Construction of Lorenz Cone with Invariant Cone Using Dikin Ellipsoid for Dynamical Systems

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Abstract
In this paper, some special Lorenz cones are constructed using Dikin ellipsoid and some hyperplane. We also study the structure of the constructed cones, especially the eigenvalues structure of the related matrix in the formula of ellipsoid. These novel Lorenz cones which locate in positive orthant by its construction are potential candidates to design invariant cone for a given dynamical system. It provides more flexibility for practitioner to choose more cones in the application for system stability analysis.

Keywords: Linear Systems, Invariant Set, Dikin Ellipsoid, Lorenz Cone.

1. Introduction

The history of the analysis of invariant properties for dynamical system goes back 1890s, when mathematician Lyapunov firstly introduced his theory on ordinary differential equations. This theory was later on named by Lyapunov theory or stability theory. This theory proves that the stability of a dynamical system, i.e. all trajectories will approach a fixed point as time is going to infinity, can be transformed to analyze the properties of a function that is named Lyapunov candidate function. Stability theory is related to the concept of invariant set. Blanchini [5] provides an excellent survey paper about invariance of dynamical system. Positively invariant set is an important concept that is widely used in many areas e.g., control theory, electronic systems, economics, etc. e.g., see [6, 12, 14]. Given a set and a dynamical system, verifying whether the set is an invariant set of the given system is an interesting topic in this field. A general equivalent condition is given by Nagumo, e.g., [13]. The explicit conditions for linear system and some common sets are derived by Hováth, et [3, 4, 9, 10]. The discrete and continuous system are usually considered separately for invariant sets. Preserving the invariance from a continuous system to a discrete system by using certain discretization methods is studied by Hováth et [8].

In this paper, we construct some special Lorenz cone using Dikin ellipsoid and hyperplane, and the structures of the constructed cones are studied. The novelty of this method...
is that linking the mathematical optimization tools to the invariant set. The motivation of the construction is to design more potential invariant cone within positive orthant, which is usually a common requirement in practical application.

**Notation and Conventions.** We use the following notation and conventions to avoid unnecessary repetitions

- The inertia of a matrix is denoted by \( \text{inertia}(Q) = \{a, b, c\} \) that indicates the number of positive, zero, and negative eigenvalues of \( Q \), respectively.
- The basis in \( \mathbb{R}^n \) is denoted by \( e_1 = \{1, 0, \ldots, 0\}, e_2 = \{0, 1, \ldots, 0\}, \ldots, e_n = \{0, 0, \ldots, 1\} \). And we let \( e = (1, 1, \ldots, 1) \).
- The aim of using \( x[k] \) to indicate the discrete state variable is to distinguish with the \( k \)-th coordinate, denoted by \( x_k \), of variable \( x \).

2. Preliminaries

2.1. Invariant Sets

In this paper, the linear discrete and continuous systems are described as follows:

\[
\begin{align*}
    x_{k+1} &= A_d x_k, \quad (1) \\
    \dot{x}(t) &= A_c x(t), \quad (2)
\end{align*}
\]

where \( x_k, x_{k+1}, x(t) \in \mathbb{R}^n \) are the state variables, and \( A_d, A_c \in \mathbb{R}^{n \times n} \) are constant coefficient matrices.

Followed on the linear systems above, the invariant sets for the corresponding discrete and continuous forms are introduced.

**Definition 2.1.** A set \( S \in \mathbb{R}^n \) is called an **invariant set** for the discrete system (1) if \( x_k \in S \) implies \( x_{k+1} \in S \), for all \( k \in \mathbb{N} \). A set \( S \in \mathbb{R}^n \) is called an **invariant set** for the continuous system (2) if \( x(0) \in S \) implies that \( x(t) \in S \), for all \( t \geq 0 \).

In fact, the set \( S \) in Definition 2.1 is usually refereed to as **positively invariant set** in literatures, as we can see that it only considers the forward time or nonnegative time. Since we only consider positively invariant set in this paper, we call it invariant set for simplicity.

**Definition 2.2.** An operator \( A \) is called **invariant** on a set \( S \in \mathbb{R}^n \) if \( AS \subset S \).

An immediate connection between Definition 2.1 and 2.2 is that a set \( S \) is an invariant set for the linear discrete system (1) if and only if \( A_d \) is invariant on \( S \); \( S \) is an invariant set for the linear continuous system (2) if and only if \( e^{A_c t} \) is invariant on \( S \).

\[^2\text{Here recall that } e^{At} = \sum_{k=1}^{\infty} \frac{(At)^k}{k!}.\]
2.2. Hyperplane, Ellipsoid, Lorenz Cone, and Dikin Ellipsoid

In this subsection, the definitions and formulas of some common types in \( \mathbb{R}^n \), namely, hyperplane, ellipsoid, Lorenz cone, and Dikin ellipsoid are introduced.

**Definition 2.3.** A hyperplane, denoted by \( \mathcal{H} \in \mathbb{R}^n \), is represented as either

\[
\mathcal{H} = \mathcal{H}(a, \alpha) = \{ x \in \mathbb{R}^n | a^T x = \alpha, \alpha \in \mathbb{R} \},
\]

or equivalently

\[
\mathcal{H} = \mathcal{H}(x_0, H) = \{ x \in \mathbb{R}^n | x = x_0 + Hz, z \in \mathbb{R}^{n-1} \},
\]

where \( x_0 \) is a point in \( \mathcal{H} \) and \( H \) consists of all basis, denoted by \( h_1, h_2, \ldots, h_{n-1} \), of the complementary space of \( a \), i.e., \( H = [h_1, h_2, \ldots, h_{n-1}] \in \mathbb{R}^{n \times n-1}, a^T H = 0, H^T a = 0 \), and span\( \{a, H\} = \mathbb{R}^n \).

The vector \( a \) in (3) is called the normal vector of the hyperplane \( \mathcal{H} \). The matrix \( H \) in (4) is called the complementary matrix of the vector \( a \). Moreover, if \( h_1, h_2, \ldots, h_{n-1} \) are mutually orthonormal, i.e., \( h_i^T h_j = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta, then we have \( H^T H = I_{n-1} \), and we call \( H \) the orthonormal complementary matrix of the vector \( a \). Formula (3) is straightforward and can be seen in many literatures. We will use Formula (3) in this paper since it considers a hyperplane in the affine plane.

