Addition-deletion theorem for free hyperplane arrangements and combinatorics

Takuro Abe

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Abstract

In the theory of hyperplane arrangements, the most important and difficult problem is the combinatorial dependency of several properties. In this article, we prove that Terao’s celebrated addition-deletion theorem for free arrangements is combinatorial, i.e., whether you can apply it depends only on the intersection lattice of arrangements. The proof is based on a classical technique. Since some parts are already completed recently, we prove the rest part, i.e., the combinatoriality of the addition theorem. As a corollary, we can define a new class of free arrangements called the additionally free arrangement of hyperplanes, which can be constructed from the empty arrangement by using only the addition theorem. Then we can show that Terao’s conjecture is true in this class. As an application, we can show that every ideal-Shi arrangement is additionally free, implying that their freeness is combinatorial.

1 Introduction

Let \( V = \mathbb{K}^\ell \), \( S = \text{Sym}(V^*) = \mathbb{K}[x_1, \ldots, x_\ell] = \bigoplus_{d=0}^\infty S_d \) with the standard polynomial degree grading, and \( \text{Der } S = \bigoplus_{i=1}^\ell S \partial_{x_i} \). An arrangement of hyperplanes \( \mathcal{A} \) is a finite set of linear hyperplanes in \( V \). In the study of hyperplane arrangements, the most interesting and difficult problem is to determine whether some property of \( \mathcal{A} \) depends only on combinatorics. For example, when \( \mathbb{K} = \mathbb{C} \), the cohomology ring \( H^*(V \setminus \bigcup_{H \in \mathcal{A}} H; \mathbb{Z}) \) is shown to be combinatorial in [10]. However in general, to determine whether some
property of $\mathcal{A}$ is combinatorial or not is a hard problem. In this article, we will show that an important property of algebra of $\mathcal{A}$ is combinatorial, which completes the recent sequential works by the author in [1], [2], [3] and [4]. For that, let us introduce a notation for algebra of $\mathcal{A}$.

For each $H \in \mathcal{A}$, let $\alpha_H$ denote a fixed defining linear form of $H$. Now the logarithmic derivation module $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := \{ \theta \in \text{Der } S \mid \theta(\alpha_H) \in S\alpha_H \quad (\forall H \in \mathcal{A}) \}.$$

$D(\mathcal{A})$ is an $S$-graded reflexive module of rank $\ell$, but not a free $S$-module in general. We say that $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$ if there is a homogeneous $S$-basis $\theta_1, \ldots, \theta_\ell$ for $D(\mathcal{A})$ such that $\deg \theta_i = d_i$ for all $i$. Here we say that a non-zero derivation $\theta \in \text{Der } S$ is homogeneous of degree $d$ if $\theta(\alpha) \in S_d$ for all $\alpha \in V^*$. For example, the Euler derivation $\theta_E := \sum_{i=1}^{\ell} x_i \partial_{x_i}$, which is always contained in $D(\mathcal{A})$, is homogeneous of degree 1.

Because of several important results which relate the algebra $D(\mathcal{A})$, topology and combinatorics of $\mathcal{A}$, the logarithmic derivation module and free arrangements have been being studied. For example, see Theorem 2.1 in [5]. However, it is not easy to check the freeness. The most useful result to check the freeness is the following addition-deletion theorem due to Terao in 1980.

**Theorem 1.1 (Addition-deletion theorem, [14])**

Let $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}^H := \{L \cap H \mid L \in \mathcal{A}'\}$. Then two of the following three imply the third:

1. $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-1}, d_\ell)$.
2. $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.
3. $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (d_1, \ldots, d_{\ell-1})$.

In particular, all the three above are true if both $\mathcal{A}$ and $\mathcal{A}'$ are free.

Though Theorem 1.1 is proved almost 40 years before, it is the most useful to construct free arrangements, and to check the freeness and non-freeness even now. Related to this the relation between the freeness and combinatorics of $\mathcal{A}$ is well-studied too. Namely, the intersection lattice $L(\mathcal{A})$ is defined by

$$L(\mathcal{A}) := \{ \cap_{B \in \mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A} \}.$$ 

This is considered to be combinatorial information of $\mathcal{A}$. Then Terao’s conjecture asks whether the freeness depends only on $L(\mathcal{A})$. Recently, the
relation between Theorem 1.1 and combinatorics is intensively studied. See Theorems 2.12 and 2.13 for example. In particular, the deletion theorem \((1)+(3) \Rightarrow (2)\) in Theorem 1.1 is shown to be combinatorial in Theorem 2.13. Actually, in these results, not the total structure of \(L(A)\) but the following three combinatorial objects play the key roles:

(i) The localization

\[ A_X := \{ H \in A \mid X \subset H \} \]

of \(A\) at \(X \in L(A)\).

