Quantizing data for distributed learning

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Abstract

We consider machine learning applications that train a model by leveraging data distributed over a network, where communication constraints can create a performance bottleneck. A number of recent approaches are proposing to overcome this bottleneck through compression of gradient updates. However, as models become larger, so does the size of the gradient updates. In this paper, we propose an alternate approach, that quantizes data instead of gradients, and can support learning over applications where the size of gradient updates is prohibitive. Our approach combines aspects of: (1) sample selection; (2) dataset quantization; and (3) gradient compensation. We analyze the convergence of the proposed approach for smooth convex and non-convex objective functions and show that we can achieve order optimal convergence rates with communication that mostly depends on the data rather than the model (gradient) dimension. We use our proposed algorithm to train ResNet models on the CIFAR-10 and ImageNet datasets, and show that we can achieve an order of magnitude savings over gradient compression methods.

I. INTRODUCTION

Consider a distributed machine learning application, where a set of terminal nodes observe data instances and are available both to provide this data and to computationally support an agent that wishes to train a model. A possible setup could be that the learning agent trains a neural network using a stochastic gradient descent approach leveraging the observations of the terminal nodes. The challenge we consider is that the nodes are connected to the server through a weak communication fabric. For instance, they are connected through wireless, bandwidth constrained links. The fact that communication constraints can create a performance bottleneck has been recognized in the literature, and a number of recent approaches propose to overcome this through compression of gradient updates [1], [2], [3], [4], [5], [6], [7]. In this paper we ask: can we do better (in number of bits communicated) than transmitting quantized gradients?

Our work starts from two observations. The first is that, although the nodes may continuously be observing data, not all the data are equally useful - perhaps a subset of the data is all we need. This can be thought of as a generalization of the notion of support vectors in Support Vector Machines (SVM); effectively, we ask, which of the data instances are most helpful for learning. As a simple example, we experimentally verified that training a neural network classifier on the data instances inside the triangle in Figure 1 yields approximately the same validation accuracy as when we train the classifier on the full set of data instances. The challenge is that, we would like to decide which are the instances to send in an online manner, without having access to the whole dataset in advance.

The second observation is that although the dimension of models (and their gradient updates) gets larger and larger - the dimension of data does not. Indeed, in the pursuit of improved accuracy, neural networks have evolved
from thousands to millions to hundreds of millions of parameters, echoed by a similar increase in the amount of sampled training data [8], [9], [10]. However, the dimension (number of features) of the data-instances themselves, whether images, natural signals, or clinical data, has not and is not expected to change much. Within this landscape, it makes sense to leverage the lower dimensionality of data to achieve a lower communication cost.

In this paper we propose such a scheme, that can support learning over applications where the size of gradient updates is prohibitive and the number of data samples is large. The proposed compression approach consists of two parts: (1) a selection scheme and (2) a quantization scheme. The selection scheme first decides which data instances are communicated depending on how important they are to learning; afterwards, these selected instances are quantized using our quantization scheme. The quantization scheme combines aspects of dataset quantization and gradient compensation, and aims to enable the learning agent perform gradient updates with low (or no) performance loss. In our theoretical analysis, we find that our scheme uses $O(d \log_2(h))$ bits in the worst case to achieve the optimal convergence rate, where $d$ and $h$ are the dataset and model dimensions, respectively. For large models ($h \gg d$), this is much smaller than $O(h)$, the information theoretic lower bound derived in [7] on the number of bits needed to convey quantized gradients to achieve optimal convergence rate. This is because we take advantage of the data dependency of gradients, which is ignored by the oracle used for the lower bounds in [7].

Our main contributions are:

- We exploit the fact that computed gradients are dependent on the datapoints to design DaQuSGD, a quantization approach that only communicates $O(d \log_2 h)$ bits in the worst case per gradient update, where $d$ is the
dimension of a single datapoint and $h$ is the dimension of the model parameters.

- We additionally exploit the fact that only a small portion of datapoints could be sufficient for a reliable model update in order to design an online sample selection approach that acts as a preliminary sampling step to our DaQuSGD approach. Our sample selection scheme provides savings in both communication and computation for the same performance.

- We analyze the convergence of our scheme for convex and non-convex objective functions, and prove that it achieves the same convergence rate as unquantized stochastic gradient descent.

- We show numerically on the CIFAR-10 and ImageNet datasets that DaQuSGD enables to train ResNet models that have more than 20M parameters, to the accuracy achieved by unquantized stochastic gradient descent, while using an order of magnitude less bits compared to gradient quantization schemes.

**Related work:** Stochastic gradient descent has been very successful in training neural networks [1], [2], [11]. Motivated from communication constraints, a number of works proposed quantized versions of stochastic gradient descent, see for example [11], [2], [12], [13], [3], [4], [5], [6], [7]. All these works are based on gradient quantization and their implementation typically uses $O(h)$ bits per gradient to achieve the same convergence rate as the unquantized gradient descent. In fact, it was shown in [7] that given stochastic gradients that lies in a unit $\ell_2$ ball, the minimum number of bits needed to achieve the same convergence rate as the unquantized stochastic gradient descent is $\Omega(h)$ bits per gradient.\(^1\) This line of work assumes gradient quantizers that have no memory, i.e., previously (quantized) gradients are not used in updates. Using memory with quantizers, sparsification methods were considered [14], [15], [16], [17], [18], that improve communication efficiency at the cost of increasing the gradient iterations until convergence.\(^2\)

The work in [19] considers data quantization for generalized linear models learning. However, in a generalized linear model, the gradient dimension is the same as the data dimension and the gradient in this case is only a scaled version of the data instance. In this paper, we consider data quantization for a general learning model. Our work is the first, as far as we know, that leverages dataset-based quantization to achieve the optimal convergence rate at a communication cost of $O(d\log_2(h))$ that mostly depends on the dataset and not the model dimension. To further reduce the communication cost, we propose a sample selection algorithm. There is a rich body of work on sample selection in centralized settings [20], [21], however, these schemes are either computationally expensive, cannot be applied to deep neural networks, require the knowledge of full dataset, or cannot be applied in a distributed setting. Schemes that are based on influence function [22], [23], [24] are also related. Influence functions approximate the effect of training samples on the model predictions during testing. This requires the computation of the gradient and Hessian of the model, which can be computationally expensive in deep networks. Moreover, influence functions are not well understood for deep models and non-convex functions [25] and are shown to be fragile in deep learning [26]. We present and analyze a simple sample selection scheme that requires only the computation of the forward path of the model. Our sample selection scheme can be applied for deep networks in distributed settings.

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1 The reason why the lower bound in [7] can be broken is discussed in Section III.

2 Effective implementations of these also use $O(h)$ bits per gradient iteration [14], [15], [16], [18].
This approach offers reduction in both communication and computation costs without sacrificing performance. Our work also differs from traditional signal compression [27]: our quantization does not aim to maintain the ability to reconstruct data, but instead the ability to perform gradient updates with low (or no) performance loss.

II. Setup

A learning agent wants to use data from a space \( Z \subseteq \mathbb{R}^d \) to generate a model in a space \( \mathcal{W} \subseteq \mathbb{R}^h \). The data samples are collected at distributed nodes that can communicate with the agent using at most \( R \) bits per transmission. The overall dataset \( S = (Z^{(1)}, ..., Z^{(N)}) \) consists of \( N \) samples from the space \( Z \) which is assumed to bounded, i.e., \( \|Z\|_2 \leq B \). We also assume that \( \mathcal{W} \) is bounded such that \( \|W - W'\|_2^2 \leq D^2, \forall W, W' \in \mathcal{W} \). The objective of the learning agent is to minimize the empirical risk of the output model. For a given model \( W \in \mathcal{W} \), this risk is given by

\[
L(W) = \frac{1}{N} \sum_{i=1}^{N} \ell(W, Z^{(i)}),
\]

where \( \ell : \mathcal{W} \times Z \rightarrow \mathbb{R}^+ \) is a loss function. The loss function \( \ell \) is known at all nodes in the system.

**Learning algorithm.** We are interested in gradient-based learning algorithms. Let \( \nabla L(W) \) be the gradient of \( L(W) \) and \( g_Z(W) = \nabla \ell(W, Z) \) be the gradient of the loss function (w.r.t. \( W \)). It is well known that if \( Z \in S \) is sampled uniformly at random, \( g_Z(W) \) is an unbiased stochastic gradient of \( L(W) \), i.e., satisfies \( \mathbb{E}[g_Z(W)] = \nabla L(W) \). The gradient \( g_Z(W) \) can be calculated at the node sampling \( Z \) from the dataset. Let \( \hat{g}_Z(W) \) be a general stochastic estimate of the gradient \( \nabla L(W) \), calculated by the learning agent. This can be the true stochastic gradient \( g_Z(W) \) estimated from a datapoint \( Z \), or a low-precision mapping of \( g_Z(W) \) \([5], [28], [29], [4], [7]\), or a function of both \( g_Z(W) \) and \( Z \). Using this estimate, the model is updated at iteration \( j + 1 \) as

\[
W^{(j+1)} = W^{(j)} - \eta \hat{g}_Z(W^{(j)}) \left(W^{(j)}\right),
\]

where \( \eta \) is the learning rate.

