Maximal Functions for Lacunary Dilation Structures

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Abstract

If $\mu$ is a smooth density on a hypersurface in $\mathbb{R}^n$ whose curvature never vanishes to infinite order, and $A \in GL_d(\mathbb{R})$ is a matrix whose eigenvalues all have absolute value greater than 1, let $\mu_k$ be the dilate of $\mu$ by $A^k$. We prove that $Tf = \sup_k f \ast \mu_k$ is bounded from a corresponding version of $H^1$ to weak $L^1$.

Consider $A \in GL_d(\mathbb{R})$ whose eigenvalues all have absolute value greater than 1; this defines a dilation structure on $\mathbb{R}^d$. Let $\mu$ be a smooth density supported on a compact hypersurface $M \subset \mathbb{R}^d$ whose Gaussian curvature never vanishes to infinite order (by scaling, we may assume that this hypersurface is contained in the unit ball), and define the measures $\mu_k$ by

$$\langle \mu_k, f \rangle = \langle \mu, f \circ A^k \rangle.$$ (0.1)

We will consider the maximal function

$$Mf(x) = \sup_{k \in \mathbb{Z}} |\mu_k \ast f(x)|.$$ (0.2)

Littlewood-Paley theory quickly shows that $M$ is bounded on $L^p$ for $1 < p < \infty$, but the endpoint behavior has been a topic of some interest. Extrapolation arguments show that $M : L \log L \to L^{1,\infty}$, but go no further. In a seminal paper, Christ [1] gave an endpoint result in two special cases: if $\mu$ is the surface measure on the sphere $S^{d-1} \subset \mathbb{R}^d$ and $A = 2I$, then $M$ is bounded from the real Hardy space $H^1(\mathbb{R}^d)$ to $L^{1,\infty}$; and if $M$ is the parabola $\{(t, t^2) : 1 \leq t \leq 2\} \subset \mathbb{R}^2$ and $A(x, y) = (2x, 4y)$, then $M$ is bounded from the parabolic Hardy space $H^1_p(\mathbb{R}^d)$ (defined in terms of the parabolic dilation structure) to $L^{1,\infty}$.

It is relatively simple to extend the first of these results; if $A = 2I$ and the Gaussian curvature of $M$ never vanishes, then $M : H^1 \to L^{1,\infty}$. As for the second, Seeger, Tao and Wright [4] obtained, for nearly our general hypotheses, a different endpoint space: namely that $M : L \log \log L \to L^{1,\infty}$. The construction localizes the measure in order to handle vanishing curvature of $M$. 

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Heo [3] found room in the original construction of Christ to localize the measure $\mu$, showing for the usual isotropic dilation structure $A = 2I$ that $\mathcal{M}$ is bounded from the Hardy space $H^1(\mathbb{R}^d)$ to weak $L^1$ (so long as the Gaussian curvature of $\mu$ never vanishes to infinite order); this was further generalized by Seeger and Wright [5]. The author has further extended this result with the full strength of the stopping-time argument, in order to obtain the following result:

**Theorem 0.1.** Let $M$, $\mu$, and $A$ be as above, and let $H^1_A(\mathbb{R}^d)$ be the Hardy space defined in terms of the dilation matrix $A$. Then $\mathcal{M}$ is bounded from $H^1_A(\mathbb{R}^d)$ to weak $L^1(\mathbb{R}^d)$, so long as the Gaussian curvature of $\mu$ never vanishes to infinite order; this was further generalized by Seeger and Wright [5]. The author has further extended this result with the full strength of the stopping-time argument, in order to obtain the following result:

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By the properties of that decomposition, there are constants $c$ and $C$ such that for any dyadic ‘cube’ there exists an integer $k$ such that the ‘cube’ contains a translate of $cA^kB_1(0)$ and is contained in a translate of $CA^kB_1(0)$. Thus we shall equivalently define this Hardy space in terms of a fixed grid which is invariant under $A$.

**Definition** For each $k \in \mathbb{Z}$ and $\vec{n} \in \mathbb{Z}^d$, let $Q^k_{\vec{n}} = A^k([0, 1]^d + \vec{n})$, and call $a_Q$ an atom if $a_Q$ is supported on such a set $Q$ and $\|a_Q\|_\infty \leq |Q|^{-1} = |\det A|^{-k}$; then $H^1_A$ is the completion of the finite sums $f = \sum_Q \lambda_Q a_Q$ in the norm $\|f\|_{H^1_A} = \sum_Q |\lambda_Q|$.

(Technically, as in [1] we use two such grids, the second translated in space, to decompose a finite sum of the “original” atoms into finite sums with comparable norms in the grids.)

We will need to keep track of two facts about our dilation structure: the volume and the diameter of our cubes.

**Definition** Let $a := |\det A|$; then the volume of the set $A^\tau([0, 1]^d)$ is equal to $a^\tau$ for each $\tau \in \mathbb{Z}$.

