Fuzzy Spacetime with $SU(3)$ Isometry in IIB Matrix Model

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Abstract

A group of fuzzy spacetime with $SU(3)$ isometry is studied at the two loop level in IIB matrix model. It consists of spacetime from 4 to 6 dimensions, namely from $CP^2$ to $SU(3)/U(1) \times U(1)$. The effective action scales in a universal manner in the large $N$ limit as $N$ and $N^{4/3}$ on 4 and 6 dimensional manifolds respectively. The 4 dimensional spacetime $CP^2$ possesses the smallest effective action in this class.

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1 Introduction

The investigations of the properties of the spacetime at the microscopic level have become an important physical subject since we now have a clear picture where the universe comes from and is going. At the current stage, the space is found to be almost flat and accelerating its expansion rate. It is therefore approaching a 4 dimensional de Sitter spacetime. Furthermore, the scale independent fluctuation of the cosmic microwave background at long distance scale suggests that the universe also started as a de Sitter spacetime. In order to explain why the universe evolves in such a peculiar way, we need to obtain a deeper understanding of the spacetime. It is expected that string theory plays a crucial role to understand the spacetime at the microscopic level. In order to address a time dependent issue, it is likely that we need a non-perturbative formulation of string theory such as IIB matrix model[1, 2].

In this model, Euclidean spacetime is expected to emerge out of the distributions of the eigenvalues of the 10 matrices. We can certainly imagine that the eigenvalues are homogeneously distributed on $S^4$ in 10 dimensions. Since a de Sitter space becomes an $S^4$ after the Euclidean continuation, we may interpret Euclidean spacetime a la Hartle and Hawking[3]. If we divide an $S^4$ into the two halves, we obtain an $S^3$ at the boundary. With the identification of the $S^3$ as a space, the effective action for $S^4$ in IIB matrix model determines the relative probability of the emergence of an $S^3$ out of nothing. We find it remarkable that the matrix models can accommodate a realistic spacetime in a nonperturbative way. In this sense our studies of homogeneous spacetime in IIB matrix model may shed light on the origin of the universe.

A fuzzy homogeneous spacetime $G/H$ can be embedded in matrix models by choosing background matrices as the generators of a group $G$ [4]. $G$ has to be a subgroup of $SO(10)$ and $H$ has to be a closed subgroup of $G$. We obtain Non-Commutative (NC) gauge theory on the fuzzy spacetime in this construction [5]. We can calculate an effective action on this background and investigate the large $N$ scaling behavior of it.

In this paper, we choose $G$ to be $SU(3)$ and investigate the class of the manifolds with $SU(3)$ isometry in IIB matrix model. They include $CP^2 = SU(3)/U(2)$ and $SU(3)/U(1) \times U(1)$. Each manifold is labeled by an irreducible representation of $SU(3)$. Note that $CP^2$ is a four dimensional manifold, while $SU(3)/U(1) \times U(1)$ is six dimensional. Therefore we can investigate the large $N$ scaling behavior of the effective action for the both 4 and 6 dimensional manifolds.
In a series of papers [6], we investigated the manifolds with $SU(2) \times SU(2)$ isometry and found certain instabilities associated with fuzzy $S^2 \times S^2$. Each fuzzy $S^2$ can be parameterized by $l$: the spin of a representation and $f$: a scale factor. We recall that the radius of $S^2$ is $lf$ while the NC length scale is $\sqrt{lf}$. Thus the both spin and scale factor specify the overall size of each $S^2$. In this construction $S^2 \times S^2$ can be characterized by the ratios of the spins and scale factors between the two $S^2$’s. The instability has been found under the variation of the both ratios. However it does not take place if we are constrained to have the identical scale factor for the both $S^2$’s. We thus expect that a more symmetric manifold will be stable.

In this respect $CP^2$ backgrounds are interesting. $CP^2$ can be embedded in Hermitian matrices as

$$A_i = f T_i,$$  \hspace{1cm} (1.1)

where $T_i$ are the generators of $SU(3)$ in a particular class of representations. As $CP^2$ can have the only one scale factor, it may not suffer from such an instability. The irreducible representations of $SU(3)$ from which $CP^2$ can be constructed as $SU(3)/U(2)$ are relatively well studied[7, 8]. Therefore it is interesting to investigate the large $N$ scaling behavior of the effective action of $CP^2$ and other manifolds with $SU(3)$ isometry and to see which manifold is most stable among them.

The organization of this paper is as follows. In section 2, we construct IIB matrix model on fuzzy $CP^2$. We find a universal expression for the 2-loop effective action on a homogeneous space. In section 3, we derive the effective actions on the manifolds with $SU(3)$ isometry and investigate the large $N$ scaling behavior of them. We find that they scale in a universal fashion which depends only on the dimensionality of the manifold. We argue that there is indeed a universality in the large $N$ scaling of the effective action on $G/H$. We conclude in section 4 with discussions. In Appendix A, we construct the generators and eigen-matrices of $SU(3)$. In Appendix B, we derive the 2-loop effective action on the manifolds constructed from $SU(3)$ algebra. In Appendix C, we numerically evaluate the 2-loop effective action.

## 2 IIB matrix model on fuzzy $CP^2$

### 2.1 Group theoretic construction

Let us recall a construction of fuzzy homogeneous spacetime $G/H$ and gauge field on them[4]. We pick a state $|0\rangle$ in a definite representation $G$ which is invariant under $H$. The set of all states which can be reached by multiplying elements of $G$ to $|0\rangle$ is called the orbit of $|0\rangle$. A
fuzzy homogeneous spacetime $G/H$ is constructed as the orbit of $|0\rangle$. It is represented by the irreducible representation that is descended from $|0\rangle$. The basic degrees of freedom in NC gauge theory are bi-local fields. We construct NC gauge field as the bi-local field by forming the tensor product of the relevant irreducible representation and its complex conjugate.

We take a Lie group $G$ to be $SU(3)$ in the present investigation. An irreducible representation of $SU(3)$ is labeled by a set of two integers $(p, q)$. An invariant subgroup $H$ depends on the irreducible representation. We have $U(2)$ as the invariant subgroup for a $(p, 0)$ representation. It gives rise to a four dimensional fuzzy $CP^2 = SU(3)/U(2)$. On the other hand $H$ is $U(1) \times U(1)$ for a generic $(p, q)$ representation. In this case we obtain a fuzzy flag manifold $SU(3)/U(1) \times U(1)$. It is a six dimensional NC spacetime which locally looks like $CP^2 \times S^2$. The representation $(p, p)$ may give the most symmetric six dimensional manifold.

In the large $N$ limit, the extension of the manifold becomes infinite with respect to NC scale. In such a situation, we expect that the effective action scales in a definite way. As we find such a scaling exhibits a universality which depends only on the dimensionality of the manifold, a group of the representations represents a universal class. We are thus interested in to identify such a universal manifold in the large $N$ limit.

