On a problem of M. Talagrand

Keith Frankston 1 | Jeff Kahn 2 | Jinyoung Park 3

1 Center for Communications Research - Princeton, Princeton, New Jersey, USA
2 Department of Mathematics, Rutgers University, Piscataway, New Jersey, USA
3 Department of Mathematics, Stanford University, Stanford, California, USA

Correspondence
Jinyoung Park, Department of Mathematics, Stanford University, Stanford, CA, USA.
Email: jinypark@stanford.edu

Funding information
NSF Grant, Grant/Award Numbers: DMS-1501962, DMS-1954035, DMS-1926686, CCF-1900460; BSF Grant, Grant/Award Number: 2014290.

Abstract
We address a special case of a conjecture of M. Talagrand relating two notions of “threshold” for an increasing family $\mathcal{F}$ of subsets of a finite set $V$. The full conjecture implies equivalence of the “Fractional Expectation-Threshold Conjecture,” due to Talagrand and recently proved by the authors and B. Narayanan, and the (stronger) “Expectation-Threshold Conjecture” of the second author and G. Kalai. The conjecture under discussion here says there is a fixed $L$ such that if, for a given $\mathcal{F}$, $p \in [0, 1]$ admits $\lambda : 2^V \to \mathbb{R}_+$ with

$$\sum_{S \subseteq F} \lambda_S \geq 1 \quad \forall F \in \mathcal{F},$$

and

$$\sum_{S} \lambda_S p^{|S|} \leq 1/2$$

(a.k.a. $\mathcal{F}$ is weakly $p$-small), then $p/L$ admits such a $\lambda$ taking values in $\{0, 1\}$ ($\mathcal{F}$ is $(p/L)$-small). Talagrand showed this when $\lambda$ is supported on singletons and suggested, as a more challenging test case, proving it when $\lambda$ is supported on pairs. The present work provides such a proof.

KEYWORDS
expectation threshold, fractional expectation threshold, Kahn–Kalai Conjecture, Talagrand conjectures, threshold

1 | INTRODUCTION

Given a finite set $V$, write $2^V$ for the power set of $V$ and, for $p \in [0, 1]$, $\mu_p$ for the product measure on $2^V$ given by $\mu_p(S) = p^{|S|}(1 - p)^{|V\setminus S|}$. An $\mathcal{F} \subseteq 2^V$ is increasing if $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$. For $\mathcal{G} \subseteq 2^V$
we use \((\mathcal{G})\) for the increasing family generated by \(\mathcal{G}\), namely \(\{B \subseteq V : \exists A \in \mathcal{G}, B \supseteq A\}\). As usual, if \(f : X \to \mathbb{R}\), then \(f(Y) = \sum_{x \in Y} f(x)\) for any \(Y \subseteq X\).

We assume throughout that \(\mathcal{F} \subseteq 2^V\) is increasing and not equal to \(2^V\) or \(\emptyset\). Then \(\mu_p(\mathcal{F})(= \sum (\mu_p(S) : S \in \mathcal{F}))\) is strictly increasing in \(p\), and we define the threshold, \(p_c(\mathcal{F})\), to be the unique \(p\) for which \(\mu_p(\mathcal{F}) = 1/2\). (This is finer than the original Erdős–Rényi notion, according to which \(p^* = p^*(n)\) is a threshold for \(\mathcal{F} = \mathcal{F}_n\) if \(\mu_p(\mathcal{F}) \to 0\) when \(p \ll p^*\) and \(\mu_p(\mathcal{F}) \to 1\) when \(p \gg p^*\). That \(p_c(\mathcal{F})\) is always an Erdős–Rényi threshold follows from [2].)

Thresholds have been a—maybe the—central concern of the study of random discrete structures (random graphs and hypergraphs, for example) since its initiation by Erdős and Rényi [4], with much of that effort concerned with identifying (Erdős–Rényi) thresholds for specific properties (see [1, 6])—though it was not observed until [2] that every sequence of increasing properties admits such a threshold.

The main concern of this paper is the relation between the following two notions of M. Talagrand [8–10]. (Our focus is Conjecture 1.4 and our main result is Theorem 1.6; we will come to these following some motivation.) Say \(\mathcal{F}\) is \(p\)-small if there is a \(\mathcal{G} \subseteq 2^V\) such that

\[
\langle \mathcal{G} \rangle \supseteq \mathcal{F},
\]

(i.e., each member of \(\mathcal{F}\) contains a member of \(\mathcal{G}\)) and

\[
\sum_{S \in \mathcal{G}} p^{|S|} \leq 1/2,
\]

and set \(q(\mathcal{F}) = \max\{p : \mathcal{F}\text{ is }p\text{-small}\}\). Say \(\mathcal{F}\) is weakly \(p\)-small if there is a \(\lambda : 2^V \to \mathbb{R}^+ (:= [0, \infty))\) such that

\[
\sum_{S \subseteq F} \lambda_S \geq 1 \quad \forall F \in \mathcal{F},
\]

and

\[
\sum_S \lambda_S p^{|S|} \leq 1/2,
\]

