Jet formation in the magnetospheres of supermassive black holes: analytic solutions describing energy loss through Blandford–Znajek processes

G. Menon and C. D. Dermer

1 Troy University, Troy, AL 36082, USA
2 Code 7653, Naval Research Laboratory, 4555 Overlook Avenue, SW, Washington, DC 20375, USA

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ABSTRACT
In this paper, we provide exact solutions for the extraction of energy from a rotating black hole via both the electromagnetic Poynting flux and matter currents. By appropriate choice of a radially independent poloidal function $\Lambda(\theta)$, we find solutions where the dominant outward energy flux is along the polar axis, consistent not only with a jet-like collimated outflow, but also with a weaker flux of energy along the equatorial plane. Unlike previously obtained solutions, the magnetosphere is free of magnetic monopoles everywhere.

Key words: black hole physics – MHD – radiation mechanisms: general.

1 INTRODUCTION

In 1977, Blandford & Znajek (1977) introduced the force-free, stationary, axisymmetric magnetosphere of the Kerr geometry as a possible setting for the extraction of energy from supermassive black holes at an astronomical scale. To this day, astrophysicists consider the Blandford–Znajek process as the leading mechanism for the observed phenomenon of luminous black holes (e.g. Punsly 2001; Komissarov 2004; McKinney 2005). Indeed, it is this mechanism we have focused on as well (Menon & Dermer 2005, 2006; Dermer & Menon 2009).

In an earlier paper (Menon & Dermer 2006), we were successful in providing the only known class of exact analytic solutions to the equations of force-free electrodynamics in Kerr geometry. Although finite everywhere in the magnetosphere, these solutions did not appear to be physically realistic. In particular, the currents in the magnetosphere were null vector fields suggesting that the charged particles travelled at the speed of light. Additionally, the current vector field was inward pointing, resulting in an influx of electromagnetic energy. This seemed to suggest that the solutions were not physically interesting.

In this paper, we provide a clear, physically realistic interpretation of the current density vector field. In particular, we decompose the infalling null vector field to currents, each of which have future pointing time-like velocities as these are candidates for physically realistic currents. In this decomposition, one of the currents is outgoing. This current provides a concrete mechanism for jet formation in black holes. However, the electromagnetic flux continues to be inward pointing. To remedy this, we extend the results via a symmetry transformation. Briefly, it turned out that the net current vector is proportional to the infalling principle null geodesic of the Kerr geometry. Since the Kerr geometry is of Petrov type D (Chandrashekar 1983), it is only natural to ask whether the equations of electrodynamics would allow the existence of solutions where the current vector was proportional to the second principle (outgoing) null geodesic of the Kerr geometry. As we shall show in this paper, this is indeed the case; and the existence of this dual class of solutions is a general property of the equations, not necessarily dependent on our particular solution. The dual solution to our particular solution (hereby referred to as the $\Omega_-$ solution) does allow for the extraction of energy via the electromagnetic Poynting flux.

All generalities are restricted by picking a concrete example and carrying out the energy extraction rates from matter currents and the electromagnetic Poynting flux. Here we find that the matter current and the electromagnetic Poynting flux naturally describe a polar jet for a specific choice of an arbitrary poloidal function $\Lambda$. This solution also describes a secondary local maxima in the energy-extraction rate peaking near the equatorial plane, which might correspond to outflowing Poynting flux that could drive a disc wind. Even though our past $\Omega_+$ solution (Menon & Dermer 2005) generalized the Blandford–Znajek split monopole solution, the non-existence of a magnetic monopole for our new solution is shown here by direct computation (this was not the case in the original approximate solution presented by Blandford & Znajek 1977).

2 THE $\Omega_-$ SOLUTION

In Menon & Dermer (2006), we derived the following class of exact solutions for the force-free magnetosphere of the Kerr black hole. Here, the components of the electromagnetic fields in the Boyer–Lindquist coordinate system are given by

$$ E_\phi = 0 = E_r, $$

(1)
\[ E_{\psi} = -\frac{2}{a^2} \Lambda \frac{\cos \theta}{\sin^3 \theta}, \] (2)

and
\[ B^\psi = 0, \] (3)
\[ B' = \alpha H' = \frac{2}{a} \Lambda \frac{\cos \theta}{\sqrt{\gamma} \sin \theta}, \] (4)

and
\[ \alpha B_\theta = H_\phi = \frac{2}{a^2} \Lambda \frac{\cos \theta}{\sin \theta}. \] (5)

This is consistent with
\[ \Omega_- = \frac{1}{a \sin^2 \theta} \] (6)
as defined in equation (B15). Please see Appendices A and B for the definitions of the quantities listed above. The general subscripts 1, 2, 3 in the Maxwell tensor corresponds to \( r, \theta, \phi \), respectively, in our case. Here, \( \Lambda \) is an arbitrary function of \( \theta \).

