TILINGS DEFINED BY AFFINE WEYL GROUPS

E. MEINRENKEN

Abstract. Let $W$ be a Weyl group, presented as a reflection group on a Euclidean vector space $V$, and $C \subset V$ an open Weyl chamber. In a recent paper, Waldspurger proved that the images $(\text{id} - w)(C)$ for $w \in W$ are all disjoint, with union the closed cone spanned by the positive roots. We prove that similarly, the images $(\text{id} - w)(A)$ of the open Weyl alcove $A$, for $w \in W^a$ in the affine Weyl group, are disjoint and their union is $V$.

1. Introduction

Let $W$ be the Weyl group of a simple Lie algebra, presented as a crystallographic reflection group in a finite-dimensional Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$. Choose a fundamental Weyl chamber $C \subset V$, and let $D$ be its dual cone, i.e. the open cone spanned by the corresponding positive roots. In his recent paper [2], Waldspurger proved the following remarkable result.

Theorem 1.1 (Waldspurger). The images $D_w := (\text{id} - w)(C)$, $w \in W$ are all disjoint, and their union is the closed cone spanned by the positive roots:

$$\overline{D} = \bigcup_{w \in W} D_w.$$  

For instance, the identity transformation $w = \text{id}$ corresponds to $D_{\text{id}} = \{0\}$ in this decomposition, while the reflection $s_\alpha$ defined by a positive root $\alpha$ corresponds to the open half-line $D_{s_\alpha} = \mathbb{R}_{>0} \cdot \alpha$.

The aim of this note is to prove a similar result for the affine Weyl group $W^a$. Recall that $W^a = \Lambda \rtimes W$ where the co-root lattice $\Lambda \subset V$ acts by translations. Let $A \subset C$ be the Weyl alcove, with $0 \in \overline{A}$.

Theorem 1.2. The images $V_w = (\text{id} - w)(A)$, $w \in W^a$ are all disjoint, and their union is $V$:

$$V = \bigcup_{w \in W^a} V_w.$$  

Figure 1 is a picture of the resulting tiling of $V$ for the root system $G_2$. Up to translation by elements of the lattice $\Lambda$, there are five 2-dimensional tiles, corresponding to the five Weyl group elements with trivial fixed point set. Letting $s_1, s_2$ denote the simple reflections, the lightly shaded polytopes are labeled by the Coxeter element $s_1 s_2$, $s_2 s_1$, the medium shaded polytopes by $(s_1 s_2)^2$, $(s_2 s_1)^2$, and the darkly shaded polytope by the longest Weyl group element $w_0 = (s_1 s_2)^3$.

One also has the following related statement.
Theorem 1.3. Suppose $S \in \text{End}(V)$ with $||S|| < 1$. Then the sets $V_w^{(S)} = (S - w)(A)$, $w \in W^a$ are all disjoint, and their closures cover $V$:

$$V = \bigcup_{w \in W^a} V_w^{(S)}.$$ 

Note that for $S = 0$ the resulting decomposition of $V$ is just the Stiefel diagram, while for $S = \tau \text{id}$ with $\tau \rightarrow 1$ one recovers the decomposition from Theorem 1.2.

The proof of Theorem 1.2 is in large parts parallel to Waldspurger’s [2] proof of Theorem 1.1. We will nevertheless give full details in order to make the paper self-contained.

Acknowledgments: I would like to thank Bert Kostant for telling me about Waldspurger’s result, and the referee for helpful comments. I also acknowledge support from an NSERC Discovery Grant and a Steacie Fellowship.

2. Notation

With no loss of generality we will take $W$ to be irreducible. Let $\mathfrak{R} \subset V$ be the set of roots, $\{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{R}$ a set of simple roots, and

$$C = \{x | \langle \alpha_i, x \rangle > 0, \ i = 1, \ldots, l\}$$

the corresponding Weyl chamber. We denote by $\alpha_{\text{max}} \in \mathfrak{R}$ the highest root, and $\alpha_0 = -\alpha_{\text{max}}$ the lowest root. The open Weyl alcove is the $l$-dimensional simplex defined as

$$A = \{x | \langle \alpha_i, x \rangle + \delta_{i,0} > 0, \ i = 0, \ldots, l\}.$$ 

