The High-Order Perturbation Approximate Solution of the Finite Ultrasonic Wave

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Abstract. To the nonlinear acoustic wave equation, the general used second harmonic solution is not accurate enough because all perturbated expansion equations higher than the second order are ignored during the equation solving process. The purpose of this paper is to obtain a more accurate solution, i.e., the high-order perturbation approximate solution. Firstly, the nonlinear acoustic wave equation is expanded into many inhomogeneous partial differential equations. the low-order harmonic solutions are obtained manually, then we formulate the form of the high-order harmonic solutions according to the properties of the low-order harmonic solutions. Using symbol calculation tool, we finally obtained higher up to the 14th order perturbation special solutions. Odd order solutions contain only odd order harmonics, and even order solutions contain only even order harmonics. The high-order perturbation solution of the second harmonic is finally achieved by summing up all of the second harmonic solution parts. The simulation results show that the relative amplitude ($A_2/A_1$) of the second harmonic increases and then decreases with the propagation distance, which is in agreement with experimental results. The high-order perturbation approximated solution can compensate for the theory deficiency and can be used to measure the nonlinear parameter with a good precision.

1. Introduction

The nonlinear acoustical parameter $\beta$ is often used to characterize the mechanical properties of the medium with more details because the parameter is related to the high-order elastic constants $[1]$. But during solving the nonlinear equation, all higher than the second high-order equations are ignored. So, the solution is not precise and only fits experimental results in a short propagation distance and of a low excitation level $[2]$.

Blackstock built a bridge function which fits the experiment result very well in a long distance even exceed where the shock wave forms. But the function is seldom used in measuring nonlinear parameter because it is not suitable for quantitative analysis. For the same reason, numerical method is rarely used in nonlinearity measurement either, although it can also achieve higher calculation accuracy $[3]$.

In this paper, we use perturbation method to obtain a high-order approximate perturbation solution of the second harmonic. Our simulation results can meet the experimental results well, and the high order solution can be used to measure nonlinear parameter with higher precision.

2. Nonlinear Acoustic Equations in Solid and Fluid

In this section, nonlinear acoustic equations in an isotropic solid and an ideal non-dissipative fluid are introduced but a purely longitudinal plane wave and nonlinear wave propagating in one-dimensional direction are hypothesized.
The general used nonlinear equation \[1\] is
\[
\frac{\partial^2 u}{\partial t^2} = c_0^2 (1 + \beta \frac{\partial u}{\partial x}) \frac{\partial^2 u}{\partial x^2},
\]
and it is originated from the stress tensor equation, strain tensor, the equation of motion, and the elastic energy. \(c_0\) is the wave velocity with low excitation, \(u\) and \(x\) are the oscillate displacement and the propagation direction, respectively. The nonlinear parameter \(\beta = \frac{3}{\lambda + 2\mu} \left( \frac{2}{\lambda + 2\mu} + \frac{2}{\mu} \right) \left( \frac{2}{\lambda + 2\mu} + \frac{2}{\mu} + \frac{2}{\lambda + 2\mu} \right) + 3 + \frac{6}{\lambda + 2\mu} \)
and these referred parameters are related to the linear and nonlinear elastic parameters.

For nonlinear acoustic equations in liquid and gas mediums, the nonlinear acoustic equations are \[4\]
\[
\frac{\partial^2 u}{\partial t^2} = c_0^2 (1 + \alpha u) \frac{\partial^2 u}{\partial x^2},
\]
\[
\frac{\partial^2 u}{\partial t^2} = c_0^2 (1 + \beta u) \frac{\partial^2 u}{\partial x^2},
\]
here \(\gamma\) is the heat capacity ratio of gas. \(A\) and \(B\) in equation (2) are the temperature dependent quantity of liquid and related to nonlinear effects of liquid medium, but they are not the same as the up-mentioned nonlinear elastic parameter. Their origin and meanings can be seen in ref. [4].

Equations (2) and (3) can both be changed into equation (4) by applying Taylor expansion and taking the first two terms into account.

\[
\frac{\partial^2 \xi}{\partial t^2} = c^2 (1 + 2\beta \frac{\partial \xi}{\partial a}) \frac{\partial^2 \xi}{\partial a^2},
\]
here \(\bar{\beta} = B/2A + 1 = (\gamma + 1)/2 = \beta/2\), It is clear that equation (1) and (4) have the same forms. \(\bar{\beta}\) is the nonlinear parameter for liquid medium, and \(\beta\) is the nonlinearity of solid. Mathematically, \(\beta = 2\beta\) \[5\].

