Research Article

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Absolutely strongly star-Hurewicz spaces

Abstract: A space $X$ is absolutely strongly star-Hurewicz if for each sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$ and each dense subset $D$ of $X$, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $D$ such that for each $x \in X$, $x \in \text{St}(F_n, U_n)$ for all but finitely many $n$. In this paper, we investigate the relationships between absolutely strongly star-Hurewicz spaces and related spaces, and also study topological properties of absolutely strongly star-Hurewicz spaces.

Keywords: Selection principles, Starcompact, acc, Strongly star-Menger, Absolutely strongly star-Menger, Strongly star-Hurewicz, Absolutely strongly star-Hurewicz, Alexandroff duplicate

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1 Introduction

By a space we mean a topological space. Let us recall that a space $X$ is countably compact if every countable open cover of $X$ has a finite subcover. Fleischman [9] defined a space $X$ to be starcompact if for every open cover $U$ of $X$, there exists a finite subset $F$ of $X$ such that $\text{St}(F, U) = X$, where $\text{St}(F, U) = \bigcup\{U \in U : U \cap F \neq \emptyset\}$. He proved that every countably compact space is starcompact. van Douwen et al. in [6] showed that every $T_2$ starcompact space is countably compact, but this does not hold for $T_1$-spaces (see [15, Example 2.5]). Matveev [14] defined a space $X$ to be absolutely countably compact (=acc) if for each open cover $U$ of $X$ and each dense subset $D$ of $X$, there exists a finite subset $F$ of $D$ such that $\text{St}(F, U) = X$. It is clear that every $T_2$ absolutely countably compact space is countably compact.

In [6], a starcompact space is called strongly starcompact.

Kočinac [11, 12] defined a space $X$ to be strongly star-Menger if for each sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $X$ such that $\{\text{St}(F_n, U_n) : n \in \mathbb{N}\}$ is an open cover of $X$.

Bonanzinga et al. in [2] (see also [3]) defined a space $X$ to be strongly star-Hurewicz if for each sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $X$ such that for each $x \in X$, $x \in \text{St}(F_n, U_n)$ for all but finitely many $n$. It is clear that every strongly star-Hurewicz space is strongly star-Menger.

Caserta, Di Maio and Kočinac [5] gave the selective version of the notion of acc spaces and introduced the classes of the following spaces (as special cases of a more general definition).

Definition 1.1 ([5]). A space $X$ is said to be absolutely strongly star-Menger if for each sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$ and each dense subset $D$ of $X$, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $D$ such that $\{\text{St}(F_n, U_n) : n \in \mathbb{N}\}$ is an open cover of $X$.
**Definition 1.2** ([5]). A space $X$ is said to be absolutely strongly star-Hurewicz if for each sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$ and each dense subset $D$ of $X$, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $D$ such that for each $x \in X$, $x \in \text{St}(F_n, U_n)$ for all but finitely many $n$.

From the above definitions, we have the following diagram.

\[
\begin{array}{ccc}
\text{acc} & \downarrow & \\
\text{absolutely strongly star-Hure} & \longrightarrow & \text{absolutely strongly Meng}
\end{array}
\]

\[
\begin{array}{ccc}
\text{starcompact} & \longrightarrow & \text{strongly star-Hure} \\
\downarrow & & \downarrow \\
\text{strongly star-Meng}
\end{array}
\]

The purpose of this paper is to investigate the relationships between absolutely strongly star-Hurewicz spaces and related spaces, and study topological properties of absolutely strongly star-Hurewicz spaces.

Throughout this paper, the *extent* $e(X)$ of a space $X$ is the smallest cardinal number $\kappa$ such that the cardinality of every discrete closed subset of $X$ is not greater than $\kappa$. Let $\omega$ denote the first infinite cardinal, $\omega_1$ the first uncountable cardinal, $\kappa$ the cardinality of the set of real numbers. For each ordinals $\alpha, \beta$ with $\alpha < \beta$, we write $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$, $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$ and $[\alpha, \beta) = \{\gamma : \alpha < \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [8].