We introduce a fuzzy homogeneous spacetime as a background of IIB matrix model, and calculate the effective action in a background field method. For this purpose, we expand the matrices around the background with a scale factor $f$:

$$A_\mu = f(p_\mu + a_\mu), \quad (2.1)$$

where $p_\mu$ is the background and $a_\mu$ represents NC gauge field. The background is taken as

$$p_\mu = \begin{cases} 1_{n \times n} \otimes T_\mu^{(p,q)} & \mu = 1, \ldots, 8 \\ 0 & \mu = 9, 10 \end{cases} \quad (2.2)$$

where $T_\mu^{(p,q)}$ are the $SU(3)$ generators of a $(p, q)$ representation. Here we have taken a simple reducible representation. We obtain $U(n)$ gauge theory on a fuzzy homogeneous spacetime in this way. This background can be realized by the matrices whose dimension is

$$N = n \cdot \text{dim}(p, q) = n \cdot \frac{1}{2}(p + 1)(q + 1)(p + q + 2). \quad (2.3)$$

One could consider a more general background such as

$$\sum_i \oplus \left(1_{n_i \times n_i} \otimes T_\mu^{(p_i,q_i)}\right). \quad (2.4)$$
However we consider a simple case (2.2) only in the present paper.

The gauge field is expanded by harmonic functions on the \((p, q)\) background:

\[
a_\mu = \sum_A a_\mu^{(A)} Y^{(A)},
\]

where the harmonic function matrices \(Y^{(A)}\) are the eigen-functions of \([T_3, ], [T_8, ]\) and \([T_\mu, [T_\mu, ]]\). The quantum numbers \((A)\) are determined by decomposing the gauge field into the irreducible representations. An explicit construction procedure of them is explained in appendix A. We obtain the propagators and vertices by using the expansion (2.5). By a perturbative calculation, we obtain the effective action \(\Gamma = \Gamma(p, q, \lambda^2, n)\). Here \(\lambda^2\) is a natural expansion parameter which is proportional to \(1/f^4\). It is a ’t Hooft coupling constant which should be kept fixed in the large \(N\) limit. We can determine the parameters \(\{p, q, \lambda, n\}\) by requiring that the effective action is stationary with respect to the change of them \(\delta \Gamma = 0\). Such a set constitutes a solution of IIB matrix model. Dynamical generation of fuzzy homogeneous spacetime can be investigated in this way. We can compare the extremal values of the effective action for these (stable) solutions to find the most favored one.

In this paper we carry out the loop expansion up to the 2-loop level. The tree level action does not admit a non-trivial solution. Such a solution appears when the 2-loop quantum correction is included in the effective action. The situation is the same with the backgrounds based on \(SU(2)\) algebras [6] and, as we discuss later, a common aspect for backgrounds based on Lie algebras \(G \subset SO(10)\).

In what follows, we explain the details of our evaluation of the effective action.

**Universal properties of the 2-loop effective action**

We can draw some common features of the effective action in homogeneous spacetime from a series of our studies. Here we assume the expansion (2.1) and \(p\) denotes a set of generators of a Lie algebra \(G \subset SO(10)\) of the form (2.2). We also assume that one can find a set of harmonic functions which are eigenfunctions of the adjoint operators \(P = [p, ]\). In the large \(N\) limit, the leading terms of the effective action of IIB matrix model up to the 2-loop level can be summarized as the following universal expression

\[
\Gamma = \frac{f^4}{4} C_G C_2(G, R) N + n^2 \left( tr \frac{P_1}{P_4} \right) + 2 n^3 \frac{C_G}{f^4} \left( \frac{1}{P_1^2 P_2^2 P_3^2} \right)
\]

\(\Gamma = SU(2)\) is the exception since the two loop amplitude is finite in the large \(N\) limit. We must use the exact propagators for gauge bosons and fermions to evaluate the 2-loop contributions in such a case.
where $R$ denotes an irreducible representation of a Lie algebra and

$$C_G \delta_{\rho\sigma} = f_{\mu\nu\rho} f_{\mu\nu\sigma}, \quad C_2(G, R) N = tr p_{\mu} p_{\mu}.$$  \hfill (2.7)

$f_{\mu\nu\rho}$ is the structure constant of the Lie algebra. The first, second and third terms in (2.6) are the tree, 1-loop and 2-loop contributions respectively.

The 2-loop contributions consist of the planar and non-planar contributions. In NC theory, the non-planar contributions are suppressed due to the NC phase. We argue that the upper cut-off becomes $\sqrt{l}$ instead of $l$ in the non-planar sector since the NC scale is $\sqrt{l}$. As the two loop contributions are quadratically divergent in the large $N$ limit for a 4 dimensional background, we argue that the non-planar contributions are suppressed by $\sqrt{N}$ in that case. The analogous suppressions should take place in higher dimensions. The two loop non-planar contributions will be suppressed by $N$ in comparison to the planar contributions for 6 dimensional backgrounds. We thus argue that the 2-loop contributions are always positive since the non-planar contributions can be neglected in the large $N$ limit.

The two loop level effective action can be bounded as

$$\Gamma \geq (1\text{-loop}) + 2 C_G \sqrt{\frac{C_2(G, R) N n^3}{2} \frac{1}{\langle P_1^2 P_2^2 P_3^2 \rangle}}.$$  \hfill (2.8)

after we minimize it with respect to $f$. Without the 2-loop contributions, we can obtain only trivial solutions as $f = 0$ is required to minimize the action. Therefore higher loop, at least 2-loop, corrections are necessary to obtain a fuzzy homogeneous spacetime in IIB matrix model.

3 The effective action on fuzzy spacetime with $SU(3)$ isometry.

In this section, we evaluate the effective action on the fuzzy manifolds with $SU(3)$ isometry. We set $n = 1$ for simplicity since we can easily recover the $n$ dependence as (2.6).

The tree level effective action of a $(p, q)$ representation is

$$\Gamma_{\text{tree}} = -\frac{1}{4} Tr [p_{\mu} p_{\nu}]^2$$

$$= \frac{3 f^4}{4} N \frac{1}{3} [p (p + 3) + q (q + 3) + pq].$$  \hfill (3.1)
When the background is \( CP^2 \) \((p,0)\) rep., the leading term of (3.1) in the large \( N \) limit becomes

\[
\Gamma_{\text{tree}} \simeq \frac{f^4}{2} N^2, \\
N \simeq \frac{p^2}{2}. 
\]  
(3.2)

On a 6d manifold \((p,p)\) rep., it becomes

\[
\Gamma_{\text{tree}} \simeq \frac{3f^4}{4} N^\frac{5}{3}, \\
N \simeq p^3. 
\]  
(3.3)

The leading term of the one loop effective action in the large \( N \) limit can be estimate as

\[
\Gamma_{1-\text{loop}} \propto Tr \left( \frac{1}{P^2} \right)^2 \sim \begin{cases} 
O(\log N) & \text{CP}^2 \\
O\left( N^\frac{1}{3} \right) & \text{6d} 
\end{cases}.
\]

We can neglect this term in the effective action because we shortly find that the effective action scales as \( O(N) \) on \( CP^2 \) or \( O(N^{4/3}) \) on a 6 dimensional manifold.

