The Birthday Problem and Zero-Error List Codes
Parham Noorzad, Michelle Effros, Michael Langberg, and Victoria Kostina

Abstract
As an attempt to bridge the gap between the probabilistic world of classical information theory and the combinatorial world of zero-error information theory, this paper studies the performance of randomly generated codebooks over discrete memoryless channels under a zero-error list-decoding constraint. This study allows the application of tools from one area to the other. Furthermore, it leads to an information-theoretic formulation of the birthday problem, which is concerned with the probability that in a given population, a fixed number of people have the same birthday. Due to the lack of a closed-form expression for this probability when the distribution of birthdays is not uniform, the resulting expression is not simple to analyze; in the information-theoretic formulation, however, the asymptotic behavior of this probability can be characterized exactly for all distributions.

I. INTRODUCTION
Finding channel capacity under a zero-error constraint requires fundamentally different tools and ideas from the study of capacity under an asymptotically negligible error constraint; the former is essentially a graph-theoretic problem [1], while the latter mainly relies on probabilistic arguments [2]. To obtain a better understanding of the contrast between zero-error information theory and classical information theory, we apply probabilistic tools to the study of zero-error channel coding.

The random code construction of Shannon [2] shows that for the discrete memoryless channel \((X, W(y|x), Y)\), a sequence of rate-\(R\) codebooks \((C_n)_{n=1}^\infty\), randomly generated according to distribution \(P(x)\), achieves

\[
\lim_{n \to \infty} E[P_e^n(C_n)] = 0,
\]

if \(R < I(X;Y)\). In (1), \(P_e^n(C_n)\) is the average probability of error of codebook \(C_n\). From Markov’s inequality, it follows that \(R < I(X;Y)\) suffices to ensure that

\[
\forall \epsilon \in (0, 1): \lim_{n \to \infty} Pr \left\{ P_e^n(C_n) \leq \epsilon \right\} = 1.
\]

This paper was presented in part at the 2017 IEEE International Symposium of Information Theory in Aachen.
This material is based upon work supported by the National Science Foundation under Grant Numbers 1321129, 1527524, and 1526771.
P. Noorzad was with the California Institute of Technology, Pasadena, CA 91125 USA. He is now with Qualcomm Research, San Diego, CA 92121 USA (email: parham@qti.qualcomm.com).
M. Effros and V. Kostina are with the California Institute of Technology, Pasadena, CA 91125 USA (emails: effros@caltech.edu, vkostina@caltech.edu).
M. Langberg is with the State University of New York at Buffalo, Buffalo, NY 14260 USA (email: mikel@buffalo.edu).
Our aim is to understand the behavior of randomly generated codebooks when \( \epsilon = 0 \) in (2). Specifically, we seek to find necessary and sufficient conditions on the rate \( R \) in terms of the channel \( W \) and input distribution \( P(x) \), such that the sequence of randomly generated codebooks \( (C_n)_{n=1}^{\infty} \) satisfies

\[
\lim_{n \to \infty} \Pr \{ P^n(\epsilon) = 0 \} = 1.
\]

In other words, our goal is to quantify the performance of randomly generated codebooks under the zero-error constraint. However, we do not limit ourselves to the case where the decoder only has one output. Instead, similar to works by Elias [3], [4], we allow the decoder to output a fixed number of messages. We say a codebook corresponds to a “zero-error \( L \)-list code” if for every message the encoder transmits, the decoder outputs a list of size at most \( L \) that contains that message. Similar to the zero-error list capacity problem [5], this problem can be solved using only knowledge of the distinguishability hypergraph of the channel. We discuss hypergraphs and their application to zero-error list codes in Section II. We then present our main result in Theorem 4 of Section III, where we provide upper and lower bounds on the rate of randomly generated zero-error list codes.

An important special case occurs when the channel \( W \) is an identity, that is,

\[
\forall (x, y) \in X \times Y: W(y | x) = \mathbb{1}\{y = x\}. \tag{3}
\]

In this case, our setup leads to an information-theoretic formulation of the birthday problem which we next describe.

A. The Birthday Problem

The classical birthday problem studies the probability that a fixed number of individuals in a population have the same birthday under the assumption that the birthdays are independent and identically distributed (i.i.d.). While this probability is simple to analyze when the i.i.d. distribution is uniform due to a closed-form expression, the same is not true in the non-uniform case. Numerical approximations for this probability are given in [6]–[8].

We can frame the birthday problem as a special case of our setup above. Consider the problem of channel coding over the identity channel defined by (3). Note that over this channel, a codebook corresponds to a zero-error \( L \)-list code if and only if no group of \( L + 1 \) messages are mapped to the same codeword. Associating codewords with birthdays, we obtain an information-theoretic formulation of the birthday problem: Given a randomly generated codebook (set of birthdays), what is the probability that some subset of \( L + 1 \) codewords (birthdays) are identical? In Corollary 8, we provide the precise asymptotic behavior of this probability in terms of the Rényi entropy of order \( L + 1 \).

We next describe prior works that study the birthday problem in a context similar to our work.

B. Prior Works

In [9], Rényi states a result in terms of the random subdivisions of a set, which we reformulate in Appendix B in terms of zero-error \( L \)-list codes over the identity channel. Rényi’s result differs from ours in the asymptotic regime under consideration.
Another related work is [10], where Fujiwara studies a variation of the birthday problem for the case $L = 1$ in the setting of quantum information theory. Specifically, Fujiwara determines the maximum growth rate of the cardinality of a sequence of codebooks that satisfy an “asymptotic strong orthogonality” condition.

Finally, we remark that the birthday problem also arises in the context of cryptography. For a given hash function, the quantity of interest is the number of hash function evaluations required to find a “collision”; that is, two inputs that are mapped to the same output. In this context, the default assumption is that the hash function values are uniformly distributed as this leads to the lowest collision probability [11, p. 192]. However, Bellare and Kohno [12] argue that the uniformity assumption need not hold for real-world hash functions. Thus, it is important to study the non-uniform case. In [12, Theorem 10.3], the authors provide upper and lower bounds for the collision probability in terms of a quantity they call “balance,” which is the same as Rényi entropy of order two up to the base of the logarithm.

In the next section, we provide an introduction into hypergraphs and their connection to zero-error list codes. The proofs of all of our results appear in Section IV.

II. Hypergraphs and Zero-Error List Codes

A discrete channel is a triple

$$\left(\mathcal{X}, W(y|x), \mathcal{Y}\right),$$

where $\mathcal{X}$ and $\mathcal{Y}$ are finite sets, and for each $x \in \mathcal{X}$, $W(\cdot|x)$ is a probability mass function on $\mathcal{Y}$. We say an output $y \in \mathcal{Y}$ is “reachable” from an input $x \in \mathcal{X}$ if $W(y|x) > 0$.

A hypergraph $G = (\mathcal{V}, \mathcal{E})$ consists of a set of nodes $\mathcal{V}$ and a set of edges $\mathcal{E} \subseteq 2^\mathcal{V}$, where $2^\mathcal{V}$ denotes the collection of subsets of $\mathcal{V}$. We assume that $\mathcal{V}$ is finite and each edge has cardinality at least two.

The distinguishability hypergraph of channel $W$, denoted by $G(W)$, is a hypergraph with vertex set $\mathcal{X}$ and an edge set $\mathcal{E} \subseteq 2^\mathcal{X}$ which contains collections of inputs that are “distinguishable” at the decoder. Formally, $\mathcal{E}$ consists of all subsets $e \subseteq \mathcal{X}$ that satisfy

$$\forall y \in \mathcal{Y}: \prod_{x \in e} W(y|x) = 0;$$

that is, $e \subseteq \mathcal{X}$ is an edge if no $y \in \mathcal{Y}$ is reachable from all $x \in e$. Note that $G(W)$ has the property that the superset of any edge is an edge; that is, if $e \in \mathcal{E}$ and $e \subseteq e' \subseteq \mathcal{X}$, then $e' \in \mathcal{E}$. Proposition [11] below, shows that any hypergraph $G$ with this property is the distinguishability hypergraph of some channel $W$.

