DEFINABLE CONVOLUTION AND IDEMPOTENT KEISLER MEASURES

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Abstract. We initiate a systematic study of the convolution operation on Keisler measures, generalizing the work of Newelski in the case of types. Adapting results of Glicksberg, we show that the supports of generically stable (or just definable, assuming NIP) measures are nice semigroups, and classify idempotent measures in stable groups as invariant measures on type-definable subgroups. We establish left-continuity of the convolution map in NIP theories, and use it to show that the convolution semigroup on finitely satisfiable measures is isomorphic to a particular Ellis semigroup in this context.

1. Introduction

Various notions and ideas from topological dynamics were introduced into the model-theoretic study of definable group actions by Newelski [18, 19]. A fundamental observation is that certain spaces of types over a definable group carry a natural algebraic structure of a (left-continuous) semigroup, with respect to the “independent product” of types. In a rather wide context, this operation can be extended from types to general Keisler measures on a definable group (i.e. finitely additive probability measures on the Boolean algebra of definable subsets), where it corresponds to convolution of measures. We first recall the classical setting. When $G$ is a locally-compact topological group, then the space of regular Borel probability measures on $G$ is equipped with the convolution product: if $\mu$ and $\nu$ are measures on $G$, then their product is the measure $\mu \ast \nu$ on $G$ defined via

$$
\mu \ast \nu(A) = \int_{y \in G} \int_{x \in G} \chi_A(x \cdot y)d\mu(x)d\nu(y),
$$

for an arbitrary Borel set $A \subseteq G$ (where $\chi_A$ is the characteristic function of $A$). And a measure $\mu$ is idempotent if $\mu \ast \mu = \mu$. A classical theorem of Wendel [28] shows that if $G$ is a compact topological group and $\mu$ is a regular Borel probability measure on $G$, then $\mu$ is idempotent if and only if the support of $\mu$ is a compact subgroup of $G$, and the restriction of $\mu$ to this subgroup is the (bi-invariant) Haar measure. Wendel’s result was extended to locally compact abelian groups by Rudin [24] and Cohen [5], and this line of research continued into the study of the structure of idempotent measures on (semi-)topological semigroups, in particular in the work of Glicksberg [12, 11] and Pym [22, 23].

In this paper we consider the counterpart of these developments in the definable category, i.e. for definable groups and Keisler measures on them. In particular, we aim to address the following questions.

(Q1) Under what conditions the convolution product of two global Keisler measures can be defined?

(Q2) What algebraic structures arise from idempotency of a Keisler measure?
(Q3) Is there a connection between the convolution semigroups of Keisler measures and Ellis semigroups?

We begin by reviewing some (mostly standard) material on Keisler measures in Section 2: we recall various classes of measures (invariant, Borel-definable, finitely satisfiable, finitely approximable, smooth), summarize the relationship between them (in general, as well as in NIP and stable theories) and discuss supports of measures. In particular, in Proposition 2.11 we give a topological characterization of the space of measures finitely satisfiable over a small model $M$, and in Lemma 2.10 we make a couple of observations on invariently supported measures (i.e. global measures such that all types in their support are (automorphism-)invariant over a fixed small model).

In Section 3.1 we extend the usual product $\otimes$ of Borel-definable measures to a slightly larger context. Namely, when defining $\mu \otimes \nu$, we only require the level functions of the measure $\mu$ to be Borel restricted to the support of $\nu$ (Definition 3.1). It is equivalent to the standard definition when $\mu$ is Borel-definable, but allows to evaluate the product of an arbitrary invariant measure $\mu$ with an arbitrary type $p$ for example (and this extends the usual independent product of invariant types, see Proposition 3.5). In relation to (Q1), in Section 3.2 we define the convolution operation on $*$-Borel pairs of Keisler measures in terms of this generalized product of measures (Definition 3.8) and observe some of its basic properties, in particular that it extends the independent product of arbitrary invariant types in a group (Proposition 3.11).

In Section 3.3, we begin investigating idempotent Keisler measures. In Proposition 3.17 we observe that every invariant measure on a type-definable subgroup is idempotent (the extended $\otimes$-product is needed for this to hold without any definability assumptions on the invariant measure). Mirroring the classical situation in Wendel’s theorem, the expectation is that in tame contexts all idempotent measures should arise in this way. In the case of a definably amenable NIP group, invariant measures were classified in [3]. We observe in Proposition 3.18 that a type-definable subgroup of bounded index of a definably amenable NIP group is still definably amenable (and the analysis from [3] extends to it). We also point that, as a consequence of Wendel’s theorem, idempotent measures finitely supported on realized types correspond to finite subgroups (Proposition 3.20); and that in an abelian NIP group, the class of idempotent dfs (= definable and finitely satisfiable) measures is closed under convolution (Proposition 3.21).

In Section 4 we study the supports of idempotent Keisler measures (question (Q2) above). In the proof of Wendel’s theorem (as well as Glicksberg’s proof in the abelian semitopological semigroup case [11]), an idempotent regular Borel measure $\mu$ is associated to a closed subgroup given by its support. In particular, $S(\mu)$ is a closed group and $\mu|_{S(\mu)}$ is its associated (bi-invariant) Haar measure. In the general model-theoretic context the situation is not as nice (see Examples 4.1 and 4.2). However, adapting some of Glicksberg’s work to our context, we show that if $\mu$ is definable, invariently supported and idempotent, then $(S(\mu), *)$ (with respect to the usual independent product of invariant types) is a compact, left-continuous semigroup with no closed two-sided ideals (Corollary 4.3 and Theorem 1.7). This assumption is satisfied when $\mu$ is a dfs measure in an arbitrary theory, or when $\mu$ is an arbitrary definable measure in an NIP theory. We also deduce that if $\text{sup}(\mu)$ has no proper closed left ideals, then $\mu$ is “generically” invariant restricting to its
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It follows that in abelian stable groups, the supports of the idempotent measures are precisely the closed subgroups of the convolution semigroup on the space of types (Corollary 4.18), which leads to a quick description of idempotent measures in strongly minimal groups (Example 4.19).

In Section 5 we classify idempotent measures on a stable group, demonstrating that they are precisely the invariant measures on its type-definable subgroups. More precisely, every idempotent measure is the unique invariant Keisler measure on its own (type-definable) stabilizer. Our proof relies on the results of the previous section and a variant of Hrushovski’s group chunk theorem due to Newelski [17].

Concerning question (Q3), it was observed by Newelski [18] that the convolution semigroup \((S_\times(G,G),\ast)\) on the space of global types finitely satisfiable in a small model \(G \prec G\) is isomorphic to the enveloping Ellis semigroup \(E(S_\times(G,G),G)\) of the action of \(G\) on this space of types. Ellis semigroups for definable group actions in the context of NIP theories were previously considered in e.g. [2, 3], to which we refer for a general discussion. In Section 6, under the NIP assumption, we obtain an analogous description for the convolution semigroup \((M_\times(G,G),\ast)\) on the space of global Keisler measures finitely satisfiable in a small model. Namely, in Theorem 6.10 we show that it is isomorphic to the Ellis semigroup \(E(M_\times(G,G),\text{conv}(G))\) of the action of \(\text{conv}(G)\), the convex hull of \(G\) in the space of global measures finitely satisfiable on \(G\), on this space of measures (see Remark 6.11 on why the convex hull is necessary). Our proof relies in particular on left-continuity of convolution of invariant measures in NIP theories established in Section 6.2 using approximation arguments with smooth measures.

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2. Preliminaries on Keisler measures

2.1. Basic facts about Keisler measures. For the majority of this article, we focus on global Keisler measures and their relationship to small elementary submodels. In this section we recall some of the material from [16, 13, 14, 15, 4, 9], and refer to e.g. [25, Chapter 7] for a more detailed introduction to the subject, or [26, 1] for a survey.

Given \(r_1, r_2 \in \mathbb{R}\) and \(\varepsilon \in \mathbb{R}_{>0}\), we write \(r_1 \approx_\varepsilon r_2\) if \(|r_1 - r_2| < \varepsilon\). Let \(T\) be a first order theory in a language \(\mathcal{L}\) and assume that \(\mathcal{U}\) is a sufficiently saturated model of \(T\) (we make no assumption on \(T\) unless explicitly stated otherwise). In this section, we write \(x, y, z, \ldots\) to denote arbitrary finite tuples of variables. If \(x\) is a tuple of variables and \(A \subseteq \mathcal{U}\), then \(\mathcal{L}_x(A)\) is the collection of formulas with free variables in \(x\) and parameters from \(A\), up to logical equivalence (which we identify with the corresponding Boolean algebra of definable subsets of \(\mathcal{U}^x\)). We write \(\mathcal{L}_x\) for \(\mathcal{L}_x(\emptyset)\). Given a partitioned formula \(\varphi(x;y)\), we let \(\varphi^\ast(y;x) := \varphi(x;y)\) be the partitioned formula with the roles of \(x\) and \(y\) reversed. As usual, \(S_\times(A)\) denotes the space of types over \(A\), and if \(A \subseteq B \subseteq \mathcal{U}\) then \(S_\times(B,A)\) (respectively, \(S_\times^{\text{inv}}(B,A)\)) denotes the closed set of types in \(S_\times(B)\) that are finitely satisfiable in \(A\) (respectively, invariant over \(A\)). For any set \(A \subseteq \mathcal{U}\), a Keisler measure over \(A\) in variables \(x\) is a finitely additive probability measure on \(\mathcal{L}_x(A)\). We denote...
the space of Keisler measures over $A$ (in variables $x$) as $\mathcal{M}_x(A)$. Every element of $\mathcal{M}_x(A)$ is in unique correspondence with a regular Borel probability measure on the space $S_x(A)$, and we will routinely use this correspondence. If $M_0 \preceq M \preceq \mathcal{U}$ are small models, then there is an obvious restriction map $r_0$ from $\mathcal{M}_x(M)$ to $\mathcal{M}_x(M_0)$ and we denote $r_0(\mu)$ simply as $\mu|_{M_0}$. Conversely, every $\mu \in \mathcal{M}_x(M_0)$ admits an extension to some $\mu' \in \mathcal{M}_x(M)$ (not necessarily a unique one).

The space $\mathcal{M}_x(A)$ is a compact Hausdorff space with the topology induced from $[0, 1]^{\mathcal{L}_x(A)}$. This is the coarsest topology on the set $\mathcal{M}_x(A)$ such that for any continuous function $f : S_x(A) \to \mathbb{R}$, the map $\mu \mapsto \int fd\mu$ is continuous. If $M_0 \preceq M$, then under this topology, the restriction map $r_0$ is continuous. We identify every $p \in S_x(A)$ with the corresponding Dirac measure $\delta_p \in \mathcal{M}_x(A)$, and under this identification $S_x(A)$ is a closed subset of $\mathcal{M}_x(A)$.

We recall several important properties of global measures that will make an appearance in this article.

**Definition 2.1.** Let $\mu \in \mathcal{M}_x(\mathcal{U})$ be a global Keisler measure.

1. $\mu$ is invariant if there is a small model $M \prec \mathcal{U}$ such that for any partitioned $\mathcal{L}(M)$-formula $\varphi(x; y)$ and any $b, b' \in \mathcal{U}^y$, if $b \equiv_M b'$ then $\mu(\varphi(x; b)) = \mu(\varphi(x; b'))$. In this case, we say $\mu$ is $M$-invariant.
2. Assume that $\mu$ is $M$-invariant and $\varphi(x; y)$ is a partitioned $\mathcal{L}(M)$-formula. We define the map $F^\varphi_{\mu, M} : S_y(M) \to [0, 1]$ by $F^\varphi_{\mu, M}(q) = \mu(\varphi(x; b))$, where $b \models q$ (this is well-defined by $M$-invariance of $\mu$).
3. $\mu$ is Borel-definable (respectively, definable) if there is $M \prec \mathcal{U}$ such that $\mu$ is $M$-invariant and for any partitioned $\mathcal{L}(M)$-formula $\varphi(x; y)$, the map $F^\varphi_{\mu, M}$ is Borel-measurable (respectively, continuous). In this case, we say that $\mu$ is Borel-definable over $M$ (respectively, definable over $M$).
4. $\mu$ is finitely satisfiable in $M \prec \mathcal{U}$ if for any $\mathcal{L}(\mathcal{U})$-formula $\varphi(x)$, if $\mu(\varphi(x)) > 0$ then $\mathcal{U} \models \varphi(a)$ for some $a \in M^x$. We let $\mathcal{M}_x(\mathcal{U}, M)$ denote the closed set of measures in $\mathcal{M}_x(\mathcal{U})$ which are finitely satisfiable in $M$.
5. $\mu$ is dfs if there is $M \prec \mathcal{U}$ such that $\mu$ is both definable over $M$ and finitely satisfiable in $M$. Similarly, if this is the case, we say that $\mu$ is dfs over $M$.
6. Given $\overline{a} \in (\mathcal{U}^x)^{<\omega}$, with $\overline{a} = (a_1, ..., a_n)$, the associated average measure $Av_{\overline{a}} \in \mathcal{M}_x(\mathcal{U})$ is defined by

$$Av_{\overline{a}}(\varphi(x)) := \frac{|\{i : \mathcal{U} \models \varphi(a_i)\}|}{n}$$

for any $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$.
7. $\mu$ is finitely approximated if there is $M \prec \mathcal{U}$ such that for any $\mathcal{L}(M)$-formula $\varphi(x; y)$ and any $\varepsilon \in \mathbb{R}_{>0}$, there exist $n \in \mathbb{N}_{>1}$ and $\overline{a} \in (M^x)^n$ such that for any $b \in \mathcal{U}^y$, $\mu(\varphi(x; b)) \approx_{\varepsilon} Av_{\overline{a}}(\varphi(x; b))$. In this case, we call $\overline{a}$ a $(\varphi, \varepsilon)$-approximation for $\mu$, and we say $\mu$ is finitely approximated in $M$.
8. $\mu$ is smooth if there exists a small model $M \prec \mathcal{U}$ such that for any $N$ with $M \preceq N \preceq \mathcal{U}$, there exists a unique measure $\mu' \in \mathcal{M}_x(N)$ such that $\mu'|_M = \mu|_M$. In this case, we say that $\mu$ is smooth over $M$.

These properties are related as follows.
Fact 2.2. (1) In any theory $T$, given $\mu \in \mathcal{M}_x(U)$, over any given $M \preceq U$:
(a) $\mu$ is smooth $\Rightarrow$ $\mu$ is finitely approximated \cite[Corollary 2.6]{15};
(b) $\mu$ is finitely approximated $\Rightarrow$ $\mu$ is dfs (e.g. by Fact 2.3 below);
(c) $\mu$ is definable $\Rightarrow$ $\mu$ is Borel-definable;
(d) if $\mu$ is either Borel-definable or finitely satisfiable, then $\mu$ is invariant.

