NON-NATURAL METRICS ON THE TANGENT BUNDLE

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Abstract. Natural metrics provide a way to induce a metric on the tangent bundle from the metric on its base manifold. The most studied type is the Sasaki metric, which applies the base metric separately to the vertical and horizontal components. We study a more general class of metrics which introduces interactions between the vertical and horizontal components, with scalar weights. Additionally, we explicitly clarify how to apply our and other induced metrics on the tangent bundle to vector fields where the vertical component is not constant along the fibers. We give application to the Special Orthogonal Group SO(3) as an example.

Key words. geometry, covariant derivative, tangent bundle, Sasaki, metric, Levi-Civita, manifold

1. Introduction. The study of tangent bundles and their relationship to the base manifold often rely on the Sasaki metric. However, we may gain valuable mathematical and physical insights by choosing a more general metric. For mechanical systems, tangent bundles arise naturally where the manifold is the configuration space and the Lagrangian mechanics involve the configurations and their velocities as state variables [1]. A fundamental process is damping, in which the changes on the configuration depend on changes to its velocity. Hence, we want to study metrics where this kind of interaction is considered. The Sasaki metric does not consider these kind of interactions.

The contributions of this paper are the generalization of the Sasaki metric and the derivation of the corresponding Levi-Civita connection on the tangent bundle. We also clarify the application of the results to general vector fields that are not constant along fibers.

The paper is organized as follows. A brief overview of relevant differential geometry concepts are provided in section 2, our main results are in section 3 and section 4, examples are in section 5, and the conclusions follow in section 6.

2. Background. Let $M$ be a n-dimensional differentiable manifold equipped with a Riemannian metric $g$ and $TM$ the tangent bundle of $M$. For a point $p \in M$, let $T_pM$ denote the tangent space of $M$ at $p$. A point $P \in TM$ is a pair in the set $\{(p,u) | p \in M, u \in T_pM\}$. Let $\pi : TM \to M$ be the projection map. The differential of the projection map is a smooth map denoted as $d\pi : \mathbb{TT}M \to TM$. For any vector fields $X, Y \in \mathcal{X}(M)$, the Levi-Civita connection on $M$ is denoted by $\nabla_X Y$.

From Sasaki, the tangent space $T_PTM$ is a direct sum decomposition $T_PTM = \mathcal{H}_P \oplus \mathcal{V}_P$, where $\mathcal{H}_P$ is the horizontal subspace and $\mathcal{V}_P$ is the vertical subspace [6].

To construct the subspaces, we begin by defining the exponential map on $M$. For an open neighborhood $U$ of $p := \pi(P) \in M$, the exponential map $\exp_p : T_pM \to M$ maps a neighborhood $U'$ of 0 in $T_pM$ diffeomorphically onto $U$. Let $\tau : \pi^{-1}(U) \to T_pM$ be the smooth map which parallel transports every $Y \in \pi^{-1}(U)$ from $q = \pi(Y)$ to $p$. For $u \in T_pM$, let $R_{-u} : T_pM \to T_pM$ be the translation defined by $R_{-u}(X) = X - u$ for $X \in T_pM$. Then, the connection map $K_{(p,u)} : T_{(p,u)}TM \to T_pM$ corresponding...
to the Levi-Civita connection is defined as

\[(2.1) \quad K(X)_{(p,u)} = d(\exp_p \circ R_{-u} \circ \tau)(\dot{X})_p \]

for all $X \in T_{(p,u)}TM$. The vertical subspaces is then defined as the kernel of the differential $d\pi$, while the horizontal subspace is defined as the kernel of the connection map $K$. Throughout this paper, we will use the $d\pi$ and $K$ mappings as projections on the horizontal and vertical subspaces.

A curve $\gamma : I \to TM$ in the tangent bundle is said to be horizontal if its tangent vector $\dot{\gamma}(t)$ is $\mathcal{H}_{\gamma(t)}$ for all $t \in I$. And similarly, a curve $\bar{\gamma} : I \to TM$ in the tangent bundle is said to be vertical if its tangent $\dot{\bar{\gamma}}(t)$ satisfies $\mathcal{V}_{\bar{\gamma}(t)}$ for all $t \in I$.

