Renewable Learning for Multiplicative Regression with Streaming Datasets

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Abstract

When large amounts of data continuously arrive in streams, online updating is an effective way to reduce storage and computational burden. The key idea of online updating is that the previous estimators are sequentially updated only using the current data and some summary statistics of historical raw data. In this article, we develop a renewable learning method for a multiplicative regression model with streaming data, where the parameter estimator based on a least product relative error criterion is renewed without revisiting any historical raw data. Under some regularity conditions, we establish the consistency and asymptotic normality of the renewable estimator. Moreover, the theoretical results confirm that the proposed renewable estimator achieves the same asymptotic distribution as the least product relative error estimator with the entire dataset. Numerical studies and two real data examples are provided to evaluate the performance of our proposed method.

Keywords: Multiplicative regression; Positive responses; Renewable learning; Streaming data.

1 Introduction

With the rapid development of data collecting and storage technologies, the sizes of available datasets have grown rapidly during recent years. In the era of big data, it is common that datasets continuously arrive in streams or large chunks. Faced with this kind of large-scale streaming dataset, many conventional statistical methods are challenging mainly due

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to (i) the entire dataset is too large to be held in a general computer’s memory; (ii) the historical data may no longer be accessible due to the storage burden or privacy limit. The online updating method is effective to address the two challenges, because it only needs the current block data and some summary statistics of previous data instead of historical raw data. To be more specific, the primary advantage of online updating method is that it does not require to access historical data, while it is able to provide real-time inference for making decisions. In the literature, many efforts have been devoted to develop online updating methods towards streaming datasets. For example, Schifano et al. (2016) proposed a cumulative estimating equation (CEE) estimator and a cumulatively updated estimating equation (CUEE) estimator with streaming datasets. Lee et al. (2020) studied an online updating method to correct the bias due to covariate measurement error in the framework of linear models. Luo and Song (2020) developed an incremental updating algorithm to analyze streaming data for generalized linear model. Lin et al. (2020) established a unified framework of renewable weighted sums for various online updating estimations with streaming datasets. Xue et al. (2020) proposed an online updating-based test to evaluate the proportional hazards assumption with streaming survival data. Wu et al. (2021) proposed an online updating method of survival analysis under the Cox proportional hazards model. Luo and Song (2021) studied a multivariate online regression analysis with heterogeneous streaming data. Lin et al. (2021) studied a homogenization strategy for heterogeneous streaming data. Hector et al. (2021) proposed a new big data learning method by seamlessly integrating parallel data processing and online streaming paradigm. Luo et al. (2021) proposed an online debiased lasso method for high-dimensional generalized linear models with streaming data. Shi and Luo (2021) studied a novel framework for online causal learning. Luo et al. (2022) proposed an incremental learning algorithm to analyze streaming data with correlated outcomes based on quadratic inference function. Wang et al. (2022) proposed a novel online renewable strategy for quantile regression, among others.

In practice, we often meet with positive data in economic or biomedical studies. The multiplicative regression plays an important role in modeling this kind of positive data, such as stock prices or life times. In many applications, the relative error (e.g. stock price data), rather than error itself, is the major concern. The multiplicative regression is able
to capture the size of relative error. There have been several papers on the statistical analysis with multiplicative regression in the literature. e.g., Chen et al. (2010) proposed a least absolute relative error estimation criterion for multiplicative regression model. Li et al. (2014) considered an empirical likelihood approach towards constructing confidence intervals of the regression parameters in multiplicative regression model. Chen et al. (2016) proposed a least product relative error (LPRE) estimation criterion for multiplicative regression model. Xia et al. (2016) studied the variable selection for multiplicative regression model. Faced with large-scale streaming data with positive responses, we propose a renewable learning method for multiplicative regression model. The main features of our approach are as follows: First, the renewable estimator and its variance are sequentially updated only using the current data batch and some summary statistics of historical data, instead of the historical raw data. Therefore, the proposed method can deal with the computation and storage burden due to massive blocks of data. Second, the renewable estimator is statistically equivalent to the traditional LPRE estimator that based on the entire dataset, which implies that it achieves the same asymptotic distribution as the traditional LPRE estimator. Third, the computational speed of the proposed renewable learning method is much faster than the full data method.

The remainder of this article is organized as follows. In Section 2, we briefly review some notations for the multiplicative regression model with streaming data. In Section 3, we present a renewable estimation method and review two sequential updating methods. Section 4 investigates the theoretical properties of the proposed renewable estimator. In Section 5, we conduct some numerical simulations to evaluate the performance of our method. Section 6 presents two illustrative real data examples. In Section 7, we give some conclusions and future research topics. All proofs are given in the Appendix.

## 2 Model and Notations

We consider the following multiplicative regression model (Chen et al., 2010),

\[ Y_i = \exp(\beta^T X_i) \epsilon_i, \]  

(2.1)
where $Y_i$ is a positive response variable, $X_i \in \mathbb{R}^p$ is a vector of covariates with the first component being 1 (intercept), $\beta = (\beta_1, \ldots, \beta_p)^T$ is a vector of regression parameters, and $\epsilon_i > 0$ is an error term, $i = 1 \ldots N$. To estimate the parameters in model (2.1), Chen et al. (2016) proposed a LPRE criterion

$$\ell(Y; X, \beta) = \sum_{i=1}^{N} \left\{ Y_i \exp(-\beta^T X_i) + Y_i^{-1} \exp(\beta^T X_i) - 2 \right\},$$

which is an infinitely differentiable and strictly convex function, where $Y = (Y_1, \ldots, Y_N)^T$ and $X = (X_1, \ldots, X_N)^T$. Accordingly, the score function is given by $S(Y; X, \beta) = \nabla_\beta \ell(Y; X, \beta)$, where $\nabla_\beta$ stands for the derivative of $\ell(Y; X, \beta)$ with respect to $\beta$. Specifically, the score function has the following explicit expression:

$$S(Y; X, \beta) = \sum_{i=1}^{N} \left\{ Y_i^{-1} \exp(\beta^T X_i) - Y_i \exp(-\beta^T X_i) \right\} X_i.$$

Denote the minimizer of $\ell(Y; X, \beta)$ as $\hat{\beta}_N$, satisfying $S(Y; X, \hat{\beta}_N) = 0$. Due to the convexity of $\ell(Y; X, \beta)$, the Newton-Raphson method is usually adopted to obtain the traditional LPRE estimator.

Note that streaming data with positive response is very common in many fields such as bioinformatics (Wei, 1992; Jin et al., 2003) and economic analysis (Teekens and Koerts, 1972). This brings new research opportunities, but also comes with challenges of storing and analyzing such streaming data. To be more specific, the storage burden is heavy due to large blocks data. Moreover, it is often computationally infeasible to perform statistical analysis due to the relatively limited computing resources at hand. Meanwhile, the previous data may be not accessible due to privacy concern. Therefore, it is desirable to develop a renewable learning method for the multiplicative regression model that does not require storing any historical individual-level data in the streaming data environment. Assume that $\mathcal{D}_1, \ldots, \mathcal{D}_b, \ldots$ are independent and identically distributed streaming datasets, where $\mathcal{D}_b = \{(X_{ib}, Y_{ib})\}_{i=1}^{n_b}$ is the $b$th dataset. Let $\mathcal{D}_b^* = \{\mathcal{D}_1, \ldots, \mathcal{D}_b\}$ denotes the cumulative data up to batch $b$ with $N_b = \sum_{k=1}^{b} n_k$. As mentioned by Luo and Song (2020), the key idea of renewable estimation method is that a previous estimator is sequentially updated only using the current data batch $\mathcal{D}_b$ and some summary statistics of historical data batches. To deal
with large-scale streaming data with positive response, we will propose a renewable learning method for the multiplicative regression model in next section.

3 Methods

3.1 Renewable Estimation

Let $\hat{\beta}_b$ and $\beta^*_b$ be the traditional LPRE estimators obtained from a single batch $D_b$ and the entire cumulative dataset $D^*_b$, respectively. Denote $\tilde{\beta}_b$ as a renewable estimator obtained from the current data batch $D_b$ and some summary statistics of historical data batches $D^*_{b-1}$, where an initial estimator with the first data batch is $\tilde{\beta}_1 = \hat{\beta}_1 = \beta^*_1$. For $b = 2, 3, \ldots$, a previous estimator $\tilde{\beta}_{b-1}$ is sequentially updated to $\tilde{\beta}_b$ using the current data batch $D_b$ and a summary statistic of previous data batches $D^*_{b-1}$. To illustrate the proposed method, we denote the score function on data batch $D_b$ as follows:

$$ S_b(D_b, \beta) = \sum_{i \in D_b} \{ Y_i^{-1} \exp(\beta^T X_i) - Y_i \exp(-\beta^T X_i) \} X_i, $$

and its negative gradient matrix is

$$ Q_b(D_b, \beta) = -\sum_{i \in D_b} \{ Y_i \exp(-\beta^T X_i) + Y_i^{-1} \exp(\beta^T X_i) \} X_i X_i^T. $$

For simplicity, we first consider two data batches $D_1$ and $D_2$. For the first data batch $D_1$, a LPRE $\hat{\beta}_1$ is obtained by solving $S_1(D_1, \hat{\beta}_1) = 0$. When the second data batch $D_2$ arrives, the traditional LPRE estimator $\hat{\beta}^*_2$ satisfies the following aggregated score equation,

$$ S_1(D_1, \hat{\beta}^*_2) + S_2(D_2, \hat{\beta}^*_2) = 0. $$

(3.1)

