LONG TIME STABILITY OF A CLASSICAL EFFICIENT SCHEME FOR AN INCOMPRESSIBLE TWO-PHASE FLOW MODEL

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Abstract. In this article we consider the implicit Euler scheme for a homogeneous two-phase flow model in a two-dimensional domain and with the aid of the discrete Gronwall lemma and of the discrete uniform Gronwall lemma we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero.

Key words. semi-implicit scheme, long-time stability, incompressible two-phase flow, discrete attractors

AMS subject classifications. 35Q30,35Q35,35Q72

1. Introduction.

It is well known that the 2D incompressible flow can be extremely complicated with possible chaos and turbulent behavior, [16]. Although some of the features of this turbulent or chaotic behavior may be deduced via analytic means, it is widely believed that numerical methods are indispensable for obtaining a better understanding of these complicated phenomena.

Let us recall that the incompressible Navier–Stokes equations govern the motion of single-phase fluids, such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [10]. For instance, this approach is used in [3] to describe cavitation phenomena in a flowing liquid. The model consists of the Navier–Stokes equations coupled with the phase-field system, [3,10,9,11]. In the isothermal compressible case, the existence of a global weak solution is proved in [8]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity $\mathbf{u}$ and the order parameter $\phi$. This system can be written as a Navier–Stokes equation coupled with a convective Allen-Cahn equation, [10]. The associated initial and boundary value problem was studied in [10], in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor $A$. They also established the existence of an exponential attractor $E$. This entails that $A$ has a finite fractal dimension, which is estimated in [10] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated, although some important issues remain unresolved, [10]. In the case of binary fluids, the analysis is even more complicated and the mathematical study is still at its infancy, as noted in [10].

In this article, we consider a homogeneous two-phase flow model in a two-dimensional domain, we discretize in time using the implicit Euler scheme and with the aid of the discrete Gronwall lemma and of the discrete uniform Gronwall lemma we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero. Our work has been...
inspired by previous results of one of the authors and her collaborators. In [19], for example, the authors considered the implicit Euler scheme for the 2D Navier–Stokes equations and proved that the numerical scheme was \( H^1 \)-uniformly stable in time. In a later article (see [6]), the authors used the theory for multi-valued attractors to prove the convergence of the discrete attractors to the global attractor of the continuous system as the time-step parameter approached zero. In [17], the author considered the implicit Euler scheme for the two-dimensional magnetohydrodynamics equations and showed that the scheme was \( H^2 \)-stable. Similar results were obtained in [18] and [7], where the authors proved not only the long-time stability of the implicit Euler scheme for the two-dimensional Rayleigh-Benard convection problem, and the thermo-hydraulics equations, respectively, but also the convergence of the global attractors generated by the numerical scheme to the global attractor of the continuous system as the time-step approaches zero.

Let us mention that although we drew our inspiration from [6, 19, 17, 18], the problem we treat here does not fall into the framework of these references. Besides the usual nonlinear term of the conventional Navier–Stokes system, the model considered here contains another (stronger) nonlinear term that results from the coupling of the convective Allen-Cahn equation and the Navier–Stokes system. Because of this, the analysis of the numerical scheme considered in this work tends to be more complicated and subtle than that of the 2D Navier–Stokes system studied in [19].

The article is divided as follows. In the next section, we recall from [10] the incompressible homogeneous two-phase flow and its mathematical setting. In Section 3 we study the stability of a time discretization scheme for the model. More precisely, we prove that the scheme is uniformly bounded in \( Y \) and \( V \), provided that the time-step is small enough. In Section 4 we recall the theory of the so-called multi-valued attractors, and then we apply it to our model.

2. A two phase flow model and its mathematical setting.

2.1. Governing equations. In this article, we consider a model of homogeneous incompressible two-phase flow with singularly oscillating forces. More precisely, we assume that the domain \( \Omega \) of the fluid is a bounded domain in \( \mathbb{R}^2 \). Then, we consider the system

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu_1 \Delta u + (u \cdot \nabla) u + \nabla p &= g - K \text{div}(\nabla \phi \otimes \nabla \phi), \\
\text{div } u &= 0, \\
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi + \mu &= 0, \\
\mu &= -\nu_2 \Delta \phi + \alpha f(\phi),
\end{aligned}
\]

in \( \Omega \times (0, +\infty) \).

In (2.1), the unknown functions are the velocity \( u = (u_1, u_2) \) of the fluid, its pressure \( p \) and the order (phase) parameter \( \phi \). The quantity \( \mu \) is the variational derivative of the following free energy functional

\[
F(\phi) = \int_{\Omega} \left( \frac{\nu_2}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) \, ds,
\]
where, e.g., $F(r) = \int_0^r f(\zeta) d\zeta$. Here, the constants $\nu_1 > 0$ and $K > 0$ correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient respectively, $\nu_2$, $\alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, $\nu_2$ is related with the thickness of the interface separating the two fluids. Hereafter, as in [10], we assume that $\nu_2 \leq \alpha$.

In (2.1), $g$ is an external time-dependent volume force and we have assumed the density equal to one.

We endow (2.1) with the boundary condition

$$u = 0, \quad \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \partial \Omega \times (0, +\infty),$$

where $\partial \Omega$ is the boundary of $\Omega$ and $\eta$ is its outward normal.

The initial condition is given by

$$(u, \phi)(0) = (u_0, \phi_0) \text{ in } \Omega.$$ (2.4)

### 2.2. Mathematical setting.

We first recall from [10] the weak formulation of (2.1)–(2.4). Hereafter, we assume that the domain $\Omega$ is bounded with a smooth boundary $\partial \Omega$ (e.g., of class $C^2$). We also assume that $f \in C^1(\mathbb{R})$ satisfies

$$\begin{align*}
&\lim_{|r| \to +\infty} f'(r) > 0, \\
&|f'(r)| \leq c_f(1 + |r|^m), \quad \forall r \in \mathbb{R},
\end{align*}$$

(2.5)

where $c_f$ is some positive constant and $m \in [1, +\infty)$ is fixed. It follows from (2.5) that

$$|f(r)| \leq c_f(1 + |r|^{m+1}), \quad \forall r \in \mathbb{R}.$$ (2.6)

If $X$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$, we will denote the induced norm by $| \cdot |_X$, while $X^*$ will indicate its dual. We set

$$V = \left\{ u \in C^\infty_c(\Omega) : \text{ div } u = 0 \text{ in } \Omega \right\}.$$

We denote by $H$ and $V$ the closure of $V$ in $(L^2(\Omega))^2$ and $(H_0^1(\Omega))^2$ respectively. The scalar product in $H$ is denoted by $\langle \cdot, \cdot \rangle_{L^2}$ and the associated norm by $| |_{L^2}$. Moreover, the space $V$ is endowed with the scalar product

$$\langle (u, v) \rangle = \sum_{i=1}^2 (\partial_x u, \partial_x v)_{L^2}, \quad \| u \| = ((u, u))^{1/2},$$

and we have the Poincaré inequality

$$|u|_{L^2}^2 \leq c_{\Omega} \| u \|_V^2, \quad \forall u \in V.$$ (2.7)

We now define the operator $A$ by

$$Au = -P \Delta u, \quad \forall u \in D(A) = H^2(\Omega) \cap V,$$

where $P$ is the Leray-Helmholtz projector of $L^2(\Omega)$ onto $H$. Then $A$ is a self-adjoint positive unbounded operator in $H$ which is associated with the scalar product defined
above. Furthermore, $A^{-1}$ is a compact linear operator on $H$ and $|A \cdot |_{L^2}$ is a norm on $D(A)$, equivalent to the $H^2$-norm.

Note that from (2.5), we can find $\gamma > 0$ such that

\[
\lim_{|r| \to +\infty} f'(r) > 2\gamma > 0.
\]

We define the linear positive unbounded operator $A_\gamma$ on $L^2(\Omega)$ by:

\[
A_\gamma \phi = -\Delta \phi + \gamma \phi, \quad \forall \phi \in D(A_\gamma),
\]

where

\[
D(A_\gamma) = \left\{ \phi \in H^2(\Omega); \frac{\partial \rho}{\partial \eta} = 0 \text{ on } \partial \Omega \right\}.
\]

Note that $A^{-1}_\gamma$ is a compact linear operator on $L^2(\Omega)$ and $|A_\gamma \cdot |_{L^2}$ is a norm on $D(A_\gamma)$ that is equivalent to the $H^2$-norm.