**Definition 2.4.** An ellipsoid, denoted by \( \mathcal{E} \in \mathbb{R}^n \), is represented as

\[
\mathcal{E} = \mathcal{E}(Q, p, \rho) = \{ x \in \mathbb{R}^n | x^T Q x + 2p^T x + \rho \leq 1 \},
\]

where \( Q > 0 \), and \( \rho = p^T Q^{-1} p \). A standard ellipsoid, denoted by \( \mathcal{E}^* \in \mathbb{R}^n \), is represented as

\[
\mathcal{E}^* = \mathcal{E}(\tilde{Q}, 0, 0) = \{ x \in \mathbb{R}^n | a_1 x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2 \leq 1 \} = \{ x \in \mathbb{R}^n | x^T \tilde{Q} x \leq 1 \}
\]

where \( \tilde{Q} = \text{diag} \{a_1, a_2, \ldots, a_n\} \), with \( a_i > 0 \), for \( i = 1, 2, \ldots, n \).

**Definition 2.5.** A Lorenz cone, denoted by \( \mathcal{C}_L \in \mathbb{R}^n \), is represented as

\[
\mathcal{C}_L = \mathcal{C}_L(Q, p, \rho) = \{ x \in \mathbb{R}^n | x^T Q x + 2p^T x + \rho \leq 0 \},
\]

where \( Q \) is a symmetric matrix with inertia\( (Q) = \{n-1, 0, 1\} \), \( p \in \mathbb{R}^n \), and \( \rho = p^T Q^{-1} p \). A standard Lorenz cone, denoted by \( \mathcal{C}_L^* \in \mathbb{R}^n \), is represented as

\[
\mathcal{C}_L^* = \{ x \in \mathbb{R}^n | x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq x_n^2, x_n \geq 0 \} = \{ x \in \mathbb{R}^n | x^T \tilde{I} x \leq 0, x^T \tilde{I} e_n \leq 0 \}
\]

where \( \tilde{I} = \text{diag} \{1, 1, \ldots, 1, -1\} \), and \( e_n = (0, \ldots, 0, 1)^T \).

**Remark 2.6.** The center of \( \mathcal{E} \) in the form of (3) is \( -Q^{-1} p \). The vertex of \( \mathcal{C}_L \) is \( -Q^{-1} p \), and the axis of \( \mathcal{C}_L \) in the form of (7) is \( \{ x \in \mathbb{R}^n | x = -Q^{-1} p + \alpha e_n \} \), where \( \alpha \in \mathbb{R} \).
In fact, one can prove that each Lorenz cone can be mapped into a standard Lorenz cone by certain transformation, e.g., \([6]\). We notice that a Lorenz cone in the form of \((7)\) consists of two branches, one of which is centrosymmetric to the other one with respect to the vertex. A standard Lorenz cone \(C^*_L\) in the form of \((8)\) is a convex set and a self-dual cone, i.e., the dual cone \(C_L^*\) is coincidence with itself. Also, the formula of \(C^*_L \cup (-C^*_L)\) is given as \(\{x \in \mathbb{R}^n | x^T \bar{I} x \leq 0\}\).

The relationships between general and standard ellipsoids and between general and standard Lorenz cones are given in the following lemma.

**Lemma 2.7.** There exists two nonsingular matrices \(P, \tilde{P}\), such that
\[
\tilde{P}^{-1} \mathcal{E}^* = P^{-1} \mathcal{E} + (QP)^{-1} p,
\]
where \(\mathcal{E}\) and \(\mathcal{E}^*\) are defined as \((5)\) and \((6)\), respectively. There exists a nonsingular matrix \(\tilde{P}\), such that
\[
C_L^* \cup (-C_L^*) = \tilde{P}^{-1} C_L + (QP)^{-1} p,
\]
where \(C_L^*\) and \(C_L^*\) are given as \((7)\) and \((8)\), respectively.

**Proof.** We only present the proof of \((9)\), the proof of \((10)\) is analogous. Since \(Q \succ 0\), there exists an orthogonal matrix \(U\) consisting of all the eigenvectors \(\{\lambda_i\}\) of \(Q\), such that \(U^T QU = \text{diag}\{\lambda_1, \ldots, \lambda_n\}\). Denoting \(Q_1 = \text{diag}\{\frac{1}{\sqrt{\lambda_1}}, \ldots, \frac{1}{\sqrt{\lambda_n}}\}\), we have \(Q_1^T U^T QU Q_1 = \bar{I}\). We let \(P = UQ_1\) and \(\tilde{P} = \text{diag}\{\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}\}\), both of which are nonsingular, then \((5)\) and \((6)\) can be respectively rewritten as
\[
\mathcal{E} = \{x \in \mathbb{R}^n | (P^{-1} x + \bar{I} P^T p)^T (P^{-1} x + \bar{I} P^T p) \leq 1\}, \quad \text{and} \quad \mathcal{E}^* = \{x \in \mathbb{R}^n | (\tilde{P}^{-1} x)^T (\tilde{P}^{-1} x) \leq 1\}.
\]
Noting that \(P \bar{I} P^T = Q^{-1}\) implies \(\bar{I} P^T = (QP)^{-1}\), we deduce \((9)\) immediately.

**Definition 2.8.** \([3, 10]\) A Dikin Ellipsoid, denoted by \(\mathcal{E}_D \in \mathbb{R}^n\), is represented as
\[
\mathcal{E}_D = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{(x - c_i)^2}{c_i^2} \leq 1 \right\} = \{x \in \mathbb{R}^n | (x - c)^T C^{-2} (x - c) \leq 1\},
\]
where \(c = (c_1, c_2, \ldots, c_n)^T\), \(C = \text{diag}\{c_1, c_2, \ldots, c_n\}\), and \(c_i > 0\), for \(i = 1, 2, \ldots, n\).

The point \(c\) is the center of Dikin ellipsoid according to \((11)\). In fact, the ellipsoid \((11)\) was introduced by Dikin and widely used in designing some mathematical optimization algorithms, e.g., affine scaling interior point methods \([2, 16]\). A common property of every Dikin ellipsoid is that it is constantly in the positive octant of \(\mathbb{R}^n\) including its boundary. This is a key property to design mathematical optimization algorithms.

**Lemma 2.9.** \([3, 10]\) Assume the Dikin ellipsoid \(\mathcal{E}_D\) is given as \((11)\) and let \(x \in \mathcal{E}_D\), then \(x \geq 0\).

