Directed Logic Program Proportions

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Abstract

Analogy-making is at the core of human intelligence and creativity with applications to such diverse tasks as commonsense reasoning, learning, language acquisition, and story telling. This paper studies directed analogical proportions between logic programs of the form \( P \) transforms into \( Q \) as \( R \) transforms into \( S \)—in symbols, \( P \rightarrow Q : R \rightarrow S \)—as a mechanism for deriving similar programs by analogy-making. The idea is to instantiate an abstract algebraic framework of analogical proportions recently introduced by the author in the domain of logic programming. Technically, we define proportions in terms of modularity where we derive abstract forms of concrete programs from a ‘known’ source domain which can then be instantiated in an ‘unknown’ target domain to obtain analogous programs. To this end, we introduce algebraic operations for syntactic logic program composition and concatenation. Interestingly, our work suggests a close relationship between modularity, generalization, and analogy which we believe should be explored further in the future. In a broader sense, this paper is a further step towards an algebraic theory of logic-based analogical reasoning and learning with potential applications to fundamental AI-problems like commonsense reasoning and computational learning and creativity.

KEYWORDS: Analogical Reasoning, Computational Learning and Creativity, Commonsense Reasoning, Logic Programming

1 Introduction

Analogy-making is at the core of human intelligence and creativity with applications to such diverse tasks as commonsense reasoning, learning, language acquisition, and story telling (see, e.g., Hofstadter (2001), Hofstadter and Sander (2013), Gust et al. (2008), Boden (1998), Sowa and Majumdar (2003), Winston (1980), and Wos (1993)). This paper studies directed analogical proportions between logic programs of the form \( P \) transforms into \( Q \) as \( R \) transforms into \( S \)—in symbols, \( P \rightarrow Q : R \rightarrow S \)—as a mechanism for deriving similar programs by analogy-making, where logic programs are rule-based systems written in the Horn fragment of predicate logic with applications to various AI-related problems, e.g., knowledge representation, planning, and diagnosis (cf. Baral (2003)).

In the literature, computational learning usually means learning from (a massive amount of) data. For example, in ‘deep learning’ artificial neural networks (ANNs) extract abstract features from data sets (cf. Goodfellow et al. (2016)) and, on the symbolic
side, inductive logic programs (ILPs) are provided with positive and negative examples of the target concept to be learned (cf. Muggleton (1991)). Another characteristic feature of current machine learning systems is the focus on goal-oriented problem solving—a typical task of ANNs is the categorization of the input data (e.g., finding cats in images) and ILPs try to construct logic programs from given examples which partially encode the problem to be solved (e.g., adding numbers or sorting lists).

The emphasis in this paper is different as we believe that theory generation is equally important to artificial intelligence—and may even be more important for artificial general intelligence than problem-solving—and deserves much more attention. In our framework, ‘theory generation’ means the generation of novel programs in an ‘unknown’ target domain via analogical transfer—realized by directed logic program proportions via generalization and instantiation—from a ‘known’ source domain. This approach is similar to ILP in that novel programs are derived from experience represented as knowledge bases consisting of ‘known’ programs. However, it differs significantly from ILP on how novel programs are constructed from experience—while in ILP the construction is goal-oriented and thus guided by partial specifications in the form of given examples, in our setting programs are derived by analogy-making to similar programs (without the need for concrete examples). For instance, we may ask—by analogy to arithmetic—what it means to ‘multiply’ two arbitrary lists (cf. Example 29) or to reverse ‘even’ lists (cf. Examples 12 and 30). Here, contrary to ILP, we do not expect a supervisor to provide the system with examples explaining list ‘multiplication’ or ‘evenness’ of lists, but instead we assume that there are arithmetic programs operating on numbers (i.e. numerals)—programs defining multiplication and evenness of numbers—which we can transfer to the list domain. For this, we instantiate Antić (2021a)’s abstract algebraic framework of analogical proportions and study directed logic program proportions of the form ‘$P$ transforms into $Q$ as $R$ transforms into $S$’, in symbols $P \to Q :: R \to S$, implementing analogy-making as is illustrated in the following running example.

**Example 1**

Imagine two domains, one consisting of numbers (or numerals) and the other made up of lists. We know from basic arithmetic what it means to add two numerals. Now suppose we want to transfer the concept of addition to the list domain. We can then ask—by analogy—the following question: What does it mean to ‘add’ two lists? We can transform this question into the following directed analogical equation:

$$Nat \to Plus :: List \to Z.$$  \hspace{1cm} (1)

In our framework, $Nat$, $Plus$, and $List$ will be logic programs, and $Z$ will be a program variable standing for a program which is obtained from $List$ as $Plus$ is obtained from $Nat$. That is, solutions to (1) will be programs implementing ‘addition of lists’. The

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1 This is Sir Michael Atiyah’s answer to the question of how he selects a problem to study: ‘I think that presupposes an answer. I don’t think that’s the way I work at all. Some people may sit back and say, ‘I want to solve this problem’ and they sit down and say, ‘How do I solve this problem?” I don’t. I just move around in the mathematical waters, thinking about things, being curious, interested, talking to people, stirring up ideas; things emerge and I follow them up. Or I see something which connects up with something else I know about, and I try to put them together and things develop. I have practically never started off with any idea of what I’m going to be doing or where it’s going to go. I’m interested in mathematics; I talk, I learn, I discuss and then interesting questions simply emerge. I have never started off with a particular goal, except the goal of understanding mathematics.’ (cf. Gowers (2000))
idea is to derive an abstract form $\text{Plus}(Z)$ as a generalization of the concrete program $\text{Plus}$ such that

$$
\text{Plus} = \text{Plus}(\text{Nat}).
$$

That is, we decompose $\text{Plus}$ into modules and generalize every instance of the module $\text{Nat}$ in $\text{Plus}$ by a program variable $Z$. We can then instantiate the form $\text{Plus}(Z)$ with $\text{List}$ to obtain a plausible solution to (12), that is, a program for ‘addition of lists’:

$$
\text{Nat} \rightarrow \text{Plus}(\text{Nat}) :: \text{List} \rightarrow \text{Plus}(\text{List}).
$$

It is important to emphasize that in order to be able to decompose the program $\text{Plus}$ into $\text{Nat}$, we need to introduce novel algebraic operations on logic programs (Section 3). We will return to this example, in a more formal manner, in Examples 11 and 29.

Interestingly, our work suggests a close relationship between modularity, generalization, and analogy which we believe should be explored further in the future. In a broader sense, this paper is a further step towards an algebraic theory of logic-based analogical reasoning and learning in knowledge representation and reasoning systems, with potential applications to fundamental AI-problems like commonsense reasoning and computational learning and creativity.