If $X$ is a vector field on $M$, then there is a unique vector field $\dot{X^h}$ on $TM$ called the horizontal lift of $X$ and a unique vector field $\dot{X^v}$ on $TM$ called the vertical lift of $X$ such that

\[(2.2) \quad d\pi(\dot{X^h}_P) = X_{\pi(P)}, \quad K(\dot{X^h}_P) = 0_{\pi(P)} \]
\[(2.3) \quad d\pi(\dot{X^v}_P) = 0_{\pi(P)}, \quad K(\dot{X^v}_P) = X_{\pi(P)} \]

for all $P \in TM$. A result of the tangent space decomposition is that any tangent vector $\dot{X} \in T_pTM$ can be decomposed into its horizontal and vertical components $\dot{X} = \dot{X}^h + \dot{X}^v$ where $A = d\pi(\dot{X})$, $B = K(\dot{X}) \in T_{\pi(P)}M$.

It is important to note that the standard results and our results in section 3 rely on vector fields that only change along horizontal curves. We will denote these type of vector fields as **lift-decomposable vector fields**.

**Definition 2.1.** A vector field $\dot{X} \in \mathfrak{X}(TM)$ is **lift decomposable** if it only changes along horizontal curves. Then any vector field $\dot{X}$ can be decomposed locally around $(p,u) \in TM$ as $\dot{X} = \dot{X}^h + \dot{X}^v$ for $A, B \in T_pM$.

**Remark 2.2.** Lift-decomposable vector fields are constant along the fibers in the that sense $d\pi(\dot{X}(p,u))_{(p,u)} = d\pi(\dot{X}(p,u'))_{(p,u')}$ and similarly for the connection map $K$ of $\dot{X}$ for any $p \in M, u, u' \in T_pM$, and $\dot{X} \in \mathfrak{X}(TM)$.

In general, lift-decomposable vector fields may be too limiting. In section 4, we show how to extend the results for lift-decomposable vector fields to any general vector fields that may change along both horizontal and vertical curves.

As shown in [2], the Lie bracket of horizontal and vertical lifts on $TM$ are given by the following

\[(2.3) \quad \begin{align*}
[\dot{X}^h, \dot{Y}^v]_{(p,u)} &= 0 \\
[\dot{X}^h, \dot{Y}^v]_{(p,u)} &= (\nabla_{\dot{X}}\dot{Y})^v_p \\
[\dot{X}^h, \dot{Y}^h]_{(p,u)} &= [X, Y]^h_p - (\mathfrak{R}(X,Y)u)^v_p 
\end{align*} \]

for all vector fields $X, Y \in \mathfrak{X}(M)$, $(p,u) \in TM$, $\dot{X}^h, \dot{X}^v, \dot{Y}^h, \dot{Y}^v$ are the respective horizontal and vertical lifts, and $\mathfrak{R}$ is the curvature tensor on $M$. Note that the vector fields are lift decomposable.
A metric $\bar{g}$ on the tangent bundle is said to be natural with respect to $g$ on $M$ if
\begin{align}
\bar{g}_{(p,u)}(X^h, Y^h) &= g_p(X, Y) \\
\bar{g}_{(p,u)}(X^h, Y^v) &= 0
\end{align}
for all $X, Y \in \mathfrak{X}(M)$ and $(p, u) \in TM$. The Sasaki metric, first introduced in [6], is a special natural metric that has been widely used to study the relationship between the base manifold and its tangent bundle. The Sasaki metric is given as
\begin{align}
\bar{g}_{(p,u)}(X^h, Y^h) &= g_p(X, Y) \\
\bar{g}_{(p,u)}(X^h, Y^v) &= 0 \\
\bar{g}_{(p,u)}(X^v, Y^v) &= g_p(X, Y).
\end{align}
The Kozul formula on $(M, g)$ is given by
\begin{align}
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)
\end{align}
for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

