Aperiodic Sampled-Data Distributed Observer Design

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Abstract

This paper deals with state estimation of linear systems using distributed observers with local sampled-data measurement and aperiodic communication. Each observer node perceives partial information of the system to be observed, but does not satisfy the observability condition. Consequently, distributed observers are designed to estimate the state of the to-be-observed system by time-varying sampling and asynchronous communication. Additionally, explicit upper bounds on allowable sampling periods for convergent estimation errors are given. Finally, a numerical example is provided to illustrate the validity of the theoretical results.

Keywords: Sampled-data control, Distributed observers, Jointly observable systems, Linear time-invariant systems.

1. Introduction

For a given linear time-invariant (LTI) system, the distributed state estimation problem intends to asymptotically estimate the system’s state by a group of node observers with partial measurements and information sharing over a network (Mitra & Sundaram, 2018; Wu, Isidori, & Lu, 2022; Welikala & Cassandras, 2022). Typically, an LTI system to be observed takes the following form:

\[ \dot{x}(t) = Ax(t), \]

where \( x(t) \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{n \times n} \) are the state and system matrix, respectively. It is noted that a node observer can measure partial state information of the system in (1), namely,

\[ y_i(t) = C_i x(t), \quad i \in V, \]

where \( V \) is the node set; \( y_i(t) \in \mathbb{R}^{p_i} \) and \( C_i \in \mathbb{R}^{p_i \times n} \) are the measurement output and output matrix induced by the \( i \)-th node observer, respectively.

In Olfati-Saber (2007), a distributed algorithm using a group of sensors over an undirected graph was designed to estimate the state of an LTI system subject to noises under the assumption that each pair \((A, C_i)\) is observable. A distributed observer over a general directed graph was proposed in Su and Huang (2011) to solve the cooperative output regulation problem, and the more general case of switching topologies was considered in Su and Huang (2012). In these remarkable results, not all the node observers can access the signal \( x(t) \), implying that \( C_i \in \{0_{n \times R^n}, I_n\} \). However, the node observers can still reconstruct the full state \( x(t) \). To advance these studies, Park and Martins (2016) further proposed a distributed estimation scheme in which each node observer estimates the system’s state using partial output signals. Another type of distributed observers was constructed in Mitra and Sundaram (2018) under the locally detectable assumption in the sense that the pair \( (A, C_N) \) is observable, where \( C_N \) contains the output matrix of the \( i \)-th node observer and all its neighbors. Wang and Morse (2017) extended the results of Park and Martins (2016) and Mitra and Sundaram (2018) to strongly connected graphs, crafting a reduced-order continuous-time distributed observer, and removed some of the limitations in Park and Martins (2016). Afterward, distributed observers for the discrete-time case was considered in Wang, Liu, Morse, and Anderson (2019).

It is worth to mention that Wang and Morse (2017) and Wang, Liu, Morse, and Anderson (2019) designed the distributed observer under the jointly observable assumption. Admittedly, the jointly observable assumption is the mildest possible restriction as it allows the pair \( (A, C_i) \) to be unobservable for each node. In the meantime, the node observers can still reconstruct the system’s state through local interactions. Apparently, Wang and Morse (2017) and Wang, Liu, Morse, and Anderson (2019) still need all \( C_i \neq 0 \), which is a weakened jointly observable assumption. To relax this assumption, Han, Trentelman, Wang, and Shen (2018a) used the Kalman observable canonical decomposition to design a full state distributed observer under the jointly observable assumption without \( C_i \neq 0 \). Subsequently, Kim, Lee, and Shim (2019) improved the design of the distributed observer in Han, Trentelman, Wang, and Shen (2018a) and Han, Trentelman, Wang, and Shen (2018b) by mixing the linear matrix inequality (LMI)-based approach with the reduced-order observer form.
Wang and Huang (2018) and Baldi, Azzollini, and Ioannou (2020) developed learning-based approaches in the designing distributed observers to adaptively estimate the state and parameters of a linear leader system. More recently, the study has stepped into designing distributed observers for the leader in nonlinear dynamics (Wu, Isidori, & Lu, 2022). It should be noted that all these references are concerned with either continuous-time or discrete-time systems, while more practical sampled-data estimation has not been fully investigated despite some pioneering and remarkable results on non-uniform sampling distributed estimation as in Li, Phillips, and Sanfelice (2017) and Sferlazza, Tarbouriech, and Zaccarian (2021). For example, the Round-Robin aperiodic sampled measurements scheme studied in Sferlazza, Tarbouriech, and Zaccarian (2021) largely exploits the sequential nature of the measurement in distributed estimation problem and complements the recently linear-matrix-inequalities-like results constructed in Li, Phillips, and Sanfelice (2017).

Moreover, existing results in Su and Huang (2012); Kim, Lee, and Shim (2019) and Wu, Isidori, and Lu (2022) incorporated with the distributed estimation have demonstrated that communication networks also play a crucial role in designing and analyzing stability. For example, an analytical relationship between the system matrix $A$ in (1) and the minimum real part of the Laplacian matrix’s eigenvalues was provided for the continuous-time distributed estimation problem in Su and Huang (2012) under the assumption that at least one observer can observe the state of the leader. A sufficient condition was given in Kim, Lee, and Shim (2019) and Wu, Isidori, and Lu (2022) for the linear and nonlinear system cases under jointly observable assumption, respectively. What’s more, some researchers have noticed the impact of the sampling period on consensus behaviour of second-order systems (Yu, Zheng, Chen, Ren, & Cao, 2011; Huang, Duan, & Chen, 2016), revealing that the consensus can not be achieved for any sampling period if there exists one eigenvalue of the Laplacian matrix with a nonzero imaginary part. The interactions among sampling periods, topology, reference signals, and related observability have been fully revealed, and all non-pathological / pathological sampling periods are given in Wang, Su, and Chen (2021). Apparently, it is imperative to investigate the state estimation problem of linear systems using distributed observers with local sampled-data measurement and aperiodic communication.

In this paper, motivated by the studies mentioned above, the distributed estimation problem of linear systems is investigated using local sampling information and computation. The main contributions are summarized as follows:

1. The design of distributed observers with aperiodic sampled-data information is proposed to estimate the state of the to-be-observed system over jointly observable assumption.

2. A maximum allowable sampling period is given to guarantee the convergence of the estimation error. As long as the sampling periods of all node observers are smaller than this maximum allowable bound, the estimation error will tend to zero exponentially. In addition, we give an algorithm to calculate the explicit upper bound of the sampling periods using a hybrid system technique.

