Little-used Mathematical Structures in Quantum Mechanics

I. Galilei Invariance and the welcher Weg Problem

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Abstract
Results of the welcher Weg experiment of Dürr, Nonn and Rempe are explained by using ray representations of the Galilei group. The key idea is that the state of the incoming atom be regarded as belonging to an irreducible unitary ray representation of this group. If this is the case, interaction with an interferometer with a which-way detector must split this state into the direct sum of two states belonging to representations with different internal energies. (While the zero of internal energy is arbitrary, the difference between two internal energies is well-defined and is invariant under unitary transformations.) The state of the outgoing atom will then be a superposition of two mutually orthogonal states, so that there will be no interference. Neither complementarity nor entanglement plays a role in this explanation. Furthermore, in atom interferometry it is not enough for a quantum eraser to erase the internal energy difference; to restore interference, two copies of a representation have to be collapsed into one. In a direct sum of copies of the same representation, copies of the same state will still be orthogonal. These assertions may be testable, and two new atom interferometry experiments are suggested. One of them is an ‘own-goal’ experiment which may decisively refute the explanation offered here, and restore the aura of mystery that this paper tries to dispel.
This is the first of three papers on mathematical structures in quantum mechanics that have been known for four to six decades, but have not been fully exploited by physicists. These structures reveal new possibilities for the interpretation of experiments that have been performed, and suggest new experiments that may shed light on unresolved or disputed problems in the foundations of quantum mechanics. The present paper re-examines the welcher Weg problem from the viewpoint of Galilei invariance. The accompanying paper examines physical implications of the existence of inequivalent (irreducible unitary) representations of the canonical commutation relations for a finite number of degrees of freedom. The last paper will investigate consequences of the existence of dense sets of analytic functions on some concrete Hilbert spaces used in physics. The existence of such sets was first noted by Stone in the late 1920s or early 1930s; Hilbert spaces of analytic functions were explicitly constructed by Bargmann in 1961.

1 The welcher Weg problem

The welcher Weg problem appears to have been suggested by the Feynman Lectures on Physics. In Chapter 1 of Part III of these Lectures [12], Feynman prepares his undergraduate audience for the shock of quantum mechanics by discussing two double-slit gedankenexperiments with electrons. The first experiment is the standard one; an interference pattern is gradually built up as electrons strike the detection screen. In the second experiment, ‘which way’ an electron takes is determined by shining a light beam on it at the slits. The pattern that builds up now no longer shows interference. The loss of interference is explained by the uncertainty principle. Feynman goes on to affirm that “No one has ever found (or even thought of) a way around the uncertainty principle”.

However, in 1991 Scully, Englert and Walther (hereafter SEW) claimed that “…we have found a way around this position-momentum uncertainty obstacle. . . That is, we have found a way. . . to obtain which-path or particle-like information without scattering or otherwise introducing large uncontrolled phase factors into the interfering beams.” They went on to state that, in their gedankenexperiment, “The principle of complementarity is manifest

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1By quantum mechanics we mean von Neumann’s Hilbert space formulation of it. The conceptual subtleties that underlie the accompanying paper cannot be expressed in Dirac’s formulation.
although the position-momentum uncertainty relation plays no role” [21].

The way around that SEW had found consisted of using cold atoms as particles in a one-particle interference experiment. An excited atom (traveling at a low speed) may be induced to emit a photon in a resonant cavity in its path. If the parameters are right, emission of the photon will have a negligible effect on the atom’s linear momentum, and therefore on its de Broglie wavelength; the effect on the interference pattern should be negligible. At the same time, the emitted photon will reveal which path the atom took.

In 1998, Dürr, Nonn and Rempe (hereafter DNR) performed the actual experiment [8] (now regarded as definitive) which, as we shall see, differed from the one suggested by SEW in some essential aspects. Their findings agreed with the expectations of SEW; the mere possibility of detection of the path taken by an atom resulted in the loss of interference, and the effect could not be explained by an appeal to the position-momentum uncertainty relation. The title that DNR gave to their paper was “Origin of quantum-mechanical complementarity probed by a ‘which-way’ experiment in an atom interferometer”.

