Reduced dimension model for heat transfer of ground heat exchanger in permafrost

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Abstract. In this work, we present reduced dimensional model for heat transfer processes of ground heat exchanger in permafrost. A mathematical model is described by a coupled system of equations for heat transfer in the ground subdomain and in heat exchanger (pipes). Because radius of the pipes is very small compared to the size of surrounding ground, we write heat transfer problem in pipes as reduced dimensional equation, where we consider pipes as one-dimensional lines. We present a computational algorithm and numerical results for model problem.

1. Introduction

For stabilization of foundations the methods of ground freezing are often used. Freezing of ground can be made by using of special installations-refrigerators or by seasonal cooling installations. They can significantly influence on temperature of ground and, consequently, their thermomechanical characteristics.

Seasonal cooling systems are installed for the purpose of forming a mass of frozen soil under building foundations and engineering constructions to prevent their thawing in summer-autumn period. In addition, there is another category of widely-spread piping systems — heat pumps. They are used for pumping heat out of ground to heat buildings (geothermal heat pump) with ability to switch from heating mode in winter period to condensing mode in summer period. Technical device in which heat exchange between two media having different temperatures is called a heat exchanger. Their mathematical models are also described by ground-pipe systems.

Ground-pipe systems use a temperature difference between the ground and the buried pipes, for heating and cooling buildings, for formation of a frozen rock solid that increase thermal mechanical characteristics and are applied to increase a stability of foundations under permafrost conditions. A coolant is used as a heat carrier in pipes. The coolant is a gas with a low boiling (evaporation) point. It remains in a gaseous state under normal conditions and becomes a liquid under high pressure.

A mathematical model for such processes consists of heat and mass transfer equation and considers a phase transition of porous moisture to ice. For such models, there are two approaches for phase transition. The first is when the boundary of the phase transition is fixed at the grid level and a calculation of heat transfer can occur in the region occupied by thawed zone, and the second approach is based on use of the two-phase Stefan model. The second approach is
more appropriate. In a two–phase model a boundary of the phase transition is represented by the transition zone [1, 2, 3, 4].

In the computational implementation of models of heat transfer in permafrost, it is necessary to construct geometric domains and computational grids in such a way that the geometric domain of pipes is resolved by mesh. Let us note that computational grids for planning a construction and a pipe system location must be unstructured. The computational grids for such regions will have areas of local condensation in pipe areas and will lead to very large systems of equations at discretization. The alternative approach proposed in this paper is based on reducing of model dimensionality on the pipe and reducing it to a one-dimensional hydraulic model, and further under assumption of the continuity of temperature field, we will approximate it using the methods similar to those used in modeling of fractions in oil and gas layers, or to be precise, a method of discrete model of fraction [5, 6, 7, 8, 9]. The obtained discrete problem will contain a system of linear equations for each time layer for a temperature in each node of a computational grid. The system of equations obtained in this way for finding a temperature distribution can be complicated by adding a convective heat transfer (heat and mass transfer). To calculate the filtration velocity in the ground the authors use the Darcy’s law and an equation of mass conservation for the case of incompressible fluid [10, 11]. To count a phase transition, i.e. an appearance of a frozen ground zone in the areas of pipes freezing and a surface of soil, where the ambient temperature is set by the Robin boundary condition, an ambient air temperature is periodic and well described by sinusoid. For numerical solution of the problem of pore water filtration in the ground, a method of fictitious regions in frozen ground is used which was studied by the authors in [12].

In this paper we constructed a new mathematical model and a computational algorithm for solving problems in ground–pipe systems. The work consists of three parts. In the first part the authors present a mathematical model for heat transfer in the ground taking into account the phase transition. In the second part we derive a reduced-dimensional mathematical model for describing a temperature distribution in the pipe. In the third part, an associated model and a computational algorithm for heat and mass transfer problem for ground–pipe system is given, taking into account a presence of filtration in ground. At the end the authors provide calculations of a model problem using proposed model and a computational algorithm.

