Strong coupling behavior of the neutron resonance mode in unconventional superconductors

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(Dated: May 11, 2014)

We analyze whether and how the neutron resonance mode in unconventional superconductors is affected by higher order corrections in the coupling between spin excitations and fermionic quasiparticles and find that in general such corrections cannot be ignored. In particular, we show that in two spatial dimensions \((d = 2)\) the corrections are of same order as the leading, one-loop contributions demonstrating that the neutron resonance mode in unconventional superconductors is a strong coupling phenomenon. The origin of this behavior lies in the quantum-critical nature of the low energy spin dynamics in the superconducting state and the feedback of the resonance mode onto the fermionic excitations. While quantum critical fluctuations occur in any dimensionality \(d \leq 3\), they can be analyzed in a controlled fashion by means of the \(\varepsilon\)-expansion \((\varepsilon = 3 - d)\), such that the leading corrections to the resonance mode position are small. Regardless of the strong coupling nature of the resonance mode we show that it emerges only in a close proximity of the phase of the superconducting gap function \(\Delta\) that vanishes at the quantum critical point, where the magnetic correlation length \(\xi\) diverges. As a result, precisely at those points

\[
\text{Im} \chi(\mathbf{Q}, \omega) = Z_{\text{res}} \delta(\omega - \Omega_{\text{res}}) + \text{Im} \chi_{\text{inc}}(\omega),
\]

where \(Z_{\text{res}}\) is the spectral weight of the resonance mode while the imaginary part of the incoherent part \(\chi_{\text{inc}}(\omega)\) vanishes for \(|\omega| < 2\Delta\).

A promising explanation for the resonance mode that permits detailed comparison with experiment was obtained within an one-loop approach\(^{24,25}\). Within this approach, collective excitations of the superconductor are sensitive to the coherence factors of the BCS-like wave function. The coherence factors determine scattering-matrix elements for (i) interactions between Bogoliubov quasiparticles and (ii) interactions between quasiparticles and the pair condensate. In the case of spin-spin coupling (where the scattering matrix is odd under time reversal), the latter processes leads to the emergence of the resonance mode if the phase of the superconducting gap function \(\Delta_{\mathbf{k}}\) takes distinct phases at momenta \(\mathbf{k}_F\) and \(\mathbf{k}_F + \mathbf{Q}\) (assuming that both belong to the Fermi surface).

Unconventional superconductivity often occurs in close proximity of competing states with long-range order. Consequently, the concomitant quantum criticality requires an investigation of the microscopic structure of the superconducting state (and in particular, of the resonance mode) that goes beyond the usual one-loop approach. It is well known, that itinerant systems in the vicinity of a spin-density-wave quantum-critical point are characterized by the energy scale \(\omega_{\text{sf}} \propto \xi^{-2}\) of the normal state spin excitation spectrum\(^{23,26}\) that vanishes at the quantum critical point, where the magnetic correlation length \(\xi\) diverges. As a result, precisely at those points

PACS numbers:

The emergence of a resonance mode in the inelastic spin excitation spectrum below the superconducting transition temperature has become an important indicator for unconventional superconductivity in a range of correlated materials. First observed\(^{12,13}\) in YBa\(_2\)Cu\(_3\)O\(_{7-\delta}\), the phenomenon occurs in other cuprate superconductors\(^{26,27}\), in heavy-electron superconductors\(^{9,11}\), and in iron-based materials\(^{12,13}\). Below \(T_c\), one observes essentially two effects in the inelastic neutron spectrum: (i) the low-energy spectral weight is suppressed for energies \(\omega < 2\Delta\), where \(\Delta\) is the magnitude of the superconducting gap; and (ii) a sharp peak occurs at \(\Omega_{\text{res}} < 2\Delta\) that is centered around a finite momentum \(\mathbf{Q}\). Usually, \(\mathbf{Q}\) coincides with the ordering vector of a nearby antiferromagnetic state. For \(T = 0\), the imaginary part of the dynamic susceptibility at the momentum \(\mathbf{Q}\) can be described as

\[
\text{Im} \chi(\mathbf{Q}, \omega) = Z_{\text{res}} \delta(\omega - \Omega_{\text{res}}) + \text{Im} \chi_{\text{inc}}(\omega),
\]

where \(Z_{\text{res}}\) is the spectral weight of the resonance mode while the imaginary part of the incoherent part \(\chi_{\text{inc}}(\omega)\) vanishes for \(|\omega| < 2\Delta\).

A promising explanation for the resonance mode that permits detailed comparison with experiment was obtained within an one-loop approach\(^{24,25}\). Within this approach, collective excitations of the superconductor are sensitive to the coherence factors of the BCS-like wave function. The coherence factors determine scattering-matrix elements for (i) interactions between Bogoliubov quasiparticles and (ii) interactions between quasiparticles and the pair condensate. In the case of spin-spin coupling (where the scattering matrix is odd under time reversal), the latter processes leads to the emergence of the resonance mode if the phase of the superconducting gap function \(\Delta_{\mathbf{k}}\) takes distinct phases at momenta \(\mathbf{k}_F\) and \(\mathbf{k}_F + \mathbf{Q}\) (assuming that both belong to the Fermi surface).

Unconventional superconductivity often occurs in close proximity of competing states with long-range order. Consequently, the concomitant quantum criticality requires an investigation of the microscopic structure of the superconducting state (and in particular, of the resonance mode) that goes beyond the usual one-loop approach. It is well known, that itinerant systems in the vicinity of a spin-density-wave quantum-critical point are characterized by the energy scale \(\omega_{\text{sf}} \propto \xi^{-2}\) of the normal state spin excitation spectrum\(^{23,26}\) that vanishes at the quantum critical point, where the magnetic correlation length \(\xi\) diverges. As a result, precisely at those points
on the Fermi surface that are connected by the magnetic ordering vector (i.e., \( k_F \) and \( k_F + Q \)) the quasiparticle lifetime for energies above \( \omega_{sf} \) deviates from the standard Fermi-liquid result\(^{27,28}\). In two- or three-dimensional systems, these sets of points are referred to as hot spots or hot lines of the Fermi surface, respectively. So far, it is unclear whether or not quantum-critical fluctuations that are relevant at higher energies \( \omega > \omega_{sf} \) and contribute to the incoherent contribution \( \text{Im} \chi^{\text{inc}}_Q (\omega) \) in Eq. (1) lead to any feedback on the spectral features of the resonance mode. For example, if higher order vertex corrections to the dynamic spin susceptibility are governed by excitations with energies smaller than \( \omega_{sf} \) (and thus behave similar to Fermi-liquid quasiparticles), then the weak coupling picture is expected to be robust. On the other hand, if such virtual excitations are quantum critical, i.e. have typical energies larger than \( \omega_{sf} \), the analysis becomes more subtle.\(^{29,30}\)

Another open issue is related to the sensitivity of the resonance mode with respect to the variation of the phase of the superconducting order parameter on the Fermi surface. Whether or not this is the case if one takes into account higher orders in perturbation theory needs to be explored. The relevance of strong coupling behavior for the resonance mode is also suggested by the observation of a nearly universal ratio of \( \Omega_{res} \) and \( \Delta \) in a wide range of systems,\(^{31}\) which one would not expect from weak coupling theory.

In this paper we evaluate self-energy and vertex corrections to the dynamic spin susceptibility in the superconducting state and determine higher order corrections to the neutron-resonance mode. For \( d = 2 \), we find that both, self-energy and vertex corrections, cause significant changes in the resonance and cannot be ignored, except for very weak coupling strength. Near a magnetic quantum-critical point these corrections are of same order as the leading one-loop result, revealing that the resonance mode is a strong coupling phenomenon. Self-energy corrections are primarily caused by singularities in the fermionic spectrum that were caused by the resonance mode in the first place. In contrast, vertex corrections are dominated by quantum-critical fluctuations contributing to \( \text{Im} \chi^{\text{inc}}_Q (\omega) \), due to the fact that virtual processes lead to the emergence of the resonance mode. In order to develop a controlled theory of the resonance mode, we perform an \( \varepsilon \)-expansion around the upper critical dimension \( d_{uc} = 3 \), that reveals how quantum-critical fluctuations affect the dynamic spin susceptibility as function of the dimensionality of the system. These results demonstrate that the theory of Refs. \(^{14-23}\) is applicable for three-dimensional superconductors including moderately anisotropic materials. On the other hand, for \( d = 2 \) the neutron resonance mode is a strong coupling phenomenon and our results show that no controlled theory for the effect exists so far. Finally, we demonstrate that higher-order vertex corrections only lead to a resonance mode if the phases of the gap at \( \Delta_{k_F} \) and \( \Delta_{k_F+Q} \) are distinct.

I. THE SPIN FERMION MODEL

Consider an unconventional superconductor in the vicinity of a spin density wave instability. Low-energy spin excitations of the system (i.e. paramagnons) can be described\(^{27,28}\) in terms of a spin-1 boson \( S_q \) that is characterized by the dynamic spin susceptibility

\[
\chi_q (\omega) = \frac{1}{r_0 + c_s (q - Q)^2 - \Pi_q (\omega)},
\]

where \( q \) and \( \omega \) are the wavevector and frequency, \( Q \) is the antiferromagnetic ordering vector and \( r_0 \) determines the distance to the instability. The spin dynamics is described by the self-energy \( \Pi_q (\omega) \). Hereafter, \( \chi_q (\omega) \) refers to the retarded susceptibility, while \( \chi_q (\omega_n) \) is used for the corresponding Matsubara function. Similar notations are used for fermionic Green’s functions and self-energies. The spin dynamics, encoded in \( \Pi_q (\omega) \), is a consequence of coupling of the collective spin degrees of freedom \( S_q \) to low-energy particle-hole excitations. At low energies and in the normal state, the dominant contribution to the imaginary part of \( \Pi_q (\omega) \) for \( q \approx Q \) comes from the fermionic quasiparticles in the vicinity of the hot spots (or hot lines) on the Fermi surface (defined by the relation \( \varepsilon_{k_F+Q} = \varepsilon_{k_F} \), where \( \varepsilon_k \) is the bare fermionic single-particle dispersion measured relative to the Fermi energy). In this paper, we consider commensurate magnetic order, where \( 2Q \) is equal to a reciprocal lattice vector, i.e. \( \varepsilon_{k+2Q} = \varepsilon_k \).

Let \( \psi^\dagger_{k\alpha} \) be the creation operator of a fermion with the momentum \( k \) and spin index \( \alpha \). Coupling between spin fluctuations and fermionic quasiparticles is described by the following term in the Hamiltonian\(^{27,28}\)

\[
\mathcal{H}_{int} = g \int d^dx \mathbf{S} \cdot (\psi^\dagger_{\alpha} \sigma_{\alpha\beta} \psi_{\beta}),
\]

where operators are given in real space and \( \sigma \) is the vector of the Pauli matrices. A microscopic derivation of this model is possible in the limit of weak electron-electron interactions and may be based upon a partial resummation of diagrams in the particle-hole spin-triplet channel. In this case, it is usually not permissible to approach the regime in the close proximity of the magnetic critical point, which for generic Fermi surface shape requires a threshold strength of the interaction. However, we may consider this spin-fermion model as a phenomenological theory of low-energy quasiparticles coupled to spin fluctuations that is valid only at energies small compared to the initial electron bandwidth. At low energies, fermions near the hot spots determine the spin dynamics and are crucial for the spin-fluctuation-induced pairing state. In this case the electronic spectrum near the hot spots may be linearized, \( \varepsilon_k = v_k \cdot k \) (where \( v_k \) is the quasiparticle velocity at momentum \( k \)). In what follows, we assume that \( g \) is small compared to the corresponding fermionic scales, which implies smallness of the dimensionless pa-
rameter $\gamma$

$$\frac{\gamma}{N} = \frac{g^2}{4\pi v_\parallel v_\perp k_F^d N^{-2}} \ll 1. \quad (4)$$

Here $v_\parallel$ and $v_\perp$ are the projections of $v_k$ at the hot spot onto directions that are parallel and perpendicular to $Q$, respectively, and $N$ is the number of hot spots (lines), or generally the number of fermion flavors that couple to the spin excitations. In what follows, we assume $v_\parallel = v_\perp = v_F/\sqrt{2}$ and use $v_F = |v|$, see Fig. 2.

The spin-fermion model can be described by an effective action that in the superconducting state takes the form

$$S = -\frac{1}{2} \int \Psi_k^\dagger \hat{G}^{-1}_{0,k} \Psi_k + \frac{1}{2} \int \chi_{0,q} S_q \cdot S_{-q} + g \int_{k,k'} (\Psi_k^\dagger \hat{\alpha} \Psi_k) \cdot S_{k-k'}, \quad (5)$$

where

$$\Psi_k = \left( \begin{array}{l} \psi_{k+} \\ \psi_{k-} \\ \psi_{-k+}^T \\ \psi_{-k-}^T \end{array} \right)^T$$

is the extended Gor'kov-Nambu-spinor. Here, we use the following notations

$$\hat{\alpha} = \begin{pmatrix} \sigma^i \\ 0 \sigma^y \sigma^y \sigma^y \end{pmatrix} \quad \text{and} \quad \hat{\beta} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -1_2 \end{pmatrix}. \quad (6)$$

The matrices $\sigma^i$ are the usual Pauli matrices and in Eq. (5) we combine the Matsubara frequencies and momenta into $\hat{k} = (k, i\omega_n)$ and use the short-hand notation

$$\int \ldots = T \sum_n \int \frac{d^d k}{(2\pi)^d} \ldots. \quad$$

In the basis of the extended spinor $\Psi_k$, the bare fermion propagator is given by

$$\hat{G}_{0,k}^{-1} = i\omega_n \mathbf{1} - \varepsilon_k \hat{\beta}. \quad (7)$$

The corresponding self-energy matrix in the superconducting state can be written as

$$\hat{\Sigma}_k = i\omega_n (1 - Z_k) \mathbf{1} + \delta \varepsilon_k \hat{\beta} + \Phi \hat{\alpha}^\Delta + \Phi^* \hat{\alpha}^{\Delta^*}, \quad (8)$$

where we defined the matrices

$$\hat{\alpha}^\Delta = \begin{pmatrix} 0 & i\sigma^y \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\alpha}^{\Delta^*} = \begin{pmatrix} 0 & 0 \\ -i\sigma^y & 0 \end{pmatrix}. \quad (9)$$

Using this definitions the dressed Green’s function can be expressed as

$$\hat{G}_k^{-1} = \hat{G}_{0,k}^{-1} - \hat{\Sigma}_k. \quad (10)$$

Explicitly, we obtain for the matrix Green’s function:

$$\hat{G}_k = \frac{i\omega_n Z_k \delta \varepsilon_k + (\varepsilon_k + \delta \varepsilon_k) \hat{\beta} + \Phi \hat{\alpha}^\Delta}{(i\omega_n Z_k)^2 - (\varepsilon_k + \delta \varepsilon_k)^2 - \Phi^2} \begin{pmatrix} g_k^{(p)} & 0 & -\mathcal{F}_k \\ 0 & g_k^{(p)} - \mathcal{F}_k & 0 \\ 0 & -\mathcal{F}_k & g_k^{(h)} \end{pmatrix}. \quad (11)$$

The resulting gap function $\Delta_k = \Phi_k Z_k$ will, as usual, be determined from the solution of the corresponding self-consistency equations.

