Cartwright-type and Bernstein-type theorems
for functions analytic in a cone

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Abstract
Cartwright-type and Bernstein-type theorems, previously known only for functions of exponential type in $\mathbb{C}^n$, are extended to the case of functions of arbitrary order in a cone.

1 Introduction
We use standard notations of multidimensional complex analysis.

Let $C$ be an open cone in $\mathbb{C}^n$ with vertex at the origin. By $[\rho, \sigma, C]$ we denote the class of all functions $f(z)$, holomorphic in $C$ and satisfying the estimate

$$\limsup_{|z| \to \infty, z \in C} \frac{\log |f(z)|}{|z|^\rho} \leq \sigma, \quad |z|^2 = |z_1|^2 + \ldots + |z_n|^2.$$ 

For entire functions we write simply $[\rho, \sigma]$. Thus $[1, \sigma]$ is the class of entire functions of exponential type not exceeding $\sigma$ in $\mathbb{C}^n$.

By $[\rho, \infty, C]$ we denote $\bigcup_{\sigma>0} [\rho, \sigma, C]$.

In this paper we make systematical use of the possibility of a "good" approximation of a function analytic in a cone by entire functions with control of growth. In the case of dimension 1 such an approximation was constructed by M.V.Keldysh [K], For the case of several variables, the result showing the possibility of such an approximation is due to the second author [R]. To formulate this result we introduce some notations.

Let $\omega$ and $\varphi$ be plurisubharmonic functions in $\mathbb{C}^n$, both possessing the "non - oscillating" property

$$(u)^{[1]}(z) \leq -A(-u)^{[1]}(z) + B,$$

where by $u^{[r]}(z)$ we denote $\sup\{u(w) : |z - w| < r\}$. Assume also that $\varphi(z) \geq 0$, $\log(1 + |z|) = o(\varphi(z))$, $|z| \to \infty$. For $\varepsilon \geq 0$ we denote by $\Omega_\varepsilon$ the set

$$\Omega_\varepsilon = \{z \in \mathbb{C}^n : \omega(z) < -\varepsilon \varphi(z)\}.$$
and suppose that
\[ \forall \varepsilon_1 > \varepsilon_2 : \inf \{|z_1 - z_2| : z_1 \in \Omega_{\varepsilon_1}, z_2 \in \mathbb{C}^n \setminus \Omega_{\varepsilon_2}\} > 0, \]
which is a kind of smoothness condition on \( \omega \) and \( \varphi \).

**Theorem A.** Let \( f(z) \) be an analytic function in \( \Omega_0 \) satisfying the estimate
\[ |f(z)| \leq C_f e^{C_f \varphi(z)}, \ z \in \Omega_0. \]
Then for each \( \varepsilon > 0 \) and each \( N \geq 1 \) there exists such an entire function \( g(z) \) that
\[ |f(z) - g(z)| \leq Ce^{-N\varphi(z)}, \ z \in \Omega_{\varepsilon}, \]
\[ |g(z)| \leq Ce^{\max(N,C_f)\left(\frac{2}{\varepsilon^2} \omega + \varphi\right)(z)}, \ z \in \mathbb{C}^n, \]
where \( C \) does not depend on \( N \).

We first apply Theorem A to prove results concerning "boundedness" of functions analytic in a cone. It is natural to call theorems of this kind Cartwright-type theorems. Results of such kind were known before only for entire functions of exponential type.

**Definition.** Let \( E \) and \( F \) be subsets of \( \mathbb{R}^n \), \( E \) being measurable. The set \( E \) is called **relatively dense with respect to** \( F \), if for some positive constants \( L \) and \( \delta \) and every \( x \in F \)
\[ |E \cap B(x,L)| \geq \delta. \]
Here \( |A| \) denotes the Lebesgue measure of a (measurable) set \( A \), and \( B(x,L) \) is the ball \( \{y \in \mathbb{R}^n : |x - y| < L\} \). The values of \( L \) and \( \delta \) are called the density parameters.

**Definition.** Let \( E \) and \( F \) be subsets of \( \mathbb{R}^n \), \( E \) being measurable. The set \( E \) is called **relatively dense of order** \( \rho \) **with respect to** \( F \), if its image under the map \( x_j \mapsto x_j^\rho, \ j = 1, \ldots, n \), is relatively dense with respect to the image of \( F \).