(2) Assuming $T$ is NIP, given $\mu \in \mathcal{M}_x(U)$, over any $M \preceq U$ we have additionally:
(a) $\mu$ is invariant $\Rightarrow$ $\mu$ is Borel-definable \cite[Corollary 4.9]{14}, or \cite[Proposition 7.9]{24};
(b) $\mu$ is dfs $\Rightarrow$ $\mu$ is finitely approximated \cite[Theorem 3.2]{14}.

(3) Assuming $T$ is stable, given any $\mu \in \mathcal{M}_x(U)$ we have moreover:
(a) $\mu$ is finitely approximated (see e.g. \cite[Lemma 4.3]{14} for a direct proof);
(b) for every $L$-formula $\varphi(x; y)$, there exist types $(p_i)_{i \in \omega}$ in $S_x(U)$ and $(r_i)_{i \in \omega}, r_i \in [0, 1]$ such that $\sum r_i = 1$, and taking $\mu' := \sum r_i \cdot p_i$ we have $\mu(\varphi(x; b)) = \mu'(\varphi(x; b))$ for all $b \in U^\nu$ \cite[Lemma 1.7]{16};
(c) If $T$ is $\omega$-stable, then there exist $(p_i)_{i \in \epsilon}$ in $S_x(U)$ and $(r_i)_{i \in \omega}, r_i \in [0, 1]$ such that $\sum r_i = 1$ and $\mu = \sum r_i \cdot p_i$ (same as the proof of \cite[Lemma 1.7]{16}), using boundedness of the global rank).

We have the following characterization of definability (see e.g. \cite[Proposition 4.4]{1}).

Fact 2.3. The following are equivalent for $\mu \in \mathcal{M}_x(U)$ and $M \preceq U$.

(1) The measure $\mu$ is definable over $M$.

(2) For any partitioned $L(M)$-formula $\varphi(x; y)$ and any $\epsilon > 0$, there exist formulas $\Phi_1(y), ..., \Phi_n(y)$ such that each $\Phi_i(y) \in L_y(M)$, the collection \{ $\Phi_i(y) : i \leq n$ \} forms a partition of $U^y$, and if $\models \Phi_i(c) \land \Phi_i(c')$, then $|\mu(\varphi(x, c)) - \mu(\varphi(x, c'))| < \epsilon$.

(3) For every partitioned formula $\varphi(x; y) \in L(M)$ and every $n \in \mathbb{N}_{\geq 1}$ there exist some $L_y(M)$-formulas $\Phi_i^{x, y}(y)$ with $i \in I_n := \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ such that:
(a) the collection \{ $\Phi_i^{x, y}(U) : i \in I_n$ \} forms a covering of $U_y$ (but not necessarily a partition);
(b) For every $i \in I_n$ and $b \in U_y$, if $U \models \Phi_i^{x, y}(b)$ then $|\mu(\varphi(x, b)) - i| < \frac{1}{n}$.

This easily implies the following.

Fact 2.4. If $M \preceq N \preceq U$ and $\mu \in \mathcal{M}_x(N)$ is definable over $M$, then there exists a unique extension $\mu' \in \mathcal{M}_x(U)$ of $\mu$ which is definable over $M$ (given by the same definition schema $\Phi$ as in Fact 2.3, and denoted $\mu|_U$).

In an NIP theory, every measure over a small model can be extended to a smooth measure over a slightly larger elementary extension (\cite[Theorem 3.16]{16}, or \cite[Proposition 7.9]{24}).

Fact 2.5. Let $T$ be an NIP theory. Let $M \preceq U$ and $\mu \in \mathcal{M}_x(M)$. Then $\mu$ admits a smooth extension. I.e., there exist some $\nu \in \mathcal{M}_x(U)$ and some small $M \preceq N \preceq U$ such that $\nu$ is smooth over $N$ and $\nu|_M = \mu$.

Definition 2.6. Given a Keisler measure $\mu \in \mathcal{M}_x(A)$, the support of $\mu$ is
$$S(\mu) = \{p \in S_x(A) : \mu(\varphi(x)) > 0 \text{ for any } \varphi(x) \in p\}.$$
Types in $S(\mu)$ are sometimes called \textit{weakly random} with respect to $\mu$ in the literature.

We recall some properties of supports, with proofs for the sake of completeness.

**Proposition 2.7.** Let $\mu \in \mathcal{M}_x(A)$.

(1) Then for any $\varphi(x) \in \mathcal{L}_x(A)$ such that $\mu(\varphi(x)) > 0$, there exists some $q \in S(\mu)$ such that $\varphi(x) \in q$. In particular, $S(\mu) \neq \emptyset$.

(2) $S(\mu)$ is a closed subset of $S_x(A)$ and $\mu(S(\mu)) = 1$ (and $S(\mu)$ is the smallest set of types under inclusion with this property).

\textbf{Proof.} (1) Without loss of generality, $\varphi(x) \equiv x = x$. Otherwise, we reiterate the proof with the normalization of $\mu$ to the definable set $\varphi(x)$, i.e., considering the Keisler measure $\mu'$ defined by $\mu'(\psi(x)) := \frac{\mu(\psi(x) \land \varphi(x))}{\mu(\varphi(x))}$ for all $\psi(x)$. Assume that $S(\mu) = \emptyset$, then for every type $p \in S_x(A)$, there exists some $\varphi_p(x) \in p$ such that $\mu(\varphi_p(x)) = 0$. Then, $\mu(\neg \varphi_p(x)) = 1$ for every $p \in S_x(A)$, hence for any $n$ and $p_1,\ldots,p_n \in S_x(A)$, we have $\bigcap_{i=1}^n \neg \varphi_{p_i}(x) \neq \emptyset$. Then $K = \bigcap_{p \in S_x(A)} \neg \varphi_p(x) \neq \emptyset$ by compactness of $S_x(A)$. By if $q \in K$, then in particular $\neg \varphi_q(x) \in q$ — a contradiction.

(2) Assume that $p \not\in S(\mu)$. Then, there exists a formula $\varphi_p(x)$ such that $\varphi_p(x) \in p$ and $\mu(\varphi_p(x)) = 0$. Then,

$$S_x(A) \setminus S(\mu) = \bigcup_{p \not\in S(\mu)} \varphi_p(x).$$

Therefore, $S(\mu)$ is closed. Assume that $\mu(S_x(M) \setminus S(\mu)) > 0$. By regularity of $\mu$, there exists a clopen $C \subseteq S_x(A) \setminus S(\mu)$ with positive measure. But by (1) we must have $C \cap S(\mu) = \emptyset$, a contradiction. \hfill \square

**Proposition 2.8.** Let $A \subseteq B \subseteq U$ and $\mu \in \mathcal{M}_x(B)$ be arbitrary. Let $r : S_x(B) \to S_x(A)$, $q \mapsto q|_A$ be the restriction map. Then:

(1) $r(S(\mu)) = S(\mu|_A)$;

(2) the measure $\mu|_A$ is the pushforward of $\mu$ along $r$, i.e. $r^*(\mu) = \mu|_A$.

\textbf{Proof.} (1) The map $r$ is a continuous surjection between compact Hausdorff spaces. By Proposition 2.7(2), $r(S(\mu))$ is compact (hence, closed), as the continuous image of a compact set. Clearly $r(S(\mu)) \subseteq S(\mu|_A)$, and as $r(S(\mu))$ is closed it suffices to show that $r(S(\mu))$ is a dense subset of $S(\mu|_A)$. Indeed, assume that $\varphi(x) \in \mathcal{L}_x(A)$ and $\varphi(x) \cap S(\mu|_A) \neq \emptyset$. Then $\mu|_A(\varphi(x)) > 0$, hence $\mu(\varphi(x)) > 0$, and by Proposition 2.7(1) there exists some $q \in S(\mu)$ with $\varphi(x) \in q$. Hence $\varphi(x) \in r(q)$, and so $r(S(\mu)) \cap \varphi(x) \neq \emptyset$. And (2) is clear. \hfill \square

**Definition 2.9.** We say that $\mu \in \mathcal{M}_x(U)$ is \textit{invariantly supported} if there exists some small model $M \prec U$ such that every type $p \in S(\mu)$ is $M$-invariant.

**Lemma 2.10.** Let $\mu \in \mathcal{M}_x(U)$.

(1) If $\mu$ is finitely satisfiable, then $\mu$ is invariantly supported.

(2) If $\mu$ is invariantly supported, then $\mu$ is invariant.

(3) If $T$ is NIP, the $\mu$ is invariant if and only if it is invariantly supported.

(4) In some theory, there exist a definable measure $\mu \in \mathcal{M}_x(U)$ and $p \in S(\mu)$ such that $p$ is not invariant (over any small set).
Proof. (1) Clearly if $\mu$ is finitely satisfiable in $M \prec U$, then every $p \in S(\mu)$ is also finitely satisfiable in $M$.

(2) Let $M \prec U$ be a small model such that every $p \in S(\mu)$ is invariant over $M$. If $\mu$ is not invariant over $M$, then there exist some $\varphi(x,y) \in L_{xy}$ and some $b \equiv_M b'$ in $U^\mu$ such that $\mu(\varphi(x,b)) \neq \mu(\varphi(x,b'))$. But then $\mu(\varphi(x,b) \triangle \varphi(x,b')) > 0$, hence $\varphi(x,b) \triangle \varphi(x,b') \in p$ for some $p \in S(\mu)$ by Proposition 2.7 — contradicting $M$-invariance of $p$.

(3) ($\Rightarrow$) holds by (2). For ($\Leftarrow$), we note that if $\mu$ is invariant over $M \prec U$, then every global type $p \in S(\mu)$ doesn’t divide over $M$ (given $\varphi(x,b) \in p$ and an $M$-indiscernible sequence $(b_i)_{i \in \omega}$ in $U^\mu$ such that $b_i \equiv_M b$, we have that $\mu(\varphi(x,b_i)) = \mu(\varphi(x,b)) = \varepsilon > 0$ for all $i$; but then $\mu(\bigwedge_{i \leq k} \varphi(x,b_i)) > 0$ for every $k \in \omega$, so in particular $\neq 0$, by a standard probability lemma, see e.g. [25, Lemma 7.5]), hence $p$ is invariant over $M$ by [14 Proposition 2.1(ii)].

(4) Let $T$ be the theory of the random graph, in a language with a single binary relation. We let $\mu(\bigwedge_{i < k} E(x,b_i)^\nu) = \frac{1}{2^k}$ for every $k \in \omega$, pairwise distinct $b_i \in U$ and $t_i \in \{0,1\}$, and $\mu(x = b) = 0$ for every $b \in U$. By quantifier elimination, this determines a measure $\mu \in \mathfrak{M}_\omega(U)$. This $\mu$ is clearly definable over $\emptyset$, and the support of $\mu$ consists of all non-realized types in $S_\omega(U)$. However, it is easy to construct by transfinite induction a non-realized type $p \in S_\omega(U)$ which is not invariant over any small $M \prec U$.

The space of measures $\mathfrak{M}_\omega(U)$ can be naturally viewed as a closed convex subset of a real topological vector space (of all bounded real-valued measures). Given $M \prec U$, we identify $M^\omega$ with the set $\{\delta_a : a \in M^\omega\} \subseteq \mathfrak{M}_\omega(U)$, and let $\text{conv}(M^\omega)$ denote the convex hull. We have the following topological characterization of finite satisfiability for measures.

**Proposition 2.11.** Let $\mu \in \mathfrak{M}_\omega(U)$ and $M \prec U$ a small model. Then $\mu$ is finitely satisfiable in $M$ if and only if $\mu$ is in the closure of $\text{conv}(M^\omega)$ (viewed as a subset of $\mathfrak{M}_\omega(U)$).

**Proof.** Assume $\mu$ is finitely satisfiable in $M$. Let $U$ be a basic open subset of $\mathfrak{M}_\omega(U)$ containing $\mu$. Say

$$U = \bigcap_{i=1}^n \{ \mu \in \mathfrak{M}_\omega(U) : r_i < \mu(\varphi_i(x)) < s_i \}$$

for some $n \in \mathbb{N}$, $\varphi_1(x), \ldots, \varphi_n(x) \in \mathcal{L}_x(U)$ and $r_1, \ldots, r_n, s_1, \ldots, s_n \in [0,1]$. The collection $\{\varphi_1(x), \ldots, \varphi_n(x)\}$ generates a finite Boolean subalgebra of $\mathcal{L}_x(U)$. Let $\theta_1(x), \ldots, \theta_n(x)$ be its atoms, and let $\Theta := \{ \theta_j(x) : \mu(\theta_j(x)) > 0 \}$. As $\mu$ is finitely satisfiable in $M$, for each $\theta_j(x) \in \Theta$, there exists some $a_j \in M^\omega$ such that $\models \theta_j(a_j)$. Let $\nu := \sum_{\theta_j \in \Theta} \mu(\theta_j(x)) \delta_{a_j} \in \mathfrak{M}_\omega(U)$. Then we have $\mu(\varphi_i(x)) = \nu(\varphi_i(x))$ for all $1 \leq i \leq n$ (note that $a_j \models \theta_i \iff i = j$), so $\nu \in U \cap \text{conv}(M^\omega)$. Hence $\mu \in \text{cl}(\text{conv}(M^\omega))$.

Conversely, suppose $\mu \in \text{cl}(\text{conv}(M^\omega))$ and let $\psi(x) \in \mathcal{L}_x(U)$ be such that $\mu(\psi(x)) > 0$. Consider the open set $U_\psi := \{ \nu \in \mathfrak{M}_\omega(U) : 0 < \nu(\psi(x)) \}$ containing $\mu$. Since $\mu$ is in the closure of $\text{conv}(M^\omega)$, there exists some $\mu_\psi = \sum_{i=1}^n r_i \delta_{a_i}$, where $a_i \in M^\omega$ for all $i$ and $\mu_\psi \in U_\psi$. But then $U \models \psi(a_i)$ for at least one $i$. \qed

3. **Definable convolution and idempotent Keisler measures**
3.1. Extended product of measures. We begin by defining a slight generalization of the product of measures that encompasses both the usual independent product of Borel-definable measures and the standard Morley product of invariant types (without any definability assumptions), and also allows to take products of $G$-invariant measures in arbitrary theories. This is accomplished by slightly tweaking the domain of the integral in the usual definition of the $\otimes$-product.

**Definition 3.1.** Let $\mu \in \mathcal{M}_x(\mathcal{U})$, $\nu \in \mathcal{M}_y(\mathcal{U})$, and $\varphi(x,y,\overline{c}) \in \mathcal{L}_{xy}(\mathcal{U})$. We say that the triple $(\mu, \nu, \varphi)$ is Borel if there exists some $N \prec \mathcal{U}$ such that:

1. $\overline{N} \subseteq \mathcal{N}$;
2. for any $q \in S(\nu|_N)$ and $d, d' \in \mathcal{U}^q$ with $d, d' \models q$, we have that $\mu(\varphi(x, d, \overline{c})) = \mu(\varphi(x, d', \overline{c}))$;
3. the map $F_{\mu,N}^\varphi : S(\nu|_N) \to [0,1]$ is Borel, where $F_{\mu,N}^\varphi(q) = \mu(\varphi(x, d, \overline{c}))$ for some/any $d \models q$.

