ANTIHOLOMORPHIC INVOLUTIONS AND
SPHERICAL SUBGROUPS OF REDUCTIVE GROUPS

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Abstract. We investigate qualitative and quantitative properties of anti-holomorphic involutions on some spherical varieties.

The classification of anti-holomorphic involutions of connected reductive complex algebraic groups $G$ is well-established. In this article, we study restrictions of such mappings onto subgroups of $G$. More precisely, we are interested in the subgroups of $G$ called spherical - groups including e.g. parabolic and symmetric subgroups of $G$.

Spherical subgroups of a given connected reductive group can be classified by means of combinatorial objects (see [Lun, Lo, BP, CF]). In terms of these invariants, we establish existence and uniqueness criteria about anti-holomorphic involutions on spherical homogeneous spaces, extending results obtained in [ACF, A2].

As proved by Luna in [Lun], the classification of all spherical subgroups of $G$ relies only on that of the so-called spherically closed subgroups of $G$. For these latter, the aforementioned combinatorial classification is really convenient since the involved objects are built on the Dynkin diagram of $G$. After having proved that this class of subgroups $H$ of $G$ is preserved by anti-holomorphic involutions $\sigma$ of $G$, we study qualitative and quantitative properties of so-called $\sigma$-equivariant real structures on $G/H$ and their wonderful compactifications.

As in [A2], a peculiar automorphism of Dynkin diagram plays a central role in the present work; we recall its definition and properties in Section 1.

Once the basic material concerning spherical subgroups of $G$ and their invariants is recalled, we study, in Section 2, properties of $\sigma$-conjugates of spherical subgroups of $G$. In particular, we show that a subgroup $H$ of $G$ is spherically closed (resp. wonderful) if and only if so is $\sigma(H)$; see Proposition 2.5 (resp. Proposition 2.13). Theorem 2.11 enables us to decide whether a spherical subgroup $H$ of $G$ is conjugate...
to $\sigma(H)$; if the latter happens to hold, we are able to prove a uniqueness statement concerning $\sigma$-equivariant real structure on $G/H$; see Proposition 2.8.

In Section 3, we investigate how $\sigma$-equivariant real structures are carried over through geometrical operations like Cartesian products, parabolic inductions.

In the last section, we focus on the so-called wonderful compactifications $X$ of spherically closed homogeneous spaces. Theorem 4.19 provides a criterion for a $\sigma$-equivariant real structure to exist on $X$. As an application of this result, we obtain, in particular, that (almost) all primitive self-normalizing spherical subgroups $H$ of $G$ are conjugate to $\sigma(H)$; see Theorem 4.21. Finally, we study the real parts of wonderful varieties equipped with a $\sigma$-equivariant real structure (Theorem 4.25) and we conclude our work by presenting several examples illustrating how these loci can be diverse and various.

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0.1. Notation. Let $G$ be a complex connected reductive group and $\sigma$ be an anti-holomorphic involution of $G$.

We fix a Borel subgroup $B$ of $G$ and we choose a maximal torus $T \subset B$ stable by $\sigma$. We denote the related set of simple roots by $S$ and the Weyl group $N_G(T)/T$ by $W$. Let $\mathcal{X}(T)$ be the character group of $T$. Then $\sigma$ defines an automorphism $\sigma^\text{T}$ on $\mathcal{X}(T)$ while setting

$$\sigma^\text{T}(\chi) = \overline{\chi \circ \sigma} \quad \text{for } \chi \in \mathcal{X}(T).$$

Given a representation $(V, \rho)$ of $G$, we denote the corresponding $\sigma$-twisted module by $V^\sigma$. Specifically, $V^\sigma$ is given by the complex conjugate $\overline{V}$ of $V$ equipped with the $G$-module structure

$$g \mapsto \overline{\rho(\sigma(g))} \quad \text{for any } g \in G.$$
Consider the subgroup of $W$ generated by the simple reflections associated to the elements of $S_1$ and let $w_\bullet$ denote its element of maximal length. Following [A1], we set 

$$
\varepsilon_\sigma(\alpha) = \begin{cases} 
\omega(\alpha) & \text{if } \alpha \in S_1 \\
-w_\bullet(\alpha) & \text{if } \alpha \in S_0.
\end{cases}
$$

**Theorem 1.1 ([A1]).** (i) The map $\varepsilon_\sigma$ is an automorphism of $S$. Further, it is induced by a self-map on $X(T)$ (still denoted by $\varepsilon_\sigma$).

(ii) If $n_\bullet \in N_G(T)$ represents $w_\bullet$ then $n_\bullet \sigma(B)n_\bullet^{-1} = B$.

(iii) If $V$ is a simple $G$-module of highest weight $\lambda$ then $V^\sigma$ is also a simple $G$-module; its highest weight equals $\varepsilon_\sigma(\lambda)$. Further, 

$$
\varepsilon_\sigma(\lambda) = w_\bullet(\sigma^T(\lambda)).
$$

**Proof.** See more precisely Theorem 3.1 and its proof in loc. cit.. \endproof

2. $\sigma$-conjugates of spherical subgroups of $G$

In the following, $H$ denotes a spherical subgroup of $G$. Without loss of generality, we assume that $BH$ is open in $G$.

Set 

$$
\mu_\sigma : G/H \longrightarrow G/\sigma(H), \quad gH \longmapsto \sigma(g)\sigma(H).
$$

2.1. As usual, we call the set of $B$-stable prime divisors of $G/H$ the set colors of $G/H$ and denote it by $\mathcal{D}$.

Consider the natural epimorphism $\pi : G \rightarrow G/H$. Given $D \in \mathcal{D}$, $\pi^{-1}(D)$ is a $B$-stable prime divisor of $G$. We can and do assume now that $G$ is simply connected. Then there exists a unique $B$-eigenfunction $f_D$ in $\mathbb{C}[G]$ defining $\pi^{-1}(D)$ and such that $f_D$ equals 1 on the center of $G$. We define the $B$-weight $\omega_D$ of the color $D$ to be the $B$-weight of $f_D$.