**Definition.** A set \( E \subset \mathbb{R}^n \) is called an **\( \varepsilon \) - net** for a set \( F \subset \mathbb{R}^n \) if for every \( x \in F \) there exists such a point \( y \in E \) that
\[ |x - y| < \varepsilon. \]

**Definition.** A set \( E \subset \mathbb{R}^n \) is called an **\( \varepsilon \) - net of order** \( \rho \) for a set \( F \subset \mathbb{R}^n \) if its image under the map \( x_j \mapsto x_j^\rho, \ j = 1, \ldots, n \), is an \( \varepsilon \) - net for the image of \( F \) under this map.

Note that \( \varepsilon \) - nets may be discrete sets.
Given \( \eta \in (0, 1) \), denote by \( C(\eta) \) the cone in the positive hyperoctant \( \mathbb{R}_+^n \) defined by the relation

\[
C(\eta) = \left\{ x \in \mathbb{R}_+^n : \min_{j=1,\ldots,n} x_j \geq \eta \max_{j=1,\ldots,n} x_j \right\}.
\]

The results on entire functions of exponential type are formulated as follows:

**Theorem B** [L1]. Let a set \( E \) be relatively dense with respect to \( \mathbb{R}_+^n \). Then each entire function \( f \in [1, \infty) \) bounded on \( E \) is bounded on \( C(\eta) \) for every fixed \( \eta \in (0, 1) \).

Moreover, for each \( \sigma \in (0, \infty) \) there exists such a finite value \( \Delta = \Delta(E, \sigma, \eta) \), that

\[
\sup_{x \in C(\eta) \setminus B(0, R)} |f(x)| \leq \Delta \cdot \sup_{x \in E} |f(x)|
\]

for some \( R = R(f, \eta) < \infty \).

**Theorem B’** [L1]. Let \( E \) be an \( \varepsilon \)-net for \( \mathbb{R}_+^n \), and let a number \( \eta \in [0, 1) \) be given.

Then there exists such a number \( \sigma_0 = \sigma_0(n, E, \varepsilon, \eta) > 0 \) that for every \( \sigma \in (0, \sigma_0) \), each function \( f \in [1, \sigma] \) bounded on \( E \) is bounded on \( C(\eta) \).

Moreover, there is such a finite value \( \Delta = \Delta(E, \sigma, \varepsilon, \eta) \), that

\[
\sup_{x \in C(\eta) \setminus B(0, R)} |f(x)| \leq \Delta \cdot \sup_{x \in E} |f(x)|
\]

for some \( R = R(f, \eta) < \infty \).

Below we formulate analogues of the above theorems for functions holomorphic in cones. Note that we will be able to consider also functions of order \( \rho \) different from 1.

For \( \rho \geq 1 \) put \( W_\rho = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : -\frac{\pi}{2\rho} < \arg z_j < \frac{\pi}{2\rho}, \ j = 1, \ldots, n\} \). Note that \( W_1 = \mathbb{C}_+^n \) (\( \mathbb{C}_+ \) stands for the right halfplane) and that \( W_\rho \cap \mathbb{R}^n = \mathbb{R}_+^n \) for each \( \rho \).

Our first result is

**Theorem 1.** Let \( E \) be a relatively dense set of order \( \rho \geq 1 \) with respect to \( \mathbb{R}_+^n \).

Then for every \( \eta \in (0, 1) \) each function \( f \in [\rho, \infty, W_\rho] \), which is analytic in a neighborhood of the origin and bounded on \( E \) is bounded on \( C(\eta) \).

Moreover, for each \( \sigma \in (0, \infty) \) there exists such a finite value \( \Delta = \Delta(E, \sigma, \eta) \), that

\[
\sup_{x \in C(\eta) \setminus B(0, R)} |f(x)| \leq \Delta \cdot \sup_{x \in E} |f(x)|
\]

for some \( R = R(f, \eta) < \infty \).
The corresponding result for $\varepsilon$-nets is

**Theorem 2.** Let $E$ be an $\varepsilon$-net of order $\rho \geq 1$ for $\mathbb{R}^n_+$ and let a number $\eta \in (0,1)$ be given.

