Solution of Dirac equation for modified Poschl Teller plus trigonometric Scarf potential using Romanovsky polynomials method

I. Prastyaningrum1, C. Cari2, A. Suparmi2
1Teaching and Education Institute PGRI Madiun Jl Setia Budi 85 Madiun 63118
2Physics Department of Post Graduate Program Sebelas Maret University
Jl. Ir. Sutami 36A Kentingan Jebres Surakarta 57126, INDONESIA

E-mail: arka.prameswara@gmail.com

The approximation analytical solution of Dirac equation for Modified Poschl Teller plus Trigonometric Scarf Potential are investigated numerically in terms of finite Romanovsky Polynomial. The combination of two potentials are substituted into Dirac Equation then the variables are separated into radial and angular parts. The Dirac equation is solved by using Romanovsky Polynomial Method. The equation that can reduce from the second order of differential equation into the differential equation of hypergeometry type by substituted variable method. The energy spectrum is numerically solved using Matlab 2011. Where the increase in the radial quantum number nr and variable of modified Poschl Teller Potential causes the energy to decrease. The radial and the angular part of the wave function also visualized with Matlab 2011. The results show, by the disturbance of a combination between this potential can change the wave function of the radial and angular part.

1. Introduction
Since Dirac equation as relativistic wave Eq. was formulated by P.A.M Dirac in 1928, the exact solution of Dirac equation for some quantum potentials plays a fundamental role in relativistic quantum mechanics [1]. It is well known that the exact energy eigenvalues of the bound state play an important role in quantum mechanics. In particular, the Dirac equation which describes the motion of a spin-1/2 particle has been used in solving many problems of nuclear and high-energy physics [1].

The exact solution of some non-central potentials for l-wave have been investigated intensively by some authors [2], On 2004, Reity, Rubish and Myhalyna has investigated Dirac equation for vector and scalar potential [3], Victor M. Villalba finds the solution in the presence of a gravitational instanton on 2005 [4], and Bakkeshizadeh and Vahidi solving the Dirac equation for Coulomb and NAD potential [5]. In general, the non-central potential is as a function of radial and angular positions simultaneously. These potentials are constructed by the combination of radial shape or nonshape invariance. The potentials that can be used such as hyperbolic Scarf Potential, Manning Rosen, Rosen-Morse, etc. The bound state energy spectra of these potentials have been investigated by Super symmetry method [6], Nikivorof-Uvarof [7], Laplace Transformation Method [8], Romanovsky polynomials [9], WKB Method [10]. In this paper will be solved the Dirac equation for non-central potential in term of Romanovsky polynomials by numerical method. The Dirac equation of non-central potentials is solved using separation variable method when the non-central potential is separable. We must set the Dirac equation like the hypergeometric term and then make a differential
equation in term of Romanovsky polynomials. The last we will get a complicated equation of energy eigenvalue that can’t be solved by an analytical method so we must use the numerical method to find the solution. Besides that, we also get angular wave function and that must be solved by Matlab 2011.

2. Review of Formulas for the Polynomial Solution to the Generalized Hypergeometric Equation

The method used to solve the Dirac equations in the limit of spin and pseudospin symmetries is the Romanovski polynomials. The Romanovsky polynomials method were discovered in 1884 by Routh in the form of complexified Jacobi polynomials on the unit circle in the complex plane and were then rediscovered as real polynomials by Romanovsky in a statistics framework [11]. The Romanovsky polynomials are built from the generalized hypergeometric equation is given,

$$\sigma(x) \frac{d^2y_n(x)}{dx^2} + \tau(x) \frac{dy_n(x)}{dx} - \lambda y_n(x) = 0$$  \hspace{1cm} (1)

where

$$\sigma(x) = ax^2 + bx + c; \quad \tau = dx + e \quad \text{and} \quad \lambda_n = -(n(n-1) + 2n(1-p))$$  \hspace{1cm} (2)

Eq. (1) is described in the textbook by NikiforovUvarov [12] where it is cast into self-adjoint form and its weight function, $w(x)$, satisfies the so-called Pearson differential equation [12]. The corresponding polynomials to the weight function are built up from the Rodrigues representation that is given as,

$$D_n^{(p,q)}(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} ((1+x^2)^n w(x))$$  \hspace{1cm} (3)

For Romanovsky polynomial, the values of parameters are,

$$a = 1, \quad b = 0, \quad c = 1, \quad d = 2(1-p); \quad e = q, \quad p > 0; \quad y_n(x) = D_n^{(p,q)}(x); \quad p = -\beta > n$$  \hspace{1cm} (4)

