Nonlinear superpositions and Ermakov systems

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Abstract

The theory of superposition rules for solutions of a Lie system of first-order differential equations is extended to deal with analogous systems of second-order and the theory is illustrated with the very rich example of Ermakov-like systems.

1 Introduction

The characterization of non-autonomous systems of first-order differential equations
\[
\frac{dx^i}{dt} = Y^i(t, x), \quad i = 1, \ldots, n,
\] (1.1)
admitting a superposition rule is due to Lie \cite{1} and such a problem has been receiving very much attention during the last thirty years because of its very important applications in physics \cite{2}-\cite{12}. The theory has recently been revisited from a more geometric approach in \cite{14} where the rôle of the superposition function is played by an appropriate connection. The main point is that this new approach allows us to consider partial superposition of solutions as well and, furthermore, it also allows superposition of solutions of a given system in order to obtain solutions of a new system. Our aim here is to show how such a superposition rule may be understood from a geometric viewpoint in a very simple but interesting case, the so called Ermakov-Pinney system \cite{15,16} as well as for other generalizations of it. Such a system is made of second-order differential equations but the theory developed by Lie can easily be adapted to deal with such SODE systems. We find in this way room for implicit nonlinear superposition rules in the terminology of \cite{17,18} and the so called Ermakov-Lewis invariants \cite{19} appear in a natural way as functions defining the foliation associated to the superposition rule. Moreover all reduction techniques developed for Lie systems \cite{6,12} are also valid in these cases.
2 Systems of differential equations admitting a superposition rule

The superposition rule for solutions of (1.1) is determined by a function \( \Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n \),

\[
x = \Phi(x(1), \ldots, x(m); k_1, \ldots, k_n),
\]

(2.2)
such that the general solution can be written, for sufficiently small \( t \), as

\[
x(t) = \Phi(x(1)(t), \ldots, x(m)(t); k_1, \ldots, k_n),
\]

(2.3)
where \( \{x(a)(t) \mid a = 1, \ldots, m\} \) is a fundamental set of particular solutions of the system (1.1) and \( k = (k_1, \ldots, k_n) \) is a set of \( n \) arbitrary constants associated with each particular solution. As a consequence of the Implicit Function Theorem, the function \( \Phi(x(1), \ldots, x(m); \cdot) : \mathbb{R}^n \to \mathbb{R}^n \) can be, at least locally around generic points, inverted, so we can write

\[
k = \Psi(x(0), \ldots, x(m))
\]

(2.4)
for a certain function \( \Psi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n \). Hereafter in order to handle a short notation we start writing \( x(0) \) instead of \( x \). The function \( \Psi \), also called superposition function, provides us with a foliation which is invariant under permutations of the \( (m + 1) \) variables. The functions \( \Phi \) and \( \Psi \) are related by:

\[
k = \Psi(\Phi(x(1), \ldots, x(m); k_1, \ldots, k_n), x(1), \ldots, x(m)).
\]

(2.5)
The fundamental property of the superposition function \( \Psi \) is that as

\[
k = \Psi(x(0)(t), x(1)(t), \ldots, x(m)(t)),
\]

(2.6)
the function \( \Psi(x(0), \ldots, x(m)) \) is constant on any \( (m + 1) \)-tuple of solutions of the system (1.1). This implies that the ‘diagonal prolongations’ \( \tilde{Y}(t, x(0), \ldots, x(m)) \) of the \( t \)-dependent vector field \( Y(t, x) = Y^i(t, x) \partial/\partial x^i \), given by

\[
\tilde{Y}(t, x(0), \ldots, x(m)) = \sum_{a=0}^{m} Y_a(t, x(a)), \quad t \in \mathbb{R},
\]

where

\[
Y_a(t, x(a)) = \sum_{i=1}^{n} Y^i(t, x(a)) \frac{\partial}{\partial x^i(a)}
\]

(2.7)
are \( t \)-dependent vector fields on \( \mathbb{R}^{n(m+1)} \) which are tangent to the level sets of \( \Psi \), i.e. the components \( \Psi^i \) are constants of motion. The level sets of \( \Psi \) corresponding to regular values define a \( n \)-codimensional foliation \( \mathcal{F} \) on an open dense subset \( U \subset \mathbb{R}^{n(m+1)} \) and the family \( \{\tilde{Y}(t), t \in \mathbb{R}\} \) of vector fields in \( \mathbb{R}^{n(m+1)} \) consists of vector fields tangent to the leaves of this foliation.

Remark that, as pointed out in [14], for each \( (x(1), \ldots, x(m)) \in \mathbb{R}^{nm} \) there is one point \( (x(0), x(1), \ldots, x(m)) \) on the level set \( \mathcal{F}_k \) of this foliation \( \mathcal{F} \) corresponding
to \( k = (k_1, \ldots, k_n) \in \mathbb{R}^n \), namely, \( (\Phi(x_{(1)}, \ldots, x_{(m)}; k), x_{(1)}, \ldots, x_{(m)}) \in \mathcal{F}_k \) (cf. (2.5)); then, the projection onto the last \( m \) factors

\[
\text{pr} : (x_{(0)}, x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R}^{n(m+1)} \mapsto (x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R}^{nm}
\]

induces diffeomorphisms on the leaves \( \mathcal{F}_k \) of \( \mathcal{F} \). Such a foliation gives us the superposition principle without referring to the function \( \Psi \): if we fix the point \( x_{(0)}(0) \) (i.e. we choose a \( k = (k_1, \ldots, k_n) \)) and \( m \) solutions \( x_{(1)}(t), \ldots, x_{(m)}(t) \), then \( x_{(0)}(t) \) is the unique point in \( \mathbb{R}^n \) such that \( (x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) \) belongs to the same leaf of \( \mathcal{F} \) as \( (x_{(0)}(0), x_{(1)}(0), \ldots, x_{(m)}(0)) \). This means that it is only \( \mathcal{F} \) really matters for the superposition rule.