### A. Normal-state behavior

In the normal state spin fluctuations can decay into gapless electron-hole excitations which leads to overdamped spin dynamics in agreement with observations obtained in various neutron scattering experiments. The corresponding dynamic susceptibility

$$\chi_q(\omega) = \frac{1}{r + e_s (q - Q)^2 + i\gamma \omega}, \quad (12)$$

where $\gamma$ is given by Eq. (13), can be obtained by evaluating the bosonic self-energy (13)

$$\Pi_q = -2g^2 \int \frac{d^d k}{(2\pi)^d} g_k^{(p)} g_{0,k+q}^{(p)}. \quad (14)$$

In order to calculate the one-loop diagrams it is convenient to linearize the spectrum around the hot spots...
\[ \mathbf{k} = \mathbf{k}_F + \mathbf{p} \] with \(|\mathbf{p}| \ll k_F\), which dominate the integrals. As can be seen in Figure 2, we can linearize \( \varepsilon_k \approx k_{F} \mathbf{p} = v_F \cdot \mathbf{p} \), where \( v_F \) is the Fermi velocity at the corresponding hot spot. Each of the \( N \) hot spots contribute equally, which allows us to focus on one of them. Along the same lines we can also linearize the connected hot spot at \( \mathbf{k}'_F = \mathbf{k}_F + \mathbf{Q} \) via

\[ \begin{align*}
\varepsilon_{k \approx k_F} &= v_F \cdot \mathbf{p} = v_{\perp} p_{\perp} + v_{\parallel} p_{\parallel}, \\
\varepsilon_{k + \mathbf{Q} \approx k'_F} &= v_F \cdot \mathbf{p} = v_{\perp} p_{\perp} - v_{\parallel} p_{\parallel}.
\end{align*} \tag{13} \]

The velocities \( v_{\perp} \) and \( v_{\parallel} \) are the perpendicular and parallel projections of \( v_F \) on \( \mathbf{Q} \). Introducing new integration variables \( \varepsilon = v_{\perp} p_{\perp} + v_{\parallel} p_{\parallel}, \varepsilon' = v_{\perp} p_{\perp} - v_{\parallel} p_{\parallel} \) it is now possible to approximate

\[ \frac{1}{L^2} \sum_k f(\varepsilon_k, \varepsilon_k + \mathbf{Q}, \Delta_k, \Delta_k + \mathbf{Q}) = \frac{N}{8\pi^2 v_{\perp} v_{\parallel}} \int d\varepsilon' d\varepsilon \frac{f(\varepsilon, \varepsilon', \Delta_k, \Delta_k + \mathbf{Q})}{1 + \frac{d}{2} \varepsilon}. \tag{14} \]

The \( \pm \) signs refer to different gap symmetries; we consider \( \Delta_k + \mathbf{Q} = \pm \Delta_k \mathbf{F} + \mathbf{Q} \) to be constant around the hot spots. In order to simplify our calculations we will set \( v_{\parallel} = v_{\perp} = v_F / \sqrt{2} \) in future calculation, which is a suitable approximation for many known unconventional superconductors like Bi-2212 (compare with Fig. 2).

Under the assumption that we can neglect the momentum dependence of the self-energy near the hot spots, the self-energy \( \Pi_{\mathbf{Q}}(\omega) \) yields

\[ \Pi_{\mathbf{Q}}(\omega) = \Pi_{\mathbf{Q}}(0) - i\gamma \omega. \]

The static contribution \( \Pi_{\mathbf{Q}}(0) \) renormalizes the bare “mass” \( r_0 \rightarrow r = r_0 - \Pi_{\mathbf{Q}}(0) \) and determines the correlation length \( \xi \) via \( r = c_s \xi^2 \).

In two dimensions \( (d = 2) \), coupling of normal-state fermionic quasiparticles with overdamped spin fluctuations leads to renormalization of the fermionic spectrum. Already at one-loop level, one finds non-trivial behavior of the fermionic self-energy at the hot-spots\(^{23}\):

\[ \Sigma_{\mathbf{k}_F}(i\omega_n) = \frac{-3g^2 \text{sign}(\omega_n)}{2\pi v_F \sqrt{c_s} \gamma} \left( \sqrt{\omega_{\text{sf}} + |\omega_n|} - \sqrt{\omega_{\text{sf}}} \right). \tag{15} \]

Here the frequency \( \omega_{\text{sf}} = r / \gamma \) plays the role of the crossover scale. Indeed, for energies below \( \omega_{\text{sf}} \) the self-energy \( \Sigma \) may be approximated by the Fermi-liquid-like expression \( \Sigma(i\omega_n) = -i\omega_n \lambda \) with the dimensionless coupling constant \( \lambda_{d=2} = 3g^2 / (4\pi v_F \sqrt{c_s} \gamma) \). However, at higher energies \( |\omega_n| > \omega_{\text{sf}} \) the fermionic spectrum exhibits non-Fermi-liquid behavior as the self-energy \( \Sigma \) on the imaginary axis becomes proportional to the square root of the frequency, \( \Sigma(i\omega_n) \propto i \text{sign}(\omega_n)|\omega_n|^{1/2} \).

For our subsequent analysis, it will be important to determine the fermionic self-energy for arbitrary dimensions \( d \ll 3 \) using the \( \varepsilon \)-expansion with the small parameter

\[ \varepsilon = 3 - d. \tag{16} \]

Similarly to Eq. \( \text{(15)} \), we find the non-Fermi-liquid behavior at high energies

\[ \Sigma^{(\nu)}(i\omega_n) = \begin{cases} -i\omega_n \lambda & \text{if } |\omega_n| \ll \omega_{\text{sf}} \\
-\omega_n \left| \frac{\Omega}{\omega_n} \right|^{|\nu|/2} & \text{if } |\omega_n| \gg \omega_{\text{sf}}, \end{cases} \tag{17} \]

that is characterized by the coupling constant

\[ \lambda = \left( 1 - \frac{\varepsilon}{2} \right) \left( \frac{\tilde{\Omega}}{\omega_{\text{sf}}} \right)^{\varepsilon/2}, \tag{18} \]

and the energy scale

\[ \Omega = \gamma^{-1} \left[ \frac{3g^2 K_{d-1}}{4v_F^{1-\varepsilon/2}(1 - \frac{\varepsilon}{2}) \sin \frac{\pi \varepsilon}{2}} \right]^{2/\varepsilon}, \tag{19} \]

where \( K_d = 2^{1-d} \pi^{-d/2} \Gamma(d/2) \) contains the information about the surface of a unit sphere in \( d \) dimensions. On the real axis this yields in the non-Fermi liquid regime

\[ \Sigma^{(\nu)}(\omega) \propto -\omega \left| \frac{\omega_0}{\omega} \right|^{\varepsilon/2} e^{i\pi \text{sign}(\omega)/4}. \tag{20} \]

For \( d = 3 \), we find \( \Sigma(i\omega_n) = -i3g^2 \text{sign}(\omega_n) \log(|\omega_0 / |\omega_n||) \) with the characteristic frequency \( \omega_0 = c_s q_0^2 / \gamma \), where \( |\mathbf{q}| < q_0 \) is the bosonic momentum cutoff. On the real axis this becomes

\[ \Sigma^{(\nu)}(\omega) \propto -\omega \log \left| \frac{\omega_0}{\omega} \right| - i\frac{\pi}{2} |\omega|. \tag{21} \]

Note, that Eq. \( \text{(21)} \) holds only for momenta on the hot lines, in contrast to Ref. \( \text{[3]} \), where within the marginal Fermi-liquid phenomenology the same frequency dependence is assumed everywhere on the Fermi surface.

The above results for the normal-state fermionic dynamics demonstrate that the upper critical dimension for non-Fermi liquid behavior of the fermionic spectrum at the hot-spots is \( d_{\text{nl}} = 3 \). Near three dimensions we can develop an \( \varepsilon \)-expansion which is controlled for arbitrary \( N \). As we show below, in the limit \( \varepsilon \rightarrow 1 \) (i.e. for \( d = 2 \)) the \( \varepsilon \)-expansion is not reliable anymore. One might hope that an expansion with respect to \( 1/N \) can be developed. As shown in Refs. \( \text{[28,33]} \) for \( d = 2 \) and gapless fermions in the normal state, the usual loop expansion does not correspond to an expansion in \( 1/N \), making a controlled expansion in \( 1/N \) a complicated task, amounting to the summation of all planar diagrams. An important question is whether the dynamics in the superconducting state, where fermions are gapped, is still plagued by similar problems.

### B. Pairing instability

In order to investigate the emergence of the resonance mode, we will consider the spin-fermion model deep in the superconducting state. For this we need an estimate of
the superconducting gap amplitude at low temperatures. Here, we obtain this quantity by combining the numerical solution of Ref. 35 with the linearized gap equations near \( T_c \) (for varying dimension \( d \)). Since these gap equations were solved elsewhere,\(^{35,40}\) we merely summarize the key results to make the article self-contained and in order to introduce the notation used throughout this paper.

In the superconducting state we express anomalous averages through the self-energy \( \Phi_k \) and determine this quantity, along with the associated gap function \( \Delta_k = \Phi_k / Z_k \) self-consistently. Since the dominant contribution to the bosonic self-energies comes from the hot spots, one obtains for these momenta \( \Delta_k + Q = \pm \Delta_k \). In the case of cuprate superconductors, the minus sign corresponds to the d-wave pairing. In the case of the iron-based superconductors, the minus sign corresponds to the \( s\_z \) state or a d-wave state, depending on the typical spin-momentum vector \( Q \).

For \( d = 2 \), the gap equation determining \( \Delta_k \) was solved in Refs. \(^{35,40}\). It was found that the amplitude of the gap function \( \Delta(T \ll T_c) \) is proportional to the instability temperature \( T_c \), with \( 2\Delta / T_c \approx 5 \). Thus, in what follows we will merely determine \( T_c \) and use it as an estimate for the gap amplitude in the superconducting state. Related pairing problems with singular pairing interactions were studied in Ref. 40. In what follows we summarize the key results for quantum-critical pairing as a function of \( \epsilon \).

The one-loop fermionic self-energy matrix in Nambu-space follows from Eq. 5:

\[
\hat{\Sigma}_k = g^2 \int q \sum_{i=1}^{3} \hat{\alpha}^i \chi_q \hat{G}_{k-q} \hat{\alpha}^i = 3g^2 \int q \chi_q \hat{G}_{k-q}. \tag{22}
\]

Using Eq. 7 and this self-energy we obtain the functions:

\[
Z_k = 1 - \frac{3g^2}{2i\omega_n} \int q \chi_q (\hat{G}^{(p)}_{k-q} + \hat{G}^{(h)}_{k-q}),
\]

\[
\delta \epsilon_k = \frac{3g^2}{2} \int q \chi_q (\hat{G}^{(p)}_{k-q} - \hat{G}^{(h)}_{k-q}),
\]

\[
\Phi_k = 3g^2 \int q \chi_q \hat{F}_{k-q}. \tag{23}
\]

The normal and anomalous Green’s function in the superconducting state are thus given by Eq. 10. The self-energies near the hot spots are weakly momentum-dependent and therefore we assume the dispersion correction \( \delta \epsilon_k = 0 \) for the determination of the superconducting transition temperature, because the frequency dependence is dominant in the \( Z_k \approx Z(i\omega_n) \) term. Integrating over fermionic energies \( \epsilon_k \) then yields the linearized Eliashberg equations\(^{31,32}\) [noting that \( \phi_{kF+Q}(i\omega_n) = -\phi_{kF}(i\omega_n) \)]

\[
\Phi_{kF}(i\omega_n) = \pi T \sum_m D(i\omega_n - i\omega_m) \frac{\Phi_{kF+Q}(i\omega_m)}{|\omega_m|Z(i\omega_m)}, \tag{24}
\]

\[
Z_{kF}(i\omega_n) = 1 + \frac{\pi T}{\omega_n} \sum_m D(i\omega_n - i\omega_m) \text{sign}(\omega_m).
\]

that determine \( T_c \). The self-energies are evaluated at the momenta \( k_F \) and \( k_F + Q \), which is suppressed in the notation. The effective coupling function in Eq. (24) is given by the integral

\[
D(i\omega_n) = \frac{3g^2}{4\pi^2 v_F} \int \frac{d^{d-1}q_{\parallel}}{(2\pi)^{d-1}} \frac{1}{r + |\omega_n| + \epsilon_{q_{\parallel}}}. \tag{25}
\]

Here integration over momenta is performed over the \( d-1 \) components of the bosonic momentum that are parallel to the Fermi surface\(^{39}\). The result of the integration is given by

\[
D(i\omega_n) = \frac{1 - \epsilon/2}{2\pi} \left[ \frac{\bar{\Omega}}{\omega_{sf} + |\omega_n|} \right]^{\epsilon/2}, \tag{26}
\]

with the energy scale \( \bar{\Omega} \) defined in Eq. 19.

The Matsubara gap function \( \Delta_n = \Phi(\omega_n)/Z(\omega_n) \) obeys the linearized equation

\[
\Delta_n = \pi T \sum_m D(i\omega_n - i\omega_m) \frac{\Delta_m}{\omega_m} - \frac{\Delta_n}{\omega_n} \text{sign}(\omega_m). \tag{27}
\]

It is convenient to bring this equation to the form

\[
\Delta_n = \frac{1 - \epsilon/2}{2\pi} \left[ \frac{\bar{\Omega}}{2\pi T} \right]^{\epsilon/2} \sum_m \frac{\text{sign}(2m + 1)}{(4m + |2m - 2m|)^{\epsilon/2}} \times \left[ \frac{\Delta_m}{2m + 1} - \frac{\Delta_n}{2n + 1} \right] \text{sign}(2m + 1). \tag{28}
\]

At the quantum critical point, where \( \omega_{sf} = 0 \), it holds that the transition temperature must be determined by
FIG. 4: \( \lambda \) dependence of the universal function \( C_d(\lambda) \) for \( d = 2 \). The dashed curve is an exponential fit to the expected weak-coupling behavior and the dots represent the numerical values of the strong coupling calculation.

a critical value of the coefficient in front of the Matsubara sum. Then the ratio \( \Omega/T_c \) should take a universal value yielding \( T_c \propto \Omega \). Away from the critical point, the transition temperature may be written in the form

\[
T_c = \Omega C_\varepsilon(\lambda),
\]

with universal function \( C_\varepsilon(\lambda) \) of the dimensionless coupling constant \( \lambda \) defined in Eq. (18). For \( \lambda \ll 1 \), we recover the BCS behavior \( T_c \propto \Omega \lambda^{-1/2} \exp(-1/\lambda) \). However, in this regime magnetic correlations are so short-ranged that our continuum theory is no longer the appropriate starting point. On the other hand, if the coupling constant is larger than unity, the pairing is quantum-critical and \( T_c \propto \Omega \). In Fig. 3 we show the numerical dependence of the strong coupling limit \( C_\varepsilon(\infty) \) as a function of the dimensional expansion parameter \( \varepsilon \). From the numerical solution of the gap equation we find for the case of two dimensions \( C_{d=2}(\infty) = 0.198 \) (the full numerical dependence on \( \lambda \) is shown in Fig. 4). Although these results are obtained in the limit of large \( \lambda \), the calculation is well controlled in the limit of small \( \varepsilon \). In our subsequent analysis we therefore use \( \Delta \approx \Omega \) in the regime of strong magnetic correlations (i.e. for \( \lambda > 1 \)).

