Incompressible Quantum Hall Fluid

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Abstract

After review the quantum Hall effect on the fuzzy two-sphere $S^2$ and Zhang and Hu’s 4-sphere $S^4$, the incompressible quantum Hall fluid on $S^2$, $S^4$ and torus are discussed respectively. Next, the corresponding Laughlin wavefunctions on $S^2$ are also given out. The ADHM construction on $S^4$ is discussed. We also point out that on torus, the incompressible quantum Hall fluid is related to the integrable Gaudin model and the solution can be given out by the Yang Bethe ansatz.

PACS: 11.90.+t, 11.25.-w
Keywords: quantum Hall effect, incomressible quantum Hall fluid, noncommutative geometry.

1 Introduction

Quantum Hall Effects (QHE) have been an important research subject in condensed matter physics and theoretical physics. A decade before, Laughlin [1] proposed the incompressible quantum Hall fluid (QHF) formulation for fractional quantization of the Hall effect (FQHF). Soon later Haldane [2] considered a 2 dimensional electron gas of $N$ particles moving on a spherical surface of radius $R$ in a radial (Dirac monopole) magnetic field $B = \frac{\hbar S}{e R^2}$, where $2S$ is an integer as required by Dirac’s quantization condition. This describes a translational (actually rotational) invariant version of the incompressible quantum Hall fluid (IQHF).

Last year, Susskind et al. constructed the noncommutative Chern-Simons field theory for the QHF on $\mathbb{R}^2$ [3] and $S^2$ [4]. Zhang and Hu [5] generalized the QHE to a 4 dimensional sphere with noncommutative geometry. In the $\mathbb{R}^2$ case, Polychronokas [7] proposed a regularized version of the noncommutative theory on the plane in the form of a finite Chern-Simons matrix model with boundary field $\psi$ (equivalently a Wilson loop [8]). Hellerman and Raamsdonk [9] suggested the corespondent Laughlin-type wavefunctions described by Chern-Simons matrix. Karabali and Sakita [10] gave an explicit form of Laughlin wavefunction in terms of the energy eigenfunctions of Calogero model. These works attract the interesting for QHE and QHF in connection with the string and brane theory [11, 12].

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The purpose of this paper is to generalize the description of the QHF which has been obtained on the noncommutative (N.C.) plane \( \mathbb{R}^2 \) and fuzzy two-sphere to Zhang and Hu’s 4-sphere and torus. In the next section, we first review the Haldane’s description of the QHE on 2-sphere, the noncommutative algebra and the Moyal structure of the Hilbert space is also shown in this section. After review Zhang and Hu’s generalization of the QHE to 4-sphere, we give out the noncommutative algebra with the Moyal structure on fuzzy 4-sphere in section 3. In section 4, by using the method suggested by Susskind [3] and Polychronakos [7, 13], we construct the N.C. Chern-Simons theory to describe the incompressible QHF on \( S^2 \) and give out the algebra structure on \( S^2 \). The corresponding incompressible QHF theory for Zhang and Hu’s 4-sphere is given in section 5. In section 6, we construct the description of the incompressible QHF on torus and relate it to the integrable Gaudin model which can be solved by the Bethe-Yang ansatz. In the last section, we shortly give some subjects which will be investigated later.

2 The one particle wavefunctions in LLL on the first Hopf fibration \( S^2 \)

On the two-dimensional sphere \( S^2 = (S^3 \sim SU(2))/ (S^1 \sim U(1)) \), the Hamiltonian of a single particle with charge \( e \) moving around a Dirac monopole is

\[
H = \frac{|\vec{\Lambda}|^2}{2MR^2} = \frac{\omega_C|\vec{\Lambda}|^2}{2\hbar S},
\]

here \( M \) is the effective mass, \( R \) is the radius of \( S^2 \) and \( \omega_C = \frac{eB}{M} \) is the cyclotron frequency. \( \vec{\Lambda} = \vec{r} \times [-i\hbar \nabla + e\vec{a}(\vec{r})] \) is the orbital angular momentum with the \( U(1) \) gauge field \( \vec{a} \) which the particle coupled with: \( \nabla \times \vec{a} = B\hat{r} \) and \( B = \frac{\hbar S}{eB^2} \) is the magnetic field strength and \( \hat{r} = \frac{\vec{r}}{|\vec{r}|} \). The commutation of the dynamical angular momentum \( \vec{\Lambda} \) is \([\Lambda^a, \Lambda^b] = i\hbar \epsilon^{abc} \Lambda^c \) which is not closed. This Hamiltonian is rotation invariant and the corresponding group manifold is \( S^3 \). The Euler angles \( \alpha, \beta, \gamma \) of \( SU(2) \) is generated by the total angular momentum \( \vec{L} \):

\[
\vec{L} = \vec{r} \times [-i\hbar \nabla + e\vec{a}(\vec{r})] + \hbar S \frac{\vec{r}}{|\vec{r}|} \equiv \vec{\Lambda} + \hbar \vec{S} = \vec{\Lambda} + \hbar \vec{S}^2
\]

which satisfies the commutation relations [3]

\[
[L^a, T^b] = i\hbar \epsilon^{abc} T^c, \quad \vec{T} = \vec{L}, \vec{r} \text{ or } \vec{\Lambda}
\]

