THE EDIT DISTANCE FUNCTION AND SYMMETRIZATION

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Abstract. The edit distance between two graphs on the same labeled vertex set is the size of the symmetric difference of the edge sets. The distance between a graph, $G$, and a hereditary property, $\mathcal{H}$, is the minimum of the distance between $G$ and each $G' \in \mathcal{H}$. The edit distance function of $\mathcal{H}$ is a function of $p \in [0,1]$ and is the limit of the maximum normalized distance between a graph of density $p$ and $\mathcal{H}$.

This paper utilizes a method due to Sidorenko [Combinatorica 13(1), pp. 109-120], called “symmetrization”, for computing the edit distance function of various hereditary properties. For any graph $H$, $\text{Forb}(H)$ denotes the property of not having an induced copy of $H$. This paper gives some results regarding estimation of the function for an arbitrary hereditary property. This paper also gives the edit distance function for $\text{Forb}(H)$, where $H$ is a cycle on 9 or fewer vertices.

1. Introduction

The study of the edit distance in graphs originated independently by Axenovich, Kézdy and the author [6], Alon and Stav [2] and, in a different formulation, by Richer [18]. Since then, there has been a great deal of study on the edit distance itself and on the so-called edit distance function.

1.1. The edit distance function. The edit distance between graphs $G$ and $G'$ on the same labeled vertex set is $|E(G) \Delta E(G')|$ and is denoted $\text{dist}(G,G')$. The distance between a graph $G$ and a property $\mathcal{H}$ is

$$\text{dist}(G, \mathcal{H}) := \min \{ \text{dist}(G, G') : V(G) = V(G'), G' \in \mathcal{H} \} .$$

The edit distance function of a property $\mathcal{H}$, denoted $ed_{\mathcal{H}}(p)$, measures the maximum distance of a density $p$ graph from $\mathcal{H}$. Formally,

$$(1) \quad ed_{\mathcal{H}}(p) = \lim_{n \to \infty} \max \left\{ \text{dist}(G, \mathcal{H}) : |V(G)| = n, |E(G)| = \left\lceil p \binom{n}{2} \right\rceil \right\} / \binom{n}{2}$$

if this limit exists.

A hereditary property is a family of graphs that is closed under the taking of induced subgraphs. It is natural to study the edit distance of graphs from hereditary properties because if $H$ is an induced subgraph of $G$ and $H'$ is an induced subgraph of $G'$, then $\text{dist}(H, H') \leq \text{dist}(G, G')$.

A hereditary property $\mathcal{H}$ is trivial if there is an $n_0$ such that $\mathcal{H}$ has no $n_0$-vertex graph (hence, no $n$-vertex graph for $n \geq n_0$). Otherwise, it is nontrivial. If $\mathcal{H}$ is a

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nontrivial hereditary property, then it has an $n$-vertex graph for all natural numbers $n$. Throughout this paper, all graph properties will be nontrivial hereditary properties.

In [8], a result of Alon and Stav [2] is generalized to show that the limit in (1) does indeed exist for nontrivial hereditary properties and, furthermore, that is the limit of the expectation of the edit distance function for random graphs with the appropriate edge-probability:

$$ed_H(p) = \lim_{n \to \infty} E[\text{dist}(G(n, p), \mathcal{H})]/\binom{n}{2}. $$

It is explicitly shown in [8] that, for any nontrivial hereditary property $H$, the function $ed_H(p)$ is continuous and concave down. Hence, it achieves its maximum at a point we define to be $(p_H^*, d_H^*)$. It should be noted that, for some hereditary properties, $p_H^*$ might be an interval.

For every hereditary property $H$, there is a family of graphs that are minimal with respect to taking induced subgraphs, which we call forbidden graphs. We denote $F(H)$ to be the minimal (with respect to vertex-deletion) set of graphs $H$ for which

$$H = \bigcap_{H \in F(H)} \text{Forb}(H).$$

If $H = \bigcap_{H \in F(H)} \text{Forb}(H)$, then we denote $\overline{H}$ to be the hereditary property that is $\overline{H} = \bigcap_{H \in F(H)} \text{Forb}(\overline{H})$. I.e., $H \in F(H)$ if and only if $\overline{H} \in F(\overline{H})$. Note that $\overline{H}$ does not denote the complement of $H$ as a set.

For background on the edit distance function, applications thereof and theoretical background, we direct the reader to Balogh and the author [8], Alon and Stav [2, 3, 4, 5], Axenovich, Kézdy and the author [6], and Axenovich and the author [7]. The theoretical background upon which this is based can be traced to papers by Prömel and Steger [15, 16, 17], Bollobás and Thomason [9, 10], and Alekseev [1], among others.

1.2. Main results. The main results of this paper are Theorem 1 and Theorem 2 but we also develop a general theory and specific techniques which enable one to compute the edit distance function.

In Theorem 1 we provide bounds on the edit distance function for hereditary properties that forbid a clique. We later cite the fact that $ed_H(p) = ed_{\overline{H}}(p)$ (in Theorem 10(v)) and can be applied to hereditary properties that forbid an independent set.

**Theorem 1.** Let $H$ be a nontrivial hereditary property such that $F(H)$ contains a complete graph and let $h$ be the minimum positive integer such that $H \subseteq \text{Forb}(K_h)$. Let $\chi$ be the chromatic number of $H$ and $m$ be the smallest positive integer such that $F(H)$ contains a complete multipartite graph with $m$ parts. Clearly, $\chi \leq m \leq h$.

$$\min \left\{ \frac{p}{\chi - 1}, \frac{1 - p}{\chi - 1} + \frac{2p - 1}{m - 1} \right\} \leq ed_H(p) \leq \min \left\{ \frac{p}{\chi - 1}, 1 - p + \frac{2p - 1}{m - 1} \right\}. $$

In particular,

$$ed_{\overline{\text{Forb}(K_h)}}(p) = \frac{p}{\chi - 1}. $$
In Theorem 2, equation (2) is a trivial result, equation (3) was proven by Marchant and Thomason [13]. Some related results for $C_4$ were obtained by Alon and Stav [3]. Thomason [20] reports that Marchant [12] has proven equation (4) and (6). We note that the problem considered in [13] and in [12] is not edit distance but can be shown to be equivalent.

**Theorem 2.** Let $C_h$ denote the cycle on $h$ vertices.

\[
ed_{\text{Forb}}(C_3)(p) = \frac{p}{2}
\]

(3) \[
ed_{\text{Forb}}(C_4)(p) = p(1 - p)
\]

(4) \[
ed_{\text{Forb}}(C_5)(p) = \min\left\{\frac{p}{2}, \frac{1 - p}{2}\right\}
\]

(5) \[
ed_{\text{Forb}}(C_6)(p) = \min\left\{p(1 - p), \frac{1 - p}{2}\right\}
\]

(6) \[
ed_{\text{Forb}}(C_7)(p) = \min\left\{\frac{p}{2}, \frac{p(1 - p)}{1 + p}, \frac{1 - p}{3}\right\}
\]

(7) \[
ed_{\text{Forb}}(C_8)(p) = \min\left\{\frac{p(1 - p)}{1 + p}, \frac{1 - p}{3}\right\}
\]

(8) \[
ed_{\text{Forb}}(C_9)(p) = \min\left\{\frac{p}{2}, \frac{1 - p}{4}\right\}
\]

(9) \[
ed_{\text{Forb}}(C_{10})(p) = \min\left\{\frac{p(1 - p)}{1 + 2p}, \frac{1 - p}{4}\right\}, \text{ for } p \in [1/7, 1].
\]

**Corollary 3.** Let $C_h$ denote the cycle on $h$ vertices. Then,

\[
\left(p_{\text{Forb}}(C_h), d_{\text{Forb}}(C_h)\right) = \begin{cases}
(1, 1/2), & \text{if } h = 3; \\
(1/2, 1/4), & \text{if } h = 4; \\
(1/2, 1/4), & \text{if } h = 5; \\
(1/2, 1/4), & \text{if } h = 6; \\
(\sqrt{2} - 1, 3 - 2\sqrt{2}), & \text{if } h = 7; \\
(\sqrt{2} - 1, 3 - 2\sqrt{2}), & \text{if } h = 8; \\
(1/3, 1/6), & \text{if } h = 9; \\
((\sqrt{3} - 1)/2, (2 - \sqrt{3})/2), & \text{if } h = 10;
\end{cases}
\]

The rest of the paper is organized as follows: Section 2 gives some of the general definitions for the edit distance function, such as colored regularity graphs. Section 3 gives some theorems with which the edit distance function can be estimated. Section 4 contains the proof of Theorem 1. Section 5 defines and categorizes so-called $p$-core colored regularity graphs introduced by Marchant and Thomason [13]. Section 6 discusses the symmetrization method in general. Section 7 proves Theorem 2 regarding cycles. Section 8 gives some concluding remarks, a conjecture and acknowledgements.