Lie’s main result [1] can be expressed as follows:

**Theorem:** The system (1.1) on a differentiable manifold \( N \) admits a superposition rule if and only if the \( t \)-dependent vector field \( Y(t, x) \) can be locally written in the form

\[
Y(t, x) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x)
\]

where the vector fields \( X_\alpha, \alpha = 1, \ldots, r \), close on a \( r \)-dimensional real Lie algebra, i.e. there exist \( r^3 \) real numbers \( c_{\alpha\beta\gamma} \) such that

\[
[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X_\gamma, \quad \forall \alpha, \beta = 1, \ldots, r.
\]  

(2.8)

The number \( m \) of solutions involved in the superposition rule for the Lie system defined by (2.8) with generic \( b_\alpha(t) \) is the minimal \( k \) such that the diagonal prolongations of \( X_1, \ldots, X_r \) to \( N^k \) are linearly independent at (generically) each point: the only real numbers solution of the linear system

\[
\sum_{a=1}^{r} c_\alpha X_\alpha(x_{(a)}) = 0, \quad a = 1, \ldots, k
\]

at a generic point \( (x_{(1)}, \ldots, x_{(k)}) \) is the trivial solution \( c_\alpha = 0, \alpha = 1, \ldots, m \), for \( k = m \), and there are nontrivial solutions for \( k < m \). Then, the superposition function \( \Psi \) is made up of \( t \)-independent constants of motion for the prolonged \( t \)-dependent vector field \( \tilde{Y} \). We shall call them first integrals.

A possible generalization consists on considering foliations which are not of codimension \( n \), or even choosing different Lie systems with the same associated Lie algebra for defining the prolonged vector field. These facts will be illustrated with several examples.
3 SODE Lie systems

A system of second-order differential equations

\[ \ddot{x}^i = f_i(t, \dot{x}), \quad i = 1, \ldots, n, \]

can be studied through the system of first-order differential equations

\[
\begin{align*}
\frac{dx^i}{dt} &= v^i \\
\frac{dv^i}{dt} &= f_i(t, x, v)
\end{align*}
\]

with associated \( t \)-dependent vector field

\[ X = v^i \frac{\partial}{\partial x^i} + f_i(t, x, v) \frac{\partial}{\partial v^i}. \]

We call SODE Lie systems those for which \( X \) is a Lie system, i.e. it can be written as a linear combination with \( t \)-dependent coefficients of vector fields closing a finite-dimensional real Lie algebra. There are many interesting examples of such SODE Lie systems and next section is devoted to introduce some particular examples.

3.1 The 1-dim harmonic oscillator with time-dependent frequency

The equation of motion is \( \ddot{x} = -\omega^2(t)x \), with associated system

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2(t)x
\end{align*}
\]  

(3.9)

and \( t \)-dependent vector field

\[ X = v \frac{\partial}{\partial x} - \omega^2(t)x \frac{\partial}{\partial v}, \]

which is a linear combination \( X = X_2 - \omega^2(t)X_1 \), with

\[ X_1 = x \frac{\partial}{\partial v}, \quad X_2 = v \frac{\partial}{\partial x} \]

such that

\[ [X_1, X_2] = 2X_3, \quad [X_1, X_3] = -X_1, \quad [X_2, X_3] = X_2, \]
where $X_3$ is the vector field given by

$$X_3 = \frac{1}{2} \left( x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v} \right).$$

Therefore $X$ defines a Lie system with associated Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Actually, the vector fields $X_i$ are the fundamental vector fields corresponding to the usual linear action and the basis of $\mathfrak{sl}(2,\mathbb{R})$ given by the following traceless $2 \times 2$ real matrices

$$a_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad a_3 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.10)$$

This system has no first integrals, i.e. there are not $t$-independent constants of motion.

### 3.2 The 2-dim isotropic harmonic oscillator with time-dependent frequency

The system of equations of motion is

$$\begin{cases} \ddot{x}_1 = -\omega^2(t)x_1 \\ \ddot{x}_2 = -\omega^2(t)x_2 \end{cases} \quad (3.11)$$

with associated system

$$\begin{cases} \dot{x}_1 = v_1 \\ \dot{v}_1 = -\omega^2(t)x_1 \\ \dot{x}_2 = v_2 \\ \dot{v}_2 = -\omega^2(t)x_2 \end{cases}$$

and $t$-dependent vector vector field

$$X = v_1 \frac{\partial}{\partial x_1} - \omega^2(t)x_1 \frac{\partial}{\partial v_1} + v_2 \frac{\partial}{\partial x_2} - \omega^2(t)x_2 \frac{\partial}{\partial v_2},$$

which is a linear combination, $X = X_2 - \omega^2(t)X_1$, with

$$X_1 = x_1 \frac{\partial}{\partial v_1} + x_2 \frac{\partial}{\partial v_2}, \quad X_2 = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2},$$

such that

$$[X_1, X_2] = 2X_3, \quad [X_1, X_3] = -X_1, \quad [X_2, X_3] = X_2,$$
where the vector field $X_3$ is defined by

$$X_3 = \frac{1}{2} \left( \frac{x_1}{\partial x_1} - \frac{v_1}{\partial v_1} + \frac{x_2}{\partial x_2} - \frac{v_2}{\partial v_2} \right).$$

Once again $X$ defines a Lie system with associated Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. However, in the preceding case there is no constant of motion and we shall see that in this last one $x_1v_2 - x_2v_1$ is a constant of the motion.

