ON THE PRINCIPLE OF STATIONARY ISOENERGETIC ACTION

Božidar Jovanović

Abstract. We present several variants of the Maupertuis principle, both on the exact and the non exact symplectic manifolds.

Dedicated to the memory of Academician Anton Bilimović (1879-1970)

CONTENTS

1. Introduction 1
2. Principle of stationary isoenergetic action in a phase space 3
3. The Maupertuis principle on non-exact symplectic manifolds 7
4. Isoenergetic hypersurfaces of contact type 12
5. Examples: contact flows and integrable systems 14
References 18

1. Introduction

1.1. The principle of least action, or the principle of stationary action, says that the trajectories of a mechanical system can be obtained as extremals of a certain action functional. It is one of the basic tools in physics being applied both in classical and quantum setting.

Consider a Lagrangian system \( (Q, L) \), where \( Q \) is a configuration space and \( L(q, \dot{q}, t) \) is a Lagrangian, \( L : TQ \times \mathbb{R} \to \mathbb{R} \). Let \( q = (q_1, \ldots, q_n) \) be local coordinates on \( Q \). The motion of the system is described by the Euler–Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \quad i = 1, \ldots, n.
\] (1.1)

The solutions of the Euler–Lagrange equations are exactly the critical points of the action integral

\[
S_L(\gamma) = \int_a^b L(q, \dot{q}, t)dt
\]

in a class of curves \( \gamma : [a, b] \to Q \) with fixed endpoints \( \gamma(a) = q_0, \gamma(b) = q_1 \) (the Hamiltonian principle of least action (1834), e.g., see [28]).

2010 Mathematics Subject Classification. 37J05, 37J55, 70H25, 70H30.
The Legendre transformation $FL: TQ \to T^*Q$ is defined by
\begin{equation}
(1.2) \quad FL(q, \xi, t) \cdot \eta = \frac{d}{ds}\bigg|_{s=0} L(q, \xi + s\eta, t) \iff p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \ldots, n,
\end{equation}
where $\xi, \eta \in T_qQ$ and $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are canonical coordinates of the cotangent bundle $T^*Q$. In order to have a Hamiltonian description of the dynamics (see the section below), we suppose that the Legendre transformation (1.2) is a diffeomorphism. The corresponding Lagrangian $L$ is called hyperregular [21].

If the Lagrangian $L$ does not depend on time then the equations (1.1) possess the energy first integral
\begin{equation}
(1.3) \quad E(q, \dot{q}) = FL(q, \dot{q}) \cdot \dot{q} - L(q, \dot{q}) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L.
\end{equation}

In that case we have

**Theorem 1.1** (the Maupertuis principle). **Suppose that** $h$ **is a regular value of** $E$. **Among all curves** $q = \gamma(\tau)$ **connecting two points** $q_0$ **and** $q_1$ **and parametrized so that the energy has a fixed value** $E = h$, **the trajectory of the equations of dynamics (1.1) is an extremal of the reduced action**
\begin{equation}
(1.4) \quad S(\gamma) = \int_a^b FL\left(q(\tau) \frac{dq}{d\tau}\right) \cdot \frac{dq}{d\tau} d\tau = \int_a^b \frac{\partial L}{\partial \dot{q}}(\tau) \cdot \frac{dq}{d\tau} d\tau, \quad q_0 = \gamma(a), \quad q_1 = \gamma(b)
\end{equation}

It is important to note that the interval $[a, b]$, parametrizing the curve $q = \gamma(\tau)$, is not fixed and it can be different for different curves being compared, while the energy must be the same.

Contrary to the Hamiltonian principle, the **Maupertuis principle, or principle of stationary isoenergetic action** determines the shape of a trajectory but not the time. In order to determine the time, we have to use the energy constant.

Historically, a variant of Theorem 1.2 was the first variational approach to mechanics. It is attributed to Maupertuis (1744), Euler (1744) and Jacobi (1842), who gave an important geometric interpretation of the principle (see [28]).

**1.2.** The classical proofs of the Maupertuis principle can be found in [28, 36, 2]. In Serbian, see the second volume of Bilimović’s course in Theoretical mechanics [4], or Dragović and Milinković’s monograph [10].

Weinstein [34] and Novikov [25] formulated multi-valued variational principles that provided the study of the existence of periodic orbits on non exact symplectic manifolds. We feel a need to present these results, along with the classical ones, in a unified way.

In the first part of the paper, we derive the principle of stationary isoenergetic action, both on the exact (Section 2) and the non-exact symplectic manifolds (Section 3). The variants of the Maupertuis principle presented in Section 3 are our small contribution to the subject. They slightly differ from the existing variational principles formulated either for closed trajectories, or formulated without imposing the constraint given by the energy.

In the second part of the paper we point out a contact interpretation of the Maupertuis principle (Sections 4, 5). There, it is illustrated how some of the well known properties of the system of harmonic oscillators, the Kepler problem (Moser’s regularization) and the Neumann system (relationship with a geodesic flow on an
ellipsoid), have natural descriptions within a framework of the contact geometry. We believe that one should expect other interesting relations between the contact structures and integrable systems as well.

It is a great pleasure to dedicate this paper to Anton Bilimović, since his work has fundamentally influenced the development of Serbian theoretical mechanics.

2. Principle of stationary isoenergetic action in a phase space

2.1. Hamiltonian equations. Let \( L(q, \dot{q}, t) \) be a hyperregular Lagrangian. We can pass from velocities \( \dot{q_i} \) to the momenta \( p_j \) by using the standard Legendre transformation \((1.2)\). In the coordinates \((q, p)\) of the cotangent bundle \( T^* \mathcal{Q} \), the equations of motion \((1.1)\) read:

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n,
\]

where the Hamiltonian function \( H(q, p, t) \) is the Legendre transformation of \( L \):

\[
H(q, p, t) = E(q, \dot{q}, t)\big|_{q=F^{-1}(q, p, t)} = \mathcal{F} L(q, \dot{q}, t) - L(q, \dot{q}, t)\big|_{q=F^{-1}(q, p, t)}.
\]

Let \( dp = \sum_i p_i dq_i \) be the canonical 1-form and \( \omega = dp \wedge dq = \sum dp_i \wedge dq_i \) the canonical symplectic form of the cotangent bundle \( T^* \mathcal{Q} \). The system of equations \((2.1)\) is Hamiltonian, that is the vector field

\[
X_H(q, p) = (\partial H/\partial p_1, \ldots, \partial H/\partial p_n, -\partial H/\partial q_1, \ldots, -\partial H/\partial q_n)
\]

can be defined by

\[
\iota_{X_H} \omega (\cdot) = \omega (X_H, \cdot) = -dH (\cdot).
\]

2.2. Characteristic line bundles. More generally, consider a \( 2n \)-dimensional symplectic manifold \( \mathcal{P} \) with a closed, non-degenerate 2-form \( \omega \). Let \( H : \mathcal{P} \times \mathbb{R} \to \mathbb{R} \) be a smooth, in general time dependent, function. Consider the corresponding Hamiltonian equation

\[
\dot{x} = X_H,
\]

where the Hamiltonain vector field \( X_H(x, t) \) is defined by \((2.2)\).

If the Hamiltonian \( H \) does not depend on time, it is the first integral of the system. Let \( M \) be a regular connected component of the invariant variety \( H = h \), which means \( dH|_M \neq 0 \).

Since \( dH (\xi) = 0 \), \( \xi \in T_x M \), from \((2.2)\) we see that \( X_H \) generates the symplectic orthogonal of \( T_x M \) for all \( x \in M \) — the characteristic line bundle \( \mathcal{L}_M \) of \( M \). It is the kernel of the form \( \omega \) restricted to \( M \):

\[
\mathcal{L}_M = \{ \xi \in T_x M \mid \omega (\xi, T_x M) = 0, \; x \in M \}.
\]

Note that \( \mathcal{L}_M \) is determined only by \( M \) and not by \( H \). If \( F \) is another Hamiltonian defining \( M \), \( M \subset F^{-1}(c) \), \( dF|_M \neq 0 \), then the restrictions of the Hamiltonian vector fields \( X_H \) and \( X_F \) to \( M \) are proportional.