**Quantization.** At the \( j \)-th iteration, a node \( s_j \) has access to a datapoint \( Z^{(j)} \). The learning agent generates the stochastic gradient \( \hat{g}_Z(W^{(j-1)}) \) by receiving information from the \( s_j \)-th node through an \( R \)-bit quantizer \( Q_j \), where we assume that \( Q_j \) has access to both the current datapoint \( Z^{(j)} \), and the latest model \( W^{(j-1)} \). An \( R \)-bit quantizer consists of mappings \( (Q^e_j, Q^d_j) \), with an encoder mapping \( Q^e_j : Z \times \mathcal{W} \rightarrow \{0,1\}^R \) and a decoder mapping \( Q^d_j : \{0,1\}^R \rightarrow \mathbb{R}^h \), where \( R \in \{0, R\} \). Note that in our setup we allow the encoder to transmit zero bits (by picking \( R = 0 \)) for some samples, which corresponds to not sending the sample. The overall quantizer \( Q_j = Q^d_j \circ Q^e_j \) captures the combined effect of the encoder and decoder. Note that in the quantization schemes that directly quantize gradients, the encoder mapping \( Q^e_j \) first computes the stochastic gradient \( g_Z(W^{(j-1)}) \) from the unquantized datapoint \( Z^{(j)} \) and then maps \( g_Z(W^{(j-1)}) \) to \( R \) bits. In this paper, we explore an alternative approach where \( Q^e_j \) can make use of the datapoint, \( Z^{(j)} \), directly. We highlight that we do not assume any extra information at the distributed nodes, i.e., they still only have access to the latest model and a full-precision datapoint.
Assumptions on the loss function. In the following sections, we assume that the function $\ell(W, Z)$ is $C_w$-smooth in $W$ for all $Z \in Z$, i.e., the gradient $g_Z(W) = \partial \ell(W, Z) / \partial W$ is $C_w$-Lipschitz continuous in $W$ for all $Z \in Z$. In other words

$$
\|g_Z(W) - g_Z(W')\|_2 \leq C_w \|W - W'\|_2, \quad \forall Z \in Z.
$$

Moreover, the gradient $g_Z(W)$ is assumed to be $C_z$-Lipschitz continuous in $Z$ for all $W \in W$, i.e.,

$$
\|g_Z(W) - g_Z'(W)\|_2 \leq C_z \|Z - Z'\|_2, \quad \forall W \in W.
$$

### III. Preliminaries and Motivation

**Convergence of stochastic gradient descent.** In our setup, the learning agent aims to learn a hypothesis of dimension $h$, using stochastic estimates of the gradient of the empirical risk function $L$, where these stochastic estimates are conveyed with a very low precision representation $\{0, 1\}^R$, that is, $R \ll h$. In this work, we are interested in providing convergence guarantees when learning from low precision estimates.

It is well known that for smooth convex risk functions, if stochastic gradient is performed with unbiased stochastic gradients that have bounded variance $\sigma^2$, then it converges with $O\left(\frac{\sigma}{\sqrt{n}}\right)$ [30]. The exact theorem is provided in Appendix B for completeness. It was also shown in [31] that under the mentioned assumptions, $O\left(\frac{1}{\sqrt{n}}\right)$ is a lower bound on the convergence rate of stochastic gradient descent. Hence, throughout the paper we refer to $O\left(\frac{1}{\sqrt{n}}\right)$ as the order-optimal convergence rate; we take away that to achieve it, it suffices to construct a quantized stochastic gradient $\hat{g}_Z(W)$ that satisfies the unbiasedness and bounded variance, where the variance bound should be $O(\sigma^2)$ (the variance cannot grow with other system parameters such as $h, d$). We follow this well known technique to prove the order-optimal convergence of DaQuSGD in Section IV.

Similar to the smooth convex risk functions case, for smooth non-convex risk functions, the stochastic gradient descent converges to a local optimal point provided that the stochastic gradients are unbiased and have bounded variance. The main theorem statement [31], is reiterated in Appendix B for completeness.

**Direct gradient quantization is not always efficient.** The work in [7] shows that if the gradients $g_Z(W)$ of the risk function $L(W)$ lie in a unit $\ell_2$ ball and the quantizer has no memory, then the minimum number of bits required for quantizing the gradient to guarantee optimal convergence rate (same as unquantized gradient descent) is $\Omega(h)$ bits per iteration; this lower bound is derived using an oracle that has access only to $g_Z(W)$ and not $Z$. For memory-based quantizers, practical implementations typically require $\Omega(h)$ bits per iteration, even when sparsification methods [2], [32], [14] are used with quantization [18]

In the following, we provide a simple illustrating example where the $\Omega(h)$ lower bound does not hold (an additional example is also provided in Appendix C). This example do not suffer from the $\Omega(h)$ lower bound since the possible gradient values do not span the whole unit $\ell_2$ ball; the gradients live in a low dimensional space through their implicit dependence on $Z$.

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3While (3) is a standard assumption [6], [13], [7], (4) is particular to our case since we relate gradients to data. We observed in our numerical evaluation that the ratio $\frac{\|g_Z(W) - g_Z(W')\|_2}{\|Z - Z'\|_2}$ was upper bounded by 1 when calculated using the sampled $Z$ and $Z_Q$ as its quantized version, see Appendix A for additional discussion.
Example. Consider the simple case where \( d = 1, h \gg d \), and the loss function is \( \ell(W, Z) = \frac{Z}{\sqrt{h}} 1^T W \), where \( \mathcal{W} = \{ W \in \mathbb{R}^h ||W||_2 \leq 1 \} \), and \( Z = \{ Z \in \mathbb{R} | |Z| \leq 1 \} \). In this case, the best bound we can get is \( \|g_Z(W)\|_2 \leq 1 \).

Consider a quantizer that only sends 1 bit per data sample to the learning agent and operates as follows. At each time \( j \), \( Q^e_j, Q^d_j \) are chosen to be

\[
Q^e_j = Q^e(x) = \begin{cases} 
1 & \text{if with probability } \frac{x + 1}{2}, \\
0 & \text{if with probability } \frac{1-x}{2}.
\end{cases}
\]

\[
Q^d_j(x) = Q^d(x) = 2x - 1.
\]

The stochastic gradient is chosen to be \( \frac{z_Q^{(j)}}{\sqrt{h}} 1 \), where \( z_Q^{(j)} \) is the output of the decoder \( Q^d_j \). It is not difficult to see that the constructed gradient is an unbiased estimate of \( \nabla L(W) \) and has a variance that is bounded by 1; we also prove it in Appendix C. Hence, using this unbiased gradient estimate, we can achieve the optimal convergence rate. In this example the gradient takes values only on the line segment \( \left\{ \frac{Z}{\sqrt{h}} 1 | |Z| \leq 1 \right\} \); it does not span the whole unit \( \ell_2 \) ball.

IV. DATASET-QUANTIZED STOCHASTIC GRADIENT DESCENT (DAQUSGD)

This section presents DAQUSGD, our proposed approach for dataset-based quantized stochastic gradient descent, summarized in Algorithm I.

The encoder \( Q^e \) consists of two components: a datapoint quantization step, followed by a gradient correction step. Given a datapoint \( Z \), a node applies a data quantizer \( \text{DataQ}(\cdot, m) \) parameterized by an integer \( m \), to create \( Z_Q = \text{DataQ}(Z, m) \), the quantized version of \( Z \). \( \text{DataQ} \) uniformly quantizes each feature of \( |Z| \) into \( m \) quantization levels as well as implicitly communicates the sign of each feature. At a high level, the quantizer \( \text{DataQ} \) can produce \( Z_Q \) such that \( \|g_{Z_Q}(W) - g_Z(W)\|_2 \) is reasonably small. However, the quantized estimate \( g_{Z_Q}(W) \) is no longer an unbiased estimate of \( L(W) \). To fix this, the next step sends a very low-precision amendment to \( g_{Z_Q}(W) \) to ensure that the central node has access to an unbiased estimate of \( \nabla L(W) \) with bounded variance. Although the error \( \Delta = g_Z(W) - g_{Z_Q}(W) \) exists in a large dimensional space \( \mathbb{R}^h \), the fact that \( \|g_{Z_Q}(W) - g_Z(W)\|_2 \) is small enables us to quantize \( \Delta \) with a small number of bits (either 1 bit if the terminal nodes and the algorithm have shared randomness or \( \lceil \log_2(h) \rceil + 1 \) otherwise). Thus the main load of the quantization is done in the data space of dimension \( d \ll h \). We next describe the two components.

A. Datapoint quantization [DataQ]

\( \text{DataQ}(\cdot, m) \) is parametrized by a tuning parameter \( m \) capturing the number of levels used to quantize each feature (see also pseudocode in Appendix D). Given a dataset sample \( Z \in \mathbb{R}^d \), where \( ||Z||_2 \leq B \), we express it as \( Z = Z^+ - Z^- \), where \( Z^+ = \max(Z, 0) \) is a vector that captures the magnitude of the positive elements of \( Z \) (negative values are replaced by zero) and similarly \( Z^- = \max(-Z, 0) \) for the negative elements. We create the non-negative concatenation vector \( \tilde{Z} = [Z^+, Z^-]^T \in \mathbb{R}^{2d} \), that inherits the following properties from \( Z \): (i) \( \tilde{Z} \) has the same \( \ell_2 \) norm as \( Z \); (ii) \( \tilde{Z} \) implicitly captures the sign of the \( i \)-th element of \( Z \) by observing whether the \( i \)-th component of the subvector \( Z^+ \) is zero or not; (iii) If \( ||Z||_2 \leq B \), then \( 0 \leq \tilde{Z} \leq B \) elementwise. Then, for each
Algorithm 1 DaQuSGD

1: Initialize $W^{(0)}$, hyperparameter: $m$ number of levels in data quantizer; Suppose $\eta_j$ follows a certain learning rate schedule.

