AN OBSTRUCTION TO ASYMPTOTIC SEMISTABILITY 
AND APPROXIMATE CRITICAL METRICS 

TOSHIKI MABUCHI 

1. Introduction 

For a polarized algebraic manifold $(M, L)$ with a Kähler metric of constant scalar curvature in the class $c_1(L)_\mathbb{R}$, we consider the Kodaira embedding 

$$\Phi_{|L_m|} : M \hookrightarrow \mathbb{P}(V_m), \quad m \gg 1,$$

where $V_m := H^0(M, \mathcal{O}(L^m))^*$. Even when a linear algebraic group of positive dimension acts nontrivially and holomorphically on $M$, we shall show that the vanishing of an obstruction to asymptotic Chow-semistability allows us to generalize Donaldson’s construction \cite{Donaldson} of approximate solutions for equations of critical metrics\footnote{In (2.6) below, $\omega = c_1(L; h)$ is called a critical metric if $K(q, h)$ is a constant function on $M$. The same concept was later re-discovered by Luo \cite{Luo} (see \cite{Mabuchi}).} of Zhang \cite{Zhang}. This generalization plays a crucial role in our forthcoming paper \cite{Mabuchi}, in which the asymptotic Chow-stability for $(M, L)$ above will be shown under the vanishing of the obstruction, even when $M$ admits a group action as above.

2. Statement of results 

Throughout this paper, we assume that $L$ is an ample holomorphic line bundle over a connected projective algebraic manifold $M$. Let $n$ and $d$ be respectively the dimension of $M$ and the degree of the image $M_m := \Phi_{|L^m|}(M)$ in the projective space $\mathbb{P}(V_m)$ with $m \gg 1$. Then to this image $M_m$, we can associate a nonzero element $\hat{M}_m$ of $W_m := \{\text{Sym}^d(V_m)\}^{\otimes n+1}$ such that its natural image $[\hat{M}_m]$ in $\mathbb{P}(W_m)$ is the Chow point associated to the irreducible reduced algebraic cycle $M_m$ on $\mathbb{P}(V_m)$. For the natural action of $H_m := \text{SL}(V_m)$ on $W_m$ and also on $\mathbb{P}(W_m)$, the subvariety $M_m$ of $\mathbb{P}(V_m)$ is said to be Chow-stable or Chow-semistable, according as the orbit $H_m \cdot \hat{M}$ is closed in $W_m$ or the origin of $W_m$ is not in the closure of $H_m \cdot \hat{M}$ in $W_m$. Fix an increasing sequence 

$$(2.1) \quad m(1) < m(2) < m(3) < \cdots < m(k) < \cdots$$

of positive integers $m(k)$. For this sequence, we say that $(M, L)$ is asymptotically Chow-stable or asymptotically Chow-semistable, according as for some $k_0 \gg 1$, the subvariety $M_{m(k)}$ of $\mathbb{P}(V_{m(k)})$ is Chow-stable or Chow-semistable for all $k \geq k_0$. 

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Let Aut⁰(M) denote the identity component of the group of all holomorphic automorphisms of M. Then the maximal connected linear algebraic subgroup G of Aut⁰(M) is the identity component of the kernel of the Jacobi homomorphism
\[ \alpha_M : \text{Aut}^0(M) \rightarrow \text{Aut}^0(\text{Alb}(M)), \] (cf. [3]).