Finally, for \( d = 3 \) the power-law dependence of the transition temperature (29) becomes

\[
T_c (d = 3) \propto \exp(-\pi/g), \quad (\lambda \gg 1),
\]

which is fully consistent with earlier results. In a recent publication, it was shown that momentum dependent self-energies correct the numerical values of Eq. (29), yet do not modify the \( \bar{\Omega} \) dependence. Here, we ignore these effects in the determination of the pairing amplitude. This is justified since we are only interested in order of magnitude of the pairing gap.

II. ANALYSIS OF THE RESONANCE MODE

Now we discuss the implications of the above picture for the behavior of the resonance mode in the vicinity of a magnetic quantum-critical point.

The analysis of the resonance mode as a spin-exciton in the superconducting state, caused by scattering between quasiparticles and the condensate, was investigated in Refs. [14,19] and based on the determination of the leading contribution to the bosonic spin self-energy. Key concepts for the emergence of the resonance mode can be carried over from the analysis of the leading order terms. To this end, we follow Abanov and Chubukov [12] and discuss the emergence of a resonance mode in the superconducting state. Generally, the imaginary part of bosonic self-energy \( \Pi(q, \omega) \) vanishes at \( T = 0 \) for frequencies \( |\omega| < 2\Delta \), where \( \Delta = |\Delta_{k_F}| \) is the amplitude of the superconducting gap at the hot spot. Within weak coupling theory holds that \( \Im\Pi(q, \omega) \) grows continuously at \( \omega = \pm 2\Delta \). A key quantity for our analysis is therefore the height of this discontinuity:

\[
D \equiv \lim_{\delta \to 0^+} \Im\Pi(q, 2\Delta + \delta).
\]

Once \( D > 0 \) the discontinuity in the imaginary part of \( \Pi(q, \omega) \) translates into a logarithmic divergence of its real part at \( 2\Delta \):

\[
\Re\Pi(q, \omega \sim 2\Delta) = -\frac{D}{\pi} \ln \left( \frac{|\omega - 2\Delta|}{2\Delta} \right). \tag{32}
\]

Within one-loop approximation the susceptibility \( \Pi(q, \omega) \) with self-energy (32) yields Eq. (1). The resulting energy of the resonance mode is

\[
\Omega_{\text{res}} = 2\Delta \left( 1 - e^{-\frac{\pi D}{2\Delta}} \right) \tag{33}
\]
with spectral weight

\[ Z_{\text{res}} = \frac{2\pi^2\Delta}{D} e^{-\frac{q}{\Delta}} = \frac{2\pi^2\Delta}{D} \left( 1 - \frac{\Omega_{\text{res}}}{2\Delta} \right). \]  

(34)

The resonance energy is bound to occur below the particle-hole continuum that sets in at \( \omega = 2\Delta \), while the imaginary part of the incoherent contributions \( \chi^{\text{inc}}_q(\omega) \) vanishes for \( |\omega| < 2\Delta \), see Fig. 3. Below we will see that at one-loop order, the discontinuity is given by \( D_0 = \pi \gamma \Delta \) such that \( Z^0_\chi = \frac{2\pi}{\gamma} \left( 1 - \frac{\Omega_{\text{res}}}{2\Delta} \right) \). The above results are correct as long as \( \Omega_{\text{res}} \) is of order \( \Delta \). However, in the limit \( \lambda \to \infty \) it was shown that \( \Omega_{\text{res}} \approx \sqrt{\omega_0} \Delta \approx \Delta/\lambda \) is determined by the leading low-frequency dependence of \( \text{Re} \Pi_q(\omega) \approx \gamma \omega^2/\Delta \). Here, we focus on the former regime.

Our analysis of corrections to the spin-susceptibility that go beyond the leading order still yields that \( \text{Im} \Pi_q(\omega) < 2\Delta) = 0 \). The emerging discontinuity \( D \) is then solely responsible for all of the qualitative features of the model, including Eqs. (1) and (33). In order to determine the self-energy \( \Pi_q(\omega) \) of the collective spin excitations, we start from the action Eq. (5) and integrate out the gapped fermions, leading to a theory of the collective spin modes:

\[ S = \frac{1}{2} \int \chi^{-1}_{q,n} S_q \cdot \bar{S}_{-q} - \frac{1}{2} \text{tr} \ln \left( -\beta \hat{G}_0^{-1} \right) \]

\[ + \frac{1}{2} n \sum_{n=1}^{\infty} \text{tr} \left[ (\hat{G}_0 \bar{\alpha} \cdot S)^n \right]. \]  

(35)

The usual skeleton expansion follows from expanding the logarithm. The overall factor \( \frac{1}{2} \) in front of the second term is a consequence of the fact that \( \Psi \) and \( \Psi^\dagger \) are not independent Grassmann fields since we had to extend the Nambu spinor due to the spin-changing interaction and one has to be careful in integrating out the fermionic degrees of freedom. Here, we use the identity\( ^{10} \)

\[ \int D\eta e^{-\frac{1}{2} \eta^T A \eta} = \sqrt{\det(A)}, \]  

(36)

where \( \eta \) is a Grassmann vector and \( A \) a quadratic matrix. It is possible to write our path integrals in this form by using the symmetry

\[ \Psi_{k'} = \Psi^T_{-k'} \hat{O} \quad \text{with} \quad \hat{O} = \left( \begin{array}{cc} 0 & I_2 \\ I_2 & 0 \end{array} \right) \]  

(37)

We then obtain

\[ \int D[\Psi_k] e^{-\frac{1}{2} \sum_{k,k'} \Psi_{k'} A_{k'-k} \Psi_k} \]

\[ = \int D[\Psi_k] e^{-\frac{1}{2} \sum_{k,k'} \Psi_{k'}(\hat{O} A_{k'-k}) \Psi_k} \]

\[ = \int D[\Psi_k] e^{-\frac{1}{2} \sum_{k,k'} \Psi_{k'}(\hat{O} A_{k'-k}) \Psi_k} \]

\[ = \sqrt{\det(\hat{O} A')} = e^{\frac{1}{4} \text{tr} \ln(A')}, \]  

(38)

where we define \( A'_{k'-k} = A_{-k'k} \) and use that the determinant of \( \hat{O} \) is 1. The expansion of the logarithm leads to the known perturbation series and it is easy to see that we are allowed to replace \( A' \) with the initial matrix \( \hat{A} \).

In summary, the only difference to the usual integration over two independent Grassmann fields is the factor \( \frac{1}{2} \) in front of the \( \text{tr} \ln(\ldots) \) term of the effective action.

In the superconducting state the propagator matrix \( G_0 \) should be replaced by \( \hat{G}_k \) to make the theory self-consistent.

**A. Resonance mode at one-loop**

Within the one-loop approximation the bosonic self-energy will be of order \( g^2 \). The corresponding contribution to the action is given by

\[ \delta S^{(2)} = \frac{g^2}{4} \text{tr} \left[ (\hat{G} \bar{\alpha} \cdot S)^2 \right] \]

\[ = \frac{g^2}{4} \int_{k,q} S_{-q} \tilde{S}_q \text{tr} \left( \hat{G}_k \bar{\alpha} \cdot \hat{G}_{k+q} \bar{\alpha} \cdot \hat{G}_{k+q} \right) \]

\[ = -\frac{1}{2} \int_q S_{-q} : S_q \Pi_q^{(2)}(i\omega_n). \]  

(39)

Here, the one-loop boson self-energy is

\[ \Pi_q^{(2)}(i\omega_n) = -g^2 \int_k \left( \hat{g}_k^{(p)} \hat{g}_{k+q}^{(p)} + \mathcal{F}_k \bar{\mathcal{F}}_{k+q}^* \right) \]

\[ = \mathcal{F}_k \bar{\mathcal{F}}_{k+q} + \mathcal{F}_k \bar{\mathcal{F}}_{k+q}. \]  

(40)

The self-energy \( \Pi_q^{(2)}(i\omega_n) \) is \( g^2 \) times the spin-susceptibility of fermions in the BCS theory. Using the standard mean-field approach (which here amounts to setting \( Z_k = 1, \delta \epsilon_k = 0 \) and \( \Phi_k = \Delta_k \) constant in frequency) we find that on the real axis

\[ \text{Im} \Pi_q^{(2)}(\omega) = 2g^2 \int \frac{d^d k}{(2\pi)^d} \int \frac{d\epsilon}{\pi} \left( f(\epsilon) - f(\epsilon + \omega) \right) \]

\[ \times \left[ \text{Im} \hat{g}_k^{(p)}(\epsilon) \text{Im} \hat{g}_{k+q}^{(p)}(\epsilon + \omega) \right. \]

\[ \left. + \text{Im} \mathcal{F}_k(\epsilon) \text{Im} \mathcal{F}_{k+q}^{*}(\epsilon + \omega) \right]. \]

(41)

The fermionic propagators in the superconducting state can be written as

\[ \hat{g}_k^{(p)}(\omega) = \frac{v_k^2}{\omega + i0 - \xi_k} + \frac{v_k^2}{\omega + i0 + \xi_k} \]

with the coherence factors \( v_k^2 = 1/2(1 + \epsilon_k/\xi_k) \), \( u_k^2 = 1/2(1 - \epsilon_k/\xi_k) \) and superconducting dispersion \( \xi_k = \sqrt{\epsilon_k^2 + \Delta_k^2} \approx \sqrt{\epsilon_k^2 + \Delta^2} \) at the hot spots. For zero
temperature and positive $\omega > 0$ Eq. \[(41)\] yields
\[
\text{Im} \Pi_{q}^{(2)}(\omega) = 2\pi g^2 \int \frac{d^d k}{(2\pi)^d} \left[ u_k u_{k+q} \right] \delta\left(\omega - \epsilon_k - \epsilon_{k+q}\right).
\] \[(42)\]

Since the self-energy for negative frequencies $\omega < 0$ can be easily obtained from $\text{Im} \Pi_{q}(-\omega) = -\text{Im} \Pi_{q}(\omega)$ we will restrict further calculations to $\omega > 0$. To analyze the resonance mode near the antiferromagnetic ordering vector, we evaluate $\Pi_{q}^{(0)}(\omega)$ at $q = Q$. The integral in Eq. \[(42)\] is dominated by fermions near the hot spots on the Fermi surface. Consequently, $\text{Im} \Pi_{Q}^{(0)}(\omega) < 2\Delta = 0$ leading to a spin gap in the spectrum of the resonance mode.

Near the $2\Delta$ threshold the imaginary part of the bosonic self-energy \[(42)\] exhibits a discontinuity. Within the one-loop calculation, the height of the discontinuity is given by \[(43)\] 
\[
D_0 = \pi \gamma \Delta,
\] with $\gamma$ from Eq. \[(1)\]. This result occurs for sign-changing gap $\Delta_{k_F} = -\Delta_{k_F + Q}$. It is straightforward to analyze Eq. \[(42)\] for the more general pairing state with $\Delta_{k_F} = \Delta_1 e^{i\epsilon_1}$ and $\Delta_{k_F + Q} = \Delta_2 e^{i\epsilon_2}$. Now the discontinuity occurs at $\omega = \Delta_1 + \Delta_2$ and is given by
\[
D_0 = \pi \gamma \sqrt{\Delta_1 \Delta_2} \sin^2 \left(\frac{\epsilon_1 - \epsilon_2}{2}\right).
\] \[(44)\]
The resonance occurs as long as the gap amplitude of both states connected by $Q$ is finite and the phases of the pairing states are distinct. For $\omega > 2\Delta$ the imaginary part of the boson self-energy will grow linearly with $\omega$ until it saturates when it reaches the band-width of the fermions. This general behavior can be seen in the numerical plot shown in Fig. \[5\].

B. Higher order corrections to the resonance mode

In Ref. \[27\] it was shown that for $d = 2$, vertex corrections in the spin-fermion model lead to logarithmic divergences in the normal state. Evaluating the spin-fermion vertex corrections in the superconducting state one finds that this logarithmic divergency is cut off at the scale of the superconducting gap $\Delta$. On the other hand, an analysis of the gap equation for spin-fluctuation-induced pairing yields for arbitrary $d < 3$ that quantum critical excitations with $\omega > \omega_{sf}$ are important for the value of the transition temperature $T_c$. Therefore, we examine the higher orders in perturbation theory in more detail. Specifically, we are interested in corrections to the discontinuity $D$ of Eq. \[(31)\]. Diagrammatically these corrections are given by
\[
\delta \Pi_{q}(\omega_n) = \text{Im} \Pi_{q}^{(1)}(\omega_n) + \text{Im} \Pi_{q}^{(2)}(\omega_n).
\] \[(45)\]

Here, the wavy lines correspond to Eq. \[(2)\] with the one-loop bosonic self-energy. The fermionic lines are the mean-field Green’s functions used in the previous section.

1. Self-energy corrections

The first diagram in Eq. \[(45)\] takes into account the self-energy corrections to the fermionic Green’s functions. To the leading order these are calculated in Appendix \[A\].

