MECHANICAL ENGINEERING | RESEARCH ARTICLE

Bending analysis of P-FGM plates resting on nonuniform elastic foundations and subjected to thermo-mechanical loading

Bachir Bouderba¹,² and Berrabah Hamza Madjid²,³*

Abstract: In a thermal environment, using the formulations of the refined shear deformation theory, the flexion of the functionally graduated rectangular plates (P-FGM) based on nonuniform elastic foundations is presented. The layers resting on nonuniform elastic foundation are subjected to symmetrical distributed loads. Nonlinear variations in the in-plane displacements through the thickness of the plate are based on the choice of the displacement field of the present theory. Without including shear correction factor, the present theory satisfies the free transverse shear stress conditions on the top and bottom surfaces of the plate. The present refined shear deformation theory contains only four unknowns. According to a power law of the volume fraction, the properties of the materials of this plate vary continuously across the thickness. Comparisons with other results have been made nonuniform foundation. The effects of the index and the dimensions of the P-FGM plate and of the elastic

ABOUT THE AUTHOR

Berrabah Hamza Madjid in Algeria, is one of the best-known scientists in materials science. He is a full professor at Relizane University in Relizane, Algeria. He was a graduate of the State University of Djillali Liabes -Sidi Bel Abbes- since 1999 and obtained his doctorate in 2011 at the State University of Djillali Liabes -Sidi Bel Abbes-, Algeria. Since 2014, he had become a doctor of science (Dr Habilitation) promoted by the State University of Djillali Liabes -Sidi Bel Abbes- Algeria.

PUBLIC INTEREST STATEMENT

In recent years, most research has been directed toward studying the use and development of Functional Materials Classification (FGM) for engineering applications. FGM materials have been used in several fields such as aerospace, nuclear, etc. Applications of microelectronics engineering, where materials have to work under extreme conditions temperature environments. It is also important for these materials to maintain their structural integrity, with minimal failures due to material mismatch. This manuscript focuses on Bending Analysis of P-FGM Plates Resting on Nonuniform Elastic Foundations and Subjected To Thermo-Mechanical Loading. The shape of the structure is important to increase the rigidity of the structures. In a thermal environment, using the formulations of the refined shear deformation theory, the flexion of the functionally graduated rectangular plates (P-FGM) based on nonuniform elastic foundations is analyzed. Nonlinear variations in the in-plane displacements through the thickness of the plate are based on the choice of the displacement field of the present theory.
foundation parameters on the thermomechanical behavior are studied. Numerical results show accuracy comparable to other theories in the literature.

**Subjects:** Mechanical Engineering; Mechanics of Solids; Mechanical Engineering Design

**Keywords:** nonuniform elastic foundation; bending; P-FG plates; refined plate theory; thermo-mechanical loading

1. **Introduction**

In recent years, materials with functional gradation (FGM) are defined as a class of composites which have a constant variation in the properties of the material from one given surface to another, eradicating stress concentration in laminate composites. In addition, FGMs are made up of a non-homogeneous isotropic mixture such as metal and ceramic. The reason why FGMs are particularly interesting is that certain types of FGM structures can be created and can adapt to various operating conditions. The increasing rates of FGM applications require precise models with precisely predictable responses (Zenkour & Hafed, 2020).

The multiterm Galerkin method has been applied to analyzing of the instability of a thick through-the-thickness functionally graded material (FGM) rectangular plates under the practical cases of thermal and mechanical loading conditions by Bateni et al. (2013) to derive the critical buckling loads/temperatures and buckled shapes of the plate.

The bending, buckling and free vibration responses of higher order functional gradation (FG) beams based on an elastic foundation with two parameters (Winkler–Pasternak) studied by Sayyad and Ghugal (2018) using a new inverse hyperbolic beams theory. The properties of the materials of the beam are classified in the direction of the thickness according to the distribution of the laws of power.

Dong et al. (2017) studied the local buckling of an infinitely thin rectangular composite plate symmetrically laminated retained by a Winkler foundation without tension and subjected to a uniform shear load in the plane.

An analytical spectral stiffness method is proposed by Liu et al. (2020) for the efficient and accurate buckling analysis of rectangular plates on Winkler foundation subject to general boundary conditions. The method combines the advantages of superposition method, stiffness-based method and the Wittrick–Williams algorithm. First, exact general solutions of the governing differential equation of plate buckling considering both elastic foundation and biaxial loading are derived by using a modified Fourier series.

Liu et al. (2020) have used the unified modeling method for the dynamic analysis of LPG-reinforced FGP plates based on a Winkler–Pasternak foundation with elastic boundary conditions.

The method of Kantorovich extended to several terms is used to solve the problem of bending of the plate with functional gradation (parallelogram) resting on the elastic foundation Winkler under a transverse load uniformly distributed. The formulations are based on the classic plate theory (CPT) with the physical neutral surface is considered as the reference plane for different configurations of tight edges, simply supported and free which are considered by Ahmed Hassan and Kurgan (2020).

Bakora and Tounsi (2015) have studied the thermomechanical behavior after buckling of thick plates with functional gradation resting on elastic foundations.

Beldjellili et al. (2016) presented the hygro-thermomechanical flexion of S-FGM plates based on variable elastic foundations using a trigonometric theory of plates with four variables.
A simple and efficient refined plate theory with four variables for the analysis of buckling of functionally graded plates was studied by Bellifa et al. (2017).

The buckling analysis of annular sector plates with functional gradation subjected to uniform plane compression loads on the basis of the three-dimensional theory of elasticity is studied. In addition, the influence of elastic foundations of the complete or partial Winkler type is also taken into account. Normal compression loads in the plane were applied to the radial, circumferential or all edges of the annular sector plates. The properties of the material vary continuously across the thickness of the plate according to a power law distribution while the Poisson ratio is assumed to be constant (Asemi et al., 2014).

A refined theory of plates with four unknowns for advanced plates based on elastic foundations in a hygrothermal environment is to be studied by Alkhateeb and his co-author (Alkhateeb & Zenkour, 2014).

In 2013, Thai et al. (2013) have used a simple refined theory for bending, buckling, and vibration of thick plates resting on elastic foundation.

The FSDT accounts for the shear deformation effect by the way of a linear variation of in-plane displacements through the thickness. A shear correction factor is therefore required (Bellifa et al., 2016; Bouderra et al., 2016; Youcef et al., 2018).

In his study, Duc (2013) has proposed an analytical investigation on the nonlinear dynamic response of eccentrically stiffened functionally graded double curved shallow shells resting on elastic foundations and being subjected to axial compressive load and transverse load. The formulations are based on the classical shell theory taking into account geometrical nonlinearity, initial geometrical imperfection and the Lekhnitsky smeared stiffeners technique with Pasternak type elastic foundation. The nonlinear equations are solved by the Runge-Kutta and Bubnov-Galerkin methods. Obtained results show effects of material and geometrical properties, elastic foundation and imperfection on the dynamical response of reinforced FGM shallow shells.

Kiani and Eslami (2014a) have studied the linear and nonlinear stability behavior of a thin circular FGM plate subjected to the uniform temperature rise and the constant angular velocity loadings. In their study, Properties of the FGM media were distributed across the thickness based on a power law form. Each property of the metal or ceramic constituents was considered to be the function of temperature based on the Touloukian model. General equilibrium equations for such conditions were obtained based on the classical plate theory.

In the literature, studies of the axisymmetric thermally induced vibrations of a circular plate made of functionally graded materials (FGMs) based on the uncoupled thermoelasticity assumptions, was considered by Kiani and Eslami (2014b).

The nonlinear response of eccentrically stiffened FGM cylindrical panels on elastic foundation subjected to mechanical loads based on Bubnov-Galerkin method, the Lekhnitsky smeared stiffeners technique with Pasternak type elastic foundation and stress function have presented by Duc and Quan (2014).

Kiani and Eslami (2015) have presented the thermal postbuckling of imperfect circular functionally graded material plates: Examination of Voigt, Mori–Tanaka, and Self-Consistent schemes. In their study, they have showed that response of a perfect clamped FGM plate is of the bifurcation type of buckling with stable postbuckling equilibrium branch, whereas imperfect clamped and perfect/imperfect simply supported FGM plates do not reveal the bifurcation type of instability through the nonuniform heating process. Furthermore, amplitude of initial
imperfection is an important factor on the equilibrium path of FGM circular plates, especially for simply supported ones.

Analytical approach to investigate the nonlinear dynamic response and vibration of imperfect functionally graded materials (S-FGM) thick circular cylindrical shells surrounded on elastic foundation using the third order shear deformation shell theory was considered by Duc et al. (2015a). The S-FGM shells are subjected to mechanical, damping and thermal loads. The Galerkin method and fourth-order Runge–Kutta method are used to calculate natural frequencies, nonlinear frequency-amplitude relation and dynamic response of the shells.

Conical shell panels made of functionally graded materials (FGMs) are rather commonly used by structural engineers. However, due to their complex geometric shape, there are only a few studies on conical shell panels made from FGMs. The linear stability analysis of eccentrically stiffened FGM conical shell panels reinforced by mechanical and thermal loads on elastic foundations based on Classical shell theory and Lekhnitsky's smeared stiffeners technique to set the balance equations and linear stability was investigated by Duc et al. (2015b).

Kiani (2016) has presented a free vibration of carbon nanotube reinforced composite plate on point Supports using Lagrangian multipliers.

Duc (2016a) has presented an analytical investigation on nonlinear thermal dynamic behavior of imperfect functionally graded (S-FGM) circular cylindrical shells eccentrically reinforced by outside stiffeners and surrounded on elastic foundations using the Reddy's third order shear deformation shell theory in thermal environment. The stress function and the Bubnov–Galerkin method are applied. Unlike previous publications, he was proposed a general formulation for forces and moments which allow the nonlinear dynamic of shear deformable eccentrically stiffened shell to be studied taking into account the thermal stress in both the shells and the stiffeners.

The nonlinear dynamic response of higher order shear deformable sandwich functionally graded circular cylindrical shells with outer surface-bonded piezoelectric actuator on elastic foundations subjected to thermo-electro-mechanical and damping loads using the stress function, the Galerkin method and the fourth-order Runge–Kutta method was analyzed by Duc (2016b).

Elliptical cylindrical shell is one of shells with special shape. Therefore, Duc et al. (2017) have studied the nonlinear dynamic response and vibration of imperfect eccentrically stiffness functionally graded elliptical cylindrical shells on elastic foundations using both the classical shell theory (CST) and Airy stress functions method with motion equations using Volmir's assumption.

Gholami and Ansari (2017) have studied the large deflection geometrically nonlinear analysis of functionally graded multilayer graphene platelet-reinforced polymer composite rectangular plates.

