TOPOLOGICAL ISOMORPHISM FOR RANK-1 SYSTEMS

By

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Abstract. We define the Polish space \( R \) of non-degenerate rank-1 systems. Each non-degenerate rank-1 system can be viewed as a measure-preserving transformation of an atomless, \( \sigma \)-finite measure space and as a homeomorphism of a Cantor space. We completely characterize when two non-degenerate rank-1 systems are topologically isomorphic. We also analyze the complexity of the topological isomorphism relation on \( R \), showing that it is \( F_\sigma \) as a subset of \( R \times R \) and bi-reducible to \( E_0 \). We also explicitly describe when a non-degenerate rank-1 system is topologically isomorphic to its inverse.

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1 Introduction

1.1 Two open questions in ergodic theory. One motivation of the work in this paper comes from two open questions concerning the conjugacy action on the group of invertible measure-preserving transformations. Let Aut$(X, \mu)$ denote the group of invertible, measure-preserving transformations of a standard Lebesgue space, taken modulo null sets and equipped with the weak topology. With the topology comes a notion of genericity. A subset of Aut$(X, \mu)$ is generic if its complement is a countable union of nowhere dense sets. We then say that a generic transformation satisfies a certain property if that property defines a generic subset of Aut$(X, \mu)$.

The group Aut$(X, \mu)$ acts on itself by conjugation, and the two open questions come from complementary aspects of this action. The first question deals with the orbit structure of the action. How can one determine whether two elements of Aut$(X, \mu)$ are conjugate, i.e., in the same orbit? As conjugacy is the natural choice for a notion of isomorphism in this context, this question is frequently referred to as the isomorphism problem for invertible measure-preserving transformations.

The second question deals with stabilizers. An informal version of this question is as follows. For a random element of Aut$(X, \mu)$, what properties is the stabilizer of that transformation likely to have? It is clear that the stabilizer of an element of Aut$(X, \mu)$ under the conjugacy action is exactly the centralizer of that element in the group Aut$(X, \mu)$. Thus, this problem is sometimes called the centralizer problem for invertible measure-preserving transformations.

1.1.1 The isomorphism problem. The isomorphism problem was first formulated by von Neumann, who, together with Halmos, gave a very important partial solution; see [11]. They showed that for ergodic transformations with pure point spectrum, the spectrum is a complete invariant. That is, two ergodic transformations with pure point spectrum are conjugate if and only if they have the same spectrum. Another very important partial solution was given by Ornstein, who showed in [19] that for Bernoulli shifts, entropy is a complete invariant.

In the past few decades, however, there have been several papers showing that certain types of “nice” classifications are impossible for all of Aut$(X, \mu)$ (or even
for a generic subset of Aut(X, μ)). One type of “nice” classification is connected with Borel reducibility, and the strongest of the anti-classification results of this type was given by Foreman and Weiss in 2004. Building on the work of Hjorth [13], they showed in [9] that no generic subset of Aut(X, μ) is classifiable by countable structures.

Another type of “nice” classification deals with a description of the conjugacy (isomorphism) relation on Aut(X, μ) as a subset of Aut(X, μ) × Aut(X, μ). Hjorth showed in [13] that this is non-Borel; and Foreman, Rudolph, and Weiss showed in [8] that when restricted to the (generic) class of ergodic transformations, the isomorphism relation is still not Borel.

In the same paper, Foreman, Rudolph, and Weiss showed that for a different generic class of transformations, the rank-1 transformations, the isomorphism relation is Borel. Unfortunately, their proof gives no explicit way of determining when two rank-1 transformations are isomorphic and no bound on the Borel complexity of the isomorphism relation on rank-1 transformations.

It would be very nice have an explicit method of determining when two rank-1 transformations are (measure-theoretically) isomorphic.

### 1.1.2 The centralizer problem

We are interested in the size and structure of the centralizer of a generic transformation. For T ∈ Aut(X, μ), the centralizer C(T) contains \( \{ T^i_{i \in \mathbb{Z}} \} \), since Aut(X, μ) is a topological group. An important result of Jonathan King [16] is that for rank-1 transformations, there is equality. Since a generic transformation is rank-1, this implies that the centralizer of a generic transformation is abelian and, in some sense, as small as possible.

On the other hand, recent results of King [17], de Sam Lazaro–de la Rue [21], Ageev [1], Stepin–Eremenko [5], Tikhonov [23], Melleray–Tsankov [18], and Solecki [22] show that the centralizer of a generic transformation is large and structurally rich. For example, Stepin and Eremenko showed that every compact abelian group embeds into the centralizer of a generic transformation. Melleray and Tsankov showed that the centralizer of a generic transformation is extremely amenable. Solecki showed that every element of the centralizer of a generic transformation is in a 1-parameter subgroup of that centralizer.

It would be very nice to know whether there exists a Polish group G to which the centralizer of a generic transformation is isomorphic. Of course, if there is such a G, it would be nice to know what it is.

Because rank-1 transformations are generic, questions about the centralizer of a generic transformation are the same as the corresponding questions for a generic rank-1 transformation. Because of King’s weak closure theorem [16],...
these questions are the same as the corresponding questions about \( \{T^i : i \in \mathbb{Z}\} \), for generic rank-1 \( T \).

1.1.3 Rank-1 systems. There are various definitions of rank-1 transformation in the literature, not all of which are equivalent. The two most common definitions involve cutting and stacking transformations and symbolic rank-1 systems. Every cutting and stacking transformation, except those isomorphic to odometers, can be realized as (i.e., is isomorphic to) a symbolic rank-1 system. For further information regarding the many definitions of rank-1 transformations and connections between them, see Ferenczi’s survey article [6].

In the symbolic definition of rank-1, one constructs a collection of concrete rank-1 systems, each of which can be viewed either as a measure-preserving transformation of a standard Lebesgue space or as a homeomorphism of a Cantor space. One then defines a measure-preserving transformation to be rank-1 if it is measure-theoretically isomorphic to a concrete rank-1 system. Analogously, one could define a homeomorphism to be rank-1 if it is topologically isomorphic to a concrete rank-1 system.

For the open questions described above, one would be interested in answering the following questions. When are two concrete rank-1 systems measure-theoretically isomorphic. What measure-theoretic self-isomorphisms exist for particular concrete rank-1 systems?

The topological analogues would be as follows. When are two concrete rank-1 systems topologically isomorphic? What topological self-isomorphisms exist for particular concrete rank-1 systems? In this paper, we answer these topological analogues completely.

It should be noted that the systems that we are concerned with (which we call non-degenerate rank-1 systems) are more general than the constructive symbolic rank-1 systems in Ferenczi’s article. Our definition omits a restriction that guarantees that the rank-1 system can be viewed as a measure-preserving transformation of a standard Lebesgue space. Without this restriction, the rank-1 system can be viewed as a measure-preserving transformation of a measure (but not necessarily probability) space. Omitting this restriction causes no problems, since we are interested in viewing rank-1 systems only as homeomorphisms of Cantor space.

1.2 Main results. The non-degenerate rank-1 systems we consider are all Bernoulli subshifts, i.e., closed subsets of \( \{0, 1\}^\mathbb{Z} \) that are invariant under the shift map \( \sigma \). For each non-degenerate rank-1 system \((X, \sigma)\), there is a particular \( V \in \{0, 1\}^\mathbb{N} \) associated to \((X, \sigma)\). It is of a certain form (a non-degenerate rank-1
word) and is such that

\[ X = \{ x \in \{0, 1\}^\mathbb{Z} : \text{every finite subword of } x \text{ is a subword of } V \}. \]

The correspondence between non-degenerate rank-1 words and non-degenerate rank-1 systems is one-to-one, and we consider the collection \( \mathcal{R} \) of all non-degenerate rank-1 words as a space of codes for all non-degenerate rank-1 systems. We define a natural (Polish) topology on \( \mathcal{R} \) and regard \( \mathcal{R} \) with this topology as the space of all non-degenerate rank-1 systems.

Our first main theorem gives a complete characterization of when two non-degenerate rank-1 systems are topologically isomorphic. We define a simple form of block code, called a replacement scheme, for non-degenerate rank-1 systems and call an isomorphism coming from a replacement scheme stable.

**Theorem 1.1.** Let \( \phi \) be a topological isomorphism between two non-degenerate rank-1 systems. Then there exists \( i \in \mathbb{Z} \) such that \( \phi \circ \sigma^i \) is a stable isomorphism.

As an immediate consequence, we are able to characterize all self-isomorphisms of a non-degenerate rank-1 system.

**Theorem 1.2.** Let \( (X, \sigma) \) be a non-degenerate rank-1 system and \( \tau \) an automorphism of \( X \) that commutes with \( \sigma \). Then \( \tau = \sigma^i \) for some \( i \in \mathbb{Z} \).

This result was proved in a slightly more general context by the second author in [12].

Every non-degenerate rank-1 word is generated by an infinite sequence of finite words. Our analysis also allows us to identify a canonical generating sequence for a non-degenerate rank-1 word. The significance of this canonical generating sequence is that if two non-degenerate rank-1 systems are isomorphic, then their respective generating sequences eventually match up in some specific way; see Proposition 3.20. We are therefore able to determine the exact complexity of the topological isomorphism relation on non-degenerate rank-1 systems in the Borel reducibility hierarchy. Recall that \( E_0 \) is the eventual agreement relation on \( \{0, 1\}^\mathbb{N} \), i.e., the equivalence relation on \( \{0, 1\}^\mathbb{N} \) defined by

\[ xE_0y \text{ if and only if there exists } n \text{ such that } x(m) = y(m) \text{ for all } m > n. \]

**Theorem 1.3.** The topological isomorphism relation for non-degenerate rank-1 systems is Borel bi-reducible with \( E_0 \). Also, the topological isomorphism relation is \( F_\sigma \) as a subset of \( \mathcal{R} \times \mathcal{R} \).
Finally, we provide a satisfactory solution to the topological version of the inverse problem for non-degenerate rank-1 systems.

**Theorem 1.4.** Let $X$ be a non-degenerate rank-1 system and $\{v_n\}_{n \in \mathbb{N}}$ be its canonical generating sequence. Then $(X, \sigma)$ is topologically isomorphic to its inverse $(X, \sigma^{-1})$ if and only if for all but finitely many $n \in \mathbb{N}$, $v_{n+1}$ is built symmetrically from $v_n$.

### 1.3 Topological conjugacy of minimal systems.

The results of this paper, especially Theorem 1.3, can also be viewed as a contribution to the study of the topological isomorphism problem for Bernoulli subshifts. In symbolic dynamics, this problem is better known as the topological conjugacy problem. The objective is to understand the complexity of the topological isomorphism (or conjugacy) relation when it is restricted to important classes of Bernoulli subshifts.

Clemens [2] proved that topological conjugacy for all Bernoulli subshifts is a universal countable Borel equivalence relation. Gao, Jackson, and Seward [10] and, independently, Clemens [3] showed the same for free Bernoulli subshifts. However, the complexity of topological conjugacy for minimal subshifts is an open question. Gao, Jackson, and Seward have shown in [10] that $E_0$ is a lower bound for this problem. Theorem 1.3 implies that topological conjugacy for minimal rank-1 systems has the same complexity as $E_0$.

### 1.4 Organization and terminology.

This paper is organized as follows. In Section 2, we define rank-1 words and rank-1 systems, and prove all the necessary preliminary results about them. In particular, we give a complete analysis of the collection of all finite words from which a rank-1 word is built and present several applications of this analysis. One application is to the definition of the canonical generating sequence for a rank-1 word. Another is the definition of a complete metric on the space $\mathcal{R}$. In Section 3, we prove the main results of this paper.

We denote the set of all natural numbers, including zero, by $\mathbb{N}$.

Three types of words appear in this paper. By a **bi-infinite word**, we mean an element of $\{0, 1\}^\mathbb{Z}$. We frequently use the lower case letters $x$, $y$, and $z$ to denote bi-infinite words. By an **infinite word**, we mean an element of $\{0, 1\}^\mathbb{N}$. We frequently use the upper case letters $U$, $V$, and $W$ to denote infinite words. By a **finite word**, we mean a finite sequence from the alphabet $\{0, 1\}$. We frequently use the lower case letters $u$, $v$, and $w$ to denote finite words.

The symbol $vw$ denotes the concatenation of finite words $v$ and $w$. Similarly, for $n \in \mathbb{N}$, $v^n$ denotes the concatenation of $n$ copies of the finite word $v$. We are...
principally interested in the set $\mathcal{F} = \{ \text{finite words that begin and end with 0} \}$.

2 The Polish space of non-degenerate rank-1 systems

This section contains the foundational results about rank-1 words and systems that allow us to analyze isomorphisms between non-degenerate rank-1 systems and understand the complexity of the isomorphism relation on the space of such systems. which we do in Section 3. We now give an overview of what the rest of this section contains.

2.1 Overview. In Subsection 2.2, we define a rank-1 word and show how it gives rise to a rank-1 system $(X, \sigma)$. We show that there are three types of rank-1 systems: degenerate if $X$ is finite; minimal if $X$ is a Cantor space on which the shift acts minimally; and non-minimal if $X$ is a Cantor space with a fixed point, i.e., a point fixed by the shift.

We define $\mathcal{R}$ to be the collection of rank-1 words that give rise to non-degenerate rank-1 systems and endow it with a natural topology. As the map sending an element of $\mathcal{R}$ to the non-degenerate rank-1 system it gives rise to is one-to-one (Proposition 2.37 below), we can view $\mathcal{R}$ as a space of codes for non-degenerate rank-1 systems. We show that although there is another reasonable choice for a space of codes for non-degenerate rank-1 systems, these two spaces of codes are Borel isomorphic (with a natural isomorphism).

In Subsection 2.3, we analyze the collection $A_V$ of finite words from which a rank-1 word $V$ is built. There is a natural partial order $\preceq$ on $A_V$. We show that $(A_V, \preceq)$ is a lattice and that a certain natural subset of $A_V$ is linearly ordered by $\preceq$. This subset is infinite if and only if $V \in \mathcal{R}$, i.e., the system associated to $V$ is non-degenerate; and in the case $V \in \mathcal{R}$, it gives rise to the canonical generating sequence of $V$. We then use our analysis of $A_V$ and this certain subset to define a complete metric on $\mathcal{R}$ that is compatible with the natural topology of $\mathcal{R}$.

In Subsection 2.4, we show that certain structural aspects of a non-degenerate rank-1 word $V$ imply similar structural aspects for each element of the non-degenerate rank-1 system $X$. The key notion is that of an expected occurrence of a finite word $v$ in an infinite word $V$ (or a bi-infinite word $x$).

If $V \in \mathcal{R}$ is built from a finite word $v$, there is a unique way to view $V$ as a collection of disjoint occurrences of $v$ interspersed with 1s: an occurrence of $v$ in $V$ is said to be expected if it is an element of that collection. We show that an occurrence of $v$ is expected implies that each element $x$ of the rank-1 system $X$ associated to $V$ also has this property, i.e., can be viewed in a unique way as a
collection of disjoint occurrences of \( v \) interspersed with 1s. An occurrence of \( v \) in \( x \) is **expected** if it is an element of that collection.

The ability to analyze elements of a non-degenerate rank-1 system according to the location of their expected occurrences of \( v \) is an essential tool in the technical proof of our main result Theorem 3.5.

### 2.2 Rank-1 words and systems.

#### 2.2.1 Definition of a rank-1 word and its associated rank-1 system.

**Definition 2.1.** Let \( V \in \{0, 1\}^\mathbb{N} \). We say that \( V \) is **built from** \( v \in \mathcal{F} \) if there exists a sequence \( \{a_i\}_{i \geq 1} \) of natural numbers such that \( V = v_1^{a_1}v_2^{a_2}v_3^{a_3} \ldots \).

Let \( A_V = \{ v \in \mathcal{F} : V \text{ is built from } v \} \). We say that \( V \) is **rank-1** if \( A_V \) is infinite.

**Remarks.**
1. If \( V \) is built from a \( v \in \mathcal{F} \), then \( V \) has infinitely many occurrences of 0.
2. If \( V \) is rank-1, then for each \( n \in \mathbb{N} \), there exists some \( v \in A_V \) such that \( |v| \geq n \).
3. If \( V \) is rank-1, then for each \( n \in \mathbb{N} \), there exists some \( v \in A_V \) with at least \( n \) occurrences of 0.

Let \( V \) be a rank-1 word. We construct a dynamical system \((X, \sigma)\) as follows. Let

\[
X = \{ x \in \{0, 1\}^\mathbb{Z} : \text{every finite subword of } x \text{ is a subword of } V \}.
\]

Clearly, \( X \subseteq \{0, 1\}^\mathbb{Z} \) is closed. Define the shift \( \sigma : X \to X \) by \( \sigma(x)(i) = x(i+1) \).

Clearly, \( \sigma \) is a homeomorphism of \( X \). We call \((X, \sigma)\) the **rank-1 system** associated to \( V \).

**Remark.** There exists a canonical measure associated to each rank-1 system \((X, \sigma)\): define \( \mu(\{ x \in X : x(0) = 0 \}) = 1 \) and extend \( \mu \) uniquely to a \( \sigma \)-finite, shift-invariant measure \( \mu \) on \( X \). A few comments about this canonical measure appear after the proof of Proposition 2.4.