We introduce the bilinear operators $B_0, B_1$ (and their associated trilinear forms $b_0, b_1$) as well as the coupling mapping $R_0$, which are defined from $D(A) \times D(A)$ into $H$, $D(A) \times D(A_\gamma)$ into $L^2(\Omega)$, and $L^2(\Omega) \times D(A^{-1}_\gamma)$ into $H$, respectively. More precisely, we set

\[
(B_0(u, v), w) = \int_\Omega [(u \cdot \nabla)v] \cdot w \, dx = b_0(u, v, w), \quad \forall u, v, w \in D(A),
\]

\[
(B_1(u, \phi), \psi) = \int_\Omega [(u \cdot \nabla)\phi] \psi \, dx = b_1(u, \phi, \psi), \quad \forall u \in D(A), \phi, \psi \in D(A_\gamma),
\]

\[
(R_0(\mu, \phi), w) = \int_\Omega \mu [\nabla \phi \cdot w] \, dx = b_1(w, \phi, \mu), \quad \forall w \in D(A), \quad (\mu, \phi) \in L^2(\Omega) \times D(A^{-1}_\gamma).
\]

Note that

\[
R_0(\mu, \phi) = \mathcal{P} \mu \nabla \phi,
\]

and

\[
|b_0(u, v, w)| \leq c_b |u|^{1/2}_{L^2} ||u||^{1/2}_{L^2} ||v||^{1/2}_{L^2} ||w||^{1/2}_{L^2}, \quad \forall u, v, w \in V,
\]

\[
|b_0(u, v, w)| \leq c_b |u|^{1/2}_{L^2} |Au|^{1/2}_{L^2} ||w||_{L^2} \forall u \in D(A), v \in V, w \in H,
\]

\[
|b_0(u, v, w)| \leq c_b |u|^{1/2}_{L^2} ||v||^{1/2}_{L^2} |Au|^{1/2}_{L^2} ||w||_{L^2} \forall u \in V, v \in D(A), w \in H,
\]

\[
b_0(u, v, v) = 0, \forall u, v \in V,
\]

the last equation implying

\[
b_0(u, v, w) = -b_0(u, w, v), \quad \forall u, v, w \in V.
\]

Similar inequalities are valid for the trilinear form $b_1$:

\[
|b_1(u, \phi, \psi)| \leq c_b |u|^{1/2}_{L^2} ||u||^{1/2}_{L^2} ||\phi||^{1/2}_{L^2} ||\psi||^{1/2}_{L^2}, \forall u \in V, \phi, \psi \in H^1(\Omega),
\]
cretization of (2.24) using the fully implicit Euler scheme,\n8 u \in D(A), \phi \in H^1(\Omega), \psi \in L^2(\Omega),\n(2.17) \quad |b_1(u, \phi, \psi)| \leq c_b|u|^{1/2}_{L^2}|A u|^{1/2}_{L^2}||\phi||_{L^2}, \forall u \in V, \phi \in D(A_\gamma), \psi \in L^2(\Omega),
(2.18) \quad b_1(u, \phi, \psi) = 0, \quad b_1(u, \phi, f_\gamma(\phi)) = 0, \quad \forall u \in V, \phi \in H^1(\Omega),\n(2.19) \quad b_1(u, \phi, \psi) = -b_1(u, \psi, \phi), \quad \forall u \in V, \phi, \psi \in H^1(\Omega).

Now we define the Hilbert spaces Y and V by\n(2.20) \quad Y = H \times H^1(\Omega), \quad V = V \times D(A_\gamma),\nendowed with the scalar products whose associated norms are\n(2.21) \quad \|(u, \phi)|^2_Y = K^{-1}|u|_{L^2}^2 + \nu_2(|\nabla \phi|_{L^2}^2 + \gamma|\phi|_{L^2}^2) =: K^{-1}|u|_{L^2}^2 + \nu_2|\phi|_{L^2}^2,\n(2.22) \quad \|(u, \phi)|^2_V = |u|^2 + |A_\gamma \phi|_{L^2}^2.

We also set\n(2.23) \quad f_\gamma(r) = f(r) - \alpha^{-1}\nu_2\gamma r\nand observe that \(f_\gamma\) still satisfies \(2.8\) with \(\gamma\) in place of \(2\gamma\) since, \(\nu_2 \leq \alpha\). Also, its primitive, \(F_\gamma(r) = \int^r f_\gamma(\zeta)d\zeta,\) is bounded from below.

Throughout this article, we will denote by \(c\) a generic positive constant depending on the domain \(\Omega\).

Using the notations above, we rewrite (2.1)–(2.3) as (see \[10\] for the details)\n\[
\begin{align*}
\frac{du}{dt} + \nu_1 A u + B_0(u, u) - KR_0(\nu_2 A_\gamma \phi, \phi) &= g, \quad \text{a.e. in } \Omega \times (0, +\infty), \\
\mu &= \nu_2 A_\gamma \phi + \alpha f_\gamma(\phi), \quad \text{a.e. in } \Omega \times (0, +\infty), \\
\frac{d\phi}{dt} + \mu + B_1(u, \phi) &= 0, \quad \text{a.e. in } \Omega \times (0, +\infty).
\end{align*}
\]
(2.24)

The weak formulation of (2.24) was proposed and studied in \[10\] \[9\], and the existence and uniqueness of solution was proved.

3. A time discretization of (2.24). In this article we consider a time discretization of (2.24) using the fully implicit Euler scheme,\n\[
\begin{align*}
\frac{u^{n+1} - u^n}{\tau} + \nu_1 A u^n + B_0(u^n, u^n) - KR_0(\nu_2 A_\gamma \phi^n, \phi^n) &= g^n, \\
\mu^n &= \nu_2 A_\gamma \phi^n + \alpha f_\gamma(\phi^n), \\
\frac{\phi^{n+1} - \phi^n}{\tau} + \mu^n + B_1(u^n, \phi^n) &= 0, \\
u^0 &= u_0, \phi^0 = \phi_0,
\end{align*}
\]
(3.1)
and prove that the attractors generated by the above system converge to the attractor generated by the continuous system (2.24) as the time-step converges to zero. To prove the existence of the discrete attractors we need to use the theory of the multi-valued attractors, that we discuss in Subsection 4.11.

Throughout the article, we assume that \(g \in L^\infty(R_+; H)\) and we let \(\|g\|_{L^\infty} := \|g\|_{L^\infty(R_+; H)}\).
3.1. Y-Uniform Boundedness. We begin with one of our main results, which proves the uniform boundedness of the approximate solution \((u^n, \phi^n)\) in \(Y\). Once the \(Y\)-uniform stability is established, the \(V\)-uniform boundedness follows right away (see Proposition 1 below).

**Theorem 1.** Let \((u^n, \phi^n)\) be a solution of (3.4). Then there exists \(\kappa > 0\) such that for every \(k > 0\), we have

\[
(\text{3.2}) \| (u^n, \phi^n) \|_Y^2 \leq (1 + \kappa k)^{-n} Q^n (\| (u_0, \phi_0) \|_Y) + \rho_0^2 \left[ 1 - (1 + \kappa k)^{-n} \right], \forall n \geq 0,
\]

where the monotonically increasing function \(Q\) is independent of \(n\), and \(\rho_0\) (given in (3.3) below), is independent of the initial data.

Moreover, there exists \(K_1 = K_1 (\| (u_0, \phi_0) \|_Y, \| g \|_\infty)\) such that for every \(k > 0\), we have

\[
(\text{3.3}) \quad \| (u^n, \phi^n) \|_Y \leq K_1, \forall n \geq 0,
\]

and for every \(i = 1, \ldots, n\) there exist \(M_1 = M_1 (\| (u^{n-1}, \phi^{n-1}) \|_Y, \| g \|_\infty, (n - i + 1)k)\) and \(M_2 = M_2 (\| (u^{n-1}, \phi^{n-1}) \|_Y, \| g \|_\infty, (n - i + 1)k)\), increasing in their arguments, such that

\[
(\text{3.4}) \quad k \sum_{j=1}^{n} \left( \frac{\nu_l}{2K} \| u^n \|^2 + 2|\mu^n|_L^2 \right) \leq M_1,
\]

\[
(\text{3.5}) \quad k \sum_{j=1}^{n} |A_\gamma (\phi^j)|_L^2 \leq M_2.
\]

**Proof.** Taking the scalar product of the first equation of (3.1) with \(2ku^n\) in \(L^2\) and using the relation

\[
2(\varphi - \psi, \varphi)_{L^2} = |\varphi|_{L^2}^2 - |\psi|_{L^2}^2 + |\varphi - \psi|_{L^2}^2,
\]

and the skew property (2.18), we obtain

\[
(\text{3.7}) \quad |u^n|^2_{L^2} - |u^{n-1}|^2_{L^2} + |u^n - u^{n-1}|_{L^2}^2 + 2\nu_1 k \| u^n \|^2
- 2Kk(b_1(u^n, \phi^n, \nu_2 A_\gamma \phi^n) = 2k(g^n, u^n)l.
\]

Using the second equations of (3.1) and of (2.18), we have \(b_1(u^n, \phi^n, \nu_2 A_\gamma \phi^n) = b_1(u^n, \phi^n, \mu^n)\) and thus (3.7) becomes

\[
(\text{3.8}) \quad |u^n|^2_{L^2} - |u^{n-1}|^2_{L^2} + |u^n - u^{n-1}|_{L^2}^2 + 2\nu_1 k \| u^n \|^2
- 2Kk(b_1(u^n, \phi^n, \mu^n) = 2k(g^n, u^n)l.
\]

Multiplying the third equation of (3.1) by \(2k\mu^n\) and integrating we obtain

\[
(\text{3.9}) \quad 2(\phi^n - \phi^{n-1}, \mu^n)l + 2k|\mu^n|_{L^2}^2 + 2k(b_1(u^n, \phi^n, \mu^n) = 0.
\]

Dividing (3.8) by \(K\) and adding the resulting equation to (3.9), we find

\[
\frac{1}{K} \left[ |u^n|_{L^2}^2 - |u^{n-1}|_{L^2}^2 + |u^n - u^{n-1}|_{L^2}^2 + \frac{2\nu_1}{K} k \| u^n \|^2 \right]
+ 2(\phi^n - \phi^{n-1}, \mu^n)l + 2k|\mu^n|_{L^2}^2 = \frac{2}{K} k(g^n, u^n)l.
\]
Using the second equation of (3.1), (2.9) and (2.23), we obtain
\[ 2(\phi^n - \phi^{n-1}, \mu^n)l = \nu_2 (\|\phi^n\|_\gamma^2 - \|\phi^{n-1}\|_\gamma^2) + 2\alpha(\phi^n - \phi^{n-1}, F_\gamma(\phi^n))l. \]

