Effective Hamiltonian for non-minimally coupled scalar fields

Emine Meşe¹, Nurettin Pirinçcioğlu¹, Irfan Açıkgoz¹ and Figen Binbay¹

¹ Department of Physics, Dicle University, TR21280, Diyarbakır, Turkey

Abstract

Performing a relativistic approximation as the generalization to a curved spacetime of the flat space Klein-Gordon equation, an effective Hamiltonian which includes non-minimal coupling between gravity and scalar field and also quartic self-interaction of scalar field term is obtained.

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1 Introduction

The Hamilton operator for quantum optics in gravitational fields is derived using minimal coupling and obtained that electric field as well as the dipole are operationally defined by measured quantities. In contrast, the magnetic dipole coupling can be modified by the gravitational field [1]. This effect is important for the interaction of mesoscopic quantum system with gravitational fields. An intensive review on interaction of mesoscopic systems including an overview of classical gravitational waves is given by Kiefer and Weber [2]. There are many other works can be given which explored these effects [3], [4], [5]. The gravitational interaction is distinguished by the fact that it interacts universally with all forms of energy. It dominates on a large scales (cosmology) and for compact objects (such as neutron stars and black holes). In large scales in curved spaces nonminimal coupling term is to be expected [6], [7].

In this work generalization to a curved spacetime of the flat space Klein-Gordon (KG) equation is considered and quartic self-interaction term in addition to nonminimal coupling effects is included. The motivation for choosing a quartic self-interaction comes from the followings: Energy of an homogenous superconductor is characterised by Ginzburg-Landau model [2] [8] [9] [10] [11]. This is exactly the same as the quartic self-interaction term.

The following section represent the derived Hamiltonian for the KG equation in curved space time including quartic self-interaction term, and the paper is completed with a short conclusion.

2 Non-minimally coupled scalar fields

We consider a generic complex scalar field \( \phi(x) \) with squared-mass \( m^2 \) and quartic coupling \( \lambda \). Its equation of motion is given by

\[
g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{m^2 c^2}{\hbar^2} \phi - \zeta \mathcal{R} \phi - \frac{\lambda}{\hbar^2 c^2} |\phi|^2 \phi = 0
\]  

where the dimensionless parameter \( \zeta \) stands for the direct coupling of \( \phi \) to curvature scalar \( \mathcal{R} \). The derivatives in the first term are meant to be covariant with respect to both general
coordinate and gauge transformations:
\[ \nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha - i \frac{q_\phi}{\hbar c} A_\mu V^\nu \]  

(2)

for \( V_\mu = \nabla_\mu \phi \equiv \partial_\mu \phi \) with \( q_\phi \) being the electric charge of \( \phi \) and \( \Gamma^\nu_{\mu\lambda} \) the connection coefficients.

For a thorough understanding of the energetics and dynamics of mesoscopic quantum systems represented by \( \phi \), it suffices to treat gravitational and electromagnetic fields as classical backgrounds. In post-Newtonian approximation, the most general parameterization of the gravitational field \( g_{\mu\nu}(x) \) (created by entire matter and energy surrounding the mesoscopic structure) is provided by the PPN formalism [12]:

\[
\begin{align*}
g_{00} &= - \left[ 1 - 2 \frac{U}{c^2} + 2 \beta \left( \frac{U}{c^2} \right)^2 \right] \\
g_{ij} &= \left[ 1 + 2 \gamma \frac{U}{c^2} \right] \delta_{ij} \\
g_{0i} &= 0, \quad \forall \, i, j = 1, 2, 3.
\end{align*}
\]

(3)

and \( g_{0i} = 0 \), for \( \forall \, i, j = 1, 2, 3 \). This parameterization rests on the assumption that entire surrounding matter is at rest as otherwise there would be a nonvanishing \( g_{0i} \) induced by flow of matter. The parameters \( \beta \) and \( \gamma \) (each of which equals unity in Einstein gravity) measure, respectively, nonlinearity in gravitational interactions and amount of spacetime curving induced by unit mass. Here \( U \) is nothing but the instantaneous Newton potential

\[ U(\vec{x}, t) = G_N \int d^3\vec{x}' \frac{\rho_m(\vec{x}', t)}{|\vec{x} - \vec{x}'|} \]

(4)

where \( \rho_m(\vec{x}', t) \) is the rest mass density of the matter distribution generating \( g_{\mu\nu}(x) \) above.

After inserting (3) in (1), and going to non-relativistic limit (by discarding high-frequency modes with \( \omega = mc^2/\hbar \) as in [1]) the equation of motion (1) can be recast into the form

\[ i\hbar \partial_t \phi = H \phi \]

(5)

where \( \phi(x) = e^{i\frac{mc^2}{\hbar} t} \phi(x) \), and

\[
H = \left( 1 - \frac{\vec{p}^2}{4m^2c^2} - (2\gamma + 1) \frac{U}{c^2} \right) \frac{\vec{p}^2}{2m} - q_\phi A_0 - mU - \left( \frac{1}{2} - \beta \right) \frac{mU^2}{c^2} - \frac{i}{\hbar} \frac{(2\gamma + 1)}{2mc^2} \vec{g} \cdot \vec{p} - \frac{3\pi G_N \hbar^2}{mc^2} \left( \gamma - \frac{4}{3} (2\gamma - 1) \zeta \right) \rho_m + \frac{\lambda}{2mc^2} |\phi|^2
\]

(6)
with $\vec{g} = -\vec{\nabla}U$ being the gravitational acceleration, and $\vec{p} = -i\hbar \left( \vec{\nabla} - \frac{iq\phi}{\hbar c} \vec{A} \right)$ the mechanical momentum.

The Hamiltonian $H$ given in (6) is hermitian with respect to the Schrödinger scalar product. All time-dependent hermiticity-violating terms are canceled due to the time dependent transformation as it is done in [1]. This Hamiltonian includes the kinetic energy terms of the system to the first relativistic correction to the coupling of the matter field to the gravity. The term represents the coupling of the matter includes gravitational Darwin term. It is clear to see that the gravitational Darwin term is modified by the influence of nonminimal coupling with comparing the Hamiltonian $H$ with those obtained by Lammerzahl in [1]. In addition Equation (6) includes the quartic self-interaction term which can be considered as energy term for superconductor characterised by Landau-Ginzburg model.

3 Conclusion

In this study we have shown that nonminimal coupling effects modifies the gravitational Darwin term. In addition quartic self-interaction is represented in the Hamiltonian. Since quartic self-interaction term in Hamiltonian (6) is similar to the energy expansion of an homogenous superconductor characterised by Ginzburg-Landau model, a future work of exploring gravitational field effects on the critical temprature $T_c$ of superconductor can be suggested.

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