It is only natural to expect the net current vector field to follow a geodesic under force-free conditions. This is indeed the case:
\[
I = -\frac{2}{a^2 \sqrt{\gamma}} \frac{d}{d\theta} \left[ \frac{\Lambda}{\sin^3 \theta} \right] n.
\] (7)

Here \( n \) is the in-falling principle null geodesic of the Kerr geometry. Explicitly,
\[
n = \frac{r^2 + a^2}{\Delta} \partial_r - \partial_\theta + \frac{a}{\Delta} \partial_\phi.
\] (8)

A simple calculation will reveal that the solution presented above satisfies Maxwell’s equations (equation B1) and the force-free condition, equation (B2). The above solution is well defined everywhere in the magnetosphere. In particular, since our solution satisfies the Znajek regularity condition (equation B16), the fields are well defined at the event horizon, as we will explicitly verify by going into the Kerr–Schild coordinate system (see Subsection 2.1). Also, the apparent singularity at the poles is removed by the transformation \( \Lambda \rightarrow \sin^3 \theta \Lambda \).

However, the above solution, as it stands, lacks any meaningful physical interpretation. Physically realistic charges cannot flow along null geodesics. This problem will be remedied in the remainder of this paper by decomposing the null current vector into time-like vector fields that are possible worldlines of charged particles in the magnetosphere. In Menon & Dermer (2006), it was the deduction of a viable expression for \( \Omega_- \) that immediately gave us the expressions for all the fields and currents. Therefore, we shall refer to the complete solution listed above as the \( \Omega_- \) solution.

### 2.1 The \( \Omega_- \) solution in the Kerr–Schild coordinate system

Transforming the Maxwell tensor \( F_{\mu \nu} \) into the Kerr–Schild coordinate system (see Appendix A), we see that
\[
F_{t\theta} = F_{\phi \theta} = F_{\phi \theta} = 0,
\] (9)
\[
F_{\theta \rho} = -E_{\rho},
\] (10)
\[
F_{\phi \theta} = \sqrt{\gamma} B',
\] (11)
and
\[
I = \frac{2}{a^2 \sqrt{\gamma}} \frac{d}{d\theta} \left[ \frac{\Lambda}{\sin^3 \theta} \right] \partial_\theta.
\]

Thus we see that the fields and currents are well defined on the event horizon \( r = r_+ \) as well. This is necessarily so since we had insisted on the Znajek regularity condition given by equation (B16) in the derivation of our solution (Menon & Dermer 2006).

### 2.2 The electromagnetic Poynting flux

Blandford & Znajek (1977) computed the expression for the energy extracted from a rotating Kerr black hole via the electromagnetic Poynting vector for the force-free, stationary, axisymmetric magnetosphere. In the 3+1 notation, the rate of electromagnetic energy extraction becomes
\[
\frac{dE_{\text{EM}}}{dr} = -\int H_\phi \Omega^- B' \sqrt{\gamma} \, dA.
\] (12)

For the case of our \( \Omega_- \) solution, the above expressions give
\[
\frac{dE_{\text{EM}}}{dr} = -\frac{8\pi}{a^2} \int_0^{\pi} \frac{\Lambda^2 \cos^2 \theta}{\sin^3 \theta} \, d\theta \leq 0.
\] (13)

That is, if stationary electromagnetic fields can indeed transfer energy to and from a black hole, the \( \Omega_- \) solution in particular does not allow for energy extraction. Instead, the black hole behaves as an energy sink. We will come back to this point when we consider a dual solution.

### 3 A TIME-LIKE DECOMPOSITION OF THE NULL CURRENT

#### 3.1 Region I

Region I of the Kerr black hole is defined by the condition \( r > r_+ \), and the region of space–time given by \( r_+ < r < r_0 \) will be referred to as region II. Define vector fields \( V_I \) and \( W_I \) in region I by
\[
V_I = \frac{(r^2 + a^2)}{\sqrt{\rho^2 \Delta}} \partial_r + \frac{a}{\Delta} \partial_\phi
\] (14)

and
\[
W_I = \frac{(3l - 1) \left[ (r^2 + a^2) \partial_r + a \partial_\phi \right] + \Delta \partial_\phi}{\sqrt{\rho^2 \Delta \sqrt{3l(3l - 2)}}}
\] (15)

Here, \( l \equiv l(\theta) \) is the energy collimation factor such that \( l(\theta) > 2/3 \) everywhere. Then,
\[
g(V_I, V_I) = -1 \quad \text{and} \quad \lim_{r \to \infty} g(V_I, \partial_r) = -1,
\]

and
\[
g(W_I, W_I) = -1
\]

and
\[
\lim_{r \to \infty} g(W_I, \partial_\phi) = -\frac{(3l - 1)}{\sqrt{3l(3l - 2)}} < 0.
\]

That is, \( V_I \) and \( W_I \) are future pointing time-like in region I, and are candidate proper velocities of charged particles. Therefore, in region I we can write the current vector as the flow of two oppositely charged time-like currents:
\[
I_I = I_{IA} + I_{IB},
\]

where
\[
I_{IA} = -\frac{6l}{a^2 \sin \theta \sqrt{\rho^2 \Delta}} \frac{d}{d\theta} \left[ \frac{\Lambda}{\sin^3 \theta} \right] V_I
\]

and
\[
I_{IB} = \frac{2\sqrt{3l(3l - 2)}}{a^2 \sin \theta \sqrt{\rho^2 \Delta}} \frac{d}{d\theta} \left[ \frac{\Lambda}{\sin^3 \theta} \right] W_I.
\]