And so, the weight function is showed as.

$$w(x) = (1+x^2)^{-p}e^{q \tan^{-1}(x)}$$  \hspace{1cm} (5)

By setting Eq. (1) with Romanovski polynomial parameter, we find the new differential equation, called the differential equation of Romanovski polynomial type.

$$(1+x^2) \frac{d^2D_n^{(p,q)}(x)}{dx^2} + (2x(1-p) + q) \frac{dD_n^{(p,q)}(x)}{dx} - [(n(n-1) + 2n(1-p))D_n^{(p,q)}(x) = 0$$  \hspace{1cm} (6)

which is hypergeometric type equation with parameters expressed in equation (4).The Romanovski polynomials obtained from equations (3) and (5) are,

$$D_n^{(p,q)}(x) = \frac{1}{(1+x^2)^{-p}e^{q \tan^{-1}(x)}} \frac{d^n}{dx^n} ((1+x^2)^n (1+x^2)^{-p}e^{q \tan^{-1}(x)})$$  \hspace{1cm} (7)

3. Find The Eigenvalues and Eigenfunctions in terms of Romanovski Polynomial

The Radial part of one dimension Dirac equation with modified Poschl Teller with the centrifugal term plus trigonometric Scarf potential is showed in Eq. (8),

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial R(r)}{\partial r} \right) - r^2 \left( (E + M)t^2 \left( \frac{k}{\sinh^2tr} - \frac{\eta(\eta+1)}{\cosh^2tr} \right) + (E^2 - M^2) \right) R(r) = \lambda R(r)$$  \hspace{1cm} (8)

Where $\lambda = l(l+1)$ and by making an approriate change of variable, $\cosh 2tr = ix$ and $R(r) = \frac{x^{(r)}}{r}$, so the Eq. (8) can be rewritten as.

$$(1+x^2) \frac{d^2\tilde{\chi}(r)}{dx^2} + x \frac{d\tilde{\chi}(r)}{dx} - \left( \frac{k(k-1)(E+M)+l(l+1)+\eta(\eta+1)(E+M)}{2(x^2+1)} \right) \tilde{\chi}(r) = 0$$  \hspace{1cm} (9)

To solve Eq.(9) in terms of Romanovski polynomial, substitute Eq.(5) in equation (9) as,
\[ \chi_n = (1 + x^2)^{\frac{p}{2}} e^{-\frac{q}{2} \tan^{-1} x} D_n(\beta, \alpha)(x) \]  

(10)

By inserting Eq.(10) into Eq.(9) and setting the coefficient of \( \frac{1}{2(x^2+1)} \) to be zero, we obtain,

\[ - \{\kappa(\kappa - 1)(E + M) + \eta(\eta + 1)(E + M)\} + 2\beta(\beta - 1) - \frac{q^2}{2} = 0 \]  

(11)

\[ -\alpha + 2\alpha\beta - \{\kappa(\kappa - 1)(E + M) + \eta(\eta + 1)(E + M)\} = 0 \]  

(12)

The Eq. (12) can be reduced to the Romanovsky equation like in Eq. (6) and then comparing the parameters we obtain the relation,

\[ (2\beta + 1) = 2(−p + 1) \]  

(13a)

\[ \alpha = −q \]  

(13b)

From Eq. (13a), (13b) we have the \( \alpha \) and \( \beta \) value,

\[ \beta = \sqrt{\kappa(\kappa - 1)(E + M) + \frac{1}{4} + \frac{l(l + 1)}{4} + \eta(\eta + 1)(E + M) + \frac{1}{4}} \]  

(14)

\[ \alpha = \lambda \left( \sqrt{\kappa(\kappa - 1)(E + M) + \frac{1}{4} + \frac{l(l + 1)}{4} + \eta(\eta + 1)(E + M) + \frac{1}{4}} \right) \]  

(15)

And finally, we obtain the energy spectra and the radial wave function by combine Eq. (14), Eq.(15), Eq.(10), Eq.(3), and Eq.(5),

\[ (E^2 - M^2) = -4t^2 \left( \sqrt{\kappa(\kappa - 1)(E + M) + \frac{1}{4} + \frac{l(l + 1)}{4} + \eta(\eta + 1)(E + M) + \frac{1}{4} + 2n} \right)^2 + \frac{l(l + 1)t^2}{d_0} \]  

(16)

And the result of the wave function of the nth level is

\[ X_{n;l} = \sqrt{1 + x^2}^\beta e^{−\alpha\tan^{-1}x} R_{n}^{(−\beta,−\alpha)}(x) \]  

(17)