The imaginary parts of the normal and anomalous self-energies
\[
\Delta \Sigma_{k}^{(p)}(\omega) = \frac{\omega}{\pi} \quad \Delta \Phi_{k}(\omega) = \frac{\omega}{\pi}.
\] \[(46)\]

are gapped near the hot spots for frequencies $|\omega| < \Delta + \Omega_{res}$, see also Ref. \[16\]. Excluding the strong coupling limit $\lambda \gg 1$ [see the discussion following Eq. \[(34)\]] we find that the excitations around $\omega \sim \pm \Delta$ are well separated from the continuum yielding sharp quasiparticle resonances. The minimal excitation energy $\Delta'$ is determined by the real part of the self-energy \[(40)\] and may be smaller than the mean-field gap $\Delta$ at the hot spots. For zero temperature we can evaluate \[(41)\] for external momentum $Q$ to
\[
\text{Im} \Pi_{Q}^{(2)}(\omega) = 2g^2 \int \frac{d^d k}{(2\pi)^d} \int \frac{d\nu}{\pi} \int_{-\Delta'}^{\Delta'} \frac{d\nu'}{\pi} \left[ \text{Im} \Sigma_{k}^{(h)}(\nu) \text{Im} \Sigma_{k+Q}^{(h)}(\nu + \nu') + \text{Im} \Phi_{k}(\nu) \text{Im} \Phi_{k+Q}(\nu + \nu') \right].
\] \[(47)\]

Obviously, there is still a spin gap of $2\Delta'$, which is as usual determined by twice the minimal excitation energy of the fermionic spectrum. The one-loop fermionic self-energies near the hot spot $k_F$ are functions that depend on the dispersion on the opposite side $\epsilon_{k_F + Q}$, see Appendix \[A\]. Note: Our approach takes into account leading momentum and frequency corrections which arise due to the interaction of the superconducting fermions with the collective boson mode, but not two-loop corrections in the fermionic self-energy. In the considered parameter regime these momentum and frequency dependencies are weak and in \[(47)\] we see that the contributions to the discontinuity come from fermions with $\nu \approx \Delta, \nu - \omega \approx -\Delta$ which lie around the hot spot. Therefore we expand to leading order in momentum and frequency
\[
Z_{k}(\omega) = Z_0 + Z_f(\omega - \Delta) + Z_m \xi_{k+Q}^2,
\] \[
\Delta_k(\omega) = \Delta + \Delta_f(\omega - \Delta) + \Delta_m \xi_{k+Q}^2,
\] \[(48)\]

where the coefficients $Z_0, Z_f, \text{etc.}$ are computed numerically.
FIG. 6: Discontinuity $D$ containing self-energy corrections relative to one-loop jump $D_0$ for parameter range $\lambda \lesssim 1$.

With the help of the self-energy functions, we find for the fermionic Green’s functions in Eq. (47)

$$\text{Im} \Sigma_k^{(h)}(\omega) = -\pi \left[ \frac{\nu^2}{Z_k} \delta(\omega - \sqrt{\frac{\epsilon_k + \delta\epsilon_k}{Z_k}}^2 + |\Delta_k|^2) + \frac{\nu^2}{Z_k} \delta(\omega + \sqrt{\frac{\epsilon_k + \delta\epsilon_k}{Z_k}}^2 + |\Delta_k|^2) \right]$$

and correspondingly for the anomalous propagators. Here, one has to rescale the energy $\epsilon_k \rightarrow (\epsilon_k + \delta\epsilon_k)/Z_k$ in the coherence factors as well. Since we are interested in the discontinuity of (47) at $\omega \approx 2\Delta$ we can expand the arguments of the two delta functions around $\nu \approx \Delta$ and evaluate the frequency integration. The integration is greatly simplified by the usual spectrum linearization around hot spots. Performing this analysis, we find that the minimal excitation energy of the particle-hole spectrum is still $2\Delta$ such that the spin gap of $2\Delta$ is unaffected by the self-energy corrections. As a result we find for the discontinuity

$$D = \begin{cases} \frac{D_0}{(1-\nu_m^2)(1-\Delta_f)} & \text{for } \Delta_{k_f+q} = -\Delta_{k_p} \\ 0 & \text{for } \Delta_{k_f+q} = \Delta_{k_p} \end{cases}$$  \quad (49)$$

The ratio $D/D_0$ is shown in Fig. 6 as a function of $\lambda$ for $N = 8$, see Appendix A for further details. Since $\Delta_f, \nu_m \sim \lambda^2$ only for $\lambda \ll 1$ the self-energy corrections are of order one for the physical regime $\lambda$ of order unity. In the limit of large $N$ the parameters $\nu_m, \Delta_f \sim 1/N$ (for arbitrary $g$), see Appendix A. The result suggests that using an $1/N$ expansion, the self-energy corrections can be calculated controllably. However, previous results, show that for $d = 2$ there are problems with the $1/N$ expansion in the normal state with a gapless fermionic spectrum. It is unclear whether these problems persist in the superconducting state discussed here.

2. Vertex-Corrections

Now we turn in the examination of vertex corrections. Performing the perturbation theory in the extended spinor-space, indicated by the double-lined propagator matrices, we can express them as

$$\delta \Pi_{k+q}^{(h)}(i\omega_n) = \sum_{k, q} \chi_q(i\nu_k) A_k(i\Omega_m) B_{k+q}(i\Omega_m + i\omega_n) C_{k+q}(i\Omega_m + i\omega_n + i\nu_k) D_{k+q}(i\Omega_m + i\nu_k),$$ \quad (50)

where in the following fermionic Matsubara frequencies will be written with capital letters and bosonic ones with small letters. Due to spin rotation symmetry we can restrict ourselves to the zz-component of the bosonic self-energy. The sum $\{A, B, C, D\}$ has to be executed over all possible combinations of Gor’kov-Nambu Green’s functions with arrow conservation at each vertex

$$\begin{align*}
\{A, B, C, D\} & = \mathcal{G}^{(p)} \mathcal{G}^{(p)} \mathcal{G}^{(p)} \mathcal{G}^{(p)} + \mathcal{G}^{(h)} \mathcal{G}^{(h)} \mathcal{G}^{(h)} \mathcal{G}^{(h)} \\
& + \mathcal{F}^{*} \mathcal{F} \mathcal{F}^{*} + \mathcal{F}^{*} \mathcal{F} \mathcal{F}^{*} \\
& + \mathcal{F}^{*} \mathcal{G}^{(p)} \mathcal{G}^{(p)} + \mathcal{G}^{(p)} \mathcal{F} \mathcal{F}^{*} \mathcal{G}^{(p)} \\
& + \mathcal{G}^{(p)} \mathcal{G}^{(p)} \mathcal{F} \mathcal{F}^{*} + \mathcal{F}^{*} \mathcal{G}^{(p)} \mathcal{G}^{(p)} \mathcal{F}^{*} \\
& + \mathcal{F}^{*} \mathcal{F} \mathcal{G}^{(h)} \mathcal{G}^{(h)} + \mathcal{G}^{(h)} \mathcal{F} \mathcal{F}^{*} \mathcal{G}^{(h)} \\
& + \mathcal{G}^{(h)} \mathcal{G}^{(h)} \mathcal{F} \mathcal{F}^{*} + \mathcal{F}^{*} \mathcal{G}^{(h)} \mathcal{G}^{(h)} \mathcal{F}^{*} \\
& + \mathcal{F}^{*} \mathcal{G}^{(p)} \mathcal{G}^{(p)} + \mathcal{G}^{(p)} \mathcal{F} \mathcal{G}^{(h)} \mathcal{F}^{*} \\
& + \mathcal{F}^{*} \mathcal{F} \mathcal{G}^{(h)} + \mathcal{G}^{(h)} \mathcal{F} \mathcal{G}^{(p)} \mathcal{F}^{*},
\end{align*}$$ \quad (51)
eral procedure to systematically examine the behavior of
continuity up to three-loop order and we present a gen-
generally. In Appendix E we explicitly analyze the dis-
account vertex corrections.

This argument can be extended to higher orders in per-
particles at hot spots do not couple to any gapless excitations.

σ
χ
Q(ω) = r − \Pi_Q^{(2)}(ω) = r − γ Δ \int f(\frac{1}{2}) \sim r, which leads to a non-critical
dependence for the discontinuity corrections. Further-
more, the spectral weight of the resonance is exponen-
tially suppressed as can be seen in (34), therefore the
δD contributions from the resonance region of the in-
ternal bosonic line are small compared to \D. Thus, a
systematic expansion in the coupling parameter \g is jus-
tified in the weak coupling regime \omega_d \gg \Delta, such that
vertex corrections to the discontinuity |δD| \ll \D are
suppressed by higher powers of the coupling parameter \g.
As stressed earlier, in this limit the magnetic correla-
tion length \xi \sim 1/\sqrt{\T} is small and our continuum
theory is not the appropriate starting point.

In the strong coupling regime \omega_d \ll (\lambda \gg 1), we
show in Appendix D [see Eq. (18)] that the the discon-

Strong coupling behavior for d = 2

The emergence of the discontinuity \D = \D_0 + \delta D
in the imaginary part of the bosonic self-energy still
hinges on the symmetry of the superconducting order
parameter. In particular, we find similarly to (49) that
there is no discontinuity for conventional gap symme-
tries \delta D_{-\text{wave}} = 0, see Appendix D for details. The
physical reason behind this result is the fact that due to
phase space restrictions (for the bosonic propagator being
sharply-peaked around \q \sim \Q), the fermionic quasiparti-
cles at hot spots do not couple to any gapless excitations.
This argument can be extended to higher orders in per-
turbation theory. Therefore, we expect, based on the
optical theorem (43), that the absence of the discontinui-
ity for s-wave pairing, as indicated in Eq. (44), is valid
generally. In Appendix D we explicitly analyze the dis-
continuity up to three-loop order and we present a gen-
eral procedure to systematically examine the behavior of
the discontinuity depending on the superconducting gap
symmetry in arbitrary order perturbation theory.

Hereafter, we discuss the sign-changing pairing
\Delta_k + \Q = -\Delta_{k'}. In the weak coupling case \omega_d \gg \Delta
(\lambda \ll 1), the characteristic scale of the internal bosonic
propagator in the continuum region is \chi_Q(\omega) = r − \Pi_Q^{(2)}(\omega) = r − \gamma Δ \int f(\frac{1}{2}) \sim r, which leads to a non-critical
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FIG. 7: Possible diagrams for the next-order self-energy \delta \Pi
in the superconducting state. Note the arrow conservation
at each vertex indicating the energy-momentum conservation
of the theory.

which are shown in Fig. (7. The correction to the bosonic
self-energy \delta \Pi_Q^\delta(\omega) is evaluated in Appendices C
and D at \T = 0. Here, we are focusing on the correction to the
discontinuity (43)

δD = \lim_{\delta \to 0^+} \text{Im} \Pi_Q(2\Delta + \delta), \quad (52)

Since the lowest possible particle-hole excitations \epsilon_k +
\epsilon_{k+\Q} connected by the magnetic ordering vector \q
lie directly at the hot spots and therefore the fermionic
quasiparticles remain gapped with \Delta, we find that the
spin gap in \text{Im} \Pi_Q(\omega) remains 2\Delta even after taking into
account vertex corrections.

Below we discuss the strong coupling behavior of the
spin resonance in \d = 2 dimensions and a systematic
approach to regularize the theory with an \epsilon-expansion
around the upper critical dimension \d_{uc} = 3.

Strong coupling behavior for \d = 2

The emergence of the discontinuity \D = \D_0 + \delta D
in the imaginary part of the bosonic self-energy still
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parameter. In particular, we find similarly to (49) that
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tially suppressed as can be seen in (34), therefore the
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ternal bosonic line are small compared to \D. Thus, a
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suppressed by higher powers of the coupling parameter \g.
As stressed earlier, in this limit the magnetic correla-
tion length \xi \sim 1/\sqrt{\T} is small and our continuum
theory is not the appropriate starting point.

In the strong coupling regime \omega_d \ll \Delta (\lambda \gg 1), we
show in Appendix D [see Eq. (18)] that the the discon-
continuity correction can be written in the form

δD = \frac{\D_0}{\sqrt{\kappa}} \chi(\frac{\omega_d}{\Delta}, \Lambda), \quad (53)

where the \g-dependence is confined to the dimensionless
function \kappa(\frac{\omega_d}{\Delta}, \Lambda). Here, we defined the dimensionless
parameter

\hat{\Delta} = \frac{c_s \Delta}{v_F \gamma} \sim \frac{c_s T_c}{v_F \gamma} = \frac{9}{\s} \gamma C_2(\lambda). \quad (54)

At the same time, \frac{\omega_d}{\Delta} \sim \frac{\omega_d}{\lambda} = \frac{1}{\kappa(\frac{\omega_d}{\Delta}, \Lambda)}$, therefore the
function \kappa depends only on the coupling constant \lambda and

δD = \frac{\D_0}{\sqrt{\kappa}} \hat{\kappa}(\lambda). \quad (55)

The function \hat{\kappa} can be computed numerically and is
shown in Fig. (\ref{fig:7}) for \n = 8. It vanishes for small \lambda \ll 1.
Thus in the weak coupling regime vertex corrections are
suppressed by a higher power in the perturbative param-
eter \g and the one-loop calculation is controlled. In the
other limit \lambda \gg 1 the function is constant \hat{\kappa}(\infty) \approx -5.3.
Therefore, \delta D \sim \D_0 due to quantum-critical spin fluctua-
tions at energy scales \omega > \omega_d. Technically, in the strong
coupling limit these fluctuations determine both the res-
one and the continuum region \omega > 2\Delta in Eq. (50).
The corresponding scales in the weak and strong cou-
pling regimes are displayed in Fig. (8) where red regions
display the quantum-critical contributions. The vertex
corrections to the resonance mode are dominated by the
continuum region, where the physics is similar to that of
the normal state. Therefore, it is not obvious that a
\frac{1}{\sqrt{\n}}-expansion of those corrections is permissible. In ad-
dition, the numerical values of \delta D are not small, even for
the physically relevant case of \n = 8 hot spots. Thus,
we conclude that the perturbation expansion is not con-
trolled in \d = 2.
The leading order vertex correction at the hot spots in the strong coupling limit $\lambda \to 0$ is given by

$$\delta \Gamma(i\Omega, i\omega) = \text{fig},$$

where $q = (i\nu, q), k = (i\Omega, k_F)$ and $Q = (i\omega, Q)$. As only the bosonic field depends on the $z$ component of the momentum (assuming fermions are restricted to the two-dimensional xy-plane) we can integrate the bosonic propagator over $q_z$. As a result we find (see Appendix F for details)

$$\delta \Gamma \sim g^{-2-1} \int dx dy d\tilde{\nu} \left( \frac{1}{\Delta(x^2 + y^2) - \frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta}} \right)^{\frac{1}{g^2}} x f(x, y, i\tilde{\Omega}, i\tilde{\omega}),$$

where $x = \tilde{\nu}k_F + \tilde{\nu}/\Delta, y = \tilde{\nu}k_F + Q + \tilde{\nu}/\Delta, \tilde{\nu} = \nu/\Delta$ and $\tilde{\nu} = \tilde{\nu}/\Delta, \tilde{\omega} = \omega/\Delta$ are the external frequencies in units of the superconducting gap. In the superconducting state the polarization operator on the imaginary axis can be split into the resonance contribution (for $|\omega| < 2\Delta$) where $\Pi_Q(\omega) \sim \omega^2/\Delta$ and the continuum contribution (for $|\omega| > 2\Delta$) with $\Pi_Q(\omega) \sim \gamma\omega$, see Ref. 10. In both cases, the ratio $\Pi_Q(i\tilde{\nu}\Delta)/\gamma\Delta$ is independent of the coupling constant $g$, confining the $g$ dependence of the integral in [17] to the parameter $\Delta \sim g^{1-2d}$. This term dominates the behavior of the bosonic propagator if

$$\hat{\Delta}(x^2 + y^2) > \frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta}.$$ 

Since this condition only holds for a small region of the phase space, we can show that these potentially critical contributions yield even higher powers of $g$ for $d > 2$.