(which becomes \(p\)-small if we restrict the range of \(\lambda\) to \(\{0, 1\}\)), and set \(q_f(\mathcal{F}) = \max\{p : \mathcal{F}\text{ is weakly }p\text{-small}\}\). As in [5] we refer to \(q(\mathcal{F})\) and \(q_f(\mathcal{F})\), respectively, as the expectation-threshold and fractional expectation-threshold of \(\mathcal{F}\). (Note the former is used slightly differently in [7].) Notice that

\[
q(\mathcal{F}) \leq q_f(\mathcal{F}) \leq p_c(\mathcal{F}).
\]

(The first inequality is trivial and the second holds since, for \(\lambda\) as in (3), (4) and \(Y\) drawn from \(\mu_p\),

\[
\mu_p(\mathcal{F}) \leq \sum_{F \in \mathcal{F}} \mu_p(F) \sum_{S \subseteq F} \lambda_S \leq \sum_S \lambda_S \mu_p(Y \supseteq S) = \sum_S \lambda_S p^{|S|} \leq 1/2.
\]

In particular, each of \(q, q_f\) is a lower bound on \(p_c\), and these turn out to be easily understood (and to agree up to constant) in many cases of interest; see [5]. The next two conjectures—respectively, the
main conjecture (conjecture 1) of [7] and a sort of LP relaxation thereof suggested by Talagrand [10, conjecture 8.3]—say that these bounds are never far from the truth.

Conjecture 1.1. There is a universal $K$ such that for every finite $V$ and increasing $\mathcal{F} \subseteq 2^V$, 

$$p_c(F) \leq K q(F) \log |V|.$$ 

Conjecture 1.2. There is a universal $K$ such that for every finite $V$ and increasing $\mathcal{F} \subseteq 2^V$, 

$$p_c(F) \leq K q_f(F) \log |V|.$$ 

Talagrand [10, conjecture 8.5] also proposes the following strengthening of Conjecture 1.2, in which $\ell(F)$ is the maximum size of a minimal member of $\mathcal{F}$.

Conjecture 1.3. There is a universal $K$ such that for every finite $V$ and increasing $\mathcal{F} \subseteq 2^V$, 

$$p_c(F) < K q_f(F) \log \ell(F).$$ 

Conjecture 1.3 is proved in [5], to which we also refer for discussion of the very strong consequences that originally motivated Conjecture 1.1, but follow just as easily from Conjecture 1.2.

Turning, finally, to the business at hand, we are interested in the following conjecture of Talagrand [10, conjecture 6.3], which says that the parameters $q$ and $q_f$ are in fact not very different.

Conjecture 1.4. There is a fixed $L$ such that, for any increasing $\mathcal{F}$, 

$$q(F) \geq q_f(F)/L.$$ 

(That is, weakly $p$-small implies $(p/L)$-small.) This of course implies equivalence of Conjectures 1.2 and 1.1, as well as of Conjecture 1.3 and the corresponding strengthening of Conjecture 1.1; in particular, in view of [5], Conjecture 1.4 would now supply a proof of Conjecture 1.1. (Post-[5] this implication is probably the best motivation for Conjecture 1.4, but the authors have long been interested in the conjecture for its own sake, as it would be a striking instance of a broad, natural class of examples where the passage from an integer problem to its fractional counterpart has only a minor effect on behavior.)

The following mild reformulation of Conjecture 1.4 will be convenient.

Conjecture 1.5. There is a fixed $J$ such that for any $V, p \in [0, 1]$, and $\lambda : 2^V \setminus \emptyset \to \mathbb{R}^+$, 

$$\left\{ U \subseteq V : \sum_{S \subseteq U} \lambda_S \geq \sum_S \lambda_S (Jp)^{|S|} \right\},$$ 

is $p$-small.

(To get Conjecture 1.4 from Conjecture 1.5, let $L$ be the $J$ of Conjecture 1.5; let $\lambda, p$ be as in (3), (4); set $q = p/L$; and notice that $\mathcal{F} \subseteq \{ U \subseteq V : \sum_{S \subseteq U} \lambda_S \geq \sum_{S \lambda_S (Jp)^{|S|}} \}$. For the reverse implication, let $J = 2L$, with $L$ as in Conjecture 1.4, and $q = Lp$ (which is $q_f(F)/L$).)

As Talagrand observes, even simple instances of Conjecture 1.4 are not easy to establish. He suggests two test cases, which in the formulation of Conjecture 1.5 become:
(i) \( V = \binom{[n]}{2} = E(K_n) \) and (for some \( k \) \( \lambda \) is the indicator of \( \{ \text{copies of } K_k \text{ in } K_n \} \);

(ii) \( \lambda \) is supported on two-element sets.

(He does prove Conjecture 1.5 for \( \lambda \) supported on singletons; see Proposition 2.1 for a quantified version that will be useful in what follows.)

The quite specific (i) above was treated in [3]. Here we dispose of the much broader (ii):

**Theorem 1.6.** Conjecture 1.5 holds when \( \text{supp}(\lambda) \subseteq \binom{V}{2} \); in other words, there is a \( J \) such that for any graph \( G = (V, E) \), \( p \in [0, 1] \) and \( \lambda : E \to \mathbb{R}^+ \),

\[
\left\{ U \subseteq V : \lambda(G[U]) \geq J^2 \lambda(G)p^2 \right\},
\]

is \( p \)-small (where \( G[U] \) is the subgraph induced by \( U \)).

(We could of course take \( G = K_n \), but find thinking of a general \( G \) more natural.)