#### 2.2.2 Three types of rank-1 words (and rank-1 systems).

Some rank-1 words give rise to completely uninteresting rank-1 systems. Proposition 2.4 below shows that three distinct types of rank-1 words give rise to three distinct types of rank-1 systems.

**Proposition 2.2.** Suppose \( V \) is a rank-1 word and \((X, \sigma)\) the rank-1 system associated to \( V \). If \( x \in X \) contains an occurrence of 0, then \( x \) contains an occurrence of every finite subword of \( V \).
Proof. Let $\alpha$ be a finite subword of $V$. There exists some $v$ from which $V$ is built that contains an occurrence of $\alpha$. Every occurrence of 0 in $V$ is a part of an occurrence of $v$ in $V$. It follows that every occurrence of 0 in any $x \in X$ is a part of an occurrence of $v$ in $x$. Thus, if $x \in X$ contains an occurrence of 0, then it also contains an occurrence of $v$ and, therefore, an occurrence of $\alpha$. \hfill $\square$

Corollary 2.3. Let $(X, \sigma)$ be a rank-1 system, and suppose $x \in X$ contains an occurrence of 0.
(a) The orbit of $x$ is dense in $X$.
(b) There are infinitely many occurrences of 0 in $x$.

Proposition 2.4. Let $V$ be a rank-1 word and $(X, \sigma)$ the rank-1 system associated to $V$.
(a) If there exists a word $v$ such that $V = vvv \ldots$, then $X$ is finite.
(b) If there exists no word $v$ such that $V = vvv \ldots$, but there exists some $n \in \mathbb{N}$ such that $1^n$ is not a subword of $V$, then $(X, \sigma)$ is a Cantor minimal system.
(c) If for each $n \in \mathbb{N}$, the word $1^n$ is a subword of $V$, then $X$ is a Cantor space with exactly one periodic point, the bi-infinite constant word 1.

Proof. Proof of (a). Suppose $V = vvv \ldots$. It is clear that if $x \in \{0, 1\}^\mathbb{Z}$ is of the form $(\ldots vvv \ldots)$, then $x \in X$. Since such an $x$ contains an occurrence of 0, Corollary 2.3 implies that the orbit of $x$ is dense. But the orbit of $x$ is finite, so $X$ must also be finite.

Proof of (b). Suppose that there exists no word $v$ such that $V = vvv \ldots$, but there exists an $n \in \mathbb{N}$ such that $1^n$ is not a subword of $V$. We claim that $X$ is non-empty. Indeed, consider the bi-infinite word $y \in \{0, 1\}^\mathbb{Z}$ defined by

$$y(i) = \begin{cases} V(i) & \text{for } i \geq 0, \\ 1 & \text{for } i < 0. \end{cases}$$

Then consider the sequence of bi-infinite words $\{\sigma^n(y)\}_{n \in \mathbb{N}}$. Since $\{0, 1\}^\mathbb{Z}$ is compact, this sequence must have a limit point in $\{0, 1\}^\mathbb{Z}$, and it is easy to check that this limit point must be in $X$.

We now know that $X$ is a non-empty, closed subset of $\{0, 1\}^\mathbb{Z}$. To show that $X$ is a Cantor space, it suffices to show that $X$ has no isolated points. To show that the system $(X, \sigma)$ is minimal, it suffices to show that the orbit of each $x \in X$ is dense in $X$.

Note that since each subword of $V$ of length $n$ contains an occurrence of 0, each subword of each $x \in X$ of length $n$ also contains an occurrence of 0. Thus,
each \( x \in X \) contains an occurrence of 0 and, by Corollary 2.3, the orbit of each \( x \in X \) is dense.

To show that \( X \) has no isolated points, it suffices to show that if an element of \( X \) has an occurrence of a finite word \( \alpha \) beginning at position \( i \), then at least two distinct elements of \( X \) have an occurrence of \( \alpha \) beginning at \( i \). Suppose \( x \in X \) has an occurrence of \( \alpha \) beginning at \( i \). The finite word \( \alpha \) must be a subword of \( V \) and hence a subword of some \( v \) from which \( V \) is built. Since \( V \) is built from \( v \) and \( V \) is not periodic, there exist distinct \( a, b \in \mathbb{N} \) such that \( v 1^a 0 \) and \( v 1^b 0 \) are subwords of \( V \). Since \( x \) contains an occurrence of 0, Proposition 2.2 implies that both \( v 1^a 0 \) and \( v 1^b 0 \) are subwords of \( x \). Applying appropriate powers of the shift to \( x \) gives two distinct elements of the orbit of \( x \) that have an occurrence of \( \alpha \) beginning at \( i \).

Proof of (c). Suppose that for each \( n \in \mathbb{N} \), the word \( 1^n \) is a subword of \( V \). It is clear that \( X \) contains the bi-infinite constant word 1, and that this is word is a fixed point of \( \sigma \). We need to show that \( X \) has no isolated points (and hence, is a Cantor space) and that no element of \( X \) that has an occurrence of 0 is periodic.

Consider the bi-infinite word \( z \in \{0, 1\}^\mathbb{Z} \) defined by

\[
z(i) = \begin{cases} 
V(i) & \text{for } i \geq 0, \\
1 & \text{for } i < 0.
\end{cases}
\]

We claim that \( z \in X \). Indeed, every finite subword of \( z \) that ends before position 0 is of the form \( 1^n \) and so is a subword of \( V \). Every finite subword of \( z \) that begins after position \( -1 \) is clearly a subword of \( V \). It remains to show that each finite subword of \( z \) that begins before 0 and ends after \( -1 \) is a subword of \( V \). To this end, it suffices to show that if \( V \) is built from \( v \), then for arbitrarily large \( n \in \mathbb{N} \), the word \( 1^n v \) is a subword of \( V \). But if \( V \) is built from \( v \) and \( n > |v| \), then \( V \) must have an occurrence of \( 1^n \) and this occurrence must be followed by \( v \) (since \( V \) is built from \( v \) and the word \( 1^n \) cannot be in the middle an occurrence of \( v \) when \( n > |v| \).) Thus \( z \in X \).

We claim that \( X \) has no isolated points. To show that no \( x \in X \) is isolated, it suffices to show that if \( x \) has an occurrence of a finite word \( \alpha \) beginning at position \( i \), then there exist two distinct elements of \( X \) that have an occurrence of \( \alpha \) beginning at \( i \). Suppose that \( x \) has an occurrence of \( \alpha \) beginning at \( i \). The finite word \( \alpha \) must be a subword of \( V \) and hence a subword of some \( v \) from which \( V \) is built. Since \( V \) is built from \( v \) and each \( 1^n \) is a subword of \( V \), there exist \( a, b \in \mathbb{N} \) such that \( v 1^a 0 \) and \( v 1^b 0 \) are subwords of \( V \). By Proposition 2.2, both \( v 1^a 0 \) and \( v 1^b 0 \) are subwords of \( z \). There thus exist two distinct elements of the orbit of \( x \) that have an occurrence of \( \alpha \) beginning at \( i \), i.e., \( x \) is not an isolated point.

We claim that no \( x \in X \) that has an occurrence of 0 is periodic. Let \( x \in X \)
have an occurrence of 0. By Corollary 2.3, the orbit of \( x \) is dense in \( X \). Since the bi-infinite constant word 1 is in \( X \) but not in the orbit of \( x \), the orbit of \( x \) must be infinite. Therefore, \( x \) is not periodic. □

**Definition 2.5.** (1) We call a rank-1 word **degenerate** if it satisfies the hypothesis of (a) and a rank-1 system **degenerate** if it satisfies the conclusion of (a). In this case, the canonical measure \( \mu \) has atoms.

(2) We call a rank-1 word **minimal** if it satisfies the hypothesis of (b) and a rank-1 system **minimal** if it satisfies the conclusion of (b). In this case, the canonical measure \( \mu \) is atomless and finite. If the measure \( \mu \) is normalized so that \( \mu(X) = 1 \), then \( (X, \mu, \sigma) \) is a measure-preserving transformation.

(3) We call a rank-1 word **non-minimal** if satisfies the hypothesis of (c) and a rank-1 system **non-minimal** if it satisfies the conclusion of (c). In this case, the canonical measure \( \mu \) is atomless and might be finite or infinite.

**Corollary 2.6.** Let \((X, \sigma)\) be a non-degenerate rank-1 system, Then each \( x \in X \) that contains an occurrence of 0 is non-periodic.

### 2.2.3 The space of non-degenerate rank-1 words.

Let \( R' \) denote the set of rank-1 words and \( R \) denote the set of non-degenerate rank-1 words. For \( v \in \mathcal{F} \), let

\[
A^v = \{ V \in R' : V \text{ is built from } v \} = \{ V \in R' : v \in A_V \}.
\]

Let \( T' \) be the topology on \( R' \) generated by \( \{ A^v : v \in \mathcal{F} \} \). Let \( T \) be the topology that \( R \) inherits as a subset of \( R' \) (with topology \( T' \)). Since \( \mathcal{F} \) is countable, \((R, T)\) is separable. In Subsection 2.3.5 below, we describe a complete metric on \( R \) that is compatible with \( T \); hence, \((R, T)\) is a Polish space.

There is, in fact, another way to see that \((R, T)\) is Polish, which we now describe briefly. One can check that \( R' \) is an \( F_{\sigma\delta} \) subset of \( \{0, 1\}^\mathbb{N} \). There is a standard way to give a Polish topology to an \( F_{\sigma\delta} \) subset of a Polish space: it involves refining the topology on the original Polish space in order to make the specified subset \( G_\delta \) in the new topology, hence making it Polish (see, e.g., [15, Lemmas 13.2 and 13.3]). One can check that, in our case, this procedure gives precisely the topology \( T' \) to \( R' \). Since \( R' \setminus R \) is countable, \( R \) is a \( G_\delta \) subset of \( R' \) and hence is Polish with the induced topology, which is \( T \).

### 2.2.4 The Borel isomorphism between two coding spaces.

Each non-degenerate rank-1 word is associated to a unique non-degenerate rank-1 system
(and vice versa, by Proposition 2.37). Thus we can view $\mathcal{R}$ as a (Polish) space of codes for all non-degenerate rank-1 systems.

There is, however, another standard coding for non-degenerate rank-1 systems, as Bernoulli subshifts. Consider
\[ K(\{0, 1\}^\mathbb{Z}) = \{X \subseteq \{0, 1\}^\mathbb{Z} : X \text{ is a closed (compact) subset of } \{0, 1\}^\mathbb{Z}\}. \]

With either the Hausdorff metric topology or the Vietoris topology, $K(\{0, 1\}^\mathbb{Z})$ becomes a Polish space; see, for example, [15, Theorem 4.25]. Furthermore, the set
\[ S = \{X \in K(\{0, 1\}^\mathbb{Z}) : X \text{ is invariant under the shift}\} \]

is a closed subspace of $K(\{0, 1\}^\mathbb{Z})$, and therefore is also a Polish space. Intuitively, $S$ is the space of all Bernoulli subshifts. Let $\mathcal{R}^*$ be the subspace of $S$ consisting of all non-degenerate rank-1 systems. We now verify that $\mathcal{R}^*$ is a Borel subset of $S$, and therefore is a standard Borel space.

To see that $\mathcal{R}^*$ is Borel, consider the map $\Phi : \mathcal{R} \to S$ defined by
\[ \Phi(V) = \{x \in \{0, 1\}^\mathbb{Z} : \text{ every subword of } x \text{ is a subword of } V\}, \]
and note that $\mathcal{R}^*$ is exactly the range of $\Phi$. It is easy to verify that $\Phi$ is a Borel map. By Proposition 2.37, it is also one-to-one. By a theorem of Luzin–Suslin (see, for example, [15, Theorem 15.1 and Corollary 15.2]), we conclude that $\mathcal{R}^*$ is Borel and that $\Phi$ is a Borel isomorphism between $\mathcal{R}$ and $\mathcal{R}^*$.

### 2.3 Finite words from which a specified rank-1 word is built.

We are interested in the structure of the set of finite words from which a specified rank-1 word is built.

**Definition 2.7.** Let $v, w \in \mathcal{F}$.

(a) We say $w$ is **built from** $v$, and write $v \preceq w$, if there exist $r \geq 1$ and $a_1, a_2, \ldots, a_{r-1} \in \mathbb{N}$ such that $w = v^1 a_1 v^2 a_2 \ldots v^1 a_{r-1} v$. 

(b) We say $w$ is **built simply** from $v$, and write $v \preceq_s w$, if $w$ is built from $v$ and $a_1 = a_2 = \cdots = a_{r-1}$.

If $v \preceq w$ but $v \neq w$, we write $v \prec w$. It is easy to check that that the relation $\preceq$ is a partial order on $\mathcal{F}$. Although the relation $\preceq_s$ is not transitive, we do have the following simple but useful lemma, which we state without proof but illustrate in Figure 1.

**Lemma 2.8.** Let $u, v, w, u' \in U$.

(a) If $u \preceq v \prec w \preceq w$ and $u \preceq_s w$ and $v \preceq_s u'$, then $u \preceq_s u'$.

(b) If $u \preceq v \preceq w$ and $u \preceq_s w$, then $u \preceq_s v$ and $v \preceq_s w$. 

Therefore, occurrences of 0 in \( V \) in Figure 1. Lemma 2.8 (a) and (b): solid arrows stand for \( \preceq \), two-head arrows for \(<\), and dotted arrows for \( \preceq_s \).

### 2.3.1 The lattice \((A_V, \preceq)\)

For the rest of Section 2.3.1, we fix a rank-1 word \( V \). Recall that \( A_V = \{ v \in \mathcal{F} : V \text{ is built from } v \} \). Since \( V \) is rank-1, \( A_V \) is infinite. Since \((\mathcal{F}, \preceq)\) is a partial order, so is \((A_V, \preceq)\).

Recall that a partial order is a lattice if every two elements have a least upper bound and a greatest lower bound. The greatest lower bound of two elements \( v \) and \( w \) is called the meet of \( v \) and \( w \) and is denoted by \( v \wedge w \); the least upper bound of \( v \) and \( w \) is called the join and is denoted by \( v \vee w \). Our next objective is to show that \((A_V, \preceq)\) is a lattice.

Let \((i_n : n \in \mathbb{N})\) enumerate, in an order-preserving way, the positions of the occurrences of 0 in \( V \). Since \( V \) is rank-1, \( V \) begins with an occurrence of 0. Therefore, \( i_0 = 0 \). Define \( L_V : \mathbb{N}_{>0} \rightarrow \mathbb{N} \) by \( L_V(n) = i_n - i_{n-1} - 1 \). Note that \( L_V(n) \) is the number of 1s in \( V \) between the occurrence of 0 at position \( i_{n-1} \) and the occurrence of 0 at position \( i_n \). When \( V \) is clear from the context, we write \( L \) for \( L_V \).

We state the following Lemma without proof.

**Lemma 2.9.** Let \( v = 01^{a_1}01^{a_2} \ldots 1^{a_{r-1}}0 \) and \( w = 01^{b_1}01^{b_2} \ldots 1^{b_{r-1}}0 \).

(a) \( v \in A_V \) if and only if \( L_V(i) = a_k \) for all \( 0 < k < r \) and \( i \equiv k \mod r \).
(b) If \( v, w \in A_V \), then \( v \preceq w \) if and only if \( s \) is a multiple of \( r \).

We now show that \((A_V, \preceq)\) is a lattice.
Proposition 2.10. For $v, w \in A_V$, there exist $u, u' \in A_V$ such that
(a) $u \preceq v \preceq u'$ and $u \preceq w \preceq u'$;
(b) if $\alpha \in A_V$ with $v \preceq \alpha$ and $w \preceq \alpha$, then $u' \preceq \alpha$; and
(c) if $\alpha \in A_V$ with $\alpha \preceq v$ and $\alpha \preceq w$, then $\alpha \preceq u$.

Proof. Let $v = 01^{a_1}01^{a_2} \ldots 01^{a_{r-1}}0$ and $w = 01^{b_1}01^{b_2} \ldots 01^{b_{r-1}}0$, and suppose that $v, w \in A_V$.

We first define $u$ and $u'$ and then show that $u, u' \in A_V$. Let $t = \gcd(r, s)$. For $0 < k < t$, let $c_k = L(k)$. Now let $u = 01^{c_1}01^{c_2}0 \ldots 01^{c_{r-1}}0$. Let $t' = \text{lcm}(r, s)$. For $0 < k < t'$, let $d_k = L(k)$. Now let $u' = 01^{d_1}01^{d_2}0 \ldots 01^{d_{r-1}}0$.

We claim that $u \in A_V$. By Lemma 2.9(a), it suffices to show that $L(i) = L(k)$ for all $0 < k < t$ and $i \equiv k \mod t$. Suppose $i \equiv k \mod t$ and $0 < k < t$. Choose $l \in \mathbb{N}$ such that $i = k + lt$. Since $t = \gcd(r, s)$, there exist $m, n \in \mathbb{N}$ such that $t = mr - ns$. This implies that $i + lns = k + lm$. Since $i \not\equiv 0 \mod s$, $L$ is constant on the congruence class of $i \mod s$. Thus, $L(i) = L(i + lns)$. Since $k \not\equiv 0 \mod r$, $L$ is constant on the congruence class of $k \mod r$. Thus, $L(k) = L(k + lm)$. Since $i + lns = k + lm$, we have $L(i) = L(k)$.