**Definition** Let $r$ denote the minimum of the absolute values of the eigenvalues of $A$, and let $n$ denote the size of the largest block in the Jordan decomposition of $A$ whose eigenvalue has absolute value $r$. Then for $\tau \leq 0$, the diameter of the set $A^\tau([0, 1]^d)$ is comparable to $r^\tau |\tau|^n$.

## 1 Stopping Time Construction

Here we will prove a more general version of the stopping-time lemma used in [1] to construct the exceptional set for the decomposition.

Now if $D_\sigma$ is the dyadic grid of sidelength $2^\sigma$, we would like to define the grid $R_{\sigma, \tau}$ to be the image of $D_\sigma$ under $A^\tau$. (Actually, we will replace $A$ with a constant integer multiple
such that \( A^n \) maps \( B(0,1) \) into \( B(0,\frac{1}{2}) \). This requires us to divide our maximal function \( M \) into \( n \) pieces, which of course does not affect our result.) We further let \( \mathcal{R} \) denote the union of all \( \mathcal{R}_{\sigma,\tau} \), and \( \mathcal{R}_0 \) denote the strictly anisotropically dilated cubes \( \bigcup_\tau \mathcal{R}_{0,\tau} \) (the supports of our atoms \( a_Q \)).

Note that we have defined \( \sigma \) differently from [1] in this more general context. Rather than designating the length of the side parallel to the \( x \)-axis, we have let it denote the number of isotropic dilations that separate \( q \) from an element of \( \mathcal{R}_0 \). In what follows, we will only need to consider \( \sigma \leq 0 \).

As in [1], we will build an exceptional set out of tendrils as well as other components. The first step is the standard anisotropic Whitney decomposition:

**Lemma 1.1.** For any \( \alpha > 0 \) and any collection of cubes \( Q \) belonging to \( \mathcal{R}_0 \), with associated positive scalars \( \lambda_Q \), there exists a collection of pairwise disjoint cubes \( S \subset \mathcal{R}_0 \) such that

1. \( \sum_{Q \subset S^*} \lambda_Q \leq C\alpha |S| \) for all \( S \in \mathcal{R}_0 \)
2. \( \sum |S| \leq \alpha^{-1} \sum \lambda_Q \)
3. \( \left\| \sum_{Q \not\subset S^* \gamma S} \lambda_Q \frac{\chi_Q}{|Q|} \right\|_{\infty} \leq \alpha. \)

The proof of this lemma is mostly standard, though we must use \( S^* \) in places because we cannot count on the ‘children’ of \( S \) to be nested in \( S \). We can safely discard the \( Q \) not belonging to any of these \( S^* \), by condition 3, although it is not true that each remaining \( Q \) belongs to \( S^* \) for a unique \( S \in \mathcal{S} \); each \( S \) whose double contains \( Q \) will have identical dimensions. (Technically speaking, this is a consequence of the standard proof of this theorem.)

We define the tendril \( T(q) \) of any \( q \in \mathcal{R}_{\sigma,\tau} \) by

\[
T(q) := q^{**} + \bigcup_{k \leq \tau(q)+2} \text{supp } \mu_k
\]

and note that \( \chi_{q^*} * \mu_j \) is supported in \( T(q) \). (Here \( q^* \) is the expansion of \( q \) by a factor of 2, and \( q^{**} \) is the expansion of \( q \) by a factor of 4.) Note as well that \( T(q) \) is the image of \( q' + \bigcup_{k \leq 2} \text{supp } \mu_k \) under \( A^\tau \) for some \( q' \in \mathcal{R}_{\sigma,0} \), and that therefore \( |T(q)| \lesssim 2^\tau |A|^\tau \).

Now we will generalize the stopping-time lemma from [1]:

**Lemma 1.2.** We are given \( \alpha > 0 \), a finite collection \( \mathcal{S} \) of pairwise disjoint \( S \in \mathcal{R}_0 \), and a finite collection \( \mathcal{Q} \) of \( Q \in \mathcal{R}_0 \), such that each \( Q \in \mathcal{Q} \) is contained in \( S^* \) for some \( S \in \mathcal{S} \). Corresponding to each \( Q \in \mathcal{Q} \) we are also given \( \lambda_Q > 0 \). Then there exist a measurable \( E \subset \mathbb{R}^3 \) and a function \( \kappa : \mathcal{Q} \to \mathbb{Z} \) such that
i) \(|E| \leq C\alpha^{-1} \sum \lambda_Q + C \sum |S|

ii) \(\chi_Q \ast \mu_j\) is supported in \(E\) for all \(Q \in \mathcal{Q}\) and all \(j < \kappa(Q)\)

iii) If \(Q \subset S^*\), then \(\kappa(Q) > \tau(S)\)

iv) For any \(\tau \in \mathbb{Z}\) and \(\sigma \leq 0\) and \(q \in \mathcal{R}_{\sigma, \tau}\),
\[
\sum_{Q \subset q^* \setminus \kappa(Q) \leq \tau} \lambda_Q \leq C\alpha|T(q)|. \tag{1.2}
\]

Proof. The proof proceeds via a double induction on \(\sigma\) and \(\tau\). We will partition \(\mathcal{Q}\) into two subcollections \(\mathcal{C}_1\) and \(\mathcal{C}_2\). To each \(Q \in \mathcal{C}_1\) we will associate a \(q \in \mathcal{R}\) with \(Q \subset q^*\), and we will (initially) set \(\kappa(Q) = \tau(q) + 1\). Each \(Q \in \mathcal{C}_2\) will have \(\kappa(Q) = \tau(S) + 1\), where \(S^*\) contains \(Q\). (As mentioned above, this uniquely determines \(\tau(S)\).)

