Affine Toda field theory on a half-line

E. Corrigan\textsuperscript{1}, P.E. Dorey\textsuperscript{1}, R.H. Rietdijk\textsuperscript{1}, R. Sasaki\textsuperscript{2}

\textsuperscript{1}Department of Mathematical Sciences
University of Durham, Durham DH1 3LE, England

\textsuperscript{2}Uji Research Center
Yukawa Institute for Theoretical Physics
Kyoto University, Uji 611, Japan

Abstract

The question of the integrability of real-coupling affine toda field theory on a half-line is addressed. It is found, by examining low-spin conserved charges, that the boundary conditions preserving integrability are strongly constrained. In particular, for the $a_n$ ($n > 1$) series of models there can be no free parameters introduced by the boundary condition; indeed the only remaining freedom (apart from choosing the simple condition $\partial_1 \phi = 0$), resides in a choice of signs. For a special case of the boundary condition, it is argued that the classical boundary bound state spectrum is closely related to a consistent set of reflection factors in the quantum field theory.

April 1994
1. Introduction

More than ten years ago Cherednik [1] formulated an algebraic approach to factorisable scattering on a half-line ($x^1 \leq 0$). The general set up, rephrased field theoretically, is as follows. Firstly, the dynamical system under consideration is integrable on the full line with all that entails in terms of factorisability of the S-matrix. Secondly, a natural assumption is that when restricted to the half line, the particle content (mass spectrum), and the S-matrices describing their mutual interactions, are exactly the same as those on the full line. Thirdly, when a particle hits the boundary it is assumed to be reflected elastically (up to rearrangements among mass degenerate particles). The compatibility of the reflections and the scatterings constitutes the main algebraic condition which generalises the Yang-Baxter equation. In other words, the effect of the boundary is local and coded into a set of reflection factors

$$|a, -\theta_a >_{\text{out}} = K_a(\theta_a)|a, \theta_a >_{\text{in}}, \quad (1.1)$$

where $a$ labels the particle, and $\theta_a$ is its rapidity.

More recently, Ghoshal and Zamolodchikov [2] have remarked that sine-Gordon theory on a half line may be quantum-integrable provided the boundary condition at $x^1 = 0$ is carefully chosen. Specifically, they checked that in addition to the energy (momentum is no longer conserved since translational symmetry is lost) there is a combination of spin $\pm 3$ charges which is also conserved, provided the boundary condition takes the form

$$\frac{\partial \phi}{\partial x^1} = \frac{a}{\beta} \sin \beta \left( \frac{\phi - \phi_0}{2} \right) \quad \text{at} \quad x^1 = 0, \quad (1.2)$$

where $a$ and $\phi_0$ are arbitrary constants, and $\beta$ is the sine-Gordon coupling constant. The condition (1.2) with $\phi_0 = 0$ or $\phi_0 = \pi/2$ has appeared in classical considerations of boundary terms by Cherednik, Sklyanin and Tarasov [1,3]. However, Ghoshal and Zamolodchikov have given reasons, based on a proposal for the scattering theory, for believing that the theory with a boundary condition can indeed depend on the extra parameter $\phi_0$.

One of the purposes of this letter is to ask a similar question in real coupling affine Toda field theory, in which it might be expected that the complications introduced by the boundary at $x^1 = 0$ are less severe than they are in sine-Gordon theory. This is because the latter has a degenerate pair of particles (the soliton and anti-soliton, distinguished only by topological charge), and a non-perturbative spectrum of breathers, whilst the former has a non-degenerate spectrum of real scalars with a correspondingly simple scattering
theory on the full line \[4,5,6,7,8\]. Some work has been done by Fring and Köberle \[9\] and by Sasaki \[10\] on the analysis of solutions to the real coupling affine Toda scattering, including reflections at the boundary. In this work, the Yang-Baxter equation plays no rôle and a bootstrap principle is invoked instead. However, as was pointed out in ref\[10\], there are infinitely many solutions to the relevant equations for these models. Until now, no attempt has been made to attribute any of them to the presence of specific boundary terms.

