Linear and Nonlinear Heat Equations on a $p$-Adic Ball

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Abstract

We study the Vladimirov fractional differentiation operator $D^\alpha_N$, $\alpha > 0, N \in \mathbb{Z}$, on a $p$-adic ball $B_N = \{ x \in \mathbb{Q}_p : |x|_p \leq p^N \}$. To its known interpretations via restriction from a similar operator on $\mathbb{Q}_p$ and via a certain stochastic process on $B_N$, we add an interpretation as a pseudo-differential operator in terms of the Pontryagin duality on the additive group of $B_N$. We investigate the Green function of $D^\alpha_N$ and a nonlinear equation on $B_N$, an analog the classical porous medium equation.

Key words: $p$-adic numbers; Vladimirov’s $p$-adic fractional differentiation operator; $p$-adic porous medium equation; mild solution of the Cauchy problem

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1 Introduction

The theory of linear parabolic equations for real- or complex-valued functions on the field $\mathbb{Q}_p$ of $p$-adic numbers including the construction of a fundamental solution, investigation of the Cauchy problem, the parametrix method, is well-developed; see, for example, the monographs [16, 21]. In such equations, the time variable is real and nonnegative while the spatial variables are $p$-adic. There are no differential operators acting on complex-valued functions on $\mathbb{Q}_p$, but there is a lot of pseudo-differential operators. A typical example is Vladimirov’s fractional differentiation operator $D^\alpha, \alpha > 0$; see the details below. This operator (as well as its multidimensional generalization, the so-called Taibleson operator) is a $p$-adic counterpart of the fractional Laplacian $(-\Delta)^{\alpha/2}$ of real analysis.

Already in real analysis, an interpretation of nonlocal operators on bounded domains is not straightforward; see [3] for a survey of various possibilities. In the $p$-adic case, Vladimirov (see [19]) defined a version $D^\alpha_N$ of the fractional differentiation on a ball $B_N = \{x \in \mathbb{Q}_p : |x|_p \leq p^N\}$ as follows. One takes a test function on $B_N$, extends it onto $\mathbb{Q}_p$ by zero, applies $D^\alpha$, and restricts the resulting function to $B_N$. Then it is possible to consider a closure of the obtained operator, for example, on $L^2(B_N)$.

In [16] (Section 4.6), a probabilistic interpretation of this operator was given. Let $\xi_\alpha(t)$ be the Markov process with the generator $D^\alpha$ on $\mathbb{Q}_p$. Suppose that $\xi_\alpha(0) \in B_N$ and denote by $\xi^{(N)}_\alpha(t)$ the sum of all jumps of the process $\xi_\alpha(\tau), \tau \in [0, t]$ whose $p$-adic absolute values exceed $p^N$. Consider the process $\eta_\alpha(t) = \xi_\alpha(t) - \xi^{(N)}_\alpha(t)$. Due to the ultrametric inequality, the jumps of $\eta_\alpha$ never exceed $p^N$ by absolute value, so that the process remains almost surely in $B_N$. It is proved in [16] that the generator of the Markov process $\eta_\alpha$ on $B_N$ equals (on test function) $D^\alpha_N - \lambda I$ where

$$\lambda = \frac{p^{\alpha+1} - 1}{p^{\alpha(1-N)}}.$$

In [16] (Theorem 4.9) the corresponding heat kernel is given explicitly.

In this paper we find an analytic interpretation of the latter operator using harmonic analysis on $B_N$ as an (additive) compact Abelian group (this group property, just as the above probabilistic construction, is of purely non-Archimedean nature and has no analogs in the classical theory of partial differential equations). We give an interpretation of $D^\alpha_N - \lambda I$ as a pseudo-differential operator on $B_N$, then consider it as an operator on $L^1(B_N)$ and study its Green function, the integral kernel of its resolvent. The choice of $L^1(B_N)$ as the basic space is motivated by applications to nonlinear equations.

The first model example of a nonlinear parabolic equation over $\mathbb{Q}_p$ is the $p$-adic analog of the classical porous medium equation:

$$\frac{\partial u}{\partial t} + D^\alpha(\Phi(u)) = 0, \quad u = u(t, x), \quad t > 0, x \in \mathbb{Q}_p,$$

where $\Phi$ is a strictly monotone increasing continuous real function on $\mathbb{R}$. Its study was initiated in [13]. Here we consider this equation on $B_N$, taking the operator $D^\alpha_N$ instead of $D^\alpha$:

$$\frac{\partial u}{\partial t} + D^\alpha_N(\Phi(u)) = 0.$$
As in [13], our study of Eq. (1.2) is based on general results by Crandall – Pierre [10] and Brézis – Strauss [6] enabling us to consider this equation in the framework of nonlinear semigroups of operators. Following [3] we consider Eq. (1.2) also in $L^\gamma(B_N), 1 < \gamma \leq \infty$.

An important motivation of the present work is provided by the $p$-adic model of a poropus medium introduced in [14, 15].

2 Preliminaries

2.1. $p$-Adic numbers [19].

Let $p$ be a prime number. The field of $p$-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \text{ if } x = p^n \frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and $m, n$ are prime to $p$. $\mathbb{Q}_p$ is a locally compact topological field. By Ostrowski’s theorem there are no absolute values on $\mathbb{Q}$, which are not equivalent to the “Euclidean” one, or one of $| \cdot |_p$.