3. Levi-Civita Connection for Lift-Decomposable Vector Fields. In this section, we assume that all vector fields on $TM$ are lift decomposable. We define a non-natural metric $\bar{g}_{(p,u)}$ on the tangent bundle as
\begin{align}
\bar{g}_{(p,u)}(X^h, Y^h) &= m_1g_p(X, Y) \\
\bar{g}_{(p,u)}(X^h, Y^v) &= m_2g_p(X, Y) \\
\bar{g}_{(p,u)}(X^v, Y^v) &= m_3g_p(X, Y)
\end{align}
where $g_p(X, Y)$ is the metric on the manifold $M$ at point $p$. $X^h, X^v, Y^h, Y^v$ are the respective horizontal and vertical lifts of the vector fields $X, Y \in \mathfrak{X}(M)$, $(p, u) \in TM$ and $m_1, m_2, m_3 \in \mathbb{R}$. The scalars $m_1, m_2, m_3$ must be chosen such that $m_1, m_3 > 0$ and $m_1m_3 - m_2^2 > 0$.

Proposition 3.1. Given a Riemannian manifold $(M, g)$ and $m_1, m_2, m_3$ chosen such that $m_1, m_3 > 0$ and $m_1m_3 - m_2^2 > 0$, the metric $\bar{g}$ defined in (3.1) is a Riemannian metric on $TM$.

Proof. The metric $\bar{g}$ must be an inner product on $T_{(p,u)}TM$ at each point $(p, u) \in TM$. The symmetry and linearity properties can be verified through simple calculations. To show positive definiteness, we consider a tangent vector $\bar{Z} \in T_{(p,u)}TM$ where $\bar{Z} = X^h + Y^v$ for $X, Y \in T_pM$. Then
\[ \bar{g}_{(p,u)}(\bar{Z}, \bar{Z}) = m_1g_p(X, X) + 2m_2g_p(X, Y) + m_3g_p(Y, Y). \]
The metric can be bound from below by using the Cauchy-Schwarz inequality such that
\[ \bar{g}_{(p,u)}(\bar{Z}, \bar{Z}) \geq m_1\|X\|^2 - 2m_2\|X\|\|Y\| + m_3\|Y\|^2 \]
where \( \| \cdot \| \) is the norm with respect to \( g \). The above equation can be rewritten in matrix notation as

\[
\tilde{g}_{(p,u)}(Z, Z) \geq \begin{bmatrix} \|X\| \\ \|Y\| \end{bmatrix}^T \begin{bmatrix} m_1 & -m_2 \\ -m_2 & m_3 \end{bmatrix} \begin{bmatrix} \|X\| \\ \|Y\| \end{bmatrix}.
\]

From the above inequality, \( \tilde{g} \) must be positive definite since the middle matrix is positive definite by \( m_1, m_3 > 0 \) and \( m_1m_3 - m_2^2 > 0 \).

The metric in (3.1) allows us to choose from a class of metrics on the tangent bundle with different horizontal and vertical subspaces along with their Levi-Civita connections.

Using the metric defined in (3.1), the Kozul formula in (2.6), and the relations from (2.3), we derive properties of the corresponding Levi-Civita connection \( \nabla \) on \( TM \) for horizontal and vertical lifts (the proof closely mirrors those found in [2, 4, 5, 6]).

**Proposition 3.2.** Given a Riemannian manifold \((M, g)\) and its tangent bundle \( TM \) equipped with the metric in (3.1), the Levi-Civita connection \( \nabla \) on \( TM \) satisfies

(i) \( 2\tilde{g} \left( \nabla_X Y^h, Z^h \right) = 2m_1 g(\nabla_X Y, Z) + 2m_2 g(\mathfrak{R}(u) Y, Z) \)

(ii) \( 2\tilde{g} \left( \nabla_X Y^h, Z^v \right) = 2m_2 g(\nabla_X Y, Z) - m_3 g(\mathfrak{R}(X, Y) u, Z) \)

(iii) \( 2\tilde{g} \left( \nabla_X Y^v, Z^h \right) = 2m_3 g(\nabla_X Y, Z) - m_3 g(\mathfrak{R}(u) Y, X) \)

(iv) \( 2\tilde{g} \left( \nabla_X Y^v, Z^v \right) = 2m_3 g(\nabla_X Y, Z) - 2m_2 g(\mathfrak{R}(u) Y, Z) \)

(v) \( 2\tilde{g} \left( \nabla_X Y^v, Z^h \right) = m_3 g(\mathfrak{R}(u) Y, Z) \)

(vi) \( 2\tilde{g} \left( \nabla_X Y^v, Z^v \right) = 0 \)

(vii) \( 2\tilde{g} \left( \nabla_X Y^h, Z^h \right) = 2m_2 g(\nabla_X Y, Z) + 2m_2 g(\mathfrak{R}(u) Y, Z) \)

(viii) \( 2\tilde{g} \left( \nabla_X Y^h, Z^v \right) = 0 \)

for all vector fields \( X, Y, Z \in \mathfrak{X}(M) \).

