On the convergence of exotic formal series solutions of an ODE. A proof by the implicit mapping theorem

R. R. Gontsov\textsuperscript{1}, I. V. Goryuchkina\textsuperscript{2}

Abstract

We propose a sufficient condition of the convergence of a complex power type formal series of the form \( \varphi = \sum_{k=1}^{\infty} \alpha_k(x^{i\gamma})x^k \), where \( \alpha_k \) are functions meromorphic at the origin and \( \gamma \in \mathbb{R} \setminus \{0\} \), that satisfies an analytic ordinary differential equation (ODE) of a general type. An example of a such type formal solution of the third Painlevé equation is presented and the proposed sufficient condition is applied to check its convergence.

1 Introduction

We consider a non-linear ODE of order \( n \),
\[
F(x, y, \delta y, \ldots, \delta^n y) = 0,
\]
where \( \delta = x(d/dx) \) and \( F(x, y_0, y_1, \ldots, y_n) \) is a holomorphic function near \( 0 \in \mathbb{C}^{n+2} \). Suppose that the equation (1) possesses a formal series solution \( y = \varphi \) of the form
\[
\varphi = \sum_{k=1}^{\infty} \alpha_k(x^{i\gamma})x^k, \quad i = \sqrt{-1}, \quad \gamma \in \mathbb{R} \setminus \{0\},
\]
where \( \alpha_k(t) \) are meromorphic functions at the origin:
\[
\alpha_k(t) = t^{-r_k} \sum_{\ell=0}^{\infty} \alpha_{k\ell} t^\ell, \quad \alpha_{k\ell} \in \mathbb{C}, \quad r_k \in \mathbb{Z},
\]
with some common punctured disc \( \mathbb{D} \setminus \{0\} = \{0 < |t| \leq R\} \) of convergence.

The series (2) will be called exotic, in the terminology of A. D. Bruno \[2\]. In particular, the Painlevé III, V, VI equations possess formal solutions of such type \[3\], \[4\], \[9\], \[13\] (in the last paper \[13\] by Shimomura the convergence of such formal solutions of the fifth Painlevé equation is established). Thus, our present goal is to obtain some general condition for the convergence of an exotic formal series solution of the equation (1). In this sense, our work continues a series of articles \[6\], \[7\], \[8\] where similar questions were studied for generalized formal power series
\[
\varphi^{pow} = \sum_{k=0}^{\infty} c_k x^{\lambda_k}, \quad \text{Re} \lambda_0 \leq \text{Re} \lambda_1 \leq \ldots, \quad \text{Re} \lambda_k \to +\infty,
\]

\textsuperscript{1}Institute for Information Transmission Problems of RAS, Bolshoy Karetny per. 19, build.1, Moscow 127051 Russia. Moscow Power Engineering Institute, Krasnokazarmennaya 14, Moscow 111250 Russia. E-mail: gontsovrr@gmail.com.

\textsuperscript{2}Keldysh Institute of Applied Mathematics of RAS, Miusskaya sq. 4, Moscow 125047 Russia. E-mail: igoryuchkina@gmail.com.
and for formal Dulac series

\[
\varphi^{\text{Dul}} = \sum_{k=1}^{\infty} P_k(\ln x)x^k, \quad P_k \in \mathbb{C}[t].
\]

Those articles were inspired by the original paper by B. Malgrange [12] on the Maillet theorem for classical formal power series solutions of a non-linear ODE.

Note that the series (2) can be written in the form of a complex power type series,

\[
\varphi = \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \alpha_{k\ell} x^{k+i\gamma(\ell-r_k)},
\]

however, this differs from a generalized power series \(\varphi^{\text{pow}}\) since for each \(k \geq 1\), there are infinite number of power exponents with the same real part \(k\) and, therefore, the set of power exponents of such a series cannot be ordered with respect to the real part increasing.

We propose the following sufficient condition of the convergence of (2), which is the main result of the present paper.

**Theorem 1.** Let (2) be a formal solution of the equation (1):

\[
F(x, \Phi) = 0, \quad \Phi := (\varphi, \delta \varphi, \ldots, \delta^n \varphi),
\]

such that \(\frac{\partial F}{\partial y_n}(x, \Phi) \neq 0\). Furthermore, let each exotic formal series \(\frac{\partial F}{\partial y_j}(x, \Phi)\) be of the form

\[
\frac{\partial F}{\partial y_j}(x, \Phi) = a_j(x^{i\gamma})x^m + b_j(x^{i\gamma})x^{m+1} + \ldots, \quad j = 0, 1, \ldots, n,
\]

with the same \(m\) for all \(j\), where \(a_n \neq 0\) and

\[
\text{ord}_0 a_j \geq \text{ord}_0 a_n, \quad j = 0, 1, \ldots, n.
\]

Then in any open sector \(S\) with the vertex at the origin, of opening less than \(2\pi\) and of sufficiently small radius, such that \(S \subset \{\arg x > (-1/\gamma) \ln R\}\) if \(\gamma > 0\) (and \(S \subset \{\arg x < (-1/\gamma) \ln R\}\) if \(\gamma < 0\)), the series (2) converges uniformly in \(S\), thus representing there a branch of a holomorphic function.