3. Compared with the existing results in Su and Huang (2011) and Ding, Han, and Guo (2013), the proposed study relaxes observable assumption into the jointly observable assumption. Each node observer can asymptotically synthesize with the state of the to-be-observed system only using its (limited) measurements and its neighbors’ estimation.

The rest of this paper is organized as follows: In Section 2, the problem is formulated, and some standard assumptions and lemmas are introduced. Section 3 is devoted to the design of distributed observers followed by a simulation example in Section 4. Finally, we conclude this paper in Section 5.

**Notation**: $\| \cdot \|$ denotes both the Euclidian norm of a vector and Euclidean induced matrix norm (spectral norm) of a matrix. $\mathbb{R}$ is the set of real numbers. $\mathbb{N}$ denotes all natural numbers. $\mathbb{Z}$ ($\mathbb{Z}_+$) is the set of all (positive) integers. $I_n$ denotes the $n \times n$ identity matrix. For $A \in \mathbb{R}^{m \times n}$, $\text{Ker}(A) = \{ x \in \mathbb{R}^n | Ax = 0 \}$ and $\text{Im}(A) = \{ y \in \mathbb{R}^m | y = Ax \}$ for some $x \in \mathbb{R}^n$ denote the kernel and range of $A$, respectively. For a subspace $\mathcal{V} \subset \mathbb{R}^n$, the orthogonal complement of $\mathcal{V}$ is denoted as $\mathcal{V}^\perp = \{ x \in \mathbb{R}^n | x^T v = 0, \forall v \in \mathcal{V} \}$. $\otimes$ denotes the Kronecker product of matrices. For $b_i \in \mathbb{R}^{n_i \times p}$, $i = 1, \ldots, m$, $\text{col}(b_1, \ldots, b_m) \triangleq [u_1^T \ldots u_m^T]^T$. For $a_i \in \mathbb{R}^{p \times n_i}$, $i = 1, \ldots, m$, $\text{row}(a_1, \ldots, a_m) \triangleq [a_1 \ldots a_m]$. For $X_1 \in \mathbb{R}^{n_1 \times m_1}, \ldots, X_k \in \mathbb{R}^{n_k \times m_k}$, $\text{diag}(X_1, \ldots, X_k) \triangleq \begin{bmatrix} X_1 & \cdots & \cdots & \cdots & X_k \end{bmatrix}$.

### 2. Problem Formulation and Assumptions

In this section, we will formulate **Jointly Observable Tracking Problem** for linear multi-agent systems, and introduce a design framework for solving this problem.

#### 2.1. Node Observers

We consider jointly observable network with the system in (1) as the to-be-observed LTI system and the following systems as the node observers:

$$\dot{\xi}_i(t) = A \xi_i(t) + Bu_i(t), \quad i \in \mathcal{V}, \quad (3)$$

where $\xi_i(t) \in \mathbb{R}^n$ is the state of the $i$-th node observer and $u_i(t) \in \mathbb{R}^m$ is the control input.
In this paper, we consider the design of control input $u(t)$ through aperiodic sampled information as follows:

$$ u(t) = k_i(\xi_i(t), \hat{x}_i(t)), \quad (4a) $$

$$ \dot{\hat{x}}_i(t) = g_i(\hat{x}_i(t), y_i(t), \sum_{j \in N_i} \hat{x}_j(t)), \quad t \in [t_k, t_{k+1}), \quad (4b) $$

where $t_0 = 0$ is the initial time and $k \in \mathbb{N}$, $\hat{x}_i(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ and the local analog-to-digital converter samples $\hat{x}_i(t)$ to yield the discrete-time signal $\hat{x}_i(t_k)$, the discrete-time signal $y_i(t_k) \in \mathbb{R}^m$ is the measurement sensed and sampled by the $i$-th agent at time instant $t_k$; $k_i(\cdot)$ and $g_i(\cdot)$ are functions to be designed later, for $i \in \mathcal{V}$. For every $k \in \mathbb{N}$, the difference between two adjacent sampling moments is

$$ t_{k+1} - t_k \triangleq h_k $$

where $h_k \in (0, h_{\text{max}})$ with $h_{\text{max}}$ being some positive number to be determined. In addition, the sampling instants are monotone increasing sequences satisfying $\lim_{k \to \infty} t_k = \infty$. It should be noted that $h_{\text{max}}$ is called the maximum allowable sampling period in Laila, Nesić, and Teel (2002); Karafyllis and Jiang (2007); Nesić, Teel, and Carnevale (2009); Sferlazza, Tarbouriech, and Zaccarian (2018); Oishi and Fujikawa (2010) and Sferlazza, Tarbouriech, and Zaccarian (2021), which is a very challenging and imperative quantity to compute.

In order to formulate our problem, we now introduce some basic knowledge of graphs. As in Mitra and Sundaram (2018) and Wang and Morse (2017), the system composed of (1) and (3) can be viewed as a multi-agent system with the to-be-observed system and $N$ node observers. The network topology among the node observer system is described by a graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} \triangleq \{1, \ldots, N\}$ and $\mathcal{E} \subseteq [\mathcal{V}]^2$, which is the 2-element subsets of $\mathcal{V}$. Here, the $i$-th node is associated with the $i$-th node observer for $i = 1, \ldots, N$. Let $N_i \triangleq \{j \in \mathcal{E} \mid (i, j) \in \mathcal{E}\}$ denote the neighborhood set of agent $i$. The weighted adjacency matrix of a digraph $\mathcal{G}$ is a nonnegative matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ii} = 0$ and $a_{ij} > 0$ $\iff$ $(j, i) \in \mathcal{E}$. Let $\mathcal{L}$ be the Laplacian matrix on graph $\mathcal{G}$, where $l_{ij}$ is the $(i,j)$-th entry of the Laplacian matrix $\mathcal{L}$ with $l_{ii} \equiv \sum_{j=1}^N a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. More details on graph theory can be found in Godsil and Royle (2013).

2.2. Problem Formulation

Now we can formulate the Jointly Observable Tracking Problem as follows:

**Problem 1 (Jointly Observable Tracking Problem).** Consider the system in (1) and (3). Find a distributed control law in (4) such that for any $\xi_i(0) \in \mathbb{R}^n$, $\hat{x}_i(0) \in \mathbb{R}^n$ and $x(0) \in \mathbb{R}^n$, the closed-loop system satisfies

$$ \lim_{t \to \infty} (\xi_i(t) - x(t)) = 0, \quad i \in \mathcal{V}. $$

It should be noted that, in Problem 1, each $i$-th node observer only can locally sense the signal $y_i(t_k) = C_i x_i(t_k)$ at discrete-time instant $t_k$ with $k \in \mathbb{N}$ and $i \in \mathcal{V}$. Apparently, $y_i(t)$ is the partial local measurement from the lumped output $y(t) \triangleq \text{col}(y_1, \ldots, y_N)$ of the to-be-observed system in (1), for $i \in \mathcal{V}$. Compared with the existing work in Su and Huang (2011) and Ding, Han, and Guo (2013), Problem 1 removes the assumption that the full state or observable state of the to-be-observed system is still available for some node observer.

