Smoothed Correlators For Symmetric Double-Well Matrix Models: Some Puzzles and Resolutions

E. Brézin\textsuperscript{1} and N. Deo\textsuperscript{2}
\textsuperscript{1} Laboratoire de Physique Théorique
Ecole Normale Supérieure
24 rue Lhomond 75231,
Paris Cedex 05, France
\textsuperscript{2} Jawaharlal Nehru Centre
for Advanced Scientific Research
Bangalore 560064, India
Edouard.Brezin@physique.ens.fr
ndeo@jncasr.ac.in

Abstract
Some puzzles which arise in matrix models with multiple cuts are presented. They are present in the smoothed eigenvalue correlators of these models. First a method is described to calculate smoothed eigenvalue correlators in random matrix models with eigenvalues distributed in a single-cut, previous know results are reproduced. The method is extended to symmetric two-cut random matrix models. The correlators are written in a form suitable for application to mesoscopic systems. Connections are made with the smooth correlators derived using the Orthogonal Polynomial (OP) method. A few interesting observations are made regarding even and odd density-density correlators and cross-over correlators in $Z_2$ symmetric random matrix models.
1 Introduction

Matrix models have been used in a wide variety of applications, starting from quantum chaotic systems to condensed matter, QCD and string theory. The recent period has seen a large increase in our understanding of the properties of these models. In this work we have been interested in highlighting some unusual properties of two-cut random matrix models that have arisen in our study. The results are unexpected as they are not seen in matrix models when the density of eigenvalues has a connected support. Indeed there it is well-known \[1, 2\] that the correlator is universal, i.e. independent of the specific potential \(V\) which defines the probability measure. This is the basis for the theory under the universality of conductance fluctuations in mesoscopic systems \[3\]. At first sight one is tempted to think that this universality persists when the potential is such that the support splits into disconnected segments. But it is found that, if indeed it is again universal, it belongs to a different universality class. If the standard large \(N\)-limit (the random matrices are \(N \times N\)) yields the smoothed correlation functions up here to an arbitrary constant, different methods report different results for this constant. Furthermore there are differences between these correlators when the size \(N\) of the matrices is an even or an odd integer. It is a rather intriguing phenomenon and, for instance, it is not clear how the naive renormalization group approach \[4\] which consisted of integrating out one line and one row, could deal with such situations. We attempt here to understand and to give a unified picture of these results.

The paper is divided as follows. It starts by establishing the notation and conventions and describes completely the method used for the model with a single-cut density of eigenvalues. Previously known results are reproduced \[1, 2, 3\]. Then the method is extended to the model with two-cuts in the density of eigenvalues, restricted to symmetric potentials. Afterwards we develop the formalism to include asymmetric potentials. Here an arbitrariness remains as the constraint on the filling factor of the two parts of the support is not fixed at leading order in the large \(N\)-limit. The large \(N\)-equations for the correlator leave us with an undetermined constant \(C\). Previous methods using the orthogonal polynomials and loop equations give different results for this constant. The orthogonal polynomial method is briefly outlined and the resulting correlators are sensitive to the even and oddness of the number of eigenvalues. Further, the constant \(C\) is different for the even and odd correlators found by the orthogonal polynomial method and that found by the loop equations. The conclusion summarizes these
results and attempts to give an explanation of these puzzles.

2 Notation, Conventions

We establish the notations and conventions and develop a method, which we extend to the two-cut model, to derive eigenvalue correlators for random matrix models with a single cut density of eigenvalues.

Let us work with the unitary invariant ensemble of random Hermitian $N \times N$ matrices, with a probability distribution

$$P(M) = \frac{1}{Z} \exp(-NTrV(M)).$$

Define the operator for the density of eigenvalues

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \lambda_i)$$

and

$$\bar{\rho}(x) = \langle \rho(x) \rangle = \int P(M) \frac{1}{N} Tr\delta(x - M) dN^2 M.$$

Since $P(M)$ gives a factor $\exp(-N^2 \int V(x) \rho(x) dx)$ we have

$$\frac{\delta \bar{\rho}(x)}{\delta V(y)} = -N^2 <\rho(x)\rho(y)>_c.$$  

In the large $N$ limit, we know that

$$\int_{a}^{b} \frac{\bar{\rho}(y)}{x-y} dy = \frac{1}{2} V'(x).$$

The solution is found through the averaged resolvent

$$G(z) = \langle \frac{1}{N} Tr \frac{1}{z-M} \rangle = \int_{a}^{b} \frac{\bar{\rho}(y)}{z-y} dy$$

$$= \frac{1}{2} V'(z) - P(z) \sqrt{(z-a)(z-b)}.$$  

Then

$$\pi \bar{\rho}(\lambda) = P(\lambda) \sqrt{(\lambda-a)(b-\lambda)}.$$  

We then need to express $P$ as a functional of $V$. There are various equivalent expressions, and in the following several will be needed. We follow here [3]
and begin by multiplying eq. \((2.6)\) by \(\frac{\sqrt{(z-a)(z-b)}}{z-u}\) and integrate \(z\) over a large circle \(C\), see Fig. 1.