For the maximal compact subgroup (Z) of H⁰(M, O(T¹⁰M)) associated to the Lie subgroup Z of Aut⁰(M). For the isotropy subgroup, denoted by ˜Sₘ, of Hₘ at the point [Mₘ] ∈ P(Wₘ), we have a natural isogeny
\[ tₘ : ˜Sₘ \rightarrow Sₘ, \]
where Sₘ is an algebraic subgroup of G. For Zₘ := t⁻¹ₘ(Z), we have a Zₘ-action on M naturally induced by the Z-action on M. Since the Z-action on M is liftable to a holomorphic bundle action on L (see for instance [7], the restriction of tₘ to Zₘ defines an isogeny of Zₘ onto Z. The vector space Vₘ is viewed as the line bundle Oₘ(−1) with the zero section blow-down to a point, while the line bundle Oₘ(−1) coincides with L⁻ᵐ when restricted to M. Hence, the natural ˜Sₘ-action on Vₘ induces a bundle action of Zₘ on Vₘ which covers the Zₘ-action on M. Infinitesimally, each X ∈ 3 induces a holomorphic vector field X' ∈ H⁰(Lᵐ, O(T¹⁰Lᵐ)) on Lᵐ. Since the C*bundle L \ {0} associated to L is an m-fold unramified covering of the C*bundle Lᵐ \ {0}, the restriction of X' to Lᵐ \ {0} naturally induces a holomorphic vector field X'' on L \ {0}. Since X'' extends to a holomorphic vector field on L, the mapping X → X'' defines inclusions
\[ \rho_m : 3 \rightarrow H^0(L, O(T^{1,0}L)), \quad m = 1, 2, \ldots, \]
inducing lifts, from M to L, of vector fields in 3. For a sequence as in (2.1), we say that the isotropy actions for (M, L) are stable if there exists an integer k₀ ≥ 1 such that
\[ \rho_m(k) = \rho_m(k₀), \quad \text{for all } k ≥ k₀. \]

For the maximal compact subgroup (Zₘ)c of Zₘ, take a (Zₘ)c-invariant Hermitian metric λ for Lᵐ. By the theory of equivariant cohomology ([11, 13]), we define (see [15, 13]):
\[ \mathcal{C}\{c^{n+1}_1; L^m\}(X) := \frac{\sqrt{-1}}{2\pi} (n+1) \int_M \lambda^{-1}(Xλ) c_1(L^m; \lambda)^n, \quad X ∈ 3, \]
where Xλ is as in [13], (1.4.1). Then the C-linear map \( \mathcal{C}\{c^{n+1}_1; L^m\} : 3 \rightarrow \mathbb{C} \) which sends each X ∈ 3 to \( \mathcal{C}\{c^{n+1}_1; L^m\}(X) \in \mathbb{C} \) is independent of the choice of h. The following gives an obstruction to asymptotic Chow-semistability (see [5, 15, 16] for related results):

**Theorem A.** For a sequence as in (2.1), assume that (M, L) is asymptotically Chow-semistable. Then for some k₀ ≥ 1, the equality \( \mathcal{C}\{c^{n+1}_1; L^{m(k)}\} = 0 \) holds for all k ≥ k₀. In particular, for this sequence, the isotropy actions for (M, L) are stable.

The following modification of a result in [7] shows that, as an obstruction, the stability condition (2.3) is essential, since the vanishing of (2.4) is straightforward from (2.3).
Theorem B. For sufficiently large \((n+2)\) distinct integers \(m_k, k = 0, 1, \ldots, n+1\), suppose that \(\rho_{m_0} = \rho_{m_1} = \cdots = \rho_{m_{n+1}}\). Then \(C\{x_1^{n+1}; L^{m_k}\} = 0\) for all \(k\).

If \(\dim Z = 0\), by setting \(m(k) = k\) in (2.1) for all \(k \geq 0\), we see that \(\rho_m\) are trivial for all \(m \gg 1\), and consequently (2.3) holds. Note also that Donaldson’s result \([3]\) treating the case \(\dim G = 0\) depends on his construction of approximate solutions for equations of critical metrics of Zhang \([20]\). In Theorem C down below, assuming (2.3), we generalize Donaldson’s construction to the case \(\dim G > 0\).