2 Logic Programs

In this section, we recall the syntax and semantics of logic programming by mainly following the lines of [Apt (1990)].

2.1 Syntax

An (unranked first-order) language $L$ consists of a set $P_{sL}$ of predicate symbols, a set $F_{sL}$ of function symbols, a set $C_{sL}$ of constant symbols, and a denumerable set $V = \{z_1, z_2, \ldots \}$ of variables. Terms and atoms are defined in the usual way. Substitutions and (most general) unifiers of terms and (sets of) atoms are defined as usual.

Let $L$ be a language. A (Horn logic) program over $L$ (or $L$-program) is a set of rules of the form

$$
A_0 \leftarrow A_1, \ldots, A_k, \quad k \geq 0,
$$

where $A_0, \ldots, A_k$ are atoms over $L$. It will be convenient to define, for a rule $r$ of the form (2), $\text{head}(r) = \{A_0\}$ and $\text{body}(r) = \{A_1, \ldots, A_k\}$, extended to programs by $\text{head}(P) = \bigcup_{r \in P} \text{head}(r)$ and $\text{body}(P) = \bigcup_{r \in P} \text{body}(r)$. In this case, the size of $r$ is $k$ denoted by $\text{sz}(r)$. A fact is a rule with empty body and a proper rule is a rule which is not a fact. We denote the facts and proper rules in $P$ by $\text{facts}(P)$ and $\text{proper}(P)$, respectively. We define the skeleton inductively as follows: (i) for an atom $p(\bar{t})$, define $sk(p(\bar{t})) := p$; (ii) for a rule $r$ of the form (2), define

$$
sk(r) := sk(A_0) \leftarrow sk(A_1), \ldots, sk(A_k);
$$

finally, (iii) define the skeleton of a program $P$ rule-wise as $sk(P) := \{sk(r) \mid r \in P\}$.
A program \( P \) is \emph{ground} if it contains no variables and we denote the grounding of \( P \) which contains all ground instances of the rules in \( P \) by \( \text{gnd}(P) \). We call any bijective substitution a \emph{renaming}. The set of all \emph{variants} of \( P \) is defined by \( \text{variants}(P) = \bigcup_{\theta} \text{renaming} P[\theta] \). The \emph{main predicate} of a program is given by the name of the program in lower case letters if not specified otherwise. We will sometimes write \( P(p) \) to make the main predicate \( p \) in \( P \) explicit and we will occasionally write \( P(\bar{x}) \) to indicate that \( P \) contains variables from \( \bar{x} = (x_1, \ldots, x_n) \), \( n \geq 1 \). We denote the program constructed from \( P(p) \) by replacing every occurrence of the predicate symbol \( p \) with \( q \) by \( P[p/q] \).

\textbf{Example 2}

Later, we will be interested in the basic data structures of numerals, lists, and (binary) trees. The programs for generating numerals and lists are given by

\[
\begin{align*}
\text{Nat}(x) & := \left\{ \begin{array}{l}
\text{nat}(0) \\
\text{nat}(s(x)) \leftarrow \text{nat}(x).
\end{array} \right. \\
\text{List}(u, x) & := \left\{ \begin{array}{l}
\text{list}([]) \\
\text{list}([u \mid x]) \leftarrow \text{list}(x).
\end{array} \right.
\end{align*}
\]

As is customary in logic programming, \([ \ ]\) and \([u \mid x]\) is syntactic sugar for \( \text{nil} \) and \( \text{cons}(u, x) \), respectively. The program for generating (binary) trees is given by

\[
\text{Tree}(u, x, y) := \left\{ \begin{array}{l}
\text{tree}(\text{void}) \\
\text{tree}(t(u, x, y)) \leftarrow \text{tree}(x), \\
\text{tree}(y).
\end{array} \right.
\]

For instance, the tree consisting of a root \( a \), and two leafs \( b \) and \( c \) is symbolically represented as \( \text{tree}(a, \text{tree}(b, \text{void}, \text{void}), \text{tree}(c, \text{void}, \text{void})) \). The skeleton of \( \text{Tree} \) is given by

\[
\text{sk(Tree)} = \left\{ \begin{array}{l}
\text{tree} \\
\text{tree} \leftarrow \text{tree}
\end{array} \right. \tag{3}
\]

We will frequently refer to the programs above in the rest of the paper.

\subsection*{2.2 Semantics}

An \emph{interpretation} is any set of of ground atoms. We define the \emph{entailment relation}, for every interpretation \( I \), inductively as follows: (i) for a ground atom \( A \), \( I \models A \) if \( A \in I \); (ii) for a set of ground atoms \( B \), \( I \models B \) if \( B \subseteq I \); (iii) for a ground rule \( r \) of the form \( \text{head} \subseteq \text{body} \), \( I \models r \) if \( I \models \text{body}(r) \) implies \( I \models \text{head}(r) \); and finally (iv) for a ground program \( P \), \( I \models P \) if \( I \models r \) holds for each rule \( r \in P \). In case \( I \models \text{gnd}(P) \) we call \( I \) a \emph{model} of \( P \). The set of all models of \( P \) has a least element with respect to set inclusion called the \emph{least model} of \( P \) and denoted by \( \text{LM}(P) \). We call a ground atom \( A \) a \emph{logical} consequence of \( P \), in symbols \( P \models A \), if \( A \) is contained in the least model of \( P \) and we say that \( P \) and \( R \) are \emph{(logically) equivalent} if \( \text{LM}(P) = \text{LM}(R) \).

\section*{3 Horn Algebras}

Our framework of analogical proportions between logic programs will be built on top of an algebra of logic programs which allows us to decompose programs into simpler modules. For this, it will be useful to introduce in this section two novel algebraic operations for logic program composition (Section 3.1) and concatenation (Section 3.2).
Notation 3
In the rest of the paper, $P$ and $R$ denote logic programs over some joint language $L$.

### 3.1 Composition

The rule-like structure of logic programs induces naturally a compositional structure which allows us to decompose programs rule-wise.

**Definition 4**
We define the (sequential) composition of $P$ and $R$ by

$$P \circ R := \left\{ \begin{array}{l}
  \text{head}(r\theta) \leftarrow \text{body}(S\theta) \\
  r \in P \\
  S \subseteq \text{variants}(R) \\
  \text{head}(S\theta) = \text{body}(r\theta) \\
  \theta = \text{mgu}(\text{body}(r), \text{head}(S))
\end{array} \right\}.$$

Roughly, we obtain the composition of $P$ and $R$ by resolving all body atoms in $P$ with the ‘matching’ rule heads of $R$. This is illustrated in the next example, where we construct the even from the natural numbers via composition.

