Higher Spin Currents with Arbitrary $N$

in the $\mathcal{N} = 1$ Stringy Coset Minimal Model

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Abstract

In the $\mathcal{N} = 1$ supersymmetric coset model based on $(A^{(1)}_{N-1} \oplus A^{(1)}_{N-1}, A^{(1)}_{N-1})$ at level $(k,N)$, the lowest $\mathcal{N} = 1$ higher spin supercurrent with spins-$\left(\frac{5}{2}, 3\right)$, in terms of two independent numerator WZW currents, is reviewed. By calculating the operator product expansions (OPE) between this $\mathcal{N} = 1$ higher spin supercurrent and itself, the next two $\mathcal{N} = 1$ higher spin supercurrents can be generated with spins-$\left(\frac{7}{2}, 4\right)$ and $\left(4, \frac{9}{2}\right)$. These four currents are polynomials of degree 3, 4, 4, 4 in the first numerator WZW currents with level $k$. The complete nonlinear OPE of the lowest $\mathcal{N} = 1$ higher spin supercurrent for general $N$ is obtained. The three-point functions involving two scalar primaries with one spin-2 current or spin-3 current are calculated in the large $N$ limit for all values of the ’t Hooft coupling. In particular, the light states that appeared in the case when the second level was fixed by 1 are no longer light ones because the eigenvalues are finite in the large $N$ limit.
1 Introduction

The $W_N (= W_{A_{N-1}})$ minimal model conformal field theory (CFT) is dual, in the ’t Hooft $\frac{1}{N}$ expansion, to the higher spin theory of Vasiliev on the $AdS_3$ coupled to one complex scalar \cite{1,2}. See the recent review papers \cite{3,4} for this duality. The CFTs have the following coset form

$$\frac{\widehat{SU(N)}_k \oplus \widehat{SU(N)}_1}{\widehat{SU(N)}_{k+1}}.$$  \hfill (1.1)

The higher spin-$s$ currents from polynomial (of degree $s$ with $s = 2, 3, \cdots, N$) combinations of the individual numerator $SU(N)$ currents of spin-1 can be constructed \cite{5}. The diagonal denominator $SU(N)$ currents commute with these higher spin currents. One of the levels for the spin-1 current is fixed by the positive integer $k$ and the other is fixed by 1 in the numerator of the coset CFT (1.1). Using the isomorphism \cite{5} between the coset construction and the Drinfeld-Sokolov reduction, the extension of above duality to the triality is described in \cite{2}. One of the specialty on the above coset model (1.1) is that the additional currents that are generated in the OPEs for general level $l$ in the second numerator $SU(N)$ current become null fields for the level $l = 1$ and hence decouple \cite{6}.

A more general diagonal coset with levels $(k, l)$ is expected to have many additional extra currents. The corresponding algebra should be larger than the conventional $W_N$ algebra. Recently, the two dimensional gauge theory coupled to the adjoint fermions for $k = l = N$ is described in \cite{7}. This coset has $\mathcal{N} = (2, 2)$ supersymmetry \cite{8}. It is known in \cite{5} that the Virasoro primary field, corresponding to the highest weight of a highest weight module of the affine Kac Moody algebra, has conformal dimension $-\frac{1}{2}$ for the adjoint representation at level $N$ which is a dual Coxeter number of $SU(N)$. By Sugawara construction, the spin-$\frac{3}{2}$ current, a superpartner of spin-2 stress energy tensor, can be obtained from the above adjoint free fermions of spin-$\frac{1}{2}$. Similarly, the second spin-$\frac{3}{2}$ current can be obtained from the second adjoint free fermions of spin-$\frac{1}{2}$ for the coset with the levels $(k = N, l = N)$.

For the levels $(k, l = N)$ when the second level is fixed by $N$, the $\mathcal{N} = (1, 1)$ supersymmetry is preserved \cite{9,10}. The coset CFTs are described as

$$\frac{\widehat{SU(N)}_k \oplus \widehat{SU(N)}_N}{\widehat{SU(N)}_{k+N}}.$$  \hfill (1.2)

The coset central charge $c(N, k)$ can be calculated as

$$c(N, k) = \frac{(N^2 - 1)}{2} \left[ 1 - \frac{2N^2}{(k+N)(k+2N)} \right] < \frac{(N^2 - 1)}{2}, \quad k = 1, 2, \cdots \hfill (1.3)$$
In \[11\], the higher spin currents with spins
\[
\left(\frac{5}{2}, 3\right), \left(\frac{7}{2}, 4\right), \left(4, \frac{9}{2}\right), \left(4, \frac{9}{2}\right), \left(\frac{9}{2}, 5\right), \left(\frac{11}{2}, 6\right), \left(6, \frac{13}{2}\right)
\]
besides the \(\mathcal{N}=1\) super stress energy tensor with spins-(\(\frac{3}{2}, 2\)) are constructed for \(N=3\) with levels \((k, l = 3)\) by reconsidering the previous works in \[12\]. The six additional super currents (or twelve currents in components) arise. Furthermore, the general \(N\) expressions for the currents with spins-(\(\frac{3}{2}, 2\)) and (\(\frac{5}{2}, 3\)) are obtained explicitly. The lowest model \((k = 1)\) with \(c = \frac{(3N+1)(N-1)}{2(2N+1)}\) in the series of coset models \([13]\) has “minimal” \(\mathcal{N}=1\) supersymmetric \(W_N\) algebra where there are currents of spins-(\(\frac{5}{2}, 2\)), (\(\frac{5}{2}, 3\)), \cdots, (\(N - \frac{1}{2}, N\)) \([13]\). As the \(k\) increases, the additional currents start to appear and the final algebra is associative for all values of \(c\). It was expected \([13]\) that this algebra is determined by considering the limit model of the series which is a simple model with \(c = \frac{N^2-1}{2}\) describing \((N^2 - 1)\) fermions in the adjoint representation of \(SU(N)\). In other words, \(k \rightarrow \infty\) limit model with fixed \(N\).

In this paper, the generalization to \(N\) for the first three super currents in the list (1.4) is constructed (to describe the three-point functions with scalars in the large \(N\) limit as one of the reasons). Actually, the first higher spin super current in (1.4) was already given in \([11]\). The explicit calculations of OPEs between the currents with spins-(\(\frac{5}{2}, 3\)) are rather involved. Among three OPEs, the OPE between spin-3 current and itself can be obtained from the previous results in \([15]\). Therefore, the second- and first-order poles of the OPE between spin-\(\frac{5}{2}\) current and spin-3 current and the first-order pole of the OPE between spin-\(\frac{5}{2}\) current and itself should be determined. In doing these calculations, the fully normal ordering products are needed to describe the zero mode calculations.

In section 2, the two \(\mathcal{N}=1\) higher spin super currents with spins-(\(\frac{7}{2}, 4\)) and (\(4, \frac{9}{2}\)) are constructed by calculating the OPEs between the \(\mathcal{N}=1\) higher spin super current with spins-(\(\frac{5}{2}, 3\)) and itself. The OPE of the \(\mathcal{N}=1\) lowest higher spin super current can be obtained for general \(N\) up to the normalizations of above two higher spin super currents. Note the presence of the \(\mathcal{N}=1\) higher spin super current of spins-(\(4, \frac{9}{2}\)), in the right hand side of OPE, which is one of the additional currents compared to the above minimal \(\mathcal{N}=1\) supersymmetric \(W_N\) algebra.

In section 3, the three-point functions with scalars including the spins-2 or 3 current are described. The conformal dimension for \((f; f)\) \([1]\) becomes nonzero even in the large \(N\) limit.

In section 4, the summary of this work is presented and some discussions on the future directions are given.

In Appendices, the detailed descriptions appearing in sections 2, 3 and 4 are presented.

\[\text{Recently, in } [14], \text{ the } \mathcal{N}=1 \text{ minimal model holography corresponding to the above “minimal” } \mathcal{N}=1 \text{ supersymmetric } W_N \text{ algebra is found.}\]
2 The three $\mathcal{N} = 1$ higher spin (Casimir) super currents

In this section, the higher spin currents will be constructed for general $N$ explicitly. The fermion fields $\psi^a(z)$ of spin-$\frac{1}{2}$, where the $SU(N)$ adjoint index $a$ runs from 1 to $(N^2 - 1)$, are used for the construction of spin-1 current with level $N$ of the coset model (1.2). The fundamental OPE of this fermion field is expressed as

$$\psi^a(z) \psi^b(w) = -\frac{1}{(z-w)} \frac{1}{2} \delta^{ab} + \cdots. \quad (2.1)$$

The Kac-Moody current of spin-1, $J^a(z)$, is defined as follows:

$$J^a(z) \equiv f^{abc} \psi^b \psi^c(z). \quad (2.2)$$

The standard OPE of this spin-1 current with level $N$ is obtained from (2.1) and (2.2)

$$J^a(z) J^b(w) = -\frac{1}{(z-w)^2} N \delta^{ab} + \frac{1}{(z-w)} f^{abc} J^c(w) + \cdots. \quad (2.3)$$

To match with the convention of [15], the different normalization from that of [11] is used in this paper. The following OPE is obtained from (2.1) and (2.2)

$$\psi^a(z) J^b(w) = \frac{1}{(z-w)} f^{abc} \psi^c(w) + \cdots. \quad (2.4)$$

The other spin-1 current of level $k$ living in the other $SU(N)$ factor of our coset model (1.2) satisfies the following OPE

$$K^a(z) K^b(w) = -\frac{1}{(z-w)^2} k \delta^{ab} + \frac{1}{(z-w)} f^{abc} K^c(w) + \cdots. \quad (2.5)$$

The normalization is corrected appropriately and matches with that in [15]. The diagonal spin-1 current, $(J^a + K^a)(z)$, living in the denominator of the coset (1.2), satisfies similar OPE with the level $(N + k)$ from (2.3) and (2.5). The OPE between $J^a(z)$ and $K^b(w)$ contains only regular terms.

The spin-2 stress energy tensor, via the Sugawara construction, is expressed as

$$T(z) = -\frac{1}{4N} J^a J^a(z) - \frac{1}{2(k+N)} K^a K^a(z) + \frac{1}{2(k+2N)} (J^a + K^a)(J^a + K^a)(z), \quad (2.6)$$

and the central charge (1.3) depends on $N$ and $k$ as follows:

$$c = \frac{(N^2 - 1)}{2} \left[ 1 - \frac{2N^2}{(k+N)(k+2N)} \right] < \frac{(N^2 - 1)}{2}, \quad k = 1, 2, \cdots. \quad (2.7)$$
The fourth-order pole of \( T(z) T(w) \) with \( \frac{2}{3} \) provides \( \frac{1}{2} \) with (2.7). The superpartner of the spin-2 current (2.6) of spin-\( \frac{3}{2} \) can be described as follows [11]:

\[
G(z) = - \frac{\sqrt{2k}}{3\sqrt{N(k + N)(k + 2N)}} \psi^a \left( J^a - \frac{3N}{k} K^a \right) (z). \tag{2.8}
\]

The relative coefficient in (2.8) can be fixed using either the OPE of the diagonal current \((J^a + K^a)(z)\) and \(G(w)\) does not contain any singular terms or this spin-\( \frac{3}{2} \) current should transform as a primary field under the stress energy tensor (2.6). See also the description of \([3]\) on the construction of the coset spin-\( \frac{3}{2} \) current (in the subsection 7.3.2 of \([3]\)). For \( N = 2 \), the form of spin-\( \frac{3}{2} \) current was obtained in \([9]\). The overall normalization in (2.8) is fixed by the highest singular term \( \frac{1}{(z - w)^{\frac{1}{2}}} \) in the OPE \( G(z) \) \( G(w) \). The central term here is given by \( \frac{2}{3} \), where \( c \) is given by (2.7). In this analysis, the OPEs, (2.1), (2.3), (2.4), and (2.5), are used.

How should the other higher spin currents be determined? The spin-3 current can be described as the cubic terms in the combination of spin-1 currents, \( J^a(z) \) and \( K^a(z) \), in the numerator of our coset model (1.2). The explicit coefficients which depend on \( N \) and \( k \) were fixed previously in \([6]\). The primary spin-3 current is, by plugging the values \( k_1 = N \) and \( k_2 = k \) into (2.8) of \([6]\),

\[
W(z) = \frac{i \, d^{abc}}{6N(k + N)(2N + k)\sqrt{6(2N + k)(2N + 5N)(N^2 - 4)}} \left[ k(k + N)(2k + N)J^a J^b J^c - 6N(k + N)(2k + N)J^a K^b K^c + 18N^2(k + N)J^a K^b K^c - 6N^3K^a K^b K^c \right] (z). \tag{2.9}
\]

What about the superpartner of this spin-3 current? This spin-\( \frac{5}{2} \) current is obtained from the method used in the construction of \( W(z) \) (2.9). Or from the \( \mathcal{N} = 1 \) supersymmetry and the OPE \( G(z) \) \( W(w) \) together with (2.8) and (2.9) (the second-order singular term of this OPE), the following spin-\( \frac{5}{2} \) current is completely determined [11]:

\[
U(z) = \frac{i \, d^{abc}}{\sqrt{50N(k + N)(2k + N)(k + 2N)(2k + 5N)(-4 + N^2)}} \left[ k(2k + N)\psi^a J^b J^c - 5N(2k + N)\psi^a J^b K^c + 10N^2\psi^a K^b K^c \right] (z). \tag{2.10}
\]

Equivalently, the coefficients in (2.10) are fixed using the OPE \( G(z) \) \( U(w) \) and reading off the first-order singular term, which is nothing but the spin-3 current \( W(w) \)\( ^2\).

\( ^2 \)In this calculation, the following identities can be derived \( d^{abc} J^a J^b J^c (z) = -6N d^{abc} \psi^a \partial \psi^b J^c (z) \) and \( d^{abc} J^a J^b K^c (z) = f^{abc} d^{cde} \psi^a \psi^b J^d K^c (z) - 2N d^{abc} \psi^a \partial \psi^b K^c (z) \). In the realization (2.2), the number of independent terms for \( d^{abc} J^a J^b K^c (z) \) is two.
The other higher spin currents, by calculating the OPEs between the currents \( W(z) \) and \( U(z) \), are determined. Consider the OPE \( W(z) W(w) \). This OPE for \( N = 3 \) was obtained previously and was expected in the more general coset model (1.2) in [11]:

\[
W(z) W(w) = \frac{1}{(z-w)^6} + \frac{1}{(z-w)^4} 2T(w) + \frac{1}{(z-w)^3} \partial T(w) + \frac{1}{(z-w)^2} \left[ \frac{3}{10} \partial^2 T + \frac{32}{22 + 5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + P_4^{uw} + P_4^{ww} \right](w) + \frac{1}{(z-w)} \left[ \frac{1}{15} \partial^3 T + \frac{1}{2} \frac{32}{22 + 5c} \partial \left( T^2 - \frac{3}{10} \partial^2 T \right) \right]
\]

where the spin-4 current consisting of the values \( k \) in the first line of (2.14) can be distributed to other independent terms. See also (A.5).

Furthermore, the other primary spin-4 current, which is located at the second-order pole of \( N \) is given by (2.11), can be described as

\[
P_4^{uw}(z) = \frac{8(10 - 7c)}{(4c + 21)(10c - 7)} \left[ - \frac{7}{10} \partial^2 T + \frac{17}{22 + 5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G \partial G \right](z).
\]

The completely symmetric traceless symbol of rank-4 \([15]\) is introduced as follows: \( d^{abcd} = d^{abe} d^{cde} + d^{ace} d^{bde} + d^{ade} d^{bce} - \frac{4(N^2 - 4)}{N(N+2)} (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}). \) If this relation is used in (2.14), then each five term in the first line of (2.14) can be distributed to other independent terms. See also (A.5).
Consider the OPE $U(z) W(w)$ which was expected in the more general coset model (1.2) in [11] 4:

$$U(z) W(w) = \frac{1}{(z-w)^4} \frac{3}{\sqrt{6}} G(w) + \frac{1}{(z-w)^3} \left( - \frac{3}{\sqrt{6}} \partial G(w) \right) + \left( \frac{1}{12} \right) \frac{3}{\sqrt{6}} \partial^2 G + \frac{11\sqrt{6}}{4c+21} \left( GT - \frac{1}{8} \partial^2 G \right) + O_2(z) (w) + \cdots.$$ (2.15)

From the second order pole of (2.15), the spin-$\frac{7}{2}$ current can be expressed as

$$O_2(z) = \text{second-order pole}(z) - \frac{1}{12} \frac{3}{\sqrt{6}} \partial^2 G(z) - \frac{11\sqrt{6}}{4c+21} \left( GT - \frac{1}{8} \partial^2 G \right) (z).$$ (2.16)

The following quantities are introduced to express concisely

$$Q^a(z) \equiv d^{abc} J^b J^c(z), \quad R^a(z) \equiv d^{abc} K^b K^c(z), \quad S^a(z) \equiv d^{abc} K^b J^c(z),$$

$$Q(z) \equiv d^{abc} J^b J^c(z), \quad S(z) \equiv d^{abc} K^b K^c(z).$$ (2.17)

As noted in [16], for infinite limit of $k$, $Q^a(z)$ plays the role of superpartner of spin-$\frac{3}{2}$ field, $\phi^a(z) \equiv \frac{2}{3} d^{abc} \psi^b J^c(z)$ and spin-$\frac{5}{2}$ field, $\Phi^a(z) \equiv \frac{2}{3} d^{abc} (2\partial \psi^b J^c - \psi^b \partial J^c)(z)$. In other words, the OPE $G(z) Q^a(w)$ for infinite $k$ contains the second-order pole with this spin-$\frac{3}{2}$ field, $\phi^a(z)$, and the first-order pole with the combination of the spin-$\frac{5}{2}$ field, $\Phi^a(z)$, and the derivative of spin-$\frac{3}{2}$ field, $\phi^a(z)$.

The next step is to determine how to obtain the second-order pole of OPE $U(z) W(w)$ together with (2.10) and (2.9). At first, the OPE between the first term of $U(z)$, $d^{abc} \psi^a J^b J^c(z) = \psi^a Q^a(z)$, and the first term of $W(w)$, $d^{def} J^d J^e J^f(w) = J^d Q^d(w)$ should be calculated. The OPE consists of two parts. One is to calculate the OPE between $\psi^a Q^a(z)$ and $J^d(x)$ and the other is to calculate the OPE between $\psi^a Q^a(z)$ and $Q^d(w)$. The OPE of $J^d(x)$ and $\psi^a(z)$ can be derived from (2.4) and the OPE between $J^d(x)$ and $Q^a(z)$ is needed. This was obtained in [15], where the level $N$ is inserted,

$$J^a(z) Q^b(w) = - \frac{1}{(z-w)^2} 3N d^{abc} J^c(w) + \frac{1}{(z-w)^3} f^{abc} Q^c(w) + \cdots.$$ (2.18)

4 When the order of the OPE (2.15) is changed, the explicit form for the OPE can be derived as follows: $W(z) U(w) = \frac{1}{(z-w)^3} \frac{3}{\sqrt{6}} G(w) + \frac{1}{(z-w)^2} \left( \frac{1}{4c+21} \right) \frac{3}{\sqrt{6}} \partial G(w) + \frac{1}{(z-w)^2} \left( \frac{1}{4c+21} \right) \frac{11\sqrt{6}}{4c+21} \left( GT - \frac{1}{8} \partial^2 G \right) + O_2(z) (w) + \cdots$.
It is straightforward to obtain the second-order pole, $-5Nd_{abc}(\psi^a J^b)Q^c(w)$, from the first part. The following fully normal ordered product can be obtained as follows:

\[ d_{abc}(\psi^a J^b)Q^c(w) = d_{abc}\psi^a J^b Q^c(w) - \frac{5}{2}(N^2 - 4)\psi^a \partial^2 J^a(w) + \frac{7}{2}(N^2 - 4)\partial\psi^a \partial J^a(w). \] 

(2.19)

Moreover, to obtain the second part of the OPE, the following OPE from (2.4) should be used

\[ Q^c(z) \psi^a(w) = -\frac{1}{(z - w)^2}Nd_{acd} \psi^d(w) + \frac{1}{(z - w)} \left[-2Nd_{ace} \partial\psi^d - 2d_{def} \psi^f J^e \right](w) + \cdots. \] 

(2.20)

The following OPE, which appeared in [15], should be used

\[ Q^a(z) Q^b(w) = \frac{1}{(z - w)^4} 6N(N^2 - 4)\delta^{ab} - \frac{1}{(z - w)^3} 6(N^2 - 4)f_{abc}J^c(w) \]

\[ - \frac{1}{(z - w)^2} \left[ Nd_{abc} Q^c + 6Nd_{ace}d_{bde} J^d J^e \right](w) \]

\[ + \frac{1}{(z - w)} \left[-3Nd_{ace}d_{bde} \partial J^d J^e + f_{ace}d_{bde} Q^c J^d + d_{ace}d_{bde} J^c Q^e \right] \]

\[ - 3Nd_{ace}d_{bde} J^d \partial J^e \right](w) + \cdots. \] 

(2.21)

The second-order pole contributed from the second part of the OPE leads to the intermediate result, $-5Nd_{abc} d_{cde} J^a(\psi^d J^e)J^b(w) - 5Nd_{abc}d_{cde} J^a J^b \psi^d J^e(w)$. The expression $d_{abc} d_{cde} J^a(\psi^d J^e)J^b(w)$ can be simplified further. Using the identity [15]

\[ d_{abc} d_{bde} f_{cfa} = \frac{(N^2 - 4)}{N} f_{def}, \] 

(2.22)

one can rearrange this normal ordered product by moving the field $J^b$ to the left. The identities in [15]

\[ f_{abc} f_{dbc} = 2N\delta^{ad}, \quad d_{abc} d_{bde} = \frac{2}{N}(N^2 - 4)\delta^{ad}, \] 

(2.23)

are used. Therefore, the final result can be expressed as

\[ d_{abc} d_{cde} J^a(\psi^d J^e)J^b(w) = d_{abc} \psi^a J^b Q^c(w) - (N^2 - 4)\partial^2 \psi^a J^a(w) - (N^2 - 4)\psi^a \partial^2 J^a(w) \]

\[ + 2(N^2 - 4)\partial\psi^a \partial J^a(w). \] 

(2.24)
Using (2.24), (2.19) and the identity \( \partial_{a} \partial_{b} J^a J^b \psi^d f^e(w) = \partial_{a} \partial_{b} \psi^d J^a J^b f^e(w) + 2(N^2 - 4) \partial \psi^a \partial J^a(w) - (N^2 - 4) \partial^2 \psi^a J^a(w) \), which can be checked by moving the field \( \psi^d \) to the left, the final second-order pole of OPE \( \psi^d Q^a(z) J^d Q^d(w) \) can be summarized as

\[
\{ \psi^d Q^a J^d Q^d \}_{2}(w) = -15N d^{abc} \psi^a J^b Q^c(w) + \frac{45}{2} N(N^2 - 4) \psi^a \partial^2 J^a(w) - \frac{75}{2} N(N^2 - 4) \partial \psi^a \partial J^a(w).
\]

(2.25)

The other OPEs can be obtained similarly. In Appendix B, the eleven contributions including (2.25) are presented explicitly. Obviously, the OPE between the first term of \( U(z) \) and the last term of \( W(w) \) does not contain the singular term. In particular, the OPE \( \psi^d R^a(z) Q(w) \) found in (B.4) is most involved. The normal ordered products in (B.1) to (B.11) are not fully normal ordered. Some nestings of these higher ordered products are presented in Appendix B using (2.22) and (2.23). Furthermore, the field \( \psi^a(z) \) should be placed to the left of the field \( J^b(z) \) and the field \( K^c(z) \) is located at the right hand side of \( J^d(z) \).

