Quasi-cyclic Flexible Regenerating Codes

Bernat Gastón, Jaume Pujol, and Mercè Villanueva
Department of Information and Communications Engineering, Universitat Autònoma de Barcelona
{Bernat.Gaston | Jaume.Pujol | Merce.Villanueva}@uab.cat

Abstract

In a distributed storage environment, where the data is placed in nodes connected through a network, it is likely that one of these nodes fails. It is known that the use of erasure coding improves the fault tolerance and minimizes the redundancy added in distributed storage environments. The use of regenerating codes not only make the most of the erasure coding improvements, but also minimizes the amount of data needed to regenerate a failed node.

In this paper, a new family of regenerating codes based on quasi-cyclic codes is presented. Quasi-cyclic flexible minimum storage regenerating (QCFMSR) codes are constructed and their existence is proved. Quasi-cyclic flexible regenerating codes with minimum bandwidth constructed from a base QCFMSR code are also provided. These codes not only achieve optimal MBR parameters in terms of stored data and repair bandwidth, but also for an specific choice of the parameters involved, they can be decreased under the optimal MBR point.

Quasi-cyclic flexible regenerating codes are very interesting because of their simplicity and low complexity. They allow exact repair-by-transfer in the minimum bandwidth case and an exact pseudo repair-by-transfer in the MSR case, where operations are needed only when a new node enters into the system replacing a lost one.

I. INTRODUCTION

The availability problem of stored data is an essential issue, which has been studied intensively lately [1]. The increasing use of distributed storage systems (DSS), like cloud storage, to a massive scale has changed the paradigm of data storage. The amount of stored data in a DSS must be minimized and the problem of storage device failures must be addressed.

The motivation for using erasure coding in DSS comes from the need to keep the information available even when storage device failures occur and, at the same time, to reduce the stored amount of data. Erasure codes in DSS allow to achieve high fault tolerance (resistance to storage device failures without losing information), requiring less storage overhead (redundancy added to repair the files) that the one required by a data replication scheme [2], also known as backups. There are the obvious cost savings from purchasing less hardware to store the data, but there are also significant savings from the fact that this also reduces data centers size, the power for running less hardware, among with other savings [3].

The use of erasure coding in DSS minimizes the amount of data stored in the system, but introduces what is known as the code repair problem [4]: how to maintain the encoded representation given by the erasure code when storage device failures occur, minimizing the amount of stored data per node \( \alpha \) and the bandwidth used to regenerate one node \( \gamma \). To maintain the same redundancy when a storage node (device) leaves the system, a newcomer has to
join the system, access some existing nodes, download data from them, and replace the lost node. This operation is not an exception, in fact, it is usual.

This paper is organized as follows. In Section II, the repair problem is exposed and some of the most known techniques used to minimize it are shown. In Section III the quasi-cyclic flexible minimum storage regenerating codes are constructed, and their bounds and properties are proved. In Section IV a technique used to produce minimum bandwidth codes from minimum storage codes is formally explained and it is applied to the proposed minimum storage codes to produce quasi-cyclic flexible regenerating codes with minimum bandwidth. These codes are compared with other regenerating codes with the same parameters. Finally, in Section V the conclusions of this work are exposed.

II. MINIMIZING THE REPAIR PROBLEM

There are various techniques used to minimize the bandwidth needed to regenerate a failed node. In this section, we define the problem, and we review some of the most well known techniques used in DSS to solve this problem.

Let $M$ be the size of a file. In a replication scheme, if each node stores $\alpha$ data units, with $\alpha \ll M$, the newcomer must download $\alpha$ data units to replace the lost node. However, in an erasure coding scheme, the same newcomer must download $M$ data units to store only $\alpha$.

Denote $\mathbb{F}_q$ the finite field of $q$ elements. A linear code is a linear map from $\mathbb{F}_q^k$ to $\mathbb{F}_q^n$ which converts vectors of $k$ coordinates over $\mathbb{F}_q$ to vectors of $n$ coordinates over the same field called codewords. Note that $k$ is the dimension and $n$ the length of the linear code, so the rate of the code is $R = k/n$. Let $d$ be the minimum distance of the linear code. It is well known that $d \leq n - k + 1$ [5]. When one or more coordinates of a codeword are lost, it is possible to obtain the original vector over $\mathbb{F}_q^k$ by using an algorithm that requires at least $n - d + 1$ correct coordinates of the codeword, this procedure is called decoding. We can say that a $[n, k, d]$ linear code encodes $k$ symbols of $\mathbb{F}_q$ into $n$ symbols of $\mathbb{F}_q$ and repairs $d - 1$ erasures. From now on, we assume that the codes used are linear and are designed to repair erasures. The classical decoding using linear codes always needs and uses $n - d + 1$ correct coordinates no matter if the codeword has one or more (up to $d - 1$) erasures. Now, the next question arises, is it possible to repair one single erasure requiring less than $n - d + 1$ coordinates?

Take a file of size $M$ and split it into $k$ pieces of size $M/k$ over $\mathbb{F}_q$ organized as a vector $v = (v_1, \ldots, v_k)$. Encode this vector using a $[n, k, d]$ code which produces a codeword $c = (c_1, \ldots, c_n)$. Assume that each one of the coordinates $c_i$ is stored in a different storage node $s_i$, $i = 1, \ldots, n$, of a DSS. Note that any subset of $n - d + 1$ of these nodes is enough to recover the file. Moreover, the fault tolerance of the DSS is $d - 1$, since it is the maximum number of node failures that can be tolerated by the DSS, where tolerated means that the DSS is still able to recover any piece of the original stored information. If the code used is a Maximum Distance Separable (MDS) code [5], which is considered the best code in terms of storage efficiency, the amount of redundancy in the system is minimum for a given fault tolerance and $d = n - k + 1$, so the file can be recovered by connecting and downloading the data from any $k$ storage nodes.

The goal is to minimize the amount of stored data per node $\alpha$ and the repair bandwidth $\gamma$, that is, the amount
of data required to regenerate a node. These parameters $\alpha$ and $\gamma$ can be reduced by using at least two different techniques, which will be summarized in the next two subsections.

A. Locally Repairable Codes

Let $C$ be an $[n, k, d]$ linear code over $\mathbb{F}_q$. We say that a coordinate $i$ of $C$ has repair degree $r_i$ if we can recover any symbol at coordinate $i$ by accessing at least $r_i$ other codewords symbols. The repair degree of $C$, $r$, is the maximum of $r_i$, $i = 1, \ldots, n$, and $C$ is called an $r$ locally repairable code ($r$-LRC). Note that in a $r$-LRC any message symbol can be recovered by accessing at most $r$ other codeword symbols.

In [6], it is shown that the minimum distance $d$ of a code is upper bounded by $d \leq n - k - \lceil \frac{k}{r} \rceil + 2$, which means that as $r$ increases (approaching to $k$) $d$ decreases. It is also known that it is a good metric for repair cost [7], [8]. Moreover, note that the MDS codes have degree $r = k$. Locally repairable codes (LRC) try to keep $r$ at very low rates, this means that for LRC it is a goal that $r \ll k$. Moreover, the LRC codes are not MDS.

