I. INTRODUCTION

It is well known that the Friedmann-Lemaître-Robertson-Walker solution of Einstein’s field equations of general relativity admits three possible geometries for the three-dimensional physical space, isomorphic to \( \mathbb{R}^3 \), \( H^3 \), or \( S^3 \), where \( \mathbb{R}^3 \) represents an open Euclidean space, \( H^3 \) represents an open hyperboloid of negative curvature, and \( S^3 \) represents a closed 3-sphere of constant positive curvature \([1]\). Among these possible geometries, only \( S^3 \) represents a closed universe with compact geometry. Moreover, observationally, the cosmic microwave background spectra mapped by the space observatory Planck now prefers a positive curvature, or \( S^3 \), at more than 99% confidence level \([2, 3]\).

On the other hand, Bell’s argument against possible local-realistic models for quantum correlations is set within a flat and immutable spacetime \([4]\). Indeed, in Chapter 7 of his book \([5]\), while exploring possible strategies to overcome his argument, Bell wonders: “The space time structure has been taken as given here. How then about gravitation?”

Considering this shortcoming of Bell’s argument and other physical considerations in the foundations of quantum mechanics, in a series of works since 2007 \([6–11]\) we have proposed the following experimentally testable hypothesis:

**Hypothesis 1:** The strong quantum correlations observed in Nature can be understood as manifestly local, realistic, and deterministic correlations if we model the three-dimensional physical space as a closed and compact quaternionic 3-sphere \( S^3 \) using geometric (or Clifford) algebra \([12]\) rather than as a flat and open space \( \mathbb{R}^3 \) using “vector algebra.”

This hypothesis has far reaching consequences, which we have explored extensively in \([6–11]\). We have also proposed a macroscopic experiment in \([8]\) that may be able to verify this hypothesis \([13]\). Our goal here is to explicitly derive one of these consequences, namely the strong correlations predicted by the entangled singlet state within the local-realistic framework of Bell \([3]\). To that end, we begin by defining a quaternionic 3-sphere whose tangent spaces are locally \( \mathbb{R}^3 \):

\[
S^3 := \left\{ \mathbf{q}(\theta, \mathbf{r}) = \varrho_r \times \left[ \cos \left( \frac{\theta}{2} \right) + \mathbf{J}(\mathbf{r}) \sin \left( \frac{\theta}{2} \right) \right] \right\} \\
\| \mathbf{q}(\theta, \mathbf{r}) \| = \varrho_r,
\]

where \( \mathbf{J}(\mathbf{r}) = I_3 \mathbf{r} \) is a unit bivector (or pure quaternion) rotating about a unit axis vector \( \mathbf{r} \in \mathbb{R}^3 \) with rotation angle \( 0 \leq \theta < 4\pi \), \( \varrho_r \) is the radius of the 3-sphere, and \( I_3 = \mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z = \mathbf{e}_x e_y e_z \) is the standard trivector in the Clifford algebra \( \text{Cl}_{3,0} \) of orthogonal directions in \( \mathbb{R}^3 \) \([12]\). The bases of the bivector \( \mathbf{J}(\mathbf{r}) \) forms the even subalgebra of \( \text{Cl}_{3,0} \):

\[
J_i J_j = -\delta_{ij} - \sum_k \epsilon_{ijk} J_k.
\]

It is easy to verify that quaternions \( \mathbf{q}(\theta, \mathbf{r}) \) appearing in \((1)\) respect the rotational symmetries exhibited by spinors:

\[
\mathbf{q}(\theta + 2\kappa\pi, \mathbf{r}) = (-1)^\kappa \mathbf{q}(\theta, \mathbf{r}) \quad \text{for } \kappa = 0, 1, 2, 3, \ldots
\]

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Thus, we can use $q(\eta_{xy}, r)$ to represent states of a physical system that returns to itself only after even multiples of $2\pi$ rotations. Given two unit vectors $x$ and $y$ and a rotation axis $r$, each of the quaternions in $S^3$ can be factorized into a product of corresponding bivectors $J(x)$ and $J(y)$ [which can be expanded using the basis bivectors in (2)] as follows:

$$q(\eta_{xy}, r) = -J(x)J(y) = -(I_3x)(I_3y)$$

$$= -(I_3)^2xy = xy = x \cdot y + x \wedge y$$

$$= \cos(\eta_{xy}) + J(r) \sin(\eta_{xy}),$$

where $\eta_{xy}$ is the angle between $x$ and $y$, $xy$ is the geometric product between $x$ and $y$, and $J(r)$ is identified with $\frac{\pi \eta_{xy}}{||x \cdot y||}$. Comparing $q(\eta_{xy}, r)$ in (4) with $q(\theta, r)$ in (1), we see that the rotation angle $(\theta)$ of $q(\theta, r)$ is twice the angle $\eta_{xy}$ between the vectors $x$ and $y$ in any factorization such as that in (4):

$$\theta = 2\eta_{xy}.$$  

Consequently, the spinorial sign changes exhibited by quaternions as shown in (4) can be expressed also using $\eta_{xy}$ as

$$q(\eta_{xy}, r + \kappa \pi, r) = (-1)^\kappa q(\eta_{xy}, r) \quad \text{for } \kappa = 0, 1, 2, 3, \ldots$$

As we shall demonstrate below, relation (3) exhibits the key property that induces the singlet correlations within $S^3$.

**II. LOCAL, REALISTIC, AND DETERMINISTIC MEASUREMENT FUNCTIONS**

Now, for our purposes here, it is convenient to express the unit quaternions defined in (1) as pairs of products of two bivectors such as $D(a)L(s_1)$ and $D(b)L(s_2)$, with bivectors $D(a) = I_3a$ and $D(b) = I_3b$ representing the detectors used by Alice and Bob and bivectors $L(s_1) = I_3s_1$ and $L(s_2) = I_3s_2$ representing the spins being detected by them in a Bell-test experiment [14]. The conservation of angular momentum requires the total spin to respect the condition

$$-L(s_1) + L(s_2) = 0 \iff L(s_1) = L(s_2) \iff s_1 = s_2 = s.$$  

Evidently, in the light of the product rule (2) for the unit bivectors, the above condition is equivalent to the condition

$$L(s_1)L(s_2) = I_3sI_3s = (I_3)^2ss = -1.$$  

Next, we introduce the following two sign functions analogous to the ones introduced by Bell in his local model [4]:

$$\mu_1 = \text{sign}(a \cdot s_1^i) = \pm 1 \quad \text{and} \quad \mu_2 = \text{sign}(s_1^i \cdot b) = \pm 1,$$