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It is easily verified that both $I_{II}$ and $I_{Ib}$ are divergence free. Consequently, we may think of them as separately flowing currents of charge. Naturally, $I_{Ia}$ describes an outgoing current. For completeness, note that in Kerr–Schild coordinates

$$V_{I} = \frac{[(r^{2} + a^{2})\partial_{r} + a\partial_{\phi}]}{\sqrt{\rho^{2}(4 - \Delta)}}$$

and

$$W_{I} = \frac{[3l(r^{2} + a^{2})\partial_{r} + 3al\partial_{\phi} + \Delta\partial_{t}]}{\sqrt{3\rho^{2}\Delta(l(3l - 2))}}.$$

It is important to note, that in our decomposition, even in the Kerr–Schild coordinate system, $V_{I}$ and $W_{I}$ are not well defined at $r_{+}$. There is a deeper reason why this happens regardless of the decomposition if we require an outgoing current in region I. Consider an arbitrary decomposition of the net current vector into terms proportional to future pointing time-like vectors in region I. At least one of these vectors say $T$ must be of the form

$$T = X + \psi^{2}\partial_{t},$$

where $\psi$ is an arbitrary function of space–time coordinates and $X$ has no other radial components so that the net radial component is positive. The only future pointing causal vector at $r_{+}$ such that its radial component is greater than or equal to zero is of the form

$$V_{+} = d^{2} \left[ (r_{+}^{2} + a^{2})\partial_{t} + a\partial_{\phi} \right],$$

where $d \in R$ is a constant (see O’Neill 1995). Then we have

$$\lim_{r \to r_{+}} \psi^{2} = 0$$

and

$$\psi_{r} \to 0$$

for some non-zero constant $d$ (since a unit time-like vector field $T$ cannot abruptly become the 0 vector), i.e. $T$ is null at the event horizon (since $V_{+}$ is null), and hence cannot be normalized. This is the reason why $V_{I}$ and $W_{I}$ are not well defined at $r_{+}$.

### 3.2 Region II

Here, since we are including $H_{+}$, our decomposition must be valid at the horizon as well. Consequently, all expressions will be given in the Kerr–Schild coordinate system. Additionally, the radial component of all the vectors in this decomposition must be inward pointing to agree with the causality conditions of the interior geometry (unlike in region I where we wanted an outflowing current). As we shall see, by construction, the decomposition here will be well defined at the horizon. In region II, we define vector fields $V_{II}$ and $W_{II}$ by

$$V_{II} = \frac{[(r^{2} + a^{2})\partial_{r} + a\partial_{\phi} + (\Delta - 2a)\partial_{t}]}{\sqrt{\rho^{2}(4 - \Delta)}}$$

and

$$W_{II} = \frac{[l(r^{2} + a^{2})\partial_{r} + lal\partial_{\phi} + (\Delta - l)\partial_{t}]}{\sqrt{\rho^{2}(2 - \Delta)}}.$$

Then,

$$g(V_{II}, V_{II}) = -1 = g(W_{II}, W_{II}),$$

and

$$\sqrt{\rho^{2}(4 - \Delta)}g(V_{II}, -\partial_{t}) = -\rho^{2} = \sqrt{\rho^{2}(2 - \Delta)}g(W_{II}, -\partial_{t}).$$

i.e. $V_{II}$ and $W_{II}$ are future pointing time-like in region II, and are candidate proper velocities of charged particles. Therefore, in region II we can write the current vector as the flow of two oppositely charged, infalling time-like currents:

$$I_{II} = I_{IIa} + I_{IIb},$$

where

$$I_{IIa} = -\frac{2\sqrt{\rho^{2}(4 - \Delta)}}{a^{2}\rho^{3}\sin^{2}\theta} \left[ \frac{\Delta}{\sin^{3}\theta} \right] V_{II},$$

$$I_{IIb} = \frac{2\sqrt{\rho^{2}(2 - \Delta)}}{a^{2}\rho^{3}\sin^{2}\theta} \left[ \frac{\Delta}{\sin^{3}\theta} \right] W_{II}.$$

### 3.3 Separating the regions outside the event horizon

Collectively, $I_{I} = I$ and $I_{II} = I$ give a meaningful description of the net current everywhere in the magnetosphere. The regions we chose might lead us to believe that the difference in the character of our decomposition must occur at the event horizon. This is not necessarily the case. $V_{II}$ and $W_{II}$ continue to be future pointing time-like in some open interval outside the event horizon, i.e. there exist $\delta > 0$ such that $V_{II}$ and $W_{II}$ are future pointing time-like in $r \leq r_{+} + \delta$ (for a fixed $I$, the surface given by $\tilde{r} = r_{+}$ is compact). Therefore, in the region given by $r \leq r_{+} + \delta$, we set

$$I = I_{II} = I_{IIa} + I_{IIb},$$

and for $r > r_{+} + \delta$, we set

$$I = I_{I} = I_{Ia} + I_{Ib}.$$

In fact, the new regions have further advantages, in that

$$\lim_{r \to r_{+} + \delta} I_{Ia} \text{ and } \lim_{r \to r_{+} + \delta} I_{Ib}$$

are well defined.

