ON WEAK ASSOCIATED REFLEXIVITY OF WEIGHTED SOBOLEV SPACES OF THE FIRST ORDER ON REAL LINE

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Abstract. We study associate and double associate spaces of two-weighted Sobolev spaces of the first order on real half-line and we show that unlike the notion of duality the associativity is divided into two cases which we call "strong" and "weak" ones with the division of the second associativity into four cases. On the way we prove that the Sobolev space of compactly supported functions possess weak associated reflexivity and the double weak-strong associate space is vacuous. The case of power weights was recently characterized by reduction to Cesàro or Copson type spaces [18].

1. Introduction

Let $1 < p < \infty, m \in \mathbb{N}$ and let $W^{p,m}, W^{p,m}_0$ and $H^{p,m}$ be classical Sobolev spaces (see [1] Chapter 3), where $W^{p,m}_0$ and $H^{p,m}$ are completions of $C^\infty_0$ and $C^m$, respectively, with regard to the norm

$$
\|f\|_{m,p} := \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_p^p \right)^{\frac{1}{p}}.
$$

Moreover, $W^{p,m} = H^{p,m}$ [1] Theorem 3.16. If $N = \sum_{0 \leq |\alpha| \leq m} 1$ then the dual of $W^{p,m}$ is a closed subspace of vector Lebesgue space $L^{p'}_N$, where $p' = \frac{p}{p-1}$. It implies reflexivity of $W^{p,m}$ as well as $W^{p,m}_0$ on the base of general criterion of reflexivity of Banach spaces [1] Theorem 1.17] and weak compactness of a ball in $W^{p,m}$ which follows from [17] § 4, Theorem 2. General form of arbitrary linear bounded functional $L \in (W^{p,m})'$ is given by [1] Theorem 3.8 with implicit formula for the norm $\|L\|$. Alternatively, $W^{-m,p'} = (W^{p,m}_0)'$ is constructed as completion of the set of functionals $V := \{L_v; v \in L^{p'}\} \subset (W^{p,m}_0)'$, $L_v(u) := \langle u, v \rangle := \int u(x)v(x)dx$ with respect to the norm

$$
\|v\|_{-m,p'} := \sup_{0 \neq u \in W^{p,m}_0} \frac{|\langle u, v \rangle|}{\|u\|_{m,p}}.
$$

Similar results are known for the Sobolev-Orlicz spaces (see [1] and literature therein).

Generally, elements of $(W^{p,m})', (W^{p,m}_0)'$ are distributions of positive order. We learn out the case when duality is replaced by associativity and limit ourselves to the study of the two-weight Sobolev spaces of the first order on the real line. The motivation to characterize associative spaces is that it gives the principle of duality which allows to reduce a problem of the boundedness of a
linear operator, say from Sobolev space to Lebesgue space, to a more manageable problem for its conjugate operator (see, for examples [3, 12, 20]).

Now we provide basic definitions. Let \( I := (a, b) \subseteq \mathbb{R} \) be an open interval of the real axis and let \( \mathcal{M}(I) \) be the set of all Lebesgue measurable functions on \( I \). For \( 1 \leq p < \infty \) we denote \( L^p(I) \subseteq \mathcal{M}(I) \) the usual Lebesgue space with the norm \( \| f \|_{L^p(I)} := \left( \int_I |f|^p \right)^{1/p} \). Let \( \mathcal{V}_p(I) := \{ v \in L^p_{\text{loc}}(I) : v \geq 0, \| v \|_{L^1(I)} \neq 0 \} \) be the set of weight functions (weights) and let \( v_0, v_1 \in \mathcal{V}_1(I) \). Denote \( W^{1,1}_{\text{loc}}(I) \) the space of all functions \( u \in L^1_{\text{loc}}(I) \), which distributional derivatives \( Du \) belong to \( L^1_{\text{loc}}(I) \). We study the weighted Sobolev space

\[
W^1_p(I) := \{ u \in W^{1,1}_{\text{loc}}(I) : \| u \|_{W^1_p(I)} < \infty \},
\]

where

\[
\| u \|_{W^1_p(I)} := \| v_0 u \|_{L^p(I)} + \| v_1 Du \|_{L^p(I)},
\]

and the subspaces \( W^{0,0}_p(I) \subseteq W^{0,1}_p(I) \subseteq W^1_p(I) \), where the second is the closure in \( W^1_p(I) \) of a subspace \( W^{0,1}_p(I) \) of all absolutely continuous functions \( AC(I) \) of the form

\[
W^{0,1}_p(I) := \{ f \in AC(I) : f(0) = 0, \text{ supp } f \text{ compact in } I, \| f \|_{W^1_p(I)} < \infty \}.
\]

Let \( (X, \| \cdot \|_X) \) be a normed space of measurable functions on \( I \). \( X \) is called an ideal space provided it satisfies the property: if \( |f| \leq |g| \) a.e. on \( I \) and \( g \in X \), then \( f \in X \) and \( \| f \| \leq \| g \| \). Put

\[
\mathcal{D}_X := \left\{ g \in \mathcal{M}(I) : \int_I |fg| < \infty \text{ for all } f \in X \right\}.
\]

For any \( g \in \mathcal{D}_X \) we define the functionals

\[
J_X(g) := \sup_{0 \neq f \in X} \frac{\int_I |fg|}{\| f \|_X} \quad \text{and} \quad J_X(g) := \sup_{0 \neq f \in X} \frac{\int_I |fg|}{\| f \|_X}
\]

and the associated spaces

\[
X'_s := \{ g \in \mathcal{M}(I) : \| g \|_{X'_s} := J_X(g) < \infty \},
\]

\[
X'_w := \{ g \in \mathcal{M}(I) : \| g \|_{X'_w} := J_X(g) < \infty \},
\]

which we call "strong" and "weak" associated spaces, respectively. A standard problem for an ideal space \((X, \| \cdot \|_X)\) is characterization of the "strong" associated space (or the Köthe dual) (see [2, Chapter 1]). Observe that \( J_X(g) = \mathcal{J}_X(g) \) for an ideal space \( X \). For a non-ideal space \( J_X(g) \) and \( \mathcal{J}_X(g) \) might be different (see [16] for examples). In particular, any weighted Sobolev space \( X \in \{ W^{1,1}_p(I), W^{0,1}_p(I), W^1_p(I) \} \) is an example for which it might be \( J_X(g) \neq \mathcal{J}_X(g) \) [15, 16].

Let \( X \in \{ W^{0,1}_p(I), W^{0,1}_p(I), W^1_p(I) \} \). A complete characterization of the associate spaces \( X'_s \) and \( X'_w \) is obtained in [16] Sections 5, 6]. Besides, it was recently discovered that for power weight functions \( v_0 \) and \( v_1 \) the spaces \( X'_s \) and \( X'_w \) coincide with Cesàro or Copson type spaces.

It appears a natural problem to characterize "double associate" spaces of the form \([X'_s]'_w\), \([X'_s]'_w\), \([X'_w]'_s\), \([X'_w]'_s\). Complete analysis of the problem for the Sobolev spaces with power weights and the Cesàro or Copson type spaces is given in [13, 18].
The main goal of the paper is to establish “weak” associated reflexivity of the Sobolev space $X = \tilde{W}_p^1(I)$ if $1 < p < \infty$. For the reflexivity of the “strong” and “weak” weighted Cesàro and Copson type spaces see [19] and [14], respectively.

In the next section we provide technical tools to deal with weighted Sobolev spaces and their associated. In particular, we remind characterization of $X_s'$ and $X_w'$ from [15], when $X = \tilde{W}_p^1(I)$ and show that $[X_s]' = [X_w]'$ is still open. However, for the power weights all $[X_s]' = [X_w]'$ are discribed [18].

