On the Formulation of an Elastic-Plastic Beam Model: the Pre-Integration Idea

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Abstract. This paper provides a formulation of an elastic-plastic beam model. The constitutive behaviour of material after elasticity limit is considered perfect plastic. Although this is not a general plastic behaviour, it allows to minimize mathematical difficulties to fulfil the idea: pre-integration through the thickness extended to material non-linearity. Three types of deformation mechanisms are found: fully elastic, plasticization over only one side or both of sides. The complete kinematic set of equations are obtained, together with the analytical stress trends. Yield stress-continuity equations are used to separate the elastic and plastic domains. These domains define a quasi-state diagram; Allowing to associate the normal force and bending moment acting on the section to the elastic-plastic state within the section. Note that in the case of a rectangular section, the solution is given in a fully analytical form, otherwise the domains are solved numerical solutions. But these last, may be obtained once for all, for any asymmetrical section.

1. Introduction
The ultimate load-bearing capacity of structures, the processing through plastic deformation and the progressive crashing are among the interests that induced the necessity to extend the structural behaviour over the elastic-plastic response of materials. This scenario is usual in the production of components used in various fields, such as automotive, aerospace and transport industries. In mechanical engineering, the case of combined bending and traction stresses in plane is frequently encountered, particularly in the metal-forming of rods and metal wires [1,2,3,4,5]. The analytical study in the elastic-plastic field is troublesome [6], certainly due to the missing of the Principle of superimposition, due to non-linearity. For this reason, most of the models take into account only the bending stress, whatever is the non-linear constitutive law [7].

This paper considers the influence of the axial load in the behaviour of plastic deformation, i.e. the combined mechanism of axial and bending loads. This has an effect on the position of the neutral axis, making it possible that plasticity occur only in one side (upper or lower) or on both sides as in the case of pure bending. The prediction of the final shape after the tensile-bending process and release, where springback happens, is of particular interest. The results in terms of residual stresses and elastic return, moves considerably away from the solution given by bending loading alone.

In the Computer-Aided-Engineering the elastic-plasticity of metallic materials, or in general the inelastic behaviour, causes nonlinear force-displacement responses. The main trouble is that the material response, through its constitutive-relation, changes in the bulk according to the stress reached. This implies that the solution is not reached on a one-shot step, but by an incremental loading procedure. At each step of the increments, it is necessary to compute the actual stiffness deduced by
the tangent material operator (moduli matrix) [8]. However, a numerical integration over the bulk (the section in a beam) is required, therefore a sampling over the thickness is carried out. There are two key points, often not sufficiently emphasized. The first is that this work-flow requires a sufficient number of sampling trough the thickness, which becomes burdensome especially for slender beams. The second is that non a-priori predictability about where plasticity will manifest and how quickly the yield front changes. This last implies that along the curvilinear abscissa a huge number of elements are required. At the best of authors knowledge, there is no way to avoid these complications, because the direct connection between loads and displacements is only possible in the elastic case.

A shortcut often considered in beam structures is the lumped plastic-hinge method [9,10] that, although unable to accurately describe the elastic-plastic state within the sections, is capable to estimate the overall collapse of a frame structure. This analysis takes the name of first order plastic hinge.

Another peculiarity inherent to elastic-plastic events, is the need to follow the load-history path, due to the partial irreversibility of the deformation process. In fact, the same final forces and moments acting on the section can be reached by countless different paths, which although differ for stress-strain field. Therefore, the final state (stress, deformation, displacements) is generally not a state function. To avoid these difficulties, proportional load-paths are always accounted, but this may be unrealistic if a more complex load history applies.

The model proposed in this paper allows to pre-compute over the thickness the behaviour of each section being beyond its elastic limit. Therefore, the procedure provides a way to built-up a beam element, pre-integrated through the thickness, whose actual elastic-plastic response is only governed by section loads. In other words, the stress state and the kinematic behaviour of the section only depends on two variables: axial load and bending moment. Three elastic-plastic cases of deformation-mechanics are possible: totally elastic, one-side plasticization (unilateral, upper or lower) and two-side plasticization (bilateral). In the case of monotonic and proportional loading, a state diagram emerges. It returns the deformation-mechanism given the values of the axial and the bending loads, i.e. one-shot calculation. Considering non-trivial path-loadings, the step history must be followed but the cited-on state diagram is still useful to solve each step.