There is also a qualitative difference between generalized power series solutions and exotic series solutions of (1). Since \(\varphi^{\text{pow}}\) converges in any sufficiently small sector with the vertex at the origin and of opening less than \(2\pi\), with no restriction on its location (see [6]), singular points of a solution represented by \(\varphi^{\text{pow}}\) cannot accumulate to the origin along a ray or ray-like curve. However, they can accumulate along spirals (for example, like in the case of Painlevé equations [1, 13]). On the other hand, singular points of a solution represented by (2) can accumulate to the origin along a ray that is not contained in the domain from Theorem 1. Indeed, if \(t = a, |a| > R\), is a singular point of some \(\alpha_k(t)\) then the equation \(x^{i\gamma} = a\) has an infinite number of solutions located on the ray \(\arg x = (-1/\gamma) \ln |a|\) and having the origin as a limit point (again, an illustration of this situation is provided by Painlevé equations, see [10]).
We note that the condition of Theorem 1 is similar to the corresponding sufficient conditions of the convergence of generalized formal power series solutions and formal Dulac series solutions of \([1]\) obtained in \([6]\) and \([8]\) (all they generalize Malgrange’s condition \([12]\) of the convergence of a classical formal power series solution and require the linearization of the equation along a formal solution to be Fuchsian in some sense). We prove Theorem 1 in a series of lemmas adapting the main idea of Malgrange, an application of the implicit mapping theorem for Banach spaces, to the situation under consideration.

2 An ODE in a reduced form

A starting point in the proof of Theorem 1 is a reduction of the initial equation to a special form, which is similar to what Malgrange did studying classical formal power series solutions.

Lemma 1. Under the assumptions of Theorem 1, there exists \(N \in \mathbb{N}\) such that the transformation

\[
y = \sum_{k=1}^{N} \alpha_k(x^{i\gamma})x^k + x^N u
\]

(3)

reduces the equation \([1]\) to an equation of the form

\[
\sum_{j=0}^{n} a_j(x^{i\gamma})(\delta + N)^ju = x M(x, x^{i\gamma}, u, \delta u, \ldots, \delta^n u),
\]

(4)

where the function \(M(x, t, u_0, \ldots, u_n)\) is meromorphic in \(t\) and holomorphic in the rest of variables, near \(0 \in \mathbb{C}^{n+3}\).

Proof. For any \(N \in \mathbb{N}\), let us write the formal solution \(\varphi\) in the form

\[
\varphi = \sum_{k=1}^{N} \alpha_k(x^{i\gamma})x^k + x^N \psi =: \varphi_N + x^N \psi.
\]

Respectively,

\[
\Phi = \Phi_N + x^N \Psi, \quad \Psi = (\psi, (\delta + N)\psi, \ldots, (\delta + N)^n \psi).
\]

Then the Taylor formula implies

\[
0 = F(x, \Phi_N + x^N \Psi) = F(x, \Phi_N) + x^N \sum_{j=0}^{n} \frac{\partial F}{\partial y_j}(x, \Phi_N)\psi_j + \\
\quad + \frac{x^{2N}}{2} \sum_{i,j=0}^{n} \frac{\partial^2 F}{\partial y_i \partial y_j}(x, \Phi_N)\psi_i\psi_j + \ldots,
\]

(5)

where \(\psi_j = (\delta + N)^j \psi\).

Definition. Let us define the valuation of \(\varphi\) as

\[
\text{val} \varphi := \min\{k \mid \alpha_k \neq 0\}.
\]
Again, by the Taylor formula,
\[
\frac{\partial F}{\partial y_j}(x, \Phi) - \frac{\partial F}{\partial y_j}(x, \Phi_N) = x^N \sum_{i=0}^{n} \frac{\partial^2 F}{\partial y_i \partial y_j}(x, \Phi_N) \psi_i + \ldots,
\]
furthermore \(\text{val}(\psi_i) \geq 1\) for all \(i\). If we choose \(N > m\) (recall that the integer \(m \geq 0\) comes from the condition of Theorem 1) then there will hold\(^3\)
\[
\frac{\partial F}{\partial y_j}(x, \Phi_N) = a_j(x^{i\gamma})x^m + b_j(x^{i\gamma})x^{m+1} + \ldots,
\]
for each \(j = 0, 1, \ldots, n\), that is, the leading coefficient \(a_j\) will be preserved if one substitutes the finite sum \(\Phi_N\) instead of \(\Phi\) in \(\frac{\partial F}{\partial y_j}\). Now from the relation (5) it follows that
\[
\text{val} F(x, \Phi_N) \geq m + N + 1.
\]
Hence, dividing (5) on \(x^{m+N}\), one obtains
\[
\sum_{j=0}^{n} a_j(x^{i\gamma})(\delta + N)^j \psi = x M(x, x^{i\gamma}, \psi, \delta \psi, \ldots, \delta^n \psi),
\]
where \(M(x, t, u_0, \ldots, u_n)\) is a function meromorphic in \(t\) and holomorphic in the rest of variables, near \(0 \in \mathbb{C}^{n+3}\). This finishes the proof of the lemma. \(\Box\)

**Remark 1.** According to the conditions of Lemma 1, one can assume that \(a_n \equiv 1\) and the rest of \(a_j\)’s to be holomorphic near the origin.