It seems not impossible that the ideas underlying Theorem 1.6 can be extended to give Conjecture 1.4 in full, but we do not yet see this.

The rest of the paper is devoted to the proof of Theorem 1.6. The most important part of this turns out to be a version of the “unweighted” case—so \( \lambda \) is the indicator of some graph \( G \) and the \( U \)’s in (7) are (roughly) those for which \( G[U] \) is abnormally large—though deriving Theorem 1.6 from this still needs some ideas. The overall approach is explained at some length in Section 2, culminating in the precise statement of what we need from the unweighted case (Theorem 2.2). Section 3 then proves Theorem 1.6 assuming Theorem 2.2, and the proof of Theorem 2.2 itself is given in Section 4. We will say something about strategy for the latter—which we consider the heart of the matter—when we are in a position to do so intelligibly (see following (34)).

## 2 | ORIENTATION

We use \([n]\) for \( \{1, 2, \ldots, n\} \), \( 2^X \) for the power set of \( X \), and \( \binom{X}{r} \) for the family of \( r \)-element subsets of \( X \), and recall from above that \( \langle A \rangle \) is the increasing family generated by \( A \subseteq 2^X \). For a set \( X \) and \( p \in [0, 1] \), \( X_p \) is the “\( p \)-random” subset of \( X \) in which each \( x \in X \) appears with probability \( p \) independent of other choices. We assume throughout that \( p \) has been specified and often omit it from our notation.

For \( A \subseteq 2^V \), the cost of \( A \) (w.r.t. our given \( p \)) is \( C(A) = \sum_{S \in A} p^{|S|} \). We say \( A \) covers \( B \subseteq 2^V \) if \( \langle A \rangle \supseteq B \), set

\[
C^*(B) = \min\{C(A) : A \text{ covers } B\},
\]

and say \( B \) can be covered at cost \( \gamma \) if \( C^*(B) \leq \gamma \). So “\( B \) \( p \)-small” is the same as \( C^*(B) \leq 1/2 \), and each of Conjecture 1.5, Theorem 1.6 says (roughly) that the collection of subsets \( U \) of \( V \) for which \( \sum_{S \subseteq U} \lambda_S \) is much larger than the “natural” value, \( \mathbb{E}[\sum_{S \subseteq V} \lambda_S] = \sum \lambda_S p^{|S|} \), admits such a “cheap” cover. Talagrand’s proof for singletons, to which we turn next, provides a first, simple illustration of this, and what we do in the rest of the paper amounts to producing such a cover for the collection in (8). Singletons. In the above language, Talagrand’s result for \( \lambda \) supported on singletons becomes:

**Proposition 2.1.** For all \( \zeta : V \to \mathbb{R}^+ \) and \( J > 2e \),

\[
C^*(\{ U \subseteq V : \zeta(U) \geq J\zeta(V)p \}) < 2e/(J - 2e).
\]
(The dependence on $J$ is best possible up to constants; e.g. take $|V| = J, p = J^{-2}$ and $\zeta \equiv 1$. The switch from $\lambda$ to $\zeta$ will be convenient when we come to use the proposition; see (17.).)

**Proof.** We may take $V = [n]$ and assume $\zeta$ is nonincreasing (and positive) and $Jp \leq 1$ (since the statement is trivial when $Jp > 1$). Define $R$ by

$$\frac{1}{Rp} = \left\lfloor \frac{1}{Jp} \right\rfloor =: a.$$ 

We claim that the collection

$$A = \bigcup_{k \geq 1} \binom{\lfloor ak \rfloor}{k},$$

covers the family in (9); this gives the proposition since the left-hand side (l.h.s) of (9) is then at most

$$C(A) = \sum_{k \geq 1} \binom{\lfloor ak \rfloor}{k} p^k < \sum_{k \geq 1} (\frac{e}{R})^k < \frac{e}{R - e} < \frac{2e}{J - 2e},$$

(the last inequality holding since $Jp \leq 1$ implies $R > J/2$.)

To see that the claim holds, observe that its failure implies the existence of some $U = \{u_1 < u_2 < \cdots < u_{\ell} \} \subseteq [n]$ with $\zeta(U) \geq J \zeta(V)p$ such that $|U \cap \lfloor ak \rfloor| < k$ for all $k > 0$. But then $u_i > ia$ for all $i \in [\ell]$, yielding the contradiction

$$\zeta(V) > \sum_{i=0}^{\ell-1} \sum_{j \in [a]} \zeta(j + ia) \geq a \zeta(U) \geq \zeta(V).$$

\[\blacksquare\]

**Toward doubletons.** Graphs here are always simple and are mainly thought of as sets of edges; thus $|G|$ is $|E(G)|$. We use $\nabla_G(v)$ or $\nabla_v$ for $\{e \in E(G) : v \in e\}$; so the degree of $v$ is $d_v = |\nabla_v|$. (We also use $N_G(v)$ for the neighborhood of $v$ in $G$.)

