Private Streaming with Convolutional Codes

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Abstract

Recently, information-theoretic private information retrieval (PIR) from coded storage systems has gained a lot of attention, and a general star product PIR scheme was proposed. In this paper, the star product scheme is adopted, with appropriate modifications, to the case of private (e.g., video) streaming. It is assumed that the files to be streamed are stored on $n$ servers in a coded form, and the streaming is carried out via a convolutional code. The star product scheme is defined for this special case, and various properties are analyzed for two channel models related to straggling and byzantine servers, both in the baseline case as well as with colluding servers. The achieved PIR rates for the given models are derived and for the first model shown to be asymptotically optimal, when the number of stripes in a file is large. The second scheme introduced in this work is shown to be the equivalent of block convolutional codes in the PIR setting. For the Byzantine server model, it is shown to outperform the trivial scheme of downloading stripes of the desired file separately without memory.

I. INTRODUCTION

Private information retrieval (PIR) studies the problem when a user wants to retrieve a file from a storage system without revealing the identity of the file in question to the storage servers. The original problem was introduced in [1], [2], and more recently the problem setting was extended to the case where the files are stored on the servers in an encoded form rather than merely being replicated [3]–[5]. The capacity of PIR for replicated storage was derived in [6] and for coded storage in [7]. For the case of colluding servers, i.e., servers that cooperate to determine the index of the requested file, the capacity was derived in [8] and for colluding and byzantine servers in [9]. In [10], [11] the respective capacities of symmetric PIR were derived. In [12], a so-called star product PIR scheme was introduced. The scheme works with any linear code as a storage code and retrieval code, and achieves the highest rate when both codes are generalized Reed-Solomon codes.

Currently, Netflix and Youtube alone are occupying more than 50% of Internet downstream traffic. Motivated by this huge increase in multi-media streaming, we will consider private streaming suitable for distributed systems sharing encoded streams. In a wider context, this is related to the problem of private stream search (PSS), which has been considered, e.g., in [13]–[15], typically using cryptographic assumptions, and allows the user to privately learn the contents of the servers. As in most works on information-theoretic PIR, we assume the user knows these contents and is able to query each server for linear combinations of files. In this paper, we require information-theoretic privacy, namely that the servers gain zero information on the index of the file being requested for streaming, based on the query received from the user.

Since convolutional codes are suitable for streaming and the star product scheme is efficient and flexible for PIR [16], [17], we will design a scheme that makes use of both of these approaches. By the use of convolutional codes, the presented scheme is related to PIR from databases encoded with non-MDS codes, for which constructions that achieve the MDS PIR capacity without collusion [7] exist [18], [19].

Convolutional codes are sensitive to burst errors but good at handling well-distributed errors. As burst errors are unlikely on, e.g., an additive white Gaussian noise (AWGN) channel, they exhibit good performance compared to block codes on such channels and have a lower bit error rate than comparable block codes with the same rate. See [20 Section V] for more details.

The main contributions of this paper are the following.

- To the best of the authors’ knowledge, information-theoretically private streaming is considered for the first time.
- Memory is introduced into the star product PIR scheme by a block convolutional structure, improving the performance of the decoder for a large class of channels.
- Two schemes for different channels, namely a block erasure channel and a non-bursty channel, e.g., an AWGN channel, are given. Both can operate on the same database and the user can adapt the queries according to the current channel conditions.

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The achieved PIR rates are derived and for the block erasure scheme shown to be asymptotically optimal for the considered model. For the Byzantine server model, the introduced scheme is shown to outperform the trivial scheme of downloading stripes of the desired file separately without memory.

This paper is structured as follows. In Section II, we briefly introduce block convolutional codes and the star product PIR scheme of [2]. In Section III we describe the combination of block convolutional codes with the star product PIR scheme and show that the achievable asymptotic (in the number of stripes and files) PIR rate is equal to the conjectured PIR capacity of a coded scheme with \( t \)-collusion. In Section IV the scheme is adapted for a block erasure channel in which the user does not receive replies in a given number of consecutive iterations of the protocol. In Section V we introduce a scheme for non-bursty channels based on the decoding algorithm of [20].

II. PRELIMINARIES

We denote by \([a,b]\) the set of integers \(\{i \mid a \leq i \leq b\}\) and \([b] = [1, b] \). If \(c\) and \(d\) are vectors of the same length \(n\), we define their star product as the coordinate-wise product

\[
c \ast d = (c_1d_1, \ldots , c_nd_n)\
\]

Further, if \(\mathcal{C}\) and \(\mathcal{D}\) are linear codes of the same length, we define their star product to be the linear code

\[
\mathcal{C} \ast \mathcal{D} = \langle c \ast d \mid c \in \mathcal{C}, d \in \mathcal{D} \rangle.
\]

Throughout the paper, \(\mathbb{F}\) will denote an arbitrary finite field.

A. Convolutional Codes

**Definition 1** (Convolutional code). Let \(G_1, \ldots , G_{M+1} \in \mathbb{F}^{k \times n}\) and \(\text{rank}(G_1) = k\). Define an \((n, k)\) convolutional code \(\mathcal{C}_c\) as

\[
Y_i = \sum_{j=1}^{M+1} X_{i-j+1}G_j,
\]

where \(X_0, \ldots , X_{-M+1} = 0\) and \(X_j \in \mathbb{F}^k\).

We refer to \(M\) as the memory of \(\mathcal{C}_c\), and if \(M = 1\), we say that \(\mathcal{C}_c\) is a unit memory (UM) code. In this paper, we consider terminated convolutional codes, i.e., \(Y\) is not a semi-infinite vector, but \(Y = (Y_1, Y_2, \ldots , Y_{t+M})\) where \(Y_i\) is defined as in (1).

An \((n, k)\)-code denotes a linear block code of length \(n\) and dimension \(k\). A generalized Reed–Solomon (GRS) code \(RS(n, k, v)\) is an \((n, k)\)-code with minimum distance \(d = n - k + 1\) and generator matrix

\[
G = \left(\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{-1} & \alpha_2^{-1} & \cdots & \alpha_n^{-1}
\end{array}\right) \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix},
\]

where \(\alpha_1, \ldots \alpha_n \in \mathbb{F}\) are distinct evaluation points and the \(v_j\)'s are all non-zero. If the choice of the \(v_j\)'s is not important, we sometimes write \(RS(n, k)\) code.