(ii) The restriction

\[ A^H := \{ L \cap H \mid L \in A \setminus \{H\} \} \]

of \(A\) onto \(H\), which is an arrangement in \(H = \mathbb{K}^{\ell-1}\). Also, for \(X \in L(A^H)\), define

\[ A^H_X := (A_X)^H = (A^H)_X. \]

It is easy to show the second equality above.

(iii) The characteristic polynomial \(\chi(A; t)\) of \(A\), see Definition 2.2.

To state the main result in this article, let us introduce the following new combinatorial property:

**Definition 1.2**

Let \(H \in A\). We say that \(A\) is divisional along \(H\) if \(\chi(A^H_X; t) \mid \chi(A_X; t)\) for all \(X \in L(A^H)\). We say that \(A\) is locally (resp. globally) divisional along \(H\) if \(\chi(A^H_X; t) \mid \chi(A_X; t)\) for all \(X \in L(A^H) \setminus \{\cap_{Y \in A^H} Y\}\) (resp. \(\chi(A^H; t) \mid \chi(A; t))\).

By the divisibility along \(H\), we can prove a combinatorial version of Theorem 1.1 as follows:

**Theorem 1.3 (Combinatorial addition-deletion theorem)**

Let \(H \in A\), \(A' := A \setminus \{H\}\) and \(A^H := \{ L \cap H \mid L \in A' \}\). Then two of the following four imply the third and fourth:

(i) \(A\) is free.

(ii) \(A'\) is free.

(iii) \(A^H\) is free, and \(A\) is globally divisional along \(H\).

(iv) \(A\) is divisional along \(H\).

In particular, all the four above are true if (iii) is true.
In Theorem 1.3, (iii) $\Rightarrow$ (i), (ii), (iv) is the division theorem in [2] (Theorem 2.12). (i) + (ii) $\Rightarrow$ (iii) + (iv) follows from Theorem 1.1 and Lemma 3.2. (i) + (iv) $\Rightarrow$ (ii) + (iii) follows from [3] (Theorem 2.13). So the rest part is (ii) + (iv) $\Rightarrow$ (i) + (iii), i.e., whether the addition theorem ((2)+(3) $\Rightarrow$ (1) in Theorem 1.1) is combinatorial or not. This is the main part of this article, as follows.

**Theorem 1.4**
For $H \in \mathcal{A}$, let $\mathcal{A}' := \mathcal{A} \setminus \{H\}$. Assume that $\mathcal{A}'$ is free. Then $\mathcal{A}$ is free if and only if $\mathcal{A}$ is divisional along $H$. Hence whether $\mathcal{A}$ is free or not depends only on $L(\mathcal{A})$ when $\mathcal{A}'$ is free.

From the viewpoint of Theorems 1.3 and 1.4, the addition-deletion theorem may be regarded as the result that implies the freeness of all the triple $\mathcal{A}, \mathcal{A} \setminus \{H\}, \mathcal{A}^H$ by (A) the freeness of one of them, and (B) one combinatorial condition. So the following is the new interpretation of Theorem 1.1 from this viewpoint.

**Theorem 1.5 (Free triple theorem)**
For $H \in \mathcal{A}$, assume that one of the following holds:

(i) $\mathcal{A}$ is free and $\mathcal{A}$ is divisional along $H$.

(ii) $\mathcal{A} \setminus \{H\}$ is free and $\mathcal{A}$ is divisional along $H$.

(iii) $\mathcal{A}^H$ is free and $\mathcal{A}$ is globally divisional along $H$.

Then all of $\mathcal{A}, \mathcal{A} \setminus \{H\}, \mathcal{A}^H$ are free.

Also, we can define the following class of free arrangements, which is very natural to consider, and by which we have constructed a lot of free arrangements.

**Definition 1.6 (Additionally free arrangements)**
The set $\mathcal{AF}$ consists of arrangements $\mathcal{A}$ such that there is a filtration

$$\emptyset = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n = \mathcal{A}$$

such that $\mathcal{A}_i$ is divisional along $H_i$, where $\mathcal{A}_i \setminus \mathcal{A}_{i-1} = \{H_i\}$ for $i = 1, \ldots, n$. Such a filtration is called the **additional filtration** of $\mathcal{A}$, and an arrangement $\mathcal{A} \in \mathcal{AF}$ is called an **additionally free arrangement**.

**Remark 1.7**
The additional filtration is the same as the **free filtration** introduced in [6]. Because we call $\mathcal{AF}$ the set of additionally free arrangements, we use the terminology “additional filtration” here.
In other words, an arrangement is additionally free if and only if that can be constructed from the empty arrangement by using the addition theorem, which is a really natural class of free arrangements. In fact, we can show the following contribution to Terao’s conjecture in terms of additionally free arrangements.

