One-loop effective action for
$SU(2)$ gauge theory on $S^3$

Bas van den Heuvel$^1$

Instituut-Lorentz for Theoretical Physics,
University of Leiden, PO Box 9506,
NL-2300 RA Leiden, The Netherlands.

Abstract: We consider the effective theory for the low-energy modes of $SU(2)$
gauge theory on the three-sphere. By explicitly integrating out the high-
energy modes, the one-loop correction to the hamiltonian for this problem
is obtained. We calculate the influence of this correction on the glueball
spectrum.

1 Introduction

Non-perturbative effects in gauge theories can be studied using finite volumes $[1, 2, 3]$. In
a small volume asymptotic freedom implies that the coupling constant is small: this means
that we can use standard perturbation theory. By increasing the volume, we can study the
onset of non-perturbative phenomena. The effects that we want to study are related to the
multiple vacuum structure of the theory. For increasing volume, the wave functional starts
to spread out over the configuration space in those directions where the potential energy
is lowest, i.e. in the direction of the low-energy modes of the gauge field. In particular, it
will flow over the instanton barrier that connects gauge copies of the vacuum.

As long as the non-perturbative effects manifest themselves appreciably only in a small
number of low-lying energy modes, this can be described adequately using a hamiltonian
formulation. It is hence our strategy to split up the gauge field in orthogonal modes and to
reduce the dynamics of this infinite number of degrees of freedom to a quantum mechanical
problem with a finite number of modes.

In our approach, we impose the Coulomb gauge by restricting the gauge fields to a
so-called fundamental domain $[4, 5]$. The spreading out over configuration space means
that the wave functional will become sensitive to the boundary conditions that have to be
imposed on the boundary of this fundamental domain. In these boundary conditions the
dependence on the $\theta$-angle will show up.

$^1$e-mail: bas@lorentz.LeidenUniv.nl
For more details on this method the reader is referred to [6], where we used the lowest order effective hamiltonian in a variational calculation of the spectrum. In the present letter we perform the one-loop calculation, that is, we integrate out the high-energy modes in the path integral to obtain the correction to the lowest order hamiltonian. We will subsequently use this new hamiltonian to find the glueball spectrum.

2 The effective theory

We will briefly review the technical set up for the analysis on the three-sphere. For details and more motivation, we again refer to [6] and references therein. Let \( n_\mu \) be the normal vector on the three-sphere. We define two orthonormal framings on \( S^3 \) by

\[
e^\alpha_\mu = \eta^\alpha_\mu n_\nu, \quad \bar{e}^\alpha_\mu = \bar{\eta}^\alpha_\mu n_\nu,
\]

where we used the 't Hooft \( \eta \) symbols [7]. These \( \eta \) symbols occur in the multiplication rules for the unit quaternions:

\[
\sigma_\mu \bar{\sigma}_\nu = \eta^\alpha_\mu \sigma_\alpha, \quad \bar{\sigma}_\mu \sigma_\nu = \bar{\eta}^\alpha_\mu \sigma_\alpha,
\]

where \( \sigma_\mu \) and their conjugates \( \bar{\sigma}_\mu = \sigma_\mu^\dagger \) are defined by

\[
\sigma_\mu = (1, i\vec{\tau}), \quad \bar{\sigma}_\mu = (1, -i\vec{\tau}).
\]

We choose to write a gauge field on \( S^3 \) \( (n_\mu A_\mu = 0) \) with respect to the framing \( e^i_\mu \):

\[
A_\mu = A^i_\mu e^i_\mu = A^a_\mu \sigma^a_\mu / 2
\]

We introduce a number of \( su(2) \) angular momentum operators. We define \( L^i = L^1 = \frac{i}{2} e^i_\mu \partial_\mu \) and \( L^2 = \frac{i}{2} \bar{e}^i_\mu \partial_\mu \). These operators generate the \( SU(2) \times SU(2) \) symmetry of \( S^3 \) and satisfy \( \vec{L}^2 = \vec{L}^2 \). We introduce a spin operator \( \vec{S} \) by \( (S^a_\mu) i = -i\varepsilon_{aij} A_j \) and an isospin operator \( \vec{T} \) by \( T^a = \text{ad} (\tau_a / 2) \). We also define \( K_i = L_i + S_i \) and \( J_i = K_i + T_i \).

