Free vibration analysis of laminated sandwich plates under thermal loading

A Garg and H D Chalak

Department of Civil Engineering, National Institute of Technology Kurukshetra, India-136119

E-mail: amang321@gmail.com

Abstract: In present work, free vibration analysis of laminated sandwich plates is carried out under thermal loading. C-0 finite element based higher-order zigzag theory is used during analysis. Third-order variation of in-plane displacement field is assumed while for transverse displacement field, quadratic variation is used for core region and is taken constant for the face layers. Present model is free from any kind of penalty requirements. Also, the proposed model satisfies transverse shear stress continuity condition at interface and same is zero at top and bottom faces. Present results are compared with those available in literature. New results for sandwich plates are proposed in the present work which will serve as benchmark for future studies.

1. INTRODUCTION

Laminated composite and sandwich structures are gaining popularity in constructing various structures in field of aerospace, aeronautics, automobile, civil, marine, defence etc. [1]. These structures are subjected to thermal loadings during their service life period. Thermal conditions alter the behavior of the laminates when compared with those predicted without using hygro-thermal loadings. A number of review works are available on the analysis of laminated composite and sandwich structures under thermal loading [2-6]. Recently, authors Garg and Chalak [7] presented a comprehensive review on the analysis of laminated composite and sandwich structures under hygro-thermal loading. From that work, it is seen that the free vibration analysis of laminated composite and sandwich plates under thermal conditions using a transverse displacement field based higher-order zigzag theory (HOZT) is not carried out.

In present work, recently proposed HOZT [8] is used for the free vibration analysis of laminated composite and sandwich plates under thermal conditions. the theory assumes third-order variation of in-plane displacement field. Parabolic variation of transverse displacement field is assumed for core region while the same is taken constant for face layers. The present theory satisfies all the important necessary conditions such as transverse shear stress continuity condition at interface and zero transverse shear stress at top and bottom surfaces of the plate. Nine-noded C-0 finite element having eleven degrees of freedom per node is used during analysis. Also, the present model is free from any kind of penalty function requirements. The present results are compared with those available in literature in order to study the applicability of the present model.

2. MATHEMATICAL MODELLING

Cubic variation of displacement field along x- and y-axis can be stated as (Figure 1):
\[ U^{(x)} = u^{(0)} + x\psi^{(x)} + \sum_{i=1}^{n(u)-1} (z - z_i^{(u)}) H(z - z_i^{(u)}) E_i^{(xu)} + \sum_{j=1}^{n(l)-1} (z - z_j^{(l)}) H(z - z_j^{(l)}) \]

\[ V^{(y)} = v^{(0)} + y\psi^{(y)} + \sum_{i=1}^{n(u)-1} (z - z_i^{(u)}) H(z - z_i^{(u)}) E_i^{(yu)} + \sum_{j=1}^{n(l)-1} (z - z_j^{(l)}) H(z - z_j^{(l)}) \]

(1)

\[ W(z) = l^{(1)} w^{(u)} + l^{(2)} w^{(0)} + l^{(3)} w^{(l)} (\text{for core}) + w^{(u)} (\text{for upper face layer}) = w^{(u)} (\text{for lower face layer}) \]

(3)

For an orthotropic lamina, the stress-strain relationship can be written as:

\[ \sigma \text{net} = [\bar{Q}_k][\bar{e}]_t - [\alpha] \Delta T \text{ or } \bar{\sigma} = [\bar{Q}_k][\bar{e}]_{\text{net}} \]

(4)

Where, \( [\bar{e}]_{\text{net}} = [\bar{e}] - [\bar{e}]_t \)