In 2018, Duc et al. (2018) have presented an analytical approach to investigate the mechanical and thermal buckling of functionally graded materials sandwich truncated conical shells resting on Pasternak elastic foundations, subjected to thermal load and axial compressive load using the rst-order shear deformation theory, Lekhnitskii smeared Steiner technique and the adjacent equilibrium criterion. In their study, Shells are reinforced by closely spaced stringers and rings, in which the material properties of shells and Steiners are graded in the thickness direction following a general sigmoid law distribution and a general power law distribution. Four models of coated shell-Steiner arrangements are investigated.
In his study, Kiani (2018) has proposed a theory to determine the thermal post-buckling of temperature dependent sandwich plates with FG-CNTRC face sheets based on the first-order shear deformation theory and von Kármán type of geometrical nonlinearity.

Asymmetric thermal buckling of temperature-dependent annular FGM plates on a partial elastic foundation discussed by Bagheri et al. (2018).

Nonlinear bending of third-order shear deformable carbon nanotube/fiber/polymer multiscale laminated composite rectangular plates with different edge supports was analyzed by Gholami and Ansari (2018).

The nonlinear dynamic response and vibration of functionally graded carbon nanotubes (FG-CNT) reinforced composite elliptical cylindrical shells resting on elastic foundations in thermal environments using an analytical solution based on the classical shell theory with the geometrical nonlinearity in von Kármán, the Airy stress function, Galerkin method, and Runge–Kutta method was analyzed by Dat et al. (2019).

Gholami et al. (2019) presented the nonlinear bending analysis of nanoplates made of FGMs based on the most general strain gradient model and 3D elasticity theory.

Asymmetric nonlinear bending analysis of polymeric composite annular plates reinforced with graphene nanoplatelets was investigated by Gholami and Ansari (2019).

Nonlinear buckling and post-buckling of imperfect piezoelectric S-FGM circular cylindrical shells with metal-ceramic-metal layers in thermal environment using Reddy’s third-order shear deformation shell theory have been investigated by Khoo et al. (2019).

Javani et al. (2019) have presented deals with the large amplitude thermally induced vibrations of an annular plate made of functionally graded materials (FGMs). One surface of the plate is subjected to rapid surface heating while the other surface is either thermally insulated or kept at reference temperature. The material properties of the constituents are assumed to be temperature dependent. In their study, they have used the rule of mixtures to obtain the properties of the graded media. They have based on the aid of the von Kármán kinematic assumptions and first order shear deformation theory, generalized differential quadrature (GDQ) method, Newton–Raphson method, Newmark time marching scheme and iterative Crank–Nicolson methods.

A nonuniform rational B-spline isogeometric finite element formulation presented by Kiani (2020) to analyze the thermal buckling behavior of composite laminated skew plates reinforced by graphene platelets. Formulation is based on the first-order shear deformation plate theory.

Ansari et al. (2020) presented the nonlinear bending analysis of arbitrary-shaped porous nanocomposite plates using a novel numerical approach.

Gholami and Ansari (2021) have studied the thermal postbuckling of temperature-dependent functionally graded nanocomposite annular sector plates reinforced by carbon nanotubes.

The most commonly used plate theory is the classical plate theory, which is based on the assumptions of Love–Kirchhoff, according to which a line normal to the average plan of the plate remains perpendicular after deformation, which amounts to neglect the effects of deformation in transverse shears. Since this model does not take into account the transverse shear effect, it gives imprecise results for thick plates. First-order shear deformation theory (Reissner, 1945; Mindlin, 1951) extended the classical plate theory by taking into account the transverse shear effect, in this case stresses and strains are constant through the thickness of the plate, which requires the introduction of one of the factors of correction (K= 5/6).
To overcome the limits of first-order theories, several authors propose higher-order shear deformation theories (HSDT). The models are based on a nonlinear distribution of the fields according to the thickness of the plate (Ambartsumian, 1958; Kaczkowski, 1968; Panc, 1975; Reissner, 1975; Levinson, 1980; Murthy, 1981; Reddy, 1984; Touratier, 1991; Karama et al., 2003; Aydogdu, 2009; Zenkour, 2009).

In this paper, the refined shear deformation theory is applied in analyzing the flexion gradually functional rectangular plates (FGM) based on nonuniform elastic foundations for a nonlinear variation of displacement in the plane through the thickness according to a power-law of the volume fraction. The elastic foundation is modelled as a nonuniform foundation.

A simplified refined shear deformation theory (RSDT) based on the Sobhy (2020) is presented here. This theory is based on the principle of virtual work; it has a strong similarity with the classical theory plates in many aspects, does not require a shear correction factor, and gives a correct description of the transverse shear stress through the thickness while fulfilling the condition of zero transverse shear stress at the upper and lower surfaces of the plate. By dividing the deflection into bending, and shear components, the number of unknowns and governing equations of the present theory is reduced to four as against five or more unknown in the corresponding theories (FSDT; PSDT; TSDT). The main advantage of this theory is that, in addition to including the shear deformation effect, the displacement field is modeled with only 4 unknowns as the case of the CPT and which is even less than the first order shear deformation theory (FSDT).

The effectiveness of the present theory is demonstrated and the results are compared with numerical models for P-FGM plates resting on nonuniform elastic foundations found in the literature.

2. Theoretical formulation

The geometry and coordinate system of FG plate is shown in Figure 1.

2.1. Basic assumptions and kinematics

The displacement field of the present formulation is obtained based on the following assumptions:
(1) The transverse displacements are partitioned into bending and shear components; (2) The in-plane displacement is partitioned into extension, bending, and shear components; (3) The bending parts of the in-plane displacements are similar to those given by classical plate theory (CPT); (4) The shear parts of the in-plane displacements give rise to the nonlinear variations of shear strains and hence to shear stresses through the thickness of the plate in such a way that the shear stresses vanish on the top and bottom surfaces of the plate (Shimpi, 2002; Shimpi et al., 2003; Sobhy, 2020).
Based on the assumptions made in the preceding section, the displacement field can be obtained

\[ u(x, y, z) = u_0(x, y) - z \frac{\partial w_b}{\partial x} - \psi(z) \frac{\partial w_e}{\partial x} \]
\[ v(x, y, z) = v_0(x, y) - z \frac{\partial w_b}{\partial y} - \psi(z) \frac{\partial w_e}{\partial y} \]
\[ w(x, y, z) = w_0(x, y) + w_s(x, y) \]  

where \( u, v, w \) are displacements in the \( x, y, z \) directions, \( u_0, v_0 \) and \( w_0, w_s \) are mid-plane displacements and \( \psi(z) \) is a shape function that represents the distribution of the transverse shear strain and stress through the thickness, as presented in Table 1.

The displacement field of the Classical thin Plate Theory (CPT) is obtained easily by setting \( \psi(z) = 0 \). The displacement of the First-order Shear Deformation Plate Theory (FSDT) is obtained by setting \( \psi(z) = z \).

Undefomed geometry of an element and deformation of a transverse normal are shown in Figure 2.

Note that the function \( \Psi(z) = z \psi(z) \) satisfies the following conditions

\[ \psi(z) = \left[ h \tan^{-1} \left( \frac{z}{h} \right) - \frac{z}{15} \left( \frac{4z}{h} \right)^2 \right] ; \quad \frac{d\psi(z)}{dz} = 0 \quad \text{for} \quad z = \pm \frac{z}{h} \]  

The kinematic relations can be obtained as follows

\[
\begin{align*}
\{ \varepsilon_x \} &= \{ \varepsilon_x^{(0)} \} + z \{ k_x^{(0)} \} + \psi(z) \{ k_x \}, \\
\{ \varepsilon_y \} &= \{ \varepsilon_y^{(0)} \} + z \{ k_y^{(0)} \} + \psi(z) \{ k_y \}, \\
\{ \gamma_{xy} \} &= \{ \gamma_{xy}^{(0)} \} + \psi(z) \{ \gamma_{xy} \}, \\
\{ \gamma_{yz} \} &= \{ \gamma_{yz}^{(0)} \} + \psi(z) \{ \gamma_{yz} \}
\end{align*}
\]

where

### Table 1. Different shear shape strain functions and the transverse displacement \( w \) of a HSDT

| Model | \( \psi(z) \) function | The transverse displacement \( w \) |
|-------|------------------------|----------------------------------|
| Ambartsumian (1958) | \( \psi(z) = \frac{1}{2} \left( \frac{h}{w_x} - \frac{h}{w_y} \right) \) | \( w_0(x, y) \) |
| Kaczkowski (1968), Panc (1975), Reissner (1975) | \( \psi(z) = \frac{1}{2} \left( 1 - \frac{4z h}{3w_x w_y} \right) \) | \( w_0(x, y) \) |
| Levinson (1980), Murthy (1981) and Reddy (1984) | \( \psi(z) = z \left( 1 - \frac{4z h}{3w_x w_y} \right) \) | \( w_0(x, y) \) |
| Tournier (1991) | \( \psi(z) = \frac{1}{2} \sin \left( \frac{\pi z}{h} \right) \) | \( w_0(x, y) \) |
| Soldatos (1992) | \( \psi(z) = h \sinh \left( \frac{z}{h} \right) - z \cosh \left( \frac{z}{h} \right) \) | \( w_0(x, y) \) |
| Karama et al. (2003) and Aydogdu (2009) | \( \psi(z) = z e^{-\alpha (z/h)^2} = z e^{-\alpha (z/h)^2 / \alpha} \), \( \forall \alpha > 0 \). | \( w_0(x, y) \) |
| Present | \( z - \left[ h \tan^{-1} \left( \frac{z}{h} \right) - \frac{z}{15} \left( \frac{4z}{h} \right)^2 \right] \) | \( w_0(x, y) + w_s(x, y) \) |
Figure 2. (a) Undeformed geometry of an element and deformation of a transverse normal according to (b) the classical (CPT), (c) first-order (FSDT) and (d) third-order plate theories (TSDT).

\[ \begin{align*}
\{k_x^0 & \} = \begin{bmatrix}
\frac{\partial^2 u_0}{\partial x^2} \\
\frac{\partial^2 v_0}{\partial y^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{bmatrix} \\
\{k_y^0 & \} = \begin{bmatrix}
\frac{\partial^2 u_0}{\partial y^2} \\
\frac{\partial^2 v_0}{\partial x \partial y} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{bmatrix} \\
\{k_{xy}^0 & \} = \begin{bmatrix}
-\frac{\partial^2 w_0}{\partial x \partial y} \\
-\frac{\partial^2 w_0}{\partial x \partial y} \\
2\frac{\partial^2 w_0}{\partial x \partial y}
\end{bmatrix}
\end{align*} \]

and

\[ g(z) = 1 - \frac{dw(z)}{dz} \]

2.2 Constitutive equations

The plate is subjected to a sinusoidally distributed load \(Q(x, y)\) and a temperature field \(T(x, y, z)\). The material properties \(P\) of the FG plate, such as Young’s modulus \(E\), Poisson’s ratio \(\nu\), and thermal expansion coefficient \(\alpha\) are given according to the formulation

\[ P(z) = P_M + P_{CM} \left(\frac{1}{2} + \frac{z}{h}\right)^k \]

where \(P_C\) and \(P_M\) are the corresponding properties of the ceramic and metal, respectively, and \(k\) is the volume fraction exponent which takes values greater than or equal to zero.