The operators \( \vec{S}^2, \vec{L}^2, L^3 \) and \( \vec{\Lambda}^2 \) can be diagonalized simultaneously and their common eigenfunctions are

\[
\Psi^J_{m,S} = D^J_{m,S}(\alpha, \beta, \gamma), \quad J \geq |S|, m = -J, \ldots, J,
\]

here \( D^J_{m,S} \)'s are the finite rotation matrix elements and the indices \( J, m \) and \( S \) are the eigenvalues of the operators \( \vec{L}^2, L^3 \) and \( \vec{S}^2 \) respectively. The Euler angles \( \omega = (\alpha, \beta, \gamma) \)
equal $\alpha = \phi, \beta = \theta$ and $\gamma = \gamma(\phi, \theta)$ which has the $U(1)$ gauge freedom. In LLL ($J = S$ with energy $\frac{1}{2} \hbar \omega_C$), these wavefunctions become

$$\Psi^S_m = D^S_{m,-S}(\alpha, \beta, \gamma), \quad m = -S, \ldots, S,$$

$$= (-1)^{S+m} \sqrt{\frac{2S!}{(S+m)!(S-m)!}} u^{S-m} v^{S+m},$$

$$= (1 + |\xi|^2)^{-\frac{S}{2}} e^{iS\gamma} \sqrt{\frac{2S!}{(S+m)!(S-m)!}} u^{S} v^{S},$$

(5)

where the geodesic projection coordinate $\xi = \tan(\frac{\theta}{2}) \exp(-i\phi)$ and $u, v$ are the spinor variables

$$u = \cos \frac{\theta}{2} \exp(i\frac{i\phi + i\gamma}{2}) = D_{\frac{1}{2}, -\frac{1}{2}}^{\phi, \theta, \gamma}(\phi, \theta, \gamma),$$

$$v = \sin \frac{\theta}{2} \exp(-i\frac{i\phi + i\gamma}{2}) = D_{\frac{1}{2}, \frac{1}{2}}^{\phi, \theta, \gamma}(\phi, \theta, \gamma).$$

(6)

For this Hopf fibration (bundle) $S^2 = SU(2)/U(1)$, our base space turns to be a Kähler manifold with the metric defined as

$$ds^2 = \frac{4d\xi d\overline{\xi}}{(1 + |\xi|^2)^2},$$

(7)

and the corresponding symplectic structure (or the Kähler form) is

$$\Omega = 2i \frac{d\xi \wedge d\overline{\xi}}{(1 + |\xi|^2)^2} = 2i \frac{\partial^2 K}{\partial \xi \partial \overline{\xi}} d\xi \wedge d\overline{\xi},$$

(8)

where $K = \ln(1 + |\xi|^2)$ is the Kähler potential.

The Hilbert space $\mathcal{H}_N$ on this Hopf fibration $S^2$ is composed by the $N = 2S + 1$ one particle wavefunctions $\Psi^S_m, (m = -S, \ldots, S)$ around the Dirac monopole $g$ ($S = ge$). The operators acting on these $2S + 1$ states are covariant $(2S + 1) \times (2S + 1)$ matrices, which as the irreducible tensorial set should be (after normalized)

$$(X^J_M)_{m,m'} = \left( \begin{array}{ccc} S & J & S \\ -m & M & m' \end{array} \right), \quad 0 \leq J \leq 2S, \quad 0 \leq M \leq J,$$

(9)

where $:::;$ denotes the $3j$-symbol. These operators constitute the right module $\mathcal{A}_N$ of N. C. algebra $\mathcal{A}_N$ on fuzzy sphere simply with the matrix product $\mathcal{A}_N$:

$$\sum_m (X^J_{m_1})_{mn}(X^J_{m_2})_{nm} = \sum_{J,M,J_1,J_2} \left( \begin{array}{ccc} J_1 & J_2 & J \\ m_1 & m_2 & M \end{array} \right) \left\{ \begin{array}{ccc} J_1 & J_2 & J \\ S & S & S \end{array} \right\} (X^J_M)_{in},$$

(10)

where $:::;$ is the $6j$-symbol. This matrix product is equivalent to the Moyal product on fuzzy sphere $\mathcal{G}$. First, the coherent states which were found by Peremolov $\mathcal{G}$ on the coset space $J^2 = S^2$ in geodesic projection coordinates are

$$|\omega\rangle \equiv |\theta, \phi\rangle = \sum_m D^S_{m,-S}(\phi, \theta, -\phi)|S, m\rangle,$$

(11)
which satisfy
\[ \frac{N}{8\pi^2} \int d\omega |\omega| = 1, \quad N = 2S + 1. \] (12)

The Symbol of the operator \( X_M^J \) is then defined as
\[
D_M^J(\theta, \phi) \equiv (\omega|X_M^J|\omega) = D_{M,-J}^{J}(\phi, \theta, -\phi)
= \sqrt{\frac{2J!}{(J+M)!(J-M)!}} u^{J+M} v^{J-M}
\] (13)
and
\[
X_M^J = \frac{2S+1}{4\pi} \int d\omega D_M^J(\theta, \phi) |\omega(\theta, \phi)|. \] (14)

The symbol of the product of two operators \( X_{M_1}^{J_1} \) and \( X_{M_2}^{J_2} \) equals the "star product" (Moyal product) of two symbols, namely
\[
D_{M_1}^{J_1} \star D_{M_2}^{J_2} = \sum_{J,M} (-1)^{J_2-J_1-M} (2J + 1) 
\times \left( \begin{array}{ccc} J_1 & J_2 & J \\ M_1 & M_2 & M \end{array} \right) \left\{ \begin{array}{ccc} J_1 & J_2 & J \\ S & S & S \end{array} \right\} D_M^J. \] (15)

This is a special case for the \(*\) product of the right module of \( A_N \) on \( S^2 \) [14].