A variation of a curve \( \gamma : [a, b] \to M \) is a mapping \( \Gamma : [a, b] \times [0, \epsilon] \to M \), such that \( \gamma(t) = \Gamma(t, 0) \), \( t \in [a, b] \). Denote \( \gamma_s(t) = \Gamma(t, s) \) and \( \delta \gamma(t) = \frac{d}{ds}|_{s=0} \gamma_s(t) \in T_{\gamma(t)} M \).
From Cartan’s formula we get (e.g., see Griffiths [12]):

**Lemma 2.1.** Let \((M, \alpha)\) be a manifold endowed with a 1-form \(\alpha\), \(\gamma : [a, b] \to M\) be an immersed curve and \(\Gamma\) be a variation of \(\gamma\). The Lie derivative of the form \(\Gamma^* \alpha\) in the direction of \(\partial/\partial s\) at the points \([a, b] \times \{0\}\) is equal to

\[
\frac{d}{ds} \Gamma^* \alpha|_{(t,0)} = \Gamma^* (i_{\delta \gamma(t)} d\alpha) + d\Gamma^* (\alpha(\delta \gamma(t))).
\]

**Theorem 2.1.** Assume that the symplectic form \(\omega\) is exact: \(\omega = d\alpha\). Let \(M\) be a regular component of the invariant hypersurface \(H^{-1}(h)\). The integral curves \(\gamma : [a, b] \to M\) of the characteristic line bundle \(L_M\) are extremals of the (reduced) action functional

\[
A(\gamma) = \int_a^b \alpha = \int_a^b \alpha(\dot{\gamma}) dt
\]

in the class of variations \(\gamma_s(t)\) such that \(\alpha(\delta \gamma(a)) = \alpha(\delta \gamma(b)) = 0\).

The proof is a direct consequence of Lemma 2.1. We have

\[
\frac{d}{ds} \left( \int_{\gamma_s} \alpha \right) |_{s=0} = \int_a^b \omega(\delta \gamma(t), \dot{\gamma}(t)) dt + \alpha(\delta \gamma(b)) - \alpha(\delta \gamma(a)).
\]

The expression above is equal to zero for all variations \(\gamma_s(t)\) if and only if \(\dot{\gamma}\) is in the kernel of the form \(\omega = d\alpha\) restricted to \(M\). That is, \(\gamma(t)\) is an integral curve of the line bundle \(L_M\).

2.3. Applying Theorem 2.1 to the symplectic space \((T^*Q, dp \wedge dq)\) we obtain Poincaré’s formulation of the Maupertuis principle in a phase space [27].

**Theorem 2.2.** If the Hamiltonian function \(H = H(q, p)\) does not depend on time, then the phase trajectories of the canonical equations (2.1) lying on the regular connected component \(M\) of the surface \(\{H(q, p) = h\}\) are extremals of the reduced action

\[
A(\gamma) = \int_{\gamma} p dq
\]

in the class of curves \(\gamma\) lying on \(M\) and connecting the subspaces \(T_{q_0}^* Q\) and \(T_{q_1}^* Q\).

Note that Theorem 1.1 follows from Theorem 2.2 (e.g., see Arnold [2]). Suppose that the Hamiltonian system (2.1) is a Legendre transformation of the Lagrangian system (1.1). The main observation is that if \(\gamma(\tau)\) is a configuration space curve parametrized such that \(E(\gamma, d\gamma/d\tau) = h\), then the lifted curve \(\bar{\gamma} = FL(\gamma, d\gamma/d\tau)\) lies on \(M\) and the reduced actions (1.4) and (2.5) for \(\bar{\gamma}\) and \(\gamma\) are equal: \(S(\bar{\gamma}) = A(\gamma)\) (see Fig. 1).

2.4. Jacobi’s metric. Consider a natural mechanical system on \(Q\) defined by the Lagrangian function:

\[
L(q, \dot{q}) = T + B - V = \frac{1}{2} \sum_{ij} K_{ij} \dot{q}_i \dot{q}_j + \sum_i B_i \dot{q}_i - V(q).
\]

Here \(ds^2 = \sum_{ij} K_{ij} dq_i dq_j\) is a Riemannian metric on \(Q\), \(V(q)\) is a potential function and \(\theta = \sum_i B_i dq_i\) is a 1-form defining a gyroscopic (or magnetic) field \(\sigma = d\theta\) (see Section 3).
The energy of the system (1.3) is the sum of the kinetic and the potential energy

\[ E(q, \dot{q}) = T + V = \frac{1}{2} \sum_{ij} K_{ij} \dot{q}_i \dot{q}_j + V(q). \]

In the region of the configuration space \( Q_h \) where \( V(q) < h \), we can define the Jacobi metric

\[ ds^2_J = 2(h - V(q)) \, ds^2 = 2(h - V(q)) \sum_{ij} K_{ij} dq_i dq_j. \]

The following version of the Maupertuis principle for Lagrangians of the form (2.6) is well known (e.g., see Kozlov [19]).

**Theorem 2.3.** Among all curves \( q = \gamma(\tau) \) connecting the points \( q_0, q_1 \in Q_h \) and parametrized so that the energy has a fixed value \( E = h \), the trajectory of the equations of dynamics (1.1) with Lagrangian (2.6) is an extremal of the integral

\[ S(\gamma) = \int_{\gamma} ds_J + \theta. \]

In particular, if there are no gyroscopic forces, the trajectories of the system within \( Q_h \), up to reparametrization, are geodesic lines of the Jacobi metric \( ds^2_J \).

Indeed, in order to guarantee a fixed value of the energy

\[ E = T + V = \frac{1}{2} \sum_{ij} K_{ij} \frac{dq_i}{d\tau} \frac{dq_j}{d\tau} + V(q) = \frac{1}{2} \left( \frac{ds}{d\tau} \right)^2 + V(q) = h, \]
the parameter $\tau$ of the curve $q = \gamma(\tau)$ must be proportional to the length $d\tau = ds/\sqrt{2(h-V)}$. Therefore

$$
\int_a^b \frac{\partial L}{\partial \dot{q}}(\tau) \frac{dq}{d\tau} d\tau = \int_a^b \left( \sum_i K_i \frac{dq_i}{d\tau} + \sum_i B_i \frac{dq_i}{d\tau} \right) d\tau
= \int_a^b \left( 2(h-V(q)) + \sum_i B_i \frac{dq_i}{d\tau} \right) d\tau = \int_s s_f + \theta.
$$

**Remark 2.1.** The variational principle stated in Theorem 2.3 is used in the study of periodic trajectories of natural mechanical systems with exact magnetic fields (see [31] and references therein). Note also that the Maupertuis principle for a configuration space $Q$ being a Banach space can be found in [21, 30].

2.5. The Hamiltonian principle of least action. Consider a Poincaré–Cartan 1-form $pdq - Hdt$ on the extended phase space $T^*Q \times \mathbb{R} \to \mathbb{R}$, where $H : T^*Q \times \mathbb{R} \to \mathbb{R}$ is a Hamiltonian function. The phase trajectories of the canonical equations (2.1) are extremals of the action (2.9)

$$
A_H(\gamma) = \int_{t_0}^{t_1} pdq - Hdt
$$
in the class of curves $\gamma(t) = (q(t), p(t), t)$ connecting the subspaces $T_{q_0}Q \times \{t_0\}$ and $T_{q_1}Q \times \{t_1\}$ (Poincaré’s modification of the Hamiltonian principle of least action [27]). Namely, a vector $(\xi, 1) \in T_{(q,p)}(T^*Q)$ belongs to $\ker d(pdq - Hdt)$ at $(q, p, t)$ if and only if $\xi = X_{H}(q, p, t)$ (see [2, 21]).

Obviously, we can replace $(T^*Q, dp \wedge dq)$ by an arbitrary exact symplectic manifold $(P, \omega = d\alpha)$. In particular, if we consider the action $A_H(\gamma) = \int_0^1 \alpha - Hdt$ on the free loop space $\Omega(P) = C^\infty(S^1, P)$, $S^1 = \mathbb{R}/\mathbb{Z}$ of $P$ and $H$ is 1-periodic in $t$-variable, then the critical points of $A_H$ are 1-periodic orbits of the equation (2.3).