We claim that $u' \in A_V$. Again by Lemma 2.9(a), it suffices to show that $L(i) = L(k)$ for all $0 < k < t'$ and $i \equiv k \mod t'$. Since $k \not\equiv 0 \mod r$ and $t' = \text{lcm}(r, s)$, either $k \not\equiv 0 \mod r$ or $k \not\equiv 0 \mod s$. If $k \not\equiv 0 \mod r$, then $L$ is constant on the congruence class of $i \mod r$. Thus, $L(i) = L(k)$. If $k \not\equiv 0 \mod s$, then $L$ is constant on the congruence class of $i \mod s$. Thus, $L(i) = L(k)$ in all cases.

The proofs of (a), (b), and (c) follow immediately from the definition of $u$ and $u'$ and Lemma 2.9(b).

We have one more proposition whose proof is closely connected with that of Proposition 2.10.

Proposition 2.11. If $v, w \in A_V$ are incomparable, then
(a) $(v \land w) \preceq v \prec (v \lor w)$;
(b) $(v \land w) \preceq w \prec (v \lor w)$; and
(c) $(v \land w) \preceq_s (v \lor w)$.

If, moreover, $V = v1^a v1^a v \ldots$, then $V = (v \land w)1^a(v \land w)1^a(v \land w) \ldots$.

Proof. Parts (a) and (b) are obvious. We now prove part (c).

Let $v, w \in A_V$ be incomparable. We adopt the notation given in the proof of Proposition 2.10. To show that $u' = v \lor w$ is built simply from $u = v \land w$, it suffices to show that $L(i) = L(t)$ if $0 < i < t'$ and $i \equiv 0 \mod t$.

Since $v$ and $w$ are incomparable, $t \not\equiv r$ and $t \not\equiv s$. Suppose that $0 < i < t'$ and $i \equiv 0 \mod t$. We want to show that $L(i) = L(t)$. We know that $i \not\equiv 0 \mod t'$.
Therefore, since \( t' = \text{lcm}(r, s) \), either \( i \not\equiv 0 \mod r \), or \( i \not\equiv 0 \mod s \). Without loss of generality, assume that \( i \not\equiv 0 \mod s \).

Suppose \( i \not\equiv 0 \mod s \). Let \( l \in \mathbb{N} \) be such that \( t = t + lt \). Let \( m, n \in \mathbb{N} \) be such that \( t = mr - ns \). Then \( i + ins = t + lmr \). Since \( i \not\equiv 0 \mod s \), \( L \) is constant on the congruence class of \( i \mod s \). Thus, \( L(i) = L(i + ins) \). Since \( t = \gcd(r, s) \), but \( t \not\equiv r \), we have \( t \not\equiv 0 \mod r \), which implies that \( L \) is constant on the congruence class of \( t \mod r \). Thus, \( L(t) = L(t + lmr) \). Since \( i + ins = t + lmr \), we have \( L(i) = L(t) \).

We have proved (c), i.e., that \((v \land w) \leq_s (v \lor w) \). Now suppose that \( V = v1^a v1^a \ldots \). To show that \( V = (v \land w)1^a(v \land w)1^a \ldots \), it suffices to show that \( L(i) = a \) if \( i \equiv 0 \mod t \).

If \( i \equiv 0 \mod r \), then \( L(i) = a \). Thus, if \( i \equiv 0 \mod t' \), then \( i \equiv 0 \mod r \) (since \( t' \) is a multiple of \( r \)); hence, \( L(i) = a \).

But also \( u \preceq_s u' \), and this implies that \( L(i) \) is the same for all \( i \) satisfying \( i \equiv 0 \mod t \) and \( 0 < i < t' \). Since \( V \) is built from \( u' \), \( L(i) \) is the same for all \( i \) satisfying \( i \equiv 0 \mod t \) and \( i \not\equiv 0 \mod t' \). Now \( r \equiv 0 \mod t \), \( r \not\equiv 0 \mod t' \), and \( L(r) = a \). Thus, if \( i \equiv 0 \mod t \) and \( i \not\equiv 0 \mod t' \), then \( L(i) = a \).

The last two paragraphs show that if \( i \equiv 0 \mod t \), then \( L(i) = a \).

\[ \square \]

**Corollary 2.12.** Suppose \( V \) is degenerate and that \( V = v1^a v1^a \ldots \), with \( |v| \) as small as possible. If \( w \in A_V \) and \( |v| \leq |w| \), then \( w = v1^a v1^a \ldots v1^a v \) and, therefore, \( v \preceq_s w \).

### 2.3.2 The total order \((B_V, \preceq)\).

**Definition 2.13.** Let \( V \in \{0, 1\}^\mathbb{N} \). We say that \( V \) is **built fundamentally** from \( v \) if \( v \in A_V \) and for all \( u, u' \in A_V \) with \( u \prec v \prec u' \), \( u \not\preceq_s u' \).

Let \( B_V = \{ v \in U : V \text{ is built fundamentally from } v \} \). Note that if \( V \in \mathcal{R}' \), then \( 0 \in B_V \).

For the rest of Subsection 2.3.2, we fix a rank-1 word \( V \). We have two important propositions about \( B_V \).

**Proposition 2.14.** Every element of \( B_V \) is comparable to every element of \( A_V \).

**Proof.** Suppose that \( v, w \in A_V \) are incomparable. Then, by Proposition 2.11, \((v \land w) \prec v \prec (v \lor w) \) with \((v \land w) \preceq_s (v \lor w) \), so \( v \not\in B_V \). \[ \square \]

**Corollary 2.15.** \((B_V, \preceq)\) is a total order.

**Proposition 2.16.** The set \( B_V \) is infinite if and only if \( V \in \mathcal{R} \), i.e., if and only if \( V \) is non-degenerate.
Proof. Suppose $V$ is degenerate. Let $v$ be of minimal length such that $V = v1^a v1^a v \ldots$ for some $a \in \mathbb{N}$. To show $B_V$ is finite, it suffices to show that $w \notin B_V$ if $w \in A_V$ with $|v| < |w|$. Suppose $w \in A_V$ with $|v| < |w|$. By Corollary 2.12, $w = v1^a v1^a v \ldots v1^a v$. Clearly, $v \prec w \prec v1^a w$ and $v \preceq s w1^a w$ and $w1^a w \in A_V$. Therefore, $w \notin B_V$.

Now suppose that $V$ is non-degenerate. To show $B_V$ is infinite, it suffices to show that for each $v \in A_V$, there exists some $w \in B_V$ satisfying $v \preceq w$. Let $v \in A_V$, and choose $w \in A_V$ to be maximal (with respect to $\preceq$) such that $v \preceq w$. (The existence of such a $w$ follows from the fact that $V$ is not periodic.)

We claim that $w \in B_V$. Suppose, towards a contradiction, that there exist $u, u' \in A_V$ such that $u \prec w \prec u'$ and such that $u \preceq s u'$. If $v = w$, then $u \prec v \prec u'$ and $u \preceq s u'$. By Lemma 2.8(b), this implies $v \preceq s u'$, which contradicts the maximality of $w$.

Suppose, then, that $v \prec w$. If $v$ is comparable to $u$, then either $u \preceq v \prec u'$ or $v \preceq u \prec w \prec u'$. In either case, Lemma 2.8 implies that $v \preceq s u'$, which contradicts the maximality of $w$.

Suppose, then, that $u$ and $v$ are incomparable. (The situation is illustrated in Figure 2.) By Proposition 2.11, $(v \wedge u) \prec u \prec (v \lor u)$ and $(v \wedge u) \preceq s (v \lor u)$. Thus $(v \wedge u) \prec u \prec (v \lor u) \preceq s u'$ with $(v \wedge u) \preceq s (v \lor u)$ and $u \preceq s u'$. By Lemma 2.8(a), $(v \wedge u) \preceq s u'$. We now have $(v \wedge u) \prec v \prec w \prec u'$, with $(v \wedge u) \preceq s u'$. By Lemma 2.8(b), $v \preceq s u'$, which contradicts the maximality of $w$.

Figure 2. The proof of Proposition 2.16.

In fact, we have proved the following corollary.
Corollary 2.17. If $V$ is non-degenerate and $v \in A_V$, then there exists some $w \in B_V$ such that $v \preceq w$.

2.3.3 The canonical generating sequence of a non-degenerate rank-1 word.

We now have that for all $V \in \mathcal{R}$, $B_V$ is an infinite set, totally ordered by $\preceq$. Let $(v_n : n \in \mathbb{N})$ be an enumeration of $B_V$ such that for $n, m \in \mathbb{N}$, $n \leq m$ if and only if $v_n \preceq v_m$. We call the sequence $\{v_n\}_{n \in \mathbb{N}}$ the canonical generating sequence for $V$.

2.3.4 Sets of the form $A^0$ and sets of the form $B^0$.

Let $A^0 = \{V \in \mathcal{R}' : V \text{ is built fundamentally from } v\} = \{V \in \mathcal{R}' : v \in B_V\}$.

Remarks. (1) $\mathcal{R}' = A^0 = B^0$.
(2) For all $v$, $B^0 \subseteq A^0$.
(3) If $v \preceq w$, then $A^w \subseteq A^v$.
(4) If $B^0 \cap B^w \neq \emptyset$, then $v$ is comparable to $w$.

Proposition 2.18. If $v < w$ and $B^v \cap A^w \neq \emptyset$, then $A^w \subseteq B^v$.

Proof. Let $v < w$ and $V \in B^v \cap A^w$. Suppose, towards a contradiction, that $W \in A^w$, but $W \notin B^v$. Then $w \in A_W$, but $v \notin B_W$. Note that $v \in A_W$, since $W \in A^w \subseteq A^v$. Since $v \notin B_W$, there exist $u, u' \in A_W$ such that $u < v < u'$ and $u \preceq_s u'$. Note that $u \in A_V$ (since $u < v$ and $v \in A_V$), but that $u'$ need not be an element of $A_V$. There are three possibilities: either $u' \preceq_s w$, $w \preceq u'$, or $u'$ and $w$ are incomparable.

If $u' \preceq_s w$, then $u' \in A_V$. Thus, $u, u' \in A_V$, with $u < v < u'$ and $u \preceq_s u'$. This is a contradiction with $v \in B_V$.

If $w \preceq u'$, then $u < v < w \preceq u'$, with $u \preceq_s u'$. By Lemma 2.8, this implies that $u \preceq_s w$. Since $u, w \in A_V$, this is a contradiction with $v \in B_V$.

Finally, if $u'$ and $w$ are incomparable (the situation is illustrated in Figure 3) then, since $u'$ and $w$ are both in $A_W$, we have $(u' \vee w), (u' \wedge w) \in A_W$. Thus $u < v \preceq (u' \wedge w) < u' < (u' \vee w)$, with $u \preceq_s u'$ and $(u' \wedge w) \preceq_s (u' \vee w)$. By Lemma 2.8(a), $u \preceq_s (u' \vee w)$. Now $u < v < w < (u' \vee w)$, with $u \preceq_s (u' \vee w)$. By Lemma 2.8(b), $u \preceq_s w$. Since $u, w \in A_V$, this is a contradiction with $v \in B_V$. \qed

Corollary 2.19. If $v \preceq w$ and $B^v \cap B^w \neq \emptyset$, then $B^w \subseteq B^v$.

Proposition 2.20. The topology generated by $\{A^v : v \in \mathcal{F}\}$ is the same as the topology generated by $\{B^v : v \in \mathcal{F}\}$. 
Proof. It suffices to show the following.

(a) If $v \in F$ and $V \in A^v$, there exists $w \in F$ such that $V \subseteq B^w \subseteq A^v$.

(b) If $v \in F$ and $V \in B^v$, there exists $w \in F$ such that $V \subseteq A^w \subseteq B^v$.

Suppose $v \in F$ and $V \in A^v$. Choose $w \in B_V$ such that $v \subseteq w$. Clearly, $B^w \subseteq A^w \subseteq A^v$. Then $V \subseteq B^w$, since $w \in B_V$. This proves (a).

Suppose $v \in F$ and $V \in B^v$. Choose $w \in B_V$ such that $v \subseteq w$. Clearly, $B^v \cap A^w \neq \emptyset$. By Proposition 2.18, $A^w \subseteq B^v$. Then $V \subseteq B^w \subseteq A^w$, since $w \in B_V$. This proves (b).

\[ \square \]

2.3.5 A complete metric on $\mathcal{R}$. Define $d : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{R}$ by

\[ d(V, W) = 2^{-\sup\{ |v| : V, W \in B^v \}}. \]

Remarks. (1) If $V, W \in \mathcal{R}$, then $V, W \in B^0$; therefore, $0 \leq d(V, W) \leq 1/2$.

(2) By Proposition 2.20, $d$ generates the topology $T$ on $\mathcal{R}$.

Recall that an ultrametric space is a pair $(\mathcal{R}, d)$, in which $\mathcal{R}$ is a set and $d : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{R}$ satisfies

(1) $d(U, V) \geq 0$ for all $U, V \in \mathcal{R}$;
(2) $d(U, V) = 0$ if and only if $U = V$;
(3) $d(U, V) = d(V, U)$ for all $U, V \in \mathcal{R}$;
(4) for all $U, V, W \in \mathcal{R}$,

\[ d(U, W) \leq \max\{ d(U, V), d(V, W) \}. \]

Proposition 2.21. $(\mathcal{R}, d)$ is a complete ultrametric space.
Proof. Conditions (1) and (2) for an ultrametric space follow immediately from the definition of $d$.

If $V \in \mathcal{R}$, then $B_V$ is infinite; thus, $d(V, V) = 0$. If $d(U, V) = 0$, then there exist arbitrarily long words that are common initial segments of $U$ and $V$, i.e., $U = V$. This establishes condition (3).

Clearly, if $U = V$ or $V = W$, condition (4) is satisfied. If neither $U = V$ nor $V = W$, let $v$ be the longest word satisfying $U, V \in B^v$ and $w$ the longest word satisfying $V, W \in B^w$. Without loss of generality, assume $|v| \leq |w|$. Since $B^v \cap B^w \neq \emptyset$, we know that $v \preceq w$. By Corollary 2.19, $B^w \subseteq B^v$, and hence $W \in B^v$. Now $U, W \in B^v$; thus, $d(U, W) \leq d(U, V)$, and condition (4) follows.

To prove that $d$ is complete, let $\{V_n\}$ be a $d$-Cauchy sequence in $\mathcal{R}$ and $B = \{ v : \text{for sufficiently large } n, V_n \in B^v \}$. We claim that $(B, \preceq)$ is a total order. If $v, w \in B$ then, for sufficiently large $n$, $V_n \in B^v \cap B^w$. Thus, $B^v \cap B^w \neq \emptyset$, and therefore, $v$ and $w$ are comparable.

We claim that $B$ contains arbitrarily long words $v$. To this end, let $M \in \mathbb{N}$. Find $N$ such that for $n, m \geq N$, the distance between $V_n$ and $V_m$ is at most $2^{-M}$. Thus, for $n, m \geq N$, there exists some $v$ of length at least $M$ such that $V_n, V_m \in B^v$. Choose $v$ of minimal length such that $|v| \geq M$ and such that $V_n \in B^v$ for some $n \geq N$. It follows from Corollary 2.19 and the minimality of $|v|$ that if $m \geq N$, then $V_m \in B^v$. Thus, $v$ is an element of $B$.

We now have an infinite set $B$ on which $\preceq$ is a total order. Let $V$ be the unique infinite word such that each element of $B$ is an initial segment of $V$. We claim that $V$ is rank-1. It suffices to show that $B \subseteq A_V$. If $v \in B$, there exist arbitrarily long initial segments of $V$ that are built from $v$; namely, the elements of $B$ that are longer than $v$. Thus, $v \in A_V$.

By Proposition 2.16, to prove that $V$ is not degenerate, it suffices to show that $B \subseteq B_V$. Let $v \in B$, and choose $w \in B$ such that $v \prec w$. Choose $n \in \mathbb{N}$ so large that $V_n \in B^v \cap B^w$. Clearly, $B^w \cap A^w \neq \emptyset$. By Proposition 2.18, $A^w \subseteq B^v$. Also $V \in B^v$, since $V \in A^w$. Therefore, $v \in B_V$.

Finally, to prove that $V$ is the $d$-limit of the sequence $\{V_n\}$, it suffices to show that $B_V \subseteq B$. Let $v \in B_V$. Choose $w \in B$ such that $v \prec w$. Also $w \in B_V$, since $B \subseteq B_V$; hence $B^v \cap B^w \neq \emptyset$. By Corollary 2.19, $B^w \subseteq B^v$. For sufficiently large $n \in \mathbb{N}$, we know that $V_n \in B^w \subseteq B^v$. Therefore, $v \in B$. \qed
2.4 Expectedness in rank-1 words and elements of rank-1 systems.

2.4.1 Expectedness in a rank-1 word. For the rest of Section 2.4.1, we fix $V \in \mathcal{R}$ and $v \in A_V$. Let $(X, \sigma)$ be the rank-1 system associated to $V$, and let $v = 01a_101a_2 \ldots 1^{a_r-1}0$.

If $V(i) = 0$ and $|\{ j < i : V(j) = 0\}|$ is a multiple of $r$, then $V$ has an occurrence of $v$ beginning at position $i$. Such an occurrence of $v$ in $V$ is called expected. Since $V$ is built from $v$, each occurrence of 0 in $V$ is part of exactly one expected occurrence of $v$. It is possible for an occurrence of $v$ in $V$ to be unexpected, i.e., not expected. It is easy to see that such an occurrence of $v$ must overlap exactly two expected occurrences of $v$ in $V$.

