A Noise Reduced Hard Thresholding method for Noisy Sparse Reconstruction

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Abstract. The problem of noisy sparse signal recovery has been studied in this paper. The generalized expectation maximization (GEM) algorithm provides solutions over state of the art convex relaxation methods for sparse signal recovery by considering the noisy measurements as missing data. However, in the case of images, artifacts in the reconstructed signal renders it to have low quality, especially based on qualitative performance metrics and so are unsuitable for further application specific processing. The problem may be reduced significantly by minimizing the total variation in the signal considering the constraints created by the statistics of noise in the reconstructed image. A modified iterative GEM for low noise reconstruction is presented and analyzed. Even though time consuming, simulations and numerical analysis show significantly higher performance metrics, especially PSNR and Structural Similarity, which imply reconstructed images may be suitable for further processing.

1. Introduction

The popular Shannon/Nyquist sampling theorem requires that a signal needs to be sampled at approximately twice the highest frequency components for accurate reconstruction. In certain applications like digital imaging and video, the Nyquist rate tends to be very high and hence, the number of samples acquired per seconds also increases. Storage and transmission of such high volume data becomes difficult, leading to the requirement of compression. Increasing the sampling rate also turns out to be expensive in terms of its computational complexity. The method of Compressive Sensing overcome these disadvantages wherein accurate reconstruction of the signal is possible even if the sampling frequency is less than twice the maximum frequency. Signals are acquired compressed, using non adaptive linear projection methods and are reconstructed using an optimization techniques like convex relaxation, greedy pursuit and probabilistic models [1–3].

Among various thresholding methods for the reconstruction of sparse signals, the GEM hard thresholding [4] uses the principle of noise variance. The sparse signal and the noise variance are found out by the GEM algorithm and the signal is reconstructed by probabilistic model with $L_0$ - norm. The reconstructed images using GEM algorithm is observed to exhibit low quality
in terms of qualitative metrics, rendering such reconstructed images unsuitable for further application specific processing. This paper aims to increase the overall reconstructed image quality especially in the higher quantization levels by reducing noise effects in the reconstructed signal based on multi-grid noise reduction algorithms [5]. Performance based on Structural Similarity Index [12] is also reported.

2. Background Theories

2.1. The GEM Sparse Recovery Algorithm

Compressive sensing theory enables a sparse or compressible signal to be acquired with far fewer measurements than what is required as per the Nyquist/Shannon sampling theory. A signal is generally modeled as

\[ y = Ax + b \]  

where \( y = [y_1, y_2, \ldots, y_n]^T \), \( A \) is the \( NXM \) sensing matrix, \( x \) is the unknown sparse vector of dimension \( MX1 \) containing \( r \) non zero elements (\( rM \)), \( b \) is the additive white Gaussian noise with zero mean and covariance matrix \( \sigma^2 \). Here the noise variance is assumed to be unknown.

In this model there are two unknown parameters \( x \) and \( \sigma^2 \). Then the set of unknown parameters can be written as

\[ \Theta = (x, \sigma^2) \in \Theta_r \]  

where \( \Theta_r \) is the parameter space. Parameter space means it is the set of all possible combinations of values for all the different parameters in the model. This can be written as

\[ \Theta_r = S_r x(\theta, +\alpha) \]  

where \( S_r \) is the sparse signal parameter space and \( r \) is the sparsity level. Here, assuming that the sparsity level is known and the noise variance \( \sigma^2 \) is positive, the elements of \( y \) are quantized into code word \( b \), represented as \([b_1, b_2b_N]^T\), where \( b_i \) indicate the quantization interval.

\[ y_i \in D(b_i) = [l(b_i), u(b_i)] = [l_i, u_i], l_i < u_i, i = 1, 2N \]  

where \( l_i \) and \( u_i \) are the lower and upper boundaries of the quantization interval. Our aim is to estimate the parameters \( \theta \) from the quantized data \( b \) and the unobserved or the missing data \( y \). For finding the missing data the concept of GEM algorithm was developed [4]. First the joint distribution of the observed data \( b \) and the missing data \( y \) given the unknown parameters \( \theta \) are solved for [4]. Then the marginal log likelihood function of \( \theta \) is obtained by integrating \( y \) from the joint distribution of \( b \) and \( y \).

\[ L(\theta) = ln(P_{b/\theta}(b/\theta)) \]  

For the computation of \( L(\theta) \) the noise variance \( \sigma^2 \) is positive and set the parameter space accordingly in (3). The maximum likelihood (ML) estimate of \( \theta \) is found out by maximizing \( L() \). The exact ML estimate requires a combinatorial search and it is difficult task.