3 LLL wave functions as the sections of the 2nd Hopft bundle

On the 2nd Hopf bundle (fibration), Zhang and Hu’s 4 dimensional fuzzy sphere
\[ S^4 = (S^7 \sim Sp(4)/SU(2))/(S^3 \sim SU(2)) \], which is equivalent to \( SO(5)/(SO(3) \otimes SO(3)) \),
the Hamiltonian of one particle is
\[
H = \frac{\hbar^2}{2MR^2} \sum_{a<b} \Lambda_{ab}^2. \] (16)

This Hamiltonian is similar as (11) in the \( S^2 \) case, but here the particle is coupling with a
\( SU(2) \) gauge field \( A_a \). \( \Lambda_{ab} \) is the dynamical angular momentum given by \( \Lambda_{ab} = -i(x_a D_b - x_b D_a) \), where \( D_a = \partial_a + A_a \) is the covariant derivative and corresponding field strength is \( f_{ab} = [D_a, D_b] \). The symmetry of \( S^4 \) is \( SO(5) \), but \( \Lambda_{ab} \) does not satisfy the commutation rules of \( SO(5) \). The one particle angular momentum of the Yang’s \( SU(2) \) monopole is \( L_{ab} = \Lambda_{ab} - f_{ab} \) which obey the \( SO(5) \) commutation rules.

The irreducible representaiton of \( SO(5) \) is labeled by two integers \( (r_1, r_2) \) with \( r_1 \geq r_2 \geq 0 \) and for such representation, the Casimir operator and its dimensionality are given by \( C(r_1, r_2) = \sum_{a<b} L_{ab}^2 = r_1^2 + r_2^2 + 3r_1 + r_2 \) and \( D(r_1, r_2) = \frac{1}{6}(1 + r_1 - r_2)(1 + 2r_2)(2 + r_1 + r_2)(3 + 2r_1) \) respectively. The \( SU(2) \) gauge potential is valued by the \( SU(2) \) Lie algebra \([I_i, I_j] = i\epsilon_{ijk} I_k \) and the corresponding Casimir is \( \sum_i I_i^2 = I(I + 1) \) which specifies the dimensions of the \( SU(2) \) representation in the monopole potential. So for a given \( I \), from the Hamiltonian (16), we can read out that the degeneracy of the energy level is given by the dimensionality of the corresponding irreducible representation \( D(r_1, r_2) \).
The ground state, which is the lowest $SO(5)$ level labeled by $(r_1 = I = \frac{p}{2}, r_2 = I = \frac{q}{2})$, is $N = \frac{1}{6}(p+1)(p+2)(p+3)$ fold degenerate, here $N$ also indicates the instanton number. The explicit expressions of the ground state wave functions of the Hilbert space $\mathcal{H}_N$ in the spinor coordinates were given by Zhang and Hu in Ref. [5]:

$$\Psi_{r_1,r_2}^{k_1,k_2,k_3,k_4,I,I_z}(\theta, \alpha, \beta, \gamma, \alpha_I, \beta_I, \gamma_I) = \sqrt{\frac{p!}{m_1!m_2!m_3!m_4!}} \Psi_1^{m_1} \Psi_2^{m_2} \Psi_3^{m_3} \Psi_4^{m_4},$$

where $r_1 = r_2 = I = \frac{p}{2}$, $p = m_1 + m_2 + m_3 + m_4$ and $k_1 = \frac{m_1 + m_2}{2}$, $k_2 = \frac{m_1 - m_2}{2}$, $k_3 = \frac{m_3 + m_4}{2}$, $k_4 = \frac{m_3 - m_4}{2}$ are the eigenvalues of the angular momentum of the stable (keeps $\vec{r}$ invariant) subgroup $SO(3) \otimes SO(3)$ and $I, I_z$ are the eigenvalues of the $SU(2)$ isospin.