As demonstrated hereinafter, the model solution is completely analytical for rectangular beam sections. For any mono-symmetrical cross-sections (e.g. circular or I-beams) the solution requires an a-priori numerical integration for the determination of the boundaries among the elastic-plastic domains, but the state function loads-displacements still exists.

2. The Elastic-Plastic Deformation

In this section, the governing equations of the elastoplastic beam are obtained. The material is isotropic, with elastic-perfectly plastic behaviour, same under tension or compression (figure 1). The cross-section is symmetrical along the $y$-axis (figure 2) and its shape keeps constant along the Cartesian abscissa. The kinematic mechanism of the displacements field is not influenced by elastic-plastic behaviour; therefore, in a planar bending the engineering type 2D-beam model shows up as:

$$ U(x, y) = u(x) - y \psi(x) $$

$$ V(x, y) = v(x) $$

The positive directions of the displacement components are shown in figure 2. The beam model follows Bernoulli-Euler hypotheses: the shear strain is assumed null and the strain-displacements relations are:

$$ \epsilon(x, y) = u' - y \psi' $$

$$ \gamma(x, y) = 0 \quad (\rightarrow \psi = v') $$

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The differential equilibrium is independent of the elastic-plastic behaviour; the normal force, the shear one and the bending moment derivatives are governed by:

\[ N' = -q_x \] (5)

\[ T' = -q_y \] (6)

\[ M' = -T \] (7)

With an elastic-perfectly plastic behaviour three deformation mechanism are possible (figure 3): totally elastic, one-side (upper or lower) plasticization and two-side plasticization. Each of them corresponds to a different kinematic set of equations (to be found) which characterizes the deformation mechanism.

\[ \sigma \]
\[ \sigma_y \]
\[ -\varepsilon_y \]
\[ \varepsilon_y \]
\[ -\sigma_y \]
\[ \varepsilon \]

**Figure 1.** Elastic-perfectly plastic behaviour of material

2.1. The totally elastic case

The governing equations for the full elastic case are straightforward and, e.g., can be found in [11]. They are reported here for completeness. The normal force \( N \), the shear force \( T \) and the bending moment \( M \) are coupled with the derivatives of displacement components as:

\[ N = EAu' \] (8)

\[ T = -EI\varepsilon'' \] (9)

\[ M = EI\varepsilon'' \] (10)

Where \( A \) and \( I \) are geometric constants, respectively, the cross-section area and the moment of inertia, that for one-symmetrical section can be computed using the knowledge of the width function \( b(y) \) and the quantities \( h^+, h^- \) (figure 4):

\[ A = \int_{h^-}^{h^+} b(y) \, dy \quad I = \int_{h^-}^{h^+} y^2 b(y) \, dy \] (11)

Furthermore, the normal stress is expressed by the binomial Navier’s formula:
\[ \sigma(x, y) = \frac{N}{A} - \frac{M}{I} y \]  

(12)

Figure 2. Positive displacement directions

Figure 3. Possible deformation mechanisms

2.2. Upper side plasticization
In this case the section is in partial plasticization at the upper side, identified by the unknown ordinate \( y_{p+} \) (figure 5). This quantity represents the limit of the plasticization front. This case is not comprehensive of all combination of \((N, M)\). Being the resulting forces and moments directed as the reference system, it is simple to prove that the only combinations that trigger this plasticization case requires that \((N > 0, M < 0)\) and \((N < 0, M > 0)\), the first caused by traction the second by compression yielding. Therefore, the constitutive equation can be written as:

\[ \sigma = \begin{cases} 
\text{sign}(\sigma_y) |\sigma_y|, & y \geq y_{p+} \\
E(u' - y v''), & y \leq y_{p+} 
\end{cases} \]  

(13)

Where the \text{sign} \ function is defined as:
\[ \text{sign}^+(\sigma_y) = \begin{cases} +1, & (N > 0, M < 0) \\ -1, & (N > 0, M < 0) \end{cases} \quad (14) \]

Imposing the equivalence between the internal stress and the section force and moment resultant and using the equations (3,4,13) one obtains:

\[ N(x) = \text{sign}^+(\sigma_y)|\sigma_y|A_{p+} + EA_e u' - ES_e v'' \quad (15) \]

\[ M(x) = \text{sign}^+(\sigma_y)|\sigma_y|S_e - ES_e u' + EI_e v'' \quad (16) \]