where the subscripts 1 and 2 on $\mu$ and $s$ refer to the observation stations of Alice and Bob (cf. Fig. 1), and the superscript $i$ on $s$ indicates its initial direction at the source with respect to their chosen detector directions. The spin direction $s = s_1 = s_2$ acts as a hidden variable, just as in Bell’s local model [4, 12]. The function $\mu_1 = \text{sign}(a \cdot s_1^i)$
can be understood as follows. If, initially (i.e., before the detection process defined by the measurement functions to be specified below), the two unit vectors \(a\) and \(s_1^1\) happen to be pointing through the same hemisphere of \(S^2 \rightarrow \mathbb{R}^3\) centered at the origin of \(s_1^1\), then \(\mu_1 = \text{sign}(a \cdot s_1^1)\) will be equal to +1, and if the two unit vectors \(a\) and \(s_1^2\) happen to be pointing through the opposing hemispheres of \(S^2\) centered at the origin of \(s_1^2\), then \(\mu_1 = \text{sign}(a \cdot s_1^2)\) will be equal to −1, provided that \(a \cdot s_1^i \neq 0\). If \(a \cdot s_1^i\) happens to be zero, then \(\mu_1 = \text{sign}(a \cdot s_1^i)\) will be assumed to be equal to the sign of the first nonzero component of \(a\) from the set \(\{x, a_y, a_z\}\). And likewise for the function \(\mu_2 = \text{sign}(s_2^2 \cdot b)\).

With above preliminaries in mind, we can now state the central theorem, proved in several different ways in [11].

**Theorem 1:** The strong quantum mechanical correlations predicted by the entangled singlet state can be understood as classical, local, realistic, and deterministic correlations among the pairs of limiting scalar points \(\mathcal{A}(a, s_1) = \pm 1\) and \(\mathcal{B}(b, s_2) = \pm 1\) of a quaternionic 3-space, or \(S^3\), assumed to be a model of the three-dimensional physical space.

The proof of this theorem requires us to compute the correlations while preserving the geometrical properties of \(S^3\):

\[
E(a, b) = \int_{S^2} \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) p(s) \, ds = -\cos(\eta_{ab}),
\]

(12)

where \(S^2\) is the base manifold of \(S^3\) [10], the function \(p(s)\) specifies a probability distribution of the spin direction \(s\) over \(S^2\), and the functions \(\mathcal{A}(a, s_1)\) and \(\mathcal{B}(b, s_2)\) encode local physical interactions taking place during the detection processes at the two ends of the experiment, producing results observed by Alice and Bob. They are defined as follows:

\[
S^3 \ni \mathcal{A}(a, s_1) = \lim_{s_1 \rightarrow \mu_1 a} \{ -D(a) L(s_1) \}
\]

(13)

\[
= \lim_{s_1 \rightarrow \mu_1 a} \{ -I_3 a S_1 \}
\]

(14)

\[
= \lim_{s_1 \rightarrow \mu_1 a} \{ a \cdot s_1 + I_3 (a \times s_1) \}
\]

(15)

\[
= \lim_{s_1 \rightarrow \mu_1 a} \{ \cos(\eta_{as_1}) + (I_3 r_1) \sin(\eta_{as_1})\}
\]

(16)

\[
= \lim_{s_1 \rightarrow \mu_1 a} \{ +q(\eta_{as_1}, r_1)\}
\]

(17)

where \(\eta_{as_1}\) is the angle between the spin direction \(s_1\) and the detector direction \(a\) chosen by Alice as depicted in Fig. 1, and \(r_1 = \frac{a \times s_1}{|a \times s_1|}\) is the normalized rotation axis of the quaternion \(q(\eta_{as_1}, r_1)\) in \(S^3\). Thus, given the definition (11) of \(\mu_1\), the spin direction \(s_1\) will tend to \(+a\) if initially the two unit vectors \(a\) and \(s_1\) happen to be pointing through the same hemisphere of \(S^2\) centered at the origin of \(s_1\), and otherwise the spin direction \(s_1\) will tend to \(-a\). Consequently, as \(s_1 \rightarrow \mu_1 a = \pm a\) during the detection process by Alice so that the angle \(\eta_{as_1} \rightarrow 0\) or \(\pi\), the binary value of the observed result by Alice is obtained because \(\cos(\eta_{as_1}) \rightarrow \pm 1\) and \(\sin(\eta_{as_1}) \rightarrow 0\), giving

\[
\mathcal{A}(a, s_1) \rightarrow +\mu_1 = \pm 1.
\]

(18)

Similarly, the measurement function for Bob is defined as

\[
S^3 \ni \mathcal{B}(b, s_2) = \lim_{s_2 \rightarrow \mu_2 b} \{ +L(s_2) D(b) \}
\]

(19)

\[
= \lim_{s_2 \rightarrow \mu_2 b} \{ +I_3 s_2 S_2 \}
\]

(20)

\[
= \lim_{s_2 \rightarrow \mu_2 b} \{ -s_2 \cdot b - I_3 (s_2 \times b) \}
\]

(21)

\[
= \lim_{s_2 \rightarrow \mu_2 b} \{ -\cos(\eta_{bs_2}) - I_3 r_2 \sin(\eta_{bs_2})\}
\]

(22)

\[
= \lim_{s_2 \rightarrow \mu_2 b} \{ -q(\eta_{bs_2}, r_2)\}
\]

(23)

where \(\eta_{bs_2}\) is the angle between the spin direction \(s_2\) and the detector direction \(b\) chosen by Bob as depicted in Fig. 1, and \(r_2 = \frac{s_2 \times b}{|s_2 \times b|}\) is the normalized rotation axis of the quaternion \(q(\eta_{bs_2}, r_2)\) in \(S^3\). Consequently, as \(s_2 \rightarrow \mu_2 b = \pm b\) during the detection process by Bob so that the angle \(\eta_{bs_2} \rightarrow 0\) or \(\pi\), the binary value of the observed result by Bob is obtained because \(\cos(\eta_{bs_2}) \rightarrow \pm 1\) and \(\sin(\eta_{bs_2}) \rightarrow 0\), giving

\[
\mathcal{B}(b, s_2) \rightarrow -\mu_2 = \mp 1.
\]

(24)

It follows from (18) and (24) that, in general, for the choices \(a \neq b\), the product of the results observed by Alice and Bob will fluctuate between \(-1\) and \(+1\):

\[
\mathcal{A}(a, s_1) \mathcal{B}(b, s_2) = \left[ \lim_{s_1 \rightarrow \mu_1 a} \{ -D(a) L(s_1) \} \right] \left[ \lim_{s_2 \rightarrow \mu_2 b} \{ +L(s_2) D(b) \} \right] = -\mu_1 \mu_2 = - (\pm 1)(\pm 1) = \mp 1.
\]

