Queuing model using sojourn time distribution with single working vacation and vacation interruption

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Abstract. Queuing that occurs due to service delays caused by the server is a result of server interference. One of the server interruptions was called the vacation incident. This paper analyzes the queue caused by a lack of servers. To solve this problem, the Sojourn Time distribution model is used at the time the service is distributed to just one server (M / G / I), with single working vacation and vacation interruption. This model is then used to look for models of old customers waiting in queuing (W), number of customers in the system (L), and opportunities for server busyness (ρ). The values of L, W, and ρ use secondary data for one server that has an arrival pattern that has Poisson distribution and service time exponentially distributed, then the values are substituted into the model (W′(θ)). The results of the model analysis show that the server is not busy or does not need to add a server.

1. Introduction

Queuing due to delays in service is an event that often occurs in everyday life, this can be caused by the number of customers in the queue or can because the server is slow in serving customers. To avoid this, efforts are needed to prevent the emergence of queues in the system. If this queue continues to occur it will cause the customer’s sojourn time in the system to increase, it will be detrimental to both the customer and the service owner. To avoid this, efforts are needed to prevent the appearance of queues in the system, due to server interruptions.

The service delay caused by the server is a result of server interruption that is not operating. One of the server interruptions is not operating due to maintenance or repair. Unoperated servers are referred to as vacation events, while the occurrence of taking care of once a day is called single working vacation. Many customers queue up so the server must immediately terminate the vacation called vacation interference.

According to [4], the queuing process is a process related to the arrival of customers to a queue or server system, then waiting in a queue until the waiter selects the customer according to the service discipline, and finally the customer leaves the queue system after the service is finished. The discussion in queuing theory in determining the customer sojourn time is, by determining the corresponding time distribution model. One indicator in determining the suitable model used in queuing theory is to determine the service time distribution.

In the general case, service time is always exponentially distributed, but in some cases service time does not always have an exponential distribution. If this happens, it is necessary to use a more general distribution so that all types of distribution can be applied, so that the distribution of the sojourn time used is the distribution of sojourn time for service times that are randomly distributed.

Other previous studies have been done a lot including [6], discussing the distribution of sojourn time on the M / G / 1 model where this model only discusses the determination of the sojourn time distribution model. [8], discuss the M / G / 1 / MWV queue model in this model the server does more than one
vacation. [1], discuss the GI / M / 1 / SWV and GI / Geo / 1 / SWV models on this model, the server only does one vacation.

Various studies on residence time distribution have been carried out. In this study, we will analyze the queue model due to the lack of a server, using the residence time distribution model \( W^*(\theta) \) with a single work holiday and vacation disruption when the service distributes one server. To obtain a numerical solution using secondary data, the model of the old customer waits in the queue (W), the number of customers in the system (L), and the chance of busyness of the server (\( \rho \)) is resolved with the help of C ++ software applications. The resulting L, W, and \( \rho \) values are included in the distribution model of Sojourn Time \( W^*(\theta) \) with single working vacation and vacation interruption to analyze the queue caused by a lack of servers.

2. Research Model

2.1. System Equations at Steady State.

According to [2], Steady state condition is a state of the system that does not depend on the initial state or time that has passed. If a system has reached a steady state condition then the chance of \( n \) customers in the system at time \( t \) is \( P_n(t) \) is not dependent on time. Assuming \( S_0 \) is the service time when the vacation is happening, \( S_1 \) is the service time when the server is available, the arrival rate of \( \lambda \), and the vacation rate \( v \), and for example \( \lambda, v, S_0, S_1 \) are free. Dropping [5], the events of \( A_1, A_2, \ldots, A_n \) are said to be mutually independent events if \( A_i \cap A_j = \emptyset \ ; i \neq j \). Given \( N(t) \) is the number of customers in the system at \( t \), and \( \xi(t) \) is the number of servers that work or not vacationing at time \( t \).

When,

\[ \xi(t) = \begin{cases} 
0 & \text{the system in the period of working vacation at time } t \\
1 & \text{the system in the normal service period at time } t
\end{cases} \]

so \( \{N(t), \xi(t), t \geq 0\} \) for every \( i = 0,1 \) is Quasi Birth Death Process (QBD) in state space \( \Omega = \{(0,0)\} \cup \{(n, i) | n \geq 1, i = 0,1\} \). With states (n, 1), for \( n \geq 1 \) shows that the system is in a normal busy state, state (n, 0), for \( n \geq 1 \) indicates that the system is in the working vacation period, and there are \( n \) customers in the queue. If state \( (1,1) \), states that in the system there is one customer and there is one server that is not doing vacation.

