GEOMETRIC LOCAL SYSTEMS ON VERY GENERAL CURVES AND
ISOMONODROMY

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ABSTRACT. We show that the minimum rank of a non-isotrivial local system of geometric origin on a suitably general $n$-pointed curve of genus $g$ is at least $2 \sqrt{g + 1}$. We apply this result to resolve conjectures of Esnault-Kerz and Budur-Wang. The main input is an analysis of stability properties of flat vector bundles under isomonodromic deformations, which additionally answers questions of Biswas, Heu, and Hurtubise.

1. INTRODUCTION

1.1. Overview. We work over the complex numbers $\mathbb{C}$. The main result of this paper, Theorem 1.2.5, is that an analytically very general $n$-pointed curve of genus $g$ (defined in Definition 1.2.3) does not carry any non-isotrivial polarizable integral variations of Hodge structure of rank less than $2 \sqrt{g + 1}$. In particular, an analytically very general $n$-pointed curve of genus $g$ carries no geometric local systems of rank less than $2 \sqrt{g + 1}$ with infinite monodromy, as we show in Corollary 1.2.7. This is a strong restriction on the topology of smooth proper maps to an analytically very general curve, and contradicts conjectures of Esnault-Kerz [EK21, Conjecture 1.1] and Budur-Wang [BW20, Conjecture 10.3.1], as explained in Corollary 1.2.10.

The above results rely on an analysis of stability properties of isomonodromic deformations of flat vector bundles with regular singularities, and require correcting a number of errors in the literature on this topic. We next state our main results on stability properties of isomonodromic deformations of flat vector bundles. Let $C_0$ be the central fiber of a family of curves $\mathcal{C} \to \Delta$ with $\Delta$ a contractible domain, and let $(E_0, \nabla_0)$ be a vector bundle with flat connection on $C_0$. Recall that, loosely speaking, the isomonodromic deformation of $(E_0, \nabla_0)$ is the deformation $(\mathcal{E}, \nabla)$ of $(E_0, \nabla_0)$ to $\mathcal{C}/\Delta$, such that the monodromy of the connection is constant.

In Corollary 1.3.3, we construct a flat vector bundle on a smooth proper curve over $\mathbb{C}$, whose isomonodromic deformations to a nearby curve are

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never semistable. (See Definition 3.4.4 for precise definitions.) The construction arises from the “Kodaira-Parshin trick,” and contradicts earlier claimed theorems of Biswas, Heu, and Hurtubise ([BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2]), which imply that such a construction is impossible. See Remark 5.1.9 for a discussion of the errors in those papers.

As a complement to this example, we show in Theorem 6.1.1 that any logarithmic flat vector bundle admits an isomonodromic deformation to a nearby curve which is close to semistable, in a suitable sense, and moreover is (parabolically) semistable if the rank is small compared to the genus of the curve. While our results contradict those of [BHH16, BHH21, BHH20], our methods owe those papers a substantial debt. Biswas, Heu, and Hurtubise pitch the question of isomonodromically deforming a vector bundle to a semistable vector bundle (see Question 1.3.1) as an analogue of Hilbert’s 21st problem, also known as the Riemann-Hilbert problem.

The semistability property of Theorem 6.1.1 is also the main input to our Hodge-theoretic main results, mentioned above. The applications to polarizable variations of Hodge structures come from the fact that flat vector bundles underlying polarizable variations are rarely (parabolically) semistable, due to well-known curvature properties of Hodge bundles.

1.2. Main Hodge-theoretic results. Results from this subsection, §1.2, as well as the next, §1.3, will be proven later in the paper, as detailed in §1.6. For convenience, throughout the paper, our main results will primarily be stated for hyperbolic curves.

Definition 1.2.1. Let $C$ be a curve over $\mathbb{C}$ of genus $g$ and $D \subset C$ a reduced effective divisor of degree $n$. Call $(C, D)$ hyperbolic if $C$ is a smooth proper connected curve and either $g \geq 2$ and $n \geq 0$, $g = 1$ and $n > 0$, or $g = 0$ and $n > 2$. We call an $n$-pointed curve $(C, x_1, \ldots, x_n)$ hyperbolic if $(C, x_1 + \cdots + x_n)$ is hyperbolic.

Remark 1.2.2. Equivalently, $(C, D)$ is hyperbolic if and only if it has no infinitesimal automorphisms, i.e., $H^0(C, T_C(-D)) = 0$.

We will also work with the following analytic notion of a (very) general general point.

Definition 1.2.3. A property holds for an analytically general point of a complex orbifold $X$, if there exists a nowhere dense closed analytic subset $S \subset X$ so that the property holds on $X - S$. We say that a property holds for an analytically very general point if, locally on $X$, there exists a countable collection
of nowhere dense closed analytic subsets such that the property holds on the complement of their union. If \( \mathcal{M}_{g,n} \) is the analytic moduli stack of \( n \)-pointed curves of genus \( g \), we say that a property holds for an analytically (very) general \( n \)-pointed curve if it holds for an analytically (very) general point of \( \mathcal{M}_{g,n} \).

**Remark 1.2.4.** From the definition, it may appear that “analytically very general” is a local notion, while “analytically general” is a global notion. However, being “analytically general” also has the following equivalent local definition, which is more similar to the definition of “analytically very general”: locally on \( X \), there exists a nowhere dense closed analytic subset such that the property holds on the complement of this subset.

The main geometric consequence of this work is the following constraint on the rank of non-isotrivial polarizable variations of Hodge structure (defined in §4) on an analytically very general curve:

**Theorem 1.2.5.** Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \). Suppose \( (C, x_1, \cdots, x_n) \) is an analytically very general \( n \)-pointed hyperbolic curve of genus \( g \), and \( V \) is a \( \mathcal{O}_K \)-local system on \( C \setminus \{x_1, \cdots, x_n\} \) with infinite monodromy. Suppose additionally that for each embedding \( \iota : \mathcal{O}_K \to \mathbb{C} \), \( V \otimes_{\mathcal{O}_K, \iota} \mathbb{C} \) underlies a polarizable complex variation of Hodge structure. Then,

\[
\text{rk}_{\mathcal{O}_K}(V) \geq 2\sqrt{g + 1}.
\]

**Remark 1.2.6.** Note that a result analogous to Theorem 1.2.5 does not hold for variations without an underlying \( \mathcal{O}_K \)-structure. Indeed, every smooth proper curve of genus at least 2 admits a polarizable complex variation of Hodge structure of rank 2 with infinite monodromy, arising from uniformization (see e.g. [Sim88, bottom of p. 870]).

Let \( X \) be a smooth variety. We say a complex local system \( V \) on \( X \) is of geometric origin if there exists a dense open \( U \subset X \), and a smooth proper morphism \( f : Y \to U \) such that \( V|_U \) is a direct summand of \( R^if_*\mathbb{C} \) for some \( i \geq 0 \). As local systems of geometric origin satisfy the hypotheses of Theorem 1.2.5 we have:

**Corollary 1.2.7.** Let \( (C, x_1, \cdots, x_n) \) be an analytically very general hyperbolic \( n \)-pointed curve of genus \( g \). If \( V \) is a local system on \( C \setminus \{x_1, \cdots, x_n\} \) of geometric origin and with infinite monodromy, then \( \dim_{\mathbb{C}} V \geq 2\sqrt{g + 1} \).

We will prove Corollary 1.2.7 in §7.3. As a consequence of Corollary 1.2.7 we obtain the following concrete geometric corollary:
Corollary 1.2.8. If $(C, x_1, \ldots, x_n)$ is an analytically very general hyperbolic $n$-pointed genus $g$ curve, then any non-isotrivial abelian scheme over $C \setminus \{x_1, \cdots, x_n\}$ has relative dimension at least $\sqrt{g+1}$. Similarly, any relative smooth proper curve over $C \setminus \{x_1, \cdots, x_n\}$ has genus at least $\sqrt{g+1}$.

We will prove Corollary 1.2.8 in § 7.4. In Proposition 7.6.1, we prove a variant of the above corollary with a stronger bound on the genus when the abelian scheme has real multiplication, corresponding to a map from $C \setminus \{x_1, \cdots, x_n\}$ to a Hilbert modular stack.

Remark 1.2.9. It is a well-known conjecture that integral local systems underlying a polarizable variation of Hodge structure are of geometric origin—see e.g. [Sim97, Conjecture 12.4] for a precise statement. Theorem 1.2.5 verifies this conjecture for local systems of rank less than $2\sqrt{g+1}$ on an analytically very general $n$-pointed hyperbolic curve of genus $g$, as local systems with finite monodromy arise from geometry.

We are grateful to Hélène Esnault for pointing out the following consequence of Corollary 1.2.7 to us. We let

$$\mathcal{M}_{B,r}(C \setminus \{x_1, \cdots, x_n\}) := \text{Hom}(\pi_1(C \setminus \{x_1, \cdots, x_n\}), GL_r(C)) \sslash GL_r(C)$$

be the character variety parametrizing conjugacy classes of semisimple representations of $\pi_1(C \setminus \{x_1, \cdots, x_n\})$ into $GL_r(C)$. See e.g. [Sik12] for a useful primer on character varieties.

Corollary 1.2.10. Let $(C, x_1, \cdots, x_n)$ be an analytically very general hyperbolic $n$-pointed curve of genus $g$. Then if $1 < r < 2\sqrt{g+1}$, the local systems of geometric origin are not Zariski-dense in the character variety $\mathcal{M}_{B,r}(C \setminus \{x_1, \cdots, x_n\})$.

Remark 1.2.11. Corollary 1.2.10 contradicts conjectures of Esnault-Kerz [EK21, Conjecture 1.1] and Budur-Wang [BW20, Conjecture 10.3.1], which imply the density of geometric local systems in the character variety of any smooth complex variety.

We will prove Corollary 1.2.10 in § 7.5.

In what follows, we say a flat vector bundle has unitary monodromy if the associated monodromy representation $\rho : \pi_1(C) \to GL_n(C)$ has image with compact closure. We will deduce the above results from Theorem 1.2.12 below, using that a discrete subset of the image of a unitary $\rho$ is finite.

Theorem 1.2.12. Let $(C, x_1, \cdots, x_n)$ be an $n$-pointed hyperbolic curve of genus $g$. Let $(E, \nabla)$ be a flat vector bundle on $C$ with $\text{rk} \, E < 2\sqrt{g+1}$ and with regular singularities at the $x_i$. If an isomonodromic deformation of $(E, \nabla)$ to an analytically
general nearby \(n\)-pointed curve underlies a polarizable complex variation of Hodge structure, then \((E, \nabla)\) has unitary monodromy.

1.3. **Main results on isomonodromic deformations.** As remarked in §1.1, the Hodge-theoretic results of §1.2 arise from an analysis of the Harder-Narasimhan filtrations of isomonodromic deformations of flat vector bundles on curves. Our first such result is a counterexample to [BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2], which demonstrates that the situation is somewhat more complicated than was previously believed — there exist irreducible flat vector bundles whose isomonodromic deformations are never semistable.

Specifically, [BHH16] ask the following question.

**Question 1.3.1 ([BHH16, p. 123]).** Let \(X\) be a smooth proper curve, and \(D \subset X\) a reduced effective divisor. Given a flat vector bundle \((E, \nabla)\) on \(X\), with regular singularities along \(D\), let \((E', \nabla')\) be the isomonodromic deformation of \((E, \nabla)\) to an analytically general nearby curve \((X', D')\). Is \(E'\) semistable?

The main claim of [BHH16] is that Question 1.3.1 has a positive answer if \((E, \nabla)\) has irreducible monodromy and the genus of \(X\) is at least 2. However, the following results answer Question 1.3.1 in the negative, even in this case. See Remark 5.1.9 for a discussion of the errors in previous claims that Question 1.3.1 had a positive answer.

We use \(\mathcal{M}_{g,n}\) to denote the analytic moduli stack of smooth proper curves with geometrically connected fibers and \(n\) distinct marked points.

**Theorem 1.3.2.** Let \(g \geq 2\) be an integer. There exists a vector bundle with flat connection \((\mathcal{F}, \nabla)\) on \(\mathcal{M}_{g,1}\) such that for each fiber \(C\) of the forgetful morphism \(\mathcal{M}_{g,1} \to \mathcal{M}_g\), the restriction of \((\mathcal{F}, \nabla)\) to \(C\)

1. has semisimple monodromy and
2. is not semistable.

We also have the following variant, where the vector bundle has irreducible monodromy, instead of just semisimple monodromy.

**Corollary 1.3.3.** Let \(C\) be a smooth projective curve of genus at least 2. There exists an irreducible flat vector bundle \((E, \nabla)\) on \(C\), whose isomonodromic deformations to a nearby curve are never semistable.

**Proof.** The restriction \((\mathcal{F}, \nabla)|_C\) from [Theorem 1.3.2] provides a semisimple flat vector bundle, each of whose flat summands has degree zero; by [Theorem 1.3.2](2),
its isomonodromic deformation to a nearby curve is never semistable. Hence one of the irreducible summands of \((\mathcal{F}, \nabla)|_C\) satisfies the statement of the corollary. \(\square\)

In a positive direction, we have have the following result, showing that the isomonodromic deformation of any semisimple flat vector bundle to an analytically general nearby curve is close to being semistable, and moreover it is semistable if the rank is small.

**Theorem 1.3.4.** Let \((C, D)\) be hyperbolic of genus \(g\) and let \((E, \nabla)\) be a flat vector bundle on \(C\) with regular singularities along \(D\), and irreducible monodromy. Suppose \((E', \nabla')\) is an isomonodromic deformation of \((E, \nabla)\) to an analytically general nearby curve, with Harder-Narasimhan filtration \(0 = (F')^0 \subset (F')^1 \subset \cdots \subset (F')^m = E'\). For \(1 \leq i \leq m\), let \(\mu_i\) denote the slope of \(\text{gr}_{HN}^i E' := (F')^i / (F')^{i-1}\). Then the following two properties hold.

1. If \(E'\) is not semistable, then for every \(0 < i < m\), there exists \(j < i < k\) with
   \[\text{rk} \text{gr}_{HN}^{i+1} E' \cdot \text{rk} \text{gr}_{HN}^k E' \geq g + 1.\]
2. We have \(0 < \mu_i - \mu_{i+1} \leq 1\) for all \(i < m\).

In other words, the consecutive associated graded pieces of the generic Harder-Narasimhan filtration have slope differing by at most one, and, if there are multiple pieces of the generic Harder-Narasimhan filtration, many of them must have large rank relative to \(g\).

**Theorem 1.3.4** is a special case of **Theorem 6.1.1** below, where we allow certain parabolic structures on the vector bundle \(E\). These more general results are required for our Hodge-theoretic applications.

**Remark 1.3.5.** **Theorem 1.3.4** also holds without the hyperbolicity assumption, as we will explain. Nevertheless, it is convenient to make the assumption so that curves have no infinitesimal automorphisms. In this case isomonodromic deformations are somewhat better behaved, see [Heu10, p. 518].

We now explain the proof of **Theorem 1.3.4** in the case \((C, D)\) is not hyperbolic. Suppose \((C, D)\) is not hyperbolic, so either \(g = 1, n = 0\) or \(g = 0, n \leq 2\). In this case the fundamental group \(\pi_1(C - \{x_1, \ldots, x_n\})\) is abelian. This implies any irreducible representation of \(\pi_1(C - \{x_1, \ldots, x_n\})\) is 1-dimensional, so the corresponding flat vector bundle is a line bundle. In this case, \(E\) and \(E'\) are semistable, so **Theorem 1.3.4** still holds.
As a corollary, we are able to salvage the main theorem of [BHH16] for flat vector bundles whose rank is small relative to $g$, using the AM-GM inequality. The following corollary can be deduced directly from Theorem 6.1.1 and AM-GM. It is also a special case of Corollary 6.1.2.

**Corollary 1.3.6.** Let $(C, D)$ be a hyperbolic curve of genus $g$. Let $(E, \nabla)$ be a flat vector bundle on $C$ with regular singularities along $D$, and suppose that $\text{rk}(E) < 2\sqrt{g + 1}$. Then an isomonodromic deformation of $E$ to an analytically general nearby curve is semistable.

As remarked above, [BHH16] claims an analogous theorem with no bound on the rank of $E$; our Corollary 1.3.3 implies such a bound is necessary. Our methods are heavily inspired by those of [BHH16], but our technique requires some new input from Clifford’s theorem for vector bundles.

These results appear to be new even for vector bundles with finite monodromy; we give some example applications in this case.

**Example 1.3.7.** In this example, we describe what Theorem 1.3.4(2) tells us about splitting types of certain vector bundles on $\mathbb{P}^1$. Suppose we are given a finite group $G$ and a finite $G$-cover $f : X \to \mathbb{P}^1$. Consider the flat vector bundle $E := f_* \mathcal{O}_X$ on $\mathbb{P}^1$, with the connection $\nabla$ induced by the exterior derivative $d : \mathcal{O}_X \to \Omega^1_X$, which has regular singularities along the branch locus of $f$. Let $F$ be a summand of $E$ with irreducible monodromy and let $F'$ be an isomonodromic deformation of $F$ to a very general nearby pointed genus 0 curve. Since every vector bundle on $\mathbb{P}^1$ is a sum of line bundles, we can write $F' = \mathcal{O}_{\mathbb{P}^1}(a_1)^{b_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{b_m}$, with $a_1 < a_2 < \cdots < a_m$ and $b_1 > 0$. Then Theorem 1.3.4(1) tells us nothing, but Theorem 1.3.4(2) tells us that the $a_i$ are consecutive, i.e., $a_{i+1} = a_i + 1$ for $1 \leq i \leq m - 1$.

Such $F'$ appear as a summand in $f'_* \mathcal{O}_{X'}$, where $f' : X' \to \mathbb{P}^1$ is a general $G$-cover of $\mathbb{P}^1$.

**Example 1.3.8.** We now give a sample application of Corollary 1.3.6 to semistability of Tschirnhausen bundles of finite covers. Consider a family $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\gamma} B$ where $\mathcal{X} \to B$, $\mathcal{Y} \to B$ and $\mathcal{X} \to B$ are smooth proper curves with geometrically connected fibers, $\beta$ is finite locally free of degree $d$, $\beta \circ \alpha$ is an $S_\beta$ cover which is the Galois closure of $\beta$. Suppose further $\beta \circ \alpha$ is branched over a divisor $\mathcal{D} \subset \mathcal{Y}$ consisting of $n$ disjoint sections over $B$ so that $(\mathcal{Y}, \mathcal{D})$ is a relative hyperbolic curve of genus $g$ over $B$. Assume the map $B \to \mathcal{M}_{g,n}$ induced by $\gamma$ is dominant.
We obtain a flat vector bundle $E := (\beta \circ \alpha)_* \mathcal{O}_X$ on $\mathcal{Y}$ with regular singularities along $\mathcal{D}$. One can decompose $E = \bigoplus_{\rho} E_{\rho}^{\dim \rho}$ as a sum of flat bundles with irreducible monodromy. Let $F$ denote one such summand corresponding to the standard representation of dimension $d - 1$. The flat vector bundle $\beta_* \mathcal{O}_X$ decomposes as $\mathcal{O}_Y \oplus F$. The dual $T$ of $F$ is known as the Tschirnhausen bundle. By Corollary 1.3.6, the restriction of $T$ to a general fiber of $\mathcal{Y} \to B$ is semistable whenever $d - 1 < 2\sqrt{g + 1}$.

Previous results on stability of $T$ were established in [DP16, Theorem 1.5]. If $h$ denotes the genus of $\mathcal{Z} \to B$, they proved the restriction of $T$ to a general fiber of $\mathcal{Y} \to B$ is semistable whenever $h \geq d g + d(d - 1)^2 g$ [DP16, Remark 3.16].

1.4. Motivation. Our main motivation comes from the following question. Let $f : X \to Y$ be a map of algebraic varieties. What are the restrictions on the topology of $f$? Our Corollary 1.2.7 places a very strong restriction on the topology of morphisms to an analytically very general curve $C$ of genus $g$. For example, it implies that if $f : X \to C$ is a proper morphism with smooth generic fiber and bad reduction at $n$ analytically very general points of $C$, then any non-isotrivial monodromy representation occurring in the cohomology of $X/C$ has dimension at least $2\sqrt{g + 1}$.

We became interested in this question and its connection to isomonodromy while trying to understand [BHH16]. In that paper Biswas, Heu, and Hurtubise raise Question 1.3.1 asking whether it is possible to isomonodromically deform irreducible flat vector bundles to achieve semistability, by analogy to Hilbert’s 21st problem (also known as the Riemann-Hilbert problem).

Hilbert’s 21st problem, as answered by Bolibruch [Bo95] (correcting earlier work of Plemelj), poses the question of whether every monodromy representation can be realized by a Fuchsian system. Esnault and Viehweg generalize this question to higher genus in [EV99]: they ask when an irreducible representation can be realized as the monodromy of a flat vector bundle $(E, \nabla)$ with regular singularities at infinity, with $E$ semistable.

In Esnault-Viehweg’s formulation, the complex structure on the underlying curve is fixed, and the residues of the differential equation at regular singular points are modified to achieve semistability. Flipping this around, Biswas-Heu-Hurtubise’s analogue asks if semistability can be achieved by modifying the complex structure and fixing the residues. They claim that this is always possible in the logarithmic, parabolic, and irregular settings, in [BHH16] [BHH20] [BHH21]. After discovering the Hodge-theoretic counterexample to these claims in Corollary 1.3.3, we proved Theorem 1.3.4 as
an attempt (1) to understand to what extent Biswas, Heu, and Hurtubise’s
Question 1.3.1 has a positive answer, and (2) to apply the cases when there is
a positive answer to the analysis of variations of Hodge structure on curves.