Then the final second-order pole, which consists of thirteen terms, is presented as follows:

\[
\{ U W \}_{2}(w) = c_1 d^{abc} \psi^a J^b Q^c(w) + c_2 d^{abc} \psi^a J^b S^c(w) + c_3 d^{abc} \psi^a J^b R^c(w) + c_4 d^{abc} \psi^a K^b S^c(w) + c_5 d^{abc} \psi^a J^b K^c(w) + c_6 d^{abc} \psi^a J^b K^c(w) + c_7 \psi^a \partial^2 J^a(w) + c_8 \partial \psi^a \partial J^a(w) + c_9 f^{abc} \psi^a K^b \partial K^c(w) + c_{10} f^{abc} \psi^a J^b K^c(w) + c_{11} \partial^2 \psi^a K^a(w) + c_{12} \partial \psi^a \partial K^a(w) + c_{13} \psi^a \partial^2 K^a(w),
\]

(2.26)

where the coefficient functions are determined as

\[
\begin{align*}
c_1 &= -15BCkN(k + N)(2k + N)^2(2k + N), \\
c_2 &= -60BCN^3(k + N)(k + 2N)(6k + 5N), \\
c_3 &= 75BCN^2(k + N)(2k + N)^2(2k + 2N), \\
c_4 &= 480BCN^4(k + N)(k + 2N), \\
c_5 &= 30BCN^2(k + N)(2k + N)^2(2k + N), \\
c_6 &= -240BCN^3(k + N)(2k + N)(k + 2N), \\
c_7 &= \frac{15}{2} BCk(-4 + N^2)N(k + N)(2k + N)(k + 2N)(6k + 11N), \\
c_8 &= -\frac{15}{2} BCk(-4 + N^2)N(k + N)(2k + N)(k + 2N)(10k + 13N), \\
c_9 &= 240BC(-4 + N^2)N^2(k + N)(2k + N)(k + 2N), \\
c_{10} &= -30BC(-4 + N^2)N(k + N)(2k + N)(k + 2N)(10k + 13N), \\
c_{11} &= -60BC(-4 + N^2)N^2(k + N)(2k + N)(k + 2N)(2k + 3N),
\end{align*}
\]

8
\[ c_{12} = 60BC(-4 + N^2)N^2(k + N)(2k + N)^2(k + 2N), \]
\[ c_{13} = -30BC(-4 + N^2)N^2(k + N)(2k + N)(k + 2N)(2k + 9N). \] (2.27)

Furthermore the following \((N, k)\) dependent functions, which are overall normalization constants of \(W(z)\) (2.9) and \(U(z)\) (2.10), respectively, are introduced
\[
B(N, k) \equiv \frac{i}{6\sqrt{6}N(k + N)(k + 2N)(2k + 5N)(-4 + N^2)}, \quad C(N, k) \equiv \frac{i}{5\sqrt{2}N(k + N)(k + 2N)(2k + 5N)(-4 + N^2)}. \] (2.28)

The presence of the first nonderivative terms of (2.26) was expected in [11]. Each term contains \(\psi^a(z)\) or its derivative term. The \(c_1\)-term in (2.26) can be seen from the result of [10]. The \(c_7\), \(c_8\)-terms do not contain the current \(K^a(z)\).

Therefore, the spin-\(\frac{7}{2}\) current is obtained from the formula (2.16), (2.17), (2.25), (2.27), (2.6), (2.8) and (2.28). Some identities in (B.23) of Appendix B can be used to obtain the composite field \(G T(z)\) that appears in (2.16).

What happens in next order pole? The explicit form for the spin-\(\frac{9}{2}\) current can be derived as follows:
\[
O_{\frac{9}{2}}(w) = -\text{first-order pole}(w) + \left(\frac{1}{60}\right)\frac{3}{\sqrt{6}}\partial^3G(w) + \left(\frac{3}{11}\right)\left(\frac{\sqrt{6}}{4c + 21}\right)\partial \left(G T - \frac{1}{8}\partial^2G\right)(w) + \left(\frac{3}{11}\right)\partial^2O_{\frac{9}{2}}(w) - \frac{12\sqrt{6}}{7(10c - 7)}\left(\frac{4}{3}T\partial G - G\partial T - \frac{4}{15}\partial^2 G\right)(w). \] (2.29)

The next step is to determine how to obtain the first-order pole of OPE \(U(z)\) \(W(w)\). As done in previous second-order pole, the first-order pole from the OPE between \(\psi^a Q^a(z)\) and \(J^d Q^d(w)\) can be obtained. Using (2.18), (2.20) and (2.21), the first-order pole consists of
\[
-5Nd^{abc}d^{cde}\partial(\psi^a J^b)(J^d J^e)(w) - 5Nd^{abc}d^{cde}J^a \partial(\psi^d J^e)J^b(w) - 5Nd^{abc}d^{cde}J^a J^b \partial(\psi^d J^e)(w). \] (2.30)

In (C.1), the following nontrivial normal ordered products are obtained
\[
\begin{align*}
\quad \quad d^{abc}d^{cde}\partial(\psi^a J^b)(J^d J^e)(w) &= d^{abc}\partial(\psi^a J^b Q^e(w) + d^{abc}\psi^a J^b Q^e(w) + 3\partial^3(\psi^a J^b)(w), \\
\quad \quad d^{abc}d^{cde}J^a \partial(\psi^d J^e)J^b(w) &= -\frac{2}{3}(N^2 - 4)J^a \partial^3(\psi^a(w) + d^{abc}d^{cde}J^a J^b \partial(\psi^d J^e)(w), \\
\quad \quad d^{abc}d^{cde}J^a J^b \partial(\psi^d J^e)(w) &= d^{abc}d^{cde}\psi^d J^a J^b J^e \partial J^e(w) + \frac{2(N^2 - 4)}{N}f^{abc}\partial(\psi^a J^b J^c(w) + d^{abc}d^{cde}\partial^2(\psi^a J^b J^c(w) - (N^2 - 4)\partial^3(\psi^a J^a(w), \quad (2.30)
\end{align*}
\]
where the OPEs of (2.18) and (2.20) are used. Some properties between the $d$ symbol and $f$ symbol are used. In Appendix C, the contributions are presented explicitly. The normal ordered products in (C.11) are not fully normal ordered. Some nestings of these higher ordered products are presented in Appendix C.

Then the final first-order pole, which consists of thirty nine terms, is presented as follows:

$$
\{U^W\}_{1}(w) = c_1 f^{abc} d^{cde} \psi^a J^d Q^b K^c + c_2 f^{abc} d^{cde} \psi^b J^d f^c R + c_3 d^{abc} d^{def} f^{eabh} \psi^a J^d f^j J^h K^c 
+ c_4 f^{abc} d^{cde} \psi^d J^a Q^b K^c + c_6 d^{abc} d^{def} f^{eabh} \psi^j f^j J^h K^d K^c 
+ c_7 d^{abc} d^{def} f^{eabh} \psi^a J^f J^h K^d K^c + c_8 f^{abc} d^{cde} \psi^a J^d K^c S^b + c_9 f^{abc} d^{cde} \psi^d J^a K^c S^b 
+ c_{10} f^{abc} d^{cde} \psi^b J^d J^e S^a + c_{13} d^{abc} d^{def} f^{eabh} \psi^a J^b K^d K^f K^g + c_{14} f^{abc} d^{cde} \psi^a K^d K^e S^b 
+ e_1 d^{abc} \partial \psi^a J^b Q^c + e_2 d^{abc} \psi^a \partial J^b Q^c + e_3 \partial \psi^a \partial^2 J^a + e_4 \psi^a \partial^3 J^a + e_6 \partial^2 \psi^a \partial J^a 
+ e_7 \partial \psi^a K^a + e_8 d^{abc} d^{cde} \psi^d J^e J^a K^c + e_9 f^{abc} \partial \psi^a J^b \partial K^c + e_{10} f^{abc} \partial^2 \psi^a J^b K^c 
+ e_{11} f^{abc} \partial \psi^a \partial J^b K^c + e_{12} \partial \psi^a \partial^2 K^a + e_{14} d^{abc} \psi^a J^b \partial S^c + e_{15} \psi^a \partial^3 K^a 
+ e_{16} d^{abc} d^{cde} \psi^d J^a J^b \partial K^c + e_{17} d^{abc} d^{cde} \partial \psi^d J^a J^b K^c + e_{18} d^{abc} \partial \psi^a J^b R^c 
+ e_{19} d^{abc} \psi^a \partial J^b R^c + e_{20} d^{abc} d^{cde} \psi^a J^b \partial J^c \partial K^e + e_{22} d^{abc} \partial \psi^a J^b S^c 
+ e_{23} d^{abc} \psi^a \partial J^b S^c + e_{24} d^{abc} d^{cde} \psi^d J^a J^b \partial K^c + e_{26} d^{abc} d^{cde} \partial \psi^d J^a J^b K^c 
+ e_{27} d^{abc} d^{cde} \psi^d J^a K^b \partial K^c + e_{28} d^{abc} \psi^a \partial K^b S^c + e_{29} f^{abc} \partial \psi^a K^b \partial^2 K^c 
+ e_{30} d^{abc} \partial \psi^a K^b S^c + e_{32} f^{abc} \partial \psi^a \partial K^b \partial K^c + e_{33} d^{abc} \psi^a K^b \partial S^c,
$$

(2.31)

where the coefficient functions are presented in (C.22) of Appendix C. Compared to the previous spin-$\frac{7}{2}$ current, where the field $d^{abc} \psi^a J^b Q^c(z)$ appears in (2.26), the nonderivative term among $c_1$-$c_{14}$ terms in (2.31) contains $K^a(z)$ term. Furthermore, the nonderivative terms contain a single $f$ symbol. Because the spin-1 currents in (2.3) and (2.5) has the first-order pole with the structure constant $f$ symbol, this reflects the final first-order pole in the OPE $U(z) W(w)$. Note that the OPE (2.3) contains a $f$ symbol on the right hand side. The $K^a(z)$ independent terms are given by $e_1$-$e_6$ terms. The last twenty eight derivative terms in (2.31) ($e_1$-$e_{33}$ terms) can be seen from the $\partial \{U^W\}_{-2}(w)$ with (2.26). This fact is consistent with the result of [16] where the field $\rho_{\frac{7}{2}}(z)$ in (3.22) of [16] has no nonderivative term.

Therefore, the spin-$\frac{5}{2}$ current is obtained from the formula (2.29), (2.16), (2.17), (2.31), (C.22), (C.8) and (2.28). Some identities in (C.20) and (C.21) of Appendix C can be used to obtain the composite field $T \partial G(z)$ and $G \partial T(z)$, respectively.

Now move the following OPE

$$
U(z) U(w) = \frac{1}{(z-w)^5} \frac{2c}{5} + \frac{1}{(z-w)^3} 2T(w) + \frac{1}{(z-w)^2} \partial T(w)
$$

10
where the primary spin-4 field is

\[ P_{4u}^u(z) = -\frac{3(2c-83)}{(4c+21)(10c-7)} \left[ -\frac{7}{10} \partial^2 T + \frac{17}{22+5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G \partial^2 G \right] (z). \] (2.33)

Then the other spin-4 field belonging to \( \mathcal{N} = 1 \) supermultiplet is expressed as

\[ P_{4u}^u(z) = \text{first-order pole}(z) - \frac{3(2c-83)}{(4c+21)(10c-7)} \left[ -\frac{7}{10} \partial^2 T + \frac{17}{22+5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G \partial^2 G \right] (z) - P_{4u}^u(z). \] (2.34)

The OPE of the first-term of (2.10) and itself can be obtained. For the OPE \( \psi^a Q^a(z) \psi^b Q^b(w) \), the following OPE is needed.

\[ \psi^a Q^a(z) \psi^b(w) = \frac{1}{(z-w)} \left[ -\frac{1}{2} Q^b - 2d^{abc} f_{d} cde \psi^a \psi^e J^d - N d^{abc} \psi^a \partial \psi^e \right] (w) + \cdots. \] (2.35)

The second part of this OPE leads to the following first-order pole

\[ \{ \psi^a Q^a, Q^b \}_{-1}(w) = \left[ -5N d^{abc} d^{def} \partial (\psi^e J^f) J^d - 5N d^{abc} d^{def} \partial (\psi^e J^f) \right] (w). \] (2.36)

In the normal ordering of \( (\psi^a J^b)(\psi^d J^e) \), one should be careful about the signs due to the fermionic property [17]. That is,

\[ (\psi^a J^b)(\psi^d J^e) = \psi^a J^b \psi^d J^e + \{(\psi^d J^e), (\psi^a J^b)\} - \{(\psi^d J^e), \psi^a\} J^b + \psi^a[(\psi^d J^e), J^b]. \] (2.37)

Furthermore, the following nontrivial normal ordered products in (D.1) with (2.35) and (2.36) can be obtained

\[ Q^a Q^a(w) = d^{abc} d^{def} \psi^a J^b \psi^d J^e (w) - 6(N^2 - 4) \partial^2 J^a J^a (w), \]

\[ d^{abc} d^{def} \psi^a J^b \psi^d J^e (w) = d^{abc} d^{def} \psi^a J^b \psi^d J^e (w) - \frac{2}{3} (N^2 - 4) \psi^a \partial^2 \psi^a (w), \]

\[ d^{abc} (\psi^a \partial \psi^b) Q^c (w) = d^{abc} d^{def} \psi^a J^b \psi^d J^e (w) + 3(N^2 - 4) \partial \psi^a \partial^2 \psi^a (w) - \frac{5}{3} (N^2 - 4) \psi^a \partial^3 \psi^a (w), \]

\[ d^{abc} d^{def} \partial (\psi^a J^b)(\psi^d J^e)(w) = d^{abc} d^{def} \partial \psi^a J^b \psi^d J^e (w) + d^{abc} d^{def} \psi^a \partial J^b \psi^d J^e (w) + \frac{N^2 - 4}{N} \partial^2 J^a J^a (w) - \frac{2}{N} \partial J^a J^a (w) + \frac{4}{3} (N^2 - 4) \partial \psi^a \partial^2 \psi^a (w) + \frac{14}{3} (N^2 - 4) \psi^a \partial^3 \psi^a (w), \]

\[ f^{abc} d^{def} (\psi^d \psi^a J^e)(w) = -f^{abc} d^{def} \psi^a (\psi^d J^e) J^d J^e (w) + N d^{abc} d^{def} \partial \psi^a \psi^d J^e J^b (w) + N d^{abc} d^{def} \partial \psi^a J^b \psi^d J^e (w) + 2N d^{abc} d^{def} \partial (\psi^a J^b)(\psi^d J^e) (w) + \frac{4}{3} (N^2 - 4) \psi^a \partial^2 \psi^a (w) + \frac{7}{3} (N^2 - 4) f^{abc} \psi^a \partial (\partial \psi^b J^e) (w). \] (2.38)
Note that the first term of the fifth equation in (2.38) should be rearranged to obtain the final result. The two nonderivative terms will appear in the final results below.\footnote{The following simplification can be made \( f^{abc}d^{def}b^fg\psi^d\psi^eJ^fJ^g(w) = f^{abc}d^{def}b^fg\psi^d\psi^eJ^fJ^g(w) + 2(N^2 - 4)f^{abc}\psi^d\partial(\psi^bJ^c)(w) - 3(N^2 - 4)f^{abc}\psi^d\partial(\psi^b\partial J^c)(w) + Nf^{abc}d^{def}b^fg\psi^d\psi^eJ^fJ^g(w) - Nf^{abc}d^{def}b^fg\psi^d\psi^eJ^fJ^g(w) - Nf^{abc}d^{def}b^fg\psi^d\psi^eJ^fJ^g(w) - Nf^{abc}d^{def}b^fg\psi^d\psi^eJ^fJ^g(w) + \frac{1}{2}(N^2 - 4)f^{abc}\psi^d\psi^e\partial^2 J^fJ^g(w).\) }

The final first-order pole, which consists of thirty-eight terms, can be constructed

\[
\{U \ U\}_{-1}(w) = c_1 d^{abc}J^aJ^bQ^c + c_2 f^{abc}d^{def}\psi^dJ^cJ^f + c_3 f^{abc}J^aJ^bR^c + c_4 d^{abc}d^{def}f^{emb}J^bK^c + c_5 d^{abc}K^bJ^c + c_6 d^{abc}d^{def}f^{eck}\psi^dJ^bK^fK^g
\]

Eventually, the correct spin-4 currents, which are the elements of \(D_{17}\) of Appendix D. In particular, the OPE \(\psi^aR^a(z) \psi^bQ^b(w)\) found in (2.34) is most involved. The nonderivative \(c_1\)-term in (2.39) appears also in \(16\). The expression which does not contain the current \(K^a(z)\) consists of \(c_{17}, c_{27}, e_{17}, e_{37}, e_{67}, e_{87}, e_{97}, e_{117}, e_{117}\) and \(e_{18}\)-terms. The nonderivative terms with a single \(f\) symbol can be obtained from the contractions between the spin-1 currents. As observed previously, the first-order pole between these currents has a \(f\) symbol. Also the quadratic expression in \(\psi^a\) appears because they do not play the role of the contraction and remain unchanged. On the other hand, the nonderivative terms without \(f\) symbol can be obtained from the contractions between the spin-\(\frac{1}{2}\) currents and therefore there is no \(\psi^a\) factor in their expressions.

Therefore, the spin-4 current is obtained from the formula (2.34), (2.17), (2.39), (2.33), (D.17), (2.6), (2.8) and (2.28). Some identities in (A.3) of Appendix A can be used to obtain the composite field \(G \partial G(z)\) that appears in (2.33).

Eventually, the correct spin-4 currents, which are the elements of \(N = 1\) super currents,
\[ O_4(w) = \left( -\frac{1}{\sqrt{6}} P_{uu}^w + \sqrt{6} P_{ww}^w \right)(w), \quad O_4'(w) = \frac{1}{8} \left( \frac{16}{7} \sqrt{\frac{2}{3}} P_{uu}^w - \frac{4}{7} \sqrt{6} P_{ww}^w \right)(w), \quad (2.40) \]

where the equations (2.13) and (2.34) are needed.

The main observation in this section is that the pole structures (2.26), (2.31), and (2.39) are obtained explicitly. Together with the pole structure (2.14) found in [15], they provide the WZW field contents for \( O_{\frac{7}{2}}(z), O_4(z), O_{4'}(z) \) and \( O_{\frac{9}{2}}(z) \) given in (2.16), (2.29) and (2.40).

The spin-3/2 current \( G(z) \) in (2.8) determines the superpartner for component field in any primary superfield, the OPE \( G(z) O_{\frac{7}{2}}(w) \) should determine the current \( O_4(w) \). Similarly, the OPE \( G(z) O_{4'}(w) \) should lead to the current \( O_{\frac{9}{2}}(w) \). It would be interesting to check these two facts explicitly for consistency. As described in the beginning of this section, the three \( N = 1 \) higher spin super currents with spins-(\( \frac{5}{2}, 3 \)), (\( \frac{7}{2}, 4 \)) and (\( 4, \frac{9}{2} \)) are constructed.

3 The three-point functions with scalars in the large \( N \) limit

In this section, the three-point functions with scalars will be determined for the spins \( s = 2, 3 \) because the normalization for these currents are known completely. For \( s = 4, 4' \), the normalization for these currents are not fixed because the OPEs between each current and itself are not constructed and the highest singular terms, \( \frac{1}{(z-w)^8} \) and \( \frac{1}{(z-w)^{8'}} \), are not determined.

The conformal dimension \( h(0; f) \) [11] and its large \( N \) limit can be derived as follows (the formula for the dimension can be found in [18, 19]):

\[
h(0; f) = \frac{N^2 - 1}{2N} \left[ \frac{1}{N + N} - \frac{1}{N + N + k} \right] = \frac{N^2 - 1}{4N^2} \left[ 1 - \frac{2N}{2N + k} \right] \to \frac{(1 - \lambda)}{4(1 + \lambda)}, \quad (3.1) \]

where the eigenvalue of the quadratic Casimir operator of \( SU(N) \), \( \frac{N^2 - 1}{2N} \), is used. The ’t Hooft coupling constant is

\[
\lambda = \frac{N}{k + N}. \quad (3.2) \]

The level \( N \) appears in the two denominators of (3.1), whereas the level \( k \) appears in the second denominator of (3.1). At the final stage of (3.1), after substituting \( k = \frac{1 - \lambda}{\lambda} N \), the large \( N \) limit was taken. There is no contribution from the trivial representation.