The linear constitutive relations are:
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
n_{xy}
\end{bmatrix} = \begin{bmatrix}
\eta_{11} & \eta_{12} & 0 \\
\eta_{12} & \eta_{22} & 0 \\
0 & 0 & \eta_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x - \alpha \Delta T \\
\varepsilon_y - \alpha \Delta T \\
\gamma_{xy}
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
\tau_{yz} \\
n_{zx}
\end{bmatrix} = \begin{bmatrix}
\eta_{44} & 0 \\
0 & \eta_{55}
\end{bmatrix} \begin{bmatrix}
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]

where \((\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \gamma_{xy})\) and \((\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xy})\) are the stress and strain components, respectively. Using the material properties defined in Eq. (5), stiffness coefficients, \(\eta_i\), can be expressed as

\[
\eta_{11} = \eta_{22} = \frac{E(z)}{1 - \nu^2},
\]

\[
\eta_{12} = \nu \eta_{11},
\]

\[
\eta_{44} = \eta_{55} = \eta_{66} = \frac{E(z)}{2(1 + \nu)},
\]

where \(\Delta T = T_f - T_i\) in which \(T_i = T_0\) is the reference temperature.

The applied temperature distribution \(T(x, y, z)\) through the thickness is assumed, to be

\[
T(x, y, z) = T_1(x, y) + \frac{z}{h} T_2(x, y) + \frac{1}{\pi} \sin \left( \frac{\pi z}{h} \right) T_3(x, y).
\]

A polynomial temperature distribution is a combination of three parts, a constant in \(T_1\), a linear in \(T_2\) and sinusoidal in \(T_3\), see, (Benyamina et al., 2018; Bouderba & Benyamina, 2018; Bouderba & Berrabah, 2020; Bouderba et al., 2013; Carrera, 2002; Mashat et al., 2020; Tungikar & Rao, 1994; Zenkour, 2010; Zenkour et al., 2013).

### 2.3 Governing equations

The governing equations of equilibrium can be derived by using the principle of virtual displacements. The principle of virtual work in the present case yields

\[
\int_{\Omega} \left[ \begin{array}{ccc}
\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} \\
+ \tau_{yz} \delta \gamma_{yz} + \tau_{zx} \delta \gamma_{zx}
\end{array} \right] d\Omega dz - \int_{\Omega} (Q - R_e) \delta w d\Omega = 0
\]

where \(\Omega\) is the top surface, and \(R_e\) is the density of reaction force of foundation. For the Pasternak foundation model (Behravan Rad, 2012)

\[
R_e = K_1(x)w - K_2(x)\nabla^2 w
\]

and
where \(K_0, J_0\) are a constant and \(\xi\) is a varied parameter. \(K_1\) is the Winkler foundation stiffness and \(K_2\) is the effect of the shear interactions of the vertical elements, and \(\nabla^2\) is the Laplace operator in \(x\) and \(y\). Note that, if \(\xi = 0\), the elastic foundation becomes Pasternak foundation and if the shear layer foundation stiffness is neglected, the Pasternak foundation becomes the Winkler foundation, the foundation is homogeneous and isotropic.

Substituting Equations (3) and (6) into Eq. (9) and integrating through the thickness of the plate, Eq (9) can be rewritten as

\[
\int_{\Omega} \left[ N_x \delta \varepsilon_x^0 + N_y \delta \varepsilon_y^0 + N_{xy} \delta \tau_{xy}^0 + M_x \delta k_x^0 + M_y \delta k_y^0 + M_{xy} \delta k_{xy}^0 \right] \, d\Omega = \int_{-h/2}^{h/2} \left( \sigma_x, \sigma_y, \tau_{xy} \right) \left\{ \begin{array}{l} 1 \\ z \\ \psi(z) \end{array} \right\} \, dz. \tag{11}
\]

The stress resultants \(N, M,\) and \(S\) are defined by

\[
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x^b \\ M_y^b \\ M_{xy}^b \\ M_x^s \\ M_y^s \\ M_{xy}^s \end{bmatrix} = \int_{-h/2}^{h/2} \left( \sigma_x, \sigma_y, \tau_{xy} \right) \left\{ \begin{array}{l} 1 \\ z \\ \psi(z) \end{array} \right\} \, dz, \tag{12a}
\]

and

\[
\left( S_{xz}^s, S_{yz}^s \right) = \int_{-h/2}^{h/2} \left( \tau_{xz}, \tau_{yz} \right) g(z) \, dz. \tag{12b}
\]

Substituting Eq. (6) into Eq. (12) and integrating through the thickness of the plate, the stress resultants are given as

\[
\begin{bmatrix} N \\ M^b \\ M^s \end{bmatrix} = \begin{bmatrix} A & B & B^T \\ B & D & D^T \\ B^T & D^T & H^T \end{bmatrix} \begin{bmatrix} \varepsilon \\ k_x^0 \\ k_y^0 \end{bmatrix} - \begin{bmatrix} N^T \\ M_{b}^T \\ M_{s}^T \end{bmatrix}, \quad S = A^T \gamma \tag{13}
\]

where

\[
N = \left( N_x, N_y, N_{xy} \right)^T, \quad M^b = \left( M_x^b, M_y^b, M_{xy}^b \right)^T,
\]

\[
M^s = \left( M_x^s, M_y^s, M_{xy}^s \right)^T. \tag{14a}
\]
\[ N^T = \{N_x', N_y', 0\}^T, M_{0T}^T = \{M_{x0}^T, M_{y0}^T, 0\}^T, \]

\[ M_{0T}^T = \{M_{x0}^T, M_{y0}^T, 0\}^T. \]

\[ \varepsilon = \{\varepsilon_x', \varepsilon_y', \gamma_{xy}'\}^T, k^b = \{k_{xx}^b, k_{xy}^b, k_{yy}^b\}^T, \]

\[ k^s = \{k_{xx}^s, k_{xy}^s, k_{yy}^s\}^T. \]

\[ A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}, \]

\[ D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}. \]

\[ B^s = \begin{bmatrix} B_{11}^s & B_{12}^s & 0 \\ B_{12}^s & B_{22}^s & 0 \\ 0 & 0 & B_{66}^s \end{bmatrix}, D^s = \begin{bmatrix} D_{11}^s & D_{12}^s & 0 \\ D_{12}^s & D_{22}^s & 0 \\ 0 & 0 & D_{66}^s \end{bmatrix}, \]

\[ H^s = \begin{bmatrix} H_{11}^s & H_{12}^s & 0 \\ H_{12}^s & H_{22}^s & 0 \\ 0 & 0 & H_{66}^s \end{bmatrix}. \]

\[ S = \{S_{y}, S_{x}\}^T, \gamma = \{\gamma_{y}, \gamma_{x}\}^T, A^s = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix} \]

where \( A_{ij}, B_{ij}, \) etc., are the plate stiffness, defined by

\[ \begin{align*}
A = & \begin{bmatrix} A_{11} & A_{12} & D_{11} & B_{11}^s & H_{11}^s \\ A_{12} & A_{22} & D_{12} & B_{12}^s & H_{12}^s \\ D_{11} & D_{12} & D_{11} & D_{12}^s & D_{12}^s \\ H_{11}^s & H_{12}^s & D_{12}^s & D_{22} & H_{22}^s \\ B_{11}^s & H_{12} & D_{66} & D_{66} & D_{66} \\ H_{12} & H_{22} & D_{66} & D_{66} & H_{66} \\ D_{66} & D_{66} & D_{66} & D_{66} & H_{66} \\ H_{66} & H_{66} & H_{66} & H_{66} & H_{66} \end{bmatrix} \\
= & \begin{bmatrix} 1, z, z^2, 1, \psi(z), \varphi \psi(z), \psi^2(z) \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}z \\ z^2 \\ \frac{1}{2}z^3 \\ \frac{1}{4}z^4 \end{bmatrix} \end{align*} \]

and

\[ \begin{align*}
A_{44}^s = & A_{55}^s = \int_{-h/2}^{h/2} E(z) \left(1 + \frac{1}{2}z^2\right) g(z) dz, \\
\xi_{11} = & \frac{E(z)}{1 - \nu^2}, \end{align*} \]

The stress and moment resultants, \( N_x' = N_y', M_{x0}^T = M_{y0}^T, M_{x0}^T = M_{y0}^T \) due to thermal loading are defined, respectively, by
\[
\left\{ \begin{array}{c}
N^T_x \\
M^T_x \\
M^T_y \\
M^T_z \\
E(z) \\
\alpha(z) \end{array} \right\} \left\{ \begin{array}{c}
1 \\
z \\
y(z) \end{array} \right\} \text{d}z. \\
\] (16)

Governing equations of equilibrium can be derived from Eq. (11) integrating the displacement gradients by parts and determining the coefficients \( \delta u_0, \delta v_0, \delta w_y \) and \( \delta w_z \) zero separately.