(25)
On the other hand, for the choice $b = a$, we have $\mu_2 = \mu_1$ with $s_2 = s_1$, and perfect anti-correlation will be observed:

$$\mathcal{A}(a, s_1) \mathcal{B}(a, s_2) = \left[ \lim_{s_1 \to \mu_1 a} \{- D(a) L(s_1)\} \right] \left[ \lim_{s_2 \to \mu_1 a} \{+ L(s_2) D(a)\} \right] = -(\mu_1)^2 = -1. \quad (26)$$

It is important to note that the functions $\mathcal{A}(a, s_1)$ and $\mathcal{B}(b, s_2)$ defined in (13) and (19) are manifestly local-realistic in the sense espoused by Einstein and formalized by Bell in [4]. Apart from the hidden variable $s_1$, the result $\mathcal{A} = \pm 1$ depends only on the measurement direction $a$, chosen freely by Alice, regardless of Bob’s actions. And, analogously, apart from the hidden variable $s_2$, the result $\mathcal{B} = \pm 1$ depends only on the measurement direction $b$, chosen freely by Bob, regardless of Alice’s actions. In particular, the function $\mathcal{A}(a, s_1)$ does not depend on $b$ or $\mathcal{B}$ and the function $\mathcal{B}(b, s_2)$ does not depend on $a$ or $\mathcal{A}$. Moreover, the hidden variables $s_1$ and $s_2$ do not depend on $a$, $b$, $\mathcal{A}$, or $\mathcal{B}$.

### III. COMPUTING THE SINGLET CORRELATIONS WITHIN 3-SPHERE

It is also important to note that the conservation of zero spin angular momentum [12] requires the equality $s_1 = s_2$ to hold during the free evolution of the spins, but not necessarily during their detection processes [10]. Consequently, in analogy with the quantum mechanical predictions, the expectation values of these results will vanish:

$$\mathcal{E}(a) = \int_{S^2} \mathcal{A}(a, s_1) p(s) \, ds = 0 \quad (28)$$

and

$$\mathcal{E}(b) = \int_{S^2} \mathcal{B}(b, s_2) p(s) \, ds = 0, \quad (29)$$

where the subscripts 1 and 2 on the spin directions are retained for clarity even though they are the same direction.

The question now is: What will be the correlations within $S^3$ between the results $\mathcal{A}(a, s_1)$ and $\mathcal{B}(b, s_2)$ observed jointly but independently by Alice and Bob, in coincident counts, at a spacelike distance from each other? To answer this question, we can compute the correlations using the following standard formula for expectation value used in [4]:

$$\mathcal{E}(a, b) = \int_{S^2} \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) p(s) \, ds. \quad (30)$$

However, there is nothing special about the order of the product $\mathcal{A}(a, s_1) \mathcal{B}(b, s_2)$ of the results observed by Alice and Bob that appear in this formula. We could equally use the reverse order $\mathcal{B}(b, s_2) \mathcal{A}(a, s_1)$ to compute the correlations. In fact, the geometrical symmetry of the experimental setup shown in Fig. 1 suggests the following symmetric formula:

$$\mathcal{E}(a, b) = \int_{S^2} \left[ \frac{1}{2} \left\{ \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) + \mathcal{B}(b, s_2) \mathcal{A}(a, s_1) \right\} \right] p(s) \, ds. \quad (31)$$

Since $\mathcal{A}(a, s_1) \mathcal{B}(b, s_2) = \mathcal{B}(b, s_2) \mathcal{A}(a, s_1) = \frac{1}{2} \left\{ \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) + \mathcal{B}(b, s_2) \mathcal{A}(a, s_1) \right\}$, this formula is equivalent to the one seen in [30] that single out the order of the product. Indeed, in analogy with (25) it is easy to verify that

$$\mathcal{B}(b, s_2) \mathcal{A}(a, s_1) = \left[ \lim_{s_2 \to \mu_b a} \{+ L(s_2) D(b)\} \right] \left[ \lim_{s_1 \to \mu_1 a} \{- D(a) L(s_1)\} \right] = -\mu_2 \mu_1 = -\mu_1 \mu_2 = \mp 1, \quad (32)$$

which follows from the definitions (13) and (19). Consequently, the equivalence of formulae (30) and (31) follows:

$$\frac{1}{2} \left\{ \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) + \mathcal{B}(b, s_2) \mathcal{A}(a, s_1) \right\} = \frac{1}{2} \left\{ -\mu_1 \mu_2 - \mu_2 \mu_1 \right\} = -\mu_1 \mu_2 = \mp 1. \quad (33)$$
Next, to prove Theorem 1, we must use the “product of limits equal to limits of product” rule giving the equalities

\[
\mathcal{A}(a, s_1) \mathcal{B}(b, s_2) = \left[ \lim_{s_1 \to \mu_1 a} \{ - D(a) L(s_1) \} \right] \left[ \lim_{s_2 \to \mu_2 b} \{ + L(s_2) D(b) \} \right] = \lim_{s_1 \to \mu_1 a} \lim_{s_2 \to \mu_2 b} \left\{ - D(a) L(s_1) L(s_2) D(b) \right\} \quad (34)
\]

and

\[
\mathcal{B}(b, s_2) \mathcal{A}(a, s_1) = \left[ \lim_{s_2 \to \mu_2 b} \{ + L(s_2) D(b) \} \right] \left[ \lim_{s_1 \to \mu_1 a} \{ - D(a) L(s_1) \} \right] = \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \left\{ - L(s_2) D(b) D(a) L(s_1) \right\}. \quad (35)
\]

The last equality can be simplified by multiplying its right-hand side with \( L(s_2) \) from the left and from the right in its numerator and denominator to give

\[
\mathcal{B}(b, s_2) \mathcal{A}(a, s_1) = \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \left\{ - \frac{L(s_2) L(s_2) D(b) D(a) L(s_1) L(s_2)}{L(s_2) L(s_2)} \right\} \quad (36)
\]

\[
= \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \left\{ - D(b) D(a) L(s_1) L(s_2) \right\}, \quad (37)
\]

where in the last step we have used the fact that bivectors square to \(-1\): \( L(s_2) L(s_2) = I_{3}s_2 I_{3}s_2 = (I_{3})^2 s_2 s_2 = -1 \). Consequently, using the equality \( (27) \), we see that the limit relation \( (35) \) reduces to the following convenient form:

\[
\mathcal{B}(b, s_2) \mathcal{A}(a, s_1) = \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \left\{ - D(b) D(a) L(s_1) L(s_2) \right\}. \quad (38)
\]