To isolate the lowest energy levels, we write

\[
V_{\text{cl}}(A) = -\frac{1}{2\pi^2} \int_{S^3} \frac{1}{2} \text{tr}(F^2_{ij}) = -\frac{1}{2\pi^2} \int_{S^3} \text{tr}(A_i \mathcal{M}_{ij} A_j) + \mathcal{O}(A^3),
\]

where the quadratic fluctuation operator \( \mathcal{M} \) can be rewritten as

\[
\mathcal{M}_{ij} = (\vec{K}^2 - \vec{L}^2)_{ij}
\]

The zero-modes of \( \mathcal{M} \) correspond to pure-gauge modes of the gauge field. The 18 dimensional space \( B(c, d) \) given by

\[
B_\mu(c, d) = \left( e^a_\mu e^i_\mu + d^a_\mu d^i_\mu \right) \sigma^a_\mu / 2 = \left( e^a_\mu + d^a_\mu V^i_\mu \right) e^i_\mu \sigma^a_\mu / 2
\]

is the eigenspace of \( \mathcal{M} \) corresponding to its lowest positive eigenvalue 4, whereas the next eigenvalue is 9. The tunnelling path is \( c^a_i = -u \delta^a_i, \ d^a_i = 0 \) with \( u \) running from 0 to 2. For
$u = 1$ it passes through the sphaleron, which is a saddle point of the energy functional. The energy functional for these 18 modes is given by

$$V_{\text{cl}}(c, d) \equiv -\frac{1}{2\pi^2} \int_{S^3} \frac{1}{2} \text{tr}(F_{ij}^2) = V_{\text{cl}}(c) + V_{\text{cl}}(d) + \frac{1}{3} (\text{tr}(X) \text{tr}(Y) - \text{tr}(XY)), \quad (8)$$

$$V_{\text{cl}}(c) = 2 \text{tr}(X) + 6 \det c + \frac{1}{4} (\text{tr}^2(X) - \text{tr}(X^2)), \quad (9)$$

with the symmetric matrices $X$ and $Y$ given by $X = cc^T$ and $Y = dd^T$. The lowest order hamiltonian for these modes is

$$R H(c, d) = -\frac{f^2}{2} \left( \frac{\partial^2}{\partial c_i^a \partial c_i^a} + \frac{\partial^2}{\partial d_i^a \partial d_i^a} \right) + \frac{1}{f} V_{\text{cl}}(c, d) \quad (10)$$

with $f = \frac{g^2}{2\pi^2}$, and $R$ the reinstated radius of the sphere.

### 3 Gauge fixing

We will impose the background gauge condition on the high-energy modes. Consider a general gauge field on $S^3$:

$$A_\mu = (A_0, A_i), \quad (11)$$

where $A_0$ is the time component of the gauge field and $A_i$ are the space components with respect to the framing $e_i^\mu$. We will now project out the background field $B(c, d)$. Let $P_S$ be the projector on the constant scalar modes, and let $P_V = P_c + P_d$ be the projector on the $(c, d)$-space. We define the background field $B$ and the quantum field $Q$ by respectively

$$B_\mu = (PA)_\mu = (P_S A_0, (P_V A)_i), \quad (12)$$

$$Q_\mu = A_\mu - B_\mu. \quad (13)$$

We define the gauge fixing function $\chi$ by

$$\chi = (1 - P_S) D_\mu (PA) A_\mu + P_S A_0. \quad (14)$$

We use $\chi$ to impose the background gauge condition: $\chi = 0$ is equivalent to $B_0 = 0$ and $D_\mu(B) Q_\mu = 0$. We perform the standard manipulations with the partition function: after introducing Faddeev-Popov ghosts and expanding the classical action up to second order in $Q$ we obtain

$$Z = \int DB_k D'Q_\mu D'\psi D'\bar{\psi} \exp \left[ \frac{1}{g_0^2} \int \text{tr} \left\{ -D_\mu(B)(1 - P)D_\mu(B) \right\} \psi \right.$$  