## 2 On absolutely strongly star-Hurewicz spaces

In this section, first we give some examples showing relationships between absolutely strongly star-Hurewicz spaces and related spaces. The results and examples extend and improve some results from [16].

**Example 2.1.** There exists a Tychonoff absolutely strongly star-Hurewicz space $X$ which is not acc.

**Proof.** Let $X = ([0, \omega] \times [0, \omega]) \setminus \{(0, \omega)\}$ be the subspace of the product space $[0, \omega] \times [0, \omega]$. Clearly, $X$ is a Tychonoff space. But it is not countably compact, since $\{(\omega, n) : n \in \omega\}$ is a countable discrete closed subset of $X$. Hence $X$ is not acc.

Now we show that $X$ is absolutely strongly star-Hurewicz. To this end, let $\{U_n : n \in \mathbb{N}\}$ be a sequence of open covers of $X$. Let

$$D = [0, \omega) \times [0, \omega).$$

Then $D$ is a dense subspace of $X$ and every dense subset of $X$ includes $D$, since every point of $D$ is isolated. Thus it suffices to show that there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $D$ such that for each $x \in X$, $x \in \text{St}(F_n, U_n)$ for all but finitely many $n$. For each $n \in \mathbb{N}$, let

$$K_n = ([0, \omega] \times [0, n-1]) \cup ([0, n-1] \times [0, \omega]).$$

Then $K_n$ is the union of finitely many compact subsets. For each $n \in \mathbb{N}$, we can find a finite subset $F_n$ of $D$ such that $K_n \subseteq \text{St}(F_n, U_n)$. Thus the sequence $(F_n : n \in \mathbb{N})$ witnesses for $(U_n : n \in \mathbb{N})$ that $X$ is absolutely strongly star-Hurewicz. In fact, $\bigcup_{n \in \mathbb{N}} K_n = X$. For any $x \in X$, there exists $n_0 \in \mathbb{N}$ such that $x \in K_{n_0}$, thus $x \in \text{St}(F_n, U_n)$ for each $n > n_0$, which shows that $X$ is absolutely strongly star-Hurewicz.

**Example 2.2.** There exists a Tychonoff countably compact (hence strongly star-Hurewicz) space $X$ which is not absolutely strongly star-Hurewicz.
Proof. Let \( X = [0, \omega_1) \times [0, \omega_1) \) be the product space of \([0, \omega_1) \) and \([0, \omega_1) \). Clearly, \( X \) is countably compact, hence strongly star-Hurewicz.

We show that \( X \) is not absolutely strongly star-Hurewicz. For each \( \alpha < \omega_1 \), let

\[
U_\alpha = [0, \alpha) \times (\alpha, \omega_1] \quad \text{and} \quad D = [0, \omega_1) \times [0, \omega_1),
\]

For each \( n \in \mathbb{N} \), let

\[
U_n = \{ U_\alpha : \alpha < \omega_1 \} \cup \{ D \}.
\]

Let us consider the sequence \( (U_n : n \in \mathbb{N}) \) of open covers of \( X \) and the dense subset \( D \) of \( X \). Let \( (F_n : n \in \mathbb{N}) \) be any sequence of finite subsets of \( D \). We only show that there exists a point \( x \in X \) such that \( x \notin St(F_n, U_n) \) for all \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \), let \( \alpha_n = \sup \{ \alpha : \alpha \in \pi(F_n) \} \), where \( \pi : [0, \omega_1) \times [0, \omega_1) \rightarrow [0, \omega_1) \) is the projection. Then \( \alpha_n < \omega_1 \), since \( F_n \) is finite. Let \( \beta = \sup \{ \alpha_n : n \in \mathbb{N} \} \). Then \( \beta < \omega_1 \). If we pick \( \alpha' > \beta \), then \( \langle \alpha', \omega_1 \rangle \notin St(F_n, U_n) \) for all \( n \in \mathbb{N} \), since, for every \( U_\beta \in U_n \), if \( \langle \alpha', \omega_1 \rangle \in U_\beta \), then \( \beta > \alpha' \); for each \( \beta > \alpha' \), \( U_\beta \cap F_n = \emptyset \), which shows that \( X \) is not absolutely strongly star-Hurewicz.