For a given time-independent Hamiltonian $H : P \to \mathbb{R}$ with a regular level set $H^{-1}(h)$, the periodic orbits having all positive periods and energy $h$ can be obtained by the use of modified action:

$$
A_{H,h}(\gamma, \lambda) = \int_0^1 \alpha(\gamma) dt - \lambda \int_0^1 (H(\gamma(t)) - h) dt,
$$
defined on the space $\Omega(P) \times \mathbb{R}^+$ (see [29, 34]). The critical points $(\gamma, \lambda)$ of $A_{H,h}$ correspond to $\lambda$-periodic orbits $x(t) = \gamma(t/\lambda)$ that lie on the energy hypersurface $H^{-1}(h)$. Moreover, Weinstein defined actions $A_H$ and $A_{H,h}$ when the symplectic form is not exact as well [35].

The Lagrangian analogue of the functional (2.10) is

$$
S_{L,h}(\gamma, \lambda) = \int_0^1 \lambda L(\gamma, \dot{\gamma}/\lambda) dt + \lambda h, \quad \gamma \in \Omega(Q), \ \lambda > 0
$$
(see [9]). The pair $(\gamma, \lambda)$ is a critical point of $S_{L,h}$ if and only if $q(t) = \gamma(t/\lambda)$ is a $\lambda$-periodic solution of the Euler–Lagrange equation (1.1) with energy $h$.

Variational principles related to the action (2.9), which arise by a reduction process are given in [8].
3. The Maupertuis principle on non-exact symplectic manifolds

3.1. Magnetic flows. Consider a natural mechanical system given by the Lagrangian function (2.6). After the Legendre transformation, it takes the form (2.1) with the Hamiltonian function

\begin{equation}
H(q, p) = \frac{1}{2} \langle p - \theta, p - \theta \rangle + V(q)
\end{equation}

where \( K_{ij} \) is the inverse of the metric tensor \( K_{ij} \).

The transformation

\begin{equation}
T_\theta : (q, p) \mapsto (q, p - \theta)
\end{equation}

is a symplectomorphism between \( (T^*Q, dp \wedge dq) \) and a "twisted" cotangent bundle \( (T^*Q, dp \wedge dq + \pi^*\sigma) \), where \( \pi : T^*Q \rightarrow Q \) is the natural projection and \( \sigma = d\theta \).

In new coordinates, also denoted by \( (q, p) \), the Hamiltonian (3.1) takes the usual form, the sum of the kinetic and the potential energy:

\begin{equation}
H(q, p) = \frac{1}{2} \langle p, p \rangle + V(q) = \frac{1}{2} \sum_{ij} K_{ij} p_i p_j + V(q),
\end{equation}

while the equations of motion take "non-canonical" form:

\begin{equation}
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} + \sum_{j=1}^{n} F_{ij} \frac{\partial H}{\partial p_j},
\end{equation}

where \( \sigma = \sum_{1 \leq i < j \leq n} F_{ij}(q) dq_i \wedge dq_j \). The equations are Hamiltonian with respect to the symplectic form \( \omega = dp \wedge dq + \pi^*\sigma \).

One can consider the system (3.3) associated to a non-exact 2-form \( \sigma \) as well (for example, the motion of a particle in a magnetic monopole field [21]). In this case, the Lagrangian (2.6) is defined only locally. Nevertheless, it is very interesting that the Hamiltonian (Weinstein [34] and Tuynman [33]) and the Maupertuis principles (Novikov [26]) of least action can be still defined.

3.2. Multivalued reduced action. Let \( (P, \omega) \) be a non exact symplectic manifold and let \( M = H^{-1}(h) \) be a regular isoenergetic hypersurface. The main observation concerning the Maupertuis principle can be stated as follows (see [26, 19] for the reduced action 2.3).

Let \( U \subset P \) be a region where \( \omega \) is exact and let \( \omega = da_1 = da_2 \). Consider a variation \( \gamma(t) = \Gamma(t, s), t \in [0, 1], s \in [0, \epsilon] \) with fixed endpoints of a curve \( \gamma \) lying in \( M \cap U \). Then

\[ \int_{\gamma_t} a_1 - \int_{\gamma_0} a_1 = -\int_{[0,1] \times [0,\epsilon]} \Gamma^* da_1 = -\int_{[0,1] \times [0,\epsilon]} \Gamma^* da_2 = \int_{\gamma_t} a_2 - \int_{\gamma_0} a_2. \]

Therefore

\[ \int_{\gamma_t} a_1 - \int_{\gamma_0} a_1 = \int_{\gamma_t} a_2 - \int_{\gamma_0} a_2 \]

and, although \( \int_{\gamma} a_i \) depends on the form \( a_i \), the derivative

\[ \frac{d}{ds}|_{s=0} \int_{0}^{1} \alpha(\gamma_s) dt = \int_{0}^{1} \omega(\delta\gamma(t), \dot{\gamma}(t)) dt \]

does not depend on \( a_i, i = 1, 2 \). One can define an appropriate multi-valued functional on a space of paths with fixed endpoints, such that an extremal (if exist)
is exactly the integral curve of the characteristic foliation on $M$. However, as in the case of the symplectic homology (see [13]), the situation simplifies in the aspherical case which is considered below.

### 3.3. Aspherical symplectic manifolds

The symplectic manifold $(P, \omega)$ is **aspherical** if $\omega$ vanishes on $\pi_2(P)$. Of course, if $\omega$ is exact or $\pi_2(P) = 0$, then $(P, \omega)$ is aspherical.

Consider the equation (2.3), where $H$ does not depend on time. Let $M$ be a regular component of $H^{-1}(h)$ and $c : [0, 1] \to P$ be an immersed curve with endpoints $x_0 = c(0) \in M$ and $x_1 = c(1) \in M$. Define $\Omega^{h}_{c}(x_0, x_1)$ as the space of regular paths that are homotopic to $\tilde{c}$ in $P$:

$$\Omega^{h}_{c}(x_0, x_1) = \{ \gamma : [0, 1] \to M \mid \gamma(0) = x_0, \gamma(1) = x_1, \dot{\gamma}(t) \neq 0, t \in [0, 1], \gamma \sim_P c \}.$$  

The space of all regular paths connecting $x_0$ and $x_1$ and laying in $M$ is the union

$$\Omega^{h}(x_0, x_1) = \bigcup_{c} \Omega^{h}_{c}(x_0, x_1),$$

where we take representatives $c$ for all non-homotopic paths (in $P$) connecting $x_0$ and $x_1$.

If we suppose that $(P, \omega)$ is symplectically aspherical then we can define a single-valued reduced action:

$$A : \Omega^{h}(x_0, x_1) \to \mathbb{R}, \quad A(\gamma)\mid_{\Omega^{h}_{c}(x_0, x_1)} = \int_{D} f^{*}_{c}\omega,$$

where

$$D = \{ z \mid z \in \mathbb{C}, |z| \leq 1 \}$$

is the unit disk, $f_{c} : D \to P$ is an arbitrary mapping that is smooth for $|z| < 1$, continuous on $D$ and $\gamma(t) = f_{c}(\exp(\sqrt{-1}\pi t))$, $c(t) = f_{c}(\exp(\sqrt{-1}\pi(2-t)))$, $t \in [0, 1]$.

That is, $f(D)$ is a surface with the boundary $\partial D = \gamma \cdot c^{-1}$.

**Figure 2.**

Since $\gamma \sim_P c$ we can always find a mapping $f$ with required properties. From $\omega|_{\pi_2(P)} = 0$, the value $A(\gamma)$ does not depend on the choice of $f$.

**Theorem 3.1.** The integral curves $\gamma : [0, 1] \to M$ of the characteristic line bundle $\mathcal{L}_M$ that connect $x_0$ and $x_1$ are extremals of the reduced action (34).
particular position with minimal action, such that the multi-valued Poincaré action \( (2.9) \) can be seen as a composition on a torus \( T \) of \( \text{ker} \omega \). Then the parameters \( \beta \) are 2-forms, representing integrals cohomology classes. We take decomposition with minimal \( n \). Then the parameters \( \mu_a \) are independent over \( Q \), in particular \( \mu = \mu_1 + \cdots + \mu_n \neq 0 \). To \( \omega \) we associate the 1-form

\[
\lambda = \sum_{a=1}^{n} \mu_a dy^a
\]
on a torus \( T^n = \{ (\exp(\sqrt{-1}y^1), \ldots, \exp(\sqrt{-1}y^n)) \} \). It can be consider as a differential of a multi-valued function \( \lambda \) on \( T^n \): \( \lambda = d\Lambda \). Also, for \( a = 1, \ldots, n \), let us define principal \( S^1 \)-bundles

\[
S^1 \longrightarrow Y_a
\]
(3.5)
having the connections \( \theta^a \) with the curvature forms \( \beta^a \) (see Kobayashi [18]).

