On Exceptional Sets in the Metric Poissonian Pair Correlations problem

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Abstract

Let \((a_n)\) be a strictly increasing sequence of positive integers, denote by \(A_N = \{a_n : n \leq N\}\) its truncations, and let \(\alpha \in [0, 1]\). We prove that if the additive energy \(E(A_N)\) of \(A_N\) is in \(\Omega(N^3)\), then the sequence \((\langle \alpha a_n \rangle)_n\) of fractional parts of \(\alpha a_n\) does not have Poissonian pair correlations (PPC) for almost every \(\alpha\) in the sense of Lebesgue measure. Conversely, it is known that \(E(A_N) = O(N^{3-\varepsilon})\), for some fixed \(\varepsilon > 0\), implies that \((\langle \alpha a_n \rangle)_n\) has PPC for almost every \(\alpha\). This note makes a contribution to investigating the energy threshold for \(E(A_N)\) to imply this metric distribution property. We establish, in particular, that there exist sequences \((a_n)\) such that

\[
E(A_N) = \Theta\left(\frac{N^3}{\log(N) \log(\log N)}\right)
\]

such that the set of \(\alpha\) for which \((\alpha a_n)\) does not have PPC is of full Lebesgue measure. Moreover, we show that for any fixed \(\varepsilon > 0\) there are sequences \((a_n)\) with \(E(A_N) = \Theta\left(\frac{N^3}{\log(N) \log(\log N)^{1+\varepsilon}}\right)\) satisfying that the set of \(\alpha\) for which the sequence \((\langle \alpha a_n \rangle)_n\) does not have PPC is of full Hausdorff dimension.

1 Introduction

The theory of uniform distribution modulo 1 dates back, at least, to the seminal paper of Weyl [16]. Weyl showed, inter alia, that for any fixed irrational \(\alpha \in \mathbb{R}\) and integer \(d \geq 1\) the sequences \((\langle \alpha n^d \rangle)_n\) are uniformly distributed modulo 1. However, in recent years various authors [2, 7, 10, 11, 12, 14, 15] have been investigating a more subtle distribution property of such sequences - namely, whether the asymptotic distribution of the pair correlations has a property which is called Poissonian, and defined as follows:

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Definition. Let \( \| \cdot \| \) denote the distance to the nearest integer. A sequence \((\theta_n)_n\) in \([0, 1]\) is said to have (asymptotically) Poissonian pair correlations, if for each \( s \geq 0 \) the pair correlation function

\[
R_2([-s, s], (\theta_n)_n, N) := \frac{1}{N} \# \left\{ 1 \leq i \neq j \leq N : \| \theta_i - \theta_j \| \leq \frac{s}{N} \right\}
\]

(1)
tends to \( 2s \) as \( N \to \infty \). Moreover, let \((a_n)_n\) denote a strictly increasing sequence of positive integers. If no confusion can arise, we write

\[
R([-s, s], \alpha, N) := R_2([-s, s], (\alpha a_n)_n, N)
\]

and say that a sequence \((a_n)_n\) has metric Poissonian pair correlations if \((\alpha a_n)_n\) has Poissonian pair correlations for almost all \( \alpha \in [0, 1] \) in the sense of Lebesgue measure.

It is known that if a sequence \((\theta_n)_n\) has Poissonian pair correlations, then it is uniformly distributed modulo 1, cf. \([1, 8]\). Yet, the sequences \((\langle \alpha n^d \rangle)_n\) do not have Poissonian pair correlations for any \( \alpha \in \mathbb{R} \) if \( d = 1 \). For \( d \geq 2 \), Rudnick and Sarnak \([10]\) proved that \((n^d)_n\) has metric Poissonian pair correlations (metric PPC). For alternative proofs, we refer the reader to Heath-Brown \([7]\) and the work of Marklof and Strömbergsson \([9]\). Given these results, it is natural to investigate which properties of a sequence of integers \((a_n)_n\) implies the metric PPC of \((a_n)_n\). Partial answers are known, e.g. it follows from work of Boca and Zaharescu \([3]\) that \((P(n))_n\) has metric PPC if \( P \) is any polynomial with integer coefficients of degree at least two. An interesting general result in this direction is due to Aistleitner, Larcher, and Lewko \([2]\) who used a Fourier analytic approach combined with a bound on GCD sums of Bondarenko and Seip \([4]\) to relate the metric PPC of \((a_n)_n\) with its combinatoric properties. For stating it, let \((a_n)_n\) denote henceforth a strictly increasing sequence of positive integers and denote the set of the first \( N \) elements of \((a_n)_n\) by \( A_N \). Moreover, define the additive energy \( E(I) \) of a finite set integers \( I \) via

\[
E(I) := \sum_{\substack{a,b,c,d \in I \ \text{a}, \ b, c, d \equiv \epsilon \}} 1.
\]

In the following, let \( \mathcal{O} \) and \( o \) denote the standard Landau symbols/O-notation.

A main finding of \([2]\) is the implication that if the truncations \( A_N \) satisfy

\[
E(A_N) = \mathcal{O}(N^{3-\varepsilon})
\]

(2)
for some fixed \( \varepsilon > 0 \), then \((a_n)_n\) has metric PPC. Note that \((\#I)^2 \leq E(I) \leq (\#I)^3 \) where \( \#I \) denotes the cardinality of \( I \subset \mathbb{Z} \). Roughly speaking, a set \( I \) has

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1The subscript 2 in \( R_2 \) indicates that relations of second order, i.e. pair correlations, are counted.

2It is worthwhile to mention that the case \( d = 2 \) is of particular interest for its connection to mathematical physics, see \([10]\) for further references.
large additive energy if and only if it contains a “large” arithmetic progression
like structure. Indeed, if \((a_n)_n\) is a geometric progression or of the form \((n^d)_n\)
for \(d \geq 2\), then \((2)\) is satisfied. Furthermore, note that the metric PPC
property may be seen as a sort of pseudorandomness; in fact, for a given sequence
of \([0,1]\)-uniformly distributed, and independent random variables \((\theta_n)_n\), one has
\[
\lim_{N \to \infty} R([-s,s],(\theta_n)_n,N) = 2s
\]
for every \(s \geq 0\) almost surely.

Wondering about the optimal bound for the additive energy of the truncations
\(A_N\) to imply the metric PPC property of \((a_n)_n\), the two following questions were
raised in \([2]\) where we use the convention that \(f = \Omega(g)\) means for \(f, g: \mathbb{N} \to \mathbb{R}\)
there is a constant \(c > 0\) such that \(g(n) > cf(n)\) holds for infinitely many \(n\).

