Variation of the light-like particle energy and its critical curve equations

W. B. Belayev
Center for Relativity and Astrophysics,
185 Box, 194358, Sanct-Petersburg, Russia
e-mail: wbelayev@yandex.ru

We consider variation of energy of the light-like particle in Riemann space-time, find lagrangian, canonical momenta and forces. Equations of the critical curve are obtained by the nonzero energy integral variation in accordance with principles of the calculus of variations in mechanics. This method is shown to not lead to violation of conformity of varied curve to the null path in contradistinction of the interval variation. Though found equations are differ from standard form of geodesics equations, for the Schwarzschild space-time their solutions coincide with each other to within parameter of differentiation.

Keywords: variations; null geodesics; energy integral; extremal curve; Schwarzschild space-time

I. INTRODUCTION

One of postulates of general relativity is claim that in gravity field in the absence of other forces the word lines of the material particles and light rays are geodesics. In differential geometry a geodesic line is defined as a curve, whose tangent vector is parallel propagated along itself [1]. Differential equations of geodesic, which is a path of extremal length, can be found also by the variation method with the aid of the virtual displacements of coordinates $x^i$ on a small quantity $\omega^i$. When we add variation to material particle coordinate, the meaning of the time-like interval slow changes, though that leaves it time-like.

Finding of differential equations of the null geodesic, corresponding to the light ray motion, by calculus of variations, described in [2]. In space-time with metrical coefficients $g_{ij}$ it is considered variation of their first integral

$$ h = g_{ij} \frac{dx^i}{d\mu} \frac{dx^j}{d\mu}, $$

where $\mu$ is affine parameter. Deriving variation for extremum determination we must admit arbitrary small displacements of coordinates. The variation of integral of $h$ expanded in multiple Taylor series is written as

$$ \delta I = \int_{\mu_0}^{\mu_1} \sum_{n=1}^{\infty} \frac{\partial^n g_{ij}}{\partial \beta_1(n) x^i \ldots \partial \beta_4(n) x^j} \frac{dx^i}{d\mu} \frac{dx^j}{d\mu} (\omega^1)^{\beta_1(n)} \ldots (\omega^4)^{\beta_4(n)} + g_{ij} \left( 2 \frac{dx^i}{d\mu} \frac{d\omega^j}{d\mu} + \frac{d\omega^i}{d\mu} \frac{d\omega^j}{d\mu} \right) d\mu, $$

where $\mu_0, \mu_1$ are meanings of affine parameter in points, which are linked by found geodesic, and it is fulfilled $\beta_1(n) + \ldots + \beta_4(n) = n$. With finding geodesic equations the sum of terms containing variations of coordinates and their derivatives by $\mu$ in first power is equated to null, that gives the geodesic equations in form

$$ \frac{d^2 x^\lambda}{d\mu^2} + \Gamma^\lambda_{ij} \frac{dx^i}{d\mu} \frac{dx^j}{d\mu} = 0, $$

where $\Gamma^\lambda_{ij}$ are Christoffel symbols:

$$ \Gamma^\lambda_{ij} = \frac{1}{2} g^{\lambda \gamma} (g_{\gamma,j,i} + g_{\gamma,i,j} - g_{ij,\gamma}). $$

Here a comma denotes partial differentiation.

Other terms of series in (1.2), containing small quantities $\omega^i, d\omega^j/d\mu$ in more high powers or their products and being able to have nonzero values, don’t take into account. Thus such method admits violation of condition $h = 0$, which means that with certain coordinates variations the interval a prior becomes time-like or space-like. Since this interval accords with the light ray motion, that leads to the Lorentz-invariance violation in locality, namely, anisotropies.
The possibility of Lorentz symmetry break for the photon in vacuum by effects from the Plank scale is studied in [3, 4]. At the contrary, it is shown in [5] that a fundamental space-time discreteness need not contradict Lorentz invariance, and causal set’s discreteness is in fact locally Lorentz invariant. Experiments [6] show exceptionally high precision of constancy of light speed confirmed a Lorentz symmetry in locality, and astrophysical tests don’t detect isotropic Lorentz violation [4].

In the method of calculus of variations in the large [7] ones are considered as possible paths along the manifold disregarding kind of interval, not as the trajectories of physical particles. This approach exceeds the limits of classical variational principle in mechanics, according as which virtual motions of the system are compared with cinematically possible motions.

Approximating time-like interval conforming in general relativity to the material particle motion between fixed points to null leads in physical sense to unlimited increase of its mass, and the space-like interval doesn’t conform to move of any object. In this connection it should pay attention on speculation that discreteness at the Planck scale reveals maximum value of momentum for fundamental particles [8].