### 4 EXTRACTION OF ENERGY FROM MATTER CURRENTS

Here we focus on the outgoing current in region I:

$$I_{Ia} = \frac{2\sqrt{3l(3l - 2)}}{a^{2}\sin\theta\sqrt{\rho^{2}\Delta}} \left[ \frac{\Delta}{\sin^{3}\theta} \right] W_{I}.$$

It is easy to see that $I_{Ia}$ is divergence free. We now use this conserved current, which is the only outflowing current in this decomposition, to construct an expression for the extraction of matter energy from the black hole. We define the charge density $\rho_{c}$ by

$$\rho_{c} = \alpha I_{Ia},$$

and the current 3-vector $J$ by

$$J^{i} = \alpha I_{Ia}^{i}$$

for $i = 1, 2, 3, \nabla_{\mu} I_{Ia}^{\mu} = 0$ implies that

$$\partial_{t}\rho_{c} + \nabla \cdot J = 0.$$

Now, assuming that at every point $I_{Ia}$ is comprised of only one species of charged particle with charge $q$ and mass $m$, we can define the mass density of the current as

$$\rho_{m} = \frac{m}{q}\rho_{c}.$$
so that \( \rho_m \geq 0 \). This happens when
\[
\frac{1}{q} \frac{d}{d\theta} \left[ \Lambda \frac{\cos \theta}{\sin^4 \theta} \right] \geq 0. 
\]
(16)

Then,
\[
\mathbf{d} \mathbf{r} M = - \left[ \int_{r \to \infty} \frac{m}{q} g(J \cdot n) dA - \int_{r \to r_+} \frac{m}{q} g(J \cdot n) dA \right].
\]

In the above equation, \( n \) is the outward pointing normal, meaning \( n = \partial_r / \sqrt{g_{rr}} \) when \( r \to \infty \) and \( n = -\partial_r / \sqrt{g_{rr}} \) when \( r \to r_+ \), and \( dA = \sqrt{|g_{rr} \gamma |} d\theta \) \( d\phi \). Therefore, the matter extraction of energy \( \mathcal{E}_M \) from the black hole is given by
\[
\mathbf{d} \mathbf{r} \mathcal{E}_M = \int \frac{m}{q} I_{\theta} \rho^2 \sin \theta d\theta d\phi.
\]
(17)

For our particular current \( I_{\theta} \), we get
\[
\frac{d}{dt} \mathcal{E}_M = \frac{4\pi}{a^2} \int \frac{m}{q} \frac{d}{d\theta} \left[ \frac{\Lambda \cos \theta}{\sin^4 \theta} \right] d\theta,
\]
which from equation (16) is greater than zero, thus allowing for extraction of matter energy from the black hole. Of course, the above equation is meaningful only when \( \Lambda \) is able to absorb the infinity produced by \( (\sin^4 \theta)^{-1} \), which is an easy task since \( \Lambda \) is any arbitrary function of \( \theta \). To understand the collimation effects, it is important to note that
\[
\frac{d^2 \mathcal{E}_M}{d\theta dA} = \frac{2}{q} \frac{m}{a} \frac{d}{d\theta} \left[ \frac{\Lambda \cos \theta}{\sin^4 \theta} \right] \frac{1}{\sqrt{\Sigma}} \sin \theta
\]
\[
= \frac{2}{q} \frac{m}{a} \frac{d}{d\theta} \left[ \frac{\Lambda \cos \theta}{\sin^4 \theta} \right] \frac{1}{r^2} \sin \theta.
\]

Therefore, we see that \( (\sin^4 \theta)^{-1} \) factor gives the currents a preferential polar jet-like feature. Here, unlike the case of the stationary electromagnetic fields, the mechanism by which charged particles carry energy from the black hole is apparent from the nature of the outgoing currents. If indeed there are regions of antiparticle currents, we must make sure to modify the sign of \( q \) appropriately in that region.

5 THE LORENTZ FACTOR AT LARGE DISTANCES

Since the Boyer–Lindquist coordinates are asymptotically flat, from the \( r \) component of \( W_t \) in equation (15), we see that the Lorentz factor \( \Gamma \) of the ejected mass at infinity becomes
\[
\Gamma(\theta) = \frac{3l-1}{\sqrt{3l(3l-2)}}.
\]

The above equation can be inverted to give
\[
l = \frac{1}{3} \left[ 1 + \frac{\Gamma}{\sqrt{\Gamma^2 - 1}} \right].
\]

As \( \Gamma \to 1 \), the energy collimation factor \( l \to \infty \), and when \( l \to 2/3 \) the Lorentz factor \( \Gamma \to \infty \). In particular, for any finite \( l \), the ejected particles are so energetic that it never comes to rest even infinitely far away from the black hole. Clearly, the collimation effects on \( l \) stem from our freedom in choosing a judicious \( l \). The strength of the jets at the poles are also compensated by the intensity of the emitted particles.

6 SYMMETRY PROPERTIES OF THE FORCE-FREE EQUATIONS

Consider a complete description of the fields and currents given by quantities \( \rho, J, E, D, B \) and \( H \). For each of these quantities will be define a dual object (the dual of a quantity \( A \) will be indicated by \( \tilde{A} \)) such that all the dual objects collectively describe a force-free, stationary, axisymmetric magnetosphere in Kerr geometry. The dual quantity will be very simply related to the original quantity, and yet, collectively, the physical content of the dual solutions will not be equivalent to the original one. The general features of this construction involve looking at the poloidal components of an object separately from the toroidal and the zeroth component of the covariant formalism. The reason for this should be fairly clear: much like the background geometry, our assumptions require quantities to be time-independent (affecting the zeroth component of a vector) and axisymmetric (affecting the toroidal component of a vector).