The absolute value $|x|_p, x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \text{ if and only if } x = 0;$$

$$|xy|_p = |x|_p \cdot |y|_p;$$

$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

The latter property called the ultra-metric inequality (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ in the topology determined by the metric $|x - y|_p$, as well as many unusual geometric properties. Note also the following consequence of the ultra-metric inequality: $|x + y|_p = \max(|x|_p, |y|_p)$, if $|x|_p \neq |y|_p$.

The absolute value $|x|_p$ takes the discrete set of non-zero values $p^N, N \in \mathbb{Z}$. If $|x|_p = p^N$, then $x$ admits a (unique) canonical representation

$$x = p^{-N} \left(x_0 + x_1 p + x_2 p^2 + \cdots\right), \quad (2.1)$$

where $x_0, x_1, x_2, \ldots \in \{0, 1, \ldots, p - 1\}, x_0 \neq 0$. The series converges in the topology of $\mathbb{Q}_p$. For example,

$$-1 = (p - 1) + (p - 1)p + (p - 1)p^2 + \cdots, \quad | - 1 |_p = 1.$$  

We denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. $\mathbb{Z}_p$, as well as all balls in $\mathbb{Q}_p$, is simultaneously open and closed.

Proceeding from the canonical representation (2.1) of an element $x \in \mathbb{Q}_p$, one can define the fractional part of $x$ as the rational number

$$\{x\}_p = \begin{cases} 
0, & \text{if } N \leq 0 \text{ or } x = 0; \\
0^{-N} \left(x_0 + x_1 p + \cdots + x_{N-1} p^{N-1}\right), & \text{if } N > 0.
\end{cases}$$

The function $\chi(x) = \exp(2\pi i \{x\}_p)$ is an additive character of the field $\mathbb{Q}_p$, that is a character of its additive group. It is clear that $\chi(x) = 1$ if and only if $|x|_p \leq 1$. 

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Denote by \( dx \) the Haar measure on the additive group of \( \mathbb{Q}_p \) normalized by the equality 
\[
\int_{\mathbb{Q}_p} dx = 1.
\]
The additive group of \( \mathbb{Q}_p \) is self-dual, so that the Fourier transform of a complex-valued function \( f \in L^1(\mathbb{Q}_p) \) is again a function on \( \mathbb{Q}_p \) defined as
\[
(\mathcal{F}f)(\xi) = \int_{\mathbb{Q}_p} \chi(x\xi) f(x) \, dx.
\]
If \( \mathcal{F}f \in L^1(\mathbb{Q}_p) \), then we have the inversion formula
\[
f(x) = \int_{\mathbb{Q}_p} \chi(-x\xi) \tilde{f}(\xi) \, d\xi.
\]

In order to define distributions on \( \mathbb{Q}_p \), we have to specify a class of test functions. A function \( f: \mathbb{Q}_p \to \mathbb{C} \) is called locally constant if there exists such an integer \( l \geq 0 \) that for any \( x \in \mathbb{Q}_p \)
\[
f(x + x') = f(x) \quad \text{if} \quad \|x'\| \leq p^{-l}.
\]
The smallest number \( l \) with this property is called the exponent of local constancy of the function \( f \).

Typical examples of locally constant functions are additive characters, and also cutoff functions like
\[
\Omega(x) = \begin{cases} 
1, & \text{if } \|x\| \leq 1; \\
0, & \text{if } \|x\| > 1. 
\end{cases}
\]
In particular, \( \Omega \) is continuous, which is an expression of the non-Archimedean properties of \( \mathbb{Q}_p \).

Denote by \( \mathcal{D}(\mathbb{Q}_p) \) the vector space of all locally constant functions with compact supports. Note that \( \mathcal{D}(\mathbb{Q}_p) \) is dense in \( L^q(\mathbb{Q}_p) \) for each \( q \in [1, \infty) \). In order to furnish \( \mathcal{D}(\mathbb{Q}_p) \) with a topology, consider first the subspace \( \mathcal{D}^l_N \subset \mathcal{D}(\mathbb{Q}_p) \) consisting of functions with supports in a ball
\[
B_N = \{ x \in \mathbb{Q}_p : \|x\|_p \leq p^N \}, \quad N \in \mathbb{Z},
\]
and the exponents of local constancy \( \leq l \). This space is finite-dimensional and possesses a natural direct product topology. Then the topology in \( \mathcal{D}(\mathbb{Q}_p) \) is defined as the double inductive limit topology, so that
\[
\mathcal{D}(\mathbb{Q}_p) = \lim_{N \to \infty} \lim_{l \to \infty} \mathcal{D}^l_N.
\]
If \( V \subset \mathbb{Q}_p \) is an open set, the space \( \mathcal{D}(V) \) of test functions on \( V \) is defined as a subspace of \( \mathcal{D}(\mathbb{Q}_p) \) consisting of functions with supports in \( V \). For a ball \( V = B_N \), we can identify \( \mathcal{D}(B_N) \) with the set of all locally constant functions on \( B_N \).

The space \( \mathcal{D}'(\mathbb{Q}_p) \) of Bruhat-Schwartz distributions on \( \mathbb{Q}_p \) is defined as a strong conjugate space to \( \mathcal{D}(\mathbb{Q}_p) \).

In contrast to the classical situation, the Fourier transform is a linear automorphism of the space \( \mathcal{D}(\mathbb{Q}_p) \). By duality, \( \mathcal{F} \) is extended to a linear automorphism of \( \mathcal{D}'(\mathbb{Q}_p) \). For a detailed
theory of convolutions and direct products of distributions on $\mathbb{Q}_p$ closely connected with the theory of their Fourier transforms see [1, 16, 19].