Next we give an example of a Tychonoff absolutely strongly star-Menger space which is not absolutely strongly star-Hurewicz by using the following result from [4]. Recall that a family of sets is almost disjoint (a.d., for short) if the intersection of any two distinct elements of the family is finite. Let \( \mathcal{A} \) be an a.d. family of infinite subsets of \( \omega \). Put \( \Psi(\mathcal{A}) = \mathcal{A} \cup \omega \) and topologize \( \Psi(\mathcal{A}) \) as follows: the points of \( \omega \) are isolated and a basic neighborhood of a point \( a \in \mathcal{A} \) takes the form \( \{ a \} \cup (a \setminus F) \), where \( F \) is a finite set of \( \omega \). \( \Psi(\mathcal{A}) \) is called a \( \mathcal{A} \)-space (see [8, 10]). It is well known that \( \mathcal{A} \) is a maximal almost disjoint family (m.a.d. family, for short) iff \( \Psi(\mathcal{A}) \) is pseudocompact.

We make use of two of the cardinals defined in [7]. Define \( \omega_1^\omega_0 \omega \) as the set of all functions from \( \omega \) to itself. For all \( f, g \in \omega_1^\omega_0 \omega \), we say \( f \preceq^* g \) if and only if \( f(n) \leq g(n) \) for all but finitely many \( n \). The unbounding number, denoted by \( b \), is the smallest cardinality of an unbounded subset of \( (\omega_1^\omega_0 \omega, \preceq^*) \). The dominating number, denoted by \( d \), is the smallest cardinality of a cofinal subset of \( (\omega_1^\omega_0 \omega, \preceq^*) \). It is not difficult to show that \( \omega_1 \leq b \leq d \leq \omega \) and it is known that \( \omega_1 < b = c \), \( \omega_1 < d = c \) and \( \omega_1 \leq b < d = \omega \) are all consistent with the axioms of ZFC (see [7] for details).

**Lemma 2.3** ([4, Proposition 2]). The following conditions are equivalent:

1. \( \Psi(\mathcal{A}) \) is strongly star-Menger;
2. \( |\mathcal{A}| < d \).

**Remark 2.4.** From the proof of Proposition 2 in [4], it is not difficult to see that the above conditions are equivalent to (3) \( \Psi(\mathcal{A}) \) is absolutely strongly star-Menger.

**Lemma 2.5** ([4, Proposition 3]). The following conditions are equivalent:

1. \( \Psi(\mathcal{A}) \) is strongly star-Hurewicz;
2. \( |\mathcal{A}| < b \).

**Remark 2.6.** From the proof of Proposition 3 in [4], it is not difficult to see that the above conditions are equivalent to (3) \( \Psi(\mathcal{A}) \) is absolutely strongly star-Hurewicz.

**Example 2.7.** There exists a Tychonoff absolutely strongly star-Menger space \( X \) which is not absolutely strongly star-Hurewicz.

**Proof.** Let \( X = \Psi(\mathcal{A}) = \omega \cup \mathcal{A} \) be the Isbell-Mrówka space, where \( \mathcal{A} \) is the almost disjoint family of infinite subsets of \( \omega \) with \( |\mathcal{A}| = b \). Then \( X \) is absolutely strongly star-Menger by Lemma 2.3 and Remark 2.4 above. However \( X \) is not absolutely strongly star-Hurewicz by Lemma 2.5 and Remark 2.6 above. Thus we complete the proof.