One sees that the reduced equation (4) possesses an exotic formal series solution
\[
\psi = \sum_{k=1}^{\infty} c_k(x^{i\gamma}) x^k, \quad c_k(t) = a_{k+N}(t) = t^{-\nu_k} \sum_{\ell \geq 0} c_{k\ell} t^\ell,
\]
moreover, this is its unique formal solution of such a form.

**Lemma 2.** The formal series (6) is a unique exotic series satisfying (4).

**Proof.** The differential operator \(\delta\) acts on the series \(\psi\) in the following way:
\[
\delta \psi = \sum_{k=1}^{\infty} \left( k + i\gamma \delta_t \right) c_k(t)_{t=x^{i\gamma}} x^k, \quad \delta_t = t(d/dt),
\]
whence
\[
(\delta + N)^j \psi = \sum_{k=1}^{\infty} \left( k + N + i\gamma \delta_t \right)^j c_k(t)_{t=x^{i\gamma}} x^k.
\]
\(^3\)There will be another requirement for choosing \(N\), which is not used in the proof of Lemma 1 and which we explain later, in Lemma 2.
Hence, each coefficient $c_k$ is a solution of the (non-homogeneous) linear differential equation
\[
\sum_{j=0}^{n} a_j(t) (k + N + i\gamma\delta_t)^j c_k(t) = \tilde{c}_k(t), \quad k = 1, 2, \ldots,
\]  
(7)
where $\tilde{c}_k$ is a known function meromorphic at the origin, which is determined by the previous $\tilde{c}_1, \ldots, \tilde{c}_{k-1}$ (for example, $\tilde{c}_1(t) = M(0,t,0,\ldots,0)$, etc.).

By assumptions (see Remark 1), the linear differential operator on the left hand side of (7) is Fuchsian at the point $t = 0$. Its exponents at this point are the roots of the polynomial
\[
P(\lambda) = \sum_{j=0}^{n} a_j(0) (k + N + i\gamma\lambda)^j.
\]
Assuming additionally $N$ to be large enough in such a way that $P(\lambda)$ has no integer roots (this is the second requirement to $N$) we get that any non-trivial local solution of the corresponding homogeneous equation
\[
\sum_{j=0}^{n} a_j(t) (k + N + i\gamma\delta_t)^j y(t) = 0
\]
ramifies at the Fuchsian singular point $t = 0$. Hence, $c_k(t)$ is a unique solution of (7) meromorphic at $t = 0$. \hfill \Box

**Lemma 3.** For the pole order $\nu_k$ of each coefficient $c_k(t)$ of the series (6) at $t = 0$, the following estimate holds: $\nu_k \leq k\mu$, where $\mu$ is the pole order of $M(x,t,u_0,\ldots,u_n)$ at $t = 0$.

**Proof.** Since the polynomial $P(\lambda) = \sum_{j=0}^{n} a_j(0) (k + N + i\gamma\lambda)^j$ has no integer roots (in particular, $P(-\nu_k) \neq 0$) from (7) it follows that $\nu_k$ is equal to the pole order of $\tilde{c}_k(t)$ at $t = 0$. Hence, as $\tilde{c}_1(t) = M(0,t,0,\ldots,0)$, one has $\nu_1 \leq \mu$ and further proceeds by induction as follows.

The Taylor terms of the function $M$ that contribute to the term $\tilde{c}_k(x^{i\gamma})x^k$ of the exotic series $x M(x,x^{i\gamma},\psi,\delta\psi,\ldots,\delta^n\psi)$, are those of the form
\[
c x^r(x^{i\gamma}s)\psi^{k_0}(\delta\psi)^{k_1} \ldots (\delta^n\psi)^{k_n}, \quad 0 \leq r \leq k - 1,
\]
with the additional restriction $k_0 + k_1 + \ldots + k_n \leq k - 1 - r$. Therefore, denoting
\[
\delta^j\psi = \sum_{k=1}^{\infty} (k + i\gamma\delta_t)^j c_k(t)|_{t=x^{i\gamma}}, \quad x^k =: \sum_{k=1}^{\infty} c_k^j(x^{i\gamma}) x^k
\]
one sees that $\tilde{c}_k(t)$ is a linear combination of functions of the form
\[
t^s(c_1^0 \ldots c_{i_0}^0) (c_1^1 \ldots c_{i_1}^1) \ldots (c_{m_1}^1 \ldots c_{m_{k_1}}^1) \ldots (c_{m_1}^n \ldots c_{m_{k_n}}^n),
\]
Furthermore
\[
s \geq -\mu, \quad \sum_{i=1}^{k_0} l_i + \sum_{i=1}^{k_1} m_i + \ldots + \sum_{i=1}^{k_n} n_i \leq k - 1.
\]
By a natural inductive assumption, for \( l < k \) the pole order of each \( c^l_1 \) at \( t = 0 \), which is equal to \( \nu_l \), does not exceed \( l\mu \), hence

\[
\nu_k \leq \mu + \sum_{i=1}^{k_0} l_i \mu + \sum_{i=1}^{k_1} m_i \mu + \cdots + \sum_{i=1}^{k_n} n_i \mu \leq \mu + (k-1)\mu = k\mu.
\]

\[\square\]

Remark 2. It follows from the inductive determination of \( c_k \)'s as solutions of linear ODEs with the same homogeneous part, the initial requirement for them to be holomorphic in the common punctured disc \( \overline{D} \setminus \{0\} \) is, in fact, an internal property of these functions.