**2. Development of the proofs**

2.1. **Notation.** All graphs are simple. If $S$ and $T$ are sets, then $S + T$ denotes the disjoint union of $S$ and $T$. If $G_1$ and $G_2$ are graphs, then $G_1 + G_2$ denotes the
disjoint union of the graphs and \( G_1 \lor G_2 \) denotes the join. If \( v \) and \( w \) are adjacent vertices in a graph, we denote the edge between them to be \( vw \).

2.2. Colored regularity graphs. A colored regularity graph (CRG), \( K \), is a simple complete graph, together with a partition of the vertices into black and white \( V(K) = VW(K) + VB(K) \) and a partition of the edges into black, white and gray \( E(K) = EW(K) + EG(K) + EB(K) \). We say that a graph \( H \) embeds in \( K \), (writing \( H \hookrightarrow K \)) if there is a function \( \varphi : V(H) \to V(K) \) so that if \( h_1h_2 \in E(H) \), then either \( \varphi(h_1) = \varphi(h_2) \in VB(K) \) or \( \varphi(h_1)\varphi(h_2) \in EB(K) \cup EG(K) \) and if \( h_1h_2 \not\in E(H) \), then either \( \varphi(h_1) = \varphi(h_2) \in VW(K) \) or \( \varphi(h_1)\varphi(h_2) \in EW(K) \cup EG(K) \).

For a hereditary property of graphs, we denote \( K(\mathcal{H}) \) to be the subset of CRGs such that no forbidden graph maps into \( K \). That is, \( K(\mathcal{H}) = \{ K : H \not\hookrightarrow K, \forall H \in \mathcal{F}(\mathcal{H}) \} \).

In a CRG, \( K \), vertex \( v \) is twin to vertex \( w \) if their neighborhoods are the same. That is, they are twin if (a) \( v \) and \( w \) and \( vw \) have the same color and (b) whenever \( x \in V(K) - \{ v, w \} \), the edges \( vx \) and \( wx \) are the same color.

We say that a CRG, \( K' \) is formed by the partition of a vertex \( v \) if \( V(K') = V(K) \cup \{ v' \} \) where, for every \( x \in V(K) \), the edge \( v'x \) has the same color in \( K' \) as \( vx \) has in \( K \). All other edges in \( K' \) inherit the same color as in \( K \). We say that \( K'' \) is formed by the fusion of equivalent vertices \( v \) and \( v' \) by letting \( V(K'') = V(K') - (\{ v, v' \}) \cup \{ v'' \} \) where, for every \( x \in V(K) \), the edge \( v''x \) has the same color as both \( vx \) and \( v'x \).

Two CRGs, \( K \) and \( K' \) are said to be equivalent if \( K' \) can be constructed from \( K \) by the partition of vertices or fusion of twin vertices. A CRG is reduced if it has no pair of equivalent vertices. A CRG, \( K' \) is an equipartition of CRG, \( K \) if there is an integer \( \ell \) such that each vertex in \( K \) is partitioned into exactly \( \ell \) vertices.

A CRG \( K' \) is said to be a sub-CRG of \( K \) if \( K' \) can be obtained by deleting vertices of \( K \).

2.3. The \( f \) and \( g \) functions. For every hereditary property, \( \mathcal{H} \), the function \( ed_{\mathcal{H}}(p) \) in \([1]\), measures not only the maximum normalized edit distance among density-\( p \) graphs but also the expectation of the normalized distance from \( G(n,p) \). That is, Alon and Stav \([2]\) prove that

\[
ed_{\mathcal{H}}(p) = \lim_{n \to \infty} E\{ \text{dist}(G(n,p), \mathcal{H}) \} / \binom{n}{2}.
\]

The normalized distance of \( G(n,p) \) from a hereditary property is well-defined because the distance from \( G(n,p) \) to \( \mathcal{H} \) is concentrated around its mean.

For every CRG, \( K \), we associate two functions of \( p \in [0,1] \). The function \( f \) is linear in \( p \) and \( g \) is found by the solution of a quadratic program. Let \( K \) have a total of \( k \) vertices \( \{ v_1, \ldots, v_k \} \), and let \( M_K(p) \) be a matrix such that the entries are:

\[
[M_K(p)]_{ij} = \begin{cases} 
p, & \text{if } v_i \lor v_j \in VW(K) \cup EW(K); 
1 - p, & \text{if } v_i \lor v_j \in VB(K) \cup EB(K); 
0, & \text{if } v_i \lor v_j \in EG(K). \end{cases}
\]

The normalized distance of \( G(n,p) \) from a hereditary property is well-defined because the distance from \( G(n,p) \) to \( \mathcal{H} \) is concentrated around its mean.
Then, we can express the $f$ and $g$ functions over the domain $p \in [0,1]$ as follows, with $VW = VW(K)$, $VB = VB(K)$, $EW = EW(K)$ and $EB = EB(K)$:

$$f_K(p) = \frac{1}{k^2} \left[ p(\lvert VW \rvert + 2 \lvert EW \rvert) + (1 - p)(\lvert VB \rvert + 2 \lvert EB \rvert) \right]$$

(10)  

$$g_K(p) = \begin{cases} 
\min \ x^T M_K(p) x \\
\text{s.t. } x^T \mathbf{1} = 1 \\
x \geq 0 
\end{cases}$$

(11)

If we denote $\mathbf{1}$ to be the vector of all ones, then $f_K(p) = (\frac{1}{k} \mathbf{1})^T M_K(p) (\frac{1}{k} \mathbf{1})$. So, $f_K(p) \geq g_K(p)$.

**Fact 4.** The function $g$ is invariant under equivalence classes of CRGs. That is, if $K$ and $K'$ are equivalent CRGs, then $g_K(p) = g_{K'}(p)$ for all $p \in [0,1]$.

We can use both the $f$ and $g$ functions of CRGs to compute the edit distance function.

**Theorem 5** ([8]). For any nontrivial hereditary property $\mathcal{H}$,

$$ed_H(p) = \inf_{K \in \mathcal{K}(\mathcal{H})} g_K(p) = \inf_{K \in \mathcal{K}(\mathcal{H})} f_K(p).$$

**Remark 6.** Marchant and Thomason [13] prove that, in fact, $ed_H(p) = \min_{K \in \mathcal{K}(\mathcal{H})} g_K(p)$. That is, for every $p \in [0,1]$, there is a CRG, $K \in \mathcal{K}(\mathcal{H})$, such that $ed_H(p) = g_K(p)$.

A sub-CRG, $K'$, of a CRG, $K$, is a component if, for all $v \in V(K')$ and all $w \in V(K) - V(K')$, the edge $vw$ is gray. Theorem 7 allows the computation of $g_K$ from the $g$ functions of its components.

**Theorem 7.** Let $K$ be a CRG with components $K^{(1)}, \ldots, K^{(\ell)}$. Then

$$(g_K(p))^{-1} = \sum_{i=1}^{\ell} (g_{K^{(i)}}(p))^{-1}.$$

**Proof.** The matrix $M_K(p)$ is a block-diagonal matrix. Let $g_i = g_{K^{(i)}}(p)$ for $i = 1, \ldots, \ell$ and $g = g_K(p)$. We may first assign the total weights of the vertices in each component. Then, the relative weights of the vertices in each component is defined by that component’s $g$ function.