**Lemma 2.2.** There are at most two colors of $G/H$ with the same $B$-weight.

**Proof.** Let $B \subsetneq P$ be a minimal parabolic subgroup of $G$. By [Lu1] (see precisely Section 1.4 therein), the set of colors $D \in \mathcal{D}$ such that $P \cdot D \neq D$ is of cardinality at most 2. Observe that $P \cdot D \neq D$ if and only if $(\omega_D, \alpha) \neq 0$ where $\alpha$ stands for the simple root of the Levi subgroup of $P$ containing $T$. The lemma follows. \endproof

For later use, we gather some obvious remarks in the following lemma.

**Lemma 2.3.**

(1) The subgroup $\sigma(H)$ of $G$ is spherical.

(2) The map $\mu_\sigma$ yields a bijection between the $B$-colors of $G/H$ and the $\sigma(B)$-colors of $G/\sigma(H)$. 

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(3) Let $D$ be a $\sigma(B)$-color of $G/\sigma(H)$. The function $\overline{f_D} \circ \sigma$ defines a $B$-stable prime divisor of $G$ that is, the color $\sigma^{-1}(D)$ of $G/H$.

Proof. Note that $\pi \circ \mu_\sigma = \sigma \circ \pi_\sigma$ with $\pi_\sigma : G \to G/\sigma(H)$ being the natural projection. The lemma follows readily. □

2.2. Let $N_G(H)$ be the normalizer of $H$ in $G$. The group of $G$-automorphisms of $G/H$ can be identified to $N_G(H)/H$. The group $N_G(H)$ thus acts naturally on the set $\mathcal{D}$ of colors of $G/H$; this action reads on the functionals $f_D$, $D \in \mathcal{D}$, as the right regular action.

Lemma 2.4. Let $n \in N_G(H)$ and $\varphi_n$ denote the corresponding transform on $\mathcal{D}$. Then $\varphi_n^2$ acts trivially on $\mathcal{D}$.

Proof. Once noticed that $\varphi_n$ permutes the colors of same $B$-weight, one thus concludes by invoking Lemma 2.2. □

The spherical closure $\overline{H}$ of $H$ is defined as the subgroup of $N_G(H)$ which fixes the set $\mathcal{D}$ pointwise; the group $\overline{H}$ is a spherical subgroup of $G$ containing $H$. The group $H$ is called spherically closed if $H = \overline{H}$. By [Lu1] (see more precisely Section 6 therein), the classification of all spherical subgroups of $G$ relies on that of spherically closed subgroups of $G$.

Example 1. (1) Let $G = SL_2$. The $G$-variety $G/T$ is isomorphic to the complement of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$; it is spherical and has two colors, both of weight equals the fundamental weight. The normalizer of $T$ in $G$ exchanges these two colors; since $N_G(T)/T \simeq \mathbb{Z}_2$, the subgroup $T$ of $G$ is spherically closed.

(2) Let $G = SO_{2n+1}$ and $H$ be the stabilizer of a non-isotropic line in $\mathbb{P}^{2n}$. Then $G/H$ is spherical and has only one color. It follows that $H \neq \overline{H} = N_G(H)$.

Proposition 2.5. Let $H$ be a spherically closed subgroup of $G$. The subgroup $\sigma(H)$ of $G$ is spherically closed.

Proof. Let us first remark that the normalizer of $\sigma(H)$ in $G$ equals $\sigma(N_G(H))$. Consider now an element $n \in \sigma(N_G(H))$ which fixes every $\sigma(B)$-color of $G/\sigma(H)$. We shall prove that $\sigma(n) \in H$.

Let $f_D \in \mathbb{C}[G]$ define the equation of $\pi_\sigma^{-1}(D)$ with $D$ being a $\sigma(B)$-color of $G/\sigma(H)$. By assumption on $n$, we have: $n \cdot f_D = f_D$. It follows
that
\[(\sigma(n) \cdot (f_D \circ \sigma))(g) = (f_D \circ \sigma)(g \sigma(n)) = f_D(\sigma(g)n) = (n \cdot f_D)(\sigma(g)) = f_D(\sigma(g)) = (f_D \circ \sigma)(g).\]

As stated in Lemma 2.3, we have: 
\[f_D \circ \sigma = f_{\sigma^{-1}(D)}\]
and the set \(D\) consists of the elements \(\sigma^{-1}(D)\) with \(D\) a \(\sigma(B)\)-color of \(G/\sigma(H)\). We thus obtain that \(\sigma(n) \in N_G(H)\) fixes every color of \(G/H\) and since \(H\) is supposed to be spherically closed, this gives as desired: \(\sigma(n) \in H\).

\[\square\]

**Corollary 2.6.** Let \(H\) be a spherically closed subgroup of \(G\). Suppose there exists \(a \in G\) such that \(\sigma(H) = aHa^{-1}\). Let
\[\mu : G/H \rightarrow G/H, \quad gH \mapsto \sigma(g)aH.\]
If \(\sigma\) defines a split real form of \(G\) then \(\mu\) is involutive.

**Proof.** In the case under consideration, the map \(\mu\) induces a bijection on \(D\). Specifically,
\[f_D = f_{\mu(D)} \circ r_a \circ \sigma\]
with \(r_a : G \rightarrow G\) being the right multiplication by \(a\). This implies that
\[\omega_D = \omega_{\mu(D)} \circ \sigma.\]
But \(\omega_{\mu(D)} \circ \sigma = \omega_{\mu(D)}\); see e.g. Theorem 1.1 (iii).

As a consequence, a color of \(G/H\) is mapped by \(\mu\) to a color of \(G/H\) of same \(B\)-weight hence \(\mu^2\) fixes every color of \(G/H\) by Lemma 2.2. Since \(\sigma\) is an involution of \(G\), we have: \(H = \sigma^2(H) = \sigma(a)Ha^{-1}\sigma(a)^{-1}\) and in turn \(\sigma(a)a \in N_G(H)\). From \(\overline{H} = H\), it thus results that \(\sigma(a)a \in H\).
The corollary follows.

\[\square\]

**Example 2.** Let \(G = SL_2\) and \(H = T = \overline{T}\). The map \(\mu\) stated in Corollary 2.6 is clearly an involution for every \(\sigma\) since \(\sigma(T) = T\).