On the other hand, the conformal dimension \( h(f; 0) \) and its large \( N \) limit can be derived as follows:

\[
h(f; 0) = \frac{N^2 - 1}{2N} \left[ \frac{1}{N + k} + \frac{1}{N + N} \right] = \frac{N^2 - 1}{4N^2} \left[ 1 + \frac{2N}{N + k} \right] \to \frac{1 + 2\lambda}{4}, \quad (3.3) \]
where the level $k$ appears in the first denominator of (3.3), whereas the level $N$ appears in the second denominator of (3.3). At the final step, the large $N$ limit was taken similarly. The trivial representation does not contribute to the conformal dimension.

The Virasoro zero mode (acting on the above primaries) eigenvalue is determined as follows. For the primary $(0; f)$, the zero mode $K_0^a$ corresponding to the numerator current with level $k$ of the coset model vanishes \cite{20}. That is, the eigenvalue is zero when this zero mode acts on the state corresponding to this primary field. On the other hand, the ground state transforms as a fundamental representation with respect to the zero mode $J_0^a$ corresponding to the numerator current with the level $N$ of the coset. The nonzero contribution arises from the first and third terms of spin 2 stress energy tensor (2.6). The result can be expressed as

$$\left(-\frac{1}{4N}J_0^a J_0^a + \frac{1}{2(k + 2N)}J_0^a J_0^a\right)|f> = \left(-\frac{1}{4N} + \frac{1}{2(k + 2N)}\right)(-N)|f>$$

$$= \frac{(1 - \lambda)}{4(1 + \lambda)}|f> = h(0; f)|f>.$$  \hspace{1cm} (3.4)

Note that because $J_0^a J_0^a = \delta^{ab} \text{Tr}(T^a T^b) = -\delta^{aa} = -(N^2 - 1)$, the large $N$ limit for the eigenvalue equation leads to $-N$. Furthermore, the extra $\frac{1}{N}$ factor arises due to the fact that the eigenvalue (not a trace) is needed. Therefore, the final contribution to the fundamental representation provides $-N$ \cite{21, 15}. At the final stage of (3.4), the identity (3.1) is used. The result (3.4) indicates that the Virasoro zero mode eigenvalue can be fixed by the conformal dimension of the scalar primary operator.

On the other hand, for the primary $(f; 0)$, the zero mode $K_0^a$ equals to $-J_0^a$. Or the eigenvalue equation of the zero mode of the diagonal denominator current in the coset has zero eigenvalue. Furthermore, the ground state transforms as a fundamental representation with respect to $K_0^a$ and as an antifundamental representation with respect to $J_0^a$. The nonzero contribution arises from the first and second terms of spin 2 Virasoro current (2.6). The result can be derived

$$\left(-\frac{1}{4N}J_0^a J_0^a - \frac{1}{2(k + N)}J_0^a J_0^a\right)|f> = \left(-\frac{1}{4N} - \frac{1}{2(k + N)}\right)(-N)|f>$$

$$= \frac{1}{4}(1 + 2\lambda)|f> = h(f; 0)|f>.$$ \hspace{1cm} (3.5)

Although the generators in the antifundamental representation have the extra minus signs, compared to those in the fundamental representation, the final result has no extra minus sign because the number of power of the $SU(N)$ generator $T^a(= K_0^a)$ is even. The identity (3.3) is used. Again, the Virasoro zero mode eigenvalue is fixed by the conformal dimension of the scalar primary operator.
The three-point functions with two real scalars, from (3.5) and (3.4), are

\[
< \mathcal{O}_+ \mathcal{O}_+ \mathcal{T} > = \frac{1}{4}(1 + 2\lambda), \quad < \mathcal{O}_- \mathcal{O}_- \mathcal{T} > = \frac{(1 - \lambda)}{4(1 + \lambda)},
\]  

where the scalar primaries correspond to \( \mathcal{O}_+ = (f; 0) \otimes (f; 0) \), \( \mathcal{O}_+ = (f; 0) \otimes (f; 0) \), \( \mathcal{O}_- = (0; f) \otimes (0; f) \) and \( \mathcal{O}_- = (0; f) \otimes (0; f) \), respectively. The normalization for the spin 2 current is described as

\[
<T(z) T(w) > = \frac{1}{(z-w)^4} \left[ \frac{1}{2}N^2 \frac{(1-\lambda)(1+2\lambda)}{2(1+\lambda)} \right]
\]  
in the large \( N \) limit. By absorbing the \((N, \lambda)\) dependent term into the spin 2 current-spin 2 current correlator (or equivalently dividing this two point function by \( \frac{1}{2}N^2 \frac{(1-\lambda)(1+2\lambda)}{2(1+\lambda)} \)),

\[
< \mathcal{O}_+ \mathcal{O}_+ \mathcal{T} > = \frac{1}{2N} \sqrt{\frac{1}{2}N^2 \frac{(1+\lambda)(1+2\lambda)}{(1-\lambda)}}, \quad < \mathcal{O}_- \mathcal{O}_- \mathcal{T} > = \frac{1}{2N} \sqrt{\frac{(1-\lambda)}{(1+\lambda)(1+2\lambda)}}
\]  

where \( < T(z) T(w) > = \frac{1}{(z-w)^4} \). Purely \( \lambda \)-dependent parts in each case of (3.7) are proportional to each other inversely. In other words, the product of these leads to \( \frac{1}{4N^2} \).

Furthermore, the conformal dimension \( h(f; f) [\Pi] \), which goes to zero in the large \( N \) limit when the second level is 1 and forms a continuum of light states near the vacuum, and its large \( N \) limit can be determined similarly.

\[
h(f; f) = \frac{N^2 - 1}{2N} \left[ \frac{1}{N+k} - \frac{1}{N+N+k} \right] = \frac{(N^2 - 1)}{2(N+k)(2N+k)} \to \frac{\lambda^2}{2(1+\lambda)}. \]  

The trivial representation corresponds to the numerator current with the level \( N \) in the coset. The appearance of \( N \) (from the level) in the second denominator of (3.8) gives rise to a nontrivial \( N \) factor in the numerator and this cancels the same factor in the denominator. Therefore, in the large \( N \) limit, the conformal dimension is nonzero. The states are no longer light. The above can be expressed as \( h(f; f) = h(f; 0) + h(0; f) - \frac{N^2 - 1}{2N} \) with (3.1) and (3.3).

The Virasoro zero mode eigenvalue can be determined similarly. For the primary \( (f; f) \), the zero mode \( J_0^a \) vanishes. More precisely, the eigenvalue becomes zero. The ground state transforms as a fundamental representation with respect to \( K_0^a \). The nonzero contribution arises from the second and third terms of (2.6). The result can be expressed as

\[
\left( -\frac{1}{2(k+N)} K_0^a K_0^a + \frac{1}{2(k+2N)} K_0^a K_0^a \right) |f > = \left( -\frac{1}{2(k+N)} + \frac{1}{2(k+2N)} \right) (-N) |f > = \frac{\lambda^2}{2(1+\lambda)} |f > = h(f; f) |f >.
\]  

In the last line of (3.9), the relation (3.8) is used.
One expects that there is a nonzero three-point function. The three-point function with two real scalars, from (3.9), are
\[
\langle \mathcal{O} \mathcal{O} T \rangle = \frac{\lambda^2}{2(1 + \lambda)}, \quad (3.10)
\]
where the scalar primaries correspond to \( \mathcal{O} = (f; f) \otimes (f; f) \) and \( \mathcal{O} = (\bar{f}; \bar{f}) \otimes (\bar{f}; \bar{f}) \), respectively. By rescaling the stress energy tensor as done before, the above (3.10) becomes
\[
\langle \mathcal{O} \mathcal{O} T \rangle = \frac{\lambda^2}{N\sqrt{(1 + \lambda)(1 - \lambda)(1 + 2\lambda)}}, \quad (3.11)
\]
where \( \langle T(z) \ T(w) \rangle = \frac{1}{(z-w)^4} \).

The spin 3 zero mode eigenvalue can be determined similarly. For the primary \((0; f)\) with vanishing \(K_a^0\) (i.e. the eigenvalue is zero), the nontrivial contribution from the spin 3 current (2.9) together with (2.28) can be obtained
\[
B(N,k)(k+N)(2k+N)d^{abc}J_a^0J_b^0J_c^0|f> = \frac{i}{6\sqrt{6}N^2(1+\lambda)}\sqrt{(2-\lambda)}(iN^2)|f> = -\frac{1}{6\sqrt{6}(1+\lambda)}\sqrt{(2-\lambda)}(2+3\lambda)|f>. \quad (3.12)
\]
The eigenvalue \(iN^2\) in (3.12) can be determined using \(d^{abc}\text{Tr}(T^aT^bT^c) = -\frac{i}{2}d^{abd}d^{abc}\text{Tr}(T^dT^c) \to iN^2\) after the extra \(\frac{1}{N}\) is multiplied.

On the other hand, for the primary \((f; 0)\) with \(K_a^0 = -J_a^0\) (i.e. the eigenvalue for the diagonal current is zero), the following result can be derived
\[
B(N,k)(k+N)(2k+N)(2k+5N)d^{abc}J_a^0J_b^0J_c^0|f> = \frac{i}{6\sqrt{6}N^2(1+2\lambda)}\sqrt{(2+3\lambda)}(2-\lambda)|f> = \frac{1}{6\sqrt{6}N^2(1+2\lambda)}\sqrt{(2+3\lambda)}(-iN^2)|f>. \quad (3.13)
\]
In this case, each term in (2.9) contributes to the eigenvalue equation. The extra minus sign in the eigenvalue of the first line of (3.13) is due to the fact that the ground state transforms as an antifundamental representation with respect to \(J_a^0\).

Then the three-point functions with scalars can be expressed as, from (3.12) and (3.13),
\[
\langle \mathcal{O}_+ \mathcal{O}_+ W \rangle = \frac{1}{6\sqrt{6}}(1+2\lambda)\sqrt{(2+3\lambda)}(2-\lambda),
\]
\[
\langle \mathcal{O}_- \mathcal{O}_- W \rangle = -\frac{1}{6\sqrt{6}}(1+\lambda)\sqrt{(2-\lambda)}(2+3\lambda). \quad (3.14)
\]
Each three-point function in (3.14) contains the corresponding two point function in (3.6), respectively. The other \( \lambda \) dependent part of one three-point function appears in the other three-point function inversely. The normalization for the spin 3 current is as follows:

\[
\langle W(z) W(w) \rangle = \frac{1}{(z-w)^6} \left[ \frac{1}{2} N^2 (1-\lambda)(1+2\lambda) \right].
\]

By different normalization, the above three-point function become

\[
\langle \mathcal{O}^+ \mathcal{O}^+ W \rangle = \frac{1}{6N} \sqrt{\frac{(1+\lambda)(1+2\lambda)(2+3\lambda)}{(1-\lambda)(2-\lambda)}},
\]

\[
\langle \mathcal{O}^- \mathcal{O}^- W \rangle = -\frac{1}{6N} \sqrt{\frac{(1-\lambda)(2-\lambda)}{(1+\lambda)(1+2\lambda)(2+3\lambda)}}.
\] (3.15)

The \( \lambda \)-dependent parts in each case of (3.15) are proportional to each other inversely.

For the primary \((f;f)\) with vanishing \( J_0^\perp \) (i.e. the eigenvalue is zero), the nontrivial contribution in the spin 3 zero mode from (2.9) together with (2.28) can be obtained

\[
B(N,k)\langle -1 \rangle 6N^3 d^{abc} K_0^a K_0^b K_0^c |f\rangle = - \frac{i\lambda^3}{N^2(1+\lambda)\sqrt{6(2-\lambda)(2+3\lambda)}} (iN^2 |f\rangle >
\]

\[
= \frac{\lambda^3}{(1+\lambda)\sqrt{6(2-\lambda)(2+3\lambda)}} |f\rangle >.
\] (3.16)

From (3.16), the three-point functions with scalars can be expressed as

\[
\langle \mathcal{O} O W \rangle = \frac{\lambda^3}{(1+\lambda)\sqrt{6(2-\lambda)(2+3\lambda)}}.
\] (3.17)

By different normalization with \( \langle W(z) W(w) \rangle = \frac{1}{(z-w)^6} \), the above three-point function (3.17) becomes

\[
\langle \mathcal{O} O W \rangle = \frac{\lambda^3}{N \sqrt{(1+\lambda)(1-\lambda)(1+2\lambda)(2+3\lambda)(2-\lambda)}}.
\] (3.18)

The factor in (3.11) appears in (3.18). In other words, \( \langle \mathcal{O} O W \rangle = \frac{\lambda}{\sqrt{(2+3\lambda)(2-\lambda)}} \langle \mathcal{O} O T \rangle \).

Therefore, the three-point functions with scalars are completely determined for \( s = 2, 3 \) in the large \( N \) limit. How should the other three-point functions corresponding to \( s = 4, 4' \)? The unnormalized three-point functions can be obtained using the prescriptions in this section. As described in the beginning of this section, because the normalizations for \( O_4(z) \) and \( O_{4'}(z) \) are not determined in this paper, the right hand side of the three-point functions are not fixed. Moreover, compared to the \( W_N \) minimal model, the result (3.8) for the conformal dimension (and corresponding three-point function in (3.11)) indicates that the corresponding states can survive in the large \( N \) limit. The states are no longer light.
4 Conclusions and outlook

In this paper, the explicit expressions for the $\mathcal{N} = 1$ higher spin currents of spins-$\left(\frac{7}{2}, 4\right)$ and $\left(4, \frac{9}{2}\right)$, which are the second and third elements in (1.4), are constructed. These are obtained from the OPE between the $\mathcal{N} = 1$ lowest higher spin super current of spins-$\left(\frac{5}{2}, 3\right)$, which is the first element in (1.4), and itself.

Furthermore, the three OPEs (2.11), (2.15) and (2.32) can be expressed as a single $\mathcal{N} = 1$ super OPE, via Appendix E, as follows:

\[
\hat{W}(Z_1) \hat{W}(Z_2) = \frac{1}{z_{12}^2} \frac{c}{15} + \frac{\theta_{12}}{z_{12}^3} \hat{T}(Z_2) + \frac{1}{z_{12}^3} \frac{1}{3} D\hat{T}(Z_2) + \frac{\theta_{12}}{z_{12}^3} \frac{2}{3} \partial\hat{T}(Z_2) + \frac{1}{z_{12}^2} \frac{1}{3} T^2(Z_2)
\]

\[
+ \frac{\theta_{12}}{z_{12}^2} \left[ \frac{(2c + 5)}{2(4c + 21)} \partial^2\hat{T} + \frac{22}{(4c + 21)} \hat{T}D\hat{T} + \frac{1}{\sqrt{6}} \hat{O}_{\frac{7}{2}} \right] (Z_2)
\]

\[
+ \frac{1}{z_{12}^2} \left[ \frac{2(18c + 1)}{(4c + 21)(10c - 7)} \hat{T}D\hat{T} + \frac{4c^2 - c - 37}{(4c + 21)(10c - 7)} D\partial^2\hat{T}
\]

\[
- \frac{2(2c - 83)}{(4c + 21)(10c - 7)} \hat{T}\partial\hat{T} + \frac{1}{7\sqrt{6}} D\hat{O}_{\frac{7}{2}} + \frac{2}{\sqrt{6}} \hat{O}_4 \right](Z_2)
\]

\[
+ \frac{\theta_{12}}{z_{12}^2} \left[ \frac{16(7c - 10)}{(4c + 21)(10c - 7)} \hat{T}D\partial\hat{T} + \frac{4c^2 - 29c + 3}{3(4c + 21)(10c - 7)} \partial^3\hat{T}
\]

\[
+ \frac{8(18c + 1)}{(4c + 21)(10c - 7)} \hat{T}\partial\hat{T} + \frac{4}{7\sqrt{6}} \partial\hat{O}_{\frac{7}{2}} + \frac{1}{\sqrt{6}} D\hat{O}_4 \right](Z_2) + \cdots \tag{4.1}
\]

where the $\mathcal{N} = 1$ super currents have the following component fields [11]

\[
\hat{W}(Z) = \frac{1}{\sqrt{6}} U(z) + \theta W(z),
\]

\[
\hat{O}_{\frac{7}{2}}(Z) = O_{\frac{7}{2}}(z) + \theta O_3(z),
\]

\[
\hat{O}_4(Z) = O_4(z) + \theta O_2(z), \quad Z = (z, \theta), \tag{4.2}
\]

and $z_{12} = z_1 - z_2 - \theta_1 \theta_2, \theta_{12} = \theta_1 - \theta_2, D = \partial_9 + \theta \partial_2$ and $\partial = \partial_2$. Because the $c$-dependent coefficients in the linear superfield terms of (4.1) approach to constant value and those in the quadratic superfield terms approach to $\frac{1}{c}$ when $c \to \infty$, the corresponding classical algebra can be obtained from (4.11) with appropriate coefficients found in this limit. This will correspond to the asymptotic symmetry algebra of $AdS_3$ bulk theory.

The $\mathcal{N} = 1$ super fusion rule of (4.1) can be summarized as

\[
[\hat{W}] [\hat{W}] = [\hat{I}] + [\hat{O}_{\frac{7}{2}}] + [\hat{O}_4]. \tag{4.3}
\]

The coupling constants, $c_{\hat{O}_{\frac{7}{2}}}^I$ and $c_{\hat{O}_4}^I$, appearing in front of $[\hat{O}_{\frac{7}{2}}]$ and $[\hat{O}_4]$, respectively, are not determined in this expression (4.3) because the normalizations of $\hat{O}_{\frac{7}{2}}(Z)$ and $\hat{O}_4(Z)$ (4.2).
are not fixed. These normalizations can be fixed only if the OPEs $\hat{O}_7(Z_1) \hat{O}_7(Z_2)$ and $\hat{O}_4(Z_1) \hat{O}_4(Z_2)$ are calculated. A further study should determine these OPEs. These coupling constants should vanish when $N = 3$ and $c = \frac{10}{7}$ (or $k = 1$). See also the recent developments \[22, 23\] where the approach 1 of \[5\] is used to fix the normalizations. They make the most general ansatz for the OPEs of the currents with each other and solve the Jacobi identities at the level of OPEs rather than modes.

The three-point functions with two scalars and a single conserved current of spins 2, 3 are obtained. The new observation is that the conformal dimension and the three-point function including the state $(f; f)$ or its complex conjugate representation behave nonzero eigenvalue equation in the large $N$ limit. This arises from the fact that the difference between the level $N + k$ for the diagonal WZW current and the level $k$ for the first numerator WZW current is given by $N (= N + k - k)$. Recall that when the level for the second numerator WZW current is equal to 1, the above difference leads to $1 (= 1 + k - k)$. Therefore, the previous light states are no longer light and they participate in the nonzero states even in the large $N$ limit.

It is not known what the spin contents are in the coset model \[12\]. How do the extra spin contents arise in addition to the spin contents in the minimal $\mathcal{N} = 1$ super $W_N$ algebra? In \[12\], the character technique was used to extract the spin contents for $c = 4$ eight fermion model (i.e. $k \to \infty$ limit with $N = 3$ in \[12\]). As a warmup, the analysis for $N = 4$ with $k \to \infty$ can be done. Finding the general $N$ character technique will be an open problem. This will be a generalization of the recent construction found in \[14\] because the spin contents for the minimal $\mathcal{N} = 1$ supersymmetric $W_N$ algebra are contained.

Compared to the $\mathcal{N} = 2$ $\mathcal{W}_{N+1}$ algebra studied in \[24, 23, 25\], the OPE of the $\mathcal{N} = 1$ lowest higher spin super current looks similar but the right hand side of \[4.3\] in the present case does not contain the $\mathcal{N} = 1$ lowest higher spin super current. Suppose that $\hat{W}(Z_2)$ can arise in the right hand side of \[4.3\]. Then the super current $\hat{W}(Z_2)$ can appear in $\frac{\hat{O}_7}{\hat{O}_7}$ term. However, due to the symmetry in $\hat{W}(Z_1) \hat{W}(Z_2)$ (same fermionic super currents), after reversing the arguments $Z_1$ and $Z_2$ and expanding around $Z_2$, the same expression $\hat{W}(Z_2)$ will appear with an opposite sign. This suggests that the super current $\hat{W}(Z_2)$ should vanish.

What about the OPEs $\hat{W}(Z_1) \hat{O}_7(Z_2)$ and $\hat{W}(Z_1) \hat{O}_4'(Z_2)$? At the moment, one expects that these OPEs can be expressed in terms of other $\mathcal{N} = 1$ super currents (the explicit calculations will be very complicated). In Appendix $E$, the two $\mathcal{N} = 1$ super OPEs are presented by replacing the $k$-dependent coefficient functions with the central charge when $N = 3$. Note that the structure constant $c$ has a factor $(7c - 10)$ in the present case and those constant for the minimal $\mathcal{N} = 1$ $W_N$ algebra was studied in \[14\]. One expects that this structure constant will depend on both the central charge $c$ and $N$ and have still a factor
(7c − 10) for general N. Furthermore, the new $\mathcal{N} = 1$ higher spin currents should appear when the spins of two super currents in the super OPE increase. It would be interesting to determine the complete set of (super) currents for the coset (1,2). At the moment, it will be very difficult to determine the field contents in terms of WZW currents but the generalization of the character technique for general N can be used to determine the spin contents at least.