It can be easy to conclude that there exists no agent that can access the signal $x(t)$. Therefore, a key technique in solving the Jointly Observable Tracking Problem is the Sampled-Data Distributed Observer defined as follows:

**Definition 1 (Sampled-Data Distributed Observer).** A dynamic of the form in (4b) is called a sampled-data distributed observer of the $i$-th node observer for the to-be-observed system in (1) if, given a communication topology $\mathcal{G}$, there exists globally defined function $g_i(\cdot)$ and a positive constant $h_{\text{max}}$, such that for any initial condition $\hat{x}_i(0) \in \mathbb{R}^n$ and $x(0) \in \mathbb{R}^n$, and any sequence $\{t_k, k \in \mathbb{N}\}$ satisfying $h_k \in (0, h_{\text{max}})$,

$$ \lim_{t \to \infty} (\hat{x}_i(t) - x(t)) = 0, \quad i \in \mathcal{V}. $$

2.3. Assumptions and Some Lemmas

To move on, we need these assumptions as follows:

**Assumption 1.** The pair $(A, B)$ is controllable.

**Assumption 2.** $\mathcal{G}$ is strongly connected directed graph.

**Assumption 3.** The system in (1) is jointly observable in the sense that $(A, C)$ is observable with $C = \text{col}(C_1, \ldots, C_N)$.

**Remark 1.** For $i \in \mathcal{V}$, we assume that the observability index of $(A, C_i)$ is $v_i$, such that rank($O_i$) = $v_i$, where $O_i \in \mathbb{R}^{p_i \times n}$ is the observability matrix and defined as following $O_i = \text{col}(C_i, C_i A, \ldots, C_i A^{v_i-1})$. For $i \in \mathcal{V}$, the observable subspace and unobservable subspace of $(A, C_i)$ are defined as $\text{Im}(O_i^T) \subset \mathbb{R}^n$ and $\ker(O_i) \subset \mathbb{R}^n$, respectively, and satisfy $\ker(O_i)^T = \text{Im}(O_i^T)$.

For $i \in \mathcal{V}$, let $V_i = \text{row}(V_{ai}, V_{oi}) \in \mathbb{R}^{n \times n}$ be an orthogonal matrix such that $V_i^T V_i = I_n$. Let $V_{ai} \in \mathbb{R}^{n \times (n-v_i)}$ be a matrix such that all the columns of $V_{ai}$ are from an orthogonal basis of the $\ker(O_i)$ satisfying $\text{Im}(V_{ai}) = \ker(O_i)$. Let $V_{oi} \in \mathbb{R}^{n \times v_i}$ be a matrix such that all the columns of $V_{oi}$ are from an orthogonal basis of the $\ker(O_i)$ satisfying $\text{Im}(V_{oi}) = \text{Im}(O_i^T)$.

For $i \in \mathcal{V}$, the matrices $A$ and $C_i$ of the system in (1) yield the Kalman observability decomposition as follows:

$$ V_i^T A V_i \equiv \begin{bmatrix} A_{ii} & A_{i} \\ 0 & A_o \\ 0 & A_{oi} \end{bmatrix}, \quad (5a) $$

$$ C_i V_i \equiv \begin{bmatrix} 0 & C_{oi} \end{bmatrix}, \quad (5b) $$
where the pair \((A_{oi}, C_{oi})\) is observable, \(A_{oi} \in \mathbb{R}^{v_i \times u_i}, A_{i} \in \mathbb{R}^{(n-v_i) \times v_i}, u_i \in \mathbb{R}^{(n-v_i) \times (n-v_i)}\) and \(C_{oi} \in \mathbb{R}^{p_i \times v_i}\) admit the following matrix decompositions: \(A_{oi} = V_{oi}^T A_{oi}, A_{i} = V_{oi}^T A_{oi}, C_{oi} = C_{oi} V_{oi}.\)

**Remark 2.** Let \(C_{oi} = \text{diag}(C_{oi1}, \ldots, C_{oii}), A_{i} = \text{diag}(A_{i1}, \ldots, A_{ii}), V_{oi} = \text{diag}(V_{oi1}, \ldots, V_{oii}), V_{oii} = \text{diag}(V_{oi1}, \ldots, V_{oii}), A_{oi} = \text{diag}(A_{oi1}, \ldots, A_{oii})\) and \(A_{oi} = \text{diag}(A_{oi1}, \ldots, A_{oii}).\)

Before proceeding, we review some lemmas proposed in Kim, Lee, and Shim (2019) and Zhang, Li, Qu, and Lewis (2015), which will play important roles in analyzing the convergence of the estimation error.

**Lemma 1.** (Zhang, Li, Qu, & Lewis, 2015) Suppose that the communication network \(\mathcal{G} = (\mathcal{V}, \mathcal{E})\) is strongly connected. Let \(x\) and \(y\) be such that \(L\) is not equal to zero and \(L^T y = 0\). Then, \(\Theta = \text{diag}(\theta_1, \ldots, \theta_N) > 0\) with \(\theta_i = \frac{1}{\tau_i}\) and \(\hat{C} = \Theta L + L^T \Theta \geq 0\).

**Lemma 2.** (Kim, Lee, & Shim, 2019) Suppose that the communication network \(\mathcal{G} = (\mathcal{V}, \mathcal{E})\) is strongly connected. Then, the following statements are equivalent:

1. The system in (1) is jointly observable;
2. The matrix \(V_{oi}^T (\hat{C} \otimes I_n) V_{u}\) is positive definite;
3. The matrix \(V_{oi}^T (\hat{C} \otimes I_n) V_{u}\) is nonsingular.

To better present our design and results, we first introduce some notions related to matrices.

**Remark 3.** Let \(\theta_m \triangleq \min\{\theta_1, \ldots, \theta_N\}\) and \(\theta_M \triangleq \max\{\theta_1, \ldots, \theta_N\}\). Let \(\lambda_1\) and \(\lambda_L\) denote the minimum and maximum eigenvalues of \(V_{oi}^T (\hat{C} \otimes I_n) V_{u}\), respectively, where \(V_{u}\) can be found in Remark 2.

Before introducing our main results, we first establish the following lemma, and its proof can be found in Appendix 5.