Our next aim is to extend Corollary 2.6 to any involution \(\sigma\). To this end, we shall study a further special class of spherical subgroups of \(G\).

2.3. Throughout this paragraph, \(H\) denotes a spherically closed subgroup of \(G\).

Let \(H^\#\) denote the intersection of the kernels of all characters of \(H\).

The group \(H^\#\) was introduced by Luna in [Lun]; in the following, we gather some of its properties. See Section 6 in loc. cit. for details.

\[\text{This statement generalizes the first assertion of Theorem 4.12 in [ACF]}\]
The quotient $H/H^\#$ is diagonalizable and

$$\mathcal{X}(H) = \mathcal{X}(H/H^\#).$$

Moreover, $G/H^\#$ is a quasi-affine spherical $G \times H/H^\#$-variety;

$$H = H^\#;$$

the sets of colors of $G/H$ and $G/H^\#$ are in bijective correspondence and the simple decomposition of its coordinate ring as a $G \times H/H^\#$-module is given in terms of the functionals $f_D$, $D \in \mathcal{D}$. By abuse of notation, let $D$ also denote the $B \times H/H^\#$-weights $(\omega_D, \chi_D)$ of the $f_D$'s. We thus have:

$$\mathbb{C}[G/H^\#] = \bigoplus_{\lambda \in \mathbb{N}} V(\lambda)$$

where $V(\lambda)$ denotes the simple $G \times H/H^\#$-module of weight $\lambda$.

**Lemma 2.7.** Under the assumption $\sigma(H) = aHa^{-1}$ with $a \in G$, the following assertions hold.

1. $\sigma(H)^\# = aH^\#a^{-1}$.
2. Let $\mu^\#: G/H^\# \to G/H^\#$, $gH^\# \mapsto \sigma(g)aH^\#$. Then 
   $$(f_D \circ \mu^\#) \circ \mu^\# = f_D \quad \text{ for all } D \in \mathcal{D}.$$ 

**Proof.** By definition, $\sigma(H)^\#$ is the intersection of the kernels of all characters of $\sigma(H)$. Obviously, there is a bijection between the characters of $H$ and those of $\sigma(H)$. Given a character $\chi$ of $\sigma(H)$, let $\chi_H$ denote the corresponding character of $H$. Observe that the kernel of $\chi$ equals $a(ker \chi_H)a^{-1}$, the first assertion follows.

To prove the second assertion, let us consider the $B \times H/H^\#$-weight of $(f_D \circ \mu^\#) \circ \mu^\#$. Straightforward computations show that this character equals $(\omega_D, \chi_D)$. We can thus invoke the multiplicity freeness of the $B \times H/H^\#$-module $\mathbb{C}[G/H^\#]$ to conclude. \hfill \Box

**Proposition 2.8.** Let $H \subset G$ be spherical and $\sigma(H) = aHa^{-1}$ with $a \in G$. If further $H \subset G$ is spherically closed then the map

$$\mu : G/H \longrightarrow G/H, \quad gH \mapsto \sigma(g)aH$$

is the unique $\sigma$-equivariant real structure on $G/H$ (up to automorphism of $G/H$).

**Proof.** The map $\mu^2$ defines a $G$-equivariant automorphism of $G/H$ hence yields a bijection of $\mathcal{D}$. Note that $(\mu^\#)^2(gH^\#) = g\sigma(a)aH^\#$ and $\sigma(a)a \in N_G(H^\#)$ since $\sigma$ is an involution of $G$; see the end of the proof of Corollary 2.6 for similar remark and justification.

By the above lemma, $(\mu^\#)^2$ fixes the set $\mathcal{D}$ pointwise; this implies that $\sigma(a)a$ is an element of the spherical closure of $H^\#$ - which is $H$. It follows that $\sigma(a)a \in H$ and in turn $\mu^2$ is the identity map on $G/H$. 

To prove the uniqueness assertion, we consider a further $\sigma$-equivariant real structure on $G/H$, say $\mu'$. The map $\mu \circ \mu'$ is thus a $G$-automorphism of $G/H$ hence it is given by an element, say $n$ of $N_G(H)$. Arguing similarly as in the proof of Lemma 2.7, we can show that $n \in H$. This yields as desired that $\mu \circ \mu'$ is the identity map on $G/H$. □

2.4. Let now $H$ be any spherical subgroup of $G$.

Following [LV], to $G/H$ we attach three combinatorial invariants (so called Luna-Vust invariants): its set of colors $D = D(G/H)$, its weight lattice $\mathcal{X} = \mathcal{X}(G/H)$ and its valuation cone $\mathcal{V} = \mathcal{V}(G/H)$. The lattice $\mathcal{X}$ consists of the $B$-weights of the function field $\mathbb{C}(G/H)$ of $G/H$; the valuation cone $\mathcal{V}$ is the set of $G$-invariant $\mathbb{Q}$-valued valuations of $\mathbb{C}(G/H)$.

Any valuation $v$ defines a homomorphism 

$$\mathbb{C}(G/H) \rightarrow \mathbb{Q}, \quad \rho : f \mapsto v(f)$$

and in turn $v$ induces an element $\rho_v$ of $\mathcal{V} := \text{Hom}(\mathcal{X}(G/H), \mathbb{Q})$; see loc. cit. for details. This yields in particular two maps:

$$\mathcal{V} \rightarrow V, \quad v \mapsto \rho_v \quad \text{and} \quad D \mapsto \rho_D$$

where $\rho_D := \rho_{v_D}$ and $v_D$ is the valuation of the divisor $D$.

The first map happens to be injective hence we can and do regard $\mathcal{V}$ within $V$. The second map may not be injective; the set $\mathcal{D}$ is thus equipped with the map $\mathcal{D} \rightarrow V$ together with an additional map $D \mapsto G_D$ with $G_D \subseteq G$ being the stabilizer of the color $D$.