2: for $j = 1$ to $n$ do

3: Node selected in the $j$-th iteration:

4: Sample datapoint $Z^{(j)}$

5: $Z_Q^{(j)}, b_Z^{(j)} \leftarrow$ DataQ($Z^{(j)}, m$);

6: $b_Z^{(j)}$ is the lossless binary compression of $Z_Q^{(j)}$

7: $\Delta \leftarrow g_{Z^{(j)}}(W^{(j-1)}) - g_{Z^{(j)}}(W^{(j-1)})$

8: $b^{(j)} \leftarrow$ GradCorrQ($\Delta, m$);

9: $b^{(j)}$ is binary compression of estimate $\tilde{\Delta}$ of $\Delta$

10: Send $b_Z^{(j)}$ and $b^{(j)}$ to the learning agent

11: Learning Agent:

12: Reconstruct $Z_Q^{(j)}$ and $\tilde{\Delta}$ from $b_Z^{(j)}$ and $b^{(j)}$

13: $\tilde{g} \leftarrow g_{Z_Q^{(j)}}(W^{(j-1)}) + \tilde{\Delta}$

14: $W^{(j)} \leftarrow W^{(j-1)} - \eta_j \tilde{g}$

15: Broadcast $W^{(j)}$ to all nodes

end for

coordinate $i$, we choose $m$ equally spaced quantization levels in the interval $[0, B]$, where the $i$-th quantization level is $q_i = \frac{iB}{m-1}$, $i \in [m-1]$. Now, $\forall j \in [d]$, let

$$
a_j(Z) = \arg\max_{i \in [m-1]} \{ q_i | q_i \leq |Z^+_j| \},
$$

$$
b_j(Z) = \arg\max_{i \in [m-1]} \{ q_i | q_i \leq |Z^-_j| \}. 
$$

(6)

The integer vectors $a = [a_1, \cdots, a_d]^T$ and $b = [b_1, \cdots, b_d]^T$ capture the indices of the levels that are just below the values in $\tilde{Z}$; DataQ maps the values in $\tilde{Z}$ to exactly the values indexed by $a$ and $b$. In particular, $Z^+$ is quantized to $a \frac{B}{m-1}$ and $Z^-$ is quantized to $b \frac{B}{m-1}$. As a result, we get that

$$
Z_Q = \text{DataQ}(Z, m) = (a - b) \frac{B}{m-1}.
$$

Communication Cost. The quantized point $Z_Q$ generated by DataQ is uniquely represented by the integer vectors $a$ and $b$. Thus, the communication cost equals the number of bits needed for lossless compression of $a$ and $b$. Let us define the set $S$ to be

$$
S = \left\{ (w, v) \in \mathbb{N}^d \times \mathbb{N}^d \mid \|w\|_1 + \|v\|_1 \leq (m-1)^2 \right\}. 
$$

(7)

We will argue that for any $(a, b)$ generated by DataQ as in (6), $(a, b) \in S$ with probability 1. As a result, we only need at most $\lceil \log_2(|S|) \rceil$ bits per sample to communicate $(a, b)$ to the central node. We almost proved Theorem 1 (the complete proof is provided in Appendix E) that provides an upper bound on the communication cost.
Remark 1 (Bounded Second Moment). If the second moment satisfies $\mathbb{E}[\|Z\|^2] \leq B^2$, but $\|Z\|_2$ is not bounded almost surely, we can send $\|Z\|_2$ with full precision and then use DataQ to quantize $Z/\|Z\|_2$, with $B = 1$. This adds an overhead of sending (only) one scalar in full precision; moreover, we can now guarantee that $\mathbb{E}[\|Z\|^2] \leq (1 + \sqrt{2})^2$ instead of the universal bound stated on $\|Z\|_2$ in Theorem 1. Additionally, we have that $\mathbb{E}[\|Z - Z_Q\|_\infty] \leq \frac{B}{m-1}$.

Remark 2 (Splitting $Z$). From Theorem 1, we have that the dependency of the number of bits on the dataset dimension $d$ is $\Omega(\log_2(d))$ when $m$ is small. However, without splitting $Z$ into $Z^+, Z^-$, we cannot represent the quantized values of $Z$, with a set of positive integers similar to $S$, without directly sending the signs in $Z$ which requires at least $d$ bits. Note that for small values of $d$, splitting might require larger number of bits than without splitting, however, we are interested in how the number of bits grow with $d$.

Remark 3 (Quantization of gradient using DataQ). DataQ with stochastic quantization \cite{5,33,34} if applied to the gradient $g_Z(W)$ with $m = \sqrt{n} + 1$ would result in $\hat{g}_Z(W)$ that satisfies $\mathbb{E}[\hat{g}_Z(W)g_Z(W)] = g_Z(W)$, $\|\hat{g}_Z(W) - g_Z(W)\|_2 \leq 2B$, hence, achieves the optimal convergence rate of $O(\frac{1}{\sqrt{n}})$. This uses at most $\log_2(2h) + 2h \log_2(3e) = O(h)$ bits, thus, achieving the communication lower bound in \cite{7}. Unlike the results in \cite{5,29} which provide a guarantee on communication cost in terms of expectation, this provides a uniform upper bound on the required number of bits.

B. Gradient correction [GradCorrQ]

We here describe the gradient correction procedure GradCorrQ (see also the pseudocode in Appendix D) that augments the quantized gradient estimated from $Z_Q = \text{DataQ}(Z,m)$. Using \cite{4} and Theorem 1, we get

$$\|g_Z(W) - g_{Z_Q}(W)\|_2 \leq C_z\|Z - Z_Q\|_2$$

$$\leq C_zB\frac{\sqrt{d}}{m-1}, \quad \forall W \in \mathcal{W}. \quad (8)$$

Let $\Delta$ be the error in computing the gradient using $Z_Q$ defined as $\Delta = g_Z(W) - g_{Z_Q}(W)$. The node that sent $Z_Q$ to the learning agent also communicates an estimate of $\Delta$, by quantizing it as follows. The learning agent and the node agree on a random number generator that generates $i^* \in [1 : h]$ uniformly at random. If there is no common

\footnote{Instead of mapping to a level below the feature, we map it either above or below with a probability depending on the distance.}
randomness, then the node can send $i^*$ to the agent using $\lceil \log_2(h) \rceil$ bits. The node also sends 1 bit $e_g$ generated as follows

$$
e_g = \begin{cases} 0 \text{ w.p. } & \frac{C_z B \sqrt{2} \sigma^2}{2C_z B \frac{\Delta_i}{m-1}}, \\ 1 \text{ w.p. } & \frac{C_z B \sqrt{2} \sigma^2 + \Delta_i}{2C_z B \frac{\Delta_i}{m-1}}. \end{cases}$$

(9)

This, in a way, uses random coordinate selection and stochastic quantization [5]. The agent constructs an estimate $\hat{\Delta}$ and uses it to create the quantized stochastic gradient as follows

$$\tilde{g}_Z(W) = g_{Z_1}(W) + \hat{\Delta},$$

where

$$\hat{\Delta}_i = \begin{cases} (2e_g - 1)C_z B h \frac{\sqrt{d}}{m-1} & \text{if } i = i^*, \\ 0 & \text{otherwise}. \end{cases}$$

(10)

Lemma [1] which is proved in Appendix [1] shows the unbiasedness and variance properties of the gradient estimate. From this, Theorem [2] which summarizes our result, follows.

**Lemma 1.** The quantized stochastic gradient $\tilde{g}_Z(W)$ is an unbiased estimate of $\nabla L(W)$ and satisfies $\|\tilde{g}_Z(W) - \nabla L(W)\|_2 \leq C_z B \left(2 + (h+1)\frac{\sqrt{d}}{m-1}\right)$.

**Theorem 2.** (1) Let $W$ be convex and let $\ell(W, Z)$ be a convex function satisfying the loss function assumptions in Section [II]. Let $\|Z\|^2_2 \leq B^2$ and $\sup_{W \in W} \|W - W^{(0)}\|^2_2 = \bar{D}^2$, where $W^{(0)} \in W$ is the initial model. Assume that the stochastic gradient descent uses the quantized gradients $\tilde{g}_Z(W)$ obtained through DaQuSGD, with step size $\eta = \left(\frac{C_w + \frac{1}{\gamma}}{\gamma}\right)^{-1}$, where $\gamma = \frac{\bar{D}}{\sigma} \sqrt{\frac{2}{n}}$. Then

$$\mathbb{E} \left[ L \left( \frac{1}{n} \sum_{i=1}^n W^{(i)} \right) \right] - L(W^*) \leq \bar{D} \tilde{\sigma} \sqrt{\frac{2}{n}} + \frac{C_w}{n} \bar{D}^2,$$

with $\tilde{\sigma} = C_z B \left(2 + (h+1)\frac{\sqrt{d}}{m-1}\right)$.

(2) Let $\ell(W, Z)$ be a function (possibly non-convex) satisfying the loss function assumptions in Section [II] and $\|\nabla L(W)\|_2 \leq \bar{D}$, $\forall W \in W$ [5]. Let $L(W^{(0)}) - L(W^*) = D_0$, where $W^{(0)} \in W$ is the initial model and $W^*$ is the optimal model. Then, DaQuSGD with step size $\eta = \min\{C_w^{-1}, \gamma\}$, where $\gamma = \frac{1}{\bar{D}} \sqrt{\frac{2D_0}{C_w}}$, satisfies

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \|\nabla L(W^{(i)})\|^2_2 \right] \leq 2\tilde{\sigma} \sqrt{\frac{2C_w D_0}{n}} + \frac{2D_0 C_w}{n},$$

(11)

where $\tilde{\sigma}$ is as before, and the expectation is taken over the random selection of points from the dataset, and the randomness in DaQuSGD.

