ON THE K-THEORY OF THE CYCLIC QUIVER VARIETY

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1. Introduction and notations.

In [N1] Nakajima has defined a new class of varieties, called quiver varieties, associated to any quiver. In the particular case of the Dynkin quiver of finite type $A$, these varieties are related to partial flag manifolds. For any quiver with no edge loops, he proved in [N2] that the corresponding simply laced Kac-Moody algebra acts on the top homology groups via a convolution product. The purpose of this paper is to compute the convolution product on the equivariant $K$-groups of the cyclic quiver variety, generalizing the previous works [GV], [V]. It is expected that equivariant $K$-groups should give an affinization of the quantized enveloping algebra of the corresponding Kac-Moody algebra. Similar algebras, called toroidal algebras, have already been studied in the cyclic case (e.g. [GKV], [VV]). Surprisingly the convolution operators we get do not satisfy exactly the relations of the quantized toroidal algebra. We obtain a twisted version, denoted by $U_{q,t}$, of the latter. Fix an integer $n \geq 3$ and two parameters $q,t \in \mathbb{C}^\times$. Then, $U_{q,t}$ is the complex unital associative algebra generated by $x^{\pm}_{i,k}$, $x^{\pm}_{i,k}$, $h^{\pm}_{i,\pm l}$, where $i \in \mathbb{Z}/n\mathbb{Z}$, $k \in \mathbb{Z}$, $l \in \mathbb{N}$. The relations are expressed in term of the formal series

$$x^+_i(z) = \sum_{k \in \mathbb{Z}} x^+_{i,k} \cdot z^{-k}, \quad x^-_i(z) = \sum_{k \in \mathbb{Z}} x^-_{i,k} \cdot z^{-k} \quad \text{and} \quad h^+_i(z) = \sum_{k \geq 0} h^+_{i,\pm k} \cdot z^{\mp k},$$

as follows

$$h^+_{i,0} h^-_{i,0} = 1, \quad [h^+_i(z), h^-_j(w)] = 0$$

$$(1 - q^{-1}t^{-1}) [x^+_i(z), x^-_j(w)] = \delta_{ij} \epsilon(z/w) (h^+_i(w) - h^-_i(z))$$

$$(z - qt w) \bigtriangledown h^+_i(z) x^+_j(w) = (qtz - w) \bigtriangledown x^+_j(w) h^+_i(z)$$

$$(t z - w) \bigtriangledown h^+_{i+1}(z) x^+_i(w) = (z - qw) \bigtriangledown x^+_i(w) h^+_{i+1}(z)$$

$$(z^\varepsilon - qt w^\varepsilon) x^+_i(z) x^+_j(w) = (qtz^\varepsilon - w^\varepsilon) x^+_j(w) x^+_i(z)$$

$$(t z^\varepsilon - w^\varepsilon) x^+_i(z) x^+_j(w) = (z^\varepsilon - qw^\varepsilon) x^+_j(w) x^+_i(z)$$

$$(t z^\varepsilon - w^\varepsilon) x^+_i(z) x^+_j(w) = (z^\varepsilon - qw^\varepsilon) x^+_j(w) x^+_i(z)$$

$$(q^{-1}t^\varepsilon + 1) x^+_i(z_1) x^+_j(z_2) x^+_i(z_3) x^+_j(w) - (q^\varepsilon t^\varepsilon + 1) x^+_i(z_1) x^+_j(z_2) x^+_i(z_3) x^+_j(w) x^+_i(z_4) +$$

\[ \text{[Other terms]} \]
Let us consider the map $B, i, j \in \mathcal{A}$ triple $(\epsilon, \tau) \in \{+1, -1\}$. Observe that we recover the usual Drinfeld relations after the specialization $q = t$. Our work relies very much on [N2],[N3]. In Section 2 we make several recollections. Theorem 1 is not stated explicitly in [N2] but Nakajima told us it was known to him. We mention it here since it extends and clarifies some result in [L]. The convolution product is computed in Sections 3 and 4. Another reason to give both Theorem 1 and Theorem 2 is the general philosophy mentioned in [GKV] and in Nakajima’s works that convolution algebras associated to moduli spaces of torsion free sheaves on surfaces should lead to affinization of affine algebras and their quantum analogues.

2. Quiver varieties.

2.1. Let us recall a few definitions from [N2]. Fix a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$. Let $\text{Rep}(\Gamma)$ be the set of isomorphism classes of finite dimensional representations of $\Gamma$ and let $L \in \text{Rep}(\Gamma)$ be the obvious 2-dimensional representation. Let $X(\Gamma) = \{S_k \mid k = 1, 2, ..., n\}$ be the set of the simple representations of $\Gamma$. Given finite dimensional vector spaces $V, W$ set

$$M_{W,V} = \text{Lin}(V, L \otimes V) \oplus \text{Lin}(W, \wedge^2 L \otimes V) \oplus \text{Lin}(V, W).$$

The group $GL(V)$ acts on $M_{W,V}$ as follows: for any $g \in GL(V)$,

$$g \cdot (B, i, j) = ((1 \otimes g) \circ B \circ g^{-1}, (1 \otimes g) \circ i, j \circ g^{-1}).$$

Let us consider the map

$$\mu_{W,V} : M_{W,V} \to \text{Lin}(V, \wedge^2 L \otimes V), \quad (B, i, j) \mapsto B \wedge B - i \circ j,$$

where $B \wedge B$ is the projection of $B \circ B \in \text{Lin}(V, L^{\otimes 2} \otimes V)$ to $\text{Lin}(V, \wedge^2 L \otimes V)$. A triple $(B, i, j) \in \mu_{W,V}^{-1}(0)$ is stable if and only if there is no nontrivial subspace $V' \subseteq \text{Ker} j$ such that $B(V') \subseteq L \otimes V'$. Let $\mu_{W,V}^{-1}(0)^s$ be the set of stable triples. The group $GL(V)$ acts freely on $\mu_{W,V}^{-1}(0)^s$ (see [N2, Lemma 3.10]). Suppose now that $W, V$ are $\Gamma$-modules. Then $M_{W,V}$ is endowed with an obvious $\Gamma$-action. Let $\mu_{W,V}^{-1}(0)^s, \Gamma \subseteq \mu_{W,V}^{-1}(0)^s$ be the fixpoint variety. Let $GL_\Gamma(V)$ and $V_k = \text{Lin}_\Gamma(S_k, V)$ be the $\Gamma$-invariants. Let

$$\Lambda_{W,V} = \mu_{W,V}^{-1}(0)^s, \Gamma / GL_\Gamma(V) \quad \text{and} \quad \Lambda_{W,V} = \mu_{W,V}^{-1}(0)^s / GL_\Gamma(V)$$

be the Nakajima varieties. By construction, $\Lambda_{W,V}$ is a smooth variety (or the empty set). Put $\Lambda W = \bigcup V \Lambda_{W,V}$ and $N W = \bigcup V N_{W,V}$. Consider the morphism $\pi : \Lambda_{W,V} \to N_{W,V}$ induced by the chain of maps

$$\mu_{W,V}^{-1}(0)^s, \Gamma \subseteq \mu_{W,V}^{-1}(0)^s \to \mu_{W,V}^{-1}(0)^s / GL_\Gamma(V).$$
2.2. Let us now recall a few results from [N2], [L]. A triple \((B, i, j) \in \mu_{W,V}^1(0)\) is **costable** if and only if there is no subspace \(V' \subseteq V\) such that

\[
B(V') \subseteq L \otimes V', \quad \text{Im } i \subseteq \wedge^2 L \otimes V' \quad \text{and} \quad V' \neq V.
\]

Let \(\mu_{W,V}^{-1}(0)^c\) be the set of costable triples and put

\[
\mu_{W,V}^{-1}(0)^{s,c} = \mu_{W,V}^{-1}(0)^c \cap \mu_{W,V}^{-1}(0)^s.
\]

For any triple \(\xi = (B, i, j) \in \Lambda_{W,V}\) the **dual**, \(\xi^*\), is the triple \((B^*, j^*, i^*) \in \Lambda_{W^*,V^*}\), where

\[
B^* = \wedge^2 L \otimes B^t, \quad j^* = \wedge^2 L \otimes j^t, \quad i^* = \wedge^2 L \otimes i^t,
\]

and \(t\) stands for the transposed map. Since \(W^* \cong W\) and \(V^* \cong V\) as \(\Gamma\)-modules, we will view \(\xi^*\) as an element of \(\Lambda_{W,V}\). Then, \(\xi \in \mu_{W,V}^{-1}(0)^s\) if and only if \(\xi^* \in \mu_{W,V}^{-1}(0)^c\).

**Lemma 1.** Fix \(\xi \in \mu_{W,V}^{-1}(0)^\Gamma\).

(i) If \(\xi\) is stable or costable then the stabilizer of \(\xi\) in \(GL_\Gamma(V)\) is trivial.

(ii) Moreover, the stabilizer of \(\xi\) in \(GL_\Gamma(V)\) is trivial and the orbit \(GL_\Gamma(V) \cdot \xi\) is closed if and only if \(\xi\) is stable and costable.

(iii) The orbit \(GL_\Gamma(V) \cdot \xi\) is closed if and only if there are subspaces \(V_1, V_2 \subseteq V\) and triples \(\xi_1 \in \mu_{W,V_1}^{-1}(0)^{s,c,\Gamma}, \xi_2 \in \mu_{(0),V_2}^{-1}(0)^\Gamma\), such that

\[
V = V_1 \oplus V_2, \quad \xi = \xi_1 \oplus \xi_2 \quad \text{and} \quad GL_\Gamma(V_2) : \xi_2 \text{ is closed}.
\]

Moreover the splitting \(\xi = \xi_1 \oplus \xi_2\) is unique.