We say that the ordered pair $(\mu, \nu)$ is Borel if $(\mu, \nu, \varphi)$ is Borel for any $\varphi(x,y,\overline{c}) \in \mathcal{L}_{xy}(\mathcal{U})$.

**Definition 3.2.** Assume that $(\mu, \nu)$ is Borel, and let $N$ be any small elementary submodel of $\mathcal{U}$ witnessing this (as in Definition 3.1). Then we define the product measure $\mu \overset{\otimes}{\otimes} \nu \in \mathcal{M}_{xy}(\mathcal{U})$ as follows:

$$
\mu \overset{\otimes}{\otimes} \nu(\varphi(x,y,\overline{c})) = \int_{S(\nu|_N)} F_{\mu,N}^\varphi \, d\nu_N,
$$

with the notation from Definition 3.1 where $\nu_N$ is the restriction of the regular Borel measure $\nu|_N$ to the compact set $S(\nu|_N)$.

We need to check that $\otimes$ is well-defined.

**Proposition 3.3.** Assume that $(\mu, \nu, \varphi)$ is Borel. Then, $\mu \overset{\otimes}{\otimes} \nu(\varphi(x,y,\overline{c}))$ does not depend on the choice of $N$ (as in Definition 3.1).

**Proof.** This proof is essentially the same as for $\otimes$ (see e.g. [25, Proposition 7.19]). Assume that $(\mu, \nu, \varphi)$ is Borel with respect to both $M$ and $N$. We may assume that $M \subseteq N$ (taking a common extension). By Proposition 2.3 let $r : S(\nu_M) \to S(\nu_N)$ be the restriction map; then $F_{\mu,M}^\varphi \circ r = F_{\mu,N}^\varphi$ and the pushforward of the measure $\nu_N$, namely $r^*(\nu_N)$, is equal to $\nu_M$. Hence we have:

$$
\int_{S(\nu_M)} F_{\mu,M}^\varphi \, d(\nu_M) = \int_{S(\nu_M)} F_{\mu,M}^\varphi \, dr^*(\nu_N) = \int_{S(\nu_N)} (F_{\mu,M}^\varphi \circ r) \, d\nu_N = \int_{S(\nu_N)} F_{\mu,N}^\varphi \, d\nu_N.
$$

We recall the independent product of invariant types (see e.g. [25, Section 2.2]).

**Fact 3.4.**

1. Assume $M \prec \mathcal{U}$ is a small submodel, $p \in S^\mathcal{U}_x(p, M)$ and $\mathcal{U}' \supseteq \mathcal{U}$. There there exists a unique type $p' \in S^\mathcal{U}'_x(p', M)$ extending $p$. Then for any $A \subseteq \mathcal{U}'$, we write $p|_A$ to denote $p'|_A$.

2. Assume that $p \in S_x(\mathcal{U}), q \in S_y(\mathcal{U})$ and $p$ is invariant. Then $p \otimes q := tp(a,b/\mathcal{U}) \in S_{xy}(\mathcal{U})$ for some/any $b \models q$ and $a \models p|_{\mathcal{U}b}$ (in some $\mathcal{U}' \supseteq \mathcal{U}$; this is well-defined by (1)).
(3) If \( p, q \in S_x(\mathcal{U}, M) \) (respectively, \( p, q \in S^{\text{inv}}_x(\mathcal{U}, M) \)), then \( p \otimes q \in S_y(\mathcal{U}, M) \) (respectively, \( p \otimes q \in S^{\text{inv}}_y(\mathcal{U}, M) \)).

The product \( \tilde{\otimes} \) extends both the independent product on invariant types and the product of Borel definable probability measures in arbitrary theories.

**Proposition 3.5.**  
(1) Let \( \mu \in \mathcal{M}_x(\mathcal{U}) \) and \( \nu \in \mathcal{M}_y(\mathcal{U}) \). Assume that \( \mu \) is Borel-definable. Then, \( \mu \otimes \nu = \mu \tilde{\otimes} \nu \).

(2) If \( \mu \in \mathcal{M}_x(\mathcal{U}) \) is invariant and \( q \in S_x(\mathcal{U}) \) is arbitrary, then \((\mu, \delta_q)\) is Borel and \( \mu \tilde{\otimes} \delta_q \) is well-defined.

(3) Let \( p \in S_x(\mathcal{U}) \) and \( q \in S_y(\mathcal{U}) \), and \( p \) is invariant. Then \( \delta_{p \tilde{\otimes} q} = \delta_p \tilde{\otimes} \delta_q \), where \( p \otimes q \) is the free product (see Fact 3.4).

**Proof.**  
(1) It is easy to see that \( \int_{S_y(N)} F^\varphi d(\nu|N) = \int_{S(\varphi|N)} F^\varphi d\nu_N \) as long as the integral on the left hand side is well-defined — which is the case when \( \mu \) is Borel definable.

(2) Let \( \varphi(x, y) \in L(\mathcal{U}) \), and let \( N \prec \mathcal{U} \) containing all the parameters from \( \varphi \) be such that \( \mu \) is invariant over \( N \). Note that the map \( F^\varphi_\mu : S_y(N) \rightarrow [0, 1] \) need not be Borel definable. However \( S(\delta_q|N) \) is a single point as \( q \) is a type, hence \( F^\varphi_\mu \mid S(\delta_q|N) \) is trivially Borel.

(3) By (2), \((\delta_p, \delta_q)\) is Borel. Let \( N \prec \mathcal{U} \) witness this, and let \( b \in \mathcal{U}^y, b \models q|_N \). Then

\[
\delta_p \tilde{\otimes} \delta_q(\varphi(x; y)) = \int_{\sup(\delta_q|_N)} F^\varphi_\delta d(\delta_q) = F^\varphi_{\delta_p}(q|_N) = \begin{cases} 1 & \varphi(x, b) \in p, \\ 0 & \neg \varphi(x, b) \in p. \end{cases}
\]

That is, \( \delta_p \tilde{\otimes} \delta_q(\varphi(x; y)) = 1 \) if and only if \( \varphi(x, y) \in \text{tp}(a, b/N) \) for some/any \( b \models q|_N \) and \( a \models q|_N \).

From now on we will simply write \( \otimes \) instead of \( \tilde{\otimes} \) to denote this extended operation when there is no ambiguity involved.

**Definition 3.6.** We say that \( \mu, \nu \in \mathcal{M}_x(\mathcal{U}) \) (\( \otimes \)-)commute if both \((\mu, \nu)\) and \((\nu, \mu)\) are Borel, and \( \mu \otimes \nu = \nu \otimes \mu \).

We recall some facts about commuting measures.

**Fact 3.7.** Assume that \( \mu \in \mathcal{M}_x(\mathcal{U}), \nu \in \mathcal{M}_y(\mathcal{U}) \) and \( M \prec \mathcal{U} \).

(1) [15, Theorem 2.5] Assume that \( \mu \) is Borel-definable over \( M \) and \( \nu \) is smooth over \( M \). Then, for any \( \varphi(x; y) \in L_{xy}(M) \), we have that,

\[
\int_{S_y(M)} F^\varphi_\nu d(\nu|M) = \int_{S_x(M)} F^{\varphi_\mu}_\nu d(\mu|M).
\]

In particular, \( \mu \otimes \nu = \nu \otimes \mu \).

(2) [6, Proposition 2.13] or [6, Proposition 2.10]) If \( \mu \) and \( \nu \) are finitely approximated over \( M \), then \( \mu \otimes \nu = \nu \otimes \mu \).

### 3.2. Definable convolution.
Throughout this section, we let \( T \) be a first order \( L \)-theory expanding a group. We let \( \mathcal{G} \) be a sufficiently saturated model of \( T \), and \( G \) denotes a small elementary submodel. We use letters \( x, y \) to denote singleton variables, i.e. of the sort on which the group is defined. For any formula \( \varphi(x, \overline{\tau}) \in L_x(\mathcal{G}) \), we let \( \varphi'(x, y, \overline{\tau}) = \varphi(x \cdot y, \overline{\tau}) \).
Definition 3.8. Let $\mu, \nu \in \mathcal{M}_x(\mathcal{G})$, and let $\nu_y$ denote the measure in $\mathcal{M}_y(\mathcal{G})$ such that for any $\varphi(y) \in \mathcal{L}_y(\mathcal{G})$, $\nu_y(\varphi(y)) = \nu(\varphi(x))$.

1. We say that $(\mu, \nu)$ is $*$-Borel if for every formula $\varphi(x, \overline{y}) \in \mathcal{L}_x(\mathcal{G})$, the triple $(\mu, \nu_y, \varphi')$ is Borel. We say that $\mu$ is $*$-Borel if the pair $(\mu, \nu)$ is $*$-Borel.

2. If $(\mu, \nu)$ is $*$-Borel, then we define the (definable) convolution product of $\mu$ with $\nu$ as follows:

$$
\mu * \nu(\varphi(x, \overline{y})) = \mu \otimes \nu_y(\varphi'(x, y, \overline{y})) = \int_{\mathcal{L}(\mathcal{G})} F_{\mu}^{\varphi'} \, d\nu_G(y),
$$

where $G$ is some/any small submodel of $\mathcal{G}$ witnessing that $(\mu, \nu_y, \varphi')$ is Borel and $\nu_G(y)$ is the Borel measure $\nu_y$ restricted to $S(\nu_y|G)$ (as in Definition 3.2). We will routinely write this product simply as $\int F_{\mu}^{\varphi'} \, d\nu$ when there is no possibility of confusion.

Note that we are integrating over translates with respect to the right action of $\mathcal{G}$, and in general throughout the article, when speaking about $\mathcal{G}$-invariance and related notion, we will typically consider the action of $\mathcal{G}$ on the right. This choice is made to make sure that this definition correctly extends Newelski’s product of types (Proposition 3.11), but of course all of our results hold with respect to left actions modulo obvious modifications. First we check that the convolution operation indeed defines a measure.

Fact 3.9. Let $\mu, \nu \in \mathcal{M}_x(\mathcal{G})$. If $(\mu, \nu)$ is $*$-Borel, then $\mu * \nu$ is a Keisler measure.

Proof. Clearly $\mu * \nu(x = x) = 1$ and $\mu * \nu(\neg \varphi(x)) = 1 - \mu * \nu(\varphi(x))$. Assume that $\psi_1(x) \land \psi_2(x) = 0$. Let $\theta(x; y) = \psi_1(x \cdot y) \lor \psi_2(x \cdot y)$, and let $G \prec \mathcal{G}$ contain all of the parameters. Then for any $q \in S(\nu|G)$ and $b \models q$ we have $F_{\mu}^{\theta}(q) = \mu(\theta(x; b)) = \mu(\psi_1(x \cdot b) \lor \psi_2(x \cdot b))$. As $\psi_1(x) \land \psi_2(x) = 0$ implies $\psi_1(x \cdot b) \land \psi_2(x \cdot b) = 0$, we have

$$
F_{\mu}^{\theta}(q) = \mu(\psi_1(x \cdot b)) + \mu(\psi_2(x \cdot b)) = F_{\mu}^{\psi_1}(q) + F_{\mu}^{\psi_2}(q).
$$

Then

$$
(\mu * \nu)(\psi_1(x) \land \psi_2(x)) = \int_{\mathcal{L}(\mathcal{G})} F_{\mu}^{\psi_1} \, d\nu_G = \int_{\mathcal{L}(\mathcal{G})} \left(F_{\mu}^{\psi_1} + F_{\mu}^{\psi_2}\right) \, d\nu_G
$$

$$
= \int_{\mathcal{L}(\mathcal{G})} F_{\mu}^{\psi_1} \, d\nu_G + \int_{\mathcal{L}(\mathcal{G})} F_{\mu}^{\psi_2} \, d\nu_G = (\mu * \nu)(\psi_1(x)) + (\mu * \nu)(\psi_2(x)).
$$

This notion of convolution extends the notion of the product of invariant types extensively studied by Newelski [18, 19] and others from the point of view of topological dynamics. The following is easy using Fact 3.4.

Fact 3.10. Let $G \prec \mathcal{G}$ be a small model. Given $p, q \in S_x^{inv}(\mathcal{G}, G)$, we define $p * q := tp(a \cdot b|G) \in S_x^{inv}(\mathcal{G}, G)$, for some/any $(a, b) \models p \otimes q$ in a larger monster model. Then $(S_x^{inv}(\mathcal{G}, G), *)$ is a semigroup, with multiplication continuous in the left coordinate: for each $q \in S_x^{inv}(\mathcal{G}, G)$, the map $- * q : S_x^{inv}(\mathcal{G}, G) \to S_x^{inv}(\mathcal{G}, G)$ is continuous. And $(S_x(\mathcal{G}, G), *)$ is a closed sub-semigroup.

Proposition 3.11. Let $\delta : S_x(\mathcal{G}, G) \to \mathcal{M}_x(\mathcal{G}, G)$ be the map $\delta(p) = \delta_p$. Then $\delta$ is a topological embedding, and $\delta_{p * q} = \delta_p * \delta_q$ for any $p, q \in S_x(\mathcal{G}, G)$.
Proof. Clearly $\delta$ is a topological embedding. Now let $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$ be arbitrary, then by Proposition 3.5(3) we have
\[
\delta_{p,q}(\varphi(x)) = \delta_{p_{\mathcal{G}}, q_{\mathcal{G}}}(\varphi \cdot y) = \delta_{p, \delta q_{\mathcal{G}}} = \delta_p \delta_q(\varphi(x)).
\]

The next lemma follows by straightforward computations.

**Proposition 3.12.** Let $\mu, \mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m \in \mathcal{M}_x(\mathcal{G})$ be arbitrary, and assume that the pairs $(\mu_i, \nu_j)$ are $\ast$-Borel for all $1 \leq i \leq n, 1 \leq j \leq m$. Let $a, a_1, \ldots, a_n \in \mathcal{G}$ and $r_1, \ldots, r_n, s_1, \ldots, s_m \in \mathbb{R}_{\geq 0}$ be such that $\sum_{i=1}^n r_i = \sum_{j=1}^m s_j = 1$. Then:

1. $\mu \ast \delta_e = \delta_e \ast \mu = \mu$,
2. $\delta_a \ast \delta_b = \delta_{ab}$,
3. $\delta_a \ast \mu)(\varphi(x)) = \mu(\varphi(a \cdot x))$ for any $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$,
4. $\left(\sum_{i=1}^n r_i \cdot \mu\right) \ast \left(\sum_{j=1}^m s_j \cdot \nu\right) = \sum_{i,j=1}^{n,m} r_i \cdot s_j \cdot (\mu \ast \nu)$,
5. $\left(\sum_{i=1}^n r_i \cdot \delta_a\right) \ast \mu)(\varphi(x)) = \sum_{i=1}^n r_i \cdot \mu(\varphi(a \cdot x))$ for any $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$.

Finally, we observe that the following properties of measures are preserved under convolution.

**Proposition 3.13.** Let $\mu, \nu \in \mathcal{M}_x(\mathcal{G})$ be such that $(\mu, \nu)$ is $\ast$-Borel, and $G \prec \mathcal{G}$.

