Quantum advantage by relational queries about physically realizable equivalence classes

Karl Svozil

Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstrasse 8-10/136, 1040 Vienna, Austria
svozil@tuwien.ac.at
http://tph.tuwien.ac.at/~svozil

Abstract. Relational quantum queries are sometimes capable to effectively decide between collections of mutually exclusive elementary cases without completely resolving and determining those individual instances. Thereby the set of mutually exclusive elementary cases is effectively partitioned into equivalence classes pertinent to the respective query. In the second part of the paper, we review recent progress in theoretical certifications (relative to the assumptions made) of quantum value indeterminacy as a means to build quantum oracles for randomness.

Keywords: quantum computation, partitioning of cases, quantum parallelism, quantum random number generators.

1 Quantum (dis-)advantages

Contemporary quantum information theory appears to be challenging yet far from being fully comprehended, worked out and mature. It is based on quantum mechanics, a theory whose semantics has been notoriously debated almost from its inception, while its syntax – its formalism, and, in particular, the rules of deriving predictions – are highly successful, accepted and relied upon. Depending on temperament and metaphysical inclination, its proponents admit that nobody understands quantum mechanics [15, 10], maintain that there is no issue whatsoever [12, 16], one should not bother too much [11, 7] about its meaning and foundations, and rather shut up and calculate [25, 24].

By transitivity or rather reduction, quantum information theory inherits quantum mechanics’ apparent lack of consensus, as well as a certain degree of cognitive dissonance between applying the formalism while suffering from an absence of conceptual clarity [27]. Strong hopes, claims and promises [33] of quantum supremacy are accompanied by the pertinent question of what exactly, if at all, could make quantum information and computation outperform classical physical resources. Surely many nonclassical quantum features present themselves as being useful or decisive in this respect; among them complementarity, coherence (aka parallelism), entanglement, or value indeterminacy (aka contextuality). But if and how exactly those features will contribute or enable future algorithmic advances still remains to be seen.
The situation is aggravated by the fact that, although the quantum formalism amounts to linear algebra and functional analysis, some of its most important theorems are merely superficially absorbed by the community at large: take, for example, Gleason’s theorem [17], and extensions thereof [30, 5]. Another example is Shor’s factoring algorithm [29, Chapter 5] whose presentations often suffer from the fact that its full comprehension requires a nonsuperficial understanding of number theory, analysis, as well as quantum mechanics; a condition seldom encountered in a single (wo)man. Moreover, often one is confronted with confusing opinions: for instance, the claim that quantum computation is universal with respect to either unitary transformations or first-order predicate calculus is sometimes confused with full Turing universality. And the plethora of algorithms collected into a quantum algorithm zoo [19] is compounded by the quest of exactly why and how quantum algorithms may outperform classical ones.

It also may very well be that the necessity to maintain coherence throughout a quantum computation will impose an exponential overhead of “physical stuff” necessary to achieve this goal. This could well compensate or even outweigh quantum parallelism; that is, the exponential simultaneous co-representability of (coherent superpositions of) classical mutually exclusive cases of a computation. Nevertheless, in what follows we shall consider the feasibility for speedups from such quantum computational strategies involving quantum parallelism.

2 Suitable partitioning of cases

One quantum feature called “quantum parallelism,” which is often presented as a possible quantum resource not available classically, is the capacity of $n$ quantum bits to encode $2^n$ classically mutually exclusive distinct classical bit states at once, that is, simultaneously: $$|\Psi\rangle = \sum_{i=0}^{2^n-1} \psi_i |i\rangle,$$ where the index $i$ runs through all $2^n$ possible combinations of $n$ classically mutually exclusive bit states $\{0, 1\}$, $|i\rangle$ are elements of an orthonormal basis in $2^n$-dimensional Hilbert space, and $\psi_i$ represent probability amplitudes whose absolute squares sum up to 1. This seemingly absurd co-representability of contradicting classical states was the motivation for Schrödinger’s cat paradox [31].