Using the definition of $\tilde{\sigma}$ in Theorem 2 we can find the condition required for optimal convergence using DaQuSGD. The following corollary summarizes the results of this section.

**Corollary 1.** For $m = h\sqrt{d}$, DaQuSGD achieves the optimal convergence rate, using at most $1 + \log_2(h) + 2\log_2(h\sqrt{d}) + 2d \log_2(e(1 + h^2/2))$ bits per iteration. 

Note that the assumptions in Section [II] imply that $\|\nabla L(W)\|_2$ is bounded almost surely.
In this part, we aim to further reduce the communication cost of DaQuSGD using sample selection. The high level incentive is the following: If a data sample contributes minimally to the model learning, we do not transmit it. This can intuitively be viewed as a generalization of support vectors in SVM; we want to only send samples that are necessary for designing the classifier, and we also want to decide which samples are necessary in an online way, without the knowledge of the full dataset. Note that, samples that are transmitted are still quantized using DaQuSGD.

Our sample selection method assumes terminal nodes have enough resources to perform the forward pass of the model that we want to train at the central node. First, for the new data sample in the $i$-th iteration, we apply the last version of the model available at the terminal node $W^{(i-1)}$ on the new data sample $Z$ to compute the loss $\ell(W^{(i-1)}, Z)$. Using this loss, we quantize and transmit the point to the central node only if the loss exceeds a threshold, i.e., if $\ell(W^{(i-1)}, Z) > \ell^{(i)}_{th}$, where $\ell^{(i)}_{th}$ is the threshold in iteration $i$.

The intuition behind this loss thresholding approach is that for classification tasks, samples close to the classification boundary (points in P2 in Fig. 2) will typically have higher loss (important for the classifier design), while the samples that are well away from the boundary will exhibit smaller loses and thus have a minimal effects on the model (P1 and P3 in Fig. 2).

Formally, in the $i$-th iteration of the learning algorithm, by applying the selection based on thresholding, the algorithm minimizes the risk function $\hat{L}(W) = \frac{1}{N} \sum_{i=1}^{N} \max\{\ell^{(i)}_{th}, \ell(W^{(i)}, Z^{(i)})\}$. Hence, by decreasing the threshold $\ell^{(i)}_{th}$ as the algorithm progresses, the algorithm approaches the optimal point for a convex loss function (or a local optimal for non-convex loss function). Ideally, we want the loss of points that are not communicated to not be detrimental to the convergence rate of the algorithm. This cannot be guaranteed in general. The following theorem (proved in Appendix G) provides a sufficient condition on designing the thresholds $\{\ell^{(i)}_{th}\}_{i=1}^{n}$ such that our thresholding approach can converge with rate $O(\frac{1}{\sqrt{n}})$, same as the vanilla SGD.

**Theorem 3.** Let $W$ be convex, $L(W)$ be convex, $C_w$-smooth, and $\sup_{W \in \mathcal{W}} \|W - W^{(0)}\|_2^2 = D^2$, where $W^{(0)} \in \mathcal{W}$.

![Fig. 2: Illustrative example for sample selection.](image-url)
is the initial hypothesis. Let \( W^* = \arg \min_{W \in W} L(W) \) and assume that \( L(W^*) \geq 0 \). Assume that stochastic gradient descent is performed with stochastic gradients \( \hat{g}(W) \) that satisfy (i) \( E[\hat{g}(W)] = \nabla L(W) \) (unbiasedness), and (ii) \( E[\|\hat{g}(W) - \nabla L(W)\|^2] \leq \tilde{B}^2 \) (bounded variance). Additionally, assume that the loss thresholds satisfy that \( \sum_{i=1}^{n} \sqrt{\ell_{th}^{(i)}} \leq \sqrt{n} \), then at iteration \( n \), if the step size is \( \eta = \min\left\{ C_w^{-1}, \left( \sqrt{nB} \right)^{-1} \right\} \), we have

\[
E \left[ L\left( \frac{1}{n} \sum_{i=1}^{n} W^{(i)} \right) \right] - L(W^*) \leq \\
\frac{\tilde{B}(\tilde{D}/2 + 1) + 2\tilde{D}\sqrt{2C_w}}{\sqrt{n}} + \frac{2C_w\tilde{D}\sqrt{2C_w}}{nB}.
\]

Hence, this approach reduces communication cost, and computation cost (since backpropagation is only applied on a subset of the samples) without sacrificing the order of the convergence rate. In fact, we show numerically, in Section [VI] that thresholding can provide an improvement in terms of the convergence.

**Remark 4 (Sample Selection + DaQuSGD).** Theorem 3 assumes only the unbiasedness and bounded variance properties of the gradient estimates. Thus, we can apply the sample selection through thresholding on top of our DaQuSGD quantization approach described earlier in Section [IV] without a penalty in the order of convergence, since it generates gradient estimates with these properties at the central node.

### VI. Experimental Evaluation

This section evaluates the practical gain DaQuSGD achieves in terms of communicated bits for training ResNet models on the CIFAR-10 and ImageNet datasets.

**ImageNet Experimental Setup.** For our ImageNet experiments, we train a ResNet-50 \( (h = 25,557,032) \) on the ImageNet \( (d = 150,528) \) dataset. We use a learning rate schedule consisting of a base learning rate of \( 1e-1 \) with a piece-wise decay of 0.25 introduced every 30 epochs and batch size of 128. The networks were trained using Stochastic Gradient Descent (SGD) with momentum of 0.9.

**CIFAR-10 Experimental Setup.** For CIFAR-10 \( (d = 3,072) \), we train a ResNet-18 \( (h = 11,173,962) \). We use a learning rate schedule consisting of a base learning rate of \( 1e-3 \) with a piece-wise decay of 0.1 introduced at epoch 80 and batch size of 128. Similar to the ImageNet models, the ResNet-18 networks were trained using SGD with momentum of 0.9.

**DaQuSGD Implementation.** We make some implementation adjustments on the DaQuSGD algorithm theoretically analyzed for convergence in Section [IV]. Our implementation differs as follows.

First, instead of scaling each data vector of the current batch by its own \( \ell_2 \)-norm before applying \( \text{DataQ} \), we scale all of them using \( \max_{i \in [1:128]} \|Z^{(i)}\|_\infty \), the maximum feature value observed in the batch. This helps retain more values close to their original unquantized values and allows to get away with communicating a single full-precision scaling factor instead of sending a factor for each datapoint.

We observed in our experiments that the gradient correction term \( \text{GradCorrQ} \) provided meaningful gains only when the quantizer \( \text{DataQ} \) used a very small number of bits per feature (less than three bits per feature). Thus in our experiments, we opted to not use \( \text{GradCorrQ} \), and instead use \( \text{DataQ} \) with 3 or more bits per feature. As a result,
terminal nodes only need to calculate the loss function and do not need to run backpropagation. This is well-suited for applications where the terminal nodes mainly function as sensors, and can perhaps support the learning agent by preprocessing the data and deciding which are more useful for learning, but not through implementing a full backpropagation step.

When sample selection is applied, the threshold at the beginning of each epoch is set to be $0.2$ of the average loss of the transmitted samples of the previous epoch.

**Comparison to direct gradient quantization.** On the ImageNet dataset, we compare the performance of our proposed DaQuSGD with sample selection against the state-of-the-art gradient sparsification and quantization...
approach QTopK-SGD with memory [18] and momentum [36]. QTopK-SGD quantizes the top $K\%$ gradient values per model block using 16 bits each and performs error compensation by keeping memory of the difference between the true and compressed gradients.

The ResNet-50 model was trained on the ImageNet dataset for 90 epochs using our DaQuSGD approach and 150 epochs using the sparse QTopK-SGD approach.

Figure 3a shows the growth in the communication budget expended during training versus the change in test accuracy (recorded at every epoch). We observe that our proposed approach offers a saving of up to a factor of 7 over the sparsification and quantization based method for gradient compression QTopK-SGD for the same accuracy.

Figure 3b illustrates that QTopK-SGD requires a higher communication budget to converge to the same accuracy for the DaQuSGD with sample selection. This higher communication budget is manifested through the larger number of training epochs (33% more than what DaQuSGD utilized) in order to achieve the same accuracy. Larger number of epochs translates to both a longer training time as well as additional computational cost. That is, DaQuSGD with sample selection does not require memory, and saves complexity in two ways: it allows to converge to the same top-1 accuracy using fewer epochs and it only requires to compute the gradient for a subset of samples at every epoch.

For training ResNet-18 over the CIFAR-10 dataset, our proposed model DaQuSGD and QSGD was trained for 100 epochs; QTopK-SGD was trained for 150 epochs. On CIFAR-10, we observed even more significant gains for our DaQuSGD approach over the QTopK-SGD and the QSGD [5] as seen in Fig. 4a and Fig. 4b. In particular, our approach provided a factor of 14 reduction in the communication cost over QTopK-SGD and a factor of 300 over QSGD, to achieve the same top-1 accuracy.