1.5. Idea of proof. To prove Theorem 1.2.5 we first reduce to proving Theorem 1.2.12
using that discrete compact spaces are finite. We then prove Theorem 1.2.12
by showing that any flat vector bundle satisfying the hypotheses of the the-
orem is forced to be (parabolically) semistable on an analytically general
curve, whence the Hodge filtration consists of a single piece by Corollary 4.1.8.
The polarization then gives a definite Hermitian form preserved by the mon-
odromy, and hence the monodromy is unitary. The key issue, which follows
from Theorem 6.1.1 is therefore to show that low rank flat vector bundles
are parabolically semistable on an analytically general curve.

To prove Theorem 6.1.1 we assume we have a flat vector bundle \((E, \nabla)\)
on our hyperbolic curve \((C, D)\), and consider an isomonodromic deforma-
tion to a nearby curve. To this end, we use the deformation theory of this
flat vector bundle with its Harder-Narasimhan filtration, which is governed
by a variant of the Atiyah bundle. We show that if the Harder-Narasimhan
filtration does not satisfy the conclusion of Theorem 6.1.1 then there is a
direction along which we can deform the curve so that the filtration is de-
stroyed. Indeed, if the filtration persisted, deformation theory provides us
with a map from \(T_C(-D)\) to a certain parabolically semistable subquotient
of \(\text{End}(E)\) which vanishes on \(H^1\). Taking the Serre duals gives a semistable
coparabolic vector bundle of low rank and large coparabolic slope which is
not generically globally generated. In the end, we rule this out by a variant
of Clifford’s theorem for vector bundles.

1.6. Organization of the paper. In §2, we review background on parabolic
bundles. In §3, we give background on Atiyah bundles, parabolic Atiyah
bundles, and isomonodromic deformations. In §4 we give background
on complex variations of Hodge structures and their associated parabolic
Higgs bundles. Experts can likely skip these three sections. In §5, we
prove Theorem 1.3.2 and Corollary 1.3.3 providing counterexamples to ear-
lier published claims about semistability of isomonodromic deformations.
In §6 we prove the main results on isomonodromic deformations, Theorem 1.3.4
and Corollary 1.3.6 (and their generalizations Theorem 6.1.1 and Corollary 6.1.2).
This is the technical heart of the paper. In §7 we prove the main conse-
quences for variations of Hodge structure, Theorem 1.2.5 Corollary 1.2.7
Corollary 1.2.10 and Theorem 1.2.12. Finally, §8 lists some questions moti-
vated by our results.
For readers unfamiliar with the theory of parabolic bundles, we suggest first considering the case of Theorem 1.2.5 where \( V \) is assumed to have unipotent monodromy at infinity. In this case one may replace parabolic stability with the usual notion of stability for vector bundles, simplifying the proof; in particular, one may use Corollary 1.3.6 in place of Corollary 6.1.2.

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2. Background on parabolic bundles

We now review some basics on parabolic sheaves and parabolic bundles, primarily following the notation of [BHH20, §2.3]. Some useful references include [Ses82, Part 3, §1], [Yok95, §1 and §3], and [BY96, §2]. Let \( C \) be a smooth proper curve and \( D \subset C \) a reduced divisor. Loosely speaking, a parabolic bundle is a vector bundle on \( C \) together with an additional filtration of the fibers over points of the given divisor, weighted by an increasing sequence of real numbers in \([0, 1)\).

2.1. Definition of parabolic bundles.

**Definition 2.1.1.** Let \( E \) be a vector bundle over a curve \( C \). Let \( D = x_1 + \cdots + x_n \subset C \) denote a divisor, with the \( x_i \) distinct. A *quasiparabolic structure* on \( E \) over \( D \) is a strictly decreasing filtration of subspaces

\[ E_{x_j} = E^1_j \supseteq E^2_j \supseteq \cdots \supseteq E^{n_j+1}_j = 0 \]

for each \( 1 \leq j \leq n \). A *parabolic structure* on \( E \) over \( D \) is a quasiparabolic structure together with \( n \) sequences of real numbers

\[ 0 \leq \alpha^1_j < \alpha^2_j < \cdots < \alpha^{n_j}_j < 1 \]

for \( 1 \leq j \leq n \). A *parabolic bundle* is a vector bundle with a parabolic structure. The collection \( \{\alpha^i_j\} \) are called the *weights* of the parabolic bundle. We say the
parabolic bundle has \textit{rational weights} if all \(a_j^i \) are rational numbers. We often notate the data of a parabolic bundle simply as \(E_\ast\) instead of \((E, \{E_j^i\}, \{a_j^i\})\).

\textbf{Definition 2.1.2 (Parabolic degree and slope).} Let \(E_\ast := (E, \{E_j^i\}, \{a_j^i\})\). We define the \textit{parabolic degree} of \(E_\ast\) as

\[
\text{par-deg}(E_\ast) := \deg(E) + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \alpha_j^i \dim(E_j^i/E_j^{i+1}).
\]

We define the \textit{parabolic slope} as \(\mu_\ast(E_\ast) := \text{par-deg}(E_\ast) / \text{rk}(E_\ast)\).

\section{2.2. Definition of parabolic sheaves.}

In order to later apply Serre duality, we will not only need parabolic bundles, but also so-called coparabolic bundles. Coparabolic bundles are examples of parabolic sheaves, which we define next. The definitions in this subsection follow \cite{Yok95} and \cite{BY96}.

\textbf{Definition 2.2.1.} Let \(X\) be a scheme and \(D \subset X\) an effective Cartier divisor. Let \(\mathcal{R}\) denote the category whose objects are real numbers with a single morphism \(i^{\alpha, \beta} : \alpha \to \beta\) if \(\alpha \geq \beta\) and no morphisms otherwise. Let \(\mathcal{M}_X\) denote the category of sheaves of \(\mathcal{O}_X\)-modules. A \(\mathcal{R}\)-\textit{filtered} \(\mathcal{O}_X\)-module is a functor \(E : \mathcal{R} \to \mathcal{M}_X\). Notationally, we use \(E_\ast\) to denote the functor \(E\), so \(E_\alpha := E(\alpha)\). We write \(i^{\alpha, \beta}_E := E(i^{\alpha, \beta})\).

\textbf{Example 2.2.2.} Define \(E[\alpha]_\ast\) as the \(\mathcal{R}\)-filtered \(\mathcal{O}_X\)-module given by \(E[\alpha]_\beta := E_{\alpha+\beta}\) and \(i^{\beta, \gamma}_{E[\alpha]} := i^{\beta+\alpha, \gamma+\alpha}_E\). Let \(i^{\alpha, \beta}_{E[\alpha]} : E[\alpha]_\ast \to E[\beta]_\ast\) denote the natural transformation whose value on \(\gamma\) is \(i^{\alpha+\gamma, \beta+\gamma}_E\). For \(f : E_\ast \to F_\ast\), we use \(f[\alpha] : E[\alpha]_\ast \to F[\alpha]_\ast\) for the natural induced map. By abuse of notation, we will frequently write \(E\) in place of \(E_0\).

\textbf{Definition 2.2.3.} A \textit{parabolic sheaf} on \(X\) with respect to \(D\) is a \(\mathcal{R}\)-filtered \(\mathcal{O}_X\)-module \(E_\ast\) equipped with an isomorphism

\[
j_E : E_\ast \otimes \mathcal{O}_X(-D) \simm E[1]_\ast
\]

such that \(i_{E, 0}^{[1, 0]} \circ j_E = \text{id}_{E_\ast} \otimes i_D : E_\ast \otimes \mathcal{O}_X(-D) \to E_\ast\), where \(i_D : \mathcal{O}_X(-D) \to \mathcal{O}_X\) is the natural inclusion.

A natural transformation of parabolic \(\mathcal{O}_X\)-modules \(f : E_\ast \to F_\ast\) is a \textit{parabolic morphism} if

\[
\begin{array}{ccc}
E_\ast \otimes \mathcal{O}_X(-D) & \xrightarrow{f \otimes \text{id}} & F_\ast \otimes \mathcal{O}_X(-D) \\
\downarrow j_E & & \downarrow j_F \\
E[1]_\ast & \xrightarrow{f[1]} & F[1]_\ast
\end{array}
\]

(2.1)
commutes.

Let $\mathcal{H}om(E_*, F_*)$ denote the set of parabolic morphisms. We let $\mathcal{H}om(E_*, F_*)$ denote the sheaf of homomorphisms defined by taking $\mathcal{H}om(E_*, F_*) (U) := \text{Hom}(E_*|_U, F_*|_U)$. We also define a parabolic sheaf $\mathcal{H}om(E_*, F_*)$ by taking $\mathcal{H}om(E_*, F_*)_\alpha := \mathcal{H}om(E_*, F[\alpha]_*)$ and the $i^{\alpha, \beta}_{\mathcal{H}om(E_*, F_*)}$ for $\alpha \geq \beta$ to be the natural maps induced by $i^{\alpha, \beta}_*: F[\alpha]_* \rightarrow F[\beta]_*$.

We define $\text{End}(E_*) := \text{Hom}(E_*, E_*)$ and use $\mathcal{E}nd(E_*) := \mathcal{H}om(E_*, E_*)$.

**Example 2.2.4.** Any parabolic vector bundle $(E, \{E^i_j\}, \{\alpha^i_j\})$ defines a parabolic sheaf as follows. For $0 \leq \alpha < 1$, define

$$E_\alpha := \cap_{j=1}^n \ker(E \rightarrow E_{x_j}/E^\beta(a, j)).$$

where $\beta(a, j) = \min(i : \alpha^i_j \geq \alpha)$, for $\alpha \leq \max_i(a^i_j)$ and taking $\beta(a, j) = n_j + 1$ for $1 > \alpha > \max_i(a^i_j)$. For $n \in \mathbb{Z}$ set $E_{x+n} := E_x(-nD)$ and take $i^{\alpha, \beta}_E$ as the natural inclusions. We will refer to parabolic vector bundles and their associated parabolic sheaves interchangeably.

**Remark 2.2.5.** It follows from the definition of $\text{End}(E_*)$ that $\mathcal{E}nd(E_*) \subset \mathcal{E}nd(E)$ is the coherent subsheaf corresponding to those endomorphisms preserving the quasiparabolic filtration $\{E^i_j\}$ at each point $x_j$ of $D$.

**Example 2.2.6.** Every vector bundle $E$ defines a parabolic bundle on $X$ with respect to $D$ by taking the quasiparabolic structure $E_{x_j} = E^1_j \subset E^2_j = 0$ with $a^1_j = 0$. In turn, this defines a parabolic sheaf by [Example 2.2.4]. We say such a parabolic bundle as trivial parabolic structure.

**Remark 2.2.7.** If $V$ is a vector bundle on a scheme $X$ and $E_*$ is a parabolic sheaf, we write a map $V \rightarrow E_*$ to denote a map $V_* \rightarrow E_*$, where $V_*$ is the parabolic sheaf corresponding to $V$ as in [Example 2.2.6].

In addition to parabolic vector bundles, we will also require coparabolic vector bundles, in order to use Serre duality. The essential idea is that while the parabolic sheaves associated to parabolic vector bundles are unchanged in intervals of the form $[a^i, a^{i+1}]$, coparabolic vector bundles are unchanged in intervals of the form $[a^i, a^{i+1}]$. I.e., parabolic vector bundles can be viewed as lower semicontinuous functors, taking the discrete topology on the set of isomorphism classes of vector bundles, while coparabolic vector bundles can be viewed as upper semicontinuous functors, see [BY96, Figure 1].
**Definition 2.2.8** (Coparabolic vector bundles, [BY96, Definition 2.3]). Let $E_*$ denote a parabolic vector bundle, viewed as a parabolic sheaf. The associated **coparabolic vector bundle**, denoted $\hat{E}_*$, is the parabolic sheaf defined by

$$\hat{E}_\alpha := \operatorname{colim}_{\beta > \alpha} E_\beta$$

The colimit above is the union taken over the inclusions given by $i_{E_\beta}^\alpha$.

**Definition 2.2.9** (Coparabolic degree and slope). If $F_*$ is a coparabolic bundle of the form $F_* = \hat{E}_*$, for $E_*$ a parabolic vector bundle, then the **coparabolic degree** of $F_*$ is defined by $\text{copar-deg}(F_*) := \text{par-deg}(E_*)$ and the coparabolic slope of $F_*$ is defined by $\mu_*(F_*) := \mu_*(E_*)$.

**Example 2.2.10.** Given a vector bundle $V$, one can define an associated parabolic bundle $E_*$ with the trivial parabolic structure, as in [Example 2.2.6]. We call $\hat{E}_*$ as the coparabolic bundle associated with trivial coparabolic structure.

## 2.3. Induced subbundles and quotient bundles.

Let $E_* := (E, \{E_i^j\}, \{\alpha^i_j\})$ be a parabolic bundle. Any subbundle $F \subset E$ has an induced parabolic structure as follows, see [Ses82, Part 3, §1.A, Definition 4]. The quasiparabolic structure over $x_j$ on $F$ is obtained from the filtration

$$F_{x_j} = E_{x_j}^1 \cap F_{x_j} \supset E_{x_j}^2 \cap F_{x_j} \supset \cdots \supset E_{x_j}^{n_{x_j}+1} \cap F_{x_j} = 0$$

by removing redundancies. For the weight associated to $F_j^i \subset F_{x_j}$ one takes

$$\max_{k,1 \leq k \leq n_j} \{ \alpha^k_j : F_j^i = E_j^k \cap F_{x_j} \}.$$

Similarly, any quotient $E \to Q$ with kernel $F$ has an induced parabolic structure given as follows. The quasiparabolic structure over $x_j$ on $Q$ is obtained from

$$Q_{x_j} = (E_{x_j}^{n_{x_j}+1} + F_{x_j})/F_{x_j} \supset (E_{x_j}^{n_{x_j}} + F_{x_j})/F_{x_j} \supset \cdots \supset (E_{x_j}^1 + F_{x_j})/F_{x_j} = 0$$

by removing redundancies. For the weight associated to a subspace $Q_j^i \subset Q_{x_j}$ one takes

$$\max_{k,1 \leq k \leq n_j} \{ \alpha^k_j : Q_j^i = (E_j^k + F_{x_j})/F_{x_j} \}.$$
2.4. Stability of parabolic and coparabolic bundles.

Definition 2.4.1 (Parabolic stability). A parabolic vector bundle \( E_\star \) is \emph{parabolically semistable} (respectively, \emph{parabolically stable}) if \( \mu_\star(F_\star) \leq \mu_\star(E_\star) \) (respectively \( \mu_\star(F_\star) < \mu_\star(E_\star) \)) for all parabolic subbundles \( F_\star \subset E_\star \) (with the induced parabolic structure as described above).

Definition 2.4.2 (Coparabolic stability). A coparabolic bundle \( E_\star \) is \emph{coparabolically semistable} if \( E_\star \) is of the form \( \hat{F}_\star \) for \( F \) a semistable parabolic bundle.

Remark 2.4.3. Note that parabolic and coparabolic stability are both defined with respect to parabolic bundles.

Remark 2.4.4. This remark will not be needed in what follows. It turns out that if \( n > 0 \), a coparabolic bundle \( E_\star = \hat{F}_\star \) is coparabolically semistable if for any injection from a parabolic vector bundle \( G_\star \hookrightarrow E_\star, \mu_\star(G_\star) < \mu_\star(E_\star) \). That is, although the definition only gives \( \mu_\star(G_\star) \leq \mu_\star(E_\star) \), equality of slopes is not possible when \( n > 0 \).

The reason for this is that any such map \( G_\star \to E_\star \) also induces a map \( G[-\varepsilon]_\star \to E_\star \to F_\star \) for some sufficiently small \( \varepsilon > 0 \). We then have \( \mu_\star(G_\star) < \mu_\star(G[-\varepsilon]_\star) \leq \mu_\star(F_\star) = \mu_\star(E_\star) \).

Lemma 2.4.5. Suppose \( E_\star \) is a parabolic bundle and \( F_\star \subset E_\star \) is a parabolic subbundle with the induced subbundle structure. If \( Q_\star = E_\star/F_\star \) then \( \text{par-deg}(E_\star) = \text{par-deg}(Q_\star) + \text{par-deg}(F_\star) \). In particular, a parabolic bundle \( E_\star \) is semistable (respectively, stable) if and only if for every quotient bundle \( Q_\star \), \( \mu_\star(E_\star) \leq \mu_\star(Q_\star) \), (respectively \( < \mu_\star(Q_\star) \)).

Proof. The second statement follows from the first, because \( \frac{\text{par-deg}(F_\star)}{\text{rk}(F_\star)} < \frac{\text{par-deg}(E_\star) - \text{par-deg}(F_\star)}{\text{rk}(E_\star) - \text{rk}(F_\star)} > \frac{\text{par-deg}(E_\star)}{\text{rk}(E_\star)} \), and similarly where one replaces the inequalities with equalities.

The first statement is stated in [Ses82, Part 3, §1.A, p. 69, Remark 3], together with the definition of exact sequence of parabolic bundles [Ses82, Part 3, §1.A, p. 68].

2.5. Harder-Narasimhan filtrations. It is a standard fact that parabolic vector bundles have a Harder-Narasimhan filtration, and its proof is similar to the construction of Harder-Narasimhan filtrations of vector bundles, see [Ses82, Part 3, §1.B, Theorem 8].

2.6. Serre duality. If \( E_\star \) is a parabolic sheaf on a scheme \( X \), we have \( H^0(C, E_\star) := \text{Hom}(\mathcal{O}_X, E_\star) = \text{Hom}(\mathcal{O}_X, E_0) = \Gamma(X, E) \), and one can define the higher cohomology groups by taking the corresponding right derived functor, as in
[Yok95] p. 130], where, more generally, \( \text{Ext}^i(E_*, F_*) \) is defined for parabolic \( \mathcal{O}_X \) modules. In general we have \( H^i(X, E_*) = H^i(X, E_0) \).

To state Serre duality, we need the notion of parabolic tensor product and duality.

**Definition 2.6.1.** For \( F_* \) a parabolic sheaf, define \( F_*^\vee := \mathcal{H}om(F_*, \mathcal{O}_X)_* \).

The following gives a useful alternate description of parabolic dualization.

**Lemma 2.6.2** (Parabolic dualization, cf. [Yok95] (3.1]). If \( F_* \) arises from a parabolic vector bundle, we have \( F_*^\vee \cong \tilde{F}_a(-a) \).

**Definition 2.6.3** (Parabolic tensor product, cf. [Yok95] Example 3.2]). Let \( \tau : X - D \to X \) denote the inclusion. Suppose \( E_*^\vee, F_*^\vee \) are both parabolic modules so that each \( F_*^\vee \) and \( E_*^\vee \) is locally free, define \( (E_*^\vee \otimes F_*^\vee)^{\alpha} := \sum_{a_1 + a_2 = \alpha} E_{a_1}^\vee \otimes F_{a_2}^\vee \), viewed as a subbundle of \( \tau^*(E \otimes F) \). We take \( i_{E_*^\vee \otimes F_*} \) to be the natural inclusion map.

**Remark 2.6.4.** In [Yok95] p. 136-137], the tensor product of two arbitrary parabolic sheaves is defined, but the definition is more difficult to state, and we will not require this greater generality.

The next lemma states that parabolic tensor products and duals interact in the usual way with degree. We omit the proof, which is a matter of unwinding definitions.

**Lemma 2.6.5.** For \( E_* \) and \( F_* \) two parabolic bundles, \( \deg E_* \otimes F_* = \deg E_* \operatorname{rk} F_* + \deg F_* \operatorname{rk} E_* \) and \( \deg E_*^\vee = -\deg E_* \).

**Proposition 2.6.6** (Serre duality). Suppose \( X \) is a smooth projective \( n \)-dimensional variety over an algebraically closed field \( k \) and let \( \omega_X \) denote the dualizing sheaf on \( X \). For all parabolic vector bundles \( E_* \), we have a canonical isomorphism

\[
H^i(X, E_*) \cong H^{n-i}(X, \tilde{E}_*^\vee \otimes \omega_X(D))^\vee.
\]

**Proof.** The version in [Yok95] Proposition 3.7] states \( \text{Ext}_X^i(E_*, F_* \otimes \omega_X(D)) \cong H^{n-i}(X, \tilde{F}_*^\vee \otimes \tilde{E}_*^\vee)^\vee \). Using [Yok95] Lemma 3.6], and taking \( E_* = \mathcal{O}_X \), we find \( H^i(X, F_* \otimes \omega_X(D)) \cong H^{n-i}(F_*^\vee \otimes \tilde{E}_*^\vee)^\vee \). Now, taking \( E_* \) as in the statement to be \( F_* \otimes \omega_X(D) \), we find \( F_*^\vee = E_*^\vee \otimes \omega_X(D) \) and so \( H^i(X, E_*)_\cong H^{n-i}(X, E_*^\vee \otimes \omega_X(D))^\vee \). \[\square\]
3. BACKGROUND ON ATIYAH BUNDLES AND ISOMONODROMIC DEFORMATIONS

3.1. The Atiyah bundle of a filtered vector bundle. We begin by defining the Atiyah bundle. Let \( C \) be a smooth projective curve. Following [Gro65 16.8.1], for \( E \) a vector bundle on \( C \), define \( \text{Diff}^1(E, E) \) as follows: for \( U \subset C \) open, \( \text{Diff}^1(E, E)(U) \) is the set of \( C \)-linear endomorphisms \( \tau \) of \( E(U) \), such that for each \( f \in \mathcal{O}_C(U), v \in E(U) \), we have that
\[
\tau_f : v \mapsto \tau(fv) - f\tau(v)
\]
is \( \mathcal{O}_C \)-linear. Here \( \tau_f \) measures the failure of \( \tau \) to be \( \mathcal{O}_C \)-linear, in that \( \tau_f \) is zero for all \( f \) if and only if \( \tau \) is \( \mathcal{O}_C \)-linear.