An independent set of a hypergraph $G = (\mathcal{V}, \mathcal{E})$ is a subset $\mathcal{I} \subseteq \mathcal{V}$ such that no subset of $\mathcal{I}$ is in $\mathcal{E}$. For the channel $(\mathcal{X}, W(y|x), \mathcal{Y})$, an independent set of $G(W)$ corresponds to a collection of inputs $\mathcal{I} \subseteq \mathcal{X}$ for which there exists an output $y \in \mathcal{Y}$ that is reachable from any $x \in \mathcal{I}$. The hypergraph $G$ is complete multipartite if there exists a partition $\{\mathcal{I}_j\}_{j=1}^k$ of $\mathcal{V}$ such that each $\mathcal{I}_j$ is an independent set, and for every subset $e \subseteq \mathcal{V}$, either $e \in \mathcal{E}$, or $e \subseteq \mathcal{I}_j$ for some $1 \leq j \leq k$.

1The attempt of finding such inputs is referred to as the birthday attack in cryptography [11, p. 187].
As an example, consider a deterministic channel \((X, W(y|x), Y)\), where for some mapping \(\varphi: X \to Y\),
\[ W(y|x) = 1\{y = \varphi(x)\}. \]
For this channel, \(G(W)\) is a complete multipartite hypergraph. Specifically, the sets \(\{\varphi^{-1}(y)\}_{y \in Y}\) are the independent components of \(G\), where for \(y \in Y\),
\[ \varphi^{-1}(y) := \{x \in X|\varphi(x) = y\}. \]
The next proposition gives a complete characterization of hypergraphs \(G\) that correspond to the distinguishability hypergraphs of arbitrary and deterministic channels, respectively.

**Proposition 1.** Consider a hypergraph \(G = (V, E)\). Then there exists a discrete channel \((X, W(y|x), Y)\) such that \(G = G(W)\) if and only if the superset of every edge of \(G\) is an edge. Furthermore, there exists a deterministic channel \(W\) such that \(G = G(W)\) if and only if \(G\) is complete multipartite.

Given the connection between channels and hypergraphs in Proposition 1, we now find a graph-theoretic condition for a mapping to be a zero-error list code. We present this condition in Proposition 2. Prior to that, we define necessary notation.

For positive integers \(i\) and \(j\) with \(j \geq i\), \([i : j]\) denotes the set \(\{i, \ldots, j\}\). When \(i = 1\), we denote \([1 : j]\) by \([j]\). For example, \([1] = \{1\}\) and \([2 : 4] = \{2, 3, 4\}\). For any set \(A\) and nonnegative integer \(k \leq |A|\), define the set
\[ \binom{A}{k} := \{B|B \subseteq A, |B| = k\}. \]
An \((M, L)\) list code for the channel \((X, W(y|x), Y)\) consists of an encoder
\[ f: [M] \to X, \]
and a decoder
\[ g: Y \to \bigcup_{\ell=1}^{L} \binom{[M]}{\ell}. \]
The pair \((f, g)\) is an \((M, L)\) zero-error list code for channel \(W\) if for every \(m \in [M]\) and \(y \in Y\) satisfying \(W(y|f(m)) > 0\), we have \(m \in g(y)\).

Proposition 2 provides a necessary and sufficient condition for the existence of an \((M, L)\) zero-error list code for \(W\) in terms of its distinguishability hypergraph \(G(W)\).

**Proposition 2.** Consider a discrete channel \((X, W(y|x), Y)\), positive integers \(M\) and \(L\), and an encoder
\[ f: [M] \to X. \]
For this encoder, a decoder
\[ g: Y \to \bigcup_{\ell=1}^{L} \binom{[M]}{\ell} \]
exists such that the pair \((f, g)\) is an \((M, L)\) zero-error list code for \(W\) if and only if the image of every \((L+1)\)-subset \(\{m_\ell\}_{\ell=1}^{L+1}\) of \([M]\) under \(f\) is an edge of \(G(W)\).
Proposition 2 reduces the existence of an \((M, L)\) zero-error list code to the existence of an encoder with a certain property. Because of this, henceforth we say a mapping \(f : [M] \to \mathcal{X}\) is an \((M, L)\) zero-error list code for the channel \(W\) if it satisfies the condition stated in Proposition 2.

For each positive integer \(n\), the \(n\)th extension channel of \(W\) is the channel 
\[
(X^n, W^n(y^n|x^n), Y^n),
\]
where
\[
W^n(y^n|x^n) := \prod_{t \in [n]} W(y_t|x_t).
\]
An \((M, L)\) zero-error list code for \(W\) is referred to as an \((M, n, L)\) zero-error list code for \(W\). It is possible to show that the distinguishability hypergraph of \(W^n\), \(G(W^n)\), equals \(G_n(W)\), the \(n\)th co-normal power of \(G(W)\) [13]. For any positive integer \(n\) and any hypergraph \(G = (V, E)\), the \(n\)th co-normal power of \(G\), which we denote by \(G^n\), is defined on the set of nodes \(V^n\) as follows. For each \(k \geq 2\), the \(k\)-subset \(e = \{v_1^n, \ldots, v_k^n\} \subseteq V^n\) is an edge of \(G^n\) if for at least one \(t \in [n]\), \(\{v_{1t}, \ldots, v_{kt}\} \in E\). This definition is motivated by the fact that \(k\) codewords are distinguishable if and only if their components are distinguishable in at least one dimension.

III. RANDOM ZERO-ERROR LIST CODES

Fix a sequence of probability mass functions \((P_n(x^n))_{n=1}^{\infty}\), where \(P_n\) is defined over \(\mathcal{X}^n\). Our aim here is to study the performance of the sequence of random codes \(F_n : [M_n] \to \mathcal{X}^n\) over the channel \((\mathcal{X}, W(y|x), \mathcal{Y})\), where
\[
F_n(1), \ldots, F_n(M_n)
\]
are \(M_n\) i.i.d. random variables, and
\[
\forall m \in [M_n]: \Pr \{F_n(m) = x^n\} := P_n(x^n).
\]
We seek to find conditions on the sequence \((M_n)_{n=1}^{\infty}\) such that
\[
\lim_{n \to \infty} \Pr \{F_n \text{ is an } (M_n, n, L) \text{ zero-error list code for } W\} = 1.
\]
Theorem 4 below, provides the desired conditions. The conditions rely on a collection of functions of the pair \((G^n(W), P_n)\), denoted by
\[
(\theta^{(t)}_{L+1}(G^n(W), P_n))_{t=1}^{L+1},
\]
which we next define.

Consider a hypergraph \(G = (V, E)\). For any positive integer \(k\), let \(v[k] = (v_1, \ldots, v_k)\) denote an element of \(V^k\). For all \(v[k] \in V^k\) and every nonempty subset \(S \subseteq [k]\), define \(v_S := (v_j)_{j \in S}\). Let \(P\) be a probability mass function on \(V\) and set
\[
P(v_S) := \prod_{j \in S} P(v_j).
\]
In addition, for each positive integer \(k \geq 2\), define the mapping \(\sigma_k : V^k \to 2^V\) as
\[
\sigma_k(v[k]) := \{v_1, \ldots, v_k\}.
\]
In words, $\sigma_k$ maps each vector $v_{[k]} \in \mathcal{V}^k$ to the set containing its distinct components. For example, if $v_{[k]} = (v, \ldots, v)$ for some $v \in \mathcal{V}$, then $\sigma_k(v_{[k]}) = \{v\}$. When the value of $k$ is clear from context, we denote $\sigma_k$ with $\sigma$.