B. Regenerating Codes

Let $\mathbb{F}_q$ be a base field with $q$ elements and let $\mathbb{F}_{q^t}$ be an extended field of $\mathbb{F}_q$ with $q^t$ elements. This means that each element of $\mathbb{F}_{q^t}$ is composed by $t$ elements of $\mathbb{F}_q$ called coordinates. Using an $[n, k, d]$ code over $\mathbb{F}_{q^t}$, we can encode an information vector $v \in \mathbb{F}_{q^t}^k$ and produce a codeword $c \in \mathbb{F}_{q^t}^n$. This kind of codes are called array codes, and each coordinate of $v$ or $c$ is called an array coordinate. A well known example of an array code is the EVENODD code [9].

In [10], a solution to minimize the bandwidth used to regenerate a failed node is proposed. This solution consists of using regenerating codes. The main idea of this kind of codes is the use of a network multicast technique for data transmission optimization, called network coding [11], in conjunction with array codes. The regenerating codes use linear codes seen over the extended field to minimize the amount of stored data and use network coding and the codes seen over the base field, to minimize the bandwidth used to regenerate a failed node.

Let $C$ be a $[n, k, r]$ regenerating code, where the length $n$ is the total number of nodes in the system; the dimension $k$ is the value such that any $k$ nodes contain the minimum amount of information necessary to reconstruct the file; and the cardinal of the set of helper nodes $r$ is the number of nodes necessary to regenerate one failed node. Let $v \in \mathbb{F}_{q^t}^k$ be the information vector representing the file to be stored in a DSS. Let $c \in \mathbb{F}_{q^t}^n$ be the corresponding codeword after encoding $v$ using $C$. Note that if each array coordinate of $c$ is stored in a node, and the code used is MDS, the parameters $n, k, d$ of the regenerating code coincide with the parameters $n, k, d$, of the linear code. Moreover, in this case, $C$ is called a Minimum Storage Regenerating (MSR) code, since it minimizes the storage overhead and $d = n - k + 1$. However, it is also possible to store more data in the same storage node, for example by adding an extra set of elements over the base field, producing more redundancy, which could be used to regenerate a failed node requiring less repair bandwidth. Using this technique, it is possible to minimize the repair problem at the cost of some extra overhead but maintaining $d = n - k + 1$. When $C$ achieves the minimum $\alpha$ such
that $\alpha = \gamma$, $C$ is called a Minimum Bandwidth Regenerating (MBR) code, and it can be seen that the parameters $n$, $k$ and $d$ do not coincide with the ones defined for a linear code.

Regenerating codes assume the data reconstruction condition: any $k$ nodes must be enough to recover the file, which means that the minimum distance must be $d = n - k + 1$, so it is necessary to have $\alpha k \geq M$. Moreover, if $d = n - k + 1$ and $\alpha k = M$, we have an MDS code. Another condition is that the regeneration of any node in the system must require less repair bandwidth than the total file size $M$, that is $\gamma < M$. If each helper node sends $\beta$ data units, the repair bandwidth used is $\gamma = \beta r$. Then, $r$ must achieve $k < r < n$, unlike in LRC, since the set of possible newcomers can not be greater than $n - 1$, and if $r = k$, there is no possible optimization in the repair bandwidth. As we will see, in order to decrease the repair bandwidth by connecting to $r$ nodes, each helper node must send less than $\alpha$ data units, so they send only a linear combination of their coordinates over the base field.

Note the difference between LRC and regenerating codes. In both cases, the repair degree is the number of helper nodes necessary to regenerate a failed node. However, in LRC, each coordinate of a codeword $c$ is stored in one node, and when we access to one helper node it means that we download the entire coordinate contained in it. Therefore, in order to decrease $\gamma$ we need $r \ll k$. In regenerating codes, to maintain the distance $d = n - k + 1$, we need $r \geq k$. Thus, $c$ is composed by array coordinates, the solution is to increase $r$ but downloading less than an entire array coordinate from each helper node. In other words, LRC are the result of creating new codes adapted to a distributed storage systems, while regenerating codes are classical codes in which a network coding technique is used to reduce the repair bandwidth.

In [10], the authors modelize the life of a DSS with a graph called information flow graph. This graph has three types of weighted edges: internal storage edges that represent the amount of data stored per node $\alpha$; regenerating edges that represent the amount of data sent by each one of the helper nodes $\beta$; and edges that connect the Source (the file to be stored) and the Data Collector (the user who access the file) with the storage nodes. From this graph, one can compute the mincut between the Source ($S$) and the Data Collector ($DC$) denoted by $\text{mincut}(S, DC)$ and, after an optimization process, minimize $\alpha$ and $\gamma$ resulting in a threshold function.

Despite it is not the aim of this paper to focus on information flow graphs, Figure II illustrates one of these information flow graphs for a $[4, 2, 3]$ regenerating code. In general, the mincut equation is $\text{mincut}(S, DC) \geq \sum_{i=0}^{k-1} \min((r-i)\beta, \alpha) \geq M$, which means that for each newcomer up to $k$, it is composed by either the weight of the set of regenerating edges or the weight of the internal storage edge of the newcomer. Then, the optimization over $\alpha$ and $\gamma$ gives the function

$$
\alpha^*(r, \gamma) = \begin{cases} 
\frac{M}{k}, & \gamma \in [f(0), +\infty) \\
\frac{M - g(i)\gamma}{k - r}, & \gamma \in [f(i), f(i - 1)) \\
i = 1, \ldots, k - 1,
\end{cases}
$$

where

$$
f(i) = \frac{2Mr}{(2k - i - 1)i + 2k(r - k + 1)} \quad \text{and}
$$
Figure 1. Information flow graph corresponding to a $[4, 2, 3]$ regenerating code.

Figure 2. Optimal tradeoff curve between $\alpha$ and $\gamma$ for a $[10, 5, 9]$ regenerating code.

$$g(i) = \frac{(2r - 2k + i + 1)i}{2r}.$$ 

For an specific set of parameters $n$, $k$ and $r$ in function (1), it is possible to find the optimal tradeoff curve representing the minimization of $\alpha$ and $\gamma$ as shown in Figure 2 for a $[10, 5, 9]$ regenerating code. The optimal parameters of the MSR code are given by the point with the minimum $\alpha$, while the optimal parameters of the MBR code are given by the point with the minimum $\gamma$. Between these two extremal points, there exist some internal points which represent the limits of the intervals that appear in the threshold function (1).

To achieve this tradeoff curve, multiple techniques have been used, like interference alignment [12], product-matrix construction [13], or designs [14], among others. Behind these techniques, there are two main ideas: the use of array codes, which allows the DSS to treat the data inside a node as a set of small independent coordinates over the base field; and the use of network coding to send linear combinations of these coordinates through the network.