Geodesic line must be extremal [1], and the test particle moves along it only in the absence of non-gravity forces. Should photon have some rest mass variations of its path don’t give different kinds of intervals, but this assumption doesn’t confirm by experiments [9]. We examine choosing of energy so in order that application of variational principle to its integral for deriving of the isotropic critical curves equations would not lead to considering non-null paths.

II. DEFINITION OF ENERGY AND ITS VARIATION

The interval in Riemann space-time is written in form

$$ds^2 = \rho^2 g_{11} dx^1 dx^2 + 2 \rho g_{1p} dx^1 dx^p + g_{pq} dx^p dx^q,$$

(2.1)

where $\rho$ is some quantity, whose meaning is assumed to be equal 1. Putting down $x^1$ as time coordinate, $x^p$ ($p = 2, 3, 4$) as space coordinates and considering $\rho$ as energy of light-like particle with $ds = 0$ we present it as

$$\rho = \left\{ -g_{1p} \frac{dx^p}{d\mu} + \sigma \left( \left( g_{1p} \frac{dx^p}{d\mu} \right)^2 - g_{11} g_{pq} \frac{dx^p}{d\mu} \frac{dx^q}{d\mu} \right)^{1/2} \right\} \left( g_{11} \frac{dx^1}{d\mu} \right)^{-1},$$

(2.2)

where $\sigma$ takes meaning $\pm 1$.

With denotation of the velocity four-vector components as $v^i = dx^i/d\mu$ energy variation will be

$$\delta \rho = \frac{\partial \rho}{\partial x^\lambda} \delta x^\lambda + \frac{\partial \rho}{\partial v^\lambda} \delta v^\lambda.$$

(2.3)

After substitution

$$\sigma \left[ \left( g_{1p} \frac{dx^p}{d\mu} \right)^2 - g_{11} g_{pq} \frac{dx^p}{d\mu} \frac{dx^q}{d\mu} \right]^{1/2} = g_{11} \frac{dx^1}{d\mu}$$

(2.4)

the partial derivatives with respect to coordinates are written as

$$\frac{\partial \rho}{\partial x^\lambda} = \frac{1}{g_{11} v^1} \left[ -\frac{\partial g_{1p}}{\partial x^\lambda} v^p + \frac{1}{2v_1} \left( 2 \frac{\partial g_{1p}}{\partial x^\lambda} g_{1q} - \frac{\partial g_{11}}{\partial x^\lambda} g_{pq} - \frac{\partial g_{pq}}{\partial x^\lambda} g_{11} \right) v^p v^q \right] - \frac{1}{g_{11}} \frac{\partial g_{11}}{\partial x^\lambda}.$$

(2.5)

This expression is reduced to

$$\frac{\partial \rho}{\partial x^\lambda} = -\frac{1}{2v_1 v^1} \frac{\partial g_{ij}}{\partial x^\lambda} v^i v^j.$$

(2.6)

The partial derivatives with respect to components of the velocity four-vector are

$$\frac{\partial \rho}{\partial v^\lambda} = -\frac{v_\lambda}{v_1 v^1}.$$

(2.7)
For the particle, moving in empty space, lagrangian is taken in form
\[ L = -\rho, \]  
and conforms to relation [10]:
\[ \rho = v^\lambda \frac{\partial L}{\partial v^\lambda} - L, \]  
which is integral of the motion. Obtained derivatives give meanings of canonical momenta
\[ p_\lambda = \frac{\partial L}{\partial v^\lambda} = \frac{v_\lambda}{v^1 v_1}, \]  
and forces
\[ F_\lambda = \frac{\partial L}{\partial x^\lambda} = \frac{1}{2 v^1 v_1} \frac{\partial g_{ij}}{\partial x^\lambda} v^i v^j. \]  

III. EQUATIONS OF ISOTROPIC CRITICAL CURVE

Motion equations are found from variation of energy integral
\[ S = \int_{\mu_0}^{\mu_1} \rho d\mu, \]  
where \( \mu_0 \) and \( \mu_1 \) are fixed points. Energy \( \rho \) has non-zero value, its variations leave interval to be light-like, and application of standard variational procedure yields Euler-Lagrange equations
\[ \frac{d}{d\mu} \frac{\partial \rho}{\partial x^\lambda} - \frac{\partial \rho}{\partial x^\lambda} = 0. \]  

Critical curve equations are obtained by substitution of partial derivatives (2.6) and (2.7) in these equations. For the time coordinate we have
\[ \frac{dv^1}{d\mu} + \frac{v^1}{2v_1} \frac{\partial g_{ij}}{\partial x^1} v^i v^j = 0. \]  