We next define functions of the pair $(G, P)$ that are instrumental in characterizing the performance of random codebooks over channels with zero-error constraints. For every positive integer $L$, define the quantity $I_{L+1}(G, P)$ as

$$I_{L+1}(G, P) := -\frac{1}{L} \log \sum_{v_{[L+1]}: \sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[L+1]}),$$

where $\log$ is the binary logarithm. Note that in (5),

$$\sum_{v_{[L+1]}: \sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[L+1]})$$

equals the probability of independently selecting $L + 1$ vertices of $G$, with replacement, that are indistinguishable. The negative sign in (5) results in the nonnegativity of $I_{L+1}(G, P)$; division by $L$, as we show in Proposition 3, makes it comparable to the Rényi entropy of order $L + 1$ [14], which is defined as

$$H_{L+1}(P) := -\frac{1}{L} \log \sum_{v \in \mathcal{V}} (P(v))^{L+1}.$$ 

We now define the sequence of functions $(\theta_{L+1}^{(n)}(G, P))_{n \in [L+1]}$. This sequence arises from the application of a second moment bound in the proof of Theorem [4], given in Subsection [V-D]. Set $\theta_{L+1}^{(L+1)}(G, P) := I_{L+1}(G, P)$, and for $\ell \in [L]$, let

$$\theta_{L+1}^{(\ell)}(G, P) := 2I_{L+1}(G, P) + \frac{1}{L} \log \sum_{v_{[\ell]}} P(v_{[\ell]}) \left[ \sum_{v_{[\ell+1:L+1]}: \sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[\ell+1:L+1]}) \right].$$

The following proposition describes a number of properties that the sequence $(\theta_{L+1}^{(\ell)}(G, P))_{\ell \in [L+1]}$ satisfies.

**Proposition 3.** For every hypergraph $G = (\mathcal{V}, \mathcal{E})$, probability mass function $P$ on $\mathcal{V}$, and positive integer $L$, the following statements hold.

(i) For all $\ell \in [L+1]$,

$$0 \leq \theta_{L+1}^{(\ell)}(G, P) \leq I_{L+1}(G, P).$$

(ii) We have

$$0 \leq I_{L+1}(G, P) \leq H_{L+1}(P).$$

Let $\text{supp}(P)$ denote the support of $P$. Then

$$I_{L+1}(G, P) = 0 \iff \forall e \subseteq \text{supp}(P): (2 \leq |e| \leq L + 1 \implies e \notin \mathcal{E}).$$

$$I_{L+1}(G, P) = H_{L+1}(P) \iff \forall e \subseteq \text{supp}(P): (2 \leq |e| \leq L + 1 \implies e \in \mathcal{E}).$$

(iii) For every positive integer $n \geq 2$, define the probability mass function $P^n$ on $\mathcal{V}^n$ as

$$\forall v^n \in \mathcal{V}^n: P^n(v^n) := \prod_{t \in [n]} P(v_t).$$

Then for all $\ell \in [L+1]$,

$$\theta_{L+1}^{(\ell)}(G^n, P^n) = n\theta_{L+1}^{(\ell)}(G, P).$$
where \( G^n \) is the \( n \)th co-normal power of \( G \) defined in Section II.

For \( L = 1 \), \( I_{L+1}(G, P) \) has further properties which we discuss in Appendix A.

We next state our main result which provides upper and lower bounds on the cardinality of a randomly generated codebook that has zero error.

**Theorem 4.** Consider a channel \( (X, W(y|x), Y) \) and a sequence of probability mass functions \( (P_n(x^n))_{n=1}^{\infty} \). If

\[
\lim_{n \to \infty} M_n^{L+1} 2^{-L I_{L+1}(G^n(W), P_n)} = 0, \tag{7}
\]

then

\[
\lim_{n \to \infty} \Pr \{ F_n \text{ is an } (M_n, n, L) \text{ zero-error list code} \} = 1. \tag{8}
\]

Conversely, assuming (8), then for some \( \ell \in [L+1] \),

\[
\lim_{n \to \infty} M_n^{\ell} 2^{-L \theta_{L+1}(G^n(W), P_n)} = 0. \tag{9}
\]

In Theorem 4 if a channel \( W \) and a sequence of probability mass functions \( (P_n(x^n))_{n=1}^{\infty} \) satisfy

\[
\max_{\ell \in [L+1]} 1 \leq \frac{1}{\ell} \theta_{L+1}(G^n(W), P_n) = \frac{1}{L+1} I_{L+1}(G^n(W), P_n), \tag{10}
\]

for sufficiently large \( n \), then (7), in addition to being sufficient for (8), is necessary as well. In the next corollary, we present a sufficient condition under which (10) holds. To describe this condition precisely, we require the next definition.

Consider a hypergraph \( G = (V, E) \) and a probability mass function \( P \) on \( V \). Let \( V_P = \text{supp}(P) \) and \( E_P \subseteq E \) be the set of all edges whose vertices lie in \( V_P \). We then refer to the hypergraph \( G_P := (V_P, E_P) \) as the subhypergraph of \( G \) induced by \( P \). For a fixed \( n \), a sufficient condition for (10) to hold is for the subhypergraph of \( G^n(W) \) induced by \( P_n \) be complete multipartite. (Recall definition from Section II.) This results in the next corollary. The proof of this corollary, together with the sufficient condition for (10), appears in Subsection IV-E.

**Corollary 5.** Consider a channel \( (X, W(y|x), Y) \) and a sequence of probability mass functions \( (P_n(x^n))_{n=1}^{\infty} \). If

for sufficiently large \( n \), the subhypergraph of \( G^n(W) \) induced by \( P_n \) is complete multipartite, then

\[
\lim_{n \to \infty} \Pr \{ F_n \text{ is an } (M_n, n, L) \text{ zero-error list code} \} = 1 \iff \lim_{n \to \infty} M_n^{L+1} 2^{-L I_{L+1}(G^n(W), P_n)} = 0.
\]

One scenario where the sufficient condition of Corollary 5 holds automatically for all \( n \geq 1 \) is when \( G(W) \) is complete multipartite. This is stated in the next lemma.

**Lemma 6.** If \( G \) is a complete multipartite hypergraph, then for all \( n \geq 2 \), so is \( G^n \).

In the case where \( G(W) \) is not complete multipartite, in order to obtain a simpler version of Theorem 4 we assume that the codebook distribution is not only i.i.d. across messages, but also over time. In addition, we assume that the message set cardinality grows exponentially in the blocklength. Formally, we fix a probability mass function
on $\mathcal{X}$ and a rate $R \geq 0$. Then, in Theorem 4, by setting $P_n := P^n$ and $M_n := \lfloor 2^{nR} \rfloor$ for all positive integers $n$, and applying Parts (i) and (iii) of Proposition 3, we get the following corollary.

**Corollary 7.** Consider a channel $(\mathcal{X}, W(y|x), \mathcal{Y})$ and a probability mass function $P$ on $\mathcal{X}$. If

$$ R < \frac{L}{L+1} I_{L+1}(G, P), $$

then

$$ \lim_{n \to \infty} \Pr \{ F_n \text{ is an } (2^{nR}, n, L) \text{ zero-error list code for } W \} = 1. \quad (11) $$

Conversely, if (11) holds, then

$$ R < LI_{L+1}(G, P). \quad (12) $$

Note that in Corollary 7, if $G(W)$ is complete multipartite, as in the next example, then using Corollary 5, the upper bound (12) can be improved to

$$ R < \frac{L}{L+1} I_{L+1}(G, P). $$

We next apply Corollary 5 to the identity channel $W = (\mathcal{X}, 1\{y = x\}, \mathcal{X})$. Per the informal discussion in the Introduction, applying our result to this channel gives the exact asymptotic behavior of the probability of coinciding birthdays in the birthday problem. We now formalize this connection.

