Relation between Hall resistance and the diamagnetic moment of Fermi electrons

P. Středa

1Institute of Physics, Academy of Sciences of the Czech Republic, Cukrovarnická 10, CZ - 162 53 Praha

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General thermodynamical arguments are used to relate the Hall current to the part of the magnetic moment originated in "macroscopic current loops". The Hall resistance is found to depend only on the electron properties in the vicinity of the Fermi energy, which is the essential advantage of the presented treatment. The obtained relation is analyzed by using Landauer-Blüttiker-like view to the electron transport. As one of the possible applications, the Hall resistance of the periodically modulated two-dimensional electron system in strong magnetic fields is briefly discussed.

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Relation between Hall effect and the diamagnetic moment of carriers has been studied for decades and often it has been the subject of the controversial discussion. The basic thermodynamical arguments are trivial. The diamagnetic moment per unit volume $\vec{M}$ is defined by the first derivative of the grand canonical potential $\Omega$ with respect of the magnetic field $B$

$$\vec{M} = -\frac{\vec{B}}{B} \left( \frac{\partial \Omega}{\partial B} \right)_\mu$$

where $N$ and $\mu$ are carrier concentration and chemical potential, respectively. Magnetic moment is supposed to be parallel with the magnetic field direction and the zero temperature is considered for simplicity. Identifying $\vec{J}$ with a current the linear response to the applied electric field $\vec{E} \equiv \nabla \phi / e$ immediately gives

$$\vec{J} = -ec \left( \frac{\partial \vec{M}}{\partial \mu} \right) \times \vec{E},$$

where $-e$ ($e > 0$) denotes the electron charge. Since the resulting current $\vec{J}$ is perpendicular to the applied electric field the tendency to identify it with the Hall current $\vec{J}$ often appears. However, it is not so simple as has been generally accepted that at least surface diamagnetic currents have to be taken into account to obtain correct values of the magnetization and that they also substantially influence the non-diagonal components of transport coefficients $\sigma_{ij}$.

For the sake of the simplicity the following treatment is limited to two-dimensional electron systems in perpendicularly applied magnetic fields.

It has been proved by several different methods that the direct relation between Hall current and magnetization, Eq. (3), can be applied whenever the chemical potential $\mu$ is located within the energy gap of the electron energy spectrum $[1, 2, 3, 4, 5, 6, 11, 12]$. For periodic systems the Maxwell relation

$$\left( \frac{\partial M}{\partial \mu} \right) \equiv \left( \frac{\partial N}{\partial B} \right)_\mu$$

together with the condition that the magnetic flux per unit cell $\Phi$ is a rational multiple of the flux quantum $\Phi_0 \equiv h c / e$, gives quantized values of the Hall conductance

$$\sigma_Q = -ec \left( \frac{\partial N}{\partial B} \right)_\mu = -\frac{e^2}{h} i \ ; \ \text{for } \mu \text{ in gap} \ .$$

Integer $i$ has to satisfy Diophantine equation $[12, 13, 14]$

$$\nu \equiv \frac{N}{eb^2} = i + s \frac{q}{p} \ ; \ \frac{p}{q} \equiv \frac{\Phi}{\Phi_0} ,$$

where $\nu = N/(eb^2)$ is filling factor and integer $s$ is additional gap quantum number $[14]$. For zero modulation $s = 0$, there are just $i$ fully occupied Landau levels below the Fermi energy and the expression Eq. (6) represents integer quantum Hall effect $[12]$. Only recently the predicted non-trivial sequence of quantum Hall values in periodically modulated systems has been observed $[16]$.

The diamagnetic moment can alternatively be evaluated by using expectation values of the corresponding operator, i.e.

$$\vec{M} \equiv -\frac{e}{2c} \text{Tr} \left[ (\mu - E) \hat{\vec{r}} \times \hat{\vec{v}} \right] ,$$

where $\hat{\vec{r}}$ and $\hat{\vec{v}}$ are electron coordinate and velocity operators, respectively, $\Theta(\mu - E)$ is the Heaviside step function. Using the Landau gauge for the vector potential, $\vec{A} \equiv (0, xB, 0)$, the single-electron Hamiltonian representing a two-dimensional electron system in the perpendicular magnetic field, say in $\hat{\vec{z}}$ direction, obeys the following form

$$H = \frac{p_z^2}{2m^*} + \frac{(py + xB)^2}{2m^*} + V_b(x, y) + V_{conf}(x) ,$$

where the confining potential $V_{conf}(x)$ defines width of the strip and $V_b(x, y)$ is a background potential. Periodic boundary conditions applied along $\hat{\vec{y}}$ direction allow to describe energy spectrum in the form of branches $\epsilon_\beta(k)$.