Thus one can obtain the equilibrium equations associated with the present shear deformation theory,

\[
\delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \quad \delta v_0 : \frac{\partial N_y}{\partial x} + \frac{\partial N_x}{\partial y} = 0. \\
\] (17a)

\[
\delta w_y : \frac{\partial^2 M_y}{\partial x^2} + 2 \frac{\partial^2 M_y}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - R_y + Q = 0, \\
\delta w_z : \frac{\partial^2 M_z}{\partial x^2} + 2 \frac{\partial^2 M_z}{\partial x \partial y} + \frac{\partial^2 M_z}{\partial y^2} + \frac{\partial S_z}{\partial y} + \frac{\partial S_y}{\partial y} - R_y + Q = 0. \\
\] (17b)

Substituting Eq. (13) into Eq. (17), we obtain the following equation,

\[
A_{11} d_{111} u_0 + A_{66} d_{22} u_0 + (A_{12} + A_{66}) d_{12} v_0 \\
- B_{11} d_{111} w_y - (B_{12} + 2B_{66}) d_{122} w_y \\
- (B_{12}^* + 2B_{66}^*) d_{123} w_y - B_{11} d_{111} w_y = F_1. \\
\] (18a)

\[
A_{22} d_{22} v_0 + A_{66} d_{11} v_0 + (A_{12} + A_{66}) d_{12} u_0 \\
- B_{22} d_{222} w_y - (B_{12} + 2B_{66}) d_{112} w_y \\
- (B_{12}^* + 2B_{66}^*) d_{113} w_y - B_{22} d_{222} w_y = F_2. \\
\] (18b)

\[
B_{11} d_{111} u_0 + (B_{12} + 2B_{66}) d_{122} u_0 \\
+ (B_{12}^* + 2B_{66}^*) d_{122} u_0 \\
- D_{11} d_{111} w_y - 2(D_{12} + 2D_{66}) d_{112} w_y \\
- D_{22} d_{222} w_y - D_{11} d_{111} w_y \\
- 2(D_{12} + 2D_{66}) d_{112} w_y - D_{22} d_{222} w_y = F_3. \\
\] (18c)

\[
B_{11}^* d_{111} u_0 + (B_{12}^* + 2B_{66}^*) d_{122} u_0 \\
+ (B_{12} + 2B_{66}) d_{122} u_0 \\
- D_{11}^* d_{111} w_y - 2(D_{12} + 2D_{66}) d_{112} w_y \\
- D_{22}^* d_{222} w_y - H_{11} d_{111} w_y \\
- 2(H_{12} + 2H_{66}) d_{112} w_y - H_{11}^* d_{112} w_y \\
+ A_{55} d_{11} w_y + A_{44} d_{12} w_y = F_4. \\
\] (18d)

where \( \{F\} = \{F_1, F_2, F_3, F_4\} \) is a generalized force vector, \( d_{ij}, d_{il} \) and \( d_{jkl} \) are the following differential operators:
The components of the generalized force vector \( \{ F \} \) are given by

\[
F_1 = \frac{\partial N_x^T}{\partial x}, \quad F_2 = \frac{\partial N_y^T}{\partial y},
\]
(20a)

\[
F_3 = R_\theta + Q - \frac{\partial^2 M_x^{bt}}{\partial x^2} - \frac{\partial^2 M_y^{bt}}{\partial y^2},
\]
\[
F_4 = R_\theta + Q - \frac{\partial^2 M_x^{gt}}{\partial x^2} - \frac{\partial^2 M_y^{gt}}{\partial y^2},
\]
(20b)

3. Exact solutions for P-FG plates

Rectangle plates are generally classified according to the type of substrate used. We are here concerned with the exact solution of Eqs. (18) For a simply supported P-FG plate. To solve this problem, Navier assumed that the transverse mechanical and temperature loads, \( Q \) and \( T_i \) in the form of \( a \) in the double Fourier series as

\[
\begin{bmatrix} Q \\ T_i \end{bmatrix} = \begin{bmatrix} Q_0 \\ t_i \end{bmatrix} \sin(\lambda x) \sin(\mu y), \quad (i = 1, 2, 3)
\]
(21)

where \( \lambda = \pi/a, \mu = \pi/b, Q_0 \) and \( t_i \) are constants.

Following the Navier solution procedure, we assume the following solution form for \( u_0, v_0, w_0 \) and \( w_s \) that satisfies the boundary conditions,

\[
\begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ w_s \end{bmatrix} = \begin{bmatrix} U \cos(\lambda x) \sin(\mu y) \\ V \sin(\lambda x) \cos(\mu y) \\ W_0 \sin(\lambda x) \sin(\mu y) \\ W_s \sin(\lambda x) \sin(\mu y) \end{bmatrix},
\]
(22)

where \( U, V, W_0, \) and \( W_s \) are arbitrary parameters to be determined subjected to the condition that the solution in Eq. (22) satisfies governing Eqs. (18). One obtains the following operator equation,

\[
[F] \{ \Delta \} = \{ \tilde{F} \},
\]
(23)

where \( \{ \Delta \} = \{ U, V, W_0, W_s \}^T \) and \([F]\) is the symmetric matrix given by

\[
[F] = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} & \Gamma_{34} \\ \Gamma_{14} & \Gamma_{24} & \Gamma_{34} & \Gamma_{44} \end{bmatrix},
\]
(24)

in which
\[ \Gamma_{11} = -(A_{11}z^2 + A_{66}h^2), \]
\[ \Gamma_{12} = -\mu (A_{12} + A_{66}), \]
\[ \Gamma_{13} = \mu [B_{11}z^2 + (B_{12} + 2B_{66}) \mu^2], \]
\[ \Gamma_{14} = \mu [B_{11}z^2 + (B_{12} + 2B_{66}) \mu^2]. \]  \hspace{1cm} (25a)

\[ \Gamma_{22} = -(A_{66}z^2 + A_{22}h^2), \]
\[ \Gamma_{23} = \mu [(B_{12} + 2B_{66}) z^2 + B_{22}h^2], \]
\[ \Gamma_{24} = \mu [(B_{12} + 2B_{66}) z^2 + B_{22}h^2], \]
\[ \Gamma_{33} = -\left( D_{11}z^2 + 2(D_{12} + 2D_{66}) z^2 \mu^2 + D_{22}h^2 \right), \]
\[ + K_1 + K_2z^2 + K_3h^2 \]  \hspace{1cm} (25b)

\[ \Gamma_{34} = -\left( D_{11}z^2 + 2(D_{12} + 2D_{66}) z^2 \mu^2 \right), \]
\[ + D_{22}h^2 \mu^2 + K_1 + K_2z^2 + K_3h^2 \]  \hspace{1cm} (25c)

\[ \Gamma_{44} = -\left( H_{11}z^2 + 2(H_{11} + 2H_{66}) z^2 \mu^2 + H_{22}h^2 \right), \]
\[ + A_{55}z^2 + A_{44}h^2 + K_1 + K_2z^2 + K_3h^2 \]  \hspace{1cm} (25d)

The components of the generalized force vector \( \{\hat{F}\} = \{\hat{F}_1, \hat{F}_2, \hat{F}_3, \hat{F}_4\} \) are given by

\[ \hat{F}_1 = \lambda (A^t_1 t_1 + B^t_1 t_2 + \alpha B^t_1 t_3), \]
\[ \hat{F}_2 = \mu (A^t_1 t_1 + B^t_1 t_2 + \alpha B^t_1 t_3), \]
\[ \hat{F}_3 = -Q_0 - h(\lambda^2 + \mu^2) (B^t_1 t_1 + D^t_1 t_2 + \alpha D^t_1 t_3), \]
\[ \hat{F}_4 = -Q_0 - h(\lambda^2 + \mu^2) (B^t_1 t_1 + D^t_1 t_2 + \alpha D^t_1 t_3). \]  \hspace{1cm} (26)

Where

\[ \{A^t, B^t, D^t\} = \left\{ \frac{E(z)}{1-\nu} \alpha(z) \{1, z, z^2\} \right\} dz. \]  \hspace{1cm} (27a)

\[ \{\alpha B^t, \alpha D^t\} = \left\{ \frac{E(z)}{1-\nu} \alpha(z) \psi(z) \{1, z\} \right\} dz. \]  \hspace{1cm} (27b)

\[ \{\beta B^t, \beta D^t, \beta F^t\} = \left\{ \frac{E(z)}{1-\nu} \alpha(z) \psi(z) \{1, z, z^2\} \right\} dz \]  \hspace{1cm} (27c)

In which

\[ Z = \frac{z}{h}, \psi(z) = \frac{\psi(z)}{h} \text{ and } \psi(z) = \frac{1}{\pi} \sin \left( \frac{\pi z}{h} \right) \]

4. Results and discussion
In this section, numerical examples are presented and discussed to verify the accuracy of the present theory in the bending analysis of P-FGM rectangular plates resting on nonuniform elastic foundations in the thermal environment. Comparisons are made with various plate
theories available in the literature. The description of various displacement models is given in Table 2.

The P-FGM plate is taken to be made of Titanium and Zirconia with the following material properties "Table 3":

The reference temperature is taken by \( T_0 = 25 \, ^\circ\text{C} \) (room temperature). Numerical results are presented in terms of nondimensional stresses and deflection. The various nondimensional parameters used are

- Center deflection
  \[
  w = \frac{10^2}{Q_0} w(q, \frac{y}{L}),
  \]

- Inplane stress
  \[
  \sigma_x = \frac{1}{10^2Q_0} \sigma_x(q, \frac{y}{L}, \frac{z}{h}),
  \]

- Inplane shear stress
  \[
  \tau_{xy} = \frac{1}{10^2Q_0} \tau_{xy}(0, 0, \frac{z}{h}),
  \]

- Transverse shear stress
  \[
  \tau_{xz} = -\frac{1}{10^2Q_0} \tau_{xz}(0, \frac{y}{L}, 0)
  \]

- Thickness coordinate
  \[
  z = \frac{z}{h}, K_0 = \frac{\sigma_x K_1}{D}, J_0 = \frac{\sigma_x K_2}{D}, D = \frac{h^2E_c}{12(1-\nu)}
  \]

| Table 2. Displacement models |
|-----------------------------|
| **Model** | **Theory** | **Unknown functions** |
| CPT | Classical plate theory deformation | 3 |
| FSDT | First-order shear deformation theory (Reissner, 1945 and Mindlin, 1951) | 5 |
| PSDT | Parabolic shear deformation theory (Reddy, 2000) | 5 |
| TSDDT | Trigonometric shear deformation theory (Touratier, 1991, Zenkour, 2009) | 5 |
| Present | Present refined shear deformation theory | 4 |

| Table 3. Material properties used in the P-FG plate (Bouderba, 2018) |
|-----------------------------|
| **Properties** | **Metal: Titanium (Ti-6Al-4 V)** | **Ceramic: Zirconia(ZrO2)** |
| \( E \) (GPa) | 66.2 | 117 |
| \( \alpha(10^{-6}/\text{C}) \) | 10.3 | 7.11 |
| \( v \) | 1/3 | 1/3 |
Table 4. Effect of the volume fraction exponent on the nondimensional displacements and stresses of an P-FG rectangular plates resting on elastic foundation and subjected to mechanical loading ($a = 10h, b = 3a, q_0 = 100, f = 0, \xi = 0$)