The following convention will be helpful. Given a graph $G$ on $V$, we associate with each $U \subseteq V$ a “weighted subset” $D(U) = D_G(U)$ of $E(G)$ that assigns to each $e$ the weight $|e \cap U|/2$. (We also use $D_v$ or $D_G(v)$ for $D(\{v\})$.) We then have (or define), for any $\lambda : G \rightarrow \mathbb{R}^+$,

$$\lambda(D(U)) = \frac{1}{2} \sum_{v \in U} \lambda(\nabla_v),$$

(e.g. $|D(U)| = \frac{1}{2} \sum_{v \in U} d_v$). Notice that

$$\mathbb{E} \lambda(G[|p]) = \mathbb{E} \lambda(D(V_P))p,$$

(e.g. $\mathbb{E}|G[|P] = \mathbb{E}|D(V_P)|p$, so $\lambda(D(U))p$ is a natural benchmark against which to measure $\lambda(G[U])$.

As mentioned at the end of Section 1, the heart of our argument deals with the unweighted case of Theorem 1.6, where we are given some (simple) graph $G$ on $V$, and the collection in (8) becomes the set of $U$’s for which $G[U]$ is atypically large. It is here that we are concerned with the production of covers, which are then available for use in the weighted case.

For the derivation of Theorem 1.6, we will decompose $G$ into subgraphs $G_1, G_2, \ldots$ so that the $\lambda$-values of the edges within a $G_i$ are roughly equal, and show that for each “heavy” $U$ (meaning one with large $\lambda(G[U])$, as in (8)), there is some $i$ for which $G_i[U]$ is large. We then plan to appeal to the
unweighted case to cover, for each $i$, the $U$’s that are “heavy” for $G_i$—a plan made delicate by the need to sum the contributions of many $G_i$’s to the l.h.s. of (2).

To deal with this we need a little more than the unweighted version of Theorem 1.6, as follows. For $J, \mu, T \in \mathbb{R}^+$, define

$$C^*_J(\mu, T),$$

(10)

to be the infimum of those $\gamma$’s for which, for every $p$ and (simple) graph $G$ (on $V$) with $|G|p^2 \leq \mu$,

$$\{ U \subseteq V : |G[U]| \geq \max\{ T, J|D_G(U)|p \} \},$$

(11)

can be covered at cost $\gamma$.

The technical-looking requirement involving $D_G$ is a crucial feature of our argument: for derivation of Theorem 1.6, we will need cost bounds that improve as the “target” $T$ grows, even if $T/\mu$ does not, which need not be the case without this extra condition (e.g. it’s not hard to see that if $G$ is the union of $(K_p)^{-1}$ disjoint copies of $K_{1, m}$, and $T = mp = K\mu$, then, no matter how large $m$ is, $\{ U \subseteq V : |G[U]| \geq T \}$—or the smaller $\{ U \subseteq V : G[U] \cong K_{1, mp} \}$—cannot be covered at cost less than $1/K$). License to use the condition will be provided by the reduction to the unweighted case in Section 3.

Our central result is:

**Theorem 2.2.** For any $\mu$ and $T = cJ^2 \mu$ with

$$c \geq 256e/J, \ J \geq 8e,$$

(12)

and $J_1 = J/(8e)$,

$$C^*_J(\mu, T) \leq 32c^{-1} \min\left\{ J_1^{-2}, J_1^{-\sqrt{T}/16} \right\}.$$ 

(13)

(Here and throughout we don’t worry about getting good constants, trying instead to keep the argument fairly clean.) As already mentioned, the proof of Theorem 2.2 is given in Section 4, following the derivation of Theorem 1.6, to which we now turn.

### 3 | PROOF OF THEOREM 1.6

Here we assume Theorem 2.2 and prove the following quantified version of Theorem 1.6.

**Theorem 3.1.** For any graph $G$ on $V$, $\lambda : G \to \mathbb{R}^+$ and

$$R \geq 4096\sqrt{2}e,$$

(14)

the set

$$U_0 = \{ U \subseteq V : \lambda(G[U]) \geq R^2 \lambda(G)p^2 \},$$

can be covered at cost $O(1/R)$.

**Proof.** We take $G, \lambda, R$ to be as in the theorem, use $D(U)$ for $D_G(U)$ (defined in Section 2), and assume throughout that

$$U \in U_0.$$
We first observe that it is enough to prove the theorem assuming

the only positive values taken by $\lambda$ are $\theta_i := 2^{-i}$, $i = 1, 2, \ldots$, \hspace{1cm} (15)

with (14) slightly weakened to

$$R \geq 4096e.$$ \hspace{1cm} (16)

Then for a general $\lambda$ (which we may of course scale to take values in $[0, 1]$) and $\lambda'$ given by

$$\lambda'_S = \max\{\theta_i : \theta_i \leq \lambda_S\},$$

$U_0$ as in the theorem is contained in the corresponding collection with $\lambda$ and $R^2$ replaced by $\lambda'$ and $R^2/2$ (which supports (16)), since $U \in U_0$ implies $2\lambda'(G[U]) > \lambda(G[U]) \geq R^2 \lambda(G)p^2 \geq R^2 \lambda'(G)p^2$. So we assume from now on that $\lambda$ and $R$ are as in (15) and (16) (respectively).

Note also that Proposition 2.1, with $\zeta(v) = \lambda(Dv)$ (for which we have $\zeta(V) = \sum \zeta(v) = \frac{1}{2} \sum \lambda(\nabla_v) = \lambda(G)$ and $\zeta(U) = \lambda(D(U))$), says that the set

$$\{U \subseteq V : \lambda(D(U)) \geq R\lambda(G)p\},$$

admits a cover of cost less than $6/R$. So we specify such a cover as a first installment on the cover $G$ (that we are constructing for Theorem 3.1), and it then becomes enough to show that

$$U^{**} := \{U \in U_0 : \lambda(D(U)) < R\lambda(G)p\},$$

can be covered at cost $O(1/R)$; in fact we will show

$$C^*(U^{**}) = O(R^{-2}).$$ \hspace{1cm} (18)

We assume from now on that $U \in U^{**}$.