2.2. Vladimirov’s operator [1, 16, 19].

The Vladimirov operator $D^\alpha$, $\alpha > 0$, of fractional differentiation, is defined first as a pseudo-differential operator with the symbol $|\xi|_p^\alpha$:

$$(D^\alpha u)(x) = \mathcal{F}_x^{-1} \left[ |\xi|_p^\alpha \mathcal{F}_y u \right], \quad u \in \mathcal{D}(\mathbb{Q}_p), \quad (2.2)$$

where we show arguments of functions and their direct/inverse Fourier transforms. There is also a hypersingular integral representation giving the same result on $\mathcal{D}(\mathbb{Q}_p)$ but making sense on much wider classes of functions (for example, bounded locally constant functions):

$$(D^\alpha u)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha - 1}} \int_{\mathbb{Q}_p} |y|_p^{-\alpha - 1}[u(x - y) - u(x)] \, dy. \quad (2.3)$$

The Cauchy problem for the heat-like equation

$$\frac{\partial u}{\partial t} + D^\alpha u = 0, \quad u(0, x) = \psi(x), \quad x \in \mathbb{Q}_p, t > 0,$$

is a model example for the theory of $p$-adic parabolic equations. If $\psi$ is regular enough, for example, $\psi \in \mathcal{D}(\mathbb{Q}_p)$, then a classical solution is given by the formula

$$u(t, x) = \int_{\mathbb{Q}_p} Z(t, x - \xi)\psi(\xi) \, d\xi$$

where $Z$ is, for each $t$, a probability density and

$$Z(t_1 + t_2, x) = \int_{\mathbb{Q}_p} Z(t_1, x - y)Z(t_2, y) \, dy, \quad t_1, t_2 > 0, \quad x \in \mathbb{Q}_p.$$

The ”heat kernel” $Z$ can be written as the Fourier transform

$$Z(t, x) = \int_{\mathbb{Q}_p} \chi(\xi x)e^{-t|\xi|_p^\alpha} \, d\xi. \quad (2.4)$$

See [16] for various series representations and estimates of the kernel $Z$.

As it was mentioned in Introduction, the natural stochastic process in $B_N$ corresponds to the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} + (D_N^\alpha u)(t, x) - \lambda u(t, x) = 0, \quad x \in B_N, t > 0; \quad (2.5)$$

$$u(0, x) = \psi(x), \quad x \in B_N, \quad (2.6)$$

where the operator $D_N^\alpha$ is defined by restricting $D^\alpha$ to functions $u_N$ supported in $B_N$ and considering the resulting function $D^\alpha u_N$ only on $B_N$. Note that $D_N^\alpha$ defines a positive definite selfadjoint operator on $L^2(B_N)$, $\lambda$ is its smallest eigenvalue.
Under certain regularity assumptions, for example if $\psi \in \mathcal{D}(B_N)$, the problem (2.5)-(2.6) possesses a classical solution
\[ u(t, x) = \int_{B_N} Z_N(t, x - y)\psi(y) \, dy, \quad t > 0, \, x \in B_N, \]
where
\[ Z_N(t, x) = e^{\lambda t}Z(t, x) + c(t), \]
(2.7)
\[ c(t) = p^{-N} - p^{-N}(1 - p^{-1})e^{\lambda t} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}t^n\frac{p^{-N\alpha_n}}{1 - p^{-\alpha_n-1}}. \]

Another interpretation of the kernel $Z_N$ was given in [8].

It was shown in [13] that the family of operators
\[ (T_N(t)u)(x) = \int_{B_N} Z_N(t, x - y)\psi(y) \, dy \]
is a strongly continuous contraction semigroup on $L^1(B_N)$. Its generator $A_N$ coincides with $D_N^\alpha - \lambda I$ at least on $\mathcal{D}(B_N)$. More generally, this is true in the distribution sense on restrictions to $B_N$ of functions from the domain of the generator of the semigroup on $L^1(Q_p)$ corresponding to $D^\alpha$.

3 Harmonic analysis on the additive group of a $p$-adic ball

Let us consider the $p$-adic ball $B_N$ as a compact subgroup of $Q_p$. As we know, any continuous additive character of $Q_p$ has the form $x \mapsto \chi(\xi x), \xi \in Q_p$. The annihilator $\{ \xi \in Q_p : \chi(\xi x) = 1 \text{ for all } x \in B_N \}$ coincides with the ball $B_{-N}$. By the duality theorem (see, for example, [18], Theorem 27), the dual group $\widehat{B}_N$ to $B_N$ is isomorphic to the discrete group $Q_p / B_{-N}$ consisting of the cosets
\[ p^m(r_0 + r_1p + \cdots + r_{N-m-1}p^{N-m-1}) + B_{-N}, \quad r_j \in \{0, 1, \ldots, p-1\}, \quad m \in \mathbb{Z}, \, m < N. \]
(3.1)

Analytically, this isomorphism means that any nontrivial continuous character of $B_N$ has the form $\chi(\xi x), x \in B_N$, where $|\xi|_p > p^{-N}$ and $\xi \in Q_p$ is considered as a representative of the class $\xi + B_{-N}$. Note that $|\xi|_p$ does not depend on the choice of a representative of the class.