**Definition 3.1.1** (The Atiyah bundle, see [BHH17, p. 5]). Let \( E \) be a vector bundle on a curve \( C \). Define the Atiyah bundle
\[
\text{At}_C(E) \subset \text{Diff}^1(E, E)
\]
as the subsheaf with sections on an open set \( U \subset C \) given as follows. Let \( \text{At}_C(E)(U) \) consist of those \( C \)-endomorphisms \( \tau \in \text{Diff}^1(E, E)(U) \) such that for each \( f \in \mathcal{O}_C(U), v \in E(U) \), the endomorphism of \( E \) defined by
\[
\tau_f : v \mapsto \tau(fv) - f\tau(v)
\]
is multiplication by a section \( \delta\tau(f) \in \mathcal{O}_C(U) \).

One can also construct Atiyah bundles associated to filtered bundles.

**Definition 3.1.2** (Atiyah bundle of a filtered vector bundle). Let \( P^\bullet := (0 = P^0 \subset P^1 \subset \cdots \subset P^m = E) \) be a filtration on \( E \). We define
\[
\text{At}_C(E, P^\bullet) \subset \text{At}_C(E)
\]
to be the subsheaf consisting of those endomorphisms that preserve \( P^\bullet \).

**Remark 3.1.3.** From the definition, (see also [BHH16 (2.7)]), there is a short exact sequence
\[
0 \to \mathcal{E}nd(E, P^\bullet) \xrightarrow{i} \text{At}_C(E, P^\bullet) \xrightarrow{\delta} \mathcal{T}_C \to 0,
\]
where \( \mathcal{E}nd(E, P^\bullet) \subset \mathcal{E}nd(E) \) is the subsheaf of \( \mathcal{O}_C \)-linear endomorphisms preserving \( P^\bullet \), \( i \) is the evident inclusion, and \( \delta \) sends a differential operator \( \tau \) to the derivation
\[
\delta\tau : f \mapsto \delta\tau(f)
\]
defined in **Definition 3.1.1**.
Remark 3.1.4. There is an alternate, perhaps more geometric, description of $\text{At}_C(E, P^\bullet)$. Namely, the filtration $P^\bullet$ gives a restriction of the structure group of $E$ to a parabolic subgroup $P \subset \text{GL}_n$, and hence gives rise to a natural $P$-torsor $p : \Pi \to C$ over $C$, which is a subscheme of the frame bundle of $E$ (i.e. it consists of those frames which are compatible with $P^\bullet$). The tangent exact sequence

$$0 \to T_{\Pi/C} \to T_{\Pi} \to p^* T_C \to 0$$

naturally admits a $P$-linearization (for the $P$-action on $\Pi$) and hence descends to a short exact sequence on $C$, which is precisely (3.1).

Next, we introduce Atiyah bundles with respect to divisors.

Definition 3.1.5 (Atiyah bundle of a filtered vector bundle with respect to a divisor). Let $D \subset C$ be a reduced effective divisor. The Atiyah bundle $\text{At}_{(C,D)}(E, P^\bullet)$ is defined as the preimage

$$\text{At}_{(C,D)}(E, P^\bullet) := \delta^{-1}(T_C(-D)),$$

where $\delta$ is the map appearing in Sequence (3.1) and where $T_C(-D) \hookrightarrow T_C$ is the natural inclusion.

If $P^\bullet = (0 = P^0 \subset P^1 = E)$ is the trivial filtration, we omit it from the notation, e.g. we will use the notation $\text{At}_{(C,D)}(E)$ in place of $\text{At}_{(C,D)}(E, 0 \subset E)$ when convenient.

The following alternate viewpoint on connections will be useful.

Proposition 3.1.6. Suppose $E$ is a vector bundle on $C$ and $D \subset C$ a reduced effective divisor. There is a natural bijection between splittings of the Atiyah exact sequence

$$0 \to \text{End}(E, P^\bullet) \to \text{At}_{(C,D)}(E, P^\bullet) \to T_C(-D) \to 0. \tag{3.2}$$

and flat connections on $E$ with regular singularities along $D$ and preserving $P^\bullet$, given by adjointness. That is, given a connection

$$\nabla : E \to E \otimes \Omega^1_C(\log D)$$

preserving $P^\bullet$, we may by adjointness view $\nabla$ as a map $q^\nabla : T_C(-D) \to \text{End}_C(E)$. This map factors through $\text{At}_{(C,D)}(E)$ and yields a splitting of (3.2). Moreover, this correspondence between flat connections and splittings is bijective.

Proof. This is a matter of unwinding definitions. $\square$

We will pass freely between $\nabla$ and $q^\nabla$ and refer to each as a connection.
3.2. The parabolic Atiyah bundle. We next recall the generalization of the Atiyah bundle to the parabolic setting.

Let $C$ be a curve with reduced divisor $D = x_1 + \cdots + x_n$ and let $E_\ast$ be a parabolic vector bundle on $(C, D)$. It will useful to recall the explicit description of $\text{End}(E_\ast)$ given in Remark 2.2.5. To define the parabolic Atiyah bundle, we also need the following definition.

**Definition 3.2.1.** Let $\text{End}_j(E_\ast)$ denote the image of the natural inclusion $\text{End}(E_\ast)_{x_j} \to \text{End}(E)_{x_j}$, viewed as a subspace of $\text{End}(E)_{x_j}$.

**Remark 3.2.2.** We can write $\text{End}(E) / \text{End}(E_\ast)$ as a direct sum of skyscraper sheaves supported on the $x_j$. More precisely, for $\text{End}_j(E_\ast)$ the image of $\text{End}(E)_{x_j} / \text{End}_j(E_\ast)$ in the fiber over $x_j$, there is an exact sequence (3.3)

$$0 \longrightarrow \text{End}(E) \longrightarrow \text{End}(E) \longrightarrow \bigoplus_{j=1}^n \text{End}(E)_{x_j} / \text{End}_j(E_\ast) \longrightarrow 0.$$ 

3.2.3. The residue homomorphism. To define the Atiyah bundle associated to $E_\ast$, we essentially want to carve it out from the Atiyah bundle by requiring our differential operators to preserve the quasiparabolic filtration over each $x_j$. For this, we need a certain homomorphism $\phi_j : \text{At}_{(C,D)}(E)_{x_j} \to \text{End}(E)_{x_j}$, referred to as a residue homomorphism. The residue homomorphism is defined in, for example, [BHH20 (2.9)], and we also recall it now.

Given any smooth proper curve $C$ and reduced divisor $D \subset C$ with $x_j$ in the support of $D$, the residue homomorphism at $x_j$ is a map $\phi_j : \text{At}_{(C,D)}(E)_{x_j} \to \text{End}(E)_{x_j}$ defined as follows. There is a commutative diagram of vector spaces

$$(3.4)\quad 0 \longrightarrow \text{End}(E)_{x_j} \longrightarrow \text{End}(E)_{x_j} \longrightarrow \text{At}_{(C,D)}(E)_{x_j} \longrightarrow T_C(-D)_{x_j} \longrightarrow 0$$

The map $\beta_j$ is induced from the natural inclusion of invertible sheaves, and hence vanishes. Therefore, $\gamma_j \circ \alpha_j = 0$, which means $\alpha_j$ factors through $\text{End}(E)_{x_j}$. This produces the desired map $\phi_j : \text{At}_{(C,D)}(E)_{x_j} \to \text{End}(E)_{x_j}$, satisfying the property that $\mu_j \circ \phi_j = \alpha_j$. By definition of $\alpha_j$, the restriction of $\alpha_j$ to $\text{End}(E)_{x_j}$ is the identity. That is, $\phi_j \circ v_j = \text{id}$. This gives us a splitting $\text{At}_{(C,D)}(E)_{x_j} \simeq \text{End}(E)_{x_j} \oplus T_C(-D)_{x_j}$. 

Now, let $z$ denote a uniformizer of the local ring of $C$ at $x_j$ and let $z\frac{\partial}{\partial z}$ denote the corresponding section of $T_C(-D)$ at $x_j$. Observe that this is independent of the choice of $z$. Given $q^\nabla : T_C(-D) \to \text{At}_{(C,D)}(E)$ a connection with regular singularities, let $q^\nabla_x(z\frac{\partial}{\partial z}) \in \text{At}_{(C,D)}(E) \simeq \mathcal{E}nd(E)_{x_j} \oplus T_C(-D)_{x_j}$ denote the image of $z\frac{\partial}{\partial z}$ under $q^\nabla$ restricted to the fiber $x_j$. Define the residue $	ext{Res}(\nabla)(x_j) \in \mathcal{E}nd(E)_{x_j}$ as the projection of $q^\nabla_x(z\frac{\partial}{\partial z}) \in \mathcal{E}nd(E)_{x_j} \oplus T_C(-D)_{x_j}$ to $\mathcal{E}nd(E)_{x_j}$.

Having defined the map $\phi_j$ above, we next define the Atiyah bundle associated to a parabolic bundle.

**Definition 3.2.4** (Atiyah bundle of a parabolic bundle). Let $D \subset C$ be a reduced effective divisor and let $E_* = (E, \{E_j\}, \{a_i^j\})$ be a parabolic bundle on $(C, D)$. For $\phi_j$ the residue homomorphism, as defined above in §3.2.3, let $\hat{\phi}_j : \text{At}_{(C,D)}(E) \to \mathcal{E}nd(E)_{x_j} / \text{End}_j(E_*)$ denote the composition

$$\text{At}_{(C,D)}(E) \to \text{At}_{(C,D)}(E)_{x_j} \xrightarrow{\phi_j} \mathcal{E}nd(E)_{x_j} \to \mathcal{E}nd(E)_{x_j} / \text{End}_j(E_*) = \mathcal{E}nd(E)_{x_j} / \text{End}_j(E_*)$$

Define $\text{At}_{(C,D)}(E_*)$ as the coherent subsheaf of $\text{At}_{(C,D)}(E)$ given by

$$\text{At}_{(C,D)}(E_*) := \ker \left( \frac{\mathcal{E}nd(E)_{x_j} / \text{End}_j(E_*)}{\oplus_{j=1}^n \mathcal{E}nd(E)_{x_j} / \text{End}_j(E_*)} \right).$$

Similarly, for $P^* := (0 = P^0 \subset P^1 \subset \cdots \subset P^m = E)$ a filtration on $E$, we let $\text{At}_{(C,D)}(E_*, P^*) \subset \text{At}_{(C,D)}(E_*)$ denote the coherent subsheaf consisting of those endomorphisms that preserve $P^*$.

**Remark 3.2.5.** Using Definition 3.2.4 and (3.1), we find that $\text{At}_{(C,D)}(E_*, P^*)$ fits into a short exact sequence

$$(3.5) \quad 0 \to \mathcal{E}nd(E_*, P^*) \to \text{At}_{(C,D)}(E_*, P^*) \to T_C(-D) \to 0.$$

By comparing (3.5) for a filtration $P^*$ and the trivial filtration, we obtain the short exact sequence

$$(3.6) \quad 0 \to \text{At}_{(C,D)}(E_*, P^*) \to \text{At}_{(C,D)}(E_*) \to \mathcal{E}nd(E_*) / \mathcal{E}nd(E_*, P^*) \to 0,$$

where $\mathcal{E}nd(E_*, P^*) \subset \mathcal{E}nd(E_*)$ is the coherent subsheaf consisting of those endomorphisms preserving the filtration $P^*$. 
3.3. Parabolic structure and connections. We continue our review of Atiyah bundles by recalling the parabolic bundle associated to a connection, and a constraint on irreducibility of connections.

Definition 3.3.1. Let \( q^\nabla : T_C(-D) \to At_{(C,D)}(E) \) be a connection on \( E \) with regular singularities along \( D \). Let \( \text{Res}(\nabla)(x_j) \in \mathcal{E}nd(E)_{x_j} \) denote the residue of \( \nabla \) at \( x_j \), described in \( \S 3.2.3 \). Suppose the eigenvalues of \( \text{Res}(\nabla)(x_j) \) are \( \eta_j^1, \ldots, \eta_j^s \). Let

\[
\lambda_j^i := \text{Re}(\eta_j^i) - \lfloor \text{Re}(\eta_j^i) \rfloor \in [0,1)
\]

denote the fractional part of the real part of \( \eta_j^i \). Reorder the \( \lambda_j^i \) and remove repetitions so that

\[
0 \leq \lambda_j^1 < \lambda_j^2 < \cdots < \lambda_j^{n_j} < 1.
\]

Define \( E_j^i \subset E_{x_j} \) as the sum of all generalized eigenspaces of \( \text{Res}(\nabla)(x_j) \) such that the fractional part of the real part of the associated eigenvalue is \( \geq \lambda_j^i \). The data \( (E, \{E_j^i\}, \{\lambda_j^i\}) \) is the parabolic bundle associated to the connection \( \nabla \). By [BHH20, Lemma 4.1], the connection \( q^\nabla : T_C(-D) \to At_{(C,D)}(E) \) factors through \( At_{(C,D)}(E_*) \subset At_{(C,D)}(E) \) and we denote the induced map by \( q^\nabla : T_C(-D) \to At_{(C,D)}(E_*) \) as well. If the real part of each \( \eta_j^i \) lies in \([0,1)\), then \( (E_*, \nabla) \) is called the Deligne canonical extension of \( (E|_{C \setminus D}, \nabla|_{C \setminus D}) \).

Proposition 3.3.2. Let \( (E, \nabla : E \to E \otimes \Omega^1_C(\log D)) \) be a flat vector bundle on \( C \) with regular singularities along \( D \), and let \( E_* \) denote the parabolic bundle associated to \( \nabla \), as in \( \text{Definition 3.3.1} \). Suppose the monodromy representation \( \rho \) associated to \( (E, \nabla)|_{C \setminus D} \) via the Riemann-Hilbert correspondence is irreducible. Let \( q^\nabla : T_C(-D) \to At_{(C,D)}(E_*) \) be the corresponding splitting of the Atiyah exact sequence via \( \text{Proposition 3.1.6} \) and \( \text{Definition 3.3.1} \). Then for any nontrivial filtration \( P^* \) of \( E \), the composition

\[
T_C(-D) \xrightarrow{q^\nabla} At_{(C,D)}(E_*) \to At_{(C,D)}(E_*) / At_{(C,D)}(E_*, P^*) \simeq \mathcal{E}nd(E_*) / \mathcal{E}nd(E_*, P^*)
\]

is nonzero.

Proof. Assume not. Then \( q^\nabla \) has image in \( At_{(C,D)}(E_*, P^*) \), and hence yields a splitting of \( (3.5) \). Using \( \text{Proposition 3.1.6} \), the corresponding connection with regular singularities on \( E \) preserves \( P^1 \) and hence yields a flat connection on \( P^1 \) with regular singularities along \( D \), whose monodromy is a sub-representation of the monodromy representation \( \rho \) associated to \( (E, \nabla)|_{C \setminus D} \).
But this contradicts the assumption that $\rho$ is irreducible. (See [BHH16, Proof of Proposition 5.3] for a similar argument in the non-parabolic setting.)

\[\square\]

3.4. **Isomonodromic deformations.** We next recall the notion of isomonodromic deformation. We also define the notion of an “isomonodromic deformation to an analytically general nearby curve” which appears in many of our main results.

**Notation 3.4.1.** Let $\mathcal{C}, \Delta$ be complex manifolds, and let $\pi : \mathcal{C} \to \Delta$ be a proper holomorphic submersion with connected fibers of relative dimension one, with $\Delta$ contractible. Let

$$s_1, \ldots, s_n : \Delta \to \mathcal{C}$$

be disjoint sections to $\pi$, and let $\mathcal{D}$ be the union

$$\mathcal{D} := \bigcup_i \text{im}(s_i).$$

Given a point $0 \in \Delta$, let $(C, D) := (\pi^{-1}(0), \pi^{-1}(0) \cap \mathcal{D})$ and further assume $(C, D)$ is hyperbolic.

**Lemma 3.4.2.** With notation as in Notation 3.4.1, let

$$(E, \nabla : E \to E \otimes \Omega^1_C(\log D))$$

be a flat vector bundle on $C$ with regular singularities along $D$. Such a logarithmic flat vector bundle extends canonically to a logarithmic flat vector bundle

$$(\mathcal{E}, \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_C(\log \mathcal{D}))$$

on $\mathcal{C}$ with regular singularities along $\mathcal{D}$.

**Proof.** This follows from Deligne’s work on differential equations with regular singularities [Del70] and is explained in [Heu10, Theorem 3.4], following work of Malgrange [Mal83a, Mal83b]. In particular, [Mal83a, Theoreme 2.1] explains the case where $C$ has genus zero, and the general case is similar. We now recapitulate the proof.

The restriction of $(E, \nabla)$ to $C \setminus D$ is a flat vector bundle and hence gives rise to a locally constant sheaf of $C$-vector spaces

$$\mathcal{V} := \ker(\nabla)$$

on $C \setminus D$. As $\Delta$ is contractible, the inclusion

$$C \setminus D \hookrightarrow \mathcal{C} \setminus \mathcal{D}$$

is a homotopy equivalence; thus $\mathcal{V}$ extends uniquely (up to canonical isomorphism) to a local system $\tilde{\mathcal{V}}$ on $\mathcal{C} \setminus \mathcal{D}$.
Manin’s local results on extending flat vector bundles across divisors \cite[Proposition 5.4]{Del70} imply that there is a canonical extension, which is unique, up to unique isomorphism,
\[
(\tilde{\nabla} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_C(\log \mathcal{D}))
\]
of
\[
(\tilde{\nabla} \otimes_C \mathcal{E}_{\mathcal{C} \setminus \mathcal{D}}, \text{id} \otimes d)
\]
to a flat vector bundle on $\mathcal{C}$ with regular singularities along $\mathcal{D}$, equipped with an isomorphism $(\mathcal{E}, \tilde{\nabla})|_{\mathcal{C}} \simeq (E, \nabla)$.

Using the above, we are ready to define isomonodromic deformations.

**Definition 3.4.3 (Isomonodromic Deformation).** With notation as in Notation 3.4.1, let $D = x_1 + \cdots + x_n$, so that $(C, D)$ is an $n$-pointed hyperbolic curve of genus $g$. Let $(E, \nabla)$ be a flat vector bundle on $C$ with regular singularities at the $x_i$. We call the extension $(\mathcal{E}, \tilde{\nabla})$ as in Lemma 3.4.2 the isomonodromic deformation of $(E, \nabla)$. If $\Delta = \mathcal{T}_{g,n}$ is the universal cover of of the analytic stack $\mathcal{M}_{g,n}$, and $\mathcal{C} \to \Delta$ is the universal curve, we call the isomonodromic deformation over such $\Delta$ the universal isomonodromic deformation.

**Definition 3.4.4.** With notation as in Definition 3.4.3, let $\Delta$ be the universal cover of $\mathcal{M}_{g,n}$. We use an isomonodromic deformation to a nearby curve to denote the restriction of $(\mathcal{E}, \tilde{\nabla})$ to any fiber of $\mathcal{C} \to \Delta$. We use an isomonodromic deformation to an analytically general nearby curve to denote the restriction of $(\mathcal{E}, \tilde{\nabla})$ to a general fiber of $\mathcal{C} \to \Delta$, i.e., a fiber in the complement of a nowhere dense closed analytic subset.

**Remark 3.4.5.** The construction of Lemma 3.4.2 is functorial: given a commutative diagram
\[
\begin{array}{ccc}
D & \to & \mathcal{D} \\
\downarrow & & \downarrow \\
C & \to & \mathcal{C} \\
\downarrow & & \downarrow \pi \\
0 & \to & \Delta
\end{array}
\]
and a flat vector bundle $(E, \nabla)$ on $C$ with regular singularities along $D$, the isomonodromic deformation over $\Delta'$ pulls back to the isomonodromic deformation over $\Delta$.

**Example 3.4.6 (Families of families, essentially in \cite{Dor01}).** With notation as in Notation 3.4.1, suppose $\mathcal{D} = \emptyset$, and let $\tilde{h} : \mathcal{X} \to \mathcal{C}$ be a proper
holomorphic submersion. Let $X = h^{-1}(C)$, and let $h = \tilde{h}|_X$. Then for each $i \geq 0$, $R^i h_* \Omega^•_{dR,X/C}$ with its Gauss-Manin connection is the isomonodromic deformation of $R^i h_* \Omega^•_{dR,X/C}$ with its Gauss-Manin connection.

**Remark 3.4.7.** Using residues of the connection, we were able to associate to $(E, \nabla)$ a certain parabolic bundle $E_*$ in [Definition 3.3.1]. This induces the structure of a relative parabolic bundle on the isomonodromic deformation $(E_*, \tilde{\nabla})$ of $(E, \nabla)$, which we denote $\mathcal{E}_*$, as explained in [BHH20, §4.3]. Let $\mathcal{E}_t$ denote the fiber of $\mathcal{E} \to \Delta$ over the point $t \in \Delta$ defining the isomonodromic deformation. By [BHH20, Lemma 4.2], the parabolic weights and the associated dimensions of the graded parts of the quasiparabolic structure corresponding to those weights on $\mathcal{E}|_{\mathcal{E}_t}$ are independent of the point $t \in \Delta$. Indeed, the parabolic structure on $\mathcal{E}|_{\mathcal{E}_t}$ is exactly the one associated to the natural connection on $\mathcal{E}|_{\mathcal{E}_t}$ obtained by restricting $\tilde{\nabla}$.