**Question 1.** Is it possible for a strictly increasing sequence \((a_n)_n\) of positive
integers with \(E(A_N) = \Omega(N^3)\) to have metric PPC?

**Question 2.** Do all increasing strictly sequences \((a_n)_n\) of positive integers with
\(E(A_N) = o(N^3)\) have metric PPC?

Both questions were answered in the negative by Bourgain whose proofs can
be found in \([2]\) as an appendix, without giving an estimate on the measure of
the set that was used to answer Question 1, and without a quantitative bound
on \(E(A_N)\) appearing in the negation of Question 2. However, a quantitative
analysis, as noted in \([15]\), shows that the sequence Bourgain constructed for
Question 2 satisfies
\[
E(A_N) = O\left(\frac{N^3}{(\log \log N)^{1/4 + \varepsilon}}\right)
\]
for any fixed \(\varepsilon > 0\). Moreover, Nair posed the problem\(^3\) whether the sequence
of prime numbers \((p_n)_n\), ordered by increasing value, has metric PPC. Recently,
Walker \([15]\) answered this question in the negative. Thereby he gave a signifi-
cantly better bound than \([11]\) for the additive energy \(E(A_n)\) for a sequence \((a_n)_n\)
not having metric PPC - since the additive energy of the truncations of \((p_n)_n\) is
in \(\Theta((\log N)^{-1} N^3)\) where \(f = \Theta(g)\), for functions \(f, g,\) means that \(f = \mathcal{O}(g)\)
and \(g = \mathcal{O}(f)\) holds. The main objective of our work is to improve upon these
answers to Questions 1, and Question 2.

For a given sequence \((a_n)_n\), we denote by \(\text{NPPC}((a_n)_n)\) the (“exceptional”) set of all \(\alpha \in (0, 1)\) such that the pair correlation function \([11]\) does not tend to
\(2s\), as \(N\) tends to infinity, for some \(s \geq 0\).

**Theorem A (Bourgain, \([2]\)).** Suppose \((a_n)_n\) is a strictly increasing sequence
of positive integers. If \(E(A_N) = \Omega(N^3)\), then \(\text{NPPC}((a_n)_n)\) has positive
Lebesgue measure.

\(^3\)This problem was posed at the problem session of the ELAZ conference in 2016.
We prove the following sharpening.

**Theorem 1.** Suppose \((a_n)_n\) is a strictly increasing sequence of positive integers. If \(E(A_N) = \Omega(N^3)\), then \(\text{NPPC}((a_n)_n)\) has full Lebesgue measure.

Moreover, we lower the known energy threshold, and estimate the Hausdorff dimension of the exceptional set from below. For stating our second main theorem, we denote by \(\mathbb{R}_{>x}\) the set of real numbers exceeding a given \(x \in \mathbb{R}\), and recall that for a function \(g: \mathbb{R}_{>1} \to \mathbb{R}_{>0}\) the lower order of infinity \(\lambda(g)\) is defined by

\[
\lambda(g) := \liminf_{x \to \infty} \frac{\log g(x)}{\log x}.
\]

**Remark.** This notion arises naturally in the context of Hausdorff dimensions. Roughly speaking, it quantifies the (lower) asymptotic growth rate of a function.

**Theorem 2.** Let \(f: \mathbb{R}_{>0} \to \mathbb{R}_{>2}\) be a function increasing monotonically to \(\infty\), and satisfying \(f(x) = \mathcal{O}((\log x)^{-7/3}x^{1/3})\). Then, there exists a strictly increasing sequence \((a_n)_n\) of positive integers with \(E(A_N) = \Theta((f(N))^{-1}N^3)\) such that if

\[
\sum_{n \geq 1} \frac{1}{nf(n)}
\]

diverges, then for Lebesgue almost all \(\alpha \in [0, 1]\)

\[
\limsup_{N \to \infty} R([-s, s], \alpha, N) = \infty
\]

holds for any \(s > 0\); additionally, if \(\mathcal{E}\) converges and \(\sup \{f(2x)/f(x): x \geq x_0\}\) is strictly less than 2 for some \(x_0 > 0\), then \(\text{NPPC}((a_n)_n)\) has Hausdorff dimension at least \((1 + \lambda)^{-1}\) where \(\lambda\) is the lower order of infinity of \(f\).

We record an immediate consequence of Theorem 2 by using the convention that the \(r\)-folded iterated logarithm is denoted by \(\log_r(x)\), i.e. \(\log_1(x) := \log(x)\) and \(\log_{r+1}(x) := \log(\log_r(x))\).

**Corollary 1.** Let \(r\) be a positive integer. Then, there is a strictly increasing sequence \((a_n)_n\) of positive integers with

\[
E(A_N) = \Theta\left(\frac{N^3}{\log(N)\log_2(N)\ldots\log_r(N)}\right)
\]

such that \(\text{NPPC}((a_n)_n)\) has full Lebesgue measure. Moreover, for any \(\varepsilon > 0\) there is a strictly increasing sequence \((a_n)_n\) of positive integers with

\[
E(A_N) = \Theta\left(\frac{(\log_r(N))^{-\varepsilon}N^3}{\log(N)\log_2(N)\ldots\log_r(N)}\right)
\]

such that \(\text{NPPC}((a_n)_n)\) has full Hausdorff dimension.
The proof of Theorem 2 connects the metric PPC property to the notion of “optimal regular systems" from Diophantine approximation. It uses, among other things, a Khintchine-type theorem due to Beresnevich. Furthermore, despite leading to better bounds, the nature of the sequences underpinning Theorem 2 is much simpler than the nature of those sequences previously constructed by Bourgain [2] (who used, inter alia, large deviations inequalities form a probability theory), or the sequence of prime numbers studied by Walker [15] (who relied on estimates, derived by the circle-method, on the exceptional set in Goldbach-like problems).

In the converse direction, there has been remarkable progress, due to a work of Bloom, Chow, Gafni, Walker - who improved under the assumption that the sequence is not “too sparse” the power saving bound (2) to a saving of a little more than the square of a logarithm. More precisely, their result is as follows.

