\sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}} - \pi_r N F' + \pi_\phi^2 (D^{-1}F)' - \frac{\pi_r N F'}{2 [F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})]^{3/2}}, \]
\[ K_{12} = \frac{\partial F_1}{\partial \pi_r} = \frac{F N}{\sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}} - \frac{\pi_r^2 F^2 N^2}{4 [F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})]^{3/2}}, \]
\[ K_{21} = \frac{\partial F_2}{\partial \pi_r} = -q F'' + \frac{\pi_r (NF)'' + \pi_\phi^2 (D^{-1}F)''}{2 \sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}} + \frac{[F' + \pi_\phi^2 (D^{-1}F)']^2}{4 [F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})]^{3/2}}, \]
\[ K_{22} = \frac{\partial F_2}{\partial \pi_r} = -\frac{\pi_r (NF)'}{\sqrt{F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})}} + \pi_r N F' + \pi_\phi^2 (D^{-1}F)' - \frac{\pi_r N F'}{2 [F(1 + \pi_r^2 N + \pi_\phi^2 D^{-1})]^{3/2}}. \]  

Taking into account the motion of the particle in an equilibrium orbit, we use a condition \( \pi_r = \frac{d\pi_r}{dt} = 0 \) to constrain the trajectory of the particle. Using the constrain and calculating the eigenvalue, we get the exponent

\[
\lambda^2 = \frac{1}{4} N \left[ F' + \pi_\phi^2 (D^{-1}F) \right]^2 - \frac{1}{2} N \left[ F'' + \pi_\phi^2 (D^{-1}F)'' \right] - \frac{q F'' N}{\sqrt{F(1 + \pi_\phi^2 D^{-1})}}. 
\]

It is clearly that the exponent is affected by the angular momentum. When \( \pi_\phi = 0 \), it implies that the contribution of the angular momentum is neglected. Since the contribution of the angular momentum plays an important role in the Lyapunov exponent, we do not neglect it in this paper.

### III. CHAOS BOUND AND ITS VIOLATION IN KISELEV BLACK HOLE

We use a probe particle with mass \( m \) moving around a charged Kiselev black hole to investigate the chaos bound. When the particle is neutral, a force on it is a centrifugal force, and this force is related to the angular momentum of the particle. This angular momentum causes the change of the effective potential, but the angular momentum is not sufficient to make the particle’s orbit closer to the horizon [16]. When the particle is charged with \( q \), one can adjust the charge mass ratio to make it close to or away from the horizon. In [45, 49], the authors studied the violation of the bound in the near-horizon regions through the expansions at the event horizon. Here we investigate the influence of the angular momentum on the bound in the near-horizon region and at a certain distance from the horizon.
The metric of the black hole is given by the metric (2), where

\[ F(r) = N(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\alpha}{r^{3\omega+1}}, \quad C(r) = r^2, \quad D(r) = r^2 \sin^2 \theta, \quad (12) \]

\( M \) and \( Q \) are the mass and electric charge of the black hole, respectively. \( \alpha \) is a normalization factor and \( \omega \) is a state parameter characterizing the anisotropic fluid. The metric (2) describes an asymptotically flat solution when \(-\frac{1}{3} \leq \omega < 0\) and a non-asymptotically flat solution when \(-1 < \omega < -\frac{1}{3}\). Some classical black hole solutions can be obtained by the specific values of \( \omega \) and \( \alpha \). The metric of Reissner-Nordström-(Anti)-de Sitter black holes is recovered when \( \omega = -1 \) and \( 3\alpha \) plays the role of the cosmological constant. When \( \omega = -\frac{1}{3} \), the metric describes a topological Reissner-Nordström solution. When \( \alpha = 0 \), it describes a Reissner-Nordström solution. Although \( \omega \) can have many values, our interest is focused on the special cases where \( \omega = -\frac{1}{2} \) and \( \omega = -\frac{2}{3} \). Its electromagnetic potential is \( A_t = \frac{Q}{r} \). The event horizon \((r_+)\) is determined by \( F(r) = 0 \). In the equatorial plane, we get \( \theta = \frac{\pi}{2} \) and \( D(r) = r^2 \).

|   | L  | 0      | 1      | 2      | 3      | 5      | 10     | 20     |
|---|-----|--------|--------|--------|--------|--------|--------|--------|
| 0 | α=0.020 | 1.378386 | 1.379215 | 1.381689 | 1.385770 | 1.398472 | 1.451301 | 1.588752 |
| 0 | α=0.060 | 1.519514 | 1.520442 | 1.523206 | 1.527749 | 1.541780 | 1.598667 | 1.741000 |
| 0 | α=0.096 | 1.666155 | 1.667152 | 1.670118 | 1.674985 | 1.689946 | 1.749770 | 1.749770 |

