Steiner symmetry in the minimization of the principal positive eigenvalue of an eigenvalue problem with indefinite weight

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Abstract
In [1] the authors, investigating a model of population dynamics, find the following result. Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded smooth domain. The weighted eigenvalue problem $-\Delta u = \lambda m u$ in $\Omega$ under homogeneous Dirichlet boundary conditions, where $\lambda \in \mathbb{R}$ and $m \in L^\infty(\Omega)$, is considered. The authors prove the existence of minimizers $\tilde{m}$ of the principal positive eigenvalue $\lambda_1(m)$ when $m$ varies in a class $\mathcal{M}$ of functions where average, maximum, and minimum values are given. A similar result is obtained in [2] when $m$ is in the class $\mathcal{G}(m_0)$ of rearrangements of a fixed $m_0 \in L^\infty(\Omega)$. In our work we establish that, if $\Omega$ is Steiner symmetric, then every minimizer in [1, 2] inherits the same kind of symmetry.

Keywords: population dynamics, eigenvalue problem, optimization, Steiner symmetry.

2010 MSC: 47A75, 35J25, 35Q80.

Declarations of interest: none.

1. Introduction and main results

In this paper we consider the eigenvalue problem with indefinite weight

\[
\begin{cases}
-\Delta u = \lambda m(x)u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded smooth domain with boundary $\partial \Omega$, the weight $m$ belongs to $L^\infty(\Omega)$ and $\lambda \in \mathbb{R}$. Under the assumption that the set $\{ x \in \Omega : m(x) > 0 \}$ has positive Lebesgue measure, problem (1) admits a smallest positive eigenvalue $\lambda_1(m)$ which we will call the principal positive eigenvalue (see Section 2). We are interested in a symmetry aspect of the minimization of $\lambda_1(m)$ when $m$ is chosen in an appropriate class of bounded functions.

Problem (1) originates from the study of the following model in mathematical biology examined by Cantrell and Cosner in [1]

\[
\begin{cases}
v_t = d\Delta v + [m(x) - cv]v & \text{in } \Omega \times (0, \infty), \\
v(x,0) = v_0(x) \geq 0 & \text{for } x \in \overline{\Omega}, \\
v(x,t) = 0 & \text{on } \partial \Omega \times (0, \infty).
\end{cases}
\]

In (2) $v(x,t)$ represents the population density of a species inhabiting the region $\Omega$ in position $x$ at time $t$ surrounded by the hostile region $\mathbb{R}^N \setminus \Omega$. $v_0$ is the initial density and $c, d$ are positive constants describing the limiting effects of crowding and the diffusion rate of the population, respectively. The function $m(x)$ represents the local grow rate of the population, $m(x)$ is positive on favorable habitats and negative in unfavorable ones. In [1] it has been shown that (2) predicts persistence for the population if $\lambda_1(m) < 1/d$. 
As a consequence, determining the best spatial arrangement of favorable and unfavorable regions, among a fixed class of environmental configurations, results in minimizing $\lambda_1(m)$ over the corresponding class of weights. In [1] model [2] has been extensively investigated. In particular the existence of a minimizer $\tilde{m}$ in a class of functions where average, maximum, and minimum values are given has been established (see [1, Theorem 3.9]). In [2] Cosner et al. considered the minimization and maximization of $\lambda_1(m)$ over the class of all weights which are equimeasurable to a fixed bounded function. Two Lebesgue measurable functions $f, g : \Omega \to \mathbb{R}$ are said equimeasurable if the superlevel sets $\{x \in \Omega : f(x) > t\}$ and $\{x \in \Omega : g(x) > t\}$ have the same Lebesgue measure for all $t \in \mathbb{R}$. For a fixed $f \in L^\infty(\Omega)$ we call the set $\mathcal{G}(f) = \{g : \Omega \to \mathbb{R} : g$ is measurable and $f$ and $g$ are equimeasurable $\}$ class of rearrangements of $f$. A systematic treatment of this subject can be found in [3]. Concerning the minimization of $\lambda_1(m)$, in [2] the authors show the existence of an optimal weight $\tilde{m}$ in a class $\mathcal{G}(m_0)$, where $m_0 \in L^{\infty}(\Omega)$ is fixed, and prove that if $\Omega$ is a ball, then the minimizing weight $\tilde{m}$ is a radially decreasing function on $\Omega$ (see [2, Theorem 2.1 and Theorem 2.3]).