**Proof:** Part (i) is proved in [N2, Lemma 3.10]. Part (ii) \(\Leftarrow\) is proved in [N2, Proposition 3.24]. Part (ii) \(\Rightarrow\) is stated without proof in [L, Section 2]. Let us prove it. Given a triple \(\xi = (B, i, j) \in \mu_{W,V}^{-1}(0)^\Gamma\), let \(\xi^0 = (B^0, i^0, j^0)\) be a representative of the unique closed orbit in \(\overline{GL_\Gamma(V) \cdot \xi}\). Fix a one-parameter subgroup \(\lambda : \mathbb{C}^\times \to GL_\Gamma(V)\) such that

\[
(2.1) \quad \xi^0 = \lim_{t \to 0} (\lambda(t) \cdot \xi).
\]

For all \(n \in \mathbb{Z}\) set

\[
V^n = \{v \in V \mid (\lambda(t) - 1) v = t^n v, \quad \forall t\}.
\]

Since the limit (2.1) is well-defined we have

\[
\text{Im } i \subseteq \wedge^2 L \otimes \bigoplus_{n \geq 0} V^n, \quad \bigoplus_{n \geq 1} V^n \subseteq \text{Ker } j, \quad B(V^n) \subseteq L \otimes \bigoplus_{m \geq n} V^m \quad \forall n.
\]

Thus,

\[
\xi \text{ is costable } \Rightarrow \quad V = \bigoplus_{n \geq 0} V^n
\]

\[
\xi \text{ is stable } \Rightarrow \quad \bigoplus_{n \geq 1} V^n = \{0\}.
\]

(2.2)
Hence, if $\xi \in \mu_{W,V}^{-1}(0)^{\ast,c,\Gamma}$ then $\xi = \xi^0$, i.e. the orbit $GL_\Gamma(V) \cdot \xi$ is closed. Part $(iii)$ $\Rightarrow$ is proved in [L, Lemma 2.30]. The decomposition $\xi = \xi_2 \oplus \xi_2$ is unique since $V_1$ (resp. $V_2$) is the smallest (resp. the biggest) subspace of $V$ such that

$$B(V_1) \subseteq L \otimes V_1 \text{ and } \text{Im}(i) \subseteq V_1 \text{ (resp. } B(V_2) \subseteq L \otimes V_2 \text{ and } V_2 \subseteq \text{Ker } j).$$

Let us prove Part $(iii) \Leftarrow$. Hence, fix a triple $\xi \in \mu_{W,V}^{-1}(0)$ admitting a splitting $\xi = \xi_2 \oplus \xi_2$ as in $(iii)$. Let $\xi^0$ be a representative of the unique closed orbit in $GL_\Gamma(V) \cdot \xi$. Fix a splitting $V = V_1^0 \oplus V_2^0$, $\xi^0 = \xi_1^0 \oplus \xi_2^0$, as in Part $(iii) \Rightarrow$. Fix a one-parameter subgroup $\lambda : \mathbb{C}^\times \to GL_\Gamma(V)$ such that $\lim_{t \to 0}(\lambda(t) \cdot \xi) = \xi^0$. Put

$$\check{V}_a^0 = \lim_{t \to 0} \lambda(t)(V_a) \text{ and } \check{\xi}_a^0 = \lim_{t \to 0}(\lambda(t) \cdot \xi_a) \quad \forall a = 1, 2.$$  

1. First, suppose that $\lambda(t)(V_1) = V_1$ for all $t$, and hence that $V_1^0 = V_1$. The characterization of $\check{V}_a^0$ and $\check{\xi}_a^0$ above implies that $V_1^0 \subseteq \check{V}_1^0$ and $V_2^0 \subseteq \check{V}_2^0$. Now, $\xi_1^0 \in GL_\Gamma(V_1) \cdot \xi_1 = GL_\Gamma(V_1) \cdot \xi_1$, and thus $V_1^0 = \check{V}_1^0 = V_1$. Then, a dimension count gives $V_2^0 \subseteq V_2$, and, in particular, $V = V_1^0 \oplus V_2^0$. Consider the unipotent group

$$U_\lambda = \{ u \in GL_\Gamma(V) \mid \lim_{t \to 0}(\lambda(t)u\lambda(t)^{-1}) = 1 \}.$$ 

Up to conjugating $\lambda$ by an element $u \in U_\lambda$ such that $u(V_a^0) = V_a$ if $a = 1, 2$, we can suppose that $\lambda = \lambda_1 \times \lambda_2$ where $\lambda_0$ is a one-parameter subgroup of $GL_\Gamma(V_a)$. Then, since $\xi_1, \xi_2$ have closed orbits, we get

$$\xi^0 = \lim_{t \to 0}(\lambda_1(t) \cdot \xi_1 \oplus \lambda_2(t) \cdot \xi_2) \in GL_\Gamma(V_1) \cdot \xi_1 \oplus GL_\Gamma(V_2) \cdot \xi_2 \subseteq GL_\Gamma(V) \cdot \xi.$$ 

2. Generally, fix an element $u \in U_\lambda$ such that $u(V_1^0) = V_1$ and consider the one-parameter subgroup $\lambda'(t) = u\lambda(t)u^{-1}$. We have

$$\lambda'(t)(V_1) = V_1 \quad \forall t \quad \text{and} \quad \lim_{t \to 0}(\lambda'(t) \cdot \xi) = u\xi^0.$$ 

Thus, Part 1 implies that $u\xi^0 \in GL_\Gamma(V) \cdot \xi$. Hence the orbit of $\xi$ is closed. \hfill \Box

2.3. Fix $W, V \in \text{Rep}(\Gamma)$. Fix a torus $T \subset GL_\Gamma(L)$ and put $G_W = GL_\Gamma(W) \times T$. The obvious action of $T$ on $L$ induces an action of $G_W$ on $M_{W,V}$ as follows

$$(g, z) \cdot (B, i, j) = ((z \otimes 1) \circ B, (z \otimes 1) \circ i \circ g^{-1}, g \circ j), \quad \forall (g, z) \in G_W.$$ 

This action descends to an action of $G_W$ on the variety $\Lambda_{W,V}$. On the other hand the action of $\Gamma$ on $L = \mathbb{C}^2$ extends to $\mathbb{P}^2$ in such a way that the line at infinity, $l_\infty$, is preserved. Let $X_{W,V}$ be the set of isomorphism classes of pairs $(E, \phi)$ where $E$ is a $\Gamma$-equivariant torsion free sheaf on $\mathbb{P}^2$ such that $H^1(\mathbb{P}^2, E(-l_\infty)) \simeq V$ as a $\Gamma$-module and $\phi$ is a $\Gamma$-invariant isomorphism

$$\phi : E|_{l_\infty} \xrightarrow{\sim} O_{l_\infty} \otimes W.$$ 

Put $X_W = \bigcup_V X_{W,V}$. Let $(e_x, e_y)$ be the canonical basis of $L$. For any triple $\xi = (B, i, j) \in \mu_{W,V}^{-1}(0)^\Gamma$ we consider the following $\Gamma$-equivariant complex of sheaves on $\mathbb{P}^2$:

$$(2.3)_\xi \quad O_{\mathbb{P}^2}(-l_\infty) \otimes V \xrightarrow{\alpha_\xi} O_{\mathbb{P}^2} \otimes ((L \otimes V) \oplus W) \xrightarrow{\beta_\xi} O_{\mathbb{P}^2}(l_\infty) \otimes \wedge^2 L \otimes V,$$
where 

\[ a_\xi = \left( zB - x e_x - y e_y \right), \quad b_\xi = \left( zB - x e_x - y e_y, zi \right), \]

and \( x, y, z \) are homogeneous coordinates on \( \mathbb{P}^2 \) such that \( l_\infty = \{ z = 0 \} \). Let \( H^0_\xi, H^1_\xi \), and \( H^2_\xi \), be the cohomology sheaves of the complex \((2.3)_\xi\). Then (see [N3, Lemma 2.6])

(a) \( H^0_\xi = \{ 0 \} \),
(b) \( \xi \in \mu_{W,V}^{-1}(0)^{c,\Gamma} \) if and only if the map \( b_\xi \) is surjective,
(c) \( \xi \in \mu_{W,V}^{-1}(0)^{s,\Gamma} \) if and only if the map \( a_\xi \) is fiberwise injective.

Considering the complex \((2.3)_\xi\) simultaneously for all \( \xi \) we get a complex of sheaves on \( \mathbb{P}^2 \times \mu_{W,V}^{-1}(0)^{s,\Gamma} \). Since the group \( GL_\Gamma(V) \) acts freely on \( \mu_{W,V}^{-1}(0)^{s,\Gamma} \), this complex descends to a complex on \( \mathbb{P}^2 \times \Lambda_{W,V} \), denoted by \( C \). The cohomology sheaves \( H^0(C) \) and \( H^2(C) \) vanish by (a), (b), and the sheaf \( \mathcal{U} = H^1(C) \) is \( \Gamma \)-equivariant. Since \( H^0(C) \) vanishes, there is a \( \Gamma \)-invariant isomorphism

\[ \Phi : \mathcal{U}_{|_{\Lambda_{W,V}}} \cong H^1(C_{|_{\Lambda_{W,V}}}) \cong O_{l_\infty \times \Lambda_{W,V}} \otimes W. \]

A pair \((\mathcal{U}, \Phi)\) is said to be universal if for any family of \( \Gamma \)-equivariant torsion free sheaves \( \mathcal{E} \) on \( \mathbb{P}^2 \) parametrized by a variety \( S \) such that

(d) for all \( s \in S \) we have a \( \Gamma \)-invariant isomorphism \( H^1(\mathbb{P}^2, \mathcal{E}_{|\mathbb{P}^2 \times \{s\}}(-l_\infty)) \cong V \),
(e) we have a \( \Gamma \)-invariant trivialisation \( \phi : \mathcal{E}_{|l_\infty \times S} \cong O_{l_\infty \times S} \otimes W, \)

there is a unique map \( i_S : S \to \Lambda_{W,V} \) with \( (i^*_S \mathcal{U}, i^*_S \Phi) \cong (\mathcal{E}, \phi) \). Let \( X^0_{W,V} \) and \( X^0_V \) be the subsets of isomorphism classes of pairs \((\mathcal{E}, \phi)\) such that \( \mathcal{E} \) is locally free. For any \( z \in T \) let \( \alpha(z) \) be the corresponding automorphism of \( \mathbb{P}^2 \). The group \( G_W \) acts on \( X_{W,V} \) in such a way that

\[ (g, z) \cdot (\mathcal{E}, \phi) = (\alpha(z)^* \mathcal{E}, g \circ \alpha(z)^* \phi), \quad \forall (g, z) \in G_W. \]

This action preserves the subset \( X^0_{W,V} \subseteq X_{W,V} \). For any variety \( X \) let \( S^n X \) be the \( n \)-th symmetric product and put \( S X = \bigcup_n S^n X \).

**Theorem 1.** There are bijections of \( G_W \)-sets

\[ \Lambda_{W,V} \cong X_{W,V} \quad \text{and} \quad N_W \cong X^0_{W,V} \times (SC^2)^\Gamma \]

such that the map \( \pi : \Lambda_W \to N_W \) is identified with the map

\[ (\mathcal{E}, \phi) \mapsto (\mathcal{E}^{**}, \phi, \sup(\mathcal{E}^{**} / \mathcal{E})). \]

Moreover \((\mathcal{U}, \Phi)\) is a universal pair.