Since \(G(z) \approx \frac{1}{2}\) at infinity

(i).

\[
\oint_{C} \frac{G(z)\sqrt{(z-a)(z-b)}}{z-u} \frac{dz}{2i\pi} = 1
\]  \hspace{1cm} (2.8)

(ii).

\[
\oint_{C} \frac{P(z)\sqrt{(z-a)(z-b)}}{z-u} \frac{dz}{2i\pi} = P(u)(u-a)(u-b)
\]  \hspace{1cm} (2.9)

(iii).

\[
\oint_{C} \frac{V'(z)\sqrt{(z-a)(z-b)}}{z-u} \frac{dz}{2i\pi} = \frac{1}{\pi} \int_{a}^{b} \frac{V'(x)\sqrt{(x-a)(b-x)}}{x-u} dx
\]

\[+ P(u)(u-a)(b-u). \]  \hspace{1cm} (2.10)

Therefore we obtain from eq. \((2.4)\)

\[
1 = \frac{1}{2} V'(u)\sqrt{(u-a)(u-b)}
\]

\[- \frac{1}{2\pi} \int_{a}^{b} \frac{V'(x)\sqrt{(x-a)(b-x)}}{x-u} dx + P(u)(u-a)(b-u). \]  \hspace{1cm} (2.11)

Let \(u\) approach the real axis on the cut. The integral in eq. \((2.11)\) has an imaginary part which cancels the first term of the right hand side which is pure imaginary. The real part of eq. \((2.11)\) gives

\[
P(\lambda)(\lambda-a)(b-\lambda) = 1 + \frac{1}{2\pi} \int_{a}^{b} \frac{V'(x)\sqrt{(x-a)(b-x)}}{x-\lambda} dx
\]

\[i.e. \hspace{1cm} \rho(\lambda) = \frac{1}{\pi} \frac{1}{\sqrt{(\lambda-a)(b-\lambda)}} \]

\[\left[1 + \frac{1}{2\pi} \int_{a}^{b} \frac{V'(x)\sqrt{(x-a)(b-x)}}{x-\lambda} dx \right]. \]  \hspace{1cm} (2.13)
Now we vary $V$, let us first ignore the variation of $a$ and $b$ (it is proved to be right below). Then
\[
\frac{\delta \bar{\rho}(\lambda)}{\delta V(\mu)} = \left( \frac{\partial \bar{\rho}(\lambda)}{\partial V(\mu)} \right)_{a,b} + \left( \frac{\partial \bar{\rho}(\lambda)}{\partial a} \right)_{V,b} \frac{\delta a}{\delta V(\mu)} + \left( \frac{\partial \bar{\rho}(\lambda)}{\partial b} \right)_{V,a} \frac{\delta b}{\delta V(\mu)}
\]
(2.14)
on the right hand side (r.h.s.) $\bar{\rho}$ is being treated as a function of $V,a,b$ as given on the r.h.s. of eq. (2.13). We show later that $\left( \frac{\partial \bar{\rho}}{\partial a} \right)_{V,b} = 0$. $\left( \frac{\partial \bar{\rho}}{\partial a} \right)_{V,b}$ is of course not the total derivative of $\bar{\rho}$ w.r.t. $a$. Then
\[
\frac{\delta \bar{\rho}(\lambda)}{\delta V(\mu)} = \frac{1}{2\pi^2} \frac{1}{\sqrt{(\lambda - a)(b - \lambda)}} \frac{\partial}{\partial \mu} \sqrt{(\mu - a)(b - \mu)}
\]
(2.15)
and one verifies easily that the result is symmetric under exchange of $\lambda$ and $\mu$ as it should. Note that the potential $V$ has disappeared from the correlator, except indirectly through the end points $a$ and $b$ of the cut. This universality follows here trivially from the linearity of the $(\rho, V)$ relation. Terms of the type
\[
\left( \frac{\delta \bar{\rho}}{\partial a} \right)_{V,b} \frac{\delta a}{\delta V(\mu)}
\]
(2.16)
have been ignored. The claim is that they vanish, but that’s the only (slightly) tricky part. The representation eq. (2.13) is appropriate, among several other possibilities, because if one differentiates inside the integral with respect to $a$, it is still a meaningful integral. So let us calculate
\[
\left( \frac{\partial \bar{\rho}}{\partial a} \right)_{V,b} = -\frac{1}{2} \frac{1}{\lambda - a} \bar{\rho}(\lambda) + \frac{1}{2\pi^2} \frac{1}{(\lambda - a)(b - \lambda)} \frac{\partial}{\partial \mu} \sqrt{(\mu - a)(b - \mu)}
\]
(2.17)
To prove that this is zero, let us return to eq. (2.6) and multiply it by $\sqrt{\frac{z - b}{z - a} \frac{1}{z - u}}$ and integrate again over a circle of large radius. Then
\[
\oint_c G(z) \sqrt{\frac{z - b}{z - a} \frac{1}{z - u}} dz = \oint_c \frac{dz}{z^2} = 0
\]
(2.18)
as for large $z$, $G(z) = \frac{1}{z}$, $\sqrt{\frac{z - b}{z - a}} \approx 1$ and $\frac{1}{z - u} \approx \frac{1}{z}$. While the second and third terms become
\[
- \oint_c \frac{P(z)(z - b)}{(z - u)} \frac{dz}{2i\pi} = -P(u)(u - b)
\]
(2.19)
and
\[ \frac{1}{2} \oint_c \sqrt{\frac{z - b}{z - a}} \frac{dz}{z - u} \frac{V'(z)}{2i\pi} = \frac{1}{2} \sqrt{\frac{u - b}{u - a}} V'(u) - \frac{1}{2} \int_a^b \sqrt{\frac{b - x}{x - a}} \frac{V'(x)}{x - u} dx. \] (2.20)

Taking \( u = \lambda + i\epsilon \) and using \( \frac{1}{\alpha - i\epsilon} = P\alpha + i\pi\delta(\alpha) \), the integral in eq. (2.20) has a part which cancels the first term leaving
\[ \frac{1}{2} \oint_c \sqrt{\frac{z - b}{z - a}} \frac{dz}{z - u} \frac{V'(z)}{2i\pi} = -\frac{1}{2} \int_a^b \sqrt{\frac{b - x}{x - a}} \frac{V'(x)}{x - u} dx. \] (2.21)

Repeating all the steps which led to eq. (2.15) i.e. combining eq. (2.18), eq. (2.19), eq. (2.21) one finds an expression for \( \tilde{\rho} \) from
\[ \frac{1}{2\pi} \int_a^b \sqrt{\frac{b - x}{x - a}} \frac{V'(x)}{x - \lambda} dx = P(\lambda)(b - \lambda) \] (2.22)

which is
\[ \tilde{\rho}(\lambda) = \frac{1}{(\lambda - a)^2} \sqrt{(\lambda - a)(b - \lambda)} \frac{1}{2} \int_a^b \sqrt{\frac{b - x}{x - a}} \frac{1}{x - \lambda} V'(x) dx \] (2.23)

thus proving that \( \left( \frac{\partial \tilde{\rho}}{\partial \alpha} \right)_{V,b} = 0 \). This completes the proof for the single-cut correlator.