Let $\alpha_i$ denote the total weight that the optimal solution of (11) assigns to the vertices of $K^{(i)}$. Then, we obtain the following optimization problem:

$$g = \begin{cases} 
\min \ \alpha_1^2 g_1 + \cdots + \alpha_\ell^2 g_\ell \\
\text{s.t. } \alpha_1 + \cdots + \alpha_\ell = 1 \\
\alpha_1, \ldots, \alpha_\ell \geq 0 
\end{cases}$$

Using the method of Lagrange multipliers, we see that the solution is $\alpha_i = \lambda / g_i$ for $i = 1, \ldots, \ell$ and $\lambda^{-1} = \sum_{i=1}^{\ell} g_i^{-1}$. Substituting these values gives the theorem statement. \(\square\)
Theorem 7 can be applied directly to CRGs that have only gray edges. Since the \( g \) function for a white vertex is \( p \) and the \( g \) function for a black vertex is \( 1 - p \), we have Corollary 8.

**Corollary 8.** If \( K \) is a CRG all of whose edges are gray, then
\[
g_K(p) = \left( \frac{|VW(K)|}{p} + \frac{|VB(K)|}{1 - p} \right)^{-1}.
\]

Proposition 9 gives the edit distance function for some special CRGs that have no gray edges.

**Proposition 9.** Let \( K \) be a CRG on \( k \) vertices and no gray edges as follows:
- If all vertices are white and all edges are black, then \( g_K(p) = \min \{ p, 1 - p + (2p - 1)/k \} \).
- If all vertices are black and all edges are white, then \( g_K(p) = \min \{ p + (1 - 2p)/k, 1 - p \} \).

### 3. Estimation of the edit distance function

Denote \( K(r,s) \) to be the CRG with \( r \) white vertices, \( s \) black vertices and all gray edges. Let \( \mathcal{H} \) be a hereditary property with \( \mathcal{H} = \bigcap_{H \in \mathcal{F}(\mathcal{H})} \text{Forb}(H) \). The notion of \((r,s)\)-colorability is discussed by Alon and Stav where they focus on hereditary properties that are complement-invariant.

The **chromatic number** of \( \mathcal{H} \), denoted \( \chi(\mathcal{H}) \) or just \( \chi \), is \( \min \{ \chi(H) : H \in \mathcal{F}(\mathcal{H}) \} \). The **complementary chromatic number** \( \chi^*(\mathcal{H}) \) or \( \chi^* \), is \( \min \{ \chi(H) : H \in \mathcal{F}(\mathcal{H}) \} \). The **binary chromatic number** is \( \max \{ k + 1 : \exists r, s, r + s = k, H \not\in K(r,s), \forall H \in \mathcal{F}(\mathcal{H}) \} \).

The **clique spectrum** of \( \mathcal{H} \) is the set
\[
\Gamma(\mathcal{H}) \overset{\text{def}}{=} \{(r,s) : H \not\in K(r,s), \forall H \in \mathcal{F}(\mathcal{H})\}.
\]

The clique spectrum has a number of useful properties. For example, it is monotone in the sense that if \((r,s) \in \Gamma(\mathcal{H})\) and \(0 \leq r' \leq r\) and \(0 \leq s' \leq s\), then \((r',s') \in \Gamma(\mathcal{H})\). As a result, the clique spectrum of a hereditary property can be expressed as a Young tableau. An extreme point of the clique spectrum \( \Gamma \) is a pair \((r,s) \in \Gamma\) for which both \((r + 1, s) \notin \Gamma\) and \((r, s + 1) \notin \Gamma\). Let \( \Gamma^* \) denote the extreme points of clique spectrum \( \Gamma \). Figure 3 shows the clique spectrum of the cycle \( C_9 \) expressed as a Young tableau, with the extreme points of the clique spectrum marked.

#### 3.1. Approximating \( \mathit{ed}_{\mathcal{H}}(p) \) by \( \gamma_{\mathcal{H}}(p) \)

Corollary 8 gives that \( g_{K(r,s)}(p) = \frac{p(1-p)}{r(1-p) + sp} \), which follows directly from Theorem 7. Define the function \( \gamma_{\mathcal{H}}(p) \) as follows:
\[
\gamma_{\mathcal{H}}(p) \overset{\text{def}}{=} \min \{ g_{K(r,s)}(p) : (r,s) \in \Gamma(\mathcal{H}) \} = \min \left\{ \frac{p(1-p)}{r(1-p) + sp} : (r,s) \in \Gamma(\mathcal{H}) \right\}.
\]

\footnote{Unfortunately, the term “cochromatic number” is taken. It should be noted that the cochromatic number, although its definition resembles that of \( \chi_B \), is not the same parameter.}
Figure 1. The clique spectrum of $C_9$ expressed as a Young tableau. The extreme points of the clique spectrum are labeled.

Clearly, $ed_H(p) \leq \gamma_H(p)$. Moreover, $\gamma_H(p) = \min \{ g_{K(r,s)}(p) : (r, s) \in \Gamma^*(H) \}$; i.e., only $(r, s)$ that are extreme points of the clique spectrum need to be used to compute $\gamma$. The value of the function $\gamma_H(p)$ is that it is computable for any hereditary property.

3.2. Basic observations on $ed_H(p)$. The following is a summary of basic facts about the edit distance function. Item (iii) comes from Alon and Stav [2]. Item (iv) comes from [8]. The remaining items are trivial.

**Theorem 10.** Let $H$ be a nontrivial hereditary property with chromatic number $\chi$, complementary chromatic number $\overline{\chi}$, binary chromatic number $\chi_B$ and edit distance function $ed_H(p)$.

(i) If $\chi > 1$, then $ed_H(p) \leq p/(\chi - 1)$.
(ii) If $\overline{\chi} > 1$, then $ed_H(p) \leq (1 - p)/(\overline{\chi} - 1)$.
(iii) $ed_H(1/2) = 1/(2(\chi_B - 1)) = \gamma_H(1/2)$.
(iv) $ed_H(p)$ is continuous and concave down.
(v) $ed_H(p) = ed_H(1 - p)$.

There are a number of immediate corollaries of Theorem 10 that help estimate the edit distance functions. Some of the most useful are summarized in Corollary 11 and we leave the proof of them to the reader.

**Corollary 11.** Let $H$ be a nontrivial hereditary property with binary chromatic number $\chi_B$. Let $(r, s)$ be extreme points in the clique spectrum of $H$ such that $r + s = \chi_B$.

(i) If $\chi = \chi_B$, then $ed_H(p) = p/(\chi - 1)$ for all $p \in [0, 1/2]$.
(ii) If $\overline{\chi} = \overline{\chi_B}$, then $ed_H(p) = (1 - p)/(\chi_B - 1)$ for all $p \in [1/2, 1]$.
(iii) If $r \geq s$, then $p_H^* \geq 1/2$.
(iv) If $r \leq s$, then $p_H^* \leq 1/2$.
(v) For any $(r, s)$ in the clique spectrum, $d_H^* \leq (\sqrt{r} + \sqrt{s})^{-2}$.