The normalized Haar measure on $B_N$ is $p^{-N} \, dx$. The normalization of the Haar measure on $Q_p / B_{-N}$ can be made in such a way (the normalized measure will be denoted $d\mu(x + B_{-N})$) that the equality
\[ \int_{Q_p} f(x) \, dx = \int_{Q_p / B_{-N}} \left( p^N \int_{B_{-N}} f(x + h) \, dh \right) \, d\mu(x + B_{-N}) \]
(3.2)
Fourier transform is extended, via duality, to the mapping from \( \mathcal{D}(\mathbb{Q}_p) \); see [4], Chapter VII, Proposition 10; [12], (28.54). With this normalization, the Plancherel identity for the corresponding Fourier transform also holds; see [12], (31.46)(c).

On the other hand, the invariant measure on the discrete group \( \mathbb{Q}_p/B_N \) equals \( \delta \)-measures concentrated on its elements multiplied by a coefficient \( \beta \). In order to find \( \beta \), it suffices to compute both sides of (3.2) for the case where \( f \) is the indicator function of the set \( \{ x \in \mathbb{Q}_p : |x - p^{-N-1}|_p \leq p^{-N} \} \). Then the left-hand side equals \( p^{-N} \) while the right-hand side equals \( \beta \). Therefore \( \beta = p^{-N} \).

The Fourier transform on \( B_N \) is given by the formula

\[
(\mathcal{F}_N f)(\xi) = p^{-N} \int_{B_N} \chi(x\xi) f(x) \, dx, \quad \xi \in (\mathbb{Q}_p \setminus B_{-N}) \cup \{0\},
\]

where the right-hand side, thus also \( \mathcal{F}_N f \), can be understood as a function on \( \mathbb{Q}_p/B_{-N} \).

The fact that \( \mathcal{F} : \mathcal{D}(\mathbb{Q}_p) \rightarrow \mathcal{D}(\mathbb{Q}_p) \), implies that \( \mathcal{F} \) maps \( \mathcal{D}(B_N) \) onto the set of functions on the discrete set \( \hat{B}_N \) having only a finite number of nonzero values. This set \( \mathcal{D}(\hat{B}_N) \) with a natural locally convex topology can be seen as the set of functions on \( \hat{B}_N = \mathbb{Q}_p/B_{-N} \). The conjugate space \( \mathcal{D}'(\hat{B}_N) \) consists of all functions on \( \hat{B}_N \) (see, for example, [11]). Therefore the Fourier transform is extended, via duality, to the mapping from \( \mathcal{D}'(B_N) \) to \( \mathcal{D}'(\hat{B}_N) \). A theory of distributions on locally compact groups covering the case of \( B_N \) was developed by Bruhat [7]. To study deeper the operator \( D^0_N \), we need, within harmonic analysis on \( B_N \), a construction similar to the well-known construction of a homogeneous distribution on \( \mathbb{Q}_p \), [19].

Let us introduce the usual Riez kernel on \( \mathbb{Q}_p \),

\[
f_\alpha^{(N)}(x) = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha-1}} |x|_p^{\alpha-1}, \quad \text{Re} \, \alpha > 0, \quad \alpha \not\equiv 1 \pmod{\frac{2\pi i}{\log p} \mathbb{Z}}.
\]

Using the formula [19]

\[
\int_{|x|_p \leq p^N} |x|_p^{\alpha-1} \, dx = \frac{1 - p^{-1}}{1 - p^{-\alpha}} p^{\alpha N},
\]

we introduce a distribution from \( \mathcal{D}'(B_N) \) setting

\[
\langle f_\alpha^{(N)}, \varphi \rangle = \frac{1 - p^{-1}}{1 - p^{-\alpha - 1}} p^{\alpha N} \varphi(0) + \frac{1 - p^{-\alpha}}{1 - p^{-\alpha - 1}} \int_{B_N} [\varphi(x) - \varphi(0)] |x|_p^{\alpha-1} \, dx, \quad \varphi \in \mathcal{D}(B_N).
\]

(3.3)

For \( \text{Re} \, \alpha > 0 \), this gives

\[
\langle f_\alpha^{(N)}, \varphi \rangle = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha - 1}} \int_{B_N} |x|_p^{\alpha-1} \varphi(x) \, dx.
\]

On the other hand, the distribution (3.3) is holomorphic in \( \alpha \not\equiv 1 \pmod{\frac{2\pi i}{\log p} \mathbb{Z}} \). Therefore \( f_\alpha^{(N)} \) makes sense for any \( \alpha > 0 \). Noticing that

\[
\frac{1 - p^{-1}}{1 - p^{-\alpha - 1}} p^{-\alpha N} = \frac{p - 1}{p^\alpha + 1 - p^{-\alpha N + \alpha}} = \lambda
\]
(see Introduction), so that
\[
\langle f^{(N)}_{-\alpha}, \varphi \rangle = \lambda \varphi(0) + \frac{1 - p^\alpha}{1 - p^{-\alpha - 1}} \int_{B_N} |\varphi(x) - \varphi(0)| x^{-\alpha-1} dx.
\] (3.4)

The emergence of \( \lambda \) in (3.4) “explains” its role in the probabilistic construction of a process on \( B_N \) ([16], Theorem 4.9).