3 The Double Well

Now let us extend the result to eigenvalues distributed in two disjoint bands \( (-a, b) \). Let us first restrict ourselves to even potentials i.e.
\[ P(M) = Z^{-1} \exp(-NTrV(M)) \]
\[ P(-M) = P(M) \] (3.1)

which implies for the resolvent
\[ G(-z) = -G(z). \] (3.2)

Since we restrict ourselves to even potentials we cannot take a functional derivative of \( \rho(\lambda) \) with respect to an arbitrary \( V(\mu) \), but we can fold the integrations over the positive part of the spectrum and then vary the potential.
Now
\[ TrV(M) = N \int_{-\infty}^{+\infty} d\lambda \rho(\lambda)V(\lambda) \]
\[ = N \int_{0}^{\infty} d\lambda V(\lambda)[\rho(\lambda) + \rho(-\lambda)]. \] (3.3)

Consequently
\[ -\frac{1}{N^2} \frac{\delta \bar{p}(\lambda)}{\delta V(\mu)} = <\rho(\lambda)\rho(\mu)>_c + <\rho(\lambda)\rho(-\mu)>_c, \] (3.4)
where use has been made of
\[ \frac{\delta V'(x)}{\delta V(\mu)} = \delta'(x - \mu). \] (3.5)

In the large N limit again
\[ 2G(z) = V'(z) - P(z)\sqrt{\sigma(z)} \] (3.6)
with \( \sigma(z) \equiv (z^2 - a^2)(z^2 - b^2) \). Note that this equation determines uniquely \( P(z), a \& b \); indeed take
\[ \text{deg}V = 2n \]
\[ \rightarrow \text{deg}[P] = 2n - 3; \]

\( P(z) \) has to be odd
\[ P(z) = \alpha_1 z + \alpha_2 z^3 + \cdots + \alpha_{n-1} z^{2n-3} \] (3.7)
we thus have \((n - 1) + 2\) unknowns. Since \( G(z) \approx_{z \rightarrow \infty} \frac{1}{z} \) we have to fix the coefficients of eq. (3.6) at infinity from \( z^{2n-1}, z^{2n-3}, \ldots, z^1, z^{-1} \rightarrow (n + 1) \) conditions. Therefore no "filling" parameter creeps into the problem (although the question of spontaneous symmetry maybe eliminated by the assumptions here).

Now we take eq. (3.6), multiply by \( \frac{\sqrt{\sigma(z)}}{z(z-u)} \) and integrate over a large circle in the z plane (using \( \sqrt{\frac{\sigma(z)}{z^2-u^2}} \) also has been checked to give the same equation), see Fig. 2. We obtain
\[ \begin{align*}
2 &= \frac{V'(u)\sqrt{\sigma(u)}}{u} - \frac{2i}{2\pi} \int_{b}^{a} \frac{V'(x)\sqrt{\sigma(x)}}{x(x-u)} dx \\
&+ \frac{2i}{2\pi} \int_{-a}^{-b} \frac{V'(x)\sqrt{\sigma(x)}}{x(x-u)} \frac{P(u)\sigma(u)}{u} \end{align*} \] (3.8)
which simplifies to

\[
2 + \frac{P(u)\sigma(u)}{u} = \frac{V'(u)\sqrt{|\sigma(u)|}}{u} - \frac{1}{\pi} \int_b^a \frac{V'(x)\sqrt{\sigma(x)}}{x} \left( \frac{1}{x - u} + \frac{1}{x + u} \right). \tag{3.9}
\]

We now let \( u = \lambda + i\epsilon \) approach the cut, say the one on the right (it doesn’t matter)

\[
2 + \frac{P(\lambda)\sigma(\lambda)}{\lambda} = \frac{V'(\lambda)\sqrt{\sigma(\lambda)}}{\lambda} - \frac{1}{\pi} \int_b^a \frac{V'(x)\sqrt{\sigma(x)}}{x} \left( \frac{1}{x - \lambda - i\epsilon} + \frac{1}{x + \lambda} \right) \tag{3.10}
\]

In the last integral use \( \frac{1}{\alpha - i\epsilon} = \frac{PP}{\alpha} + i\pi\delta(\alpha) \) and we obtain

\[
2 + \frac{P(\lambda)\sigma(\lambda)}{\lambda} = -\frac{1}{\pi} \int_b^a \frac{V'(x)\sqrt{\sigma(x)}}{x} \left( \frac{1}{x - \lambda} + \frac{1}{x + \lambda} \right) dx \tag{3.11}
\]

Let us take the derivative with respect to \( V(\mu) \) (\( \mu > 0 \) by definition of \( V \)).

\[
\frac{\sigma(\lambda)}{\lambda} \frac{\delta P(\lambda)}{\delta V(\mu)} = \frac{1}{\pi} \frac{\partial}{\partial \mu} \sqrt{\sigma(\mu)} \left( \frac{1}{\mu - \lambda} + \frac{1}{\mu + \lambda} \right) \tag{3.12}
\]

(assuming that we can show here as usual that a counterpart of \( \frac{\delta P}{\delta \sigma} V, \frac{\delta \sigma}{\delta V} \) vanish, see Appendix A for a proof, the exact same steps can be followed here). Then

\[
\bar{\rho}(\lambda) = \frac{1}{2\pi} \sqrt{\sigma(\lambda)} |P(\lambda)| \tag{3.13}
\]