**Example 5**
Reconsider the program $Nat$ of Example 2 generating the natural numbers. By composing the only proper rule in $Nat$ with itself, we obtain

$$\{\text{nat}(s(x)) \leftarrow \text{nat}(x)\} \circ \{\text{nat}(s(x)) \leftarrow \text{nat}(x)\} = \{\text{nat}(s(s(x))) \leftarrow \text{nat}(x)\}.$$

Notice that this program, together with the single fact in $Nat$, generates the even numbers. Let us therefore define the program

$$Even := (\text{facts}(Nat) \cup \text{proper}(Nat))^{2} \text{[nat/even]} = \left\{ \begin{array}{l}
  \text{even}(0), \\
  \text{even}(s(s(x))) \leftarrow \text{even}(x)
\end{array} \right\}, \quad (4)$$

We will come back to this program in Example 12.

The following example shows that, unfortunately, composition is not associative.

**Example 6**
Consider the Horn rule

$$r := a \leftarrow b, c,$$

and the Horn programs

$$P := \left\{ \begin{array}{l}
  b \leftarrow b \\
  c \leftarrow b, c
\end{array} \right\} \quad \text{and} \quad R := \left\{ \begin{array}{l}
  b \leftarrow d \\
  b \leftarrow e \\
  c \leftarrow f
\end{array} \right\}.$$  

A simple computation yields

$$\{r\}(PR) = \left\{ \begin{array}{l}
  a \leftarrow d, f \\
  a \leftarrow e, f \\
  a \leftarrow d, e, f
\end{array} \right\} \neq \left\{ \begin{array}{l}
  a \leftarrow d, f \\
  a \leftarrow e, f
\end{array} \right\} = (\{r\}P)R.$$

\(^2\) We write $X \subseteq_k Y$ in case $X$ is a subset of $Y$ consisting of $k$ elements.
3.2 Concatenation

In many cases, a program is the ‘concatenation’ of two or more simpler programs on an atomic level. A typical example is the program

\[
\text{Length} := \begin{cases} \\
\text{length}([], 0) \\
\text{length}([u \mid x], s(y)) \leftarrow \text{length}(x, y) \\
\end{cases}
\]

which is, roughly, the ‘concatenation’ of \textit{List} in the first and \textit{Nat} in the second argument modulo renaming of predicate symbols (cf. Example 2). This motivates the following definition.

\textbf{Definition 7}

We define the \textit{concatenation} of \(P\) and \(R\) inductively as follows:

1. For atoms \(p(s)\) and \(p(t)\), we define

\[p(s) \cdot p(t) := p(s, t),\]

extended to sets of atoms \(B\) and \(B'\) by

\[B \cdot B' := \{ A \cdot A' \mid A \in B, A' \in B' : sk(A) = sk(A') \}.\]

2. For rules \(r\) and \(r'\) with \(sk(r) = sk(r')\), we define

\[r \cdot r' := (\text{head}(r) \cdot \text{head}(r')) \leftarrow (\text{body}(r) \cdot \text{body}(r')).\]

3. Finally, we define the concatenation of \(P\) and \(R\) by

\[P \cdot R := \{ r \cdot r' \mid r \in P, r' \in R : sk(r) = sk(r') \}.
\]

We will often write \(PR\) instead of \(P \cdot R\) in case the operation is understood from the context.

We can now formally deconcatenate the list program from above as

\[
\text{Length} = \text{List}[[\text{list}/\text{length}] \cdot \text{Nat}[[\text{nat}/\text{length}]]].
\]

We will return to deconcatenations of this form in Section 4 (cf. Example 11).

\textbf{Theorem 8}

Concatenation is associative.

\textbf{Proof}

We know that the concatenation of words is associative. From this we deduce

\[A \cdot (A' \cdot A'') = (A \cdot A') \cdot A'', \text{ for any atoms } A, A', A'' \text{ with } sk(A) = sk(A') = sk(A'').\]

This implies

\[B \cdot (B' \cdot B'') = \{ A \cdot (A' \cdot A'') \mid A \in B, A' \in B', A'' \in B'', sk(A) = sk(A') = sk(A'') \} = \{ (A \cdot A') \cdot A'' \mid A \in B, A' \in B', A'' \in B'', sk(A) = sk(A') = sk(A'') \} = (B \cdot B') \cdot B'', \text{ for any sets of atoms } B, B', B''.\]
From this, we deduce the associativity of rule concatenation:

\[
\begin{align*}
    p \cdot (q \cdot r) &= (\text{head}(p) \cdot (\text{head}(q) \cdot \text{head}(r))) \leftarrow (\text{body}(p) \cdot (\text{body}(q) \cdot \text{body}(r))) \\
    &= ((\text{head}(p) \cdot \text{head}(q)) \cdot \text{head}(r)) \leftarrow (\text{body}(p) \cdot \text{body}(q)) \cdot \text{body}(r)) \\
    &= (p \cdot q) \cdot r.
\end{align*}
\]

We have

\[
\text{sk}(r) = \text{sk}(s) \iff \text{sk}(r \cdot s) = \text{sk}(r) = \text{sk}(s),
\]

which finally implies the associativity of concatenation via

\[
P \cdot (Q \cdot R) = \bigcup_{p \in P, q \in Q, r \in R}^{sk(p) = sk(q) = sk(r)} (p \cdot (q \cdot r)) = (P \cdot Q) \cdot R.
\]

We are now ready to define Horn algebras.

**Definition 9**
The Horn algebra over \( L \) consists of all \( L \)-programs together with all unary and binary operations on programs introduced above, including composition, concatenation, union, the least model operator, et cetera.

### 4 Logic Program Forms

Recall from Example 1 that we wish to derive abstract generalizations of concrete programs, which can then be instantiated to obtain similar programs. We formalize this idea via logic program forms as follows.

**Definition 10**
In the rest of the paper, we assume that we are given program variables \( X, Y, Z \ldots \) as placeholders for concrete programs. Let \( \mathcal{A} \) be a Horn algebra over some language \( L \). A (logic program) form over \( \mathcal{A} \) (or \( \mathcal{A} \)-form) is any well-formed expression built up from \( L \)-programs, program variables, and all algebraic operations on programs from above including substitution. More precisely, \( \mathcal{A} \)-forms are defined by the grammar

\[
F ::= P \mid Z \mid F \cup F \mid F \circ F \mid F \cdot F \mid F \sigma \mid LM(F) \mid \text{head}(F) \mid \text{body}(F) \mid \text{facts}(F) \mid \text{proper}(F),
\]

where \( P \in \mathcal{A} \) is an \( L \)-program, \( Z \) is a program variable, and \( \sigma \) is a substitution. We denote the set of all \( \mathcal{A} \)-forms with variables among \( \bar{Z} \) by \( \mathcal{A}[\bar{Z}] \). We will denote forms by boldface letters.