### 3.3 Milne–Pinney equation

We call Milne–Pinney equation the second-order non-linear differential equation \cite{16,20}:

$$\ddot{x} = -\omega^2(t)x + \frac{k}{x^3}, \quad \text{(3.12)}$$

where $k$ is a constant. It describes the time-evolution of an isotonic oscillator \cite{21,22}, i.e. an oscillator with inverse quadratic potential \cite{23}. This oscillator shares with the harmonic one the property of having a period independent of the energy \cite{24}, i.e. they are isochronous systems, and in the quantum case they have a equispaced spectrum \cite{25}.

The corresponding system of first-order differential equations is

$$\begin{cases} 
\dot{x} = v \\
\dot{v} = -\omega^2(t)x + \frac{k}{x^3}
\end{cases}$$

and the associated $t$-dependent vector field

$$X = v \frac{\partial}{\partial x} + \left( -\omega^2(t)x + \frac{k}{x^3} \right) \frac{\partial}{\partial v}.$$

This is a Lie system because it can be written as

$$X = L_2 - \omega^2(t)L_1,$$

where

$$L_1 = x \frac{\partial}{\partial v}, \quad L_2 = \frac{k}{x^3} \frac{\partial}{\partial v} + v \frac{\partial}{\partial x},$$

are such that

$$[L_1, L_2] = 2L_3, \quad [L_3, L_2] = -L_2, \quad [L_3, L_1] = L_1$$

with

$$L_3 = \frac{1}{2} \left( x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v} \right),$$
i.e. they span a 3-dimensional real Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Actually, one can show that they are the fundamental vector fields associated with the basis (3.10) relative to the following action of the group $SL(2, \mathbb{R})$ on a point $(x_0, v_0) \in \mathbb{R}^2$:

If the matrix $A$ in $SL(2, \mathbb{R})$ is given by

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then $\Phi(A, (x, v)) = (\bar{x}, \bar{v})$ with

$$\begin{cases} 
\bar{x} = \text{sign}(x) \sqrt{\frac{k + [(\beta v + \alpha x)(\delta v + \gamma x) + k(\delta \beta/x^2)]^2}{(\delta v + \gamma x)^2 + k(\delta/x)^2}} \\
\bar{v} = \sqrt{(\delta v + \gamma x)^2 + k\delta^2 x^2 \left(1 - \frac{x^2}{\delta^2 x^2}\right)}
\end{cases}$$

(3.13)

with $\text{sign}(x) = x/|x|$.

### 3.4 Ermakov system

Consider the system [26][27]

$$\begin{cases} 
\dot{x} = v_x \\
\dot{v}_x = -\omega^2(t)x \\
\dot{y} = v_y \\
\dot{v}_y = -\omega^2(t)y + \frac{1}{y^3}
\end{cases}$$

with associated $t$-dependent vector field

$$X = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} - \omega^2(t)x \frac{\partial}{\partial v_x} + \left(-\omega^2(t)y + \frac{1}{y^3}\right) \frac{\partial}{\partial v_y},$$

which is a linear combination with time-dependent coefficients, $X = -\omega^2(t)X_1 + X_2$, of the vector fields

$$X_1 = x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial v_y}, \quad X_2 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \frac{1}{y^3} \frac{\partial}{\partial v_y}.$$
This system is made up by two Lie systems, which correspond, respectively, to the examples of the 1-dimensional harmonic oscillator (3.9) and the Milne equation (3.12), both closing on a \( \mathfrak{sl}(2, \mathbb{R}) \) algebra with \( X_3 \) given by

\[
X_3 = \frac{1}{2} \left( x \frac{\partial}{\partial x} - v_x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial y} - v_y \frac{\partial}{\partial v_y} \right).
\]

3.5 Generalized Ermakov system

It is the system given by [18, 28, 29, 30, 31, 32, 33]:

\[
\begin{align*}
\ddot{x} &= \frac{1}{x^3} f(y/x) - \omega^2(t)x \\
\ddot{y} &= \frac{1}{y^3} g(y/x) - \omega^2(t)y
\end{align*}
\]

that for the choice \( f(u) = 0 \) and \( g(u) = 1 \) reduces to the Ermakov system.