The distance measure of interest for convolutional codes is the extended row distance \(d^e_r\), which determines the minimum number of errors required for an error burst of \(\ell\) blocks to occur. For UM codes this distance can be lower bounded by the designed extended row distance

\[
d^e_r = d_1 + (\ell - 1)d_n + d_2, \ \ell \geq 1,
\]

where \(d_1, d_2,\) and \(d_n\) denote the distances of the codes generated by \(G_1, G_2,\) and \([G_1^T, G_2^T]^T\) respectively. In [20], a decoding algorithm is given, which is guaranteed to be successful if the number of errors does not exceed half the designed extended row distance for any \(\ell\), i.e., it decodes successfully if

\[
\sum_{j=0}^{\ell+s} w_H(w_j) < \frac{d_r}{2}, \ \forall s \in [\ell + M], \ell \in [0, \ell + M - s],
\]

where \(w_j\) denotes the error vector of the \(j\)-th block. We refer to an \((n, k)\) UM code for which \(d_n, d_1\) and \(d_2\) achieve the Singleton bound for block codes as an optimal \((n, k)\) UM code.
B. Star Product PIR

We review the star product scheme for PIR from an arbitrary storage code, as introduced in [12]. Let $C$ be an $(n, k)$ code (the storage code) with generator matrix $G \in \mathbb{F}^{k \times n}$, storing $m$ files $X^1, \ldots, X^m \in \mathbb{F}^k$. This means that each server $j \in [n]$ stores a column $Y_j$ of the matrix $Y = XG \in \mathbb{F}^{m \times n}$, where $X \in \mathbb{F}^{m \times k}$ is a data matrix, whose $i$-th row $X^i$ represents the $i$-th file. The scheme we will describe allows a user to retrieve the file $X^i$ without disclosing the index $i$.

Let $D$ be a code of the same length $n$ as $C$. Let $D \in \mathbb{F}^{m \times n}$ be a matrix whose $m$ rows are i.i.d. uniformly random codewords of $D$. The query for the $j$-th server is given by

$$q_j^i = D_{j} + e_i E_{1,j},$$

where $e_i$ denotes the $i$-th standard basis vector and $E = E_{1,\cdot} \in \mathbb{F}^{1 \times n}$, where the $\cdot$ as index means that we consider all columns.

The servers now respond with the standard inner product of their $(m \times 1)$ stored vector $Y_j$ and the query vector $q_j^i$ which they received, so the response of the $j$-th server is the symbol

$$r_j^i = \langle q_j^i, Y_j \rangle = \sum_{s=1}^{m} D_{s,j} Y_j^s + E_{1,j} Y_j^i \in \mathbb{F}.$$

Considering the $n$ responses obtained as a vector in $\mathbb{F}^{1 \times n}$, we can write it as

$$r^i = \sum_{s=1}^{m} (D_{s,1} Y_1^s, \ldots, D_{s,n} Y_n^s) + (E_{1,1} Y_1^i, \ldots, E_{1,n} Y_n^i) \in C \ast D + E \ast Y^i.$$  \hspace{1cm} (5)

Assuming $E$ has weight $w_{\text{E}}(E) < d_{C \ast D}$, erasure decoding in $C \ast D$ now allows us to retrieve the vector $E \ast Y^i$, which depends only on the desired file $Y^i$. The rate achieved by this scheme is given by

$$R^*_{\text{PIR}} = \frac{n - (k + t - 1)}{n},$$

where $t$ is the number of colluding servers, i.e., the maximal number of servers that can exchange their queries such that the scheme is still private (see [12] for details). If the response is corrupted by channel erasures or a bounded number of malicious servers, the user first decodes the response in $C \ast (D + E)$, as in [17]. This is discussed further in Section V.

III. PIR from Convolutional Codes

In this section, it is shown how a large file can be streamed with asymptotically (with respect to the number of files) no rate loss compared to the retrieval of stripes without memory, by designing, as per user’s request, the retrieved symbols such that they are codewords of a convolutional code.

A. Storage Code

Denote by $m$ the number of files $X^s \in \mathbb{F}^{d \ell}$, $s \in [m]$, and by $n$ the number of servers. The files are split into $\ell$ stripes $X^s_i \in \mathbb{F}^k$ and encoded with an $RS(n, k)$ storage code $C$ with evaluation points $\alpha_j$, $j \in [n]$. The $j$-th server stores the $j$-th symbol of every encoded stripe $Y^s_i \in \mathbb{F}^n$ (see Figure 1).

B. Query

We query for a linear combination of $M + 1$ stripes in each block and design the queries such that the responses are codewords of a convolutional code. Let $D$ be an $RS(n, t)$ code; the matrix $D \in \mathbb{F}^{(M+1)m \times n}$ as in (4); and $J \subset [n]$ with $|J| \leq d_{C \ast D} - 1$. The query for the $j$-th server is given by

$$q_j^i = D_{j} + e_{sz} + E_{z+1,j}, \ z \in [0, M],$$

where $e_i$ is the $i$-th standard basis vector and the matrix $E \in \mathbb{F}^{M+1 \times n}$ is given by

$$E_{z+1,j} = \left\{ \begin{array}{ll} \alpha_j^{zk} & , \text{if } j \in J \\ 0 & , \text{otherwise} \end{array} \right.$$ \hspace{1cm} (8)

C. Response

The protocol consists of $\ell + M$ iterations in each of which the servers respond with the inner product of the query and a vector containing the stored symbols of $M + 1$ stripes of each file, depending on the iteration. In iteration $\xi$ the response of server $j$ is given by

$$r_{\xi,j}^i = \langle q_j^i, (Y_{\xi,j}, Y_{\xi-1,j}, \ldots, Y_{\xi-M,j})^T \rangle,$$

where $Y_{-M+1} = \cdots = Y_0 = Y_{\ell+1} = \cdots = Y_{\ell+M} = 0$ and $Y_{\xi} = X_{\xi}G$ denotes the matrix storing the $\xi$-th part of every file.

1This notation is chosen to be consistent with the later sections when $E$ will be a matrix.
2Note that $E_{z+1,j}$ is a scalar.
D. Decoding

The response is given by

$$r^j_\xi = \sum_{z=0}^{M} \sum_{s=0}^{m} Y^z_{\xi-z} \ast D_{zm+1} + Y^1_{\xi-z} \ast E_{z+1}.$$  \hspace{1cm} (10)

An illustration of the responses for UM codes is given in Figure 2.

**Lemma 1.** Let $|J| \geq k$. Given the the responses $\{r^1_\xi, r^2_\xi, \ldots, r^J_\xi\}$ the file $X^i$ can be recovered.