The leading term of the two loop effective action in the large \( N \) limit is evaluated as

\[
\Gamma_{2-\text{loop}} = \frac{6}{f^4} F_3 \equiv \frac{6}{f^4} \left\langle \frac{1}{P_1^2 P_2^2 P_3^2} \right\rangle, 
\]  
(3.4)

where the detailed calculations are explained in appendix B. In this way, we obtain the effective action in the large \( N \) limit as

\[
\Gamma = \Gamma_{\text{tree}} + \Gamma_{2-\text{loop}} \\
= \frac{f^4 N}{4} [p(p+3) + q(q+3) + pq] + \frac{6}{f^4} F_3. 
\]  
(3.5)

We can now explore the behavior of the effective action. Firstly, we investigate \( F_3 \) of (3.4) to determine the scaling behavior for various representations. We have numerically estimated \( F_3 \) in appendix C. Fig.1 shows \( F_3 \) against \( N \). We first observe that \( F_3 \) of the \((p,0)\) representations approaches a constant in the large \( N \) limit. This value is estimated as

\[
F_3 \sim 1.197 + \frac{1.03}{p} - \frac{5.4}{p^2} + \frac{6.8}{p^3} - \frac{2.9}{p^4}. 
\]  
(3.6)

We next observe that \( F_3 \) of the \((p,p)\) representations behaves as \( O(N) \). Thirdly we find that \( F_3 \) of the \((p,q)\) representations where \( 0 < q < p \) behaves like that of \( U(q+1) \) gauge theory.
in the large $N$ limit when $q$ is fixed. It is because it approach a constant which is consistent with the 2-loop effective action of $U(q+1)$ gauge theory on $CP^2$:

$$(q+1)^3F_3. \tag{3.7}$$

By assuming that we have correctly identified the large $N$ scaling behavior of $F_3$ for various representations, we can obtain the large $N$ limit of the effective actions after identifying the suitable 't Hooft couplings for $CP^2$ and 6d manifolds. In the $CP^2$ case, the action in the large $N$ limit is

$$\Gamma = \frac{1}{2\lambda^2} + 6\lambda^2 F_3,$$

$$\lambda^2 = \frac{1}{f^4N}. \tag{3.8}$$

In a 6d manifold of the $(p,p)$ representations, it is

$$\Gamma = N^{\frac{3}{4}} \frac{3}{4\lambda^2} + 6\lambda^2 \frac{F_3}{N},$$

$$\lambda^2 = \frac{1}{f^4N^{\frac{3}{4}}}. \tag{3.9}$$
Because of the different large $N$ scaling behaviors of the effective actions, we find that the $CP^2$ background is preferable to the 6d manifold.

After identifying the 't Hooft coupling, we can minimize the effective action with respect to it. We can use (2.8) to determine the minimum of the effective action:

$$\Gamma \geq \Gamma_{\text{min}} \equiv 2\sqrt{\Gamma_{\text{tree}}\Gamma_{\text{2-loop}}}.$$  \hfill (3.10)

![Figure 2: $\Gamma_{\text{min}}/N$ against $N$.](image)

Fig.2 shows $\Gamma_{\text{min}}/N$ against $N$. We can observe that the effective action on the fuzzy $CP^2$ in the large $N$ limit is the smallest in this class with $SU(3)$ symmetry as it approaches a constant. This value can be estimated by using (3.6) as

$$\frac{\Gamma}{N} \simeq 3.79.$$  \hfill (3.11)

The 't Hooft coupling at this minimum is

$$\lambda^2 \simeq 0.26.$$  \hfill (3.12)

We remark here that (3.11) is comparable to the minimum of the effective action of the fuzzy $S^2 \times S^2$ background at the most symmetric point [6]:

$$\frac{\Gamma_{S^2 \times S^2}}{N} \simeq 3.61.$$  \hfill (3.13)
Although we believe that the estimate (3.13) is accurate, our estimate (3.11) suffers considerable uncertainty since it is derived from our numerical investigation up to $N \sim 100$. As we observe in Table 1 that $F_3$ is gradually decreasing, we cannot determine the lower bound of the effective action of $CP^2$ yet. Within these limitations, we can still conclude that the fuzzy $CP^2$ background is stable in its class and its effective action is comparable to that of fuzzy $S^2 \times S^2$.

Here we summarize our findings for the backgrounds with $SU(3)$ symmetry. The effective action becomes $O(N)$ for the $(p, 0)$ representations in the large $N$ limit. On the other hand the $(p, p)$ representations give the effective action $O(N^{3/4})$. We recall here that the $(p, 0)$ representations give a 4-dimensional NC spacetime while the $(p, p)$ representations gives a 6-dimensional one in the large $N$ limit. Since the both effective actions are positive, the $(p, 0)$ representations are favored over the $(p, p)$ representations in the large $N$ limit. We also have an observation for the $(p, q)$ representations with $q << p$. In this case the $(p, q)$ representations behave like a direct product of the $(p, 0)$ representations and the $(q + 1) \times (q + 1)$ identity matrix. In such a case, we effectively obtain $U(q + 1)$ gauge theory on $CP^2$ and the effective action is proportional to $(q + 1)^3N$. We thus argue that the effective action is minimized for $q = 0$. Therefore the $(p, 0)$ representations are a solution of IIB matrix model as long as $SU(3)$ symmetry is not broken. We conclude that a four dimensional fuzzy $CP^2$ is singled out by IIB matrix model within the manifolds with $SU(3)$ symmetry.

One of our goals of this paper is to investigate the scaling behavior of the effective action of this class of spacetime in the large $N$ limit. Let us recall the situation for the manifolds constructed from $SU(2)$ algebras [6]. The four dimensional fuzzy $S^2 \times S^2$ makes the effective action to be $O(N)$, and a six dimensional spacetime $S^2 \times S^2 \times S^2$ gives $O(N^{3/4})$ action. These scaling behaviors can be derived from the power counting of the higher loop contributions. We also assumed that the leading quantum corrections cancel due to supersymmetry. Such an assumption can be justified since the quantum corrections do cancel for commuting backgrounds and the commutators of the backgrounds reduce the degrees of divergences. In our identification of the 't Hooft couplings, we used the fact that the three point vertices scales as $1/\sqrt{N}$ in the large $N$ limit.

We argue that the same scaling rule holds in general. In fact our reasoning to identify the scaling behavior of the effective action does not depend on the details of a particular Lie algebra. In particular, the large $N$ scaling rule of the three point vertices are the consequence of our normalization of the two point vertices to be $O(1)$. Therefore it must hold in generic
Lie algebra. In fact we have numerically found, at the 2-loop level, that a 4-dimensional fuzzy $CP^2$, namely the $(p, 0)$ representation gives $O(N)$ effective action and a 6-dimensional fuzzy flag manifold, namely the $(p, p)$ representation gives $O(N^{\frac{4}{3}})$ behavior. These findings support our argument that any four dimensional fuzzy homogeneous spacetime gives $O(N)$ effective action and six dimensional one gives $O(N^{\frac{4}{3}})$ action.

