Abstract

We reformulate the Shiromizu, Maeda and Sasaki (SMS) braneworlds within the framework of the five-dimensional Einstein equations. In many applications of the braneworld Einstein field equations, the Weyl term is attributed to the bulk, thus splitting the non-Einsteinian terms into ‘bulk’ and ‘brane’ terms. Here by employing standard geometrical identities, we show that such a split is non-unique, since they get mixed up in different formulations. An important consequence of this non-uniqueness is that even though the full brane-bulk systems in all such formulations are completely equivalent, important differences can arise were one to truncate different formulations by throwing away the associated ‘bulk’ terms. This is particularly likely to be the case in more general anisotropic/inhomogeneous settings with non-AdS bulks, in which the common truncation of the SMS (which throws away the Weyl term) would not coincide with the full system. We emphasize that rather than providing support for any particular truncation, these differences show clearly the dangers of using any truncated equations and provide a strong argument in favour of studying the full brane-bulk system. The different formulations we provide also permit different ways of approaching the full brane-bulk system which may greatly facilitate its study. An example of this is the second-order nature of the formulations given here as opposed to SMS’s formulation which is third-order.

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1 Introduction

Braneworlds are often studied within the framework of the 5-d Einstein Field Equations (EFE), a prime example being the general formulation of Shiromizu, Maeda and Sasaki (SMS) [1]. According to this, the observed universe is confined to a 4-d brane, which is a particular timelike hypersurface embedded within a 5-d bulk. On the brane, the 10 4-d Einstein field Equations (EFE) are replaced by the 10 SMS braneworld EFE (BEFE) [1]. There are two important ways in which these BEFE differ from the EFE. Firstly, these BEFE do not constitute a closed system, since they contain an unspecified ‘electric’ Weyl tensor term, which can only be specified in terms of the 5-d bulk. Secondly, they contain a term quadratic in the energy-momentum tensor of the brane, which can have important consequences for the evolution of very early universe models [2, 3]. This term originates from the junction conditions which follow from the assumption that the brane is a thin matter sheet embedded in the bulk, about which there is a $\mathbb{Z}_2$ reflection symmetry [1]. Given the complexity of the full SMSbrane-bulk system of equations, the usual approaches in studies of the braneworld models effectively involve throwing away the Weyl term using some plausibility arguments, or leaving all or part of it as some free unknown. However, it is generally accepted that in principle none of these approaches are entirely satisfactory [4].

Here we reformulate the SMS braneworlds by means of geometrical identities. Whereas the non-Einsteinian terms in SMS’s BEFE are often interpreted as a ‘bulk’ term (the Weyl term) which is then often discarded and ‘brane’ terms to be kept, we demonstrate by our reformulations that such a split of the non-Einsteinian terms of the BEFE into ‘bulk’ and ‘brane’ parts is non-unique. Clearly all such reformulations and splits (including SMS’s) give completely equivalent systems, so long as the full brane-bulk systems is considered. Important differences would, however, arise were one to truncate different formulations by throwing away the corresponding ‘bulk’ parts. This demonstrates clearly that different truncations result in different imprints on the residual ‘braneworld physics’ and therefore all truncations are unsatisfactory. Only the study of the full brane-bulk system is free from such ambiguities.

The different formulations given here also provide different ways of studying the full system which may be more convenient to study the brane-bulk system than SMS’s formulation. An example of this is that many of our formulations are second-order unlike the third-order SMS system.

In Sec 2 we summarize the steps involved in deriving the SMS braneworld EFE. In Sec 3 we use a brief account of the origin of the junction conditions to demonstrate some of the non-uniqueness in the split of the non-Einsteinian terms in the BEFE into ‘bulk’ and ‘brane’ terms. In Sec 4 we give two formulations of the BEFE with no explicit quadratic terms, the second of which has no explicit Weyl term either. This demonstrates how completely equivalent formulations of the full system would give different (in general unsatisfactory) truncated braneworld equations were one to remove the corresponding ‘bulk terms’. We conclude in Sec 5.
2 Braneworld Einstein Field Equations in brief