In Section 2.4.3, below, we show that if $x \in X$, there exists a unique collection of occurrences of $v$ in $x$ (called the expected occurrences of $v$ in $x$) such that each occurrence of 0 is part of exactly one element of that collection. We also show that for each $i \in \mathbb{Z}$, the set of all $x \in X$ that have an expected occurrence of $v$ beginning at position $i$ is a clopen subset of $X$. The key to these results is the observation that it is possible to determine whether an occurrence of $v$ in $V$ (say it begins at $i$) is expected by knowing what $V$ looks like “close” to position $i$; in particular, we do not need to count the number of 0s in $V$ before position $i$.

Recall that $\{i_n\}_{n \in \mathbb{N}}$ enumerates, in an order-preserving way, the positions of the occurrences of 0 in $V$. Recall also that $L : \mathbb{N}_{>0} \to \mathbb{N}$ is defined by $L(n) = i_n - i_{n-1} - 1$. If $V$ is built from $v$, then, from Lemma 2.9(a), $L(n) = a_k$ for all $0 < k < r$ and $n \equiv k \mod r$. Thus, for each $0 < k < r$, $L$ is constant on each congruence class of $k \mod r$. Since $V \in \mathcal{R}$ is not periodic, neither is $L$. Thus $L$ is not constant on the congruence class of 0 ($\mod r$). In fact, we can show that if $V$ is built from $v$, the function $L$ fails to be constant on the congruence class of 0 ($\mod r$) somewhat regularly.

**Proposition 2.22.** There exists $t \in \mathbb{N}$ such that no $t$ consecutive elements of the sequence $\{L(r), L(2r), L(3r), \ldots\}$ are equal.

**Proof.** First we show that for sufficiently large $d$, there cannot even be two consecutive elements of $\{L(r), L(2r), L(3r), \ldots\}$ both equal to equal $d$. To this end, simply choose $w$ such that $V$ is built from $w$ and such that $w$ has more than $r$ occurrences of 0. Then let $D$ be the maximal number of consecutive 1s in $w$. Since $w$ has more occurrences of 0 than does $v$, every occurrence of $w$ in $V$ intersects at least two expected occurrences of $v$. This implies that at least one of each pair of two consecutive elements of $\{L(r), L(2r), L(3r), \ldots\}$ must be less than or
equal to \( D \). Therefore, if \( d > D \), there cannot be two consecutive elements of \( \{ L(r), L(2r), L(3r), \ldots \} \) that all equal \( d \).

Suppose, towards a contradiction, that there is no bound on the number of consecutive elements of \( \{ L(r), L(2r), L(3r), \ldots \} \) that are equal. By the preceding paragraph, there must be a single value \( d \) such that there is no bound on the number of consecutive elements of \( \{ L(r), L(2r), L(3r), \ldots \} \) that equal \( d \). Now let \( \alpha = v1^d \), and notice that for each \( n \), \( \alpha^n \) is a subword of \( V \). This implies the existence of a periodic \( x \in X \) that contains a 0, namely, \( \{ \cdots aaaa \cdots \} \), contradicting Corollary 2.6.

For the rest of Subsection 2.4.1, we fix \( t \in \mathbb{N} \) such that no \( t \) consecutive elements of the sequence \( \{ L(r), L(2r), L(3r), \ldots \} \) are equal.

**Corollary 2.23.** The following statements (about \( n \in \mathbb{N} \)) are equivalent.

(i) \( n \equiv 0 \mod r \);

(ii) \( L(n + r), L(n + 2r), \ldots, L(n + tr) \) are not all equal.

We now describe a uniform way to check whether a particular occurrence of \( v \) in \( V \) is expected.

**Corollary 2.24.** Suppose \( V \) has an occurrence of \( v \) beginning at position \( i \). Let \( \alpha = 01^b_101^b_20 \ldots 01^b_t \) be the subword of \( V \) of length \( 2t|v| \) that begins at \( i \).

(a) If \( b_l \geq |v| \) for some \( 0 < l \leq s \), then the occurrence of \( v \) in \( V \) beginning at \( i \) is expected if and only if \( l \equiv 0 \mod r \).

(b) If \( b_l < |v| \) for all \( 0 < l \leq s \), then \( s > rt \) and the occurrence of \( v \) in \( V \) beginning at \( i \) is expected if and only if \( b_r, b_{2r}, \ldots, b_{tr} \) are not all equal.

**Proof.** Let \( n \in \mathbb{N} \) be such that \( i = in \). Note that \( b_l = L(n + l) \) for \( 0 < l < s \).

Proof of (a). If \( b_l \geq |v| \), then \( L(n + l) \geq |v| \). This implies that \( L(n + l) > a_k \) for each \( 0 < k < r \). Thus, \( n + l \not\equiv k \mod r \) for each \( 0 < l < r \). Therefore, \( n + l \equiv 0 \mod r \). Thus, \( n \equiv 0 \mod r \) if and only if \( l \equiv 0 \mod r \). The occurrence of \( v \) in \( V \) beginning at position \( in \) is expected if and only if \( n \equiv 0 \mod r \). Therefore, the occurrence of \( v \) in \( V \) beginning at \( in = i \) is expected if and only if \( l \equiv 0 \mod r \).

Proof of (b). Suppose that \( b_l < |v| \) for all \( 0 < l \leq s \). Simple counting shows that \( |\alpha| \leq t|v| + t(|v| - 1) < 2t|v| \) if \( s \leq tr \). Therefore, \( s > tr \).

The occurrence of \( v \) in \( V \) beginning at position \( in \) is expected if and only if \( n \equiv 0 \mod r \). Furthermore, by Corollary 2.23, \( n \equiv 0 \mod r \) if and only if \( L(n + r), L(n + 2r), \ldots, L(n + tr) \) are not all equal. On the other hand, \( L(n + r) = b_r, \ldots, L(n + tr) = b_{tr} \). Therefore, the occurrence of \( v \) in \( V \) beginning at \( i \) is expected if and only if \( b_r, b_{2r}, \ldots, b_{tr} \) are not all equal. \( \square \)
The crucial fact about Corollary 2.24 is that to determine whether an occurrence of \( v \) in \( V \) beginning at position \( i \) is expected, it suffices to know the subword of \( V \) of length \( 2r|v| \) that begins at \( i \).

2.4.2 **Expectedness in an element of a rank-1 system.** For the rest of Subsection 2.4.2, we fix a non-degenerate rank-1 system \((X, \sigma)\) and \( v \in \mathcal{F} \). Let \( v = 01^{a_1}01^{a_2} \ldots 1^{a_r-1}0 \).

**Definition 2.25.** We say that \( x \in X \) is **built from** \( v \) if there exists a collection \( E_v \) of occurrences of \( v \) in \( x \) such that each 0 in \( x \) is part of exactly one element of \( E_v \).

**Remarks.**

(1) If the collection \( E_v \) witnesses that \( x \) is built from \( v \), i.e., if \( E_v \) is a collection of occurrences of \( v \) in \( x \) such that each 0 in \( x \) is part of exactly one element of \( E_v \), then distinct elements of \( E_v \) must actually be disjoint (since \( v \) begins with 0).

(2) If \( x \) is the constant word 1, i.e., \( x \) contains no occurrence of 0, then \( x \) is built from \( v \) vacuously.

A priori, there are three types of elements of \( x \in X \) that have an occurrence of 0:

(a) \( x \) has a first occurrence of 0;
(b) \( x \) has a last occurrence of 0; or
(c) \( x \) has neither a first nor last occurrence of 0.

By Corollary 2.3 no \( x \in X \) can satisfy both (a) and (b) above. For each of these possibilities we have a lemma, which we state here without proof.

**Lemma 2.26.** Suppose \( x \in X \) has a first occurrence of 0 and that \( E_v \) witnesses that \( x \) is built from \( v \). The following statements (about \( i \in \mathbb{Z} \)) are equivalent.

(i) \( x(i) = 0 \), and \(|\{ j < i : x(j) = 0 \}| \) is a multiple of \( r \).
(ii) Some element of \( E_v \) begins at position \( i \) in \( x \).

**Lemma 2.27.** Suppose \( x \in X \) has a last occurrence of 0 and that \( E_v \) witnesses that \( x \) is built from \( v \). The following statements (about \( i \in \mathbb{Z} \)) are equivalent.

(i) \( x(i) = 0 \), and \(|\{ j \geq i : x(j) = 0 \}| \) is a multiple of \( r \).
(ii) Some element of \( E_v \) begins at position \( i \) in \( x \).

If \( x \in X \) has an occurrence of 0 but neither a first nor a last occurrence of 0, let \( \{ i_n \}_{n \in \mathbb{Z}} \) enumerate, in an order-preserving way, the positions of the occurrences of
0 in x. Define \( L_x : \mathbb{Z} \rightarrow \mathbb{N} \) by \( L_x(n) = i_n - i_{n-1} - 1 \), i.e., \( L_x(n) \) is the number of 1s between the 0 at position \( i_{n-1} \) in x and the 0 at position \( i_n \) in x. Note that since x is not periodic, neither is \( L_x \).

**Lemma 2.28.** Suppose \( x \in X \) has an occurrence of 0 but neither a first nor last occurrence of 0. Suppose, furthermore, that \( E_v \) witnesses that x is built from v. Then the following statements (about \( n \in \mathbb{Z} \)) are equivalent.

(i) Some element of \( E_v \) begins at position \( i_n \) in x.

(ii) The function \( L_x \) is not constant on the congruence class of \( n \) (mod r) but is constant on each other congruence class mod r.

**Proposition 2.29.** If \( x \in X \) is built from v, there exists a unique collection \( E_v \) of occurrences of v in x such that each 0 in x is part of exactly one element of \( E_v \).

**Proof.** If \( x \) has no occurrence of 0, the only possibility for \( E_v \) is the empty set. That the proposition holds in case \( x \) has an occurrence of 0 follows immediately from Lemmas 2.26, 2.27, and 2.28.

**Definition 2.30.** Let \( x \in X \) be built from v. We say that an occurrence of v in x is **expected** if it belongs to \( E_v \) and **unexpected** otherwise.

The next corollary follows immediately from Definition 2.25, Proposition 2.29, and Definition 2.30.

**Corollary 2.31.** Suppose \( x \in X \) is built from v and that \( u \preceq v \). Then x is built from u, and every expected occurrence of u in x is completely contained in an expected occurrence of v.

### 2.4.3 Connections between a rank-1 word and elements of its associated system.

For the rest of Subsection 2.4.3, we fix \( V \in \mathbb{R} \) and let \((X, \sigma)\) be the non-degenerate rank-1 system associated to \( V \).

**Proposition 2.32.** If \( V \) is built from v, then each \( x \in X \) is built from v.

**Proof.** Suppose \( V \) is built from v, and let \( x \in X \). We first define the collection \( E_v \) of occurrences of v in x. Suppose \( x \) has an occurrence of v beginning at position \( i \). Let \( \alpha = 01^{b_1}01^{b_2}0\ldots01^{b_s} \) be the subword of x of length \( 2r|v| \) that begins at \( i \).

(a) If \( b_k \geq |v| \) for some \( 0 < k \leq s \), then the occurrence of v in V beginning at \( i \) belongs to \( E_v \) if and only if \( k \equiv 0 \mod r \).

(b) If \( b_k < |v| \) for all \( 0 < k \leq s \), then \( s > rt \); and the occurrence of v in V beginning at \( i \) belongs to \( E_v \) if and only if \( b_r, b_{2r}, \ldots, b_{rt} \) are not all equal.
That each occurrence of 0 is a part of exactly one element of $E_v$ follows from Corollary 2.24. Indeed, suppose $x$ has an occurrence of 0 at position $i$. Let $\beta$ be the subword of $x$ beginning at $i - |v| + 1$ and ending at $i + 2t|v| - 1$. Note that if $x$ has an occurrence of $v$ (say it begins at $i - r$) that contains the occurrence of 0 at $i$, then the subword of $x$ of length $2t|v|$ and beginning at $i - r$ is completely contained in the occurrence of $\beta$ beginning at $i - |v| + 1$.

Let $j \in \mathbb{N}$ be such that $V$ has an occurrence of $\beta$ beginning at position $j - |v| + 1$. Note that $V$ has an occurrence of 0 at position $j$. Note, further, that if $V$ has an occurrence of $v$ (say it begins at $j - r$) that contains the occurrence of 0 at $j$, then the subword of $V$ of length $2t|v|$ and beginning at $j - r$ is completely contained in the occurrence of $\beta$ beginning at $j - |v| + 1$.

It is clear that $x$ has an occurrence of $v$ beginning at position $i - r$ that contains the occurrence of 0 at position $i$ if and only if $V$ has an occurrence of $v$ beginning at position $j - r$ that contains the occurrence of 0 at position $j$. Moreover, by Corollary 2.24, such an occurrence of $v$ in $x$ belongs to $E_v$ if and only if the corresponding occurrence of $v$ in $V$ is expected.

Since the occurrence of 0 in $V$ at position $j$ is contained in exactly one expected occurrence of $v$, the occurrence of 0 in $x$ at position $i$ is contained in part of exactly one element of $E_v$. \hfill \qed

To formulate our next result, we require the following definition.

**Definition 2.33.** (a) For each finite word $\alpha$ and $i \in \mathbb{Z}$,

$$U_{\alpha,i} = \{ x \in X : V \text{ has an occurrence of } \alpha \text{ beginning at position } i \}.$$  

(b) For each $v$ such that $V$ is built from $v$,

$$E_{v,i} = \{ x \in X : V \text{ has an expected occurrence of } v \text{ beginning at position } i \}.$$  

Corollary 2.34 follows immediately from the definition of $E_v$ in the proof of Proposition 2.32.

**Corollary 2.34.** If $V$ is built from $v$ and $i \in \mathbb{Z}$, then $E_{v,i}$ is a clopen subset of $X$.

We also have the following result.

**Proposition 2.35.** For non-empty and open $U \subseteq X$, there exist some $v$ from which $V$ is built fundamentally and some $j \in \mathbb{Z}$ such that $E_{v,j} \subseteq U$.

**Proof.** Let $\alpha$ be a finite word and $i \in \mathbb{Z}$ be such that $U_{\alpha,i} \subseteq U$ is non-empty. Since $U_{\alpha,i}$ is non-empty, there exists $k \in \mathbb{N}$ such that $V$ has an occurrence of
\(\alpha\) beginning at position \(k\). Choose \(v\) to be such that \(V\) is built fundamentally from \(v\) and \(|v| \geq k + |\alpha|\). Note that the occurrence of \(v\) in \(V\) beginning at position 0 completely contains the occurrence of \(\alpha\) in \(V\) beginning at \(k\). Thus, if any \(x \in X\) has an occurrence of \(v\) beginning at position \(l\), \(x\) has an occurrence of \(\alpha\) beginning at position \(l + k\). To complete the proof, let \(j = i - k\) and notice that \(E_{\alpha,j} \subseteq U_{\alpha,i} \subseteq U\).

**Proposition 2.36.** If \(x \in X\) is not periodic but is built from \(v\), then \(v\) is an initial segment of \(V\).

**Proof.** Choose \(w\) such that \(V\) is built from \(w\) and \(|v| \leq |w|\). By Proposition 2.32, \(x\) is built from \(w\). To show that \(v\) is an initial segment of \(V\), it suffices to show the existence of \(i \in \mathbb{Z}\) such that \(x\) has an occurrence of \(v\) beginning at position \(i\) and also an occurrence of \(w\) beginning at \(i\). Let \(r\) be the number of occurrences of 0 in \(v\) and \(s\) the number of occurrences of 0 in \(w\).

If \(x\) has a first occurrence of 0, say, at position \(i\), then by Lemma 2.26, \(x\) has an occurrence of \(v\) beginning at \(i\) and also an occurrence of \(w\) beginning at \(i\).

If \(x\) has a last occurrence of 0, then by Corollary 2.3, \(x\) has infinitely many occurrences of 0. Let \(i\) be such that \(x(i) = 0\) and \(|\{j \geq i : x(j) = 0\}| = rs\). By Lemma 2.27, \(x\) has an occurrence of \(v\) beginning at position \(i\) and also an occurrence of \(w\) beginning at \(i\).

Finally, suppose \(x\) has an occurrence of 0 but neither a first nor last occurrence of 0. It is given that \(x\) is built from \(v\). By Lemma 2.28, there exists \(k \in \mathbb{Z}\) such that \(L_x\) is not constant on the congruence class of \(k \pmod{r}\) but is constant on each other congruence class mod \(r\). Similarly, since \(x\) is built from \(w\), there exists \(l\) such that \(L_x\) is not constant on the congruence class of \(l \pmod{s}\) but is constant on each other congruence class mod \(s\). We claim that there exists \(n \in \mathbb{Z}\) such that \(n \equiv k \pmod{r}\) and \(n \equiv l \pmod{s}\). Indeed, if no such \(n\) exists, then, for every \(m \in \mathbb{Z}\), either \(m\) is not congruent to \(k\) mod \(r\) or \(m\) is not congruent to \(l\) mod \(s\). In either case, \(L_x(m) = L_x(m + rs)\). This implies that \(L_x\) is periodic, which implies that \(x\) is periodic, contrary to assumption. It now follows from Lemma 2.28 that some element of \(E_v\) begins at position \(i_n\), and some element of \(E_w\) begins at position \(i_n\). Thus \(x\) has an occurrence of \(v\) beginning at position \(i_n\) and an occurrence of \(w\) beginning at \(i_n\).

