Generation of higher-dimensional isospectral-nonisospectral integrable hierarchies associated with a new class of higher-dimensional column-vector loop algebras

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Abstract: We construct a new class of higher-dimensional column-vector loop algebras. Based on it, a method for generating higher-dimensional isospectral-nonisospectral integrable hierarchies is proposed. As an application, we derive a generalized nonisospectral integrable Schrödinger hierarchy which can be reduced to the famous derivative nonlinear Schrödinger equation. By using the higher-dimensional column-vector loop algebras, we obtain an expanded isospectral-nonisospectral integrable Schrödinger hierarchy which can be reduced to many classical and new equations, such as the expanded nonisospectral derivative nonlinear Schrödinger system, the heat equation, the Fokker-Plank equation which has a wide range of applications in stochastic dynamic systems. Furthermore, we deduce a $\mathbb{Z}_N$ nonisospectral integrable Schrödinger hierarchy, which means that the coupling results are extended to an arbitrary number of components. Additionally, the Hamiltonian structures of these hierarchies are discussed by using the quadratic form trace identity.

Key words: Generalized integrable Schrödinger hierarchies; Higher-dimensional isospectral-nonisospectral integrable hierarchies; Higher-dimensional column-vector loop algebras; Hamiltonian structure

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1. Introduction

Integrable nonlinear evolution equations have attracted wide attention in mathematics and theoretical physics because they successfully describe and explain nonlinear phenomena in natural science. Among them, nonisospectral equations are of interests in some sense. Many scholars discussed a large number of nonisospectral deformations of classical integrable systems. There are some nonisospectral soliton equations that have been demonstrated to be able to describe solitary waves in a certain type of nonuniform media. There are also some important soliton systems admit nonisospectral linear representations, such as the Bianchi system and the Ernst equation, which means that it is helpful to discover some geometric properties of themselves. Moreover, for the inverse scattering transform of nonisospectral equations, there are many excellent research results. Therefore, how to generate nonisospectral equations is a crucial research topic.

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There are many powerful ways to derive the integrable hierarchy of soliton equations, including the Lax pair method proposed by Magri [18], the method that Ma and Qiao came up with to generate isospectral and nonisospectral integrable hierarchies by applying the generalized Lax representations [19, 22], the approach put forward by Tu [23] and later called Tu scheme [24, 25]. Among them, the Tu scheme has been proven to be a simple and effective method. Many integrable hierarchies and the corresponding Hamiltonian structures as well as other properties were obtained by using the Tu scheme [26–32]. However, it is not difficult to find that the integrable systems generated by the Tu scheme were usually presented under the case of isospectral problems. In order to obtain richer nonisospectral integrable systems and the corresponding Hamiltonian structures as well as related symmetries, Zhang, et.al proposed an approach for generating nonisospectral integrable hierarchies based on the above methods, we introduce a scheme for generating higher-dimensional isospectral-nonisospectral integrable hierarchies related to a new class of N-dimensional column-vector loop algebras. Based on the method, we derive a nonisospectral generalized Schrödinger hierarchy and the corresponding expanded isospectral-nonisospectral integrable hierarchies. It follows that many isospectral and nonisospectral integrable systems can be obtained by reducing these hierarchies. Among them, three representative equations as:

i) The nonisospectral derivative nonlinear Schrödinger equations

\[
\begin{align*}
q_t &= i q_{xx} + (q^2 r)_x - (\frac{q}{2} q_x + q^2 r + q_x \partial^{-1}(qr))k_0(t) + (q + 2 x q_x)k_1(t), \\
q_i &= -ix_{xx} + (q^2 q)_x + (-\frac{i}{2} x + qr^2 + r_x \partial^{-1}(qr))k_0(t) + (r + 2 x r_x)k_1(t),
\end{align*}
\]

which can be reduced to the famous derivative nonlinear Schrödinger equation

\[
i q_t + q_{xx} \mp i(q^2 q)_x = 0.
\]

ii) The expanded nonisospectral derivative nonlinear Schrödinger system

\[
\begin{align*}
q_1 &= i q_{1xx} + (q^2 q)_x - \varepsilon q_x \partial^{-1}(q_1 q_2) - \varepsilon q_2 \partial^{-1}(q_1 q_2) - \frac{1}{2} q_1, \\
q_2 &= -i q_{2xx} + (q^2 q)_x - \varepsilon q_x \partial^{-1}(q_1 q_2) - \varepsilon q_2 \partial^{-1}(q_1 q_2) + \frac{i}{2} q_1,
\end{align*}
\]

where

\[
U_6 = \begin{pmatrix}
-q_1 \partial^{-1}(q_1 r_1 + q_2 r_2) - \varepsilon q_2 \partial^{-1}(q_1 r_2 + q_2 r_1) - \frac{1}{2} q_1 \\
-q_2 \partial^{-1}(q_1 r_2 + q_2 r_1) - \varepsilon q_2 \partial^{-1}(q_1 r_2 + q_2 r_1) + \frac{i}{2} q_1
\end{pmatrix},
\quad U_7 = \begin{pmatrix}
q_1 + 2 x q_1 \\
q_2 + 2 x q_2
\end{pmatrix}.
\]

iii) The linear equation

\[
q_2 t = \frac{i}{2} q_1 q_{2xx} + i \eta_0(t) q_{2x} - \frac{i}{2} q_2 k_0(t) + (2 x q_2 + 2 q_2) k_1(t),
\]

which can be reduced to the heat equation

\[
w_t = w_{xx}.
\]
and the Fokker-Plank equation
\[ w_t = w_{xx} + w + x w_x. \]

Under obtaining the expanded isospectral-nonisospectral integrable systems, their some properties including symmetry, Bäcklund transformations, exact solutions, and so on, could be discussed in our future work [34–37].

2. Generation of higher-dimensional nonisospectral integrable hierarchies

Assume \( \tilde{C}^l \) is a \( l \)-dimensional complex linear space \( \tilde{C}^l = \{ X = (x_1, \ldots, x_l)^T, x_j = \sum_{m \geq 0} a_{jm} \lambda^m, m = 0, \pm 1, \pm 2 \ldots \} \). The linear space \( \tilde{C}^l \) is a Lie algebra when \( A, B, C \in \tilde{C}^l \) satisfy
\[ [A, B] = -[B, A], \quad [\alpha A + \beta B, C] = \alpha [A, C] + \beta [B, C], \quad [[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \]

Introducing a linear functional on \( \tilde{C}^l \) for symmetries constant matrix \( F = (f_{ij})_{l \times l} \) as follows:
\[ \{ a, b \} = a^T F b, \tag{1} \]
which needs to satisfy
\[ \{ A, B \} = \{ B, A \}, \quad \{ \alpha A + \beta B, C \} = \alpha \{ A, C \} + \beta \{ B, C \}, \quad \{ [A, B], C \} = \{ A, [B, C] \}, \quad A, B, C \in \tilde{C}^l. \]

For a \( \Lambda \in \tilde{C}^l \), we introduce the functional
\[ W = \{ V, U_\lambda \} + \{ \Lambda, V_x - [U, V] \}, \]
where \( U, V \) satisfy \( V_x = [U, V] \). In follows that one has \( \frac{\delta}{\delta u_i} \{ V, U_\lambda \} = \frac{\delta}{\delta u_i} W, \quad i = 1, 2, \ldots, s \), which can be used to deduce the quadratic-form identity
\[ \frac{\delta}{\delta u_i} \{ V, \frac{\partial U}{\partial \lambda} \} = \lambda^{-1} \frac{\partial}{\partial \lambda} \lambda^i \{ V, \frac{\partial U}{\partial u_i} \}, \quad i = 1, 2, \ldots, s. \tag{2} \]

Suppose \( \{ e_1(n), \ldots, e_l(n) \} \) is a set of basis of \( \tilde{C}^l \), where \( e_1(n) = (\lambda^{N_1 n \pm x}, 0, \ldots, 0)^T, \quad e_2(n) = (0, \lambda^{N_2 n \pm x}, 0, \ldots, 0)^T, \quad \ldots, \quad e_l(n) = (0, \ldots, 0, \lambda^{N_l n \pm x})^T, \quad x = 0, \pm 1, \ldots, \pm (N_i - 1). \)

**Definition 1.** One basis element \( R \in \tilde{C}^l \) is called pseudoregular when the following conditions hold:
(1) \( \tilde{C}^l = \text{Ker ad } R \bigoplus \text{Im ad } R \);
(2) \( \text{ker ad } R \) is commutative, where \( \text{Ker ad } R = \{ x | x \in \tilde{C}^l, [x, R] = 0 \}, \quad \text{Im ad } R = \{ x | \exists y \in \tilde{C}^l, x = [y, R] \} \).

**Definition 2.** For any basis element \( e_i(n)(i = 1, 2, \ldots, l) \), its gradation can be defined by
\[ \text{deg}(e_i(n)) = N_i n \pm x. \tag{3} \]
Obviously, for \( g \in \tilde{G} \), \( g \) can be expressed by \( g = \sum_{n} k_n e_i(n) =: \sum_{n} g_n \), here \( k_n \) are constants. \( g \) can be decomposed into two parts as follows:

\[
g_+ = \sum_{n \geq \mu} g_n, \quad g_- = \sum_{n < \mu} g_n,
\]

and call \( g_+ \) the positive part of \( g \), \( \mu \in \mathbb{Z} \) is some integer chosen.

In the following, we present the steps for generating nonisospectral integrable hierarchies and its extended integrable hierarchies by using the above definitions and notation (see [38–40]).

**Step 1:** Selecting the spectral problems associated with the column-vector loop algebra \( \tilde{C}^l \):

\[
\begin{cases}
\psi_x = U \psi, \quad U = R(n) + u_1 e_1(n) + u_2 e_2(n) + \cdots + u_l e_l(n), \\
\psi_t = V \psi, \quad V = A_1 e_1(n) + A_2 e_2(n) + \cdots + A_l e_l(n), \\
\lambda_t = \sum_{i \geq 0} k_i(t) \lambda^{-N} e_1^n,
\end{cases}
\]

where \( N_i \in \mathbb{Z} \), the potential functions \( u_1, u_2, \cdots, u_l \in \mathbb{S} \) (the Schwartz space), and \( R(n), e_1(n), e_2(n), \cdots, e_l(n) \in \tilde{C}^l \) satisfy that

(a) \( R, e_1, e_2, \cdots, e_l \) are linear independent;
(b) \( R \) is pseudoregular;
(c) \( \deg(R(n)) \geq \deg(e_i(n)), i = 1, 2, \cdots, l. \)

**Step 2:** In order to deduce the recurrence equation, we solve the following nonisospectral stationary zero curvature equation for \( A_i, i = 1, 2, \cdots, l. \):

\[
V_x = \frac{\partial U}{\partial \lambda} \lambda_t + [U, V].
\]

It follows that the corresponding compatibility condition of the spectral problems \( \Box \) is obtained

\[
\frac{\partial U}{\partial u_t} u_t + \frac{\partial U}{\partial \lambda} \lambda_t - V_x + [U, V] = 0.
\]

\( \Box \) can be broken down into

\[
-V_{+x}^{(n)} + \frac{\partial U}{\partial \lambda} \lambda_{+,+}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - \frac{\partial U}{\partial \lambda} \lambda_{-,+}^{(n)} - [U, V_-^{(n)}],
\]

where

\[
V_{+}^{(m)} = \lambda^{N,m} V - V_-^{(m)} = \sum_{i=0}^{m} \sum_{j=1}^{l} A_{ij} e_j(N_i m - i \pm x),
\]

\[
\lambda_{+,+}^{(m)} = \lambda^{N, \pm m} \lambda_t - \lambda_{-,+}^{(m)} = \sum_{i=0}^{m} k_i(t) \lambda^{N_i, m - N_i \pm x}, \quad x = 0, 1, \cdots, N_i - 1, m < n.
\]

**Step 3:** By observing the result of equation \( \Box \), we choose a modified term \( \Delta_n \in \tilde{C}^l \) so that

\[
V^{(n)} = (\lambda^{N,n} V)_+ + \Delta_n =: V_+^{(n)} + \Delta_n,
\]

then

\[-V_x^{(n)} + \frac{\partial U}{\partial \lambda} \lambda_{+,+}^{(n)} + [U, V^{(n)}] = B_1 e_1 + B_2 e_2 + \cdots + B_l e_l.\]
where \( B_i(i = 1, 2, \cdots, l) \) are functions in variables.