Each of the states given by a branch index $\beta$ and the wave number $k$ has its own center of the mass, $X_\beta(k)$, defined
as the expectation value of the $x$ coordinate. Because
of the infinite strip length the expectation values of the
operator $-yv_x$ entering the expression Eq. (7) are not
well defined. Nevertheless, as the direct consequence of
the current conservation law it can be proved, that it
gives the same contribution to the magnetic moment as
the operator $xv_y$.

In order to find relation between Hall current and the
magnetic moment in the general case, let us decompose
the magnetic moment into two parts

$$\tilde{M} \equiv \tilde{M}_i + \tilde{M}_a .$$

The part denoted as $\tilde{M}_i$ originates in microscopic current
loops determined by the relative particle motion with
respect of the center of the mass. The remaining part $\tilde{M}_a$
appears due to the presence of the macroscopic currents
and it is defined as follows

$$M_a = \frac{e}{\hbar} \sum_{\beta,k} \Theta (\mu - \varepsilon_\beta (k)) \ X_\beta (k) \ v_\beta (k) ,$$

$$v_\beta (k) = \frac{1}{\hbar} \frac{d\varepsilon_\beta (k)}{dk} ,$$

where $v_\beta (k)$ denotes velocity expectation values.

![Diagram](image)

FIG. 1: Schematic view to the hollow cylinder formed from
the two dimensional strip. Magnetic field $\vec{B}$ is perpendicular
to the strip. The magnetic moment $\Delta \tilde{M}_a$, parallel to the
cylinder axis, represents the effect of the macroscopic current
loops determined by the relative particle motion with
respect to the center of mass. The remaining part $\tilde{M}_a$
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and it is defined as follows

$$M_a = \frac{e}{\hbar} \sum_{\beta,k} \Theta (\mu - \varepsilon_\beta (k)) \ X_\beta (k) \ v_\beta (k) ,$$

$$v_\beta (k) = \frac{1}{\hbar} \frac{d\varepsilon_\beta (k)}{dk} ,$$

where $v_\beta (k)$ denotes velocity expectation values.

To clear up the role of the both parts, $\tilde{M}_i$ and $\tilde{M}_a$,
let us consider a hollow cylinder formed from the strip
as has been already suggested by Laughlin to explain
integer quantum Hall effect [17]. As illustrated in Fig. 1,
the part $\tilde{M}_i$ remains perpendicular to the strip surface,
while contributions of macroscopic loops, representing
extended states, give rise to the magnetic moment $\Delta \tilde{M}_a$
parallel with the cylinder axis. In the equilibrium $\Delta \tilde{M}_a$
vanesishes because of the zero total current. Any electron
transfer to state with opposite velocity direction gives
rise to non-zero current as well as non-zero $\Delta \tilde{M}_a$. Since
the mass-center positions of different states generally do
not coincide the transfers are leading to non-equilibrium
charge distribution defining an electric field which is in
average perpendicular to the current flow.

The above discussion leads to the conclusion that the
Hall current for the strip of the width $w$ is controlled
by only the part of the magnetic moment defined by
Eq. (9). Replacing $\tilde{M}$ in Eq. (3) by $\tilde{M}_a$ gives the
following expression for the Hall conductance

$$G_H (\mu) \equiv \frac{1}{R_H (\mu)} = - \frac{e^2}{2\pi w} \left( \frac{\partial M_a}{\partial \mu} \right) \delta$$

$$= \frac{e^2}{2\pi w} \sum_{\beta} \delta (\mu - \varepsilon_\beta (k)) \ X_\beta (k) v_\beta (k) \ dk =$$

$$\frac{e^2}{\hbar} \sum_{j=1}^{N_p} \frac{X_\beta (k_{j+\beta}) - X_\beta (k_{j-\beta})}{w} ,$$

where $j$ counts pair states $k_{j+\beta}$ and $k_{j-\beta}$ at the Fermi
energy $\mu$ having opposite velocity directions and $N_p$ is
their number. The considered boundary conditions imply
that $G_H$ is just equal to the inverse value of the Hall
resistance $R_H$.