Some the solving of Eq. (20) and Eq. (21) are

\[ R_0^{(−\beta_0,−\alpha_0)} = 1 \]  

(18a)

\[ X_{0;l} = \sqrt{1 + x^2}^\beta e^{−\alpha\tan^{-1}x} \]  

(18b)

\[ R_1^{(−\beta_1,−\alpha_1)} = 2x(\beta + 1) - \alpha \]  

(19a)

\[ X_{1;l} = \sqrt{1 + x^2}^\beta e^{−\alpha\tan^{-1}x}(2x(\beta + 1) - \alpha) \]  

(19b)

4. Solution of the Angular Part

The angular Dirac equation with non-central potential given as,

\[ -\left[ \frac{\cot \theta \partial P(\theta)}{P(\theta)} + \frac{1}{\cos \theta} \frac{\partial^2 P(\theta)}{\partial \theta^2} + \frac{1}{\phi(\phi)} \frac{\partial^2 \phi(\phi)}{\partial \phi^2} \right] + \left( E + M \right) \left( \frac{b^2 + a(a-1)}{\sin^2 \theta} - \frac{2b(a-1)\cos \theta}{\sin^2 \theta} \right) = \lambda \]  

(20)

Where,

\[ \frac{\partial^2 \phi(\phi)}{\partial \phi^2} = m^2 \]  

(21)

By inserting Eq. (21) and Eq.(10) into Eq. (20), with \( \lambda = l(l + 1) \) and setting \( \cos \theta = ix \) we have angular wave function in the Romanovsky Polynomials form,

\[ (x^2 + 1)\frac{\partial^2 D_n^{(p,q)}(x)}{\partial x^2} + (2x(\beta + 1) - \alpha) \frac{\partial D_n^{(p,q)}(x)}{\partial x} - \left[ \frac{a\beta x^2 + \beta^2 + 1}{x^2(E + M)} + \frac{2b(a-1)}{x^2(E + M)} \right] \]  

(23)

\[ l(l + 1) \frac{D_n^{(p,q)}(x)}{\partial x} \]
By setting the coefficient of \( \frac{1}{(1+x^2)} \) to be zero we get the \( \alpha \) and \( \beta \) value, that are

\[
\beta = \frac{(E+M)2b(a-\frac{1}{2})i}{(E+M)2b(a-\frac{1}{2})i}
\]

(24a)

\[
\alpha = \frac{(E+M)2b(a-\frac{1}{2})i}{\beta}
\]

(24b)

Because of this condition the solution of Eq. (23) is given as,

\[
(x^2 + 1) \frac{d^2 D_n^{(p,q)}(x)}{dx^2} + (2x(\beta + 1) - \alpha) \frac{dD_n^{(p,q)}(x)}{dx} - [\beta^2 - \beta + l(l + 1)]D_n^{(p,q)}(x)
\]

(25)

Eq. (25) is a second order differential equation of Romanovsky Polynomial polar term. The Eq. (25) can be combined with Eq. (9), and then we obtain, the \( \alpha \) and \( \beta \) value.

\[
\beta = \sqrt{(E+M)(b + (a - \frac{1}{2}))^2 + m^2 - \frac{1}{4}(E+M)^2} + \sqrt{(E+M)(b - (a - \frac{1}{2}))^2 + m^2 - \frac{1}{4}(E+M)^2} = \sqrt{\mu + \mu - \frac{2}{(E+M)^2}}
\]

(26)

\[
\alpha = -i\sqrt{(E+M)(b + a - \frac{1}{2})^2 + m^2 - \frac{1}{4}(E+M)^2} + \sqrt{(E+M)(b - a + \frac{1}{2})^2 + m^2 - \frac{1}{4}(E+M)^2} = -i[\mu + \mu - \frac{2}{(E+M)^2}]
\]

(27)

The quantum orbital number can be found by combine Eq. (28a), Eq. (29b), and Eq. (26),

\[
l = \sqrt{\frac{(E+M)(b + (a - \frac{1}{2}))^2 + m^2 - \frac{1}{4}(E+M)^2}{(E+M)(b - (a - \frac{1}{2}))^2 + m^2 - \frac{1}{4}(E+M)^2}} + n_l = \sqrt{\frac{\mu + \mu - \frac{2}{(E+M)^2}}{\mu + \mu - \frac{2}{(E+M)^2}}} + n_l
\]

(28)