The above analysis yields for the correction to the discontinuity

$$\frac{D - D_0}{D_0} \propto \gamma^\alpha,$$

with the exponent

$$\alpha = \frac{2}{\epsilon} - 2 > 0.$$

The result agrees with our previous findings for $d = 2$ (i.e. for $\epsilon = 1$ where the exponent $\alpha$ vanishes). However, for

Systematic expansion in $\epsilon = 3 - d$ for strong coupling limit

The above results show that unless $\lambda \ll 1$ no controlled perturbative expansion exists in $d = 2$. Formally the higher order corrections are of order $1/N$ with $N$ the number of fermion species, yet previous results for $d = 2$ demonstrated that the standard loop-expansion can not be understood as an expansion in $1/N$. To avoid this problem (albeit in a different context) Moss et al. suggested in case of a related problem to combine the $1/N$-expansion with a further expansion in the parameter $z_b - z_b^*$, where $z_b$ is the dynamical critical exponent of the boson field and $z_b^*$ is the value of $z_b$ where quantum-critical corrections become logarithmic. For the problem of fermions coupled to a fluctuating transverse gauge field, discussed in Ref. 14, $z_b^* = 2$. Adapting this approach to our problem yields $z_b^* = 1$. However, such an approach is somewhat problematic for the determination of the resonance mode as the boson-dynamics is supposed to be the result of the calculation, i.e. we want to determine the relevant value of $z_b$. If indeed the running boson propagator that determines $D$ would be governed by the resonance mode itself, we would have a consistent theory, as the resonance mode is indeed characterized by a dynamic scaling exponent $z_b = 1$. However, our results for $d = 2$ clearly demonstrate that higher order corrections to the resonance mode have their origin in normal-state quantum-critical excitations with $z_b = 2$. On the other hand, our result [17] shows that the upper critical dimension of the quantum-critical behavior is $d_{uc} = 3$. Therefore, we propose to use the $\epsilon$-expansion instead of the expansion of Ref. 14.

The leading order vertex correction at the hot spots in the strong coupling limit $\lambda \to 0$ is given by

$$\delta \Gamma(i\Omega, i\omega) = \text{fig},$$

where $q = (i\nu, q), k = (i\Omega, k_F)$ and $Q = (i\omega, Q)$. As only the bosonic field depends on the $z$ component of the momentum (assuming fermions are restricted to the two-dimensional xy-plane) we can integrate the bosonic propagator over $q_z$. As a result we find (see Appendix F for details)

$$\delta \Gamma \sim g^{-2-1} \int dx dy d\tilde{\nu} \left( \frac{1}{\Delta(x^2 + y^2) - \frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta}} \right)^{\frac{1}{g^2}} x f(x, y, i\tilde{\Omega}, i\tilde{\omega}),$$

where $x = \tilde{\nu}k_F + \tilde{\nu}/\Delta, y = \tilde{\nu}k_F + Q + \tilde{\nu}/\Delta, \tilde{\nu} = \nu/\Delta$ and $\tilde{\nu} = \tilde{\nu}/\Delta, \tilde{\omega} = \omega/\Delta$ are the external frequencies in units of the superconducting gap. In the superconducting state the polarization operator on the imaginary axis can be split into the resonance contribution (for $|\omega| < 2\Delta$) where $\Pi_Q(\omega) \sim \omega^2/\Delta$ and the continuum contribution (for $|\omega| > 2\Delta$) with $\Pi_Q(\omega) \sim \gamma\omega$, see Ref. 10. In both cases, the ratio $\Pi_Q(i\tilde{\nu}\Delta)/\gamma\Delta$ is independent of the coupling constant $g$, confining the $g$ dependence of the integral in [17] to the parameter $\Delta \sim g^{1-2d}$. This term dominates the behavior of the bosonic propagator if

$$\hat{\Delta}(x^2 + y^2) > \frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta}.$$ 

Since this condition only holds for a small region of the phase space, we can show that these potentially critical contributions yield even higher powers of $g$ for $d > 2$.

The above analysis yields for the correction to the discontinuity

$$\frac{D - D_0}{D_0} \propto \gamma^\alpha,$$

with the exponent

$$\alpha = \frac{2}{\epsilon} - 2 > 0.$$
small ε the exponent α is large, such that vertex corrections are small (exactly for d = 3 they are exponentially small). A similar treatment can be performed for the self-energy corrections, which is not of importance as they can be controlled by a large N theory. Thus, while the resonance mode for d = 2 cannot be determined in a controlled fashion, systems with three-dimensional spin excitation spectrum can be well described in terms of the weak coupling theory.

III. SUMMARY

In summary, we investigated the role of higher order corrections for the theoretical description of the resonance mode within the spin fermion model. Within this model the occurrence of a resonance mode can be traced back to the emergence of a discontinuity in the imaginary part of the bosonic, spin-self energy at twice the gap value Δ at the hot spots of the Fermi surface.

First, we explicitly show that even if one includes higher order corrections the resonance mode only emerges if the phase of the superconducting gap function ∆_{k_F} and ∆_{k_F+Q} are distinct. Thus, we expect the one-loop result

\[ D \sim \sin^2 \left( \frac{\phi_1 - \phi_2}{2} \right) \]

for the height of the discontinuity to be valid more generally. Here, \( \phi_1 \) and \( \phi_2 \) are the phases of the superconducting order parameter at hot spots connected by the magnetic ordering vector \( Q \). This behavior makes the resonance mode a powerful tool to investigate the inner structure of the pairing condensate.

Second, we find for the two-loop vertex and self-energy correction to \( D \) the following result

\[ \frac{\delta D}{D_0} = \frac{g^2}{N} \cdot f(g), \]

where \( D_0 \) is the one-loop result. The function \( f(g) \) is determined by the boson dynamics. In the strong coupling limit \( \lambda \gg 1 \) (or \( \xi = \sqrt{\delta / T} \gg \frac{2\mu^2 g}{q} \)) and in the case of two dimensions, this function scales as a power law \( f(g) \sim 1/g^2 \) implying that in two dimensions the neutron resonance mode in unconventional superconductors is a strong coupling phenomenon. The origin of this behavior is the emergence of quantum-critical fluctuations at intermediate energies \( \omega_d \lesssim \omega \sim \Delta \) that dominate the low energy spin dynamics in the superconducting state. While such quantum critical fluctuations occur for all dimensions \( d \lesssim 3 \), they can be analyzed in a controlled fashion by means of the \( \varepsilon \)-expansion with the small parameter \( \varepsilon = 3 - d \). Quantum critical fluctuations to the resonance mode are now governed by power law behavior, yet the resulting exponents are such that the leading corrections are small. They become of order unity only for \( \varepsilon \to 1 \), i.e. for \( d = 2 \).

Our findings have implications for a number of unconventional superconductors: In case of heavy electron superconductors, such as CeCoIn\(_5\)\cite{12,13} and the iron-based systems\cite{14,15}, there are numerous indications that these are three dimensional, albeit moderately anisotropic materials. This implies that we expect the one-loop description of Refs. \[14,19\] to be valid for these materials. The situation is different for the copper-oxide high-temperature superconductor\cite{16,17} and for the organic charge transfer salts\cite{18,19}, that are strongly anisotropic and behave in many ways as two-dimensional systems. Here, our findings imply that a quantitative description of the resonance mode requires going beyond the leading one-loop order. In case of the cuprates, it is tempting to speculate that the observed universal ratio of the resonance mode and the pairing gap in a range of different materials is related to the strong coupling behavior of the resonance mode revealed here. A possible scenario is that higher order corrections modify the one-loop result \( \Omega_{\text{res}}/\Delta = f(\lambda) \) in a way that the function \( f(\lambda \gg 1) \) approaches an universal value. In case of the organic superconductor\cite{20}, no neutron measurement of the resonance mode has so far been reported. Numerous experiments support however the existence of an unconventional superconducting state with sign changing order parameter\cite{21}, i.e. one expects a resonance mode in these systems as well. Our results imply that the highly anisotropic quasi two-dimensional organics should behave similar to the cuprates.

Acknowledgments

The authors are grateful to A. V. Chubukov and M. Khodas for discussions. This work was supported by the Deutsche Forschungsgemeinschaft through DFG-SPP 1458 “Hochtemperatursupraleitung in Eisenpniktiden”.

Appendix A: Self-energies in the superconducting state

To investigate the momentum and frequency dependence of the fermionic self-energy and their influence on the resonance mode, we calculate the one-loop self-energy in the superconducting state

\[ \Sigma_k^{(p)}(i\Omega_m) = \sum_{q} \chi_q(i\omega_n) G_k^{(p)}(i\Omega_m + i\omega_n), \]
which we call the propagator self-energy. In a similar fashion we define the hole \( \Sigma^{(h)} \) and anomalous self-energy \( \Phi \) by replacing \( G^{(p)} \) with \( G^{(h)} \) or \( \mathcal{F} \). We can now express the renormalization factors:

\[
Z_k = 1 - \frac{\Sigma^{(p)}_k + \Sigma^{(h)}_k}{2i\omega_n}, \quad \delta \epsilon_k = \frac{\Sigma^{(p)}_k - \Sigma^{(h)}_k}{2}.
\]

After performing the Matsubara sum, we find for the imaginary part of the one-loop particle self-energy on the real axis

\[
\text{Im} \Sigma^{(p)}_k(\Omega) = -\frac{3g^2}{L^2} \sum_q \frac{du^2_k + Q}{v}\text{Im} \chi_q(\Omega - \xi_{k+q})\theta(\Omega - \xi_{k+q}) \approx (u^2_k + Q) \text{Im} \chi_q(-\Omega - \xi_{k+q})\theta(-\Omega - \xi_{k+q}).
\]

We see that the imaginary part of the normal self-energy is zero for \(|\Omega| < \Delta + \Omega_{\text{res}}\) due to the gapped spectrum of both the resonance mode and the fermionic quasiparticles at the hot spots. The same holds for the anomalous self-energy. For \( \lambda \leq 1 \) this means that the fermionic excitations around \( \Omega \approx \Delta \) are well defined. Therefore, we only have to analyze the real part of the self-energies if we are interested in their influence to the discontinuity. Using the Kramers-Kronig relation and the asymmetry of \( \text{Im} \chi \) in frequency we find for the real part of the self-energy

\[
\text{Re} \Sigma^{(p)}_k(\Omega) = -\frac{3g^2}{L^2} \sum_q \frac{du^2_k + Q}{v}\text{Im} \chi_q(\epsilon - \xi_{k+q})\frac{(u^2_k + v^2_k)\epsilon + (u^2_k + v^2_k\Omega)}{\epsilon^2 - \Omega^2}.
\]

Since we want to consider small derivations from the hot spots \( \delta k = k - k_F \) and from the AF wave vector \( p = q - Q \), we expand

\[
\epsilon_{k+q} = \epsilon_{k_F} + Q \cdot (p + \delta k) = v_F p_\perp + \epsilon_{k+Q},
\]

where \( p_\perp \) is the component of \( p \) perpendicular to the Fermi surface. We introduce dimensionless variables \( x = v_F p_\perp/\Delta, y = v_F p_\parallel/\Delta, z = \epsilon/\Delta, \delta = \epsilon_{k+Q}/\Delta, \tilde{\Omega} = \omega/\Delta \) and the dimensionless RPA spin susceptibility

\[
\chi_{x,y}(z) = \gamma \Delta \chi_q(z : \Delta)_{(q-Q^2)} = \frac{x}{\epsilon_F}(x^2 + y^2) = \frac{\gamma \Delta}{r + c_\Delta \frac{\Delta}{\epsilon_F}(x^2 + y^2) - \Pi^{(2)}_q(\nu : \Delta)} = \frac{1}{\frac{\omega_{\text{sf}}}{\Delta} + \tilde{\Delta}(x^2 + y^2) - \Pi^{(2)}_q(\nu : \Delta)}. \quad (A5)
\]

Here, we defined the dimensionless pairing parameter \( \tilde{\Delta} = \frac{\omega_{\text{sf}}}{\sqrt{\epsilon_F}} \). Finally we can express the self-energies as

\[
\text{Re} \Sigma^{(p/h)}_k(\Omega) = \text{Re} \Sigma^{(+/-)}_k(\Omega) = -\frac{3\Delta}{2\pi^2 N} \int dx dy \int_0^\infty dz \text{Im} \chi_{x,y}(z) \frac{\tilde{\Omega} z \pm (x + \delta)^2 + 1}{(z + \sqrt{(x + \delta)^2 + 1})^2 - \tilde{\Omega}^2}.
\]

where \( \Delta = |\Delta| \) is assumed to be real, which we can locally choose at the hot spot. Note that the anomalous self-energy \( \Phi_k \) switches sign for \( k \rightarrow k + Q \) just like the superconducting gap. We approximate the spin susceptibility for positive \( z \) as

\[
\text{Im} \chi_{x,y}(z) = 2\pi e^{-\frac{\omega_{\text{sf}}}{\Delta}} - \Delta(x^2 + y^2)\delta(z - \tilde{\Omega}_{\text{res}}) + \frac{z}{\frac{\omega_{\text{sf}}}{\Delta} + \tilde{\Delta}(x^2 + y^2)^2 + z^2} \theta(z - 2), \quad (A7)
\]

where the first term describes the resonance at \( \tilde{\Omega}_{\text{res}} = 2(1 - e^{-\frac{\omega_{\text{sf}}}{\Delta}} - \Delta(x^2 + y^2)) \) and the second term uses the normal state behavior to express the continuum region. Using \( \tilde{\Delta} \approx T_c \) it is possible to express (here \( d = 2 \))

\[
\frac{\omega_{\text{sf}}}{\Delta} = \frac{1}{4\lambda^2 C_2(\lambda)}, \quad \tilde{\Delta} = \frac{9}{N^2 C_2(\lambda)}. \quad (A8)
\]
depending only on the dimensionless coupling constant $\lambda$ and the function $C_2(\lambda)$ defined in [29]. Expanding the formulas (A6) around $\Omega = \pm 1$ and $\delta \approx 0$ it is possible to determine numerically the coefficients in

$$Z_k(\Omega) = 1 - \frac{\text{Re}\Sigma_k^{(p)}(\Omega) + \text{Re}\Sigma_k^{(h)}(\Omega)}{2\Omega} \approx Z_0 + Z_f(|\Omega| - \Delta) + Z_m \varepsilon_{k+Q}^2,$$

$$\delta \varepsilon_k(\Omega) = \frac{\text{Re}\Sigma_k^{(p)}(\Omega) - \text{Re}\Sigma_k^{(h)}(\Omega)}{2} \approx \nu_m \varepsilon_{k+Q} + O(\varepsilon_{k+Q}^2, (|\Omega| - \Delta)\cdot \varepsilon_{k+Q}),$$

$$\Phi_k(\Omega) \approx \Phi_0 + \Phi_f(|\Omega| - \Delta) + \Phi_m \varepsilon_{k+Q}^2,$$

$$\Delta_k(\Omega) = \frac{\Phi_k(\Omega)}{Z_k(\Omega)} \approx \frac{\Phi_0}{Z_0} + \frac{Z_0\Phi_f - \Phi_0 Z_f(|\Omega| - \Delta) + Z_0\Phi_m - \Phi_0 Z_m}{\Delta_0 \approx \Delta} \varepsilon_{k+Q}^2.$$

For $\lambda \lesssim 1$ all parameters are small compared to 1 and the expansion is a good approximation around the hot spots, but for $\lambda \gg 1$ the frequency parameters $Z_f, \Phi_f, \Delta_f$ become large, because the resonance energy $\Omega_{\text{res}} \ll \Delta$ is small compared to the gap and the real part of the self-energies at $\omega = \Omega_{\text{res}} + \Delta \approx \Delta$ develops a resonance near the expansion region.

