Relativistic hydrodynamics from Boltzmann equation with modified collision term

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Abstract

Generalizing the collision term in the relativistic Boltzmann equation to include nonlocal effects, and using Grad’s 14-moment approximation for the single-particle distribution function, we derive evolution equations for the relativistic dissipative fluid dynamics and compare them with the corresponding equations obtained in the standard Israel-Stewart and related approaches. Significance of this generalization on hydrodynamic evolution is demonstrated in the framework of one-dimensional scaling expansion.

1 Introduction

Relativistic dissipative hydrodynamics has been quite successful in explaining the spectra and azimuthal anisotropy of particles produced in heavy-ion collisions at the RHIC and LHC \(^1\). Apart from its applications, theoretical formulation of relativistic dissipative hydrodynamics is quite a challenging task by itself. The first-order dissipative fluid dynamics commonly known as relativistic Navier-Stokes (NS) theory involves parabolic differential equations and suffers from acausality and instability. The second-order or Israel-Stewart (IS) theory \(^2\) with its hyperbolic equations restores causality but may not guarantee stability. The correct formulation of relativistic dissipative fluid dynamics is still unresolved and is currently under intense investigation \(^3\) \(^4\).

It is essential to note that all formulations of the second-order dissipative hydrodynamics employing the Boltzmann equation (BE) make a strict assumption of locality in the configuration space for the collision term \(^2\). This means, the collisions that increase or decrease the number of particles with a given momentum \(p\), in an infinitesimal space-time volume element, are assumed to occur at the same point \(x^\mu\). This makes the collision integral a purely local functional of the single-particle phase-space distribution function \(f(x, p)\) independent of the derivatives \(\partial^\mu f\). Although \(f(x, p)\) may not vary significantly over the length scale of a single collision event, its variation over the length scales extending over many inter-particle spacings may not be negligible. Including the gradients of \(f(x, p)\) in the collision term gives rise to different evolution equations for the dissipative quantities.

In this article, we provide a new formulation of the dissipative hydrodynamic equations within kinetic theory by using a nonlocal collision term in the Boltzmann equation: \(p^\mu \partial_\mu f = C[f]\).

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Figure 1: Collisions $kk' \rightarrow pp'$ and $pp' \rightarrow kk'$ separated by a spacetime interval $\xi^\mu$ within an infinitesimal fluid element of size $dR$, containing a large number of particles represented by dots.

## 2 Non-locality in collision term

For two-body elastic collisions, the collision term is

$$C[f] = \frac{1}{2} \int dp' dk' \ W_{pp' \rightarrow kk'} \ (f_k f_{k'} \ f_{p'} f_p - f_p f_{p'} \ f_k f_{k'}) $$

where $W_{pp' \rightarrow kk'}$ is the collisional transition rate, $f_p \equiv f(x, p)$, $\tilde{f}_p \equiv 1 - r f(x, p)$ with $r = 1, -1, 0$ for Fermi, Bose, and Boltzmann gas and $dp = d\mathbf{p}/[(2\pi)^3 E_p]$. The first and second terms in Eq. (1) refer to the processes $kk' \rightarrow pp'$ and $pp' \rightarrow kk'$, respectively. These processes are traditionally assumed to occur at the same space-time point $x^\mu$ with an underlying assumption that $f(x, p)$ is constant not only over a region characterizing a single collision, but also over an infinitesimal fluid element of size $dR$, large compared to the interparticle separation. It is important to note that the space-time points at which the above two processes occur may be separated by a small interval $\xi^\mu$ within $d^4R$ (see Fig. 1). With this realistic viewpoint, the second term in Eq. (1) involves $f(x - \xi, p) f(x - \xi, p') f(x - \xi, k) f(x - \xi, k')$, which on Taylor expansion up to second order in $\xi^\mu$, results in the modified BE [3]

$$p^\mu \partial_\mu f = C[f] + \partial_\mu (A^\mu f) + \partial_\mu \partial_\nu (B^{\mu\nu} f).$$

The momentum dependence of coefficients,

$$A^\mu = \frac{1}{2} \int dp' dk' \ \xi^\mu \ W_{pp' \rightarrow kk'} \ f_{p'} \ f_k \ f_{k'} \ f_p,$$

$$B^{\mu\nu} = -\frac{1}{4} \int dp' dk' \ \xi^\mu \xi^{\nu} \ W_{pp' \rightarrow kk'} \ f_{p'} \ f_k \ f_{k'} \ f_p,$$