**Proof.** The Kozul formula on the tangent bundle is used repeatedly to find the properties of the Levi-Civita connection.

(i) The statement follows from the Kozul formula in the first equation. Then substituting properties from (2.3) and (3.1), we obtained the second equation. The third equation follows from the fact that six of the terms produce the Kozul formula on \( M \). Lastly, we obtained the fourth equation by combining the Riemannian curvature tensor dependent terms such that \( Z \) is isolated using the properties of the curvature tensor.

\[
2\tilde{g} \left( \nabla_X Y^h, Z^h \right) = X^h \tilde{g} \left( Y^h, Z^h \right) + Y^h \tilde{g} \left( Z^h, X^h \right) - Z^h \tilde{g} \left( X^h, Y^h \right)
\]

\[
- \tilde{g} \left( X^h, \left[ Y^h, Z^h \right] \right) + \tilde{g} \left( Y^h, \left[ Z^h, X^h \right] \right) + \tilde{g} \left( Z^h, \left[ X^h, Y^h \right] \right)
\]

\[
= m_1 g(Y, Z) + m_1 g(Z, X) - m_1 g(X, Y) - m_1 g(X, [Z, Y])
\]

\[
+ m_2 g(X, \mathfrak{R}(Y, Z) u) + m_1 g(Y, [Z, X]) - m_2 g(Y, \mathfrak{R}(Z, X) u)
\]

\[
+ m_3 g(Z, [X, Y]) - m_2 g(Z, \mathfrak{R}(X, Y) u)
\]

\[
= 2m_1 g(\nabla_X Y, Z) + m_2 g(X, \mathfrak{R}(Y, Z) u) - m_2 g(Y, \mathfrak{R}(Z, X) u)
\]

\[
- m_2 g(Z, \mathfrak{R}(X, Y) u)
\]

\[
= 2m_1 g(\nabla_X Y, Z) + 2m_2 g(\mathfrak{R}(u) Y, Z)
\]
(ii) The statement is obtained in a similar fashion to (i). The first equation is the Kozul formula. The second equation is obtained by substituting properties from (2.3) and (3.1) followed by the expansion of the derivative of the metric terms using the metric compatibility. Note that by (3.1), we can choose $g(X, Y)$ to be purely horizontal or vertical. Thus, $Z^\circ g(X, Y) = 0$. Finally, the last equation is obtained by expanding the Lie Bracket and combining terms.

$$2\bar{g}\left(\nabla_{X^h} Y^h, Z^\circ\right) = X^h\bar{g}\left(Y^h, Z^\circ\right) + Y^h\bar{g}\left(Z^\circ, X^h\right) - Z^\circ\bar{g}\left(X^h, Y^h\right)$$

$$-\bar{g}\left(X^h, \left[Y^h, Z^\circ\right]\right) + \bar{g}\left(Y^h, \left[Z^\circ, X^h\right]\right) + \bar{g}\left(Z^\circ, \left[X^h, Y^h\right]\right)$$

$$= m_2g(Z, \nabla_X Y) + m_2g(X, \nabla_Y Z) + m_2g(Y, \nabla_X Z) + m_2g(Z, \nabla_Y Z)$$

$$- m_2g(X, \nabla_Y Z) + m_2g(Y, -\nabla_X Z) + m_2g(Z, [X, Y]) - m_3g(Z, \mathfrak{R}(X, Y)u)$$

$$= 2m_2g(\nabla_X Y, Z) - m_3g(\mathfrak{R}(X, Y)u, Z)$$

(iii)-(vii) are analogous to (ii).

