CONVEXIFICATION FOR A 3D INVERSE SCATTERING PROBLEM WITH THE MOVING POINT SOURCE*

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Abstract. For the first time, we develop in this paper the globally convergent convexification numerical method for a Coefficient Inverse Problem for the 3D Helmholtz equation for the case when the backscattering data are generated by a point source running along an interval of a straight line and the wavenumber is fixed. Thus, by varying the wavenumber, one can reconstruct the dielectric constant depending not only on spatial variables but the wavenumber (i.e. frequency) as well. Our approach relies on a new derivation of a boundary value problem for a system of coupled quasilinear elliptic partial differential equations. This is done via an application of a special truncated Fourier-like method. First, we prove the Lipschitz stability estimate for this problem via a Carleman estimate. Next, using the Carleman Weight Function generated by that estimate, we construct a globally strictly convex cost functional and prove the global convergence to the exact solution of the gradient projection method. Finally, our theoretical finding is verified via several numerical tests with computationally simulated data. These tests demonstrate that we can accurately recover all three important components of targets of interest: locations, shapes and dielectric constants. In particular, large target/background contrasts in dielectric constants (up to 10:1) can be accurately calculated.

Key words. Coefficient inverse scattering problem, point sources, Carleman weight function, globally convergent numerical method, data completion, Fourier truncation

AMS subject classifications. 35R25, 35R30

1. Introduction. We consider a 3D Coefficient Inverse Problem (CIP) for the Helmholtz equation in the case when the wavenumber (i.e. frequency) is fixed and the backscattering boundary data for the inversion are generated by the point source moving along an interval of a straight line. We develop analytically and test computationally the so-called convexification globally convergent numerical method for this CIP. We call a numerical method for a CIP globally convergent if there is a theorem, which claims that this method delivers at least one point in a sufficiently small neighborhood of the correct solution without any advanced knowledge of this neighborhood. In other words, a good first guess is not required.

The coefficient of the Helmholtz equation, i.e. the spatially distributed dielectric constant, is the subject of the solution of our CIP. In principle, the case when the source is moving and the frequency is fixed enables one to consider a physically realistic problem when the dielectric constant depends not only on spatial variables but on the frequency as well. Indeed, if we repeat those measurements for an interval of frequencies, then we can find the dependence of the dielectric constant on both spatial variables and the frequency. But we assume here that the dielectric constant depends only on spatial variables and not on the frequency. The analytical part of this paper is devoted to the derivation of the method and its convergence analysis. In the numerical

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part, we demonstrate the numerical performance of our technique for the case of imaging of dielectric constants of targets, which mimic antipersonnel land mines and improvised explosive devices (IEDs).

Unlike this paper, previously the convexification method was constructed for some CIPs for the Helmholtz equation only for the case of a single direction of the incident plane wave with the wavenumber running over a certain interval [24, 25, 26]. We demonstrate in our numerical studies below that in the moving source case, the convexification method accurately images all three components of targets of interest: locations, shapes and the target/background contrasts in the dielectric constant. This is unlike the above mentioned previously studied case of a single direction of the incident plane wave, which ensured only first and third components, while shapes were not accurately imaged.

One of strengths of the convexification is that it works only with the non over-determined data. This means that the number \( m \) of independent variables in the data equals the number \( n \) of independent variables in the unknown coefficient, \( m = n \). In particular, in our CIP \( m = n = 3 \). On the other hand, there are some globally convergent numerical methods for CIPs, which work with the case \( m > n \). In this regard we refer to, e.g., [14, 15, 16].

From the applied standpoint, we are oriented towards the problem of the detection and identification of antipersonnel land mines and IEDs. Reconstructions of dielectric constants from experimentally collected backscattering data for targets mimicking these explosive devices and buried under the ground were studied in [25, 33], where a single direction of the incident plane wave was used. Even though we do not consider here the case of buried targets, explosives can often be located in the air, and we model this case. In this regard we refer to the work [30], which analyses the experimental data collected in the field from explosive-like targets by engineers of the US Army Research Laboratory. Some targets in [30] are located in air and some are buried in the ground. Both here and in [30] the spatially dependent dielectric constants are subject to the solutions of CIPs. It was stated in [30] (page 33) that even though the knowledge of the dielectric constant alone is insufficient to identify an explosive, one can still hope that this knowledge might serve as an important piece of information, additional to the conventional ones, to help better identify explosives and thus, to decrease the false alarm rate.

Any CIP is both nonlinear and ill-posed. Therefore, a conventional least squares cost functional for this problem is, as a rule, non convex; see, e.g., [10, 12] for some works in which least squares cost functionals are applied to solve CIPs. The non convexity, combined with the ill-posedness, causes the presence of many local minima and ravines; see, e.g., [39] for a convincing numerical example of multiple local minima. Since a minimization procedure can stop at any local minimum, there is no guarantee that the solution obtained via the minimization process applied to that functional is indeed close to the correct one. The only exception is sometimes the case when the starting point of iterations is located in a sufficiently small neighborhood of the correct solution. We call the latter local convergence. However, a good first guess about the solution is rarely available in applications.

The above motivates this research group to work on the convexification approach. The roots of the convexification are in the method of Carleman estimates for CIPs. This method was originated in the work [9]. The idea of [9] led to many publications of many authors. Since this paper is not a survey of the method of [9], we refer for brevity only to the books [6, 8, 18] and the survey [19]. We also note that initially the method of [9] was created exclusively for proofs of uniqueness and stability theorems.
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In the convexification, one constructs a weighted Tikhonov-like functional. The weight is the Carleman Weight Function, i.e. the function involved as the weight in the Carleman estimate for the corresponding PDE operator. Given a convex bounded set $D(\beta) \subset H$ of an arbitrary diameter $\beta > 0$ in an appropriate Hilbert space $H$, one can choose the parameter $\lambda > 0$ of the Carleman Weight Function such that the strict convexity of the functional on that set is ensured. Thus, the phenomenon of local minima does not occur. Furthermore, starting from the publication [3], all works on the convexification include theorems, which claim convergence of the gradient projection method of the minimization of that functional to the correct solution of the corresponding CIP if starting from an arbitrary point of $D(\beta)$. Given that the diameter $\beta > 0$ of $D(\beta)$ is an arbitrary one, this is global convergence.

Initial publications on the convexification [7, 17] were only theoretical ones. More recently, however, the work [3] has clarified some points, which were preventing one from numerical studies. As a result, the most recent works [24, 25, 26, 27, 28], so as the current one, contain both analytical and numerical studies of the convexification. In particular, an accurate performance of the convexification on experimental backscattering data was demonstrated in [25, 26]. One of important conclusions of these numerical studies is that even though the theory requires large values of the parameter $\lambda$, accurate numerical results can be obtained for reasonable values $\lambda \in [1, 3]$.

We also refer to publications [4, 5], where a different version of the convexification is used to develop globally convergent numerical methods for CIPs for some hyperbolic PDEs. Just as in the convexification, the Carleman Weight Functions play a pivotal role in [4, 5].

As to the CIPs with the fixed wavenumber, we refer to numerical procedures developed during a long standing effort by the group of Novikov since about 1988 [34]; also, see, e.g. [1, 2, 35, 36]. The statements of CIPs in these publications are different from ours. These reconstruction techniques are also different from the convexification. By the above definition, they can also be called “globally convergent numerical methods”, since neither of them requires a good first guess for the solution while still delivering the exact coefficient in the end. An interesting feature of [1, 36] is that these publications consider the case of the non overdetermined data for the reconstruction of the potential of the Schrödinger equation at high values of the wavenumber. Another noticeable feature of [1] is that the data there are phaseless. Corresponding numerical results can be found in [1, 2].

The main new elements of this paper are:

1. It is the first time when the convexification is applied to a CIP for the Helmholtz equation in the case when the data are generated by the moving point source and the wavenumber is fixed.
2. We prove the Lipschitz stability estimate for an overdetermined boundary value problem for an auxiliary system of coupled quasilinear elliptic PDEs. This result is interesting in its own right.
3. In the proof of the central theorem about the global strict convexity of our above mentioned weighted Tikhonov-like functional we do not subtract the boundary data from the solution of that system of quasilinear elliptic PDEs. In other words, we do not arrange boundary conditions for that difference to be equal to zero.
4. We prove the Lipschitz stability of minimizers of our weighted Tikhonov-like functional with respect to small perturbations of the data as well as “Lipschitz-like” convergence rate of the gradient projection method (the latter

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converges globally). These results are stronger than those of all previous works on the convexification [24, 25, 26, 27, 28], where the weaker Hölder stability of minimizers and the “Hölder-like” convergence rates were proven.

5. The numerical results are new.

We now discuss one important issue related to this paper. We refer to Remarks 3.1 for a further discussion of this issue. It is well known that the CIP under consideration is a \textit{substantially challenging one}. This is the fundamental underlying reason why we actually replace our CIP with an approximate one, which we call “approximate mathematical model”. Indeed, even the uniqueness of our CIP can be proven, at least at this particular moment of time, only within the framework of this model: it follows immediately from Theorem 3.2, also; see section 2. In fact, the major part of our analytical treatment works \textit{only within the framework of our approximate mathematical model}. This model amounts to the truncation of a certain Fourier series. However, due to both the ill-posedness and the nonlinearity of our CIP, we cannot provide an analytical estimate as \( N \to \infty \) of the accuracy of the solution of the CIP which results from that approximation. Here, \( N \) is the number of terms of that truncated Fourier series. Nevertheless, we analyze numerically the \( N \)--dependence of the approximation accuracy of a function generated by the solution of the forward problem. Based on this analysis, we choose the number \( N \) numerically and consider this choice as an optimal one, see subsection 7.2.2 and Figure 2a.

In our numerical studies (subsections 7.2.1 and 7.2.3), we generate the data for our CIP via the solution of the forward problem without any truncation of any Fourier series. Next, we apply our method to these data to find those \( N \) Fourier coefficients “pretending” that these data are governed by our approximate mathematical model. We believe that a good accuracy of our reconstruction results justifies our model from the numerical standpoint.

We point out that a similar situation is quite \textit{typical} in the field of Inverse Problems: when a very challenging problem is replaced with an approximate mathematical model for the sake of a numerical method. Corresponding numerical results are usually good ones. In this regard, we refer to some works of other authors [13, 15, 16] as well as to our previous works on the convexification [24, 25, 26, 27, 28]. In addition, the above mentioned works [1, 2, 35, 36] actually also use certain truncations of Fourier type transforms at some steps, i.e., just as ourselves, they use some approximate mathematical models. The computational experience of all these cited works, including the current paper, tells one that the number \( N \), including its analogs, can be successfully chosen numerically.

At the start of our work on this CIP, we had two choices: either not to work on it, due to the above mentioned substantial challenges, or to figure out an approximate mathematical model and then to develop a globally convergent numerical method. Keeping in mind the importance of the above discussed application, we have chosen the second option.

The paper is organized as follows. In the next section, we detail the CIP we work with. Then, in section 3 an auxiliary boundary value problem is derived for a system of coupled quasilinear elliptic PDEs and the Lipschitz stability of this problem is proven. Next, in section 4 the above mentioned weighted Tikhonov-like functional is constructed. The central theorem about the global strict convexity of this functional is proven in section 5. In addition, the Lipschitz stability of minimizers is also proven in section 5. In section 6 we establish the global convergence of the gradient projection method and present its convergence rate. Numerical studies are described in section 7.
2. Statement of the Coefficient Inverse Problem. We model the propagation of the electric wave field by the Helmholtz equation instead of the Maxwell’s equations. This modeling was numerically justified in Appendix of the paper [29]. Such a mathematical model is true at least for rather simple medium consisting of a homogeneous background and a few embedded inclusions. Besides, good accuracies of reconstructions obtained by our research group from experimental data in publications [25, 32, 33], where the Helmholtz equation was used to model the wave propagation process, speak in favor of this modeling.

Denote \( x = (x, y, z) \in \mathbb{R}^3 \). Let the number \( R > 0 \). We define the cube \( \Omega \subset \mathbb{R}^3 \) as
\[
(2.1) \quad \Omega = \{ x : |x|, |y|, |z| < R \}.
\]
Let \( \Gamma \subset \partial \Omega \) be the lower part of the boundary of \( \Omega \) where measurements of the backscatter data are conducted,
\[
(2.2) \quad \Gamma := \{ x : |x|, |y| < R, z = -R \}.
\]
Let \( c := c(x) \in [1, \infty) \) be a sufficiently smooth function that represents the dielectric function of the medium. We assume that
\[
(2.3) \quad \begin{cases} 
  c(x) \geq 1 & \text{in } \mathbb{R}^3, \\
  c(x) = 1 & \text{in } \mathbb{R}^3 \setminus \Omega.
\end{cases}
\]
Here, \( k > 0 \) is the wavenumber. The function \( c \) is the spatially distributed and \( k \)-dependent dielectric constant. The second assumption in (2.3) means that we have vacuum outside of the domain of interest \( \Omega \).

Let \( a \) and \( d \) be two numbers such that \( d > R \) and \( a > 0 \). We define the line of sources as
\[
(2.4) \quad L_{\text{src}} := \{ (\alpha, 0, -d) : -a \leq \alpha \leq a \}.
\]
Obviously, this line is parallel to the \( x \)-axis. The distance from \( L_{\text{src}} \) to \( \Gamma \) is \( d \), and the length of our line of sources is \( 2a \). Since \( d > R \), then \( L_{\text{src}} \cap \Omega = \emptyset \). Thus, for each \( \alpha \in [-a, a] \) the corresponding point source is \( x_\alpha := (\alpha, 0, -d) \in L_{\text{src}} \).