**Lemma 3.** Consider the following sampled-data system
\[ \hat{x}(t) = A_{oi} x(t) - \gamma V_{oi}^T (\hat{C} \otimes I_n) V_{u} x(t), t \in [t_{k+1}, t_k), (6) \]
where \(z(t) = \text{col}(z_1(t), \ldots, z_N(t))\) with \(z_i(t) \in \mathbb{R}^{n-v_i}, i \in \mathcal{V}\). Suppose Assumptions 1 and 2 hold. Then, for all \(\gamma > \gamma_{\text{max}}\), and \(\tau \in (0, \tau_0)\), the system in (6) is exponentially stable at the origin for all \(h \in (0, T)\) over \(\mathbb{N}\), where
\[ \gamma_{\text{max}} = \frac{2 \theta_M \sum_{i \in \mathcal{V}} \| A_{oi} \|}{\lambda_1} \text{ and } \tau_0 = \frac{c_1}{c_2}, (7) \]
with
\[ c_1 = \frac{\gamma \lambda_1}{\theta_M} - 2 \sup_{i \in \mathcal{V}} \| A_{oi} \|, \]
\[ c_2 = \sup_{i \in \mathcal{V}} (\| A_{oi} \| + \| V_{oi} \| \| L \|). \]
We are set up to state the main results.

3. **Main Results**

This section devotes to the design and analysis of aperiodic sampled-data distributed observers.

3.1. **Aperiodic Sampled-Data Distributed Observers Design**

We now introduce the following linear distributed observer as follows: \(\forall i \in [t_k, t_{k+1})\)
\[ \hat{x}_i(t) = A_i \hat{x}_i(t) + L_i (C_i \hat{x}_i(t_k) - y_i(t_k)) + \gamma M_i \sum_{j \in \mathcal{N}_i} (\hat{x}_j(t_k) - \hat{x}_i(t_k)), i \in \mathcal{V}, \]
where \(t_0\) is the initial time, \(t_{k+1} - t_k = h_k\) over \(k \in \mathbb{N}\), \(\gamma \geq \gamma_{\text{max}}\), and \(h_k \in (0, h_{\text{max}})\) with \(\gamma_{\text{max}}\) and \(h_{\text{max}}\) being some positive numbers to be determined and given in Algorithm 1.

\[ L_i = V_{l_i} \begin{bmatrix} 0 \\ L_{oi} \end{bmatrix} \text{ and } M_i = V_{l_i} \begin{bmatrix} I_{n-v_i} & 0 \\ 0 & 0 \end{bmatrix} V_{oi}^T, (10) \]
and \(L_{oi} \in \mathbb{R}^{v_i}\) is chosen such that \(\hat{A}_i \triangleq A_{oi} + L_{oi} C_{oi}\) is Hurwitz, for \(i \in \mathcal{V}\). In order to calculate the maximum allowable sampling period \(h_{\text{max}}\), we first give the following notations:
\[ \tau_1(\chi, \kappa) := \left\{ \begin{array}{ll} \frac{\psi}{\kappa}, & \text{if } \chi > \kappa; \\ \frac{\psi}{\kappa}, & \text{if } \chi = \kappa; \\ \frac{\psi}{\chi}, & \text{if } \chi < \kappa; \end{array} \right. \]
\[ (11) \]
where \(r = \sqrt{\frac{\chi^2}{\kappa^2} - 1}, \kappa = \sup_{i \in \mathcal{V}} \| L_{oi} C_{oi} \|\) and \(\chi \geq \chi_{\text{max}}\) with \(\chi_{\text{max}} = \max\{\chi_1, \ldots, \chi_N\}\) and \(\chi_i\) being
\[ \chi_i = \| \hat{A}_i^T(sI - \hat{A}_i)^{-1} L_{oi} C_{oi} \|_{\infty}. \]
\[ (12) \]

**Algorithm 1** Maximum allowable sampling period \(h_{\text{max}}\)

1. Select \(\Theta = \text{diag}(\theta_1, \ldots, \theta_N)\) such that \(\Theta L + L^T \Theta\) is semi-positive definite matrix.
2. Choose positive constant \(\gamma > \gamma_{\text{max}}\) using (7).
3. Compute \(\tau_0 = \frac{c_1}{c_2}\) given in (7) and (8).
4. Select \(\chi \geq \chi_{\text{max}}\) using (12).
5. Calculate \(\tau_1(\chi, \kappa)\) using (11).
6. Let \(h_{\text{max}} = \min\{\tau_1(\chi, \kappa), \tau_0\}\).

**Remark 4.** The jointly observable distributed observer for continuous-time case was considered in Kim, Lee, and Shim (2019). In terms of our notation, the convergence of the observer proposed in Kim, Lee, and Shim (2019) can be guaranteed if \(\gamma\) was chosen such that
\[ \gamma \geq \frac{2 \theta_M \sum_{i \in \mathcal{V}} \| A_{oi} \| + \psi}{\lambda_1}, \]
where \(\psi\) is bigger than the maximum among the real parts of the eigenvalues of \(A_{oi} + L_{oi} C_{oi}\). When \(h \to 0\), the sampled-data distributed observer in (9) will reduce to the continuous-time case. Clearly, in our design, \(\gamma_{\text{max}}\) defined in (7) is smaller than \(\frac{2 \theta_M \sum_{i \in \mathcal{V}} \| A_{oi} \| + \psi}{\lambda_1}\).
3.2. The Convergence Analysis

For \( i \in \mathcal{V} \), let \( \hat{x}_i(t) = \tilde{x}_i(t) - x(t) \) be the estimation error of the \( i \)-th observer. Then, for all \( t \in [t_k, t_{k+1}) \), we have

\[
\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + L_i C_i \tilde{x}_i(t_k) + \gamma M_i \sum_{j \in \mathcal{N}_i} (\tilde{x}_j(t_k) - \hat{x}_i(t_k))
\]

\[
= A\hat{x}_i(t) + L_i C_i \tilde{x}_i(t_k) - \gamma M_i \sum_{j=1}^{\mathcal{N}} L_{ij} \tilde{x}_j(t_k),
\]

(13)

where \( L_{ij} \) is the \((i,j)\)-th entry of the Laplacian matrix \( \mathcal{L} \). Let \( \tilde{x}_{oi} = V_{oi}^T \tilde{x}_i \) and \( \hat{x}_{oi} = V_{oi}^T \hat{x}_i \), for \( i \in \mathcal{V} \). Then, we have the following system from (3) and (13), \( \forall t \in [t_k, t_{k+1}) \),

\[
\dot{\hat{x}}_{oi}(t) = A_{oi}\hat{x}_{oi}(t) + A_{ri}\tilde{x}_r(t) - \sum_{j=1}^{\mathcal{N}} L_{oj}^T [V_{uj}^T \tilde{x}_u(t_k) + V_{oij}^T \tilde{x}_{oi}(t_k)],
\]