Then there exists such a number $\sigma_0 = \sigma_0(n, E, \varepsilon, \eta) > 0$ that for every $\sigma \in [0, \sigma_0)$, each function $f \in [\rho, \sigma, W_{\rho}]$ which is analytic near the origin and bounded on $E$ is bounded on $C(\eta)$.

Moreover, there is such a finite value $\Delta = \Delta(E, \sigma, \varepsilon, \eta)$, that

$$\sup_{x \in C(\eta) \setminus B(0,R)} |f(x)| \leq \Delta \cdot \sup_{x \in E} |f(x)|$$

for some $R = R(f, \eta) < \infty$.

**Remark.** Theorem B' and respectively Theorem 2 may be formally slightly strengthened by assuming the set $E$ to be an $\varepsilon$-net not for the whole $\mathbb{R}^n_+$ but for its relatively dense subset.

**Remark.** To the best of our knowledge, Theorems 1 and 2 are new in the case $\rho > 1$ even for entire functions.

**Remark.** There are examples showing sharpness (in a certain sense) of theorems B and B'; for instance, it was shown [L1] that the results fail to hold if we do not truncate the cone $C(\eta)$ by the ball $B(0,R)$, and that the value of $R$ cannot be chosen independent of $f \in [1, \sigma]$, etc. The same examples with obvious modifications play a similar role for theorems 1 and 2.

Next we mention V. Bernstein-type theorems for entire functions of finite order. By this we mean results giving conditions on sets sufficient for calculation of the (radial) indicator. Remind that the radial indicator of a function $f(z) \in [\rho, \infty)$ is defined as follows:

$$h_f(z) = \limsup_{w \to z} \limsup_{t \to \infty} \frac{\log |f(tw)|}{t^\rho}.$$ 

For the case of dimension 1 the first lim sup (regularization) may be omitted. We refer to [Ro] for the properties of the radial indicator.

V. Bernstein [Be] was the first to give a sufficient condition on a set $E$ on a ray which guarantees that

$$h_f(1) = \limsup_{t \to \infty, t \in E} \frac{\log |f(t)|}{t^\rho}.$$ 

The references to the further results in this direction are given in [L2]. We mention below results of the first author concerning entire functions in $\mathbb{C}^n$. 

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Definition. Let \( \varepsilon(R), R \in \mathbb{R}_+ \), be a function monotonically decreasing to zero as \( R \to \infty \). A set \( E \subset \mathbb{R}^n \) is called an \( \varepsilon(R) - \text{net} \) for a set \( F \subset \mathbb{R}^n \) if for each \( x \in F \) there exists \( y \in E \) such that

\[
|x - y| \leq \varepsilon(|x|).
\]

Definition. A set \( E \subset \mathbb{R}^n \) is called an \( \varepsilon(R) - \text{net of order } \rho \) for a set \( F \subset \mathbb{R}^n \) if its image under the map \( x_j \mapsto x_j^\rho, j = 1, \ldots, n \), is an \( \varepsilon(R) - \text{net} \) for the image of \( F \) under this map.

Theorem C \[L2\]. Let a set \( E \) be an \( \varepsilon(R) - \text{net of order } \rho \in (0, \infty) \) for some cone \( C(\eta_0) \).

Then the relation

\[
h_f\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) = \lim_{\eta \to 0} \limsup_{|x| \to \infty, x \in E \cap C(\eta)} \frac{\log |f(x)|}{|x|^\rho}
\]

holds for every function \( f \in [\rho, \infty) \).

Theorem C yields the following uniqueness result.

Theorem D \[L2\]. Let \( E \) be as in theorem C and let

\[
\limsup_{|x| \to \infty, x \in E} \frac{\log |f(x)|}{|x|^\rho} = -\infty
\]

for some function \( f \in [\rho, \infty) \).

Then \( f(z) \equiv 0 \).

The cones \( W_\tau \) were defined above for \( \tau \geq 1 \). Now we would like to extend the definition to all \( \tau > \frac{1}{2} \). For \( \tau \in (\frac{1}{2}, 1) \) define \( W_\tau \) to be the same as \( W_{\frac{2\tau}{2\tau - 1}} \).

Our theorem 3 below is an analogue of theorem C for functions holomorphic in cones.

**Theorem 3.** Let a set \( E \) be an \( \varepsilon(R) - \text{net of order } \rho \geq \frac{1}{2} \) for some cone \( C(\eta_0) \).

