1. Introduction. Let $k$ be an algebraic number field of degree $d$ over $\mathbb{Q}$, $v$ a place of $k$ and $k_v$ the completion of $k$ at $v$. We select two absolute values from the place $v$. The first is denoted by $\| \|$ and defined as follows:

(i) if $v | \infty$ then $\| \|$ is the unique absolute value on $k_v$ that extends the usual absolute value on $\mathbb{Q}_\infty = \mathbb{R}$,

(ii) if $v | p$ then $\| \|$ is the unique absolute value on $k_v$ that extends the usual $p$-adic absolute value on $\mathbb{Q}_p$.

The second absolute value is denoted by $| |_v$ and defined by $|x|_v = \|x\|_v^{d_v/d}$ for all $x$ in $k_v$, where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree. If $\alpha \neq 0$ is in $k$ then these absolute values satisfy the product formula

$$\prod_v |\alpha|_v = 1.$$  

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $\overline{\mathbb{Q}}^\times$ the multiplicative group of nonzero elements in $\overline{\mathbb{Q}}$. The absolute, logarithmic Weil height (or simply the height)

$$h : \overline{\mathbb{Q}}^\times \to [0, \infty)$$

is defined as follows. Let $\alpha$ be a nonzero algebraic number; we select an algebraic number field $k$ containing $\alpha$, and then

$$h(\alpha) = \sum_v \log^+ |\alpha|_v,$$

where the sum on the right of (1.2) is over all places $v$ of $k$. It can be shown that $h(\alpha)$ is well defined because the right hand side of (1.2) does not depend
on the field $k$. By combining (1.1) and (1.2) we obtain the useful identity

$$2h(\alpha) = \sum_v |\log |\alpha|_v|,$$

where $| \cdot |$ (an absolute value without a subscript) is the usual archimedean absolute value on $\mathbb{R}$.

Let $\text{Tor}(\overline{\mathbb{Q}}^\times)$ denote the torsion subgroup of $\overline{\mathbb{Q}}^\times$ and write

$$\mathcal{G} = \overline{\mathbb{Q}}^\times/\text{Tor}(\overline{\mathbb{Q}}^\times)$$

for the quotient group. If $\zeta$ is a point in $\text{Tor}(\overline{\mathbb{Q}}^\times)$, then it is immediate from (1.2) that $h(\alpha) = h(\zeta\alpha)$ for all points $\alpha$ in $\overline{\mathbb{Q}}^\times$. Thus $h$ is constant on each coset of the quotient group $\mathcal{G}$, and so we may regard the height as a map

$$h : \mathcal{G} \to [0, \infty).$$

The height has the following well known properties (see [1, Section 1.5]):

(i) $h(\alpha) = 0$ if and only if $\alpha$ is the identity element in $\mathcal{G}$,

(ii) $h(\alpha^{-1}) = h(\alpha)$ for all $\alpha$ in $\mathcal{G}$,

(iii) $h(\alpha\beta) \leq h(\alpha) + h(\beta)$ for all $\alpha$ and $\beta$ in $\mathcal{G}$.

These conditions imply that the map $(\alpha, \beta) \mapsto h(\alpha\beta^{-1})$ defines a metric on the group $\mathcal{G}$ and therefore induces a metric topology. Our objective in this paper is to determine the completion of $\mathcal{G}$ with respect to this metric.

Let $r/s$ denote a rational number, where $r$ and $s$ are relatively prime integers and $s$ is positive. If $\alpha$ is in $\overline{\mathbb{Q}}^\times$ and $\zeta_1$ and $\zeta_2$ are in $\text{Tor}(\overline{\mathbb{Q}}^\times)$, then all roots of the two polynomial equations

$$x^s - (\zeta_1\alpha)^r = 0 \quad \text{and} \quad x^s - (\zeta_2\alpha)^r = 0$$

belong to the same coset in $\mathcal{G}$. If we write $\alpha^{r/s}$ for this coset, we find that

$$(r/s, \alpha) \mapsto \alpha^{r/s}$$

defines a scalar multiplication in the abelian group $\mathcal{G}$. This shows that $\mathcal{G}$ is a vector space (written multiplicatively) over the field $\mathbb{Q}$ of rational numbers. Moreover, we have (see [1, Lemma 1.5.18])

$$h(\alpha^{r/s}) = |r/s|h(\alpha).$$

Therefore the map $\alpha \mapsto h(\alpha)$ is a norm on the vector space $\mathcal{G}$ with respect to the usual archimedean absolute value $| \cdot |$ on its field $\mathbb{Q}$ of scalars. From these observations we conclude that the completion of $\mathcal{G}$ is a Banach space over the field $\mathbb{R}$ of real numbers. It remains now to give an explicit description of this Banach space.

Let $Y$ denote the set of all places $y$ of the field $\overline{\mathbb{Q}}$. Let $k \subseteq \overline{\mathbb{Q}}$ be an algebraic number field such that $k/\mathbb{Q}$ is a Galois extension. At each place $v$
of $k$ we write

$$Y(k, v) = \{ y \in Y : y \mid v \}$$

for the subset of places of $Y$ that lie over $v$. Clearly, we can express $Y$ as the disjoint union

$$Y = \bigcup_v Y(k, v),$$

where the union is over all places $v$ of $k$. If $y$ is a place in $Y(k, v)$ we select an absolute value $\| y \|$ from $y$ such that the restriction of $\| y \|$ to $k$ is equal to $\| v \|$. As the restriction of $\| v \|$ to $\mathbb{Q}$ is one of the usual absolute values on $\mathbb{Q}$, it follows that this choice of the normalized absolute value $\| y \|$ does not depend on $k$.

In Section 2 we show that each subset $Y(k, v)$ can be expressed as an inverse limit of finite sets. This determines a totally disconnected, compact, Hausdorff topology in $Y(k, v)$. Then (1.6) implies that $Y$ is a totally disconnected, locally compact, Hausdorff space. Again the topology in $Y$ does not depend on the field $k$. We also show that the absolute Galois group $\text{Aut}(\mathbb{Q}/k)$ acts transitively and continuously on the elements of each compact, open subset $Y(k, v)$.