Set $G_i = \{e \in G : \lambda(e) = \theta_i\}$. For (18) we first show that, for each $U \in U^{**}$, some $G_i[U]$ must be “large,” meaning $U$ belongs to $U'_i$, defined in (25), and then bound the costs of the $U'_i$’s using Theorem 2.2.

From this point we use $D_i(U)$ for $D_{G_i}(U)$. We observe that for any $H \subseteq G$,

$$\lambda(H) = \sum_i \theta_i |H \cap G_i|,$$

and abbreviate

$$w_i = \lambda(G_i) = \theta_i |G_i|, \quad w = \lambda(G) = \sum w_i.$$ 

Given $U$, define $L = L(U)$, $K = K(U)$, $L_i = L_i(U)$ and $K_i = K_i(U)$ by

$$\lambda(D(U)) = Lwp,$$

$$\lambda(G[U]) = KLwp^2,$$

$$|D_i(U)| = L_i |G_i| p,$$

\hspace{1cm} (19)
The parameters $K, L, K_i, L_i$ keep track of how far $\lambda(D(U)), \lambda(G(U)), |D_i(U)|$, and $|G_i[U]|$ exceed their natural values; thus, for example, $L_i$ is large if $|D_i(U)|$ is atypically large, in which case we hope to cover $U$ using Proposition 2.1 (where weights are degrees in $G_i$); while $K_i$ large means $|G_i[U]|$ is large relative to what one might “expect” given $|D_i(U)|$, putting us in the territory of Theorem 2.2.

We have

$$L_{wp} = \sum \theta_i |D_i(U)| = \sum L_i w_i p,$$

and

$$KL_{wp}^2 = \sum \theta_i |G_i[U]| = \sum K_i L_i w_i p^2.$$

Since $U \in U_0$, we have

$$\sum K_i L_i w_i \geq R^2 w,$$  

while $U \in U^*$ gives

$$L < R.$$  

Note also that, with

$$I = I(U) = \{i : K_i > R/2\},$$

we have

$$\sum \{K_i L_i w_i : i \in I\} > R^2 w/2,$$

as follows from (22) and (using (21) and (23))

$$\sum \{K_i L_i w_i : i \notin I\} \leq (R/2)L w < R^2 w/2.$$  

Now let $E_i = |G_i| p^2 = \mathbb{E}[G_i[V_p]]$ and, for integer $\alpha$,

$$E_{\alpha} = \{i : E_i \in (2^{\alpha-1}, 2^\alpha]\}.$$  

We arrange the $i$’s in an array, with columns indexed by $\alpha$’s (in increasing order) and column $\alpha$ consisting of the indices in $E_{\alpha}$, again in increasing order. (So $w_i$’s within a column decrease as we go down. Note column lengths may vary.) Define $B_\beta$ to be the set of indices in row $\beta$ (Table 1).

Set $y_i = \theta_i 2^\alpha / p^2$ (for $i \in E_\alpha$) and $y = \sum_{i \geq 1} y_i$, noting that

$$y_i / 2 < w_i \leq y_i.$$  

Set

$$c_\beta^* = \frac{3}{2^{\beta-1}} R^2 / 16 \quad (\beta \geq 1).$$
and \(c_i = c^*_\beta\) if \(i \in B_\beta\). Let \(w^*_\beta\) and \(y^*_\beta\) be (respectively) the sums of the \(w_i\)'s and \(y_i\)'s over \(i \in B_\beta\), and notice that

\[
y^*_{\beta+1} \leq y^*_\beta / 2 \quad \text{for } \beta \geq 1,
\]

(since \(i = B_{\beta+1} \cap E_\alpha\)—where we abusively use \(i\) for \(\{i\}\)—implies \(i > j := B_\beta \cap E_\alpha\), whence \(2y_i \leq y_j\)).

**Claim 3.2.** For each \(U \in U^*\) there is an \(i \in I(U)\) with \(K_i(U)L_i(U) > c_i\).

**Proof.** With \(\sum^*\) denoting summation over \(I(U)\), we have (using (24) at the end)

\[
\sum^* c_i w_i \leq \sum c^*_\beta w^*_\beta \leq \sum c^*_\beta y^*_\beta
\]

\[
\leq y^*_1 (c^*_1 + c^*_2 / 2 + c^*_3 / 2^2 + \cdots)
\]

\[
\leq y (c^*_1 + c^*_2 / 2 + c^*_3 / 2^2 + \cdots)
\]

\[
\leq (R^2 / 4)y < (R^2 / 2)w < \sum^* K_i(U)L_i(U)w_i.
\]

\[\blacksquare\]

It follows that if, for each \(i\), \(G_i\) covers

\[
U_i := \{ U \subseteq V : i \in I(U); K_i(U)L_i(U) > c_i \},
\]

then \(\bigcup G_i\) covers \(U^*\); so we have

\[
C^*(U^*) \leq \sum_i C^*(U_i),
\]

(26)

