We discuss the issues concerning different lattice regularization techniques for the matter Hamiltonian in the framework of loop quantum gravity. Our analysis is implemented in the simplified cosmological formalism and neglecting internal degrees of freedom, the results, however, are general. We found that polymer point holonomy representations are disfavored if constructing canonical quantum general relativity. To omit the related problems with general covariance in violation of the equivalence principle at the level of the quantum generally-relativistic corrections, all the fundamental fields should be regularized using holonomy-flux representations. Our results and the proposed solution shed new light on the Thiemann’s procedure of lattice regularization.

I. INTRODUCTION

I.1. Preface

Quantum gravity represents the idea of constructing the fusion of the principles of general relativity (GR) with a possible quantum nature of the gravitational field, or, from a different perspective, a possible quantum characteristic of the geometry of spacetime. The quantum theory, which construction captures the restrictions of the strong equivalence principle (EP) is loop quantum gravity (LQG) [1] (a detailed introduction to this model, with several extensions and applications, is presented in [2]). In this theory, the gravitational field is described in terms of Ashtekar variables [3], taking the form analogous to the SU(2) Yang-Mills field. It is regularized on a lattice and quantized in the way, which previously has been successfully applied to quantum chromodynamics (QCD) [4].

Thinking about physical, i.e. possibly-detectable predictions, we shall focus not only on quantum gravity but rather on quantum GR, being a postulated theory linking GR and quantum field theories (QFT) of matter interactions. By the fusion of GR with QFT, we understand a fundamental theory, which at the same time connects both formalisms, and imposes the restrictions of GR on QFT and vice versa. Therefore, to begin any meaningful discussion about the phenomenology of canonical quantum general relativity (CQGR) we need a consistent generally-relativistic theory of quantum matter coupled to LQG.

This article aims to verify theoretically if any postulated formulation of CQGR fulfills these requirements. As we will see, the answer to this question is negative. The associated problems, however, disqualify neither the methodology nor the main constructions of the only completely formulated candidate for CQGR, defined in detail in [2]. We hope that the inaccuracies between this description, and our understanding of the proper formulation of CQGR, will help to elaborate improvements, which solve all these problems. As we will see, to present our arguments, we need to investigate the generally-relativistic aspects of the matter sector of the theory. The troublesome issues are already visible when checking the quantum generally-relativistic corrections corrections to bosonic fields in a simple cosmological framework, even when neglecting all the internal degrees of freedom. Moreover, this setting is enough to allow us to formulate general hints for future improvements. Therefore, the core of our analysis is presented in the formalism associated with the most popular cosmological toy model based on loop quantization, called loop quantum cosmology (LQC) [5, 6].

The method of the matter coupling to canonical LQG has been first introduced in [7], as an extension to the program of quantization of the gravitational field [1]. Only a few different methods have been presented so far, usually being only small modifications or enhancements of the original idea — see [8] for an early review. Let us also mention at this point the probably most popular loop-related model including matter degrees of freedom, which is based on a significantly different method. In this model [9], usually called the dressed metric approach, techniques of QFT on curved spacetime are applied to quantize a scalar field on the semiclassical geometry obtained from a loop-quantized Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Strictly speaking, however, this cannot be considered as an attempt to adjust restrictions and methods of CQGR to cosmology, being contradictory to several principles of GR and QFT. For instance, constructing a Fock space on the expectation value of the Hamiltonian operator introduces
a kind of ‘a double quantization’, while neglecting quantum generally-relativistic corrections in the matter sector, restricts GR only to gravitation.

A model introduced as a cosmological reduction of CQGR is called quantum reduced loop gravity (QRLG) \[^{10}\]. It would be selected entirely as a framework for our analysis, if it was not demonstrated in \[^{11}\] that this approach is incorrectly defined. Nonetheless, the analysis related to the bosonic matter in \[^{12}\] and \[^{13}\] are general, directly based on the procedure in \[^{2}\] \[^{7}\], and the associated results are not connected with the constructional defects of QRLG. Investigating these results, one can recognize the structure of quantum corrections coming from the gravitational degrees of freedom coupled to matter. This structure turns out to be the issue revealing the problems with general covariance of quantum corrections in CQGR. As we will see, the method proposed to couple the scalar field is the example violating general covariance, while the one proposed to couple the vector field appears to be the example of a correct approach.

I.2. Structure and conventions

Our article is organized as follows. In Sec. \[^{1}\] we introduce the formalism for lattice field theories. We define the lattice regularization for all fundamental bosonic fields. In Sec. \[^{11}\] quantization of the cosmological framework is introduced. We select the maximally simplified formulation of the loop-based theory that captures the essential issues of our analysis. These are the principles of GR and methods of QFT, which at the semiclassical level verify the consistency of the laws of CQGR. This verification is realized in Sec. \[^{IV}\]. We demonstrate which aspects of this postulated theory are violated in the only model that attempts to be a complete formulation of CQGR. We find that general covariance of several quantum corrections is not completely realized, while other corrections appear to be entirely correct. This problem turns out to concern only the methods that can be freely modified not changing any significant constructions of the analyzed framework, while the irreproachable structures of this framework suggest how to improve the problematic issues. Finally, we propose a procedure possibly resolving the quantum general covariance confusion and in Sec. \[^{V}\] we conclude our results.

In this article, we consider the standard framework of canonical LQG. We introduce \(3 + 1\) decomposition of a manifold \(M\) representing spacetime foliated into Cauchy hypersurfaces \(\Sigma_i\) \[^{14}\] \[^{15}\]. We use the tetrad formalism to introduce the internal flat space with an SU(2) symmetry, orthogonal to an SO(1, 3) Lorentz symmetry group of spacetime. We also assume the simplified structure of vierbeins introducing the time gauge. We use the convention, in which the gravitational coupling constant is defined as \(\kappa = 16\pi G\), while \(g_{\mu}^2\) and \(g_{\nu}^2\) are the scalar and vector fields’ coupling constants, respectively. Notice that we normalized the speed of light to \(c = 1\). For convenience, we also define the following symbol, \(k := \frac{1}{2}\gamma\hbar\kappa = 8\pi\gamma l_p^2\), being a fundamental constant for the canonical DeWitt quantization \[^{16}\] in LQG, where \(\gamma\) and \(l_p\) are the Immirzi parameter and the Planck length, respectively. In several expressions, repeated indices written in \((\ )\) brackets are not summed. All the indices not inserted into the brackets are summed according to the Einstein summation convention.

II. REGULARIZATION

II.1. Lattice gauge theories

The Standard Model of particle physics is based on the Yang-Mills theory \[^{17}\] with a gauge group \(G_{\text{SM}} := U(1) \times SU(2) \times SU(3)\). The U(1) sector acts on bosons associated with the weak hypercharge symmetry as well as through (after the symmetry breaking) the electromagnetic interaction. The SU(2) group describes the weak isospin gauge. Finally, the SU(3) model of gluon interactions describes the color symmetry in QCD. The gauge scalar field transforms under the electroweak \(U(1) \times SU(2)\) group. Notice that after the symmetry breaking, one still needs a theory describing the scalar field without an associated internal symmetry (for instance to describe the scalar potential in pure electromagnetism).

Besides the bosonic interactions represented by the gauge fields, the main role in the Standard Model play fermions. They transform under the whole triplet of the \(G_{\text{SM}}\) gauge group. However, fermions, by definition, transform under the Lorentz transformations like spinors, so they interact with spacetime differently than bosons. Hence, since in GR we treat spacetime not as a static background — like in QFT, but as a dynamical field, the presence of fermions requires a significantly different description. It is well known that the fermionic field is not coupled only to the metric tensor, but also to the torsion tensor. Moreover, fermions prevent vanishing of the torsion field even at microscopic scales \[^{18}\], therefore any complete theory of the matter-spacetime interactions has to describe both metric and torsion coupling to the Standard Model.
In this article, for simplicity, we focus only on the analysis of the bosonic fields. It allows us to construct a generally covariant and metric independent quantum representation of matter fields coupled to a torsion-free and non-perturbative quantum representation of spacetime — LQG. It is worth noting that although some models of fermions coupled to a quantum gravitational field have been already analyzed \cite{19, 20}, this issue still remains one of the least investigated areas of LQG.

Considering all the gauge fields of the bosonic sector of the Standard Model coupled to LQG, it is natural to discuss only the U(1) and SU(2) symmetries in the defining representation, corresponding to the electroweak interactions. The gravitational field, written in terms of Ashtekar variables \cite{14}, transforms in the SU(2) representation. Considering an additional free gluon dynamics in the absence of quarks could be interpreted as an art for art’s sake.

However, there are interesting models of SU(3) electroweak-like lattice theories, considering both the defining and adjoint representations \cite{21, 22}. Since there are no significant difficulties that would prevent us from the extension of this study to the $\mathfrak{su}(3)$ transformations in both representations, we can consider the model invariant under either a defining or adjoint representation of any group \{U(1), SU(2), SU(3)\} $\supseteq G \supseteq G_{SM}$. Considering such theory makes our argumentation valid also for a future, more complete analysis of gluon-quark interactions coupled to LQG. Anyhow, in the context of our article, this generalization is only formal, since our conclusions about quantum generally-relativistic corrections to semiclassical dynamics of any vector field are independent of the internal structure.

We define the Yang-Mills derivative $\partial$, covariant with respect to the $\mathfrak{su}(N)$ algebra representation for $N = 2, 3$, as follows,

$$\partial X^I := dX^I + f_{IJK} A^J X^K.$$  

Here, $X := X^I \tau^I$ is some $\mathfrak{su}(N)$-valued tensor, $A := A^I \tau^I$ denotes a vector potential $\tau^I$ and $f^{IJK}$ are an algebra generator and a structure constant, respectively, while $I, J, ...$ label directions in the internal space — see Appendix \ref{sec:app} for details. The abelian limit of this theory is well known, hence in this subsection we omit its explicit formulation.

Let $l : [0, L] \to \Sigma_t$ be a smooth path parameterized by $s \in [0, L]$ and located inside the constant time surface $\Sigma_t$ constructed by the ADM decomposition \cite{14, 15}. Let us define an embedding of $l(s)$ in $\Sigma_t$ and introduce a dimensionless parameter $\nu \in (0, 1)$ such that $l_\nu(s) := l(\nu s)$ and $L \to L_\nu := L \nu$. The parallel transport equation for a vector $u(s)$ along $l_\nu(s)$ reads

$$\partial_{l_\nu} u(s) = \frac{d}{ds} u(s) + A(l_\nu(s)) u(s) = 0.$$  \hspace{1cm} (2)

It has the following solution, $u(s) = (h_{l_\nu(s)})^{-1}(0)^I$ where

$$h_{l_\nu} := \mathcal{P} \exp \left( \int_0^{L_\nu} ds A(l_\nu(s)) \right).$$  \hspace{1cm} (3)

is the SU(N) holonomy.

**II.2. Yang-Mills theory**

Let us begin introducing a physical picture of the Yang-Mills theory, based on the objects defined in the previous subsection. We decompose $\mathfrak{su}(N)$-valued tensor fields in a four-dimensional coordinate basis of the manifold $M$. Next, we define the action for the vector field $A_\nu$,

$$S(\Delta) := -\frac{1}{4 g_4^2} \int_M d^4x \sqrt{-\gamma} g^{\mu\nu} \gamma^{\xi\pi} F^I_{\mu\xi} F^I_{\nu\pi},$$  \hspace{1cm} (4)

where $g_{\mu\nu}$, $g^{\mu\nu}$ and $g$ are the metric tensor, its inverse, and determinant, respectively. The coupling constant is denoted as $g_4^2$, while $F^I_{\mu\nu}$ is the curvature of $A^I_\mu$,

$$F^I_{\mu\nu} := \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + f_{IJK} A^J_\mu A^K_\nu.$$  \hspace{1cm} (5)

\(^1\) We use the standard convention in physics, interchangeably calling $A_\mu$ a vector potential or a connection \cite{17}.

\(^{ii}\) Notice that $(h_{l_\nu})^{-1} = h_{l_\nu^{-1}}$ is the inverse holonomy. We use the standard notation for LQG \cite{2} without the minus sign in the exponent in expression (3), i.e. defining the holonomy as a solution of the parallel transport equation for a one form, which is analogous, but not identical to (2).
Introducing the 3+1 splitting of spacetime, the Legendre transform of \([4]\) leads to the fully constrained system for the canonical pair of variables, \(A^I_a\) and
\[
E^a_I = \frac{\sqrt{q}}{g^2} \epsilon^a_0 q^{ab} F^I_{ab}. 
\]
(6)

Notice that we wrote this expression in the form after the ADM splitting, with \(q\) denoting determinant of \(q_{ab}\) metric on \(\Sigma\), and \(\epsilon^a_0 = (1/N, -N^a/N)\) being the upper row of the vierbein matrix \(\epsilon^a_I\) (here ‘\(I\)’ represents directions in the Minkowski space) in the tetrad formalism\(^1\). The total Hamiltonian is expressed as the sum of three groups of first class constraints,
\[
H_{I}^{(\Delta)} = G^{(\Delta)} + V^{(\Delta)} + H^{(\Delta)},
\]
(7)
each one generating a different type of invariance. The Gauss constraint,
\[
G^{(\Delta)} := \int_{\Sigma_t} d^3x A^I_{t} \epsilon^I_a = -\int_{\Sigma_t} d^3x A^I_{a} \Sigma_a E^a_I,
\]
(8)
is a generator of \(\mathfrak{su}(N)\) symmetry. Notice that we introduced here the covariant derivative \(\Sigma_a\) that acts like \(\partial_a\) on the gauge indices and on the tensorial indices like
\[
\Sigma_a X^b_c = \nabla_a X^b_c := \partial_a X^b_c + \Gamma^b_{da} X^d_c - \Gamma^d_{ca} X^a_d,
\]
(9)
where \(\Gamma^b_{ca}\) is the Levi-Civita connection on \(\Sigma_t\). The vector constraint (called also the diffeomorphism constraint),
\[
V^{(\Delta)} := \int_{\Sigma_t} d^3x N^a \epsilon^I_a = \int_{\Sigma_t} d^3x N^a E^a_I E^b_I,
\]
(10)
imposes the invariance under spatial diffeomorphism transformations. Finally, the Hamiltonian constraint (called also the scalar constraint),
\[
H^{(\Delta)} := \int_{\Sigma_t} d^3x N H^{(\Delta)} = \frac{g^2}{2} \int_{\Sigma_t} d^3x N \frac{1}{\sqrt{q}} q_{ab} \left( E^a_I E^b_I + B^a_I B^b_I \right),
\]
(11)
generates the time reparametrization symmetry. Notice that we introduced the ‘magnetic field’,
\[
B^a_I := \frac{1}{2g^2} \epsilon^{abc} F^I_{bc},
\]
(12)
where \(\epsilon^{abc} := \sqrt{q} \epsilon^{abc}\) and \(\epsilon^{abc}\) is the Levi-Civita tensor. It is worth noting that both \(E^a_I\) and \(B^a_I\) are not vectors, but \(\mathfrak{su}(N)\)-valued objects of weight one — a vector density and a pseudovector density, respectively. As a result, quantities in \([6]\) and \([12]\) scale properly according to the scaling of the integration measure \(d^3x\) (with \(d^3x \sqrt{q}\) being an invariant measure on \(R^3\), while \(\sqrt{q}\) is a scalar density having weight one).