**Remark 2.8.** Assuming \( \omega_1 < b = c \), the space \( X = \Psi(\mathcal{A}) \) with \( |\mathcal{A}| = \omega_1 \) is absolutely strongly star-Hurewicz by Lemma 2.5 and Remark 2.6 above. This space shows that there exists a Tychonoff absolutely strongly star-Hurewicz space \( X \) such that \( \epsilon(X) = \omega_1 \), since \( \mathcal{A} \) is a discrete closed subset of \( X \) with \( |\mathcal{A}| = \omega_1 \). However the author does not know if there exists an example in ZFC showing that there exists a Tychonoff absolutely strongly star-Hurewicz space \( X \) such that \( \epsilon(X) \geq c \). Quite recently, M. Sakai proved that the answer to this question in negative.
In the following, we study topological properties of absolutely strongly star-Hurewicz spaces. Assuming $\omega_1 < b = c$, the space $X = \Psi(A)$ with $|A| = \omega_1$ is absolutely strongly star-Hurewicz. This space shows that a closed subspace of a Tychonoff absolutely strongly star-Hurewicz space $X$ need not be absolutely strongly star-Hurewicz, since $A$ is a discrete closed subset of $X$ with $|A| = \omega_1$. Next we give a stronger example.

**Example 2.9.** There exists a Tychonoff absolutely strongly star-Hurewicz space having a regular-closed $G_\delta$-subspace which is not absolutely strongly star-Hurewicz.

**Proof.** Let $S_1 = [0, \omega_1) \times [0, \omega_1]$ be the product of $[0, \omega_1)$ and $[0, \omega]$. Since $[0, \omega_1)$ is acc by Theorem 1.8 in [14], then $S_1$ is acc by Theorem 2.3 in [14], hence $S_1$ is absolutely strongly star-Hurewicz.

Let $S_2 = [0, \omega_1) \times [0, \omega_1]$ be the space $X$ of Example 2.2. Then $S_2$ is not absolutely strongly star-Hurewicz.

Let $\pi : [0, \omega_1) \times \{ \omega \} \to [0, \omega_1) \times \{ \omega_1 \}$ be a map defined by $\pi((\alpha, \omega)) = (\alpha, \omega_1)$ for each $\alpha \in \omega_1$, and let $X$ be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $(\alpha, \omega)$ of $S_1$ with $\pi((\alpha, \omega))$ of $S_2$ for every $\alpha < \omega_1$.

Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then $\varphi(S_2)$ is a regular-closed subspace of $X$. For each $n \in \omega$, let

$$U_n = \varphi([0, \omega_1) \times [0, \omega_1]) \cup ([0, \omega_1) \times (n, \omega_1)).$$

Then $U_n$ is open in $X$ and $\varphi(S_2) = \bigcap_{n \in \omega} U_n$. Thus $\varphi(S_2)$ is a regular-closed $G_\delta$-subspace of $X$. However $\varphi(S_2)$ is not absolutely strongly star-Hurewicz, since it is homeomorphic to $S_2$.

To show that $X$ is absolutely strongly star-Hurewicz, we show that $X$ is acc, since every acc space is absolutely strongly star-Hurewicz. To this end, let $U$ be an open cover of $X$. Let $S$ be the set of all isolated points of $[0, \omega_1)$ and let

$$D = \varphi((S \times [0, \omega)) \cup (S \times S)).$$

Then $D$ is a dense subset of $X$ and every dense subset of $X$ includes $D$. Thus it is sufficient to show that there exists a finite subset $F$ of $D$ such that $St(F, U) = X$. Since $\varphi(S_1)$ is homeomorphic to $S_1$ and consequently $\varphi(S_1)$ is acc, there exists a finite subset $F_1$ of $\varphi((S \times [0, \omega))$ such that

$$\varphi(S_1) \subseteq St(F_1, U).$$

On the other hand, since $[0, \omega_1) \times [0, \omega_1)$ is countably compact and thus it is acc by Theorem 1.2 in [1]. Hence $\varphi([0, \omega_1) \times [0, \omega_1))$ is acc, since $\varphi([0, \omega_1) \times [0, \omega_1))$ is homeomorphic to $[0, \omega_1) \times [0, \omega_1)$. Thus there exists a finite subset $F_2$ of $\varphi(S \times S)$ such that

$$\varphi([0, \omega_1) \times [0, \omega_1)) \subseteq St(F_2, U).$$

If we put $F = F_1 \cup F_2$. Then $F$ is a finite subset of $D$ such that $St(F, U) = X$, which shows that $X$ is acc, and thus absolutely strongly star-Hurewicz.