The state $|\Psi\rangle$ “carrying” all these classical cases in parallel is not directly accessible or “readable” by any physical operational means. And yet, it can be argued that its simultaneous representation of classically exclusive cases can be put to practical use indirectly if certain criteria are met:

- first of all, there needs to be a quantum physical realizable grouping or partitioning of the classical cases, associated with a particular query of interest; and
- second, this aforementioned query needs to be realizable by a quantum observable.

In that way, one may attain knowledge of a particular feature one is interested in; but, unlike classical computation, (all) other features remain totally unspecified and unknown. There is no “free quantum lunch” here, as a total specification
of all observables would require the same amount of quantum queries as with classical resources. And yet, through coherent superposition (aka interference) one might be able to “scramble” or re-encode the signal in such a way that some features can be read off of it very efficiently – indeed, with an exponential (in the number of bits) advantage over classical computations which lack this form of rescrambling and re-encoding (through coherent superpositions). However, it remains to be seen whether, say, classical analog computation with waveforms, can produce similar advantages.

For the sake of a demonstration, the Deutsch algorithm [26, Chapter 2] serves as a Rosetta stone of sorts for a better understanding of the formalism and respective machinery at work in such cases. It is based on the four possible binary functions $f_0, \ldots, f_3$ of a single bit $x \in \{0, 1\}$: the two constant functions $f_0(x) = 1 - f_3(x) = 0$, as well as the two nonconstant functions: the identity $f_1(x) = x$ and the not $f_3(x) = (x + 1) \mod 2$, respectively. Suppose that one is presented with a black box including in- and output interfaces, realizing one of these classical functional cases, but it is unknown which one. Suppose further that one is only interested in their parity; that is, whether or not the encoded black box function is a constant function of the argument. Thereby, with respect to the corresponding equivalence relation of being “(not) constant in the argument” the set of functions $\{f_0, \ldots, f_3\}$ is partitioned into $\{\{f_0, f_3\}, \{f_1, f_2\}\}$.

A different way of looking at this relational encoding is in terms of zero knowledge proofs: thereby nature is in the role of an agent which is queried about a property/proposition, and issues a correct answer without disclosing all the details and the fine structure of the way this result is obtained.

Classically the only way of figuring this (“constant or not”) out is to input the two bit-state cases, corresponding to two separate queries. If the black box admits quantum states, then the Deutsch algorithm presents a way to obtain the answer (“constant or not”) directly in one query. In order to do this one has to perform three successive steps [32, 36]:

– first one needs to scramble the classical bits into a coherent superposition of the two classical bit states. This can be done by a Hadamard transformation, or a quantum Fourier transformation;
– second, one has to transform the coherent superposition according to the binary function which is encoded in the box. This has to be done while maintaining reversibility; that is, by taking “enough” auxiliary bits to maintain bijectivity/permutation; even if the encoded function is many-to-one (eg, constant).
– third, one needs to unscramble this resulting state to produce a classical output signal which indicates the result of the query. As all involved transformations need to be unitary and thus reversible the latter task can again be achieved by an (inverse) Hadamard transformation, or an (inverse) quantum Fourier transformation.

This structural pattern repeats itself in many quantum algorithms suggested so far. It can be subsumed into the threefold framework: “spread (the classical state into a coherent superposition of classical states) — transform (according
to some functional form pertinent to the problem or query considered) — fold
(onto partitions of classical states which can be accessed via quantum queries
and yield classical signals).”

Besides the (classical) pre- and post-processing of the data, Shor’s algo-

rithm \[29, \text{Chapter 5}\] has a very similar structure in its quantum (order-finding)

core: It creates a superposition of classically mutually exclusive states \(i\) via a gen-

eralized Hadamard transformation. It then processes this coherent superposition

of all \(i\) by computing \(x^i \mod n\), for some (externally given) \(x\) and \(n\), the num-

ber to be factored. And it finally “folds back” the expanded, processed state by

applying an inverse quantum Fourier transform, which then (with high proba-

bility) conveniently yields a classical information (in one register) about the period

or order; that is, the least positive integer \(k\) such that \(x^k = 1(\mod n)\) holds.