**Theorem B (Bloom, Chow, Gafni, Walker [5].** Let \((a_n)_n\) be a strictly increasing sequence of positive integers. Suppose there is an \(\varepsilon > 0\) and a \(C = C(\varepsilon) > 0\) such that

\[
E(A_N) = \mathcal{O}_\varepsilon \left( \frac{N^3}{(\log N)^{2+\varepsilon}} \right), \quad \delta(N) \geq \frac{C}{(\log N)^{2+2\varepsilon}}
\]

where \(\delta(N) := N^{-1} \# (A_N \cap \{1, \ldots, N\})\). Then, \((a_n)_n\) has metric PPC.

## 2 First main theorem

Let us give an outline of the proof of Theorem 1. For doing so, we begin by sketching the reasoning of Theorem A: As it turns out, except for a set of neglectable measure, the counting function in (1) can be written as a function that admits a non-trivial estimate for its \(L^1\)-mean value. The \(L^1\)-mean value is infinitely often too small on sets whose measure is uniformly bounded from below. Thus, there exists a sequence of sets \((\Omega_r)_r\), of \(\alpha \in [0, 1]\) such that \(R([-s, s], \alpha, N)\) is too small for every \(\alpha \in \Omega_r\) for having PPC and Theorem A follows.

Our reasoning for proving Theorem 1 is building upon this argument of Bourgain while we introduce new ideas to construct a sequence of sets \((\Omega_r)_r\) that are “quasi (asymptotically) independent” - meaning that for every fixed \(t\) the relation \(\lambda(\Omega_r \cap \Omega_t) \leq \lambda(\Omega_r)\lambda(\Omega_t) + o(1)\) holds as \(r \to \infty\). Roughly speaking, applying a suitable version of the Borel-Cantelli lemma, combined with a sufficiently careful treatment of the \(o(1)\) term, will then yield Theorem 1. However, before proceeding with the details of the proof we collect in the next paragraph some tools from additive combinatorics that are needed.

### 2.1 Preliminaries

We start with a well-know result relating, in a quantitative manner, the additive energy of a set of integers with the existence of a (relatively) dense subset with...
small difference set where the difference set $B - B := \{b - b' : b, b' \in B\}$ for a set $B \subseteq \mathbb{R}$.

**Lemma 1** (Balog-Szeméredi-Gowers lemma, [13, Thm 2.29]). Let $A \subseteq \mathbb{Z}$ be a finite set of integers. For any $c > 0$ there exist $c_1, c_2 > 0$ depending only on $c$ such that the following holds. If $E(A) \geq c(\#A)^3$, then there is a subset $B \subseteq A$ such that

1. $\#B \geq c_1 \#A$,
2. $\#(B - B) \leq c_2 \#A$.

Moreover, we recall that for $\delta > 0$ and $d \in \mathbb{Z}$ the set

$$B(d, \delta) := \{\alpha \in [0, 1] : \|d\alpha\| \leq \delta\}$$

is called Bohr set. These appear frequently in additive combinatorics. The following two simple observation will be useful.

**Lemma 2.** Let $B \subseteq \mathbb{Z}$ be a finite set of integers. Then,

$$\lambda \left( \left\{ \alpha \in [0, 1] : \min_{d \in (B - B) \setminus \{0\}} \|d\alpha\| < \frac{\varepsilon}{\#(B - B)} \right\} \right) \leq 2\varepsilon$$

for every $\varepsilon \in (0, 1)$ where $\lambda$ is the Lebesgue measure.

**Proof.** By observing that the set under consideration is contained in

$$\bigcup_{m, n \in B \atop m \neq n} B \left( m - n, \frac{\varepsilon}{\#(B - B)} \right),$$

and $\lambda \left( B \left( m - n, \frac{\varepsilon}{\#(B - B)} \right) \right) = \frac{2\varepsilon}{\#(B - B)}$, the claim follows at once. \qed

**Lemma 3.** Suppose $A$ is a finite intersection of Bohr sets, and $B$ is a finite union of Bohr sets. Then, $A \setminus B$ is the union of finitely many intervals.

Furthermore, we shall use the Borel-Cantelli lemma in a version due to Erdős-Rényi.

**Lemma 4** (Erdős-Rényi). Let $(A_n)_n$ be a sequence of Lebesgue measurable sets in $[0, 1]$ satisfying

$$\sum_{n \geq 1} \lambda(A_n) = \infty.$$

Then,

$$\lambda \left( \limsup_{n \to \infty} A_n \right) \geq \limsup_{N \to \infty} \left( \frac{\sum_{n \leq N} \lambda(A_n)}{\sum_{m, n \leq N} \lambda(A_n \cap A_m)} \right)^2.$$
Moreover, let us explain the main steps in the proof of Theorem 1. Let
\( \varepsilon := \varepsilon(j) := \frac{1}{10} c_1^2 \) be for \( j \in \mathbb{N} \) where the constant \( c_1 \) is specified later-on, and fix \( j \) for now. In the first part of the argument, we show how a sequence - that is constructed in the second part of the argument - with the following crucial (but technical) properties implies the claim. For every fixed \( j \), we find a corresponding \( s = s(j) \) and construct a sequence \( (\Omega_r)_r \) of exceptional values \( \alpha \) satisfying the following properties:

(i) For all \( \alpha \in \Omega_r \), the pair correlation function admits the upper bound
\[
R([-s, s], \alpha, N) \leq 2\bar{c}s
\]  
for some absolute constant \( \bar{c} \in (0, 1) \), depending on \( (a_n) \) only.

(ii) For all integers \( r > t \geq 1 \), the relation
\[
\lambda(\Omega_r \cap \Omega_t) \leq \lambda(\Omega_r) \lambda(\Omega_t) + 2\varepsilon \lambda(\Omega_t) + \mathcal{O}(r^{-2})
\]  
holds.

(iii) Each \( \Omega_r \) is the union of finitely many intervals (hence measurable).