The chief aim of the present paper is to provide an explanation of the phenomena observed by DNR that is based on the theory of ray representations of the inhomogeneous Galilei group. This theory can describe atoms in different energy levels, but there is an unexpected subtlety. The explanation, which also predicts phenomena that cannot be accounted for by the DNR model, is quite independent of any notion of complementarity. Additionally, two new experiments are suggested; one of them, fittingly called the own-goal experiment, may refute the proposed explanation quite decisively. The other may discriminate between the DNR explanation and the one offered here.

The plan of the work is as follows: In Sec. 2 we review, briefly, the gedankenexperiment of SEW. In Sec. 3 we review the experiment of DNR. This is followed by Sec. 4, the main section of this paper. In it we use the theory of ray representations of the Galilei group to explain the experimental results of DNR without using any notion of complementarity. In Sec. 5 we compare our explanation with that of DNR, which is modelled on a moving two-level atom. The new experiments are suggested in Sec. 6, the last section of the paper. The first, discussed in 6.1, is the own-goal experiment that may...

2There have been quite a number of works on the subject before SEW, between SEW and DNR, and after DNR. We have chosen to focus on DNR and SEW for reasons mentioned above. References to earlier works will be found in SEW and DNR. References to some later works may be found in a 2010 article by Ferrari and Brauneccker [11].
decisively refute the explanation based on Galilei invariance. The second is a variant of the DNR experiment, and includes a quantum eraser which may enable it to distinguish between the Galilei invariance and the DNR explanations. There are two Appendices. In the first, we collect together the key definitions and formulae of the theory of ray representations, and recapitulate the notion of unitary equivalence for these representations. In the second, we provide some references to (i) quantum optics and the manipulation of particles by light, and (ii) complementarity and uncertainty, subjects which form the experimental and theoretical backdrops to this paper.

2 The gedankenexperiment of Scully, Englert and Walther

The experimental scheme of SEW is a double-slit interferometer modified by the placement of a resonant cavity just before each slit, shown schematically in Fig. 1. The two cavities, each tuned to resonate at 21 GHz and prepared in the zero-photon state, together constitute the which-way detector. The two paths are marked 1 and 2 in the figure. Formulae (2.1) and (2.2) are taken from the section on Gedanken experiments illustrating complementarity in [21].

If the resonant cavities are not present, the state vector of the atom, after it emerges from the double slit, will be given by Eq. (4) of [21], namely (we adhere to their notation):

$$\Psi(r) = \frac{1}{\sqrt{2}}[\psi_1(r) + \psi_2(r)]|i\rangle.$$ (2.1)
In (2.1), \( r \) is the coordinate of the centre of mass and \( |i\rangle \) the internal state of the atom. The subscripts 1 and 2 refer to paths 1 and 2 (Fig. 1). When the which-way detector is present, and the atom has emitted a photon in one of the cavities, (2.1) changes to (Eq. (6) in [21])

\[
\Psi(r) = \frac{1}{\sqrt{2}}[\psi_1(r)|1_10_2\rangle + \psi_2(r)|0_11_2\rangle]|b\rangle,
\]

(2.2)

where \( |0_11_2\rangle \) is the state of the which-way detector with no photon in cavity 1 and one photon in cavity 2, and similarly for \( |1_10_2\rangle \). The atom is prepared in the state \( |a\rangle \) and makes the transition \( a \rightarrow b \) in the which-way detector. The authors write that: “Please note that unlike (4) this \( \Psi(r) \) is not a product of two factors, one referring to the atomic and the other to photonic degrees of freedom. The system and the detector have become entangled by their interaction.” The orthogonality condition \( \langle 0_11_2|1_10_2\rangle = 0 \) will now cause the interference term in this \( |\Psi(r)|^2 \) to vanish.

The validity of (2.2) may be disputed, but we shall not enter into this dispute, because (2.2) is not used by DNR to interpret the results of the experiment they actually performed. We now turn to this experiment.