Affine Toda field theory \[4,11\] is a theory of \(r\) scalar fields in two-dimensional Minkowski space-time, where \(r\) is the rank of a compact semi-simple Lie algebra \(g\). The classical field theory is determined by the lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi) \tag{1.3}
\]

where

\[
V(\phi) = \frac{m^2}{\beta^2} \sum_{i=0}^r n_i e^{\beta \alpha_i \cdot \phi} \tag{1.4}
\]

In (1.4), \(m\) and \(\beta\) are real constants, \(\alpha_i\) \(i = 1, \ldots, r\) are the simple roots of the Lie algebra \(g\), and \(\alpha_0 = -\sum_i n_i \alpha_i\) is an integer linear combination of the simple roots; it corresponds to the extra spot on an extended (untwisted or twisted) Dynkin-Kac diagram for \(\hat{g}\). The coefficient \(n_0\) is taken to be one. For the theory on a half line, (1.3) will be replaced by

\[
\bar{\mathcal{L}} = \theta(-x^1) \mathcal{L} - \delta(x^1) \mathcal{B}, \tag{1.5}
\]

where \(\mathcal{B}\), a functional of the fields but not their derivatives, represents the boundary term. In other words, at the boundary \(x^1 = 0\)

\[
\frac{\partial \phi}{\partial x^1} = -\frac{\partial \mathcal{B}}{\partial \phi} \tag{1.6}
\]

There is some evidence, outlined below, to suggest that the generic form of the boundary term is given by

\[
\mathcal{B} = \frac{m}{\beta^2} \sum_{0}^{r} A_i e^{\frac{\beta}{2} \alpha_i \cdot \phi}, \tag{1.7}
\]

where the coefficients \(A_i\), \(i = 0, \ldots, r\) are a set of real numbers. The condition (1.7) is clearly a generalisation of (1.2). However, there are situations in which the coefficients are constrained. For example, for the affine Toda field theories based upon the \(a_n^{(1)}\) series of Lie algebras the sequence of conserved charges includes all spins (except zero) modulo
Except for $a_1^{(1)}$, which corresponds to the sinh-Gordon model, each of these theories has conserved charges of spin $\pm 2$. Requiring a combination of these to be preserved in the presence of the boundary term requires the form $\text{(1.7)}$ with the further constraint:

$$\text{either } |A_i| = 2, \text{ for } i = 0, \ldots, n \text{ or } A_i = 0 \text{ for } i = 0, \ldots, n. \quad (1.8)$$

On the other hand, requiring a combination of spin $\pm 3$ charges to be preserved leads to $\text{(1.7)}$ with no further constraints on the coefficients. This is perhaps surprising. Note however that the spin two (or other even spin) charges for the theories on the whole line play a special rôle since they distinguish particles from their antiparticles, a fact which follows from a general feature of the eigenvalues of the conserved quantities, namely:

$$q_s^\bar{a} = (-)^{s+1}q_s^a. \quad (1.9)$$

It is possible, therefore, that a generic boundary condition fails to distinguish between the two particles of a conjugate pair. Consequently, it is also expected that the reflection of particles from the boundary will not be diagonal unless the extra constraints $\text{(1.8)}$ are satisfied. If this curious behaviour as a function of the $A_i$ survives a full analysis of all the conserved charges, then it is reminiscent of behaviour at the reflectionless points of the sine-Gordon scattering matrices. At those points (which occur at special values of the coupling $\beta$) there are extra conserved charges which serve to distinguish the soliton from the anti-soliton, and cause the scattering to be diagonal.

Another feature of $\text{(1.7)}$ is that, generically, this choice of $B$ in $\text{(1.6)}$ does not permit $\phi = 0$ as a solution. The two exceptions to this are:

$$\text{either } A_i = A_n \text{ for } i = 0, \ldots, r \text{ or } A_i = 0 \text{ for } i = 0, \ldots, r. \quad (1.10)$$

As will be seen below, in the case of $a_n^{(1)}$ the first of these exceptions already has some interesting features in the sense of calculable boundary bound states.

2. Spin 2 charges on a half-line

There are sophisticated procedures, based on the existence of the Lax pair representation (see, for example, ref[12]) for obtaining the classical conserved quantities of affine Toda field theory. However, these do not appear to have been adapted to the half-line. Therefore, a pedestrian approach leading directly to $\text{(1.7)}$ and $\text{(1.8)}$ will be adopted here,
in the expectation that a fuller (and more satisfying) treatment will be found in the future. For related discussions of the problem on the full line, see for example refs [13].

The spin ±3 densities corresponding to the spin ±2 charges for the whole line may be described by the general formulae (using light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$):

$$T_{\pm 3} = \frac{1}{3} A_{abc} \partial_{\pm} \phi_a \partial_{\pm} \phi_b \partial_{\pm} \phi_c + B_{ab} \partial_{\pm}^2 \phi_a \partial_{\pm} \phi_b,$$

where the coefficients $A_{abc}$ are completely symmetric and the coefficients $B_{ab}$ are antisymmetric. For constructing conserved quantities, the densities must satisfy

$$\partial_{\pm} T_{\pm 3} = \partial_{\pm} \Theta_{\pm 1}$$

and explicit calculation reveals

$$\Theta_{\pm 1} = -\frac{1}{2} B_{ab} \partial_{\pm} \phi_a V_b, \quad V_b = \frac{\partial V}{\partial \phi_b};$$