On the other hand, if \((\alpha, \beta)\) is the pair corresponding to \(i\) (i.e., \(i\) is the \(\beta\)th entry in column \(\alpha\) of our array), then (see (10), (11) for \(C^*_J\))

\[
C^*(U_i) \leq C^*_{R/2}(2^\alpha, T_{\alpha, \beta}),
\]

where \(T_{\alpha, \beta} = \max\{c^*_\beta 2^{\alpha-1}, 1\}\). (To see this, note \(|G_i|p^2 = E_i \leq 2^\alpha\), and \(U \in U_i\) implies, using (19) and (20),

\[
|G_i[U]| = K_i(U)L_i(U)|G_i|p^2 \begin{cases} > c_i |G_i|p^2 > c^*_\beta 2^{\alpha-1} \\ = K_i|D_i(U)|p > (R/2)|D_i(U)|p. \end{cases}
\]
So, with $\alpha$ and $\beta$ ranging over integers and positive integers respectively, (18) will follow from

$$
\sum C_{R/2}(2^\alpha, T_{\alpha,\beta}) = O(R^{-2}).
$$

(27)

**Proof of (27).** For $T_{\alpha,\beta} = 1$ we bound $C_{R/2}(2^\alpha, T_{\alpha,\beta})$ by $2^\alpha$, using the trivial

$$
C^*(\mu, 1) \leq \mu,
$$

(28)

(since $\{\{x, y\} : xy \in G\}$ itself covers the set in (11)), which—since $T_{\alpha,\beta} = 1$ iff $2^\alpha \leq 32R^{-2}(2/3)^{\beta-1}$—bounds the contribution of such pairs to the sum in (27) by

$$
\sum_{\beta} \sum_{\alpha : T_{\alpha,\beta} = 1} 2^\alpha \leq 64R^{-2} \sum_{\beta} (2/3)^{\beta-1} = 3 \cdot 64R^{-2}.
$$

(29)

For $T_{\alpha,\beta} > 1$ we use Theorem 2.2 with $T = T_{\alpha,\beta} (= c^*_\beta 2^{\alpha-1})$, $\mu = 2^\alpha$, $J = R/2$, and (thus)

$$
c = T/(\mu J^2) = c^*_\beta/(2J^2) = (3/2)^{\beta-1}/8.
$$

Note that (16) gives $J \geq 8e$ and $c \geq 256e/J$, so (12) holds.

(Here, finally, we see the role of $C^*_J$ mentioned in the paragraph following (11): for a given $\beta$ we may be summing over many $\alpha$’s with the same $T/\mu$, so it is crucial that we have cost bounds that shrink with $T$ when this ratio is held constant.)

For each integer $s \geq 0$ let $T_s = \{(\alpha, \beta) : T_{\alpha,\beta} \in (2^s, 2^{s+1}]\}$. For each $\beta \geq 1$ there is a unique $\alpha$ such that $(\alpha, \beta) \in T_s$, and every $(\alpha, \beta)$ with $T_{\alpha,\beta} > 1$ is in some $T_s$. Let $f(s) = \min\{J_1^{-2}, J_1^{-2/3}\}$. Then for fixed $s$, we have (see (13))

$$
\sum_{(\alpha, \beta) \in T_s} C^*_J(2^\alpha, T_{\alpha,\beta}) \leq \sum_{\beta} 32c^{-1}f(s) = \sum_{\beta} 256\left(\frac{2}{3}\right)^{\beta-1}f(s) < 3 \cdot 256f(s),
$$

(30)

and summing over all $s$ we get

$$
\sum_{T_{\alpha,\beta} > 1} C^*_J(2^\alpha, T_{\alpha,\beta}) < \sum_{s \geq 0} 768f(s) = \sum_{s \geq 0} 768 \min\left\{J_1^{-2}, J_1^{-2/3}\right\} = O(J_1^{-2}).
$$

(31)

Finally, combining (31) and (29) gives (27).

\[\Box\]

4 | PROOF OF THEOREM 2.2

Aiming for simplicity, we just bound the cost in (13) assuming

$$
T = 2^{2k+3},
$$

for some positive integer $k$ and

$$
c = T/(\mu J^2) \geq 64e/J,
$$

(32)
showing that in this case
\[ C_j^*(\mu, T) \leq 8e^{-1}J_i^{-2^{k-1}-1}. \] (33)

(Recall from the theorem statement that \( J_1 = J/(8e) \).)

Before proving this, we show it implies Theorem 2.2, which, since \( C_j^*(\mu, t) \) is decreasing in \( t \), just requires showing that the right-hand side (r.h.s.) of (13) bounds \( C_j^*(\mu, T_0) \) for some \( T_0 \leq T \).

If \( T < 32 \) this follows from the trivial (28), since \( \mu = T/(cJ^2) \leq 32 \) and the bound in (13). Suppose then that \( T \geq 32 \) and let \( T_0 = c_0J^2 \mu \) be the largest integer not greater than \( T \) of the form \( 2^{k+3} \) (with positive integer \( k \)). We then have \( c_0 > c/4 \) (supporting (32)) and \( 2^{k-1} > \sqrt{T_0}/8 > \sqrt{T}/16 \), and it follows that the bound on \( C_j^*(\mu, T_0) \) given by (33) is less than the bound in (13).