(3)

can be made explicit by expressing them in terms of the available tensors $p^\mu$ and the metric $g^{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$ as $A^\mu = a p^\mu$ and $B^{\mu\nu} = b_1 g^{\mu\nu} + b_2 p^\mu p^\nu$. The scalar coefficients $a$, $b_1$ and $b_2$ are functions of $x^\mu$. 
The conserved particle current, $N^\mu$, and the energy-momentum tensor, $T^{\mu\nu}$ have the standard form [2]:

$$N^\mu = \int dp \ p^\mu f = nu^\mu + n^\mu,$$

$$T^{\mu\nu} = \int dp \ p^\mu p^\nu f = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}. \quad (4)$$

Conservation of current, $\partial_\mu N^\mu = 0$ and energy-momentum tensor, $\partial_\mu T^{\mu\nu} = 0$, yield the fundamental evolution equations for $n$, $\epsilon$ and $u^\mu$

$$Dn + n\partial_\mu u^\mu + \partial_\mu n^\mu = 0,$$

$$De + (\epsilon + P + \Pi)\partial_\mu u^\mu - \pi^{\mu\nu}\nabla(\mu)u_\nu = 0,$$

$$(\epsilon + P + \Pi)Du^\alpha - \nabla^\alpha(P + \Pi) + \Delta^\alpha_\mu \partial_\mu \pi^{\mu\nu} = 0. \quad (5)$$

Conservation of current and energy-momentum implies vanishing zeroth and first moments of the collision term $C_m[f]$ leading to three constraint equations for the coefficients $(a, b_1, b_2)$ [4],

$$\partial_\mu a = 0, \quad \partial^2 (b_1 a_0) + \partial_\mu \partial_\nu (b_2 I^{\mu\nu}) = 0,$$

$$u_\alpha \partial_\nu (b_2 I^{\mu\nu}) + u_\alpha \partial^2 (b_1 n a_0) = 0. \quad (6)$$

In order to obtain the second-order evolution equations for dissipative quantities, we consider second moment of the modified BE, Eq. [2]

$$\int dp \ p^\alpha p^\beta \partial_\gamma f = \int dp \ p^\alpha p^\beta [C[f] + p^c \partial_\gamma (a f) + \partial^2 (b_1 f_0) + (p \cdot \partial)^2 (b_2 f_0)]. \quad (7)$$

Using Grad’s 14-moments approximation for the single particle distribution function in orthogonal basis,

$$f = f_0 + f_0 \tilde{f}_0 (\lambda_\Pi \Pi + \lambda_n n_\alpha p^\alpha + \lambda_\pi \pi_\alpha \beta p^\alpha p^\beta), \quad (8)$$

we finally obtain the following evolution equations for the dissipative fluxes

$$\Pi = \tilde{a}_\Pi N^\mu - \beta_\Pi \tau_\Pi \tilde{\Pi} + \tau_\Pi n \cdot \tilde{u} - \delta_\Pi n \cdot \Pi + \lambda_\Pi n \cdot \nabla \alpha$$

$$+ \lambda_\Pi \pi^{\mu\nu} \sigma^{\mu\nu} + \lambda_{\Pi\omega} \tilde{u} \cdot \tilde{u} + \lambda_{\Pi\omega} \omega^{\mu\nu} \omega^{\mu\nu} + (8 \text{ terms}), \quad (9)$$

$$n^\mu = \tilde{a}_n N^\mu - \beta_n \tau_n \tilde{n}^{(\mu)} + \lambda_n n_\nu \omega^{\mu\nu} - \delta_n n^{(\mu} \theta + l_n \nabla^{(\sigma} \Pi - l_n \Pi \nabla^{(\nu} \Pi - l_n \nabla^{(\sigma} \Pi - l_n \Pi \nabla^{(\nu} \Pi - n \Pi \pi^{\sigma\nu} \partial_\gamma \pi^{\gamma\nu}$$

$$- \tau_{\Pi n} \Pi \tilde{n}^{} - \tau_{\Pi n} \pi^{\mu\nu} \tilde{u}_\nu + \lambda_{\Pi n} n_\nu \pi^{\mu\nu} + \lambda_{\Pi n} \Pi n^\mu + \lambda_{\Pi n} \omega^{\mu\nu} \tilde{u}_\nu$$

$$+ \lambda_{\Pi n} \Delta^\alpha_\mu \partial_\mu \omega^{\gamma\nu} + (9 \text{ terms}), \quad (10)$$