Put \(N_m := \dim C V_m - 1\). Let \(h\) be a Hermitian metric for \(L\) such that \(\omega = c_1(L; h)\) is a Kähler metric on \(M\). By the inner product
\[
(\sigma, \sigma')_h := \int_M <\sigma, \sigma'>_h \omega^n, \quad \sigma, \sigma' \in V_m^*,
\]
on \(V_m^* = H^0(M, O(L^m))\), we choose a unitary basis \(\{\sigma_{0}^{(m)}, \sigma_{1}^{(m)}, \ldots, \sigma_{N_m}^{(m)}\}\) for \(V_m^*\). Here, \(<\sigma, \sigma'>_h\) denotes the function on \(M\) obtained as the the pointwise inner product of the sections \(\sigma, \sigma'\) by the Hermitian metric \(h^m\) on \(L^m\). Put
\[
K(q, h) := \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\sigma_i^{(m)}\|_h^2,
\]
where \(\|\sigma\|_h^2 := <\sigma, \sigma>_h\) for all \(\sigma \in V_m^*\) and we set \(q := 1/m\). We then have the asymptotic expansion of Tian-Zelditch (cf. \([18], [19]\)) for \(m \gg 1\):
\[
K(q, h) = 1 + a_1(\omega)q + a_2(\omega)q^2 + a_3(\omega)q^3 + \ldots,
\]
where \(a_i(\omega), i = 1, 2, \ldots,\) are smooth functions on \(M\). Then \(a_1(\omega) = \sigma_\omega/2\) (cf. \([11]\)) for the scalar curvature \(\sigma_\omega\) of \(\omega\). Put \(C_q := \{m^n c_1(L)^n |M|/n!\}^{-1}(N_m + 1)\). Then

Theorem C. For a Kähler metric \(\omega_0\) in the class \(c_1(L)_{\mathbb{R}}\) of constant scalar curvature, choose a Hermitian metric \(h_0\) for \(L\) such that \(\omega_0 = c_1(L, h_0)\). For a sequence as in (2.1), assume that the isotropy actions for \((M, L)\) are stable, i.e., (2.3) holds. Put \(q = 1/m(k)\). Then there exists a sequence of real-valued smooth functions \(\varphi_k, k = 1, 2, \ldots,\) on \(M\) such that \(h(\ell) := h_0 \exp(-\sum_{k=1}^{\ell} q^k \varphi_k)\) satisfies \(K(q, h(\ell)) - C_q = O(q^{\ell+2})\) for each nonnegative integer \(\ell\).

The last equality \(K(q, h(\ell)) - C_q = O(q^{\ell+2})\) means that there exist a positive real constant \(A = A_\ell\) independent of \(q\) such that \(\|K(q, h(\ell)) - C_q\|_{C^0(M)} \leq A_\ell q^{\ell+2}\) for all \(0 \leq q \leq 1\) on \(M\). By \([19]\), for every nonnegative integer \(j\), a choice of a larger constant \(A = A_{j, \ell} > 0\) keeps Theorem C still valid even if \(C^0(M)\)-norm is replaced by \(C^j(M)\)-norm.

3. An obstruction to asymptotic semistability

The purpose of this section is to prove Theorems A and B. Fix a sequence as in (2.1), and in this section, any kind of stability is considered with respect to this sequence.
Proof of Theorem A: Assume that $(M,L)$ is asymptotically Chow-semistable, i.e., for some $k_0 \gg 1$, the subvariety $M_m(k)$ of $\mathbb{P}(V_m(k))$ is Chow-semistable for all $k \geq k_0$. Then the isotropy representation of $Z_m(k)$ on the line $\mathbb{C} \cdot M_m(k)$ is trivial (cf. [15], (3.5) (cf. [16]; [20], (1.5)), we obtain the required equality

\[(3.1)\quad \mathcal{C}\{c_1^{n+1}; L^m(k)\}(X) = 0, \quad X \in \mathfrak{z},\]

for all $k \geq k_0$. For $\lambda$ in (2.4), by setting $h := \lambda^{1/m}$, we have a Hermitian metric $h$ for $L$. Put $\chi_m := \mathcal{C}\{c_1^{n+1}, L^m\}/m^{n+1}$ for positive integers $m$. Then by the Leibniz rule,

\[(3.2)\quad \chi_m(X) = \frac{\sqrt{-1}}{2\pi} (n+1) \int_M h^{-1}(Xh)_{\rho_m} c_1(L;h)^n, \quad X \in \mathfrak{z},\]

where the complexified action $(Xh)_{\rho_m}$ of $X$ on $h$ as in [13], (1.4.1), is taken via the lifting $\rho_m$ in (2.2). Then by (3.1),

$$\chi_m(k_0) = \chi_m(k_{0+1}) = \cdots = \chi_m(k) = \cdots;$$

and since lifts in (2.2), from $M$ to $L$, of holomorphic vector fields in $\mathfrak{z}$ are completely characterized by $\chi_m$ (cf. [7]), we obtain (2.3), as required. \hfill \Box