$$+ \frac{1}{2} F_{\mu\nu}^2(B) - 2(D_\mu F_{\mu\nu})(B)Q_\nu + Q_\mu W_{\mu\nu}(B)Q_\nu \right], \quad (15)$$

with

$$\begin{cases} 
W_{00} = -D_{\rho}^2(B) \\
W_{0i} = -W_{i0} = -2 \text{ ad } (\dot{B}_i) \\
W_{ij} = -2 \text{ ad } (F_{ij}(B)) - (D_{\rho}^2(B))_{ij} + 2\delta_{ij} 
\end{cases} \quad (16)$$
Remember that the covariant derivative $D_i(B)$ acting on vectors (or tensors) gives extra connection terms (due to $S^3$ being a curved manifold), e.g.

$$
(D_i(B)C)_j = \partial_i C_j + [B_i, C_j] - \varepsilon_{ijk} C_k
$$

(17)

$$
= (-2iL_i + iB^a_iT^a - iS^a_i)_{jk} C_k
$$

(18)

The primed integration means that we have excluded the $(c, d)$-modes from the integration over the vector field $Q_k$, and the constant modes from the integration over the scalar fields $\psi, \bar{\psi}$ and $Q_0$.

The action contains a term $J_\nu Q_\nu$ with $J_\nu = (D_\mu F_{\mu\nu})(B)$. Since $B$ need not satisfy the equations of motion, this term does not vanish. When expanding the path integral in Feynman diagrams, this term will give rise to extra diagrams, where $J$ acts as a source. It can be shown that $J$ will only contribute to terms in the effective lagrangian that we will consider to give only small corrections: they are at least of the order $c^2d^4$ or $c^4d^2$ and we will ignore them.

Dropping the term linear in $Q_\mu$, we obtain from $Z$ the effective action

$$
S_{\text{eff}}^{\text{EUCL}}[B] = S_{\text{cl}}^{\text{EUCL}}[B] - \ln \det' (-D_\mu(B)(1 - P)D_\mu(B)) + \frac{1}{2} \ln \det' (W_{\mu\nu})
$$

(19)

4 The effective potential

As the one-loop computation is the central result for this paper we present here a few of the details that are crucial to understand why we were able to complete this calculation. For computing the effective potential, $B$ is considered to be independent of time. This effective potential is determined up to fourth order in the fields $c$ and $d$ and up to sixth order in the tunnelling parameter $u$, where we expect the physics to be most sensitive to the precise shape of the potential. We use $A(B)$ to denote any of the operators occurring in (19). They are of the form

$$
A(B) = -\partial_0^2 - \partial_s^2 + \tilde{A}(B).
$$

(20)

In order to neatly perform the dimensional regularisation, we introduced a laplacian term for an $\varepsilon$-dimensional torus of size $L$ that we attached to our space $S^3 [8]$. The scale $L$ should of course be chosen proportional to the radius $R$ of the three-sphere. The assumption $[\partial_0, \tilde{A}(B)] = 0$ allows us to write

$$
\ln \det(A) = -\frac{T}{2\sqrt{\pi}} \left( \frac{L}{2\sqrt{\pi}} \right)^\varepsilon \int_0^\infty ds s^{-3/2-\varepsilon/2} \operatorname{tr} \left( e^{-s\tilde{A}} \right),
$$

(21)

where we took the time periodic with period $T$. The operator $\tilde{A}$ can be expressed in terms of the angular momentum operators defined above, the functions $V^j_i$ and the constants $c_i^a$ and $d_i^a$. To take the trace, we need a basis of functions. For the scalar operators (the ghost operator and $W_{00}$), we can use $|l m_L m_R \otimes |m_l m_j \rangle$ or equivalently $|l m_R; j m_j \rangle$, where $m_L$, $m_R$ and $m_j$ correspond to the $z$-components of $\vec{L}_1$, $\vec{L}_2$ and $\vec{J}$ respectively. Here $l = 0, 1, 2, 1, \ldots$
and \( j = |l - 1|, \ldots, l + 1 \). For the vector operator \( W_{ij} \), we can use \(|lm_R; km_k \rangle \otimes |1 m_l \rangle\) or \(|lm_R; k; j m_j \rangle\), where the bounds on the various quantum numbers are obvious. Note that the \( c \) and \( d \) modes correspond to the vector modes with \((l, k) = (0, 1)\) and \((l, k) = (1, 0)\) respectively. For the scalar operators, the trace must not be taken over the \( l = 0 \) functions, whereas for the vector operator the trace must not include the \( c \) and \( d \) modes. Note that for the case of the ghost operator, the operator \((1 - P)\) (see eq. \(19\)) causes some intermediate vector modes to be projected to zero.