Solutions are obtained in two steps: expectation (E) - step and maximization (M) - step. In the expectation step, the missing data are estimated and in maximization step the likelihood
function is increased under the assumption that the missing data is known. Assuming that the parameter estimate \( \hat{p} = (x(p), \hat{y}(p)) \) is known, where \( p \) indicate the iteration index, the E-step can be found out by finding the expected complete data log likelihood function of (5).

\[
Q(\hat{\theta}(p)) = E_{y/b} \left[ \ln \frac{y/b}{y/b} \right] = -\frac{1}{2N} \ln(2\pi\sigma^2) - E_{y/b} [(y - H_s)^T(y - H_s)/b, (p)]/(2\sigma^2)
\]

(6)

Expectation step reduces the Bayesian minimum mean square error and variance using the mean and variance of the truncated pdf [6]. Similarly the M-step increases the expected complete data log likelihood function. Here finding the new parameter estimate by the equation

\[
x^{(p+1)} = Tr(s^{(p)} + (1/c^2)H^T(\hat{y}^{(p)} - H_s))
\]

(7)

c being the step size coefficient which should satisfy the inequality

\[
c \geq \rho H
\]

(8)

where \( \rho H \) denotes the largest singular value of \( H \) and \( T_r \) in (7) is the hard thresholding operator. So we get the new parameter \( (p + 1) = (s^{(p + 1)}, \hat{y}^{(p + 1)}) \). The iteration between \( E \) and \( M \) is repeated until two consecutive sparse signal estimates \( x^p \) and \( x^{(p+1)} \) do not differ significantly. The signal is reconstructed by \( l_0 \) - norm which counts the number of non-zero entries in \( s \). This optimization can recover a sparse signal exactly. However, the PSNR of such recovered signal turns out to be low, rendering the signal not suitable for further processing.

2.2. Noise Reduction Methods

The ROF Model [8] considers reducing the total variation or integral of the absolute gradient, thereby reducing unwanted detail preserving edge informations. Given an input signal \( x \), the total variation denoising aims to find an approximate \( y \), with smaller total variation than \( x \), but at the same time close to that of \( x \). Sum of square errors being a measure of closeness, the total variation denoising problem reduces to minimizing the function

\[
E(x, y) = \lambda V(y)
\]

(9)

where \( \lambda \) is termed the regularization parameter and plays a critical role in the denoising algorithm. Differentiating this equation with respect to \( y \) a Lengrange equation is arrived at which can be solved for to obtain a solution for \( y \). Various approaches for arriving at a solution has been reported in literature [5–7].

3. The Proposed Model

The sparse signal obtained from the GEM Hard Threshold may be modeled as in 10 to contain an unknown noise \( \eta \).

\[
Y(x, y) = X(x, y) + \eta(x, y)
\]

(10)

The aim is to compute \( x \) of the nonlinear equation 11 with different LaGrange multipliers \( \lambda \), so as to satisfy the constrained condition in 12.
\[ L_\lambda(x) = -\nabla \left( \frac{\nabla x}{\sqrt{(\nabla x)^2 + \varepsilon}} \right) + \lambda (x - y) = 0 \] 

\[ F(x) = \frac{1}{2} (\| x - y \|^2 - \sigma^2) = 0 \]

where \( \sigma \) is the variance between the noisy image and the original image and \( \lambda \) is the regularization parameter.

The algorithm is iterative; by first computing weighted \( \lambda \) to further, solve the linear equation (11) to obtain the unique solution \( x_\lambda_k \). The steps continue until no significant difference in values of \( x \) is observed. Numerical examples in the subsequent sections prove this solution to produce higher PSNR to recovered sparse signals, and also higher values of Structural Similarity Index, thereby rendering them useful to further processing.