**Proof:** 1. Given a triple \( \xi = (B, i, j) \in \mu_{W,V}^{-1}(0)^{c,\Gamma} \), let \( \xi^0 \) be a representative of the unique closed orbit in \( GL_\Gamma(V) \cdot \xi \). Fix a splitting \( \xi^0 = \xi^0_1 \oplus \xi^0_2 \), \( V = V_1 \oplus V_2 \), as in Lemma 1(iii), and fix a one-parameter subgroup \( \lambda : \mathbb{C}^\times \to GL_\Gamma(V) \) such that \( \xi^0 = \lim_{t \to 0} (\lambda(t) \cdot \xi) \). From (2.2) we have

\[ V = \bigoplus_{n \geq 0} V^n, \quad \text{where} \quad V^n = \{ v \in V \mid \lambda(t) \cdot v = t^n v, \quad \forall t \}. \]
Then $\xi^0$ is the triple $(B^0, i^0, j)$, where $B^0 = \lim_{t \to 0} (\lambda(t) \cdot B)$ and $i^0$ is the component of the map $i$ in $\text{Lin}_\Gamma(W, \wedge^2 L \otimes V^0)$. Moreover,
\[
W_1 = W, \quad V_1 = V^0, \quad \xi_1^0 = (B^0_{/V_1}, i^0, j),
\]
\[
W_2 = \{0\}, \quad V_2 = \bigoplus_{n \geq 1} V^n, \quad \xi_2^0 = (B_{/V_2}, 0, 0).
\]
Thus, we have a short exact sequence of complexes
\[
0 \to (2.3)_{\xi_2^0} \to (2.3)_{\xi} \to (2.3)_{\xi_1^0} \to 0.
\]
Moreover, $\mathcal{H}^2_{\xi} = \{0\}$ since $\xi$ is costable and $\mathcal{H}^1_{\xi_2^0} = \{0\}$ since $(2.3)_{\xi_2^0}$ is a Koszul complex. Thus,
\[
(2.4) \quad 0 \to \mathcal{H}^1_{\xi} \to \mathcal{H}^1_{\xi_1^0} \to \mathcal{H}^2_{\xi_2^0} \to 0
\]
is an exact sequence. Since $\xi_1^0$ is stable and costable, Claims (b) and (c) imply that $\mathcal{H}^1_{\xi_1^0}$ is a locally free sheaf. Since $\mathcal{H}^2_{\xi_2^0}$ has a finite length, the sheaf $\mathcal{H}^1_{\xi}$ is torsion free and its double dual is $\mathcal{H}^1_{\xi_1^0}$. Since $\mathcal{H}^0_{\xi}$ vanishes we have a $\Gamma$-isomorphism
\[
\phi_\xi : \mathcal{H}^1_{\xi|l_\infty} = \mathcal{H}^1((2.3)_{\xi|l_\infty}) \sim \mathcal{O}_{l_\infty} \otimes W.
\]
Since $\mathcal{H}^1_{\xi}$ is torsion free and is trivial on $l_\infty$, we have $H^i(\mathbb{P}^2, \mathcal{H}^1_{\xi}(-l_\infty)) = \{0\}$ if $i = 0, 2$ (see [N3, Lemma 2.3]). Tensoring $(2.3)_{\xi}$ with $\mathcal{O}_{\mathbb{P}^2}(-l_\infty)$ we obtain a complex of acyclic sheaves quasi-isomorphic to $\mathcal{H}^1_{\xi}(-l_\infty)$. It gives $H^1(\mathbb{P}^2, \mathcal{H}^1_{\xi}(-l_\infty)) \simeq V$. Hence we have a map
\[
\mu^{-1}_{W,V}(0)^x, \xi \mapsto (\mathcal{H}^1_{\xi}, \phi_\xi),
\]
which descends to a map $\Psi : \Lambda_{W,V} \to X_{W,V}$.

2. It is proved in [N3, Section 2.1] that $\Psi$ is surjective. The case of families is identical. Let us sketch it for the convenience of the reader. Fix an algebraic variety $S$, put $\mathbb{P} = \mathbb{P}^2 \times S$, and let $\rho : \mathbb{P} \to S$ be the projection. Fix a family $\mathcal{E}$ of $\Gamma$-equivariant torsion free sheaves on $\mathbb{P}^2$ parametrized by $S$ endowed with $\Gamma$-isomorphisms
\[
\psi : R^1\rho_* (\mathcal{E}(-l_\infty)) \sim \mathcal{O}_S \otimes V \quad \text{and} \quad \phi : \mathcal{E}|_{l_\infty \times S} \sim \mathcal{O}_{l_\infty \times S} \otimes W.
\]
There is a (relative) Beilinson spectral sequence converging to $\mathcal{E}(-l_\infty)$ with $E_1$-term
\[
E_1^{i,j} = \mathcal{O}_{\mathbb{P}^2}(-il_\infty) \otimes R^j\rho_* (\mathcal{E} \otimes \Omega^i_{\mathbb{P}/S}((i-1)l_\infty)),
\]
where $\Omega^i_{\mathbb{P}/S}$ is the relative bundle of $i$-forms, and $i, j = 0, 1, 2$. Then, [N3, Lemma 2.3] states that the spectral sequence degenerates at the $E_2$ term, giving a $\Gamma$-equivariant complex
\[
\mathcal{O}_{\mathbb{P}^2}(-l_\infty) \boxtimes R^1\rho_* (\mathcal{E}(-2l_\infty)) \Rightarrow \mathcal{O}_{\mathbb{P}^2} \boxtimes R^1\rho_* (\mathcal{E} \otimes \Omega^1_{\mathbb{P}/S} \otimes l_\infty) \Rightarrow \mathcal{O}_{\mathbb{P}^2} \boxtimes R^1\rho_* (\mathcal{E}(-l_\infty)).
\]
such that \( \ker a = \text{coker} b = \{0\} \) and \( \ker b / \text{im} a = \mathcal{E} \). Now, \( \psi \) and \( \phi \) give \( \Gamma \)-isomorphisms

\[
R^1 \rho_* (\mathcal{E}(-l_\infty)) \simeq R^1 \rho_* (\mathcal{E}(-2l_\infty)) \simeq \mathcal{O}_S \otimes V
\]

\[
R^1 \rho_* (\mathcal{E} \otimes \Omega^1_{\mathcal{E}/S}) \simeq \mathcal{O}_S \otimes (L \otimes V \otimes W),
\]

and the maps \( a, b \), can be expressed as \( a_\xi, b_\xi \), in (2.3). Let \( (\mathcal{U}, \Phi') \) be the pull-back of \( (\mathcal{U}, \Phi) \) to \( \mathbb{P}^2 \times \mu_{W,V}^{-1}(0)^s \Gamma \). We have constructed a map \( j_S : S \to \mu_{W,V}^{-1}(0)^s \Gamma \) such that \( (\mathcal{E}, \phi) \simeq (j_S \mathcal{U}', j_S^* \Phi') \). If the sheaf \( R^1 \rho_* (\mathcal{E}(-l_\infty)) \) is no longer trivial but satisfies (d), fix an open covering \( (S_k)_k \) of \( S \) such that \( R^1 \rho_* (\mathcal{E}(-l_\infty)) \mid_{S_k} \) is trivial for all \( k \). The corresponding maps \( S_k \to \mu_{W,V}^{-1}(0)^s \Gamma \to \Lambda_{W,V} \) glue together. Let \( i_S \) be the resulting map.

3. Let us prove that \( \Psi \) is injective. Fix \( \xi^1, \xi^2 \in \mu_{W,V}^{-1}(0)^s \Gamma \) such that \( \Psi(\xi^1) \simeq \Psi(\xi^2) \). By [OSS, Lemma II.4.1.3] the isomorphism lifts to an isomorphism of the complexes (2.3) \( (2.3) \xi_1 \simeq (2.3) \xi_2 \). This isomorphism is a triple \( (\alpha, \beta, \gamma) \in GL_T(V) \times GL_T(L \otimes V \otimes W) \times GL_T(V) \). Considering the fibers at the points \([1 : 1 : 0], [0 : 1 : 0] \), and \([1 : 0 : 0] \), we get \( \alpha = \gamma \) and \( \beta(u \otimes v) = u \otimes \alpha(v) \) for all \( u \in L, v \in V \). Then, since the isomorphism is compatible with the trivialisations \( \phi_{\xi^1} \) and \( \phi_{\xi^2} \), the map \( \beta \) fixes \( W \). Hence, \( \xi^2 = \alpha \cdot \xi^1 \). We are done. In particular, the map \( i_S \) associated to a family parametrized by \( S \) is unique.

4. Now, recall that points in \( N_{W,V} \) are in bijection with closed \( GL_T(V) \)-orbits in \( \mu_{W,V}^{-1}(0)^\Gamma \). By Claims (b), (c), we know that \( \Psi \) maps \( \mu_{W,V}^{-1}(0)^s \Gamma / GL_T(V) \) to \( X^0_{W,V} \). Thus the second claim in the theorem follows from Lemma 1(iii). By construction \( \pi \) maps the \( GL_T(V) \)-orbit of a triple \( \xi \in \mu_{W,V}^{-1}(0)^s \Gamma \) to the unique closed orbit in \( GL_T(V) \cdot \xi \). Thus the third claim follows from the exact sequence (2.4).

\( \square \)

REMARKS. 1. Observe that (d) and (e) guarantee that the family \( \mathcal{E} \) is flat over \( S \), by [H], since all geometric fibers of \( \mathcal{E} \) over \( S \) have the same Hilbert polynomial.

2. Observe also that Part 3 of the proof implies that elements of \( X_{W,V} \) do not have automorphisms.

3. Finally, observe that the map \( \pi \) above is the same as the one used in [B].

COROLLARY. The fibers of the map \( \pi : \Lambda_{W,V} \to N_{W,V} \) are isomorphic to the \( \Gamma \)-fixpoint sets of a product of punctual Quot-schemes. In particular, \( \pi \) is a proper map.