Using the probability limit is obtained

\[ P_{n,i} = P(N = n, \xi = i) = \lim_{t \to \infty} P\{N(t) = n, \xi(t) = i\} \quad (n,i) \in \Omega \]

To get the probability limit it is resolved using per QGDP, from this solution the system equation is obtained at steady state, that is when it occurs:

1. Customers in the system and servers are on vacation,

\[ (\lambda + v) P_{0,0} = P_{1,0}(0) + P_{1,1}(0) \] (1)

2. One customer on the system when the server is on vacation,

\[ \frac{d}{dx} P_{1,0}(x) = \lambda P_{0,0}S_0(x) - (\lambda + v) P_{1,0}(x) \] (2)

3. There are \( n \) customers on the system when the server is on vacation,

\[ \frac{d}{dx} P_{n,0}(x) = \lambda P_{n-1,0}S_0(x) - (\lambda + v) P_{n,0}(x) ; n \geq 2 \] (3)

4. One customer on the system when the server is in normal service,

\[ \frac{d}{dx} P_{1,1}(x) = P_{2,0}(0)S_1(x) + vP_{1,0}S_1(x) + \lambda P_{0,1}S_1(x) - \lambda P_{1,1}(x) + P_{2,1}(0)S_1(x) \] (4)

5. There are \( n \) customers in the system when the server is in normal service

\[ \frac{d}{dx} P_{n,1}(x) = P_{n+1,0}(0)S_1(x) + vP_{n,0}S_1(x) + \lambda P_{n-1,1}(x) - \lambda P_{n,1}(x) + P_{n+1,1}(0)S_1(x) \]
Because the change from a server that is doing a vacation to the server under normal conditions depends on \( v \) and \( \lambda \), then the chance of no customers in the system when the server is in normal service is,

\[
\lambda P(v, \lambda) = \frac{v P(v, \lambda)}{\lambda}
\]

\[
= \frac{v}{\lambda}
\]

(6)

2.2. Laplace-Stieltjes Transforms (LST)

According to [7], suppose that \( F(t) \) is a function well defined from \( t \geq 0 \) and \( s \) is a complex number. If it follows the stieltjes integral

\[
\int_0^t \lambda e^{-st} dF(t) = \int_0^t e^{-st} f(t) dt
\]

(7)

The system equation when the steady state is defined is that \( n \) is the number of customers in the system and \( \xi \) is the number of servers that work where \( i \in \{0.1\} \), then:

a. By using the laplace stieltjes transformation (lst) the equation is obtained

\[
P_{0,0}^\prime(\theta) = \int_0^\infty e^{-\theta x} P_{0,n}(x) dx
\]

(8)

b. Using the Probability generating function (pgf) the equation is obtained

\[
P^\prime(z, \theta) = \sum_{n=1}^\infty P^\prime_{0,n}(\theta) z^n,
\]

(9)

Determine the opportunity for the absence of customers in the system when the server is on vacation, namely by integrating equations (2) and (4) then substituting it to equation (1), obtained

\[
P(v, \lambda) = \frac{v}{\lambda}
\]

(10)

To get the chance of no customers in the system when the server is on vacation, substitute equation (10) into equation (6), obtained

\[
P_{0,1} = \frac{v}{\lambda + v - \lambda \xi(v + \lambda)(v + \lambda)(v + \lambda)}
\]

(11)

By integrating equation (4), it is then solved using laplace stieltjes transformation and probability generating function, then the chances of the server being vacationed are

\[
P_{0}(z, \theta) = \frac{P_{0,0}(z) \xi(z)(v + \lambda - \lambda)}{(\theta - v)(\theta - v + \lambda + 2z)}
\]

(12)

Using the same method, to get the laplace stieltjes transformation equation from server service opportunities under normal conditions, obtained

\[
P^\prime(z, \theta) = \frac{P_{0,0}(z)(v + \lambda)(v + \lambda)(v + \lambda)}{(\theta - v - \lambda)(\theta - v + \lambda + 2z)}
\]

(13)