The base space $S^4$ can be parameterized by the following coordinate systems

$$x_1 = \sin \theta \sin \frac{\beta}{2} \sin(\alpha - \gamma),$$
$$x_2 = -\sin \theta \sin \frac{\beta}{2} \cos(\alpha - \gamma),$$
$$x_3 = -\sin \theta \cos \frac{\beta}{2} \sin(\alpha + \gamma),$$
$$x_4 = \sin \theta \cos \frac{\beta}{2} \cos(\alpha + \gamma),$$
$$x_5 = \cos \theta,$$

where $\theta, \beta \in [0, \pi)$ and $\alpha, \gamma \in [0, 2\pi)$. Next, the orbital coordinates $x_a$ ($a = 1, \cdots, 5$), which describe the point on the $S^4$ by $x_a = Rx_a$, is related the spinor coordinates $\Psi_\alpha$ ($\alpha = 1, 2, 3, 4$) through the relation

$$x_a = \bar{\Psi}_\Gamma_a \Psi = \bar{\Psi}_\alpha (\Gamma_a)_{\alpha \alpha'} \Psi_{\alpha'}, \quad \sum_{\alpha} \bar{\Psi}_\alpha \Psi_\alpha = 1,$$

where $\Gamma_a, (a = 1, \cdots, 5)$ are the five $4 \times 4$ Dirac matrices which satisfy the Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. The explicit solution of eq. (20) is given by Zhang and Hu [5]:

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{\frac{1 + x_5}{2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \sqrt{\frac{1}{2(1 + x_5)}} (x_4 - ix_i \sigma_i) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $(u_1, u_2)$ is an arbitrary two components complex vector satisfying $\sum_a \bar{u}_a u_a = 1$ and it has a rotation $SU(2)$ symmetry which maps to the same point $x_a$ on $S^4$. The corresponding group manifold is $S^3$ and the direction of the $SU(2)$ isospin is specified by the Euler angles $\alpha_I, \beta_I, \gamma_I$, which describe the spinors in $\Psi$:

$$u_1 = \cos \frac{\beta_I}{2} \exp(i\frac{\alpha_I + \gamma_I}{2}), \quad u_2 = \sin \frac{\beta_I}{2} \exp(-i\frac{\alpha_I - \gamma_I}{2}).$$

The geometric connection gives out the $SU(2)$ gauge potential $A_a$ of the Yang’s monopole defined on $S^4$:

$$A_a = \frac{-i}{1 + x_5} \eta_{ab}^i x_b I_i, \quad \eta_{ab}^i = \epsilon_{iabc} + \delta_{ia} \delta_{ib} - \delta_{ib} \delta_{ia},$$

where $I_i = \frac{a_i}{2}$ and $\eta_{ab}^i$ is known as the t’Hooft tensor.
By using the parameterization of $S^4$ and the choice of the isospin coordinates, the one-particle wavefunction can be rewritten as:

$$
\Psi^{(r_1,r_2)}_{k_1,k_1z,k_2z,r_2,\rho} = (\sin \theta)^{-1}(1 - \cos \theta)^{k_1 + \frac{1}{2}}(1 + \cos \theta)^{-k_2 - \frac{1}{2}}
\times P_{r_1+1+k_2-k_1}^{k_1+\frac{1}{2},-k_2-\frac{1}{2}}(\cos \theta)U_{k_1,k_1z,k_2z,r_2,\rho},
$$

where $P_n^{\alpha,\beta}$ is the Jacobi polynomial and

$$
U_{k_1,k_1z,k_2z,r_2} = \sum_{m,l_z}(k, m; I, I_z|k_2, k_2z)D_{m,k_1z}^{k_1}(\alpha, \beta, \gamma)D_{l_z}^{l_2}(\alpha_x, \beta_x, \gamma_x)|I, I'_z). \tag{25}
$$

These one-particle wavefunctions are the Yang’s $SU(2)$ monopole harmonics, i.e. the spherical harmonics on the coset space $SO(5)/SU(2)$, which is locally isomorphic to the sphere $S^4 \times S^3$.

The Hilbert space $\mathcal{H}_N$ on this second Hopf bundle (fibration) $S^4$ is composed by all these one-particle wavefunctions. The completeness of these one-particle wavefunctions is ensured by the following relation:

$$
\sum_{p \geq r_1 \geq r_2 \geq 0} D[r_1, r_2] = \frac{1}{6}(p + 1)(p + 2)(p + 3)^2 = N \times N, \tag{26}
$$

where $D[r_1, r_2]$ is the dimensionality the representation of the $SU(2)$ gauge group given before which indicates the degeneracy of the energy level.

In the LLL level, using the one-particle wavefunctions as the basis of the Hilbert space $\mathcal{H}_N$, the operators acting on these $N$ states are covariant $N \times N$ matrices and they can be also expressed by the Yang’s $SU(2)$ monopole harmonics $T^{(r_1,r_2)}_{k_1,k_1z,k_2z,r_2,\rho}$ and $p \geq r_1 \geq r_2 \geq 0$, $J_1, J_2$ in Yang’s lattice $(\frac{1}{2}, \frac{2}{2}), (\frac{2}{2}, \frac{1}{2}), (\frac{3}{2}, 0), (0, \frac{3}{2})$, $p = r_1 + r_2, q = r_1 - r_2$, but herein the spinor $|I, I_z\rangle$ composed from $u_1, u_2$ is replaced by operators $\Sigma^r_{\rho}$ fused from Pauli matrices $\sigma_i$. These operators also constitute the right module of the N.C. algebra $\mathcal{A}_N$ on $S^4$.