Proof of Theorem B: For $q := \text{l.c.m}\{m_k; k = 0, 1, \ldots, n+1\}$, we take a $q$-fold unramified cover $\nu: \tilde{Z} \to Z$ between algebraic tori. Then the $Z$-action on $M$ naturally induces a $\tilde{Z}$-action on $M$ via this covering. Since $\nu$ factors through $Z_{m_k}$, the lift, from $M$ to $L^{m_k}$, of the $Z_{m_k}$-action naturally induces a lift, from $M$ to $L^{m_k}$, of the $\tilde{Z}$-action. The assumption

\[(3.3)\quad \rho_{m_0} = \rho_{m_1} = \cdots = \rho_{m_{n+1}}\]

shows that the lifts, from $M$ to $L^{m_k}$, $k = 0,1,\ldots, n+1$, of the $\tilde{Z}$-action come from the same infinitesimal action of $\mathfrak{z}$ as vector fields on $L$. For brevity, the common $\rho_{m_k}$ in (3.3) will be denoted just by $\rho$. Then the proof of [6], Theorem 5.1, is valid also in our case, and the formula in the theorem holds. By $Z_{m_k} \subset \text{SL}(V_{m_k})$ and by its contragredient representation, the $\tilde{Z}$-action on $V_{m_k}^* = H^0(M, \mathcal{O}(L^{m_k}))$ comes from an algebraic group homomorphism: $\tilde{Z} \to \text{SL}(V_{m_k}^*)$. Hence, by the notation in (3.2) above,

$$\int_M h^{-1}(Xh)_{\rho} c_1(L;h)^n = 0 \quad \text{for all } X \in \mathfrak{z}, \quad \text{i.e., } \mathcal{C}\{c_1^{n+1}; L^{m_k}\} = 0 \quad \text{for all } k,$$

as required. \hfill \Box

4. PROOF OF THEOREM C

Throughout this section, we assume that the first Chern class $c_1(L)_{\mathbb{R}}$ admits a Kähler metric of constant scalar curvature. Then a result of Lichnérowicz [11] (see also [9]) shows that $G$ is a reductive algebraic group, and consequently the identity component of the center of $G$ coincides with $Z$ in the introduction. Let $K$ be a maximal compact subgroup of $G$. Then the maximal compact subgroup $Z_c$ of $Z$ satisfies

\[(4.1)\quad Z_c \subset K.\]
For an arbitrary $K$-invariant Kähler metric $\omega$ on $M$ in the class $c_1(L)_R$, we write $\omega$ as the Chern form $c_1(L; h)$ for some Hermitian metric $h$ for $L$. Let $\Psi(\varphi, \omega)$ denote the power series in $\varphi$ given by the right-hand side of (2.7). Then

\[
(4.2) \quad \int_M \{ \Psi(\varphi, \omega) - C_q \} \omega^n = \int_M \left\{ -C_q + \frac{n!}{m^n} \sum_{i=0}^{N_m} \| \sigma_i^{(m)} \|_h^2 \right\} \omega^n = 0.
\]

Let $h_0$ be a Hermitian metric for $L$ such that $\omega_0 := c_1(L; h_0)$ is a Kähler metric of constant scalar curvature on $M$. We write

\[
\omega_0 = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta} g_{\alpha\beta} dz^\alpha \wedge dz^\beta,
\]

for a system $(z^1, z^2, \ldots, z^n)$ of holomorphic local coordinates on $M$. In view of [10] (see also [9]), replacing $\omega_0$ by $g^*\omega_0$ for some $g \in G$ if necessary, we may assume that $\omega_0$ is $K$-invariant. Let $D_0$ be the Lichnérówicz operator, as defined in [2], (2.1), for the Kähler manifold $(M, \omega_0)$. Since $\omega_0$ has a constant scalar curvature, $D_0$ is a real operator. Let $\mathcal{F}$ denote the space of all real-valued smooth $K$-invariant functions $\varphi$ such that $\int_M \varphi \omega_0^n = 0$.