(\( \lambda > 0 \))

\[
\frac{\delta \bar{\rho}(\lambda)}{\delta V(\mu)} = \left( \frac{1}{2\pi} \sqrt{\sigma(\lambda)} \right) \lambda \frac{\partial}{\partial \mu} \sqrt{(\mu^2 - b^2)(a^2 - \mu^2)} \frac{\lambda^2 - \mu^2}{\mu^2 - \lambda^2} - \frac{1}{\pi \sqrt{\sigma(\lambda)}} \left( \frac{\lambda}{\mu^2 - \lambda^2} \right)^2 [2\lambda^2 \mu^2 - (\lambda^2 + \mu^2)(a^2 + b^2) + 2a^2b^2] \tag{3.14}
\]
Let us check immediately the $b = 0$ limit
\[ \lambda \sqrt{\|\sigma(\lambda)\|} \to \frac{1}{\sqrt{(a^2 - \lambda^2)}} \] (3.15)
and the rest looks unfamiliar; but if we remember that we are computing
\[ \rho_2(\lambda, \mu) + \rho_2(\lambda, -\mu) \] (3.16)
and
\[ \frac{a^2 - \lambda \mu}{(\lambda - \mu)^2} + \frac{a^2 + \lambda \mu}{(\lambda + \mu)^2} = -2 \frac{[2\lambda^2 \mu^2 - a^2(\lambda^2 + \mu^2)]}{(\lambda^2 - \mu^2)^2}, \] (3.17)
we check that this result agrees as expected for $b = 0$ with the single cut result. Therefore for a symmetric double well, assuming no spontaneous symmetry breaking, we have the undisputable answer for $\rho_2(\lambda, \mu) + \rho_2(\lambda, -\mu)$ i.e. eq. (3.14). Note that, as expected, the short distance behaviour of $\rho_2(\lambda, \mu)$ is the same as for the single well with only one cut.

### 4 Asymmetric Double Well

In order to extract $\rho_2(\lambda, \mu)$ alone we have to consider arbitrary potentials instead of restricting ourselves to even ($Z_2$) symmetric potentials as we have done in the above section.

Again let us start with
\[ \rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \] (4.1)
with
\[ \bar{\rho}(\lambda) = <\rho(\lambda)> . \] (4.2)

Since the weight contains
\[ \exp(-NTrV) = \exp(-N^2 \int V(\lambda)\rho(\lambda)d\lambda) \] (4.3)
\[ \rho_2^c(\lambda, \mu) = <\rho(\lambda)\rho(\mu)>_c = -\frac{1}{N^2} \frac{\delta\rho(\lambda)}{\delta V(\mu)} \] (4.4)
\begin{equation}
\bar{\rho}(\lambda) = -\frac{1}{\pi} \text{Im} G(\lambda + i0) \tag{4.5}
\end{equation}

\begin{equation}
G(z) = \frac{1}{N} \text{Tr} \frac{1}{z - M} \tag{4.6}
\end{equation}

\begin{equation}
2G(z) = V'(z) - P(z) \sqrt{\sigma(z)} \tag{4.7}
\end{equation}

\begin{equation}
\sigma(z) = \prod_{i=1}^{4} (z - a_i). \tag{4.8}
\end{equation}

See Fig. 3.

The support of the eigenvalues consists of the two segments \([a_1, a_2]\) and \([a_3, a_4]\), (we assume that they are labelled by increasing order); the positivity of \(\rho(\lambda)\) is satisfied provided the polynomial \(P(z)\) has an odd number of zeroes between \(a_2\) and \(a_3\). Contrary to the two previous cases (4.7) is not sufficient to determine fully the polynomial \(P(z)\) and the four endpoints of the two cuts. Counting parameters and equations one sees readily that we miss one parameter, which we can take as the filling factor of one of the two wells. This factor remains undetermined at this level of the large \(N\)-limit, and we would have to return to a minimization of the free energy to fix it. However, since this parameter is not fixed at this leading order, we may ignore it and proceed as before for finding the leading order of the correlator.

We denote
\begin{equation}
\epsilon_\lambda = +1 \quad a_3 < \lambda < a_4 \\
-1 \quad a_1 < \lambda < a_2 \tag{4.9}
\end{equation}

then
\begin{equation}
\sqrt{\sigma(\lambda \pm i0)} = \pm i \epsilon_\lambda \sqrt{\sigma(\lambda)}. \tag{4.10}
\end{equation}

Therefore
\begin{equation}
2\pi \rho(\lambda) = \epsilon_\lambda \sqrt{\sigma(\lambda) |P(\lambda)|} \tag{4.11}
\end{equation}

and positivity implies that \(\text{sign} P(\lambda) = \epsilon_\lambda\). By multiplication by \(\frac{\sigma(z)}{(z-u)(z-v)}\) integration over a large circle, we obtain

\begin{equation}
2 + \frac{P(u)\sigma(u) - P(v)\sigma(v)}{u - v} = \frac{V'(u)\sqrt{\sigma(u)} - V'(v)\sqrt{\sigma(v)}}{u - v} - \frac{1}{\pi} \left( \int_{a_3}^{a_4} - \int_{a_1}^{a_2} \right) \frac{dx V'(x) \sqrt{\sigma(x)}}{(x-u)(x-v)} \tag{4.12}
\end{equation}
Let $u = \lambda + ie$ and $v = \mu + i\eta$ then

$$2 + \frac{P(\lambda)\sigma(\lambda) - P(\mu)\sigma(\mu)}{\lambda - \mu} = \frac{1}{\pi} \left( \int_{a_1}^{a_2} - \int_{a_3}^{a_4} \right) \frac{V'(x)\sqrt{|\sigma(x)|}}{(x - \lambda)(x - \mu)} \quad (4.13)$$

$$R(\lambda, \nu) = \frac{\delta P(\lambda)}{\delta V(\nu)} \quad (4.14)$$

$$\frac{\sigma(\lambda)R(\lambda, \nu) - \sigma(\mu)R(\mu, \nu)}{\lambda - \mu} = \frac{1}{\pi} \frac{\partial}{\partial \nu} \frac{\sqrt{|\sigma(\nu)|}}{\nu - \lambda} \quad (4.15)$$