**Proposition 2.37.** Let \(V\) and \(W\) be distinct elements of \(\mathcal{R}\), and let \((X, \sigma)\) and \((Y, \sigma)\) be the rank-1 systems associated to \(V\) and \(W\), respectively. Then \(X\) and \(Y\) do not share any non-periodic points. In particular, \(X \neq Y\).
Proof. Let \( x \in X \cap Y \) be non-periodic. By Proposition 2.32, if \( V \) is built from \( v \), \( x \) is built from \( v \); and by Proposition 2.36, \( v \) is an initial segment of \( W \). Since \( V \) is built from arbitrarily long words, \( V \) and \( W \) agree on arbitrarily long initial segments and thus are equal. \( \square \)

3 Isomorphisms between non-degenerate rank-1 systems.

3.1 Stable isomorphisms and replacement schemes.

Definition 3.1. Let \((X, \sigma)\) and \((Y, \nu)\) be non-degenerate rank-1 systems. A topological isomorphism (or isomorphism) from \((X, \sigma)\) to \((Y, \sigma)\) is a homeomorphism \( \phi : X \to Y \) that commutes with \( \sigma \).

We want to understand when two non-degenerate rank-1 systems are isomorphic. More generally, we want to identify all isomorphisms between non-degenerate rank-1 systems.

Definition 3.2. It follows immediately from Definition 3.2 that an isomorphism \( \phi : X \to Y \) between non-degenerate rank-1 systems is stable if there exist \( v, w \in F \) such that for all \( x \in X \),

1. \( x \) is built from \( v \) and \( \phi(x) \) is built from \( w \); and
2. \( x \) has an expected occurrence of \( v \) beginning at position \( k \) if and only if \( \phi(x) \) has an expected occurrence of \( w \) beginning at \( k \).

For stable isomorphisms, we call the pair \((v, w)\) a replacement scheme for \( \phi \).

An isomorphism \( \phi \) is stable if and only if \( \phi^{-1} \) is stable, and \((v, w)\) is a replacement scheme for \( \phi \) if and only if \((w, v)\) is a replacement scheme for \( \phi^{-1} \).

In Theorem 3.5 below, we show that every isomorphism between non-degenerate rank-1 systems is nearly stable. First, however, we describe conditions on \( V \) and \( W \) that are necessary and sufficient for the existence of a stable isomorphism between \((X, \sigma)\) and \((Y, \sigma)\).

Definition 3.3. Let \( V, W \in \mathcal{R} \) and \( v, w \in F \). We say the pair \((v, w)\) is a replacement scheme for \( V \) and \( W \) if

1. \( V \) is built from \( v \) and \( W \) is built from \( w \); and
2. \( V \) has an expected occurrence of \( v \) beginning at position \( k \) if and only if \( W \) has an expected occurrence of \( w \) beginning at \( k \).

Proposition 3.4. Let \( V, W \in \mathcal{R} \) and \( v, w \in F \). The pair \((v, w)\) is a replacement scheme for \( V \) and \( W \) if and only if \((v, w)\) is a replacement scheme for some stable isomorphism \( \phi : X \to Y \).
Proof. We first prove the forward implication. If \((v, w)\) is a replacement scheme for \(V\) and \(W\), we can obtain \(W\) from \(V\) by replacing every expected occurrence of \(v\) with \(w\) and then adding or deleting occurrences of 0 as necessary. For example, if the length of \(v\) minus the length of \(w\) is \(l \geq 0\), then \(W\) can be obtained from \(V\) by replacing every expected occurrence of \(v\) by \(w1^l\). There exists a unique function \(\phi : X \rightarrow \{0, 1\}^\mathbb{Z}\) such that for each \(x \in X\), \(\phi(x)\) can be obtained from \(x\) in the same way that \(W\) can be obtained from \(V\) (in our example, by replacing every expected occurrence of \(v\) with \(w1^l\)). It is then straightforward to check that \(\phi\) is a homeomorphism from \(X\) to \(Y\). It is obvious from the definition of \(\phi\) that \(\phi\) commutes with \(\sigma\) and that \((v, w)\) is a replacement scheme for \(\phi\).

We now prove the reverse implication. Suppose \((v, w)\) is a replacement scheme for an isomorphism \(\phi : X \rightarrow Y\). Let \(\{v_m\}_{m \in \mathbb{N}}\) and \(\{w_n\}_{n \in \mathbb{N}}\) be the canonical generating sequences for \(V\) and \(W\), respectively.

We claim that there exists \(x \in X\) that has an expected occurrence of \(v_m\) beginning at position 0 for each \(m \in \mathbb{N}\). For each \(m \in \mathbb{N}\), choose \(x_m \in X\) that has an expected occurrence of \(v_m\) beginning at 0. Note that this implies that \(x_m\) has an expected occurrence of \(x_n\) beginning at 0 for all \(n \leq m\). Since \(X\) is compact, the sequence \(\{x_m\}_{m \in \mathbb{N}}\) has a convergent subsequence. Let \(x \in X\) be the limit of a convergent subsequence. It follows from Corollary 2.34 that for all \(m \in \mathbb{N}\), \(x\) has an expected occurrence of \(v_m\) beginning at 0.

Now consider \(\phi(x)\), and let \(U\) be the infinite word in \(\phi(x)\) that begins at 0. To show that \((v, w)\) is a replacement scheme for \(V\) and \(W\), it suffices to show that \(U = W\).

We first claim that \(U\) is rank-1. It is easy to see that to each \(v_m\) such that \(v < v_m\), there exists a corresponding word \(v'_m\) from which \(U\) is built. For example, if the length of \(v\) minus the length of \(w\) is \(l \geq 0\) and \(v_m = v1^{a_1}w1^{a_2} \ldots 1^{a_{l+1}}w\), then \(U\) is built from the word \(v'_m = w1^{a_1+l}w1^{a_2+l} \ldots 1^{a_{l+1}}w\). Since there are infinitely many \(v_m\) such that \(v < v_m\), \(U\) is rank-1.

We next claim that \(U\) contains every finite subword of \(W\). Indeed, since \(x\) has an occurrence of \(V\) beginning at 0, \(x\) has expected occurrences of \(v\) beginning at arbitrarily large \(k\). Since \(w\) contains an occurrence of 0, there exist occurrences of 0 that occur in \(\phi(x)\) at arbitrarily large \(k\). This implies that there are expected occurrences of each \(w_n\) that occur in \(\phi(x)\) at arbitrarily large \(k\). Thus \(U\) contains an occurrence of each \(w_n\) and thus every finite subword of \(W\).

Let \((Z, \sigma)\) be the non-degenerate rank-1 system corresponding to \(U\). Since \(U\) contains every finite subword of \(W\), \(Y \subseteq Z\). Since a rank-1 system is non-degenerate if and only if it is uncountable (see Proposition 2.4), and \(Y\) is non-degenerate, \(Z\) must be non-degenerate. Now, from Proposition 2.37, it follows
that $W = U$.

We are now ready to state the main theorem of this paper, that modulo a power of the shift, every isomorphism between non-degenerate rank-1 systems is stable.

**Theorem 3.5.** If $\phi : X \to Y$ is an isomorphism between non-degenerate rank-1 systems, then for some $q \in \mathbb{Z}$, the isomorphism $\phi \circ \sigma^q : X \to Y$ is stable.

As no minimal rank-1 system can be isomorphic to a non-minimal rank-1 system, it suffices to prove the theorem in two cases: when both $(X, \sigma)$ and $(Y, \sigma)$ are minimal; and when both $(X, \sigma)$ and $(Y, \sigma)$ are non-minimal. We do this in Sections 3.2 and 3.3, respectively. First, we give some implications.

**Corollary 3.6.** If $(X, \sigma)$ is a non-degenerate rank-1 system and $\tau$ is an automorphism of $X$ that commutes with $\sigma$, then $\tau = \sigma^i$ for some $i \in \mathbb{Z}$.

**Proof.** The map $\tau$, being an automorphism, is an isomorphism between $(X, \sigma)$ and $(X, \sigma)$. By Theorem 3.5, there exists $q \in \mathbb{Z}$ such that $\tau \circ \sigma^q$ is a stable isomorphism. Let $(v, w)$ be a replacement scheme for $\tau \circ \sigma^q$. By Proposition 3.4, $(v, w)$ is a replacement scheme for $V$ and $V$. This implies that $v = w$, which in turn, implies that $\tau \circ \sigma^q$ is the identity map. Thus, $\tau = \sigma^{-q}$.

Before proceeding with the proof of Theorem 3.5, we state one more important corollary, which gives a nice characterization of when $V, W \in \mathcal{R}$ are isomorphic. This characterization is used in Section 3.4 to determine the complexity of the isomorphism relation on $\mathcal{R}$ and in Section 3.5 to characterize when a non-degenerate rank-1 system is isomorphic to its inverse.

**Corollary 3.7.** Let $V, W \in \mathcal{R}$. Then $V$ is isomorphic to $W$, i.e., there exists an isomorphism $\phi : X \to Y$, if and only if there exists a replacement scheme $(v, w)$ for $V$ and $W$.

**Proof.** By Proposition 3.4, if $(v, w)$ is a replacement scheme for $V$ and $W$, there exists an isomorphism $\phi : X \to Y$. On the other hand, by Theorem 3.5, if $\phi : X \to Y$ is an isomorphism, there exists $q \in \mathbb{Z}$ such that $\phi \circ \sigma^q$ is stable. Since $\phi \circ \sigma^q : X \to Y$ is stable, there exists some $(v, w)$ that is a replacement scheme for $\phi \circ \sigma^q$. By Proposition 3.4, this $(v, w)$ is also a replacement scheme for $V$ and $W$. $\square$
3.2 Proof of Theorem 3.5 in the minimal case.

3.2.1 The setup. Let $(X, \sigma)$ and $(Y, \sigma)$ be minimal non-degenerate rank-1 systems, and let $\phi : X \to Y$ be an isomorphism. Let $V$ and $W$ be the rank-1 words that give rise to $(X, \sigma)$ and $(Y, \sigma)$, respectively. Let $X_{\text{max}}$ be the largest number of consecutive 1s that occur in $V$ (this is the same as the largest number of 1s that occur in an element of $X$). Let $Y_{\text{max}}$ be the largest number of consecutive 1s that occur in $W$.

Using basic techniques, we will choose $v_1, w_2, v_3 \in \mathcal{F}$ and $r, s \in \mathbb{Z}$ satisfying all of the following conditions:

1. $v_1, v_3 \in A_V$ and $w_2 \in A_W$;
2. $v_1 \preceq v_3$;
3. $|v_1| \geq X_{\text{max}}$ and $|v_1| \geq Y_{\text{max}}$.
4. for all $x \in X$ and all $k \in \mathbb{Z}$,
   a) if $x$ has an expected occurrence of $v_3$ beginning at position $k$, then $\phi(x)$ has an expected occurrence of $w_2$ beginning at position $k + r$;
   b) if $\phi(x)$ has an expected occurrence of $w_2$ beginning at position $k$, then $x$ has an expected occurrence of $v_1$ beginning at position $k + s$.

Choose $v_1 \in A_V$ such that $|v_1| > X_{\text{max}}$ and $|v_1| > Y_{\text{max}}$, and consider the non-empty open set $\phi(E_{v_1,0}) \subseteq Y$. By Proposition 2.35, there exist $w_2 \in A_W$ and $j \in \mathbb{Z}$ such that $E_{w_2,j} \subseteq \phi(E_{v_1,0})$. Note that if $\phi(x)$ has an occurrence of $w_2$ beginning at position $j$, then $x$ has an expected occurrence of $v_1$ beginning at position 0. Since $\phi$ commutes with $\sigma$, if $\phi(x) \in Y$ has an occurrence of $w_2$ beginning at any position $k \in \mathbb{Z}$, then $x$ has an expected occurrence of $v_1$ beginning at position $k - j$. Let $s = -j$. Next note that if $\phi(x) \in Y$ has an occurrence of $w_2$ beginning at any $k \in \mathbb{Z}$, then $x$ has an expected occurrence of $v_1$ beginning at $k + s$.

Now, since $\phi^{-1}(E_{w_2,0}) \subseteq X$ is non-empty and open, by Proposition 2.35, there exist $v_3' \in A_V$ and $i \in \mathbb{Z}$ such that $E_{v_3',i} \subseteq \phi(E_{w_2,0})$. Let $v_3 = v_3' \lor v_1$, and note that $v_1 \preceq v_3$. Also, since $v_3' \preceq v_3$, it follows from Corollary 2.31 that $E_{v_3,i} \subseteq E_{v_3',i}$. Thus $E_{v_3,i} \subseteq \phi(E_{w_2,0})$. If $x$ has an occurrence of $v_3$ beginning at position $i$, then $\phi(x)$ has an expected occurrence of $w_2$ beginning at position 0. Since $\phi$ commutes with $\sigma$, if $x \in X$ has an occurrence of $v_3$ beginning at any position $k \in \mathbb{Z}$, then $\phi(x)$ has an expected occurrence of $w_2$ beginning at position $k - i$. Let $r = -i$, and note that if $x \in X$ has an occurrence of $v_3$ beginning at any position $k \in \mathbb{Z}$, then $\phi(x)$ has an expected occurrence of $w_2$ beginning at position $k + r$.

Figure 4 illustrates the information we have so far. The arrows indicate forcing of occurrences of words.
3.2.2 What we need. To prove Theorem 3.5, we need to find finite words $v$ and $w$ and an integer $q$ such that for all $x \in X$ and all $k \in \mathbb{Z}$,

1. $x$ is built from $v$ and $\phi(x)$ is built from $w$, and
2. $x$ has an expected occurrence of $v$ beginning at position $k$ if and only if $\phi(x)$ has an expected occurrence of $w$ beginning at position $k + q$.

The desired $v$ is the word $v_3$. Proposition 2.32 implies that since $v \in A_V$, each $x \in X$ is built from $v$.

3.2.3 First steps. Each $\phi(x) \in Y$ consists of expected occurrences of $w_2$ interspersed with 1s. Each expected occurrence of $w_2$ in $\phi(x)$ forces an expected occurrence of $v_1$ in $x$; and, by Corollary 2.31, that expected occurrence of $v_1$ is contained in an expected occurrence of $v_3$. Thus we can associate to each expected occurrence of $w_2$ in $\phi(x)$ an expected occurrence of $v_3$ in $x$. Notice that if an expected occurrence of $w_2$ in $\phi(x)$ beginning at $j$ is associated to an expected occurrence of $v_3$ in $x$ beginning at $k$, then $k \leq j + s < k + |v_3|$ or, equivalently, $0 \leq (j - k) + s < |v_3|$.

We can fix an expected occurrence of $v_3$ in any $x \in X$ (say it begins at position $k$) and then consider the collection of expected occurrences of $w_2$ in $\phi(x)$ associated to it. In Proposition 3.11, we show that the structure of that collection is, in some sense, independent of $x$ and $k$.

Lemma 3.8. Suppose $x$ has an expected occurrence of $v_3$ beginning at $k$, which is followed by $1^a$ and then by another expected occurrence of $v_3$. Suppose also that the expected occurrence of $v_3$ that contains the expected occurrence of $v_1$ beginning at position $k + r + s$ is followed by $1^b$ and then by another expected occurrence of $v_3$. Then $a = b$. 
Proof. We know that \( x \) has an expected occurrence of \( v_1 \) beginning at position \( k + r + s \). Thus, there is also an expected occurrence of \( v_1 \) beginning at position \( k + r + s + |v_3| + b \). Also, there is an expected occurrence of \( v_1 \) beginning at position \( k + |v_3| + a + r + s \). If \( a \neq b \), then these two expected occurrences of \( v_1 \) are distinct, and hence disjoint. This clearly implies that \( |a - b| \geq |v_1| \), which cannot happen, since \( a \) and \( b \) are non-negative and \( |v_1| > X_{\text{max}} \).

The next claim can be proved with a similar argument. We state it without proof.

Lemma 3.9. Suppose \( x \) has an expected occurrence of \( v_3 \) beginning at position \( k \), which is preceded by \( 1^a \) and then by another expected occurrence of \( v_3 \). Suppose also that the expected occurrence of \( v_3 \) that contains the expected occurrence of \( v_1 \) beginning at \( k + r + s \) is preceded by \( 1^b \) and then by another expected occurrence of \( v_3 \). Then \( a = b \).

Lemma 3.10. \( 0 \leq r + s < |v_3| \).

Proof. Suppose \( 0 > r + s \) or \( r + s \geq |v_3| \). Choose \( x \in X \) and \( k \in \mathbb{Z} \) such that \( x \) has an expected occurrence of \( v_3 \) beginning at position \( k \). Then \( x \) has an expected occurrence of \( v_1 \) beginning at position \( k + r + s \), which is contained in an expected occurrence of \( v_3 \) in \( x \); say it begins at position \( k' \). Note that \( k \neq k' \). Applying Lemmas 3.8 and 3.9 repeatedly (infinitely many times) shows that \( x \) is periodic with period dividing \( |k - k'| \). This contradicts Corollary 2.6.

Figure 5 illustrates the information we have so far.