Finally, we can take a step towards the quantization of this model. CQGR based on LQG is defined in terms of lattice-regularized variables that are invariant under spatial diffeomorphisms and internal symmetry transformations\(^2\). At the quantum level they are transformed into operators that act on the Hilbert space, constructed in the way to be invariant under actions of the quantized constraints defined in \([8]\) and \([10]\). The Hamiltonian constraint operator (HCO) is formulated rewriting \([11]\) into an object dependent on the lattice-regularized variables. They are the curved-background\(^3\) Wilson loops\(^4\),
\[
\mathcal{W}(\Sigma^I, h_{l'\cup l}) = -\frac{\varepsilon^2}{2T} F^I_{ab} \dot{e}^a_b \dot{e}^b_a + O(\varepsilon^4),
\]
(13)
where \(h_{l'\cup l}\) is the holonomy of a loop that begins at the initial (or final) point of a path \(l\), goes along this path and returns to the same point along \(l'\). Notice that in \([13]\) we used the notation introduced in Appendix A. The second

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\(^1\) It is worth noting that \(\epsilon^I_a = 0\) (which can be considered as the definition of the time gauge). The \(\epsilon^I_a\) dreibein and its inverse, \(\epsilon^a_I\), play the roles of the transition matrices between the Euclidean space with the Cartesian basis (and with \(\delta_{ij} = \epsilon^a_i \epsilon^b_j\), representing the flat metric) and the curved \(\Sigma_t\) space (with \(q_{ab} := \epsilon^a_i \epsilon^b_j\)).

\(^2\) See \([22]\) for the discussion about the apparent difference between the Hamiltonian that one obtains from \([11]\) and the Kogut-Susskind Hamiltonian\(^2\) — both formulated in terms of traces of gauge holonomies of a loop.
type of the lattice-regularized variables are the fluxes of the $E_I^a = \frac{1}{2} \epsilon^{abc} E^I_{bc}$ field, constructed by smearing two-forms $\star E^I_{bc}$ (Hodge dual to $E_I^a$) over some two-dimensional surfaces $S$,

$$\mathcal{E}_I(S) := \int_S \star E^I = \int_S n_a E_I^a,$$

where $n_a := \epsilon_{abc} dx^b \wedge dx^c$ is the normal to $S$.

In what follows, we are going to discuss the simplest physical model revealing inconsistencies between different lattice regularization procedures. To present and solve these problems in the case of the scalar and vector fields, it is enough to focus on the $U(1)$ electromagnetic example given by the Hamiltonian constraint in (11), neglecting the internal indices. Consequently, instead of the canonical fields in (6) and (12), we are going to use the electric and magnetic fields, respectively. These fields are defined in the following way,

$$E^a_I := \sqrt{q} g^{ab} q_{0b} F_{\mu b},$$  

$$B^a_I := \frac{1}{2g} \epsilon^{abc} F_{bc}.$$  

The vector constraint simplifies to $V(\Delta) = \int_{\Sigma_i} d^3x N^a E_{ab} E^b_i$, while the Gauss one is not present. All the results concerning regularization of the electromagnetic field are straightforwardly generalizable to the SU(N) case.

### II.3. GR in terms of Ashtekar variables

Let us focus now on the gauge field that plays a unique role in CQGR, which in the free field’s sector of the action is not coupled to any other dynamical object (unlike matter fields). This is the gravitational field. Starting with the Einstein-Hilbert action,

$$S^{(gr)} := \frac{1}{\kappa} \int_M d^4x \sqrt{-g} R,$$

where $R$ is the Ricci scalar, while the gravitational coupling constant reads, $\kappa = 16\pi G$, we are going to rewrite the classical theory of gravity into the form of a gauge theory. We perform the ADM decomposition, introducing Ashtekar variables [3],

$$A^i_a := \frac{1}{2} \epsilon_{ijk} \Gamma^i_{jka} + \gamma K^i_a,$$

being the Ashtekar-Barbero connection, and the densitized dreibein defined as,

$$E^a_i := \sqrt{q} e^a_i.$$  

Here, $\Gamma^i_{jka}$ is the spin connection, $K^i_a := \frac{1}{2} \Gamma^i_{0ka}$ is the dreibein-contracted extrinsic curvature, while $\gamma$ denotes the Immirzi parameter. It is worth noting that the Poisson brackets between Ashtekar variables (with respect to the ADM variables — the metric $q_{ab}$ and its conjugated momentum) is canonical up to a constant,

$$\{ A^i_a(t, x), E^b_j(t, y) \} = \frac{\gamma \kappa}{2} \delta^b_a \delta^i_j \delta^3(x - y).$$

Next, we define the curvature of the Ashtekar-Barbero connection, analogously to the Yang-Mills curvature in [5],

$$F^i_{ab} := \partial_a A^i_b - \partial_b A^i_a + \epsilon_{ijk} A^j_a A^k_b.$$  

Then, applying the Legendre transform, we obtain the total Hamiltonian in the form identical to [7], however, having a different structure of the constraints. The gravitational $\mathfrak{su}(2)$ equivalents to [8], [10], and [11] are

$$G^{(gr)} := \frac{1}{\gamma \kappa} \int_{\Sigma_i} d^3x A^i_D_a E^a_i,$$

$$V^{(gr)} := \frac{1}{\gamma \kappa} \int_{E_i} d^3x N^a F^i_{ab} E^b_i.$$
and
\[
H^{(\varphi)} := \frac{1}{\kappa} \int_{\Sigma_t} d^3 x \, N \left( \frac{1}{\sqrt{q}} \left( F_{ab}^i - (\gamma^2 + 1) \epsilon_{ilm} K_a^l \epsilon^{ijk} E_{lj} E_{k}^b \right) \right),
\]

respectively. It is worth mentioning that analogously to (8) and (9), \( D_a \) acts like the covariant derivative in (1), i.e. on the gauge indices it acts as follows,
\[
D_a X^i = \partial_a X^i + \epsilon_{ijk} A^j a X^k,
\]
while on the tensorial indices like \( \nabla_a \). Notice that the difference with respect to the structure of the \( su(2) \) Yang-Mills theory is explicit only in (24). Therefore, constructing generally covariant and gauge invariant variables (analogous to (13) and (14)) simplifies the further constructions of Hilbert spaces for matter fields (since matter Hamiltonians contain gravitational degrees of freedom) and unifies the entire formalism.

Let us discuss how these generally covariant and gauge invariant variables are introduced. The elements of the Hamiltonian constraint are the \( F_{ab}^i \) curvature and \( E^i_{aj} \) field, which are regularized in the way presented in (13) and (14), respectively. The additional object, not appearing in (11), is the extrinsic curvature. The lattice-regularized, i.e. the invariant representation of \( K^a_i \), is constructed taking the integrated densitized trace,
\[
K := \int_{\Sigma_t} d^3 x K^a_i E^a_i.
\]

Finally, it is useful to introduce another lattice-related object that will be applied in Sec. III to tame the singularities appearing in the Hamiltonian constraints in the form of \( 1/\sqrt{q} \). Expanding the holonomy in (3) along a link \( l^a \), we obtain the equation linking the holonomy with the connection (here, to complete analysis in Sec. II.2, written for a general Yang-Mills field),
\[
\text{tr}(\tau^I h^L) = -\frac{\epsilon}{T} A^j a L^a + O(\epsilon^2).
\]
This closes the remarks concerning lattice gauge theories.

II.4. Methods of scalar fields coupling to LQG

Looking for a diffeomorphism-invariant representation of the scalar field, we have to rely on a different strategy than for vector fields. Let us consider the real scalar field \( \varphi \). A generalization to the complex case, \( \phi = \varphi_1 + i\varphi_2 \), with two species of real fields is straightforward. Also, the introduction of an internal symmetry is not problematic and can be done as follows, \( \varphi = \varphi^+ \tau^I \). However, conversely to the lattice formulation of the Yang-Mills field and its canonically conjugated momentum, the scalar field and its momentum \( \pi \) are not a one-form and a vector density, respectively. They are a scalar and a pseudoscalar density, respectively. Therefore, it is not natural to define their geometrical representations smearing them along a link and over a surface, respectively. This is why a point-solid duality appears to be more appropriate to smear geometrically \( \varphi \) and \( \pi \).

As in the previous subsection, let us discuss the simplest model of the scalar field, which, without loss of generality, discloses the problems sourced by the choice of the point holonomy representation. We consider then a massless singlet field. The mass term, i.e. the quadratic potential, extending this model to the Klein-Gordon field, trivially couples to gravity only by multiplication with \( \sqrt{q} \), and it does not have to be discussed separately. The same coupling is assumed for any other power of a polynomial potential. Consequently, we define the following simple action
\[
S^{(\varphi)} := \frac{1}{2g^2} \int_M d^4 x \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi.
\]

The Legendre transform results in the total Hamiltonian expressed as a sum of two first class constraints,
\[
H_T^{(\varphi)} = V^{(\varphi)} + H^{(\varphi)}.
\]

The vector and scalar constraints are given by the formulas,
\[
V^{(\varphi)} := \int_{\Sigma_t} d^3 x N^a \partial_a \varphi \pi
\]
and
\[ H^{(\phi)} := \frac{1}{2} \int_{\Sigma_t} d^3x \, N \left( \frac{\partial^2}{\partial \phi^2} \phi + \frac{\sqrt{q}}{g^2} q^{ab} \partial_a \phi \partial_b \phi \right), \] (31)
respectively, while the momentum canonically conjugated to \( \varphi \) reads,
\[ \pi = \frac{\sqrt{q}}{g^2} e^\mu_0 \partial_\mu \phi. \] (32)

Since the scalar field has simpler structure than the vector field, one can try to construct a Fock-like representation for the former one. However, to define annihilation and creation operators proportional to \( \varphi \pm i\lambda \pi \), one has to replace the scalar density \( \pi \) with a scalar. Following the idea in [2, 20], one can consider a point-related scalar object,
\[ \Pi(x) := \int_R d^3y \delta(x,y) \pi(y), \] (33)
based on the \textit{a priori} smeared momentum,
\[ \pi(x) := \sum_{y \in R} \delta^3(x-y) \Pi(y). \] (34)
As a result, one can correctly define scalar-valued ladder operators proportional to \( \varphi \pm i\lambda \Pi \), with \( \lambda \) having dimension \( \text{length}^{-2} \), and be able to establish the Fock space as a GNS Hilbert space — see [2]. However, this construction is valid as long as we consider a discrete geometry, but it diverges in the continuous limit. In particular, it seems to be problematic to apply this formal model to cosmology, where the semi-classical limit of the gravitational field coupled to a quantum scalar inflaton field is expected to have a particular importance. As argued in Sec. IV.2, approximating CQGR with QFT on curved spacetime, the continuous limit, having a physical meaning, has to be correctly defined.

The alternative approach is to consider a holonomy-like representation [7, 20], defining the object analogous to (3),
\[ u_x := \exp(i\varepsilon \varphi \phi(x)). \] (35)
Choosing the ‘position representation’ of the momentum operator (defined with respect to smeared \( \pi(x) \) inside a region centered at \( x \)) allows to interpret the regulator \( \varepsilon \) as an eigenvalue of operator \( -i\hbar \int d^3y \delta(x,y) \partial_\partial \phi \) [12]. The point holonomy representation in (36), however, does not allow us to construct a Fock space. One has to consider the polymer representation [25–27] with states having a similar form to \( u_x \). This model has a well-defined continuous limit, but it requires introducing a polymer distribution of states. The scalar field defined at points forms a structure having different features than Yang-Mills fields defined along paths (including the gravitational one). Finally, what we demonstrate in Sec. IV.3, quantum corrections coming from the gravitational degrees of freedom in the scalar field’s Hamiltonian in the point holonomy representation break general covariance at the level of quantum corrections. This becomes evident already in the simplified cosmological models, in which one can compare the inverse volume corrections arising from Hamiltonians of the gravitational field [28] and vector field [13], with the ones from the scalar field in the polymer representation [12].

Finally, there is another line of research attempting to resolve the mentioned problems, however, it is not related the idea of constructing CQGR. In this approach one defines Fock space representations for matter fields on semiclassical LQC. We present it in the next subsection.

II.5. Phenomenological models contradictive with CQGR

At the end of this section, we would like to briefly discuss quantum cosmological models not related directly to LQG. All of them were introduced as different approaches to the matter quantization issue in LQC. Following the review of these phenomenological theories [29], we can recognize three different models: the effective constraints, dress metric, and separate universe quantization approaches.