3. Perturbation Method to Solve the Nonlinear Acoustic Equation

The perturbation expansion expression of equation (1) can be written as
\[
\frac{\partial^2 u}{\partial t^2} = c_0^2 (1 + \beta u) \frac{\partial^2 u}{\partial x^2},
\]
Substituting (5) into equation (1), expanded equations can be obtained:
\[
\beta^0: \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]
\[
\beta^1: \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]
\[
\beta^2: \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]
\[
\ldots,
\]
\[
\beta^{n+1}: \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]
The solution of the linear equation (6) is the fundamental frequency signal. Others perturbed expansion equation are all non-homogeneous partial differential equations, and their special solutions are high-order harmonics. The non-homogeneous terms of the non-homogeneous equations can be regarded as the source of the higher harmonics. Which means that the low harmonics couple together and generate the sum frequency high harmonics and the difference frequency low harmonics. For example, if \(n=8\), the 8th high harmonic can be obtained by the fundamental (or the 1st) and 7th, the 2nd and 6th, the 3rd and
5th, 4th and 4th harmonics (refer to the Product to Sum Formula of Sines and Cosines). The sum frequency harmonics are all the 8th order, but the difference frequency harmonic order can be the 2nd, 4th, and 6th order. The solutions of the even order perturbed expansion equations contain only even order harmonics. And it is easy to know that the solutions of the odd order perturbed expansion equations contain only odd order harmonics.

Firstly, the fundamental signal, i.e., the solution of linear equation (6) is

\[ u_1 = A_1 \cos(\omega t - kx) \]  

(10)

\( A_1 \) and \( \omega \) are displacement amplitude and angular frequency respectively. \( k \) is the wave number.

Substituting equation (10) into (7), we can obtain

\[ \frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} = -c^2k^2 A_1^2 \sin[2(\omega t - kx)] \]  

(11)

This equation is the nonhomogeneous equation of the second harmonic. Let \( \theta = \omega t - kx \), we substitute

\[ \theta = at - kx \]  

(12)

into Eq. (11), \( P_1 = 0 \) and \( 4kC_1 = \beta k^2 A_1^2 / 2 \) must be satisfied in order to guarantee the identity of this equation with arbitrary \( x \) and \( t \). Besides, all harmonics can not be generated at the source position (\( x=0 \)), then \( C_2 = D_2 = 0 \). According to Eq. (5), the second order perturbation solution of the second harmonic can be written as

\[ u'_2 = \frac{1}{8} k^2 A_1^2 \beta \cos 2\theta \]  

(13)

Equation (13) is the generally used second order perturbation solution of the second harmonic. This solution is not accurate enough for ignoring all higher than the second order expanded equations and can bring error to the nonlinear parameter measurement result, especially in the long propagation distance and of the high excitation.

Substituting Equations (10) and (13) into (8), the 3rd order perturbed expansion equation can be written as

\[ \frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} = \frac{k^4 A_1^4}{8} \left(-3kx \sin 3\theta + kx \sin \theta + \frac{3}{2} \cos \theta - \frac{5}{2} \cos 3\theta \right) \]  

(14)

Its special solution should include the 3rd harmonic and the 1st harmonic. We assume the 3rd and 1st harmonic solutions are

\[ u_{31} = (C_{31}x + C_{312}) \cos 3\theta + (D_{31}x + D_{312}) \sin 3\theta \]  

(15)

\[ u_{31} = (C_{31}x^2 + C_{312}x) \cos \theta + (D_{31}x^2 + D_{312}x) \sin \theta \]  

(16)

\( C \) and \( D \) are the underdetermined coefficients. Substituting equation (15) into (8), \( D_{01} = 0 \), \( C_{31} = 0 \), \( 12kC_{31} = 3k^4 A_1^4 / 8 \) and \( 2C_{31} - 6kD_{31} = 5k^4 A_1^4 / 16 \) should be satisfied for arbitrary \( x \) and \( t \) simultaneously. In the same way, the 1\textsuperscript{st} harmonic solution of the 3rd order perturbed expansion equation can also be obtained. The special solution of the 3rd order perturbed expansion equation, including the 1\textsuperscript{st} harmonic and the 3\textsuperscript{rd} harmonic, is

\[ u_3 = A_1^4 k^4 \left[ \left(-k x^2 \cos \theta / 32 + (x \sin \theta / 16) + (k x^2 \cos 3\theta / 32 - (x \sin 3\theta / 24) \right) \right] \]  