| $k$ | $K_0$ | $J_0$ | Theory          | $\mathcal{W}$ | $\sigma_z$ | $\tau_{xy}$ | $\tau_{xz}$ |
|-----|-------|-------|-----------------|----------------|------------|-------------|-------------|
| 0   | 100   | 100   | Present         | 0.082269       | 0.049172   | 0.070082    | 0.041420    |
|     |       |       | Mudhaffar et al. (2021) | 0.08226       | 0.04920    | 0.07009     | 0.04356     |
|     |       |       | Sayyad and Ghugal (2019) | 0.08227       | 0.04927    | 0.07068     | 0.03700     |
|     |       |       | Zidi et al. (2014) | 0.08228       | 0.04919    | 0.06972     | 0.04116     |
|     |       |       | Reddy (2000) | 0.08228       | 0.04919    | 0.06972     | 0.04116     |
|     |       |       | Touratier (1991) | 0.08227       | 0.04919    | 0.06972     | 0.04246     |
|     |       |       | Mindlin (1951) | 0.08228       | 0.04890    | 0.06986     | 0.03293     |
|     |       |       | CPT            | 0.08201       | 0.05035    | 0.07193     | -            |
| 1   | 100   | 100   | Present         | 0.084176       | 0.045715   | 0.052165    | 0.032356    |
|     |       |       | Mudhaffar et al. (2021) | 0.08417       | 0.04575    | 0.05217     | 0.03403     |
|     |       |       | Sayyad and Ghugal (2019) | 0.08416       | 0.04580    | 0.05181     | 0.02924     |
|     |       |       | Zidi et al. (2014) | 0.08418       | 0.04574    | 0.05190     | 0.03214     |
|     |       |       | Reddy (2000) | 0.08417       | 0.04574    | 0.05190     | 0.03214     |
|     |       |       | Touratier (1991) | 0.08418       | 0.04574    | 0.05191     | 0.03318     |
|     |       |       | Mindlin (1951) | 0.08418       | 0.04546    | 0.05198     | 0.02572     |
|     |       |       | CPT            | 0.08399       | 0.04683    | 0.05353     | -            |
| 2   | 100   | 100   | Present         | 0.084563       | 0.045383   | 0.047944    | 0.029706    |
|     |       |       | Mudhaffar et al. (2021) | 0.08456       | 0.04541    | 0.04794     | 0.03131     |
|     |       |       | Sayyad and Ghugal (2019) | 0.08453       | 0.04549    | 0.04776     | 0.02633     |
|     |       |       | Zidi et al. (2014) | 0.08457       | 0.04539    | 0.04770     | 0.02951     |
|     |       |       | Reddy (2000) | 0.08457       | 0.04539    | 0.04770     | 0.02951     |
|     |       |       | Touratier (1991) | 0.08457       | 0.04540    | 0.04770     | 0.03050     |
|     |       |       | Mindlin (1951) | 0.08457       | 0.04515    | 0.04781     | 0.02303     |
|     |       |       | CPT            | 0.08437       | 0.04655    | 0.04931     | -            |
| 5   | 100   | 100   | Present         | 0.084912       | 0.046520   | 0.045744    | 0.027554    |
|     |       |       | Mudhaffar et al. (2021) | 0.08491       | 0.04655    | 0.04574     | 0.02910     |
|     |       |       | Sayyad and Ghugal (2019) | 0.08487       | 0.04665    | 0.04616     | 0.02450     |
|     |       |       | Zidi et al. (2014) | 0.08491       | 0.04652    | 0.04549     | 0.02735     |
|     |       |       | Reddy (2000) | 0.08491       | 0.04652    | 0.04549     | 0.02735     |
|     |       |       | Touratier (1991) | 0.08492       | 0.04656    | 0.04549     | 0.02832     |
|     |       |       | Mindlin (1951) | 0.08491       | 0.04630    | 0.04568     | 0.02085     |
|     |       |       | CPT            | 0.08471       | 0.04780    | 0.04714     | -            |

(Continued)
Table 4 shows the effects of the volume fraction exponent \( p \) on the nondimensional transverse displacements and stresses of P-FGM rectangular \((b/a = 3)\) plate resting on two-parameter elastic foundation \((K_0 = J_0 = 100)\) and subjected to a mechanical load. The numerical results are obtained for various volume fraction exponents \((k)\) and elastic foundation parameters \((K_0\) and \(J_0\)) in which \((K_0)\) is the Winkler foundation parameter and \((J_0)\) is the Pasternak foundation parameter. Numerical results obtained using the present refined shear deformation theory are compared with classical plate theory (CPT), FSDT of Mindlin (1951), trigonometric shear deformation theory (TSDT) of Touratier (1991), parabolic shear deformation theory (PSDT) of Reddy (2000), HSDT of [Zidi et al. (2014), Sayyad and Ghugal (2019), and Mudhaffar et al. (2021)]. Present results are in excellent agreement with those presented by other researchers. The inclusion of the Winkler foundation parameter yields higher magnitude results than those with the inclusion of Pasternak foundation parameters. It is point out that as the volume fraction exponent increases for P-FGM plates, the transverse displacement will increase. The inverse effect of the volumic fraction exponent is observed on the stresses. Stresses decrease with the increase in the volume fraction exponent.

In Tables 5, Table 6, and Table 7, we have studied the variation of the Effect of the volume fraction exponent and linear elastic foundation parameters on the dimensional and stresses of an P-FG rectangular plate for different values of \( k \) and two fixed values of \( K_0 \) and \( J_0 \) (0 and 100), the comparison of the Present refined shear deformation theory (not introduce the shear factor) with the other theories shows a convergence between all nondimensional quantities such as \( \bar{w}, \bar{\sigma}_x, \bar{\tau}_{xy}, \) and \( \tau_{zz}, \) for a value of zero of \( k \) the change of the value of \( J_0 \) from 0 to 100 causes the change of buckling mode in spite of the variation of the value of \( K_0 \) is illustrated in Table 5, in addition, the variation of \( \tau_{xy} \) is amplified compared to other nondimensional quantities. Table 6 gives almost the same results as Table 5, which shows the influence of elastic foundation parameters on these nondimensional quantities especially for the Winkler model which is the simplest model for elastic foundation which describes the interactions of the plate and the foundation in the most appropriate way possible and the Pasternak model which has improved these interactions. For Table 7, shows Effects of side-to-thickness ratio and type of elastic foundation parameters on the dimensionless deflection of an P-FG rectangular plate, for fixed values of \((K_0 = 0 \) and \( J_0 = 0)\) and the increase of \( k \) the influence of the ratio \( (a/h) \) is significant, on the other hand the significance is weak for values other than 0 (i.e. for \( K_0 \) and \( J_0 \)).

Table 8 shows the effects of the aspect ratio \((a/b)\) and side-to-thickness ratio \((a/h)\) on the nondimensional transverse displacements and stresses of P-FGM plate \((k = 2)\) resting on two-parameter elastic foundation \((K_0 = J_0 = 100)\) and subjected to a mechanical load and including the effect of the temperature field \((t = t_1 = t_2 = t_3)\).
### Table 5. Effect of the volume fraction exponent and linear elastic foundation parameters on the dimensionless and stresses of an P-FG rectangular plate

\( a = 10h, b = 2a, q_0 = 100, t = 0, \xi = 0.3 \)

| \( k \) | \( K_0 \) | \( J_0 \) | Theory | \( \bar{w} \) | \( \bar{\sigma}_x \) | \( \bar{\tau}_{xy} \) | \( \bar{\tau}_{xw} \) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | Present | 0.68134 | 0.42410 | 0.86253 | -0.38432 |
| | | | PSDT* | 0.68134 | 0.42408 | 0.86253 | -0.38180 |
| | | | TSDT* | 0.68131 | 0.42424 | 0.86240 | -0.39400 |
| | | | FSDT* | 0.68135 | 0.42148 | 0.86459 | -0.30558 |
| | | | CPT* | 0.65704 | 0.42148 | 0.86459 | - |
| 100 | 0 | 0 | Present | 0.43145 | 0.26773 | 0.51300 | -0.22858 |
| | | | PSDT* | 0.43148 | 0.26769 | 0.51299 | -0.22708 |
| | | | TSDT* | 0.43145 | 0.26779 | 0.51296 | -0.23434 |
| | | | FSDT* | 0.43147 | 0.26602 | 0.51422 | -0.18175 |
| | | | CPT* | 0.42159 | 0.26964 | 0.52179 | - |
| 0 | 100 | 0 | Present | 0.096328 | 0.058930 | 0.10590 | -0.047223 |
| | | | PSDT* | 0.09632 | 0.05889 | 0.10590 | -0.04687 |
| | | | TSDT* | 0.09632 | 0.05892 | 0.10589 | -0.04837 |
| | | | FSDT* | 0.09632 | 0.05851 | 0.10615 | -0.03751 |
| | | | CPT* | 0.09583 | 0.06042 | 0.10958 | - |
| 100 | 100 | 0 | Present | 0.089046 | 0.054451 | 0.097724 | -0.043529 |
| | | | PSDT* | 0.08904 | 0.05441 | 0.09772 | -0.04325 |
| | | | TSDT* | 0.08904 | 0.05444 | 0.09772 | -0.04464 |
| | | | FSDT* | 0.08903 | 0.05406 | 0.09795 | -0.03462 |
| | | | CPT* | 0.08860 | 0.05584 | 0.10116 | - |
| 1 | 100 | 100 | Present | 0.091887 | 0.050910 | 0.073065 | -0.034166 |
| | | | PSDT* | 0.09187 | 0.05086 | 0.07306 | -0.03393 |
| | | | TSDT* | 0.09187 | 0.05089 | 0.07305 | -0.03502 |
| | | | FSDT* | 0.09187 | 0.05052 | 0.07321 | -0.02716 |

(Continued)
| $k$ | $K_0$ | $J_0$ | Theory | $\bar{W}$ | $\bar{\sigma}_x$ | $\tau_{yx}$ | $\tau_{xz}$ |
|-----|-------|-------|--------|-----------|----------------|-------------|-------------|
| 2   | 100   | 100   | Present | 0.092453  | 0.050580      | 0.067189    | −0.031389   |
|     |       |       | PSDT*  | 0.09245   | 0.05055       | 0.06719     | −0.03117    |
|     |       |       | TSDT*  | 0.09246   | 0.05057       | 0.06718     | −0.03221    |
|     |       |       | FSDT*  | 0.09245   | 0.05022       | 0.06738     | −0.02434    |
| 5   | 100   | 100   | Present | 0.092992  | 0.051937      | 0.064229    | −0.029124   |
|     |       |       | PSDT*  | 0.09299   | 0.05189       | 0.06413     | −0.02892    |
|     |       |       | TSDT*  | 0.09299   | 0.05191       | 0.06412     | −0.02992    |
|     |       |       | FSDT*  | 0.09298   | 0.05160       | 0.06460     | −0.02205    |
| $\infty$ | 100   | 100   | Present | 0.094398  | 0.032590      | 0.058159    | −0.025917   |
|     |       |       | PSDT*  | 0.09439   | 0.05756       | 0.05815     | −0.02574    |
|     |       |       | TSDT*  | 0.09439   | 0.05758       | 0.05814     | −0.02656    |
|     |       |       | FSDT*  | 0.09439   | 0.05719       | 0.05829     | −0.02060    |