3 Banach spaces of exotic series

To prove Theorem 1, one should prove the convergence of the formal exotic series solution \( \psi \) of the reduced equation (4). The idea is, using the implicit mapping theorem for Banach spaces, to prove the existence of a convergent exotic series solution of (4) and then, from uniqueness proved in Lemma 2, to deduce the convergence of \( \psi \).

For \( \nu \geq 1 \), we consider the space \( \mathcal{O}_\nu(\overline{D} \setminus \{0\}) \) of functions holomorphic in a punctured disc and having a pole of order at most \( \nu \) at \( t = 0 \). Expanding \( f \in \mathcal{O}_\nu(\overline{D} \setminus \{0\}) \) into a Laurent series,

\[
f = \frac{1}{t^\nu} \sum_{k \geq 0} f_k t^k,
\]

we define its norm \( \|f\| \) by

\[
\|f\| = \frac{1}{r^\nu} \sum_{k \geq 0} |f_k| R^k, \quad 0 < r < R.
\]

Lemma 4. The space \( \mathcal{O}_\nu(\overline{D} \setminus \{0\}) \) with the norm \( \| \cdot \| \) is a Banach space.

Proof. Since \( \max_{r \leq |t| \leq R} |f(t)| \leq \|f\| \), any fundamental sequence \( f^{(m)} \in \mathcal{O}_\nu(\overline{D} \setminus \{0\}) \) is also fundamental with respect to the norm of uniform convergence and, therefore, converges uniformly to a function \( F \) holomorphic in the annulus \( \{r \leq |t| \leq R\} \): \( F = \sum_{k \in \mathbb{Z}} c_k t^k \). Due to the Cauchy formula for the coefficients of a Laurent series,

\[
c_k = \frac{1}{2\pi i} \int_{|t|=\rho} \frac{F(t)}{t^{k+1}} dt, \quad r < \rho < R,
\]

each coefficient \( c_k \) of \( F \) is equal to the limit of the sequence of the corresponding coefficients \( c^{(m)}_k \) of \( f^{(m)} \), hence \( c_k = 0 \) for \( k < -\nu \) and \( F \in \mathcal{O}_\nu(\overline{D} \setminus \{0\}) \). Thus, it remains to prove that \( \|f^{(m)} - F\| \to 0 \) (\( m \to \infty \)). Since for any \( \varepsilon > 0 \) there exists \( m_0 \) such that \( \|f^{(m)} - f^{(n)}\| \leq \varepsilon \) for \( m, n \geq m_0 \):

\[
\frac{1}{r^\nu} \sum_{k \geq 0} |c^{(m)}_{k-\nu} - c^{(n)}_{k-\nu}| R^k \leq \varepsilon,
\]
one has

\[ \frac{1}{r^\nu} \sum_{k=0}^{K} |c_{k-\nu}^{(m)} - c_{k-\nu}^{(n)}| R^k \leq \varepsilon \]

for any \( K \geq 0 \). Taking the limit in the above inequality with respect to \( n \to \infty \), we obtain

\[ \frac{1}{r^\nu} \sum_{k=0}^{K} |c_{k-\nu}^{(m)} - c_{k-\nu}^{(n)}| R^k \leq \varepsilon, \]

which holds for every \( K \geq 0 \). Therefore,

\[ \frac{1}{r^\nu} \sum_{k \geq 0} |c_{k-\nu}^{(m)} - c_{k-\nu}^{(n)}| R^k \leq \varepsilon \]

for \( m \geq m_0 \). This means that \( \|f^{(m)} - F\| \leq \varepsilon \) for \( m \geq m_0 \) and, therefore, \( \|f^{(m)} - F\| \to 0 \) \( (m \to \infty) \).

Now keeping in mind Lemma 3 we put \( \nu_k = k\mu \) and introduce the spaces

\[ H^j = \left\{ \theta = \sum_{k=1}^{\infty} p_k(x^\gamma) x^k \mid p_k \in \mathcal{O}_\nu_k(T \setminus \{0\}), \|\theta\|_j = \sum_{k=1}^{\infty} \|(k + N + i\gamma \delta_t)^j p_k\| < +\infty \right\}, \]

\( j = 0, 1, \ldots, n \),

of exotic series. These turn out to be Banach spaces which further allows us to apply the implicit mapping theorem for a properly chosen operator in these very spaces, to prove the existence of a convergent exotic series solution of \( \mathbb{1} \).