2. Heat transfer in the ground

We consider a mathematical model describing the heat transfer processes in the frozen and thawed soils. This model takes into account the presence of phase transitions of pore water in the ground at a given temperature phase transition \( T^* \) in domain \( \Omega = \Omega^- \cup \Omega^+ \):

\[
\begin{align*}
\Omega^+(t) &= \{x | x \in \Omega, \ T(x, t) > T^* \}, \\
\Omega^-(t) &= \{x | x \in \Omega, \ T(x, t) < T^* \}.
\end{align*}
\]

Here \( \Omega^+ \) is a domain occupied by the liquid phase, where the temperature is higher than the phase transition temperature and \( \Omega^- \) is a domain occupied by the solid phase, phase transition occurs at the interface \( S \).

For the simulation of heat transfer processes with phase transitions, we use the Stefan model [1, 2]

\[
(\alpha(\phi) + \rho^+ L \phi') \frac{\partial T}{\partial t} - \text{div} (\lambda(\phi) \text{grad} T) = f, \tag{1}
\]

where \( L \) is the specific heat of the phase transition, \( m \) is porosity, \( \rho^+, c^+, \lambda^+ \) and \( \rho^-, c^-, \lambda^- \) are density, specific heat capacity and the thermal conductivity of melted and frozen zones, respectively. We have the following coefficients

\[
\alpha(\phi) = \rho^- c^- + \phi (\rho^+ c^+ - \rho^- c^-), \quad \lambda(\phi) = \lambda^- + \phi (\lambda^+ - \lambda^-),
\]
\[ c^- \rho^- = (1 - m)c_{sc}\rho_{sc} + mc_i\rho_i, \quad \lambda^- = (1 - m)\lambda_{sc} + m\lambda_i, \]
\[ c^+ \rho^+ = (1 - m)c_{sc}\rho_{sc} + mc_w\rho_w, \quad \lambda^+ = (1 - m)\lambda_{sc} + m\lambda_w. \]

and

\[ \phi = \begin{cases} 
0, & \text{when } T < T^*, \\
1, & \text{when } T > T^*. 
\end{cases} \]

The indexes \( sc, w, i \) denote the solid skeleton, water, and ice, respectively.

In practice, the phase transitions occur in a small temperature range \([T^* - \Delta, T^* + \Delta]\). As the function \( \phi \) we take \( \phi_\Delta \):

\[ \phi_\Delta = \begin{cases} 
T - T^* + \Delta, & T \leq T^* - \Delta, \\
\frac{T^* - \Delta}{2\Delta}, & T^* - \Delta < T < T^* + \Delta, \\
1, & T \geq T^* + \Delta. 
\end{cases} \]

Therefore, we get the following equation for nonlinear parabolic temperature in the domain \( \Omega \):

\[ (\alpha(\phi_\Delta) + \rho_L \phi'_\Delta) \frac{\partial T}{\partial t} - \text{div}(\lambda(\phi_\Delta) \text{grad} T) = f. \] (2)

The equation (2) is supplemented with the initial condition:

\[ T(x, 0) = T_0(x), \quad x \in \Omega, \] (3)

and boundary conditions

\[ -k \frac{\partial T}{\partial n} = \alpha(T - T_{air}), \quad x \in \Gamma_1, \quad -k \frac{\partial T}{\partial n} = 0, \quad x \in \Gamma_2, \quad T = T_1, \quad x \in \Gamma_3, \] (4)

where \( \alpha \) is heat convection coefficient, \( T_{air} \) is outdoor temperature.

### 3. Heat and mass transfer in the heat exchanger

To calculate the thermal regime in the pipe system (heat exchanger), we use a reduced dimension model. In this model, we assume that the coolant, that have a low freezing temperature, circulates in the cylindrical tube. We suppose, that the coolant does not freeze under the temperatures considered in the model. Then, for a heat transfer in a pipes, we have following equation in the cylindrical coordinates

\[ c_p \rho_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + v_r \frac{\partial T}{\partial \tau} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( r \lambda_p \frac{\partial T}{\partial r} \right) - \frac{\partial}{\partial \tau} \left( \lambda_p \frac{\partial T}{\partial \tau} \right) = 0, \] (5)

where \( c_p, \rho_p \) and \( \lambda_p \) are specific heat capacity, density and the thermal conductivity of coolant, \( T \) is temperature of the coolant in pipe, \( v_r \) is velocity of the coolant in the radial direction and \( v_r \) is velocity of the coolant in the direction of flow.