(viii) The statement follows from the result that the Lie bracket of two vertical vector fields vanish and that $g(\cdot, \cdot)$ can be chosen to be purely horizontal or vertical.

$$2\bar{g}\left(\nabla_{X^v} Y^v, Z^v\right) = X^v\bar{g}\left(Y^v, Z^v\right) + Y^v\bar{g}\left(Z^v, X^v\right) - Z^v\bar{g}\left(X^v, Y^v\right)$$

$$-\bar{g}\left(X^v, \left[Y^v, Z^v\right]\right) + \bar{g}\left(Y^v, \left[Z^v, X^v\right]\right) + \bar{g}\left(Z^v, \left[X^v, Y^v\right]\right)$$

$$= m_3X^v\bar{g}(Y, Z) + m_3Y^v\bar{g}(Z, X) - m_3Z^v\bar{g}(X, Y)$$

$$-\bar{g}\left(X^v, 0\right) + \bar{g}\left(Y^v, 0\right) + \bar{g}\left(Z^v, 0\right)$$

$$= 0$$

Next, we extract the explicit form of the horizontal and vertical components of the Levi-Civita connection on the tangent bundle from Proposition 3.2. To do so, we first present a useful lemma.

**Lemma 3.3.** Let $\bar{f}$ be a function $\bar{f} : T_{(p, u)}TM \to T_{(p, u)}TM$ such that

$$d\pi\left(\bar{f} \circ \bar{X}\right) = \frac{m_3d\pi\left(\bar{X}\right) - m_2K\left(\bar{X}\right)}{m_1m_3 - m_2^2}$$

$$K\left(\bar{f} \circ \bar{X}\right) = \frac{-m_2d\pi\left(\bar{X}\right) + m_1K\left(\bar{X}\right)}{m_1m_3 - m_2^2}$$

for all vector fields $\bar{X} \in \mathfrak{X}(TM)$, $(p, u) \in TM$, and $m_1, m_2, m_3 \in \mathbb{R}$ such that $m_1, m_3 > 0$ and $m_1m_3 - m_2^2 > 0$. Then

$$g_{(p, u)}(\bar{f} \circ \bar{X}, \bar{Y}) = g_{(p, u)}(\bar{X}, \bar{f} \circ \bar{Y})$$

$$= g_p(d\pi(\bar{X}), d\pi(\bar{Y})) + g_p(K(\bar{X}), K(\bar{Y})).$$

**Proof.** The claim follows directly from the definitions of $\bar{f}$ and $\bar{g}$. □

**Lemma 3.3** is important in that if there is an expression for $\bar{g}$ with one known tangent vector, the horizontal and vertical components of the unknown tangent vector can be extracted through the metric instead of the $d\pi$ and $K$ mappings.
Remark 3.4. The results of Lemma 3.3 can be better understood in local coordinates using matrix operations. To illustrate the point, we assume \( g \) to be the natural Euclidean inner product on \( M \), then

\[
\bar{g}_{(p,u)}(\bar{X},\bar{Y}) = \left[ \frac{d\pi}{K}(\bar{X}) \right]^T \begin{bmatrix} m_1 & m_2 & m_3 \\ m_2 & m_3 & m_1 \\ m_3 & m_1 & m_2 \end{bmatrix} \left[ \frac{d\pi}{K}(\bar{Y}) \right]_p
\]

(3.4)

\[
= \left[ \frac{d\pi}{K}(\bar{X}) \right]^T \mathcal{M} \left[ \frac{d\pi}{K}(\bar{Y}) \right]_p
\]

where \( \mathcal{I}_n \) is the \( n \times n \) identity matrix, \( \bar{X}, \bar{Y} \in T_{(p,u)}TM, (p,u) \in TM \), and \( m_1, m_2, m_3 \in \mathbb{R} \) such that \( m_1, m_3 > 0 \) and \( m_1 m_3 - m_2^2 > 0 \). Since \( \mathcal{M} \) is positive definite, its inverse \( \mathcal{M}^{-1} \) exists. Thus, the function \( \bar{f} \) can be interpreted (in matrix notation) as

\[
\bar{f} \circ \bar{X} = \mathcal{M}^{-1} \left[ \frac{d\pi}{K}(\bar{X}) \right].
\]

When \( \bar{f} \) acts on a tangent vector in (3.4), we recover the identity matrix and the simple pairing of the horizontal and vertical components.