**Lemma 5.** Each \( H^j, j = 0, 1, \ldots, n \), is a Banach space.

**Proof.** Let \( \{\theta^{(m)}\} \) be a fundamental sequence in \( H^j \), that is, for any \( \varepsilon > 0 \) there exists \( m_0 \) such that \( \|\theta^{(m)} - \theta^{(n)}\|_j \leq \varepsilon \) for \( m, n \geq m_0 \):

\[ \sum_{k=1}^{\infty} \|(k + N + i\gamma \delta_t)^j (p_k^{(m)} - p_k^{(n)})\| \leq \varepsilon. \] (8)

It follows that for each fixed \( k \) one has

\[ \|(k + N + i\gamma \delta_t)^j (p_k^{(m)} - p_k^{(n)})\| \leq \varepsilon, \]

that is, \( \{(k + N + i\gamma \delta_t)^j p_k^{(m)}\}_{m=1}^{\infty} \) is a fundamental sequence in \( \mathcal{O}_\nu_k(T \setminus \{0\}) \). Hence, by Lemma 4 it converges to some \( q_k \in \mathcal{O}_\nu_k(T \setminus \{0\}) \). From \( \mathbb{3} \) one deduces that for any fixed \( K \geq 1 \)

\[ \sum_{k=1}^{K} \|(k + N + i\gamma \delta_t)^j (p_k^{(m)} - p_k^{(n)})\| \leq \varepsilon \]

whenever \( m, n \geq m_0 \), therefore

\[ \sum_{k=1}^{K} \|(k + N + i\gamma \delta_t)^j p_k^{(m)} - q_k\| \leq \varepsilon \]
for \( m \geq m_0 \). Since the last inequality holds for any \( K \), one has

\[
\sum_{k=1}^{\infty} \| (k + N + i\gamma \delta_t)^j p_k^{(m)} - q_k \| \leq \varepsilon
\]

for \( m \geq m_0 \). Expanding \( q_k \) into the Laurent series in \( D \setminus \{0\} \),

\[
q_k(t) = \sum_{\ell \geq -\nu_k} q_{k\ell} t^{\ell},
\]

one sees that

\[
\tilde{q}_k(t) = \sum_{\ell \geq -\nu_k} \frac{q_{k\ell}}{(k + N + i\ell\gamma)^j} t^{\ell} \in \mathcal{O}_{\nu_k}(D \setminus \{0\})
\]

and \((k + N + i\gamma \delta_t)^j \tilde{q}_k = q_k\), that is, \( \theta^{(m)} \) converges to \( \theta = \sum_{k=1}^{\infty} \tilde{q}_k(x^{i\gamma}) x^k \) in \( H^j \). \[\square\]

**Lemma 6.** The Banach spaces \( H^j \) possess the following properties.

i) The following inclusions hold:

\[ H^n \subset \ldots \subset H^1 \subset H^0 \subset \mathcal{O}(S_1), \]

where \( S_1 \) is any open sector with the vertex at the origin, of opening less than \( 2\pi \) and of radius \( 1 \), such that

\[
S_1 \subset \{ (-1/\gamma) \ln R < \arg x < (-1/\gamma) \ln r \} \quad \text{if} \quad \gamma > 0,
\]

\[
S_1 \subset \{ (-1/\gamma) \ln r < \arg x < (-1/\gamma) \ln R \} \quad \text{if} \quad \gamma < 0.
\]

Furthermore, \( \| \theta \|_{j-1} \leq \| \theta \|_j \) for any \( \theta \in H^j \).

ii) For any \( \theta_1, \theta_2 \in H^0 \), there hold \( \theta_1 \theta_2 \in H^0 \) and \( \| \theta_1 \theta_2 \|_0 \leq \| \theta_1 \|_0 \| \theta_2 \|_0 \).

iii) For any \( \theta \in H^j \) one has \( \delta \theta \in H^{j-1} \) and the operator \( \delta : H^j \to H^{j-1} \) is continuous.

**Proof.** i) Let \( \theta = \sum_{k=1}^{\infty} p_k(x^{i\gamma}) x^k \in H^j \). Expanding

\[
(k + N + i\gamma \delta_t)^j p_k = \frac{1}{i^{\nu_k}} \sum_{\ell \geq 0} p_{k\ell} t^{\ell}, \quad \| (k + N + i\gamma \delta_t)^j p_k \| = \frac{1}{r^{\nu_k}} \sum_{\ell \geq 0} |p_{k\ell}| R^{\ell},
\]

one has

\[
(k + N + i\gamma \delta_t)^{j-1} p_k = \frac{1}{i^{\nu_k}} \sum_{\ell \geq 0} \frac{p_{k\ell}}{k + N + i(\ell - \nu_k)\gamma} t^{\ell},
\]

whence the estimate \( \| (k + N + i\gamma \delta_t)^{j-1} p_k \| \leq (1/k) \| (k + N + i\gamma \delta_t)^j p_k \| \) follows. This implies \( \| \theta \|_{j-1} \leq \| \theta \|_j \) and \( H^j \subset H^{j-1}, \ j = 1, \ldots, n. \)