(3.11) \[ 2\alpha(\phi^n - \phi^{n-1}, F_\gamma(\phi^n))l = 2\alpha F_\gamma(\phi^n) - 2\alpha F_\gamma(\phi^{n-1}) + 2\alpha R^n_\gamma, \]

where

(3.12) \[ F_\gamma(\phi^n) = \int_\Omega F_\gamma(\phi^n(x))dx, \]

\[ R^n_\gamma = -\int_0^1 \int_\Omega \left[ f_\gamma(\phi^{n-1}(x) + t(\phi^n(x) - \phi^{n-1}(x))) - f_\gamma(\phi^n(x)) \right] (\phi^n(x) - \phi^{n-1}(x)) dt dx, \]

(3.13) and thus

\[ 2(\phi^n - \phi^{n-1}, \mu^n)l = \nu_2 (\|\phi^n\|_\gamma^2 - \|\phi^{n-1}\|_\gamma^2) + 2\alpha F_\gamma(\phi^n) - 2\alpha F_\gamma(\phi^{n-1}) + 2\alpha R^n_\gamma. \]

Combining the above relation with (3.10) we find

(3.14) \[ \frac{1}{K} |u^n|_{L^2}^2 - |u^{n-1}|_{L^2}^2 + |u^n - u^{n-1}|_{L^2}^2 + \nu_2 (\|\phi^n\|_\gamma^2 - \|\phi^{n-1}\|_\gamma^2) \]

\[ + \frac{2\nu_1}{K} k|\mu^n|_{L^2}^2 + 2\alpha F_\gamma(\phi^n) - 2\alpha F_\gamma(\phi^{n-1}) + 2\alpha R^n_\gamma + 2k|\mu^n|_{L^2}^2 = \frac{2}{K} k(g^n, u^n). \]

Multiplying the third equation of (3.1) by 2k\phi^n and integrating we obtain (using the second equation of (3.1))

(3.15) \[ |\phi^n|_{L^2}^2 - |\phi^{n-1}|_{L^2}^2 + |\phi^n - \phi^{n-1}|_{L^2}^2 + 2k\nu_2 |\phi^n|_\gamma^2 + 2\alpha k(f_\gamma(\phi^n), \phi^n)l = 0. \]

Adding (3.14) and (3.15) we find

(3.16) \[ \frac{1}{K} |u^n|_{L^2}^2 - |u^{n-1}|_{L^2}^2 + |u^n - u^{n-1}|_{L^2}^2 + \nu_2 (\|\phi^n\|_\gamma^2 - \|\phi^{n-1}\|_\gamma^2) \]

\[ + \frac{2\nu_1}{K} k|\mu^n|_{L^2}^2 + 2k|\phi^n|_{L^2}^2 + 2k|\phi^{n-1}|_{L^2}^2 + 2\alpha F_\gamma(\phi^n) - 2\alpha F_\gamma(\phi^{n-1}) + 2\alpha R^n_\gamma + 2k|\mu^n|_{L^2}^2 + 2\alpha k(f_\gamma(\phi^n), \phi^n)l + 2\alpha R^n_\gamma = \frac{2}{K} k(g^n, u^n). \]

Using the Cauchy–Schwarz inequality and the Poincaré inequality (2.7), we majorize the right-hand side of (3.16) by

\[ \frac{2}{K} k(g^n, u^n)l \leq \frac{2}{K} k|g^n|_{L^2} |u^n|_\gamma l \leq \frac{2}{K} \sqrt{c_1} k|g^n|_\gamma |u^n|_\gamma l \]

\[ \leq \frac{\nu_1}{K} k|u^n|_{L^2}^2 + \frac{c_0}{\nu_1 K} k|g^n|_{L^2}^2. \]

Hereafter, we assume that the potential function \( f \) satisfies the following additional condition:

(3.18) \[ f'(r) \geq -\frac{1}{2\alpha}, \quad \forall r \in R. \]
Now, using the mean value theorem and recalling (2.23), as well as (3.18), relation (3.13) yields
\[
2\alpha R^n_n \geq -\frac{1}{2} \| \phi^n - \phi^{n-1} \|_{L^2}^2 - \nu_2 \gamma \| \phi^n - \phi^{n-1} \|_{L^2}^2.
\]

Relations (3.16), (3.17), (3.19) and (2.21) give
\[
\begin{align*}
\frac{1}{K} \left[ |u^n|^2_{L^2} - |u^{n-1}|_{L^2}^2 + |u^n - u^{n-1}|^2_{L^2} \right] + \nu_2 \| \phi^n \|_{L^2}^2 - \nu_2 \| \phi^{n-1} \|_{L^2}^2 + \nu_2 \| \phi^n - \phi^{n-1} \|_{L^2}^2 \\
\left. + \frac{v_1}{K} k\| u^n \|_{L^2}^2 + 2k\nu_2 \| \phi^n \|_{L^2}^2 + 2k\mu_1 \| \phi^n \|_{L^2}^2 + 2\alpha k(f_\gamma(\phi^n), \phi^n)l \leq \frac{c_0}{v_1} k|g^n|^2_{L^2}. \right]
\end{align*}
\]

Now, for any \( n \geq 1 \), let
\[
E^n = \frac{1}{K} |u^n|^2_{L^2} + \nu_2 \| \phi^n \|_{L^2}^2 + 2\alpha \mathcal{F}_\gamma(\phi^n) + \| \phi^n \|_{L^2}^2 + 2\alpha C_F, |\Omega|,
\]
where \( C_F \) is taken large enough to ensure that \( E^n \geq 0 \) (recall that \( F_\gamma \) is bounded from below by a constant independent of \( \nu_2 \) and \( \alpha \)). We rewrite (3.20) in the form
\[
E^n - E^{n-1} + \kappa kE^n \leq k\Lambda^n,
\]
where \( \kappa \in (0, 1) \) is to be determined and
\[
\Lambda^n = \frac{v_1}{K} k\| u^n \|_{L^2}^2 + \frac{\kappa}{K} |u^n|^2_{L^2} - (2 - \kappa) \nu_2 \| \phi^n \|_{L^2}^2 + \kappa \| \phi^n \|_{L^2}^2 + 2\alpha C_F, |\Omega| \\
- 2\| \mu_1 \|_{L^2}^2 + \frac{c_0}{v_1} \| g^n \|_{L^2}^2 + 2\alpha \left[ \kappa (F_\gamma(\phi^n) - f_\gamma(\phi^n), \phi^n, 1) l - (1 - \kappa) (f_\gamma(\phi^n), \phi^n, 1) l \right].
\]

Now note that owing to assumption (2.23), we have (for any \( r \in R \))
\[
f_\gamma(r)r \geq c_f 2|f_\gamma(r)|(1 + |r|) - c_f 2(1 + \alpha^{-1} \nu_2),
\]
\[
F_\gamma(r) - f_\gamma(r)r \leq c_f'(1 + \alpha^{-1} \nu_2)|r|^2 + c_f',
\]
\[
|F_\gamma(r)| \leq |f_\gamma(r)|(1 + |r|) + c_1,
\]
where \( c_f, c_\gamma, c_f', c_1 \) are positive, sufficiently large constants that depend on \( f \) only.

Using the above inequalities and the Poincaré inequality (2.21), we obtain the following bound on \( \Lambda^n \):
\[
\Lambda^n \leq -\frac{1}{K}(v_1 - \kappa c_0)|u^n|^2 - (2 - \kappa)\nu_2\| \phi^n \|_{L^2}^2 - \left[ 2 - \frac{\kappa}{\nu_2 \gamma} (1 + \nu_2 \gamma + 2c_f'(\alpha + \nu_2)) \right] \nu_2 \gamma |\phi^n|^2_{L^2} \\
- 2\| \mu_1 \|_{L^2}^2 + \frac{c_0}{v_1} |g^n|^2_{L^2} - (1 - \kappa)c_\gamma \alpha (|F_\gamma(\phi^n)|, 1) l + c_2,
\]

where
\[
c_2 = [(1 - \kappa)(c_1 c_\gamma \alpha + c_f \alpha + c_f \nu_2) + 2\alpha c_F, + 2\alpha c_f''] |\Omega|.
\]
Choosing $\kappa \in (0, 1)$ as

\[
\kappa = \min \left\{ \frac{\nu_1}{2c_\Omega}, \frac{\nu_2\gamma}{1 + \nu_2\gamma + 2c_f'(\alpha + \nu_2)} \right\},
\]

relation (3.26) gives

\[
\Lambda^n \leq -\frac{\nu_1}{2c_\Omega} \|u^n\|^2 - \nu_2\|\phi^n\|^2 - 2|\mu^n|_{L^2}^2 + \frac{c_\Omega}{\nu_1 c_\Omega'} \|g^n\|^2_{L^2} - (1 - \kappa)c_*\alpha(|F_\gamma(\phi^n)|, 1)l + c_2,
\]

and combining it with (3.22), we obtain

\[
E^n - E^{n-1} + \kappa k E^n + \frac{1}{2} k \left( \frac{c_\Omega}{\nu_1 c_\Omega'} \|g^n\|^2_{L^2} + 2k|\mu^n|_{L^2}^2 \right) + c_3 k |F_\gamma(\phi^n)|_{L^1} \leq \frac{c_\Omega}{\nu_1 c_\Omega'} k \|g^n\|^2_{L^2} + c_2 k.
\]

Neglecting some positive terms, we obtain

\[
E^n \leq \frac{1}{\beta} E^{n-1} + \frac{1}{\beta} k \left( \frac{c_\Omega}{\nu_1 c_\Omega'} \|g^n\|^2_{L^2} + c_2 \right),
\]

where

\[
\beta = 1 + \kappa k.
\]