The effective constraints method [30, 31] does not define any QFT for matter, but it rather introduces unspecified perturbations around the classical cosmological matter density. The structure of these perturbations is then restricted
by a closeness of the constraint algebra. Therefore, this effective model is a priori not contradictive with CQGR, unless the formulation of LQC is not a cosmological limit of CQGR. In the latter case, however, one can always repeat the procedures of the effective constraints method around a different cosmological framework obtained from LQG. Anyhow, the result, by definition, does not provide any insight into the structure of the matter sector.

The dressed metric approach, originated by the idea in [3] and applied both to the scalar [32] and vector [32] fields, defines the standard Fock space known from models of QFT in curved spacetime. This is realized by choosing the expectation value of HCO in LQC as a background. However, this approximation omits the corrections coming from the gravitational degrees of freedom in the matter field’s HCO, being of the same order of significance as the cosmological corrections to the metric field and the inflaton field — see Sec. IV.1 This way, not only the essence of the non-perturbative approach to quantum cosmology is neglected by this effective theory, but particular quantum corrections are omitted. It is worth noting that a specific variant of this approach, called the hybrid quantization [34], additionally assumes left and right multiplication of the total HCO by quantized factor $q^{-1/4}$. This affects the structure of all the gravitational corrections, resulting in a new model, which cannot be understood as a directly quantized cosmological limit of classical GR.

Finally, the separate universe quantization [35] is the quantization of the long-wavelength gravitational modes over LQC background. This method uses the long-wavelength approximation to construct a loop quantization for both the background and the perturbations. It is an improvement when compared to the dressed metric approach, in which the standard QFT procedure is broken introducing separate quantization for the background variables (loop quantization) and the perturbative degrees of freedom (Fock quantization). However, the separate universe quantization generates a different problem, neglecting the specific structure of quantum matter fields with the corresponding corrections and, what is more important from the perspective of this idea, it also neglects the gravitational corrections to the matter sector.

Concluding, all these effective approaches are not compatible with the standard canonical procedures of QFT on a lattice. In our opinion, the only model that has a possibility to be a cosmological limit of CQGR, would be a modified version of the second proposal. This is theoretically possible under the following restrictions: (i) all gravitational degrees of freedom must be quantized via the same (loop) method, and (ii) all gravitational corrections to the matter sector must have the same structure for each element in the Hamiltonian. Fock quantization would be then constructible on the curved background, defined applying a kind of a renormalization procedure to the expectation value of gravitational degrees of freedom in HCO, removing the gravitational corrections from the matter sector.

III. KINEMATICS

III.1. Cosmological models

Let us define a simple cosmological model constructed from the Einstein’s gravity, classically reduced to the Bianchi I symmetry, coupled with bosonic matter and quantized via the loop technique. The kinematics of the gravitational sector of this model is directly related to the Bianchi I extension of LQC [36], QRLG [10], and cosmological coherent quantum gravity (CCQG) [37]. From the perspective of this article, it is not important neither if the reduction of Ashtekar variables to their diagonal variant is done before or after quantization, nor if the Lorentzian sector of the Hamiltonian is or is not proportional to the Euclidean one [4]. What is important, is to keep the general covariance of the matter sector unchanged from the classical perspective of GR — both before and after the lattice regularization, as well as at the quantum level.

The only complete formulation of the kinematics of the matter sector of CQGR has been presented in [2, 7]. We follow directly this formulation, which has been already applied to a reduced cosmological model in the failed framework of QRLG. Neglecting constructional problems of this model [11], we can safely adapt the formulation of the kinematics of the scalar field [12] and the vector field [13] to our purpose. Finally, since LQC is the most popular cosmological setting related to LQG, we show explicitly how our toy model is linked to the Bianchi I extension of LQC.

Let us now begin to discuss the elements and factors constituting the gravitational Hamiltonian operator in the simplified framework over a cubic lattice. A natural candidate for the cosmological sector of LQG appears to be QRLG. This model has been constructed to capture the part of degrees of freedom of the full theory, which corresponds to any gravitational framework having the diagonal metric tensor and triads. As we pointed out in [11], substantial shortcomings have been unnoticed in the original construction of this model. In particular, the theory is overconstrained...
and conversely to what is stated in [10] [39], is neither \((U(1))^3\) invariant, nor Cartesian frame diffeomorphic invariant — these gauge symmetries became fixed in the ‘quantum reduction procedure’. Nonetheless, the semiclassical limit\(^1\) derived from the kinematics of the gravitational HCO postulated by QRLG,

\[
\langle \hat{H}_{\text{QRLG}} \rangle = -\frac{2}{\gamma^2 \kappa} \sum_v \mathbb{N}_v \mathcal{V}_v \sqrt{E_1^0 \delta_1^0 \delta_2^0 E_2^0 \delta_3^0} \frac{1}{|E_1^0|} \prod_{k \neq i} \frac{\sin(\varepsilon_{ik} \hat{A}_k^0 \delta_k^0)}{\varepsilon_{ik}} \left[ 1 + \mathcal{O}\left( \frac{1}{(j(\gamma))} \right) \right],
\]

with particular assumptions [28] [40] added when deriving the continuum limit, reproduces the outcome of the Bianchi I extension of LQC [36]. Here, \(\mathcal{V}_v\) is the volume element of the homogeneous patch around each node \(v\), and \(\hat{A}_k^0 \delta_k^0\) is the diagonal form of the Ashtekar connection, obtained simultaneously fixing SU(2) and diffeomorphism symmetries [39]. Analogously, \(E_i^0\) denotes the diagonal densitized dreibein, reintroduced by the correspondence principle — replacing the eigenvalue of operator \(\hat{E}_1^0\) with its classical equivalent. A similar (but not identical) expression appears when the Lorentzian term of the gravitational sector of HCO is taken into account [38]. Moreover, similar (but again slightly different) results have been found while solving the action of HCO from LQG on the coherent-complexifier states [37].

We are, however, much more interested in a different type of the next to the leading order kinematical corrections of these cosmological theories. As we will see, the essential issue of this article is the form of the corrections coming from the lattice regularization — the so-called ‘inverse volume corrections’. It is worth noting that the presence of them was first pointed in the isotropic model of LQC in [41]. As demonstrated in [42], these corrections have a significant contribution to the dynamics of the primordial universe.

Aside from differences in the corrections coming from the expansion of the trigonometric functional in the gravitational sector of the scalar constraint\(^2\), for instance from the functional \(\sin(\varepsilon_{ik} \hat{A}_k^0 \delta_k^0)/\varepsilon_{ik}\) in expression [37], the structure of the inverse volume corrections remains the same\(^3\) up to a constant factor (which also varies from one model to another). These corrections come from the action of the operator

\[
\hat{h}_a^{-1}[\hat{V}, \hat{h}_a] \tag{38}
\]

and are present also in the matter sector, where the expression analogous to (38) appears,

\[
\hat{h}_a^{-1}[\hat{V}^n, \hat{h}_a], \tag{39}
\]

where \(n\) is a positive rational number. Precisely, the inverse volume corrections are the result of the action of the volume operator (the square root of derivatives with respect to the connection) on a basis state, being initially modified by the action (the multiplication) of the gravitational holonomy operator \(\hat{h}_a\).

Writing explicitly the expression for the scalar constraint for gravity in the holonomy-volume-regularized form,

\[
H^{(gr)} = \int_{\Sigma_t} d^3x \mathcal{N}(x) \left( \mathcal{H}^{(gr)}_{\text{Eucl}}(x) + \mathcal{H}^{(gr)}_{\text{Lor}}(x) \right), \tag{40}
\]

where

\[
\mathcal{H}^{(gr)}_{\text{Eucl}}(x) = \frac{2\gamma^2}{\kappa^2} \lim_{\varepsilon \to 0} \epsilon^{abc} \left( \frac{1}{\varepsilon^2} \left( h_{ab}(x) - h^{-1}_{ab}(x) \right) \frac{1}{\varepsilon} h^{-1}_{c}(x) \left\{ \hat{V}(x), h_c(x) \right\} \right), \tag{41}
\]

and

\[
\mathcal{H}^{(gr)}_{\text{Lor}}(x) = -\frac{2\gamma^2}{\kappa^2} \lim_{\varepsilon \to 0} \epsilon^{abc} \left( \frac{1}{\varepsilon} h^{-1}_{a}(x) \left\{ \hat{K}(x), h_a(x) \right\} \frac{1}{\varepsilon} h^{-1}_{c}(x) \left\{ \hat{V}(x), h_c(x) \right\} \right), \tag{42}
\]

it is easy to recognize the terms introducing the corrections related to the expression (38). It is also worth mentioning that both in the case of QRLG and LQC, only the Euclidean term in (41) contributes to the HCO, which matrix element on coherent states is written in [37].

\(^1\) Appropriate coherent states are defined in Sec. [11] [39].
\(^2\) The form of the trigonometric functional slightly varies from one model to another, but always remains expandable to a power series of connections.
\(^3\) This statement is true as long as the volume operator is an eigenoperator of basis states for a given model. This happens in the case of both LQC and QRLG. Considering a coherent state expansion around a hexavalent node with perpendicular links [43], the answer will depend on the restrictions imposed on the possible corrections. Taking the ‘always-cuboidal lattice’, the issue is reduced to the one in LQC and QRLG, while allowing to have also non-perpendicular perturbations of links, complexity of this problem increases drastically. Let us neglect this issue for now, which is rather a secondary problem.
The structure of the lattice corrections in the matter sector is also directly readable from the regularized form of Hamiltonians, already before quantization. The vector field contributions to the gravitational Hamiltonian in \[40 \] \[42 \] reads,

\[
H^{(\Delta)} = \int_{\Sigma_t} d^3 x N(x) \left( \mathcal{H}^{(\Delta)}_{\text{mom}}(x) + \mathcal{H}^{(\Delta)}_{\text{mag}}(x) \right),
\]

where

\[
\mathcal{H}^{(\Delta)}_{\text{mom}}(x) = \frac{2^7 \kappa^2}{(\gamma \kappa)^3} \lim_{\varepsilon \to 0} \mathcal{E}^a(x) \left( \tau^a \frac{1}{\varepsilon} h^{-1}_a(z) \left\{ \mathbf{V}^z(z), h_e(z) \right\} \right)
\]

\[
\times \int d^3 y \delta^3(x-y) \mathcal{E}^b(y) \left( \tau^b \frac{1}{\varepsilon} h^{-1}_b(y) \left\{ \mathbf{V}^z(y), h_e(y) \right\} \right)
\]

and

\[
\mathcal{H}^{(\Delta)}_{\text{mag}}(x) = \frac{2^7 \kappa^2}{(\gamma \kappa)^2} \lim_{\varepsilon \to 0} \mathcal{B}^a(x) \left( \tau^a \frac{1}{\varepsilon} h^{-1}_a(x) \left\{ \mathbf{V}^z(x), h_e(x) \right\} \right)
\]

\[
\times \int d^3 y \delta^3(x-y) \mathcal{B}^b(y) \left( \tau^b \frac{1}{\varepsilon} h^{-1}_b(y) \left\{ \mathbf{V}^z(y), h_e(y) \right\} \right).
\]

The regularized scalar field’s Hamiltonian is given by,

\[
H^{(\varphi)} = \int_{\Sigma_t} d^3 x N(x) \left( \mathcal{H}^{(\varphi)}_{\text{mom}}(x) + \mathcal{H}^{(\varphi)}_{\text{der}}(x) + \mathcal{H}^{(\varphi)}_{\text{pot}}(x) \right),
\]

where

\[
\mathcal{H}^{(\varphi)}_{\text{mom}}(x) = \frac{2^{21} \kappa^2}{3^3 (\gamma \kappa)^6} \lim_{\varepsilon \to 0} \pi(x) \epsilon_{ijk} \epsilon^{abc} \int d^3 z \delta^3(x-z) \left( \tau^a \frac{1}{\varepsilon} h^{-1}_a(z) \left\{ \mathbf{V}^z(z), h_e(z) \right\} \right)
\]

\[
\times \left( \tau^b \frac{1}{\varepsilon} h^{-1}_b(z) \left\{ \mathbf{V}^z(z), h_e(z) \right\} \right) \right)
\]

\[
\times \int d^3 y \delta^3(x-y) \pi(y) \epsilon_{imn} \epsilon^{def} \int d^3 z' \delta^3(y-z') \left( \tau^d \frac{1}{\varepsilon} h^{-1}_d(z') \left\{ \mathbf{V}^z(z'), h_e(z') \right\} \right)
\]

\[
\times \left( \tau^e \frac{1}{\varepsilon} h^{-1}_e(z') \left\{ \mathbf{V}^z(z'), h_e(z') \right\} \right) \right)
\]

\[
\mathcal{H}^{(\varphi)}_{\text{der}}(x) = \frac{2^{17} \kappa^2}{3^4 (\gamma \kappa)^4} \lim_{\varepsilon \to 0} \partial_{\alpha} \varphi(x) \epsilon_{ijk} \epsilon^{abc} \left( \tau^a \frac{1}{\varepsilon} h^{-1}_a(x) \left\{ \mathbf{V}^z(x), h_e(x) \right\} \right)
\]

\[
\times \left( \tau^b \frac{1}{\varepsilon} h^{-1}_b(x) \left\{ \mathbf{V}^z(x), h_e(x) \right\} \right) \right)
\]

\[
\times \int d^3 y \delta^3(x-y) \partial_{\alpha} \varphi(y) \epsilon_{imn} \epsilon^{def} \left( \tau^d \frac{1}{\varepsilon} h^{-1}_d(y) \left\{ \mathbf{V}^z(y), h_e(y) \right\} \right)
\]

\[
\times \left( \tau^e \frac{1}{\varepsilon} h^{-1}_e(y) \left\{ \mathbf{V}^z(y), h_e(y) \right\} \right) \right)
\]

and

\[
\mathcal{H}^{(\varphi)}_{\text{pot}}(x) = \frac{1}{2^7 \kappa^2} \sqrt{q(x)} \mathcal{V} \left( \varphi(x) \right) \approx \frac{1}{2^7 \kappa^2} \lim_{\varepsilon \to 0} \mathcal{V} \left( \varphi(x) \right) \frac{1}{\varepsilon^3} \mathbf{V}(x, \varepsilon).
\]

Here, we reintroduced the potential term to show explicitly that the ambiguity in the choice of the form of the potential functional, \[40 \] \[42 \] does not influence the quantum generally-relativistic corrections. This is because the volume operator in cosmological variants of LQG is an eigenoperator of the cubic/cuboidal states.