**Definition 3.4.8.** Given two parabolic bundles $F_* = (F, \{F_i\}, \{\alpha_i^j\})$ and $E_* = (E, \{E_i^j\}, \{\beta_i^j\})$ with respect to $D = x_1 + \cdots + x_n$, we say $F_*$ is refined by $E_*$ if $F = E$, and for each $F_i^j$ there is some $i'$ with $F_i^j = E_i'^j$, under the identification $E_{x_j} = F_{x_j}$ and $\alpha_{ij}' = \beta_{ij}'$. In the case that the parabolic structure of $F_*$ at $x_{j_1}, \ldots, x_{j_k}$ are trivial, i.e., they are of the form $F_{x_{j_i}} = F_{x_{j_i}}^1 \supset F_{x_{j_i}}^2 = 0$, we consider $F$ as a parabolic bundle with respect to $D \setminus \{x_{j_1}, \ldots, x_{j_k}\}$.

**Remark 3.4.9.** Continuing with the notation of [Remark 3.4.7] for any parabolic bundle $F_*$ refined by $E_*$, there is a corresponding isomonodromic deformation of $(F_*, \nabla)$ over $\Delta$ given by taking the parabolic structure from [Remark 3.4.7] and forgetting part of the parabolic structure on $\mathcal{E}_*$. As an important special case, the trivial parabolic structure on $E$ described in [Example 2.2.6] is refined by the parabolic bundle $E_*$ arising from [Definition 3.3.1].

### 3.5. Deformation theory of isomonodromic deformations

We now analyze the infinitesimal deformation theory of isomonodromic deformations.

**Notation 3.5.1.** Let $\text{Art}_C$ be the category of local Artin $C$-algebras. Let $C$ be a smooth proper curve over $C$, $D \subset C$ a reduced effective divisor with $D = x_1 + \cdots + x_n$, and

$$ (E, \nabla : E \to E \otimes \Omega^1_C(\log D)) $$

a flat vector bundle on $C$ with regular singularities along $D$. Let $P^•$ be a filtration of $E$. 

---

**Remark 3.4.7.** Using residues of the connection, we were able to associate to $(E, \nabla)$ a certain parabolic bundle $E_*$ in [Definition 3.3.1]. This induces the structure of a relative parabolic bundle on the isomonodromic deformation $(E_*, \tilde{\nabla})$ of $(E, \nabla)$, which we denote $\mathcal{E}_*$, as explained in [BHH20, §4.3]. Let $\mathcal{E}_t$ denote the fiber of $\mathcal{E} \to \Delta$ over the point $t \in \Delta$ defining the isomonodromic deformation. By [BHH20, Lemma 4.2], the parabolic weights and the associated dimensions of the graded parts of the quasiparabolic structure corresponding to those weights on $\mathcal{E}|_{\mathcal{E}_t}$ are independent of the point $t \in \Delta$. Indeed, the parabolic structure on $\mathcal{E}|_{\mathcal{E}_t}$ is exactly the one associated to the natural connection on $\mathcal{E}|_{\mathcal{E}_t}$ obtained by restricting $\tilde{\nabla}$.

**Definition 3.4.8.** Given two parabolic bundles $F_* = (F, \{F_i\}, \{\alpha_i^j\})$ and $E_* = (E, \{E_i^j\}, \{\beta_i^j\})$ with respect to $D = x_1 + \cdots + x_n$, we say $F_*$ is refined by $E_*$ if $F = E$, and for each $F_i^j$ there is some $i'$ with $F_i^j = E_i'^j$, under the identification $E_{x_j} = F_{x_j}$ and $\alpha_{ij}' = \beta_{ij}'$. In the case that the parabolic structure of $F_*$ at $x_{j_1}, \ldots, x_{j_k}$ are trivial, i.e., they are of the form $F_{x_{j_i}} = F_{x_{j_i}}^1 \supset F_{x_{j_i}}^2 = 0$, we consider $F$ as a parabolic bundle with respect to $D \setminus \{x_{j_1}, \ldots, x_{j_k}\}$.

**Remark 3.4.9.** Continuing with the notation of [Remark 3.4.7] for any parabolic bundle $F_*$ refined by $E_*$, there is a corresponding isomonodromic deformation of $(F_*, \nabla)$ over $\Delta$ given by taking the parabolic structure from [Remark 3.4.7] and forgetting part of the parabolic structure on $\mathcal{E}_*$. As an important special case, the trivial parabolic structure on $E$ described in [Example 2.2.6] is refined by the parabolic bundle $E_*$ arising from [Definition 3.3.1].
**Definition 3.5.2** (Deformations of a curve with divisor). Let
\[
\text{Def}_{(C,D)} : \text{Art}_C \to \text{Set}
\]
be the functor sending a local Artin $C$-algebra $(A, m, \kappa)$ (so $m$ is the maximal ideal and $\kappa$ is the residue field) to the set of flat deformations of $(C, D)$ over $A$. More precisely, it assigns to $A$ the set of those $(\mathcal{E}, \mathcal{D}, q, f)$ where $q : \mathcal{E} \to \text{Spec} A$ is a flat morphism, $\mathcal{D} \subset \mathcal{E}$ is a relative Cartier divisor over $\text{Spec} A$ and $f : C \to \mathcal{E}$ is a map inducing an isomorphism $C \to \mathcal{E} \times_{\text{Spec} A} \text{Spec} \kappa$ taking $D$ isomorphically to $\mathcal{D} \times_{\text{Spec} A} \text{Spec} \kappa$.

**Proposition 3.5.3.** With notation as in **Definition 3.5.2**, there is a canonical and functorial bijection
\[
\text{Def}_{(C,D)}(C[\varepsilon]/\varepsilon^2) \sim H^1(C, T_C(-D)).
\]

*Proof.* This is standard, see [Ser06, Proposition 3.4.17]. □

We next generalize the above to describe the deformation theory of filtered vector bundles on curves.

**Definition 3.5.4** (Deformations of a parabolic filtered vector bundle). Let
\[
\text{Def}_{(C,D,E*,P^*)} : \text{Art}_C \to \text{Set}
\]
be the functor sending $A$ to the set of flat deformations of $(C, D, E, P^*)$ over $A$. More precisely, it assigns to $A$ the set of those $(\mathcal{E}, \mathcal{D}, q, f, \mathcal{E}, \{\mathcal{E}_i\}, \mathcal{P}^*, \psi)$ where $(\mathcal{E}, \mathcal{D}, q, f)$ is a flat deformation of $(C, D)$ over $A$ as in **Definition 3.5.2**, $\mathcal{E}$ is a vector bundle on $\mathcal{E}$, $\oplus_i \mathcal{E}_i$ are sub-bundles of $\mathcal{E}|_\mathcal{D}$, $\mathcal{P}^*$ is a filtration of $\mathcal{E}$ by sub-bundles, and $\psi : f^*(\mathcal{E}, \mathcal{P}^*) \to (E, P)$ is an isomorphism of filtered vector bundles on $C$ inducing an isomorphism $\mathcal{E}_{ij}|_{x_j} \sim E_{ij}$ for each $i, j$.

**Proposition 3.5.5.** Let $(E, P^*)$ be a filtered parabolic vector bundle on a curve $C$. Let $D \subset C$ be a reduced effective divisor. There is a canonical and functorial bijection
\[
\text{Def}_{(C,D,E*,P^*)}(C[\varepsilon]/\varepsilon^2) \sim H^1(C, \text{At}_{(C,D)}(E, P^*)).
\]

*Proof.* In the non-parabolic case, this is explained in [BHH16, §2.2]. The parabolic case is described in [BHH20, Lemma 3.2] for the case of a 1-step filtration $P^* = (0 \subset P^1 \subset E)$, and the case of arbitrary length filtrations is analogous. □

**Remark 3.5.6.** If $P^*$ is trivial, we omit it from the notation. In particular,
\[
\text{Def}_{(C,D,E_*)}(C[\varepsilon]/\varepsilon^2) \sim H^1(C, \text{At}_{(C,D)}(E_*)).
\]
Now, begin with a flat vector bundle \((E, \nabla)\) with regular singularities along \(D \subset C\), and let \(E_\star\) denote the associated parabolic bundle defined in Definition 3.3.1. There is an evident natural transformation
\[
\text{Forget} : \text{Def}_{(C, D, E_\star)} \rightarrow \text{Def}_{(C, D)}
\]
given by forgetting \(E_\star\). The construction of isomonodromic deformations yields a section
\[
\text{iso} : \text{Def}_{(C, D)} \rightarrow \text{Def}_{(C, D, E_\star)}
\]
to this map (which depends on \(\nabla\)), as we now spell out.

**Proposition 3.5.7.** With notation as above, let \(\delta : \text{At}_{(C, D)}(E_\star) \rightarrow T_C(-D)\) be the natural quotient map, and
\[
q^\nabla : T_C(-D) \rightarrow \text{At}_{(C, D)}(E_\star)
\]
be the section to \(\delta\) described in Definition 3.3.1 arising from \(\nabla\) via Proposition 3.1.6 and \([BHH20, \text{Lemma } 4.1]\). Under the natural identifications
\[
\text{Def}_{(C, D)}(C[\varepsilon]/\varepsilon^2) \xrightarrow{\sim} H^1(C, T_C(-D))
\]
and
\[
\text{Def}_{(C, D, E_\star)}(C[\varepsilon]/\varepsilon^2) \xrightarrow{\sim} H^1(C, \text{At}_{(C, D)}(E_\star))
\]
arising from Proposition 3.5.3 and Proposition 3.5.5 the two squares below commute:
\[
\begin{array}{ccc}
\text{Def}_{(C, D, E_\star)}(C[\varepsilon]/\varepsilon^2) & \xrightarrow{\sim} & H^1(C, \text{At}_{(C, D)}(E_\star)) \\
\text{Forget} \downarrow & & \delta_\star \downarrow \\
\text{Def}_{(C, D)}(C[\varepsilon]/\varepsilon^2) & \xrightarrow{\sim} & H^1(C, T_C(-D)) \\
\end{array}
\qquad
\begin{array}{ccc}
\text{Def}_{(C, D, E_\star)}(C[\varepsilon]/\varepsilon^2) & \xrightarrow{\sim} & H^1(C, \text{At}_{(C, D)}(E_\star)) \\
\text{iso} \downarrow & & (q^\nabla)_\star \downarrow \\
\text{Def}_{(C, D)}(C[\varepsilon]/\varepsilon^2) & \xrightarrow{\sim} & H^1(C, T_C(-D)) \\
\end{array}
\]

**Proof.** This is explained in \([BHH20, \text{Lemma } 3.1 \text{ and Lemma } 4.3]\). Also see \([BHH16, \S 2.2 \text{ and } \S 4.1]\) for the non-parabolic case. \(\square\)

We recall one additional result describing when a filtration extends to a deformation.

The proof of the following lemma in the non-parabolic case is explained following the proof of \([BHH16, \text{Lemma } 3.1]\).

**Lemma 3.5.8.** Suppose we are given \((C, D, E_\star, \nabla)\) as in Proposition 3.5.7 and a filtration \(P^*\) of \(E\) as in Notation 3.5.1. Assume further we have a first-order deformation \((\mathcal{E}, \mathcal{D})\) of \((C, D)\) corresponding to an element \(s \in \text{Def}_{(C, D)}(C[\varepsilon]/\varepsilon^2) \xrightarrow{\sim} H^1(C, T_C(-D))\). With \(q^\nabla\) as in Proposition 3.5.7, suppose \(q^\nabla(s)\) corresponds to
a deformation \((C, D, \mathcal{E}_*)\) of \((C, D, E_*)\) in which \(P^\bullet \subset E\) admits an extension to a filtration \(\mathcal{P}^\bullet\) of \(\mathcal{E}\). Then

\[
q^\nabla(s) \in \ker \left( H^1(C, \text{At}_{(C,D)}(E_*)) \to H^1(C, \mathcal{E}\text{nd}(E_*)/\mathcal{E}\text{nd}(E_*, P^\bullet)) \right).
\]

**Proof.** Note that the map

\[
H^1(C, \text{At}_{(C,D)}(E_*)) \to H^1(C, \mathcal{E}\text{nd}(E_*)/\mathcal{E}\text{nd}(E_*, P^\bullet))
\]

is induced by the surjection of sheaves \(\text{At}_{(C,D)}(E_*) \to \mathcal{E}\text{nd}(E_*)/\mathcal{E}\text{nd}(E_*, P^\bullet)\) from \((3.6)\). In the above situation, \(q^\nabla(s) \in H^1(C, \text{At}_{(C,D)}(E_*))\) is in the image of the natural map

\[
H^1(C, \text{At}_{(C,D)}(E_*, P^\bullet)) \to H^1(C, \text{At}_{(C,D)}(E_*))
\]

and the composition

\[
H^1(C, \text{At}_{(C,D)}(E_*, P^\bullet)) \to H^1(C, \text{At}_{(C,D)}(E_*)) \to H^1(C, \mathcal{E}\text{nd}(E_*)/\mathcal{E}\text{nd}(E_*, P^\bullet))
\]

vanishes. Indeed, this composition is part of the long exact sequence in cohomology induced by the short exact sequence of sheaves \((3.6)\). Therefore, \(q^\nabla(s) \in \ker \left( H^1(C, \text{At}_{(C,D)}(E_*)) \to H^1(C, \mathcal{E}\text{nd}(E_*)/\mathcal{E}\text{nd}(E_*, P^\bullet)) \right)\).

\[\square\]

4. **Hodge-theoretic preliminaries**

We briefly recall the definition of a variation of Hodge structure, and some standard positivity and semistability results for the (parabolic Higgs) bundles associated to such variations. In particular, [Lemma 4.1.5](#) which shows the first filtered piece of the Hodge filtration tends to have positive parabolic degree, is crucial to our semistability arguments. The properties in [Proposition 4.1.4](#) will also be used repeatedly in this paper.

4.1. **Complex variations of Hodge structure.** Let \(X\) be a smooth irreducible complex variety.

**Definition 4.1.1** (Polarizable complex variations of Hodge structure). A complex variation of Hodge structure on \(X\) is a triple \((V, V^{p,q}, D)\), where \(V\) is a \(C^n\) complex vector bundle on \(X\), \(V = \bigoplus V^{p,q}\) is a direct sum decomposition, and \(D\) is a flat connection satisfying Griffiths transversality:

\[
D(V^{p,q}) \subset A^{1,0}(V^{p,q}) \oplus A^{0,1}(V^{p,q}) \oplus A^{1,0}(V^{p-1,q+1}) \oplus A^{0,1}(V^{p+1,q-1}).
\]

A polarization on \((V, V^{p,q}, D)\) is a flat Hermitian form \(\psi\) on \(V\) such that the \(V^{p,q}\) are orthogonal to one another under \(\psi\), and such that \((-1)^p\psi\) is positive definite on each \(V^{p,q}\). A polarizable complex variation of Hodge structure is a complex variation of Hodge structure which admits a polarization.
We call the holomorphic flat vector bundle \((E, \nabla) := (\ker(D) \otimes \mathcal{O}, \text{id} \otimes d)\) the holomorphic flat vector bundle associated to the complex variation of Hodge structure. The filtration \(F^p V := \oplus_{i \geq p} V^{i,q}\) of \(V\) induces a decreasing Hodge filtration \(F^{ullet} V\) by holomorphic sub-bundles, such that

\[
\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_X.
\]

If \(V\) is a complex local system on \(X\) which is isomorphic to \(\ker(D)\) for some polarizable complex variation of Hodge structure \((V, V^{p,q}, D)\), we say that \(V\) underlies a polarizable complex variation of Hodge structure.

For the next definition, recall that in, in the case of curves, we defined the residues of a connection with regular singularities in § 3.2.3. In the case of higher dimensional varieties see [Del70, p. 53].

**Definition 4.1.2** (Deligne canonical extension [Del70, Remarques 5.5(i)]). Let \(\overline{X}\) be a smooth projective variety containing \(X\) as a dense open subvariety with simple normal crossings complement \(Z\). Let \((E, \nabla)\) be a flat holomorphic vector bundle on \(X\). The Deligne canonical extension \((\overline{E}, \nabla : \overline{E} \to \overline{E} \otimes \Omega^1_X(\log Z))\) of \((E, \nabla)\) to \(\overline{X}\) is the unique flat vector bundle on \(\overline{X}\) with regular singularities along \(Z\), equipped with an isomorphism \((\overline{E}, \nabla)|_X \cong (E, \nabla)\), characterized by the property that all eigenvalues of its residues along components of \(Z\) have real parts lying in \([0, 1)\).

**Definition 4.1.3** (The associated Higgs bundle). Let \((E, F^{ullet}, \nabla)\) be a holomorphic vector bundle \(E\) on a smooth variety \(\overline{X}\), with a flat connection \(\nabla\) with regular singularities along a simple normal crossings divisor \(Z \subset \overline{X}\), and a decreasing filtration \(F^{ullet}\) by holomorphic sub-bundles satisfying the Griffiths transversality condition

\[
\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_X(\log Z).
\]

The associated Higgs bundle is the pair \((\oplus_i \text{gr}_{F^i}^E, E, \theta)\), where the Higgs field

\[
\theta := \bigoplus_i (\theta_i : \text{gr}_{F^i}^E \to \text{gr}_{F^{i-1}}^E \otimes \Omega^1_X(\log Z))
\]

is the \(\mathcal{O}_{\overline{X}}\)-linear map induced by \(\nabla\).

The vector bundle \(E\) canonically has the structure of a parabolic bundle \(E_*\) (if \(X\) is a curve, this structure is described in Definition 3.3.1 and it is described in general in [AHL19, Proposition 5.4]). This structure induces the structure of a parabolic bundle on \(\oplus_i \text{gr}_{F^i}^E, E_*\), via §2.3 as a direct sum of subquotients of \(E\), preserved by \(\theta\). We refer to the pair \((\oplus_i \text{gr}_{F^i}^E, E_*, \theta)\) with its
parabolic structure as the \textit{parabolic Higgs bundle} associated to the variation of Hodge structure.

We collect some basic facts about polarizable complex variations of Hodge structure, the canonical extensions thereof, and their associated Higgs bundles:

\textbf{Proposition 4.1.4.} Let $\overline{X}$ be a smooth projective curve, $Z \subset \overline{X}$ a reduced divisor, and let $X = \overline{X} \setminus Z$. Let $(V, V^p,q, D)$ be a polarizable complex variation of Hodge structure on $X$, and let $(E, F^\bullet, \nabla)$ be the holomorphic flat vector bundle associated to this variation of Hodge structure, with its Hodge filtration. Let $(\overline{E}, \overline{\nabla})$ be its Deligne canonical extension. Let $\overline{E}_*$ be the parabolic bundle associated to $(\overline{E}, \overline{\nabla})$, as defined in Definition 3.3.1.

1. The local system $V := \ker(\nabla)$ associated to $(E, \nabla)$ is semisimple.
2. The local system $V$ may be canonically decomposed as
   \[ V \simeq \bigoplus_i L_i \otimes W_i, \]
   where the $L_i$ are pairwise non-isomorphic irreducible complex local systems on $X$, and each $W_i$ is a complex vector space. Each $L_i$ underlies a polarizable complex variation of Hodge structure, and each $W_i$ carries a complex polarized Hodge structure, both unique up to shifting the grading, and compatible with the variation carried by $V$.
3. $\overline{E}_*$ has parabolic degree zero.
4. There exists a canonical extension of $F^\bullet$ to $\overline{E}$, such that $(\overline{E}, F^\bullet, \overline{\nabla})$ satisfies the Griffiths transversality condition (4.2).
5. The parabolic Higgs bundle $(\bigoplus_i \text{gr}_i^\bullet, \overline{E}_*, \theta)$ associated to $(\overline{E}_*, F^\bullet, \overline{\nabla})$ is parabolically polystable of degree zero. That is, there exist a collection of parabolic vector bundles $E^i_*$ with $\deg(E^i_*) = 0$ and maps $\theta^i : E^i_* \to E^i_* \otimes \Omega^1_{\overline{X}}(\log Z)$ so that both $(\bigoplus_i \text{gr}_i^\bullet, \overline{E}_*, \theta) = (\bigoplus_i (E^i_*, \theta^i))$ and for any $\theta^i$-stable proper sub-bundle $H_* \subset E^i_*$ with its induced parabolic structure, \text{par-deg} $H_* < 0$.

\textbf{Proof.} The proof of (1) is explained in [Del87, 1.11-1.12] and (2) is [Del87, 1.13]. The proof of (3) follows from [EV86, B.3] while (4) is explained in e.g. [Bru17, Section 7]. Finally, (5) is due to Simpson [Sim90, Theorem 5]. See the discussion in the introduction of [AHL19] for a nice summary of this and related topics. \hfill $\Box$
The next lemma is crucial in the proof of our main result since the positivity it gives for $F^i E_*$ will contradict our later results on semistability, unless the Hodge filtration has only a single part. The connection to semistability is spelled out below in Corollary 4.1.8.

**Lemma 4.1.5.** Let $C$ be a smooth proper curve, $Z \subset C$ a reduced effective divisor, and $(V, V^{p,q}, D)$ a polarizable complex variation of Hodge structure on $C \setminus Z$. Let $(E_*, F^*, \nabla)$ be the Deligne canonical extension of the associated flat holomorphic vector bundle to $C$ with its canonical parabolic structure. Let $i$ be maximal such that $F^i E_*$ is non-trivial, where $F^*$ is the Hodge filtration, and suppose that the Higgs field

$$
\theta_i : F^i E_* \to \text{gr}^{i-1} E_* \otimes \Omega^1_C(\log Z)
$$

is non-zero. Then $F^i E_*$ has positive parabolic degree.