We use signs $=$ and $\equiv$ for determining new quantities. We write $A \lesssim B$, if $A \leq cB$ with some positive constant $c$, which depends only on $p$. $A \approx B$ is equivalent to $A \lesssim B \lesssim A$. Symbols $\mathbb{N}$ and $\mathbb{Z}$ are used for the sets of natural and integer numbers, respectively. Denotation $\chi_E$ means the characteristic function (indicator) of a set $E$. Uncertainties of the form $0 \cdot \infty$, $\infty$ and $\frac{0}{0}$ are taken to be zero. Symbol $\square$ stands for the end of a proof. If $1 < p < \infty$, then $p' := \frac{p}{p-1}$.

2. SOBOLEV SPACES AND THEIR ASSOCIATED

Let $1 < p < \infty$. Suppose for simplicity that $I = (0, \infty)$ and there exists $c \in (0, \infty)$ for which

$$\|v_1^{-1}\|_{L^{p'}(0,c)} \|v_0\|_{L^{p'}(0,c)} = \|v_1^{-1}\|_{L^{p'}(c,\infty)} \|v_0\|_{L^{p'}(c,\infty)} = \infty.$$  \hfill (2.1)

Then by [7] Lemma 1.6 $\tilde{W}_p^1(0,\infty) = W_p^1(0,\infty)$ and by the Oinarov–Otelbaev construction [7], [15], [16] there exist unique strictly increasing absolutely continuous functions $a(t)$ and $b(t)$ such that

$$\lim_{t \to 0} a(t) = \lim_{t \to 0} b(t) = 0, \quad \lim_{t \to \infty} a(t) = \lim_{t \to \infty} b(t) = \infty, \quad a(t) < t < b(t) \quad (t > 0),$$

$$\int_{a(t)}^{b(t)} v_1^{-p'} = \int_{t}^{b(t)} v_1^{-p'}, \quad t > 0,$$  \hfill (2.2)

(equilibrium condition) and

$$\left( \int_{a(t)}^{b(t)} v_1^{-p'} \right)^{1/p'} \left( \int_{t}^{b(t)} v_0^{-p'} \right)^{1/p} = 1, \quad t > 0.$$  \hfill (2.3)

Put

$$V_1(t) := \int_{\Delta(t)} v_1^{-p'}, \quad V_1^{\pm}(t) := \int_{\Delta^{\pm}(t)} v_1^{-p'},$$

$$\Delta(t) := (a(t), b(t)), \quad \Delta^{-}(t) := (a(t), t), \quad \Delta^{+}(t) := (t, b(t))$$

and let $a^{-1}(t)$ be the function reverse to $a(t)$. Define

$$\mathcal{G}(g) := \left( \int_{0}^{\infty} v_1^{-p'}(t) \left( \int_{t}^{a^{-1}(t)} \frac{g(x)}{V_1(x)} \int_{a(x)}^{t} v_1^{-p'} dx \right) \left( \int_{t}^{b(t)} v_1^{-p'} \right) \bigg{)}^{1/p'},$$

$$\mathcal{G}(g) := \left( \int_{0}^{\infty} V_1^{p'}(t) \left( \int_{t}^{a^{-1}(t)} \frac{g(x)}{V_1(x)} dx \right) \left( \int_{t}^{b(t)} v_1^{-p'} \right) \right)^{1/p'},$$

$$\mathcal{G}(g) := \left( \int_{0}^{\infty} \left( \int_{t}^{a^{-1}(t)} |g(x)| dx \right) \left( \int_{t}^{b(t)} v_1^{-p'} \right) \right)^{1/p'}.$$
and notate $W^1_p := W^1_p(0, \infty)$, $W^0_p := \overset{\circ}{W}^1_p(0, \infty)$, $\overset{\circ}{W}^1_p := W^0_p(0, \infty)$.

**Theorem 2.1.** [3 Theorem 3.1], [15 Theorem 4.1], [15 Theorem 4.5] Let $1 < p < \infty$ and $g \in L^1_{\text{loc}}(0, \infty)$. Suppose that $v_0, v_1 \in V_p(0, \infty)$, $\frac{1}{v_1} \in L^p_{\text{loc}}(0, \infty)$ and the condition (2.1) is satisfied. Then

$$J_{W^1_p}(g) = J_{W^0_p}(g) \approx G(g).$$

If $X = W^1_p$ or $X = \overset{\circ}{W}^1_p$, then

$$X'_s = \{g \in L^1_{\text{loc}}(0, \infty) : G(g) < \infty, \|g\|_{X'_s} \approx G(g)\}.$$

Secondly,

$$J_{\overset{\circ}{W}^1_p}(g) \approx G(g) + \mathcal{G}(g), \quad (2.4)$$

and if $X = \overset{\circ}{W}^1_p$, then

$$X'_w = \{g \in L^1_{\text{loc}}(0, \infty) : G(g) + \mathcal{G}(g) < \infty, \|g\|_{X'_w} \approx G(g) + \mathcal{G}(g)\}.$$

Also, $J_{W^1_p}(g) < \infty$ if and only if $G(g) < \infty$ and $J_{W^1_p}(g) \approx G(g) + \mathcal{G}(g)$.

**Remark 2.2.** Let $v_0 = v_1 \equiv 1$. Then we can open the right hand side of (1.1) for $W^{1,p}(0, \infty)$, using [16 Example 7.2]. Namely, we have

$$\|v\|_{-1,p'} \approx \left(\int_0^\infty \left| t^{\frac{n}{p} + \frac{1}{2}} \int_0^t v^p \, dt \right| \right)^{\frac{1}{p'}}.$$
Remark 2.4. From \([2,4]\) we obtain Hölder’s type inequality (see \([2, \text{Theorem } 2.4]\)) in \(W^1_p\) and \(W^{p',1/v_1}\): if \(1 < p < \infty\) then

\[
\left| \int_0^\infty f(t) g(t) \right| \lesssim \|f\|_{W^p_p} \|g\|_{W^{p',1/v_1}} \quad \text{for any } f \in W^1_p \text{ and } g \in W^{p',1/v_1}.
\]

The norm in \(W^{p',1/v_1}\) admits an alternative formulation in terms of a sequence \(\{\eta_k\}_{k \in \mathbb{Z}}\) of the form:

\[
\eta_0 = 1, \quad \eta_k = a^{-1}(\eta_{k-1}) \quad (k \in \mathbb{N}), \quad \eta_k = a(\eta_{k+1}) \quad (-k \in \mathbb{N}).
\]

To be able to declare it in the next lemma we denote

\[
G^{(\delta)}(t) := [V_1(t)]^{\delta} \int_t^{a^{-1}(t)} \frac{g(x)}{V_1(x)} \left( \int_{a(x)}^t v_1^{-\nu'} dx \right)^{1-\delta} dx, \quad \delta = 0, 1,
\]

and observe that for \(t \in [\eta_{k-1}, \eta_k]\)

\[
G^{(\delta)}(t) = G^{(\delta)}_{1,k}(t) + G^{(\delta)}_{2,k}(t), \quad (2.5)
\]

\[
G^{(\delta)}_{1,k}(t) := V_1^{\delta}(t) \int_{\eta_{k-1}}^{\eta_k} \frac{g(x)}{V_1(x)} \left( \int_{a(x)}^t v_1^{-\nu'} dx \right)^{1-\delta} dx,
\]

\[
G^{(\delta)}_{2,k}(t) := V_1^{\delta}(t) \int_{\eta_{k-1}}^{a^{-1}(t)} \frac{g(x)}{V_1(x)} \left( \int_{a(x)}^t v_1^{-\nu'} dx \right)^{1-\delta} dx.
\]