### III.2. States space

Before discussing the form of the semiclassical quantum-geometrical corrections in the matter sector of cosmologically-related models of CQGR, let us specify how this quantum theory is constructed. Let us also clear that despite a different construction of the Hilbert space in the standard formulation of LQC \[53 \], one can easily extend the framework described below to this popular cosmological model, consequently obtaining the same conclusions.
We consider the system of minimally coupled bosonic matter and gravity, with the Hilbert space defined as follows,
\[ \mathcal{H}_{\text{kin}} := \mathcal{H}^{(\text{gr})}_{\text{kin}} \otimes \mathcal{H}^{(\Delta)}_{\text{kin}} \otimes \mathcal{H}^{(\phi)}_{\text{kin}}. \]  

The SU(N) Yang-Mills sector is labeled by \( \mathcal{H}^{(\Delta)}_{\text{kin}} \). This Hilbert space is defined in analogy to the one for the SU(2)-invariant gravitational field in LQG, \( \mathcal{H}^{(\text{gr})}_{\text{kin}} \) — see [11]. In both cases, it is the space of cylindrical functions of holonomies of the gauge connections. Also, in both cases, the basis states are the invariant spin network states, \( |\Gamma; j_{\nu}, i_{v}\rangle \) for the SU(N)-invariant vector field, and \( |\Gamma; j_{l}, i_{v}\rangle \) for the SU(2)-invariant gravitational field, respectively. They are labeled by quantum numbers (spins) \( j_{\nu} \) and \( j_{l} \), respectively. They carry the notion of the irreducible representations of the appropriate gauge groups at each link \( l \). Then, to preserve the gauge invariance, corresponding intertwiners are attached at each node \( v \), being denoted as \( l_{v} \) and \( i_{v} \), respectively. As we already mentioned at the end of Sec. II.2 for this article, it is enough to restrict our analysis of the Yang-Mills field to the U(1) case with the trivial intertwiners — C-numbers representing positions on the phase circle. These, being the elements commuting with any other structures in the Hilbert space, can be then neglected (quantum states are, by definition, specified up to a number). The same happens in the case of the intertwiners of reduced gravitational models. Both in LQC and CCQG intertwiners are not included in the definition of U(1)-fixed states, while, as we argued in [11], they were incorrectly reinstated in the original construction of QRLG. To restore the accurate formulation of this model, the intertwiners must be removed from the states in the improved version of this model. Summarizing, as a result of the simplification of all gauge symmetries to U(1), we will henceforth consider the states \( |\Gamma; j_{l}\rangle \in \mathcal{H}^{(\Delta)}_{\text{kin}} \) and \( |\Gamma; j_{l}\rangle \in \mathcal{H}^{(\psi)}_{\text{kin}} \).

Next, let us sketch the lattice quantization for a point-regularized field, which is a qualitatively different method. The Hilbert space selected in this method, called the point-holonomy representation, is defined as
\[ \mathcal{H}^{(\phi)}_{\text{kin}} := \left\{ a_1 U_{\pi_1} + \ldots + a_n U_{\pi_n} : a_i \in \mathbb{C}, n \in \mathbb{N}, \pi_i \in \mathbb{R} \right\}, \]

where the wave function reads,
\[ U_{z}(\varphi) := \langle \varphi | U_{\{\pi_1,\ldots,\pi_n\},\{\pi_{\pi},\pi_{\pi}\}} \rangle := e^{i\sum_{v \in \Sigma} \pi_{\pi_{\pi}}}. \]

This definition explicitly preserves the rotational symmetry of the scalar field at each point.

Considering the single-point state located at \( v \), \( \langle \varphi | v; U_{\pi} \rangle := e^{i\pi_{\pi}}, \) the action of the canonical operators in the Schrödinger-DeWitt representation is trivially defined in terms of the exponentiated field, shifting the state as follows,
\[ e^{i\pi_{\pi_{\pi}}\varphi_{v}} |v; U_{\pi} \rangle := e^{i\pi_{\pi_{\pi}}} |v; U_{\pi} \rangle = |v \cup v'; U_{\pi_{\pi_{\pi}}} \rangle. \]
FIG. 2. The simplified basic state of bosonic fields for cubic lattice (e.g. in LQC)

Analogous action for the momentum operator corresponding to the $\varepsilon$-smeared momentum $\pi$ around the $v$ node, $\Pi(v) := \int d^3w \delta_{v,w} \pi(w)$, turns out to be the eigenvalue equation set on the basis states,

$$\Pi(v')|v; U_\pi\rangle := -i\hbar \frac{\partial}{\partial \phi(v')} |v; U_\pi\rangle = \hbar \pi_{v',w} \delta_{v,v'} |v; U_\pi\rangle.$$  

(54)

The inner product of the single-point states is as simple and symmetric as the form of the canonical operators, reading

$$\langle v; U_\pi|v'; U_{\pi'}\rangle = \delta_{v,v'} \delta_{\pi,\pi'}.$$

(55)

Details of this formulation of the point-holonomy model over the cuboidal lattice are explained in [12]. In the case of the general lattice, however, this model has been already proposed earlier, [25–27].

Concluding, the basis states are selected as $\mathcal{H}_{kin} \ni \{\Gamma; j_l, 2^l, U_\pi\} := \{\Gamma; j_l\} \otimes \{\Gamma; 2^l\} \otimes \{\Gamma; U_\pi\}$. Considering a single hexavalent node state $c_v \in \Gamma$, we can express this structure in the graphical form represented by FIG. 1 (within the dashed frame). Here, $j_l^{(i)}$ and $2^l^{(i)}$ are the spin numbers associated to the links $i_{p,q,r}^{(i)}$. The scalar field state is represented by the point holonomy $e^{i\pi_{p,q,r}}$ (attached at each node $v_{p,q,r} \in \Gamma$) with the real coefficient $\pi_{p,q,r}$.

It is worth mentioning that analogous structure in LQC would be represented by a normalized (in the way preserving the symmetry of the anisotropic Bianchi I model) hexavalent node state in FIG. 2. This selection of the node state can be directly related to the formulation of the full theory [2] when the basis states at collinear links are homogeneous — precisely when $\forall_j j_{x,y,z}^{(i)} = j_{x,y,z}^{(i)}$ and $\forall_j j_{x,y,z}^{(i)} = j_{x,y,z}^{(i)}$. This symmetry, however, excludes inhomogeneities from the formalism of LQC

The normalization in QRLG is performed differently. In this model, it is done while taking the continuum limit [28, 44] of the action of HCO. It is worth noting that this procedure is correct, but it can be realized in an even more straightforward way. One can simply divide each link in the middle, creating two paths. Consequently, the proper division of the Cauchy hypersurface into cuboids or cubes should be done in the manner that we proposed in FIG. 3

---

1 We always keep in mind that our model has to be connected to CQGR. Therefore, we a priori exclude the phenomenological models, which by construction violate general covariance (even if restricted only to the cosmological degrees of freedom). In this case, since the volume operator acts on all the links emanated from a node, different quantum numbers attached to any collinear pair of links in FIG. 2 would clearly generate different eigenvalue than the eigenvalue of the volume operator acting only inside the ‘dashed cube’.

2 This resembles the analysis in the original construction of LQG [2] (when the action of the flux operator of the densitized dreibein is introduced).
Finally, notice that already at first glance, the structure of the state represented by FIG. 1 (or its modifications in FIG. 2 and FIG. 3) reveals the methodological inconsistency in the construction of this multi-matter coupling model — the scalar field is lattice-regularized in a different manner than the other fields. The inconsistency, however, is present only in the construction, and not in the usage of any methods excluding themselves. The former problem can be questioned then only due to its apparent overcomplexity, while the latter one would be self-contradictory — compare our critical comments in II.4 and II.5, respectively. Anyhow, we do not exclude a priori even the proposals recalled in II.5. However, demonstrating in Sec. IV.3 that there are the corrections missing in the dynamics derived from the phenomenological models, proves that these models cannot be classified as the cosmological limit of CQGR.

III.3. Gravitational coherent states

Let us begin with a general introduction. To make our analysis clear, we initiate derivation using the notation typical to LQC, introduced in Ref. [5] and later adapted to the anisotropic model in Ref. [36],

$$\tilde{A}_i^a(t) := \frac{1}{l_0^i} \tilde{c}_i^a(t) \, \psi_a^i,$$  \hfill (56)

$$\tilde{E}_i^a(t) := \frac{l_0^i}{V_0} \, \tilde{q}_i^a(t) \, \sqrt{\psi} \, \psi_a^i.$$ \hfill (57)

Matrices $\psi_a^i$ and $\psi_i^a$ represent constant orthonormal frame and co-frame fields, respectively, associated with Cartesian coordinates $x^a$ in the Euclidean space. The determinant of the fiducial metric $\psi_{ab}$ in (57) compensates the density weight of $\tilde{E}_i^a(t)$. The fiducial length $l_0^i$ and the corresponding volume of the fiducial cell, $V_0 := l_0^1 l_0^2 l_0^3$, have been introduced to remove any manifestations of fiducial geometry from the Poisson brackets that reads,

$$\{ \tilde{c}_i^a(t), \tilde{q}_j^b(t) \} = \frac{K_7}{2} \delta_i^j \delta^a_b.$$ \hfill (58)

We consider the semiclassical limit of the quantum theory of gravity and matter, which is based on LQG and is defined via the Hamiltonian $\hat{H}$. By the semiclassical limit, we understand solving the Ehrenfest theorem based on the coherent states, $|\rangle \in \mathcal{H}_{kin}$, defined as the tensor product of the coherent states for different canonical pairs of fields.
The Ehrenfest theorem reads,

\[ \frac{d}{dt} \langle \hat{O} \rangle - \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle = \frac{1}{i\hbar} \left\langle [\hat{O}, \hat{H}] \right\rangle, \]

where the states factorize as follows,

\[ \left| \right\rangle = \left| \right\rangle^{(gr)} \otimes \left| \right\rangle^{(matt)} = \left| \right\rangle^{(gr)} \otimes \bigotimes_{\Phi} \left| \right\rangle^{(\Phi)}, \]

while the symbol \( \Phi \) represents any matter field. Assuming that canonical variables are not explicitly time-dependent, we can neglect term \( \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle \).

The normalized Bianchi I coherent states for the gravitational sector are defined as

\[ \left| \right\rangle^{(gr)} := \sum_{v}^{3} \left| \right\rangle^{\mu_{v}} \right\rangle \left( \left( \phi_{0}^{(v)}(\hat{A}) \right) \right)^{-\frac{1}{2}} \left| \right\rangle^{\nu_{v}}(\hat{A}), \]

where the shadow-like state (we keep the naming convention used in [45]) with \( d \)-width Gaussian distribution around eigenvalue of the densitized dreibein operator at the node \( v \) reads,

\[ |\nu_{v}(\hat{A}) := \sum_{\mu_{v}}^{3} \exp \left[ -\frac{1}{2 \sigma^{2}} \left( \mu_{v}^{2} - p^{2}(v) \right)^{2} \right] \exp \left[ -i \left( \mu_{v}^{2} - p^{2}(v) \right) z^{2}(v) \right] |\mu_{v}(v) \rangle. \]

The link excitation states introduced in [5, 36] are given by the expression,

\[ |\mu_{v}(v) := \exp \left[ i \frac{\mu_{v}^{2}}{2} z^{2}(v) \right], \quad \mu_{v} \in \mathbb{Z}. \]

Notice that this definition has been formulated in analogy to the expression of the reduced holonomy,

\[ \tilde{h}_{i}^{(v)}(v) := \exp \left( \int_{0}^{v} \nu^{(v)} ds \hat{A}_{i}^{v} \right) = e^{\nu_{v}^{(v)} z^{v}}. \]

Finally, actions of the lattice-regularized LQC variables in [56] and [57] read,

\[ \tilde{\hat{c}}^{(i)} |\mu_{v}(v) := - \frac{2}{\nu_{v}} \text{tr}(\hat{c}^{(i)}(v)) |\mu_{v}(v) = \frac{i}{\nu_{v}^{(i)}} \left( |\mu_{v}(v) - \nu_{v}^{(i)} \rangle - |\mu_{v}(v) + \nu_{v}^{(i)} \rangle \right) \]

and

\[ \tilde{\hat{p}}^{(i)} |\mu_{v}(v) := - i k \frac{\partial}{\partial \hat{c}^{(i)}} |\mu_{v}(v) = \frac{\nu_{v}^{(i)} k}{2} |\nu_{v}(v) \rangle, \]

respectively.

It is worth mentioning that the formalism of QRLG appears to be more complicated, with all derivations performed on the states of LQG, thus on Wigner matrices. The recent analysis of this model [11], however, suggests that correcting the reduction and quantizing the reduced phase space, one gets an analog of LQC with a simplified notation. Let us simplify that notation even more, replacing the reduced variables in [56] and [57] by

\[ \hat{A}_{a}^{i}(t) := A_{(a)}^{i}(t) 0 \epsilon_{a}^{i} = \frac{1}{\epsilon \epsilon^{(i)}(t)} \]

and

\[ \hat{E}_{i}^{a}(t) := E_{(i)}^{a}(t) \epsilon_{a}^{i} = \frac{1}{\epsilon^{2}} \hat{p}^{(i)}(t), \]

respectively, where \( \epsilon \) is the small regularization parameter. It has the meaning of the length of the edge of the (normalized) basic cell (the dashed cube in FIG. 2 or FIG. 3). Notice that the Poisson brackets related to the symplectic structure restricted to the basic cell having volume \( \epsilon^{3} \), remains unchanged when comparing to formula...
The main difference between the reduced variables \([67]\) and \([68]\) as preserving the original weights of Ashtekar variables.