We investigated whether IIB matrix model has a fuzzy $S^2 \times S^2$ solution at the 2-loop level previously. The most symmetric $S^2 \times S^2$ solution turns out to be unstable along some directions of their moduli parameters. They describe the relative sizes of the two spheres. The instability drive the symmetric $S^2 \times S^2$ to the asymmetric one. Fortunately we find fuzzy $CP^2$ has no such instability. The extremal value of the effective action is comparable to that of the symmetric $S^2 \times S^2$. We thus obtain a new evidence for the existence of a symmetric stable 4-dimensional spacetime in IIB matrix model.

4 Conclusions

In this paper we have investigated the effective action of IIB matrix model on fuzzy $CP^2$ and the related manifold with $SU(3)$ isometry at the two loop level. Since the backgrounds constructed by using $SU(3)$ algebra contain the manifolds with different dimensionality such as $CP^2$ (4d) and a 6d manifold, we can compare the minimum of the effective action of the 4d and 6d dimensional backgrounds like [6] in our investigation of the stability of $CP^2$.

We have investigated the large $N$ scaling behavior of the effective action. The action scales as $N$ on $CP^2$ and $N^{\frac{4}{3}}$ on a 6d manifold respectively. The effective action of the $(p, q)$ representations where $p > q$ with fixed $q$ also scales as $N$, since it behaves like $U(q + 1)$ gauge theory of $CP^2$. From these results, we have found that $CP^2$ minimizes the effective action among the backgrounds which are constructed by $SU(3)$ algebra. We conclude that the fuzzy $CP^2$ background is a solution in IIB matrix model and stable as far as $SU(3)$ symmetry is not broken.

These scaling behaviors are in accord with other 4d manifolds like $S^2 \times S^2$ and $T^2 \times T^2$ and also a 6d manifold $S^2 \times S^2 \times S^2$[6, 11]. These facts support our contention that the effective action of a compact manifold embedded in IIB matrix model has the universal scaling behavior: it scales as $N$ and $N^{\frac{4}{3}}$ on a 4d and 6d manifold respectively.

We have also compared the minimum of the effective actions of $CP^2$ with that of $S^2 \times S^2$. We have observed the effective action of $CP^2$ is comparable to that of $S^2 \times S^2$. As we have
observed in Table 1 that the 2-loop effective action on $CP^2$ is gradually decreasing, we cannot
determine the lower bound of it yet. Therefore we cannot say which is smaller even at the
two loop level. To answer this question, it is desirable to obtain an asymptotic expression of
the 2-loop effective action on $CP^2$ like such an expression on $S^2$ which is obtained from the
Wigner’s 6$j$ symbols. Such an effort may be useful to determine whether higher symmetry
of the background may lower the effective action or not.

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Appendix A

Construction of background

A fundamental representation of SU(3) is 3-dimensional. The Lie group generators can
be written by Gell-Mann matrices $\lambda_\mu$ as $t_\mu = \lambda_\mu/2$. We take Gell-Mann matrices as the
following form

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad
\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (A.1)
$$

We denote state vectors on which these generators act as $|a\rangle, |b\rangle, \ldots$, here indices $a, b, \ldots$
run from 1 to 3. These vectors have the following components

$$
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad
|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad
|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (A.2)
$$

The Cartan matrices are $t_3$ and $t_8$. They act on $|a\rangle$ as the following way

$$
t_3|1\rangle = \frac{1}{2}|1\rangle, \quad t_3|2\rangle = \frac{-1}{2}|2\rangle, \quad t_3|3\rangle = 0 \cdot |3\rangle,
$$

$$
t_8|1\rangle = \frac{1}{2\sqrt{3}}|1\rangle, \quad t_8|2\rangle = \frac{1}{2\sqrt{3}}|2\rangle, \quad t_8|3\rangle = \frac{-1}{\sqrt{3}}|3\rangle. \quad (A.3)
$$
The raising/lowering operators are

\[ j_1^{\pm} = t_4 \pm it_5, \quad j_2^{\pm} = t_6 \pm it_7 \]  

(A.4)

and they act on the state vectors as

\[ j_1^{\pm} : |3\rangle \leftrightarrow |1\rangle, \quad j_2^{\pm} : |2\rangle \leftrightarrow |3\rangle, \quad \text{otherwise gives zero}. \]  

(A.5)

A general SU(3) representation is labeled by a set of two integers \((p, q)\) and have the dimension \(\text{dim}(p, q) = (p + 1)(q + 1)(p + q + 2)/2\). The fundamental representation is denoted as \((1, 0)\). The \((p, q)\) representation can be constructed from \((1, 0)\) by forming tensor products.

As the first example, we construct the \((2, 0)\) representation. The \((2, 0)\) state vectors are constructed from the tensor products of the two sets of the \((1, 0)\) vectors:

\[ |v^{(2,0)}\rangle = |a\rangle |b\rangle + |b\rangle |a\rangle. \]  

(A.6)

We should take an appropriate normalization factor in the above expression. The symmetric property of this tensor product is represented by a Young tableau \(\begin{array}{c}
1 \\
2 \\
p
\end{array}\). A single box \(\Box\) denotes the \((1, 0)\) vector. The \((2, 0)\) generators which act on the state vectors are the tensor products of \((1, 0)\) generators \(t_\mu\) and the \(3 \times 3\) unit matrix \(\mathbb{1}_3\):

\[ T^{(2,0)}_\mu = t_\mu \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes t_\mu. \]  

(A.7)

To obtain the explicit matrix representation of the generators, we need to calculate the matrix elements

\[ \langle v^{(2,0)} | T^{(2,0)} | v^{(2,0)} \rangle. \]  

(A.8)

In this way, we can write down the generators as \(6 \times 6\) matrices. An extension to the \((p, 0)\) representation is easily obtained by tensoring \(p\) sets of the fundamental representations. The \((p, 0)\) state vectors up to the normalization factor are given by totally-symmetrized tensor products of the \((1, 0)\) vectors

\[ |v^{(p,0)}\rangle = \prod_{i=1}^{p} |a_i\rangle + \text{permutations for } \{a_i\} \]  

(A.9)

Its symmetric property is represented by the Young tableau: \(\begin{array}{c}
1 \\
2 \\
p \\
\end{array}\).
The representations of the generators which act on these \((p,0)\) state vectors are

\[
T^{(p,0)}_{\mu} = \sum_{i=0}^{p-1} (1_3 \otimes)^i t_\mu (\otimes 1_3)^{p-1-i}
\]  

(A.10)

To obtain an explicit matrix representation of the generators, we need to calculate the matrix elements

\[
\langle v^{(p,0)} | T^{(p,0)} | v^{(p,0)} \rangle.
\]  

(A.11)

In this way, we can write down the generators as \((p+1)(p+2)/2 \times (p+1)(p+2)/2\) matrices.