Since the flow in pipes is characterized by significant velocities of the coolant in longitudinal direction, it is possible to neglect the radial flow velocity, \( v_r = 0 \). Then we multiply by the radius \( r \) and integrate the resulting equation along the axis \( r \in [0, R] \)

\[ \int_0^R c_p \rho_p r \frac{\partial T}{\partial t} dr + \int_0^R c_p \rho_p r v_r \frac{\partial T}{\partial \tau} dr - \int_0^R \frac{1}{r} \frac{\partial}{\partial r} \left( r \lambda_p \frac{\partial T}{\partial r} \right) dr + \lambda_p r \frac{\partial T}{\partial r} \bigg|_0^R = 0. \] (6)
Next, we determine the average temperature of the coolant in a pipe

\[ T_p = \frac{2}{R^2} \int_0^R r \, d r, \]

and suppose following interface condition between pipe and surrounding ground

\[ \lambda_p \frac{\partial T_p}{\partial r} \bigg|_0^R = -R \alpha_p (T - T_p). \]

Therefore, we obtain following reduced dimension (one–dimensional) equation in a pipe

\[ \pi R^2 c_p \rho_p \left( \frac{\partial T_p}{\partial t} + \nu \frac{\partial T_p}{\partial \tau} \right) - \pi R^2 \frac{\partial}{\partial \tau} \left( \lambda_p \frac{\partial T_p}{\partial \tau} \right) = 2 \pi R \alpha_p (T - T_p). \]

(7)

4. Coupled heat transfer problem and finite element approximation

A mathematical model is described by a coupled system of equations for heat transfer in the ground subdomain and in heat exchanger (pipes)

\[ \left( \alpha (\phi_\Delta) + \rho_l L \phi_\Delta' \right) \frac{\partial T}{\partial t} - \text{div} (\lambda (\phi_\Delta) \text{grad} \, T) = f, \quad x \in \Omega \]

(8)

\[ \pi R^2 c_p \rho_p \left( \frac{\partial T_p}{\partial t} + \nu \frac{\partial T_p}{\partial \tau} \right) - \pi R^2 \frac{\partial}{\partial \tau} \left( \lambda_p \frac{\partial T_p}{\partial \tau} \right) = q_i, \quad x \in \gamma_i, \]

(9)

where \( \gamma_i \) is the \( i \)-th pipe.

For including thermal effects of the ground, we define following heat source at the pipe locations.

\[ f = \sum_{i=1}^{n} q_i \delta(x - x_i), \]

(10)

where \( q_i \) is power unit length of \( i \)th heat sink, \( n \) is quantity of heat sinks and \( \delta \) is the Dirac delta function determining the position of point sources in a ground. Here, we use the Dirac delta function, since the diameter of the pipes is much smaller than the ground size.

Assuming an ideal thermal contact between the outer surface of the pipes and the surrounding ground, the power of the heat sinks is written as follows

\[ q_i = 2 \pi R^i \alpha_p (T - T_p), \]

(11)

where \( \alpha_p \) is heat convection coefficient between the pipe surface and surrounding ground, \( R \) is pipe radius, \( T \) and \( T_p \) are the ground and coolant temperatures.