Select \(\tau_0\) larger than \(\tau(Q)\) for all \(Q \in \mathcal{Q}\), such that \(\alpha|A|^\tau > \sum_{Q \in \mathcal{Q}} \lambda_Q\). Initialize \(\tau = \tau_0 - 1\) and \(\sigma = 0\), and define \(Q(0, \tau_0) = \mathcal{Q}\); we will define collections \(Q(\sigma, \tau)\) by removing elements at each step so that \(Q(\sigma, \tau) \subset Q(\sigma + 1, \tau) \subset Q(\sigma', \tau + 1)\) for any \(\tau \leq \tau_0\) and any \(\sigma\) and \(\sigma' \leq 0\).

Furthermore, we will define \(\Lambda_{\sigma, \tau}(q) = \sum_{Q \subset q^* \setminus \kappa(Q) \leq \tau} \lambda_Q\) for each \(q \in \mathcal{R}_{\sigma, \tau}\).

For each \(\tau\) in descending order from \(\tau_0\), proceed by descent on \(\sigma\); for each fixed \(\sigma\), select all \(q \in \mathcal{R}_{\sigma, \tau}\) such that \(\Lambda_{\sigma, \tau}(q) > \alpha 2^\sigma|A|^\tau\). Any \(Q\) contained in \(q^*\) for a selected \(q\) is classified into \(\mathcal{C}_1\) and assigned to one of the selected \(q\); \(\kappa(Q)\) is defined to be \(\tau(q) + 1\). (Again, the assignment of \(q\) will not be unique, but the assignment of \(\kappa(Q)\) is.) Then \(Q(\sigma - 1, \tau)\) is defined to consist of all \(Q \in \mathcal{Q}(\sigma, \tau)\) which were not classified at this step.

Eventually, \(-\sigma\) is so large that no \(Q \in \mathcal{Q}\) can be contained in \(q^*\) for any \(q \in \mathcal{R}_{\sigma, \tau}\). When this happens, all unassigned \(Q\) with the dimensions \(\tau(Q) = \tau\) are classified into \(\mathcal{C}_2\), assigned to some \(S\) with \(Q \subset S^*\), and given \(\kappa(Q) = \tau(S) + 1\). Then \(\tau\) is incremented down by 1, \(Q(0, \tau - 1)\) is defined to consist of the remaining unclassified \(Q\) (i.e. all \(Q \in \mathcal{Q}(-\infty, \tau)\) such that \(\tau(Q) < \tau\)), and we start descending in \(\sigma\) again. This process repeats until we reach \(\tau\) smaller than \(\tau(Q)\) for all \(Q\), at which point all of \(\mathcal{Q}\) has been classified.

Throughout this process, we have ensured the usual stopping-time condition \(\Lambda_{\sigma, \tau}(q) \leq C\alpha 2^\sigma|A|^\tau \leq C\alpha|T(q)|\) for all \(q \in \mathcal{R}\), since otherwise a parent of \(q\) would have been chosen instead. (This is true as well if \(\sigma = 0\), since in that case we may consider the anisotropic parents of \(q\).) Of course, the left-hand side of (1.2) is precisely \(\Lambda_{\sigma, \tau}(q)\), so condition (iv) is verified.

Now we define \(E_2\) to be the union of all the tendrils \(T(q)\) for all \(q\) selected in the process and \(E_1\) to be the union of all the quadruples \(S^{**}\) of \(S \in \mathcal{S}\). We already know that \(|E_2|\)
is appropriately bounded in size. For $|E_1|$, each $Q$ is assigned to one of at most $3^d$ cubes $q$. Therefore, summing over the selected $q$,

$$ |E_1| \leq \sum_q |T(q)| \lesssim \sum_q \alpha^{-1} \Lambda_{\sigma(q),\tau(q)}(q) \lesssim \alpha^{-1} \sum_{Q \in \mathcal{C}_1} \lambda_Q. $$

Therefore, with $E = E_1 \cup E_2$, we have satisfied condition (i). Condition (ii) is clearly true for $Q \in \mathcal{C}_1$, and since $\mu$ is supported in the unit ball, it is trivial to show for $Q \in \mathcal{C}_2$ as well. Unfortunately, condition (iii) need not hold; however, we can repair this by replacing the current $\kappa$ with $\max(\kappa(Q), \tau(S) + 1)$. This does not affect (i); (ii) is still true because $\kappa(Q)$ is either unchanged or replaced with $\tau(S) + 1$, which we noted is fine; it makes (iii) trivially true; and it preserves (iv) because fewer $Q$ will now be summed over on the left. Thus we are done.