Let \( \gamma(t) \), \( t \in [t_0, t_1] \) be a piece-wise smooth, closed curve on \( P \). Recall, a piece-wise smooth curve \( \tilde{\gamma}(t) \subset Y_a \) is a horizontal lift of \( \gamma \) if \( \rho_a \circ \tilde{\gamma}(t) = \gamma(t) \) and \( \theta^a (\frac{d}{dt} \tilde{\gamma}(t)) = 0 \), whenever the velocity vector is defined. The holonomy \( \text{Hol}^a(\gamma) \) is an element \( g \in S^1 \), such that \( g \cdot \tilde{\gamma}(t_0) = \tilde{\gamma}(t_1) \).

**Lemma 3.1** ([33]). Let \( \gamma_s(t) = \Gamma(t, s) \) be a variation of \( \gamma : [0, 1] \rightarrow P \) with fixed endpoints and let \( c : [0, 1] \rightarrow P \) be an arbitrary curve connecting \( x_0 = \gamma(0) \) and \( x_1 = \gamma(1) \). We have a family of closed orbits \( \tilde{\gamma}_s = \gamma_s \cdot c^{-1} \). The derivative of \( \text{Hol}^a(\tilde{\gamma}_s) \) is given by:

\[
\left. \frac{d}{ds} \text{Hol}^a(\tilde{\gamma}_s) \right|_{s=0} = \int_0^1 \beta^a(\tilde{\gamma}(t), \delta\gamma(t))dt \cdot \frac{\partial}{\partial y^a}.
\]
Consider the equation (2.3), where $\partial H/\partial t = 0$. Let $M$ be a regular component of $H^{-1}(h)$ and $\Omega^b(x_0, x_1)$ be a space of regular paths $\gamma : [0, 1] \to M$ that connect points $x_1$ and $x_2$.

For every $\gamma \in \Omega^b(x_0, x_1)$ define a piece-wise smooth, closed path $\bar{\gamma} = \gamma \cdot c^{-1} : [0, 2] \to M$, where $c \in \Omega^b(x_0, x_1)$ is fixed. We call

$$A_{\bar{\gamma}} : \Omega^b(x_0, x_1) \to \mathbb{T}^n, \quad \gamma \mapsto (\text{Hol}^1(\bar{\gamma}), \ldots, \text{Hol}^n(\bar{\gamma}))$$

a torus valued reduced action. From Lemma 3.1 we have:

$$\lambda \left( \frac{d}{ds} A_{\bar{\gamma}}(\gamma), |s=0 \right) = \sum_{a=1}^n \mu_a \int_0^1 \beta^a(\dot{\gamma}(t), \delta \gamma(t)) dt = \int_0^1 \omega(\gamma(t), \delta \gamma(t)) dt.$$

Whence, we obtain the following principle of stationary isoenergetic action on the non-exact symplectic manifolds.

**Theorem 3.2.** A curve $\gamma \in \Omega^b(x_0, x_1)$ is an integral curve of the characteristic line bundle $\mathcal{L}_M$ if and only if

$$\frac{d}{ds} (A_{\bar{\gamma}}(\gamma)) |_{s=0} = \lambda \left( \frac{d}{ds} A_{\bar{\gamma}}(\gamma), |s=0 \right) = 0$$

for all variations $\gamma_s \in \Omega^b(x_0, x_1)$.

For the completeness of the exposition we include:

**Proof of Lemma 3.1.** In local trivializations

$$\rho^{-1}(U_i) \cong U_i \times S^1(x_i, y_i \text{ mod } 2\pi),$$

we have local connection 1-forms $\alpha_i$ on $U_i$ such that $\theta = \alpha_i + dy_i$ (the index $a$ is omitted). The transition functions between fiber coordinates and connection 1-forms are given by

$$y_j = y_i + g_{ij}(x), \quad \alpha_i = \alpha_j + dg_{ij}, \quad g_{ij} : U_i \cap U_j \to S^1.$$

On the other hand, the curvature 2-form is invariant: $\beta = da_i = da_j$.

Suppose $\gamma_s([t_0, t_1]) \subset U_i$, $s \in [0, 1]$. The local expression for $\dot{\gamma}_s$ reads

$$\dot{\gamma}_s(t) = (\gamma_s(t), y_i(t, s)), \quad \theta_i \left( \frac{d}{dt} \dot{\gamma}(t) \right) = 0 \iff \alpha_i(\dot{\gamma}(t)) + y_i(t, s) = 0.$$

Therefore

$$y_i(t_1, s) = y_i(t_0, s) - \int_{t_0}^{t_1} \alpha_i(\dot{\gamma}_s(t)) dt.$$

By taking the differential of (3.7) at $s = 0$ and applying (2.4) we get

$$\delta y_i(t_1) + \alpha_i(\delta \gamma(t_1)) = \delta y_i(t_0) + \alpha_i(\delta \gamma(t_0)) + \int_{t_0}^{t_1} \beta(\dot{\gamma}(t), \delta \gamma(t)) dt,$$

where $\delta y_i(t) = \left. \frac{d}{ds} y_i(t, s) \right|_{s=0}, \delta \gamma(t) = \left. \frac{d}{ds} \gamma_s(t) \right|_{s=0}$.

Now, assume $t_0 < t'_0 < t_1 < t'_1$, $\gamma_s([t_0, t_1]) \subset U_i$ and $\gamma_s([t'_0, t'_1]) \subset U_j$. The transformations (3.5) imply

$$\delta y_i(t) + \alpha_i(\delta \gamma(t)) = \delta y_j(t) + \alpha_j(\delta \gamma(t)), \quad t \in [t'_0, t'_1].$$
By combining (3.8) and (3.9), it follows
\begin{equation}
(3.10) \quad \delta y_k(t'_1) + s_c(\delta \gamma(t'_1)) = \delta y_k(t_0) + s_c(\delta \gamma(t_0)) + \int_{t_0}^{t'_1} \beta(\gamma(t), \delta \gamma(t))dt.
\end{equation}

Let $\gamma_0 = \gamma_s \cdot e^{-1} : [0, 2] \to M$ and let $U_1, \ldots, U_i$ be local charts, such that $\gamma_0([t_{i-1}, t_i]) \subset U_i, \quad 0 = t_0 < t_1 < \cdots < t_k = 1 < t_{k+1} < \cdots < t_i = 2, \quad s \in [0, \epsilon].$

From the relation (3.10) and $\delta \gamma(0) = \delta \gamma(1) = 0 = \delta \gamma(t) = 0, \quad t \in [1, 2], \text{ we get}$
\begin{equation}
\delta \gamma_k(1) - \delta \gamma_k(0) = \int_0^1 \beta(\gamma(t), \delta \gamma(t))dt, \quad \delta \gamma_2 - \delta \gamma_k(1) = 0.
\end{equation}

We can suppose that the horizontal lifts of all curves start from the same point in $Y.$ Then $\gamma_1(0) = 0.$ This proves the statement. \qed

3.5 Reduced action for magnetic flows. Let us return to the magnetic equations (3.3), where $H(q, p)$ is an arbitrary smooth function and $\sigma$ is not exact. Let $M$ be a regular component of $H(q, p)^{-1}(h)$ and let $\pi : T^*Q \to Q$ be the natural projection.