By making use of the integral formula about the Jacobi polynomial and the product of the $SU(2)$ $D$-functions, the matrix elements of the operator $T^R_J (R = [r_1, r_2], J = (j_1, j_1z; j_2, j_2z; r_2, \rho))$ are read as:

$$
\left\langle \begin{array}{c|c|c}
\frac{p}{K^1} & R & \frac{p}{K^2}
\end{array} \right| T^R_J \left| \begin{array}{c|c|c}
\frac{p}{K^1} & J & \frac{p}{K^2}
\end{array} \right\rangle = \left\langle \begin{array}{c|c|c}
\frac{p}{K^1} & R & \frac{p}{K^2}
\end{array} \right| J \left| \begin{array}{c|c|c}
\frac{p}{K^1} & J & \frac{p}{K^2}
\end{array} \right\rangle \times \left\langle \begin{array}{c|c|c|c|c|c}
K^1_1 & j_1 & K^2_1 & I & r_1 & I \\
K^2_1 & j_2 & K^1_2 & I & r_2 & I
\end{array} \right\rangle \times \left\langle \begin{array}{c|c|c|c|c|c}
K^1_{1z} & j_{1z} & K^2_{1z} & I & r_1 & I \\
K^2_{1z} & j_{2z} & K^1_{2z} & I & r_2 & I
\end{array} \right\rangle, \tag{27}
$$

where $\left\langle \begin{array}{c|c|c}
\frac{p}{K^1} & R & \frac{p}{K^2}
\end{array} \right| J \left| \begin{array}{c|c|c}
\frac{p}{K^1} & J & \frac{p}{K^2}
\end{array} \right\rangle$ is independent of the $SU(2)$ magnetic quantum numbers, and it is expressed by the product of two factors. One is the contribution of the integral with respect to the variables $\alpha, \beta, \gamma, \alpha_I, \beta_I$ and $\gamma_I$ and the other is given by the integral part of $\theta$.

Similar as in the $S^2$ case, the Moyal product of the two operators $T^{R_1}_{J_1}$ and $T^{R_2}_{J_2}$ are also closed in $\mathcal{A}_N$:

$$
T^{R_1}_{J_1} \ast T^{R_2}_{J_2} = \sum_{R,J} \frac{1}{N} \left( \begin{array}{c|c|c}
R & R_1 & R_2 \\
J & J_1 & J_2
\end{array} \right) \left\langle \begin{array}{c|c|c}
\frac{p}{2} & \frac{p}{2} & \frac{p}{2}
\end{array} \right| T^R_J, \tag{28}
$$
here \( R \equiv (r_1, r_2), J \equiv \begin{pmatrix} J_1 & J_2 \\ J_{1z} & J_{2z} \end{pmatrix} \). This algebra is also a quasitriangular algebra.

4 Two incompressible quantum Hall fluid on \( S^2 \)

When a D0-brane enters a D2-brane, it can dissolve into magnetic flux, and its density is equivalent to a magnetic field on the membrane while the D particle currents result in the electric field. Through the T-duality, the D0-brane plays the role of the electron which is the ends of strings on D2-brane \([4, 18, 19, 20, 21]\). This give a configuration of the branes and strings - the quantum Hall soliton and the low energy dynamics display the fractional QHE which can be modeled by a noncommutative Chern-Simons theory \([4]\).

In \([3]\), Susskind proposed the non-commutative Chern-Simons theory to describe the area preserving gauge transformation of the incompressible electron gas on a constant magnetic field. Here the area preserving gauge is:

\[
X_i = x_i + \epsilon_{ij} \frac{\hat{A}_j}{2\pi\rho_0},
\]

where \( X_i \) is the noncommutative target space coordinate which describes the positions of the electrons, \( x_i \) is the comoving coordinates of the incompressible electron gas. \( \hat{A}_i \) is the gauge field and \( \theta = \frac{1}{2\pi\rho_0} \) is the noncommutative parameter. And the action of the noncommutative Chern-Simons theory is:

\[
S = \frac{k}{4\pi} \int d^3x \epsilon_{\mu\nu\rho} \hat{A}_\mu \partial_\nu \hat{A}_\rho + \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho,
\]

where \( k \equiv B\theta = \frac{B}{2\pi\rho_0} = \frac{1}{\nu} \) and \( \nu \) is called the filling fraction.

Up to a total divergent, this noncommutative Chern-Simons theory is equivalent to the matrix model:

\[
S = \int dt \frac{B}{2} \text{Tr} \left\{ \epsilon_{ab}(\dot{X}_a + i[A_0, X_a])X_b + 2\theta A_0 \right\},
\]

where \( X_a, (a = 1, 2) \) are two (infinite) target hermitian ”matrices” of the matrix theory \([7]\). \([\cdot, \cdot] \) is the matrix commutator and \( \text{Tr} \) represents the (matrix) trace over the Hilbert space. \( A_0 \) is the Lagrangian multiplier which will generate the Gaussian constraint condition of the gauge transformation. This action describes an incompressible fluid of infinite particles on the plane in a constant magnetic field \( B \). The action of the finite \( N \) particles system which describes the dynamics of a Quantum Hall droplet is proposed by Polychronakos \([7]\):

\[
S = \int dt \frac{B}{2} \text{Tr} \left\{ \epsilon_{ab}(\dot{X}_a + i[A_0, X_a])X_b + 2\theta A_0 - \omega X_a^2 \right\} + \Psi^\dagger (i\dot{\Psi} - A_0 \Phi),
\]

here \( X_a \) is represented by \( N \times N \) matrices in matrix theory. \( \Psi \) is a complex \( N \)-vector which comes from the droplet boundary states. The potential term \( \omega X_a^2 \) serves as a spatial regulator and it breaks the translation invariance of the action.