**Proof:** By (9) the vectors $E_{z+1}$ are designed such that for any $c \in C \ast E_{z+1}$, $z \in [0, M]$ it holds that $c_j = 0$, $\forall j \notin J$. As $|J| \leq d_{C \ast D} - 1$ erasure decoding in $C \ast D$ recovers the vector

$$\sum_{z=0}^{M} E_{z+1} \ast Y^i_{\xi-z} = \sum_{z=0}^{M} X^i_{\xi-z} \ast G_{C \ast E_{z+1}}$$

in each iteration, where the $G_{C \ast E_{z+1}}$ are generator matrices of the storage code $C$ with column multipliers $E_{z+1}$. Since $|J| \geq k$, each $G_{C \ast E_{z+1}}$ is of rank $k$ and it follows that given the set $\{X^1_{\xi-M}, \ldots, X^j_{\xi-M}\}$, the stripe $X^i_1$ can be determined uniquely. In the first iteration $X_{1-M} \ast \cdots \ast X_{0} = 0$ so $X_1$ can be recovered and recovery of the remaining stripes follows by induction.

As both $C$ and $D$ are GRS codes, the distance of the star product $C \ast D$ is given by $d_{C \ast D} = n - (k + t - 1) + 1$ and it follows that at most $d_{C \ast D} - 1 = n - (k + t - 1)$ symbols can be downloaded in each iteration. As there are $\ell + M$ iterations and the number of stripes in a file is $\ell$, the PIR rate is given by

$$R_{PIR} \leq \frac{\ell(n - (k + t - 1))}{(\ell + M)n},$$  \hspace{1cm} (11)

which approaches the PIR rate of 12 given in (4) for $\ell \rightarrow \infty$. The highest PIR rate in this setting is achieved for $|J| = k$ and $n = 2k + t - 1$. For the trivial case of memory $M = 0$ and $t = 1$ collusion the scheme is a repeated application of the star product scheme 12 and therefore asymptotically achieves the capacity of PIR from MDS coded databases without collusion 7 for $m \rightarrow \infty$.

IV. PROTECTING AGAINST BLOCK ERASES

In the previous section, we showed how to design queries such that the symbols of the desired file recovered from the responses are symbols of a code of higher dimension and memory $M$. While this setting asymptotically achieves the same PIR rate as a comparable system that downloads blocks without memory, its advantages might not be obvious. In this section, we utilize the construction to design a PIR scheme that is able to stream files consisting of many stripes in the presence of burst block erasures, i.e., iterations where all the responses of the servers are lost. Since we are interested in streaming applications, decoding should be possible without a big delay and without querying for more data or retransmission of blocks. Therefore, we consider a sliding decoding window of $N$ blocks and denote the maximum burst length of block erasures in a window by $e$. To protect against these erasures, more symbols of each block have to be retrieved privately in each iteration than in the setting of the previous section.
Lemma 2. The number of symbols \( \gamma \) privately retrieved in each non-erased block has to satisfy
\[
\gamma \geq \frac{Nk}{N - \epsilon}.
\]

Proof: Losing \( \epsilon \) consecutive blocks out of \( N \) blocks leaves \( (N - \epsilon) \frac{Nk}{N - \epsilon} = Nk \) retrieved symbols in that window, the minimal number to recover the corresponding \( Nk \) message symbols.

Trivially \( M \geq \epsilon \) has to hold, since otherwise a burst of \( M + 1 \) block erasures would make the received symbols independent of some stripe of the file and recovery impossible.

A. Query

The queries are similar to Section III-B, but by Lemma 2 it has to hold that
\[
|J| \geq \frac{Nk}{N - \epsilon}.
\]

The set \( J \) has to be chosen such that recovery of the file is possible in the presence of block erasures.

Definition 2. Let \( G \) be the generator matrix of a convolutional code of memory \( M = \epsilon \) and \((n, k)\) component codes generated by \( G_z \), \( z \in [0, M] \). We say that a set \( J \subset [n] \) has the recovering property if
\[
\text{rank} \left( G \bigg| J \right) = Nk
\]
for any \( R = [\xi - N + 1, \xi]; \xi \in [\ell + M] \), where \( G \bigg| J \) denotes the restriction of \( G \) to the positions in \( J \) in each block and to the blocks indexed by \( R \).

This assures that a burst of \( \epsilon \) block erasures can be recovered while still being within the window of \( N \) blocks. In our setting, the matrices \( G_z \) will be generator matrices of \( C \times E_{z+1} \). In the Appendix, we will show that when \( G_z \) generate a Reed-Solomon code, then the recovering property is equivalent to a rather simple algebraic criterion. We also show that codes with sets satisfying the recovering property exist.

B. Decoding

Decoding the queries to obtain the respective stripes of the requested file consists of two main steps: erasure decoding to obtain the linear combination of desired symbols and recovering the stripes from these symbols.

Theorem 1. Let \( J \) be a set with the recovering property as in Definition 2 and let \( n \geq k + t - 1 + |J| \). For any set \( R = [\xi - N + 1, \xi]; \xi \in [\ell + M] \), the stripes \( \{X^{i}_{\xi - N + 1}, \ldots, X^{i}_{\xi}\} \) can be recovered from the responses \( r^{i}_{s}, s \in R \).

Proof: The code \( C \times D \) has distance \( d_{C \times D} = n - (k + t - 1) + 1 \geq |J| + 1 \), and it follows that the vector
\[
\sum_{z=0}^{M} E_{z+1} \ast Y_{\xi - z}^{i} = \sum_{z=0}^{M} X^{i}_{\xi - z} \cdot G_{C \times E_{z+1}}
\]
can be recovered for any \( \xi \in R \). By Definition 2 the matrix generating these vectors has rank \( Nk \) and thus all \( N \) stripes in this window can be recovered.
C. Performance

Lemma 3. The PIR rate is given by

$$R_{PIR}^b = \left(1 - \frac{\epsilon}{N}\right) \frac{\ell(n - (k + t - 1))}{(\ell + \epsilon)n},$$

with equality for $\gamma = \frac{Nk}{N-\epsilon}$.