For finding of other three equations of motion the second term of (2.2) is presented in form
\[ \frac{d}{d\mu} \frac{\partial c}{\partial x^\lambda} = \frac{1}{v^1(v_1)^2} \left[ (g_{1p} v_\lambda - g_{p\lambda} v_1) \frac{dv^p}{d\mu} - \left( \frac{\partial g_{ij}}{\partial x^\lambda} v_1 - \frac{\partial g_{ij}}{\partial x^\lambda} v_1 \right) v^i v^j \right] + \frac{g_{11} v_1 v_\lambda + g_{p\lambda} v^p v_1}{(v^1)^2 v_1^2} \frac{dv^1}{d\mu}. \]  

Replacement of derivative \( dv^1/d\mu \) here on its expression, obtained from (3.3), and substitution found terms in Euler-Lagrange equations gives
\[ (g_{p\lambda} v_1 - g_{1p} v_\lambda) \frac{dv^p}{d\mu} + \left[ \frac{1}{2v_1} \frac{\partial g_{ij}}{\partial x^1} (g_{11} v_1 v_\lambda + g_{p\lambda} v^p v_1) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^\lambda} v_1 + \frac{\partial g_{ij}}{\partial x^\lambda} v_1 - \frac{\partial g_{1i}}{\partial x^\lambda} v_\lambda \right] v^i v^j = 0. \]  

These equations contain accelerations corresponded to the space coordinates and coupled with (3.3) describe motion of the test light-like particle along critical curve. They don’t coincide to usual form (1.3) of the null geodesics equations.
IV. ISOTROPIC CURVES IN SCHWARZSCHILD SPACE-TIME

Central gravity field of a point body is described by the Schwarzschild metric. This is solution of Einstein’s field equations for space-time, which is empty except central point. We chose units so, that a light velocity constant $c$ is rendered to 1. At spherical coordinates $x^i = (t, r, \theta, \phi)$ the line element is

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1} dr^2 - r^2 d\theta^2 + \sin^2 \theta d\phi^2,$$

(4.1)

where $\alpha$ is constant. Motion equations (3.3) and (3.5) lead to

$$\frac{d^2t}{d\mu^2} = 0$$

(4.2)

and

$$\frac{d^2r}{d\mu^2} + \frac{\alpha}{2r^2} \left(1 - \frac{\alpha}{r}\right) \left(\frac{dt}{d\mu}\right)^2 - \frac{3\alpha}{2r(r - \alpha)} \left(\frac{dr}{d\mu}\right)^2 - (r - \alpha) \left(\frac{d\theta}{d\mu}\right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\mu}\right)^2 = 0,$$

(4.3)

$$\frac{d^2\theta}{d\mu^2} + \frac{2r - 3\alpha}{r(r - \alpha)} \frac{dr}{d\mu} \frac{d\theta}{d\mu} + \frac{1}{2} \sin^2 \theta \left(\frac{d\phi}{d\mu}\right)^2 = 0,$$

(4.4)

$$\frac{d^2\phi}{d\mu^2} + \frac{2r - 3\alpha}{r(r - \alpha)} \frac{dr}{d\mu} \frac{d\phi}{d\mu} + 2 \cot \theta \frac{d\theta}{d\mu} \frac{d\phi}{d\mu} = 0.$$

(4.5)

Metric (4.1) for the isotropic curve yields

$$\left(1 - \frac{\alpha}{r}\right) \left(\frac{dt}{d\mu}\right)^2 - \left(1 - \frac{\alpha}{r}\right)^{-1} \left(\frac{dr}{d\mu}\right)^2 - r^2 \left(\frac{d\theta}{d\mu}\right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\mu}\right)^2 = 0.$$

(4.6)

Equations (4.2) and (4.5) are solvable by quadrature:

$$\frac{dt}{d\mu} = A,$$

(4.7)

$$\frac{d\phi}{d\mu} = \frac{h}{r^2 \sin^2 \theta} \left(1 - \frac{\alpha}{r}\right),$$

(4.8)

where $A, h$ are constants. Substituting obtained solutions in equation (4.6) with $\theta = \pi/2$ we find

$$\frac{dr}{d\mu} = \pm \left[\left(1 - \frac{\alpha}{r}\right)^2 A^2 - \left(\frac{h}{r}ight)^2 \left(1 - \frac{\alpha}{r}\right)^3\right]^{1/2}.$$

(4.9)

Though found equations (3.3) and (3.5) for the Schwarzschild space-time are differ from standard form of null geodesic equations, their solutions coincide with each other to within parameter of differentiation

$$d\mu = d\mu_s \left(1 - \frac{\alpha}{r}\right)^{-1},$$

(4.10)

where $\mu$ corresponds with standard solution [2].