### 2 Proof of Main Theorem

We will use the following lemma from Seeger, Tao and Wright ([4], Lemma 2.5):

**Lemma 2.1.** Let $\psi \in C^\infty([-1, 1]^{d-1})$ be real-valued, with $\sup_{|\alpha| \leq 3} |\partial^\alpha \psi(x)| \leq A \leq 1$ on $[-1, 1]^{d-1}$. Suppose $|\det \psi'(y_0)| \geq \beta$, and $Q \subset [-1, 1]^{d-1}$ is a cube of side length $\epsilon_1 \beta$ containing $y_0$, where $\epsilon_1 \leq [10(d-1)4A]^{-1}$. Let $\chi \in C^\infty(Q)$ with $\|\partial^\alpha \chi\|_{\infty} \leq c_\alpha (\epsilon_1 \beta)^{-|\alpha|}$. Define the measure $\nu$ on $\mathbb{R}^d$ by

$$ \langle \nu, f \rangle = \int \chi(y)f(y, \psi(y))dy $$

and define its reflection by $\langle \tilde{\nu}, f \rangle = \langle \nu, f(-\cdot) \rangle$. Then there are constants $C_{\alpha}$ so that

$$ |\partial_x^\alpha [\nu * \tilde{\nu}] (x)| \leq C_{\alpha} \beta^{d-3-2|\alpha|} |x|^{-1-|\alpha|}. \quad (2.1) $$

As in [1], we will use this regularity of the kernel convolved with its reflection to obtain an especially strong $L^2$ bound on a part of our operator. In [3], this idea was combined with that of partitioning $\mu$ into pieces of small support, and setting aside those which are “bad” in a certain sense; so long as we can bound the contribution of these “bad” pieces in $L^1$, we may assume quantitative conditions on the remaining pieces.

For each $s$, we use a smooth partition of unity to write $\mu = \sum_{\rho \in I_s} \mu_\rho^s$, where $\mu_\rho^s$ is supported on a ball $B_\rho^s$ of diameter $2^{-\epsilon s}$, and $|I_s| \lesssim 2^{(d-1)\epsilon s}$. (We take $0 < \epsilon \ll \log(r)$, which ensures that $B_\rho^s$ will have large diameter compared to the atoms it will be convolved with.)

There are two types of pieces that we’d like to exclude. As in [3], we will set aside those pieces on which the curvature falls below $2^{-\epsilon s}$ (since the size of the curvature is used in Lemma 2.1). We will also need a certain “transversality” condition; the intersection of $Q \in \mathcal{R}_{0, \tau}$ with a typical piece of $M$ should have measure comparable to the volume of $Q$ divided by the diameter of $Q$, and we will use this fact.
Definition Let

\[ I_1^s := \{ \rho : \min_{x \in \text{supp} (\mu^s_{\rho})} |K(x)| < 2^{-\epsilon s} \}, \tag{2.2} \]

where \( K \) is the Gaussian curvature of the manifold, and for \( 0 < \zeta \ll \epsilon \) let

\[ I_2^s := \{ \rho : \exists Q \in \mathcal{R}_0 \text{ such that } \mu^s_{\rho}(Q) > 2^{\zeta s}|Q|\text{diam}(Q)^{-1} \}. \tag{2.3} \]

Then we claim that the contribution of these sets in \( L^1 \) will be summable:

**Lemma 2.2.** There exists \( \eta > 0 \) such that \( |I_1^s \cup I_2^s| \lesssim 2^\left( (d-1)\epsilon - \eta \right)s \).

**Proof.** The bound on \( |I_1^s| \) follows from the nonvanishing curvature of \( M \), as shown in [3].

For the bound on \( |I_2^s| \), we first identify the direction of slowest contraction under the dilation group, noting that Jordan blocks can induce a logarithmic factor. Consider the real Jordan form of \( A \); among the blocks whose eigenvalues have norm \( r \), choose the one whose block size \( n \) is maximal. (A \( 2n \times 2n \) complex Jordan block counts as size \( n \) here.)

In the corresponding eigenspace, there exists a unit vector \( \vec{v} \) and a subspace \( W \) (of dimension 1 or 2, depending on whether it is a real or complex eigenvalue) such that \( A^\tau \vec{v} \approx r^\tau |\tau|^n \) and the distance from \( r^{-\tau} |\tau|^{-n} A^\tau \vec{v} \) to \( W \) tends to 0 as \( \tau \to -\infty \).

Now if \( \vec{N}_x \) is the normal vector to \( M \) at \( x \), and \( |\langle \vec{N}_x, \vec{v} \rangle| \geq c > 0 \) for all \( x \in M \cap B_\rho \), then \( \mu^s_{\rho}(M \cap Q) \lesssim c^{1-d}|\det A|^n|A^\tau \vec{v}|^{-1} \). The assumption on the curvature ensures that the estimate follows.

