Gauge invariant regularization of QCD on the Light Front with the lattice in transverse space coordinates

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Abstract

It is introduced the gauge invariant regularization of quantum chromodynamics (QCD), adjusted to modeling nonperturbative vacuum effects in QCD on the light front (LF) via modeling the dynamics of zero Fourier modes of fields on the LF.

1 Introduction

Quantization of field theory on the light front (LF) [1], i.e. on the hyperplane $x^+ = 0$ in the coordinates

$$x^\pm = (x^0 \pm x^3)/\sqrt{2}, \quad x^\pm = x^1, x^2,$$

where $x^0, x^1, x^2, x^3$ are Lorentz coordinates and the $x^+$ plays the role of the time, requires special regularization of the theory. The LF momentum operator $P_-$ (the generator of translations along the $x^-$ axis) is nonnegative for the states with nonnegative energy and mass:

$$P_- = (P_0 - P_3)/\sqrt{2} \geq 0 \quad \text{for} \quad p_0 \geq 0, \ p^2 \geq 0.$$ 

The vicinity of its minimal eigenvalue, $p_- = 0$, corresponds both to ultraviolet and infrared domains of momenta in Lorentz coordinates. Quantizing field theory on the LF one finds singularities at $p_- \to 0$. And the regularization of these singularities may affect the description of both ultraviolet and infrared momenta physics, in particular, correct description of vacuum effects.

Usual ways of the regularization of the $p_- \to 0$ singularities are the following:

(a) the cutoff $|p_-| \ (|p_-| \geq \varepsilon > 0)$,

(b) "DLCQ" regularization, i.e. the space cutoff in the $x^-$, $|x^-| \leq L$, plus the periodic boundary conditions on fields $x^-$, that leads to the discretization of the $P_-$ spectrum: $p_- = \frac{\pi n}{L}$, $n = 0, 1, 2, \ldots$ The Fourier mode of the field with the $p_- = 0$ ("zero mode") is separated here from other modes. In canonical formalism zero mode turns out to be dependent on other modes due to constraints (for gauge field theory see [2, 3]).

Both ways of the regularization break Lorentz symmetry, and the regularization (a) violates also gauge invariance in gauge field theory. This can lead to difficulties with the renormalization

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of the theory, and also to a nonequivalence of results, obtained with LF and with usual ("equal
time") quantization. In the framework of perturbation theory it was shown \[4, 5\] that to restore
the symmetry and the above mentioned equivalence it is necessary to add to the regularized
LF Hamiltonian some special "counterterms".

However for the Quantum Chromodynamics (QCD) one can expect effects nonperturbative
in coupling constant, in particular, vacuum condensates. Applying the regularization of the
type (a) where one excludes zero modes, we get the absence of such condensates. The regul-
larization (b) leads to canonically constrained and dynamically not independent zero modes.
With these zero modes one again cannot correctly describe condensates \[6, 7\]. The study of
this problem in (1+1)-dimensional quantum electrodynamics suggested some way to introduce
correct description of condensates in the regularization (b), at least semiphenomenologically,
using zero modes as independent variables \[6, 7\].

In the present paper we review briefly our new parametrization of gauge fields on the lattice
in "transversal" space coordinates on the LF. This parametrization is convenient for separate
treatment of zero modes of fields on the LF and gives a way to introduce gauge invariant
regularization of the theory. Then we limit ourselves by QCD(2+1) model in coordinates close
to the LF and perform the limiting transition to the LF Hamiltonian keeping the dynamical
independence of zero modes of fields. We apply this Hamiltonian for simple example of mass
spectrum calculation.

2 The definition of gauge fields on the "transverse" lattice

The gluon part of QCD Lagrangian in continuous space has the following form:

\[
\mathcal{L} = -\frac{1}{2} Tr F_{\mu \nu} F^{\mu \nu}.
\]  

where

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]
\]

and the gluon vector fields \(A_\mu(x)\) are \(N \times N\) Hermitian traceless matrices. Under SU(N) gauge
transformations the \(A_\mu(x)\) transform as follows:

\[
A_\mu(x) \rightarrow \Omega(x) A_\mu(x) \Omega^+(x) + \frac{i}{g} \Omega(x) \partial_\mu \Omega^+(x).
\]  

Here the \(\Omega(x)\) are \(N \times N\) matrices, corresponding to the SU(N) gauge transformation.

In the LF Hamiltonian approach one uses continuous coordinates \(x^+, x^-\) and introduces, as
an ultraviolet regulator, the lattice in transversal coordinates. Gauge invariance is maintained
via appropriate use of Wilson lattice method \[10\], describing gauge fields by matrices related
to lattice links.

If one uses the unitary matrices for these link variables and constructs the Hamiltonian,
one needs to apply the "transfer matrix" method, described in the paper \[11, 12\]. However this
method is not accommodated to the LF and to the corresponding choice of the gauge \(A_- = 0\).

To overcome this difficulty we propose the modification of these link variables, introducing
nonunitary matrices of special form, where only zero modes are related with links and nonzero
modes are related with the sites, belonging to these links. Using these lattice variables we can represent the complete regularization of the theory in gauge invariant form.