$$\pi^{\mu\nu} = \tilde{a}_\pi N^{\mu\nu} - \beta_\pi \tau_\pi \tilde{\pi}^{(\mu\nu)} + \tau_{\pi n} n^{(\mu} \tilde{u}^{\nu)} + l_{\pi n} \nabla^{(\sigma\nu} \Pi + \lambda_{\pi n} \pi^{(\mu\nu} \tilde{u}^{\nu)}$$

$$- \lambda_{\pi n} n^{(\mu} \nabla^{(\nu)} \alpha - \tau_{\pi n} \pi^{(\mu\nu} \tilde{u}^{\nu)} + \lambda_{\pi n} \pi^{(\mu\nu} \omega^{\rho\nu)}$$

$$+ \lambda_{\pi n} \omega^{(\mu\nu)} + \chi_{\pi n} \Pi \nabla^{(\mu} \omega^{\nu) b_2} + \chi_{\Pi n} \nabla^{(\mu} \tilde{u}^{\nu) b_2}, \quad (11)$$
in the usual notation [4]. The “8 terms” (“9 terms”) involve second-order, linear scalar (vector) combinations of derivatives of $b_1, b_2$. We observe that in the above equations, the nonlocal coefficients $(a, b_1, b_2)$ modify the standard NS as well as the IS terms. This method is also able to generate all possible second-order terms allowed by symmetry.
3 Results and discussion

To illustrate the numerical significance of the non-locality in the collision term through the new dissipative hydrodynamic equations derived here, we consider Bjorken evolution of a massless Boltzmann gas ($\epsilon = 3P$) at vanishing net baryon number density [5].

In terms of the coordinates $(\tau, x, y, \eta)$ where $\tau = \sqrt{t^2 - z^2}$ and $\eta = \tanh^{-1}(z/t)$, the initial four-velocity becomes $u^\mu = (1, 0, 0, 0)$. In this scenario $\Pi = 0 = n^\mu$ and the equation for $\pi = -\tau^2 \pi^\eta$ reduces to

$$\frac{\pi}{\tau \pi} + \beta_\pi \frac{d\pi}{d\tau} = \beta_\pi \frac{4}{3\tau} - \lambda \frac{\pi}{\tau} - \psi \frac{db_2}{d\tau},$$

(12)

where the coefficients are

$$\beta_\pi = \tilde{a} + \frac{4b_2}{a\beta\eta}, \quad \beta_\pi = \frac{2}{3} \tilde{a} P, \quad \psi = \frac{22 \tilde{a} \beta - 2b_1 \beta^2 - 20b_2}{3a\beta\eta} P.$$  

(13)

The coupled differential equations (12), (6) and (12) are solved simultaneously for a variety of initial conditions relevant for RHIC and LHC.

Figure 2 (left panel), shows the evolution of several quantities for a particular choice of initial conditions. $T$ decreases monotonically to the crossover temperature $T_c \simeq 170$ MeV at time $\tau \simeq 10$ fm/c. Parameters $b_1, b_2$ vary smoothly to zero at large times indicating reduced but still significant presence of nonlocal effects at late times. This is also evident in Fig. 2 (right panel), where the pressure anisotropy $P_L/P_T = (P - \pi)/(P + \pi/2)$ shows marked deviation from IS. Although the shear pressure $\pi$ vanishes rapidly indicating approach to ideal fluid dynamics, the $P_L/P_T$ is far from unity. Faster isotropization for initial $a > 0$ may be attributed to a smaller effective shear viscosity in the modified NS equation.
Figure 3 shows the evolution of $P_L/P_T$ for isotropic initial pressure configuration, at various $\eta/s$ for the LHC energy regime. With small initial corrections ($\sim 10\%$ to NS and $\sim 20\%$ to the IS terms) due to $a$, $b_1$, $b_2$, nonlocal hydrodynamics (solid lines) exhibits appreciable deviation from the (local) IS theory (dashed lines). The above results clearly demonstrate the importance of the nonlocal effects, which should be incorporated in transport calculations as well.

4 Summary

To summarize, we have derived viscous hydrodynamic equations by introducing a nonlocal generalization of the collision term in the Boltzmann equation. The Navier-Stokes as well as Israel-Stewart equations are modified and new terms are obtained in the evolution equations of the dissipative quantities. The method presented is able to generate all possible terms that are allowed by symmetry. Within one-dimensional scaling expansion, we find that nonlocality of the collision term has a rather strong influence on the evolution of the viscous medium via hydrodynamic equations.

References

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