By equation (7) the relation between shear force and displacement results. In the latter equations the following geometric quantities appear: the area of the (upper) plastic region \( A_{p+} \), the area of the elastic region \( A_e \), the static moment of the elastic region \( S_e \), and the moment of inertia of the elastic region \( I_e \). These geometrical quantities are expressed by:

\[ A_e(x) = \int_{y_p(x)}^{h^+} b(y) \, dy \quad ; \quad A_{p+}(x) = \int_{y_p(x)}^{h^+} b(y) \, dy \quad (17) \]

\[ S_e(x) = \int_{y_p(x)}^{h^+} y b(y) \, dy \quad ; \quad I_e(x) = \int_{h^-}^{y_p(x)} y^2 b(y) \, dy \quad (18) \]

It is worth noting that, differently from the full elastic case, the quantities in equations (17,18) are not constant along \( x \)-axis, but they change inasmuch they depend on the trend of \( y_p(x) \). Extracting the expression of \( u' \) and \( v'' \) by equations (15,16), the stress trend is found using equation (13), obtaining:

\[ \sigma(x,y) = \begin{cases} \text{sign}^+(\sigma_{sn})|\sigma_{sn}|, & y \geq y_{p+} \\ 1 + \frac{S_e^2}{I_e A_e - S_e^2} N \frac{A_{p+}}{A_e} + \frac{S_e}{I_e A_e - S_e^2} M - \text{sign}^+(\sigma_{sn})|\sigma_{sn}| \frac{A_{p+}}{A_e} \left( A_{p+} + \frac{A S_e^2}{I_e A_e - S_e^2} \right) + \frac{S_e N + A_e M - \text{sign}^+(\sigma_{sn})|\sigma_{sn}| A S_e}{I_e A_e - S_e^2}, & y \leq y_{p+} \end{cases} \quad (19) \]
The unknown $y_{p+}$ is solved imposing that for $y = y_{p+}$ (figure 5) the equations in (19) are equal:

$$M(S_e - A_e y_{p+}) + N(I_e - S_e y_{p+}) + sign^+(\sigma_{sn})|\sigma_{sn}|A(I_e + S_e y_{p+}) = 0$$  \hspace{1cm} (20)

From which it is evident that the trend of the plasticization front $y_{p+}$ depends on the trend of $N(x)$, $M(x)$ other than the cross-section shape.

2.3. Lower side plasticization
The equations that govern the case of lower side (figure 6) plasticization is formally identical to the previous one unless the replacement of the $sign^+$ function with:

$$sign^-(\sigma_y) = \begin{cases} +1, & (N > 0, M > 0) \\ -1, & (N < 0, M < 0) \end{cases}$$  \hspace{1cm} (21)

and the area $A_{p+}$ with $A_{p-}$. 

Figure 4. Totally elastic case: nomenclature

Figure 5. Upper side plasticization: nomenclature
In this case, the geometrical quantities are function of the trend of the ordinate $y_{p-}(x)$:

$$A_e(x) = \int_{y_{p-}(x)}^{h^+} b(y) \, dy \quad ; \quad A_p^+(x) = \int_{h^-}^{y_{p-}(x)} b(y) \, dy$$

$$S_e(x) = \int_{y_{p-}(x)}^{h^+} y \, b(y) \, dy \quad ; \quad I_e(x) = \int_{y_{p-}(x)}^{y_{p+}} y^2 \, b(y) \, dy$$

2.4. Two-side plasticization

When a two-side plasticization holds, two ordinates $y_{p+}, y_{p-}$ appear to manage the plasticization front trend (figure 7). The constitutive equation becomes:

$$\sigma = \begin{cases} 
\text{sign}^\pm(\sigma(y)|\sigma(y)|, \ y \geq y_{p+} \\
E(u' - y'v'), \ y_{p-} \leq y \leq y_{p+} \\
-\text{sign}^\pm(\sigma(y)|\sigma(y)|, \ y \leq y_{p-}
\end{cases}$$

Where:

$$\text{sign}^\pm(\sigma(y)) = \begin{cases} 
+1, \quad (N > 0, M < 0) \text{ or } (N < 0, M < 0) \\
-1, \quad (N > 0, M > 0) \text{ or } (N < 0, M > 0)
\end{cases}$$