Then the relation

\[
h_f\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) = \lim_{\eta \to 0} \limsup_{|x| \to \infty, x \in E \cap C(\eta)} \frac{\log |f(x)|}{|x|^\rho}
\]

holds for every function \( f \in [\rho, \infty, W_\tau), \tau > 0 \).

Note that while the indicator of an entire function of finite type \( \sigma \) is bounded below by \(-\sigma \) \[Ro\], the (regularized) indicator of a function holomorphic in a cone needs not
to be bounded from below. Hence the corresponding uniqueness result holds only if the cone \( W_\tau \), in which our function is defined, is wide enough.

**Theorem 4** Let \( E \) be as in theorem 3 with \( \rho > 2 \) and let

\[
\limsup_{|x| \to \infty, \ x \in E} \frac{\log |f(x)|}{|x|^{\rho}} = -\infty
\]

for some function \( f \in [\rho, \infty, W_\tau), \ \tau \leq \frac{\rho}{2} \).

Then \( f(z) \equiv 0 \).

## 2 Some remarks concerning cones in \( \mathbb{C}^n \)

In this paper we will deal mainly with two types of cones in \( \mathbb{C}^n \).

One of them, \( W_\tau \), is defined in the previous section. We introduce another one.

For \( t > 0 \) denote by \( ||.||_t \) a norm in \( \mathbb{C}^n \) given by

\[
||z||_t = \max_{j=1,\ldots,n} \{|\text{Re}z_j|, |\text{Im}z_j|t\}.
\]

By \( Y_t(\eta), \ \eta \in [0,1) \), we denote the cone in \( \mathbb{C}^n \) given by

\[
Y_t(\eta) = \left\{ z \in \mathbb{C}^n_+ : \min_{j=1,\ldots,n} \text{Re}z_j \geq \eta||z||_t \right\}.
\]

Note that for all \( t > 0 \) the intersection of \( Y_t(\eta) \cap \mathbb{R}^n_+ \) is exactly the real cone \( C(\eta) \).

Obviously, \( Y_t(0) = \mathbb{C}^n_+ \).

The geometry of the cone \( Y_t(\eta) \) is very simple. We just observe that the ray \( \ell = \{\xi(1,\ldots,1), \xi > 0\} \) lies on the complex line \( \mathcal{L} = \{z_1 = \ldots = z_n\} \) which has the largest intersection with \( Y_t(\eta) \):

\[
\mathcal{L} \cap Y_t(\eta) = \{ w \in \mathbb{C} : |\arg w| < \arctan t\eta \}.
\]

One easily sees that, given a number \( \tau > 0 \), it is possible to choose such \( t \) and \( \eta \) that

\[
Y_t(\eta) \subset W_\tau
\]

and

\[
\mathcal{L} \cap Y_t(\eta) = \mathcal{L} \cap W_\tau.
\]
We would like to write each of the two types of cones in the form \( \{ z \in \mathbb{C}^n : u(z) < 0 \} \) for some plurisubharmonic function \( u(z) \) in \( \mathbb{C}^n \). For \( Y_t(\eta) \) we can take \( u(z) \) to be of order 1:
\[
u(z) = \max_{j=1,\ldots,n} (-Re z_j) + \eta \| z \|_t,
\]
while for \( W_\tau \) with \( \tau \in (1/2,1) \) one can take function of any order \( \rho \in [\tau,1) \):
\[
u(z) = \max_{j=1,\ldots,n} u_j(z),
\]
where
\[
u_j = \begin{cases} |z_j|^\rho \sin(\rho (\arg z_j - \pi / 2)) & \text{if } |\arg z_j| < \pi / 2 + \pi / 2\rho; \\
|z_j|^\rho & \text{if } |\arg z_j| \in [\pi / 2 + \pi / 2\rho, \pi]. \end{cases}
\]

3 Proof of Theorems 1 and 2

Proof of Theorem 1. The idea of the proof is to approximate the function \( f(z) \) by an entire function \( g(z) \) with the help of theorem A, apply theorem B to \( g(z) \) and derive the required estimates for \( f(z) \).