1. If $\mu, \nu$ are definable over $G$, then $\mu \ast \nu$ is definable over $G$.
2. If $\mu, \nu$ are finitely satisfiable over $G$, then $\mu \ast \nu$ is finitely satisfiable over $G$.
3. If $\mu, \nu$ are finitely approximated over $G$, then $\mu \ast \nu$ is finitely approximated over $G$.
4. If $\mu(x = b) = 0$ for every $b \in \mathcal{G}$, then $\mu \ast \nu(x = b) = 0$ for every $b \in \mathcal{G}$.

**Proof.** Claims (1), (2) and (3) are slight variations on the preservation of the corresponding properties with respect to $\otimes$ (see e.g. [15] Lemma 1.6) or [6] Proposition 2.6 for (1) and (2), and [6] Proposition 2.13 or [6] Proposition 2.10 for (3)).

We note that
\[
\mu \ast \nu(x = b) = \mu \otimes \nu(b) = \int_{\mathcal{L}_x(\mathcal{G})} F_{\mu}^\nu(b) \, d\nu_G.
\]
And $F_{\mu}^\nu(q) = \mu(x \cdot c = b)$ for some $c$ such that $\mu(c) = q$. Then, $\mu(x \cdot c = b) = \mu(x = bc^{-1}) = 0$ by assumption. Therefore, $\int F_{\mu}^\nu \, d\nu_G = \int 0 \, d\nu_G = 0$.

**3.3. Idempotent measures.** We continue working in a theory expanding a group, and begin with some standard definitions.

**Definition 3.14.** Let $\mu \in \mathcal{M}_x(\mathcal{G})$.

1. We say that $\mu$ is idempotent if $\mu$ is $\ast$-Borel and $\mu \ast \mu = \mu$.
2. We say that $\mu$ is right-invariant if for any formula $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$ and any $a \in \mathcal{G}$, we have $\mu(\varphi(x)) = \mu(\varphi(x \cdot a))$.

**Definition 3.15.** Let $\mathcal{H}$ be a type-definable subgroup of $\mathcal{G}$, where $H(x)$ is the partial type defining the domain of $\mathcal{H}$ (which we associate with the closed set of types implying $H$). Then $\mathcal{H}$ is definably amenable if there exists a measure $\mu \in \mathcal{M}_x(\mathcal{G})$ such that $\mu(H) = 1$, and for any $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$ and $a \in \mathcal{H}$ we have $\mu(\varphi(x)) = \mu(\varphi(x \cdot a))$. In this case, we call $\mu$ right $\mathcal{H}$-invariant.
Remark 3.16. For NIP groups, existence of a right-invariant measure on $\mathcal{H}$ is equivalent to the existence of a left invariant measure on $\mathcal{H}$ (as well as a bi-invariant measure, see [K, Lemma 6.2]).

Proposition 3.17. Let $\mathcal{H}$ be a type-definable, definably amenable subgroup of $\mathcal{G}$, defined by a partial type $H(x)$. Suppose that $\mu \in M_x(\mathcal{G})$ is right $\mathcal{H}$-invariant. Then $\mu$ is idempotent. Moreover, if $\nu$ is another measure such that $\nu(H(x)) = 1$, then $(\mu, \nu)$ is $*$-Borel and $\mu * \nu = \mu$.

Proof. We show that for any measure $\nu \in M_x(\mathcal{G})$ such that $\nu(H(x)) = 1$, $(\mu, \nu)$ is $*$-Borel and $\mu * \nu = \mu$. For ease of notation, we will identify $\nu$ with $\nu_y$. Fix a formula $\varphi(x) \in L_x(\mathcal{G})$. Let $\mathcal{G}$ be a small elementary submodel of $\mathcal{G}$ containing the parameters of $H(x)$ and $\varphi(x)$. Fix some $q \in \text{sup}(\nu|_{\mathcal{G}}) \subseteq S_y(\mathcal{G})$, then $q \vdash H(y)$. If not, then $q \in S_y(\mathcal{G}) \setminus H(y)$. Since $H(y)$ is closed, $S_y(\mathcal{G}) \setminus H(y)$ is open, hence $S_y(\mathcal{G}) \setminus H(y) = \bigcup_{i \in I} \psi_i$ for some index set $I$ and $\psi_i \subseteq L_y(\mathcal{G})$. Then $\psi_i(y) \in q$ for some $i$ and since $q \in S(\nu|_{\mathcal{G}})$, we know that $\nu(\psi_i(y)) > 0$. But this is a contradiction since $\nu(H(y)) = 1$ and $\psi_i(y)$ is disjoint from $H(y)$. Therefore, if $b \in \mathcal{G}$ and $b = q$, then $b \in \mathcal{H}$. Now, the function $F_{\mu,\mathcal{G}}^\varphi(y) = \mu(\varphi(x \cdot b)) = \mu(\varphi(x))$ by right $\mathcal{H}$-invariance of $\mu$, hence $(\mu, \nu)$ is $*$-Borel. And $\mu * \nu = \mu$ as

$$\mu * \nu(\varphi(x)) = \int_{\text{sup}(\nu|_{\mathcal{G}})} F_{\mu,\mathcal{G}}^\varphi d\nu_G = \int_{\text{sup}(\nu|_{\mathcal{G}})} \mu(\varphi(x)) d\nu_G = \mu(\varphi(x)).$$

In particular, $(\mu, \mu)$ is Borel and $\mu * \mu = \mu$. \hfill $\square$

The expectation is that in tame situations, all idempotent measures are of this form for some type-definable subgroup. We will show that this is indeed the case when $\mathcal{G}$ is a stable group in Section 5, but for now we discuss some examples in which idempotent measures arise.

If $\mathcal{G}$ is a definably amenable group, and $\mathcal{H}$ is a type-definable subgroup of finite index (hence definable), then $\mathcal{H}$ is definably amenable (if $\mu$ is a right-invariant measure on $\mathcal{G}$, then $\mu_{\mathcal{H}}(\varphi(x)) := [\mathcal{G} : \mathcal{H}] \cdot \mu(\varphi(x) \cap H(x))$ gives a right-invariant measure on $\mathcal{H}$). This generalizes to subgroups of bounded index when $\mathcal{G}$ is NIP.

Proposition 3.18. Assume that $\mathcal{G}$ is definably amenable and NIP, and let $\mathcal{H}$ be a type-definable subgroup of $\mathcal{G}$ of bounded index. Then $\mathcal{H}$ is also definably amenable.

Proposition 3.18 follows from a slightly generalized construction of the $\mathcal{G}$-invariant measures $\mu_y$ from [E, L, K] in NIP groups. We will use some properties of the absolute type-definable connected component $\mathcal{G}^0$, the intersection of all type-definable subgroups of $\mathcal{G}$ of bounded index, and refer to the aforementioned texts for further details. To be compatible with our set up for convolutions, we work with $\mathcal{G}$ acting on the right. Let $\mathcal{G} \prec \mathcal{G}$ be a small model such that $\mathcal{G}^0$ is type-definable over $\mathcal{G}$. As usual, $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}^0$ is the surjective group homomorphism with $\pi(y)$ depending only on $\text{tp}(g/\mathcal{G})$. Then $\mathcal{G}/\mathcal{G}^0$ is a compact Hausdorff topological group with respect to the logic topology, i.e. a subset $X$ of $\mathcal{G}/\mathcal{G}^0$ is closed if $\pi^{-1}(X)$ is type-definable. The induced map $S_\pi(\mathcal{G}) \to \mathcal{G}/\mathcal{G}^0$ is continuous. With respect to this topology, closed subgroups of $\mathcal{G}/\mathcal{G}^0$ are in a bijective correspondence with type-definable subgroups of $\mathcal{G}$ of bounded index (equivalently, containing $\mathcal{G}^0$). Namely, if $K$ is a closed subgroup of $\mathcal{G}/\mathcal{G}^0$, then $\mathcal{H} := \pi^{-1}(K)$ is a type-definable set containing $\mathcal{G}^0 = \pi^{-1}(e_K)$, and is a group since $\pi$ is a group homomorphism (and vice versa).
Also, if \( \mathcal{H} \subseteq \mathcal{G} \) is type-definable, then \( K := \pi(\mathcal{H}) \subseteq \mathcal{G}/\mathcal{G}^{00} \) is a closed subgroup (as \( \pi : S_x(G) \to \mathcal{G}/\mathcal{G}^{00} \) is a closed map).

Recall that a global type \( p \in S_x(G) \) is strongly \( f \)-generic over \( G \) if \( p \cdot g \) is \( G \)-invariant for every \( g \in \mathcal{G} \). If \( \mathcal{G} \) is definably amenable and \( G \) is an arbitrary small model, there there exists a type strongly \( f \)-generic over \( G \) (see [1]). Moreover, as every right translate of a strong \( f \)-generic over \( G \) is again a strong \( f \)-generic over \( G \) and \( \mathcal{G}^{00} \) is a normal subgroup of \( G \), we can always find one with \( p(x) \vdash \mathcal{G}^{00}(x) \).

**Proof of Proposition 3.21.** Let \( K := \pi(\mathcal{H}) \), then \( \pi^{-1}(K) = \mathcal{H} \) (by the fourth isomorphism theorem for groups), hence \( K \) is a closed subgroup of \( \mathcal{G}/\mathcal{G}^{00} \). Denote by \( \nu \) the right-invariant Haar measure on Borel subsets of \( K \) normalized by \( \nu(K) = 1 \).

Let \( p \in S_x(G) \) be a strong \( f \)-generic over \( G \), with \( p \vdash \mathcal{G}^{00} \), so in particular \( p \vdash \mathcal{H} \) and \( p \) is \( \mathcal{G}^{00} \)-invariant. For a formula \( \varphi(x) \in L_x(\mathcal{G}) \), let

\[
A_{\varphi,p} := \{ g \in K : \varphi(x) \in p \cdot g \}.
\]

Then \( A_{\varphi,p} \) is a Borel subset of \( K \) (as \( A_{\varphi,p} = K \cap \{ g \in \mathcal{G}/\mathcal{G}^{00} : \varphi(x) \in p \cdot g \} \), and the latter set is Borel by [13] Proposition 5.6). We define

\[
\mu_{p,\nu}(\varphi(x)) := \nu(A_{\varphi,p}).
\]

Then we have the following.

- \( \mu_{p,\nu} \) is a Keisler measure with \( \mu_{p,\nu}(H) = 1 \).
  
  It is easy to check that \( \mu_{p,\nu} \) is a measure. And by regularity, \( \mu_{p,\nu}(H) = \inf \{ \mu_{p,\nu}(\psi(x)) : H(x) \vdash \psi(x), \psi(x) \in L_x(\mathcal{G}) \} \), and as \( p \vdash H \vdash \psi \) for all such \( \psi \), we have that \( A_{\psi,p} = K \) by definition, hence \( \mu_{p,\nu}(\psi) = 1 \).

- \( \mu_{p,\nu} \) is right \( \mathcal{H} \)-invariant (as \( \mu_{p,\nu}(\varphi(x) \cdot g) = \nu(A_{\varphi,g,p}) = \nu(A_{\varphi,p} \cdot \pi(g)) = \nu(A_{\varphi,p}) = \mu_{p,\nu}(\varphi(x)) \) by \( K \)-invariance of \( \nu \), as \( \pi(g) \in K \).

Hence \( \mathcal{H} \) is definably amenable, witnessed by \( \mu_{p,\nu} \).

**Question 3.19.** Is Proposition 3.18 true without the NIP assumption?

Classification of measures supported on finite subsets of \( \mathcal{G} \) follows from Wendel’s theorem.

**Proposition 3.20.** If \( \mu \) is a measure on \( \mathcal{G} \) whose support is a finite collection of realized types, then \( \mu \) is idempotent if and only if \( \mu = \frac{1}{|\mathcal{H}|} \sum_{a \in \mathcal{H}} \delta_a \) for some finite subgroup \( \mathcal{H} \) of \( \mathcal{G} \).

**Proof.** (\( \Rightarrow \)) is by Proposition 3.17

(\( \Rightarrow \)) Assume that \( S(\mu) = \{ a_1, ..., a_n \} = A \subseteq \mathcal{G} \). As \( \mu \) is idempotent, \( S(\mu) \) is closed under multiplication (if not, then there exists \( c \in \mathcal{G} \setminus A \) such that \( c = a_i \cdot a_j \) and \( c \) for some \( i, j \); then \( \mu(x = c) = 0 \), but \( \mu \ast \mu(x = c) > 0 \)). Therefore \( A \) is closed under products. As any finite subset of a group closed under products is a subgroup, \( A \) is a compact group, and \( \mu|_A \) is an idempotent measure on \( A \). Therefore, by [28] Theorem 1, \( \mu|_A \) is the unique Haar measure on the subgroup \( S(\mu|_A) \) of \( A \). But as \( S(\mu) = A \), we conclude that \( \mu = \frac{1}{|\mathcal{H}|} \sum_{a \in A} \delta_a \).

Finally, we observe a sufficient condition for idempotence to be preserved under convolution.

**Proposition 3.21.** (1) Assume that \( \mathcal{G} \) is abelian, \( \mu, \nu \in \mathcal{M}_x(\mathcal{G}) \) are idempotent, and \( \mu, \nu \otimes \)-commute. Then \( \mu \ast \nu \) is idempotent.
In particular, if $\mathcal{G}$ is NIP and abelian, and both $\mu, \nu$ are idempotent and dfs, then $\mu * \nu$ is idempotent and dfs.

Proof. Fix a formula $\varphi(x) \in \mathcal{L}(\mathcal{G})$ and assume that $G \preceq \mathcal{G}$ witnesses that both $(\mu, \nu, \varphi)$ and $(\nu, \mu, \varphi)$ are Borel. Then

$$\mu * \nu(\varphi(x)) = \mu_x \otimes \nu_y(\varphi(x, y)) = \nu_y \otimes \mu_x(\varphi(x, y)).$$

By change of variables and abelianity of $\mathcal{G}$, we can conclude

$$= \nu_x \otimes \mu_y(\varphi(y, x)) = \nu_x \otimes \mu_y(\varphi(x, y)) = \nu * \mu(\varphi(x)).$$

Now, let $\lambda = \mu * \nu$. Using associativity of $\otimes$,

$$\lambda * \lambda = \mu * \nu * \mu * \nu = \mu * \mu * \nu * \nu = \mu * \nu = \lambda.$$

(2) follows from (1), Facts 2.2(2b) and 4.1d and Proposition 3.12. \(\square\)

4. Supports of idempotent measures

In this section, we will show that if $\mu$ is definable, invariantly supported (see Definition 3.4) and idempotent, then $(S(\mu), *)$ is a compact, left-continuous semigroup with no closed two-sided ideals. This assumption is satisfied when $\mu$ is a dfs measure in an arbitrary theory (by Fact 2.10(1)), and when $\mu$ is an arbitrary definable measure in an NIP theory (by Fact 2.10(3)).

We begin by considering two examples, which illustrate in particular that the support of an idempotent dfs Keisler measure need not be a group in general.