\( \square \)

3. The convolution algebra.

3.1. For any linear algebraic group \( G \) and any quasi-projective \( G \)-variety \( X \) let \( K_G(X) \) and \( R(G) \) be respectively the complexified Grothendieck group of \( G \)-equivariant coherent sheaves on \( X \) and the complexified Grothendieck group of finite dimensional \( G \)-modules. If \( \mathcal{E} \) is a \( G \)-equivariant coherent sheaf on \( X \), let \( \mathcal{E} \) denote also its class in \( K_G(X) \). Similarly, for any \( V \in \text{Rep} (G) \), let \( V \) denote also the class of \( V \) in \( R(G) \). Recall that if \( X' \subseteq X \) is a closed subvariety, then \( K_G(X') \) is identified with the complexified Grothendieck group of the Abelian category of
coherent sheaves on $X$ supported on $X'$. Then, let $[X'] \in K_G(X)$ be the class of the structural sheaf $\mathcal{O}_{X'}$ of $X'$. Suppose now that $X$ is smooth and that $\pi : X \to Y$ is a proper $G$-equivariant map, with $Y$ a quasi-projective (possibly singular) $G$-variety. Let $Z = X \times_Y X$ be the fiber product, endowed with the diagonal action of $G$. If $1 \leq i, j \leq 3$, let $p_{ij} : X^3 \to X^2$ be the projection along the factor not named. Obviously, $Z = p_{13}(p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z))$. Thus, we have the convolution map
\[
\ast : K_G(Z) \times K_G(Z) \to K_G(Z), \quad (x, y) \mapsto Rp_{13*}((p_{12}^*x) \otimes L (p_{23}^*y)).
\]
It is known that $(K_G(Z), \ast)$ is an associative algebra with unit $[\Delta]$, where $\Delta \subseteq X \times X$ is the diagonal, see [CG]. Similarly we consider the map
\[
\ast : K_G(Z) \times K_G(X) \to K_G(X), \quad (x, y) \mapsto Rp_{1*}(x \otimes L (p_2^*y)),
\]
where $p_1$ and $p_2$ are the projections of $X \times X$ onto its factors. Then, $K_G(X)$ is a $K_G(Z)$-module. Now, suppose that $G$ is reductive and that $z \in G$ is a central element such that the fixpoint subvariety $X^z$ is compact (and smooth). Let $\iota : X^z \hookrightarrow X$ and $q_1 : X \times X^z \to X$ be the obvious maps. Let $K_G(X)_z$ and $R(G)_z$ be the localized rings with respect to the maximal ideal associated to $z$. The direct image morphism $R_{\iota*} : K_G(X^z)_z \to K_G(X)_z$ is invertible by the Localization Theorem. Hence, since $q_1$ is proper there is a well-defined convolution product
\[
\ast : K_G(X)_z \times K_G(X)_z \to K_G(X)_z, \quad (x, y) \mapsto Rq_1* (x \otimes L (R_{\iota*})^{-1}(y)),
\]
and the Kunneth theorem [CG, Theorem 5.6.1] holds. In particular, if $[\Delta] \in K_G(X) \otimes_{R(G)} K_G(X)$ then $K_G(X)_z$ is a projective $R(G)_z$-module and
\[
K_G(X \times X)_z \simeq K_G(X)_z \otimes_{R(G)_z} K_G(X)_z.
\]
For more details on the convolution product see [CG]. We introduce two more notations. For any equivariant vector bundle $\mathcal{E}$ and any $z \in \mathcal{C}$ set $\Lambda_z \mathcal{E} = \bigoplus_i (-z)^i \Lambda^i \mathcal{E}$, where $\Lambda^i$ stands for the usual wedge power, and let $D \mathcal{E}$ be the determinant. If $z$ is a formal variable it is well-known that $\Lambda_z, D$, induce maps $D : K_G(X) \to K_G(X)$, $\Lambda_z : K_G(X)[[z]] \to K_G(X)[[z]]$, such that for all $\mathcal{E}, \mathcal{F} \in K_G(X)$ we have
\[
D(\mathcal{E} + \mathcal{F}) = D(\mathcal{E})D(\mathcal{F}) \quad \text{and} \quad \Lambda_z (\mathcal{E} + \mathcal{F}) = \Lambda_z (\mathcal{E})\Lambda_z (\mathcal{F}).
\]
Moreover, if $\mathcal{E}, \mathcal{F} \in K_G(X)$ have the same rank then $\Lambda_z \mathcal{E}(\Lambda_z \mathcal{F})^{-1}$ has an expansion in $K_G(X)[[z]]$ and $K_G(X)[[z^{-1}]]$. For simplicity we put $\Lambda \mathcal{E} = \Lambda_1 \mathcal{E}$. If $V, W$ are $G$-modules such that $\Lambda W$ is non zero, let $\Lambda (V - W)$ be the class of $\Lambda V(\Lambda W)^{-1}$ in the fraction field of $R(G)$.

3.2. We use the convolution product in the following situation. Consider the fiber product $Z_W = \Lambda_W \times \nabla_W \Lambda_W$. Given $V^a \in \text{Rep}(\Gamma)$ and $\xi^a = (B^a, i^a, j^a) \in \mu_{W, V^a}(0)^3$, $a = 1, 2$, put $\xi = (\xi^1, \xi^2)$ and form the complex
\[
\text{Lin}_\Gamma(V^1, L \otimes V^2) \oplus (3.1)_\xi \text{Lin}_\Gamma(V^1, V^2) \xrightarrow{a_\xi} \text{Lin}_\Gamma(W, \Lambda^2 L \otimes V^2) \xrightarrow{b_\xi} \text{Lin}_\Gamma(V^1, \Lambda^2 L \otimes V^2) \oplus \text{Lin}_\Gamma(V^1, W)
Lemma 2. If \( z \) is general, then \( \Lambda^*_W \) is contained in \( \pi^{-1}(0) \). In particular, it is compact.

Proof: It suffices to prove that \( N^*_W(V) = \{ 0 \} \). Given a representative \( \xi \) of a closed orbit fixed by \( z \), we consider the splitting \( \xi = \xi^1 + \xi^2 \) introduced in Lemma 1(iii). Then, \( \xi^2 \) is identified with a point in \( (SC^2)^\Gamma \). Since \( q_0 \) and \( t_0 \) are not roots of unity, necessarily \( \xi^2 = 0 \). Thus, we can suppose that \( \xi = (B, i, j) \) is stable and costable. Suppose \( V \neq \{ 0 \} \). Fix \( g \in GL_1(V) \) such that \( g \cdot \xi = z \cdot \xi \). Put \( B = B_x \otimes e_x + B_y \otimes e_y \).

By costability there is a non-zero element \( v \in V \) of the form \( mi(w) \) where \( w \in W \) and \( m \) is a word in \( B_x, B_y \), such that \( B(v) = 0 \). Since \( q_0B_x = gB_xg^{-1}, t_0B_y = gB_yg^{-1}, q_0t_0i = g i \), and since \( z \) is general, we have \( g(v) = \lambda v \) with \( \lambda \neq 1 \). Now we have also \( jg = j \) and, thus, \( j(v) = 0 \). This is in contradiction with the stability of \( \xi \).

Thus, if \( z \in T \) is general by Section 3.1 we have an isomorphism

\[
K_{G_W}(\Lambda^*_W \times \Lambda^*_W) \simeq \text{End}_{R(G_W)} K_{G_W}(\Lambda^*_W).
\]

3.3. For any \( k \) let \( C^+_k \subseteq \Lambda^*_W \times \Lambda^*_W \) be the variety of the pairs \( (\xi^1, \xi^2) \in \Lambda^*_W \times \Lambda^*_W \), where \( \xi^a = (B^a, i^a, j^a), a = 1, 2 \), is such that \( V^2 \) admits a \( B^2 \)-stable \( \Gamma \)-submodule isomorphic to \( V^1 \), containing \( \text{Im}(i^2) \), with \( V^2/V^1 \simeq S_k \), and \( \xi^1 \) is isomorphic to the restriction of \( \xi^2 \) to \( V^1 \). Let \( C^-_k \) be obtained by exchanging the components of \( C^+_k \). The varieties \( C^+_k \) are called Hecke correspondences. They are closed subvarieties of \( Z_W \) (see [N2]). Let \( V = \mu^{-1}_{W,1}(0)^a, \Gamma \times GL_1(V) \) be the universal bundle on \( \Lambda^*_W \) and let \( W = \mathcal{O}_{\Lambda^*_W} \otimes W \) be the trivial sheaf. The sheaves \( V, W \), are obviously \( G_W \times \Gamma \)-equivariant. Let \( V^a \) be the pull-back of \( V \) by the \( a \)-th projection \( p_a : \Lambda^*_W \times \Lambda^*_W \to \Lambda^*_W \), and let \( W \) denote also the trivial sheaf \( \mathcal{O}_{\Lambda^*_W \times \Lambda^*_W} \otimes W \). The restriction of \( V^1 \) to \( C^+_k \) is a subbundle of \( V^2 \) (resp. the restriction of \( V^2 \) to \( C^-_k \) is a subbundle of \( V^1 \)). Let \( L^\pm_k \) be the quotient sheaf: it is the extension by zero to \( Z_W \) of an invertible sheaf on \( C^\pm_k \). Put

\[
\theta = L^* - \wedge^2 L^* - 1 \in R(G_W \times \Gamma).
\]
Fix $q,t \in R(T)$ such that $L = tS^{-1} + qS$. For any $\Gamma$-module $V$ or any $\Gamma$-equivariant sheaf $\mathcal{E}$ put

$$V_k = \text{Lin}_\Gamma(S_k, V), \quad v_k = \text{dim} V_k, \quad \mathcal{E}_k = \text{Lin}_\Gamma(S_k, \mathcal{E}).$$

If $s \in \mathbb{Z}$ and $k \in \mathbb{Z}/n\mathbb{Z}$, let us define the following classes in $K_{G_\mathcal{W}}(Z_W)$:

$$\Omega_{k,s}^+ = (\mathcal{L}_k^+) \cdot D(\theta V^2 + (\wedge^2 L^*) W)_k^*$$

$$\Omega_{k,s}^- = (\mathcal{L}_k^-)^{s+h_k} \gamma_k,$$

where $h_k = \text{dim}((\theta^* V^1 + W)_k$ and $\gamma_k = q^{v_k} t^{-v_k} t^{-v_k-1}$. Let $\Theta_k^\pm (z)$ be the expansion of

$$\Theta_k(z) = (-1)^{h_k} \gamma_k \Lambda(z) ((\wedge^2 L^* - 1)(\theta V^* + W^*))_k$$

in $K_{G_\mathcal{W}}(Z_W)[[z^{-1}]]$ (an element in $K_{G_\mathcal{W}}(A_W)$ is identified with its direct image in $K_{G_\mathcal{W}}(Z_W)$ via the diagonal embedding $A_W \hookrightarrow Z_W$). More precisely, let $\Theta_{k,s}^\pm \in K_{G_\mathcal{W}}(Z_W)$ be such that

$$\Theta_k^\pm (z) = \sum_{s \in \mathbb{N}} \Theta_{k,s}^\pm z^{\mp s}.$$

**3.4.** We now consider the cyclic quiver only. Thus, put $\gamma = \begin{pmatrix} e^{2i\pi/n} & 0 \\ 0 & e^{-2i\pi/n} \end{pmatrix}$ and $\Gamma = \{ \gamma^k \mid k \in \mathbb{Z}/n\mathbb{Z} \}$. Fix a generator $S$ of $X(T)$ and set $S_k = S^\otimes k$ for all $k \in \mathbb{Z}$. We fix $T = \mathbb{C}^\times \times \mathbb{C}^\times$. Let us consider the sum

$$\varepsilon_k = \sum_V (-1)^{v_S} [\Delta_{W,V}],$$

where $[\Delta_{W,V}] \in K_{G_\mathcal{W}}(Z_W)$ is the class of the diagonal. The following theorem is the main result of the paper.

**Theorem 2.** If $z = (q_0,t_0)$ is general then the map

$$x_{k,s}^+ \mapsto \Omega_{k,s}^+ \ast \varepsilon_k \ast \varepsilon_{k-1}, \quad x_{k,s}^- \mapsto \Omega_{k,s}^- \ast \varepsilon_{k-1} \ast \varepsilon_k, \quad h_{k,s}^\pm \mapsto -\Theta_{k,s}^\pm \ast \varepsilon_{k-1} \ast \varepsilon_{k+1},$$

extends uniquely to an algebra homomorphism from $U_{q_0,t_0}$ to the specialization of $K_{G_\mathcal{W}}(Z_W)$ at the point $z$. 