First, we formulate the forward problem. Let \( k = \text{const.} > 0 \) and assume that the function \( c \) is known. For each source position \( x_\alpha \in L_{\text{src}} \) the forward problem is:
\[
(2.5) \quad \Delta u + k^2 c(x) u = -\delta(x - x_\alpha) \quad \text{in } \mathbb{R}^3, \\
(2.6) \quad \lim_{r \to \infty} r (\partial_r u - iku) = 0 \quad \text{for } r = |x - x_\alpha|, i = \sqrt{-1}.
\]
Conditions (2.5)–(2.6) form the Helmholtz equation with the Sommerfeld radiation condition at the infinity. Let \( u_0(x, \alpha) \) be the solution of (2.5)–(2.6) with \( c = 1 \),
\[
(2.7) \quad u_0(x, \alpha) = \frac{\exp(i k |x - x_\alpha|)}{4\pi |x - x_\alpha|}.
\]
Using the Helmholtz equation for \( u_{0,\alpha} = u_0(x, \alpha) \), we obtain from (2.5)–(2.6)
\[
\begin{align*}
\Delta (u - u_{0,\alpha}) + k^2 (u - u_{0,\alpha}) &= -k^2 (c(x,k) - 1) u \quad \text{in } \mathbb{R}^3, \\
\lim_{r \to \infty} r [\partial_r (u - u_{0,\alpha}) - i k (u - u_{0,\alpha})] &= 0 \quad \text{for } r = |x - x_\alpha|.
\end{align*}
\]
In view of the fact that $c(x) = 1$ in $\mathbb{R}^3 \setminus \Omega$, we thus find that the solution $u$ to the system (2.5)–(2.6) satisfies the so-called Lippmann–Schwinger equation (see, e.g. [11, Section 8.2]), which reads for all $x \in \mathbb{R}^3$ as

$$
(2.8) \quad u(x,\alpha) = u_0(x,\alpha) + k^2 \int_{\Omega} \frac{\exp (ik |x-x'|)}{4\pi |x-x'|} \left( c(x') - 1 \right) u(x',\alpha) \, dx'.
$$

We now pose the CIP which we solve in this paper. The schematic diagram of measurements for this problem is illustrated in Figure 1.

Fig. 1: A schematic diagram of data collection for our CIP. The wave field is generated by point sources $x_{\alpha} \in L_{src}$. We measure the backscattering wave field at an array of detectors on the lower side $\Gamma$ of the cube $\Omega$.

**Coefficient Inverse Problem (CIP).** Given $k > 0$, determine the coefficient $c(x)$ for $x \in \Omega$ in the system (2.5)–(2.6), assuming that the following function $F(x, x_{\alpha})$ is given

$$
(2.9) \quad F(x, x_{\alpha}) = u(x,\alpha) \quad \text{for} \quad x \in \Gamma, x_{\alpha} \in L_{src},
$$

where $u(x,\alpha)$ is the solution to (2.5)–(2.6).

Physically, to reconstruct the dielectric function $c$ of objects in $\Omega$, one sends the incident wave field from the source $x_{\alpha}$. This wave scatters when hitting the objects. Then, one measures the backscattering wave on the square $\Gamma$. And the data (2.9) are used to reconstruct the unknown dielectric constant inside the cube $\Omega$.

Uniqueness of this CIP is a long standing open problem. Currently uniqueness can be proven by the method of [9] only if the right hand side of equation (2.5) is not vanishing in $\Omega$. Nevertheless, uniqueness within the framework of our approximate mathematical model (Remarks 3.1) follows immediately from Theorem 3.2.

Remarks 2.1.

1. In this work, we are not interested in a specification of smoothness condition imposed on the function $c(x)$. Thus, $c(x)$ is supposed to be sufficiently smooth with respect to $x$. Some particular discussions concerning this matter can be found in, e.g., [24] and references therein, where the smoothness of $c$ is essential for the asymptotic behavior of the solution $u$ to the forward problem (2.5)–(2.6). We also note that in studies of CIPs the smoothness conditions are usually not of a considerable concern, see e.g. [37, Theorem 4.1].

2. To solve the forward problem (2.5)–(2.6) using the integral equation (2.8) for all $x \in \Omega$, we rely on numerical methods commenced in [41]. This way enables us to
extract information of $u(x, \alpha)|_{\Gamma}$, and by repeating this process for each $\alpha \in [-a, a]$ we obtain computationally the simulated data (2.9).

3. An Auxiliary System of Coupled Quasilinear Elliptic Equations.

3.1. An equation without the unknown coefficient. Observe that since $L_{\text{src}}$ is located outside of $\overline{\Omega}$, then the point source $x_\alpha = (\alpha, 0, -d)$ is not in $\overline{\Omega}$. Hence, (2.5)–(2.6) imply that for each $\alpha \in [-a, a]$

\begin{equation}
\Delta u + k^2 c(x) u = 0 \quad \text{in } \Omega.
\end{equation}

We now define the function $\log u(x, \alpha)$. The conformal Riemannian metric generated by the function $c(x)$ is

\[ d\tau = \sqrt{c(x)|dx|}, \quad |dx| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}. \]

Following [21], we assume that geodesic lines generated by this metric and originated at sources $x_\alpha \in L_{\text{src}}$ are regular. In other words, for each point $x \in \mathbb{R}^3$ and for each point $x_\alpha \in L_{\text{src}}$ there exists unique geodesic line $\Gamma(x, x_\alpha)$ connecting them. A sufficient condition of the regularity of geodesic lines was established in [38] and can also be found in [21]. The travel time along $\Gamma(x, x_\alpha)$ is

\[ \tau(x, x_\alpha) = \int_{\Gamma(x, x_\alpha)} \sqrt{c(\xi)}ds. \]

Tentatively, we denote $u = u(x, k, \alpha)$. It was established in [21] that, under certain conditions imposed on $c(x)$, which we do not discuss here (Remark 2.1), asymptotic behavior of the function $u(x, k, \alpha)$ at $k \to \infty$ is:

\begin{equation}
\label{eq:3.2}
u(x, k, \alpha) = A(x, \alpha) \exp(ik\tau(x, x_\alpha)) [1 + O(1/k)] \quad \text{for all } (x, \alpha) \in \overline{\Omega} \times [-a, a],
\end{equation}

where the function $A(x, \alpha) > 0$. Let $k > 1$ be a number. Assuming that $k$ is sufficiently large, and that $k \geq \overline{k}$, we obtain from (3.2) that $u(x, k, \alpha) \neq 0$ for all $(x, k, \alpha) \in \overline{\Omega} \times [\overline{k}, \infty) \times [-a, a]$. Denoting the term $O(1/k)$ in (3.2) as $O(1/k) = s(x, k, \alpha)$, we naturally assume that $|s(x, k, \alpha)| < 1$. Hence, using (3.2), we uniquely define the function log $u(x, k, \alpha)$ as

\begin{equation}
\log u(x, k, \alpha) = ik\tau(x, x_\alpha) + \ln A(x, \alpha) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} s^n(x, k, \alpha),
\end{equation}

for all $(x, k, \alpha) \in \overline{\Omega} \times [\overline{k}, \infty) \times [-a, a]$. In (3.2) the series is obviously taken from the power series expansion of the function log$(1 + s(x, k, \alpha))$.

Now, suppose that $k$ is not so large, but still $k \geq \overline{k} > 0$ where the number $\overline{k}$ is such that

\begin{equation}
\label{eq:3.4}u(x, k, \alpha) \neq 0 \quad \text{for all } (x, k, \alpha) \in \overline{\Omega} \times [\overline{k}, \infty) \times [-a, a].
\end{equation}

Then, using an idea of [22], we define the function $\varphi(x, k, \alpha)$ as

\begin{equation}
\varphi(x, k, \alpha) = - \int_{k}^{\infty} \frac{\partial_x u(x, \eta, \alpha)}{u(x, \eta, \alpha)} d\eta + \log u(x, k, \alpha), \quad \forall (x, k, \alpha) \in \overline{\Omega} \times [\overline{k}, \infty) \times [-a, a],
\end{equation}

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where \(\log u(x, k, \alpha)\) is defined in (3.3). Hence, \((\partial_k u - \imath k \partial \varphi)(x, k, \alpha) = 0\). Multiplying both sides of the latter by \(e^{-\varphi}\), we obtain \(\partial_k (ue^{-\varphi})(x, k, \alpha) = 0\). Since \(\varphi(x, k, \alpha) = u(x, k, \alpha)\), then \(u(x, k, \alpha) = \exp(\varphi(x, k, \alpha))\). This uniquely defines the function \(\log u(x, k, \alpha) = \varphi(x, k, \alpha)\) as long as (3.4) holds. Finally, we note that we use below only derivatives of the function \(u(x, k, \alpha)\), which means that we do not use \(\log u\) “directly”.

In all our above cited previous publications about numerical methods for CIPs for the Helmholtz equation we have not observed numerically such values of the function \(|u(x, k, \alpha)|\) which would be close to zero. The same is true for the current paper. Thus, we assume below that the fixed number \(k\) we work with is such \(k \in [\underline{k}, \infty)\). Hence, by (3.4)–(3.5), the function \(\log u(x, k, \alpha) = \varphi(x, k, \alpha)\) is uniquely defined. Thus, we assume below that

\[
\log u(x, \alpha) \neq 0 \quad \text{for all } (x, \alpha) \in \Omega \times [-a, a].
\]

We set

\[
\log u_0(x, \alpha) = \imath k|x - x_0| - \ln (4\pi |x - x_0|).
\]

Denote \(v_0(x, \alpha) = u(x, \alpha)/u_0(x, \alpha)\) and define the function \(v(x, \alpha)\) as

\[
v(x, \alpha) = \log v_0(x, \alpha) = \log u(x, \alpha) - \log u_0(x, \alpha) \quad \text{for } x \in \Omega, \alpha \in [-a, a].
\]

Obviously,

\[
\nabla v(x, \alpha) = \frac{\nabla v_0(x, \alpha)}{v_0(x, \alpha)}, \quad \Delta v(x, \alpha) = \frac{\Delta v_0(x, \alpha)}{v_0(x, \alpha)} - \left(\frac{\nabla v_0(x, \alpha)}{v_0(x, \alpha)}\right)^2.
\]

Using (3.8), we obtain the equation for \(v\):

\[
\Delta v + (\nabla v)^2 + 2
\nabla v \cdot \nabla (\log u_0(x, \alpha)) = -k^2(c(x, k) - 1), \quad x \in \Omega.
\]

Differentiating (3.9) with respect to \(\alpha\), we obtain

\[
\Delta \partial_\alpha v + 2 \nabla v \cdot \nabla \partial_\alpha v + 2 \nabla \partial_\alpha v \cdot \tilde{x}_\alpha + 2 \tilde{x}_\alpha \cdot \nabla v = 0 \quad \text{for all } x \in \Omega.
\]

Recall that \(x - x_\alpha = (x - \alpha, y, z + d)\). We have the following notations in (3.10):

\[
\hat{x}_\alpha = \frac{\imath k(x - x_\alpha)}{|x - x_\alpha|} - \frac{x - x_\alpha}{|x - x_\alpha|^2},
\]

\[
\tilde{x}_\alpha = \frac{\imath k}{|x - x_\alpha|^3} \left((x - \alpha)^2 - y^2 - (z + d)^2, (x - \alpha) y, (x - \alpha) z\right)
\]

\[
- \frac{1}{|x - x_\alpha|^3} \left((x - \alpha)^2 - y^2 - (z + d)^2, 2(x - \alpha) y, 2(x - \alpha) z\right).
\]

The notion behind this differentiation is to get rid of the \(\alpha\)-independent dielectric function \(c\) in (3.9) and thus, the auxiliary equation depends only on \(v\) and \(\partial_\alpha v\) is presented in (3.10). This approach is actually very similar with the first step of the method of [9, 19], which, however, was initially proposed only for proofs of uniqueness theorems. To deal with the variable \(\alpha\) in (3.10), we rely below on a special orthonormal basis with respect to \(\alpha\) to reduce (3.10) to a system of coupled elliptic quasilinear PDEs.
3.2. A special Fourier basis. To approximately solve the auxiliary problem (3.10), we use a truncated Fourier series. To do this, we use a special orthonormal basis in $L^2(-a,a)$ denoted by $\{\Psi_n(\alpha)\}_{n=0}^{\infty}$, $\alpha \in (-a,a)$. This basis was first constructed in [23].

For each $n \in \mathbb{N}$, let $\varphi_n(\alpha) = \alpha^n e^\alpha$ for $\alpha \in [-a,a]$. Observe that the set $\{\varphi_n(\alpha)\}_{n \in \mathbb{N}}$ is linearly independent and complete in $L^2(-a,a)$. Using the classical Gram–Schmidt orthonormalization procedure, we obtain the orthonormal basis $\{\Psi_n(\alpha)\}_{n \in \mathbb{N}}$ in $L^2(-a,a)$. See, e.g. This basis possesses the following main properties [23]:

- $\Psi_n \in C^\infty[-a,a]$ for all $n \in \mathbb{N}$;
- Let $s_{mn} = \langle \Psi_n', \Psi_m \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(-a,a)$. Then the square matrix $S_N = (s_{mn})_{m,n=0}^{N-1}$ is invertible for any $N$ since

$$s_{mn} = \begin{cases} 1 & \text{if } n = m, \\
0 & \text{if } n < m. \end{cases}$$

We note that neither classical orthogonal polynomials nor the classical basis of trigonometric functions do not hold the second property. This is because in any of these two the first column of the integer $N \geq 1$ the matrix $S_N$ would be identically zero. By virtue of this property, the matrix $S_N$ is actually an upper diagonal matrix with $\det(S_N) = 1$. Hence, the inverse matrix $S_N^{-1}$ exists.