Imposing shear stress continuity condition in transverse direction at layer interface, zero or transverse shear stress free condition at top and bottom of plate and conditions of \( U = u^{(u)}, V = v^{(u)} \) at top of the plate and \( U = u^{(l)}, V = v^{(l)} \) at bottom of the plate,

\[ \zeta^{(x)}, \zeta^{(y)}, \mu^{(x)}, \mu^{(y)}, E_i^{(1x)}, E_i^{(1y)}, E_j^{(1x)}, E_j^{(1y)}, (\partial w_u/\partial x), (\partial w_u/\partial y), (\partial w_l/\partial x), (\partial w_l/\partial y) \]

can be written in form of displacements \( u^{(0)}, v^{(0)}, \psi^{(x)}, \psi^{(y)}, u^{(u)}, u^{(l)}, u^{(u)}, u^{(l)}, v^{(0)}, v^{(l)} \) as:

\[ [B] = \begin{bmatrix} \zeta^{(x)} \eta^{(x)} & \zeta^{(y)} \eta^{(y)} & E_1^{(xu)} & E_2^{(xu)} & \cdots & E_{n(u)-1}^{(xu)} & E_1^{(yu)} & E_2^{(yu)} & \cdots & E_{n(u)-1}^{(yu)} \\ \frac{\partial w^{(u)}}{\partial x} & \frac{\partial w^{(l)}}{\partial x} & \frac{\partial w^{(u)}}{\partial y} & \frac{\partial w^{(l)}}{\partial y} & \frac{\partial v^{(u)}}{\partial x} & \frac{\partial v^{(l)}}{\partial x} & \frac{\partial v^{(u)}}{\partial y} & \frac{\partial v^{(l)}}{\partial y} \end{bmatrix}^T [u^{(0)}, v^{(0)}, \psi^{(x)}, \psi^{(y)}, u^{(u)}, u^{(l)}, v^{(u)}, v^{(l)}] \]
or \[ [B] = [A][F] \] (5)

Elements of [A] are function of material properties. Since last four entries in [B] represents derivative of transverse displacement field at top and bottom of plate in terms of elements of \{F\} helps in eliminating problem associated with C-1 continuity requirements.

Using Eq. (3), (6) & (7) the generalised displacement vector \{\nu\} can be written as:

\[
U = b^{(1)}u^{(0)} + b^{(2)}v^{(0)} + b^{(3)}\psi^{(x)} + b^{(4)}\psi^{(y)} + b^{(5)}u^{(u)} + b^{(6)}v^{(u)} + b^{(7)}u^{(l)} + b^{(8)}v^{(l)}
\]

(6)

\[
V = c^{(1)}u^{(0)} + c^{(2)}v^{(0)} + c^{(3)}\psi^{(x)} + c^{(4)}\psi^{(y)} + c^{(5)}u^{(u)} + c^{(6)}v^{(u)} + c^{(7)}u^{(l)} + c^{(8)}v^{(l)}
\]

(7)

Coefficients \(b^{(i)}\)'s and \(c^{(i)}\)'s are function of thickness coordinates, material properties and unit step function.

Applying four additional conditions of satisfying displacement along \(x\)- and \(y\)- directions at top and bottom of plate, derivatives of transverse displacements can be written in form of nodal field variables appearing in Eq. (5). Hence, Eq. (6) & (7) is free from any kind of C-1 continuity requirements without defining new field variables or incorporating any kind of penalty function.

Using Eq. (3), (6) & (7) the generalised displacement vector \{\nu\} can be written as:

\[
\{\nu\} = \{u^{(0)}v^{(0)}w^{(0)}\psi^{(x)}\psi^{(y)}u^{(u)}v^{(u)}w^{(u)}u^{(l)}v^{(l)}w^{(l)}\}^T
\]

With the help of unknowns, linear train-displacement relationships can be written as:

\[
\{\varepsilon\}_{net(6x1)} = \left[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \frac{\partial w}{\partial x} \frac{\partial u}{\partial y} \frac{\partial w}{\partial x} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} \right] or = [H]_{(6x33)}\{\varepsilon\}_{(33x1)}
\]

(8)