The derivation of the SMS braneworld EFE begins with the split of 5-d spacetime w.r.t a timelike hypersurface $\mathcal{Y}$ with spacelike normal $n^A$. The 5-d metric $g_{AB}$ is thus split according to

$$g_{AB} = \left( \begin{array}{cc} \beta_1 \beta_1 + \alpha^2 & \beta_b \\ \beta_a & h_{ab} \end{array} \right),$$

where $h_{ab}$ is the metric induced on the hypersurface. The extrinsic curvature of the hypersurface relative to the embedding spacetime is

$$K_{ab} = -\frac{1}{2\alpha} \delta h_{ab},$$

where the hypersurface derivative is given by $\delta \equiv \frac{\partial}{\partial z} - L_\beta$, with $L_\beta$ denoting the Lie derivative w.r.t $\beta_i$. In normal coordinates, $\alpha = 1$ and $\beta_i = 0$, so $\delta = \frac{\partial}{\partial z}$.

The three projections of the Riemann tensor are respectively the Gauss, Codazzi and Ricci equations

$$\mathcal{R}_{abcd} = R_{abcd} - 2K_{a[c}K_{d]b},$$

$$\mathcal{R}_{\perp a b c} = -2D_eK_{b[c},$$

$$\mathcal{R}_{\perp a \perp b} = \frac{1}{\alpha}(\delta K_{ab} - D_bD_a\alpha) + K_a^cK_{cb},$$

where the tensorial objects represented by the capital Roman and Calligraphic letters represent the usual quantities in 4-d and 5-d respectively. $D_\alpha$ denotes the 4-d covariant derivative, and where we have used $O_{...\alpha...} = O_{...A...}h_A^\alpha$ for projections onto the hypersurface $\mathcal{Y}$, and $O_{...\perp...} \equiv O_{...A...}n^A$ for projections onto the normals. As an example of this notation, we use $\mathcal{C}_{\alpha\beta} = \mathcal{C}_{\perp\alpha\perp\beta} = \mathcal{C}_{ABCD}h_A^\alpha h_B^\beta n^C n^D$ for the ‘electic’ part of the Weyl tensor. Contracting each of these equations once results in

$$\mathcal{R}_{bd} - \mathcal{R}_{\perp b \perp d} = R_{bd} - (K K_{bd} - K_b^cK_{cd}),$$

$$G_{\perp} = \mathcal{C}_{\perp} = -(D_bK^b_a - D_aK),$$

$$\mathcal{R}_{\perp \perp} = \frac{1}{\alpha}(\delta K - D^2\alpha) - K \odot K,$$

where $K \odot K \equiv K_{ij}K^{ij}$. Contracting the Gauss equation a second time gives

$$-2G_{\perp \perp} = \mathcal{R} - 2\mathcal{R}_{\perp \perp} = R - K^2 + K \odot K.$$  \hspace{1cm} (9)

The above equations are also related to the projections of the Einstein tensor $G_{AB}$ as shown.

Now to obtain 4-d BEFE, one begins by constructing the 4-d Einstein tensor from the contracted Gauss equation (6) and the doubly-contracted Gauss equation (9):

$$G_{ab} = G_{ab} - \mathcal{R}_{\perp a \perp b} + \mathcal{R}_{\perp \perp} h_{ab} + K K_{ab} + K_a^cK_{bc} - \frac{K^2 - K \odot K}{2} h_{ab}.$$  \hspace{1cm} (10)

SMS now take three steps in order to derive their formulation of the BEFE, and a fourth step is then often used in practice.

**Step I:** Using the definition of the Weyl tensor, $\mathcal{R}_{\perp a \perp b}$ is replaced by the ‘electric’ part of the Weyl tensor $\mathcal{E}_{ab} \equiv \mathcal{C}_{\perp a \perp b}$ and extra terms built from the projections of $\mathcal{R}_{AB}$.