The reduced holonomy becomes,

\[
h^{(i)}(v) = \exp \left( \int_0^\varepsilon ds \bar{A}^i_v \tau^i v_v(s) \right) = e^{\varepsilon \tau^i}.
\]  

(69)

Next, we define the link excitation states in analogy to \([63]\),

\[
|m_v^{(i)}⟩ := \exp \left[ im_v^{(i)} \epsilon^{(i)} \right], \quad 2m_v^{(i)} \in \mathbb{Z}.
\]  

(70)

Then, the actions of the lattice-regularized variables are

\[
\hat{c}^{(i)} |m_v^{(i)}⟩ := -\frac{2}{\varepsilon} \text{tr}(\tau^{(i)} \hat{h}^{(i)}) |m_v^{(i)}⟩ = \frac{i}{\varepsilon} \left( |m_v^{(i)} - \frac{1}{2}⟩ - |m_v^{(i)} + \frac{1}{2}⟩ \right)
\]  

(71)

and

\[
\hat{p}^{(i)} |m_v^{(i)}⟩ := -ik \frac{\partial}{\partial c^{(i)}} |m_v^{(i)}⟩ = m_v^{(i)} \hat{k} |m_v^{(i)}⟩.
\]  

(72)

This way, we obtained an analogue of LQC, directly related to LQG, not introducing any additional rescaling or redensitizing procedure. Consequently, \(m_v^{(i)}\) is related to the spin number \(j_v^{(i)}\) in LQG (introduced in Sec. III.1 and III.2) via the following equation,

\[
\hat{j}_v^{(i)} = |m_v^{(i)}⟩.
\]  

(73)

Finally, to center the basic cell states at nodes, thus to go from cubulation in FIG. 2 to the one in FIG. 3 we split all the link states as follows,

\[
|m_v^{(i)}⟩ = \exp \left[ im_v^{(i)} \epsilon^{(i)} \right] \exp \left[ i\bar{m}_v^{(i)} \epsilon^{(i)} \right] = |\bar{m}_v^{(i)}⟩ \otimes |\bar{m}_v^{(i)}⟩,
\]  

(74)

where the link \(l^{(i)}(v) : [0, 1] \rightarrow \Sigma_v\) started at the point \(v\) has been split in half,

\[
l^{(i)}(v) = \bar{l}^{(i)}(v) \left[ \bar{l}^{(i)}(v + \epsilon^{(i)}) \right]^{-1}.
\]  

(75)

Expression \(v \mp \epsilon^{(i)}\) labels the nearest node along the negatively/positively-oriented \(i\)-th direction. This way, two paths, \(\bar{l}^{(i)}(v)\) and \(\left[ \bar{l}^{(i)}(v + \epsilon^{(i)}) \right]^{-1}\), having the following properties,

\[
\begin{align*}
l^{(i)}(v)(0) &= \bar{l}^{(i)}(v)(0) = \left[ \bar{l}^{(i)}(v) \right]^{-1}(0),  \\
l^{(i)}(v)(1/2) &= \bar{l}^{(i)}(v)(1) = \left[ \bar{l}^{(i)}(v + \epsilon^{(i)}) \right]^{-1}(0),  \\
l^{(i)}(v)(1) &= \left[ \bar{l}^{(i)}(v + \epsilon^{(i)}) \right]^{-1}(1) = \bar{l}^{(i)}(v + \epsilon^{(i)})(0),
\end{align*}
\]  

(76)

have been created. As long as the link is a section and we factorize the lattice into cubes, the quantum numbers are split proportionally \(\bar{m}_v^{(i)} = \bar{m}_v^{(i)} + \epsilon^{(i)} = 1/2 m_v^{(i)}\). Let us remind that the non-trivially commuting elements of the quantized gravitational Hamiltonian \([40]\) are node-oriented volume operators acting on all the links emanated from a considered node (with the quantum number possibly modified by holonomy operators). This is why we define the averaged link states having the same symmetry as the volume operator,

\[
|\bar{m}_v^{(i)}⟩ := \frac{1}{2} |m_v^{(i)}⟩ - |m_v^{(i)}⟩ = |\bar{m}_v^{(i)}⟩ \otimes |\bar{m}_v^{(i)}⟩ = \exp \left[ \frac{i}{2} \left( m_v^{(i)} \epsilon^{(i)} + m_v^{(i)} \epsilon^{(i)} \right) \right] = \exp \left[ i\bar{m}_v^{(i)} \epsilon^{(i)} \right].
\]  

(77)

\(^1\) Precisely, one can define a number density — from \([72]\) we know that the quantum numbers correspond to momenta — and integrate it along the section of length \(\varepsilon\).
The lattice-regularized canonical variables have the following actions on the normalized states,

\[
\hat{c}^{(i)} |\bar{m}^{(i)}_v\rangle = -\frac{2}{\varepsilon} \text{tr} \left( \tau^{(i)} h^{(i)}_{\bar{g}} \right) \left( |\bar{m}^{(i)}_v\rangle \otimes |\bar{m}^{(i)}_v\rangle \right) = \frac{i}{\varepsilon} \left( |\bar{m}^{(i)}_{v-e^{(i)}}, -\varepsilon\rangle - |\bar{m}^{(i)}_{v-e^{(i)}}, +\varepsilon\rangle \right)
\]

(78)

and

\[
\hat{p}^{(i)} |\bar{m}^{(i)}_v\rangle = \bar{m}^{(i)}_v k |\bar{m}^{(i)}_v\rangle.
\]

(79)

Notice that to derive (78), one needs to act holonomy operator on whole links. In other words, the action on a half-link restricts the standard holonomy to a holonomy along a half-link,

\[
h^{(i)} = e^{\varepsilon A_v^{(i)} \phi_v^{(i)}} \rightarrow \frac{1}{2} h^{(i)} := e^{\varepsilon A_v^{(i)} \phi_v^{(i)}} = e^{\frac{1}{2} \varepsilon c^i}.
\]

(80)

The coherent states analogous to (61) (and similar to the one chosen to define the semiclassical limit in QRLG [40]) are given by the formula,

\[
|\bar{c}^{(i)}\rangle := \sum_\nu \sum_i^3 \left[ \left( \bar{c}^{(i)}(A) |\bar{c}^{(i)}(A)\rangle \right)^{-\frac{1}{2}} \langle \bar{c}^{(i)}(A) | \bar{c}^{(i)}(A) \right) \otimes \left( \bar{c}^{(i)}(A) |\bar{c}^{(i)}(A)\rangle \right)^{-\frac{1}{2}} \langle \bar{c}^{(i)}(A) | \bar{c}^{(i)}(A) \right].
\]

(81)

The corresponding link-oriented shadow coherent state becomes,

\[
|\bar{c}^{(i)}_v(A)\rangle := \sum_{\bar{m}^{(i)}_v} \exp \left[ -\frac{1}{2d^2} \left( \bar{m}^{(i)}_v - \frac{p^{(i)}_v}{k} \right)^2 \right] \exp \left[ -i \left( \bar{m}^{(i)}_v - \frac{p^{(i)}_v}{k} \right) c^{(i)}_v \right] |\bar{m}^{(i)}_v\rangle.
\]

(82)

This procedure can be also normalized to adjust to the node states in (77). The normalized, node-centered coherent states are given then by the expression,

\[
|\bar{c}^{(i)}\rangle := \sum_\nu \sum_i^3 \left[ \left( \bar{c}^{(i)}(A) |\bar{c}^{(i)}(A)\rangle \right)^{-\frac{1}{2}} \langle \bar{c}^{(i)}(A) | \bar{c}^{(i)}(A) \right),
\]

(83)

where the link-oriented, node-centered shadow state becomes,

\[
|\bar{c}^{(i)}_v(A)\rangle := \sum_{\bar{m}^{(i)}_v} \exp \left[ -\frac{1}{2d^2} \left( \bar{m}^{(i)}_v - \frac{p^{(i)}_v}{k} \right)^2 \right] \exp \left[ -i \left( \bar{m}^{(i)}_v - \frac{p^{(i)}_v}{k} \right) c^{(i)}_v \right] |\bar{m}^{(i)}_v\rangle.
\]

(84)

It is worth noting that both LQC and our simplified, node-symmetrized model lead to the same expectation values of the canonical operators. In the reduced case, one gets

\[
\langle \bar{c}^{(i)}(A) | \hat{c}^{(i)} | \bar{c}^{(i)}(A)\rangle = \langle \bar{c}^{(i)}_v(A) | \hat{c}^{(i)}_v(A) | \bar{c}^{(i)}_v(A)\rangle = \frac{2}{\varepsilon} \exp \left[ -\left( \frac{\varepsilon}{2d^2} \right)^2 \sum_{\bar{m}^{(i)}_v} \exp \left[ -\frac{1}{d^2} \left( \bar{m}^{(i)}_v - \frac{p^{(i)}_v}{k} \right)^2 \right] \sin \left[ \frac{\varepsilon}{2} c^{(i)}_v + i\varepsilon \left( \frac{p^{(i)}_v}{k} - \bar{m}^{(i)}_v \right) \right] \right],
\]

(85)

while to obtain analogous result for LQC one just need to do the following replacement,

\[
\bar{m}^{(i)}_v \rightarrow \frac{1}{2} \mu^{(i)}_v.
\]

(86)

Substituting the appropriate correspondence principle,

\[
\bar{m}^{(i)}_v \rightarrow \frac{p^{(i)}_v}{k},
\]

(87)

the result can be recast in the simple form,

\[
\langle \hat{c}^{(i)} \rangle = c^{(i)} (1 + O(\varepsilon^2)).
\]

(88)

Analogously, the expectation value of the reduced flux operator is,

\[
\langle \hat{p}^{(i)} \rangle = p^{(i)}.
\]

(89)

This way, we introduced the framework very similar to LQC (also related to QRLG and CCQG). It can be easier compared to LQG due to the same densities of canonical variables and due to the possibility of introducing inhomogeneities in the form of different quantum numbers. The latter has been achieved constructing node-centered single cell states, visualized by FIG. 3.
IV. QUANTUM CORRECTIONS

IV.1. Ehrenfest theorem

In the previous section, we demonstrated how the structure of the coherent states removes differences in the semi-classical results coming from different cosmological models. Therefore, from now on, we generalize the analysis, considering the matrix elements taken with respect to these coherent states, not necessarily specifying the model. Precisely, the results below are true for LQC, QRLG, CCGR, and similar theories, thus most clearly expressible in our simplified framework. Consequently, dynamics of the whole system is captured in the set of equations resulting from the Ehrenfest theorem,

$$\frac{d\langle c \rangle^{(gr)}}{dt} = \frac{1}{\hbar} \left\langle [\hat{c}, \hat{H}^{(gr)}] \right\rangle^{(gr)} + \frac{1}{\hbar} \left\langle [\hat{c}, \hat{H}^{(matt)}] \right\rangle^{(gr)} = \frac{1}{\hbar} \left\langle [\hat{c}, \hat{H}^{(gr)}] \right\rangle^{(gr)} + \Delta_c^{(matt)}, \quad (90)$$

$$\frac{dp}{dt} = \frac{1}{\hbar} \left\langle [\hat{p}, \hat{H}^{(gr)}] \right\rangle^{(gr)}, \quad (91)$$

$$\frac{d\langle \phi \rangle^{(matt)}}{dt} = \frac{1}{\hbar} \left\langle [\phi, \hat{H}^{(matt)}] \right\rangle^{(matt)} = \frac{1}{\hbar} \left\langle [\phi, \hat{H}^{(matt)}] \right\rangle^{(matt)} + \Delta_{\phi}^{(gr)}. \quad (92)$$

Here, likewise in (60), the symbol $\phi$ represents any matter field (precisely the canonical field variable or its corresponding conjugate momentum). To simplify the equations, the indices labeling directions, and the position of operators were omitted. We also assumed that all the gravitational and matter fields have no explicit time dependence and their evolution is encoded only in the equations of motion.

The quantum generally-relativistic corrections both in (90) and (92) are of the same order in the inverse spin expansion,

$$\Delta_c^{(matt)} \propto \Delta_{\phi}^{(gr)} \propto \frac{1}{m^2}. \quad (93)$$

Notice that for simplicity we assumed here the large quantum number approximation$^1$

$$|\tilde{m}| \gg 1. \quad (94)$$

The second type of modifications to classical dynamics comes from the quantum-gravitational corrections $\delta_c$, $\delta_c$, and $\delta_p$, which arise from the terms

$$\frac{d\langle c \rangle^{(gr)}}{dt} = \frac{1}{\hbar} \frac{dc}{dt} (1 + \delta_c), \quad (95)$$

$$\left\langle [\hat{c}, \hat{H}^{(gr)}] \right\rangle^{(gr)} = i\hbar \frac{\delta H^{(gr)}}{\delta p} (1 + \delta_c), \quad (96)$$

and

$$\left\langle [\hat{p}, \hat{H}^{(gr)}] \right\rangle^{(gr)} = -i\hbar \frac{\delta H^{(gr)}}{\delta c} (1 + \delta_p), \quad (97)$$

respectively. They have a qualitatively different structure than the quantum corrections in (93),

$$\delta_c \propto \delta_c \propto \delta_p \propto \varepsilon^2. \quad (98)$$

Another difference between these corrections is in the fact that the gravitational corrections are functionals of the connection, $\delta_c = \delta_c[c]$, $\delta_c = \delta_c[c]$, $\delta_p = \delta_p[c]$, while the generally-relativistic ones depend only on quantum numbers (thus indirectly on the reduced flux via the correspondence principle in (87)). This can be written as,

$$\frac{\partial}{\partial c} \Delta_c^{(matt)} = \frac{\partial}{\partial c} \Delta_{\phi}^{(gr)} = 0. \quad (99)$$

$^1$ This approximation relates the single fiducial cell formula with the Hamiltonian on $\Sigma_c$ in the continuum limit$^2$. Generalization to any value of $\tilde{m}$ is possible, but it requires a redefinition of the coherent states. Heuristically, this is done replacing $\tilde{m}$ with $\tilde{m}_c := \tilde{m}/\varepsilon$ in the definition (77). Precise formulation of this method will be presented in forthcoming articles.
Notice also that neglecting the evolution of the gravitational degrees of freedom in (92), one finds

\[
\frac{d\langle \Phi \rangle^{(\text{matt})}}{dt} = \frac{1}{\hbar} \langle [\hat{\Phi}, \hat{H}^{(\text{matt})}] \rangle^{(\text{matt})}, \quad \Delta^{(\text{gr})} = 0, \tag{100}
\]

which is the Ehrenfest theorem for the lattice-regularized QFT on classical curved spacetime.