Proof: By definition, $Nk$ information symbols have to be downloaded in each window of $N$ blocks. In each round $d_{c,t} - 1$ symbols of the $\gamma$ desired symbols can be downloaded. The PIR rate is hence given by

$$R_{PIR} = \frac{\ell}{\ell + \epsilon} \frac{Nk}{d_{c,t} - 1} n \leq \frac{\ell}{\ell + \epsilon} \frac{k(n - (k + t - 1))}{\frac{Nk}{N-\epsilon} n} = \left(1 - \frac{\epsilon}{N}\right) \frac{\ell(n - (k + t - 1))}{(\ell + \epsilon)n}.$$

Figure 3 gives a comparison of the rates of Lemma 3 and a trivial scheme based on the scheme of [12], that protects against $\epsilon$ consecutive block erasures by downloading each block $\epsilon + 1$ times, resulting in a PIR rate of $R_{PIR}^\epsilon = \frac{\ell}{\ell + \epsilon} R_{PIR}^\star$. In Figure 5 the number of consecutive erasures within a decoding window is fixed to $\epsilon = 3$. For a window size of $N = 4$ the minimal number of symbols to download in each iteration of the convolutional scheme is $\frac{Nk}{N-\epsilon} = 4k$ and it follows that the PIR rates of both schemes are the same, except for the rate loss due to termination. As the PIR rate of the trivial scheme does not depend on the decoding window size, it remains constant. The PIR rate of the convolutional scheme approaches the PIR rate $R_{PIR}^\star$, i.e., the conjectured asymptotic PIR capacity of a coded scheme with $t$-collusion [12] for increasing decoding window size $N$ and number of stripes $\ell$ is

$$R_{PIR}^b = \left(1 - \frac{\epsilon}{N}\right) \frac{\ell(n - (k + t - 1))}{(\ell + \epsilon)n} \xrightarrow{N,\ell \rightarrow \infty} R_{PIR}^\star.$$

In Figure 3 the ratio between the number of consecutive block erasures $\epsilon$ and the decoding window size $N$ is fixed to $\epsilon = 0.1N$. For $N = 2$ the PIR rate of the trivial scheme and the convolutional scheme coincide, except for a rate loss due to termination, as the single remaining block after $\epsilon = \frac{2}{3} = 1$ block erasures allows only for a trivial scheme. For larger window size, the PIR rate of the convolutional scheme decreases as the number of consecutive block erasures and therefore the required memory increases, while the PIR rate of the trivial scheme decreases with $\frac{\epsilon}{N}$. In Figure 5 the decoding window size is fixed to $N = 12$. For $\epsilon = 0$ both schemes are the same, as a convolutional code with memory $M = 0$ results in separate downloading of the stripes. For $\epsilon = 11$ the difference between the PIR rates of the two schemes is solely due to the rate loss caused by termination.

D. Examples

For ease of understanding, we give two examples of the described scheme for specific parameters. Example 1 shows that the window size has to be chosen sufficiently large to allow for a non-trivial scheme and a gain in PIR rate. Example 2 describes each step of the scheme in detail for specific parameters and gives a class of explicit locators for which the set $J$ has the recovering property from Definition 2.

Example 1. Consider the case where $\epsilon = 1$ and $N = 2$. In this case Lemma 2 gives $\gamma = 2k$ and the PIR rate for $\ell \rightarrow \infty$ is $R_{PIR}^b = \frac{\ell}{\ell + 1} R_{PIR}^\star$, where $R_{PIR}^\star$ is the rate achieved by the scheme in [12]. In this case, the same result can be achieved with a trivial scheme that downloads each block twice.

Example 2. Let $m = 3$, $M = 1$, $n = 6$, $k = 2$, $t = 1$, $N = 3$ and $\epsilon = 1$. Let $D \in \mathbb{F}^{6 \times 6}$ be a random matrix with 6 i.i.d. random codewords from an RS$(t,n)$ code as rows and $J = \{4,5,6\}$. Assume the user wants to retrieve the second file $X^2$. With (5) the query matrix is given by

$$D + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{F}^{6 \times 6}.$$
The query $q_j^2$ for the $j$-th server is given by the $j$-th column.

In the first iteration the user obtains $X^2_1$. Now assume the second block is lost. In the third and fourth iteration the nodes return $r^3_{3,j} = \langle q_j^2, (Y^3_{3,j}, Y^3_{2,j})^T \rangle$ and $r^4_{4,j} = \langle q_j^2, (Y^4_{4,j}, Y^3_{3,j})^T \rangle$. The user receives

\[ r^3 = \sum_{s=1}^{m} (D_s \ast Y^s_3 + D_{M+s} \ast Y^s_2) + (0, 0, Y^2_{3,4}, 0, Y^2_{3,5}, 0, Y^2_{3,6}) + \alpha Y^2_{2,4}, Y^2_{2,5}, Y^2_{2,6}) \]

\[ r^4 = \sum_{s=1}^{m} (D_s \ast Y^2_4 + D_{M+s} \ast Y^3_3) + (0, 0, Y^2_{4,4}, 0, Y^2_{4,5}, 0, Y^2_{4,6}) + \alpha Y^2_{3,4}, Y^2_{3,5}, Y^2_{3,6}) \]
The distance of \( C \times D \) is \( d_{C \times D} = 4 \) and treating positions 4 – 6 as erasures gives

\[
(Y_{3,(4:6)}^2 + \alpha_4^2 \cdot Y_{2,(4:6)}^2, Y_{4,(4:6)}^2 + \alpha_4^2 \cdot Y_{3,(4:6)}^2) = (X_2^2, X_3^2, X_4^2),
\]

\[
\begin{pmatrix}
\alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\
1 & 1 & 1 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
1 & 1 & 1
\end{pmatrix},
\]

(13)

where \( \alpha_{1:6} = (\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2, \alpha_5^2, \alpha_6^2) \). If this matrix has full rank, the files \( X_2^2, X_3^2 \) and \( X_4^2 \) can be recovered. Whether it does have full rank depends on the choice of evaluation points and we will now show that we can choose evaluation points such that this matrix is invertible. Let us assume that the field size \(|\mathbb{F}| > 3\). Let \( \alpha_4, \alpha_5, \alpha_6 \in \mathbb{F} \) be such that their squares \( \alpha_j^2 \) are all distinct. Assume for a contradiction that the matrix

\[
A = \begin{pmatrix}
\alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\
1 & 1 & 1 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
1 & 1 & 1
\end{pmatrix}
\]

does not have full rank, but satisfies \( xA = 0 \) for some non-zero row vector \( x = [x_1, \ldots, x_6] \). Denoting

\[
A' = \begin{pmatrix}
\alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\
1 & 1 & 1 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
1 & 1 & 1
\end{pmatrix}
\]

and studying the first and the last three columns of \( A \) separately, we get that

\[
[x_1, \ldots, x_4]A' = [x_3, \ldots, x_6]A' = 0.
\]