2.3. Sojourn Time Distribution Model

According to [3], a system where the customer has completed the part and the service started again for the next service in the queue, then the service time is randomly distributed. Sojourn Time from the customer that came in case \( i \); \( i = 1,2,3,4 \) denoted by \( W_i \) and defined:

\[
W_i(\theta) = P_{i-1}(case - i) E[e^{-\theta W_i}|case - i]
\]

Model is used to find the customer's residence time by multiplying case-i expectations of opportunities during the i-case. Assuming that vacation time is distributed exponentially, where vacation time is
represented by \( V \) which is distributed exponentially, if \( S_0 \) is the service time during the working holiday period with \( fkp S_0(x) \) and \( lst S_0^*(\theta) \), and if \( S_t \) is the service time during the normal service period with \( fkp S_t(x) \) and \( lst S_t^*(\theta) \). To determine the model from transit time, there are a number of things that must be considered, namely by paying attention to the Laplace-Stieltjes Transformation from the transit time using the First In First Out service from customers which is divided into several cases:

Case 1. Customers come when the system is empty and the server is on vacation, the customers who come cannot be directly served so there is a service delay. If this happens the server can serve customers at a slow rate or the server can stop vacation and serve customers with normal service rates. So the formula used for this model is the customer sojourn time when the server makes a vacation plus the customer's sojourn time when the server is available, that is

\[
w_1^*(\theta) = P_{0,0} \left( \text{Pr}(S_0 < V) E[e^{-\theta S_0}|S_0 < V] + \text{Pr}(S_0 \geq V) E[e^{-\theta(S_0+S_t)}|S_0 \geq V] \right)
= P_{0,0} \left( S_0^*(\theta + v) + \frac{v S_t^*(\theta - S_0^*(\theta + v))}{\theta + v} \right)
\]

(14)

Case 2. Customers come when the system is empty and the server is available so the formula used for this model is the customer's sojourn time when the server is in normal condition,

\[
w_2^*(\theta) = P_{0,1} \left( \text{Pr}(S_0 = 0) E[e^{-\theta S_0}|S_0 = 0] \right)
= P_{0,1} \times S_0^*(\theta)
\]

(15)

Case 3. Customers come during the working vacation period when the server is busy. To get the appropriate sojourn time distribution, use the Total Probability Theorem. This case occurs if the server is doing a vacation but in the system as many as \( n \) customers are queuing so the server must immediately end the vacation. In this model vacation interruption occurs, so the formula used is the sum of customer service times when normal conditions are added when the server is vacationing, then

\[
w_3^*(\theta) = \sum_{n=1}^{\infty} \int_{x=0}^{\infty} P_{n,0}(x) \text{Pr}(S_{R,0} < V|S_{R,0} = x) x(S_0^*(\theta))^n e^{-\theta x} dx + \sum_{n=1}^{\infty} \int_{x=0}^{\infty} P_{n,0}(x) \int_{y=0}^{\infty} \text{Pr}(V = y|S_{R,0} = x) dy(S_0^*(\theta))^n+1 e^{-\theta y} dx
\]

by using (9), obtained

\[
w_3^*(\theta) = \frac{\theta + v S_0^*(\theta)}{\theta + v} P_0^*(S_0^*(\theta), \theta + v) + \frac{v S_t^*(\theta))^n}{\theta + v} P_0^*(S_t^*(\theta), 0)
\]

(16)

Case 4. Customers come during the normal service period to see the server busy. Customers come as many as \( n \) customers queuing in the system and the server is available, so the formula used is the sum of the service times during normal conditions, then

\[
w_4^*(\theta) = \sum_{n=1}^{\infty} \int_{x=0}^{\infty} P_{n,1}(x) (S_0^*(\theta))^n e^{-\theta x} dx = \sum_{n=1}^{\infty} P_{n,1}^*(\theta) (S_0^*(\theta))^n
= P_1^*(S_0^*(\theta), \theta)
\]

(17)

Because in practice it cannot be estimated that the model must be used, so that a combination of models is made to facilitate decision making. To declare the unpredictable sojourn time condition (\( W \)), the sojourn time model of a customer that will be used is a combination of the

\[
w^*(\theta) = \frac{w_1^*(\theta) + w_2^*(\theta) + w_3^*(\theta) + w_4^*(\theta)}{w_1^*(0) + w_2^*(0) + w_3^*(0) + w_4^*(0)}
\]

By combining the equations (14), (15), (16), and (17) and equations (10), (11), (12), and (13), to obtain the sojourn time distribution model (\( W^*(\theta) \)), is
\[ W^*(\theta) = \frac{P_{0,0}(\lambda(\theta - \lambda)S_0^*(\theta + v)(\theta + v - vS_1^*(\theta)) + \theta S_1^*(\theta)[\lambda^2 + v(\theta + \lambda + v)]}{\lambda(v + \theta)(\theta - \lambda + \lambda S_1^*(\theta))} \]  

(18)

3. Results and Discussion

The vacation time is assumed to be exponential in distribution, the results of this study the resulting formula is suitable for vacation time which has an exponential distribution, the total time the server does vacation is 0.493611 so \( v = 0.493611 \).