(*) Taken from Bouderba (2018)
Table 6. Effect of the volume fraction exponent and linear elastic foundation parameters on the dimensionless and stresses of an P-FG rectangular plate models

| $k$ | $K_0$ | $J_0$ | Theory | $\tilde{w}$ | $\tilde{\sigma}_x$ | $\tau_{xy}$ | $\tau_{xz}$ |
|-----|-------|-------|--------|-------------|----------------|-------------|-------------|
| 0   | 0     | 0     | Present | 2.1764     | 0.19592       | -0.49721    | -0.37916    |
|     |       |       | PSDT*  | 2.1982     | 0.19106       | -0.46854    | -0.37714    |
|     |       |       | TSDT*  | 2.1762     | 0.19592       | -0.49742    | -0.38826    |
| 100 | 0     | 0     | Present | 1.3781     | -0.30358      | -1.6140     | 0.11832     |
|     |       |       | PSDT*  | 1.3921     | -0.31344      | -1.5960     | 0.12040     |
|     |       |       | TSDT*  | 1.3781     | -0.30378      | -1.6139     | 0.12172     |
| 0   | 100   | 0     | Present | 0.30768    | -0.97058      | -2.9139     | 0.69757     |
|     |       |       | PSDT*  | 0.31080    | -0.98704      | -2.9098     | 0.70349     |
|     |       |       | TSDT*  | 0.30770    | -0.97098      | -2.9138     | 0.71573     |
| 100 | 100   | 0     | Present | 0.28443    | -0.98498      | -2.9400     | 0.70937     |
|     |       |       | PSDT*  | 0.28729    | -1.0016       | -2.9362     | 0.71507     |
|     |       |       | TSDT*  | 0.28443    | -0.98538      | -2.9397     | 0.72767     |
| 1   | 100   | 100   | Present | 0.23893    | -0.77435      | -2.3872     | 0.53224     |
|     |       |       | PSDT*  | 0.24161    | -0.79081      | -2.3854     | 0.53861     |
|     |       |       | TSDT*  | 0.23889    | -0.77461      | -2.3872     | 0.56615     |
| 2   | 100   | 100   | Present | 0.23601    | -0.68577      | -2.3167     | 0.50929     |
|     |       |       | PSDT*  | 0.23872    | -0.70196      | -2.3152     | 0.51525     |
|     |       |       | TSDT*  | 0.23605    | -0.68590      | -2.3167     | 0.52354     |
| 5   | 100   | 100   | Present | 0.24427    | -0.59047      | -2.4015     | 0.53252     |
|     |       |       | PSDT*  | 0.24692    | -0.60679      | -2.4000     | 0.53858     |
|     |       |       | TSDT*  | 0.24426    | -0.59086      | -2.4010     | 0.54889     |
| $\infty$ | 100   | 100   | Present | 0.26417    | 0.85866       | -2.5031     | 0.62288     |
|     |       |       | PSDT*  | 0.26667    | -0.51623      | -2.5006     | 0.62782     |
|     |       |       | TSDT*  | 0.26417    | -0.50018      | -2.5029     | 0.63900     |

(*) Taken from Boudierba (2018)
Table 7. Effects of side-to-thickness ratio and type of elastic foundation parameters on the dimensionless deflection of an P-FG rectangular plate using present refined shear deformation theory ($q_0 = 100$, $t = 10$, $b = 2a$, $\varepsilon = 0.3$)

| $k$ | $K_0$ | $\lambda_0$ | Elastic foundation | $a/h$ |
|-----|-------|-------------|--------------------|-------|
|     |       |             | 5                  | 10    | 20   | 50    |
| 0   | 0     | 0           | Parabolic          | 6.72950 | 2.17620 | 1.03690 | 0.71779 |
|     |       |             | Sinusoidal         | 6.72950 | 2.17620 | 1.03690 | 0.71779 |
| 100 | 100   | 100         | Parabolic          | 0.74677 | 0.26416 | 0.12890 | 0.08982 |
|     |       |             | Sinusoidal         | 0.95290 | 0.33594 | 0.16378 | 0.11411 |
| 1   | 0     | 0           | Parabolic          | 6.69090 | 2.32020 | 1.22620 | 0.91966 |
|     |       |             | Sinusoidal         | 6.69090 | 2.32020 | 1.22620 | 0.91966 |
| 100 | 100   | 100         | Parabolic          | 0.58341 | 0.22138 | 0.11980 | 0.09048 |
|     |       |             | Sinusoidal         | 0.74998 | 0.28382 | 0.15350 | 0.11590 |
| 3   | 0     | 0           | Parabolic          | 7.22880 | 2.51580 | 1.33420 | 1.00320 |
|     |       |             | Sinusoidal         | 7.22880 | 2.51580 | 1.33420 | 1.00320 |
| 100 | 100   | 100         | Parabolic          | 0.57495 | 0.22098 | 0.12034 | 0.09117 |
|     |       |             | Sinusoidal         | 0.74098 | 0.28402 | 0.15545 | 0.11705 |
| 5   | 0     | 0           | Parabolic          | 7.68250 | 2.65240 | 1.39090 | 1.03740 |
|     |       |             | Sinusoidal         | 7.68250 | 2.65240 | 1.39090 | 1.03740 |
| 100 | 100   | 100         | Parabolic          | 0.59145 | 0.22614 | 0.12189 | 0.09161 |
|     |       |             | Sinusoidal         | 0.76256 | 0.29089 | 0.15665 | 0.11772 |
| $\infty$ | 0 | 0         | Parabolic          | 9.98950 | 3.36970 | 1.71350 | 1.24960 |
|     |       |             | Sinusoidal         | 9.98950 | 3.36970 | 1.71350 | 1.24960 |
| 100 | 100   | 100         | Parabolic          | 0.65884 | 0.24429 | 0.12737 | 0.09356 |
|     |       |             | Sinusoidal         | 0.85260 | 0.31546 | 0.16438 | 0.12072 |
Table 8. Effect of the temperature, aspect ratio \((a/b)\) and side-to-thickness ratio \((a/h)\) on the nondimensional displacements and stresses of an P-FG plates resting on elastic foundation \((K = 2. K_0 = J_0 = 100, q_0 = 100, \xi = 0.3)\)

| \(a/h\) | \(b/a\) | \(t\) | \(\bar{w}\) | \(\bar{d}_x\) | \(\tau_{xy}\) | \(\tau_{xz}\) |
|---|---|---|---|---|---|---|
| 5  | 1  | 0  | 0.05031 | 0.00737 | 0.01510 | −0.01140 |
|    |    | 5  |         |         |         |         |
|    |    | 10 |         |         |         |         |
| 2  | 0  | 0  | 0.09337 | 0.01174 | 0.01515 | −0.01426 |
|    | 5  |    |         |         |         |         |
|    | 10 |    |         |         |         |         |
| 3  | 0  | 0  | 0.09900 | 0.01204 | 0.01086 | −0.01359 |
|    | 5  |    |         |         |         |         |
|    | 10 |    |         |         |         |         |
| 10 | 1  | 0  | 0.04935 | 0.03283 | 0.07043 | −0.02635 |
|    | 5  |    |         |         |         |         |
|    | 10 |    |         |         |         |         |
| 2  | 0  | 0  | 0.09245 | 0.05058 | 0.06719 | −0.03139 |
|    | 5  |    |         |         |         |         |
|    | 10 |    |         |         |         |         |
| 3  | 0  | 0  | 0.09826 | 0.05154 | 0.04770 | −0.02971 |
|    | 5  |    |         |         |         |         |
|    | 10 |    |         |         |         |         |

(Continued)
Table 8. (Continued)

| $a/h$ | $b/a$ | $t$ | $\bar{W}$ | $\sigma_x$ | $\tau_{yx}$ | $\tau_{xz}$ |
|-------|-------|-----|-----------|-------------|-------------|-------------|
| 20    | 1     | 0   | 0.04907   | 0.13532     | 0.29375     | -0.05479    |
|       |       | 5   |           |             |             |             |
|       |       | 10  |           |             |             |             |
| 2     | 0     | 0   | 0.09221   | 0.20643     | 0.27618     | -0.06441    |
|       | 5     |     |           |             |             |             |
|       | 10    |     |           |             |             |             |
| 3     | 0     | 0   | 0.09806   | 0.20998     | 0.19555     | -0.06084    |
|       | 5     |     |           |             |             |             |
|       | 10    |     |           |             |             |             |
| 50    | 1     | 0   | 0.04898   | 0.85299     | 1.85810     | -0.13863    |
|       | 5     |     |           |             |             |             |
|       | 10    |     |           |             |             |             |
| 2     | 0     | 0   | 0.09213   | 1.29780     | 1.73970     | -0.16229    |
|       | 5     |     |           |             |             |             |
|       | 10    |     |           |             |             |             |
| 3     | 0     | 0   | 0.09799   | 1.31930     | 1.23090     | -0.15300    |
|       | 5     |     |           |             |             |             |
|       | 10    |     |           |             |             |             |

(Continued)
Table 8. (Continued)

| a/h | b/a | t  | \( w \) | \( \sigma_x \) | \( \tau_{xy} \) | \( \tau_{xz} \) |
|-----|-----|----|--------|-------------|-------------|-------------|
| 100 | 1   | 0  | 0.04897| 3.41650     | 7.44560     | −0.27774    |
|     |     | 5  |        |             |             |             |
|     |     | 10 |        |             |             |             |
| 2   | 0   | 0  | 0.09212| 5.19500     | 6.96640     | −0.32487    |
|     |     | 5  |        |             |             |             |
|     |     | 10 |        |             |             |             |
| 3   | 0   | 0  | 0.09799| 5.28080     | 4.92820     | −0.30629    |
|     |     | 5  |        |             |             |             |
|     |     | 10 |        |             |             |             |
In Figure 3, the influence of dimensionless center deflection ($w$) of a rectangular P-FG plate ($k = 2$) for different side-to-thickness ratio ($a/h = 5$ to $100$) is clear, this influence is significant for a value is equal to $5$ is especially for the sinusoidal type ($t_1 = 0, t_2 = t_3 = 10$) on the other hand we see no influence (a level) in the case ($t_1 = 0, t_2 = t_3 = 0$).

For Figure 4, the concentration of the inplane stress is illustrated in the upper and lower surfaces, for the linear and parabolic types the variation of the stress is almost the same, on the other hand there is a difference with the sinusoidal type for values of ($\sigma_x = -0.5$ and $0.5$).
In Figure 5, the dimensionless inplane shear stress \( (\tau_{xz}) \) is zero in the upper and lower parts of the plate, on the other hand it is maximum for a value between (0.05 and 0.15) for all the linear, parabolic and sinusoidal types. The present theory satisfies the traction free boundary conditions at the top and bottom surfaces of the P-FGM plate.