As \( A' \) is a Vandermonde matrix, any three of its rows are independent, so \( x' A' = 0 \) implies that \( x' \) is either the zero vector or has full support. As we know that \( x = [x_1, \ldots, x_6] \) is not the zero vector, it follows that \( x_1 \) is also non-zero, and after scaling we may assume that \( x_1 = 1 \). As \( A' \) has a one-dimensional left null space that contains both \( [x_1, \ldots, x_4] \) and \( [x_3, \ldots, x_6] \), we must have \( [x_3, \ldots, x_6] = t[x_1, \ldots, x_4] \) for some \( t \in \mathbb{F} \). We can therefore write

\[
[x_1, x_2, x_3, x_4] = [1, s, t, ts]
\]

for some \( s, t \in \mathbb{F}_q \). The linear system of equations

\[
[1, s, t, ts] \begin{pmatrix}
\alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\
1 & 1 & 1 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
1 & 1 & 1
\end{pmatrix} = 0
\]

implies that

\[
0 = \alpha_j^2 + sa_j^3 + t + tsa_j = (\alpha_j^2 + t)(1 + sa_j)
\]

holds for \( j = 4, 5, 6 \). But since \( \alpha_j^2 \) were distinct for different \( j \), at most one of the points may satisfy \( \alpha_j^2 + t = 0 \), and at most one of them may satisfy \( 1 + sa_j = 0 \). This is a contradiction and it follows that any set of locators with distinct squares has the recovering property. By \([11]\) the PIR rate for \( \ell \to \infty \) is given by

\[
R_{\text{PIR}}^b = \frac{\frac{2}{3} \cdot \frac{6 - 2}{6}}{4} = \frac{4}{9}.
\]

For these parameters, the PIR rate of the trivial scheme is given by

\[
R_{\text{PIR}} = \frac{\frac{1}{2} \cdot \frac{6 - 2}{6}}{2} = \frac{2}{6} < R_{\text{PIR}}^b.
\]
V. PIR with Byzantine Servers and Convolutional Codes

In this section, we consider incorrectly received responses, due to either byzantine servers or errors during transmission. We focus on constructions that result in a convolutional code of memory $M = 1$, i.e., UM codes. For these codes, the decoder introduced in [20] can efficiently decode up to half the designed extended row distance, by a combination of bounded minimum distance (BMD) decoding in the blocks and trellis-based decoding with the Viterbi algorithm. A key step in this algorithm is decoding blocks in the cosets given by successfully decoded neighboring blocks. It is therefore imperative for a good performance to design the code such that these cosets have good distance properties. In the following, we describe a scheme that achieves this goal in the PIR setting.

A. Query

We query for two stripes in each block (i.e., unit memory $M = 1$) and design the queries such that when one block can be decoded and both stripes can be recovered, the neighboring blocks have good distance properties in the corresponding cosets.

Let $D \in \mathbb{F}^{2m \times n}$ be as in [3] and $D$ be an $RS(n,t)$ code. The query for the $j$-th server is given by

$$q^j_\ell = D_{\ell,j} + e_i E_1 + e_{m+i} E_2,$$

where $E_1 = [a_j^{k-k}]$, $E_2 = [a_j^{k+t-1}]$ and $e_i$ is the $i$-th standard basis vector.

B. Response

The response to one query consists of $\ell + 1$ parts. In iteration $\xi$ the response of server $j$ is given by

$$r^\xi_j = \sum_{s=1}^{m} \left( \sum_{t \in C^*_\xi} \sum_{e \in E_1} \sum_{e \in E_2} (D_s \star Y^s_{\xi,j} + D_{m+s} \star Y^s_{\xi-1,j}) + E_1 \star Y^1_{\xi,j} + E_2 \star Y^2_{\xi-1,j} + w_{\xi,j} \right),$$

where $w_{\xi,j}$ denotes the error vector of iteration $\xi$.

Lemma 4. The codes $C_\star(D+E_1+E_2)$, $C_\star(D+E_1)$, and $C_\star(D+E_2)$ have respective distances $d_{C_\star(D+E_1+E_2)} = n - 3k - t + 2$ and $d_{C_\star(D+E_1)} = d_{C_\star(D+E_2)} = n - 2k - t + 2$. The codes $C \star D$, $C \star E_1$, and $C \star E_2$ intersect trivially.

Proof: An $RS(n,k,1)$ code is the evaluation of all polynomials $f(z)$ with $\deg(f(z)) \leq k - 1$ at the evaluation points $\alpha_j$. Multiplying any polynomials corresponding to the codes $C$, $D$, $E_1$ and $E_2$ gives

$$f_C(z) \cdot (f_D(z) + u_{-k} z^{-k} + u_{k+t-1} z^{k+t-1}) = \sum_{i=0}^{k+t-2} u_i z^i + \sum_{i=-k}^{-1} u_i z^i + \sum_{i=k+t-1}^{2k+t-2} u_i z^i = z^{-k} \sum_{i=0}^{3k+t-2} u_{i-k} z^i,$$

where $u_i \in \mathbb{F}$. Evaluating this polynomial at $\alpha_j, j \in [n]$, gives a codeword of $C \star (D+E_1+E_2) = RS(n,3k+t-1,[a_j^{-k}])$. By the same argument, it holds that $C \star (D+E_1) = RS(n,2k+t-1,[a_j^{-k}])$ and $C \star (D+E_2) = RS(n,2k+t-1,1)$. The distances follow from the Singleton bound and the trivial intersection from the distinct powers in the polynomials.

To illustrate an example for explicit parameters.

Example 3. Let $n = 10$, $k = 2$ and $t = 2$. The defined matrices are given by:

$$G_C = G_D = \left( \begin{array}{cccccc} 1 & 1 & \cdots & 1 & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_9 & \alpha_{10} \end{array} \right),$$

$$E_1 = \left( \begin{array}{cccc} \alpha_1^{-2} & \alpha_2^{-2} & \cdots & \alpha_9^{-2} & \alpha_{10}^{-2} \end{array} \right), \quad E_2 = \left( \begin{array}{cccc} \alpha_1^3 & \alpha_2^3 & \cdots & \alpha_9^3 & \alpha_{10}^3 \end{array} \right).$$
The large number of states makes trellis decoding of the convolutional code infeasible. In [20] an algorithm combining BMD decoding in the blocks and Viterbi decoding on a reduced trellis is given, with decoding complexity only cubic in \( n \), if the complexity of the block decoder is quadratic in \( n \). We give a brief description of this algorithm and show how it can be applied to decode the responses.

