General Relativity and collineations

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Abstract. In order to get information from the field equations in General Relativity one makes simplifying assumptions which restrict spacetime metric. These assumptions can be geometric and physical. Collineations are the geometric assumptions which concern the “symmetries” of the metric defined by the Lie symmetries of either the metric itself or the tensors defined form the metric. Because the metric fixes the Einstein tensor, and through Einstein field equations, the energy momentum tensor $T_{ab}$, the effects of collineations extend to all parts of General Relativity. The purpose of this article is to give a brief overview of certain effects of collineations on the Geometry, the Kinematics and the Physics of General Relativity.

1. Introduction
In General Relativity one can distinguish the symmetries in two large classes. Symmetries which are related to dynamic quantities such as physical fields, dynamic equations, etc. and symmetries which are referred to the geometrical quantities, that is, the metric and all other geometric objects obtained from it (connection, curvature tensor, Ricci tensor etc.). The first set of symmetries we call simply “symmetries” whereas the second set we call collineations.

A collination is defined by a relation of the form:

$$\mathcal{L}_\xi \Phi = \Lambda$$

(1)

where $\xi^a$ is the symmetry or collination vector, $\Phi$ is any of the quantities $g_{ab}, \Gamma^b_{bc}, R_{ab}, R^b_{bcd}$ and geometric objects constructed from them and $\Lambda$ is a tensor with the same index symmetries as $\Phi$. By demanding specific forms for the quantities $\Phi$ and $\Lambda$ one finds all the well known collineations. For example $\Phi_{ab} = g_{ab}$ and $\Lambda_{ab} = 2\psi g_{ab}$ defines the Conformal Killing vectors (CKV) and specializes to a Special Conformal Killing vector (SCKV) when $\psi_{;ab} = 0$, to a Homothetic vector field (HVF) when $\psi =$constant and to a Killing vector (KV) when $\psi = 0$. When $\Phi_{ab} = R_{ab}$ and $\Lambda_{ab} = 2\psi R_{ab}$ the symmetry vector $\xi^a$ is called a Ricci Inheritance Collineation (RIC) and specializes to a Ricci Collineation (RC) when $\Lambda_{ab} = 0$. When $\Phi_{ab} = T_{ab}$ and $\Lambda_{ab} = 2\psi T_{ab}$, where $T_{ab}$ is the energy momentum tensor, the vector $\xi^a$ is called a Matter
Inheritance Collineation (MIC) and specializes to a Matter collineation (MC) when $\Lambda_{ab} = 0$. The function $\psi$ in the case of CKVs is called the conformal factor and in the case of inheriting collineations the \textit{inheriting factor}.

Collineations are not independent because of identities relating the Lie derivatives of the metric tensors. Indeed we have [1]:

\begin{align*}
L_X g_{ab} &= 2 \psi g_{ab} + 2 H_{ab} \quad (2) \\
L_X \Gamma^a_{bc} &= \frac{1}{2} g^{ad}[(L_X g_{bd})_{;c} + (L_X g_{be})_{;d} - (L_X g_{de})_{;b}] \quad (3) \\
L_X R_{abcd} &= (L_X \Gamma^a_{bc})_{;d} - (L_X \Gamma^a_{db})_{;c} \quad (4) \\
L_X R_{ab} &= (L_X \Gamma^c_{ab})_{;c} - (L_X \Gamma^c_{ac})_{;b} \quad (5)
\end{align*}

For example a KV or a HVF is a RC or a MC but not the opposite. A RC or a MC which is not a KV or a HVF (and in certain cases a SCKV) is called proper. There are many types of collineations and most of them have been classified in a diagram which exhibits clearly their relative properness [2, 3].

One could safely say that the higher collineations are not yet widely known/used/understood in practice and are considered as a rather “new case”. One reason for this is that there has not yet been developed a unified method which will study all collineations simultaneously as members of one (or perhaps a few) class(es) and allow a measure of their relative action and strength in specific situations.