Its faces are indexed by the proper subsets $I \subset \{0, \ldots, l\}$, where $A_I$ is given by inequalities $\langle \alpha_i, x \rangle + \delta_{i,0} > 0$ for $i \not\in I$ and equalities $\langle \alpha_i, x \rangle + \delta_{i,0} = 0$ for $i \in I$. Each $A_I$ has codimension
$|I|$. In particular, $A_i = A_{\{i\}}$ are the codimension 1 faces, with $\alpha_i$ as inward-pointing normal vectors. Let $s_i$ be the affine reflections across the affine hyperplanes supporting $A_i$,

$$s_i: x \mapsto x - (\langle \alpha_i, x \rangle + \delta_i,0)\alpha_i^\vee, \quad i = 0, \ldots, l,$$

where $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i, \alpha_i \rangle$ is the simple co-root corresponding to $\alpha_i$. The Weyl group $W$ is generated by the reflections $s_1, \ldots, s_l$, while the affine Weyl group $W^a$ is generated by the affine reflections $s_0, \ldots, s_l$. The affine Weyl group is a semi-direct product

$$W^a = \Lambda \rtimes W$$

where the co-root lattice $\Lambda = \mathbb{Z}[\alpha_1^\vee, \ldots, \alpha_l^\vee] \subset V$ acts on $V$ by translations. For any $w \in W^a$, we will denote by $\tilde{w} \in W$ its image under the quotient map $W^a \to W$, i.e. $\tilde{w}(x) = w(x) - w(0)$, and by $\lambda_w = w(0) \in \Lambda$ the corresponding lattice vector.

The stabilizer of any given element of $A_I$ is the subgroup $W_I$ generated by $s_i$, $i \in I$. It is a finite subgroup of $W^a$, and the map $w \mapsto \tilde{w}$ induces an isomorphism onto the subgroup $W_I$ generated by $\tilde{s}_i$, $i \in I$. Recall that $W_I$ is itself a Weyl group (not necessarily irreducible): its Dynkin diagram is obtained from the extended Dynkin diagram of the root system $\Phi$ by removing all vertices that are in $I$.

3. **The top-dimensional polytopes**

For any $w \in W^a$, the subset

$$V_w = (\text{id} - w)(A)$$

is the relative interior of a convex polytope in the affine subspace $\text{ran}(\text{id} - w)$. Let

$$W^a_{\text{reg}} = \{ w \in W^a | \text{id} - w \text{ is invertible} \}$$

and $W_{\text{reg}} = W \cap W^a_{\text{reg}}$, so that $w \in W_{\text{reg}} \iff \tilde{w} \in W_{\text{reg}}$. The top dimensional polytopes $V_w$ are those indexed by $w \in W_{\text{reg}}$, and the faces of these polytopes are $V_{w,I} := (\text{id} - w)(A_I)$. For $w \in W_{\text{reg}}$ and $i = 0, \ldots, l$ let

$$n_{w,i} := (\text{id} - \tilde{w}^{-1})^{-1}(\alpha_i).$$

**Lemma 3.1.** For all $w \in W^a_{\text{reg}}$, the open polytope $V_w$ is given by the inequalities

$$\langle n_{w,i}, \xi + \lambda_w \rangle + \delta_i,0 > 0$$

for $i = 0, \ldots, l$. The face $V_{w,I} = (\text{id} - w)(A_I)$ is obtained by replacing the inequalities for $i \in I$ by equalities.

**Proof.** For any $\xi = (\text{id} - w)x \in V$, we have

$$\langle \alpha_i, x \rangle = \langle (\text{id} - \tilde{w}^{-1})^{-1}\alpha_i, (\text{id} - \tilde{w})x \rangle = \langle n_{w,i}, (\text{id} - \tilde{w})x \rangle = \langle n_{w,i}, \xi + \lambda_w \rangle,$$

since $\tilde{w}^{-1}$ is the transpose of $\tilde{w}$ under the inner product $\langle \cdot, \cdot \rangle$. This gives the description of $V_w$ and of its faces $V_{w,I}$. \qed

**Lemma 3.2.** Suppose $w \in W^a_{\text{reg}}$, $i \in \{0, \ldots, l\}$. Then

$$V_{w,i} = V_{\sigma,i} \subset \text{ran}(\text{id} - \sigma)$$

with $\sigma = ws_i$. In particular, $\sigma$ is an affine reflection, and $n_{w,i}$ is a normal vector to the affine hyperplane $\text{ran}(\text{id} - \sigma)$. One has $\langle n_{w,i}, \alpha_i^\vee \rangle = 1$. 

