Comment on “Statistical Distribution for Generalized Ideal Gas of Fractional-Statistics Particles”, Phys. Rev. Lett. 73, 922 (1994)

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In Ref. [1] Haldane introduced the fruitful concept of fractional exclusion statistics (FES). One of the most influential papers in which the thermodynamics of FES systems was deduced is Ref. [2]. Unfortunately, some important, but eventually subtle, properties of the exclusion statistics parameters were overlooked in the original paper and in all the papers after that, including Ref. [2]. This omission makes the thermodynamics of FES systems inconsistent when mutual exclusion statistics is manifesting in the system (see Ref. [3] for details).

By this Comment I want to point-out this error—an error that persisted for such a long time—and to give the correct statistical mechanics interpretation of FES. For brevity, I shall use the notations and definitions of Ref. [2], mainly without explanations. Let us assume that we have a FES system containing the particle species of Ref. [2], mainly without explanations. Let us assume that we have a FES system consisting of the particle species indexed by \( i = 0, 1, \ldots \). In the formalism of Ref. [2], the number of configurations and the entropy of the system are given by (see Eqs. 4 and 6 of [2])

\[
W = \prod_i N_i \alpha_i^{N_i - 1},
\]

and

\[
Z = W \prod_i e^{\beta(\mu_i - \epsilon_i)},
\]

respectively, where \( G_i \) and \( N_i \) are the number of states and the number of particles of species \( i \). Maximizing \( Z \) one gets the system of equations

\[
(1 + w_j) \prod_j \left( \frac{w_j}{1 + w_j} \right)^{\alpha_j} = e^{\beta(\epsilon_i - \mu_i)} \tag{1a}
\]

\[
\sum_j \left( \delta_{ij} w_j + \beta_j \right) n_j = 1, \tag{1b}
\]

where \( n_i = N_i / G_i \) and \( \beta_j = \alpha_j G_j / G_i \). As shown in [3], if there are \( \alpha_{ij} \neq 0 \) for \( i \neq j \), then the system would be physically inconsistent since the populations \( n_i \) would depend on the decomposition of the original system into the subsystems \( i = 0, 1, \ldots \) which is in general arbitrary to a large extent—this being the standard coarse-graining procedure. In the same paper I corrected this inconsistency by a conjecture: I replaced \( \alpha_{ij} \) by \( \tilde{\alpha}_{ij} \), which, for \( i \neq j \) are proportional to the dimension of the space on which they act; concretely,

\[
\tilde{\alpha}_{ii} = c_{ii} \quad \text{and} \quad \tilde{\alpha}_{ij} = G_i c_{ij}, \quad \text{for} \quad i \neq j. \tag{2}
\]

With these new statistics parameters, the system of equations,

\[
\beta(\mu_i - \epsilon_i) + \ln \left( 1 + \tilde{n}_i \right) = \sum_{j(i \neq j)} G_j \ln(1 + \tilde{n}_j) c_{ji}, \tag{3}
\]

for the single-particle level populations replaces Eqs. (1), which are the same as Eqs. (10) and (11) in Ref. [2].

In Ref. [4] I showed that FES is manifesting in general interacting systems and I calculated the exclusion statistics parameters. In this way I showed that the mutual exclusion statistics parameters satisfy indeed the conjecture introduced in [3].

Finally, in Ref. [5] I analysed the basic properties of the exclusion statistics parameters using a very simple and intuitive model. In this model I assumed, like above, that we have a system consisting of the particle species \( i = 0, 1, \ldots \). If we split one of the species, say species \( j \), into a number of sub-species, \( j_0, j_1, \ldots \), then all the parameters \( \tilde{\alpha}_{ki} \), with both, \( k \) and \( l \) different from \( j \), remain unchanged, whereas the rest of the parameters must satisfy the relations

\[
\tilde{\alpha}_{ij} = \tilde{\alpha}_{ij_0} = \tilde{\alpha}_{ij_1} = \ldots, \quad \text{for any} \quad i, i \neq j. \tag{4a}
\]

\[
\tilde{\alpha}_{ji} = \tilde{\alpha}_{j_0i} + \tilde{\alpha}_{j_1i} + \ldots, \quad \text{for any} \quad i, i \neq j. \tag{4b}
\]

\[
\tilde{\alpha}_{ij} = \tilde{\alpha}_{j_0i} + \tilde{\alpha}_{j_1i} + \ldots = \cdots \tag{4c}
\]

These are the general properties that have to be satisfied by the exclusion statistics parameters in any system of FES.

Notice now that the conjecture introduced in Ref. [3] satisfies Eqs. (4). Eventually in any physical system one can do the coarse graining finely enough, so that the exclusion statistics parameters satisfy Eq. (2), as happened in the systems analysed in Ref. [4]. In such a case the particle populations of the energy levels are given by Eqs. (3).

Equations (1) (or 10 and 11 in Ref [2]) are valid only if \( \tilde{\alpha}_{ij} = 0 \) for any \( i \neq j \). In such a case the result is identical to the result of [3].

[1] F. D. M. Haldane, Phys. Rev. Lett. 67, 937 (1991).
[2] Y.-S. Wu, Phys. Rev. Lett. 73, 922 (1994).
[3] D. V. Anghel, J. Phys. A: Math. Theor. 40, F1013 (2007), arXiv:0710.0724.
[4] D. V. Anghel, Phys. Lett. A 372, 5745 (2008), arXiv:0710.0728.
[5] D. V. Anghel, arXiv:0906.4836.