Example 4.1. Let $T = T_{doag}$ be the complete theory of a divisible ordered abelian group in the language $\{+, -, 0, 1\}$. Let $\mathcal{G}$ be a monster model of $T$ and consider $G := \mathbb{Q}$ as an elementary substructure in the natural way. Let $p_\infty$ be the unique global type finitely satisfiable in $G$ and extending $\{x > a : a \in \mathbb{Q}\}$. Let $p_{-\infty}$ be the unique global type finitely satisfiable in $G$ and extending $\{x < a : a \in \mathbb{Q}\}$. Let $\mu := \frac{1}{2} \delta_{p_{-\infty}} + \frac{1}{2} \delta_{p_\infty}$, we claim that $\mu, \delta_{p_{-\infty}}, \delta_{p_\infty} \in \mathcal{M}_0(\mathcal{G})$ are idempotent. By Proposition 3.12, the product $\delta_\alpha * \delta_\beta$ for $\alpha, \beta \in \{p_{-\infty}, p_\infty\}$ is finitely satisfiable in $\mathbb{Q}$. Then, using Proposition 3.12 it is not hard to verify the following calculation:

$$\mu * \mu = \left(\frac{1}{2} \delta_{p_{-\infty}} + \frac{1}{2} \delta_{p_\infty}\right) * \left(\frac{1}{2} \delta_{p_{-\infty}} + \frac{1}{2} \delta_{p_\infty}\right) = \frac{1}{4} \left(p_{-\infty} * p_{-\infty}\right) + \frac{1}{4} \left(p_{-\infty} * p_\infty\right) + \frac{1}{4} \left(p_\infty * p_{-\infty}\right) + \frac{1}{4} \left(p_\infty * p_\infty\right)$$

$$= \frac{1}{4} p_{-\infty} + \frac{1}{4} p_\infty + \frac{1}{4} p_{-\infty} + \frac{1}{4} p_\infty = \frac{1}{2} p_{-\infty} + \frac{1}{2} p_\infty = \mu.$$
of $\bar{\mu}$ in $G$). As the types are determined by the cuts in the circular order, it follows that for every $a \in S^1$ there are exactly two types $a_+, a_-(x) \in S(\bar{\mu})$ determined by whether $C(a + \epsilon, x, b)$ holds for every infinitesimal $\epsilon$ and $b \in G$, or $C(b, x, a - \epsilon)$ holds for every infinitesimal $\epsilon$ and $b \in G$, respectively. It follows that $(\sup(\bar{\mu}), \ast) \cong S^1 \times \{+,-\}$ with multiplication defined by:

$$a_\delta \ast b_\gamma = (a \cdot b)_\delta$$

for all $a, b \in S^1$ and $\delta, \gamma \in \{+,-\}$. Again, $(S(\mu), \ast)$ is not a group.

Next we establish various properties of $(S(\mu), \ast)$ when $\mu$ is a global idempotent measure which is definable and invariantly supported. Given $S_1, S_2 \subseteq S_\varepsilon(G)$, we write $S_1 \ast S_2 := \{p_1 \ast p_2 \in S_\varepsilon(G) : p_i \in S_i\}$ (under the assumption that all such products are defined, i.e. assuming $(p_1, p_2)$ is Borel for all $p_i \in S_i$). The assumption of being invariantly supported in the lemmas below is only needed to ensure that $S(\mu) \ast S(\mu)$ is defined (Fact 3.10).

**Proposition 4.3.** Let $\mu, \nu \in M_\varepsilon(G)$. Assume that $\mu$ is definable, and both $\mu$ and $\nu$ are invariantly supported. Then:

1. $S(\mu) \ast S(\nu) \subseteq S(\mu \ast \nu)$;
2. $S(\mu) \ast S(\nu)$ is a dense subset of $S(\mu \ast \nu)$.

**Proof.** (1) Assume that $p \in S(\mu), q \in S(\nu)$, and let $\varphi(x) \in p \ast q$. Choose $G \prec G$ such that $\mu$ is definable over $G$, $p, q$ are finitely satisfiable in $G$, and $G$ contains all the parameters from $\varphi$. We need to show that $\mu \ast \nu(\varphi(x)) > 0$. Now,

$$\mu \ast \nu(\varphi(x)) = \int_{S(\nu|G)} F^{\varphi}_{\mu,G} d\nu_{G}$$

Since $\mu$ is definable, the map $F^{\varphi}_{\mu,G} : S(\nu|G) \to [0, 1]$ is continuous. Therefore, it suffices to find some $r \in S(\nu|G)$ such that $F^{\varphi}_{\mu,G}(r) > 0$. Consider $r := q|G$. Then, $F^{\varphi}_{\mu,G}(q|G) = \mu(\varphi(x \cdot b))$, where $b \models q|G$. Then, $\varphi(x \cdot b) \in p$ and since $p \in S(\mu)$, we have that $\mu(\varphi(x \cdot b)) > 0$. Hence, $F^{\varphi}_{\mu}(q|G) > 0$ and so $\mu \ast \nu(\varphi(x)) > 0$.

(2) By (1), we already know that $S(\mu) \ast S(\nu) \subseteq S(\mu \ast \nu)$. Fix some $r \in S(\mu \ast \nu)$ and a formula $\varphi(x) \in r$. Assume that $\varphi(x) \in r$. We need to find $p \in S(\mu)$ and $q \in S(\nu)$ such that $\varphi(x) \in p \ast q$. Choose $G$ such that $\mu$ is definable over $G$, all types in $S(\mu),$ $S(\nu)$ are invariant over $G$, and $G$ contains the parameters of $\varphi(x)$. Since $\varphi(x) \in r$ and $r$ is in the support of $\mu \ast \nu$, we know that $\mu \ast \nu(\varphi(x)) > 0$. Therefore, $\int_{S(\nu|G)} F^{\varphi}_{\mu,G} d\nu_{G} > 0$, and so there exists some $t \in S(\nu|G)$ such that $F^{\varphi}_{\mu,G}(t) > 0$. If $c \models t$, then $\mu(\varphi(x \cdot c)) > 0$. So, by Proposition 2.8(1), there exists $p \in S(\mu)$ such that $\varphi(x \cdot c) \in p$. By Proposition 2.8 we let $q \in S(\nu)$ be such that $q|G = t$. By construction, we then observe that $\varphi(x) \in p \ast q$. \hfill \Box

**Corollary 4.4.** Assume that $\mu$ is definable, invariantly supported and idempotent. Then $(S(\mu), \ast)$ is a compact Hausdorff (with the subspace topology) semigroup which is left-continuous, i.e. the map $- \ast q : S(\mu) \to S(\mu)$ is continuous for each $q \in S(\mu)$.

**Proof.** By Proposition 4.3(2), $S(\mu)$ is a compact Hausdorff space. By Proposition 4.3, $S(\mu) \ast S(\mu) \subseteq S(\mu \ast \mu) = S(\mu)$. Now, choose some $G \prec G$ such that $\mu$ is definable over $G$, and all types in $S(\mu)$ are invariant over $G$. Then $(S(\mu), \ast)$ is a sub-semigroup of $(S^\mu(G), \ast)$ and $\ast$ is left-continuous by Fact 3.10. \hfill \Box

We now define some global functions which mimic the map $y \mapsto \int f(x \cdot y) d\mu(x)$. 
Definition 4.5. Let $\mu \in \mathfrak{M}_x(G)$ be definable, and fix $\varphi(x) \in \mathcal{L}_x(G)$. We then define the global function $D_{\mu}^{\varphi} : S_{\mu}(G) \to [0,1]$ via $p \mapsto \mu(\varphi(x \cdot c))$, for some/any $c \models p|_G$ and $G \prec G$ small and containing the parameters of $\varphi(x)$.

Note that for any formula $\varphi(x) \in \mathcal{L}_x(G)$, the map $D_{\mu}^{\varphi}$ is continuous: $D_{\mu}^{\varphi} = F_{\mu,G}^\varphi \circ r$, where $r : S_{\mu}(G) \to S_{\mu}(G)$ is the restriction map, and $F_{\mu,G}^\varphi$ is continuous by definability of $\mu$. The next two results are adapted from Glicksberg’s work on semi-topological semigroups into the general model theory context. In particular, see [12, 11].

Proposition 4.6. Let $\mu \in \mathfrak{M}_x(G)$ be definable, invariantly supported and idempotent, and $\varphi(x) \in \mathcal{L}_x(G)$ arbitrary. Assume that $D_{\mu}^{\varphi} |_{S(\mu)}$ attains a maximum at $q \in S(\mu)$ (exists as this is a continuous function on a compact set). Then for any $p \in S(\mu)$, we have that $D_{\mu}^{\varphi}(q) = D_{\mu}^{\varphi}(p \ast q)$.

Proof. Fix a small model $G_0 \prec G$ such that $\mu$ is definable over $G_0$, and $G_0$ contains the parameters of $\varphi(x)$. Let $b \models q|_{G_0}$ and let $\theta(x;y) := \varphi(x \cdot y \cdot b)$. Now fix a larger submodel $G \prec G$ such that $G_0 b \subseteq G$. Let $\delta := \mu(\varphi(x \cdot b))$. Observe that then for any $t \in S(\mu|_G)$, $a \models t$, and $t \in S(\mu)$ such that $t|_G = t$, we have $F_{\mu,G}^\varphi(t) = \mu(\varphi(x \cdot a \cdot b) = \mu(\varphi(x \cdot (a \cdot b))) = D_{\mu}^{\varphi}(t \ast q) \leq D_{\mu}^{\varphi}(q) = \delta$ (by the assumption on $q$). We conclude that for any $t \in S(\mu|_G)$, $F_{\mu,G}^\varphi(t) \leq \delta$. On the other hand,

$$\delta = D_{\mu}^{\varphi}(q) = \mu(\varphi(x \cdot b)) = \mu \ast \mu(\varphi(x \cdot b)) = \mu \ast \mu_{\mu}(\theta(x; y))$$

$$= \int_{S(\mu|_G)} F_{\mu,G}^\varphi d\mu_G.$$ 

Therefore, $F_{\mu}^\varphi = \delta$ almost everywhere (with respect to $\mu_G$). Since both maps are continuous, they are equal on $S(\mu|_G)$. Finally, for any $p \in S(\mu)$ and $a \models p$, we have:

$$D_{\mu}^{\varphi}(q) = \delta = F_{\mu,G}^\varphi(p|_G) = \mu(\varphi((x \cdot a \cdot b)) = \mu(\varphi(x \cdot (a \cdot b))) = D_{\mu}^{\varphi}(p \ast q),$$

as wanted. \hfill $\square$

Theorem 4.7. Let $\mu \in \mathfrak{M}_x(G)$ be definable, invariantly supported and idempotent. Let $I \subseteq S(\mu)$ be a closed two-sided ideal. Then, $I = S(\mu)$.

Proof. If $I$ is dense in $S(\mu)$, then $I = S(\mu)$. So we may assume that $I$ is not dense in $S(\mu)$.

Therefore, there exists some $\varphi(x) \in \mathcal{L}_x(G)$ such that $\varphi(x) \cap S(\mu) \neq \emptyset$ and $\varphi(x) \cap I = \emptyset$. Let $G \prec G$ contain the parameters of $\varphi$, and such that $\mu$ is definable and invariantly supported over $G$.

Claim 4.8. There exists some $q \in S(\mu)$ such that $D_{\mu}^{\varphi}(q) > 0$.

Proof. Assume not. Let $p, q \in S(\mu)$ be arbitrary. Let $b \models q|_{G_0}, a \models p|_{G_0}$. Then $\mu(\varphi(x \cdot b)) = D_{\mu}^{\varphi}(q) = 0$ by assumption, hence $\models \neg \varphi(a \cdot b)$ as $p \in S(\mu)$, so $\varphi(x) \notin p \ast q$.

Consider now the continuous characteristic function $\chi_\varphi : S(\mu) \to \{0,1\}$. By Proposition 4.3 (2) and the previous paragraph, $\chi_\varphi$ vanishes on a dense subset $S(\mu) \ast S(\mu)$ of $S(\mu)$, hence $\chi_\varphi$ vanishes on $S(\mu)$. But this contradicts the choice of $\varphi$. \hfill $\square$
Proposition 4.14. Assume that Example 4.13. For example, the measure \( \tilde{\mu} \) is idempotent and minimal (i.e. Definition 4.12. We let \( \mu \) for any \( p, q \) for any \( \mu \). Then, there exists a minimal left ideal \( I \) closed). We let \( J \) (which is automatically continuous). Then, there exists a minimal left ideal \( J \) (which is automatically closed). We let \( J(I) = \{ i \in I : i^2 = i \} \) be the set of idempotents in \( I \).

(1) \( J(I) \) is non-empty.
(2) For every \( p \in I \) and \( i \in J(I) \), we have that \( p \cdot i = p \).
(3) \( I = \bigcup \{ i \cdot I : i \in J(I) \} \), where the union is over disjoint sets, and each set \( i \cdot I \) is a group with identity \( i \).
(4) \( I \cdot q \) is a minimal right ideal for all \( q \in S \).

Assume that \( \mu \in \mathfrak{M}_x(G) \) is definable, invariantly supported and idempotent. Then \((S(\mu), \ast)\) is a semigroup satisfying the assumption of Fact 4.11 by Corollary 4.3.

Definition 4.12. We let \( I_\mu \) denote the minimal (closed) left ideal of \((S(\mu), \ast)\) (it exists by Fact 4.11. We say that \( \mu \) is minimal if \( I_\mu = S(\mu) \).

In particular, if \( \mu \) is minimal, then \( S(\mu) \) is a disjoint union of subgroups.

Example 4.13. For example, the measure \( \tilde{\mu} \) considered in Example 4.12 is minimal.

Proposition 4.14. Assume that \( \mu \in \mathfrak{M}_x(G) \) be definable, invariantly supported, idempotent and minimal (i.e. \( I_\mu = S(\mu) \)). Let \( \varphi(x) \in \mathcal{L}_x(G) \) be any formula. Then for any \( p, q \in S(\mu) \), we have that \( D_\mu(p) = D_\mu(q) \).
Proof. By Fact 4.11, \( S(\mu) = \bigcup \{ i \in S(\mu) : i \in J(I_\mu) \} \). By continuity, \( D^\varphi_p \) attains a maximum at some \( p \in S(\mu) \). Let now \( q \in S(\mu) = I_\mu \) be arbitrary. Then \( q \in i \ast I_\mu \) for some \( i \in J(I_\mu) \). Also \( i \ast p \in i \ast I_\mu \) as \( I_\mu = S(\mu) \). As \( i \ast I_\mu \) is a group by Fact 4.11(3), there exists some \( r \in i \ast I_\mu \) such that \( r \ast (i \ast p) = q \). But then, applying Proposition 4.10 we have

\[
D^\varphi_p(q) = D^\varphi_p((r \ast (i \ast p)) = D^\varphi_p(r \ast (i \ast p)) = D^\varphi_p(q).
\]

As \( q \in S(\mu) \) was arbitrary, this shows the proposition. \( \square \)

**Proposition 4.15.** Assume that \( \mu \in \mathcal{M}_e(G) \) be definable, invariantly supported, idempotent and minimal. Then for every \( \varphi(x) \in L_x(G) \), \( \mu(\varphi(x)) = D^\varphi_p(p) \) for any \( p \in S(\mu) \).