The equalities \( (34) \) and \( (38) \) are easy to verify by taking limits explicitly on both sides. Nevertheless, for completeness of our derivation, we have proved them in Appendix A below. With them, we can now derive the correlations using

\[
\mathcal{E}(a, b) = \int_{S^2} \frac{1}{2} \left[ \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) + \mathcal{B}(b, s_2) \mathcal{A}(a, s_1) \right] p(s) \, ds
\]

\[
= \int_{S^2} \frac{1}{2} \left[ \lim_{s_1 \to \mu_1 a} \lim_{s_2 \to \mu_2 b} \left\{ - D(a) L(s_1) L(s_2) D(b) \right\} + \lim_{s_1 \to \mu_1 a} \lim_{s_2 \to \mu_2 b} \left\{ - D(b) D(a) L(s_1) L(s_2) \right\} \right] p(s) \, ds. \quad (39)
\]

Now, before proceeding further we have to make a choice. Either we ignore the fact that the spin angular momentum is conserved during the free evaluation of the spins (cf. Fig. 1), which amounts to ignoring the conditions \( (9) \) and \( (10) \), or set \( L(s_1) L(s_2) = -1 \) respecting the conservation law. We first proceed by ignoring this condition, which leads to

\[
\mathcal{E}(a, b) = \int_{S^2} \frac{1}{2} \left[ \lim_{s_1 \to \mu_1 a} \lim_{s_2 \to \mu_2 b} \left\{ - D(a) L(s_1) L(s_2) D(b) \right\} + \lim_{s_1 \to \mu_1 a} \lim_{s_2 \to \mu_2 b} \left\{ - D(b) D(a) L(s_1) L(s_2) \right\} \right] p(s) \, ds \quad (40)
\]

\[
= \int_{S^2} \frac{1}{2} \left\{ - \mu_1 \mu_2 \left( \mathcal{D}(a) \mathcal{L}(a) \mathcal{L}(b) \mathcal{D}(b) \right) + \left\{ - \mu_1 \mu_2 \left( \mathcal{D}(b) \mathcal{D}(a) \mathcal{L}(a) \mathcal{L}(b) \right) \right\} \right\} p(s) \, ds \quad (41)
\]

\[
= \int_{S^2} \frac{1}{2} \left\{ - \mu_1 \mu_2 I_3 a I_3 a I_3 b I_3 b \right\} + \left\{ - \mu_1 \mu_2 I_3 b I_3 a I_3 a I_3 b \right\} \left( I_3 a \right)^2 \mathcal{L}(b)^2 \, p(s) ds \quad (42)
\]

\[
= \int_{S^2} \frac{1}{2} \left\{ - \mu_1 \mu_2 \left( I_3 a \right)^2 \mathcal{L}(b)^2 \right\} + \left\{ - \mu_1 \mu_2 \left( I_3 b \right)^2 \mathcal{L}(a)^2 \mathcal{L}(b)^2 \right\} \, p(s) ds \quad (43)
\]

\[
= \int_{S^2} \frac{1}{2} \left\{ - \mu_1 \mu_2 \left( I_3 a \right)^2 \mathcal{L}(b)^2 \right\} + \left\{ - \mu_1 \mu_2 \left( I_3 b \right)^2 \mathcal{L}(a)^2 \mathcal{L}(b)^2 \right\} \, p(s) ds \quad (44)
\]

\[
= \int_{S^2} \frac{1}{2} \left\{ - \mu_1 \mu_2 \right\} \, p(s) ds \quad (45)
\]

\[
= \int_{S^2} \left\{ - \mu_1 \mu_2 \right\} \, p(s) ds \quad (46)
\]

\[
= \int_{S^2} \left\{ - \text{sign}(a \cdot s_1^3) \text{sign}(b \cdot s_2^3) \right\} \, p(s) ds, \quad (47)
\]

where in the step from \( (44) \) to \( (45) \) we have used \( \left( I_3 a \right)^2 = \left( I_3 b \right)^2 = -1 \), and in the last step the expressions for \( \mu_1 \) and \( \mu_2 \) are substituted from \( (11) \). Needless to say, we could have inferred the last two steps immediately from \( (26) \), but have preferred to carry out longer calculation to demonstrate that our derivation is consistent and reproduces a known
In Fig. 2 these weak correlations are depicted by saw-tooth shaped dashed straight lines. By contrast, the solid cosine curve in Fig. 2 depicts the strong singlet correlations predicted by our 3-sphere model as well as quantum mechanics.

Bell and Peres claim that it leads to the following weak correlations that are incapable of violating Bell inequalities:

\[
E(a, b) = \int_{S^2} \left\{- \text{sign}(a \cdot s_1^i) \text{sign}(b \cdot s_2^i) \right\} p(s) \, ds = \begin{cases} 
-1 + \frac{2}{\pi} \eta_{ab} & \text{if } 0 \leq \eta_{ab} \leq \pi, \\
+3 - \frac{2}{\pi} \eta_{ab} & \text{if } \pi \leq \eta_{ab} \leq 2\pi.
\end{cases}
\] (48)

In Fig. 2 these weak correlations are depicted by saw-tooth shaped dashed straight lines. By contrast, the solid cosine curve in Fig. 2 depicts the strong singlet correlations predicted by our 3-sphere model as well as quantum mechanics. They follow if the spin angular momentum is assumed to be conserved during the free evolution of the constituent spins by setting \(L(s_1) L(s_2) = -1\) in the above derivation of the correlations, as we now proceed to demonstrate:

\[
E(a, b) = \int_{S^2} \frac{1}{2} \left[ A(a, s_1) B(b, s_2) + B(b, s_2) A(a, s_1) \right] p(s) \, ds
\] (49)

\[
= \int_{S^2} \frac{1}{2} \left[ \lim_{s_2 \to \mu_2 b} \left\{ -D(a) L(s_1) L(s_2) D(b) \right\} + \lim_{s_2 \to \mu_2 b} \left\{ -D(b) D(a) L(s_1) L(s_2) \right\} \right] p(s) \, ds
\] (50)

\[
= \int_{S^2} \frac{1}{2} \left[ \lim_{s_1 \to \mu_1 a} \left\{ D(a) D(b) \right\} + \lim_{s_1 \to \mu_1 a} \left\{ D(b) D(a) \right\} \right] p(s) \, ds
\] (51)

\[
= \int_{S^2} \frac{1}{2} \left[ \left\{ I_3 a I_3 b \right\} + \left\{ I_3 b I_3 a \right\} \right] p(s) \, ds
\] (52)

\[
= \frac{1}{2} \left[ \left\{ (I_3)^2 a b \right\} + \left\{ (I_3)^2 b a \right\} \right] \int_{S^2} p(s) \, ds
\] (53)