As described in the introduction, the $\mathcal{N} = 2$ supersymmetric extension can arise for $k = N$ where the coset is given by

$$\frac{\tilde{SU}(N) \oplus \tilde{SU}(N)}{SU(N)_{2N}}.$$  \hspace{1cm} (4.4)

It would be interesting to determine the $\mathcal{N} = 2$ higher spin super currents as well as the $\mathcal{N} = 2$ super stress energy tensor with spins $\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 2$. In this case, it is not clear how to take the large $N$ limit with fixed 't Hooft coupling constant because the $k$ dependence disappears and the 't Hooft coupling constant becomes $\lambda = \frac{1}{2}$ \cite{21}. In doing this, the first step is to construct the $\mathcal{N} = 2$ superconformal algebra realized by the WZW currents living in the coset (1,2) with the substitution of $k = N$ where $c = \frac{N^2 - 1}{3}$. The two independent spin $\frac{3}{2}$ currents can be obtained from two independent free fermions of spin $-\frac{1}{2}$. Obviously, the $U(1)$ spin 1 current in the $\mathcal{N} = 2$ superconformal algebra can be constructed from the product of these two fermions. The next step is to determine the higher spin currents, $(2, \frac{5}{2}, \frac{5}{2}, 3), \cdots, (s, s + \frac{1}{2}, s + \frac{1}{2}, s + 1), \cdots$ in terms of WZW currents. The structure for $N = 3$ will be useful to determine the lower $\mathcal{N} = 2$ higher spin super currents in terms of the WZW currents living in this specific coset. It would be interesting to see whether the recent findings with $\mathcal{N} = 1$ supersymmetry in \cite{14} can arise in the coset model (4.4) or not. Although they have different supersymmetries, they share the common 't Hooft coupling constant $\lambda = \frac{1}{2}$. The first step in this direction is to understand the precise relation between $\mathcal{N} = 2$ $\mathcal{W}_3$ algebra and the $\mathcal{N} = 1$ $\mathcal{W}_3$ algebra and to see how the latter can be embedded in the former.

As described in \cite{5}, one of the levels is equal to 1 for the equivalence between the coset construction and Drinfeld-Sokolov reduction and the triality in \cite{2} is based on this particular case. Because the coset model (1,2) in this paper has the second level $N$ which is not equal to 1, it is not clear how to observe the isomorphism between these two constructions. It would be interesting to study Drinfeld-Sokolov reduction for the more general coset model.

One application of the result of this paper is to analyze the $\mathcal{N} = 1$ higher spin super currents living in the coset model with orthogonal group. In particular, the holographic minimal model with $\mathcal{N} = 1$ supersymmetry is described in \cite{26,27} where the coset is expressed as the orthogonal groups. In this case, although the second level of the numerator WZW current is equal to 1, the spin $\frac{3}{2}$ current is obtained. It would be interesting to construct the
\( \mathcal{N} = 1 \) higher spin super currents when the second level is given by the dual Coxeter number of \( SO(2N) \). Similar analysis can be done in the context of \([28, 29, 30, 31]\). By applying to the present methods, the construction of \( \mathcal{N} = 1 \) higher spin currents when the second level is given by the dual Coxeter number of \( SO(2N) \). Furthermore, it would be interesting to determine the \( \mathcal{N} = 2 \) higher spin currents by restricting the first level to the dual Coxeter number of \( SO(2N) \) also.

The asymptotic symmetry (for the lowest super current) of the \( \mathcal{N} = 1 \) higher spin AdS\(_3\) gravity can be read off from the two dimensional CFT results obtained thus far. The three-point functions from the CFT computations should correspond to the three-point functions in the AdS\(_3\) bulk theory. The bulk theory would have higher spin gauge symmetry in AdS\(_3\) string theory because the central charge in this coset model is proportional to \( N^2 \) rather than \( N \). Furthermore, the light states in \([1, 2, 3]\) are no longer light ones because the conformal dimension of spin 2 for the state \((f; f)\) has finite value in the large \( N \) limit. It would be highly nontrivial to find the AdS\(_3\) bulk string theory. One direction to find the AdS\(_3\) string theory was described in the context of large \( \mathcal{N} = 4 \) holography in \([32, 33]\). The two-dimensional CFT has more supersymmetry and the extra transverse space (in type IIB string theory) has a \( S^1 \) factor. It would be interesting to see any relations between the coset model in this paper and the coset model in \([32]\).

Because there are fermionic currents including the spin-\(\frac{3}{2}\) current \((2.8)\), the three-point functions with a bosonic operator, a fermionic operator (superpartner of scalar operator) and a fermionic current can be constructed. It would be interesting to describe the fermionic operators explicitly and see how to construct the corresponding three-point functions with two fermionic operators and a single higher spin current with integer spin \([34, 35]\).

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Appendix A  The second-order pole in the OPE $W(z) W(w)$
and some expressions relevant to spin 4 field

In (2.11), to extract the primary spin 4 current, the second-order pole is needed to calculate. Because this second order pole of $W(z) W(w)$ for the arbitrary levels $(k_1, k_2)$ in the coset model (1.2) was found in [15], the corresponding expression can be obtained by simply substituting $k_1 = N$ and $k_2 = k$ into the primary spin 4 current in [15]. Each fourteen term with (2.17) is presented as follows:

\[
\{Q Q\}_{-2} = -18NQ^aQ^a - 54N(N^2 - 4)J^a \partial^2 J^a, \\
\{Q Q^a K^a\}_{-2} = -6N(2d^{abc}J^aQ^bK^c - 3(N^2 - 4)\partial^2 J^a K^a), \\
\{Q J^a S^a\}_{-2} = -6NQ^a S^a, \\
\{Q^a K^a Q\}_{-2} = -6N(2d^{abc}J^aQ^bK^c - 3(N^2 - 4)J^a \partial^2 K^a), \\
\{Q^a K^a Q^b K^b\}_{-2} = d^{abcd} \left[ \frac{2}{3} N J^a J^c K^d - 3N J^a J^b K^c K^d \right] \\
+ d^{abcd}d^{ede} \left[ -kJ^a J^c J^d + 2N J^a J^b K^c K^d \right] + \frac{8(N^2 - 4)}{N^2 + 1} J^a J^a J^b K^b \\
- \frac{12(N^2 - 4)}{(N^2 + 1)} J^a J^a K^b K^b + 6k(N^2 - 4)\partial^2 J^a J^a + \frac{8(N^2 - 4)(N^2 + 2)}{N^2 + 1} f^{abc} J^a \partial J^b K^c, \\
\{Q^a K^a J^b S^b\}_{-2} = -(N + 2k)d^{abc}J^bQ^bK^c + 2NQ^a S^a \\
- 3N d^{abc} J^a K^b S^c + 3(N + 2k)(N^2 - 4)\partial^2 J^a K^a, \\
\{Q^a K^a S\}_{-2} = -3(N + k)Q^a S^a, \\
\{J^a S^a Q\}_{-2} = -6NQ^a S^a, \\
\{J^a S^a Q^b K^b\}_{-2} = -3N d^{abc} J^a K^b S^c + 2NQ^a S^a - (N + 2k)d^{abc} J^a K^b Q^c \\
+ 3(N + 2k)(N^2 - 4)\partial^2 J^a J^a, \\
\{J^a S^a J^b S^a\}_{-2} = d^{abcd} \left[ \frac{2}{3} N K^a K^b K^c J^d - (N + 2k)K^a K^b J^c J^d \right] \\
+ d^{abcd}d^{ede} \left[ -NK^a K^b K^c K^d + 2kK^a K^b J^c J^d \right] \\
+ \frac{8(N^2 - 4)}{N^2 + 1} K^a K^a K^b J^b - \frac{4(N^2 - 4)(N + 2k)}{N(N^2 + 1)} K^a K^a J^b J^b + 2(N^2 - 4)(N + 2k)\partial^2 K^a K^a \\
- \frac{2(N^2 - 4)(6k + 9N + 6kN^2 + 5N^3)}{3(N^2 + 1)} \partial^2 K^a J^a
\]
provides the complete expression for the second order pole of the lemma that appeared in (2.37) should be used).

For the calculation of \( W \), the field \( G \partial G \) into Appendix (G.6) in [15] is used

\[
T^2(z) = \frac{k^2}{16N^2(2N+k)^2} \left[ -4N\partial^2 J^a J^a(z) + J^a J^a J^b J^b(z) \right]
+ \frac{N^2}{4(N+k)^2(2N+k)^2} \left[ -2(k+N)\partial^2 K^a K^a(z) + K^a K^b K^b(z) \right]
+ \frac{Nk}{4N(N+k)(2N+k)^2} J^a J^b K^b(z)
+ \frac{1}{(2N+k)^2} \left[ J^a K^a J^b K^b(z) + N\partial^2 J^a K^a(z) - \frac{N}{2} \partial^2 K^a K^a(z) + f^{abc} J^a \partial K^b K^c(z) \right]
- \frac{k}{2} J^a \partial^2 J^a(z) - f^{abc} J^a \partial J^b K^c(z)
- \frac{N}{2(N+k)^2(2N+k)^2} \left[ J^a K^a K^b K^b(z) - (N+k)\partial^2 J^a K^a(z) \right]
+ J^b K^a K^b K^b(z) - (2N+k)J^a \partial^2 K^a(z) + 2f^{abc} J^a K^b \partial K^c(z)
- \frac{k}{4N(2N+k)^2} \left[ J^a K^a J^b J^b(z) - 2N J^a \partial^2 K^a(z) \right]
+ J^a J^a J^b K^b(z) - 3N\partial^2 J^a K^a(z) + 2f^{abc} J^a \partial J^b K^c(z) \right].
\] (A.2)

The field \( G \partial G(z) \) can be obtained from the following identities (again the rearrangement lemma that appeared in (2.37) should be used)

\[
(\psi^a J^a) \partial(\psi^b J^b) = \psi^a J^a \partial(\psi^b J^b) - \frac{N}{2} \psi^a \partial^2 \psi^a + \frac{3}{4} \partial^2 J^a J^a - \frac{9}{4} \partial(J^a \partial J^a),
\]

23
\[(\psi^a J^a) \partial (\psi^b K^b) = \psi^a \partial \psi^b J^a K^b + \psi^a \psi^b J^a \partial K^b - \frac{3}{4} \partial^2 J^a K^a - \frac{3}{2} \partial J^a \partial K^a,\]
\[(\psi^a K^a) \partial (\psi^b J^b) = \psi^a \partial \psi^b J^a K^b + \psi^a \psi^b \partial J^a K^a - \frac{3}{2} \partial J^a \partial K^a - \frac{3}{4} J^a \partial^2 K^a,\]
\[(\psi^a K^a) \partial (\psi^b K^b) = \psi^a \psi^b K^a \partial K^b + \psi^a \psi^b \partial J^a K^b + \frac{1}{3} k \psi^a \partial^2 \psi^b K^c + \frac{1}{2} f^{abc} \psi^a \partial^2 \psi^b K^c + \frac{1}{2} \partial J^a \partial K^a + \frac{k}{2} \partial \psi^a \partial^2 \psi^a,\]
\[(A.3)\]

and it is expressed as
\[G \partial G(z) = \frac{2k^2}{9N(k + N)(k + 2N)} \left[ \psi^a J^a \partial (\psi^b J^b) - \frac{N}{2} \psi^a \partial^3 \psi^a + \frac{3}{4} \partial^2 J^a J^a - \frac{9}{4} \partial(J^a \partial J^a) \right.\]
\[\left. - \frac{3N}{k} \left( \psi^a \partial \psi^b J^a K^b + \psi^a \psi^b J^a \partial K^b - \frac{3}{4} \partial^2 J^a K^a - \frac{3}{2} \partial J^a \partial K^a \right.\]
\[\left. + \psi^a \partial \psi^b J^a K^b + \psi^a \psi^b \partial J^a K^a - \frac{3}{2} \partial J^a \partial K^a - \frac{3}{4} J^a \partial^2 K^a \right) + \frac{9N^2}{k^2} \left( \psi^a \psi^b K^a \partial K^b \right.\]
\[\left. + \psi^a \partial \psi^b K^a K^b + \frac{1}{3} k \psi^a \partial^3 \psi^a - \frac{1}{2} f^{abc} \psi^a \partial^2 \psi^b K^c + \frac{1}{2} \partial J^a \partial K^a \right.\]
\[\left. - \frac{1}{4} \partial^2 K^a K^a - \frac{1}{2} \partial K^a \partial K^a + \frac{1}{2} \partial^2 J^a J^a + \frac{k}{2} \partial \psi^a \partial^2 \psi^a \right].\]
\[(A.4)\]

As noticed in the footnote 3, the first five terms in (2.14) can be expressed as |[11]|
\[-\frac{3(N^2 - 4)(N^2 - 3)}{N^2 + 1} \partial K^a \partial K^a(z) + \frac{2(N^2 - 4)(N^2 - 3)}{N^2 + 1} \partial^2 K^a K^a(z). \tag{A.5}\]

The explicit coefficient functions, which depend on \(N\) and \(k\), in (2.14) can be read off from the results [15]

\[
c_1 = -\frac{k(1 + N^2)(-5k^2 - 15kN - 8N^2 + k^2N^2 + 3kN^3)}{4(-4 + N^2)N(k + 2N)(-3 + N^2)D(N, k)},
\]

\[
c_2 = \frac{2(1 + N^2)(-36k^3 - 201k^2N - 215kN^2 + 20k^3N^2 - 40N^3 + 85k^2N^3 + 75kN^4)}{9(-4 + N^2)(k + 2N)(2k + 5N)(-3 + N^2)D(N, k)},
\]

\[
c_3 = \frac{(-4 + N^2)(k + 2N)(2k + 5N)}{N},
\]

\[
c_4 = \frac{2N(1 + N^2)(-25k^2 - 107kN - 72N^2 + 5k^2N^2 + 15kN^3)}{3(-4 + N^2)(k + 2N)(2k + 5N)(-3 + N^2)D(N, k)},
\]

\[
c_5 = -\frac{N(1 + N^2)(-5k^2 - 15kN - 8N^2 + k^2N^2 + 3kN^3)}{2(-4 + N^2)(k + 2N)(2k + 5N)(-3 + N^2)D(N, k)},
\]

\[
c_6 \equiv \frac{n_1}{d_1},
\]

\[
n_1 = k(-162k^3 - 657k^2N - 849kN^2 - 12k^3N^2 - 312N^3 - 114k^2N^3 - 218kN^4
\]
\[+ 14k^3N^4 - 136N^5 + 67k^2N^5 + 75kN^6),
\]

\[
d_1 = 6(-4 + N^2)N(k + 2N)(2k + 5N)(-3 + N^2)D(N, k),
\]

\[
c_7 \equiv \frac{n_2}{d_2},
\]

\[
n_2 = 4(99k^3 + 309k^2N + 332kN^2 - 32k^3N^2 + 112N^3 - 142k^2N^3 - 194kN^4
\]
\[+ 5k^3N^4 - 64N^5 + 25k^2N^5 + 30kN^6),
\]

\[
d_2 = 3(-4 + N^2)(k + 2N)(2k + 5N)(-3 + N^2)D(N, k),
\]

\[
c_8 = \frac{2N}{3(-4 + N^2)(2k + N)(2k + 5N)},
\]

\[
c_9 \equiv \frac{n_3}{d_3},
\]

\[
n_3 = -4N(-25k^3 - 295k^2N - 652kN^2 - 20k^3N^2 - 432N^3 - 66k^2N^3 - 10kN^4
\]
\[+ 5k^3N^4 + 96N^5 + 25k^2N^5 + 30kN^6),
\]

\[
d_3 = (4 + N^2)(k + N)(k + 2N)(2k + 5N)(-3 + N^2)D(N, k),
\]

\[
c_{10} \equiv \frac{n_4}{d_4},
\]

\[
n_4 = 3N(-10k^4 - 60k^3N - 177k^2N^2 - 8k^4N^2 - 261kN^3 - 48k^3N^3 - 152N^4
\]
\[- 86k^2N^4 + 2k^4N^4 - 42kN^5 + 12k^3N^5 + 24N^6 + 23k^2N^6 + 15kN^7),
\]

\[
d_4 = (4 + N^2)(k + N)(2k + N)(k + 2N)(2k + 5N)(-3 + N^2)D(N, k),
\]

\[
c_{11} = -\frac{k(-27k^2 - 57kN - 24N^2 + 7k^2N^2 + 13kN^3)}{N^2(k + 2N)(-3 + N^2)D(N, k)}.
\]
The large $N$ limit with fixed 't Hooft coupling constant $\lambda$ in (3.2) provides the following limiting values on the coefficient functions

\begin{align*}
c_1 & \to \frac{(-1 + \lambda)}{20N^3(1 + \lambda)}, & c_2 & \to -\frac{2\lambda(4 + \lambda)}{9N^3(1 + \lambda)(2 + 3\lambda)}, \\
c_3 & \to \frac{\lambda^2}{N^3(1 + \lambda)(2 + 3\lambda)}, & c_4 & \to \frac{2\lambda^2}{3N^3(1 + \lambda)(2 + 3\lambda)}, \\
c_5 & \to -\frac{\lambda^2}{10N^3(1 + \lambda)}, & c_6 & \to -\frac{(-1 + \lambda)(14 + 11\lambda)}{30N^3(1 + \lambda)(2 + 3\lambda)}, \\
c_7 & \to \frac{4\lambda}{3N^3(2 + 3\lambda)}, & c_8 & \to -\frac{2\lambda^2}{3N^3(2 + \lambda)(2 + 3\lambda)}, \\
c_9 & \to \frac{N^3(-2 + \lambda)(2 + 3\lambda)}{4\lambda^2}, & c_{10} & \to -\frac{3\lambda^2(2 + 2\lambda + \lambda^2)}{5N^3(-2 + \lambda)(1 + \lambda)(2 + 3\lambda)}, \\
c_{11} & \to \frac{(-1 + \lambda)(7 + 6\lambda)}{5N^4(1 + \lambda)(1 + 2\lambda)}, & c_{12} & \to -\frac{8\lambda(-4 + 9\lambda + 10\lambda^2)}{15N^4(1 + \lambda)(1 + 2\lambda)(2 + 3\lambda)}, \\
c_{13} & \to -\frac{4\lambda^2(3 + 2\lambda)}{5N^4(1 + \lambda)(1 + 2\lambda)(2 + 3\lambda)}, & c_{14} & \to -\frac{8\lambda^2(5 + 21\lambda + 14\lambda^2)}{5N^4(-1 + \lambda)(1 + \lambda)(1 + 2\lambda)(2 + 3\lambda)}, \\
c_{15} & \to \frac{2\lambda^2(3 + 3\lambda + 2\lambda^2)}{5N^4(-1 + \lambda)(1 + \lambda)(1 + 2\lambda)}, & c_{16} & \to \frac{2\lambda(19 + 16\lambda)}{45N(1 + \lambda)(2 + 3\lambda)}, \\
\end{align*}
Let us describe the second-order singular terms in the OPE:

\[ c_{17} \rightarrow -\frac{2\lambda(3 + 2\lambda)}{5N(1 + \lambda)(2 + 3\lambda)}, \quad c_{18} \rightarrow \frac{2\lambda(3 + 2\lambda)}{15N(1 + \lambda)(2 + 3\lambda)}, \quad c_{19} \rightarrow \frac{8\lambda^2 (11 + 19\lambda + 10\lambda^2)}{5N^4(-1 + \lambda)(1 + \lambda)(1 + 2\lambda)(2 + 3\lambda)}. \] (A.7)

Because the coset model (121) in this paper is different from that in [15], some of the missing terms in [15] can survive in (A.7): two terms with coefficients \( c_5 \) and \( c_{10} \). The terms with the coefficients, \( c_{11}, c_{12}, c_{13}, c_{14}, c_{15} \) and \( c_{19} \) do not contribute to the final eigenvalue equation because the \( \frac{1}{N} \) behavior of those terms is suppressed.

**Appendix B  The second-order pole in the OPE \( U(z) W(w) \) and other expression relevant to spin \( \frac{7}{2} \) field**

Let us describe the second-order singular terms in the OPE \( U(z) W(w) \). They come from the following 11 OPEs:

\[
\begin{align*}
\{\psi^a Q^a Q\}_{-2} &= -5Nd_{abc}(\psi^a J^b)Q^c - 5Nd_{abc}d_{cde}J^a(\psi^d J^e)J^b, \\
-5Nd_{abc}d_{cde}J^a J^b \psi^d J^e, \\
\{\psi^a Q^a Q^b K^b\}_{-2} &= -5Nd_{abc}d_{cde}(\psi^a J^b)J^d K^e - 5Nd_{abc}d_{cde}J^a \psi^d J^e K^b, \\
\{\psi^a Q^a J^b S^b\}_{-2} &= -5Nd_{abc}d_{aef} S^c, \\
\{\psi^a R^a Q\}_{-2} &= d_{abc}d_{cde} \left[ f^dab f^cgh (\psi^i J^b K^c) J^f - f^dal f^egh f^hbi \partial (\psi^i K^c) J^f \right] + f^dab f^cgh J^d \psi^i J^b K^c - f^dal f^egh f^hbi J^e \partial (\psi^i K^c) + f^dab f^cgh J^d \psi^i J^b K^c - f^dal f^egh f^hbi J^e \partial (\psi^i K^c) + f^dab f^cgh J^d \psi^i J^b K^c
\end{align*}
\] (B.1)

\[
\begin{align*}
\{\psi^a R^a Q^b K^b\}_{-2} &= d_{abc}d_{cde} \left[ -k^dfe(\psi^a J^b) (J^d J^e) + f^fca f^dag (\psi^i J^b K^g) J^e - f^fca f^dag f^hbi J^d \psi^i J^b K^g - f^fca f^dag f^hbi J^d \partial (\psi^i K^g) + f^fca f^dag K^d (\psi^i K^g) - f^fca f^dag K^d (\psi^i K^g) J^f + f^fca f^dag K^d J^e \psi^i J^b K^g + f^fca f^dag K^d J^e \psi^i J^b K^g \right],
\end{align*}
\] (B.4)
In (B.3), no rearrangement is needed because the field $\psi$ does not contain $\psi^a$. The nonderivative first term of (B.12) will be contributed to the $c_d$ term in (2.26) by moving $\psi^d$ to the left. The second term of (B.2) should be rearranged to obtain the final expression. In (B.3), no rearrangement is needed because the field $S^d(z)$ does not contain $\psi^a(z)$ or $J^b(z)$ from (2.17). In order to simplify the expression (B.4) further, one uses the identity

$$d^{a b d} = -N d^{d e f a}.$$
\[ f^{hae} d^{ebf} f^{fcg} d^{gh} = \frac{N}{2} (d^{ace} d^{dec} - d^{ade} d^{bce}) - \frac{N}{2} d^{abe} d^{cde}, \quad (B.13) \]

which will be used continually in this paper. The normal ordered products, using the previous method, are expressed as

\[
\begin{align*}
    d^{abc}(\psi^a R^b) F^c & = d^{abc}\psi^a J^b R^c - (N^2 - 4) \partial^2 (\psi^a K^a), \\
    f^{abc}(\psi^a K^b) F^c & = -N \partial^2 (\psi^a K^a) + f^{abc} f^a \partial (\psi^b K^c), \\
    d^{def} d^{ceh} J^d (\psi^h K^e) F^f & = d^{def} d^{ceh} J^d J^f \psi^h K^c + \frac{N^2 - 4}{N} f^{abc} J^a \partial (\psi^b K^c), \\
    d^{abc}(\psi^a K^b) Q^c & = d^{abc} \psi^a K^b Q^c + 2(N^2 - 4) \partial \psi^a \partial K^a - (N^2 - 4) \psi^a \partial^2 K^a, \\
    d^{abc} d^{def} f^{ebh} f^{dga} (\psi^g J^h K^e) F^f & = N(N^2 - 4) \partial^2 (\psi^a K^a) - \frac{N}{2} d^{abc} d^{ade} J^d J^b \psi^a K^e \\
    & + \frac{N}{2} d^{abc} d^{ade} J^d J^b \psi^a \psi^b K^e + \frac{N}{2} d^{abc} d^{ade} J^d J^e \psi^a K^b, \\
    d^{abc} d^{def} f^{dgh} f^{eah} (\psi^h J^g K^e) F^f & = N(N^2 - 4) \partial^2 (\psi^a K^a) - \frac{N}{2} d^{abc} d^{ade} J^d J^a \psi^b K^e \\
    & + \frac{N}{2} d^{abc} d^{ade} J^d J^a \psi^b \psi^d K^e + \frac{N}{2} d^{abc} d^{ade} J^d J^b \psi^d K^e. \quad (B.14)\end{align*}
\]

The first, second, seventh, ninth, tenth and twenty first terms of (B.14) are related to the first, second, third, fourth, and fifth equation of (B.14), respectively. Partial results are presented due to the space of this paper. Other terms, which were not presented here should be further rearranged to arrive at the final expression. As noticed previously, the nonderivative term of first equation in (B.14) will contribute to c3-term of (2.26), the nonderivative term of third and fourth equations in (B.14) will be combined to c5-term of (2.26). The \( \psi^a \) appearing in the nonderivative terms in the fifth and sixth equations of (B.14) should be moved to the left.