The following theorem combines Proposition 3.2 and Lemma 3.3 to extract the explicit form of the horizontal and vertical components of the Levi-Civita connection \( \nabla \bar{X} \bar{Y} \) on \( TM \) for any vector fields \( \bar{X}, \bar{Y} \in \mathfrak{X}(TM) \).

**Theorem 3.5.** Let \((M, g)\) be a Riemannian manifold and \( \nabla \) be the Levi-Civita connection on the tangent bundle \((TM, g)\) equipped with the metric (3.1). Then

(i) \( d\pi \left( \nabla_X \bar{Y} \right) = \nabla_X Y + \frac{1}{m_1 m_3 - m_2^2} \left( m_2 m_3 \mathfrak{R}(u, X) Y + m_2 m_1 \mathfrak{R}(X, Y) u \right) \)

(ii) \( K \left( \nabla_X \bar{Y} \right) = \frac{1}{m_1 m_3 - m_2^2} \left( -m_2 \mathfrak{R}(u, X) Y - m_1 \mathfrak{R}(X, Y) u \right) \)

(iii) \( d\pi \left( \nabla_Y \bar{X} \right) = \frac{1}{m_1 m_3 - m_2^2} \frac{m_1 m_3}{2} \mathfrak{R}(u, Y) X \)

(iv) \( K \left( \nabla_Y \bar{X} \right) = \nabla_Y X - \frac{1}{m_1 m_3 - m_2^2} \left( \frac{m_2}{2} \mathfrak{R}(u, Y) X \right) \)

(v) \( d\pi \left( \nabla_X \bar{X} \right) = \frac{1}{m_1 m_3 - m_2^2} \frac{m_2}{2} \mathfrak{R}(u, X) Y \)

(vi) \( K \left( \nabla_X \bar{X} \right) = -\frac{1}{m_1 m_3 - m_2^2} \left( \frac{m_2}{2} \mathfrak{R}(u, X) Y \right) \)

(vii) \( d\pi \left( \nabla_Y \bar{Y} \right) = 0 \)

(viii) \( K \left( \nabla_Y \bar{Y} \right) = 0 \)

for all vector fields \( X, Y \in \mathfrak{X}(M) \) and \((p,u) \in TM \).

**Proof.** Proposition 3.2 provides an expression for \( \bar{g}_{(p,u)} \left( \nabla \bar{X} \bar{Y}, \cdot \right) \) for any vector fields \( \bar{X}, \bar{Y} \in \mathfrak{X}(TM) \) at a point \((p,u) \in TM \) where the second argument of \( \bar{g} \) can be chosen arbitrarily. Thus, we chose a purely horizontal and vertical field to extract the components of the connection. For any arbitrary vector field \( Z \in \mathfrak{X}(M) \) and \( \bar{f} \) defined in Lemma 3.3

\[
g_{(p,u)} \left( \nabla \bar{X} \bar{Y}, \bar{f} \circ \bar{Z} \right) = g_p \left( d\pi \left( \nabla \bar{X} \bar{Y} \right), Z \right)
\]

\[
g_{(p,u)} \left( \nabla \bar{X} \bar{Y}, \bar{f} \circ \bar{Z} \right) = g_p \left( K \left( \nabla \bar{X} \bar{Y} \right), Z \right).
\]
The results of this section allows us to compute the Levi-Civita connection on the tangent bundle for any vector fields $X, Y \in \mathfrak{X}(TM)$ that are lift decomposable. However, lift-decomposable vector fields do not span the space of all possible smooth vector fields. In general, vector fields on the tangent bundle may change along both horizontal and vertical curves. In the next section, we show how to extend the results for lift-decomposable vector fields to any general vector field.