However, solving equation (3.1) requires revisiting the previous data batch $D_1$. To derive a renewable estimator that does not need to revisit $D_1$, we take the first-order Taylor expansion of $S_1(D_1, \hat{\beta}^*_2)$ at the estimator $\tilde{\beta}_1$,

$$ S_1(D_1, \tilde{\beta}_1) + Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \hat{\beta}^*_2) + O_p \left( \| \hat{\beta}^*_2 - \tilde{\beta}_1 \|^2 \right) + S_2(D_2, \hat{\beta}^*_2) = 0. $$

If $\min \{ n_1, n_2 \}$ is large enough, both $\beta^*_2$ and $\tilde{\beta}_1$ are consistent estimators of the true value $\beta_t$ (Chen et al., 2016). After ignoring the error term $O_p \left( \| \hat{\beta}^*_2 - \tilde{\beta}_1 \|^2 \right)$, we can derive a renewable estimator $\tilde{\beta}_2$ satisfying
\[ S_1(D_1, \tilde{\beta}_1) + Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \tilde{\beta}_2) + S_2(D_2, \tilde{\beta}_2) = 0. \]

Due to the fact that \( S_1(D_1, \tilde{\beta}_1) = 0 \), the renewable estimator \( \tilde{\beta}_2 \) satisfies the following estimating equation:

\[ Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \tilde{\beta}_2) + S_2(D_2, \tilde{\beta}_2) = 0. \quad (3.2) \]

In a similar way, the traditional LPRE estimator \( \hat{\beta}_3^* \) satisfies the following aggregated score equation after data batch \( D_3 \) arrives,

\[ S_1(D_1, \hat{\beta}_3^*) + S_2(D_2, \hat{\beta}_3^*) + S_3(D_3, \hat{\beta}_3^*) = 0. \]

Taking the first-order Taylor expansion of \( S_1(D_1, \hat{\beta}_3^*) \) and \( S_2(D_2, \hat{\beta}_3^*) \) at \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \), respectively, we obtain

\[
S_1(D_1, \hat{\beta}_1) + Q_1(D_1, \hat{\beta}_1)(\hat{\beta}_1 - \hat{\beta}_3^*) + O_p \left( \| \hat{\beta}_3^* - \hat{\beta}_1 \|^2 \right) + S_2(D_2, \hat{\beta}_2) \\
+ Q_2(D_2, \hat{\beta}_2)(\hat{\beta}_2 - \hat{\beta}_3^*) + O_p \left( \| \hat{\beta}_3^* - \hat{\beta}_2 \|^2 \right) + S_3(D_3, \hat{\beta}_3^*) = 0. \quad (3.3)
\]

The error terms \( O_p \left( \| \hat{\beta}_3^* - \hat{\beta}_1 \|^2 \right) \) and \( O_p \left( \| \hat{\beta}_3^* - \hat{\beta}_2 \|^2 \right) \) in (3.3) could be asymptotically ignored if \( \min\{n_1, n_2, n_3\} \) is large enough. Removing such error terms, it is straightforward to deduce that

\[ S_1(D_1, \tilde{\beta}_1) + Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \tilde{\beta}_3^*) + S_2(D_2, \tilde{\beta}_2) + Q_2(D_2, \tilde{\beta}_2)(\tilde{\beta}_2 - \tilde{\beta}_3^*) + S_3(D_3, \tilde{\beta}_3^*) = 0. \]

In view of \( S_1(D_1, \tilde{\beta}_1) = 0 \) and (3.2), we have

\[ Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \tilde{\beta}_3^*) - Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \tilde{\beta}_2) + Q_2(D_2, \tilde{\beta}_2)(\tilde{\beta}_2 - \tilde{\beta}_3^*) + S_3(D_3, \tilde{\beta}_3^*) = 0. \]

By merging the terms \( Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \tilde{\beta}_3^*) \) and \(-Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \tilde{\beta}_2)\), we obtain this expression:

\[ Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_2 - \tilde{\beta}_3^*) + Q_2(D_2, \tilde{\beta}_2)(\tilde{\beta}_2 - \tilde{\beta}_3^*) + S_3(D_3, \tilde{\beta}_3^*) = 0. \]
Therefore, the renewable estimator $\tilde{\beta}_3$ is a solution to the following equation:

$$Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_2 - \tilde{\beta}_3) + Q_2(D_2, \tilde{\beta}_2)(\tilde{\beta}_2 - \tilde{\beta}_3) + S_3(D_3, \tilde{\beta}_3) = 0.$$ 

Similarly, we introduce a renewable estimator $\tilde{\beta}_b$ satisfying the incremental estimating equation:

$$\sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k)(\tilde{\beta}_b - \tilde{\beta}_k) + S_b(D_b, \tilde{\beta}_b) = 0. \quad (3.4)$$

For convenience, we denote the aggregated negative gradient matrix $\sum_{k=1}^b Q_k(D_k, \tilde{\beta}_k)$ as $\tilde{Q}_b$. Based on (3.4), the renewable estimator $\tilde{\beta}_b$ can be easily solved via the following Newton-Raphson iterations,

$$\tilde{\beta}_b^{(m+1)} = \tilde{\beta}_b^{(m)} - \left\{ \tilde{Q}_b^{(m)} + Q_b(D_b, \tilde{\beta}_b^{(m)}) \right\}^{-1} \tilde{S}_b^{(m)},$$

where the adjusted score $\tilde{S}_b^{(m)} = \tilde{Q}_b^{(m)}(\tilde{\beta}_b^{(m)} - \tilde{\beta}_b^{(m)}) + S_b(D_b, \tilde{\beta}_b^{(m)})$ is updated over iterations. To speed up the calculations, we may avoid updating the negative gradient matrix $Q_b(D_b, \tilde{\beta}_b^{(m)})$ at each iteration. As suggested by Luo and Song (2020), we obtain the following iterative formula by replacing $\tilde{\beta}_b^{(m)}$ with $\tilde{\beta}_b^{(m)}$ in $Q_b(D_b, \tilde{\beta}_b^{(m)})$,

$$\tilde{\beta}_b^{(m+1)} = \tilde{\beta}_b^{(m)} - \left\{ \tilde{Q}_b^{(m)} + Q_b(D_b, \tilde{\beta}_b^{(m)}) \right\}^{-1} \tilde{S}_b^{(m)}.$$

### 3.2 Sequential Updating Methods

Note that Schifano et al. (2016) proposed a general cumulative estimating equation (CEE) approach, which can be directly applied to the multiplicative regression model with streaming data. For comparison, we provide some details on the CEE estimator $\tilde{\beta}_b^{cee}$ for model (2.1), which is given by

$$\tilde{\beta}_b^{cee} = (\tilde{Q}_b^{cee} + Q_b^{cee})^{-1}(\tilde{Q}_b^{cee} \tilde{\beta}_b^{cee} + Q_b^{cee} \tilde{\beta}_b), \quad \tilde{Q}_b^{cee} = \sum_{k=1}^b Q_k^{cee}, \quad b = 1, 2, \ldots.$$
Here $\tilde{\beta}_0^{cee} = 0_{p \times 1}$, $\tilde{Q}_0^{cee} = 0_{p \times p}$ and $Q_b^{cee} = -\nabla_\beta S_b(\mathcal{D}_b, \tilde{\beta}_b)$ is the negative gradient matrix of data batch $\mathcal{D}_b$. For symbol simplicity, we denote

$$C_b(\mathcal{D}_b, \beta) = S_b(\mathcal{D}_b, \beta)S_b(\mathcal{D}_b, \beta)^T = \sum_{i \in \mathcal{D}_b} \left\{ Y_i^{-1} \exp(\beta^T X_i) - Y_i \exp(-\beta^T X_i) \right\}^2 X_i X_i^T,$$

and $C_b^{cee} = C_b(\mathcal{D}_b, \hat{\beta}_b)$. Set $\tilde{V}_0^{cee} = 0_{p \times p}$, from equation (18) of Schifano et al. (2016), the variance of $\tilde{\beta}_b^{cee}$ is

$$\tilde{V}_b^{cee} = (\tilde{Q}_{b-1}^{cee} + Q_b^{cee})^{-1} \left\{ \tilde{Q}_{b-1}^{cee} \tilde{V}_b^{cee} (\tilde{Q}_{b-1}^{cee})^T + Q_b^{cee} V_b^{cee} (Q_b^{cee})^T \right\} \times \left\{ (\tilde{Q}_{b-1}^{cee} + Q_b^{cee})^{-1} \right\}^T, \quad b = 1, 2, \ldots,$$

where $V_b^{cee} = \left\{ Q_b^{cee} C_b^{cee^{-1}} Q_b^{cee^T} \right\}^{-1}$ presented by Chen et al. (2016) is the estimated variance of $\tilde{\beta}_b$ from the $b$th data batch.

To further reduce bias of the CEE estimator, a cumulatively updated estimating equation (CUEE) estimator was proposed by Schifano et al. (2016). Denote $Q_b^{cuee} = -\nabla_\beta S_b(\mathcal{D}_b, \tilde{\beta}_b)$ and $C_b^{cuee} = C_b(\mathcal{D}_b, \tilde{\beta}_b)$, where $\tilde{\beta}_b$ is a CEE estimator. From equations (22) and (23) of Schifano et al. (2016), with initial $\tilde{Q}_0^{cuee} = Q_0^{cuee} = 0_{p \times p}$ and $\tilde{V}_0^{cuee} = 0_{p \times p}$, the CUEE estimator and its corresponding variance matrix are

$$\tilde{\beta}_b^{cuee} = (\tilde{Q}_{b-1}^{cuee} + Q_b^{cuee})^{-1} \left\{ \sum_{k=1}^{b-1} Q_k^{cuee} \tilde{\beta}_k + Q_b^{cuee} \tilde{\beta}_b - \sum_{k=1}^{b-1} S_k(\mathcal{D}_k, \tilde{\beta}_k) - S_b(\mathcal{D}_b, \tilde{\beta}_b) \right\},$$

and

$$\tilde{V}_b^{cuee} = (\tilde{Q}_{b-1}^{cuee} + Q_b^{cuee})^{-1} \left\{ \tilde{Q}_{b-1}^{cuee} \tilde{V}_b^{cuee} (\tilde{Q}_{b-1}^{cuee})^T + Q_b^{cuee} V_b^{cuee} (Q_b^{cuee})^T \right\} \times \left\{ (\tilde{Q}_{b-1}^{cuee} + Q_b^{cuee})^{-1} \right\}^T, \quad for \quad b = 1, 2, \ldots,$$

where $Q_b^{cuee} = \sum_{k=1}^b Q_k^{cuee}$ and $V_b^{cuee} = \left\{ Q_b^{cuee} C_b^{cuee^{-1}} Q_b^{cuee^T} \right\}^{-1}$.