The obtained result can be rederived by using
Landauer-Büttiker-like view to the electron transport.
The studied finite sample is supposed to be attached
via ideal leads to the source and drain. Difference of
the source and drain chemical potentials, $\Delta \mu$, induces
the current which depends on the occupation of states
within ideal leads. Assuming scattering leading to uni-
form occupation of the outgoing channels within ideal
leads, represented by transmission and reflection proba-
bilities, $t$ and $r = 1 - t$, respectively, we get

$$J \equiv \frac{e}{\hbar} \sum_{j=1}^{N_p} \left[ \Delta \varepsilon_\beta (k_{j+\beta}) - r \Delta \varepsilon_\beta (k_{j-\beta}) \right] = \frac{e}{\hbar} N_p t \Delta \mu .$$

The energy intervals $\Delta \varepsilon_\beta (k_{j+\beta}) = \Delta \mu$ of incoming states
just equals to that of outgoing states $\Delta \varepsilon_\beta (k_{j-\beta})$. Denoting
chemical potential of the drain as $\mu_0$, the effective
potential within the ideal lead on the drain side equals to
$\mu_0 + t \Delta \mu / 2$, while that on source side is $\mu_0 + (1 + r) \Delta \mu / 2$.
Identifying their difference with the voltage drop the
above outlined standard textbook procedure gives the
following expression for the sample resistance [13]

$$R = \frac{r \Delta \mu}{eJ} = \frac{\hbar}{e^2 N_p t} ,$$

where the ratio $t/r$ defines the relaxation time.
The current $J$ can alternatively be expressed as a function of the difference between effective chemical potentials of the incoming and outgoing channels, $\mu^+ \equiv \mu_0 + \Delta \mu$ and $\mu^- \equiv \mu_0 + r \Delta \mu$, respectively. The expression Eq. (13) then obeys the following form

$$J \equiv \frac{e}{h} \sum_{j=1}^{N_p} \left[ X_\beta(k_{j,\beta}^+) - X_\beta(k_{j,\beta}^-) \right],$$

where $X^+$ and $X^-$ denote mean positions of incoming and outgoing states, respectively. The difference $\mu^+ - \mu^-$ determines the strength of the voltage across the strip and it is the same at both ideal leads, that close to the source and that at the drain side. It implies that the expression Eq. (15) defines a Hall resistance. However, for microscopic systems the measured voltage difference depends not only on the properties of the studied system but also on the voltage detection techniques. To find the relation between $\mu^+ - \mu^-$ and measured voltage drop is thus non-trivial problem.

To proceed further, let us consider a macroscopic system composed of the parallel microscopic strips. Assuming no particle transfer between them and a gradient of the chemical potential perpendicular to the current flow, conditions for the standard measurement on the Hall bar samples are ensured. Constant gradient gives the same difference $\mu^+ - \mu^-$ within each of the strips and the substitution

$$\frac{\mu^+ - \mu^-}{X^+ - X^-} \rightarrow \frac{d\mu}{dx} \rightarrow e \mathcal{E}_x,$$

immediately gives Eq. (12) defining the Hall resistance of the macroscopic strip.

To illuminate the discussion let us apply the above presented general results to one particular example, the periodically modulated system in the strong magnetic field giving three magnetic flux quanta per unit cell, i.e. $p/q = 3$. Potential modulation

$$V_b(x, y) = V_0 \left( \cos K x + \cos K y \right); K \equiv \frac{2\pi}{a},$$

is assumed to be weak, i.e. $V_0$ is much less than the Landau level spacing $\hbar \omega_c$ ($\omega_c \equiv eB/m^*c$), $a$ denotes the lattice constant. The typical energy dispersion for the lowest Landau level, which is broadened by weak modulation, is shown in Fig. 2.

The confining potential $V_{\text{conf}}(x)$ was modeled by $V_c[(x_L - x)/a]^4$ for $x < x_L$, by $V_c[(x_R - x)/a]^4$ for $x > x_R$, and it was zero for $x_L < x < x_R$. The region of the periodic potential within the interval $(x_{iL}, x_{iR})$ has been surrounded by strips of zero potential that destroy interference between magnetic edge states located at opposite edges through the "bulk" states, the situation expected in macroscopic systems. The same model has been considered in the already published paper [14], where energy spectra as a function of the mass center are presented.