\(^3\)A dual cone of a cone \(C\) is defined as \(\{y \in \mathbb{R}^n | y^T x \geq 0, \text{ for all } x \in C\}\).
Proof. For every $i$, we have \( \frac{(x_i-c_i)^2}{c_i^2} \leq \sum_{i=1}^{n} \frac{(x_i-c_i)^2}{c_i^2} \leq 1. \) This yields $-c_i \leq x_i - c_i \leq c_i$, i.e., $0 \leq x_i \leq 2c_i$.

Let us consider the hyperplane $H$ given as (3) with the normal vector $a$, and assume $H$ intersects through the center of the Dikin ellipsoid $E_D$ given as (11), then we have
\[
H = H(a, a^T c) = \{ x \in \mathbb{R}^n | a^T x = a^T c \},
\]
and the intersection $H \cap E_D$ is also an ellipsoid.

**Lemma 2.10.** Let a hyperplane $H$ and a Dikin ellipsoid $E_D$ be given as (12) and (11), respectively. Then $H \cap E_D = \{ x \in \mathbb{R}^n | x = c + Hz \}$, where $z \in \mathbb{R}^{n-1}$ satisfies $z^T H^T C^{-2} Hz \leq 1$, and $H$ is a complementary matrix of the vector $a$.

**Proof.** According to the second formula of $H$ given as (4), and let $x_0 = c$, we have $x = c + Hz$. Substituting $x = c + Hz$ into (11), this lemma is immediate.

**Definition 2.11.** [17] A matrix $D \in \mathbb{R}^{n \times n}$ is called an arrowhead matrix if it has the following form
\[
D = \begin{bmatrix}
\alpha & p \\
p^T & B
\end{bmatrix},
\]
where $\alpha \in \mathbb{R}$, $p \in \mathbb{R}^{n-1}$, and $B = \text{diag}\{b_1, b_2, \ldots, b_{n-1}\}$. Here we assume $b_1 \geq b_2 \geq \cdots \geq b_{n-1}$.

**Lemma 2.12.** [17] The following properties of the arrowhead matrix $D$ given as (13) are true:

1. The characteristic polynomial of $D$ is
\[
\det(\lambda I - D) = (\lambda - \alpha) \prod_{k=1}^{n-1} \left( \lambda - b_k \right) - \sum_{j=1}^{n-1} |p_j|^2 \prod_{k=1, k \neq j}^{n-1} \left( \lambda - b_k \right).
\]

2. The eigenvalues of $D$ are real and satisfying the following condition
\[
\lambda_1 \geq b_1 \geq \lambda_2 \geq b_2 \geq \cdots \geq b_{n-1} \geq \lambda_n.
\]

3. **Construction of Novel Lorenz Cones**

In this section, we will construct some novel standard Lorenz cones and derive the corresponding explicit formulas for some special cases. In particular, these novel standard Lorenz cones are constructed by using a Dikin ellipsoid as its base or using the intersection of a Dikin ellipsoid and a hyperplane as its base. Since the Lorenz cones we considered are standard, i.e., the vertices are the origin, we have that the cones are constantly in the positive octant in $\mathbb{R}^n$ including its boundary. Also, the properties of the constructed Lorenz cones, especially the structures of the eigenvalues of the matrices involved in the cone formulas, are studied. The motivation of these novel cones is that they are considered as candidate invariant sets for dynamical systems in the positive orthant.\(^4\)

\(^4\)A set $B$ is refereed to as a base of a cone $C$ if for any $x \in C$, there exists $\hat{x} \in B$, such that $x = \lambda \hat{x}$ for some $\lambda > 0$. 

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Definition 3.1. [11] Let $X \in \mathbb{R}^{m \times n}$. The vectorization of $X$, denoted by $\text{vec}(X)$, is writing all columns of $X$ into a single vector, i.e., $\text{vec}(X) = (x_{11}, ..., x_{m1}, x_{12}, ..., x_{m2}, ..., x_{1n}, ..., x_{mn})^T$.

Lemma 3.2. Let $\{v_i\}_{i=1}^n$ be a basis of $\mathbb{R}^n$. Then $\text{vec}(v_iv_j^T)$, $1 \leq i, j \leq n$, is a basis of $\mathbb{R}^{n^2}$.

Corollary 3.3. Let $a \in \mathbb{R}^n$, and $H = [h_1, h_2, ..., h_{n-1}] \in \mathbb{R}^{n \times (n-1)}$ be a complementary matrix of the vector $a$. Then the following vectors are a basis of $\mathbb{R}^{n^2}$:

$$
\text{vec}(aa^T), \text{vec}(ah_i^T), \text{vec}(h_ia^T), \text{and vec}(h_ih_j^T), \text{where } 1 \leq i, j \leq n-1.
$$

(16)

Lemma 3.4. Let $a \in \mathbb{R}^n$, and $H = [h_1, h_2, ..., h_{n-1}] \in \mathbb{R}^{n \times (n-1)}$ be a complementary matrix of the vector $a$. Assume $H^TXH = 0 \in \mathbb{R}^{(n-1) \times (n-1)}$, where $X \in \mathbb{R}^{n \times n}$, then there exist $\mu \in \mathbb{R}$, and $z_1, z_2 \in \mathbb{R}^{n-1}$, such that

$$
X = \mu aa^T + az_1^TH + Hz_2a^T.
$$

(17)

If $X$ is further assumed to be symmetric, then $z_1 = z_2$.

Proof. According to [11], the equation $H^TXH = 0$ can be solved by the following equation

$$(H^T \otimes H^T)\text{vec}(X) = \text{vec}(H^TXH) = \text{vec}(0) \in \mathbb{R}^{(n-1)^2},
$$

(18)

where $M \otimes N$ is the Kronecker product, e.g., [11]. According to [18], we have that $\text{vec}(X)$ is in $\ker(H^T \otimes H^T)$, i.e., the kernel of $H^T \otimes H^T$. We note that $\text{vec}(h_ih_j^T)$, for $i, j = 1, ..., n-1$, are in the range space of $H^T \otimes H^T$. Then, according to Corollary 3.3, we have that $\text{vec}(aa^T), \text{vec}(ah_i^T)$, and $\text{vec}(h_ia^T)$, for $i = 1, ..., n-1$, are in $\ker(H^T \otimes H^T)$. Thus $\text{vec}(X)$ can be represented as a linear combination of $\text{vec}(aa^T), \text{vec}(ah_i^T)$, and $\text{vec}(h_ia^T)$. Then (17) is immediate by writing this linear representation in a matrix form. If $X$ is a symmetric matrix, then we have $a(z_1 - z_2)^TH = H(z_1 - z_2)a^T$. By the linear independence of $\text{vec}(ah_i^T)$, and $\text{vec}(h_ia^T)$, for $i = 1, 2, ..., n-1$, we have $z_1 = z_2$. \qed