The dual charges and current are defined by
\[
\tilde{\rho} = \rho, \quad \tilde{J}_T = J_T, \quad \tilde{J}_P = -J_P.
\]
(18)

Therefore, if we define
\[
\tilde{D} = D,
\]
(19)

we see that Gauss’s theorem will be naturally satisfied since
\[
\nabla \cdot \tilde{D} = \nabla \cdot D = \rho = \tilde{\rho}.
\]
(20)

For the other inhomogeneous Maxwell’s equation to hold, we define
\[
\tilde{B}_P = H_P, \quad \tilde{H}_T = -H_T.
\]
(21)

Then,
\[
(\nabla \times \tilde{H})_T = (\nabla \times \tilde{H}_T)_T = (\nabla \times H_T)_T = J_T = \tilde{J}_T.
\]
(22)

and
\[
(\nabla \times \tilde{H})_P = (\nabla \times \tilde{H}_T)_P = (\nabla \times H_T)_P = -J_P = \tilde{J}_P.
\]
(23)

The second equality in the above equation holds only because the fields are axisymmetric, for example
\[
(\nabla \times \tilde{H})' = e^{\epsilon k} \left( \tilde{\partial}_\rho \tilde{H}_\rho - \tilde{\partial}_\phi \tilde{H}_\phi \right).
\]
(24)

Therefore, for time-independent solutions, it follows from equations (22) and (23) that
\[
-\partial_t \tilde{D} + \nabla \times \tilde{H} = J.
\]
(25)

Now let consider the homogeneous Maxwell’s equations. Having defined \( \tilde{H} \) and \( \tilde{D} \), we have no more freedom in picking \( B \) and \( \tilde{E} \). It is not difficult to see that
\[
\tilde{B}_P = B_P, \quad \tilde{B}_T = -B_T,
\]
(26)

and
\[
\tilde{E} = E.
\]
(27)

Clearly, the curl of \( \tilde{E} \) vanishes, and once again due to axisymmetry, the divergence of \( \tilde{B} \) is trivial as well. Therefore, we have shown that the new quantities satisfy Maxwell’s equations. It is just a matter of simple calculation to show that the dual fields and currents are force free. Therefore, there exists a 3-vector \( \tilde{\omega} \) such that \( \tilde{E} = -\tilde{\omega} \times \tilde{B} \). It turns out that \( \tilde{\omega} = \omega \).

From equation (21) we see that if \( H_\rho \) satisfies the Znajek regularity condition, \( H_\rho \) will not (unless \( H_\rho = -H_\rho = 0 \)). Therefore, if we are using the dual solution to describe the external magnetosphere, we must separate the regions at \( r = r_+ + \delta \) as explained in Subsection 3.3.
7 THE $\tilde{\Omega}_-\,$ SOLUTION

The dual solution to the $\Omega_-\,$ solution presented earlier is given by

$$\tilde{\Omega}_- = \frac{1}{a \sin^2 \theta},$$

(28)

and

$$\tilde{E}_\phi = 0 = \tilde{E}_r,$$

(29)

$$E_{\theta} = -\frac{2}{a^2} \Lambda \frac{\cos \theta}{\sin^4 \theta},$$

(30)

and

$$B^\theta = 0$$

(31)

Finally,

$$I^\nu = \frac{2}{a^2 a^2 \sqrt{\Lambda}} \frac{d}{d \theta} \left[ \Lambda \frac{\cos \theta}{\sin^4 \theta} \right] I^\nu,$$

(34)

where $I^\nu$ is the principal outgoing null geodesic of the Kerr geometry given by

$$I^\nu = \left( \frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right).$$

(35)

We will refer to the set of equations above as the $\tilde{\Omega}_-\,$ solution.

7.1 Extraction of energy from the electromagnetic Poynting flux

These solutions do allow for the extraction of energy via the electromagnetic Poynting flux from the black hole. From equation (12) we see that the rate of energy extraction

$$\frac{d^2 \tilde{E}_M}{dt^2} = -\int (-H_\phi) \Omega B^\nu \sqrt{\gamma} dA = -\frac{d\tilde{E}_M}{dt} > 0.$$  

(36)

For the case of our $\tilde{\Omega}_-\,$ solution, the above expressions give

$$\frac{d^2 \tilde{E}_M}{dt^2} = \frac{8 \pi}{a^2} \int_0^\pi \Lambda^2 \frac{\cos \theta}{\sin^3 \theta} \frac{d\theta}{d \theta} > 0.$$  

(37)

The collimation effects from the electromagnetic fields are given by

$$\frac{d^2 \tilde{E}_M}{dA dt} \approx \frac{4}{a^2} \frac{\Lambda^2 \cos^2 \theta}{\sin^4 \theta} \frac{1}{r^2}.$$  

(38)