By Schifano et al. (2016), the consistency of CEE and CUEE estimators are established under a strong regularity condition, i.e., the number of data batches $b$ is of order $O(n_k^j)$, for $j < 1/3$ and all $k = 1, \ldots, b$. However, this condition is not always valid for streaming data, because $n_k$ is typically small, but $b$ grows at a high rate. We will compare the proposed renewable estimator with the CEE and CUEE estimators via numerical simulations.
4 Theoretical Properties

For notational simplicity, we denote

\[ C(\beta) = \mathbb{E}_\beta\{S(Y; X, \beta)S(Y; X, \beta)^T\}, \tag{4.1} \]

and

\[ Q(\beta) = -\mathbb{E}_\beta\{\nabla_\beta S(Y; X, \beta)\}. \tag{4.2} \]

To establish the consistency and asymptotic normality of the renewable estimator, we need the following regularity conditions.

(C.1) The true parameter \( \beta_t \) lies in the interior of a compact set \( \Theta \subset \mathbb{R}^p \).

(C.2) The terms \( C(\beta) \) and \( Q(\beta) \) are positive-definite for all \( \beta \in N_\delta(\beta_t) \), where \( N_\delta(\beta_t) = \{ \beta : \| \beta - \beta_t \| \leq \delta \} \) is a neighborhood around true value \( \beta_t \), and \( \delta \) is a positive constant.

(C.3) \[ \sup_{\beta \in \Theta} \frac{1}{N} \sum_{i=1}^{N} \left\{ Y_i \exp(-\beta^T X_i) + Y_i^{-1} \exp(\beta^T X_i) \right\}^2 \| X_i \|^2 = O_P(1). \]

Conditions (C.1) and (C.2) are regularity conditions, (e.g., Chen et al., 2016). Condition (C.3) is used to establish the consistency of the renewable estimator \( \tilde{\beta}_b \), together with its asymptotic distribution.

We first establish the consistency of the renewable estimator \( \tilde{\beta}_b \) towards the true value \( \beta_t \).

**Theorem 1** If the conditions (C.1)-(C.3) hold and \( N_b = \sum_{k=1}^b n_k \to \infty \), then the renewable estimator \( \tilde{\beta}_b \) given in equation (3.4) is consistent to \( \beta_t \), i.e. \( \tilde{\beta}_b - \beta_t \xrightarrow{P} 0 \).

To conduct statistical inference, we present the asymptotic normality of the renewable estimator \( \tilde{\beta}_b \) in the following theorem.

**Theorem 2** If the conditions (C.1)-(C.3) hold and \( N_b = \sum_{k=1}^b n_k \to \infty \), then the renewable estimator \( \tilde{\beta}_b \) has a mean-zero asymptotic normal distribution:

\[ \sqrt{N_b}(\tilde{\beta}_b - \beta_t) \xrightarrow{d} N(0, \mathbb{G}^{-1}(\beta_t)), \]

where \( \xrightarrow{d} \) denotes convergence in distribution, \( \mathbb{G}(\beta_t) = Q^T(\beta_t)C^{-1}(\beta_t)Q(\beta_t) \), and \( C(\beta_t) \) and \( Q(\beta_t) \) are given in (4.1) and (4.2), respectively.
It is worth mentioning that the asymptotic covariance of the renewable estimator \( \tilde{\beta}_b \) given in Theorem 2 is exactly the same as that of the traditional LPRE \( \hat{\beta}_b^* \) (Theorem 3 in Chen et al., 2016) based on the full data. This implies that the renewable estimator achieves the same efficiency as the traditional LPRE \( \hat{\beta}_b^* \). Using the aggregated matrix \( \tilde{Q}_b = \sum_{k=1}^{b} Q_k(D_k, \tilde{\beta}_k) \) and \( \tilde{C}_b = \sum_{k=1}^{b} C_k(D_k, \tilde{\beta}_k) \), we calculate the estimated asymptotic covariance matrix as

\[
\tilde{\Sigma}_b(\beta_t) = \left( N_b^{-1} \tilde{G}_b \right)^{-1} = N_b \left( \tilde{Q}_b^T \tilde{C}_b^{-1} \tilde{Q}_b \right)^{-1},
\]

where \( \tilde{G}_b = \tilde{Q}_b^T \tilde{C}_b^{-1} \tilde{Q}_b \). The estimated asymptotic variance matrix for the renewable estimator \( \tilde{\beta}_b \) is

\[
\tilde{V}(\tilde{\beta}_b) = \frac{1}{N_b} \tilde{\Sigma}_b(\beta_t) = \left( \tilde{Q}_b^T \tilde{C}_b^{-1} \tilde{Q}_b \right)^{-1}.
\]

In Algorithm 1, we summarize the procedure of the renewable estimation for multiplicative regression with streaming data.

5 Simulation Study

In this section, we conduct some simulations to demonstrate the effectiveness of our proposed method. The true value of \( \beta \) is chosen as \( \beta_t = (0.2, -0.2, 0.2, -0.2, 0.2)^T \). Denote the covariate \( \mathbf{X} = (1, \mathbf{X}^T)^T \) with \( \mathbf{X} = (X_1, \ldots, X_4)^T \), i.e. \( p = 5 \). We consider two cases for the error term: \( \log(\epsilon) \) follows \( N(0, 1) \), and \( \log(\epsilon) \) follows uniform distribution over \((-2, 2)\). Moreover, we choose four cases for the covariate \( \mathbf{X} \),

**Case 1.** \( \mathbf{X} \sim N(0, \Sigma) \), where \( \Sigma_{ij} = 0.5^{|i-j|} \).

**Case 2.** \( \mathbf{X} \sim 0.5N(1, \Sigma) + 0.5N(-1, \Sigma) \), where \( \Sigma_{ij} = 0.5^{|i-j|} \).

**Case 3.** \( \mathbf{X} \sim t_5(0, \Sigma) \), i.e., \( \mathbf{X} \) follows a multivariate \( t \) distribution with degree of freedom \( \chi = 5 \) and covariance matrix \( \Sigma_{ij} = 0.5^{|i-j|} \).

**Case 4.** \( \mathbf{X} = (X_1, \ldots, X_4)^T \), where \( X_i \)’s are independently and identically distributed exponential random variables with the probability density function \( f(x) = e^{-x} \).
Algorithm 1: Renewable Estimation

**Input:** Sequentially arrived datasets $\mathcal{D}_1, \ldots, \mathcal{D}_b, \ldots$;

**Output:** $\tilde{\beta}_b$ and $\tilde{V}(\tilde{\beta}_b)$, for $b = 1, 2, \ldots$;

1. Initialize: set initial values $\tilde{\beta}_{\text{init}} = 0$, $\tilde{Q}_0 = 0_{p \times p}$ and $\tilde{C}_0 = 0_{p \times p}$;

2. for $b = 1, 2, \ldots$ do 
   
   3. Load the dataset $\mathcal{D}_b$;
   
   4. repeat
      
      5. $\tilde{\beta}_b^{(m+1)} = \tilde{\beta}_b^{(m)} - \left\{ \tilde{Q}_{b-1} + Q_b(\mathcal{D}_b, \tilde{\beta}_{b-1}) \right\}^{-1} \left\{ \tilde{Q}_{b-1}(\tilde{\beta}_{b-1} - \tilde{\beta}_b^{(m)}) + S_b(\mathcal{D}_b, \tilde{\beta}_b^{(m)}) \right\};$
   
      6. until convergence;
   
   7. Update $\tilde{Q}_b = \tilde{Q}_{b-1} + Q_b(\mathcal{D}_b, \tilde{\beta}_b)$ and $\tilde{C}_b = \tilde{C}_{b-1} + C_b(\mathcal{D}_b, \tilde{\beta}_b)$;
   
   8. Calculate $\tilde{V}(\tilde{\beta}_b) = \left\{ \tilde{Q}_b^T \tilde{C}_b^{-1} \tilde{Q}_b \right\}^{-1};$
   
   9. Release the dataset $\mathcal{D}_b$ from the memory.

10. end

11. Output $\tilde{\beta}_b$ and $\tilde{V}(\tilde{\beta}_b)$, for $b = 1, 2, \ldots$. 


For comparison, we compare our renewable estimator with three competing estimators, which include

(i) the traditional LPRE estimator obtained from the entire data (Chen et al., 2016);

(ii) the CEE estimator (Schifano et al., 2016);

(iii) the CUEE estimator (Schifano et al., 2016).

