ON NORMAL CONTACT PAIRS

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ABSTRACT. We consider manifolds endowed with a contact pair structure. To such a structure are naturally associated two almost complex structures. If they are both integrable, we call the structure a normal contact pair. We generalize the Morimoto’s Theorem on product of almost contact manifolds to flat bundles. We construct some examples on Boothby–Wang fibrations over contact-symplectic manifolds. In particular, these results give new methods to construct complex manifolds.

1. INTRODUCTION

A contact pair on a manifold is a pair of one-forms \( \alpha_1 \) and \( \alpha_2 \) of constant and complementary classes, for which \( \alpha_1 \) induces a contact form on the leaves of the characteristic foliation of \( \alpha_2 \), and vice versa. This notion, considered in [2, 4], was firstly introduced in [12] by the name bicontact and further studied in [1].

In [5] we considered the notion of contact pair structure on a manifold \( M \), that is a contact pair \((\alpha_1, \alpha_2)\) together with a tensor field \( \phi \) on \( M \), of type \((1, 1)\), such that \( \phi^2 = -Id + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2 \) and \( \phi(Z_1) = \phi(Z_2) = 0 \), where \( Z_1 \) and \( Z_2 \) are the Reeb vector fields of the pair. This is a special type of \( f \)-structure with complemented frame (see [10, 21, 25]).

In this paper, we associate to a contact pair structure the almost complex structures \( J = \phi - \alpha_2 \otimes Z_1 + \alpha_1 \otimes Z_2 \) and \( T = \phi + \alpha_2 \otimes Z_1 - \alpha_1 \otimes Z_2 \). This can be seen as a generalization of the almost complex structure used in almost contact geometry to define normality (see [11] and the references therein). Nevertheless our structure is more intrinsic in that, for its definition, we do not need to consider the manifold \( M \times \mathbb{R} \) as in the case of the almost contact structures. A natural problem is the study of the integrability condition for these almost complex structures and we call a contact pair structure normal, if the associated almost complex structures are both integrable. An interesting feature of this structure is that, under the assumption that \( \phi \) is decomposable, there are almost contact structures induced on the leaves (which are contact manifolds) of the characteristic foliations, and then a natural problem is to relate the normality of the whole structure to that of the induced structures (in the sense of almost contact manifolds).

One could expect a general result similar to that of Morimoto [20], which says that on a product of manifolds, each of them endowed with an almost contact structures, there is a natural almost complex structure which is integrable if and only if the almost contact structures are normal.

In our case this is not true in full generality, since there are interesting counterexamples showing that the contact pair structure \((\alpha_1, \alpha_2, \phi)\) on \( M \) can be more complicated: even if \( M \) is locally the product of two contact manifolds, the tensor field \( \phi \) is not the sum of two tensors on the factors.

Anyway, we can generalize Morimoto’s result in the context of flat bundles, already used in [18, 7] to construct new examples of symplectic pairs.

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By performing the Boothby–Wang fibration over a manifold endowed with a contact-symplectic pair \([3]\) (which can be thought as a special almost contact structure), we are able to construct \(S^1\)-invariant contact pair structures on the total space and we show that under some hypothesis, the contact pair structure is normal if and only if the contact-symplectic pair on the base is normal as almost contact structure. This is an even counterpart of the constructions given by Morimoto (resp. Hatakeyama) of normal contact structures on Boothby–Wang fibration over a complex (resp. almost Kähler) manifold.

Furthermore, the flat bundles and the Boothby–Wang fibrations yielding normal contact pairs give new constructions of complex manifolds.

In the sequel we denote by \(\Gamma(B)\) the space of sections of a vector bundle \(B\). For a given foliation \(F\) on a manifold \(M\), we denote by \(T_F\) the subbundle of \(TM\) whose fibers are given by the distribution tangent to the leaves. All the differential objects considered are supposed to be smooth.

2. Preliminaries on contact pairs and contact pair structures

In this section we firstly give the notions concerning contact pairs which are useful for our purpose, next we recall the definition and the properties of contact pair structures. A manifold endowed with a contact pair was called \(\text{bicontact}\) in \([12]\). Here we maintain the notations of \([2, 4]\) and we refer to \([2, 3, 4, 5, 6, 7]\) for further informations and several examples of such structures.

**Definition 1** \([2, 4, 12]\). A pair \((\alpha_1, \alpha_2)\) of 1-forms on a \((2h + 2k + 2)\)-dimensional manifold \(M\) is said to be a contact pair of type \((h, k)\) if:

i) \(\alpha_1 \wedge (d\alpha_1)^h \wedge \alpha_2 \wedge (d\alpha_2)^k\) is a volume form,

ii) \((d\alpha_1)^{h+1} = 0\) and \((d\alpha_2)^{k+1} = 0\).

Since the form \(\alpha_1\) (resp. \(\alpha_2\)) has constant class \(2h + 1\) (resp. \(2k + 1\)), the distribution \(\text{Ker} \alpha_1 \cap \text{Ker} d\alpha_1\) (resp. \(\text{Ker} \alpha_2 \cap \text{Ker} d\alpha_2\)) is completely integrable and then it determines the so-called characteristic foliation \(F_1\) (resp. \(F_2\)) whose leaves are endowed with a contact form induced by \(\alpha_2\) (resp. \(\alpha_1\)).

To a contact pair \((\alpha_1, \alpha_2)\) of type \((h, k)\) are associated two commuting vector fields \(Z_1\) and \(Z_2\), called \(\text{Reeb vector fields}\) of the pair, which are uniquely determined by the following equations:

\[
\begin{align*}
\alpha_1(Z_1) &= \alpha_2(Z_2) = 1, \quad \alpha_1(Z_2) = \alpha_2(Z_1) = 0, \\
i_{Z_1}d\alpha_1 &= i_{Z_2}d\alpha_2 = i_{Z_2}d\alpha_1 = i_{Z_1}d\alpha_2 = 0,
\end{align*}
\]

where \(i_X\) is the contraction with the vector field \(X\). In particular, since the Reeb vector fields commute, they determine a locally free \(\mathbb{R}^2\)-action, called the \(\text{Reeb action}\).

The kernel distribution of \(d\alpha_1\) (resp. \(d\alpha_2\)) is also integrable and then it defines a foliation whose leaves inherit a contact pair of type \((0, k)\) (resp. \((h, 0)\)).