**Theorem 1.8**

\( \mathcal{A} \) is free if \( \mathcal{A} \in \mathcal{AF} \). In particular, Terao’s conjecture is true in the class of additionally free arrangements.

It was known that Terao’s conjecture is true in the class of inductively free arrangements (Definition 2.14) which can be constructed by using the addition and restriction theorem. In [2], the new class so called the divisionally free arrangement is defined (Definition 2.15) in which Terao’s conjecture is true, and can be constructed by only using the division theorem (Theorem 2.12). Additionally free arrangements are the other direction of this generalization. Actually, we can show that the freeness of ideal-Shi arrangements depends only on combinatorics by using the additional freeness (Theorem 5.1). Moreover, we define the largest class of free arrangements \( \mathcal{SF} \) in which Terao’s conjecture is true. The class \( \mathcal{SF} \) is called the stair-free arrangements, constructed in §4 by combining both additionally and divisionally free arrangements.

The organization of this article is as follows. In §2 we recall several results on arrangements. In §3 we prove Theorem 1.4. In §4 we study additionally free arrangements and its extension joining with the divisionally free arrangements. In §5 we apply our main result to ideal-Shi arrangements, proving that their freeness depends only on combinatorics.

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## 2 Preliminaries

In this section let us collect several definitions and results on arrangements which will be used in this article. See [11] for general reference. First let us recall several combinatorics of arrangements.

**Definition 2.1**

For the intersection lattice

\[
L(\mathcal{A}) := \{ \cap_{H \in B} H \mid B \subset A \}
\]
of an arrangement $\mathcal{A}$, define
\[ L_k(\mathcal{A}) := \{ X \in L(\mathcal{A}) \mid \text{codim}_V X = k \}. \]
The rank $r(\mathcal{A})$ of an arrangement $\mathcal{A}$ is defined by
\[ r(\mathcal{A}) := \text{codim} \bigcap_{H \in \mathcal{A}} H. \]
For $X \in L(\mathcal{A})$, the localization $\mathcal{A}_X$ of $\mathcal{A}$ at $X$ is defined as
\[ \mathcal{A}_X := \{ H \in \mathcal{A} \mid X \subset H \}. \]
For $H \in \mathcal{A}$, the restriction $\mathcal{A}^H$ of $\mathcal{A}$ onto $H$ is defined as
\[ \mathcal{A}^H := \{ H \cap L \mid L \in \mathcal{A} \setminus \{ H \} \}. \]
Let
\[ 0_X := \cap_{H \in \mathcal{A}_X} H \in L(\mathcal{A}_X) \]
for $X \in L(\mathcal{A})$. So $\text{rank}(\mathcal{A}_X) = \text{codim}_V 0_X$.

Next recall the most important combinatorial invariant of $\mathcal{A}$.

**Definition 2.2**

1. The Möbius function $\mu : L(\mathcal{A}) \to \mathbb{Z}$ is defined by $\mu(V) = 1$, and by
\[ \mu(X) = - \sum_{Y \in L(\mathcal{A}), \ X \subseteq Y \subseteq V} \mu(Y) \]
for $Y \in L(\mathcal{A}) \setminus \{ V \}$.

2. The characteristic polynomial $\chi(\mathcal{A}; t)$ is defined by
\[ \chi(\mathcal{A}; t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}. \]

**Remark 2.3**

By definition $\chi(\mathcal{A}; t)$ is a combinatorial invariant of $\mathcal{A}$. Moreover, when $\mathbb{K} = \mathbb{C}$, $(-t)^{\ell} \chi(\mathcal{A}; -1/t)$ coincides with the topological Poincaré polynomial of $\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. So it is a topological invariant too.

To compute $\chi(\mathcal{A}; t)$, the following deletion-restriction formula is the most useful. For the proof, see Corollary 2.57 in [11] for example.
Theorem 2.4
For $H \in \mathcal{A}$, it holds that

$$\chi(\mathcal{A}; t) = \chi(\mathcal{A} \setminus \{H\}; t) - \chi(\mathcal{A}^H; t).$$

We also use the following property, which directly follows from the definition of Möbius function in Definition 2.2 (1).

Proposition 2.5
When $\mathcal{A}$ is not empty, $\chi(\mathcal{A}; t)$ is divisible by $t - 1$. Define the reduced characteristic polynomial $\chi_0(\mathcal{A}; t) := \chi(\mathcal{A}; t)/(t - 1)$.

The localization behaves nicely in the sense of freeness as follows, see Theorem 4.37 in [11] for the proof.

Theorem 2.6
If $\mathcal{A}$ is free, then $\mathcal{A}_X$ is free for all $X \in L(\mathcal{A})$.

We will see the relation between the localization and freeness for details in the following.