These methods are compared from two aspects of estimation efficiency and computation speed. The results for estimation efficiency include: the estimated bias (BIAS) given by the sample mean of the estimates minus the true value $\beta_t$, the sampling standard error (SSE) of the estimates, the mean of the estimated standard errors (ESE) and the empirical 95% coverage probabilities (CP) towards the true value $\beta_t$. In addition, the computation efficiency is assessed by computation time (C.Time) and running time (R.Time), where C.Time includes time on data loading and algorithm execution, and R.time only accounts time for algorithm execution. All the simulation results in Tables 1–8 are implemented in the R programming language based on 500 replications.

5.1 Evaluation of Estimation Efficiency

5.1.1 Scenario 1: Fixed $N_B$ and Varying Batch Size $n_b$

We first consider a scenario with fixed $N_B$ and varying batch size $n_b$. Specifically, we generate $B$ data blocks consisting of $N_B = 10^5$ independent observations, where each data batch has $n_b$ observations. In Tables 1-4, we present the results for $\beta_1$ (intercept) and $\beta_2$ ($\beta_i$’s are similar to $\beta_2$ and omitted, $i = 3, 4, 5$). From Tables 1-4 we can see that the proposed renewable estimator performs comparably with the traditional LPRE estimator based on the entire data. Moreover, the bias of the CUEE is much smaller than that of the CEE estimator, while its SSE is much larger than that of the CEE as $n_b$ decreases to 50. The coverage probabilities of the CUEE estimator are mostly dropped below 90% as $n_b$ decreases to 50. This confirms that if the condition $B = O(n_b^j)$, $j < 1/3$ is not satisfied, the CEE and CUEE methods do not have valid asymptotic distributions for inference. In contrast, the
proposed method achieves valid and efficient statistical efficiency, and its coverage probability is around 95% under the chosen settings.

5.1.2 Scenario 2: Fixed Batch Size $n_b$ and Varying $B$

Now we consider another scenario with fixed batch size $n_b$ and varying $B$. For convenience, we fix batch size $n_b = 100$, and $N_B$ is varying from $10^3$ to $10^5$. In Tables 5-8, we present the estimation results for $\beta_1$ (intercept) and $\beta_2$, respectively ($\beta_i$'s are similar to $\beta_2$ and omitted, $i = 3, 4, 5$). It is clear to see that the SSE and ESE of all estimators decrease as $B$ increases, which verifies the consistency of the four estimators. Moreover, it is shown that the four estimators are unbiased, the SSE is close to the ESE, and the coverage probabilities are satisfactory when $B = 10$ and 100. However, as $B$ increases to $10^3$, the SSE and ESE of the CUEE are significantly larger than that of other competitors. Although the ESE of the CEE is smaller than the proposed method, its coverage probability is much smaller compared with the full data LPRE when $\log(\epsilon)$ follows $N(0, 1)$. On the contrary, the proposed renewable method always exhibits the similar performance to the traditional LPRE estimator as $B$ increases from 10 to $10^3$, which confirms the stability and effectiveness of the proposed renewable estimator.

5.2 Evaluation of Computation Efficiency

To assess the computation efficiency, we report the CPU time (in seconds) for the LPRE, CEE, CUEE and the proposed method. We consider the following two scenarios: (a) varying $B$ (fixed $N_B = 10^7$); (b) varying $N_b$ (fixed $B = 10^3$). All computations are carried out on a laptop running R programming language with 16GB random-access memory (RAM). The results are the mean CPU time of ten replications. Tables 9 and 10 report C.Time and R.Time for two scenarios mentioned above with Case 1 and $\log(\epsilon) \sim N(0, 1)$. From Table 9, we can see that the proposed method is always much faster than the other three competitors. Moreover, the C.Time of the proposed method is less than 12 seconds, while the traditional LPRE requires more than 500 seconds when the number of batches $B$ increases to $10^4$. This fast computation of the proposed method does not lose statistical efficiency. In addition, the
CUEE takes more computing time compared with the CEE. The main reason is that the CEE estimator does not require an additional step to calculate the intermediary estimator $\hat{\beta}_b$ and the aggregated matrix $\sum_{k=1}^{b} S_k(D_k, \hat{\beta}_k)$. As shown in Table 10, compared with the other three competitors, our proposed renewable learning method has significant computational advantages both in R.Time and C.Time as $N_B$ increases to $10^5$.

6 Real Data Analysis

6.1 The Bike Sharing Data

In this section, we apply our proposed method to the bike sharing dataset, which is publicly available at http://archive.ics.uci.edu/ml/datasets/Bike+Sharing+Dataset. The streaming datasets arrive monthly during the 24-month period from January 2011 to December 2012, where $B = 24$ and $N_b = 17379$. We consider four covariates: a binary variable “workingday” ($X_1$) to indicate whether a certain day is a working day or not (1 = working day; 0 = non-working day), three continuous variables: temperature ($X_2$), humidity ($X_3$) and windspeed ($X_4$). The square of the number of bikes rented hourly is used as the response. Similar to the simulation studies, we also compare our proposed method with the CEE, CUEE and traditional LPRE method. We report the estimated parameters, standard errors and $p$-values in Table 11. As we can see, all $p$-values are sufficiently small ($\ll 0.05$), which indicates that all covariates are significant towards the response. It is seen from Table 11 that the number of rented bikes in non-working days is more than that of working days. The temperature and windspeed have positive influences on the number of rented bikes, and the humidity has a negative effect. Additionally, the CUEE has slight larger standard errors than those of the CEE and our proposed renewable method, which is in line with the simulation results.

6.2 The Electric Power Consumption Data

We apply our proposed method to an electric power consumption dataset, which contains 2,049,280 completed measurements for a house located at Sceaux between December 2006 and November 2010. We consider the scenario where the data arrive monthly
during the 48-month period with $B = 48$ data batches. The data is publicly available at http://archive.ics.uci.edu/ml/datasets/Individual+household+electric+power+consumption. For the analysis, the minute-averaged current intensity (in ampere) is used as the response. We consider three covariates: active electrical energy in the kitchen ($X_1$, in watt-hour), active electrical energy in the laundry room ($X_2$, in watt-hour), and active electrical energy for an electric water-heater and an air-conditioner ($X_3$, in watt-hour). All covariates are centered and scaled with mean 0 and variance 1. Similar to Section 6.1, we present the estimated coefficients, standard errors and $p$-values in Table 12. As shown in Table 12, three online updating estimates are unbiased and all standard errors are significantly small. In addition, all $p$-values are small enough to indicate that each covariate is significant towards the response. It is not surprising that there is more electric current through the water-heater and air-conditioner than through the laundry room and kitchen. This is because the power of the water-heater and air-conditioner is relatively large.

7 Concluding Remarks

In this paper, we proposed a renewable estimation method for the multiplicative regression model in the streaming data environment. The consistency and asymptotic normality of the renewable estimator were established. The simulation studies showed that the proposed renewable estimator was desirable compared with the CEE and CUEE estimator. In addition, the proposed estimator was asymptotically equivalent with the traditional LPRE estimator based on the entire data available. Two real data examples illustrated the effectiveness of our proposed renewable method.

There are several important topics to investigate further in the future. First, the paper mainly considered the problem of parameter estimation and statistical inference for multiplicative regression with streaming data. However, the online variable selection was not considered in the online updating context. This problem is interesting especially when accessing historical data is limited. Second, it is worth to extend our proposed renewable method to other survival models, such as the Cox model of Cox (1972) and the additive hazards model in Lin and Ying (1994).
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Appendix

Lemma 1 If the conditions (C.1) and (C.2) hold, we have

\[ S_b(D_b, \beta_t) = O_p(\sqrt{n_b}). \]  

(A.1)

Proof. Direct calculation yields that

\[ \mathbb{E}\{S_b(D_b, \beta_t)\} = 0, \]  

(A.2)

and

\[
\text{Var}\{S_b(D_b, \beta_t)\} = \mathbb{E}\left[ \sum_{i \in D_b} \left\{ Y_i^{-1} \exp(\beta_t^T X_i) - Y_i \exp(-\beta_t^T X_i) \right\}^2 X_i X_i \right]
\]

\[ \leq \mathbb{E}\left[ \sum_{i \in D_b} \left\{ Y_i^{-1} \exp(\beta_t^T X_i) + Y_i \exp(-\beta_t^T X_i) \right\}^2 X_i X_i \right]
\]

\[ = O_P(n_b), \]  

(A.3)

where the last equality is due to the condition (C.3). Combining (A.2), (A.3) and Chebyshev’s inequality, (A.1) follows. This ends the proof.

Proof of Theorem 1. We define a function

\[ f_b(\beta) = -\frac{1}{N_b} \sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k)(\beta - \tilde{\beta}_{b-1}) + \frac{1}{N_b} S_b(D_b, \beta). \]

According to (3.2), the renewable estimator \( \tilde{\beta}_2 \) satisfies

\[ f_2(\tilde{\beta}_2) = 0. \]  

(A.4)

Based on (A.1) and the fact that \( \tilde{\beta}_1 - \beta_t = o_p(1) \), it can be verified that

\[ f_2(\beta_t) = \frac{1}{N_2} Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_1 - \beta_t) + \frac{1}{N_2} S_2(D_2, \beta_t) = o_p(1), \]  

(A.5)
as $N_2 \to \infty$. By taking a difference between equations (A.5) and (A.4), we can get
\[
 f_2(\beta_t) - f_2(\tilde{\beta}_2) = \frac{1}{N_2} Q_1(D_1, \tilde{\beta}_1)(\tilde{\beta}_2 - \beta_t) - \frac{1}{N_2} S_2(D_2, \tilde{\beta}_2) + \frac{1}{N_2} S_2(D_2, \beta_t)
\]
\[
 = o_p(1). \tag{A.6}
\]

In addition, by taking the first-order Taylor expansion of $S_2(D_2, \tilde{\beta}_2)$ around $\beta_t$, we obtain
\[
 S_2(D_2, \tilde{\beta}_2) = S_2(D_2, \beta_t) - Q_2(D_2, \beta_t)(\tilde{\beta}_2 - \beta_t) + O_p(\|\tilde{\beta}_2 - \beta_t\|). \tag{A.7}
\]

In view of (A.6) and (A.7), we can derive
\[
 f_2(\beta_t) - f_2(\tilde{\beta}_2) = \frac{1}{N_2} \left\{ Q_1(D_1, \tilde{\beta}_1) + Q_2(D_2, \beta_t) \right\} (\tilde{\beta}_2 - \beta_t) + O_p \left( \frac{1}{N_2} \|\tilde{\beta}_2 - \beta_t\| \right)
\]
\[
 = o_p(1).
\]

Under the condition (C.2), we know that $\frac{1}{N_2} \left\{ Q_1(D_1, \tilde{\beta}_1) + Q_2(D_2, \beta_t) \right\}$ is positive-definite. Therefore, we have $\tilde{\beta}_2 - \beta_t \overset{P}{\to} 0$ as $N_2 \to \infty$.