In the fuzzy two-sphere \( S^2 \), the action is similar as the droplet in \( \mathbb{R}^2 \):

\[
S = \int dt \frac{B}{2} \left\{ \epsilon_{ab}(\dot{X}_a + i[A_0, X_a])X_b + 2\theta A_0 \right\} + \Psi^\dagger (i\dot{\Psi} - A_0 \Psi).
\]
here $\Psi$ is a complex $N$-vector which defines the $CP(N-1)$ model manifold [22] and it has a left global $U(N)$ symmetry and a right local $U(1)$ gauge symmetry which can be mapped to the gauge field $A$ (statistical gauge field) for D0 fluid by the Seiberg-Witten map. In this action, the term $\frac{eB}{2} \epsilon_{ab} \dot{X}_a X_b$ is the Lorentz force which gives out the Chern-Simons part in the action as in plane $\mathbb{R}^2$ and $A_0$ is still the Lagrangian multiplier.

The Gaussian constraint equation can be obtained by varying the action (33) with respect to the Lagrangian multiplier $A_0$:

$$G \equiv -iB[X_1, X_2] + \Psi \Psi^\dagger - B\theta = 0. \quad (34)$$

The traceless part of this equation gives out the commutation relation of $X_1$ and $X_2$ (moment map equation):

$$[X_1, X_2] = i\theta(1 - |\Psi\rangle\langle\Psi|). \quad (35)$$

This equation is the D-Flat equation in Supersymmetric Yang-Mills theory. Here we have adopt an oversimplified notation, i.e. in moving frame, for the $S^2$ without $\Psi$ source, the orbital angular momentum $L_i = \epsilon_{ij} L_{jr}$ is equivalent to the Killing vector $\nabla_j$ along $S^2$ surface:

$$[\Lambda_+, \Lambda_-] \simeq J_r \equiv -S \sim BR^2 \sim \theta.$$  

For the right module $A_n$ on $S^2$ without quasiparticle

$$[X_\pm, \star A] \simeq \nabla_\pm A, \quad A \in A_n,$$

$$[X_1, X_2] = \theta, \quad X_\pm \equiv \frac{1}{\sqrt{2}}(X_1 \pm iX_2), \quad (37)$$

here $\theta^2 \sim \frac{L^2}{r^2} \sim 2S + 1$ which is the quantum effect.

In the plane case, the explicit expressions of the Laughlin wavefunctions was given in [9]. Upon quantization, the Gaussian constraint (34) and the moment map equation (35) require the physical states to be singlets of $SU(n)$. The ground state being a completely antisymmetric $SU(n)$ sigulet with $\Psi$ and $X_-$ has the following form:

$$|\Phi_{gr}\rangle = [e^{i1 \cdots iN} \Psi^\dagger_{i_1} (\Psi^\dagger X_{-})_{i_2} \cdots (\Psi^\dagger X_{(N-1)})_{i_N}]^k|0\rangle, \quad (38)$$

where $X_\pm = \frac{1}{\sqrt{2}}(X_1 \pm iX_2)$ which satisfy the constrain equation (34) and $|0\rangle$ is annihilated by $X_+$'s and $\Psi$'s, while the excited states can be written as

$$|\Phi_{exc}\rangle = \prod_{i=1}^{N} (\text{Tr} X_-) c_i [e^{i1 \cdots iN} \Psi^\dagger_{i_1} (\Psi^\dagger X_{-})_{i_2} \cdots (\Psi^\dagger X_{(N-1)})_{i_N}]^k|0\rangle, \quad (39)$$

On the fuzzy two-sphere $S^2$, the Killing vectors on $S^2$ $\Lambda_\pm \sim X_+ \sim D^{\pm}_{\pm1}$ act on the state $|\Psi\rangle \in H_n$ which is equivalent to the symbol $D_{N,-S}^S$. The Hamiltonian of the system is

$$H = \frac{\omega_C (\hat{I}^2 - \hat{S}^2)}{2\pi S} \quad (40)$$

For the Incompressible Quantum Hall Fluid, the system now contains $n$ particles and the moduli space is $S^{2\otimes n}/S_n$ with the hyperKahler metric

$$ds^2 = \sum_{j=1}^{n} \frac{d\zeta_j d\bar{\zeta}_j}{(1 + |\zeta_j|^2)^2}, \quad (41)$$
and the simplectic form
\[ \omega = \sum_{j=1}^{n} 2i \frac{d\zeta_j \wedge d\bar{\zeta}_j}{(1 + |\zeta_j|^2)^2}. \]  

(42)

The ground state on \( S^2 \) is expressed by the Vandermonde determinant of the spinor coordianates \( |\Psi_{i,j}| \approx \prod_{i<j}(u_iu_j - u_ju_i) \), where \( \Psi_{i,j} \equiv D^S_{i-j} - s(\alpha_j, \beta_j, \gamma_j) \), while in the limit of \( S^2 \to \mathbb{C}^1 \) plane, the quantum \( (X_{-})_{ij} = \delta_{ij} \partial_t - \frac{id}{\zeta_j}(1 - \delta_{ij}) \), \( X_+ = \text{diag}(z_1, \ldots, z_n) \), and the excited state \( |\Phi_{\text{exc}}\rangle \) as the eigenfunction of the Hamiltonian \( = \text{Tr}(X^2) \) is expressed by Jack polynomial.