The main result of our paper is the extension of the symmetry result obtained in [2] to the more general case of a Steiner symmetric domain. Roughly speaking, a set is Steiner symmetric if it is symmetric and convex relative to a hyperplane and a function is Steiner symmetric if any of its superlevel set is Steiner symmetric (for a precise definition see Section 2).

In what follows we denote a point $x \in \mathbb{R}^N$ by $(x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and assume it is Steiner symmetric with respect to the hyperplane $T = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$. Let $m_0 \in L^{\infty}(\Omega)$ such that $\{x \in \Omega : m_0(x) > 0\}$ has positive measure. Then every minimizer $\check{m}$ of the problem

$$\inf\{\lambda_1(m) : m \in \mathcal{G}(m_0)\}$$

is Steiner symmetric relative to $T$.

As a corollary we obtain a Steiner symmetry result for the minimizers of Theorem 3.9 in [1]. We denote by $|E|$ the measure of any Lebesgue measurable set $E \subseteq \mathbb{R}^N$ and, when $N = 1$, we also write $|E|_1$. We identify two measurable sets $E, F$ that are equal up to a nullset, i.e. if $|E \setminus F \cup F \setminus E| = 0$.

**Corollary 1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and assume it is Steiner symmetric with respect to the hyperplane $T = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$. Let $\mathcal{M} = \{m(x) \in L^{\infty}(\Omega) : -m_2 \leq m(x) \leq m_1 \text{ almost everywhere in } \Omega, m(x) > 0 \text{ on a set of positive measure and } \int_\Omega m(x)dx = m_3\}$, where $m_1, m_2$ and $m_3$ are constants with $m_1$ and $m_2$ positive and $-m_2 m_1 \leq m_3 \leq m_1 |\Omega|$. Then every measurable set $E \subseteq \Omega$ such that $\check{m} = m_1 \chi_E - m_2 \chi_{\Omega \setminus E} \in \mathcal{M}$ and $\lambda_1(\check{m}) = \inf\{\lambda_1(m) : m \in \mathcal{M}\}$ is Steiner symmetric relative to $T$.

In Section 2 we introduce some definitions and results that we will use in the proofs of Theorem 1 and Corollary 1 in Section 3.

**2. Preliminaries**

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. By $H^1_0(\Omega)$ and $W^{2,2}(\Omega)$ we denote the usual Sobolev spaces and we use the norm $\|u\|_{H^1_0(\Omega)} = \int_\Omega |\nabla u|^2\,dx$ (see [4]). We consider problem (1) in weak form: $u \in H^1_0(\Omega)$ is a weak solution of (1) if

$$\int_\Omega \nabla u \cdot \nabla \varphi\,dx = \lambda \int_\Omega mu\varphi\,dx \quad \forall \varphi \in C^\infty_0(\Omega).$$

A nontrivial solution of (1) is called an eigenfunction associated to the eigenvalue $\lambda$. In [5] it has been shown that, if $|\{x \in \Omega : m(x) > 0\}| > 0$, then there exists a smallest positive eigenvalue $\lambda_1(m)$ that we

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1 We follow the notation of Theorem 3.9 in [1] except for the replacement of $m_0$ by $m_3$, $\check{m}$ by $\check{m}$ and $\lambda_1^+$ by $\lambda_1$. For any set $A$, $\chi_A$ denotes its usual characteristic or indicator function.
call the principal positive eigenvalue of \( \lambda_1(m) \). Moreover, \( \lambda_1(m) \) is simple and any eigenfunction is one-signed in \( \Omega \). By classical regularity results, any eigenfunction \( u \) related to \( \lambda_1(m) \) belongs to \( H^1_0(\Omega) \cap W^{2,2}(\Omega) \cap C^{1,\beta}(\overline{\Omega}) \) for every \( \beta \in (0,1) \) (see [4]). The principal positive eigenvalue \( \lambda_1(m) \) has a variational characterization also known as the Courant-Fischer Principle

\[
\frac{1}{\lambda_1(m)} = \max_{u \in H^1_0(\Omega)} \frac{\int_{\Omega} mu^2 \, dx}{\int_{\Omega} |\nabla u|^2 \, dx}
\]