If any newcomer is able to exactly replicate the lost node, we say that the DSS has the exact repair property. Otherwise, if the newcomers store a linear combination that does not reduce the dimension of $C$ but it is not exactly the same as the included in the lost node, we say that the DSS has the functional repair property [15]. Exact repair is much more desirable than functional repair, since despite the number of failed nodes that the DSS has repaired over an interval of time, it is possible to use systematic encoding of the information and keep this systematic representation over the time. This means that there is always one accessible copy of the original file.
stored in the DSS. It is worth to mention that in [16] it is proved that the interior points of the tradeoff curve are not achievable using exact repair.

We say that a DSS has the un-coded repair property if it is possible to replace a failed node without doing any linear operation in the newcomer neither in the helper nodes. There exist uncoded constructions for the MBR point like the ones shown in [13] and [14]. However, for the MSR point, there only exist uncoded constructions using functional repair [17].

III. QUASI-CYCLIC FLEXIBLE MSR CODES

In this section, we describe the quasi-cyclic flexible minimum storage regenerating (QCFMSR) codes in detail. Specifically, in Subsection III-A, we show how to construct them and some of their properties; in Subsection III-B, we see how to regenerate a failed node; in Subsection III-C, we prove their existence by showing that the data reconstruction condition is achieved; and finally, in Subsection III-D, we describe an example of a [6, 3, 4] QCFMSR code.

A. Code Construction

Let $C$ be an array code of length $n = 2k$ and dimension $k$ over $\mathbb{F}_{q^2}$ constructed from the nonzero coefficients $\zeta_1, \ldots, \zeta_k$ over $\mathbb{F}_q$, and for which the encoding is done over the base field $\mathbb{F}_q$ in the following way. An information vector $v \in \mathbb{F}^k_{q^2}$ is seen as a vector $v = (v_1, \ldots, v_n)$ over $\mathbb{F}_q$, and is encoded into $c \in \mathbb{F}^n_{q^2}$ seen as a codeword over $\mathbb{F}_q$, where the redundancy coordinates $\rho_1, \ldots, \rho_n$ are given by the following equation:

$$
\rho_i = \sum_{j=i+1}^{k+i} \zeta_{j-i} v_j \quad i = 1, \ldots, n,
$$

where $\zeta_l \in \mathbb{F}_q \setminus \{0\}$ for $l = 1, \ldots, k$ and $j = i + 1, \ldots, k + i \mod n$. The rate of the code is $R = 1/2$ and the encoding over $\mathbb{F}_q$ is done by using a quasi-cyclic code [5] as we will see later. Quasi-cyclic codes are known by their simplicity for encoding-decoding operations.

A [2k, k, r] QCFMSR code over $\mathbb{F}_{q^2}$ is a regenerating code constructed from the array code $C$. Take a file of size $M$ and split it into $k$ pieces over $\mathbb{F}_{q^2}$, or equivalently, into $n = 2k$ pieces over $\mathbb{F}_q$ organized as a vector $v = (v_1, \ldots, v_n)$ over $\mathbb{F}_q$. The [2k, k, r] QCFMSR code over $\mathbb{F}_{q^2}$ is composed by a set of $n = 2k$ storage nodes, denoted by $\{s_1, s_2, \ldots, s_n\}$, where each storage node $s_i$, $i = 1, \ldots, n$, stores two coordinates over $\mathbb{F}_{q^2}$, $(v_i, \rho_i)$, which can be seen as one array coordinate over $\mathbb{F}_{q^2}$. The size of each coordinate over $\mathbb{F}_q$ is $M/2k$ and the size of each array coordinate stored in $s_i$ is $\alpha = M/k$.

Let $S$ be the set of all subsets of $\{1, \ldots, n\}$ of size $k$. Let $D$ be an $n \times n$ matrix over $\mathbb{F}_q$ and let $s = \{i_1, \ldots, i_k\} \in S$. Let $D^{(s)}$ denote the $i$th column vector of $D$ and $D^s$ denote the $n \times k$ submatrix of $D$ given by the $k$ columns determined by the set $s$.
Let $F = (I|Z)$ be a $n \times 2n$ matrix, where $I$ is the $n \times n$ identity matrix, and $Z$ is a $n \times n$ circulant matrix defined from the nonzero coefficients $\zeta_1, \ldots, \zeta_k$ as follows:

$$Z = \begin{pmatrix} 0 & 0 & \cdots & 0 & \zeta_k & \zeta_{k-1} & \cdots & \zeta_1 \\ \zeta_1 & 0 & \cdots & 0 & \zeta_k & \cdots & \zeta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \zeta_k & \zeta_{k-1} & \cdots & \zeta_1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_k & \cdots & \zeta_2 & \zeta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta_k & \zeta_{k-1} & \zeta_{k-2} & \cdots & 0 \end{pmatrix}. \tag{3}$$

The matrix $F$ represents the array code $C$, so also the QCFMSR code constructed from $C$. Each row is the encoding of one coordinate over the base field $\mathbb{F}_q$, and each node is represented by two columns, one from $I$ and another one from $Z$. Actually, the node $s_i$, which stores $(v_i, \rho_i)$, is also given by

$$(v_i, \rho_i) = (v_{I^i}, vZ^i).$$

Note that the information coordinates are represented by the identity matrix $I$, while the redundancy coordinates are represented by the circulant matrix $Z$.

Circulant matrices have been deeply studied because of their symmetric properties [18]. Moreover, $F$ can be seen as a generator matrix of a double circulant code over $\mathbb{F}_q$ [5]. Double circulant codes are a special case of quasi-cyclic codes, which are a family of quadratic residue codes. Quasi-cyclic codes have already been used for distributed storage [19], which points out the significance of these codes for DSS.
Figure 3 shows the construction of a QCFMSR code. First, the file is split into $n$ symbols over $\mathbb{F}_q$. Then, these symbols are encoded using $F$ and producing $2n$ symbols over $\mathbb{F}_q$. Finally, each two symbols are stored together in one node, this creates the array code with coordinates over $\mathbb{F}_q$ that can be seen as array coordinates over $\mathbb{F}_{q^2}$.

**B. Node Regeneration**

In this subsection, we show how to regenerate a failed node $s_i$, which stores $(v_i, \rho_i)$, minimizing the required repair bandwidth. Actually, we can just follow the next algorithm:

1) Download the information coordinates $v_j$, $j = i + 1, \ldots, i + k \mod n$, from the next $k$ nodes. Note that due to the circulant scheme, the next node of $s_n$ is $s_1$. From these information coordinates, compute the redundancy coordinate $\rho_i$ of the newcomer.

2) Download the redundancy coordinate $\rho_{i-1}$ from the previous node, following the same circulant scheme. Solving a simple equation, obtain the information coordinate $v_i$ of the newcomer.

It can be seen that $r = r_i = k + 1$ for any $s_i, i = 1, \ldots, n$, and when the repair problem is faced, it is clear that QCFMSR codes are optimal in terms of the tradeoff curve given by the threshold function (1) for $r = k + 1$. Note that QCFMSR codes are in fact a family of regenerating codes because $r > k$. However, unlike regenerating codes, for these flexible regenerating codes the set of $r$ helping nodes is not any but an specific set of remaining nodes with cardinality $r$. In other words, the set of nodes which is going to send data to an specific newcomer is fixed.

