Size limiting in Tsallis statistics.

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February 14, 2008

Abstract

Power law scaling is observed in many physical, biological and socio-economical complex systems and is now considered as an important property of these systems. In general, power law exists in the central part of the distribution. It has deviations from power law for very small and very large step sizes. Tsallis, through non-extensive thermodynamics, explained power law distribution in many cases including deviation from the power law, both for small and very large steps. In case of very large steps, they used heuristic crossover approach.

In real systems, the size is limited and thus, the size limiting factor is important. In the present work, we present an alternative model in which we consider that the entropy factor $q$ decreases with step size due to the softening of long range interactions or memory. This explains the deviation of power law for very large step sizes. Finally, we apply this model for distribution of citation index of scientists and examination scores and are able to explain the entire distribution including deviations from power law.

I. Introduction

Only recently physicists started to study the natural systems as a whole rather than in parts [1-6] and are interested in holistic properties of these systems normally called “Complex Systems”. These systems are difficult to
understand from the basic principles. The difficulties in understanding these systems arise from the fact that, in most of the cases, a large number of elementary interactions is taking place at the same time for a large number of components. Further, these systems are in constant evolution and do not have an equilibrium state [1]. Power law scaling [7,8] is observed in many biological [9-11], physical [2,12-20] and socio-economical complex systems [21-29] and it is now considered an important property of them. As socio-economical systems also have almost the same characteristics, physicists are also studying some of these systems.

In 1988, Tsallis [30] presented non-extensive thermodynamics in which he incorporated long range interactions and long memory effects. He proposed a generalized definition of entropy \( S_q \):

\[
S_q = C \left( 1 - \frac{\sum_{i=1}^{W} p_i^q}{q - 1} \right)
\]

(1)

where \( C \) is a positive constant, and \( W \) is the total number of microscopic possibilities of the system. \( q \) is an entropic index, which plays a central role and is related to long range interactions and long memory effect in a network. This expression recovers the usual Boltzmann-Gibbs entropy \( -C \sum_{i=1}^{W} p_i \ln p_i \) in the limit \( q \to 1 \), i.e. in short range interactions [31]. The size frequency distribution function \( N(x) \) is given through

\[
\frac{dN(x)}{dx} = -\lambda N(x)
\]

(2)

where \( \lambda \) is a positive constant. \( N(x) \) is the frequency probability of step size \( x \).

This gives

\[
N(x) = N_0 \exp(-\lambda x)
\]

(3)
where $N_0$ is a normalization constant, thus we have an exponential decay which is exactly the case of Boltzmann statistics considering short range interactions. However for $q > 1$, a more generalized equation [32] holds giving:

$$\frac{dN(x)}{dx} = -\lambda N^q(x) \quad (4)$$

hence

$$N(x) = \frac{N_0}{[1 + (q - 1)\lambda x]^\frac{1}{q-1}} \quad (5)$$

or in an alternative form:

$$N(x) = \frac{N_0}{[1 + \beta x]^\alpha} \quad (6)$$

where $\beta = (q - 1)\lambda$ and $\alpha = 1/(q - 1)$.

For relatively large values of $x$, the distribution becomes

$$N(x) \approx \text{const.} x^{-\alpha} \quad (7)$$
i.e. a power law. In this case, a $\log N(x) vs. \log(x)$ plot exhibits a straight line for large values of $x$. Power law distribution can not continue forever in real systems. It has to be truncated in some way to avoid infinite variance.

Recently, we have shown that by gradually truncating power law distribution after certain critical value, we are able to explain the entire distribution including very large steps in financial and physical complex systems [33-35]. Although this model explains empirical results for large step sizes, it has an undesirable discontinuity at critical step size. Also, this model fails for small steps.