Forms generalize logic programs and induce transformations on programs in the obvious way by replacing program variables with concrete programs. This means that we can interpret logic program forms as ‘meta-terms’ over the algebra of logic programs with programs as ‘constants’, program variables as variables, and algebraic operations on programs as ‘function symbols’. This is illustrated in the following examples.

**Example 11**
The program \( \text{Plus} \) of Example 1 for the addition of numerals is given by

\[
\text{Plus} := \begin{cases} 
\text{plus}(0, y, y) \\
\text{plus}(s(x), y, s(z)) \leftarrow \\
\text{plus}(x, y, z)
\end{cases}.
\]
Recall from Example 1 that we wish to derive a form **Plus** from **Plus** which abstractly represents addition. Notice that **Plus** is, essentially, the concatenation of the program **Nat** (Example 2) in the first and last argument together with a middle part. Formally, we have

\[
\text{Plus} = \text{Nat}(x)[\text{nat/plus}] \cdot \left\{ \begin{array}{ll}
\text{plus}(y, y), \\
\text{plus}(y) \leftarrow \text{plus}(y)
\end{array} \right\} \cdot \left\{ \begin{array}{l}
\text{plus}, \\
\text{proper(Nat)(z)[nat/plus]}
\end{array} \right\}.
\]

We therefore define the form **Plus**\((Z\langle q\rangle(\vec{x}))\), where \(Z\) is a program variable, \(q\) stands for the main predicate symbol in \(Z\), and \(\vec{x}\) is a sequence of variables, by

\[
\text{Plus}(Z\langle q\rangle(\vec{x})) := Z[q/\text{plus}] \cdot \left\{ \begin{array}{l}
\text{plus}(y, y), \\
\text{plus}(y) \leftarrow \text{plus}(y)
\end{array} \right\} \cdot \left\{ \begin{array}{l}
\text{plus}, \\
\text{proper}(Z)[q/\text{plus}, \vec{x}/\vec{z}]
\end{array} \right\}.
\]

Here \(\vec{z}\) is a sequence of fresh variables distinct from the variables in \(\vec{x}\). We can think of **Plus** as a generalization of **Plus** where we have abstracted from the concrete data type **Nat**. In fact, **Plus** is an instance of **Plus**:

\[
\text{Plus} = \text{Plus}(\text{Nat}(x)).
\]

Similarly, instantiating the form **Plus** with the program **List**\((u, x)\) for constructing the data type of lists (cf. Example 2) yields the program\(^3\)

\[
\text{plus}([], y, y), \\
\text{plus}([u | x], y, [u | z]) \leftarrow \\
\text{plus}(x, y, z),
\]

which is the program **PlusList** for appending lists from Example 11, e.g., we have

\[
\text{Plus}(\text{List}) \models \text{plus}([a, b], [c, d], [a, b, c, d]).
\]

As a further example, we want to define the ‘addition’ of (binary) trees by instantiating the form **Plus** with **Tree**. Note that we now have multiple choices: since **Tree**\((u, x, y)\) contains two bound variables (i.e., \(x\) and \(y\)), we have two possibilities. Let us first consider the program

\[
\text{Plus}(\text{Tree}(u, x, x)) = \left\{ \begin{array}{l}
\text{plus(void, y, y)}, \\
\text{plus}(t(u, x, x), y, t(u, z, z)) \leftarrow \\
\text{plus}(x, y, z).
\end{array} \right\}.
\]

This program ‘appends’ the tree in the second argument to each leaf of the symmetric tree in the first argument. Notice that all of the above programs are syntactically almost identical, e.g., we can transform **Plus** into **PlusList** via a simple rewriting of terms. The next program shows that we can derive programs from **Plus** which syntactically differ

\(^3\) Here we have instantiated the sequence of variables \(\vec{x}\) with \(\vec{x} = (u, z)\).
more substantially from the above programs. Concretely, the program

\[
\text{Plus}(\text{Tree}(u, x_1, x_2)) = \begin{cases} 
\text{plus}(\text{void}, y, y), \\
\text{plus}(t(u, x_1, x_2), y, t(u, z_1, z_2)) \leftarrow \\
\text{plus}(x_1, y, z_1), \\
\text{plus}(x_2, y, z_1), \\
\text{plus}(x_1, y, z_2), \\
\text{plus}(x_2, y, z_2).
\end{cases}
\]

is logically equivalent to program (8). However, in some situations this more complicated representation is beneficial. For example, we can now remove the second and third body atom to obtain the more compact program

\[
\begin{align*}
\text{plus}(\text{void}, y, y), \\
\text{plus}(t(u, x_1, x_2), y, t(u, z_1, z_2)) \leftarrow \\
\text{plus}(x_1, y, z_1), \\
\text{plus}(x_2, y, z_2).
\end{align*}
\]

This program, in analogy to program (8), 'appends' the tree in the second argument to each leaf of the not necessarily symmetric tree in the first argument and thus generalizes (8).

**Example 12**

In Example 5 we have constructed the program Even, representing the even numbers, from Nat by inheriting its fact and by iterating its proper rule once. By replacing Nat in (4) by a program variable Z, we arrive at the form

\[
\text{Even}(Z) := \text{facts}(Z) \cup (\text{proper}(Z) \circ \text{proper}(Z)).
\]  

(9)

We can now instantiate this form with arbitrary programs to transfer the concept of 'evenness' to other domains. For example, consider the program Reverse for reversing lists given by

\[
\text{Reverse} := \text{Reverse}_0 \cup \text{Plus}(\text{List}(u, x)),
\]

where

\[
\text{Reverse}_0 := \begin{cases} 
\text{reverse}([\ ], [\ ]), \\
\text{reverse}([u \mid x], y) \leftarrow \\
\text{reverse}(x, z), \\
\text{plus}(z, [u], y)
\end{cases}
\]

By instantiating the form Even with Reverse, we obtain the program

\[
\text{Even}(\text{Reverse}) = \begin{cases} 
\text{reverse}([\ ], [\ ]), \\
\text{reverse}([u_1, u_2 \mid x], [u_3 \mid y]) \leftarrow \\
\text{reverse}(x, z), \\
\text{plus}(z, [u_2], [u_3 \mid w]), \\
\text{plus}(w, [u_1], y), \\
\text{plus}([\ ], y, y), \\
\text{plus}([u_1, u_2 \mid x], y, [u_1, u_2 \mid z]) \leftarrow \\
\text{plus}(x, y, z).
\end{cases}
\]
One can verify that this program reverses lists of even length. Similarly, if \textit{Sort} is a program for sorting lists, then \textit{Even(Sort)} is a program for sorting ‘even’ lists and so on.