3.2.4 The key proposition. From the setup in Subsection 3.2.1, we know that if \( x \) has an expected occurrence of \( v_3 \) beginning at position \( k \), then \( \phi(x) \) has
an expected occurrence of $w_2$ beginning at position $k + r$. From Lemma 3.10, we know that $0 \leq r + s < |v_3|$. 

**Proposition 3.11.** Suppose $r'$ satisfies $0 \leq r' + s < |v_3|$ and is such that there exist $\bar{x} \in X$ and $\bar{k} \in \mathbb{Z}$ such that $\bar{x}$ has an expected occurrence of $v_3$ beginning at position $\bar{k}$ and $\phi(\bar{x})$ has an expected occurrence of $w_2$ beginning at $\bar{k} + r'$. Then, for every $x \in X$ and $k \in \mathbb{Z}$, if $x$ has an expected occurrence of $v_3$ beginning at $k$, then $\phi(x)$ has an expected occurrence of $w_2$ beginning at $k + r'$.

**Proof.** Suppose, towards a contradiction, that there exists $r'$ satisfying the hypothesis but not the conclusion of the proposition. Either $r > r'$ or $r < r'$. The arguments that lead to a contradiction in these two cases are almost identical, and we give only one of them. Suppose $r < r'$.

Let $r_0$ be the largest natural number less than $r'$ that satisfies both the hypothesis and the conclusion of the proposition. Note that $r_0 \geq r$. Let $r_1$ be the smallest natural number greater than $r_0$ that satisfies the hypothesis but not the conclusion of the proposition. Note that $r_0 \leq r'$. Choose $x_0, k_0, x_1, k_1$ so that all of the following hold:

1. $x_0$ has an expected occurrence of $v_3$ beginning at position $k_0$;
2. $\phi(x_0)$ has expected occurrences of $w_2$ beginning at positions $k_0 + r_0$ and $k_0 + r_1$;
3. $x_1$ has an expected occurrence of $v_3$ beginning at position $k_1$; and
4. $\phi(x_1)$ has an expected occurrence of $w_2$ beginning at position $k_1 + r_0$, but not at position $k_1 + r_1$.

Let $a$ be such that the expected occurrence of $w_2$ in $\phi(x_0)$ beginning at $k_0 + r_0$ is followed by $1^a$ and then by another expected occurrence of $w_2$. Let $b$ be such that the expected occurrence of $w_2$ in $\phi(x_1)$ beginning at $k_1$ is followed by $1^b$ and then by another expected occurrence of $w_2$.

We claim that $a = r_1 - r_0 - |w_2|$, i.e., the first expected occurrence of $w_2$ in $\phi(x_0)$ after the expected occurrence of $w_2$ beginning at $k_0 + r_0$ is the one beginning at $k_0 + r_1$. If not, there is an expected occurrence of $w_2$ between those beginning at $k_0 + r_0$ and $k_0 + r_1$; say it begins at $k + \hat{r}$, with $r_0 < \hat{r} < r_1$. If whenever $x \in X$ has an occurrence of $v_3$ at $k$, $\phi(x)$ has an expected occurrence of $w_2$ beginning at position $k + \hat{r}$, we have a contradiction with the maximality of $r_0$. On the other hand, if, for such $x$, $\phi(x)$ does not have an expected occurrence of $w_2$ beginning at $k + \hat{r}$, we have a contradiction with the minimality of $r_1$. Therefore, $a = r_1 - r_0 - |w_2|$.

Note that $b \neq a$; otherwise, $\phi(x_1)$ has an expected occurrence of $w_2$ beginning at position $k_1 + r_1$. Figure 6 illustrates the situation.

We now claim that $|b - a| \geq |v_1|$. First, notice that since $\phi(x_0)$ has an occurrence of $w_2$ beginning at $k_0 + r_1$, there is an expected occurrence of $v_1$ in $x_0$.
beginning at $k_0 + r_1 + s$. As $0 \leq r_1 + s < |v_3|$, this expected occurrence intersects, and thus is contained in, the expected occurrence of $v_3$ beginning at $k_0$ in $x_0$. This clearly implies that $x_1$ has an expected occurrence of $v_1$ beginning at position $k_1 + r_1 + s = k_1 + r_0 + |w_2| + a + s$. But since $\phi(x_1)$ has an occurrence of $w_2$ beginning at position $k_1 + r_0 + |w_2| + b$, there is also an expected occurrence of $v_1$ in $x_1$ beginning at position $k_1 + r_0 + |w_2| + b + s$. Since $a \neq b$, these two expected occurrences of $v_1$ are distinct, and hence disjoint. This clearly implies that $|b - a| \geq |v_1|$.

Since $a$ and $b$ are both non-negative integers, either $a \geq |v_1|$ or $b \geq |v_1|$. This contradicts the assumption that $|v_1| > Y_{\text{max}}$. □

3.2.5 Proof of Theorem 3.5 in the minimal case. Choose $x \in X$ and $k \in \mathbb{Z}$ such that $x$ has an expected occurrence of $v_3$ beginning at position $k$. Let $w$ be the smallest subword of $\phi(x)$ containing every expected occurrence of $w_2$ beginning at position $k + r'$, for some $0 \leq r' + s < |v_3|$. In other words, $w$ is the smallest subword of $\phi(x)$ containing every expected occurrence of $w_2$ that is associated to the expected occurrence of $v_3$ in $x$ beginning at $k$. Let $q$ be such that this occurrence of $w$ in $\phi(x)$ begins at position $k + q$. Note that if an expected occurrence of $w_2$ in $\phi(x)$ is not associated to the expected occurrence of $v_3$ beginning

Figure 6. The proof of Proposition 3.11.
at $k$ in $x$, then it is disjoint from the occurrence of $w$ at $k + q$.

It follows from Proposition 3.11 that $w$ and $q$ are independent of the choice of $x$ and $k$. Thus, if $x' \in X$ has an expected occurrence of $v_3$ beginning at position $k' \in \mathbb{Z}$, $\phi(x')$ has an occurrence of $w$ beginning at position $k' + q$. Moreover, an expected occurrence of $w_2$ in $\phi(x)$ is contained in the occurrence of $w$ beginning at $k' + q$ if it is associated to the expected occurrence of $v_3$ beginning at $k$ in $x$; otherwise, it is disjoint from the occurrence of $w$ at $k' + q$.

We have now defined $w$ and $q$ and stated the relevant facts. To prove the theorem, we must show that for all $x \in X$ and all $k \in \mathbb{Z}$,

(1) $\phi(x)$ is built from $w$, and

(2) $x$ has an expected occurrence of $v_3$ beginning at position $k$ if and only if $\phi(x)$ has an expected occurrence of $w$ beginning at position $k + q$.

To show that $\phi(x)$ is built from $w$, we need to define a collection of expected occurrences of $w$. We say an occurrence of $w$ in $\phi(x)$ beginning at position $j$ is expected if and only if $x$ has an expected occurrence of $v_3$ beginning at position $j - q$. Recall that an expected occurrence of $w_2$ in $\phi(x)$ is contained in an expected occurrence of $w$ beginning at $j$ if it is associated to the expected occurrence of $v_3$ beginning at $j - q$ in $x$; otherwise, it does not intersect the expected occurrence of $w$ beginning at $j - q$ in $x$. Thus every occurrence of $0$ is contained in exactly one expected occurrence of $w$, and $\phi(x)$ is built from $w$.

Condition (2) above follows immediately from the definition of expected occurrence of $w$. This completes the proof of the Theorem 3.5 in the minimal case.

### 3.3 Proof of Theorem 3.5 in the non-minimal case

The proof of Theorem 3.5 is more intricate in the non-minimal case than in the minimal case, although significant parts of the argument are essentially identical.

**Lemma 3.12.** Let $(X, \sigma)$ be a non-minimal rank-1 system, and let $k \in \mathbb{Z}$.

(a) There exists a unique $z \in X$ such that $z$ has a first occurrence of $0$ at position $k$.

(b) There exists a unique $z \in X$ such that $z$ has a last occurrence of $0$ at position $k$.

**Proof.** We prove only (b); the proof of (a) is similar.

Choose $x \in X$ which has an occurrence of $0$. For each $n \in \mathbb{N}$, the word $01^n$ is a subword of $V$ and hence, by Proposition 2.2, a subword of $x$; say $x$ has an occurrence of $01^n$ beginning at position $k_n$. For each $n \in \mathbb{N}$, $\sigma^{k_n-k}(x)$ has an occurrence of $01^n$ beginning at $k$. Passing to a subsequence if necessary, we may
assume that \( \{\sigma^{k_n-k}(x)\}_{n \in \mathbb{N}} \) converges to some \( z \in X \). It is clear that \( z \) has a last occurrence of 0 at position \( k \).

Suppose \( z \) and \( z' \) are distinct elements of \( X \) that each have a last occurrence of 0 beginning at \( k \). It is clear that \( z(i) = z'(i) \) for all \( i \geq k \). Suppose \( i < k \). Let \( v \in \mathcal{F} \) be such that \( V \) is built from \( v \) and such that \( |v| > k - i \). It follows from Proposition 2.29 that both \( z \) and \( z' \) must have an occurrence of \( v \) beginning at \( k - |v| + 1 \). The occurrence of \( v \) in each of \( z \) and \( z' \) must contain the position \( i \), implying that \( z(i) \neq z'(i) \). Thus, \( z = z' \). \( \square \)

We now begin the proof of Theorem 3.5 in the non-minimal case.

Let \((X, \sigma)\) and \((Y, \sigma)\) be non-minimal rank-1 systems, and fix an isomorphism \( \phi : X \to Y \).

### 3.3.1 Preliminary setup and some lemmas.
Using the techniques of Subsection 3.2.1, we can choose \( v_0, w_0 \in \mathcal{F} \) and \( t, t' \in \mathbb{Z} \) satisfying all of the following conditions.

1. \( v_0 \in V \) and \( w_0 \in W \);
2. for all \( x \in X \) and \( k \in \mathbb{Z} \),
   a. if \( x \) has an expected occurrence of \( v_0 \) beginning at position \( k \), then \( \phi(x) \) has an occurrence of 0 at position \( k + t \);
   b. if \( \phi(x) \) has an expected occurrence of \( w_0 \) beginning at position \( k \), then \( x \) has an occurrence of 0 at position \( k + t' \).

We use this preliminary setup (i.e., choices for \( v_0, w_0, t, t' \)) to prove some important lemmas. Then we produce a different setup (the only thing we retain from this initial setup is the choice of \( t \)) that we use for the main argument.

There are three elements of \( X \) to which we give specific names; \( z_0 \) denotes the bi-infinite constant word 1, \( z_1 \) denotes the element of \( X \) that has a first occurrence of 0 at position 0, and \( z_2 \) denotes the element of \( X \) that has a last occurrence of 0 at 0.

**Lemma 3.13.**

1. \( \phi(z_1) \) has a first occurrence of 0.
2. \( \phi(z_2) \) has a last occurrence of 0.

**Proof.** We prove only (b); the proof of (a) is similar.

Since \( z_2 \) has an occurrence of 0, it has an expected occurrence of \( v_0 \). This forces an occurrence of 0 in \( \phi(z_2) \). Suppose, towards a contradiction, that \( \phi(z_2) \) has no last occurrence of 0. Then there exist arbitrarily large \( k \) for which \( \phi(z_2) \) has an expected occurrence of \( w_0 \) beginning at position \( k \). For each such \( k \), we know that \( z_2 \) must have an occurrence of 0 at position \( k + t' \). This contradicts the fact that \( z_2 \) has a last occurrence of 0. \( \square \)
Lemma 3.14. Let $f \in \mathbb{Z}$ be such that $\phi(z_1)$ has a first occurrence of 0 at position $f$. Let $g \in \mathbb{Z}$ be such that $\phi(z_2)$ has a last occurrence of 0 at position $g$. There exists $A \in \mathbb{N}$ such that for all $a \geq A$, $x \in X$, and $k \in \mathbb{Z}$, $01^a0$ occurs in $x$ beginning at $k$ if and only if $01^{a+f-g}0$ occurs in $\phi(x)$ beginning at $k+g$.

Proof. We prove that for sufficiently large $a$, whenever $x \in X$ has an occurrence of $01^a0$ beginning at position $k$, $\phi(x)$ has an occurrence of $01^{a+f-g}0$ beginning at position $k+g$. Similar reasoning shows that for sufficiently large $a$, $x$ has an occurrence of $01^a0$ beginning at position $k$ whenever $\phi(x)$ has an occurrence of $01^{a+f-g}0$ beginning at position $k+g$ (thus completing the proof).

Note that $z_0 \in X \cap Y$ and that $\phi(z_0) = z_0$, since $z_0$ is the unique fixed point of both $(X, \sigma)$ and $(Y, \sigma)$.

Suppose, towards a contradiction, that there exist sequences $\{x_n\} \subseteq X$, $\{k_n\} \subseteq \mathbb{Z}$, $\{a_n\} \subseteq \mathbb{N}$ such that
(a) $a_n \to \infty$;
(b) $x_n$ has an occurrence of $01^{a_n}0$ beginning at $k_n$; and
(c) $\phi(x_n)$ does not have an occurrence of $01^{a_n+f-g}0$ beginning at $k_n+g$.

For each $n$, there is an occurrence of $01^{a_n}$ in $\sigma^{k_n}(x_n)$ beginning at position 0. Passing to a subsequence, if necessary, we may assume that $\{\sigma^{k_n}(x_n)\}$ converges. It must converge to $z_2$; for the element to which it converges must have a last occurrence of 0 at 0. Thus, $\phi(\sigma^{k_n}(x_n)) \to \phi(z_2)$.

Also, for each $n$, there is an occurrence of $1^a0$ in $\sigma^{k_n+a_n+1}(x_n)$ ending at position 0. Again passing to a subsequence, if necessary, we may assume that $\{\sigma^{k_n+a_n+1}(x_n)\}$ converges. It must converge to $z_1$; for the element to which it converges must have a first occurrence of 0 at 0. Thus, $\phi(\sigma^{k_n+a_n+1}(x_n)) \to \phi(z_1)$.

Choose $l \in \mathbb{N}$ so large that
(1) $l \geq |w_0|$;
(2) $l \geq -l' - g$ (equivalently, $0 \leq g + l + l'$); and
(3) $l \geq f + l'$ (equivalently $f - l + l' \leq 0$);
and $n \in \mathbb{N}$ so large that
(1) $\phi(\sigma^{k_n}(x_n))$ has an occurrence of $01^l$ beginning at position $g$; and
(2) $\phi(\sigma^{k_n+a_n+1}(x_n))$ has an occurrence of $1^l0$ ending at position $f$.

Since $\phi$ and $\sigma$ commute,
(1) $\phi(x_n)$ has an occurrence of $01^l$ beginning at position $k_n + g$; and
(2) $\phi(x_n)$ has an occurrence of $1^l0$ ending at position $k_n + a_n + 1 + f$.

We choose $x_n$, $k_n$, and $a_n$ such that $\phi(x_n)$ does not have an occurrence of $01^{a_n+f-g}0$ beginning at $k_n + g$ (and ending at $k_n + a_n + 1 + f$). This implies that there must be some occurrence of 0 in $\phi(x_n)$ at position $i$, with $k_n + g + l < i \leq
$k_n + a_n + f - l$. But this occurrence must belong to an expected occurrence of $w_0$; say this expected occurrence of $w_0$ begins at $j$. Since $l \geq |w_0|$ (in particular, since $1^l$ is not a subword of $w_0$), $k_n + g + l < j \leq k_n + a_n + f - l$. Figure 7 illustrates the situation.

Figure 7. The proof of Lemma 3.14.

The occurrence of $w_0$ in $\phi(x_n)$ beginning at position $j$ forces an occurrence of 0 in $x$ at position $j + t'$. Clearly,

$$k_n + g + l + t' < j + t' \leq k_n + a_n + f - l + t'.$$

Since $0 \leq g + l + t'$ and $f - l + t' \leq 0$,

$$k_n < j + t' \leq k_n + a_n.$$

This is a contradiction, since $x_n$ has an occurrence of 0 at position $j + t'$, and $x_n$ has an occurrence of $01^a_0$ beginning at position $k_n$. \hfill \square

### 3.3.2 Main setup.

Using basic techniques, we choose $v_1, w_2, v_3 \in \mathcal{F}$ and $r, s \in \mathbb{Z}$ satisfying all of the following (recall that $t$ has already been chosen).

1. $v_1, v_3 \in A_V$ and $w_2 \in A_W$.
2. $v_1 \not\preceq v_3$.
3. $|v_1| > t - g$ (equivalently, $t < |v_1| + g$) and $|v_1| > A + |f - g|$.
4. For all $x \in X$ and all $k \in \mathbb{Z}$,
   (a) if $x$ has an expected occurrence of $v_3$ beginning at position $k$, then $\phi(x)$ has an expected occurrence of $w_2$ beginning at position $k + r$;
   (b) if $\phi(x)$ has an expected occurrence of $w_2$ beginning at position $k$, then $x$ has an expected occurrence of $v_1$ beginning at position $k + s$; and
   (c) if $x$ has an expected occurrence of $v_1$ beginning at position $k$, then $\phi(x)$ has an occurrence of 0 at position $k + t$. 
Since $V$ is built from $v_0$, and $v_0$ is such that $\phi(x)$ has an occurrence of 0 beginning at position $k + t$ if $x$ has an occurrence of $v_0$ beginning at position $k$, we can choose $v_1 \in B_V$ such that $|v_1| > \max\{|v_0|, t - g, A + |f - g|\}$. Note that $v_0 < v_1$, since $|v_1| > |v_0|$ and $v_1 \in B_V$. Thus, if $x$ has an expected occurrence of $v_1$ beginning at $k$, then $x$ has an expected occurrence of $v_0$ beginning at $k$, which implies that $\phi(x)$ has an occurrence of 0 at $k + t$.