4. Simulation Results and Discussions

Two dimensional signals have been considered to test the method with a peak signal-to-noise ratio (PSNR) and Structural Similarity Index (SSIM) as performance metrics. Images of size 256X256 were considered and testing was performed on an intel dual core machine with 2Gb RAM and Matlab(R)2011b. The two unknown parameters i.e; sparse signal and the noise variance was computed by the proposed Low-Noise GEM algorithm. The smoothening parameter are kept fixed; upon achieving the residue to the constrained function, both the diffusion equation and the Lagrange multiplier are stopped. The signal was then reconstructed using \( l_0 \) norm technique and empirical Bayesian signal estimate. Comparisons on PSNR and SSIM indexes of the proposed algorithm against the state of the art GEM hard thresholding is reported and discussed.

| Quantization Level | PSNR GEM | SSIM GEM | PSNR Proposed | SSIM Proposed |
|---------------------|----------|----------|---------------|---------------|
| 3                   | 55.2     | 0.39     | 76.52         | 0.56          |
| 4                   | 56.5     | 0.43     | 76.78         | 0.61          |
| 8                   | 58.3     | 0.48     | 78.02         | 0.69          |
| 16                  | 60.4     | 0.53     | 79.37         | 0.73          |

The Bird image in Figure 1 shows (a) the original image (b) sampled and reconstructed using the GEM hard threshold and (c) the proposed low-noise hard threshold solving for the Lagrange Multiplier in 100 iterations. Atificats introduced in (b) is significantly reduced in (c), as visible to the naked eye. Significant improvement in qualitative metrics is also observed. However, some noise with the Salt and Pepper nature is observed in lower quantization levels. This effect is significantly reduced at higher quantization levels as seen in Figure 2(b). Again, since the number of divisions to solve the non-linear equations are lower, a faster convergence is observed
at lower bins. This might also enable faster and easy processing in the event of system on chip developments for standalone systems.

Figure 3 show results pertaining to simulations on complex valued biomedical images. It
is observed that results correspond to observations as in natural images. Such significant improvement renders compressed sense reconstructed bio-medical feasible for clinical applications. It may be observed as in Table that at higher quantization levels like 16 and more, the GEM algorithm provides equally good PSNRs as with the proposed method. However, at lower quantization levels, the proposed method provides considerably better PSNRs, make them usable for further processing.

| Quantization Level | PSNR GEM | SSIM GEM | PSNR Proposed | SSIM Proposed |
|--------------------|----------|----------|---------------|---------------|
| 3                  | 54.8     | 0.54     | 72.39         | 0.64          |
| 4                  | 55.5     | 0.61     | 73.88         | 0.73          |
| 8                  | 58.0     | 0.69     | 75.87         | 0.81          |
| 16                 | 60.4     | 0.75     | 77.03         | 0.89          |

Figure 3: Complex Valued Brain Image
Table 3: PSNR and SSIM values of Brain

| Quantization Level | PSNR GEM | SSIM GEM | PSNR Proposed | SSIM Proposed |
|--------------------|----------|----------|---------------|---------------|
| 3                  | 62.8     | 0.58     | 72.78         | 0.67          |
| 4                  | 63.5     | 0.62     | 74.64         | 0.70          |
| 8                  | 65.2     | 0.71     | 76.44         | 0.79          |
| 16                 | 67.4     | 0.79     | 77.55         | 0.87          |

5. Conclusions

The issue of accurate recovery of noise corrupted sparse signal has been reviewed in this paper. The generalized expectation-maximization (GEM) hard thresholding reconstruction algorithm which automates noise parameter estimation from quantized measurements, reconstructs noisy sparse signals, however, with artifacts. This might render the reconstructed images not performing up to expectations in further application specific processing. This problem, as reported, can be reduced by minimizing the total variance in the reconstructed signal. An iterative Low-Noise GEM Hard Threshold, which minimizes the variance in the signal has been proposed. Simulations on both natural and biomedical images show that increased PSNR and signal quality can be achieved at comparable computational complexity, thus paving way for better Compressive Sensing applications to be explored.

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