The nontrivial normal ordered products in (B.5) can be expressed as

\[
\begin{align*}
    d^{abc} d^{def} f^{fcg} f^{dah} (\psi^h J^b K^e) F^f & = -\frac{N}{2} d^{abc} d^{ade} J^d J^e \psi^a K^b + \frac{N}{2} d^{abc} d^{ade} J^d J^b \psi^a K^e \\
    & + \frac{N}{2} d^{abc} d^{ade} J^d J^a \psi^b K^e + 2N(N^2 - 4) \partial \psi^a \partial K^a \\
    & - (N^2 - 4) f^{abc} J^a \partial (\psi^b K^c) - 2N(N^2 - 4) \partial^2 (\psi^a K^a), \\
    d^{def} d^{ceh} K^d (\psi^h K^c) F^f & = d^{def} d^{ceh} K^d J^f \psi^h K^c + \frac{N^2 - 4}{N} f^{abc} K^a \partial (\psi^b K^c), \\
    d^{abc} d^{def} f^{dgh} f^{eah} (\psi^h J^g K^e) F^f & = -2N(N^2 - 4) \partial \psi^a \partial K^a + (N^2 - 4) f^{abc} J^a \partial (\psi^b K^c) \\
    & - \frac{N}{2} d^{abc} d^{ade} J^d J^a \psi^b K^e + \frac{N}{2} d^{abc} d^{ade} J^d J^a \psi^d K^b \\
    & + \frac{N}{2} d^{abc} d^{ade} J^d J^e \psi^a K^b, \\
    d^{abc} d^{def} K^d (\psi^a K^c) F^f & = d^{abc} d^{ade} J^d \psi^a K^e K^b + \frac{N^2 - 4}{N} f^{abc} K^a \partial (\psi^b K^c). \quad (B.15)\end{align*}
\]
The second, eighth, tenth, and fourteenth terms of (B.5) are related to the first, second, third, and fourth equation of (B.15), respectively. The \(c_1\)-term of (2.26) arises from the first term of (B.5) and the second and fourth equations of (B.15) contribute to the \(c_6\)-term of (2.26). The \(c_3\)-term of (2.26) can be obtained from the third equation of (B.15). The \(c_2\)-term can be seen from the last term of (B.5). All the \(\psi^a\) field appearing to the right of \(J^b\) in (B.15) should be moved to the left.

Similarly, the normal ordered products in (B.6) can be expressed as

\[
d_{abc} d_{def} f^{dag} f^{ech} (\psi^a J^b K^h) K^f = -\frac{N}{2} d_{abc} d^{cde} \psi^a J^d K^e K^b + \frac{N}{2} d_{abc} d^{cde} \psi^a J^d K^b K^e + \frac{N}{2} d_{abc} d^{cde} \psi^a J^b K^d K^e,
\]

\[
d_{abc} (\psi^a K^b) S^c = d_{abc} d^{cde} \psi^a J^b K^d K^e - \frac{N}{2} (N + 2k)(N^2 - 4) \partial^2 \psi^a K^a, \quad (B.16)
\]

The first, sixth and eighth terms of (B.6) are related to the first, second, and third equation of (B.16), respectively. The nonderivative term of the first equation in (B.16) appears in the \(c_2\)-, \(c_6\)-terms of (2.26) and the second equation of (B.16) contributes to the \(c_4\)-term of (2.26).

The normal ordered products in (B.7) and (B.8) are expressed as

\[
d_{abc} d^{dch} (\psi^a J^b K^h) K^f = d_{abc} d^{cde} \psi^a J^b K^d K^e, \quad (B.17)
\]

coming from the second term of (B.7) and

\[
d_{abc} d^{fah} (\psi^b K^h K^c) J^f = d_{abc} d^{cde} \psi^b J^c K^d K^e, \quad (B.18)
\]

coming from the first term of (B.8), respectively. These two expressions play the role of \(c_2\)-term in (2.26).

The following OPEs can be seen from (B.9)

\[
d_{abc} (\psi^a K^b) Q^c = d_{abc} d^{cde} \psi^a J^d J^e K^b + 2(N^2 - 4) \partial \psi^a \partial K^a - (N^2 - 4) \psi^a \partial^2 K^a, \quad (B.19)
\]

The first and second terms of (B.9) are related to the first and second equations of (B.19), respectively. Note that the last expression of (B.19) goes to the \(c_2\)-term of (2.26) while the first relation of (B.19) provides the \(c_5\)-term of (2.26). The third, and last terms of (B.9)
contribute to the $c_2$, $c_4$-terms of (2.26), respectively. The $\psi^b$ in the last equation of (B.19) should be moved to the left.

In (B.10), the following OPE is used

$$S^a(z)S^b(w) = \frac{1}{(z-w)^4} N^2(N^2 - 4)(N + 2k)\delta^{ab} - \frac{1}{(z-w)^2} 2N(N^2 - 4)(N + 2k)f^{abc}K^c(w)$$

$$- \frac{1}{(z-w)^2} [N^2d^{abc}S^c(w) + 2(N + 2k)d^{ace}d^{bde}K^cK^d(w)]$$

$$+ \frac{1}{(z-w)^2} [-\psi K^cK^d(w) + f^{ace}d^{bed}S^cK^d(w) + f^{ade}d^{bed}K^cS^e(w)$$

$$- (N + 2k)d^{ace}d^{bde}K^dK^c(w)] + \cdots.$$  \hspace{1cm} (B.20)

The normal ordered products in (B.10) are expressed as

$$d^{abc}d^{def}f^{dag}f^{ebh}(\psi^aK^bK^c)K^f = -(N + k)(N^2 - 4)\partial^2(\psi^aK^bK^c) + \frac{N}{2}f^{abc}d^{def}d^{gah}\psi^aK^bK^c$$

$$- \frac{N}{2}d^{abc}d^{gah}\psi^aK^dK^e + \frac{N}{2}d^{abc}d^{gah}d^{def}\psi^aK^bK^dK^e$$

$$+ \frac{N}{2}d^{abc}d^{gah}d^{def}\psi^aK^dK^eK^b,$$

$$d^{abc}d^{def}f^{dag}f^{ech}(\psi^aK^bK^h)K^f = -(N^2 - 4)f^{abc}d^{def}d^{gah}(\psi^aK^bK^c) - \frac{N}{2}d^{abc}d^{def}d^{gah}\psi^aK^dK^e$$

$$- \frac{N}{2}d^{abc}d^{def}d^{gah}\psi^aK^dK^b + \frac{N}{2}d^{abc}d^{def}d^{gah}\psi^aK^bK^dK^e$$

$$+ \frac{N}{2}d^{abc}d^{def}d^{gah}\psi^aK^dK^bK^e,$$

$$f^{abc}\partial/(\psi^aK^b)K^c = N\partial^2(\psi^aK^bK^c) + f^{abc}K^a\partial/(\psi^bK^c).$$  \hspace{1cm} (B.21)

The first, second, and third terms of (B.10) are related to the first, second and third equation of (B.21), respectively. The nonderivative term of the first equation in (B.21) appears in $c_4$-term of (2.26). The $c_6$-term of (2.26) is obtained from the last term of (B.10) by rearranging the fields. Moreover, the $c_2$-term of (2.26) can be obtained from the last line of (B.10).
\[ d^{abc}(\psi^a K^b) S^c = d^{abc} d^{cde} \psi^a K^b K^c K^e - \frac{1}{N} (N^2 - 4)(N + 2k) \partial^2 \psi^a K^a. \]  

(B.22)

The second, third, and first terms of (B.11) are related to the first, second, and third equation of (B.22), respectively. The \(c_4\)-term of (2.26) is obtained from the above nonderivative terms in (B.22).

Therefore, collecting all the nontrivial terms in (B.1)-(B.11) using the extra rearrangements for the composite fields described previously with (B.13), (B.17), (B.18) and (B.20), the final second-order pole of the OPE \(U(z) W(w)\) is presented as (2.26). The nonderivative terms can be seen easily and the derivative terms are collected from various places.

To express \(GT(w)\) with (2.6) and (2.8) explicitly, the following identities are needed

\[
\begin{align*}
(\psi^a J^a)(J^b J^b) &= \psi^a J^a J^b J^b + N \psi^a \partial^2 J^a - 6N \partial^2 \psi^a J^a, \\
(\psi^a J^a)(J^b K^b) &= \psi^a J^a J^b K^b + 2f^{abc} \psi^a \partial J^b K^c - \frac{3}{2} N \partial^2 \psi^a K^a, \\
(\psi^a K^a)(J^b J^b) &= \psi^a J^b J^a K^b + 2N \psi^a \partial^2 K^a - N \psi^a \partial K^a, \\
(\psi^a K^a)(J^b K^b) &= \psi^a J^a K^b K^b - f^{abc} \psi^a \partial J^b K^c - f^{abc} \psi^a K^b \partial K^c - \frac{k}{2} \partial^2 \psi^a J^a N \psi^a \partial^2 K^a, \\
(\psi^a K^a)(K^b K^b) &= \psi^a K^a J^b K^b - (N + k) \partial^2 \psi^a K^a. 
\end{align*}
\]

(B.23)

where the rearrangement lemma (A.15) of [36].

The large \(N\) limit with fixed \(\lambda\) in (3.2) for the coefficient functions in (2.26) leads to

\[
\begin{align*}
c_1 &\to \frac{(-2 + \lambda)(-1 + \lambda)}{4\sqrt{3}N^{5/2}\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_3 &\to \frac{(-2 + \lambda)\lambda}{4\sqrt{3}N^{5/2}\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_5 &\to \frac{(-2 + \lambda)}{2\sqrt{3}N^{5/2}\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_7 &\to \frac{(1 + \lambda)(6 + 5\lambda)}{8\sqrt{3}N^{5/2}\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_9 &\to \frac{\lambda(2 + \lambda)}{4\lambda^2}, \\
c_{11} &\to \frac{\lambda(2 + \lambda)}{\sqrt{3}N\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_{13} &\to \frac{\lambda(2 + 7\lambda)}{2\sqrt{3}N\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_2 &\to \frac{(-6 + \lambda)\lambda}{\sqrt{3}N^{5/2}(2 + 3\lambda)}, \\
c_4 &\to \frac{(-6 + \lambda)\lambda}{\sqrt{3}N^{5/2}(2 + 3\lambda)}, \\
c_6 &\to \frac{(-6 + \lambda)\lambda}{\sqrt{3}N^{5/2}(2 + 3\lambda)}, \\
c_8 &\to \frac{(1 + \lambda)(10 + 3\lambda)}{8\sqrt{3}N\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_{10} &\to \frac{(1 + \lambda)(10 + 3\lambda)}{2\sqrt{3}N^{5/2}\sqrt{1 + \lambda(2 + 3\lambda)}}, \\
c_{12} &\to \frac{(2 + \lambda)\lambda}{\sqrt{3}N\sqrt{1 + \lambda(2 + 3\lambda)}. 
\end{align*}
\]

(B.24)
Appendix C  The first-order pole in the OPE $U(z)$ $W(w)$ and coefficient functions of spin $\frac{9}{2}$ field

As in previous Appendix B, the first-order pole of $U(z) W(w)$ can be derived

\[
\begin{align*}
\{\psi^a Q^a Q\}_{-1} &= -5N d^{abc} d^{cde} \partial(\psi^a J^b)(J^d J^c) - 5N d^{abc} d^{cde} J^a \partial(\psi^d J^c) J^b \\
-5N d^{abc} d^{cde} J^a J^b \partial(\psi^d J^c), & \quad (C.1) \\
\{\psi^a Q^a K^b b\}_{-1} &= -5N d^{abc} c^{cde} K^a \partial(\psi^d J^c) J^b - 5N d^{abc} d^{cde} K^a J^b \partial(\psi^d J^c), & \quad (C.2) \\
\{\psi^a J^b S^b\}_{-1} &= -5N d^{abc} d^{cde} \partial(\psi^a J^b) K^d K^e, \quad (C.3) \\
\{\psi^a R^a Q^b K^b\}_{-1} &= d^{abc} d^{cde} f^{dabc} f^{ebh} \partial(\psi^a J^b)(J^d J^c) - N \delta^{bd} \partial(\psi^a K^c)(J^d J^f) \\
- f^{aeh} J^d((\psi^a J^b K^c) J^f) - f^{ebh} J^d((\psi^a J^c K^b) J^f) - N \delta^{eb} J^d(\partial(\psi^a K^c) J^f) & \quad (C.4) \\
- f^{aeh} J^d J^c \psi^b J^b K^c + f f^{ah} f^{bhi} J^d J^e \partial(\psi^i K^c) - f f^{bhi} J^d J^c \psi^a J^b K^c \\
- N \delta^{fb} J^d J^e \partial(\psi^a K^c) & \quad (C.5) \\
\{\psi^a R^a J^b S^b\}_{-1} &= d^{abc} d^{cde} f^{dabc} f^{ebh} \partial(\psi^a J^b)(J^d J^c) - f^{aeh} J^d((\psi^a J^b K^c) J^f) \\
- N \delta^{bd} \partial(\psi^a K^c)(J^d J^f) - f^{aeh} J^d((\psi^a J^c K^b) J^f) - f^{ebh} J^d((\psi^a J^d K^b) J^f) & \quad (C.6) \\
- f^{aeh} J^d J^c \psi^b J^b K^c + f f^{ah} f^{bhi} J^d J^e \partial(\psi^i K^c) & \quad (C.7) \\
- f^{bhi} J^d J^c \psi^a J^b K^c \\
- N \delta^{fb} J^d J^e \partial(\psi^a K^c) & \quad (C.8)
\end{align*}
\]

Then by counting the $N$ behavior from (B.24), the zero mode eigenvalue equation can be analyzed once the superpartners of scalar fields are found.
The second relation of (2.30) determines the following normal ordered product in (C.2):

\[
\{ \psi^a S^a, J^b b^b \}_{-1} = d^{abc} d^{def} \left[ -f^{dag} (\psi^g K^b K^c) J^d (\psi^g K^b K^c) \right] - \frac{1}{3} (N^2 - 4) \psi^a \partial^2 K^a, \tag{C.9}
\]

\[
\{ \psi^a, J^b b^b \}_{-1} = d^{abcdef} \left[ -f^{dag} (\psi^g K^b K^c) (K^e K^f) - (N + 2k) d^{cdf} J^d \partial(\psi^a K^b) K^f 
- f^{ega} J^d (\psi^g K^b K^c) K^f - (N + 2k) d^{cdf} J^d \partial(\psi^a K^b) - f^{ega} J^d (\psi^g K^b K^c) K^f \right], \tag{C.10}
\]

\[
\{ \psi^a S^a, J^b b^b \}_{-1} = d^{abc} d^{def} \left[ -f^{dag} (\psi^g K^b K^c) (K^e K^f) - f^{ega} J^d (\psi^g K^b K^c) (K^e K^f) 
- (N + 2k) d^{cdf} J^d \partial(\psi^a K^b) K^f - f^{ega} K^d (\psi^g K^b K^c) K^f - (N + 2k) d^{cdf} K^d K^e \partial(\psi^a K^b) 
- f^{ega} K^d K^e \psi^a K^b K^c \right]. \tag{C.11}
\]

The second relation of (2.30) determines the following normal ordered product in (C.2):

\[
d^{abc} d^{cde} K^a \partial(\psi^d J^e) J^b = -\frac{2}{3} (N^2 - 4) K^a \partial^2 \psi^a + d^{abc} d^{cde} K^a J^b \partial(\psi^d J^e). \tag{C.12}
\]

The \( K^a \) in (C.12) can be moved to the right easily. No normal ordering procedure in (C.3) is needed.

The normal ordered products appearing in (C.4) are summarized as

\[
d^{abc} (\partial(\psi^a K^b)) Q^c = d^{abc} \partial \psi^a K^b Q^c + d^{abc} \psi^a \partial K^b Q^c + (N^2 - 4) \partial \psi^a \partial^2 K^a 
- \frac{1}{3} (N^2 - 4) \psi^a \partial^2 K^a, \]

\[
d^{abc} d^{def} f^{eah} J^d (\psi^h J^b K^c) J^f = d^{abc} d^{def} f^{eah} J^d J^f \psi^h J^b K^c + (N^2 - 4) f^{abc} J^a \partial^2 (\psi^b K^c) 
+ \frac{1}{2} N d^{abc} d^{cde} J^d \partial(\psi^c J^a K^b) - \frac{1}{2} N d^{abc} d^{cde} J^d \partial(\psi^e J^a K^b) 
+ \frac{1}{2} N d^{abc} d^{cde} J^d \partial(\psi^d J^a K^b), \]

\[
d^{abc} d^{def} f^{ebh} J^d (\psi^h J^b K^c) J^f = \frac{1}{2} N d^{abc} d^{cde} J^d \partial(\psi^c J^a K^b) + \frac{1}{2} N d^{abc} d^{cde} J^d \partial(\psi^d J^a K^b) 
- \frac{1}{2} N d^{abc} d^{cde} J^d \partial(\psi^e J^a K^b) + d^{abc} d^{def} f^{ebh} J^d J^f \psi^d J^h K^c, \]

\[
d^{abc} d^{def} f^{ehh} J^d (\psi^h J^b K^c) J^f = -\frac{1}{2} (N^2 - 4) f^{abc} J^a \partial^2 (\psi^b K^c) - N d^{abc} d^{cde} J^a J^b \partial(\psi^d K^e), \]

\[
d^{abc} d^{def} f^{ebi} J^d (\psi^i J^b K^c) J^f = \frac{N^2 - 4}{2N} f^{abc} J^a \partial^2 (\psi^b K^c) + d^{abc} d^{cde} J^d J^e \partial(\psi^a K^b), \]

\[
d^{abc} d^{def} f^{dag} (\psi^g J^b K^c) (J^e, J^f) = 5(2N - 4) f^{abc} \psi^a \partial J^b \partial K^c + 2N d^{abc} d^{cde} J^e J^f \partial K^b 
- 2(2N - 4) f^{abc} \psi^a \partial J^b \partial K^c - N d^{abc} \psi^b Q^b \partial K^c 
- \frac{4}{3} N (N^2 - 4) \psi^a \partial^3 K^a + d^{abc} f^{dag} (\psi^g J^b) K^c Q^d, \]

\[
d^{abc} d^{cde} (\psi^j J^d) J^a \partial K^e = d^{abc} d^{cde} \psi^j J^d J^a \partial K^e - \frac{N^2 - 4}{N} f^{abc} \psi^a \partial J^b \partial K^c 
- (N^2 - 4) \partial^2 \psi^a \partial K^a, \]

\[
d^{abc} d^{def} f^{dga} (\psi^a J^g K^c) (J^e, J^f) = d^{abc} f^{dga} (\psi^b J^g) K^c Q^d - 2(2N - 4) f^{abc} \psi^a J^b \partial K^c \]
\[ + Nd^{abc} e^{cde} \psi^b J^d J^a \partial K^e - Nd^{abc} e^{cde} \psi^d J^b J^a \partial K^e \]
\[ - Nd^{abc} e^{cde} \psi^d J^e J^a \partial K^b + 2Nd^{abc} \psi^b Q^d \partial K^c \]
\[ - 2(N^2 - 4) f^{abc} \psi^a \partial J^b \partial K^c. \]  

(C.13)

The first, fifth, seventh, sixth, eighth, third, and fourth terms of (C.4) are related to the first, second, third, fourth, fifth, sixth, and eighth equation of (C.14), respectively. The \(c_1\), \(c_2\), \(c_3\), and \(c_4\)-terms of (2.31) can be obtained from the second, third, sixth, and last equations of (C.13), respectively. Further rearrangement for the fields is needed. In particular, the last equation of (C.13): \(d^{abc} f^{dag}(\psi^b J^a) K^c Q^d(z)\) \(^6\).