Proof. For any orthogonal transformation \( g \in O(V) \) and any reflection \( s \in O(V) \), the dimension of the fixed point set of the orthogonal transformations \( g, gs \) differs by \( \pm 1 \). Since \( \tilde{w} \) fixes only the origin, it follows that \( \tilde{\sigma} \) has a 1-dimensional fixed point set. Hence \( \text{ran}(\text{id} - \sigma) \) is an affine hyperplane, and \( \sigma \) is the affine reflection across that hyperplane. Since \( s_i \) fixes \( A_i \), we have \( V_{w,i} = (\text{id} - w)(A_i) = (\text{id} - ws_i)(A_i) = V_{\sigma,i} \cap \text{ran}(\text{id} - \sigma) \). By definition \( n_{w,i} = \tilde{w}^{-1}n_{w,i} = \alpha_i \). Hence

\[-2\langle n_{w,i}, \alpha_i \rangle = \left| n_{w,i} - \alpha_i \right|^2 - \left| n_{w,i} \right|^2 = \left| \tilde{w}^{-1}n_{w,i} \right|^2 - \left| n_{w,i} \right|^2 = 0. \]

The following Proposition indicates how the top-dimensional polytopes \( V_{w,i} \) are glued along the polytopes of codimension 1.

**Proposition 3.3.** Let \( \sigma \in W^a \) be an affine reflection, i.e. \( \text{ran}(\text{id} - \sigma) \) is an affine hyperplane. Consider

\[ \xi \in V_\sigma \setminus \bigcup_{|I| \geq 2} V_{\sigma,I}. \]

Then there are two distinct indices \( i, i' \in \{0, \ldots, l\} \) such that \( \xi \in V_{\sigma,i} \cap V_{\sigma,i'} \). Furthermore, \( w = \sigma s_i \) and \( w' = \sigma s_{i'} \) are both in \( W^a_{\text{reg}} \), so that \( V_{w,i} = V_{\sigma,i} \) and \( V_{w',i'} = V_{\sigma,i'} \), and the polytopes \( V_w, V_{w'} \) are on opposite sides of the affine hyperplane \( \text{ran}(\text{id} - \sigma) \).

Proof. Let \( n \) be a generator of the 1-dimensional subspace \( \ker(\text{id} - \tilde{\sigma}) \). Then \( n \) is a normal vector to \( \text{ran}(\text{id} - \sigma) \). The pre-image \( (\text{id} - \sigma)^{-1}(\xi) \subset V \) is an affine line in the direction of \( n \). Since \( \xi \in V_\sigma \), this line intersects \( A_i \), hence it intersects the boundary \( \partial A \) in exactly two points \( x, x' \). By \( \mathbb{1} \), \( x, x' \) are contained in two distinct codimension 1 boundary faces \( A_i, A_{i'} \). Since \( n \) is ‘inward-pointing’ at one of the boundary faces, and ‘outward-pointing’ at the other, the inner products \( \langle n, \alpha_i \rangle \), \( \langle n, \alpha_{i'} \rangle \) are both non-zero, with opposite signs. Let \( w = \sigma s_i \) and \( w' = \sigma s_{i'} \). We will show that \( w \in W^a_{\text{reg}} \), i.e. \( \tilde{w} \in W_{\text{reg}} \) (the proof for \( w' \) is similar). Let \( z \in V \) with \( \tilde{w}z = z \). Then \( \tilde{\sigma}^{-1}z = \tilde{s}_iz \), so

\[ (\text{id} - \tilde{\sigma}^{-1})(z) = (\text{id} - \tilde{s}_i)(z) = \langle \alpha_i, z \rangle \alpha_{i'}^\lor. \]

The left hand side lies in \( \text{ran}(\text{id} - \tilde{\sigma}) \), which is orthogonal to \( n \), while the right hand side is proportional to \( \alpha_i \). Since \( \langle n, \alpha_i \rangle \neq 0 \) this is only possible if both sides are 0. Thus \( z \) is fixed under \( \tilde{\sigma} \), and hence a multiple of \( n \). On the other hand we have \( \langle \alpha_i, z \rangle = 0 \), hence using again that \( \langle n, \alpha_i \rangle \neq 0 \) we obtain \( z = 0 \). This shows \( \ker(\text{id} - \tilde{w}) = 0 \).