(14a)

\[
\dot{\hat{x}}_{oi}(t) = A_{oi}\hat{x}_{oi}(t) + L_i C_i \tilde{x}_i(t_k), \quad i \in \mathcal{V}.
\]

(14b)

Let \( \tilde{x}_o = \text{col}(\tilde{x}_1, \ldots, \tilde{x}_N) \), \( \tilde{x}_o = \text{col}(\hat{x}_1, \ldots, \hat{x}_N) \) and \( L_o = \text{diag}(L_{o1}, \ldots, L_{oN}) \). Then, the system in (14) can be put into the following compact form, \( \forall t \in [t_k, t_{k+1}) \),

\[
\dot{\hat{x}}_o(t) = A_o\hat{x}_o(t) + L_o C_o \tilde{x}_o(t),
\]

(15a)

\[
\dot{\hat{x}}_o(t) = A_o\hat{x}_o(t) + L_o C_o \tilde{x}_o(t),
\]

(15b)

where \( A_o, A_r, A_{oi}, A_{ri}, C_o \) and \( V_o \) can be found in Remark 2.

In order to analyze the convergence of the system in (15), we first establish the following lemma to investigate the stability of the system in (15b) using the emulation-based approach in Nesic, Teel, and Carnevale (2009).

**Lemma 4.** For the system in (15b), suppose Assumption 3 holds. Choose \( \epsilon, \chi_{\text{max}}, \chi \) and \( \tau_1 \) according to (11) and (12). Then, for all \( \chi \geq \chi_{\text{max}} \) and \( \tau \in (0, \tau_1) \), the system in (15b) is exponentially stable at the origin for all \( h_k \in (0, \tau) \), \( k \in \mathbb{N} \).

Proof: Let \( e(t) = \hat{x}_o(t_k) - \tilde{x}_o(t_k) \) be the sampling-induced error, for any \( t \in [t_k, t_{k+1}) \) over \( k \in \mathbb{N} \) and \( i \in \mathcal{V} \). Then, the dynamic in (15b) can be rewritten in the following manner:

\[
\begin{align*}
\hat{x}_o(t) &= f(\hat{x}_o(t), e(t)), \quad \forall t \in [t_k, t_{k+1}); \\
\dot{e}(t) &= g(\hat{x}_o(t), e(t)), \quad \forall t \in [t_k, t_{k+1}); \\
e(t_k^+) &= 0, \quad k \in \mathbb{N}^*;
\end{align*}
\]

(16)

where \( A_d = \text{diag}(A_1, \ldots, A_N) \), \( A_o = A_{oi} + L_o C_o \), \( i \in \mathcal{V} \),

\[
f(\tilde{x}_o, e) = \tilde{A}_d \tilde{x}_o + L_o C_o e, \quad \text{and} \quad g(\tilde{x}_o, e) = -f(\tilde{x}_o, e).
\]

The dynamic in (16) with \( h_k \in [\epsilon, \tau_1] \) can be modeled by the following hybrid system:

\[
\begin{align*}
\dot{x}_o &= f(\tilde{x}_o, e), \\
\dot{e} &= g(\tilde{x}_o, e), \\
\tau &= 1 \\
\tilde{x}_o^+ &= \tilde{x}_o \\
e^+ &= 0, \quad \tau \in [\epsilon, \infty), \ \text{jump dynamics}; \\
\tau^+ &= 0
\end{align*}
\]

(17)

where \( \tau \in \mathbb{R}_+ \) is a clock state, and \( \epsilon \) is an arbitrary small positive number. With \( \tau_1(\chi, \epsilon) \) defined in (11), let \( \lambda \in (0, 1) \) and \( \phi : [0, \tau_1] \rightarrow \mathbb{R} \) be the solution of the following equation:

\[
\dot{\phi} = -2\epsilon\phi - \chi(\phi^2 + 1), \quad \phi(0) = \lambda^{-1}.
\]

According to Carnevale, Teel, and Nesic (2007), \( \phi \in [\lambda, \lambda^{-1}] \). For \( i \in \mathcal{V}, A_i \) is Hurwitz, for any positive number \( \chi \geq \chi_{\text{max}} \), where \( \chi_{\text{max}} \) is defined in (12), there exists a positive definite matrix \( P_i \in \mathbb{R}^{n_i \times n_i} \) such that

\[
\begin{bmatrix}
\tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{1}{\lambda} \tilde{A}_i^T \tilde{A}_i + \rho \Phi_i^T L_i^T P_i - \chi L_i^T P_i
\end{bmatrix} < 0.
\]

(18)

Then, we can choose a Lyapunov function as follows:

\[
U(\tau, \tilde{x}_o, e) = \tilde{x}_o^T P_d \tilde{x}_o + \phi(\tau e^T e),
\]

where \( P_d = \text{diag}(P_1, \ldots, P_N) \) with \( P_i \) giving in the matrix equation in (18). On the jump domain in (17), it is noted that

\[
U(\tau^+, \tilde{x}_o^+, e^+) = \tilde{x}_o^{+T} P_d \tilde{x}_o^+ + \phi(\tau^+)(e^+)^T e^+.
\]

In addition, from the flow dynamics in (17), we have

\[
\dot{U} = \tilde{x}_o^T \left[ P_d \tilde{A}_d + \frac{1}{\lambda} \tilde{A}_d^T \tilde{A}_d \right] \tilde{x}_o + 2 \tilde{x}_o^T P_d L_o C_o e + \frac{1}{\chi} \tilde{x}_o^T \left[ L_o C_o \right] \tilde{x}_o + 2 \tilde{x}_o^T P_d L_o C_o e + \frac{1}{\chi} \tilde{x}_o^T \left[ L_o C_o \right] \tilde{x}_o \leq \rho^* U,
\]

where \( \rho^* \) is a positive constant. Besides, the fact that \( \tilde{A}_d \) is Hurwitz leads to that the system \( \tilde{x}_o = f(\tilde{x}_o, 0) \) is exponentially stable. Therefore, we have system (15b) is exponentially stable from Theorem 2 in Nesic, Teel, and Carnevale (2009).