\textit{Example 13}

The program for checking list membership is given by

$$\text{Member} := \left\{ \begin{array}{l} \text{member}(u, [u \mid x]) \\ \text{member}(u, [v \mid x]) \leftarrow \text{member}(u, x) \end{array} \right\}.$$  

Notice the syntactic similarity between the program \textit{List} of Example 2 and the second arguments in \textit{Member}—in fact, we can deconcatenate \textit{Member} as follows:

$$\text{Member} = \left\{ \begin{array}{l} \text{member}(u) \\ \text{member}(u) \leftarrow \text{member}(u) \end{array} \right\} \cdot \left\{ \begin{array}{l} \text{member}([u \mid x]) \\ \text{member}([v \mid x]) \leftarrow \text{member}(x) \end{array} \right\}.$$  

The second factor can be expressed in terms of \textit{List} via

$$\{\text{member}([u \mid x])\} = (\text{proper} (\text{List}(u, x)) \circ \text{body}(\text{proper} (\text{List}(u, x)))) [\text{list/member}]$$

and

$$\{\text{member}([v \mid x]) \leftarrow \text{member}(x)\} = \text{proper} (\text{List}(v, x)) [\text{list/member}].$$

This yields the form \textit{Member}(\textit{Z}(u, \vec{x})(q)), where \vec{x} is a (possibly empty) sequence of variables, given by

$$\left\{ \begin{array}{l} \text{member}(u) \\ \text{member}(u) \leftarrow \text{member}(u) \end{array} \right\} \cdot \left\{ \begin{array}{l} \text{proper}(Z(u, x)) \circ \text{body}(\text{proper}(Z(u, x))) \\ \text{proper}(Z(u, x)) [u/v] \end{array} \right\} [q/\text{member}].$$

We can now ask—by analogy—what ‘membership’ means in the numerical domain. For this, we compute

$$\text{Member}(\textit{Nat}(u)) = \left\{ \begin{array}{l} \text{member}(u, s(u)) \\ \text{member}(u, s(v)) \leftarrow \text{member}(u, v) \end{array} \right\}.$$  

One can easily check that this program computes the ‘less than’ relation between numerals.

\section{5 Directed Logic Program Proportions}

This is the main section of the paper. Recall from Example 1 that we want to formalize analogical reasoning and learning in the logic programming setting via directed analogical proportions between programs. For this, we instantiate here \textit{Antič (2021a)}’s abstract algebraic framework of analogical proportions within the Horn algebra of logic programs from above using logic program forms.

Let us first recall \textit{Antič (2021a)}’s framework. In the rest of the paper, we may assume some ‘known’ source domain \(A\) and some ‘unknown’ target domain \(B\), both Horn algebras over some languages \(L_A\) and \(L_B\), respectively. We may think of the source domain \(A\) as
our background knowledge—a repertoire of programs we are familiar with—whereas $\mathcal{B}$ stands for an unfamiliar domain which we want to explore via analogical transfer from $\mathcal{A}$.

For this we will consider directed analogical equations of the form ‘$P$ transforms into $Q$ as $R$ transforms into $Z$’—in symbols, $P \rightarrow Q :: R \rightarrow Z$—where $P$ and $Q$ are programs of $\mathcal{A}$, $R$ is a program of $\mathcal{B}$, and $Z$ is a program variable. The task of learning logic programs by analogy is then to solve such equations and thus to expand our knowledge about the intimate relationships between (seemingly unrelated) programs, that is, solutions to directed analogical equations will be programs of $\mathcal{B}$ which are obtained from $R$ in $\mathcal{B}$ as $Q$ is obtained from $P$ in $\mathcal{A}$ in a mathematically precise way (Definition 16). Specifically, we want to functionally relate programs via rewrite rules as follows. Recall from Example 1 that transforming $\mathsf{Nat}$ into $\mathsf{Plus}$ means transforming $\mathsf{Id}(\mathsf{Nat})$ into $\mathsf{Plus}(\mathsf{Nat})$, where $\mathsf{Id}(Z) := Z$ and $\mathsf{Plus}(Z)$ are forms. We can state this transformation more pictorially as the rewrite rule $\mathsf{Id} \rightarrow \mathsf{Plus}$. Now transforming the program $\mathsf{List}$ ‘in the same way’ means to transform $\mathsf{Id}(\mathsf{List})$ into $\mathsf{Plus}(\mathsf{List})$, which again is an instance of $\mathsf{Id} \rightarrow \mathsf{Plus}$. Let us make this notation official.

**Notation 14**
We will always write $\mathbf{F}(\tilde{Z}) \rightarrow \mathbf{G}(\tilde{Z})$ or $\mathbf{F} \rightarrow \mathbf{G}$ instead of $(\mathbf{F}, \mathbf{G})$, for any pair of forms $\mathbf{F}$ and $\mathbf{G}$ containing program variables among $\tilde{Z}$.

The above explanation motivates the following definition.

**Definition 15**
Define the set of *justifications* of two programs $P, R \in \mathcal{A}$ in $\mathcal{A}$ by

$$Jus_\mathcal{A}(P, R) := \left\{ F \rightarrow G \in \mathcal{A}[\tilde{Z}] \times \mathcal{A}[\tilde{Z}] \mid P = F(\tilde{E}) \ 	ext{and} \ R = G(\tilde{E}) \ 	ext{for some} \ \tilde{E} \in \mathcal{A}[\tilde{Z}] \right\}.$$  

For instance, $Jus(\mathsf{Nat}, \mathsf{Plus}(\mathsf{Nat}))$ and $Jus(\mathsf{List}, \mathsf{Plus}(\mathsf{List}))$ both contain the justification $Z \rightarrow \mathsf{Plus}(Z)$.

We are now ready to state the main definition of the paper as an instance of (Antić 2021a Definition 5).