By $\mathcal{D}(G/H_1) = \mathcal{D}(G/H_2)$, we just mean that there is a bijection $\varphi : \mathcal{D}(G/H_1) \rightarrow \mathcal{D}(G/H_2)$ such that $\rho_D = \rho_{\varphi(D)}$ and $G_D = G_{\varphi(D)}$ for every $D \in \mathcal{D}(G/H_1)$.

A spherical homogeneous space is uniquely determined (up to $G$-isomorphism) by its Luna-Vust invariants; see Losev’s results in [Lo]. In case $G/H$ is affine, these three invariants can be replaced by a single one: the weight lattice $\Gamma = \Gamma(G/H)$, that is the set given by the highest weights of the coordinate ring of $G/H$ considered as a $G$-module; see again [Lo].

Thanks to [A2], the relations between the Luna-Vust invariants of $G/H$ and those of $G/\sigma(H)$ are well-understood. Specifically, we have the following description involving the automorphism $\varepsilon_\sigma$ of $S$ (see Section 1 for recollection of its definition).

**Lemma 2.9.** If $H$ is a spherical subgroup of $G$ then

1. $\mathcal{X}(G/\sigma(H)) = \varepsilon_\sigma(\mathcal{X})$,
2. $\mathcal{V}(G/\sigma(H)) = \varepsilon_\sigma(\mathcal{V})$ and

\[\text{See also [H] for analogy.}\]
\(3\) \(\mathcal{D}(G/\sigma(H)) = \{\mu_\sigma(n_{\bullet}D) : D \in \mathcal{D}\}\) equipped with the maps
\[
\mu_\sigma(n_{\bullet}D) \mapsto \varepsilon_\sigma(\rho_D) \quad \text{and} \quad \mu_\sigma(n_{\bullet}D) \mapsto n_{\bullet}\sigma(G_D)n_{\bullet}^{-1}
\]
with \(n_{\bullet}\) being a representative in \(N_G(T)\) of \(w_{\bullet}\).

\(4\) If \(H\) is also reductive then \(\Gamma(G/\sigma(H)) = \varepsilon_\sigma(\Gamma)\).

**Proof.** The three first assertions stem from Proposition 5.2, Proposition 5.3 and Proposition 5.4 in [A2] resp. whereas the fourth one follows from Theorem 2.1 in [A1] (see also the first paragraph of the proof of Theorem 6.1 therein).

**Definition 2.10.** The set \(\mathcal{D}\) of colors of a spherical homogeneous space \(G/H\) is called \(\varepsilon_\sigma\)-stable if for every \(D \in \mathcal{D}\), there exists \(D' \in \mathcal{D}\) (depending on \(D\)) such that
\[
\varepsilon_\sigma(\rho_D) = \rho_{D'} \quad \text{and} \quad n_{\bullet}\sigma(G_D)n_{\bullet}^{-1} = G_{D'}.
\]

**Theorem 2.11.** Let \(H\) be a spherical subgroup of \(G\). The subgroups \(H\) and \(\sigma(H)\) of \(G\) are conjugate whenever one of the following situations occurs.

- (1) The combinatorial invariants of \(G/H\) are \(\varepsilon_\sigma\)-stable.
- (2) The group \(H\) is reductive and the weight monoid of \(G/H\) is \(\varepsilon_\sigma\)-stable.

**Proof.** Thanks to the aforementioned Losev’s results, it suffices to prove that \(\mathcal{X} = \mathcal{X}(G/\sigma(H)), \mathcal{V} = \mathcal{V}(G/\sigma(H))\) and \(\mathcal{D} = \mathcal{D}(G/\sigma(H))\) (in the sense recalled above). Remark that \(D \mapsto \mu_\sigma(n_{\bullet}D)\) defines a bijection between \(\mathcal{D}(G/H)\) and \(\mathcal{D}(G/\sigma(H))\). The required equalities are thus given by our assumption of \(\varepsilon_\sigma\)-stability together with Lemma 2.9.

**Example 3.** Let \(H = B^-\) with \(B^-\) being the Borel subgroup of \(G\) opposite to \(B\). First, recall that since \(T\) is \(\sigma\)-stable, \(\sigma(B^-)\) is conjugate to \(B^-\). Secondly, \(\mathcal{X} = \mathcal{X} = \{0\}\) and \(\mathcal{D} = \{Bs_\alpha B^-/B^- : \alpha \in S\}\) with \(s_\alpha \in W\) being the simple reflection associated to \(\alpha\). These invariants are clearly \(\varepsilon_\sigma\)-stable; note that \(\mathcal{D}\) may not be fixed by \(\varepsilon_\sigma\) (for instance in the case of the quasi-split but non-split real form in type A).

**Remark 2.12.** In Proposition 5.4 in [A2], the \(\varepsilon_\sigma\)-stability assumption on \(\mathcal{D}\) is replaced by the stronger conditions:
\[
\varepsilon_\sigma(\rho_D) = \rho_D \quad \text{and} \quad n_{\bullet}\sigma(G_D)n_{\bullet}^{-1} = G_D.
\]
This condition leaves aside many cases as the preceding example shows.
2.5. We now turn to another class of spherical groups: wonderful subgroups of $G$. First, let us gather freely from [LV] notions and results on combinatorial invariants of spherical homogeneous spaces; for a survey of this material, one may also consult Section 30 in [T].

Equivariant embeddings of $G/H$ are classified by a finite family of couples $(C, F)$, subject to restrictions, with $C$ being a finitely generated strictly convex cone in $V$ and $F$ being a subset of $D$. In case $V$ is strictly convex (equivalently, if $N_G(H)/H$ is finite), the couple $(V, \emptyset)$ is admissible and thus corresponds to an equivariant embedding of $G/H$. This embedding is complete; it is called the canonical embedding of $G/H$.

The spherical subgroup $H \subset G$ is called wonderful if $N_G(H)/H$ is finite and its canonical embedding is smooth. Spherically closed subgroups of $G$ and normalizers of spherical subgroups of $G$ are wonderful subgroups of $G$; see [K].

Proposition 2.13. Let $H$ be a spherical subgroup of $G$. Then $H$ is wonderful if and only if $\sigma(H)$ is a wonderful subgroup of $G$.