**Step II:** The 5-d EFE are then assumed, which permits one to exchange all remaining projections of $\mathcal{R}_{AB}$ for 5-d energy-momentum terms. Only when this is carried out does (10) become a system of field equations rather than of geometrical identities. We refer to such field equations as timelike hypersurface EFE (THEFE) since at first sight they resemble the 4-d EFE. Together with the 5 Gauss and Codazzi constraints that one forms from using the EFE on (9) and (7), the 10 THEFE form a system of 15 equations in place of the 15 5-d EFE. The 5 constraints may be seen as consistency conditions on the hypersurface. For zero 5-d energy-momentum components $T_{a\perp}$, the last 4 of these correspond to 4-d energy-momentum conservation on the brane.

**Step III:** To derive braneworld EFE, which are a special subcase of THEFE, one uses normal coordinates and chooses the braneworld energy-momentum tensor ansatz

$$T_{AB} = Y_{AB}\delta(z) - \Lambda g_{AB}, \quad Y_{AB} = (T_{AB} - \lambda h_{AB}), \quad T_{AB}n^A = 0,$$  \hspace{1cm} (11)

where $T_{ab}$ is the energy-momentum of the matter confined to the brane. $\Lambda$ and $\lambda$ are 5-d and 4-d cosmological constants respectively. One then adopts junction conditions (JC) to hold across the brane, in particular

$$[K_{ab}]^+ = K_{ab}^+ - K_{ab}^- = \kappa^2 \left( Y_{ab} - \frac{Y}{h_{ab}} \right).$$  \hspace{1cm} (12)
(where we make explicit a $\kappa^2_5$ proportional to the 5-d gravitational constant). With the additional supposition of $Z_2$ symmetry\footnote{The difference in the sign of (12) between this letter and SMS’s paper is due to our use of the opposite sign convention in the definition of extrinsic curvature. We compensate for this in subsequent formulae by also defining $K_{ab} = -K^+_{ab}$ rather than $+K^+_{ab}$.} i.e $-K_{ab} \equiv K^+_{ab} = -K_{ab}$

$$K_{ab} = -\frac{\kappa^2_5}{2} \left( Y_{ab} - \frac{Y}{3} h_{ab} \right) = -\frac{\kappa^2_5}{2} \left( T_{ab} - \frac{T - \lambda}{3} h_{ab} \right)$$

(13)

as derived in the next section. The SMS braneworld EFE then read

$$G_{ab} = L_{ab}^{\text{SMS}}(T) + Q_{ab}^{\text{SMS}}(T) - \mathcal{E}_{ab},$$

(14)

where $Q_{ab}(T)$ is the quadratic term and $L_{ab}(T)$ is the linear (together with zeroth order) term in $T_{ab}$ respectively, given by

$$Q_{ab}^{\text{SMS}} = \kappa^2_5 \left[ \frac{T_{12}}{12} T_{ab} - \frac{1}{4} T_{ad} T_{db} + \left( \frac{T \circ T}{8} - \frac{T^2}{24} \right) h_{ab} \right],$$

(15)

$$L_{ab}^{\text{SMS}} = -\frac{\kappa^2_5}{2} \left( \Lambda + \frac{\kappa^2_5}{6} \lambda^3 \right) h_{ab} + \frac{\kappa^2_5}{6} \lambda T_{ab}.$$  

(16)

\textbf{Step IV:} In contrast to the 4-d EFE, the SMS braneworld EFE are not closed since $\mathcal{E}_{ab}$ is unspecified. To close the system, SMS write down further equations (which give a large third-order brane-bulk system) for the ‘evolution’ away from the timelike brane of the ‘electric’ and ‘magnetic’ parts of the manifestly 5-d Weyl tensor \cite{1}. Given the complexity of this full brane-bulk third-order system, other workers have often treated the SMS braneworld EFE alone. This involves the ad hoc prescription of the functional form of $\mathcal{E}_{ab}$. This is sometimes completely thrown away (see e.g \cite{5}). It is sometimes decomposed according to a standard procedure \cite{4}. Because the original functional form is unknown, the functional form of each of the parts defined by the decomposition is also unknown.\footnote{The unknowns are sometimes kept, e.g as the ‘Weyl charge’ for black holes in \cite{6}.} Some parts are then set to zero and others are restricted by unjustified but convenient ansätze. In particular the anisotropic stress stress part $P_{ab}$ is sometimes set to 0 (see e.g \cite{7}) or otherwise restricted \cite{8}. The radiative perfect fluid part is often kept, but is then argued to be small (despite containing an unknown factor) in the inflationary \cite{3} and perturbative \cite{4, 9} treatments. Having dealt with the Weyl term in such a way, the above form of $Q_{ab}^{\text{SMS}}$ is then often taken to be uniquely defined and is the starting-point of many works in brane cosmology.