One attempt to develop a unified study of collineations has been introduced in [4] by the introduction of the so called generic collineation defined by the equation/identity:

\[ L_\xi g_{ab} = 2(\psi g_{ab} + H_{ab}), \]

where $\psi$ and $H_{ab}$, $(H_{ab} = H_{ba}, H^a_b = 0)$ are tensor fields called the parameters of the collineation $\xi^a$. Computing the Lie derivative of each metric geometric object w.r.t. $\xi^a$ (see equations above) in terms of the $L_\xi g_{ab}$ one is able to express all collineations in terms of the quantities $\psi$ and $H_{ab}$ and their derivatives. This then allows the classification and the study of all collineations in a unified and concise manner by means of the study of the fields $\psi$ and $H_{ab}$ which are common to all of them.

Indeed in the literature one can find these conditions for $\psi$ and $H_{ab}$ (or some similar quantities) for most types of collineations, but there does not seem to exist either an organized or a complete list for all of them. Some important expressions are:

\begin{align*}
L_X R_{ab} &= -2 \psi_{;ab} - (\Box \psi) g_{ab} + H^c_{ba;ac} + H^c_{a;bac} - (\Box \psi) H_{ab} \quad (9) \\
L_X R &= -2 \psi R - 6(\Box \psi) + 2H^a_{ab} - 2R_{ab}H^{ab} \quad (10) \\
L_X G_{ab} &= -2 \psi_{;ab} + 2(\Box \psi) g_{ab} + 2\left( H^d_{(ab);d} - \frac{1}{2} g_{ab} H_{tt} \right) - (\Box \psi) H_{ab} - RH_{ab} + g_{ab} \left( H^{ab} R_{ab} \right) \quad (11)
\end{align*}

Any collineation acts on any other via the components of the generic collineation, thus all effect the kinematics and the dynamics of spacetime via identities which have to be satisfied by physical systems in that spacetime.
Example. Let $X^a$ be a CKV ($\Leftrightarrow H_{ab} = 0$). Then the following identities must be satisfied by the geometric quantities in that spacetime:

$$L_X \Gamma^a_{bc} = 2 \delta_{(b} \psi^a_{c)} - \psi^a_{;b} g_{ab}$$
$$L_X R_{ab} = -2 \psi^c_{;ab} - (\Box \psi) g_{ab}$$
$$L_X R = -2 R \psi - 6 (\Box \psi)$$
$$L_X G_{ab} = L_X T_{ab} = -2 \psi^c_{;ab} + 2 (\Box \psi) g_{ab}$$
$$L_X C^a_{bcdn} = 0.$$ 

Using Einstein equations we have the important result: $L_X T_{ab} = -2 \psi^c_{;ab} + 2 (\Box \psi)$.

2. Metrics in spacetime

In spacetime we have more metrics defined by the spacetime metric $g_{ab}$ [5, 6]. These metrics - which of course need not be Lorentzian or non-degenerate - are all second order symmetric tensors defined from the spacetime metric $g_{ab}$ by means of differentiations. Such metrics are e.g. $R_{ab}, G_{ab}$. Collineations are the symmetries of these metrics. The KV’s of the base metric are KV’s for these metrics. Therefore these new metrics have at least the same symmetries with the base metric and must be of the same form as the “base” metric but more special! E.g. $R_{ab}$ of a FRW metric is a FRW metric - in the sense that it is 1+3 decomposable with maximally symmetric 3-d subspaces. However the symmetry group of the Ricci tensor is larger from that of the space-time metric because its signature can be $(+++)$ or $(++- -)$. But there do exist space-time metrics whose Ricci (or Matter) tensor is non-degenerate, has Lorentzian character and has the same symmetry group $G$ (that is, there are no proper RCs) with the space-time metric. Such an example is the RW metric:

$$ds^2 = -dt^2 + \cosh^2 t \left( dx^2 + dy^2 + dz^2 \right).$$

whose Ricci tensor is computed to be:

$$R_{00} = -3, \quad R_{\alpha\beta} = \frac{3}{2} \cosh 2t - \frac{1}{2}.$$ 

It is easy to show that this Ricci tensor satisfies the above properties, therefore it is a perfectly legitimate RW metric! This result questions the uniqueness of the space time metric for given symmetry assumptions and requires further consideration.