Proof. As stated in Lemma 2.3, $\sigma(H) \subset G$ is spherical. By Lemma 2.9, $V(G/\sigma(H)) = \varepsilon_\sigma(V(G/H))$ and $V(G/\sigma(H)) = \varepsilon_\sigma(V(G/H))$. It follows that the valuation cone of $G/\sigma(H)$ is strictly convex if so is $V(G/H)$ and vice versa. Finally, the canonical embedding (whenever it exists) of any spherical $G/H'$ is smooth if and only if $V(G/H')$ is generated by a basis of $V(G/H')$; see Section 4 in [B]. This criterion allows to conclude.

In Section 3, we will continue our study of (involutive) $\sigma$-equivariant anti-holomorphic maps on wonderful homogeneous spaces (and even on their canonical embeddings). In order to do so, we first need to discuss qualitative results on such mappings.

3.

We investigate now how real structures and the related real parts (definition recalled right below) are carried over through geometrical operations on varieties: Cartesian product, parabolic induction.

Given a real structure $\mu$ on a complex manifold $X$, its corresponding real part $X^\mu$ is defined as follows:

$$X^\mu = \{ x \in X : \mu(x) = x \}.$$

3.1. Let $X_1$ and $X_2$ be two anti-holomorphically diffeomorphic complex manifolds, each equipped with a real structure $\mu_1$ and $\mu_2$ respectively. It is well-known that their Cartesian product inherits many
real structures: the direct product \((\mu_1, \mu_2)\) or the maps \((x_1, x_2) \mapsto (\tau^{-1}(x_2), \tau(x_1))\) defined up to an anti-holomorphic diffeomorphism \(\tau\) between \(X_1\) and \(X_2\).

3.2. Let \(P\) be any parabolic subgroup of \(G\) and let \(P = P^uL\) its Levi decomposition with \(L\) being the Levi subgroup of \(P\) containing \(T\). Given a \(L\)-variety \(X'\), one considers the fiber product \(X := G \times_P X'\) with \(P^u\) acting trivially on \(X'\). The variety \(X\) is usually called a parabolic induction of \(X'\); it is a \(G\)-variety with the natural action of \(G\).

**Remark 3.14.** Let \(P_1\) and \(P_2\) be parabolic subgroups of \(G\) such that \(P_1 \cap P_2\) contains a Levi subgroup \(L\) of both \(P_1\) and \(P_2\). Then there exists \(n \in G\) such that \(P_2 = nP_1n^{-1}\) and \([g, x] \mapsto [gn, x]\) defines an isomorphism between the \(G\)-varieties \(G \times_{P_1} X'\) and \(G \times_{P_2} X'\).

**Lemma 3.15.** Let \(P\) be a parabolic subgroup of \(G\) such that \(\sigma(P) = nPn^{-1}\) with \(n \in G\). Suppose further that the Levi factor \(L\) of \(P\) containing \(T\) is \(\sigma\)-stable. Let \(X'\) be a \(L\)-variety equipped with a \(\sigma_L\)-equivariant anti-holomorphic map \(\mu'\) (with \(\sigma_L\) being the restriction of \(\sigma\) onto \(L\)). Then

\[
(1) \quad G \times_P X' \longrightarrow G \times_P X', \quad [g, x] \longmapsto [\sigma(g)n, \mu'(x)]
\]

defines a \(\sigma\)-equivariant anti-holomorphic diffeomorphism.

**Proof.** First note that \(\sigma(P)\) is a parabolic subgroup of \(G\); it contains the Borel subgroup \(\sigma(B)\) of \(G\). Since \(\sigma(L) = L\), we can consider the parabolic inductions \(G \times_P X'\) and \(G \times_{\sigma(P)} X'\). In particular, we let the unipotent radicals \(P^u\) and \(\sigma(P)^u\) of \(P\) and \(\sigma(P)\) resp. act trivially on \(X'\). From \(\sigma(P)^u = \sigma(P)^u\), we derive for any \((p = p^ul, x) \in P^uL \times X'\) the following equalities \(\mu'(\sigma(p)x) = \mu'(\sigma(l)x) = l\mu'(x) = p\mu'(x)\).

As a consequence, the assignment \((g, x) \mapsto (\sigma(g), \mu'(x))\) defines an anti-holomorphic map from \(G \times_P X'\) to \(G \times_{\sigma(P)} X'\). Moreover, the subgroups \(P\) and \(\sigma(P)\) of \(G\) being assumed to be conjugate, the \(G\)-varieties \(G \times_P X'\) and \(G \times_{\sigma(P)} X'\) are isomorphic; see the remark above. The lemma follows. \(\square\)

**Proposition 3.16.** Let \(X'\) and \(X = G \times_P X'\) satisfy the properties stated in Proposition \([3.13]\). Suppose also that \(X\) is equipped with the diffeomorphism stated in \([4]\). If \(X\) contains fixed points w.r.t, so does \(G/P\) w.r.t \(gP \mapsto \sigma(g)P\).
Proof. By assumption, \( \sigma(P) = nPn^{-1} \) for some \( n \in G \). Let \( x = (g, z) \in X = G \times_{P} X' \) be a fixed point, we thus get: \( \sigma(g)n = gp^{-1} \). This implies that \( gP \) is a real point of \( G/P \) with respect to the real structure \( gP \mapsto \sigma(g)nP \). The proposition follows. \( \square \)

4. Wonderful varieties

We shall now be concerned with a particular class of spherical varieties: the wonderful varieties. Wonderful \( G \)-varieties are classified by combinatorial objects supported on the Dynkin diagram of \( G \) called spherical systems. The purpose of this section is to establish existence criteria of \( \sigma \)-equivariant real structures as well as quantitative properties of real loci of wonderful varieties in terms of these invariants and the automorphism \( \varepsilon_{\sigma} \) of the Dynkin diagram of \( G \).

4.1. Basic material. The canonical embedding of a wonderful homogeneous space can be intrinsically defined. Specifically, by a theorem of [Lu1], a smooth complete \( G \)-variety \( X \) is a smooth canonical embedding of a spherical homogeneous space if and only if

1. \( X \) contains an open \( G \)-orbit \( X_{G}^{o} \);
2. the complement \( X \setminus X_{G}^{o} \) consists of a finite union of prime divisors \( D_{1}, \ldots, D_{r} \) with normal crossings;
3. two points of \( X \) are on the same \( G \)-orbit if (and only if) they are contained in the same \( D_{i} \)'s.