7.2 Extraction of energy from the electromagnetic currents

In region I, the dual currents can be decomposed as follows:

$$I_t = I_{ta} + I_{tb},$$

where

$$I_{ta} = \frac{2[3l - 2]}{a^2 \sin^2 \theta \sqrt{\rho^2 \Delta}} \frac{d}{d \theta} \left[ \Lambda \frac{\cos \theta}{\sin^4 \theta} \right] V_t$$

and

$$I_{tb} = -\frac{2\sqrt{3l(3l - 2)}}{a^2 \sin \theta \sqrt{\rho^2 \Delta}} \frac{d}{d \theta} \left[ \Lambda \frac{\cos \theta}{\sin^4 \theta} \right] W_t.$$

Naturally, the extraction of matter energy stems from $I_{ta}$. Analogous to equation (17), here the matter extraction of energy $\tilde{E}_M$ from the black hole is such that

$$\frac{d^2 \tilde{E}_M}{dt^2} \approx \frac{m}{a^2} \frac{d}{d \theta} \left[ \Lambda \frac{\cos \theta}{\sin^4 \theta} \right] \frac{1}{r^2 \sin \theta}$$

and

$$\frac{d}{dt} \tilde{E}_M = -\frac{4\pi}{a^2} \int \frac{m}{q} \frac{d}{d \theta} \left[ \Lambda \frac{\cos \theta}{\sin^4 \theta} \right] d\theta$$

which is positive when $q(\theta)$ is correctly chosen so that

$$-\frac{1}{q} \frac{d}{d \theta} \left[ \Lambda \frac{\cos \theta}{\sin^4 \theta} \right] \geq 0.$$  

(39)

8 A PARTICULAR CHOICE OF $\Lambda$

Since we want energy extraction via both the electromagnetic fluxes and the matter currents, throughout this section, we will focus on the $\tilde{\Omega}_-\,$ solution. Consider the simplest case where

$$\Lambda = \Lambda_0 \sin^2 \theta.$$  

(40)

The factor of $\sin^2 \theta$ is necessary to make the fields well defined on the poles. From equation (39) we have

$$q = q_- < 0 \text{ when } 0 \leq \theta < \pi/4 \text{ and } \frac{3\pi}{4} < \theta \leq \pi,$$

and

$$q = q_+ > 0 \text{ when } \pi/4 < \theta < 3\pi/4,$$

when $\Lambda_0$ is positive, which we will now require. Let $m_-$ and $m_+$ be the corresponding mass of the particle species. Then,

$$\frac{d^2 \tilde{E}_M}{dt^2} \approx \frac{m_2 \Lambda_0}{a^2} \cos \theta$$

Clearly, this describes a jet-like solution where the rate of energy extraction is maximized at the poles. There is a second local maxima along the equatorial plane suggesting the existence of a strong accretion disc. The presence of $1/\sin \theta$ is indicative of the spheroidal coordinate system used, and the apparent divergence vanishes upon integration to give

$$\frac{d}{dt} \tilde{E}_M = \frac{4\pi \Lambda_0}{a^2} \left[ m_+ + m_+ - m_- - q_+ - q_+ \right].$$

(41)

We are not assuming that the species of charged particles in the two regions (near the poles and near the equatorial planes) are particle antiparticle pairs, although nothing precludes it in our formalism. They could in particular be currents of electrons and protons (so that the black holes remain neutral during the process of energy extraction). The electromagnetic Poynting flux, however, gives

$$\frac{d^2 \tilde{E}_M}{dA dt} \approx \frac{4}{a^2} \frac{\Lambda_0^2 \cos^2 \theta}{\sin^4 \theta}.$$  

(42)

Here, just as in the case of the matter currents, the rate of electromagnetic energy extraction is a maximum along the polar axis. Interestingly, at the equatorial plane, the electromagnetic extraction rate is trivial, suggesting a secondary mechanism for the observed glow around the accretion disc. The total extraction rate is given by

$$\frac{d^2 \tilde{E}_M}{dt^2} = \frac{16\pi \Lambda_0^2}{3a^4}.$$  

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9 MAGNETIC MONPOLES

Solutions containing magnetic monopoles as measured by an observer external to the black hole can appear in curved space–times that contain singularities. Simply put, this is because the external observer does not use an inner Gaussian sphere while using the divergence theorem to avoid the singularity. An example of this is the original split monopole solution of Blandford & Znajek (1977) wherein they discontinuously matched the magnetic fields along the equatorial plane, thus splitting the solution to avoid a magnetic monopole in the asymptotic regions of the black hole. Exactly the same feature appeared in our generalization of the split monopole solution (Menon & Dermer 2005). It is important to note that the monopole is apparent and not physical. Indeed the magnetic fields in these cases are divergence free.