As we shall see in the next section, the split into a term which ‘characterizes the bulk’ $-\mathcal{E}_{ab} = P_{ab}^{\text{SMS}}$ and a term ‘on the brane’ $L_{ab}^{\text{SMS}}(T) + Q_{ab}^{\text{SMS}}(T)$ is in fact highly non-unique.

### 3 Non-uniqueness of split of BEFE terms into ‘brane’ and ‘bulk’ terms

The Weyl term in the SMS braneworld EFE (14) has been the subject of much mystery. Our aim here is not to argue about what form the Weyl term may take in particular solutions of the system (e.g zero everywhere). Rather we show that different formulations of the full brane-bulk system exist which - although completely equivalent - lead to very different splits of the non-Einsteinian terms in the BEFE into ‘brane’ and ‘bulk’ terms. Some of these reformulations have no explicit Weyl term present in the BEFE.

To proceed we first note that the issue of the presence of a Weyl term in systems derived from splitting the Einstein equations w.r.t non-null hypersurfaces is an old one, which has nothing to do with the brane energy-momentum ansatz, since the Weyl term is present from the start in (10). At the relevant level, it also has nothing to do with the signature and dimension of the spacetime and the signature of the codimension 1 hypersurface (an example of which is the brane) w.r.t which the split is performed. Thus as discussed in \cite{10} the presence of such a Weyl term has been considered in the development of the GR Cauchy problem. As we recollect below, exactly the same procedure is used in the derivation of the junction conditions \cite{11}. \textit{The Weyl term does not occur if one chooses a formulation in which it is removed early on by use of the Ricci equation.} If this is not done, as is the case in the SMS formulation, then one requires subsequent use of Bianchi identities and the Ricci equation in order to close the system at third order.

We next provide the derivation of the junction conditions to illustrate that the suggested use of the Ricci equation is entirely natural. This derivation also demonstrates some of the different ways in which the SMS construction of the BEFE is non-unique. While such formulations are equivalent to SMS’s, each has a distinct split into ‘brane’ and ‘bulk’ terms. We will then systematically list and explain the sources of non-uniqueness.

In embedding a 4-d timelike thin matter sheet in a 5-d spacetime manifold, whereas to have well-defined geometry one requires the metric to be continuous across the thin matter sheet yielding the JC
\[ [h_{ab}]^+ = 0, \]  
(17)
discontinuities in certain derivatives of the metric are permissible. Consider the 3 projections of \( G_{AB} \), one uses (9) and (7) to obtain \( G_{a\perp} \) and \( G_{\perp a} \) and for \( G_{ab} \) one proceeds as SMS do by forming (10) and then applying the following steps:

**Step V:** The Ricci equation (5) is used to remove all the \( R_{\perp a \perp b} \).

**Step VI:** The contracted Ricci equation (8) is used to remove all the \( R_{\perp \perp} \). Thus one arrives at
\[ G_{ab} = G_{ab} - \left( \frac{\delta K_a - D_a D_b \alpha}{\alpha} + K_a d K_{bd} \right) + \left( \frac{\delta K - D^2 \alpha}{\alpha} - K \circ K \right) h_{ab} + K K_{bd} - K_a d K_{bd} - \frac{K^2 - K \circ K}{2} h_{bd}. \]  
(18)
Then passing to normal coordinates and rearranging via the definition of extrinsic curvature (2) to form the completed normal derivative \( \frac{\partial}{\partial z} (K_a - h_{ab} K) \), one obtains the following geometrical identity:
\[ G_{ab} = G_{ab} - \left( 2 K K_a + 2 K_a d K_{bd} + \frac{K^2 + K \circ K}{2} h_{ab} + \frac{\partial}{\partial z} (K_a - h_{ab} K) \right). \]  
(19)
Now performing \( \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} G_{AB} d\epsilon \) one obtains the JC
\[ [G_{a\perp}]^+ = 0, \quad [G_{a\perp}]^+ = 0, \]  
(20)
\[ [G_{ab}]^+ = [K_a - h_{ab} K]^+. \]  
(21)
If one now uses the 5-d EFE (Step II), with thin matter sheet energy-momentum
\[ Y_{AB} = \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} T_{AB} d\epsilon, \]  
(22)
one obtains the JC
\[ 0 = Y_{\perp \perp}, \quad 0 = Y_{a \perp}, \]  
(23)
from (20) and additionally, via a trace reversal, the JC (12) from (21).

Clearly this is a different choice of procedure from that used by SMS to formulate their THEFE. This illustrates that one has a number of choices as to what geometrical objects are present in the THEFE, and hence in the BEFE. The terms in the BEFE that require the knowledge of the bulk constitute the ‘bulk’ part of the bulk-brane split. The terms in question are higher-dimensional curvature terms, in particular those terms (such as the Weyl term in SMS’s formulation) which cannot be replaced via the 5-d EFE, and normal derivatives of objects such as the extrinsic curvature [which occur in Eq. (19)]. The key point is that we can change around which of these objects occur in the THEFE by using well-known geometrical identities and do so changes the form of the term quadratic in \( K_a \), and hence the term quadratic in \( T_{ab} \) when one passes to the BEFE by adopting the braneworld energy-momentum tensor ansatz (11). For example, using the set of steps at the start of this section leading to Eq. (19), the EFE and the braneworld energy–momentum ansatz lead to BEFE of the following form
\[ G_{ab} = L_{ab} + Q_{ab} + B_{ab}, \]  
(24)
with
\[ Q_{ab} = -\frac{\kappa^4}{72 \sqrt{2}} \left[ 36 T_a d T_{bd} - 15 T T_{ab} + (9 T \circ T + 5 T^2) h_{ab} \right], \]  
(25)
\[ L_{ab} = -\frac{\kappa^4}{9} [12 T_{ab} \lambda - (7 T \lambda - 8 \lambda^2) h_{ab} + \kappa^2 (2 T_{ab} - (\lambda + \Lambda) h_{ab}]], \]  
(26)
\[ B_{ab} = \frac{\partial}{\partial z} (K h_{ab} - K_{ab}) \]  
(27)
where \( L_{ab} \) are the terms linear in the brane energy–momentum and \( B_{ab} \) is the ‘bulk’ part of the split.

It is now clear that the split into \( B_{ab} \) and \( L_{ab} + Q_{ab} \) is non-unique, since the above split is clearly not the same as that due to SMS. Rather, the outcome of the split depends on which geometrical objects are used in the formulation.

With what has been clarified by the above recollection of the derivation of the JC in mind, we now provide a comprehensive list of steps that may in general be applied in the construction of BEFE.

Steps I and II together mean that the Weyl ‘bulk’ term \( \varepsilon_{ab} \) is equivalent to a Riemann ‘bulk’ term together with matter terms. This swap by itself involves no terms which are quadratic in the extrinsic curvature.
Step V says that the Riemann ‘bulk’ term is equivalent to a ‘bulk’ term containing of hypersurface derivative of the extrinsic curvature together with a $K_{ab}K^a_b$ term. Steps VI and II together say that the hypersurface derivative of the trace of the extrinsic curvature is equivalent to a matter term together with a $K \circ K$ term. The idea is to use Steps VI and II, and Step V, on arbitrary proportions (parametrized by $\mu$ and $\nu$) of $R_{\perp a \perp b}$ and of $R_{\perp \perp}$:

$$G_{ab} = G_{ab} - (1 + \nu)R_{\perp a \perp b} + (1 - \mu)R_{\perp \perp}h_{ab} + \frac{1}{2} \left[ \nu(\delta K_{ab} - D_a D_b \alpha) + \mu(\delta K - D^2 \alpha)h_{ab} \right] + \left( K \delta K_{ab} + (\nu - 1)K_a \delta K_{bd} - \frac{K^2}{2}h_{ab} + \left( \frac{1}{2} - \mu \right)K \circ K_{ab} \right).$$  \hspace{1cm} (28)