Lemma 2.5. Let \(1 < p < \infty\), \(v_0, v_1 \in V_p(0, \infty)\), \(\frac{1}{v_1} \in L^p_p(0, \infty)\) and the condition \((2.1)\) is satisfied. Then

\[
\|g\|_{W^{p',1/v_1}} \approx \sum_{k \in \mathbb{Z}} \left\{ \int_{\eta_{k-1}}^{\eta_k} v_1^{-\nu'}(t) \left| G^{(0)}_{1,k}(t) \right|^{p'} dt + \int_{\eta_{k-1}}^{\eta_k} v_1^{-\nu'}(t) \left| G^{(0)}_{2,k}(t) \right|^{p'} dt \right\}.
\]

Proof. The upper estimate follows from \((2.5)\) and

\[
\|g\|_{W^{p',1/v_1}} \lesssim \sum_{k \in \mathbb{Z}} \int_{\eta_{k-1}}^{\eta_k} v_1^{-\nu'}(t) \left\{ \left| G^{(0)}(t) \right|^{p'} + \left| G^{(1)}(t) \right|^{p'} \right\} dt.
\]

To establish the lower estimate we assume that the inequality

\[
\left| \int_0^\infty f(t) g(t) \right| \leq C \|f\|_{\infty_\infty} = C \left\{ \|f v_0\|_p + \|f' v_1\|_p \right\}
\]

holds with \(C = \|g\|_{W^{p',1/v_1}}\), and let for some \(N \in \mathbb{N}\)

\[
F_{1,N}(x) := \sum_{|k| \leq N} x_{\eta_{k-1}, \eta_k}(x) \int_{\eta_{k-1}}^{x} v_1^{-\nu'}(t) \left[ \text{sgn} \ G^{(0)}_{1,k}(t) \right] \left( \int_{a(x)}^t v_1^{-\nu'} \right)^{1-\delta} dx \left[ V_1(t) \right]^{\delta} \left| G^{(0)}_{1,k}(t) \right|^{p'-1} dt,
\]

\[
F_{2,N}(x) := \sum_{|k| \leq N} x_{\eta_{k}, \eta_{k+1}}(x) \int_{a(x)}^{\eta_k} v_1^{-\nu'}(t) \left[ \text{sgn} \ G^{(0)}_{2,k}(t) \right] \left( \int_{a(x)}^t v_1^{-\nu'} \right)^{1-\delta} dx \left[ V_1(t) \right]^{\delta} \left| G^{(0)}_{2,k}(t) \right|^{p'-1} dt.
\]

If \(f = F_{1,N} + F_{2,N}\) then

\[
\int_0^\infty g(x) f(x) dx = \sum_{|k| \leq N} \left\{ \int_{\eta_{k-1}}^{\eta_k} v_1^{-\nu'}(t) \left| G^{(0)}_{1,k}(t) \right|^{p'} dt + \int_{\eta_{k-1}}^{\eta_k} v_1^{-\nu'}(t) \left| G^{(0)}_{2,k}(t) \right|^{p'} dt \right\}. \quad (2.7)
\]
To evaluate
\[
\|F_{1,N}^{(\delta)} v_0\|_p = \sum_{|k| \leq N} \int_{a_k}^{b_k} v_0^p(x) \left| \frac{1}{V_1^{-1}(x)} \int_{b_k}^{t} v_1^{-p'}(t) \left[ \text{sgn} \ G_{1,k}^{(\delta)}(t) \right] \right|^p \, dx \times \left( \int_{a(x)}^{t} v_1^{-p'} \right)^{1-\delta} \left( V_1(t) \right)^{\delta} \left| G_{1,k}^{(\delta)}(t) \right|^{p'-1} \, dt \right|^p \, dx
\]
we apply well known characterization of weighted Hardy’s inequality [2, p. 6], in order to obtain
\[
\int_{a_k}^{b_k} v_0^p(x) \left( \int_{b_k}^{t} v_1^{-p'}(t) \left| G_{1,k}^{(\delta)}(t) \right|^{p'-1} \, dt \right)^p \, dx \leq A_1^p \int_{a_k}^{b_k} v_1^{-p'} |G_{1,k}^{(\delta)}|^{p'} dx,
\]
where (see (2.3))
\[
A_1 := \sup_{\eta_{k-1} < t < \eta_k} \left( \int_{\eta_{k-1}}^{\eta_k} v_0^p \right)^{1/p} \left( \int_{a(x)}^{t} v_1^{-p'} \right)^{1/p} \leq \left( \int_{\eta_{k-1}}^{\eta_k} v_0^p \right)^{1/p} \left( \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'} \right)^{1/p} \leq 1.
\]
Therefore, by using in the \( \delta = 1 \) - case the relation
\[
V_1(t) = 2V_1^+ (t) \leq 2 \int_{a_k}^{b_k} v_1^{-p'} \leq 2 \int_{b_k}^{b(x)} v_1^{-p'} \leq 2V_1(x) = 4V_1^- (x), \quad \eta_{k-1} < t \leq x, \quad (2.8)
\]
we have for the both \( \delta = 0, 1 \):
\[
\|F_{1,N}^{(\delta)} v_0\|_p \leq \sum_{|k| \leq N} \int_{a_k}^{b_k} v_0^p(x) \left( \int_{b_k}^{t} v_1^{-p'}(t) \left| G_{1,k}^{(\delta)}(t) \right|^{p'-1} \, dt \right)^p \, dx \leq \sum_{|k| \leq N} \int_{a_k}^{b_k} v_1^{-p'} \left| G_{1,k}^{(\delta)} \right|^{p'} =: \left[ G_{1,N}^{(\delta)}(g) \right]^{p'}.
\]
(2.9)
Analogously, we evaluate, by making use of
\[
V_1(t) = 2V_1^+ (t) \leq 2 \int_{a_k}^{b_k} v_1^{-p'} \leq 2V_1(x) = 4V_1^- (x), \quad \eta_k \leq x \leq \eta_{k+1}, \quad (2.10)
\]
that
\[
\|F_{2,N}^{(\delta)} v_0\|_p = \sum_{|k| \leq N} \int_{a_k}^{b_k} v_0^p(x) \left| \frac{1}{V_1^{-1}(x)} \int_{b_k}^{t} v_1^{-p'}(t) \left[ \text{sgn} \ G_{2,k}^{(\delta)}(t) \right] \right|^p \, dx \times \left( \int_{a(x)}^{t} v_1^{-p'} \right)^{1-\delta} \left( V_1(t) \right)^{\delta} \left| G_{2,k}^{(\delta)}(t) \right|^{p'-1} \, dt \right|^p \, dx \leq \sum_{|k| \leq N} \int_{a_k}^{b_k} v_0^p(x) \left( \int_{a(x)}^{t} v_1^{-p'}(t) \left| G_{2,k}^{(\delta)}(t) \right|^{p'-1} \, dt \right)^p \, dx \leq A_2^p \int_{a_k}^{b_k} v_1^{-p'} |G_{1,k}^{(\delta)}|^{p'} dx,
\]
where (see (2.3))
\[
A_2 := \sup_{\eta_{k-1} < t < \eta_k} \left( \int_{\eta_{k-1}}^{\eta_k} v_0^p \right)^{1/p} \left( \int_{t}^{\eta_k} v_1^{-p'} \right)^{1/p} \leq 1.
\]
Therefore,
\[
\|F_{2,N}^{(\delta)} v_0\|_p \leq \sum_{|k| \leq N} \int_{a_k}^{b_k} v_1^{-p'} \left| G_{2,k}^{(\delta)} \right|^{p'} =: \left[ G_{2,N}^{(\delta)}(g) \right]^{p'}.
\] (2.11)
Further, since