**Proof.** Assume not. By Proposition 4.14 and replacing \( \varphi(x) \) by \( \neg \varphi(x) \) if necessary, we may assume that \( \mu(\varphi(x)) > D^\varphi_p(i) \), where \( i \) is an idempotent in \( S(\mu) \). Then \( \mu(\varphi(x) \land \neg \varphi(x \cdot b)) > 0 \), where \( b \vdash i \vdash G \) and \( G \prec \mathcal{G} \) is chosen as usual. Hence there exists \( q \in S(\mu) \) such that \( \varphi(x) \land \neg \varphi(x \cdot b) = q \). Then \( \varphi(x) \in q \), and \( \neg \varphi(x) \in q \ast i \). However, \( q \ast i = q \) by Fact 4.11(2), and so we have \( \varphi(x), \neg \varphi(x) \in q \) — a contradiction. \( \square \)

A direct translation of the previous proposition then says that minimal idempotent measures are “generically” right-invariant on their supports.

**Corollary 4.16.** Assume that \( \mu \in \mathcal{M}_e(G) \) be definable, invariantly supported, idempotent and minimal. Let \( \varphi(x; b) \in L_x(G) \). Then, for any \( a \in G \) such that \( tp(a/\mathcal{G}b) \in S(\mu|_{\mathcal{G}b}) \), we have \( \mu(\varphi(x)) = \mu(\varphi(x \cdot a)) \).

Finally, we record a corollary for the case when the group \( G \) is stable and abelian.

**Remark 4.17.** \( I_\mu = S(\mu) \) if and only if for every \( p, q \) in the \( S(\mu) \) there exists \( r \in S(\mu) \) such that \( r \ast q = p \).

The following corollary is a direct consequence of Glicksberg’s theorem for semitopological semigroups (11) (note that unless the group is abelian, we only have continuity of \( * \) on the left, so we were not in the context of Glicksberg’s theorem in the earlier considerations).

**Corollary 4.18.** If \( G \) is stable, abelian and \( \mu \in \mathcal{M}_e(G) \) is idempotent, then \( \text{sup}(\mu) \) is a compact Hausdorff topological group.

**Proof.** Note that \( \mu \) is automatically dfs by Fact 2.23), hence the results of this section apply to it. We see that \( (S(\mu), *) \) is commutative, as in Proposition 3.21. Then \( * \) is both left and right-continuous. Hence \( I_\mu = S(\mu) \) by Theorem 4.7. But this is equivalent to: for every \( p, q \in S(\mu) \) there exists \( r \in S(\mu) \) such that \( r \ast q = p \). By commutativity of \( * \) and Fact 4.11, this implies that \( S(\mu) \) is a group. Finally, by a classical theorem of Ellis [8], separate continuity of multiplication implies joint continuity for (locally) compact groups. \( \square \)

Using this corollary, we can quickly describe idempotent measures in strongly minimal groups.

**Example 4.19.** Let \( G \) be a strongly minimal group. Then the idempotent measures are precisely of the following form:

1. Haar measures on finite subgroups of \( G \);

2. \( \mu = \mu(G,G) \);
(2) \( \delta_p \), where \( p \) is the unique non-algebraic type in \( S_x(G) \).

Proof. Assume that \( G \) is \( \omega \)-stable and abelian, and let \( \mu \) be an idempotent measure. By Fact 2.3c \( \mu = \sum_{i \in \omega} r_i \cdot p_i \) for some \( p_i \in S_x(G) \) and some \( r_i \in \mathbb{R}_{\geq 0} \) with \( \sum_{i \in \omega} r_i = 1 \). By Corollary 4.18 \( S(\mu) = \{ p_i : i \in \omega \} \) is a countable compact group, and every countable compact group must be finite (using existence of finite Haar measure). So in fact \( \mu = \sum_{i<n} r_i \cdot p_i \) for some \( n \in \omega \). As \( \mu|_{S(\mu)} \) is idempotent, we get that \( r_i = \frac{1}{n} \) as in Proposition 5.1.

By strong minimality, let \( p \in S_x(G) \) be the unique non-algebraic type. Note that \( p \) is clearly both left and right \( G \)-invariant, hence \( p \cdot \delta_a = \delta_a \cdot p = p \cdot p = p \) for any \( a \in G \) (by Proposition 5.17).

If \( p \not\in S(\mu) \), then \( S(\mu) \) is a subgroup of \( G \) by Proposition 3.12 and we are in the first case. If \( S(\mu) = \{ p \} \), then we are in the second case.

Otherwise \( \mu = \sum_{1 \leq i \leq n} \frac{1}{n+1} \cdot \delta_{a_i} + \frac{1}{n+1} \cdot p \) for some \( a_i \in G \) and \( n \geq 1 \). Then, using Proposition 3.12

\[
\mu = \mu \ast \mu = \sum_{1 \leq i,j \leq n} \frac{1}{n+1} \cdot \delta_{a_i \cdot a_j} + 2 \cdot \frac{n}{n+1} \cdot \delta_{a_i} + \frac{1}{(n+1)^2} \cdot p
\]

Consider the formula \( \varphi(x) := \bigwedge_{1 \leq i,j \leq n} x \neq a_i \cdot a_j \), then \( \varphi(x) \in p \). Hence on the one hand \( \mu(\varphi(x)) = \frac{1}{n+1} \), and on the other \( \mu(\varphi(x)) = \frac{2n+1}{(n+1)^2} \). But \( \frac{1}{n+1} = \frac{2n+1}{(n+1)^2} \) for any \( n \geq 1 \), a contradiction.

This example is generalized to arbitrary stable groups in the next section.

5. Idempotent measures in stable groups

In this section we classify idempotent measures on a stable group, demonstrating that they are precisely the invariant measures on its type-definable subgroups. Our proof relies on the results of the previous section and a variant of Hrushovski’s group chunk theorem due to Newelski [17]. We will assume some familiarity with the theory of stable groups (see [21] or [27] for a general reference). As before, \( G \) is amonster model for a theory extending a group.

5.1. Stabilizers of definable measures.

Definition 5.1. Given a measure \( \mu \in \mathcal{M}_x(G) \), we consider the following (left) stabilizer group associated to it:

\[
\text{Stab}(\mu) := \{ g \in G : g \cdot \mu = \mu \}
= \{ g \in G : \mu(\varphi(x)) = \mu(\varphi(g \cdot x)) \text{ for all } \varphi(x) \in \mathcal{L}(G) \}.
\]

Below we use the characterization of definability of a measure from Fact 2.3, and we follow the notation there.

Definition 5.2. Assume that \( \mu_x \in \mathcal{M}_x(G) \) is definable over a small model \( G \prec G \).

(1) Fix a formula \( \varphi(x; y) \in \mathcal{L} \) and \( n \in \mathbb{N}_{>0} \). We write \( \varphi'(x; y, z) \) to denote the formula \( \varphi(z; x; y) \), and given \( i \in I_n \), we write

\[
\Phi^{c',+}_{\geq i}(y, z) := \bigvee_{j \in I_n, j \geq i} \Phi^{c',+}_j(y, z).
\]
(2) Consider the following formula with parameters in $G$ (where $e$ is the identity of $G$):

$$\forall y \bigwedge_{i \in I_n, s \geq \frac{n}{3}} \left( \Phi_{\geq i}^{\geq 1}(y, e) \rightarrow \Phi_{\geq \left(i - \frac{n}{3}\right)}^{\geq 1}(y, z) \wedge \Phi_{\geq i}^{\geq 1}(y, z) \rightarrow \Phi_{\geq \left(i - \frac{n}{3}\right)}^{\geq 1}(y, e) \right).$$

(3) We define the following partial type over $G$:

$$\text{Stab}_\mu(z) := \bigwedge_{\varphi(x, y) \in \mathcal{L}, n \in \mathbb{N}_{>0}} \text{Stab}_{\mu, \varphi}(z).$$

Proposition 5.3. Let $\mu \in \mathcal{M}_2(G)$ be definable. Then $\text{Stab}(\mu) = \text{Stab}_\mu(G)$.

Proof. Assume $g \notin \text{Stab}(\mu)$. Then there exist some $\varphi(x, y) \in \mathcal{L}$ and $b \in G^y$ such that taking $r := \mu(\varphi(x, b)) = \mu(\varphi(x, b, e))$ and $s := \mu(\varphi(g \cdot x, b)) = \mu(\varphi(x, b, g))$ we have $r \neq s$. Say $r > s$ (the case $r < s$ is similar). We choose $n \in \mathbb{N}_{>0}$ large enough so that $|r - s| \geq \frac{4}{n}$ (so in particular $r \geq \frac{3}{n}$). As $\{\Phi_{\geq i}^{\geq 1}(G) : i \in I_n\}$ is a covering of $G^y$ by Fact 2.3a, there is some $i \in I_n$ such that $\models \Phi_{\geq i}^{\geq 1}(b, e)$, so particular $\models \Phi_{\geq i}^{\geq 1}(b, e)$. Hence $|r - i| < \frac{1}{n}$ by Fact 2.3b (hence $i \geq \frac{3}{n}$). If $\models \Phi_{\geq \left(i - \frac{1}{n}\right)}^{\geq 1}(b, g)$, then by Fact 2.3b again we must have $\mu(\varphi(x, b, g)) > i - \frac{1}{n}$, so $s > i - \frac{1}{n}$, and $r - s < \frac{4}{n}$, contradicting the choice of $n$. Hence $g \notin \text{Stab}_{\mu, \varphi}(z)$.

Assume $g \in \text{Stab}(\mu)$, and let $\varphi(x, y), b \in G^y, n \geq 1$ and $i \geq \frac{3}{n}$ in $I_n$ be arbitrary. Assume that $\models \Phi_{\geq i}^{\geq 1}(b, e)$ holds, then by Fact 2.3b we have $\mu(\varphi(x, b, e)) > i - \frac{1}{n}$.

5.2. Stable groups and group chunks. As before, $T$ is a theory extending a group in a language $\mathcal{L}$, and we let $G$ be a monster model of $T$. In this section we review some results on stable groups that will be needed for our purpose.

Fact 5.4. (see e.g. [20] Fact 1.8] + [3]) Let $G$ be a stable group and $G \prec G$ a small model. Let $H$ be a subgroup of $G$ type-definable over $G$ (by a partial type $H(x)$ over $G$). Let $S_H(G) := \{p \in S(G) : p(x) \models H(x)\}$ be the set of types over $G$ concentrated on $H$. Then the following hold.

1. For $p, q \in S_H(G)$, we have that $p \ast q$ is equal to $tp(a-b/G)$, where $a \models p, b \models q$ and $a \perp_G b$ (in the sense of forking independence).
2. The semigroup $(S_H(G), \ast)$ has a unique minimal closed left ideal $I$ (also the unique minimal closed right ideal) which is already a subgroup of $(S_H(G), \ast)$.
3. $I$ is precisely the generic types of $H$, and with its induced topology $I$ is a compact topological group (isomorphic to $H^0$).
4. $H$ admits a unique left invariant Keisler measure $\mu$ (which is also the unique right invariant Keisler measure) with $S(\mu) = I$. Viewing $\mu$ as a regular Borel measure on $S_H(G)$ and restricting to the closed set $I$, $\mu \mid_{S(\mu)}$ coincides with the Haar measure on $I$. 


In what follows, we work in the stable theory \( T_G \) in the language \( L_G \) with all of the elements of \( G \) named by new constants (obviously, \( T \) stable implies \( T_G \) is stable), and let \( \hat{G} > G \) be a larger monster model of \( T_G \). We will be following the notation from [17].

**Definition 5.5.** (1) Throughout this section, \( \Delta \) will denote a finite invariant set of formulas, i.e. formulas of the form \( \varphi(u \cdot x \cdot v, \bar{y}) \in L_G \) (so a right or left translate of an instance of \( \varphi \) is again an instance of \( \varphi \)).

(2) We write \( R_\Delta \) to denote Shelah’s \( \Delta \)-rank, note that it is invariant under two-sided translation since \( \Delta \) is.

(3) For \( P \subseteq S_x(\hat{G}) \), we let \( \text{cl}(P) \) denote the topological closure of \( P \), and \(*P\) denote the closure of \( P \) under \(*\).

(4) For \( P \subseteq S_x(\hat{G}) \), let \( \text{gen}(P) \) denote the set of \( r \in \text{cl}(\ast P) \) such that there is no \( q \in \text{cl}(\ast P) \) with \( R_\Delta(r) \leq R_\Delta(q) \) for all \( \Delta \) and \( R_\Delta(r) < R_\Delta(q) \) for some \( \Delta \).

(5) For \( P \subseteq S_x(\hat{G}) \), let \( \langle P \rangle \) denote the smallest \( \hat{G} \)-type definable subgroup of \( \hat{G} \) containing \( P(\hat{G}) \), where \( P(\hat{G}) = \{ b \in \hat{G} : b \models p \text{ for some } p \in P \} \).

In the following two facts, \( \hat{G} \) is viewed as a small elementary submodel of the stable group \( \hat{G} \models T_G \).

**Fact 5.6.** (1) [17] Fact 2.1] If \( P \subseteq S_x(\hat{G}) \) is non-empty, then \( \text{gen}(P) \) is a non-empty closed subset of \( S_x(\hat{G}) \).

(2) [17] Lemma 2.2] \( R_\Delta(p \ast q) \geq R_\Delta(p), R_\Delta(q) \) for any \( p, q \in S_x(\hat{G}) \) and \( \Delta \) (this follows by the symmetry of forking, invariance of \( R_\Delta \) under two-sided translations, and the fact that forking is characterized by drop in rank).

The following fact is [17] Theorem 2.2] applied in \( T_G \). It is stated there for strong types over \( \emptyset \), which implies our statement as the elements of a small model \( G \prec \hat{G} \) are named by constants.

**Fact 5.7.** (T stable) Let \( P \subseteq S_x(\hat{G}) \) be non-empty set of types. Then

\[
\langle P \rangle = \left\{ a \in \hat{G} : \text{tp}(a/\hat{G}) \ast \text{gen}(P) = \text{gen}(P) \text{ setwise} \right\}
\]

is a \( \hat{G} \)-type definable subgroup of \( \hat{G} \) and \( \text{gen}(P) \) is precisely the set of generic types of \( \langle P \rangle \) over \( \hat{G} \).

5.3. Classification of idempotent measures. We are ready to prove the main result of this section.

**Theorem 5.8.** Let \( G \) be a monster model of \( T \), and let \( \mu \in M_\omega(G) \) be a global Keisler measure (in particular, \( \mu \) is dfs by Fact 2.2(3a)). Then the following are equivalent:

1. \( \mu \) is idempotent;
2. \( \mu \) is the unique right-invariant (and also the unique left-invariant) measure on the type-definable subgroup \( \text{Stab}(\mu) \) of \( G \).