(5)

Furthermore, each maximizer of (5) is an eigenfunction associated to \( \lambda_1(m) \) (see Proposition 1.10 and the proof of Lemma 1.1 in [3]).

For a fixed \( m_0 \in L^\infty(\Omega) \) with \( |\{x \in \Omega : m_0(x) > 0\}| > 0 \), we consider the minimization of \( \lambda_1(m) \) as \( m \) varies in the class of rearrangements \( G(m_0) \). This problem has been studied in [2], where the authors proved an existence and characterization result (see [2, Theorem 2.1]) which we rephrase here as follows.

For any \( m \in L^\infty(\Omega) \) such that \( |\{x \in \Omega : m(x) > 0\}| > 0 \), we denote by \( u_m \) the unique positive eigenfunction associated to \( \lambda_1(m) \) normalized by \( \|u_m\|_{L^1_0(\Omega)} = 1 \).

**Proposition 1.** The minimization problem (3) admits a solution and, if \( m \) is such a solution, then there exists an increasing function \( \psi \) such that \( m = \psi(u_m) \) a.e. in \( \Omega \).

Following [3], we introduce the notion of Steiner symmetrization.

Let \( l(x') = \{x = (x_1, x') \in \mathbb{R}^N : x_1 \in \mathbb{R}\} \) for any fixed \( x' \in \mathbb{R}^{N-1} \) and let \( T \) be the hyperplane \( \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\} \).

**Definition 1.** Let \( E \subset \mathbb{R}^N \) be a measurable set.

Then the set \( E^\sharp = \{x = (x_1, x') \in \mathbb{R}^N : 2|x_1| < |E \cap l(x')|_{1},x' \in \mathbb{R}^{N-1}\} \) is said the Steiner symmetrization of \( E \) with respect to the hyperplane \( T \) and \( E \) is called Steiner symmetric if \( E^\sharp = E \).

It can easily be shown that \( |E| = |E^\sharp| \).

In the sequel, by \( \{u > c\} \) we mean the set \( \{x \in E : u(x) > c\} \).

**Definition 2.** Let \( E \subset \mathbb{R}^N \) be a measurable set of finite measure and \( u : E \to \mathbb{R} \) a measurable function bounded from below. Then the function \( u^\sharp : E^\sharp \to \mathbb{R} \), defined by \( u^\sharp(x) = \sup\{c \in \mathbb{R} : x \in \{u > c\}^\sharp\} \), is said the Steiner symmetrization of \( u \) with respect to the hyperplane \( T \) and \( u \) is called Steiner symmetric if \( u^\sharp = u \) a.e. in \( E \).

It can be proved that \( \{x \in E^\sharp : u^\sharp(x) > t\} = \{x \in E : u(x) > t\}^\sharp \) for all \( t \in \mathbb{R} \). In particular, when \( E \) is Steiner symmetric, \( u \) and \( u^\sharp \) are equimeasurable.

We remind some well known properties of the Steiner symmetrization:

i) if \( E \subset \mathbb{R}^N \) is a measurable set of finite measure, \( u : E \to \mathbb{R} \) is a measurable function bounded from below and \( \psi : \mathbb{R} \to \mathbb{R} \) is an increasing function, then \( \psi(u^\sharp) = (\psi(u))^\sharp \) a.e. in \( E \) (see [3, Lemma 3.2]);

ii) if \( E \subset \mathbb{R}^N \) is a measurable set of finite measure, \( u, v : E \to \mathbb{R} \) are two measurable functions bounded from below such that \( uv \in L^1(E) \), then the Hardy-Littlewood’s inequality holds:

\[
\int_E u(x)v(x) \, dx \leq \int_{E^\sharp} u^\sharp(x)v^\sharp(x) \, dx
\]