Note that every subset $e$ of $\mathcal{X}$ with $|e| \geq 2$ is an edge of $G(W)$. Thus for $n \geq 2$, every $e \subseteq \mathcal{X}^n$ with $|e| \geq 2$ is an edge of $G^n(W)$. Therefore, for distinct messages $m_1, \ldots, m_{L+1} \in [M_n]$, we have $F_n(m_1) = \cdots = F_n(m_{L+1})$ if and only if

$$ (F_n(m_1), \ldots, F_n(m_{L+1})) \text{ is not an edge in } G^n(W). $$

Hence Proposition 2 implies that (15) holds if and only if

$$ \lim_{n \to \infty} \Pr \{ F_n \text{ is an } (M_n, n, L) \text{ zero-error list code} \} = 1. \quad (13) $$

Now from Corollary 5, it follows that (13) holds if and only if

$$ \lim_{n \to \infty} M_n^{L+1} 2^{-LH_{L+1}(P_n)} = 0. \quad (14) $$

This proves the next corollary.

**Corollary 8.** Fix an integer $L \geq 1$, a finite set $\mathcal{X}$, and a sequence of probability mass functions $(P_n(x^n))_{n=1}^{\infty}$. For each $n$, let $F_n : [M_n] \to \mathcal{X}^n$ be a random mapping with i.i.d. values and distribution $P_n(x^n)$; that is,

$$ \forall m \in [M_n]: \Pr \{ F_n(m) = x^n \} = P_n(x^n). $$

Then we have

$$ \lim_{n \to \infty} \Pr \{ \exists m_1, \ldots, m_{L+1} \in [M_n]: F_n(m_1) = \cdots = F_n(m_{L+1}) \} = 0 \quad (15) $$

if and only if

$$ \lim_{n \to \infty} M_n^{L+1} 2^{-LH_{L+1}(P_n)} = 0. $$
In words, to guarantee the absence of collisions of \((L + 1)\)-th order, the population size \(M_n\) must be negligible compared to \(2^{\frac{\binom{L+1}{2}}{H_{L+1}(P_n)}}\).

IV. Proofs

In this section, we provide detailed proofs of our results.

A. Proof of Proposition

For each of the two cases, one direction is proved before the statement of the proposition in Section II. Here we prove the reverse direction of each case.

Suppose \(G = (V, E)\) is a hypergraph where the superset of every edge is an edge. We define a channel \((X, W(y|x), Y)\) such that \(G = G(W)\). Set

\[
X := V, \quad Y := 2^V \setminus E.
\]

Note that \(Y\) is not empty, since by definition, each edge has cardinality at least two. Define \(W\) as

\[
W(y|x) := \frac{1\{x \in y\}}{|\{\bar{y} \in Y : x \notin \bar{y}\}|}.
\]

Then for every subset \(e \subseteq X\) and every \(y \in Y\),

\[
\prod_{x \in e} W(y|x) = \prod_{x \in e} \frac{1\{x \in y\}}{|\{\bar{y} \in Y : x \notin \bar{y}\}|} \neq 0 \tag{16}
\]

if and only if \(e \subseteq y\). Since by definition of \(Y\), \(y\) is not an edge, and by assumption, the superset of every edge is an edge, (16) holds for some \(y \in Y\) if and only if \(e \notin E\). Thus \(G = G(W)\).

Next assume \(G = (V, E)\) is complete multipartite; that is, there exists a partition \(\{I_j\}_{j=1}^k\) of \(V\) such that each \(I_j\) is an independent set, and for every subset \(e \subseteq V\), either \(e \in E\), or \(e \subseteq I_j\) for some \(1 \leq j \leq k\). For this hypergraph, we define a deterministic channel

\[
(X, W(y|x), Y)
\]

such that \(G = G(W)\). Set

\[
X := V = \bigcup_{j \in [k]} I_j, \quad Y := [k],
\]

and define \(W\) as

\[
W(y|x) := 1\{x \in I_y\}.
\]

Then for every subset \(e \subseteq X\) and every \(y \in Y\),

\[
\prod_{x \in e} W(y|x) = \prod_{x \in e} 1\{x \in I_y\} \neq 0 \tag{17}
\]
if and only if \( e \subseteq I_y \). By assumption, however, every \( e \in \mathcal{E} \) is either in \( \mathcal{I}_y \) or is a subset of an independent set \( \mathcal{I}_y \) for some \( y \in Y \). Thus (17) holds for some \( y \in Y \) if and only if \( e \notin \mathcal{E} \). This completes the proof.

### B. Proof of Proposition 2

Let \((f, g)\) be an \((M, L)\) zero-error list code for channel \(W\). If \([M]\) has a subset of cardinality \(L+1\), say \(\{m_{\ell}\}_{\ell=1}^{L+1}\), such that \(\{f(m_{\ell})\}_{\ell=1}^{L+1}\) is not an edge in \(G(W)\), then for some \(y \in Y\),

\[
\prod_{\ell \in [L+1]} W(y|f(m_{\ell})) > 0.
\]

Thus for every \(\ell \in [L+1]\), \(W(y|f(m_{\ell})) > 0\), which implies \(m_{\ell} \in g(y)\). Thus \(g(y)\) contains at least \(L+1\) distinct elements, which is a contradiction.

Conversely, suppose we have an encoder \(f: [M] \to X\) that maps every \((L+1)\)-subset of \([M]\) onto an edge of \(G(W)\). For each \(y \in Y\), define the set

\[
\mathcal{M}_y := \left\{ m \in [M] | W(y|f(m)) > 0 \right\}.
\]

Suppose for some \(y^* \in Y\), \(|\mathcal{M}_{y^*}| > L\). Then \(\mathcal{M}_{y^*}\) has a subset \(A\) of cardinality \(L+1\). By assumption, \(f\) maps \(A\) to an edge of \(G(W)\), which implies that for all \(y \in Y\), including \(y = y^*\),

\[
\prod_{m \in A} W(y|f(m)) = 0.
\]

This contradicts the definition of \(\mathcal{M}_{y^*}\). Thus for all \(y \in Y\), \(|\mathcal{M}_y| \leq L\).

Now if we define the decoder as

\[
\forall y \in Y: g(y) = \mathcal{M}_y,
\]

then the pair \((f, g)\) is an \((M, L)\) zero-error list code and the proof is complete.

### C. Proof of Proposition 3

(i) We prove the nonnegativity of \(\theta_{L+1}(G, P)\) first for \(\ell = L+1\) and then for arbitrary \(\ell \in [L]\). Recall that \(\theta_{L+1}(G, P) = I_{L+1}(G, P)\). We have

\[
\sum_{v_{[L+1]}:\sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[L+1]}) \leq \sum_{v_{[L+1]} \in \mathcal{V}^{L+1}} P(v_{[L+1]}) = \left( \sum_{v \in \mathcal{V}} P(v) \right)^{L+1} = 1,
\]

which implies

\[
\theta_{L+1}(G, P) = I_{L+1}(G, P) = -\frac{1}{L} \log \sum_{v_{[L+1]}:\sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[L+1]}) \geq 0.
\]

For \(\ell \in [L]\), rewrite \(\theta_{L+1}(G, P)\) as

\[
\theta_{L+1}(G, P) = \frac{1}{L} \log \frac{\sum_{v^{(\ell)}} P(v_{[\ell]}) \left[ \sum_{v_{[L+1]}:\sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[L+1]}) \right]^2}{\sum_{v_{[L+1]}:\sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[L+1]})^2}.
\]

Note that

\[
\sum_{v^{(\ell)} \in \mathcal{V}^{\ell}} P(v_{[\ell]}) = \left( \sum_{v \in \mathcal{V}} P(v) \right)^\ell = 1.
\]
Therefore, by the Cauchy-Schwarz inequality,
\[
\sum_{v[ℓ]} P(v[ℓ]) \left[ \sum_{v[ℓ+1:L]+1:σ(v[ℓ+1])∉E} P(v[ℓ+1:L+1]) \right] \geq \left[ \sum_{v[ℓ]} P(v[ℓ]) \sum_{v[ℓ+1:L]+1:σ(v[ℓ+1])∉E} P(v[ℓ+1:L+1]) \right]^2 \geq \left[ \sum_{v[ℓ]+1:L+1:σ(v[ℓ+1])∉E} P(v[ℓ+1:L+1]) \right]^2,
\]
which implies \(θ^{(ℓ)}_{L+1}(G, P) ≥ 0\).