The gluon field components $A_+$ and $A_-$ are related with the lattice sites. Under the gauge transformations they transform according to previous formulae \(2\). Transverse components are described by the following $N \times N$ complex matrices:

$$M_\mu(x) = (I + i ga \tilde{A}_\mu(x)) U_\mu(x),$$  \hspace{1cm} (3)

where $m$ is the index of transversal components and the $\tilde{A}_\mu(x)$ are Hermitian $N \times N$ matrices, related to corresponding lattice sites, $U_\mu(x)$ are unitary $N \times N$ matrices, related to the links $(x - ae_\mu, x)$, $a$ is the parameter of the lattice (the size of the link) and the $e_\mu$ is the unit vector along the $x^\mu$ axis, $g$ is the QCD coupling constant.

We define the transformation law under gauge transformations as follows:

$$\tilde{A}_\mu(x) \to \Omega(x) \tilde{A}_\mu(x) \Omega^+(x), \hspace{0.5cm} U_\mu(x) \to \Omega(x) U_\mu(x) \Omega^+(x - ae_\mu).$$ \hspace{1cm} (4)

In consequence the matrices $M_\mu(x)$ transform like link variables \(3 \), \(9\):

$$M_\mu(x) \to \Omega(x)M_\mu(x)\Omega^+(x - ae_\mu).$$

Let us remark that the Hermicity of matrices $\tilde{A}_\mu(x)$ is kept under these gauge transformations.

Let us introduce the operator $D_-$ by the following definitions:

$$D_- \tilde{A}_\mu(x) = \partial_- \tilde{A}_\mu(x) - ig[A_-(x), \tilde{A}_\mu(x)],$$

$$D_- U_\mu(x) = \partial_- U_\mu(x) - igA_-(x)U_\mu(x) + igU_\mu(x)A_-(x - ae_\mu),$$

$$D_- M_\mu(x) = \partial_- M_\mu(x) - igA_-(x)M_\mu(x) + igM_\mu(x)A_-(x - ae_\mu).$$ \hspace{1cm} (5)

This definition of the $D_-$ has gauge invariant form under the gauge transformations, defined above.

Further we impose on the $U_\mu(x)$ the condition

$$D_- U_\mu(x) = 0,$$ \hspace{1cm} (6)

while from the $\tilde{A}_\mu(x)$ we exclude the part, which satisfies the equality $D_- \tilde{A}_\mu(x) = 0$. In the gauge $A_- = 0$ these conditions simply mean a separation of zero ($U_\mu(x)$) and nonzero ($\tilde{A}_\mu(x)$) Fourier modes of the field in the $x^-$. In general we have some gauge invariant definition of this separation.

Furthermore we can introduce the gauge invariant cutoff in $p_-$, using a cutoff in the eigenvalues $q_-$ of the $D_- : |q_-| \leq \Lambda$.

Now let us consider the naive continuous space limit $a \to 0$. We require the following relation in the fixed gauge $A_- = 0$ at $a \to 0$:

$$U_\mu(x) \to \exp i ga A_{\mu 0}(x) \to (I + i ga A_{\mu 0}(x)).$$

Here the $A_{\mu 0}(x)$ is zero mode of the field $A_\mu(x)$ in continuous space. And for the $\tilde{A}_\mu(x)$ we require that it tend to nonzero mode part of the $A_\mu(x)$. Then at nonzero $A_-$ we can get for the matrix $M_\mu(x)$ the following relation:

$$M_\mu(x) \to (I + i ga A_\mu(x) + O((ag)^2)).$$ \hspace{1cm} (7)
Indeed, at \( a \to 0 \) we have:

\[
M_\mu(x) \to \Omega(x; A_-)(I + i ga A_\mu(x))_{A_-} \to \Omega(x; A_-)(I + i ga A_\mu(x))_{A_-} = a \Omega(x; A_-)_{\partial_\mu \Omega^+ (x; A_-)} \\
\to (I + i ga A_\mu(x)),
\]

where the \( \Omega(x; A_-) \) is the matrix of the gauge transformation, which transforms the field in the gauge \( A_- = 0 \) to the field with a given \( A_- (x) \).

Let us introduce the lattice analog of the continuous space field strength \( F_{\mu\nu}(x) \). With this aim we define the following quantities \((\mu, \nu = 1, 2)\):

\[
G_{\mu\nu}(x) = -\frac{1}{ga^2} [M_\mu(x)M_\nu(x - ae_\mu) - M_\nu(x)M_\mu(x - ae_\nu)],
\]

\[
G_{+\mu}(x) = iF_{+\mu}(x), \quad G_{-\mu} = \frac{1}{ga} D_- M_\mu,
\]

\[
G_{+\mu}(x) = \frac{1}{ga} [\partial_+ M_\mu(x) - ig(A_+(x)M_\mu(x) - M_\mu(x)A_+(x - ae_\mu))].
\]

It is not difficult to show that at \( a \to 0 \) one gets \( G_{\mu\nu}(x) \to iF_{\mu\nu}(x) \), and the analogous relations are true for the \( G_{+\mu}, G_{-\mu} \).

We get the following transformation law under the gauge transformations:

\[
G_{\pm\mu}(x) \to \Omega(x) G_{\pm\mu}(x) \Omega^+(x - ae_\mu), \\
G_{\mu\nu}(x) \to \Omega(x) G_{\mu\nu}(x) \Omega^+(x - ae_\mu - ae_\nu).
\]

Having these quantities one can construct gauge-invariantly regularized action and the LF Hamiltonian of the QCD (similarly to the work \[9\]).

One can also apply the transfer matrix method of the paper \[11\] to construct the Hamiltonian on the lattice even in the gauge \( A_- = 0 \), because only zero modes are described by unitary matrices on links, and it is possible to find the necessary "transfer matrix" in \( x^+ \), in analogy with paper \[11\].

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