5 Incompressible quantum Hall fluid on \( S^4 \)

On the four-sphere \( S^4 \), we may set the brane construction as \((2,0)\) noncommutativity for electronic fluid or as IIA supergravity for \( D0 \) fluid.

On the \( S^4 \), the four Killing vectors along the surface become the basis \( \Psi_i(i = 1, \ldots, 4) \), the fundamental \( SU(2) \) monopole wave function of C. N. Yang [23, 24], which is conformally equivalent to the instanton on 4-sphere. So \( S^4 \) has a hyperKähler symplectic metric. The left global \( n \times n \) quarternion matrix \( n = \frac{1}{2} s(s + 1)(s + 2)(s + 3) \) is the ADHM [25] matrix for the normal component of the projective module of instanton, with \( n \) the Pontrjagin number, i.e. instanton number. While the right local \( SU(2) \) gauge symmetry, in the bundle \( SO(5)/SU(2) \sim \text{base 4-sphere } SO(5)/(SU(2) \otimes SU(2)) \times \text{gauge fibre } SU(2) \) is the tangential part of this normal bundle, i.e. determines the instanton potential. These are the same as in Witten’s paper [26].

Now on \( S^4 \), the D-flat equation of noncommutative ADHM constrains are:

\[ \mu_r = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \]
\[ \mu_c = IJ + [B_1, B_2]. \]  

(43)

We introduce The 4 Killing vectors the quarternion as

\[ q = \psi_i \sigma_i = \psi_1 1 + \psi_2 i + \psi_3 j + \psi_4 k, \]

here

\[ \psi_i = \psi_i (r_1, r_2; k_1, k_{1z}; k_2, k_{2z}; s, s_r) = D_{K_i}^{[\frac{1}{2}, \frac{1}{2}]}; \]
\[ \psi_1 = \psi(\frac{1}{2}, \frac{1}{2}; 0, 0; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), \]
\[ \psi_2 = \psi(\frac{1}{2}, \frac{1}{2}; 0, 0; \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \]
\[ \psi_3 = \psi(\frac{1}{2}, \frac{1}{2}; 0, 0; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), \]
\[ \psi_4 = \psi(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 0; 0; \frac{1}{2}, -\frac{1}{2}). \]  

(44)

here \( \psi(r_1, r_2; k_1, k_{1z}; k_2, k_{2z}; s, s_r) \) is the eigenfunction of \( SO(5) \) [23, 24], \( r_1, r_2 \) are the \( B_2 (SO(5)) \) Casimir; \( k_1, k_2 \) are the Casimir for the two \( SO(3) \) angular momentum in the stationary subgroup at pole \( SO(4) = SO(3) \otimes SO(3); k_{1z}, k_{2z} \) are their "magnetic" quantum number respectively; \( s, s_r \) are the (anti)selfdual \( SU(2) \) gauge symmetry for
SU(2) monopole (instanton). ψµ is the Yang’s SU(2) monopole harmonic function with \((p,q) = (1,0)\), here \((p,q)\) is the \(Sp(2) \equiv C_2\) Casimir with \(p = r_1 + r_2\), \(q = r_1 - r_2\). In the hedgehog gauge, it also represent the (anti)selfdual stationary subgroup around the radial direction, e.g. \(s_r\) is that along \(\hat{r}\). In angular momentum theory, the conventional rotation is \(D_R^{\kappa}\). For the detail of \(D\) function please confer to \[12\].

Now the Moduli space of the QHF on \(S^4\) is \(S^{4\otimes n}/S_n\). Next, we have the hyperKähler symplectic form for this noncommutative \(S^4\) in terms of these Killing vectors:

\[
\Omega = \sum_{i=1}^{n} dq_i \wedge d\bar{q}_i, \tag{45}
\]

here the \(q_i\) (the \(\psi_{ji}\)) depends on the shift of the ”center” of N.C. soliton from the north pole to \((\theta_i, \alpha_i, \beta_i, \gamma_i)\).

The ”area preserving” constrain equation with quasiparticle source \(I, J\) turns to be

\[
[X, X^\dagger] = (i + j + k)(1 - n|v\rangle\langle v|), \tag{46}
\]

where \(X = q + B = X_\mu \sigma_\mu, B = \left(\begin{array}{cc}
-B_1 & B_2 \\
-B_2^\dagger & -B_1^\dagger
\end{array}\right)\), \((i+j+k)\) are self-dual orbital momentum and we have choosen \(I, J\) to get an Weyl invariant \(\langle v| = \frac{1}{\sqrt{n}}(1, \cdots, 1)\).