(iv) For all \( r \geq 1 \), the measure \( \lambda(\Omega_r) \) is uniformly bounded from below by
\[
\lambda(\Omega_r) \geq c_1^2 \frac{8}{1}.
\]  

2.2 Proof of Theorem 1

1. Suppose there is \( (\Omega_r)_r \) satisfying (i)-(iv). Then, by using (ii), we get
\[
\sum_{r,t \leq N} \lambda(\Omega_r \cap \Omega_t) \leq 2 \sum_{2 \leq t \leq N} \sum_{1 \leq r < t} (\lambda(\Omega_r) \lambda(\Omega_t)) + 2\varepsilon N^2 + \mathcal{O}(N)
\]
\[
\leq \left( \sum_{t \leq N} \lambda(\Omega_t) \right)^2 + 2\varepsilon N^2 + \mathcal{O}(N).
\]
By recalling that \( \Omega_r = \Omega_r(\varepsilon) = \Omega_r(j) \), we let \( \Omega(j) := \limsup_{r \to \infty} \Omega_r \). By using the inequality above in combination with Lemma 4 and the bound (9), we obtain that the set \( \Omega(j) \) has measure at least
\[
\limsup_{N \to \infty} \frac{\left( \sum_{r \leq N} \lambda(\Omega_r) \right)^2}{\sum_{r,t \leq N} \lambda(\Omega_r \cap \Omega_t)} \geq \limsup_{N \to \infty} \frac{1}{1 + \frac{4\varepsilon N^2}{\left( \sum_{r \leq N} \lambda(\Omega_r) \right)}}
\]
\[
\geq \limsup_{N \to \infty} \frac{1}{1 + \frac{4\varepsilon N^2}{c_1^2}} = \frac{1}{1 + \frac{4\varepsilon c_1^2}{c_1}}.
\]
Note that due to (7) every $\alpha \in \Omega (j)$ does not have PCC. Now, letting $j \to \infty$ proves the assertion.

2. For constructing $(\Omega _r)_r$ with the required properties, let $c > 0$ such that $E (A_N) > cN^3$ for infinitely many integers $N$. By choosing an appropriate subsequence $(N_i)_i$ and omitting the subscript $i$ for ease of notation, $E (A_N) > cN^3$ holds for every $N$ occurring in this proof. Moreover, let $c_1, c_2$ and $B_N$ be as in Lemma 1 corresponding to the $c$ just mentioned. Arguing inductively, while postponing the base step, we assume that for $1 \leq r < R$, and $s = \frac{2 \varepsilon}{c_2}$ there are sets $(\Omega _r)_{1 \leq r < R}$ that satisfy the properties (i)-(iv) for all distinct integers $1 \leq r, t < R$. Let $N \geq R$. Lemma 2 implies that the set $\Omega _{\varepsilon,N} \subseteq [0, 1]$ of all $\alpha \in [0, 1]$ satisfying $\|(r - t) \alpha\| < N^{-1} s$ for some distinct $r, t \in B_N$ has measure at most $2 \varepsilon$. Setting

$$D_N := \{(r, t) \in (A_N \times A_N) \setminus (B_N \times B_N) : r \neq t\},$$

we get for $\alpha \notin \Omega _{\varepsilon,N}$ that

$$R ([-s, s], \alpha, N) = \frac{1}{N} \# \{(r, t) \in D_N : \|(r - t) \alpha\| < N^{-1} s\}.$$  

Let $\ell R$ denote the length of the smallest subinterval of $\Omega _r$ for $1 \leq r < R$, and define $C (\Omega _r)$ to be the set of subintervals of $\Omega _r$. Note that $\ell R > 0$, and $\max_{1 \leq r < R} \# C (\Omega _r) < \infty$. We divide $[0, 1)$ into

$$P := \left[1 + 2 \ell R \frac{R^2}{N^2} \max_{1 \leq r < R} \# C (\Omega _r)\right]$$

parts $\mathcal{P}_i$ of equal lengths, i.e. $\mathcal{P}_i := \left[\frac{i}{P}, \frac{i+1}{P}\right)$ where $i = 0, \ldots, P - 1$. After writing

$$\frac{1}{N} \int_{\mathcal{P}_i} \# \{(r, t) \in D_N : \|(r - t) \alpha\| \leq N^{-1} s\} \, d\alpha = \frac{1}{N} \sum_{(r, t) \in D_N} \int_{\mathcal{P}_i} 1_{[\sqrt{\frac{1}{N^2}}, \sqrt{\frac{1}{N^2}}]} (\|(r - t) \alpha\|) \, d\alpha,$$

we split the sum into two parts: one part containing differences $|r - t| > R^k P$, and a second part containing differences $|r - t| \leq R^k P$ where

$$k := \left\lfloor \frac{1}{\log 2 \log c_1^2 (1-2^{-1-c_1})} \frac{20}{s} \right\rfloor + 1.$$

Letting $1_B$ denote the characteristic function of $X \subseteq [0, 1]$, the Cauchy-Schwarz inequality implies

$$\int_{\mathcal{P}_i} 1_{[\sqrt{\frac{1}{N^2}}, \sqrt{\frac{1}{N^2}}]} (\|(r - t) \alpha\|) \, d\alpha \leq \sqrt{\frac{1}{P} \frac{2s}{N}}.$$

\footnote{The base step uses simplified versions of the arguments exploited in the induction step, and will therefore be postponed.}
Since for any $x > 0$ there are at most $2xN$ choices of $(r, t) \in D_N$ such that $|r - t| \leq x$, we obtain

$$
\frac{1}{N} \sum_{(r, t) \in D_N \cap \{|r-t| \leq PRk\}} 1_{[-sN, sN]}(\|(r - t)\alpha\|) \, d\alpha \leq 2 PRk \sqrt{\frac{2s}{PN}}
$$

which is $\leq P^{-1}R^{-k}$ if $N$ is sufficiently large. Moreover, for any $|r - t| > PRk$ we observe that

$$
\int \frac{1}{P_i} 1_{[-sN, sN]}(\|(r - t)\alpha\|) \, d\alpha \leq \frac{2s}{PN} + \frac{4}{PR^{2k}N}
$$

and $\#D_N \leq N^2 - (\#B_N)^2 \leq \tilde{c}N^2$ where $\tilde{c} := 1 - c_1^2$. Therefore, the mean value \[10\] on $P_i$ of the counting function $R$ is bounded from above by

$$
\frac{1}{N} (\#D_N)^2 \left( \frac{2s}{PN} + \frac{4}{PR^{2k}N} \right) \leq \frac{2\tilde{c}s}{P} + \frac{5}{PR^{2k}}.
$$

Hence, it follows that the measure of the set $\Delta_N (i)$ of $\alpha \in P_i$ with

$$
\frac{1}{N} \# \{(r, t) \in D_N : \|(r - t)\alpha\| \leq N^{-1}s\} \leq 2 \left( 1 - \frac{c_1^2}{2} \right) s \quad (11)
$$

admits, by the choice of $k$, the lower bound

$$
\lambda (\Delta_N (i)) \geq \frac{1}{P} - \frac{1}{P} \frac{2\tilde{c}s + 5R^{-k}}{2\left(1 - \frac{c_1^2}{2}\right)s} \geq \frac{1}{P} \left( \frac{c_2^2}{2} - \frac{c_3^2}{8} \right). \quad (12)
$$