We next prove the upper bound on \(θ^{(ℓ)}_{L+1}(G, P)\). Note that
\[
\sum_{v[ℓ]} P(v[ℓ]) \left[ \sum_{v[ℓ+1:L]+1:σ(v[ℓ+1])∉E} P(v[ℓ+1:L+1]) \right]^2 \leq \sum_{v[ℓ]} P(v[ℓ]) \left[ \sum_{v[ℓ+1:L]+1:σ(v[ℓ+1])∉E} P(v[ℓ+1:L+1]) \right] = 2^{-LI_{L+1}(G, P)}.
\]
Thus
\[
θ^{(ℓ)}_{L+1}(G, P) ≤ 2I_{L+1}(G, P) − I_{L+1}(G, P) = I_{L+1}(G, P).
\]

(ii) The inequality \(I_{k}(G, P) ≥ 0\) is proved in (i). Equality holds if and only if
\[
∀ v_{[L+1]} ∈ (\text{supp}(P))^{L+1}: σ(v_{[L+1]}) ∉ E,
\]
which is equivalent to
\[
∀ e ⊆ \text{supp}(P): (2 ≤ |e| ≤ L + 1 ⟹ e ∉ E).
\]

We next prove the upper bound on \(I_{L+1}(G, P)\). Since each edge of \(G\) has cardinality at least two, for all \(v ∈ V\), \(\{v\} ∉ E\). Thus
\[
\sum_{v_{[L+1]:σ(v_{[L+1]})∉E}} P(v_{[L+1]}) ≥ \sum_{v ∈ V} (P(v))^{L+1},
\]
which implies
\[
I_{L+1}(G, P) ≤ H_{L+1}(P),
\]
where \(H_{L+1}(P)\) is the Rényi entropy of order \(L + 1\). Equality holds if and only if
\[
∀ v_{[L+1]} ∈ (\text{supp}(P))^{L+1}: (v_1 = · · · = v_{L+1}) ∨ (σ(v_{[L+1]}) ∈ E),
\]
which is equivalent to
\[
∀ e ⊆ \text{supp}(P): (2 ≤ |e| ≤ L + 1 ⟹ e ∈ E).
\]

(iii) Fix a positive integer \(n\). Let \(E_n\) denote the set of edges of \(G^n\). Let \(v^n_{[L+1]}\) denote the vector
\[
v^n_{[L+1]} := (v^n_1, . . . , v^n_{L+1}),
\]
and \(σ_{L+1}(v^n_{[L+1]})\) denote the set
\[
σ_{L+1}(v^n_{[L+1]}) := \{v^n_1, . . . , v^n_{L+1}\}.
\]
Furthermore, let \(S ⊆ V^{L+1}\) denote the set
\[
S := \{v_{[L+1]} | σ_{L+1}(v_{[L+1]}) ∉ E\}.
\]
Note that for each \( v_{[L+1]}^n, \sigma_{L+1}(v_{[L+1]}^n) \notin \mathcal{E}_n \) if and only if
\[
\forall t \in [n] : \{v_t, \ldots, v_{(L+1)t}\} \notin \mathcal{E}.
\]
Thus
\[
\{v_{[L+1]}^n | \sigma_{L+1}(v_{[L+1]}^n) \notin \mathcal{E}_n \} = \mathcal{S}_n,
\]
which implies
\[
\sum_{v_{[L+1]}^n : \sigma_{L+1}(v_{[L+1]}^n) \notin \mathcal{E}_n} P^n(v_{[L+1]}^n) = \sum_{v_{[L+1]}^n \in \mathcal{S}_n} P^n(v_{[L+1]}^n) = \sum_{v_{[L+1]}^n \in \mathcal{S}_n} \prod_{t \in [n]} P(v_{[k]t}) = \prod_{t \in [n]} \sum_{v_{[L+1]}^n \in \mathcal{S}} P(v_{[L+1]}^n) = \left( \sum_{v_{[L+1]}^n \in \mathcal{S}} P(v_{[L+1]}^n) \right)^n,
\]
where in (18), \( v_{[L+1]}^n = (v_{1t}, \ldots, v_{(L+1)t}) \). Therefore,
\[
I_{L+1}(G^n, P^n) = nI_{L+1}(G, P).
\]

For \( \ell \in [L] \), we can write \( \theta_{L+1}^{(\ell)}(G^n, P^n) \) as
\[
\theta_{L+1}^{(\ell)}(G^n, P^n) = 2I_{L+1}(G^n, P^n) + \frac{1}{\ell} \log \sum_{(v_{[\ell]}^n, v_{[\ell+1:L+1]}^n) : \subseteq \mathcal{S}_n : \subseteq \mathcal{S}_n} P^n(v_{[\ell]}^n) P^n(v_{[\ell+1:L+1]}^n) P^n(v_{[\ell+1:L+1]}^n).
\]

Using a similar argument as above, it follows that for all \( \ell \in [L] \),
\[
\theta_{L+1}^{(\ell)}(G^n, P^n) = n\theta_{L+1}^{(\ell)}(G, P).
\]

### D. Proof of Theorem 2

We start by finding upper and lower bounds on the probability that a random mapping from \([M]\) to \(\mathcal{X}\) is an \((M, L)\) zero-error list code for the channel \(W^n\). The theorem then follows from applying our bounds to the channel \(W^n\) for every positive integer \(n\).

Consider the random mapping \(F: [M] \rightarrow \mathcal{X}\), where \((F(m))_{m \in [M]}\) is a collection of i.i.d. random variables and each \(F(m)\) has distribution
\[
\Pr \{F(m) = x\} := P(x).
\]

For every \(S \in \binom{[M]}{L+1}\), define the random variable \(Z_S\) as
\[
Z_S := 1\{\{F(m)\}_{m \in S} \notin \mathcal{E}\};
\]

\(^2\)Without loss of generality, we may assume that \(M > L\), since if \(M \leq L\), then every mapping \(f: [M] \rightarrow \mathcal{X}\) is an \((M, L)\) zero-error list code.
that is, $Z_S$ is the indicator of the event that $\{F(m)\}_{m \in S}$ is not an edge of the distinguishability hypergraph $G(W)$. Let 

$$Z := \sum_{S \in \binom{[M]}{L+1}} Z_S.$$ 

Note that by Proposition 2, $F$ is an $(M, L)$ zero-error list code if and only if $Z = 0$. The rest of the proof consists of computing a lower and an upper bound for $\Pr\{Z = 0\}$.