As we had seen above, \( n_{w,i} \) is a normal vector to \( \text{ran}(\text{id} - \sigma) \), hence it is a multiple of \( n \). By Lemma 3.2 it is a positive multiple if and only if \( \langle n, \alpha_i \rangle > 0 \). But then \( \langle n, \alpha_{i'} \rangle < 0 \), and so \( n_{w',i'} \) is a negative multiple of \( n \). This shows that \( V_w, V_{w'} \) are on opposite sides of the hyperplane \( \text{ran}(\text{id} - \sigma) \). \( \square \)

Consider the union over \( W \subset W^a \),

\[ X := \bigcup_{w \in W} V_w. \]

Thus \( \bigcup_{w \in W^a} V_w = \bigcup_{\lambda \in A} (\lambda + X) \). The statement of Theorem 1.2 means in particular that \( X \) is a fundamental domain for the action of \( A \). Figures 2 and 3 give pictures of \( X \) for the root systems \( B_2 \) and \( G_2 \). The shaded regions are the top-dimensional polytopes (i.e. the sets \( V_w \) for \( \text{id} - w \) invertible), the dark lines are the 1-dimensional polytopes (corresponding to reflections), and the origin corresponds to \( w = \text{id} \).

\[ \begin{aligned} \text{Figure 2} & \quad \text{Figure 3} \end{aligned} \]
**Proposition 3.4.** (a) The sets $\lambda + \text{int}(X)$, $\lambda \in \Lambda$ are disjoint, and $\bigcup_{\lambda \in \Lambda} \lambda + X = V$. (b) The open polytopes $V_w$ for $w \in W_{\text{reg}}$ are disjoint, and $\bigcup_{w \in W_{\text{reg}}} V_w = V$.

**Proof.** Since the collection of closed polytopes $\overline{V}_w$, $w \in W_{\text{reg}}$ is locally finite, the union $\bigcup_{w \in W_{\text{reg}}} \overline{V}_w$ is a closed polyhedral subset of $V$. Proposition 3.3 shows that a point $\xi \in V_{w,i}$ cannot contribute to the boundary of this subset unless it lies in $\bigcup_{\sigma \in W^a} \bigcup_{|I| \geq 2} V_{\sigma,I}$. We therefore see that the boundary has codimension $\geq 2$, and hence is empty since $\bigcup_{w \in W_{\text{reg}}} \overline{V}_w$ is a closed polyhedron. This proves $\bigcup_{w \in W_{\text{reg}}} V_w = V$, and also $\bigcup_{\lambda \in \Lambda} (\lambda + X) = V$ with $X$ as defined in (2). Hence the volume $\text{vol}(X)$ (for the Riemannian measure on $V$ defined by the inner product) must be at least the volume of a fundamental domain for the action of $\Lambda$:

$$\text{vol}(X) \geq |W| \text{vol}(A).$$
On the other hand, \( \text{vol}(V_w) = \text{vol}((\text{id} - w)(A)) = \det(\text{id} - w) \text{vol}(A) \), so
\[
(4) \quad \text{vol}(X) \leq \sum_{w \in W} \text{vol}(V_w) = \text{vol}(A) \sum_{w \in W} \det(\text{id} - w) = |W| \text{vol}(A)
\]
where we used the identity \([1, \text{p.134}] \sum_{w \in W} \det(\text{id} - w) = |W| \). This confirms \( \text{vol}(X) = |W| \text{vol}(A) \). It follows that the sets \( \lambda + \text{int}(X) \) are pairwise disjoint, or else the inequality \((3)\) would be strict. Similarly that the sets \( V_w, w \in W_a \) are disjoint, or else the inequality \((4)\) would be strict. (Of course, this also follows from Waldspurger’s Theorem 1.1 since \( C_w \subset D_w \).)

Hence all \( V_w, w \in W_a \) are disjoint. \( \square \)

To proceed, we quote the following result from Waldspurger’s paper, where it is stated in greater generality \([2, \text{“Lemme”}]\).