\[
= \frac{1}{2} \left[ \left\{ -a \cdot b - a \wedge b \right\} + \left\{ -b \cdot a - b \wedge a \right\} \right]
\] (54)

\[
= -a \cdot b - \frac{1}{2} \left\{ a \wedge b + b \wedge a \right\}
\] (55)

\[
= -\cos(\eta_{ab}) - I_3 \left[ \frac{1}{2} \left\{ (a \times b) + (b \times a) \right\} \right]
\] (56)

\[
= -\cos(\eta_{ab}) - 0 \quad [\text{the additive identity } 0 \text{ is the same across all grades in } Cl_{3,0}],
\] (57)

where the step (51) follows from (50) by setting \(L(s_1) L(s_2) = -1\) in (50), as required by the conservation of zero spin angular momentum. Comparing the integral in (49) with the result (57) in light of the equality (25), then leads to

\[
E(a, b) = \int_{S^2} \left\{- \text{sign}(a \cdot s_1^i) \text{sign}(b \cdot s_2^i) \right\} p(s) \, ds = -\cos(\eta_{ab}).
\] (58)
IV. COMPARISON OF THE 3-SPHERE MODEL WITH BELL’S LOCAL MODEL

In the previous section we derived two contradictory results in equations (48) and (58). Remarkably, the left-hand sides of these two equations are mathematically identical. Their right-hand sides, however, differ dramatically, and this difference is manifest from their graphs shown in Fig. 2. We then have

\[ \pi \]

To calculate the joint probabilities of observing the two results defined in (59) following Bell [4] and Peres [13], consider the spinorial sign changes have been taken into account automatically as parts of the geometrical features of \( S^3 \). But it is instructive to bring out in which step in their derivation of the correlations (48) Bell [4] and Peres [13] have ended up neglecting the spinorial sign changes. For this purpose, it is instructive to reflect on the figure 6.5 on page 161 of Peres [13], which has been used for calculating the probabilities of various measurement results jointly observed by Alice and Bob. We have reproduced that figure here for convenience as Fig. 3, with appropriate changes in notation.

Now in Bell’s local model the measurement functions of Alice and Bob are the following sign functions, as in (11):

\[ \mathcal{A}(a, s_1) = +\mu_1 = \text{sign}(a \cdot s_1^1) = \pm 1 \quad \text{and} \quad \mathcal{B}(b, s_2) = -\mu_2 = -\text{sign}(s_2^1 \cdot b) = \mp 1. \] (59)

Thus, as we saw in Section III if the initial direction \( s_1^1 \) of the two spins is uncontrollable but describable by isotropic probability distribution \( p(s) \), then the probability that the spin of particle 1 observed by Alice will be detected parallel to the vector \( a \) (regardless of whether particle 2 is detected) is unambiguously predicted by Bell’s local model to be

\[ P_1^+(a) = P_1^-(a) = \frac{1}{2}. \] (60)

And, likewise, the probability that the spin of particle 2 observed by Bob will be detected parallel to \( b \) is given by

\[ P_2^+(b) = P_2^-(b) = \frac{1}{2}. \] (61)

To calculate the joint probabilities of observing the two results defined in (59) following Bell [4] and Peres [13], consider a unit \( S^2 \) embedded in \( \mathbb{R}^3 \), cut through by the equatorial planes perpendicular to the detector directions \( a \) and \( b \), as shown in Fig. 3. We then have \( \mathcal{A}(a, s_1) = +1 \) if \( s_1 \) points through one of the hemispheres, and \( \mathcal{A}(a, s_1) = -1 \) if it points through the other hemisphere. Likewise, a second equatorial plane perpendicular to \( b \) determines the regions where \( \mathcal{B}(b, s_2) = \pm 1 \). The unit \( S^2 \) is thereby divided by the two equatorial planes into four sectors, with alternating signs for the product \( \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) \). The adjacent sectors have their surface areas proportional to \( \eta_{ab} \) and \( \pi - \eta_{ab} \) for \( 0 \leq \eta_{ab} \leq \pi \), and proportional to \( 2\pi - \eta_{ab} \) and \( \pi - (2\pi - \eta_{ab}) = \eta_{ab} - \pi \) for \( \pi \leq \eta_{ab} \leq 2\pi \), with the area of each hemisphere being \( 2\pi \). As a result, the probabilities of observing the results jointly and simultaneously work out to be

\[ P_{12}\{\mathcal{A} = +1, \mathcal{B} = +1 | \eta_{ab}\} = P_{12}^{++}(\eta_{ab}) = \begin{cases} \frac{\eta_{ab}}{2\pi} & \text{if } 0 \leq \eta_{ab} \leq \pi \\ \frac{\pi - \eta_{ab}}{2\pi} & \text{if } \pi \leq \eta_{ab} \leq 2\pi \end{cases} = P_{12}^{--}(\eta_{ab}) \] (62)

and

\[ P_{12}\{\mathcal{A} = +1, \mathcal{B} = -1 | \eta_{ab}\} = P_{12}^{+-}(\eta_{ab}) = \begin{cases} \frac{2\pi - \eta_{ab}}{2\pi} & \text{if } 0 \leq \eta_{ab} \leq \pi \\ \frac{\eta_{ab} - \pi}{2\pi} & \text{if } \pi \leq \eta_{ab} \leq 2\pi \end{cases} = P_{12}^{-+}(\eta_{ab}), \] (63)

where \( P_{12}^{+-}(\eta_{ab}), \) etc., are probabilities of observing the result +1 by Alice and −1 by Bob, etc., and the subscripts 1 and 2 label the two remote observation stations of Alice and Bob as in Fig. 1. The remaining probabilities all vanish:

\[ P_{12}^{++}(\eta_{ab}) = P_{12}^{--}(\eta_{ab}) = P_{12}^{+-}(\eta_{ab}) = P_{12}^{-+}(\eta_{ab}) = P_{12}^{00}(\eta_{ab}) = 0. \] (64)
where the superscript 0 stands for no detection. The correlations predicted by Bell’s local model thus seem to be

\[
\mathcal{E}(a, b) = \int \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) \, p(s) \, ds \approx \lim_{n \to +1} \left[ \frac{1}{n} \sum_{k=1}^{n} \mathcal{A}(a, s_k) \mathcal{B}(b, s_k) \right]
\]

\[
= \left\{ \mathcal{A}(a, s_1) \mathcal{B}(b, s_2) \right\} = 1
\]

\[
= \left\{ \left\{ (++) \times \left( \frac{\eta_{ab}}{2\pi} \right) \right\} + \left\{ (--) \times \left( \frac{\eta_{ab}}{2\pi} \right) \right\} + \left\{ (+-) \times \left( \frac{\pi - \eta_{ab}}{2\pi} \right) \right\} + \left\{ (-+) \times \left( \frac{\pi - \eta_{ab}}{2\pi} \right) \right\} \right\}
\]

\[
= \left\{ (-1 + \frac{2}{\pi} \eta_{ab}) \text{ if } 0 \leq \eta_{ab} \leq \pi \right\}
\]

\[
+ \left\{ 3 - \frac{2}{\pi} \eta_{ab} \text{ if } \pi \leq \eta_{ab} \leq 2\pi \right\}
\]

where \( n \) is the total number of experiments performed, \( k \) is the trial number, and \( p(s) = \frac{1}{n} \) is the probability density.