As in Theorem 2.2, we need not to fix endpoints in the fiber directions. Consider a class of regular curves $\gamma$ lying on $M$ and connecting the subspaces $T_{q_0}Q$ and $T_{q_1}Q,$ such that the projection $\pi(\gamma)$ is homotopic to $c:
\end{equation}
\begin{equation}
\Omega^h_c(q_0, q_1) = \{ \gamma : [0, 1] \to M | \pi(\gamma(0)) = q_0, \pi(\gamma(1)) = q_1, \pi(\gamma) \sim c \},
\end{equation}
and a class of all regular paths connecting $T_{q_0}Q$ and $T_{q_1}Q$ and lying in $M:$
\begin{equation}
\Omega^h(q_0, q_1) = \bigcup_c \Omega^h_c(q_0, q_1),
\end{equation}
where we take representatives $c : [0, 1] \to Q$ for all non-homotopic paths connecting $q_0$ and $q_1.$

**Theorem 3.3.** Assume $\sigma|_{\pi^1(Q)} = 0.$ The phase trajectories of the magnetic equations (3.3) in the class of curves $\Omega^h_c(q_0, q_1)$ are extremals of the reduced action
\begin{equation}
A : \Omega^h_c(q_0, q_1) \to \mathbb{R}, \quad A(\gamma)|_{\Omega^h_c(q_0, q_1)} = \int_0^1 pdq + \int_D f_\gamma \sigma,
\end{equation}
where $f_\gamma : D \to Q$ is smooth for $|z| < 1,$ continuous on $D$ and $\pi(\gamma(t)) = f_\gamma(\exp(\sqrt{-1}t\pi)), \quad c(t) = f_\gamma(\exp(\sqrt{-1}\pi(2 - t))), \quad t \in [0, 1].$

If $\sigma|_{\pi^1(Q)} \neq 0,$ we can use a combination of the usual reduced action and a torus valued action with respect to the form $\sigma.$ Suppose
\begin{equation}
\sigma = \sum_{a=1}^N \mu_a \beta^a,
\end{equation}
where $\beta^a$ are 2-forms, representing integrals cohomology classes in $Q$. We take decomposition with minimal $n$. As above, to $\sigma$ we associate principal $S^1$-bundles $L_n$ over $Q$ having the connections $\theta^a$ with curvature forms $\beta^a, \quad a = 1, \ldots, n.$

Let us fix $c : [0, 1] \to Q, \quad c(0) = q_0, \quad c(1) = q_1.$ For every $\gamma \in \Omega^h_c(q_0, q_1),$ we associate a pease-wise smooth, closed path $\gamma = \pi(\gamma) \cdot e^{-1} : [0, 2] \to Q.$ Define
\begin{equation}
B_{\gamma^*} : \Omega^h_c(q_0, q_1) \to \mathbb{T}^n, \quad \gamma \mapsto (\text{Hol}^1(\gamma), \ldots, \text{Hol}^N(\gamma)),
\end{equation}
where Hol is the holonomy of the bundle \( L_{\alpha} \rightarrow Q \). Let \( \nu = \sum_{a=1}^{n} \mu_d dy^a \) be a 1-form on \( \mathbb{T}^n \), considered as a differential of a multi-valued function \( \Upsilon: \mathbb{T} \rightarrow \nu \).

**Theorem 3.4.** A curve \( \gamma \in \Omega^h(q_0, q_1) \) is an integral curve of the characteristic line bundle \( L_M \) if and only if

\[
\frac{d}{ds} \left( \int_{\gamma_s} p dq - \Upsilon \circ B_{T^n}(\gamma_s) \right) |_{s=0} = 0
\]

for all variations \( \gamma_s \in \Omega^h(q_0, q_1) \).

**Remark 3.1.** For various approaches to the existence problem of closed magnetic orbits, see [9, 31] and references therein. Integrable magnetic geodesic flows on homogeneous spaces can be found in [6].

4. Isoenergetic hypersurfaces of contact type

4.1. A contact form \( \alpha \) on a \((2n+1)\)-dimensional manifold \( M \) is a Pfaffian form satisfying \( \alpha \wedge (d\alpha)^n \neq 0 \). By a contact manifold \((M, \mathcal{H})\) we mean a connected \((2n+1)\)-dimensional manifold \( M \) equipped with a nonintegrable contact (or horizontal) distribution \( \mathcal{H} \), locally defined by a contact form: \( \mathcal{H}_U = \ker \alpha|_U \), \( U \) is an open set in \( M \) [20]. A contact manifold \((M, \mathcal{H})\) is co-oriented (or strictly) contact if \( \mathcal{H} \) is defined by a global contact form \( \alpha \). For a given contact form \( \alpha \), the Reeb vector field \( Z \) is a vector field uniquely defined by

\[
i_Z \alpha = 1, \quad i_Z d\alpha = 0.
\]

4.2. In studying the existence problem of closed Hamiltonian trajectories on a fixed isoenergetic surface, Weinstein introduced the following concept [35]. An orientable hypersurface \( M \) of a symplectic manifold \((P, \omega)\) is of contact type if there exist a 1-form \( \alpha \) on \( M \) satisfying

\[
da = j^* \omega, \quad \alpha(\xi) \neq 0, \xi \in \mathcal{L}_M, \xi \neq 0
\]

where \( j: M \rightarrow P \) is the inclusion. If \((M, \alpha)\) is of contact type, since \( \mathcal{L} = \ker \omega_M \), the kernel of \( \alpha \)

\[
\mathcal{H} = \{ \xi \in T_x M \mid \alpha(\xi) = 0, x \in M \}
\]

is a \((2n-2)\)-dimensional nonintegrable distribution on which \( d\alpha = \omega \) is nondegenerate. Consequently, \( \alpha \wedge da^{n-1} \) is a volume form on \( M \) and \((M, \mathcal{H})\) is a co-oriented contact manifold.

Now, let \((P, \omega = da)\) be an exact symplectic manifold. Consider a regular component \( M \) of an isoenergetic surface \( H^{-1}(\hbar) \) (\( H \) does not depend on time). If \( \alpha(X_H)|_M \neq 0 \) then \( M \) is of contact type. We say that \( M \) is of contact type with respect to \( \alpha \).

If \( M \) is of contact type with respect to \( \alpha \) then \( \alpha \) has no zeros in some open neighborhood of \( M \). Contrary, suppose that an 1-form \( \alpha \) has no zeros in some open neighborhood of \( M \). Then, from the nondegeneracy of \( \omega \), there exists a unique vector field \( E \) such that

\[
i_E \omega = \alpha.
\]

The vector field \( E \) has no zeros. From Cartan’s formula, the condition \( i_E \omega = \alpha \) is equivalent to \( L_E \omega = \omega \), i.e., \( E \) is the Liouville vector field of \( \omega \). We have (e.g., see Libermann and Marle [20]):
Lemma 4.1. A regular connected component $M$ of an isoenergetic surface $H^{-1}(\hbar)$ is of contact type with respect to $\alpha$ if and only if the Liouville vector field defined by (4.1) is transverse to $M$.

Proof. Since $i_{E}\omega^{n} = n\alpha \wedge d\alpha^{n-1}$, the kernel of $\alpha \wedge d\alpha^{n-1}$ is the vector bundle generated by $E$. Therefore $\alpha \wedge d\alpha^{n-1}|_{M}$ is a volume form on $M$ at $x$ if and only if $E(x) \notin T_{x}M$.

Let $M$ be of contact type with respect to $\alpha$ and let $Z$ be the corresponding Reeb vector field on $M$:

$$i_{Z}d\alpha|_{M} = 0, \quad \alpha(Z) = 1.$$

Since $Z$ is a section of ker $d\alpha|_{M}$, it is proportional to $X_{H}|_{M}$: $Z = N X_{H}|_{M}$, $N \neq 0$. Consequently, the flow of $Z$ can be seen as a flow of $X_{H}|_{M}$ after a time reparametrization $dt = N d\tau$:

$$\frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} = X_{H}(x) : N(x) = Z(x), \quad x \in M.$$

Alternatively, we can change the Hamiltonian $H$. Extend $N$ to a neighborhood of $M$. Then

$$X_{N(H-h)}(x) = N(x) X_{H}(x), \quad x \in M.$$

Based on the observations (4.2), (4.3), we have the following statement.