**Remark 4.1.6.** Lemma 4.1.5 is essentially due to Griffiths in the case of real variations of Hodge structure on a smooth proper curve. In that case it follows from the curvature formula [Gri70, Theorem 5.2], and is observed there in some special cases [Gri70, Corollary 7.10]. For a more precise reference, see [Pet00, Corollary 2.2], which immediately implies the claim for real variations of Hodge structure on a smooth proper curve. However, as we were unable to find a precise reference in the case of complex variations on a quasi-projective curve, we now give a simple proof. An similar argument is given in the proof of [EK18, Theorem 3.8].

**Proof of Lemma 4.1.5** The parabolic vector bundle $F^i E_*$ with the zero Higgs field is a quotient of the parabolic Higgs bundle $(\oplus i \text{gr}^{i} E_*, \theta)$, and hence has non-negative parabolic degree by the fact that the latter is polystable of degree zero, by Proposition 4.1.4(5). It has degree zero if and only if it is a direct summand of $(\oplus i \text{gr}^{i} E_*, \theta)$ by polystability; but this is ruled out by the nonvanishing of $\theta_i$. □

**Remark 4.1.7.** By the construction of the Higgs field $\theta$, the condition that $\theta_i$ is non-zero in Lemma 4.1.5 is equivalent to the statement that $F^i E_*$ is not preserved by $\nabla$. For example, it is automatically nonzero if $(E_*, \nabla)$ has irreducible monodromy and $F^i E_*$ is a proper sub parabolic bundle of $E_*$.

We now spell out how Lemma 4.1.5 relates to semistability.

**Corollary 4.1.8.** Let $(E_*, F^*, \nabla)$ be as in Lemma 4.1.5. Then the parabolic bundle $E_*$ is not semistable.
Proof. The parabolic vector bundle $E_\ast$ has parabolic degree zero by Proposition 4.1.4(3). But by Lemma 4.1.5, $F^iE_\ast$ has positive degree, and is hence a destabilizing subsheaf. □

5. Variations of Hodge structure and the Kodaira-Parshin trick

In this section we find that variations of Hodge structure on $\mathcal{M}_{g,1}$ with monodromy which is “big” in a suitable sense provide examples of flat vector bundles on curves whose isomonodromic deformation to a nearby curve is never semistable. We then produce such variations of Hodge structure via the Kodaira-Parshin trick. This will be used to prove Theorem 1.3.2 and contradicts earlier claimed theorems of Biswas, Heu, and Hurtubise [BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2], as described further in Remark 5.1.9.

In §7, we will use that variations of Hodge structure with suitably large monodromy yield flat vector bundles which do not have isomonodromic deformations to semistable bundles. This will be used to analyze variations of Hodge structure on an analytically very general curve.

5.1. Construction of the counterexample. We now set up the proof of Theorem 1.3.2 and Corollary 1.3.3. We will construct a variation of Hodge structure over the analytic moduli stack $\mathcal{M}_{g,1}$ whose restriction to each fiber of the forgetful map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ satisfies the hypotheses of Lemma 4.1.5. We will do this via the Kodaira-Parshin trick (see [Par68, Proposition 7] and also [LV20, Proposition 7.1]), which produces a family of curves over $\mathcal{M}_{g,1}$ which is non-isotrivial when restricted to each fiber of the natural forgetful map $\mathcal{M}_{g,1} \to \mathcal{M}_g$. We give a proof appealing to [CD17], but one can also prove it using Proposition 4.1.4 and Corollary 4.1.8 as we mention in Remark 5.1.8. Because we had difficulty finding a suitable reference, we now present a version of the Kodaira-Parshin trick in families.

Proposition 5.1.1 (Kodaira-Parshin trick). Let $Y$ denote a Riemann surface of genus $g \geq 1$ with a point $p \in Y$ and let $Y^0 := Y - p$. Choose a basepoint $y \in Y^0$. Suppose $G$ is a finite center-free group with a surjection $\phi : \pi_1(Y^0, y) \to G$ which sends the loop around the puncture $p \in Y$ to a non-identity element of $G$. Then there exists a smooth proper relative dimension 1 map of analytic stacks $f : \mathcal{X} \to \mathcal{M}_{g,1}$ so that the fiber over a geometric point $[(C, p)] \in \mathcal{M}_{g,1}$ is a finite disjoint union of $G$-covers of $C$ ramified at $p$.

We will prove this below in §5.1.5.
Remark 5.1.2. In the finite disjoint union of $G$-covers appearing at the end of the statement of Proposition 5.1.1, we can explicitly identify the finite set of $G$-covers. Namely, suppose $h \in \pi_1(\mathcal{M}_{g,2})$, viewed as an automorphism of the fundamental group of a 2-pointed genus $g$ curve $\pi_1(Y, y)$. (See §5.1.3 below for an explanation of the action of $\pi_1(\mathcal{M}_{g,2})$ on $\pi_1(Y, y)$.) There is one cover associated to each map of the form $\phi_h : \pi_1(\mathcal{Y}_0, \gamma) \to G$, with $\phi_h(g) := \phi(hgh^{-1})$, modulo the following equivalence relation: we identify $\phi_h \sim \phi_g$ if they are conjugate, i.e., if there exists $m \in G$ with $\phi_h(s) = m\phi_g(s)m^{-1}$ for all $s \in \pi_1(Y, y)$.

5.1.3. Setup to prove Proposition 5.1.1. Let $\pi : \mathcal{M}_{g,2} \to \mathcal{M}_{g,1}$ be the natural forgetful map. Let $x \in \mathcal{M}_{g,2}$ be a point. Let $\tilde{x} := \pi(x) \in \mathcal{M}_{g,1}$ and let $C^0 \subset \mathcal{M}_{g,2}$ denote the fiber of $\pi^{-1}(\tilde{x})$.

Note that $C^0$ is the complement of a point in a smooth proper connected curve of genus $g$. There is a natural short exact sequence

$$1 \to \pi_1(C^0, x) \to \pi_1(\mathcal{M}_{g,2}, x) \to \pi_1(\mathcal{M}_{g,1}, \tilde{x}) \to 1$$

(5.1)

associated to the map $\mathcal{M}_{g,2} \to \mathcal{M}_{g,1}$ with fiber $C^0$. We may obtain this from the Birman exact sequence \cite[Theorem 4.6]{FM12} for mapping class groups, after identifying the fundamental group for $\mathcal{M}_{g,n}$ with the mapping class group of an $n$-times punctured genus $g$ surface. (The case $n = 0$ follows from contractibility of the universal cover of $\mathcal{M}_g$ \cite[Theorem 10.6]{FM12} with covering group given by the mapping class group, and the case of general $n$ can be deduced from the Birman exact sequence \cite[Theorem 4.6]{FM12}.)

Let $G$ be a center-free finite group and suppose further there is a surjection $\gamma : \pi_1(C^0, x) \to G$.

We assume that $\gamma$ takes the conjugacy class of the loop around the puncture of $C^0$ to a non-identity conjugacy class of $G$.

Define $\Gamma \subset \pi_1(\mathcal{M}_{g,2}, x)$ as the set of $h \in \pi_1(\mathcal{M}_{g,2}, x)$ such that there exists $\tilde{\gamma}(h) \in G$ with

$$\gamma(hgh^{-1}) = \tilde{\gamma}(h)\gamma(g)\tilde{\gamma}(h)^{-1}$$

for all $g \in \pi_1(C^0, x)$.

Lemma 5.1.4. Keeping notation as in §5.1.3, the map $\gamma$ determines a well-defined surjective homomorphism

$$\tilde{\gamma} : \Gamma \to G$$

$$h \mapsto \tilde{\gamma}(h).$$
Moreover, $\Gamma \subset \pi_1(\mathcal{M}_{g,2}, x)$ has finite index.

Proof. We first claim that $\Gamma$ contains $\pi_1(C^o, x)$ and surjects onto $G$. Indeed, for $h \in \pi_1(C^o, x)$, one may take $\tilde{\gamma}(h) = \gamma(h)$. Therefore, the surjectivity of $\gamma$ implies that $\tilde{\gamma}$ is also surjective.

Next, we claim that for each $h$, $\tilde{\gamma}(h)$ is uniquely determined. Indeed, suppose $\tilde{\gamma}(h)$ may be either $\alpha$ and $\beta$. Then we would have $\alpha \gamma(g) \alpha^{-1} = \beta \gamma(g) \beta^{-1}$. Since $\gamma$ is surjective, as shown above, we find $\alpha \beta^{-1}$ lies in the center of $G$, and therefore is trivial. So $\alpha = \beta$.

The uniqueness of $\tilde{\gamma}(h)$ just established shows that $\tilde{\gamma}$ determines a well-defined map. This is moreover a homomorphism by the above established uniqueness, because we then obtain $\tilde{\gamma}(h) \tilde{\gamma}(h') = \tilde{\gamma}(hh')$.

Finally, we claim $\Gamma$ has finite index in $\pi_1(\mathcal{M}_{g,2}, x)$. To see this, observe that there is an action of $\pi_1(\mathcal{M}_{g,2}, x)$ on the set of surjective homomorphisms $\pi_1(C^o, x) \to G$ sending $\phi : \pi_1(C^o, x) \to G$ to the map $\phi^h(g) := \phi(hgh^{-1})$. By definition, we have $h \in \Gamma$ if and only if $\gamma^h$ is conjugate to $\gamma$. In particular, $\Gamma$ contains the stabilizer of $\gamma$ under the action of $\pi_1(\mathcal{M}_{g,2}, x)$. But this stabilizer has finite index in $\pi_1(\mathcal{M}_{g,2}, x)$ because $G$ is finite and $\pi_1(C^o, x)$ is finitely generated. Therefore, there are only finitely many homomorphisms $\pi_1(C^o, x) \to G$, and, in particular, finitely many such surjective homomorphisms.

\[\square\]

5.1.5.

Proof of Proposition 5.1.1. Let $\bar{\Gamma}$ be the kernel of the map $\tilde{\gamma}$ from Lemma 5.1.4. The subgroup $\bar{\Gamma}$ corresponds to a finite étale cover $\mathcal{X}^o \to \mathcal{M}_{g,2}$. Observe that $\mathcal{M}_{g,2} \subset \mathcal{M}_{g,1} \times \mathcal{M}_{g,1}$ can be viewed as a dense open substack, and let $\mathcal{X}$ be the normalization of $\mathcal{M}_{g,1} \times \mathcal{M}_{g,1}$ in the function field of $\mathcal{X}^o$, forming the following cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}^o & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,2} & \longrightarrow & \mathcal{M}_{g,1} \times \mathcal{M}_{g,1}.
\end{array}
\]

(5.2)

Restricting the natural map $\mathcal{X} \to \mathcal{M}_{g,1} \times \mathcal{M}_{g,1}$ to a fiber $C$ of the universal curve $\mathcal{M}_{g,1} \times \mathcal{M}_{g,1} \to \mathcal{M}_{g,1}$ yields a finite disjoint union of $G$-covers of $C$, ramified only over the tautological marked point of $C$. We then take
our desired relative curve \(f : \mathcal{X} \to \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \to \mathcal{M}_{g,1}\) as the resulting composition. □

In order to use Proposition 5.1.1, we will need to know there are groups \(G\) satisfying its hypotheses. We now provide such an example.

**Example 5.1.6.** As a concrete example of a group \(G\) to which Proposition 5.1.1 applies, we can take \(G = S_3\) to be the symmetric group on three letters and identify \(\pi_1(Y^c, y)\) with the free group on the generators \(a_1, \ldots, a_g, b_1, \ldots, b_g\).

The group \(\pi_1(Y, y)\) is generated by \(a_1, \ldots, a_g, b_1, \ldots, b_g\) with the relation \(\prod_{i=1}^{g} [a_i, b_i] = 1\). Consider the surjection \(\phi : \pi_1(C^c, y) \twoheadrightarrow S_3\) sending \(a_1 \mapsto (12), b_1 \mapsto (13)\) and sending \(a_i \mapsto \text{id}, b_i \mapsto \text{id}\) for \(i > 1\). The loop around the puncture maps to \(\phi(\prod_{i=1}^{g} [a_i, b_i]) = (12)(13)(12)^{-1}(13)^{-1} = (123) \neq \text{id}\).

5.1.7. **Proof of Theorem 1.3.2.** Let \(f : \mathcal{X} \to \mathcal{M}_{g,1}\) denote the map from Proposition 5.1.1. Concretely, we can take \(G = S_3\) and the map \(\phi\) as in Proposition 5.1.1 to be that given in Example 5.1.6. Define the local system \(V := R^1 f_* \mathcal{O}\) on \(\mathcal{M}_{g,1}\), and define \(\mathcal{F}\) to be the vector bundle \(V \otimes \mathcal{O}\). Note that \(\mathcal{F}\) admits a natural (Gauss-Manin) connection \(\text{id} \otimes d\). The local system \(V\) evidently underlies a variation of Hodge structure.

Let \(C\) be a fiber of the forgetful morphism \(\mathcal{M}_{g,1} \to \mathcal{M}_g\). Let \(X := f^{-1}(C) \subset \mathcal{X}\). We claim that the flat vector bundle \((\mathcal{F}, \nabla)\) satisfies the conditions of Theorem 1.3.2, i.e. it has semisimple monodromy and \(\mathcal{F}|_C\) is not semistable.

We first check that \((\mathcal{F}, \nabla)|_C\) has semisimple monodromy. This is true for any flat vector bundle arising from the Gauss-Manin connection on the cohomology of a family of smooth proper varieties, by work of Deligne [Del71a, Théorème 4.2.6].

We now check that \((\mathcal{F}, \nabla)|_C\) is not semistable. By [CD17, Theorem 4], if \(X \to C\) is not isotrivial, \(f_* \omega_f\) is a destabilizing subsheaf of \(\mathcal{F}\). It remains to show \(X \to C\) is not isotrivial. The fiber of \(f|_X\) over a point \(x \in C\) is a finite disjoint union of finite covers of \(C\), branched only over \(x\). These fibers must vary in moduli as \(x\) varies, as there are only finitely many non-constant maps between any two curves over of genus at least 2, by de Franchis’ theorem. (See [dF13] or [SA66, Corollary 3, p. 75], for example.) □

**Remark 5.1.8.** We can also give a somewhat more involved proof of Theorem 1.3.2 using Corollary 4.1.8 in place of [CD17, Theorem 4], as we now explain. This argument inspired the Hodge-theoretic results Theorem 1.2.5 and Corollary 1.2.7, proven in §7.
With notation as in the proof of Theorem 1.3.2, $\mathcal{F}$ has degree 0 since it admits a flat connection, by Proposition 4.1.4(3). Therefore, it suffices to show $\mathcal{F}$ has a subsheaf of positive degree. The Hodge filtration exhibits $F^1\mathcal{F} \simeq (f|_{X})_*\omega_{(f|_{X})}$ as a subsheaf of $\mathcal{F}$, which is destabilizing by Corollary 4.1.8 once we verify that $\delta : F^1\mathcal{F} \twoheadrightarrow F^0\mathcal{F}/F^1\mathcal{F} \simeq R^1(f|_{X})_*\mathcal{O}_X$ is nonzero.

We now check $\delta$ is nonzero. Locally around a point of $C$, $\delta$ can be identified with the derivative of the period map [CMSP17, Theorem 5.3.4] sending a curve corresponding to a fiber of $f|_{X} : X \to C$ to the corresponding Hodge structure on its first cohomology. To show this derivative is not identically zero it suffices to show that the period map is non-constant. More concretely, by the Torelli theorem, we only need to check $f|_{X} : X \to C$ is not isotrivial. This follows by de Franchis’ theorem, as explained in the proof of Theorem 1.3.2.

Remark 5.1.9. As noted prior to its statement, Theorem 1.3.2 contradicts [BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2], which claim, for example, that any irreducible flat vector bundle on a smooth proper curve of genus at least 2 admits a semistable isomonodromic deformation. We now explain the gaps in the proofs of those results. The error in [BHH16, Theorem 1.3] occurs in [BHH16, Proposition 4.3]; the issue is that the map denoted $f^\ast \nabla$ in diagram (4.14) does not in general exist. The proof works correctly if $G = \text{GL}_2$. An identical error occurs in [BHH21, Proposition 4.4]. A different argument is given in [BHH20]. There, the error occurs in the proof of [BHH20, Proposition 5.1], in which the large diagram claimed to be commutative does not in general commute.

6. Analysis of Harder-Narasimhan filtration

6.1. Main results on isomonodromic deformations. In this section, we prove the following theorem and corollary, generalizing Theorem 1.3.4. For the statement of this theorem, recall the notion of a refinement of a parabolic bundle, introduced in Definition 3.4.8.

Theorem 6.1.1. Let $(C, D)$ be hyperbolic of genus $g$ with $D = x_1 + \cdots + x_n$, and let $(E, \nabla)$ be a flat vector bundle on $C$ with regular singularities along $D$ and irreducible monodromy. Let $E_\ast$ denote a parabolic structure on $E$ refined by the parabolic structure on $E$ associated to $\nabla$. Suppose $(E'_\ast, \nabla')$ is an isomonodromic deformation (which exists by [Remark 3.4.9]) of $(E_\ast, \nabla)$ to an analytically general nearby curve, with Harder-Narasimhan filtration $0 = (F'_0) \subset (F'_1) \subset \cdots \subset (F'_m) = E'_\ast$. For $1 \leq i \leq m$, let $\mu_i$ denote the slope of $\text{gr}_{HN}^{i} E'_\ast := (F'_i)^i/(F'_i)^{i-1}$. Then the following two properties hold.
(1) If \( E' \) is not parabolically semistable, then for every \( 0 < i < m \), there exists \( j < i < k \) with
\[
\text{rk} \, \text{gr}_E^{j+1} E'_* \cdot \text{rk} \, \text{gr}_E^k E'_* \geq g + 1.
\]

(2) We have \( 0 < \mu_i - \mu_{i+1} \leq 1 \) for all \( i < m \).

**Corollary 6.1.2.** Let \((C, D)\) be a hyperbolic curve of genus \( g \). Let \((E, \nabla)\) be a flat vector bundle on \( C \) with regular singularities along \( D \), and suppose that \( \text{rk}(E) < 2 \sqrt{g} + 1 \). Then, for \( E_* \) any parabolic structure refined by the parabolic structure on \( E \) associated to \( \nabla \), the isomonodromic deformation of \( E_* \) to an analytically general nearby curve is parabolically semistable.

The proofs are given in \( \S 6.4.9 \) and \( \S 6.4.10 \). This latter corollary salvages the main theorem of [BHH20] in the case of vector bundles of low rank when \( E_* \) is given the parabolic structure associated to \( \nabla \); it salvages the main theorem of [BHH16] in low rank when \( E_* \) is given the trivial parabolic structure (in which case it agrees with [Corollary 1.3.6]).

Crucial in this section will be the notion of generic global generation.

**Definition 6.1.3 (Generic global generation).** A vector bundle \( V \) is generically globally generated if the evaluation map \( H^0(C, V) \otimes \mathcal{O}_C \to V \) does not factor through a proper subbundle of \( V \), i.e. if the cokernel of this map is torsion.

We call a parabolic sheaf \( E_* \) generically globally generated if \( E = E_0 \) is a vector bundle which is generically globally generated.

The basic idea of the proof of [Theorem 6.1.1] will be to show that any counterexample to [Theorem 6.1.1] will produce a certain semistable parabolic vector bundle of high slope which is not generically globally generated. In order to see why this failure of generic global generation leads to a contradiction, we will need some facts about (generic) global generation of vector bundles on curves, arising from Clifford’s theorem.

**6.2. Preliminary results on high slope bundles with many sections.** We start with a bound on the dimension of the space of global sections of a vector bundle whose Harder-Narasimhan polygon has slopes between \( 0 \) and \( 2g \).

**Lemma 6.2.1.** Suppose \( V \) is a vector bundle on a smooth proper curve \( C \) with Harder-Narasimhan filtration \( 0 = N^0 \subset N^1 \subset \cdots \subset N^m = V \). Suppose
moreover that for each $i$, the slope of $\text{gr}_N^i V = N^i / N^{i-1}$ satisfies

$$0 \leq \mu(\text{gr}_N^i V) := \frac{\deg(\text{gr}_N^i V)}{\text{rk}(\text{gr}_N^i V)} \leq 2g.$$ 

Then $\dim H^0(C, V) \leq \frac{\deg V}{2} + \text{rk} V$.

**Proof.** For convenience set $W_i := \text{gr}_N^i V = N^i / N^{i-1}$. Suppose $W_1, \ldots, W_k$ have slopes $> 2g - 2$ and $W_{k+1}, \ldots, W_m$ have slopes $\leq 2g - 2$.

Using Clifford’s theorem for vector bundles [BPGN97, Theorem 2.1], for $i > k$, we have

$$\dim H^0(C, W_i) \leq \frac{\deg W_i}{2} + \text{rk} W_i.$$ 

Also, for $i \leq k$, since $W_i$ are semistable, there are no maps $W_i \to \omega_C$. Therefore, $H^1(C, W_i) = 0$ when $i \leq k$. It follows from Riemann Roch that

$$\dim H^0(C, W_i) = \deg W_i + (1 - g) \text{rk} W_i$$ 

for $i \leq k$. Summing over $i$, we get

$$\dim H^0(C, W) \leq \sum_{i=1}^m \dim H^0(C, W_i)$$

$$\leq \sum_{i=1}^k \left( \deg W_i + (1 - g) \text{rk} W_i \right) + \sum_{i=k+1}^m \left( \frac{\deg W_i}{2} + \text{rk} W_i \right)$$

$$= \sum_{i=1}^k \left( \frac{\deg W_i}{2} + \text{rk} W_i \right) + \sum_{i=1}^k \left( \frac{\deg W_i}{2} + g \text{rk} W_i \right) - g \text{rk} W_i$$

$$= \frac{\deg W}{2} + \text{rk} W + \sum_{i=1}^k \left( \frac{\deg W_i}{2} - g \text{rk} W_i \right).$$

To conclude, it is enough to show $\frac{\deg W_i}{2} - g \text{rk} W_i \leq 0$. However, since we were assuming the slope $\mu(W_i) \leq 2g$, we find $\deg W_i \leq 2g \text{rk} W_i$ and so $\frac{\deg W_i}{2} \leq g \text{rk} W_i$, as desired. \qed

The following lemma is a well known criterion for global generation, which we spell out for completeness.