4. $H \subseteq \text{Forb}(K_h)$

In this section, we prove Theorem 1 which bounds the edit distance function for hereditary properties that have no copy of a complete graph. Note that $H \subseteq \text{Forb}(K_h)$ if and only if $K_h \in \mathcal{F}(H)$. 
Proof of Theorem 1. Since \( \chi(H) = 1 \) and \( H \) is not trivial, \( \chi(H) > 1 \). If \( K \in K(H) \), then \( K \) cannot have a black vertex, otherwise \( K_h \to K \). So, we may assume that \( K \in K(H) \) has all white vertices. In every set of \( \chi \) white vertices, there must be a non-gray edge. By Turán’s theorem, this means that \( K \) has at least \( \left( k \right) - \frac{\chi - 2}{\chi - 1} \cdot \frac{k^2}{2} \) non-gray edges. Hence,

\[
ed_H(p) \geq f_K(p) \geq \frac{1}{k^2} \left[ pk + 2 \min\{p, 1 - p\} \left( \left( \left( k \right) - \frac{\chi - 2}{\chi - 1} \cdot \frac{k^2}{2} \right) \right) \right] \geq \frac{\min\{p, 1 - p\}}{\chi - 1}.
\]

In every set of \( m \) white vertices, there must be a white edge. Again, by Turán’s theorem, \( ed_H(p) \geq f_K(p) \geq \frac{p}{m - 1} \). So, \( ed_H(p) \) is bounded below by both \( p/(\chi - 1) \) and the line segment connecting the points \( \left( 1/2, \frac{1}{2(\chi - 1)} \right) \) and \( \left( 1, \frac{1}{m - 1} \right) \). Hence,

\[
ed_H(p) \geq \min \left\{ \frac{p}{\chi - 1}, \frac{1 - p}{\chi - 1} + \frac{2p - 1}{m - 1} \right\}.
\]

As to the upper bound, we give two CRGs into which no \( H \in F(H) \) can map. The first is \( K^{(1)} \leq K(\chi - 1, 0) \), the CRG with \( \chi - 1 \) white vertices and all edges gray. By Corollary 3, \( g_{K^{(1)}}(p) = p/(\chi - 1) \).

The second CRG, \( K^{(2)} \), is \( m - 1 \) white vertices and all black edges. If there were some \( H \in F(H) \) such that \( H \to K^{(2)} \), then \( H \) would be a complete \( (m - 1) \)-partite graph, which is forbidden by our choice of \( m \). By Proposition 9, \( g_{K^{(2)}}(p) = \min\{p, 1 - p + (2p - 1)/(m - 1)\} \). So,

\[
ed_H(p) \leq \min \left\{ \frac{p}{\chi - 1}, p, 1 - p + \frac{2p - 1}{m - 1} \right\}.
\]

The final statement comes from the observation that if \( H = \text{Forb}(K_h) \), then \( \chi = m = h \).

By Theorem 10[4] we have the similar result for empty graphs: Let \( H \) be a nontrivial hereditary property such that \( F(H) \) contains an empty graph and \( h \) be the minimum positive integer such that \( H \subseteq \text{Forb}(K_h) \). Let \( \chi \) be the complementary chromatic number\(^2\) of \( H \) and \( m \) be the smallest positive integer such that \( F(H) \) contains a \( m \) disjoint cliques. Clearly, \( \chi \leq m \leq h \).

\[
\min \left\{ \frac{\chi - 1}{p} + \frac{1 - 2p}{m - 1}, \frac{1 - p}{\chi - 1} \right\} \leq ed_H(p) \leq \min \left\{ \frac{1 - 2p}{m - 1}, \frac{1 - p}{\chi - 1} \right\}.
\]

In particular, \( ed_{\text{Forb}(K_h)}(p) = \frac{1 - p}{\chi - 1} \).

5. The \( p \)-core CRGs

Recall that, in Remark 6 we observed that \( ed_H(p) = \min \{ g_K(p) : K \in K(H) \} \). That is, for any hereditary property \( H \) and \( p \in [0, 1] \), there is a CRG, \( K \in K(H) \) such that \( ed_H(p) = g_K(p) \). This is found by looking at so-called \( p \)-core CRGs. A CRG, \( K \), is a \( p \)-core CRG, or simply a \( p \)-core, if \( g_K(p) < g_{K'}(p) \) for all nontrivial sub-CRGs \( K' \) of \( K \).

\(^2\)The term \( \chi(H) \) is, the smallest number, \( k \), such that no member of \( F(H) \) can be partitioned into \( k \) cliques. In fact, \( \chi(H) = \chi(\overline{H}) \).
Moreover, $p$-cores can be easily classified:

**Theorem 12** (Marchant-Thomason, [13]). Let $K$ be a $p$-core CRG.

- If $p = 1/2$, then $K$ has all of its edges gray.
- If $p < 1/2$, then $\text{EB}(K) = \emptyset$ and there are no white edges incident to white vertices.
- If $p > 1/2$, then $\text{EW}(K) = \emptyset$ and there are no black edges incident to black vertices.

The optimal solution to the quadratic program in (11) is, in some sense, regular, as described in Theorem 13. Theorem 13 is the “symmetrization” referenced in the title.

The fundamental observation is that if every optimal solution, $x^*$, of (11) has no zero entries, then

\[ M_K(p) \cdot x^* = g_K(p) \mathbf{1}, \]

where $\mathbf{1}$ is the all-ones vector. Of course, an optimal solutions having no zero entries corresponds, by definition, to a CRG being $p$-core and that the optimal solution to quadratic program in (11) is unique.

By Theorem 12 if $K$ is a $p$-core CRG, then no edge has the same color as either of its endvertices, so we can reinterpret (12) as follows:

**Theorem 13** (Marchant-Thomason, [13]). Let $K$ be a $p$-core CRG. There is a unique vector $x$ that is an optimal solution to the quadratic program in (11). For all $v \in V(K)$, let the entry of $x$ corresponding to $v$ be $x(v)$. For each $v \in V(K)$,

\[ g_K(p) = x(v) \left[ p \text{d}_W(v) + (1 - p) \text{d}_B(v) \right], \]

where

\[ \text{d}_W(v) = \begin{cases} x(v), & \text{if } v \in WV(K); \\ \sum_{vz \in EW(K)} x(z), & \text{if } v \in VB(K); \end{cases} \]

and

\[ \text{d}_B(v) = \begin{cases} x(v), & \text{if } v \in VB(K); \\ \sum_{vz \in EB(K)} x(z), & \text{if } v \in VW(K). \end{cases} \]

### 6. Computing edit distance functions using symmetrization

Theorem 13, Theorem 12, Remark 6 and the definition of $p$-cores have all of the elements in order to express $d_G(v) := 1 - d_W(v) - d_B(v)$ for any vertex $v$ in a $p$-core CRG. It is often useful and intuitive to focus on the gray neighborhood of vertices.

**Lemma 14.** Let $p \in (0, 1)$ and $K$ be a $p$-core CRG with optimal weight function $x$.

(i) If $p \leq 1/2$, then $x(v) = g_K(p)/p$ for all $v \in VW(K)$ and

\[ d_G(v) = \frac{p - g_K(p)}{p} + \frac{1 - 2p}{p} x(v), \quad \text{for all } v \in VB(K). \]

(ii) If $p \geq 1/2$, then $x(v) = g_K(p)/(1 - p)$ for all $v \in VB(K)$ and

\[ d_G(v) = \frac{1 - p - g_K(p)}{1 - p} + \frac{2p - 1}{1 - p} x(v), \quad \text{for all } v \in VW(K). \]

\footnote{Pikhurko [14] uses this term for the approach by Sidorenko [19].}
Proof. We will prove the case for $p \leq 1/2$. The case where $p \geq 1/2$ is symmetric. Let $v \in VW(K)$. By Theorem 12, all vertices are incident to $v$ via a gray edge, and by Theorem 13, $g_K(p) = p \mathbf{x}(v)$. Now let $v \in VB(K)$. By Theorem 12, $v$ has no black neighbors and

$$g_K(p) = p(1 - \mathbf{x}(v) - d_G(v)) + (1 - p)\mathbf{x}(v).$$

Solving for $d_G(v)$ gives the result. $\square$

Lemma 15. Let $p \in (0,1)$ and $K$ be a $p$-core CRG with optimal weight function $\mathbf{x}$.

(i) If $p \leq 1/2$, then $\mathbf{x}(v) \leq g_K(p)/(1 - p)$ for all $v \in VB(K)$.

(ii) If $p \geq 1/2$, then $\mathbf{x}(v) \leq g_K(p)/p$ for all $v \in VW(K)$.

Proof. We use the fact that $\mathbf{x}(v) + d_G(v) \leq 1$. Applying Lemma 14 and solving for $\mathbf{x}(v)$ gives the result. $\square$

Remark 16. From this point forward in the paper, if $K$ is a CRG under consideration and $p$ is fixed, $\mathbf{x}(v)$ will denote the weight of $v \in V(K)$ under the optimal solution of the quadratic program in equation (11) that defines $g_K$.