Step 4: The nonisospectral integrable hierarchy of evolution equations are deduced via the nonisospectral zero curvature equation

\[
\frac{\partial U}{\partial t} + \frac{\partial U}{\partial \lambda} \lambda_{i+1}^{(n)} - V_x^{(n)} + [U, V^{(n)}] = 0. \tag{8}
\]

Step 5: Constructing a new \( lN \) dimensional column-vector loop algebra \( \tilde{C}^{lN} \) \((N = 1, 2, 3, \cdots)\) based on \( \tilde{C}^l \), then extending the spectral problems \([1]\) to

\[
\begin{align*}
\psi_x &= U \psi, \quad U = R(n) + u_1 e_1(n) + u_2 e_2(n) + \cdots + u_{lN} e_{lN}(n), \\
\psi_t &= \nabla \psi, \quad \nabla = A_1 e_1(n) + A_2 e_2(n) + \cdots + A_{lN} e_{lN}(n), \\
\lambda_t &= \sum_{i \geq 0} k_i(t) \lambda^{-N_i},
\end{align*} \tag{9}
\]

where the potential functions \( u_1, \cdots, u_{lN} \in S \), and \( R(n), e_1(n), \cdots, e_{lN}(n) \in \tilde{C}^{lN} \).

Step 6: Carrying out the above steps again, the higher-dimensional isospectral-nonisospectral integrable hierarchies related to \( \tilde{C}^{lN} \) can be deduced. It follows that the Hamiltonian structures of these hierarchies could be obtained according to the trace identity.

3. A nonisospectral generalized Schrödinger hierarchy

Introducing the linear space

\( \widetilde{\mathbb{R}^3} = \{ h(n), e(n), f(n) \} \),

where

\( h(n) = (\lambda^{2n}, 0, 0)^T, \quad e(n) = (0, \lambda^{2n-1}, 0)^T, \quad f(n) = (0, 0, \lambda^{2n-1})^T. \)

\( \widetilde{\mathbb{R}^3} \) is a loop algebra because the following operation relations hold:

\[
[h(n), e(m)] = f(m + n), \quad [h(n), f(m)] = -e(m + n), \quad [e(n), f(m)] = h(m + n - 1), \quad m, n \in \mathbb{Z}.
\]

**Definition 3.** We define the commutator:

\[
[a(n), b(m)] = [(a_2 b_3 - a_3 b_2) \lambda^{-2}, 2(a_1 b_2 - a_2 b_1), 2(a_3 b_1 - a_1 b_3)]^T \lambda^{2(n+m)},
\]

where \( \forall a(n) = (a_1(n), a_2(n), a_3(n))^T, \forall b(n) = (b_1(n), b_2(n), b_3(n))^T \in \widetilde{\mathbb{R}^3}. \)

Based on the loop algebra \( \widetilde{\mathbb{R}^3} \), we introduce the linear spectral problems:

\[
\begin{align*}
\varphi_x &= M \varphi, \quad M = (-i \lambda^2 + i \alpha, q \lambda, r \lambda)^T, \\
\varphi_t &= N \varphi, \quad N = \sum_{j \geq 0} \lambda^{-2j} (a_j, b_j, c_j \lambda)^T, \\
\lambda_t &= \sum_{j \geq 0} k_j(t) \lambda^{1-2j}.
\end{align*} \tag{10}
\]

By solving stationary zero curvature equation

\[
N_x = \frac{\partial M}{\partial \lambda} \lambda_t + [M, N], \quad \tag{11}
\]

\begin{center}
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we obtain
\[
\begin{aligned}
a_{jx} &= q c_{j+1} + r b_{j+1} - 2 i k_{j+1}(t), \\
b_{jx} &= -2 i b_{j+1} + 2 i a b_j - 2 q a_j + k_j(t) q, \\
c_{jx} &= 2 i c_{j+1} - 2 i a c_j + 2 r a_j + k_j(t) r,
\end{aligned}
\] (12)
which has an equivalent form as follows:
\[
\begin{aligned}
a_{jx} &= -\frac{i}{2} q c_{jx} - \frac{i}{2} r b_{jx} + \alpha(q c_j - r b_j) + i q r k_j(t) - 2 i k_{j+1}(t), \\
c_{j+1} &= -\frac{i}{2} c_{jx} + i r a_j + \alpha c_j + \frac{i}{2} r k_j(t), \\
b_{j+1} &= \frac{i}{2} b_{jx} + i q a_j + \alpha b_j - \frac{i}{2} q k_j(t).
\end{aligned}
\]
By taking initial values
\[
b_0 = \alpha_0 q, \quad c_0 = \alpha_0 r,
\] (13)
on one has
\[
\begin{aligned}
a_0 &= -\frac{i}{2} \alpha_0 q r + i k_0(t) \partial^{-1}(q r) - 2 i k_1(t) x + \beta_0(t), \\
b_1 &= \frac{i}{2} \alpha_0 q x + \frac{\alpha_0}{r} q^2 r - q \partial^{-1}(q r) k_0(t) + 2 q x k_1(t) + i \beta_0(t) q - \frac{i}{2} q k_0(t) + \alpha_0 \alpha q, \\
c_1 &= -\frac{i}{2} \alpha_0 r x + \frac{\alpha_0}{r} q r^2 - r \partial^{-1}(q r) k_0(t) + 2 r x k_1(t) + i \beta_0(t) r + \frac{i}{2} r k_0(t) + \alpha_0 \alpha r,
\end{aligned}
\]
where \( \beta_0(t) \) is an integral constant. Note that
\[
N_{\lambda}^{(n)} = \sum_{j=0}^{n} (a_j \lambda^{-2 j}, b_j \lambda^{1-2 j}, c_j \lambda^{1-2 j}) \lambda^{2 n} = \lambda^{2 n} N - N_{\lambda}^{(n)} - \sum_{j=n+1}^{\infty} (a_j \lambda^{-2 j}, b_j \lambda^{1-2 j}, c_j \lambda^{1-2 j}) \lambda^{2 n},
\]
\[
\lambda_{\lambda}^{(n)} = \lambda^{2 n} \lambda - \lambda_{\lambda}^{(n)} = \sum_{j=0}^{n} k_j(t) \lambda^{2 n-2 j+1},
\]
then (11) can be broken down into
\[
-N_{\lambda}^{(n)} + \frac{\partial M}{\partial \lambda} \lambda_{\lambda}^{(n)} + [M, N_{\lambda}^{(n)}] = N_{\lambda}^{(n)} - \frac{\partial M}{\partial \lambda} \lambda_{\lambda}^{(n)} - [M, N_{\lambda}^{(n)}],
\] (14)
where
\[
\deg N_{\lambda}^{(n)} \geq 0, \quad \deg \frac{\partial M}{\partial \lambda} \lambda_{\lambda}^{(n)} \geq 1, \quad \deg ([M, N_{\lambda}^{(n)}]) \geq 1,
\]
\[
\deg N_{\lambda}^{(n)} \leq -1, \quad \deg \frac{\partial M}{\partial \lambda} \lambda_{\lambda}^{(n)} \leq 0, \quad \deg ([M, N_{\lambda}^{(n)}]) \leq 1.
\]
Thus, the gradation of the left-hand side of (14) is more than 0, while the right-hand side is less than 1. It follows that we take the gradations 0 and 1 in (14) yields
\[
-N_{\lambda}^{(n)} + \frac{\partial M}{\partial \lambda} \lambda_{\lambda}^{(n)} + [M, N_{\lambda}^{(n)}] = (-q c_{n+1} + r b_{n+1} + 2 i k_{n+1}(t), 2 i b_{n+1} \lambda - 2 i c_{n+1} \lambda)^T.
\]
Taking the modified term \( \triangle n = (-a_n, 0, 0)^T \) so that for \( N^{(n)} = N_{\lambda}^{(n)} + \triangle n \), then a direct calculation reads that
\[
-N_{\lambda}^{(n)} + \frac{\partial M}{\partial \lambda} \lambda_{\lambda}^{(n)} + [M, N^{(n)}] = (0, 2 q a_n \lambda + 2 i b_{n+1} \lambda, -2 r a_n \lambda - 2 i c_{n+1} \lambda)^T.
\]
Therefore, the zero curvature equation

\[-N_{x}^{(n)} + \frac{\partial M}{\partial u} u_{t} + \frac{\partial M}{\partial \lambda} \lambda_{x}^{(n)} + [M, N^{(n)}] = 0\]

admits that

\[
\begin{align*}
    u_{tn} &= \begin{pmatrix} q \\ r \end{pmatrix} \\
    &= \begin{pmatrix} 0 & \partial - 2i\alpha \\ \partial + 2i\alpha & 0 \end{pmatrix} \begin{pmatrix} c_{n} \\ b_{n} \end{pmatrix} - k_{n}(t) \begin{pmatrix} q \\ r \end{pmatrix} \\
    &= J_{1} \begin{pmatrix} c_{n} \\ b_{n} \end{pmatrix} - k_{n}(t) \begin{pmatrix} q \\ r \end{pmatrix},
\end{align*}
\]

where

\[J_{1} = \begin{pmatrix} 0 & \partial - 2i\alpha \\ \partial + 2i\alpha & 0 \end{pmatrix}.\]

By using the recursion relations \[12\], the nonisospectral hierarchies \[14\] can be rewritten as

\[
\begin{align*}
    u_{tn} &= \begin{pmatrix} q \\ r \end{pmatrix} \\
    &= \begin{pmatrix} -2q\partial^{-1}q_{c_{n+1}} + (-2i + 2q\partial^{-1}r)b_{n+1} + 4iqxk_{n+1}(t) \\ -2r\partial^{-1}rb_{n+1} + (2i + 2r\partial^{-1}q)b_{n+1} - 4irxk_{n+1}(t) \end{pmatrix} \\
    &= \begin{pmatrix} -2q\partial^{-1}q & -2i + 2q\partial^{-1}r \\ 2i + 2r\partial^{-1}q & -2r\partial^{-1}r \end{pmatrix} \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} + 4ik_{n+1}(t)x \begin{pmatrix} q \\ -r \end{pmatrix} \\
    &= J_{2} \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} + 4ik_{n+1}(t)x \begin{pmatrix} q \\ -r \end{pmatrix},
\end{align*}
\]

where

\[J_{2} = \begin{pmatrix} -2q\partial^{-1}q & -2i + 2q\partial^{-1}r \\ 2i + 2r\partial^{-1}q & -2r\partial^{-1}r \end{pmatrix}.\]