5. Conclusions
In this study and for thermo-mechanical loading, working on the bending analysis of P-FGM plates based on nonuniform elastic foundations, use refined shear deformation theory with four unknown functions, the comparison between the different theories is illustrated in the Tables 4, 5 and 6 and the Figure 6, effect of the volume fraction exponent \( k \) and linear elastic foundation parameters on the dimensionless and stresses of an P-FG rectangular plate is shown in the different values found, which shows us the significance of the change in values for nondimensional quantities, with the inclusion of the temperature field.

The variables of dimensionless shear stress and dimensionless axial stress on the upper and lower surfaces of the plate under investigation are different. The continuity of the properties of functional gradation materials eliminates the interface problems of composite materials and
hence the stress distributions are smooth. Typically, FGM plates are made from a mixture of ceramic and metal or a combination of different materials. The ceramic has a lower coefficient of thermal expansion than metal. The ceramic has greater value of the modulus of elasticity as compared to the metal. That is the reason the FGM plates do not exhibit intermediate response for strain and shear strain to pure metal and pure ceramic plate.

The type of foundation influences the bending of the plates. FG plates have got a higher capability to bear thermal stresses and hence they sustain at elevated temperatures. It should be noted that the unknown functions in the current theory are four, while the unknown functions in the other theories are five. From the performed comparisons, we conclude that the current model is in good agreement with others existing in the literature. The present theory is not only precise but also simple to analyze plate bending. The new refined shear deformation plate theory has been developed successfully with some advantages such as simpler, more efficient and high accuracy in predicting the bending behavior of the P-FGM plates resting on nonlinear elastic foundations and subjected to thermo-mechanical loading.

Acknowledgements
This research was supported by the Algerian Ministry of Higher Education and Scientific Research (MESRS) and The General Directorate of Scientific Research and Technological Development (DGDRSDT). Their support is greatly appreciated.

Funding
This work was supported by the Algerian Ministry of Higher Education and Scientific Research (MESRS) [MESRS].

Author details
Bachir Bouderba,2 Berrabah Hamza Madjid1,3 E-mail: b_hamza_2005@yahoo.fr
1 Department of Science and Technology Faculty of Science and Technology, El-Wancharissi University of Tissemsilt, Tissemsilt, Algeria.
2 Mechanical Engineering Materials and Structures Laboratory, Tissemsilt, Algeria.
3 Department of Civil Engineering, Faculty of Science and Technology, University of Relizane, Relizane, Algeria.

Disclosure statement
No potential conflict of interest was reported by the author(s).

Citation information
Cite this article as: Bending analysis of P-FGM plates resting on nonuniform elastic foundations and subjected to thermo-mechanical loading, Bachir Bouderba & Berrabah Hamza Madjid, Cogent Engineering (2022), 9: 2108576.

References
Ahmed Hassan, A. H., & Kurgan, N. (2020). Bending analysis of thin FGM skew plate resting on Winkler elastic foundation using multi-term extended Kantorovich method. Engineering Science and Technology, an International Journal, 23(4), 788–800. https://doi.org/10.1016/j.jestch.2020.03.009
AlKhatteeb, S. A., & Zenkour, A. M. (2014). A refined theory of plates with four unknowns for advanced plates based on elastic foundations in a hygrothermal environment. Composite Structures, 111, 240–248. https://doi.org/10.1016/j.compstruct.2013.12.013
Ambartsoumian, S. A. (1958). On the Theory of bending plates. Izv Otd Tech Nauk AN SSSR, 5, 69–77.
Ansari, R., Hassani, R., Gholami, R., & Rouhi, H. (2020). Nonlinear bending analysis of arbitrary-shaped porous nanocomposite plates using a novel numerical approach. International Journal of Non-Linear Mechanics, 126, 103556. https://doi.org/10.1016/j.ijnonlinmec.2020.103556
Asemi, K., Salehi, M., & Akhlaghi, M. (2014). Three-dimensional biaxial buckling analysis of functionally graded annular plate fully or partially supported on Winkler elastic foundation. Aerospace Science and Technology, 39, 426–441. https://doi.org/10.1016/j.ast.2014.04.011
Aydogdu, M. (2009). A new shear deformation theory for laminated composite plates. Composite Structures, 89(1), 94–101. https://doi.org/10.1016/j.compstruct.2008.07.008
Bagheri, H., Kiani, Y., & Eslami, M. R. (2018). Asymmetric thermal buckling of temperature dependent annular FGM plates on a partial elastic foundation. Computers & Mathematics with Applications, 75(5), 1566–1581. https://doi.org/10.1016/j.camwa.2017.11.021
Bakora, A., & Tounsi, A. (2015). Thermo-mechanical post-buckling behavior of thick functionally graded plates resting on elastic foundations. Structural Engineering & Mechanics, 56(1), 85–106. http://dx.doi.org/10.12989/sem.2015.56.1.085
Bateni, M., Kiani, Y., & Eslami, M. R. (2013). A comprehensive study on stability of FGM plates. International Journal of Mechanical Sciences, 75, 134–144. https://doi.org/10.1016/j.ijmecsci.2013.05.014
Behroozen Rad, A. (2012). Static response of 2-D functionally graded circular plate with gradient thickness and elastic foundations to compound loads. Structural Engineering and Mechanics, 44(2), 139–161. https://doi.org/10.12989/sem.2012.44.2.139
Beldjelli, Y., Tounsi, A., & Mahmoud, S. R. (2016). Hydrothermo-mechanical bending of S-FGM plates resting on variable elastic foundations using a four variables trigonometric plate theory. Smart Structures and Systems, 18(4), 755–768. https://doi.org/10.12989/ss.2016.18.4.755
Bellifia, H., Benhouch, K. H., Hadji, L., Houari, M. S. A., & Tounsi, A. (2016). Bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position. Journal of the Brazilian Society of Mechanical Sciences and Engineering, 38(1), 265–275. https://doi.org/10.1007/s40430-015-0154-0
Benyamina, A. B., Boudenba, B., & Souula, A. (2018). Bending response of composite material plates with specific properties, case of a typical FGM “Ceramic/Metal” in thermal environments. Periodica Polytechnica Civil Engineering, Paper, 11891, 1–9. https://doi.org/10.3311/PpCi.11891

Boudena, B. (2018). Bending of FGM rectangular plates resting on non-uniform elastic foundations in thermal environment using an accurate theory. Steel and Composite Structures, 27, 311–325. https://doi.org/10.1002/zamm.201800238

Boudena, B., & Benyamina, A. B. (2018). Static analysis of composite material plates “Case of a typical ceramic/metal FGM” in thermal environments. Journal of Materials and Engineering Structures, 5, 33–45. e-2170-127X http://revue.umonto.dbjM3

Boudena, B., & Berrabah, H. M. (2020). Bending response of porous advanced composite plates under thermomechanical loads. Journal of Mechanics Based Design of Structures and Machines, an International Journal, 1–21. https://doi.org/10.1080/15397734.2020.1801464

Boudena, B., Houri, M. S. A., & Tounsi, A. (2013). Thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations. Steel and Composite Structures, 14(1), 85–104. https://doi.org/10.12989/scs.2013.14.1.085

Boudena, B., Houri, M. S. A., Tounsi, A., & Mahmoud, S. R. (2016). Thermal stability of functionally graded sandwich plates using a simple shear deformation theory. Structural Engineering and Mechanics, 58(3), 397–422. https://doi.org/10.12989/sem.2016.58.3.397

Carrera, E. (2002). Temperature profile influence on layered plates response considering classical and advanced theories. AIAA Journal, 40(9), 1885–1896. https://doi.org/10.2514/2.1868

Duc, N. D., Khoa, N. D., Nguyen, P. D., & Duc, N. D. (2019). An analytical solution for nonlinear dynamic response and vibration of FG-CNT reinforced nano-composite elliptical cylindrical shells resting on elastic foundations. ZAMM - Zeitschrift fuer Angewandte Mathematik und Mechanik, 100, e201800238. https://doi.org/10.1002/zamm.201800238

Dong, J., Ma, X., Zhuge, Y., & Mills, J. E. (2017). Shear buckling analysis of laminated plates on tensionless elastic foundations. Steel and Composite Structures, 24, 697–709. https://doi.org/10.1016/j.compstruct.2018.02.038

Duc, N. D. (2010). Dynamic response of imperfect eccentrically stiffened FGM double curved shallow shells on elastic foundation. J.Composite Structures, 102, 306–314. https://doi.org/10.1016/j.compstruct.2010.03.009

Duc, N. D. (2016a). Nonlinear thermal dynamic analysis of eccentrically stiffened S-FGM circular cylindrical shells surrounded on elastic foundations using the Reddy’s third-order shear deformation shell theory. European Journal of Mechanics - A/Solids, 58, 10–30. https://doi.org/10.1016/j.euromechsol.2016.01.004

Duc, N. D. (2016b). Nonlinear thermo-electro-mechanical dynamic response of shear deformable piezoelectric Sigmoid functionally graded sandwich circular cylindrical shells on elastic foundations. Journal of Sandwich Structures and Materials, 20, 351–378. http://dx.doi.org/10.1177%2F1099362616653266

Duc, N. D., Cong, P. H., Anh, V. M., Quang, V. D., Tran, P., Tuan, N. D., & Thinh, N. H. (2016a). Mechanical and thermal stability of eccentrically stiffened functionally graded conical shell panels resting on elastic foundations and in thermal environment. J. Composite Structures, 132, 597–609. http://dx.doi.org/10.1016/j.compstruct.2015.05.072

Duc, N. D., Kim, S. E., & Chan, D. Q. (2018). Thermal buckling analysis of FGM sandwich truncated conical shells reinforced by FGM stiffeners resting on elastic foundations using FSDT. Journal of Thermal Stresses, 41(3), 331–365. https://doi.org/10.1080/01495739.2017.1398623

Duc, N. D., Nguyen, P. D., & Khoa, N. D. (2017). Nonlinear dynamic analysis and vibration of eccentrically stiffened S-FGM elliptical cylindrical shells surrounded on elastic foundations in thermal environments. Thin-Walled Structures, 117, 178–189. https://doi.org/10.1016/j.tws.2017.04.013

Duc, N. D., & Quan, T. Q. (2015a). Nonlinear response of imperfect eccentrically stiffened FGM cylindrical panels on elastic foundation subjected to mechanical loads. European Journal of Mechanics – A/Solids, 46, 60–71. https://doi.org/10.1016/j.euromechsol.2014.02.005