7. $\text{Forb}(C_h)$, $h \in \{3, \ldots, 9\}$

Thomason [20] reports that Ed Marchant has found the edit distance function for $C_5$ and $C_7$. Here we find the function for all $C_h$, $h \in \{3, \ldots, 9\}$. The proofs in this section might be substantially similar to Marchant’s.

In order to compute the edit distance function for cycles, we first make the observation that $C_3$ is a complete graph and so Theorem 1 gives Corollary 17.

Corollary 17. $\text{ed}_{\text{Forb}(C_3)}(p) = p/2$.

Furthermore, the only $p$-core for which this is achieved for $p \in (0,1)$ is $K(2,0)$.

For $C_h$, $h \geq 4$, we first take care of easy cases so that the only $p$-cores that need to be considered have all black vertices. We use Lemma 18 which establishes the upper bound and eliminates all cases except when $p \leq 1/2$ and all vertices are black.

Lemma 18. Let $h \geq 4$ and $p \in (0,1)$.

$$\gamma_{\text{Forb}(C_h)}(p) = \begin{cases} p(1 - p), & \text{if } h = 4; \\ \min \left\{ \frac{p(1-p)}{1-p+((h/3)-1)p}, \frac{1-p}{(h/2)-1} \right\}, & \text{if } h \geq 6 \text{ is even}; \\ \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1-p+((h/3)-1)p}, \frac{1-p}{(h/2)-1} \right\}, & \text{if } h \text{ is odd}. \end{cases}$$

Furthermore, if there is a $p$-core CRG, $K \in \mathcal{K}(\text{Forb}(C_h))$ such that $g_K(p) < \gamma_{\text{Forb}(C_h)}(p)$ for any $p \in (0,1)$, then $p < 1/2$ and $K$ has all black vertices.
Proof. We leave it to the reader to verify that the extreme points of the clique spectrum of \( \text{Forb}(C_h) \) are \((0, \lceil h/2 \rceil - 1), (1, \lceil h/3 \rceil - 1) \) and, if \( h \) is odd, \((2, 0)\). This establishes the value of \( \gamma_{\text{Forb}(C_h)}(p) \).

If \( h = 4 \), the classes of possible CRGs are restricted. If \( K \) has at least 2 white vertices, they are connected via a gray or black edge and so \( C_4 \) would embed in \( K \). If \( K \) has a white and at least two black vertices, then the edges between the white and black vertices are both gray and the edge between the black vertices is either gray or white and so \( C_4 \) would embed in \( K \). Thus, if \( K \) has a white vertex, then it has at most one black vertex and this is \( K(1, 1) \), the CRG that defines \( \gamma_{\text{Forb}(C_4)}(p) = p(1-p) \). If \( K \) has all white edges, then \( g_K(p) = \min\{p+(1-2p)/k, 1-p\} > p(1-p) \). So, \( ed_{\text{Forb}(C_4)}(p) = p(1-p) \).

Now, let \( h \geq 5 \). Since \( \gamma_{H}(1/2) = ed_{H}(1/2) \) for all hereditary properties and \( 0 = \gamma_{\text{Forb}(C_3)}(1) \), convexity gives that \( ed_{H}(p) = \frac{1-p}{1-h/2} \), for all \( p \geq 1/2 \).

Finally, let \( p \in (0, 1/2) \) and \( K \) be a \( p \)-core CRG such that \( C_h \not\rightarrow K \). If \( K \) has only white vertices and \( h \) is even, then \( K \approx K(1, 0) \) and \( g_K(p) = p > \gamma_H(p) \). If \( K \) has only white vertices and \( h \) is odd, then there are at most 2 white vertices and \( g_K(p) \geq p/2 \) with equality if and only if \( K \approx K(2, 0) \).

If \( K \) has both white and black vertices, then it has at most 1 white vertex because \( C_h \not\rightarrow K(2, 1) \). Furthermore, it can have at most \( \lceil h/3 \rceil - 1 \) black vertices. To see this, denote the vertices of \( C_h \) by \( \{0, 1, \ldots, h-1\} \) where \( 0 \sim 1 \sim \cdots \sim h-1 \sim 0 \). Let \( S \) consist of the members of \( \{0, \ldots, h-2\} \) that are divisible by 3. If \( h-1 \) is divisible by 3, then add \( h-2 \) to \( S \). The graph \( C_h - S \) has \( \lceil h/3 \rceil \) connected components, each of which are cliques of size 1 or 2. Thus, regardless of whether the edges are white or gray, there are at most \( \lceil h/3 \rceil - 1 \) black vertices in \( K \) and \( g_K(p) \geq \frac{p(1-p)}{1-p+(\lceil h/3 \rceil - 1)p} \), with equality if and only if \( K \approx K(1, \lceil h/3 \rceil - 1) \).

Summarizing, if \( p \in (0, 1/2) \) and \( g_K(p) = ed_{\text{Forb}(C_h)}(p) \), then \( K \) is either \( K(0, \lceil h/2 \rceil - 1) \), \( K(1, \lceil h/3 \rceil - 1) \), \( K(2, 0) \) and \( h \) is odd, or \( K \) has all black vertices (and white or gray edges).

From this point forward, we only restrict ourselves to \( p \in (0, 1/2) \) and CRGs, \( K \), with only black vertices and white or gray edges because of Lemma 18. We can immediately address 4- and 5-cycles. Corollary 19 and Corollary 20 have appeared before. Corollary 19 was proven in the proof of Lemma 18.

Corollary 19 (Marchant-Thomason 13).

\[
ed_{\text{Forb}(C_4)}(p) = p(1-p).
\]

Corollary 20 (12).

\[
ed_{\text{Forb}(C_5)}(p) = \min \left\{ \frac{p}{2}, \frac{1-p}{2} \right\}.
\]

Proof. Thanks to Lemma 18, we can restrict to \( p \in (0, 1/2) \) and \( p \)-core CRGs \( K \in \text{Forb}(C_5) \) for which the vertices are black. Let \( v_1 \) have largest weight in \( K \) and \( v_2 \) have largest weight in \( N_G(v_1) \). Let \( g \) denote \( g_K(p) \). Since \( K \) has no
triangles,
\[
\frac{2p - g}{p} + \frac{1 - 2p}{p} (x(v_1) + x(v_2)) \leq 1
\]
\[
\frac{1 - 2p}{p} (x(v_1) + x(v_2)) \leq \frac{2g - p}{p}.
\]
So, \( g > \frac{p}{2} \), a contradiction. \( \square \)

See Figure 2 and Figure 3.

![Figure 2](image1.png)

**Figure 2.** Plot of \( ed_{\text{Forb}(C_4)}(p) = p(1 - p) \). The boundary of the shaded region is \( ed_{\text{Forb}(C_4)}(p) \).

![Figure 3](image2.png)

**Figure 3.** Plot of \( ed_{\text{Forb}(C_5)}(p) = \min\{\frac{p}{2}, \frac{1-p}{2}\} \).

Proposition 21 shows that in order to find CRGs with black vertices, white or gray edges with no \( C_h \), there are many lengths of gray cycles that are forbidden in the CRG.

**Proposition 21.** Let \( p \in (0, 1/2) \) and \( K \) be a \( p \)-core CRG such that \( K \) has black vertices and white and gray edges. If \( C_h \not\to K \) then \( K \) has no gray cycle with length in \( \{\lceil h/2 \rceil, \ldots, h\}\).

**Proof.** If \( C_h \to K \), then each vertex of \( K \) receives either one or two vertices that are consecutive on the cycle. Thus, the cycle \( K \) must contain is one that corresponds to the contraction of edges of \( C_h \) that map to a single black vertex of \( K \). Since these edges form a matching, the cycle required to be in \( K \) has length at least \( \lceil h/2 \rceil \) and at most \( h \). \( \square \)

In order to deal with \( \text{Forb}(C_h) \) for \( h \geq 6 \), we use Proposition 21 along with two major lemmas. Lemma 22 is a general structural lemma and the results on \( \text{Forb}(C_h) \) that we give are immediate corollaries. It should be noted that if we write that a CRG, say, “has no gray 4-cycle,” we mean so in the subgraph sense, so it does not contain a gray \( K_4 \) either.