with the constraint

$$A_{abc} V_a + B_{ab} V_{ac} + B_{ac} V_{ab} = 0.$$  

Eq(2.4) implies

$$\frac{1}{\beta} A_{ijk} + B_{ij} C_{ik} + B_{ik} C_{ij} = 0,$$

where it is useful to define

$$A_{ijk} = A_{abc}(\alpha_i)_a(\alpha_j)_b(\alpha_k)_c, \quad B_{ij} = B_{ab}(\alpha_i)_a(\alpha_j)_b;$$

and

$$C_{ij} = \alpha_i \cdot \alpha_j.$$  

Eq(2.5) implies that $B_{ij}$ is very restricted: it is non-zero only for the $a_n^{(1)}$ cases and, in those cases, $B_{ij} = 0$ except for $j = i \pm 1 \mod n + 1$, and $B_{i\pm 1} = B_{i\pm 1} = B_{i\pm 1} = B_{i\pm 1}$, $i = 1, \ldots, n + 1$.

Rewriting the conditions (2.2) in terms of the variables $x^0, x^1$,

$$\partial_0(T_{+3} - \Theta_{+1} \pm (T_{-3} - \Theta_{-1})) = \partial_1(T_{+3} + \Theta_{+1} \mp (T_{-3} + \Theta_{-1}));$$

implies that the combination $(T_{+3} - \Theta_{+1} + T_{-3} - \Theta_{-1})$ is a candidate density for a conserved quantity on the half-line if, at $x^1 = 0$,

$$(T_{+3} + \Theta_{+1} - T_{-3} - \Theta_{-1}) = \partial_0 \Sigma_2.$$  

5
Then, provided (2.8) is satisfied, the charge $P_2$, given by

$$P_2 = \int_{-\infty}^{0} dx^1 (T_{+3} - \Theta_{+1} + T_{-3} - \Theta_{-1}) - \Sigma_2$$

(2.9)

is conserved.

Eq (2.8) is a surprisingly strong condition. Together with the definitions (2.1) and (2.3), it is found that $\Sigma_2$ does not exist unless the following two conditions hold at $x^1 = 0$:

$$A_{abc}B_a + 2B_{ab}B_{ac} + 2B_{ac}B_{ab} = 0,$$

(2.10)

$$\frac{1}{3} A_{abc}B_aB_b + 2B_{ab}V_aB_b = 0.$$

(2.11)

Both of these involve the boundary term. Comparing (2.10) with (2.4) reveals that the boundary term $B$ must be equal to

$$\frac{m}{\beta^2} \sum_{i=0}^{r} A_i e^{\frac{2}{\beta^2} \alpha_i \cdot \phi},$$

apart from an additive arbitrary constant. The second condition, eq (2.11), is nonlinear in the boundary term and therefore provides equations for the constant coefficients $A_i$ in terms of the coefficients in the potential. To analyse these equations, the term in $A_{abc}$ is best eliminated using (2.5), to yield:

$$\frac{1}{24} \sum_{ijk} (B_{ij}C_{ik} + B_{ik}C_{ij})A_iA_jA_k e_i e_j e_k = \sum_{ij} B_{ij}A_j e_i^2 e_j,$$

(2.12)

where

$$e_i = e^{\frac{2}{\beta^2} \alpha_i \cdot \phi}.$$

Comparing the coefficients of the products of exponentials in (2.12) requires either $A_i = 0$ for all $i$, or, $A_i^2 = 4$ for all $i$.

### 3. Spin 3 charges on a half-line

The appropriate candidate density for a spin 3 charge on the half-line is

$$T_{+4} - \Theta_{+2} + T_{-4} - \Theta_{-2},$$
with a corresponding charge $P_3$ given by
\[ P_3 = \int_{-\infty}^{0} dx^1 (T_{+4} - \Theta_{+2} + T_{-4} - \Theta_{-2}) - \Sigma_3, \] (3.1)

provided
\[ T_{+4} + \Theta_{+2} - T_{-4} - \Theta_{-2} = \partial_0 \Sigma_3 \] (3.2)
at $x^1 = 0$. The starting point for the discussion is the expression
\[ T_{\pm 4} = \frac{1}{4} A_{abcd} \partial_{\pm} \phi_a \partial_{\pm} \phi_b \partial_{\pm} \phi_c \partial_{\pm} \phi_d + \frac{1}{2} B_{abc} \partial^2_{\pm} \phi_a \partial_{\pm} \phi_b \partial_{\pm} \phi_c + \frac{1}{2} D_{ab} \partial^2_{\pm} \phi_a \partial^2_{\pm} \phi_b, \] (3.3)
which corresponds to a conserved charge on the whole line provided that
\[ \Theta_{\pm 2} = \frac{1}{2} B_{abc} V_b \partial_{\pm} \phi_a \partial_{\pm} \phi_c + \frac{1}{2} D_{ab} V_{ac} \partial_{\pm} \phi_b \partial_{\pm} \phi_c, \] (3.4)