**4. Proof of Theorem 2.**

In the rest of the paper $\Gamma$ and $T$ are as in Section 3.4.

**4.1.** Fix a maximal torus $T(W) \subseteq GL(T(W)$ and set $T_W = T(W) \times T$. Put $R_W = R(T_W)$ and $K_W = K(T_W(A_W))$. Let $\bar{R}_W$ be the fraction field of $R_W$ and set $\bar{K}_W = \bar{R}_W \otimes_{R_W} K_W$. Fix $z \in T$ and consider the fixpoint varieties $Z_{W}^\pm, \Lambda_{W}^\pm$. The action by convolution of $K_{G_\mathcal{W}}(Z_W)$, $K_{G_\mathcal{W}}(Z_{W}^\pm)$, on $K_{G_\mathcal{W}}(A_W)$, $K_{G_\mathcal{W}}(A_{W}^\pm)$, gives two algebra homomorphisms

$$K_{G_\mathcal{W}}(Z_W) \to \text{End}_{R(G_W)} K_{G_\mathcal{W}}(A_W)$$

$$K_{G_\mathcal{W}}(Z_{W}^\pm) \to \text{End}_{R(G_W)} K_{G_\mathcal{W}}(A_{W}^\pm).$$
The bivariant version of the Localization Theorem [CG, Theorem 5.11.10] gives isomorphisms

\[ K_{GW}(Z_W) \cong K_{GW}(Z_{\tilde{W}}) \quad \text{and} \quad K_{GW}(\Lambda_W) \cong K_{GW}(\Lambda_{\tilde{W}}) \]

which commute with the convolution product. Thus, it identifies the maps (4.1) after localization with respect to \( z \). Suppose now that \( z \) is general. Then \( \Lambda_{\tilde{W}} \subseteq \pi^{-1}(0) \) and, thus, \( Z_{\tilde{W}} \cong \Lambda_{\tilde{W}} \times \Lambda_{\tilde{W}} \). In particular \( K_{GW}(\Lambda_W) \) is a faithful \( K_{GW}(Z_W)_z \)-module by Section 3.2. Observe that (see [CG; Theorem 6.1.22] for instance)

\[ K_W = R_W \otimes_{R(\Lambda_W)} K_{GW}(\Lambda_W). \]

Observe also that, since \( R_W \) is free over \( R(\Lambda_W) \),

\[ \text{End}_{R_W}(K_W) = R_W \otimes_{R(\Lambda_W)} \text{End}_{R(\Lambda_W)} K_{GW}(\Lambda_W). \]

Moreover, \( \text{End}_{R_W}(K_W) \) is a projective \( R(\Lambda_W) \)-module by Sections 3.1 and 3.2. Thus, we have a chain of injective algebra homomorphisms

\[ K_{GW}(Z_W)_z \hookrightarrow \text{End}_{R(\Lambda_W)} K_{GW}(\Lambda_W)_z \hookrightarrow \text{End}_{R_W} K_W \hookrightarrow \text{End}_{R_W} \Lambda_W. \]

Let \( x_{k,s}^\pm \in \text{End}_{R_W} \Lambda_W \) be the operator such that

\[ x_{k,s}^\pm(x) = R_{p_1}(\Omega_{k,s}^{\pm} \otimes L_{p_2}(x)), \quad \forall x \in K_W. \]

The chain of embeddings above sends the image of \( \Omega_{k,s}^{\pm} \) in \( K_{GW}(Z_W)_z \) to \( x_{k,s}^\pm \). Hence, it suffices to compute the relations between the operators \( x_{k,s}^\pm \).

4.2. Let \( \Pi \) be the set of all partitions \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \). A partition is identified with its Ferrer diagram, whose boxes are labelled by pairs in \( \mathbb{N} \times \mathbb{N} \). Let \( A = \mathbb{C}[x, y] \) be the ring of regular functions on \( L \). The \( \Gamma \)-action on \( L \) induces an action of \( \Gamma \) on \( A \). If \( \lambda \in \Pi \) let \( J_\lambda \subset A \) be the subspace linearly spanned by the monomials \( x^i y^j \) such that \( (i, j) \notin \lambda \). The quotient space \( V_\lambda = A/J_\lambda \) is a \( \Gamma \)-module. Let us consider the linear operators \( B_\lambda^* \in \text{Lin}_\Gamma(V_\lambda, L \otimes V_\lambda) \) and \( i_\lambda^* \in \text{Lin}_\Gamma(\mathbb{C}, V_\lambda) \) such that

\[ B_\lambda^* = e_x \otimes x + e_y \otimes y \quad \text{and} \quad i_\lambda^*(1) = 1 + J_\lambda. \]

Put \( \xi_\lambda^* = (B_\lambda^*, i_\lambda^*, 0) \) and let \( \xi_\lambda = (B_\lambda, 0, j_\lambda) \in \Lambda_{\mathbb{C}, V_\lambda} \) be the dual triple (the representation \( V_\lambda \) is identified with its dual). If \( W \in \text{Rep}(\Gamma) \) is one-dimensional then \( \Lambda_{\mathbb{C}, V_\lambda} \) is isomorphic to \( \Lambda_{W, W \otimes V_\lambda} \) and, thus, \( \xi_\lambda \) is identified with a point in \( \Lambda_{W, W \otimes V_\lambda} \). The action of \( T_W \) on \( \Lambda_{W} \) has only finitely many fixpoints. More precisely, fix \( k_1, \ldots, k_w \in \mathbb{Z}/n \mathbb{Z} \) such that \( W \cong \bigoplus_{a=1}^w S_{k_a} \) as a \( \Gamma \)-module. Then we have the following lemma.

**Lemma 3.** Fix \( W, V \in \text{Rep}(\Gamma) \).

(i) We have \( \Lambda_{\mathbb{C}}^{(W)} \cong \prod_{a=1}^w \Lambda_{S_{k_a}} \).

(ii) Moreover if \( W \) is one-dimensional then \( \Lambda_{W, V} \) is the set of all the triples \( \xi_\lambda \) such that \( W \otimes V_\lambda = V \) as a \( \Gamma \)-module.

**Proof:** Part (ii) is well-known, since for all \( W \in X(\Gamma) \), the variety \( \Lambda_{W} \) is a subvariety of the Hilbert scheme of all finite length subschemes in \( \mathbb{C}^2 \) (see [N3, Section...
2.2]). Fix $\xi = (B, i, j) \in \Lambda_{W}^{T(W)}$ and $h \in T_{W}$, $h$ generic. Fix an isomorphism $T_{W} \simeq (\mathbb{C}^{*})^{w}$ and put $h = (t_{1}, t_{2}, \ldots, t_{w}) \in (\mathbb{C}^{*})^{w}$. Set $W_{a} = \text{Ker}(h - t_{a}) \subseteq W$ for all $a = 1, 2, \ldots, w$. Let $g \in GL_{\Gamma}(V)$ be such that

$$
(B, i \circ h^{-1}, h \circ j) = ((1 \otimes g) \circ B \circ g^{-1}, (1 \otimes g) \circ i, j \circ g^{-1}).
$$

If $a = 1, 2, \ldots, w$ let $V_{a} \subseteq V$ be the generalized eigenspace of $g$ associated to the eigenvalue $t_{a}^{-1}$. Then,

$$
B(V_{a}) \subseteq L \otimes V_{a} \quad \forall a.
$$

If $\bigoplus_{a} V_{a} \neq V$ then there exists a non-zero complex number $z$ such that $\text{Ker}(g - z^{-1}) \neq \{0\}$ and $z \neq t_{a}$ for all $a$. Moreover, (4.2) implies that $\text{Ker}(g - z^{-1}) \subseteq \text{Ker} j$ and is $B$-stable, which contradicts the stability of the triple $\xi$. Hence, $\xi = \bigoplus_{a} (B_{a}, i_{a}, j_{a})$ where

$$
B_{a} : V_{a} \rightarrow L \otimes V_{a} \quad i_{a} : W_{a} \rightarrow \Lambda^{2}L \otimes V_{a} \quad j_{a} : V_{a} \rightarrow W_{a}
$$

are the restrictions of $B, i, j$. In particular, $(B_{a}, i_{a}, j_{a})$ is stable. \hfill \Box

4.3. Given a multipartition $\lambda = (\lambda_{1}, \ldots, \lambda_{w}) \in \Pi^{w}$, put $V_{\lambda} = \bigoplus_{a=1}^{w} S_{k_{a}} \otimes V_{\lambda_{a}}$. The space $V_{\lambda}$ is endowed with the structure of a $\Gamma$-module as in Section 4.2. Put $B_{\lambda} = \bigoplus_{a=1}^{w} B_{\lambda_{a}} \in \text{Lin}_{\Gamma}(V_{\lambda}, L \otimes V_{\lambda})$ and $j_{\lambda} = \bigoplus_{a=1}^{w} j_{\lambda_{a}} \in \text{Lin}_{\Gamma}(W_{\lambda}, V_{\lambda})$. Let $\xi_{\lambda} = (\xi_{\lambda_{1}}, \ldots, \xi_{\lambda_{w}}) \in \Lambda_{W,V_{\lambda}}^{T(W)}$ be the class of the triple $(B_{\lambda}, 0, j_{\lambda})$. Let us consider the map $\iota_{\lambda} : \{\bullet\} \rightarrow \Lambda_{W}, \bullet \mapsto \xi_{\lambda}$, and put

$$
b_{\lambda} = R_{\iota_{\lambda}}(1) \in K_{W}, \quad T_{\lambda} = \iota^{*}_{\lambda}(T_{\xi_{\lambda}}) \in R_{W}.
$$

The elements $b_{\lambda}, \lambda \in \Pi^{w}$, form a basis of the $\tilde{R}_{W}$-vector space $\bar{K}_{W}$ by the Localization Theorem. Let $p : \Lambda_{W}^{T(W)} \rightarrow \{\bullet\}$ be the projection to a point and let $\iota : \Lambda_{W}^{T(W)} \rightarrow \Lambda_{W}$ be the embedding. The map $p$ is obviously proper. We consider the pairing

$$
\langle \ \mid \ \rangle : \quad \bar{K}_{W} \otimes \bar{K}_{W} \rightarrow \bar{R}_{W}, \quad x \otimes y \mapsto R_{p^{*}}((R_{t_{\lambda}})^{-1}(x) \otimes^{L} (R_{t_{\mu}})^{-1}(y)).
$$

**Lemma 4.** If $\lambda, \mu \in \Pi^{w}$ then $\langle b_{\lambda} | b_{\mu} \rangle = \delta_{\lambda,\mu} \Lambda(T_{\lambda}^{*})$.