**Lower Bound.** By Markov’s inequality,

$$\Pr\{ F \text{ is an } (M, L) \text{ zero-error list code} \} = \Pr\{Z = 0\} = 1 - \Pr\{Z \geq 1\} \geq 1 - \mathbb{E}[Z]. \quad (19)$$

For any $S \in \binom{[M]}{L+1}$,

$$\mathbb{E}[Z_S] = \sum_{x[L+1]: \sigma(x[L+1]) \notin E} P(x[L+1]) = 2^{-L L_{L+1}(G, P)}, \quad (20)$$

By linearity of expectation,

$$\mathbb{E}[Z] = \binom{M}{L+1} 2^{-L L_{L+1}(G, P)}. \quad (21)$$

Combining (19) and (21) gives

$$\Pr\{Z = 0\} \geq 1 - \binom{M}{L+1} 2^{-L L_{L+1}(G, P)} \geq 1 - M^{L+1} 2^{-L L_{L+1}(G, P)},$$

where the last inequality follows from the fact that

$$\binom{M}{L+1} \leq M^{L+1}.$$ 

**Upper Bound.** We apply the second moment method. By the Cauchy-Schwarz inequality,

$$\mathbb{E}[Z] = \mathbb{E}[Z 1_{\{Z \geq 1\}}] \leq \sqrt{\mathbb{E}[Z^2] \times \Pr\{Z \geq 1\}},$$

thus

$$\Pr\{Z \geq 1\} \geq \frac{(\mathbb{E}[Z])^2}{\mathbb{E}[Z^2]}$$

or

$$\Pr\{Z = 0\} \leq 1 - \frac{(\mathbb{E}[Z])^2}{\mathbb{E}[Z^2]}.$$ 

\[^{3}\text{For results regarding the distribution of } Z \text{ in the classical birthday problem scenario, we refer the reader to the work of Arratia, Goldstein, and Gordon [15], [16]. A direct application of the bounds in [15], [16] to } \Pr\{Z = 0\} \text{ leads to weaker results than those we present here.}\]
To evaluate the upper bound on $\Pr\{ Z = 0 \}$, we calculate $\mathbb{E}[Z^2]$. We have

$$
Z^2 = \left[ \sum_{S \in \binom{[M]}{L+1}} Z_S \right]^2 = \sum_{S \in \binom{[M]}{L+1}} Z_S^2 + \sum_{S, S' \in \binom{[M]}{L+1}, S \neq S'} Z_S Z_{S'} , \\
= \sum_{S \in \binom{[M]}{L+1}} Z_S + \sum_{\ell=0}^{L} \sum_{S, S' \in \binom{[M]}{L+1} |S \cap S'| = \ell} Z_S Z_{S'} . 
$$

(22)

For all $\ell \in \{0, 1, \ldots, L\}$, fix sets $S_{\ell}, S'_{\ell} \in \binom{[M]}{L+1}$ such that $|S_{\ell} \cap S'_{\ell}| = \ell$. When $\ell \in [L]$, $(F(m))_{m \in S_{\ell}}$ and $(F(m))_{m \in S'_{\ell}}$ are independent given $(F(m))_{m \in S_{\ell} \cap S'_{\ell}}$. Thus for $\ell \in [L]$,

$$
\mathbb{E}[Z_{S_{\ell}} Z_{S'_{\ell}}] = \sum_{x[\ell]} P(x[\ell]) \left[ \sum_{\substack{x[\ell+1:L+1]: \sigma(x[\ell:L+1]) \in E |S_{\ell}\cap S'_{\ell}|}} P(x[\ell+1:L+1]) \right]^2 \\
= 2^{L(\theta_{L+1}^{(\ell)}(G,P)-2I_{L+1}(G,P))} , 
$$

(23)

where in (23), we use the definition of $\theta_{L+1}^{(\ell)}(G,P)$ given by (6). When $\ell = 0$, $Z_{S_{\ell}}$ and $Z_{S'_{\ell}}$ are independent. Thus by (20),

$$
\mathbb{E}[Z_{S_0} Z_{S_0}] = (\mathbb{E}[Z_{S_0}])^2 = 2^{-2I_{L+1}(G,P)} . 
$$

(24)

Equations (22), (23), and (24) together imply

$$
\mathbb{E}[Z^2] = \binom{M}{L+1} 2^{-L I_{L+1}(G,P)} + \binom{M}{0, L+1, L+1} 2^{-2L I_{L+1}(G,P)} \\
+ \sum_{\ell=1}^{L} \binom{M}{\ell, L+1 - \ell, L+1 - \ell} 2^{L(\theta_{L+1}^{(\ell)}(G,P)-2I_{L+1}(G,P))} ,
$$

(25)

where in (25), for $\ell \in \{0, 1, \ldots, L\}$, the quantity

$$
\binom{M}{\ell, L+1 - \ell, L+1 - \ell} := \binom{M}{\ell} \binom{M-\ell}{L+1-\ell} \binom{M-L-1}{L+1-\ell} ,
$$

equals the number of pairs $(S, S')$, where $S, S' \in \binom{[M]}{L+1}$ and $|S \cap S'| = \ell$. Combining (21) with (25) now gives

$$
\frac{\mathbb{E}[Z^2]}{(\mathbb{E}[Z])^2} = \binom{M}{L+1}^{-1} 2^{L I_{L+1}(G,P)} + \binom{M}{L+1}^{-2} \binom{M}{0, L+1, L+1} \\
+ \binom{M}{L+1}^{-2} \sum_{\ell=1}^{L} \binom{M}{\ell, L+1 - \ell, L+1 - \ell} 2^{L(\theta_{L+1}^{(\ell)}(G,P))} \\
\leq \binom{M}{L+1}^{-1} 2^{L I_{L+1}(G,P)} + \sum_{\ell=1}^{L} \binom{L+1}{\ell} \binom{M}{\ell}^{-1} 2^{L(\theta_{L+1}^{(\ell)}(G,P))} .
$$

(26)
The inequality in (26) follows from the fact that for each \( \ell \in \{0, 1, \ldots, L\}, \)
\[
\left( \frac{M}{L+1} \right)^{-2} \left( \frac{M}{\ell, L+1-\ell, L+1-\ell} \right) = \left( \frac{L+1}{\ell} \right)^2 \left( \frac{M}{\ell} \right)^{-1} \times \frac{(M-L-1)!(M-L-1)!}{(M-\ell)!(M-2L-2+\ell)!} \\
= \left( \frac{L+1}{\ell} \right)^2 \left( \frac{M}{\ell} \right)^{-1} \prod_{j=\ell}^{L} \left( \frac{M-L-1+\ell-j}{M-j} \right) \\
\leq \left( \frac{L+1}{\ell} \right)^2 \left( \frac{M}{\ell} \right)^{-1}.
\]

This completes the proof of the upper bound.

The asymptotic result, as stated in Theorem 4, follows from applying, for every \( \ell \in [L+1] \), the inequality
\[
\left( \frac{M}{\ell} \right) \geq \left( \frac{M}{\ell} \right)^{\ell}.
\]

### E. Proof of Corollary 5

Corollary 5 follows from applying the next lemma to the hypergraph \( G^n(W) \) for sufficiently large \( n \).