Note that QCFMSR codes have also the exact repair property, which means that once encoded, the information and the redundancy can be represented for the whole life of the DSS by $c = (v_1, \ldots, v_n, \rho_1, \ldots, \rho_n)$, where $v_i$ and $\rho_i$ are the information and redundancy coordinates, respectively. It is shown in [20] and [12] that when $r < 2k - 3$, exact MSR codes do not exist. However, QCFMSR codes exist for $r = k + 1$ which satisfies $r < 2k - 3$ for $k > 4$. These facts illustrate the importance of the flexibility over the set of helper nodes in this construction. Moreover, despite QCFMSR codes do not achieve uncoded repair, they are very efficient regenerating one node, because they need only two simple operations on the newcomer and no operation on the helper nodes.

**C. Data Reconstruction**

In Subsection IIA we have seen that $M = \alpha k$. In this subsection, we prove that the array code over $\mathbb{F}_{q^2}$, used to construct a QCFMSR code, satisfies that $d = n - k + 1$ for some $\zeta_1, \ldots, \zeta_k$ and, as a consequence, QCFMSR codes are MDS codes over $\mathbb{F}_{q^2}$ applied to DSS, so they are MSR codes. In [21], we performed a computational search to claim the existence of QCFMSR codes. In this paper, we prove their existence theoretically.

Let $F^s = (I^s|Z^s)$ denote the $n \times n$ submatrix of $F$ determined by $s = \{i_1, \ldots, i_k\} \in S$. Let $p_s(\zeta_1, \zeta_2, \ldots, \zeta_k) \in \mathbb{F}_q[\zeta_1, \ldots, \zeta_k]$ be the multivariate polynomial associated with the determinant of $F^s = (I^s|Z^s)$.

Assume that a DC wants to obtain the file. Then, it connects to any $k$ nodes $\{s_{i_1}, \ldots, s_{i_k}\}$ and downloads $(v_{i_1}, \rho_{i_1}), \ldots, (v_{i_k}, \rho_{i_k})$, so the DC is downloading the encoding given by $F^s$. In order to obtain the file given by $v = (v_1, v_2, \ldots, v_n)$, we need $F^s$ to be full rank. Moreover, in order to satisfy the data reconstruction condition, we need $F^s$ to be full rank for all $s \in S$. Therefore, we have to prove the following two statements:
Lemma 3. Let $T \subseteq W_c$ such that $T \neq \emptyset$, then $|E(T)| \geq |T|$. 

Proof: Note that $W_c$ has $k$ vertices of degree 1 and $k$ vertices of degree $k$. We can decompose $T = T_1 \cup T_2$, where $T_1$ contains the vertices of degree 1 and $T_2$ contains the vertices of degree $k$. It is clear that $|E(T_1)| = |T_1|$ by construction, and it is easy to see that $|E(T_2)| \geq k + |T_2| - 1 \geq |T_2|$ by the circular construction of matrix $Z$ and because $k > 1$. Therefore, we can assume that $T_1 \neq \emptyset$ and $T_2 \neq \emptyset$.

If $|T_1| \leq k - 1$, since $|E(T_1) \cap E(T_2)| \leq |E(T_1)| = |T_1|$, we have that $|E(T)| = |E(T_1)| + |E(T_2) - E(T_1) \cap E(T_2)| \geq |T_1| + k + |T_2| - 1 - |T_1| \geq |T_1| + |T_2| = |T|$. On the other hand, if $|T_1| = k$, then $|E(T_1) \cap E(T_2)| \leq |T_1| - 1$ since for each different vertex $t_i \in T_2$, there exists a different vertex $t_j \in T_1$ such that $E(t_i) \cap E(t_j) = \emptyset$. Thus, we also have that $|E(T)| = |E(T_1)| + |E(T_2)| - |E(T_1) \cap E(T_2)| \geq |T_1| + k + |T_2| - 1 - |T_1| + 1 \geq |T_1| + |T_2| = |T|$. 

1) The polynomial associated with the determinant of $F^s$ is not identically zero, which means that, when it is expanded as a summation of terms, there exists at least one term with a nonzero coefficient.

2) The polynomial $p_s(\zeta_1, \zeta_2, \ldots, \zeta_k)$ associated with the determinant of $F^s$ is nonzero with high probability, for a random choice of the nonzero coefficients $\zeta_1, \ldots, \zeta_k$.

We begin by proving the first statement. It is known that there exists a relation between determinants of matrices and bipartite graphs. Let $G(W_r \cup W_c, E)$ be the bipartite graph associated with a matrix $F^s$, where each row of the matrix is represented by a vertex $w_{c_i}$ in $W_r$, and each column of $F^s$ is represented by a vertex $w_{c_i}$ in $W_c$, where $i = 1, \ldots, n$. Two vertices $w_{c_i} \in W_r, w_{c_j} \in W_c$ are adjacent if the entry in the row $i$ and column $j$ of $F^s$ is nonzero. Moreover, the weight of this edge is the nonzero value of this $i, j$th entry. Let $E(w_{c_i})$ (resp. $E(w_{c_j})$) denote the neighbors of $w_{c_i}$ (resp. $w_{c_j}$) in the graph $G$. Let $T = \{t_1, \ldots, t_m\} \subseteq W_c$ be a subset of vertices of $W_c$ or $T \subseteq W_r$ be a subset of vertices of $W_r$ indistinctly. Let $E(T)$ denote the set $\bigcup_{i=1}^{m} E(t_i)$.

**Lemma 1** (22). The polynomial associated with the determinant of $F^s$, $p_s(\zeta_1, \zeta_2, \ldots, \zeta_k)$, is not identically zero if and only if the bipartite graph $G(W_r \cup W_c, E)$ associated with $F^s$ has a perfect matching.

To prove the existence of a perfect matching in the bipartite graph $G(W_r \cup W_c, E)$ associated with $F^s$, we will use Hall’s theorem.

**Lemma 2** (23). A bipartite graph $G(W_r \cup W_c, E)$ contains a complete matching from $W_r$ to $W_c$ (resp. $W_c$ to $W_r$) if and only if it satisfies Hall’s condition, that is, for any $T \subseteq W_c$ (resp. $T \subseteq W_r$), $|T| \leq |E(T)|$. Moreover, if $|W_r| = |W_c|$, the complete matching is achieved in both directions, so it corresponds to a perfect matching.

**Lemma 3.** Let $T \subseteq W_c$ such that $T \neq \emptyset$, then $|E(T)| \geq |T|$. 

**Proof:** Note that $W_c$ has $k$ vertices of degree 1 and $k$ vertices of degree $k$. We can decompose $T = T_1 \cup T_2$, where $T_1$ contains the vertices of degree 1 and $T_2$ contains the vertices of degree $k$. It is clear that $|E(T_1)| = |T_1|$ by construction, and it is easy to see that $|E(T_2)| \geq k + |T_2| - 1 \geq |T_2|$ by the circular construction of matrix $Z$ and because $k > 1$. Therefore, we can assume that $T_1 \neq \emptyset$ and $T_2 \neq \emptyset$.