Using \[12\] again, we deduce the following recursion relation

\[
\begin{align*}
    \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}\partial + \frac{1}{2}r\partial^{-1}q\partial + \alpha + i\alpha r\partial^{-1}q & \frac{1}{2}r\partial^{-1}r\partial + i\alpha r\partial^{-1}r \\ \frac{1}{2}q\partial^{-1}q\partial + i\alpha q\partial^{-1}q & \frac{1}{2}\partial + \frac{1}{2}q\partial^{-1}r\partial + \alpha - i\alpha q\partial^{-1}r \end{pmatrix} \begin{pmatrix} c_{n} \\ b_{n} \end{pmatrix} \\
    + k_{n}(t) & \begin{pmatrix} t \partial - r\partial^{-1}(qr) \\ -\frac{1}{2}i\partial + 2q\partial^{-1}(qr) \end{pmatrix} + 2k_{n+1}(t)x \begin{pmatrix} q \\ -r \end{pmatrix} \\
    &= : L \begin{pmatrix} c_{n} \\ b_{n} \end{pmatrix} + k_{n}(t)R + k_{n+1}(t)S,
\end{align*}
\]

where

\[
\begin{align*}
    L &= \begin{pmatrix} -\frac{1}{2}\partial + \frac{1}{2}r\partial^{-1}q\partial + \alpha + i\alpha r\partial^{-1}q & \frac{1}{2}r\partial^{-1}r\partial + i\alpha r\partial^{-1}r \\ \frac{1}{2}q\partial^{-1}q\partial + i\alpha q\partial^{-1}q & \frac{1}{2}\partial + \frac{1}{2}q\partial^{-1}r\partial + \alpha - i\alpha q\partial^{-1}r \end{pmatrix}, \\
    R &= \begin{pmatrix} \frac{1}{2}t \partial - r\partial^{-1}(qr) \\ -\frac{1}{2}i\partial + 2q\partial^{-1}(qr) \end{pmatrix}, \\
    S &= 2x \begin{pmatrix} q \\ -r \end{pmatrix}.
\end{align*}
\]
Substituting (17) into (15) yields

\[
\begin{align*}
\dot{u}_n = \begin{pmatrix} q \\ r \end{pmatrix}_t = J_1 \begin{pmatrix} c_n \\ b_n \end{pmatrix}_t - k_n(t) \begin{pmatrix} q \\ r \end{pmatrix}_t \\
= & J_1 L^n \begin{pmatrix} \alpha_0 q \\ \alpha_0 r \end{pmatrix} + J_1 \sum_{j=0}^{n-1} (L^j R k_{n-1-j}(t)) + J_1 \sum_{j=0}^{n} L^j S k_{n-j}(t) - k_n(t) \begin{pmatrix} q \\ r \end{pmatrix}_t \\
= & \Phi^n \left( \begin{pmatrix} \alpha_0 q \\ \alpha_0 r \end{pmatrix} - 2i\alpha_0 q + 2i\alpha_0 r \right) + \sum_{j=0}^{n-1} \Phi^j J_1 R k_{n-1-j}(t) + \sum_{j=0}^{n} k_{n-j}(t) \Phi^j J_1 S - k_n(t) \begin{pmatrix} q \\ r \end{pmatrix},
\end{align*}
\]

where

\[
\Phi = J_1 L J_1^{-1}.
\]

(18) is a generalized nonisospectral integrable Schrödinger hierarchy. When \( n = 1 \), the nonisospectral integrable hierarchy (18) becomes

\[
\begin{align*}
q_t &= i q_{xx} + (q^2 r)_x - \left( \frac{2}{3} q x + q^2 r + q x \partial^{-1}(q r) \right) k_0(t) + (q + 2 x q_x) k_1(t), \\
r_t &= - i r_{xx} + (r^2 q)_x + \left( \frac{2}{3} r x + q r^2 + q x \partial^{-1}(q r) \right) k_0(t) + (r + 2 x r_x) k_1(t).
\end{align*}
\]

When we take \( k_0(t) = k_1(t) = 0 \), \( r = \pm q^* \), the nonisospectral system (20) reduces to the following derivative nonlinear Schrödinger equation (see [41])

\[
i q_t + q_{xx} \pm i(q^2 q^*)_x = 0.
\]

(21)

Therefore, (20) can be called nonisospectral derivative nonlinear Schrödinger equations. In the following, we would like to discuss the Hamiltonian structure of the hierarchy (18) based on the trace identity (23).

3.1. Hamiltonian structure

Denoting the trace of the square matrices \( A \) and \( B \) by \( < A, B > \), that means \( < A, B > = tr(AB) \). From (10), we have

\[
< N, \frac{\partial M}{\partial q} > = \sum_{j \geq 0} c_j \lambda^{2-2j}, \quad < N, \frac{\partial M}{\partial r} > = \sum_{j \geq 0} b_j \lambda^{2-2j}, \quad < N, \frac{\partial M}{\partial \lambda} > = \sum_{j \geq 0} (-4 i a_j + r b_j + q c_j) \lambda^{1-2j}.
\]

Substituting the above results into the trace identity

\[
\frac{\delta}{\delta u} \left( < N, \frac{\partial M}{\partial \lambda} > \right) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left( < N, \frac{\partial M}{\partial q} > \right)
\]

gives rise to

\[
\frac{\delta}{\delta u} (-4 i a_n + r b_n + q c_n) = (2 - 2n + \gamma) \begin{pmatrix} c_n \\ b_n \end{pmatrix}. \tag{22}
\]

It follows that we find that \( \gamma = -2 \) by substituting the initial value of (13) into the above equation, hence

\[
\begin{pmatrix} c_n \\ b_n \end{pmatrix} = \frac{\delta H_n}{\delta u} \quad H_n = \frac{-r b_n - q c_n + 4 i a_n}{2n}.
\]
Then, the hierarchy (16) and (18) can be written as

\[ u_{tn} = \left( \frac{q}{r} \right)_{tn} = J_1 \frac{\delta H_n}{\delta u} - k_n(t) \left( \frac{q}{r} \right) = J_2 \frac{\delta H_{n+1}}{\delta u} + 4iK_{n+1}t x \left( \frac{q}{r} \right). \] (23)

It is remarkable that when \( k_n(t) = k_{n+1}(t) = 0 \), (23) is the Hamiltonian structure of the corresponding isospectral integrable hierarchy of (18).

4. A 6 dimensional column-vector loop algebra

The generalized nonisospectral integrable Schrödinger hierarchy is derived by using the 3 dimensional loop algebra in the previous section. In order to derive the extended integrable hierarchies of the hierarchy (15), a 6 dimensional complex linear space \( \mathbb{C}^6 \) will be introduced in this section. Let us first define a commutative operation as follows:

\[
[a, b] = \begin{bmatrix} a_2 b_3 - a_3 b_2 + \varepsilon a_5 b_6 - \varepsilon a_6 b_5, 2(a_1 b_2 - a_2 b_1 + \varepsilon a_4 b_5 - \varepsilon a_5 b_4), 2(-a_1 b_3 + a_3 b_1 - \varepsilon a_4 b_6 + \varepsilon a_6 b_4) \\
2a_2 b_6 - a_6 b_2 - a_3 b_5 + a_5 b_3, 2(a_1 b_5 - a_5 b_1 - a_2 b_4 + a_4 b_2), 2(-a_1 b_6 + a_6 b_1 + a_3 b_4 - a_4 b_3) \end{bmatrix}^T,
\] (24)

where \( a = (a_1, a_2, a_3, a_4, a_5, a_6)^T \), \( b = (b_1, b_2, b_3, b_4, b_5, b_6)^T \in \mathbb{C}^6 \).

**Lemma 1.** The linear space \( \mathbb{C}^6 \) is a Lie algebra if equipped with the operation (24).

**Proof.** A basis of the Lie algebras \( gl(3) \) is given by

\[ gl(3) = span\{h, e, f\} \]

with \( h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \).

We construct an expanded Lie algebra \( gl(6) = span\{h_1, h_2, h_3, h_4, h_5, h_6\} \) corresponding to the Lie algebra \( gl(3) \), whose a set of basis is

\[
h_1 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, h_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, h_3 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix},
\]

\[
h_4 = \begin{pmatrix} 0 & \varepsilon h \\ h & 0 \end{pmatrix}, h_5 = \begin{pmatrix} 0 & \varepsilon e \\ e & 0 \end{pmatrix}, h_6 = \begin{pmatrix} 0 & \varepsilon f \\ f & 0 \end{pmatrix},
\]

which satisfies

\[
[h_1, h_2] = 2h_2, \ [h_1, h_3] = -2h_3, \ [h_1, h_4] = 0, \ [h_1, h_5] = 2h_5, \ [h_1, h_6] = -2h_6,
\]

\[
[h_2, h_3] = h_1, \ [h_2, h_4] = -2h_5, \ [h_2, h_5] = 0, \ [h_2, h_6] = h_4, \ [h_3, h_4] = 2h_6,
\]

\[
[h_3, h_5] = -h_4, \ [h_3, h_6] = 0, \ [h_4, h_5] = 2\varepsilon h_2, \ [h_4, h_6] = -2\varepsilon h_3, \ [h_5, h_6] = \varepsilon h_1,
\]

with \( \varepsilon \in \mathbb{R} \).

Let \( G_1 = span\{h_1, h_2, h_3\}, G_2 = span\{h_4, h_5, h_6\} \), then \( gl(6) = G_1 \oplus G_2 \). Denoting

\[ [G_i, G_j] = \{[A, B] | A \in G_i, B \in G_j\}, \]

where \( G_i, G_j \) are 6 dimensional column-vector loop algebras.
then \( G_1 \) and \( G_2 \) satisfy the properties of closure

\[
[G_1, G_1] \subseteq G_1, \quad [G_1, G_2] \subseteq G_2, \quad [G_2, G_2] \subseteq G_1.
\]

For \( \forall A = \sum_{i=1}^{6} a_i h_i \in gl(6) \), we suppose

\[
\delta : gl(6) \rightarrow \mathbb{C}^6, A = \sum_{i=1}^{6} a_i h_i \rightarrow a = (a_1, a_2, a_3, a_4, a_5, a_6)^T,
\]

(25)

and then one can find that \( \delta \) is an isomorphism between \( gl(6) \) and \( \mathbb{C}^6 \). Therefore, for \( \forall A, B \in gl(6) \), \( A = \sum_{i=1}^{6} a_i h_i \), \( B = \sum_{i=1}^{6} b_i h_i \), one has

\[
\delta([A, B]) = \delta((a_2 b_3 - a_3 b_2 + \varepsilon a_5 b_6 - \varepsilon a_6 b_5) h_1 + 2(a_1 b_2 - a_2 b_1 + \varepsilon a_4 b_5 - \varepsilon a_5 b_4) h_2
\]

\[
+ 2(-a_1 b_3 + a_3 b_1 - \varepsilon a_4 b_6 + \varepsilon a_6 b_4) h_3 + (a_2 b_6 - a_6 b_2 - a_3 b_5 + a_5 b_3) h_4
\]

\[
+ (a_1 b_5 - a_5 b_1 - a_2 b_4 + a_4 b_2) h_5 + 2(-a_1 b_6 + a_6 b_1 + a_3 b_4 - a_4 b_3) h_6
\]

\[
= [a_2 b_3 - a_3 b_2 + \varepsilon a_5 b_6 - \varepsilon a_6 b_5, 2(a_1 b_2 - a_2 b_1 + \varepsilon a_4 b_5 - \varepsilon a_5 b_4), 2(-a_1 b_3 + a_3 b_1 - \varepsilon a_4 b_6 + \varepsilon a_6 b_4)
\]

\[
a_2 b_6 - a_6 b_2 - a_3 b_5 + a_5 b_3, 2(a_1 b_5 - a_5 b_1 - a_2 b_4 + a_4 b_2), 2(-a_1 b_6 + a_6 b_1 + a_3 b_4 - a_4 b_3)]^T
\]

\[
= [a, b].
\]

Therefore, we conclude that \( \mathbb{C}^6 \) is a Lie algebra related to the operation (24). \( \square \)

The commutative operation (24) can be written as

\[
[a, b] = a^T R(b),
\]

where

\[
R(b) = \\
0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\
b_3 & -2b_1 & 0 & b_5 & -2b_4 & 0 \\
-b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\
0 & 2\varepsilon b_5 & -2\varepsilon b_6 & 0 & 2b_2 & -2b_3 \\
\varepsilon b_6 & -2\varepsilon b_4 & 0 & b_3 & -2b_1 & 0 \\
-\varepsilon b_5 & 0 & 2\varepsilon b_4 & -b_2 & 0 & 2b_1 
\]

It follows that \( F \) is obtained according to \( R(b) F = -(R(b) F)^T \) and \( F^T = F \):

\[
F = \\
2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 2\varepsilon & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \varepsilon \\
0 & 1 & 0 & 0 & \varepsilon & 0 
\]
which can be used to define a linear functional on the Lie algebra $\mathbb{C}^6$:

$$\{a, b\} = a^T F b, \quad a, b \in gl(6).$$  \hspace{1cm} (26)

The Hamiltonian structure of expanded integrable hierarchies can be deduced by using (26). Based on the Lie algebra $gl(6)$, we introduce the loop algebra $\widetilde{gl}(6) = \text{span}\{h_1(n), h_2(n), h_3(n), h_4(n), h_5(n), h_6(n)\}$, where the gradations of $h_i(n)(i = 1, \ldots, 6)$ are given by

$$\text{deg} h_1(n) = \text{deg} h_4(n) = 2n, \quad \text{deg} h_2(n) = \text{deg} h_3(n) = \text{deg} h_5(n) = \text{deg} h_6(n) = 2n - 1.$$