After some similar derivations, it can be easily shown that
\[
 \tilde{\beta}_k - \beta_t = o_p(1), \tag{A.8}
\]
for $k = 1, \ldots, b - 1$. According to (3.4), the renewable estimator $\tilde{\beta}_b$ satisfies
\[
 f_b(\tilde{\beta}_b) = 0. \tag{A.9}
\]

From (A.1) and (A.8), we obtain
\[
 f_b(\beta_t) = \frac{1}{N_b} \sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k)(\tilde{\beta}_{b-1} - \beta_t) + \frac{1}{N_b} S_b(D_b, \beta_t) = o_p(1), \tag{A.10}
\]
as $N_b \to \infty$. By taking a difference between (A.10) and (A.9), we get
\[
 f_b(\beta_t) - f_b(\tilde{\beta}_b) = \frac{1}{N_b} \sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k)(\tilde{\beta}_b - \beta_t) - \frac{1}{N_b} S_b(D_b, \tilde{\beta}_b) + \frac{1}{N_b} S_b(D_b, \beta_t)
\]
\[
 = o_p(1). \tag{A.11}
\]

Similar to (A.7), taking the first-order Taylor expansion of $S_b(D_b, \tilde{\beta}_b)$ around $\beta_t$, we have
\[
 S_b(D_b, \tilde{\beta}_b) = S_b(D_b, \beta_t) - Q_b(D_b, \beta_t)(\tilde{\beta}_b - \beta_t) + O_p \left( \|\tilde{\beta}_b - \beta_t\| \right). \tag{A.12}
\]
It follows from (A.11) and (A.12) that
\[
f_b(\beta_t) - f_b(\hat{\beta}_b) = \frac{1}{N_b} \left\{ \sum_{k=1}^{b-1} Q_k(D_k, \hat{\beta}_k) + Q_b(D_b, \beta_t) \right\} (\hat{\beta}_b - \beta_t) + O_p \left( \frac{1}{N_b} \| \hat{\beta}_b - \beta_t \| \right) \\
= o_p(1). \tag{A.13}
\]
Since \( \frac{1}{N_b} \left\{ \sum_{k=1}^{b-1} Q_k(D_k, \hat{\beta}_k) + Q_b(D_b, \beta_t) \right\} \) is positive-definite, we have \( \hat{\beta}_b - \beta_t \xrightarrow{P} 0 \) as \( N_b \to \infty \). This completes the proof.

**Proof of Theorem 2.** By Chen et al. (2016), we know that the traditional LPRE \( \hat{\beta}_1 \) satisfies \( \frac{1}{N_1} S_1(D_1, \hat{\beta}_1) = 0 \) and
\[
\sqrt{N_1} (\hat{\beta}_1 - \beta_t) \xrightarrow{d} N(0, G^{-1}(\beta_t)),
\]
as \( N_1 \to \infty \). Besides, using the Taylor expansion method, the score function has the following expression:
\[
\frac{1}{N_1} S_1(D_1, \beta_t) = \frac{1}{N_1} S_1(D_1, \hat{\beta}_1) + \frac{1}{N_1} Q_1(D_1, \hat{\beta}_1)(\hat{\beta}_1 - \beta_t) + O_p \left( \frac{1}{N_1} \| \beta_t - \hat{\beta}_1 \|^2 \right).
\]
Notice that \( \frac{1}{N_1} S_1(D_1, \hat{\beta}_1) = 0 \), we can get
\[
\frac{1}{N_1} S_1(D_1, \beta_t) = \frac{1}{N_1} Q_1(D_1, \hat{\beta}_1)(\hat{\beta}_1 - \beta_t) + o_p(1). \tag{A.14}
\]
From (A.4), (A.7), (A.14) and Theorem 1, we know that
\[
\frac{1}{N_2} \left\{ S_1(D_1, \beta_t) + S_2(D_2, \beta_t) \right\} = \frac{1}{N_2} \left\{ Q_1(D_1, \hat{\beta}_1) + Q_2(D_2, \tilde{\beta}_2) \right\} (\tilde{\beta}_2 - \beta_t) + o_p(1).
\]
Similarly, at the \((b-1)\)th data batch, we have
\[
\frac{1}{N_{b-1}} \sum_{k=1}^{b-1} S_k(D_k, \beta_t) = \frac{1}{N_{b-1}} \sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k)(\tilde{\beta}_k - \beta_t) + o_p(1). \tag{A.15}
\]
Based on (A.9), (A.10) and (A.13), it can be verified that
\[
- \frac{1}{N_b} \left\{ \sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k) + Q_b(D_b, \beta_t) \right\} (\tilde{\beta}_b - \beta_t) + \frac{1}{N_b} \sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k)(\tilde{\beta}_{b-1} - \beta_t) \\
+ \frac{1}{N_b} S_b(D_b, \beta_t) + o_p(1) = 0. \tag{A.16}
\]
Combining (A.15) and (A.16), it follows that

$$\frac{1}{N_b} \sum_{k=1}^{b} S_k(D_k, \beta_t) - \frac{1}{N_b} \left\{ \sum_{k=1}^{b-1} Q_k(D_k, \tilde{\beta}_k) + Q_b(D_b, \beta_t) \right\} (\tilde{\beta}_b - \beta_t) + o_p(1) = 0.$$

According to Theorem 1, all $\tilde{\beta}_k$'s are consistent, $k = 1, \ldots, b$. By the Continuous Mapping Theorem (Theorem 5.1 in Billingsley 1968), we can deduce that

$$\frac{1}{N_b} \sum_{k=1}^{b} S_k(D_k, \beta_t) - \frac{1}{N_b} \sum_{k=1}^{b} Q_k(D_k, \beta_t)(\tilde{\beta}_b - \beta_t) + o_p(1) = 0.$$

By the condition (C.2), we have

$$\sqrt{N_b}(\tilde{\beta}_b - \beta_t) = \left\{ \frac{1}{N_b} \sum_{k=1}^{b} Q_k(D_k, \beta_t) \right\}^{-1} \left\{ \frac{1}{\sqrt{N_b}} \sum_{k=1}^{b} S_k(D_k, \beta_t) \right\} + o_p(1).$$

Note that $Q_k(D_k, \tilde{\beta}_k)$ and $C_k(D_k, \tilde{\beta}_k)$ are consistent to $Q_k(D_k, \beta_t)$ and $C_k(D_k, \beta_t)$, respectively. Then we have

$$\frac{1}{N_b} \left\{ Q_b^T \tilde{C}_b^{-1} Q_b \right\} \xrightarrow{P} G(\beta_t).$$

From the central limit theorem and Slutsky's theorem, we get

$$\sqrt{N_b}(\tilde{\beta}_b - \beta_t) \overset{d}{\rightarrow} N(0, G^{-1}(\beta_t)),$$

where $G(\beta_t) = Q^T(\beta_t)C^{-1}(\beta_t)Q(\beta_t)$. This ends the proof.

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Table 1. Simulation results for the estimator $\tilde{\beta}_1$ with varying batch size $n_b$ and $\log(\epsilon) \sim N(0, 1)$.