If $|T_1| \leq k - 1$, since $|E(T_1) \cap E(T_2)| \leq |E(T_1)| = |T_1|$, we have that $|E(T)| = |E(T_1)| + |E(T_2)| - |E(T_1) \cap E(T_2)| \geq |T_1| + k + |T_2| - 1 - |T_1| \geq |T_1| + |T_2| = |T|$. On the other hand, if $|T_1| = k$, then $|E(T_1) \cap E(T_2)| \leq |T_1| - 1$ since for each different vertex $t_i \in T_2$, there exists a different vertex $t_j \in T_1$ such that $E(t_i) \cap E(t_j) = \emptyset$. Thus, we also have that $|E(T)| = |E(T_1)| + |E(T_2)| - |E(T_1) \cap E(T_2)| \geq |T_1| + k + |T_2| - 1 - |T_1| + 1 \geq |T_1| + |T_2| = |T|$. 

Figure 4. A [6, 3, 4] QCFMSR code with coordinates over $\mathbb{F}_q$ and array coordinates over $F_q^2$. 

May 2, 2014 DRAFT
following the construction given in Subsection III-A, it gives a QCFMSR code. Moreover, if more than one storage node fails, up to complexity in contrast with the one for Reed-Solomon codes, which has quadratic complexity [5].

D. Example

In this subsection, we describe the construction of a [6, 3, 4] QCFMSR code over $\mathbb{F}_{5^2}$.
First, the file is fragmented into 6 information coordinates $v = (v_1, v_2, \ldots, v_6)$. Then, each $v_i$ for $i = 1, \ldots, 6$ is stored in a node $s_i = (v_i, \rho_i)$, along with its corresponding redundancy symbol $\rho_i$, which is computed using a quasi-cyclic matrix $F$ of the following form:

$$F = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & \zeta_3 & \zeta_2 & \zeta_1 \\
\zeta_1 & 0 & 0 & \zeta_3 & \zeta_2 & \zeta_1 \\
\zeta_2 & \zeta_1 & 0 & 0 & \zeta_3 & \zeta_1 \\
\zeta_3 & \zeta_2 & \zeta_1 & 0 & 0 & \zeta_1 \\
0 & \zeta_3 & \zeta_2 & \zeta_1 & 0 & 0 \\
0 & 0 & \zeta_3 & \zeta_2 & \zeta_1 & 0 \\
\end{pmatrix}. \quad (4)
$$

By construction, the node regenerating condition is always achieved. In order to satisfy the data reconstruction condition, we need to find nonzero coefficients $\zeta_1, \zeta_2, \zeta_3$ such that $p(\zeta_1, \zeta_2, \zeta_3) \neq 0$ over $\mathbb{F}_5$. Since $p(\zeta_1, \zeta_2, \zeta_3) = \zeta_1^2 \zeta_2^2 \zeta_3^2 (-\zeta_2^2 + \zeta_3 \zeta_1) (\zeta_3^3 + \zeta_1^3) (-\zeta_1^2 \zeta_3 + \zeta_2^2 \zeta_3) (-\zeta_3^4 - \zeta_1^4)$, a possible solution over $\mathbb{F}_5$ is $(\zeta_1, \zeta_2, \zeta_3) = (1, 1, 2)$. Figure 4 shows the distribution of information and redundancy coordinates in the nodes. Each array coordinate over $\mathbb{F}_{2^q}$ is represented by one storage node. It can be seen that $d = 4 - 2 + 1$ and that $\alpha k = M$, so the quasi-cyclic flexible code is a MSR code.

Using the same argument, it is also possible to construct a $[6, 3, 4]$ QCFMSR code over $\mathbb{F}_{8^2}$ with nonzero coefficients $(\zeta_1, \zeta_2, \zeta_3) = (1, 1, z)$ over $\mathbb{F}_8$, where $z$ is a primitive element of this field. Note that there is not any $[6, 3, 4]$ QCFMSR code over $\mathbb{F}_{2^p}$, $\mathbb{F}_{3^p}$, $\mathbb{F}_{4^p}$ and $\mathbb{F}_{7^p}$.

**IV. Regenerating Codes with Minimum Bandwidth**

In this section, we show how the QCFMSR codes are converted from minimum storage codes to minimum bandwidth codes by using the same techniques used in [14] and [16], where MDS codes are used as base codes to construct minimum bandwidth codes. Note that the minimum bandwidth is achieved with the minimum $\alpha$ such that $\alpha = \gamma$. Moreover, we give some bounds and conditions for the existence of these minimum bandwidth codes constructed using QCFMSR as base codes. Specifically, in Subsections IV-A, IV-B and IV-C we show how to construct these codes in general and their properties; and in Subsection IV-D we focus on the quasi-cyclic case.

**A. Code Construction**

Let $C$ be a $[n, k, r]$ MSR code over $\mathbb{F}_{q^t}$, where any subset of $k$ storage nodes is enough to reconstruct the file. In this subsection, we explain how to construct a new $[\bar{n}, \bar{k}, \bar{r}]$ regenerating code $\bar{C}$ with minimum bandwidth over $\mathbb{F}_{q^t}$, from the base code $C$. Figure 5 shows an example of a minimum bandwidth regenerating code created from a $[6, 3, 4]$ QCFMSR code, which can be used to illustrate this construction. Despite it is possible to construct $\bar{C}$ from any regenerating code, the construction makes sense if the regenerating code $C$ is MSR, because then, $\bar{C}$ achieves the optimal parameters $\alpha$ and $\gamma$.
Lemma 7. Given $n \geq 3$, there exists a simple, undirected, and $\bar{r}$-regular graph $H(W, E)$, where $W$ is the set of vertices where $|W| = \bar{n}$ and $E$ is the set of edges where $|E| = n$, satisfying the following conditions: $|W| = \bar{n}$, $|E| = n$, $1 < \bar{r} < \bar{n}$, and $\bar{r}\bar{n} = 2n$.

Proof: Condition $\bar{r}\bar{n} = 2n$ is given by the Handshaking lemma for a simple, undirected, and $\bar{r}$-regular graph. By Erdos-Gallai degree sequence theorem, for $\bar{r} > 1$ and $\bar{r} < \bar{n}$, there exists at least a simple, undirected, and $\bar{r}$-regular graph such that $\bar{r}\bar{n} = 2n$.