**Proof:** Let $q : \Lambda_{W} \rightarrow \{\bullet\}$ be the projection. The map $q$ is not proper, but the direct image of a sheaf with compact support is well-defined. Then we have

$$
\langle b_{\lambda} | b_{\mu} \rangle = Rq_{*}(R_{t_{\lambda}}(1) \otimes^{L} R_{t_{\mu}}(1))
$$

$$
= R(q \circ \iota_{\lambda})_{*}(\iota^{*}_{\lambda}R_{t_{\mu}}(1)),
$$

by the projection formula. The lemma follows since $\iota^{*}_{\lambda}R_{t_{\mu}}(1) = \delta_{\lambda,\mu} \Lambda(T_{\lambda}^{*}).$ \hfill \Box

4.4. The Hecke correspondence $C_{k}^{\pm}$ is smooth (see [N2]). If $\xi_{\mu} = (\xi_{\mu}, \xi_{\lambda}) \in C_{k}^{\pm}$ let $N_{\mu \lambda} \in R_{W}$ be the class of the fiber of the normal bundle to $C_{k}^{\pm}$ in $\Lambda_{W} \times \Lambda_{W}$ at $\xi_{\mu} \lambda$ and let $\Omega_{\mu \lambda}^{(\pm)} \in R_{W}$ be the restriction of $\Omega_{k,s}^{(\pm)}$ to $\xi_{\mu} \lambda$. Let us use the following notation :

$$
\mu \mapsto \lambda \quad \iff \quad \xi_{\mu} \lambda \in C_{k}^{+}.
$$
Then, for all $\lambda \in \Pi^w$ we have
\[
x_{k,s}^+ (b_\lambda) = \sum_{\mu} \Omega_{\mu \lambda}^{(s)} \Lambda_{\mu \lambda} b_\mu,
\]
for some $\Lambda_{\mu \lambda} \in \bar{R}_W$, where the sum is over all the multipartitions $\mu$ such that $\mu \rightarrow^{k} \lambda$ (resp. $\lambda \rightarrow^{\mu} \mu$).

**Lemma 5.** If $\mu \rightarrow^{k} \lambda$ or $\lambda \rightarrow^{\mu} \mu$ then $\Lambda_{\mu \lambda} = \Lambda(N^*_{\mu \lambda} - T^*_\mu) \in \bar{R}_W$.

**Proof:** We have
\[
\Omega_{\mu \lambda}^{(s)} \Lambda_{\mu \lambda} = \langle b_\mu | b_\mu \rangle^{-1} \langle x_{k,s}^+ (b_\lambda) | b_\mu \rangle
\]
\[
= \langle b_\mu | b_\mu \rangle^{-1} R(p \times p)_* \left( p_2^* R_{\mu \lambda} (1) \otimes^L p_2^* R_{\mu \lambda} (1) \otimes^R \Omega_{k,s}^\pm \right)
\]
\[
= \langle b_\mu | b_\mu \rangle^{-1} (t_\mu \times t_\lambda)^* (\Omega_{k,s}^\pm)
\]
\[
= \Omega_{\mu \lambda}^{(s)} \Lambda(T^*_\mu)^{-1} \Lambda(N^*_{\mu \lambda}).
\]

Lemma 5 can be made more explicit as follows. Recall that $R(T) = \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$ and $L = tS^{-1} + qS$. Put $R(T(W)) = \mathbb{C}[X_{1}^{\pm 1}, \ldots, X_{w}^{\pm 1}]$, in such a way that $W = \sum_a X_a S_{k_a}$ in $R(T_W \times \Gamma)$.

**Lemma 6.** For all $\lambda \in \Pi^w$ the following equality holds
\[
t_{\lambda}^* V = \sum_a \sum_{(i,j) \in \lambda_a} q^i t^j X_a S_{k_a+i-j}.
\]

**Proof:** Recall that the fixpoint $\xi_{\lambda} \in \Lambda^T_{W,V_{\lambda}}$ is the class of the triple $(B_\lambda, 0, j_\lambda)$ as in Section 4.3. Recall also that the action of $GL_\Gamma(V_{\lambda})$ on $\mu^{-1}_{W,V_{\lambda}}(0)^{s,F}$ is free. Let $\rho : T_W \rightarrow GL_\Gamma(V_{\lambda})$ be the group homomorphism such that for any $g = (h, z) \in T_W$,
\[
(4.3) \quad \left( (z \otimes 1) \circ B_\lambda, h \circ j_\lambda \right) = \left( (1 \otimes \rho(g)) \circ B_\lambda \circ \rho(g)^{-1}, j_\lambda \circ \rho(g)^{-1} \right).
\]

Thus, $V_{\lambda}$ may be viewed as a $T_W \times \Gamma$-module. We must prove that the class of $V_{\lambda}$ in the Grothendieck ring is
\[
\sum a \sum_{(i,j) \in \lambda_a} q^i t^j X_a S_{k_a+i-j}.
\]

If $h = (t_1, \ldots, t_w) \in T(W)$ and $z = (q_0, t_0) \in T$ then (4.3) implies that $\rho(g)$ acts on $S_{k_a} \otimes (x^i y^j + J_\lambda)$ by the scalar $q_0^i t_0^j t_a$. The proof is finished.

For all $\lambda \in \Pi^w$ set $V_{\lambda} = \sum_a \sum_{(i,j) \in \lambda_a} q^i t^j X_a S_{k_a+i-j},
\[
R_{\lambda} = \sum_{k=1}^{w} \sum_{\mu \rightarrow^{k} \lambda} V_{\lambda/\mu} \quad \text{and} \quad I_{\lambda} = \sum_{k=1}^{w} \sum_{\lambda \rightarrow^{k} \mu} V_{\mu/\lambda},
\]
where \( V_{\lambda/\mu} = V_\lambda - V_\mu \). Recall that
\[
\theta = t^{-1}S + q^{-1}S^{-1} - 1 - q^{-1}t^{-1} \in R(T_W \times \Gamma)
\]
(see Section 3.3). For any \( \lambda \in \Pi^w \) put \( H_\lambda = \theta^* V_\lambda + W \in R(T_W \times \Gamma) \).

**Lemma 7.** For all \( \lambda \in \Pi^w \) we have \( H_\lambda = I_\lambda - qtR_\lambda \).

**Proof:** If \( \xi = (B, i, j) \in \mu_{W,V}^{-1}((0)^* \Gamma) \), then the fiber of (2.3) at the point \([x : y : z] = [0 : 0 : 1] \) is the \( \Gamma \)-equivariant complex
\[
V \xrightarrow{a_\xi} W \oplus (L \otimes V) \xrightarrow{b_\xi}(\wedge^2 L) \otimes V.
\]

By Claim (2.3) (c), there is no 0-th cohomology. Let \( H_\lambda^k \) be the \( k \)-th cohomology group corresponding to \( \xi = \xi_\lambda \), and, as usual, put \( H_\lambda^k = \text{Lin}_\Gamma(S_k, H_\lambda^k) \). We have \( H_\lambda^1 - H_\lambda^2 = H_\lambda \). Moreover there are isomorphisms
\[
p_2^{-1}(\xi_\lambda) \cap C_k^+ \cong \mathbb{P}(H_{\lambda,k}^2), \quad (\xi_1, \xi_\lambda) \mapsto \wedge^2 L^* \otimes (V_\lambda/V^1)_k,
\]
\[
p_2^{-1}(\xi_\lambda) \cap C_k^- \cong \mathbb{P}(H_{\lambda,k}^1), \quad (\xi_1, \xi_\lambda) \mapsto (\lambda_1(V^1/V_\lambda))_k,
\]
(for the second map, observe that \((L \otimes V^1)_k = (L \otimes V_\lambda)_k \)). Thus, \( T_W \times \Gamma \) acts on \( \mathbb{P}(H_{\lambda,k}^1) \) and \( \mathbb{P}(H_{\lambda,k}^2) \) with finitely many fixpoints corresponding to the triples \( \xi_\mu \) such that \( \mu \to \lambda \) and \( \lambda \to \mu \). The lemma follows. \( \square \)

Recall that for any \( V \in R(T_W \times \Gamma) \), the element \( V_k \in R(T_W) \) is such that \( V = \sum_k V_k S_k \). Moreover, for any \( \lambda \in \Pi^w \) and any \( k \) put \( v_{\lambda,k} = \text{dim} H_{\lambda,k}, h_{\lambda,k} = \text{dim} H_{\lambda,k} \), and
\[
\gamma_{\lambda,k} = q^{-v_{\lambda,k}-v_{\lambda,k-1}} t^{v_{\lambda,k}-v_{\lambda,k+1}}.
\]

**Lemma 8.** If \( \mu \to \lambda \) then
\[
\Omega_{\mu\lambda}^{(e)} = (V_{\lambda/\mu})^{+v_{\lambda,k}} \gamma_{\lambda,k} \quad \text{and} \quad \Lambda_{\mu\lambda} = \Lambda(q^{-t} V_{\lambda/\mu} R_{\mu}^* - V_{\lambda/\mu} I_{\lambda}^*)_0
\]
\[
\Omega_{\mu\lambda}^{(e)} = V_{\lambda/\mu} D(-q^{-t} H_{\lambda,k}) \quad \text{and} \quad \Lambda_{\mu\lambda} = \Lambda(q^{-t} V_{\lambda/\mu} R_{\mu}^* - V_{\lambda/\mu} I_{\lambda}^*)_0.
\]

**Proof:** Given \( V^a \in \text{Rep}(\Gamma) \) and \( \xi^a = (B^a, i^a, j^a) \in \mu_{W,V}^{-1}(0)^{1}_{W,V} \), \( a = 1, 2 \), put \( \xi = (\xi_1, \xi_2) \) and let us form the complex
\[
\begin{align*}
\text{Lin}_\Gamma(V^1, L \otimes V^2) \\
\oplus \ Lin_\Gamma(V^1, V^2) \xrightarrow{a_\xi} \ Lin_\Gamma(W, \wedge^2 L \otimes V^2) \xrightarrow{b_\xi} \ Lin_\Gamma(V^1, \wedge^2 L) \oplus \ Lin_\Gamma(V^1, W)
\end{align*}
\]

where
\[
a_\xi : f \mapsto (B^2 \circ f - f \circ B^1, f \circ i^1, -j^2 \circ f)
\]
\[
b_\xi : (g, i, j) \mapsto (B^2 \wedge g \wedge B^1 + i \circ j^1 + i^2 \circ j, tr_{V^1}(i^1 \circ j) + tr_{V^2}(i \circ j^2)).
\]
Taking the complex \((4.4)\) simultaneously for all \(\xi\) we get an equivariant complex of \(T_W\)-sheaves over \(\Lambda_W \times \Lambda_W\). Suppose now that \(V^1 \subseteq V^2\) and that \(V^2/V^1 \simeq S_k\). It is proved in [N2, Lemma 5.2] that the 0th and the 2nd cohomology sheaves vanish. Thus, the first cohomology sheaf, \(H^1\), is locally free. Moreover Nakajima has constructed a section of \(H^1\) vanishing precisely on the Hecke correspondence \(C_k^+\). Thus, if \(\mu \rightarrow \lambda\) we get
\[
N_{\mu\lambda} = (\theta^* V^*_\mu V_\lambda + qtW^*V_\lambda + V^*_\mu W - qt)_0.
\]
Similarly we get, using the complex \((3.1)\),
\[
T_\mu = (\theta^* V^*_\mu V_\mu + qtW^*V_\mu + V^*_\mu W)_0.
\]
Thus,
\[
\Lambda_{\lambda\mu} = \Lambda(V^*_{\lambda/\mu}H^*_\lambda - q^{-1}t^{-1})_0
\]
\[
\Lambda_{\mu\lambda} = \Lambda(q^{-1}t^{-1}V^*_{\lambda/\mu}H_\lambda - q^{-1}t^{-1})_0.
\]
(4.5)
The lemma follows from Lemma 7.