**Lemma 6.2.2.** Let $V$ be a semistable vector bundle on a smooth proper curve $C$, such that the slope of $V$ satisfies $\mu(V) > 2g - 1$. Then $V$ is globally generated.

**Proof.** Let $p \in C$ be a point. It suffices to show $V|_p$ is generated by global sections of $V$. Indeed, $V(-p)$ is a semistable bundle with slope $\mu(V(-p)) > 2g - 1$. By the above theorem, $V(-p)$ is globally generated, and hence so is $V|_p$.\qed
2g − 2. Hence \( H^1(C, V(-p)) = 0 \), as any map \( V(-p) \to \omega_C \) would be destabilizing. Since \( H^1(C, V(-p)) = 0 \), the sequence

\[
0 \longrightarrow V(-p) \longrightarrow V \longrightarrow V|_p \longrightarrow 0
\]

is exact on global sections, so \( H^0(C, V) \otimes \mathcal{O} \to V \to V|_p \) is surjective, as desired.  

The following result will be key to the proof of Theorem 6.1.1, via Proposition 6.3.6, as it places a constraint on the rank of a vector bundle which is not generically globally generated.

**Lemma 6.2.3.** Suppose \( V \) is a vector bundle on a smooth proper curve \( C \) with \( \mu(V) \geq 2g - 2 \) (respectively, \( > 2g - 2 \)). Assume further \( U \subset V \) is a proper subbundle \( \delta := h^0(C, V) - h^0(C, U) \), and either \( U = 0 \) or else both

1. \( \mu(U) \leq \mu(V) \), and
2. each graded piece \( \text{gr}_{\text{HN}}^i U \) of the Harder Narasimhan filtration of \( U \) satisfies \( 0 \leq \mu(\text{gr}_{\text{HN}}^i U) \leq 2g \).

Then, \( \text{rk} V \geq g(\text{rk} V - \text{rk} U) - \delta \) (respectively, \( > g(\text{rk} V - \text{rk} U) - \delta \)). In particular, if \( h^0(C, U) = h^0(C, V) \), \( \text{rk} V \geq g \) (respectively, \( \geq g + 1 \)).

**Proof.** In the case \( U = 0 \), the inequality \( \text{rk} V \geq g(\text{rk} V - \text{rk} U) - \delta \) (respectively, \( \text{rk} V > g(\text{rk} V - \text{rk} U) - \delta \)) is equivalent to \( h^0(C, V) \geq (g - 1) \text{rk} V \) (respectively \( h^0(C, V) > (g - 1) \text{rk} V \)). This holds by Riemann-Roch.

We now assume \( U \neq 0 \). Applying Lemma 6.2.1, we conclude

\[
H^0(C, U) \leq \frac{\deg U}{2} + \text{rk} U.
\]

Using property Riemann-Roch and the definition of \( \delta \),

\[
\dim H^0(C, U) + \delta = \dim H^0(C, V) \geq \deg V + (1 - g) \text{rk} V.
\]

Combining the above gives

\[
\deg V + (1 - g) \text{rk} V \leq \frac{\deg U}{2} + \text{rk} U + \delta.
\]

To simplify notation, we use \( c := \text{rk} V - \text{rk} U \) to denote the corank of \( U \) in \( V \). Rewriting (6.2), and using \( \text{rk} U = \text{rk} V - c \) and \( \mu(U) \leq \mu(V) \) gives

\[
\mu(V) \text{rk}(V) + (1 - g) \text{rk} V \leq \frac{\mu(U) \text{rk} U}{2} + \text{rk} U + \delta \leq \frac{\mu(V)}{2} (\text{rk} V - c) + \text{rk} V - c + \delta.
\]
Rearranging the terms, and multiplying both sides by 2, we obtain

\[(\mu(V) + 2) \cdot c \leq (2g - \mu(V)) \text{rk}\ V + 2\delta.\]

Since \(2g - 2 \leq \mu(V)\), we find \(2g \leq \mu(V) + 2\) and \(2g - \mu(V) \leq 2\), implying

\[2g \cdot c \leq (\mu(V) + 2)c \leq (2g - \mu(V)) \text{rk}\ V + 2\delta \leq 2 \text{rk}\ V + 2\delta.\]

Therefore, \(\text{rk}\ V \geq gc - \delta\). In particular, if \(\delta = 0\), \(\text{rk}\ V \geq g\) as \(c \geq 1\). In the case \(2g - 2 < \mu(V)\), we similarly find \(\text{rk}\ V > gc - \delta\) and \(\text{rk}\ V \geq g + 1\) when \(\delta = 0\).

\[\square\]

### 6.3. A constraint on global generation of parabolic bundles.

In this subsection, we show that semistable parabolic bundles with large slope which are not generically globally generated cannot have small rank. This is accomplished in Proposition 6.3.6. Although it is a special case of Proposition 6.3.6, we start off by stating and proving the following special, yet pivotal, case, in order to convey the main idea and orient the reader. We call it “the non-GGG lemma,” where GGG stands for “generically globally generated.”

**Proposition 6.3.1 (The non-GGG lemma).** Suppose \(V\) is a nonzero semistable vector bundle on a smooth proper curve \(C\) which is not generically globally generated.

1. If \(\mu(V) > 2g - 2\), then \(\text{rk}\ V \geq g + 1\).
2. If \(\mu(V) = 2g - 2\), then \(\text{rk}\ V \geq g\).

**Proof.** The statement is trivial when \(g = 0\), so we assume \(g \geq 0\). Let \(U \subset V\) be the saturation of the image of the evaluation map

\[H^0(C, V) \otimes \mathcal{O}_C \to V.\]

We aim to apply Lemma 6.2.3. If \(V\) is not generically globally generated, \(U \subset V\) is a proper sub-bundle of \(V\), with \(H^0(C, U) \to H^0(C, V)\) an isomorphism. Hence, we will be done by the final statement of Lemma 6.2.3 once we verify hypotheses (1) and (2) of Lemma 6.2.3.

Semistability of \(V\) implies \(\mu(U) \leq \mu(V)\), verifying (1).

We conclude by checking hypothesis (2). Using Lemma 6.2.2, we may assume \(2g - 2 \leq \mu(V) \leq 2g - 1\). Since \(\mu(V) \leq 2g - 1\) and \(V\) is semistable, each graded piece \(\text{gr}_H^j U\) of the Harder-Narasimhan filtration of \(U\) must have slope at most \(2g - 1\). Let \(j\) be maximal such that \(\text{gr}_H^j U\) is non-zero. Since \(U\) is generically globally generated, \(\text{gr}_H^j U\) has a global section, and therefore has non-negative slope. By the definition of the Harder-Narasimhan
filtration, the same is true for $gr^i_{HN} U$ for all $i$. This verifies the final hypothesis (2) of Lemma 6.2.3, so we conclude $rk V \geq g + 1$ in case (a) and $rk V \geq g$ in case (b).

We next wish to generalize Proposition 6.3.1 to the parabolic setting.

Remark 6.3.2. The main difficulty in generalizing to the parabolic setting will be that the graded parts of the Harder-Narasimhan filtration of the bundle $U$ appearing in the proof of Proposition 6.3.1 need no longer have slope bounded above by $2g - 1$. We will get around this second issue in the proof of Proposition 6.3.6 by quotienting both $U$ and $V$ by the part of the Harder-Narasimhan filtration with slope more than $2g - 2$, and applying the ensuing argument to the resulting quotients.

Before taking up this generalization, we record a couple of preliminary lemmas which will allow us to understand generic global generation of parabolic and coparabolic bundles.

Lemma 6.3.3. Suppose $W_* = (W, \{W^i_j\}, \{\alpha^i_j\})$ is a parabolic bundle on $C$ with respect to a reduced divisor $D = x_1 + \cdots + x_n$. Let $\alpha := \max_{i,j} \alpha^i_j$. Then $\mu_*(W_*) - \mu(W) \leq na$. Further, equality holds if and only if all $\alpha^i_j$ are equal to $\alpha$.

Proof. By definition, this difference $\mu_*(W_*) - \mu(W)$ is

$$\sum_{j=1}^n \sum_{i=1}^{n_j} \frac{\dim(W^i_j/W^{i+1}_j)}{\text{rk } W} \cdot \alpha^i_j.$$

To verify the inequality, splitting the contribution from each point, it suffices to show

$$\sum_{i=1}^{n_j} \frac{\dim(W^i_j/W^{i+1}_j)}{\text{rk } W} \leq \alpha.$$

Indeed, this holds because, for all $j$,

$$\sum_{i=1}^{n_j} \frac{\dim(W^i_j/W^{i+1}_j)}{\text{rk } W} \leq \sum_{i=1}^{n_j} \frac{\dim(W^i_j/W^{i+1}_j)}{\text{rk } W} = \alpha \sum_{i=1}^{n_j} \frac{\dim(W^i_j/W^{i+1}_j)}{\text{rk } W} = \alpha.$$

Finally, equality holds in the above inequality for all $j$ if and only if all $\alpha^i_j$ are equal to $\alpha$. □

Recall from Definition 2.4.2 that a coparabolic bundle $\hat{E}_*$ is defined to be semistable if $E_*$ is semistable.
Lemma 6.3.4. Let $V_*$ be a semistable coparabolic vector bundle or a semistable parabolic bundle on a curve $C$ with respect to a reduced divisor $D = x_1 + \cdots + x_n$ and $\mu_*(V_*) = r + n$. Then any vector bundle $Q$ arising as a quotient of $V$ satisfies $\mu(Q) \geq r$. Moreover, $\mu(Q) > r$ holds in the parabolic case if $n > 0$.

Proof. First we deal with the case that $V_*$ is a parabolic vector bundle. By Lemma 2.4.5 any parabolic quotient of $V_*$ (with the induced quotient structure of §2.3) has parabolic slope at least $r + n$. Therefore, to complete the parabolic case, it suffices to show that for any parabolic bundle $W_*$ on $X$, $\mu_*(W_*) - \mu(W) \leq n$, with a strict inequality if $n > 0$. The $n = 0$ case is trivial while the $n > 0$ case follows from Lemma 6.3.3 as $\alpha_i^j < 1$ for all $i, j$.

Now, suppose $V_*$ is a coparabolic bundle of the form $\hat{W}_*$ for $W_* = (W, \{W^i_j\}, \{\alpha^i_j\})$ a parabolic bundle. For any $\epsilon > 0$, there is a map $W[\epsilon]_* \rightarrow \hat{W}_*$ and hence a map $W[\epsilon]_0 \rightarrow Q$. Let $Q^\epsilon$ denote the image of this map, which we may endow with the associated quotient parabolic structure to obtain a quotient bundle $W[\epsilon]_* \rightarrow Q^\epsilon$. This implies $\mu_*(Q^\epsilon) \geq \mu_*(W[\epsilon]_*) = r + n - \epsilon$ for all $\epsilon > 0$, by Lemma 2.4.5. By Lemma 6.3.3 it follows that $\mu_*(Q^\epsilon) - \mu(Q^\epsilon) \leq n$, so $\mu(Q^\epsilon) \geq r - \epsilon$. Now $Q^\epsilon \rightarrow Q$ has torsion cokernel, so $\mu(Q) \geq \mu(Q^\epsilon) \geq r - \epsilon$ for all $\epsilon > 0$, giving the result. □

We also need the following fairly standard lemma, which has little to do with parabolic bundles.

Lemma 6.3.5. Suppose $V$ is a vector bundle on a smooth proper curve $C$ so that each graded part of the Harder-Narasimhan filtration of $V$ has slope more than $2g - 2$. Then $H^1(C, V) = 0$.

Proof. Let $0 = N^0 \subset N^1 \subset \cdots \subset N^s = V$ denote the Harder-Narasimhan filtration. From the exact sequence

$$(6.3) \quad H^1(C, N^{i-1}) \longrightarrow H^1(C, N^i) \longrightarrow H^1(C, N^i / N^{i-1}) \longrightarrow 0$$

by induction on $i$, it is enough to show $H^1(C, N^i / N^{i-1}) = 0$ for every $i \leq s$. Since $N^i / N^{i-1}$ is semistable of slope more than $2g - 2$, there are no nonzero maps $N^i / N^{i-1} \rightarrow \omega_C$, and so $H^1(C, N^i / N^{i-1}) = 0$. □

Combining the above, we are able to verify the parabolic analog of Proposition 6.3.1, namely Proposition 6.3.6 below. In this paper, we will only require the case $\delta = 0, c = 1$ of Proposition 6.3.6 so we encourage the reader to focus on that case. We include the more general version, as the proof is nearly the same, and will be useful in future work.
Proposition 6.3.6. Suppose $C$ is a smooth proper connected genus $g$ curve and $E_\ast = (E, \{E_i\}, \{a_i\})$ is a nonzero parabolic bundle $C$ with respect to $D = x_1 + \cdots + x_n$. Suppose \( \hat{E}_\ast \) is coparabolically semistable. Let $U \subset \hat{E}_0$ be a (non-parabolic) subbundle with $c = \text{rk} E_0 - \text{rk} U$ and $\delta = h^0(C, \hat{E}_0) - h^0(C, U)$.

(I) If $\mu_\ast(\hat{E}_\ast) > 2g - 2 + n$, then $\text{rk} E > gc - \delta$.

(II) If $\mu_\ast(\hat{E}_\ast) = 2g - 2 + n$, then $\text{rk} E \geq gc - \delta$.

In particular, if \( \hat{E}_\ast \) fails to be generically globally generated, and $\mu_\ast(\hat{E}_\ast) > 2g - 2 + n$, then $\text{rk} E \geq g + 1$.

Proof. We first check the cases $g = 0$ and $g = 1$. The statement is trivial when $g = 0$. Case (II) is trivial when $g = 1$, because $\text{rk} E \geq c$. Similarly case (I) is trivial when $\delta > 0$. It remains to verify case (I) when $g = 1$ and $\delta = 0$. In this case, it is enough to show that $\text{rk} U > 0$, and since $H^0(C, U) = H^0(C, \hat{E}_0)$, it is enough to show $H^0(C, \hat{E}_0) \neq 0$. By Lemma 6.3.4, $\mu(\hat{E}_0) > 0$, so Riemann-Roch implies $H^0(C, \hat{E}_0) \neq 0$.

We now assume $g \geq 2$. Let $V_\ast := \hat{E}_\ast$ denote the given coparabolic bundle. As a first step, we reduce to the case that $U$ is generically globally generated. Indeed, let $U'$ denote the saturation of the image $H^0(C, U) \otimes \mathcal{O}_C \to U \to V$. Since $h^0(C, U') \geq h^0(C, U)$ and $\text{rk} V - \text{rk} U' \geq \text{rk} V - \text{rk} U$, the result holds for $U$ if it holds for $U'$. We may therefore assume $U$ is generically globally generated.

Let $0 = N^0 \subset N^1 \subset \cdots \subset N^s = U$ denote the Harder-Narasimhan filtration of $U$. Let $t$ be the minimal index so that $\mu(N^{t+1}/N^t) \leq 2g - 2$. If no such index exists, take $t = s$. We will show that in fact $\text{rk} V/N^t > gc$ in case (I) of the statement of this proposition and $\text{rk} V/N^t \geq gc$ in case (II). We will do so by verifying the hypotheses of Lemma 6.2.3 applied to the subbundle $U/N^t \subset V/N^t$. To this end, using Lemma 6.3.4, we find

\[(6.4) \quad \mu(V/N^t) > 2g - 2 \text{ in case (I) and } \mu(V/N^t) \geq 2g - 2 \text{ in case (II).}\]

We next verify $h^0(C, V) - h^0(C, U) = h^0(C, V/N^t) - h^0(C, U/N^t)$, which implies that present value of $\delta$, $h^0(C, V) - h^0(C, U)$, agrees with the value of $\delta$ we will use in our application of Lemma 6.2.3, $h^0(C, V/N^t) - h^0(C, U/N^t)$. Indeed, $H^1(C, N^t) = 0$ by Lemma 6.3.5. Therefore, $h^0(C, U/N^t) = h^0(C, U) - h^0(C, N^t)$ and $h^0(C, V/N^t) = h^0(C, V) - h^0(C, N^t)$.

Hence,

\[h^0(C, V/N^t) - h^0(C, U/N^t) = h^0(C, V) - h^0(C, U) = \delta.\]
In the case \( t = s \), so \( N^t = \mathcal{U} \), we have \( \mathcal{U}/N^t = 0 \), and we have verified the hypotheses of Lemma 6.2.3, so we now assume \( N^t \neq \mathcal{U} \). It remains to check hypotheses (1) and (2) of Lemma 6.2.3. We first check (2). Each graded piece of the Harder-Narasimhan filtration of \( \mathcal{U}/N^t \) has slope \( \leq 2g - 2 \) by construction of \( N^t \), which verifies the upper bound in (2). We claim the lower bound in (2) follows from generic global generation of \( \mathcal{U} \). Indeed, recall we are assuming \( \mathcal{U} \) is generically globally generated, via the reduction made near the beginning of the proof. Therefore, \( H^0(C, \mathcal{U}/N^{s-1}) = H^0(C, N^s/N^{s-1}) \neq 0 \), as any quotient bundle of a generically globally generated bundle is generically globally generated. This implies \( \mu(N^s/N^{s-1}) \geq 0 \), and so \( \mu(N^j/N^{j-1}) \geq 0 \) for all \( 1 \leq j \leq t \).

Finally, we verify (1), again in the case \( t < s \), i.e., \( N^t \neq \mathcal{U} \). In the previous paragraph, we showed each graded piece of the Harder-Narasimhan filtration of \( \mathcal{U}/N^t \) has slope at most \( 2g - 2 \), so we also obtain \( \mu(\mathcal{U}/N^t) \leq 2g - 2 \). On the other hand, we have already verified that \( \mu(V/N^t) \geq 2g - 2 \) in Section 6.4, above. Hence, \( \mu(\mathcal{U}/N^t) \leq 2g - 2 \leq \mu(V/N^t) \), verifying (1).

Applying Lemma 6.2.3 to the vector bundle \( V/N_t \) with the subbundle \( \mathcal{U}/N_t \) shows
\[
\text{rk } V/N_t \geq g(\text{rk } V/N_t - \text{rk } \mathcal{U}/N_t) - \delta = g(\text{rk } V - \text{rk } \mathcal{U}) - \delta = gc - \delta,
\]
where the inequality is strict in case (I) because \( \mu(V/N^t) > 2g - 2 \). This proves cases (I) and (II) since \( V = \mathcal{E}_0 \).

The final statement holds by taking \( \mathcal{U} \) to be the saturation of the image of \( H^0(C, \mathcal{E}_0) \otimes \mathcal{O}_C \to \mathcal{E}_0 \). In this case, \( H^0(\mathcal{U}) = H^0(C, \mathcal{E}_0) \), \( \delta = 0 \), and \( c \geq 1 \) when \( \mathcal{E}_0 \) is not generically globally generated. So we get \( \text{rk } E \geq g + 1 \).

\[ \Box \]

6.4. Reduction for the proof of Theorem 6.1.1 We next prove some results in preparation for the proof of Theorem 6.1.1. Reviewing the idea of the proof, described in Section 1.5, may be helpful.

**Notation** 6.4.1. Let \( (C, D) \) be a hyperbolic curve. Let \( (E, \nabla) \) be a flat vector bundle on \( C \) with regular singularities along \( D \), whose associated monodromy representation is irreducible. Let \( E_* \) be a parabolic structure on \( E \) refined by the canonical parabolic structure associated to \( \nabla \), as in Definition 3.3.1 and Definition 3.4.8. Let \( q^\nabla : T_C(-D) \to \text{At}_{(C,D)}(E_*) \) be the splitting of the Atiyah exact sequence associated to \( \nabla \) and described in Definition 3.3.1. Let \( N_*^N \), given by \( 0 = N^0_* \subset N^1_* \subset \cdots \subset N^m_* = E_* \), be a nontrivial filtration of \( E_* \) by parabolic subbundles (with the induced parabolic structure). In particular, note \( m > 1 \). Let \( \text{gr}^j_{N_*}(E_*) := N_*^j/N_*^{j-1} \) denote the quotient parabolic bundle with the induced parabolic quotient structure as described in Section 2.3.
The parabolic bundle $\mathcal{E}nd(E_\ast) / \mathcal{E}nd(E_\ast, N^\bullet_\ast)$ has a filtration by sheaves whose associated graded sheaf is of the form
\[ \bigoplus_{1 \leq i < j \leq n} \mathcal{H}om(\text{gr}_{N_\ast}^i(E_\ast), \text{gr}_{N_\ast}^j(E_\ast))_* . \]
For $i < j$ define $E^i_j := \mathcal{H}om(\text{gr}_{N_\ast}^i(E_\ast), \text{gr}_{N_\ast}^j(E_\ast))_* .

Let $\Delta = \mathcal{I}_{g,n}$ be the universal cover of the analytic stack $\mathcal{M}_{g,n}$, and let $(\mathcal{E}, \mathcal{D})$ be the universal marked curve over $\mathcal{I}_{g,n}$. Let $0 \in \Delta$ be such that $(\mathcal{E}, \mathcal{D})_0$ is isomorphic to $(C, D)$; fix such an isomorphism. Let $(\mathcal{E}_\ast, \nabla)$ be the universal isomonodromic deformation of $(E_\ast, \nabla)$ to $\mathcal{E}$.