Lemma 4.2. The function

$$H_{0} = \frac{H - h}{E(H)}$$

has $M$ as an invariant surface and the Hamiltonian vector field $X_{H_{0}|_{M}}$ is equal to the Reeb field $Z$. If $\rho$ is any smooth function of a real variable, such that $\rho'(\lambda) = 1$, then $\rho(H_{0} + \lambda)$ has the same property. In particular, for $\rho(x) = -1/(4x)$, $\lambda = -1/2$, we get

$$H_{f} = \frac{E(H)}{4h - 4H + 2E(H)}, \quad H_{f}|_{M} = \frac{1}{2}, \quad Z = X_{H_{f}|_{M}}.$$

Proof. According to (2.2), (4.1), we have

$$\alpha(X_{F}) = \omega(E, X_{F}) = dF(E) = E(F), \quad F \in C^{\infty}(P).$$

Thus, $Z = X_{H/E(H)}|_{M}$, i.e., $N = 1/E(H)$. It is clear that $H_{0}|_{M} = 0$, while (4.3) implies $Z = X_{H_{0}|_{M}}$.

Let $\rho$ be a smooth function, such that $\rho'(\lambda) = 1$. Then $\rho(H_{0} + \lambda)|_{M} = \rho(\lambda)$ and $E(\rho(H_{0} + \lambda))|_{M} = \rho'(\lambda) E(H_{0}) = 1$.

4.3. Exact magnetic flows. Consider a natural mechanical system given by the Hamiltonian function (5.1). The canonical 1-form $pdq$ is different from zero outside the zero section $\{p = 0\}$, where we have the standard Liouville vector field $E = \sum_{i} p_{i} \partial / \partial p_{i}$ on $T^{*}Q$.

Since $E(H) = (p, p - \theta)$, a regular hypersurface $M_{h} = H^{-1}(h)$ is of contact type with respect to $pdq$ within a region

$\begin{align*}
M_{0,h} &= \{ (p - \theta, p - \theta) + 2V(q) = 2h, \langle p, p - \theta \rangle \neq 0 \} \\
&= \{ (p - \theta, p - \theta) + 2V(q) = 2h, \langle p, p \rangle \neq 2V + \langle \theta, \theta \rangle - 2h \} \subset T^{*}Q_{h}.
\end{align*}$
Note that the equation $\langle p, p - \theta \rangle = 0$, $\theta \neq 0$, defines an ellipsoid in $T_q^*Q$. Assume

$$h_* = \max_{q \in Q} \left( V(q) + \frac{1}{2} \langle \theta, \theta \rangle \right) < \infty$$

(for example, $h_*$ exists if $Q$ is compact). Then regular hypersurfaces $M = H^{-1}(h)$, for $h > h_*$, are of contact type with respect to $pdq$. The function (4.4) has the form

$$H_j(q, p) = (p - \theta, p)/(4(h - V(q)) + 2(\theta, p)).$$

In particular, if $\theta \equiv 0$, $H_j$ is the Hamiltonian function of the geodesic flow of Jacobi’s metric (2.7) and $M_j$ is the corresponding co-sphere bundle over $Q$.

**Remark 4.1.** The function $N$ in the time reparametrization (1.12) equals $N = 1/E(H)$, $E(H) = \langle p, p - \theta \rangle = 2(h - V(q))_M$. That is,

$$dt = ds/\sqrt{(h - V)},$$

which agrees with Corollary 2.2 (where the time parameter $dt$ of the original system is denoted by $dr$, $ds = ds/\sqrt{(2(h - V))} = ds_j/2(h - V)$, and $ds_j$ is the natural parameter of Jacobi’s metric).

5. Examples: contact flows and integrable systems

5.1. Harmonic oscillators. Consider the simplest integrable system - the system of $n$ independent harmonic oscillators defined by the Hamiltonian function

$$H = \sum_i F_i, \quad F_i = \frac{1}{2}(a_i q_i^2 + b_i p_i^2), \quad 1, \ldots, n,$$

in the standard symplectic linear space $\mathbb{R}^{2n}(q,p)$. Here we suppose that the products $a_i b_i$, $i = 1, \ldots, n$ are positive.

By the use of the first integrals $F_i = c_i$, a generic solution of the equations

$$\dot{q}_i = b_i p_i, \quad \dot{p}_i = -a_i q_i, \quad i = 1, \ldots, n$$

can be written in the form $q_i(t) = \sqrt{\frac{2a_i}{\omega_i}} \cos (\omega_i t + \varphi_i^0)$, $p_i(t) = -\sqrt{\frac{2a_i}{\omega_i}} \sin (\omega_i t + \varphi_i^0)$, $\omega_i = \sqrt{a_i b_i}$, where $\varphi_i^0 \in [0, 2\pi)$ are determined from the initial conditions. Assume

$$A_k = a_{r_1 + \cdots + r_{k-1} + 1} = \cdots = a_{r_1 + \cdots + r_k},$$

$$B_k = b_{r_1 + \cdots + r_{k-1} + 1} = \cdots = b_{r_1 + \cdots + r_k},$$

$$1 \leq k \leq s, \quad r_1 + \cdots + r_s = n, \quad r_0 = 0$$

and that the frequencies $\sqrt{A_1 B_1}, \sqrt{A_2 B_2}, \ldots, \sqrt{A_s B_s}$ are independent over $Q$.

Due to the $U(r_1) \times \cdots \times U(r_s)$-symmetry, the system (5.2) has additional Noether integrals

$$F_{ij}^k = A_k q_i q_j + B_k p_i p_j, \quad G_{ij}^k = q_j p_i - p_j q_i,$$

$$r_1 + \cdots + r_{k-1} + 1 \leq i < j \leq r_1 + \cdots + r_k, \quad k = 1, \ldots, s,$$

implying the non-commutative integrability of the system (25, 22). Generic trajectories fill up densely invariant $s$-dimensional invariant isotropic tori generated by the Hamiltonian vector fields of integrals

$$H_1 = F_1 + \cdots + F_{r_1}, \quad \ldots, \quad H_s = F_{r_1 + \cdots + r_{s-1} + 1} + \cdots + F_{r_1 + \cdots + r_s}.$$
However, if instead of \( p\,dq \), we take
\[
\alpha = \sum_{i=1}^{n} p_i dq_i - \frac{1}{2} d\left( \sum_{i=1}^{n} p_i q_i \right) = \frac{1}{2} \sum_{i} p_i dq_i - q_i dp_i,
\]
then \( d\alpha = d(p\,dq) = dp \wedge dq \) and the only zero of \( \alpha \) is at the origin 0. The corresponding Liouville vector field is
\[
E = \frac{1}{2} \sum_{i} q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i}.
\]

Since \( E(H) = h|_{M_h} \), the quadric \( M_h \) is of contact type with respect to \( \alpha \) and the Reeb flow on \( M_h \) is \( Z = h^{-1}X_H|_{M_h} \).

The above construction provides natural examples of contact structures on quadrics within \( \mathbb{R}^{2n} \) having the integrable Reeb flows with \( s \)-dimensional invariant tori, for any \( s = 1, \ldots, n \). The case \( s = n \) corresponds to contact commutative integrability introduced by Banyaga and Molino [3] (see also [16, 7]), while for \( s < n \) we have contact noncommutative integrability recently proposed in [14].

By taking all parameters to be positive \((a_i, b_i > 0, i = 1, \ldots, n)\), after rescaling of \( M_h \) to a sphere \( S^{2n-1} \), we get \( K \)-contact structures on a sphere \( S^{2n-1} \) given by Yamazaki (see Example 2.3 in [37]). In particular, for \( a_1 = a_2 = \cdots = a_n = b_0 = 1 \), we have the standard contact structure on a sphere \( S^{n-1} = H^{-1}(1/2) \) with the Reeb flow which defines the Hopf fibration (e.g., see [20]).