This system of second-order equations can be written as one of first-order equations by doubling the number of degrees of freedom by introducing the new variables \( v_x \) and \( v_y \):

\[
\begin{align*}
\dot{x} &= v_x \\
\dot{v}_x &= -\omega^2(t)x + \frac{1}{x^3} f(y/x) \\
\dot{y} &= v_y \\
\dot{v}_y &= -\omega^2(t)y + \frac{1}{y^3} g(y/x).
\end{align*}
\]

Such system determines the integral curves of the vector field

\[
X = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \left(-\omega^2(t)x + \frac{1}{x^3} f(y/x)\right) \frac{\partial}{\partial v_x} + \left(-\omega^2(t)y + \frac{1}{y^3} g(y/x)\right) \frac{\partial}{\partial v_y},
\]

which can be written as a linear combination

\[
X = N_2 - \omega^2(t) N_1
\]

where \( N_1 \) and \( N_2 \) are the vector fields

\[
N_1 = x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial v_y}, \quad N_2 = v_x \frac{\partial}{\partial x} \frac{1}{x^3} f(y/x) \frac{\partial}{\partial v_x} + v_y \frac{\partial}{\partial y} + \frac{1}{y^3} g(y/x) \frac{\partial}{\partial v_y}.
\]

Note that these vector fields generate a 3-dimensional real Lie algebra with a
third generator

\[ N_3 = \frac{1}{2} \left( x \frac{\partial}{\partial x} - v_x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial y} - v_y \frac{\partial}{\partial v_y} \right). \]

In fact, as \([N_1, N_2] = 2N_3, \quad [N_3, N_1] = N_1, \quad [N_3, N_2] = -N_2\),

they generate a Lie algebra isomorphic to \(\mathfrak{sl}(2, \mathbb{R})\). Therefore the system is a Lie system.

The fact that the systems of preceding examples are Lie systems with the same associated Lie algebra means that they can be solved simultaneously in the group \(SL(2, \mathbb{R})\) by the equation (see e.g. \([6,12]\))

\[ \dot{g} \ g^{-1} = \omega^2(t) \ a_1 - a_2, \]

where \(a_1 \) and \(a_2\) are given in (3.10).

4 The superposition functions for these examples

Consider first the 1-dimensional harmonic oscillator (3.9). In order to look for a superposition rule we should consider a system like in (3.11) and check whether the vector fields \(X_1\) and \(X_2\) are linearly independent in a generic point. There are many points in which one can choose non trivial coefficients \(\lambda_1, \lambda_2\) and \(\lambda_3\) such that \(\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3\) vanishes in such a point. On the contrary, if we introduce another copy and obtain the system (3.11), only from the vanishing of such vector in a point such that \(x_1 v_2 - x_2 v_1 = 0\) and \(x_3 v_1 - x_1 v_2 = 0\) we cannot say that \(\lambda_1 = \lambda_2 = \lambda_3 = 0\), therefore \(m = 2\) and consequently there is a superposition rule involving two particular solutions.

Note that such system (3.11) admits a first integral because the function \(F\) given by \(F(x_1, x_2, v_1, v_2)\) is such that \(X_2 F = 0\) iff there exists a function \(\bar{F}(\xi, v_1, v_2)\) with \(\xi = x_1 v_2 - x_2 v_1\), such that \(F(x_1, x_2, v_1, v_2) = \bar{F}(\xi, v_1, v_2)\), and then from the second condition,

\[ x_1 \frac{\partial F}{\partial v_1} + x_2 \frac{\partial F}{\partial v_2} = 0, \]

we obtain the first integral, which corresponds to the angular momentum, \(F(x_1, x_2, v_1, v_2) = x_1 v_2 - x_2 v_1\), which can be seen as a partial superposition rule. Actually, if \(x_1(t)\) is a solution of the first equation, then we obtain for
each real number $k$ the first-order differential equation for the variable $x_2$

$$x_1(t) \frac{dx_2}{dt} = k + \dot{x}_1(t)x_2,$$

from where $x_2$ can be found to be given by

$$x_2(t) = k'x_1(t) + k x_1(t) \int^t \frac{d\zeta}{x_1^2(\zeta)}.$$  \hspace{1cm} (4.14)

In order to look for the superposition rule which does not involve quadratures
we should consider three copies of the same oscillator, and the extended vector
fields $X_1$ and $X_2$ given by

$$X_1 = x_1 \frac{\partial}{\partial v_1} + x_2 \frac{\partial}{\partial v_2} + x \frac{\partial}{\partial v}, \quad X_2 = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v \frac{\partial}{\partial x}.$$ We can determine the first integrals $F$ as solutions of $X_1F = X_2F = 0$. The
condition $X_2F = 0$ says that there exists a function $\bar{F} : \mathbb{R}^5 \to \mathbb{R}^2$ such that
$F(x_1, x_2, x, v_1, v_2, v) = \bar{F}(\xi_1, \xi_2, v_1, v_2, v)$ with $\xi_1(x_1, x_2, x, v_1, v_2, v) = xv_1 - x_1v$ and $\xi_2(x_1, x_2, x, v_1, v_2, v) = xv_2 - x_2v$, and the condition $X_1F = 0$ transforms into

$$x_1 \frac{\partial \bar{F}}{\partial v_1} + x_2 \frac{\partial \bar{F}}{\partial v_2} + x \frac{\partial \bar{F}}{\partial v} = 0,$$

i.e. $\xi_1$ and $\xi_2$ are first integrals (Of course, $\xi = x_1v_2 - x_2v_1$ is also a first
integral). They produce a superposition rule, because from

$$\begin{align*}
x v_2 - x_2v &= k_1 \\
x_1v - vx_1 &= k_2
\end{align*}$$

we obtain the expected superposition rule for two solutions:

$$x = c_1 x_1 + c_2 x_2, \quad v = c_1 v_1 + c_2 v_2, \quad c_i = \frac{k_1}{k}, \quad k = x_1v_2 - x_2v_1.$$ As a second example, consider the Milne system given in (3.12) The generators
of this Lie system with algebra $\mathfrak{sl}(2, \mathbb{R})$ span a distribution of dimension two
and there is no first integral of the motion for such subsystem. By adding
the other $\mathfrak{sl}(2, \mathbb{R})$ linear Lie system appearing in the Ermakov system, the
harmonic oscillator with time dependent angular frequency, as the distribution
in the 4-dimensional space is of rank three, there is an integral of motion. The
first integral can be obtained from $L_1F = L_2F = 0$. But $L_1F$ means that
$F(x, y, v_x, v_y) = \bar{F}(x, y, \xi)$ with $\xi = xv_y - yv_x$, and then $L_2F = 0$ is written

$$v_x \frac{\partial \bar{F}}{\partial x} + v_x \frac{\partial \bar{F}}{\partial x} + x \frac{\partial \bar{F}}{\partial \xi}.$$
and we obtain the associated system of characteristics

\[
\frac{x \, dy - y \, dx}{\xi} = \frac{y^3 \, d\xi}{x} \implies \frac{d(x/y)}{\xi} + \frac{y \, d\xi}{x} = 0,
\]

from where the following first integral is found \cite{19}:

\[
\psi(x, y, v_x, v_y) = \left(\frac{x}{y}\right)^2 + \xi^2 = \left(\frac{x}{y}\right)^2 + (xv_y - yv_x)^2,
\]

which is the well-known Lewis–Ermakov invariant \cite{26,27,28}.

We can follow a similar path in the case of the generalized Ermakov system. There exists a first integral for the motion, \( F : \mathbb{R}^4 \to \mathbb{R} \), for any \( \omega^2(t) \), because this Lie system has an associated integrable distribution of rank three and the manifold is 4-dimensional.

This first integral \( F \) satisfies \( N_i F = 0 \) for \( i = 1, \ldots, 3 \), but as \( [N_1, N_2] = 2N_3 \) it is enough to impose \( N_1 F = N_2 F = 0 \). Then, if \( N_1 F = 0 \),

\[
\frac{x \, \partial F}{\partial v_x} + \frac{y \, \partial F}{\partial v_y} = 0,
\]

and according to the method of characteristics we obtain:

\[
\frac{dx}{0} = \frac{dy}{0} = \frac{dv_x}{x} = \frac{dv_y}{y}
\]

and therefore there exists a function \( \tilde{F} : \mathbb{R}^3 \to \mathbb{R} \) such that \( F(x, y, v_x, v_y) = \tilde{F}(x, y, \xi = xv_y - yv_x) \). The condition \( N_2 F = 0 \) reads now

\[
v_x \frac{\partial \tilde{F}}{\partial x} + v_y \frac{\partial \tilde{F}}{\partial y} + \left( -\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x) \right) \frac{\partial \tilde{F}}{\partial \xi}.
\]

We can therefore consider the associated system of the characteristics:

\[
\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{d\xi}{-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x)}
\]

But using that

\[
\frac{-y \, dx + x \, dy}{\xi} = \frac{dx}{v_x} = \frac{dy}{v_y},
\]

we arrive to

\[
\frac{-y \, dx + x \, dy}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x)}
\]

i.e.

\[
-\frac{y^2 d\left(\frac{z}{y}\right)}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x)}
\]
and integrating we obtain the following first-integral, with \( u = y/x \),

\[
\frac{1}{2} \xi^2 + \int^u \left[ -\frac{1}{\zeta^3} f \left( \frac{1}{\zeta} \right) + \zeta g \left( \frac{1}{\zeta} \right) \right] \, d\zeta = C.
\]

This first integral allows us to determine, by means of quadratures, a solution of one subsystem in terms of a solution of the other equation.

### 4.1 The Pinney equation revisited

We mentioned before the possibility of obtaining solutions of a given system from particular solutions of another related system. We next study a particular example, studied by Pinney long time ago \[16\]. Consider the system of first-order differential equations:

\[
\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{z} &= v_z \\
\dot{v}_x &= -\omega^2(t)x + \frac{k}{x^3} \\
\dot{v}_y &= -\omega^2(t)y \\
\dot{v}_z &= -\omega^2(t)z
\end{aligned}
\]

which corresponds to the vector field

\[
X = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + \frac{k}{x^3} \frac{\partial}{\partial v_x} - \omega^2(t) \left( x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial v_y} + z \frac{\partial}{\partial v_z} \right).
\]

The vector field \( X \) can be expressed as \( X = N_2 - \omega^2(t) N_1 \) where the vector fields \( N_1 \) and \( N_2 \) are:

\[
\begin{aligned}
N_1 &= y \frac{\partial}{\partial v_y} + x \frac{\partial}{\partial v_x} + z \frac{\partial}{\partial v_z}, \quad N_2 &= v_y \frac{\partial}{\partial y} + \frac{1}{x^3} \frac{\partial}{\partial v_x} + v_z \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v_z}.
\end{aligned}
\]

These vector fields generate a 3-dimensional real Lie algebra with the vector field \( N_3 \) given by

\[
N_3 = \frac{1}{2} \left( x \frac{\partial}{\partial x} - v_x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial y} - v_y \frac{\partial}{\partial v_y} + z \frac{\partial}{\partial z} - v_z \frac{\partial}{\partial v_z} \right).
\]

In fact, they generate a Lie algebra isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) because \( [N_1, N_2] = 2N_3, \quad [N_3, N_1] = N_1, \quad [N_3, N_2] = -N_2 \).
The distribution generated by these fundamental vector fields has rank three. Thus, as the manifold of the Lie system is of dimension six we obtain three time-independent integrals of motion.