In order to analyze the stability of the system in (15), we now use the discretization technique to transform the continuous-time system in (14) into a discrete-time system. The discretization technique is called the step-invariant transformation, which can be found in Chen and Francis (1995). The continuous-time system in (15) can be transformed into the following time-varying discrete-time system:

\[
\hat{x}_u(t_{k+1}) = A(h_k) \hat{x}_u(t_k) + g(t_k),
\]

(19a)
Then, along the trajectory of the system in (19a), we have
\[
\ddot{x}_o(t_{k+1}) = \left[ e^{Ah_k} + \int_{0}^{h_k} e^{A_{i}k} C_{o_i} d\tau \right] \ddot{x}_o(t_k), \tag{19b}
\]
where
\[
\Lambda(h_k) = e^{Ah_k} - \gamma \int_{0}^{h_k} e^{A_{i}k} d\tau V_u^T (L \otimes I_k) V_u,
\]
\[
g(t_k) = - \gamma \int_{0}^{h_k} e^{A_{i}k} d\tau V_u^T (L \otimes I_k) V_u \ddot{x}_o(t_k)
+ \int_{0}^{h_k} e^{A_{i}k} A_t \ddot{x}_o(t_k + \tau) d\tau.
\]
Then, we have the following results.

**Corollary 1.** For the system in (19), suppose Assumptions 2 and 3 hold. Choose $\gamma_{\text{max}}$ and $\tau_0$ according to (7). Then, for all $\gamma \geq \gamma_{\text{max}}$ and $\tau \in (0, \tau_0)$, the system in (19) is exponentially stable at the origin for all $h_k \in (0, \tau)$ over $k \in \mathbb{N}$, provided that $\ddot{x}_o(t_0) = 0$ over $\mathbb{N}$. Proof: As $\ddot{x}_o(t) = 0$ for all $t \geq 0$, the system in (19a) reduces to
\[
\ddot{x}_o(t_{k+1}) = \Lambda(h_k) \ddot{x}_o(t_k). \tag{20}
\]
By Lemma 3, under Assumptions 2 and 3, for all $\gamma \geq \gamma_{\text{max}}$ and $\tau \in (0, \tau_0)$, the system in (15) is exponentially stable at the origin for all $h_k \in (0, \tau), k \in \mathbb{N}$, provided that $\ddot{x}_o(t_0) = 0$ for all $t \geq 0$. Hence, we have the system in (20) is exponentially stable for all $\gamma \geq \gamma_{\text{max}}$ and $h_k \in (0, \tau)$ over $k \in \mathbb{N}$. □

**Theorem 1.** For the system in (15), suppose Assumptions 2 and 3 hold. Choose $\gamma_{\text{max}}$ and $h_{\text{max}}$ according to (7) and Algorithm 1. Then, for all $\gamma \geq \gamma_{\text{max}}$ and $\tau \in (0, h_{\text{max}})$, the system in (15) is exponentially stable at the origin for all $h_k \in (0, \tau)$ over $k \in \mathbb{N}$ such that
\[
\lim_{t \to \infty} (\dot{x}_i(t) - x(t)) = 0, \ i \in \mathbb{V},
\]
for any $x(0) \in \mathbb{R}^n$ and $\dot{x}_i(0) \in \mathbb{R}^n$.

Proof: Under Assumptions 2 and 3, choose $\gamma_{\text{max}}$ and $h_{\text{max}}$ according to (7) and Algorithm 1. Then, for all $\gamma \geq \gamma_{\text{max}}$ and $\tau \in (0, h_{\text{max}})$, the system in (20) is exponentially stable at the origin from Corollary 1. By the Converse Lyapunov Theorem ((Stein, 1952), (Rugh, 1996, Theorem 23.3) and (Bai, Fu, & Sastry, 1988)), there exists a time-varying symmetric matrix $P(k)$ over $k \in \mathbb{N}$ such that
\[
\alpha_1 I \leq P(k) \leq \alpha_2 I, \tag{21a}
\]
\[
\Lambda^T(h_k) P(k + 1) \Lambda(h_k) - P(k) \leq - \alpha_3 I, \tag{21b}
\]
for some positive constants $\alpha_1$, $\alpha_2$, and $\alpha_3$. Choose the Lyapunov function for the system in (20) as follows:
\[
U(t_k) = \dot{x}_o^T(t_k) P(k) \dot{x}_o(t_k).
\]
Then, along the trajectory of the system in (19a), we have
\[
U(t_{k+1}) - U(t_k) = \left[ \Lambda(h_k) \ddot{x}_o(t_k) + g(t_k) \right]^T P(k + 1) \left[ \Lambda(h_k) \ddot{x}_o(t_k) + g(t_k) \right]
- \ddot{x}_o^T(t_k) P(k) \ddot{x}_o(t_k)
\leq - \alpha_3 ||\ddot{x}_o(t_k)||^2 + \alpha_2 ||g(t_k)||^2 + 2\alpha_2 ||g(t_k)|| ||\ddot{x}_o(t_k)||
\leq - \frac{3\alpha_3}{4} ||\ddot{x}_o(t_k)||^2 + \frac{\alpha_2^2 + 4\alpha_2 \alpha_3}{4\alpha_3} ||g(t_k)||^2. \tag{22}
\]
By Lemma 1, under Assumptions 2 and 3, for all $\gamma \geq \gamma_{\text{max}}$ and $\tau \in (0, \tau_0)$, we have $\lim_{t \to \infty} \ddot{x}_o(t_0) = 0$ from Lemma (1), which yields that $\lim_{t \to \infty} ||\ddot{x}_o(t)|| = 0$ exponentially and $||g(t_k)||$ is bounded over $\mathbb{N}$. The system in (22) is input-to-state stable with $\frac{\alpha_2^2 + 4\alpha_2 \alpha_3}{4\alpha_3} ||g(t_k)||^2$ as the input. By Lemma 3.8 in Jiang and Wang (2001), the system in (19a) has the $K$ asymptotic gain property. Hence, there exists a class $\mathcal{K}$ function $\beta(\cdot)$ such that, for any initial condition, the solution of system (19a) satisfies
\[
\limsup_{k \to \infty} ||\ddot{x}_o(t_k)|| \leq \beta \left( \limsup_{k \to \infty} \frac{\alpha_2^2 + 4\alpha_2 \alpha_3}{4\alpha_3} ||g(t_k)||^2 \right).
\]
Therefore, $\lim_{t \to \infty} \frac{\alpha_2^2 + 4\alpha_2 \alpha_3}{4\alpha_3} ||g(t_k)||^2 = 0$ implies that $\lim_{t \to \infty} \ddot{x}_o(t_k) = 0$ exponentially. It follows from the system in (15a) that, $\forall t \in [t_k, t_{k+1})$,
\[
||\dot{x}_o(t)|| \leq ||\dot{x}_o(t_k)|| e^{||A||t} \left( 1 + \gamma \tau ||L|| ||V_u||^2 \right)
+ \gamma ||A|| \tau ||A|| ||L|| ||V_u|| ||\ddot{x}_o(t_k)||
\]
which together with $\lim_{t \to \infty} \ddot{x}_o(t_k) = 0$ and $\lim_{t \to \infty} \dot{x}_o(t_k) = 0$ yield that $\lim_{t \to \infty} \ddot{x}_o(t) = 0$. Hence, $\lim_{t \to \infty} \ddot{x}_o(t_k) = 0$ and $\lim_{t \to \infty} \ddot{x}_o(t) = 0$ lead to
\[
\lim_{t \to \infty} \ddot{x}_i(t) = 0, \ i \in \mathbb{V},
\]
from the fact that $\ddot{x}_o(t) = V_o^T \ddot{x}_i(t)$ and $\ddot{x}_o(t_k) = V_o^T \ddot{x}_i(t_k)$.