**Lemma 22.** Let \( p \in (0, 1/2) \) and \( K \) be a \( p \)-core with black vertices and white or gray edges.

(i) If \( K \) has no gray edge, then \( g_K(p) > p \).
(ii) If \( K \) has neither a gray 3-cycle nor a gray 4-cycle, then \( g_K(p) > p(1 - p) \).

(iii) If \( K \) has no gray 3-cycle, then \( g_K(p) > p/2 \).

(iv) If \( K \) has a gray 3-cycle, but no gray \( C_4^+ \) (that is, four vertices that induce 5 gray edges), then \( g_K(p) \geq \min\{2p/3, (1 - p)/3\} \).

(v) If \( K \) has no gray 4-cycle, then \( g_K(p) > p(1 - p) \) for \( p \in (0, 1/3) \).

(vi) If \( K \) has a gray \( C_4^+ \) but no gray \( C_5^+ \) (that is, five vertices that induce some 5-cycle with two chords), then \( g_K(p) > \min\{2p/3, p/(1-p)/(1+p)\} \).

(vii) If \( K \) has a gray chordless 4-cycle, but no gray \( K_{3,3}^- \) (that is, a \( K_{3,3} \) missing an edge), then \( g_K(p) > \min\{2p/3, 2p(1 - p)/(2 + p)\} \). Note that \( K_{3,3}^- \) has a 6-cycle as a subgraph.

Proof. For ease of notation, in calculations, we sometimes let \( g \) denote \( g_K(p) \).

(i) If \( K \) has no gray edges, then for any \( v \in V(K) \), \( g = p + (1 - 2p)x(v) > p \).

(ii) Let \( v_0 \in V(K) \) have the largest weight and \( N_G(v_0) = \{x_1, \ldots, x_\ell\} \), the gray neighborhood of \( v_0 \). Let \( x_i = x(v_i) \) for \( i = 0, 1, \ldots, \ell \). Since there are no gray triangles, there are no gray edges in \( N_G(v_0) \) and since there are no gray quadrangles, \( N_G(v_i) - \{v_0\} \) and \( N_G(v_j) - \{v_0\} \) are disjoint for all distinct \( i, j \in \{1, \ldots, \ell\} \). So, \( \{v_0\}, N_G(v_0) \) and each \( N_G(v_i) - \{v_0\} \), \( i = 1, \ldots, \ell \) form a family of \( \ell + 2 \) pairwise disjoint sets.

\[
x_0 + d_G(v_0) + \sum_{i=1}^{\ell} [d_G(v_i) - x_0] \leq 1
\]

\[
x_0 + d_G(v_0) + \sum_{i=1}^{\ell} \left[ \frac{p-g}{p} + \frac{1-2p}{p} x_i - x_0 \right] \leq 1
\]

\[
x_0 + d_G(v_0) + \ell \left[ \frac{p-g}{p} - x_0 \right] + \frac{1-2p}{p} d_G(v_0) \leq 1
\]

\[
x_0 + \frac{1-p}{p} d_G(v_0) + \ell \left[ \frac{p-g}{p} - x_0 \right] \leq 1.
\]

Since \( x_0 \) is the largest weight, \( \ell \geq d_G(v_0)/x_0 \) and as long as \( g \geq p(1-p) \), we have \( \frac{p-g}{p} - x_0 \geq \frac{p-g}{1-2p} > 0 \) by Lemma 15(i). Consequently,

\[
x_0 + \frac{1-p}{p} d_G(v_0) + \frac{d_G(v_0)}{x_0} \left[ \frac{p-g}{p} - x_0 \right] \leq 1
\]

\[
x_0^2 + d_G(v_0) \left[ \frac{p-g}{p} + \frac{1-2p}{p} x_0 \right] \leq x_0
\]

\[
x_0^2 + \left[ \frac{p-g}{p} + \frac{1-2p}{p} x_0 \right]^2 \leq x_0
\]

\[
(13) \quad \left( \frac{p-g}{p} \right)^2 + \left[ 2 \cdot \frac{p-g}{p} \cdot \frac{1-2p}{p} - 1 \right] x_0 + \left[ 1 + \left( \frac{1-2p}{p} \right)^2 \right] x_0^2 \leq 0.
\]
A quadratic expression of the form \( c + bx + ax^2 \) with \( a > 0 \) has a minimum value of \( c - b^2/4a \).

\[
\left( \frac{p-g}{p} \right)^2 - \frac{\left( 2 \cdot \frac{p-g}{p} \cdot \frac{1-2p}{p} - 1 \right)^2}{4 \left( 1 + \left( \frac{1-2p}{p} \right)^2 \right)} \leq 0
\]

\[
4 \left( \frac{p-g}{p} \right)^2 + 4 \left( \frac{p-g}{p} \right) \left( \frac{1-2p}{p} \right) - 1 \leq 0.
\]

So,

\[
\frac{p-g}{p} \leq \frac{1}{2} \left( \frac{1-2p}{p} + \sqrt{\left( \frac{1-2p}{p} \right)^2 + 1} \right)
\]

\[
g \geq \frac{1}{2} \left( 1 - \sqrt{1-4p+5p^2} \right).
\]

This expression is greater than \( p(1-p) \) for all \( p \in (0, 1/2) \).

(iii) By (i), we may assume that \( K \) has a gray edge, otherwise \( g_K(p) > p \).

Let \( v_1v_2 \) be a gray edge and \( x_i = x(v_i) \) for \( i = 1, 2 \). Since they have no common gray neighbor,

\[
d_G(v_1) + d_G(v_2) \leq 1
\]

\[
2 \left( \frac{p-g}{p} \right) + \frac{1-2p}{p} (x_1 + x_2) \leq 1
\]

Since \( x_1 + x_2 > 0 \), we have \( g > p/2 \).

(iv) Let \( \{v_1, v_2, v_3\} \) be a gray triangle in \( K \) where \( x_i = x(v_i) \) for \( i = 1, 2, 3 \). Since no pairs of \( v_i \) can have a common neighbor other than the remaining \( v_j \),

\[
\sum_{i=1}^{3} [d_G(v_i) - (x_1 + x_2 + x_3 - x_i)] + (x_1 + x_2 + x_3) \leq 1
\]

\[
\sum_{i=1}^{3} d_G(v_i) - (x_1 + x_2 + x_3) \leq 1
\]

\[
3 \left( \frac{p-g}{p} \right) + \frac{1-3p}{p} (x_1 + x_2 + x_3) \leq 1
\]

\[
\frac{2p}{3} + \frac{1-3p}{3} (x_1 + x_2 + x_3) \leq g.
\]

If \( p < 1/3 \), then \( g > 2p/3 \). If \( p > 1/3 \), then \( x_1 + x_2 + x_3 \leq 1 \) implies that \( g \geq (1-p)/3 \).

(v) Let \( v_0 \in V(K) \) have the largest weight. Since there are no gray quadrangles, no member of \( N_G(v_0) \) has more than one gray neighbor in \( N_G(v_0) \).