**Lemma 9.** Let \( G = (\mathcal{V}, \mathcal{E}) \) be a hypergraph and let \( P \) be a distribution on \( \mathcal{V} \). If \( G_P \) is complete multipartite, then
\[
\max_{\ell \in [L+1]} \frac{1}{\ell} \theta^{(\ell)}_{L+1}(G, P) = \frac{1}{L+1} I_{L+1}(G, P). \tag{27}
\]

**Proof.** Since \( G_P = (\mathcal{V}_P, \mathcal{E}_P) \) is complete multipartite, there exists a partition of \( \mathcal{V}_P \) consisting of independent sets. Let \( \{I_j\}_{j=1}^k \) denote such a partition. Define the distribution \( P^* \) on \([k]\) as
\[
P^*(j) := \sum_{v \in I_j} P(v).
\]

In words, \( P^*(j) \) is the weight assigned to the independent set \( I_j \) by \( P \). Since \( G_P \) is a complete multipartite hypergraph, \( \sigma(v_{[L+1]}) \notin \mathcal{E} \) if and only if there exists some \( j \) such that \( \sigma(v_{[L+1]}) \subseteq I_j \). Thus
\[
I_{L+1}(G_P, P) = -\frac{1}{L} \log \sum_{v_{[L+1]}: \sigma(v_{[L+1]}) \notin \mathcal{E}} P(v_{[L+1]}) \\
= -\frac{1}{L} \log \sum_{j=1}^k \sum_{\sigma(v_{[L+1]}) \subseteq I_j} P(v_{[L+1]}) \\
= -\frac{1}{L} \log \sum_{j=1}^k \left( P^*(j) \right)^{L+1} \\
= H_{L+1}(P^*),
\]
where \( H_{L+1} \) denotes the Rényi entropy of order \( L+1 \). Similarly, for all \( \ell \in [L+1] \) we have
\[
\theta^{(\ell)}_{L+1}(G_P, P) = 2H_{L+1}(P^*) - \frac{2L+1-\ell}{L} H_{2L+2-\ell}(P^*). \tag{28}
\]

Using (28), we see that proving (27) is equivalent to showing that for all \( \ell \in [L+1] \),
\[
\left( \sum_{j=1}^k P^*(j)^{2L+2-\ell} \right)^{\frac{1}{2L+2-\ell}} \leq \left( \sum_{j=1}^k P^*(j)^{L+1} \right)^{\frac{1}{L+1}}
\]
which follows from the well-known fact that for all \( p \geq q \), the \( q \)-norm dominates the \( p \)-norm.
Finally, note that for all $\ell \in [L + 1]$,
\[
\theta_{L+1}^{(\ell)}(G_P, P) = \theta_{L+1}^{(\ell)}(G, P).
\]
This completes the proof.

\[\square\]

F. Proof of Lemma \[6\]

Since $G$ is complete multipartite, there exists a partition of its set of vertices, say $\{I_j\}_{j=1}^k$, consisting of independent sets. Next note that for any $n \geq 2$, the set of vertices of $G^n$ is given by
\[
\bigcup_{j_1, \ldots, j_n \in [k]} I_{j_1} \times \cdots \times I_{j_n}.
\]
We show that an arbitrary subset of $V^n$, say $\{v^n_1, \ldots, v^n_\ell\}$, is an edge in $G^n$ if and only if
\[
\forall j_1, \ldots, j_n \in [k]: \{v^n_1, \ldots, v^n_\ell\} \not\subseteq I_{j_1} \times \cdots \times I_{j_n}.
\]
By definition, $\{v^n_1, \ldots, v^n_\ell\}$ is an edge in $G^n$ if and only if for some $t \in [n]$, $\{v_{1t}, \ldots, v_{\ell t}\}$ is an edge in $G$. Since $G$ is complete multipartite, the latter condition holds if and only if
\[
\exists i \in [n] \text{ such that } \forall j \in [k]: \{v_{1i}, \ldots, v_{\ell i}\} \not\subseteq I_j,
\]
which is equivalent to (29).

V. Conclusion

From Shannon’s random coding argument \[2\] it follows that if the rate of a randomly generated codebook is less than the input-output mutual information, the probability that the codebook has small probability of error goes to one as the blocklength goes to infinity. In this work, we find necessary and sufficient conditions on the rate so that the probability that a randomly generated codebook has zero probability of error goes to one as the blocklength goes to infinity. We further show that this result extends the classical birthday problem to an information-theoretic setting and provides an intuitive meaning for Rényi entropy.

APPENDIX A

Properties of $I_2(G, P)$

In this appendix, we describe two properties of $I_2(G, P)$. In the first part, we state the Motzkin-Straus theorem \[17\], which gives the maximum of $I_2(G, P)$ over all distributions $P$ for a fixed graph $G$. In the second part, we show that $I_2(G, P)$ is always less than or equal to Körner’s graph entropy \[18\].

A. The Motzkin-Straus Theorem

Consider a graph $G = (V, E)$. Motzkin and Straus \[17\] prove that
\[
\max_P I_2(G, P) = \log \omega(G),
\]
where the maximum is over all distributions $P$ defined on $\mathcal{V}$, and $\omega(G)$ is the cardinality of the largest clique in $G$. An implication of this result is Turán’s graph theorem [19], which states that

$$\omega(G) \geq \frac{|\mathcal{V}|^2}{|\mathcal{V}|^2 - 2|\mathcal{E}|}. \tag{30}$$

To see this, let $P$ be the uniform distribution on $\mathcal{V}$. Then

$$I_2(G, P) = -\sum_{v, v' : (v, v') \notin \mathcal{E}} \frac{1}{|\mathcal{V}|^2} = \log \frac{|\mathcal{V}|^2}{|\mathcal{V}|^2 - 2|\mathcal{E}|},$$

and (30) follows by the Motzkin-Straus theorem. We remark that extensions of the Motzkin-Straus theorem to hypergraphs are presented in [20]–[22].

From Proposition 3, Part (ii) it follows that for any distribution $P$ defined on a set $\mathcal{V}$,

$$\max_G I_2(G, P) = H_2(P), \tag{31}$$

where the maximum is over all graphs $G = (\mathcal{V}, \mathcal{E})$. In (31), the maximum is achieved when $G$ is the complete graph on the support of $P$.

B. Relation with Körner’s Graph Entropy

Consider a graph $G$ with vertex set $\mathcal{X}$. Let $P$ be a probability distribution on $\mathcal{X}$ and $\mathcal{Y} \subseteq 2^{\mathcal{X}}$ be the set of all maximal independent subgraphs of $G$. Let $\Delta(G, P)$ denote the set of all probability distributions $P(x, y)$ on $\mathcal{X} \times \mathcal{Y}$ whose marginal on $\mathcal{X}$ equals $P(x)$, and

$$\Pr \{ X \in Y \} = \sum_{(x, y) : x \in y} P(x, y) = 1.$$

For the graph $G$ and probability distribution $P$, Körner’s graph entropy [18] is defined by

$$H_1(G, P) = \min_{\Delta(G, P)} I(X; Y). \tag{32}$$

Our aim is to define $H_2(G, P)$ in a similar manner to how $H_2(P)$, the Rényi entropy of order 2, is defined. One way to accomplish this task is through the use of Jensen’s inequality. Applying Jensen’s inequality to Shannon entropy gives

$$H_1(P) = -\sum_x P(x) \log P(x) \geq -\log \left( \sum_x (P(x))^2 \right) = H_2(P).$$

Analogously, applying Jensen’s inequality to the mutual information in (32) gives

$$I(X; Y) = -\sum_{(x, y) : x \in y} P(x, y) \log \frac{P(x, y)P(y)}{P(x, y)} \geq -\log \left( \sum_{(x, y) : x \in y} P(x)P(y) \right).$$
Thus we define $H_2(G, P)$ as

$$H_2(G, P) = \min_{\Delta(G, P)} \left( -\log \left( \sum_{(x, y): x \in y} P(x)P(y) \right) \right)$$

$$\leq H_1(G, P).$$

Our next proposition relates $H_2(G, P)$ and $I_2(G, P)$.

**Proposition 10.** For any graph $G$ and any probability distribution $P$ defined on its vertices, $I_2(G, P) \leq H_2(G, P)$.