In Section 4 we establish the existence of a regular measure $\lambda$, defined on the Borel subsets of $Y$, that is positive on open sets, finite on compact sets, and satisfies $\lambda(\tau E) = \lambda(E)$ for all automorphisms $\tau$ in $\text{Aut}(\mathbb{Q}/k)$ and all Borel subsets $E$ of $Y$. The restriction of the measure $\lambda$ to each subset $Y(k, v)$ is unique up to a positive multiplicative constant. We construct $\lambda$ so that

$$\lambda(Y(k, v)) = \frac{[k_v : Q_v]}{[k : Q]}$$

for each Galois extension $k$ of $\mathbb{Q}$ and each place $v$ of $k$. It follows from our construction that $\lambda$ does not depend on the number field $k$. In particular, if $l$ is any finite, Galois extension of $\mathbb{Q}$, if $w$ is place of $l$ and

$$Y(l, w) = \{ y \in Y : y \mid w \},$$

then

$$\lambda(Y(l, w)) = \frac{[l_w : Q_w]}{[l : Q]}.$$
so that $\mathcal{X}$ is a co-dimension one linear subspace of $L^1(Y, \mathcal{B}, \lambda)$. For each point $\alpha$ in $\mathcal{G}$ we define a map $f_\alpha : Y \to \mathbb{R}$ by
\begin{equation}
(1.9) \quad f_\alpha(y) = \log \|\alpha\|_y.
\end{equation}
If $k$ is a finite Galois extension of $\mathbb{Q}$ that contains $\alpha$, then $y \mapsto \log \|\alpha\|_y$ is constant on each compact, open set $Y(k, v)$, and the value of this map on each set $Y(k, v)$ is nonzero for only finitely many places $v$ of $k$. It follows that $f_\alpha(y)$ is a continuous function on $Y$ with compact support. Using (1.7) and the product formula (1.1), we find that
\begin{equation}
(1.10) \quad \int_Y f_\alpha(y) \, d\lambda(y) = \sum_v \int_{Y(k, v)} \log \|\alpha\|_y \, d\lambda(y)
= \sum_v \left[\frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]}\right] \log \|\alpha\|_v = \sum_v \log |\alpha|_v = 0.
\end{equation}
This shows that $\alpha \mapsto f_\alpha(y)$ maps $\mathcal{G}$ into the subspace $\mathcal{X}$. It follows easily that
\begin{equation*}
f_{\alpha \beta}(y) = f_\alpha(y) + f_\beta(y) \quad \text{and} \quad f_{\alpha^{r/s}}(y) = (r/s)f_\alpha(y),
\end{equation*}
and therefore $\alpha \mapsto f_\alpha(y)$ is a linear map from the vector space $\mathcal{G}$ into $\mathcal{X}$. The $L^1$-norm of each function $f_\alpha$ is given by
\begin{equation}
(1.11) \quad \int_Y |f_\alpha(y)| \, d\lambda(y) = \sum_v \int_{Y(k, v)} |\log \|\alpha\|_y| \, d\lambda_v(y)
= \sum_v \left[\frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]}\right] |\log \|\alpha\|_v| = \sum_v |\log |\alpha|_v| = 2h(\alpha).
\end{equation}
This shows that the map $\alpha \mapsto f_\alpha$ is a linear isometry from the vector space $\mathcal{G}$ with norm determined by $2h$ into the subspace $\mathcal{X}$ with the $L^1$-norm. Let
\begin{equation}
(1.12) \quad \mathcal{F} = \{f_\alpha(y) : \alpha \in \mathcal{G}\}
\end{equation}
denote the image of $\mathcal{G}$ under this linear map. Then $\alpha \mapsto f_\alpha$ is a linear isometry from the vector space $\mathcal{G}$ (written multiplicatively) onto the vector space $\mathcal{F}$ (written additively). Now the completion of $\mathcal{G}$ is determined by finding the closure of $\mathcal{F}$ in $\mathcal{X}$.

**Theorem 1.** Let $\mathcal{X}$ be the co-dimension one subspace of $L^1(Y, \mathcal{B}, \lambda)$ defined by (1.8). Then $\mathcal{F}$ is dense in $\mathcal{X}$.

It is immediate from Theorem 1 that there exists an isometric isomorphism from the completion of the vector space $\mathcal{G}$ with respect to the height $2h$ onto the real Banach space $\mathcal{X}$.

The functions in the vector space $\mathcal{F}$ belong to the real vector space $C_c(Y)$ of continuous functions with compact support. Hence $\mathcal{F}$ belongs to the space $L^p(Y, \mathcal{B}, \lambda)$ for $1 \leq p \leq \infty$. Theorem 1 asserts that the closure of
\( \mathcal{F} \) in \( L^1(Y, \mathcal{B}, \lambda) \) is the co-dimension one subspace \( \mathcal{X} \). We also determine the closure of \( \mathcal{F} \) with respect to the other \( L^p \)-norms.

**Theorem 2.** If \( 1 < p < \infty \) then \( \mathcal{F} \) is dense in \( L^p(Y, \mathcal{B}, \lambda) \).

Let \( C_0(Y) \) denote the Banach space of continuous real-valued functions on \( Y \) which vanish at infinity, equipped with the sup-norm. As \( \mathcal{F} \subseteq C_c(Y) \subseteq C_0(Y) \), it is clear that the closure of \( \mathcal{F} \) with respect to the sup-norm is a subspace of \( C_0(Y) \).

**Theorem 3.** The vector space \( \mathcal{F} \) is dense in \( C_0(Y) \).

It follows from the classification of separable \( L^p \)-spaces (see [3, pp. 14–15]) that the Banach space \( L^1(Y, \mathcal{B}, \lambda) \) has a Schauder basis, or simply a basis. As \( \mathcal{X} \subseteq L^1(Y, \mathcal{B}, \lambda) \) is a closed subspace of co-dimension one, it is easy to show that \( \mathcal{X} \) also has a basis. Then it follows from a well known result of Krein, Milman and Rutman [4] that a basis for \( \mathcal{X} \) can be selected from the dense subset \( \mathcal{F} \). Thus there exists a sequence of distinct elements \( \alpha_1, \alpha_2, \ldots \) in \( \mathcal{G} \) such that the corresponding collection of functions

\[
\{ f_{\alpha_1}(y), f_{\alpha_2}(y), \ldots \}
\]

is a basis for the Banach space \( \mathcal{X} \). That is, for every function \( F \) in \( \mathcal{X} \) there exists a unique sequence of real numbers \( x_1, x_2, \ldots \) such that

\[
F(y) = \lim_{N \to \infty} \sum_{n=1}^{N} x_n f_{\alpha_n}(y)
\]

in \( L^1 \)-norm. While these remarks establish the existence of such a basis, it would be of interest to construct an explicit example of a sequence \( \alpha_1, \alpha_2, \ldots \) in \( \mathcal{G} \) such that the corresponding sequence of functions (1.13) forms a basis for \( \mathcal{X} \).

**2. Preliminary lemmas.** We have stated Theorem 1 for the Weil height on algebraic number fields. However, many of the arguments can be given in the more general setting of a field \( K \) with a proper set of absolute values satisfying a product formula. We now describe this situation.

Let \( K \) be a field and let \( v \) be a place of \( K \). That is, \( v \) is an equivalence class of nontrivial absolute values on \( K \). We write \( K_v \) for the completion of \( K \) at the place \( v \). If \( L/K \) is a finite extension of fields then there exist finitely many places \( w \) of \( L \) such that \( w | v \). In general we have

\[
\sum_{w | v} [L_w : K_v] \leq [L : K],
\]

where \( L_w \) is the completion of \( L \) at \( w \). We say that \( v \) is well behaved if the
identity
\[ \sum_{w|v} [L_w : K_v] = [L : K] \]
holds for all finite extensions \( L/K \) (see [5, Chapter 1, Section 4]).