Consider the auxiliary function $v(x,\alpha)$ that we have defined in subsection 3.1. Given $N \geq 1$, our truncated Fourier series for $v$ is written as

$$v(x,\alpha) = \sum_{n=0}^{N-1} \langle v(x, \cdot), \Psi_n(\cdot) \rangle \Psi_n(\alpha) \quad \text{for } x \in \Omega, \alpha \in [-a,a]. \tag{3.12}$$

Actually the sign “≈” should be used in (3.12). However, we use “=” for the further convenience of our work with our approximate mathematical model; see Remarks 3.1 about this model.

Remarks 3.1.

1. The representation (3.12) is an approximation of the function $v(x,\alpha)$ since the rest of the Fourier series is not counted here. Furthermore, we assume that the $\alpha$–derivative $\partial_\alpha v(x,\alpha)$ can be obtained via the term-by-term differentiation of the right-hand side of (3.12) with respect to $\alpha$. Next, we suppose that the substitution of (3.12) and its $\alpha$–derivative in the left hand side of equation (3.10) give us zero in its right hand side. In addition, we assume that the substitution of (3.12) in the left hand side of (3.9) provides us with the exact coefficient $c(x)$ in its right hand side. Finally, we impose in Section 3.2 the boundary condition (3.19) on $\partial \Omega \setminus \Gamma$. 

2. The assumptions of item 1 form our approximate mathematical model. We cannot prove convergence as $N \rightarrow \infty$. Indeed, such a result is very hard to prove due to both the nonlinearity and the ill-posedness of our CIP. Therefore, our goal below is to find spatially dependent Fourier coefficients $v_n(x) = \langle v(x, \cdot), \Psi_n(\cdot) \rangle$. The number $N$ should be chosen numerically; see subsection 7.2.2 and Figure 2a.

3. Everywhere below we work only within the framework of this approximate mathematical model. As it was pointed out in Introduction, the fundamental underlying reason why we are accepting this model is that the original CIP is an extremely challenging one.
We now substitute (3.12) into (3.10) to get
\[
\Delta \left( \sum_{n=0}^{N-1} v_n (x) \Psi_n' (\alpha) \right) + 2\nabla \left( \sum_{n=0}^{N-1} v_n (x) \Psi_n (\alpha) \right) \cdot \nabla \left( \sum_{n=0}^{N-1} v_n (x) \Psi_n' (\alpha) \right) + 2\nabla \left( \sum_{n=0}^{N-1} v_n (x) \Psi_n' (\alpha) \right) \cdot \vec{x}_\alpha + 2\vec{x}_\alpha \cdot \nabla \left( \sum_{n=0}^{N-1} v_n (x) \Psi_n (\alpha) \right) = 0.
\]

This equation is equivalent with:
\[
\sum_{n=0}^{N-1} \Psi_n' (\alpha) \Delta v_n (x) + 2 \sum_{n=0}^{N-1} \sum_{l=0}^{N-1} \Psi_n (\alpha) \Psi_l' (\alpha) \nabla v_n (x) \cdot \nabla v_l (x) + 2 \Psi_n (\alpha) \sum_{n=0}^{N-1} \vec{x}_\alpha \cdot \nabla v_n (x) = 0.
\]

Multiply both sides of (3.13) by the function \(\Psi_m (\alpha)\) for \(0 \leq m \leq N-1\) and then integrate the resulting equation with respect to \(\alpha\). We arrive at the following system of coupled quasilinear elliptic equations:
\[
\begin{align*}
\Delta V (x) + K (\nabla V (x)) &= 0 \quad \text{for} \ x \in \Omega, \\
V (x) &= \psi_0 (x) \quad \text{for} \ x \in \partial \Omega, \\
V_z (x) &= \psi_1 (x) \quad \text{for} \ x \in \Gamma, \\
K (\nabla V (x)) &= S_N^{-1} f (\nabla V (x)).
\end{align*}
\]

Here \(\psi_0 (x)\) and \(\psi_1 (x)\) are known boundary data and we explain in subsection 3.3 how to obtain them. Above, the unknown vector function \(V(x) \in \mathbb{R}^N\) is given by
\[
V (x) = \begin{pmatrix} v_0 (x) & v_1 (x) & \cdots & v_{N-1} (x) \end{pmatrix}^T.
\]

And \(S_N = (s_{mn})_{m,n=0}^{N-1}\) is the \(N \times N\) matrix that we have constructed above. The nonlinearity \(f = (f_m)_{m=0}^{N-1} \in \mathbb{R}^N\) is quadratic with respect to the first derivatives of components of \(V (x)\),
\[
f_m (\nabla V (x)) = 2 \sum_{n=0}^{N-1} \nabla v_n (x) \cdot \nabla v_l (x) \int_{-a}^{a} \Psi_m (\alpha) \Psi_n (\alpha) \Psi_l' (\alpha) d\alpha + 2 \sum_{n=0}^{N-1} \int_{-a}^{a} \Psi_m (\alpha) \Psi_n' (\alpha) \nabla v_n (x) \cdot \vec{x}_\alpha d\alpha + 2 \sum_{n=0}^{N-1} \int_{-a}^{a} \Psi_m (\alpha) \Psi_n (\alpha) \vec{x}_\alpha \cdot \nabla v_n (x) d\alpha.
\]

It follows from (3.17) and (3.18) that the vector function \(K (\nabla V)\) is quadratic with respect to components of \(\nabla V\).

The problem (3.14)–(3.16) is overdetermined since we have two boundary conditions (3.15), (3.16) instead of just one. Also, this is not a regular Cauchy problem for the system (3.14) since the Dirichlet data in (3.15) are given at the entire boundary \(\partial \Omega\). If solving problem (3.14)–(3.16), then we would find the dielectric constant \(c\) via backwards calculations. Therefore, we focus below on the solution of problem (3.14)–(3.16).
3.3. Boundary data (3.15), (3.16). We now explain how to find the boundary data for the vector \( V(x) \) in (3.15), (3.16). It follows from (2.9) and (3.12) that the Dirichlet data at \( x \in \Gamma \) for \( V(x) \) are known. As it is known, several data completion methods are heuristically applied in inverse problems with incomplete data; see, e.g. [33]. To complement the lack of the boundary data information on \( \partial \Omega \setminus \Gamma \), we use the data completion for (3.1). More precisely, we choose for each \( \alpha \),

\[
(3.19) \quad u(x,\alpha)|_{\partial \Omega} = \begin{cases} F(x,x_\alpha), & \text{if } x \in \Gamma, \\ u_0(x,\alpha), & \text{if } x \in \partial \Omega \setminus \Gamma, \end{cases}
\]

where the \( u_0(x,\alpha) \) is given in (2.7) and it is the solution of (2.5)–(2.6) for the case of the uniform background. The choice (3.19) is fairly reasonable because of the fact that \( \partial \Omega \subset (\mathbb{R}^3 \setminus \Omega) \). Recall that the second condition (3.19) is the final item of our approximate mathematical model (Remarks 3.1).

As to the data (3.16), usually measurements are performed far from the domain of interest, i.e. on the plane \( \{ z = -R' \} \), where \( R' > R \). It is time consuming to solve a CIP in a large domain. Besides, the data at the measurement plane are hard to use for an inversion algorithm since they do not look “nice”. This can be evidenced in our recent work for the experimental data; see, e.g., Figure 3a in [32]. To “move” the data closer to the target’s side, the so-called “data propagation” procedure can be applied to the measured data; see [33] for a detailed description of this procedure. By this procedure one obtains “propagated data”, i.e. an approximation of the data at our desired rectangle \( \Gamma \subset \{ z = -R \} \). Besides, the propagated data look much better than the original data: e.g. compare Figures 3a and 3b in [32]. In addition, it is clear from the data propagation procedure that one of its outcomes is an approximation of the \( z \)-derivative of the function \( u(x,k) \) at \( \Gamma \). Thus, we assume that, in addition to the Dirichlet data at \( \Gamma \), we know the Neumann boundary data \( u_z(x,\alpha) = G(x,\alpha) \) for \( x \in \Gamma, x_\alpha \in L_{src} \). Having the function \( G(x,\alpha) \) and using (3.12), one can easily find the Neumann boundary condition \( \psi_1(x) \) at \( x \in \Gamma \) in (3.16).

3.4. Lipschitz stability of the boundary value problem (3.14)–(3.16).

For any Banach space \( B \) considered below and any integer \( X > 1 \) we denote the Banach space \( B^X = B \times B \times \ldots \times B \) with the norm

\[
\|g\|_{B^X} = \left( \sum_{j=1}^{X} \|g_j\|^2_B \right)^{1/2} \quad \text{for all } g = (g_1, ..., g_K) \in B^X.
\]

Let the number \( r > R \) and the number \( \lambda > 0 \). In principle, many functions can be used as CWFs for an elliptic operator. However, a rather general one [20] depends on two large parameters, which is inconvenient for the numerical implementation. In our experience, better to choose a rather simple Carleman Weight Function (CWF). Thus, we define our CWF as

\[
(3.20) \quad \mu_\lambda(z) = \exp \left[ 2\lambda (z - r)^2 \right], z \in [-R, R].
\]

We choose \( r > R \) since one of conditions imposed on the CWF in any Carleman estimate is that its gradient should not vanish in the closed domain. Obviously, the function \( \mu_\lambda(z) \) is decreasing for \( z \in (-R, R) \) and

\[
(3.21) \quad \max_{\overline{\Omega}} \mu_\lambda(z) = \exp \left[ 2\lambda (R + r)^2 \right], \quad \min_{\overline{\Omega}} \mu_\lambda(z) = \exp \left[ 2\lambda (R - r)^2 \right].
\]
In other words, by (2.1) and (2.2) the CWF (3.20) attains its maximal value in $\overline{\Omega}$ on the part $\Gamma$ of the boundary where measurements are conducted, and it attains its minimal value on the opposite side.

Define the subspace $H^2_0(\Omega)$ of the space $H^2(\Omega)$ as:

$$H^2_0(\Omega) := \{ v \in H^2(\Omega) : v|_{\partial \Omega} = 0, \partial_x v|_{\Gamma} = 0 \}.$$  

Theorem 3.1 follows immediately from [28, Theorem 4.1].

**Theorem 3.1** (Carleman estimate). Let $\mu_\lambda(z)$ be the function defined in (3.20). Then there exist constants $\lambda_0 = \lambda_0(\Omega, r) \geq 1$ and $C = C(\Omega, r) > 0$ depending only on the domain $\Omega$ such that for every function $u \in H^2_0(\Omega)$ and for all $\lambda \geq \lambda_0$ the following Carleman estimate holds:

$$\int_{\Omega} |\Delta u|^2 \mu_\lambda(z) \, dx \geq \frac{C}{\lambda} \sum_{i,j=1}^3 \int_{\Omega} |u_{x_i x_j}|^2 \mu_\lambda(z) \, dx + C\lambda \int_{\Omega} \left[ |\nabla u|^2 + \lambda^2 |u|^2 \right] \mu_\lambda(z) \, dx.$$  

Suppose that there exist two vector functions $V^{(1)}(x)$ and $V^{(2)}(x)$ satisfying equation (3.14) with boundary conditions as in (3.15), (3.16),

$$V^{(1)}(x) = \psi^{(1)}_0(x), V^{(2)}(x) = \psi^{(2)}_0(x) \quad \text{for } x \in \partial \Omega,$$

$$V^{(2)}_z(x) = \psi^{(1)}_1(x), V^{(2)}_z(x) = \psi^{(2)}_1(x) \quad \text{for } x \in \Gamma.$$  

Suppose that there exist two vector functions $F_1, F_2 \in H^N_N(\Omega)$ satisfying boundary conditions (3.24), (3.25), i.e.

$$F_1(x) = \psi^{(1)}_0(x), F_2(x) = \psi^{(2)}_0(x) \quad \text{for } x \in \partial \Omega,$$

$$\partial_x F_1(x) = \psi^{(1)}_1(x), \partial_x F_2(x) = \psi^{(2)}_1(x) \quad \text{for } x \in \Gamma.$$  

Let $M > 0$ be a number. We assume that

$$V^{(1)}, V^{(2)}, F_1, F_2 \in G(M) = \left\{ W \in H^N_N(\Omega) : \|W\|_{H^N_N(\Omega)} < M \right\}.$$  

Note also that by the embedding theorem

$$G(M) \subset C^N_N(\overline{\Omega}) \quad \text{and} \quad \|W\|_{C^N_N(\overline{\Omega})} \leq C_1 \quad \text{for all } W \in G(M).$$  

Here and below $C_1 = C_1(\Omega, N, M) > 0$ denotes different constants depending only on listed parameters.