The structure of the quantum generally-relativistic corrections in (93) is the key element of our article. It is the only type of corrections dependent on the gravitational degrees of freedom that appears in the matter sector. We will show how this structure verifies the general covariance of the quantum perturbations and the quantum nature of the gravitational interactions.

The gravitational coupling to the matter sector in GR, is implemented by multiplication by factors \( q^{\pm 1/2} \) and/or contraction with the metric tensor, \( q_{ab} \) — see expressions (11) and (31). In the lattice-regularized contributions to the Hamiltonian, \( [43] \) and \( [46] \), this is done (both in CQGR and in the cosmological models) by the formula

\[
\text{tr} \left( \tau_h^{-1}(v) \{ V^n(v), h_{(d)}(v) \} \right), \quad n \in \mathbb{Q}_+. \tag{101}
\]

At the quantum level, this becomes the trace of the product of the \( su(2) \) generator and the operators in (39). Notice that the exact structure of the gravitational coupling (i.e. the number of terms proportional to (101) and the power of the volume operator) varies for different matter fields. Moreover, even in the same Hamiltonian, different elements can be coupled to different objects — compare (47), (48) and (49). Let us rewrite then the expression in (92) into a more specific form, which captures these differences.

We begin pointing the following relation,

\[
\left\langle \left[ \hat{\Phi}, \hat{H}^{(\text{matt})} \right] \right\rangle^{(\text{matt})}^{(\text{gr})} = \left\langle \hat{\Phi}, \left\langle \hat{H}^{(\text{matt})} \right\rangle^{(\text{gr})} \right\rangle^{(\text{matt})}. \tag{102}
\]

Therefore, the structure of \( \Delta^{(\text{gr})} \) depends only on the matrix element \( \left\langle \hat{H}^{(\text{matt})} \right\rangle^{(\text{gr})} \). Let us split then the matter sector’s Hamiltonian into contributions from different fields \( \Phi_a \)

\[
H^{(\text{matt})} = \sum_\alpha H^{(\Phi_\alpha)} = \sum_\alpha \left( H^{(\Phi_\alpha)}_{\text{one}} + H^{(\Phi_\alpha)}_{\text{two}} + \ldots \right) =: \sum_\alpha \left( \sum_{\text{elements}} H^{(\Phi_\alpha)} \right). \tag{103}
\]

The second splitting in the formula above introduced the terms \( H^{(\Phi_\alpha)}_{\text{one}}, H^{(\Phi_\alpha)}_{\text{two}}, \ldots \), labeling different elements of the Hamiltonian of a field \( \Phi_\alpha \). For instance, the Hamiltonian of \( \Phi_\Delta \) decomposes as follows, \( H^{(\Delta)} = H^{(\Phi_\Delta)}_{\text{elec}} + H^{(\Phi_\Delta)}_{\text{matt}} \).

Taking a matrix element is a linear operation, thus without loss of generality, we can focus on a single element,

\[
\left\langle \hat{H}^{(\Phi_\alpha)}_{\text{element}} \right\rangle^{(\text{gr})} = H^{(\Phi_\alpha)}_{\text{element}} \left( 1 + \delta^{(\Phi_\alpha)}_{\text{element}} + \delta^{(\Phi_\alpha)}_{\text{element}} + \ldots \right), \tag{104}
\]

where \( \delta^{(\Phi_\alpha)}_{\text{element}} \propto 1/m^2 \), \( \delta^{(\Phi_\alpha)}_{\text{element}} \propto 1/m^4 \), etc. For simplicity, let us neglect the terms of order \( 1/m^4 \) and smaller. Consequently, the quantum generally-relativistic corrections to the matter sector can be expressed as follows,

\[
\Delta^{(\text{gr})} = \frac{1}{\hbar} \sum_\alpha \sum_{\text{elements}} \left\langle \hat{\Phi}_\alpha, \hat{H}^{(\Phi_\alpha)}_{\text{element}} \right\rangle^{(\text{matt})} \delta^{(\Phi_\alpha)}_{\text{element}}, \tag{105}
\]

where linearity of a commutator has been applied. Finally, let us point out that when substituting the correspondence principle from (87), the corrections take explicit dependence on the dreibein, thus on the metric tensor. In the case of the vector field in the cosmological framework, this can be read from the expression,

\[
\left\langle \left[ \hat{\Phi}, \hat{H}^{(\Delta)} \right] \right\rangle^{(\Delta)} = \sum_\alpha \left\langle \left[ \hat{\Phi}, \hat{H}^{(\Delta)} \right] \right\rangle^{(\Delta)} = \sum_\alpha \left\langle \hat{\Phi}, \hat{H}^{(\Delta)} \right\rangle^{(\Delta)} \left( 1 + \delta^{(\Delta)}_{\langle a \rangle} \right), \tag{106}
\]

\[\text{In this general approach,} \Phi_\alpha \text{ represents any matter field. For instance, in our simplified cosmological model with bosonic matter (with the Hilbert space given in (50)), we have } \Phi_\Delta := \Delta \text{ and } \Phi_\varphi := \varphi.\]
where
\[ \bar{\delta}^{(\Delta)}_{(a)} \propto \frac{k^2}{(p(a))^2}. \] (107)

The form of the outcome above reflects the symmetry between regularized elements of the Hamiltonian in [44] and [45]. The analogous matrix element for the scalar field leads to the unsymmetrical (with respect to the metric tensor) result,
\[ \left\langle \left[ \hat{\phi}, \hat{H}^{(\phi)} \right] \right\rangle^{(\phi)} = \left\langle \left[ \hat{\phi}, \hat{H}^{(m)}_{\text{mom}} \right] \right\rangle^{(\phi)} (1 + \delta^{(\phi)}_{\text{mom}}) + \sum_{a} \left\langle \left[ \hat{\phi}, \hat{H}^{(\phi)}_{(a),\text{der}} \right] \right\rangle^{(\phi)} (1 + \delta^{(\phi)}_{(a),\text{der}}) + \left\langle \left[ \hat{\phi}, \hat{H}^{(\phi)}_{\text{pot}} \right] \right\rangle^{(\phi)}, \] (108)

where
\[ \delta^{(\phi)}_{\text{mom}} \propto \sum_{a} \frac{k^2}{(p(a))^2} \] (109)
and
\[ \delta^{(\phi)}_{(a),\text{der}} \propto \sum_{b \neq a} \frac{k^2}{(p(b))^2}. \] (110)

Notice that the correction \( k^2/(p(a))^2 \) corresponds to the scale of the single cell (we operate in the simplified framework) — compare variables (57) and (68), or holonomies (64) and (69). In other words, asking heuristically for the scale at which spacetime is discrete and setting the SI unit system value of the Planck length, \( l_P \approx 1.62 \times 10^{-35} \) m, \( \varepsilon \) takes value \( \approx 1 \) m.

### IV.2. General quantum relativity

In the previous subsection, we presented the structure of the lattice-regularized Hamiltonian of Einstein’s gravity coupled with the scalar and vector field. Considering its simplified formulation in the cosmological framework is enough to capture the LQG-originated modifications to general covariance of coordinate transformations for all the bosonic fields in the Standard Model. We aim to apply these results to inspect the generally-relativistic aspects of a candidate for the theory of all fundamental interactions.

We begin reminding a few methodological facts about the theories, being the reference for the construction of CQGR. QFT is the framework describing features of matter, which are contradictory with the interpretation of physics, being based on classical (i.e. not quantum) models. This general theory assumes (by definition) the existence of objects beyond the measurable and observable world. We are not going to speculate about the ontological nature of states and operators, which is a philosophical problem. Let us, however, point out what consists of the epistemological site of QFT, which is definitely a physical question. From the perspective of the astronomy and experimental physics, we can only ‘sense’ observables. Therefore, any issue related to an observer is at the same time related to the observables (even if the major part of formal derivations connected to the problem is expressed in terms of operators and states).

GR is the theory of this perspective — the epistemological theory of relative observers. Let us recall the [strong] formulation of EP.

The outcome of any local, nongravitational [or gravitational] test experiment is independent of the experimental apparatus’ velocity relative to the gravitational field and is independent of where and when in the gravitational field the experiment is performed.\(^1\)

\(^1\) More precisely, rewriting our simplified framework to the one elaborated for LQC [5], the following relation holds, \( \varepsilon = a l_0 \), where \( a \) is the cosmological scale factor, while \( l_0 \) is the fiducial length of the edge of the single cell. Setting the cosmological parameters for today at \( a_0 = q_{ab} = q = 1 \) (the flat space approximation), the minimal length of the single cell’s edge takes value \( l_0 = \sqrt{4\gamma l_P} \approx 5.73 \gamma \times 10^{-35} \) m, and this corresponds to the spin number \( \hat{m} = \pm 1/2 \). Conversely, setting the SI unit system length as the reference fiducial length, \( l_0 = 1 \) m, the spin number, expressing the value of the quantum corrections here and in Sec IV.3 in formulas (118), (119) and (120), takes value \( \hat{m} \approx \pm 1.52 \gamma^{-1} \times 10^{68} \). This indeed justifies that the large spin approximation, which we assumed in [54], while constructing the coherent states, is correct. Finally, let us emphasize that this result, being dependent on the unfixed value of the Immirzi parameter \( \gamma \), cannot be considered yet as a definite prediction of this model.

\(^\text{II}\) The Einstein EP, cited after [15], is based on the original Einstein’s article [18]. To formulate the so-called strong variant of EP, one needs to include the phrase in [ ] brackets.
Notice that this statement is directly related to the “outcome of experiment”, thus from the perspective of QFT, to the observable. This essential point should remind us that GR is more than a description of gravitation. Its field-theoretical formulation has to be consistent with EP, unless someone is looking for a different theory of gravity.

Finally, let us consider the hypothetical model of quantum GR that we understood as the fusion of QFT and GR. In the case of CQGR, this fusion imposes the list of restrictions on the formulation of this postulated theory. We selected here the most significant requirements for a well-defined CQGR candidate.

a) [QFT] Quantization is performed in a canonical procedure.

b) [QFT] Semiclassical limit reproduces measurable observables.

c) [QFT/GR] Gauge symmetries are preserved.

d) [GR] Equivalence principle is satisfied.

e) [GR] Theory is background independent.

Let us verify if any variant of LQG-based frameworks satisfies these restrictions, or at least, if it could be properly reformulated.

First, notice that the phenomenological models discussed in Sec. II.5 violate requirements a) and e), introducing multiple quantizations (even for the degrees of freedom related to the same field), with the semiclassical limit of one quantum theory playing the role of the background for another theory. This excludes these models from being a candidate for a cosmological limit of CQGR.

In our opinion, there is only one completely formulated candidate for CQGR, described in detail in [2]. This theory satisfies conditions a), b), and e), however, the restrictions c) and d) are cross-violated when concerning a single problem. The semiclassical limit of the matter sector derived with respect to the gravitational degrees of freedom is not generally covariant. Assuming the diffeomorphism invariance, only the classical limit can be considered as an observable, while the semiclassical corrections would have to be neglected. Assuming EP (either in Einstein or strong variant), the semiclassical limit (being an observable) would weaken general covariance that would be only...
approximately satisfied. We are going to demonstrate these statements using the cosmological framework, which we elaborated in the previous sections. However, before we do that, let us explain the physical meaning of the ‘QFT-dominated’ and ‘GR-dominated’ approximations of CQGR.

In order to describe the physical meaning of CQGR, we need to discuss separately how to understand taking the matrix element of HCO only with respect to the matter or gravitational degrees of freedom. The meaning of the low energy (semiclassical) limit of quantum matter excitations, is the classical matter fields’ theory coupled to LQG, while the meaning of the low energy (semi-flat) limit of quantum geometry excitations, is QFT on classical curved spacetime. This reminds us that verifying both the classical $\hbar \to 0$ limit and the flat $G \to 0$ limit, is the necessary, but not sufficient condition for a well-defined theory. We can illustrate this issue, generalizing the famous picture of the Bronstein cube [49], drawn in FIG. 4.

The classical and flat limit of CQGR is $l_P \to 0$. As long as we do not have any direct access to the Planck scale, we can try to look for indirect effects in the semiclassical or semi-flat limits. From the formal perspective we should ask how the theory behaves at the quantum-relativistic (length) scale $L \lesssim l_P$, when $\hbar \gtrsim 0$ or $G \gtrsim 0$. To visualize this issue, let us focus only on the plane parallel to the front face ($c = 1$) of the Bronstein cube in FIG. 4 capturing both interesting us limits, $\hbar \to 0$ and $G \to 0$. It is easy to understand that the Bronstein’s scheme is very simple, and the aspects involving quantum GR are not placed only at one point, at the upper right corner of the $c = 1$ face. To look for the possibly observational effects of quantum GR, we should investigate all the regions of this face. Notice that fixing the scale of the approximation at length $L$, satisfying $l_P \gtrsim L = \text{constant}$, is represented by a hyperbola. Let us call this curve the ‘Planck hyperbola’ — we visualized this idea in FIG. 5.