3 The experiment of Dürr, Nonn and Rempe

In the experiment of Dürr, Nonn and Rempe [8], a monochromatic beam \( A \) of \(^{85}\text{Rb}\) atoms is split into two, a transmitted beam \( C \) and a Bragg-refracted beam of the first order \( B \) using a standing light wave. The reflectivity of the ‘light crystal’ is determined by the intensity of the standing wave, and is adjusted to \( \sim 50\% \). Beams \( B \) and \( C \) are again split into two each by an identical light crystal, as shown in Fig. 2. In the absence of a which-way detector, \( D \) and \( E \) are observed to interfere, as are \( F \) and \( G \). (Throughout this section, we shall adhere strictly to the notation of [8]. \text{Acknowledgement:} \text{Fig. 2 is a simplified form of Fig. 1 of [8]; Fig. 3 is a slightly modified form of Fig. 3 in the same source.})

The ground state \( 5^2S_{1/2} \) of \(^{85}\text{Rb}\) is split into two hyperfine states with total angular momenta \( F = 2 \) and \( F = 3 \); denote these two states by \( |2\rangle \)

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3The idea of reflecting electrons through a light crystal, which became feasible only with intensities available with lasers, was thought of by Kapitza and Dirac in 1933 [17]. The reflection and refraction of matter waves by standing light waves has become known as the Kapitza-Dirac effect [7].
A microwave field at $\omega_{mw} \sim 3$ GHz will induce Rabi oscillations between the states $|2\rangle$ and $|3\rangle$. Let $|e\rangle$ denote the $5^2P_{3/2}$ excited state of $^{85}\text{Rb}$. A simplified level scheme of $^{85}\text{Rb}$ is shown in Fig. 3(a). The incident beam A is prepared in the state $|2\rangle$.

The frequency $\omega_{\text{light}}$ of the light crystal is tuned halfway between the two transitions $|2\rangle \rightarrow |e\rangle$ and $|3\rangle \rightarrow |e\rangle$, as shown in Fig. 3. It is detuned from the two transitions by amounts $\Delta_{2e}$ and $\Delta_{3e}$ that have the same absolute values but opposite signs: $\Delta_{3e} > 0, \Delta_{2e} = -\Delta_{3e} < 0$. Therefore, as in classical optics, Bragg refraction from the light crystal will shift the phase of a $|2\rangle$-beam by $\pi$, but will leave the phase of a $|3\rangle$-beam unchanged. This phase difference is used to induce a population difference between the refracted and transmitted beams.

Which-way information is stored in the beams themselves, as follows (see Fig. 3(b)). A $\pi/2$-microwave pulse at $\omega_{mw}$ is applied to the incident beam (prepared in the state $|2\rangle$) before it reaches the first light crystal. This pulse changes the incident beam A to the superposition $(|3\rangle + |2\rangle)/\sqrt{2}$. The first light crystal splits this beam and, additionally, changes the phase of the $|2\rangle$ component in the Bragg-refracted beam B by $\pi$. The transmitted beam is unaffected. A second $\pi/2$-microwave pulse now changes the population of B to $-|2\rangle$, and that of C to $|3\rangle$.

After the second microwave pulse has acted, the state vector of the beam

Figure 2: The DNR experiment without which-way detection
System is, up to normalization
\[ |\psi\rangle = |\psi_B\rangle \otimes |2\rangle + |\psi_C\rangle \otimes |3\rangle. \] (3.1)

In the above, \(|\psi_{B,C}\rangle\) describe only the state of the centre of mass. The authors write that:

Equation (3.1) shows that the internal state is correlated with the way taken by the atom. The which-way information can be read out later by performing a measurement of the internal atomic state. The result of this measurement reveals which way the atom took: if the internal state is found to be \(|2\rangle\), the atom moved along B, otherwise along C.