For example, for $\bar{r} = \bar{n} - 1$, we have the complete graph $H = K_{\bar{n}}$, and for $\bar{r} = 2$ we have the cycle graph $H = C_{\bar{n}}$. As $|E| = n$ in $H(W, E)$, it is possible to assume that each edge in $E$ corresponds to a different coordinate $c_j$ over $\mathbb{F}_q$ of a codeword $c = (c_1, \ldots, c_n) \in C$. Note that $c_j$ could be also seen as an array coordinate over the base field $\mathbb{F}_q$, but in this section, we consider $c_j$ just as a coordinate over $\mathbb{F}_q$. Given a codeword $c \in C$, since $|W| = \bar{n}$ in $H(W, E)$, we can construct a codeword $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_{\bar{n}}) \in \bar{C}$, where each array coordinate $\bar{c}_i$ corresponds to a different vertex $w_i \in W$ and contains the coordinates of $c$ given by the $\bar{r}$ edges incident to $w_i$. Moreover, since the graph is simple, any two vertices can not be connected by more than one edge, so each coordinate of $c$ is contained in two array coordinates of $\bar{c}$. As $C$ is defined over $\mathbb{F}_q$, $\bar{C}$ is defined over $\mathbb{F}_{q^{\bar{r}}}$.

In the next subsections, we prove that $\bar{C}$ is a regenerating code with minimum bandwidth. Firstly, in Subsection [IV-B] we show that $\gamma = \alpha$. Then, in Subsection [IV-C], we look for the minimum $\alpha$ such that any $\bar{k}$ array coordinates of $\bar{c} \in \bar{C}$ are enough to reconstruct the file. Note that $\bar{C}$ is a regenerating code, but not a code from the classical point of view, since $|\bar{C}| = |C|$ and $\bar{k}$ is not the dimension of the code but an integer such that $1 < \bar{k} < \bar{n}$.

Figure 5. Construction of a $[6, 2, 2]$ quasi-cyclic flexible regenerating code with minimum bandwidth from a $[6, 3, 4]$ QCFMSR code.
B. Node Regeneration

Assume that a storage node fails, which is the same as erasing one array coordinate \(c_i, i = 1, \ldots, n\) of a codeword \(c = (c_1, \ldots, c_n) \in \hat{C}\), or equivalently one vertex \(\omega_i \in W\) of \(H(W, E)\). The newcomer can replace the failed node by downloading and storing the \(\bar{r}\) coordinates of \(c\) included in each one of the \(\bar{r}\) neighbors of \(\omega_i\), and given by the corresponding \(\bar{r}\) edges incidents to \(\omega_i\). According to this regeneration process, \(\gamma = \alpha\).

Note that these regenerating codes with minimum bandwidth, have the exact repair and the uncoded repair properties. Also note that the node regeneration is given by an specific subset of \(\bar{r}\) helper nodes, so they can also be called flexible regenerating codes.

C. Data Reconstruction

Let \(E(\omega_i), i = 1, \ldots, \bar{n}\), be the set of edges incident to the vertex \(\omega_i \in W\). Let \(\bar{S}\) be the set of all subsets of \(\{1, \ldots, \bar{n}\}\) of size \(\bar{k} > 1\) and let \(\bar{s} \in \bar{S}\). For a subset \(\bar{s}\), \(|\bigcup_{i \in \bar{s}} E(w_i)| = \sum_{i \in \bar{s}} |E(w_i)| - \theta_{\bar{s}}\), where \(\theta_{\bar{s}}\) represents the intersection terms in the inclusion-exclusion formula. Since each edge is incident to two vertices, for \(\bar{s}' \subseteq \bar{s}\) of size \(|\bar{s}'| > 2\), \(|\bigcap_{i \in \bar{s}} E(w_i)| = 0\), so \(\theta_{\bar{s}} = \sum_{i < j, \ i, j \in \bar{s}} |E(w_i) \cap E(w_j)|\). Let \(\theta\) be the maximum of all \(\theta_{\bar{s}}\), \(\bar{s} \in \bar{S}\). Since \(|E(w_i) \cap E(w_j)| \leq 1\) for any \(i, j \in \{1, \ldots, \bar{n}\}\) and \(i \neq j\), we have that \(\theta \leq \left(\binom{\bar{k}}{2}\right)\).

Lemma 8. Let \(C\) be a \([n, k, \bar{r}]\) MSR code over \(\mathbb{F}_q^r\) with \(n \geq 3\). Choose \(\bar{n}, \bar{k}\) and \(\bar{r}\) such that \(\bar{r}\bar{n} = 2n, 1 < \bar{k} < \bar{n}, 1 < \bar{r} < k\) and \(k \leq \bar{k} \bar{r} - \theta\). Then, there is a \([\bar{n}, \bar{k}, \bar{r}]\) regenerating code \(\hat{C}\) over \(\mathbb{F}_q^{\bar{r}}\). Moreover, the minimum \(\alpha\) is achieved when \(k = \bar{k} \bar{r} - \theta\).

Proof: Given a file distributed using \(C\), we know that there are \(n\) nodes and that any \(k\) of those \(n\) nodes are enough to reconstruct the file. By Lemma 7, we know that if \(n \geq 3\), there exists a set of \(n\) vertices \(W\) and a set of \(n\) edges \(E\), such that it is possible to construct \(H(W, E)\) with \(1 < \bar{r} < \bar{n}\) and \(\bar{r}\bar{n} = 2n\). Then, from \(H(W, E)\) it is possible to construct a code \(\hat{C}\) as it is described in Subsection IV-A.

The conditions \(1 < \bar{k} < \bar{n}\) and \(\bar{r} < k\) are necessary because if they are not achieved, the code \(\hat{C}\) has no sense as a regenerating code. Note that a subset of cardinal \(\bar{r} < k\) coordinates of \(c\) contained in \(\bar{r}\) different nodes can regenerate a failed one, so \(\gamma < M\). Finally, in order to reconstruct the file distributed using \(\hat{C}\), any subset of \(\bar{k}\) nodes must store at least \(k\) coordinates of \(c \in C\), so \(k \leq |\bigcup_{i \in \bar{s}} E(\omega_i)|\). Since \(k \leq \bar{k} \bar{r} - \theta \leq \bar{k} \bar{r} - \theta_{\bar{s}} = |\bigcup_{i \in \bar{s}} E(\omega_i)|\) this condition is satisfied. Therefore, \(\hat{C}\) is a regenerating code.

In Subsection IV-B it is shown that \(\gamma = \alpha\). Moreover, \(\bar{r}\) is the number of coordinates of \(c\) which compose an array coordinate of \(c\). Then, the minimum \(\bar{r}\) will lead to the minimum \(\alpha\). As \(\bar{r} \geq (k + \theta)/\bar{k}\), the minimum \(\bar{r}\) is achieved when \(k = \bar{k} \bar{r} - \theta\).

As we are trying to minimize \(\alpha\), we assume the equality \(k = \bar{k} \bar{r} - \theta\) given by Lemma 8 and we establish an upper bound for the parameter \(\theta\) in Proposition 9.

Proposition 9. In the graph \(H(W, E)\) with \(k = \bar{k} \bar{r} - \theta\), we have that

1) \(\theta \leq \left(\binom{\bar{k}}{2}\right)\) if \(\bar{k} \leq \bar{r} + 1\).
2) \( \theta \leq \lfloor \frac{k}{\bar{r}+1} \rfloor (\frac{r+1}{2}) + \left( \frac{k}{2} \right)^{\mod (\frac{r+1}{2})} \) if \( \bar{r} \geq r + 1 \).