Let \( \mathcal{M}_K \) be a collection of distinct places of \( K \) and at each place \( v \) in \( \mathcal{M}_K \) let \( \| \cdot \|_v \) denote an absolute value from \( v \). We say that the collection of absolute values
\[(2.1) \quad \{ \| \cdot \|_v : v \in \mathcal{M}_K \} \]
is proper if it satisfies the following conditions:

(i) each place \( v \) in \( \mathcal{M}_K \) is well behaved,
(ii) if \( \alpha \) is in \( K^\times \) then \( \| \alpha \|_v \neq 1 \) for at most finitely many places \( v \) in \( \mathcal{M}_K \),
(iii) if \( \alpha \) is in \( K^\times \) then the absolute values in (2.1) satisfy the product formula
\[ \prod_{v \in \mathcal{M}_K} \| \alpha \|_v = 1. \]

Now suppose that (2.1) is a proper set of absolute values on \( K \) and \( L/K \) is a finite extension of fields. Let \( \mathcal{M}_L \) be the collection of places of \( L \) that extend the places in \( \mathcal{M}_K \). That is, if \( W_v(L/K) \) is the finite set of places \( w \) of \( L \) such that \( w \mid v \), then
\[ \mathcal{M}_L = \bigcup_{v \in \mathcal{M}_K} W_v(L/K). \]

At each place \( w \) in \( W_v(L/K) \) we select an absolute value \( \| \cdot \|_w \) that extends the absolute value \( \| \cdot \|_v \) on \( K \). Then we define an equivalent absolute value \( | \cdot |_w \) from the place \( w \) by setting
\[ \log |\alpha|_w = \frac{[L_w : K_v]}{[L : K]} \log \|\alpha\|_w \]
for all \( \alpha \) in \( L^\times \). In general, \( \| \cdot \|_w \) and \( | \cdot |_w \) are distinct but equivalent absolute values on \( L \). And we note that \( | \cdot |_w \) is an absolute value because
\[ 0 < \frac{[L_w : K_v]}{[L : K]} \leq 1. \]

Then it follows, as in [5, Chapter 2, Section 1], that
\[(2.2) \quad \{ | \cdot |_w : w \in \mathcal{M}_L \} \]
is a proper set of absolute values on \( L \). In particular, if \( \alpha \) is in \( L^\times \) then the absolute values in (2.2) satisfy the product formula
\[ \prod_{w \in \mathcal{M}_L} |\alpha|_w = 1. \]
We assume that $K \subseteq N$ are fields, that $N/K$ is a (possibly infinite) Galois extension, and we write $\text{Aut}(N/K)$ for the corresponding Galois group. We give $\text{Aut}(N/K)$ the Krull topology, and we briefly recall how this is defined. Let $\mathcal{L}$ denote the set of intermediate fields $L$ such that $K \subseteq L \subseteq N$ and $L/K$ is a finite Galois extension. Obviously, $\mathcal{L}$ is partially ordered by set inclusion. If $L$ and $M$ are in $\mathcal{L}$ then the composite field $LM$ is in $\mathcal{L}$, $L \subseteq LM$, $M \subseteq LM$, and therefore $\mathcal{L}$ is a directed set. For each $L$ in $\mathcal{L}$ let $\text{Aut}(L/K)$ denote the Galois group of automorphisms of $L$ that fix $K$. If $L \subseteq M$ are both in $\mathcal{L}$, we define $\pi_L^M : \text{Aut}(M/K) \to \text{Aut}(L/K)$ to be the map that restricts the domain of an automorphism in $\text{Aut}(M/K)$ to the subfield $L$. Then each map $\pi_L^M$ is a surjective homomorphism of groups and $\pi_L^L$ is the identity map. It follows that

\[ \{ \text{Aut}(L/K), \pi_L^M \} \]

is an inverse system, and $\text{Aut}(N/K)$ can be identified with the inverse (or projective) limit:

\[ \text{Aut}(N/K) = \lim_{\leftarrow L \in \mathcal{L}} \text{Aut}(L/K). \]

Thus $\text{Aut}(N/K)$ is a profinite group, and therefore is a totally disconnected, compact, Hausdorff, topological group. We write

\[ \pi_L : \text{Aut}(N/K) \to \text{Aut}(L/K) \]

for the canonical map associated with each $L$ in $\mathcal{L}$. Then $\pi_L$ is continuous and the collection of open sets

\[ \{ \pi_L^{-1}(\tau) : L \in \mathcal{L} \text{ and } \tau \in \text{Aut}(L/K) \} \]

is a basis for the Krull topology in $\text{Aut}(N/K)$.

Next we assume that $v$ is a place of the field $K$. That is, $v$ is an equivalence class of nontrivial absolute values on $K$. If $L$ is in $\mathcal{L}$ we write $W_v(L/K)$ for the set of places $w$ of $L$ such that $w \mid v$. As $L/K$ is a finite extension, it follows that $W_v(L/K)$ is a finite set. If $L \subseteq M$ belong to $\mathcal{L}$ we define connecting maps

\[ \psi_L^M : W_v(M/K) \to W_v(L/K) \]

as follows: if $w_M$ belongs to $W_v(M/K)$ then $\psi_L^M(w_M)$ is the unique place $w_L$ in $W_v(L/K)$ such that $w_M \mid w_L$. If $L \subseteq M$ are in $\mathcal{L}$ then each absolute value on $L$ extends to $M$ and therefore each connecting map $\psi_L^M$ is surjective. We give each finite set $W_v(L/K)$ the discrete topology so that each map $\psi_L^M$ is continuous. Clearly, $\psi_L^L$ is the identity map. We find that

\[ \{ W_v(L/K), \psi_L^M \} \]

is an inverse system of finite sets. Let

\[ Y(K, v) = \lim_{\leftarrow L \in \mathcal{L}} W_v(L/K) \]
denote the inverse limit and write $\psi_L : Y(K, v) \to W_v(L/K)$ for the canonical continuous map associated to each $L$ in $\mathcal{L}$. It follows, as in [2, Appendix 2, Section 2.4], that $Y(K, v)$ is a nonempty, totally disconnected, compact, Hausdorff space. Moreover (see [2, Appendix 2, Section 2.3]), the collection of open sets

\begin{equation}
\{ \psi^{-1}_L(w) : L \in \mathcal{L} \text{ and } w \in W_v(L/K) \}
\end{equation}

is a basis for the topology of $Y(K, v)$. Clearly, each subset in the collection (2.4) is also compact, and for each field $L$ in $\mathcal{L}$ we can write

$$
Y(K, v) = \bigcup_{w \in W_v(L/K)} \psi^{-1}_L(w)
$$

as a disjoint union of open and compact sets.

We recall that a map $g : Y(K, v) \to \mathbb{R}$ is \textit{locally constant} if at each point $y$ in $Y(K, v)$ there exists an open neighborhood of $y$ on which $g$ is constant.

\textbf{Lemma 1.} Let $g : Y(K, v) \to \mathbb{R}$ be locally constant. Then there exists $L$ in $\mathcal{L}$ such that for each place $w$ in $W_v(L/K)$ the function $g$ is constant on the set $\psi^{-1}_L(w)$.