**APPENDIX A**

**DISCUSSION ON THE VIABILITY OF LIPSCHITZ CONTINUITY ASSUMPTION IN 4**

The assumption in 4 is particular to our setup as we relate gradient updates to the datapoints. For this, we need that if a quantized point $Z_Q$ is close to its original point $Z$, then the gradients $g_Z(W), g_{Z_Q}(W)$ are also close. We employ the assumption of Lipschitz continuity of the gradient in $Z$ in 4 to formalize this notion. Note that this assumption is implied if \( \| \frac{\partial g_Z(W)}{\partial Z} \|_2 \leq C_z \) [30], which is the case in many loss functions (for instance, in the loss function in the example in Section III and example 2 in Appendix C below) since standard theoretical analysis in learning theory assume a bounded space for $Z$ and $W$ [37]. We also highlight that in many cases (as in the loss function in the example in Appendix C below, where $C_z = O(h)$), $C_z$ might depend on $h$ or $d$. However, from Theorem 2 in the main paper, to remove the effect of $C_z$ in the convergence rate, we can choose $m = C_z h \sqrt{d}$, which results in number of bits that is at most $1 + \log_2(h) + 2 \log_2(C_z h \sqrt{d}) + 2d \log_2(e(1 + C^2_z h^2 / 2))$, hence, the dependency in the number of bits on $C_z$ is only $\log_2(C_z)$. We observed in our numerical evaluation, on the CIFAR-10 dataset with ResNet-18, that our DaQuSGD approach gives a ratio $\frac{\|g_Z(W) - g_{Z_Q}(W)\|_2}{\|Z - Z_Q\|_2}$ that is upper bounded by 1 when calculated using the sampled $Z$ and taking $Z'$ as its quantized version $Z_Q$. 
APPENDIX B

SGD STANDARD CONVERGENCE THEOREMS

The central theorem used to prove convergence of stochastic gradient descent for smooth and convex risk functions is Theorem 6.3 in [30], which modified to our notation can be expressed as follows.

**Theorem 4 ([30], Theorem 6.3).** Let \( \mathcal{W} \) be convex, \( L(W) \) be convex, \( C_w \)-smooth, and \( \sup_{W \in \mathcal{W}} \| W - W^{(0)} \|^2 = \tilde{D}^2 \), where \( W^{(0)} \in \mathcal{W} \) is the initial hypothesis. Let \( W^* = \arg \min_{W \in \mathcal{W}} L(W) \). Assume that stochastic gradient descent is performed with stochastic gradients \( \hat{g}(W) \) that satisfy (i) \( \mathbb{E}[\hat{g}(W)] = \nabla L(W) \) (unbiasedness), and (ii) \( \mathbb{E}[\|\hat{g}(W) - \nabla L(W)\|^2] \leq \tilde{B}^2 \) (bounded variance), then at iteration \( n \), if the step size is \( \eta = \left(C_w + \frac{1}{\tilde{D}}\right)^{-1} \) and \( \gamma = \frac{\tilde{D}}{\sqrt{2n}} \),

\[
\mathbb{E} \left[ L \left( \frac{1}{n} \sum_{i=1}^{n} W^{(i)} \right) \right] - L(W^*) \leq \tilde{D} \tilde{B} \sqrt{\frac{3}{2n}} + \frac{C_w \tilde{D}^2}{n}.
\]

For smooth non-convex risk functions, the stochastic gradient descent converges to a local optimal point. The main result we use in our analysis is from [11], expressed following in our notation.

**Theorem 5 ([11]).** Let \( L(W) \) be \( C_w \)-smooth, and \( \|\nabla L(W)\|_2 \leq D, \forall W \in \mathcal{W} \). Let \( L(W^{(0)}) - L(W^*) = D_0 \), where \( W^{(0)} \in \mathcal{W} \) is the initial hypothesis and \( W^* \) is the optimal hypothesis. Assume that stochastic gradient descent is performed with stochastic gradients \( \hat{g}(W) \) that satisfy (i) \( \mathbb{E}[\hat{g}(W)] = \nabla L(W) \) (unbiasedness), and (ii) \( \mathbb{E}[\|\hat{g}(W) - \nabla L(W)\|^2] \leq \tilde{B}^2 \) (bounded variance), then at iteration \( n \) with step size \( \eta = \min\{C_w^{-1}, \gamma\} \) and \( \gamma = \frac{1}{\tilde{B}} \sqrt{\frac{2D_0}{nC_w}} \), we have that

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|\nabla L(W^{(i)})\|^2] \leq 2\tilde{B} \sqrt{\frac{2C_w D_0}{n}} + \frac{2D_0 C_w}{n},
\]

where the expectation is taken over the distribution of the stochastic gradient \( \hat{g}(W) \).

APPENDIX C

EXAMPLES THAT DIRECT GRADIENT QUANTIZATION IS NOT ALWAYS EFFICIENT

We reiterate the setup of the example studied in the main paper, below for completeness of the discussion, and then provide an extra example.

A. Example 1

Consider the simple case where \( d = 1, h \gg d \), and the loss function is \( \ell(W, Z) = \frac{Z}{\sqrt{h}} 1^T W \), where \( \mathcal{W} = \{ W \in \mathbb{R}^h : \|W\|_2 \leq 1 \} \), and \( Z = \{ Z \in \mathbb{R} : |Z| \leq 1 \} \). In this case, the best bound we can get is \( \|g_Z(W)\|_2 \leq 1 \). Consider a quantizer that only sends 1 bit per data sample to the learning agent and operates as follows. At each time \( j \), \( Q_e^e, Q_d^d \) are chosen to be

\[
Q_e^e(x) = \begin{cases} 
1 & \text{if with probability } \frac{x+1}{2}, \\
0 & \text{if with probability } \frac{1-x}{2}
\end{cases},\quad Q_d^d(x) = Q_d^d(x) = 2x - 1.
\]

Note that \( Z_Q^{(j)} = Q_d^d \in \{-1, 1\} \). Thus, we have that

\[
\mathbb{E} [Q_d^d \circ Q_e^e(x)|x] = \frac{x+1}{2} - \frac{1-x}{2} = x.
\]
Hence, we have that 
\[ \mathbb{E}(\frac{Z^{(j)}}{\sqrt{h}} 1 | Z) = \frac{Z^{(j)}}{\sqrt{h}} 1 = g_Z(W), \ \forall W \in \mathcal{W}. \] Moreover,
\[ ||\frac{Z^{(j)}}{\sqrt{h}} 1||_2 = ||\frac{1}{\sqrt{h}} 1||_2 = 1, \] where \((i)\) follows from the fact that \(Z^{(j)}_Q \in \{-1, 1\}.

\[ \text{B. Example 2} \]

Let us consider the loss function \(\ell(W, Z) = \frac{1}{\sqrt{h}} f(W^T v(Z)), \) where \(v(Z) = [1, Z, Z^2, ..., Z^k]. \) \(W = \{W \in \mathbb{R}^h | ||W||_2 \leq 1\}, \) \(Z = \{Z \in \mathbb{R} | |Z| \leq 1\}, \) and \(f\) is a general 1-smooth function. For instance, \(f\) could be the logistic regression loss, i.e.,
\[ f(W^T v(Z)) = f_y(W^T v(Z)) = \log(1 + \exp(-yW^T v(Z))). \]
We here assume that \(d = 1\) for simplicity, and include the \(d > 1\) generalization at the end of the appendix. In this case, the best bound we can get on the norm of the stochastic gradient is \(||g_Z(W)||_2 \leq 1\) as \(Z\) is allowed to take the value \(Z = 1.\)

We show next that the optimal convergence rate can be achieved with only log_2(h) bits per sample, instead of the expected \(\Omega(h)\) bits. Let \(Q^e : [-1, 1] \rightarrow \{0, 1\}, Q^d : \{0, 1\} \rightarrow [-1, 1]\) be defined as in [15]. The algorithm runs as follows. Each node transmits \(1 + \lceil \log_2(h) \rceil\) bits representing the set of values \(\{Q^e(f'(W^T v(Z)))\} \cup \{Q^e(Z^{2j})\} \lceil \log_2(k) \rceil,\) where \(f'\) is the derivative of the function \(f.\) Let \(b_j(i)\) be the \(j\)-th least significant bit in the binary representation of the integer \(i.\) The learning agent constructs \(Q(Z^i) = \prod_{j=1}^{\lceil \log_2(h) \rceil} Q^d \circ Q^e(Z^{2j}) b_j(i) \forall i \in [0 : k]\)
and the quantized stochastic gradient \(\hat{g}_Z(W) = \frac{1}{\sqrt{h}} Q^d \circ Q^e(f'(W^T v(Z)))[1, Q(Z^1), ..., Q(Z^k)]^T\)

It is easy to see from binary expansions that any number in the set \(\{1, ..., \lceil \log_2(k) \rceil\}\) can be expressed as the sum of power of two of numbers from the set \(\{1, ..., \lceil \log_2(k) \rceil\}\), where each element appears in the sum at most once. With this, we now show that \(\hat{g}_Z(W)\) is an unbiased stochastic gradient. By definition, we have that

\[ \mathbb{E}[(\hat{g}_Z(W))_i | Z] = \frac{1}{\sqrt{h}} \mathbb{E} \left[ Q^d \circ Q^e(f'(W^T v(Z))) \prod_{j=1}^{\lceil \log_2(k) \rceil} (Q^d \circ Q^e(Z^{2j}))^{b_j(i)} | Z \right] \]
\[ \overset{[i]}{=} \frac{1}{\sqrt{h}} \mathbb{E}[(Q^d \circ Q^e(f'(W^T v(Z))))/Z] \prod_{j=1}^{\lceil \log_2(k) \rceil} \mathbb{E} \left[(Q^d \circ Q^e(Z^{2j}))^{b_j(i)} | Z \right] \]
\[ \overset{[ii]}{=} \frac{1}{\sqrt{h}} \mathbb{E}[(Q^d \circ Q^e(f'(W^T v(Z))))/Z] \prod_{j=1}^{\lceil \log_2(k) \rceil} \mathbb{E} \left[Q^d \circ Q^e(Z^{2j}) | Z \right]^{b_j(i)} \]
\[ \overset{[iii]}{=} \frac{1}{\sqrt{h}} f'(W^T v(Z)) Z \prod_{j=1}^{\lceil \log_2(k) \rceil} b_j(i)^2 j = \frac{1}{\sqrt{h}} f'(W^T v(Z)) Z = g_Z(W)_i, \] (18)
where: \([i]\) follows from the fact that conditioned on \(Z\) each of the quantized values is independent; \([ii]\) follows from the fact that \(b_j(i) \in \{0, 1\}\) and therefore, we either take the expectation over the variable or have the expectation over 1; \([iii]\) follows from the fact that the quantized values using \(Q^d \circ Q^e\) are unbiased estimators of their respective unquantized values.

