Parametric dynamic analysis of a superconducting bearing system

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Abstract. The dynamics of a disk-shaped permanent-magnet rotor levitated over a high-temperature superconductor is studied. The interaction between the rotor magnet and the superconductor is modelled by assuming the magnet to be a magnetic dipole and the superconductor as a diamagnetic material. In the magneto-mechanical analysis of the superconductor part, the frozen image concept is combined with the diamagnetic image and the damping in the system was neglected. The interaction potential of the system is the combination of magnetic and gravitational potential. From the dynamical analysis, the equations of motion of the permanent magnet are stated as a function of lateral, vertical and tilt directions. The vibration behaviour of the permanent magnet is analyzed with a numerical calculation obtained by the non-dimensionalized differential equations for small initial impulses.

Introduction

Since the discovery of high temperature superconductors (HTS), many engineering applications have emerged. These applications are generally based on the interactions between the permanent magnets (PM) and superconductors [1-3]. The possibilities of engineering applications were improved by the theoretical and experimental studies [3]. One of the most important issues in the levitation applications is the stability of the levitated PM over the HTS. The response of the superconductor to the magnetic field of the PM was widely studied from an experimental point of view. However, modelling the motion of the PM/HTS systems is not easy due to the complicated hysteresis in the superconductor [4-7]. Some deficiencies also occur such as the tilt on the levitated magnet (the angle between the rotational symmetry of the PM with its vertical direction). These deficiencies cause non-linear drag on the levitated PM which in turn results in instabilities.

Studying the levitation for real geometries generally requires the combination of electromagnetic and mechanical analysis. The aim of this paper is to study the dynamics of a tilted PM levitated over a superconductor. The study performed by Sugiura et al. [8] used the frozen image concept to define the interaction potential for the PM/HTS system and also discussed the lateral, vertical and tilt effect. In order to provide a more complete dynamical analysis, vertical, radial and tilt characteristics were studied.

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Magneto-mechanical analysis

The configuration basically consists of vibration permanent magnet levitated over the high temperature superconductor. A disk shaped NdFeB PM levitated over a melt-textured YBCO. The PM disk has a diameter of 25.4 mm, height of 6.35 mm, mass of 35.6 gram and axially polarized with magnetization of 1.1 Tesla.

The governing equation of the PM levitated over the HTS is not simple due to the complicated magnetic interaction potential. This is because in actual geometries the total potential depends on both gravitational and magnetic forces. The magnetic part of the interaction for this system can be approximated in terms of the frozen-image model [4]. In this approach the PM is assumed as a magnetic dipole and HTS as the material that consists of the combination of flux pinning and diamagnetic property. In this study, the frozen-image model does not account for hysteresis in the HTS, which would result in a transfer of some part of the moment of the diamagnetic-image into a small frozen-flux moment as the PM moves closer to the HTS surface. For quasi-static motions, where the hysteresis is not a major effect, it was proven that the frozen-image concept successfully predicts the interaction force between the PM and HTS. In addition, prediction of some of the characteristics of the system by the frozen image model in the dynamical case is expected.

The interaction of the PM and superconductor system can be represented for a first order approximation by the forces between the dipoles and their images. In this system, the first dipole is represented by the PM and second dipole by its image. The total potential represents the system of the dipole and its diamagnetic mirror image. When the HTS is field cooled with the PM at a particular field cooling height \( h \), two images appear: one is a frozen-image and the other one is a diamagnetic-image. In this configuration, while the frozen image is fixed in \( h \), the diamagnetic image is able to move with the PM. Thus, the PM has the distance of \( (h + z) \) from the surface of the HTS, which is called as levitation height. The frozen and diamagnetic images are equal in magnitude and are opposite in direction to each other. Thus, the frozen image model states that the net interaction forces between the PM and its frozen and diamagnetic images are zero.

When the dipole from the PM and its frozen image is considered, \( m_1 \) is the dipole from the PM and this time \( m_2 \) is the dipole from the frozen image of the PM [3, 4]. By taking HTS surface at zero potential, with \( M \) and \( H \) as the mass and the half height of the PM, respectively. Defining \( A_1 = [r^2 + (2h + H + z)^2]^{1/2} \) and \( A_2 = r \sin \theta + (2h + H + z) \cos \theta \), the total interaction potential of the system can be written as

\[
U_{\text{tot}} = \beta \left[ \frac{1 + \cos^2 \theta}{2(2h + H + z)^3} + \frac{\cos \theta}{A_1^3} \frac{3A_2(2h + H + z)}{A_1^5} \right] + Mg(h + H/2 + z) \tag{1}
\]

where \( \theta \) is the angle between rotational magnetization symmetry axis of PM with the vertical direction and \( \beta = \mu_0 m_1 m_2 / 4\pi \). In order to simplify the geometry of the system, the origin was chosen to be the centre of mass, which is the sum of field cooling height \( h \) (distance from the surface of the superconductor to the bottom of the PM) and half of the PM height. The kinetic energy of the levitated PM above the superconductor is also given as