**Theorem 1.** The operator \( D^\alpha_N \), \( \alpha > 0 \), acts from \( \mathcal{D}(B_N) \) to \( \mathcal{D}(B_N) \) and admits, for each \( \varphi \in \mathcal{D}(B_N) \), the representations:

(i) \( D^\alpha_N \varphi = f^{(N)}_{-\alpha} \ast \varphi \) where the convolution is understood in the sense of harmonic analysis on the additive group of \( B_N \);

(ii) \[
(D^\alpha_N \varphi)(x) = \lambda \varphi(x) + \frac{1 - p^\alpha}{1 - p^{-\alpha - 1}} \int_{B_N} |y|^{-\alpha-1}_p [\varphi(x - y) - \varphi(x)] dy, \quad \alpha > 0.
\]

(iii) On \( \mathcal{D}(B_N) \), \( D^\alpha_N - \lambda I \) coincides with the pseudo-differential operator \( \varphi \mapsto \mathcal{F}^{-1}_{N}(P_{N,\alpha}\mathcal{F}_N \varphi) \) where
\[
P_{N,\alpha}(\xi) = \frac{1 - p^\alpha}{1 - p^{-\alpha - 1}} \int_{B_N} |y|^{-\alpha-1}_p [\chi(y\xi) - 1] dy.
\] (3.5)

This symbol is extended uniquely from \( (\mathbb{Q}_p \setminus B_{-N}) \cup \{0\} \) onto \( \mathbb{Q}_p/B_{-N} \).

**Proof.** Denote, for brevity, \( a_p = \frac{1 - p^\alpha}{1 - p^{-\alpha - 1}} \). Let \( x \in B_N \). Assuming that \( \varphi \) is extended by zero onto \( \mathbb{Q}_p \), we find that
\[
(D^\alpha_N \varphi)(x) = a_p \int_{\mathbb{Q}_p} |y|^{-\alpha-1}_p [\varphi(x - y) - \varphi(x)] dy = I_1 + I_2 + I_3
\]
where
\[
I_1 = a_p \int_{B_N} |y|^{-\alpha-1}_p [\varphi(x - y) - \varphi(x)] dy,
\]
\[
I_2 = a_p \int_{|y|_p > p^N} |y|^{-\alpha-1}_p \varphi(x - y) dy,
\]
\[
I_3 = -a_p \varphi(x) \int_{|y|_p > p^N} |y|^{-\alpha-1}_p dy.
\]

We get using properties of \( p \)-adic integrals [19] that
\[
I_2 = a_p \int_{|x-z|_p > p^N} |x - z|^{-\alpha-1}_p \varphi(z) dz = a_p \int_{|z|_p > p^N} |z|^{-\alpha-1}_p \varphi(z) dz = 0;
\]

\[
I_3 = -a_p \varphi(x) \int_{|y|_p > p^N} dy = 0.
\]

\[
I_1 = a_p \int_{B_N} |y|^{-\alpha-1}_p [\varphi(x - y) - \varphi(x)] dy,
\]
\[
I_2 = a_p \int_{|y|_p > p^N} |y|^{-\alpha-1}_p \varphi(x - y) dy,
\]
\[
I_3 = -a_p \varphi(x) \int_{|y|_p > p^N} |y|^{-\alpha-1}_p dy.
\]
\[ I_3 = -a_p \varphi(x) \sum_{j=N+1}^{\infty} \int_{|y|=p^j} |y|^{-\alpha - 1} dy = -a_p \varphi(x)(1 - \frac{1}{p}) \sum_{j=N+1}^{\infty} p^{-\alpha j} = \lambda \varphi(x), \]

which implies (ii). Comparing with (3.4) we prove (i).

In order to prove (3.5) we note that

\[ \mathcal{F}_N \left( D_N^\alpha \varphi - \lambda \varphi \right) (\xi) = a_p p^{-N} \int_{B_N} \chi(x) dx \int_{B_N} |y|^{-\alpha - 1} [\varphi(x-y) - \varphi(x)] dy \]

\[ = a_p p^{-N} \int_{B_N} |y|^{-\alpha - 1} dy \int_{B_N} \chi(x) |\varphi(x-y) - \varphi(x)| dx = P_{n,\alpha} (\mathcal{F}_N \varphi) (\xi), \]

\[ \xi \in \mathbb{Q}_p/B_{-N}. \]

An important consequence of the representations given in Theorem 1 is the fact that, in contrast to operators on \( \mathbb{Q}_p \), \( D_N^\alpha : \mathcal{D}(B_N) \to \mathcal{D}(B_N) \), so that we can define in a straightforward way, the action of this operator on distributions. In particular, the pseudo-differential representation remains valid on \( \mathcal{D}'(B_N) \). Below (Theorem 3) this will be used to describe the domain of the operator \( A_N \) on \( L^1(B_N) \).

4 The Green function

In Section 2 (just as in [13]) we defined the operator \( A_N \) as the generator of the semigroup \( T_N \) on \( L^1(B_N) \). We can write its resolvent \( (A_N + \mu I)^{-1} \), \( \mu > 0 \), as

\[ ((A_N + \mu I)^{-1} u) (x) = \int_0^\infty e^{-\mu t} dt \int_{B_N} Z_N(t, x - \xi) u(\xi) d\xi, \quad u \in L^1(B_N), \]  

(4.1)

where \( Z_N \) is given in (2.7).