Using recursively (3.30), we find

\[
E^n \leq \frac{1}{\beta^n} E^0 + k \sum_{i=1}^{n} \frac{1}{\beta^i} \left( \frac{c_\Omega}{\nu_1 c_\Omega'} \|g^{n+1-i}\|^2_{L^2} + c_2 \right) \leq (1 + \kappa k)^{-n} E^0 + \frac{1}{\kappa} \left( \frac{c_\Omega}{\nu_1 c_\Omega'} \|g\|_{\infty}^2 + c_2 \right) \left[ 1 - (1 + \kappa k)^{-n} \right].
\]

Now observe that, due to (2.5), we can find $C_f > 0$ such that

\[
E^n \leq C_f \left( 1 + ||(u^n, \phi^n)||_Y^2 + |\phi^n|_{L^m+2}^2 \right),
\]

and thus, relation (3.32) yields

\[
E^n \leq (1 + \kappa k)^{-n} Q^2(||(u_0, \phi_0)||_Y) + \rho_0^2 \left[ 1 - (1 + \kappa k)^{-n} \right],
\]

where $Q$ is a monotonically increasing function of the initial data, independent of $n$, and

\[
\rho_0^2 = \frac{1}{\kappa} \left( \frac{c_\Omega}{\nu_1 c_\Omega'} \|g\|_{\infty}^2 + c_2 \right).
\]

We therefore obtain

\[
E^n \leq K^2 = K^2(||(u_0, \phi_0)||_Y, \|g\|_{\infty}) := Q^2(||(u_0, \phi_0)||_Y) + \rho_0^2, \forall n \geq 0,
\]

and since $||(u^n, \phi^n)||_Y^2 \leq E^n$, relations (3.34) and (3.30) give (3.2) and (3.3), respectively.
Adding inequalities (3.29) with \( n \) from \( i \) to \( N \), dropping some positive terms, and recalling (3.33), we obtain conclusion (3.3) of the theorem.

Now using (3.1), (2.23), (2.6) and the Sobolev imbedding \( H^1(\Omega) \hookrightarrow L^{2m+2}(\Omega) \), for any \( m \), we obtain

\[
|A_{ij}\phi^n|^2 \leq \frac{3}{\nu^2} |\mu^n|^2 + \frac{6\beta^2}{\nu^2} c_f^2 |\Omega| + \frac{6\beta^2}{\nu^2} c_f^2 (\|\phi^n\|^2)^{m+1} + 3\gamma^2 |\phi^n|^2, 
\]

for some positive constant \( c_f^2 \) depending on \( c_f \). Summing with \( n \) from \( i \) to \( N \) and using (3.4) and (3.36), we obtain conclusion (3.5). This completes the proof of the theorem. \( \Box \)

As a direct consequence of (3.2), we have

**Corollary 3.1.** If

\[
0 < k < \frac{1}{\kappa},
\]

then \( B_{\mathbf{Y}}(0, \sqrt{2}\rho_0) \), the ball in \( \mathbf{Y} \) centered at 0 and radius \( \sqrt{2}\rho_0 \), is an absorbing ball for \((u^n, \phi^n)\) in \( \mathbf{Y} \).

3.2. \textbf{V-Uniform Boundedness.} We now seek to obtain uniform bounds for \((u^n, \phi^n)\) in \( \mathbf{V} \), similar to those we have already obtained in \( \mathbf{Y} \) (see (3.2) above). In order to do this, we will first use the discrete Gronwall lemma to derive an upper bound on \( \|(u^n, \phi^n)\|_\mathbf{V} \), \( n \leq N \), for some \( N > 0 \), and then we will use the discrete uniform Gronwall lemma to obtain an upper bound on \( \|(u^n, \phi^n)\|_\mathbf{V} \), \( n \geq N \).

We begin with some preliminary inequalities.

**Lemma 1.** For every \( k > 0 \), we have

\[
\|(u^n, \phi^n)\|_\mathbf{V} \leq K_2 \|(u^{n-1}, \phi^{n-1})\|_\mathbf{V} + c_3 (|f_\gamma(\phi^n)|^2 + |g^n|^2), \quad \forall n \geq 1,
\]

where \( K_2 = K_2(\|(u_0, \phi_0)\|_\mathbf{Y}, \|g\|_\infty) \) and \( c_3 > 0 \) are given below, in (3.47) and (3.48), respectively.

**Proof.** Multiplying the first equation of (3.1) by \( 2/k \), the third equation of (3.1) by \( 2\nu_2KA_\gamma(\phi^n - \phi^{n-1}) \), adding the resulting equations and integrating, we obtain (using the second equation of (3.1))

\[
\frac{1}{k} \sum_{i=0}^{n-1} |u^n - u^{n-1}|^2 + 2\nu_2 \|\phi^n - \phi^{n-1}\|^2 + \frac{1}{\kappa} k \|u^n\|^2 + \frac{2}{\kappa} k |A_\gamma\phi^n|^2 \\
- \frac{1}{\kappa} k \|u^{n-1}\|^2 - \frac{2}{\kappa} k |A_\gamma\phi^{n-1}|^2 + \frac{1}{\kappa} k |u^n - u^{n-1}|^2 + \frac{2}{\kappa} k |A_\gamma(\phi^n - \phi^{n-1})|^2 \\
+ \frac{2}{\kappa} k b_0(u^n, u^n, u^n - u^{n-1}) - 2kb_1(u^n - u^{n-1}, \phi^n, \nu_2 A_\gamma \phi^n) + 2\nu_2kb_1(u^n, \phi^n, A_\gamma(\phi^n - \phi^{n-1})) \\
= \frac{2}{\kappa} k (g^n, u^n - u^{n-1}) - 2\nu_2\alpha k(f_\gamma(\phi^n), A_\gamma(\phi^n - \phi^{n-1})).
\]

Using the Cauchy–Schwarz inequality, we majorize the right-hand side of (3.40) by

\[
\frac{2}{\kappa} k (g^n, u^n - u^{n-1}) \leq \frac{2}{\kappa} k |g^n||u^n - u^{n-1}| \leq \frac{2}{\kappa} \sqrt{\gamma^2} k |g^n||u^n - u^{n-1}| \quad \text{(by (2.7))}
\]

\[
\leq \left( \frac{1}{\nu_1 \kappa} k\right) |u^n - u^{n-1}|^2 + \frac{4c_3}{\nu_1 \kappa} k |g^n|^2,
\]

where \( \gamma = \sqrt{\nu_2} \).
\[ 2 \nu_2 \alpha k |(f_\gamma(\phi^n), A_\gamma(\phi^n - \phi^{n-1}))| \leq 2 \nu_2 \alpha k |f_\gamma(\phi^n)||A_\gamma(\phi^n - \phi^{n-1})| \]

\[ \leq \frac{\nu_2^2}{4} k |A_\gamma(\phi^n - \phi^{n-1})|^2_{L^2} + 4 \alpha^2 k |f_\gamma(\phi^n)|^2_{L^2}. \]

The nonlinear terms are bounded as follows:

\[ \frac{2}{\mathcal{K}} k b_0(u^n, u^n, u^n - u^{n-1}) = -\frac{2}{\mathcal{K}} k b_0(u^n, u^n, u^n - u^{n-1}) \quad \text{(by (2.13))} \]

\[ \leq \frac{2}{\mathcal{K}} c_0 k |u^n|_{L^2}^2 ||u^n||_{L^2} ||u^{n-1}||_{L^2} \quad \text{(by (2.1))} \]

\[ -2 \nu_1 b_1 (u^n - u^{n-1}, \phi^n, \nu_2 A_\gamma \phi^n) + 2 \nu_2 b_1 (u^n, \phi^n, A_\gamma (\phi^n - \phi^{n-1})) \]

\[ = -2 \nu_2 b_1 (u^n - u^{n-1}, \phi^n, A_\gamma \phi^n) + 2 \nu_2 b_1 (u^n, \phi^n, A_\gamma (\phi^n - \phi^{n-1})) \]

\[ \leq 2 \nu_2 c_0 k |u^n - u^{n-1}|_{L^2}^2 ||u^n - u^{n-1}||_{L^2} ||\phi^n||_{L^2} ||\phi^{n-1}||_{L^2} \quad \text{(by (2.1))} \]

\[ + 2 \nu_2 c_0 k |u^n - u^{n-1}|_{L^2}^2 ||\phi^n||_{L^2} ||\phi^{n-1}||_{L^2} \quad \text{(by (2.1))} \]

\[ \leq \frac{\nu_1}{2 \mathcal{K}} k |u^n - u^{n-1}|_{L^2}^2 + \frac{\nu_2}{2} k |A_\gamma \phi^n|^2_{L^2} + cK_1^2 k |A_\gamma \phi^n|^2_{L^2} \]

\[ + \frac{\nu_2}{2} k |A_\gamma (\phi^n - \phi^{n-1})|^2_{L^2} + cK_1^2 k ||u^n - u^{n-1}||_{L^2}^2 \quad \text{(by (3.3))}. \]