\[
T = \frac{1}{2} M (\dot{r}^2 + \dot{z}^2) + \frac{1}{2} I \dot{\theta}^2 \tag{2}
\]

where \( I \) is the radial moment of inertia and the dot over each variable, from here on, presents the time derivative of that variable. The kinetic energy also consists of the spatial components since the PM is not fixed in one point. From the Lagrangian function of the PM/HTS system the equations of motion of the system given as
\[ \ddot{r} + \frac{3\beta}{M} \left[ \frac{5rA_2(2h + H + z)}{A_1} - \frac{A_3}{A_1^3} - \frac{A_4}{A_1^5} \right] = 0 \]  
(3)

\[ \ddot{z} + g + \frac{3\beta}{M} (B_2 + B_3) = 0 \]  
(4)

\[ \ddot{\theta} + \frac{\beta}{I} \left\{ -\sin \theta \cos \theta \frac{\sin \theta}{(2h + H + 2z)^3} - \frac{3A_4(2h + H + z)}{A_1^3} - \frac{A_3}{A_1^5} \right\} = 0 \]  
(5)

where

\[ B_1 = \frac{5rA_2(2h + H + z)}{A_1^3} - \frac{A_3}{A_1^5} \]  
(6)

\[ B_2 = -\frac{(1 + \cos^2 \theta)}{(2h + H + 2z)^4} \]  
(7)

\[ B_3 = -\frac{2(2h + H + z)\cos \theta + A_2}{A_1^5} + \frac{5A_3(2h + H + z)^2}{A_1^7} \]  
(8)

and

\[ A_3 = r \cos \theta + (2h + H + z)\sin \theta \]  
(9)

\[ A_4 = r \cos \theta - (2h + H + z)\sin \theta. \]  
(10)

Here, the double dot over each variable, from here on, presents the second time derivative of that variable.

The equations given in (3), (4) and (5) represent the centre of mass motion of the PM. The PM is assumed to be cooled in the coordinates of \((r, \theta, z) = (0, 0, h)\). This assumption provides the magnetization of the PM being aligned purely in the vertical direction so that the magnetization of the frozen image can only occur in the vertical direction.

The equations from (3) to (5) are coupled non-linear differential equations and an analytical solution is not possible. It is possible to characterize the PM motion from the numerical calculation obtained by using the Runge-Kutta method. The theoretical analysis of the various vibrational frequencies such as vertical, radial and tilt will be next in discussion. In this analysis first we will directly solve the equation of the system with the Runge-Kutta method and compare the results with non-dimensionalized equations solved again with Runge-Kutta method.

**Non-dimensionalization of the equation set**

Non-dimensionalization can be explained as replacing units with suitable variables and for these units to be removed partially or completely from mathematical equality. For example, if a system owns a resonance frequency length or a time consistency then non-dimensionalization can rearrange values owned by this system. Non-linear equation applications can be given as an example for this situation. Non-dimensionalization ensures the determining of the characteristic of the system used by the natural units of the system. In other words, non-dimensionalization is used to determine parameters which are actually necessary for the analysis of the system.
The non-dimensionalized equations set are derived with the proper variable changes. Bearing in the
mind that $H_0 = h + H/2$ and by converting $r^* = r/H_0$, $z^* = z/H_0$ and $\omega^* = \omega_0 / \sqrt{K/M}$ the
parametric terms are given as

$$K = \frac{3\mu_0 m_1 m_2}{16\pi H_0^2}, \quad g_A = \frac{M}{H_0K}.$$  \hfill (11)

At first we considered starred equations, and then we omitted star superscripts in the equations. By
some arrangements, below equations are derived:

$$\ddot{r} + 4\left[\frac{5C_3(2 + z)}{C_1^{7/2}} - \frac{C_3}{C_1^{5/2}}\right] = 0$$  \hfill (12)

$$\ddot{z} + 4\left\{(1 + \cos^2 \theta) + \frac{5C_2(2 + z)^2}{C_1^{7/2}} - \frac{C_4}{C_1^{4/2}} + g_A\right\} = 0$$  \hfill (13)

$$\dot{\theta} + \frac{M\beta}{KlH_0^3} \left[\frac{\sin \theta \cos \theta}{8(1 + z)^3} + \frac{\sin \theta}{C_1^{7/2}} + \frac{3C_5(2 + z)}{C_1^{5/2}}\right] = 0$$  \hfill (14)

where

$$C_1 = r^2 + (2 + z)^2$$  \hfill (15)

$$C_2 = r[r \sin \theta + (2 + z) \cos \theta]$$  \hfill (16)

$$C_3 = r \cos \theta + (2 + z) \sin \theta$$  \hfill (17)

$$C_4 = r \sin \theta + 3(2 + z) \cos \theta$$  \hfill (18)

$$C_5 = r[r \cos \theta - (2 + z) \sin \theta].$$  \hfill (19)