Definition 2.7
Let $X \in L_k(\mathcal{A})$, and let $\emptyset_k$ denote the empty arrangement in $\mathbb{K}^\ell$. Then $\mathcal{A}_X$ decomposes into

$$\mathcal{A}_X = \mathcal{B}_X \times \emptyset_{\ell-k}$$

for some arrangement $\mathcal{B}_X$ in $V/X$ with $\text{rank}(\mathcal{B}_X) = k$, and the empty arrangement $\emptyset_{\ell-k}$ in $X$. We say that $\mathcal{B}_X$ is the essential part of $\mathcal{A}_X$.

For the proof of the following, see Proposition 4.14 in [11] for example.

Proposition 2.8
Let $X \in L_k(\mathcal{A})$ and $\mathcal{B}_X$ be the essential part of $\mathcal{A}_X$. Then $\mathcal{A}_X$ is free if and only if $\mathcal{B}_X$ is free.

Let us introduce the local version of freeness.

Definition 2.9
We say that $\mathcal{A}$ is locally free if $\mathcal{A}_X$ is free for all $X \in L(\mathcal{A}) \setminus \{0_V\}$.

It is known that $\mathcal{A}$ is locally free if and only if $\widehat{D(\mathcal{A})}$ is a vector bundle on $\mathbb{P}^{\ell-1} = \text{Proj}(S)$. We use the following famous criterion for freeness. For the proof of the arrangement version, see Theorem 4.19 in [11].
Theorem 2.10 (Saito’s criterion, [12])
\( \mathcal{A} \) is free if and only if there are homogeneous derivations \( \theta_1, \ldots, \theta_\ell \in D(\mathcal{A}) \) such that they are independent over \( S \), and that \( |\mathcal{A}| = \sum_{i=1}^\ell \deg \theta_i \).

The freeness is related to combinatorics and topology of arrangements by the following famous result.

Theorem 2.11 (Teraro’s factorization, Main Theorem in [15])
Let \( \mathcal{A} \) be free with \( \exp(\mathcal{A}) = (d_1, \ldots, d_\ell) \). Then
\[
\chi(\mathcal{A}; t) = \prod_{i=1}^\ell (t - d_i).
\]

Based on the division property in Definition 1.2 with the theory of multiarrangements, following two generalizations of Theorem 1.1 have been obtained.

Theorem 2.12 (Division theorem, Theorem 1.1, [2])
Let \( H \in \mathcal{A} \). If \( \mathcal{A}^H \) is free, and \( \chi(\mathcal{A}^H; t) \mid \chi(\mathcal{A}; t) \), then \( \mathcal{A} \) is free.

Theorem 2.13 (Theorem 1.2, [3])
Let \( \mathcal{A} \) be free, \( H \in \mathcal{A} \) and let \( \mathcal{A}' := \mathcal{A} \setminus \{H\} \). Then \( \mathcal{A}' \) is free if and only if \( \mathcal{A} \) is divisional along \( H \).

There are several class of free arrangements in which Terao’s conjecture is true. Let us recall two of them.

Definition 2.14 ([11])
Let \( \mathcal{IF}_\ell \) denote the set of arrangements in \( \mathbb{K}_\ell \) defined in the following manner.

1. \( \emptyset_\ell \in \mathcal{IF}_\ell \) for all \( \ell \geq 0 \).

2. \( \mathcal{A} \in \mathcal{IF}_\ell \) if there is \( H \in \mathcal{A} \) such that \( \mathcal{A} \setminus \{H\} \in \mathcal{IF}_\ell \), and \( \mathcal{A}^H \in \mathcal{IF}_{\ell-1} \).

The arrangement \( \mathcal{A} \) belonging to
\[ \mathcal{IF} := \bigcup_{\ell \geq 0} \mathcal{IF}_\ell \]
is called the inductively free arrangement.

Definition 2.15 ([2])
The class \( \mathcal{DF}_\ell \) of arrangements in \( \mathbb{K}_\ell \) consists of arrangements \( \mathcal{A} \) such that, there is \( X_i \in L_i(\mathcal{A}) \) (\( i = 0, \ldots, \ell - 2 \)) such that
\[
X_0 = V \supset X_1 \supset \cdots \supset X_{\ell-2},
\]
and \( \chi(A^X; t) \mid \chi(A^{X_i}; t) \) for \( i = 1, \ldots, \ell - 2 \). The set
\[
\mathcal{DF} := \bigcup_{\ell \geq 0} \mathcal{DF}_\ell
\]
is called the set of \textbf{divisionally free arrangements}, and the above flag 
\( \{X_i\}_{i=0}^{\ell-2} \) of \( A \) is called a \textbf{divisional flag} of \( A \).