Note that $\Delta_N (i)$ is the union of finitely many intervals, due to Lemma 3. So, we may take $\Delta'_N (i) \subset \Delta_N (i)$ being a finite union of intervals such that $\lambda (\Delta'_N (i))$ equals the lower bound in (12). Let

$$
\Omega_R := \Omega_N (N) := \Delta_N \setminus \Omega_{\varepsilon,N} \quad \text{where} \quad \Delta_N := \bigcup_{i=0}^{P-1} \Delta'_N (i).
$$

We are going to show now that $\Omega_R$ satisfies the properties $\text{(i) - (iv)}$. Now, $\Omega_R$ satisfies property $\text{(iv)}$ with $r = R$ since

$$
\lambda (\Omega_R) \geq \lambda (\Delta_N) - \lambda (\Omega_{\varepsilon,N}) = \frac{c_2^2}{2} - \frac{c_1^2}{8} - 2\varepsilon \geq \frac{c_1^2}{8}.
$$

Furthermore, $\Omega_R$ satisfies property $\text{(i)}$ by construction and also property $\text{(iii)}$ since all sets involved in the construction of $\Omega_R$ were a finite union of intervals.
Let \(1 \leq r < R\), and \(I\) be a subinterval of \(\Omega_r\). Then,
\[
\lambda(I \cap \Delta_N) = \sum_{i : P_i \cap I \neq \emptyset} \lambda(P_i \cap I \cap \Delta_N)
\leq \frac{2}{P} + \sum_{i : P_i \subseteq I} \lambda(P_i \cap \Delta_N)
\leq \frac{2}{P} + \sum_{i : P_i \subseteq I} \lambda(\Delta'_N (i)).
\]

By summing over all subintervals \(I \in C(\Omega_r)\), we obtain that
\[
\lambda(\Omega_r \cap \Delta_N) \leq \sum_{I \in C(\Omega_r)} \left( \frac{2}{P} + \sum_{i : P_i \subseteq I} \lambda(\Delta_N) \right)
\leq \frac{1}{R^2} + \sum_{I \in C(\Omega_r)} P \lambda(I) \frac{\lambda(\Omega_N)}{P}
= \lambda(\Omega_r) \lambda(\Omega_N) + \frac{1}{R^2}.
\]
We deduce property \(\text{(ii)}\) from this estimate and Lemma \(\text{[2]}\) via
\[
\lambda(\Omega_r \cap \Omega_R) \leq \lambda(\Omega_r \cap \Delta_N)
\leq \lambda(\Omega_r) (\lambda(\Omega_N) - \lambda(\Omega_{\varepsilon,N})) + \frac{1}{R^2} + \lambda(\Omega_r) \lambda(\Omega_{\varepsilon,N})
\leq \lambda(\Omega_r) \lambda(\Omega_R) + 2\varepsilon \lambda(\Omega_r) + O(R^{-2})
\]
This concludes the induction step. The only part missing now is the base step of the induction. For realizing it, let \(N\) denote the smallest integer \(m\) with \(E(\lambda_{m}) > cm^3\). We replace \(P_i\) in \(\text{(10)}\) by \([0, 1]\) to directly derive
\[
\int_{0}^{1} \frac{1}{N} \# \{(r, t) \in D_N : \|r - t\| \alpha \leq N^{-1} s\} \, d\alpha \leq 2\tilde{c}s,
\]
and conclude that the set \(\Omega'_{1}\) of \(\alpha \in [0, 1]\) satisfying \(\text{(11)}\) has a measure at least \(\frac{c^2}{2}\). Thus, \(\Omega_1 := \Omega'_{1} \setminus \Omega_{N,\varepsilon}\) has measure at least as large as the right hand side of \(\text{(12)}\). For property \(\text{(iii)}\) is nothing to check and that \(\Omega_1\) is a finite union of intervals follows from Lemma \(\text{[3]}\) by observing that
\[
\Omega_1 = \bigcap_{d_1, \ldots, d_{L(N)}} \left( B (d_1, N^{-1} s)^C \cup \ldots \cup B (d_{L(N)}, N^{-1} s)^C \right)
\]
where the intersection runs through any set of \(L(N) = \lfloor N2\tilde{c}s \rfloor\) tuples of differences \(d_i = r_i - t_i \neq 0\) of components of \((r_i, t_i) \in D_N\) for \(i = 1, \ldots, L(N)\).

Thus, the proof is complete.
3 Second main theorem

The sequences \((a_n)_n\) enunciated in Theorem 2 are constructed in two steps. In the first step, we concatenate (finite) blocks, with suitable lengths, of arithmetic progressions to form a set \(P_A\). In the second step, we concatenate (finite) blocks, with suitable lengths, of geometric progressions to form a set \(P_G\) and then define \(a_n\) to be the \(n\)-th element of \(P_A \cup P_G\). On the one hand, the arithmetic progression like part \(P_A\) serves to ensure, due to considerations from metric Diophantine approximation, the divergence property \((\ref{divergence})\) on a set with full measure or controllable Hausdorff dimension; on the other hand, the geometric progression like part \(P_G\) lowers the additive energy, as much as it can. For doing so, a geometric block will appear exactly before and after an arithmetic block, and have much more elements.

For writing the construction precisely down, we introduce some notation. Let henceforth \(\lfloor x \rfloor\) denote the greatest integer \(m\) that is at most \(x \in \mathbb{R}\). Suppose throughout this section that \(f\) is as in Theorem 2. We set \(P_A^{(1)}\) to be the empty set while \(P_G^{(1)} := \{1, 2\}\). Moreover, for \(j \geq 2\) we let \(P_A^{(j)}\) denote the set of \(\lfloor 2^j (f(2^j))^{-\beta} \rfloor\) consecutive integers that start with \(C_j = 2 \max\{P_G^{(j-1)}\}\), and \(P_G^{(j)}\) is such that the difference set \(P_G^{(j)} - 2C_j\) is the geometric progression \(2^i\) for \(1 \leq i \leq \lfloor (f(2^j))^{-\gamma} 2^j (1 - (f(2^j))^{-\beta}) \rfloor\) where \(0 < \gamma < \beta < 3/4\) are parameters\footnote{No particular importance should be attached to requiring \(\beta < 3/4\), or using “dyadic steps lengths \(2^j\). Doing so is for simplifying the technical details only - eventually, it will turn out that \(\beta = 2/3\) is the optimal choice of parameters in this approach. For proving this to the reader, we leave \(\gamma, \beta\) undetermined till the end of this section.} to be chosen later on. In this notation, we take