Where

\[
\{\varepsilon\} = \left[ \begin{array}{c}
\{u^{(0)}v^{(0)}w^{(0)}\psi^{(x)}\psi^{(y)}u^{(u)}v^{(u)}w^{(u)}u^{(l)}v^{(l)}w^{(l)}(\partial u^{(0)}/\partial x)(\partial u^{(0)}/\partial y)(\partial v^{(0)}/\partial x)(\partial v^{(0)}/\partial y)(\partial w^{(0)}/\partial x)(\partial w^{(0)}/\partial y)
\end{array} \right]
\]

\[
\times \left( \frac{\partial w^{(0)}}{\partial x} ) \frac{\partial w^{(0)}}{\partial y} ) (\partial \theta^{(x)}/\partial x ) (\partial \theta^{(y)}/\partial y ) (\partial \theta^{(x)}/\partial y ) (\partial \theta^{(y)}/\partial y ) (\partial u^{(u)}/\partial x ) (\partial u^{(u)}/\partial y ) \\
(\partial v^{(u)}/\partial x ) (\partial v^{(u)}/\partial y ) (\partial w^{(u)}/\partial x ) (\partial w^{(u)}/\partial y ) (\partial u^{(l)}/\partial x ) (\partial u^{(l)}/\partial y ) \\
(\partial v^{(l)}/\partial x ) (\partial v^{(l)}/\partial y ) (\partial w^{(l)}/\partial x ) (\partial w^{(l)}/\partial y ) \right]
\]

Elements of [H] are function of unit step function and \(z\).

Using nine-noded quadratic finite element having eleven degrees of freedom per node \{\nu^{(0)}, v^{(0)}, w^{(0)}, \psi^{(x)}, \psi^{(y)}, u^{(u)}, v^{(u)}, w^{(u)}, u^{(l)}, v^{(l)}, w^{(l)}\}, generalised displacement vector can be written as:
\[ Y = \sum_{i=1}^{11} N_i y_i \]  

Using above Eq., Eq. (8) can be written in term of unknowns as:
\[ \{ \varepsilon \}_b(6 \times 1) = [B](6 \times 99)\{ Y \}_b(99 \times 1) \]  

Where [B] is strain-displacement relationship in Cartesian coordinate system.

For any point in plate, displacement field can be expressed as:
\[ \{ \chi \} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = [P]\{ \chi \} \]  

Elements contained in \( \chi \) are function of unit step and \( z \).

Differentiating above Eq. with respect to time to get velocity for any point in plate can be stated as:
\[ \frac{d\{ \chi \}}{dt} = [P]\frac{d\{ \chi \}}{dt} \]  

With the help of Eq. (11) & (12), Lagrangian of plate system can be stated as:
\[ L = KE - U_s - W_t \]  

Where KE is kinetic energy, \( U_s \) is the strain energy and \( W_{ht} \) is the work done due to thermal load which can be written as:
\[ KE = \frac{1}{2} \sum_{k=1}^{n} \int \int \int \frac{d\{ \chi \}}{dt}^T \rho_k \frac{d\{ \chi \}}{dt} dxdydz \]  
\[ U_s = \frac{1}{2} \sum_{k=1}^{n} \int \int \{ \varepsilon \}_{net}^T [Q_k] \{ \varepsilon \}_{net} dxdydz \]  
\[ W_{ht} = \{ \gamma \}^T \{ P_t \} \]  

With the help of Eq. (9), displacement vector \( \{ y \} \) as per Eq. (11) can be written as:
\[ \{ y \} = [Z]\{ y \} \]  

Applying Hamilton’s principle, equation of motion for the plate system can be written as:
\[ \gamma^T Ldt = 0 \]  

With the help of Eq. (18), generalized eigen value problem is given by:
\[ [K][\Lambda] = \omega_n^2 [M][\Lambda] \]  

\( \omega_n \) is natural frequency, \{\Lambda\} is the eigen vector defining mode shapes, \([M]\) is total mass matrix, \([K]\) is total stiffness matrix given by \([K_e] - [K_t]\).

Now minimizing the potential energy as given by above Eq. with respect to \{\gamma\} as:
\[ [K_e]\{ y \} = [P_e^T] \]  

Now taking the contribution from all the plate elements, global stiffness matrix and global thermal load vector is then calculated. Finally, these equations are solved after incorporating concerned boundary condition. Now the stresses can be computed with the help of stress-strain relationship along with continuity conditions as per Eq. (5).