| Case | LPRE | CEE | CUEE | Renew |
|------|------|-----|------|-------|
|      | $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| BIAS $\times 10^{-4}$ | 1.60 | 1.59 | 1.62 | 1.67 | 1.58 | 1.50 | 0.96 | 1.60 | 1.60 | 1.60 |
| $n_b = 10^5$ | 3.49 | 3.49 | 3.48 | 3.46 | 3.49 | 3.47 | 4.18 | 3.49 | 3.49 | 3.49 |
| $n_b = 1000$ | 3.43 | 3.39 | 3.29 | 3.05 | 3.43 | 3.44 | 3.49 | 3.42 | 3.42 | 3.42 |
| $n_b = 200$ | 0.954 | 0.952 | 0.948 | 0.920 | 0.958 | 0.958 | 0.928 | 0.954 | 0.954 | 0.954 |
| $n_b = 50$ | 0.954 | 0.954 | 0.954 | 0.954 | 0.954 | 0.954 | 0.954 | 0.954 | 0.954 | 0.954 |
| BIAS $\times 10^{-4}$ | -2.34 | -2.34 | -2.34 | -2.32 | -2.34 | -1.35 | -0.29 | -2.34 | -2.34 | -2.34 |
| $n_b = 10^5$ | 3.50 | 3.50 | 3.50 | 3.47 | 3.50 | 3.84 | 9.52 | 3.50 | 3.50 | 3.50 |
| $n_b = 1000$ | 3.43 | 3.39 | 3.29 | 3.05 | 3.43 | 3.46 | 3.53 | 3.43 | 3.43 | 3.43 |
| $n_b = 200$ | 0.944 | 0.942 | 0.928 | 0.908 | 0.944 | 0.928 | 0.728 | 0.944 | 0.944 | 0.944 |
| $n_b = 50$ | 0.944 | 0.944 | 0.944 | 0.944 | 0.944 | 0.944 | 0.944 | 0.944 | 0.944 | 0.944 |
| BIAS $\times 10^{-4}$ | 2.30 | 2.31 | 2.33 | 2.42 | 2.29 | 2.16 | 1.02 | 2.28 | 2.28 | 2.31 |
| $n_b = 10^5$ | 3.30 | 3.30 | 3.29 | 3.29 | 3.30 | 3.33 | 3.86 | 3.30 | 3.30 | 3.30 |
| $n_b = 1000$ | 3.43 | 3.40 | 3.30 | 3.07 | 3.43 | 3.44 | 3.49 | 3.43 | 3.43 | 3.43 |
| $n_b = 200$ | 0.940 | 0.940 | 0.936 | 0.914 | 0.940 | 0.948 | 0.924 | 0.940 | 0.940 | 0.940 |
| $n_b = 50$ | 0.940 | 0.940 | 0.940 | 0.940 | 0.940 | 0.940 | 0.940 | 0.940 | 0.940 | 0.940 |
| BIAS $\times 10^{-4}$ | 9.25 | 9.28 | 9.44 | 9.60 | 9.23 | 9.21 | 8.27 | 9.26 | 9.27 | 9.28 |
| $n_b = 10^5$ | 7.72 | 7.72 | 7.69 | 7.59 | 7.71 | 7.72 | 10.92 | 7.72 | 7.72 | 7.72 |
| $n_b = 1000$ | 7.67 | 7.56 | 7.32 | 6.79 | 7.68 | 7.72 | 7.88 | 7.66 | 7.66 | 7.66 |
| $n_b = 200$ | 0.948 | 0.946 | 0.938 | 0.914 | 0.950 | 0.950 | 0.866 | 0.948 | 0.948 | 0.948 |
Table 2. Simulation results for the estimator $\tilde{\beta}_2$ with varying batch size $n_b$ and $\log(\epsilon) \sim N(0, 1)$.

|          | LPRE | CEE      | CUEE      | Renew |
|----------|------|----------|-----------|-------|
|          | $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| **BIAS $\times 10^{-4}$** | -1.15 | -1.13 | -1.04 | -0.74 | -1.07 | -0.78 | -0.08 | -1.15 | -1.15 | -1.15 |
| **$SSE \times 10^{-3}$** | 4.10 | 4.10 | 4.08 | 4.04 | 4.10 | 4.13 | 5.51 | 4.10 | 4.10 | 4.10 |
| **$ESE \times 10^{-3}$** | 3.96 | 3.90 | 3.75 | 3.43 | 3.96 | 3.98 | 4.05 | 3.96 | 3.95 | 3.95 |
| CP | 0.930 | 0.930 | 0.922 | 0.896 | 0.930 | 0.926 | 0.872 | 0.930 | 0.930 | 0.930 |

|          | LPRE | CEE      | CUEE      | Renew |
|----------|------|----------|-----------|-------|
|          | $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| **BIAS $\times 10^{-4}$** | 1.18 | 1.20 | 1.24 | 1.30 | 1.12 | 0.56 | -0.99 | 1.19 | 1.19 | 1.20 |
| **$SSE \times 10^{-3}$** | 3.65 | 3.66 | 3.65 | 3.62 | 3.66 | 3.83 | 7.12 | 3.65 | 3.65 | 3.65 |
| **$ESE \times 10^{-3}$** | 3.73 | 3.68 | 3.54 | 3.24 | 3.74 | 3.77 | 3.87 | 3.73 | 3.73 | 3.73 |
| CP | 0.952 | 0.950 | 0.942 | 0.916 | 0.952 | 0.948 | 0.822 | 0.952 | 0.952 | 0.952 |

|          | LPRE | CEE      | CUEE      | Renew |
|----------|------|----------|-----------|-------|
|          | $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| **BIAS $\times 10^{-4}$** | 0.76 | 0.79 | 0.82 | 1.41 | 0.59 | -0.38 | -1.28 | 1.07 | 1.09 | 0.77 |
| **$SSE \times 10^{-3}$** | 3.00 | 2.99 | 2.98 | 2.93 | 3.00 | 3.22 | 9.46 | 3.08 | 3.08 | 2.99 |
| **$ESE \times 10^{-3}$** | 3.07 | 3.00 | 2.85 | 2.57 | 3.07 | 3.12 | 3.39 | 3.06 | 3.06 | 3.06 |
| CP | 0.964 | 0.960 | 0.946 | 0.910 | 0.964 | 0.958 | 0.736 | 0.962 | 0.962 | 0.964 |

|          | LPRE | CEE      | CUEE      | Renew |
|----------|------|----------|-----------|-------|
|          | $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| **BIAS $\times 10^{-4}$** | -3.77 | -3.76 | -3.74 | -4.21 | -3.74 | -3.45 | 1.97 | -3.77 | -3.78 | -3.79 |
| **$SSE \times 10^{-3}$** | 6.96 | 6.96 | 6.93 | 6.85 | 6.97 | 7.14 | 12.82 | 6.96 | 6.96 | 6.95 |
| **$ESE \times 10^{-3}$** | 6.86 | 6.71 | 6.39 | 5.77 | 6.87 | 6.93 | 7.14 | 6.85 | 6.85 | 6.85 |
| CP | 0.954 | 0.948 | 0.940 | 0.908 | 0.954 | 0.948 | 0.852 | 0.954 | 0.954 | 0.956 |
Table 3. Simulation results for the estimator $\tilde{\beta}_1$ with varying batch size $n_b$ and $\log(\epsilon) \sim \text{Uniform}(-2, 2)$.

| Case 1 | LPRE | CEE | CUEE | Renew |
|--------|------|-----|------|-------|
| $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| $BIAS \times 10^{-4}$ | 1.58 | 1.57 | 1.57 | 1.52 | 1.56 | 1.47 | 0.35 | 1.58 | 1.58 | 1.58 |
| $SSE \times 10^{-3}$ | 3.06 | 3.06 | 3.06 | 3.07 | 3.06 | 3.05 | 3.63 | 3.06 | 3.06 | 3.06 |
| $ESE \times 10^{-3}$ | 2.98 | 2.97 | 2.97 | 2.94 | 2.98 | 2.99 | 3.02 | 2.98 | 2.98 | 2.98 |
| CP | 0.948 | 0.948 | 0.946 | 0.942 | 0.948 | 0.954 | 0.890 | 0.948 | 0.948 | 0.948 |

| Case 2 | LPRE | CEE | CUEE | Renew |
|--------|------|-----|------|-------|
| $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| $BIAS \times 10^{-4}$ | −3.59 | −3.59 | −3.61 | −3.67 | −3.56 | −3.57 | 0.59 | −3.59 | −3.59 | −3.59 |
| $SSE \times 10^{-3}$ | 2.95 | 2.95 | 2.95 | 2.96 | 2.95 | 3.16 | 8.49 | 2.95 | 2.95 | 2.95 |
| $ESE \times 10^{-3}$ | 2.98 | 2.97 | 2.97 | 2.94 | 2.98 | 3.00 | 3.06 | 2.98 | 2.98 | 2.98 |
| CP | 0.948 | 0.948 | 0.946 | 0.940 | 0.948 | 0.928 | 0.716 | 0.948 | 0.948 | 0.948 |

| Case 3 | LPRE | CEE | CUEE | Renew |
|--------|------|-----|------|-------|
| $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| $BIAS \times 10^{-4}$ | −2.62 | −2.63 | −2.66 | −2.78 | −2.62 | −2.53 | −2.49 | −2.68 | −2.63 | −2.63 |
| $SSE \times 10^{-3}$ | 2.84 | 2.84 | 2.84 | 2.85 | 2.84 | 2.85 | 3.22 | 2.84 | 2.84 | 2.84 |
| $ESE \times 10^{-3}$ | 2.98 | 2.97 | 2.97 | 2.94 | 2.98 | 2.99 | 3.03 | 2.98 | 2.98 | 2.98 |
| CP | 0.956 | 0.956 | 0.954 | 0.956 | 0.956 | 0.956 | 0.922 | 0.956 | 0.956 | 0.956 |

| Case 4 | LPRE | CEE | CUEE | Renew |
|--------|------|-----|------|-------|
| $n_b = 10^5$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |
| $BIAS \times 10^{-4}$ | −5.19 | −5.21 | −5.28 | −5.12 | −5.11 | −5.07 | −3.83 | −5.19 | −5.20 | −5.20 |
| $SSE \times 10^{-3}$ | 6.69 | 6.69 | 6.67 | 6.73 | 6.69 | 6.80 | 10.42 | 6.69 | 6.69 | 6.69 |
| $ESE \times 10^{-3}$ | 6.65 | 6.65 | 6.63 | 6.52 | 6.66 | 6.70 | 6.83 | 6.65 | 6.65 | 6.65 |
| CP | 0.940 | 0.940 | 0.940 | 0.934 | 0.940 | 0.940 | 0.830 | 0.940 | 0.940 | 0.940 |
Table 4. Simulation results for the estimator $\tilde{\beta}_2$ with varying batch size $n_b$ and $log(\epsilon) \sim Uniform(-2, 2)$.