Table 1. The position of the equilibrium orbit changes with the value of the angular momentum of the particle when \( \omega = -\frac{1}{2} \) and \( Q = 0.95 \). The event horizon is located at \( r_+ = 1.376809 \) when \( \alpha = 0.020 \), at \( r_+ = 1.517366 \) when \( \alpha = 0.060 \) and at \( r_+ = 1.663377 \) when \( \alpha = 0.096 \).
Table 2. The position of the equilibrium orbit changes with the value of the angular momentum of the particle when $\omega = \frac{-1}{2}$ and $Q = 0.99$. The event horizon is located at $r_+ = 1.398266$ when $\alpha = 0.060$, at $r_+ = 1.529366$ when $\alpha = 0.090$ and at $r_+ = 1.675008$ when $\alpha = 0.120$.

| $L$ | 0   | 1   | 2   | 3   | 5   | 10  | 20  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\alpha = 0.060$ | 1.399487 | 1.400110 | 1.401974 | 1.405067 | 1.414827 | 1.457618 | 1.584125 |
| $\alpha = 0.090$ | 1.531051 | 1.531177 | 1.533915 | 1.543746 | 1.548549 | 1.595668 | 1.726790 |
| $\alpha = 0.120$ | 1.677243 | 1.678035 | 1.680399 | 1.684295 | 1.696411 | 1.746872 | 1.882388 |

Table 3. The position of the equilibrium orbit changes with the value of the angular momentum of the particle when $\omega = \frac{-2}{3}$ and $Q = 0.95$. The event horizon is located at $r_+ = 1.388661$ when $\alpha = 0.020$, at $r_+ = 1.576836$ when $\alpha = 0.060$ and at $r_+ = 1.825580$ when $\alpha = 0.096$.

| $L$ | 0   | 1   | 2   | 3   | 5   | 10  | 20  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\alpha = 0.020$ | 1.390238 | 1.391053 | 1.393485 | 1.397495 | 1.409974 | 1.461827 | 1.596804 |
| $\alpha = 0.060$ | 1.579006 | 1.579873 | 1.582457 | 1.586702 | 1.599804 | 1.652856 | 1.786028 |
| $\alpha = 0.096$ | 1.828424 | 1.829270 | 1.831788 | 1.835921 | 1.848626 | 1.899553 | 2.026144 |

Table 4. The position of the equilibrium orbit changes with the value of the angular momentum of the particle when $\omega = \frac{-2}{3}$ and $Q = 0.99$. The event horizon is located at $r_+ = 1.450608$ when $\alpha = 0.060$, at $r_+ = 1.652985$ when $\alpha = 0.090$ and at $r_+ = 1.963428$ when $\alpha = 0.120$.

| $L$ | 0   | 1   | 2   | 3   | 5   | 10  | 20  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\alpha = 0.060$ | 1.451859 | 1.452452 | 1.454266 | 1.457163 | 1.461238 | 1.506609 | 1.624338 |
| $\alpha = 0.090$ | 1.654750 | 1.655394 | 1.657314 | 1.660482 | 1.670363 | 1.711977 | 1.826890 |
| $\alpha = 0.120$ | 1.965787 | 1.966395 | 1.968206 | 1.971190 | 1.980453 | 2.018876 | 2.122473 |

We first find the positions of the equilibrium orbits. When $\omega = \frac{-1}{2}$, Eq. (12) takes form

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \alpha \sqrt{r},$$  \hspace{1cm} (13)
and then the surface gravity is $\kappa = \frac{M}{r^2} - \frac{Q^2}{r^3} - \frac{\alpha}{r^{8/3}}$. When $\omega = -\frac{2}{3}$, Eq. (12) becomes

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \alpha r$$

and the surface gravity is $\kappa = \frac{M}{r^2} - \frac{Q^2}{r^3} - \frac{\alpha}{r^{2}}$. Using the condition $\pi_r = \frac{dx_r}{dt} = 0$ and Eqs. (9), (13) and (14), we get the expressions of the equilibrium orbits. Due to their complexity, the specific positions $r_0$ of the orbits are numerically analyzed. We order $M = 1$, $m = 1$, $q = 15$ in this paper, and get the positions of some certain orbits in Table 1-Table 4. The location of the horizon is also obtained by numerical solutions. From the tables, it is clearly found that when the state parameter, electric charge and normalization factor are fixed, the positions of the orbits gradually move away from the horizon with the increase of the angular momentum. The equilibrium orbits are very close to the horizon when the angular momentum is small ($L < 5$). When the state parameter, electric charge and angular momentum are fixed, the values of the positions of the orbits and horizon increase with the increase of the value of the normalization factor.