**Theorem 3.2** (Lipschitz stability estimate). Let $V^{(1)}(x)$ and $V^{(2)}(x)$ be two solutions of equation (3.14) with boundary conditions (3.24), (3.25). Suppose that there exist two vector functions $F_1, F_2 \in H^N_N(\Omega)$ satisfying (3.26), (3.27). Also, let (3.28) holds. Then the following Lipschitz stability estimate is valid

$$\|V^{(1)} - V^{(2)}\|_{H^N_N(\Omega)} \leq C_1 \|F_1 - F_2\|_{H^N_N(\Omega)}.$$  

**Proof.** Denote

$$Q_1(x) = V^{(1)}(x) - F_1(x), \quad Q_2(x) = V^{(2)}(x) - F_2(x),$$

$$\tilde{Q}(x) = Q_1(x) - Q_2(x), \quad \tilde{F}(x) = F_1(x) - F_2(x).$$
Then (3.18) and (3.24)-(3.32) imply that

\begin{equation}
\Delta \tilde{Q}(x) = T_1(x) \cdot \nabla \tilde{Q}(x) + T_2(x) \cdot \nabla \tilde{F}(x) - \Delta \tilde{F}(x),
\end{equation}

(3.34)

\begin{equation}
\tilde{Q}\mid_{\partial \Omega} = 0, \quad \tilde{Q}_z\mid_{r=0} = 0,
\end{equation}

(3.35)

\begin{equation}
T_1, T_2 \in C^2_A(\overline{\Omega}), \quad \|T_1\|_{C^2_A(\overline{\Omega})}, \quad \|T_2\|_{C^2_A(\overline{\Omega})} \leq C_1.
\end{equation}

Square absolute values of both sides of equation (3.33). Next, multiply the resulting equation by the CWF (3.20) and integrate over the domain \(\Omega\). Using (3.35), we obtain (3.36)

\begin{equation}
\int_\Omega \left| \Delta \tilde{Q} \right|^2 \mu_\lambda(z) \, dx \leq C_1 \int_\Omega \left| \nabla \tilde{Q} \right|^2 \mu_\lambda(z) \, dx + C_1 \int_\Omega \left( \left| \Delta \tilde{F} \right|^2 + \left| \nabla \tilde{F} \right|^2 \right) \mu_\lambda(z) \, dx.
\end{equation}

Taking into account (3.22) and (3.34) and also applying (3.23) to (3.36), we obtain for all \(\lambda \geq \lambda_0 > 1\)

\begin{equation}
C_1 \int_\Omega \left( \left| \Delta \tilde{F} \right|^2 + \left| \nabla \tilde{F} \right|^2 \right) \mu_\lambda(z) \, dx + C_1 \int_\Omega \left| \nabla \tilde{Q} \right|^2 \mu_\lambda(z) \, dx \\
\geq \frac{1}{\lambda} \sum_{i,j=1}^3 \int_\Omega \left| \tilde{Q}_{x_i x_j} \right|^2 \mu_\lambda(z) \, dx + \lambda \int_\Omega \left[ \left| \nabla \tilde{Q} \right|^2 + \left| \tilde{Q} \right|^2 \right] \mu_\lambda(z) \, dx.
\end{equation}

Choose a number \(\lambda_1 \geq \lambda_0\) such that \(\lambda_1 \geq 2C_1\). Then (3.37) implies that

\begin{equation}
C_1 \int_\Omega \left( \left| \Delta \tilde{F} \right|^2 + \left| \nabla \tilde{F} \right|^2 \right) \mu_{\lambda_1}(z) \, dx \\
\geq \frac{1}{\lambda_1} \sum_{i,j=1}^3 \int_\Omega \left| \tilde{Q}_{x_i x_j} \right|^2 \mu_{\lambda_1}(z) \, dx + \frac{\lambda_1}{2} \int_\Omega \left[ \left| \nabla \tilde{Q} \right|^2 + \left| \tilde{Q} \right|^2 \right] \mu_{\lambda_1}(z) \, dx.
\end{equation}

This inequality and (3.21) lead to:

\begin{equation}
C_1 \exp(4Rr\lambda_1) \int_\Omega \left( \left| \Delta \tilde{F} \right|^2 + \left| \nabla \tilde{F} \right|^2 \right) \, dx \\
\geq \frac{1}{\lambda_1} \sum_{i,j=1}^3 \int_\Omega \left| \tilde{Q}_{x_i x_j} \right|^2 \, dx + \frac{\lambda_1}{2} \int_\Omega \left[ \left| \nabla \tilde{Q} \right|^2 + \left| \tilde{Q} \right|^2 \right] \, dx.
\end{equation}

Hence, with a new constant \(C_1\) we have

\begin{equation}
\left\| \tilde{Q} \right\|_{H^2_\lambda(\Omega)} \leq C_1 \left\| \tilde{F} \right\|_{H^2_\lambda(\Omega)}.
\end{equation}

Next, by (3.31), (3.32) and triangle inequality

\begin{equation}
\left\| \tilde{Q} \right\|_{H^2_\lambda(\Omega)} = \left\| (V^{(1)} - F_1) - (V^{(2)} - F_2) \right\|_{H^2_\lambda(\Omega)} \\
\geq \left\| V^{(1)} - V^{(2)} \right\|_{H^2_\lambda(\Omega)} - \| F_1 - F_2 \|_{H^2_\lambda(\Omega)}.
\end{equation}

Combining this with (3.38), we obtain the target estimate (3.30) of this theorem. \(\square\)
4. Weighted Tikhonov-like Functional. For the convenience of the presentation, each \(N-D\) complex valued vector function \(W = \text{Re} W + i \text{Im} W\) is considered below as the \(2N-D\) vector function with real valued components \((\text{Re} W, \text{Im} W) := (W_1, W_2) := W \in \mathbb{R}^{2N}\). All results and proofs below are for these \(2N-D\) vector functions. For any number \(s \in \mathbb{C}\), its complex conjugate is denoted as \(\overline{s}\).

We find an approximate solution of the problem (3.14)-(3.17) via the minimization of an appropriate weighted Tikhonov-like functional with the CWF (3.20) involved in it. Due to (3.14), denote

\[
(4.1) \quad L(V)(x) = \Delta V(x) + K(\nabla V(x)).
\]

Let \(\gamma \in (0, 1)\) be the regularization parameter. We now consider the following weighted Tikhonov-like functional \(J_{\lambda, \gamma} : H_{2N}^3(\Omega) \rightarrow \mathbb{R}_+\),

\[
(4.2) \quad J_{\lambda, \gamma}(V) = \exp \left[-2\lambda (R + r)^2 \right] \int_{\Omega} |L(V)|^2 \mu_\lambda(z) \, dx + \gamma \|V\|_{H_{3}^N(\Omega)}^2.
\]

Here \(\exp \left[-2\lambda (R + r)^2 \right]\) is the balancing multiplier: to balance first and second terms in the right hand side of (4.2), see (3.21). We use the \(H_{3}^N(\Omega)\) –norm in the regularization term here since \(H_{3}^N(\Omega) \subset C_{1}^2(\overline{\Omega})\) and an obvious analog of (3.29) holds.

Assuming for a moment that the nonlinear term \(K(\nabla V(x))\) is absent in (4.1), we remark that since the Laplace operator is linear, then one can also find an approximate solution of the problem (3.14)-(3.17) by the regular quasi-reversibility method with \(\lambda = 0\) in (4.2) (see, e.g. [20]). However, if \(K(\nabla V(x)) \neq 0\), then the presence of the CWF serves three purposes: first, it controls this nonlinear term; second, it “maximizes” the influence of the important boundary data at \(z = -R\); and third, it “convexifies” the cost functional globally. These are the underlying reasons of the convexification idea. Below \((\cdot, \cdot)\) is the scalar product in the space \(H_{2N}^3(\Omega)\). Let \(M > 0\) be an arbitrary number. We define the set \(B(M) \subset H_{2N}^3(\Omega)\) as

\[
(4.3) \quad B(M) = \left\{ V \in H_{2N}^3(\Omega) : \|V\|_{H_{2N}^3(\Omega)} < M, V|_{\partial \Omega} = \psi_0, V|_{\Gamma} = \psi_1 \right\}.
\]

By (3.29), we know that

\[
(4.4) \quad B(M) \subset C_{2N}^1(\overline{\Omega}) \text{ and } \|V\|_{C_{2N}^1(\overline{\Omega})} \leq C_1 \text{ for all } V \in B(M).
\]

Minimization problem (MP). Minimize the cost functional \(J_{\lambda, \gamma}(V)\) on the set \(B(M)\).

5. Analysis of the Functional \(J_{\lambda, \gamma}(V)\).

5.1. Strict convexity on \(B(M)\). Theorem 5.1 is the central analytical result of this work. Note that in the proof of this theorem we do not “subtract” boundary conditions from the vector function \(V\), which means that we do not arrange zero boundary conditions for the difference. Hence, we do not require here that our boundary conditions should be extended in the entire domain \(\Omega\). This is a new element compared with our proofs in the previous works on the convexification [3, 25, 26, 27, 28]. Thus, the proof of Theorem 5.1 is significantly different from those of these works. Still, we use that subtraction in Theorems 5.4 and 6.1.

**Theorem 5.1.** The functional \(J_{\lambda, \gamma}(V)\) has its Fréchet derivative \(J_{\lambda, \gamma}'(V)\) at any point \(V \in B(M)\). Let \(\lambda_0 > 1\) be the number of Theorem 3.1. There exists a
sufficiently large number $\lambda_2 = \lambda_2 (M, N, r, \Omega) \geq \lambda_0$ such that the functional $J_{\lambda, \gamma}(V)$ is strictly convex on $\overline{B(M)}$ for all $\lambda \geq \lambda_2$. More precisely, for all $\lambda \geq \lambda_2$ the following inequality holds

\begin{equation}
(5.1) \quad J_{\lambda, \gamma}(V^{(2)}) - J_{\lambda, \gamma}(V^{(1)}) - J_{\lambda, \gamma}(V^{(1)}) (V^{(2)} - V^{(1)})
\end{equation}

\begin{equation}
\geq C_1 \frac{\left\| (V^{(2)} - V^{(1)}) \right\|^2}{H^2_{2N}(\Omega)} + \gamma \frac{\left\| (V^{(2)} - V^{(1)}) \right\|^2}{H^3_{2N}(\Omega)} \quad \text{for all } V^{(1)}, V^{(2)} \in \overline{B(M)}.
\end{equation}

**Proof.** Let $V^{(1)}, V^{(2)} \in \overline{B(M)}$ be two arbitrary points. Define $H^3_{2N}(\Omega) = H^3_{2N}(\Omega) \cap H^2_{0,2N}(\Omega)$; see (3.22). Denote $h = (h_1, h_2) = V^{(2)} - V^{(1)}$. Then

\begin{equation}
(5.2) \quad h \in \overline{B(2M)} \quad \text{and} \quad h \in H^3_{0,2N}(\Omega).
\end{equation}

Obviously $\left| L \left( V^{(2)} \right) \right|^2 = \left| L \left( V^{(1)} + h \right) \right|^2$. Observe that it follows from (3.14), (3.17) and (3.18) that the vector function $K(\nabla V)$ is the sum of linear and quadratic parts with respect to the gradients $\nabla v_n(x)$ of the components $v_n(x)$ of the vector function $V$. Using this as well as (4.1), we obtain

\begin{equation}
(5.3) \quad L \left( V^{(1)} + h \right) = L \left( V^{(1)} \right) + \Delta h + K_1(x) \nabla h + K_2(x, \nabla h).
\end{equation}

Here, the vector functions $K_1, K_2$ are continuous with respect to $x$ in $\overline{\Omega}$. Also, $K_1(x)$ is independent on $h$. As to the vector function $K_2(x, \nabla h)$, it is quadratic with respect to the gradients $\nabla h_n(x)$ of the components of the vector function $h$. The latter, (4.3) and (4.4) imply that

\begin{equation}
(5.4) \quad |K_2(x, \nabla h)| \leq C_1 |\nabla h|^2 \quad \text{for all } x \in \overline{\Omega}.
\end{equation}

Squaring absolute values of both sides of (5.3), we obtain

\begin{equation}
(5.5) \quad \left| L \left( V^{(1)} + h \right) \right|^2 = \left| L \left( V^{(1)} \right) \right|^2 + 2 \text{Re} \left\{ L \left( V^{(1)} \right) [\Delta h + K_1(x) \nabla h + K_2(x, \nabla h)] \right\}
\end{equation}

\begin{equation}
+ |\Delta h + K_1(x) \nabla h + K_2(x, \nabla h)|^2.
\end{equation}

In (5.5), we single out the linear, with respect to $h$, term as well as the term $|\Delta h|^2$. We obtain

\begin{equation}
(5.6) \quad \left| L \left( V^{(1)} + h \right) \right|^2 - \left| L \left( V^{(1)} \right) \right|^2
= 2 \text{Re} \left\{ L \left( V^{(1)} \right) [\Delta h + K_1(x) \nabla h] \right\} + |\Delta h|^2 + 2 \text{Re} \left\{ L \left( V^{(1)} \right) K_2(x, \nabla h) \right\}
\end{equation}

\begin{equation}
+ 2 \text{Re} \left\{ \overline{\Delta h} \cdot [K_1(x) \nabla h + K_2(x, \nabla h)] \right\} + |K_1(x) \nabla h + K_2(x, \nabla h)|^2.
\end{equation}