Lemma 3.5. Let a hyperplane $\mathcal{H}$ and a Dikan ellipsoid $\mathcal{E}_D$ be given as (13) and (11), respectively, and the base of a standard Lorenz cone $\mathcal{C}_L$ be $\mathcal{H} \cap \mathcal{E}_D$. Let $\mathcal{C}_L$ be represented as $\mathcal{C}_L = \{ x \mid x^TQx \leq 0 \}$. Then the following conditions hold:

$$
H^TDH = \beta H^TQH, \quad c^TQH = 0, \quad c^TQc = -\beta,
$$

(19)

where $\beta$ is a positive number and $H$ is a complementary matrix of $a$.

Proof. We use the representation $\mathcal{H} = \mathcal{H}(c, H) = \{ x \mid x = c + Hz \}$. Substituting $x = c + Hz$ into (11) and $\mathcal{C}_L = \{ x \mid x^TQx \leq 0 \}$, respectively, we have

$$
z^TH^TDHz - 1 \leq 0 \quad \text{and} \quad z^TH^TQHz + 2c^TQHz + c^TQc \leq 0.
$$

(20)

The lemma is immediate by comparing the two inequalities in (20). \qed
Theorem 3.6. Assume the conditions in (19) hold, then
\[
\gamma Q = D - \frac{1 + c^T Dc}{(c^T a)^2} c a^T - \frac{1}{c^T a} (a c^T D \tilde{H} + \tilde{H} D c a^T),
\]
(21)
where \( \gamma \) is a positive number, \( \tilde{D} = D - (D \tilde{H} + \tilde{H} D) \), and \( \tilde{H} = H (H^T H)^{-1} H^T \).

Proof. Since \( C_L \) is defined as \( \{ x \mid x^T Q x \leq 0 \} \), we let \( \beta = 1 \) in (19) for simplicity. According to Lemma 3.4 and the first condition in (19), there exist \( \mu \in \mathbb{R} \) and \( z \in \mathbb{R}^{n-1} \), such that
\[
Q = D + \mu a a^T + a z^T H^T + H z a^T.
\]
(22)
If we substitute (22) into the second condition in (19) and note that \( H^T a = 0 \), \( a^T H = 0 \), and \( c^T a = \alpha \), then we have
\[
z = -\frac{1}{\alpha} (H^T H)^{-1} H^T D c.
\]
(23)
If we substitute (22) and (23) into the third condition in (19), then we have
\[
\mu = -\frac{1}{\alpha^2} (1 + c^T D c - c^T (D \tilde{H} + \tilde{H} D) c).
\]
(24)
where \( \tilde{H} = H (H^T H)^{-1} H^T \).

We then substitute (23) and (24) into (22), and this theorem is immediate. \( \square \)

It is interesting to note that (23) is the same as the solution of \( H z = -\frac{1}{\alpha} D c \) solved by the least squares method. Since \( \tilde{H} \) is a symmetric matrix, all the eigenvalues of \( \tilde{H} \) are real numbers. Note that \( \tilde{H}^k = \tilde{H} \) holds for any positive integer \( k \), thus the eigenvalues of \( \tilde{H} \) are either 0 or 1 by choosing \( k = 2 \). Also, according to Sylvester’s law of inertia \[7\], we have inertia \( \{ \tilde{H} \} = \{ n - 1, 1, 0 \} \) because of \( H^T \tilde{H} H = I_{n-1} \). Thus, the matrix \( \tilde{H} \) has eigenvalues 1 with multiplier \( n - 1 \) and 0 with multiplier 1.

Remark 3.7. The vectors \( h_1, h_2, \ldots, h_{n-1} \) in Lemma 3.4 are not necessarily orthonormal. However, if we use the orthonormal basis, i.e., \( H^T H = I_{n-1} \), the previous computation may be simplified.

We now consider the Lorenz cones generated by the origin and the intersection between Dikin ellipsoids and some special hyperplanes.

3.1. The hyperplane is \( \mathcal{H}_i = \mathcal{H}(e_i, c_i) = \{ x \mid e_i^T x = c_i \} \).

Theorem 3.8. Let a Dikin ellipsoid \( E_D \) be given as (11), and a hyperplane be \( \mathcal{H}_i = \mathcal{H}(e_i, c_i) \), and the base of the standard Lorenz cone \( C_L \) be \( \mathcal{H}_i \cap E_D \). Let \( C_L \) be represented as \( C_L = \{ x \mid x^T Q_i x \leq 0 \} \). Then
\[
Q_i = D + \frac{n - 3}{c_i^2} E_i + \sum_{j=1, j \neq i}^{n} \frac{1}{c_i c_j} (E_{ij} + E_{ji}),
\]
(25)
where \( E_{ij} \) denotes an \( n \times n \) matrix that has \( ij \)-th entry 1 and other entries 0.
Substituting (26) into (21), the lemma is immediate. 

Proof. Note that the normal vector of \( H \) is \( e_i \), thus the complementary matrix of \( e_i \) can be chosen as \( H = [e_1, ..., e_{i-1}, e_{i+1}, ..., e_n] \). Then we have \( H = HH^T = I_n - E_{ii} \). Thus, we have

\[
\alpha = c_i, \quad a a^T = E_{ii}, \quad a c^T = \sum_{j=1}^n c_j E_{ij}, \quad c a^T = \sum_{j=1}^n c_j E_{ji}, \quad \text{and} \quad D H = H D = D - D E_{ii}. \tag{26}
\]

Substituting (26) into (21), the lemma is immediate. \( \square \)

For example, let \( i = 1 \) in (25), we have that \( Q_1 \) is explicitly given as follows:

\[
Q_1 = \begin{bmatrix}
\frac{n-2}{c_1^2} & -\frac{1}{c_1 c_2} & \cdots & -\frac{1}{c_1 c_n} \\
-\frac{1}{c_1 c_2} & \frac{1}{c_2^2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{c_1 c_n} & 0 & \cdots & \frac{1}{c_n^2}
\end{bmatrix}. \tag{27}
\]

Note that \( Q_1 \) is an arrowhead matrix. For any \( Q_i \) given as (25), by certain row and column transformation, it can be represented as an arrowhead matrix. In fact, the matrix \( Q_i \) given as (25) has many interesting properties.