We will later take the filtration $N^\bullet_\ast$ to be the Harder-Narasimhan filtration (cf. §2.5) of $E_\ast$.

By Proposition 3.3.2, the map $q^\nabla$ yields a non-zero map
\[ (6.5) \quad T_C(-D) \xrightarrow{q^\nabla} \text{At}_{(C,D)}(E_\ast) \to \mathcal{E}nd(E_\ast) / \mathcal{E}nd(E_\ast, N^\bullet_\ast) . \]

We now observe that if $N^\bullet_\ast$ extends to the universal isomonodromic deformation of $(C, D, E_\ast)$ on the first-order neighborhood of $(C, D)$, the induced map on first cohomology must vanish.

**Lemma 6.4.2.** Retain notation as in [Notation 6.4.1](#). If the filtration $N^\bullet_\ast$ extends to a filtration on the restriction of $(\mathcal{E}_\ast, \nabla)$ to the first-order neighborhood of $(C, D) = (\mathcal{E}, \mathcal{D})_0 \subset (\mathcal{E}, \mathcal{D})$, then the composite map
\[ H^1(C, T_C(-D)) \xrightarrow{(q^\nabla)_*} H^1(C, \text{At}_{(C,D)}(E_\ast)) \to H^1(C, \mathcal{E}nd(E_\ast)_*/ \mathcal{E}nd(E_\ast, N^\bullet_\ast)_*) . \]
induced by (6.5) is identically zero.

**Proof.** By Proposition 3.5.7 the map
\[ (q^\nabla)_* : H^1(C, T_C(-D)) \to H^1(C, \text{At}_{(C,D)}(E_\ast)) \]
induced by the connection sends a first-order deformation of the pointed curve $(C, D)$ to the corresponding first-order deformation of the triple $(C, D, E_\ast)$ obtained from isomonodromically deforming the connection $\nabla$. But given a first-order deformation $(\tilde{C}, \tilde{D}, \tilde{E}_\ast)$ of $(C, D, E_\ast)$ such that $N^\bullet_\ast \subset E_\ast$ admits an extension $N^\bullet_\ast$ to $\tilde{E}_\ast$, the corresponding element of $H^1(C, \text{At}_{(C,D)}(E_\ast))$ maps to 0 in $H^1(C, \mathcal{E}nd(E_\ast)_*/ \mathcal{E}nd(E_\ast, N^\bullet_\ast)_*)$, by Lemma 3.5.8. The assumption is precisely that this is true for all elements of $H^1(C, \text{At}_{(C,D)}(E_\ast))$ in the image of $(q^\nabla)_*$. \(\square\)
We now analyze the parabolic bundles $E_{*}^{i,j} := \mathcal{H}om(\text{gr}_{N_{*}}^{i}(E_{*}), \text{gr}_{N_{*}}^{j}(E_{*}))_{*}$, for $i < j$.

**Lemma 6.4.3.** With notation as in [Notation 6.4.1](#) for every $0 < i < m$, there exists $j, k$ with $j < i$ and $k \geq i + 1$ so that the nonzero map $T_{C}(-D) \rightarrow \mathcal{E}nd(E_{*})_{*}/\mathcal{E}nd(E_{*}, N_{*}^{i})_{*}$ induces a nonzero map $\psi_{j+1,k} : T_{C}(-D) \rightarrow E_{*}^{i+1,k}$.

**Proof.** First, recall the non-zero map $T_{C}(-D) \rightarrow \mathcal{E}nd(E_{*})/\mathcal{E}nd(E_{*}, N_{*}^{i})$ induced by $q \nabla$, produced in [Proposition 3.3.2](#). Let $j$ be maximal such that $\nabla(N^{j}) \subset N_{i} \otimes \Omega_{C}^{1}(D)$. Note that $j < i$ as the monodromy of $(E, \nabla)|_{C \setminus D}$ is irreducible, so $N_{i}$ is not a proper flat subbundle of $(E, \nabla)$, implying $\nabla(N^{j}) \not\subset N_{i} \otimes \Omega_{C}^{1}(D)$. Let $k$ be minimal such that $\nabla(N^{j+1}) \subset N_{k} \otimes \Omega_{C}^{1}(D)$. Note that $k \geq i + 1$ by the definition of $j$. By construction, the connection induces a nonzero $\mathcal{O}_{C}$-linear map of parabolic bundles

$$N_{*}^{j+1}/N_{*}^{j} \rightarrow (N_{*}^{k}/N_{*}^{j}) \otimes \Omega_{C}^{1}(D) \rightarrow (N_{*}^{k}/N_{*}^{j-1}) \otimes \Omega_{C}^{1}(D),$$

or equivalently a nonzero map

$$\psi_{j+1,k} : T_{C}(-D) \rightarrow \mathcal{H}om(\text{gr}_{N_{*}}^{j+1}(E_{*}), \text{gr}_{N_{*}}^{k}(E_{*}))_{*} = E_{*}^{j+1,k}. \Box$$

We have shown that for each $i$, there exist $j < i < k$, and a non-zero map

$$T_{C}(-D) \rightarrow E_{*}^{i+1,k} = \mathcal{H}om(\text{gr}_{N_{*}}^{i+1}(E_{*}), \text{gr}_{N_{*}}^{k}(E_{*})).$$

We next refine [Lemma 6.4.2](#) by showing that if its hypotheses are satisfied and if in addition $N_{*}^{i}$ is the Harder-Narasimhan filtration of $E_{*}$, the map on $H^{1}$ induced by $\psi_{j+1,k} : T_{C}(-D) \rightarrow E_{*}^{i+1,k}$ must also vanish.

**Lemma 6.4.4.** Use notation as in [Notation 6.4.1](#). Suppose in addition that $N_{*}^{i}$ is the Harder-Narasimhan filtration of $E_{*}$. Fix $i$ with $0 < i < m$ and let $j, k$, and

$$\psi_{j+1,k} : T_{C}(-D) \rightarrow E_{*}^{j+1,k}$$

be the data constructed in [Lemma 6.4.3](#). If the filtration $N_{*}^{i}$ extends to a filtration on the restriction of $(\mathcal{E}_{*}, \nabla)$ to the first-order neighborhood of $(C, D) = (\mathcal{E}, \mathcal{D})_{0} \subset (\mathcal{E}, \mathcal{D})$, then the map $H^{1}(C, T_{C}(-D)) \rightarrow H^{1}(C, E_{*}^{j+1,k})$ induced by $\psi_{j+1,k}$ vanishes.

For the proof, we will need the following two lemmas.

**Lemma 6.4.5.** Suppose $F_{*}, G_{*}$ are parabolic vector bundles on a smooth proper curve connected $C$ with respect to a divisor $D$ such that the Harder Narasimhan filtrations $0 = N_{*}^{0} \subset N_{*}^{1} \subset \cdots \subset N_{*}^{n} = F_{*}$ and $0 = M_{*}^{0} \subset M_{*}^{1} \subset \cdots \subset M_{*}^{0} = G_{*}$.
Then \( H^0(\mathcal{O}_* \otimes G_*) \) which in turn factors through \( T_* \) sheaves \( \text{Hom} \) \( \text{gr} \) group be a parabolic bundle \( H_* \). Lemma 6.4.6.

Proof. Observe that \( F_*^\vee \otimes G_* \) has a filtration whose associated graded pieces are \( \text{gr}^i_{N_*}(F_*)^\vee \otimes \text{gr}^j_{M_*}(G_*) \), so it is enough to show the latter have vanishing \( H^0 \).

An nonzero element of \( H^0(C, \text{gr}^i_{N_*}(F_*)^\vee \otimes \text{gr}^j_{M_*}(G_*)) \) would yield a nonzero map \( \text{gr}^i_{N_*}(F_*) \to \text{gr}^j_{M_*}(G_*) \). The saturation of the image of this map would be a parabolic bundle \( H_* \). By semistability, \( \mu_*(\text{gr}^i_{N_*}(F_*)) \leq \mu_*(H_*) \leq \mu_*(\text{gr}^j_{M_*}(G_*)) \), contradicting the assumption that \( \mu_*(\text{gr}^i_{N_*}(F_*)) > \mu_*(\text{gr}^j_{M_*}(G_*)) \).

Lemma 6.4.6. Suppose

\[
0 \to F_* \to G_* \to H_* \to 0
\]

is a short exact sequence of parabolic sheaves on a smooth proper connected curve \( C \) with respect to a divisor \( D \) and we are given a map \( E \to F_* \) inducing the 0 map \( H^1(C, E) \to H^1(C, F_*) \to H^1(C, G_*) \). If additionally \( H^0(C, H_*) = 0 \) then \( H^1(C, E) \to H^1(C, F_*) \) vanishes.

Proof. The vanishing of \( H^0(C, H_*) \) yields an injection \( H^1(C, F_*) \to H^1(C, G_*) \). If the composition \( H^1(C, E) \to H^1(C, F_*) \to H^1(C, G_*) \) vanishes then so does \( H^1(C, E) \to H^1(C, F_*) \).

We now proceed with the proof of Lemma 6.4.4.

Proof of Lemma 6.4.4. The proof is a diagram chase. Recall that the nonzero map in Lemma 6.4.3 was constructed by beginning with the map \( T_C(-D) \to \text{End}(E)_*/\text{End}(E, N^*_*) \), and then showing the induced map \( T_C(-D) \to \mathcal{H}om(N^{i+1}_x, E_*/N^{k-1})_* \) factors through a map \( T_C(-D) \to \mathcal{H}om(N^{i+1}_x, N^k_x/N^{k-1})_* \), which in turn factors through \( T_C(-D) \to \mathcal{H}om(N^{i+1}_x/ N^k_x, N^k_x/N^{k-1})_* \).

We will show each of the above three maps vanishes on \( H^1 \). By Lemma 6.4.2 \( T_C(-D) \to \text{End}(E)_* \) vanishes on \( H^1 \). Next, the injection of parabolic sheaves \( N^{i+1}_x \to E_* \) induces a surjection of parabolic sheaves \( \text{End}(E)_* \to \mathcal{H}om(N^{i+1}_x, E_*)_x \). From this, we obtain a surjection

\[
\text{End}(E)_*/\text{End}(E, N^*_*)_* \to \mathcal{H}om(N^{i+1}_x, E_*/N^{k-1})_*.
\]
Note that $H^2$ of parabolic sheaves vanishes on a curve, because the same is true for usual sheaves. We therefore obtain a surjection

$$H^1(C, \mathcal{E}nd(E_\ast)_\ast / \mathcal{E}nd(E_\ast, N^k_\ast)_\ast) \twoheadrightarrow H^1(C, \mathcal{H}om(N^{j+1}_\ast, E_\ast / N^k_\ast)_\ast).$$

Thus the composition $T_C(-D) \rightarrow \mathcal{E}nd(E_\ast)_\ast / \mathcal{E}nd(E_\ast, N^k_\ast)_\ast \rightarrow \mathcal{H}om(N^{j+1}_\ast, E_\ast / N^k_\ast)_\ast$ induces the zero map on $H^1$, because the first map does by Lemma 6.4.2.

We next show the natural map $T_C(-D) \rightarrow \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), E_\ast / N^k_\ast)_\ast$ to be described below, vanishes on $H^1$. Using Lemma 6.4.5, we find that

$$H^0(C, \mathcal{H}om(N^j_\ast, E_\ast / N^k_\ast)_\ast) = 0.$$ 

Therefore, applying Lemma 6.4.6 to the short exact sequence

$$0 \rightarrow \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), E_\ast / N^k_\ast)_\ast \rightarrow \mathcal{H}om(N^{j+1}_\ast, E_\ast / N^k_\ast)_\ast \rightarrow \mathcal{H}om(N^j_\ast, E_\ast / N^k_\ast)_\ast \rightarrow 0$$

and the map $f : T_C(-D) \rightarrow \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), E_\ast / N^k_\ast)_\ast$ shows that $f$ induces the 0 map on $H^1$.

We conclude by showing the map

$$\psi_{j+1,k} : T_C(-D) \rightarrow \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), \mathfrak{gr}^k_N(E_\ast))_\ast = E^{j+1,k}_\ast$$

vanishes on $H^1$. Again by Lemma 6.4.5

$$H^0(C, \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), E_\ast / N^k_\ast)_\ast) = 0.$$ 

Applying Lemma 6.4.6 to the exact sequence

(6.6)

$$0 \rightarrow \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), \mathfrak{gr}^k_N(E_\ast))_\ast \rightarrow \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), E_\ast / N^k_\ast)_\ast \rightarrow \mathcal{H}om(\mathfrak{gr}^{j+1}_N(E_\ast), E_\ast / N^k_\ast)_\ast \rightarrow 0$$

and the map $\psi_{j+1,k}$ shows that $\psi_{j+1,k}$ vanishes on $H^1$, as desired. 

We now show that if the map $H^1(C, T_C(-D)) \rightarrow H^1(C, E^{j+1,k}_\ast)$ vanishes, we will be able to produce a coparabolic bundle which is not generically globally generated. We will later apply Proposition 6.3.6 to this bundle to obtain Theorem 6.1.1(1).

**Lemma 6.4.7.** With notation as in Notation 6.4.1, suppose the map $H^1(C, T_C(-D)) \rightarrow H^1(C, E^{j+1,k}_\ast)$ (induced by the non-zero map $\psi_{j+1,k} : T_C(-D) \rightarrow E^{j+1,k}_\ast$ of Lemma 6.4.3) vanishes. Then the coparabolic bundle $(E^{j+1,k}_\ast)^\vee \otimes \omega_C(D)$ is not generically globally generated.
Proof. Since $\psi_{j+1,k} : T_c(-D) \to E_{\star}^{j+1,k}$ is nonzero, we obtain a nonzero Serre dual map
\begin{equation}
(E_{\star}^{j+1,k})^\vee \otimes \omega_C(D) \to \omega_C \otimes \omega_C(D),
\end{equation}
which induces the 0 map
\[H^0(C, (E_{\star}^{j+1,k})^\vee \otimes \omega_C(D)) \to H^0(C, \omega_C \otimes \omega_C(D))\]
by Serre duality [Proposition 2.6.6] and [Lemma 6.4.4]. In particular, $(E_{\star}^{j+1,k})^\vee \otimes \omega_C(D)$ is not generically globally generated. Indeed, any global section must lie in the kernel of (6.7), which has corank one in $(E_{\star}^{j+1,k})^\vee \otimes \omega_C(D)$.

This concludes our setup for proving Theorem 6.1.1(1). To prove Theorem 6.1.1(2), we will need the following generalization of Lemma 6.2.2 to the coparabolic setting.

Lemma 6.4.8. If $V_\star$ is a semistable coparabolic bundle with respect to a divisor $D = x_1 + \cdots + x_n$, of coparabolic slope $\mu_\star(V_\star) > 2g - 1 + n$, then $V_\star$ is globally generated.

Proof. Let $p \in C$ be a point. Suppose $V_\star = \hat{E}_\star$ for $E_\star$ a parabolic bundle. It suffices to show $V_\star$ is generated by global sections at $p$. Indeed, $V_\star(-p)$ is a semistable coparabolic bundle with coparabolic slope $\mu_\star(V_\star(-p)) > 2g - 2 + n$. Therefore, $\mu_\star(E_\star(p) \otimes \omega_C(D)) < 0$, implying $H^0(C, E_\star(p) \otimes \omega_C(D)) = 0$, as any global section would be destabilizing. By Serre duality [Proposition 2.6.6] (taking $E_\star(p) \otimes \omega_C(D)$ in place of the parabolic vector bundle $E_\star$ in Proposition 2.6.6), $H^1(C, V_\star(-p)) = H^0(C, E_\star(p) \otimes \omega_C(D))$ is surjective, as desired.

6.4.9. We now prove Theorem 6.1.1.

Proof of Theorem 6.1.1. We use notation as in Notation 6.4.1. We aim first to show that if $(E_\star', \nabla')$ is the isomonodromic deformation of $(E_\star, \nabla)$ to an analytically general nearby curve $C'$, and if $E_\star'$ is not semistable, then for every $i$ there are some $j < i < k$ with $\text{rk gr}_{HN}^{j+1} E_\star' \cdot \text{rk gr}_{HN}^{k} E_\star' \geq g + 1$.

By [BHH21] Lemma 5.1, the locus of bundles in a family $\mathcal{E}_\star$ on $\mathcal{C} \to \Delta$ which are not semistable form a closed analytic subset, and if a general member is not semistable, then, after passing to an open subset of $\Delta$, there...
is a filtration on $\mathcal{E}_s$ restricting to the Harder-Narasimhan filtration on each fiber. Thus after replacing $(C, D)$ with an analytically general nearby curve $(C', D')$, and replacing $(E_s, \nabla)$ with the restriction $(E'_s, \nabla')$ of the isomonodromic deformation to $(C', D')$, we may assume the Harder-Narasimhan filtration $HN^\bullet$ of $E'_s$ extends to a filtration of $\mathcal{E}_s$ on a first-order neighborhood of $C'$. We set $(E'_s)^{i/j} := \mathcal{H}om(gr_{HN}^{i}(E'_s), gr_{HN}^{j}(E'_s))_s$; note that $(E'_s)^{i/j}$ is semistable by the definition of the Harder-Narasimhan filtration.

We next verify that for every $0 < i < m$, there is some $j < i < k$ for which $(E'_s)^{i+1,j} \otimes \omega_C(D)$ is not generically globally generated. By Lemma 6.4.3 for every $0 < i < m$, there is some $j < i$ and $k \geq i + 1$ so that the map $T_{C'}(-D') \to \mathcal{E}nd(E'_s)/\mathcal{E}nd_{HN^\bullet}(E'_s)$ induces a nonzero map

$$T_{C'}(-D') \to (E'_s)^{i+1,k} := \mathcal{H}om(gr_{HN}^{i+1}(E'_s), gr_{HN}^{k}(E'_s))_s.$$  

By Lemma 6.4.4, $H^1(C', T_{C'}(-D')) \to H^1(C', (E'_s)^{i+1,k})$ vanishes. Hence, by Lemma 6.4.7, $(E'_s)^{i+1,k} \otimes \omega_C(D)$ is not generically globally generated. Note that $(E'_s)^{i+1,k} \otimes \omega_C(D)$ has slope $2g - 2 + n$ by Lemma 2.6.5.

We are finally in a position to prove Theorem 6.1.1(1). It follows from Proposition 6.3.6 that

$$\text{rk gr}_{HN}^{j+1}(E'_s) \cdot \text{rk gr}_{HN}^{k}(E'_s) = \text{rk}((E'_s)^{j+1,k}) \otimes \omega_C(D) \geq g + 1.$$  

Thus Theorem 6.1.1(1) holds.

We now conclude by verifying Theorem 6.1.1(2). By Lemma 6.4.8

$$(E'_s)^{j+1,k} \otimes \omega_C(D) = \mathcal{H}om(gr_{HN}^{j+1}(E'_s), gr_{HN}^{k}(E'_s) \otimes \omega_C(D)$$  

must have slope at most $2g - 1 + n$, since it is not generically globally generated. As $\mathcal{H}om(gr_{HN}^{j+1}(E'_s), gr_{HN}^{k}(E'_s))$ has negative parabolic slope by the definition of the Harder-Narasimhan filtration and Lemma 2.6.3, we find

$$-1 \leq \mu_*(\mathcal{H}om(gr_{HN}^{j+1}(E'_s), gr_{HN}^{k}(E'_s))) < 0.$$  

Using Lemma 2.6.5, the parabolic slope of a tensor product of parabolic vector bundles is the sum of their parabolic slopes, and taking duals negates parabolic slope. Therefore,

$$0 < \mu_*(gr_{HN}^{j+1}(E'_s)) - \mu_*(gr_{HN}^{k}(E'_s)) \leq 1.$$  

Since

$$\mu_*(gr_{HN}^{j+1}(E'_s)) \geq \mu_*(gr_{HN}^{i}(E'_s)) \geq \mu_*(gr_{HN}^{i+1}(E'_s)) \geq \mu_*(gr_{HN}^{k}(E'_s)),$$  


we also conclude

\[ 0 < \mu_*(\text{gr}^i_{HN}(E'_*)) - \mu_*(\text{gr}^{i+1}_{HN}(E'_*)) \leq 1. \]

\(\square\)

6.4.10. We now prove Corollary 6.1.2 using Theorem 6.1.1 and the AM-GM inequality.

Proof of Corollary 6.1.2. It suffices to consider the case where \((E, \nabla)\) has irreducible monodromy, as an extension of semistable parabolic bundles of the same slope is semistable.

If \((E'_*, \nabla')\) is an isomonodromic deformation of \((E_*, \nabla)\) to an analytically general nearby curve which is not semistable, it follows from Theorem 6.1.1(1) that for each \(i\), there will be \(j, k\) with \(j < i < k\) so that the Harder-Narasimhan filtration \(HN\) of \(E'_*\) satisfies \(\text{rk} \text{gr}^{j+1}_{HN} E'_* \cdot \text{rk} \text{gr}^k_{HN} E'_* \geq g + 1\). Since \(\text{rk} \text{gr}^{j+1}_{HN} E'_* + \text{rk} \text{gr}^k_{HN} E'_* \leq \text{rk} E'_* = \text{rk} E_*\), it follows from the arithmetic mean-geometric mean inequality that

\[ g + 1 \leq \text{rk} \text{gr}^{j+1}_{HN} E'_* \cdot \text{rk} \text{gr}^k_{HN} E'_* \leq \left( \frac{\text{rk} E_*}{2} \right)^2. \]

So \(\text{rk} E_* \geq 2\sqrt{g + 1}\) as desired. \(\square\)

6.4.11. We conclude by proving Theorem 1.3.4 and Corollary 1.3.6.

Proof of Theorem 1.3.4 and Corollary 1.3.6. These follow immediately from Theorem 6.1.1 and Corollary 6.1.2 respectively, taking \(E_*\) to have the trivial parabolic structure. \(\square\)

7. Variations of Hodge structure on an analytically general curve

We prove Theorem 1.2.12 in §7.1, Theorem 1.2.5 in §7.2, Corollary 1.2.8 in §7.3 and Corollary 1.2.7 in §7.4. Finally, we prove an additional application concerning maps from very general curves to Hilbert modular varieties in §7.2.