**Theorem 2.16 (Theorems 1.3 and 1.6, [2])**

\[
\mathcal{IF} \subsetneq \mathcal{DF},
\]
and \( A \in \mathcal{DF} \) is free. Moreover, Terao’s conjecture is true for divisionally free 
arrangements.

For the proof of our main result, let us recall the following key results.
For the proof, see Proposition 4.41 in [11] for example.

**Proposition 2.17 ([14])**

Let \( H \in A \). Then there is a polynomial \( B \) of degree \( |A| - |A^H| - 1 \) such that 
\( \alpha_H \nmid B \), and 
\[
\theta(\alpha_H) \in (\alpha_H, B)
\]
for all \( \theta \in D(A \setminus \{H\}) \).

The following submodule of \( D(A) \) plays a key role too in the proof of 

**Proposition 2.18 (e.g., Lemma 1.33, [17])**

For \( H \in A \), define 
\[
D_H(A) := \{ \theta \in D(A) \mid \theta(\alpha_H) = 0 \}.
\]

Then
\[
D(A) = S\theta_E \oplus D_H(A).
\]

In other words,
\[
D(A)/S\theta_E \cong D_H(A)
\]
as \( S \)-graded modules. Thus \( \theta_E \) is a part of basis when \( A \neq \emptyset \) is free. So
\( \exp(A) = (1, d_2, \ldots, d_\ell) \) if \( A \) is not empty.

Then we have the following algebro-geometric interpretation of \( \chi(A; t) \) 
by using \( D_H(A) \).
Theorem 2.19 (Theorem 4.1, [9])
Assume that $\mathcal{A}$ is locally free, $H \in \mathcal{A}$ and let $\widehat{D_H(\mathcal{A})}$ denote the sheafification of $D_H(\mathcal{A})$ onto $\mathbf{P}^{\ell-1} \simeq \text{Proj}(S)$. If $c_1(D_H(\mathcal{A}))$ denotes the Chern polynomial of $D_H(\mathcal{A})$, then
$$t^{\ell-1}c_1(\widehat{D_H(\mathcal{A})}) = \chi_0(\mathcal{A}; t).$$

3 Proof of main results

In this section we prove Theorem 1.4. First let us show the following easy results.

Lemma 3.1
Let $H \in \mathcal{A}$ and $\mathcal{A}' := \mathcal{A} \setminus \{H\}$. Assume that $\mathcal{A}'$ is free, and $X \in L(\mathcal{A}^H)$. Then $A'_X := A_X \setminus \{H\}$ is free.

Proof. First assume that $X \in L(\mathcal{A}')$ too. Then $(\mathcal{A}'_X) = A_X \setminus \{H\}$. Since $\mathcal{A}'$ is free, Theorem 2.6 shows that $\mathcal{A}'_X$ is free. Next assume that $X \not\in L(\mathcal{A}')$. Then there is $Y \in L(\mathcal{A}')$ such that $\text{codim} Y + 1 = \text{codim} X$ and that $X = Y \cap H$. Moreover, there are no $L \in \mathcal{A}'$ such that $L \cap Y = X$. Then $A_X \setminus \{H\} = (\mathcal{A}')_Y$, which is free by Theorem 2.6. □

Lemma 3.2
If $\mathcal{A}$ and $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ are both free, then $\mathcal{A}$ is divisional along $H$.

Proof. Apply Theorems 1.1, 2.11, and Lemma 3.1. □

Lemma 3.3
Assume that $\mathcal{A}$ is divisional along $H \in \mathcal{A}$. Then for any $X \in L(\mathcal{A}^H)$, $\mathcal{A}_X$ is divisional along $H$. Moreover, if $\mathcal{B}_X$ denotes the essential part of $\mathcal{A}_X$, then $\mathcal{B}_X$ is divisional along $H$ too.

Proof. Immediate by Definitions 1.2, 2.2 and 2.7. □

Now let us show the main result in this section, which is the key to prove Theorem 1.4.

Theorem 3.4
Assume that $\mathcal{A}$ is locally free, and $\mathcal{A}'$ is free. Then $\mathcal{A}$ is free if and only if $\chi(\mathcal{A}^H; t) | \chi(\mathcal{A}; t)$. 