First we note that theorem B may be reformulated for an arbitrary cone \( C \subset \mathbb{R}^n \), since it is always possible to find such an automorphism \( \psi : \mathbb{C}^n \to \mathbb{C}^n \), that \( \psi(\mathbb{R}^n_+) \subset C \), and to consider \( f(\psi(z)) \) instead of \( f(z) \) which results in an obvious recalculation of all coefficients and does not affect the order of the holomorphic function.

Due to the possibility of the transformation \( z_j \mapsto z_j^\rho \), \( j = 1, \ldots, n \) which takes cones (with vertex in the origin) into cones, particularly, \( C(\eta) \) into \( C(\eta^{1/\rho}) \), relatively dense sets of order \( \rho \) into relatively dense sets of order 1 and holomorphic functions of order \( \rho \) in \( W_\rho \) into holomorphic functions of order 1 in \( W_1 = \mathbb{C}^n^+ \) it is enough to assume \( \rho = 1 \) in what follows.

Next we define functions \( \omega \) and \( \varphi \) in the following way. Put
\[
\omega = \max_{j=1,\ldots,n} (-Re z_j), \quad \varphi = \max(\delta, \| z \|_t),
\]
where \( t > 0 \) is arbitrary, and we choose \( \delta = \log \frac{1}{\sup_{x \in E} |f(x)|} \).

Then the set \( \Omega_0 = \{ z : \omega(z) < 0 \} \) is exactly \( \mathbb{C}^n^+ \), and for \( \varepsilon \in (0,1) \) the set \( \Omega_\varepsilon = \{ z : \omega(z) < -\varepsilon \varphi(z) \} \) (which is \( Y_t(\varepsilon) \) without some neighborhood of the vertex) has the property \( \Omega_\varepsilon \cap \mathbb{R}^n_+ \supset C(\varepsilon) \setminus B(0,\varepsilon \delta) \). It is clear that the conditions of theorem A are satisfied.
Let \( f(z) \) be a given function from the class \([1, \sigma, W_1]\). By theorem A, there exists such an entire function \( g(z) \) of exponential type \( \leq K\sigma \) with \( K = K(n, \varepsilon) \) not depending on \( f \in [1, \sigma]_+ \) that \( |f(z) - g(z)| \leq e^{-\varphi}, \ z \in \Omega_\varepsilon \). Our choice of \( \delta \) implies that

\[
\sup_{x \in E \cap (C(\varepsilon) \setminus B(0, \varepsilon))} |g(x)| \leq 2 \sup_{x \in E} |f(x)|.
\]

The function \( f^*(z) = g(z - \delta) \) is an entire function belonging to the class \([1, K\sigma]\) bounded by \( 2 \sup_{x \in E} |f(x)| \) on a set \( E^* = \{ z + \delta : \ z \in E \} \) which is relatively dense with respect to the cone \( C(\varepsilon) \). By our remark above we can apply theorem B to \( f^* \). According to this theorem, for each \( \eta \in (0, \varepsilon) \) there exist such positive numbers \( \Delta \) and \( R = R(f) \) that

\[
\sup_{x \in C(\eta) \setminus B(0, R)} |f^*(x)| \leq \Delta \cdot \sup_{x \in E^*} |f^*(x)| \leq 2\Delta \cdot \sup_{x \in E} |f(x)|.
\]

For each \( \eta_1 \in (0, \eta) \) there exists such a number \( R_1 > R \) that

\[
C(\eta_1) \setminus B(0, R_1) \subset \{ z + \delta : \ z \in C(\eta) \setminus B(0, R) \}.
\]

Since

\[
|f(x) - f^*(x + \delta)| \leq \sup_{t \in E} |f(t)|
\]

for \( x \in C(\eta_1) \setminus B(0, R_1) \), and the numbers \( \varepsilon \in (0, 1), \ \eta \in (0, \varepsilon), \ \eta_1 \in (0, \eta) \) were arbitrarily chosen, we obtain the required estimate.

The theorem is proved.

**Proof of Theorem 2.** The proof repeats the proof of the previous theorem with the only difference that theorem B' is applied instead of theorem B. The corresponding value of \( \sigma_0 \) in theorem 2 differs from that in theorem B' by the factor \( \max(\eta/2, 1) \) where \( C \) is the constant from theorem A.