and
\[ B_{abc} V_b - B_{cab} V_b + D_{ba} V_{bc} - D_{bc} V_{ba} = 0 \]
\[ A_{abcd} V_a + \frac{1}{4} (B_{abe} V_{ad} + B_{acd} V_{ab} + B_{abd} V_{ac}) \]
\[ - \frac{1}{6} (D_{ad} V_{abc} + D_{ab} V_{acd} + D_{ac} V_{abd}) = 0. \] (3.5)
The analysis of (3.2) is quite complicated in this case. Firstly, there are conditions on the boundary term to match eqs(3.5):
\[ B_{abc} B_b - B_{cab} B_b + 2D_{ba} B_{bc} - 2D_{bc} B_{ba} = 0 \]
\[ A_{abcd} B_a + \frac{1}{2} (B_{abe} B_{ad} + B_{acd} B_{ab} + B_{abd} B_{ac}) \]
\[ - \frac{2}{3} (D_{ad} B_{abc} + D_{ab} B_{acd} + D_{ac} B_{abd}) = 0. \] (3.6)
These are clearly satisfied, as a consequence of (3.5), by the general boundary term (1.7).
Secondly, there is no non-linear condition to correspond to (2.11). This is because, for even spin densities, the left hand side of (3.2) contains no terms consisting merely of $x^1$ derivatives evaluated at the boundary. Hence, terms with a single factor $\partial_0 \phi$ have the opportunity of combining to the required form. A lengthy calculation reveals this to be the case, with no further restrictions on the coefficients $A_i$.

Finally, note that only one combination of the spin $\pm s$ charges can be conserved on the half-line and that the conserved combination is ‘parity even’: $P_s = Q_s + Q_{-s} - \Sigma_s$, where $Q_{\pm s}$ would be the usual conserved charges if the densities were integrated over the whole line.
4. Classical boundary bound states

With the suggested boundary condition (1.7), the equations of motion for the theory on a half-line become

\[ \partial^2 \phi = -\frac{m^2}{\beta} \sum_{i=0}^{r} n_i \alpha_i e^{2 \alpha_i \cdot \phi} \quad x^1 < 0 \]
\[ \partial_1 \phi = -\frac{m}{2\beta} \sum_{i=0}^{r} A_i \alpha_i e^{\frac{1}{2} \alpha_i \cdot \phi} \quad x^1 = 0. \]  

(4.1)

With the conventions adopted above, the total conserved energy is given by

\[ E = \int_{-\infty}^{0} \mathcal{E} dx + B, \]  

(4.2)

where \( \mathcal{E} \) is the usual energy density for Toda field theory. The competition between the two terms when \( B \) is negative permits the existence of boundary bound states.

The coupling constant \( \beta \) can be used to keep track of the scale of the Toda field \( \phi \), in which case it is appropriate to consider an expansion of the field as a power series in \( \beta \) of the following type:

\[ \phi = \sum_{k=-1}^{\infty} \beta^k \phi^{(k)}. \]  

(4.3)

Generally, the series starts at \( k = -1 \) since the leading term on the right hand side of the boundary condition is of order \( 1/\beta \), and may be non-zero. The first two terms of the series satisfy the equations:

\[ \partial^2 \phi^{(-1)} = -m^2 \sum_{i=0}^{r} n_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)}} \quad x^1 < 0 \]
\[ \partial_1 \phi^{(-1)} = -\frac{m}{2} \sum_{i=0}^{r} A_i \alpha_i e^{\frac{1}{2} \alpha_i \cdot \phi^{(-1)}} \quad x^1 = 0, \]  

(4.4)

and

\[ \partial^2 \phi^{(0)} = -m^2 \sum_{i=0}^{r} n_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)} \alpha_i \cdot \phi^{(0)}} \quad x^1 < 0 \]
\[ \partial_1 \phi^{(0)} = -\frac{m}{4} \sum_{i=0}^{r} A_i \alpha_i e^{\frac{1}{4} \alpha_i \cdot \phi^{(-1)} \alpha_i \cdot \phi^{(0)}} \quad x^1 = 0. \]  

(4.5)

Exceptionally, \( \phi^{(-1)} = 0 \) is a solution to (4.4) when the coefficients \( A_i \) are equal to \( An_i \). Otherwise, \( \phi^{(0)} \) satisfies a linear equation in the background provided by \( \phi^{(-1)} \). Since
\( \phi^{(-1)} \) represents the ‘ground’ state, it is expected to be time-independent and of minimal energy. For the \( a_n \) case, the ground state \( \phi^{(-1)} = 0 \) preserves the \( Z_{n+1} \) symmetry of the extended Dynkin diagram, and there is not expected to be a non-zero solution with the same symmetry.