**Remark 5.1.** A modification of the canonical form \( p\,dq \) given by (5.3) can be applied for starshaped hypersurfaces in \( \mathbb{R}^{2n} \). More generally, consider a regular isoenergetic hypersurface \( M_h = H^{-1} \) in \((T^* Q(q, p), dp \wedge dq)\). It is of contact type if there exist a closed 1-form \( \varphi \) on \( M_h \) such that \( p\,dq|_{X_H|_{M_h}} + \varphi(X_H|_{M_h}) \neq 0 \). Assume \( M_h \) is compact. Then the required 1-form \( \varphi \) exists if and only if \( \int_{M_h} p\,dq(X_H|_{M_h})d\mu \neq 0 \) for every invariant probability measure \( \mu \) with zero homology (see Appendix B in [9]). In particular, for a compact regular energy surface \( M_h = H^{-1}(h) \) in the standard symplectic linear space \((\mathbb{R}^{2n}(q, p), dq \wedge dq)\) we have the following sufficient conditions. Suppose:

(i) \( p\,dq(X_H|_{M_h}) > 0, \) for \( p \neq 0, (q, p) \in M; \)

(ii) \( \text{if } M \cap \{ p = 0 \} \neq \emptyset, \text{then } \frac{\partial}{\partial q} H(q, 0) \neq 0 \text{ at the points } (q, 0) \in M. \)

Then \( M_h \) is of contact type with respect to
\[
\alpha = \sum_{i=1}^{n} p_i dq_i - \epsilon \left( \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} H(q, 0) \right),
\]
for a certain parameter \( \epsilon \) (see [13]).

**5.2. The regularization of Kepler’s problem.** The motion of a particle in the central potential field is described by the Hamiltonian function
\[
H : \mathbb{R}^{2n}_* = \mathbb{R}^{2n} \setminus \{ q = 0 \} \to \mathbb{R}, \quad H(q, p) = \frac{|p|^2}{2} - \frac{c}{|q|},
\]
where \((\cdot, \cdot)\) is the Euclidean scalar product in \( \mathbb{R}^n \). Moser’s regularization of Kepler’s problem (see [23]) can be interpreted in contact terms as follows.

Let \( M_h = \{ H = h \} \subset \mathbb{R}^{2n}_* \) be an isoenergetic hypersurface. Let us interchange the roll of \( q \) and \( p \) and consider the form \( \alpha = -\sum_{i=1}^{n} q_i dp_i \) and the associated
Liouville vector field

\[ E = \sum_{i=1}^{n} q_i \frac{\partial}{\partial q_i}. \]

Since \( E(H) = \gamma/|q| \), \( M_h \) is of contact type with respect to \( \alpha \). According to Lemma 4.2, the Reeb flow on \( M_h \) can be seen as a Hamiltonian flow of

\[ H_0 = (|p|^2 - 2h)|q|/2\gamma - 1. \]

In order to get a smooth Hamiltonian we can take \( F = (H_0 + 1)^2/2 \) (see Lemma 4.2):

\[ F(q, p) = \frac{(|p|^2 - 2h)^2}{8\gamma^2} |q|^2. \]

Then \( F|M_h = \frac{1}{2} \), \( Z = X_F|_{M_h} \) and, moreover, \( X_F \) is defined on the whole \( \mathbb{R}^{2n} \).

Assume \( h < 0 \). The Hamiltonian \( F(q, p) \) can be interpreted as a geodesic flow of the metric proportional to

\[ ds_h^2 = \frac{dp_1^2 + \cdots + dp_n^2}{(2h - |p|^2)^2}. \]

It represents the round sphere metric obtained by a stereographic projection (see Moser [23]). Thus, for \( h < 0 \), there exist a compact contact manifold \( M_h = M_h \cup S^n \) (a co-sphere bundle over \( S^n \)) with a Reeb vector field \( \tilde{Z} \), which is a smooth extension of \( Z \). In particular, for \( n = 2 \), \( M_h \cong \mathbb{RP}^3 \). On \( \mathbb{RP}^3 \) we have a standard contact structure, obtained from the standard contact structure on \( S^3 \) via antipodal mapping.

Note that for \( h > 0 \), the metric \( ds_h^2 \) is defined within the ball of radius \( \sqrt{2h} \) and represents Poincaré’s model of the Lobachevsky space.

The contact regularization of the restricted 3-body problem is given in [1].

### 5.3. The Maupertuis principle and geodesic flows on a sphere

It is well known that the standard metric on a rotational surface and on an ellipsoid have the geodesic flows integrable by means of an integral polynomial in momenta of the first (Clairaut) and the second degree (Jacobi) [2]. A natural question is the existence of a metric on a sphere \( S^2 \) with polynomial integral which can not be reduced to linear or quadratic one. The first examples are given in [5]. Namely, the motion of a rigid body about a fixed point in the presence of the gravitation field admits \( \text{SO}(2) \)-reduction (rotations about the direction of gravitational field) to a natural mechanical system on \( S^2 \). Starting from the integrable Kovalevskaya and Goryachev–Chaplygin cases and taking the corresponding Jacobi’s metrics, we get the metrics with additional integrals of 4-th and 3-th degrees, respectively.

We proceed with a celebrated Neumann system. The Neumann system describes the motion of a particle on a sphere \( \langle q, q \rangle = 1 \) with respect to the quadratic potential \( V(q) = \frac{1}{2}(Aq, q) \), \( A = \text{diag}(a_1, \ldots, a_n) \) (we assume that \( A \) is positive definite). The Hamiltonian of the system is:

\[ H_N(q, p) = \frac{1}{2}(p, p) + \frac{1}{2}(Ax, x). \]

Here, the cotangent bundle of a sphere \( T^*S^{n-1} \) is realized as a submanifold \( P \) of \( \mathbb{R}^{2n} \) given by the constraints

\[ F_1 \equiv \langle q, q \rangle = 1, \quad F_2 \equiv \langle q, p \rangle = 0. \]
The canonical symplectic form on \( P \cong T^*S^{n-1} \) is a restriction of the standard symplectic form \( dp \wedge dq \) to \( P \). Let \( H : \mathbb{R}^{2n} \to \mathbb{R} \). The Hamiltonian vector field \( X_H|_P \) reads
\[
X_H(q,p)|_P = X_H(q,p) - \lambda_1 X_{F_1}(q,p) - \lambda_2 X_{F_2}(q,p), \quad (q,p) \in P,
\]
where the Lagrange multipliers are determined from the condition that \( X_H|_P \) is tangent to \( P \) (e.g., see [24]).

There is a well known Knörrer’s correspondence between the trajectories \( q(t) \) of the Neumann system (5.4) restricted to the zero level set of the integral (5.6)
\[
H(q,p) = \frac{1}{2} \left( \langle A^{-1}q,q \rangle \langle A^{-1}p,p \rangle - \langle A^{-1}q,p \rangle^2 - \langle A^{-1}q,q \rangle \right).
\]
and the geodesic lines on an ellipsoid \( E_n^{n-1} = \{ x \in \mathbb{R}^n \mid \langle x, Ax \rangle = 1 \} \) by the use of a time reparametrization and the Gauss mapping \( q = Ax/|Ax| \) [17].

Recently, by using optimal control techniques, Jurdjevic obtain a similar statement for the flow of the system defined by the Hamiltonian (5.6) [15].

We give the interpretation of Jurdjevic’s time change by the use of Maupertuis principle. Since the potential \( V(q) = -\frac{1}{2} \langle A^{-1}q,q \rangle \) is negative, the isoenergetic surface (5.7)
\[
M_0 = \{ H|_P = 0 \} \subset P \cong T^*S^{n-1}
\]
is of contact type with respect to \( pdq|_P \). The Reeb vector field \( Z \) equals to the Hamiltonian vector field of (5.8)
\[
H_J = \frac{1}{4 \langle A^{-1}q,q \rangle} \left( \langle A^{-1}q,q \rangle \langle A^{-1}p,p \rangle - \langle A^{-1}q,p \rangle^2 \right)|_P
\]
(the Hamiltonian of the corresponding Jacobi’s metric).

The Legendre transformation of a function of the form (5.8) in the presence of constraints (5.5) is given in [11] (see Theorem 2 [11] and interchange the role of the tangent and cotangent bundles of a sphere). As a result, we obtain the Lagrangian function
\[
L(q,\dot{q}) = \frac{1}{2} \langle A\dot{q},\dot{q} \rangle_{S^{n-1}}.
\]

Remarkably, after the linear coordinate transformation \( x = \sqrt{A}q \), \( L(q,\dot{q}) \) becomes the Lagrangian \( L(x,\dot{x}) = \frac{1}{2} \langle \dot{x},\dot{x} \rangle \) of the standard metric on the ellipsoid
\[
E_2^{n-1} = \{ x \in \mathbb{R}^n \mid \langle A^{-1}x,x \rangle = 1 \}.
\]

Recall that the Reeb flow on \( M_0 \) can be seen as a time reparametrization of the original Hamiltonian flow (see Remark 4.1). We can summarize the consideration above in the following statement.