Proof: Each node \( \omega_i \in W \) has \( r \) incident edges, so \( \bar{k} \) nodes have \( \bar{k}r \) edges. Now, we distinguish two cases.

Case \( \bar{k} \leq r + 1 \): In \( H(W,E) \), each vertex \( \omega_i \), \( i = 1, \ldots, \bar{n} \), shares one, and only one, edge with another vertex \( \omega_j \). Each vertex \( \omega_i \), \( i \in \bar{s} \) and \( |\bar{s}| = \bar{k} \), can share a maximum of one edge with each one of the other vertices \( \omega_j \), \( j \in \bar{s}, i \neq j \). Therefore, the maximum number of shared edges is \( \left( \frac{\bar{k}}{2} \right) \). In other words, when \( \bar{k} \leq r + 1 \), it is possible to create a complete subgraph of \( \bar{k} \) vertices in \( H(W,E) \) with \( \left( \frac{\bar{k}}{2} \right) \) edges.

Case \( \bar{k} > r + 1 \): Given \( H(W,E) \) and \( \bar{s} \), we are going to construct a subgraph which maximizes the number of shared edges. Each vertex \( \omega_i \), \( i \in \bar{s} \), can share a maximum of \( \bar{r} \) edges with the remaining vertices \( \omega_j \), \( j \in \bar{s}, i \neq j \). Therefore, the maximum number of shared edges is when we consider a complete subgraph with \( \bar{r} + 1 \) vertices and \( \left( \frac{r+1}{2} \right) \) edges. As \( \bar{k} > r + 1 \), there could be \( \lfloor \frac{\bar{k}}{\bar{r}+1} \rfloor \) complete subgraphs, each one with \( \left( \frac{r+1}{2} \right) \) edges. The vertices out of these complete subgraphs can share a maximum of \( \left( \frac{\bar{k}}{2} \right)^{\mod (\frac{r+1}{2})} \) edges, which leads to the upper bound

\[
\theta \leq \lfloor \frac{\bar{k}}{\bar{r}+1} \rfloor \left( \frac{r+1}{2} \right) + \left( \frac{\bar{k}}{2} \right)^{\mod (\frac{r+1}{2})}
\]

for \( \bar{k} > r + 1 \).

D. Quasi-cyclic flexible regenerating codes with minimum bandwidth

It is possible to use QCFMSR codes as base regenerating codes to create regenerating codes with minimum bandwidth using the above construction. In this subsection, we analyze the resulting parameters of these called quasi-cyclic flexible regenerating codes with minimum bandwidth.

Corollary 10. For \( \bar{k} \leq r + 1 \), there exists a \([\bar{n}, \bar{k}, \bar{r}]\) quasi-cyclic flexible regenerating code with minimum bandwidth constructed from a \([2k, k, k + 1]\) QCFMSR code if the set of parameters \( \{k, \bar{n}, \bar{k}, \bar{r}\} \) achieves:

\[
k = \frac{\bar{k}(2\bar{r} - \bar{k} + 1)}{2},
\]

\[
\bar{n} = \frac{2\bar{k}(2\bar{r} - \bar{k} + 1)}{\bar{r}},
\]

\(1 < \bar{r} < k\).

Proof: Straightforward from Lemmas 7, 8 and Proposition 5.

Figure 5 shows an example of a quasi-cyclic flexible regenerating code with minimum bandwidth for \( \bar{k} \leq r + 1 \) created from a \([6, 3, 4]\) QCFMSR code. Each node \( \omega_i \in W \) can be repaired downloading half node \( \omega_{i-1} \) and half node \( \omega_{i+1} \). Moreover, any \( \bar{k} = 2 \) nodes in \( \tilde{C} \) contain at least \( \bar{k} = 3 \) different coordinates of \( c \in C \), which allow us to reconstruct the file. Note that \( \alpha = \frac{2M}{r} \), which is equal to the value given in [10] for a \([6, 3, 4]\) MBR code.

Corollary 11. For \( \bar{k} > r + 1 \), there exists a \([\bar{n}, \bar{k}, \bar{r}]\) quasi-cyclic flexible regenerating code with minimum bandwidth constructed from a \([2k, k, k + 1]\) QCFMSR code if the set of parameters \( \{k, \bar{n}, \bar{k}, \bar{r}\} \) achieves:

\[
k = \bar{r} - \left\lfloor \frac{\bar{k}}{\bar{r}+1} \right\rfloor \left( \frac{\bar{r}+1}{2} \right) + \left( \frac{\bar{k}}{2} \right)^{\mod (\frac{\bar{r}+1}{2})},
\]

\[
\bar{n} = \frac{2\bar{n}}{\bar{r}}.
\]
Figure 6. Construction of a [10, 4, 2] quasi-cyclic flexible regenerating code with minimum bandwidth from a [10, 5, 6] QCFMSR code.

Figure 7. Parameters $[n, k, r]$ for some quasi-cyclic flexible regenerating codes with minimum bandwidth constructed from $[n, k, r]$ QCFMSR codes, and comparison between the $\alpha = \gamma$ values given in [10] and the ones achieved with the proposed construction.

$1 < \bar{r} < k.$

Proof: Straightforward from Lemmas 7, 8 and Proposition 9. ■

Figure 6 shows an example of a quasi-cyclic flexible regenerating code with minimum bandwidth for $\bar{k} > \bar{r} + 1$ created from a [10, 5, 6] QCFMSR code. Each node $\omega_i \in W$ can be repaired downloading half node $\omega_{i-1}$ and half node $\omega_{i+1}$. Moreover, any $\bar{k} = 4$ nodes in $\bar{C}$ contain at least $k = 5$ different coordinates of $c \in C$, which allows us to reconstruct the file. Note that $\alpha = \frac{2}{5}M$ which is less than $\frac{4}{7}M$, the value given in [10] for a [10, 5, 6] MBR code. It is worth to mention that the reason of the decreasing on the lower bound given in [10] is the flexibility on the parameter $\bar{r}$.

Figure 7 shows the parameters of some quasi-cyclic flexible regenerating codes with minimum bandwidth. The first column shows the parameters $[\bar{n}, \bar{k}, \bar{r}]$ of the quasi-cyclic flexible regenerating codes with minimum bandwidth.
The second column shows the parameters \([n, k, r]\) of the corresponding QCFMSR codes. The third and forth columns compare the minimum \(\alpha\) such that \(\alpha = \gamma\) for MBR codes as stated in \([10]\) with the one achieved by the quasi-cyclic flexible regenerating codes with minimum bandwidth. First part of the table shows cases when \(\bar{k} \leq \bar{r} + 1\), and the second part cases when \(\bar{k} > \bar{r} + 1\).

V. CONCLUSIONS

In this paper, we provide some interesting contributions to the current state of the art of erasure codes applied to distributed storage. We construct a family of regenerating codes using quasi-cyclic codes, where an specific set of helper nodes is used to repair a storage node failure. Firstly, we show a construction designed to minimize the storage per node, the resulting codes are called quasi-cyclic flexible minimum storage regenerating (QCFMSR) codes. Moreover, this construction is generalized to achieve bandwidth optimality instead of storage optimality using graph techniques.