Since the operator $D_0$ preserves the space $\mathcal{F}$, we write $D_0$ as an operator $D_0 : \mathcal{F} \to \mathcal{F}$, and the kernel in $\mathcal{F}$ of this operator will be denoted by $\text{Ker} D_0$. Let $\mathfrak{g}_c$ denote the Lie subalgebra of $\mathfrak{g}$ corresponding to the maximal compact subgroup $Z_c$ of $Z$. Then

\[
(3.3) \quad \gamma : \text{Ker} D_0 \cong \mathfrak{g}_c, \quad \eta \leftrightarrow \gamma(\eta) := \text{grad}^C_{\omega_0} \eta,
\]

where $\text{grad}^C_{\omega_0} \eta := (1/\sqrt{-1}) \sum g^{\beta\alpha} \eta_\beta \partial/\partial z^\alpha$ denotes the complex gradient of $\eta$ with respect to $\omega_0$. We then consider the orthogonal projection

\[
P : \mathcal{F} (= \text{Ker} D_0 \oplus \text{Ker} D_0^\perp) \to \text{Ker} D_0.
\]

Starting from $h(0) = h_0$ and $\omega(0) := \omega_0$, we inductively define a Hermitian metric $h(k)$ for $L$, and a Kähler metric $\omega(k) := c_1(L; h(k))$, called the $k$-approximate solution, by

\[
h(k) = h(k-1) \exp(-q^k \varphi_k), \quad k = 1, 2, \ldots,
\]

\[
\omega(k) = \omega(k-1) + \frac{\sqrt{-1}}{2\pi} q^k \bar{\partial} \partial \varphi_k, \quad k = 1, 2, \ldots,
\]

for a suitable function $\varphi_k \in \text{Ker} D_0^\perp$, where we require $h(k)$ to satisfy $K(q, h(k)) - C_q = O(q^{k+2})$. In other words, by (4.2), each $\omega(k)$ is required to satisfy the following conditions:

\[
(4.4) \quad (1 - P) \{ \Psi(q, \omega(k)) - C_q \} \equiv 0, \quad \text{modulo } q^{k+2},
\]

\[
(4.5) \quad P \{ \Psi(q, \omega(k)) - C_q \} \equiv 0, \quad \text{modulo } q^{k+2}.
\]

If $k = 0$, then $\omega(0) = \omega_0$, and by [11], both (4.4) and (4.5) hold for $k = 0$. Hence, let $\ell \geq 1$ and assume (4.4) and (4.5) for $k = \ell - 1$. It then suffices to find $\varphi_\ell \in \text{Ker} D_0^\perp$ satisfying both (4.4) and (4.5) for $k = \ell$. Put

\[
\Phi(q, \varphi) := (1 - P) \left\{ \Psi(q, \omega(\ell - 1) + (\sqrt{-1}/2\pi) q^\ell \bar{\partial} \partial \varphi) - C_q \right\}, \quad \varphi \in \text{Ker} D_0^\perp.
\]
Then by (4.4) applied to $k = \ell - 1$, we have $\Phi(q, 0) \equiv u_\ell q^{\ell + 1}$ modulo $q^{\ell + 2}$, where $u_\ell$ is a function in $\text{Ker} \, D_0^\perp$. Since $2\pi \omega(\ell - 1) = 2\pi \omega_0 + \sqrt{-1} \sum_{k=1}^{\ell - 1} q^k \partial \bar{\partial} \varphi_k$, we have $\omega(\ell - 1) = \omega_0$ at $q = 0$. Since the scalar curvature of $\omega_0$ is constant, the variation formula for the scalar curvature (see for instance [4], (2.5); [3]) shows that