After direct calculation, one has

$$[A(n), B(m)] = (a_2 b_3 - a_3 b_2 + \varepsilon a_5 b_6 - \varepsilon a_6 b_5) h_1(n + m - 1) + 2(a_1 b_2 - a_2 b_1 + \varepsilon a_4 b_5 - \varepsilon a_5 b_4) h_2(n + m)
+ 2(-a_1 b_3 + a_3 b_1 - \varepsilon a_4 b_6 + \varepsilon a_6 b_4) h_3(n + m) + (a_2 b_6 - a_6 b_2 - a_3 b_5 + a_5 b_3) h_4(n + m - 1)
+ 2(a_1 b_5 - a_5 b_1 - a_2 b_4 + a_4 b_2) h_5(n + m) + 2(-a_1 b_6 + a_6 b_1 + a_3 b_4 - a_4 b_3) h_6(n + m),$$

where $A(n), B(m) \in \widetilde{gl}(6), A(n) = \sum_{i=1}^{6} a_i h_i(n), B(m) = \sum_{i=1}^{6} b_i h_i(m)$. Furthermore, the corresponding commutator in $\mathbb{C}^6$ according to (26) is given by

$$[a(n), b(m)] = [(a_2 b_3 - a_3 b_2 + \varepsilon a_5 b_6 - \varepsilon a_6 b_5) \lambda^{-1}, 2(a_1 b_2 - a_2 b_1 + \varepsilon a_4 b_5 - \varepsilon a_5 b_4),
2(-a_1 b_3 + a_3 b_1 - \varepsilon a_4 b_6 + \varepsilon a_6 b_4), (a_2 b_6 - a_6 b_2 - a_3 b_5 + a_5 b_3) \lambda^{-1},
2(a_1 b_5 - a_5 b_1 - a_2 b_4 + a_4 b_2), 2(-a_1 b_6 + a_6 b_1 + a_3 b_4 - a_4 b_3)]^T \lambda^{2(n+m)-1}.$$  \hspace{1cm} (27)

5. A $3N$ dimensional column-vector loop algebra

In the previous section, we construct the 6 dimensional column-vector loop algebra, and then we will introduce a $3N$ dimensional complex linear space $\mathbb{C}^{3N}$ in this section. Let us first define a commutative operation as follows:

$$[a, b] = (c_1, c_2, c_3, \ldots, c_{3N-2}, c_{3N-1}, c_{3N})^T,$$  \hspace{1cm} (28)

where

$$a = (a_1, a_2, a_3, \ldots, a_{3N-2}, a_{3N-1}, a_{3N})^T, \quad b = (b_1, b_2, b_3, \ldots, b_{3N-2}, b_{3N-1}, b_{3N})^T \in \mathbb{C}^6,$$

$$c_{3k-2} = \sum_{i=1}^{k} (a_{3i-1} b_{3(-i+1+k)} - a_{3(-i+1+k)} b_{3i-1}) + \sum_{i=k+1}^{N} \sigma \varepsilon (a_{3i-1} b_{3(-i+1+k+N)} - a_{3(-i+1+k+N)} b_{3i-1}),$$

$$c_{3k-1} = \sum_{i=1}^{k} (a_{3i-2} b_{3(-i+1+k)-1} - a_{3(-i+1+k)-1} b_{3i-2}) + \sum_{i=k+1}^{N} \sigma \varepsilon (a_{3i-2} b_{3(-i+1+k+N)-1} - a_{3(-i+1+k+N)-1} b_{3i-2}),$$

$$c_{3k} = \sum_{i=1}^{k} (-a_{3i-2} b_{3(-i+1+k)} + a_{3(-i+1+k)} b_{3i-2}) + \sum_{i=k+1}^{N} \sigma \varepsilon (-a_{3i-2} b_{3(-i+1+k+N)} + a_{3(-i+1+k+N)-1} b_{3i-2}),$$

$$k = 1, 2, \ldots, N,$$

with

$$\sigma = \begin{cases} 
0, & k = N, \\
1, & 1 \leq k \leq N - 1.
\end{cases}$$
Lemma 2. The linear space $\mathbb{C}^{3N}$ is a Lie algebra if equipped with the operation $[28]$.

Proof. Introducing a $N \times N$ square matrix of the following form:

$$M(A_1, A_2, \cdots, A_N) = \begin{pmatrix}
A_1 & \varepsilon A_N & \varepsilon A_{N-1} & \cdots & \varepsilon A_4 & \varepsilon A_3 & \varepsilon A_2 \\
A_2 & A_1 & \varepsilon A_N & \cdots & \varepsilon A_5 & \varepsilon A_4 & \varepsilon A_3 \\
A_3 & A_2 & A_1 & \cdots & \varepsilon A_5 & \varepsilon A_4 & \varepsilon A_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
A_{N-2} & A_{N-3} & A_{N-4} & \cdots & A_1 & \varepsilon A_N & \varepsilon A_{N-1} \\
A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_2 & A_1 & \varepsilon A_N \\
A_N & A_{N-1} & A_{N-2} & \cdots & A_3 & A_2 & A_1
\end{pmatrix}, \quad (29)$$

where $A_m$ ($1 \leq m \leq N$) represent $N$ arbitrary square matrices of the same order. Based on the $N \times N$ square matrix (29), we construct a set of high-dimensional matrices as follows:

$$f_1 = \begin{pmatrix}
h & 0 & 0 & \cdots & 0 & 0 \\
0 & h & 0 & \cdots & 0 & 0 \\
0 & 0 & h & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & h & 0 \\
0 & 0 & 0 & \cdots & 0 & h
\end{pmatrix}, \quad f_2 = \begin{pmatrix}
e & 0 & 0 & \cdots & 0 & 0 \\
0 & e & 0 & \cdots & 0 & 0 \\
0 & 0 & e & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & e & 0 \\
e & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad f_3 = \begin{pmatrix}
f & 0 & 0 & \cdots & 0 & 0 \\
0 & f & 0 & \cdots & 0 & 0 \\
0 & 0 & f & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & f & 0 \\
f & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},$$

$$f_4 = \begin{pmatrix}
0 & e & 0 & \cdots & 0 & 0 \\
0 & 0 & e & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & e & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & e \\
0 & 0 & 0 & \cdots & e & 0
\end{pmatrix}, \quad f_5 = \begin{pmatrix}
0 & 0 & e & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & e & 0 \\
0 & 0 & 0 & \cdots & 0 & e \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & e \\
0 & 0 & 0 & \cdots & e & 0
\end{pmatrix}, \quad f_6 = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & e f \\
0 & 0 & 0 & \cdots & 0 & e f \\
0 & 0 & 0 & \cdots & e f & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & e f \\
0 & 0 & 0 & \cdots & 0 & e f
\end{pmatrix},$$

$$\cdots$$

$$f_{3N-2} = \begin{pmatrix}
0 & e h & 0 & \cdots & 0 & 0 \\
0 & 0 & e h & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & e h & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & e h \\
0 & 0 & 0 & \cdots & e h & 0
\end{pmatrix}, \quad f_{3N-1} = \begin{pmatrix}
0 & e & 0 & \cdots & 0 & 0 \\
0 & 0 & e & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & e & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & e \\
e & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad f_{3N} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & e f \\
0 & 0 & 0 & \cdots & 0 & e f \\
0 & 0 & 0 & \cdots & e f & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & e f \\
e & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.$$  

When $h, e, f$ are selected as one of the bases of the Lie algebra $gl(3)$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$[f_{3i-2}, f_{3j-2}] = [f_{3i-1}, f_{3j-1}] = [f_{3i}, f_{3j}] = 0, \quad i, j = 1, 2, \cdots, N,$$

$$[f_{3i-2}, f_{3j-1}] = \begin{cases} 2f_{3(j+i-1)-1}, & 1 \leq j \leq N - i + 1, \\
2\varepsilon f_{3(j+i-1)-N-1}, & N - i + 2 \leq j \leq N, \end{cases}$$

$$[f_{3i-2}, f_{3j}] = \begin{cases} -2f_{3(j+i-1)}, & 1 \leq j \leq N - i + 1, \\
-2\varepsilon f_{3(j+i-1)-N}, & N - i + 2 \leq j \leq N, \end{cases}$$

$$[f_{3i-1}, f_{3j-2}] = \begin{cases} f_{3(j+i-1)-2}, & 1 \leq j \leq N - i + 1, \\
\varepsilon f_{3(j+i-1)-N-2}, & N - i + 2 \leq j \leq N, \end{cases}$$

$$[f_{3i}, f_{3j-1}] = \begin{cases} 2\varepsilon f_{3(j+i-1)-N}, & 1 \leq j \leq N - i + 1, \\
2f_{3(j+i-1)-N-1}, & N - i + 2 \leq j \leq N, \end{cases}$$

$$[f_{3i}, f_{3j}] = \begin{cases} -2\varepsilon f_{3(j+i-1)}, & 1 \leq j \leq N - i + 1, \\
-f_{3(j+i-1)-N}, & N - i + 2 \leq j \leq N, \end{cases}$$

$$[f_{3i}, f_{3j-2}] = \begin{cases} -\varepsilon f_{3(j+i-1)-N-2}, & 1 \leq j \leq N - i + 1, \\
-f_{3(j+i-1)-2}, & N - i + 2 \leq j \leq N, \end{cases}$$

$$[f_{3i-1}, f_{3j-2}] = \begin{cases} -f_{3(j+i-1)-2}, & 1 \leq j \leq N - i + 1, \\
-\varepsilon f_{3(j+i-1)-N-2}, & N - i + 2 \leq j \leq N, \end{cases}$$
with $\varepsilon \in \mathbb{R}$.

It means that $f_i(i = 1, \cdots, 3N)$ are a set of bases of the Lie algebra $gl(3N) = \text{span}\{f_1, f_2, f_3, \cdots, f_{3N-2}, f_{3N-1}, f_{3N}\}$.

In fact, we can also choose a different set of bases, and the results can be deduced similarly.

Let $\mathcal{G}_k = \text{span}\{f_{3k-2}, f_{3k-1}, f_{3k}\}$, $k = 1, 2, \cdots, N$, then $gl_{3N} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_N$. Thus, we obtain

$$[\mathcal{G}_i, \mathcal{G}_j] \subseteq \mathcal{G}_{i+j-\delta N}, \quad \delta = \begin{cases} 0, & 2 \leq i + j \leq N + 1, \\ 1, & N + 2 \leq i + j \leq 2N, \end{cases} \quad i, j = 1, 2, \cdots, N.$$ 

For $\forall A \in gl(3N)$, it can be expressed by $A = \sum_{i=1}^{3N} a_i f_i$. Suppose

$$\varrho : gl(3N) \to \mathbb{C}^{3N}, \quad A = \sum_{i=1}^{3N} a_i f_i \rightarrow a = (a_1, a_2, a_3, \cdots, a_{3N-2}, a_{3N-1}, a_{3N})^T,$$

(30)

one can find that $\varrho$ is an isomorphism between $gl(3N)$ and $\mathbb{C}^{3N}$. Therefore, for $\forall A, B \in gl(3N)$, $A = \sum_{i=1}^{3N} a_i f_i$, $B = \sum_{i=1}^{3N} b_i f_i$, one has

$$\delta([A, B]) = \delta[c_1 f_1 + c_2 f_2 + c_3 f_3 + \cdots + c_{3N-2} f_{3N-2} + c_{3N-1} f_{3N-1} + c_{3N} f_{3N}]$$

$$= (c_1, c_2, c_3, \cdots, c_{3N-2}, c_{3N-1}, c_{3N})^T$$

$$= : [a, b],$$

where $c_j$ ($j = 1, 2, \cdots, 3N$) is given by $\varrho$. Hence, we conclude that $\mathbb{C}^{3N}$ equipped with the operation $\varrho$ is a Lie algebra.