(6)

(see [3, Lemma 3.3]);

iii) if \( \Omega \) is a bounded domain and \( u \in H^1_0(\Omega) \) is nonnegative, then \( u^\sharp \in H^1_0(\Omega^\sharp) \) and the Pólya-Szego’s inequality holds:

\[
\int_\Omega |\nabla u(x)|^2 \, dx \geq \int_{\Omega^\sharp} |\nabla u^\sharp(x)|^2 \, dx
\]

(7)

(see [2, Theorem 2.1]).

The proof of Theorem 1 relies on a deep result of Cianchi and Fusco which we specialize to our case (see [3, Theorem 2.6 and Proposition 2.3]).
Proposition 2. Let $\Omega \subset \mathbb{R}^N$ be a Steiner symmetric bounded domain. Let $u \in H^1_0(\Omega)$ be a nonnegative function satisfying
\[ |\{(x_1, x') \in \Omega : u^\sharp_\sharp_\sharp_\sharp(x_1, x') = 0\} \cap \{(x_1, x') \in \Omega : u^\sharp(x_1, x') < M(x')\}| = 0. \tag{8} \]
where $M(x') = \text{esssup}\{u^\sharp(x_1, x') : (x_1, x') \in \Omega \cap l(x')\}$. If equality is attained in (7), then $u^\sharp = u$ a.e. in $\Omega$.

3. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. Let $\hat{m}$ be a minimizer of problem (3), by Proposition 1 there exists an increasing function $\psi$ such that $\hat{m} = \psi(u_m)$. Therefore, by property i), the Steiner symmetry of $\hat{m}$ is an immediate consequence of the Steiner symmetry of $u_m$. Hence we only need to show that $u_m^\sharp = u_m$. By using (4) with $m = \hat{m}$, $\lambda = \lambda_1(\hat{m})$, $u = u_m$, and letting $\varphi \to u_m$ in $H^1_0(\Omega)$ we have
\[ \hat{\lambda}_1 = \lambda_1(\hat{m}) = \frac{\int_{\Omega} |\nabla u_m|^2 \, dx}{\int_{\Omega} \hat{m} u_m^\sharp \, dx}. \]
The inequalities (3), (7) and property i) yield
\[ \int\Omega \hat{m} u_m^\sharp \, dx \leq \int\Omega \hat{m}^\sharp(u_m^\sharp)^2 \, dx \quad \text{and} \quad \int\Omega |\nabla u_m|^2 \, dx \geq \int\Omega |\nabla u_m^\sharp|^2 \, dx. \]
Consequently we deduce
\[ \hat{\lambda}_1 = \frac{\int_{\Omega} |\nabla u_m|^2 \, dx}{\int_{\Omega} \hat{m} u_m^\sharp \, dx} \geq \frac{\int_{\Omega} |\nabla u_m^\sharp|^2 \, dx}{\int_{\Omega} \hat{m}^\sharp(u_m^\sharp)^2 \, dx}. \]
Exploiting (5) and the minimality of $\hat{\lambda}_1$ we can write
\[ \frac{1}{\hat{\lambda}_1} \leq \frac{\int_{\Omega} \hat{m} u_m^\sharp \, dx}{\int_{\Omega} |\nabla u_m|^2 \, dx} \leq \frac{\int_{\Omega} \hat{m}^\sharp(u_m^\sharp)^2 \, dx}{\int_{\Omega} |\nabla u_m|^2 \, dx} \leq \frac{\int_{\Omega} \hat{m}^\sharp(u_m^\sharp)^2 \, dx}{\int_{\Omega} |\nabla u_m^\sharp|^2 \, dx} = \frac{1}{\lambda_1(u_m^\sharp)} \leq \frac{1}{\hat{\lambda}_1}. \tag{9} \]
Therefore all the previous inequalities become equalities and yield
\[ K \int_{\Omega} \hat{m} u_m^\sharp \, dx = \int_{\Omega} \hat{m}^\sharp(u_m^\sharp)^2 \, dx \quad \text{and} \quad \int_{\Omega} |\nabla u_m|^2 \, dx = \int_{\Omega} |\nabla u_m^\sharp|^2 \, dx; \tag{10} \]
Furthermore, $u_m^\sharp$ is an eigenfunction associated to $\lambda_1(\hat{m}^\sharp)$. By the simplicity of $\lambda_1(\hat{m}^\sharp)$, being $u_m^\sharp$ positive in $\Omega$ and, by (10), $\|u_m^\sharp\|_{H^1_0(\Omega)} = \|u_m\|_{H^1_0(\Omega)} = 1$, we conclude that $u_m^\sharp = u_m^\sharp$. For simplicity of notation, we put $v = u_m^\sharp = u_m^\sharp$. The second identity of (10) will give our result provided we show that the hypothesis (8) of Proposition 2 with $u = u_m$ is satisfied. The rest of the proof is devoted to this task. By (9), $\hat{m}^\sharp$ is a minimizer of (3) and $v$ is the normalized positive eigenfunction associated to $\lambda_1(\hat{m}^\sharp) = \hat{\lambda}_1$. Moreover, by Proposition 1 there exists an increasing function $\Psi$ such that $\hat{m}^\sharp = \Psi(v)$. Thus $v$ satisfies the problem
\[ \begin{cases} -\Delta v = \hat{\lambda}_1 \Psi(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial \Omega. \end{cases} \tag{11} \]
Let $\Omega_+ = \{(x_1, x') \in \Omega : x_1 > 0\}$ and $C_0^\infty(\Omega_+) = \{\varphi \in C_0^\infty(\Omega_+) : \varphi \text{ is nonnegative}\}$. From (11) in weak form we have
\[ \int_{\Omega_+} \nabla v \cdot \nabla \varphi_{x_1} \, dx = \hat{\lambda}_1 \int_{\Omega_+} \Psi(v) \varphi_{x_1} \, dx \quad \forall \varphi \in C_0^\infty(\Omega_+). \]
Being $v \in W^{2,2}(\Omega)$, we can rewrite the previous equation as
\[ -\int_{\Omega_+} \nabla v_{x_1} \cdot \nabla \varphi \, dx = \hat{\lambda}_1 \int_{\Omega_+} \Psi(v) \varphi_{x_1} \, dx. \]
Adding $\lambda \int_{\Omega_+} \Psi(v)v_{x_1} \varphi \, dx$ to both sides and since $v \in C^{1, \beta}(\Omega)$, it becomes

$$- \int_{\Omega_+} \nabla v_{x_1} \cdot \nabla \varphi \, dx + \lambda \int_{\Omega_+} \Psi(v)v_{x_1} \varphi \, dx = \lambda \int_{\Omega_+} \Psi(v)(v \varphi)_{x_1} \, dx. \quad (12)$$

Let us show that $\int_{\Omega_+} \Psi(v)(v \varphi)_{x_1} \, dx \geq 0$. By Fubini’s Theorem we get

$$\int_{\Omega_+} \Psi(v)(v \varphi)_{x_1} \, dx = \int_{\mathbb{R}^{N-1}} dx' \int_0^{b(x')} \Psi(v)(v \varphi)_{x_1} \, dx_1, \quad (13)$$

where $b(x') = \frac{|\Omega \cap l(x')|}{2}$.

For any fixed $x' \in \mathbb{R}^{N-1}$, let $\alpha = \alpha(x_1)$ be a primitive of $(v \varphi)_{x_1}$ on $[0, b(x')]$. Since $\alpha(x_1)$ is continuous and $\Psi(v)$ is decreasing with respect to $x_1$, the Riemann-Stieltjes integral $\int_0^{b(x')} \Psi(v) \, d\alpha(x_1)$ is well defined (see [8, Theorem 7.27 and the subsequent note]). Moreover, by using [8, Theorem 7.8] we have

$$\int_0^{b(x')} \Psi(v)(v \varphi)_{x_1} \, dx_1 = \int_0^{b(x')} \Psi(v) \, d\alpha(x_1). \quad (14)$$