As for the inclusion \( H^0 \subset \mathcal{O}(S_1) \), for any \( x \in S_1 \) one has

\[
|x^{i\gamma}| = e^{-\gamma \arg x} \in (r, R),
\]

therefore

\[
|p_k(x^{i\gamma}) x^k| \leq \max_{r \leq |t| \leq R} |p_k(t)| \leq \| p_k \|,
\]
and the convergence of $\sum_{k=1}^{\infty} \|p_k\|$ implies the uniform convergence of $\sum_{k=1}^{\infty} p_k(x^j) x^k$ in $S_1$.

ii) For any functions $f \in O_{\nu_k}(\overline{\mathcal{D}} \setminus \{0\})$, $g \in O_{\nu_k}(\overline{\mathcal{D}} \setminus \{0\})$ one has $\|fg\| \leq \|f\| \|g\|$. Indeed, let the Laurent expansions of $f$ and $g$ in $\overline{\mathcal{D}} \setminus \{0\}$ be, respectively,

$$f = \frac{1}{t^{\nu_k}} \sum_{s \geq 0} f_s t^s, \quad g = \frac{1}{t^{\nu_k}} \sum_{s \geq 0} g_s t^s.$$

Then

$$\|fg\| = \frac{1}{t^{\nu_k + \nu_\ell}} \sum_{s \geq 0} |\sum_{i=0}^{s} f_i g_{s-i}| R_s \leq \frac{1}{t^{\nu_k + \nu_\ell}} \sum_{s \geq 0} \left( \sum_{i=0}^{s} |f_i||g_{s-i}| \right) R_s = \|f\| \|g\|.$$

Therefore, for $\theta_1 = \sum_{k=1}^{\infty} p_k(x^j) x^k$, $\theta_2 = \sum_{k=1}^{\infty} q_k(x^j) x^k \in H^0$ there holds

$$\theta_1 \theta_2 = \sum_{k=2}^{\infty} \left( \sum_{\ell=1}^{k-1} p_\ell(x^j) q_{k-\ell}(x^j) \right) x^k,$$

where $p_\ell q_{k-\ell} \in O_{\nu_k}(\overline{\mathcal{D}} \setminus \{0\})$ for each $\ell$ (since $\nu_\ell + \nu_{k-\ell} = \nu_k$). Furthermore

$$\|\theta_1 \theta_2\|_0 = \sum_{k=2}^{\infty} \left\| \sum_{\ell=1}^{k-1} p_\ell q_{k-\ell} \right\| \leq \sum_{k=2}^{\infty} \sum_{\ell=1}^{k-1} \|p_\ell\| \|q_{k-\ell}\| = \|\theta_1\|_0 \|\theta_2\|_0.$$

iii) If $\theta \in H^j$ then $(\delta + N)\theta \in H^{j-1}$ and $\|(\delta + N)\theta\|_{j-1} = \|\theta\|_j$, hence

$$\delta \theta = (\delta + N)\theta - N\theta \in H^{j-1} \quad \text{and} \quad \|\delta \theta\|_{j-1} \leq (1 + N)\|\theta\|_j,$$

that is, $\delta : H^j \rightarrow H^{j-1}$ is a continuous operator. \qed

4 Finishing the proof of Theorem 1 by the implicit mapping theorem

Theorem 1, in view of Lemma 2, will follow from the following lemma.

Lemma 7. The equation (4) possesses an exotic series solution $\theta_0 \in O(S)$, where $S$ is a sector as in Theorem 1.

Proof. Consider the mapping $h : \mathbb{C} \times H^n \rightarrow H^0$ defined by the formula

$$h(\lambda, \theta) = \sum_{j=0}^{n} a_j(x^j)(\delta + N)^j \theta - \lambda x M(\lambda x, x^j, \theta, \delta \theta, \ldots, \delta^n \theta)$$

in some neighbourhood of the point $(0, 0) \in \mathbb{C} \times H^n$, $h(0, 0) = 0$. By Lemma 6, this mapping and its derivative $\partial h/\partial \theta$ are continuous at $(0, 0)$. The local solvability of the
equation $h(\lambda, \theta) = 0$, $\theta = \theta(\lambda)$, and as a consequence, Lemma 7 will follow from the implicit mapping theorem ([5 Th. 10.2.1]) if we prove that

$$\frac{\partial h}{\partial \theta}(0, 0) = \sum_{j=0}^{n} a_j(x^{\frac{\gamma}{\lambda}})(\delta + N)^j : H^n \to H^0$$

is a linear homeomorphism. To prove this, we should first prove that for any sequence $\{q_k(t)\}$, $q_k \in O_{\nu_k}(\overline{D} \setminus \{0\})$, such that $\sum_{k=1}^{\infty} \|q_k\| \leq +\infty$, there exists a unique sequence $\{p_k(t)\}$, $p_k \in O_{\nu_k}(\overline{D} \setminus \{0\})$, such that

$$\sum_{j=0}^{n} a_j(t)(k + N + i\gamma \delta t)^j p_k(t) = q_k(t), \quad (9)$$

and $\sum_{k=1}^{\infty} \|\sigma + \iota \gamma \delta t\| \leq +\infty$.