**Proposition 3.5** (Waldspurger). Given \( w \in W \) and a proper subset \( I \subset \{0, \ldots, l\} \) there exists a unique \( q \in W_I \) such that
\[
\ker(\text{id} - wq) \cap \{ x \in V | \langle \alpha_i, x \rangle > 0 \text{ for all } i \in I \} \neq \emptyset.
\]
Following \([2]\) we use this to prove,

**Proposition 3.6.** Every element of \( V \) is contained in some \( V_w, w \in W^a \):
\[
(5) \quad \bigcup_{w \in W^a} V_w = V.
\]

**Proof.** Let \( \xi \in V \) be given. Pick \( w \in W^a_{\text{reg}} \) with \( \xi \in \overline{V}_w \), and let \( I \subset \{0, \ldots, l\} \) with \( \xi \in V_{w,I} \). Then \( x := (\text{id} - w)^{-1}(\xi) \in A_I \) is fixed under \( W^a_I \). Using Proposition 3.5 we may choose \( \tilde{q} \in W_I \) and \( n \in V \) such that
(a) \( \tilde{w}\tilde{q}(n) = n \),
(b) \( \langle \alpha_i, n \rangle > 0 \text{ for all } i \in I \)
Taking \(||n||\) sufficiently small we have \( x + n \in A \), and
\[
(\text{id} - wq)(x + n) = (\text{id} - wq)(x) + (\text{id} - \tilde{w}\tilde{q})n = (\text{id} - w)(x) = \xi.
\]
This shows \( \xi \in V_w \). \( \square \)

4. **Disjointness of the sets \( \lambda + X \)**

To finish the proof of Theorem 1.2 we have to show that the union \((5)\) is disjoint. Waldspurger’s Theorem 1.1 shows that all \( D_w = (\text{id} - w)(C), w \in W \) are disjoint. (We refer to his paper for a very simple proof of this fact.) Hence the same is true for \( V_w \subset D_w, w \in W \). It remains to show that the sets \( \lambda + X, \lambda \in \Lambda, \) with \( X \) given by \([2]\), are disjoint.

The following Lemma shows that the closure \( \overline{X} = \bigcup_{w \in W} \overline{V}_w \) only involves the top-dimensional polytopes.

**Lemma 4.1.** The closure of the set \( X \) is a union over \( W_{\text{reg}} \),
\[
\overline{X} = \bigcup_{w \in W_{\text{reg}}} \overline{V}_w.
\]
Furthermore, \( \text{int}(\overline{X}) = \text{int}(X) \).
Proof. We must show that for any $\xi \in \nabla_\sigma$, $\sigma \in W \setminus W_{\text{reg}}$, there exists $w \in W_{\text{reg}}$ such that $\xi \in \nabla_w$. Using induction, it is enough to find $\sigma' \in W$ such that $\xi \in \nabla_{\sigma'}$ and dim($\ker(\text{id} - \sigma')$) = dim($\ker(\text{id} - \sigma)$) - 1. Let $\pi : V \to \ker(\text{id} - \sigma) ^{\perp} = \text{ran}(\text{id} - \sigma)$ denote the orthogonal projection. Then $\text{id} - \sigma$ restricts to an invertible transformation of $\pi(V)$, and $\nabla_\sigma$ is the image of $\pi(\overline{A})$ under this transformation. We have
\[
\pi(\overline{A}) = \pi(\partial \overline{A}) = \bigcup_{i=0}^{l} \pi(\overline{A}_i),
\]
and this continues to hold if we remove the index $i = 0$ from the right hand side, as well as all indices $i$ for which $\dim(\pi(A_i)) < \dim(\pi(V))$. That is, for each point $x \in \pi(\overline{A})$ there exists an index $i \neq 0$ such that $x \in \pi(\overline{A}_i)$, with $\dim(\pi(A_i)) = \dim(\pi(V))$. Taking $x$ to be the pre-image of $\xi$ under $(\text{id} - \sigma)|_{\pi(V)}$, we have $\xi \in \nabla_{\sigma,i}$ with $i \neq 0$ and $\dim V_{\sigma,i} = \dim \text{ran}(\text{id} - \sigma)$. Let $\sigma' = \sigma s_i$. Then $V_{\sigma,i} = V_{\sigma',i}$, hence $\dim(\text{ran}(\text{id} - \sigma')) \geq \dim V_{\sigma,i} = \dim(\text{ran}(\text{id} - \sigma))$, which shows $\dim(\ker(\text{id} - \sigma')) \leq \dim(\ker(\text{id} - \sigma))$. By elementary properties of reflection groups, the dimensions of the fixed point sets of $\sigma, \sigma'$ differ by either $+1$ or $-1$. Hence $\dim(\ker(\text{id} - \sigma')) = \dim(\ker(\text{id} - \sigma)) - 1$, proving the first assertion of the Lemma.