![FIG. 5. Planck hyperbola, $l_P = \text{constant}$](image)

It is clear that the semiclassical/semi-flat quantum relativistic corrections can be approximated by three different perturbative theories, depending on the significance of the geometric or energetic contribution to perturbations. This can be represented by the scheme below.

$$
\begin{align*}
\text{quantum field theory} & \quad \text{on curved spacetime} \\
\text{special relativity} & \quad \text{on quantum geometry} \\
\text{quantum general relativity} & \quad \text{on quantum geometry}
\end{align*}
$$

QFT on curved spacetime (see [50] for a general introduction) is a well-established theory. From the cosmological perspective, elements of this model are applied to explain details of the inflation (introduced in [51]) — being the late
phase of the evolution of the universe before the Standard Model matter is formed, associated with the mechanism of this formation (called reheating — see [52, 54]).

The classical field theory on quantum geometry is an approximation of the early phase of the first stage of the evolution of the universe. This toy model [55, 56], assuming homogeneous and isotropic mixture of quantum geometry and classical matter, explains how the initial singularity (present in the theories related to classical geometry) is resolved by a bouncing scenario.

Concluding, we shown that both the semiclassical and semi-flat limit should not be understood only as a formal method of taking the matrix elements in expressions (90), (91) and (92). Both approximations have direct cosmological applications. Therefore, a well-defined model of CQGR should explain how these approximations are formally derived, in a way that satisfies the restrictions in [54–56].

IV.3. Violation of general covariance

Having constructed the framework and discussed the physical meanings of the models approximating CQGR, we are ready to show explicitly which constructions appear to be violating the laws of this theory.

To find the exact form of the quantum generally-relativistic corrections in expressions (106) and (108), we have to replace the connection in (111), this expression after quantization takes the form,

$$\text{tr} \left( \tau^{\hat{n}} \hat{h}^{-1}_{(\hat{n})} \left[ \hat{V}^{n}, \hat{h}_{(\hat{n})} \right] \right) = \sin \left( \frac{\hat{c}_{(\hat{n})}}{2} \right) \hat{V}^{n} \cos \left( \frac{\hat{c}_{(\hat{n})}}{2} \right) - \cos \left( \frac{\hat{c}_{(\hat{n})}}{2} \right) \hat{V}^{n} \sin \left( \frac{\hat{c}_{(\hat{n})}}{2} \right).$$  (111)

Notice that we kept here the general spatial direction, not projected yet on the Euclidean frame. The vector labeling over the connection in $\hat{c}_{(\hat{n})}$ is related to the direction-independent series representation of the cosine operator functional,

$$\cos \left( \frac{\hat{c}_{(\hat{n})}}{2} \right) = \sum_{k=0}^{\infty} (-1)^{k} \frac{(\hat{c}_{(\hat{n})} \hat{c}_{(\hat{n})})^{k}}{(2k)!}.  (112)$$

To derive the action of the operator in (111), one first needs to find the action of the volume operator. Considering the cubic cell having volume $\varepsilon^{3}$, the classical expression, in terms of simplified momenta in (68), reads,

$$\hat{V} := \frac{1}{\varepsilon^{3}} \int_{0}^{\varepsilon^{3}} d^{3}x \sqrt{p^{1}p^{2}p^{3}} = \sqrt{p^{1}p^{2}p^{3}} = \varepsilon^{3} \sqrt{\hat{q}}.  (113)$$

To avoid problems with the square root of operators after quantization, let us expand the volume operator around the coherent state, finding

$$\hat{V}^{n} = \varepsilon^{3n} \left( \langle \hat{q} \rangle \right)^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{n/2}{k} \left( \frac{\hat{q}}{\langle \hat{q} \rangle} \right)^{k},  (114)$$

where the expectation value of the $\hat{q}$ operator reads,

$$\langle \hat{q} \rangle = k^{3} \bar{m}^{1} \bar{m}^{2} \bar{m}^{3}.  (115)$$

As a next step, we derive the action of operator in (114) on the states modified by the exponentiated connection in (111) (which were already introduced in (71)), finding

$$\hat{V}^{n} \left| \bar{m}^{(i)}_v \pm \frac{1}{2} \right> \left( k^{3} \bar{m}^{1} \bar{m}^{2} \bar{m}^{3} \right)^{\frac{n}{2}} \left[ 1 \pm \frac{n}{4} \bar{m}^{(i)}_v + \frac{n(n-2)}{2^{5}} \left( \frac{1}{\bar{m}^{(i)}_v} \right)^{2} + \frac{n(n-2)(n-4)}{2^{7}} \left( \frac{1}{\bar{m}^{(i)}_v} \right)^{3} + O \left( \frac{1}{\bar{m}^{(i)}_v} \right)^{4} \right].  (116)$$

Now, it is easy to calculate the action of the quantum generally-relativistic corrections-generating operator,

$$\text{tr} \left( \tau^{(i)} \hat{h}_{(j)}^{-1} \left[ \hat{V}^{n}, \hat{h}_{(j)} \right] \right) = \left( k^{3} \bar{m}^{1} \bar{m}^{2} \bar{m}^{3} \right)^{\frac{n}{2}} \delta^{(i)}_{(j)} \left[ 1 + \frac{n^{2} - 6n + 8}{2^{5}} \frac{1}{\bar{m}^{(i)}_v} + O \left( \frac{1}{\bar{m}^{(i)}_v} \right)^{3} \right].  (117)$$
Consequently, we find the explicit value of the dimensionless corrections in (107), (109), and (110), being
\[ \delta^{(\Delta)}_{(i)} = \frac{7}{2^6} \frac{1}{(m_v(t))^{2}} \rightarrow 7\pi^{2}g^{\alpha}_{\beta} \frac{q(t)}{\xi^{4}} \frac{q(t)}{q}, \]
\[ \delta^{(\varphi)}_{\text{mom}} = \frac{7}{2^6} \sum_{i} \frac{1}{(m_v(t))^{2}} = \sum_{i} \delta^{(\Delta)}_{(i)}, \]
and
\[ \delta^{(\varphi)}_{(i),\text{der}} = \frac{65}{2^8} \sum_{j \neq i} \frac{1}{(m_v(t))^{2}} = \frac{65}{84} \sum_{j \neq i} \delta^{(\Delta)}_{(i)}, \]
respectively. Notice that the right-hand side of formula (115) expresses taking the correspondence principle, introduced in (87). The meaning of ratio \( l_{P} / \xi \) was explained in the comment at the end of Sec. IV.1 and in the related footnote.

Let us finally explain how the structure of the quantum generally-relativistic corrections reveals the violation or respecting of general covariance. As we discussed in Sec. IV.2, the GR site of CQGR imposes EP on, at least, all the observables related to matter degrees of freedom. In the formalism of GR and consequently CQGR, this is encoded in general covariance. We begin discussing this in the context of the vector field.

The vector field observables are the expectation values of \( \mathbf{E}^a \) and \( \mathbf{B}^a \). In the case of electromagnetism on curved spacetime, these are the well-known electric vector and magnetic pseudovector densities — being the physical modes of the electromagnetic wave. Another observable is the expectation value of HCO — the scalar density representing the energy density \( \mathbf{E}^a \) of the electromagnetic field. This quantity in our simplified cosmological framework takes the form,
\[ \langle \hat{H}^{(\Delta)} \rangle = \sum_{i} \langle \hat{H}^{(\Delta)}_{(i)} \rangle = \sum_{i} \langle \hat{H}^{(\Delta)}_{(i)} \rangle (1 + \delta^{(\Delta)}_{(i)}). \]
It is clear that the above modification of general covariance, encoded in the contraction with the metric tensor, is the same in the case of all the observables. Moreover, this modification is identical concerning dynamics — see (106).

Therefore, although the quantum generally-relativistic corrections change the diffeomorphism symmetry encoded in the contraction with the metric tensor (or in the case of Ashtekar variables, with the densitized dreibein), the relative local covariance for both physical quantities, \( \mathbf{E}^a \) and \( \mathbf{B}^a \), is the same.

Considering then the quantum system of the vector field and gravity, it would be beneficial to have a procedure that allows to restore the classical structure of general covariance. This would allow us to apply methods of QFT on curved spacetime to the expectation value of HCO taken only for gravitational degrees of freedom. Notice that since the Hamiltonian density of the system is not a measurable quantity, it is specified up to a constant and usually set to zero. Let us keep this convention. The ‘covarianization’ procedure can be then defined as the multiplication of the elements of the Hamiltonian by the inverse of \( (1 + \delta^{(\Delta)}_{(i)}) \). This way, the corrections violating general covariance are moved to the gravitational site of the Hamiltonian, resulting in the new ‘covarianized’ system representing the Einstein EP at quantum level. Notice, however, that on the one hand, this procedure would deform locally the diffeomorphism symmetry. On the other hand, the action of this deformation would not be breaking the structure of the Hamiltonian without any reaction. The corresponding local deformations would modify accordingly the gravitational sector of the Hamiltonian.

In the case of the scalar field, the situation is completely different. First, it is not clear which objects, except the expectation value of HCO, are the observables carrying a physical meaning. Second, the variable \( \pi \) is a pseudoscalar density, but \( \partial_{\nu} \varphi \) and \( \varphi \) are a one-form and a scalar, respectively. Therefore, the introduction of some ‘redensitization’ procedure with respect to the scalar field’s degrees of freedom is needed. Finally, let us see how the analog of (121)

\[ \langle \hat{H}^{(\varphi)} \rangle = \sum_{i} \langle \hat{H}^{(\varphi)}_{(i)} \rangle_{\text{mom}} + \langle \hat{H}^{(\varphi)}_{(i)} \rangle_{\text{der}} + \langle \hat{H}^{(\varphi)}_{(i)} \rangle_{\text{pot}} = \sum_{i} \langle \hat{H}^{(\varphi)}_{(i)} \rangle_{\text{mom}} (1 + 2\delta^{(\Delta)}_{(i)}) \]
\[ + \langle \hat{H}^{(\varphi)}_{(i)} \rangle_{\text{der}} (1 - \delta^{(\Delta)}_{(i)} + \frac{65}{84} \sum_{j \neq i} \delta^{(\Delta)}_{(j)}) + \langle \hat{H}^{(\varphi)}_{(i)} \rangle_{\text{pot}} (1 - \delta^{(\Delta)}_{(i)}) (1 + \delta^{(\Delta)}_{(i)}) + O\left(\left(\delta^{(\Delta)}_{(i)}\right)^{2}\right), \]

1 Notice that an observable does not always correspond to an explicitly measurable quantity. In the case of the expectation value of HCO, the observable is the energy density, while the measurable quantity is the difference between the energy densities related to different points or regions in spacetime.

11 We already discussed this issue in Sec. IV.2. See also [2, 20].
where \( \langle H^{(\varphi)}_{1\text{mom}} \rangle = \langle H^{(\varphi)}_{2\text{mom}} \rangle = \langle H^{(\varphi)}_{3\text{mom}} \rangle = \frac{1}{3} \langle \hat{H}^{(\varphi)} \rangle \) and \( \langle H^{(\varphi)}_{1\text{pot}} \rangle = \langle H^{(\varphi)}_{2\text{pot}} \rangle = \langle H^{(\varphi)}_{3\text{pot}} \rangle = \frac{1}{3} \langle \hat{H}^{(\varphi)} \rangle \). As in the case of any matter field, the structure of the quantum generally-relativistic corrections is the same for both the expectation value of HCO (above) and the semiclassical dynamics encoded in the Ehrenfest theorem (in \( \langle 108 \rangle \)). It is clear that the regularization-originated modifications to the diffeomorphism symmetry at the level of quantum corrections are different for different terms of the Hamiltonian of the scalar field. Even neglecting selfinteractions, the relative local covariance for both the momentum sector and the derivative sector is deformed, and thus the Einstein EP is violated at the quantum level. Consequently, considering the system of the scalar field and gravity, the covarianization procedure would not resolve the problem.

### IV.4. Covarianization

Let us finalize this article with a general remark concerning the procedure that we called the covarianization. It is based on the observation that HCO for the matter degrees of freedom expressed in terms of the variables being tensor densities that correspond to physically measurable quantities and being smeared along lattice links, is scaleable to an explicitly generally covariant form. This procedure reflects the idea to construct CQGR that satisfies all laws of GR, in other words, to improve the Thiemann’s regularization procedure with a method restoring the Einstein EP at the level of the quantum generally-relativistic corrections. The strong EP, however, would be anyhow violated both before and after scaling the Hamiltonian.

The covarianization method starts fixing the total energy of a considered system at zero, setting

\[
H = 0. \tag{123}
\]

In the case of the cosmological frameworks that are approximable by our simplified bosonic system on the cubic lattice, the Hamiltonian can be expressed as the following sum,

\[
\hat{H} = \sum_{i} \left( \hat{H}_{(i)}^{(g)} + \hat{H}_{(i)}^{(A)} + \hat{H}_{(i)}^{(\varphi)} \right) =: \sum_{i} \hat{H}_{(i)}. \tag{124}
\]

Let us assume that the scalar field is represented by the isotropic Proca Hamiltonian. The splitting of the matter sector into three orthogonal directions has been already discussed in the previous section. In the case of gravitational Hamiltonian, this a known result, explained for instance in \( \langle 10 \rangle \). Considering, as in our article, only the next to the leading order corrections, the covarianization is defined as follows,

\[
\hat{H}_{(i)} \overset{\text{covars}}{\rightarrow} \hat{H}_{(i)} \left( 1 + \delta_{(i)}^{(A)} \right)^{-1}. \tag{125}
\]

Notice two facts related to the scalar field’s sector. First, to recover general covariance at the quantum level via the procedure in \( \langle 125 \rangle \), the analogous Hamiltonian for the massless field must have the same structure as the Hamiltonian for the vector field. Second, the mass term, or any other additional potential becomes shifted down by the factor \( \left( 1 - \sum_{i} \delta_{(i)}^{(A)} \right) \). This prediction has significant theoretical consequences for the dynamics of the earliest stage of the universe described by the quantum generally-relativistic model with the massive inflaton field.