$$\Phi(q, \varphi_\ell) \equiv \Phi(q, 0) - q^{\ell + 1}(D_0 \varphi_\ell/2) \equiv (2u_\ell - D_0 \varphi_\ell) (q^{\ell + 1}/2),$$

modulo $q^{\ell + 2}$. Since $u_\ell$ is in $\text{Ker} \, D_0^\perp$, there exists a unique $\varphi_\ell \in \text{Ker} \, D_0^\perp$ such that $2u_\ell = D_0 \varphi_\ell$ on $M$. Fixing such $\varphi_\ell$, we obtain $h(\ell)$ and $\omega(\ell)$. Thus (4.4) is true for $k = \ell$.

Now, we have only to show that (4.5) is true for $k = \ell$. Before checking this, we give some preliminary remarks. Note that $C_q = 1 + O(q)$. Moreover, by (2.7), $\Psi(q, \omega) = 1 + q\{a_1(\omega) + a_2(\omega)q + \ldots\}$, and hence

$$\Psi(q, \omega(\ell)) - C_q = \Psi\left(q, \omega(\ell - 1) + (\sqrt{-1}/2\pi) q^\ell \partial \bar{\partial} \varphi_\ell\right) - C_q
\equiv \Psi(q, \omega(\ell - 1)) - C_q \equiv 0,$$

modulo $q^{\ell + 1}$.

By [17], p. 35, the $G$-action on $M$ is liftable to a bundle action of $G$ on the real line bundle $(L \cdot \bar{L})^{1/2} = (L^m \cdot \bar{L}^m)^{1/2m}$. Then the induced $K$-action on $(L \cdot \bar{L})^{1/2}$ is unique, because liftings, from $M$ to $L^m$, of the $G$-action differ only by scalar multiplications of $L^m$ by characters of $Z$. In this sense, $h(\ell)$ is $K$-invariant. Put $r := \dim_{\mathbb{C}} Z$. Then we can write $Z_m = G_m^r = \{ t = (t_1, t_2, \ldots, t_r) \in (\mathbb{C}^*)^r \}$. By the natural inclusion

$$\psi_m : Z_m \hookrightarrow H_m = \text{SL}(V_m),$$

we can choose a unitary basis $\{\tau_0, \tau_1, \ldots, \tau_{Nm}\}$ for $(V_m^*, (,)_h(\ell))$ (cf. (2.5)) such that, for some integers $\alpha_{i_0}$ with $\sum_i \alpha_{i_0} = 0$, the contragredient representation $\psi_m^*$ of $\psi_m$ is given by

$$\psi_m^*(t) \tau_i = \left( \prod_{j=1}^r t_j^{\alpha_{i_j}} \right) \tau_i, \quad i = 0, 1, \ldots, N_m,$$

for all $t \in (\mathbb{C}^*)^r = Z_m$. Now by (2.3), for some $\rho : \mathfrak{g} \hookrightarrow H^0(L, \mathcal{O}(T^{1,0}L))$, we can write $\rho_{m(k)} = \rho$ for all $k \geq k_0$. Consider the Kähler metric $\omega_m := c_1(L; h_m)$ on $M$ in the clasas $c_1(L)^{\mathbb{R}}$, where $h_m := (|\tau_0|^2 + |\tau_1|^2 + \ldots + |\tau_{Nm}|^2)^{-1/m}$. From now on, let $m = m(k)$, where $k$ is running through all integers $\geq k_0$. Put $X_j := t_j \partial/\partial t_j$. Then $\{X_1, X_2, \ldots, X_r\}$ forms a $\mathbb{C}$-basis for the Lie algebra $\mathfrak{g}$ such that, using the notation as in (3.2), we have

$$h^{-1}_m(X_j h_m)_\rho = -\frac{\sum_i \alpha_{i_0} |\tau_i|^2}{m \sum_i |\tau_i|^2}, \quad 1 \leq j \leq r, \quad \text{for } m = m(k) \text{ with } k \geq k_0,$$