It follows in particular that the closure of $\text{int}(X)$ equals that of $X$. Suppose $\xi \in \text{int}(\overline{X})$. By Proposition 3.6 there exists $\lambda \in \Lambda$ with $\xi \in \lambda + X$. It follows that $\text{int}(\overline{X})$ meets $\lambda + X$, and hence also meets $\lambda + \text{int}(\overline{X})$. Since the $\Lambda$-translates of $\text{int}(\overline{X})$ are pairwise disjoint (see Proposition 3.3), it follows that $\lambda = 0$, i.e. $\xi \in X$. This shows $\xi \in X \cap \text{int}(\overline{X}) = \text{int}(X)$, hence $\text{int}(\overline{X}) \subset \text{int}(X)$. The opposite inclusion is obvious. \qed

Since we already know that the sets $\lambda + \text{int}(X)$ are disjoint, we are interested in $X \setminus \text{int}(X) \subset \partial X = \overline{X} \setminus \text{int}(X)$. Let us call a closed codimension 1 boundary face of the polyhedron $\overline{X}$ ‘horizontal’ if its supporting hyperplane contains $V_{w,0}$ for some $w \in W_{\text{reg}}$, and ‘vertical’ if its supporting hyperplane contains $V_{w,i}$ for some $w \in W_{\text{reg}}$ and $i \neq 0$. These two cases are exclusive:

**Lemma 4.2.** Let $n$ be the inward-pointing normal vector to a codimension 1 face of $\overline{X}$. Then $\langle n, \alpha_{\text{max}} \rangle \neq 0$. In fact, $\langle n, \alpha_{\text{max}} \rangle < 0$ for the horizontal faces and $\langle n, \alpha_{\text{max}} \rangle > 0$ for the vertical faces.

**Proof.** Given a codimension 1 boundary face of $\overline{X}$, pick any point $\xi$ in that boundary face, not lying in $\bigcup_{w \in W} \bigcup_{|i| \geq 1} V_{w,i}$. Let $w \in W_{\text{reg}}$ and $i \in \{0, \ldots, l\}$ such that $\xi \in V_{w,i}$, and $n_{w,i}$ is an inward-pointing normal vector. By Proposition 3.3 there is a unique $i' \neq i$ such that $\xi \in V_{w',i'}$, where $w' = ws_is_{i'}$. Since $V_{w}, V_{w'}$ lie on opposite sides of the affine hyperplane spanned by $V_{w,i}$, and $\xi$ is a boundary point of $\overline{X}$, we have $w' \notin W$. Thus one of $i, i'$ must be zero. If $i = 0$ (so that the given boundary face is horizontal) we obtain $\langle n_{w,0}, \alpha_{\text{max}} \rangle = -\langle n_{w,0}, \alpha_0 \rangle < 0$. If $i = 0$ we similarly obtain $\langle n_{w',0}, \alpha_{\text{max}} \rangle < 0$, hence $\langle n_{w,i}, \alpha_{\text{max}} \rangle > 0$. \qed

**Lemma 4.3.** Let $\xi \in X \setminus \text{int}(X)$. Then there exists a vertical boundary face of $\overline{X}$ containing $\xi$. Equivalently, the complement $\partial X \setminus (X \setminus \text{int}(X))$ is contained in the union of horizontal boundary faces.

**Proof.** The alcove $A$ is invariant under multiplication by any scalar in $(0, 1)$. Hence, the same is true for the sets $V_w$ for $w \in W$, as well as for $X$ and $\text{int}(X)$. Hence, if $\xi \in X \setminus \text{int}(X)$ there exists $t_0 > 1$ such that $t \xi \in X \setminus \text{int}(X)$ for $1 \leq t < t_0$. The closed codimension 1 boundary face
containing this line segment is necessarily vertical, since a line through the origin intersects the affine hyperplane \( \{ x | \langle n_w, 0, x - \xi \rangle = 0 \} \) in at most one point.

**Proposition 4.4.** For any \( \xi \in X \), there exists \( \epsilon > 0 \) such that \( \xi + s \alpha_{\text{max}} \notin \text{int}(X) \) for \( 0 < s < \epsilon \).