Using the appropriate basis, we can calculate the determinants exactly for the vacuum \( u = 0 \) \((B = 0)\), and for the sphaleron configuration \( u = 1 \) \((c_0^a = -\delta_0^a \text{ and } d = 0)\). The final summation over \( l \) is expressed using the function \( \zeta(s, a) \), which is defined by

\[
\zeta(s, a) = \sum_{k=2}^{\infty} \frac{1}{(k^2 + a)^s}, \quad \text{Re}(s) > \frac{1}{2},
\]

(22)

After analytic continuation we have

\[
\zeta(s, a) = \sum_{m=0}^{\infty} \frac{(s)_m}{m!} (-a)^m (\zeta_R(2s + 2m) - 1), \quad s \neq \frac{1}{2}, -\frac{1}{2}, \ldots,
\]

(23)

where \( \zeta_R \) denotes the Riemann \( \zeta \)-function and \((s)_m\) is Pochhammer’s symbol. If \( s \) approaches one of the poles, this expansion can be used to split off the divergent term: the remainder of the series is denoted with \( \zeta_F(s, a) \). We find for the effective potentials

\[
\mathcal{V}^{(1)}(B = 0) = -18 + 3 \zeta_R(-3) - 3 \zeta_R(-1),
\]

\[
\mathcal{V}^{(1)}(\text{Sph}) = -1 - 3 \sqrt{2} - 12 \sqrt{3} + \frac{9}{2} \sqrt{6} - \frac{5}{2} \sqrt{10} + \frac{11}{4} \varepsilon - \frac{11}{8} \gamma + \frac{11}{4} \log(2)
\]

\[
+ \frac{11}{4} \log \left( \frac{L}{2\sqrt{\pi}} \right) + \zeta_F(-\frac{3}{2}, -3) + \zeta_F(-\frac{3}{2}, -2) + \zeta_F(-\frac{3}{2}, 1)
\]

\[
+ 3 \zeta_F(-\frac{1}{2}, -3) + \zeta_F(-\frac{1}{2}, -2) - 5 \zeta_F(-\frac{1}{2}, 1)
\]

(24)

The pole term \( \frac{11}{4 \varepsilon} \) is absorbed through the usual renormalisation of the coupling constant

\[
\frac{1}{g_R^2} = \frac{1}{g_0^2} + \frac{11}{12 \pi^2 \varepsilon} + \text{finite renormalisation}
\]

(26)

For a general configuration along the tunneling path, we make an expansion of the eigenvalues of the operators in \( u \). Although it is still possible to calculate the spectra exactly, a polynomial form of the effective potential is much more useful in the variational method that we will use. We used Bloch perturbation theory [9] to obtain the expansions of the eigenvalues of \( W_{ij} \). Results up to tenth order in \( u \) were obtained, where we used both MATHEMATICA and FORM in the calculations.

As can be seen from fig. 1, the expansions do not converge to the exact result at \( u = 1 \). This should come as no big surprise, since we have no reason to expect the radius of convergence of the expansion to be as large as one. To find the effective potential for larger \( u \), we write \( u = 1 + a \) and make a similar expansion of the effective potential around the sphaleron. Using the fourth order expansion in \( u \) and the first order expansion in \( a \) (i.e. the value and the slope of the potential at the sphaleron), we can construct a polynomial in \( u \) of degree six that is a good approximation to the effective potential.
We now turn to $B = B(c)$ (we still keep $d = 0$). The perturbative evaluation of the individual eigenvalues is no longer possible, but we can use the following technique. Suppose we have $\hat{A} = F + \hat{A}$ with $F = \hat{A}(0)$ such that $[F, \hat{A}] = 0$. This allows us to substitute in (21)

$$\text{tr} \left( e^{-s\hat{A}} \right) = \text{tr} \left( e^{-sF} e^{-s\hat{A}} \right) = \sum_i e^{-sF_i} \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \text{tr}_i \left( \hat{A}^n \right). \quad (27)$$