We call a smooth complete \( G \)-variety wonderful if it satisfies the aforementioned properties (1), (2) and (3).

As mentioned above, wonderful subgroups of \( G \) (and in turn wonderful \( G \)-varieties) can be classified by more convenient invariants than the Luna-Vust invariants. Let us recall how they are defined by Luna [Lu1].

One may consult also [T] for a survey.

Let \( X \) be a wonderful \( G \)-variety. Equivalently, consider a wonderful subgroup \( H \) of \( G \) and denote as previously its Luna-Vust invariants by \( \mathcal{X}, V, \mathcal{V}, \mathcal{D} \). The cone \( \mathcal{V} \) being strictly convex and simplicial, it can be defined by inequalities. More precisely, there exists a set \( \Sigma_{X} \) of linearly independent primitive elements such that

\[
\mathcal{V} = \{ v \in V : v(\gamma) \leq 0, \ \forall \gamma \in \Sigma_{X} \}.
\]

The set \( \Sigma_{X} \) is called the set of spherical roots of \( X \) (or \( G/H \)); it forms a basis of \( V \) and, in turn, it also determines \( \mathcal{X} \) entirely. Consider now the set of colors \( \mathcal{D} \). Let

\[
P_{X} = \bigcap_{D \in \mathcal{D}} G_{D}.
\]
Obviously, $P_X$ is a parabolic subgroup of $G$ containing $B$; let thus $S^p_X$ be the set of simple roots associated to $P_X$.

Finally, the third datum $A_X$ attached to $X$ is a subset of $V$. Given $\alpha \in \Sigma_X \cap S$, let

$$A_X(\alpha) = \{ \rho_D : D \in D \text{ and } P_\alpha \cdot D \neq D \} \subset V$$

where $B \subset P_\alpha$ stands for the parabolic subgroup of $G$ associated to $\alpha$. Recall that the $\rho_D$ may not be distinct; we thus regard the set $A_X(\alpha)$ as a multi-set. The set $A_X$ is defined as the union of the $A_X(\alpha)$’s with $\alpha \in \Sigma_X \cap S$.

The triple $(S^p_X, \Sigma_X, A_X)$ is called the spherical system of $X$ (or $G/H$).

Wonderful $G$-varieties are uniquely determined (up to $G$-isomorphism) by their spherical systems; see [Lo, CF].

4.2. Existence criterion for real structures.

4.2.1. Given $\sigma$, recall the definition of the associated automorphism $\varepsilon_\sigma$ of $S$ as well as its properties stated in Section 1.

**Definition 4.17.** A spherical system $(S^p, \Sigma, A)$ of $G$ is called $\varepsilon_\sigma$-stable if the sets $S^p, \Sigma$ and $A$ are stable by $\varepsilon_\sigma$.

**Lemma 4.18.** Let $H \subset G$ be wonderful with spherical system $(S^p, \Sigma, A)$. Then $\sigma(H) \subset G$ is wonderful and its spherical system is the triple $(\varepsilon_\sigma(S^p), \varepsilon_\sigma(\Sigma), \varepsilon_\sigma(A))$.

**Proof.** By Proposition 2.13, $\sigma(H) \subset G$ is wonderful. The assertion on the spherical systems follows readily from Lemma 2.9 and the recalls made at the beginning of this section. □

**Theorem 4.19.** Let $X$ be a wonderful $G$-variety with spherical system $\mathcal{S}$. There exists a $\sigma$-equivariant real structure on $X$ if and only if $\mathcal{S}$ is $\varepsilon_\sigma$-stable.

**Proof.** Suppose $\mathcal{S}$ is $\varepsilon_\sigma$-stable. Then by Proposition 2.11, we have: $\sigma(H) = aHa^{-1}$ for some $a \in G$ and, in turn, the mapping

$$\mu : G/H \to G/H, \quad gH \mapsto \sigma(g)aH$$

is well-defined. Moreover, thanks to the uniqueness of the wonderful embedding, $\mu$ can be extended to the whole $X$; see e.g. [ACF] for details.

Note that a $\sigma$-equivariant real structure $m$ on $X$ yields in particular a $\sigma$-equivariant real structure on the open $G$-orbit of $X$. The converse thus stems from Lemma 4.18 and Theorem 2.1 in [A2]. □
A spherical system of $G$ is called spherically closed if the corresponding subgroup of $G$ is spherically closed.

**Corollary 4.20.** Let $X$ be a wonderful $G$-variety with spherical system $\mathcal{S}$. If $\mathcal{S}$ is spherically closed and $\varepsilon_\sigma$-stable then $X$ can be uniquely (up to an automorphism of $X$) equipped with a $\sigma$-equivariant real structure.

**Proof.** Once we apply Theorem 4.19, we are left to prove that the map $\mu$ is involutive. This desired property of $\mu$ is fulfilled whenever $H$ is spherically closed thanks to Proposition 2.6. □

A wonderful $G$-variety $X$ is called primitive if it is neither parabolically induced nor the fiber product of wonderful varieties, meaning that $X$ cannot be written as $X_1 \times X_2 \times X_3$ with $X_i$ ($i = 1, 2, 3$) being a wonderful $G$-variety. By analogy, we call a wonderful subgroup $H \subset G$ primitive if its canonical embedding is primitive.

**Theorem 4.21.** Let $H \subset G$ be a primitive self-normalizing spherical subgroup of $G$. Then the subgroups $H$ and $\sigma(H)$ of $G$ are conjugate as soon as $(G, H, \sigma)$ is not one of the following triples.

1. $(SO_{4n}, N_G(GL_{2n}), \sigma)$;
2. $(SO_8, \text{Spin}_7, \sigma)$;
3. $(SO_8/SL_2 \cdot Sp_4, \sigma)$

where $\sigma$ defines the real form $SO_{p,q}$ with $p \leq q$ and $p, q$ odd.