| Case | LPRE  | CEE  | CUEE  | Renew |
|------|-------|------|-------|-------|
|      | $n_b = 10^5$ |       |       |       |
| 1    | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |       |
| $BIAS \times 10^{-4}$ | 1.02 | 0.95 | 0.90 | 1.00 |
| $SSE \times 10^{-3}$ | 3.55 | 3.58 | 3.51 | 4.55 |
| $ESE \times 10^{-3}$ | 3.44 | 3.37 | 3.51 | 4.55 |
| CP   | 0.948 | 0.944 | 0.944 | 0.948 |

| Case | LPRE  | CEE  | CUEE  | Renew |
|------|-------|------|-------|-------|
|      | $n_b = 10^5$ |       |       |       |
| 2    | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |       |
| $BIAS \times 10^{-4}$ | 1.03 | 1.13 | -0.87 | 1.03 |
| $SSE \times 10^{-3}$ | 3.29 | 3.32 | 3.35 | 5.43 |
| $ESE \times 10^{-3}$ | 3.24 | 3.18 | 3.35 | 5.43 |
| CP   | 0.948 | 0.938 | 0.820 | 0.948 |

| Case | LPRE  | CEE  | CUEE  | Renew |
|------|-------|------|-------|-------|
|      | $n_b = 10^5$ |       |       |       |
| 3    | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |       |
| $BIAS \times 10^{-4}$ | 0.57 | 0.57 | 0.57 | 0.57 |
| $SSE \times 10^{-3}$ | 2.65 | 2.69 | 2.69 | 7.10 |
| $ESE \times 10^{-3}$ | 2.66 | 2.56 | 2.56 | 7.10 |
| CP   | 0.942 | 0.926 | 0.758 | 0.942 |

| Case | LPRE  | CEE  | CUEE  | Renew |
|------|-------|------|-------|-------|
|      | $n_b = 10^5$ |       |       |       |
| 4    | $n_b = 1000$ | $n_b = 200$ | $n_b = 50$ |       |
| $BIAS \times 10^{-4}$ | 5.64 | 5.79 | 5.79 | 5.64 |
| $SSE \times 10^{-3}$ | 5.84 | 5.91 | 5.91 | 10.55 |
| $ESE \times 10^{-3}$ | 5.95 | 5.73 | 5.73 | 10.55 |
| CP   | 0.948 | 0.934 | 0.812 | 0.948 |
Table 5. Simulation results for the estimator $\hat{\beta}_1$ with varying $B$ and $\log(\epsilon) \sim N(0, 1)$.

| Case   | LPRE | CEE | CUEE | Renew |
|--------|------|-----|------|-------|
|        | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| $BIAS \times 10^{-4}$ | $-0.09$ | $1.31$ | $-4.03$ | $-4.67$ | $-0.12$ | $1.31$ | $-4.01$ | $0.01$ | $1.28$ |
| $SSE \times 10^{-3}$ | $34.33$ | $10.60$ | $3.39$ | $34.34$ | $10.56$ | $3.39$ | $34.32$ | $10.61$ | $3.46$ |
| $ESE \times 10^{-3}$ | $33.50$ | $10.80$ | $3.43$ | $31.67$ | $10.09$ | $3.20$ | $33.20$ | $10.85$ | $3.46$ |
| CP     | $0.948$ | $0.960$ | $0.942$ | $0.932$ | $0.938$ | $0.926$ | $0.948$ | $0.958$ | $0.942$ |
|        | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| $BIAS \times 10^{-4}$ | $-0.02$ | $1.29$ | $-4.94$ | $-4.90$ | $0.12$ | $1.29$ | $-4.70$ | $-0.01$ | $1.29$ |
| $SSE \times 10^{-3}$ | $33.91$ | $10.58$ | $3.39$ | $33.80$ | $10.55$ | $3.38$ | $33.88$ | $10.60$ | $3.48$ |
| $ESE \times 10^{-3}$ | $33.52$ | $10.80$ | $3.43$ | $31.69$ | $10.09$ | $3.19$ | $33.21$ | $10.85$ | $3.46$ |
| CP     | $0.948$ | $0.956$ | $0.942$ | $0.936$ | $0.938$ | $0.928$ | $0.948$ | $0.960$ | $0.936$ |
|        | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| $BIAS \times 10^{-4}$ | $0.16$ | $2.04$ | $-0.40$ | $-1.23$ | $0.16$ | $2.17$ | $-0.59$ | $0.14$ | $2.08$ |
| $SSE \times 10^{-3}$ | $35.31$ | $10.59$ | $3.53$ | $35.33$ | $10.58$ | $3.52$ | $35.33$ | $10.64$ | $3.68$ |
| $ESE \times 10^{-3}$ | $33.81$ | $10.80$ | $3.43$ | $32.03$ | $10.13$ | $3.21$ | $33.53$ | $10.85$ | $3.46$ |
| CP     | $0.926$ | $0.960$ | $0.944$ | $0.912$ | $0.938$ | $0.928$ | $0.920$ | $0.958$ | $0.932$ |
|        | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| $BIAS \times 10^{-4}$ | $1.31$ | $3.05$ | $5.61$ | $1.31$ | $2.55$ | $1.93$ | $1.10$ | $0.91$ | $0.93$ |
| $SSE \times 10^{-3}$ | $77.34$ | $23.79$ | $7.33$ | $77.16$ | $23.71$ | $7.29$ | $77.47$ | $23.99$ | $8.86$ |
| $ESE \times 10^{-3}$ | $74.97$ | $24.14$ | $7.66$ | $70.86$ | $22.44$ | $7.10$ | $74.45$ | $24.41$ | $7.78$ |
| CP     | $0.950$ | $0.950$ | $0.964$ | $0.938$ | $0.930$ | $0.944$ | $0.948$ | $0.946$ | $0.936$ |
Table 6. Simulation results for the estimator $\tilde{\beta}_2$ with varying $B$ and $\log(c) \sim N(0, 1)$.  

| Case | LPRE | CEE | CUBE | Renew | 
|------|------|-----|------|-------| 
|       | $B = 10$ | $B = 10^2$ | $B = 10^3$ | $B = 10^4$ | 
| 1    | $B = 10$ | $B = 10^2$ | $B = 10^3$ | $B = 10^4$ | 
| BIAS $\times 10^{-4}$ | 10.49 | 10.06 | 10.52 | 10.82 | 
| SSE $\times 10^{-3}$ | 38.29 | 38.23 | 38.21 | 38.88 | 
| ESE $\times 10^{-3}$ | 38.58 | 38.15 | 37.92 | 3.95 | 
| CP | 0.942 | 0.964 | 0.962 | 0.964 | 
| BIAS $\times 10^{-4}$ | -7.40 | -6.21 | -7.08 | -7.54 | 
| SSE $\times 10^{-3}$ | 34.19 | 34.20 | 34.17 | 34.81 | 
| ESE $\times 10^{-3}$ | 36.11 | 35.71 | 35.50 | 35.30 | 
| CP | 0.966 | 0.964 | 0.960 | 0.946 | 
| BIAS $\times 10^{-4}$ | -0.84 | -0.74 | -0.17 | -0.64 | 
| SSE $\times 10^{-3}$ | 31.15 | 31.20 | 31.21 | 31.21 | 
| ESE $\times 10^{-3}$ | 30.11 | 30.73 | 30.71 | 30.71 | 
| CP | 0.996 | 0.924 | 0.920 | 0.920 | 
| BIAS $\times 10^{-4}$ | -21.85 | -22.18 | -22.76 | -22.76 | 
| SSE $\times 10^{-3}$ | 65.35 | 65.60 | 65.30 | 65.30 | 
| ESE $\times 10^{-3}$ | 65.11 | 65.10 | 65.10 | 65.10 | 
| CP | 0.950 | 0.950 | 0.950 | 0.950 |
Table 7. Simulation results for the estimator $\hat{\beta}_1$ with varying $B$ and $\log(\epsilon) \sim \text{Uniform}(-2, 2)$.

| Case | LPRE | CEE | CUEE | Renew |
|------|------|-----|------|-------|
|      | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| **BIAS $\times 10^{-4}$** | | | | | | | | | |
| 1    | 1.52 | 1.96 | -4.02 | 1.55 | 1.81 | -3.81 | 1.64 | 1.96 | -4.02 |
| 2    | 1.87 | 1.97 | -0.75 | 2.31 | 2.14 | -0.76 | 1.73 | 1.85 | -0.51 |
| 3    | 12.06 | 3.47 | 1.89 | -12.24 | 3.41 | 1.95 | -11.89 | 2.76 | 1.52 |
| 4    | 55.07 | -8.32 | -2.22 | 56.47 | -7.80 | -2.23 | 54.70 | -7.43 | -1.47 |
| **SSE $\times 10^{-3}$** | | | | | | | | | |
| 1    | 30.12 | 8.81 | 2.74 | 30.14 | 8.81 | 2.74 | 30.13 | 8.86 | 2.83 |
| 2    | 30.18 | 8.82 | 3.03 | 30.18 | 8.82 | 3.04 | 30.16 | 8.88 | 3.18 |
| 3    | 29.76 | 9.41 | 2.98 | 29.66 | 9.38 | 2.96 | 29.84 | 9.47 | 3.00 |
| 4    | 65.06 | 20.50 | 6.68 | 65.30 | 20.56 | 6.72 | 65.17 | 20.57 | 7.44 |
| **ESE $\times 10^{-3}$** | | | | | | | | | |
| 1    | 29.77 | 9.41 | 2.98 | 29.66 | 9.38 | 2.96 | 29.84 | 9.47 | 3.00 |
| 2    | 29.76 | 9.41 | 2.98 | 29.65 | 9.38 | 2.96 | 29.84 | 9.47 | 3.00 |
| 3    | 29.78 | 9.41 | 2.98 | 29.66 | 9.37 | 2.96 | 29.86 | 9.48 | 3.00 |
| 4    | 66.50 | 21.04 | 6.65 | 65.96 | 20.86 | 6.60 | 66.72 | 21.27 | 6.74 |
| **CP** | 0.958 | 0.964 | 0.964 | 0.956 | 0.960 | 0.964 | 0.958 | 0.962 | 0.954 |
|      | 0.956 | 0.962 | 0.952 | 0.954 | 0.958 | 0.948 | 0.958 | 0.962 | 0.936 |
|      | 0.948 | 0.956 | 0.942 | 0.944 | 0.954 | 0.946 | 0.948 | 0.960 | 0.940 |
|      | 0.938 | 0.958 | 0.944 | 0.944 | 0.960 | 0.946 | 0.938 | 0.960 | 0.912 |
Table 8. Simulation results for the estimator $\hat{\beta}_2$ with varying $B$ and $\log(\epsilon) \sim Uniform(-2, 2)$.