The sum over $i$ is a sum over the eigenspaces of $F$, $F_i$ is the corresponding eigenvalue and $\text{tr}_i()$ denotes a trace within the eigenspace. For both the scalar operators we have $F = 4\vec{L}^2$ and $[\vec{L}^2, \hat{A}(c)] = 0$. The remaining problem of calculating $\text{tr}_i \left( \hat{A}^n(c) \right)$ then reduces to calculating traces of the form

$$\text{tr}_i \left( L_{i_1} \ldots L_{i_n} T^{a_1} \ldots T^{a_m} \right) \quad (28)$$

which can be done relatively easy. For the vector operator we have $F = 2\vec{L}^2 + 2\vec{K}^2$ and $[\vec{K}^2, W(c)] \neq 0$. The trace in (27) can however still be written as a sum of traces in the different eigenspaces of $F$:

$$\begin{align*}
\text{tr} \left( e^{-s\hat{A}} \right) &= \sum_i e^{-sF_i} \text{tr}_i \left( e^{-s(F_i + \hat{A})} \right) \\
&= \sum_i e^{-sF_i} \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \text{tr}_i \left( \left( F_i + \hat{A} \right)^n \right) \\
&= \sum_m \text{sp}_m \quad (29)
\end{align*}$$

Here $\text{sp}_m$ is the contribution of order $m$ in $\hat{A}$. Let $P_i$ denote the projector on the eigenspace of $F_i$, and let $T$ be given by

$$\begin{align*}
T_i &= \text{tr}_i \left( \hat{A} \right) = \text{tr}(P_i \hat{A}), \quad (30) \\
T_{ij \ldots} &= \text{tr}(P_i \hat{A} P_j \hat{A} \ldots). \quad (31)
\end{align*}$$

For a given value of $m$ we can perform the combinatorics to write $\text{sp}_m$ in terms of the $T$ functions. With $\Delta_{ji} = F_j - F_i$ we find

$$\begin{align*}
\text{sp}_1 &= \sum_i e^{-sF_i} (-s) T_i, \quad (32) \\
\text{sp}_2 &= \sum_i e^{-sF_i} \sum_{n=2}^{\infty} \frac{(-s)^n}{n!} \text{tr}_i \left( \hat{A} (F_i)^{n-2} \hat{A} \right) \\
&= \sum_i e^{-sF_i} \sum_{n=2}^{\infty} \frac{(-s)^n}{n!} T_i \Delta_{ji}^{n-2} \\
&= \sum_i T_i e^{-sF_i} \frac{s^2}{2} + \sum_{i \neq j} T_{ij} e^{-sF_i} \frac{s}{\Delta_{ji}}, \quad (33) \\
\text{sp}_3 &= \sum_{ijk} T_{ijk} e^{-sF_i} \sum_{n=3}^{\infty} \frac{(-s)^n}{n!} \sum_{m_1=0}^{n-3} \Delta_{ji}^{m_1} \Delta_{ki}^{n-3-m_1}.
\end{align*}$$
etc.

Starting from one eigenspace \( i_m \), the number of intermediate states that can be reached is finite. This allows us to extract one overall summation, and to perform the remaining finite sums. For the case at hand, we extract the summation over \( l \) and are left with an \( l \)-dependent expression in which the \( T \) functions have the form

\[
T_{k_1 k_2 \ldots} = \text{tr}_l \left( P^K_{k_1} \hat{W} P^K_{k_2} \hat{W} \cdots \right).
\]  

Since the intermediate modes in the \( T \) functions can only have \( k = l-1, \ldots, l+1 \), we can write

\[
P^K_k = a_0(k) + a_1(k)\tilde{K}^2 + a_2(k)(\tilde{K}^2)^2,
\]

for certain values of the coefficients. Note that a \( T \) function with an intermediate value \( k = 0 \) should be discarded, since this corresponds to a \( d \) mode. The combinatorics for \( sp_m \), the finite sums, the evaluation of the \( T \) functions and the final summation over \( l \) were all done in FORM.

