List coloring of matroids and base exchange properties

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Abstract. A coloring of a matroid is an assignment of colors to the elements of its ground set. We restrict to proper colorings – those for which elements of the same color form an independent set. Seymour proved that a $k$-colorable matroid is also colorable from any lists of size $k$.

We generalize this theorem to the case when lists have still fixed sizes, but not necessarily equal. For any fixed size of lists assignment $\ell$, we prove that, if a matroid is colorable from a particular lists of size $\ell$, then it is colorable from any lists of size $\ell$. This gives an explicit necessary and sufficient condition for a matroid to be list colorable from any lists of a fixed size.

As an application, we show how to use our condition to derive several base exchange properties.

1. Introduction

Let $M$ be a matroid on a ground set $E$ (we refer the reader to [5] for a background of matroid theory). A coloring of $M$ is an assignment of colors to the elements of $E$. In analogy to graph theory we say that a coloring is proper if elements of the same color form an independent set in the matroid. Via this correspondence one can define for matroids all chromatic parameters studied for graphs.

The chromatic number of a loopless matroid $M$, denoted by $\chi(M)$, is the minimum number of colors in a proper coloring of $M$. For instance, if $M$ is a graphic matroid obtained from a graph $G$, then $\chi(M)$ is the least number of colors needed to color edges of $G$ so that no cycle is monochromatic. This number is known as the arboricity of the underlying graph $G$.

Key words and phrases. Matroid coloring, List coloring, Acyclic coloring, Base exchange property.

Research partially supported by the Polish National Science Centre grant no. 2011/03/N/ST1/02918. The paper was completed during author’s stay at Freie Universität Berlin in the frame of Polish Ministry “Mobilność Plus” program.
For matroids the chromatic number can be easily expressed in terms of rank function. Extending a theorem of Nash-Williams [8] for graph arboricity, Edmonds [2] proved the following formula
\[
\chi(M) = \max_{\emptyset \neq A \subseteq E} \left\lceil \frac{|A|}{r(A)} \right\rceil,
\]
where \(r\) is the rank function of a loopless matroid \(M\) on a ground set \(E\). Some game-theoretic versions of the chromatic number of a graph were already studied for matroids, see [6, 7].

In this note we study list coloring of matroids. The concept of list coloring was initiated for graphs by Vizing [10], and independently by Erdős, Rubin and Taylor [3]. Let us recall its definition in the matroid setting.

Suppose each element \(e\) of the ground set \(E\) of a matroid \(M\) is assigned with a set (a list) of colors \(L(e)\). By the size of list assignment (or simply lists) \(L\) we mean the function \(\ell\) satisfying \(\ell(e) = |L(e)|\) for each \(e \in E\). We say that matroid \(M\) is colorable from lists \(L\) if there exists a proper coloring of \(M\), such that each element receives a color from its list. The list chromatic number (sometimes called choice number) of a loopless matroid \(M\), denoted by \(\text{ch}(M)\), is the minimum number \(k\), such that \(M\) is colorable from any lists of size at least \(k\).

Clearly, \(\text{ch}(M) \geq \chi(M)\). For graphs in general, the inequality between corresponding parameters is strict (\(\text{ch}(G)\) is not even bounded by a function of \(\chi(G)\)). A celebrated result of Galvin [4] asserts that for the line graph of a bipartite graph there is an equality. Surprisingly, Seymour [9] proved that actually equality holds for all matroids.

**Theorem 1.** For every loopless matroid \(M\) there is an equality \(\text{ch}(M) = \chi(M)\).

Seymour’s theorem can be rephrased by saying that the following conditions are equivalent:

1. matroid \(M\) is colorable from the lists \(L(e) = \{1, \ldots, k\}\),
2. matroid \(M\) is colorable from any lists of size \(k\).

Our main result is a generalization of Seymour’s Theorem 1 to the setting, where sizes of lists are still fixed, but not necessarily equal.

**Theorem 2.** Let \(M\) be a matroid and let \(\ell\) be a lists size function. Then the following conditions are equivalent:

1. matroid \(M\) is colorable from the lists \(L_{\ell}(e) = \{1, \ldots, \ell(e)\}\),
2. matroid \(M\) is colorable from any lists of size \(\ell\).

As a corollary we get a strengthening of Seymour’s Theorem 1. Namely, a \(k\)-colorable matroid is also colorable from any lists of fixed size varying between 1 and \(k\), and average size at most \(\frac{k+1}{2}\).

**Corollary 3.** Let \(M\) be a \(k\)-colorable matroid, and let \(I_1, \ldots, I_k\) be a partition of its ground set into independent sets (color classes). Define lists size function by \(\ell(e) = i\) for elements \(e \in I_i\). Then matroid \(M\) is colorable from any lists of size \(\ell\).