**Theorem 2.** The resolvent (4.1) admits the representation

\[ ((A_N + \mu I)^{-1} u) (x) = \int_{B_N} K_\mu(x - \xi) u(\xi) d\xi + \mu^{-1} p^{-N} \int_{B_N} u(\xi) d\xi, \quad u \in L^1(B_N), \mu > 0, \]  

(4.2)

where for \( 0 \neq x \in B_N \), \( |x|_p = p^m \),

\[ K_\mu(x) = \int_{p^{-N+1} \leq |\eta|_p \leq p^{-m+1}} \frac{\chi(\eta x)}{|\eta|^\alpha_p - \lambda + \mu} d\eta. \]  

(4.3)

If \( \alpha > 1 \), then for any \( x \in B_N \),

\[ K_\mu(x) = \int_{|\eta|_p \geq p^{-N+1}} \frac{\chi(\eta x)}{|\eta|^\alpha_p - \lambda + \mu} d\eta. \]  

(4.4)
The kernel $K_\mu$ is continuous for $x \neq 0$ and belongs to $L^1(B_N)$. If $\alpha > 1$, then $K_\mu$ is continuous on $B_N$. If $\alpha = 1$, then

$$|K_\mu(x)| \leq C|\log |x||^p, \quad x \in B_N.$$  \hfill (4.5)

If $\alpha < 1$, then

$$|K_\mu(x)| \leq C|x|^{\alpha - 1}, \quad x \in B_N.$$  \hfill (4.6)

**Proof.** Let us use the representation (2.7) substituting it into the equality

$$\int_{B_N} Z_N(t, x) \, dx = 1$$

(for the latter see Theorem 4.9 in [16]). We find that

$$c(t) = p^{-N} - e^\lambda p^{-N} \int_{B_N} Z(t, y) \, dy,$$

so that

$$Z_N(t, x) = e^\lambda \left[ Z(t, x) - p^{-N} \int_{B_N} Z(t, y) \, dy \right] + p^{-N}, \quad x \in B_N.$$

Let us consider the expression in brackets proceeding from the definition (2.4) of the kernel $Z$. Using the integration formula from Chapter 1, §4 of [19] we obtain that

$$Z(t, x) - p^{-N} \int_{B_N} Z(t, y) \, dy = I_1(t, x) + I_2(t, x)$$

where

$$I_1(t, x) = \int_{|\xi|_p \geq p^{-N+1}} \chi(\xi x) e^{-t|\xi|_p^p} \, d\xi,$$

$$I_2(t, x) = \int_{|\xi|_p \leq p^{-N}} [\chi(\xi x) - 1] e^{-t|\xi|_p^p} \, d\xi,$$

and $I_2(t, x) = 0$ for $x \in B_N$.

Let $|x|_p = p^m, \ m \leq N$. Then there exists such an element $\xi_0 \in \mathbb{Q}_p$, $|\xi_0|_p = p^{-m+1}$, that $\chi(\xi_0 x) \neq 0$. Then making the change of variables $\xi = \eta + \xi_0$ we find using the ultra-metric property that

$$\int_{|\xi|_p \geq p^{-m+2}} \chi(x\xi) e^{-t|\xi|_p^p} \, d\xi = \chi(x\xi_0) \int_{|\eta|_p \geq p^{-m+2}} \chi(x\eta) e^{-t|\eta|_p^p} \, d\eta,$$

so that

$$\int_{|\xi|_p \geq p^{-m+2}} \chi(x\xi) e^{-t|\xi|_p^p} \, d\xi = 0.$$
Therefore
\[ I_1(t, x) = \int_{p^{-N+1} \leq |\xi|_p \leq p^{-m+1}} \chi(x\xi) e^{-t|\xi|_p^\alpha} d\xi, \]
thus
\[ Z_N(t, x) = e^{\lambda t} \int_{p^{-N+1} \leq |\xi|_p \leq p^{-m+1}} \chi(x\xi) e^{-t|\xi|_p^\alpha} d\xi + p^{-N}, \quad |x|_p = p^m. \]

Substituting this in (4.1) and integrating in \( t \) we come to (4.2) and (4.3). Note that \( |\eta|_p^\alpha > \lambda \), as \( |\eta|_p \geq p^{-N+1} \).

If \( \alpha > 1 \), then the integral in (4.4) is convergent. For \( |x|_p = p^m \) we prove repeating the above argument that
\[ \int_{|\eta|_p \geq p^{-m+2}} \frac{\chi(\eta x)}{|\eta|_p^\alpha - \lambda + \mu} \, d\eta = 0. \]
Therefore in this case the representation (4.3) can be written in the form (4.4).

Obviously, \( K_\mu(x) \) is continuous for \( x \neq 0 \). If \( \alpha > 1 \), then there exists the limit
\[ \lim_{x \to 0} K_\mu(x) = \int_{|\eta|_p \geq p^{-N+1}} \frac{d\eta}{|\eta|_p^\alpha - \lambda + \mu} < \infty, \]
so that in this case \( K_\mu \) is continuous on \( B_N \).

Let \( \alpha < 1 \). By (4.3) and an integration formula from [19], Chapter 1, §4,
\[ K_\mu(x) = \sum_{l=-N+1}^{-m+1} \frac{1}{p^{\alpha l} - \lambda + \mu} \int_{|\xi|_p = p^l} \chi(\xi x) \, d\xi \]
\[ = (1 - \frac{1}{p}) \sum_{l=-N+1}^{-m} \frac{p^l}{p^{\alpha l} - \lambda + \mu} - \frac{p^{-m}}{p^{\alpha(-m+1)} - \lambda + \mu}, \quad |x|_p = p^m. \]

For some \( \gamma > 0 \), \( p^{\alpha l} - \lambda + \mu \geq \gamma p^{\alpha l} \). Computing the sum of a progression we obtain the estimate (4.6). Similarly, if \( \alpha = 1 \), then \( |K_\mu(x)| \leq C(-m + N) \), which gives, as \( m \to -\infty \), the inequality (4.5).