If the coefficients may be chosen to be \( A_i = A n_i \), and the ground state is assumed to be \( \phi^{(-1)} = 0 \), eqs(4.3) reduce to a diagonalisable system whose solution in terms of eigenvectors \( \rho_a \) of the mass\(^2\) matrix may be written as follows:

\[
\phi^{(0)} = \sum_{a=1}^{r} \rho_a (R_a e^{-ip_a x^1} + I_a e^{ip_a x^1}) e^{-i\omega_a x^0},
\]

where

\[
M^2 \rho_a = m^2 \sum_{0}^{r} n_i \alpha_i \otimes \alpha_i \rho_a = m_a^2 \rho_a, \quad \omega_a^2 - p_a^2 = m_a^2,
\]

and the reflection factor, denoted \( K_a \) for consistency with some earlier references, is

\[
K_a = R_a / I_a = \frac{ip_a + Am_a^2/4m}{ip_a - Am_a^2/4m}, \quad a = 1, \ldots, r.
\]

If \( A = 0 \), it is clear from (4.7) that \( K_a = 1 \) and there are no exponentially decaying solutions to the linear system. On the other hand, if \( A \neq 0 \) the reflection coefficients (4.7) have poles at

\[
p_a = -i \frac{Am_a^2}{4m},
\]

for which

\[
\omega_a^2 = m_a^2 (1 - \frac{A^2 m_a^2}{16m^2}).
\]

Thus, provided \( A^2 < 16m^2/m_a^2 \) and \( A < 0 \), the channel labelled \( a \) has a bound state, with the corresponding solution to the linear system decaying exponentially away from the boundary as \( x^1 \to -\infty \).

For the case \( a_n^{(1)} \), it has already been established that \( A^2 = 4 \), and the masses for the affine Toda theory are known to be

\[
m_a = 2m \sin\left(\frac{a\pi}{n + 1}\right),
\]

Hence, with all the \( A_i = -2 \), there are bound states for each \( a \), with

\[
\omega_a^2 = 4m^2 \sin^2\left(\frac{a\pi}{n + 1}\right) (1 - \sin^2\left(\frac{a\pi}{n + 1}\right)) = m^2 \sin^2\left(\frac{2a\pi}{n + 1}\right).
\]
Notice that there is a characteristic difference between $n$ odd and $n$ even. In the latter case, the bound-state ‘masses’ are doubly degenerate, matching the degeneracy in the particle states themselves. However, in the former case there is a four-fold degeneracy in the bound-state masses, and $\omega_{(n+1)/2} = 0$.

One of the remarkable features of the quantum affine Toda field theories based on simply-laced Lie algebras is that the quantum mass spectrum is essentially the same as the classical mass spectrum. It is therefore tempting to suppose that a similar miracle will occur for the theories on a half-line, in which case the reflection factors corresponding to the special boundary condition $A_i = -2$ (in the case of $a_n^{(1)}$) will contain poles corresponding to the bound-state masses (4.9). Since the S-matrices are known, the reflection factors are strongly constrained (but not uniquely determined) by various bootstrap relations. These are described in refs.\[9,10], in which a number of solutions have been given. It is not intended to review this material here but rather to indicate that there are solutions which match the relatively simple boundary condition and boundary states described above.

5. Quantum boundary bound states

The simplest case to consider is $a_2^{(1)}$ which contains a conjugate pair of particles with masses given by

$$m_1 = m_2 = \sqrt{3}m.$$  \hfill (5.1)

The classical reflection factors are given by (4.7), with $A = -2$. It is useful to introduce the block notation (see ref\[5] for details)

$$\langle x \rangle = \frac{\sinh(\frac{\theta}{2} + i\pi x)}{\sinh(\frac{\theta}{2} - i\pi x)}; \hfill (5.2)$$

where $h$ is the Coxeter number of the Lie algebra (in this case $h = 3$). In this notation, the classical reflection factor may be rewritten as follows:

$$\frac{ip - \frac{3m}{2}}{ip + \frac{3m}{2}} = -(1)(2). \hfill (5.3)$$

In the same notation, the S-matrix elements are given by

$$S_{11}(\theta) = S_{22}(\theta) = \frac{(2)}{(B)(2 - B)}; \quad S_{12}(\theta) = S_{11}(i\pi - \theta) = -\frac{(1)}{(1 + B)(3 - B)}. \hfill (5.4)$$
where the parameter $B$ depends on the coupling constant and has been conjectured to be