- The Ermakov invariant $I_1$ of the subsystem involving variables $x$ and $y$.
- The Ermakov invariant $I_2$ of the subsystem involving variables $x$ and $z$.
- The Wronskian $W$ of the subsystem involving variables $y$ and $z$. They define a foliation with 3-dimensional leaves. We can use this foliation for obtaining in terms of them such a superposition rule.

The Ermakov invariants read as:

$$I_1 = \frac{1}{2} \left( (yv_x - xv_y)^2 + k \left( \frac{y}{x} \right)^2 \right), \quad I_2 = \frac{1}{2} \left( (xv_z - zv_x)^2 + k \left( \frac{z}{x} \right)^2 \right),$$

and $W$ is:

$$W = yv_z - zv_y.$$

In terms of these three integrals we can obtain an explicit expression of $x$ in terms of $y, z$ and the integrals $I_1, I_2, W$:

$$x = \frac{\sqrt{2}}{W} \left( I_2 y^2 + I_1 z^2 \pm \sqrt{4I_1 I_2 - kW^2 yz} \right)^{1/2}.$$

This can be interpreted, as pointed out by Pinney [16], as saying that there is a superposition rule allowing us to express the general solution of the Milne–Pinney equation in terms of two independent solutions of the corresponding harmonic oscillator with the same time-dependent angular frequency.

5 The reduction technique for SODE Lie systems

We aim to illustrate by simple examples the usefulness of the reduction technique for Lie systems in this particular case of SODE systems. The main point is that given a Lie system in a homogeneous space for the associated group $G$, the knowledge of a particular solution allows us to reduce the problem to a new Lie system in the corresponding stability subgroup [6, 12]. Actually, if the Lie system is defined in the Lie group $G$ by the equation

$$\dot{g}(t) g^{-1}(t) = a(t), \quad g(0) = e \quad (5.15)$$

and we know a particular solution $x_1(t)$ of the associated system in a homogeneous space, we can choose a curve $\tilde{g}(t)$ in $G$ such that $\tilde{g}(0) = e$ and $\Phi(\tilde{g}(t), x_0) = x(t)$, and therefore, as also $\Phi(g(t), x_0) = x(t)$, there is a curve $h(t)$ in $G_{x_0}$ such that $g(t) = \tilde{g}(t) h(t)$. Then $h(t)$ is the solution of the Lie
system in $G_{x_0}$

$$\dot{h}(t) h^{-1}(t) = \text{Ad} (\bar{g}^{-1}(t))(a(t)) + \bar{g}^{-1}(t) \bar{g}(t), \quad h(0) = e.$$ 

Once such a Lie system has been solved the solution in the group is $g(t) = \bar{g}(t) h(t)$ and therefore the solution starting from any point $y_0$ of the associated Lie system in a homogeneous space is given by $\Phi(\bar{g}(t) h(t), y_0)$. As it is shown in the reduction technique, the transformation through the curve $\bar{g}(t)$ reduces the problem from the group $G$ to the stability group of the initial condition of the particular solution $x_0$.

Instead, we can be interested in the reduction of the Lie system to one in the stability subgroup of a previously fixed point $z_0$ and we should proceed as follows. Let $g_0$ be a fixed element in $G$ such that $\Phi(g_0, x_0) = z_0$. Then $g(t)$ is a solution of (5.15) iff $\bar{g}(t) = g(t) g_0^{-1}$ is a solution of the same equation as in (5.15) but with $\bar{g}(0) = g_0^{-1}$. Moreover, such curve $\bar{g}(t)$ can also be used to define the solution of the system in the homogeneous space starting from $x_0$ by means of $\Phi(\bar{g}(t), z_0)$. Note that if $x_1(t)$ is a particular solution in the given homogeneous space with initial condition $x_1(0) = x_0$, we can choose a curve $g_1(t)$ in $G$ such that $\Phi(g_1(t), z_0) = x_1(t)$ and then from $\Phi(g_1(t), z_0) = \Phi(\bar{g}(t), z_0) = x_1(t)$ we obtain that:

$$z_0 = \Phi(g_1^{-1}(t), \Phi(g_1(t), z_0)) = \Phi(g_1^{-1}(t), \Phi(\bar{g}(t), z_0)) = \Phi(g_1^{-1}(t) \bar{g}(t), z_0).$$

Thus, $h(t) = g_1^{-1}(t) \bar{g}(t)$ lies in the stability group of $z_0, G_{z_0}$, and, consequently, $g_1(t)$ and $\bar{g}(t)$ differ on a curve in $G_{z_0}$. Recall that $\bar{g}(t)$ satisfies the equation (5.15) but with $\bar{g}(0) = g_0^{-1}$ and therefore $h(t)$ is such that:

$$\dot{h}(t) h^{-1} = \frac{d}{dt} (g_1^{-1}(t) \bar{g}(t))(g_1^{-1}(t) \bar{g}(t))^{-1}$$

$$= \dot{g}_1^{-1}(t) g_1(t) + \text{Ad}(g_1^{-1}(t))(a(t)) = a'(t) \in T_I G_{z_0}, \quad (5.16)$$

and satisfies the initial condition $h(0) = g_1^{-1}(0) g_0^{-1}$.