The commutative operation $\varrho$ can be written as

$$[a, b] = a^T \mathcal{R}(b),$$

where

$$\mathcal{R}(b)^T = \left( \begin{array}{cccccccc} \mathcal{R}_1(b) & \varepsilon \mathcal{R}_1(b) & \varepsilon \mathcal{R}_{N-1}(b) & \cdots & \varepsilon \mathcal{R}_4(b) & \varepsilon \mathcal{R}_5(b) & \varepsilon \mathcal{R}_2(b) \\ \mathcal{R}_2(b) & \mathcal{R}_1(b) & \varepsilon \mathcal{R}_N(b) & \cdots & \varepsilon \mathcal{R}_6(b) & \varepsilon \mathcal{R}_5(b) & \varepsilon \mathcal{R}_3(b) \\ \mathcal{R}_3(b) & \mathcal{R}_2(b) & \mathcal{R}_1(b) & \cdots & \mathcal{R}_6(b) & \mathcal{R}_5(b) & \varepsilon \mathcal{R}_4(b) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{R}_{N-2}(b) & \mathcal{R}_{N-3}(b) & \mathcal{R}_{N-4}(b) & \cdots & \mathcal{R}_1(b) & \varepsilon \mathcal{R}_N(b) & \varepsilon \mathcal{R}_{N-1}(b) \\ \mathcal{R}_{N-1}(b) & \mathcal{R}_{N-2}(b) & \mathcal{R}_{N-3}(b) & \cdots & \mathcal{R}_2(b) & \mathcal{R}_1(b) & \varepsilon \mathcal{R}_N(b) \\ \mathcal{R}_N(b) & \mathcal{R}_{N-1}(b) & \mathcal{R}_{N-2}(b) & \cdots & \mathcal{R}_3(b) & \mathcal{R}_2(b) & \mathcal{R}_1(b) \end{array} \right),$$

where

$$\mathcal{R}_k(b) = \left( \begin{array}{ccc} 0 & 2b_{3k-1} & -2b_{3k} \\ b_{3k} & -2b_{3k-2} & 0 \\ -b_{3k-1} & 0 & 2b_{3k-2} \end{array} \right), \quad k = 1, 2, \cdots, N.$$
It follows that $\mathcal{F}$ can be obtained according to $\mathcal{R}(b)\mathcal{F} = -(\mathcal{R}(b)\mathcal{F})^T$ and $\mathcal{F}^T = \mathcal{F}$ as follows:

$$
\mathcal{F} = \begin{pmatrix}
\mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 & \cdots & \mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 \\
\mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 & \cdots & \mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 \\
\mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 & \cdots & \mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 & \cdots & \mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 \\
\mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 & \cdots & \mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 \\
\mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 & \cdots & \mathcal{F}_1 & \mathcal{F}_1 & \mathcal{F}_1 \\
\end{pmatrix}, \quad \mathcal{F}_1 = \begin{pmatrix} 2 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{pmatrix},
$$

which can be used to define a linear functional on the Lie algebra $\mathbb{C}^{3N}$:

$$\{a, b\} = a^T \mathcal{F} b, \quad a, b \in \mathfrak{gl}(3N). \quad (31)$$

The Hamiltonian structure of higher-dimensional integrable hierarchies can be deduced by using (31). In the following, we consider a loop algebra $\mathbb{C}^{3N}$ corresponding to the Lie algebra $\mathbb{C}^{3N}$. Based on the Lie algebra $\mathfrak{gl}(3N)$, we introduce the loop algebra $\bar{\mathfrak{g}}l(3N) = \text{span}\{f_1(n), f_2(n), f_3(n), \cdots, f_{3N-2}(n), f_{3N-1}(n), f_{3N}(n)\}$, where the gradations of $f_i(n)(i = 1, \cdots, 3N)$ are given by

$$\text{deg } f_{3k-2}(n) = 2n, \quad \text{deg } f_{3k-1}(n) = f_{3k}(n) = 2n - 1, \quad k = 1, 2, \cdots, N.$$ 

After direct calculation, we have

$$[A(n), B(m)] = \sum_{i=1}^N c_{3i-2} h_{3i-2}(n + m - 1) + \sum_{i=1}^N c_{3i-1} h_{3i-1}(n + m) + \sum_{i=1}^N c_{3i} h_{3i}(n + m),$$

where $c_j (j = 1, 2, \cdots, 3N)$ is given by (28), $A(n), B(m) \in \bar{\mathfrak{g}}l(3N)$, $A(n) = \sum_{i=1}^{3N} a_i f_i(n)$, $B(m) = \sum_{i=1}^{3N} b_i f_i(m)$. Furthermore, the corresponding commutator in $\mathbb{C}^{3N}$ according to (30) is given by

$$[a(n), b(m)] = (c_1 \lambda^{-1}, c_2, c_3, c_4 \lambda^{-1}, c_5, c_6, \cdots, c_{3N-2} \lambda^{-1}, c_{3N-1}, c_{3N})^T \lambda^{2(n+m)-1}. \quad (32)$$

By using $g$, many new loop algebras related to the loop algebra $\bar{\mathfrak{g}}l(3N)$ can be defined, which will not be repeated here.

6. An expanded isospectral integrable Schrödinger hierarchy

In order to deduce an expanded isospectral-nonisospectral integrable hierarchy of the hierarchy [15], we will use the column-vector loop algebra $\mathbb{C}^6$ of section 4. Considering the isospectral-nonisospectral problems

$$\begin{cases}
\varphi_x = U \varphi, \quad U = (-i\lambda^2 + i\alpha, q_1, r_1, 0, q_2, r_2) \;^T, \\
\varphi_t = V \varphi, \quad V = \sum_{k \geq 0} (a_k, b_k, c_k \lambda, d_k, e_k \lambda, h_k \lambda) \lambda^{-2k} + \sum_{j \geq 0} (a_j, b_j, c_j \lambda, d_j, e_j \lambda, h_j \lambda) \lambda^{-2j} =: V_1 + V_2, \\
\lambda_t = \sum_{j \geq 0} k_j(t) \lambda^{1-2j}.
\end{cases} \quad (33)$$
By solving the following isospectral stationary zero curvature equation

\[
V_{1x} = [U, V_1],
\]

we obtain the recursion equations

\[
\begin{cases}
    a_{k+1} = q_1 c_{k+1} - r_1 b_{k+1} + \varepsilon q_2 h_{k+1} - \varepsilon r_2 e_{k+1}, \\
    b_{k+1} = 2(-ib_{k+1} + i\alpha b_k - q_1 a_k - \varepsilon q_2 d_k), \\
    c_{k+1} = 2(ic_{k+1} - i\alpha c_k + r_1 a_k + \varepsilon r_2 d_k), \\
    d_{k+1} = q_1 h_{k+1} - r_2 b_{k+1} - r_1 e_{k+1} + q_2 c_{k+1}, \\
    e_{k+1} = 2(-ie_{k+1} + i\alpha c_k - q_2 a_k - q_1 d_k), \\
    h_{k+1} = 2(ih_{k+1} - i\alpha h_k + r_2 a_k + r_1 d_k),
\end{cases}
\]

which has an equivalent form as follows:

\[
\begin{cases}
    a_{k+1} = q_1(-\frac{i}{2}c_k + \alpha c_k) + \varepsilon q_2(-\frac{i}{2}h_k + \alpha h_k) - r_1(\frac{i}{2}b_k + \alpha b_k) - \varepsilon r_2(\frac{i}{2}e_k + \alpha e_k), \\
    d_{k+1} = q_2(-\frac{i}{2}c_k + \alpha c_k) + q_1(-\frac{i}{2}h_k + \alpha h_k) - r_2(\frac{i}{2}b_k + \alpha b_k) - r_1(\frac{i}{2}e_k + \alpha e_k), \\
    c_{k+1} = -\frac{i}{2}c_k + i\alpha c_k + i\varepsilon r_2 d_k + \alpha c_k, \\
    b_{k+1} = \frac{i}{2}b_k + i\alpha c_k + i\varepsilon q_2 d_k + \alpha b_k, \\
    h_{k+1} = -\frac{i}{2}h_k + i\varepsilon q_2 d_k + i\alpha c_k + \alpha h_k, \\
    e_{k+1} = \frac{i}{2}e_k + i\alpha d_k + i\varepsilon q_2 d_k + \alpha e_k.
\end{cases}
\]

We take initial values

\[
b_0 = \alpha_1 q_1, \quad c_0 = \alpha_1 r_1, \quad e_0 = \alpha_1 q_2, \quad h_0 = \alpha_1 r_2,
\]

and then

\[
a_0 = -\frac{i\alpha_1}{2}(q_1 r_1 + \varepsilon q_2 r_2) + \beta_0(t), \quad d_0 = -\frac{i\alpha_1}{2}(q_2 r_1 + q_1 r_2) + \gamma_0(t),
\]

\[
b_1 = \frac{i\alpha_1}{2} q_1 x + \alpha_1 a_1 + \frac{\alpha_1}{2} q_1 (q_1 r_1 + \varepsilon q_2 r_2) + \frac{\varepsilon \alpha_1}{2} q_2 (q_2 r_1 + q_1 r_2) + i\alpha_1 \beta_0(t) + i\varepsilon q_2 \gamma_0(t),
\]

\[
c_1 = -\frac{i\alpha_1}{2} q_2 x + \alpha_1 a_1 + \frac{\alpha_1}{2} q_2 (q_1 r_1 + \varepsilon q_2 r_2) + \frac{\varepsilon \alpha_1}{2} q_1 (q_2 r_1 + q_1 r_2) + i\alpha_1 \beta_0(t) + i\varepsilon q_2 \gamma_0(t),
\]

\[
e_1 = \frac{i\alpha_1}{2} q_2 x + \alpha_1 a_1 + \frac{\alpha_1}{2} q_2 (q_1 r_1 + \varepsilon q_2 r_2) + \frac{\alpha_1}{2} q_2 (q_2 r_1 + q_1 r_2) + i\alpha_1 \beta_0(t) + i\gamma_0(t),
\]

\[
h_1 = -\frac{i\alpha_1}{2} q_2 x + \alpha_1 a_1 + \frac{\alpha_1}{2} q_2 (q_1 r_1 + \varepsilon q_2 r_2) + \frac{\alpha_1}{2} q_2 (q_2 r_1 + q_1 r_2) + i\alpha_1 \beta_0(t) + i\gamma_0(t),
\]

\[
\vdots
\]

where \(\gamma_0(t)\) and \(\beta_0(t)\) are integral constant.