The tangent bundle of a manifold \(M\) endowed with a contact pair can be split in different ways. For \(i = 1, 2\), let \(T\mathcal{F}_i\) be the subbundle determined by the characteristic foliation of \(\alpha_i\), \(T\mathcal{G}_i\) the subbundle of \(TM\) whose fibers are given by \(\text{ker} d\alpha_i \cap \text{ker} \alpha_1 \cap \text{ker} \alpha_2\) and \(\mathbb{R}Z_1, \mathbb{R}Z_2\) the line bundles determined by the Reeb vector fields. Then:

\[
\begin{align*}
TM &= T\mathcal{F}_1 \oplus T\mathcal{F}_2 \\
TM &= T\mathcal{G}_1 \oplus T\mathcal{G}_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2.
\end{align*}
\]
Moreover we have $TF_1 = TG_1 \oplus \mathbb{R}Z_2$ and $TF_2 = TG_2 \oplus \mathbb{R}Z_1$.

In a similar way, we define symplectic pairs and contact-symplectic pairs:

**Definition 2** ([7]). A symplectic pair of type $(h, k)$, for $h, k \neq 0$, on a $2h + 2k$-dimensional manifold $M$ is a pair of closed two-forms $\omega_1, \omega_2$ such that:

i) $\omega_h \wedge \omega_k^j$ is a volume form;

ii) $\omega_1^{h+1} = 0$ and $\omega_2^{k+1} = 0$.

**Definition 3** ([2, 3]). A contact-symplectic pair of type $(h, k)$ on a $(2h + 2k + 1)$-dimensional manifold $N$ consists of a 1-form $\beta$ and a closed 2-form $\eta$ such that:

i) $\beta \wedge (d\beta)^h \wedge \eta^k$ is a volume form,

ii) $(d\beta)^{h+1} = 0$ and $\eta^{k+1} = 0$.

To a contact-symplectic pair is associated a Reeb vector field $W$, uniquely defined by the following equations:

$$\beta(W) = 1, \ i_W d\beta = i_W \eta = 0.$$  

Furthermore, let $F_1$ and $F_2$ be the characteristic foliations of $\eta$ and $\beta$ respectively, and $TF_1, TF_2$ the corresponding subbundles of $TN$. Let $RW$ the line bundle determined by the Reeb vector field and $TH$ the bundle whose fibers are given by $\ker \beta \cap \ker \eta$. Then we have the following splittings:

$$TN = TF_1 \oplus TF_2 = RW \oplus TH \oplus TF_2,$$

where $TF_1 = RW \oplus TH$. Moreover the two form $d\beta$ (resp. $\eta$) induces a symplectic form on $TH$ (resp. $TF_2$).

**The Boothby–Wang construction.** The Boothby-Wang fibration [15], associates regular contact forms to integral symplectic forms. If $(M, \omega)$ is a closed symplectic manifold and $\omega$ represents an integral class in $H^2(M; \mathbb{R})$ then there exists a principal $S^1$-bundle $\pi: E \to M$ with Euler class $[\omega]$ and a connection 1-form $\alpha$ on it with curvature $\omega$, i.e. we have $d\alpha = \pi^*\omega$. As $\omega$ is assumed to be symplectic on $M$, it follows that $\alpha$ is a contact form on the total space $E$.

If $\omega$ is an arbitrary closed 2-form representing an integral cohomology class, we can again find a connection 1-form $\alpha$ with curvature $\omega$. If $\omega$ has constant rank $2k$, then $\alpha$ has constant class $2k + 1$, that is $\alpha \wedge (d\alpha)^k \neq 0$, and $(d\alpha)^{k+1} = 0$.

This yields the following results from [7]:

**Theorem 4** ([7]). Let $M$ be a closed manifold with a symplectic pair $(\omega_1, \omega_2)$. If $[\omega_1] \in H^2(M; \mathbb{R})$ is an integral cohomology class, then the total space of the circle bundle $\pi: E \to M$, with Euler class $[\omega_1]$, carries a natural $S^1$-invariant contact-symplectic pair.

**Theorem 5** ([7]). Let $M$ be a closed manifold with a contact-symplectic pair $(\alpha, \beta)$. If $[\beta] \in H^2(M; \mathbb{R})$ is an integral cohomology class, then the total space of the circle bundle $\pi: E \to M$, with Euler class $[\beta]$, carries a natural $S^1$-invariant contact pair.

**Corollary 6** ([7]). If a closed manifold $M$ has a symplectic pair $(\omega_1, \omega_2)$ such that both $[\omega_1] \in H^2(M; \mathbb{R})$ are integral, then the fiber product of the two circle bundles with Euler classes equal to $[\omega_1]$ and $[\omega_2]$ respectively carries a natural $S^1$-invariant contact pair.

In particular the Corollary 6 affirms that starting from a symplectic pair whose two forms represent integral classes, then performing a double Boothby-Wang fibration, one obtains a contact pair on the top.
2.1. **Almost contact structures.** An almost contact structure on a manifold $M$ is a triple $(\alpha, Z, \phi)$ of a one-form $\alpha$, a vector field $Z$ and a field of endomorphisms $\phi$ of the tangent bundle of $M$, such that $\phi^2 = -\text{Id} + \alpha \otimes Z$, $\phi(Z) = 0$ and $\alpha(Z) = 1$. In particular, it follows that $\alpha \circ \phi = 0$ and that the rank of $\phi$ is $\dim M - 1$.

If a manifold $M$ carries such a structure, one can consider an almost complex structure on $M \times \mathbb{R}$. Every $Y \in \Gamma(T(M \times \mathbb{R}))$ can be written as $X + f \frac{d}{dt}$ for $X$ tangent to $M$ and $f \in C^\infty(M \times \mathbb{R})$. Then the almost complex structure is defined as follows:

$$J(X + f \frac{d}{dt}) = \phi X - fZ + \alpha(X) \frac{d}{dt}.$$

The almost contact structure is said to be normal if $J$ is integrable. The integrability condition for $J$ is equivalent to the following condition:

$$[\phi, \phi](X, Y) + 2d\alpha(X, Y)Z = 0, \forall X, Y \in \Gamma(TM),$$

where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$.