Recall the Alexandorff duplicate $A(X)$ of a space $X$. The underlying set $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $(x, 0) \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{(x, 1)\})$, where $U$ is a neighborhood of $x$ in $X$. It is well known that a $T_2$ space $X$ is countably compact iff $A(X)$ is acc (see [17, 18]). In the following, we give two examples to show that the result can not be generalized to the absolutely strongly star-Hurewicz.

**Example 2.10.** Assuming $\omega_1 < b = c$, there exists a Tychonoff absolutely strongly star-Hurewicz space $X$ such that $A(X)$ is not absolutely strongly star-Hurewicz.

**Proof.** Assuming $\omega_1 < b = c$, let $X = \omega \cup A$ be the Isbell-Mrówka space with $|A| = \omega_1$. Then $X$ is absolutely strongly star-Hurewicz by Lemma 2.5 and Remark 2.6. However $A(X)$ is not absolutely strongly star-Hurewicz. In fact, the set $A \times \{1\}$ is an open and closed subset of $A(X)$ with $|A \times \{1\}| = \omega_1$, and for each $a \in A$, the point $(a, 1)$ is isolated in $A(X)$. Hence $A(X)$ is not absolutely strongly star-Hurewicz, since every open and closed subset of an absolutely strongly star-Hurewicz space is absolutely strongly star-Hurewicz, and $A \times \{1\}$ is not absolutely strongly star-Hurewicz.
Example 2.11. There exists a Tychonoff strongly star-Hurewicz space $X$ such that $X$ is not absolutely strongly star-Hurewicz, but $A(X)$ is absolutely strongly star-Hurewicz.

Proof. Let $X = [0, \omega_1] \times [0, \omega_1]$ be the space $X$ of Example 2.2. Then $X$ is not absolutely strongly star-Hurewicz. But $X$ is strongly star-Hurewicz being starcompact. Since $X$ is countably compact, then $A(X)$ is acc (see [17, 18]), hence $A(X)$ is absolutely strongly star-Hurewicz. Thus we complete the proof.

Theorem 2.12. If $X$ is a $T_1$-space and $A(X)$ is an absolutely star-Hurewicz space, then $e(X) < \omega_1$.

Proof. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset $B$ of $X$ such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point. Thus $A(X)$ is not absolutely strongly star-Hurewicz, since every open and closed subset of an absolutely strongly star-Hurewicz space is absolutely star-Hurewicz and $B \times \{1\}$ is not absolutely strongly star-Hurewicz.

Question 1. Is the space $A(X)$ of an absolutely star-Hurewicz space $X$ with $e(X) < \omega_1$ also absolutely star-Hurewicz?

The following example shows that the continuous image of an absolutely strongly star-Hurewicz space need not be absolutely strongly star-Hurewicz.

Example 2.13. There exists a continuous mapping $f : X \to Y$ such that $X$ is absolutely strongly star-Hurewicz, but $Y$ is not absolutely strongly star-Hurewicz.

Proof. We have already noticed that the space $[0, \omega_1] \times [0, \omega_1]$ is acc. The space $X = ([0, \omega_1] \times [0, \omega_1]) \oplus [0, \omega_1]$ is acc as the discrete sum of two acc spaces. Hence $X$ is absolutely strongly star-Hurewicz.

Let $Y = [0, \omega_1] \times [0, \omega_1]$ be the space $X$ of Example 2.2. Then $Y$ is not absolutely strongly star-Hurewicz.

Let $f : X \to Y$ be a mapping defined by

$$f((\alpha, \beta)) = (\alpha, \beta)$$

for each $(\alpha, \beta) \in [0, \omega_1] \times [0, \omega_1)$

and

$$f(\alpha) = (\alpha, \omega_1)$$

for each $\alpha \in [0, \omega_1)$.