The fact that \(||\hat{g}_Z(W)||_2 \leq 1\) is obvious since the range of \(Q^d \circ Q^e\) is in \([-1, 1]^k\).
Lower bound: For simplicity we only consider symmetric schemes. We will show that any symmetric quantization scheme that satisfies \( \mathbb{E}[Q(v(Z))|Z] = v(Z) \), \( \forall Z \in \{Z \in \mathbb{R}||Z| \leq 1\} \) uses at least \( \log_2(d) \) bits. Consider a scheme that uses \( k \) bits, hence, for any \( Z \) with \( |Z| \leq 1 \), \( Q(v(Z)) \) takes one of \( 2^k \) values denoted as \( e_1, \ldots, e_{2^k} \). Let \( d_i = \mathbb{E}[Q^d(e_i)] \), where \( Q^d(e_i) \) is the decoded value of \( e_i \). Then, \( \mathbb{E}[Q(v(Z))|Z] \) is some convex combination of \( d_1, \ldots, d_{2^k} \). As a result, \( \mathbb{E}[Q(v(Z))|Z] = v(Z) \), \( \forall Z \in \{Z \in \mathbb{R}||Z| \leq 1\} \) implies that the set \( \{Z \in \mathbb{R}||Z| \leq 1\} \) is contained in the convex hull of the points \( d_1, \ldots, d_{2^k} \).

Now consider the matrix with columns \( [v(\frac{1}{n}), v(\frac{2}{n}), \ldots, v(1)] \). This matrix has full rank, and clearly \( 2^k \) is lower bounded by the rank of this matrix. Hence, \( k \geq \log_2(d) \).

Extension for \( d > 1 \): If \( d > 1 \), the algorithm can be extended by sending a quantized version of \( f'(W^T v(Z)) \) together with a quantized version of \( [Z_1^{2^j}, Z_2^{2^j}, \ldots, Z_d^{2^j}] \) for each \( j \in [1: \lfloor \log_2(k) \rfloor] \) with \( d \) bits using the DataQ with stochastic quantization that is described in Section IV. This would require \( 1 + \lfloor \log_2(k) \rfloor \) bits total. Note that when \( d > 1 \), each element of \( v(Z) \) corresponds to a monomial \( \prod_{i=1}^{d} Z_i^{x_i}, \sum_{i=1}^{d} x_i \leq k \), hence, the size of the model is \( h = O(k^d) \).

**APPENDIX D**

**Pseudocodes for DataQ and GradCorrQ**

The pseudocode for DataQ, GradCorrQ are given below in Algorithm 2, 3 respectively.

**Algorithm 2 DataQ(Z, m)**

1: Hyperparameters: Bound \( B \) on the value of \( \|Z\|_2 \).
2: \( Z^+ \leftarrow \max(Z, 0) \), \( Z^- \leftarrow \max(-Z, 0) \)
3: \( a \leftarrow \left[\frac{(m-1)Z^+}{B}\right] \), \( b \leftarrow \left[\frac{(m-1)Z^-}{B}\right] \); \( a, b \in \mathbb{N}^d \)
4: Define set \( S = \{(a, b) \in \mathbb{N}^d \times \mathbb{N}^d \mid \|a\|_1 + \|b\|_1 \leq (m-1)^2\} \).
5: \( b_Z^{(a,b)} \leftarrow \text{Index}_S ((a, b)) \)
6: **Reconstruction.** \( Z_Q \leftarrow [a - b] \frac{B}{m-1} \).
7: **Return** \( Z_Q, b_Z^{(a,b)} \)
8: **Comment:** The reconstruction step is also performed at the learning agent.

**APPENDIX E**

**Proof of Theorem**

First, we reiterate Theorem I from the paper for readability.

**Theorem.** The proposed DataQ algorithm satisfies the following statements: (1) For the integer vectors \( (a, b) \) uniquely defining \( Z_Q \), we have that \( (a, b) \in S \) with probability one; (2) DataQ uses at most \( 2 \log_2(m) + \min \{2d \log_2(e^\frac{2d+4m^2}{m^2}), m^2 \log_2(e^\frac{2d+4m^2}{m^2}) \} \) bits per sample for communication; (3) For the generated \( Z_Q \), we have that \( \|Z - Z_Q\|_\infty \leq \frac{B}{m-1} \) and \( \|Z_Q\|_2 \leq (1 + \sqrt{\frac{2}{m-1}})B \).
Algorithm 3 GradCorrQ(\(\Delta, m\))

1: **Hyperparameters**: \(C_z\), Lipschitz constant of loss function in \(Z\); \(h\), dimension of hypothesis space; \(d\), dimension of dataset.

2: Initialize \(\hat{\Delta} = 0^h\)

3: Pick integer index \(i^* \in [h]\) uniformly at random.

4: \(b(i^*) \leftarrow \) binary representation of \(i^*\) using \(\log_2(h)\) bits.

5: \(p \leftarrow \frac{\Delta_{i^*}}{2C_zB^2/m^2} + \frac{1}{2}\)

6: \(e_g \leftarrow \text{Bernoulli}(p)\)

7: \(\hat{\Delta}_{i^*} \leftarrow (2e_g - 1)C_zBh\sqrt{\frac{d}{m-1}}\)

8: **Return** \(\hat{\Delta}, (b(i^*), e_g)\)

9: **Comment**: Using \((b(i^*), e_g)\), the central node can recreate \(\hat{\Delta}\).

1) **Proof that \((a, b) \in S\):**

Recall that for datapoint \(Z\), the definition of \((a, b)\) is given by

\[
\forall j \in \{1, 2, \cdots, d\} : a_j(Z) = \arg \max_{i \in \{0, 1, \cdots, m-1\}} \left\{ q_i \middle| q_i = \frac{iB}{m-1} \leq |Z_j^+| \right\},
\]

and

\[
b_j(Z) = \arg \max_{i \in \{0, 1, \cdots, m-1\}} \left\{ q_i \middle| q_i = \frac{iB}{m-1} \leq |Z_j^-| \right\}.
\]

Thus, given the upper bound \(B\) on the \(\ell_2\)-norm of \(Z\), we have that

\[
B^2 \geq \|Z\|_2^2 = \|Z^+\|_2^2 + \|Z^-\|_2^2 \geq \sum_{j=1}^{d} \frac{B^2(a_j^2 + b_j^2)}{(m-1)^2} \geq \sum_{j=1}^{d} \frac{B^2(a_j + b_j)}{(m-1)^2},
\]

where \((i)\) follows since \(a_j, b_j \in \mathbb{N}\). From the above inequality, we now have that

\[
\sum_{j=1}^{d} (a_j + b_j) \leq (m-1)^2,
\]

which implies that \((a, b) \in S = \left\{ (w, v) \in \mathbb{N}^d \times \mathbb{N}^d \middle| \|w\|_1 + \|v\|_1 \leq (m-1)^2 \right\}\).

2) **Proof of the upper bound on number of bits:**

It suffices to show that \(\log_2(|S|) \leq 2 \log_2(m) + \min\{d \log_2(e^{2d+m^2}/2d), m^2 \log_2(e^{2d+m^2}/m^2)\}\). Note that \(|S|\) can be written as

\[
|S| = \sum_{q=0}^{(m-1)^2} \left\{ (w, v) \in \mathbb{N}^d \times \mathbb{N}^d \middle| \|w\|_1 + \|v\|_1 \leq q \right\}
\]

For a given integer \(q\), the number of positive integral solutions to the equation \(\sum_{i=1}^{d} (a_i + b_i) = q\) is a classical
counting problem and its solution is given in closed form⁶ as \((\binom{2d+q-1}{q})\) [38]. Given this, we can write \(|S|\) as

\[
|S| = \sum_{q=0}^{(m-1)^2} \binom{2d + q - 1}{q} \leq \sum_{q=0}^{(m-1)^2} \binom{2d + q}{2d} \leq \sum_{q=0}^{(m-1)^2} \binom{2d + m^2}{2d}
\]

\[
\overset{(i)}{\leq} m^2 \min \left\{ \left( \frac{2d + m^2}{2d} \right), \left( \frac{2d + m^2}{m^2} \right) \right\} \leq m^2 \min \left\{ \left( \frac{e \cdot 2d + m^2}{2d} \right), \left( \frac{e \cdot 2d + m^2}{m^2} \right) \right\}, \tag{21}
\]

where: \((i)\) is due to the symmetry of the binomial coefficient. Now by taking the logarithm of both sides, we get the intended upper bound for \(\log_2(|S|)\);

\((ii)\) uses the upper bound on the binomial coefficient based on Sterling’s bounding of the factorial.