Next we consider an extension of our construction to the \((p,p)\) representations. It is obtained by \((2p + p)\)-fold tensor products. The state vectors of the \((p,p)\) representation up to the normalization factors can be written as

\[
|v^{(p,p)}\rangle = \prod_{i=1}^{p} \left( |a_i\rangle|b_i\rangle - |b_i\rangle|a_i\rangle \right) \prod_{j=1}^{p} |a_j\rangle + \text{permutations of } \{a_i, a_j\}
\]  

(A.12)

Here the permutations between \(a_i\) and \(a_j\) should be included also. Indices \(a_i\) and \(b_i\) are antisymmetrized. Its symmetry property is represented by a Young tableau: \[
\begin{array}{cccc}
1 & 2 & \ldots & p \\
\end{array}
\]

Next we consider an extension of our construction to the \((p,0)\) representations. It is obtained by \((2p + p)\)-fold tensor products. The state vectors of the \((p,p)\) representation up to the normalization factors can be written as

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\]  

(A.13)

To obtain explicit form of the generators, we need to calculate the matrix elements

\[
\langle v^{(p,p)} | T^{(p,p)} | v^{(p,p)} \rangle.
\]  

(A.14)

In this way, we can write down the generators as \((p+1)^3 \times (p+1)^3\) matrices.

An extension to an arbitrary \((p,q)\) representation is easily obtained by forming the \((p + 2q)\)-fold tensor products. The state vectors of \((p,q)\) type can be written as

\[
|v^{(p,q)}\rangle = \mathcal{C}_{(p,q)} \prod_{i=1}^{q} \left( |a_i\rangle|b_i\rangle - |b_i\rangle|a_i\rangle \right) \prod_{j=1}^{p} |a_j\rangle + \text{permutations of } \{a_i, a_j\}
\]  

(A.15)

Here the permutations between \(a_i\) and \(a_j\) should be included also. Indices \(a_i\) and \(b_i\) are antisymmetrized. The symmetric property is given by a Young tableau: \[
\begin{array}{cccc}
1 & 2 & \ldots & p \\
\end{array}
\]

Here \(\mathcal{C}_{(p,q)}\) is a normalization constant. The representations of the generators which act on these \((p,q)\) state vectors are

\[
T^{(p,q)}_{\mu} = \sum_{i=0}^{2p+q-1} (1_3 \otimes)^i t_\mu (\otimes 1_3)^{2p+q-1-i}
\]  

(A.16)
To obtain an explicit matrix form of the generators, we need to calculate the matrix elements

\[ \langle v^{(p,q)} | T^{(p,q)} | v^{(p,q)} \rangle . \]  

(A.17)

In this way, we can write down the generators as \( N^{(p,q)} \times N^{(p,q)} \) matrices where

\[ N^{(p,q)} = \frac{(p+1)(q+1)(p+q+2)}{2}. \]  

(A.18)

Construction of matrix harmonics in SU(3) background

Suppose that we take a matrix model background to be a \((p,q)\) representation. The gauge (and adjoint fermion) fields are expanded by harmonic matrices as follows

\[ \phi = \sum_{(A)} \sum_{ms} \phi_{ms}^{(A)} Y_{ms}^{(A)}, \]  

(A.19)

where \( Y_{ms}^{(A)} \) are the matrix harmonics. The index \((A)\) denotes the sets of two integers \((p_A, q_A)\) which label the irreducible representations. They are \( N^{(p,q)} \times N^{(p,q)} \) matrices which satisfy

\[ P_3 Y_{ms}^{(A)} \equiv [p_3, Y_{ms}^{(A)}] = m Y_{ms}^{(A)}, \]
\[ P_3 Y_{ms}^{(A)} \equiv [p_8, Y_{ms}^{(A)}] = s Y_{ms}^{(A)}, \]
\[ P^2 Y_{ms}^{(A)} \equiv [p_\mu [p_\mu, Y_{ms}^{(A)}]] = \left( \frac{1}{2} p_A^2 + p_A + \frac{1}{2} q_A^2 + q_A \right) Y_{ms}^{(A)}. \]  

(A.20)

The gauge fields are constructed as bi-local fields. When the background is a \((p,q)\) representation, the bi-local state has a tensor structure \((p,q) \otimes (q,p)\). They can be decomposed into the irreducible representations, and the decomposition may have the following form

\[ (p,q) \otimes (q,p) = \sum_{n=0}^{p+q} D_n (n,n) + \sum_{l \neq m}^{p+2q} E_{ml} ((l,m) + (m,l)) \]  

(A.21)

where \( D_n \) and \( E_{lm} \) are multiplicity factors. If we take \( q = 0 \), the decomposition becomes a simple form as

\[ (p,0) \otimes (0,p) = \sum_{n=0}^{p} (n,n). \]  

(A.22)

Here we give the \( p = q = 1 \) case for another simple example

\[ (1,1) \otimes (1,1) = (2,2) + 2(1,1) + (0,0) + (3,0) + (0,3). \]  

(A.23)

Thus, in expansion (A.19), the sets of the integers \((p_A, q_A)\) run over the irreducible representations which appear in the decomposition, and \( m \) and \( s \) take the value of these irreducible representations \((p_A, q_A)\).
Now we explain how to construct such matrices in a given background. Let us describe a background (i.e. SU(3) generator of a \((p, q)\) rep.) in terms of SU\((N^{(p,q)})\) basis

\[
T_{\mu}^{(p,q)} = \sum_\alpha (A_\alpha E_\alpha + B_{-\alpha} E_{-\alpha}) + \sum_a C_a H_a
\]  

(A.24)

where \(\{E_\alpha, E_{-\alpha}, H_\alpha\}\) are Cartan’s basis which satisfy the following relations

\[
\begin{align*}
[H_\alpha, H_\beta] &= 0, \\
[H_\alpha, E_{\pm\alpha}] &= \pm \alpha_\alpha E_{\pm\alpha}, \\
[E_\alpha, E_{-\alpha}] &= \alpha^a H^a, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad (E_\alpha^\dagger = E_{-\alpha})
\end{align*}
\]

(A.25)

One can take a representation of \(T_3^{(p,q)}\) and \(T_8^{(p,q)}\) as diagonal matrices

\[
T_3^{(p,q)} = \sum_a C_a H_a, \quad T_8^{(p,q)} = \sum_b C'_b H_b
\]

(A.26)

Each \(E_\alpha\)'s can be assigned to an off-diagonal matrix which has only one non-zero component:

\[
(E_\alpha)_{ij} = \begin{cases} 1 & \text{for } (i, j) = (i_\alpha, j_\alpha), \\ 0 & \text{otherwise}. \end{cases}
\]

(A.27)

Then we have

\[
[T_3^{(p,q)}, E_\alpha] = \sum_\alpha C_\alpha \alpha^a E_\alpha, \quad [T_8^{(p,q)}, E_\alpha] = \sum_b C'_b \alpha^b E_\alpha.
\]

(A.28)

It implies that \(Y_{m,s}^{(A)}\) with \((m, s) = (\sum_a C_a, \sum_b C'_b)\) can be written as linear combinations of \(E_\alpha\)'s which have same eigenvalues of \((m, s)\). On the other hand, Cartan subalgebra \([H, H] = 0\) implies that \(Y_{m=s=0}^{(A)}\) can be obtained by linear combinations of \(H\).