(assuming once more an equivalent form of $\frac{\delta \bar{\rho}}{\delta a_i \delta \lambda}$ to be zero, see Appendix A). Hence

$$\sigma(\lambda)R(\lambda, \nu) - \frac{\epsilon_\nu}{\pi} \frac{\partial}{\partial \nu} \frac{\sqrt{|\sigma(\nu)|}}{\nu - \lambda} = \sigma(\mu)R(\mu, \nu) - \frac{\epsilon_\nu}{\pi} \frac{\partial}{\partial \nu} \frac{\sqrt{|\sigma(\nu)|}}{\nu - \mu} \quad (4.16)$$

From these equations one finds

$$\sigma(\lambda)R(\lambda, \nu) - \frac{\epsilon_\nu}{\pi} \frac{\partial}{\partial \nu} \frac{\sqrt{|\sigma(\nu)|}}{\nu - \lambda} = \frac{h_{\epsilon_\nu}(\nu)}{\sqrt{|\sigma(\nu)|}} \quad (4.17)$$

and we are left with two unknown functions $h_+$ and $h_-$ of a single variable. This gives the connected correlator

$$\rho^c_2(\lambda, \mu) = -\frac{1}{2\pi N^2} \epsilon_\lambda \sqrt{|\sigma(\lambda)|} R(\lambda, \mu), \quad (4.18)$$
i.e.

$$2\pi N^2 \rho^c_2(\lambda, \mu) = \epsilon_\lambda \frac{\sqrt{|\sigma(\lambda)|}}{\sqrt{|\sigma(\mu)|}} \left\{ \frac{\epsilon_\mu}{\pi} \frac{\partial}{\partial \mu} \frac{\sqrt{|\sigma(\mu)|}}{\mu - \lambda} + \frac{h_{\epsilon_\nu}(\mu)}{\sqrt{|\sigma(\mu)|}} \right\}. \quad (4.19)$$

From its definition the two-point correlator is symmetric under exchange of the two eigenvalues

$$\rho^c_2(\lambda, \mu) = \rho^c_2(\mu, \lambda). \quad (4.20)$$

This imposes the following constraints:

$$h_+(\mu) + h_-(\mu) = 0 \quad (4.21)$$

and

$$\pi[h_+(\mu) - h_+(\lambda)] = \sqrt{|\sigma(\lambda)|} \frac{\partial}{\partial \lambda} \frac{\sqrt{|\sigma(\lambda)|}}{\lambda - \mu} - \sqrt{|\sigma(\mu)|} \frac{\partial}{\partial \mu} \frac{\sqrt{|\sigma(\mu)|}}{\mu - \lambda}. \quad (4.22)$$
A straightforward algebra gives from there the function \( h_+ \) up to an arbitrary constant:

\[
h_+ (\lambda) = \frac{1}{\pi} (\lambda^2 - \frac{1}{2} s \lambda + C)
\]

with

\[
s = a_1 + a_2 + a_3 + a_4.
\]

We are thus left with one undetermined constant in the two-point function:

\[
4 \pi^2 N^2 \rho_2^c (\lambda, \mu) = \frac{\epsilon_\lambda \epsilon_\mu}{\sqrt{\sigma(\lambda)} \sqrt{\sigma(\mu)}} \left( \frac{\sigma(\lambda) + \sigma(\mu)}{(\lambda - \mu)^2} \right)
+ \frac{\sigma'(\lambda) - \sigma'(\mu)}{(\lambda - \mu)} + \lambda^2 + \mu^2 - \frac{s}{2} (\lambda + \mu) + 2C.
\]

Let us verify that, without any restriction on the constant \( C \), this result satisfies the normalisation condition

\[
\int d\nu \rho_2^c (\lambda, \nu) = 0,
\]

which follows from the definition of \( \rho_2^c \). Returning to (4.20)

\[
\int d\mu \frac{\partial}{\partial \mu} \sqrt{\sigma(\mu)} = \int_{a_1}^{a_2} d\mu \frac{\partial}{\partial \mu} \sqrt{\sigma(\mu)} + \int_{a_3}^{a_4} d\mu \frac{\partial}{\partial \mu} \sqrt{\sigma(\mu)} = 0
\]

since \( \sigma \) vanishes at the end points. (This point is in fact slightly delicate, since there is a non-integrable singularity at \( \mu = \lambda \). In the literature concerning the application of random matrices to the calculation of the fluctuations of the conductance in mesoscopic systems \[3\], this integration throughout the singularity is done in a routine way. A proper justification of the procedure implies returning to the true correlation function, before the smoothing which produces spurious short-distance singularities through replacements such as \( (\sin^2 x \rightarrow \frac{1}{2} x^2) \). The smoothing is produced here by the large \( N \) limit.) Next consider

\[
\int dz \frac{z^2 - sz/2}{\sqrt{\sigma(z)}}
\]

over a large circle. Note that \( \sqrt{\sigma(z)} = z^2 [1 - \frac{s}{2z} + O(\frac{1}{z^2})] \). Consequently

\[
\frac{z^2 - sz/2}{\sqrt{\sigma(z)}} = 1 + O(\frac{1}{z^2}).
\]

Since there is no coefficient of \( \frac{1}{z} \) the integral vanishes.
Shrinking the contour over the cuts we obtain
\[
\left( \int_{a_1}^{a_2} - \int_{a_3}^{a_4} \right) d\nu \frac{\nu^2 - s\nu^2/2}{\sqrt{\sigma(\nu)}} = 0.
\] (4.29)