3) **Proof that** \(|Z_j - Z_{Qj}| \leq \frac{B}{m-1}, \text{ and } \|Z\|_2 \leq \left(1 + \sqrt{\frac{d}{m-1}}\right)B:\)**

\[\|Z_j - Z_{Qj}\|_\infty \leq \frac{B}{m-1}\] is directly due to the quantization scheme in DataQ since the distance between any two quantization levels is given by \(\frac{B}{m-1}\).

Now to prove the bound on \(\|Z\|_2\), note that

\[
\|Z_Q\|_2 \overset{(i)}{\leq} \|Z\|_2 + \|Z - Z_Q\|_2
\]

\[
\leq B + \sqrt{\|Z - Z_Q\|_2^2} = B + \sqrt{\sum_{j=1}^{d} |Z_j - Z_{Qj}|^2}
\]

\[
\leq B + \sqrt{\sum_{j=1}^{d} \frac{B^2}{(m-1)^2}} = \left(1 + \sqrt{\frac{d}{m-1}}\right)B, \tag{22}
\]

where \((i)\) follows from the triangle inequality.

**APPENDIX F**

**PROOF OF LEMMA[1]**

**Lemma.** The quantized stochastic gradient \(\hat{g}_Z(W)\) is an unbiased estimate of \(\nabla L(W)\) and satisfies \(\|g_Z(W) - \nabla L(W)\|_2 \leq C_z B \left(2 + (h + 1) \frac{\sqrt{d}}{m-1}\right)\).

**Proof.** Recall that \(\Delta = g_Z(W) - g_{Z_Q}(W)\), and that, we have

\[
\hat{\Delta}_i = \begin{cases} 
0 & \text{with probability } 1 - \frac{1}{h} \\
C_z B h \frac{\sqrt{d}}{m-1} & \text{with probability } \frac{\Delta_i + C_z B \frac{\sqrt{d}}{m-1}}{2C_z B \frac{\sqrt{d}}{m-1}} - \frac{1}{h} \\
-C_z B h \frac{\sqrt{d}}{m-1} & \text{with probability } \frac{C_z B \frac{\sqrt{d}}{m-1} - \Delta_i}{2C_z B \frac{\sqrt{d}}{m-1}} - \frac{1}{h}. 
\end{cases} \tag{23}
\]

⁶The closed form relies on a standard approach in combinatorics called the “stars and bars method”.

Hence, we can prove the unbiasedness property of \( \hat{\Delta} \) by direct computation as follows

\[
\mathbb{E}[\hat{\Delta}_i|\Delta] = \mathbb{E}[\hat{\Delta}_i|\Delta_i]
= 0 + \left( C_Z B h \frac{\sqrt{d}}{m-1} \right) \frac{\Delta_i + C_Z B \frac{\sqrt{d}}{m-1} - 1}{2C_z B \frac{\sqrt{d}}{m-1} h} + \left( -C_Z B h \frac{\sqrt{d}}{m-1} \right) \frac{C_Z B \frac{\sqrt{d}}{m-1} - \Delta_i 1}{2C_z C \frac{\sqrt{d}}{m-1} h}
= \left( 2C_Z B h \frac{\sqrt{d}}{m-1} \right) \frac{2\Delta_i}{2C_z B \frac{\sqrt{d}}{m-1} h} = \Delta_i.
\]

(24)

Next, we can prove the unbiasedness of \( \tilde{g}_Z(W) \) as follows

\[
\mathbb{E}[\tilde{g}_Z(W)|Z] = \mathbb{E}_{Z_Q} [\mathbb{E}[\tilde{g}_Z(W)|Z, Z_Q]]
= (i) \mathbb{E}_{Z_Q} [\mathbb{E}[g_{Z_Q}(W) + \hat{\Delta}|Z, Z_Q]]
= (ii) \mathbb{E}_{Z_Q} [\mathbb{E}[g_{Z_Q}(W) + \Delta|Z, Z_Q]]
= (iii) \mathbb{E}_{Z_Q} [\mathbb{E}[g_{Z_Q}(W)|Z, Z_Q]]
= (iv) \mathbb{E}_{Z_Q} [\mathbb{E}[g_{Z}(W)|Z, Z_Q]]
= g_{Z}(W),
\]

(25)

where: (i) follows from the definition of \( \tilde{g}_Z(W) \); (ii) follows by the tower property of expectation and the fact that \( \hat{\Delta} \) is constructed by stochastic quantization of \( \Delta \); (iii) follows from the unbiasedness of \( \hat{\Delta} \) in (24); (iv) is due to the fact that \( \Delta = g_{Z}(W) - g_{Z_Q}(W) \).

To prove the bound on the variance, we note that

\[
\| \tilde{g}_Z(W) - \nabla L(W) \|_2 \leq (i) \| g_{Z_Q}(W) - \nabla L(W) \|_2 + \| \hat{\Delta} \|_2
\leq (i) \| g_{Z}(W) - \nabla L(W) \|_2 + \| g_{Z}(W) - g_{Z_Q}(W) \|_2 + \| \hat{\Delta} \|_2
\leq (ii) \| g_{Z}(W) - \nabla L(W) \|_2 + C_z B \frac{\sqrt{d}}{m-1} + \| \hat{\Delta} \|_2
\leq (iii) \| g_{Z}(W) - \nabla L(W) \|_2 + C_z B \frac{\sqrt{d}}{m-1} + C_z B h \frac{\sqrt{d}}{m-1}
= \left( \frac{1}{N} \sum_{i=1}^{N} \| g_{Z(i)}(W) - g_{Z}(W) \|_2 + C_z B \frac{\sqrt{d}}{m-1} + C_z B h \frac{\sqrt{d}}{m-1} \right)
\leq \frac{1}{N} \sum_{i=1}^{N} \| g_{Z(i)}(W) - g_{Z}(W) \|_2 + C_z B \frac{\sqrt{d}}{m-1} + C_z B h \frac{\sqrt{d}}{m-1}
\leq (iv) \frac{1}{N} \sum_{i=1}^{N} C_z \| Z^{(i)} - Z \|_2 + C_z B \frac{\sqrt{d}}{m-1} + C_z B h \frac{\sqrt{d}}{m-1}
\leq 2C_z B + C_z B \frac{\sqrt{d}}{m-1} + C_z B h \frac{\sqrt{d}}{m-1},
\]

(26)

where: (i) is due to the triangle inequality; (ii) follows from the \( C_z \)-Lipschitz continuity of \( g_Z \) and the definition of \( \text{Data}_Q \); (iii) follows from (23); (iv) follows due to the convexity of the norm; (v) follows from the \( C_z \)-Lipschitz continuity of \( g_Z(W) \) in \( Z \). \( \square \)
APPENDIX G

PROOF OF THEOREM 3

In this part we prove the convergence of SGD when our sample selection scheme through thresholding is applied. We denote the stochastic gradient with loss thresholding at the $j$-th iteration to be $\tilde{g}_{Z(j)}(W^{(j)})$ which is given by

$$
\tilde{g}_{Z(j)}(W^{(j)}) = \hat{g}_{Z(j)}(W^{(j)}) \mathbb{I} \left( \ell(W^{(j)}, Z^{(i)}) \leq \ell_{th} \right),
$$

where $\mathbb{I}(\cdot)$ is the indicator function. In the following we present a simple proof of our result by adapting a standard proof of SGD convergence. From convexity of $L(W)$, we have that

$$L(W^{(j)}) \leq L(W^*) + \nabla L(W^{(j)})^T (W^* - W^{(j)}). \tag{27}$$

It is well known that for convex function, smoothness implies a quadratic upper bound \[30\], i.e., we have that

$$L(W^{(j+1)}) \leq L(W^{(j)}) + \nabla L(W^{(j)})^T (W^{(j+1)} - W^{(j)}) + \frac{C_w}{2} \| W^{(j+1)} - W^{(j)} \|^2_2 \tag{28}$$

where $(a)$ follows from the definition of the model update rule using SGD in \[2\] with gradient $\tilde{g}_{Z(j)}(W^{(j)})$.

