Ghost decoupling in ’t Hooft spectrum for mesons

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(August 5, 2018)

Abstract

We show that the replacement of the “instantaneous” ’t Hooft’s potential with the causal form suggested by equal time canonical quantization in light-cone gauge, which entails the occurrence of negative probability states, does not change the bound state spectrum when the difference is treated as a single insertion in the kernel.
In 1974 G. ’t Hooft \[1\] proposed a very interesting model to describe the mesons, starting from a SU(N) Yang-Mills theory in 1+1 dimensions in the large N limit.

Quite remarkably in this model quarks look confined, while a discrete set of quark-antiquark bound states emerges, with squared masses lying on rising Regge trajectories.

The model is solvable thanks to the “instantaneous” character of the potential acting between quark and antiquark.

Three years later such an approach was criticized by T.T. Wu \[2\], who replaced the instantaneous ’t Hooft’s potential by an expression with milder analytical properties, allowing for a Wick’s rotation without extra terms.

Unfortunately this modified formulation led to a quite involved bound state equation, which may not be solved. An attempt to treat it numerically in the zero bare mass case for quarks \[3\] led only to partial answers in the form of a completely different physical scenario. In particular no rising Regge trajectories were found.

After those pioneeristic investigations, many interesting papers followed ’t Hooft’s approach, pointing out further remarkable properties of his theory and blooming into the recent achievements of two dimensional QCD, whereas Wu’s approach sank into oblivion, if not disrepute.

Still, equal time canonical quantization of Yang-Mills theories in light-cone gauge \[7\] leads precisely in 1+1 dimensions to the Wu’s expression for the vector exchange between quarks \[8\], which is nothing but the 1+1 dimensional version of the Mandelstam-Leibbrandt (ML) \[7,9\] propagator. Its causal nature, which entails the occurrence of negative probability states, makes it compulsory in order to achieve gauge invariance and renormalization in 1+(D-1) dimensions \[10,11\].

Purpose of this note is to show that indeed the difference between ’t Hooft’s \[\frac{1}{k^2}\] and Wu’s \[\frac{1}{(k_- - i \text{sign}(k_+))^2}\] potentials might in some sense be treated as a perturbation. Our main result will be that no correction due to this difference will affect the ’t Hooft’s bound state spectrum in the small coupling region, owing to a precise cancellation between “real” and “virtual” insertions. This phenomenon is analogous to the one occurring, with respect to the
same extra term, in perturbative four-dimensional calculations concerning Altarelli-Parisi \[4\] and Balitsky-Fadin-Kuraev-Lipatov \[5\] kernels. This analogy may have far-reaching consequences.

We follow here definitions and notations of refs. \[1\] and \[2\] the reader is invited to consult.

The ’t Hooft potential exhibits an infrared singularity which, in the original formulation, was handled by introducing an infrared cutoff; a quite remarkable feature of this theory is that bound state wave functions and related eigenvalues turn out to be cutoff independent. As a matter of fact in ref. \[6\], it has been pointed out that the singularity at \( k^- = 0 \) can also be regularized by a Cauchy principal value \((P)\) prescription without finding differences in gauge invariant quantities. Then, the difference between the two potentials is represented by the following distribution

\[
\Delta(k) \equiv \frac{1}{(k^- - i\varepsilon \text{sign}(k^+))^2} - P\left(\frac{1}{k^-}\right) = -i\pi \text{sign}(k^+) \delta'(k^-). \tag{1}
\]

This is the quantity we are going to treat as an insertion in the Wu’s integral equations for the quark propagator and for the bound state wave function, starting from ’t Hooft’s solutions. We stress that we shall sum exactly the same planar diagrams of refs. \[1\] and \[2\], which are the relevant ones in the large \( N \) limit.