Above discussed model is implemented in FORTRAN 90 code for calculating stresses and displacement in LCS plates under thermal condition. The obtained results are compared with those already available in literature in order to study the feasibility of proposed model.
3. RESULTS AND DISCUSSION

Convergence and Validation studies: At first, convergence and validation studies are carried out on the 10-layered square shaped angle-ply SSSS laminated composite plate \([\theta^0/-\theta^0/\theta^0/-\theta^0/\ldots]_{10}\) subjected to equal rise of temperature at its top and bottom surfaces with bi-sinusoidal variation. Material properties are taken as \(E_1/E_2 = 15, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.3356E_2, v_{12} = 0.3, v_{23} = 0.49, \alpha_1/\alpha_0 = 0.015, \alpha_2/\alpha_0 = 1, \rho = 1, \alpha_0 = 10^{-6}\). Results for non-dimensional frequencies (\(\lambda = oh\sqrt{\rho/E_2}\)) are reported in Table 1. It can be seen that the present results converge at mesh size of 12 x 12, hence, in further studies same mesh size is used. For validation, present behavior is validated with those given by Matsunaga [9] using global-local higher order theory (GLHOT). Present results are found in good agreement. It can be seen that the \(\lambda\) is maximum at \(\theta=45^\circ\) while at other values of \(\theta\), values are lower. Also, the values are symmetric about 45°. With increase in value of thickness ratio (h/a), non-dimensional frequency also increases.

Table 1. Convergence and validation studies of non-dimensional natural frequencies for 10-layered \([\theta^0/-\theta^0/\theta^0/-\theta^0/\ldots]_{10}\) square shaped SSSS laminated composite plate under equal rise of temperature at top and bottom surface (index to values indicate power to 10).

| h/a Source | 0° | 15° | 30° | 45° | 60° | 75° | 90° |
|------------|----|----|----|----|----|----|----|
| Present (4x4) | 0.1241 | 0.1342 | 0.1528 | 0.1612 | 0.1528 | 0.1342 | 0.1241 |
| Present (8x8) | 0.1233 | 0.1334 | 0.1520 | 0.1604 | 0.1520 | 0.1334 | 0.1233 |
| Present (12x12) | 0.1230 | 0.1331 | 0.1517 | 0.1601 | 0.1517 | 0.1331 | 0.1230 |
| Present (16x16) | 0.1230 | 0.1331 | 0.1517 | 0.1601 | 0.1517 | 0.1331 | 0.1230 |
| GLHOT [9] | - | 0.1328 | 0.1510 | - | 0.1595 | - | - |
| GLHOT [9] | 0.4902 | 0.5035 | 0.6040 | 0.6376 | 0.6039 | 0.5304 | 0.4902 |
| GLHOT [9] | - | 0.5286 | 0.6009 | 0.6342 | - | - | - |
| GLHOT [9] | 0.2985 | 0.3227 | 0.3651 | 0.3836 | 0.3650 | 0.3226 | 0.2985 |
| GLHOT [9] | - | 0.3202 | 0.3623 | 0.3811 | - | - | - |
| GLHOT [9] | 0.1095 | 0.1178 | 0.1309 | 0.1359 | 0.1318 | 0.1184 | 0.1099 |
| GLHOT [9] | - | 0.1163 | 0.1298 | 0.1355 | - | - | - |
| GLHOT [9] | 0.3489 | 0.3738 | 0.4034 | 0.4122 | 0.4034 | 0.3737 | 0.3489 |

Laminated sandwich plate: In this example a 21-layer angle-ply (\(\theta = 45^\circ\)) sandwich plate has been analyzed with the same temperature conditions as in the previous example. Material properties are taken as: Face sheets (total thickness \(h_0E_1/E_2 = 19, G_{12} = G_{13} = 0.52E_2, G_{23} = 0.338E_2, v_{12} = 0.32, v_{23} = 0.49, \alpha_1/\alpha_0 = 0.001, \alpha_2/\alpha_0 = 1.0;\) Core (thickness \(h_c\)) \(E_1 = 3.2 \times 10^5 GPa, E_2 = 2.9 \times 10^5 GPa, E_3 = 0.4 GPa, G_{12} = 2.4 \times 10^5 GPa G_{13} = 7.9 \times 10^2 GPa, G_{23} = 6.6 \times 10^2 GPa, v_{12} = 0.99, v_{13} = v_{23} = 3.0 \times 10^5, \alpha_1 = \alpha_2 = 1.36 \alpha_0\). Results for non-dimensional frequency are reported in Table 2. The present model is able to predict results in an efficient manner for sandwich plates.