\[
P_A := \bigcup_{j \geq 1} P_A^{(j)}, \quad P_G := \bigcup_{j \geq 1} P_G^{(j)},
\]

and denote by \(a_n\) the \(n\)-th smallest element in \(P_A \cup P_G\). For \(d \in \mathbb{Z}\) and finite sets of integers \(X, Y\), we abbreviate the number of representation of \(d\) as a difference of an \(x \in X\) and a \(y \in Y\) by \(\text{rep}_{X,Y}(d) := \# \{ (x, y) \in X \times Y : x - y = d \}\); observe that

\[
E(X) = \sum_{d \in \mathbb{Z}} \left( \text{rep}_{X,X}(d) \right)^2, \tag{13}
\]

and

\[
R([-s, s], \alpha, N) = \frac{1}{N} \sum_{d \neq 0} \text{rep}_{A_N, A_N}(d) 1_{[0, \alpha]}(\|ad\|). \tag{14}
\]

3.1 Preliminaries

We begin to determine the order of magnitude of \(E(A_N)\) for the truncations \(A_N\) of the sequence constructed above. Since the cardinality of elements in the union of the blocks \(P_G^{(j)}, P_A^{(j)}\) has about exponential growth, it is reasonable to
expect $E(A_N)$ to be of the same order of magnitude as the additive energy of the last block $P_G^{(j)} \cup P_A^{(j)}$ that is fully contained in $A_N$ - note that $J = J(N)$; i.e. to expect the magnitude of $E(P_G^{(j)} \cup P_A^{(j)})$ which is roughly equal to $E(P_A^{(j)})$.

The following proposition verifies this heuristic considerations.

**Proposition 1.** Let $(a_n)_n$ be as in the beginning of Section 3, and $f$ be as in one of the two assertions in Theorem 3. Then, $E(A_N) = \Theta(N^3(f(N))^{-3(\beta-\gamma)})$.

For the proof of Proposition 1 we need the next technical lemma.

**Lemma 5.** Let $F_j := 2^j(f(2^j))^{-\delta}$, for $j \geq 1$ and fixed $\delta \in (0, 1)$, where $f$ is as in Proposition 1. Then, $\sum_{i \leq j} F_i = O(F_j)$ and

$$\sum_{d \in \mathbb{Z}} \left( \sum_{j, i \leq j} \text{rep}_{P_G^{(j)}, P_A^{(i)}}(d) \right)^2 = O(j^6 2^{2j}).$$

**Proof.** Suppose that $f(x) = \mathcal{O}(x^{1/3}(\log x)^{-7/3})$ is such that (5) diverges. Because

$$\sum_{j \leq J+1} \frac{1}{f(2^j)} \geq \sum_{k \leq 2J} \frac{1}{kf(k)}$$

diverges as $J \to \infty$ and $(f(2^j)/f(2^{j+1}))_j$ is non-decreasing, we conclude that

$$\lim_{j \to \infty} \frac{f(2^j)}{f(2^{j+1})} = 1.$$ 

Therefore, there is an $i_0$ such that the estimate $(f(2^j))^{-1} f(2^{i+h}) < \left(\frac{3}{2}\right)^{j-i}$ holds for any $i \geq i_0$ and $h \in \mathbb{N}$. Hence,

$$\frac{1}{F_j} \sum_{i \leq j} F_i \leq o(1) + \sum_{i_0 \leq i \leq j} 2^{i-j} \left(\frac{3}{2}\right)^{j-i} = \mathcal{O}(1).$$

If $f$ is such that (1) converges and $f(2x) \leq (2-\varepsilon) f(x)$ for $x$ large enough, then we obtain by a similar argument that $\sum_{i \leq j} F_i$ is in $\mathcal{O}(F_j)$. Furthermore, $\text{rep}_{P_G^{(j)}, P_A^{(i)}}(d) = \mathcal{O}(i)$, for every $j \geq 1$, and non-vanishing for $\mathcal{O}(2^{2j})$ values of $d$ which implies the last claim.

We can now prove the proposition.

**Proof of Proposition 1.** Let $F_j = 2^j(f(2^j))^{-\beta}$, $N \geq 1$ be large and denote by $J = J(N) \geq 0$ the greatest integer $j$ such that $P_G^{(j-1)} \subseteq A_N$. Since

$$E(A_N) \geq E(P_A^{(J-1)}) = \Omega(N^3(f(N))^{-3(\beta-\gamma)}),$$

it remains to show that $E(A_N) = \mathcal{O}(N^3(f(N))^{-3(\beta-\gamma)})$. By exploiting (13),

$$E(A_N) \leq \sum_{d \in \mathbb{Z}} (\text{rep}_{A_T, A_T}(d))^2$$

where $T_N := \# \bigcup_{j \leq J} (P_A^{(j)} \cup P_G^{(j)})$. 

12
Moreover, \( \text{rep}_{A_{T_{1}I_{T_{1}}}^{(i)}}(a, b, d) = S_1(d) + S_2(d) \) where \( S_1(d) \) abbreviates the mixed sum \( \sum_{i,j \leq J} (\text{rep}_{P_A^{(i)}}(a, b, d) + \text{rep}_{P_G^{(i)}}(a, b, d)) \) and \( S_2(d) \) abbreviates the sum \( \sum_{i,j \leq J} (\text{rep}_{P_G^{(i)}}(a, b, d) + \text{rep}_{P_A^{(i)}}(a, b, d)) \). Using that for any real numbers \( a, b \), the inequality \( (a + b)^2 \leq 2(a^2 + b^2) \) holds, we obtain

\[
E(A_N) = O \left( \sum_{d \in \mathbb{Z}} (S_1(d))^2 + \sum_{d \in \mathbb{Z}} (S_2(d))^2 \right).
\]

Lemma 5 implies that \( \sum_{d \in \mathbb{Z}} (S_2(d))^2 = O((\log N)^6 N^2) \) due to \( J = O(\log N) \). Moreover, we note that \( \text{rep}_{P_A^{(i)}}(a, b, d) \) is non-vanishing for at most \( 4F_J \) values of \( d \) as \( i, j \leq J \). Since \( \text{rep}_{P_A^{(i)}}(a, b, d) \leq F_{\min(i, j)} \) holds, we deduce that

\[
\sum_{i,j \leq J} \text{rep}_{P_A^{(i)}}(a, b, d) = O \left( \sum_{j \leq J} \sum_{i \leq J} F_i \right).
\]

By Lemma 5, the right hand side is in \( O(F_J) \). Since \( \text{rep}_{P_A^{(i)}}(a, b, d) \leq 1 \), where \( i, j \leq J \), is non-vanishing for at most \( O(T_J) = O(N^2) \) values of \( d \), we obtain that

\[
\sum_{d \in \mathbb{Z}} (S_1(d))^2 = O(F_J^3 + (\log N)^6 N^2)
\]

which is in \( O(N^3(f(N))^{-3(3-\gamma)}) \). Hence, \( E(A_N) = O(N^3(f(N))^{-3(3-\gamma)}) \).