Now let \( f \in H^1(\mathbb{R}^d) \) be a finite sum \( f = \sum Q \lambda_Q a_Q \), where the \( Q \) are all elements of \( \mathcal{R}_0 \), with \( \lambda_Q > 0, \|a_Q\|_{\infty} \leq |Q|^{-1} \), and \( \int a_Q = 0 \) for all \( Q \). We want to show

\[ |\{ x : Mf(x) > 2\alpha \}| \lesssim \alpha^{-1} \sum_Q \lambda_Q. \]

We apply Lemma [11] to obtain the collection \( \mathcal{S} \subset \mathcal{R}_0 \), and note that if \( g \) denotes the sum of all \( \lambda_Q a_Q \) for \( Q \) not contained in any \( S^* \), then by [3], \( \|g\|_{\infty} \leq \alpha \) and thus \( \|Mg\|_{\infty} \leq \alpha \). Thus we may assume that all of the \( Q \) are contained in \( S^* \) for some \( S \in \mathcal{S} \).

We now apply Lemma [12] to our collection \( Q \) of such cubes and our collection \( \mathcal{S} \), obtaining the exceptional set \( E \) and the function \( \kappa : Q \to \mathbb{Z} \). Since \( |E| \lesssim \alpha^{-1} \sum_Q \lambda_Q \), it suffices to prove that

\[ |\{ x \notin E : \sup_k |\mu_k * (\sum_Q \lambda_Q a_Q)(x)| > \alpha \}| \lesssim \alpha^{-1} \sum_Q \lambda_Q. \]
By (3), we see that
\[
|\{ x \notin E : \sup \mu_k * (\sum_Q \lambda_Q a_Q)(x) > \alpha \} | \leq |\{ x : \sup_j \mu_j * (\sum_{Q: \kappa(Q) \leq j} \lambda_Q a_Q)(x) > \alpha \} |
\]
\[
\leq |\{ x : \sup_j \sum_{s=0}^{\infty} \mu_j * (\sum_{Q: \kappa(Q) = j-s} \lambda_Q a_Q)(x) > \alpha \} |.
\]

We now partition \( \mu \) as discussed above, depending on \( s \); write
\[
\mathcal{M}' f(x) := \sup_j \sum_{s=0}^{\infty} \sum_{\rho \in I_1 \cup I_2} \mu_{\rho,j}^s * (\sum_{Q: \kappa(Q) = j-s} \lambda_Q a_Q)(x),
\]
\[
\mathcal{M}'' f(x) := \sup_j \sum_{s=0}^{\infty} \sum_{\rho \notin I_1 \cup I_2} \mu_{\rho,j}^s * (\sum_{Q: \kappa(Q) = j-s} \lambda_Q a_Q)(x).
\]

Now by Lemma 2.2, we see that
\[
|\{ x : \mathcal{M}' f(x) > \alpha/2 \} | \leq \frac{2}{\alpha} \| \mathcal{M}' f \|_{L^1} \leq \frac{2}{\alpha} \sum_{j} \sum_{s=0}^{\infty} \left\| \sum_{\rho \in I_1 \cup I_2} \mu_{\rho,j}^s \right\|_1 \left\| \sum_{Q: \kappa(Q) = j-s} \lambda_Q a_Q \right\|_1
\]
\[
\lesssim \alpha^{-1} \sum_{s=0}^{\infty} 2^{-\eta s} \sum_j \sum_{\kappa(Q) = j-s} \lambda_Q \lesssim \alpha^{-1} \sum_Q \lambda_Q.
\]

Thus we only need concern ourselves with \( \mathcal{M}'' \). By Chebyshev’s inequality, it will suffice to prove that \( \| \mathcal{M}'' f \|_2 \lesssim \alpha \sum_Q \lambda_Q \). Note that
\[
|\mathcal{M}'' f(x)|^2 \leq \left( \sup_j \left\| \sum_{s=0}^{\infty} \sum_{\rho \notin I_1 \cup I_2} \left( \sum_{Q: \kappa(Q) = j-s} \lambda_Q a_Q \right) * \mu_{\rho,j}^s(x) \right\|_2 \right)^2
\]
\[
\leq \sum_j \sum_{s=0}^{\infty} \sum_{\rho \notin I_1 \cup I_2} \left( \sum_{Q: \kappa(Q) = j-s} \lambda_Q a_Q \right) * \mu_{\rho,j}^s(x)
\]
and by Minkowski’s inequality,
\[
|\mathcal{M}'' f(x)| \leq \sum_{s=0}^{\infty} \sum_{\rho \notin I_1 \cup I_2} \left( \sum_j \left| \left( \sum_{Q: \kappa(Q) = j-s} \lambda_Q a_Q \right) * \mu_{\rho,j}^s(x) \right|^2 \right)^{1/2}.\]
Therefore it suffices to prove that there exists $C < \infty$ and $\delta > 0$ such that for any $s \geq 0$, $\rho \in I_s \setminus (I^1_s \cup I^2_s)$, and $j \in \mathbb{Z}$,

$$\left\| \left( \sum_{\kappa(Q) = j - s} \lambda_Q a_Q \ast \mu^s_{\rho,j} \right) \right\|_{L^2}^2 \leq C 2^{-(2\zeta(d - 1) + \delta) s} \sum_{\kappa(Q) = j - s} \lambda_Q. \quad (2.4)$$

Finally, by scaling, we may assume $j = 0$.