Relations \( (3.40) \) and \( (3.44) \) yield

\[ \frac{2}{\mathcal{K}} |u^n - u^{n-1}|_{L^2}^2 + 2 \nu_2 ||\phi^n - \phi^{n-1}||_{L^2}^2 + \frac{\nu_1}{2 \mathcal{K}} k |u^n||_{L^2}^2 + \frac{\nu_2}{2} k |A_\gamma \phi^n|^2_{L^2} \]

\[ - \left( \frac{\nu_1}{\mathcal{K}} + \frac{2 \nu_2}{\nu_1} K_1^2 + cK_1^4 \right) k ||u^{n-1}||_{L^2}^2 - (\nu_2^2 + cK_1^2) k |A_\gamma \phi^{n-1}|_{L^2}^2 + \frac{\nu_1}{2 \mathcal{K}} k ||u^n - u^{n-1}||_{L^2}^2 \]

\[ \leq \frac{4c_0}{\nu_1 K} k |g^n|^2_{L^2} + 4 \alpha^2 k |f_\gamma(\phi^n)|^2_{L^2} \]

and neglecting some positive terms we obtain

\[ \frac{\nu_1}{2 \mathcal{K}} ||u^n||_{L^2}^2 + \frac{\nu_2}{2} k |A_\gamma \phi^n|^2_{L^2} \leq \left( \frac{\nu_1}{\mathcal{K}} + \frac{2 \nu_2}{\nu_1} K_1^2 + cK_1^4 \right) ||u^{n-1}||_{L^2}^2 \]

\[ + (\nu_2^2 + cK_1^2) |A_\gamma \phi^{n-1}|_{L^2}^2 + \frac{4c_0}{\nu_1 K} |g^n|^2_{L^2} + 4 \alpha^2 |f_\gamma(\phi^n)|^2_{L^2}. \]

Taking

\[ K_2 = 2 \left( \frac{\mathcal{K}}{\nu_1} + \frac{1}{\nu_2^2} \right) \left( \frac{\nu_1}{\mathcal{K}} + \frac{2 \nu_2}{\nu_1} K_1^2 + cK_1^4 + \nu_2^2 + cK_1^2 \right), \]

and

\[ c_3 = 2 \left( \frac{\mathcal{K}}{\nu_1} + \frac{1}{\nu_2^2} \right) \left( \frac{4c_0}{\nu_1 K} + 4 \alpha^2 \right), \]

we obtain conclusion \((3.39)\) of the lemma. □

**LEMMA 2.** For every \( k > 0 \) and for every \( n \geq 1 \) we have

\[ c_4 K_1^2 k ||(u^n, \phi^n)||_{L^2}^2 - ||(u^n, \phi^n)||_{L^2}^2 + ||(u^{n-1}, \phi^{n-1})||_{L^2}^2 \]

\[ + c_5 k \left( |f_\gamma(\phi^n)\nabla \phi^n|^2_{L^2} + |g^n|^2_{L^2} \right) \geq 0, \]
for some positive constants $c_4$ and $c_5$.

Proof.

Multiplying the first equation of (3.1) by $2kAu^n$, the third equation of (3.1) by $2kA^n_2\phi^n$, adding the resulting equations and integrating, we obtain (using the second equation of (3.1))

\[
||u^n||^2 + |A_\gamma \phi^n|^2_{L^2} - (||u^{n-1}||^2 + |A_\gamma \phi^{n-1}|^2_{L^2}) + ||u^n - u^{n-1}||^2 + |A_\gamma (\phi^n - \phi^{n-1})|^2_{L^2} \\
+ 2\nu_1 k|Au^n|_{L^2}^2 + 2k\nu_2 |A_\gamma^{3/2}\phi^n|^2_{L^2} + 2kb_0(u^n, u^n, Au^n) - 2\nu_2 A_\gamma \phi^n \\
+ 2kb_1(u^n, \phi^n, A^n_2\phi^n) = 2k(g^n, Au^n)l - 2\alpha_kf_1(\phi^n), A^n_2\phi^n).l
\]

(3.50)

Using the Cauchy-Schwarz inequality, we estimate the right-hand side of the above equality as

\[
2k(g^n, Au^n)l \leq 2|k||g^n||Au^n||l \leq \frac{\nu_1}{4} k|Au^n|^2_{L^2} + \frac{4}{\nu_1} k|g^n|^2_{L^2},
\]

(3.51)

\[
2\alpha_k|f_1(\phi^n), A^n_2\phi^n)|l = 2\alpha_k|A^{1/2}_2 f_1(\phi^n), A_\gamma^{3/2} \phi^n)|l \leq 2\alpha_k|A^{1/2}_2 f_1(\phi^n)||A_\gamma^{3/2} \phi^n|l \\
\leq c|\nabla f_1(\phi^n)|l|A_\gamma^{3/2} \phi^n|l \leq \frac{\nu_2}{3} k|A_\gamma^{3/2} \phi^n|^2_{L^2} + ck|f_1(\phi^n)|\nabla \phi^n|^2_{L^2}.
\]

(3.52)

We bound the nonlinear terms as follows:

\[
2k|b_0(u^n, u^n, Au^n)| \leq 2\nu_2 k|u^n|^2_{L^2}||u^n|||Au^n|^3_{L^2} \\
\leq \frac{\nu_1}{4} k|Au^n|^2_{L^2} + ck|u^n|^2_{L^2}||u^n||^4,
\]

(3.53)

\[
2\nu_2 k|b_1(Au^n, \phi^n, \nu_2 A_\gamma \phi^n)| \leq 2\nu_2 k|Au^n||\nabla \phi^n||L^\infty||A_\gamma \phi^n||L^2 \\
\leq c\nu_2 k|Au^n||\phi^n||^1/2|A_\gamma^{3/2} \phi^n||^1/2||A_\gamma \phi^n||^1/2||A_\gamma \phi^n||^1/2 \\
\leq \frac{\nu_2}{3} k|A_\gamma^{3/2} \phi^n|^2_{L^2} + \frac{\nu_1}{4} k|Au^n|^2_{L^2} + ck||\phi^n||^2|A_\gamma \phi^n||^4_{L^2},
\]

(3.54)

\[
2k|b_1(u^n, \phi^n, A^n_2\phi^n)| = 2k|(A^{1/2}_\gamma B_1(u^n, \phi^n), A_\gamma^{3/2}\phi^n)|l \leq 2k|A^{1/2}_\gamma B_1(u^n, \phi^n)||l|A_\gamma^{3/2}\phi^n||l \\
\leq ck||u^n||^2_{L^2}||Au^n||_{L^2}||\phi^n||^1/2|A_\gamma \phi^n||_{L^2}^2 + ck||u^n||^2_{L^2}||u^n||^1/2|A_\gamma \phi^n||_{L^2}^2 \\
+ cK_1^1 k|Au^n||_{L^2}^2 + ckf_1(\phi^n)|\nabla \phi^n||^2_{L^2} + \frac{2}{\nu_1} k|g^n|^2_{L^2},
\]

(3.55)

from which the conclusion of the lemma follows right away. \]

In order to prove the uniform boundedness of $||(u^n, \phi^n)||_{V}$ we will make use of the following two lemmas, whose proofs can be found in [15]:

**Lemma 3.** Given $k > 0$ and positive sequences $\xi_n, \eta_n$ and $\zeta_n$ such that

\[
\xi_n \leq \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta_n, \quad \text{for } n \geq 1,
\]

(3.57)
we have, for any $n \geq 2$,

$$\xi_n \leq \left( \xi_0 + \sum_{i=1}^{n} k\zeta_i \right) \exp\left( \sum_{i=0}^{n-1} k\eta_i \right). \quad (3.58)$$

**Lemma 4.** Given $k > 0$, a positive integer $n_0$, positive sequences $\xi_n$, $\eta_n$ and $\zeta_n$ such that

$$\xi_n \leq \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta_n, \quad \text{for } n \geq n_0,$$  

and given the bounds

$$\sum_{n=n_0}^{N+k_0} k\eta_n \leq a_1, \quad \sum_{n=n_0}^{N+k_0} k\zeta_n \leq a_2, \quad \sum_{n=n_0}^{N+k_0} k\xi_n \leq a_3,$$  

for any $k_0 \geq n_0$, we have,

$$\xi_n \leq \left( \frac{a_3}{Nk} + a_2 \right) e^{a_1}, \quad \forall n \geq N + n_0. \quad (3.61)$$

We are now able to prove the following:

**Proposition 1.** Let $(u_0, \phi_0) \in V$ and $(u^n, \phi^n)$ be the solution of the numerical scheme (3.7). Also, let $k$ be such that

$$k \leq \min \left\{ \frac{1}{K}, 1 \right\} =: \kappa_1. \quad (3.62)$$

Then there exists $K_3\left(\|u_0, \phi_0\|_V, \|g\|_\infty\right)$, such that

$$\|(u^n, \phi^n)\|_V \leq K_3\left(\|u_0, \phi_0\|_V, \|g\|_\infty\right), \forall n \geq 0,$$  

and for all $i = 1, \cdots, m$, we have

$$\sum_{n=1}^{m} \left( \|u^n - u^{n-1}\|^2 + |A_\gamma(\phi^n - \phi^{n-1})|_{L^2}^2 \right) \leq \frac{K_3^2 + cK_1^2K_3^4(m - i + 1)k + \frac{2}{\nu_1} \|g\|^2_\infty (m - i + 1)k}{c^2K_1^2K_3^2 + 2(2c^2 + \alpha^{-2}nu_2^2)K_3^4}(m - i + 1)k. \quad (3.64)$$

Moreover, there exists $K_4 = K_4(\|g\|_\infty)$, independent of the initial data, such that

$$\|(u^n, \phi^n)\|_Y \leq K_4(\|g\|_\infty), \forall n \geq 2N_0 + 1,$$  

where $N_0 := \lfloor T_0/k \rfloor$, with $T_0$ being the time of entering an absorbing ball for $\|(u^n, \phi^n)\|_Y$.