We next choose $w_2$ and $s$ and then $v_3$ and $r$, as was done in the minimal case.

### 3.3.3 The main argument in the non-minimal case.

We have a slightly different setup here than we did in the minimal case. Below, we state and prove one lemma and give a proof of Proposition 3.11 in the non-minimal case. Since the proof of Theorem 3.5 in the non-minimal case proceeds from that proposition in exactly the same way as in the minimal case, we do not repeat the argument here.

**Lemma 3.15.**

(1) $0 \leq r + s < |v_3|$;

(2) $0 \leq s + t < |w_2|$.

**Proof.** We prove only (a); the proof of (b) is similar.

Recall that the word $z_1 \in X$ has a first occurrence of 0 at position 0. Thus, it has an expected occurrence of $v_3$ beginning at position 0. This forces an expected occurrence of $v_1$ in $z_0$ beginning at position $r + s$. Since the first occurrence of 0 in $z_1$ is at position 0, it must be that $0 \leq r + s$.

Similarly, since $z_2$ has a last occurrence of 0 at position 0, it has an expected occurrence of $v_3$ beginning at position $1 - |v_3|$. This forces an expected occurrence of $v_1$ in $z_2$ beginning at position $1 - |v_3| + r + s$. This clearly cannot begin after position 0. Therefore, $1 - |v_3| + r + s \leq 0$. Thus, $r + s < |v_3|$.

Figure 8 illustrates the information we have so far for the non-minimal case.

We now prove Proposition 3.11 in the non-minimal case.

**Proof.** Suppose, towards a contradiction, that there exists $r'$ satisfying the hypothesis but not the conclusion of the proposition. Either $r > r'$ or $r < r'$. The arguments that lead to a contradiction in these cases are almost identical, so we give only one of them. Suppose $r < r'$.

Let $r_0$ be the largest natural number less than $r'$ that satisfies both the hypothesis and the conclusion of the proposition. Note that $r_0 \geq r$. Let $r_1$ be the smallest natural number greater than $r_0$ satisfying the hypothesis but not the conclusion of the proposition. Note that $r_0 \leq r'$.

Choose $x_0, k_0, x_1, k_1$ such that all of the following hold:

(1) $x_0$ has an expected occurrence of $v_3$ beginning at position $k_0$;
(2) $\phi(x_0)$ has expected occurrences of $w_2$ beginning at position $k_0 + r_0$ and $k_0 + r_1$;
(3) $x_1$ has an expected occurrence of $v_3$ beginning at position $k_1$;
(4) $\phi(x_1)$ has an expected occurrence of $w_2$ beginning at position $k_1 + r_0$ but not at position $k_1 + r_1$.

Let $a$ be such that the expected occurrence of $w_2$ in $\phi(x_0)$ beginning at $k_0 + r_0$ is followed by $1^a$ and then by another expected occurrence of $w_2$.

We claim that $a = r_1 - r_0 - |w_2|$, i.e., the first expected occurrence of $w_2$ in $\phi(x_0)$ after the expected occurrence of $w_2$ beginning at $k_0 + r_0$ is the one beginning at $k_0 + r_1$. If not, there exists an expected occurrence of $w_2$ between those beginning at $k_0 + r_0$ and $k_0 + r_1$; say it begins at $k + \hat{r}$, with $r_0 < \hat{r} < r_1$. If $\phi(x)$ has an expected occurrence of $w_2$ beginning at $k + \hat{r}$ whenever $x \in X$ has an expected occurrence of $v_3$ at $k$, we have contradiction with the maximality of $r_0$. If, for such $x$, $\phi(x)$ does not have an expected occurrence of $w_2$ beginning at $k + \hat{r}$, we have a contradiction with the minimality of $r_1$. Therefore, $a = r_1 - r_0 - |w_2|$.

We now claim that there is an occurrence of 0 in $\phi(x_1)$ somewhere after the expected occurrence of $w_2$ that begins at $k_1 + r_0$. Indeed, there is an expected occurrence of $v_1$ in $x_0$ beginning at position $k_0 + r_1 + s$. As $0 \leq r_1 + s < |v_3|$, this expected occurrence of $v_1$ intersects, and thus is contained in, the expected occurrence of $v_3$ beginning at position $k_0$ in $x_0$. Thus, there is also an expected occurrence of $v_1$ in $x_1$ at position $k_1 + r_1 + s$. This forces an occurrence of 0 in $\phi(x_1)$ at $k_1 + r_1 + s + t$. Recall that $r_1 - r_0 \geq |w_2|$. By Lemma 3.15, $0 \leq s + t$. These facts imply $k_1 + r_0 + |w_2| \leq k_1 + r_1 + s + t$, and thus the occurrence of 0 in $\phi(x_1)$ at position $k_1 + r_1 + s + t$ is somewhere after the expected occurrence of $w_2$ that begins at $k_1 + r_0$.

Let $b$ be such that the expected occurrence of $w_2$ in $\phi(x_1)$ beginning at $k_1$ is followed by $1^b$ and then by another expected occurrence of $w_2$. Note that $b \neq a$;
for otherwise, $x_1$ has an expected occurrence of $w_2$ beginning at $k_1 + r_1$. Also, $b < a$ gives a contradiction with the minimality of $r_1$. Thus $b > a$.

Figure 9 illustrates the information we have so far.

We now claim that $|b - a| \geq |v_1|$. There is an expected occurrence of $v_1$ in $x_0$ beginning at $k_0 + r_1 + s$. As $0 \leq r_1 + s < |v_3|$, this expected occurrence of $v_1$ intersects, and thus is contained in, the expected occurrence of $v_3$ beginning at $k_0$ in $x_0$. Thus, there is also an expected occurrence of $v_1$ in $x_1$ at position $k_1 + r_1 + s = k_1 + r_0 + |w_2| + a + s$. Since $\phi(x_1)$ has an expected occurrence of $w_2$ beginning at $k_1 + r_0 + |w_2| + b$, there is also an expected occurrence of $v_1$ in $x_1$ beginning at $k_1 + r_0 + |w_2| + b + s$. Since $a \neq b$, these two expected occurrences of $v_1$ are distinct, hence disjoint. This implies that $|b - a| \geq |v_1|$.

Since $b > a$ and $|b - a| \geq |v_1|$, we have $b \geq |v_1| > A + |f - g|$. Recall that $\phi(x_1)$ has an occurrence of $01^b0$ beginning at $k_1 + r_0 + |w_2| - 1$. Therefore, by Lemma 3.14, there is an occurrence of $01^{b-f+g}0$ in $x_1$ beginning at position $k_1 + r_0 + |w_2| - 1 - g$.

We claim that this occurrence of $01^{b-f+g}0$ is completely contained in the expected occurrence of $v_3$ beginning at $k_1$. It suffices to show that the position of the last occurrence of 0 contained in the expected occurrence of $v_3$ beginning at
position $k_1$ in $x_1$ is after position $k_1 + r_0 + |w_2| - 1 - g$. Because $x_0$ has an expected occurrence of $v_1$ beginning at $k_0 + r_0 + |w_2| + a + s$ that is completely contained in the expected occurrence of $v_3$ beginning at $k_0$, $x_1$ has an expected occurrence of $v_1$ beginning at $k_1 + r_0 + |w_2| + a + s$ that is completely contained in the expected occurrence of $v_3$ beginning at $k_1$. Since $v_1$ ends in 0, there is an occurrence of 0 at position $k_1 + r_0 + |w_2| + a + s + |v_1| - 1$ in $x$ that is completely contained in the expected occurrence of $v_3$ beginning at $k_1$. Thus, it suffices to show that

$$k_1 + r_0 + |w_2| - 1 - g < k_1 + r_0 + |w_2| + a + s + |v_1| - 1,$$

i.e.,

$$0 < a + s + |v_1| + g.$$

We know that $0 \leq a$. From Lemma 3.15, we also know that $0 \leq s + t$. We chose $v_1$ such that $t < |v_1| + g$. Thus,

$$0 \leq a + s + t < a + s + |v_1| + g,$$

as claimed.

We now know that the occurrence of $01^{b-f+g}0$ in $x_1$ beginning at position $k_1 + r_0 + |w_2| - 1 - d$ is completely contained in the expected occurrence of $v_3$ beginning at position $k_1$. This implies an occurrence of $01^{b-f+g}0$ in $x_0$ beginning at position $k_0 + r_0 + |w_2| - 1 - g$. By Lemma 3.14, there is an occurrence of $01^{b}0$ in $\phi(x_0)$ beginning at $k_0 + r_0 + |w_2| - 1$. But there is an occurrence of $01^a0$ in $\phi(x_0)$ beginning at $k_0 + r_0 + |w_2| - 1$. Thus, $a = b$, a contradiction.

This completes the proof of Theorem 3.5 in the non-minimal case.

### 3.4 The complexity of the isomorphism relation on $\mathcal{R}$.

We want to understand the complexity of the (topological) isomorphism relation on $\mathcal{R}$ as a subset of $\mathcal{R} \times \mathcal{R}$ and also as a Borel equivalence relation.

By Corollary 3.7, we know that $V$ and $W$ are isomorphic (i.e., the rank-1 systems associated to $V$ and $W$ are isomorphic) if and only if there exist $v, w \in \mathcal{F}$ such that

1. $V$ is built from $v$ and $W$ is built from $w$, and
2. $V$ has an expected occurrence of $v$ beginning at position $k$ if and only if $W$ has an expected occurrence of $w$ beginning at $k$.

We claim that given $v, w \in \mathcal{F}$, the conjunction of the two conditions above is closed. To see this, first note that the condition “$V$ is built from $v$” is equivalent to the condition “for all $n \in \mathbb{N}$, there exists a finite word ‘beginning with $V \upharpoonright n = \{ V(0), V(1), \ldots, V(n-1) \}$ that is built from $v$’”, which is a closed condition. Thus
condition (1) is closed. Next, note that if \( V \) is built from \( v \), then the truth of the statement “\( V \) has an expected occurrence of \( v \) beginning at \( k \)” depends only on the first \( k + |v| \) values of \( V \). This implies that the conjunction of the two conditions is closed.

In light of the fact that the set \( \mathcal{F} \) is countable, we have proved the following proposition.

**Proposition 3.16.** The isomorphism relation on \( \mathcal{R} \) is \( F_\sigma \) as a subset of \( \mathcal{R} \times \mathcal{R} \).

We now want to understand the complexity of the isomorphism relation on \( \mathcal{R} \) as a Borel equivalence relation. Without further elaboration, we use the most basic results about hyperfinite Borel equivalence relations. These results, together with a discussion of the notion of Borel reducibility, can be found in [4].

Recall that \( E_0 \) is the eventual agreement relation on \( \{0, 1\}^\mathbb{N} \), i.e., the equivalence relation on \( \{0, 1\}^\mathbb{N} \) defined by

\[
x E_0 y \iff \exists n \forall m > n x(m) = y(m).
\]

**Theorem 3.17.** The isomorphism relation on \( \mathcal{R} \) is Borel bi-reducible with \( E_0 \).

We prove Theorem 3.17 in two stages. In Proposition 3.21, we show that there exists a Borel reduction from \( E_0 \) to the isomorphism relation on \( \mathcal{R} \). In doing so, we describe an uncountable family of non-degenerate rank-1 systems containing Chacon’s transformation and apply the main theorem (in the form of Corollary 3.7) to determine when two systems in this family are topologically isomorphic, showing, in particular, that Chacon’s transformation is not topologically isomorphic to its inverse. It should be mentioned that this same family was described in both [7] and [14], where identical criteria are used to determine whether two systems in this family are measure-theoretically isomorphic and Chacon’s transformation is shown not to be measure-theoretically isomorphic to its inverse.

Then in Proposition 3.22, we show that the topological isomorphism relation on \( \mathcal{R} \) is hyperfinite. By a theorem of Dougherty, Jackson, and Kechris [4], this is equivalent to the existence of a Borel reduction from the topological isomorphism relation on \( \mathcal{R} \) to \( E_0 \).

**Definition 3.18.** Suppose \( v, \bar{v}, w, \bar{w} \in \mathcal{F} \) with \( \bar{v} = v^{a_1} v^{a_2} \ldots v^{a_{r-1}} v \) and \( \bar{w} = w^{b_1} w^{b_2} \ldots w^{b_{r-1}} w \). We say that \( \bar{w} \) is **built from** \( w \) in the same way that \( \bar{v} \) is **built from** \( v \) if \( a_i + |v| = b_i + |w| \) for each \( 0 < i < r \).

**Remarks.** (1) This definition is the natural finite analogue of the definition of a replacement scheme for \( V \) and \( W \) (with \( \bar{v} \) playing the role of a finite \( V \) and \( \bar{w} \) playing the role of a finite \( W \)).
(2) If \((v, w)\) is a replacement scheme for \(V\) and \(W\) and \(v \prec \tilde{v} \in A_V\), then the unique \(\tilde{w} \in \mathcal{F}\) such that \(\tilde{w}\) is built from \(w\) in the same way that \(\tilde{v}\) is built from \(v\) satisfies \(\tilde{w} \in A_W\). Moreover, \((\tilde{v}, \tilde{w})\) is a replacement scheme for \(V\) and \(W\).

**Lemma 3.19.** Suppose that \((v, w)\) is a replacement scheme for \(V\) and \(W\) and that \(v \prec \tilde{v} \in B_V\). Then there exists \(\tilde{w} \in B_W\) such that \((\tilde{v}, \tilde{w})\) is a replacement scheme for \(V\) and \(W\).

**Proof.** Choose \(\tilde{w} \in A_W\) such that \(\tilde{w}\) is built from \(w\) in the same way that \(\tilde{v}\) is built from \(v\). Then \(\tilde{w} \in A_W\), and \((\tilde{v}, \tilde{w})\) is a replacement scheme for \(V\) and \(W\). We need to show that \(\tilde{w} \in B_W\).

Suppose that \(\tilde{w} \notin B_W\). Then there exist \(u, u' \in A_W\) such that \(u \prec \tilde{w} \prec u'\) and \(u \preceq_s u'\). There are three possibilities:

1. \(w \preceq u\),
2. \(u \prec w\), or
3. \(w\) and \(u\) are incomparable.

**Case 1:** \(w \preceq u\). Let \(t (t')\) be such that \(t (t')\) is built from \(v\) in the same way that \(u (u')\) is built from \(w\). It is straightforward to check that \(t \prec \tilde{v} \prec t'\) and \(t \preceq_s t'\), which implies that \(\tilde{v} \notin B_V\).

**Case 2:** \(u \prec w\). By Lemma 2.8(b), \(w \preceq u'\). Also \(w \prec \tilde{w} \prec u'\). As in the previous case, let \(t'\) be such that \(t'\) is built from \(v\) in the same way that \(u'\) is built from \(w\). It is straightforward to check that \(v \prec \tilde{v} \prec t'\) and \(v \preceq_s t'\), which implies that \(\tilde{v} \notin B_V\).

**Case 3:** \(u\) and \(w\) are incomparable. By Proposition 2.11, \((u \land w) \prec u \prec (u \lor w)\) and \((u \land w) \preceq_s (u \lor w)\). Also, \(u \lor w \prec u'\) (since \(\tilde{w} \prec u'\) and \(\tilde{w}\) is built from each of \(u\) and \(w\)), and \(u \preceq_s u'\). Thus, by Lemma 2.8(a), \(u \land w \preceq_s u'\). Then, by Lemma 2.8(b), \(w \preceq u'\). As in the previous two cases, let \(t'\) be such that \(t'\) is built from \(v\) in the same way that \(u'\) is built from \(w\). It is straightforward to check that \(v \prec \tilde{v} \prec t'\) and \(v \preceq_s t'\), which implies that \(\tilde{v} \notin B_V\).

Thus \(\tilde{v} \in B_V\) implies \(\tilde{w} \in B_W\), as desired. □

The following proposition, which relies heavily on Lemma 3.19, shows that if two non-degenerate rank-1 systems are isomorphic, their canonical generating sequences eventually match up in a very specific way. Recall that if \(V\) is a non-degenerate rank-1 word, then the canonical generating sequence \(\{v_n\}_{n \in \mathbb{N}}\) of \(V\) is simply the enumeration of \(B_V\) satisfying \(n \leq m\) if and only if \(v_n \preceq v_m\).

**Proposition 3.20.** Let \(\{v_n\}_{n \in \mathbb{N}}\) and \(\{w_m\}_{m \in \mathbb{N}}\) be canonical generating sequences that give rise to non-degenerate rank-1 words \(V\) and \(W\), respectively.
Suppose the associated rank-1 systems are isomorphic. Then there exist \( N, M \in \mathbb{N} \) such that for all \( k \geq 0 \),

(a) \((v_{N+k}, w_{M+k})\) is a replacement scheme for \( V \) and \( W \), and

(b) \( v_{N+k+1} \) is built from \( v_{N+k} \) in the same way that \( w_{M+k+1} \) is built from \( w_{M+k} \).