\[
[F_{1,N}(x)]' = -\sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k)}(x) \left[ \frac{V_1^{-}(x)}{V_1^{+}(x)} \right]^{1-\delta} \int_{\eta_{k-1}}^{x} v_1^{-p'}(t) |\text{sgn} \ G_{1,k}^{(\delta)}(t)| dt + \sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k)}(x)
\]

\[
\times \left\{ \begin{array}{ll}
\int_{a(x)}^{t} v_1^{-p'}(x) \frac{\text{sgn} \ G_{1,k}^{(0)}(x) |G_{1,k}^{(0)}(x)|^{p'-1}}{V_1^{-}(x)} dt + \sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k)}(x) \\
\int_{a(x)}^{t} v_1^{-p'}(x) \frac{\text{sgn} \ G_{1,k}^{(0)}(x) |G_{1,k}^{(0)}(x)|^{p'-1}}{V_1^{-}(x)} dt + \sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k)}(x)
\end{array} \right.
\]

\[
[F_{2,N}(x)]' = -\sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k]}(x) \left[ \frac{V_1^{-}(x)}{V_1^{+}(x)} \right]^{1-\delta} \int_{a(x)}^{t} v_1^{-p'}(t) |\text{sgn} \ G_{2,k}^{(\delta)}(x) | dt - \sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k]}(x)
\]

\[
\times \left\{ \begin{array}{ll}
\int_{a(x)}^{t} v_1^{-p'}(x) \frac{\text{sgn} \ G_{2,k}^{(0)}(x) |G_{2,k}^{(0)}(x)|^{p'-1}}{V_1^{-}(x)} \int_{a(x)}^{t} v_1^{-p'}(x) \frac{\text{sgn} \ G_{2,k}^{(0)}(x) |G_{2,k}^{(0)}(x)|^{p'-1}}{V_1^{-}(x)} dx + \sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k]}(x) \\
\int_{a(x)}^{t} v_1^{-p'}(x) \frac{\text{sgn} \ G_{2,k}^{(0)}(x) |G_{2,k}^{(0)}(x)|^{p'-1}}{V_1^{-}(x)} \int_{a(x)}^{t} v_1^{-p'}(x) \frac{\text{sgn} \ G_{2,k}^{(0)}(x) |G_{2,k}^{(0)}(x)|^{p'-1}}{V_1^{-}(x)} dx + \sum_{|k| \leq N} \chi_{[\eta_{k-1}, \eta_k]}(x)
\end{array} \right.
\]

then

\[
\| [F_{1,N}]' v_1 \|_p \leq \left\{ \begin{array}{ll}
I_1 + [G_{1,N}^{(0)}(g)]^{p-1} + II_1, & \delta = 0, \\
I_1 + [G_{1,N}^{(1)}(g)]^{p-1}, & \delta = 1,
\end{array} \right.
\]

where

\[
P_1^p := \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^p(x) \left[ \frac{V_1^{-}(x)}{V_1^{+}(x)} \right]^{1-\delta} \left( \int_{\eta_{k-1}}^{x} v_1^{-p}(t) |V_1^{(\delta)}(t)|^{p-1} dt \right)^p dx
\]

and

\[
II_1^p := \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^p(x) \left[ \frac{V_1^{-}(x)}{V_1^{+}(x)} \right]^{1-\delta} \left( \int_{\eta_{k-1}}^{x} v_1^{-p}(t) |G_{1,k}^{(\delta)}(t)|^{p-1} dt \right)^p dx.
\]

In view of \( v_1^{-p'}(a(x)) a'(x) \leq 2 v_1^{-p'}(x) \) (see (3.7)), we obtain, by using (2.8) in the \( \delta = 1 \) case, that

\[
P_1^p \leq \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^p(x) \left[ \frac{v_1^{-p'}(x) - v_1^{-p'}(a(x)) a'(x)}{|V_1^{+}(x)|^p} \right] \left( \int_{\eta_{k-1}}^{x} v_1^{-p'} |G_{1,k}^{(\delta)}|^{p-1} dx \right)^p dx
\]

\[
\leq \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^p(x) \left[ \frac{v_1^{-p'}(x) + v_1^{-p'}(a(x)) a'(x)}{|V_1^{+}(x)|^p} \right] \left( \int_{\eta_{k-1}}^{x} v_1^{-p'} |G_{1,k}^{(\delta)}|^{p-1} dx \right)^p dx
\]

\[
\leq 3^p \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left[ \frac{V_1^{-}(x)}{V_1^{+}(x)} \right]^{1-\delta} \left( \int_{\eta_{k-1}}^{x} v_1^{-p'} |G_{1,k}^{(\delta)}|^{p-1} dx \right)^p dx.
\]
Analogously,
\[ II_2^p \leq 2^p \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left[ V_1^{-}(x) \right]^{-p} \left( \int_{\eta_{k-1}}^{x} v_1^{-p'}(x) |G^{(0)}_{1,k}(t)|^{p'-1} \right)^p \, dx. \]

On the strength of the boundedness characteristics for the Hardy operator \[^{[5, \text{p. 6}}],
\[ \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left[ V_1^{-}(x) \right]^{-p} \left( \int_{\eta_{k-1}}^{x} v_1^{-p'}(t) |G^{(\delta)}_{1,k}(t)|^{p'-1} \, dt \right)^p \, dx \leq A_1^p \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) |G^{(\delta)}_{1,k}|^{p'} \],
where
\[ A_1 := \sup_{\eta_{k-1} < t < \eta_k} \left( \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left[ V_1^{-}(x) \right]^{-p} \, dx \right)^{1/p} \left( \int_{\eta_{k-1}}^{t} v_1^{-p'} \right)^{1/p'}. \]

It holds
\[ A_1^p \leq \sup_{\eta_{k-1} < t < \eta_k} \left( \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left( \int_{\eta_{k-1}}^{x} v_1^{-p'} \right)^{p-1} \, dx \right)^{1/p} \left( \int_{\eta_{k-1}}^{t} v_1^{-p'} \right)^{1/p'}, \]
\[ \frac{1}{p-1} \sup_{\eta_{k-1} < t < \eta_k} \left[ \left( \int_{\eta_{k-1}}^{t} v_1^{-p'} \right)^{1-p} - \left( \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'} \right)^{1-p} \right] \left( \int_{\eta_{k-1}}^{t} v_1^{-p'} \right)^{p-1} \leq \frac{1}{p-1}. \]

Therefore,
\[ \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left( \int_{\eta_{k-1}}^{x} v_1^{-p'}(t) |G^{(\delta)}_{1,k}(t)|^{p'-1} \, dt \right)^p \, dx \leq \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) |G^{(\delta)}_{1,k}|^{p'} = \left[ G^{(\delta)}_{1,N}(g) \right]^{p'}, \]
that is
\[ \| [F_{1,N}^{(\delta)}]^p v_1 \|_p \lesssim \left[ G^{(\delta)}_{1,N}(g) \right]^{p'-1}, \]
and, by letting \( N \to \infty \), the estimate
\[ \| g \|_{W_{p',1}^{p',1}/v_1} \geq \sum_{k \in \mathbb{Z}} \left\{ \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(t) |G^{(0)}_{1,k}(t)|^{p'} \, dt + \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(t) |G^{(1)}_{1,k}(t)|^{p'} \, dt \right\} \quad (2.12) \]
is now performed, basing on (2.9) and (2.7).