If \( \alpha > 1 \), we can also give an interpretation of the resolvent \( (A_N + \mu I)^{-1} \) in terms of the harmonic analysis on \( B_N \). We have
\[ (A_N + \mu I)^{-1} u = (K_\mu + \mu^{-1}1) * u, \quad u \in L^1(B_N), \quad (4.7) \]
where \( 1(x) \equiv 1, \ K_\mu \) is given by (4.4), and the convolution is taken in the sense of the additive group of \( B_N \).

Denote by \( \Pi_N \) the set of all rational numbers of the form
\[ p^l \left( \nu_0 + \nu_1 p + \cdots + \nu_{-l+N-1} p^{-l+N-1} \right), \quad l < N, \]
where $\nu_j \in \{0, 1, \ldots, p-1\}$, $\nu_0 \neq 0$. As a set, the quotient group $\mathbb{Q}_p/B_N$ coincides with $\Pi_N \cup \{0\}$, and
\[
\{ \xi \in \mathbb{Q}_p : |\xi|_p \geq p^{-N+1} \} = \bigcup_{\eta \in \Pi_N} (\eta + B_N)
\]
where the sets $\eta + B_N$ with different $\eta \in \Pi_N$ are disjoint.

Taking into account the fact that $\chi(\rho x) = 1$ for $x \in B_N$, $\rho \in B_N$, we find from (4.4) that
\[
K_\mu(x) = p^{-N} \sum_{0 \neq \eta \in \mathbb{Q}_p/B_N} \frac{\chi(\eta x)}{|\eta|_p^\alpha - \lambda + \mu}.
\]

Let us describe the domain $\text{Dom } A_N$ of the generator of our semigroup $T_N(t)$ on $L^1(B_N)$ in terms of distributions on $B_N$.

**Theorem 3.** If $\alpha > 1$, then the set $\text{Dom } A_N$ consists of those and only those $u \in L^1(B_N)$, for which $f_{-\alpha}^{(N)} * u \in L^1(B_N)$ where the convolution is understood in the sense of the distribution space $\mathcal{D}'(B_N)$. If $u \in \text{Dom } A_N$, then $A_N u = f_{-\alpha}^{(N)} * u - \lambda u$ where the convolution is understood in the sense of the distributions from $\mathcal{D}'(B_N)$.

**Proof.** Let $u = (A_N + \mu I)^{-1} f$, $f \in L^1(B_N)$, $\mu > 0$. Representing this resolvent as a pseudo-differential operator, we prove that $f_{-\alpha}^{(N)} * u - \lambda u + \mu u = f$ in the sense of $\mathcal{D}'(B_N)$.

Conversely, let $u \in L^1(B_N)$, $D_B^\alpha u = f_{-\alpha}^{(N)} * u \in L^1(B_N)$ where $D_B^\alpha$ is understood in the sense of $\mathcal{D}'(B_N)$. Set $f = (D_B^\alpha - \lambda I + \mu I) u$, $\mu > 0$. Denote $u' = (A_N + \mu I)^{-1} f$. Then $u' \in \text{Dom } A_N$, and the above argument shows that
\[
(D_B^\alpha - \lambda I + \mu I)(u - u') = 0.
\]

Applying the pseudo-differential representation we see that
\[
[P_{N, \alpha}(\xi) + \mu] [(\mathcal{F}_N u)(\xi) - (\mathcal{F}_N u')(\xi)] = 0, \quad \xi \in \mathbb{Q}_p/B_N.
\]

It is seen from (3.5) that the factor $P_{N, \alpha}(\xi) + \mu$ is real-valued, strictly positive and locally constant on $B_N$. Therefore the distribution $\mathcal{F}_N u - \mathcal{F}_N u'$ is zero. Since $\mathcal{F}_N$ is an isomorphism (see [7]), we find that $u = u'$, so that $u \in \text{Dom } A_N$. ■

### 5 Nonlinear equations

Let us consider the equation (1.2) where $\Phi$ is a strictly monotone increasing continuous real function, $\Phi(0) = 0$, and the linear operator $D_B^\alpha$ is understood as the operator $A_N + \lambda I$ on $L^1(B_N)$. By the results from [10] and [6], the nonlinear operator $D_B^\alpha \circ \Phi$ is $m$-accretive, which implies the unique mild solvability of the Cauchy problem for the equation (1.2) with the initial condition $u(0, x) = u_0(x)$, $u_0 \in L^1(B_N)$; see e.g. [2] for the definitions. As in the classical case [3], this mild solution can be interpreted also as a weak solution.

Following [3], we will show that the above construction of the $L^1$-mild solution gives also $L^\gamma$-solutions for $1 < \gamma \leq \infty$. 

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Theorem 4. Let \( u(t, x), t > 0, x \in B_N, \) be the above mild solution. If \( 0 < u_0 \in L^\gamma(B_N), \) \( 1 \leq \gamma \leq \infty, \) then \( u(t, \cdot) \in L^\gamma(B_N) \) and
\[
\|u(t, \cdot)\|_{L^\gamma(B_N)} \leq \|u_0\|_{L^\gamma(B_N)}. \tag{5.1}
\]

**Proof.** The case \( \gamma = 1 \) has been considered, while the case \( \gamma = \infty \) will be implied by the inequality (5.1) for finite values of \( \gamma \) (see Exercise 4.6 in [5]).