If $\alpha$ is a contact form and $(\alpha, Z, \phi)$ an almost contact structure, we often refer to it as a contact form with structure tensor $\phi$. When the structure is normal we call it normal contact form for short.

2.2. **Contact-symplectic pairs as almost contact structures.** A contact-symplectic pair $(\beta, \eta)$ on a manifold $N$ can be viewed as a special almost contact structure (in [9] D. Blair considered similar structures) when it is endowed with an endomorphism $\psi$ of $TN$, satisfying

$$\psi^2 = -\text{Id} + \beta \otimes W,$$

where $W$ is the Reeb vector field of $(\beta, \eta)$. Such a $\psi$ always exists because on the kernel of $\beta$ the 2-form $d\beta + \eta$ is symplectic. By a standard polarization process, one can always construct such a $\psi$ and an associated metric $g$, that is a metric satisfying the following conditions:

$$g(X, \psi Y) = (d\beta + \eta)(X, Y) \text{ and } g(X, W) = \beta(X), \forall X, Y \in \Gamma(TN).$$

Since the symplectic subbundle determined by the kernel of $\beta$ can be split into two symplectic subbundles $T\mathcal{H}$ and $T\mathcal{F}_2$ as in [4], by polarization on both of them one can always construct a so called decomposable endomorphism $\psi$ which preserves the tangent spaces of the foliations (or equivalently $\psi(T\mathcal{H}) = T\mathcal{H}$ and $\psi(T\mathcal{F}_2) = T\mathcal{F}_2$) and an associated metric $g$ for which the foliations are orthogonal with respect to $g$. We do not give the details for that, since we have proven the analog of this statement for contact pair structures in [5].

**Definition 7.** An almost contact-symplectic structure on a manifold $M$ is a triple $(\beta, \eta, \psi)$, where $(\beta, \eta)$ is a contact-symplectic pair with Reeb vector filed $W$ and $\psi$ is an endomorphism of $TM$ satisfying (6).

2.3. **Contact pair structures.** This notion has been considered in [5]. We recall here the definition and some basic properties which are useful in the sequel.

**Definition 8 ([5]).** A contact pair structure on a manifold $M$ is a triple $(\alpha_1, \alpha_2, \phi)$, where $(\alpha_1, \alpha_2)$ is a contact pair and $\phi$ a tensor field of type $(1,1)$ such that:

$$\phi^2 = -\text{Id} + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2 \text{ and } \phi(Z_1) = \phi(Z_2) = 0$$

where $Z_1$ and $Z_2$ are the Reeb vector fields of $(\alpha_1, \alpha_2)$. 


Moreover we have $\alpha_i \circ \phi = 0$, $i = 1, 2$ and the rank of $\phi$ is equal to $\dim M - 2$. Recall that on a manifold $M$ endowed with a contact pair, there always exists an endomorphisms $\phi$ verifying (7). Moreover, $\phi$ can be chosen to be decomposable ([5], Proposition 5), that is:

**Definition 9 (5).** The endomorphism $\phi$ is said to be decomposable if $\phi(T \mathcal{F}_i) \subset T \mathcal{F}_i$, for $i = 1, 2$.

The condition for $\phi$ to be decomposable is equivalent to $\phi(T \mathcal{G}_i) = T \mathcal{G}_i$, $i = 1, 2$.

The following results are concerned with the structures induced on the leaves of the characteristic foliations:

**Proposition 10 (5).** If $\phi$ is decomposable, then $(\alpha_1, Z_1, \phi)$ (resp. $(\alpha_2, Z_2, \phi)$) induces a contact form with structure tensor the restriction of $\phi$ on the leaves of $\mathcal{F}_2$ (resp. $\mathcal{F}_1$).

### 3. Almost complex structures

To define a normal almost contact structure on a manifold $M$, one needs to consider an almost complex structure on $M \times \mathbb{R}$. In the case of a contact pair structure the almost complex structure can be defined in a more natural and intrinsic way on the manifold.

**Definition 11.** Let $(\alpha_1, \alpha_2, \phi)$ be a contact pair structure on a manifold $M$ and $Z_1, Z_2$ the Reeb vector fields of the pair. The almost complex structure on $M$

\[ J = \phi - \alpha_2 \otimes Z_1 + \alpha_1 \otimes Z_2, \tag{8} \]

is called the *almost complex structure associated* to $(\alpha_1, \alpha_2, \phi)$.

We can also consider a second almost complex structure

\[ T = \phi + \alpha_2 \otimes Z_1 - \alpha_1 \otimes Z_2, \tag{9} \]

which is nothing but the almost complex structure associated to the contact pair $(\alpha_2, \alpha_1, \phi)$ and commutes with $J$.

**Remark 12.** The almost complex structure induced by $T$ on $T \mathcal{G}_1 \oplus T \mathcal{G}_2$ is the same as $J$, but opposite to it on the subbundle $\mathbb{R}Z_1 \oplus \mathbb{R}Z_2$. Then the orientations induced by $J$ and $T$ are opposite. In general one can not expect that both structures are integrable since this imposes some topological obstructions, in particular on a four dimensioned closed manifold (see [19]).

Recalling that the forms $\alpha_1, \alpha_2$ are invariant by the Reeb vector fields, a straightforward calculation shows that the Nijenhuis tensor of the almost complex structure $J$ associated to the contact pair structure $(\alpha_1, \alpha_2, \phi)$ is given by:

\[ N_J(X, Y) = [\phi, \phi](X, Y) + 2d\alpha_1(X, Y)Z_1 + 2d\alpha_2(X, Y)Z_2 + \alpha_1(X)[L_{Z_2}\phi](Y) - \alpha_1(Y)[L_{Z_2}\phi](X) + \alpha_2(Y)[L_{Z_1}\phi](X) - \alpha_2(X)[L_{Z_1}\phi](Y) + [(L_{\phi X}\alpha_1)(Y) - (L_{\phi Y}\alpha_1)(X)]Z_2 + [(L_{\phi X}\alpha_2)(X) - (L_{\phi Y}\alpha_2)(Y)]Z_1, \tag{10} \]

for each $X, Y \in \Gamma(TM)$, where $L_X$ is the Lie derivative along $X$, $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$.