$$B(\beta) = \frac{\beta^2/2\pi}{1 + \beta^2/4\pi},$$

and checked to one-loop order for all simply-laced affine Toda theories [14]. The boundary condition does not distinguish the two particles and, if the two reflection factors describing reflection of either particle off the ground state of the boundary are denoted $K_0^0(\theta)$ and $K_2^0(\theta)$, it is expected that

$$K_0^0(\theta) = K_2^0(\theta).$$

In addition, the bootstrap equation [9,2] consistent with the $11 \rightarrow 2$ coupling in the theory

$$K_2^0(\theta) = K_1^0(\theta - i\pi/3)K_1^0(\theta + i\pi/3)S_{11}(2\theta) \quad (5.5)$$

must be satisfied, as must the ‘crossing-unitarity’ relation [2]

$$K_1^0(\theta)K_2^0(\theta - i\pi) = S_{11}(2\theta). \quad (5.6)$$

Since the relation (5.6) follows automatically from the bootstrap relation (5.5) and its conjugate partner [10,9], it is in fact only necessary to verify the bootstrap relations.

On the basis of the discussion in the previous section, the reflection factors are expected to contain a fixed simple pole (at $\theta = i\pi/3$) indicating the existence of the boundary bound state expected in each channel at the mass $\sqrt{3}m/2$. It is also expected that as $\beta \rightarrow 0$ the reflection factors revert to the classical expression (5.3). A ‘minimal’ hypothesis with these properties is:

$$K_1^0(\theta) = K_2^0(\theta) = -\frac{(1)(2 + B)}{B}. \quad (5.7)$$

Remarkably, this simple ansatz satisfies both the requirements, (5.6) and (5.5), as is easily verified. As $\beta \rightarrow 0$, the $\beta$-dependent factors in (5.7) give the rapidity dependent factor (2) in the classical reflection factor (5.3). This expression is not invariant under the transformation $\beta \rightarrow 4\pi/\beta$, the weak-strong coupling symmetry characteristic of the quantum affine Toda theory on the whole line. Rather, as $\beta \rightarrow \infty$, $K_1^0 \rightarrow 1$.

Each channel has a boundary bound state (associated with the pole at $\theta = i\pi/3$), and it is convenient to label these $b_1$ and $b_2$. The boundary bootstrap equation [2] defines the reflection factors for the particles reflecting from the boundary bound states. If, as is being assumed here, there remain sufficiently many charges conserved in the presence of
the boundary to ensure that the reflection off the boundary is diagonal, then the equation given by Ghoshal and Zamolodchikov simplifies dramatically. If the scattering of particle \( a \) with the boundary state \( \alpha \) has a boundary bound state pole at \( \theta = iv_{a\alpha}^\beta \), then the reflection factors for the new boundary state are given by

\[
K^\beta_b(\theta) = S_{ab}(\theta - iv_{a\alpha}^\beta)S_{ab}(\theta + iv_{a\alpha}^\beta)K^\alpha_b(\theta).
\] (5.8)

Thus, for the case in hand, the four possibilities are

\[
\begin{align*}
K^{b_1}_1 &= S_{11}(\theta + i\pi/3)S_{11}(\theta - i\pi/3)K^0_1(\theta) = S_{12}(\theta)K^0_1 \\
K^{b_2}_1 &= S_{12}(\theta + i\pi/3)S_{12}(\theta - i\pi/3)K^0_2(\theta) = S_{11}(\theta)K^0_2
\end{align*}
\] (5.9)

and

\[
\begin{align*}
K^{b_2}_1 &= S_{12}(\theta + i\pi/3)S_{12}(\theta - i\pi/3)K^0_1(\theta) = S_{11}(\theta)K^0_1 \\
K^{b_2}_2 &= S_{11}(\theta + i\pi/3)S_{11}(\theta - i\pi/3)K^0_2(\theta) = S_{12}(\theta)K^0_2.
\end{align*}
\] (5.10)

Consider the fixed pole structure of eqs(5.9). Since both \( S_{12} \) and \( K^0_1 \) have a simple pole at \( \theta = i\pi/3 \), their product has a double pole; this is not to be interpreted as a new bound state. On the other hand, \( S_{11} \) has a simple pole at \( \theta = 2i\pi/3 \) and \( K^0_2 \) has a simple pole at \( \theta = i\pi/3 \); the first of these does not indicate a new boundary bound state since for that interpretation \( \theta \) ought to lie in the range \( 0 \leq \theta \leq i\pi/2 \). However, the second pole lies in the correct range and indicates a boundary state of mass \( \sqrt{3}m \). This state has all the quantum numbers of particle 1 (the state \( b_1 \) has the quantum numbers of particle 2 each multiplied by \( 1/2 \)), and may therefore be interpreted as a particle 1 state at zero momentum, next to the boundary in its ground state. Establishing the latter relies on the fact that the particle charges and the boundary state charges are related in the quantum field theory via

\[
P^\alpha_s \cos(sv_{a\alpha}^\beta) = P^\beta_s - P^\alpha_s.
\] (5.11)

Eqs(5.10) have a similar interpretation. Consequently, the complete boundary spectrum corresponding to the symmetrical boundary condition (1.7) with \( A_1 = A_2 = -2 \) consists of a ground state, a pair of boundary states, and a tower of states constructed by gluing zero rapidity particles to either the ground state or to the boundary states \( b_1, b_2 \).