In (5.6), the term $2 \text{Re} \left\{ L \left( V^{(1)} \right) [\Delta h + K_1(x) \nabla h] \right\}$ is linear with respect to $h$. Thus, we obtain

\begin{equation}
(5.7) \quad J_{\lambda, \gamma}(V^{(1)} + h) - J_{\lambda, \gamma}(V^{(1)}) = L \text{Im} \left\{ h \right\} + \gamma \left\| h \right\|_{H^2_{2N}(\Omega)}
\end{equation}

\begin{equation}
+ e^{-2\lambda(R+r)^2} \int_\Omega \left\{ |\Delta h|^2 + 2 \text{Re} \left[ \overline{\Delta h} \cdot (K_1(x) \nabla h + K_2(x, \nabla h)) \right] \right\} \mu_\lambda(z) \, dx
\end{equation}

\begin{equation}
+ e^{-2\lambda(R+r)^2} \int_\Omega \left[ 2 \text{Re} \left\{ L \left( V^{(1)} \right) K_2(x, \nabla h) \right\} + |K_1(x) \nabla h + K_2(x, \nabla h)|^2 \right] \mu_\lambda(z) \, dx,
\end{equation}
Thus, we can rewrite (5.7) as
\[ \text{Lin} (h) = 2 \gamma \left( V^{(1)} , h \right) + 2 e^{-2 \lambda (R+r)^2} \int_{\Omega} \text{Re} \left[ \Delta h + (K_1 (x) \nabla h) \left( \Delta V^{(1)} + K \left( \nabla V^{(1)} \right) \right) \right] \mu_{\lambda} (z) \, dx. \]

Besides, it follows from (5.7) and (5.8) that
\[ \lim_{\|h\|_{H^2(\Omega)} \to 0^+} \left\{ \frac{1}{\|h\|_{H^2(\Omega)}} \left[ J_{\lambda,\gamma} \left( V^{(1)} + h \right) - J_{\lambda,\gamma} \left( V^{(1)} \right) - \text{Lin} (h) \right] \right\} = 0. \]

Hence, the functional \( \text{Lin} (h) \) is the Frechét derivative of the functional \( J_{\lambda,\gamma} \) at the point \( V^{(1)} \in B (M) \). By the Riesz theorem, there exists a unique point \( J'_{\lambda,\gamma} (V^{(1)}) \) such that
\[ J'_{\lambda,\gamma} (V^{(1)}) \in H^3_{0,2N} (\Omega) \text{ and } \text{Lin} (h) = \left( J'_{\lambda,\gamma} (V^{(1)}) , h \right) \text{ for all } h \in H^3_{0,2N} (\Omega). \]

Thus, we can rewrite (5.7) as
\[ J_{\lambda,\gamma} \left( V^{(1)} + h \right) - J_{\lambda,\gamma} \left( V^{(1)} \right) = \gamma \|h\|_{H^2(\Omega)}^2 \]
\[ + 2 e^{-2 \lambda (R+r)^2} \int_{\Omega} \left\{ |\Delta h|^2 + 2 \text{Re} \left[ \overline{\Delta h} \cdot (K_1 (x) \nabla h + K_2 (x, \nabla h)) \right] \right\} \mu_{\lambda} (z) \, dx \]
\[ + e^{-2 \lambda (R+r)^2} \int_{\Omega} \left[ 2 \text{Re} \left[ \overline{L (V^{(1)})} K_2 (x, \nabla h) \right] + |K_1 (x) \nabla h + K_2 (x, \nabla h)|^2 \right] \mu_{\lambda} (z) \, dx. \]

We now estimate from the below the term in the second line of (5.10). By the Cauchy–Schwarz inequality, (4.4) and (5.4) we find that
\[ 2 |\overline{\Delta h} \cdot (K_1 (x) \nabla h + K_2 (x, \nabla h))| \leq \frac{1}{2} |\Delta h|^2 + C_1 |\nabla h|^2. \]

Therefore,
\[ \int_{\Omega} \left\{ |\Delta h|^2 + 2 \text{Re} \left[ \overline{\Delta h} \cdot (K_1 (x) \nabla h + K_2 (x, \nabla h)) \right] \right\} \mu_{\lambda} (z) \, dx \]
\[ \geq \int_{\Omega} |\Delta h|^2 \mu_{\lambda} (z) \, dx - \frac{1}{2} \int_{\Omega} |\nabla h|^2 \mu_{\lambda} (z) \, dx - C_1 \int_{\Omega} |\nabla h|^2 \mu_{\lambda} (z) \, dx \]
\[ = \frac{1}{2} \int_{\Omega} |\Delta h|^2 \mu_{\lambda} (z) \, dx - C_1 \int_{\Omega} |\nabla h|^2 \mu_{\lambda} (z) \, dx. \]

Next, using (5.4), we estimate from the below the term in the third line of (5.10),
\[ e^{-2 \lambda (R+r)^2} \int_{\Omega} \left[ 2 \text{Re} \left[ \overline{L (V^{(1)})} K_2 (x, \nabla h) \right] + |K_1 (x) \nabla h + K_2 (x, \nabla h)|^2 \right] \mu_{\lambda} (z) \, dx \]
\[ (5.12) \quad \geq -C_1 e^{-2\lambda(R+r)^2} \int_\Omega |\nabla h|^2 \mu_\lambda (z) \, dx. \]

Thus, (5.10)–(5.12) imply

\[ (5.13) \quad J_{\lambda,\gamma} \left( V^{(1)} + h \right) - J_{\lambda,\gamma} \left( V^{(1)} \right) - \left( J'_{\lambda,\gamma} \left( V^{(1)} \right), h \right) \]
\[ \geq e^{-2\lambda(R+r)^2} \frac{1}{2} \left[ \int_\Omega |\Delta h|^2 \mu_\lambda (z) \, dx - C_1 \int_\Omega |\nabla h|^2 \mu_\lambda (z) \, dx \right] + \gamma \| h \|_{H^2_N(\Omega)}^2. \]

Now we apply the Carleman estimate (3.23) to the second line of (5.13). This use is possible due to (5.2). For brevity, we do not count the multiplier \( e^{-2\lambda(R+r)^2} \) for a while. With a constant \( \tilde{C} = \tilde{C}(\Omega, r, N) > 0 \) and a number \( \tilde{\lambda}_0 = \tilde{\lambda}_0(\Omega, r, N) \geq \lambda_0 > 1 \) depending only on listed parameters, we obtain for all \( \lambda \geq \tilde{\lambda}_0 \)

\[ (5.14) \quad \frac{1}{2} \int_\Omega |\Delta h|^2 \mu_\lambda (z) \, dx - C_1 \int_\Omega |\nabla h|^2 \mu_\lambda (z) \, dx \geq \frac{\tilde{C}}{\lambda} \sum_{i,j=1}^3 \int_\Omega |h_{x_i,x_j}|^2 \mu_\lambda (z) \, dx \]
\[ + \frac{\lambda}{\lambda_2} \int_\Omega \left[ |\nabla h|^2 + \lambda^2 |h|^2 \right] \mu_\lambda (z) \, dx - C_1 \int_\Omega |\nabla h|^2 \mu_\lambda (z) \, dx. \]

Choose the number \( \lambda_2 = \lambda_2(M, \Omega, r, N) \geq \tilde{\lambda}_0 > 1 \) depending only on listed parameters such that \( \tilde{C}\lambda_2 > 2C_1 \). Then we obtain from (5.14)

\[ (5.15) \quad \geq \frac{\tilde{C}}{\lambda_2} \sum_{i,j=1}^3 \int_\Omega |h_{x_i,x_j}|^2 \mu_\lambda_2 (z) \, dx + \frac{1}{2} \tilde{C}\lambda_2 \int_\Omega \left[ |\nabla h|^2 + \lambda_2^2 |h|^2 \right] \mu_\lambda_2 (z) \, dx \]
\[ \geq C_1 e^{2\lambda_2(R+r)^2} \| h \|_{H^2_N}^2. \]

Hence, combining (5.13)–(5.15) we arrive at

\[ J_{\lambda,\gamma} \left( V^{(1)} + h \right) - J_{\lambda,\gamma} \left( V^{(1)} \right) - \left( J'_{\lambda,\gamma} \left( V^{(1)} \right), h \right) \geq C_1 \| h \|_{H^2_N}^2 + \gamma \| h \|_{H^2_N(\Omega)}^2, \]

which is equivalent to our target estimate (5.1). \( \square \)

5.2. The minimizer of \( J_{\lambda,\gamma} (V) \) on \( \overline{B(M)} \). In Theorem 5.2 below, we state the Lipschitz continuity of the Frechét derivative \( J'_{\lambda,\gamma} (V) \) on \( \overline{B(M)} \). We omit the proof of this theorem because it is very similar to the proof of Theorem 3.1 in [3].

**Theorem 5.2.** For any \( \lambda > 0 \) the Frechét derivative \( J'_{\lambda,\gamma} (V) \) of the functional \( J_{\lambda,\gamma} (V) \) is Lipschitz continuous on the set \( \overline{B(M)} \). In other words, there exists a number \( D = D(\Omega, r, N, M, \lambda, \gamma) > 0 \) depending only on listed parameters such that for any \( V^{(1)}, V^{(2)} \in \overline{B(M)} \) the following estimate holds:

\[ \| J'_{\lambda,\gamma} \left( V^{(2)} \right) - J'_{\lambda,\gamma} \left( V^{(1)} \right) \|_{H^2_N(\Omega)} \leq D \| V^{(2)} - V^{(1)} \|_{H^2_N(\Omega)}. \]
As to the existence and uniqueness of the minimizer, they are established in Theorem 5.3. In fact, this theorem follows immediately from a combination of above Theorems 5.1 and 5.2 with Lemma 2.1 and Theorem 2.1 of [3]. Hence, we omit its proof.

**Theorem 5.3.** Let the number \( \lambda_2 = \lambda_2(M, N, r, \Omega) > 1 \) be the one in Theorem 5.1. Then for any \( \lambda \geq \lambda_2 \) and for any \( \gamma > 0 \) the functional \( J_{\lambda, \gamma}(V) \) has a unique minimizer \( V_{\min, \lambda, \gamma} \in B(M) \). Furthermore, the following inequality holds:

\[
(5.16) \quad (J_{\lambda, \gamma}'(V_{\min, \lambda, \gamma}), V_{\min, \lambda, \gamma} - Q) \leq 0 \quad \text{for all} \quad Q \in B(M).
\]

### 5.3. The distance between the minimizer and the “ideal” solution.

In accordance with the concept of Tikhonov for ill-posed problems [40], assume now that there exists the “ideal” solution \( V^* \) of problem (3.14)–(3.17) with the “ideal” noiseless data \( \psi_0^*, \psi_1^* \). It makes sense to obtain an estimate of the distance between \( V^* \) and the minimizer \( V_{\min, \lambda, \gamma} \) of the functional \( J_{\lambda, \gamma}(V) \) for the case of noisy data with the noise level \( \delta \in (0, 1) \). This is what is done in the current subsection.

To obtain this estimate, we need to “extend” the boundary data \( \psi_0, \psi_1 \) in (3.15), (3.16) inside \( \Omega \). Recall that, unlike all previous works on the convexification, we have not done this extension in the proof of our central Theorem 5.1. Thus, we assume there exists a vector function \( G(x) \in H^3(\Omega) \) satisfying boundary conditions (3.15), (3.16),

\[
G|_{\partial \Omega} = \psi_0(x), G_z|_\Gamma = \psi_1(x).
\]

On the other hand, the existence of the corresponding vector function \( G^*(x) \in H^3(\Omega) \) satisfying boundary conditions with the “ideal” data,

\[
G^*|_{\partial \Omega} = \psi_0^*(x), G^*_z|_\Gamma = \psi_1^*(x)
\]

follows from the existence of the ideal solution \( V^* \). We assume that

\[
\|G - G^*\|_{H^3(\Omega)} < \delta.
\]

In addition, we suppose that

\[
(5.20) \quad \|V^*\|_{H^3(\Omega)}, \|G^*\|_{H^3(\Omega)} < M - \delta.
\]

Using (5.19), (5.20) and the triangle inequality, we easily see that

\[
(5.21) \quad \|G\|_{H^3(\Omega)} < M.
\]

Our goal now is to estimate \( \|V_{\min, \lambda, \gamma} - V^*\|_{H^3(\Omega)} \) via the noise parameter \( \delta \).

**Theorem 5.4.** (Accuracy and stability of minimizers). Suppose that conditions (5.17)–(5.20) hold. Let \( \lambda_2 = \lambda_2(M, N, r, \Omega) > 1 \) be the number in Theorems 5.1, 5.3. Choose the number \( \lambda_3 = \lambda_2(3M, N, \Omega) > \lambda_2 > 1 \). Let \( \lambda = \lambda_3 \) and \( \gamma = \delta^2 \). Then the following accuracy estimate holds

\[
(5.22) \quad \|V_{\min, \lambda, \gamma} - V^*\|_{H^3(\Omega)} \leq C_1 \delta.
\]

**Remark 5.1.** Since the power of \( \delta \) is 1 in (5.22), then it is natural to call (5.22) “the Lipschitz stability estimate for the minimizers”, which is similar to Theorem 3.2. This estimate is obviously stronger than in all previous works on the convexification;
Let $W$ be any vector function in $W$. To the second line of (5.25) with $\tilde{\varepsilon}$ functions where $\tilde{\varepsilon}$ convexity inequality (5.1) here, setting, e.g. that $V$ vector functions called $W$. On the other hand, (5.21) and (5.23) imply that using the triangle inequality, (5.19) and (5.20), we obtain (5.26) which is claimed by Theorem 5.3.