In the last section we make a link between list coloring and base exchange properties. We use our theorem as a tool to obtain easily several such properties. The idea is to choose suitable lists, such that existence of a proper coloring guarantees a particular exchange property. The crucial point is that lists may have different sizes (ex. if size of a list is 1, then a color is already determined).
2. Proof of Theorem 2

Proof. Clearly, condition (1) follows from (2). We argue the opposite implication. Let \( L \) be a fixed list assignment of size \( \ell \). Without loss of generality we can assume that all lists \( L(e) \) are subsets of a finite set of integers \( \{1, \ldots, d\} \). Let us denote \( Q_i = \{e \in E : i \in L(e)\} \), and respectively \( Q_i^\ell \) for lists \( L_i \).

Consider matroids \( M_1, \ldots, M_d \), with \( M_i \) equal to the restriction \( M|_{Q_i} \) of \( M \) to the set \( Q_i \) (with the ground set trivially extended to \( E \)). It is straightforward that a proper coloring from lists \( L \) exists if and only if it is possible to partition the ground set \( E \) into subsets \( I_1, \ldots, I_d \) with \( I_i \) independent in the matroid \( M_i \) (\( I_i \) is the color class of \( i \)). By the Matroid Union Theorem (see [5]) such a partition exists if and only if for every subset \( A \subseteq E \) there is an inequality
\[
\begin{equation}
|A \cap Q_1| + \cdots + |A \cap Q_d| \geq |A|.
\end{equation}
\]

Analogously, a proper coloring from lists \( L_i \) exists if and only if for every subset \( A \subseteq E \) there is an inequality
\[
\begin{equation}
|A \cap Q_1^\ell| + \cdots + |A \cap Q_d^\ell| \geq |A|.
\end{equation}
\]

Thus, to prove that condition (1) implies (2) it is enough to show that for every subset \( A \subseteq E \) there is an inequality
\[
(2.1) \quad \begin{equation}
|A \cap Q_1| + \cdots + |A \cap Q_d| \geq |A|.
\end{equation}

Notice that \( \bigcup_i Q_i \) and \( \bigcup_i Q_i^\ell \) are equal as multisets, since both \( L \) and \( L_i \) are list assignments of size \( \ell \) (each \( e \in E \) belongs to each of the unions exactly \( \ell(e) \) times). We will show that the inequality (2.1) is satisfied for any sets \( Q_i \) satisfying \( \bigcup_i Q_i = \bigcup_i Q_i^\ell \) as multisets. The proof is by induction on the number of pairs of sets \( Q_k, Q_l \) such that \( Q_k \) and \( Q_l \) are incomparable in the inclusion order (none is contained in the other).

If the number of such pairs is zero, then \( Q_i \) are linearly ordered by inclusion. Let us reorder them in such a way that \( Q_1 \supset Q_2 \supset \cdots \supset Q_d \). Then the equality \( \bigcup_i Q_i = \bigcup_i Q_i^\ell \) implies that \( Q_1 = Q_1^\ell \), \( Q_2 = Q_2^\ell \), \ldots, \( Q_d = Q_d^\ell \), so the inequality (2.1) is in fact an equality.

Suppose now that there exists a pair of sets \( Q_k, Q_l \) incomparable in the inclusion order. Replace them in the family \( \{Q_i\}_1^{1, \ldots, d} \) by sets \( Q_k \cup Q_l \) and \( Q_k \cap Q_l \) to obtain a family \( \{Q_i^\ell\}_1^{1, \ldots, d} \). Since \( Q_k \cup Q_l = (Q_k \cup Q_l) \cup (Q_k \cap Q_l) \) as multisets, the sets \( Q_i^\ell \) also satisfy the multiset equality \( \bigcup_i Q_i^\ell = \bigcup_i Q_i^\ell \). Moreover, the number of pairs incomparable in the inclusion order among \( Q_i^\ell \) is lower than among sets \( Q_i \). By the inductive assumption, the inequality (2.1) holds for sets \( Q_i^\ell \). Combining it with submodularity of the rank function
\[
r(A \cap Q_k) + r(A \cap Q_l) \geq r(A \cap (Q_k \cup Q_l)) + r(A \cap (Q_k \cap Q_l)),
\]
we get inequality (2.1) for sets \( Q_i \). This completes the inductive step. \( \square \)

3. Applications

A family \( \mathcal{B} \) of subsets of a finite set \( E \), just from the definition, forms a set of bases of a matroid if it is non-empty, and if for every \( B_1, B_2 \in \mathcal{B} \) and \( e \in B_1 \setminus B_2 \) there exists \( f \in B_2 \setminus B_1 \), such that \( B_1 \cup f \setminus e \in \mathcal{B} \).