4. Levi-Civita Connection for General Vector Fields. In this section, we extend the Levi-Civita connection in section 3 to general vector fields on the tangent bundle that may change along both horizontal and vertical curves. As discussed in section 2 and section 3, the Levi-Civita connection in Theorem 3.5 is only valid for lift-decomposable vector fields. In general, vector fields $Y \in \mathfrak{X}(TM)$ at a point $(p, u) \in TM$ depend on both horizontal and vertical motions and may be expressed as

$$
\bar{Y}(p, u) = A^h_{(p, u)} + B^v_{(p, u)} + \bar{C}(p, u) + \bar{D}(p, u)
$$

(4.1)

where $A, B \in T_pM, \bar{C} \in \mathcal{H}(p, u), \bar{D} \in \mathcal{V}(p, u)$, and $\bar{C} = \bar{D} = 0$ at $(p, u)$ and along horizontal curves passing through $(p, u)$. To be more precise, $\bar{C}$ and $\bar{D}$ are the point-wise horizontal and vertical projections of the field $\bar{Y}(p', u') - A^h_{(p, u)} - B^v_{(p, u)}$ for any point $(p', u') \in TM$ in the neighborhood around $(p, u)$. It is important to note that $A$ and $B$ change along horizontal curves, and $\bar{C}$ and $\bar{D}$ change along vertical curves.

The standard results and our results in section 3 already considered how vector fields change along horizontal curves to derive the connection in Theorem 3.5. In that formulation, we ignored the motion along the vertical curves because the vector fields are lift decomposable and thus constant along those curves. Now, we must also consider changes along vertical curves to obtain the Levi-Civita connection for general vector fields on the tangent bundle.

Corollary 4.1. The Levi-Civita connection $\nabla_X \bar{Y}$ on the tangent bundle $TM$ for any general vector fields $X, Y \in \mathfrak{X}(TM)$ at a point $(p, u) \in TM$ is given by

$$
\nabla_X \bar{Y} = \nabla_{(F^h + G^v)} \left( A^h + B^v \right) + \tilde{\nabla}_G \left( \bar{C} + \bar{D} \right)
$$

(4.2)

where $\bar{Y}$ is decomposed into the components defined in (4.1) and $\bar{X} = F^h + G^v$ for $F, G \in T_pM$. The first term is the connection from Theorem 3.5 which captures changes along horizontal curves. The second term captures changes along vertical curves and does not depend on $F^h$ since $\bar{C}, \bar{D}$ are zero along any horizontal curve. The connection $\tilde{\nabla}$ is the usual connection on the flat tangent space corresponding to the choice of local coordinates.

Proof. The proof follows from the vector field decomposition in (4.1) and the properties of the Levi-Civita connection. Note that since $\bar{C}, \bar{D} = 0$ along horizontal curves, the connection $\nabla_{F^h} \left( \bar{C} + \bar{D} \right) = 0$. \qed

5. Examples. In this section, we present two applications of our results. In the first, we show that the Sasaki metric and the corresponding Levi-Civita connection on $TM$ is a special case. In the second, we apply the results on $SO(3)$ and derive the Levi-Civita connection on $TSO(3)$.

5.1. Sasaki Metric. In this example, we show that the Sasaki metric [6] and the induced Levi-Civita connection on $TM$ is a special case of our results. If we choose

$$m_1 = 1, \quad m_2 = 0, \quad m_3 = 1$$
then the metric (3.1) becomes
\[
\bar{g} (X^\hat{h}, Y^\hat{h}) = g (X, Y) \\
\bar{g} (Y^\hat{h}, Y^\hat{v}) = 0 \\
g (X^\hat{v}, Y^\hat{v}) = g (X, Y).
\]

The induced connection \( \bar{\nabla} \) on \( TM \), given by Theorem 3.5, can be shown to be equivalent to the results obtained by Kowalski in [5].

5.2. SO(3) Example. In this example, we consider the Special Orthogonal Group \( SO(3) \) equipped with a metric \( g \) and its tangent bundle \( TSO(3) \) equipped with the metric in (3.1). The Levi-Civita connection on \( SO(3) \) is given by Edelman in [3]

\[
\nabla_X Y = \dot{Y} + \frac{1}{2} R (X^T Y + Y^T X)
\]

for all vector fields \( X, Y \in \mathfrak{x} (SO(3)) \) at a point \( R \in SO(3) \) and \( \dot{Y} \) is the usual time derivative. Given left-invariant vector fields \( \bar{X}, \bar{Y} \in \mathfrak{x} (TSO(3)) \) along a curve \( \bar{\gamma} \) such that

\[
\bar{X} = \left( R \hat{\zeta}, R \hat{\eta} \right), \quad \bar{Y} = \left( R \hat{\alpha}, R \hat{\beta} \right), \quad \bar{\gamma} = (R, R \hat{\omega})
\]

where constants \( \zeta, \eta, \alpha, \beta, \omega \in \mathbb{R}^3 \), \( \hat{(\cdot)}: \mathbb{R}^3 \to \mathfrak{so}(3) \) is the hat operator which map real numbers to the Lie algebra, and \( TT_RSO(3) \to T_RSO(3) \). Then the induced Levi-Civita connection \( \bar{\nabla} \) on \( TSO(3) \), in local coordinates, is given by