For general \( B(c, d) \), not even \( \tilde{L}^2 \) commutes with the various operators. The methods described above are however sufficient. The \( T \) traces that we have to calculate now also contain the operators \( \tilde{L}_2 \) and explicit \( V \) functions, as well as projectors on different \( \tilde{L}^2 \) intermediate levels. Using the \( c \leftrightarrow d \) symmetry it is however only necessary to obtain the \( c^2d^2 \) terms. Since the precise form of the coefficients is not very illuminating, we postpone writing down the effective potential until we have performed the renormalisation.

**5 The renormalisation**

To obtain the one-loop contribution to the operator \( \dot{B}^2 \), we perform the usual expansion of the path integral in Feynman diagrams. The subtleties related to the summation over the space-momenta were dealt with in the previous section. The diagrams needed are depicted in fig. 2, where the particle in the loop is respectively a ghost, a scalar \( Q_0 \) or a vector particle. The insertions in the diagrams correspond to the operator \( \hat{A} \) defined above. The propagators in momentum space are given by

\[
\frac{\not{k}}{k_0^2 + k_\varepsilon^2 + 4l(l+1)}
\]

for scalar particles, and

\[
\frac{\not{k}}{k_0^2 + k_\varepsilon^2 + 2l(l+1) + 2k(k+1)}
\]

for vector particles. Here \( k_0 \) is the time component of the momentum and \( k_\varepsilon \) is a momentum related to the \( \varepsilon \)-dimensional torus.

The \( \dot{B}^2 \) term comes from the diagrams with two insertions. If the two time-momenta in these diagrams are denoted by \( p \) and \( p+q \), we first perform the integration over \( p \), and then expand the result in \( q^2 \). Using partial integration, these powers of \( q^2 \) can be transformed in time-derivatives acting on \( \hat{A} \) and hence on \( B \). There is also a diagram with an explicit dependence on \( \dot{B} \). It is the diagram with two insertions of the operator \( W_{0i} \), one \( Q_0 \) and one \( Q_i \) propagator.
κ₀ = 0.0566264741439181
κ₁ = -0.2453459985179565
κ₂ = 3.66869179814223
κ₃ = 0.500703203096610
κ₄ = -0.839359633413003
κ₅ = -0.84965412245339
κ₆ = -0.0650330854836428
κ₇ = -0.361712159967145
κ₈ = -2.295356861354712

Table 1: The numerical values of the coefficients.

Adding up the different contributions, we obtain the one-loop contribution to the kinetic term $c_i^a c_i^a + d_i^a d_i^a$ in the lagrangian. Demanding the renormalised kinetic part to look just like the classical term gives us the finite part of the renormalisation (26):

$$\frac{1}{g_R^2} = \frac{1}{g_0^2} + \frac{11}{12\pi^2\varepsilon} + \frac{11}{12\pi^2} \log\left(\frac{L}{2\sqrt{\pi}}\right) + \kappa_0,$$

where $\kappa_0$ can be found in table I. This renormalisation scheme can easily be related to other schemes like the $\overline{MS}$ scheme.

The finite, renormalised effective potential becomes

$$V_{\text{eff}} = \frac{2\pi^2}{g_R^2} V_{\text{cl}}(c,d) + V^{(1)}_{\text{eff}},$$

with

$$V^{(1)}_{\text{eff}}(c,d) = V^{(1)}_{\text{eff}}(c) + V^{(1)}_{\text{eff}}(d) + \kappa_7 \text{tr}(X) \text{tr}(Y) + \kappa_8 \text{tr}(XY),$$

$$V^{(1)}_{\text{eff}}(c) = \kappa_1 \text{tr}(X) + \kappa_2 \text{det}(c) + \kappa_3 \text{tr}^2(X) + \kappa_4 \text{tr}(X^2) + \kappa_5 \text{det}(c) \text{tr}(X) + \kappa_6 \text{tr}^3(X),$$

with the numerical values for $\kappa_i$ in table I. Note that the $u_5^6$ term in the effective potential along the tunnelling path uniquely determines the coefficient of the $\text{tr}(X) \text{det}(c)$ term. The $u_6^6$ term can be obtained from combinations of the three independent invariants $\text{tr}^3(X)$, $\text{tr}(X) \text{tr}(X^2)$ and $\text{tr}(X^3)$. We choose to replace the $u_6^6$ term by $\text{tr}^3(X)$, which is the simplest of these from the viewpoint of the variational calculation.