In this case a stronger property holds. For every bases \( B_1, B_2 \) and \( e \in B_1 \setminus B_2 \) there exists \( f \in B_2 \setminus B_1 \), such that both \( B_1 \cup f \setminus e \) and \( B_2 \cup e \setminus f \) are bases. It is called symmetric exchange property, and was discovered by Brualdi [1].
Surprisingly, even more is true. One can exchange symmetrically not only single elements, but also subsets. This property is known as multiple symmetric exchange. We demonstrate usefulness of Theorem 2 by using it as a tool to give easy proofs of multiple symmetric exchange property and its generalizations.

**Proposition 4.** Let $B_1$ and $B_2$ be bases of a matroid $M$. Then for every $A_1 \subset B_1$ there exists $A_2 \subset B_2$, such that $(B_1 \setminus A_1) \cup A_2$ and $(B_2 \setminus A_2) \cup A_1$ are both bases.

**Proof.** Observe that by adding parallel elements to the elements of the intersection $B_1 \cap B_2$ we can restrict to the case when bases $B_1$ and $B_2$ are disjoint.

When bases $B_1, B_2$ are disjoint, then restrict matroid $M$ to their union. Let $\ell$ be a lists size function such that $\ell|_{B_1} \equiv 1$, and $\ell|_{B_2} \equiv 2$. It is easy to check that condition (1) of Theorem 2 is satisfied.

Let $L$ be a list assignment of size $\ell$ which assigns list $\{1\}$ to elements of $A_1$, list $\{2\}$ to elements of $B_1 \setminus A_1$, and list $\{1, 2\}$ to elements of $B_2$. By Theorem 2 there exists a proper coloring from these lists. Denote by $C_1$ elements colored with 1, and by $C_2$ those colored with 2. Now $A_2 = C_2 \cap B_2$ is a good choice, since sets $(B_1 \setminus A_1) \cup A_2 = C_2$ and $(B_2 \setminus A_2) \cup A_1 = C_1$ are independent. \qed

Multiple symmetric exchange property can be slightly generalized. Instead of having a partition of one of bases into two parts we can have an arbitrary partition of it. We prove that for any such partition there exists a partition of the second basis which is consistent in two different ways.

**Proposition 5.** Let $A$ and $B$ be bases of a matroid $M$. Then for every partition $B_1 \sqcup \cdots \sqcup B_k = B$ there exists a partition $A_1 \sqcup \cdots \sqcup A_k = A$, such that $(B \setminus B_i) \cup A_i$ are bases for all $1 \leq i \leq k$.

**Proof.** As before we can assume that bases $A, B$ are disjoint, and restrict to their union. Consider matroid $M'$ equal to $M$, where elements of $B$ have $k - 1$ parallel copies. So the ground set of $M'$ equals to $B^1 \sqcup \cdots \sqcup B^{k-1} \sqcup A$. Let $\ell$ be a lists size function such that $\ell|_{B'} \equiv k - 1$, and $\ell|_{A'} \equiv k$. It is easy to check that condition (1) of Theorem 2 is satisfied.

Let $L$ be a list assignment of size $\ell$ which assigns list $\{1, \ldots, k\} \setminus \{i\}$ to elements of $B_i$ and all its copies, and list $\{1, \ldots, k\}$ to elements of $A$. By Theorem 2 there exists a proper coloring from these lists. Denote by $C_i$ elements colored with $i$. Now $A_i = C_i \cap A$ is a good partition, since sets $(B \setminus B_i) \cup A_i$ are independent. \qed

**Proposition 6.** Let $A$ and $B$ be bases of a matroid $M$. Then for every partition $B_1 \sqcup \cdots \sqcup B_k = B$ there exists a partition $A_1 \sqcup \cdots \sqcup A_k = A$, such that $(A \setminus A_i) \cup B_i$ are all bases for $1 \leq i \leq k$.

**Proof.** We assume that bases $A, B$ are disjoint, and restrict to their union. Consider matroid $M'$ equal to $M$, where elements of $A$ have $k - 1$ parallel copies. So the ground set of $M'$ equals to $A^1 \sqcup \cdots \sqcup A^{k-1} \sqcup B$. Let $\ell$ be a lists size function such that $\ell|_{A'} \equiv k$, and $\ell|_{B} \equiv 1$. It is easy to check that condition (1) of Theorem 2 is satisfied.

Let $L$ be a list assignment of size $\ell$ which assigns list $\{i\}$ to elements of $B_i$, and list $\{1, \ldots, k\}$ to elements of $A$ and all its copies. By Theorem 2 there exists a proper coloring from these lists. Denote by $C_i$ elements colored with $i$. Now let $A_i$ contain all $a \in A$ such that no copy of $a$ is colored with $i$. Then sets $(A \setminus A_i) \cup B_i$ are independent, so it is a good partition. \qed
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