6 Incompressible Quantum Hall Fluid on Torus

On the torus \(T\), we choose the following comoving frame coordinates of electrons:

\[
z_i = \frac{1}{\sqrt{2}}(x_i + iy_i), \quad \bar{z}_i = \frac{1}{\sqrt{2}}(x_i - iy_i), \quad (i = 1, 2, \cdots, N). \tag{47}\]

Then the moment map equation becomes

\[
[z_j, \bar{z}_k]_\star = \frac{1}{B}\delta_{ij}, \quad [\bar{z}_k, \star f(z)] = \partial_k f(z). \tag{48}\]

As similar in the \(S^2\) case, now on torus, the moduli space is \(T^{2\otimes n}/S_n\) and corresponding symplectic form: \(\Omega = \sum_j dz_j \wedge d\bar{z}_j\). As a Kähler manifold (with both complex and symplectic forms), the Kähler potential is given by

\[
K = \log \prod_{j \neq k} \sigma(z_j - z_k). \tag{49}\]

Then the ground state wave function will be

\[
\Psi_{\text{ground}} = \prod_{j \neq k} \sigma(z_j - z_k). \tag{50}\]

On the torus, there is an automorphism between the Wilson loop algebra \(\mathcal{A}_n\) and the Lie algebra \(sl_n(\mathbb{T})\) \[27\]. It is shown \[28\] that the level \(l\) representation of the Lie algebra
\( \text{sl}_n(\mathbb{T}) \) on the elliptic curve \( \mathbb{T} \) is

\[
E_\alpha = (-1)^{\alpha_1} \sigma_\alpha(0) \sum_j \prod_{k \neq j} \frac{\sigma_\alpha(z_{jk})}{\sigma_0(z_{jk})} \left[ \frac{l}{n} \sum_{i \neq j} \frac{\sigma_\alpha(z_{ji})}{\sigma_0(z_{ji})} - \partial_j \right],
\]

(51)

\[
E_0 = - \sum_j \partial_j,
\]

(52)

where \( \alpha \equiv (\alpha_1, \alpha_2) \in \mathbb{Z}_n \times \mathbb{Z}_n, \alpha \neq (0,0), \) \( z_{jk} = z_j - z_k, \) \( \partial_j = \frac{\partial}{\partial z_j} , \) \( E_0 \) commutes with \( E_\alpha. \)

In a more common basis, let \( E_{ij} \equiv \sum_{\alpha \neq (0,0)} (I^\alpha)_{ij} E_\alpha, \) where \( (I_{a_1, a_2})_{ab} = \delta_{a+a_1, a_2} \), we have

\[
[E_{jk}, E_{lm}] = E_{jm} \delta_{kl} - E_{lk} \delta_{jm},
\]

(53)

The Weyl reflection is realized by \( j \leftrightarrow k \) for \( E_{jk}(j \neq k). \)

The D-flat equation with source then is

\[
[D_z, \phi] = \zeta(1 - n|v\rangle \langle v|)\sigma^2(u), \quad \langle v| = (1, \cdots, 1).
\]

(54)

The \( \text{sl}(n) \) bundle, the Hamiltonian reduced by the moment map with quasiparticle as a source, turns to be a differential \( \mathcal{L} \) operator (quantum lax operator) of Gaudin model

\[
L_{ij} = \sum_{\alpha \neq (0,0)} w_\alpha(u) E_\alpha(I_\alpha)_{ij}, \quad L_{ij} \sim \phi_{ij}
\]

(55)

where

\[
w_\alpha(u) = \frac{\partial'\sigma(u)}{\sigma_\alpha(u) \sigma_0(u)}.
\]

(56)

The commutators of \( \mathcal{L} \) is

\[
[L^{(1)}(u_1), L^{(2)}(u_2)] = [r^{(1,2)}(u_1 - u_2), L^{(1)}(u_1) \ominus L^{(2)}(u_2)].
\]

(57)

where the classical Yang-Baxter matrix

\[
r(u)^{k,l}_{i,j} = \sum_{\alpha \neq (0,0)} w_\alpha(u)(I_\alpha)^{k}_{i} \otimes (I_\alpha^{-1})^{l}_{j},
\]

(58)

satisfies the classical Yang-Baxter equation

\[
[r_{12}(u_{12}), r_{13}(u_{13})] + [r_{12}(u_{12}), r_{23}(u_{23})] + [r_{13}(u_{13}), r_{23}(u_{23})] = 0.
\]

(59)

The Hamiltonian is then defined as

\[
H = \text{Tr}(L^2)
\]

(60)

and the Laughlin wave functions the eigenfunctions of this Hamiltonian

\[
\text{Tr}(L^2) \psi = (-\partial^2_z + 2 \sum_i \varphi(z - z_i(t))) \psi = \epsilon \psi,
\]

(61)

where \( \varphi(z) = \partial^2 \sigma(z) \). This is a \( q \) Lamé equation and can be solved by the Bethe-Yang Ansatz.
7 Discussion

In this paper, we have succeeded in constructing the N.C. algebra of the $S^2$ and $S^4$ and generalizing the descriptions of the incompressible QHF to the fuzzy 4-sphere $S^4$ and torus $\mathbb{T}$. On the torus, we notice that the incompressible QHF can be related to the integrable Gaudin model which can be solved by the Bethe-Yang ansatz.

As seen in section 4, after T-duality, the D0 branes which enter in D2 brane become the electrons which is the ends of the strings on D2 brane. So for other noncommutative manifolds, it is interesting to relate the matrix algebra given in this paper with some applications in D-brane dynamics.

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