Consider now the minimizer $V_{\min,\lambda,\gamma} \in \overline{B(M)}$ which is claimed by Theorem 5.3. Let $W_{\min,\lambda,\gamma} = V_{\min,\lambda,\gamma} - G \in B(2M)$. Then (5.26) implies that

\begin{equation}
J_{\lambda,\gamma} (W^* + G) - J_{\lambda,\gamma} (W + G) - J'_{\lambda,\gamma} (W + G) (W^* - W) - J'_{\lambda,\gamma} (V) \left( \tilde{V}^* - V \right),
\end{equation}

where $\tilde{V}^* = W^* + G$ and $V = W + G$. Notice that by (5.23) and (5.24) both vector functions $\tilde{V}^*, V \in B(3M)$. Hence, by Theorem 5.1 we can apply the estimate (5.1) to the second line of (5.25) with $\lambda = \lambda_3 = \lambda_2 (3M, N, r, \Omega) > 1$. Thus,

\begin{equation}
J_{\lambda,\gamma} (W^* + G) - J_{\lambda,\gamma} (W + G) - J'_{\lambda,\gamma} (W + G) (W^* - W) \geq C_1 ||W^* - W||^2_{H^2_{\Omega}(\Omega)} + \gamma ||W^* - W||^2_{H^2_{\Omega}(\Omega)}
\end{equation}

for all $W \in B(2M)$.

Using the triangle inequality, (5.19) and (5.20), we obtain

\begin{align*}
||W^* + G||_{H^2_{\Omega}(\Omega)} &= ||W^* + G^* + (G - G^*)||_{H^2_{\Omega}(\Omega)} \\
&\leq ||W^* + G^*||_{H^2_{\Omega}(\Omega)} + ||G - G^*||_{H^2_{\Omega}(\Omega)} \\
&= ||V^*||_{H^2_{\Omega}(\Omega)} + ||G - G^*||_{H^2_{\Omega}(\Omega)} < (M - \delta) + \delta = M.
\end{align*}

This means that $(W^* + G) \in B(M)$. Therefore, we use (5.16) to get

\begin{equation}
-J'_{\lambda,\gamma} (V_{\min,\lambda,\gamma}) (W^* + G) - V_{\min,\lambda,\gamma} \leq 0.
\end{equation}

Hence,

\begin{align*}
J_{\lambda,\gamma} (W^* + G) - J_{\lambda,\gamma} (V_{\min,\lambda,\gamma}) - J'_{\lambda,\gamma} (V_{\min,\lambda,\gamma}) (W^* + G) - V_{\min,\lambda,\gamma} \\
\leq J_{\lambda,\gamma} (W^* + G).
\end{align*}
Moreover, substituting this inequality in (5.27), we obtain
\[ J_{\lambda, \gamma} (W^* + G) \geq C_1 \|W^* - W_{\min, \lambda, \gamma}\|^2_{H^2_2(\Omega)}. \]

We now estimate the left hand side of (5.28). Note that the functional \( J_{\lambda, \gamma}(V) \) can be represented as
\[ J_{\lambda, \gamma}(V) = J^0_{\lambda, \gamma}(V) + \gamma \|V\|^2_{H^2_3(\Omega)}, \]
(5.29)
\[ J^0_{\lambda, \gamma}(V) = \exp \left[-2\lambda (R + r)^2 \right] \int_\Omega |L(V)|^2 \mu_\lambda(z) \, dx. \]
(5.30)
Since \( W^* + G^* = V^* \) is the ideal solution, then \( L(V^*)(x) = 0 \) for \( x \in \Omega \).

Next, using the finite increment formula and (4.1), we obtain
\[ |L(W^* + G)|^2 (x) = |L(W^* + G^* + G - G^*)|^2 (x) \]
\[ = |L(V^*) + S(G - G^*)|^2 (x) = |S(G - G^*)|^2 (x), \]
where by (3.21) and (5.19) the following estimate is valid:
\[ \exp \left[-2\lambda (R + r)^2 \right] \int_\Omega |S(G - G^*)|^2 (x) \mu_\lambda(z) \, dx \leq C_1 \delta^2. \]

This and (5.30) imply that
\[ J^0_{\lambda, \gamma}(W^* + G) \leq C_1 \delta^2. \]
(5.31)
Next, using (5.21), (5.23), (5.29) and (5.31), we obtain
\[ J_{\lambda, \gamma}(W^* + G) \leq C_1 (\delta^2 + \gamma). \]
(5.32)
Therefore, using (5.28), (5.32) and recalling that \( \gamma = \delta^2 \), we obtain
\[ \|W^* - W_{\min, \lambda, \gamma}\|^2_{H^2_2(\Omega)} \leq C_1 \delta. \]
(5.33)
Finally, using (5.19) and the triangle inequality, we obtain the following lower bound for the left-hand side of (5.33)
\[ \|W^* - W_{\min, \lambda, \gamma}\|^2_{H^2_2(\Omega)} = \|(W^* + G^*) - (W_{\min, \lambda, \gamma} + G) + (G - G^*)\|^2_{H^2_2(\Omega)} \]
\[ = \|(V^* - V_{\min, \lambda, \gamma}) + (G - G^*)\|^2_{H^2_2(\Omega)} \geq \|V^* - V_{\min, \lambda, \gamma}\|^2_{H^2_2(\Omega)} - \|G - G^*\|^2_{H^2_2(\Omega)} \geq \|V^* - V_{\min, \lambda, \gamma}\|^2_{H^2_2(\Omega)} - \delta. \]

Substituting this in (5.33), we obtain the target estimate (5.22). \( \square \)

**Corollary 5.1.** The functional \( I_{\lambda, \gamma}(W) := J_{\lambda, \gamma}(W + G) \) is strictly convex on \( B_0(2M) \) for all \( \lambda \geq \lambda_3 \), where \( \lambda_3 \) is the number defined in Theorem 5.4.

**Proof.** It follows from the proof of Theorem 5.4 and (5.1) that the following analog of (5.26) holds for all \( \lambda \geq \lambda_3 \) and for all \( W(1), W(2) \in B_0(2M) \)
\[ I_{\lambda, \gamma}(W(2)) - I_{\lambda, \gamma}(W(1)) - I'_{\lambda, \gamma}(W(1)) (W(2) - W(1)) \]
\[ \geq C_1 \|W(2) - W(1)\|^2_{H^2_2(\Omega)} + \gamma \|W(2) - W(1)\|^2_{H^2_2(\Omega)}. \square \]
6. The Globally Convergent Gradient Projection Method. Now we construct an approximation for the vector function \( W^* = V^* - G^* \) for \( W^* \in B_0(2M) \). It follows from (5.23) that \( B_0(2M) \subset H^3_{0,2N}(\Omega) \). Let \( P_{\Omega} : H^3_{0,2N}(\Omega) \to B_0(2M) \) be the orthogonal projection operator of the space \( H^3_{0,2N}(\Omega) \) on the closed ball \( B_0(2M) \). Let \( W^{(0)} \in B_0(2M) \) be an arbitrary point of the ball \( B_0(2M) \) and let \( \eta > 0 \) be a number. The gradient projection method constructs the following sequence:

\[
W^{(n)} = P_{\Omega} \left( W^{(n-1)} - \eta J' \left( W^{(n-1)} + G \right) \right), \quad n = 1, 2, \ldots
\]

It is important for computations that \( \left( W^{(n-1)} - \eta J' \left( W^{(n-1)} + G \right) \right) =: Y_{n-1}(x) \in H^3_{0,2N}(\Omega) \). Indeed, \( W^{(n-1)} \in H^3_{0,2N}(\Omega) \); also, (5.9) holds. In other words, the vector function \( Y_{n-1}(x) \) satisfies the boundary conditions \( Y_{n-1}\left|_{\partial \Omega} = \partial_y Y_{n-1} \right|_{x = 0} = 0 \).

**Theorem 6.1.** Let \( \lambda \geq \lambda_2 = \lambda_2(2M,N,r,\Omega) > 1 \), where \( \lambda_2 \) was defined in Theorem 5.1. Let \( \lambda_{\min,\gamma} \) be the minimizer of the functional \( J_{\lambda,\gamma}(W + G) \) on the set \( B_0(2M) \), the existence and uniqueness of which follow from Theorem 5.3 and Corollary 5.1. Then there exists a sufficiently small number \( \eta_0 = \eta_0(2M,N,r,\Omega,\lambda) \in (0,1) \) depending only on listed parameters such that for any \( \eta \in (0,\eta_0) \) we can find a number \( \theta = \theta(\eta) \in (0,1) \) such that the sequence \( \{W^{(n)}\}_{n \in \mathbb{N}} \) converges to \( \lambda_{\min,\gamma} \) in the \( H^3_{2N}(\Omega) \)-norm and the following convergence estimate holds:

\[
\left\| W_{\min,\gamma} - W^{(n)} \right\|_{H^3_{2N}(\Omega)} \leq \theta^n \left\| W_{\min,\gamma} - W^{(0)} \right\|_{H^3_{2N}(\Omega)}, \quad n = 1, 2, \ldots
\]

Theorem 6.1 follows immediately from the combination of Theorems 5.1–5.3 with Theorem 2.1 of [3].

**Theorem 6.2.** Let \( \lambda = \lambda_{4} = \lambda_2(2M,N,r,\Omega) > \lambda_2 \). Suppose that conditions imposed in Theorems 5.4 and 6.1 hold. Then the following convergence estimates are valid for \( n = 1, 2, \ldots \):

\[
\left\| W^* - W^{(n)} \right\|_{H^2_{2N}(\Omega)} \leq C_1 \delta + \theta^n \left\| W_{\min,\gamma} - W^{(0)} \right\|_{H^2_{2N}(\Omega)},
\]

\[
\left\| c^*(x) - c_n(x) \right\|_{L^2(\Omega)} \leq C_1 \delta + \theta^n \left\| W_{\min,\gamma} - W^{(0)} \right\|_{H^2_{2N}(\Omega)},
\]

where \( c^*(x) \) stands in the right hand side of equation (3.8) in the case when \( W^*(x) \) is replaced with \( V^*(x) = W^*(x) + G^*(x) \). The function \( v^*(x,\alpha) \) is obtained via components of the vector function \( V^*(x) \) and (3.12) and then this function is substituted in the left hand side of (3.9); see the first item of Remarks 3.1. The function \( c_n(x) \) is obtained in the same way with the only replacement of \( V^*(x) \) with \( V^{(n)}(x) = W^{(n)}(x) + G(x) \).

**Proof.** Combining (5.33) with (6.2) we obtain

\[
\left\| W^* - W^{(n)} \right\|_{H^2_{2N}(\Omega)} = \left\| (W^* - W_{\min,\gamma}) + (W_{\min,\gamma} - W^{(n)}) \right\|_{H^2_{2N}(\Omega)}
\]

\[
\leq \left\| W^* - W_{\min,\gamma} \right\|_{H^2_{2N}(\Omega)} + \left\| W_{\min,\gamma} - W^{(n)} \right\|_{H^2_{2N}(\Omega)}
\]

\[
\leq C_1 \delta + \theta^n \left\| W_{\min,\gamma} - W^{(0)} \right\|_{H^2_{2N}(\Omega)},
\]

which proves (6.3). As to (6.4), it follows from (6.3) and the construction of functions \( c^*(x), c_n(x) \) described in the formulation of Theorem 6.2. \( \square \)

**Remarks 6.1.**
1. Since the starting point \( W^{(0)} \) of the gradient projection method (6.1) is an arbitrary point of the ball \( B_0(2M) \) and since smallness conditions are not imposed on \( M \), then convergence estimates (6.3) and (6.4) mean the global convergence of the gradient projection method (6.1) to the correct solution. In other words, a good first guess about the ideal solution is no longer required. We note that in the case of a non convex functional, the global convergence of a gradient-like method cannot be guaranteed.

2. Another observation here is that due to estimate (5.22) we have “Lipschitz-like” convergence rate in (6.3), (6.4) with respect to \( \delta \). This is stronger than the “Hölder-like” convergence rates in all previous publications [24, 25, 26, 27, 28] about the convexification where \( \delta \) is replaced with \( \delta^\rho \), where \( \rho \in (0, 1) \), also, see Remark 5.1.

7. Numerical Results.

7.1. Some details of the numerical implementation. First, we note that it follows from (3.7), (3.12) and (3.19) that

\[
V |_{\partial \Gamma \setminus \Gamma_r} = 0.
\]

Let \( \{\alpha_n\}_{n=0}^{\ell-1} \) be the set of selected ode points on the segment \([-a, a] \), i.e. \( -a = \alpha_0 < \alpha_1 < \cdots < \alpha_{\ell-1} = a \), \( \alpha_n - \alpha_{n-1} = h_s \), where \( h_s > 0 \) is the grid step size. Thus, in our computations, our sources are \( \{x_{\alpha_n}\}_{n=0}^{\ell-1} = \{(\alpha_n, 0, -d)\}_{n=0}^{\ell-1} \subset \text{src} \); see (2.4). To minimize the functional \( J_{\lambda, \gamma}(V) \) in (4.2), we rewrite the involved derivatives via finite differences. The idea is to minimize the resulting functional with respect to the values of \( V \) at grid points. We use the same grid step size \( h \) in \( x, y, z \) directions. For any vector function \( u(x) \) we denote \( u_{p,q,s} = u(x_p, y_q, z_s) \) the corresponding discrete function defined at grid points.

We define the Laplace operator in finite differences as \( \Delta^h u_{p,q,s} = \partial_{xx}^h u_{p,q,s} + \partial_{yy}^h u_{p,q,s} + \partial_{zz}^h u_{p,q,s} \), where, for interior grid points of \( \Omega \) we use, e.g.,

\[
\partial_{xx}^h u_{p,q,s} = h^{-2} (u_{p+1,q,s} - 2u_{p,q,s} + u_{p-1,q,s}).
\]

Then, the gradient operator in finite difference \( \nabla^h u_{p,q,s} = (\partial_x^h u_{p,q,s}, \partial_y^h u_{p,q,s}, \partial_z^h u_{p,q,s}) \) follows. In addition, the data at \( \Gamma \) are given by \( \partial_{\nu}^h u_{p,q,0} = h^{-1} (u_{p,q,1} - u_{p,q,0}) \).