Let \( N_G(v_0) = \{x_1, x_1', \ldots, x_m, x_m'\} \cup \{x_{2m+1}, \ldots, x_{\ell} \} \), the gray neighborhood of \( v_0 \) such that for \( i = 1, \ldots, m \), \( x_i x_i' \) is a gray edge. Let \( x_i = x(v_i) \)
for \( i = 0, 1, \ldots, \ell \). Since there are no gray quadrangles, the gray neighborhoods outside of \( \{v_0\} \cup N_G(v_0) \) of distinct vertices in \( N_G(v_0) \) are distinct. Hence,

\[
x_0 + d_G(v_0) + \sum_{i=1}^{m} [d_G(v_i) + d_G(v'_i) - x_i - x'_i - 2x_0] + \sum_{j=2m+1}^{\ell} [d_G(v_j) - x_0] \leq 1
\]

\[
x_0 + d_G(v_0) + \ell \left( \frac{p-g}{p} - x_0 \right) + \sum_{i=1}^{m} \left( \frac{1-3p}{p} \right) (x_i + x'_i)
\]

\[
+ \sum_{j=2m+1}^{\ell} \left( \frac{1-2p}{p} \right) x_j \leq 1
\]

\[
\ell \left( \frac{p-g}{p} - x_0 \right) + x_0 + d_G(v_0) + \left( \frac{1-3p}{p} \right) d_G(v_0) \leq 1.
\]

Again, we use the fact that \( \ell \geq d_G(v_0)/x_0 \) and \( \frac{p-g}{p} - x_0 \geq 0 \).

\[
\frac{d_G(v_0)}{x_0} \left( \frac{p-g}{p} - x_0 \right) + x_0 + \left( \frac{1-2p}{p} \right) d_G(v_0) \leq 1
\]

\[
\left( \frac{p-g}{p} \right)^2 + \left[ \frac{p-g}{p} \cdot \frac{2-5p}{p} - 1 \right] x_0 + \left[ \frac{1-2p}{p} \cdot \frac{1-3p}{p} + 1 \right] x_0^2 \leq 0.
\]

Optimizing over \( x_0 \),

\[
\left( \frac{p-g}{p} \right)^2 - \frac{\left( \frac{p-g}{p} \cdot \frac{2-5p}{p} - 1 \right)^2}{4 \left( \frac{1-2p}{p} \cdot \frac{1-3p}{p} + 1 \right)} \leq 0
\]

\[
\left( \frac{p-g}{p} \right)^2 \left[ \frac{1-2p}{p} \cdot \frac{1-3p}{p} + 4 - \left( \frac{2-5p}{p} \right)^2 \right]
\]

\[
+ 2 \cdot \frac{p-g}{p} \cdot \frac{2-5p}{p} - 1 \leq 0
\]

\[
3 \left( \frac{p-g}{p} \right)^2 + 2 \left( \frac{2-5p}{p} \right) \left( \frac{p-g}{p} \right) - 1 \geq 0.
\]

So,

\[
\frac{p-g}{p} \leq \frac{1}{3} \left( \frac{2-5p}{p} + \sqrt{\left( \frac{2-5p}{p} \right)^2 + 3} \right)
\]

\[
g \geq \frac{2}{3} \left( 1 - p - \sqrt{1-5p + 7p^2} \right).
\]

Some calculations show that \( g > p(1-p) \) for \( p \in (0, 1/3) \).

(vi) Let the gray \( C^+_4 \) be denoted \( \{v_1, v_2, v_3, v_4\} \) such that all edges are gray except, perhaps \( v_1v_3 \). Let \( x_i = x(v_i) \) for \( i = 1, 2, 3, 4 \). Without loss of generality, let \( x_2 \geq x_4 \).
No pair \((v_i, v_j)\) can have a common gray neighbor except, perhaps \((v_2, v_4)\). Denoting \(N_G(v)\) to be the set of gray neighbors of vertex \(v\), the sets \(N_G(v_1) - \{v_2, v_4\}\), \(N_G(v_3) - \{v_2, v_4\}\) and \(N_G(v_2) - \{v_1, v_3, v_4\}\) must be disjoint. So,

\[
(d_G(v_1) - x_2 - x_4) + (d_G(v_3) - x_2 - x_4) \\
+ (d_G(v_2) - x_1 - x_3 - x_4) + (x_1 + x_2 + x_3 + x_4) \leq 1 \\
3 \cdot \frac{p-g}{p} + \frac{1-2p}{p} (x_1 + x_3) + \frac{1-3p}{p} x_2 - 2x_4 \leq 1 \\
2 + \frac{1-2p}{p} (x_1 + x_3) + \frac{1-3p}{p} x_2 - 2x_4 \leq \frac{3g}{p}.
\]

Solving for \(g\),

\[
g \geq \frac{2p}{3} + \frac{1-2p}{3} (x_1 + x_3) + \frac{1-3p}{3} x_2 - \frac{2p}{3} x_4 \\
\geq \frac{2p}{3} + \frac{1-2p}{3} (x_1 + x_3) + \frac{1-5p}{3} x_2.
\]

If \(p \leq 1/5\), then \(g > 2p/3\). If \(p > 1/5\), then we use Lemma \(15\), which gives that \(x_2 \leq g/(1-p)\). So,

\[
g \geq \frac{2p}{3} + \frac{1-2p}{3} (x_1 + x_4) + \frac{1-5p}{3} \left(\frac{g}{1-p}\right) \\
\geq \frac{p(1-p)}{1+p} + \frac{(1-2p)(1-p)}{2(1+p)} (x_1 + x_4).
\]

Consequently, \(g > p(1-p)/(1+p)\).

(vii) Let the gray 4-cycle be denoted \(\{v_1, v_2, v_3, v_4\}\) such that all edges are gray except \(v_1v_3\) and \(v_2v_4\). Let \(x_i = x(v_i)\) for \(i = 1, 2, 3, 4\). If both pairs \((v_1, v_3)\) and \((v_2, v_4)\) have common neighbors outside of \(\{v_1, v_2, v_3, v_4\}\), then a \(K_{3,3}\) is formed. So, suppose \(v_2\) and \(v_4\) have no common neighbors other than \(v_1\) and \(v_3\). Without loss of generality, let \(x_2 \geq x_4\).

The sets \(N_G(v_1) - \{v_2, v_4\}\), \(N_G(v_3) - \{v_2, v_4\}\) and \(N_G(v_2) - \{v_1, v_3\}\) must be disjoint. So,

\[
(d_G(v_1) - x_2 - x_4) + (d_G(v_3) - x_2 - x_4) \\
+ (d_G(v_2) - x_1 - x_3 - x_4) + (x_1 + x_2 + x_3 + x_4) \leq 1 \\
3 \frac{p-g}{p} + \frac{1-2p}{p} (x_1 + x_3) + \frac{1-3p}{p} x_2 - x_4 \leq 1 \\
2 + \frac{1-2p}{p} (x_1 + x_3) + \frac{1-3p}{p} x_2 - x_4 \leq \frac{3g}{p}.
\]

Solving for \(g\),

\[
g \geq \frac{2p}{3} + \frac{1-2p}{3} (x_1 + x_3) + \frac{1-3p}{3} x_2 - \frac{p}{3} x_4 \\
\geq \frac{2p}{3} + \frac{1-2p}{3} (x_1 + x_3) + \frac{1-4p}{3} x_2.
\]
If \( p \leq \frac{1}{4} \), then \( g > \frac{2p}{3} \). If \( p > \frac{1}{4} \), then we use Lemma 15, which gives that \( x_2 \leq g/(1-p) \).

\[
g \geq \frac{2p}{3} + \frac{1-2p}{3}(x_1 + x_4) + \frac{1-4p}{3} \left( \frac{g}{1-p} \right) \\
\geq \frac{2p(1-p)}{2+p} + \frac{(1-2p)(1-p)}{2+p} (x_1 + x_4).
\]

Consequently, \( g > \frac{2p(1-p)}{2+p} \).