3. The geometric effects of collineations

The geometric effects of collineations are manifold depending on the type of collineation. In the following we shall comment briefly on some of these effects and give some references where the interested reader can find more information.

3.1. Low level collineations and the simplification of the metric

The purpose of introducing the low level collineations was to simplify the metric, that is to determine the functional form of the metric in terms of a limited number of parameters. This simplification hopefully leads to simpler field equations whose solution or more general study is feasible. In the following we present a Theorem quoted by Petrov which concerns the simplification of the metric when it admits a gradient CKV.
Theorem (THEOREM of PETROV). Assume spacetime metric admits a gradient non-null CKV $\phi_a$ with conformal factor $\psi$. Then there exists a coordinate system in which the metric takes the form:

$$ds^2 = \varepsilon(x^1)g_{11}(x^1)(dx^1)^2 + [g_{11}(x^1)]^{-1}\Gamma_{\mu\nu}(x^\rho)dx^\mu dx^\nu$$

where

$$g_{11}(x^1) = 2[\psi(x^1)dx^1 + c]^{-1}\psi_a = \delta^1_a\varepsilon(x^1).$$

Generalization for a non-gradient CKV has been done in [7].

3.2. Select a unique metric from a class of metrics
Assume that spacetime admits an abelian isometry group $G_3$ acting orthogonally on 3-d spacelike sections. Then the metric is the (class of) Bianchi Type I (hypersurface orthogonal) metric:

$$ds^2 = -dt^2 + A^2_\mu(t)(dx^\mu)^2$$

Now in the class of Bianchi Type I spacetimes determine those which admit in addition a CKV. Then it has been shown [8] that there exists only one such Bianchi I spacetime with line element:

$$ds^2 = -dt^2 + \sinh^2 k t\frac{k}{2}\cosh^2 k t\frac{k}{2}dx^2 + \sinh^2 k t\frac{k}{2}\cosh^2 k t\frac{k}{2}dy^2 + \sinh^2 k t dz^2$$

This spacetime is anisotropic, rotation free, geodesic with shear and expansion.

3.3. Restrict the matter
The matter collineations $L X T_{ab} = 0$ restrict the geometry because they are computed terms of the parameters $\psi, H_{ab}$ via Einstein’s equations. Therefore when we make a symmetry assumption on the metric and a subsequent on the matter it is imperative that we check that these two assumptions are compatible.

For example if one requires spherical symmetry (three KVs hence three matter collineations) then it is well known that the resulting spacetime has vanishing heat flux (for comoving observers only!!). If one further demands staticity (that is one more unit timelike KV) - e.g. Schwarzschild spacetime- then the matter must be a perfect fluid or one of its specializations.

3.4. Geometric equations of state
This is a new way to obtain equations of state which are compatible with the geometry of spacetime. In this approach we let Geometry do the Physics and simply check a posteriori if the resulting equations of state have any physical value or not. The best way to explain the introduction of these equations of state is by a specific example

Consider the standard Friedmann Robertson Walker (FRW) models with vanishing cosmological constant and comoving observers $u^a = S^{-1}(\tau)\delta^a_0$, where $\tau = \int\frac{dt}{S(t)}$, $t$ being the standard cosmic time. For these observers the energy momentum tensor has a perfect fluid form i.e. $T_{ab} = \mu u_a u_b + p g_{ab}$ where $\mu, p$ are the energy density and the isotropic pressure measured by the observers $u^a$. This decomposition of $T_{ab}$ in the coordinates $(\tau, x^\mu)$ leads to the relations:

$$T_{00} = \mu S^2(\tau), T_{\mu\nu} = p S^2(\tau)U^2(k, x^\mu)\delta_{\mu\nu}.$$
Expression (18) is a result (a) of the symmetry assumptions of the metric and (b) the choice of observers. Using Einstein field equations we compute:

$$\mu = 3\left( S,_{\tau} \right)^2 + kS^2 S_S,_{\tau} + (S,_{\tau})^2 + kS^2$$

There remains one variable (the $S(\tau)$) free. Therefore we have to supply one more equation in order to solve the model. This extra equation is a barotropic equation of state, that is, a relation of the form $p = p(\mu)$.