After the second microwave pulse, the components B and C are incident upon a second light crystal (Fig. 2). This crystal splits each component, but does not affect their populations. Thus D and F are in state \(|2\rangle\) (the phase is no longer critical), and F and G are in state \(|3\rangle\). The state vector is now proportional to
\[ |\psi\rangle = -|\psi_D\rangle \otimes |2\rangle + |\psi_E\rangle \otimes |3\rangle + |\psi_F\rangle \otimes |2\rangle + |\psi_G\rangle \otimes |3\rangle. \] (3.2)

One sees from Fig. 2 that, in the interference region, D and E overlap in space, as do F and G, but the first pair has negligible overlap with the second pair.

However, (3.2) predicts no interference in either pair, because \(\langle 2|3 \rangle = 0\); and that is exactly what is observed. We draw the reader’s attention to a
crucial difference between the SEW equation (2.2) and the DNR equation (3.2); the latter involves no quantity that is alien to the atom.

4 Galilei invariance and quantum mechanics

At the root of our endeavour is the assumption that nonrelativistic quantum mechanics has an invariance group, which is the inhomogeneous Galilei group. In this section we shall spell out how this assumption leads to an explanation of the phenomena observed by DNR.

4.1 Particles of nonzero mass

It was established by Inönu and Wigner in 1952 that true unitary irreducible representations of the Galilei group $G$ do not have a particle interpretation [15]. The fact that free particles in nonrelativistic quantum mechanics could be described by unitary ray representations of $G$ was established by Bargmann in 1954 [1]. There is a one-parameter family of such representations, the parameter being the mass. Equivalently, one could say that the group $G$ has a one-parameter family of central extensions $\tilde{G}_m$, and a particle of mass $m \neq 0$ corresponds to a true representation of $\tilde{G}_m$. The parameter $m$ may be made explicit in the group exponent:

$$\omega(g_1, g_2) = i \exp m \gamma(g_1, g_2).$$

The group $G$ is a ten-parameter group. Its generators are $H$ (time translations), $P$ (space translations), $J$ (rotations) and $K$ (boosts). The Casimir operators of this group are

$$P^2 \text{ and } (K \times P)^2.$$  

The group $\tilde{G}_m$ is an eleven-parameter group; it has an extra generator, $I$, which commutes with every other generator and is represented by the identity matrix in any irreducible unitary representation. The commutator $[P_i, K_j] = 0$ in $G$ is replaced by $[P_i, K_j] = i\delta_{ij}mI$ in $\tilde{G}_m$, all other commutators remaining the same. As a result, the Casimir operators of $\tilde{G}_m$ differ
drastically from those of $G$. They are, in addition to $I$,
\begin{equation}
U = H - \frac{1}{2m}P^2, \quad (4.3)
\end{equation}
\begin{equation}
S^2 = \left( J - \frac{1}{m}K \times P \right)^2. \quad (4.4)
\end{equation}

The spectrum of $S^2$ is discrete; its eigenvalues are $s(s + 1)$, where $s$ is a nonnegative integer (or half-odd integer, for the covering group). That of $U$ is continuous, and fills the real line. By analogy with thermodynamics, the spectral value $u$ of $U$ in an irreducible representation is called the internal energy of the particle. An irreducible unitary representation of $\tilde{G}_m$ is characterized by the pair $(u, s)$.

### 4.2 Internal energy and equivalence of representations

In the following, we shall consider a fixed nonzero $m$, and shall omit the subscript $m$ of $\tilde{G}_m$. We shall denote true representations of $\tilde{G}$ by $\tilde{D}$, and ray representations of $G$ by $D$ instead of $(D, \omega)$; the factor system will be displayed through the mass $m$.