For estimating the measure or the Hausdorff dimension of \( \text{NPPC}((a_n)_{n}) \) from below, we recall the notion of an optimal regular system. This notion, roughly speaking, describes sequences of real numbers that are exceptionally well distributed in any subinterval, in a uniform sense, of a fixed interval.

**Definition.** Let \( J \) be a bounded real interval, and \( S = (\alpha_i)_{i} \) a sequence of distinct real numbers. \( S \) is called an optimal regular system in \( J \) if there exist constants \( c_1, c_2, c_3 > 0 \) depending on \( S \) and \( J \) only such that for any \( I \subseteq J \) there is an index \( Q_0 = Q_0(S, I) \) such that for any \( Q \geq Q_0 \) there are indices

\[
c_1Q \leq i_1 < i_2 < \ldots < i_t \leq Q
\]

satisfying \( \alpha_{i_h} \in I \) for \( h = 1, \ldots, t \), and

\[
|\alpha_{i_h} - \alpha_{i_l}| \geq \frac{c_2}{Q}
\]

for \( 1 \leq h \neq \ell \leq t \), and

\[
c_3\lambda(I)Q \leq t \leq \lambda(I)Q.
\]

Moreover, we need the following result(s) due to Beresnevich which may be thought of as a far reaching generalization of Khintchine’s theorem, and Jarník-Besicovitch theorem in Diophantine approximation.
Theorem 3 ([6 Thm. 6.1, Thm. 6.2]). Suppose $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a continuous, non-increasing function, and $S = (\alpha_i)_i$ an optimal regular system in $(0, 1)$. Let $K_S(\psi)$ denote the set of $\xi$ in $(0, 1)$ such that $|\xi - \alpha_i| < \psi(i)$ holds for infinitely many $i$. If
\[ \sum_{n \geq 1} \psi(n) \tag{18} \]
diverges, then $K_S(\psi)$ has full measure.
Conversely, if (18) converges, then $K_S(\psi)$ has measure zero and the Hausdorff dimension equals the reciprocal of the lower order of $\frac{1}{\psi}$ at infinity.

For a rational $\alpha = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, we denote by $H(\alpha)$ its (naive) height, i.e. $H(\alpha) := \max\{|p|, |q|\}$. It is well-known that the set of rational numbers in $(0, 1)$, ordered in classes by increasing height and in each class ordered by numerically values, gives rise to an optimal regular system in $(0, 1)$.

The following lemma says, roughly speaking, that this assertion remains true for the set of rationals in $(0, 1)$ whose denominators are members of a special sequence that is not too sparse in the natural numbers. The proof can be given by modifying the proof of the classical case, compare [6, Prop. 5.3]; however, we shall give the details for making this article more self-contained.

Lemma 6. Let $\vartheta : \mathbb{R}_{>0} \to \mathbb{R}_{>1}$ be monotonically increasing to infinity with $\vartheta(x) = \mathcal{O}(x^{1/4})$ and $\vartheta(2^{j+1})/\vartheta(2^j) \to 1$ as $j \to \infty$. For each $j \in \mathbb{N}$, we let
\[ B_j := \frac{2^j}{f(2^j) \sqrt{\vartheta(2^j)}}, \quad b_j := \frac{2}{3} B_j. \]

Let $S = (\alpha_i)_i$ denote a sequence running through all rationals in $(0, 1)$ whose denominators are in $M := \bigcup_{j \geq 1} \{ n \in \mathbb{N} : b_j \leq n \leq B_j \}$ such that $i \mapsto H(\alpha_i)$ is non-decreasing. Then, $S$ is an optimal regular system in $(0, 1)$.

Proof. Let $X \geq 2$. There are strictly less than $2X^2$ rational numbers in $(0, 1)$ with height bounded by $X$. We take $J = J(X)$ to be the largest integer $j \geq 1$ such that $B_j \leq X$. Then, for $X$ large enough, there are at least
\[ \sum_{J \leq j, B_j \leq q \leq B_j} \varphi(q) \geq \sum_{J \leq j} \left( \frac{1}{3\pi^2} B_j^2 + \mathcal{O}(B_j \log B_j) \right) \]
\[ \geq \frac{22J}{6\pi^2} \frac{2^{2J}}{f(2^j) \vartheta(2^j)} + \mathcal{O}(J2^j) \]
\[ > \left( \frac{X}{3\pi} \right)^2 \]
distinct such rationals in $(0, 1)$ with height not exceeding $X$. Hence, we obtain $\sqrt{\vartheta} \leq H(\alpha_i) \leq \sqrt{25\pi^2(i+1)}+1$ for $i$ sufficiently large. Let $Q \in \mathbb{N}$, $I \subseteq [0, 1]$ be a non-empty interval, and let $F$ denote the set of $\xi \in I$ satisfying the inequality
\[ \|q \xi\| < Q^{-1} \text{ with some } 1 \leq q \leq \frac{1}{1000} Q. \]  
Note that \( F \) has measure at most
\[
\sum_{q \leq \frac{1}{1000} Q} \left( \frac{2}{qQ} q \lambda(I) + \frac{2}{qQ} \right) = \frac{1}{500} \lambda(I) + O\left(\frac{\log Q}{Q}\right) < \frac{1}{400} \lambda(I)
\]
for \( Q \geq Q_0 \) where \( Q_0 = Q_0(S, I) \) is sufficiently large. Let \( \{p_i/q_i\}_{1 \leq j \leq t} \) be the set of all rationals \( p_i/q_i \in (0, 1) \) with \( q_j \in M, \frac{1}{1000} Q < q_j < Q \) that satisfy
\[
\left| \frac{p_j}{q_j} - \frac{p_{j'}}{q_{j'}} \right| > \frac{2000}{Q^2}
\]
whenever \( 1 \leq j \neq j' \leq t \). Observe that for \( J \) as above with \( X = Q \) sufficiently large, it follows that
\[
\{q \in M : b_j \leq q \leq B_j\} \subseteq \left\{ \frac{Q}{1000}, \frac{Q}{1000} + 1, \ldots, Q \right\}
\]
holds and there are hence at least \( \frac{1}{32} B_j^2 + O(B_j \log B_j) > \frac{1}{400} Q^2 \) choices of \( p_i/q_i \in (0, 1) \) with \( q_j \in M \) and \( \frac{1}{1000} Q < q_j < Q \). Due to \( \lambda(I \setminus F) > \frac{399}{400} \lambda(I) \), we conclude \( t \geq 400 \left( \frac{399}{400} \right) \lambda(I) \). Thus, taking \( c_1 := \frac{1}{1000}, c_2 := 2000, \) and \( c_3 := \frac{399}{400} \) in \((15), (16) \) and \((17)\), respectively, \( S \) is shown to be an optimal regular system.