Taking the expectation of both sides over the randomness in $\tilde{g}_{Z(j)}(W^{(j)})$, we get

$$E \left[ L(W^{(j+1)}) \right] \leq L(W^{(j)}) - \eta \nabla L(W^{(j)})^T (E \left[ \tilde{g}_{Z(j)}(W^{(j)}) \right]) + \frac{C_w \eta^2}{2} E \left[ \| \tilde{g}_{Z(j)}(W^{(j)}) \|^2_2 \right]. \tag{29}$$

Note that by definition of $\tilde{g}_{Z(j)}(W^{(j)})$, we have that the expectation of the computed gradient is $E \left[ \tilde{g}_{Z(j)}(W^{(j)}) \right] = \nabla L(W^{(j)}) - \Delta_j$, where $\Delta_j = \frac{1}{N} \sum_{i=1}^N \nabla \ell(W, Z^{(i)}) \mathbb{I} \left( \ell(W^{(j)}, Z^{(i)}) \leq \ell_{th} \right)$. Additionally, we have that

$$E \left[ (\tilde{g}_{Z(j)}(W^{(j)}))^2 \right] \leq E \left[ (\tilde{g}_{Z(j)}(W^{(j)}))^2 \right] = \| \nabla L(W) \|^2_2 + \tilde{B}^2, \tag{30}$$

Thus, by substituting this bounds on first and second moments of $\tilde{g}_{Z(j)}$ in \[29\], and using the fact that $\eta \leq \frac{1}{C_w}$, we have that

$$E \left[ L(W^{(j+1)}) \right] \leq L(W^{(j)}) - \eta \nabla L(W^{(j)})^T (\nabla L(W^{(j)}) - \Delta_j) + \frac{C_w \eta^2}{2} \| \nabla L(W^{(j)}) - \Delta_j \|^2_2 - \frac{C_w \eta^2}{2} \| \Delta_j \|^2_2$$

$$+ \frac{C_w \eta^2}{2} \Delta_j^T \nabla L(W^{(j)}) + \frac{C_w \eta^2}{2} \tilde{B}^2$$

$$= L(W^{(j)}) - \eta (1 - \frac{C_w \eta}{2}) \| \nabla L(W^{(j)}) - \Delta_j \|^2_2 - \eta (1 - C_w \eta) \Delta_j^T \nabla L(W^{(j)})$$

$$+ \frac{C_w \eta^2}{2} \tilde{B}^2 + \eta (1 - \frac{C_w \eta}{2}) \| \Delta_j \|^2_2$$

$$\leq L(W^{(j)}) - \frac{\eta}{2} \| \nabla L(W^{(j)}) - \Delta_j \|^2_2 - \eta (1 - C_w \eta) \Delta_j^T \nabla L(W^{(j)}) + \frac{\eta}{2} (\tilde{B}^2 + 2 \| \Delta_j \|^2_2). \tag{31}$$

Combining \[27\], \[31\], we get

$$E \left[ L(W^{(j+1)}) \right] \leq L(W^*) + \nabla L(W^{(j)})^T (W^* - W^{(j)}) - \frac{\eta}{2} \| \nabla L(W^{(j)}) - \Delta_j \|^2_2$$

$$- \eta (1 - C_w \eta) \Delta_j^T \nabla L(W^{(j)}) + \frac{\eta}{2} (\tilde{B}^2 + 2 \| \Delta_j \|^2_2). \tag{32}$$
By adding and subtracting $\frac{1}{2\eta}||W^{(j)} - W^*||_2^2$ to complete square we get that

$$E\left[L(W^{(j+1)}) \right] \leq L(W^*) - \frac{1}{2\eta}||W^{(j)} - W^* - \eta(\nabla L(W^{(j)}) - \Delta_j)||_2^2 + \frac{1}{2\eta}||W^{(j)} - W^*||_2^2$$

$$\quad + \Delta_j^T(W^{(j)} - W^*) - \eta (1 - \eta C_w) \Delta_j^T \nabla L(W^{(j)}) + \eta \frac{B^2}{2} ||\Delta_j||_2^2)$$

$$\leq L(W^*) - \frac{1}{2\eta}E \left[||W^{(j)} - W^* - \eta g_{Z(i)}(W^{(j)})||_2^2 \right] + \frac{1}{2\eta}||W^{(j)} - W^*||_2^2$$

$$\quad + \Delta_j^T(W^{(j)} - W^*) + \eta^2 C_w \Delta_j^T \nabla L(W^{(j)}) + \eta \left(\tilde{B}^2 + \frac{1}{2}||\Delta_j||_2^2\right)$$

$$= (a) L(W^*) - \frac{1}{2\eta}E \left[||W^{(j+1)} - W^*||_2^2 - ||W^{(j)} - W^*||_2^2 \right]$$

$$\quad + ||\Delta_j||_2||W^{(j)} - W^*||_2 + \eta ||\Delta_j||_2||\nabla L(W^{(j)})||_2 + \eta \left(\tilde{B}^2 + \frac{1}{2}||\Delta_j||_2^2\right)$$

$$\leq L(W^*) - \frac{1}{2\eta}E \left[||W^{(j+1)} - W^*||_2^2 - ||W^{(j)} - W^*||_2^2 \right]$$

$$\quad + ||\Delta_j||_2||W^{(j)} - W^*||_2 + \eta ||\Delta_j||_2||\nabla L(W^{(j)})||_2 + \eta \left(\tilde{B}^2 + \frac{1}{2}||\Delta_j||_2^2\right)$$

$$= L(W^*) - \frac{1}{2\eta}E \left[||W^{(j+1)} - W^*||_2^2 - ||W^{(j)} - W^*||_2^2 \right]$$

$$\quad + ||\Delta_j||_2 \left(||W^{(j)} - W^*||_2 + \eta ||\nabla L(W^{(j)})||_2 + \eta \left(\tilde{B}^2 + \frac{1}{2}||\Delta_j||_2^2\right)\right)$$

where: $(a)$ followed from the update rule of $W^{(j+1)}$ from $W^{(j)}$ and $g_{Z(i)}(W^{(j)})$; $(b)$ from Cauchy-Schwartz inequality. We now want to find bounds for the terms $||\Delta_j||_2$, $||W^{(j)} - W^*||_2$ and $||\nabla L(W^{(j)})||_2$. Note that from our assumptions, we have that $||W^{(j)} - W^*||_2 \leq \tilde{D}$.

For a point $Z$, let $W'_Z$ be the minimizer of $\ell(., Z)$, then we have that

$$||\nabla \ell(W, Z) - \nabla \ell(W'_Z, Z)||_2^2 \leq (a) 2C_w \left(\nabla \ell(W'_Z, Z)^T (W'_Z - W) + \ell(W, Z) - \ell(W'_Z, Z)\right)$$

$$\leq (b) 2C_w \left(\nabla \ell(W'_Z, Z)^T (W'_Z - W) + \ell(W, Z)\right),$$

where $(a)$ is a consequence of the smoothness of $\ell(W, Z)$ and $(b)$ follows from our assumption that $\ell(W, Z) \geq 0$. By making use of the above equation and the fact that $\nabla \ell(W'_Z, Z) = 0$ for a given $Z$, we get that $||\nabla \ell(W, Z)||_2^2 \leq 2C_w L(W, Z)$. Using this bound on $||\nabla \ell(W, Z)||_2^2$, we have that

$$||\Delta_j||_2 \leq \frac{1}{N} \sum_{i=1}^{N} \left\| \nabla \ell(W, Z^{(i)}) \right\|_2 \left( \ell(W^{(j)}, Z^{(i)}) \leq \ell_{th}^{(j)} \right)$$

$$\leq \frac{1}{N} \sum_{i=1}^{N} \left\| \nabla \ell(W, Z^{(i)}) \right\|_2 \left( \ell(W^{(j)}, Z^{(i)}) \leq \ell_{th}^{(j)} \right)$$

$$\leq \frac{1}{N} \sum_{i=1}^{N} \left( \sqrt{2C_w \ell(W, Z^{(i)})} \right) \left( \ell(W^{(j)}, Z^{(i)}) \leq \ell_{th}^{(j)} \right) \leq \sqrt{2C_w \ell_{th}^{(j)}},$$

Finally, note that the smoothness property implies that $||\nabla \ell(W, Z)||_2 \leq 2C_w \bar{D}$, which using the same steps as in (34) implies that $||\Delta_j||_2 \leq 2C_w \bar{D}$. Additionally, it also implies that $||\nabla L(W)||_2 \leq 2C_w \bar{D}$. 
By substituting these upper bounds in (33), we get that
\[
\mathbb{E} \left[ L(W^{(j+1)}) \right] \leq L(W^*) - \frac{1}{2\eta} \mathbb{E} \left[ \|W^{(j+1)} - W^*\|^2_2 - \|W^{(j)} - W^*\|^2_2 \right] \\
+ \|\Delta_j\|_2 \left( \|W^{(j)} - W^*\|^2_2 + \eta \|\nabla L(W^{(j)})\|_2 + \frac{\eta}{2} \|\Delta_j\|_2 \right) + \eta \tilde{B}^2 \\
= L(W^*) - \frac{1}{2\eta} \mathbb{E} \left[ \|W^{(j+1)} - W^*\|^2_2 - \|W^{(j)} - W^*\|^2_2 \right] \\
+ \left( \sqrt{2C_w \ell_{th}^{(j)}} \right) \left( \tilde{D} + 2\eta C_w \tilde{D} + \eta C_w \tilde{D} \right) + \eta \tilde{B}^2 \\
= L(W^*) - \frac{1}{2\eta} \mathbb{E} \left[ \|W^{(j+1)} - W^*\|^2_2 - \|W^{(j)} - W^*\|^2_2 \right] + \tilde{D} \sqrt{2C_w \ell_{th}^{(j)}} (1 + 3\eta C_w) + \eta \tilde{B}^2. \tag{35}
\]
And from convexity and \( \sum_{j=1}^n \sqrt{\ell_{th}^{(j)}} \leq \sqrt{n} \), we have
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^n L(W^{(j)}) \right] \leq L(W^*) + \frac{\|W^{(0)} - W^*\|^2_2}{2n\eta} + \frac{\tilde{D} \sqrt{2C_w} (1 + 3\eta C_w)}{\sqrt{n}} + \eta \tilde{B}^2. \tag{36}
\]
Substituting \( \eta = \frac{1}{B \sqrt{n}} \), we get the result.

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