We have applied the matrix \( S_N^{-1} \) in (3.17) to obtain equation (3.14). However, this is convenient only for the above theory. In computations we do not apply \( S_N^{-1} \). The resulting matrix equation is equivalent to (3.14) and analogs of the above Theorems 5.2-5.4, 6.1, 6.2 can be straightforwardly formulated for the continuous form of functional (7.2), which is the direct analog of functional (4.2). Now about the computational implementation of boundary conditions. Rather than satisfying boundary conditions exactly, we minimize the differences between boundary values of the discrete vector function \( V_{p,q,s} \) and boundary conditions. To do this, we use penalty terms with certain weights \( K_0, K_1, K_2 > 0 \). These weights are chosen numerically.

Let \( V^h = \{V_{p,q,s}^h\}_{p,q,s=0}^{2h-1} \) be the discrete version of the vector function \( V \). Thus, taking into account (3.14)–(3.16) and (7.1), for each \( N \geq 1 \) we minimize the following fully discrete form of the weighted Tikhonov-like \( J_{\lambda, \gamma}(V) \):

\[
\text{This manuscript is for review purposes only.}
(7.2)

\[ J_{\lambda,\gamma}^h (V^h) = e^{-2\lambda(R+r)^2} \sum_{p,q,s=0}^{Z_h-1} h^3 \left( \sum_{\ell=0}^{Z_h-1} S_{\alpha} \Delta^h v_{p,q,s} + f \left( \nabla^h v_{p,q,s} \right) \right)^2 + K_0 \left| V_{p,q,0}^h - \psi_{0,p,q}^h \right|^2 + K_1 \left| \partial_z^h V_{p,q,0}^h - \psi_{1,p,q}^h \right|^2 \]

\[ + \sum_{p,q=0}^{Z_h-1} h^2 \left( K_0 \left| V_{p,q,0}^h - \psi_{0,p,q}^h \right|^2 + K_1 \left| \partial_z^h V_{p,q,0}^h - \psi_{1,p,q}^h \right|^2 \right) \]

\[ + K_2 \sum_{p,q=0}^{Z_h-1} h^2 \left| w_{p,q,Z_h-1}^h \right|^2 + K_2 \sum_{q,s=0}^{Z_h-1} h^2 \left( \left| w_{0,q,s}^h \right|^2 + \left| w_{Z_h-1,q,s}^h \right|^2 \right) \]

\[ + K_2 \sum_{p,s=0}^{Z_h-1} h^2 \left( \left| w_{p,0,s}^h \right|^2 + \left| w_{p,Z_h-1,s}^h \right|^2 \right) + \gamma \sum_{p,q,s=0}^{Z_h-1} h^3 \left( \left| V_{p,q,s}^h \right|^2 + \left| \nabla^h V_{p,q,s}^h \right|^2 \right). \]

It follows from (7.2) that we consider the regularization term in the \( H^1 \) norm. Indeed, it is much more complicated to implement the \( H^3 \) norm. On the other hand, if the number of grid points is not too large, then these norms, taken in the discrete form, are “effectively” equivalent. Our numerical experience shows that the \( H^1 \) norm in the regularization term is sufficient. Terms with \( K_0, K_1, K_2 > 0 \) are those penalty terms mentioned above with respect to the boundary conditions. Since these terms are convex, then they do not ruin the strict convexity of our functional. Numbers \( K_0, K_1, K_2 \) are chosen numerically and will be specified in subsection 7.2.2.

7.2. Numerical studies.

7.2.1. Generic algorithm. For each \( \alpha \in [-a, a] \), we computationally simulate the data (2.9) by solving the Lippmann–Schwinger equation (2.8). Thus, on the grid of \( x \in \Gamma \) and \( x_0 \in L_{\text{src}} \) one has \( u_{p,q,s} = F(x_p, y_q, z_s, x_{\alpha_l}) \) for each \( l = 0, \ell - 1 \). Given \( N \) in (3.12) and \( k > 0 \), our analysis then leads to the following algorithm:

1. To generate the data \( \psi_j(x_p, y_q, z_s) \) for \( j = 0,1 \) in (3.15), (3.16), solve the Lippmann–Schwinger equation (2.8).
2. Compute the minimizer \( V_{\text{min}}^h \) of the functional \( J_{\lambda,\gamma}^h (V^h) \) in (7.2) as an approximation of \( V^h \).
3. Using the minimizer of item 2 and the special basis \( \{ \Psi_n(\alpha) \}_{n=0}^{N-1} \), construct an approximation of the auxiliary function \( v(x_p, y_q, z_s, \alpha_l) \);
4. Compute an approximation of the unknown dielectric constant \( c(x_p, y_q, z_s, k) \) by the following formulae:

\[ c_{p,q,s} = \text{mean}_{\alpha_l} \left[ \frac{\Delta^h v_{p,q,s,\alpha_l} + \left( \nabla^h v_{p,q,s,\alpha_l} \right)^2}{k^2} + 2\nabla^h v_{p,q,s,\alpha_l} \cdot \hat{x}_{p,q,s,\alpha_l} \right] + 1, \]

where \( \hat{x}_{p,q,s,\alpha_l} \) denotes the value of \( \hat{x}_a \) at \( (x_p, y_q, z_s) \) for every \( \alpha_l \); see (3.11).

We use the absolute value here since one should have \( c_{p,q,s} \geq 1 \), see (2.3).

7.2.2. Computational procedure. We now describe some details of the algorithm of section subsection 7.2.1. The domain under consideration is now the three-dimensional cube with the edge length \( R = 3 \). Besides, we choose \( a = 1, \gamma = 10^{-4} \), \( d = 7.5, N = 4, Z_h = 51 \) and \( \ell = 11 \). Also, we choose \( K_0 = 1, K_1 = 2 \) and \( K_2 = 10^{-3} \) in (7.2) for all steps. The wavenumber \( k \) is specified below. Even though our above
theory says that we need to use the gradient projection method and large values of \( \lambda \), we have discovered computationally that the simpler to implement gradient descent method and a reasonable value \( \lambda = 1.1 \) work well. So, we use them. These observations coincide with those of our previous works on the numerical studies of the convexification [24, 25, 26, 27, 28].

The choice of \( N \) can be specified as follows. We take a reference sample to be imaged. This is the one of Test 1; see Table 1 for details. Once chosen, we keep \( N \) the same for all other examples. First, we solve the Lippmann–Schwinger equation (2.8) to generate the data for the reference inclusion of Test 1. Then we obtain \( u_{\text{true}}(x, \alpha) \) and \( v_{\text{true}}(x, \alpha) \), respectively. Denote \( v_{\text{true},n}(x) \) the corresponding Fourier coefficient of \( v_{\text{true}}(x, \alpha) \) with respect to our basis \( \{ \Psi_n(\alpha) \}_{n=0}^{\infty} \). Having \( v_{\text{true},n}(x) \) numerically, we can compute the function \( v_{\text{true},n}(x, \alpha) \) in (3.12). Hence, we are able to compute the following relative \( L^\infty \)-like error:

\[
E_\infty(v_{\text{true}}) = \frac{\max_{x, \alpha} |v_{\text{true}} - v_{\text{true}}^N|}{\max_{x, \alpha} |v_{\text{true}}|} \times 100\%.
\]

We observe in Figure 2a that \( N = 4 \) is acceptable in the sense that the error \( E_\infty(v_{\text{true}}) \) is sufficiently small (around \( 5 \times 10^{-2} \)). Hence, it is intuitively clear that increasing \( N \) would only increase the computational time without providing an essential difference in results. This choice of \( N \) is in an agreement with, e.g., [?] where a truncation was done to solve an inverse boundary value problems of an elliptic equation.

As the number of point sources may not be large in practice, we compute \( v_{\text{true},n} \) by using the Gauss–Legendre quadrature method:

\[
v_{\text{true},n}(x) := \langle v_{\text{true}}(x, \cdot), \Psi_n(\cdot) \rangle_{L^2(-a,a)} \approx \sum_{l=0}^{\ell} w_l v_{\text{true}}(x, \tilde{\alpha}_l) \Psi_n(\tilde{\alpha}_l),
\]

where \( \tilde{\alpha}_l \) are the abscissae in \([-a, a]\) and \( w_l \) are the corresponding weights. Since these abscissae are fixed, we get the values of \( v_{\text{true}}(x, \tilde{\alpha}_l) \) from \( v_{\text{true}}(x, \alpha_l) \) using the spline interpolation via the built-in griddedInterpolant in MATLAB. As we compute the

---

Fig. 2: (a) The choice of \( N \) based on the relative \( L^\infty \) error. (b) Cross-section by the plane \( \{ x = 0 \} \) of the image computed in Test 2 after finishing the first step of computations.
basis \( \Psi_n \) symbolically, we know their values at the abscissae very precisely.

Following the algorithm of subsection 7.2.1, we now detail the computational procedure in the minimization process applied to the functional (7.2). Recall that the Dirichlet data at \( \partial \Omega \setminus \Gamma \) are obtained by the heuristic data completion (3.19) and this completion forms the final item of our approximate mathematical model (the first item of Remarks 3.1). Since the magnitude of the backscattering data is small and since the correct boundary data on \( \partial \Omega \setminus \Gamma \) are actually neglected by (3.19), then one can anticipate that only those targets will be reasonably imaged, which are close to the measurement side \( \Gamma \) of the cube \( \Omega \). In other words, the data propagation procedure mentioned in subsection 3.3 should estimate well distances to targets, and it was shown in section 6.2 of [33] for the case of experimentally collected data that this latter is possible. It is also clear that this very limited information indicates that our reconstruction is very challenging.

On the other hand, it follows from (4.3) and Theorem 5.1 that all iterates of the gradient descent method we use should have the same Dirichlet boundary conditions at \( \partial \Omega \) and the same Neumann boundary condition at \( \Gamma \) (i.e. at \( z = -R \)). Furthermore, our computational experience shows that since the correct boundary data on \( \partial \Omega \setminus \Gamma \) are not given, then we need to have a partly heuristic method of estimating the location and sizes of the target to be imaged. These cause our choice of the starting point of iterations as well as the choice of first and second steps of our numerical procedure described below. Nevertheless, the main point is that our choices still use only the measured data and do not rely on any information about a small neighborhood of the correct solution.

**Step 1.** This step consists of two (2) substeps. As it was pointed out in the Introduction section, the focus of our application is the detection and identification of antipersonnel land mines and IEDs. The sizes of these targets are usually small: between 5 and 15 centimeters (cm). Therefore, we search for targets in the domain \( \Omega_1 := \{ -R \leq z \leq -R + 2 \} \subset \Omega \), which means 20 cm in depth from the ground boundary. In the first substep, we choose the following linear approximation, denoted by \( V_0 (x) \in \mathbb{R}^N \), as the starting point of iterations in the minimization of the functional \( J_{h,\gamma}^V (V^h) \):

\[
\begin{align*}
V_0 (x) &= (v_{00} (x), v_{01} (x), \ldots, v_{0(N-1)} (x))^T, \\
v_{0n} &= (\psi_{0n} + \psi_{1n} (z + R)) \chi(z),
\end{align*}
\]

where \( \chi : [-R, R] \to \mathbb{R} \) is a smooth function given by

\[
\chi(z) = \begin{cases} 
\exp \left( \frac{(z+R)^2}{(z+R)^2-(R-1)^2} \right) & \text{if } z < -1, \\
0 & \text{otherwise.}
\end{cases}
\]

It is worth noting that \( \chi \) attains the maximum value of 1 at \( z = -R \) and then, we can show that \( v_{0n} (z = -R) = \psi_{0n}, \partial_z v_{0n} (z = -R) = \psi_{1n} \). Thus, the starting point \( v_{0n} \) satisfies the Dirichlet and Neumann boundary conditions at \( z = -R \). Besides, the vector function \( V_0 \) satisfies the zero Dirichlet boundary condition at \( z = R \) because \( \chi \) tends to 0 as \( z \to -1^+ \). We have numerically observed that \( \psi_{0n} = \psi_{1n} = 0 \) at \( x, y = \pm R \). Hence, the vector function defined in (7.4) satisfies the zero Dirichlet boundary conditions. On the first substep, we minimize the functional \( J_{h,\gamma}^V (V^h) \) with the starting point (7.4).

As to the step size \( \eta \) of the gradient descent method, we start from \( \eta_1 = 10^{-1} \). This \( \eta_1 \) is unchanged on the first step as well as the second step below. On each
iterative step number \( m \) the step size \( \eta_m \) is reduced by the factor of 2 if the value of the functional at the iteration number \( m \) exceeds its value on the previous iteration, i.e. if \( J_{\lambda, \gamma}^h (V_m^h) \geq J_{\lambda, \gamma}^h (V_{m-1}^h) \). Otherwise, \( \eta_{m+1} = \eta_m \). The minimization process is stopped at \( m_{\text{stop}} \) when either \( \eta_{m_{\text{stop}}} < 10^{-8} \) or \( \left| J_{\lambda, \gamma}^h (V_{m_{\text{stop}}}^h) - J_{\lambda, \gamma}^h (V_{m_{\text{stop}}-1}^h) \right| < 10^{-8} \).