The obvious choice is a linear equation of state $p = (\gamma - 1)\mu$. There are several solutions for this simple choice which are of cosmological interest. For example $\gamma = 1$ ($p = 0$) implies degeneracy of the energy momentum-tensor (dust) and the value $\gamma = \frac{4}{3}$ ($p = \frac{1}{3}\mu$) implies radiation dominated matter. Both states of matter are extreme and they have been relevant at certain stages of the evolution of the Universe. For other values of $\gamma$ one obtains intermediate states which cannot be excluded as unphysical (see e.g. [9] for a thorough review). Therefore it would be interesting to use non-linear equations of state which will deal with more complex-and physical-forms of matter. But what will be an "objective" criterion to write such equations?

It is proposed that this equation is one of the equations defined by the requirement of existence of a proper RC or a MC. Of course for every choice of observers every such equation will have a different form, which will have to be checked that it leads to physically reasonable results. From the geometric point of view this proposal is reasonable. Indeed the KVs are used to fix the general form of the metric and, because a KV is a RC and a MC, they also fix the $R_{ab}$, $T_{ab}$. Therefore the proper RCs and MCs are symmetries which contain the effects of covariant differentiation ($R_{ab}$) and Einstein field equations ($T_{ab}$). One extra advantage of this type of equations of state is that, unlike the standard ones, they are observer independent in the sense that they take a specific form only after a class of observers is selected. We shall call this type geometric equations of state. For details and results on geometric equations of state for the case of MCs in FRW spacetime the reader is referred to [10].

4. The effects of collineations on the kinematics

In order to study the effects of a collineation on the kinematics of a spacetime we express the derivative of a vector field in terms of collineation components and its bivector by the identity:

$$X_{a;b} = X_{(a;b)} + X_{(a;b)} = \left( \frac{1}{2} L_X g_{ab} + F_{a;b} = \psi_{g_{ab}} + H_{ab} + F_{ab} \right. \right)$$

Taking $X^a$ to be the 4-velocity we have that [11, 12]:

$$u_{ij} = \omega_{ij} + \sigma_{ij} + \frac{1}{2} h_{ij} - \dot{u}_i u_j = \left( \frac{1}{2} L_X g_{ab} + F_{a;b} = \psi_{g_{ab}} + H_{ab} + F_{ab} \right. \right)$$

which computes the kinematic variables $\omega_{ij}$, $\sigma_{ij}$, $h_{ij}$, $\dot{u}_i$ in terms of the collineation quantities $\psi$, $H_{ab}$. The above effect kinematics at various levels, some of which are shortly referred below.

4.1. The frozen in property

Consider the mathematical identity:

$$L_X u_i = \psi u_i + \alpha X_i + \beta v_i \right) \right)$$

where $\dot{v}_i u^i = v_i X^i = 0$.

We say that $X^i$ is frozen along $u^i$ iff $\beta = 0$. Geometrically this condition means that the integral curves of the vector fields $X^i$, $u^i$ form a 2-d timelike surface (e.g. a string). The
physical meaning of the frozen in condition is that the flow lines of \(u^i\) are preserved under Lie transport along integral lines of \(X^i\) and leads to dynamical conservation laws for the observers \(u^i\).

### 4.2. Determination of the metric

At the level of the metric we have the following result[7]:

**Proposition.** The following statements are equivalent:

- **Physical definition** Kinematically the FRW spacetime is a fluid of observers \(u^i\) such that \(\omega_{ab} = \sigma_{ab} = 0, \dot{u}^i = 0\) and dynamically is a perfect fluid that is \(q_i = 0, \pi_{ij} = 0\).

- **Symmetry definition**

The FRW is a spacetime which admits a timelike CKV such that:

1. It is reduced to a KV for a conformally related metric
2. The 3-spaces normal to that vector are spaces of constant curvature (which admit 6 spacelike KVs.)