Thus, now we assume that \( 1 < \gamma < \infty. \) It is sufficient to prove (5.1) for \( u_0 \in D(B_N). \)

Indeed, if that is proved, we approximate in \( L^\gamma(B_N) \) an arbitrary function \( u_0 \in L^\gamma(B_N) \) by a sequence \( u_{0,j} \in D(B_N). \) For the corresponding solutions \( u_j(t, x) \) we have
\[
\|u_j(t, \cdot)\|_{L^\gamma(B_N)} \leq \|u_{0,j}\|_{L^\gamma(B_N)}. \tag{5.2}
\]

Since our nonlinear semigroup consists of operators continuous on \( L^1(B_N), \) we see that, for each \( t \geq 0, u_j(t, \cdot) \rightarrow u(t, \cdot) \) in \( L^1(B_N). \) By (5.2), the sequence \( \{u_j(t, \cdot)\} \) is bounded in \( L^\gamma(B_N). \) These two properties imply the weak convergence \( u_j(t, \cdot) \rightarrow u(t, \cdot) \) in \( L^\gamma(B_N) \) (see Exercise 4.16 in [5]).

Next, we use the weak lower semicontinuity of the \( L^\gamma \)-norm (see Theorem 2.11 in [17]), that is the inequality
\[
\liminf_j \|u_j(t, \cdot)\|_{L^\gamma(B_N)} \geq \|u(t, \cdot)\|_{L^\gamma(B_N)}.
\]
Passing to the lower limit in both sides of (5.2), we come to (5.1).

Let us prove (5.1) for \( u_0 \in D(B_N), 1 < \gamma < \infty. \) By the Crandall-Liggett theorem (see [2] or [9]), \( u(t, x) \) is obtained as a limit in \( L^1(B_N), \)
\[
u(t, \cdot) = \lim_{k \to \infty} \left( I + \frac{t}{k} D_N^\alpha \circ \Phi \right)^{-k} u_0,
\]
that is \( u(t, \cdot) = \lim_{k \to \infty} u_k \) where \( u_k \) are found recursively from the relation
\[
\frac{t}{k + 1} D_N^\alpha \circ \Phi(u_{k+1}) + u_{k+1} = u_k. \tag{5.3}
\]

Under our assumptions, \( u(t, x) > 0 \) (this follows from Theorem 4 in [10]). The nonlinear operator \( (I + \frac{t}{k} D_N^\alpha \circ \Phi)^{-1} \) is also positivity preserving (Proposition 1 in [10]), so that \( u_k > 0 \) for all \( k. \)

Note that the operator \( D_N^\alpha \) commutes with shifts while the equation (5.3) for \( u_{k+1} \) has a unique solution in \( L^1(B_N). \) As a result, if \( u_0 \in D(B_N), \) then all the functions \( u_k \) belong to \( D(B_N). \)

Rewriting (5.3) in the form
\[
\left( \frac{t}{k + 1} \right)^{-1} (u_{k+1} - u_k) = -D_N^\alpha \circ \Phi(u_{k+1}), \tag{5.4}
\]
multiplying both sides by \( u_{k+1}^{-1} \) and integrating on \( B_N \) we find that
\[
\left( \frac{t}{k + 1} \right)^{-1} \int_{B_N} (u_{k+1} - u_k) u_{k+1}^{-1} dx = - \int_{B_N} u_{k+1}^{-1} D_N^\alpha \circ \Phi(u_{k+1}) dx. \tag{5.5}
\]
Let \( w = u_{k+1}^{\gamma-1} \). Then \( w \in \mathcal{D}(B_N) \). It follows from (5.4) that \( D_N^\alpha \Phi(u_{k+1}) \in \mathcal{D}(B_N) \). Also we have \( \Phi(u_{k+1}) \in \mathcal{D}(B_N) \), so that \( \Phi(u_{k+1}) \) belongs to the domain of a selfadjoint realization of the operator \( D_N^\alpha \) in \( L^2(B_N) \). Therefore we can transform the integral in the right-hand side of (5.5) as follows:

\[
\int_{B_N} u_{k+1}^{\gamma-1} D_N^\alpha \circ \Phi(u_{k+1}) \, dx = \int_{B_N} \Phi(w^{\gamma-1}) D_N^\alpha(w) \, dx. \tag{5.6}
\]

The right-hand side of (5.6) is nonnegative by Lemma 2 of [3]. Now it follows from (5.5) that

\[
\int_{B_N} u_{k+1}^\gamma \, dx \leq \int_{B_N} u_k u_{k+1}^{\gamma-1} \, dx.
\]

Applying the Hölder inequality we find that

\[
\int_{B_N} u_{k+1}^\gamma \, dx \leq \left( \int_{B_N} u_k^\gamma \, dx \right)^{1/\gamma} \left( \int_{B_N} u_{k+1}^{\gamma-1} \, dx \right)^{\frac{\gamma-1}{\gamma}},
\]

which implies the inequality

\[
\|u_{k+1}\|_{L^\gamma(B_N)} \leq \|u_k\|_{L^\gamma(B_N)}
\]

and, by induction, the inequality

\[
\|u_{k+1}\|_{L^\gamma(B_N)} \leq \|u_0\|_{L^\gamma(B_N)}.
\]

Passing to the limit, we prove (5.1). \( \blacksquare \)

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