Similarly, in view of \( V_1^{-}(x) = \frac{1}{t} V_1(x) \geq \frac{1}{t} V_1^{-}(a(x)) = \frac{1}{t} V_1(a(x)) \) (for \( \delta = 1 \)),
\[ \| [F_{2,N}^{(\delta)}]^p v_1 \|_p \leq \begin{cases} I_2 + II_2, & \delta = 0, \\ I_2 + [G^{(2)}_{1,N}(g)]^{p'-1}, & \delta = 1, \end{cases} \]
where
\[ I_2^p := \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left[ \frac{[V_1^{-}(x)]^p}{[V_1^{-}(x)]^p} \right] \left( \int_{a(x)}^{\eta_k} v_1^{-p'}(t) \left( \int_{a(x)}^{t} v_1^{-p'} \right)^{1-\delta} \left[ V_1(t) \right]^\delta |G^{(\delta)}_{2,k}(t)|^{p'-1} \, dt \right)^p \, dx \]
and
\[ II_2^p := \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left[ \frac{[V_1^{-}(x)]^p}{[V_1^{-}(x)]^p} \right] \left( \int_{a(x)}^{\eta_k} v_1^{-p'}(a(x)) a'(x) \right)^p \left( \int_{a(x)}^{\eta_k} v_1^{-p'} |G^{(0)}_{2,k}|^{p'-1} \, dx \right)^p \, dx, \]
we obtain analogously to the previous case (see also (2.11) for \( \delta = 1 \)):
\[ I_2^p \leq \sum_{|k| \leq N} \int_{\eta_{k-1}}^{\eta_k} v_1^{-p'}(x) \left| v_1^{-p'}(x) - v_1^{-p'}(a(x)) a'(x) \right|^p \left( \int_{a(x)}^{\eta_k} v_1^{-p'} |G^{(\delta)}_{2,k}|^{p'-1} \right)^p \, dx \]
Lemma 2.6. Let by letting

By characteristics for the Hardy inequality [5, p. 6],

Basing on Lemma 2.5 one can prove the following

where

We have

Therefore,

which, in combination with (2.11) and (2.7), yields the estimate

by letting \( N \to \infty \). Thus, (see also (2.12)) the required lower bound is now confirmed. \( \square \)

Basing on Lemma 2.5 one can prove the following

Lemma 2.6. Let \( 1 < p < \infty \), \( v_0, v_1 \in \mathcal{V}_p(0, \infty) \), \( \frac{1}{v_1} \in L^p_{\text{loc}}(0, \infty) \) and the condition (2.1) is satisfied. Then the space \( \mathcal{W}_{p',1/v_1} \) is dense in \( \mathcal{W}_{p',1/v_1} \).

Proof. Let \( g \in \mathcal{W}_{p',1/v_1} \). Then \( \|g\|_{\mathcal{W}_{p',1/v_1}} < \infty \) by (2.6). Therefore,

(2.13)

Let \( g_N := \chi_{[\eta_{-N}, \eta_N]} g \) with some \( N \in \mathbb{N} \). Then \( g_N \in \mathcal{W}_{p',1/v_1} \). Indeed,

\[
G(|g_N|)^{p'} = \left\{ \int_{0}^{\eta_{-N}} + \int_{\eta_{-N}}^{\eta_N} + \int_{\xi_N}^{\infty} \right\} v^{-p'}_1(x) \left( \int_{a(x)}^{b(x)} |g_N| \right)^{p'} dx,
\]
where
\[
\int_0^{\eta_{N-1}} v_1^{-p'}(x) \left( \int x^{-1}(x) |X_{\eta_{N-1}}| g(x) \right)^{p'} \, dx = 0 = \int_{\eta_N}^{\infty} v_1^{-p'}(x) \left( \int x^{-1}(x) |X_{\eta_{N-1}}| g(x) \right)^{p'} \, dx.
\]

The assertion follows from the fact that
\[
\int_{\eta_{N-1}}^{\eta_N} v_1^{-p'}(x) \left( \int x^{-1}(x) |X_{\eta_{N-1}}| g(x) \right)^{p'} \, dx \leq \int_{\eta_{N-1}}^{\eta_{N+1}} v_1^{-p'} \left( \int |X_{\eta_{N-1}}| g(x) \right)^{p'} \, dx < \infty.
\]

Denote \( G_{i,k}(t) := H_{i,k}^{(\delta)}(g(t), \eta_{N-1}) \). We can write
\[
\|g - g_N\|_{W_{p',1/v_1}}^p = \sum_{i=1,2} \sum_{\delta=1,2} \sum_{k \in \mathbb{Z}} \int_{\eta_{k-1}}^{\eta_N} v_1^{-p'}(t) \left| H_{i,k}^{(\delta)}(t) - H_{i,k}^{(\delta)}(g(t)) \right|^{p'} \, dt.
\]

This approves the statement of Lemma in view of (2.13). \( \square \)

Now we can make an addition to the last assertion of Theorem 2.1.

**Remark 2.7.** Let \( X = W_{p}^1 \). Then, by Theorem 2.1
\[
X'_{w} = \left\{ g \in W_{p',1/v_1} : \|g\|_{X'_{w}} \approx \|g\|_{W_{p',1/v_1}} < \infty \right\}.
\]

It follows from here that \( X'_{w} \subseteq W_{p',1/v_1} \), and the inclusion can be strict, since there are examples of \( g_0 \in W_{p',1/v_1} \) when \( g_0 \notin W_{p',1/v_1} \) (see [15] Remark 5.5).

Indeed, if \( g_0 \in X'_{w} \) then, by [15] Theorem 2.5,
\[
\|g_0\|_{X'_{w}} = J_X(g_0) < \infty \iff \infty > J_X(g_0) = \|g_0\|_{W_{p',1/v_1}} = \infty,
\]
which is a contradiction.

Let
\[
X'_{\text{ext}} := \left\{ g \in W_{p',1/v_1} : \text{there exists} \{g_k\} \subseteq X'_{w} \text{ such that} \right. \lim_{k \to \infty} \|g - g_k\|_{W_{p',1/v_1}} = 0 \text{ and } \left. \|g\|_{X'_{\text{ext}}} := \lim_{k \to \infty} \|g_k\|_{X'_{w}} \right\}.
\]

Notice that the definition of \( X'_{\text{ext}} \) is independent of a choice of \( \{g_k\} \). Then
\[
X'_{\text{ext}} \hookrightarrow W_{p',1/v_1} \text{ and } \|g\|_{W_{p',1/v_1}} \leq \|g\|_{X'_{\text{ext}}}.
\]
Conversely, let \( g \in \mathcal{W}_{p',1/v_1} \). Then, by Lemma 2.6 there exists \( \{g_k\} \subset \mathcal{W}_{p',1/v_1} \subset X'_w \) such that \( \|g\|_{\mathcal{W}_{p',1/v_1}} = \lim_{k \to \infty} \|g_k\|_{\mathcal{W}_{p',1/v_1}} = \|g\|_{X'_w} \). Hence, \( g \in X'_w \) and we have \( \mathcal{W}_{p',1/v_1} \subset X'_w \) and \( \|g\|_{X'_w} = \|g\|_{\mathcal{W}_{p',1/v_1}} \). Thus,

\[ X'_w = \mathcal{W}_{p',1/v_1} \]

with equality of the norms.

The next technical statement is used in Corollary 2.9 to prove \([X'_w]' = \{0\}\).

**Lemma 2.8.** Let \( 1 < p < \infty \), \([c,d] \subset (0, \infty)\) and \( h \in L^1([c,d])\). Then for any \( \varepsilon > 0 \) there exists \( g \in \mathcal{W}_{p',1/v_1} \) such that \( |g| = |h| \) on \([c,d]\) and \( \|g\|_{\mathcal{W}_{p',1/v_1}} < \varepsilon \).