Noting

\[
V_{1+}^{(n)} = \sum_{k=0}^{n} (a_k, b_k, c_k, d_k, e_k, h_k) T \lambda^{2(n-k)}, \quad V_{1-}^{(n)} = \sum_{k=n+1}^{\infty} (a_k, b_k, c_k, d_k, e_k, h_k) T \lambda^{2(n-k)},
\]
it follows that (34) can be decomposed into \(- (V^{(n)}_{1,+})_x + [U, V^{(n)}_{1,+}] = (V^{(n)}_{1,-})_x - [U, V^{(n)}_{1,-}]\). The gradation of the left-hand side \(\geq 0\), while the right-hand side \(\leq 1\). Hence, one has
\[
- (V^{(n)}_{1,+})_x + [U, V^{(n)}_{1,+}] = (-q_1 c_{n+1} + r_1 b_{n+1} - \varepsilon q_2 h_{n+1} + \varepsilon r_2 e_{n+1}, 2i b_{n+1} \lambda, -2 i c_{n+1} \lambda,
- q_1 h_{n+1} + r_2 b_{n+1} - q_2 e_{n+1} + r_1 e_{n+1}, 2i e_{n+1} \lambda, -2 i h_{n+1} \lambda)^T.
\]

Taking the modified term \(\overline{\Delta}_n = (-a_n, 0, 0, -d_n, 0, 0)^T\) so that for \(V^{(n)}_1 = V^{(n)}_{1,+} + \overline{\Delta}_n\), then
\[
- V^{(n)}_{1,x} + [U, V^{(n)}_1] = [0, 2(i b_{n+1} + q_1 a_n + \varepsilon q_2 d_n), -2(i c_{n+1} + r_1 a_n + \varepsilon r_2 d_n),
0, 2(i e_{n+1} + q_2 a_n + q_1 d_n), -2(i h_{n+1} + r_2 a_n + r_1 d_n)]^T \lambda.
\]

Furthermore, by solving the zero curvature equation
\[
U_t - V^{(n)}_{1,x} + [U, V^{(n)}_1] = 0,
\]
we obtain the extended isospectral integrable Schrödinger hierarchy as follows:
\[
\begin{pmatrix}
q_1 \\
r_1 \\
q_2 \\
r_2
\end{pmatrix}_n =
\begin{pmatrix}
-2i b_{n+1} - 2q_1 a_n - 2\varepsilon q_2 d_n \\
2i c_{n+1} + 2r_1 a_n + 2\varepsilon r_2 d_n \\
-2i e_{n+1} - 2q_2 a_n - 2q_1 d_n \\
2i h_{n+1} + 2r_2 a_n + 2r_1 d_n
\end{pmatrix} =
\begin{pmatrix}
b_{n+1} \\
c_{n+1} \\
e_{n+1} \\
h_{n+1}
\end{pmatrix} + 2i \alpha
\]
(37)

6.1. Hamiltonian structure

From \(\boxplus\), one has
\[
\frac{\partial U}{\partial q_1} = (0, \lambda, 0, 0, 0, 0)^T, \quad \frac{\partial U}{\partial r_1} = (0, \lambda, 0, 0, 0, 0)^T, \quad \frac{\partial U}{\partial q_2} = (0, 0, 0, \lambda, 0, 0)^T,
\]
\[
\frac{\partial U}{\partial r_2} = (0, 0, 0, 0, \lambda)^T, \quad \frac{\partial U}{\partial \lambda} = (-2i \lambda, q_1, r_1, 0, q_2, r_2)^T,
\]
\[
\{V_1, \frac{\partial U}{\partial q_1}\} = \{V_1, \frac{\partial U}{\partial q_2}\} = \sum_{k \geq 0} (c_k + h_k) \lambda^{2-2k}, \quad \{V_1, \frac{\partial U}{\partial q_2}\} = \{V_1, \frac{\partial U}{\partial \lambda}\} = \sum_{k \geq 0} (c_k + \varepsilon h_k) \lambda^{2-2k},
\]
\[
\{V_1, \frac{\partial U}{\partial r_1}\} = \sum_{k \geq 0} (b_k + e_k) \lambda^{2-2k}, \quad \{V_1, \frac{\partial U}{\partial r_2}\} = \sum_{k \geq 0} (b_k + \varepsilon e_k) \lambda^{2-2k},
\]
\[
\{V_1, \frac{\partial U}{\partial \lambda}\} = \sum_{k \geq 0} (-4i a_k - 4id_k + q_1 c_k + q_1 h_k + q_2 c_k + q_2 h_k + q_2 d_k + q_1 l_k + r_2 e_k + \varepsilon r_2 e_k) \lambda^{1-2k}.
\]

It follows that the quadratic form trace identity
\[
\frac{\delta}{\delta u} \{V_1, \frac{\partial U}{\partial \lambda}\} = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left\{ \begin{pmatrix} V_1, \frac{\partial U}{\partial q_1} \\ V_1, \frac{\partial U}{\partial q_2} \\ V_1, \frac{\partial U}{\partial r_1} \\ V_1, \frac{\partial U}{\partial r_2} \end{pmatrix} \right\},
\]

admits that
\[
\frac{\delta}{\delta u} [2(a_n + d_n) + (b_n + e_n)q + (c_n + h_n)r + b_n u_1 + c_n u_2] = (2 - 2n + \gamma) \begin{pmatrix} c_n + h_n \\ b_n + e_n \\ c_n + \varepsilon h_n \\ b_n + \varepsilon e_n \end{pmatrix}.
\]
It follows that the Hamiltonian structure of the expanding isospectral integrable hierarchy (37) is obtained as

\[
\left( \begin{array}{c} c_n + h_n \\ b_n + e_n \\ c_n + \varepsilon h_n \\ b_n + \varepsilon e_n \end{array} \right) = \frac{\delta H_n}{\delta u},
\]

where

\[
H_n = \frac{4i(a_n + d_n) - r_1(b_n + e_n) - r_2(b_n + \varepsilon e_n) - q_1(c_n + h_n) - q_2(c_n + \varepsilon h_n)}{2^n}.
\]

It follows that the Hamiltonian structure of the expanding isospectral integrable hierarchy is obtained as follows:

\[
u_t = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix} \bigg|_{t_n} = \begin{pmatrix} c_n + h_n \\ b_n + e_n \\ c_n + \varepsilon h_n \\ b_n + \varepsilon e_n \end{pmatrix} = J_3 \frac{\delta H_n}{\delta u},
\]

where

\[
J_3 = \frac{1}{\varepsilon - 1} \begin{pmatrix} 0 & \varepsilon \partial - 2i\varepsilon \alpha & 0 & -\partial + 2i\alpha \\ \varepsilon \partial + 2i\varepsilon \alpha & 0 & -\partial - 2i\alpha & 0 \\ -\partial - 2i\alpha & 0 & \partial + 2i\alpha & 0 \end{pmatrix}.
\]

By using the recurrence equations, we obtain

\[
u_t = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix} \bigg|_{t_n} = \begin{pmatrix} c_{n+1} + h_{n+1} \\ b_{n+1} + e_{n+1} \\ c_{n+1} + \varepsilon h_{n+1} \\ b_{n+1} + \varepsilon e_{n+1} \end{pmatrix} = J_4 \frac{\delta H_{n+1}}{\delta u},
\]

where

\[
J_4 = \frac{2}{\varepsilon - 1} \begin{pmatrix} \varepsilon J_{41} & J_{42} \\ J_{42} & J_{41} \end{pmatrix},
\]

with

\[
J_{41} = \begin{pmatrix} q_2 \partial^{-1} q_1 + q_1 \partial^{-1} q_2 - q_1 \partial^{-1} q_1 - \varepsilon q_2 \partial^{-1} q_2 & -i + q_1 \partial^{-1} r_1 + \varepsilon q_2 \partial^{-1} r_2 - q_1 \partial^{-1} r_2 - q_2 \partial^{-1} r_1 \\ i + r_1 \partial^{-1} q_1 + \varepsilon r_2 \partial^{-1} q_2 - r_1 \partial^{-1} q_2 - r_2 \partial^{-1} q_1 & r_1 \partial^{-1} r_2 + r_2 \partial^{-1} r_1 - r_1 \partial^{-1} r_1 - \varepsilon r_2 \partial^{-1} r_2 \end{pmatrix},
\]

\[
J_{42} = \begin{pmatrix} q_1 \partial^{-1} q_1 + \varepsilon q_2 \partial^{-1} q_2 - \varepsilon q_1 \partial^{-1} q_2 - \varepsilon q_2 \partial^{-1} q_1 & i - q_1 \partial^{-1} r_1 - \varepsilon q_2 \partial^{-1} r_2 + \varepsilon q_1 \partial^{-1} r_2 + \varepsilon q_2 \partial^{-1} r_1 \\ -i - r_1 \partial^{-1} q_1 - \varepsilon r_2 \partial^{-1} q_2 + \varepsilon r_1 \partial^{-1} q_2 + \varepsilon r_2 \partial^{-1} q_1 & r_1 \partial^{-1} r_1 + \varepsilon r_2 \partial^{-1} r_2 - \varepsilon r_1 \partial^{-1} r_2 - \varepsilon r_2 \partial^{-1} r_1 \end{pmatrix}.
\]

By using the recurrence equations, one has

\[
\begin{pmatrix} c_{n+1} + h_{n+1} \\ b_{n+1} + e_{n+1} \\ c_{n+1} + \varepsilon h_{n+1} \\ b_{n+1} + \varepsilon e_{n+1} \end{pmatrix} = L_1 \begin{pmatrix} c_n + h_n \\ b_n + e_n \\ c_n + \varepsilon h_n \\ b_n + \varepsilon e_n \end{pmatrix} + i\beta_n(t) \begin{pmatrix} r_1 + r_2 \\ q_1 + q_2 \\ r_1 + \varepsilon r_1 \\ q_1 + \varepsilon q_2 \end{pmatrix} + i\gamma_n(t) \begin{pmatrix} q_1 + \varepsilon r_2 \\ r_1 + \varepsilon r_2 \\ q_1 + \varepsilon q_2 \\ \varepsilon q_1 + \varepsilon q_2 \end{pmatrix},
\]

\(17\)}
where

\[ L_1 = \frac{2}{\varepsilon - 1} \begin{pmatrix} L_{11} & L_{12} \\ \varepsilon L_{12} & L_{11} \end{pmatrix}, \]

with

\[ L_{11} = \begin{pmatrix} -\frac{1}{\varepsilon + 1} q_1 \partial_x + \frac{1}{\varepsilon + 1} q_1 \partial_y + i a \partial_x q_1 + i a \partial_y q_1 \\ -i r_1 \partial_x - r_1 \partial_y + i r_2 \partial_x - i r_2 \partial_y - i \gamma - r_1 - i \gamma - r_2 \end{pmatrix}, \]

\[ L_{12} = \begin{pmatrix} -\frac{1}{\varepsilon + 1} q_1 \partial_x + \frac{1}{\varepsilon + 1} q_1 \partial_y + i a \partial_x q_1 + i a \partial_y q_1 \\ -i r_1 \partial_x - r_1 \partial_y + i r_2 \partial_x - i r_2 \partial_y - i \gamma - r_1 - i \gamma - r_2 \end{pmatrix}. \]

Therefore, we obtain the bi-Hamiltonian structure of the extended isospectral integrable Schrödinger hierarchy

\[ u_{\varepsilon} = J_3 \frac{\delta H_n}{\delta u} = J_4 \frac{\delta H_{n+1}}{\delta u} \]

\[ = J_4 L_1 \begin{pmatrix} c_n + h_{n+1} \\ b_n + e_{n+1} \\ c_n + \varepsilon h_{n+1} \\ b_n + \varepsilon e_{n+1} \end{pmatrix} + i \beta_n(t) J_4 \begin{pmatrix} r_1 + r_2 \\ q_1 + q_2 \\ r_1 + \varepsilon r_2 \\ q_1 + \varepsilon q_2 \end{pmatrix} + i \gamma_n(t) J_4 \begin{pmatrix} r_1 + r_2 \\ q_1 + q_2 \\ r_1 + \varepsilon r_2 \\ q_1 + \varepsilon q_2 \end{pmatrix} \]

\[ = (J_4 L_1^2) \alpha_1 + \sum_{j=0}^{n} i \beta_{n-j} (J_4 L_1^2) \begin{pmatrix} r_1 + r_2 \\ q_1 + q_2 \\ r_1 + \varepsilon r_2 \\ q_1 + \varepsilon q_2 \end{pmatrix} + \sum_{j=0}^{n} i \gamma_{n-j} (J_4 L_1^2) \begin{pmatrix} r_1 + r_2 \\ q_1 + q_2 \\ r_1 + \varepsilon r_2 \\ q_1 + \varepsilon q_2 \end{pmatrix}. \]

When \( n = 1, \alpha_1 = 2, \alpha = 0 \), the integrable hierarchy reduces to

\[ u_1 = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} i q_1 x + (q_1 r_1 + \varepsilon q_2 r_1 + 2 \varepsilon q_1 q_2 r_2) x + i \beta_0(t) q_1 x + i \varepsilon \gamma_0(t) q_2 x \\ -i r_1 x + (r_1 q_1 + \varepsilon r_2 q_1 + 2 \varepsilon q_1 q_2 r_2) x + i \beta_0(t) r_1 x + i \varepsilon \gamma_0(t) r_2 x \\ i q_2 x + (q_1 r_1 + \varepsilon q_2 r_1 + 2 \varepsilon q_1 q_2 r_2) x + i \beta_0(t) q_2 x + i \varepsilon \gamma_0(t) q_1 x \\ -i r_2 x + (r_1 q_1 + \varepsilon r_2 q_1 + 2 \varepsilon q_1 q_2 r_2) x + i \beta_0(t) r_2 x + i \varepsilon \gamma_0(t) r_1 x \end{pmatrix}, \]

which is an expanded isospectral derivative nonlinear Schrödinger system.