Therefore
\[
2\pi N^2 \left( \int_{a_1}^{a_2} d\mu \rho_2^\xi + \int_{a_3}^{a_4} d\mu \rho_2^\xi \right) = C \frac{\epsilon_\lambda}{\sqrt{\sigma(\lambda)}} \left[ \int_{a_3}^{a_4} d\mu \frac{d\mu}{\sqrt{\sigma(\mu)}} - \int_{a_1}^{a_2} d\mu \frac{d\mu}{\sqrt{\sigma(\mu)}} \right].
\] (4.30)

Again taking a very large circle
\[
\oint \frac{dz}{\sqrt{\sigma(z)}} = 0
\] (4.31)
since the coefficient of \( \frac{1}{z} \) vanishes. Therefore, shrinking the circle,
\[
\int_{a_3}^{a_4} \frac{d\mu}{\sqrt{\sigma(\mu)}} = \int_{a_1}^{a_2} \frac{d\mu}{\sqrt{\sigma(\mu)}}
\] (4.32)
which shows that the normalization is correct for any value of \( C \).

Let us specialize to the symmetric double-well \( (a_1 = -b, a_2 = -a, a_3 = a \) and \( a_4 = b ) \)
\[
|\sigma(\lambda)| = (\lambda^2 - a^2)(b^2 - \lambda^2).
\] (4.33)

After a few lines
\[
2\pi^2 \rho_2^\xi(\lambda, \mu) = \frac{\epsilon_\lambda \epsilon_\mu}{\sqrt{|\sigma(\lambda)||\sigma(\mu)|}} \left[ \frac{1}{(\mu - \lambda)^2} \right] \frac{1}{(\mu - \lambda)^2} [C(\mu - \lambda)^2 + \lambda \mu(\mu - a^2 - b^2) + a^2 b^2].
\] (4.34)

Several remarks (i). The result is manifestly symmetric under \( \lambda \leftrightarrow \nu \) (ii). \( C \) remains an unknown function of \( a \& b \) (iii). If we assume that \( C \) vanishes when \( b=0 \) then since
\[
\lim_{a \to 0} \sqrt{|\sigma(\lambda)|} = \epsilon_\lambda \lambda(a^2 - \lambda^2)
\] (4.35)
we recover the single band result. (iv). The cross-correlator \( \rho_2^\xi(\lambda, -\nu) \) is simply eq. (134) with \( \nu \) replaced by \(-\nu\). Combining eq. (134) with \( \rho_2^\xi(\lambda, -\nu) \) we reproduce the result eq. (134) in section 3 (v). We also note that for \( C = -\frac{1}{2}((a^2 + b^2) - (a + b)^2 E(k)/K(k)) \) we recover the connected density-density correlator derived from the connected Green’s function (where \( E(k) \) and \( K(k) \) are complete elliptic integrals of first and second kind and \( k = \frac{2\sqrt{ab}}{(a+b)} \)) in eq. (15) of ref. [5].
5 Orthogonal Polynomials, The Kernel, Odd and Even N

This section follows the notation of ref. 6. Let us calculate the Kernel $K_N(\mu, \nu)$ for \(N\) even and \(N\) odd

$$K_N(\mu, \nu) = \sqrt{\frac{R_N}{N}} \left[ \psi_N(\mu)\psi_{N-1}(\nu) - \psi_{N-1}(\mu)\psi_N(\nu) \right] \left( \mu - \nu \right). \quad (5.1)$$

The asymptotic ansatz for $\psi_n(\lambda)$ for $N \to \infty$ but $N - n$ finite is

$$\psi_n(\lambda) = \frac{1}{\sqrt{f(\lambda)}} \left[ \cos(N\zeta - (N - n)\phi + \chi + (-1)^n\eta)(\lambda) + O\left(\frac{1}{N}\right) \right]$$

$$f(\lambda) = \frac{\pi}{2\lambda} \left( \frac{b^2 - a^2}{2} \right) \sin 2\phi(\lambda)$$

$$\zeta'(\lambda) = -\pi \rho(\lambda)$$

$$\cos 2\phi(\lambda) = \frac{\lambda^2 - (a^2 + b^2)/2}{(b^2 - a^2)/2}$$

$$\cos 2\eta(\lambda) = \frac{b \cos \phi(\lambda)}{\lambda}$$

$$\sin 2\eta(\lambda) = \frac{a \sin \phi(\lambda)}{\lambda}. \quad (5.2)$$

Substituting

$$\psi_N(\lambda) = \frac{1}{\sqrt{f(\lambda)}} \left[ \cos(N\zeta - (N - N)\phi + \chi + (-1)^N\eta)(\lambda) \right] \quad (5.3)$$

$$\psi_{N-1}(\lambda) = \frac{1}{\sqrt{f(\lambda)}} \left[ \cos(N\zeta - (N - N + 1)\phi + \chi + (-1)^{(N-1)}\eta)(\lambda) \right] \quad (5.4)$$

we get

$$K_N(\mu, \nu) = \sqrt{\frac{R_N}{N(\mu - \nu)\sqrt{f(\mu)f(\nu)}}} \cos Nh(\mu) \cos Nh(\nu)$$

$$\times \left[ \cos \phi(\nu) \cos 2\eta(\nu) - (-1)^N \sin \phi(\nu) \sin 2\eta(\nu) \right.$$

$$\left. - \cos \phi(\mu) \cos 2\eta(\mu) + (-1)^N \sin \phi(\mu) \sin 2\eta(\mu) \right]$$

$$+ \sin Nh(\nu) \cos Nh(\mu)(\sin \phi(\nu) \cos 2\eta(\nu) + (-1)^N \sin 2\eta(\nu) \cos \phi(\nu))$$

$$- \sin Nh(\mu) \cos Nh(\nu)(\sin \phi(\mu) \cos 2\eta(\mu) + (-1)^N \sin 2\eta(\mu) \cos \phi(\mu)) \quad (5.5)$$
where
\[ Nh(\mu) = (N\zeta + \chi + (-1)^N \eta)(\mu). \] (5.6)