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Proof. The statement is true if $A' = \emptyset$. Assume not in the following. If $A$ and $A'$ are free, then Lemma 3.2 implies that $A$ is divisional along $H$. So it suffices to show that $A$ is free if $A'$ is free, $A$ is locally free and $\chi(A^H; t) | \chi(A; t)$. If $\exp(A') = (1, d_2, \ldots, d_\ell)$ with $d_2 \leq \cdots \leq d_\ell$, then Theorem 2.11 implies that

$$\chi_0(A'; t) = \prod_{i=2}^{\ell} (t - d_i).$$

Since $\chi(A^H; t) | \chi(A; t)$, Theorem 2.4 shows that $\chi(A^H; t) | \chi(A'; t)$ too, and thus $|A'| - |A^H| = d_j$ for some $j$. We may assume that $d_j < d_{j+1}$, or $j = \ell$. Let $I := \{2, \ldots, \ell\} \setminus \{j\}$. Then by Theorems 1.1 and 2.11

$$\begin{align*}
\chi_0(A; t) &= (t - d_j - 1) \prod_{i \in I} (t - d_i), \\
\chi_0(A^H; t) &= \prod_{i \in I} (t - d_i).
\end{align*}$$

Let $\theta_1 = \theta_E, \theta_2, \ldots, \theta_\ell$ be a homogeneous basis for $D(A')$ with $\deg \theta_i = d_i$. If $\theta_k \not\in D(A)$ for some $k$ with $d_k = d_j$, then we can show that $A$ is free. Namely, by Proposition 2.17 there is a homogeneous polynomial $B$ of degree $|A'| - |A^H| = d_j$ such that

$$\theta(\alpha_H) \in (\alpha_H, B)$$

for all $\theta \in D(A')$. So $\theta \in D(A')$ is in $D(A)$ if $\deg \theta < d_j$. Now assume that $\theta_k \not\in D(A)$ for some $k$ with $d_k = d_j$. We may assume that $k = j$. Then $\theta_j(\alpha_H) = B$ modulo $\alpha_H$. Let $\theta_i(\alpha_H) = a_i \alpha_H + b_i B$. Then replacing $\theta_i$ by $\theta_i - b_i \theta_j$, we may assume that $\theta_i \in D(A)$ if $i \neq j$. Thus Theorem 2.10 shows that $\theta_1, \theta_2, \ldots, \theta_{j-1}, \alpha_H \theta_j, \theta_{j+1}, \ldots, \theta_\ell$ form a basis for $D(A)$.

Assume not, i.e.,

$$D' := \oplus_{i=2}^j S \theta_i \subset D(A).$$

By replacing $\theta_i$ ($i = 2, \ldots, j$) by $\theta_i - (\theta_i(\alpha_H)/\alpha_H) \theta_E$, we may assume that $\theta_i \in D_H(A)$ for $i = 2, \ldots, j$. By composing the inclusion $D(A) \subset D(A')$ and the canonical projection $D(A') \to D'$, we have a surjection

$$p : D(A) \to D'$$

which has a canonical section by the above. Hence

$$D(A) = S \theta_E \oplus D' \oplus M$$

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for some $S$-submodule $M$ of $D(A)$. By Proposition 2.18
\[ D_H(A) \simeq D(A)/S\theta_E = D' \oplus M. \]

Since $A$ is locally free, Theorem 2.19 implies that
\[ \chi_0(A; t) = t^{\ell - 1}c_{1/t}(\widetilde{D_H(A)}). \]

Since $D_H(A) \simeq D' \oplus M$, it holds that
\[ c_{1/t}(\widetilde{D_H(A)}) = c_{1/t}(\widetilde{D'})c_{1/t}(\widetilde{M}). \]

Noting that $D' = \bigoplus_{i=2}^{j}S[-d_i]$,
\[ t^{\ell - 1}c_{1/t}(\widetilde{D_H(A)}) = t^{\ell - j}c_{1/t}(\widetilde{M}) \prod_{i=2}^{j} (t - d_i). \]

Here
\[ \text{rank}(\widetilde{M}) = \text{rank}(\widetilde{D_H(A)}) - \text{rank}(\widetilde{D'}) = \ell - j. \]

Since $\widetilde{D_H(A)}$ is locally free, so is $\widetilde{M}$. Hence $\deg c_{t}(\widetilde{M}) = \ell - j$. So $c(t) := c_{1/t}(\widetilde{M})t^{\ell - j} \in \mathbb{Z}[t]$. Therefore
\[ \chi_0(A; t) = (t - d_j - 1) \prod_{i=1}^{\ell} (t - d_i) = c(t) \prod_{i=2}^{j} (t - d_i). \]

Assume that the multiplicity of $d_j$ in $\chi_0(A'; t)$ is $a \geq 1$. Then $d_j < d_{j+1}$ or $j = \ell$ implies that $C(t) := c(t) \prod_{i=2}^{j} (t - d_i)$ is divisible by $(t - d_j)^a$, but $\chi_0(A; t)$ is not, a contradiction. □

**Proof of Theorem 1.4** The statement is true if $A' = \emptyset$. Assume not in the following. If $A$ is free, then $A$ is divisional along $H$ by Lemma 3.2. So it suffices to show that $A$ is free if $A'$ is free, and $A$ is divisional along $H$.