Again, the relations between the resultants and the displacements for this case are:

$$N = \text{sign}^\pm(\sigma(y)|\sigma(y)|(A_{p+} - A_{p-}) + EA_eu' - ES_ev''$$

$$M = \text{sign}^\pm(\sigma(y)|\sigma(y)|(S_{p+} - S_{p-}) - ES_eu' + EI_ev''$$

Where:
\[
\begin{align*}
A_e(x) &= \int_{y_p(x)}^{y_p+(x)} b(y) \, dy ; \quad A_{p+}(x) &= \int_{y_p(x)}^{h^+} b(y) \, dy ; \quad A_{p-}(x) &= \int_{y_p(x)}^{h^-} b(y) \, dy \\
S_e(x) &= \int_{y_p(x)}^{y_p+(x)} y b(y) \, dy ; \quad S_{p+}(x) &= \int_{y_p(x)}^{h^+} y b(y) \, dy ; \quad S_{p-}(x) &= \int_{y_p(x)}^{h^-} y b(y) \, dy \\
I_e(x) &= \int_{y_p(x)}^{y_p+(x)} y^2 b(y) \, dy
\end{align*}
\]

Obtaining the expression of \( u' \) and \( v'' \) by equations (26,27), the stress trend is given by equation (24):

\[
\sigma(x, y) = \begin{cases} 
\text{sign}^\pm(\sigma_{sn})|\sigma_{sn}|, & y \geq y_{p+} \\
\left(1 + \frac{S_e^2}{I_e A_e - S_e^2}\right)N + \frac{S_e}{I_e A_e - S_e^2}M - \text{sign}^\pm(\sigma_{sn})\frac{|\sigma_{sn}|}{A_e}[(A_{p+} - A_{p-}) + \\
\frac{(A_{p+} - A_{p-})S_e - A_e(S_{p+} - S_{p-})}{I_e A_e - S_e^2} - y \frac{S_eN + A_eM}{I_e A_e - S_e^2} + \\
-y \frac{\text{sign}^\pm(\sigma_{sn})|\sigma_{sn}|[(A_{p+} - A_{p-})S_e - A_e(S_{p+} - S_{p-})]}{I_e A_e - S_e^2}] - \frac{\text{sign}^\pm(\sigma_{sn})|\sigma_{sn}|}{I_e A_e - S_e^2} - y \frac{\text{sign}^\pm(\sigma_{sn})|\sigma_{sn}|}{I_e A_e - S_e^2}, & y_{p-} \leq y \leq y_{p+} \\
\end{cases}
\]

Figure 7. Two-side plasticization: nomenclature

The unknowns \( y_{p+}, y_{p-} \) are found imposing that for \( y = y_{p+} \) and \( y = y_{p-} \) the second of equation (31) must be equal to the first or the third, respectively. Therefore, two stress-continuity equations allow to solve the two above mentioned unknowns:
\[ M(S_e - y_{p+}A_e) = N[S_e y_{p+} - I_e] - \text{sign}(\sigma_{sn})|\sigma_{sn}| [(A_{p+} - A_{p-})(S_e y_{p+} - I_e) + \\
+ (S_{p+} - S_{p-})(S_e - y_{p+}A_e) - l_e A_e + S_e^2] \]

\[ M(S_e - y_{p-}A_e) = N[S_e y_{p-} - I_e] - \text{sign}(\sigma_{sn})|\sigma_{sn}| [(A_{p+} - A_{p-})(S_e y_{p-} - I_e) + \\
+ (S_{p+} - S_{p-})(S_e - y_{p-}A_e) - l_e A_e + S_e^2] \]

2.5. Fully plastic state

The fully plastic condition is a special case of the previous two-side plasticization where the elastic area is null and the ordinates \( y_{p+} \) and \( y_{p-} \) coincide. Therefore, the only unknown ordinate \( y_p \) appears (figure 8). Considering null the elastic region in equation (26,27):

\[ N = \text{sign}(\sigma_y)|\sigma_y|(A_{p+} - A_{p-}) \]  

(33) \[ M = \text{sign}(\sigma_y)|\sigma_y|(S_{p+} - S_{p-}) \]  

(34)  

Where the expression of the geometrical quantities \( A_{p+}, A_{p-}, S_{p+}, S_{p-} \) are simply found imposing \( y_{p+} = y_{p-} = y_p \) in the equation (28,29).