The Nijenhuis tensor $N_T$ of the almost complex structure defined in (9), is obtained from $N_J$ by interchanging the role of the forms $\alpha_1, \alpha_2$ and their Reeb vector fields. Then, for each $X, Y \in$
\(\Gamma(TM)\) we have:

\[
N_T(X,Y) = [\phi,\phi](X,Y) + 2d\alpha_1(X,Y)Z_1 + 2d\alpha_2(X,Y)Z_2 - \alpha_1(X)L_{Z_1}\phi)(Y) \\
+ \alpha_1(Y)L_{Z_2}\phi)(X) - \alpha_2(Y)L_{Z_1}\phi)(X) + \alpha_2(X)L_{Z_2}\phi)(X) - \alpha_2(X)L_{Z_1}\phi)(X)
\]

\[
- [(L_{\phi X}\alpha_1)(Y) - (L_{\phi Y}\alpha_1)(X)]Z_2 - [(L_{\phi Y}\alpha_2)(X) - (L_{\phi X}\alpha_2)(Y)]Z_1.
\]

(11)

The vanishing of both \(N_J\) and \(N_T\) is equivalent to the vanishing of their sum and their difference. Since \([L_{Z_1}\phi)(X)\) is in the kernel of \(\alpha_1\) and \(\alpha_2\) for every \(X \in \Gamma(TM)\), the integrability of both \(J\) and \(T\) is equivalent to the following system:

\[
\begin{cases}
[\phi,\phi](X,Y) + 2d\alpha_1(X,Y)Z_1 + 2d\alpha_2(X,Y)Z_2 = 0 \\
- \alpha_1(X)L_{Z_2}\phi)(Y) + \alpha_1(Y)L_{Z_2}\phi)(X) - \alpha_2(Y)L_{Z_1}\phi)(X) + \alpha_2(X)L_{Z_1}\phi)(X) = 0 \\
[(L_{\phi X}\alpha_1)(Y) - (L_{\phi Y}\alpha_1)(X)]Z_2 = 0 \\
[(L_{\phi Y}\alpha_2)(X) - (L_{\phi X}\alpha_2)(Y)]Z_1 = 0,
\end{cases}
\]

(12)

for every \(X,Y \in \Gamma(TM)\). Now, putting \(Y = Z_i\) in the first equation, one obtains \(L_{Z_i}\phi = 0\), which implies the second equation. Applying \(\alpha_i\) to \(N_J(\phi X,Y)\) gives the last equations.

These observations yield the following theorem:

**Theorem 13.** The integrability of both \(J\) and \(T\) is equivalent to the following equation:

\[
[\phi,\phi](X,Y) + 2d\alpha_1(X,Y)Z_1 + 2d\alpha_2(X,Y)Z_2 = 0 \quad \forall X,Y \in \Gamma(TM).
\]

(13)

By using the splitting (11), the equation (13) is equivalent to the following system:

\[
[\phi,\phi](X,Y) + 2d\alpha_1(X,Y)Z_1 = 0 \quad \forall X,Y \in \Gamma(TF_2),
\]

(14)

\[
[\phi,\phi](X,Y) + 2d\alpha_2(X,Y)Z_2 = 0 \quad \forall X,Y \in \Gamma(TF_1),
\]

(15)

\[
[\phi,\phi](X,Y) = 0 \quad \forall X \in \Gamma(TF_1), \forall Y \in \Gamma(TF_2).
\]

(16)

In analogy with the case of the almost contact structures we give the following definition:

**Definition 14.** A contact pair structure \((\alpha_1, \alpha_2, \phi)\) on a manifold \(M\) is said to be a normal contact pair if the Nijenhuis tensors \(N_J\) and \(N_T\) vanish identically.

The equation (13) states exactly the normality of the contact pair structure.

3.1. **Decomposable \(\phi\) and induced contact structures.** In this case we already remarked that the contact pair structure induces contact forms with structure tensor \(\phi\), on the leaves of the characteristic foliations \(F_1\) and \(F_2\) (Proposition 10). Applying Theorem 13 we have:

**Corollary 15.** Let \((\alpha_1, \alpha_2, \phi)\) be a contact pair with decomposable \(\phi\). The structure is normal if and only if the induced structures are normal and (16) is satisfied.

**Proof.** When \(\phi\) is decomposable, (14) and (15) are equivalent to the normality of the induced structures. \(\square\)

A partial converse of this corollary is the following:

**Corollary 16.** If \(\phi\) is decomposable and both characteristic foliations are normal for the induced structures, then \(J\) is integrable if and only if \(T\) is integrable.
Proof. Let us suppose that \( J \) is integrable. We want to prove that (14), (15) and (16) are satisfied. The first two equations are a consequence of the normality of the induced structures. Moreover, this implies

\[
[L_{Z_1}\phi](X) = 0 \quad \forall X \in \Gamma(TF_2),
\]

(17)

\[
[L_{Z_2}\phi](X) = 0 \quad \forall X \in \Gamma(TF_1).
\]

Because \( N_J \) vanishes, for \( i = 1, 2 \) we have \( N_J(X, Z_i) = 0 \) for every \( X \). Combining this with (17) and (18), we obtain \( L_{Z_i}\phi = 0 \). This implies that for \( X, Y \) tangent to different foliations

\[
0 = N_J(X, Y) = [\phi, \phi](X, Y),
\]

which gives (16). We argue similarly for \( T \) and this completes the proof. \( \square \)

An immediate consequence is the Theorem of Morimoto for a product of contact manifolds (see [20]). If \( J \) and \( T \) are the almost complex structures defined in (8) and (9) respectively, then we have:

**Corollary 17** ([20]). Suppose that \((M_1, \alpha_1, \phi_1)\) and \((M_2, \alpha_2, \phi_2)\) are contact manifolds with structure tensor \( \phi_1 \) and \( \phi_2 \) respectively. Then the contact pair structure \((\alpha_1, \alpha_2, \phi_1 \oplus \phi_2)\) on \( M_1 \times M_2 \) is normal if and only if \((\alpha_1, \phi_1)\) and \((\alpha_2, \phi_2)\) are normal as almost contact structures.