The standard parametrization of a group element $g$ of $G$ is
\[ g = (b, a, v, R), \]
where $b$ is a time translation, $a$ a space translation, $v$ a pure Galilei transformation or boost and $R$ a rotation. With this parametrization, we may write an element $\tilde{g}$ of $\tilde{G}$ as
\[ \tilde{g} = (\theta, b, a, v, R), \]
where $\theta \in \mathbb{R}$. For simplicity\footnote{Nonzero spins present no difficulties; only the formulae become longer (see [19]).} we shall restrict ourselves to representations $\tilde{D}_{(u,s)}$ of $\tilde{G}$ with $s = 0$. Let $\mathfrak{H} = L^2(\mathbb{R}^3, dp)$ and $\psi \in \mathfrak{H}$. The representation $\tilde{D}_{(u,0)}$, which we shall write as $\tilde{D}_u$, is defined by
\begin{equation}
\tilde{D}_u(\theta, b, a, v, R)\psi(p) = e^{i[\theta+Eb+pa]}\psi(R^{-1}(p-mv)), \quad (4.5)
\end{equation}
where $E$, the total energy, is a spectral value of $H$. Using (4.3), the representation (4.5) may be written as
\begin{equation}
\tilde{D}_u(\theta, b, a, v, R)\psi(p) = e^{iub}e^{i[\theta+(p^2/2m)b+pa]}\psi(R^{-1}(p-mv)),
\end{equation}
which may be written more compactly as
\[ \tilde{D}_u(\theta, g)\psi(p) = e^{iub} \tilde{D}_0(\theta, g)\psi(p). \] (4.6)

For any \( u \), the operators \( \{\tilde{D}(0, g)\} \) constitute a ray representation \( \{D_u(g)\} \) of \( G \). Setting \( \theta = 0 \) in (4.6), we find, by a slight change of notation, that
\[ D_{w'}(g) = e^{iub} D_{w'-u}(g), \] (4.7)
where \( w' \) is arbitrary. Since the phase factor on the right depends only on \( g \), (4.7) establishes that \( D_{w'} \) and \( D_{w'-u} \) are equivalent. This mathematical equivalence represents the physical fact that the zero of energy is arbitrary.

Consider now the direct sum \( D_u \oplus D_{w'} \). Using (4.7), we arrive at the formula
\[ D_u(g) \oplus D_{w'}(g) = e^{iub} [D_0(g) \oplus D_{w'-u}(g)]. \] (4.8)
This shows that although the representations \( D_u \) and \( D_{w'} \) are equivalent for all \( u, u' \), the representation \( D_u(g) \oplus D_{w'}(g) \) is equivalent to \( D_0(g) \oplus D_{w'-u}(g) \), and not to \( D_0(g) \oplus D_0(g) \) for \( u' \neq u \); although the zero of energy can be chosen arbitrarily, the same zero has to be chosen for all energies, so that energy differences remain physically meaningful.

Let \( D_u, D_{u'} \) be irreducible on \( \mathcal{H}_1, \mathcal{H}_2 \) respectively, \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) and \( \Pi_1, \Pi_2 \) the projections \( \Pi_1 : \mathcal{H} \to \mathcal{H}_1, \Pi_2 : \mathcal{H} \to \mathcal{H}_2 \). For \( \psi \in \mathcal{H} \), set
\[ \psi_1 = \Pi_1\psi, \quad \psi_2 = \Pi_2\psi. \] (4.9)
Clearly,
\[ (\psi_1, \psi_2) = 0. \] (4.10)
This orthogonality is due to the fact that the states \( \psi_1 \) and \( \psi_2 \) belong to different subrepresentations of the direct sum; it would continue to hold even when \( u = u' \).

### 4.3 Loss of interference in the DNR experiment

The electronic configuration of an atom has a countable number of discrete electronic energy levels, of which we shall only be concerned with two or three at a time. All of these have the same mass, say \( m \), in nonrelativistic quantum

\[ ^5 \text{This fact, which has no parallel in the theory of true representations, seems to have been first noticed by Lévy-Leblond in 1963 [18].} \]
mechanics. The spin of the atom as a whole will play no explicit role in our considerations, and therefore we shall set $s = 0$ to reduce notational clutter. We now introduce Galilei invariance into our considerations via the following crucial assumption:

**Assumption 1** The states of an atom in a given energy level belong to an irreducible ray representation $D_u$, with mass $m$, of the Galilei group $G$. (The zero of $u$ is arbitrary.)