The Wu’s integral equation for the quark self-energy in the Minkowski momentum space is

\[
\Sigma(p; \eta) = \frac{ig^2}{\pi^2} \frac{\partial}{\partial p_+} \int \frac{dk_+ dk_-}{k_+ k_-} \left[ P\left(\frac{1}{k_- - p_-}\right) + i\eta \pi \text{sign}(k_+ - p_+) \delta(k_- - p_-) \right] \\
\cdot \frac{k_-}{k^2 + m^2 - k_- \Sigma(k; \eta) - i\varepsilon}, \tag{2}
\]

where \( \eta \) is a real parameter which will be used in the sequel as a counter of insertions and here should be set equal to 1.

Its exact solution with appropriate boundary conditions reads

\[
\Sigma(p; \eta) = \frac{1}{2p_-} \left( \left[p^2 + m^2 + (1 - \eta) \frac{g^2}{\pi}\right] - \left[p^2 + m^2 - (1 - \eta) \frac{g^2}{\pi}\right] \right) \\
\cdot \sqrt{1 - \frac{4\eta g^2 p^2}{\pi(p^2 + m^2 - (1 - \eta) \frac{g^2}{\pi} - i\varepsilon)^2}}. \tag{3}
\]
One can immediately realize that ’t Hooft’s and Wu’s solutions are recovered for $\eta = 0$ and $\eta = 1$ respectively.

The dressed quark propagator turns out to be

$$S(p; \eta) = -\frac{ip_-}{m^2 + 2p_+p_- - p_-\Sigma(p; \eta)}.$$  (4)

Wu’s bound state equation in Minkowski space, using light-cone coordinates, is

$$\psi(p, r) = -\frac{ig^2}{\pi^2}S(p; \eta)S(p - r; \eta) \int dk_+ dk_- \left[ P\left(\frac{1}{(k_- - p_-)^2}\right) + \right.
\left. + i\eta\pi\text{sign}(k_+ - p_+)\delta'(k_- - p_-)\right] \psi(k, r).$$  (5)

We are here considering for simplicity the equal mass case and $\eta$ should be set equal to 1.

Let us denote by $\phi_k(x)$, $0 \leq x = \frac{p_-}{r} \leq 1$, $r_- > 0$, the ’t Hooft’s eigenfunction corresponding to the eigenvalue $\alpha_k$ for the quantity $-\frac{2r_+r_-}{M^2}$, where $M^2 = m^2 - \frac{g^2}{\pi}$. Those eigenfunctions are real, of definite parity under the exchange $x \to 1 - x$ and vanishing outside the interval $0 < x < 1$:

$$\phi_k(x) = \int dp_+ \frac{r_-}{M^2} \psi_k(p_+, p_-, r),$$
$$\psi_k = \frac{1}{i\pi}\phi_k(x) \frac{M^4}{M^2 + 2r_-p_+x - i\epsilon} \frac{1 - \alpha_k x(1 - x)}{M^2 - \alpha_k M^2(1 - x) - 2r_-p_+(1 - x) - i\epsilon}.$$  (6)

They are solutions of eq.(5) for $\eta = 0$ and form a complete set.

We are interested in a first order calculation in $\eta$. Of course this procedure is to be considered in a heuristic way; moreover it is likely to be sensible only in the weak coupling region $\frac{g^2}{\pi} < M^2$. The integral equation (5), after first order expansion in $\eta$ of its kernel, becomes

$$\psi(p_+, p_-, r) = \frac{ig^2}{\pi^2} \frac{p_-}{M^2 + 2p_+p_- - i\epsilon} \frac{p_- - r_-}{M^2 + 2(p_+ - r_+)(p_- - r_-) - i\epsilon}
\left. \cdot \left[ \left(1 - \frac{\eta g^2 M^2}{\pi}\right)[(M^2 + 2p_+p_- - i\epsilon)^{-2} + (M^2 + 2(p_+ - r_+)(p_- - r_-) - i\epsilon)^{-2}] \right] \right)
\cdot \int dk_+ dk_- \frac{1}{(k_- - p_-)^2} \psi(k_+, k_-, r) -
- i\eta \int dk_+ dk_- \text{sign}(k_+ - p_+)\delta'(k_- - p_-)\psi(k_+, k_-, r).$$  (7)
We integrate this equation over $p_+$ with $r_- > 0$ and search for solutions with the same support properties of ‘t Hooft’s ones. We get