**Lemma 3.9.** For every \( Q_i \) given as (25), we have \( \det(Q_i) = -\prod_{k=1}^n \frac{1}{c_k} \).

Proof. We only consider \( Q_1 \) given as (27). By choosing \( \lambda = 0 \) and \( A = Q_1 \) in (14), we have

\[
\det(-Q_1) = -\frac{n-2}{c_1^2} \prod_{k=2}^n \frac{1}{c_k^2} - \sum_{j=2}^n \frac{1}{c_1^2 c_j} \prod_{j=2}^n \frac{1}{c_k} = (-1)^n \frac{n-2}{c_1^2} \prod_{k=2}^n \frac{1}{c_k} - (-1)^n \frac{n}{c_1^2} \prod_{k=2}^n \frac{1}{c_k}
\]

\[
= (-1)^{n-1} \frac{n}{c_1^2} \prod_{k=2}^n \frac{1}{c_k} = (-1)^{n-1} \prod_{k=1}^n \frac{1}{c_k}
\]

Noting that that \( \det(Q_1) = (-1)^n \det(-Q_1) \), this lemma is immediate. \( \square \)

**Lemma 3.10.** For every \( Q_i \) given as (25), we have \( \text{inertia}(Q_i) = \{n - 1, 0, 1\} \).

Proof. Without loss of generality, we only consider \( Q_1 \). According to the second statement in Lemma 2.12 we have \( \lambda_{n-1} \geq \min_{j=2, ..., n} \{c_j^{-2}\} > 0 \). Also, the determinant of \( Q_i \) is negative according to Lemma 3.9 thus we have that \( \lambda_n \) is negative. This proof is complete. \( \square \)

**Lemma 3.11.** A lower bound and an upper bound of the eigenvalues of \( Q_i \) given as (25) are

\[
\frac{1}{2} \left( \frac{n-2}{c_i^2} + \frac{1}{c_i^2} + \frac{1}{c_i} \sqrt{\left( \frac{n-2}{c_i^2} - \frac{c_i}{c_o^2} \right)^2 + 4 \sum_{j=1, j \neq i}^n \frac{1}{c_j^2}} \right), \tag{28}
\]

and

\[
\frac{1}{2} \left( \frac{n-2}{c_i^2} + \frac{1}{c_i^2} + \frac{1}{c_i} \sqrt{\left( \frac{n-2}{c_i^2} - \frac{c_i}{c_s^2} \right)^2 + 4 \sum_{j=1, j \neq i}^n \frac{1}{c_j^2}} \right), \tag{29}
\]

respectively, where \( c_o = \max_{j=1, j \neq i}^n \{c_j\} \) and \( c_s = \min_{j=1, j \neq i}^n \{c_j\} \).
Proof. Without loss of generality, we only consider $Q_1$ given as (27). In our proof, we let $Q_1$ be divided into two functions, $Q_1 = f(x) + g(x)$, where $x \in \mathbb{R}$, $f(x)$, and $g(x)$ are as follows:

$$f(x) = \frac{n - 2 - x}{c_1^2} E_{11} + \sum_{j=2}^{n} \frac{1}{c_j} E_{jj}, \quad \text{and} \quad g(x) = \frac{x}{c_1^2} E_{11} + \sum_{j=2}^{n} -\frac{1}{c_j c_j} (E_{ij} + E_{ji}).$$

For any $x \in \mathbb{R}$, we have $\lambda_n(Q_1) \geq \lambda_n(f(x)) + \lambda_n(g(x))$ and $\lambda_1(Q_1) \leq \lambda_1(f(x)) + \lambda_1(g(x))$. Hence,

$$\lambda_1(Q_1) \leq \min_{x \in \mathbb{R}} \{\lambda_1(f(x)) + \lambda_1(g(x))\}, \quad \text{and} \quad \lambda_n(Q_1) \geq \max_{x \in \mathbb{R}} \{\lambda_n(f(x)) + \lambda_n(g(x))\}.$$ 

Note that $f(x)$ is a diagonal matrix for any $x \in \mathbb{R}$, then its eigenvalues are $\lambda_1(f(x)) = \max\{(n - 2 - x)c_1^{-2}, c_0^{-2}\}$ and $\lambda_n(f(x)) = \min\{(n - 2 - x)c_1^{-2}, c_0^{-2}\}$. Note that the rank of $g(x)$ is 1 for any $x \in \mathbb{R}$, and its characteristic polynomial is $\lambda^{n-2}(\lambda^2 - xc_1^{-2}\lambda - \sum_{j=2}^{n}(c_j)^{-2})$. Thus, we have

$$\lambda_1(g(x)) = \frac{1}{2} \left( \frac{x}{c_1^2} + \sqrt{\left( \frac{x}{c_1^2} \right)^2 + 4 \sum_{j=2}^{n} \left( \frac{1}{c_j c_j} \right)^2} \right),$$

and

$$\lambda_n(g(x)) = \frac{1}{2} \left( \frac{x}{c_1^2} - \sqrt{\left( \frac{x}{c_1^2} \right)^2 + 4 \sum_{j=2}^{n} \left( \frac{1}{c_j c_j} \right)^2} \right).$$

We now consider the following four cases:

1. If $\frac{n - 2 - x}{c_1^2} \geq \frac{1}{c_1^2}$, i.e., $x \leq n - 2 - \frac{c_1^2}{c_1^2}$, then

$$\lambda_1(f(x)) + \lambda_1(g(x)) = \frac{n - 2}{c_1^2} + \frac{2}{c_1} \left( \frac{\sum_{i=2}^{n} \frac{1}{c_i^2}}{\frac{x}{c_1} + \sqrt{\frac{x^2}{c_1^4} + 4 \sum_{i=2}^{n} \frac{1}{c_i^2}}} \right).$$

2. If $\frac{n - 2 - x}{c_1^2} \leq \frac{1}{c_1^2}$, i.e. $x \geq n - 2 - \frac{c_1^2}{c_1^2}$, then

$$\lambda_1(f(x)) + \lambda_1(g(x)) = \frac{1}{c_2^2} + \frac{1}{2c_1} \left( \frac{x}{c_1} + \sqrt{\frac{x^2}{c_1^4} + 4 \sum_{i=2}^{n} \frac{1}{c_i^2}} \right).$$