On the other hand, if \( A_1 = A_2 = 2 \), the classical reflection data has no boundary bound states and the classical reflection coefficient (5.3) is replaced by its inverse. In this case, a candidate for the reflection factors in the quantum field theory is

\[
K^0_1(\theta) = K^0_2(\theta) = \frac{(3 - B^2)}{(2)(1 - B^2)}.
\] (5.12)

12
This clearly satisfies all the bootstrap conditions and there are no physical strip poles corresponding to boundary bound states. As $\beta \to \infty$, these reflection factors tend to unity.

In order to generalise (5.14) to other members of the $a_n$ series, it is useful to have some new notation. It is convenient to introduce a pair of new blocks:

$$< x > = \frac{(x + \frac{1}{2})}{(x - \frac{1}{2} + \frac{B}{2})}, \quad \tilde{< x >} = \frac{(x - \frac{1}{2})}{(x + \frac{1}{2} - \frac{B}{2})}. \quad (5.13)$$

These are related to the notation $[x]$ introduced in \[10\] via

$$[x] = < x > \tilde{< x >}. \quad (5.14)$$

In terms of (5.14), the quantities $S(2\theta)$ can be conveniently manipulated, since

$$\{x\}(2\theta) = [x](\theta)/[h - x](\theta), \quad (5.15)$$

where

$$\{x\} = \frac{(x - 1)(x + 1)}{(x - 1 + B)(x + 1 - B)}$$

is the basic building block from which all the S-matrices of simply-laced affine Toda field theories are constructed \[5\].

In terms of the new blocks, eq(5.7) may be rewritten as

$$K_0^1 = \frac{< \frac{1}{2} >}{< \frac{1}{2} >} = \frac{< \frac{1}{2} >}{< h - \frac{1}{2} >},$$

which is in a suitable form to generalise. Following the bootstrap, using it recursively to define all the other reflection factors, leads to the general expression

$$K_0^a = \frac{< a - \frac{1}{2} >}{< h - a + \frac{1}{2} >} \frac{< a - 1 - \frac{1}{2} >}{< h - a + 1 + \frac{1}{2} >} \cdots \frac{< \frac{1}{2} >}{< h - \frac{1}{2} >} = K_0^{h-a}. \quad (5.16)$$

Moreover,

$$K_0^a \to -(a)(h - a), \quad \beta \to 0$$

and, for each $a$, $K_0^a \to 1$ as $\beta \to \infty$. The limit $\beta \to 0$ yields the classical reflection factor (4.7), corresponding to particle $a$ in the field theory based on $a_n^{(1)}$. The generalisation of (5.12) is obtained by replacing $< x >$ by $\tilde{< x >}$ in (5.16).
6. Summary

Arguments have been given which strongly suggest that affine Toda field theory will remain integrable on a half-line, provided the boundary condition is carefully chosen. For the $a_n$ series of Toda theories, the spin two charges provide a strong constraint. If they are to be conserved in the presence of the boundary, the boundary condition has at most a discrete ambiguity. Without these conserved charges, some particles would not be distinguished from their anti-particles in the quantum theory. For a particularly symmetrical form of the boundary condition, the classical boundary bound states have been used as a guide to the construction of simple reflection factors, consistent with all the requirements of the bootstrap, and, from these, a picture of the spectrum of the theory has been built. This was given in detail for the case $a_2$, but may be deduced from the general expressions for $K^0_a$ in the other cases. However, as may be inferred from ref[10], these solutions are not unique and the picture based on them is therefore tentative.

There are many questions left unanswered. For example, the general expressions (5.16) contain a variety of poles not all of which correspond to boundary bound states. Those that cannot be interpreted in terms of boundary states require an explanation in terms of Landau-type singularities, similar to the Coleman-Thun mechanism [15], which served to explain the multiple poles in the S-matrix itself [4]. For example, referring back to eq(5.9), the double pole in $K^0_1$ at $\theta = i\pi/3$ can be understood in this way. However, a full analysis of these singularities will need perturbation theory to be set up on the half-line.