In a single-particle interference experiment with a two-arm interferometer, the geometry of the apparatus forces the incoming wave function to split into the sum of two components. (This is to be regarded as an empirical fact.) Suppose now that one of these components is induced to make a transition to a different energy level by interaction with a which-way detector. Then the wave function of the atom should become a superposition of two different components with different energies, but essentially the same momentum, and therefore the same kinetic energy. Most of the energy difference has therefore to be attributed to the internal energy. However, as the internal energy is constant in an irreducible representation, this implies that interaction with the interferometer with which-way detector splits the representation of the incoming wave into a direct sum of two different irreducibles, which differ in their internal energies by the energy difference of the two atomic levels. We shall condense this into a proposition:

**Proposition 1** Let the incoming particle be in a state $\psi_{\text{in}} \in D_u$. If a which-way detector is present in one of the arms of the interferometer, then the outgoing state of the particle, $\psi_{\text{out}}$, will belong to the representation $D_u \oplus D_{u'}, u \neq u'$. In the notation of (4.9), $\psi_{\text{out}}$ will be given by

$$\psi_{\text{out}} = \psi_1 + \psi_2,$$

(4.11) and the orthogonality relation (4.10) will hold.

In the DNR experiment (as in the SEW gedankenexperiment) the which-way detector changes the energy state of the atom, and has little effect on its wavelength. If assumption [1] is valid, then proposition [1] will be valid to a very good approximation.

Equation (4.10) will explain the loss of interference. In Feynman’s gedankenexperiment, loss of interference was a statistical phenomenon due to the uncertainty principle; by contrast, loss of interference due to the orthogonality (4.10) may be called a *dynamical* phenomenon.
5 Comparison of DNR and Galilei invariance explanations

The fundamental difference between the explanations offered by SEW, on the one hand, and DNR and Galilei invariance, on the other hand is the following. In the former, there is no interference because the two states of the which-way detector – which is physically distinct from the atom – are orthogonal; in the latter, there is no interference because the two states of the atom are orthogonal. At first sight it might appear that the DNR explanation is just the Galilei invariance explanation pared down to the essentials: a moving two-level atom. The identifications $\psi_u = \psi_B \otimes |2\rangle$ and $\psi'_u = \psi_C \otimes |3\rangle$ turn (3.1) and (4.11) into each other.

However, the explanations offered by Galilei invariance and by DNR are not equivalent. This inequivalence may be discussed using the notion of quantum erasure in atom interferometry. The term quantum erasure refers to the deletion of which-way information from an atom. If an atom provides which-way information by jumping from state $j$ to state $k$, this information may be ‘erased’ by forcing it to jump back from state $k$ to state $j$. If, in the DNR scheme, atoms in state $|3\rangle$ are forced to jump back into the state $|2\rangle$, the expression (3.1) for $|\psi\rangle$ will become

$$|\psi\rangle = |\psi_B\rangle \otimes |2\rangle + |\psi_C\rangle \otimes |2\rangle,$$

i.e., interference will be restored. In the Galilei invariance explanation, however, replacing $\psi'_u$ by $\psi_u$ in $\psi_{\text{out}}$ given by (4.11) will not change the orthogonality condition ($\psi_1, \psi_2$) = 0. To restore interference, a quantum eraser has to collapse the representation $D_u \oplus D_{u'}$ to a single irreducible $D$.

Whether or not this happens is testable in the laboratory, and is discussed in Sec. 6.2.

6 Experimental tests

The two experiments suggested below may discriminate between the various explanations of the results of which-way experiments in atom interferometry.

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6The notion of quantum erasure was introduced by Scully and Drühl in 1982 [20]. Walborn et al carried out an experiment with photons in 2002 [23]. Quantum erasure in atom interferometry was discussed by SEW. Our remarks on quantum erasure apply only to atom interferometry; Galilei invariance can say little that is useful about photons.
6.1 Own-goal experiment

The setup is the same as that shown in Fig. 1 with two important differences:

1. There is only one cavity; cavity 2 is not present.

2. Cavity 1 is prepared in a coherent state, so that an atom entering it will decay with probability 1.

Condition 2 above means that which-way information cannot be obtained in this experiment; the addition of extra photons should make no detectable difference to a cavity prepared in a coherent state. If interference is not observed under this arrangement, it would support the Galilei invariance argument, and throw doubt on the SEW version of complementarity. If, however, interference is observed, then it would decisively refute the Galilei invariance argument.