![Figure 8. Fully plastic state: nomenclature](image)

3. The quasi-State Diagram

The kinematic equations are now fully deduced for the three possible deformation-mechanics. In order to apply the pre-integration, it is necessary to deduce from the mere knowledge of \((N,M)\) how the section plasticizes. For this purpose, it is unavoidably to solve the continuity stress equations (20,32,33,34) which imply the computation of some integrals.

The case of rectangular cross-section \( b \times h \) allows a full and straightforward analytical solution that is reported in the following. The steps are not explained here because they are long, and they could mask the basic idea of this paper (only lengthy algebraic manipulations are required).

The equations (33,34) provide the limit parabola that corresponds to collapse (fully yield region):

\[ m + n^2 = 1 \]  

(35)
where a convenience notation involving dimensionless variables is adopted:

\[ n = \frac{N}{N_p} = \frac{N}{bh\sigma_y} \quad ; \quad m = \frac{M}{M_p} = \frac{4M}{bh^2\sigma_y} \]  

(36)

In which \( N_p, M_p \) are the values of the resultant force and moments that, alone, cause the entire plasticization of the cross-section (collapse).

Solving the equation (20), the trend of the plasticization front \( y_{p+} \) is found:

\[ \frac{y_{p+}(x)}{h} = 2 - \frac{3m(x)}{2(1 - n(x))} \]  

(37)

Imposing \( y_{p+} = \frac{h}{2} \) the boundary of incipient yielding of the upper-side is obtained:

\[ m = \frac{2}{3}(1 - n) \]  

(38)

Similarly, for the case of lower side plasticization the incipient lower yielding gives:

\[ \frac{y_{p-}(x)}{h} = -2 + \frac{3m(x)}{2(1 - n(x))} \]  

(39)

\[ m = \frac{2}{3}(1 + n) \]  

(40)

In the same way, for bilateral plasticization, the equation (32) provides:

\[ \frac{y_{p+}(x)}{h} = \frac{1}{2} \left[-n(x) + \sqrt{3(1 - n^2(x) - m(x))}\right] \]  

(41)

\[ \frac{y_{p-}(x)}{h} = \frac{1}{2} \left[-n(x) - \sqrt{3(1 - n^2(x) - m(x))}\right] \]  

(42)

Imposing \( y_{p+} = \frac{h}{2} \), \( y_{p-} = -\frac{h}{2} \) the incipient two-side plasticization boundary is given:

\[ m = \frac{2}{3}(1 - n)(1 + 2n) \]  

(43)

Figure 9 shows the perfect coincidence of these analytical results when compared with the numerical calculation of the equations (20,32,33,34).
Therefore, given only the values of a pair \((N, M)\), besides the elementary elastic solution, it derives the plasticization type, to which the respective kinematic equations are associated, and the trend of the plasticization front. It is worth noting that these results are lawful only for monotone and proportional path loading, this is why it is a quasi-state diagram. For complex histories a step analysis is required. At each step one should carefully account whether an additional plasticization occurs or an elastic return holds. Although, the quasistate diagram remains useful to analyses this occurrence.

4. Conclusions
This paper addresses a complete formulation of beam model extended to elastic-perfect plastic behavior. The material behaves in the same way under traction or compression. The cross-section keeps constant along the Cartesian abscissa and one-symmetry holds along the y-axis.

Three different deformation mechanics appear possible: fully elastic, one-side plasticization (upper or lower) and two-side plasticization. For all of these a different set of kinematic equations are derived, together with their analytical stress trend. For each of the elastic-plastic cases one or two stress-continuity equations appear; these equations are fundamental, since they allow to return the location of the plasticization front. This makes it possible to predict, having the end forces applied: the plasticization front trend, the type of plasticity and the kinematic response. For a monotonic and proportional path loading the diagram develops like a state function. The history-loading can be followed again using this graph, observing when a further plasticization occurs or an elastic unloading holds. For the case of a rectangular beam section the quasi-state diagram is solved analytically. For other cross-sections it is always possible to build up the diagram after the numerical solutions of the governing equations. These numerical solutions, however, can be carried out once for all, for any cross section.

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