7. An expanded nonisospectral integrable Schrödinger hierarchy

In this section, we will discuss an expanded nonisospectral integrable hierarchy of the hierarchy. From the isospectral-nonisospectral problem, the nonisospectral stationary zero curvature equation

\[ V_{xx} = \frac{\partial U}{\partial x} \lambda_t + [U, V_2] \]

gives rise to

\[ \begin{align*}
    a_{xj} &= q_1 c_{j+1} - r_1 b_{j+1} + \varepsilon q_2 h_{j+1} - \varepsilon r_2 e_{j+1} - 2 i k_{j+1}(t), \\
    b_{xj} &= 2(-i b_{j+1} + i a b_k - q_1 a_j - \varepsilon q_2 d_j) + q_1 k_j(t), \\
    c_{xj} &= 2(i c_{j+1} - i a c_j + r_1 a_j + \varepsilon r_2 d_j) + r_1 k_j(t), \\
    d_{xj} &= q_1 h_{j+1} - r_2 b_{j+1} - r_1 e_{j+1} + q_2 e_{j+1}, \\
    e_{xj} &= 2(-i e_{j+1} + i a e_j - q_2 a_j - q_1 d_j) + q_3 k_j(t), \\
    h_{xj} &= 2(i h_{j+1} - i a h_j + r_2 a_j + r_1 d_j) + r_2 k_j(t),
\end{align*} \]
which has an equivalent form as follows:

\[
\begin{align*}
  a_{jx} &= q_1(-\frac{i}{2}c_{jx} + a\epsilon_j) + q_2(-\frac{i}{2}h_{jx} + a\eta_j) - r_1(\frac{i}{2}b_{jx} + a\eta_j) - \varepsilon r_2(\frac{i}{2}e_{jx} + a\epsilon_j) \\
  &\quad + i(q_1r_1 + q_2r_2)k_j(t) - 2ik_{j+1}(t), \\
  d_{jx} &= q_2(-\frac{i}{2}c_{jx} + a\epsilon_j) + q_1(-\frac{i}{2}h_{jx} + a\eta_j) - r_2(\frac{i}{2}b_{jx} + a\eta_j) - r_1(\frac{i}{2}e_{jx} + a\epsilon_j) \\
  &\quad + i(q_1r_2 + q_2r_1)k_j(t), \\
  c_{j+1} &= -\frac{i}{2}c_{jx} + ir_1a_j + i\varepsilon r_2d_j + a\epsilon_j + \frac{i}{2}r_1k_j(t), \\
  b_{j+1} &= \frac{i}{2}b_{jx} + iq_1a_j + i\varepsilon q_2d_j + a\eta_j - \frac{i}{2}q_1k_j(t), \\
  h_{j+1} &= -\frac{i}{2}h_{jx} + ir_2a_j + ir_1d_j + a\eta_j + \frac{i}{2}r_2k_j(t), \\
  e_{j+1} &= \frac{i}{2}e_{jx} + iq_2d_j + iq_2a_j + a\epsilon_j - \frac{i}{4}q_2k_j(t).
\end{align*}
\]

By taking

\[
b_0 = \alpha_1q_1, \quad c_0 = \alpha_1r_1, \quad e_0 = \alpha_1q_2, \quad h_0 = \alpha_1r_2,
\]

one has

\[
\begin{align*}
  a_0 &= -\frac{i\alpha_1}{2}(q_1r_1 + q_2r_2) + i\partial^{-1}(q_1r_1 + q_2r_2)k_0(t) - 2ixk_1(t) + \eta_0(t), \\
  d_0 &= -\frac{i\alpha_1}{2}(q_2r_1 + q_1r_2) + i\partial^{-1}(q_1r_2 + q_2r_1)k_0(t) + \eta_0(t), \\
  b_1 &= \frac{i\alpha_1}{2}q_{1x} + \alpha_2q_1 + \frac{\alpha_1}{2}q_1(q_1r_1 + q_2r_2) + \frac{\alpha_1}{2}q_2(q_2r_1 + q_1r_2) + iq_1\eta_0(t) + i\varepsilon q_2\bar{\eta}_0(t) \\
  &\quad + (-q_1\partial^{-1}(q_1r_1 + q_2r_2) - q_2\partial^{-1}(q_1r_2 + q_2r_1) - \frac{i}{2}q_1k_1(t) + 2xq_1k_1(t), \\
  c_1 &= -\frac{i\alpha_1}{2}r_{1x} + \alpha_2r_1 + \frac{\alpha_1}{2}r_1(q_1r_1 + q_2r_2) + \frac{\alpha_1}{2}r_2(q_2r_1 + q_1r_2) + ir_1\eta_0(t) + i\varepsilon r_2\bar{\eta}_0(t) \\
  &\quad + (-r_1\partial^{-1}(q_1r_1 + q_2r_2) - r_2\partial^{-1}(q_1r_2 + q_2r_1) + \frac{i}{2}r_1k_1(t) + 2xr_1k_1(t), \\
  e_1 &= \frac{i\alpha_1}{2}q_{2x} + \alpha_2q_2 + \frac{\alpha_1}{2}q_2(q_1r_1 + q_2r_2) + \frac{\alpha_1}{2}q_1(q_2r_1 + q_1r_2) + iq_2\eta_0(t) + i\varepsilon q_2\bar{\eta}_0(t) \\
  &\quad + (-q_2\partial^{-1}(q_1r_1 + q_2r_2) - q_1\partial^{-1}(q_1r_2 + q_2r_1) - \frac{i}{2}q_2k_1(t) + 2xq_2k_1(t), \\
  h_1 &= -\frac{i\alpha_1}{2}r_{2x} + \alpha_2r_2 + \frac{\alpha_1}{2}r_2(q_1r_1 + q_2r_2) + \frac{\alpha_1}{2}r_1(q_2r_1 + q_1r_2) + ir_2\eta_0(t) + i\varepsilon r_2\bar{\eta}_0(t) \\
  &\quad + (-r_2\partial^{-1}(q_1r_1 + q_2r_2) - r_1\partial^{-1}(q_1r_2 + q_2r_1) + \frac{i}{2}r_2k_1(t) + 2xr_2k_1(t),
\end{align*}
\]

where \(\eta_0(t)\) and \(\bar{\eta}_0(t)\) are integral constants.

Noting

\[
V_{2,+}^{(m)} = \sum_{j=0}^{m}(a_j, b_j\lambda, c_j\lambda, d_j, e_j\lambda, h_j\lambda)^T\lambda^{2(m-j)}, \quad V_{2,-}^{(m)} = \sum_{j=m+1}^{\infty}(a_j, b_j\lambda, c_j\lambda, d_j, e_j\lambda, h_j\lambda)^T\lambda^{2(m-j)},
\]
it follows that can be decomposed into \(-(V_{2,+}^{(m)})_x + [U, V_{2,+}^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda^{(m)}_{t,+} = (V_{2,-}^{(m)})_x - [U, V_{2,-}^{(m)}] - \frac{\partial U}{\partial \lambda} \lambda^{(m)}_{t,-} \). The gradation of the left-hand side ≥ 0, while the right-hand side ≤ 1. Hence, one has

\[-(V_{2,+}^{(m)})_x + [U, V_{2,+}^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda^{(m)}_{t,+} = (-q_1 c_{m+1} + r_1 b_{m+1} - \varepsilon q_2 h_{m+1} + \varepsilon r_2 m_{m+1} + 2i k_{m+1}(t), 2i b_{m+1} \lambda, -2i c_{m+1} \lambda, -q_1 h_{m+1} + r_2 b_{m+1} - q_2 c_{m+1} + r_1 e_{m+1}, 2i e_{m+1} \lambda, -2i h_{m+1} \lambda)^T.\]

Taking the modified term \(\overline{\Delta}_m = (-a_m, 0, 0, -d_m, 0, 0)^T\) so that for \(V_2^{(m)} = V_2^{(m)} + \overline{\Delta}_m\), then we obtain

\[-V_2^{(m)} + [U, V_2^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda^{(m)}_{t,+} = 0, 2i b_{m+1} + q_1 a_m + \varepsilon q_2 d_m, -2i c_{m+1} + r_1 a_m + \varepsilon r_2 d_m, 0, 2i e_{m+1} + q_2 a_m + q_1 d_m, -2i h_{m+1} + r_2 a_m + r_1 d_m)^T.\]

Furthermore, by solving the nonisospectral zero curvature equation

\[U_t - V_2^{(m)} + [U, V_2^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda^{(m)}_{t,+} = 0,\]

we obtain the extended nonisospectral integrable Schrödinger hierarchy as follows:

\[u_{tm} = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}_{tm} = \begin{pmatrix} -2ib_{m+1} - 2q_1 a_m - 2\varepsilon q_2 d_m \\ 2ic_{m+1} + 2r_1 a_m + 2\varepsilon r_2 d_m \\ -2i e_{m+1} - 2q_2 a_m - 2q_1 d_m \\ 2ih_{m+1} + 2r_2 a_m + 2r_1 d_m \end{pmatrix} = \begin{pmatrix} b_{m+1} \\ c_{m+1} \\ e_{m+1} \\ h_{m+1} \end{pmatrix} = \begin{pmatrix} b_{m+1} \lambda \\ c_{m+1} \lambda \\ e_{m+1} \lambda \\ h_{m+1} \lambda \end{pmatrix} = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}_{tm}. \tag{46}\]

### 7.1. Hamiltonian structure

Similar to section 6.1, the bi-Hamiltonian structure of the expanded nonisospectral integrable hierarchy \([48]\) is obtained as follows:

\[u_{tn} = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}_{tn} = J_3 \begin{pmatrix} c_{m+1} + h_{m+1} \\ b_{m+1} + e_{m+1} \\ c_{m+1} + \varepsilon h_{m+1} \\ b_{m+1} + \varepsilon e_{m+1} \end{pmatrix} + 4i k_{m+1}(t)x \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}_{tn} = J_3 \frac{\delta H_{m+1}}{\delta u} - k_{m}(t) \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}. \tag{47}\]

where \(J_3\) and \(J_4\) are given by \([38]\) and \([39]\), respectively.

By using the recursion equations \([44]\), one has

\[
\begin{pmatrix}
    c_{m+1} + h_{m+1} \\
    b_{m+1} + e_{m+1} \\
    c_{m+1} + \varepsilon h_{m+1} \\
    b_{m+1} + \varepsilon e_{m+1}
\end{pmatrix} = L_1 \begin{pmatrix}
    c_{m+1} + h_{m+1} \\
    b_{m+1} + e_{m+1} \\
    c_{m+1} + \varepsilon h_{m+1} \\
    b_{m+1} + \varepsilon e_{m+1}
\end{pmatrix} + i \overline{\eta}_m(t) \begin{pmatrix}
    r_1 + r_2 \\
    q_1 + q_2 \\
    r_1 + \varepsilon r_2 \\
    q_1 + \varepsilon q_2
\end{pmatrix} + \frac{i}{2} k_{m}(t) \begin{pmatrix}
    r_1 + r_2 \\
    q_1 + q_2 \\
    r_1 + \varepsilon r_2 \\
    q_1 + \varepsilon q_2
\end{pmatrix}.
\]
where $L_1$ is given by (40).