We prove by induction on $\text{rank}(A) \geq 0$. We may assume that $\text{rank}(A) = \ell$ by Definition 2.7. The statement is true if $\ell \leq 3$ by 11, or Theorems 2.11 and 2.12. Assume that $\ell \geq 4$. By Theorem 3.4, it suffices to show that $A$ is locally free. Since $A'$ is free, $A_X = (A')_X$ is free too if $X \not\subset H$. So let $X \in L_i(A) \backslash \{0_V\}$ with $X \subset H$. By Lemma 3.1, $A_X \backslash \{H\}$ is free. Let $B_X$ be the essential part of $A_X$ as in Definition 2.7. In other words, $A_X = B_X \times \mathbb{Q}_{\ell-i}$, and $i = \text{rank}(B_X) < \text{rank}(A) = \ell$. Then by Proposition 2.8 $B_X \backslash \{H\}$ is free too. Since $B_X$ is of rank $i < \ell$, divisional along $H$ by Lemma 3.3 and $B_X \backslash \{H\}$ is free, the induction hypothesis on $\ell$ shows that $B_X$ is free. Thus $A$ is locally free, and Theorem 3.4 completes the proof. □
Proof of Theorems 1.3 and 1.5 Immediate by Theorems 1.1, 1.4, 2.11, 2.12, and 2.13.

Finally we pose several conjectures related to Theorem 1.4.

Conjecture 3.5
For $H \in \mathcal{A}$, let $\mathcal{A}' := \mathcal{A} \setminus \{H\}$. Then

1. $\mathcal{A}$ is free if $\mathcal{A}'$ is free, and $\mathcal{A}$ is globally divisional along $H$.

2. Assume that

\[ \chi_0(\mathcal{A}; t) = (t - d_2 - 1) \prod_{i=3}^{\ell} (t - d_i), \]

\[ \chi_0(\mathcal{A}'; t) = (t - d_2) \prod_{i=3}^{\ell} (t - d_i) \]

for some integers $d_2, \ldots, d_\ell$. Then $\mathcal{A}$ and $\mathcal{A}'$ are both free.

Note that Conjecture 3.5 (1) and (2) are true if $\ell \leq 3$ by Theorem 1.4 or Theorem 2.12.

4 Additionally and stair-free arrangements of hyperplanes

Theorem 1.8 is clear by Theorem 1.4. Let us show the relation among inductively, divisionally and additionally free arrangements.

Proposition 4.1
It holds that

\[ \mathcal{AF} \supset \mathcal{IF} \subset \mathcal{DF}. \]

Moreover, $\mathcal{DF} \not\subset \mathcal{AF}$.

Proof. $\mathcal{IF} \subset \mathcal{AF}$ is clear by Definitions 1.6 and 2.14. $\mathcal{IF} \subset \mathcal{DF}$ follows by Theorem 2.16. Moreover, by Lemma 3.13 in [8], we know that $\mathcal{A}(G_{31}) \in \mathcal{DF} \setminus \mathcal{AF}$. Here $G_{31}$ is one of the finite unitary reflection groups classified and labeled by Shephard and Todd in [13]. So $\mathcal{A}(G_{31})$ is the corresponding unitary reflection arrangement in $\mathbb{C}^4$. □

Now we have two class $\mathcal{DF}$ and $\mathcal{AF}$ of free arrangements which are larger than $\mathcal{IF}$. Since their definition is easy to deal with, we can join these two to obtain the new largest class of free arrangements in which Terao’s conjecture is true.
Definition 4.2 (Stair-free arrangements)
The set $\mathcal{SF}_\ell$ consists of hyperplane arrangements $\mathcal{A}$ in $\mathbb{K}^\ell$ which satisfy the following; there are $H_1^i, \ldots, H_{\ell}^i, H_i$ $(i = 3, \ldots, \ell - 1)$ in $L_{\ell-i}(A)$ such that

1. $\mathcal{A}_\ell := A$, $\mathcal{A}_i := (\mathcal{A}_{i+1} \setminus \{H_j^i\}_{j=1}^{i+1})^{H_i}$. So $H_j^i, H_i \in \mathcal{A}_{i+1}$.

2. $\mathcal{A}_i^k := \mathcal{A}_i \setminus \{H_j^{i-1}\}_{j=1}^k$ is divisional along $H_j^{i-1}$ for all $i, k$.

3. $\chi(\mathcal{A}; t)$ divides $\chi(\mathcal{A}_i^{i+1}; t)$.

$\mathcal{A} \in \mathcal{SF}$ is called a **stair-free arrangement of hyperplanes**.

By Theorems 1.4 and 2.12, the following is clear.

Theorem 4.3

1. $\mathcal{A}$ is free if $\mathcal{A} \in \mathcal{SF}$.