![Graph](image)

FIG. 1: The influence of the angular momentum on the chaos bound, where $\omega = -\frac{1}{2}$ and $Q = 0.95$. The violation of the bound occurs at $L > 7.25$ when $\alpha = 0.020$ and at $9.27 < L < 52.41$ when $\alpha = 0.060$. When $\alpha = 0.096$, there is no violation.
FIG. 2: The influence of the angular momentum on the chaos bound, where \( \omega = -\frac{1}{2} \) and \( Q = 0.99 \). The violation of the bound occurs at \( L > 6.85 \) when \( \alpha = 0.060 \), at \( L > 8.25 \) when \( \alpha = 0.090 \) and at \( 10.51 < L < 58.95 \) when \( \alpha = 0.120 \).

We insert the expressions of \( F(r) \), \( N(r) \) and \( D(r) \) into Eq. (11), and then numerically calculate the values of the Lyapunov exponent at the equilibrium orbits \( r = r_0 \) and the surface gravity. The chaos bound is analyzed in Figure 1-Figure 4. From the figures, the following phenomena are found.

1. When the values of the state parameter, normalization factor and angular momentum are fixed, different values of the electric charge produce different values of the Lyapunov exponent. When the electric charge is relatively small, no matter how the angular momentum is taken, the bound can not be violated. The violation of the bound only occurs when the electric charge is large enough.

2. The angular momentum of the particle affects the bound. When the value of the state parameter is fixed, and the electric charge is large enough, the angular momentum of the some specific values violates the bound. For example, the violation occurs when \( \alpha = 0.020 \), \( L > 7.25 \) and \( \alpha = 0.060 \), \( 9.27 < L < 52.41 \) in Figure 1. A maximum value of \( \lambda^2 - \kappa^2 \) occurs at \( L = 26.00 \) when \( \alpha = 0.060 \).

3. The normalization factor affects the bound. When the values of the state parameter,
FIG. 3: The influence of the angular momentum on the chaos bound, where $\omega = -\frac{2}{3}$ and $Q = 0.95$. The violation of the bound occurs at $L > 7.42$ when $\alpha = 0.020$ and at $10.91 < L < 35.28$ when $\alpha = 0.060$. When $\alpha = 0.096$, there is no violation.

electric charge and angular momentum are fixed, the maximum value of $\lambda^2 - \kappa^2$ decreases with the increase of the normalization factor. It is more likely to cause the violation of the bound when the value of the normalization factor is relatively small. The value of the angular momentum corresponding to the maximum value of $\lambda^2 - \kappa^2$ varies with the value of the normalization factor. In Figure 1 and Figure 3, we find that when $Q = 0.95$ and $\alpha > 0.096$, no matter how the value of the angular momentum increases, there is no violation. This phenomenon also occurs in Figure 2 and Figure 4.

4. The state parameter affects the bound. When $Q = 0.99$ and $\alpha = 0.12$, the angular momentum taking the specific values ($10.51 < L < 58.95$) leads to that the violation occurs at $\omega = -\frac{1}{2}$ and does not appears at $\omega = -\frac{2}{3}$.

5. The values of $\lambda^2 - \kappa^2$ described by the red points in the figures are greater than 0. In fact, there is $\lambda^2 - \kappa^2 < 0$ when the value of the angular momentum increases to a certain value, which shows that the bound is not violated in these regions.

Therefore, the state parameter, normalization factor and electric charge and the angular momentum jointly affect the chaos bound. The angular momentum of the particle plays an
FIG. 4: The influence of the angular momentum on the chaos bound, where $\omega = -\frac{2}{3}$ and $Q = 0.99$. The violation of the bound occurs at $L > 7.35$ when $\alpha = 0.060$ and at $10.00 < L < 75.10$ when $\alpha = 0.090$. There is no violation when $\alpha = 0.120$.

important role in the violation of the bound. When the value of the angular momentum is small, the equilibrium orbit is very close to the event horizon. From the figures, it is found that the bound is violated at a certain distance from the horizon and not violated in the near-horizon region. In [45], the authors found the violation of the bound in the near-horizon regions through the Taylor expansions of the metric functions on the horizon. A sufficiently large charge of the black hole also causes the violation in the near-horizon region [49]. When the charge mass ratio of the particle is large enough, we may get a result consistent with them.

IV. CONCLUSIONS

In this paper, we investigated the chaos bound in the near-horizon region and at a certain distance from the horizon of the charged Kiselev black hole. The angular momentum of the charged particle plays an important role in the investigation. It affects not only the value of the Lyapunov exponent, but also the position of the equilibrium orbit. By fixing the charge
mass ratio and changing the value of the angular momentum, we got the position which can be in the near-horizon region and at a certain distance from the horizon. Our result shows that when the electric charge is large enough and the angular momentum takes the specific values, the bound is violated at a certain distance from the horizon, and there is no violation in the near-horizon region. The small value of the normalization factor is more likely to cause the violation.

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