If we choose $\mu = t$ it follows from Eq. (4.7) that $A = 1$. Canonical momenta (2.10) and forces (2.11) under considering constraint on $\theta$ are

$$p_1 = 1, \quad p_2 = \pm \frac{1}{(1 - \frac{\alpha}{r})} \sqrt{1 - \frac{h^2}{r^2} \left(1 - \frac{\alpha}{r}\right)}, \quad p_3 = 0, \quad p_4 = -h;$$

(4.11)
\[ F_1 = F_3 = F_4 = 0, \quad F_2 = \frac{\alpha}{r^2} \left( 1 - \frac{2}{\alpha r} \right) - \frac{h^2}{r^3} + \frac{\alpha h^2}{2 r^4}. \]  

(4.12)

A nonzero component of the contravariant vector of canonical force is

\[ F^2 = -\frac{\alpha}{r^2} + \frac{h^2}{r^3} \left( 1 - \frac{\alpha}{r} \right) \left( 1 - \frac{\alpha}{2 r} \right). \]  

(4.13)

In so far as Newtonian limit of gravity theory with gravitational constant \( G \) and mass \( M \) requires \( \alpha = 2GM \), the first term of \( F^2 \) yields twice Newton gravity force. That conforms to light deflection in central gravity field [11], which is twice meaning being given by Newton gravity theory.

**V. CONCLUSION**

Presented form of energy allows applying of method of lagrangian for analysis of light-like particle motion. Virtual displacements of coordinates retain conformity of the trajectory of the light-like particle to the null path in Riemann space-time, not leaded to Lorentz-invariance violation. Obtained extremal isotropic curve equations for the Schwarzschild metric give the same solution, which follows from standard null geodesics equations to within appropriate parameter. At that found equations as whole are different from one, which are produced by variational method being applied to the path.

---

[1] Landau L.D. and Lifshitz E.M., Classical Theory of Fields, Fourth Revised English Edition, Oxford: Pergamon. 1975.
[2] McVittie G.C., General Relativity and Cosmology, London, Chapman and Hall Ltd. 1956.
[3] A. Kostelecky and A. Pickering, Vacuum Photon Splitting in Lorentz-Violating Quantum Electrodynamics, Phys. Rev. Lett. 91, (2003) 031801, hep-ph/0212382 T. Jacobson, S. Liberati, D. Mattingly, F.W. Stecker, New limits on Planck scale Lorentz violation in QED, Phys.Rev.Lett. 93 (2004) 021101, astro-ph/0309681.
[4] A. Kostelecky, M. Mewes, Astrophysical Tests of Lorentz and CPT Violation with Photons, arXiv:0809.2846.
[5] F. Dowker, J. Henson, R. D. Sorkin, Quantum Gravity Phenomenology, Lorentz Invariance and Discreteness, Mod.Phys.Lett. A19 (2004) 1829-1840, gr-qc/0311055 L. Bombelli, J. Henson, R. D. Sorkin, Discreteness without symmetry breaking: a theorem, gr-qc/0605006.
[6] H. Mueller et al., Relativity tests by complementary rotating Michelson-Morley experiments, Phys.Rev.Lett. 99 (2007) 050401, arXiv:0706.2031 P. L. Stanwix et al., Improved test of Lorentz Invariance in Electrodynamics using Rotating Cryogenic Sapphire Oscillators, Phys.Rev. D74 (2006) 081101 gr-qc/0609072, Antonini et al., Test of constancy of speed of light with rotating cryogenic optical resonators, Phys. Rev. A. 71, (2005) 050101.
[7] Marston Morse, The Calculus of Variations in the Large, Colloquium Publications of the American Mathematical Society, vol. 18, New York, 1934.
[8] G.Amelino-Camelia, Doubly Special Relativity, Nature 418, (2002)34, gr-qc/0207049 J. Magueijo and L. Smolin, Lorentz invariance with an invariant energy scale, Phys. Rev. Lett. 88 (2002) 190403, hep-th/0112090.
[9] Goldhaber A.S. and Nieto M.M., Photon and Graviton Mass Limits, arXiv:0809.1003.
[10] Landau L.D. and Lifshitz E.M., Vol. 1. Mechanics, 3ed., Pergamon, 1976.
[11] A. Einstein, Die Grundlage der allgemeinen Relativit"{a}tstheorie, Ann. Phys. 49 (1916) 769-822.