| Case | LPRE | CEE | CUEE | Renew |
|------|------|-----|------|-------|
|      | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| **BIA$S \times 10^{-4}$** | -21.12 | 4.00 | -1.10 | -22.51 | 3.90 | -1.06 | -21.15 | 3.44 | -0.48 | -21.15 | 4.02 | -1.10 |
| **$SSE \times 10^{-3}$** | 33.64 | 10.52 | 3.32 | 33.72 | 10.54 | 3.31 | 33.67 | 10.54 | 3.63 | 33.68 | 10.52 | 3.32 |
| **$ESE \times 10^{-3}$** | 34.39 | 10.87 | 3.44 | 34.18 | 10.80 | 3.42 | 34.52 | 10.97 | 3.48 | 34.37 | 10.87 | 3.44 |
| **CS** | 0.962 | 0.966 | 0.958 | 0.954 | 0.964 | 0.954 | 0.962 | 0.962 | 0.936 | 0.962 | 0.966 | 0.958 |
| **Case 2** | LPRE | CEE | CUEE | Renew |
| $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| **BIA$S \times 10^{-4}$** | 19.84 | -2.10 | -0.76 | 20.55 | -2.13 | -0.82 | 19.85 | -1.32 | -0.66 | 19.71 | -2.13 | -0.76 |
| **$SSE \times 10^{-3}$** | 32.21 | 10.26 | 3.26 | 32.23 | 10.27 | 3.25 | 32.24 | 10.30 | 3.39 | 32.21 | 10.26 | 3.26 |
| **$ESE \times 10^{-3}$** | 32.47 | 10.24 | 3.24 | 32.28 | 10.18 | 3.22 | 32.59 | 10.35 | 3.27 | 32.44 | 10.24 | 3.24 |
| **CS** | 0.956 | 0.948 | 0.942 | 0.952 | 0.944 | 0.944 | 0.952 | 0.960 | 0.938 | 0.954 | 0.946 | 0.942 |
| **Case 3** | LPRE | CEE | CUEE | Renew |
| $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| **BIA$S \times 10^{-4}$** | 0.11 | -0.74 | -1.29 | 2.04 | -0.63 | -1.46 | 0.09 | -0.15 | -0.61 | 0.47 | -0.80 | -1.29 |
| **$SSE \times 10^{-3}$** | 26.50 | 8.29 | 2.76 | 26.52 | 8.37 | 2.80 | 26.42 | 8.74 | 3.77 | 26.45 | 8.29 | 2.76 |
| **$ESE \times 10^{-3}$** | 26.69 | 8.42 | 2.66 | 26.30 | 8.28 | 2.62 | 26.85 | 8.65 | 2.77 | 26.63 | 8.42 | 2.66 |
| **CS** | 0.940 | 0.960 | 0.944 | 0.936 | 0.958 | 0.944 | 0.942 | 0.950 | 0.912 | 0.936 | 0.960 | 0.946 |
| **Case 4** | LPRE | CEE | CUEE | Renew |
| $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ | $B = 10$ | $B = 100$ | $B = 10^3$ |
| **BIA$S \times 10^{-4}$** | -41.34 | 6.31 | 2.04 | -42.09 | 6.16 | 1.95 | -41.60 | 4.98 | 0.68 | -40.81 | 6.34 | 2.04 |
| **$SSE \times 10^{-3}$** | 59.29 | 19.41 | 6.10 | 59.47 | 19.46 | 6.16 | 59.42 | 19.67 | 7.20 | 59.23 | 19.40 | 6.10 |
| **$ESE \times 10^{-3}$** | 59.60 | 18.81 | 5.95 | 58.69 | 18.53 | 5.86 | 60.00 | 19.16 | 6.08 | 59.46 | 18.81 | 5.95 |
| **CS** | 0.946 | 0.944 | 0.942 | 0.938 | 0.938 | 0.940 | 0.948 | 0.938 | 0.904 | 0.944 | 0.944 | 0.942 |
Table 9. The computing time in seconds for Case 1 with varying $B$ and fixed the cumulative sample size $N_b = 10^7$.

| B   | LPRE C.Time | CEE C.Time | CUEE C.Time | Renew C.Time |
|-----|-------------|------------|-------------|--------------|
| $10$ | 12.689      | 11.841     | 14.743      | 9.751        |
| $10^2$ | 17.922     | 11.446     | 14.425      | 9.075        |
| $10^3$ | 48.489     | 10.642     | 13.101      | 8.526        |
| $10^4$ | 511.851    | 14.463     | 17.330      | 11.439       |

Table 10. The computing time in seconds for Case 1 with varying the cumulative sample size $N_b$ and fixed $B = 10^3$.

| N   | LPRE C.Time | CEE C.Time | CUEE C.Time | Renew C.Time |
|-----|-------------|------------|-------------|--------------|
| $10^6$ | 4.360     | 1.471      | 1.767       | 1.172        |
| $5 \times 10^6$ | 35.748   | 5.501      | 6.819       | 4.362        |
| $10^7$ | 58.028    | 10.889     | 13.514      | 8.707        |
| $5 \times 10^7$ | 385.513  | 56.306     | 69.927      | 44.129       |
| $10^8$ | 978.540   | 121.692    | 148.298     | 93.962       |
Table 11. Simulation results from various estimators for the bike sharing data with $B = 24$ and $N_b = 17379$.

|                | LPRE                  | CEE                   |
|----------------|-----------------------|-----------------------|
|                | est   | sd     | p-value    | est   | sd     | p-value    |
| Intercept      | 2.2142 | 0.0280 | $< 10^{-9}$ | 2.2145 | 0.0256 | $< 10^{-9}$ |
| Workingday     | -0.0342 | 0.0102 | $8.55 \times 10^{-4}$ | -0.0334 | 0.0095 | $4.36 \times 10^{-4}$ |
| Temperature    | 1.4525 | 0.0261 | $< 10^{-9}$ | 1.4538 | 0.0243 | $< 10^{-9}$ |
| Humidity       | -1.1379 | 0.0279 | $< 10^{-9}$ | -1.1404 | 0.0254 | $< 10^{-9}$ |
| Windspeed      | 0.1816 | 0.0428 | $2.22 \times 10^{-5}$ | 0.1837 | 0.0405 | $5.84 \times 10^{-6}$ |

|                | CUEE                  | Renew                 |
|                | est   | sd     | p-value    | est   | sd     | p-value    |
| Intercept      | 2.2139 | 0.0266 | $< 10^{-9}$ | 2.2169 | 0.0263 | $< 10^{-9}$ |
| Workingday     | -0.0329 | 0.0100 | $1.05 \times 10^{-3}$ | -0.0344 | 0.0099 | $5.01 \times 10^{-4}$ |
| Temperature    | 1.4506 | 0.0249 | $< 10^{-9}$ | 1.4507 | 0.0248 | $< 10^{-9}$ |
| Humidity       | -1.1377 | 0.0266 | $< 10^{-9}$ | -1.1404 | 0.0263 | $< 10^{-9}$ |
| Windspeed      | 0.1812 | 0.0414 | $1.22 \times 10^{-5}$ | 0.1826 | 0.0412 | $9.26 \times 10^{-6}$ |
Table 12. Simulation results from various estimators for the electric power consumption data with $B = 48$ and $N_b = 2049280$.

|                        | LPRE |          |          | CEE    |          |          |
|------------------------|------|----------|----------|--------|----------|----------|
|                        | est  | sd       | p-value  | est    | sd       | p-value  |
| Intercept              | 1.1162 | 0.0004  | $< 10^{-9}$ | 1.1207 | 0.0004  | $< 10^{-9}$ |
| Kitchen                | 0.2205 | 0.0004  | $< 10^{-9}$ | 0.2201 | 0.0004  | $< 10^{-9}$ |
| Laundry Room           | 0.2045 | 0.0005  | $< 10^{-9}$ | 0.2058 | 0.0005  | $< 10^{-9}$ |
| Water-heater and Air-conditioner | 0.6326 | 0.0004  | $< 10^{-9}$ | 0.6299 | 0.0003  | $< 10^{-9}$ |
|                        |      |          |          |        |          |          |
|                        | est  | sd       | p-value  | est    | sd       | p-value  |
| CUEE                   | 1.1237 | 0.0005  | $< 10^{-9}$ | 1.1160 | 0.0004  | $< 10^{-9}$ |
| Kitchen                | 0.2197 | 0.0004  | $< 10^{-9}$ | 0.2205 | 0.0004  | $< 10^{-9}$ |
| Laundry Room           | 0.2041 | 0.0004  | $< 10^{-9}$ | 0.2047 | 0.0005  | $< 10^{-9}$ |
| Water-heater and Air-conditioner | 0.6287 | 0.0004  | $< 10^{-9}$ | 0.6326 | 0.0004  | $< 10^{-9}$ |