**Proof.** If \( \xi \in \text{int}(X) \) there is nothing to show, hence suppose \( \xi \in X \setminus \text{int}(X) \). Suppose first that \( \xi \) is not in the union of horizontal boundary faces of \( X \). Then there exists an open neighborhood \( U \) of \( \xi \) such that \( U \cap X = U \cap X \). All boundary faces of \( X \) meeting \( \xi \) are vertical, and their inward-pointing normal vectors \( n \) all satisfy \( \langle n, \alpha_{\text{max}} \rangle > 0 \). Hence, \( \xi + s \alpha_{\text{max}} \in \text{int}(U \cap X) = \text{int}(X) \subset X \) for \( s > 0 \) sufficiently small.

For the general case, suppose that for all \( \epsilon > 0 \), there is \( s \in (0, \epsilon) \) with \( \xi + s \alpha_{\text{max}} \notin \text{int}(X) \). We will obtain a contradiction. Since \( \xi \) is contained in some vertical boundary face, one can choose \( t > 1 \) so that \( \xi' := t \xi \in X \setminus \text{int}(X) \), but \( \xi' \) is not in the closure of the union of horizontal boundary faces. Given \( \epsilon > 0 \), pick \( s \in (0, \epsilon) \) such that \( \xi + \frac{s}{t} \alpha_{\text{max}} \notin \text{int}(X) \). Since \( \text{int}(X) \) is invariant under multiplication by scalars in \( (0,1) \), the complement \( V \setminus \text{int}(X) \) is invariant under multiplication by scalars in \( (1, \infty) \), hence we obtain \( \xi' + s \alpha_{\text{max}} \notin \text{int}(X) \). This contradicts what we have shown above, and completes the proof.

**Proposition 4.5.** The sets \( \lambda + X \) for \( \lambda \in \Lambda \) are disjoint.

**Proof.** Suppose \( \xi \in (\lambda + X) \cap (\lambda' + X) \). By Proposition 4.4, we can choose \( s > 0 \) so that \( \xi + s \alpha_{\text{max}} \in (\lambda + \text{int}(X)) \cap (\lambda' + \text{int}(X)) \). Since the \( \Lambda \)-translates of \( \text{int}(X) \) are disjoint, it follows that \( \lambda = \lambda' \).

This completes the proof of Theorem 1.2. We conclude with some remarks on the properties of the decomposition \( V = \bigcup_{w \in W^a} V_w \).

**Remarks 4.6.** (a) The group of symmetries \( \tau \) of the extended Dynkin diagram (i.e. the outer automorphisms of the corresponding affine Lie algebra) acts by symmetries of the decomposition \( V = \bigcup_{w \in W^a} V_w \), as follows. Identify the nodes of the extended Dynkin diagram with the simple affine reflections \( s_0, \ldots, s_l \). Then \( \tau \) extends to a group automorphism of \( W^a \), taking \( s_i \) to \( \tau(s_i) \). This automorphism is implemented by a unique Euclidean transformation \( g: V \to V \) i.e. \( gwg^{-1} = \tau(w) \) for all \( w \in W^a \). Then \( g \) preserves \( A \), and consequently \( gV_w = g(\text{id} - w)(A) = (\text{id} - \tau(w))(A) = V_{\tau(w)}, \quad w \in W^a \).

(b) It is immediate from the definition that the Euclidean transformation \( -w: V \to V, \quad x \mapsto -wx \) takes \( V_{w^{-1}} \) into \( V_w \):

\[
-w(V_{w^{-1}}) = V_w.
\]

(c) For any positive root \( \alpha \), let \( s_\alpha \) be the corresponding reflection. Then \( (\text{id} - s_\alpha)(\xi) = \langle \alpha, \xi \rangle \alpha^\vee \), where \( \alpha^\vee \) is the co-root corresponding to \( \alpha \). Hence \( D_{s_\alpha} \) is the relative interior of the line segment from \( 0 \) to \( \lambda \alpha^\vee \), where \( \lambda \) is the maximum value of the linear functional \( \xi \mapsto \langle \alpha, \xi \rangle \) on the closed alcove \( \overline{A} \). This maximum is achieved at one of the vertices. Let \( \varpi_i^\vee, \ldots, \varpi_l^\vee \) be the fundamental co-weights, defined by \( \langle \alpha_i, \varpi_j^\vee \rangle = \delta_{ij} \) for \( i, j = 1, \ldots, l \).