4.5. We now prove the relations. Let \(\Theta_{\lambda,k}^\pm(z)\) be the expansion of
\[
\Theta_{\lambda,k}(z) = (-1)^{h_{\lambda,k}^\gamma_{\lambda,k}}\Lambda_z((qt - 1)H_{\lambda,k}^*)
\]
in \(R(T_W)[[z^{\pm 1}]]\). Then, consider the operator \(h_k^\pm(z)\) such that
\[
h_k^\pm(z)(b_\lambda) = \Theta_{\lambda,k}^\pm(z) b_\lambda.
\]
Put \(x_k^\pm(z) = \sum_{s \in \mathbb{Z}} x_{k,s}^\pm z^{-s}\) for all \(k\).

**Lemma 9.** The following relation holds
\[
(1 - q^{-1}t^{-1})[x_k^+(z), x_i^-(w)] = \delta_{kl} \epsilon(z/w)(h_k^+(z) - h_k^-(z)).
\]

**Proof:** Suppose first that \(\lambda, \mu, \alpha\) and \(\beta\) are such that
\[
\mu \rightarrow \alpha, \quad \lambda \rightarrow \alpha, \quad \beta \rightarrow \lambda \quad \text{and} \quad \beta \rightarrow \mu.
\]
If \(\lambda \neq \mu\) then \(V_\alpha + V_\beta = V_\mu + V_\lambda\). Thus,
\[
N^*_{\alpha\alpha} + N^*_{\lambda\alpha} - T^*_\alpha = (\theta V_\beta V^*_\alpha + q^{-1}t^{-1}WV^*_\alpha + V_\beta W^* - 2q^{-1}t^{-1})_0
\]
\[
= N^*_{\beta\lambda} + N^*_{\beta\mu} - T^*_\beta,
\]
and so \(\Lambda_{\mu\alpha} \Lambda_{\alpha\lambda} = \Lambda_{\mu\beta} \Lambda_{\beta\lambda}\). Moreover, a direct computation gives \(\Omega_{\mu\alpha}^{(o)} \Omega_{\alpha\lambda}^{(o)} = \Omega_{\mu\beta}^{(o)} \Omega_{\beta\lambda}^{(o)}\). Hence,
\[
[x_k^+, x_i^-](b_\lambda) = \delta_{kl} C b_\lambda.
\]
for some $C \in \tilde{R}_W$. Let us now compute the constant $C$. Put $k = l$ and $\mu = \lambda$. Lemma 8 gives
\[
\Lambda_{\lambda\beta} \Lambda_{\beta\lambda} = \Lambda (q^{-1} t^{-1} (V_{\lambda/\beta} R^*_\beta + V^*_\lambda I_\lambda) - V^*_\lambda R_\beta - V_{\lambda/\beta} I_\lambda)_{(0)}
\]
\[
\Lambda_{\lambda\alpha} \Lambda_{\alpha\lambda} = \Lambda (q^{-1} t^{-1} (V_{\alpha/\lambda} R^*_\alpha + V^*_\alpha I_\alpha) - V^*_\alpha R_\lambda - V_{\alpha/\lambda} I_\alpha)_{(0)}
\]
and
\[
\Omega^{(0)}_{\lambda\beta} \Omega^{(0)}_{\beta\lambda} = \gamma_{\lambda,k} D (-q^{-1} t^{-1} V^*_\lambda H_\lambda)_{(0)}
\]
\[
\Omega^{(0)}_{\lambda\alpha} \Omega^{(0)}_{\alpha\lambda} = \gamma_{\alpha,k} D (-q^{-1} t^{-1} V^*_\alpha H_\alpha)_{(0)}
\]
Recall that, with the notations in Section 3.1, we have $D(\mathcal{E}) \Lambda(\mathcal{E}^*) = (-1)^{\dim \mathcal{E}} \Lambda(\mathcal{E})$. Hence we obtain
\[
[x^+_{k,s}, x^-_{k,t}] (b_\lambda) = (-1)^{h_{\lambda,s}} \sum_{\lambda_{\alpha,k}} \gamma_{\lambda,k} \left( -q t \sum_{\lambda_{\alpha,k}} V^{s+t}_{\alpha/\lambda,k} \frac{\Lambda (q^{-1} t^{-1} V_{\alpha/\lambda} R^*_\alpha + q t V_{\alpha/\lambda} I^*_\alpha)_{(0)}}{\Lambda (V_{\alpha/\lambda} R^*_\alpha + V_{\alpha/\lambda} I^*_\alpha)_{(0)}} + \sum_{\beta_{\lambda,k}} V^{s+t}_{\lambda/\beta,k} \frac{\Lambda (q^{-1} t^{-1} V_{\lambda/\beta} R^*_\beta + q t V_{\lambda/\beta} I^*_\beta)_{(0)}}{\Lambda (V_{\lambda/\beta} R^*_\beta + V_{\lambda/\beta} I^*_\beta)_{(0)}} \right) b_\lambda.
\]
Hence, Lemma 9 follows from Lemma 7 and the following fact.

**FACT.** Let $q$ and $a_i, i \in I$, be formal variables. Fix a partition $I = I_1 \amalg I_2$. For all $s \in \mathbb{Z}$ consider the following 1-form
\[
\omega_s = z^{s-1} \left( \prod_{j \in I_1} \frac{1 - q^{-1} z/a_j}{1 - z/a_j} \right) \left( \prod_{j \in I_2} \frac{1 - q z/a_j}{1 - z/a_j} \right) dz.
\]
Then,
\[
\frac{\text{res}_0 \omega_s + \text{res}_\infty \omega_s}{1 - q^{-1}} = -q \sum_{i \in I_2} a_i^s \left( \prod_{j \in I_1} \frac{1 - q^{-1} a_i/a_j}{1 - a_i/a_j} \right) \left( \prod_{j \in I_2 \setminus \{i\}} \frac{1 - q a_i/a_j}{1 - a_i/a_j} \right) + \sum_{i \in I_1} a_i^s \left( \prod_{j \in I_1 \setminus \{i\}} \frac{1 - q^{-1} a_i/a_j}{1 - a_i/a_j} \right) \left( \prod_{j \in I_2} \frac{1 - q a_i/a_j}{1 - a_i/a_j} \right).
\]
\[
\square
\]
**LEMMA 10.** We have
\[
(w - qz)^{+1} x^\pm_k (w) x^\pm_k (z) = (qw - z)^{+1} x^\pm_k (z) x^\pm_k (w)
\]
\[
(tw - z) x^+_{k+1} (w) x^+_{k+1} (z) = (qz - w) x^+_k (z) x^+_k (w)
\]
\[
(w - tz) x^-_{k+1} (w) x^-_{k+1} (z) = (z - qw) x^-_k (z) x^-_k (w)
\]
\[
x^\pm_l (w) x^\pm_k (z) = x^\pm_k (z) x^\pm_l (w) \quad \text{if} \quad l \neq k, k \pm 1.
\]

Proof: For all \( k, l \) set
\[
A_{\nu} = \{ \mu \in \Pi^w | \nu \rightarrow^{k} \mu \rightarrow^{l} \lambda \} \quad \text{and} \quad A'_{\nu} = \{ \mu' \in \Pi^w | \nu \rightarrow^{k} \mu' \rightarrow^{l} \lambda \}.
\]
Then we have
\[
x_{k}^{+}(w)x_{k}^{+}(z)(b_{\lambda}) = \sum_{\nu} \sum_{\mu \in A_{\nu}} \epsilon(w^{-1}V_{\mu/\nu})\epsilon(z^{-1}V_{\lambda/\mu}) \Omega^\mu \Lambda^\mu b_{\nu}
\]
\[
x_{k}^{+}(z)x_{k}^{+}(w)(b_{\lambda}) = \sum_{\nu} \sum_{\mu' \in A'_{\nu}} \epsilon(w^{-1}V_{\lambda/\mu'})\epsilon(z^{-1}V_{\mu'/\nu}) \Omega^{\mu'} \Lambda^{\mu'} b_{\nu},
\]
where \( \Lambda^\mu = \Lambda_{\nu \mu} \Lambda_{\mu \lambda} \) and \( \Omega^\mu = \Omega^{(0)}_{\nu \mu} \Omega^{(0)}_{\mu \lambda} \). In particular if we have
\[
(4.6) \quad V_{\lambda/\mu} = V_{\mu'/\nu} \quad \text{and} \quad V_{\mu/\nu} = V_{\lambda/\mu'},
\]
then
\[
(4.7) \quad \Lambda^\mu = \Lambda(\theta V_{\nu/\nu}V_{\lambda/\mu}^* - \theta V_{\mu'/\nu}V_{\lambda/\mu})_0 \Lambda^{\mu'}
\]
\[
\Omega^\mu = D(\theta V_{\nu/\mu})_k D(\theta V_{\lambda/\mu}), \Omega^{\mu'}.
\]
Suppose first that \( l \neq k \pm 1 \). Then, there is a bijection \( A_{\nu} \rightarrow A'_{\nu}, \mu \mapsto \mu' \), such that (4.6) holds. If \( l = k \) then (4.7) gives
\[
(1 - V_{\lambda/\mu}^* V_{\mu/\nu})(1 - q^{-1}t^{-1}V_{\lambda/\mu}^* V_{\mu'/\nu})\Lambda^\mu = (1 - V_{\lambda/\mu}^* V_{\mu/\nu})(1 - q^{-1}t^{-1}V_{\lambda/\mu}^* V_{\mu'/\nu})\Lambda^{\mu'},
\]
and \( V_{\mu/\nu}^{-2} \Omega^\mu = V_{\lambda/\mu}^{-2} \Omega^{\mu'} \). Thus
\[
(w - qtz) \epsilon(w^{-1}V_{\mu/\nu})\epsilon(z^{-1}V_{\lambda/\mu}) \Omega^\mu \Lambda^\mu = (qtw - z) \epsilon(w^{-1}V_{\lambda/\mu})\epsilon(z^{-1}V_{\mu'/\nu}) \Omega^{\mu'} \Lambda^{\mu'}.
\]
If \( l \neq k, k \pm 1 \) then (4.7) gives \( \Lambda^\mu = \Lambda^{\mu'} \), \( \Omega^\mu = \Omega^{\mu'} \), and the first claim of the lemma follows. Suppose now that \( l = k + 1 \). Set
\[
B_{\nu} = A_{\nu} \setminus \{ \mu \in A_{\nu} | V_{\lambda/\mu} = tS^{-1}V_{\mu/\nu} \}
\]
\[
B'_{\nu} = A'_{\nu} \setminus \{ \mu' \in A'_{\nu} | V_{\lambda/\mu'} = qSV_{\mu'/\nu} \}.
\]
Then,
\[
(4.6) \quad V_{\lambda/\mu} = V_{\mu'/\nu} \quad \text{and} \quad V_{\mu/\nu} = V_{\lambda/\mu'},
\]
\[
(4.7) \quad \Lambda^\mu = \Lambda(\theta V_{\nu/\nu}V_{\lambda/\mu}^* - \theta V_{\mu'/\nu}V_{\lambda/\mu})_0 \Lambda^{\mu'}
\]
\[
\Omega^\mu = D(\theta V_{\nu/\mu})_k D(\theta V_{\lambda/\mu}), \Omega^{\mu'}.
\]
Now, there is a bijection \( B_{\nu} \rightarrow B'_{\nu}, \mu \mapsto \mu' \), such that (4.6) holds. Then, (4.7) implies that if \( \mu \in B_{\nu} \) then
\[
(1 - t^{-1}V_{\mu'/\nu}^* V_{\lambda/\mu,k}) \Lambda^\mu = (1 - q^{-1}tV_{\mu'/\nu}^* V_{\lambda/\mu,k}) \Lambda^{\mu'},
\]
and \( q^{-l} V_{\nu, \mu} \Omega^\mu = t^{-l} V_{\lambda, \mu} \Omega^\mu \). The formulas for the operators \( x_k^+ \) are proved in a similar way. It suffices to observe that if \( \mu, \mu' \in A'_\nu \), and (4.6) holds, then we have the following identities which are very similar to (4.7):