**Proof.** We first find \( N, M \in \mathbb{N} \) such that \((v_N, w_M)\) is a replacement scheme for \( V \) and \( W \). Then we complete the proof by showing that if \((v_{N+k}, w_{M+k})\) is a replacement scheme for \( V \) and \( W \), then \( v_{N+k+1} \) is built from \( v_{N+k} \) in the same way that \( w_{M+k+1} \) is built from \( w_{M+k} \), and \((v_{N+k+1}, w_{M+k+1})\) is a replacement scheme for \( V \) and \( W \).

By Corollary 3.7, there must exist a replacement scheme \((v, w)\) for \( V \) and \( W \). By Propositions 2.14 and 2.16, there exists \( \tilde{v} \in \mathcal{B}_V \) such that \( v \prec \tilde{v} \). By Lemma 3.19, there exists \( \tilde{w} \in \mathcal{B}_W \) such that \((\tilde{v}, \tilde{w})\) is a replacement scheme for \( V \) and \( W \). Let \( N \) be such that \( \tilde{v} = v_N \) and \( M \) be such that \( \tilde{w} = w_M \). Then \((v_N, w_M)\) is a replacement scheme for \( V \) and \( W \).

Suppose now that \((v_N + k, w_M + k)\) is a replacement scheme for \( V \) and \( W \). Let \( w' \) be the word built from \( w_{M+k} \) in the same way that \( v_{N+k+1} \) is built from \( v_{N+k} \). Note that \((v_{N+k+1}, w')\) is a replacement scheme for \( V \) and \( W \). By the proof of Lemma 3.19, \( w' \in \mathcal{B}_W \), and thus \( w' = w_{M+k+1} \) for some \( l > 0 \). It remains to show that \( l = 1 \).

If \( l > 1 \), then \( w_{M+k} \prec w_{M+k+1} \prec w_{M+k+l} \). Let \( v' \) be the word built from \( v_{N+k} \) in the same way that \( W_{M+k+1} \) is built from \( W_{M+k} \). By the proof of Lemma 3.19, \( v' \in \mathcal{B}_V \). However, this cannot be, since \( v_{N+k} \prec v' \prec v_{N+k+1} \). Thus \( l = 1 \). \( \square \)

### 3.4.1 The reduction from \( E_0 \) to \( \mathcal{R} \)

In the proof of Proposition 3.21 below, we describe an uncountable family of non-degenerate rank-1 systems, indexed by infinite strings of 0s and 1s. The classical Chacon transformation corresponds to the infinite string consisting exclusively of 1s, and its inverse corresponds to the infinite string consisting exclusively of 0s. The heart of the proposition is that two systems in this family are (topologically) isomorphic if and only if their corresponding infinite strings differ in only finitely many positions. In particular, this implies that Chacon’s transformation is not (topologically) isomorphic to its inverse.

**Proposition 3.21.** There exists a Borel reduction from \( E_0 \) to the topological isomorphism relation on \( \mathcal{R} \).

**Proof.** We construct a Borel function from \( \{0, 1\}^\mathbb{N} \) to \( \mathcal{R} \) such that \( \alpha \in \{0, 1\}^\mathbb{N} \) and \( \beta \in \{0, 1\}^\mathbb{N} \) are \( E_0 \)-related (i.e., agree on all but finitely many coordinates) if and only if their images are topologically isomorphic.
For \( \alpha \in \{0, 1\}^\mathbb{N} \), we produce \( V \) as follows. First let \( v_0 = 0 \) and then define inductively
\[
v_{m+1} = \begin{cases} v_m 1v_m v_m, & \text{if } \alpha(m) = 0; \\ v_m v_m 1v_m, & \text{if } \alpha(m) = 1. \end{cases}
\]
Note that each \( v_m \) is a proper initial segment of \( v_{m+1} \). Let \( V \) be the limit of the \( v_m \).

It is clear that \( V \) is built from each \( v_m \) and thus is rank-1. We show below that, in fact, \( A_V = \{ v_m : m \in \mathbb{N} \} = B_V \). We use the fact that there exists no \( w \in V \) such that \( v_m \prec w \prec v_{m+1} \) (this follows from the way that \( v_{m+1} \) is built from \( v_m \)).

We first claim that \( A_V = \{ v_m : m \in \mathbb{N} \} \). We have already shown above that \( A_V \subseteq \{ v_m : m \in \mathbb{N} \} \). Suppose that \( V \) is built from \( v \) but \( v \notin \{ v_m : m \in \mathbb{N} \} \). Let \( m \) be the largest integer such that \( v_m \prec v \) (note that such an \( m \) exists, since \( 0 = v_0 \prec v \)). By assumption, \( v \neq v_{m+1} \) and \( v_{m+1} \neq v \). Also, \( v \neq v_{m+1} \); for otherwise, we would have \( v_m \prec v \prec v_{m+1} \). Thus, \( v \) and \( v_{m+1} \) are incomparable. By Proposition 2.11, \( (v \land v_{m+1}) \prec v_{m+1} \prec (v \lor v_{m+1}) \) and \( (v \land v_{m+1}) \preceq_s (v \lor v_{m+1}) \). Thus, by Lemma 2.8, \( (v \land v_{m+1}) \preceq_s v_{m+1} \). Since \( v_m \preceq v \lor v_{m+1} \prec v_{m+1} \), it must be that \( v_m = v \land v_{m+1} \). Thus \( v_m \preceq_s v_{m+1} \), which contradicts the way that \( v_{m+1} \) is built from \( v_m \).

We now have shown that \( A_V = \{ v_m : m \in \mathbb{N} \} \). That \( B_V = \{ v_m : m \in \mathbb{N} \} \) follows from the fact that no \( v_{m+1} \) is built simply from \( v_m \). Since \( B_V \) is infinite, \( V \) is non-degenerate, and thus, in \( \mathcal{R} \); see Proposition 2.16.

It is clear that this map is continuous and, therefore, Borel.

Let \( \alpha \in \{0, 1\}^\mathbb{N} \) produces the sequence of words \( \{ v_m : m \in \mathbb{N} \} \) and the infinite word \( V \). Let \( \beta \in \{0, 1\}^\mathbb{N} \) produce the sequence of words \( \{ w_n : n \in \mathbb{N} \} \) and the infinite word \( W \). We need to show that \( \alpha \) and \( \beta \) are \( E_0 \)-related if and only if \( V \) and \( W \) are isomorphic (by Corollary 3.7, this happens if and only if there is a replacement scheme for \( V \) and \( W \)).

If \( \alpha \) and \( \beta \) are \( E_0 \)-related, there exists \( N \in \mathbb{N} \) such that \( \alpha(n) = \beta(n) \), for all \( n \geq N \). It is easy to check that this implies that \( (v_N, w_N) \) is a replacement scheme for \( V \) and \( W \).

Now suppose there is a replacement scheme for \( V \) and \( W \). To show that \( \alpha(n) = \beta(n) \), for sufficiently large \( n \geq N \), it suffices to show that for sufficiently large \( n \), \( (v_n, w_n) \) is a replacement scheme for \( V \) and \( W \). Indeed, if both \( (v_n, w_n) \) and \( (v_{n+1}, w_{n+1}) \) are replacement schemes for \( V \) and \( W \), then \( w_{n+1} \) must be built from \( w_n \) in the same way that \( v_{n+1} \) is built from \( v_n \). This clearly implies that \( \alpha(n) = \beta(n) \).

We first claim that there exists \( n \in \mathbb{N} \) such that \( (v_n, w_n) \) is a replacement scheme for \( V \) and \( W \). By assumption, there exists a replacement scheme for \( V \) and \( W \). Also, as shown above, \( A_V = \{ v_m : m \in \mathbb{N} \} \), and \( A_W = \{ w_n : n \in \mathbb{N} \} \). Let \( m, n \in \mathbb{N} \)}
be such that \((v_m, w_n)\) is a replacement scheme for \(V\) and \(W\). We need to show that \(m = n\). Note that \(|v_n| = |w_n|\) and that \(|v_m| = |w_m|\). If \(m < n\), \(V\) has an expected occurrence of \(v_m\) beginning at position \(|v_m|\) or \(|v_m| + 1\) (depending on whether \(\alpha(m) = 0\) or \(\alpha(m) = 1\)), and \(W\) does not have an expected occurrence of \(w_n\) at either \(|v_m|\) or \(|v_m| + 1\) (the second expected occurrence of \(w_n\) in \(W\) begins either at \(|w_n|\) or \(|w_n| + 1\)). Since \((v_m, w_n)\) is a replacement scheme for \(V\) and \(W\), this is impossible, i.e., \(m \geq n\). A similar argument shows \(m \leq n\).

We now claim that if \((v_n, w_n)\) is a replacement scheme for \(V\) and \(W\), then \((v_{n+1}, w_{n+1})\) is a replacement scheme for \(V\) and \(W\). For a replacement scheme \((v_n, w_n)\) for \(V\) and \(W\), \(v_n \prec v_{n+1} \in B_V\) and thus, by Proposition 3.19, \((v_{n+1}, w_{n'})\) is also a replacement scheme for \(V\) and \(W\) for some \(n'\). The same argument used in the preceding paragraph shows that \(n' = n + 1\). Thus, \((v_{n+1}, w_{n+1})\) is a replacement scheme for \(V\) and \(W\). \(\square\)

3.4.2 Hyperfiniteness.

**Proposition 3.22.** The topological isomorphism relation on \(\mathcal{R}\) is hyperfinite.

**Proof.** We show that the isomorphism relation on \(\mathcal{R}\) is a countable union of finite equivalence relations. In order to define the equivalence relations, we need a norm on replacement schemes. Suppose \((v, w)\) is a replacement scheme for \(V\) and \(W\) and that \(V = v_1^{a_1}v_2^{a_2}\ldots\) and \(W = w_1^{b_1}w_2^{b_2}\ldots\). Note that since \((v, w)\) is a replacement scheme, \(a_i + |v| = b_i + |w|\), for each \(i \geq 1\). We now define

\[
||v, w|| = |v| + \min\{a_i : i \geq 1\} = |w| + \min\{b_i : i \geq 1\}.
\]

Note that if \((u, v)\) is a replacement scheme for \(U\) and \(V\) and \((v, w)\) is a replacement scheme for \(V\) and \(W\), then \((u, w)\) is a replacement scheme for \(U\) and \(W\) and

\[
||u, v|| = ||v, w|| = ||u, w||.
\]

For \(k \geq 1\) and \(V, W \in \mathcal{R}\), we say \(V \sim_k W\) if and only if there exists a replacement scheme \((v, w)\) for \(V\) and \(W\) such that \(v \in B_V\), \(w \in B_W\), and \(||v, w|| \leq k\).

We first show that each \(\sim_k\) is an equivalence relation. It is clear that \(\sim_k\) is reflexive and symmetric. To show that \(\sim_k\) is transitive, let \(U, V, W \in \mathcal{R}\) with \(U \sim_k V\) and \(V \sim_k W\). Let \((u, \tilde{v})\) be a replacement scheme witnessing that \(U \sim_k V\) and \((v, \tilde{w})\) a replacement scheme witnessing that \(V \sim_k W\). Without loss of generality, assume that \(|v| \leq |\tilde{v}|\). Since \(v, \tilde{v} \in B_V\), \(v \preceq \tilde{v}\); see Proposition 2.20. Now, by Lemma 3.19, there exists \(\tilde{w} \in B_W\) such that \((\tilde{v}, \tilde{w})\) is a replacement scheme. Now \((u, \tilde{w})\) is a replacement scheme for \(U\) and \(W\), and \(||u, \tilde{w}|| = ||(u, \tilde{v})||\). Thus \(U \sim_k W\).
It is easy to see that each $\sim_k$ equivalence class is finite (i.e., $\sim_k$ is a finite equivalence relation). Indeed, let $V \in \mathcal{R}$ and observe that if $(v, w)$ witnesses that $V \sim_k W$, then $v, w \in \mathcal{F}$ with $|v|, |w| \leq k$. As there exist only finitely many pairs $(v, w)$ such that $v, w \in \mathcal{F}$ and $|v|, |w| \leq k$, there exist only finitely many $W \in \mathcal{R}$ such that $V \sim_k W$.

Finally, we show that the isomorphism relation on $\mathcal{R}$ is the union of the equivalence relations $\sim_k$. It is clear that if $V \sim_k W$, then $V$ and $W$ are isomorphic. On the other hand, if $V, W \in \mathcal{R}$ are isomorphic, then, by Corollary 3.7, there exists some replacement scheme $(v, w)$ for $V$ and $W$. Choose $\tilde{v} \in B_V$ such that $v \preceq \tilde{v}$. By Lemma 3.19, there exists $w \in B_W$ such that $(\tilde{v}, \tilde{w})$ is a replacement scheme for $V$ and $W$. Then $V \sim_{||\tilde{v}, \tilde{w}||} W$.

We have shown that each $\sim_k$ is a finite equivalence relations and that their union is the isomorphism relation on $\mathcal{R}$, i.e., the isomorphism relation on $\mathcal{R}$ is hyperfinite. □

### 3.5 The inverse problem for non-degenerate rank-1 systems

We want to know when a non-degenerate rank-1 system $(X, \sigma)$ is (topologically) isomorphic to its inverse $(X, \sigma^{-1})$.

**Definition 3.23.** For various objects $o$, we define the reverse of $o$, denoted by $\overline{o}$, as follows.

1. For a finite word $\alpha = (a_0, a_1, \ldots, a_n)$, $\overline{\alpha} = (a_n, \ldots, a_1, a_0)$.
2. For an bi-infinite word $x \in \{0, 1\}^\mathbb{Z}$, $\overline{x}$ is the unique bi-infinite word such that $\overline{x}(k) = x(-k)$ for all $k \in \mathbb{Z}$.
3. For a set of bi-infinite words $X$, $\overline{X} = \{\overline{x} : x \in X\}$.
4. For $V \in \mathcal{R}$ with canonical generating sequence $\{v_n\}_{n \in \mathbb{N}}$, $\overline{V}$ is the unique infinite word such that $\overline{v_n}$ is an initial segment of $\overline{V}$ for each $n \in \mathbb{N}$.

**Remarks.**

1. If $V \in \mathcal{R}$ has canonical generating sequence $\{v_n\}_{n \in \mathbb{N}}$, then $\overline{V}$ lies in $\mathcal{R}$ and has canonical generating sequence $\{\overline{v_n}\}_{n \in \mathbb{N}}$.
2. If $X$ is the non-degenerate rank-1 system associated $V$, then $\overline{X}$ is the non-degenerate rank-1 system associated to $\overline{V}$. Moreover, $(\overline{X}, \sigma)$ is topologically isomorphic to $(X, \sigma^{-1})$.

**Definition 3.24.** For $w = v_1^{a_1} v_2^{a_2} \ldots v_r^{a_r} v$, we say that $w$ is built symmetrically from $v$ if $a_i = a_{r-i}$ for all $0 < i < r$.

It is straightforward to check that for $v, w \in \mathcal{F}$ with $v \preceq w$, the following are equivalent.

1. $w$ is built symmetrically from $v$. 

Proposition 3.25. Let $V \in \mathbb{R}$ with canonical generating sequence $\{v_n\}_{n \in \mathbb{N}}$. The following are equivalent.

1. $(X, \sigma)$ is topologically isomorphic to $(X, \sigma^{-1})$.
2. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $v_{n+1}$ is built symmetrically from $v_n$.

Proof. Suppose $N \in \mathbb{N}$ is such for all $n \geq N$, $v_{n+1}$ is built symmetrically from $v_n$. It is straightforward to check that $(v_N, \overline{v_N})$ is a replacement scheme for $V$ and $\overline{V}$. Thus, $(X, \sigma)$ and $(\overline{X}, \sigma)$ are topologically isomorphic, which implies that $(X, \sigma)$ and $(X, \sigma^{-1})$ are topologically isomorphic.

Now suppose $(X, \sigma)$ is isomorphic to $(X, \sigma^{-1})$. Then $(X, \sigma)$ and $(\overline{X}, \sigma)$ are topologically isomorphic. By Corollary 3.7, there exists a replacement scheme $(v, \overline{v})$ for $V$ and $\overline{V}$. Choose $M \in \mathbb{N}$ so that $|v| < |v_M|$. Then for all $m \geq M$, $v \prec v_m$ (because $|v| < |v_m|$ and $v_m$ is comparable with every element of $A_V$). By Lemma 3.19, there exists $n$ such that $(v_m, \overline{v_n})$ is a replacement scheme for $V$ and $\overline{V}$. It is straightforward to check that this implies $m = n$. Let $v_{n+1} = v_n1^{a_1}v_n1^{a_2}\ldots 1^{a_r-2}v_n1^{a_r-1}v_n$. By the definition of the reverse of a finite word,

$$\overline{v_{n+1}} = \overline{v_n}1^{a_r-1}\overline{v_n}1^{a_r-2}\ldots 1^{a_2}\overline{v_n}1^{a_1}\overline{v_n}.$$ 

But $(v_n, \overline{v_n})$ is also a replacement scheme for $V$ and $\overline{V}$. Thus,

$$\overline{v_{n+1}} = \overline{v_n}1^{a_1}\overline{v_n}1^{a_2}\ldots 1^{a_r-2}\overline{v_n}1^{a_r-1}\overline{v_n}.$$ 

Therefore, $\overline{v_{n+1}}$ is built symmetrically from $\overline{v_n}$, which implies that $v_{n+1}$ is built symmetrically from $v_n$. \qed

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