**Definition 16**
A directed program equation in $(\mathcal{A}, \mathcal{B})$ is an expression of the form ‘$P$ transforms into $Q$ as $R$ transforms into $Z$’—in symbols,

$$P \rightarrow Q :: R \rightarrow Z,$$

where $P$ and $Q$ are source programs from $\mathcal{A}$, $R$ is a target program from $\mathcal{B}$, and $Z$ is a program variable. Given a target program $S \in \mathcal{B}$, define the set of justifications of $P \rightarrow Q :: R \rightarrow S$ in $(\mathcal{A}, \mathcal{B})$ by

$$Jus_{(\mathcal{A}, \mathcal{B})}(P \rightarrow Q :: R \rightarrow S) := Jus_\mathcal{A}(P, Q) \cap Jus_\mathcal{B}(R, S).$$

We say that $S$ is a solution to (10) in $(\mathcal{A}, \mathcal{B})$ if $Jus_{(\mathcal{A}, \mathcal{B})}(P \rightarrow Q :: R \rightarrow S)$ is a subset maximal set of justifications with respect to $S$, that is, iff for any program $S' \in \mathcal{B}$,

$$Jus_{(\mathcal{A}, \mathcal{B})}(P \rightarrow Q :: R \rightarrow S) \subseteq Jus_{(\mathcal{A}, \mathcal{B})}(P \rightarrow Q :: R \rightarrow S')$$

implies

$$Jus_{(\mathcal{A}, \mathcal{B})}(P \rightarrow Q :: R \rightarrow S') \subseteq Jus_{(\mathcal{A}, \mathcal{B})}(P \rightarrow Q :: R \rightarrow S).$$
In this case, we say that \( P, Q, R, S \) are in \textit{directed logic program proportion} in \((A, B)\) written as
\[
(A, B) \models P \rightarrow Q :: R \rightarrow S.
\]

\textit{Notation 17}

We will always write \( A \) instead of \((A, A)\) et cetera.

Roughly, a program \( S \) in the target domain is a solution to a directed program equation of the form \( P \rightarrow Q :: R \rightarrow Z \) iff there is no other target program \( S' \) whose transformation from \( R \) is more similar to the transformation of \( P \) into \( Q \) in the source domain expressed in terms of maximal sets of algebraic justifications. Analogical equations formalize the idea that analogy-making is the task of transforming different objects from the source to the target domain in ‘the same way’ or as Pólya (1954) puts it:

Two systems are analogous if they agree in clearly definable relations of their respective parts.

In our formulation, the ‘parts’ are the programs \( P, Q, R, S \) and the ‘definable relations’ are represented by rewrite rules of the form \( F \rightarrow G \) relating \( P, Q \) and \( R, S \) in ‘the same way’ via maximal sets of justifications.

\textit{Notation 18}

Notice that any justification \( F(\bar{Z}) \rightarrow G(\bar{Z}) \) of \( P \rightarrow Q :: R \rightarrow S \) in \((A, B)\) must satisfy
\[
P = F(\bar{E}_1) \quad \text{and} \quad Q = G(\bar{E}_1) \quad \text{and} \quad R = F(\bar{E}_2) \quad \text{and} \quad S = G(\bar{E}_2),
\]
for some \( \bar{E}_1 \in A[\bar{Z}] \) and \( \bar{E}_2 \in B[\bar{Z}] \). We sometimes write \( F \xrightarrow{\bar{E}_1 \rightarrow \bar{E}_2} G \) to make the witnesses \( \bar{E}_1, \bar{E}_2 \) and their transition explicit. This situation can be depicted as follows:

\[
\begin{array}{ccc}
P & \rightarrow & Q :: & R & \rightarrow & S. \\
\bar{Z}/\bar{E}_1 & & & \bar{Z}/\bar{E}_2 \\
\bar{Z}/\bar{E}_1 & & & \bar{Z}/\bar{E}_2 \\
F(\bar{Z}) & & & G(\bar{Z})
\end{array}
\]

\textit{Example 19}

Consider the directed equation of Example 1 given by
\[
\text{Nat} \rightarrow P \cdot \text{List} :: \text{List} \rightarrow Z.
\]
(12)

This equation asks for a list program \( S \) which is obtained from \text{List} as the program \text{Plus} on numerals is obtained from \text{Nat}. In Example 29 we will see that the program for concatenating lists is a solution to (12).

---

4 This is why ‘copycat’ is the name of a prominent model of analogy-making 

\[\text{Holstaler and Mitchell 1995}.\] See Correa et al. (2012).
6 Properties of Logic Program Proportions

We summarize here Antić (2021a)’s most important properties of analogical equations and proportions interpreted in the logic programming setting from above.

6.1 Characteristic Justifications

Computing all justifications of an analogical proportion is complicated in general, which fortunately can be omitted in many cases.

**Definition 21**
We call a set \( J \) of justifications a characteristic set of justifications of \( P \to Q :: R \to S \) in \((A, B)\) iff \( J \) is a sufficient and necessary set of justifications of \( P \to Q :: R \to S \) in \((A, B)\), that is, iff

\[
J \subseteq \text{Jus}_{(A,B)}(P \to Q :: R \to S) \iff (A, B) \models P \to Q :: R \to S. \tag{14}
\]

In case \( J = \{ F \to G \} \) is a singleton set satisfying (14), we call \( F \to G \) a characteristic justification of \( P \to Q :: R \to S \) in \((A, B)\). Moreover, we say that \( J \) is a trivial set of justifications in \((A, B)\) iff every justification in \( J \) justifies every directed proportion \( P \to Q :: R \to S \) in \((A, B)\), that is, iff

\[
J \subseteq \text{Jus}_{(A,B)}(P \to Q :: R \to S) \quad \text{for all} \quad P, Q \in A \text{ and } R, S \in B.
\]

In this case, we call every justification in \( J \) a trivial justification in \((A, B)\). We say that \( P \to Q :: R \to S \) is a trivial proportion in \((A, B)\) iff \((A, B) \models P \to Q :: R \to S \) and \( \text{Jus}_{(A,B)}(P \to Q :: R \to S) \) consists only of trivial justifications (cf. Example 22).

**Example 22**
The forms

\[
tr_1(X, Y) := (X \cap Y) \cup (X - Y) \quad \text{and} \quad tr_2(X, Y) := (X \cap Y) \cup (Y - X)
\]

justify any proportion \( P \to Q :: R \to S \), which shows that \( tr_1 \overset{(P,Q)\to(R,S)}{\longrightarrow} tr_2 \) is a trivial justification. This example shows that trivial justifications may contain useful information about the underlying structures—in this case, it encodes the trivial observation that any two programs \( P \) and \( Q \) are symmetrically related via \( P = (P \cap Q) \cup (P - Q) \) and \( Q = (P \cap Q) \cup (Q - P) \).