Finally, it is also unclear whether every boundary condition preserving combinations of classically conserved charges will preserve quantum integrability. One logical possibility is that the affine Toda quantum theory with a boundary must respect the affine diagram symmetry, in which case the variety of boundary conditions would be greatly reduced (for example (1.2) would not be allowed except for $\phi_0 = 0$). Another possibility is that a symmetry breaking of this type, being local to the boundary, influences the form of the reflection factors, eq(1.1), but otherwise, as originally envisaged by Cherednik, preserves the particle scattering far from the boundary.

7. Acknowledgements

One of us (EC) wishes to thank the British Council and the Japan Society for the Promotion of Science for the opportunity to meet another of us (RS). RH Rietdijk wishes to thank the United Kingdom Science and Engineering Research Council for postdoctoral support.
References

[1] I. V. Cherednik, ‘Factorizing particles on a half line and root systems’, Theor. Math. Phys. 61 (1984) 977.

[2] S. Ghoshal and A. B. Zamolodchikov, ‘Boundary S matrix and boundary state in two-dimensional integrable quantum field theory’, Rutgers preprint RU-93-20; hep-th/9306002.

S. Ghoshal, ‘Boundary state boundary S matrix of the sine-Gordon model’, Rutgers preprint RU-93-51; hep-th/9310188.

[3] E. K. Sklyanin, ‘Boundary conditions for integrable equations’, Funct. Anal. Appl. 21 (1987) 164;

E. K. Sklyanin, ‘Boundary conditions for integrable quantum systems’, J. Phys. A21 (1988) 2375;

V. O. Tarasov, ‘The integrable initial-value problem on a semiline: nonlinear Schrödinger and sine-Gordon equations’, Inverse Problems 7 (1991) 435.

[4] A. E. Arinshtein, V. A. Fateev, A. B. Zamolodchikov, ‘Quantum S-matrix of the 1+1 dimensional Toda chain’, Phys. Lett. B87 (1979) 389.

[5] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, ‘Affine Toda field theory and exact S-matrices’, Nucl. Phys. B338 (1990) 689;

H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, ‘Multiple poles and other features of affine Toda field theory’, Nucl. Phys. B356 (1991) 469.

[6] P. Christe and G. Mussardo, ‘Integrable systems away from criticality: the Toda field theory and S matrix of the tricritical Ising model’, Nucl. Phys. B330 (1990) 465;

P. Christe and G. Mussardo, ‘Elastic S-matrices in (1+1) dimensions and Toda field theories’, Int. J. Mod. Phys. A5 (1990) 4581.

[7] P. E. Dorey, ‘Root systems and purely elastic S-matrices, I & II’, Nucl. Phys. B358 (1991) 654; Nucl. Phys. B374 (1992) 741.

[8] G. W. Delius, M. T. Grisaru and D. Zanon, ‘Exact S-matrices for non simply-laced affine Toda theories’, Nucl. Phys. B282 (1992) 365;

E. Corrigan, P. E. Dorey and R. Sasaki, ‘On a generalised bootstrap principle’, Nucl. Phys. B408 (1993) 579–599.

[9] A. Fring and R. Köberle, ‘Factorized scattering in the presence of reflecting boundaries’, Sao Paulo preprint USP-IFQSC-TH-93-06; hep-th/9304141.

A. Fring and R. Köberle, ‘Affine Toda field theory in the presence of reflecting boundaries’, Sao Paulo preprint USP-IFQSC-TH-93-12; hep-th/9309142.

[10] R. Sasaki, ‘Reflection bootstrap equations for Toda field theory’, Kyoto preprint YITP/U-93-33; hep-th/9311027.

[11] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, ‘Two-dimensional generalised Toda lattice’, Comm. Math. Phys. 79 (1981) 473.
[12] V. G. Drinfel’d and V. V. Sokolov, *J. Sov. Math.* 30 (1984) 1975;
D. I. Olive and N. Turok, ‘The Toda lattice field theory hierarchies and zero-curvature conditions in Kac-Moody algebras’, *Nucl. Phys.* B265 (1986) 469.

[13] G. W. Delius, M. T. Grisaru and D. Zanon, ‘Quantum conserved currents in affine Toda theories’, *Nucl. Phys.* B385 (1992) 307;
M. R. Niedermaier, ‘The quantum spectrum of conserved charges in affine Toda theories’, Munich preprint MPI-Ph/93-92; hep-th/9401078.

[14] H. W. Braden and R. Sasaki, ‘The S-matrix coupling dependence for a, d and e affine Toda field theory’, *Phys. Lett.* B255 (1991) 343;
R. Sasaki and F. P. Zen, ‘The affine Toda S-matrices vs perturbation theory’, *Int. J. Mod. Phys.* 8 (1993) 115.

[15] S. Coleman and H. Thun, *Comm. Math. Phys.* 61 (1978) 31.