\textit{Proof.} At each point $y$ in $Y(K, v)$ there exists a field $L^{(y)}$ in $\mathcal{L}$ and a place $w^{(y)}$ in $W_v(L^{(y)}/K)$ such that $y$ is contained in $\psi^{-1}_{L^{(y)}}(w^{(y)})$ and $g$ is constant on the open set $\psi^{-1}_{L^{(y)}}(w^{(y)})$. By compactness there exists a finite collection of fields $L^{(1)}, \ldots, L^{(J)}$ in $\mathcal{L}$, and for each integer $j$ a corresponding place $w^{(j)}$ in $W_v(L^{(j)}/K)$, such that

$$
Y(K, v) \subseteq \bigcup_{j=1}^J \psi^{-1}_{L^{(j)}}(w^{(j)}),
$$

and $g$ is constant on each open set $\psi^{-1}_{L^{(j)}}(w^{(j)})$. Let $L = L^{(1)} \cdots L^{(J)}$ be the composite field, which is obviously in $\mathcal{L}$. If $w$ is a place of $L$ then there exists an integer $j$ such that

$$
\psi^{-1}_L(w) \cap \psi^{-1}_{L^{(j)}}(w^{(j)})
$$

is not empty. As $L$ is a finite extension of $L^{(j)}$, we conclude that $\psi^{-1}_{L^{(j)}}(w) = w^{(j)}$, and therefore

\begin{equation}
\psi^{-1}_L(w) \subseteq \psi^{-1}_{L^{(j)}}(w^{(j)}).
\end{equation}

Then (2.5) implies that $g$ is constant on $\psi^{-1}_L(w)$.

Let $C(Y(K, v))$ denote the real Banach algebra of real-valued continuous functions on $Y(K, v)$ with the supremum norm. Let $LC(Y(K, v)) \subseteq C(Y(K, v))$ denote the subset of locally constant functions.
**Lemma 2.** The subset $LC(Y(K,v))$ is a dense subalgebra of $C(Y(K,v))$.

**Proof.** It is obvious that $LC(Y(K,v))$ is a subalgebra of $C(Y(K,v))$, and that $LC(Y(K,v))$ contains the constant functions. Now suppose that $y_1$ and $y_2$ are distinct points in $Y(K,v)$. Let $U_1$ be an open neighborhood of $y_1$, and $U_2$ an open neighborhood of $y_2$, such that $U_1$ and $U_2$ are disjoint. Then there exists a field $L$ in $\mathcal{L}$ and a place $w$ in $W_v(L/K)$ such that

$$y_1 \in \psi^{-1}_L(w) \quad \text{and} \quad \psi^{-1}_L(w) \subseteq U_1.$$ 

As $\psi^{-1}_L(w)$ is both open and compact, the characteristic function of the set $\psi^{-1}_L(w)$ is a locally constant function that separates the points $y_1$ and $y_2$. Then it follows from the Stone–Weierstrass theorem that the subalgebra $LC(Y(K,v))$ is dense in $C(Y(K,v))$. ■

We select an absolute value from the place $v$ of $K$ and denote it by $\| \|_v$. If $L$ is in $\mathcal{L}$ and $w$ is a place in $W_v(L/K)$, we select an absolute value $\| \|_w$ from $w$ such that the restriction of $\| \|_w$ to $K$ is equal to $\| \|_v$. As

$$N = \bigcup_{L \in \mathcal{L}} L,$$

it follows that each point $(w_L)$ in $Y(K,v)$ determines a unique absolute value on the field $N$. That is, each point $(w_L)$ in $Y(K,v)$ determines a unique place $y$ of $N$ such that $y \mid v$.

Now suppose $y$ is a place of $N$ such that $y \mid v$. Select an absolute value $\| \|_y$ from $y$ such that the restriction of $\| \|_y$ to the subfield $K$ is equal to $\| \|_v$. If $L$ is in $\mathcal{L}$ then the restriction of $\| \|_y$ to $L$ must equal $\| \|_w$ for a unique place $w_L$ in $W_v(L/K)$. Thus each place $y$ of $N$ with $y \mid v$ determines a unique point $(w_L)$ in the product

$$\prod_{L \in \mathcal{L}} W_v(L/K)$$

such that $y \mid w_L$ for each $L$. It is trivial to check that

$$\psi^M_L(w_M) = w_L$$

whenever $L \subseteq M$ are in $\mathcal{L}$. Therefore each place $y$ of $N$ with $y \mid v$ determines a unique point $(w_L)$ in the inverse limit $Y(K,v)$. In view of these remarks we may identify $Y(K,v)$ with the set of all places $y$ of $N$ that lie over the place $v$ of $K$. In this way we determine a totally disconnected, compact, Hausdorff topology in the set of all places $y$ of $N$ that lie over the place $v$ of $K$.

**3. Galois action on places.** Next we recall that the Galois group $\text{Aut}(N/K)$ acts on the set $Y(K,v)$ of all places of $N$ that lie over $v$. More
precisely, if $\tau$ is in $\text{Aut}(N/K)$ and $y$ is in $Y(K,v)$, then the map
\begin{equation}
\alpha \mapsto \|\tau^{-1}\alpha\|_y
\end{equation}
is an absolute value on $N$, and the restriction of this absolute value to $K$ is clearly equal to $\|\cdot\|_v$. Therefore (3.1) determines a unique place $\tau y$ in $Y(K,v)$. That is, the identity
\begin{equation}
\|\tau^{-1}\alpha\|_y = \|\alpha\|_{\tau y}
\end{equation}
holds for all $\alpha$ in $N$, for all $\tau$ in $\text{Aut}(N/K)$, and for all places $y$ in $Y(K,v)$. It is immediate that $1y = y$ and $(\sigma \tau)y = \sigma(\tau y)$ for all $\sigma$ and $\tau$ in $\text{Aut}(N/K)$. Thus $(\tau, y) \mapsto \tau y$ defines an action of the group $\text{Aut}(N/K)$ on the set $Y(K,v)$. Moreover, $\text{Aut}(N/K)$ acts transitively on $Y(K,v)$ (see [7, Chapter II, Proposition 9.1]).

**Lemma 3.** The function $(\tau, y) \mapsto \tau y$ from $\text{Aut}(N/K) \times Y(K,v)$ onto $Y(K,v)$ is continuous.

**Proof.** Let $L$ be in $\mathcal{L}$ and $w$ in $W_v(L/K)$. In view of (2.4) we must show that
\[
\{ (\tau, y) \in \text{Aut}(N/K) \times Y(K,v) : \tau y \in \psi^{-1}_L(w) \}
\]
is open in $\text{Aut}(N/K) \times Y(K,v)$ with the product topology. For $w$ in $W_v(L/K)$ we define
\[
E_w = \{ (\sigma, z) \in \text{Aut}(L/K) \times W_v(L/K) : \sigma z = w \}.
\]
Then we have
\[
\{ (\tau, y) \in \text{Aut}(N/K) \times Y(K,v) : \tau y \in \psi^{-1}_L(w) \}
\]
\[
= \{ (\tau, y) \in \text{Aut}(N/K) \times Y(K,v) : \pi_L(\tau)\psi_L(y) = w \}
\]
\[
= \bigcup_{(\sigma, z) \in E_w} \{ (\tau, y) \in \text{Aut}(K/k) \times Y(K,v) : \pi_L(\tau) = \sigma \text{ and } \psi_L(y) = z \}
\]
\[
= \bigcup_{(\sigma, z) \in E_w} \pi^{-1}_L(\sigma) \times \psi^{-1}_L(z),
\]
which is obviously an open subset of $\text{Aut}(N/K) \times Y(K,v)$. $\blacksquare$