7.1. The proof of Theorem 1.2.12. We start with the following lemma, which gives a useful criterion for showing a monodromy representation is unitary.

Lemma 7.1.1. Suppose \((C, x_1, \ldots, x_n)\) is an \(n\)-pointed hyperbolic curve and \((E, \nabla)\) is a flat vector bundle on \(C \setminus \{x_1, \ldots, x_n\}\) underlying a polarizable complex variation of Hodge structure. Let \((\overline{E}_*, \overline{\nabla})\) be the Deligne canonical extension of \((E, \nabla)\)
to a flat vector bundle on $C$ with regular singularities at the $x_i$, with the parabolic structure associated to $\nabla$. If $E_\ast$ is semistable, then the representation of $\pi_1(C \setminus \{x_1, \cdots, x_n\})$ associated to $(E, \nabla)$ is unitary.

**Proof.** By Proposition 4.1.4(2), we may write $V := \ker(\nabla)$ as

$$V := \bigoplus_i L_i \otimes W_i$$

where the $L_i$ each have irreducible monodromy and also underlie polarizable variations of Hodge structure, and the $W_i$ are constant complex Hodge structures. It suffices to show the representation associated to each $L_i$ has unitary monodromy. We may therefore reduce to the case that $V = L_i$ and assume that $(E, \nabla)$ has irreducible monodromy.

Let $i$ be maximal such that $F^iE_\ast$ is non-zero. Since $E_\ast$ is semistable, it follows from Lemma 4.1.5 that the natural map

$$F^iE_\ast \to F^{i-1}E_\ast / F^iE_\ast \otimes \omega_C$$

induced by the connection is zero, i.e. the connection preserves $F^iE_\ast$. By irreducibility of the monodromy of $(E, \nabla)$, we must have that $F^iE_\ast$ equals $E_\ast$. But in this case $(E, \nabla)$ is unitary, as the monodromy preserves the polarization, a definite Hermitian form. $\square$

7.1.2. We now recall the setup of Theorem 1.2.12. Let $(C, x_1, \cdots, x_n)$ be an $n$-pointed hyperbolic curve of genus $g$. Let $(E, \nabla)$ be a flat vector bundle on $C$ with $\text{rk} E < 2\sqrt{g + 1}$ such that $(E, \nabla)$ has regular singularities at the $x_i$. Our goal is to show that if an isomonodromic deformation of $(E, \nabla)$ to an analytically general nearby $n$-pointed curve underlies a polarizable complex variation of Hodge structure, then $(E, \nabla)$ has unitary monodromy.

**Proof of Theorem 1.2.12** As the hypothesis is about the restriction of $(E, \nabla)$ to $C \setminus \{x_1, \cdots, x_n\}$, we may without loss of generality assume $(E, \nabla)$ is the Deligne canonical extension of $(E, \nabla)|_{C \setminus \{x_1, \cdots, x_n\}}$.

Endow $E$ with the parabolic structure associated to $\nabla$, and denote the corresponding parabolic bundle by $E_\ast$. After replacing $(C, x_1, \cdots, x_n)$ with an analytically general nearby curve, and $(E, \nabla)$ with its isomonodromic deformation to this curve, we may assume by Corollary 6.1.2 that $E_\ast$ is semistable. Thus $(E, \nabla)$ has unitary monodromy by Lemma 7.1.1. $\square$

7.2. The proof of Theorem 1.2.5. The proof of Theorem 1.2.5 follows from the integrality assumption and the following lemma.
Lemma 7.2.1. Suppose $\Gamma$ is a group, $K$ is a number field, and $\rho$ is a representation $\rho : \Gamma \to \text{GL}_m(O_K)$. 
If for each embedding $\iota : K \hookrightarrow \mathbb{C}$ the representation $\rho \otimes_{O_{K,\iota}} \mathbb{C}$ is unitary, then $\rho$ has finite image.

Proof. Indeed, for $\iota : K \hookrightarrow \mathbb{C}$ an embedding, let $\rho_\iota : \Gamma \to \text{GL}_m(\mathbb{C})$ be the corresponding representation $\rho \otimes_{O_{K,\iota}} \mathbb{C}$. First, $\prod \rho_\iota : \Gamma \to \prod \text{GL}_m(\mathbb{C})$ has compact image by the definition of being unitary. Moreover, the image of $\text{GL}_m(O_K) \hookrightarrow \prod \mathbb{C}$ is discrete, since the difference of any two distinct elements of $O_K$ has norm at least 1. Hence the image of $\prod \rho_\iota$ is discrete and compact, and therefore finite. $\square$

Let $K$ be a number field with ring of integers $O_K$. Let $(C, x_1, \ldots, x_n)$ be an analytically general hyperbolic $n$-pointed curve of genus $g$, and let $V$ be a $O_K$-local system on $C \setminus \{x_1, \ldots, x_n\}$ with infinite monodromy. Suppose that for each embedding $\iota : O_K \hookrightarrow \mathbb{C}, V \otimes_{O_{K,\iota}} \mathbb{C}$ underlies a polarizable complex variation of Hodge structure. Our goal is to prove Theorem 1.2.5, which states that $\text{rk}_{O_K}(V) \geq 2\sqrt{g+1}$.

Proof of Theorem 1.2.5. We use $\mathcal{T}_{g,n}$ to denote the universal cover of $\mathcal{M}_{g,n}$. For a fixed representation $\rho : \pi_1(C \setminus \{x_1, \ldots, x_n\}) \to \text{GL}_m(O_K)$ let $T_\rho$ denote the set of $[(C', x'_1, \ldots, x'_n)] \in \mathcal{T}_{g,n}$, for which the associated $O_K$-local system $V$ has the following property: for each embedding $\iota : O_K \hookrightarrow \mathbb{C}, V \otimes_{O_{K,\iota}} \mathbb{C}$ underlies a polarizable complex variation of Hodge structure on $C' \setminus \{x'_1, \ldots, x'_n\}$. Let $M_\rho$ denote the image of $T_\rho$ under the covering map $\mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$.

Our goal is to show that an analytically very general point of $\mathcal{M}_{g,n}$ lies in the complement of the union of the $M_\rho$, where $\rho$ ranges over the set of representations of $\pi_1(C \setminus \{x_1, \ldots, x_n\}) \to \text{GL}_r(O_K)$, with infinite image, for $K$ a number field, and $r < 2\sqrt{g+1}$. Since there are only countably many such representations $\rho$, it is enough to show that an analytically very general point lies in the complement of $M_\rho$ for fixed $\rho$. Since $(C, x_1, \ldots, x_n)$ is
hyperbolic, \( \mathcal{T}_{g,n} \rightarrow \mathcal{M}_{g,n} \) is a covering space of countable degree, and so the image of a closed analytic set is locally contained in a countable union of closed analytic subsets. It therefore suffices to show that for any \( \rho \) with infinite monodromy and \( \text{rank} < 2\sqrt{g + 1} \), \( T_\rho \) is contained in a closed analytic subset of \( \mathcal{T}_{g,n} \).

We now show such \( T_\rho \) as above are contained in a closed analytic subset of \( \mathcal{T}_{g,n} \). Indeed, suppose \( V \) is the local system associated to \( \rho \) on some curve \( C \setminus \{x_1, \ldots, x_n\} \), and that for each embedding \( \iota: \mathcal{O}_K \rightarrow C \), \( V \otimes \mathcal{O}_K, \iota C \) underlies a polarizable complex variation of Hodge structure. It is enough to show this complex polarizable variation of Hodge structure does not extend to an analytically general nearby curve. Indeed, if it did, Theorem 1.2.12 implies \( \prod_{\iota: \mathcal{O}_K \rightarrow C} \rho, \iota \) has unitary monodromy, and Lemma 7.2.1 implies its monodromy is finite. This contradicts our assumption that \( \rho \) has infinite monodromy. \( \square \)

7.3.

**Proof of Corollary 1.2.7** Let \( g \geq 1 \) be an integer and let \( (C, x_1, \ldots, x_n) \) be an analytically general hyperbolic \( n \)-pointed curve of genus \( g \). Let \( U \subset C \setminus \{x_1, \ldots, x_n\} \) be a dense Zariski-open subset. Let \( f: Y \rightarrow U \) be a smooth proper morphism, \( i \geq 0 \) an integer, and suppose \( V \) is a complex local system on \( (C, x_1, \ldots, x_n) \) with infinite monodromy such that \( V|_U \) is a summand of \( R^i f_* C \). Then we wish to show that \( \dim \mathbb{C} V \geq 2\sqrt{g + 1} \).

It suffices to show that \( V \) satisfies the hypotheses of Theorem 1.2.5. The existence of an \( \mathcal{O}_K \)-structure follows from the fact that \( R^i f_* C \) has a \( \mathbb{Z} \)-structure. Let \( W \) be the corresponding \( \mathcal{O}_K \)-local system. All that remains is to verify that for each embedding \( \iota: \mathcal{O}_K \rightarrow C \), the corresponding complex local system \( W, \iota := W \otimes \mathcal{O}_K, \iota C \) underlies a polarizable complex variation of Hodge structure. But each such embedding yields a summand \( W, \iota|_U \) of \( R^i f_* C \), Galois-conjugate to the original embedding \( V|_U \subset V|_U \subset R^i f_* C \). Any such summand underlies a polarizable complex variation of Hodge structure, by Proposition 4.1.4(2). Now \( W, \iota|_U \) extends from \( U \) to \( C \setminus \{x_1, \ldots, x_n\} \) and so the same is true for the corresponding complex polarizable variation of Hodge structure by [Sch73, Corollary 4.11], which completes the proof. \( \square \)

7.4.

**Proof of Corollary 1.2.8** The statement for relative curves \( h: S \rightarrow C \setminus \{x_1, \ldots, x_n\} \) reduces to the statement for abelian varieties upon taking the relative Jacobian \( \text{Pic}^0_h \), as the Torelli theorem implies that \( h \) is isotrivial if \( \text{Pic}^0_h \) is. Hence,
After passing to a finite étale cover of part [Del71b, Corollaire 4.1.2] implies that the first part of the Hodge filtration parametrizing conjugacy classes of semisimple representations $F$ of $\mathbb{C}$ fibers this local system is identified with the first homology of $f$ upon choosing a topological identification $\mathbb{C}$. Not Zariski dense in $\mathbb{C}$ is also trivial.

7.5. We next prove that local systems of geometric origin with low rank are not Zariski dense in the character variety of an analytically very general genus $g$ curve. Recall that we use $\mathcal{M}_{B,r}(X)$ for the character variety parametrizing conjugacy classes of semisimple representations

$$\rho : \pi_1(C \{x_1, \ldots, x_n\}) \rightarrow \text{GL}_r(\mathbb{C}).$$

Proof of Corollary 1.2.10. Let $(C, x_1, \ldots, x_n)$ be an analytically very general hyperbolic $n$-pointed curve of genus $g$, and fix an integer $r$ with $1 < r < 2\sqrt{g + 1}$. Note that we must have $g \neq 0$ because there are no integers $r$ with $1 < r < 2\sqrt{0 + 1}$. By Corollary 1.2.7, the points of $\mathcal{M}_{B,r}(C \{x_1, \ldots, x_n\})$ of geometric origin correspond to representations $\rho$ with finite image whenever $r < 2\sqrt{g + 1}$. We wish to show these finite image representations are not Zariski dense in $\mathcal{M}_{B,r}(C \{x_1, \ldots, x_n\})$. This follows from by Lemma 7.5.1 upon choosing a topological identification $C \{x_1, \ldots, x_n\} \simeq \Sigma_{g,n}$.

Lemma 7.5.1. Let $\Sigma_{g,n}$ be a topological $n$-punctured genus $g$ surface with basepoint $p \in \Sigma_{g,n}$. For $r > 1$, the set of representations $\rho : \pi_1(\Sigma_{g,n}, p) \rightarrow \text{GL}_r(\mathbb{C})$ with finite image are not Zariski dense in the character variety $\mathcal{M}_{B,r}(\Sigma_{g,n}, p)$.

Proof. It suffices to prove non-density of representations with finite image in the framed character variety

$$\mathcal{M}_{B,r}^{\square}(\Sigma_{g,n}) := \text{Hom}(\pi_1(\Sigma_{g,n}, p), \text{GL}_r(\mathbb{C})).$$

For $s \in \mathbb{Z}_{>0}$, let $V_s \subset \mathcal{M}_{B,r}^{\square}(\pi_1(\Sigma_{g,n}, p))$ be the closed subvariety consisting of those representations $\rho$ such that

$$[\rho(x)^s, \rho(y)^s] = 1$$

for all $x, y \in \pi_1(\Sigma_{g,n}, p)$. 
By Jordan’s theorem ([Jor78, p. 91] or [CR66, Theorem 36.13]) on finite subgroups of GL$_r$(C), there is some constant $m_r$ such that for each finite subgroup $G \subset$ GL$_r$(C), there exists an abelian normal subgroup $H \subset G$ of index at most $m_r$. Hence $V_{(m, r)}$ contains all representations with finite image. Thus it suffices to show $V_s$ is not all of $\mathcal{M}_{B, r}^\square(\Sigma_{g, n})$ for any $s > 0$.

We now write
\[ \pi_1(\Sigma_{g, n}, p) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g, \gamma_1, \ldots, \gamma_n \mid \prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{j=1}^{n}\gamma_j \right\rangle \]
for the standard presentation of the fundamental group. If $g \geq 2$, let $\rho : \pi_1(\Sigma_{g, n}, p) \to$ GL$_2$(C) be the representation defined by
\[ a_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad a_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]
and sending all other generators to the identity. If $g = 1$, hyperbolicity of $\Sigma_{g, n}$ implies $n > 0$. Then $\pi_1(\Sigma_{g, n}, x)$ is free on $n + 1$ generators, and we may set $\rho$ to be a representation sending two of the generators to the matrices above, and all other generators to the identity. Then $\rho$ does not lie in $V_s$ for any $s$ because $\rho(a_1^s)$ does not commute with $\rho(a_2^s)$ for any integer $s > 0$. This completes the case $r = 2$. For the case that $r > 2$, the representation $\rho \oplus \text{triv}^{\oplus r-2}$ lies outside of $V_s$ for all $s$, for the same reason. \[ \square \]

7.6. An application to Hilbert modular varieties. Let $K$ be a totally real number field of degree $h$ over $\mathbb{Q}$, $\mathcal{O}_K$ its ring of integers, and use $\mathcal{X}_K$ to denote the Hilbert modular stack parameterizing $h$-dimensional principally polarized abelian varieties $A$ with an injection $\mathcal{O}_K \hookrightarrow \text{End}(A)$. Using Corollary 1.2.8, we find there are no nonconstant maps from an analytically very general $n$-pointed curve of genus $g$ to the moduli stack of principally polarized abelian varieties of dimension $h$ whenever $h \leq \sqrt{g+1}$, or equivalently when $g > h^2$. We now show that the analogous statement for Hilbert modular stacks holds whenever $g \geq 1$. In particular, this improved bound is independent of $h$. We thank Alexander Petrov for pointing out the following application.

**Proposition 7.6.1.** Let $\mathcal{X}_K$ be the Hilbert modular stack associated to a totally real field $K$. Let $(C, x_1, \ldots, x_n)$ be an analytically very general hyperbolic $n$-pointed curve of genus $g \geq 1$. Any map $\phi : C \setminus \{x_1, \ldots, x_n\} \to \mathcal{X}_K$ is constant.
Proof. We first use the map $\phi$ to produce a rank 2 integral local system on $C \setminus \{x_1, \ldots, x_n\}$. A map $\phi : C \setminus \{x_1, \ldots, x_n\} \to X_K$ yields an abelian scheme $f : A \to C \setminus \{x_1, \ldots, x_n\}$ with an $\mathcal{O}_K$-action. In particular, $R^1 f_* \mathbb{Z}$ is a complex PVHS of rank $2h$ with an $\mathcal{O}_K$-action. If $K$ has degree $h$, then this rank $2h$ local system may not be a $\mathcal{O}_K$-local system, as its fibers may only be locally free, as opposed to free. However, after passing to a finite extension $K'$ of $K$, such as the Hilbert class field of $K$, any such rank 2 locally free module becomes a free module, and so $R^1 f_* \mathbb{Z} \otimes \mathcal{O}_K \mathcal{O}_{K'}$ is an $\mathcal{O}_{K'}$-local system of rank 2.

We now show the map $\phi$ must be constant. By construction, for any embedding $\mathcal{O}_{K'} \to \mathbb{C}$, the local system $R^1 f_* \mathbb{Z} \otimes \mathcal{O}_K \mathcal{O}_{K'} \mathbb{C}$ underlies a $\mathbb{C}$-PVHS. Hence by Theorem 1.2.5, $R^1 f_* \mathbb{Z} \otimes \mathcal{O}_K \mathcal{O}_{K'}$ has finite monodromy, as

$$2 = \text{rk} \mathcal{O}_K R^1 f_* \mathbb{Z} \otimes \mathcal{O}_K \mathcal{O}_{K'} < 2\sqrt{g+1}.$$ 

Hence, the same holds for $R^1 f_* \mathbb{Z}$. This implies the map $\phi$ is constant using the theorem of the fixed part [Del71, Corollaire 4.1.2] (see the proof of Corollary 1.2.8 for a similar argument).

□

8. Questions

We conclude with some open questions related to our results.

8.1. Bounds.

Question 8.1.1. Is the bound of $2\sqrt{g+1}$ appearing in Corollary 1.3.6, Theorem 1.2.5 and Theorem 1.2.12 sharp? If not, can one explicitly construct low rank geometric variations of Hodge structure with infinite monodromy on a general curve or general n-pointed curve? Do there exist counterexamples to the above results if one replaces $2\sqrt{g+1}$ by a linear function of $g$?

We have no reason to believe the bound is sharp. The Kodaira-Parshin trick (as used in §5, for example) is one source of variations of Hodge structure on $\mathcal{M}_{g,n}$ of rank bounded in terms of $g, n$, but it is not the only one. For example, the representations constructed in [KST16] are cohomologically rigid and hence underlie integral variations of Hodge structure by [EG18, Theorem 1.1] and [Sim92, Theorem 3]. Assuming Simpson’s motivicity conjecture ([Sim92, Conjecture, p. 9]) these constructions are geometric in nature, though this is not clear from the construction. Of course it would be extremely interesting to prove that these representations arise from algebraic geometry.
It the representations constructed in [KS16] have rank growing exponentially in $g$ [KS16, Corollary 4.5]. It is natural, given our results, to ask if one can use their methods to produce representations of smaller rank.

We also raise a related question about bounds on maps to the moduli space of curves.

**Question 8.1.2.** Fix an integer $g \geq 2$. What is the smallest integer $h \geq 2$ for which the generic genus $g$ curve, i.e., the generic fiber of $\mathcal{M}_{g,1} \to \mathcal{M}_g$, has a non-constant map to $\mathcal{M}_h$?

**Remark 8.1.3.** Since a map $C \to \mathcal{M}_h$ corresponds to a family of smooth curves of genus $h$ over $C$, by considering the associated family of Jacobians, it follows from Corollary 1.2.8 that $h \geq \sqrt{g+1}$. The Kodaira-Parshin trick [Proposition 5.1.1] does not a priori apply to construct maps from the generic curve to $\mathcal{M}_h$, because as written it produces disconnected covers. But one can apply a variant where one takes a cover defined by a characteristic quotient of the fundamental group to show there is some (fairly large) value of $h$ for which the generic genus $g$ curve has a non-constant map to $\mathcal{M}_h$. See [McM00, Theorem 1.4] for more details.

### 8.2. Non-abelian Hodge loci.

Let $(C, x_1, \ldots, x_n)$ be an $n$-pointed curve, $V$ a $\mathbb{Z}$-local system on $C \setminus \{x_1, \ldots, x_n\}$ with quasi-unipotent local monodromy around the $x_i$, and let $(E, \nabla)$ be the associated flat vector bundle. We refer to the locus $H_V$ in $\mathcal{T}_{g,n}$ where the corresponding isomonodromic deformation of $(E, \nabla)$ underlies a polarizable variation of Hodge structure as a *non-abelian Hodge locus*. By analogy to the famous result on algebraicity of Hodge loci of Cattani-Deligne-Kaplan [CDK95], it is natural to ask:

**Question 8.2.1 (Compare to [Sim97, Conjecture 12.3]).** Let $Z$ be an irreducible component of $H_V$. Is the image of $Z$ in $\mathcal{M}_{g,n}$ algebraic?

This would follow if all $\mathbb{Z}$-local systems which underlie polarizable variations of Hodge structure arise from geometry, which is perhaps a folk conjecture (and is conjectured explicitly in [Sim97, Conjecture 12.4]). Just as [CDK95] provides evidence for the Hodge conjecture, a positive answer to Question 8.2.1 would provide evidence for this conjecture.

When we refer to an analytically very general curve we mean in the sense of Definition 1.2.3. A positive answer to Question 8.2.1 would allow us to replace this with the usual algebraic notion of a very general curve in Theorem 1.2.5. It seems plausible that one can make this replacement in Corollary 1.2.7 without requiring input from Question 8.2.1 using the main result of [CDK95].
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