1) Decode each received block in \( C_\alpha = C \ast (D + E_1 + E_2) \), an RS\((n, 3k + t - 1)\) code of distance \( d_\alpha = n - 3k - t + 2 \).
2) From the blocks successfully decoded in step 1) decode \( l_F \) steps forward and \( l_B \) backward (see [20]) in the respective coset \( C \ast (D + E_1) \) or \( C \ast (D + E_2) \). By Lemma 4 these are \( RS(n, 2k + t - 1) \) codes and can therefore be decoded up to half their minimum distance \( d_1 = d_2 = n - 2k - t + 2 \).
3) Build a reduced trellis and find the maximum-likelihood path with the Viterbi algorithm.
4) By Lemma 4 the codes \( C \ast D, C \ast E_1 \) and \( C \ast E_2 \) intersect trivially, and it follows that the parts of the file \( X^i \) can be recovered uniquely from the codeword corresponding to the most likely path.

**Theorem 2.** If (3) holds, where \( d_\alpha^i \) is given by (2) with \( d_\alpha = n - 3k - t + 2 \) and \( d_1 = d_2 = n - 2k + t + 2 \), decoding of the responses is successful and the file \( X^i \) is decoded correctly.

**Proof:** By [20] the maximum likelihood path will be in the reduced trellis if (3) holds, which depends on the distance \( d_\alpha \) in each block and the distances \( d_1 \) and \( d_2 \) in the corresponding cosets of the neighboring blocks. For the code given by the responses \( \{ r_1^i, \ldots, r_{l+1}^i \} \) these are shown in Lemma 4. If the path is contained in the trellis, the Viterbi decoder will find it, as it is an ML decoder.

**Corollary 1.** The PIR rate of the scheme is

\[
R_{PIR} = \frac{\ell k}{(\ell + 1)n},
\]

with \( n > 3k + t - 1 \) and it has error correction capability similar to an optimal \((n - (k + t - 1), k)\) UM-code.

This result is similar to the scheme of [17] which also has error/erasure correction capability similar to an optimal (MDS) block code of shorter length. This allows for a direct comparison, i.e., in any non-private setting where a block convolutional code performs better than a comparable block code, our scheme will perform better when the privacy requirement is introduced.

D. Combination of Block Erasures and Byzantine servers

In this section we briefly discuss the combination of the two introduced schemes to a scheme that protects against both considered error models, i.e., block erasures and Byzantine servers/channel errors. A direct combination is not possible as the methods used to recover the symbols of the desired file are different (erasure decoding vs. trivially intersecting codes). However, it is possible to give conditions under which the scheme for Byzantine servers also protects against single block erasures, i.e., the case of \( M = \epsilon = 1 \). By observing that the algorithm of [20] does not require termination, it is apparent that if the condition for successful decoding (3) is fulfilled between any two erasures, the erased blocks can also be recovered from the correctly decoded neighboring blocks. For a higher number of consecutive block erasures this is not possible, as the decoding algorithm of [20] is designed only for unit memory codes, i.e., memory \( M = 1 \), and trivially \( \epsilon \leq M \) has to hold. A generalization of the algorithm seems possible, although the derivation of decoding guarantees might be difficult. Similar to the algorithm for unit memory, it would consists of three main steps: decoding of each block separately, decoding of the \( M \) preceeding and successive blocks in the corresponding cosets and trellis decoding. In the PIR setting, problems arise from the second step, as it is not clear how to design the queries such that decoding one reply can be used to obtain higher distance in both of the neighboring blocks. For the case of \( M = 1 \), as shown in Example 3 the codes \( C \ast E_1 \) and \( C \ast E_2 \) are given by the lowest and highest powers, respectively, of the Vandermonde matrix that generates the code in each block. Therefore,
decoding a block in one of the cosets gives another Reed-Solomon code and hence a large distance. For larger memory it is unclear how to arrange these codes such that the former as well as the sequential blocks have good distance properties if decoded in the respective cosets.

VI. CONCLUSION

In this paper, we have considered information-theoretical private streaming by combining the star product PIR scheme [12] with a block convolutional structure, thereby introducing the known benefits of codes with memory into the decoding of privately streamed/downloaded data. We introduced two schemes for different channels, i.e., a block erasure channel and a non-bursty channel (e.g., AWGN), that are suitable for streaming/downloading files, when the file size is larger than the packet size communicated in each iteration. Both work on the same database and the user can adapt to changing channel conditions by designing queries accordingly. The PIR rates of both schemes were derived and shown to improve upon a trivial scheme based on separate retrieval of stripes.

Future work includes the combination of the two schemes and design of an additional outer code to improve the error-correction performance.

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APPENDIX

PROOF OF EXISTENCE

In this section, we explore the existence of locators with the recovering property of Definition 2, when the block matrices are generator matrices of a Reed-Solomon code. First we formally define the matrix obtained by restricting the generator matrix of a convolutional code to the $N - \epsilon$ non-erased blocks and $|J| = \gamma$ code positions. We set $\epsilon = M$, as this is the largest
number of erased blocks that can be corrected by the convolutional code. In addition, we consider decoding windows of length $N = 2M + 1$, which is clearly the shortest window that can allow for decoding the $M = \epsilon$ erased blocks. For simplicity, we permute rows and columns to give the matrices $G_i$ in ascending order in each block. Note that this does not change the rank of the matrix.

**Definition 3.** Let $V_i$ be the diagonal matrix with the entries $[\alpha^i_j]_{1 \leq j \leq \gamma}$ and

$$G_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_\gamma \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_\gamma^{k-1} \end{pmatrix} \cdot V_{i-1} \in \mathbb{F}^{k \times \gamma}.$$  

Define

$$A = \begin{pmatrix} G_1 & G_1 & \cdots & G_1 \\ G_2 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ G_{M+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ G_{M+1} & \cdots & \cdots & \cdots \end{pmatrix}.$$  

Now, $\{\alpha_j\}_{j=1, \ldots, \gamma}$ has the recovering property if and only if $A$ has rank $Nk = (2M + 1)k$. The rank of $A$ can be related to the dimension of intersections of different generalized Reed-Solomon codes.