2. It holds that $\mathcal{IF} \subset \mathcal{DF} \cup \mathcal{AF} \subset \mathcal{SF}$.

3. Terao’s conjecture is true in the class of stair-free arrangements.

When you apply the division theorem, the arrangement moves vertically, i.e., the direction of dimensions, and the addition theorem horizontally, i.e., the direction of the cardinality of hyperplanes. If you connect these paths, then it looks like a stair. That is the reason of the name in Definition 4.2. A typical example of a stair-free arrangement is the Catalan and Shi arrangements, which will be explained in §5. In [7], it is shown that the freeness of Catalan arrangement is combinatorial. Seeing the proof, [7] shows that Catalan arrangement is stair-free.

Related to these new classes, we have the following conjecture.

Conjecture 4.4

1. $\mathcal{IF} \subset \mathcal{AF}$.

2. $\mathcal{DF} \not\subset \mathcal{AF}$.

3. $\mathcal{DF} \cup \mathcal{AF} \subset \mathcal{SF}$.
5 Applications to root systems

In this section we apply Theorem 1.4 to explicit examples, showing the combinatorial dependency of their freeness.

In this section let $\mathbb{K} = \mathbb{R}$, thus $V = \mathbb{R}^\ell$. Let $W$ an irreducible crystallographic Weyl group of rank $\ell$ acting on $V$. Let $\Phi$ be the corresponding root system to $W$, $\Phi^+$ the set of positive roots, and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ the set of simple roots.

A lot of arrangements related to $W$ are check to be free, and their freeness are often proved to be combinatorial. For example, if $\mathcal{A}_W$ denotes the set of all reflecting hyperplanes of reflections in $W$, then $\mathcal{A}_W \in \mathcal{IF}$, thus its freeness depends only on combinatorics. To review some of them, let us introduce some notation.

For $\alpha \in \Phi^+$, let $H_\alpha$ denote the reflecting hyperplane of $\alpha$, and

\[ H_\alpha^k := \{\alpha = k\} \subset V \]

be an affine hyperplane for $k \in \mathbb{Z}$. Let $cH_\alpha^k$ be the cone $\alpha = kz$ of $H_\alpha^k$ by the new coordinate in $z$, i.e., $cH_\alpha^k$ is a hyperplane in $\mathbb{R}^{\ell+1} = \text{Spec}(S[z])$. Then for $m \in \mathbb{Z}_{>0}$, the Catalan arrangement $\text{Cat}^m$ is defined by

\[ \text{Cat}^m := \{z = 0\} \cup \{cH_\alpha^k\}_{\alpha \in \Phi^+, -m \leq k \leq m}, \]

and the Shi arrangement $\text{Shi}^m$ is defined by

\[ \text{Shi}^m := \{z = 0\} \cup \{cH_\alpha^k\}_{\alpha \in \Phi^+, -m+1 \leq k \leq m}. \]

Both are free by [16], and their freeness depends only on the lattice by [2] for Shi arrangements, and [7] for Catalan arrangements. Also, we say that $I \subset \Phi^+$ is a lower ideal if $\alpha \in I$, $\beta \in \Phi^+$, $\alpha - \beta \in \sum_{i=1}^\ell \mathbb{Z}_{\geq 0} \alpha_i$ implies $\beta \in I$. Then it was shown that $\mathcal{A}_I := \{H_\alpha \mid \alpha \in I\}$ is free in [5]. Moreover, if we define the ideal Shi arrangements

\[
\begin{align*}
\text{Shi}^m_{+I} & := \text{Shi}^m \cup \{cH_\alpha^{-m} \mid \alpha \in I\}, \\
\text{Shi}^m_{-I} & := \text{Shi}^m \setminus \{cH_\alpha^m \mid \alpha \in I\},
\end{align*}
\]

then it is shown in Theorem 1.6 in [6] that they are both free, and the freeness of $\text{Shi}^m_{+I}$ depends only on combinatorics. So the left problems is that of $\text{Shi}^m_{-I}$. Since ideal-Shi arrangements have a free filtration by Theorem 1.5 in [6], which is nothing but the additional filtration, Theorem 1.6 implies the following.

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Theorem 5.1

It holds that

\[ \text{Shi}^m_{+I} \in \mathcal{AF}. \]

Thus, the freeness of \( \text{Shi}^m_{+I} \) depends only on \( L(\text{Shi}^m_I) \) for all lower ideal \( I \subset \Phi^+ \) and \( m \in \mathbb{Z}_{\geq 0} \).

Since \( \text{Shi}^m_{\Phi^+} = \text{Cat}^m \) and \( \text{Shi}^m_{\emptyset} = \text{Shi}^m \), Theorem 5.1 gives a uniform proof of the combinatorial dependency of their freeness.

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