System boundaries give rise to edge state branches composed of pair states. Among them magnetic branches are the most important since states of each pair having opposite velocities are located at opposite strip edges [14, 19] and they thus substantially contribute to the magnetic moment. Magnetic edge-state branches cross energy gaps of infinite systems and the number of their crossing with the line of the fixed energy per each edge just equals to the absolute value of the integer $i$ satisfying the Diophantine equation, Eq. (6). Whenever $\mu$ crosses only magnetic edge branches the evaluation of the expression Eq. (12) gives quantum Hall values in the limit of the infinite strip width [14,19].

![Energy spectrum of the lowest broadened Landau level](image)

**FIG. 2: Energy spectrum of the lowest broadened Landau level, $p/q = 3$, $V_0 = 0.2 \cdot \hbar \omega_c$. Two branches just above the lowest magnetic subband are composed of non-magnetic edge states while edge state branches above the central subband are formed by magnetic edge states.**

Energy branches composed of "bulk" states which are spread within the strip interior are forming $p = 3$ magnetic subbands. Number of branches increases with rising width of the strip. To eliminate the details depending on the position of interior edges, the averaging procedure over the values of $x_{iL}$ and $x_{iR}$ has been applied to measurable quantities. By subtracting values obtained for two cases, for which the width of the periodic parts differs by the lattice constant, the contributions of the studied quantities per elementary cell has been obtained. Numerical calculations show that the averaged quantities per unit cell are practically independent on the form of the confining potential and that the increase of the total strip-width above ten lattice constant does not change the results, at least for the studied example. Let us further note, that the briefly outlined numerical procedure allows evaluation of the derivatives with respect of the
magnetic field since, at least in principle, it can be applied for any value of $B$.

The averaged value of the total magnetic moment $\bar{M}$ has been established by using the thermodynamic relation, Eq. (1), with the grand canonical potential having the following explicit form

$$\langle \Omega \rangle = \left\langle \sum_{\beta,k} \Theta (\mu - \varepsilon_\beta (k)) \left[ \varepsilon_\beta (k) - \mu \right] \right\rangle,$$  

(18)

where angular brackets denote the above described averaging. The obtained results for the derivative of the magnetic moment with respect of the chemical potential are shown in Fig. 3. As expected "bulk" states give diamagnetic contributions while the edge state contributions have paramagnetic character corresponding expected values of the quantum Hall effect.

The averaged value of the total magnetic moment $\bar{M}$ is approximately determined by the filling factor of this subband $\nu_e = 3\nu - 1 [\nu \epsilon (1/3, 2/3)]$, i.e. $\langle G_H \rangle \approx -e^2 \nu_e / h$. Opposite to the lower subband giving rise electron-like Hall effect the contribution of the upper subband has hole-like character. For higher magnetic fields giving $p/q = 2n + 1$, the dependence of the Hall conductance on the energy $\mu$ has qualitatively similar structure. There appear $n$ electron-like Hall peaks bellow central region and $n$ dips in the Hall conductance above. The more rich structures, to which a separate publication will be devoted, can be expected for fractions $p/q$ for which $q$ differs from the unity.

The derived formula, Eq. (12), expresses Hall resistance of the macroscopic systems in terms of Fermi electron properties. In the presented form its validity is limited to the systems where the scattering events lead to the uniform occupation of conducting channels, by another words if the scattering in macroscopic systems can be characterized by a single energy-dependent relaxation time. In principle the same result should be obtained by using quantum theory of the linear response, i.e. via the Kubo formula. As has been already mentioned the results coincide in the case of the non-dissipative quantum Hall regime \[3\] \[11\]. It is also trivial to prove that the same results are obtained for free electron gas in the weak field limit for which Landau level quantization is smeared out. To prove it in the general case is the challenge for future theoretical studies.

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FIG. 3: Averaged derivatives of the total magnetic moment $M$ (dashed line) and $M_a$ (full line) with respect of the chemical potential $\mu$ for $p/q = 3$, $V_0 = 0.2h\omega_c$ and the strip width around $16 \cdot a$. The full line coincides with the $\mu$-dependence of $-(h/e^2) < G_H >$.
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