In our case here, since \( B' = B' \), as long as \( \Lambda \) is the same in regions I and II, we will not introduce any magnetic monopoles as a result of using the dual solution in the external region I \( (r > r_+) \). However, it is important to pick \( \Lambda \) carefully if we are to exclude magnetic monopoles in every closed region of space–time. Our choice of \( \Lambda \) given by equation (40) is one such choice. Indeed,

\[
\int_{r=\text{const}} g(B, n) \, dA = \frac{4\pi}{a} \int_{0}^{\pi} \Lambda \cos \theta \, d\theta = \frac{4\pi\Lambda_0}{a} \int_{0}^{\pi} \cos \theta \sin^2 \theta \, d\theta = 0.
\]

10 CONCLUSION

In our search for global analytic solutions describing the fields and currents in the force-free magnetosphere of a Kerr black hole, we specialized to time-independent, azimuthally symmetric solutions written in terms of a radially independent poloidal function \( \Omega \). We found exact results in terms of a dual solution to the \( \Omega_+ \) function that was shown earlier (Menon & Dermer 2005) to satisfy exactly the constraint equation for the black hole magnetosphere. Particular choices of an arbitrary poloidal function \( \Lambda \) give jet-like features for the matter or electromagnetic currents, but lack a physical basis for extracting energy from the spin of the black hole. This limitation of the global solutions may be overcome by taking into account actual physical conditions in the vicinity of the black hole, in particular the presence of an equatorial accretion disc where the force-free condition breaks down, either due to the dominant pressure of the accreting plasma or to currents flowing in the accretion disc. Another difficulty in applying global solutions to force-free magnetospheres is that physical effects are neglected that will certainly play a role in the actual system, for example, streaming instabilities from charged particle currents; or magnetic reconnection events at boundaries between regions of oppositely directed magnetic intensity, as found in the X-point of pulsar magnetospheres (e.g. Gruzinov 2006), which might have analogues in the black hole magnetosphere, are neglected. Mass flows and currents in the accretion disc will imply boundary conditions at the equator that will reduce uncertainty in the choice of \( \Lambda \).

There are two important clarifications that we must make before we conclude this article. In Menon & Dermer (2006), we had claimed that it is impossible to extract energy when \( \Omega = \Omega_+ \). The reference in this case is to the energy extracted via the electromagnetic flux alone; and it still holds true here (as it must). In this paper, as we have seen, it is possible to extract energy via matter currents when \( \Omega = \Omega_+ \). The extraction of energy via electromagnetic flux also does occur when \( \Omega = \Omega_+ \). The price to pay in this case is in the discontinuity of \( H_\rho \). This in turn produces a delta function current at the membrane joining the two regions. The analysis of this current on the membrane requires further study.

Outside of a few mild constraints, the functions \( l \) and \( \Lambda \) are arbitrary functions of \( \theta \). It is not clear whether astrophysical black holes permit a wide variety of magnetospheres, or if there is some other mechanism restricting the large degrees of freedom the decomposed currents and the fields have. None the less, we have constructed a specific, exact solution to the Blanford–Znajek mechanism that extracts energy from the black hole.

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APPENDIX A: KERR GEOMETRY ESSENTIALS

For completeness, we define the various Kerr coordinates used. For asymptotic analysis, the Boyer–Lindquist coordinates are preferred, while the horizon and the interior region \( (r \leq r_+) \) is analysed using the usual Kerr–Schild coordinate system.

A1 Boyer–Lindquist coordinates

In the Boyer–Lindquist coordinates \( \{ t, r, \theta, \varphi \} \) of the Kerr geometry, the metric takes the form

\[
ds^2 = (\beta^2 - \alpha^2) \, dt^2 + 2\beta \eta \, d\varphi \, dt + \gamma_{rr} \, dr^2 + \gamma_{\theta\theta} \, d\theta^2 + \gamma_{\varphi\varphi} \, d\varphi^2,
\]

where the metric coefficients are given by

\[
\beta^2 - \alpha^2 = g_{tt} = -1 + \frac{2Mr}{\rho^2}, \quad \beta \eta = g_{\eta\varphi} = -\frac{2Mr \sin^2 \theta}{\rho^2}, \quad \gamma_{rr} = \frac{\rho^2}{\Delta}, \quad \gamma_{\theta\theta} = \frac{\Sigma^2 \sin^2 \theta}{\rho^2}, \quad \gamma_{\varphi\varphi} = \frac{\Sigma^2 \sin^2 \theta}{\rho^2}.
\]

Here,

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2 \quad \text{and} \quad \Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta.
\]
Additionally
\[ \alpha^2 = \frac{\rho^2 \Delta}{\Sigma^2}, \quad \beta^2 = \frac{\beta^2}{\gamma_{\theta\phi}} \]
and
\[ \sqrt{-g} = \alpha \sqrt{\gamma} = \rho^2 \sin \theta. \]

The parameters \( M \) and \( a \) are the mass and angular momentum per unit mass, respectively, of the Kerr black hole. The horizons \( r_+ \) are located at \( r_+ = M \pm \sqrt{M^2 - a^2}. \)

### A2 Kerr–Schild coordinates

Kerr–Schild coordinates are given by the transformation
\[
\begin{bmatrix}
    df \\
    d\vec{r} \\
    d\theta \\
    d\phi
\end{bmatrix} = \begin{bmatrix}
    1 & G & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & H & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    dr \\
    d\tau \\
    d\theta \\
    d\phi
\end{bmatrix}, \tag{A1}
\]

where
\[
G = \frac{r^2 + a^2}{\Delta} \quad \text{and} \quad H = \frac{a}{\Delta}. \tag{A2}
\]

In this frame, the metric becomes
\[
g_{\mu\nu} = \begin{bmatrix}
    z - 1 & 1 & 0 & -za \sin^2 \theta \\
    1 & 1 & 0 & -a \sin^2 \theta \\
    0 & 0 & 0 & \rho^2 \\
    -za \sin^2 \theta & -a \sin^2 \theta & 0 & \Sigma^2 \sin^2 \theta / \rho^2
\end{bmatrix}, \tag{A3}
\]

where \( z = 2Mr/\rho^2 \). Components of vectors transform as
\[
\tilde{X}^\mu = A^\mu_\nu X^\nu,
\]

and 1-forms transform as
\[
\tilde{X}_\mu = (A^{-1})^\nu_\mu X_\nu,
\]

where the matrix \( A \) is given in equation (A1).