The following lemma is a useful characterization of characteristic justifications in terms of injectivity.
Lemma 23 (Uniqueness Lemma)
For any justification $\mathbf{F}(\vec{Z}) \rightarrow \mathbf{G}(\vec{Z})$ of $P \dashv \vdash Q :: R \dashv \vdash S$ in $(\mathbb{A}, \mathbb{B})$, if there is a unique $\vec{E} \in \mathbb{B}^{\mid Z \mid}$ such that $R = \mathbf{F}(\vec{E})$, then $\mathbf{F} \rightarrow \mathbf{G}$ is a characteristic justification of $P \dashv \vdash Q :: R \dashv \vdash S$ in $(\mathbb{A}, \mathbb{B})$.

Proof
Since $\mathbf{F}(\vec{Z}) \rightarrow \mathbf{G}(\vec{Z})$ is a justification of $P \dashv \vdash Q :: R \dashv \vdash S$ in $(\mathbb{A}, \mathbb{B})$ by assumption, there are sequences of programs $\vec{E}_1 \in \mathbb{A}^{\mid Z \mid}$ and $\vec{E}_2 \in \mathbb{B}^{\mid Z \mid}$ satisfying (11), where $\vec{E}_2$ is uniquely determined by assumption. Consequently, given any program $\vec{S}' \in \mathbb{B}$, $\mathbf{F} \rightarrow \mathbf{G}$ is a justification of $P \dashv \vdash Q :: R \dashv \vdash \vec{S}'$ in $(\mathbb{A}, \mathbb{B})$ iff $\vec{S}' = \mathbf{G}(\vec{E}_2) = \vec{S}$, which shows that $\mathbf{F} \rightarrow \mathbf{G}$ is indeed a characteristic justification.

6.2 Functional Solutions
The following reasoning pattern—which roughly says that functional dependencies are preserved across (different) domains—will often be used in the rest of the paper.

Theorem 24 (Functional Solutions)
For any $\mathbb{A} \cap \mathbb{B}$-form $\mathbf{G}(Z)$, we have

$$(\mathbb{A}, \mathbb{B}) \models P \dashv \vdash \mathbf{G}(P) :: R \dashv \vdash \mathbf{G}(R), \quad \text{for all } P \in \mathbb{A} \text{ and } R \in \mathbb{B}.$$  

Proof
The justification $Z \rightarrow \mathbf{G}(Z)$ is a characteristic justification of $P \dashv \vdash \mathbf{G}(P) :: R \dashv \vdash \mathbf{G}(R)$ in $(\mathbb{A}, \mathbb{B})$ by Lemma 23 as the form $\text{id}(Z) = Z$ is injective in $\mathbb{A}$ and $\mathbb{B}$. □

Functional solutions are plausible since transforming $P$ into $\mathbf{G}(P)$ and $R$ into $\mathbf{G}(R)$ is a direct implementation of ‘transforming $P$ and $R$ in the same way’, and it is therefore surprising that functional solutions can be nonetheless ‘unexpected’ and therefore ‘creative’ as will be demonstrated in Section 7.

Remark 25
An interesting consequence of Theorem 24 is that in case $Q \in \mathbb{A} \cap \mathbb{B}$ is a constant program contained in both domains $\mathbb{A}$ and $\mathbb{B}$, we have

$$(\mathbb{A}, \mathbb{B}) \models P \dashv \vdash Q :: R \dashv \vdash Q, \quad \text{for all } P \in \mathbb{A} \text{ and } R \in \mathbb{B}, \quad (15)$$

characteristically justified by Theorem 24 via $Z \rightarrow Q$. This can be intuitively interpreted as follows: every program in $\mathbb{A} \cap \mathbb{B}$ has a ‘name’ and can therefore be used to form logic program forms, which means that it is in a sense a ‘known’ program. As the framework is designed to compute ‘novel’ or ‘unknown’ programs in the target domain via analogy-making, (15) means that ‘known’ target programs can always be computed.

The following result summarizes some useful consequences of Theorem 24.

Corollary 26
For any source program $P \in \mathcal{A}$, target program $R \in \mathcal{B}$, and joint programs $Q, S \in \mathcal{A} \cap \mathcal{B}$, the following proportions hold in $(\mathcal{A},\mathcal{B})$:

\[
\begin{align*}
P \rightarrow P^c &:: R \rightarrow R^c \\
\nearrow \hspace{1cm} \searrow \hspace{1cm} \nearrow \\
\searrow \hspace{1cm} \nearrow \hspace{1cm} \searrow
\end{align*}
\]

P \rightarrow P \cup Q :: R \rightarrow R \cup Q

P \rightarrow Q \circ P \circ S :: R \rightarrow Q \circ R \circ S

P \rightarrow Q \cdot P \cdot S :: R \rightarrow Q \cdot R \cdot S

P \rightarrow \text{facts}(P) :: R \rightarrow \text{facts}(R)

P \rightarrow \text{head}(P) :: R \rightarrow \text{head}(R)

P \rightarrow \text{body}(P) :: R \rightarrow \text{body}(R)

P \rightarrow \text{LM}(P) :: R \rightarrow \text{LM}(R).

The following result is an instance of Antić (2021a, Theorem 3).

**Theorem 27**

For any logic programs $P, Q \in \mathcal{A}$ and $R \in \mathcal{B}$, we have

\[
(A, B) \models P \rightarrow P :: R \rightarrow R \quad \text{(reflexivity),} \tag{16}
\]

\[
\mathcal{A} \models P \rightarrow Q :: P \rightarrow Q \quad \text{(outer reflexivity).} \tag{17}
\]

### 7 Examples

In this section, we demonstrate the idea of learning logic programs by analogy via directed logic program proportions by giving some illustrative examples.

**Example 28**

Let $A = \{a, b\}$ and $B = \{c, d\}$ be propositional alphabets, and let $\mathcal{A}$ and $\mathcal{B}$ for the moment be the identical spaces of all propositional programs over $A \cup B$. Consider the following directed equation:

\[
\{a \leftarrow b\} \rightarrow \left\{ \begin{array}{c} b \\ a \leftarrow b \end{array} \right\} :: \{c \leftarrow d\} \rightarrow Z. \tag{18}
\]

Here we have at least two candidates for the solution $S$. First, we can say that the second program in (18) is obtained from the first by adding the fact $b$, in which case we expect—by analogy—that

\[
S = \left\{ \begin{array}{c} b \\ c \leftarrow d \end{array} \right\} \tag{19}
\]

is a solution to (18). Define the form

\[
\mathcal{G}(Z) := Z \cup \{b\}. \tag{20}
\]