Proof of (33). We have \(|G|^2 \leq \mu, T = 2^{2k+3} (= cJ^2 \mu \text{ with } J \text{ as in (12) and } c \text{ as in (32)})\), and, with \( U^* := \{ U \subseteq V : |G[U]| > \max\{T, J|DG(U)|p\} \} \), (34)

want to show that \( C^*(U^*) \) is no more than the bound in (33).

Here, finally, we come to specification of a cover, \( \mathcal{G} \). Each member of \( \mathcal{G} \) will be a disjoint union of stars (a.k.a. a star forest), where for present purposes a star at \( v \) in \( W (\subseteq V) \) is some \( \{v\} \cup S \subseteq W \) with \( S \subseteq NG(v) \). (Where convenient we will also refer to this as the “star \( (v, S) \)”. We say such a star is good if
\[ |S| \geq Jd_p/4. \] (35)

Given a positive integer \( L \), we define
\[ L^v = \max\{L, [Jd_p/4]\} \],
\[ L^v = \max\{L, [Jd_p/4]\} \],
(36)

and say a star \( (v, S) \) is \( L \)-special if \( |S| = L^v \).

For positive integers \( b \) and \( L \), let \( \mathcal{G}(b, L) (\subseteq 2^V) \) consist of all disjoint unions of \( b \) \( L \)-special stars in \( G \). We will specify a particular collection \( \mathcal{C} \) of pairs \( (b, L) \) and set
\[ \mathcal{G} = \cup\{\mathcal{G}(b, L) : (b, L) \in \mathcal{C}\}. \]

Theorem 2.2 is then given by the following two assertions.

Claim 4.1. \( \mathcal{G} \) covers \( U^*. \)

Claim 4.2. \( C(\mathcal{G}) \) is at most the bound in (33).

Set (with \( i \in [k] \) throughout) \( L_i = 2^{i-1} \) and
\[ \delta_i = \max\{2^{-(i+2)}, 2^{i-k-3}\} \geq 1/(8L_i), \] (37)

and notice that
\[ \sum \delta_i \leq \sum 2^{-(i+2)} + \sum 2^{i-k-3} \leq 1/2. \] (38)
Let
\[ b_i = \delta_i 4^{-i} T \geq 2^{k-i}. \] (39)

Finally, set
\[ C = \{(b_i, L_i) : i \in [k]\}. \]

Proof of Claim 4.1. We are given \( U \in U^* \) and must show it contains a member of \( G \). Let \( U_0 = U \) and for \( j = 1, \ldots \) until no longer possible do: let \( (v_j, S_j) \), with \( S_j = N_G(v_j) \cap U_{j-1} \), be a largest good star in \( U_{j-1} \), and set \( d_j = |S_j| \) and \( U_j = U_{j-1} \setminus \{v_j\} \cup S_j \).

The passage from \( U_{j-1} \) to \( U_j \) deletes at most \( d_j^2 \) edges that contain vertices of \( S_j \) of \( U_{j-1} \)-degree at most \( d_j \); any other edge deleted in this step contains \( u \in S_j \) with \( U_{j-1} \)-degree less than \( \delta_j \) (or \( u \), having \( U_{j-1} \)-degree greater than \( d_j \), would have been chosen in place of \( v_j \)); and of course each vertex \( u \) of the final \( U_j \) has \( U_j \)-degree less than \( \delta_j \). We thus have
\[ |G[U]| \leq \sum_j d_j^2 + \sum_{v \in U} Jd_v p/4 \leq \sum_j d_j^2 + |G[U]|/2 \] (using the second bound in (34)), so
\[ \sum_j d_j^2 \geq |G[U]|/2 \geq T/2. \] (40)

Set
\[ B_i = \begin{cases} \{j : d_j \in [2^{i-1}, 2^i)\} & \text{if } i \in [k-1], \\ \{j : d_j \geq 2^{k-1}\} & \text{if } i = k. \end{cases} \]

(It may be worth noting that, while the \( d_j \)'s are decreasing, the degrees corresponding to \( B_i \) increase with \( i \).) In view of (40), either \( |B_k| \geq 1 \) or (using (38))
\[ \sum_{i \in [k-1]} |B_i| 4^i \geq T/2 \geq \sum_{i \in [k-1]} \delta_i T = \sum_{i \in [k-1]} b_i 4^i. \]

Since \( b_k = 1 \), it follows that for some \( i \in [k] \) we have
\[ |B_i| \geq b_i. \] (41)

On the other hand, since \( j \in B_i \) implies \( |S_j| \geq L'_i \) (= \( \max\{L_i, [Jd_v p/4]\}\)), the set \( \bigcup \{S_j \cup \{v_j\} : j \in B_i\} \) contains some \( W \in G(b_i, L_i) \subseteq G \) whenever \( i \) is as in (41). This completes the proof of Claim 4.1. \[ \blacksquare \]

Proof of Claim 4.2. We first bound the costs, say \( C(b, L) \), of the collections \( G(b, L) \). Given \( (b, L) \), set
\[ q_v = p \left( \frac{ed_v p}{L_v} \right)^{L_v}. \]
Then \( q_v \) bounds the total cost of the set of \( L \)-special stars at \( v \) (using \( \left( \frac{d_v}{L^v} \right) \leq (ed_v/L^v)^{L^v} \)), and it follows that

\[
C(b, L) \leq \sum \left\{ \prod_{v \in B} q_v : B \in \binom{V}{b} \right\} .
\]