We pick our time orientation for the Kerr geometry such that the null vector field \(-\partial_\tau\) is future pointing everywhere.

### APPENDIX B: EQUATIONS OF ELECTRODYNAMICS IN STATIONARY SPACE–TIMES

We only state the relevant equations of electrodynamics of stationary space–times. For a detailed development, see Dermer & Menon (2009). Maxwell’s equations can be written as
\[
\nabla \bullet F^{\mu\nu} = 0, \quad \text{and} \quad \nabla \times F^{\mu\nu} = I^\mu. \tag{B1}
\]

Here \( F^{\mu\nu} \) is the Maxwell stress tensor, \( I^\mu \) is the 4-vectors of the electric current and \( \nabla \) is the covariant derivative of the geometry. \( \bullet \) is the 2-form defined by
\[
\bullet F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu
\nu\rho\sigma} F_{\rho\sigma}. \tag{B2}
\]

Here, \( \epsilon^{\mu
\nu\rho\sigma} \) is the completely antisymmetric Levi–Civita tensor density of space–time such that \( \epsilon_{0123} = \sqrt{-g} = \alpha \sqrt{\gamma} \) and \( \gamma \) along with the other relevant Kerr quantities are defined in Appendix A. In the 3+1 formalism, where \( \partial_\theta \) is the asymptotically stationary time-like killing vector field, \( E \) and \( B \) are defined so that
\[
F_{\mu \nu} = \begin{bmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & \sqrt{\gamma} B^3 & -\sqrt{\gamma} B^3 \\
E_2 & -\sqrt{\gamma} B^3 & 0 & \sqrt{\gamma} B^1 \\
E_3 & \sqrt{\gamma} B^2 & -\sqrt{\gamma} B^1 & 0
\end{bmatrix}. \tag{B3}
\]

We also define dual vectors \( D \) and \( H \) by
\[
\star F_{\mu \nu} = \begin{bmatrix}
0 & H_1 & H_2 & H_3 \\
-H_1 & 0 & \sqrt{\gamma} D^3 & -\sqrt{\gamma} D^2 \\
-H_2 & -\sqrt{\gamma} D^3 & 0 & \sqrt{\gamma} D^1 \\
-H_3 & \sqrt{\gamma} D^2 & -\sqrt{\gamma} D^1 & 0
\end{bmatrix}. \tag{B4}
\]

Naturally, \( F \) and \( \star F \) are not independent. They are related by
\[
\alpha D = E - \beta \times B \tag{B5}
\]
and
\[
H = \alpha B - \beta \times D. \tag{B6}
\]

Here,
\[
(A \times B)^i \equiv \epsilon^{ijk} A_j B_k, \tag{B7}
\]
where \( \epsilon^{ijk} \) is the Levi–Civita tensor of our absolute space defined as \( r^0 = \text{constant} \). Also, \( \beta \) is the shift dual vector given by \( \beta = \beta_{\phi} \partial_\phi \). Naturally, the spatial coordinates are given by \( (r, \theta, \phi) \), and the 3-vectors \( E, B, D, H \) live in this absolute space. Now, Maxwell’s equations can be rewritten as
\[
\tilde{\nabla} \cdot B = 0, \tag{B8}
\]
\[
\tilde{\nabla} \times E = 0, \tag{B9}
\]
\[
\tilde{\nabla} \cdot D = \rho_c, \tag{B10}
\]
and
\[
-\partial_\tau D + \tilde{\nabla} \times H = J, \tag{B11}
\]
where \( \rho_c = \alpha \gamma \) and \( J^\mu = \alpha F^\mu \). Here \( \rho_c \) is the charge density and \( J \) is the electric 3-current. \( \tilde{\nabla} \) is the covariant derivative of the 3-space with the induced metric. The force-free condition that we will enforce is
\[
F_{\mu \nu} I^\nu = 0. \tag{B12}
\]
This condition takes the form
\[
E \cdot J = 0 \tag{B13}
\]
and
\[
\rho_c E + J \times B = 0. \tag{B14}
\]

For the case of a stationary, axisymmetric, force-free magneto-sphere, it is easy to show that there exists \( \omega = \Omega_\phi \gamma \) such that \( E = -\omega \times B \). \tag{B15}

Additionally, Znajek (1977) showed that
\[
H_{\phi} \bigg|_{r_+} = \frac{\sin^2 \theta}{\alpha} B^i (2Mr\Omega - a) \bigg|_{r_+} \tag{B16}
\]
is the required condition in the Boyer–Lindquist coordinates that the otherwise bounded fields must satisfy so that they continue to be well defined in the Kerr–Schild coordinates at the event horizon. Equation (B16) is referred to as the Znajek regularity condition.

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