**4. The invariant measure.** In this section it will be convenient to write $G = \text{Aut}(N/K)$. Let $\mu$ denote a Haar measure on the Borel subsets of the compact topological group $G$ normalized so that $\mu(G) = 1$. If $F$ is in $C(Y(K,v))$ and $z_1$ is a point in $Y(K,v)$ then it follows from Lemma 3 that $\tau \mapsto F(\tau z_1)$ is a continuous function on $G$ with values in $\mathbb{R}$. Let $z_2$ be a second point in $Y(K,v)$. Because $G$ acts transitively on $Y(K,v)$, there exists $\eta$ in $G$ so that $\eta z_2 = z_1$. Then using the translation invariance of Haar
Weil height

measure we get

\[(4.1) \quad \int_G F(\tau z_1) \, d\mu(\tau) = \int_G F(\tau \eta z_2) \, d\mu(\tau) = \int_G F(\tau z_2) \, d\mu(\tau).\]

It follows that the map \(I_v : C(Y(K, v)) \to \mathbb{R}\) given by

\[(4.2) \quad I_v(F) = \int_G F(\tau z_v) \, d\mu(\tau)\]

does not depend on the point \(z_v\) in \(Y(K, v)\).

Let \(\mathcal{M}_K\) be a collection of distinct places of \(K\) and at each place \(v\) in \(\mathcal{M}_K\) let \(\|\|_v\) denote an absolute value from \(v\). We assume that

\[\{\|\|_v : v \in \mathcal{M}_K\}\]

is a proper collection of absolute values. Again we assume that \(N/K\) is a (possibly infinite) Galois extension of fields. Let \(Y\) be defined by the disjoint union

\[(4.3) \quad Y = \bigcup_{v \in \mathcal{M}_K} Y(K, v).\]

Thus \(Y\) is the collection of all places \(y\) of \(N\) such that \(y \mid v\) for some place \(v\) in \(\mathcal{M}_K\). It follows that \(Y\) is a nonempty, totally disconnected, locally compact, Hausdorff space.

Let \(C_c(Y)\) denote the real vector space of continuous functions \(F : Y \to \mathbb{R}\) having compact support. If \(F\) belongs to \(C_c(Y)\) then there exists a finite subset \(S_F \subseteq \mathcal{M}_K\) such that \(F\) is supported on the compact set

\[\bigcup_{v \in S_F} Y(K, v).\]

In particular, we have \(I_v(F) = 0\) for almost all places \(v\) of \(\mathcal{M}_K\). Therefore we define \(I : C_c(Y) \to \mathbb{R}\) by

\[(4.4) \quad I(F) = \sum_{v \in \mathcal{M}_K} \int_G F(\tau z_v) \, d\mu(\tau),\]

where \(z_v\) is a point in \(Y(K, v)\) for each place \(v\) in \(\mathcal{M}_K\). By our previous remarks the value of each integral on the right of (4.4) does not depend on \(z_v\), and only finitely many of those integrals are nonzero. Hence there is no question of convergence in the sum on the right of (4.4).

Theorem 4. There exists a \(\sigma\)-algebra \(\mathcal{Y}\) of subsets of \(Y\), that contains the \(\sigma\)-algebra \(\mathcal{B}\) of Borel sets in \(Y\), and a unique, regular measure \(\lambda\) defined on \(\mathcal{Y}\), such that

\[(4.5) \quad I(F) = \int_Y F(y) \, d\lambda(y)\]
for all $F$ in $C_c(Y)$. Moreover, the measure $\lambda$ satisfies the following conditions:

(i) If $\eta$ is in $G$ and $F$ is in $L^1(Y, \mathcal{Y}, \lambda)$ then

\begin{equation}
\int_{Y(K,v)} F(\eta y) \, d\lambda(y) = \int_{Y(K,v)} F(y) \, d\lambda(y)
\end{equation}

at each place $v$ in $\mathcal{M}_K$.

(ii) If $E$ is in $\mathcal{Y}$ then

\[ \lambda(E) = \inf \{ \lambda(U) : E \subseteq U \subseteq Y \text{ and } U \text{ is open} \}. \]

(iii) If $E$ is in $\mathcal{Y}$ then

\[ \lambda(E) = \sup \{ \lambda(V) : V \subseteq E \text{ and } V \text{ is compact} \}. \]

(iv) If $E$ is in $\mathcal{Y}$ and $\lambda(E) = 0$ then every subset of $E$ is in $\mathcal{Y}$.

Proof. Clearly, (4.4) defines a positive linear functional on $C_c(Y)$. By the Riesz representation theorem (see [8, Theorems 2.14 and 2.17]), there exists a $\sigma$-algebra $\mathcal{Y}$ of subsets of $Y$, containing the $\sigma$-algebra $\mathcal{B}$ of Borel sets in $Y$, and a regular measure $\lambda$ defined on $\mathcal{Y}$, such that

\begin{equation}
I(F) = \int_Y F(y) \, d\lambda(y)
\end{equation}

for all $F$ in $C_c(Y)$. If $\eta$ is in $G$ and $F$ is in $C_c(Y)$, then by the translation invariance of the Haar measure $\mu$ we have

\begin{equation}
\int_{Y(K,v)} F(\eta y) \, d\lambda(y) = \int_G F(\eta \tau z) \, d\mu(\tau) = \int_G F(\tau z) \, d\mu(\tau)
\end{equation}

at each place $v$ in $\mathcal{M}_K$. Initially (4.8) holds for all functions $F$ in $C_c(Y)$. As $C_c(Y)$ is dense in $L^1(Y, \mathcal{Y}, \lambda)$ (see [8, Theorem 3.14]), it follows in a standard manner that (4.8) also holds for functions $F$ in $L^1(Y, \mathcal{Y}, \lambda)$.

The properties (ii), (iii) and (iv) attributed to $\lambda$ all are consequences of the Riesz theorem. 

Because the Haar measure $\mu$ satisfies $\mu(G) = 1$, it is immediate from (4.2) and (4.5) that $\lambda(Y(K,v)) = 1$ at each place $v$ in $\mathcal{M}_K$. As the places in $\mathcal{M}_K$ are well behaved, we obtain a further identity for the $\lambda$-measure of basic open sets in each subset $Y(K,v)$.