Table 2. Non-dimensional natural frequency of angle-ply \([\theta/\theta/\ldots/\theta]_{10} [Core]\ldots/\theta/\theta/\ldots/\theta]_{10}\) sandwich square plate (\(\theta = 45^\circ\)) having simply supported boundary conditions for different thickness ratio

| h/h Source | 0.01 | 0.02 | 0.05 | 0.1 | 0.2 |
|------------|------------------|------------------|------------------|------------------|------------------|
| Present | 0.2418 | 0.2426 | 0.2422 | 0.2427 | 0.2428 |
| GLHOT [9] | 0.2426 | - | - | - | - |
| Present | 0.9509 | 0.9509 | 0.9509 | 0.9509 | 0.9509 |
| GLHOT [9] | 0.9509 | - | - | - | - |
| Present | 0.2180 | 0.2180 | 0.2180 | 0.2180 | 0.2180 |
| GLHOT [9] | 0.2180 | - | - | - | - |
| Present | 0.3858 | 0.3858 | 0.3858 | 0.3858 | 0.3858 |
| GLHOT [9] | 0.3858 | - | - | - | - |
Laminated sandwich plate with different boundary conditions: In this example, five-layer cross-ply (0°/90°/Core/90°/0°) sandwich plate has been solved for different boundary conditions for \(h/h_f = 0.3\) for the same loading conditions and material properties as used in the previous example. The results for same with different boundary conditions (at least two boundaries of the plate are simply supported) are reported in Table 3. It can be seen that the minimum value of \(\lambda\) is observed for SSFF boundary condition while maximum for SSCC condition. Thus, boundary condition widely affects the behavior of the plate.

Table 3. Non-dimensional natural frequency of a simply supported laminated square sandwich plate (0°/90°/Core/90°/0°) for different thickness ratio.

| Boundary Condition | \(h/a\) | \(0.01\) | \(0.02\) | \(0.05\) | \(0.10\) | \(0.20\) |
|-------------------|---------|--------|--------|--------|--------|--------|
| SSSS              | 1.096 \(^2\) | 0.4337 \(^2\) | 0.2500 \(^1\) | 0.8099 | 0.2108 |
| SSCC              | 0.1359 \(^2\) | 0.5341 \(^2\) | 0.2935 \(^1\) | 0.8863 | 0.2195 |
| SCCCC             | 0.1742 \(^2\) | 0.6728 \(^2\) | 0.3458 \(^1\) | 0.9731 \(^1\) | 0.2319 |
| SSSF              | 0.8058 \(^1\) | 0.3176 \(^2\) | 0.3100 \(^1\) | 0.5977 \(^1\) | 0.1565 |
| SSSF              | 0.7850 \(^1\) | 0.3091 \(^2\) | 0.1788 \(^1\) | 0.5831 \(^1\) | 0.1520 |

4. CONCLUSION

In present work, an attempt has been made to carry out the free vibration analysis of laminated composite sandwich plates under thermal conditions using C-0 HOZT. The present theory satisfies all the important conditions such as transverse shear stress free condition at bottom and top surfaces of the plate along with the continuity of the same at interfaces. The present FE based model is free from any requirements of penalty function and is thus computationally efficient. The present model is able to predict results with great accuracy without any post-processing technique or penalty requirements. A number of problems are solved in order to study the dependency of the free vibration behavior of the plate on various parameters such as angle of ply, thickness of plate, boundary condition etc. Thus, the present model can be applied for research work and industrial applications on same.

ACKNOWLEDGMENTS

First author thanks NIT Kurukshetra and MHRD, GoI for providing financial assistance through PhD scholarship grant (2K17/NITK/PHD/6170004) for research work. The authors also thank Dr. Anupam
Chakrabarti (Professor, IIT Roorkee) for allowing to use computational facility.

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