### 4.3. Kinematic physical properties of the fluid

The effects of collineations on the kinematic and physical properties of a fluid are many and have been considered many times in the literature mainly for the low level collineations up to CKVs [15, 12]. Here we quote only the following result:

**Proposition.** Suppose spacetime admits a spacelike CKV. Then:

I. If the fluid is irrotational \((\omega_{ab} = 0)\) then the fluid is frozen along the integral curves of the CKV.

II. If the fluid is rotational then the fluid is frozen along the integral curves of the CKV iff the CKV coincides with the vorticity vector \(\omega^i\).

### 5. The 1+1+2 decomposition - double congruences

When the 4-velocity and the collineation are not parallel (and without loss of generality are normal) one can use both vector fields to project along them and normal to them by means of the screen projection operator:

\[
p_{ab} = g_{ab} + u_a u_b - n_a n_b
\]

where \(n_a\) is the unit vector along the collineation.

The new projection operator introduces the 1+1+2 (local) decomposition of tensorial quantities [4, 13]. A new characteristic vector entering the scheme is the screen vector:

\[
N^a = p_b^a L_n u^b
\]

which vanishes iff \(u^a\) is frozen in along \(X^a\). The screen vector is closely related with the physics of string fluids (see [14] for details).

The 1+1+2 decomposition of the vector \(n^a\) gives new “kinematic” quantities through the identity:

\[
p_c^a p_b^d n_{cd} = R_{ab} + S_{ab} + \frac{1}{2} E p_{ab}.
\]

It is possible to express (see [4, 15]) the kinematic quantities of \(n^a\) in terms of the collineation parameters \(\psi, H_{ab}\). The 1+1+2 decomposition of Ricci’s identity for \(n^a\) leads to new “refined” propagation and constraint equations which we shall not refer here due to the lack of space.
What is more interesting is the 1+1+2 decomposition of the energy momentum tensor $T_{ab} = \mu u_a u_b + p h_{ab} + 2 q (u_a u_b) + \pi_{ab}$ which leads to new dynamical variables $\nu, \gamma, Q^a, P^a, D_{ab}$ through the identities:

$$q^a = \nu a + Q^a; \quad v = q^a n_a, \quad Q^a = p_b q^b$$

$$\pi_{ab} = \gamma (n_a n_b - \frac{1}{2} p_{ab}) + 2 P_{(a} n_{b)} + D_{ab}$$

$$\gamma = \pi_{ab} n^a n^b; \quad P^a = p_b \pi^{b} n^c,$$

$$D_{ab} = (p_c p^d - \frac{1}{2} p^c p_{ab}) n_c.$$ 

Apart of their many applications the new dynamical variables have been used in covariant perturbation methods (see [16] for literature and details).

In the 1+1+2 decomposition the conservation equations become:

$$\dot{\mu} + (\mu + p) \theta + (\nu + 2 \nu E + p a Q_{a;b} + Q^a (\ln \xi)_{\alpha} + 2 Q^a \dot{\mu}_a + \frac{3}{2} \gamma (\ln \xi) + 2 \sigma_{ab} P^a n^b + \sigma_{ab} D^{ab} = 0$$

$$(p + \gamma)(-\mu + p + 4 \gamma) E + 2 \nu E + p a P_{a;b} + P^a (\dot{\mu}_a - 2 n_a) = 0$$

$$p_a [(\mu + p - \frac{\gamma}{2}) \dot{\mu}_a + n_c + \dot{Q}_c + 4 \frac{\gamma}{3} Q^a (\omega_{cd} + \sigma_{cd}) Q^d + 2 \nu \sigma_{cd} n^d + \dot{\sigma}_c + \delta_{cd} P_{cd}$$

$$+ 2 E P_c + D_{c..d} - \frac{3}{2} \gamma (\ln \xi)_{\alpha} - \frac{1}{2} \gamma \xi] = 0.$$ 