On simplifying further

\[ K_N(\mu, \nu) = \frac{\sqrt{R_N}}{2N(\mu - \nu)\sqrt{f(\mu)f(\nu)}} \]
\[ \times \frac{[\cos(Nh(\mu) + Nh(\nu)) + \cos(Nh(\mu) - Nh(\nu))]}{\sqrt{f(\mu)f(\nu)}} \]
\[ \times \left( \frac{b}{\nu} \cos^2 \phi(\nu) - (-1)^N \frac{a}{\nu} \sin^2 \phi(\nu) - \frac{b}{\mu} \cos^2 \phi(\mu) + (-1)^N \frac{a}{\mu} \sin^2 \phi(\mu) \right) \]
\[ + \left( \sin(Nh(\nu) + Nh(\mu)) + \sin(Nh(\nu) - Nh(\mu)) \right) \]
\[ \times \left( \frac{b}{\nu} + (-1)^N \frac{a}{\nu} \right) \sin \phi(\nu) \cos \phi(\nu) \]
\[ - \left( \sin(Nh(\mu) + Nh(\nu)) + \sin(Nh(\mu) - Nh(\nu)) \right) \]
\[ \times \left( \frac{b}{\mu} + (-1)^N \frac{a}{\mu} \right) \sin \phi(\mu) \cos \phi(\mu) \] (5.7)

Squaring and averaging we get after some tedious algebra

\[ <K^2_N(\mu, \nu)> = \frac{R_N}{4N^2(\mu - \nu)^2 f(\mu)f(\nu)2\nu\mu(b^2 - a^2)^2/4} \]
\[ \times \left[ 2\nu\mu \left( \frac{b^2 - a^2}{2} - \nu^2 \mu^2 (2ab(-1)^N + a^2 + b^2) \right) \right. \]
\[ + \left( \frac{a^2 + b^2}{2} (\nu^2 + \mu^2)(2ab(-1)^N + a^2 + b^2) \right) \]
\[ - 2ab(-1)^N a^2b^2 - (a^2 + b^2) \left( \frac{b^4 + a^4}{2} \right) \]
\[ - \left( \nu^2 + \mu^2 \right) \left( \frac{b^2 - a^2}{2} \right) + \frac{2}{4} (b^2 - a^2)^2(a^2 + b^2) \]. (5.8)

Simplifying for $N$ even,

\[ <K_N(\mu, \nu)^2> = \frac{R_N(a + b)^2}{4N^2(\mu - \nu)^2 f(\mu)f(\nu)\nu\mu(b^2 - a^2)} \]
\[ \times \left[ \nu\mu(b^2 + a^2) - \nu^2 \mu^2 - a^2b^2 + (\nu - \mu)^2 ab \right] \]

(5.9)

while for $N$ odd

\[ <K_N(\mu, \nu)^2> = \frac{R_N(a - b)^2}{4N^2(\mu - \nu)^2 f(\mu)f(\nu)\nu\mu(b^2 - a^2)} \]

14
\[
\times [\nu \mu (b^2 + a^2) - \nu^2 \mu^2 - a^2 b^2 - (\nu - \mu)^2 ab].
\]

Note that \( R_{N_{\text{even}}} (a + b)^2 = A(a + b)^2 = \frac{(a-b)^2}{4} (a + b)^2 = \frac{(a^2 - b^2)^2}{4} \) and \( R_{N_{\text{odd}}} (a - b)^2 = B(a - b)^2 = \frac{(a^2 - b^2)^2}{4} \). Comparing this expression with that found by the previous method of section 4, we find that \( C = (-1)^N \nu \mu ab \). The standard large N-limit techniques of analyzing matrix models like the loop equation method ref. \[1, 5\] and renormalization group ref. \[4\] assume a smooth behavior with respect to N at large N. The result that C differs for odd or even N by terms of order one suggests that these methods may need to be revisited in the context of random matrix models with eigenvalue distributions with gaps.

6 Conclusion

To conclude we have outlined a method which reproduces known results for the single cut model and extended it to the two-cut random matrix model. The two-point density-density correlator contains a derivative part familiar from the single cut model but in addition contains a non-trivial non-derivative piece. It is further seen that different methods give different values for the two-point correlator. The orthogonal polynomial method is briefly outlined and gives different values for the non-derivative piece for even and odd eigenvalues. The loop equation method gives a different result. The difference in the results are in the non-derivative part of the two-point density-density correlator. The method outlined unifies these differences in a constant C which takes different values. Different values of C found from the orthogonal polynomial and loop equation methods are identified.

This raises several questions regarding the analysis of this model. One possibility is that the even-odd differences may require some care in handling the large N techniques of random matrix models e.g. loop equations and renormalization group. Another question relates to spontaneous breaking of the \( Z_2 \) symmetry in the large N limit. In this context, for the \( Z_2 \) symmetric random matrix models with two wells an infinite family of solutions which break the \( Z_2 \) symmetry and have the same free energy as the \( Z_2 \) symmetric solution but different connected correlators have been identified in ref. \[7\]. It would be interesting to compare whether the different solutions noted here correspond to some of the multiple solutions of ref. \[7\]. Finally let us note that when the number of connected components for the support of the
eigenvalues changes, one finds a new universality class for the correlators. It is thus not completely obvious that it is legitimate to use the simple one cut function in the application to mesoscopic fluctuations. It seems interesting to us that this simple system, namely N charges confined by a symmetric double-well with a logarithmic repulsion between the charges, exhibits such rich behavior.