where in the numerator and the denominator, the sum is taken over all integers $i$ such that $0 \leq i \leq N_m$. From (2.3) and Theorem B, using the notation as in (3.2), we obtain

$$\int_M h(\ell)^{-1}(X_j h(\ell))_\rho \omega(\ell)^n = 0, \quad 1 \leq j \leq r.$$
By \( \int_M h_0^{-1}(X_jh_0) \rho/\int_M \omega_0^m = 0 \), we have \( \eta_j := h_0^{-1}(X_jh_0) \in \text{Ker} D_0 \). Then \( \gamma(\eta_j) = \sqrt{-1} X_j \). Hence \( \{\eta_1, \eta_2, \ldots, \eta_r\} \) is an \( \mathbb{R} \)-basis for \( \text{Ker} D_0 \). Since \( \Psi(q, \omega(\ell)) \equiv C_q \mod q^{\ell+1} \), it follows that

\[
(4.8) \quad -C_q + \frac{n!}{m^n} \sum_{i=0}^{N_m} ||\tau_i||^2_{h(\ell)} \equiv v_\ell q^{\ell+1} \mod q^{\ell+2}
\]

where the equivalence just above follows from (4.7). The last integrand is rewritten as

\[
q^{\ell+1} \int_M \eta_j v_\ell \omega_0^m \equiv C_q \int_M \frac{\sum_i \alpha_{ij} ||\tau_i||^2_{h(\ell)}}{m \sum_i ||\tau_i||^2_{h(\ell)}} \{\omega(\ell)\}^n \equiv C_q \int_M h_m^{-1}(X_jh_m)_\rho \{\omega(\ell)\}^n
\]

\[
\equiv C_q \int_M \{h_m^{-1}(X_jh_m)_\rho - h(\ell)^{-1}(X_jh(\ell))_\rho\} \{\omega(\ell)\}^n,
\]

where the equivalence just above follows from (4.7). The last integrand is rewritten as

\[
h_m^{-1}(X_jh_m)_\rho - h(\ell)^{-1}(X_jh(\ell))_\rho = X_j \log \{h_m/h(\ell)\} = -\frac{1}{m} X_j \log \left( \frac{n!}{m^n} \sum_{i=0}^{N_m} ||\tau_i||^2_{h(\ell)} \right)
\]

\[
\equiv -q X_j \log(C_q + v_\ell q^{\ell+1}) \equiv -C_q^{-1}(X_jv_\ell)q^{\ell+2} \equiv 0, \mod q^{\ell+2}.
\]

Therefore, \( \int_M \eta_j v_\ell \omega_0^m = 0 \) for all \( j \). From \( v_\ell \in \text{Ker} D_0 \), it now follows that \( v_\ell = 0 \). This shows that (4.5) is true for \( k = \ell \), as required.

\[\square\]

5. Concluding remarks

As in Donaldson’s work \[3\], the construction of approximate solutions in Theorem C is a crucial step to the approach of the stability problem for a polarized algebraic manifold with a Kähler metric of constant scalar curvature. Actually, in a forthcoming paper \[14\], this construction allows us to prove the following:
Theorem. For a sequence as in (2.1), assume that the isotropy actions for \((M, L)\) are stable. Assume further that \(c_1(L)_R\) admits a Kähler metric of constant scalar curvature. Then for this sequence, \((M, L)\) is asymptotically Chow-stable.

Moreover, if a sequence (2.1) exists in such a way that (2.3) holds, then the same argument as in the case \(\dim G = 0\) (cf. [3]) is applied, and we can also show the uniqueness, modulo the action of \(G\), of the Kähler metrics of constant scalar curvature in the polarization class \(c_1(L)_R\). We finally remark that, if \(\dim G = 0\), the asymptotic Chow-stability implies the asymptotic stability in the sense of Hilbert schemes (cf. [17], p.215). Hence the result of Donaldson [3] follows from the theorem just above.

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043 Japan