Then the computations

\[
\mathcal{G}(\{a \leftarrow b\}) = \left\{ \begin{array}{c} b \\ a \leftarrow b \end{array} \right\} \quad \text{and} \quad \mathcal{G}(\{c \leftarrow d\}) = S \tag{21}
\]

show that $S$ is indeed a solution by Theorem [24] that is, we have

\[
\{a \leftarrow b\} \rightarrow \mathcal{G}(\{a \leftarrow b\}) :: \{c \leftarrow d\} \rightarrow \mathcal{G}(\{c \leftarrow d\}). \tag{22}
\]
However, what if we separate the two domains by saying that $A$ and $B$ are the spaces of propositional programs over the disjoint alphabets $A$ and $B$, respectively? In this case, $Z \rightarrow G(Z)$ is no longer a valid justification of (22) as $\{b\}$ in (20) is not contained in $A \cap B$. This makes sense since, in this case, the ‘solution’ $S$ contains the fact $b$ alien to the target domain $B$. Thus the question is whether we can redefine $G$, without using the fact $b$, so that (21) holds. Observe that $b$ is also the body of $a \leftarrow b$, which motivates the following definition:

$$G'(Z) := Z \cup \text{body}(Z).$$

A simple computation shows that $G'$ satisfies

$$G'(\{a \leftarrow b\}) = \left\{ \begin{array}{l} b \\ a \leftarrow b \end{array} \right\},$$

which means that we can compute a solution $S'$ of (18) via Theorem 24 as

$$S' := G'(\{c \leftarrow d\}) = \left\{ \begin{array}{l} d \\ c \leftarrow d \end{array} \right\}.
$$

**Example 29**

Reconsider the situation in Example 11 where we have derived the abstract form of Plus generalizing addition. As a consequence of Theorem 24, we have the following directed logic program proportion

$\text{Nat} \rightarrow \text{Plus(Nat)} :: \text{List} \rightarrow \text{Plus(List)}.$

(23)

This proportion formalizes the intuition that ‘numbers are to addition what lists are to list concatenation.’ Similarly, we have

$\text{Nat} \rightarrow \text{Plus(Nat)} :: \text{Tree} \rightarrow \text{Plus(Tree)}.$

Without going into technical details, we want to mention that a similar procedure as in Example 11 applied to a program for multiplication yields a form $\text{Times}(Z(q)(\bar{x}))$ such that $\text{Times}(\text{List}(u,x))$ is a program for ‘multiplying’ lists, e.g.,

$$\text{Times}(\text{List}(u,x)) \models \text{times}([a,a],[b,b],[b,b,b,b]).$$

We then have the following directed logic program proportion as an instance of Theorem 24

$\text{Nat} \rightarrow \text{Times(Nat)} :: \text{List} \rightarrow \text{Times(List)}.$

In other words, addition is to multiplication what list concatenation is to list ‘multiplication.’

**Example 30**

In Example 5 we have constructed $\text{Even}$ from $\text{Nat}$ via composition and in Example 12

---

5 For simplicity, we omit here the variables $u$ and $x$ from notation, that is, we write $\text{Nat}$ and $\text{List}$ instead of $\text{Nat}(x)$ and $\text{List}(u,x)$, respectively.
we have then derived the abstract form Even generalizing ‘evenness.’ As a consequence of Theorem 24, we have the following directed logic program proportion:

\[ \text{Nat} \rightarrow \text{Even}(\text{Nat}) :: \text{Reverse} \rightarrow \text{Even}(\text{Reverse}). \]

This shows that the (seemingly unrelated) program for reversing lists of even length shares the syntactic property of ‘evenness’ with the program for constructing the even numbers.

**Example 31**

In Example 13 we have derived the abstract form Member generalizing ‘membership’ and we have asked the following question: What does ‘membership’ mean in the numerical domain? We can now state this question formally in the form of the following directed logic program equation:

\[ \text{List} \rightarrow \text{Member} :: \text{Nat} \rightarrow \text{Z}. \]

As a consequence of Theorem 24, we have the following directed logic program proportion:

\[ \text{List}(u, x) \rightarrow \text{Member}(\text{List}(u, x)) :: \text{Nat}(u) \rightarrow \text{Member}(\text{Nat}(u)), \]

where Member(Nat(u)) is the program computing the numerical ‘less than’ relation of Example 13.

8 Conclusion

This paper studied directed analogical proportions between logic programs for logic-based analogical reasoning and learning in the setting of logic programming. This enabled us to compare logic programs possibly across different domains in a uniform way which is crucial for AI-systems. For this, we defined the composition and concatenation of logic programs and showed, by giving some examples, that syntactically similar programs have similar decompositions. This observation led us to the notion of logic program forms which are proper generalizations of logic programs. We then used forms to formalize directed analogical proportions between logic programs—as an instance of Antić (2021a)’s model—as a mechanism for deriving novel programs in an ‘unknown’ target domain via analogical transfer—realized by generalization and instantiation—from a ‘known’ source domain. In a broader sense, this paper is a further step towards an algebraic theory of logic-based analogical reasoning and learning in knowledge representation and reasoning systems, with potential applications to fundamental AI-problems like commonsense reasoning and computational learning and creativity.

Future Work

In this paper we have demonstrated the utility of our framework of directed logic program proportions for learning logic programs by analogy with numerous examples. The main task for future research is to develop methods for the algorithmic computation of solutions to directed program equations as defined in this paper. At its core, this requires algebraic methods for logic program decomposition (and deconcatenation). This task turns out to be non-trivial even for the propositional case (cf. Antić (2021c)).
Composition and concatenation are interesting operations on programs in their own right and a comparison to other operators for program modularity (cf. [Bugliesi et al. (1994) and Brogi et al. (1999)] remains as future work. A related question is whether these operations are sufficient for modeling all plausible analogies in logic programming or whether further operations are needed (‘completeness’). It is important to emphasize that in the latter case, adding novel operations to the framework does not affect the general formulations of the core Definitions 10 and 16.

In this paper we have restricted ourselves to Horn programs. In the future we plan to adapt our framework to extended classes of programs as, for example, higher-order (cf. Chen et al. (1993) and Miller and Nadathur (2012)) and non-monotonic logic programming under the stable model or answer set semantics (Gelfond and Lifschitz 1991) and extensions thereof (cf. Brewka et al. (2011)). For this, we will define the composition and concatenation of answer set programs (Antić 2021b) which is non-trivial due to negation as failure occurring in rule bodies (and heads).

Finally, a formal comparison of analogical reasoning and learning as defined in this paper with other forms of reasoning and learning, most importantly inductive logic programming (Muggleton 1991), is desirable as this line of research may lead to an interesting combination of different learning methods.

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