At the MSR point, we provide an exact repair solution for all parameters achieving \(r = k + 1\) and \(n = 2k\). This construction is minimum according to the MSR point in the fundamental tradeoff curve. Moreover, QCFMSR codes have a very simple regenerating algorithm that approaches to the repair-by-transfer property. In our solution, the helper nodes do not need to do any linear combination among their symbols. The only linear combination is done in the newcomer to obtain the symbols the first time that it enters into the system. As far as we are concerned, this is the first construction achieving this repair simplicity for the MSR point. We also claim that such codes exist with high probability.

From an industrial point of view, it is interesting to have codes with high rates, since these are the ones desired for actual data centers. Despite there are constructions with equal \([25]\) or higher \([26]\) rate than QCFMSR codes, their other properties (uncoded repair at the helper nodes, low decoding and repairing complexity, good rate, low repair degree \(r = k + 1\) and exact repair) makes them very interesting.

We also use a technique shown in \([14]\) and \([16]\) to construct codes with minimum bandwidth using graphs and an existing MSR regenerating code. We analyze and prove this construction giving bounds on the parameters of those codes. This construction gives the minimum possible bandwidth \(\gamma = \alpha\) achieved by an specific set of helper nodes and it has the repair-by-transfer property. Finally, we show that QCFMSR codes can be used as base codes to construct the quasi-cyclic flexible regenerating codes with minimum bandwidth. We provide the conditions needed on the parameters \(\{n, k, d, \bar{n}, \bar{k}, \bar{r}\}\) for both cases, when \(\bar{k} \leq \bar{r} + 1\) and \(\bar{k} > \bar{r} + 1\).

ACKNOWLEDGMENT

The authors would like to thank Professor Alexandros G. Dimakis for his helpful and valuable comments and suggestions to this paper. This work has been partially supported by the Spanish MICINN grant TIN2010-17358, the Spanish Ministerio de Educación FPU grant AP2009-4729 and the Catalan AGAUR grant 2009SGR1224.
REFERENCES

[1] C. Plaxton, R. Rajaraman, and A. Richa, “Accessing nearby copies of replicated objects in a distributed environment,” in Proceedings of the ACM Symposium on Parallel Algorithms and Architectures, 1997, pp. 311–320.

[2] H. Weatherspoon and J. Klabatowicz, “Erasure coding vs. replication: A quantitative comparison,” in In Proceedings of the First International Workshop on Peer-to-Peer Systems (IPTPS) 2002, 2002, pp. 328–338.

[3] C. Huang, H. Simitci, Y. Xu, A. Ongus, B. Calder, P. Gopalan, J. Li, and S. Yekhanin, “Erasure coding in windows azure storage,” in Proc. of the USENIX technical conference, 2012.

[4] D. Papailiopoulos and A. Dimakis, “Locally repairable codes,” in Proc. of the IEEE ISIT, 2012.

[5] F. MacWilliams and N. Sloane, The Theory of Error-Correcting Codes. New York, USA: North-Holland Publishing, 1977.

[6] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, “On the locality of codeword symbols,” IEEE Transactions on Information Theory, vol. 58, pp. 6925–6934, Nov 2012.

[7] F. Oggier and A. Datta, “Self-repairing homomorphic codes for distributed storage systems,” in Infocom, The 30th IEEE International Conference on Computer Communications, 2011.

[8] D. Papailiopoulos, J. Luo, A. Dimakis, C. Huang, and J. Li, “Simple regenerating codes: network coding for cloud storage,” in IEEE International Conference on Computer Communications, 2012.

[9] A. Thangaraj and C. Sankar, “Evenodd: an optimal scheme for tolerating double disk failures in raid architectures,” in Proceedings of the 21st Annual International Symposium on Computer Architecture, 2002.

[10] A. Dimakis, P. Godfrey, M. Wainwright, and K. Ramchandran, “Network coding for distributed storage systems,” IEEE Trans. Inf. Theory, vol. 56, no. 9, pp. 4539–4551, 2010.

[11] R. Ahlswede, N. Cai, S.-Y. Li, and R. Yeung, “Network information flow,” IEEE Trans. Inf. Theory, vol. 46, pp. 1204–1216, 2000.

[12] N. B. Shah, K. V. Rashmi, P. V. Kumar, and K. Ramchandran, “Interference alignment in regenerating codes for distributed storage: Necessity and code constructions,” IEEE Trans. Inf. Theory, 2011.

[13] K. Rashmi, N. Shah, and P. Kumar, “Optimal exact-regenerating codes for distributed storage at the MSR and MBR points via a product-matrix construction,” IEEE Trans. Inf. Theory, vol. 57, no. 8, pp. 5227–5239, August, 2011.

[14] S. E. Rouayheb and K. Ramchandran, “Fractional repetition codes for repair in distributed storage systems,” in Proceedings of the Allerton conference on Communication, Control, and Computing, 2010.

[15] A. Dimakis, K. Ramchandran, and Y. Wu, “A survey on network codes for distributed storage,” IEEE Trans. Inf. Theory, vol. 99, pp. 1204–1216, 2011.

[16] N. Shah, K. Rashmi, P. Kumar, and K. Ramchandran, “Distributed storage codes with repair-by-transfer and non-achievability of interior points on the storage-bandwidth tradeoff,” IEEE Trans. Inf. Theory, vol. 99, 2011.

[17] Y. Hu, P. P. C. Lee, and K. W. Shum, “Analysis and construction of functional regenerating codes with uncoded repair for distributed storage systems,” arXiv:1208.2787v2 [cs.IT].

[18] P. Davis, Circulant Matrices. New York, USA: John Wiley and Sons, 1979.

[19] M. Blaum, J. Brady, J. Bruck, and J. Menon, “Quasicyclic mds codes for distributed storage with efficient exact repair,” in Proceedings of the IEEE Information Theory Workshop, 2011.

[20] N. B. Shah, K. V. Rashmi, P. V. Kumar, and K. Ramchandran, “Explicit codes minimizing repair bandwidth for distributed storage,” in Proceedings IEEE ITW, Cairo, 2010.

[21] B. Gastón, J. Pujol, and M. Villanueva, “Quasi-cyclic minimum storage regenerating codes for distributed data compression,” in Proceedings of the Data Compression Conference, 2011.

[22] R. Motwani and P. Raghavan, Randomized Algorithms. Cambridge University Press, 1995.

[23] P. Hall, “On representatives of subsets,” J. London Math. Soc., vol. 10, pp. 26–30, 1935.

[24] R. Zippel, “An explicit separation of relativised random polynomial time and relativised deterministic polynomial time,” Journal Information Processing Letters, vol. Volume 33 Issue 4, December 1989.

[25] I. Tamo, Z. Wang, and J. Bruck, “Mds array codes with optimal rebuilding,” in Proceedings IEEE ISIT, 2011.

[26] D. Papailiopoulos and A. Dimakis, “Distributed storage codes through hadamard designs,” in Proc. of the IEEE ISIT, 2011.