Following the above observation, we first take all commutators \([T^3, E]\) and \([T^8, E]\) to find quantum numbers \(m\) and \(s\) of each \(E\)'s. Next we determine suitable linear combinations in the matrix basis which possess the same \(m\) and \(s\). Then we obtain matrix harmonics which correspond to the irreducible representations in the decomposition (A.21).

One way to determine such linear combinations is to use the raising/lowering operators. The decomposition (A.21) contains the irreducible representation \((p_A, q_A) = (p + 2q, p - q)\). The value \(p + 2q\) is the maximum value of \(p_A\) in this decomposition. The highest weight state is unique in each irreducible representation, and \(p + 2q\) is the largest number in the decomposition. Then there should be only one matrix base corresponding to such a state whose eigenvalues are \(m = \frac{1}{2}(p+2q+p-q) = 2p + q/2\) and \(s = \frac{1}{2\sqrt{3}}(p+2q+p-q-2(p-q)) =\)
3q/2\sqrt{3}$. Therefore a matrix base with the eigenvalues $m_0 \equiv 2p + q/2$ and $s_0 \equiv 3q/2\sqrt{3}$ is
uniquely identified with the highest weight state of $(p + 2q, p - q)$. Next we carry out the
operations of the lowering operators and generate sets of independent combinations of the
matrix basis with $m'(< m_0)$ and $s' (\neq s_0)$. After suitable orthogonalizations, they form
the state vectors with quantum number $m'$ and $s'$. Some of these belong to the $(p + 2q, p - q)$ rep.
and form $Y_{m's'}^{(p+2q,p-q)}$. Others belong to different irreducible representations and form $Y_{m's'}^{(A')}$. 
In this way, we can identify all $(A') \neq (p + 2q, p - q)$ which appear in the decomposition
(A.21).

There is another way to obtain suitable combinations of the matrix basis more straight-
forwardly. First we collect matrix basis with the same quantum numbers $m$ and $s$ and
denote this set of basis as $\{w_i\}$. Next we diagonalize the Casimir operator $P^2$ whose matrix
elements are

$$P_{ij}^2 = tr(w_i^\dagger P^2 w_j) . \quad (A.29)$$

A different eigenvalue of $P^2$ correspond to a different $(A)$ of $Y_{m's}^{(A)}$, and $Y_{m's}^{(A)}$ themselves are
obtained as the eigenvectors. This method is useful if one has automatic computation tools
for linear algebra, like mathematica or maple etc.

**An explicit example**

We give an explicit construction of a background (generators) and the matrix harmonics in
a simple case. We consider the $(2,0)$ representation.

An expression of the state vectors of the $(2,0)$ representation is the following

$$|1\rangle^{(2,0)} = |a\rangle |a\rangle , \quad |2\rangle^{(2,0)} = \frac{|a\rangle |b\rangle + |b\rangle |a\rangle}{\sqrt{2}} , \quad |3\rangle^{(2,0)} = |b\rangle |b\rangle ,$$

$$|4\rangle^{(2,0)} = \frac{|a\rangle |c\rangle + |c\rangle |a\rangle}{\sqrt{2}} , \quad |5\rangle^{(2,0)} = \frac{|b\rangle |c\rangle + |c\rangle |b\rangle}{\sqrt{2}} ,$$

$$|6\rangle^{(2,0)} = |c\rangle |c\rangle \quad (A.30)$$

where $|a\rangle, |b\rangle$ and $|c\rangle$ are the state vectors of the fundamental representation.

The $SU(3)$ generators of the $(2,0)$ representation are

$$T_3^{(2,0)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & \frac{1}{\sqrt{3}} \\ -1 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} , \quad T_8^{(2,0)} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\ 1 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{2}{\sqrt{3}} \end{pmatrix} ,$$
To construct matrix harmonics, we define off-diagonal matrix bases as

\[
T_1^{(2,0)} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_2^{(2,0)} = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_4^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_5^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_6^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix},
\]

\[
T_7^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}.
\]  

To construct matrix harmonics, we define off-diagonal matrix bases as

\[
\begin{pmatrix} 0 & E_{\alpha_1} & E_{\alpha_2} & E_{\alpha_3} & E_{\alpha_4} & E_{\alpha_5} \\ E_{-\alpha_1} & 0 & E_{\alpha_6} & E_{\alpha_7} & E_{\alpha_8} & E_{\alpha_9} \\ E_{-\alpha_2} & E_{-\alpha_6} & 0 & E_{\alpha_{10}} & E_{\alpha_{11}} & E_{\alpha_{12}} \\ E_{-\alpha_3} & E_{-\alpha_7} & E_{-\alpha_{10}} & 0 & E_{\alpha_{13}} & E_{\alpha_{14}} \\ E_{-\alpha_4} & E_{-\alpha_8} & E_{-\alpha_{11}} & E_{-\alpha_{13}} & 0 & E_{\alpha_{15}} \\ E_{-\alpha_5} & E_{-\alpha_9} & E_{-\alpha_{12}} & E_{-\alpha_{14}} & E_{-\alpha_{15}} & 0 \end{pmatrix}.
\]  

This notation means that $E_{\alpha_1}$ is given by the form:

\[
E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

and so on.

Following the decomposition

\[
(2, 0) \otimes (0, 2) = (2, 2) + (1, 1) + (0, 0),
\]

we construct $Y^{(2,2)}$, $Y^{(1,1)}$ and $Y^{(0,0)}$ using the above matrix bases and diagonal matrices.
Finally there is the singlet correspond to \((0, 0)\):

\[
Y_{0,0}^{(0,0)} = \frac{1}{\sqrt{6}} \mathbf{1}_6.
\]
The 2-loop contribution to the effective action is calculated with these harmonics. The planar contribution is

$$6n^3 \sum_{(n_1,n_2,n_3)=1}^{2} \sum_{m_1,s_1} \sum_{m_2,s_2} \sum_{m_3,s_3} \frac{tr(Y^{(n_1,n_1)}_{m_1,s_1} Y^{(n_2,n_2)}_{m_2,s_2} Y^{(n_3,n_3)}_{m_3,s_3})tr(Y^{(n_3,n_3)\dagger}_{m_3,s_3} Y^{(n_2,n_2)\dagger}_{m_2,s_2} Y^{(n_1,n_1)\dagger}_{m_1,s_1})}{n_1(n_1+1)n_2(n_2+1)n_3(n_3+1)}$$

(A.38)

in $U(n)$ gauge theory. On the other hand, the non-planar contribution is

$$-6n \sum_{(n_1,n_2,n_3)=1}^{2} \sum_{m_1,s_1} \sum_{m_2,s_2} \sum_{m_3,s_3} \frac{tr(Y^{(n_1,n_1)}_{m_1,s_1} Y^{(n_2,n_2)}_{m_2,s_2} Y^{(n_3,n_3)}_{m_3,s_3})tr(Y^{(n_3,n_3)\dagger}_{m_3,s_3} Y^{(n_2,n_2)\dagger}_{m_2,s_2} Y^{(n_1,n_1)\dagger}_{m_1,s_1})}{n_1(n_1+1)n_2(n_2+1)n_3(n_3+1)}.$$  

(A.39)

By substituting the explicit form of $Y^{(n,n)}_{ms}$, we obtain the planar contribution:

$$6n^3 \frac{42605}{41472}$$  

(A.40)

and the non-planar contribution:

$$-6n \frac{1115}{41472}.$$  

(A.41)

**Appendix B**

In this appendix, we evaluate the two loop effective action of IIB matrix model in a fuzzy background which is made from a $(p,q)$ rep. of the $SU(3)$ generators (3.4).