6.2 Modified DNR experiment

Suppose that the own-goal experiment has been performed, and that its result supports the Galilei invariance explanation. Then the following experiment would be of considerable interest.

Consider the beams D, E, F and G of Fig. 2. After the two $\pi/2$-microwave pulses have acted as described (Sec. 3), D, F will be in state $|2\rangle$, and E, G in state $|3\rangle$.

Let now a $\pi$-microwave pulse be applied to E. It will change the state of E to $|2\rangle$. The experiment will consist of looking for interference in the pair D, E. Then:

1. If interference is not observed, it would be additional support for the Galilei invariance argument (as opposed to the two-level atom argument): the direct sum of two copies of the same representation has not collapsed into a single irreducible representation.

2. If interference is observed, it would mean either that (a) the Galilei invariance argument is valid, but the two component representations have collapsed into one; or that (b) the Galilei invariance argument is invalid.

\textsuperscript{7} The experiment could also be carried out with the beams G, F. It does not matter whether the state of E is changed to $|2\rangle$, or that of D to $|3\rangle$. 
In the opinion of the present author, observation of interference in this experiment will have significant theoretical implications, but it would be premature to speculate.

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**Appendix A: Ray representations**

The following collection of definitions and formulae is tailored to our needs. More detailed summaries may be found in [18], [19] and [22]. A complete mathematical account will be found in Bargmann’s original paper [1].

Following Wigner and Bargmann, we shall denote operator rays by bold-face symbols: an operator ray is a collection \( A = \{ e^{i\alpha} A | \alpha \in \mathbb{R}, \text{A fixed} \} \), where \( A \) is any operator on \( \mathfrak{h} \). An operator \( B \in A \) is a representative of the ray. Operator rays can be multiplied, and a family of operator rays \( \{ D(g) | g \in G \} \) is said to form a ray representation of the group \( G \) if

\[
D(g_1)D(g_2) = D(g_1g_2) \quad \forall \ g_1, g_2 \in G. \tag{A.1}
\]

As rays cannot be added, it would be more convenient to work with operators. Bargmann showed how this could be done without sacrificing generality.

If \( D(g), D(g') \) are representatives of \( D(g), D(g') \) respectively, then one must have

\[
D(g_1)D(g_2) = \omega(g_1, g_2)D(g_1g_2) \tag{A.2}
\]

where \( \omega(g_1, g_2) \) is a complex number of modulus unity. It has to satisfy the condition

\[
\omega(g_1, g_2g_3)\omega(g_2, g_3) = \omega(g_1, g_2)\omega(g_1g_2, g_3) \tag{A.3}
\]

which follows from the associativity of multiplication in \( G \), and the condition

\[
\omega(e, e) = 1 \tag{A.4}
\]

which follows from \( D(e) = I \). A continuous complex-valued function of modulus unity on \( G \times G \) that satisfies (A.3) and (A.4) is called a factor system of \( G \). Equations (A.2), (A.3) and (A.4) are the defining relations of ray representations in terms of operators.
Let \( \varphi \) be a continuous complex-valued function of modulus unity on \( G \). If \( D(g) \) is a representative of the ray \( D(g) \), then so is

\[
D'(g) = \varphi(g) D(g). \tag{A.5}
\]

If the \( D(g) \) satisfy (A.2), then the \( D'(g) \) satisfy

\[
D'(g_1)D'(g_2) = \omega'(g_1, g_2) D'(g_1 g_2) \tag{A.6}
\]

where

\[
\omega'(g_1, g_2) = \frac{\varphi(g_1) \varphi(g_2)}{\varphi(g_1 g_2)} \omega(g_1, g_2). \tag{A.7}
\]