Then, the transformation given by $g_1^{-1}(t)$ changes the initial equation in $G$ associated to $a(t)$ into a new equation in $G_{z_0}$ associated with $a'(t)$ independently of the initial condition of the particular solution $x_1(t)$. When such stability group $G_{z_0}$ has a solvable Lie algebra $\mathfrak{g}_{z_0}$, we can solve the equation in $G_{z_0}$ for any $a'(t)$.

Conversely, if $h(t)$ is a curve in $G_{z_0}$ solution of the Lie system in $G_{z_0}$

$$\dot{h}(t) h^{-1}(t) = a'(t), \quad (5.17)$$

satisfying the initial condition $h(0) = g_1^{-1}(0) g_0^{-1}$, then $g(t) = g_1(t) h(t) g_0$ is the curve solution of our initial Lie system. On the other side, the equations defining Lie systems are right-invariant and therefore if $h'(t)$ is a curve solution of
but with the usual initial condition \( h'(0) = e \), then the solution of (5.17) with initial condition \( h(0) = g_1^{-1}(0)g_0^{-1} \) is \( h(t) = h'(t)g_1^{-1}(0)g_0^{-1} \). Therefore, the solution of the initial Lie system is given by \( g(t) = g_1(t)h'(t)g_1^{-1}(0) \).

Let us apply this theoretical development to three particular cases of this reduction procedure: a reduction of a harmonic oscillator with a time-dependent frequency using one particular solution, a reduction of a Milne–Pinney equation through a particular solution, and a reduction of a Milne–Pinney equation by means of a particular solution of a harmonic oscillator with the same time-dependent frequency.

Consider first the simple example (3.9) of the harmonic oscillator with a time-dependent frequency and assume that a particular solution \( x_1(t) \) is known. This is a Lie system in \( \mathbb{R}^2 \) which is a homogeneous space for the group \( SL(2, \mathbb{R}) \). The orbit of the point \((1, 0)\) under the linear action is all \( \mathbb{R}^2 \), the stability group of such point being the 1-dimensional Lie subgroup generated by \( v \partial/\partial x \). We can choose the matrix

\[
g_1(t) = \begin{pmatrix} x_1(t) & 0 \\ \dot{x}_1(t) & x_1^{-1}(t) \end{pmatrix}
\] (5.18)

as a curve mapping the point \((1, 0)\) onto the given solution. Then we should make the change of coordinates corresponding to the transformation in the homogeneous manifold induced by \( g_1^{-1}(t) \):

\[
\begin{pmatrix} x \\ v_x \end{pmatrix} = \begin{pmatrix} x_1(t) & 0 \\ \dot{x}_1(t) & x_1^{-1}(t) \end{pmatrix} \begin{pmatrix} z \\ v_z \end{pmatrix},
\]

and then we obtain the new second-order differential equation which is a first-order system in \( \dot{z} \),

\[
x_1(t) \ddot{z} + 2 \dot{x}_1(t) \dot{z} = 0,
\]

and allows us to find the general solution by means of a quadrature, because if \( u = \dot{z} \), the general solution of \( x_1(t) \ddot{u} + 2 \dot{x}_1(t) u = 0 \) is \( u = k/(x_1(t))^2 \), for any constant and then we obtain a second quadrature the expression (4.14). The fact that the new equation is solvable is related to the fact that the new equation is related with the solvability of \( g_{z_0} \). This allows us to obtain its solutions by quadratures. Also, note that this change \( x = x_1(t) z \) corresponds to the traditional d’Alembert method of reduction of order (see e.g. [34]).

Consider now the example (3.12) of the Milne–Pinney equation. This is a homogeneous case if we consider that either \( x > 0 \) or \( x < 0 \) and we restrict ourselves to one of these cases. Now, given a particular solution \( x_1(t) \) of the Milne–Pinney equation the \( g_1(t) \) constructed with \( x_1(t) \) as in (5.18) transforms the point \( z_0 = (1, 0) \) into a solution \((x_1(t), \dot{x}_1(t))\) of this differential equation.
Thus, as in the last case, we can use again \( g_1^{-1}(t) \) to transform the initial equation in the group given by \( a(t) \) into a new one given by \( a'(t) \) that actually is a Lie equation in \( g_{\text{so}} \). This new one will be solvable because \( g_{\text{so}} \) is solvable in this case.