(i)

\[
d\pi (\bar{\nabla}_{\bar{X}} \bar{Y}) = R \left( \hat{\zeta} \hat{\alpha} + \frac{1}{2} \left( \hat{\zeta}^T \hat{\alpha} + \hat{\alpha}^T \hat{\zeta} \right) \right)
\]

\[
- R \frac{m_2 m_3}{8(m_1 m_3 - m_2^2)} \left( 2 \left[ \left[ \hat{\omega}, \hat{\zeta} \right], \hat{\alpha} \right] + [\left[ \hat{\zeta}, \hat{\alpha} \right], \hat{\omega}] \right)
\]

\[
- R \frac{m_3^2}{8(m_1 m_3 - m_2^2)} \left[ [\hat{\omega}, \hat{\beta}], \hat{\zeta} \right]
\]

\[
- R \frac{m_2^3}{8(m_1 m_3 - m_2^2)} [\left[ \hat{\omega}, \hat{\eta} \right], \hat{\alpha} \right]
\]

(ii)

\[
K (\bar{\nabla}_{\bar{X}} \bar{Y}) = R \left( \hat{\zeta} \hat{\beta} + \frac{1}{2} \left( \hat{\zeta}^T \hat{\beta} + \hat{\beta}^T \hat{\zeta} \right) \right)
\]

\[
+ R \frac{m_2 m_3}{8(m_1 m_3 - m_2^2)} \left[ [\hat{\omega}, \hat{\beta}], \hat{\zeta} \right]
\]

\[
+ R \frac{1}{8(m_1 m_3 - m_2^2)} \left( 2m_2^2 \left[ \left[ \hat{\omega}, \hat{\zeta} \right], \hat{\alpha} \right] + m_1 m_3 \left[ \left[ \hat{\zeta}, \hat{\alpha} \right], \hat{\omega} \right] \right)
\]

\[
+ R \frac{m_2 m_3}{8(m_1 m_3 - m_2^2)} [\left[ \hat{\omega}, \hat{\eta} \right], \hat{\alpha} \right].
\]

In the general case where \( \omega = \omega(t) \), \( \alpha = \alpha(\omega) \), \( \beta = \beta(\omega) \), an additional term is required to account for changes in the vertical subspace along the curve \( \bar{\gamma} \) (see Corollary 4.1). The connection on \( TSO(3) \) for this vector field is given by
(i) 
\[ d\pi (\overline{\nabla}_\mathcal{X} \mathcal{Y}) = \ldots + R \left( \frac{\partial \alpha}{\partial \omega} \right) \hat{\eta} \]

(ii) 
\[ K (\overline{\nabla}_\mathcal{X} \mathcal{Y}) = \ldots + R \left( \frac{\partial \beta}{\partial \omega} \right) \hat{\eta}. \]

where \( \overline{\nabla} \) is the usual directional derivative on \( \mathbb{R}^3 \). Both results can be validated by the metric compatibility requirement along their respective curves.

6. Conclusions. In this paper, we study the relationship between Riemannian manifolds and their tangent bundle. Namely, we see that a manifold equipped with a metric and Levi-Civita connection induces a metric and Levi-Civita connection on its tangent bundle by the natural decomposition of the tangent bundle into the horizontal and vertical subspaces. We then defined a non-natural metric on the tangent bundle and derived the corresponding Levi-Civita connection. In addition, we showed explicitly how to extend the results to vector fields that are not constant along the fibers. As a validation of our results, we see that under special conditions the non-natural metric reduces to the Sasaki metric and the corresponding Levi-Civita connection agrees with the results of Kowalski.

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