As noted, these weak correlations are depicted by dashed straight lines in Fig. 2. In our derivation (48), within \( S^3 \), they follow if the conservation of spin angular momentum encoded in the condition \( L(s_1) L(s_2) = -1 \) is neglected, which amounts to neglecting the spinorial sign changes intrinsic to the quaternions that constitute the 3-sphere. On the other hand, if the spinorial sign changes are taken into account in the derivation by respecting the condition \( L(s_1) L(s_2) = -1 \), then the resulting correlations are \( \mathcal{E}(a, b) = -\cos(\eta_{ab}) \), as we derived in (65). They are depicted by the sinusoidal curve in Fig. 2. We can appreciate this in more detail as follows. Using the mathematical identity

\[
-\cos(\eta_{ab}) = \frac{1}{2} \sin^2\left(\frac{\eta_{ab}}{2}\right) + \frac{1}{2} \sin^2\left(\frac{\eta_{ab}}{2}\right) - \frac{1}{2} \cos^2\left(\frac{\eta_{ab}}{2}\right) - \frac{1}{2} \cos^2\left(\frac{\eta_{ab}}{2}\right)
\]
the correlations we derived in [53] can be expressed as

\[
\mathcal{E}(\mathbf{a}, \mathbf{b}) = -\cos(\eta_{ab}) \\
= \frac{[(++) \times \left( \frac{1}{2} \sin^2 \left( \frac{\eta_{ab}}{2} \right) \right)] + [(-- \times \left( \frac{1}{2} \sin^2 \left( \frac{\eta_{ab}}{2} \right) \right)] + [(++) \times \left( \frac{1}{2} \cos^2 \left( \frac{\eta_{ab}}{2} \right) \right)] + [(-- \times \left( \frac{1}{2} \cos^2 \left( \frac{\eta_{ab}}{2} \right) \right)]}{[\frac{1}{2} \sin^2 \left( \frac{\eta_{ab}}{2} \right) + \frac{1}{2} \sin^2 \left( \frac{\eta_{ab}}{2} \right) + \frac{1}{2} \cos^2 \left( \frac{\eta_{ab}}{2} \right) + \frac{1}{2} \cos^2 \left( \frac{\eta_{ab}}{2} \right)]}.
\]

(70)

Comparing this expression with [67] we thus see that, for \( 0 \leq \eta_{ab} \leq \pi \), the probability of obtaining the joint result \( \mathcal{A} = ++ \) is not \( \frac{\eta_{ab}}{2\pi} \) but \( \frac{1}{2} \sin^2 \left( \frac{\eta_{ab}}{2} \right) \) if the spinorial sign changes are taken into account. In other words, occasionally what may be deemed to be a result \( \mathcal{A} = ++ \) according to the reasoning in Bell [4] and Peres [13], would in fact be either \( \mathcal{A} = + - \) or \( \mathcal{A} = - + \), thereby changing the number of times the result \( \mathcal{A} = ++ \) occurs from \( \frac{\eta_{ab}}{2\pi} \)-many times to \( \frac{1}{2} \sin^2 \left( \frac{\eta_{ab}}{2} \right) \)-many times. We can appreciate this by considering the example of the result \( \mathcal{A} \) defined in [13]:

\[
S^3 \ni \mathcal{A}(\mathbf{a}, \mathbf{s}_1) = \lim_{\mathbf{s}_1 \to \mu_{1} \mathbf{a}} \{ -\mathbf{D}(\mathbf{a}) \mathbf{L}(\mathbf{s}_1) \} = \lim_{\mathbf{s}_1 \to \mu_{1} \mathbf{a}} \{ + \mathbf{q}(\eta_{a s_1}, \mathbf{r}_1) \}.
\]

(72)

Here the spin \( \mathbf{L}(\mathbf{s}_1) \) is a bivector or pure quaternion, and corresponds to a binary rotation — i.e., rotation by \( \pi \) [13]. We can recognize this by considering the corresponding non-pure quaternion and noticing that \( \mathbf{q}(\theta = \pi, \mathbf{s}_1) = \mathbf{L}(\mathbf{s}_1) \):

\[
\mathbf{q}(\theta, \mathbf{s}_1) = \cos \left( \frac{\theta}{2} \right) + \mathbf{L}(\mathbf{s}_1) \sin \left( \frac{\theta}{2} \right).
\]

(73)

But the product of two rotations by \( \pi \) about the same axis \( \mathbf{s}_1 \) is a rotation by \( 2\pi \), and that is not an identity operation but subject to spinorial sign changes specified in [53]. Indeed, any product of two unit bivectors, such as \( \mathbf{L}(\mathbf{s}_1) \mathbf{L}(\mathbf{s}_2) \) for \( \mathbf{s}_1 = \mathbf{s}_2 \), is equal to \(-1\), which is precisely our condition (9) for the conservation of angular momentum. Thus the condition \( \mathbf{L}(\mathbf{s}_1) \mathbf{L}(\mathbf{s}_2) = -1 \) enforces sign changes in the results (72) because they depend on \( \mathbf{L}(\mathbf{s}_1) \), thereby inducing changes also in the products of results such as \( \mathcal{A} = ++ \rightarrow + - \), etc. Consequently, the probability of occurring the joint result \( \mathcal{A} = ++ \) changes from \( \frac{\eta_{ab}}{2\pi} \) to \( \frac{1}{2} \sin^2 \left( \frac{\eta_{ab}}{2} \right) \), as we noted by comparing the expressions (72) and (71).