**Theorem 5.** If $L$ is in $\mathcal{L}$ and $w$ is a place in $W_v(L/K)$, then

\begin{equation}
\lambda(\psi_L^{-1}(w)) = \frac{[L_w : K_v]}{[L : K]}
\end{equation}
Proof. Let $\tau$ be in $G$. Then
\begin{equation}
\tau\psi^{-1}_L(w) = \{\tau y \in Y(K, v) : \psi_L(y) = w\} \\
= \{y \in Y(K, v) : \pi_L(\tau^{-1})\psi_L(y) = w\} \\
= \{y \in Y(K, v) : \psi_L(y) = \pi_L(\tau)w = \psi^{-1}_L(\pi_L(\tau)w)\}.
\end{equation}
Now let $w_1$ and $w_2$ be distinct places in $W_v(L/K)$. Select $\tau$ in $G$ so that $\pi_L(\tau)w_2 = w_1$. Then (4.10) implies that
\[\tau\psi^{-1}_L(w_2) = \psi^{-1}_L(w_1),\]
and using (4.6) we find that
\[\lambda(\psi^{-1}_L(w_2)) = \lambda(\psi^{-1}_L(w_1)).\]
Because
\begin{equation}
Y(K, v) = \bigcup_{w \in W_v(L/K)} \psi^{-1}_L(w)
\end{equation}
is a disjoint union of $|W_v(L/K)|$ distinct sets, the sets on the right of (4.11) all have equal $\lambda$-measure, and $\lambda(Y(K, v)) = 1$, we conclude that
\begin{equation}
\lambda(\psi^{-1}_L(w)) = |W_v(L/K)|^{-1}.
\end{equation}
As $v$ is well behaved we have
\begin{equation}
[L : K] = \sum_{v \in W_v(L/K)} [L_w : K_v].
\end{equation}
Because $L/K$ is a Galois extension, all local degrees $[L_w : K_v]$ for $w$ in $W_v(L/K)$ are equal, and we conclude from (4.13) that
\begin{equation}
|W_v(L/K)| = \frac{[L : K]}{[L_w : K_v]}.
\end{equation}
The identity (4.9) now follows from (4.12) and (4.14). □

Let $LC_c(Y)$ be the algebra of locally constant, real-valued functions on $Y$ having compact support. Clearly, $LC_c(Y) \subseteq C_c(Y)$.

Lemma 4. Let $g$ belong to $LC_c(Y)$. Then there exists $L$ in $\mathcal{L}$ such that for each place $w$ in $\mathcal{M}_L$ the function $g$ is constant on the set $\psi^{-1}_L(w)$.

Proof. Let $S_g \subseteq \mathcal{M}_K$ be a finite set of places of $K$ such that the support of $g$ is contained in the compact set
\[V_g = \bigcup_{v \in S_g} Y(K, v).\]
For each place $v$ in $S_g$ we apply Lemma 1 to the restriction of $g$ to $Y(K, v)$. Thus there exists a field $L^{(v)}$ in $\mathcal{L}$ such that for each place $w'$ in $W_v(L^{(v)}/K)$,
the function \( g \) is constant on \( \psi_{L(v)}^{-1}(w') \). Let \( L \) be the compositum of the finite collection of fields 
\[
\{ L(v) : v \in S_g \}.
\]
Clearly, \( L \) belongs to \( \mathcal{L} \).

Let \( w \) be a place in \( \mathcal{M}_L \). If \( w \mid v \) and \( v \notin S_g \), then \( g \) is identically zero on \( \psi_L^{-1}(w) \), and in particular it is constant on this set. If \( w \mid v \) and \( v \in S_g \), then \( w \mid w' \) for a unique place \( w' \) in \( W_v(L(v)/K) \). Because 
\[
\psi_{L(v)}^{-1}(w) \subseteq \psi_L^{-1}(w'),
\]
and \( g \) is constant on \( \psi_{L(v)}^{-1}(w') \), it is obvious that \( g \) is constant on \( \psi_L^{-1}(w) \).

**Lemma 5.** For \( 1 \leq p < \infty \) the set \( LC_c(Y) \) is dense in \( L^p(Y, \mathcal{B}, \lambda) \). Moreover, \( LC_c(Y) \) is dense in \( C_0(Y) \) with respect to the sup-norm.

**Proof.** Let \( 1 \leq p < \infty \). Because \( C_c(Y) \) is dense in \( L^p(Y, \mathcal{B}, \lambda) \), it suffices to show that if \( F \) is in \( C_c(Y) \) and \( \varepsilon > 0 \), then there exists a function \( g \) in \( LC_c(Y) \) such that 
\[
\left\{ \int_Y |F(y) - g(y)|^p \, d\lambda(y) \right\}^{1/p} < \varepsilon.
\]
Let \( S_F \subseteq \mathcal{M}_K \) be a nonempty, finite set of places such that \( F \) is supported on the compact set 
\[
V_F = \bigcup_{v \in S_F} Y(K, v).
\]
For each \( v \) in \( S_F \) we apply Lemma 2 to the restriction of \( F \) to \( Y(K, v) \). Thus there exists a locally constant function \( g_v : Y(K, v) \to \mathbb{R} \) such that 
\[
\sup_{y \in Y(K, v)} \{|F(y) - g_v(y)| : y \in Y(K, v)\} < |S_F|^{-1/p} \varepsilon.
\]
Now define \( g : Y \to \mathbb{R} \) by 
\[
g(y) = \begin{cases} 
g_v(y) & \text{if } y \in Y(K, v) \text{ and } v \in S_F, \\
0 & \text{if } y \in Y(K, v) \text{ and } v \notin S_F. \end{cases}
\]
Then \( g \) is locally constant and supported on the compact set \( V_F \). Therefore \( g \) belongs to \( LC_c(Y) \). As \( \lambda(Y(K, v)) = 1 \) at each place \( v \) in \( \mathcal{M}_K \), we get 
\[
\left\{ \int_Y |F(y) - g(y)|^p \, d\lambda(y) \right\}^{1/p} < \left\{ \sum_{v \in S_F} |S_F|^{-1/p} \varepsilon^p \right\}^{1/p} \leq \varepsilon.
\]
This proves the first assertion of the lemma.

As \( C_c(Y) \) is dense in \( C_0(Y) \) with respect to the sup-norm, the second assertion of the lemma follows by the same argument. In this case we select...
the locally constant functions \( g_v : Y(K, v) \to \mathbb{R} \) so that
\[
\sup \{|F(y) - g_v(y)| : y \in Y(K, v)\} < \varepsilon.
\]
Then we define \( g : Y \to \mathbb{R} \) as in (4.16). Again we find that \( g \) belongs to \( LC_c(Y) \), and the inequality
\[
\sup \{|F(y) - g(y)| : y \in Y\} < \varepsilon
\]
is obvious. ■

5. The completion of \( \mathcal{G} \). In this section we return to the situation considered in the introduction. We let \( K = \mathbb{Q}, N = \overline{\mathbb{Q}}, \) and we let \( \mathcal{M}_Q \) be the set of all places of \( \mathbb{Q} \). Then \( Y \) is the set of all places of \( \overline{\mathbb{Q}} \), and \( Y \) is a nonempty, totally disconnected, locally compact, Hausdorff space. By Theorem 4 there exists a \( \sigma \)-algebra \( \mathcal{Y} \) of subsets of \( Y \), containing the \( \sigma \)-algebra \( \mathcal{B} \) of Borel sets in \( Y \), and a measure \( \lambda \) on \( \mathcal{Y} \), satisfying the conclusions of that result. The basic identity (1.7) is verified by Theorem 5. Then the map
\[
(5.1) \quad \alpha \mapsto f_\alpha(y)
\]
defined by (1.9) is a linear map from the \( \mathbb{Q} \)-vector space
\[
\mathcal{G} = \overline{\mathbb{Q}}^\times / \text{Tor}(\overline{\mathbb{Q}}^\times)
\]
(written multiplicatively) into the vector space \( C_c(Y) \). The identity (1.10) implies that each function \( f_\alpha(y) \) belongs to the closed subspace \( \mathcal{X} \subseteq L^1(Y, \mathcal{B}, \lambda) \) defined by (1.8). It follows from basic properties of the height, and in particular (1.4), that
\[
\alpha \mapsto 2h(\alpha)
\]
defines a norm on \( \mathcal{G} \) with respect to the usual archimedean absolute value on \( \mathbb{Q} \). Then (1.11) shows that (5.1) defines a linear isometry of \( \mathcal{G} \) into the subspace \( \mathcal{X} \).