Duc, N. D., Tuan, N. D., Tran, P., Dao, N. T., & Dat, N. T. (2015a). Nonlinear dynamic analysis of Sigmoid functionally graded circular cylindrical shells on elastic foundations using the third order shear deformation theory in thermal environments. International Journal of Mechanical Sciences, 101–102, 338–348. http://dx.doi.org/10.1016/j.ijmecsci.2015.08.018

Gholami, R., & Ansari, R. (2017). Large deflection geometrically nonlinear analysis of functionally graded multilayer graphene platelet-reinforced polymer composite rectangular plates. Composite Structures, 180, 760–771. https://doi.org/10.1016/j.compstruct.2017.08.053

Gholami, R., & Ansari, R. (2018). Nonlinear bending of third-order shear deformable carbon nanotube/fiber/polymer multiscale laminated composite rectangular plates with different edge supports. The European Physical Journal Plus, 133(7), 282. http://dx.doi.org/10.1140/epjp/i2018-12103-2

Gholami, R., & Ansari, R. (2019). Asymmetric nonlinear bending analysis of polymeric composite annular plates reinforced with graphene nanoplatelets. International Journal for Multiscale Computational Engineering, 17(1), 45–63. https://doi.org/10.1615/IntMultCompEng.2019029156

Gholami, R., Ansari, R., Gholami, R., & Rouhi, H. (2019). Nonlinear bending analysis of nanoplates made of FGMs based on the most general strain gradient model and 3D elasticity theory. The European Physical Journal Plus, 134(4), 167. https://doi.org/10.1140/epjp/i2019-12501-x

Javani, M., Kiani, Y., & Esiami, M. R. (2019). Large amplitude thermally induced vibrations of temperature dependent annular FGM plates. Composites Part B: Engineering, 163, 371–383. https://doi.org/10.1016/j.compositesb.2018.11.018

Kaczkowski, Z. (1968). “Plates. Statistical Calculations”. Arkady.

Karama, M., Afaq, K. S., & Mistou, S. (2003). Mechanical behaviour of laminated composite beam by the new multi-layered laminated composite structures model with transverse shear stress continuity. International Journal of Solids and Structures, 40(6), 1525–1546. https://doi.org/10.1016/S0020-7683(02)00647-9

Khoo, N. D., Thiem, H. T., & Duc, N. D. (2019). Nonlinear buckling and post-buckling of imperfect piezoelectric...
S-FGM circular cylindrical shells with metal-ceramic-metal layers in thermal environment using Reddy's third-order shear deformation shell theory. Journal of Mechanics of Advanced Materials and Structures, 26 (3), 248–259. https://doi.org/10.1080/15376494.2017.1341583

Kiani, Y. (2016). Free vibration of carbon nanotube reinforced composite plate on point Supports using Lagrangian multipliers. Meccanica, 51(6), 1353–1367. https://doi.org/10.1007/s11012-016-0466-3

Kiani, Y. (2018). Thermal post-buckling of temperature dependent sandwich plates with FG-CNTRC face sheets. Journal of Thermal Stresses, 41(7), 866–882. https://doi.org/10.1080/01495379.2018.1425645

Kiani, Y. (2020). NURBS-based thermal buckling analysis of graphene platelet reinforced composite laminated skew plates. Journal of Thermal Stresses, 43(1), 90–108. https://doi.org/10.1080/01495379.2019.1673687

Kiani, Y., & Esfandiari, M. R. (2014a). Nonlinear thermo-inertial stability of thin circular FGM plates. Journal of the Franklin Institute, 351(2), 1057–1073. https://doi.org/10.1016/j.jfranklin.2013.09.013

Kiani, Y., & Esfandiari, M. R. (2014b). Based on the uncoupled thermoelasticity assumptions, axisymmetric thermally induced vibrations of a circular plate made of functionally graded materials (FGMs) are analyzed. Source: Journal of Thermal Stresses, 37(24), 1495–1518. Publisher: Taylor and Francis Ltd https://doi.org/10.1080/01495379.2014.917259

Kiani, Y., & Esfandiari, M. R. (2015). thermal postbuckling of imperfect circular functionally graded material plates: Examination of Voigt, Mori-Tanaka, and Self-Consistent schemes. Journal of Pressure Vessel Technology, 137(2), 021201111. https://doi.org/10.1115/1.4026993

Levinson, M. (1980). An accurate simple theory of the statics and dynamics of elastic plates. Mechanics Research Communications, 7(6), 343–350. https://doi.org/10.1016/0098-6431(80)90049-X

Liu, X., Liu, X., & Zhou, W. (2020). An analytical spectral stiffness method for buckling of rectangular plates on Winkler foundation subject to general boundary conditions. Applied Mathematical Modelling, 86, 36–53. https://doi.org/10.1016/j.apm.2020.05.010

Liu, J., Deng, X., Wang, Q., Zhong, R., Xiong, R., & Zhao, J. (2020). A unified modeling method for dynamic analysis of GPL-reinforced FGP plate resting on Winkler-Pasternak foundation with elastic boundary conditions. Composite Structures, 244, 112217. https://doi.org/10.1016/j.compositescientific.2020.112217

Mashat, D. S., Zenkour, A. M., & Radwan, A. F. (2020). A quasi-3D higher-order plate theory for bending of FG plates resting on elastic foundations under hydro-thermo-mechanical loads with porosity. European Journal of Mechanics - A/Solids, 82, 103985. https://doi.org/10.1016/j.euromechsol.2020.103985

Mindlin, R. D. (1951). Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates. Journal of Applied Mechanics, 18(1), 31–38. https://doi.org/10.1100/1978-1-461-8865-4.4, 29

Mudhaffar, I. M., Tounsi, A., Chikhi, A., Al-Osta, M. A., Al-Zahrani, M. M., & Al-Dulaij, S. U. (2021). Hygro-thermo-mechanical bending behavior of advanced functionally graded ceramic metal plate resting on a viscoelastic foundation. Structures, 33, 2177–2189. https://doi.org/10.1016/j.istruc.2021.05.090

Murthy, V. V. (1981). An improved transverse shear deformation theory for laminated anisotropic plates. NASA Technical Paper 1903.

Panc, V. (1975). Theories of elastic plates. Prague: Academia. https://doi.org/10.1007/978-94-010-1906-4

Reddy, J. N. (1984). A simple higher-order theory for laminated composite plates. Journal of Applied Mechanics, 51(4), 745–752. https://doi.org/10.1115/1.3167719

Reddy, J. N. (2000). Analysis of functionally graded plates. International Journal for Numerical Methods in Engineering, 47(1–3), 663–684. https://doi.org/10.1002/(SICI)1097-0207(20000310)47:1<663::AID-NME785>3.0.CO;2-8

Reer, E. (1984). The effect of transverse shear deformation on the bending of elastic plates. ASME Journal of Applied Mechanics, 12(2), 69–77. Corpus ID: 124143526. https://doi.org/10.1115/1.4009435

Reer, E. (1985). On transverse bending of plates, including the effect of transverse shear deformation. International Journal of Solids and Structures, 11, 569–573. https://doi.org/10.1016/0020-7681(75)90030-X

Soyyod, A. S., & Ghugal, Y. M. (2018). An inverse hyperbolic theory for FG beams resting on Winkler-Pasternak elastic foundation. Advances in Aircraft and Spacecraft Science, 5, 671–689. https://doi.org/10.4006/2091-0002.02018.990030-X

Soyyod, A. S., & Ghugal, Y. M. (2019). Effects of nonlinear hygrothermoelastic loading on bending of FGM rectangular plates resting on two-parameter elastic foundation using four-unknown plate theory. Journal of Thermal Stresses, 42(2), 213–232. https://doi.org/10.1080/01495379.2018.1469962

Shimpi, R. P. (2002). Refined plate theory and its variants. AIAA Journal, 40(11), 137–146. https://doi.org/10.2514/2.1622

Shimpi, R. P., Arya, H., & Naik, N. K. (2003). A higher order displacement model for the plate analysis. Journal of Reinforced Plastics and Composites, 22(18), 1667–1688. https://doi.org/10.1177/0731684403027618

Sobhy, M. (2020). Size dependent hygro-thermal buckling of porous FGM sandwich microplates and microbeams using a novel four-variable shear deformation theory. International Journal of Applied Mechanics, 12 (2050017), 30. https://doi.org/10.1142/S1758825120500179

Soldatos, K. P. (1992). A transverse shear deformation theory for homogeneous monoliconic plates. Acta Mechanica, 94(3–4), 195–220. http://dx.doi.org/10.1007/BF01207698

Thai, H. T., Choi, D. H., & Choi, D.-H. (2013). A simple refined theory for bending, buckling, and vibration of thick plates resting on elastic foundation. International Journal of Mechanical Sciences, 73, 40–52. https://doi.org/10.1016/j.ijmecsci.2013.03.017

Touratier, M. (1991). An efficient standard plate theory. International Journal of Engineering Science, 29(8), 901–916. https://doi.org/10.1016/0022-2229(91)90165-Y

Tungik, V. B., & Rao, K. M. (1994). Three dimensional exact solution of thermal stresses in rectangular composite laminates. Composite Structures, 27(4), 419–430. https://doi.org/10.1016/0263-8223(94)90268-2

Youcef, D. O., Kaci, A., Benzaïr, A., Bousahl, A. A., & Tounsi, A. (2018). Dynamic analysis of nanoscale objects including surface stress effects. Smart Structures and Systems, 21, 65–74. https://doi.org/10.1177/1545225516686833

Zenkour, A. M. (2009). The refined sinusoidal theory for FGM plates on elastic foundations. International Journal of Mechanical Sciences, 51, 869–880. https://doi.org/10.1016/j.ijmecsci.2009.09.026
Zenkour, A. M. (2010). Hygro-thermo-mechanical effects on FGM plates resting on elastic foundations. *Composite Structures*, 93(1), 234–238. https://doi.org/10.1016/j.compositestruct.2010.04.017

Zenkour, A. M., Allam, M. N. M., & Radwan, A. F. (2013). Effects of hygrothermal conditions on cross-ply laminated plates resting on elastic foundations. *Archives of Civil and Mechanical Engineering*, 14, 144–159. http://dx.doi.org/10.1016/j.acme.2013.07.008

Zenkour, A. M., & Hafed, Z. S. (2020). Bending response of functionally graded piezoelectric plates using a two-variable shear deformation theory. *Advances in Aircraft and Spacecraft Science*, 7, 115–134. https://doi.org/10.12989/aas.2020.7.2.115

Zidi, M., Tounsi, A., Houari, M. S. A., Bedia, E. A. A., & Beg, O. A. (2014). Bending analysis of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory. *Aerospace Science and Technology*, 34, 24–34. https://doi.org/10.1016/j.ast.2014.02.001