**Step VII:** We are furthermore free to choose to characterize the bulk in terms of hypersurface derivatives $\delta$ (which are normal derivatives $\frac{\partial}{\partial z}$ in normal coordinates) of objects which may be related to the extrinsic curvature by use of the metric tensor. Relating the hypersurface derivatives of these new objects to those of the extrinsic curvature requires taking metrics inside or outside the hypersurface derivative. Examples of such moves, which all follow from the product rule and the definition of extrinsic curvature, are (i)

$$\delta K = h^{ab}\delta K_{ab} + 2\alpha K \circ K.$$ \hspace{1cm} (29)

(ii) We could choose to work with objects with raised indices e.g. $K^\alpha _{\beta}$:

$$\delta K_{ab} = \delta(h_{ac}K^c_b) = h_{ac}\delta K^c_a - 2\alpha K_{ac}K^c_b.$$ \hspace{1cm} (30)

(iii) We could define an object by removing a portion $\eta$ of the trace from the extrinsic curvature:

$$K^{(\eta)}_{ab} \equiv K_{ab} - \eta K h_{ab}.$$ Then

$$\delta K_{ab} = \delta K^{(\eta)}_{ab} + \eta \delta K_{ab} + 2\alpha K_{ab}. \hspace{1cm} (31)$$

**Step VIII:** We are also free to choose to characterize the bulk in terms of the normal derivatives of densitized objects such as $h^{\xi}K_{ab}$ and $h^{\xi}K$ for $h = \text{det} h_{ab}$. Then the identities relating the hypersurface derivatives of these objects to those of $K_{ab}$ are

$$\delta K_{ab} = h^{-\xi_{\mu}}\delta(h^{\xi}K_{ab}) + 2\alpha \xi_{\mu}KK_{ab},$$ \hspace{1cm} (32)

$$\delta K = h^{-\xi_{\mu}}\delta(h^{\xi}K) + 2\alpha \xi_{\mu}K^2.$$ \hspace{1cm} (33)

To illustrate that trace-removed and densitized objects are entirely natural, we may recall (out of many examples in the literature) the ‘gravitational momenta’ $p_{ab} = -\sqrt{h}(K_{ab} - \xi K h_{ab})$. Also the move whereby $h_{ab}$ is taken inside the derivative to form a complete normal derivative in Eq. (19) may be interpreted as the $\eta = 1$ case of (iii).

We note from this thorough consideration of possible ‘bulk’ terms that all 4 ‘brane’ terms quadratic in the extrinsic curvature in the THEFE can be changed independently. There are thus many bulk characterizations such that all 4 of these terms, and hence $Q_{ab}(T)$, are zero. In the next section we illustrate this point with the help of two examples.

## 4 Examples of BEFE with No Quadratic Terms

The diversity of splits into ‘bulk’ and ‘brane’ terms ensures that the truncation Step IV, whereby whichever bulk term is present is neglected, produces all possible combinations of quadratic terms depending on the choice of split employed. Here we illustrate by simple examples that were any such truncation used, then which particular truncation it is could lead to big differences in the remaining ‘braneworld physics’. We do this by building splits into ‘bulk’ and ‘brane’ terms in which no $Q_{ab}$ at all is left in the ‘brane’ term. Thus for example rather than brane FLRW cosmology (which includes $\rho^2$ terms) [2, 3, 4] one would obtain standard FLRW cosmology (with a $\rho$ term alone).