**Proposition 5.1 ([15]).** Under the time substitution
\[
dt = dt/2(A^{-1}q,q)
\]
and the linear transformation \( x = \sqrt{A}q \), the \( q \)-components of the trajectories of the system defined by the Hamiltonian function (5.8) that lie on the zero energy level (5.7), become geodesic lines of the standard metric on the ellipsoid \( E_2^{n-1} \).

Further interesting examples of transformations related to the Maupertuis Principle, which map a given integrable system into another one are given in [32].
Acknowledgments. I am grateful to Professor Jurdjevic for providing a preprint of the paper [15] and to the referee for valuable remarks. This research was supported by the Serbian Ministry of Science Project 174020, Geometry and Topology of Manifolds, Classical Mechanics and Integrable Dynamical Systems.

References

[1] P. Albers, U. Frauenfelder, O. van Koert, G. P. Paternain, The contact geometry of the restricted 3-body problem, Comm. Pure Appl. Math. 65 (2012), no. 2, 229–263. arXiv:1010.2140v1 [math.SG].
[2] V. I. Arnol’d, Математические методы классической механики, Москва, Наука 1974 (Russian). English translation: V. I. Arnol’d, Mathematical methods of classical mechanics. Springer-Verlag, 1978.
[3] A. Banyaga, P. Molino, Геометрия дес контакт полностью интегрируемы тороидального типа, Семинар Гастон Дарбуа, Монтпелье (1991-92), 1-25 (French). English translation: A. Banyaga, P. Molino, Complete Integrability in Contact Geometry, Penn State preprint PM 197, 1996.
[4] A. Билимович, Рациональная механика. Том 2 (Механика системы), Научная книга, Београд 1951 (Serbian).
[5] A. В. Болсинов, В. В. Коллов, А. Т. Фоменко, Принцип Мопертюи и геодезические потоки на сфере, возникающие из интегрируемых случаев динамики твердого тела, Успехи Мат. Наг. 50 (1995), н. 3, 3-32 (Russian); English translation: A. V. Bolsinov, V. V. Kollov, A. T. Fomenko, The Maupertuis principle and geodesic flow on the sphere arising from integrable cases in the dynamic of a rigid body, Russian Math. Surv. 50 (1995).
[6] A. V. Bolsinov, B. Jovanović, Magnetic Flows on Homogeneous Spaces, Com. Mat. Helv., 83 (2008), no. 3, 679700, arXiv: math-ph/0609005.
[7] C. P. Boyer, Completely integrable contact Hamiltonian systems and toric contact structures on $S^2 \times S^3$, SIGMA 7 (2011), 058, 22 pages, arXiv: 1101.5587 [math.SG].
[8] H. Cendra, J. E. Marsden, S. Pekarsky, T. S. Ratiu, Variational principles for Lie-Poisson and Hamilton–Poincaré equations, Moscow Math. J. 3 (2003), no. 3, 833-867.
[9] G. Contreras, L. Macarini, G.P. Paternain, Periodic orbits for exact magnetic flows on surfaces, Int. Math. Res. Not. 8 (2004) 361-387.
[10] B. Драговић, Д. Милинковић, Анализа на многоструктурностима, Математички факултет, Београд (Serbian).
[11] Y. Fedorov, B. Jovanović, Hamiltonization of the Generalized Veselova LR System, Reg. Chaot. Dyn. 14 (2009), no. 4-5, 495–505.
[12] P. A. Griffits, Exterior differential systems and the calculus of variations, Progress in Mathematics, 25. Birkhäuser, Boston, Mass., 1983.
[13] H. Hofer, E. Zehnder, Symplectic Invariants and Hamiltonian Dynamics, Birkhäuser 1994.
[14] B. Jovanović, Noncommutative integrability and action angle variables in contact geometry, to appear in J. Sympl. Geometry, arXiv:1103.3611 [math.SG].
[15] V. Jurdjevic, Optimal control on Lie groups and integrable Hamiltonian systems, Regul. Chaotic Dyn. 16 (2011), no. 5, 514-535.
[16] B. Khesin, S. Tabachnikov, Contact complete integrability, Regular and Chaotic Dynamics, Special Issue: Valery Vasilievich Kozlov 60 (2010), arXiv:0910.0375 [math.SG].
[17] H. Knörrer, Geodesics on quadrics and a mechanical problem of C. Neumann, J. Reine Angew. Math. 334 (1982), 69–78.
[18] S. Kobayashi, Principal fibre bundles with the 1-dimensional toroidal group Tohoku Math. J. (2)9 (1956), 29-45.
[19] B. В. Коллов, Вариационное исчисление в целом и классическая механика, Успехи Мат. Наук 40 (1982), н. 2 (242), 33-60 (Russian).
[20] P. Libermann, C. Marle, Symplectic Geometry, Analytical Mechanics, Riedel, Dordrecht, 1987.
[21] J. E. Marsden, T. S. Ratiu, Introduction to mechanics and symmetry, 2nd edition, Springer 1999.
THE MAUPERTUIS PRINCIPLE

[22] A. S. Milchenko, A. T. Fomenko, Обобщенный метод Лиувилля интегрирования hamiltonovskikh систем, Функт. анализ и его прилож. 12(2) (1978), 46–56 (Russian); English translation: A. S. Mishchenko, A. T. Fomenko, Generalized Liouville method of integration of Hamiltonian systems, Funkts. Anal. Prilozh. 12, No.2, 46-56 (1978) Funkt. Anal. Appl. 12, 113–121 (1978)

[23] J. Moser, Regularization of Kepler’s Problem and the Averaging Method on a Manifold, Comm. Pure and Appl. Math. 23 (1970) 606–636.

[24] J. Moser, Geometry of quadric and spectral theory, In: Chern Symposium 1979, Berlin–Heidelberg–New York, 147–188, 1980.

[25] N. N. Nekhoroshev, Переменные действия–угол и их обобщения, Tr. Mosk. Mat. O.-va. 26 (1972), 181–198 (Russian). English translation: N. N. Nekhoroshev, Action-angle variables and their generalization, Trans. Mosc. Math. Soc. 26, 180–198 (1972).

[26] S. P. Novikov, Гамильтонов формализм и многочленный аналог теории Морса, Успехи Мат. Наук 37 (1982), N. 5(227), 3-49 (Russian).

[27] H. Poincaré, Les méthodes nouvelles de la mécanique céleste III. Invariant intégraux. Solutions périodiques du deuxième genre. Solutions doublement asymptotiques, Paris, Gauthier-Villars, 1899 (French).

[28] L. S. Polak (ед.), Вариационные принципы механики (Сборник статей классиков науки), Физматгиз, Москва (1959) (Russian).

[29] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, Comm. Pure. Appl. Math. 31 (1978), 157-184.

[30] G. Romano, R. Barretta, A. Barretta, On Maupertuis principle in dynamics, Rep. Math. Phys. 63 (2009) No. 3, 331-346.

[31] A. I. Taimanov, Periodic magnetic geodesics on almost every energy level via variational methods, Regular and Chaotic Dynamics 15 (2010), 598-605. arXiv:1001.2077

[32] A. V. Tsiganov, The Maupertuis principle and canonical transformations of the extended phase space, J. Nonlinear Math. Phys. 8 (2001), no. 1, 157-182.

[33] G. M. Tuynman, Un principe variationnel pour les variétés symplectiques, C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), no. 3, 339-342 (French).

[34] A. Weinstein, Bifurcations and Hamiltonian’s principle, Math. Z. 159 (1978) 235-248.

[35] A. Weinstein, On the hypotheses of Rabinowitz’ periodic orbit theorems, J. Differential Equations 33 (1979), no. 3, 353-358.

[36] E.T. Whittaker, A treatise on the analytic dynamics of particles and rigid bodies, Cambridge, 1904.

[37] T. Yamazaki, A construction of K-contact manifolds by a fiber join, Tohoku Math. J. 51 (1999) 433-446.

Mathematical Institute SANU, Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11000 Belgrade, Serbia
E-mail address: bozaj@mi.sanu.ac.rs