In this calculation, we make use of the following relation:

$$\sum_{m} P_{\mu} Y_{m}^{(r,s)\dagger} P_{\nu} Y_{m}^{(r,s)} = - \sum_{m} Y_{m}^{(r,s)\dagger} P_{\nu} P_{\mu} Y_{m}^{(r,s)},$$

(B.1)

where the subscript $(r, s)$ denotes an irreducible representation of $SU(3)$ and $m$ denotes the eigenvalues of the Cartan subalgebra in the $(r, s)$ representation. We first note that the harmonic matrices of $SU(3)$ obey the orthogonal relations:

$$Tr \left( Y_{m}^{(r,s)\dagger} Y_{m'}^{(r',s')\dagger} \right) = \delta_{(r,s),(r',s')} \delta_{m,m'}.$$  

(B.2)

Let us perform a unitary transformation on $Y_{m}^{(r,s)}$:

$$Y_{m}^{(r,s)} \rightarrow U Y_{m}^{(r,s)} U^\dagger = \sum_{n} u_{mn} Y_{n}^{(r,s)};$$

$$Y_{m}^{(r,s)\dagger} \rightarrow \left( U Y_{m}^{(r,s)} U^\dagger \right)^\dagger = \sum_{n} u_{mn}^* Y_{n}^{(r,s)\dagger},$$

(B.3)
where $U$ is an $N \times N$ unitary matrix and $u_{mn}$ is the unitary transformation represented in the $m$ bases. Under (B.3), (B.2) is transformed as

$$\text{Tr} \left( Y_m(r,s) \dagger Y_m(r',s') \right) \to \sum_{n,n'} u_{mn}^* u_{m'n'} \text{Tr} \left( Y_n(r,s) \dagger Y_n(r',s') \right)$$

$$= \sum_{n,n'} u_{mn}^* u_{m'n'} \delta_{nn'} = (uu^\dagger)_{m'm} \delta_{(r,s),(r',s')}.$$  \hfill (B.4)

Since (B.2) is apparently invariant under (B.3), we can obtain

$$(uu^\dagger)_{m'm} = \delta_{m'm}. \hfill (B.5)$$

Using this relation, we can show that $\sum_m Y_m(r,s) \dagger Y_m(r,s)$ is invariant under (B.3):

$$\sum_m Y_m(r,s) \dagger Y_m(r,s) \to \sum_{m,n,n'} u_{mn}^* u_{m'n'} Y_n(r,s) \dagger Y_n(r,s)$$

$$= \sum_n Y_n(r,s) \dagger Y_n(r,s)$$  \hfill (B.6)

Since $P_\mu$ are the generators of $SU(3)$ transformation, (B.6) is equivalent to

$$P_\mu \left( \sum_m Y_m(r,s) \dagger Y_m(r,s) \right) = 0. \hfill (B.7)$$

From this formula, we can obtain (B.1).

We introduce the wave functions and averages as

$$\Psi_{123} \equiv \text{Tr} \left( Y_{m_1}(r_{1,s_1}) Y_{m_2}(r_{2,s_2}) Y_{m_3}(r_{3,s_3}) \right),$$

$$\langle X \rangle_P \equiv \sum_{(r_1,s_1),m_1} \Psi_{123} X \Psi_{123},$$

$$P_\mu Y_{m_1}(r_{1,s_1}) \equiv [P_\mu, Y_{m_1}^{(r_{1,s_1})}]. \hfill (B.8)$$

Where the sum of $(r_i, s_i)$ runs over the representations which are made from the product of $(p, q)$ and $(q, p)$. We introduce the following quantity

$$f_{\mu\nu\rho} f_{\mu\nu\sigma} = C_G \delta_{\rho\sigma}, \hfill (B.9)$$

where $C_G$ is a constant which assumes $C_G = 2$ for $SU(2)$ and $C_G = 3$ for $SU(3)$. With these preparations, we can calculate the two loop effective action almost the same way as the fuzzy sphere case.
We expand quantum fluctuations in terms of the harmonic matrices:

\[ a^\mu = \sum_{(r,s),m} a^{(r,s)}_m Y^{(r,s)}_m, \]

fermion \[ \varphi = \sum_{(r,s),m} \varphi^{(r,s)}_m Y^{(r,s)}_m, \]

anti-ghost \[ b = \sum_{(r,s),m} b^{(r,s)}_m Y^{(r,s)}_m, \]

ghost \[ c = \sum_{(r,s),m} c^{(r,s)}_m Y^{(r,s)}_m. \] (B.10)

Then the propagators are derived from the kinematic terms:

\[ \langle a^\mu a^\nu \rangle = \sum_{(r,s),m} \left( P^2 \delta_{\mu\nu} + 2i f_{\mu\nu\rho} P^\rho \right)^{-1} Y^{(r,s)}_m Y^{(r,s)}_m, \]

\[ \langle \varphi \bar{\varphi} \rangle = \sum_{(r,s),m} (-\Gamma_{\mu} P_{\mu})^{-1} Y^{(r,s)}_m Y^{(r,s)}_m, \]

\[ \langle cb \rangle = \sum_{(r,s),m} \frac{1}{P^2} Y^{(r,s)}_m Y^{(r,s)}_m. \] (B.11)

We exclude the singlet state \((0,0)\) in the propagator. To calculate the leading contributions in the large \(N\) limit, we expand the boson and the fermion propagators as

\[ \left( P^2 \delta_{\mu\nu} + 2i f_{\mu\nu\rho} P^\rho \right)^{-1} \simeq \frac{\delta_{\mu\nu}}{P^2} - \frac{2i f_{\mu\nu\rho} P^\rho}{P^4} + \frac{4 I_{\mu\nu}(P)}{P^6}, \]

\[ (-\Gamma_{\mu} P_{\mu})^{-1} \simeq \frac{\Gamma_{\mu} P_{\mu}}{P^2} + \frac{i f_{\mu\nu\rho} \Gamma^{\nu\rho} P_{\mu} P_{\rho}}{2 P^4}. \] (B.12)

We have introduced the following tensor

\[ I_{\mu\nu} \equiv f_{\tau\mu\rho} f_{\tau\nu\sigma} P_{\rho} P_{\sigma}. \] (B.13)

Using these propagators, we can calculate the contributions to the two loop effective action from various interaction vertices as follows.