This concludes the proof of Lemma 22.

\[ \square \]

Corollary 23.

\[ ed_{\text{Forb}(C_6)}(p) = \min \left\{ p(1-p), \frac{1-p}{2} \right\}. \]

Proof. Lemma 18 gives that the function stated above is \( \gamma_{\text{Forb}(C_6)}(p) \) and so \( ed_{\text{Forb}(C_6)}(p) \leq \min \left\{ p(1-p), \frac{1-p}{2} \right\} \). By Lemma 18 we only need to consider \( p \in (0, \frac{1}{2}) \) and \( K \) being a black-vertex \( p \)-core CRG in \( K(\text{Forb}(C_6)) \) for which \( g_K(p) < \gamma_{\text{Forb}(C_6)}(p) \). By Proposition 21, \( K \) has neither a 3-cycle nor a 4-cycle. Lemma 22 gives that \( g_K(p) \geq p(1-p) \). So, there is no such \( K \) and the corollary follows.

\[ \square \]

Corollary 24.

\[ ed_{\text{Forb}(C_7)}(p) = \min \left\{ p^2, \frac{p(1-p)}{2} + \frac{1-p}{3} \right\}. \]

Proof. The function stated above is \( \gamma_{\text{Forb}(C_7)}(p) \). Let \( p \in (0, \frac{1}{2}) \) and suppose \( K \) is a black-vertex \( p \)-core CRG in \( K(\text{Forb}(C_7)) \) for which \( g_K(p) < \gamma_{\text{Forb}(C_7)}(p) \). By Proposition 21, \( K \) has no gray 4-cycle.

Since \( K \) has no gray 4-cycle, then by Lemma 22, either \( g_K(p) > p(1-p) \) or \( K \) has a gray 3-cycle. In terms of the former, it is trivial that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_7)}(p) \) for \( p \in (0, \frac{1}{2}) \), so we assume that \( G \) has a gray 3-cycle.

If \( K \) has a gray 3-cycle but no \( C_4^+ \), then by Lemma 22, we have \( g_K(p) > \min\{2p/3, (1-p)/3\} \). Straightforward calculations verify that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_7)}(p) \) for \( p \in (0, \frac{1}{2}) \).

\[ \square \]

Corollary 25.

\[ ed_{\text{Forb}(C_8)}(p) = \min \left\{ \frac{p(1-p)}{1+p}, \frac{1-p}{3} \right\}. \]

Proof. The proof is the same as for Corollary 24.

\[ \square \]

Corollary 26.

\[ ed_{\text{Forb}(C_9)}(p) = \min \left\{ \frac{p}{2}, \frac{1-p}{3} \right\}. \]
Proof. The function stated above is \( \gamma_{\text{Forb}(C_4)}(p) \). Let \( p \in (0, 1/2) \) and suppose \( K \) is a black-vertex \( p \)-core CRG in \( K(\text{Forb}(C_9)) \) for which \( g_K(p) < \gamma_{\text{Forb}(C_9)}(p) \). By Proposition \( \ref{prop:9-core} \), \( K \) has no gray \( C_5^{+} \).

Since \( K \) has no gray \( C_5^{+} \), then by Lemma \( \ref{lem:gray-4-cycle} \), either \( g_K(p) > \min\{2p/3, (1-p)/(1+p)\} \) or \( K \) has no gray \( C_4^{+} \). In terms of the former, straightforward calculations verify that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_9)}(p) \) for \( p \in (0, 1/2) \), so we assume that \( G \) has no gray \( C_4^{+} \).

If \( K \) has no gray \( C_4^{+} \), then by Lemma \( \ref{lem:gray-4-cycle} \), either \( g_K(p) > \min\{2p/3, (1-p)/3\} \) or \( K \) has no gray 3-cycle. In terms of the former, it is trivial that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_9)}(p) \) for \( p \in (0, 1/2) \), so we assume that \( G \) has no gray 3-cycle. If that is the case, however, Lemma \( \ref{lem:gray-4-cycle} \) gives that \( g_K(p) > p/2 \), a contradiction. So, there is no such \( K \) for which \( g_K(p) < \gamma_{\text{Forb}(C_9)}(p) \) and the corollary follows. \( \square \)

Corollary 27. \[ \text{ed}_{\text{Forb}(C_{10})}(p) = \min \left\{ \frac{p(1-p)}{1+2p}, \frac{1-p}{4} \right\}, \] if \( p \in [1/7, 1] \).

Proof. The function stated above is \( \gamma_{\text{Forb}(C_{10})}(p) \). Let \( p \in (0, 1/2) \) and suppose \( K \) is a black-vertex \( p \)-core CRG in \( K(\text{Forb}(C_{10})) \) for which \( g_K(p) < \gamma_{\text{Forb}(C_{10})}(p) \). By Proposition \( \ref{prop:10-core} \), \( K \) has no gray \( C_5^{+} \).

Since \( K \) has no gray \( C_5^{+} \), then by Lemma \( \ref{lem:gray-4-cycle} \), either \( g_K(p) > \min\{2p/3, (1-p)/(1+p)\} \) or \( K \) has no gray \( C_4^{+} \). In terms of the former, straightforward calculations verify that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_{10})}(p) \) for \( p \in [1/7, 1/2] \), so we assume that \( K \) has no gray \( C_4^{+} \).

If \( K \) has no gray \( C_4^{+} \), then by Lemma \( \ref{lem:gray-4-cycle} \), either \( g_K(p) > \min\{2p/3, (1-p)/3\} \) or \( K \) has no gray 3-cycle. In terms of the former, it is trivial that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_{10})}(p) \) for \( p \in [1/7, 1/2] \), so we assume that \( K \) has no gray 3-cycle.

If \( K \) has no gray 3-cycle, then by Lemma \( \ref{lem:gray-4-cycle} \), either \( g_K(p) > p(1-p) \) or \( K \) has a gray 4-cycle. In terms of the former, it is trivial that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_{10})}(p) \) for \( p \in (0, 1/2) \), so we assume \( K \) has a 4-cycle, but since it cannot be \( C_4^{+} \), it must be a gray chordless 4-cycle.

If \( K \) has a chordless gray 4-cycle, then by Lemma \( \ref{lem:gray-4-cycle} \), either \( g_K(p) > \min\{2p/3, 2p(1-p)/(2+p)\} \) or \( K \) has a gray \( K_{3,3} \). In terms of the former, straightforward calculations verify that this is a contradiction to \( g_K(p) < \gamma_{\text{Forb}(C_{10})}(p) \) for \( p \in [1/7, 1/2] \), so we assume that \( K \) has a gray \( K_{3,3} \). However, as observed in Lemma \( \ref{lem:gray-4-cycle} \), this contains a gray 6-cycle, which is a contradiction to \( K \in K(\text{Forb}(C_{10})) \). \( \square \)

Remark 28. See Figures \( \ref{fig:ed} \) for plots of the edit distance functions described in Corollaries \( \ref{cor:9-core} \), \( \ref{cor:10-core} \), \( \ref{cor:11-core} \), \( \ref{cor:12-core} \) and \( \ref{cor:13-core} \).

8. Conclusions

8.1. Forb(\( G(n_0, p_0) \)). We provide a conjecture with some interesting implications. Recall that \( G(n, p) \) denotes the Erdős-Rényi random graph on \( n \) vertices with edge-probability \( p \). The hereditary property \( \mathcal{H} = \text{Forb}(G(n_0, p_0)) \) is a random variable.
Conjecture 29. Fix $p_0 \in (0, 1)$ and let $\mathcal{H} = \text{Forb}(G(n_0, p_0))$. Then
\[
ed_H(p) = (1 + o(1)) \frac{2 \log_2 n_0}{n_0} \min \left\{ \frac{p}{- \log_2 (1 - p_0)}, \frac{1 - p}{- \log_2 (1 - p_0)} \right\}
\]
with probability approaching 1 as $n_0 \to \infty$.

The functions that define this bound are of the form $p/(\chi - 1)$ and $(1 - p)/(\chi - 1)$. Conjecture 29 was proved for the case $p_0 = 1/2$ by Alon and Stav [3]. If it is true in general, then it implies that $p^*_H = \frac{\log(1 - p_0)}{\log p_0 (1 - p_0)}$, which is only equal to $p_0$. 
itself when $p_0 \in \{0, 1/2, 1\}$. Recall that $ed_H(p) = \lim_{n \to \infty} \text{dist}(G(n, p), H)/\binom{n}{2}$ and it achieves its maximum at $p_0^H$. Informally, the conjecture implies that it is harder to edit away copies of $G(n, p_0)$ from $G(n, p_0^*)$ than it is from $G(n, p_0)$. This seems to be rather counterintuitive.

If Conjecture 29 is false, then it implies that there is more information about the structure of random graphs than is revealed by just the chromatic numbers.

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