**Lemma 5.** For $A$ as in Definition 3 it holds that

$$\text{rk}(A) = (2M + 1)k \iff \langle G_1 \rangle + \sum_{i=1}^{M} (\langle G_1 \rangle \cap \langle G_{-M} \rangle) V_i \text{ is a direct sum}.$$  

**Proof:** Multiplying $A$ from the right by a transformation matrix gives

$$A' = \begin{pmatrix} G_1 & \cdots & G_1 \\ \vdots & \ddots & \vdots \\ G_{M+1} & \cdots & G_{M+1} \end{pmatrix} \begin{pmatrix} -V_{-M} & \cdots & V_{-M} \\ \vdots & \ddots & \vdots \\ -V_0 & \cdots & V_0 \end{pmatrix} = \begin{pmatrix} -G_{-M+3} & 0 & \cdots & 0 & 0 \\ 0 & -G_{-M+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_0 & 0 \\ 0 & 0 & \cdots & 0 & G_1 \end{pmatrix}.$$  

The full rank of the transformation matrix follows from its upper diagonal structure, so $\text{rk}(A) = \text{rk}(A')$. Now $\text{rk}(A') < (2M + 1)k$ if and only if there exists an $x$ such that $x \cdot A' = 0$.

The last $\gamma$ columns of the equation $xA' = 0$ now reads

$$0 = y_1G_1 + y_2G_2 + \cdots + y_MG_M + 1,$$  

where $y_i = x_{(M+1)k+1, (M+1)k}$. The first $M\gamma$ columns show that $y_iG_i \in \langle G_{i-M-1} \rangle$ for $i \geq 2$, so writing $c_i = y_iG_i$ we get a nontrivial solution to the equation $0 = c_1 + c_2 + \cdots + c_M$, where $c_i \in \langle G_1 \rangle$ and $c_i \in \langle G_i \rangle \cap \langle G_{i-M-1} \rangle = \langle (G_i \cap (G_{-M}) \rangle V_i$ for $i \geq 2$. It follows that if $\langle G_1 \rangle + \sum_{i=1}^{M} (\langle G_1 \rangle \cap \langle G_{-M} \rangle) V_i$ is not a direct sum, there is a linear combination of vectors from the respective subspaces such that (16) is fulfilled and therefore a vector $x$ with $x \cdot A' = 0$ exists. If it is a direct sum, the only solution of (16) is $y_1 = y_2 = \cdots = y_M = 0$ and therefore $x = 0$.

We now give an explicit method to choose locators with the recovering property for arbitrary $M = \epsilon$ and $k$, where $N = 2M + 1$ and $\frac{2M+1}{M+1}k \leq \gamma \leq N - k$. In particular, we show that a field with such locators always exists.

**Lemma 6.** Let $\{\alpha_j\}_{j \in [\gamma]} \subseteq \mathbb{F}$ be a set of locators with $\text{ord}(\alpha_j)\mid Mk + \gamma$ for all $j \in [\gamma]$. Then for $A$ as in Definition 3 it holds that $\text{rk}(A) = Nk$ if and only if the set

$$\{(\alpha^{i-k}_{1}, \ldots, \alpha^{i-k}_{\gamma}) : 2 \leq i \leq M + 1, 1 \leq j \leq k - \gamma \} \cup \{(\alpha^i_1, \ldots, \alpha^i_\gamma) : 0 \leq j < k\}$$  

is linearly independent.
Proof: As the locators were chosen such that $\alpha_{\gamma}^{Mk+\gamma} = 1$, we get that $G_i + G_{i-M-1}$ is generated by the vectors
\[
\{(\alpha_1^{(i-1)k+j}, \ldots, \alpha_{\gamma}^{(i-1)k+j}) : 0 \leq j < k\} \cup \{(\alpha_1^{(i-M-2)k+j}, \ldots, \alpha_{\gamma}^{(i-M-2)k+j}) : 0 \leq j < k\}
\]
which is the set of evaluation vectors of $\gamma$ consecutive powers, and thus an independent set of vectors. It follows that $(G_i \cap G_{i-M-1})$ is generated by the intersection
\[
\{(\alpha_1^{(i-1)k+j}, \ldots, \alpha_{\gamma}^{(i-1)k+j}) : 0 \leq j < k\} \cap \{(\alpha_1^{(i-2)k+j+\gamma}, \ldots, \alpha_{\gamma}^{(i-2)k+j+\gamma}) : 0 \leq j < k\},
\]
are very likely to yield regenerating sets even over fields with these assumptions. (The matrices $G_i$ is direct, or in other words if the union of their generating sets is linearly independent.

To apply Theorem 3 directly for an explicit construction, we need to work over a field $\mathbb{F}_q$ with elements of multiplicative order $Mk + \gamma$, so $Mk + \gamma|q - 1$. In the case of unit memory, we have larger flexibility to choose the field size, in that we only need $\gamma|q - 1$. Therefore, we will study the unit memory case next. However, we will see in Table 1 that random assignments are very likely to yield regenerating sets even over fields without these assumptions.

Theorem 3. Let $\alpha = \alpha_1 \in \mathbb{F}$ be an element of order $Mk + \gamma$, and let $\alpha_i = \alpha^i$ for $1 \leq i \leq \gamma$. Then $\text{rk}(A) = Nk$, where $A$ is as defined in Definition 3.

Proof: By Lemma 6 we need to show that the vectors
\[
\{(\alpha_1^{ik-j}, \ldots, \alpha_{\gamma}^{ik-j}) : 2 \leq i \leq M + 1, 1 \leq j \leq 2k - \gamma\} \cup \{(\alpha_1^{ia}, \ldots, \alpha_{\gamma}^{ia}) : 0 \leq j < k\}
\]
are linearly independent. This set of vectors can also be written as
\[
\{(x, x^2, \ldots, x^\gamma) : x \in \{\alpha^{ik-j} : 2 \leq i \leq M + 1, 1 \leq j \leq 2k - \gamma\} \cup \{\sigma^j : 0 \leq j < k\}\},
\]
and since $\sigma$ was chosen so that $\sigma^j$ takes different values for all $0 \leq j \leq Mk + \gamma$, these are indeed $M(2k - \gamma) + k$ different vectors of the form $(x, x^2, \ldots, x^\gamma)$. We know that any set of $\gamma$ such vectors are independent, and since
\[
M(2k - \gamma) + k = k(2M + 1) - \gamma M \leq k(2M + 1) \left(1 - \frac{M}{M+1}\right) = \frac{k(2M + 1)}{M+1} \leq \gamma,
\]
the vectors are indeed linearly independent.

To apply Theorem 3 directly for an explicit construction, we need to work over a field $\mathbb{F}_q$ with elements of multiplicative order $Mk + \gamma$, so $Mk + \gamma|q - 1$. In the case of unit memory, we have larger flexibility to choose the field size, in that we only need $\gamma|q - 1$. Therefore, we will study the unit memory case next. However, we will see in Table 1 that random assignments are very likely to yield regenerating sets even over fields without these assumptions.