\[
\begin{align*}
\Lambda_{\lambda \mu} \Lambda_{\mu \nu} & = \Lambda (\theta V_{\nu, \mu} V_{\lambda, \mu} - \theta V_{\mu, \nu} V_{\lambda, \mu}) \Lambda_{\lambda \mu} \Lambda_{\mu \nu} \\Omega^{(0)}_{\lambda \mu} \Omega^{(0)}_{\mu \nu} & = D(\theta V_{\nu, \mu}) D(\theta V_{\lambda, \mu}) \Omega^{(0)}_{\lambda \mu} \Omega^{(0)}_{\mu \nu}.
\end{align*}
\]

\( \square \)

**Lemma 11.** We have

\[
\begin{align*}
h^+_i(w) x^+_k(z) & = x^+_k(z) h^+_i(w) \quad \text{if} \quad l \neq k, k \pm 1 \\
(w - qtz) h^+_k(w) x^+_k(z) & = (qtw - z) x^+_k(z) h^+_k(w) \\
(tw - z) h^+_{k+1}(w) x^+_k(z) & = (qz - w) x^+_k(z) h^+_{k+1}(w).
\end{align*}
\]

**Proof:** For all \( \lambda, \mu \in \Pi^w \) we have

\[
\Theta_{\lambda, \mu}(w) = (-1)^{h_{\lambda, l} + h_{\mu, l} + 1} \gamma_{\lambda, l}^{-1} \gamma_{\mu, l}^{-1} \Lambda_w ((\theta^* - \theta) V_{\lambda, \mu}) \Theta_{\lambda, \mu}(w).
\]

Thus,

\[
h^+_k(w) x^+_k(z) (1 - wz^{-1}) (qt - wz^{-1}) = (1 - wz^{-1}) (1 - twz^{-1}) x^+_k(z) h^+_k(w),
\]

and

\[
(tw - z) h^+_{k+1}(w) x^+_k(z) = (qz - w) x^+_k(z) h^+_{k+1}(w).
\]

\( \square \)

**Lemma 12.** We have

\[
t^{\pm 1} x^+_k(z_1) x^+_k(z_2) x^+_{k+1}(w) + ((qt)^{\pm 1} + 1) x^+_k(z_1) x^+_{k+1}(w) x^+_k(z_2) + q^{\pm 1} x^+_{k+1}(w) x^+_k(z_1) x^+_k(z_2) + \{z_1 \leftrightarrow z_2\} = 0.
\]

**Proof:** Fix \( l = k + 1 \), fix \( \lambda, \omega \in \Pi^w \), and consider the following sets

\[
A = \{\mu, \nu \in \Pi^w | \omega \rightarrow^k \nu \rightarrow^k \mu \rightarrow^k \lambda\}
\]

\[
A' = \{\mu', \nu' \in \Pi^w | \omega \rightarrow^k \nu' \rightarrow^k \mu' \rightarrow^k \lambda\}
\]

\[
A'' = \{\mu'', \nu'' \in \Pi^w | \omega \rightarrow^k \nu'' \rightarrow^k \mu'' \rightarrow^k \lambda\}.
\]

1. First, suppose that \( A, A' \) and \( A'' \) are nonempty. Then, there are bijections

\[
A \xrightarrow{\sim} A', \quad (\mu, \nu) \mapsto (\mu', \nu')
\]

\[
A \xrightarrow{\sim} A'', \quad (\mu, \nu) \mapsto (\mu'', \nu'').
\]
such that
\[ V_{\nu/\omega} = V_{\mu'/\nu'}, \quad V_{\mu/\nu} = V_{\nu'/\omega}, \quad V_{\lambda/\mu} = V_{\lambda'/\mu'}; \]
\[ V_{\nu/\omega} = V_{\lambda'/\mu'}, \quad V_{\mu'/\nu} = V_{\nu'/\omega}, \quad V_{\lambda'/\mu} = V_{\mu'/\nu'}. \]

Put
\[ \Lambda^{\nu\mu} = \Lambda_{\omega\nu} \Lambda_{\nu\mu} \Lambda_{\mu\lambda}, \quad \text{and} \quad \Omega^{\nu\mu} = \Omega^{(0)}_{\omega\nu} \Omega^{(0)}_{\nu\mu} \Omega^{(0)}_{\mu\lambda}. \]

Then, (4.5) gives
\[
\Lambda^{\nu\mu} = \Lambda^{\nu''\prime\mu''} \Lambda \left( \theta V_{\lambda/\nu} V_{\nu'/\omega} - \theta V_{\lambda'/\nu'} V_{\nu'/\omega} \right)_0 \\
= \Lambda^{\nu''\prime \mu'} \Lambda \left( \theta V_{\lambda/\nu} V_{\nu'/\omega} - \theta V_{\lambda'/\nu'} V_{\nu'/\omega} \right)_0 \\
\Omega^{\nu\mu} = \Omega^{\nu''\prime \mu'} \left( D(-\theta V_{\nu'/\omega})_k \right)^2 D(-\theta V_{\nu'/\lambda})_t \\
= \Omega^{\nu''\prime \mu'} D(-\theta V_{\nu'/\omega})_k D(-\theta V_{\nu'/\mu})_t.
\]

Thus,
\[ \Omega^{\nu\mu} \Lambda^{\nu\mu} = q^2 \Lambda (q^{-1} V_{\lambda/\nu, k} V_{\nu'/\omega, l} - t V_{\lambda/\nu, k} V_{\nu'/\omega, l}) \Omega^{\nu''\prime \mu''} \Lambda^{\nu''\prime \mu''} \\
= -q \Lambda (q^{-1} V_{\lambda/\nu, k} V_{\nu'/\omega, l} - t V_{\lambda/\nu, k} V_{\nu'/\omega, l}) \Omega^{\nu''\prime \mu'} \Lambda^{\nu''\prime \mu'}. \]

Put
\[
E(z_1, z_2) = \epsilon(w^{-1} V_{\nu'/\omega}) \epsilon(z_1^{-1} V_{\mu/\nu}) \epsilon(z_2^{-1} V_{\lambda/\mu}) \Omega^{\nu\mu} \Lambda^{\nu\mu} \\
E'(z_1, z_2) = \epsilon(w^{-1} V_{\nu'/\nu'}) \epsilon(z_1^{-1} V_{\nu'/\omega}) \epsilon(z_2^{-1} V_{\lambda/\mu'}) \Omega^{\nu'\mu'} \Lambda^{\nu'\mu'} \\
E''(z_1, z_2) = \epsilon(w^{-1} V_{\lambda/\mu''}) \epsilon(z_1^{-1} V_{\mu'/\nu''}) \epsilon(z_2^{-1} V_{\nu'/\omega}) \Omega^{\nu''\mu''} \Lambda^{\nu''\mu''}. 
\]

Then,
\[
E'(z_1, z_2) = -q^{-1} \frac{z_1 - tw_{z_1}}{z_1 - w_q^{-1} w} E(z_1, z_2) \\
E''(z_1, z_2) = q^{-2} \frac{(z_1 - tw_{z_1})(z_2 - tw_{z_2})}{(z_1 - w_q^{-1} w)(z_2 - w_q^{-1} w)} E(z_1, z_2).
\]

Using (4.5) it is easy to see that the expression \( E(z_1, z_2) \) has the form
\[ E(z_1, z_2) = \frac{z_1 - w_q^{-1} t z_2}{z_1 - z_2} S, \]
where the factor \( S \) is symmetric in \( z_1, z_2 \). Moreover, observe that
\[
qt + \frac{(z_1 - tw_{z_1})(z_2 - tw)}{(z_1 - w_q^{-1} w)(z_2 - w_q^{-1} w)} = \\
= \frac{(z_1 - tw_{z_1})(z_2 - qt z_1)}{(z_1 - w_q^{-1} w)(z_2 - z_1)} + \frac{(z_2 - tw_{z_2})(z_1 - qt z_2)}{(z_2 - w_q^{-1} w)(z_1 - z_2)}. 
\]

Thus we finally get
\[ E(z_1, z_2) + (q + t^{-1}) E'(z_1, z_2) + qt^{-1} E''(z_1, z_2) + \{ z_1 \leftrightarrow z_2 \} = 0. \]
2. If $A, A'$ or $A''$ is the empty set then either $A' \neq \emptyset$ or $A = A' = A'' = \emptyset$. Moreover, if $A' \neq \emptyset$ then, either $A \simeq A'$ and $A'' = \emptyset$, or $A \simeq A''$ and $A' = \emptyset$. Then proceed as in part 1.

\[ \square \]

Acknowledgements. Part of this work was done while the second author was visiting the Institute for Advanced Study at Princeton. The second author is grateful to G. Lusztig for his kind invitation. We are also grateful to V. Ginzburg for his interest and his encouragements.

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