Each linear differential operator $L_k = \sum_{j=0}^{n} a_j(t)(k + N + i\gamma \delta t)^j$ is locally factorized into a product of the first order linear differential operators near its Fuchsian singular point $t = 0$,

$$L_k = (i\gamma)^n(\delta + b_1(t)) \ldots (\delta + b_n(t)), \quad b_i \in O(\overline{D})$$

(see, for example, [11 §19.3]). Furthermore, the exponents of $L_k$ at this point, $\lambda_1 = -b_1(0), \ldots, \lambda_n = -b_n(0)$, are the roots of the polynomial $P(\lambda) = \sum_{j=0}^{n} a_j(0)(k + N + i\gamma \lambda)^j$ and thus are not integers (for the simplicity of notation, we do not put here the index $k$ to the objects related to $L_k$).

The equation (9) takes the form

$$(i\gamma)^n(\delta + b_1(t)) \ldots (\delta + b_n(t)) p_k = q_k.$$

To understand an idea, let us study it in the case $n = 1$. A general solution of the equation

$$\delta t y + b_1(t)y = 0$$

has the form

$$y(t) = c e^{-\int \frac{b_1(t)}{t} dt} = c t^{\lambda_1} \tilde{b}_1(t), \quad c \in \mathbb{C}, \quad \tilde{b}_1 \in O(\overline{D}), \quad \tilde{b}_1(0) \neq 0,$$

therefore

$$(\delta t + b_1(t)) p_k = q_k$$

implies a general expression for $p_k$:

$$p_k(t) = c t^{\lambda_1} \tilde{b}_1(t) + t^{\lambda_1} \tilde{b}_1(t) \int \frac{q_k(t)}{b_1(t)} t^{-\lambda_1-1} dt.$$

Since the power exponent $\lambda_1$ is non-integer, the above expression represents a function meromorphic at $t = 0$ only if $c = 0$. Expanding $q_k(t)/\tilde{b}_1(t)$ into a Laurent series near the point $t = 0$,

$$\frac{q_k(t)}{\tilde{b}_1(t)} = \sum_{\ell > -\nu_k} q_k \ell t^\ell,$$
we have
\[ p_k(t) = \tilde{b}_1(t) \sum_{\ell \geq -v_k} \frac{q_{k\ell}}{\ell - \lambda_1} t^\ell \in \mathcal{O}_{\nu_k}(\mathcal{D} \setminus \{0\}). \]

Using a similar reasoning, one concludes by induction for a general \( n \) that the equation \( k \) has a unique solution \( p_k \in \mathcal{O}_{\nu_k}(\mathcal{D} \setminus \{0\}) \). To prove the convergence of \( \sum_{k=1}^{\infty} \|(k + N + i\gamma_0 t)^n p_k(t)\| \), we write
\[
(k + N + i\gamma_0 t)^n p_k(t) = q_k(t) - \sum_{j=0}^{n-1} a_j(t) (k + N + i\gamma_0 t)^j p_k(t).
\]

From the proof of Lemma 6 (i) it follows that
\[
\|(k + N + i\gamma_0 t)^j p_k\| \leq \frac{1}{k^{n-j}} \|(k + N + i\gamma_0 t)^n p_k\|,
\]

hence
\[
\|(k + N + i\gamma_0 t)^n p_k\| \leq \|q_k\| + C \left( \frac{1}{k} + \ldots + \frac{1}{k^n} \right) \|(k + N + i\gamma_0 t)^n p_k\|.
\]

Therefore,
\[
\|(k + N + i\gamma_0 t)^n p_k\| \leq \left( 1 - \frac{C}{k} - \ldots - \frac{C}{k^n} \right)^{-1} \|q_k\|
\]
for \( k \) large enough, and the convergence of the series \( \sum_{k=1}^{\infty} \|q_k\| \) implies that of \( \sum_{k=1}^{\infty} \|(k + N + i\gamma_0 t)^n p_k\| \). This proves the bijectivity of \( \partial h/\partial \theta(0, 0) \).

From the above it follows that if \( \theta_1 = \sum_{k=1}^{\infty} p_k(x^{i\gamma}) x^k \in H^n \), \( \theta_2 = \sum_{k=1}^{\infty} q_k(x^{i\gamma}) x^k \in H^0 \) and \( \partial h/\partial \theta(0, 0) \theta_1 = \theta_2 \), then \( \|(k + N + i\gamma_0 t)^n p_k\| \leq A\|q_k\| \), that is, \( \|\theta_1\|_n \leq A\|\theta_2\|_n \). This, with Lemma 6, implies that \( \partial h/\partial \theta(0, 0) : H^n \rightarrow H^0 \) is a linear homeomorphism.