**Proof.** Firstly, we show that for \( g \) with \( \text{supp} \subset [c,d] \) it holds

\[
\|g\|_{\mathcal{W}_{p',1/v_1}}^p \lesssim \left[ V_1(c) \right]^{p+1} \int_c^d \frac{g}{V_1} \left| v_{1}^{-p'}(t) V_1' \right| \int_t^d \frac{g}{V_1} \left| \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \right| \, dx \, dt.
\]  

(2.14)

We start from the functional \( \mathcal{G}(g) \), for which it holds, by the triangle inequality, that

\[
\mathcal{G}(g \chi_{[c,d]}) \leq \left( \int_a^c \frac{g}{V_1} \left| v_{1}^{-p'}(t) V_1' \right| \int_t^d \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \, dx \, dt \right)^{1/p'} 
\]

\[
+ \left( \int_a^c \frac{g}{V_1} \left| v_{1}^{-p'}(t) V_1' \right| \int_t^d \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \, dx \, dt \right)^{1/p'}.
\]

Since for any \( \alpha > 0 \)

\[
\int_a^t v_{1}^{-p'} \left[ V_1^+ \right]^{\alpha} \leq \int_a^t v_{1}^{-p'} \left[ \int_a^t v_{1}^{-p'} \right]^{\alpha} \, dx \leq \left[ V_1(t) \right]^{\alpha+1},
\]  

(2.15)

we have

\[
\int_a^c v_{1}^{-p'}(t) V_1' \left| \int_t^d \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \, dx \right|^{p'} \, dt 
\]

\[
= \int_a^c v_{1}^{-p'}(t) V_1' \left| \int_c^{d} \frac{g(x)}{V_1(x)} \, dx \right|^{p'} \, dt + \int_a^c v_{1}^{-p'}(t) V_1' \left| \int_t^d \frac{g(x)}{V_1(x)} \, dx \right|^{p'} \, dt 
\]

\[
\leq \left[ V_1(c) \right]^{p+1} \left[ \int_c^d \frac{g}{V_1} \right]^{p'} + \int_c^d v_{1}^{-p'}(t) V_1' \left| \int_t^d \frac{g(x)}{V_1(x)} \, dx \right|^{p'} \, dt.
\]  

(2.16)

By the substitution \( y = a^{-1}(t) \) and in view of (3.7) and \( V_1^+(a(y)) \leq V_1(y) \),

\[
\int_a^d v_{1}^{-p'}(t) V_1' \left| \int_a^{-1}(t) \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \, dx \right|^{p'} \, dt = \int_a^d v_{1}^{-p'}(t) V_1' \left| \int_a^{-1}(t) \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \, dx \right|^{p'} \, dt 
\]

\[
\leq \int_c^d v_{1}^{-p'}(a(y)) V_1' \left| \chi_{[c,d]}(y) a(y) \right| \int_y^d \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \, dx \, dy \leq \int_c^d v_{1}^{-p'}(a(y)) V_1' \left| \chi_{[c,d]}(y) \right| \int_y^d \frac{g(x)}{V_1(x)} \, dx \, dy 
\]

\[
\leq \int_c^d v_{1}^{-p'}(y) V_1' \left| \int_y^d \frac{\chi_{[c,d]}(x) g(x)}{V_1(x)} \, dx \right|^{p'} \, dy = \int_c^d v_{1}^{-p'}(y) V_1' \left| \int_y^d \frac{g(x)}{V_1(x)} \, dx \right|^{p'} \, dy.
\]
Let \( \{ a_i \}_{i=0}^{n-1} \) be a partition of \([c, d]\) such that \( \int_{a_i}^{a_{i+1}} |h| = n^{-1} \int_{c}^{d} |h| \) and suppose \( \beta_i \in [a_i, a_{i+1}] \) are such that \( \int_{a_i}^{\beta_i} |h| = \int_{\beta_i}^{a_{i+1}} |h|, \ i \in \{0, \ldots, n-1\} \). Put

\[
g := \chi_{[c,d]} - \sum_{i=0}^{n-1} \chi_{[\beta_i, \alpha_{i+1}]}.
\]
Then \( \tilde{g} \in \mathcal{W}_{p',1/v_1} \), \( |\tilde{g}| = |h| \) on \([c, d]\), \( f_{\alpha_i}^{\alpha_i+1} \frac{\tilde{g}}{V_1} = 0 \) for \( i = 0, \ldots, n - 1 \) and (see \( \text{(2.14)} \))

\[
||\tilde{g}||^p_{\mathcal{W}_{p',1/v_1}} \leq \int_c^d v_1^{-p'}(x) V_1^{p'}(x) \left| \int_x^d \frac{\tilde{g}}{V_1} \right|^{p'} dx = \sum_{i=0}^{n-1} \int_{\alpha_i}^{\alpha_i+1} v_1^{-p'}(x) V_1^{p'}(x) \left| \int_x^{\alpha_i} \frac{\tilde{g}}{V_1} \right|^{p'} dx \\
= \sum_{i=0}^{n-1} \int_{\alpha_i}^{\alpha_i+1} v_1^{-p'}(x) V_1^{p'}(x) \left[ \int_x^{\alpha_i} \frac{\tilde{g}}{V_1} \right]^{p'} dx \leq \sum_{i=0}^{n-1} (\int_{\alpha_i}^{\alpha_i+1} |h| )^{p'} \int_{\alpha_i}^{\alpha_i+1} v_1^{-p'} V_1^{p'} \\
= n^{-p'} \left( \int_c^d |h| \right) \sum_{i=0}^{n-1} \int_{\alpha_i}^{\alpha_i+1} v_1^{-p'} V_1^{p'} = n^{-p'} \left( \int_c^d |h| \right) \int_c^d v_1^{-p'} V_1^{p'} < \varepsilon^{p'}. 
\]

\( \square \)

**Corollary 2.9.** Let \( f \in \mathcal{M}(0, \infty) \). If \( \text{meas} \{ x \in (0, \infty) : f(x) \neq 0 \} > 0 \) then \( J_{\mathcal{W}_{p',1/v_1}}(f) = \infty \).

**Proof.** Let \( f \neq 0 \). There is a segment \([c, d] \subset (0, \infty) \) such that \( c < d \) and \( \text{meas} \{ (c, d) \cap \{ x \in (0, \infty) : f(x) \neq 0 \} \} > 0. \) Fix an arbitrary \( \varepsilon > 0 \). By Lemma \( 2.8 \) there exists \( \tilde{g} \in \mathcal{W}_{p',1/v_1} \) with \( \text{supp} \tilde{g} \subset [c, d] \) such that \( ||\tilde{g}||_{\mathcal{W}_{p',1/v_1}} < \varepsilon \) and \( |\tilde{g}| = 1 \) on \((c, d)\). Then

\[
J_{\mathcal{W}_{p',1/v_1}}(f) \geq \int_c^d \frac{|f \tilde{g}|}{||\tilde{g}||_{\mathcal{W}_{p',1/v_1}}} \geq \varepsilon^{-1} \int_c^d |f|. 
\]

\( \square \)

3. Main result

We start with auxiliary assertions needed to prove the main result.

**Lemma 3.1.** Let \( 1 < p < \infty, v_0, v_1 \in \mathcal{V}_p(0, \infty), \frac{1}{v_1} \in L^p_{\text{loc}}(0, \infty) \), and the condition \( \text{(2.1)} \) is satisfied. Then

\[
L^p_{v_1/v_0}(0, \infty) \subset \mathcal{W}_{p',1/v_1} \quad (3.1)
\]

and

\[
||g||_{\mathcal{W}_{p',1/v_1}} \lesssim ||g||_{p',1/v_0} \quad (3.2)
\]

for any \( g \in L^p_{1/v_0}(0, \infty) \).