By [8] Theorems 7.31 and 7.8 there exists a point $x_0$ in $[0, b(x')]$ such that

$$- \int_0^{b(x')} \Psi(v) \, d\alpha(x_1) = -\Psi(v(0, x')) \int_0^{x_0} d\alpha(x_1) - \Psi(v(b(x'), x')) \int_{x_0}^{b(x')} d\alpha(x_1)$$

$$= -\Psi(v(0, x')) \int_0^{x_0} (v \varphi)_{x_1} \, dx_1 - \Psi(v(b(x'), x')) \int_{x_0}^{b(x')} (v \varphi)_{x_1} \, dx_1.$$

Computing the integrals and recalling that $\varphi \in C^\infty_0(\Omega_+)$, $v$ is positive and $\Psi(v)$ is decreasing, we conclude that

$$- \int_0^{b(x')} \Psi(v) \, d\alpha(x_1) = v(x_0, x') \varphi(x_0, x') [\Psi(v(b(x'), x')) - \Psi(v(0, x'))] \leq 0.$$

Therefore, by the previous inequality and (14) it follows

$$\int_0^{b(x')} \Psi(v)(v \varphi)_{x_1} \, dx_1 \geq 0,$$

for any $x' \in \mathbb{R}^{N-1}$ and, in turn, from (13) we obtain

$$\int_{\Omega_+} \Psi(v)(v \varphi)_{x_1} \, dx \geq 0.$$

Hence, by (12), $v_{x_1}$ satisfies the differential inequality

$$\Delta v_{x_1} + \lambda \Psi(v)v_{x_1} \geq 0 \quad \text{in } \Omega_+$$

in weak form. Then, applying [8] Theorem 2.5.3 and being $v_{x_1} \leq 0$ in $\Omega_+$, we conclude that either $v_{x_1} \equiv 0$ or $v_{x_1} < 0$. The former would lead to the contradiction $v \equiv 0$ in $\Omega_+$. Consequently, we have $v_{x_1} < 0$ in $\Omega_+$. Similarly it can be shown that $v_{x_1} > 0$ in $\Omega_- = \{(x_1, x') \in \Omega : x_1 < 0\}$. Thus

$$\{(x_1, x') \in \Omega : v_{x_1}(x_1, x') = 0\} \cap \{(x_1, x') \in \Omega : v(x_1, x') < M(x')\} = \emptyset,$$

where $M(x') = \text{esssup} \{v(x_1, x') : (x_1, x') \in \Omega \cap l(x')\}$. Hence, by Proposition 2 with $u = u_{\tilde{m}}$, we find $u_{\tilde{m}} = u_{\tilde{m}}$ and finally $\tilde{m}^2 = \tilde{m}$. This proves the theorem.
Remark 1. The counterpart of this theorem in the case of the fractional Laplacian operator has been proved in [10]. It is somewhat surprising that, in the fractional setting, the proof is much more simple.

We state the following lemma in order to prove Corollary [11].

Lemma 2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $m_1, m_2 > 0$ and $m_3$ be constants such that $-m_2|\Omega| < m_3 \leq m_1|\Omega|$. Then, the set of functions $\mathcal{N} = \{m_11_E - m_2\chi_{\Omega \setminus E} : E \subseteq \Omega \text{ is measurable, with } m_1|E| - m_2|\Omega \setminus E| = m_3\}$ coincides with the class of rearrangements $\mathcal{G}(m_0)$, where $m_0$ is an arbitrary fixed element of $\mathcal{N}$.