**Proof.** We apply the criterion stated in Theorem 4.19. To this end, we consult the list of the groups $H \subset G$ under consideration and their spherical systems given in [BCE]. Recall the definition of the automorphism $\varepsilon_\sigma$ stated in Section 1 for convenience, one may also consult Table 5 in [O] where $\varepsilon_\sigma$ together with the Satake diagrams are given. The proof is thus reduced to check case-by-case which spherical systems under consideration are $\varepsilon_\sigma$-stable for a given $\sigma$.

We end up with the spherical systems numbered as (34), (36) and (37) in [BCE], that is, with the groups $(SO_8/SL_2 \cdot Sp_4), (SO_8, \text{Spin}_7)$ and $(SO_{4n}, N_G(GL_{2n}))$. Indeed, their spherical roots are $\{2\alpha_1, 2\alpha_2, \alpha_3+\alpha_4\}$, $\{2\alpha_1+2\alpha_2+\alpha_3+\alpha_4\}$ and $\{\alpha_1+2\alpha_2+\alpha_3, \ldots, \alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}, 2\alpha_n\}$ respectively. Note that these sets are not $\varepsilon_\sigma$-stable if $\sigma$ is the involution stated in the theorem. □

**Corollary 4.22.** Let $G$ be a simple group and $X$ be an affine homogeneous spherical $G$-variety with weight monoid $\Gamma$. Let $d_0$ denote the codimension of a generic $G_0$-orbit on $X$. If $X$ can be equipped with a $\sigma$-equivariant real structure then $d_0 = \text{rk} \Gamma$. 

Proof. We proceed by contradiction: suppose that $d_0 \neq \text{rk} \Gamma$. Write $X = G/H$. The triples $(G, H, \sigma)$ are given in Theorem 6.4 in [A1]. One thus observes that $(G, N_G(H), \sigma)$ are exactly the triples stated in Theorem [4.19]. Since $H$ and $\sigma(H)$ are conjugate subgroups of $G$ if and only if so are their normalizers in $G$, we can conclude the proof of the corollary by invoking Theorem 4.19. \hfill \Box

4.2.2. The importance of the assumptions we made in the statements of the previous paragraph is reflected in the following examples.

Example 4. Let $G = SL_{n+1}$ with $(n \geq 2)$ and $P \subset G$ be the standard parabolic subgroup associated to the simple roots $\alpha_1$ and $\alpha_2$. Consider the variety $X = G \times_{P} X'$ with $X'$ being the $SL_3$-variety $\mathbb{P}^2 \times (\mathbb{P}^2)^*$. The varieties $X$ and $X'$ have a single spherical root, namely the root $\alpha_1 + \alpha_2$. Let $n$ be odd and $\sigma$ define the quasi-split but non-split real form of $G$. If $n > 2$ then the spherical system of $X$ is not $\varepsilon_\sigma$-stable since $\varepsilon_\sigma(\alpha_1 + \alpha_2) = \alpha_{n-1} + \alpha_n$. Note that here $\sigma(P)$ and $P$ are not conjugated subgroups of $G$.

Example 5. Let $G = SL_4$ with $\sigma$ defining the quasi-split but non-split real form of $G$. Consider the spherical system of $G$ given by the triple $\emptyset, \{\alpha_1, \alpha_2 + \alpha_3\}, \emptyset$. This spherical system is not $\varepsilon_\sigma$-stable since $\varepsilon_\sigma(\alpha_1) = \alpha_3$. The associated wonderful $G$-variety equals $X = X_1 \times_{G/P} X_2$ where $P$ is the standard parabolic subgroup of $G$ associated to the simple root $\alpha_3$, $X_1$ (resp. $X_2$) is the parabolic induction of $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{P}^2 \times (\mathbb{P}^2)^*$) from the parabolic subgroup of $G$ with Levi subgroup of semisimple part $SL_2(\alpha_1)$ (resp. $SL_3(\alpha_2, \alpha_3)$).

Remark 4.23. A spherical subgroup $H$ of $G$ whose spherical closure has a $\varepsilon_\sigma$-stable spherical system may not be conjugate to $\sigma(H)$, as the following example shows.

Example 6. Let $G = SL_3$ with $\sigma$ defining the compact real form of $G$. Consider the standard $G$-module $\mathbb{C}^3$ equipped its canonical basis $\{e_1, e_2, e_3\}$. Let $V$ (resp. $H$) denote the line generated by (resp. the stabilizer in $G$ of) $e_3$. The fiber bundle $X = G \times_H V$ is thus a spherical affine $G$-variety whose spherical system $(\emptyset, \{\alpha_1 + \alpha_2\}, \emptyset)$ is $\varepsilon_\sigma$-stable. The coordinate ring of $X$ equals $\oplus_{\Gamma} V(\lambda)$ where $\Gamma = \mathbb{N}(\omega_1 + \omega_2) + \mathbb{N}\omega_2$. Since the weight $\omega_2$ is mapped to $\omega_1$ by $\varepsilon_\sigma$, the generic stabilizer $H_0$ of the variety $X$ is not conjugate to $\sigma(H_0)$.

4.3. Quantitative properties of real structures. Let $X$ be a wonderful $G$-variety of rank $r$ with spherical system $(S^r, \Sigma, A)$. Recall from the definition of wonderful varieties, that the $G$-orbits of $X$ are indexed by the subsets of $\{1, \ldots, r\}$ or equivalently by the subsets of $\Sigma$. 


Further, given $I \subset \{1, \ldots, r\}$, the closure of the corresponding $G$-orbit within $X$ is a wonderful $G$-variety $X_I$. Specifically, we have

$$X_I = \bigcap_{i \in I} D_i$$

and the spherical system of $X_I$ is $(S^p, \Sigma_I, A_I)$ where

$$\Sigma_I = \{\gamma_i \in \Sigma : i \notin I\}$$

and $A_I$ stands for the union of the $A(\alpha)$'s such that $\alpha \in \Sigma_I$; see e.g Subsection 1.2 in [BL] for details. Moreover, $X_I$ is obtained by parabolic induction from the parabolic subgroup $P_I$ of $G$ containing $B^-$ and associated to the set of simple roots

$$S_I = S^p \cup \text{Supp } \Sigma_I.$$

Here \text{Supp } $\Sigma_I$ denotes the support of $\Sigma_I$, that is the subset of $S$ defined by the $\alpha$'s such that there exists $\gamma \in \Sigma_I$ with $\gamma = \sum_{\beta \in S} a_\beta \beta$ and $a_\alpha \neq 0$.