3.3. Application to Jointly Observable Tracking Problem

In this sub-section, we will apply this distributed observer (13) to solve the Jointly Observable Tracking Problem of linear multi-agent systems.

We now consider the design of control input as follows:
\[
u(t_k) = K(x_i(t_k) - \dot{x_i}(t_k)), \tag{23a}
\]
\[
\dot{x}_i(t) = A \dot{x}_i(t) + L_i (C \dot{x}_i(t_k) - y_i(t_k))
+ \gamma M_i \sum_{j \in N_i} (\dot{x}_j(t_k) - \dot{x}_i(t_k)), \ i \in \mathbb{V}, \tag{23b}
\]
where $t_0 = 0$ is the initial time, choose $\gamma \geq \gamma_{\text{max}}$ and $h_k \in (0, h_{\text{max}})$ with $\gamma_{\text{max}}$ and $h_{\text{max}}$ being given in (7) and Algorithm 1, $k \in \mathbb{N}$ and $K$ is the feedback gain such that $A + BK$ is Hurwitz. Then, we have the following theorem.

**Theorem 2.** Consider the system in (1) and (3), suppose Assumptions 1 and 2 hold. For any initial conditions $x_i(0) \in \mathbb{R}^n$, $\dot{x}_i(0) \in \mathbb{R}^n$ and $x(0) \in \mathbb{R}^n$, Problem 1 is solvable by the following control law in (23) with $K$ being chosen such that $A + BK$ is Hurwitz, for $i \in \mathbb{V}$. 6
Proof: Let $\hat{\xi}_i(t) = \xi_i(t) - x(t)$, for $i \in \mathcal{V}$. Then, we have

$$\dot{\hat{\xi}}_i(t) = A\hat{\xi}_i(t) + Bu_i(t),$$

(24)

Substituting the control law in (23a) into the system in (24) yields the following dynamics of the estimation error:

$$\dot{\hat{\xi}}_i(t) = A\hat{\xi}_i(t) + BK(\xi_i(t) - \hat{x}_i(t)) = (A + BK)\hat{\xi}_i(t) - BK\hat{x}_i(t), \quad i \in \mathcal{V}.$$  

(25)

By Theorem 1, under Assumptions 2, and 3, there exists positive $h_{\text{max}} > 0$ such that for any sampling periods $h_k \in (0, h_{\text{max}})$ over $\mathbb{N}$ and sufficiently large enough $\gamma$, $\hat{x}_i(t)$ converges to zero exponentially as $t \to \infty$, for $i \in \mathcal{V}$. Moreover, $A + BK$ is Hurwitz. Thus, the system in (25) can be viewed as a stable system with $-BK\hat{x}_i(t)$ as the input, this input converges to zero as $t \to \infty$, for $i \in \mathcal{V}$. Hence, for any initial condition $\hat{\xi}_i(0)$, $\lim_{t \to \infty} \hat{\xi}_i(t) = 0$, $i \in \mathcal{V}$. \hfill \Box

4. Numerical Example

![Figure 1: Communication topology $\mathcal{G}$](image)

In this example, we consider a linear distributed system composed of a to-be-observed system and five node observers as shown in Figure 1. The dynamic of the to-be-observed system takes the form in (1) with

$$A = \begin{bmatrix} 0 & 0.1 & 0 \\ -0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$  

The partition $y_1(t_k), y_2(t_k)$ and $y_3(t_k)$ of the augmented output $y(t) = \text{col}(y_1(t), y_2(t), y_3(t), y_4(t), y_5(t)) = Cx(t)$ can be measured by 1-th, 2-th and 3-th node observers as shown in Figure 1 at time instant $t_k$, respectively. Whereas $y_4(t) = 0$ and $y_5(t) = 0$ further imply that 4-th and 5-th node observers can not measure anything of the to-be-observed system in (1). Therefore, it can be evaluated that none of the pair $(A, C_i)$ is observable, while $(A, C)$ is observable. The dynamic of the node observer system is in (3) with

$$A = \begin{bmatrix} 0 & 0.1 & 0 \\ -0.1 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$  

Choose the following matrix based on Kalman observability decomposition as follows:

$$V_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

(26)

It can be easily verified that the topology in Figure 1 satisfies Assumption 2. Let $\theta = \text{col}(1, 1, 1, 1, 1)$. Hence, $\lambda_2 = 5$ and $\lambda_1 = 1$. Then, it can be calculated from (7) and (12) that $\gamma_{\text{max}} = 0.2$, $\chi_{\text{max}} = 4.8016$ and $\kappa = 4.4721$.

Let $\gamma = 0.4$ and $\chi = 4.802$, which further gives $\tau_0 = 0.0822$, $\tau_1(\chi, \kappa) = 0.5721$ and $h_{\text{max}} = 0.0822$ from Algorithm 1 and equation (7). Thus, we can design a control law composed of (9) and (23) with the following parameters: $K^T = \text{col}(22.6, 46.7, -49.5), L_{\text{col}} = \text{col}(-4 - 2), L_{\text{col}} = \text{col}(-4 - 2), L_{\text{col}} = -2.5$. Then, from (10) and (26), we have

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$M_4 = I_3, \quad M_5 = I_3, \quad L_1 = \text{col}(-2, -4), \quad L_2 = \text{col}(4, 2, 0),$$

$$L_3 = \text{col}(0, 0, -2.5), \quad L_4 = \text{col}(0, 0, 0), \quad L_5 = \text{col}(0, 0, 0).$$

Simulation is conducted with the following initial condition: $x(0) = \text{col}(1, 2, 3), \hat{x}_i(0) = \text{col}(0, 0, 0)$ and $\hat{\xi}_i(0) = \text{col}(0, 0, 0)$, for $i \in \mathcal{V}$.