4 Proof of Theorems 3 and 4

**Proof of Theorem 3.** Given a function \( F(z) \) analytic in \( W(\tau) \) and of order \( \rho \), denote \( h_F(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \) by \( H_F \) and let \( H_F(E) \) be the corresponding limit calculated over the set \( E \). It is obvious that

\[
H_F(E) \leq H_F.
\]

We need to prove the converse.
The way to do it is to use theorem A to find an entire function $g \in [\rho, \infty)$ with the properties

$$H_g = H_f$$

and

$$H_g(E) = H_f(E)$$

and use theorem C for entire functions to prove that

$$H_g(E) = H_g,$$

which yields the desired relation.

Assume that $\rho \geq 1$ first. Then one can take $\rho = 1$ by the same arguments as before. We consider two cases. Assume first that $H_f(E) > -\infty$. Since multiplication of our function by $e^{A(z_1 + \ldots + z_n)}$ does not affect the investigated property of the set $E$, we can always assume that $H_f(E) > 0$. Hence an entire function $g(z)$ uniformly approximating $f(z)$ in a cone containing the ray $\ell = \{(t, \ldots, t), \ t > 0\}$ will have the same values of $H_g(E)$ and $H_g$, which yields the desired relation. Thus it is enough to construct such a function.

Take $N = 1$, $\omega(z) = \max_{j=1,\ldots,n} (-\Re z_j) + \eta' ||z||_t$, for such $t$ and $\eta'$ that the cone $\Omega_0 = \{z : \omega(z) < 0\}$ is contained in $W_\tau$, and take $\varphi(z) = \max(\delta, ||z||_t)$. For $\epsilon \in (0, 1-\eta')$ the set $\Omega_\epsilon = \{z : \omega(z) < -\epsilon \varphi(z)\}$ (which is $Y_t(\eta' + \epsilon)$ without a neighborhood of the vertex) has the property

$$\Omega_\epsilon \bigcap \mathbb{R}^n_+ \supset C(\eta' + \epsilon) \setminus B(0, \epsilon \delta).$$

Applying theorem A, we are done.

Now consider the case $H_f(E) = -\infty$. We need to prove that $H_f = -\infty$. We choose $\omega$ and $\varphi$ to be the same as above. Fix some $\epsilon \in (0, 1-\eta)$ and denote by $g_N$ the entire function of finite type corresponding to the choice of $N \geq 1$ in theorem A. Note that for any such function

$$H_{g_N}(E) = H_{g_N}.$$  

The entire function $g_N(z)$ satisfies $|f(z) - g(z)| < e^{-N||z||\tau}$ on $\Omega_\epsilon$, in particular, on $C(z')$ for $|x|$ large enough. We have

$$H_{g_N} \leq \lim_{\eta \to 0} \lim_{|x| \to \infty} \sup_{x \in E \bigcap C(\eta)} \frac{\log(|f(x)| + |g_N(x) - f(x)|)}{|x|} \leq \max(-N, H_f(E)) = -N,$$
and

\[
H_f \leq \lim_{\eta \to 0} \limsup_{|x| \to \infty, x \in C(\eta)} \frac{\log(|g_N(x)| + |g_N(x) - f(x)|)}{|x|}
\]

\[
\leq \max(-N, H_{g_N}(E))
\]

\[
= \max(-N, H_{g_N})
\]

\[
= -N.
\]

Since \(N\) was arbitrary, we conclude that \(H_f = -\infty\). The theorem is proved in this case.

The proof in the case \(\rho \in (\frac{1}{2}, 1)\) follows the same scheme as above with the only difference that we do not pass over to the order 1. To apply theorem A, for \(\omega(z)\) we take the function \(u(z)\) mentioned in the end of section 2, and set \(\varphi(z) = \max_{j=1,\ldots,n} (|z_j|^{\rho}, \delta)\).

Remark. In the case \(\rho \geq 1\) it is also possible to give another proof of Theorem 3 based on Theorem 2 (and thus using Theorem A indirectly).

Proof of Theorem 4. Since the conditions of Theorem 4 imply that the rays \(\ell = \{(t, \ldots, t), \ t > 0\}\) and \(e^{i\pi} \ell\) both belong to the cone where our function is holomorphic, the result follows from Theorem 3 and the properties of the indicator \([Ro]\) Ch. 3, §5.

5 Acknowledgments

The authors would like to thank S.Yu.Favorov, A.Yu.Rashkovskii, L.I.Ronkin and M.L.Sodin for helpful discussions.

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