Let \( c_i \in \mathbb{N} \) be the coefficients of \( \alpha_{\text{max}} \) relative to the simple roots: \( \alpha_{\text{max}} = \sum_{i=1}^l c_i \alpha_i \). Then the non-zero vertices of \( A \) are \( \varpi_i^\vee / c_i \). Similarly let \( a_i \in \mathbb{Z}_{\geq 0} \) be the coefficients of \( \alpha \), so that \( \alpha = \sum_{i=1}^l a_i \alpha_i \). Then the value of \( \alpha \) at the \( i \)-th vertex of \( \overline{A} \) is \( a_i / c_i \), and \( \lambda \) is the maximum of those values. Two interesting cases are: (i) If \( \alpha = \alpha_{\text{max}} \), then all
\(a_i/c_i = 1\), and \(\alpha^\vee = \alpha\). That is, the open line segment from the origin to the highest root always appears in the decomposition. (ii) If \(\alpha = \alpha_i\), then \(a_i = 1\) while all other \(a_j\) vanish. In this case, one obtains the open line segment from the origin to \(\frac{1}{c_i}\alpha^\vee\).

(d) Every \(V_w\) contains a distinguished ‘base point’. Indeed, let \(\rho \in V\) be the half-sum of positive roots, and \(h^\vee = 1 + \langle \alpha_{\text{max}}, \rho \rangle\) the dual Coxeter number. Then \(\rho/h^\vee \in A\), and consequently \(\rho/h^\vee - w(\rho/h^\vee) \in V_w\).

5. Proof of Theorem 1.3

The proof is very similar to the proof of Proposition 3.4, hence we will be brief. Each \(V_w(S) = (S - w)(A)\) is the interior of a simplex in \(V\), with codimension 1 faces \(V_{w,i} = (S - w)(A_i)\). As in the proof of Lemma 3.1 we see that

\[n_{w,i} = (S - w^{-1})^{-1} \alpha_i\]

is an inward-pointing normal vector to the \(i\)-th face \(V_{w,i}^{(S)}\). For \(S = 0\) this simplifies to

\[n_{w,i}^{(0)} = -w \alpha_i\]

If \(w' = ws_i\) we have \(V_{w,i}^{(S)} = V_{w',i}^{(S)}\), so that \(n_{w,i}^{(S)}\) and \(n_{w',i}^{(S)}\) are proportional. Since \(n_{w,i}^{(0)} = -n_{w',i}^{(0)}\), it follows by continuity that \(n_{w,i}^{(S)}\) is a negative multiple of \(n_{w',i}^{(S)}\). As a consequence, we see that \(V_w^{(S)}, V_{w'}^{(S)}\) are on opposite sides of affine hyperplane supporting \(V_{w,i}^{(S)} = V_{w',i}^{(S)}\). Arguing as in the proof of Proposition 3.4, this shows that

\[\bigcup_{w \in W} V_w^{(S)} = V.\]

Letting \(X^{(S)} = \bigcup_{w \in W} V_w^{(S)}\), it follows that \(V = \bigcup_{\lambda \in A} (\lambda + X^{(S)})\). Hence \(\text{vol}(X^{(S)}) \geq |W| \text{vol}(A)\).

But

\[\text{vol}(X^{(S)}) \leq \sum_{w \in W} \text{vol} \left((S - w)(A)\right)\]

\[= \text{vol}(A) \sum_{w \in W} |\text{det}(S - w)|\]

\[= \text{vol}(A) \sum_{w \in W} |\text{det}(\text{id} - S w^{-1})| = |W| \text{vol}(A),\]

using \([11, p.134]\). It follows that \(\text{vol}(X^{(S)}) = |W| \text{vol}(A)\), which implies (as in the proof of Proposition 3.3) that all int\((V_w^{(S)}) = V_w^{(S)}\) are disjoint. This completes the proof.

Remark 5.1. Theorem 1.3 and its proof, go through for any \(S\) in the component of 0 in the set \(\{S \in \text{End}(V) \mid \text{det}(S - w) \neq 0 \ \forall w \in W\}\). For instance, the fact that \(\text{det}(\text{id} - S w^{-1}) > 0\) follows by continuity from \(S = 0\). On the other hand, if e.g. \(S\) is a positive matrix with \(S > 2 \text{id}\), the result becomes false, since then (cf. \([11, p.134]\)) \(\sum_{w \in W} |\text{det}(S - w)| = \sum_{w \in W} \text{det}(S - w) = \text{det}(S)|W|\).
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University of Toronto, Department of Mathematics, 40 St George Street, Toronto, Ontario M4S2E4, Canada
E-mail address: mein@math.toronto.edu