Finally, one can ask if the idea of the covarianization can be generalized to all fundamental fields, including fermions, and any ADM spacetime symmetries. Let us first discuss the second issue, which also related to the question of whether this procedure is entirely correct. Notice that the modification of the Hamiltonian in \( \langle 125 \rangle \) is metric-dependent. Therefore, it is a local procedure. This is also directly visible through the locality of the node-related operator in \( \langle 117 \rangle \). Therefore, the fundamental issue concerning this method is whether the local modification of the Hamiltonian is allowed. In our opinion, the answer to this question is rather positive.

The arguments supporting the correctness of the covarianization are the following. This procedure introduces local deformations to the diffeomorphism symmetry two orders below the leading term of the action of the quantum generally-relativistic corrections-generating operator — see \( \langle 116 \rangle \). This is exactly the same order of modifications, as the modifications of the classical Hamiltonian introduced by the lattice regularization through the approximation in \( \langle 27 \rangle \) leading to formula \( \langle 101 \rangle \). Therefore, the covarianization, if introduced during the regularization step, can

\[\text{It is worth mentioning that the equation of motion for the Proca field with the associated Lorenz gauge condition (being a Lorentz invariant condition) is the Klein-Gordon equation. The article introducing the related lattice quantization is in preparation. Meanwhile, one can test our idea in the phenomenologically-oriented numerical simulations, considering the electric cosmological reference frame.}\]
be understood as the method of restoring the Einstein EP to the system. Specifically, replacing the curvature of the Ashtekar connection in the gravitational sector with the loop holonomy via the relation in \( [13] \), one applies the approximation that neglects terms of order \( \varepsilon^2 \) over a constant, while replacing the Ashtekar connection in all contributions to the scalar constraint, via the relation in \( [27] \), one neglects only linear terms in the regulator over a constant. This way, the deformation of general covariance is introduced in both operations, however, the latter one is one order more significant. The object, keeping this more significant modification after quantization, is the operator in \( [117] \), corrections of those we subtract in the covarianization.

Notice next, that any change of the Hamiltonian, different than a global multiplication by a constant, breaks general covariance unless it is a globally defined local modification, covariant in indices. Our procedure restricts this condition. Moreover, the covarianization, modifying equally the scalar constraint of the whole system, can be understood as the selection of the observer’s perspective on the energy density. This procedure then necessarily deforms the Hamiltonian in the spatial indices of matter field’s observables, which play the role of observers. Finally, the aftermaths of our method occur accordingly in the gravitational and cosmological constant sector of the scalar constraint, as well as in all the potential terms, hence it is not a phenomenological deformation imposed ‘by hand’ only to selected elements.

Let us assume then that the covarianization is a properly defined procedure, restoring the Einstein EP to CQGR. Let us also assume that the scalar field is represented by the Hamiltonian having analogous form to the Hamiltonian of the vector field, with the matter degrees of freedom being symmetric in the spatial indices. The scalar constraint of the system capturing all fundamental interactions can be then expressed in the following way,

\[
H = \int d^3x N \frac{1}{\kappa\sqrt{q}} \left( (F_{cd}^a - (\gamma^2 + 1)\epsilon_{ilm}K^l_a K^m_d)\epsilon^{ijk}E^a_j E^b_k \right) + \frac{\delta^2}{2\sqrt{q}} q_{cd}(E^a_i E^b_j + E^a_j E^b_i) + 2\sqrt{q} \Lambda q_{cd}^{ab} + \mathcal{H}^{(\nu)}_{cd} q_{ab} + \mathcal{H}^{(\nu)ab}_{cd} \delta_{da} \delta_{db} =: H_{ab}^{\delta a} \delta_{db}.
\]

What is remarkable in our approach, is the fact that the lattice-regularized fermionic sector\(^1\) i.e. the analog of \( [43] \) and \( [46] \), reads,

\[
H(v) = \int d^3x N \frac{1}{\sqrt{q}} \left[ \epsilon_{ijk} \epsilon^{abc} \left( \tau^j h^{-1}(x) \left\{ \mathbf{V}^2(x), h_i(x) \right\} \right) \tau^k \left( \frac{1}{\varepsilon} h^{-1}(x) \left\{ \mathbf{V}^2(x), h_\delta(x) \right\} \right) \right] (\text{fermionic})_{k}^e \delta_{da} \delta_{db},
\]

where, ‘fermionic’ denotes the Dirac field’s degrees of freedom. Consequently, the related quantum generally-relativistic corrections take analogous, but antisymmetric form, when compared to \( [121] \). Let us emphasize that in general both scalar and vector field’s Hamiltonian, as well as the cosmological term’s contribution, when written in the form \( H_{ab}^{\delta a} \delta_{db} \), are symmetric in the pairs of spatial indices, while both the fermionic and gravitational sector are antisymmetric.

Keeping our assumption, the covarianization (when restricted to the leading order corrections) is defined as follows,

\[
H_{ab}^{\delta a} \delta_{db} (v) \xrightarrow{\text{covar.}} H_{ab}^{\delta a} (v) \left( 1 + \frac{1}{2} \delta_{ab} (v) \right) \left( 1 + \frac{1}{2} \delta_{ab} (v) \right)^{-1}.
\]

Notice that the general expression, recovering the Einstein EP at all orders of the quantum generally-relativistic corrections, can be written as an infinite series, readable from the exact form of the eigenvalue approximated in \( [117] \). In our opinion, this procedure ought to be added to Thiemann’s regularization method when defining the regularization of the matter degrees of freedom. This way one would not modify the methodology of introducing the holonomy-flux representation in LQG \([12,2]\). Moreover, since the regularization of the matter degrees of freedom defines a local lattice representation, any additional local modification of this step would not affect the diffeomorphism symmetry. Finally, when added in the same manner to all matter terms, covarianization could be interpreted as a kind of ‘antireregularization’, moving lattice-originated corrections from the massless matter sector to potential terms and the gravitational sector.

Before concluding, let us emphasize the locality issue in the idea of the covarianization form one more perspective. We can explicitly demonstrate that our incertitude in taking this procedure as a definite method of restoring the Einstein EP is independent of any other possibly questionable issues concerning formula \( [128] \). This can be done investigating the isotropic limit of the covarianization,

\[
H \xrightarrow{\text{covar.}} H \left( 1 + \delta (A) \right)^{-1},
\]

---

\(^1\) The regularization of the Dirac field is explained in \([2]\), in chapter 12.3.2 Fermionic sector — notice that the divergent quantity \( 1/\sqrt{q} \) in formula \( (12.3.13) \) is reabsorbed to the nominator in formula \( (12.3.18) \), changing the power of the volume operator as follows, \( \mathbf{V} \to \mathbf{V}^{2/3} \). The description of this procedure is the reprint of the one introduced in \([7] \), with analogous reabsorption between expressions \( (3.12) \) and \( (3.17) \). Consequently, the fermionic contribution to the Hamiltonian can be expressed as in \([27] \).
where $\delta(\Delta) := \frac{1}{3} \sum_{n} \delta(\Delta_n)$. It is clear that the correction $\delta(\Delta)$ is derived averaging local quantities, thus it is local as well. Therefore, to be precise, expressions (129) and (125) must be identified as explicitly local formulas, analogously to (128).

The only possibility to consider (129) as a global procedure is limited to the framework of homogeneous and isotropic LQC. Selecting the minisuperspace — the single cell-restricted theory, local corrections become simultaneously global. Then, one will be able to formulate QFT on the effective covariantized background, keeping in mind that several phenomenological models based on LQC, should not be analyzed in connection with our method. This is because the covarianization is motivated by the violation of the Einstein EP at the quantum level, and thus this principle (understood in the way that we specified in our article) should not be violated by any other simplification or method needed to construct considered phenomenological model — see our comments in Sec. II.5.

V. CONCLUSIONS

Our article revealed the problems with the accurate respecting of general covariance, understood in the way comparable to the original Einstein’s formulation of principles of GR. These issues are the outcomes of chosen regularization procedures. Selecting different methods to regularize variables, i.e. the line and point holonomy expansions, not surprisingly occurs in artificial differences in the quantum generally-relativistic corrections. Moreover, neglecting lattice corrections of different orders during the regularization, warns us that the corrections arising from the worse approximated terms may be artificial as well.

We pointed out that the problems in the point holonomy regularization applied to the scalar field are present in the internal relations between the terms of the related scalar constraint. This is clear when comparing with the regularized expressions for vector fields and even for fermions. What is linked with this issue, we emphasized that to know if the general covariance relates proper matter variables with gravitational degrees of freedom, one needs to first identify the Dirac observables in the expression of the Hamiltonian. This is clear in the case of electromagnetism and, in our opinion, is also possible in the whole electrodynamics. In the case of the scalar field, however, we are not certain which variables represent measurable physical quantities. Therefore, to select proper candidates for the scalar field’s observables, we suggest to construct the related lattice representation in terms of isotropic variables analogous to the electric and magnetic field.

In the major part of our manuscript, we operated in the simplified cosmological framework, neglecting also internal structure of fields. The conclusions elaborated in this formalism are, however, general, but what we do not know is whether we found all the inconsistencies related to the discussed topic. Once performing a more comprehensive analysis, other issues may occur. It is also worth mentioning that the framework we selected, is not only a formal simplification. Effects derived taking expectation value of only matter degrees of freedom in the cosmological sector of CQGR effectively approximate results linked to the earliest phase in the evolution of the universe. Therefore, even if we did not discuss in detail how to quantize matter degrees of freedom and how to study connected dynamics, one can simply replace all the matter field’s expectation values with classical quantities, obtaining the theory having yet a physical importance.

Finally, we proposed the effective method of restoring the original classical concept of the Einstein EP in CQGR. We found that this procedure leads to non-trivial effects. Let us mention that if one wishes to study physical cosmology choosing the framework of homogenous isotropic LQC, needs to be careful. Several effective procedures linked with this toy model are explicitly in contradiction to the general perspective of CQGR motivating the covarianization. Moreover, this modification is expected to have many subtle effects. For instance, taking the Friedmann equation’s perspective, the expression for the pressure (that is derivable from the Hamiltonian constraint), will be appropriately modified.

It is also worth mentioning again that covarianization is not a procedure, which can be imposed arbitrarily. It generates local deformations, although this is needed to remove the deformations earlier introduced. The question about the correctness of this idea and related applicational restrictions, we leave as an open problem.

Appendix A: Defining and adjoint representations of $su(2)$ and $su(3)$

In this article we discuss a general model of bosonic fields based on the Yang-Mills theory with gauge symmetries corresponding to representations of the SU(N) Lie algebras for $N \in \{2, 3\}$, subsequently denoted by $su(N)$. We use the following conventions.

The generators of $su(N)$ in the defining representation are labeled by $t^I$, where $I, J, ... = 1, 2, ..., N^2 - 1$. They are the traceless, hermitian matrices of order $n := \text{order}(t) = N$. We fix the normalization for $N = 2$ as $t^I := -\frac{1}{2} \sigma^I$, where $\sigma^I$ are the Pauli matrices.
where $\sigma^I$ are Pauli matrices. Consequently, the general normalization is set to

$$\text{tr}(t^I t^J) = -\frac{1}{2} \delta^{IJ}. \quad (A1)$$

These generators satisfy the following commutation relation,

$$[t^I, t^J] = f_{IJK} t^K, \quad (A2)$$

where the structure constants $f_{IJK}$ are totally antisymmetric tensors.

The generators of $su(N)$ in the adjoint representation are denoted by $T^I$ and defined by the structure constants, $(T^I)_{JK} := -f_{IJ}K$. They are the matrices of order $n = N^2 - 1$. As previously, the commutation relation is specified by the relation,

$$[T^I, T^J] = f_{IJK} T^K. \quad (A3)$$

Consequently, the normalization reads,

$$\text{tr}(T^I T^J) = -N \delta^{IJ}. \quad (A4)$$

The $su(N)$-valued tensor fields’ gauge dependence is introduced by the contraction with the generators $x = x^I t^I$ or $X = X^I T^I$, where the coefficients of the related decomposition can be determined by the relations $x^I = -2 \text{tr}(x t^I)$ and $X^I = -\frac{1}{N} \text{tr}(X t^I)$, for the defining and adjoint representations, respectively.

In what follows, it is convenient to unify the notation above independently on the chosen representation. Let us consider only $su(2)$ and $su(3)$ defining and adjoint representations. We define a generator $\tau \in \{t, T\}$ and the associated factor

$$T := \frac{12}{7} \left( \frac{n^2}{N} \right) - \frac{51}{7} \frac{n}{N} + \frac{53}{7}. \quad (A5)$$

Henceforth, the normalization of generators is unified to

$$\text{tr}(\tau^I \tau^J) = -\frac{1}{T} \delta^{IJ}, \quad (A6)$$

where the coefficients of the decomposition of tensors $x^I \tau^I$ and $X^I \tau^I$ becomes fixed to the forms $x^I = -T \text{tr}(x \tau^I)$ and $X^I = -T \text{tr}(X \tau^I)$, respectively.

\footnote{When discussing the $su(2)$ generators’ dependence in gravity expressed in Ashtekar variables, we slightly modify this notation. The generators of $su(2)$ in this particular case are denoted by $\tau^I := -\frac{1}{4} \sigma^I$.}
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