**Proof.** It is clear that in this case \( \phi \) is decomposable. If the almost contact structures on \( M_1 \) and \( M_2 \) are normal, then (14) and (15) are verified. Equation (16) is automatically satisfied if \( X \) and \( Y \) are tangent to different foliations because the manifold is a product and the vector fields can be supposed to commute. The converse is true by Corollary 15. \( \square \)

We give now an example of a manifold endowed with a normal contact pair, with decomposable \( \phi \) and where the induced structures are normal, but the manifold is not itself a product of two contact manifolds:

**Example 18.** Let \( M = \tilde{S}L_2 \) be the universal covering of the identity component of the isometry group of the hyperbolic plane \( \mathbb{H}^2 \) endowed with an invariant normal contact form \( \alpha \) (see [16]) and \( N = M \times M \). It is well known that \( N \) admits cocompact irreducible lattices \( \Gamma \) (see [13]). This means that \( \Gamma \) does not admit any subgroup of finite index which is a product of two lattices of \( M \). The manifold \( N \) can be endowed with the obvious contact pair structure and by the invariance of the contact forms by \( \Gamma \), the contact pair descends to the quotient and is normal. Even if the local structure is like a product, globally the foliations can be very interesting in the sense that both could have dense leaves.

Now we want to investigate deeply the condition \( L_{Z_i}\phi = 0 \), for \( i = 1, 2 \), since this condition is the analog of the \( K \)-contact condition for the almost contact structures. In the proof of Corollary 16 we saw that, if the induced structures are normal, the condition \( L_{Z_i}\phi = 0 \) is necessary to the integrability of both almost complex structures. One can ask if this condition together with the integrability of one of the almost complex structures is weaker than the integrability of both of them. We begin with the following proposition:

**Proposition 19.** Let \( M \) be a manifold endowed with a contact pair structure \((\alpha_1, \alpha_2, \phi)\) together with a decomposable \( \phi \) and suppose that the almost complex structure \( J \) associated to the pair is integrable. Let \( T \) be the almost complex structure associated to \((\alpha_2, \alpha_1, \phi)\). Then the following properties are equivalent:
(1) $T$ is integrable;
(2) $L_{Z_i} \phi = 0$;
(3) $L_{Z_2} \phi = 0$.

**Proof.** Suppose that both almost complex structures are integrable, then we have already seen in
the proof of Theorem 13 that this implies $L_{Z_i} \phi = 0$, $i = 1, 2$.

Conversely, since $J$ is integrable, for every $X \in \Gamma(TM)$ we have

$$0 = N_J(X, Z_2) = \phi([L_{Z_1} \phi](X)) - [L_{Z_2} \phi](X),$$

which implies that $[L_{Z_2} \phi](X) = 0$ if and only if $[L_{Z_1} \phi](X) = 0$. It remains to show that $T$
also integrable. This can be easily seen by calculating its Nijenhuis tensor $N_T(X, Y)$. One has just
to remark that when $X, Y$ are tangent to the same foliation, since $\phi$ is decomposable and $Z_1, Z_2$
are not in $\ker \alpha_1 \cap \ker \alpha_2$, then the equations obtained are exactly (14) and (15). Again, by the
decomposability of $\phi$, if $X$ and $Y$ are tangent to different foliations, one obtains (16).

Combining Theorem 13 and Proposition 19 we obtain the following theorem:

**Theorem 20.** Let $(\alpha_1, \alpha_2, \phi)$ be a contact pair structure on a manifold $M$ with a decomposable $\phi$
and such that $L_{Z_i} \phi = 0$ (resp. $L_{Z_2} \phi = 0$), then the following conditions are equivalent:

i) $J$ is integrable;
ii) $T$ is integrable;
iii) the induced structures are normal and $[\phi, \phi](X, Y) = 0$ for all $X \in \Gamma(TF_1)$ and for all
$Y \in \Gamma(TF_2)$.

Moreover these equivalent conditions imply $L_{Z_2} \phi = 0$ (resp. $L_{Z_1} \phi = 0$).

### 3.2. Non Morimoto case. In general when $\phi$ is decomposable, if the induced structures are normal, the
condition $L_{Z_i} \phi = 0$ for $i = 1, 2$ does not imply the normality of the whole structure. The
following examples show that the situation in the general case can be more complicated. In particular,
they show that there are contact pair structures with decomposable $\phi$ and normal induced
structures but, unlike the Morimoto construction, the contact pair structure is not normal. There
neither $J$ nor $T$ is integrable and (16) is not satisfied.

**Example 21.** Consider the simply connected Lie group $G$ with structure equations:

$$d\omega_1 = d\omega_6 = 0 \quad d\omega_2 = \omega_5 \wedge \omega_6$$

$$d\omega_3 = \omega_1 \wedge \omega_4 \quad d\omega_4 = \omega_1 \wedge \omega_5 \quad d\omega_5 = \omega_1 \wedge \omega_6,$$

where the $\omega_i$'s form a basis for the cotangent space of $G$ at the identity.

The pair $(\omega_2, \omega_3)$ is a contact pair of type $(1, 1)$ with Reeb vector fields $(X_2, X_3)$, the $X_i$'s being
dual to the $\omega_i$'s. Now define $\phi$ to be zero on the Reeb vector fields and

$$\phi(X_5) = X_6 \quad \phi(X_6) = -X_5 \quad \phi(X_1) = X_4 \quad \phi(X_4) = -X_1.$$ 

Since the kernel of $\omega_2 \wedge d\omega_2$ is generated by $X_1, X_3, X_4$, it is clearly preserved by $\phi$. The same
holds for the kernel of $\omega_3 \wedge d\omega_3$. Moreover $\phi$ is easy verified to be invariant under the flows of
the Reeb vector fields. The induced structures are normal, but not the whole structure because it is
well known that this Lie algebra does not admit any complex structure.

Since the structure constants of the group are rational, there exist lattices $\Gamma$ such that $G/\Gamma$ is
compact and then we obtain nilmanifolds carrying the same type of structure.
Example 22. The Lie group having the following structure equations admits invariant complex structures (see [24]):
\[ d\omega_1 = d\omega_2 = d\omega_3 = 0 \ , \ d\omega_4 = \omega_1 \wedge \omega_2 \ , \ d\omega_5 = \omega_1 \wedge \omega_3 \ , \ d\omega_6 = \omega_2 \wedge \omega_4 . \]
The pair \((\omega_5, \omega_6)\) is a contact pair of type \((1, 1)\). A straightforward calculation shows that every invariant contact pair structure of type \((1, 1)\) with invariant and decomposable \(\phi\) has normal induced structures but the whole structure is not normal since \([1\overline{6}]\) is not satisfied.