Now we can proceed to the proof of Theorem 2.

### 3.2 Proof of Theorem 2

We argue in two steps depending on whether or not the series \((5)\) converges. Proposition \((1)\) implies the announced \( \Theta \)-bounds on the additive energy of \( A_N \), in both cases.

(i) Suppose \((5)\) diverges, and fix \( s > 0 \). Let \( \vartheta : \mathbb{R}_{>0} \to \mathbb{R}_{>1} \) be monotonically increasing to infinity with \( \vartheta(x) = \mathcal{O}(x^{1/s}) \) such that
\[
\psi(n) := \frac{1}{nf(n) \vartheta(n)}
\]
satisfies the divergence condition \((18)\). Thus, \( \vartheta\left(2^j\right)/\vartheta\left(2^{j-1}\right) \to 1 \) as \( j \to \infty \). Hence, \( S = (\alpha_i) \) from Lemma \((6)\) is an optimal regular system. Furthermore, if \( \alpha_i = \frac{m_i}{n_i} \), then \( i \geq cn^2 \) holds true with a constant \( c = c(f, \vartheta) > 0 \) due to \( b_j \leq n \leq B_j \), for some integer \( J \), and
\[
\sum_{j \leq J-1} \sum_{b_j \leq m \leq B_j} \phi(m) = \Theta(B_j^2).
\]
Therefore, \( \psi(i) \leq c^{-1} n^{-2} (f(cn^2) \vartheta(cn^2))^{-1} \). The growth assumption on \( f \) and
the growth bound \( \vartheta(x) = \mathcal{O}(x^{1/s}) \) yields that if \( j \) is large enough, then \( b_j \leq
n ≤ B_j implies cn^2 > 2^j and hence we obtain ψ(i) ≤ c^{-1}n^{-2}(f(2^j)\vartheta(2^j))^{-1}.

Combining these considerations, we have established that

\[ n\psi(i) = O\left(2^{-j}(\vartheta(2^j))^{-1/2}\right). \]

Moreover, for a function g : \mathbb{N} → \mathbb{R}_{>0}, we let \( E_g \) denote the set of \( α \in (0, 1) \)
such that for infinitely many \( j \) there is some \( n \) with \( b_j ≤ n ≤ B_j \) satisfying

\[ ||nα|| = O(2^{-j}g(j)). \]

Set \( h(j) := (\vartheta(2^j))^{-1/2} \). Applying Theorem 3 with ψ as in (19), implies that \( E_h \) has full measure. Therefore, for any \( α \in E_h \) we get

\[ ||nα|| ≤ n|α - α_i| = O\left(2^{-j}(\vartheta(2^j))^{-1/2}\right) \tag{20} \]

for infinitely many \( j \). Now if \( b_j ≤ n ≤ B_j \) for \( j \) sufficiently large and \( n, α \) as in (20), then it follows that by taking any integer \( m ≤ (f(2^j))^{7/6}(\vartheta(2^j))^{-1/6} \) also the multiples

\[ nm ≤ 2^j (f(2^j))^{7/6}(\vartheta(2^j))^{-1/6} \]

satisfy that \( 1_{[s, t]}(||(α(mn)||) = 1 \) where \( T_j = O\left(2^j\left(f(2^j)\right)^{-7}\right) \) is as in the Proof of Proposition 1. If additionally \( γ - 1 ≥ -β \) holds, then we obtain that

\[ \text{rep}_{A_T, A_{T_j}}(mn) ≥ 1/2 \left(f(2^j)\right)^{-β} \]

holds for \( j \) sufficiently large. By (14), we obtain

\[ R([-s, s], α, T_j) ≥ C\left(f(2^j)^{2γ-β}(\vartheta(2^j))^{1/3} \right) \]

for infinitely many \( j \) where \( C > 0 \) is some constant. For the optimal choice of the parameters \( β, γ > 0 \), we are therefore led to find the maximal \( β \) such that \( 2γ - β ≥ 0 \) and \( γ - 1 ≥ -β \) is satisfied. The (unique) solution is \( β = 2/3 \) and \( γ = 1/3 \). Hence, (6) follows for \( α \in E_h \).

(ii) Suppose the series \( (5) \) converges. We keep the same sequence as in step (i) while taking \( \vartheta(x) = 1 + \log(x) \), as we may. The arguments of step (i) show that any \( α \in E_h \), where \( h(j) = j^{-1/2} \), satisfies \( 9 \); now the conclusion is that \( E_h \) has Hausdorff dimension equal to the reciprocal of

\[ \liminf_{x→∞} \frac{-\log(ψ(x))}{\log x} = 1 + \liminf_{x→∞} \frac{\log f(x)}{\log x}. \]

Thus, the proof is complete.

Concluding remarks We would like to mention two open problems related to this article. The first problem concerns extensions of Theorem 1:

Problem 1. Let \( (a_n)_n \) be an increasing sequence of positive integers with \( E(A_N) = Ω\left(N^{3}\right) \). Has the complement of NPPC\((a_n)_n\) Hausdorff dimension zero; or is it, in fact, empty?

The second problem is related to Corollary 1:

Problem 2. How large has \( E(A_N) \) to be for ensuring that NPPC\((a_n)_n\) has full Lebesgue measure?
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