3. If $\frac{n - 2 - x}{c_1^2} \geq \frac{1}{c_0^2}$, i.e., $x \leq n - 2 - \frac{c_0^2}{c_0^2}$, then

$$\lambda_n(f(x)) + \lambda_n(g(x)) = \frac{1}{c_2^2} - \frac{2}{c_1} \left( \frac{\sum_{j=2}^{n} \frac{1}{c_j^2}}{\frac{x}{c_1} + \sqrt{\frac{x^2}{c_1^4} + 4 \sum_{j=2}^{n} \frac{1}{c_j^2}}} \right).$$
4. If \( \frac{n-2-x}{c_1} \leq \frac{1}{c_2} \), i.e., \( x \geq n - 2 - \frac{c^2}{c_2} \), then

\[
\lambda_n(f(x)) + \lambda_n(g(x)) = \frac{n-2}{c_1} - \frac{1}{2c_1} \left( \frac{x}{c_1} + \sqrt{\frac{x^2}{c_1^2} + 4 \sum_{i=2}^{n} \frac{1}{c_i^2}} \right). \tag{37}
\]

Since functions (36) and (35) are increasing functions and functions (37) and (34) are decreasing functions, we have

\[
\arg\min_{x \in \mathbb{R}} \{ \lambda_1(f(x)) + \lambda_1(g(x)) \} = \{ n - 2 - \frac{c^2}{c_2} \} \tag{38}
\]

and

\[
\arg\max_{x \in \mathbb{R}} \{ \lambda_n(f(x)) + \lambda_n(g(x)) \} = \{ n - 2 - \frac{c^2}{c_2} \}. \tag{39}
\]

Then substituting \( x = n - 2 - \frac{c^2}{c_2} \) into (36) (or (37)), and substituting \( x = n - 2 - \frac{c^2}{c_2} \) into (34) (or (35)), the lower and upper bounds (28) and (29) are immediate. This proof is complete.

3.2. The hyperplane is \( \mathcal{H} = \mathcal{H}(e, e^T c) = \{ x \in \mathbb{R}^n | e^T x = c^T e \} \).

First of all, we compute an orthogonal complementary basis of \( e \). Obviously, \( \{ e_1 - e_2, e_1 - e_3, ..., e_1 - e_n \} \) is a basis of the complementary space of \( e \). We use this basis to construct an orthogonal complementary basis of \( e \) given as follows:

\[
h_i = -\frac{1}{\sqrt{i(i+1)}} \sum_{k=1}^{i} e_k + \frac{\sqrt{i}}{\sqrt{i+1}} e_{i+1}, \text{ where } i = 1, 2, ..., n-1, \tag{40}
\]

which can be explicitly written as

\[
H = [h_1, h_2, ..., h_{n-1}] = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots \\
\frac{1}{\sqrt{12}} & \cdots \\
\vdots & \ddots & \ddots
\end{bmatrix}. \tag{41}
\]

For simplicity, we denote the \( i \)th row of \( H \) by \( p_i \), where \( i = 1, 2, ..., n \). Without loss of generality, assume \( i > j \), we have

\[
p_i^T p_j = -\frac{1}{i} + \sum_{k=i+1}^{n} \frac{1}{(k-1)k} = -\frac{1}{n} \quad \text{and} \quad p_i^T p_i = \frac{i-1}{i} + \sum_{k=i+1}^{n} \frac{1}{(k-1)k} = 1 - \frac{1}{n}. \tag{42}
\]

Thus, we have

\[
HH^T = I - \frac{1}{n} ee^T. \tag{43}
\]
Since $(HH^T)H = H$, i.e., $(HH^T)h_i = h_i$, for $i \in I(n-1)$, we have that $h_i$ is an eigenvector of $HH^T$. Also, note that the rank of $HH^T$ is $n - 1$, thus the eigenvalues of $HH^T$ is 1 with multiplier $n - 1$ and 0 with multiplier 1.

**Theorem 3.12.** Let a Dikin ellipsoid $E_D$ be in the form of (11), and a hyperplane be $H = H(e, e^Tc)$. Then the Lorenz cone $C_L$ generated by the origin and $H \cap E_D$ is represented as $C_L = \{x | x^T Q x \leq 0\}$, and

$$Q = D + \frac{n-1}{(e^Tc)^2} ee^T - \frac{1}{e^Tc}(ee^T D + Dce^T) = D + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{n-1}{(e^Tc)^2} - \frac{1}{e^Tc} \left( \frac{1}{c_i} + \frac{1}{c_j} \right) \right) E_{ij}. \quad (44)$$

**Proof.** According to (43), we have $	ilde{H} = H H^T = I - \frac{1}{n} ee^T$, which yields $D \tilde{H} = D - \frac{1}{n} Dee^T$, and $\tilde{H} D = D - \frac{1}{n} ee^T D$. Thus $\tilde{D} = D - (D \tilde{H} + \tilde{H} D) = \frac{1}{n}(Dee^T + ee^T D) - D$. Then we have

$$1 + c^T \tilde{D} c = 1 + \frac{1}{n}(c^T Dee^T c + c^T ee^T D c) - c^T Dc = 1 - n + \frac{2\alpha}{n} \sum_{i=1}^{n} \frac{1}{c_i}. \quad (45)$$

$$ec^T \tilde{D} H + \tilde{H} Dce^T = ec^T D - \frac{1}{n} e(c^T De)c^T + Dce^T - \frac{1}{n} e(e^T Dc)e^T$$

$$= ec^T D + Dce^T - \left( \frac{2}{n} \sum_{i=1}^{n} \frac{1}{c_i} \right) ee^T. \quad (46)$$

Substituting (45) and (46) into (21), this lemma is immediate. The proof is complete. \[\square\]

We now present an interesting result about the eigenvalue and eigenvector structures of a class of rank 1 matrix.

**Lemma 3.13.** If the rank of a square matrix $A$ is 1, the matrix $A$ has at most 1 nonzero eigenvalues with multiplier 1.