According to the result of Morimoto (Corollary [1\overline{7}]), the manifolds carrying contact pair structures in the previous examples can not be, even locally, products of manifolds endowed with normal contact forms.

3.3. Contact pairs of type \((h, 0)\). In the particular case of a manifold \(M\), endowed with a contact pair structure \((\alpha_1, \alpha_2, \phi)\) of type \((h, 0)\), the 1-form \(\alpha_2\) is closed and the Nijenhuis tensors of the almost complex structures \(J\) and \(T\) associated to the pair simplify further. Moreover the tensor \(\phi\) is automatically decomposable because \(\alpha_2 \circ \phi = 0\) implies that \(\phi(T\mathcal{F}_2) \subset T\mathcal{F}_2\). Since \(\Gamma(T\mathcal{F}_1)\) is generated by \(Z_2\) and \(\phi(Z_2) = 0\), we also have \(\phi(T\mathcal{F}_1) \subset T\mathcal{F}_1\).

The following is a variation of the Theorem [2\overline{0}] for contact pairs of type \((h, 0)\):

**Theorem 23.** Let \(M\) be a manifold endowed with a contact pair structure \((\alpha_1, \alpha_2, \phi)\) of type \((h, 0)\), such that \(L_{Z_2} \phi = 0\). Then \((\alpha_1, \alpha_2, \phi)\) is a normal contact pair if and only if \((\alpha_1, \phi)\) induced on every leaf of \(\mathcal{F}_2\) is normal.

Defining normality for a contact manifold \((M, \alpha)\) with structure tensor \(\phi\), is the same as considering the contact pair \((\alpha, dt)\) on \(M \times \mathbb{R}\) and asking for its almost complex structure to be integrable. This is exactly the local situation of the previous theorem.

**Remark 24.** A manifold endowed with a normal contact pair of type \((h, 0)\) can be viewed as an even analog of a cosymplectic manifold.

We end this section with the following example:

**Example 25.** Let us consider the simply connected nilpotent Lie group \(Nil^4\), having the following structure equations:
\[ d\omega_1 = d\omega_4 = 0 \ , \ d\omega_2 = \omega_1 \wedge \omega_4 \ , \ d\omega_3 = \omega_2 \wedge \omega_4 \ . \]
The pair \((\omega_3, \omega_1)\) is a contact pair of type \((1, 0)\). Since the structure constants of the group are rational, then there exist cocompact lattices and the corresponding nilmanifold are endowed with a contact pair structure and hence with an almost complex structure. Nevertheless this contact pair can not be normal since no such nilmanifold admits complex structures. This can be seen for example by saying that such a nilmanifold has first Betti number \(b_1 = 2\) (see [2\overline{1}]) and if it is complex with even first Betti number then it must be Kähler by [1\overline{4}]. But the only nilmanifolds which are Kähler must be Tori (see [8]) and this is not the case.

3.4. Remarks on bicontact Hermitian manifolds. Contact pairs appeared firstly in [1\overline{2}], where they arose in the context of the Hermitian geometry with the name bicontact.

More precisely a bicontact Hermitian manifold is a Hermitian manifold \((M, J, g)\) together with a unit vector field \(U\) such that \(U\) and \(V = JU\) are infinitesimal automorphisms of the Hermitian structure. Let \(u\) and \(v\) be the covariant forms of \(U\) and \(V\) respectively. The bicontact manifold \(M\) is said to be of bidegree \((1, 1)\) if \(du\) is of bidegree \((1, 1)\) and in this case \(dv\) is of bidegree \((1, 1)\) too.
Actually, a bicontact Hermitian manifold \((M, J, g, U)\) of bidegree \((1, 1)\) can be regarded as a manifold endowed with a normal contact pair structure \((u, v, \phi)\), where \(\phi = J + v \otimes u - u \otimes V\) is decomposable, together with a metric \(g\) which is compatible in the sense of \([5]\). This easily follows from Propositions 2.7 and 2.8 of \([12]\) and by the fact that the bidegree \((1, 1)\) of \(du\) implies the decomposability of \(\phi\). By using Propositions \([19, 10]\) and the local model for a contact pair (see \([2, 4]\)), Theorem 4.4 of \([12]\) can be restated in terms of normal contact pairs with decomposable \(\phi\). Moreover, Theorem 4.4 of \([12]\) implies the necessary condition of Corollary \([15]\).

4. CONSTRUCTIONS ON FLAT BUNDLES

Flat bundles are fibre bundles with a foliation transverse and complementary to the fibre and have been useful to construct symplectic pairs in \([7]\). In the same paper was pointed out that one can use these bundles to construct contact pairs. We describe the general construction of flat bundles and then we specialize to contact pair structures.

Let \(B\) and \(F\) be two connected manifolds, and let \(\rho: \pi_1(B) \to \text{Diff}(F)\) be a representation of the fundamental group of \(B\) in the group of diffeomorphisms of \(F\). The suspension of \(\rho\) defines a horizontal foliation (whose holonomy is \(\rho\)) on the fiber bundle \(\pi: M_\rho \to B\) with fiber \(F\) and total space

\[ M_\rho = (\tilde{B} \times F)/\pi_1(B), \]

where \(\pi_1(B)\) acts on the universal covering \(\tilde{B}\) by covering transformations and on \(F\) via \(\rho\). We have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{B} \times F & \xrightarrow{\pi_\rho} & (\tilde{B} \times F)/\pi_1(B) \\
Q \downarrow & & \downarrow \pi \\
\tilde{B} & \xrightarrow{p} & B
\end{array}
\]

where \(Q\) is the projection on the first factor, \(p\) the covering projection, \(\pi\) the projection of the bundle and \(\pi_\rho\) the quotient map.

Let us consider contact manifolds \((B, \alpha_1, Z_1, \phi_1)\) and \((F, \alpha_2, Z_2, \phi_2)\) with structure tensors \(\phi_1\) and \(\phi_2\) respectively. Instead of taking a representation of \(\pi_1(B)\) in \(\text{Diff}(F)\), we take a representation \(\rho\) in \(\text{Cont}(F, \phi_2)\), the group of contactomorphisms preserving \(\phi_2\), and we construct the flat bundles by using this representation.