As our first example, we take as the primary object the antidensitized extrinsic curvature $K_{ab} \equiv \frac{K_{ab}}{\sqrt{h}}$ so that the bulk is (in part) characterized by its normal derivatives. Thus, using (28), (29), (32) and the 5-d Einstein field equations, we obtain

$$G_{ab} = L_{ab} + B_{ab}, \hspace{1cm} (34)$$

with

$$L_{ab} = \frac{T_{ab}}{3} + \frac{1}{6}(5T_{\perp \perp} - T)h_{ab}, \hspace{0.5cm} B_{ab} = -2\xi_{ab} + \sqrt{h} \left( \frac{\partial K_{ab}}{\partial z} - \frac{1}{2}h^{cd}\frac{\partial K_{cd}}{\partial z} h_{ab} \right).$$ \hspace{1cm} (35)$$

This example may be reformulated, using (29), (31) and (32), so that the primary objects are the ‘gravitational momenta’, in which case

$$B_{ab} = -2\xi_{ab} - \frac{1}{\sqrt{h}} \left( \frac{\partial p_{ab}}{\partial z} + \left( \frac{1}{3} \frac{\partial p}{\partial z} - \frac{1}{2}h^{cd}\frac{\partial p_{cd}}{\partial z} \right) h_{ab} \right).$$ \hspace{1cm} (36)
Truncating these equations by neglecting the bulk term $B_{ab}$ and assuming perfect fluid matter on the brane with equation of state $P = (\gamma - 1)\rho$, we obtain the corresponding braneworld Friedmann equation

$$\frac{3}{a^2}(\dot{a}^2 + k) = \kappa_5^2 \left[ \rho \left( \frac{-1}{3} + \frac{\gamma}{2} \right) + \frac{\Lambda - \lambda}{3} \right],$$  (37)

which clearly does not possess a quadratic term in $\rho$.

One may consider the BEFE formulation in Eqs (24-27) as having a ‘bulk’ term which contains no $E_{ab}$ at all. We then ask if it is possible to find a formulation in which neither $Q_{ab}(T)$ nor $E_{ab}$ feature. With these restrictions in mind, we found the following example. By considering as our primary object the densitized extrinsic curvature with one index raised $\Gamma_{ab}^c \equiv \sqrt{\kappa} \kappa_{ab}$, we found the corresponding BEFE to have a ‘bulk’ term composed entirely of normal derivatives:

$$G_{ab} = L_{ab} + B_{ab},$$  (38)

$$L_{ab} = T_{ab} + \frac{1}{2} \left( T_{\perp\perp} - \frac{T}{3} \right) h_{ab}, \quad B_{ab} = \frac{1}{\sqrt{\kappa}} \left( \frac{1}{2} \frac{\partial K}{\partial z} h_{ab} - \frac{\partial K}{\partial z} h_{cb} \right).$$  (39)

Truncating this and using $P = (\gamma - 1)\rho$, the corresponding braneworld Friedmann equation is

$$\frac{3}{a^2}(\dot{a}^2 + k) = \kappa_5^2 \left[ \rho \left( \frac{1}{3} + \frac{\gamma}{2} \right) + \frac{2\Lambda + \lambda}{3} \right],$$  (40)

which again is devoid of a $\rho^2$ term.

The explicit absence of quadratic terms in the above examples is due to choosing variables in which the quadratic terms have been entirely incorporated into derivatives off the hypersurface, knowledge of which one would assume would require unavailable knowledge about the bulk. Thus the quadratic terms are implicitly present in the full systems above, as they must be since these systems are equivalent to that of SMS. However, were one to truncate the ‘bulk-like’ terms in these formulations, one would find that one had inadvertently thrown away the ‘brane-like’ quadratic terms as well! We emphasize that we are not advocating that any of these Friedmann equations arising by truncation of bulk terms should be taken seriously. On the contrary, the aim of these examples is to demonstrate with simple calculations (rather than involving perturbative methods or anisotropic models) the fact that in general truncations result in inequivalent residual ‘braneworld physics’. Thus truncations should be avoided in the study of the SMS braneworld.