Now a sign change in a bivector or pure quaternion, such as \( \mathbf{L}(\mathbf{s}_1) \), means a change in its sense of rotation. In other words, a counterclockwise rotation, say \(+ \mathbf{L}(\mathbf{s}_1) \), changes to a clockwise rotation, \(- \mathbf{L}(\mathbf{s}_1) \). This can be expressed as

\[
+ \mathbf{L}(\mathbf{s}_1) = +I_3 \mathbf{s}_1 \rightarrow - \mathbf{L}(\mathbf{s}_1) = -I_3 \mathbf{s}_1 = I_3(- \mathbf{s}_1).
\]

(74)

Thus a spinorial sign change in the bivector \(+ \mathbf{L}(\mathbf{s}_1) \) can be expressed equivalently as a sign change in the direction of its axis vector \( \mathbf{s}_1 \). Indeed, a counterclockwise rotation about \( + \mathbf{s}_1 \) represented by \(+ \mathbf{L}(\mathbf{s}_1) \) is the same as the clockwise rotation about \(- \mathbf{s}_1 \). But that implies that sign changes in the local result such as (72) will induce changes such as

\[
+ \text{sign}(\mathbf{a} \cdot \mathbf{s}_1) \rightarrow - \text{sign}(\mathbf{a} \cdot \mathbf{s}_1)
\]

(75)

in the integrand on the left-hand side of (158), so that some of its values will change from ++ to +−, etc., under the spinorial sign changes seen in [53], with precisely which changes would occur dictated by the condition \( \mathbf{L}(\mathbf{s}_1) \mathbf{L}(\mathbf{s}_2) = -1 \) for the conservation of spin angular momentum we imposed in the step (71) of our derivation of the correlations (58).

V. PROPOSED MACROSCOPIC TEST OF THE 3-SPHERE HYPOTHESIS

In [57] we have proposed a macroscopic experiment that may be able to detect the signatures of the above spinorial sign changes under \( 2\pi \) rotations in the form of strong singlet correlations derived in the previous sections. If realized, the experiment would determine whether Bell inequalities are violated for the manifestly local-realistic 3-sphere model we considered above. Needless to say, the proposed experiment has the potential to transform our understanding of the relationship between classical and quantum physics. It is based on a macroscopic variant of the local model considered by Bell [4] and Peres [13], we discussed above, but differs from it in one important respect. It involves measurements of the actual spin angular momenta of two fragments of an exploding bomb rather than their normalized spin values \pm 1. The latter are to be computed only after all runs of the experiment are completed, which can be executed either in the outer space or in a terrestrial laboratory. In the latter case the effects of gravity and air resistance would complicate matters, but it may be possible to choose experimental parameters judiciously enough to compensate for such effects.

With this assumption, consider a “bomb” made out of a hollow toy ball of diameter, say, three centimeters. The thin hemispherical shells of uniform density that make up the ball are snapped together at their rims in such a manner
that a slight increase in temperature would pop the ball open into its two constituents with considerable force [7]. A small lump of density much greater than the density of the ball is attached on the inner surface of each shell at a random location, so that, when the ball pops open, not only would the two shells propagate with equal and opposite linear momenta orthogonal to their common plane, but would also rotate with equal and opposite spin momenta about a random axis in space, as shown in Fig. 4. The volume of the attached lumps can be as small as a cubic millimeter, whereas their mass can be comparable to the mass of the ball. This will facilitate some $10^6$ possible spin directions for the two shells, whose outer surfaces can be decorated with colors to make their rotations easily detectable [5].

Now consider a large ensemble of such balls, identical in every respect except for the relative locations of the two lumps (affixed randomly on the inner surface of each shell). The balls are then placed over a heater—one at a time—at the center of the experimental setup [13], with the common plane of their shells held perpendicular to the horizontal direction of the setup. Although initially at rest, a slight increase in temperature of each ball will eventually eject its two constituent balls after the explosion, reducing their precession and nutation effects considerably. Consequently, we assume that during the narrow time window of the detection process the contributions of precession and nutation

\[ \mathcal{E}(a, b) = \int_{S^2} \left\{ -\text{sign}(a \cdot s_1^k) \text{sign}(b \cdot s_2^k) \right\} p(s) \, ds \approx \lim_{n \to \infty} \left[ \frac{1}{n} \sum_{k=1}^{n} \left\{ -\text{sign}(a \cdot s_1^k) \text{sign}(b \cdot s_2^k) \right\} \right], \quad (76) \]

together with

\[ \mathcal{E}(a) = \lim_{n \to \infty} \left[ \frac{1}{n} \sum_{k=1}^{n} \{\text{sign}(+s_1^k \cdot a)\} \right] = 0 \quad \text{and} \quad \mathcal{E}(b) = \lim_{n \to \infty} \left[ \frac{1}{n} \sum_{k=1}^{n} \{\text{sign}(-s_2^k \cdot b)\} \right] = 0, \quad (77) \]

where $n$ is the total number of experiments performed and $k$ specifies a trial number. As we discussed in the previous sections, naïve computation of (76) by Bell [4] and Peres [13] gives the weak correlations [13], whereas the quaternionic 3-sphere model predicts strong correlations [13] by taking the spinorial properties of the 3-sphere [11] into account.

Let us now turn to the practical problem of determining the direction of rotation of a bomb fragment. In order to minimize the contributions of precession and nutation about the rotation axis of the fragment, the bomb may be composed of two flexible squashy balls instead of a single ball. The two balls can then be squeezed together at the start of a run and released as if they were two parts of the same bomb. This will retain the spherical symmetry of the two constituent balls after the explosion, reducing their precession and nutation effects considerably. Consequently, we assume that during the narrow time window of the detection process the contributions of precession and nutation

\[ \mathcal{E}(a) = \lim_{n \to \infty} \left[ \frac{1}{n} \sum_{k=1}^{n} \{\text{sign}(+s_1^k \cdot a)\} \right] = 0 \quad \text{and} \quad \mathcal{E}(b) = \lim_{n \to \infty} \left[ \frac{1}{n} \sum_{k=1}^{n} \{\text{sign}(-s_2^k \cdot b)\} \right] = 0, \quad (77) \]

where $n$ is the total number of experiments performed and $k$ specifies a trial number. As we discussed in the previous sections, naïve computation of (76) by Bell [4] and Peres [13] gives the weak correlations [13], whereas the quaternionic 3-sphere model predicts strong correlations [13] by taking the spinorial properties of the 3-sphere [11] into account.
are negligible. In other words, during this narrow time window the individual spins will remain confined to the plane perpendicular to the horizontal direction of the setup. This is because we will then have $\mathbf{s} = \mathbf{r} \times \mathbf{p}$, with $\mathbf{r}$ specifying the location of the massive lump in the constituent ball and $\mathbf{p}$ being the ball’s linear momentum. We can now exploit these physical constraints to determine the direction of rotation of a constituent ball unambiguously, as follows.