**Proof.** On the strength of

\[
V_1^+(t) \leq \int_t^{b(x)} v_1^{-p'} \leq V_1(x) = 2V_1^-(x), \quad t \leq a^{-1}(t) \quad (3.3)
\]

it holds

\[
||g||_{\mathcal{W}_{p',1/v_1}} \lesssim \left( \int_0^\infty v_1^{-p'}(t) \left( \int_t^{a^{-1}(t)} |g(x)| dx \right)^{p'} dt \right)^{1/p'}. 
\]

Then \( (3.2) \) will follow from

\[
\left( \int_0^\infty v_1^{-p'}(t) \left( \int_t^{a^{-1}(t)} |g(x)| dx \right)^{p'} dt \right)^{1/p'} \leq C ||g||_{p',1/v_0}. \quad (3.4)
\]
Consider the dual to (3.3) inequality
\[
\left( \int_0^\infty v_0^p(y) \left( \int_{a(y)}^{b(y)} |f| \right)^p dy \right)^{1/p} \leq C \| f \|_{p,v_1},
\]
which is a consequence of
\[
\left( \int_0^\infty v_0^p(y) \left( \int_{a(y)}^{b(y)} |f| \right)^p dy \right)^{1/p} \leq C \| f \|_{p,v_1}.
\]

It is known [15, Theorem 3.1] that
\[ C_1 \approx \mathcal{A} := \sup_t \left( \int_{a(t)}^{b(t)} v_1^{-p'} \right)^{1/p'} \left( \int_t^{a^{-1}(t)} v_0^p \right)^{1/p} \leq V_1(a^{-1}(t))^{1/p'} V_0(a^{-1}(t))^{1/p} = 1. \]

Put
\[ V_0(t) := \int_{a(t)}^{b(t)} v_0^p, \quad V_0^+(t) := \int_0^{\Delta^+(t)} v_0^p. \]

We have by (3.3)
\[ V_1^+(t) \leq V_1(a^{-1}(t)), \quad \int_t^{a^{-1}(t)} v_0^p \leq \int_t^{b(a^{-1}(t))} v_0^p = V_0^+(a^{-1}(t)). \]

Therefore, by (2.3),
\[ \mathcal{A}_a(t) := \left( \int_{a(t)}^{b(t)} v_1^{-p'} \right)^{1/p'} \left( \int_t^{a^{-1}(t)} v_0^p \right)^{1/p} \leq V_1(a^{-1}(t))^{1/p'} V_0(a^{-1}(t))^{1/p} = 1. \]

Analogously,
\[ \mathcal{A}_b(t) := \left( \int_{a(t)}^{b(t)} v_1^{-p'} \right)^{1/p'} \left( \int_t^{b^{-1}(t)} v_0^p \right)^{1/p} \leq V_1(b^{-1}(t))^{1/p'} V_0(b^{-1}(t))^{1/p} = 1. \]

Thus,
\[ \mathcal{A} \approx \sup_{t>0} \left[ \mathcal{A}_a(t) + \mathcal{A}_b(t) \right] \lesssim 1 \]
and (3.2) follows. \[\square\]

**Corollary 3.2.** Let \(1 < p < \infty\) and \(f \in \mathcal{D}_{W,1/v_1}^p\) (see (1.2)). Under the conditions of Lemma 3.1 the embedding (3.1) entails \(f \in L_{p,v_0}^p(0, \infty)\) and
\[ \infty > J_{W,1/v_1}^p(f) \gtrsim \| f \|_{p,v_0}. \] \(\text{(3.5)}\)

**Proof.** By Lemma 3.1
\[
J_{W,1/v_1}^p(f) = \sup_{0 \neq g \in \mathcal{D}_{W,1/v_1}^p} \frac{\left| \int_0^\infty g f \right|}{\| g \|_{W,1/v_1}^p} \gtrsim \sup_{0 \neq g \in L_{p,v_1}^p(0, \infty)} \frac{\left| \int_0^\infty g f \right|}{\| g \|_{p,v_1}^p} = \| f \|_{p,v_0}.
\]

\[\square\]

**Lemma 3.3.** Let \(1 < p < \infty\). Under the conditions of Lemma 3.1 if \(J_{W,1/v_1}^p(f) < \infty\) then \(f = \tilde{f}\) a.e., where \(\tilde{f} \in AC_{\text{loc}}(0, \infty)\) and
\[ \infty > J_{W,1/v_1}^p(f) \gtrsim \| \tilde{f} \|_{p,v_1}. \] \(\text{(3.6)}\)
Proof. Let 
\[ g_\phi(x) := \frac{d\phi}{dx}, \quad \phi \in C_0^\infty(0, \infty). \]
Show that \( g_\phi \in \mathcal{W}_{p',1/v_1} \). It is sufficiently to prove the inequalities \( \mathcal{G}(g_\phi) \lesssim \|\phi\|_{p',1/v_1} \) and \( \mathcal{G}(g_\phi) \lesssim \|\phi\|_{p',1/v_1} \). By taking into account the equality
\[
v_{1}^{-p'}(a(x))a'(x) + v_{1}^{-p'}(b(x))b'(x) = 2v_{1}^{-p'}(x), \tag{3.7}
\]
which follows from the equilibrium condition (2.2), we write
\[
\int_t^{a^{-1}(t)} \frac{g_\phi(x)}{V_1(x)} \left( \int_a(x) v_{1}^{-p'} \right) dx = -\phi(t) + \int_t^{a^{-1}(t)} \phi(x) \left( \frac{v_{1}^{-p'}(a(x))a'(x)}{V_1^{-1}(x)} \right. \\
+ \frac{v_{1}^{-p'}(x) f_a(x) v_{1}^{-p'}}{[V_1^{-1}(x)]^2} \left. - \frac{v_{1}^{-p'}(a(x))a'(x) f_a(x) v_{1}^{-p'}}{[V_1^{-1}(x)]^2} \right) dx \leq |\phi(t)| + 5 \int_t^{a^{-1}(t)} \frac{v_{1}^{-p'}(x) |\phi(x)|}{V_1^{-1}(x)} dx. \tag{3.8}
\]
Thus,
\[
\mathcal{G}(g_\phi) \lesssim \|\phi\|_{p',1/v_1} + \left( \int_0^\infty v_{1}^{-p'}(t) \left( \int_t^{a^{-1}(t)} \frac{v_{1}^{-p'}(x) |\phi(x)|}{V_1^{-1}(x)} dx \right)^{p'} dt \right)^{1/p'}.
\]
Put \( h = v_{1}^{-1} |\phi| \) and consider dual to
\[
\left( \int_0^\infty v_{1}^{-p'}(t) \left( \int_t^{a^{-1}(t)} \frac{v_{1}^{-p'}(x) h(x)}{V_1^{-1}(x)} dx \right)^{p'} dt \right)^{1/p'} \leq C\|h\|_{p'} \tag{3.9}
\]
inquiry
\[
\left( \int_0^\infty \frac{v_{1}^{-p'}(x)}{[V_1^{-1}(x)]^p} \left( \int_a(x) v_{1}^{-1}(t)|\psi(t)| dt \right)^{p} dx \right)^{1/p} \leq C\|\psi\|_{p},
\]
which follows from
\[
\left( \int_0^\infty \frac{v_{1}^{-p'}(x)}{[V_1^{-1}(x)]^p} \left( \int_a(x) v_{1}^{-1}(t)|\psi(t)| dt \right)^{p} dx \right)^{1/p} \leq C_2\|\psi\|_{p}.
\]
By the criteria for the boundedness of Hardy–Steklov operators [21 Theorem 1],
\[
C_2 \approx \mathcal{A} := \sup_t \left( \int_a(x) v_{1}^{-p'} \right)^{1/p'} \left( \int_{b^{-1}(t)}^{a^{-1}(t)} \frac{v_{1}^{-p'}}{V_1^{-1}(x)} \right)^{1/p}.
\]
Since \( \int_b^{-1}(t) v_{1}^{-p'} [V_1^{-1}]^{-p} \lesssim V_1^{-1-p}(t) \) (see [16 (5.18)]) we have \( \mathcal{A} \lesssim 1 \). This yields \( \mathcal{G}(g_\phi) \lesssim \|\phi\|_{p',1/v_1} < \infty \).