Therefore, we have

\[
\begin{align*}
  u_{t_m} &= J_4 \frac{\delta H_m}{\delta u} - k_m(t)U_5 = J_4 \frac{\delta H_{m+1}}{\delta u} + 4i k_{m+1}(t) x U_4 \\
  &= J_4 L_1 \frac{\delta H_m}{\delta u} + i \beta_m(t) J_4 U_1 + i \gamma_m(t) J_4 U_2 + \frac{i}{2} k_m(t) J_4 U_3 + 4i k_{m+1}(t) x U_4 \\
  &= (J_4 L_1^{m+1} + 1 + \sum_{j=0}^{m} i \gamma_{m-j}(t) J_4 L_1^j) U_1 + \sum_{j=0}^{m} i \gamma_{m-j}(t) J_4 L_1^j U_2 + \sum_{j=0}^{m} \frac{i}{2} k_{m-j}(t) J_4 L_1^j U_3 + 4i k_{m+1}(t) x U_4,
\end{align*}
\]

where

\[
U_1 = \begin{pmatrix} r_1 + r_2 \\ q_1 + q_2 \\ r_1 + r_2 \\ q_1 + q_2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} r_1 + r_2 \\ q_1 + q_2 \\ \varepsilon r_1 + \varepsilon r_2 \\ \varepsilon q_1 + \varepsilon q_2 \end{pmatrix}, \quad U_3 = \begin{pmatrix} q_1 + q_2 \\ r_1 - q_2 \\ r_1 + r_2 \\ q_1 - q_2 \end{pmatrix}, \quad U_4 = \begin{pmatrix} q_1 \\ -r_1 \\ q_2 \\ -r_2 \end{pmatrix}, \quad U_5 = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}.
\]

By solving the zero curvature equation

\[
\frac{\partial \overline{U}}{\partial t} + \frac{\partial U}{\partial x} \lambda^{(m)}_{k+} - V^{(m)}_x - V^{(n)}_x + [U, V^{(m)} + V^{(n)}] = 0,
\]

we deduce the expanded isospectral-nonisospectral integrable Schrödinger hierarchy

\[
\begin{align*}
  u_{t_m} &= J_4 \left[ (L_1^{m+1} + 1 + \sum_{j=0}^{m} i \gamma_{m-j}(t) L_1^j) U_1 + \sum_{j=0}^{m} i \gamma_{m-j}(t) L_1^j U_2 + \sum_{j=0}^{m} \frac{i}{2} k_{m-j}(t) L_1^j U_3 \right] + 4i k_{m+1}(t) x U_4.
\end{align*}
\]

We now discuss some reductions of the hierarchy (48).

- **When $n = 1$, $\alpha_1 = 2$, $\alpha = 0$, the integrable hierarchy (48) reduces to**

\[
\begin{align*}
  u_4 &= \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix} = \begin{pmatrix}
  i q_{1x} + (q_1^2 r_1 + \varepsilon q_1^2 r_1 + 2 \varepsilon q_1 q_2 r_2)_x + i \beta_0(t) q_1 x + i \varepsilon \gamma_0(t) q_2 x \\
  -r_1 q_{1x} + (r_1^2 q_1 + \varepsilon r_1^2 q_1 + 2 \varepsilon q_1 q_2 r_2)_x + i \beta_0(t) r_1 x + i \varepsilon \gamma_0(t) r_2 x \\
  i q_{2x} + (q_1^2 q_2 + \varepsilon q_1^2 q_2 + 2 q_1 q_2 r_1)_x + i \beta_0(t) q_2 x + i \varepsilon \gamma_0(t) q_1 x \\
  -r_2 q_{2x} + (r_1^2 q_2 + \varepsilon r_1^2 q_2 + 2 q_1 q_2 r_1)_x + i \beta_0(t) r_2 x + i \varepsilon \gamma_0(t) r_1 x
\end{pmatrix} + k_0(t) (U_0)_x + k_1(t) U_7,
\end{align*}
\]

where

\[
U_0 = \begin{pmatrix}
  -q_1 \partial^{-1}(q_1 r_1 + \varepsilon q_2 r_2) - q_2 \partial^{-1}(q_1 r_2 + q_2 r_1) - \frac{i}{2} q_1 \\
  -r_1 \partial^{-1}(q_1 r_1 + \varepsilon q_2 r_2) - q_2 \partial^{-1}(q_1 r_2 + q_2 r_1) + \frac{i}{2} r_1 \\
  -q_2 \partial^{-1}(q_1 r_1 + q_2 r_2) - q_1 \partial^{-1}(q_1 r_2 + q_2 r_1) - \frac{i}{2} q_2 \\
  -r_2 \partial^{-1}(q_1 r_1 + q_2 r_2) - r_1 \partial^{-1}(q_1 r_2 + q_2 r_1) + \frac{i}{2} r_2
\end{pmatrix}, \quad U_7 = \begin{pmatrix}
  q_1 + 2 x q_{1x} \\
  r_1 + 2 x r_{1x} \\
  q_2 + 2 x q_{2x} \\
  r_2 + 2 x r_{2x}
\end{pmatrix}.
\]

- **When $k_0(t) = k_1(t) = 0$, $\beta_0(t) = \overline{\beta}_0(t)$, $\gamma_0(t) = \overline{\gamma}_0(t)$, the system (50) reduces to (12), and thus (50) is an expanded nonisospectral derivative nonlinear Schrödinger system.**

- **When $q_1 = r_1 = r_2 = 0$, $\alpha = 0$, the hierarchy (48) reduces to the linear equation**

\[
q_{2x} = \frac{i}{2} q_1 q_{2xx} + i \beta_0(t) q_{2x} - \frac{i}{2} q_2 k_0(t) + (2 x q_{2x} + 2 q_2) k_1(t).
\]
• When $\overline{\beta}_0(t) = k_0(t) = k_1(t) = 0$, $\alpha_1 = -2i$, (51) becomes the heat equation

$$w_t = w_{xx}. \quad (52)$$

• When $\overline{\beta}_0(t) = k_0(t) = 0$, $k_1(t) = \frac{1}{2}$, $\alpha_1 = -2i$, (51) becomes the Fokker-Plank equation

$$w_t = w_{xx} + w + xw_x, \quad (53)$$

which has a wide range of applications in stochastic dynamic systems.

## 8. A $Z_N^\varepsilon$ nonisospectral integrable Schrödinger hierarchy

In order to deduce a $Z_N^\varepsilon$ nonisospectral integrable hierarchy related to (51), we will use the column-vector loop algebra $\mathbb{C}^{3N}$ of section 5. Considering the $3N$ dimensional nonisospectral problems

$$\begin{cases}
\varphi_x = \overline{U} \varphi, \quad \overline{U} = (-i\lambda^2 + i\alpha, q_1\lambda, r_1\lambda, 0, q_2\lambda, r_2\lambda, 0, q_3\lambda, r_3\lambda, \cdots, 0, q_N\lambda, r_N\lambda)^T, \\
\varphi_t = \nabla \varphi, \quad \nabla = \sum_{l \geq 0} (a_{1l}, b_{1l}, c_{1l}, a_{2l}, b_{2l}, c_{2l}, a_{3l}, b_{3l}, c_{3l}, \cdots, a_{Nl}, b_{Nl}, c_{Nl})^T \lambda^{2l}, \\
\lambda_t = \sum_{l \geq 0} k_l(t) \lambda^{l-2l}.
\end{cases} \quad (54)$$

By solving the nonisospectral stationary zero curvature equation

$$\nabla_x = [\overline{U}, \nabla] + \frac{\partial \overline{U}}{\partial \lambda} \lambda_t, \quad (55)$$

we obtain the recursion equations as follows:

$$\begin{cases}
a_{kl,x} = -2ik_{l+1} + \sum_{i+j=k+1, 1 \leq i,j \leq k} (q_i c_{j,l+1} - r_j b_{j,l+1}) + \sum_{m+n=k+N+1, k+l \leq m, n \leq N} \sigma \varepsilon (q_m c_{n,l+1} - r_m b_{n,l+1}), \\
b_{kl,x} = q_k k_l(t) - 2ib_{l+1} + 2i\alpha b_{l+1} - \sum_{i+j=k+1, 1 \leq i,j \leq k} 2q_i a_{j,l+1} - \sum_{m+n=k+N+1, k+l \leq m, n \leq N} 2\sigma \varepsilon q_m a_{n,l+1}, \\
c_{kl,x} = r_k k_l(t) + 2ic_{l+1} - 2i\alpha c_{l+1} + \sum_{i+j=k+1, 1 \leq i,j \leq k} 2r_i a_{j,l+1} + \sum_{m+n=k+N+1, k+l \leq m, n \leq N} 2\sigma \varepsilon r_m a_{n,l+1}, \\
k = 1, 2, \cdots, N, \quad l \geq 0,
\end{cases} \quad (56)$$

where

$$\sigma = \begin{cases} 
1, & 1 \leq k \leq N - 1 \\
0, & k = N
\end{cases}.$$  

Noting

$$\begin{aligned}
\nabla_+^{(m)} &= \sum_{l=0}^{m} (a_{1l}, b_{1l}, c_{1l}, a_{2l}, b_{2l}, c_{2l}, a_{3l}, b_{3l}, c_{3l}, \cdots, a_{Nl}, b_{Nl}, c_{Nl})^T \lambda^{2l}, \\
\nabla_-^{(m)} &= \sum_{l=m+1}^{\infty} (a_{1l}, b_{1l}, c_{1l}, a_{2l}, b_{2l}, c_{2l}, a_{3l}, b_{3l}, c_{3l}, \cdots, a_{Nl}, b_{Nl}, c_{Nl})^T \lambda^{2l},
\end{aligned}$$

it follows that (53) can be decomposed into $-(\nabla_+^{(m)})_x + [\overline{U}, \nabla_+^{(m)}] + \frac{\partial \overline{U}}{\partial \lambda} \lambda_+ = (\nabla_-^{(m)})_x - [\overline{U}, \nabla_-^{(m)}] - \frac{\partial \overline{U}}{\partial \lambda} \lambda_-$. Taking the modified term $\tilde{\Delta}_m = (-a_{1m}, 0, 0, -a_{2m}, 0, 0, \cdots, -a_{Nm}, 0, 0)^T$ so that for $\nabla^{(m)} = \nabla_+^{(m)} + \tilde{\Delta}_m$, then the zero curvature equation

$$U_t - V_2^{(m)} + [U, V_2^{(m)}] + \frac{\partial \overline{U}}{\partial \lambda} \lambda_+ = 0,$$
gives rise to the $Z^N_\varepsilon$ nonisospectral integrable Schrödinger hierarchy:

$$u_{tm} = \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \\ \vdots \\ q_N \\ r_N \end{pmatrix}_{tm},$$

(57)

where

$$(q_k)_{tm} = -2ib_{k,m+1} - \sum_{i+j=k+1, 1 \leq i,j \leq k} 2q_i a_j,m - \sum_{s+n=k+N+1, k+1 \leq s,n \leq N} 2\sigma \varepsilon q_s a_n,m = (b_{k,m})_x - 2i\alpha b_{k,m} - k_m(t)q_k,$$

$$(r_k)_{tm} = 2ic_{k,m+1} - \sum_{i+j=k+1, 1 \leq i,j \leq k} 2r_i a_j,m - \sum_{s+n=k+N+1, k+1 \leq s,n \leq N} 2\sigma \varepsilon r_s a_n,m = (c_{k,m})_x + 2i\alpha c_{k,m} - k_m(t)r_k,$$

$k = 1, 2, \cdots, N.$

9. Conclusions and discussions

To the best of our knowledge, in most of the existing literature, the generation of extended integrable hierarchies is only developed to finite dimensions. This paper presented an efficient method for generating higher-dimensional isospectral-nonisospectral integrable hierarchies, that is, introducing high-dimensional isospectral-nonisospectral problems by using the higher-dimensional column-vector loop algebras constructed by us. Based on the method, we obtained the generalized nonisospectral integrable Schrödinger hierarchy [18] and its expanded isospectral-nonisospectral integrable Schrödinger hierarchy [19]. By considering the reduction of these hierarchies, we obtained many well-known and new equations. Moreover, the bi-Hamiltonian structures of these hierarchies were discussed via the Tu scheme. Finally, we obtain the $Z^N_\varepsilon$ nonisospectral integrable Schrödinger hierarchy [57], which can be reduced to the extended nonisospectral integrable Schrödinger hierarchy [16] and the generalized nonisospectral integrable Schrödinger hierarchy [15] respectively when $N = 1, 2$.

In section 4, we only considered one of the bases of Lie algebras $gl(3)$. Actually, we can also choose a different set of bases, and the corresponding results can be deduced similarly. In addition to the application of a specific class of spectral problems discussed in this paper, the ideas and methods used in this paper are also universal for other isospectral and nonisospectral problems.

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