$$
\phi(x, r) = \frac{g^2}{\pi M^2} \frac{x(1-x)}{1-\alpha x(1-x) - i \epsilon} \left[ (1 - \eta) \frac{g^2}{\pi M^2} \frac{x^2 + (1-x)^2}{(1-\alpha x(1-x) - i \epsilon)^2} \right]
$$

\[ P \int_0^1 \frac{dy}{(y-x)^2} \phi(y, r) - \frac{\alpha \eta}{2} \int d\xi \log \frac{1-x}{1-\alpha \xi - i \epsilon} \psi'(\xi, x, r) \], \quad (8)

where $'$ means derivative with respect to $x$.

It is now straightforward to check that ‘t Hooft’s solution $\psi_k(p_+, p_-, r)$ is indeed a solution also of this equation when $\alpha$ is set equal to $\alpha_k$, for any value of $\eta$, in particular for $\eta = 1$, thanks to a precise cancellation of the contributions coming from the propagators (“virtual” insertions) against the extra term due to the modified form of the “potential” (“real” insertion). In other words the extra piece of the kernel at $\alpha = \alpha_k$ vanishes when acting on $\psi_k$.

As a matter of fact, taking ‘t Hooft’s equation into account, we get

$$
(\alpha_k - \alpha) \phi_k(x) \left[ 1 - \frac{\eta g^2}{\pi M^2 [1-\alpha x(1-x) - i \epsilon]^2} \left( (1-x)^2 + x^2 \left[ 1 + \frac{1-\alpha x(1-x)}{1-\alpha_k x(1-x) - i \epsilon} \right] \right) \right] =
$$

\[ = \frac{\eta g^2}{\pi M^2} \phi'_k(x) \log \frac{1-\alpha_k x(1-x) - i \epsilon}{1-\alpha x(1-x) - i \epsilon}. \quad (9) \]

There are no corrections from a single insertion in the kernel to ‘t Hooft eigenvalues and eigenfunctions. We stress that this result does not depend on their detailed form, but only on their general properties.

In conclusion the ghosts which are responsible of the causal behaviour of the ML propagator [12], do not modify the bound state spectrum, as their “real” contribution cancels against the “virtual” one in propagators. Wu’s equation for colorless bound states, although much more involved than the corresponding ‘t Hooft’s one, might still apply. This is at least the heuristic lesson one learns from a single insertion in the kernel and is in agreement with a similar mechanism occurring in four-dimensional perturbative QCD [13].

Higher order insertions, although worth to be explored, do not lead to a trivial problem owing to the non linear character of the integral equation for the propagator. Moreover the
validity of our result is limited to the small coupling region; for $\frac{g^2}{\pi M^2} > 1$ the theory is likely to be in a different phase (see for instance [13]) and Wu’s equation may lead to a quite different physical scenario.

In QCD$_4$ consistency with renormalization procedure for general Green’s functions strongly suggests the ML prescription to regularize infrared singularities; in QCD$_2$ the ML option may not be strictly compelling as the theory becomes super-renormalizable.

Planarity plays a crucial role in our considerations; as a matter of fact the same ghost contribution cancels also in Wilson loops provided the large $N$ limit is considered [14]. Indeed cross diagrams are order $\frac{1}{N^2}$ with respect to planar ones, which in turn are unaffected by the ML prescription. Planarity and ghost cancellation might be deeply related, an argument which is worth of further investigation.

ACKNOWLEDGEMENTS. We thank G. Nardelli for many useful discussions. L.G. acknowledges a INFN post-doctoral fellowship.
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