Let \((\alpha_1, \alpha_2, \phi_1 \oplus \phi_2)\) be the contact pair structure of \(B \times F\), \(J\) its almost complex structure and \(T\) the almost complex structure of \((\alpha_2, \alpha_1, \phi_1 \oplus \phi_2)\). Then, by Morimoto’s result (Corollary \([17]\), \(J\) is integrable if and only if \((B, \alpha_1, Z_1, \phi_1)\) and \((F, \alpha_2, Z_2, \phi_2)\) are normal and this if and only if \(T\) is integrable.

The manifold \(\tilde{B} \times F\) is naturally endowed with a contact pair structure \((\tilde{\alpha}_1, \alpha_2, \tilde{\phi}_1 \oplus \phi_2)\) where \(\tilde{\alpha}_1\) and \(\tilde{\phi}_1\) are the lift to \(\tilde{B}\) of \(\alpha_1\) and \(\phi_1\) respectively. The almost complex structure \(\tilde{J}\) associated to it, is the lift of \(J\) and then it is integrable if and only if \(J\) is integrable.

Since \((\tilde{\alpha}_1, \alpha_2, \tilde{\phi}_1 \oplus \phi_2)\) is invariant by the action of \(\pi_1(B)\), the total space of the flat bundle

\[ M_\rho = (\tilde{B} \times F)/\pi_1(B) \]

is endowed with a contact pair structure, denoted by \((\tilde{\alpha}_1, \alpha_2, \tilde{\phi}_1 \oplus \phi_2)\). The almost complex structure \(\tilde{J}\) descends to the quotient and it defines the almost complex structure \(J_\rho\) of \((\tilde{\alpha}_1, \alpha_2, \tilde{\phi}_1 \oplus \phi_2)\). Then \(J_\rho\) is integrable if and only if its lift \(\tilde{J}\) is integrable. Starting with \(T\), we obtain the almost complex structure \(T_\rho\) associated to \((\alpha_2, \tilde{\alpha}_1, \tilde{\phi}_1 \oplus \phi_2)\).

The above discussion yields the following theorem:
\textbf{Theorem 26.} Let \((B, \alpha_1, Z_1, \phi_1)\) and \((F, \alpha_2, Z_2, \phi_2)\) be two connected contact manifolds with structure tensors \(\phi_1\) and \(\phi_2\) respectively and \(\rho\) any representation of \(\pi_1(B)\) in \(\text{Cont}(F, \phi_2)\). Then the flat bundle \(M_\rho = (\tilde{B} \times F) / \pi_1(B)\), is naturally endowed with a contact pair structure \((\tilde{\alpha}_1, \alpha_2, \tilde{\phi}_1 \oplus \phi_2)_\rho\). This contact pair structure is normal if and only if \((B, \alpha_1, \phi_1)\) and \((F, \alpha_2, \phi_2)\) are normal.

By choosing normal contact forms on \(B\) and \(F\) and a non trivial representation \(\rho\), this construction furnishes examples of complex manifolds which are locally but not globally product of contact manifolds as in Morimoto’s theorem (see Corollary \([17]\)). Here is an explicit example:

\textbf{Example 27.} Consider a closed manifold \(F\) endowed with a normal \(K\)-contact structure, for example Sasakian, \((\alpha_2, Z_2, \phi_2, g)\). In this case the one parameter group of diffeomorphisms \(\{\varphi_t, t \in \mathbb{R}\}\) generated by the flow of the Reeb vector field \(Z_2\) is a non trivial subgroup of \(\text{Cont}(F, \phi_2)\). Pick any element \(\varphi_a\) which is not the identity. Let \(B = \text{Nil}^3 / \Gamma\) where \(\text{Nil}^3\) is the Heisenberg group of upper triangular real \((3 \times 3)\) matrices

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

and \(\Gamma\) the discrete subgroup consisting of all its elements with integer entries. Then \(B\) is a closed 3-manifold for which \(\pi_1(B) = \Gamma\), since \(\text{Nil}^3\) is simply connected. The invariant normal contact structure \((\omega, Z, \Phi)\) on \(\text{Nil}^3\) given by \(\omega = dz - xdy\) and \(\Phi\) defined by \(\Phi(\frac{\partial}{\partial z}) = -\left(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z}\right)\) descends to \(B\) as a normal contact structure \((\alpha_1, Z_1, \phi_1)\) (see \([16]\)). Now choose the representation \(\rho : \pi_1(B) \to \text{Cont}(F, \phi_2)\) defined by

\[
\rho \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} = (\varphi_a)^p,
\]

for all integers \(p, q\) and \(r\). In the same way, we can find other examples by using the Geiges’ classification \([16]\) and \([23]\).

5. \textbf{Constructions on Boothby–Wang Fibrations}

In this section we use the Boothby–Wang fibration to construct \(S^1\)-invariant contact pair structures on the total space of a principal circle bundle over a base space endowed with a contact-symplectic pair structure.

For a given closed manifold \(B\) endowed with a contact-symplectic pair \((\beta, \eta)\), if \(\eta\) has integral cohomology class, as showed in Section\([2]\) one can construct a Boothby–Wang fibration and obtain as total space a manifold \(M\) endowed with a contact pair \((\alpha_1, \alpha_2)\) where \(\alpha_2 = \pi^* (\beta)\) and \(\alpha_1\) is the connection form of the bundle and then \(d\alpha_1 = \pi^* (\eta)\), where \(\pi\) is the bundle projection. Let \(Z_1\) and \(Z_2\) be the Reeb vector fields of \((\alpha_1, \alpha_2)\). Then \(Z_1\) is tangent to the action and \(Z_2\) is the horizontal lift of the Reeb vector field \(W\) of \((\beta, \eta)\) with respect to the connection form \(\alpha_1\).