**Proof.** Using (3.39), we infer from (3.49)

$$\|(u^n, \phi^n)\|_V^2 \leq c_4K_1^2k \left( K_2\|(u^{n-1}, \phi^{n-1})\|_V^2 + c_3(|f_\gamma(\phi^n)|_{L^2}^2 + |g^n|_{L^2}^2) \right)^2 + \|(u^{n-1}, \phi^{n-1})\|_V^2 + c_5k \left( |f_\gamma'(\phi^n)\nabla\phi^n|_{L^2}^2 + |g^n|_{L^2}^2 \right)(u^{n-1}, \phi^{n-1})\|_V^2 + \left[ 1 + 2c_4K_1^2K_3^2k(\|(u^{n-1}, \phi^{n-1})\|_V^2 + 2c_5k \left( |f_\gamma'(\phi^n)\nabla\phi^n|_{L^2}^2 + |g^n|_{L^2}^2 \right) \right] \times \left( 2c_5 \right)^2 $$  

which we rewrite in the form

$$\xi_n \leq \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta_n, \quad (3.67)$$
with

\[ \xi_n = \|(u^n, \phi^n)\|_V^2, \quad \eta_n = 2c_4 K_f^2(\|(u^n, \phi^n)\|_V^2), \]
\[ \zeta_n = 2c_2c_4 K_f^2((f_1(\phi^n))^2_2 + g^n)^2_2 + c_5 \left((f_1(\phi^n))_2^2 + (g^n)^2_2\right). \]

Using (3.20), (3.4), (3.5) and recalling (3.3) and the Sobolev imbedding \( H^1(\Omega) \hookrightarrow L^q(\Omega) \), for any \( 1 \leq q \leq \infty \), we obtain

\[ |f_1(\phi^n)|^2_2 \leq \alpha_f^2 \left(|\Omega| + K_1^{2(m+1)}\right) + 2\alpha \nu_2^2 K_1^2, \]
\[ |f_1(\phi^n)|\nabla \phi^n|^2_2 \leq \epsilon \nu_2^2 K_1^2|\phi^n|^2_2 + 2(2\epsilon^2 + \alpha \nu_2^2)K_1^2. \]

Relations (3.69), (3.70), together with (3.4), (3.5), and conclusion (3.58) of Lemma 3 give

\[ \xi_n = \|(u^n, \phi^n)\|_V^2 \leq K_2^2(\|(u_0, \phi_0)\|_V, \|g\|_\infty, 2N_0K), \quad \forall n = 1, \cdots, 2N_0, \]

for some continuous function \( K_2(\cdot, \cdot, \cdot) \), increasing in all its arguments.

In order to derive an upper bound on \( \|(u^n, \phi^n)\|_V \), \( n \geq 2N_0 \), we apply Lemma 4 to (3.67). In order to do so, we recall that \( \|(u^n, \phi^n)\|_Y < \rho_0 \), for \( n \geq N_0 \),

\[ \xi_n = \|(u^n, \phi^n)\|_V^2 \leq \left(\frac{1}{T_0} + a_2\right) \epsilon \alpha_1 =: K_3^2(\|g\|_\infty), \quad \forall n \geq 2N_0 + 1, \]

which is exactly (3.64). Combining (3.72) with (3.71), we obtain conclusion (3.63).

Taking the sum of (3.56) with \( n \) from \( i \) to \( m \) and using (3.63) and (3.70) gives conclusion (3.64) and thus the proof of Proposition 2 is complete. \( \square \)

4. Convergence of Attractors. In this section we address the issue of the convergence of the attractors generated by the discrete system (3.1) to the attractor generated by the continuous system (2.24). Whereas for the continuous system (2.24) one can prove both the existence and uniqueness of the solution (see [10])—and, therefore, define a global attractor—, for the discrete system (3.1) one can prove (using Proposition 1) the uniqueness of the solution provided that \( k \leq \kappa(\|(u_0, \phi_0)\|_Y) \), for some \( \kappa(\|(u_0, \phi_0)\|_Y) > 0 \). Since the time restriction depends on the initial data, one cannot define a single-valued attractor in the classical sense, and this is why we need to use the attractor theory for the so-called multi-valued mappings. Multi-valued dynamical systems have been investigated by many authors (see, e.g., [1], [2], [5], [12], [13], [14]), but in this article we use the tools developed in [6] (see also, [7]) to study the convergence of the discrete (multi-valued) attractors to the continuous (single-valued) attractor. For convenience, we recall those results in Subsection 4.1 and then we apply them to the two-phase flow model in Subsection 4.2.

4.1. Attractors for multi-valued mappings. Throughout this subsection, we consider \( (H, |\cdot|) \) to be a Hilbert space and \( T \) to be either \( R^+ = [0, \infty) \) or \( N \).

Definition 4.1. A one-parameter family of set-valued maps \( S(t) : 2^H \rightarrow 2^H \) is a multi-valued semigroup (m-semigroup) if it satisfies the following properties:

(S.1) \( S(0) = I_{2^H} \) (identity in \( 2^H \));
(S.2) \( S(t + s) = S(t)S(s), \) for all \( t, s \in T \).
Moreover, the m-semigroup is said to be **closed** if \( S(t) \) is a closed map for every \( t \in T \), meaning that if \( x_n \to x \) in \( H \) and \( y_n \in S(t)x_n \) is such that \( y_n \to y \) in \( H \), then \( y \in S(t)x \).

**Definition 4.2.** The **positive orbit** of \( B \), starting at \( t \in T \), is the set

\[
\gamma_t(B) = \bigcup_{\tau \geq t} S(\tau)B,
\]

where

\[
S(t)B = \bigcup_{x \in B} S(t)x.
\]

**Definition 4.3.** For any \( B \in \mathcal{P}H \), the set

\[
\omega(B) = \bigcap_{t \in T} \overline{\gamma_t(B)}
\]

is called the **\( \omega \)-limit set** of \( B \).

**Definition 4.4.** A nonempty set \( B \in \mathcal{P}H \) is **invariant** for \( S(t) \) if

\[
S(t)B = B, \quad \forall t \in T.
\]

**Definition 4.5.** A set \( B_0 \in \mathcal{P}H \) is an **absorbing set** for the m-semigroup \( S(t) \) if for every bounded set \( B \in \mathcal{P}H \) there exists \( t_B \in T \) such that

\[
S(t)B \subset B_0, \quad \forall t \geq t_B.
\]

**Definition 4.6.** A nonempty set \( C \in \mathcal{P}H \) is **attracting** if for every bounded set \( B \) we have

\[
\lim_{t \to \infty} \text{dist}(S(t)B, C) = 0,
\]

where \( \text{dist}(\cdot, \cdot) \) is the **Hausdorff semidistance**, defined as

\[
(4.1) \quad \text{dist}(B, C) = \sup_{b \in B} \inf_{c \in C} |b - c|, \forall B, C \subset H.
\]

**Definition 4.7.** A nonempty compact set \( A \in 2^X \) is said to be the **global attractor** of \( S(t) \) if \( A \) is an invariant attracting set.

**Definition 4.8.** Given a bounded set \( B \in \mathcal{P}H \), the Kuratowski measure of **noncompactness** \( \alpha(B) \) of \( B \) is defined as

\[
\alpha(B) = \inf \{ \delta : B \text{ has a finite cover by balls of diameter less than } \delta \}.
\]

The following theorem, whose proof can be found in [6], gives conditions under which a global attractor exists.

**Theorem 2.** Suppose that the closed m-semigroup \( S(t) \) possesses a bounded absorbing set \( B_0 \in \mathcal{P}H \) and

\[
(4.2) \quad \lim_{t \to \infty} \alpha(S(t)B_0) = 0.
\]
Then \( \omega(B_0) \) is the global attractor of \( S(t) \).

For the purpose of this article, we need to introduce the notion of discrete \( m \)-semigroups. More precisely, we have the following:

**Definition 4.9.** Given a set-valued map \( S : 2^H \to 2^H \), we define a discrete \( m \)-semigroup by

\[
S(n) = S^n, \quad \forall n \in N,
\]

and we will denote it by \( \{S\}_{n \in N} \) (instead of \( \{S^n\}_{n \in N} \)).

**Remark 4.1.** Given two nonempty sets \( B, C \in 2^H \), we write

\[
B - C = \{b - c : b \in B, c \in C\} \quad \text{and} \quad |B| = \sup_{b \in B} |b|.
\]

In order to prove the convergence of the attractors generated by the discrete system (3.1) to the attractor generated by the continuous system (2.24) we will use the following result, whose proof can be found in [6] (see also [20], [18], [7]).

**Theorem 3.** Let \( S(t) \) be a closed \( m \)-semigroup, possessing the global attractor \( A \), and for \( \kappa_0 > 0 \), let \( \{S_k, 0 < k \leq \kappa_0\}_{n \in N} \) be a family of discrete closed \( m \)-semigroups, with global attractor \( A_k \). Assume the following:

1. **(H1) [Uniform boundedness]**: there exists \( \kappa_1 \in (0, \kappa_0] \) such that the set
   \[
   K = \bigcup_{k \in [0, \kappa_1]} A_k
   \]
   is bounded in \( H \);
2. **(H2) [Finite time uniform convergence]**: there exists \( t_0 \geq 0 \) such that for any \( T^* > t_0 \),
   \[
   \lim_{k \to 0} \sup_{x \in A_k, nk \in [t_0, T^*]} |S^0_k x - S(nk)x| = 0.
   \]

Then

\[
\lim_{k \to 0} \text{dist}(A_k, A) = 0,
\]

where \( \text{dist} \) denotes the Hausdorff semidistance defined in (4.1).

**4.2. Application: The two-phase flow model.** Here we will prove that there exists \( \kappa_1 > 0 \) such that if \( 0 < k \leq \kappa_1 \), the system (3.1) generates a closed discrete \( m \)-semigroup \( \{S_k\}_{n \in N} \) with global attractors \( A_k \), that will converge to \( A \) in the sense of Theorem 3.