Thus, basing on the implicit mapping theorem for the mapping \( h \) we conclude that there exist \( \lambda_0 > 0 \) and
\[
\theta_{\lambda_0} = \sum_{k=1}^{\infty} p_k(x^{i\gamma}) x^k \in H^n
\]
such that
\[
\sum_{j=0}^{n} a_j(x^{i\gamma})(\delta + N)^j \theta_{\lambda_0} = \lambda_0 x M(\lambda_0 x, x^{i\gamma}, \theta_{\lambda_0}, \delta \theta_{\lambda_0}, \ldots, \delta^n \theta_{\lambda_0}). \tag{10}
\]

Finally, define an operator
\[
i_{\lambda_0} : \theta_{\lambda_0} \mapsto \sum_{k=1}^{\infty} p_k(x^{i\gamma}) (x/\lambda_0)^k
\]
from \( H^0 \) to \( \mathcal{O}(S) \), \( S = \lambda_0 S_1 \), which clearly commutes with \( \delta \): \( \delta(i_{\lambda_0} \theta) = i_{\lambda_0} (\delta \theta) \). Then applying it to the both sides of the equality \( \tag{10} \) we obtain that the exotic series \( \theta_0 = i_{\lambda_0} (\theta_{\lambda_0}) \in \mathcal{O}(S) \) is a solution of \( \tag{1} \). \( \square \)
5 An example: the third Painlevé equation

Let us consider the third Painlevé equation

\[-x^2 y y'' + x^2 (y')^2 - x y' + a x y^3 + b x y + c x^2 y^4 + d x^2 = 0,\]  \hspace{1cm} (11)

where \(a, b, c, d\) are complex parameters. Using general methods exposed in [2] one can find an exotic formal series solution \(y = \varphi\) of the form

\[\varphi = -\frac{4C \gamma^2 x^{2i\gamma}}{4(c\gamma^2 + a^2)x^{2i\gamma} - 4aC \gamma x^{i\gamma} + C^2} x^{-1} + \sum_{k=1}^{\infty} c_k(x^{i\gamma}) x^k,\]

where \(C \in \mathbb{C}\) and \(\gamma \in \mathbb{R}\) are arbitrary nonzero constants and functions \(c_k(t)\) are meromorphic at \(t = 0\) (since the equation (11) is not only analytical but even polynomial, for applying Theorem 1 there is no requirement for an exotic formal series solution to begin with a strictly positive power of \(x\)).

Let us rewrite the equation (11) by means of the operator \(\delta\) and get the equation

\[-y \delta^2 y + (\delta y)^2 + a x y^3 + b x y + c x^2 y^4 + d x^2 = 0,\]

that is, \(F(x, y, \delta y, \delta^2 y) = 0\), where

\[F(x, y_0, y_1, y_2) = -y_0 y_2 + y_1^2 + a x y_0^3 + b x y_0 + c x^2 y_0^4 + d x^2.\]

The partial derivatives of \(F\) along the formal solution \(\varphi\) are

\[\frac{\partial F}{\partial y_2}(x, \Phi) = \left(\frac{4\gamma^2}{C} x^{i\gamma} + \ldots\right) x^{-1} + \ldots,\]

\[\frac{\partial F}{\partial y_1}(x, \Phi) = \left((8 - 8i) x^{i\gamma} + \ldots\right) x^{-1} + \ldots,\]

\[\frac{\partial F}{\partial y_0}(x, \Phi) = (-8i x^{i\gamma} + \ldots) x^{-1} + \ldots.\]

Since they all begin with the same power \(x^{-1}\) and the order of their leading coefficient at \(t = 0\) (with respect to \(t = x^{i\gamma}\)) is the same (= 1), by Theorem 1 the series \(\varphi\) converges in a sectorial domain near the origin.

References

[1] Brezhnev, Yu. V., A \(\tau\)-function solution of the sixth Painlevé transcendent, Theor. Math. Phys. 161:3 (2009), 1616–1633.

[2] Bruno, A. D., Exotic expansions of solutions to an ordinary differential equation, Doklady Math. 76:2 (2007), 729–733.

[3] Bruno, A. D., Goryuchkina, I. V., All asymptotic expansions of solutions to the sixth Painlevé equation, Doklady Math. 76:3 (2007), 851–855.
[4] Bruno, A. D., Parusnikova, A. V., Local expansions of solutions to the fifth Painlevé equation, *Doklady Math.* **83**:3 (2011), 348–352.

[5] Dieudonné, J. Foundations of Modern Analysis. Academic Press, New York, 1960.

[6] Gontsov, R. R., Goryuchkina, I. V., On the convergence of generalized power series satisfying an algebraic ODE, *Asympt. Anal.* **93**:4 (2015), 311–325.

[7] Gontsov, R. R., Goryuchkina, I. V., The Maillet–Malgrange type theorem for generalized power series, *Manuscripta Math.* **156**:1 (2018), 171–185.

[8] Gontsov, R. R., Goryuchkina, I. V., On the convergence of formal Dulac series satisfying an algebraic ODE, *Sb. Math.* **210** (2019), to appear.

[9] Guzzi, D., Tabulation of Painlevé 6 transcendents, *Nonlinearity* **25** (2012), 3235–3276.

[10] Guzzi, D., Poles distribution of PVI transcendents close to a critical point, *Physica D* **241** (2012), 2188–2203.

[11] Ilyashenko, Yu. S., Yakovenko, S. Yu. Lectures on Analytic Differential Equations. Grad. Stud. Math. **86**, AMS, 2008.

[12] Malgrange, B., Sur le théorème de Maillet, *Asympt. Anal.* **2** (1989), 1–4.

[13] Shimomura, S., Critical behaviours of the fifth Painlevé transcendents and the monodromy data, *Kyushu J. Math.* **71** (2017), 139–185.