In discussing folding-unfolding phenomena that occurs in proteins, Tsallis et. al. [36] argued that with the increase of temperature, increases thermal motion, which in turn decreases long memory or long range interactions
and finally decreases entropy index $q$. Thus $q$ approaches to 1. At low temperatures, the distribution function, which shows power law becomes exponential at relatively high temperatures. For a fixed temperature, $q$ is considered to be a constant. In order to consider long range departure, they assume a crossover to another type of behavior and modify Equation (4) as

$$
\frac{dN(x)}{dx} = -\mu_r N^r(x) - (\lambda - \mu_r) N^*(x) \tag{8}
$$

$\mu_r$ is very small compared to $\lambda$. That gives a crossover between two different power laws (respectively characterized by $q$ and $r$) or from power law to normal distribution within a nonextensive scenario, which is definitely a case for many complex systems and gives multifractality. The cut-off is sharpest when $r = 1$. In this case

$$
N(x) = \frac{N_0}{\left(1 - \frac{\lambda}{\mu_1} + \frac{\lambda e^{(q-1)\mu_1 x}}{\mu_1} \right)^{1/q}} \tag{9}
$$

Although cross-over behavior as suggested by Tsallis can avoid an infinite variance, in the present work, we are looking for another possibility, i.e. truncation of power law due to finite size in real systems which in fact is not a cross-over behavior. We therefore suggest an alternative approach to address the long time ($t$) or long distance ($x$) departures. We consider that entropy factor $q$ decreases with step size ($x$) due to the softening of long range interactions or memory effects due to finite size in real systems which arises because of physical limitation of the component or the system itself. Thus $q$ depends on the step size. This is similar as anharmonic terms are important for calculating potential energy in lattice vibrations.

Finally, we apply this model for the distribution of the citation index of scientists and examination scores of an entrance examination and compare it with Tsallis approach [36].

II. The model

The physical limiting factor is of a very small importance for small steps, while it is necessary for larger steps. Entropy index $q$ is equal to 1 in the absence of long memory or long range interactions. Thus the information
about these interactions is given through \((q - 1)\). We consider that this factor approaches to zero for very large values of \(x\) due to finite size in real systems. In general, for this, we propose

\[
(q(x) - 1) = \frac{(q_0 - 1)}{1 + \sum_j \theta_j x^j} \tag{10}
\]

where \(q_0\) and \(q(x)\) are values of entropy index \(q\) for step size zero and step size \(x\) respectively. \(\theta_i\) and \(i\) are adjustable parameters, depending on the size limiting.

To simplify, we propose an exponential decay i.e.

\[
q(x) - 1 = (q_0 - 1) \exp(-(\theta x)^i) \tag{11}
\]

where \(\theta\) and \(i\) show the rate of decrease of the importance of these interactions with the increase of step size \(x\). The higher value of \(i\) indicates a sharper cut-off.

For very large values of \(x\), \(q(x)\) approaches to 1 and thus gives normal distribution as required through central limit theorem. In the present model the distribution function is given through:

\[
N(x) = N_0[1 + (q_0 - 1)\lambda x \exp(-(\theta x)^i)]^{-\left(\frac{\exp((\theta x)^i))}{(q_0 - 1)}\right)} \tag{12}
\]

In Figure 1 we compare \(N(x)\) vs. \(x\) in Tsallis approach through Equation (9) and present approach through Equation (12) for very large steps. Under the present model, the gradual truncation of the power law can be adjusted from very sharp to very slow through the value of \(i\) without interfering in power law behavior in the central part of the distribution. This is not possible in Tsallis approach. For larger values of \(\mu_r\) (line II of Figure 1), we can get a sharp cut-off, but then it deviates significatory from power law in the central part.
Figure 1 – Theoretical distribution of Tsallis and present Model in log-log scale. We consider: $N_0 = 1.10^8$; $\lambda = 0.005$; $q_0 = 1.5$. Curves I and II are through Tsallis model considering $r = 1$ and $\mu_r = 1.10^{-4}$ and $1.10^{-3}$ respectively. Curves A, B, C, D, E, F, and G are through present model considering $i = 1/3$ and $\theta = 3.10^{-7}$ (Curve A), $i = 1/2$ and $\theta = 3.10^{-6}$ (Curve B), $i = 1$ and $\theta = 3.10^{-5}$ (Curve C), $i = 2$ and $\theta = 1.10^{-4}$ (Curve D), $i = 3$ and $\theta = 1.5.10^{-4}$ (Curve E), $i = 4$ and $\theta = 1.8.10^{-4}$ (Curve F) and $i = 5$ and $\theta = 2.10^{-4}$ for Curve G.