Eq. (29) is a solving of quantum orbital number, that value used to find angular wave function, radial wave function, and energy eigenvalue. When we were found the quantum orbital number, to determine the angular wavefunction, Eq. (26), (27) and Eq. (28) are inserted into Eq.(3) and Eq.(5) so that we obtain the weight function \( w(x) \) and the Romanovsky polynomials \( D_n^{(p,q)}(x) \). From the weight function and the Romanovsky polynomials we obtain the angular wave function, are

\[
R(\beta,-\alpha) = \frac{1}{(1+x^2)^{\beta-\alpha} tan^{-1}(x)} \int (1+x^2)^{\beta-\alpha} e^{-\alpha tan^{-1}(x)} dx = 1
\]

(29a)

\[
P_{l,m} = \sqrt{(1+x^2)^{\beta-\alpha} tan^{-1}(x)} \int (1+x^2)^{\beta+1} e^{-\alpha tan^{-1}(x)} dx = (E+M)^2 + \frac{\mu + \mu - \frac{2}{(E+M)^2}}{2}(\beta + 1) + i[\mu + \mu - \frac{2}{(E+M)^2}]
\]

(29b)

\[
R(\beta,-\alpha) = \frac{1}{(1+x^2)^{\beta-\alpha} tan^{-1}(x)} \int (1+x^2)^{\beta-\alpha} e^{-\alpha tan^{-1}(x)} dx = 1
\]

(30a)

\[
P_{l,m} = \sqrt{(1+x^2)^{\beta-\alpha} tan^{-1}(x)} \int (1+x^2)^{\beta+1} e^{-\alpha tan^{-1}(x)} dx = 2x(\frac{\mu + \mu - \frac{2}{(E+M)^2}}{2}) + i[\mu + \mu - \frac{2}{(E+M)^2}]
\]

(30b)

5. Result and Discussion

In this section, we discuss several results obtained in the previous section. From the energy eigenvalue in Eq. (16) and the orbital quantum number in Eq. (28), by using Matlab, we have the numerical solution of the energy eigenvalue, with parameters \( n, M, k, \eta, l, \) and \( t \). Where there is two variation \( \kappa \) and \( \eta \), we get the energy eigenvalue that be showed at the table below
Table 1. Energy eigenvalue with $\kappa$ variation

| $n$ | $M$ | $\kappa$ | $\eta$ | $l$ | $t$ | Energy (eV) |
|-----|-----|----------|--------|----|----|-------------|
| 1   | 1   | 1        | 2      | 2  | 2  | -0.3265     |
| 1   | 1   | 2        | 2      | 2  | 2  | -0.6162     |
| 1   | 1   | 3        | 2      | 2  | 2  | -0.7499     |
| 1   | 1   | 4        | 2      | 2  | 2  | -0.8241     |
| 1   | 1   | 5        | 2      | 2  | 2  | -0.8697     |

Table 2. Energy eigenvalue with $\eta$ variation

| $n$ | $M$ | $\kappa$ | $\eta$ | $l$ | $t$ | Energy (eV) |
|-----|-----|----------|--------|----|----|-------------|
| 1   | 1   | 1        | 2      | 2  | 2  | -0.0298     |
| 1   | 1   | 2        | 2      | 2  | 2  | -0.5491     |
| 1   | 1   | 2        | 3      | 2  | 3  | -0.7792     |
| 1   | 1   | 2        | 4      | 2  | 3  | -0.8724     |
| 1   | 1   | 2        | 5      | 2  | 3  | -0.9223     |

**Figure 1.** Graph of energy eigenvalue versus $\kappa$ and $\eta$.

From the Figure 1 can be showed that if the $\kappa$ and $\eta$ value is increase so the energy eigenvalue is decrease. With little increasing of $\kappa$ and $\eta$ can make little change of energy eigenvalue. By varying parameters corresponding to value of $\kappa$ and $\eta$, some of the radial wave function are showed in Eq. (18) and Eq. (19). Radial wave function under the influences of modified Poschl Teller potential and trigonometric Scarf potential. Poschl Teller potential are affected by potential constant $\kappa$ and $\eta$. The eigen function from ground state condition can be plot from the solution of $X_{0,l}$, the eigen function can be seen at Figure 2.

**Figure 2.** Eigenfunction of ground state condition and first energy with $t$ variation.

**Figure 3.** Visualisation of angular wave function without disturbance factor.
Wavefunction explains about probability density to find of the electron because statistical interpretation about wave function basically shows that the result of measuring in the quantum system unpredictable. From the Figure 2 left, that explain about ground state condition, we can see that with the increase of \( t \) constant, the amplitude of wave function decreased, and with the increase of the \( r \) value, the amplitude of wave function also increased. That means, by the increasing of amplitude, the probability density to find of the electron is bigger. The fact also happens when the first condition (Figure 2 right). But the differences from Figure 2 left and Figure 2 right, in Figure 2 left, by the small increase of \( r \) value, causes a small increase in amplitude, but from Figure 2 right, by the small increase of \( r \) value causes a bigger increase in amplitude. From the solved special function and by using the variation of \( b \) and \( a \) value, we can get some angular wave function. The visualisation of the angular wave function result be showed by Figure 3 and Figure 4.