**Discussion**

Obtaining the numerical solutions of the equation of motion for the PM levitated above the
superconductor provides an insight for the stability analysis. Because of the non-linear nature of the
effective force components of the magnetic systems, the magnetic bearings act like non-linear
systems. In order to find physical solutions of such systems, the linearization procedure is generally
used. One of the common techniques for the stability analysis is to use the non-dimensionalization
under the initial constant impulse applied to the system. From the numerical calculations, the model
predicted the resonance frequency with a reasonable accuracy for a typical vertical vibration amplitude
of the PM as a function of time.

Although the impulse, which is applied vertically, does not affect the other directions, when a small
displacement is applied to the PM either from radial or tilt, there exists vibrational modes in the
directions of radial and tilt and also negligibly small amount in the vertical. Figure 1 shows the
theoretical vibration spectrum, where the displacement was applied in the radial direction. The
appearance of both radial and tilt vibration in the same frequency spectrum indicates the strong cross-
couplings among these directions. This indicates that all the vibration frequency modes are actually
the combination of pure vibration frequency modes such as vertical, lateral and even the angular
directions. However, in the vertical direction this agreement is not too strong, and the agreement gets
worse especially in the tilt direction. This difference in vertical direction is also due to the hysteresis effect, which acts as damper in the system. This is because the hysteresis occurs more in the vertical direction than that of the radial, so the difference in between the theory and the experiments is high for the vertical direction compared to the radial. In addition, as shown in figure 2, a non sinusoidal shape of the radial vibration amplitude as a function of time indicates that there is actually radial and tilt displacement of the permanent magnet.

![Figure 1](image1.png)

**Figure 1.** Resonance frequencies in the radial and tilt directions from the FFT analysis.

![Figure 2](image2.png)

**Figure 2.** The amplitude of vibration in radial and tilt directions.

From the dynamical point of view, the response of the damping forces is not dominant compared to the forces due to effective potential of the system. For this reason, the analysis presented in the model
is assumed to be reliable for studying the motion along the vertical, radial and tilt. The eccentricity between the magnetic centre and the mass centre gets larger if the inhomogeneities from the magnet and superconductor are too much. This behaviour can be observed in figure 3 that the resonance frequency as a function of cooling height gets unrealistic for the case of very low and very high cooling heights. This can be explained in terms of the frozen image model assumption that the permanent magnet was assumed as a dipole which creates unrealistic results if the dipole is not far away from its image.

![Resonance Frequency vs Field Cooling Height](image1)

**Figure 3.** Resonance frequency as a function of field cooling height.

The amplitude of the radial and tilt vibration modes as a function of cooling height shows non linear behaviour. As shown in figure 4, the amplitude is changing with the cooling height in a parabolic function. This may be attributed to frozen image model that the dipole acts in appropriate cooling height. However a real explanation of this behaviour still needs further theoretical analysis.

![Amplitude vs Field Cooling Height](image2)

**Figure 4.** Amplitude of vibration as a function of field cooling height.
Preliminary results obtained for non-dimensionalized equations show that vertical vibration amplitude as a function of time is scaled. As shown in figure 5, the vertical resonance frequency predicted for 10 mm cooling height is 5 Hz, which is far different than the one obtained in direct solution shown in figure 3. In order to explain these unexpected results, the set of equations needs to be analyzed by Taylor series of expansion. At this stage, these preliminary results also show that the non-dimensionalization must be carried out with the Taylor expansion of the equations of the motion to investigate initial condition dependence on the parametric equations.

![Graph](image)

**Figure 5.** Resonance frequency in the vertical direction for non-dimensional equation set.

**Conclusion**
In this paper, the dynamics of a levitated PM over a superconductor was studied. The dynamic behaviour of the PM was investigated with the combination of mechanical and electromagnetic considerations. The governing equations of the motion of the PM levitated over the HTS were simplified by approximating the magnetic part of the interaction for this system in terms of the advance frozen-image model approach. In this approach, the PM was assumed to be a dipole and the HTS as a diamagnet, in which the image dipole resides.

In conclusion, for designing a superconducting bearing for applications, the geometric and magnetic considerations must be taken into account. The theoretical results show non-sinusoidal shape of the radial vibration amplitude as a function of time. This indicates that there are actually radial and tilt displacements of the permanent magnet. Also, the non-linear behaviour of the amplitude of radial resonance frequency still needs further theoretical analysis, either considering frozen image model or other existing models. In order to explain these unexpected results, the set of equations needs to be analyzed by Taylor series of expansion. At this stage, these preliminary results also show that the non-dimensionalization must be carried out with the Taylor expansion of the equations of the motion to investigate initial condition dependence on the parametric equations.

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