In order to find the effects of a collineation on the dynamics we need the 1+1+2 decomposition of the derivative:

$$\psi_{ab} = \lambda_{\psi} u_a u_b - 2 k_{\psi} (u_a n_b) - 2 S_{\psi} (u_a n_b) + \gamma_{\psi} n_a n_b + 2 P_{\psi} (n_a n_b) + D_{\psi} a + \frac{1}{2} \alpha_{\psi} p_{ab}. \quad (26)$$

To find the effects of a collineation on the dynamics take the lie derivative of the Einstein equations and 1+1+2 decompose them. This is achieved with the following steps:

1. Consider:

$$L_X G_{ab} = L_X T_{ab} \quad (27)$$

2. Express $L_X G_{ab}$ in terms of $\psi, H_{ab}$ and their derivatives (see earlier formula).

3. Express $L_X T_{ab}$ in terms of 1+1+2 dynamic variables $L_X (\mu, p, \gamma, Q^a, P^a, D_{ab})$ for a generic type of matter.

4. Equate the results and take the irreducible parts by projecting respectively with the following tensors:

$$u_a u_b, \quad u_a n_b, \quad n_a n_b, \quad u_a p_b, \quad n_a p_b, \quad P_{ab}, \quad \left( p_c p^d - \frac{1}{2} p_{ab} p^{cd} \right)$$
The result is the generic set of equations (that is, the field equations for the generic collineation and the generic type of matter). Obviously these equations are very involved and the space in the present article does not suffice to write them down explicitly. To give an idea of these equations we refer the (00) equation which gives the propagation \( \mu \) of the energy density \( \mu \) along the collineation:

\[
\ddot{\mu} + \mu \frac{2\psi}{X} - \left( \frac{r}{3} \right) p H_{11} + \left( \frac{\gamma}{4X} \right) (3H_{22} - H_{11}) = \left( \frac{1}{2X} \right) [\lambda_{A} + \gamma_{A} + \alpha_{A} - 4\gamma_{\psi} + 4\alpha_{\psi}] + \left( \frac{\nu}{X} X^T \right) H_{21} + Q_{c}N^{c} - 2Q^{c} \omega_{cd} n^{l} - D^{cd} H_{cd} - \left( \frac{1}{3} \right) P^{c} H_{c2}
\]

where we have used the notation:

\[ H_{11} = H_{ab} u^{a} u^{b}, \quad H_{12} = H_{ab} u^{a} n^{b}, \quad H_{22} = H_{ab} n^{a} n^{b}. \]  

In fact it turns out that all field equations for the scalars are of the form:

\[ \ddot{S} + ... = 0 \]

where \( S = p, \gamma, \nu \) and for the tensorial quantities \( Q^{a}, P^{a}, D_{ab} \) they are of the form:

\[ p_{a}^{k} p_{b}^{l} \ddot{S}^{ab} + ... = 0 \]

We observe the "similarity" of these equations. There are many uses of this similarity in dealing with problems in practice. We emphasize that these equations hold for any collineations and any type of matter. Therefore it is possible and desirable to develop an algebraic computing program which will produce the result (i.e. the field equations) for any given data (matter and/or collineation). This we plan to do in a future publication.

However a special case concerning the application the method for a spacelike CKV and any fluid can be found in [13]. In the same work one can find the reduction of the results in the special case the fluid is an anisotropic fluid with energy momentum tensor \( T^{ab} = \mu u^{a} u^{b} + p_{\parallel} n^{a} n^{b} + p_{\perp} p_{ab} \).

References

[1] Latin indices run over the four coordinates in space-time. Greek indices take values different from 1. For instance, if \( I = 2 \), then \( \mu = 0, 1, 3 \). The signature of the space-time metric \( g_{ab} \) is \((-++++)\). Semi-colon "\(^{,}\)" denotes covariant differentiation w.r.t. space-time metric and "\(^{,\gamma}\)" w.r.t. the 3-metric \( g_{\alpha\beta}(x^{\mu}) \). The Ricci tensor is defined by \( R_{ab} \equiv R^{c}_{abc} \). The Weyl tensor is defined by, \( C_{abcd} = R^{c}_{abcd} - 2R^{c}_{[a} R_{bd]} + \frac{2}{3} R^{c}_{[a} S^{d]} \).

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