Proof. Clearly, for any element of $\mathcal{N}$, the set $E$ has measure $e = (m_2|\Omega| + m_3)/(m_1 + m_2)$. We recall that a class of rearrangements is an equivalence class with respect to the equimeasurability relation between functions. First, we show that the elements of $\mathcal{N}$ are all equimeasurable. This follows immediately from the identity

$$|\{x \in \Omega : f(x) > t\}| = \begin{cases} |\Omega| & \text{if } t < -m_2, \\ e & \text{if } -m_2 \leq t < m_1, \\ 0 & \text{if } t \geq m_1 \end{cases}$$

(15)

for each $f \in \mathcal{N}$. Now, let $f$ be a measurable function which satisfies (15). We will show that $f \in \mathcal{N}$ and this will complete the proof. For abbreviation, by $\{f > t\}$ we mean $\{x \in \Omega : f(x) > t\}$ and similarly for $\{f = t\}$ and $\{f \geq t\}$. Applying elementary measure theory to the identity $\{f \geq t\} = \bigcap_{k=1}^{\infty} \{f > t - 1/k\}$ for $t = m_1, -m_2$ and using (15) we find $||f \geq m_1|| = e$ and $||f \geq -m_2|| = |\Omega|$. Finally, from $||f = t|| = ||f \geq t|| - ||f > t||$ and (15) again for $t = m_1, -m_2$, we get $||f = m_1|| = e$ and $||f = -m_2|| = |\Omega| - e$, which imply $f \in \mathcal{N}$.

Proof of Corollary [11]. It is a straightforward consequence of the proof of Theorem 3.9 in [1] (see the first line after (3.13) on page 311), Lemma [2] Theorem [1] and the claim after Definition [2].

Remark 2. The biological meaning of our results in quite clear. If the region $\Omega$ is Steiner symmetric, for the population to survive the best scenario is given when the favorable habitats are located far from the boundary $\partial \Omega$ and arranged in a Steiner symmetrical fashion. As a Steiner symmetric set is convex relative to a direction, the favorable region cannot be made, at least in that direction, of many disconnected pieces. In other words, in should not be very fragmented. This conclusion is consistent with the findings of [1] and with the experimental evidences in Ecology.

Acknowledgment. The authors are partially supported by the research project Integro-differential Equations and Non-Local Problems, funded by Fondazione di Sardegna (2017).

References

References

[1] R. Cantrell, C. Cosner, Diffusive logistic equations with indefinite weights: population models in disrupted environments, Proc. R. Soc. Edinb. Sect A 112 (3-4) (1989) 293-318. doi:10.1017/s030821050001876x
[2] C. Cosner, F. Cuccu, G. Porru, Optimization of the first eigenvalue of equations with indefinite weights, Advanced Nonlinear Studies 13 (1) (2013) 79–95. doi:10.1515/ans-2013-0105
[3] P. W. Day, Rearrangements of measurable functions, Dissertation (Ph.D.), California Institute of Technology, 1970. doi:10.7907/6V22-F375
[4] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, Heidelberg, 1977. doi:10.1007/978-3-642-66233-0
[5] D. de Figueiredo, Positive solutions of semilinear elliptic problems, Vol. 957, Differential equations, Lecture Notes in Mathematics, A. Dold, B. Eckmann (eds), Springer, 1982, pp. 34–87. doi:10.1007/BFb0066233
[6] F. Brock, Rearrangements and applications to symmetry problems in PDE, Vol. 4, Edited by M. Chipot, Elsevier BV, 2007, Ch. 1, pp. 1–60. doi:10.1016/s1874-5733(07)80004-0
[7] A. Cianchi, N. Fusco, Steiner symmetric extremals in Pólya-Szegö-type inequalities, Adv. Math. 203 (2006) 673–728. doi:10.1016/j.aim.2005.05.007
[8] T. M. Apostol, Mathematical Analysis, Addison-Wesley, 1974.

[9] P. Pucci, J. Serrin, Maximum Principles for Elliptic Partial Differential Equations in Handbook of Differential Equations: Stationary Partial Differential Equations, Vol. 4, Edited by M. Chipot, Elsevier BV, 2007, Ch. 6, pp. 355–483. doi:10.1016/S1874-5733(07)80009-X

[10] C. Anedda, F. Cuccu, S. Frassu, Steiner symmetry in the minimization of the first eigenvalue of a fractional eigenvalue problem with indefinite weight, Canadian Journal of Mathematics (2020) 1–29. doi:10.4153/S0008414X200000267