As far as Shor’s factoring algorithm is concerned, everything else is computed
classically.

Whether or not this strategy to find “quantum oracles” corresponding to ar-

bitrary partitions of classical cases is quantum feasible remains to be seen. There

appears to be an \textit{ad hoc} counterexample, as there is no speedup for generalized

parity \[14\]; at least with the means considered.

3 Quantum oracles for random numbers

Let me, for the sake of presenting another quantum resource, contemplate one

example for which, relative to the assumptions made, quantum “computation”

outperforms classical recursion theory: the generation of (allegedly) irreducibly

indeterministic numbers; or sequences thereof \[4\]. A recent extension of the

Kochen-Specker theorem \[1, 3, 5\] allowing partial value assignments suggests the

following algorithm: Suppose one prepares a quantized system capable of three

or more mutually exclusive outcomes, formalized by Hilbert spaces of dimension

three and higher, in an arbitrary pure state. Then, relative to certain reasonable

assumptions (for value assignments and noncontextuality), this system cannot

be in any defined, determined property in any other direction of Hilbert state

not collinear or orthogonal to the vector associated with the prepared state \[30,

18\]: the associated classical truth assignment cannot be a total function. The

proof by contradiction is constructive and involves a configuration of intertwin-

ing quantum contexts (aka orthonormal bases). Figure 1 depicts a particular

configuration of quantum observables, as well as a particular one of their faith-

ful orthogonal representations \[22\] in which the prepared and measured states

are an angle

\[\arccos \left( \langle a | b \rangle \right) = \arccos \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{\pi}{2}\]
apart \[5, \text{Table 1}\].

Whenever one approaches quantum indeterminacy from the empirical, inductive

side, one has to recognize that, without \textit{a priori} assumptions, formal proofs

of (in)computability, and more so algorithmic incompressibility (aka randomness

\[23\]) are blocked by reduction to the halting problems and similar \[35\]. The

best one can do is to run tests, such as Borel normality and other criteria, on
Fig. 1. Greechie orthogonality diagram of a logic [5, Fig. 2, p. 1020-8] realizable in $\mathbb{R}^3$ with the true–implies–value indefiniteness (neither true nor false) property on the atoms $|a\rangle$ and $|b\rangle$, respectively. The 8 classical value assignments require $|a\rangle$ to be false. Therefore, if one prepares the quantized system in state $|a\rangle$, the second state $|b\rangle$ cannot have any consistent classical value assignment – it must be value indeterminate/indefinite.

finite sequences of random number generators [9, 2] which turn out to be consistent with the aforementioned value indefiniteness and quantum indeterminacy.

4 Afterthoughts on assumptions

Let me, as a substitute for a final discussion, mention a caveat: as all results and certifications hold true relative to the assumptions made, different assumptions and axioms may change the perceptual framework and results entirely. One might, for instance, disapprove of the physical existence of states and observables beyond a single vector or context [34, 6]. Thereby, the problem of measuring other contexts would be relegated to the general measurement problem of coherent superpositions [21]. In this case, as von Neumann, Wigner and Everett have pointed out, by “nesting” the measurement objects and the measurement apparatus in larger and larger systems [13], the assumption of universal validity of the quantum state evolution would result in a mere epistemic randomness; very much like the randomness encountered in, and the second law of classical statistical physics. From that perspective quantum randomness might turn out to be valid “for all practical purposes” [7] through interaction with a huge number of (uncontrollable) degrees of freedom in the environment of a quan-
tized system in a coherent state, “squeezing” out this coherence very much like a balloon losing gas [8].

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