6 Variational Results and Conclusions

We calculated the effect of the high-energy modes on the dynamics of the low-energy modes. It resulted in a renormalisation of the coupling constant and a correction to the potential in the effective hamiltonian.

With the obtained hamiltonian, we can repeat the variational approximation of the spectrum. The results remain qualitatively the same: the lowest-lying scalar ($j = 0$) and tensor ($j = 2$) levels can be found in the same sectors as before, although the values of the
energy have changed. Results for the lowest glueball masses can be found in fig. 3. Around 
\( f = 0.25 \) the mass ratio \( m_{2^+}/m_{0^+} \) is roughly 1.5, which compares nicely with the lattice 
results [10].

Just as for the lowest-order hamiltonian, we used Temple’s inequality [11] to obtain 
lower bounds for the energy levels. This convinced us as before that we obtained accurate 
results. However, due to the more complicated structure of the potential, a larger number 
of basis vectors is required.

The onset of the influence of the boundary can be seen at \( f = 0.2 \). One of the issues 
raised in [6] was the level of localisation of the wave function around the sphaleron. This is 
related to the question whether the assumption is true that only the boundary conditions 
at and near the sphalerons are felt. We argued that this was determined by the rise of the 
potential in the transverse directions. The one-loop correction to the \( \text{tr}(Y) \) term in the 
potential at the \( c \)-sphaleron, which can be expressed in \( \kappa_1, \kappa_7 \) and \( \kappa_8 \), is such that it results 
in a lesser degree of localisation. Beyond \( f = 0.3 \) the approximation breaks down as can 
be seen for instance by the crossing of the scalar and tensor glueball. A fuller study into 
this localisation, as well as more details and results will be presented in the near future.

7 Acknowledgment

The author wishes to thank Pierre van Baal for many helpful discussions on the subject.

References

[1] J. Koller and P. van Baal, Nucl. Phys. B302 (1988) 1; P. van Baal, Acta Phys. Pol. 
B20 (1989) 295.
[2] P. van Baal and N. D. Hari Dass, Nucl. Phys. B385 (1992) 185.
[3] P. van Baal and B.M. van den Heuvel, Nucl. Phys. B417 (1994) 215.
[4] M.A. Semenov-Tyan-Shanskii and V.A. Franke, J. Sov. Math. 34 (1986) 1999.
[5] D. Zwanziger, Nucl. Phys. B412 (1994) 657 and references therein.
[6] B.M. van den Heuvel, Phys. Lett. B 368 (1996) 124.
[7] G. ’t Hooft, Phys. Rev. D14 (1976) 3432.
[8] M. Lüscher, Ann. Phys. 142 (1982) 359.
[9] C. Bloch, Nucl. Phys. 6 (1958) 329.
[10] B. Carpenter, C. Michael and M.J. Teper, Phys. Lett. B 198 (1987) 511; C. Michael, 
G.A. Tickle and M.J. Teper, Phys. Lett. B 207 (1988) 313.
[11] M. Reed and B. Simon, Methods of modern mathematical physics, vol. 4 (Academic 
Press, New York, 1978)
Figure captions

Figure 1: Expansion of $\psi'(u)$ in the tunneling parameter $u$. We dropped the $\varepsilon$ and $\log(L)$ dependent parts. We have drawn the expansion up to order $u^4$, $u^6$, $u^8$ and $u^{10}$. Longer dashes correspond to higher order expansions. The horizontal line at $u = 1$ denotes the exact result at the sphaleron.

Figure 2: Topology of the different Feynman diagrams.

Figure 3: Glueball masses for $\theta = 0$ as a function of the coupling constant. The lower and upper drawn curves are the masses of resp. the first scalar ($0^+$) and tensor ($2^+$) glueball. The dotted lines denote the perturbative results.