Theorem 4. Let $M = 1$, $\gamma = \frac{q}{2}k$ be an integer and $\text{ord}(\alpha_i)|\gamma \quad \forall i \in [\gamma]$. Then for $A$ as in Definition 3 it holds that $\text{rk}(A) = 3k$.

Proof: By Lemma 5 it holds that $\text{rk}(A) = 3k$ if $(G_1) + (G_2) \cap (G_0)$ is a direct sum, which is equivalent to $\dim((G_0) \cap (G_1) \cap (G_2)) = \dim((G_1) \cap (G_2) \cap (G_3)) = 0$.

The matrices $G_1$ and $G_2$ are given by
\[
G_1 = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{\gamma} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_{\gamma}^{k-1}
\end{pmatrix}
\]
\[
G_2 = \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{\gamma} \\
\alpha_1^{\gamma-1} & \alpha_2^{\gamma-1} & \cdots & \alpha_{\gamma}^{\gamma-1} \\
\alpha_1^{\gamma} & \alpha_2^{\gamma} & \cdots & \alpha_{\gamma}^{\gamma} \\
\alpha_1^{2k-1} & \alpha_2^{2k-1} & \cdots & \alpha_{\gamma}^{2k-1}
\end{pmatrix}
\]
\[
\overset{(a)}{=} \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{\gamma} \\
\alpha_1^{\gamma-1} & \alpha_2^{\gamma-1} & \cdots & \alpha_{\gamma}^{\gamma-1} \\
\alpha_1^{\gamma} & \alpha_2^{\gamma} & \cdots & \alpha_{\gamma}^{\gamma} \\
\alpha_1^{2k-1} & \alpha_2^{2k-1} & \cdots & \alpha_{\gamma}^{2k-1}
\end{pmatrix}
\]
TABLE I: Results of computer search for different parameters. The column $P_{\text{full}}$ gives the probability of the rank of $A$ being full obtained from checking 10000 random choices of locators from the respective field.

| $k$ | $M = \epsilon$ | $N$ | $q$ | $P_{\text{full}}$ |
|-----|----------------|-----|-----|------------------|
| 2   | 1              | 3   | 16  | $> 0.99$        |
|     | 2              | 1   | 3   | 64              | $> 0.99$ |
|     | 2              | 1   | 3   | 256             | $> 0.99$ |
| 4   | 1              | 3   | 16  | 0.93            |
| 4   | 1              | 3   | 64  | 0.98            |
| 4   | 1              | 3   | 256 | $> 0.99$        |
| 8   | 1              | 3   | 64  | 0.95            |
| 8   | 1              | 3   | 256 | $> 0.99$        |
| 16  | 1              | 3   | 64  | 0.98            |
| 16  | 1              | 3   | 256 | $> 0.99$        |
| 3   | 2              | 5   | 16  | 0.69            |
| 3   | 2              | 5   | 64  | 0.89            |
| 3   | 2              | 5   | 256 | $> 0.99$        |
| 6   | 2              | 5   | 64  | 0.98            |
| 6   | 2              | 5   | 256 | $> 0.99$        |
| 9   | 2              | 5   | 64  | 0.97            |
| 9   | 2              | 5   | 256 | $> 0.99$        |
| 18  | 2              | 5   | 256 | $> 0.99$        |
| 4   | 3              | 7   | 64  | 0.98            |
| 4   | 3              | 7   | 256 | $> 0.99$        |
| 8   | 3              | 7   | 64  | 0.98            |
| 8   | 3              | 7   | 256 | $> 0.99$        |
| 16  | 3              | 7   | 256 | $> 0.99$        |

where $(a)$ holds because $\text{ord}(\alpha_i) | \gamma$. As $\langle G_1 \rangle \cup \langle G_2 \rangle$ contains all rows of a $\gamma \times \gamma$ Vandermonde matrix it spans the entire space $\mathbb{F}^\gamma$ and therefore $\dim(\langle G_1 \cap \langle G_2 \rangle) \leq 2k - \gamma$. It follows that a complete basis of the intersection is given by

$$\text{basis}(\langle G_1 \cap \langle G_2 \rangle) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_\gamma^{k-1} \end{pmatrix}. $$

By the same argument

$$\text{basis}(\langle G_2 \cap \langle G_3 \rangle) = \begin{pmatrix} \alpha_1^k & \alpha_2^k & \cdots & \alpha_\gamma^k \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_\gamma^{k-1} \end{pmatrix}$$

and by the linear independence of the rows of Vandermonde matrices it follows that

$$\dim((\langle G_1 \cap \langle G_2 \rangle) \cap (\langle G_2 \cap \langle G_3 \rangle)) = \dim(\langle G_1 \cap \langle G_2 \rangle \cap \langle G_3 \rangle) = 0.$$

With Theorem 4 we now have an explicit construction for the considered case by choosing the locators $\alpha_i$ from a multiplicative group of size $\gamma$. It remains to be shown that a field with a multiplicative group of that order exists.

**Lemma 7.** For any $\gamma$ there exists a field $\mathbb{F}_q$ such that there is a choice of locators for which the matrix $A$ as in Theorem 4 is of full rank.

**Proof:** By Theorem 4 the matrix $A$ is always of full rank if the locators are chosen to be of order $\gamma$. A field $\mathbb{F}_q$ contains a multiplicative group of order $\gamma$ if $\gamma | q - 1$, where $q$ is a power of a prime. By Dirichlet’s theorem there are infinitely many primes of the form $p = l + m\gamma$, if $\gamma$ and $l$ are coprime. For $l = 1$ any $\gamma$ is coprime and it follows that for any $\gamma$ there are infinitely many primes $p$ such that $\gamma | p - 1$.

For applications in data storage, the most interesting fields to consider are those of characteristic 2. Whether a field $\mathbb{F}_2$ for which a construction as described in Theorem 4 is possible exists, depends on the existence of a Mersenne number $M_p = 2^p - 1$ such that $\gamma | M_p$. As all Mersenne numbers are odd, so are all their divisors and it follows that the construction over a field $\mathbb{F}_{2^r}$ is only feasible for odd $\gamma$, i.e., for $4 | k$.

For certain parameters Theorem 4 gives an explicit choice of locators such that the matrix $A$ as defined in Definition 3 is of full rank. In general it is an open problem whether matrices of such structure are of full rank, however as shown in Table I, computer searches suggest that the probability is high for a random choice of locators from a sufficiently large field.