The following normal ordered products from (C.3) are obtained

\[
d^{abc} d^{def} f^{fgi}(\psi^a J^b K^g)(J^d J^e) = 4(N^2 - 4) f^{abc} \partial \psi^a \partial J^b \partial K^c - 3(N^2 - 4) f^{abc} \psi^a \partial J^b \partial K^c + \frac{4}{3} N(N^2 - 4) \psi^a \partial^3 K^a - Nd^{abc} e^{cde} \psi^d J^b J^e \partial K^a \]
\[ - Nd^{abc} e^{cde} \psi^d J^e \partial K^a + d^{abc} d^{def} f^{fgi}(\psi^a J^b K^g) J^d J^e, \]
\[
d^{abc} d^{def} f^{eaf}(\psi^a J^b J^c K^f) J^f = d^{abc} d^{def} f^{eaf} J^f \psi^a J^b K^d K^c + (N^2 - 4) f^{abc} K^a \partial^2 (\psi^b K^c) \]
\[ + \frac{N}{2} d^{abc} d^{def} K^d \partial (\psi^e J^a K^b) - \frac{N}{2} d^{abc} d^{def} K^d \partial (\psi^a J^b K^e) \]
\[ + \frac{N}{2} d^{abc} d^{def} K^d \partial (\psi^a J^b K^e), \]
\[ d^{abc} d^{def} f^{ebh}(\psi^a J^b J^c K^f) J^f = d^{abc} d^{def} f^{ebh} J^f \psi^a J^b J^c K^d + \frac{N}{2} d^{abc} d^{def} K^d \partial (\psi^e J^a K^b) \]
\[ + \frac{N}{2} d^{abc} d^{def} K^d \partial (\psi^a J^b K^e) - \frac{N}{2} d^{abc} d^{def} K^d \partial (\psi^a J^b K^e), \]
\[ d^{abc} d^{def} f^{ebi}(\psi^a J^b K^f)(\partial(\psi^c K^e)) J^f = -\frac{1}{2} (N^2 - 4) f^{abc} K^a \partial^2 (\psi^b K^c) - Nd^{abc} d^{def} K^a J^b \partial (\psi^d K^e), \]
\[ d^{abc} d^{def} \delta^{eb}(\partial(\psi^a K^c)) J^f = \frac{N^2 - 4}{2N} f^{abc} K^a \partial^2 (\psi^b K^c) + d^{abc} d^{def} K^d J^e \partial (\psi^a J^b K^c). \]  

(C.14)

The second, third, fifth, fourth and sixth terms of (C.5) are related to the first, second, third, fourth, fifth equations of (C.14), respectively. The \(c_6\), and \(c_7\)-terms of (2.31) can be obtained from the second, and third equations of (C.14), respectively. The product \(f^{abc} d^{def} \psi^d J^e Q^a K^b(z)\) can be expressed as \(c_1\) - and \(c_4\)-terms of (2.31) using the Jacobi identity.

The expression in (C.6) can be simplified as

\[d^{abc} d^{def} f^{dag}(\psi^b J^b K^c)(K^c K^f) = d^{abc} d^{def} f^{dag} \psi^a J^b K^c K^e K^f - Nd^{abc} \partial (\psi^a J^b) S^c, \]
\[d^{abc} d^{def} f^{dbg}(\psi^a J^b K^c)(K^c K^f) = d^{abc} d^{def} f^{dbg} \psi^a J^b K^c K^e K^f - Nd^{abc} \partial (\psi^a J^b) S^c, \]

\[This becomes d^{abc} f^{dag}(\psi^b J^b K^c) Q^d(z) = f^{dag} d^{abc} \psi^b J^b K^c Q^d(z) - 2(N^2 - 4) f^{abc} \partial (\psi^a J^b) K^c(z) + N d^{abc} d^{def} \psi^b J^d J^a K^e(z) - (N^2 - 4) f^{abc} \partial (\psi^a J^b) K^e(z) - N d^{abc} d^{def} \psi^a J^b \partial J^a K^e(z) + 2Nd^{abc} \partial (\psi^a J^b K^c(z) + \frac{1}{2} (N^2 - 4) f^{abc} \partial^2 (\psi^a J^b K^c(z).\]
\[ d^{abc} d^{def} f^{gh} f^{daeg} \partial(\psi^h K^e) (K^e K^f) = -N d^{abc} \partial \psi^a K^b S^c - N d^{abc} \psi^a \partial K^b S^c + \frac{1}{3} (N^2 - 4) (N + 2k) \partial^3 \psi^a K^a, \]

\[ d^{abc} d^{def} f^{ecg} J^d (\psi^a J^b K^g) K^f = d^{abc} d^{def} f^{ecg} J^d \psi^a J^b K^f K^g + N d^{abc} d^{def} J^d (\psi^a J^b K^e). \]  

(C.15)

The first, fourth, second, and sixth terms of (C.6) are related to the first, second, third and fourth equation of (C.15), respectively. The \( c_8 \)- and \( c_{10} \)-terms of (2.31) can be obtained from the first, second, and last equations of (C.15), respectively. Further rearrangement should be done when \( \psi^a \) is located to the right of \( J^b \) in (C.15). The product \( d^{abc} d^{def} f^{ecg} \psi^a J^d J^b K^f K^g (z) \) can be expressed as \( c_6 \) and \( c_7 \)-terms of (2.31) using the Jacobi identity.

Similarly, the normal ordered products in (C.7) can be reduced to

\[ d^{abc} d^{def} f^{ecg} (\psi^a J^b K^g) (K^e K^f) = d^{abc} d^{def} f^{ecg} \psi^a J^b K^g K^e K^f + 2N d^{abc} \partial (\psi^a J^b) S^c, \]

\[ d^{abc} d^{def} f^{ecg} K^d (\psi^a J^b K^g) K^f = d^{abc} d^{def} f^{ecg} \psi^a J^b K^d K^f K^g + N d^{abc} d^{def} J^d \partial (\psi^a J^b K^e). \]  

(C.16)

The second and fourth terms of (C.7) are related to the first and second equation of (C.16), respectively. The \( c_8 \), \( c_9 \), and \( c_{13} \)-terms of (2.31) can be obtained from the first and second equations of (C.16), respectively. Note that the normal ordered product \( d^{abc} d^{def} f^{ecg} \psi^a J^b K^g K^e K^f (z) \) can be expressed as \( c_8 \) and \( c_9 \)-terms of (2.31) using the Jacobi identity.

For the normal ordered products in (C.8), the rearrangement can be made as follows:

\[ d^{abc} d^{def} f^{daeg} (\psi^g K^b K^c) (J^f) = d^{abc} d^{def} f^{daeg} \psi^g K^b K^c J^f + 2N d^{abc} d^{def} \psi^d J^e \partial (K^a K^b), \]

\[ d^{abc} d^{def} f^{ecg} J^d (\psi^g K^b K^c) J^f = d^{abc} d^{def} f^{ecg} J^d J^f \psi^g K^b K^c + N d^{abc} d^{def} J^d \partial (\psi^e K^a K^b). \]

The \( c_{10} \)-term of (2.31) can be obtained from these two equations by moving the field \( \psi^g \) to the left.

The expression in (C.9) leads to

\[ d^{abc} \partial (\psi^a K^b) Q^e = d^{abc} \partial \psi^a K^b Q^c + d^{abc} \psi^a \partial K^b Q^e + (N^2 - 4) \partial \psi^a \partial^2 K^a - \frac{1}{3} (N^2 - 4) \psi^a \partial^3 K^a, \]

\[ d^{def} f^{fag} d^{ghi} (\psi^a K^h K^i) (J^d J^e) = d^{def} f^{fag} d^{ghi} \psi^a J^d J^e K^h K^i - 2N d^{abc} d^{def} \psi^d J^e \partial (K^a K^b), \]

\[ d^{abc} d^{def} f^{daeg} K^f (\psi^g K^b K^c) J^e = d^{abc} d^{def} f^{daeg} K^f \psi^g K^b K^c + N d^{abc} d^{def} K^d \partial (\psi^e K^a K^b). \]

The second, and third equations provide the \( c_8 \), \( c_{10} \)-terms of (2.31), respectively.
The following simplifications are applied to the normal ordered products in (C.10)

\[d^{abc}d^{def} f^{dag}(\psi^a K^b K^c)(K^e K^f) = f^{dag} d^{abc} d^{def} \psi^a (K^b K^c) K^e K^f + \frac{2}{3} (N^2 - 4)(N + 2k) \partial^3 \psi^a K^a + \frac{1}{4} (N + 2k)(N^2 - 4) \partial^2 \psi^a \partial K^a K^f + \frac{1}{4} N (N + 2k)(N^2 - 4) f^{abc} \partial \psi^a \partial K^b K^c - \frac{1}{4} N d^{abc} \partial \psi^a S^b K^c - \frac{1}{4} N d^{abc} \partial \psi^a K^b \partial S^c,
\]

\[d^{def} d^{eag} J^d \partial (\psi^a K^g) K^f = d^{def} d^{eag} J^d \partial (\psi^a K^g) K^f - \frac{k}{3N} (N^2 - 4) J^a \partial^3 \psi^a - \frac{N^2 - 4}{2N} f^{abc} J^a \partial^2 (\psi^b K^c),
\]

\[d^{def} f^{eag d^ghi} J^d (\psi^a K^h K^i) K^f = d^{def} f^{eag d^ghi} J^d (\psi^a K^h K^i) K^f + \frac{1}{2N} (N^2 - 4)(N + 2k) f^{abc} J^a \partial^2 (\psi^b K^c) + \frac{1}{2N} (N + 2k)(N^2 - 4) f^{abc} \partial \psi^a \partial K^b K^c - \frac{1}{4} N d^{abc} \partial \psi^a K^b \partial S^c, \tag{C.18}
\]

The nonderivative terms of first and third equations provide the $c_8$, $c_{14}$-terms of (2.31), respectively. Further rearrangement is needed. For example, the first equation of (C.18). All the $\psi^a$ located at the right hand side of $J^b$ should be moved to the left to obtain the final expression of (2.31).

Finally, the following normal ordered products in (C.11) are obtained

\[d^{def} d^{eag} K^d \partial (\psi^a K^g) K^f = d^{def} d^{eag} K^d \partial (\psi^a K^g) K^f - \frac{k}{3N} (N^2 - 4) K^a \partial^3 \psi^a - \frac{N^2 - 4}{2N} f^{abc} K^a \partial^2 (\psi^b K^c),
\]

\[d^{def} f^{eag d^ghi} K^d (\psi^a K^h K^i) K^f = d^{def} f^{eag d^ghi} K^d (\psi^a K^h K^i) K^f - \frac{1}{2N} (N^2 - 4)(N + 2k) f^{abc} K^a \partial^2 (\psi^b K^c) - \frac{1}{2N} (N + 2k)(N^2 - 4) f^{abc} \partial \psi^a \partial K^b K^c + \frac{1}{4} N d^{abc} \partial \psi^a K^b \partial S^c),
\]

\[d^{def} d^{egh} K^d \partial (\psi^g J^h) J^f = -\frac{2}{3} (N^2 - 4) K^a \partial^3 \psi^a + d^{abc} d^{eac} K^a J^b \partial (\psi^d J^e) \tag{C.19}
\]

The nonderivative term contributes to the $c_{14}$ term of (2.31).

In summary, collecting all the nontrivial terms in (C.11) using the extra rearrangements for the composite fields described previously, the final first-order pole of the OPE $U(z) U(w)$ is presented as (2.31). The following identities are used, together with (C.17) and (C.19),

\[f^{abc} \partial \psi^a J^b \partial J^c = -\frac{N}{2} \partial^2 \psi^a \partial J^a + N \partial \psi^a \partial^2 J^a,
\]

37
\[ f^{abc} \psi^a \partial^2 J^b K^c = -f^{abc} \psi^a J^b K^c - 2f^{abc} \psi^a \partial J^b K^c, \]
\[ d^{abc} d^{cde} \partial \psi^d J^a K^e K^b = d^{abc} d^{cde} \partial \psi^d J^a K^b K^c + \frac{N^2 - 4}{N} f^{abc} \partial \psi^a J^b \partial K^c. \]

The following results are used to express \( T \partial G(w) \)

\[
\begin{align*}
(J^a J^b) \partial (\psi^b J^b) &= J^a J^b \partial (\psi^b J^b) - 3N \psi^a \partial^2 J^a - 2N \psi^a \partial^3 J^a, \\
(J^a K^a) \partial (\psi^b J^b) &= J^a \partial (\psi^b J^b)K^a - \frac{3}{2} N \psi^a \partial^2 K^a - N \psi^a \partial^3 K^a, \\
(K^a K^a) \partial (\psi^b J^b) &= \partial (\psi^b J^b) K^a K^a, \\
(J^a J^b) \partial (\psi^b K^b) &= J^a J^b \partial (\psi^b K^b) + f^{abc} J^a \partial^2 (\psi^b K^c) - 2N \partial^3 \psi^a K^a - 4N \partial^2 \psi^a \partial K^a - 2N \partial \psi^a \partial^2 K^a, \\
(K^a K^a) \partial (\psi^b K^b) &= K^a K^a \partial (\psi^b K^b) - f^{abc} K^a \partial^2 (\psi^b K^c) - N \partial^2 \psi^a \partial K^a - (2N + 2k) \partial^2 \psi^a J^a - \frac{1}{2} \partial^2 \psi^a \partial^2 J^a, \\
(J^a K^a) \partial (\psi^b K^b) &= J^a K^a \partial (\psi^b K^b) - \frac{1}{2} f^{abc} J^a \partial^2 (\psi^b K^c) + \frac{1}{2} f^{abc} K^a \partial^2 (\psi^b K^c) - \frac{k}{2} \partial^2 \psi^a \partial J^a + \frac{k}{3} \partial^3 \psi^a J^a + 5N \partial^2 \psi^a \partial K^a + \frac{N}{2} \partial^2 \psi^a \partial^2 K^a.
\end{align*}
\]

The composite field \( G \partial T(w) \) can be obtained from

\[
\begin{align*}
(\psi^a J^a) \partial (J^b J^b) &= \psi^a J^a \partial (J^b J^b) + \frac{N}{3} \psi^a \partial^3 J^a - \frac{10N}{3} \partial^3 \psi^a J^a - 3N \partial^2 \psi^a \partial J^a, \\
(\psi^a K^a) \partial (J^b J^b) &= \psi^a K^a \partial (J^b J^b) + 2N \partial^2 \psi^a \partial K^a - \frac{2}{3} N \psi^a \partial^3 K^a, \\
(\psi^a J^a) \partial (J^b K^b) &= \psi^a J^a \partial (J^b K^b) + \frac{1}{2} f^{abc} \psi^a \partial^2 (J^b K^c) + \frac{1}{2} f^{abc} J^a \partial^2 (\psi^b K^c) - \frac{4}{3} N \partial^3 \psi^a K^a - \frac{5}{2} N \partial^2 \psi^a \partial K^a - N \partial \psi^a \partial^2 K^a, \\
(\psi^a K^a) \partial (J^b K^b) &= \psi^a K^a \partial (J^b K^b) - \frac{1}{2} f^{abc} \psi^a \partial^2 (J^b K^c) + \frac{1}{2} f^{abc} K^a \partial^2 (\psi^b K^c) + \frac{N}{2} \partial^2 \psi^a \partial K^a + N \partial \psi^a \partial^2 K^a + \frac{N}{2} \partial^2 \psi^a \partial^2 K^a, \\
(\psi^a J^a) \partial (K^b K^b) &= \psi^a J^a \partial (K^b K^b), \\
(\psi^a K^a) \partial (K^b K^b) &= \psi^a K^a \partial (K^b K^b) - \frac{2}{3} (N + k) \partial^3 \psi^a K^a - (N + k) \partial^2 \psi^a \partial K^a.
\end{align*}
\]

The coefficient functions, which appear in (2.31) are determined completely

\[
\begin{align*}
c_1 &= 5BCN(k + N)(2k + N)^2(k + 6N), \\
c_2 &= 10BCkN(k + N)(2k + N)^2, \\
c_3 &= 10BCkN(k + N)(2k + N)^2, \\
c_4 &= 5BCN(k + N)(2k + N)^2(k + 6N),
\end{align*}
\]
\[\begin{align*}
c_6 &= -120BCN^2(k + N)(2k + N)(k + 2N), \\
c_7 &= -120BCN^2(k + N)(2k + N)(k + 2N), \\
c_8 &= -30BCN^3(2k^2 + 13kN + 12N^2), \\
c_9 &= 30BCN^3(2k + N)(3k + 4N), \\
c_{10} &= -30BCN^2(k + N)(2k + N)(k + 2N), \\
c_{13} &= -60BCN^4(2k + N), \\
c_{14} &= -60BCN^4(3k + 2N), \\
e_1 &= -15BCkN(k + N)(2k + N)^2(k + 2N), \\
e_2 &= -15BCkN(k + N)(2k + N)^2(k + 2N), \\
e_3 &= -15BCk(-2 + N)(N(2 + N)(k + N)(2k + N)^2(k + 2N), \\
e_4 &= \frac{5}{2}BCk(-2 + N)(N(2 + N)(k + N)(2k + N)(k + 2N)(6k + 11N), \\
e_5 &= -15BCk(-2 + N)(N(2 + N)(k + N)(2k + N)(k + 2N)(2k + 9N), \\
e_6 &= -20BC(-2 + N)N^2(2 + N)(k + N)(2k + N)(k + 2N)(k + 3N), \\
e_7 &= 5BCN^2(k + N)(2k + N)^2(2 + 6N), \\
e_8 &= -30BC(-2 + N)N(2 + N)(k + N)(2k + N)(k + 2N)(2k + 11N), \\
e_9 &= -30BCk(-2 + N)(N(2 + N)(k + N)(2k + N)(k + 2N)(18k + 25N), \\
e_{10} &= -90BC(-2 + N)N(2 + N)(k + N)(2k + N)^2(k + 2N), \\
e_{11} &= 30BC(-2 + N)N^2(2 + N)(k + N)(2k + N)^2(k + 2N), \\
e_{12} &= -15BCN^3(8k^3 + 30k^2N + 45kN^2 + 22N^3), \\
e_{13} &= -10BC(-2 + N)N^2(2 + N)(2k + N)(k + 2N)(2k^3 + 15k^2N + 37kN^2 + 26N^3), \\
e_{14} &= 30BCN^2(k + N)(2k + N)^2(k + 2N), \\
e_{15} &= 45BCN^2(k + N)(2k + N)^2(k + 2N), \\
e_{16} &= 45BCN^2(k + N)(2k + N)^2(k + 2N), \\
e_{17} &= 10BCN^2(k + N)(2k + N)^2(7k + 12N), \\
e_{18} &= 15BCN^2(k + N)(2k + N)(k + 2N)(2k + 5N), \\
e_{19} &= 60BCN^3(k + N)(k + 2N)(2k + 3N), \\
e_{20} &= -150BCN^3(k + N)(2k + N)(k + 2N), \\
e_{21} &= -15BCN^3(k + N)(2k + N)^2(k + 2N), \\
e_{22} &= -150BCN^3(k + N)(2k + N)(k + 2N), \\
e_{23} &= 5BCN^2(k + N)(2k + N)^2(k + 6N), \\
e_{24} &= -240BCN^3(k + N)(2k + N)(k + 2N), \\
e_{25} &= -240BCN^3(k + N)(2k + N)(k + 2N), \\
e_{26} &= 60BCN^4(6k^2 + 21kN + 14N^2), \\
e_{27} &= 120BC(-2 + N)N^2(2 + N)(k + N)(2k + N)(k + 2N), \\
e_{28} &= 240BCN^4(k + N)(k + 2N), \\
e_{29} &= 120BC(-2 + N)N^3(2 + N)(2k + N)(3k + 4N), \\
e_{30} &= 60BCN^4(2k^2 + 3kN + 2N^2), \\
e_{31} &= \frac{(-2 + \lambda)\lambda(1 + 5\lambda)}{12\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}, \\
e_{32} &= \frac{(-2 + \lambda)(-1 + \lambda)\lambda}{6\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}, \\
e_{33} &= \frac{(-2 + \lambda)\lambda(1 + 5\lambda)}{12\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}, \\
e_{34} &= \frac{(-2 + \lambda)(-1 + \lambda)\lambda}{6\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}, \\
\end{align*}\]

and their large \(N\) limit with \(\mathbb{312}\) can be expressed as

\[\begin{align*}
c_1 &\rightarrow \frac{(-2 + \lambda)\lambda(1 + 5\lambda)}{12\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}, \\
c_3 &\rightarrow \frac{(-2 + \lambda)(-1 + \lambda)\lambda}{6\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}, \\
c_4 &\rightarrow \frac{(-2 + \lambda)\lambda(1 + 5\lambda)}{12\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}, \\
c_2 &\rightarrow \frac{(-2 + \lambda)(-1 + \lambda)\lambda}{6\sqrt{3N^{7/2}(1 + \lambda)^{3/2}(2 + 3\lambda)}}. \\
\end{align*}\]
From these results (C.23), it would be interesting to see that the zero mode eigenvalue equa-
tion for the spin-\(\frac{3}{2}\) current can be done similarly once the superpartners of scalar fields are determined.