### 2.1 Proof for $s = 0$

In order to handle our anisotropic dilation structure, as in [1] we must consider elements of $\mathcal{R}_0$ as subsets of isotropic dyadic cubes, since the bounds we will obtain (by convolving the measure $\mu$ with its reflection) depend on the Euclidean distance.

For $s = 0$, we will consider each $Q \in \mathcal{R}_{0,\tau}$ with $\kappa(Q) = 0$ as a subset of an isotropic cube $q \in \mathcal{R}_{\sigma,0}$ such that $Q \subset q^*$ and $\text{diam}(Q) \approx r^\tau |\tau|^n \approx 2^\sigma$. Define

$$A_q(x) = \sum_{Q \subset q^*: \kappa(Q) = 0, \text{diam}(Q) \approx 2^\sigma} \lambda_Q a_Q(x) \quad \text{and} \quad \lambda_q = \sum_{Q \subset q^*: \kappa(Q) = 0, \text{diam}(Q) \approx 2^\sigma} \lambda_Q.$$

(To avoid double-counting, we actually modify this a bit: we partition the cubes $Q \in \mathcal{R}_{0,\tau}$ among the $q$ of the appropriate size.)

**Lemma 2.3.** For any $q \in \mathcal{R}_{\sigma,0}$, $\|A_q \ast \mu^s_{\rho,0}\|_{\infty} \leq C 2^{-\sigma + \zeta s} \lambda_q$, and $\|A_q \ast \mu^s_{\rho,0}\|_1 \leq C 2^{(\zeta + (1 - d) s)} \lambda_q$.

**Proof.** Since $\|a_Q\|_{\infty} \leq |Q|^{-1}$, the definition of $I^2_s$ implies that for any $Q$ contained in $q$,

$$\|a_Q \ast \mu^s_{\rho,0}\|_{\infty} \lesssim |Q|^{-1} \mu^s_{\rho,0}(Q) \leq 2^{\zeta \text{diam}(Q)^{-1}} = 2^{-\sigma + \zeta s},$$

which implies the first inequality. The second follows from the fact that $A_q \ast \mu^s_{\rho,0}$ is supported on a set of measure $\leq 2^{\sigma + \epsilon(1 - d) s}$.

This immediately implies a first estimate for any $q$ and $q'$ with $\sigma(q') \geq \sigma(q)$,

$$\left| \langle A_q \ast \mu^0_{\rho,0}, A_{q'} \ast \mu^0_{\rho,0} \rangle \right| \leq C 2^{-\sigma'} \lambda_q \lambda_{q'}. \quad (2.5)$$
We will obtain a second estimate on this quantity (which is stronger when \( q \) and \( q' \) are distant but weaker when they are near) using our assumption on the curvature of \( M \cap B_\rho \) (from the definition of \( I_1^2 \)) and Lemma 2.1.

**Lemma 2.4.** If \( q \) and \( q' \) are separated by a Euclidean distance of \( d(q, q') \geq 2^\sigma', \) then

\[
|\langle A_q \ast \mu_{\rho,0}^s, A_{q'} \ast \mu_{\rho,0}^s \rangle| \leq C 2^{\sigma' + \epsilon s(5-d)} d(q, q')^{-2} \lambda_q \lambda_{q'}.
\]  

**Proof.** By translation, we may assume that \( q' \) is centered at 0.

\[
\langle A_q \ast \mu_{\rho,0}^s, A_{q'} \ast \mu_{\rho,0}^s \rangle = \langle A_q, A_{q'} \ast \mu_{\rho,0}^s \rangle.
\]

Let \( \varphi(x) = \mu_{\rho,0}^s \ast \tilde{\mu}_{\rho,0}^s(x); \) by Lemma 2.1 we see that \( |\varphi(x)| \lesssim 2^{\epsilon s(3-d)} |x|^{-1} \) and \( |\nabla \varphi(x)| \lesssim 2^{\epsilon s(5-d)} |x|^{-2} \). Now for any \( x \in q, \)

\[
A_{q'} \ast \varphi(x) = \sum_{Q \text{ assigned to } q'} \lambda_Q \int a_Q(y) \varphi(x - y) \, dy
\]