In the analysis of the queuing model there are several assumptions that must be met, namely the arrival pattern which is poisson distribution, service time that is generally distributed either in the form of exponential distribution, erlang or other distribution, and vacation time which is exponentially distributed. Based on the results of testing the type of distribution using SPSS software, the results of the arrival pattern are poisson distribution with \( \lambda = 27.4 \) arrivals / hour, and service time is exponentially distributed with \( S_1 = 46.97\approx47 \) documents, Figure 1 and Figure 2.

![Figure 1. Test Distribution of the arrival pattern](image1.png)

![Figure 2. Service Time Distribution Test](image2.png)

The vacation time is assumed to be exponential in distribution, the results of this study the resulting formula is suitable for vacation time which has an exponential distribution, the total time the server does vacation is 0.493611 so \( V = 0.493611 \), Table 1.

| Vacation to | Start a Vacation | End of Vacation | Old Vacation |
|-------------|------------------|----------------|-------------|
| 1           | 8:30:45          | 9:00:22        | 0:29:37     |

Table 1. Vacation Time

According to [7], If in the Laplace-Stieltjes transformation the distribution function is known to have an exponential distribution, the transformation becomes:

\[ F_X(s) = \int_0^\infty e^{-st}dF_X(t) = \int_0^\infty e^{-st} \lambda e^{-\lambda t} = \frac{\lambda}{\lambda + s} \]  

(19)

Because the service time is exponentially distributed, then substitute equation (19) into equation (18) so that the sojourn time distribution model is obtained \((W^*(\theta))\), namely
Then to find out the customer waiting time \( W \) of equation (20), so that it does not contain a variable \( \theta \), from

\[
W = \lim_{\theta \to \infty} \frac{P_0(\theta - \lambda) \frac{\lambda}{\lambda + s_0} (\theta + \nu - \nu S_1(\theta)) + \theta - \frac{\lambda}{\lambda + s_1} [\lambda^2 + \nu(\theta + \lambda + \nu)]}{\lambda(\nu + \theta)(\theta - \lambda + \lambda \frac{\lambda}{\lambda + s_1})}
\]

be,

\[
W = \frac{\lambda \nu (1 - \lambda E[S_1])}{\lambda^2 + \lambda \nu - \lambda^2 (1 + \nu E[S_1])} \frac{2\lambda(2\lambda + s_1) + 2\nu(2\lambda + s_0)}{(2\lambda + s_0)(2\lambda + s_1)}
\]

In the next stage Little Law’s is used to determine the number of customers in the system (L), and the server’s busy schedule (\( \rho \)) is,

\[
L = \frac{\lambda^2 (1 - \lambda E[S_1])}{\lambda^2 + \lambda \nu - \lambda^2 (1 + \nu E[S_1])} \frac{2\lambda(2\lambda + s_1) + 2\nu(2\lambda + s_0)}{(2\lambda + s_0)(2\lambda + s_1)}
\]

and

\[
\rho = \frac{\lambda}{s_1}
\]

To get a numeric solution, with the help of c++ software, substitute the parameters that have been obtained from the data into the models equation (21), (22), and (23). From the processing results, the average waiting time for documents in the system is 0.287174 hours, the average number of documents in the system is 7.897284 documents \( \approx 8 \) documents, and the server busyness is 0.585106. so the server is not busy or does not need additional servers.

4. Conclusion
From the results of this study we obtained a sojourn time distribution model \( W^*(\theta) \), in service time which exponentially distributed one server with single working vacation and vacation interruption. Based on the results of data analysis by substituting parameters into the sojourn time distribution model \( W^*(\theta) \), the average waiting time of documents in the system (W), the average number of documents in the system (L), and the opportunity to get busy server (\( \rho \)) so that it can be concluded that the server is not busy or does not need to add a server.

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