In order to do that, we define, for \( k > 0 \), the multi-valued map \( S_k : 2^Y \to 2^Y \) as follows: for every \( \tilde{v} = (\tilde{u}, \tilde{\phi}) \in Y \),

\[
S_k \tilde{v} = \{v = (u, \phi) \in V : v \text{ solves } (4.3) \text{ below with time-step } k \}:
\]

\[
\begin{cases}
  u + v_1 kAu + kB_0(u, u) - KkR_0(v_2 A_\gamma \phi, \phi) = \tilde{u} + kg,
  \\
  \mu + v_2 A_\gamma \phi + \alpha f_\gamma(\phi) = \tilde{\mu},
  \\
  \phi + k\mu + kB_1(u, \phi) = \tilde{\phi}.
\end{cases}
\]

(4.3)

Using the same ideas as in [6] (see also [7]), one can prove the following:
Theorem 4. The multi-valued map $S_k$ associated with the implicit Euler scheme (3.1) generates a closed discrete $m$-semigroup $\{S_k\}_{k \in N}$.

Proposition 2. Let $k \leq \kappa_1$, where $\kappa_1$ is given in Proposition 1. Then there exists a constant $R_1 > 0$ such that for every $R \geq 0$ and $\|(u_0, \phi_0)\|_Y \leq R$, there exists $N_1 = N_1(R, k) \geq 1$ such that

$$\|S_k^n(u_0, \phi_0)\|_Y \leq R_1, \quad \forall n \geq N_1. \tag{4.4}$$

Thus, the set

$$B_1 = \{(u, \phi) \in V : \|(u, \phi)\|_Y \leq R_1\}$$

is a $V$-bounded absorbing set for $\{S_k\}_{k \in N}$, for $k \in (0, \kappa_1]$.

Proposition 3. For every $k \in (0, \kappa_1]$, there exists the global attractor $A_k$ of the $m$-semigroup $\{S_k\}_{k \in N}$.

Remark 4.2. Since the global attractor $A_k$ is the smallest closed attracting set of $Y$, Proposition 2 also implies

$$A_k \subset B_1, \forall k \in (0, \kappa_1], \tag{4.5}$$

and thus

$$\bigcup_{k \in (0, \kappa_1]} A_k \subset B_1. \tag{4.6}$$

From relation (4.6) we can see that condition $(H1)$ of Theorem 3 is satisfied. In order to prove that condition $(H2)$ is satisfied we define, for any function $\psi$ and for any $k > 0$, the following:

$$\psi_k(t) = \psi^n, \quad t \in [(n-1)k, nk), \tag{4.7}$$

$$\tilde{\psi}_k(t) = \psi^n + \frac{t-nk}{k}(\psi^n - \psi^{n-1}), \quad t \in [(n-1)k, nk). \tag{4.8}$$

With the above notations, the system (3.1) can be rewritten as follows, for $t \in [(n-1)k, nk)$:

$$\begin{aligned}
\frac{d\tilde{u}_k(t)}{dt} &= \nu_1 A\tilde{u}_k(t) + B_0(\tilde{u}_k(t), \tilde{u}_k(t)) - KR_0(\nu_2 A, \tilde{\phi}_k(t), \tilde{\phi}_k(t)) = g + g_k(t), \\
\frac{d\tilde{\phi}_k(t)}{dt} &= \nu_2 A\gamma \phi_k + \alpha f_k(\phi_k) + B_1(\tilde{u}_k(t), \tilde{\phi}_k(t)) = B_1(\tilde{u}_k(t), \tilde{\phi}_k(t)) - B_1(u_k(t), \phi_k(t)),
\end{aligned} \tag{4.9}$$

where

$$g_k(t) = \nu_1 A(\tilde{u}_k(t) - u_k(t)) + B_0(\tilde{u}_k(t) - u_k(t), \tilde{u}_k(t)) + B_0(u_k(t), \tilde{u}_k(t) - u_k(t)) - K R_0(\nu_2 A, \phi_k(t), \tilde{\phi}_k(t) - \phi_k(t)) \tag{4.10}$$

Subtracting (4.9) from (2.24) and setting

$$\xi_k(t) = u(t) - \tilde{u}_k(t), \quad \eta_k(t) = \phi(t) - \tilde{\phi}_k(t), \tag{4.11}$$
and thus, setting \( N^* = \left\lfloor T^*/k \right\rfloor \) and using (4.21) and (3.64), we obtain
\[
\|g_k\|_{L^2(0,T^*;V')}^2 = \sum_{n=1}^{N^*+1} \int_{(n-1)k}^{nk} \|g_k(t)\|_{V'}^2 \, dt \leq kK_6(T^*),
\]
which proves (4.14).

Now let \( \phi \in D(A_\gamma) \) be such that \(|A_\gamma \phi|_{L^2} \leq 1 \), and let \( t \in [(n-1)k, nk) \) be fixed. Using (2.19), (2.17) and (3.3), we obtain

\[
|b_1(h_k(t) - u_k(t), \tilde{\phi}_k(t), \phi)| \leq cK_1\|u^n - u^{n-1}\|,
\]

which proves (4.14).

We also have

\[
|(A_\gamma(\tilde{\phi}_k(t) - \phi_k(t)), \phi)|_{L^2} \leq |\tilde{\phi}_k(t) - \phi_k(t)|_{L^2} |A_\gamma \phi|_{L^2} \leq |\phi^n - \phi^{n-1}|_{L^2}.
\]

Gathering relations (4.23) and (4.26) we obtain

\[
\|h_k(t)\|_{D(A_\gamma)} \leq c(K_1 + K_3 + K_8)(\|u^n - u^{n-1}\| + |A_\gamma(\phi^n - \phi^{n-1})|_{L^2}),
\]

and recalling (4.27) and (4.64) we obtain (4.15). This completes the proof of the lemma.

We are now in a position to prove that condition (H2) of Theorem 3 is satisfied.

**PROPOSITION 4** (Finite time uniform convergence). For any \( T^* > 0 \) we have

\[
\lim_{k \to 0} \sup_{(u_0, \phi_0) \in A_k, \eta_k \in [0, T^*]} \|S_k^n(u_0, \phi_0) - S(nk)(u_0, \phi_0)\|_{\mathcal{Y}} = 0.
\]

**Proof.** Multiplying the first equation of (4.12) by \( \xi(t) \) and integrating we obtain

\[
\frac{1}{2} \frac{d}{dt} |\xi_k(t)|_{L^2}^2 + \nu_1 \|\xi_k(t)\|^2 + b_0(\xi_k(t), u(t), \xi_k(t)) - \mathcal{K} \left( b_1(\xi_k(t), \phi(t), \nu_2 A_\gamma \eta_k(t)) + b_1(\xi_k(t), \eta_k(t), \nu_2 A_\gamma \tilde{\phi}_k(t)) \right) = -(g_k(t), \xi_k(t))_{L^2}.
\]

Using (2.11) and (2.17) we bound the nonlinear terms as follows:

\[
|b_0(\xi_k(t), u(t), \xi_k(t))| \leq c_6 |\xi_k(t)|_{L^2} \|\xi_k(t)\| \|u(t)\|
\]

\[
\leq \frac{\nu_1}{8} |\xi_k(t)|^2 + c|\xi_k(t)|^2 \|u(t)\|^2,
\]

\[
\mathcal{K} |b_1(\xi_k(t), \phi(t), \nu_2 A_\gamma \eta_k(t))| \leq c_6 \nu_2 |\xi_k(t)|_{L^2}^{1/2} |\xi_k(t)|^{1/2} \|\phi(t)\|^{1/2} |A_\gamma \phi(t)|_{L^2}^{1/2} |A_\gamma \eta_k(t)|_{L^2}
\]

\[
\leq \frac{\nu_1}{4} |A_\gamma \eta_k(t)|_{L^2}^2 + \frac{\nu_1}{8} |\xi_k(t)|^2 + c|\xi_k(t)|^2 \|\phi(t)\|^2 |A_\gamma \phi(t)|_{L^2}^2,
\]

\[
\mathcal{K} |b_1(\xi_k(t), \eta_k(t), \nu_2 A_\gamma \tilde{\phi}_k(t))| \leq c_6 \nu_2 |\xi_k(t)|_{L^2}^{1/2} |\xi_k(t)|^{1/2} \|\eta_k(t)\|^{1/2} |A_\gamma \eta_k(t)|_{L^2}^{1/2} |A_\gamma \tilde{\phi}_k(t)|_{L^2}
\]

\[
\leq \frac{\nu_1}{4} |A_\gamma \eta_k(t)|_{L^2}^2 + \frac{\nu_1}{8} |\xi_k(t)|^2 + c|\xi_k(t)|_{L^2} |\eta_k(t)| |A_\gamma \tilde{\phi}_k(t)|_{L^2}^2.
\]

Using the Cauchy-Schwarz inequality, we bound the right-hand side of (4.20) as

\[
|(g_k(t), \xi_k(t))_{L^2}| \leq \|g_k(t)\|_{V'} \|\xi_k(t)\| \leq \frac{\nu_1}{8} |\xi_k(t)|^2 + c\|g_k(t)\|_{V'}^2.
\]
Relations (4.29)–(4.33) imply
\[
\frac{d}{dt} \|\xi_k(t)\|_{L^2}^2 + \nu_1 \|\xi_k(t)\|^2 \leq c\|\xi_k(t)\|_{L^2}^2 \|u(t)\|^2 + \frac{\nu_2^2}{2}\|A_\gamma \eta_k(t)\|_{L^2}^2 + c\|\xi_k(t)\|_{L^2}^2 \|\phi(t)\|^2 \|A_\gamma \phi(t)\|_{L^2}^2 \nonumber \\
+ c\left(\|\xi_k(t)\|_{L^2}^2 + \|\eta_k(t)\|^2\right) \|A_\gamma \tilde{\phi}_k(t)\|_{L^2}^2 + c\|g_k(t)\|_{L^2}^2.
\]
(4.34)