= \[
\sum_{Q \text{ assigned to } q'} \lambda_Q \int a_Q(y) [\varphi(x - y) - \varphi(x)] \, dy
\]

by the cancellation of \( a_Q, \) and therefore

\[
|A_{q'} \ast \varphi(x)| \lesssim \sum_{Q \text{ assigned to } q'} \lambda_Q \left( \operatorname{diam}(Q)^{-1} \sup_{x \in q} |\nabla \varphi(x)| \right) \|a_Q\|_1 \lesssim \lambda_q 2^{\sigma' + \epsilon s(5-d)} d(q, q')^{-2}
\]

which implies the result. \( \square \)

Now

\[
\left\| \left( \sum_{\kappa(Q)=0} \lambda_Q a_Q \right) \ast \mu_{\rho,0}^0 \right\|_2^2 = \left\| \sum_q A_q \ast \mu_{\rho,0}^0 \right\|_2^2
\]

\[
\leq 2 \sum_{\sigma \leq \sigma' \leq 0} \sum_{q \in \mathcal{R}_{\sigma,0}} \sum_{q' \in \mathcal{R}_{\sigma',0}} \left| \langle A_q \ast \mu_{\rho,0}^0, A_{q'} \ast \mu_{\rho,0}^0 \rangle \right|
\]

\[
= \left( 2 \sum_{q'} \sum_{q \in \mathcal{R}^{**}} \right) + \left( 2 \sum_{q'} \sum_{q' \in \mathcal{R}^{**}} \right) = I + II.
\]

For the sum \( I, \) we use the trivial bound (2.5) and property (iv) of Lemma 1.2 (note that \( |T(q')| \approx 2^{\sigma'} \)) to see

\[
I \leq C \sum_{q'} \sum_{q \in \mathcal{R}^{**}} 2^{-\sigma'} \lambda_q \lambda_{q'}
\]

\[
\leq C \sum_{q'} \alpha \lambda_{q'} \leq C \alpha \sum_{\kappa(Q)=0} \lambda_Q
\]
as desired. For the sum \( II \), note that \( 2^{\sigma'} \leq d(q,q') \leq 2^{3} \) for all \( q,q' \). Thus by (2.6),

\[
II \leq C \sum_{q'} \sum_{m=\sigma'}^{3} \sum_{q:d(q,q') \approx 2^{m}} 2^{\sigma'} 2^{-2m} \lambda_q \lambda_{q'}.
\]

Since the \( q \) with \( d(q,q') \approx 2^{m} \) are contained in one of \( 4^{d} \) isotropic cubes \( q'' \in \mathcal{R}_{m,0} \) (or one of at most \( 32^{d} \) cubes in \( \mathcal{R}_{0,0} \), if \( 1 < m \leq 3 \)), we apply property (iv) to those isotropic cubes and see

\[
II \leq C \sum_{q'} \sum_{m=\sigma'}^{3} 2^{\sigma'} 2^{-2m} \alpha 2^{m} \lambda_{q'}
\]

\[
\leq C \alpha \sum_{q'} \lambda_{q'} \sum_{m=\sigma'}^{3} 2^{\sigma'-m} = C \alpha \sum_{\kappa(Q)=0} \lambda_{Q}
\]

and the proof is complete for \( s = 0 \).

### 2.2 Proof for \( s > 0 \)

We must gain the factor of \( 2^{-(2(\sigma(d-1)+\delta)s) \text{ that appears in (2.5); we will do this by taking advantage of the better bounds in (1.2) of Lemma 1.2 for elements of } \mathcal{R}_{0,-s}, \text{ as compared to } \mathcal{R}_{\sigma,0}. \text{ We will therefore simply rewrite } q \in \mathcal{R}_{\sigma,0} \text{ as a union of cubes in some } \mathcal{R}_{\theta,-s} \text{ whenever it comes time to apply that bound. However, we run out of room to do this for pairs of cubes } (q,q') \text{ that are too far apart; in that case, we instead will use the Whitney decomposition (Lemma 1.1).}

Again, we assign \( Q \) with \( \kappa(Q) = -s \) to an isotropic \( q \in \mathcal{R}_{\sigma,0} \) of approximately equal diameter, but we also assign it to an element of \( \mathcal{R}_{\theta,-s}, \) where \( \theta \) is the largest integer such that an element of \( \mathcal{R}_{\sigma,0} \) can contain an element of \( \mathcal{R}_{\theta,-s}. \) That is to say, \( 2^{\theta} r^{-s}s^{n} \approx 2^{\sigma}. \) (Note that \( Q \) will indeed be contained in one of these, as it has been anisotropically dilated at least \( s \) times.)