**Proof.** The dimension of the kernel space of $Ax = 0$ is $n - 1$. The basis in this kernel space can be eigenvalues of $A$ corresponding to 0. Also, note that the sum of all eigenvalues is equal to the trace of $A$ that may be zero, in which case, all eigenvalues are 0. For example, $A = E_{12}$. The eigenvalues are all 0 with eigenvectors $e_1, e_2, ..., e_n$. \[\square\]

**Corollary 3.14.** Let two vectors $a, c \in \mathbb{R}^n$, then the eigenvalues of $ac^T$ are $a^T c$ with multiplier 1 and 0 with multiplier $n-1$. An eigenvector corresponding to $a^T c$ is $a$. The eigenvectors corresponding to 0 can be the complementary basis of $c$.

**Corollary 3.15.** Let a nonzero vector $a \in \mathbb{R}^n$, then the matrix $aa^T$ is rank 1, and the eigenvalues of $aa^T$ are $\|a\|^2$ with multiplier 1 and 0 with multiplier $n-1$. An eigenvector corresponding to $\|a\|^2$ is $a$. The eigenvectors corresponding to 0 can be the complementary basis of $a$.

**Corollary 3.16.** The eigenvalues of $ee^T$ is $n$ with multiplier 1 and eigenvector $e$, and 0 with multiplier $n-1$ and eigenvectors $e_1 - e_2, e_1 - e_3, ..., e_1 - e_n$.
Lemma 3.17. \( \det(\beta I + \alpha ee^T) = (1 + n\frac{\alpha}{\beta})\beta^n. \)

Proof. Let \( T_n = \det(\beta I + \alpha ee^T). \) Writing \( T_n \) by definition, we can find the following iteration formula:

\[
T_n = (\beta + \alpha)T_{n-1} + \sum_{j=2}^{n} (-1)^{1+j} \alpha M_{ij} = (\beta + \alpha)T_{n-1} - (n-1)\alpha^2\beta^{n-2}. \tag{47}
\]

Let \( F_n = \frac{T_n}{\beta^n}, \) then \( F_n = \frac{\beta + \alpha}{\beta}F_{n-1} - \frac{n-1}{n^2}\beta^2. \) It is easy to prove that \( F_n - F_{n-1} = \frac{\alpha}{\beta}. \) Then this lemma is immediate. \( \square \)

Corollary 3.18. \( \det(I + \alpha ee^T) = 1 + n\alpha. \)

Lemma 3.19. If \( c_1 = c_2 = \ldots = c_n = c, \) then eigenvalues of \( Q \) in \( (44) \) is \( \frac{1}{c} \) with multiplier \( n-1 \) and \( \frac{1}{mc} \) with multiplier \( 1. \)

Proof. Since \( c_1 = c_2 = \ldots = c_n = c, \) we have \( Q = \frac{1}{c}(I - \frac{n+1}{n}ee^T). \) According to Lemma 3.17, we have \( \det(Q - \lambda I) = 0 \) to yield the characteristics polynomial is \( (1 - \lambda^2)^{n-1}(\frac{1}{n} - \lambda^2) = 0. \) The the lemma is immediate. \( \square \)

3.3. Tangent cone

Lemma 3.20. Let \( A, B, C, D \in \mathbb{R}^n. \) Assume \( AC \perp CB, CD \perp AB, \) and \( D \in AB. \) Then \( \|AB\|^2 = \|AC\|^2 + \|BC\|^2, \) \( \|AC\|^2 = \|AD\|\|AB\|, \) and \( \|BC\|^2 = \|BD\|\|AB\|. \)

Proof. Note that \( \overrightarrow{AB} = \overrightarrow{CB} - \overrightarrow{CA} \) and \( \overrightarrow{CB} \perp \overrightarrow{CA}, \) the first equation is immediate. The remaining two are not trivial to prove, even they looks natural in 2 or 3 dimension. Now we present a rigorous proof.

We denote \( x_A, x_B, x_C, \) and \( x_D \) the coordinates of \( A, B, C \) and \( D \) in \( \mathbb{R}^n, \) respectively. Since \( D \in AB, \) there exists \( 0 \leq \lambda \leq 1 \) such that \( x_D = \lambda x_A + (1 - \lambda)x_B. \) Since \( AC \perp BC \) and \( CD \perp AB, \) we have

\[
(\lambda x_A + (1 - \lambda)x_B - x_C)^T(x_A - x_B) = 0, \quad (x_A - x_C)^T(x_B - x_C) = 0. \tag{48}
\]

Expanding two equations in \( (48) \) and plugging \( x_A^Tx_B - x_C^Tx_x = x_B^tx_C - \|x_C\|^2, \) which is obtained from the second equation in \( (48), \) into the first equation in \( (48), \) we have

\[
\lambda = \frac{\|x_B - x_C\|^2}{\|x_A - x_B\|^2} = \frac{\|BC\|^2}{\|AB\|^2}. \tag{49}
\]

Also, \( DA = (1 - \lambda)(x_B - x_A) \) and \( BA = x_B - x_A, \) then

\[
\|AD\|\|AB\| = (1 - \lambda)(x_B - x_A)^T(x_B - x_A) = (1 - \lambda)\|x_A - x_B\|^2 = \|AB\|^2 - \|BC\|^2 = \|AC\|^2. \tag{50}
\]

Similarly, the third equation in this lemma is easy to obtain. \( \square \)
Lemma 3.21. The distance of a point $\bar{x} \in \mathbb{R}^n$ to a hyperplane $S = \{a^Tx = \alpha\}$ is

$$\text{dist}(x, S) = \frac{|a^T\bar{x} - \alpha|}{\|a\|}.$$  \hfill (51)

Theorem 3.22. Assume the ellipsoid $E = \{x \in \mathbb{R}^n | (x - c)^T(x - c) \leq 1\}$, then the Lorenz cone $C$ is

$$Q = I - \frac{1}{\|c\|^2 - 1}cc^T.$$  \hfill (52)

Proof. Since the ellipsoid $E$ is a $n$-sphere, the hyperplane through the intersection of the ellipsoid and Lorenz cone (note that this intersection is a $n - 1$-dimensional ellipsoid) is $\{x \in \mathbb{R}^n | c^Tx = \alpha\}$. We now compute the value of $\alpha$. According to Lemma 3.20, we obtain that the distance from origin to this hyperplane is $\|c\| - \frac{1}{\|c\|}$.