Similarly,
\[
\int_t^{a^{-1}(t)} \frac{g_\phi(x)}{V_1(x)} dx = \frac{\phi(a^{-1}(t))}{V_1^{-1}(a^{-1}(t))} - \frac{\phi(t)}{V_1^{-1}(t)} + \int_t^{a^{-1}(t)} \phi(x) v_{1}^{-p'}(x) v_{1}^{-p'}(a(x))a'(x) dx \\
\quad \leq \frac{|\phi(a^{-1}(t))|}{V_1^{-1}(a^{-1}(t))} + \frac{\phi(t)}{V_1^{-1}(t)} + \int_t^{a^{-1}(t)} |\phi(x)| v_{1}^{-p'}(x) v_{1}^{-p'}(a(x))a'(x) dx.
\]
Since \( 2v_{1}^{-p'}(a^{-1}(t)) v_{1}^{-p'}(a^{-1}(t))' \geq v_{1}^{-p'}(t) \) (see [3.7] with \( x = a^{-1}(t) \)) and \( V_1(a^{-1}(t)) \geq V_1^{-1}(t) = \frac{1}{2} V_1^{-1}(t) \), we have
\[
\int_0^\infty v_{1}^{-p'}(t) V_1^{-1}(t) \left[ \frac{|\phi(a^{-1}(t))|}{V_1^{-1}(a^{-1}(t))} + \frac{\phi(t)}{V_1^{-1}(t)} \right]^{p'} dt
\]
\[ \lesssim \int_0^\infty [a^{-1}(t)]' |\phi(a^{-1}(t))v_1^{-1}(a^{-1}(t))|^{p'} dt + \int_0^\infty |\phi(t)v_1^{-1}(t)|^{p'} dt \simeq \|\phi\|_{p',v_1}. \]

Further,
\[ \int_0^\infty v_1^{-p'}(t) V_t^{p'}(t) \left( \int_t^{a^{-1}(t)} |\phi(x)| \frac{v_1^{-p'}(x) + v_1^{-p'}(a(x))a'(x)}{V_1^{-1}(x)} dx \right)^{p'} dt \]
\[ \leq \int_0^\infty v_1^{-p'}(t) \left( \int_t^{a^{-1}(t)} |\phi(x)| \frac{v_1^{-p'}(x) + v_1^{-p'}(a(x))a'(x)}{V_1^{-1}(x)} dx \right)^{p'} dt \]
\[ \leq 3 \int_0^\infty v_1^{-p'}(t) \left( \int_t^{a^{-1}(t)} |\phi(x)| \frac{v_1^{-p'}(x)h(x)}{V_1^{-1}(x)} dx \right)^{p'} dt \simeq \int_0^\infty v_1^{-p'}(t) \left( \int_t^{a^{-1}(t)} \frac{v_1^{-p'}(x)h(x)}{V_1^{-1}(x)} dx \right)^{p'} dt \]
(see (3.9)). Therefore, \( \mathcal{G}(g_\phi) \lesssim \|\phi\|_{p',1/v_1} < \infty \) and
\[ \|g_\phi\|_{W_{p',1/v_1}} \lesssim \|\phi\|_{p',1/v_1}. \] (3.10)

It follows from (3.10) that
\[ \sup_{0 \neq \phi \in C_0^\infty(0,\infty)} \left| \int_0^\infty f \phi' \right| \lesssim \sup_{0 \neq \phi \in C_0^\infty(0,\infty)} \left| \int_0^\infty fg_\phi \right| \]
\[ \leq \sup_{g \in W_{p',1/v_1}} \left| \int_0^\infty fg \right| = J_{W_{p',1/v_1}}(f) < \infty. \] (3.11)

Put \( \Lambda \phi := \int_0^\infty f \phi', \phi \in C_0^\infty(0,\infty) \). On the strength of (3.11), \( |\Lambda \phi| \lesssim \|\phi\|_{p',1/v_1} \). By the Hahn–Banach theorem there exists an extension \( \tilde{\Lambda} \in (L_{p',1/v_1}(0,\infty))^* \) of \( \Lambda \). By the Riesz representation theorem, there exists \( u \in L^p_{v_1}(0,\infty) \) such that \( \tilde{\Lambda} h = -\int_0^\infty uh \), \( h \in L_{p',1/v_1}(0,\infty) \). It implies
\[ -\int_0^\infty u \phi = \int_0^\infty f \phi', \quad \phi \in C_0^\infty(0,\infty), \] (3.12)
this means that \( u \) is a distributional derivative of \( f \). Then by [3] Theorem 7.13 the function \( f \) a.e. coincides with a function \( \tilde{f} \in AC_{loc}(0,\infty) \) and \( u = \tilde{f}' \). It follows from (3.12) that
\[ J_{W_{p',1/v_1}}(f) \geq \sup_{0 \neq \phi \in C_0^\infty(0,\infty)} \left| \int_0^\infty fg_\phi \right| \geq \sup_{0 \neq \phi \in C_0^\infty(0,\infty)} \left| \int_0^\infty \tilde{f}' \phi \right| = \|\tilde{f}'\|_{p,v_1}. \]

Our main result of the paper reads the following

**Theorem 3.4.** Let \( 1 < p < \infty \) and \( f \in \mathcal{D}_{W_{p',1/v_1}} \). Then \( J_{W_{p',1/v_1}}(f) < \infty \) if and only if \( f = \tilde{f} \) a.e., \( \tilde{f} \in \tilde{W}_p^1 \) and \( \|f\|_{W_{p'}^1} \approx J_{W_{p',1/v_1}}(f) \). Thus,
\[ W_p^{\infty} = [W_{p',1/v_1}]_w = [W_{p'}^{\infty}]_w. \]

**Proof.** The **sufficient** part of Theorem follows from Remark 2.4.

**Necessity.** It is already proved that \( \tilde{f} \in W_p^1 \) (see (3.5), (3.6)). Let \( E := \{x \in (0,\infty) : |\tilde{f}(x)| > 0\} \). Since \( |\tilde{f}| \) is continuous on \( (0,\infty) \), then \( E \) is open set. Suppose that \( \text{mes}(b,\infty) \cap E) > 0 \) for any \( b \in (0,\infty) \). Then there exists a sequence of segments \( \{[a_k,b_k]\}_k \subset (0,\infty) \) such that \( b_k < a_{k+1} and
\( m_k := \min_{x \in [a_k, b_k]} |\tilde{f}(x)| > 0 \). Put \( \theta_k := \frac{1}{km_k(b_k - a_k)} \). By Lemma 2.8 there is \( g_k \in W'_{p', 1/v_1} \) such that 
\[ \|g_k\|_{W'_{p', 1/v_1}} < 2^{-k} \text{ and } |g_k| = \theta_k \text{ on } (a_k, b_k). \]
Put \( g := \sum_{k=1}^{\infty} g_k \). Then \( \|g\|_{W'_{p', 1/v_1}} \leq 1 \) and 
\[ \int_0^{\infty} |\tilde{f}g| \geq \sum_{k=1}^{\infty} \theta_k m_k(b_k - a_k) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty, \]
which contradicts with \( J_{W'_{p', 1/v_1}}(f) < \infty \).

Similarly we show that \( \tilde{f}(0) = 0 \). Thus, \( \text{supp } \tilde{f} \subset [0, \infty) \) is compact.

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