![Figure 2: Sampling period vs. the sampling time](image)

Figure 2 shows the time-varying sampling periods vs sampling time. Figure 3 shows the estimation errors of all node observers. Figure 4 shows the tracking error of all node observers, which all converge to zero.

5. Conclusions

In this paper, a distributed state estimation problem with the jointly observable assumption has been investigated by using the sampled-data information. A maximum allowable sampling period for all node observers is
For any $t \in [t_k, t_{k+1})$, the time derivative of $U(t)$ along (6) can be evaluated as

$$
\dot{U}(t) = 2 \sum_{i=1}^{N} \theta_i z_i^T(t) A_{ui}^T z_i(t) - \gamma z^T(t) [V_u^T (\hat{\mathcal{L}} \otimes I_n) V_u] z(t) - \gamma z^T(t) [V_u^T (\hat{\mathcal{L}} \otimes I_n) V_u] [z(t_k) - z(t)].
$$

(29)

Under Assumptions 2 and 3, the matrix $V_u^T (\hat{\mathcal{L}} \otimes I_n) V_u$ is positive definite from Lemma 2. Then, for any $t \in [t_k, t_{k+1})$, we have

$$
\dot{U}(t) \leq 2 \theta_M \sup_{i \in V} \|A_{ui}\| \|z(t)\|^2 - \gamma \lambda_L \|z(t)\|^2 + \gamma \lambda_L \|z(t)\| \|z(t) - z(t_k)\|.
$$

(30)

From (6) and (28), for any $t \in [t_k, t_{k+1})$, we have

$$
\|\dot{z}(t)\| \leq \sup_{i \in V} \|A_{ui}\| \|z(t)\| + \gamma \sup_{i \in V} \|V_u\| \|L\| \|z(t_k)\|
\leq \frac{\sup_{i \in V} \|A_{ui}\| + \gamma \|V_u\| \|L\| \|U_M(t_k)\|}{\sqrt{\theta_m}}.
$$

(31)

where $U_M(t_k) = \max_{s \in [t_k, t_{k+1})} U(z(s))$. It is noted that, for any $t \in [t_k, t_{k+1})$,

$$
\|z(t) - z(t_k)\| \leq \int_{t_k}^{t} \|\dot{z}(s)\| ds
\leq \frac{\sup_{i \in V} \|A_{ui}\| + \gamma \|V_u\| \|L\| \|U_M(t_k)\|}{\sqrt{\theta_m}}.
$$

(32)

It can be easily evaluated from (28), (30), (31) and (32) that

$$
\dot{U}(t) \leq -c_1 U(t) + c_2 (t - t_k) \sqrt{U(t)} \sqrt{U_M(t_k)}
\leq -c_1 U(t) + c_2 h_k \sqrt{U(t)} \sqrt{U_M(t_k)}, \quad t \in [t_k, t_{k+1}),
$$

where $c_1$ and $c_2$ are given in (8). In what follows, we will show that the following equation is satisfied

$$
\max_{s \in [t_k, t_{k+1})} U(s) = U(t_k), \quad \forall h_k \in (0, \tau), \quad \tau < \tau_0.
$$

(34)

If equation (34) is not true, we can assume that there exists a time instant $t' \in [t_k, t_{k+1})$ such that $U(t') > U(t_k)$. For any $\gamma > \gamma_{\max}$, it is noted from (33) and (7) that

$$
\dot{U}(t_k) \leq -c_1 U(t_k) < 0, \quad \forall z(t_k) \neq 0.
$$

(35)

Thus $U(t)$ will decrease near the time instant $t_k$. Hence, there exists a time instant $t'' \in [t_k, t')$ such that

(a) $U(t'') = U(t_k)$, (b) $\dot{U}(t') > 0$,
(c) $U(t) \leq U(t'')$, $\forall t \in [t_k, t'')$.

(36)

Then, equations (33) and (36) imply

$$
\dot{U}(t'') \leq -c_1 U(t'') + c_2 (t'' - t_k) \sqrt{U(t'')} \sqrt{U_M(t_k)}
$$

Appendix A

Proof: Define the following Lyapunov function for the system in (6)

$$
U(z(t)) = \sum_{i=1}^{N} \theta_i \|z_i(t)\|^2,
$$

(27)

where $\theta_i$ is given in Algorithm 1. Then, $U(z(t))$ satisfies the following property

$$
\theta_m \|z(t)\|^2 \leq U(z(t)) \leq \theta_M \|z(t)\|^2.
$$

(28)
The fact that $\tau < \tau_0$ leads to $\dot{U}(t^+) \leq 0$, contradicting the second inequality in (36). Hence, we have (34), which yields the following equation from (33)

$$U(t) \leq -c_1 U(t) + c_2 \tau \sqrt{U(t) U(t_k)}, \forall t \in [t_k, t_{k+1}). \quad (37)$$

Motivated by Qian and Du (2012), let $\eta(t) = \sqrt{U(t) U(t_k)}$. The time derivative of $\eta(t)$ during time interval $[t_k, t_{k+1})$ along (37) can be evaluated as

$$\dot{\eta}(t) \leq -\frac{c_1}{2} \eta(t) + \frac{c_2 \tau}{2}, \forall t \in [t_k, t_{k+1}). \quad (38)$$

By using the comparison lemma in Khalil (2002), from equation (38),

$$\eta(t) \leq e^{-\frac{c_1}{2} (t-t_k)} \left(1 - \frac{c_2 \tau}{c_1} \right) + \frac{c_2 \tau}{c_1} \frac{c_1}{c_1} \forall t \in [t_k, t_{k+1}).$$

It is noted that $\eta(t_k) = 1$ and $c_1 - c_2 \tau < 0$, then we have

$$\lim_{t \to t_k^+} \eta(t) = e^{-\frac{c_1}{2} (t_k+1-t_k)} \left(1 - \frac{c_2 \tau}{c_1} \right) + \frac{c_2 \tau}{c_1} \frac{c_1}{c_1} = \eta(t_k+1) \leq \eta(t_k), \quad (39)$$

In addition, since $U(t)$ is continuous for all $t \geq 0$,

$$\lim_{t \to t_k^+} \eta(t) = \sqrt{U(t_k+1) U(t_k)},$$

which together with equation (39) yields

$$U(t_k+1) \leq \rho^2 U(t_k).$$

Equation (39) and $b_k \in (0, \tau_0)$ lead to $0 < \rho < 1$. Therefore, we have $U(t_k)$ converges to zero as $k$ tends to infinity (exponentially), which together with equation (34) further implies $\lim_{t \to \infty} U(t) = 0$ exponentially. Finally, from equation (28), the system in (6) is exponentially stable at the origin. \qed

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