**Theorem 4.24.** Let $X$ be a spherically closed wonderful $G$-variety endowed with a $\sigma$-equivariant real structure $\mu$. Let $r$ denote the rank of $X$ and $(S^p, \Sigma, A)$ be its spherical system. The real points of $X$ are located on its $G$-orbits $G \cdot x_I$ ($I \subset \{1, \ldots, r\}$) such that

1. $\Sigma_I = \varepsilon_\sigma(\Sigma_I)$ and
2. $S^*_\bullet \subset S_I$.

In particular, if $\sigma$ defines the compact real form of $G$ then the real points w.r.t $\mu$ are located on the open $G$-orbit of $X$.

**Proof.** Given $I \subset \{1, \ldots, r\}$, consider the corresponding $G$-orbit. Suppose this orbit has a real point w.r.t $\mu$ then obviously so does its closure $X_I$ within $X$.

As recalled, $X_I$ is a wonderful $G$-variety with set of spherical roots equals $\Sigma_I$. Therefore, this set has to be $\varepsilon_\sigma$-stable by Theorem 4.19. This proves the assertion stated in (1).

To prove that condition (2) has to be satisfied, recall that $X_I$ is parabolically induced from $P_I$. Further, $X_I$ is also spherically closed; see Section 2.4 in [BP]. By the uniqueness statement (Proposition 2.8), the real structure of $X_I$ is that described in Lemma 3.15. We can thus apply Proposition 3.10. In particular, $G/P_I$ has a real point w.r.t real structure $gP_I \mapsto G/P_I$. This together with the last assertion of Theorem 2.9 imply (2). The theorem follows. \qed

A converse of the above theorem reads as follows.

**Proposition 4.25.** Let $X$ be a wonderful $G$-variety endowed with a $\sigma$-equivariant real structure $\mu$. If the set $S^p_X$ contains all compact roots
af $G$, w.r.t $\sigma$ then every $\mu$-stable $G$-orbit of $X$ contains real points w.r.t $\mu$.

Proof. Recall that the projective $G$-orbit of $X$ is isomorphic to $G/P_X$ where $B^- \subset P_X \subset G$ is associated to $S_X^p$. Since there exists a $\sigma$-equivariant real structure on $X$ by assumption, the spherical system of $X$ is $\varepsilon_{\sigma}$-stable; see Theorem 4.19. In particular, $S_X^p$ is $\varepsilon_{\sigma}$-stable. This implies that $P$ and $\sigma(P)$ are conjugate subgroups of $G$. Since $P_X^- \subset G$ is self-normalizing hence spherically closed, the real structure of $X$ restricted onto $G/P_X$ is the mapping $gP \mapsto \sigma(g)P$; see 2.8. Finally, thanks to the assumption made on $S_X^p$, the base point $eP \in G/P$ is a real point w.r.t. $\mu$, namely $\sigma(P) = P$. The rest of the proof just mimics that of Theorem 3.10 in [ACF]. □

The three following examples show that we may encounter various situations for the set of real points of wonderful varieties.

Example 7. Let $G$ be of type $E_8$. Then $\varepsilon_{\sigma}$ is trivial for every $\sigma$; see e.g. Table 5 in [O]. Consider the nilpotent orbit $O$ of characteristic $(00000010)$. Let $H \subset G$ be the stabilizer of $e \in P(g)$ where $e \in O$. As pointed out in Appendix B of [BCF], $H \subset G$ is wonderful and even spherically closed. Since $\varepsilon_{\sigma}$ is trivial, the spherical system of $G/H$ is always $\varepsilon_{\sigma}$-stable. Thanks to Theorem 4.19 $G/H$ can be equipped with a (unique) $\sigma$-equivariant real structure $\mu_{\sigma}$. Further, $S^p = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$; see again [BCF]. Therefore $S^p_{G/H}$ fullfills the property of Proposition 4.25 whenever $\sigma$ defines the real form $E_{VIII}, E_{IX}$ and in turn $G/H$ has real points w.r.t $\mu_{\sigma}$, for these involutions $\sigma$. This is in accordance with Djokovic’s tables ([D]) stating that $O$ has a real form w.r.t $E_{VIII}, E_{IX}$.

Example 8. Let $G$ be of type $E_6$ and let $H \subset G$ be the normalizer of a nilpotent element in the adjoint orbit $O$ of characteristic $(000100)$. The orbit $O$ is spherical and $H$ is wonderful; its spherical system is given by the triple $(\emptyset, \Sigma = \{\alpha_1 + \alpha_6, \alpha_3 + \alpha_5, \alpha_2 + \alpha_4\}, \emptyset)$; see [BCF] for details. Observe that this spherical system is $\varepsilon_{\sigma}$-stable for every $\sigma$. Since $S^p = \emptyset$, the assumption of Proposition 4.25 is not fullfilled either for the real forms $E_{III}, E_{IV}$ or for the compact form of $E_6$. In another hand, by Djokovic’s tables, we know that the nilpotent orbit $O$ under consideration does not have any real point in the aforementioned real forms of $E_6$.

Example 9. Let $G = G_1 \times G_1$ with $G_1$ being a simple group. Then the $G$-variety $G_1 \simeq G$ is spherical. If $G_1$ is adjoint then $G_1 \simeq G$ is even wonderful and $S^p = \emptyset$, so $S^p$ does not always fulfill
the condition of Proposition 4.25. Equip $G$ with the involution $\sigma = (\sigma_1, \sigma_1)$ where $\sigma_1$ is any anti-holomorphic involution of $G_1$. This case gives an example where there are always real points in $G/H$ w.r.t $\sigma$, whatever $\sigma_1$ is.

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