**Proof:** (2) implies (1) by Proposition 5.14, and we show that (1) implies (2).

Let \( \mu \in M_\omega(G) \) be an idempotent measure, by Fact 2.2(3a) \( \mu \) is definable over some small model \( G \prec G \) by Proposition 5.3.

By Corollary 4.4, \( S(\mu) \) is a closed subset of \( S_x(\hat{G}) \) and is closed under \(*\), hence \( \text{cl}(\ast S(\mu)) = S(\mu) \) and \( \text{gen}(S(\mu)) \subseteq S(\mu) \).
We claim that \( \text{gen}(S(\mu)) \) is a two-sided ideal in \((S(\mu),\ast)\). Indeed, let \( r \in \text{gen}(S(\mu)) \) and \( q \in S(\mu) \) be arbitrary. If \( r \ast q \) is not in \( \text{gen}(S(\mu)) \), then there exists some \( p \in S(\mu) \) with \( R(\Delta(p)) \geq R(\Delta(r \ast q)) \geq R(\Delta(r)) \) for all \( \Delta \) and some inequality strict (by Fact 5.6(2)), contradicting \( r \in \text{gen}(S(\mu)) \). But also if \( q \ast r \) is not in \( \text{gen}(S(\mu)) \), then there exists some \( p \in S(\mu) \) with \( R(\Delta(p)) \geq R(\Delta(q \ast r)) \geq R(\Delta(r)) \) and some inequality strict, again by Fact 5.6(2), contradicting \( r \in \text{gen}(S(\mu)) \). Hence \( \text{gen}(S(\mu)) = S(\mu) \) by Theorem 4.7.

We now fix a larger monster model \( \hat{\mathcal{G}} \supset \mathcal{G} \) as above (and view \( \mathcal{G} \) as a small elementary submodel of it). Then, by Fact 5.7 we have that

\[
\hat{\mathcal{H}} := \langle S(\mu) \rangle = \{ a \in \hat{\mathcal{G}} : a \models p \text{ for some } p \in S(\mu) \}
\]

is a \( \mathcal{G} \)-type-definable subgroup of \( \hat{\mathcal{G}} \) and \( S(\mu) = \text{gen}(S(\mu)) \) is precisely the set of generic types of \( \hat{\mathcal{G}} \) restricted to \( \mathcal{G} \). Note that the definition of \( \hat{\mathcal{H}} \) a priori uses all of the parameters in \( \mathcal{G} \), and we need to argue that it can be defined over a subset of \( \mathcal{G} \) that is small with respect to \( \mathcal{G} \). Let \( \mathcal{H}(x) \) be a partial type over \( \mathcal{G} \) defining \( \hat{\mathcal{H}} \), i.e. \( H(\hat{\mathcal{G}}) = \hat{\mathcal{H}} \). Given \( p \in S_\mu(\mathcal{G}) \), we let \( \hat{p} \in S_\mu(\hat{\mathcal{G}}) \) be its unique \( \mathcal{G} \)-definable extension, and let \( \hat{\mu} \in \mathcal{M}_x(\hat{\mathcal{G}}) \) be the unique \( \mathcal{G} \)-definable extension of \( \mu \) (by Fact 2.4). We have the following sequence of observations.

1. \( p \ast q = r \iff \hat{p} \ast \hat{q} = \hat{r} \) for any \( p, q, r \in S_\mu(\mathcal{G}) \).
2. The same holds for measures, in particular \( \hat{\mu} \) is an idempotent of \( (\mathcal{M}_x(\hat{\mathcal{G}}), \ast) \).
3. \( \text{Stab}_\mu(\hat{\mathcal{G}}) = \text{Stab}(\hat{\mu}) \) (by Proposition 5.3 and definability of the measure).
4. \( S(\hat{\mu}) = \{ \hat{p} : p \in S(\mu) \} \).
5. The generics of \( \mathcal{H}(x) \) over \( \hat{\mathcal{G}} \) are precisely \( \{ \hat{p} : p \text{ is a generic of } \mathcal{H} \text{ over } \mathcal{G} \} \).
6. By stability, every generic \( r \) of \( \mathcal{H}(x) \) over \( \hat{\mathcal{G}} \) does not fork over \( \mathcal{G} \), so it is definable over \( \mathcal{G} \) and \( r | \mathcal{G} \) is a generic of \( H(x) \) over \( \mathcal{G} \), hence \( r = (r | \mathcal{G}) \).
7. Hence \( S(\hat{\mu}) \) is precisely the set of the generics of \( \mathcal{H}(x) \) over \( \hat{\mathcal{G}} \), in particular \( (S(\hat{\mu}), \ast) \) is a topological group by Fact 5.4(3).
8. Then \( \hat{\mu} \) restricted to \( (S(\hat{\mu}), \ast) \) (viewed as a regular Borel measure) is right \( \ast \)-invariant.

By (6), \( (S(\hat{\mu}), \ast) \) is a group, so for any \( p \in S(\hat{\mu}) \), \( p^{-1} \) is well-defined. By regularity, it suffices to check \( \ast \)-invariance for formulas. Let \( \varphi(x, \bar{b}) \in \mathcal{L}_x(\hat{\mathcal{G}}) \). Then for any \( p \in S(\hat{\mu}) \) we have

\[
\hat{\mu}(\varphi(x, \bar{b}) \ast p) = \hat{\mu}(\{ q \ast p : \varphi(x, \bar{b}) \in q \})
\]
group on finitely satisfiable measures in NIP theories. Recall that $M \cdot \mu \rightarrow \pi$ group action described above as $\pi E$ from $M$ of measures carries a natural structure of a real topological vector space induced via $G$ and $G$. When the action map $\pi \{ s \}$ be the space of functions from $s X$. For the rest of this section, we will denote elements of conv($G \rangle$ is (cl ($\pi$ is continuous. Therefore, we can consider the Ellis semigroup of this semigroup action, namely $E (\mathcal{M}_x (G, G), conv(G))$. 

Fact 6.1 (Newelski [18]). There exists a semigroup isomorphism (which is also a homeomorphism of compact spaces) $E (\mathcal{S}_x (G, G), G) \cong (S_x (G, G), \ast)$. 

In this section, we provide an analogous description for the convolution semigroup on finitely satisfiable measures in NIP theories. Recall that $\mathcal{M}_x (G, G) \subseteq \mathcal{M}_{\pi_x} (G)$ is the collection of global measures finitely satisfiable in $G$, and this space of measures carries a natural structure of a real topological vector space induced from $\mathcal{M}_x (G)$. We identify $G$ with the set $\{ \delta_g : g \in G \} \subseteq \mathcal{M}_x (G, G)$, and let $\text{conv}(G)$ denote the convex hull of $G$. There is a natural semigroup action of $\text{conv}(G)$ on $\mathcal{M}_x (G, G)$: for any $\sum_{i=1}^n r_i \delta_{g_i} \in \text{conv}(G)$ (with $g_i \in G$ and $r_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^n r_i = 1$), $\mu \in \mathcal{M}_x (G, G)$ and $\varphi (x) \in L_x (G)$, we define $\left( \sum_{i=1}^n r_i \delta_{g_i} \right) \cdot \mu \in \mathcal{M}_x (G, G)$ by

$$\left( \sum_{i=1}^n r_i \delta_{g_i} \right) \cdot \mu (\varphi (x)) = \sum_{i=1}^n r_i \mu (g_i \cdot x).$$

For the rest of this section, we will denote elements of $\text{conv}(G)$ as $k$, the semigroup action described above as $\pi : \text{conv}(G) \times \mathcal{M}_x (G, G) \to \mathcal{M}_x (G, G)$, and the map $\mu \mapsto \pi (k, \mu)$ as $\pi_k$. It is not difficult to see that for every $k \in \text{conv}(G)$, the map $\pi_k$ is continuous. Therefore, we can consider the Ellis semigroup of this semigroup action, namely $E (\mathcal{M}_x (G, G), \text{conv}(G))$. 

Remark 5.9. Some of these results can be generalized for idempotent measures in NIP groups, and we hope to address it in future work. 

6. DESCRIBING THE CONVOLUTION SEMIGROUP ON FINITELY SATISFIABLE MEASURES AS AN ELLIS SEMIGROUP

6.1. Dynamics. We begin this section by recalling the construction of the Ellis semigroup. Let $X$ be a compact Hausdorff space and $S$ be a semigroup acting on $X$ by homeomorphisms. In particular, there is a map $\pi : S \times X \to X$ such that for each $s \in S$, the map $\pi_s : X \to X, x \mapsto \pi(s, x)$ is a homeomorphism. Let $X^S$ be the space of functions from $X$ to $X$ equipped with the product topology. Then, $\{ \pi_s : s \in S \}$ is naturally a subset of $X^S$. Finally, the Ellis semigroup of the action $(X, S, \pi)$ is $(\text{cl} (\{ \pi_s : s \in S \}), \circ)$, where we take the closure of $\{ \pi_s : s \in S \}$ in $X^S$. When the action map $\pi$ is be clear, we will denote this semigroup as $E(X, S)$.

Let now $T$ be a first order theory expanding a group, $G$, a saturated model of $T$, and $G \prec G$, a small elementary substructure. Recall that $S_x (G, G)$ denotes the set of global types finitely satisfiable in $G$. There is a natural action of $G$ on $S_x (G, G)$ via $g \cdot p = \{ \varphi (x) : \varphi (g^{-1} \cdot x) \in p \}$.

Fact 6.1 (Newelski [18]). There exists a semigroup isomorphism (which is also a homeomorphism of compact spaces) $E (S_x (G, G), G) \cong (S_x (G, G), \ast)$. 

In the rest of this section, we will denote elements of $\text{conv}(G)$ as $k$, the semigroup action described above as $\pi : \text{conv}(G) \times \mathcal{M}_x (G, G) \to \mathcal{M}_x (G, G)$, and the map $\mu \mapsto \pi (k, \mu)$ as $\pi_k$. It is not difficult to see that for every $k \in \text{conv}(G)$, the map $\pi_k$ is continuous. Therefore, we can consider the Ellis semigroup of this semigroup action, namely $E (\mathcal{M}_x (G, G), \text{conv}(G))$. 


We will show that if $T$ is NIP, then this Ellis semigroup $E(G, G)$ is isomorphic to the convolution semigroup of global measures which are finitely satisfiable in $G$, i.e. $(\mathcal{M}_x(G, G), *)$ (Theorem 6.11). We demonstrate that these two semigroups are isomorphic by considering the map $\rho : \mathcal{M}_x(G, G) \to \mathcal{M}_x(G, G)^{\mathcal{M}_x(G, G)}$ defined by $\rho(\nu) = \rho_\nu := \nu * \varepsilon$, and proving that the image of $\rho$ is precisely the Ellis semigroup. Before continuing, we observe that $\rho$ is well-defined, and that $\mathcal{M}_x(G, G)$ is a semigroup by recalling the following facts.

**Fact 6.2.** Let $T$ be NIP and assume that $\mu \in \mathcal{M}_x(G, G)$. Then:

1. $\mu$ is Borel-definable over $G$ (by Fact 2.2(2a));
2. for any $\nu \in \mathcal{M}_x(G, G)$, $\mu * \nu \in \mathcal{M}_x(G, G)$ (by Proposition 5.15(2));
3. the operation $*$ on $\mathcal{M}_x(G, G)$ is associative, hence $(\mathcal{M}_x(G, G), *)$ is a semigroup (by associativity of $\odot$).

Hence the map $\rho : \mathcal{M}_x(G, G) \to \mathcal{M}_x(G, G)^{\mathcal{M}_x(G, G)}$ is well-defined. In the next subsection we show that it is also left-continuous.

### 6.2. Left-continuity of convolution

We begin with a general continuity result in arbitrary NIP theories. Let $T$ be an NIP theory, $\mathcal{U}$ a monster model of $T$, and $M$ a small elementary substructure of $\mathcal{U}$.

**Proposition 6.3 (T NIP).** Let $M \prec \mathcal{U}$ and let $\mathcal{M}_x^{\text{inv}}(\mathcal{U}, M)$ be the closed set of global $M$-invariant measures (Definition 2.7). If $\nu \in \mathcal{M}_y(\mathcal{U})$ and $\varphi(x; y)$ is any partitioned $L_{xy}(\mathcal{U})$ formula, then the map $- \otimes \nu(\varphi(x; y)) : \mathcal{M}_x^{\text{inv}}(\mathcal{U}, M) \to [0, 1]$ is continuous.

**Proof.** Choose $N_0 < \mathcal{U}$ small and such that $M \leq N_0$, and $N_0$ contains the parameters of $\varphi$. Then, choose a small $N < \mathcal{U}$ such that $N_0 \leq N$ and there exists some $\hat{\nu} \in \mathcal{M}_y(\mathcal{U})$ such that $\hat{\nu}|_{N_0} = \nu|_{N_0}$ and $\hat{\nu}$ is smooth over $N$ (by Fact 2.5). Fix $\varepsilon \in \mathbb{R}_{>0}$, by Fact 2.2(1a) let $\bar{b} = (b_1, \ldots, b_n)$ be some $(\varphi^*, \varepsilon)$-approximation for $\hat{\nu}$ over $N$ (where $\varphi^*(y; x) = \varphi(x; y)$ and $\bar{b}$ is some element in $(N^y)^{<\omega}$, see Definition 2.7(7)). Note that every $\mu \in \mathcal{M}_x^{\text{inv}}(\mathcal{U}, M)$ is invariant over both $N_0$ and $N$. Then we have (the last equality holds as in Proposition 6.3):

$$
\mu \otimes \nu(\varphi(x; y)) = \int_{S_y(N_0)} F_{\mu, N_0}^\varphi d(\nu|_{N_0}) = \int_{S_y(N_0)} F_{\mu, N_0}^\varphi d(\hat{\nu}|_{N_0}) = \int_{S_y(N)} F_{\mu, N}^\varphi d(\hat{\nu}|_{N}).
$$

As $\hat{\nu}$ is smooth over $N$, by Fact 3.7(1) we have

$$
\int_{S_y(N)} F_{\mu, N}^\varphi d(\hat{\nu}|_{N}) = \int_{S_y(N)} F_{\mu, N}^{\varphi^*} d(\mu|_{N}).
$$

Note that $F_{\alpha, x, N}^{\varphi^*}(p) = \frac{1}{n} \sum_{i=1}^{n} \chi_{\{r \in S_y(N); \varphi(x, b_i) \in r\}}(p)$ for every $p \in S_x(N)$, where $\chi$ is the characteristic function. Now, using that $\bar{b} \subseteq N$ is a $(\varphi^*, \varepsilon)$-approximation for $\hat{\nu}$, we have the following (note that we identify $\varphi(x, b_i)$ with the set of types satisfying it over $N$ in the first step, and over $\mathcal{U}$ in the second step).

$$
\int_{S_y(N)} F_{\alpha, N}^{\varphi^*} d(\mu|_{N}) \approx_{\varepsilon} \int_{S_y(N)} F_{\alpha, x, N}^{\varphi^*} d(\mu|_{N})
$$

$$
= \int_{S_y(N)} \left( \frac{1}{n} \sum_{i=1}^{n} \chi_{\varphi(x, b_i)} \right) d(\mu|_{N})
$$
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \int_{S_x(N)} \chi_{\varphi(x,b_i)} d(\mu|N) \right) = \frac{1}{n} \sum_{i=1}^{n} \mu|N(\varphi(x,b_i)) \\
= \frac{1}{n} \sum_{i=1}^{n} \int_{S_x(U)} \chi_{\varphi(x,b_i)} d\mu.
\]

Clearly, each map \( \int \chi_{\varphi(x,b_i)} : \mathcal{M}_x(U) \to [0,1] \) is continuous by the definition of the topology on the space of measures. Therefore, each map \( \int \chi_{\varphi(x,b_i)} : \mathcal{M}_{xuv}(U, M) \to [0,1] \) is continuous, hence their sum is continuous as well. Since the choice of \( \tilde{b} \) is independent of the choice of \( \mu \), we have

\[
\sup_{\mu \in \mathcal{M}_{xuv}(U, M)} |\mu \otimes \nu(\varphi(x,y))| - \frac{1}{n} \sum_{i=1}^{n} \int_{S_x(U)} \chi_{\varphi(x,b_i)} d\mu < \epsilon.
\]

Therefore, the map \( - \otimes \nu(\varphi(x,y)) \) is a uniform limit of continuous functions and hence continuous. \( \square \)

Now, we apply this to our group theoretic context. Let again \( T \) be an NIP theory expanding a group, \( \mathcal{G} \) a monster model of \( T \), and \( G < \mathcal{G} \) a small model.