(17)

Similarly, the special solution of the 4\textsuperscript{th} order perturbed expansion equation, including the 4\textsuperscript{th} and the second harmonic, is

\[ u_4 = A_1^4 k^4 \left(-\frac{1}{96} k^2 x^2 \cos 2\theta + \frac{3}{64} k^4 x^2 \sin 2\theta + \frac{1}{24} \cos 2\theta + \frac{1}{96} k^2 x^2 \cos 4\theta - \frac{1}{32} k^4 x^2 \sin 4\theta - \frac{7}{384} \cos 4\theta \right) \]  

(18)

From the form of these low order special solutions, we can write the form of the high order solution as
Here $\text{fix}(Y)$ means rounding $Y$ to the nearest integers towards zero. The $n$th harmonic can be derived from harmonics lower than $n$th order. Using Symbolic Math Tool, we can obtain

\begin{align}
    u_{n2} &= A_1^k x^2 \left[ \frac{k^2 x^4}{3072} - \frac{85 k^2 x^4}{6144} + \frac{175 k x^4}{192} + \frac{325 k x^4}{6144} \right] \cos \theta + \left( \frac{k^2 x^4}{192} + \frac{325 k x^4}{6144} \right) \sin \theta, \\
    u_{n4} &= A_1^k x^4 \left[ -\frac{k^4 x^8}{184320} + \frac{91 k^4 x^8}{36864} + \frac{6935 k^4 x^8}{147456} + \frac{79 k^4 x^8}{3072} \right] \cos 2\theta + \left( \frac{23 k^4 x^8}{122880} + \frac{569 k^4 x^8}{36864} + \frac{18397 k x^8}{294912} \right) \sin 2\theta, \\
    u_{n6} &= A_1^k x^6 \left[ -\frac{k^6 x^{12}}{17694720} + \frac{491 k^6 x^{12}}{5898240} + \frac{18565 k^6 x^{12}}{2359296} - \frac{2094175 k^6 x^{12}}{28311552} + \frac{23881 k^6 x^{12}}{884736} \right] \cos 2\theta + \left( \frac{56 k^6 x^{12}}{294912} + \frac{12679 k^6 x^{12}}{11796480} + \frac{232043 k^6 x^{12}}{7077888} + \frac{4433377 k x^{12}}{56623104} \right) \sin 2\theta.
\end{align}

Because only the second harmonic will be discussed, other order harmonics are not listed here. A more accurate solution of the nonlinear acoustic equation (1) can be obtained by summing up all of the second harmonics up-mentioned, but the nonlinear parameter $\beta$ should be considered refer to equation (5). With the increase of perturbation order, the summation solution can meet the real situation well.

### 4. Simulation Results

We simulate the second harmonic solution of the longitudinal nonlinear wave in infinite water ($\beta = 7$ or $\beta = 5$), and obtain the up to 14th order harmonics. The main frequency of the excitation signal is 2.25MHz. The convergence process of the second harmonic relative displacement amplitudes ($A_2/A_1$) vary with propagation distance is shown in figure 1. Solid lines P2, P4, P6, … are the corresponding order relative displacement amplitudes ($A_n/A_1$). The more order harmonics solved, the better the relative amplitude of the second harmonic meet experimental results (circle). Because the accuracy is no longer significantly improving, only the top 14th harmonics solutions are presented in figure 1.

![Figure 1](image_url)  
**Figure 1.** The relative displacement $A_2/A_1$ vary with the propagation distance. Circle: the experimental results. Lines P2, P4, P6, … are the corresponding order perturbation solutions of the second harmonic.

### 5. Conclusion

Based on perturbation method, we obtain a high order solution of the nonlinear acoustic equation, the relative displacement amplitude $A_2/A_1$ increases first and then decreases with the propagation distance, which is more accurate than the commonly used second order solution. The results can reduce the limited...
conditions of nonlinearity measurement such as near propagation distance and low excitation, can help to improve the measurement accuracy of nonlinear parameter by making full use of experimental data.

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