**Appendix B: Self-energies corrections for the discontinuity**

Using (11) and $\text{Im}\Sigma_k(|\omega| < \Delta') = 0$ the imaginary part of the polarization operator for zero temperature including the dressed propagators can be written as

$$\text{Im}\Pi^{(2)}_{\mathbf{q}}(\omega) = \frac{2g^2}{L^2} \sum_k \int_{\Delta'}^{\omega - \Delta'} d\lambda \frac{\lambda}{\pi} \text{Im}\Sigma^{(h)}_k(\lambda) \text{Im}\Sigma^{(h)}_{k+\mathbf{q}}(\lambda + \omega) + \text{Im}\mathcal{F}_k(\lambda) \text{Im}\mathcal{F}^*_0(\lambda + \omega),$$

where $\Delta' \lesssim \Delta$ is the minimal excitation energy around the hot spot. Inserting the imaginary parts of the propagator we find for the $\mathcal{GG}$ contribution for $\omega \approx 2\Delta$ and external momentum $\mathbf{Q}$

$$\text{Im}\Pi^{(2)}_{\mathbf{q},\mathbf{Q}}(\omega) = \frac{2g^2}{L^2} \sum_k \int_{\Delta'}^{\omega - \Delta'} d\lambda \frac{\lambda}{\pi} \frac{v_k(\lambda)^2 w_{k+\mathbf{Q}}(\lambda - \omega)^2}{Z_k(\lambda) Z_{k+\mathbf{Q}}(\lambda - \omega)^2} \delta \left( \lambda - \sqrt{\left(\frac{\varepsilon_k + \varepsilon_{k+\mathbf{Q}}}{Z_k(\lambda)}\right)^2 + \Delta_k(\lambda)^2} \right) \times \delta \left( \lambda - \omega + \sqrt{\left(\frac{\varepsilon_{k+\mathbf{Q}} + \varepsilon_{k+\mathbf{Q}}}{Z_{k+\mathbf{Q}}(\lambda - \omega)}\right)^2 + \Delta_{k+\mathbf{Q}}(\lambda - \omega)^2} \right)$$

Since the momentum $\mathbf{k} \approx \mathbf{k}_f$ and frequencies $\lambda \approx \Delta, \lambda - \omega \approx -\Delta$ are still restricted around the hot spots, we perform a frequency expansion in the $\delta$ distributions and use the relations in (A9)

$$\sqrt{\left(\frac{\varepsilon_k + \varepsilon_{k+\mathbf{Q}}}{Z_k(\lambda)}\right)^2 + \Delta_k(\lambda)^2} \overset{\lambda \approx \Delta}{\approx} \sqrt{\left(\frac{\varepsilon_k + \varepsilon_{k+\mathbf{Q}}}{Z_k(\Delta)}\right)^2 + \Delta_k(\Delta)^2} + \frac{\alpha_k}{Z_k(\Delta)} \sqrt{\left(\frac{\varepsilon_k + \varepsilon_{k+\mathbf{Q}}}{Z_k(\Delta)}\right)^2 + \Delta_k(\Delta)^2} \cdot (\lambda - \Delta)$$

$$\sqrt{\left(\frac{\varepsilon_{k+\mathbf{Q}} + \varepsilon_{k+\mathbf{Q}}}{Z_{k+\mathbf{Q}}(\lambda - \omega)}\right)^2 + \Delta_{k+\mathbf{Q}}(\lambda - \omega)^2} \overset{\omega \approx \Delta}{\approx} \sqrt{\left(\frac{\varepsilon_{k+\mathbf{Q}} + \varepsilon_{k+\mathbf{Q}}}{Z_{k+\mathbf{Q}}(\lambda - \omega)}\right)^2 + \Delta_{k+\mathbf{Q}}(-\Delta)^2} + \alpha_{k+\mathbf{Q}} \cdot (\lambda - \omega + \Delta)$$

Using these relations we write

$$\delta \left( \lambda - \sqrt{\left(\frac{\varepsilon_k + \varepsilon_{k+\mathbf{Q}}}{Z_k(\lambda)}\right)^2 + \Delta_k(\lambda)^2} \right) \delta \left( \lambda - \omega + \sqrt{\left(\frac{\varepsilon_{k+\mathbf{Q}} + \varepsilon_{k+\mathbf{Q}}}{Z_{k+\mathbf{Q}}(\lambda - \omega)}\right)^2 + \Delta_{k+\mathbf{Q}}(\lambda - \omega)^2} \right)$$

$$= \frac{\delta(\lambda - \beta_k)\delta(\lambda - \omega + \beta_{k+\mathbf{Q}})}{(1 - \alpha_k)(1 - \alpha_{k+\mathbf{Q}})}$$
with
\[ \hat{\beta}_k = \sqrt{\frac{x_k + 2\xi_k}{2\xi_k(\Delta)}} + \Delta_k(\Delta) - \alpha_k \Delta \frac{1}{1 - \alpha_k}. \] (B4)

Evaluating the frequency integration we find
\[ \text{Im}\Pi^{(2)}_{\nu(\omega)}(\omega) = \frac{2g^2\pi}{L^2} \sum_k v_k(\hat{\beta}_k)^2 u_k + Q(\hat{\beta}_k + Q)^2 \frac{\delta(\omega - \hat{\beta}_k - \hat{\beta}_k + Q)}{(1 - \alpha_k)(1 - \alpha_k + Q)} \] (B5)

Now, we expand \( \hat{\beta}_k \) till second order in \( \epsilon_k, \epsilon_k + Q \ll \Delta \)
\[ \hat{\beta}_k \approx \Delta + \frac{2\Delta \epsilon_k^2 + \Delta_m \epsilon_k + Q}{2\Delta \epsilon_k^2(1 - \Delta)} + \frac{\Delta_m \epsilon_k + Q}{1 - \Delta}. \] (B6)

In the considered regime \( \lambda \leq 1 \) the last term is negligible, because \( |\Delta_m Z_0| |\Delta| \ll |\tau_m| \). We see that the minimal excitation energy \( \Delta' = \Delta \) remains the same even including self-energy corrections and the discontinuity still appears at \( 2\Delta \). Also the momentum contributing to the discontinuity \( \epsilon_k = \epsilon_k + Q = 0 \) are again restricted to the hot spots. After the usual linearization \( \epsilon = \epsilon_k, \epsilon' = \epsilon_k + Q \) we substitute \( x = (\epsilon + \nu_m \epsilon')/\sqrt{2\Delta Z_0^2(1 - \Delta)} \) and \( y = (\epsilon' + \nu_m \epsilon)/\sqrt{2\Delta Z_0^2(1 - \Delta)} \) and find for the \( \mathcal{G} \mathcal{G} \) contribution of the discontinuity
\[ D_{\mathcal{G} \mathcal{G}} = \lim_{\delta \to 0^+} \text{Im}\Pi^{(2)}_{\nu(\omega)}(2\Delta + \delta) \begin{aligned} &= \frac{g^2 N}{2\pi v_F^2 Z_k(\beta_k)^2 Z_k + Q(\beta_k + Q)^2(1 - \alpha_k)(1 - \alpha_k + Q)} \times \frac{2\Delta Z_0^2}{1 - \nu_m^2} \lim_{\delta \to 0^+} \int dx dy \delta(x - x^2 - y^2) \end{aligned} \]
\[ = \frac{D_0}{2(1 - \Delta^2)(1 - \nu_m^2)} \] (B7)

The anomalous \( \mathcal{F} \mathcal{F}^* \) contribution gives the same contribution, but depending on the gap symmetry we find
\[ D_{\mathcal{F} \mathcal{F}^*} = \frac{D_0}{2(1 - \nu_m^2)(1 - \Delta)} \text{sign}(\Delta_{kF}/\Delta_{kF + Q}). \] (B8)

Therefore, the discontinuity vanishes again for the s-wave symmetry \( \Delta_{kF} = \Delta_{kF + Q} \).

**Appendix C: Evaluation of the Matsubara summation**

In order to get the correction \( \delta D \) to the discontinuity we have to calculate the imaginary part of \( \delta\Pi_Q(\omega) \). Thus, we have to execute and analytically continue the double Matsubara summation
\[ Q_{k,q}(i\omega_n) = T^2 \sum_{\Omega_m, \nu_k} A_k(i\Omega_m) B_{k+Q}(i\Omega_m + i\omega_n) C_{k+Q+Q}(i\Omega_m + i\omega_n + i\nu_k) D_{k+Q}(i\Omega_m + i\nu_k) \chi(i\nu_k), \] (C1)
where \( A, B, C, D \) are the different combinations of fermionic propagators in the superconducting state. For \( T = 0 \) the fermionic \( f(\epsilon) = \theta(-\epsilon) \) and bosonic distributions functions \( g(\epsilon) = -g(-\epsilon) \) severaly restrict the phase space for the considered case \( \omega > 0 \). Using the identities for the fermionic Green’s functions
\[ A^R_A(x) = \frac{u_{\epsilon_k}^2}{x - \xi_k + i0}, \quad \text{Im} A^R_A(x) = \mp \pi [u_{\epsilon_k}^2 \delta(x - \xi_k) + v_{\epsilon_k}^2 \delta(x + \xi_k)], \] (C2)
\[ A_k(x) = \text{Re} A^R_A(x) = \mathcal{P} \frac{u_{\epsilon_k}^2}{x - \xi_k} + \mathcal{P} \frac{v_{\epsilon_k}^2}{x + \xi_k}, \]
it can be shown that the analytical continuation of (C1) yields

\[ \text{Im} Q(\omega) = v_A^2 u_{B,k+Q}^2 \delta(\omega - \xi_k - \xi_k + Q) \int_0^\infty dx \text{Im} \chi_q^R(-x) C_{k+Q}(-x + \xi_k + Q) D_{k+Q}(-x - \xi_k) \]

\[ - u_C^2 u_{D,k+Q}^2 \delta(\omega - \xi_k - \xi_k + Q) \int_0^\infty dx \text{Im} \chi_q^R(x) A_k(-x - \xi_k + Q) B_{k+Q}(-x + \xi_k + Q) \]

\[ + v_A^2 u_{C,k+Q}^2 \text{Im} \chi_q^R(-\omega + \xi_k + \xi_k + Q) B_{k+Q}(-\omega + \xi_k + Q) D_{k+Q}(\omega - \xi_k + Q) \theta(\omega - \xi_k - \xi_k + Q) \]

\[ - u_B^2 u_{Q,k+Q}^2 \text{Im} \chi_q^R(\omega - \xi_k - \xi_k + Q) A_k(-\omega + \xi_k + Q) C_{k+Q}(-\omega + \xi_k + Q) \theta(\omega - \xi_k - \xi_k + Q) \]

\[ - \pi v_B^2 u_{Q,k+Q}^2 v_C^2 D_{k+Q} A_k(-\omega - \xi_k + Q) \delta(\omega - \xi_k - \xi_k + Q) \]

\[ - \pi v_B^2 u_{C,k+Q}^2 u_{Q,k+Q}^2 C_{k+Q} \delta(\omega - \xi_k + Q) \delta(\omega - \xi_k + Q) \]

\[ - \pi v_B^2 u_{C,k+Q}^2 u_{Q,k+Q}^2 D_{k+Q}(-\omega - \xi_k + Q) \delta(\omega - \xi_k - \xi_k + Q) \]

\[ + \pi v_B^2 u_{C,k+Q}^2 u_{Q,k+Q}^2 D_{k+Q} \delta(\omega - \xi_k + Q) \delta(\omega - \xi_k + Q) \]

\[ + \pi v_B^2 u_{C,k+Q}^2 u_{Q,k+Q}^2 C_{k+Q} \delta(\omega - \xi_k - \xi_k + Q) \delta(\omega - \xi_k - \xi_k + Q) \]  \hspace{1cm} (C3)

Appendix D: Evaluation of the momentum-integration for discontinuity from vertex corrections

The imaginary part of \( \Pi_{VC}(Q, \omega) \) can be obtained from (C3) by using a similar linearization as explained in Section I.A:

\[
\left( \frac{1}{T^2} \right)^2 \sum_{k,q \approx Q} f(\varepsilon_k, \varepsilon_{k+Q}, \varepsilon_{k+Q}, \Delta_k, \Delta_{k+Q}, \Delta_{k+Q+q}, \Delta_{k+q+Q}) g([q - Q]^2) \\
= N \left( \frac{1}{8\pi^2 v_{\perp} v_{\parallel}} \right)^2 \int d\epsilon \int d\epsilon' \int d\lambda \int d\lambda' f(\epsilon, \epsilon', \epsilon + \lambda, \epsilon' + \lambda', \Delta, \pm \Delta, \Delta, \pm \Delta) g \left( \frac{(\lambda + \lambda')^2}{2v_{\perp}} + \frac{(\lambda - \lambda')^2}{2v_{\parallel}} \right) \\
\approx N \left( \frac{1}{4\pi^2 v_{\perp}^2} \right)^2 \int d\epsilon \int d\epsilon' \int d\lambda \int d\lambda' f(\epsilon, \epsilon', \epsilon + \lambda, \epsilon' + \lambda', \Delta_{kF}, \pm \Delta_{kF}, \pm \Delta_{kF}) g \left( \frac{\lambda^2 + \lambda'^2}{v_{\perp}^2} \right),
\]

\hspace{1cm} (D1)

describing small derivations from the hot spots by \( p = k - k_F \) and from the AF wave vector by \( p' = q - Q \). Using this approximation we can write

\[
\text{Im} \Pi_{VC}(Q, \omega) = -g^4 N \sum_{\{A,B,C,D\}} \left( \frac{1}{8\pi^2 v_{\perp} v_{\parallel}} \right)^2 \int d\epsilon \int d\epsilon' \int d\lambda \int d\lambda' \text{Im} Q(\omega),
\]