Since \( \kappa(Q) = -s \) implies \( \tau(Q) < -s, \) the diameter of \( Q \) is at most \( r^{-s}s^{n}, \) and thus \( \theta \leq 0. \)

Now by (iv), for \( q' \in \mathcal{R}_{\theta,-s}, \)

\[
\sum_{Q \subset q', \kappa(Q) = -s, \text{diam}(Q) \approx \text{diam}(q')} \lambda_{Q} \leq C \alpha |T(q')| = C \alpha 2^{\theta} |A|^{-s}
\]

and there are \( \lesssim 2^{d(\sigma-\theta)} |A|^{s} \) such \( q' \) contained in \( q \in \mathcal{R}_{\sigma,0}, \) so that

\[
\sum_{Q \subset q', \kappa(Q) = -s, \text{diam}(Q) \approx \text{diam}(q')} \lambda_{Q} \leq C \alpha 2^{d\sigma+(1-d)\theta}. \quad (2.7)
\]
This gains a factor of \((r^{-s}s^n)^{d-1}\) over the direct estimate for \(q \in R_{\sigma,0}\), which we can use to offset our losses due to decreased curvature or due to alignment with the direction of slowest contraction. First, we see by Lemma 2.3 that

\[
I = 2 \sum_{q'} \sum_{q \subset q'} \left| \langle A_q * \mu_{s,0}^p, A_{q'} * \mu_{s,0}^p \rangle \right|
\]

\[
\lesssim \sum_{q'} \sum_{Q \subset Q'} 2^{\zeta s - \epsilon (d-1)s} \lambda_Q 2^{-\sigma' + \zeta s} \lambda_{q'}
\]

\[
\lesssim \sum_{q'} \lambda_q 2^{-\sigma' + 2\zeta s - \epsilon (d-1)s} \sum_{Q \subset Q'} \lambda_Q
\]

\[
\lesssim \sum_{q'} \lambda_q 2^{-\sigma' + 2\zeta s - \epsilon (d-1)s} \alpha 2^{\sigma' (r^{-s}s^n)^{1-d}}
\]

\[
\lesssim 2^{-(2\epsilon (d-1)+\delta)s} \alpha \sum_Q \lambda_Q,
\]

so long as \(\epsilon \ll \log(r)\), \(\zeta \ll \epsilon\) and \(\delta \ll \epsilon\).

This same gain will help us for the part of \(II\) where we can write \(q \in R_{m,0}\) as the union of elements of \(R_{\theta,-s}\) with \(\theta \leq 0\). However, this leaves us with a number of terms that we cannot handle in this manner: those with \(d(q,q') \geq r^{-s}s^n\). (We separate out these terms before putting the absolute values on inner products; just set aside all \(q, q'\) with that much distance between them.) These terms we will deal with separately, aggregating them instead via the cubes \(S\).

First, we will consider the pairs with \(d(q,q') \leq r^{-s}s^n\); we call this \(II'\), and by Lemma 2.4 and the fact that \(T(q') \lesssim 2^m (r^{-s}s^n)^{1-d}\) for \(q' \in R_{\theta,-s}\) here, we see that

\[
II' \leq \sum_{q'} \sum_{m=\sigma'} \sum_{q : d(q) \approx 2^m} \left| \langle A_q * \mu_{s,0}^p, \tilde{\mu}_{s,0}^p, A_{q'} \rangle \right|
\]

\[
\lesssim \sum_{q'} \sum_{m=\sigma'} \sum_{q : d(q) \approx 2^m} 2^{\sigma' + \epsilon s (5-d)} \lambda_q \lambda_{q'}
\]

\[
\lesssim \sum_{q'} \sum_{m=\sigma'} 2^{\sigma' + \epsilon s (5-d)} \lambda_{q'} \sum_{m=\sigma'} 2^{2\sigma' - 2m} \cdot \alpha 2^{m (r^{-s}s^n)^{d-1}}
\]

\[
\lesssim \alpha (r^{-s}s^n)^{d} 2^{\epsilon (5-d)s} \sum_{q'} \lambda_{q'}
\]

\[
\lesssim 2^{-(2\epsilon (d-1)+\delta)s} \alpha \sum_Q \lambda_Q.
\]

(Note that we used \(\epsilon \ll \log(r)\) in the final step.)
We still have to bound the sum
\[ III = \left| \sum_{q' \in q} \sum_{d(q) \geq r^{-s} n} \langle A_q * \mu^s_{\rho,0}, A_{q'} * \mu^s_{\rho,0} \rangle \right|. \]

If \( d(S) \) is the distance from \( S \) to \( q' \), then again
\[
\left| \mu^s_{\rho,0} * \mu^s_{\rho,0} * \sum_{Q \subset S, \kappa(Q) = -s} \lambda_Q a_Q(x) \right| \leq C \sum_{Q \subset S, \kappa(Q) = -s} r^s |s|^n 2^{(5-d)\epsilon s} d(S)^{-2} \lambda_Q \leq r^s |s|^n 2^{(5-d)\epsilon s} \alpha |S| d(S)^{-2}
\]
using property (1) from the Whitney decomposition. Now, since the \( S \) are disjoint, we can replace the sum over \( S \) with \( \int_{r^{-s} n}^{5} t^{-2} t^{d-1} dt \leq C(1 + s) \). Thus
\[
III \lesssim \alpha \sum_{Q} \lambda_Q r^{-s} n 2^{(5-d)\epsilon s} (1 + s)
\]
which as above is good enough.

**Remark** The reason that this argument works in \( H^1 \) and not in \( L^1 \) is that we used the cancellation of atoms to prove (2.6).

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