Then $f$ is a continuous one-to-one mapping, which completes the proof.

Recall from [13] or [14] that a continuous mapping $f : X \to Y$ is varpseudoopen provided $\text{int}_Y f(U) \neq \emptyset$ for every nonempty open set $U$ of $X$. In [14], it was proved that a continuous varpseudoopen image of an acc space is acc. Similarly, we may show the following result.

Theorem 2.14. A continuous varpseudoopen image of an absolutely strongly star-Hurewicz space is absolutely strongly star-Hurewicz.

Proof. Suppose that $X$ is an absolutely strongly star-Hurewicz space and $f : X \to Y$ is a continuous varpseudoopen onto map. Let $(U_n : n \in \mathbb{N})$ be a sequence of open covers of $Y$ and $D$ a dense subset of $Y$. For each $n \in \mathbb{N}$, let $V_n = \{f^{-1}(U) : U \in U_n\}$. Then $(V_n : n \in \mathbb{N})$ is a sequence of open covers of $X$, and $f^{-1}(D)$ a dense subset of $X$, since $f$ is varpseudoopen. Hence there exists a sequence $(E_n : n \in \mathbb{N})$ of finite subsets of $f^{-1}(D)$ such that for each $x \in X$, $x \in \text{St}(E_n, V_n)$ for all but finitely many $n$. For each $n \in \mathbb{N}$, let $F_n = f(E_n)$. Then $(F_n : n \in \mathbb{N})$ is a sequence of finite subsets of $D$ such that for each $y \in Y$, $y \in \text{St}(F_n, U_n)$ for all but finitely many $n$, which shows that $Y$ is absolutely strongly star-Hurewicz.

Question 2. Find an inner characterization of those spaces $X$ for which $f(X)$ is absolutely strongly star-Hurewicz for each continuous mapping $f$.

Since an open map is varpseudoopen, we have the following result by Theorem 2.14.
Theorem 2.15. Let $X$ and $Y$ be two spaces. If $X \times Y$ is absolutely strongly star-Hurewicz, then $X$ and $Y$ are absolutely strongly star-Hurewicz.

The following remark shows that the converse of Theorem 2.15 need not be true even if the product of an absolutely strongly star-Hurewicz space and a compact space.

Remark 2.16. The product of an absolutely strongly star-Hurewicz space and a compact space need not be absolutely strongly star-Hurewicz. In fact, the space $X = [0, \omega_1] \times [0, \omega_1]$ of Example 2.2 is not absolutely strongly star-Hurewicz. The first factor is acc by [14, Theorem 1.8], hence it is absolutely strongly star-Hurewicz, and the second is compact. Matveev showed that the product of a $T_2$ acc space with a first countable compact space is acc (see [14, Theorem 2.3]). However, the author does not know if the product of an absolutely strongly star-Hurewicz space and a first countable compact space is absolutely strongly star-Hurewicz.

Next we turn to consider preimages. We show that the preimage of an absolutely strongly star-Hurewicz space under a closed 2-to-1 continuous map need not be absolutely strongly star-Hurewicz.

Example 2.17. Assuming $\omega_1 < b = c$, there exists a closed 2-to-1 continuous map $f : X \to Y$ such that $Y$ is an absolutely strongly star-Hurewicz space, but $X$ is not absolutely strongly star-Hurewicz.

Proof. Let $Y = \Psi(A) = \omega \cup A$ be the space $X$ of Example 2.10. Then $Y$ is absolutely strongly star-Hurewicz.

Let $X$ be the space $A(Y)$ of Example 2.10. Then $X$ is not absolutely strongly star-Hurewicz.

Let $f : X \to Y$ be the projection. Then $f$ is a closed 2-to-1 continuous map, which completes the proof.

Remark 2.18. The space $[0, \omega_1] \times [0, \omega_1]$ in Remark 2.16 also shows that the preimage of an absolutely strongly star-Hurewicz space under an open perfect map need not be absolutely strongly star-Hurewicz.

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