**Proof.** Let $P(x, y) \in \Delta(G, P)$. Then

$$\sum_{x \in y} P(x)P(y) = \sum_{x, x'} P(x)P(x') \sum_{y} P(y|x') 1\{x, x' \in y\}. \quad (33)$$

Since every $y$ is an independent subgraph of $G$, if $x, x' \in y$, then $(x, x') \notin \mathcal{E}$. Thus

$$1\{x, x' \in y\} \leq 1\{(x, x') \notin \mathcal{E}\},$$

which implies

$$\sum_{y} P(y|x') 1\{x, x' \in y\} \leq 1\{(x, x') \notin \mathcal{E}\}.$$

By (33), we have

$$\sum_{x \in y} P(x)P(y) \leq \sum_{x, x'} P(x)P(x') 1\{(x, x') \notin \mathcal{E}\}.$$

Calculating the logarithm of both sides and maximizing the left hand side over $\Delta(G, P)$ gives $H_2(G, P) \geq I_2(G, P)$. \hfill \Box

**APPENDIX B**

**CONNECTION TO RÉNYI’S RESULT**

We next state Rényi’s result [9, Equation (5.3)] in the context of zero-error list codes over the identity channel.

For the identity channel setting described above, fix $\epsilon \in (0, 1)$, positive integer $L \geq 1$, and probability mass function $P$ on $\mathcal{X}$. For every positive integer $n$ and $x^n \in \mathcal{X}^n$, define

$$P_n(x^n) := \prod_{i=1}^{n} P(x_i). \quad (34)$$

For positive integers $M$ and $n$, let $\Phi_{M,n}: [M] \rightarrow \mathcal{X}^n$ be a random mapping with i.i.d. values $\Phi_{M,n}(1), \ldots, \Phi_{M,n}(M)$, each with distribution $P_n(x^n)$. Define $n_{L+1}(M, \epsilon)$ as the least positive integer $n$ for which

$$\Pr \{ \Phi_{M,n} \text{ is an } (M, n, L) \text{ zero-error list code} \} \geq 1 - \epsilon.$$

In [9], Rényi states that

$$\lim_{M \rightarrow \infty} \frac{\log M}{n_{L+1}(M, \epsilon)} = \frac{L}{L + 1} \cdot H_{L+1}(P). \quad (35)$$
We next describe a connection between Rényi’s result and our result concerning the birthday problem. Consider the scenario where the distribution of each codeword is given by (34), the size of the message set equals

\[ M_n := \lfloor 2^{nR} \rfloor \]

for some \( R \geq 0 \), and (13) holds. We show that both (14) and (35) imply

\[ R \leq \frac{L}{L+1} \cdot H_{L+1}(P). \]  

(36)

First note that for \( P_n \) given by (34), (14), together with Part (iii) of Proposition 3, gives

\[ \lim_{n \to \infty} \left[ (L+1) \log \lfloor 2^{nR} \rfloor - nLH_{L+1}(P) \right] = -\infty, \]

which directly leads to (36).

Next we show that (35) implies (36). First note that for all \( n \), \( F_n \) has the same distribution as \( \Phi_{M_n,n} \). In addition, by assumption, for sufficiently large \( n \),

\[ \Pr \{ F_n \text{ is an} (M_n, n, L) \text{ zero-error list code} \} \geq 1 - \epsilon. \]

Therefore, by the definition of \( n^*_L+1 \), we have

\[ n \geq n^*_L+1(M_n, \epsilon). \]

Thus if \( R > 0 \),

\[ \lim_{M \to \infty} \frac{\log M}{n^*_L+1(M, \epsilon)} = \lim_{n \to \infty} \frac{\log \lfloor 2^{nR} \rfloor}{n^*_L+1(\lfloor 2^{nR} \rfloor, \epsilon)} \geq \lim_{n \to \infty} \frac{1}{n} \log \lfloor 2^{nR} \rfloor = R. \]

Applying (35) completes the proof.

ACKNOWLEDGMENTS

The first author thanks Ming Fai Wong for helpful discussions regarding an earlier version of Theorem 4.

REFERENCES

[1] C. E. Shannon, “The zero error capacity of a noisy channel,” IRE Trans. Inf. Theory, vol. 2, no. 3, pp. 8–19, 1956.
[2] ———, “A mathematical theory of communication,” Bell Syst. Tech. J., vol. 27, pp. 379–423, 623–656, 1948.
[3] P. Elias, “Zero error capacity under list decoding,” IEEE Trans. Inf. Theory, vol. 34, no. 5, pp. 1070–1074, 1988.
[4] ———, “Error-correcting codes for list decoding,” IEEE Trans. Inf. Theory, vol. 37, no. 1, pp. 5–12, 1991.
[5] J. Korner and K. Marton, “On the capacity of uniform hypergraphs,” IEEE Trans. Inf. Theory, vol. 36, no. 1, pp. 153–156, 1990.
[6] M. H. Gail, G. H. Weiss, N. Mantel, and S. J. O’Brien, “A solution to the generalized birthday problem with application to allozyme screening for cell culture contamination,” J. Appl. Prob., vol. 16, no. 2, pp. 242–251, 1979.
[7] T. S. Nunnikhoven, “A birthday problem solution for nonuniform birth frequencies,” Am. Stat., vol. 46, no. 4, pp. 270–274, 1992.
[8] C. Stein, “Application of Newton’s identities to a generalized birthday problem and to the Poisson binomial distribution,” Stanford University — Department of Statistics, Tech. Rep. 354, September 1990.
[9] A. Rényi, “On the foundations of information theory,” Rev. Int. Statist. Inst., vol. 33, no. 1, pp. 1–14, 1965.
[10] A. Fujiwara, “Quantum birthday problems: Geometrical aspects of quantum random coding,” IEEE Trans. Inf. Theory, vol. 47, no. 6, pp. 2644–2649, 2001.
[11] A. Joux, Algorithmic Cryptanalysis, 1st ed. Chapman & Hall/CRC, 2009.
[12] M. Bellare and T. Kohno, “Hash function balance and its impact on birthday attacks,” in Advances in Cryptology – EUROCRYPT ’04, C. Cachin and J. Camenisch, Eds. Springer-Verlag, 2004.

4More precisely, (14) gives (36) with strict inequality.
[13] G. Simonyi, “Graph entropy: A survey,” *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, vol. 20, pp. 399–441, 1995.

[14] A. Rényi, “On measures of information and probability,” in *Proc. Fourth Berkeley Symp. on Math. Statist. and Prob.*, vol. 1. University of California Press, 1961, pp. 547–561.

[15] R. Arratia, L. Goldstein, and L. Gordon, “Two moments suffice for Poisson approximations: The Chen-Stein method,” *Ann. Prob.*, vol. 17, no. 1, pp. 9–25, 1989.

[16] ———, “Poisson approximation and the Chen-Stein method,” *Stat. Sci.*, vol. 5, no. 4, pp. 403–434, 1990.

[17] T. S. Motzkin and E. G. Straus, “Maxima for graphs and a new proof of a theorem of Turán,” *Canad. J. Math.*, vol. 17, pp. 533–540, 1965.

[18] J. Körner, “Coding of an information source having ambiguous alphabet and the entropy of graphs,” in *Trans. 6th Prague Conf. Information Theory*. Academia, 1973, pp. 411–425.

[19] M. Aigner, “Turán’s graph theorem,” *Am. Math. Monthly*, vol. 102, no. 9, pp. 808–816, 1995.

[20] V. T. Sós and E. G. Straus, “Extremals of functions on graphs with applications to graphs and hypergraphs,” *J. Combin. Theory Ser. B*, vol. 32, no. 3, pp. 246–257, 1982.

[21] P. Frankl and V. Rödl, “Hypergraphs do not jump,” *Combinatorica*, vol. 4, no. 2-3, pp. 149–159, 1984.

[22] S. R. Bulò and M. Pelillo, “A generalization of the Motzkin-Straus theorem to hypergraphs,” *Optim. Lett.*, vol. 3, no. 2, pp. 287–295, 2009.