### Appendix D: The first-order pole in the OPE \(U(z) U(w)\) and coefficient functions of spin 4 field

The first-order pole in \(U(z) U(w)\), consisting of nine terms, are summarized as

\[
\{\psi^a Q^a \psi^b Q^b\}_{-1} = -\frac{1}{2} Q^a Q^a - 2 f^{abc} d^{cde} (\psi^d \psi^e J^c) Q^b - N d^{abc} (\psi^a \partial \psi^c) Q^b \\
+ 5 N d^{abc} d^{cde} \psi^a \partial (\psi^d J^e) J^b + 5 N d^{abc} d^{cde} \psi^b \partial (\psi^d J^e),
\]

(D.1)

\[
\{\psi^a R^a \psi^b R^b\}_{-1} = -\frac{1}{2} d^{abc} d^{def} \left[ \frac{1}{2} \delta^{cd} (J^a K^b) (J^e J^f) - f^{dbg} (\psi^a \psi^g K^c) (J^e J^f) \\
+ f^{ega} \psi^d (\psi^g J^b K^c) J^f - f^{ega} f^{ghb} \psi^d \partial (\psi^h K^c) J^f + f^{ega} \psi^d J^e \psi^g J^b K^c \\
- f^{ega} f^{gbh} \psi^d J^e \partial (\psi^h K^c) + f^{ebg} \psi^d (\psi^a J^g K^c) J^f + N \delta^{eb} \psi^d \partial (\psi^a K^c) J^f \\
+ N \delta^{fb} \psi^d J^e \partial (\psi^a K^c) + \partial \psi^d J^e \partial (\psi^a K^c) \right],
\]

(D.4)

\[
\{\psi^a S^a \psi^b S^b\}_{-1} = d^{abc} d^{def} \left[ -\frac{1}{2} \delta^{cd} (J^a K^b) (J^e K^f) - f^{dbg} (\psi^a \psi^g K^c) (J^e K^f) \\
+ f^{ega} \psi^d (\psi^g J^b K^c) K^f - f^{ega} f^{ghb} \psi^d \partial (\psi^h K^c) K^f + f^{ega} \psi^d J^e \psi^g J^b K^c \\
+ \delta^{cd} k^{a} \psi^d J^e \partial (\psi^d J^e) + f^{ebg} \psi^d (\psi^a J^g K^c) K^f + N \delta^{eb} \psi^d \partial (\psi^a K^c) K^f \right],
\]

(D.5)

\[
\{\psi^a Q^a \psi^b S^b\}_{-1} = d^{abc} d^{def} \left[ -\frac{1}{2} \delta^{cd} (J^a K^b) (J^e K^f) - f^{dbg} (\psi^a \psi^g K^c) (J^e K^f) \\
+ f^{ega} \psi^d (\psi^g J^b K^c) K^f + \delta^{cd} k^{a} \psi^d \partial (\psi^d J^e) + f^{ega} \psi^d J^e \psi^g J^b K^c \\
+ \delta^{cd} k^{a} \psi^d K^e \partial (\psi^a J^b), \right]
\]

(D.6)

\[
\{\psi^a S^a \psi^b Q^b\}_{-1} = d^{abc} d^{def} \left[ -\frac{1}{2} \delta^{cd} (K^a K^b) (J^e J^f) + f^{ega} \psi^d (\psi^g K^b K^c) J^f \\
+ f^{ega} \psi^d J^e \psi^g K^b K^c \right],
\]

(D.7)

\[
\{\psi^a S^a \psi^b R^b\}_{-1} = d^{abc} d^{def} \left[ -\frac{1}{2} \delta^{cd} (K^a K^b) (J^e K^f) + f^{ega} J^e \psi^g K^b K^c \\
+ (N + 2k) \delta^{cd} J^e \psi^b \partial (\psi^e K^f) + f^{ega} \psi^d (\psi^g K^b K^c) K^f \right],
\]

(D.8)
\[+(N + 2k)\delta^{cd}\psi^a \partial(\psi^b K^f) K^b + f^{f ga} \psi^d K^e \psi^g K^b K^c + (N + 2k)\delta^{cd}\psi^a K^b \partial(\psi^c K^f)\].

The normal ordered product in third equation of (D.2) can be simplified as
\[d^{abc} d^{cde} (\psi^a \partial \psi^b) J^d K^e = d^{abc} d^{cde} J^d \psi^a \partial \psi^b K^e + \frac{N^2 - 4}{2N} \partial^2 J^a K^a.\] (D.10)

The first and second terms of (D.2) contribute to the \(c_3\)-, \(c_{10}\)-terms of (2.39), respectively. The second equation of (D.2) should be rearranged.

In (D.3), the nonderivative two terms contribute to \(c_{11}\)-, \(c_{12}\)-terms of (2.39), respectively. In (D.3), the following expressions are obtained

\[d^{abc} d^{def} f^{efg} \psi^d (\psi^g J^b K^c), J^f = (N^2 - 4) f^{abc} \psi^a \partial^2 (\psi^b K^c) + N d^{abc} d^{cde} \psi^d \partial(\psi^e J^a K^b)
+ \frac{N}{2} d^{abc} d^{cde} \psi^d \partial(\psi^e J^b K^c) - \frac{N}{2} d^{abc} d^{cde} \psi^d \partial(\psi^e J^a K^c)
- \frac{N}{2} d^{abc} d^{cde} \psi^d \partial(\psi^e J^a K^b) + N d^{abc} d^{cde} \psi^d \partial(\psi^e J^a K^b)
+ N d^{abc} d^{cde} \psi^d \partial(\psi^e J^b K^c) + d^{abc} d^{def} f^{efg} \psi^d J^f J^g \psi^a K^c,
\]

\[d^{abc} d^{def} f^{efg} \psi^d (\psi^g J^b K^c), J^f = \frac{N^2 - 4}{2N} f^{abc} \psi^a \partial^2 (\psi^b K^c) - N d^{abc} d^{cde} \psi^d J^e \partial(\psi^b K^a),
\]

\[d^{abc} d^{def} f^{efg} \psi^d (\psi^g J^b K^c), J^f = \frac{N^2 - 4}{2N} f^{abc} \psi^a \partial^2 (\psi^b K^c) + d^{abc} d^{cde} \psi^d J^e \partial(\psi^a K^b),
\]

\[d^{abc} d^{def} f^{efg} \psi^d (\psi^g J^b K^c), J^f = f^{abc} d^{cde} \psi^d \partial^2 (\psi^b K^c) + 2N d^{abc} d^{cde} \psi^d \partial(\psi^a J^b K^c)
- N d^{abc} d^{cde} \psi^d \partial(\psi^e J^b K^c) - N d^{abc} d^{cde} \psi^d \partial(\psi^a J^e K^b)
- N d^{abc} d^{cde} \psi^d \partial(\psi^a J^e K^b) - N d^{abc} d^{cde} \psi^d \partial(\psi^a J^e K^b) + 3(N^2 - 4) f^{abc} \psi^a \partial(\psi^b K^c)
+ \frac{9}{2} (N^2 - 4) f^{abc} \psi^a \partial(\psi^b K^c) - 2(N^2 - 4) \partial^2 J^a K^a
+ N d^{abc} d^{cde} \psi^d \partial(\psi^a J^b K^c).
\]

The third, seventh, fourth, eighth and second terms of (D.4) are related to the first, second, third, fourth and fifth equation of (D.11), respectively. The \(c_3\)- and \(c_{10}\)-terms of (2.39) can
The fourth, eighth, third, second, seventh terms of (D.5) are related to the first, second, third, fourth, and sixth equations of (D.12), respectively. Note that the first term of fourth equation of (D.12) is rearranged and represented the fifth equation of (D.12). In other words, the fifth equation of (D.12) comes from the nonderivative term in fourth equation of (D.12). Furthermore, the other two terms appearing in the fourth equation of (D.12) should be rearranged.

The fourth, eighth, third, second, seventh terms of (D.5) are related to the first, second, third, fourth, and sixth equations of (D.12), respectively. Note that the first term of fourth equation of (D.12) is rearranged and represented the fifth equation of (D.12). In other words, the fifth equation of (D.12) comes from the nonderivative term in fourth equation of (D.12). Furthermore, the other two terms appearing in the fourth equation of (D.12) should be rearranged. The $c_4$, $c_7$- and $c_9$-terms of (2.39) can be obtained from the nonderivative expressions of (D.5). The normal ordered product $f^{abc} d^{def} f^{ebg} \psi^d J^f J^g \psi^a K^c(z)$ can be expressed as $c_4$- and $c_9$-terms using the Jacobi identity.

The expression in (D.6) (the second and third equations) leads to the following normal
ordered product
\[ f^{abc} d^{cde} (\psi^d \psi^a) K^e) S^b = f^{abc} d^{cde} (\psi^d \psi^a) K^e S^b - \frac{N^2 - 4}{2N} (N + 2k) \partial^2 J^a K^a, \]
\[ d^{abc} d^{def} f^{feg} \psi^d (\psi^a J^b K^c) K^f = N d^{abc} d^{cde} \psi^a \partial (\psi^a J^b K^c) \]
\[ + d^{abc} d^{def} f^{feg} \psi^d (\psi^a J^b K^f) K^g. \]
\[ (D.13) \]
This contributes to the \( c_6 \)-term of (2.39). Moreover, the first term of (D.6) can contribute to the \( c_5 \)-term of (2.39).

For the normal ordered product in (D.7), the relation holds
\[ d^{abc} d^{def} f^{feg} \psi^d (\psi^a K^b K^c) J^f = f^{abc} d^{cde} \psi^d J^b J^c S^a. \]
\[ (D.14) \]
This plays the role of \( c_{12} \)-term of (2.39). The first term of (D.7) contributes to \( c_{11} \)-term of (2.39). The last term of (D.7) contributes to \( c_{12} \)-term of (2.39).

The following identity can be used in (D.8)
\[ S^a R^a = d^{abc} J^a K^b S^c - \frac{1}{N} (N + 2k) (N^2 - 4) J^a \partial^2 K^a, \]
\[ d^{abc} d^{def} f^{feg} \psi^d (\psi^a K^b K^c) K^f = \frac{N + 2k}{2N} (N^2 - 4) f^{abc} \psi^a \partial^2 (\psi^b K^c) - N d^{abc} \psi^a \partial (\psi^b S^c) \]
\[ + f^{abc} d^{cde} \psi^d \psi^a K^e S^b. \]
\[ (D.15) \]
The first and last terms of (D.8) are related to the first and second equation of (D.15), respectively. The nonderivative term of second equation in (D.15) goes to \( c_{16} \)-term of (2.39).

Finally, the normal ordered products in (D.9) are summarized as
\[ f^{feg} d^{ghi} \psi^d (\psi^a K^b K^i) K^f = -\frac{1}{2N} (N + 2k) (N^2 - 4) f^{abc} \psi^a \partial^2 (\psi^b K^c) + N d^{abc} \psi^a \partial (\psi^b S^c) \]
\[ + f^{abc} d^{cde} \psi^d K^e S^b, \]
\[ S^a S^a = d^{abc} K^a K^b S^c - \frac{2}{N} (N^2 - 4) (N + 2k) K^a \partial^2 K^a, \]
\[ d^{def} d^{feg} \psi^d \partial (\psi^a K^g) K^f = -\frac{k}{3N} (N^2 - 4) \psi^a \partial^3 \psi^a - \frac{N^2 - 4}{2N} f^{abc} \psi^a \partial^2 (\psi^b K^c) \]
\[ + d^{abc} d^{cde} \psi^a K^b \partial (\psi^d K^e). \]
\[ (D.16) \]
The second, first, and third terms of (D.9) are related to the first, second, and third equation of (D.16), respectively. Obviously, the nonderivative terms of first and second equations in (D.16) goes to \( c_{16} \)-, \( c_{17} \)-terms of (2.39), respectively.

Collecting all the nontrivial terms in (D.1)-(D.9) using the extra rearrangements for the composite fields with (D.10), (D.13) and (D.14), the final first-order pole of the OPE
ordered products:

\[ d^{abc} d^{cde} \partial \psi^a \psi^d J^b J^e = d^{abc} d^{cde} \partial \psi^a \psi^d J^b J^e + \frac{N^2 - 4}{N} \left( \frac{1}{2} \partial J^a \partial J^a + \frac{N}{3} \psi^a \partial^3 \psi^a + N \partial \psi^a \partial^2 \psi^a \right), \]

\[ d^{abc} d^{cde} \psi^a \psi^d \partial J^b J^e = -d^{abc} d^{cde} \psi^a \psi^d J^b \partial J^e + \frac{N^2 - 4}{2N} \left( J^a \partial^2 J^a + \frac{4}{3} N \psi^a \partial^3 \psi^a \right), \]

\[ d^{abc} d^{cde} \psi^a \psi^d J^b \partial J^d = -d^{abc} d^{cde} \psi^a \psi^d J^b \partial J^d. \]

\[ f^{abc} J^a \partial J^b K^c = -f^{abc} J^a \partial J^b K^c + N \partial^2 J^a K^a, \]

\[ f^{abc} \psi^a \partial^2 \psi^b K^c = \frac{1}{2} \partial^2 J^a K^a - f^{abc} \partial \psi^a \partial \psi^b K^c, \]

\[ f^{abc} \psi^a \partial^2 \psi^b J^c = J^a \partial^2 J^a + \frac{4}{3} N \psi^a \partial^3 \psi^a, \]

\[ f^{abc} \psi^a \partial^2 \psi^b J^c = \frac{1}{2} \partial J^a \partial J^a + \frac{1}{3} N \psi^a \partial^3 \psi^a - N \partial^2 \psi^a \partial \psi^a, \]

\[ d^{abc} d^{cde} \partial \psi^a \psi^d J^b J^e = -d^{abc} d^{cde} \partial \psi^a \psi^d J^b J^e. \]

The coefficient functions in \((2.39)\) can be obtained

\[
\begin{align*}
c_1 &= -\frac{1}{2} C^2 k^2 (2k + N)^2, & c_2 &= 2 C^2 k^2 (2k + N)^2, \\
c_3 &= 5 C^2 k N (2k + N)^2, & c_4 &= -25 C^2 N^2 (2k + N)^2, \\
c_5 &= 50 C^2 N^3 (2k + N), & c_6 &= -50 C^2 N^2 (2k + N)(2k + 3N), \\
c_7 &= \frac{-25}{2} C^2 N^2 (2k + N)^2, & c_9 &= 5 C^2 N (2k + N)^2 (-2k + 5N), \\
c_{10} &= 30 C^2 k N (2k + N)^2, & c_{11} &= -10 C^2 k N^2 (2k + N), \\
c_{12} &= -10 C^2 N^2 (2k + N)(4k + 5N), & c_{13} &= 5 C^2 k N (2k + N)^2, \\
c_{16} &= -100 C^2 N^3 (2k + 3N), & c_{17} &= -50 C^2 N^4, \\
e_1 &= -C^2 k N (2k + N)^2 (18k + 25N), & e_3 &= -C^2 k N (2k + N)^2 (6k + 25N), \\
e_6 &= -3 C^2 k N (2k + N)^2, & e_7 &= 25 C^2 N^3 (2k + N)(6k + 11N), \\
e_8 &= 4 C^2 k^2 (-2 + N)(2 + N)(2k + N)^2, & e_9 &= -\frac{1}{2} C^2 k (-4 + N^2)(2k + N)^2 (4k + 25N), \\
e_{10} &= -\frac{5}{4} C^2 (-4 + N^2) N (2k + N)(28k^2 + 124k N + 95N^2), \\
e_{11} &= -C^2 k (6k - 25N)(-4 + N^2)(2k + N)^2, \\
e_{12} &= -\frac{5}{2} C^2 (-4 + N^2)(2k + N)(4k + 5N), & e_{14} &= \frac{5}{2} C^2 N^2 (2k + N)^2 (8k + 5N), \\
e_{16} &= 25 C^2 (-4 + N^2) N^2 (2k + N)(2k + 5N), \\
e_{17} &= -C^2 k (-4 + N^2) N (2k + N)^2 (51k + 25N), \\
e_{18} &= -\frac{1}{3} C^2 k (-4 + N^2) N (2k + N)(242k^2 + 271k N + 175N^2),
\end{align*}
\]
\begin{align*}
e_{20} &= \frac{25}{2} C^2 (-4 + N^2) N (2k + N)^2, & e_{21} &= -\frac{75}{2} C^2 N^3 (2k + N)^2, \\
e_{22} &= \frac{25}{2} C^2 N^3 (2k + N) (10k + 21N), \\
e_{23} &= -\frac{25}{2} C^2 (-4 + N^2) N (2k + N) (4k^2 + 8kN + 11N^2), \\
e_{24} &= -25 C^2 (-4 + N^2) N (2k + N) (2k^2 + 9kN + 8N^2), \\
e_{25} &= -\frac{5}{2} C^2 N^2 (2k + N)^2 (22k + 35N), & e_{28} &= 5 C^2 N^2 (2k + N)^2 (8k + 15N), \\
e_{29} &= 25 C^2 N^2 (2k + N) (-2k^2 - 3kN + N^2), \\
e_{30} &= \frac{5}{2} C^2 N^3 (-36k^2 - 28kN + 35N^2), & e_{34} &= 5 C^2 (-4 + N^2) N (2k + N)^2 (2k + 5N), \\
e_{36} &= -\frac{5}{2} C^2 N^2 (2k + N)^2 (14k + 15N),
\end{align*}

and their large \(N\) limit with fixed \(\lambda\) in (3.2) reduces to

\begin{align*}
c_1 &\to \frac{(-2 + \lambda)(-1 + \lambda)^2}{100 N^3 (1 + \lambda)(2 + 3\lambda)}, & c_2 &\to \frac{(-2 + \lambda)(-1 + \lambda)^2}{25 N^3 (1 + \lambda)(2 + 3\lambda)}, \\
c_3 &\to \frac{(-2 + \lambda)(-1 + \lambda)\lambda}{10 N^3 (1 + \lambda)(2 + 3\lambda)}, & c_4 &\to \frac{(-2 + \lambda)\lambda^2}{2 N^3 (1 + \lambda)(2 + 3\lambda)}, \\
c_5 &\to \frac{\lambda^3}{N^3 (1 + \lambda)(2 + 3\lambda)}, & c_6 &\to \frac{\lambda^3 (2 + \lambda)}{N^3 (1 + \lambda)(2 + 3\lambda)}, \\
c_7 &\to \frac{(-2 + \lambda)\lambda^2}{4 N^3 (1 + \lambda)(2 + 3\lambda)}, & c_9 &\to \frac{(-2 + \lambda)\lambda(-2 + 7\lambda)}{10 N^3 (1 + \lambda)(2 + 3\lambda)}, \\
c_8 &\to \frac{3(-2 + \lambda)(-1 + \lambda)\lambda}{5 N^3 (1 + \lambda)(2 + 3\lambda)}, & c_{11} &\to \frac{(-1 + \lambda)\lambda^2}{5 N^3 (1 + \lambda)(2 + 3\lambda)}, \\
c_{12} &\to \frac{\lambda^2 (4 + \lambda)}{5 N^3 (1 + \lambda)(2 + 3\lambda)}, & c_{13} &\to \frac{(-2 + \lambda)(-1 + \lambda)\lambda}{10 N^3 (1 + \lambda)(2 + 3\lambda)}, \\
c_{16} &\to \frac{2\lambda^2 (2 + \lambda)}{N^3 (-2 + \lambda)(1 + \lambda)(2 + 3\lambda)}, & c_{17} &\to \frac{\lambda^4}{N^3 (-2 + \lambda)(1 + \lambda)(2 + 3\lambda)}, \\
e_1 &\to \frac{(-2 + \lambda)(-1 + \lambda)(18 + 7\lambda)}{50 N^2 (1 + \lambda)(2 + 3\lambda)}, & e_{3} &\to \frac{(-2 + \lambda)(-1 + \lambda)(6 + 19\lambda)}{50 N^2 (1 + \lambda)(2 + 3\lambda)}, \\
e_6 &\to \frac{3(-2 + \lambda)(-1 + \lambda)^2}{50 N^2 (1 + \lambda)(2 + 3\lambda)}, & e_{7} &\to \frac{\lambda^2 (6 + 5\lambda)}{2 N^2 (1 + \lambda)(2 + 3\lambda)}, \\
e_8 &\to \frac{2(-2 + \lambda)(-1 + \lambda)^2}{25 N (1 + \lambda)(2 + 3\lambda)}, & e_{9} &\to \frac{(-2 + \lambda)(-1 + \lambda)(-48 + 23\lambda)}{100 N (1 + \lambda)(2 + 3\lambda)}, \\
e_{10} &\to \frac{\lambda(-28 - 68\lambda + \lambda^2)}{40 N (1 + \lambda)(2 + 3\lambda)}, & e_{11} &\to \frac{(-2 + \lambda)(-1 + \lambda)(-6 + 31\lambda)}{50 N (1 + \lambda)(2 + 3\lambda)}, \\
e_{12} &\to \frac{(-2 + \lambda)\lambda (4 + \lambda)}{20 N^2 (1 + \lambda)(2 + 3\lambda)}, & e_{14} &\to \frac{(-2 + \lambda)\lambda(-8 + 3\lambda)}{20 N^2 (1 + \lambda)(2 + 3\lambda)},
\end{align*}
\( e_{16} \rightarrow \frac{\lambda^2}{2N(1+\lambda)}, \quad e_{17} \rightarrow \frac{(-2 + \lambda)(-1 + \lambda)(-32 + 7\lambda)}{50(1+\lambda)(2+3\lambda)}, \)

\( e_{18} \rightarrow \frac{(-1 + \lambda)(242 - 213\lambda + 146\lambda^2)}{150(1+\lambda)(2+3\lambda)}, \quad e_{20} \rightarrow \frac{(-2 + \lambda)^2}{4N^2(1+\lambda)(2+3\lambda)}, \)

\( e_{21} \rightarrow \frac{3(-2 + \lambda)^2}{4N^2(1+\lambda)(2+3\lambda)}, \quad e_{22} \rightarrow \frac{\lambda^2(10 + 11\lambda)}{4N^2(1+\lambda)(2+3\lambda)}, \)

\( e_{23} \rightarrow \frac{\lambda(4 + 7\lambda^2)}{4N(1+\lambda)(2+3\lambda)}, \quad e_{24} \rightarrow \frac{\lambda(2 + 5\lambda + \lambda^2)}{2N(1+\lambda)(2+3\lambda)}, \)

\( e_{25} \rightarrow \frac{(-2 + \lambda)\lambda(22 + 13\lambda)}{20N^2(1+\lambda)(2+3\lambda)}, \quad e_{28} \rightarrow \frac{(-2 + \lambda)\lambda(8 + 7\lambda)}{10N^2(1+\lambda)(2+3\lambda)}, \)

\( e_{29} \rightarrow \frac{\lambda(-2 + \lambda + 2\lambda^2)}{2N^2(1+\lambda)(2+3\lambda)}, \quad e_{30} \rightarrow \frac{\lambda^2(-36 + 44\lambda + 27\lambda^2)}{20N^2(1+\lambda)(2+3\lambda)}, \)

\( e_{34} \rightarrow \frac{(-2 + \lambda)\lambda}{10N(1+\lambda)}, \quad e_{36} \rightarrow \frac{(-2 + \lambda)\lambda(14 + \lambda)^2}{20N^2(1+\lambda)(2+3\lambda)}. \) (D.18)

Therefore, the analysis for the zero mode eigenvalue equation can be obtained from these limiting values (D.18) on the coefficient functions after the normalization is fixed.