After the first substep, we obtain the numerical coefficient of \( c \), denoted by \( \hat{c}(x) \). Then we complete the first step of computations by getting rid of possible artifacts in \( \hat{c} \) via a postprocessing, which forms the second substep. This is done by replacing the function \( \hat{c} \) with the function \( c_{\text{temp}} \), where

\[
c_{\text{temp}}(x) = \begin{cases} \hat{c}(x) & \text{if } |\hat{c}(x)| \geq 0.7 \max_x |\hat{c}(x)|, \\ 0 & \text{otherwise}. \end{cases}
\]

Lastly, the function \( c_{\text{temp}} \) is smoothed by a Gaussian filter. We will shortly specify the Gaussian procedure in the end of Step 2.

**Step 2.** We have numerically observed that, in the first step, we can find a good approximation for the location of the target and a somewhat good approximation of its shape. The second step is for an improvement of the values of the function \( c(x) \). We start this step by plugging \( c_{\text{temp}} \) in the Lippmann-Schwinger equation \( (2.8) \), solving it, and thus, obtaining a new vector function \( \tilde{v}(x, \alpha) \). Then we find first \( N \) Fourier components of \( \tilde{v}(x, \alpha) \), denoted by \( \tilde{v}_n(x) \), as in \( (3.12) \). Note that after Step 1, we can see where the object is located by an evaluation of 2D cross-sections of the image obtained after Step 1; see e.g. Figure 2b. This way enables us to narrow the domain of our search, i.e. we look for the object in \( \Omega_2 := \{-b_x \leq x \leq b_x, -b_y \leq y \leq b_y, -R \leq z \leq -b_z \} \subset \Omega_1 \), for some numbers \( b_x, b_y, b_z > 0 \). We rely on this information to get a smooth function, denoted by \( \tilde{\chi}(x) \), with \( \tilde{v}_n \) and ensure the boundary conditions during the minimization. Recall the function \( \chi \) in \( (7.6) \) which is essentially generated by the function \( \exp \left( \frac{t^2}{t^2 - 1} \right) \) for \( t \in [0, 1] \) tending to 0 as \( t \to 1^+ \) and attaining the maximum 1 at \( t = 0 \). Moreover, the derivative of this function vanishes at \( t = 0 \). Thereby, the starting point of iterations of the gradient descent method for the second stage of the minimization of the functional \( J_{\lambda, \gamma}^h (V^h) \) is chosen, as follows:

\[
V_1(x) = \begin{pmatrix} v_{10}(x) & v_{11}(x) & \cdots & v_{1(N-1)}(x) \end{pmatrix}^T \quad \text{for } v_{1n} = \tilde{v}_n(x) \tilde{\chi}(x),
\]

where

\[
\tilde{\chi}(x) = \begin{cases} \exp \left( \frac{x^2}{x^2 - b_x^2} \right) \exp \left( \frac{y^2}{y^2 - b_y^2} \right) \exp \left( \frac{(z+R)^2}{(z+R)^2 - (R-b_z)^2} \right) & \text{if } x \in \Omega_2, \\ 0 & \text{otherwise}. \end{cases}
\]

The second step allows us to obtain more accurate values of \( c(x) \) inside of the inclusion to be imaged. Our reconstruction is concluded after we smooth the final solution \( c_{\text{final}}(x) \) by the Gaussian filtering via the `smooth3` function in MATLAB. In particular, we find \( c_{\text{comp}}(x) \) as \( c_{\text{comp}}(x) = \text{smooth}(|c_{\text{final}}(x)| (1 + \hat{p})) \), where \( c_{\text{final}}(x) \) is the function \( c(x) \) obtained in the last iterative step of the minimization procedure of Step 2. This procedure is similar to the one of smoothing \( c_{\text{temp}}(x) \) on Step 1. Note that due to \( (7.3) \), \( c_{\text{final}}(x) > 0 \). Here, finding the value of \( \hat{p} \) is based upon the maximal value of \( |c_{\text{final}}(x)| \). This maximal value is computed with a good accuracy. Next, however, we need to smooth the function \( c_{\text{final}}(x) \) using the Gaussian filtering. The smoothed
Table 1: Description of numerical examples and the corresponding relative errors between $\max(c_{\text{true}})$ and $\max(c_{\text{comp}})$.

| Test | Inclusion | 1 | 2 | 3 | 4 | 5 |
|------|-----------|---|---|---|---|---|
|       | Sphere    | Ellipsoid | Ellipsoid | Rectangle | Sphere (noise) |
| $\max(c_{\text{true}})$ | 2 | 5 | 10 | 2 | 2 |
| $\max(c_{\text{comp}})$ | 1.8873 | 5.1886 | 9.3461 | 2.1834 | 1.8782 |
| Error (7.7) | 5.64% | 3.77% | 6.54% | 9.17% | 6.09% |

Table 2: Comparison of the location information between the true and computed objects. The true lowest point of all true objects is fixed at $(0, 0, -2.8)$.

| Test | True center | Computed center | Lowest point |
|------|-------------|----------------|--------------|
| 1    | $(0, 0, -2.5)$ | $(0, 0, -2.4)$ | $(0, 0, -2.823)$ |
| 2    | $(0, 0, -2)$  | $(-0.01, -0.02, -1.98)$ | $(0, 0, -2.81)$ |
| 3    | $(0, 0, -2)$  | $(-0.02, -0.03, -2.13)$ | $(0, 0, -2.58)$ |
| 4    | $(0, 0, -2.5)$ | $(0, -0.01, -2.35)$ | $(-0.01, 0, -2.79)$ |
| 5    | $(0, 0, -2.5)$ | $(-0.01, 0, -2.39)$ | $(0, 0, -2.83)$ |

version of $c_{\text{final}}(x)$ always has a lower maximal value. Therefore, we find such a number $\tilde{p} \geq 0$ that $\max(c_{\text{comp}}(x)) = (1 + \tilde{p})\max(\text{smooth}(c_{\text{comp}}(x)))$. Hence, the value of $\tilde{p}$ varies in every single test. As an example, in Test 1 $\max|c_{\text{final}}(x)| = 1.8873$. But the smoothed version $\text{smooth}(|c_{\text{final}}(x)|)$ without $\tilde{p}$ attains the maximal value 1.6446, while with $\tilde{p} = 0.3765$ it moves back to the value 1.8873. The above forms the second step of the numerical solution of our CIP.

Remark 7.1. We point out that the computational procedure described above does not use any advanced knowledge of a small neighborhood of the correct solution. In other words, our reconstruction method converges globally; see Introduction for the definition of the global convergence.

We now mention that to run the minimization procedure, we need to compute the gradient $J'_\lambda\gamma$ in (6.1) of the discrete functional $J_{\lambda\gamma}$ in (7.2). We have discovered that having the expression via an explicit formula, significantly reduces the computational time. We have derived such a formula using the technique of Kronecker deltas, which has been outlined in [31]. For brevity we do not provide this formula here.

7.2.3. Reconstruction results. To this end, the dimensionless spatial variables are defined as $x' = x/(10 \text{ cm})$. This means that, for instance, in our reference Test 1, where the mine-like target is ball-shaped with the radius 0.3, its actual diameter is 6 cm. Therefore, with $R = 3$ considered in subsection 7.2.2 we suppose to look for a possible explosive in a cubic area $0.216 \text{ m}^3$, where $m$ stands for meters. We set in (2.4) $d = 7.5$, which means that the distance between the line of sources, whose length is 20 cm, and the ground surface should be 45 cm, which is realistic for some ground penetrating radars.

Now, since $k = 2\pi/\nu$, where $\nu$ is the wave length, then, after the change of variables $x' = x/(10 \text{ cm})$ in the Helmholtz equation (2.5), the dimensionless number $k = 2\pi \cdot (10 \text{ cm})/\nu$. Following the previous publication of this group about the work with experimental data [33], we use here the frequency of 3.15 GHz, which means that
the wavelength $\nu = 9.5$ cm. Therefore, the dimensionless value of $k$ we work with is

$$k = 2\pi \cdot (10 \text{ cm}) / (9.5 \text{ cm}) = 6.6.$$ 

In our Tests 1–5 listed below, we want to accurately reconstruct all three components of targets: locations, target/background contrasts and shapes.

In all our tests, the mine-like objects are such that the lowest point of their front surfaces is fixed at $(x, y, z) = (0, 0, -2.8)$. However, when running the minimization procedure, we do not assume a knowledge of neither the locations nor the shapes of inclusions. We only assume that any inclusion of our interest is located close to the ground surface $\Gamma$, which concerns the choice of the smooth function $\chi$ in subsection 7.2.2.

It was demonstrated in [33] that the data propagation procedure mentioned in subsection 3.3 estimates quite accurately the $z$–coordinate of the front surface of the target. Hence, we indeed can assume that the plane $\{z = -R\}$ is close to the target to be imaged. On the other hand, if it would be far, then the image would be worse. Indeed, in this case the backscattering data at $\Gamma$ would be less sensitive to the presence of the inclusion and, at the same time, the transmitted data at $\Pi \cap \{z = R\}$ are artificially set not to be sensitive to this presence.

In the reconstruction results, we are concerned with the relative error between $\max(c_{\text{true}}(x))$ and $\max(c_{\text{comp}}(x))$, where $c_{\text{comp}}(x)$ is the computed function $c(x)$. More precisely, we define this error as

$$E_{\text{max}} = \frac{|\max(c_{\text{true}}(x)) - \max(c_{\text{comp}}(x))|}{\max(c_{\text{true}}(x))} \times 100\%.$$ 

Values of $\max(c_{\text{true}}(x)), \max(c_{\text{comp}}(x))$ and $E_{\text{max}}$ for all five tests are tabulated in Table 1. In the following, we depict both three-dimensional true and computed inclusions by using the isosurface function in MATLAB with the associated isovalue being 5% of the maximal value. As to the locations of computed inclusions, we briefly report them in Table 2.

**Test 1. Ball-shaped inclusion.** In this test, we examine our numerical method for the case of a ball-shaped object with the dielectric constant in it $c_{\text{true}} = 2$. Images
of the true object and its reconstruction are presented in Figure 3. We observe in
Figure 3a that the shape of the reconstructed object is imaged accurately. We also
obtain that the lowest point of the computed one is at (0, 0, −2.823), while it should
be (0, 0, −2.8) for the true solution as we have set up above. So, these are close. The
size of the computed inclusion is slightly larger than the correct one. Together with
the approximate dielectric constant, we conclude that location, shape and \( \max(c) \)
are reconstructed accurately.

**Tests 2–3. Ellipsoids.** The next two tests, Tests 2 and 3, are about ellipsoidal
targets with high target/background contrast levels of 5 and 10. Here, we consider the
ellipsoids with principal semi-major axis and two semi-minor axes, respectively, being
0.8 and 0.3. The objects are centered at (0, 0, −2). The computational results show
that the sizes of reconstructed ellipsoidal targets with \( c_{true} = 5, 10 \) decrease when
\( c_{true} \) increases. Here is an explanation on this. We have computationally observed
that the area of the region on \( \Gamma \), where the data concentrate around the maximal
absolute value, decreases when \( c_{true} \) increases from 5 to 10. We still have observed
the ellipsoidal shape of the reconstructed target; see Figure 4c. Besides, the center
of the computed one with \( c_{true} = 10 \) is close to the true position and the number
\( \max(c_{comp}(x)) \) is still good; see, e.g., Tables 1 and 2. Therefore, the results for
\( c_{true} = 10 \) still accommodate the above listed main purposes of our reconstructions.
Last but not least, the reconstruction of the number \( c_{true} = 5 \) is very accurate.

**Remark 7.2.** We point out that inclusions with high contrasts of, e.g., Tests 2–3
are unlikely to be imaged by conventional techniques.

**Test 4. Rectangular prism.** In this test, we consider a rectangular prism with
dielectric constant \( c_{true} = 2 \). This object is of the size 1.2 × 1.2 × 0.6 (W×L×H),
which is realistic for the above-mentioned targets. The reconstruction result for this
case is displayed on Figure 5. The lowest point of the reconstructed object is located
at the point (0, 0, −2.79) while the correct point is (0, 0, −2.8). We see that this result
still fulfills the main purpose of our reconstructions listed above in this section; see
Tables 1 and 2 and Figure 5. It should be noticed that this prism’s boundary regularity
is mathematically weaker than the one of the spherical object. This explains why we
obtain a smooth shape rather than the one with a sharp boundary.
Fig. 5: Reconstruction of a rectangular prism with the correct value of the coefficient $c_{\text{true}} = 2$ (Test 4). The maximal value $\max(c_{\text{comp}})$ of the computed coefficient $c$ is 2.1834. (a) True image. (b) Reconstructed image.

Test 5. Ball-shaped inclusion with a noise in the data. In the last Test 5, we concentrate on the reconstruction with noisy data $F$ and $G$ defined, respectively, in (2.9) and subsection 3.3. In doing so, we simply add a random multiplicative noise to the simulated Dirichlet and Neumann data obtained when solving the Lippmann–Schwinger equation:

$$F_{\text{noise}}(x, \alpha) = F(x, \alpha) (1 + \delta_{\text{rand}}), \quad G_{\text{noise}}(x, \alpha) = G(x, \alpha) (1 + \delta_{\text{rand}}).$$

Here, $\delta \in (0, 1)$ represents the noise level and “rand” is a random number uniformly distributed in the interval $(-1, 1)$. Thus, this addition is pointwise with respect to both spatial points and the point source. Given $\delta = 0.05$, which corresponds to the 5% noise, we choose to use this noisy data in our Test 1. One can see from Figure 3c that our reconstruction still has a good performance.

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