Both (A.2) and (A.6) describe operator multiplication in the same ray representation (A.1) of \( G \). We therefore define two factor systems \( \omega(g_1, g_2) \) and \( \omega'(g_1, g_2) \) to be equivalent if there exists a continuous complex-valued function \( \varphi \) of modulus unity on \( G \) such that (A.7) is satisfied. It is easily verified that this is a true equivalence relation. It partitions the set of factor systems of \( G \) into equivalence classes. The main problem of the theory of ray representations is to determine the set of equivalence classes of factor systems of \( G \). A factor system \( \omega \) may be written as

\[
\omega(g, g') = \exp i \xi(g, g'), \tag{A.8}
\]

where \( \xi \) is a real-valued function on \( G \times G \), called a group exponent.

A ray representation may be specified by a set of operator representatives and an equivalence class of factor systems. We shall write the pair as \( (D, \omega) \).

The factor system \( \omega \) is said to be equivalent to unity if there exists a \( \varphi \) such that \( \omega' = 1 \). In this case the ray representation is equivalent to a true representation.

### 6.3 Unitary equivalence of ray representations

Let \( G \) be a group that admits a nontrivial factor system, and let \( (D, \omega) \) and \( (D', \omega') \) be two ray representations of \( G \):

\[
D(g_1)D(g_2) = \omega(g_1, g_2) D(g_1 g_2) \tag{A.9}
\]

\[
D'(g_1)D'(g_2) = \omega'(g_1, g_2) D'(g_1 g_2) \tag{A.10}
\]
The factor systems $\omega, \omega'$, and therefore the representations $(D, \omega)$, $(D', \omega')$, are equivalent if there exists a continuous function $\varphi(g)$ on $G$ such that (A.7) is satisfied.

Next, let $U$ be a unitary operator and $\zeta(g)$ a continuous real-valued function with $\zeta(e) = 0$ such that

$$UD(g)U^{-1} = e^{i\zeta(g)}D'(g) \quad \forall \ g \in G.$$  \hfill (A.11)

Multiplying (A.9) by $U$ from the left and $U^{-1}$ on the right, using (A.11), setting $\varphi(g) = \exp i\zeta(g)$ and using (A.7), we recover (A.10). This shows that the unitary transformation $U$ transforms the representation $(D, \omega)$ into the representation $(D', \omega')$. The two ray representations are equivalent; their operator representatives can be transformed into each other by a unitary transformation, but the representative of a transformed operator ray has to be chosen in accordance with (A.11).

The equivalence (A.11) has been called projective equivalence by Lévy-Leblond [18].

Appendix B: Further references

For a recent introduction to quantum optics, see the textbook by Fox [13]. The monograph by Haroche and Raimond [14] gives more detailed accounts of the topics they cover.

The Nobel lectures of Chu [5] and Cohen-Tannoudji [6] provide very readable introductions to the manipulation of atoms by light.

Jaeger, Shimony and Vaidman [16], and independently, Englert [10] have obtained a duality relation between path determination and fringe visibility in interferometry without using any Robertson-Heisenberg inequality derived from a pair of noncommuting self-adjoint operators. It was suggested, on this basis, that “the duality relation is logically independent of the uncertainty relation” [10]. This suggestion was controverted by Dürr and Rempe, who derived the duality relation from the Robertson-Heisenberg uncertainty relation for a suitably-chosen pair of self-adjoint operators [9]. It should, however, be recalled that spectral theorems, going back to Hilbert and von Neumann, ensure that for any subset $S$ of real numbers, there exist an infinity of self-adjoint operators that have $S$ as their common spectrum, so that the construction of Dürr and Rempe may be highly nonunique. An introduction
to spectral theory – enough to justify the above statement – may be found in Appendix A6 of [22].

On the other hand, conditions other than the spectrum may inhibit certain quantities, such as time in quantum mechanics and the phase of a quantized electromagnetic field, from being described by self-adjoint operators. Recent reviews of the time-energy and the number-phase uncertainty relations may be found in Busch [3] and Busch et al [4] respectively.

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