Now multiplying the second equation of (4.12) by \(\nu_2 A_\gamma \eta_k(t)\) and integrating we obtain
\[
\frac{\nu_2}{2} \frac{d}{dt} \|\eta_k(t)\|_{L^2}^2 + \nu_2^2 \|A_\gamma \eta_k(t)\|_{L^2}^2 + b_1(\xi_k(t), \phi(t), \nu_2 A_\gamma \eta_k(t)) + b_1(\tilde{u}_k(t), \eta_k(t), \nu_2 A_\gamma \eta_k(t)) \nonumber \\
= -\alpha(f_\gamma(\phi(t)) - f_\gamma(\tilde{\phi}_k(t)), \nu_2 A_\gamma \eta_k(t)) - (h_k(t), \nu_2 A_\gamma \eta_k(t))_{L^2}.
\]
(4.35)

Using (2.17) we bound the nonlinear terms as follows:
\[
|b_1(\xi_k(t), \phi(t), \nu_2 A_\gamma \eta_k(t))| \leq c \nu_2 \|\xi_k(t)\|_{L^2} \|\xi_k(t)\|_{L^2} \|\phi(t)\|_{L^2} + \|A_\gamma \phi(t)\|_{L^2} \|A_\gamma \eta_k(t)\|_{L^2} \nonumber \\
\leq \frac{\nu_2^2}{8} \|A_\gamma \eta_k(t)\|_{L^2}^2 + \frac{\nu_1}{4K} \|\xi_k(t)\|_{L^2}^2 + c\|\xi_k(t)\|_{L^2}^2 \|\phi(t)\|_{L^2} \|A_\gamma \phi(t)\|_{L^2}^2,
\]
(4.36)

\[
|b_1(\tilde{u}_k(t), \eta_k(t), \nu_2 A_\gamma \eta_k(t))| \leq c \nu_2 \|\tilde{u}_k(t)\|_{L^2} \|\tilde{u}_k(t)\|_{L^2} \|\eta_k(t)\|_{L^2} + \|A_\gamma \eta_k(t)\|_{L^2}^3 \|\eta_k(t)\|_{L^2} \nonumber \\
\leq \frac{\nu_2^2}{8} \|A_\gamma \eta_k(t)\|_{L^2}^2 + c\|\tilde{u}_k(t)\|_{L^2}^2 \|\tilde{u}_k(t)\|_{L^2}^2 \|\eta_k(t)\|_{L^2}^2.
\]
(4.37)

Using the Cauchy-Schwarz inequality, we bound the right-hand side of (4.35) as
\[
\alpha \|f_\gamma(\phi(t)) - f_\gamma(\tilde{\phi}_k(t)), \nu_2 A_\gamma \eta_k(t)\|_{L^2} \leq \nu_2 \|f_\gamma(\phi(t)) - f_\gamma(\tilde{\phi}_k(t))\|_{L^2} \|A_\gamma \eta_k(t)\|_{L^2} \nonumber \\
\leq \frac{\nu_2^2}{8} \|A_\gamma \eta_k(t)\|_{L^2}^2 + c\|f_\gamma(\phi(t)) - f_\gamma(\tilde{\phi}_k(t))\|_{L^2} \leq \frac{\nu_2^2}{8} \|A_\gamma \eta_k(t)\|_{L^2}^2 + cK_2^2 \|\eta_k(t)\|_{L^2} \nonumber \\
|(h_k(t), \nu_2 A_\gamma \eta_k(t))_{L^2}| \leq \nu_2 \|h_k(t)\|_{D(A_\gamma)'} \|A_\gamma \eta_k(t)\|_{L^2} \nonumber \\
\leq \frac{\nu_2^2}{8} \|A_\gamma \eta_k(t)\|_{L^2}^2 + c\|h_k(t)\|_{D(A_\gamma)'}^2.
\]
(4.39)

Relations (4.35)–(4.39) yield
\[
\frac{\nu_2}{2} \frac{d}{dt} \|\eta_k(t)\|_{L^2}^2 + \frac{\nu_1}{2K} \|\xi_k(t)\|^2 + \frac{\nu_2^2}{2} \|A_\gamma \eta_k(t)\|_{L^2}^2 \leq c\|\xi_k(t)\|_{L^2}^2 \|u(t)\|^2 + \|\xi_k(t)\|_{L^2}^2 \|\phi(t)\|^2 \|A_\gamma \phi(t)\|_{L^2}^2 \nonumber \\
+ c\left(\|\xi_k(t)\|_{L^2}^2 + \|\eta_k(t)\|^2\right) \|A_\gamma \tilde{\phi}_k(t)\|_{L^2}^2 + c\|g_k(t)\|_{L^2}^2 \|\tilde{\phi}_k(t)\|_{L^2}^2.
\]
(4.40)

Dividing (4.34) by \(K\) and adding the resulting equation to (4.40) we obtain
\[
\frac{d}{dt} \|\xi_k(t), \eta_k(t)\|_{L^2}^2 + \frac{\nu_1}{2K} \|\xi_k(t)\|^2 + \frac{\nu_2^2}{2} \|A_\gamma \eta_k(t)\|_{L^2}^2 \leq c\|\xi_k(t)\|_{L^2}^2 \|u(t)\|^2 + \|\xi_k(t)\|_{L^2}^2 \|\phi(t)\|^2 \|A_\gamma \phi(t)\|_{L^2}^2 \nonumber \\
+ c\left(\|\xi_k(t)\|_{L^2}^2 + \|\eta_k(t)\|^2\right) \|A_\gamma \tilde{\phi}_k(t)\|_{L^2}^2 + c\|g_k(t)\|_{L^2}^2 \|\tilde{\phi}_k(t)\|_{L^2}^2 + c\|h_k(t)\|_{L^2}^2 \|\tilde{\phi}_k(t)\|_{L^2}^2 \nonumber \\
+ c\|g_k(t)\|_{L^2}^2 \|\tilde{\phi}_k(t)\|_{L^2}^2 + c\|h_k(t)\|_{L^2}^2 \|\tilde{\phi}_k(t)\|_{L^2}^2.
\]
(4.41)
where
\[ \mathcal{G}(t) = c \left( \|u(t)\|^2 + \|\phi(t)\|^2 + |A\gamma\phi(t)|^2_{L^2} + |A\gamma\bar{\phi}(t)|^2_{L^2} + |\bar{u}_k(t)|^2_{L^2} + \|\bar{u}_k(t)\|^2 + K_2^2 \right). \]

By the Gronwall Lemma and using the fact that \( \xi_k(0) = \eta_k(0) = 0 \), we obtain
\[ (4.42) \| (\xi_k(t), \eta_k(t))_Y \| \leq c \int_0^t \exp \left( \int_{\tau}^t \mathcal{G}(s) \, ds \right) \left( \|g_k(\tau)\|^2_{L^2} + \|h_k(\tau)\|^2_{D(A, Y)} \right) \, d\tau. \]

Using the fact that the solution \((u, \phi)\) of the continuous problem is uniformly bounded in \( Y \) for all \( t \geq 0 \) (cf. [10]), and recalling (4.18) and (3.63), we obtain
\[ (4.43) \int_{\tau}^t \mathcal{G}(s) \, ds \leq c_6, \]
for some constant \( c_6 = c_6(T^*) > 0 \).

Relations (4.42), (4.43), (4.14) and (4.15) give
\[ (4.44) \| (\xi_k(t), \eta_k(t))_Y \| \leq k c_7, \]
and thus
\[ (4.45) \lim_{k \to 0} \sup_{(u_0, \phi_0) \in \mathcal{A}_k} \sup_{nk \in [0, T^*]} \| S^k_{nk}(u_0, \phi_0) - S(nk)(u_0, \phi_0) \|_Y \]
\[ = \lim_{k \to 0} \sup_{(u_0, \phi_0) \in \mathcal{A}_k} \sup_{nk \in [0, T^*]} \sup_{(u^n, \phi^n) \in S^k_{nk}(u_0, \phi_0)} \| (u^n, \phi^n) - (u(nk), \phi(nk)) \|_Y \]
\[ = \lim_{k \to 0} \sup_{(u_0, \phi_0) \in \mathcal{A}_k} \sup_{nk \in [0, T^*]} \sup_{(u^n, \phi^n) \in S^k_{nk}(u_0, \phi_0)} \| (\bar{u}_k(nk), \bar{\phi}_k(nk)) - (u(nk), \phi(nk)) \|_Y \]
\[ = \lim_{k \to 0} \sup_{(u_0, \phi_0) \in \mathcal{A}_k} \sup_{nk \in [0, T^*]} \sup_{(u^n, \phi^n) \in S^k_{nk}(u_0, \phi_0)} \| (\xi_k(nk), \eta_k(nk)) \|_Y = 0, \]
which concludes the proof of the lemma.

Having proved that conditions (H1) and (H2) of Theorem 3 are satisfied we also obtain that the discrete attractors converge to the continuous attractor as the time-step approaches zero. More precisely, we have the following:

**Theorem 5.** The family of attractors \( \{\mathcal{A}_k\}_{k \in (0, \kappa_1]} \) converges, as \( k \to 0 \), to \( \mathcal{A} \), in the following sense:

\[ \lim_{k \to 0} \text{dist}(\mathcal{A}_k, \mathcal{A}) = 0, \]

where dist denotes the Hausdorff semidistance in \( Y \), namely
\[ \text{dist}(\mathcal{A}_k, \mathcal{A}) = \sup_{x_k \in \mathcal{A}_k} \inf_{x \in \mathcal{A}} \| x_k - x \|_Y. \]

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