**Lemma 6.** Let \( k \) be an algebraic number field and let \( v \mapsto t_v \) be a real-valued function defined on the set of all places \( v \) of \( k \). If
\[
(5.2) \quad \sum_v t_v \log |\alpha|_v = 0
\]
for all \( \alpha \) in \( k^\times / \text{Tor}(k^\times) \), then the function \( v \mapsto t_v \) is constant.

**Proof.** Let \( S \) be a finite set of places of \( k \) containing all archimedean places, and assume that the cardinality of \( S \) is \( s \geq 2 \). We write \( \mathbb{R}^s \) for the \( s \)-dimensional real vector space of column vectors \( \mathbf{x} = (x_v) \) having rows indexed by places \( v \) in \( S \). In particular, we write \( \mathbf{t} = (t_v) \) for the column vector in \( \mathbb{R}^s \) formed from the values of the function \( v \mapsto t_v \) restricted to \( S \).
And we write \( u = (u_v) \) for the column vector in \( \mathbb{R}^s \) such that \( u_v = 1 \) for each \( v \) in \( S \).

Let 
\[
U_S(k) = \{ \eta \in k : |\eta|_v = 1 \text{ for all } v \notin S \}
\]
denote the multiplicative group of \( S \)-units in \( k \). By the \( S \)-unit theorem (stated as [6, Theorem 3.5]), there exist multiplicatively independent elements \( \xi_1, \ldots, \xi_{s-1} \) in \( U_S(k) \) which form a fundamental system of \( S \)-units. Write
\[
M = ([k_v : \mathbb{Q}_v] \log \| \xi_r \|_v)
\]
for the associated \((s - 1) \times s\) real matrix, where \( r = 1, \ldots, s - 1 \) indexes rows and \( v \) in \( S \) indexes columns. As the \( S \)-regulator does not vanish, the matrix \( M \) has rank \( s - 1 \). Hence the null space
\[
N = \{ x \in \mathbb{R}^s : Mx = 0 \}
\]
has dimension 1. From the product formula we have \( Mu = 0 \). Therefore \( N \) is spanned by the vector \( u \). By hypothesis we have \( Mt = 0 \), and it follows that \( t \) is a scalar multiple of \( u \). That is, the function \( v \mapsto t_v \) is constant on \( S \). As \( S \) is arbitrary the lemma is proved.

We now prove Theorem 1. Let \( \mathcal{E}_1 \) denote the closure of \( \mathcal{F} \) in \( \mathcal{X} \). As \( \mathcal{F} \) is a vector space over the field \( \mathbb{Q} \), it follows that \( \mathcal{E}_1 \) is a vector space over \( \mathbb{R} \), and therefore \( \mathcal{E}_1 \) is a closed linear subspace of \( \mathcal{X} \). If \( \mathcal{E}_1 \) is a proper subspace then it follows from the Hahn–Banach theorem (see [9, Theorem 3.5]) that there exists a continuous linear functional \( \Phi : \mathcal{X} \to \mathbb{R} \) such that \( \Phi \) vanishes on \( \mathcal{E}_1 \), but \( \Phi \) is not the zero linear functional on \( \mathcal{X} \). We will show that such a \( \Phi \) does not exist, and therefore we must have \( \mathcal{E}_1 = \mathcal{X} \).

Let \( \Phi : \mathcal{X} \to \mathbb{R} \) be a continuous linear functional that vanishes on \( \mathcal{E}_1 \), but \( \Phi \) is not the zero linear functional on \( \mathcal{X} \). It follows from (1.8) that \( \mathcal{X}^\perp \subseteq L^\infty(Y, \mathcal{B}, \lambda) \) is the one-dimensional subspace spanned by the constant function 1. As the dual space \( \mathcal{X}^* \) can be identified with the quotient space \( L^\infty(Y, \mathcal{B}, \lambda)/\mathcal{X}^\perp \), there exists a function \( \varphi(y) \) in \( L^\infty(Y, \mathcal{B}, \lambda) \) such that \( \varphi(y) \) and the constant function 1 are linearly independent, and
\[
\Phi(F) = \int_Y F(y)\varphi(y) \, d\lambda(y)
\]
for all \( F \) in \( \mathcal{X} \). Because \( \Phi \) vanishes on \( \mathcal{E}_1 \) we have
\[
\int_Y f_\alpha(y)\varphi(y) \, d\lambda(y) = 0
\]
for each function \( f_\alpha \) in \( \mathcal{F} \).

Now let \( k \) be a number field in \( \mathcal{L} \) and let \( \alpha \) be in \( k^\times / \text{Tor}(k^\times) \subseteq \mathcal{G} \). From (4.9) and (5.3) we find that
\(0 = \sum_v \left\{ \int_{\psi_{k^{-1}(v)}} \log \|\alpha\|_y \varphi(y) \lambda(y) \right\}
\)
\(= \sum_v \left\{ \int_{\psi_{k^{-1}(v)}} \varphi(y) \lambda(y) \right\} \log \|\alpha\|_v \)
\(= \sum_v \left\{ \lambda(\psi_{k^{-1}(v)})^{-1} \int_{\psi_{k^{-1}(v)}} \varphi(y) \lambda(y) \right\} \log |\alpha|_v. \)

It follows from Lemma 6 that the function
\(v \mapsto \lambda(\psi_{k^{-1}(v)})^{-1} \int_{\psi_{k^{-1}(v)}} \varphi(y) \lambda(y)\)
is constant on the set of places \(v\) of \(k\). We write \(c(k)\) for this constant.

Let \(k \subseteq l\) be number fields in \(L\), and let \(v\) be a place of \(k\). Using (4.9) and (4.14) we have
\(\lambda(\psi_{k^{-1}(v)}) = |W_v(l/k)| \lambda(\psi_l^{-1}(w))\)
for all places \(w\) in the set \(W_v(l/k)\). This leads to the identity
\(c(l) = |W_v(l/k)|^{-1} \sum_{w \in W_v(l/k)} \left\{ \lambda(\psi_l^{-1}(w))^{-1} \int_{\psi_l^{-1}(w)} \varphi(y) \lambda(y) \right\} \)
\(= \lambda(\psi_{k^{-1}(v)})^{-1} \sum_{w \in W_v(l/k)} \left\{ \int_{\psi_l^{-1}(w)} \varphi(y) \lambda(y) \right\} \)
\(= \lambda(\psi_{k^{-1}(v)})^{-1} \int_{\psi_{k^{-1}(v)}} \varphi(y) \lambda(y) = c(k). \)

Thus there exists a real number \(C\) such that \(C = c(k)\) for all fields \(k\) in \(L\).