III. Distribution of citation Index

Now we apply this model to describe the distribution of citation index of the scientists. The citation index of a scientist is the total number of times that his articles are cited in other articles. The citation patterns of scientific publications form a rather complex network [37]. Here nodes are published
papers. The citation of an article is an interaction of a scientific work with other scientific works. Most of the articles are cited in the proper group only. However many articles go beyond it and are of the interest of others. Some pioneer articles are cited for many decades. Thus, citation index arises from both short and long range interactions and can be treated through statistical distribution based on Tsallis entropy concept.

The fact that a scientist is cited more times facilitates him to get more financial help to his research projects and better students. Some other small groups also came in his influence. This, in turn, contributes to form a better and larger group. In physical terms these effects produce long range interactions. A pioneer work is also cited just to complete introduction of a problem, and thus is cited for a larger time [42], although the problem is not directly connected to it. This gives a long range memory

In case of the scientist’s citation index, unfortunately, reliable information is available only for some of the most cited physicists or chemists. There are many scientists with the same name and do not exist a rigid control to separate them. This can make a significant error for low cited scientists. Thus, it is not possible to have a complete statistical analysis as we did in the case of scientific publications [38] and found that non-extensive thermodynamical distribution (Tsallis statistics) is valid over eight orders of magnitude ($10^{-4}$ to $10^4$). It is therefore interesting to construct a Zipf plot [39] in case of citation index of scientists, in which the number of citations of the $n^{th}$ most cited scientists out of an ensemble of $M$ scientists is plotted versus rank $n$. By its very definition, the Zipf plot is closely related to the cumulative large $x$ tail of the citation distribution. This plot is therefore well suited for determining the large $x$ of the citation distribution. This plot also smoothes out the fluctuations in the high-citation tail and thus facilitates quantitative analysis.

Given an ensemble of $M$ scientists and the corresponding number of citations for each of these scientists in rank order $Y_1 \geq Y_2 \geq Y_3 \ldots \geq Y_n \geq \ldots Y_M$, then the number of citations of the $n^{th}$ most-cited scientist $Y_n$ may be estimated by the criterion [39]:

$$\int_{Y_n}^{\infty} N(x)dx = n$$  \hspace{1cm} (13)

This specifies that there are $n$ scientists out of the ensemble of $M$ which
are cited at least $Y_n$ times. From the dependence of $Y_n$ on $n$ in a Zipf plot, one can test whether it agrees with a hypothesized form for $N(x)$.

We analyze citation index of (a) the most-cited Brazilian physicists and chemists and (b) Internationally most cited physicists and chemists. By Brazilian scientists we mean all scientists who are working in Brazil or have a permanent working address in Brazil. All physicists (chemists) including Brazilian physicists publish their work in the same Journals and work almost on the same problems. Physics, like any other basic science, is the same all over the world. In case of the internationally most cited physicists, we have distribution function only for a few physicists ($\approx 1000$). Considering the most cited Brazilian physicists with the same parameters, we widely extend this range to roughly 100,000 most cited physicists. Thus the distribution of citation index of scientists is more critically discussed.