\hspace{1cm} (D2)

where we set \( \varepsilon_k = \epsilon, \varepsilon_{k+Q} = \epsilon', \varepsilon_{k+q+Q} = \epsilon + \lambda, \varepsilon_{k+q} = \epsilon' + \lambda' \) and \( (q - Q)^2 = \frac{\lambda^2 + \lambda'^2}{v_{\perp}^2} \). Later it will be useful to use the following symmetries of the momentum integration before the linearization of the spectrum

\[
(k, k + Q + q) \leftrightarrow (k + Q, k + q), \\
(k, k + Q) \leftrightarrow (k + Q + q, k + q).
\]

\hspace{1cm} (D3)

After the linearization of the spectrum the coherence-factors

\[
v_{G(p),k}^2 = \frac{1}{2} \left( 1 + \frac{\varepsilon_k}{\xi_k} \right), \\
v_{G(p),k}^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_k}{\xi_k} \right), \\
v_F^2, k = -\frac{1}{2} \frac{\Delta_k}{\xi_k},
\]

\hspace{1cm} (D4)
can be written as
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\otimes & u_{A,e}^2 = u_{C,e}^2 & v_{A,e}^2 = v_{C,e}^2 & u_{B,e}^2 = u_{D,e}^2 & v_{B,e}^2 = v_{D,e}^2 \\
G(p) & \frac{1}{2} \left( 1 + \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) & \frac{1}{2} \left( 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) & \frac{1}{2} \left( 1 + \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) & \frac{1}{2} \left( 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) \\
G(b) & \frac{1}{2} \left( 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) & \frac{1}{2} \left( 1 + \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) & \frac{1}{2} \left( 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) & \frac{1}{2} \left( 1 + \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \right) \\
F & \frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} & -\frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} & \pm \frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} & \pm \frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \\
F^* & -\frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} & \frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} & \pm \frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} & \pm \frac{1}{2} \frac{|\Delta_{k_F}|^2}{\sqrt{\epsilon^2 + |\Delta_{k_F}|^2}} \\
\hline
\end{array}
\]  
\tag{D4}
\]
where the different signs of the coherence-factors for the anomalous Green’s functions occur for the different gap symmetries $\Delta_{k_F+\mathbf{Q}} = \pm \Delta_{k_F}$. Because we see from the $\{A, B, C, D\}$ sum, that we have only combinations of $F$ and $F^*$ in the diagrams, there will always occur combinations $\Delta_{k_F} \cdot \Delta_{k_F}^* = |\Delta_{k_F}|^2 = \Delta^2$ and it is allowed to assume $\Delta_{k_F} = \Delta_{k_F}^* = \Delta$ to be real. Therefore, we can simplify the above table
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\otimes & u_{A,e}^2 & v_{A,e}^2 & u_{B,e}^2 & v_{B,e}^2 & u_{C,e}^2 & v_{C,e}^2 \\
G(p) & u_2^2 & v_2^2 & u_2^2 & v_2^2 & u_2^2 & v_2^2 \\
G(b) & u_2^2 & v_2^2 & u_2^2 & v_2^2 & u_2^2 & v_2^2 \\
F & u_e v_e & u_e v_e & u_e v_e & u_e v_e & u_e v_e & u_e v_e \\
F^* & -u_e v_e & u_e v_e & u_e v_e & u_e v_e & u_e v_e & u_e v_e \\
\hline
\end{array}
\]
\tag{D5}
where we defined using $\xi_e = \sqrt{\epsilon^2 + \Delta^2}$
\[
u_2^2 = \frac{1}{2} \left( 1 + \frac{\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \right) = \frac{1}{2} \left( 1 - \frac{\epsilon}{\xi_e} \right),
\]
\[
u_2^2 = \frac{1}{2} \left( 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \right) = \frac{1}{2} \left( 1 + \frac{\epsilon}{\xi_e} \right). \tag{D6}
\]

We now have set the framework for the momentum integration and continue in examining the discontinuity of the imaginary part at $\omega = 2\Delta + \delta$ with $\delta \rightarrow 0$. Because there are four different kinds of terms in $\text{Im} Q(\omega)$, we have to analyze them separately. At first we will examine the terms with $\theta$-function, then the terms with $\delta$-function and $x$-integration, thereafter the terms with 3 coherence-factors and at least the terms with 4 coherence-factors.

**Terms with $\theta$-functions**

Using the above linearization we obtain for the first of the terms with the $\theta$-function:
\[
\text{Im} \Pi_{C,\theta,1}(\omega) = -g^4 N \left( \frac{1}{4\pi^2} \right)^2 \sum_{\{A,B,C,D\}} \int \frac{d\epsilon}{2\pi} d\lambda d\lambda' \text{Im} \chi_{\lambda,\lambda'}(-\omega + \xi_e + \xi_{e+\lambda}) v_{A,e}^2 u_{C,e+\lambda}^2 B_{e'}(\omega - \xi_e) D_{e'+\lambda'}(-\omega + \xi_{e+\lambda}) \theta(\omega - \xi_e - \xi_{e+\lambda}).
\]

Due to the $\theta$-function the phase space of the $(\epsilon, \lambda)$-integration will be restricted strongly at the discontinuity $\omega \rightarrow 2\Delta + \delta$:
\[
\lim_{\delta \rightarrow 0} \int \frac{d\epsilon}{2\pi} d\lambda \theta(\omega - \xi_e - \xi_{e+\lambda}) f(\epsilon, \epsilon + \lambda) = \lim_{\delta \rightarrow 0} \int d\lambda \theta(2\Delta + \delta - \xi_e - \xi_{e+\lambda}) f_{\delta}(\epsilon, \epsilon + \lambda)
\approx \lim_{\delta \rightarrow 0} \int d\lambda \theta(\epsilon^2 + (\epsilon + \lambda)^2) f_{\delta}(\epsilon, \epsilon + \lambda)
\approx \lim_{\delta \rightarrow 0} \int d\lambda \theta(\epsilon^2 + (\epsilon + \lambda)^2) f_{\delta}(\sqrt{2\delta\Delta}, \sqrt{2\delta\Delta})
= \lim_{\delta \rightarrow 0} 2\pi \Delta f_{\delta}(\sqrt{2\delta\Delta}, \sqrt{2\delta\Delta}) \cdot \delta. \tag{D7}
\]

Thus, these terms will vanish if the remaining function $f_{\delta}(0,0)$ is non-singular. Using the asymmetry of $\text{Im} \chi_{\lambda,\lambda'}(x) = -\text{Im} \chi_{\lambda,\lambda'}(-x)$ and the symmetry [D3], taking the limit $\epsilon = 0, \lambda = 0$ and putting in the fermionic Green’s functions.
we obtain for the sum of both terms with $\theta$ function

$$\text{Im}\Pi_{C,0}(Q, 2\Delta + \delta) = 2\pi g^4 N\Delta \left( \frac{1}{8\pi^2 v_{\parallel} v_{\perp}} \right)^2 \cdot \delta \cdot \int d\epsilon' d\lambda' \text{Im} \chi_{0, \lambda' - \epsilon'}(0) \times$$

$$P \left( \begin{array}{c}
\frac{v_{A,0} u_{B, \epsilon} u_{C,0} u_{D, \lambda}}{\Delta + \delta - \xi(\lambda)} \\
\frac{v_{A,0} u_{B, \epsilon} u_{C,0} u_{D, \lambda}}{\Delta + \delta - \xi(\lambda)} \\
\end{array} \right) \left( \begin{array}{c}
2\Delta + \delta \\
2\Delta + \delta \\
\end{array} \right)$$

With the help of the symmetry (D3) we can perform the summation over the possible diagrams in (51) and find that for both $d$-wave and $s$-wave pairing the sum over the different combinations of the coherence factors are similar up to a sign

$$\sum_{\{A,B,C,D\}} \left( v_{A,0} u_{B, \epsilon} u_{C,0} u_{D, \lambda} + u_{A, \lambda} u_{B,0} u_{C,\epsilon} u_{D,0} \right) = \frac{\epsilon' \lambda'}{2\xi' \xi},$$

$$\sum_{\{A,B,C,D\}} \left( v_{A,0} u_{B, \epsilon} u_{C,0} u_{D, \lambda} + u_{A, \lambda} u_{B,0} u_{C,\epsilon} u_{D,0} \right) = -\frac{\epsilon' \lambda'}{2\xi' \xi}.$$
where we defined the $f$ functions, which can be calculated for the $d$-wave case as

$$f_{uu}(\lambda, \lambda') = \sum_{\{A,B,C,D\}} (v_{A,0}^2 u_{B,0}^2 u_{C,0}^2 v_{D,0}^2 + u_{A,0}^2 v_{B,0}^2 v_{C,0}^2 v_{D,0}^2) = \frac{\lambda \lambda' + (-\Delta + \xi \lambda)(\Delta + \xi \lambda')}{2 \xi \lambda \epsilon},$$

$$f_{uv}(\lambda, \lambda') = \sum_{\{A,B,C,D\}} (v_{A,0}^2 u_{B,0}^2 u_{C,0}^2 v_{D,0}^2 + v_{A,0}^2 u_{B,0}^2 u_{C,0}^2 v_{D,0}^2) = \frac{-\lambda \lambda' + (-\Delta + \xi \lambda)(-\Delta + \xi \lambda')}{2 \xi \lambda \epsilon},$$

$$f_{vu}(\lambda, \lambda') = \sum_{\{A,B,C,D\}} (v_{A,0}^2 u_{B,0}^2 v_{C,0}^2 v_{D,0}^2 + u_{A,0}^2 v_{B,0}^2 v_{C,0}^2 v_{D,0}^2) = \frac{-\lambda \lambda' + (\Delta + \xi \lambda)(\Delta + \xi \lambda')}{2 \xi \lambda \epsilon},$$

$$f_{vv}(\lambda, \lambda') = \sum_{\{A,B,C,D\}} (v_{A,0}^2 u_{B,0}^2 v_{C,0}^2 v_{D,0}^2 + v_{A,0}^2 u_{B,0}^2 v_{C,0}^2 v_{D,0}^2) = \frac{-\lambda \lambda' + (\Delta + \xi \lambda)(-\Delta + \xi \lambda')}{2 \xi \lambda \epsilon}. \quad (D11)$$

Fortunately, it is easy to show that these $f$ functions vanish in the $s$-wave case. As it will be shown in the following calculation all contributions to the discontinuity from the vertex correction contain these $f$ functions and therefore we can immediately assess that also for the vertex corrected theory the phase sensitivity of the discontinuity is conserved. For this purpose we will restrict our further calculations to the $d$-wave case and using (D11) we find

$$\delta_2 = 4\pi \Delta g^4 N \left( \frac{1}{8\pi^2 v_{\perp} v_{\parallel}} \right)^2 \int d\lambda d\lambda' \int_0^\infty dz \Im \chi_{\lambda,\lambda'}(z) \left( \frac{\lambda \lambda' + z^2}{[(\Delta - z)^2 - \xi \lambda][(\Delta + z)^2 - \xi \lambda']} \right) \quad (D12)$$

Using (D9), the symmetry $\chi_{\lambda,\lambda'}(x) = \chi_{\lambda',\lambda}(-x)$ and the functions in (D11) we easily obtain for the discontinuity contribution from the terms with the $\delta$ function and 3 or 4 coherence factors

$$\delta_3 = \lim_{\delta \to 0} \Im \Pi_{VC,\delta + 3cf}(Q, 2\Delta + \delta) = 2\pi^2 \Delta g^4 N \left( \frac{1}{8\pi^2 v_{\perp} v_{\parallel}} \right)^2 \int d\lambda d\lambda' \left[ \chi_{\lambda,\lambda'}(\Delta + \xi \lambda) \left( \frac{f_{uv}(\lambda, \lambda')}{2\Delta - \xi \lambda - \xi \lambda'} + \frac{f_{vu}(\lambda, \lambda')}{-2\Delta - \xi \lambda + \xi \lambda'} \right) \right. 
+ \left. \chi_{\lambda',\lambda}(\xi \lambda' - \Delta) \left( \frac{f_{vv}(\lambda, \lambda')}{2\Delta - \xi \lambda' - \xi \lambda} + \frac{f_{vv}(\lambda, \lambda')}{2\Delta - \xi \lambda' + \xi \lambda} \right) \right] \quad (D13)$$

$$\delta_4 = \lim_{\delta \to 0} \Im \Pi_{VC,\delta + 4cf}(Q, 2\Delta + \delta) = -2\pi^3 \Delta g^4 N \left( \frac{1}{8\pi^2 v_{\perp} v_{\parallel}} \right)^2 \int d\lambda d\lambda' \Im \chi_{\lambda,\lambda'}(\Delta + \xi \lambda) \delta(2\Delta + \xi \lambda - \xi \lambda') f_{vv}(\lambda, \lambda') \quad (D14)$$

At this point of the calculation it will be useful to apply the approximation that the parallel and perpendicular Fermi velocities are equal, see Eq. (D7). Thus, the bosonic propagator will be an even function in $\lambda$ and $\lambda'$ and odd terms in the $f$ functions will vanish due to the antisymmetry of the complete integrand. This allows us to simplify

$$f_{uu}(\lambda, \lambda') = \frac{(-\Delta + \xi \lambda)(\Delta + \xi \lambda')}{2 \xi \lambda \epsilon},$$

$$f_{uv}(\lambda, \lambda') = \frac{(-\Delta + \xi \lambda)(-\Delta + \xi \lambda')}{2 \xi \lambda \epsilon},$$

$$f_{vu}(\lambda, \lambda') = \frac{(\Delta + \xi \lambda)(\Delta + \xi \lambda')}{2 \xi \lambda \epsilon},$$

$$f_{vv}(\lambda, \lambda') = \frac{(\Delta + \xi \lambda)(-\Delta + \xi \lambda')}{2 \xi \lambda \epsilon}. \quad (D15)$$