Let \(g\) belong to \(LC_c(Y)\). By Lemma 4 there exists a number field \(l\) in \(L\) such that \(g\) is constant on \(\psi_l^{-1}(w)\) for each place \(w\) of \(l\). Therefore
\(\int_Y g(y) \varphi(y) \lambda(y) = \sum_w \left\{ \int_{\psi_l^{-1}(w)} g(y) \varphi(y) \lambda(y) \right\} \)
\(= C \sum_w \left\{ \lambda(\psi_l^{-1}(w)) g(\psi_l^{-1}(w)) \right\} \)
\(= C \sum_w \left\{ \int_{\psi_l^{-1}(w)} g(y) \lambda(y) \right\} = C \int_Y g(y) \lambda(y). \)

By Lemma 5 the set \(LC_c(Y)\) is dense in \(L^1(Y, B, \lambda)\), and we conclude from (5.6) that
\(\int_Y F(y) \varphi(y) \lambda(y) = C \int_Y F(y) \lambda(y). \)
for all $F$ in $L^1(Y, \mathcal{B}, \lambda)$. This shows that $\varphi(y) = C$ in $L^\infty(Y, \mathcal{B}, \lambda)$, and so contradicts our assumption that $\varphi(y)$ and the constant function 1 are linearly independent. Hence the continuous linear functional $\Phi$ does not exist, and therefore $\mathcal{E}_1 = \mathcal{X}$. This proves Theorem 1.

6. Proof of Theorems 2 and 3. We suppose that $1 < p < \infty$ and write $\mathcal{E}_p$ for the closure of $\mathcal{F}$ in $L^p(Y, \mathcal{B}, \lambda)$. As before, $\mathcal{E}_p$ is a closed linear subspace. By the Hahn–Banach theorem it suffices to show that if $\Phi : L^p(Y, \mathcal{B}, \lambda) \to \mathbb{R}$ is a continuous linear functional that vanishes on $\mathcal{E}_p$, then in fact $\Phi$ is identically zero on $L^p(Y, \mathcal{B}, \lambda)$.

Let $p^{-1} + q^{-1} = 1$, and let $\varphi(y)$ be an element of $L^q(Y, \mathcal{B}, \lambda)$ such that $\Phi(F) = \int_Y F(y) \varphi(y) d\lambda(y)$ for all $F$ in $L^p(Y, \mathcal{B}, \lambda)$. We assume that $\Phi$ vanishes on $\mathcal{E}_p$, and then we have

$$\int_Y f_\alpha(y) \varphi(y) d\lambda(y) = 0$$

for each function $f_\alpha$ in $\mathcal{F}$.

Let $k$ be a number field in $\mathcal{L}$ and let $\alpha$ be in $k^\times / \text{Tor}(k^\times) \subseteq \mathcal{G}$. As before, we apply (4.9) and (5.3) to obtain the identity (5.4). Then Lemma 6 implies that the function

$$v \mapsto \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} \varphi(y) d\lambda(y)$$

is constant on the set of places $v$ of $k$. Now, however, we apply Hölder’s inequality and find that

$$\sum_v \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} |\varphi(y)|^q d\lambda(y) \leq [k : \mathbb{Q}] \int_Y |\varphi(x)|^q d\lambda(y) < \infty.$$ 

This shows that the constant value of the function (6.2) is zero. Thus we have

$$\int_{\psi_k^{-1}(v)} \varphi(y) d\lambda(y) = 0$$

for all $k$ in $\mathcal{L}$ and for all places $v$ of $k$. It follows using Lemma 4 that

$$\int_Y g(y) \varphi(y) d\lambda(y) = 0$$

for all $g$ in $LC_c(Y)$. By Lemma 5 the set $LC_c(Y)$ is dense in $L^p(Y, \mathcal{B}, \lambda)$, and we conclude that the continuous linear functional $\Phi$ is identically zero. This completes the proof of Theorem 2.
Next we suppose that $E_\infty$ is the closure of $F$ in $C_0(Y)$. Again it suffices to show that if $\Phi : C_0(Y) \to \mathbb{R}$ is a continuous linear functional that vanishes on $E_\infty$, then $\Phi$ is identically zero on $C_0(Y)$. If $\Phi$ is such a linear functional, then by the Riesz representation theorem (see [8, Theorem 6.19]) there exists a regular signed measure $\nu$, defined on the $\sigma$-algebra $\mathcal{B}$ of Borel sets in $Y$, such that

$$\Phi(F) = \int_Y F(y) \, d\nu(y)$$

for all $F$ in $C_0(Y)$. Moreover, we have $\|\Phi\| = \|\nu\|$, where $\|\Phi\|$ is the norm of the linear functional $\Phi$ and $\|\nu\|$ is the total variation of the signed measure $\nu$. We assume that $\Phi$ vanishes on $E_\infty$, and therefore

$$\int_Y f_\alpha(y) \, d\nu(y) = 0$$

for each function $f_\alpha$ in $F$. By arguing as in the proof of Theorem 2, we conclude that for each number field $k$ in $\mathcal{L}$ the function

$$v \mapsto \lambda(\psi_k^{-1}(v))^{-1}\nu(\psi_k^{-1}(v)),$$

defined on the set of all places $v$ of $k$, is constant. As

$$\sum_v |\lambda(\psi_k^{-1}(v))^{-1}\nu(\psi_k^{-1}(v))| \leq [k : \mathbb{Q}] \sum_v |\nu(\psi_k^{-1}(v))| \leq [k : \mathbb{Q}] \|\nu\| < \infty,$$

we conclude that the value of the constant function (6.3) is zero. This shows that

$$\nu(\psi_k^{-1}(v)) = 0$$

for all $k$ in $\mathcal{L}$ and for all places $v$ of $k$. It follows as before that

$$\Phi(g) = \int_Y g(y) \, d\nu(y) = 0$$

for all $g$ in $LC_c(Y)$. As $LC_c(Y)$ is dense in $C_0(Y)$ by Lemma 5, we find that $\Phi$ is identically zero on $C_0(Y)$. This proves Theorem 3.

References

[1] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, Cambridge Univ. Press, New York, 2006.
[2] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
[3] W. B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. 1, W. B. Johnson and J. Lindenstrauss (eds.), Elsevier, New York, 2001, 1–84.
[4] M. Krein, D. Milman and M. Rutman, *A note on bases in Banach space*, Comm. Inst. Sci. Math. Méc. Univ. Kharkoff (4) 16 (1940), 106–110 (in Russian).
[5] S. Lang, *Fundamentals of Diophantine Geometry*, Springer, New York, 1983.
[6] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, PWN–Polish Sci. Publ., Warszawa, 1974.
[7] J. Neukirch, *Algebraic Number Theory*, Springer, New York, 1999.

[8] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[9] —, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.

Department of Mathematics  
University of Texas  
Austin, TX 78712, U.S.A.  
E-mail: allcock@math.utexas.edu  
vaaler@math.utexas.edu 

Received on 24.6.2008  
(5748)