The form of angular wave function related to the direction of angular momentum and describing depending on probability density to the angular. The general definition of angular wave function same with the radial wave function, but both have differences. Radial wave function about far or near electron from the nucleus, if angular wave function correlation with around of electron. For the value of \( m_l = 0 \), electron can be found on the \( z \) axis. But for the value of \( m_l = \pm 1 \), electron can be found on the \( x \) and \( y \) shape. If \( m_l = 1 \), the electron counter clock wise, but if \( m_l = -1 \) the electron opposite.

The form of angular wave function it influenced by the value of \( \nu \) and \( q \) that give different influence to form of angular wave function. Increasing \( q \) value but the is constant, the noise that be given by Rosen Morse potential can make smaller wave function on the \( x, y, \) and \( z \) direction. But if the increasing \( \nu \) value with the \( q \) constant, the noise that be given by Rosen Morse potential can make bigger wave function. If \( \nu \) and \( q \) value not constant, all the wave function can be smaller. The changes of angular wave function to the \( \nu \) and \( q \) value explained that the noise from Rosen Morse potential make change to probability on the angular function, but not influence to the direction of electron circle.

6. Conclusion

In this paper, we study the Dirac equation for modified Poschl Teller plus trigonometric Scarf potential. The radial wave function part is obtained approximately from Eq.(19) and the angular part from Eq.(30). The result under influences by modified Poschl Teller plus trigonometric Scarf potentials changes the angular and radial part of the wavefunction. The energy eigenvalue can be obtained via Romanovsky Polynomials method in Eq. (16) and the equation of orbital quantum number \( l \) as expressed in Eq. (28). Both correlated between quantum number. The energy spectrum is also numerically solved using Matlab. From the value of energy eigenvalue, we found that by a little increased of \( \gamma \) can make large change of energy eigenvalue. Than from wave function we also know that the probability density to find of electron under influences the amplitude of wave function.
References

[1] M. Azizi, N. Salehi and A. Akbar 2013 High Energy Physics, vol 2013 ISRN Article ID 310392, 6 pages.
[2] Cari and Suparmi 2012 International Journal of Applied Physics and Mathematics 2159-161.
[3] Reity, Rubish and Myhalyna 2014 Proceedings of Institute of Mathematics of NAS of Ukraine, 50 1429–1434.
[4] Victor M. Villalba 2005 Journal of Physics Conference Series 24 136-140.
[5] S. Bakkeshizadeh and V. Vahidi 2012 Exact Solution of the Dirac Equation for the Coulomb Potential Plus NAD Potential by Using the Nikiforov-Uvarov Method vol 6 Adv. Studies Theor. Phys, p 733 – 742.
[6] Cari, Suparmi and H.Marini 2012 Penentuan Spektrum Energi dan Fungsi Gelombang Potensial Morse dengan Koreksi Sentrifugal Menggunakan Metode SWKB dan Operator SUSY vol 2 Indonesian Journal of Applied Physics, p 112.
[7] Ikot, A. N., and Akpabio 2010 Approximate Solution of the Schrödinger Equation with Rosen-Morse Potential Including the Centrifugal Term vol 2 Applied Physics Research, pp. 202-208.
[8] Eshghi, Hamzavi, and S. M. Ikhdair 2012 Exact Solutions of a Spatially Dependent Mass Dirac Equation for Coulomb Field plus Tensor Interaction via Laplace Transformation Method vol 2012 Advances in High Energy Physics Article ID 873619.
[9] Castillo, David 2009 Exactly Solvable Potentials and Romanovski Polynomials in Quantum Mechanics. Instituto de Física Universidad Autónoma de San Luis Potosi. arXiv:0808.1642v2 [math-ph].
[10] Sadeghi, Pahlavani, Naderi and Banijamali 2005 Solution of the Relativistic Dirac Equation for Woods-Saxon potential Proceedings QPF Sept. 22.
[11] H.J. Weber 2008 Connections between Romanovski and other polynomials arXiv:0706.3153v1[math.CA].
[12] A. F. Nikiforov and U. B. Uvarov, Special Function in Mathematical Physics, Birkhausa, Basel 1988