In this way, we start with:

\[
\Phi(g_1^{-1}(t), (x, v_z)) = (z, v_z). \tag{5.19}
\]

The new equation in the group determined by the curve \( a'(t) \) constructed with \( g_1^{-1}(t) \),

\[
\dot{h}(t) h^{-1}(t) = \text{Ad}(g_1^{-1}(t))(a(t)) + \dot{g}_1^{-1}g_1(t) = a'(t), \tag{5.20}
\]

turns out to be in this case:

\[
\dot{h}(t) h^{-1}(t) = a'(t) = \frac{1}{x_1^2(t)}(k a_1 - a_2).
\]

If we consider a new variable \( \tau \) defined by

\[
\tau = \int_0^t \frac{d\zeta}{x_1^2(\zeta)}, \tag{5.21}
\]

the new equation in the group is given by:

\[
\frac{dh(\tau)}{d\tau} h^{-1}(\tau) = k a_1 - a_2, \tag{5.22}
\]

then the solution with \( h(0) = e \) is \( h(\tau) = \exp((k a_1 - a_2)\tau) \) and we arrive to the solution for the original Lie system \( g(t) = g_1(t)h(\tau)g_1^{-1}(0) \). Thus, the solution for the time evolution is:

\[
g(t) = \begin{pmatrix} x_1(t) & 0 \\ \dot{x}_1(t) & x_1^{-1}(t) \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}\tau) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\tau) \\ -\sqrt{k} \sin(\sqrt{k}\tau) & \cos(\sqrt{k}\tau) \end{pmatrix} \begin{pmatrix} x_1^{-1}(0) & 0 \\ -\dot{x}_1(0) & x_1(0) \end{pmatrix} = \begin{pmatrix} x_1(t) \cos(\sqrt{k}\tau) & \frac{x_1(t) \sin(\sqrt{k}\tau)}{\sqrt{k}} \\ -\sqrt{k} \sin(\sqrt{k}\tau) x_1(t) + \cos(\sqrt{k}\tau) \dot{x}_1(t) & \frac{x_1(t) \sin(\sqrt{k}\tau) \dot{x}_1(t)}{\sqrt{k}} \end{pmatrix} \begin{pmatrix} x_1^{-1}(0) & 0 \\ -\dot{x}_1(0) & x_1(0) \end{pmatrix}.
\]

Now, if \( \Phi \) denotes the action for the Milne–Pinney equation and we define:

\[
\begin{pmatrix} A \\ B \end{pmatrix} = \Phi \begin{pmatrix} x_1^{-1}(0) & 0 \\ -\dot{x}_1(0) & x_1(0) \end{pmatrix}, \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \tag{5.23}
\]

then the set of solutions of the Milne–Pinney equation is written in terms of \( A, B \) as:

\[
x(t) = \sqrt{\frac{(B^2 + \frac{k}{\sqrt{k}} + A^2 k + (-B^2 - \frac{k}{\sqrt{k}} + A^2) \cos(2\sqrt{k}\tau(t)) + 2AB \sqrt{k} \sin(2\sqrt{k}\tau(t)) x_1^2(t)}{2k}}
\]
As a final example, consider once again the Milne–Pinney equation but assume that a particular solution $x_1(t)$ of the time-dependent frequency harmonic oscillator with the same frequency as the Milne–Pinney equation is known. Recall that it has been shown that both equations are related with the same Lie system in the same group $G$. In this case, once again $g_1(t)$ given by (5.18) transforms $(1,0)$ into a particular solution $(x_1(t),\dot{x}_1(t))$ of the time-dependent harmonic oscillator as a first-order differential equation. Thus, the transformation induced in $G$ changes the initial differential equation in $G$ corresponding to the harmonic and the Pinney equation, which is the same one, into a new one in the stability group $G_{z_0}$. Now, the new equation in the group $G_{z_0}$ is given by:

$$\dot{h}(t)h^{-1}(t) = \text{Ad} (g_1^{-1}(t))(a(t)) + \dot{g}_1(t)g_1^{-1}(t) = -\frac{1}{x_1(t)}a_2$$

(5.24)

and making use of a reparametrization given by (5.21) we obtain that the new equation is:

$$\frac{dh(\tau)}{d\tau} h^{-1}(\tau) = -a_2,$$

(5.25)

and the solution with $h(0) = e$ is:

$$h(\tau) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix},$$

(5.26)

from where we obtain the solution of our Lie system:

$$g(t) = \begin{pmatrix} x_1(t) & 0 \\ \dot{x}_1(t) & x_1^{-1}(t) \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^{-1}(0) & 0 \\ -\dot{x}_1(0) & x_1(0) \end{pmatrix} = \begin{pmatrix} x_1(t) & x_1(0)x_1(t) \\ \dot{x}_1(t)\tau + x_1^{-1}(t) & 1/x_1(0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dot{x}_1(0) & x_1(0) \end{pmatrix} = \begin{pmatrix} x_1(t)/x_1(0) - x_1(t)\dot{x}_1(0)\tau & x_1(0)x_1(t) \\ \dot{x}_1(t)/x_1(0) - \dot{x}_1(0)/x_1(t) - \tau\dot{x}_1(t)\dot{x}_1(0) & x_1(0)(\dot{x}_1(t)\tau + x_1^{-1}(t)) \end{pmatrix}.$$ 

If we introduce the parameters $A$ and $B$ by

$$\begin{pmatrix} A \\ B \end{pmatrix} = \Phi \begin{pmatrix} x_1^{-1}(0) & 0 \\ -\dot{x}_1(0) & x_1(0) \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix},$$

(5.27)

where $\Phi$ denotes the action for the Milne–Pinney equation, we obtain:
\[ x(t) = \frac{x_1(t)}{A} \sqrt{A^4 + 2A^3B\tau(t) + (A^2B^2 + k)\tau(t)^2} \]

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