### Appendix E The OPEs between \( \hat{T}(Z) \) and \( \hat{W}(Z) \) and other OPEs in \( \mathcal{N} = 1 \) superspace

The \( \mathcal{N} = 1 \) superconformal algebra is described as the super OPE

\[
\hat{T}(Z_1) \hat{T}(Z_2) = \frac{1}{z_{12}^6} \frac{c}{6} + \frac{\theta_{12}}{z_{12}^3} \frac{3}{2} \hat{T}(Z_2) + \frac{1}{2} \frac{\theta_{12}}{z_{12}^2} D \hat{T}(Z_2) + \frac{\theta_{12}}{z_{12}} \partial \hat{T}(Z_2) + \cdots, \quad (E.1)
\]

where \( z_{12} = z_1 - z_2 - \theta_1 \theta_2, \theta_{12} = \theta_1 - \theta_2, D = \partial_\theta + \theta \partial_z \) and \( \partial = \partial_z \). The super stress energy tensor is

\[
\hat{T}(Z) = \frac{1}{2} G(z) + \theta T(z), \quad Z = (z, \theta). \quad (E.2)
\]

The primary superfield of dimension-\( \frac{5}{2} \),

\[
\hat{W}(Z) = \frac{1}{\sqrt{6}} U(z) + \theta W(z), \quad (E.3)
\]

satisfies

\[
\hat{T}(Z_1) \hat{W}(Z_2) = \frac{\theta_{12}}{z_{12}^2} \frac{5}{2} \hat{W}(Z_2) + \frac{1}{2} \frac{\theta_{12}}{z_{12}^2} D \hat{W}(Z_2) + \frac{\theta_{12}}{z_{12}} \partial \hat{W}(Z_2) + \cdots. \quad (E.4)
\]
Together with (E.1) and (E.4), the previous OPEs, (2.11), (2.15) and (2.32) are summarized as following single $\mathcal{N} = 1$ super OPE

\[
\hat{W}(Z_1) \hat{W}(Z_2) = \frac{1}{z_{12}^5} \frac{c}{15} \hat{T}(Z_2) + \frac{1}{z_{12}^3} \frac{3}{1} D \hat{T}(Z_2) + \frac{\theta_{12}}{z_{12}^3} \frac{2}{3} \hat{\theta} \hat{T}(Z_2) + \frac{1}{z_{12}^2} \frac{2}{3} \hat{\theta}^2 (Z_2)
\]

\[
+ \frac{\theta_{12}}{z_{12}^2} \left[ \frac{1}{4} \hat{\theta}^2 \hat{T} + \frac{1}{(4c + 21)} \left( 2 \hat{T} D \hat{T} - \frac{1}{4} \partial^2 \hat{T} \right) + \frac{1}{\sqrt{6}} \hat{O}_{\hat{z}^2} \right] (Z_2)
\]

\[
+ \frac{1}{z_{12}} \left[ \frac{1}{20} D \partial^2 \hat{T} + \frac{9}{2(22 + 5c)} \left( D \hat{T} D \hat{T} - \frac{3}{10} D \partial^2 \hat{T} \right) - \frac{2(2c - 83)}{2(4c + 21)(10c - 7)} \left( -\frac{7}{10} D \partial^2 \hat{T} + \frac{17}{(22 + 5c)} \left( D \hat{T} D \hat{T} - \frac{3}{10} D \partial^2 \hat{T} \right) + 4 \hat{T} \partial \hat{T} \right) + \frac{1}{7\sqrt{6}} D \hat{O}_{\hat{z}^2} + \frac{2}{\sqrt{6}} \hat{O}_4 \right] (Z_2)
\]

\[
+ \frac{12}{7(10c - 7)} \left( \frac{8}{3} D \partial \hat{T} \hat{\partial} - 2 \hat{T} D \partial \hat{T} - \frac{8}{15} \partial \hat{T} \hat{T} \right) + \frac{4}{7\sqrt{6}} D \hat{O}_{\hat{z}^2} + \frac{1}{\sqrt{6}} D \hat{O}_4 \right] (Z_2) + \cdots
\]

\[
= \frac{1}{z_{12}^5} \frac{c}{15} + \frac{\theta_{12}}{z_{12}} \left[ \frac{1}{3} D \hat{T}(Z_2) + \frac{1}{3} \frac{2}{3} \hat{\theta} \hat{T}(Z_2) + \frac{1}{3} \frac{2}{3} \hat{\theta}^2 (Z_2)
\]

\[
+ \frac{\theta_{12}}{z_{12}^2} \left[ \frac{1}{(4c + 21)(10c - 7)} D \hat{T} D \hat{T} + \frac{2(2c - 83)}{(4c + 21)(10c - 7)} D \partial^2 \hat{T}
\]

\[
+ \frac{2(2c - 83)}{(4c + 21)(10c - 7)} \hat{T} \partial \hat{T} + \frac{1}{7\sqrt{6}} D \hat{O}_{\hat{z}^2} + \frac{2}{\sqrt{6}} \hat{O}_4 \right] (Z_2)
\]

\[
+ \frac{\theta_{12}}{z_{12}} \left[ \frac{16(7c - 10)}{(4c + 21)(10c - 7)} \hat{T} D \partial \hat{T} + \frac{4(2c - 29c + 3)(2c - 83)}{3(4c + 21)(10c - 7)} \partial \hat{T}
\]

\[
+ \frac{8(18c - 1)}{(4c + 21)(10c - 7)} D \hat{T} \partial \hat{T} + \frac{4}{7\sqrt{6}} \partial \hat{O}_{\hat{z}^2} + \frac{1}{\sqrt{6}} D \hat{O}_4 \right] (Z_2) + \cdots,
\]

(E.5)

where the identity $\frac{1}{z_{12}^5} = \frac{1}{z_{12}^5} - n \frac{\theta_{12}}{z_{12}} (n = 1, \cdots, 6)$ is used and the following relations for the quasi primaries with (E.2) can be used

\[
T^2 - \frac{3}{10} \partial^2 T = \left( D \hat{T} D \hat{T} - \frac{3}{10} D \partial^2 \hat{T} \right) |_{\theta = 0},
\]

\[
GT - \frac{1}{8} \partial^2 G = \left( 2 \hat{T} D \hat{T} - \frac{1}{4} \partial^2 \hat{T} \right) |_{\theta = 0},
\]

\[
\frac{4}{3} T \partial G - G \partial T - \frac{4}{15} \partial^2 G = \left( \frac{8}{3} D \hat{T} \partial \hat{T} - 2 \hat{T} D \partial \hat{T} - \frac{8}{15} \partial \hat{T} \hat{T} \right) |_{\theta = 0}.
\]

(E.6)

When the higher spin $\mathcal{N} = 1$ super currents $\hat{O}_{\hat{z}^2}(Z)$ and $\hat{O}_4(Z)$ vanish, then the above OPE (E.5) becomes the one for the “minimal” $\mathcal{N} = 1$ super $W_3$ algebra [37] and see also [5] where the same convention is used. Note that the central charge is given by (2.7).
The other $\mathcal{N} = 1$ super OPEs can be obtained from the corresponding OPEs with component approach. Because

$$\hat{O}_Z^2(Z) = O_2^Z(z) + \theta O_4(z), \quad (E.7)$$

the $\mathcal{N} = 1$ super OPE $\hat{W}(Z_1) \hat{O}_Z^2(Z_2)$ can be obtained from the OPEs, $U(z) O_4^Z(w)$, $U(z) O_4^Z(w)$, $W(z) O_4^Z(w)$ and $W(z) O_4^Z(w)$. The $\mathcal{N} = 1$ super primary field $[E.7]$ satisfies the OPE similar to $(E.4)$. The $\theta_{12}$ independent terms are originating from the first OPE while the $\theta_{12}$ dependent terms are originating from the third OPE. Other OPEs can be located at various places appropriately (in $\mathcal{N} = 1$ supersymmetric way).

As done in $[E.6]$, recalling the following relations for the quasi primaries

$$GU - \frac{\sqrt{6}}{3} \partial W = \sqrt{6} \left( 2 \hat{T} \hat{W} - \frac{1}{3} \partial D \hat{W} \right) |_{\theta = 0},$$
$$TW - \frac{3}{14} \partial^2 W = \left( D \hat{T} D \hat{W} - \frac{3}{14} \partial^2 D \hat{W} \right) |_{\theta = 0},$$
$$G \partial U - \frac{5}{3} \partial GU - \frac{\sqrt{6}}{7} \partial^2 W = \sqrt{6} \left( 2 \hat{T} \partial \hat{W} - \frac{10}{3} \partial \hat{T} \hat{W} - \frac{1}{7} \partial^2 \hat{W} \right) |_{\theta = 0},$$
$$TU - \frac{1}{4} \partial^2 U = \left( 2 \hat{T} D \hat{W} - \frac{1}{6} \partial^2 \hat{W} \right) |_{\theta = 0},$$
$$GW - \frac{1}{6 \sqrt{6}} \partial^2 U = \sqrt{6} \left( D \hat{T} \hat{W} - \frac{1}{4} \partial^2 \hat{W} \right) |_{\theta = 0}, \quad (E.8)$$

where the relations $(E.2)$ and $(E.3)$ are used, the second $\mathcal{N} = 1$ super OPE with fixed $N = 3$ can be expressed as

$$\hat{W}(Z_1) \hat{O}_Z^2(Z_2) = \frac{\theta_{12}}{z_{12}^4} \left[ - \frac{(-36 + c)(-10 + 7c)}{5(20 + 3c)(21 + 4c)} \sqrt{6} \hat{W} \right] (Z_2)$$
$$+ \frac{1}{z_{12}^2} \left[ - \frac{2(-36 + c)(-10 + 7c)}{5(20 + 3c)(21 + 4c)} \sqrt{6} D \hat{W} \right] (Z_2) + \frac{\theta_{12}}{z_{12}^4} \left[ - \frac{(-36 + c)(-10 + 7c)^2}{3(20 + 3c)(21 + 4c)} \frac{\sqrt{6}}{5} \partial \hat{W} \right] (Z_2)$$
$$+ \frac{\sqrt{6}}{(52c)(20 + 3c)(21 + 4c)} \left[ \frac{\sqrt{6}}{3(-36 + c)(-10 + 7c)} \left( 6 \hat{T} \hat{W} - D \partial \hat{W} \right) + \sqrt{6} \hat{O}_4 \right] (Z_2)$$
$$+ \left[ - \frac{(-36 + c)(-10 + 7c)}{5(20 + 3c)(21 + 4c)} \frac{\sqrt{6}}{10} \partial^2 \hat{W} + \frac{5}{\sqrt{6}} \frac{1}{8} (\hat{O}_4 - D \hat{O}_4) + \frac{1}{\sqrt{6}} D \hat{O}_4 \right] (Z_2)$$
$$- \frac{2(-36 + c)(-10 + 7c)(47 + 14c)}{3(52c)(37 + 2c)(20 + 3c)(21 + 4c)} \sqrt{6} \left( D \hat{T} \hat{W} - \frac{1}{4} \partial^2 \hat{W} \right)$$
$$- \frac{2 \sqrt{6}(-36 + c)(5 + 6c)(-10 + 7c)}{5(52c)(37 + 2c)(20 + 3c)(21 + 4c)} \left( 2 \hat{T} D \hat{W} - \frac{1}{6} \partial^2 \hat{W} \right) \right] (Z_2)$$

49
\[ + \frac{1}{z_{12}} \left[ \frac{1}{16} \cdot \frac{1}{14} \cdot \frac{1}{14} \right] (2 - 36 + c)(-10 + 7c) \left( \frac{3}{2} \sqrt{6} \partial^2 W - D \partial^2 \check{W} \right) + \frac{1}{8} \partial \left( 6 \check{T} \check{W} - D \partial \check{W} \right) + \frac{3}{8} \sqrt{6} \partial \check{O}_v \right] = 1 super primary field (E.10) satisfies the OPE similar to (E.4).

Further simplifications of (E.9) can be made by collecting the coefficient functions in the same field content, as in (E.5) but the present form is more useful eventhough it is rather complicated. Each \( N = 1 \) super primary field (E.10) satisfies the OPE similar to (E.4).

As done in (E.6) and (E.8), from the following identifications between the composite fields in the component approach and the corresponding super fields at vanishing \( \theta \),

\[ TO_\frac{Z}{2} \cdot \frac{3}{16} \partial^2 O_\frac{Z}{2} = \left( \frac{3}{16} \partial^2 \check{O}_\frac{Z}{2} \right) |_{\theta=0} \]

\[ GP_{\frac{v}{u}} = \frac{4}{9} \sqrt{6} \partial O_\frac{Z}{2} - \frac{56}{56} \partial^2 O_\frac{Z}{2} = \left( \frac{56}{56} \partial^2 \check{O}_\frac{Z}{2} + 4 \check{D} \check{O}_4 - \frac{1}{9} \check{D} \partial \check{O}_4 - \frac{1}{9} \partial \partial \check{O}_4 \right) |_{\theta=0} \]

\[ GP_{\frac{w}{w}} = \frac{2}{9} \sqrt{3} \partial O_\frac{Z}{2} - \frac{12}{14} \sqrt{3} \partial^2 O_\frac{Z}{2} = \left( \frac{12}{14} \partial^2 \check{O}_\frac{Z}{2} + 7 \check{D} \check{O}_4 - \frac{2}{9} \check{D} \partial \check{O}_4 - \frac{4}{9} \partial \partial \check{O}_4 \right) |_{\theta=0} \]

\[ GO_{\frac{Z}{2}} + \frac{1}{4} \sqrt{6} \partial P_{\frac{Z}{2}} - \frac{6}{4} \partial P_{\frac{Z}{2}} = \left[ 2 \check{T} \check{O}_2 + \frac{1}{28} \partial \left( \check{D} \check{O}_2 + 14 \check{O}_4 \right) - \frac{1}{14} \partial \left( 4 \check{D} \check{O}_2 + 7 \check{O}_4 \right) \right] |_{\theta=0} \]
\[
TP_{4}^{uu} - \frac{1}{6}\partial^2 P_{4}^{uu} = \frac{\sqrt{6}}{7} \left[ D\tilde{T}D\tilde{O}_z + 14 D\tilde{T}\hat{O}_4 - \frac{1}{6} \partial^2 (D\tilde{O}_z + 14 \hat{O}_4) \right] |_{\theta=0},
\]

\[
TP_{4}^{ww} - \frac{1}{6}\partial^2 P_{4}^{ww} = \frac{2}{7}\sqrt{\frac{2}{3}} \left[ 4 D\tilde{T}D\tilde{O}_z + 7 D\tilde{T}\hat{O}_4 - \frac{1}{6} \partial^2 (4D\tilde{O}_z + 7 \hat{O}_4) \right] |_{\theta=0},
\]

and further relations

\[
G\partial O_z - \frac{7}{3} \partial G O_z + \frac{1}{9\sqrt{6}} \partial^2 P_{4}^{uu} - \frac{1}{3} \sqrt{\frac{2}{3}} \partial^2 P_{4}^{ww} = \left[ 2 \tilde{T}\partial O_z - \frac{14}{3} \partial T \tilde{O}_z + \frac{1}{63} \partial^2 (D\tilde{O}_z + 14 \hat{O}_4) \right] - \frac{2}{63} \partial^2 (4D\tilde{O}_z + 7 \hat{O}_4) |_{\theta=0},
\]

\[
GO_z - \frac{2}{63} \sqrt{\frac{2}{3}} \partial^2 P_{4}^{uu} + \frac{1}{21\sqrt{6}} \partial^2 P_{4}^{ww} = \left[ 2 \tilde{T}D\hat{O}_4 - \frac{4}{441} \partial^2 (D\tilde{O}_z + 14 \hat{O}_4) + \frac{1}{441} \partial^2 (4D\tilde{O}_z + 7 \hat{O}_4) \right] |_{\theta=0},
\]

where the relations (2.40), (1E.2), (1E.7) and

\[
\hat{O}_4(Z) = O_4(z) + \theta O_2(z)
\]

are used, the following \(N = 1\) super OPE with fixed \(N = 3\) can be obtained

\[
\tilde{W}(Z_1) \tilde{O}_4(Z_2) = \frac{1}{z_{12}^{\frac{3}{2}}} \left[ \frac{2(21 + 4c)(16 + 5c)}{15(5 + 2c)} \sqrt{6} \tilde{O}_z \right] (Z_2)
\]

\[
+ \frac{\theta_{12}}{z_{12}^{\frac{3}{2}}} \left[ \frac{2(-8 + 43c)\sqrt{6}}{15(5 + 2c)} \left( D\tilde{O}_z + 14 \hat{O}_4 \right) + \frac{(156 + 149c + 10c^2)}{15(5 + 2c)} \frac{1}{7}\sqrt{\frac{2}{3}} \left( 4D\tilde{O}_z + 7 \hat{O}_4 \right) \right] (Z_2)
\]

\[
+ \frac{1}{z_{12}^{\frac{3}{2}}} \left[ \frac{2(21 + 4c)(16 + 5c)}{2} \frac{\sqrt{6}}{7} \tilde{O}_z \right.
\]

\[
+ \frac{1}{z_{12}^{\frac{3}{2}}} \left[ \frac{2(-8 + 43c)3\sqrt{6}}{15(5 + 2c)} \frac{\sqrt{6}}{7} \tilde{O}_z \right. + \frac{(-36 + c)(-7 + 10c)}{60(5 + 2c)} \sqrt{6} D\tilde{O}_4 \left] (Z_2) \right.
\]

\[
+ \frac{1}{z_{12}^{\frac{3}{2}}} \left[ \frac{2(-8 + 43c)3\sqrt{6}}{15(5 + 2c)} \frac{\sqrt{6}}{7} \tilde{O}_z \right. + \frac{(-36 + c)(-7 + 10c)}{60(5 + 2c)} \sqrt{6} D\tilde{O}_4 \right]
\]

\[
+ \frac{1}{z_{12}^{\frac{3}{2}}} \left[ \frac{2(21 + 4c)(16 + 5c)}{2} \frac{3}{56} \sqrt{6} \partial^2 \tilde{O}_z - \frac{(-36 + c)(-7 + 10c)}{60(5 + 2c)} \frac{1}{3} \sqrt{6} D\partial \tilde{O}_4 \right]
\]

\[
+ \frac{4(16 + 5c)(94 + 9c)}{5(5 + 2c)(53 + 2c)} \sqrt{6} \left( D\tilde{T}\hat{O}_4 - \frac{3}{16} \partial^2 \hat{O}_4 \right) + \sqrt{6} \hat{O}_4
\]

\[
- \frac{\sqrt{\frac{2}{3}}}{5(5 + 2c)(53 + 2c)(20 + 3c)} \left[ \frac{2}{7} (\hat{T}D\tilde{O}_4 + 14 \hat{T}\hat{O}_4) + \frac{4}{9} D\partial \hat{O}_4 - \frac{1}{56} \partial^2 \hat{O}_4 \right]
\]

\[
- \frac{\sqrt{\frac{3}{2}}}{5(5 + 2c)(53 + 2c)(20 + 3c)} \left[ \frac{2}{7} (\hat{T}D\tilde{O}_4 + 7 \hat{T}\hat{O}_4) - \frac{2}{9} D\partial \hat{O}_4 - \frac{1}{14} \partial^2 \hat{O}_4 \right]
\]
\[
\theta_{12} \left[ \frac{2(-8 + 43c)}{15(5 + 2c)} \left( \frac{1}{12} \frac{\sqrt{6}}{7} \partial^2 (D\hat{\phi}_z + 14\hat{\phi}_4) \right) + \frac{156 + 149c + 10c^2}{15(5 + 2c)} \left( \frac{1}{127} \frac{2}{3} \partial^2 (4D\hat{\phi}_z + 7\hat{\phi}_4) + \frac{2}{11} \frac{2}{3} (D\hat{\phi}_z + \hat{\phi}_6) \right) \\
+ \frac{11\sqrt{6}}{5(5 + 2c)} \frac{2}{5} \partial \left( 2\hat{T}\hat{\phi}_z + \frac{1}{28} \partial (D\hat{\phi}_z + 14\hat{\phi}_4) - \frac{1}{14} \partial (4D\hat{\phi}_z + 7\hat{\phi}_4) \right) \\
+ \frac{4}{45(-1 + c)(5 + 2c)(53 + 2c)(61 + 2c)(20 + 3c)} \left( D\hat{T}D\hat{\phi}_2 + 14 D\hat{T}\hat{\phi}_4 \right) - \frac{4}{15(-1 + c)(5 + 2c)(53 + 2c)(61 + 2c)(20 + 3c)} \left( D\hat{T}D\hat{\phi}_2 + 14 D\hat{T}\hat{\phi}_4 \right) \right)
\]

where the \( \mathcal{N} = 1 \) primary super fields appearing in (E.12) are given

\[
\hat{O}_4^\dagger (Z) = O_4^\dagger (z) + \theta O_6(z), \quad \hat{\phi}_4^\dagger (Z) = O_6^\dagger (z) + \theta O_6^\dagger (z).
\]  

Further simplifications of (E.12) can be made by collecting the coefficient functions in the same field content, as in (E.5). Each \( \mathcal{N} = 1 \) super primary field in (E.11) or (E.13) satisfies the OPE similar to (E.4).

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