**Proposition 6.4.** Let \( \nu \in \mathcal{M}_x(\mathcal{G}, G) \). Then the map \( - \ast \nu : \mathcal{M}_x(\mathcal{G}, G) \to \mathcal{M}_x(\mathcal{G}, G) \) is continuous.

**Proof.** Let \( U \) be a basic open subset of \( \mathcal{M}_x(\mathcal{G}, G) \). That is, there exist formulas \( \varphi_1(x), ..., \varphi_n(x) \) in \( L_x(\mathcal{G}) \) and real numbers \( r_1, ..., r_n, s_1, ..., s_n \in [0,1] \) such that

\[
U = \bigcap_{i=1}^{n} \{ \mu \in \mathcal{M}_x(\mathcal{G}, G) : r_i < \mu(\varphi_i(x)) < s_i \}.
\]

Then we have

\[
\left( - \ast \nu \right)^{-1}(U) = \bigcap_{i=1}^{n} \{ \mu \in \mathcal{M}_x(\mathcal{G}, G) : r_i < \mu \ast \nu(\varphi_i(x)) < s_i \}
\]

\[
= \bigcap_{i=1}^{n} \{ \mu \in \mathcal{M}_x(\mathcal{G}, G) : r_i < \mu_x \otimes \nu_y(\varphi_i(x \cdot y)) < s_i \}
\]

\[
= \bigcap_{i=1}^{n} \left( - \otimes \nu_y(\varphi_i(x \cdot y)) \right)^{-1}(\{ \mu \in \mathcal{M}_x(\mathcal{G}, G) : r_i < \mu(\varphi_i(x)) < s_i \}).
\]

Therefore, by continuity of the map \( - \otimes \nu(\varphi(x \cdot y)) \) (Proposition 6.3), the preimage of \( U \) under \( - \ast \nu \) is a finite intersection of open sets, and therefore open. \( \square \)

6.3. **The isomorphism.** In this subsection we show that the map \( \rho : \mathcal{M}_x(\mathcal{G}, G) \to \mathcal{E}(\mathcal{G}, G) = E(\mathcal{M}_x(\mathcal{G}, G), \text{conv}(G)) \) given by \( \rho(\nu) = \rho_\nu = \nu \ast - \) is an isomorphism. We begin by recalling the topology on \( \mathcal{M}_x(\mathcal{G}, G)^{\mathcal{M}_x(\mathcal{G}, G)} \).

**Remark 6.5.** The topology on \( \mathcal{M}_x(\mathcal{G}, G)^{\mathcal{M}_x(\mathcal{G}, G)} \) is generated by the basic open sets of the form

\[
U = \bigcap_{i=1}^{n} \{ f : \mathcal{M}_x(\mathcal{G}, G) \to \mathcal{M}_x(\mathcal{G}, G) | r_i < f(\nu_i)(\psi_i(x)) < s_i \},
\]

with \( n \in \mathbb{N}, r_i, s_i \in \mathbb{R}, \psi_i(x) \in L_x(\mathcal{G}) \), and \( \nu_i \in \mathcal{M}_x(\mathcal{G}, G) \) (with possible repetitions of \( \nu_i \)'s and \( \psi_i \)'s).
Lemma 6.6. The map $\rho$ is injective.

Proof. Note that for every $\nu \in \mathcal{M}_x(\mathcal{G}, G)$, $\rho_\nu(\delta_e) = \nu$, where $e$ is the identity of $\mathcal{G}$. $\square$

Lemma 6.7. If $\mu \in \mathcal{M}_x(\mathcal{G}, G)$, then $\rho_\mu \in \text{cl} \{ \{ \pi_k : k \in \text{conv}(G) \} \}$. So $\rho(\mathcal{M}_x(\mathcal{G}, G)) \subseteq E(\mathcal{M}_x(\mathcal{G}, G), \text{conv}(G))$.

Proof. Let $U$ be an open subset of $\mathcal{M}_x(\mathcal{G}, G)^{\mathcal{M}_x(\mathcal{G}, G)}$ containing $\rho_\mu$. It is a union of basic open sets (see Remark 6.5), hence we can choose some $n \in \mathbb{N}$, a sufficiently small $\varepsilon > 0$ and some $\psi_1(\cdot), \ldots, \psi_n(\cdot) \in L_x(U)$ and $\nu_1, \ldots, \nu_n \in \mathcal{M}_x(\mathcal{G}, G)$ such that

$$B_\varepsilon := \bigcap_{i=1}^n \{ f : |f(\nu_i)(\psi_i(x)) - \rho_\mu(\psi_i)(\psi_i(x))| < \varepsilon \} \subseteq U.$$

Let $H_0 \prec \mathcal{G}$ be a small model containing $\mathcal{G}$ and the parameters of $\psi_1, \ldots, \psi_n$. By Fact 2.4 we can choose a small model $H \prec \mathcal{G}$ and measures $\nu_i \in \mathcal{M}_x(\mathcal{G})$ such that:

- $G \subseteq H_0 \subseteq H \prec \mathcal{G}$;
- $\nu_i|_{H_0} = \nu_i|_{H_0}$, for all $1 \leq i \leq n$;
- $\nu_i$ is smooth over $H$, for all $1 \leq i \leq n$.

Take some $0 < \varepsilon_0 < \frac{\varepsilon}{2}$. Recall from Section 3.2 that $\psi'(x; y) = \psi(x \cdot y) \in L_x(y(H_0))$. By Fact 2.2.1, let $\tilde{b}_i = (b_{ij} : 1 \leq j \leq m_i) \in H^{\leq \omega}$ be a $((\psi_i')^*, \varepsilon_0)$-approximation for $\tilde{\nu}_i$. Then, using that $\mu$ is invariant over both $H_0$ and $H$ and $\tilde{\nu}_i$ is smooth over $H$ as in Proposition 6.3, for every $1 \leq i \leq n$ we have:

$$\rho_\mu(\nu_i)(\psi_i(x)) = \mu * \nu_i(\psi_i(x)) = \mu \otimes \nu_i(\psi_i(x \cdot y))$$

$$= \int_{S_x(H_0)} F_{\mu, H_0}^{\psi_i} d(\nu_i|_{H_0}) = \int_{S_x(H_0)} F_{\nu_i, H_0}^{\psi_i} d(\tilde{\nu}_i|_{H_0})$$

$$= \int_{S_x(H)} F_{\nu_i, H}^{\psi_i} d(\tilde{\nu}_i|_{H}) = \int_{S_x(H)} F_{\psi_i, H}^{\nu_i} d(\mu|_{H})$$

$$\approx \varepsilon_0 \int_{S_x(H)} F_{\psi_i, H}^{(\psi_i')^*} d(\mu|_{H}) = \frac{1}{m_i} \sum_{j=1}^{m_i} \mu(\psi_i(x \cdot b_{ij})).$$

Let $\Psi = \{ \psi_i(x \cdot b_{ij}) : 1 \leq i \leq n, 1 \leq j \leq m_i \}$. Since $\mu$ is finitely satisfiable in $\mathcal{G}$, we can find some $k_\mu \in \text{conv}(G)$ such that $k_\mu(\theta(x)) = \mu(\theta(x))$ for each $\theta(x) \in \Psi$ (see Proposition 2.11). We claim that then $\pi_{k_\mu}$ is in $B_\varepsilon$. This follows directly from running the equations above in reverse: for each $1 \leq i \leq n$ we have (using that $k_\mu$ is obviously invariant over $G$, hence also over $H_0$)

$$\int_{S_x(H)} F_{\psi_i, H}^{(\psi_i')^*} d(\mu|_{H}) \approx \varepsilon_0 \int_{S_x(H)} F_{\psi_i, H}^{(\psi_i')^*} d(\mu|_{H})$$

$$= \int_{S_x(H)} F_{\psi_i, H}^{(\psi_i')^*} d(\mu|_{H}) = \frac{1}{m_i} \sum_{j=1}^{m_i} \mu(\psi_i(x \cdot b_{ij})).$$

Hence $\rho_\mu(\nu_i)(\psi_i(x)) \approx 2 \varepsilon_0 \pi_{k_\mu}(\nu_i)(\psi_i(x))$ for each $1 \leq i \leq n$, so $\pi_{k_\mu} \in B_\varepsilon \subseteq U$ and we are finished. $\square$

Lemma 6.8. $\rho(\mathcal{M}_x(\mathcal{G}, G)) = E(\mathcal{M}_x(\mathcal{G}, G), \text{conv}(G))$. 

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Proof. Let \( f \in E(\mathcal{M}_x(G, G), \text{conv}(G)) \) be arbitrary. Then \( f \in \text{cl}(\{\pi_k : k \in \text{conv}(G)\}) \), and so there exists a net \((k_i)_{i \in I}\) with \( k_i \in \text{conv}(G) \) such that \( \lim_{i \in I} \pi_{k_i} = f \). Then, using Remark 6.4, for every \( \psi(x) \in L_x(G) \) and \( \nu \in \mathcal{M}_x(G, G) \) we have

\[
\lim_{i \in I} \pi_{k_i}(\nu)(\psi(x)) = f(\nu)(\psi(x)).
\]

Consider \( \delta_e \), where \( e \in G \) is the identity. Let \( \mu_f := f(\delta_e) \in \mathcal{M}_x(G, G) \). We claim that the net \((k_i)_{i \in I}\) converges to \( \mu_f \) in \( \mathcal{M}_x(G, G) \). Indeed, for any \( \psi(x) \in L_x(G) \) we have

\[
\lim_{i \in I} \pi_{k_i}(\psi(x)) = \lim_{i \in I} \pi_{k_i}(\delta_e)(\psi(x)) = f(\delta_e)(\psi(x)) = \mu_f(\psi(x)).
\]

Next, we claim that for any \( \nu \in \mathcal{M}_x(G, G) \), we have that \( f(\nu) = \rho_{\mu_f}(\nu) \). Indeed, first we have

\[
f(\nu) = \lim_{i \in I} \pi_{k_i}(\nu) = \lim_{i \in I}[\pi_{k_i} \circ \rho_{\nu}](\delta_e) = \lim_{i \in I}\rho_{k_i \ast \nu}(\delta_e) = \lim_{i \in I}[k_i \ast \nu].
\]

The map \(- \ast \nu : \mathcal{M}_x(G, G) \to \mathcal{M}_x(G, G)\) is continuous by Proposition 6.4 hence it commutes with \( \text{conv}\). Therefore,

\[
\lim_{i \in I}[k_i \ast \nu] = [\lim_{i \in I}k_i] \ast \nu = \mu_f \ast \nu = \rho_{\mu_f}(\nu).
\]

We conclude that \( f = \rho_{\mu_f} = \mu_f \ast -\). \(\square\)

Lemma 6.9. The map \( \rho^{-1} : E(\mathcal{M}_x(G, G), \text{conv}(G)) \to \mathcal{M}_x(G, G) \) is a continuous bijection.

Proof. The map \( \rho^{-1} \) is a well-defined bijection by Lemmas 6.6 and 6.8. Let \( U \) be a basic open subset of \( \mathcal{M}_x(G, G) \), say

\[
U = \bigcap_{i=1}^{n} \{ \mu \in \mathcal{M}_x(G, G) : r_i < \mu(\varphi_i(x)) < s_i \}
\]

for some \( n \in \mathbb{N} \), \( \varphi_i(x) \in L_x(U) \) and \( r_i, s_i \in [0, 1] \). Then,

\[
(\rho^{-1})^{-1}(U) = \bigcap_{i=1}^{n} \{ f \in E(\mathcal{M}_x(G, G), \text{conv}(G)) : r_i < f(\delta_e)(\varphi_i(x)) < s_i \}.
\]

This is a restriction of a basic open subset (see Remark 6.3) to \( E(\mathcal{M}_x(G, G), \text{conv}(G)) \), hence open in the subspace topology. \(\square\)

Theorem 6.10. The map \( \rho : (\mathcal{M}_x(G, G), \ast) \to E(\mathcal{M}_x(G, G), \text{conv}(G)) \) is a homeomorphism which respects the semigroup operation, and therefore an isomorphism.

Proof. The map \( \rho \) is a homeomorphism since, by Lemma 6.9, \( \rho^{-1} \) is a continuous bijection between compact Hausdorff spaces. And note that \( \rho(\mu \ast \nu)(\lambda) = (\mu \ast \nu) \ast \lambda = \mu \ast (\nu \ast \lambda) = \rho_{\mu}(\nu \ast \lambda) = \rho_{\mu} \circ \rho_{\nu}(\lambda) \), hence \( \rho(\mu \ast \nu) = \rho_{\mu} \circ \rho_{\nu} \). \(\square\)

Remark 6.11. On the other hand, if \( T \) is NIP, then

\[
E(\mathcal{M}_x(G, G), G) \cong E(S_x(G, G), G),
\]

and so \( \cong (S_x(G, G), \ast) \) by Fact 6.1. For a countable \( G \prec G \), this is an immediate consequence of the corresponding observation in the context of tame metrizable dynamical systems (see e.g. [10] Theorem 1.5); and for an arbitrary small \( G \prec G \), an approximation argument with smooth measures (as in Lemma 6.7) can be adapted. As typically \( (\mathcal{M}_x(G, G), \ast) \not\cong (S_x(G, G), \ast) \), we see that it was crucial to consider
the action of $\text{conv}(G)$ rather than $G$ in our characterization of $(\mathcal{M}_c(G, G), \ast)$ as an Ellis semigroup.

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