For the numerical solution of the problem, we use a finite–element method [13]. Let \( W = H_1(\Omega) \), then we have following variational formulation: find \( T \in W \) such that

\[
\int_{\Omega} \left( \alpha (\phi_\Delta) + \rho_l L \phi_\Delta' \right) \frac{\partial T}{\partial t} \, w \, d \Omega + \int_{\Omega} \lambda (\phi_\Delta) \text{grad} \, T \cdot \text{grad} \, w \, d \Omega \\
+ \pi R^2 \sum_{i=1}^{n} \int_{\gamma_i} c_p \rho_p \left( \frac{\partial T_p}{\partial t} + \nu \cdot \text{grad}_r T_p \right) \, w \, d s + \pi R^2 \sum_{i=1}^{n} \int_{\gamma_i} \lambda_p \text{grad}_r T_p \cdot \text{grad}_r w \, d s = 0, \quad w \in W, \quad (12)
\]
where $\gamma_i$ is the $i$–th pipe.

Here, for approximation of the coupled system of equations (8) – (9), we suppose that the temperature on the ground and pipes at the similar nodes are same. This approximation method is inspired by the well–known discrete fracture model (DFM) that used for fluid flow problems in the fractured porous medium. In this method, we construct unstructured grid, and resolve one–dimensional pipe lines by construction conforming mesh [6, 7, 8, 9].

5. Numerical results and conclusion

In this section, we present results of the numerical simulation for the three–dimensional problem. As a model parameters, we use following properties of the ground

- for melt zone $\rho^+ c^+ = 2.772 \cdot 10^6 \text{ [J/(m}^3 \cdot \text{K}]}$, $\lambda^+ = 1.94 \text{ [W/(m} \cdot \text{K}]}$
- for frozen zone $\rho^- c^- = 2.052 \cdot 10^5 \text{ [J/(m}^3 \cdot \text{K}]}$, $\lambda^- = 2.14 \text{ [W/(m} \cdot \text{K}]}$
- for phase transition $\rho^+ L = 125.532 \cdot 10^6 \text{ [J/(m}^3 \cdot \text{K}]}$ and $\delta = 1.0$

For the pipe system, the following parameters is used: $v_\tau = 1.0 \text{ [m/s]}$, $\rho_p c_p = 1.763 \cdot 10^6 \text{ [J/(m}^3 \cdot \text{K}]}$, $R = 0.05 \text{ [m]}$ and the temperature of the coolant at the inlet is set equal to the air temperature.

We perform numerical simulation in the computational domain with size 10 meters in the direction $X$, 10 meters in the direction $Y$ and 10 meters in the direction $Z$. On the top boundary, we set boundary conditions that simulate the presence of the warm construction of dimensions $5 \times 4$ meters. This condition is given in the form of a Dirichlet boundary condition with a constant temperature $20^\circ C$. At a depth of 1 meter under the structure installed a system of pipes with circulating coolant. We present results for the 1 year simulation of the heat processes with time step $\tau = 1$ day. For numerical simulations using finite–element method, we construct unstructured tetrahedral grid with 650k cells and 105k vertices (Fig. 1). The ground has an initial temperature $T_0 = -3.0^\circ C$. On the top boundary, we set Robyn type boundary conditions that taking into account the amplitude of the air temperature fluctuation, which is of a regional characteristic and varies from $T_w = -50^\circ C$ in winter to $T_s = 30^\circ C$ in summer.

![Computational geometry and unstructured grid](image)

**Figure 1.** Computational geometry and unstructured grid

The numerical results of the heat transfer problem solution using proposed method are presented in Fig. 2 and 3. Fig. 2 illustrates the slice at the depth 1 meters. The slices
are presented for several initial time layers and illustrates the dynamics of the movement of the coolant in the pipe and the beginning of freezing of the surrounding ground. In Fig. 3, we shown a dynamics of freezing of soil at different simulation times. From the results, we see that the temperature under a construction remains negative for all simulation time and help to prevent soils melt.

The presented results illustrate the efficiency of the proposed mathematical model and numerical algorithm for heat transfer processes in the ground with pipes systems. This model help to significantly reduce the dimension of the computational problem that arises after space approximations, since the model for pipes is written in the form of a one-dimensional equation.

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Figure 3. Temperature distribution a) winter, b) spring, c) summer, d) autumn

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