The following result is the analog for contact pair structures of the construction used in \([20]\)\([17]\):

\textbf{Theorem 28.} The total space \(M\) of a Boothby–Wang fibration over a closed base space \(B\) endowed with an almost contact-symplectic structure \((\beta, \eta, \psi)\), where \([\eta]\) \(\in H^2(B, Z)\), is naturally endowed with a \(S^1\)-invariant contact pair structure \((\alpha_1, \alpha_2, \phi)\). Moreover, if \(\psi\) is decomposable so is \(\phi\).
Proof. With the previous notations, let \((\alpha_1, \alpha_2)\) be the contact pair on \(M\), with Reeb vector fields \(Z_1, Z_2\). For any tangent vector \(Y\) of \(B\) at \(q = \pi(p)\), we denote by \(Y_p^*\) the horizontal lift (with respect to the connection form \(\alpha_1\)) of \(Y\) at \(p \in M\). Let \(\phi\) be the endomorphism of \(TM\) defined as follows
\[
\phi_pX = (\psi_sX)_p^*,
\]
for every \(X \in T_pM\), \(\pi^s\) being the differential of the projection \(\pi\).

The triple \((\alpha_1, \alpha_2, \phi)\) is a contact pair structure on \(M\). To see that, we first remark that \(\phi(X^*) = (\psi X)^* = (\pi_s X)^* = X - \alpha_1(X)Z_1\). Then we have
\[
\phi^2(X) = \phi(\phi^*P^*X) = \phi^2P^*X = (\pi_s X + \beta(\pi_s X)W)^* = -X + \alpha_1(X)Z_1 + \alpha_2(X)Z_2.
\]
Moreover, we have \(\phi Z_1 = 0\), because \(\pi_s Z_1 = 0\) and \(\phi Z_2 = \phi(W^*) = (\psi W)^* = 0\), because \(\psi W = 0\) by the definition of almost contact-symplectic structure. If \(\psi\) is decomposable, the decomposability of \(\phi\) can be easily verified on lifted vector fields. Observe that \(L_{Z_1} \phi = 0\) by construction. \(\square\)

Now we want to relate the normality of the contact pair structure on the total space to that of the almost contact-symplectic structure on the base. With the previous notations we have:

Lemma 29. Let \(B\) be a closed manifold endowed with an almost contact-symplectic structure \((\beta, \eta, \psi)\) with \([\eta] \in H^2(B, \mathbb{Z})\) and \(M\) the total space of the corresponding Boothby–Wang fibration, endowed with the \(S^1\)-invariant contact pair structure \((\alpha_1, \alpha_2, \phi)\) of Theorem 28. Then the almost complex structure \(J\) associated to \((\alpha_1, \alpha_2, \phi)\) is integrable if and only if the following conditions on the base are satisfied:

\[
\begin{align}
-2\eta(\psi X, \psi Y) + 2\eta(X, Y) - d\beta(\psi X, Y) - d\beta(X, \psi Y) &= 0 \quad (19) \\
[\psi, \psi](X, Y) + 2d\beta(X, Y) + \eta(\psi X, Y) + \eta(X, \psi Y) &= 0 \quad (20) \\
L_W \psi &= 0 \quad (21)
\end{align}
\]

Proof. The tensor \(N_J\) vanishes if and only if \(N_J(Z_1, X^*) = 0\) and \(N_J(X^*, Y^*) = 0\) for every lifted vector fields \(X^*, Y^*\) (with respect to the connection form \(\alpha_1\)) and for the vertical vector field \(Z_1\). A straightforward calculation shows that \(N_J(Z_1, X^*) = 0\) is equivalent to (21) and \(N_J(X^*, Y^*) = 0\) is equivalent to (19) and (20). \(\square\)

As a consequence of the above lemma we have:

Theorem 30. With the same assumptions as in Lemma 29, if \(\eta\) is invariant under \(\psi\), that is \(\eta(\psi X, \psi Y) = \eta(X, Y)\), the \(S^1\)-invariant contact pair structure on the total space of the Boothby–Wang fibration has integrable \(J\) if and only if the almost contact-symplectic structure on the base is a normal almost contact structure.

Proof. If \(\eta\) is invariant under \(\psi\), the conditions (19), (20) and (21) reduce to the following system
\[
\begin{align}
L_W \psi &= 0 \\
-d\beta(\psi X, Y) - d\beta(X, \psi Y) &= 0 \\
[\psi, \psi](X, Y) + 2d\beta(X, Y) &= 0.
\end{align}
\]

The third equation implies the others and it is exactly the condition for \((\beta, W, \psi)\) to be a normal almost contact structure. \(\square\)
Theorem 31. With the same assumptions as in Lemma 29, let us suppose that \( \eta \) is invariant under \( \psi \) and that \( \psi \) is decomposable. Then the \( S^1 \)-invariant contact pair structure on the total space of the Boothby–Wang fibration is normal if and only if the almost contact-symplectic structure on the base is a normal almost contact structure.

Proof. Theorem 28 implies that \( \psi \) is decomposable and \( L_{Z_1} \phi = 0 \). By Theorem 20, the normality of the pair is equivalent to the integrability of \( J \), which follows from Theorem 30. \( \square \)

We end this section with some examples:

Example 32. Taking for example a flat bundle where the base space is a closed Kähler manifold with integral Kähler class (that is a projective variety) and the fiber is a closed normal contact manifold, yields a contact symplectic pair verifying the assumptions of Theorem 31.

Example 33. If the almost contact-symplectic structure \((\beta, \eta, \psi)\) has decomposable \(\psi\) and is endowed with an associated metric as in Subsection 2.2, then the assumptions of Theorem 31 are satisfied.

Example 34. Using the double Boothby-Wang fibration over a closed manifold \( B \) endowed with a symplectic pair \((\omega_1, \omega_2)\) such that \([\omega_i] \in H^2(B, \mathbb{Z})\) and a complex structure \( J \) preserving the tangent spaces of the foliations and compatible, on each leaf, with the symplectic form induced by the pair, also gives an example for the Theorem 31.

An interesting example of the former situation, already used in [7], is given by the quotient of a polydisc \( \mathbb{H}^2 \times \mathbb{H}^2 \) by an irreducible lattice of the identity component of its isometry group, where \( \mathbb{H}^2 \) is the hyperbolic plane. In this case the pair is given by the Kähler forms on each factor and the corresponding cohomology classes are integral. More generally one could consider a product of \( n \) copies of \( \mathbb{H}^2 \).

Remark 35. Again with the Boothby–Wang fibration we obtain new constructions of closed complex manifolds.

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