Thus, the hyperplane through $E \cap C$ is $S = \{x \in \mathbb{R}^n | c^Tx = \|c\|^2 - 1\}$. \hfill (53)

According to [1], there exist parameters $z \in \mathbb{R}^{n-1}, \lambda, \mu$, such that

$$D = I + \lambda cc^T + cz^TH^T + Hzc^T, \quad 0 = -c - \mu c - (\|c\|^2 - 1)Hz,$$ \hfill (54)

$$0 = \|c\|^2 - 1 - \lambda(\|c\|^2 - 1)^2 + 2\mu(\|c\|^2 - 1).$$ \hfill (55)

From the second and third equation in (54), we have

$$Hz = \frac{-1}{\|c\|^2 - 1}((1 + \mu)c, \quad \lambda(\|c\|^2 - 1) - 2\mu = 1.$$ \hfill (56)

Substitute (56) into the first equation in (54), the theorem is immediate. \hfill \Box

4. Invariance of Lorenz Cone

A tangent cone of a convex set $C \subset \mathbb{R}^n$ at point $x \in C$ is defined as

$$T_C(x) = \{z \in \mathbb{R}^n | \lim_{h \to 0} \inf \frac{\text{dist}(x + hz, C)}{h} = 0\}$$ \hfill (57)

where $\text{dist}(x, C) = \inf_{y \in C} \|x - y\|$. It is easy to see that $T_C(x) = \mathbb{R}^n$ if $x$ is located interior of $C$. If $x$ is on the boundary of $C$ with smooth neighborhood, then the tangent cone is the affine half-space which is obtained by parallelling the tangent line at $x$ with respect to $C$ to be through origin.

Lemma 4.1. [9, 13] [Nagumo] Let $C$ be convex and closed set in $\mathbb{R}^n$, then $C$ is invariant with respect to dynamical system (2) if and only if $\forall x \in \partial C$, then $Ax \in T_C(x)$, where $T_C(x)$ is tangent cone of $C$ at $x$. 

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This is an elegant and intuitive conclusion. We can understand this lemma in the following way: the right side, i.e. $Ax$ in dynamical system is actually the slope of tangent line of trajectory at point $x$, since the left side is the derivative of trajectory. If the trajectory starts from one point in $C$, then the unique possibility to move out from this set must be through some point on the boundary. This lemma states that the slope is in the tangent cone at $x$, which forces the trajectory to move back to $C$.

Based on Nagumo lemma, it is easy to derive the sufficient and necessary condition that one ellipsoid or cone is positively invariant with respect to dynamical system (2). For simplicity, we remove the suffix $c$ in $A_c$ in (2).

Lemma 4.2. Let an ellipsoid be defined as $E = \{x \in \mathbb{R}^n | x^T P x \leq 1\}$, an cone be defined as $C = \{x \in \mathbb{R}^n | x^T Q x \leq 0\}$, then $E$ or $C$ is a positively invariant set with respect to dynamical system if and only if $\langle Ax, P x \rangle \leq 0$ or $\langle Ax, Q x \rangle \leq 0$ for $x$ on the boundary of the set, where $\langle a, b \rangle$ is the inner product of $a$ and $b$.

Proof. We just prove the ellipsoid case, the proof for cone is almost same. It is easy to see that the outer normal at $x \in \partial E$ is $2P x$, and since the boundary of an ellipsoid is smooth, the tangent cone at $x$ is $T_E(x) = \{y | \langle y, P x \rangle \leq 0\}$.

Then by Nagumo lemma, dynamical system (2) is positively invariant on $E$ if and only if $Ax \in T_E(x)$ for any $x$ on the boundary of $E$. Thus, this lemma was proved.

According to [15], it concluded that the dynamical system (2) is positively invariant for the ice cream cone $C_0 = \{x \in \mathbb{R}^n | x_1^2 + \cdots + x_{n-1}^2 \leq x_n^2\}$ if and only if there exists $a \in R$ such that

$$Q_n A + A^T Q_n + a Q_n \leq 0,$$

where $Q_n = \text{diag}(1,\ldots,1,-1)$, and the inequality means semi-negative definite.

Lemma 4.3. Let cone be defined as $C = \{x \in \mathbb{R}^n | x^T Q x \leq 0\}$, then dynamical system is positively invariant on $C$ if and only if there exists $a \in \mathbb{R}$ such that

$$Q A + A^T Q + a Q \leq 0,$$

where the inequality means semi-negative definite.

Proof. There exists one nonsingular transformation $P$ such that $C = PC_0$. Thus, $\forall x \in C_0$, there exists $x^* \in C$ such that $x^* = P x$. Since $x^*$ satisfies dynamical system equation, we have

$$(x^*)' = A x^* \iff (P x)' = A P x \iff x' = P^{-1} A P x.$$

Thus, dynamical system (2) is positively invariant on $C$ is equivalent that the right dynamical system is positive invariant on $C_0$. By the previous lemma and $P^T Q P = Q_n$, there exist $a \in R$ such that

$$Q_n P^{-1} A P + (P^{-1} A P)^T Q_n + a Q_n \leq 0$$
$$P^T Q P P^{-1} A P + P^T A P P^{-1} Q P + a P^T Q P \leq 0$$
$$P^T (Q A + A^T Q + a Q) P \leq 0$$
In fact, the last “inequality” is equivalent with (60). If (60) is true, then for any $x$,

$$x^T P^T (QA + A^T Q + aQ)Px = (Px)^T (QA + A^T Q + aQ)(Px) \leq 0. \quad (63)$$

The other hand, for any $x$, since $P$ is singular, there exists a $y$ such that $P^{-1}x = y$, i.e. $x = Py$. Then

$$x^T (QA + A^T Q + aQ)x = y^T P^T (QA + A^T Q + aQ)Py \leq 0. \quad (64)$$

The proof is completed.

Figure 1 gives the shape of the constructed Lorenz cone by using the Dikin ellipsoid and two different hyperplanes.

Figure 1: A framework to derive invariance conditions for continuous systems.

5. Conclusion

In this paper, we study the cone based invariant sets for a given dynamical system. We construct some special Lorenz cones using Dikin ellipsoid and some hyperplanes. Dikin ellipsoid is originally introduced from mathematical optimization to design the polynomial time linear algorithm. We use this tool into the construction to ensure that the constructed Lorenz cone is located in positive orthant, which is usually a common requirement in the real world application. We also study the structure of the constructed cones, especially the eigenvalues structure of the related matrix in the formula of ellipsoid. The novelty of this paper is building the link between optimization and invariant sets. It also provides more flexibility for other researchers either in theory or in practice to choose more Lorenz cones for analysis.
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