For a given value of \( \varphi := \sum_{v \in V} q_v \), the r.h.s. of (42) is largest when the \( q_v \)'s are all equal (this just uses \( xy \leq [(x + y)/2]^2 \)), whence

\[
C(b, L) \leq \left( \frac{|V|}{b} \right) \left( \frac{\varphi}{|V|} \right)^b \leq \left( \frac{e\varphi}{b} \right)^b .
\]

Recalling (36), we have

\[
q_v \leq \frac{d_v \rho^2 \cdot e \left( \frac{4e}{J} \right)^{L-1}}{L} ,
\]

so (since \( |G|\rho^2 \leq \mu \))

\[
\varphi \leq 2\mu \cdot \frac{e \left( \frac{4e}{J} \right)^{L-1}}{L} .
\]

Now using (43) and (44), recalling that \( T = cJ^2 \mu, L_i = 2^{i-1}, b_i = \delta_i 4^{-i} T = \delta_i T/(4L_i^2) \) and \( J_1 = J/(8e) \), and for the moment omitting the subscript \( i \), we have (with the final inequality (45) justified below)

\[
C(b, L) \leq \left[ \frac{2e^2 \mu 4L^2}{L \delta T} \left( \frac{4e}{J} \right)^{L-1} \right]^b
= \left[ 8e^2 L \cdot \frac{1}{cJ^2 \delta} \left( \frac{4e}{J} \right)^{L-1} \right]^b
= \left[ c^{-1} L / \frac{2\delta}{cJ^2} \left( \frac{4e}{J} \right)^{L+1} \right]^b
\leq \left[ \frac{c}{4} \cdot J_1^{L+1} \right]^{-b} .
\]

For (45), or the equivalent

\[
2^{L+1} \delta \geq L ,
\]

it is enough to show \( 2^{L+1} \geq L^2 \) (since \( \delta \geq 1/(8L) \); see (37)), which is true for positive integer \( L \).

Finally, returning to Claim 4.2 (and recalling that \( L \) and \( b \) in the display ending with (45) are really \( L_i \) and \( b_i \)), we have

\[
C(G) = \sum_{i=1}^{k} C(b_i, L_i) \leq \sum_{i=1}^{k} \left[ \frac{c}{4} \cdot J_1^{L+1} \right]^{-b_i} .
\]

We use \( b_i \geq 2^{k-i} \) (see (39)) and \( L_i = 2^{i-1} \) to bound the r.h.s. of (47) by

\[
\sum_{i=1}^{k} \left[ \frac{cJ_1^{2^{k-i+1}}}{4} \right]^{-2^{k-i}} = \sum_{i=1}^{k} J_1^{-2^{k-i}} \left[ \frac{cJ_1}{4} \right]^{-2^{k-i}} = \sum_{j=0}^{k-1} \left( \frac{cJ_1}{4} \right)^{2^{k-i+1}} \left[ \frac{cJ_1}{4} \right]^{-1} \left( \frac{cJ_1}{4} \right)^{1-2^j} ;
\]
but, since \( cJ_1/4 \geq 2 \) (using (32) and \( J_1 = J/(8e) \)), the last expression in (48) is less than the bound 
\( 8c^{-1}J_1^{-2^{k-1}-1} \) in (33).

\[ \blacksquare \]

**ACKNOWLEDGMENTS**

The authors were supported by NSF grant DMS-1501962 and BSF Grant 2014290. The second author was also supported by NSF grant DMS-1954035, and the third directly by NSF grant DMS-1926686 and indirectly by NSF grant CCF-1900460.

**REFERENCES**

1. B. Bollobás, Random graphs, Cambridge Studies in Advanced Mathematics, 2nd ed., Vol 73, Cambridge University Press, Cambridge, UK, 2001.
2. B. Bollobás and A. Thomason, *Thresholds functions*, Combinatorica 7 (1987), 35–38.
3. B. DeMarco and J. Kahn, *Note on a problem of M. Talagrand*, Random Struct. Algorithms 47 (2015), no. 4, 663–668.
4. P. Erdős and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 5 (1960), 17–61.
5. K. Frankston, J. Kahn, B. Narayanan, and J. Park, *Thresholds versus fractional expectation-thresholds*, Ann. Math. 194 (2021), no. 2, 475–495.
6. S. Janson, T. Łuczak, and A. Rucinski, Random graphs, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, NY, 2000.
7. J. Kahn and G. Kalai, *Thresholds and expectation thresholds*, Comb. Probab. Comput. 16 (2007), 495–502.
8. M. Talagrand, “Are all sets of positive measure essentially convex?” Geometric aspects of functional analysis, J. Lindenstrauss and V. Milman (eds.), Springer, New York, NY, 1995, pp. 295–310.
9. M. Talagrand, The generic chaining, Springer Monographs in Mathematics, Springer-Verlag, Berlin, Germany, 2005.
10. M. Talagrand, Are many small sets explicitly small? Proc. 2010 ACM Int. Sympos. Theory Comput, New York, 2010, pp. 13–35.

**How to cite this article:** K. Frankston, J. Kahn, and J. Park, *On a problem of M. Talagrand*, Random Struct. Algorithms. 61 (2022), 710–723. https://doi.org/10.1002/rsa.21077