Acknowledgements:
EB would like to thank Ivan Kostov for discussions. ND thanks the Abdus Salam ICTP, Trieste, Italy; Ecole Normale Supérieure, Paris, France; Isaac Newton Institute, Cambridge, England; and NEC, Princeton, USA; for support and hospitality where part of this work was done. Special thanks goes to the Raman Research Institute for support and encouragement. Thanks also to C. Dasgupta, S. Jain, H. R. Krishnamurthy, N. Kumar, R. Nityananda, N. Mukunda, T. V. Ramakrishnan, C. N. R. Rao and B. S. Shastry for encouragement and discussions.

Appendix A

Eq. (4.15) was derived under the assumption that a counterpart of \((\frac{\delta \rho}{\delta a})_{V,b_i} = 0\). Here we prove this result for the asymmetric double-well. (Following a similar procedure we can show that an equivalent form of \((\frac{\delta \rho}{\delta a})_{V,b_i} = 0\) from which eq. (3.12) follows for the symmetric double well). From eq. (4.15) it is easy to see that we have to prove the following equation equivalent to \((\frac{\delta \rho}{\delta a})_{V,b_i} = 0\) in the single well problem

\[
\frac{-2\pi \bar{\rho}(\lambda)\epsilon_{\lambda}}{\lambda - \mu} + 2\pi \bar{\rho}(\mu)\epsilon_{\mu} \frac{\delta \sqrt{\sigma(\mu)}}{\delta a} = \frac{1}{\pi} \int_{a}^{b} V'(x) \frac{\delta \sqrt{\sigma(x)}}{\delta a} \delta a \frac{1}{\pi} \int_{c}^{d} V'(x) \frac{\delta \sqrt{\sigma(x)}}{\delta a} \delta a.
\]

Let us take

\[
2G(z) = V'(z) - P(z) \sqrt{(z-a)(z-b)(z-c)(z-d)}.
\]

Multiple by \(\frac{1}{(z-u)(z-v)} \sqrt{\sigma(z)} \) and integrate over a large circle. The first term

\[
2 \oint_{c} \frac{G(z) \sqrt{\sigma(z)}}{(z-u)(z-v)(z-a)} = \oint_{c} \frac{1}{z^2} dz = 0
\]
as for large $z$ \( G(z) \approx \frac{1}{z} \), \( \sqrt{\sigma(z)} \approx z^2 \), \( \frac{1}{(z-u)(z-v)} \approx \frac{1}{z} \) and \( \frac{1}{z-a} \approx \frac{1}{z} \). The third term becomes

\[
\oint_c \frac{P(z)\sigma(z)}{(z-u)(z-v)(z-a)} \frac{dz}{2\pi i} = \frac{P(u)(u-b)(u-c)(u-d) - P(v)(v-b)(v-c)(v-d)}{(u-v)}.
\]

(A.5)

While the second term is

\[
\oint_c \frac{V'(z)\sqrt{\sigma(z)}}{(z-u)(z-v)(z-a)} \frac{dz}{2\pi i} = \frac{\sqrt{\sigma(u)V'(u)}}{(u-a)(u-v)} + \frac{\sqrt{\sigma(v)V'(v)}}{(v-a)(v-u)} + \frac{1}{\pi i} \int_a^b \frac{V'(x)\sqrt{\sigma(x)}}{(x-u)(x-v)(x-a)} dx + \frac{1}{\pi i} \int_c^d \frac{V'(x)\sqrt{\sigma(x)}}{(x-u)(x-v)(x-a)} dx.
\]

(A.6)

On using \( u = \lambda + i\epsilon \) and \( v = \mu + i\epsilon \) with \( \lambda, \mu \) on the right hand cut, \( \frac{1}{x-\lambda-i\epsilon} = \frac{P}{x-\lambda} + i\pi\delta(x-\lambda) \) and \( \sqrt{\sigma(x)} = i\epsilon|\sqrt{\sigma(x)}| \) the second integral simplifies to

\[
\oint_c \frac{V'(z)\sqrt{\sigma(x)}}{(z-u)(z-v)(z-a)} \frac{dz}{2\pi i} = \frac{\int_a^b \frac{V'(x)|\sqrt{\sigma(x)}|}{(x-\lambda)(x-\mu)(x-a)} dx}{\pi} - \frac{\int_c^d \frac{V'(x)|\sqrt{\sigma(x)}|}{(x-\lambda)(x-\mu)(x-a)} dx}{\pi}.
\]

(A.7)

Combining these three terms and simplifying we get eq. (A.2) which is what is needed in order to get eq. (4.15).

References

[1] J. Ambjorn, J. Jurkiewicz and Yu. M. Makeenko, Phys. Lett. B251 (1990) 517

[2] E. Brézin and A. Zee, Nucl. Phys. B402 (1993) 613

[3] C. W. J. Beenakker, Nucl. Phys. B422 (1994) 515; Phys. Rev. Lett. 70 (1993) 1155

[4] E. Brézin and J. Zinn-Justin, Phys. Lett. B288 (1992) 54

[5] J. Ambjorn and G. Akemann, J. Phys. A29 (1996) L555

17
[6] N. Deo, *Nucl. Phys.* **B504** (1997) 609

[7] R. C. Brower, N. Deo, S. Jain and C. I. Tan, *Nucl. Phys.* **B405** (1993) 166
Figure captions:

Fig. 1. The complex z-plane with one-cut and contour used for evaluating the two-point density-density correlator for the one-cut random matrix model.

Fig. 2. The complex z-plane with two-cuts and contour used for evaluating the two-point density-density correlator for the two-cut random matrix model.

Fig. 3. The complex z-plane with asymmetric two-cuts and contour used for evaluating the two-point density-density correlator for the asymmetric two-cut random matrix model.
Fig. 3 The complex $z$-plane with two asymmetric cuts and contour used for evaluating the two-point density-density correlator for the asymmetric two-cut random matrix model.
Fig. 1 The complex $z$-plane with one-cut and contour used for evaluating the two-point density-density correlator for the one-cut random matrix model.
Fig. 2 The complex z-plane with two-cuts and contour used for evaluating the two-point density-density correlator for the two-cut random matrix model.