In Figure 2 we plot citation number ($Y_n$) versus rank ($n$) for first 205 Brazilian physicists in 1999 [40] and compare with Tsallis and the present model. For the present model we use the following parameters $N_0 = 1.5$, $q_0 = 1.39$, $\lambda = 0.0055$ and $\theta = 2.5 \times 10^{-4}$ and $i = 1$. For Tsallis statistics (Equation 9), the parameters used are $N_0 = 1.65$, $q_0 = 1.39$, $\lambda = 0.0055$, $\mu_r = 3.10^{-4}$ and $r = 1$. 
Figure 2 Zipf plot of the number of citation of the $n^{th}$ ranked Brazilian physicist $Y_n$ versus rank $n$ on a double logarithmic scale.
In Figure 3, we plot citation number \( Y_n \) versus rank \( n \) for 1120 most cited physicists over the period 1981-June 1997 [41] and compare it with the theoretical curve with the same value of \( q_0 \) and \( \lambda \) as used in Figure 2. We changed the value of constant \( N_0 \) from 1.5 to 95 as total number of physicists is much larger in this case. We are considering Brazilian physicists citations roughly 1.5% of the total citations, which is reasonable. The value of \( \theta \) changes from \( 2.5 \times 10^{-4} \) to \( 3 \times 10^{-5} \). This shows that size limiting factors are much more important for Brazilian physicists compared to internationally most cited physicists which are mostly from U.S.A. This is perhaps due to...
the absence of basic infrastructure and large research laboratories in Brazil to work on important problems particularly in experimental physics. It is interesting to note that 8 out of 10 most cited Brazilian physicists are working in theoretical physics. In case of Tsallis statistics we use the same basic parameters \(q\) and \(\lambda\) as in Figure 2. We change \(N_0\) from 1.65 to 135 and \(\mu\) from \(3.10^{-4}\) to \(3.8.10^{-4}\). We again observe a good agreement both in ours and Tsallis model. Note that we are able to explain both distributions with the same values of basic parameters.

**Figure 4** Zipf plot of the number of citation of the \(n^{th}\) ranked Brazilian chemist \(Y_n\) versus rank \(n\) on a double logarithmic scale.
Figure 5 Zipf plot of the number of citation of the $n^{th}$ ranked International chemist $Y_n$ versus rank $n$ on a double logarithmic scale.

In Figure 4, we plot citation number ($Y_n$) versus rank ($n$) for the first 119 Brazilian chemists in 1999 [40]. We compare it with present model and Tsallis model. We consider following parameters for the present model $N_0 = 0.8$, $q_0 = 1.35$, $\lambda = 0.006$ and $\theta = 2.2.10^{-4}$ and $i = 1$. For Tsallis statistics we use $N_0 = 0.8$, $q_0 = 1.35$, $\lambda = 0.006$ and $\mu_r = 1.10^{-3}$ and $r = 1$.

In Figure 5, we plot the citation number versus rank for the first 10838 chemists [39], and compare this plot with present model considering $N_0 = 180$ and $\theta = 2.5.10^{-5}$ and $i = 1$ In case of Tsallis statistics we use $N_0 = 190$ and $\mu_r = 3.2.10^{-4}$ and $r = 1$. The other parameters are the same as in Figure 4.

The values of $N_0$ show that citations of Brazilian chemists is roughly 0.4% of the total citations. The value of $\theta$ changes from $2.2.10^{-4}$ to $2.5.10^{-5}$. The adjustment is good both for ours and Tsallis model.
We found that our approach considering the softening of long range interactions, as well as Tsallis approach considering cross-over behavior (Equation 9), gives almost the same results and can explain the entire empirical curve including deviations for small and very large steps. Thus, the model presented in this work is an alternative approach. The present approach is interesting as parameter $\theta$ is related to size limitations of the system and thus can be more informative.

IV. Distribution of an Entrance Examination Scores

Recently we studied the statistical distribution of the student’s performance, which is measured through their marks, in the university entrance examination (Vestibular) of UNESP (Universidade Estadual Paulista) in Brazil, for the years 1998, 1999 and 2000. To our surprise, we observed long ubiquitous power law tails in place of normal distribution in physical and biological sciences [29, 35]. In humanities we have almost normal distribution. These power law tails in physical and biological sciences exist independently of economical, teaching, and study conditions [29]. This shows that the power law tails are due to the nature of the subject itself. These observations are interesting as they treats education as a complex system and bring out the relative importance of the different factors on science and mathematics education at high school level, which is today, an issue of great concern in our society.