4-gauge boson vertex is

\[ V_4 = -\frac{1}{4} Tr [a_\mu, a_\nu]^2. \] (B.14)

The leading contribution to the two loop effective action is

\[ <-V_4> = -45F_1 - 42C_G G_1 + 3C_G G_2. \] (B.15)

Here

\[ F_1 = \left< \frac{1}{P^4 P_2^4} \right>_P, \]
\[ \begin{align*}
G_1 &= \left\langle \frac{1}{P_1^2 P_2^2} \right\rangle_p, \\
G_2 &= \left\langle \frac{P_3}{P_1^2 P_2^4} \right\rangle_p. 
\end{align*} \tag{B.16} \]

Ghost vertex is
\[ V_g = Tr b [p_\mu, [a_\mu, c]]. \tag{B.17} \]

Their contribution is
\[ \frac{1}{2} < V_g V_g >= F_2 + 4H_2. \tag{B.18} \]

Here
\[ \begin{align*}
F_2 &= \left\langle \frac{P_2 \cdot P_3}{P_1^2 P_2^2 P_3^2} \right\rangle_p, \\
H_2 &= \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^4 P_2^2 P_3^2} \right\rangle_p, 
\end{align*} \tag{B.19} \]

and
\[ \begin{align*}
P_i \cdot I(j) \cdot P_k &= P_i^{\mu} I(\mu, P_j) P_k^\nu. 
\end{align*} \tag{B.20} \]

3-gauge boson vertex is
\[ V_3 = -Tr P_\mu a_\nu [a_\mu, a_\nu] . \tag{B.21} \]

Their contribution is
\[ \begin{align*}
\frac{1}{2} < V_3 V_3 > &= 9F_1 - 9F_2 + C_G (6F_3 + 2G_1 + G_2) \\
&+ 32H_1 - 36H_2 - 16H_3 + 12H_4 - 4H_5. 
\end{align*} \tag{B.22} \]

Newly introduced functions are defined as
\[ \begin{align*}
F_3 &= \left\langle \frac{1}{P_1^2 P_2^2 P_3^2} \right\rangle_p, \\
G_1' &= G_1 - \frac{1}{N} Tr \left[ \left( \frac{1}{P^2} \right)^3 \right], \\
H_1 &= \left\langle \frac{P_1 \cdot I(2) \cdot P_3}{P_1^2 P_2^2 P_3^2} \right\rangle_p, \\
H_3 &= \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^4 P_2^2 P_3^2} \right\rangle_p, \\
H_4 &= \left\langle \frac{P_1 \cdot I(2) \cdot P_3}{P_1^4 P_2^2 P_3^2} \right\rangle_p, \\
H_5 &= \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^2 P_2^2 P_3^4} \right\rangle_p. 
\end{align*} \tag{B.23} \]
In SU(3), we can evaluate the following quantity as

\[
\frac{1}{N} Tr \left[ \left( \frac{1}{p^2} \right)^3 \right] = \frac{1}{N} \sum_{(r,s),m} \frac{1}{2} \left( r + 1 \right) \left( s + 1 \right) \left( r + s + 2 \right) \left[ \frac{1}{2} \left( r(r+2) + s(s+2) \right) \right]^3. \tag{B.24}
\]

Fermion vertex is

\[
V_f = -\frac{1}{2} Tr \bar{\phi} \Gamma_\mu [a_\mu, \varphi]. \tag{B.25}
\]

Their contribution is

\[
\frac{1}{2} \langle V_f V_f \rangle = -64F_2 + (-8C_GG'_1 + 4C_GG_2 + 8C_GF_3 + 32H_4) \\
-16C_GF_3 + 48C_GG'_1 + -8C_GF_2 + 64H_2 + 64H_3. \tag{B.26}
\]

After summing up (B.15), (B.18), (B.22) and (B.26), we find the 2-loop effective action:

\[
\Gamma_{2-loop} = 2CGF_3 + 32H_1 + 32H_2 + 48H_3 + (12 + 32)H_4 - 4H_5 \\
= 2CGF_3. \tag{B.27}
\]

It is because

\[
H_1 + H_2 = 0, \\
H_3 + H_4 = 0, \\
H_3 - H_5 = 0. \tag{B.28}
\]

Since we have used the common properties of SU(2) and SU(3), the result (B.27) is valid for SU(2) and SU(3) and consistent with the fuzzy sphere’s results.

**Appendix C**

In this appendix, we calculate \( F_3 \) in (B.27) numerically. A practical way to calculate \( F_3 \) is to use Monte-Carlo simulation [9]. Our strategy is to construct a Gaussian matrix model to calculate it:

\[
F_3 = \left\langle \frac{1}{P_1^2P_2^2P_3^2} \right\rangle_p \\
= \int dadbdc Tr(abc) Tr(cba) \exp \left( -\frac{1}{2} [a, p^\mu]^2 - \frac{1}{2} [b, p^\mu]^2 - \frac{1}{2} [c, p^\mu]^2 \right). \tag{C.1}
\]

We can use the heat-bath algorithm to calculate this correlator. The result is shown in Table 1. We estimate the statical errors using a jackknife method [10, 9].
Table 1: The results of $F_3$ using Monte-Carlo simulation.

| $SU(3)$ rep. | $N$ | $F_3$      |
|--------------|-----|------------|
| (1,0)        | 3   | 0.69152 +/- 0.00056 |
| (2,0)        | 6   | 1.02763 +/- 0.00064 |
| (3,0)        | 10  | 1.15620 +/- 0.00069 |
| (4,0)        | 15  | 1.21168 +/- 0.00072 |
| (5,0)        | 21  | 1.23653 +/- 0.00072 |
| (6,0)        | 28  | 1.24858 +/- 0.00071 |
| (7,0)        | 36  | 1.25357 +/- 0.00073 |
| (8,0)        | 45  | 1.25474 +/- 0.00086 |
| (9,0)        | 55  | 1.25222 +/- 0.00091 |
| (10,0)       | 66  | 1.25201 +/- 0.00088 |
| (11,0)       | 78  | 1.25188 +/- 0.00091 |
| (12,0)       | 91  | 1.24959 +/- 0.00091 |

Table 2: The results of $F_3$ using the harmonic matrices.

| $SU(3)$ rep. | $N$ | $F_3$      |
|--------------|-----|------------|
| (1,0)        | 3   | 0.691358   |
| (2,0)        | 6   | 1.027320   |
| (3,0)        | 10  | 1.156321   |
| (4,0)        | 15  | 1.211689   |
| (5,0)        | 21  | 1.236921   |
| (6,0)        | 28  | 1.248420   |
The other way to calculate $F_3$ is to use the harmonic matrices. We can obtain these matrices on the computer using the method explained in appendix A. The result is shown in Table 2.

Since Table 2 shows the exact results, this calculation is preferable to the Monte-Carlo. But we have used the Monte-Carlo method, because the exact evaluation requires more computer power than the Monte-Carlo. Nevertheless we can use Table 2 to check Table 1. We can thus claim that the Monte-Carlo method gives the correct results.
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