Since only the directions of rotation are relevant for computing the correlation function $I(I)$, it would be sufficient for our purposes to determine only the direction of the vector $\mathbf{s}$ at each end of the setup. This can be accomplished by arranging three (or more) successive laser screens perpendicular to the horizontal path of the constituent balls, say about half a centimeter apart, and a few judiciously situated cameras around them. To facilitate the detection of the rotation of a ball as it passes through the screens, the surface of the balls can be decorated with distinctive marks, such as dots of different sizes and colors. Then, when a ball passes through the screens, the entry points of a specific mark on the ball can be recorded by the system of cameras. Since the ball would be spinning while passing through the screens, the entry points on the screens would be located at different relative positions on the screens. The rotation axis of the ball can therefore be determined unambiguously by determining the plane spanned by the entry points and the right-hand rule. In other words, the rotation axis can be determined as the orthogonal direction to the plane spanned by the entry points, with the sense of rotation determined by the right-hand rule. This procedure of determining the direction of rotation can be followed through manually, or it can be automated with the help of a computer software. Finally, the horizontal distance from the center of the setup to the location of the middle of the screens can be taken as the distance of the rotation axis from the center of the setup. This distance would help in establishing the simultaneity of the spin measurements at the two ends of the setup.

Undoubtedly, there would be many sources of errors in a mechanical experiment such as this. But if it is performed carefully enough, then our discussion above strongly suggests that it will refute the prediction $\langle 48 \rangle$, which is based on an incorrect calculation by Bell $[4]$ and Peres $[13]$, and vindicate the prediction $\langle 58 \rangle$ derived within the 3-sphere model.

**Appendix A: Proofs of the Equalities (34) and (38)**

In this appendix we prove the equalities (34) and (38), which amounts to proving that the “product of limits equal to limits of product” rule holds. To that end, we begin with the left-hand side of (34):

$$\lim_{\mu_1 \to a} \left\{ - D(a) L(s_1) \right\} \lim_{\mu_2 \to b} \left\{ + L(s_2) D(b) \right\},$$

(34)

which commutes with all other elements of $\text{Cl}_{3,0}$ and squares to $-1$. Similarly, the right-hand side of (34) simplifies to

$$\lim_{\mu_1 \to a} \left\{ - D(a) L(s_1) \right\} \left\{ + L(s_2) D(b) \right\},$$

(35)

where we have used the fact that all vectors involved in the model are unit vectors and the fact that the pseudoscalar $I_3$ commutes with all other elements of $\text{Cl}_{3,0}$ and squares to $-1$. Similarly, the right-hand side of (34) simplifies to

$$\lim_{\mu_2 \to b} \left\{ - D(a) L(s_1) \right\} \left\{ + L(s_2) D(b) \right\}.$$

(36)

Since only the directions of rotation are relevant for computing the correlation function $I(I)$, it would be sufficient for our purposes to determine only the direction of the vector $\mathbf{s}$ at each end of the setup. This can be accomplished by arranging three (or more) successive laser screens perpendicular to the horizontal path of the constituent balls, say about half a centimeter apart, and a few judiciously situated cameras around them. To facilitate the detection of the rotation of a ball as it passes through the screens, the surface of the balls can be decorated with distinctive marks, such as dots of different sizes and colors. Then, when a ball passes through the screens, the entry points of a specific mark on the ball can be recorded by the system of cameras. Since the ball would be spinning while passing through the screens, the entry points on the screens would be located at different relative positions on the screens. The rotation axis of the ball can therefore be determined unambiguously by determining the plane spanned by the entry points and the right-hand rule. In other words, the rotation axis can be determined as the orthogonal direction to the plane spanned by the entry points, with the sense of rotation determined by the right-hand rule. This procedure of determining the direction of rotation can be followed through manually, or it can be automated with the help of a computer software. Finally, the horizontal distance from the center of the setup to the location of the middle of the screens can be taken as the distance of the rotation axis from the center of the setup. This distance would help in establishing the simultaneity of the spin measurements at the two ends of the setup.

Undoubtedly, there would be many sources of errors in a mechanical experiment such as this. But if it is performed carefully enough, then our discussion above strongly suggests that it will refute the prediction $\langle 48 \rangle$, which is based on an incorrect calculation by Bell $[4]$ and Peres $[13]$, and vindicate the prediction $\langle 58 \rangle$ derived within the 3-sphere model.
Analogously, we can also prove the equality (A17) by simplifying its left-hand side and right-hand side, as follows:

\[
\lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ + \mathbf{L}(s_2) \mathbf{D}(b) \} \left[ \lim_{s_1 \to \mu_1 a} \{ - \mathbf{D}(a) \mathbf{L}(s_1) \} \right] = \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ + \mathbf{I}_3 s_2 I_3 b \} \left[ \lim_{s_1 \to \mu_1 a} \{ - I_3 a I_3 s_1 \} \right]
\]  

(A12)

\[
= \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ - (I_3)^2 s_2 b \} \left[ \lim_{s_1 \to \mu_1 a} \{ - (I_3)^2 a s_1 \} \right]
\]  

(A13)

\[
= \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ - s_2 b \} \left[ \lim_{s_1 \to \mu_1 a} \{ + a s_1 \} \right]
\]  

(A14)

\[
= - \mu_2 b b [ + \mu_1 a a ]
\]  

(A15)

\[
= - \mu_1 \mu_2
\]  

(A16)

and

\[
\lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ - \mathbf{D}(b) \mathbf{D}(a) \} \left[ \mathbf{L}(s_1) \mathbf{L}(s_2) \right] = \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ - I_3 b I_3 a \} \left[ I_3 s_1 I_3 s_2 \right]
\]  

(A17)

\[
= \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ - (I_3)^2 b a \} \left[ (I_3)^2 s_1 s_2 \right]
\]  

(A18)

\[
= \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ + b a \} \left[ - s_1 s_2 \right]
\]  

(A19)

\[
= \lim_{s_2 \to \mu_2 b} \lim_{s_1 \to \mu_1 a} \{ - b a s_1 s_2 \}
\]  

(A20)

\[
= - \mu_1 \mu_2 b a a b
\]  

(A21)

\[
= - \mu_1 \mu_2 b b
\]  

(A22)

\[
= - \mu_1 \mu_2
\]  

(A23)

Since the right-hand sides of (A16) and (A17) are equal, “the product of limits equal to limits of product” rule holds.

**Note added to proof:** While in this paper we have concentrated exclusively on the singlet correlations involving spin angular momenta, in the Bell-test experiments such as [14] what is usually observed are correlations among the polarization states of entangled photons. But the 3-sphere model we have discussed above easily accommodates such photon polarization states with appropriate changes in notation, as we have demonstrated in [16] and Chapter 8 of [6].

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