In our earlier works [29], we took marks of the students in a block of subjects, i.e. physics, chemistry and mathematics together for physical sciences and physics, chemistry and biology together for biological sciences. Thus it is not possible to make a detailed quantitative analysis. Further these are optional subjects for the examination. Thus the student’s interest for a particular area is also an important factor.

To confirm our observations, in the present work, we analyze the statistical distribution of the marks obtained by students in individual subjects i.e. in Physics, Mathematics, and Portuguese as native language in the Air Force Academy entrance examination in Brazil. These students don’t have any special interest for any of these subjects as they like to have a military career in the Air Force. Thus the statistical distribution can give better information about the peculiar nature of each subject at high school level.
Physics and Mathematics are areas of systematic study and depend much on regular study. To understand a chapter, the students need to know the material given in the earlier chapters. A student who understands well the first chapter has better conditions to understand the second and subsequent chapters. The one, who didn’t understand the first chapter, will find many difficulties in understanding the subsequent chapters. This gives a kind of positive feedback or long term memory effect, the reason behind power law [8, 31]. In case of Portuguese, being a native language, each chapter is more or less independent. Further being native language they learn Portuguese in a natural way. Although in Portuguese there is also some dependence of understanding earlier materials, it is not as strong as in Physics and Mathematics.

Figure 6 – Distribution of marks obtained by students in log-log scale in Physics for years 2003 to 2006 considering together.
We compare the marks obtained by the students in Physics, Mathematics, and Portuguese as native language. To show clearly the validity of power law we plotted log (frequency) vs. log (marks). In figure 6, we compare the distribution for Physics with present and Tsallis approach for all the years 2003 to 2006 together to have a better idea of the distribution. The distribution for an individual year, i.e. 2003, 2004, 2005 or 2006 is almost the same as all the years together. The parameters of distribution are: $N_0 = 4350; \lambda = 0.09$ and $q_0 = 1.4$. $\theta = 1.265.10^{-2}$ and $i = 12$ for our model and $\mu_r = 0.006$, and $r = 1$ for Tsallis model. In figure 7, we did the same for Mathematics. The parameters of distribution are: $N_0 = 5250; \lambda = 0.1$; and $q_0 = 1.3$. $\theta = 1.265.10^{-2}$ and $i = 9$ for our model and $\mu_r = 0.01$, and $r = 1$ for Tsallis model. We shifted the origin axis to $x_m$, i.e. we use $(x - x_m)$ in place of $x$ both in Equations (9) and (12), where $x_m$ is the mark for maximum frequency. We took $x_m = 38$ for Physics and 39 for Mathematics. We observe that the distribution in cut-off region is better given through present approach. In figure 8, we compare the distribution of marks obtained in Portuguese with Normal distribution and found that Normal distribution explains the distribution satisfactory. The parameters for Normal distribution are: $N_0 = 148700$, $\mu = 52.64$ and $\sigma = 26.19$. 


Figure 7 – Distribution of marks obtained by students in log-log scale in Mathematics for years 2003 to 2006 considering together.
Figure 8 - Distribution of marks obtained by students in Portuguese as native language in log-log scale for years 2003 to 2006 considering together.

V. Discussion

In case of scientist’s citation index, there is no visible limit and therefore the cut-off is slower and can be explained both through Tsallis or present approach. In case of examination score there is a visible limit. No one can get more than the maximum marks. This makes the size limiting very strong and gives a sharp cut-off. As size limiting exists in all real systems, the present approach is appropriate.

In conclusion, in the present paper, we presented a statistical distribution considering that the entropic index \((q - 1)\), which gives information about long range interactions and/or memory effects, decreases with step size. This distribution gives a power law in the central part and deviates for very small
and very large steps as really observed in most of the complex systems and thus can explain the entire distribution. This distribution is interesting as it eliminates the necessity of truncating power law phenomenologically [33].

Acknowledgments

We are thankful to an anonymous referee for useful suggestions.
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Normal distribution

Empirical points