We now introduce dimensionless integration variables $x = \tilde{x}, y = \tilde{y}, \nu = \tilde{\nu}$, the dimensionless dispersion $\tilde{\epsilon} = \sqrt{x^2 + 1}$ and the dimensionless RPA spin susceptibility

$$\tilde{\chi}_{x,y}(\nu) = \gamma_{\Delta} \chi_{x,y,\Delta}(\nu \cdot \Delta) = \gamma_{\Delta} \frac{1}{r + c_s \frac{(x \Delta)^2 + (y \Delta)^2}{\epsilon^2} - \Pi_\Delta^{(2)}(\nu \cdot \Delta)} = \frac{1}{\gamma_{\Delta} + \Delta(x^2 + y^2) - \Pi_\Delta^{(2)}(\nu \cdot \Delta)}, \quad (D16)$$

which is a similar definition as in Eq. (A5). Analogously, we defined the dimensionless pairing parameter $\hat{\Delta} = \frac{\epsilon \Delta}{c_s \tilde{\epsilon}}$. Substituting these variables in the above formulas for $\delta_1, \delta_2$ and $\delta_3$, evaluating the remaining $\delta$-function and simplifying
the expressions leads to
\[ \delta D = \delta_2 + \delta_3 + \delta_4 = \frac{D_0}{N} \cdot \kappa(\frac{\omega_{sf}}{\Delta}, \Delta) \]
with the dimensionless function
\[ \kappa(\frac{\omega_{sf}}{\Delta}, \Delta) = \frac{C}{\pi^2} \int_0^\infty dx \, dx \, d\nu \, \text{Im} \tilde{\chi}_{x,y}(\nu) P \left[ \frac{\nu^2}{(1 - \nu^2)^2 - \xi_x^2} \right] \]
\[ \quad \times \left[ \frac{1}{(1 - \nu^2)^2 - \xi_x^2} \right] \left[ \frac{1}{(1 - \nu^2)^2 - \xi_y^2} \right] \]
\[ \quad + \frac{2}{\pi} \int_0^\infty dx \, dy \, P \left[ \frac{\tilde{\chi}_{x,y}(1 + \tilde{\xi}_x) (1 + \tilde{\xi}_y)^2}{\xi_x - (2 + \tilde{\xi}_x)^2} + \frac{\chi_{x,y} (\tilde{\xi}_y - 1)}{\xi_y} \right] \left[ \frac{1}{(1 - \tilde{\xi}_y)^2 - \xi_y^2} \right] \]
\[ \quad - \int_0^\infty dx \, \text{Im} \tilde{\chi}_{x,y} \sqrt{(2 + \tilde{\xi}_x)^2 - 1} \left( 1 + \tilde{\xi}_x \right) \left[ \frac{1}{\tilde{\xi}_x} \right] \left[ \frac{1}{\tilde{\xi}_x} \right] \left[ \frac{1}{(2 + \tilde{\xi}_x)^2 - 1} \right] . \] (D18)

**Numerical analysis of the vertex correction of discontinuity**

At this point of the calculation we need a good input for the 1-loop spin susceptibility in the superconducting state in order to estimate the numerical function \( \kappa(r', \Delta) \). From (D18) we see that both the resonance region \( \omega < 2\Delta \) and the continuum region \( \omega > 2\Delta \) contribute to the discontinuity correction. In the continuum region \( \omega > 2\Delta \) we can approximate the self-energy to be similar to the normal state polarization operator \( \Pi_Q(\omega) \approx -i\gamma \theta(\omega - 2\Delta) \) with an additional spin gap below \( 2\Delta \), because for \( \omega > \Delta \) the superconducting and the normal conducting properties are quite similar. In the resonance region \( \omega < 2\Delta \) we use the known results from the one-loop calculations and estimate the dimensionless spin susceptibility to be
\[ \tilde{\chi}_{x,y}(\nu < 2) = -\frac{1}{\pi \nu - \Omega_{\text{res}} + i0} \] with \( \Omega_{\text{res}} = 2\pi \left( 1 - \frac{\tilde{\Omega}_{\text{res}}}{2} \right) \), (D19)
where dimensionless resonance energy is defined as \( \tilde{\Omega}_{\text{res}} = 2(1 - e^{-\frac{\Delta^2 + \Lambda(x^2 + y^2)}{2}}) \). The imaginary part of the \( \tilde{\chi}_{x,y} \) is then just given by Eq. (D15). As already stressed the two external parameters \( \frac{\Delta^2 + \Lambda}{2} \) and \( \Delta \) can be expressed in terms of \( C_2(\lambda) \) and \( \lambda \) using Eq. (D18). Using these approximations it is possible to determine numerically the function \( \kappa(\frac{\omega_{sf}}{\Delta}, \Delta) = \tilde{\kappa}(\lambda) \).

In the limit of large \( \lambda \gg 1 \) the resonance energy \( \Omega_{\text{res}} \ll \Delta \) at \( q = Q \). For small \( x, y \ll \Delta^{-1} \sim 1 \), this will lead to a different dependence of the resonance energy \( \Omega_{\text{res}} \sim \sqrt{\omega_{sf}/\Delta + \Lambda(x^2 + y^2)} \) and of the spectral weight \( \tilde{Z}_{\text{res}} \sim \frac{\omega_{sf}/\Delta + \Lambda(x^2 + y^2)}{\omega_{sf}/\Delta + \Lambda(x^2 + y^2)} \). For \( \lambda \gg 1 \), \( \omega_{sf} \ll \Delta \) and consequently the spectral weight for small \( x, y \ll \Delta^{-1} \) will be larger than for the exponential dependence in Eq. (D19). Nevertheless, as this different behavior is restricted to a small \( x, y \) phase space we do not expect qualitative changes of the occurring integrals in (D18).

**Appendix E: Phase sensitivity at three-loop**

Let us consider the possible three-loop diagrams without the diagrams containing the one-loop self-energy correction that was already calculated in Eq. (46), see Fig. 10. Note that we have no diagrams containing a bosonic line with particle-hole bubble since they are already included in the self-consistent two-loop diagram (50). We want to show that the discontinuity for the s-wave symmetry \( \Delta_{k_F+Q} = \Delta_{k_F} \) vanished also for these diagrams. As can be seen in the previous Appendix D the restriction to external frequency \( \omega = 2\Delta \) and momenta \( Q \) constrain two fermionic propagators connected to the left or right external boson line to lie directly at the hot spots. In the two-loop calculation this allowed us to combine all important information about the coherence factors and different combinations of normal and anomalous Green’s functions in the functions defined in Eq. (D11). From now on we will always restrict the two propagators on the left side to the hot spots. The first diagram in Fig. 10 is given by
\[ \Pi^{(3)}_{(a)} \sim \sum_{i,j} \sum_{k,q,q'} \chi_i \chi_{q'} \text{tr} \left[ \hat{G}_{k+q} \cdot \chi_{k+q} \hat{G}_{k+q} + \hat{G}_{k+q} \cdot \chi_{k+q} \hat{G}_{k+q} + \hat{G}_{k+q} \cdot \chi_{k+q} \hat{G}_{k+q} \right] \]
\[ = \sum_{k,q,q'} \sum_{\{A,B,C,D,E,F\}} \chi_i \chi_{q'} A_{k+q} B_{k+q} C_{k+q} + q D_{k+q} + q E_{k+q} + q F_{k+q} \] (E1)}
where $Q = (\omega, \mathbf{Q})$ is the external frequency and momentum. In the second step we performed the trace over the Nambu and spin degrees of freedom and found again all possible diagrams $\{A, B, C, D, E, F\}$ with arrow conservation at each vertex. Assuming that only the scattering between fermions around the hot spots contribute (so $\chi_q$ is strongly peaked around $\mathbf{Q}$) we are able to linearize the dispersions. Thus we get for the discontinuity contribution from the diagram (a)

$$
\delta D_{(a)}^{(3)}(\omega) \sim \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 \chi_1(\{\epsilon_i\}) \chi_2(\{\epsilon_i\}) \sum_{\{A, B, C, D, E, F\}} \nu^2_{A,0} \nu^2_{B,0} \nu^2_{C,\epsilon_1} \nu^2_{D,\epsilon_2} \nu^2_{E,\epsilon_3} \nu^2_{F,\epsilon_4} + \ldots \tag{E2}
$$

where we set the two left propagators $A$ and $B$ at the hot spots and $\mathbf{k}$. There are several other contributions containing different combinations of the coherence factors of $C, D, E, F$ and we linearized the momenta in such a way that the fermionic propagators have independent momentum integration variables $\epsilon_1, \ldots, \epsilon_4$. The bosonic modes now contain the information about the complicated dependence of the different momentum integrations. Nevertheless, the explicit form of this dependence is not of importance, because the sum over all possible diagrams $\{A, B, C, D, E, F\}$, which can be obtained by evaluating the trace in (E1), vanishes for the s-wave case as can be shown by coherence factors similar to (D5) and (D6). This behavior is not depending on the combination of coherence factors we wrote in (E2) and performing the Nambu and spin traces over the remaining diagrams in Fig. 10 we can show that the combination of Green’s functions $\{A, B, C, D, E, F\}$ are always the same, except of an overall factor. We also calculated several four-loop diagrams in a similar manner and found the same result. Summarizing, we could show that the phase sensitivity of the discontinuity of the bosonic self-energy is conserved up to three-loop order and presented a procedure to systematically examine the behavior of the discontinuity depending on the superconducting gap symmetry in arbitrary order perturbation theory.

Appendix F: $\epsilon$ expansion for vertex

The most straightforward way to see the suppression of the vertex correction in higher powers of $g$ in $d = 3 - \epsilon$ dimensions for the strong coupling case $r \ll \gamma \Delta$ is to look at the vertex

$$
\delta \Gamma_{k, Q} = \frac{1}{2}[G(p)G(p)] \tag{F1}
$$

with external momenta at the hot spots $k = k_F$ and the magnetic ordering vector $Q$ on the imaginary axis. We will only consider the $G^{(p)}G^{(p)}$ combination of Green’s functions for zero temperatures and at the phase transition $r = 0$,.
resulting in
\[
\delta \Gamma_{k_F, Q}(i\Omega, i\omega) \sim g^3 \int d^3q \int d\nu \frac{1}{c_s(q - Q)^2 - \Pi_Q(i\nu)} \frac{i(\Omega + \nu) + \varepsilon_{k_F+q} \Gamma(i\nu + \omega + \nu) + \varepsilon_{k_F+q+Q} \Gamma'(i\nu + \omega + \nu)^2}{(\Omega + \nu)^2 + \chi^2 + \Delta^2 (\Omega + \omega + \nu)^2 + \lambda'^2 + \Delta^2}.
\]

(F2)

Here, we assumed that only the bosonic propagator has a momentum dependence in the additional dimension \(q_z\) and linearized the remaining two dimensional \((q_x, q_y)\) integration in the usual way. The first integration can be performed by \(dq_z = \Omega_1 - d\Omega z\epsilon\), where \(\Omega_1 = 2\pi d/\Gamma(d/2)\) is the solid angle in \(d\) dimensions. A renormalization of the energy scales
\[
\frac{x}{\Delta}, y = \frac{\lambda'}{\Delta}, \tilde{\nu} = \frac{\nu}{\Delta}, \tilde{\omega} = \frac{\omega}{\Delta}, \tilde{\Omega} = \frac{\Omega}{\Delta}
\]
yields the vertex
\[
\delta \Gamma_{k_F, Q}(i\Omega, i\omega) \sim g^3 \Delta^{\frac{2+\epsilon}{2}} \int dx dy d\tilde{\nu} \left( \frac{1}{\Delta(x^2 + y^2) - \frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta}} \right)^{\frac{1+\epsilon}{2}} \frac{i(\tilde{\Omega} + \tilde{\nu}) + x}{(\tilde{\Omega} + \tilde{\nu})^2 + x^2 + 1} \frac{i(\tilde{\Omega} + \tilde{\nu} + \tilde{\omega}) + y}{(\tilde{\Omega} + \tilde{\omega} + \tilde{\nu} + \tilde{\omega})^2 + y^2 + 1}.
\]

(F3)

In the last step we used \(\gamma \sim g^2\) and the result from the pairing instability \(\Delta \sim g^{\frac{4-2\epsilon}{4}}\). The bosonic self-energy can be approximated on the imaginary axis as
\[
\frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta} \approx \begin{cases} \tilde{\nu}^2 & \text{for } \tilde{\nu} < 2 \\ -|\tilde{\nu}| & \text{for } \tilde{\nu} > 2 \end{cases}
\]

(F4)

and is therefore just a function not depending on \(g\). The only \(g\)-dependence in the integrand of (F3) is hidden in the parameter \(\Delta \sim \frac{\Delta}{\gamma} \sim g^{\frac{4-2\epsilon}{4}}\). There can be critical contributions where
\[
\left( \frac{1}{\Delta(x^2 + y^2) - \frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta}} \right)^{\frac{1+\epsilon}{2}} \sim \left( \frac{1}{\Delta} \right)^{\frac{1+\epsilon}{2}},
\]

(F5)

for the resonance and continuum region, which can be separated from the non-critical terms
\[
\delta \Gamma_{k_F, Q}(i\Omega, i\omega) = a(i\Omega, i\omega) \cdot g^{\frac{2-1}{2}} + \delta \Gamma_{k_F, Q}^{\text{crit,res}}(i\Omega, i\omega) + \delta \Gamma_{k_F, Q}^{\text{crit,con}}(i\Omega, i\omega)
\]

(F6)

Here, the dimensionless function \(a(i\Omega, i\omega)\) is not \(g\) dependent. These terms are for \(\epsilon < 1\) not critical, because the vertex correction \(\frac{\delta \Gamma}{\Gamma_0} \sim g^{\frac{2-2}{2}}\) is suppressed by a higher power in \(g\) compared to the bare vertex \(\Gamma_0 = g\). The question is how to estimate the critical contributions. For the resonance region \(\tilde{\nu} < 2\) the condition (F5) is fulfilled for
\[
\Delta(x^2 + y^2) > \frac{\Pi_Q(i\tilde{\nu}\Delta)}{\gamma\Delta} \approx \tilde{\nu}^2 |\tilde{\nu}| < \sqrt{\Delta} \sim g^{\frac{4-2\epsilon}{4}}
\]

and gives an exponentially small integration area for the \(\tilde{\nu}\) integration. Thus the most critical contributions from the above integral (F3) of the resonance region are
\[
\delta \Gamma_{k_F, Q}^{\text{crit,res}}(i\Omega, i\omega) \sim g^{\frac{2-1}{2}} \int d\tilde{\nu} \left( \frac{1}{\Delta} \right)^{\frac{1+\epsilon}{2}} \sim g^{\frac{2-1}{2}} \Delta^{\frac{1-\epsilon}{2}} \sim g^{\frac{2+2\epsilon-3}{2}}
\]

(F8)

In an analogue treatment for the continuum region the critical contributions can analogue be estimated to at least of order
\[
\delta \Gamma_{k_F, Q}^{\text{crit,con}}(i\Omega, i\omega) \sim g^{\frac{1+2\epsilon-5}{2}}.
\]

(F9)
Therefore, for small $\varepsilon < 1$ the so-called critical contributions of the vertex corrections

$$\frac{\delta \Gamma_{\text{crit,res}}}{g} \sim g^\alpha_1(\varepsilon)$$

$$\frac{\delta \Gamma_{\text{crit,con}}}{g} \sim g^\alpha_2(\varepsilon)$$

with $\alpha_1(\varepsilon), \alpha_2(\varepsilon) > 0$

are suppressed by a higher order in $g$ and therefore negligible.