It is important to note here that in the case of the SMS formulation with AdS bulk (which has $E_{ab} = 0$), the full and the truncated systems coincide. In this case SMS’s formulation (together with its $\rho^2$ term) is equivalent to all the full formulations. However it is not typical for the bulk to have $E_{ab} = 0$, which means that this convenient adaptation is of limited use. By the very same argument, we caution that perhaps some 5-d spacetimes possess embedded $B_{ab} = 0$ hypersurfaces for some $B_{ab} \neq E_{ab}$ which would amount to the full brane-bulk system admitting solutions containing branes with a non-SMS quadratic term. Furthermore, were this to occur for some $B_{ab}$ corresponding to no BEFE quadratic terms, then this would amount to the full brane-bulk system admitting solutions containing branes with no associated quadratic term. Since each 5-d spacetime contains an infinity of embedded 4-d timelike hypersurfaces for which any of the $B_{ab} = 0$ conditions might hold, proving or disproving the above possibilities is a difficult geometrical problem.

5 Discussion

Whereas the particular SMS formulation (often just the BEFE with $E_{ab}$ or $P_{ab}$ thrown away) has often been taken to be the starting point for GR-based brane cosmology, we have shown that there are many choices of formulation of braneworlds by use of geometrical identities. Whereas these formulations are clearly equivalent for the full brane-bulk system, in each case different BEFE terms have a manifest ‘bulk’ origin because the geometrical identities used mix up ‘bulk’ and ‘brane’ terms. Then were one to throw away the ‘bulk’ term in each case, one would obtain inequivalent truncated systems. It is important to bear in mind that one does not a priori know whether in general it is more or less dangerous to throw away one type of $B_{ab}$ (e.g $E_{ab}$) than any other. In SMS’s formulation, the quadratic term is expressed in terms of the energy-momentum residing on the brane, whereas the Weyl term is a portion of a higher-dimensional tensor, so one might feel justified in throwing away the one but not the other. But in other formulations both of these terms are replaced by combinations of other quadratic terms and decidedly bulk-like derivatives off the hypersurface. Indeed we have shown that there exist formulations in which both of these terms are replaced entirely by derivatives off the hypersurface. Thus our point is that there is no clear concept of which truncations are or are not dangerous. Furthermore, such truncations lead to inequivalent residual ‘braneworld physics’ as exemplified by the braneworld Friedmann equations with no $\rho^2$ term in them. Thus we have an argument against performing any truncations at all, and that includes:

$$\frac{3}{a^2}(\dot{a}^2 + k) = \kappa_5^2 \left[ \rho \left( \frac{-1}{3} + \frac{\gamma}{2} \right) + \frac{\Lambda - \lambda}{3} \right],$$
throwing away $E_{ab}$ or $P_{ab}$. Our examples serve as a warning that to understand the SMS braneworld, one must consider the full brane-bulk system.

Finally, having argued in favour of the study of the full brane-bulk system, we note that the availability of formulations pointed out in this letter may greatly facilitate this study. SMS’s formulation of the full brane-bulk system is third-order (in the metric) since it includes evolution equations for the ‘electric’ and ‘magnetic’ parts of the Weyl tensor. Our point is that this system has this form only because the Ricci equation (5) is not used early on in deriving the formalism. Thus the ‘electric’ Weyl term remains within the BEFE’s as an extra unknown, and Bianchi identities together with the Ricci equation are required to evolve it [which in turn involves the ‘magnetic’ Weyl term as yet another unknown]. We suggest that before a detailed study of the brane-bulk system in this particular third-order formulation of SMS is carried out, it is well worth investigating the reformulations which can be obtained along the lines of this letter. From the point of view of PDE theory, knowing precisely which reformulations are available for a given system is of central importance toward providing theorems. In particular here, some formulations of the brane-bulk systems are closed at second order due to the early use of the Ricci equation (5) in deriving these formulations. These second-order formulations include our BEFE (24–27), the GR Cauchy problem analogue in [10] and our second example (38, 39). We emphasize that each such formulation consists of just 15 at most second-order equations (the BEFE together with the Gauss–Codazzi constraints) rather than the much larger number of mostly third-order equations in SMS’s formulation.

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