Homogeneous and Isotropic Spacetime in Conformal Scalar-Tensor Gravity

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The background field equations for homogeneous and isotropic spacetime are derived in conformal scalar-tensor gravity. The background temporal evolution of the scalar field, i.e. particle masses, and satisfies an equation which is identical in form to the Friedmann equation of the standard cosmological model in general relativity. In a static background spacetime the scalar field (logarithmic) time-derivative replaces the ‘Hubble function’. It is also shown that linear perturbations are governed by equations which are identical to those obtained in general relativity, but with their evolution stemming from the scalar field dynamics.

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I. INTRODUCTION

A non-vacuum homogeneous and isotropic spacetime is described in general relativity (GR) by the dynamical Friedmann-Robertson-Walker (FRW), which can be either expanding or contracting. In this theoretical framework redshift of light from distant objects is commonly observed, and cosmic evolution in general, are driven by the temporal evolution of the monotonically increasing masses of fundamental particles.

We work in a mostly-positive metric signature convention, and adopt units such that $\hbar = c = 1$. Our theoretical approach is outlined in section II, and the model describing a maximally-symmetric static spacetime is presented in section III, and summarized in section IV.

II. THEORETICAL FRAMEWORK

The theoretical framework adopted here is formally in the class of generalized Brans-Dicke (BD) theories where the matter lagrangian explicitly depends on the scalar field, (as in Bergmann-Wagoner gravity [2, 3]), i.e. both the effective gravitational ‘constant’ $G$ and particle masses are spacetime-dependent. Specified to inertial/gravitational phenomena, the action described in [1] reduces to

$$\mathcal{I}_{CSM} = \int \left[ \frac{1}{6} (|\phi|^2 R + \phi^* \phi - \lambda |\phi|^4 + \mathcal{L}_M(\phi, \phi^*, g_{\mu\nu}) \right] \sqrt{-g} d^4 x.$$  

Here, $\phi$ and $g_{\mu\nu}$ are the scalar and metric fields respectively, $\lambda$ is a dimensionless self-coupling parameter, the ordinary derivative of $\phi$ with respect to $x^\mu$ is denoted $\phi_\mu \equiv \phi_{,\mu}$. The lagrangian density of the non-pure-scalar field sector, $\mathcal{L}_M$, is assumed to depend on $\phi$ via Yukawa-like couplings of the form $\lambda_{\mu, i} \psi \phi \psi + \text{h.c.}$, where $\lambda_{\mu,i}$ is the dimensionless Yukawa-like coupling of the $i$th particle species, i.e. particle masses are proportional to $|\phi|$, in analogy with the standard model of particle physics. This lagrangian describes gravitational and inertial phenomena [1], indeed the main interest of the current work. Eq. (1) is conformally-invariant under local re-scaling, $\phi \rightarrow \phi / \Omega^{-1}$ and $g_{\mu\nu} \rightarrow g_{\mu\nu} \Omega^2$, where $\Omega(x)$ is an arbitrary function of spacetime. The action for a pointlike massive particle $\mathcal{L}_p \propto |\phi| \int g_{\mu\nu} \frac{dx^\mu}{\Omega} \frac{dx^\nu}{\Omega} d^4 x$ is similarly invariant under this transformation, where $\xi$ is the affine parameter.

Variation of Eq. (1) with respect to $g_{\mu\nu}$ and $\phi^*$ results in the generalized Einstein equations, and scalar field equation, respectively,

$$\frac{1}{3} |\phi|^2 G^\mu_\nu = T^\mu_\nu M - \lambda \delta^\mu_\nu |\phi|^4$$ \hspace{0.5cm} (2)

$$\frac{\phi R}{6} - \Box \phi - 2\lambda |\phi|^2 \phi + \frac{\partial \mathcal{L}_M}{\partial \phi^*} = 0,$$ \hspace{0.5cm} (3)

and the generalized energy momentum (non-) conservation then follows, e.g. [4, 5]

$$T^\mu_\nu_{M,\mu\nu} = \mathcal{L}_M,\phi \phi^* + \mathcal{L}_M,\phi^* \phi^*.$$ \hspace{0.5cm} (4)

The effective energy-momentum tensor $\Theta^\mu_\nu$ associated with the scalar field is

$$\Theta^\mu_\nu = \frac{1}{3} \delta^\mu_\nu (\phi^* \Box \phi + \phi \Box \phi^* - \phi^*_\mu \phi^\mu) + \frac{1}{3} (2 \phi^*_\mu \phi^\mu + 2 \delta^\mu_\nu \phi^* \phi^\nu - \phi^*_\mu \phi^\mu - \phi^*_\nu \phi^\nu).$$ \hspace{0.5cm} (5)

Here and throughout, $f^\nu_\mu \equiv (f_\mu)^\nu$, with $f_\mu$ denoting covariant derivatives of $f$, the covariant Laplacian is $\Box f$, and $(T_M)_{\mu\nu} \equiv -\frac{1}{2} \frac{\delta \sqrt{-g} \mathcal{L}_M}{\delta g^\mu_\nu}$ is the energy-momentum tensor. Eq. (5), which is not independent of (2) & (3), implies that energy-momentum (of matter alone) is generally not conserved, which is expected in the case that $\Lambda$ or particle masses are spacetime-dependent. Finally,
combining Eq. (3) with the trace of Eq. (2), we obtain the following consistency relation

$$\phi \frac{\partial L_M}{\partial \phi^a} + \frac{\partial L_M}{\partial \phi} = T_M.$$  \hfill (6)

### III. BACKGROUND EQUATIONS AND LINEAR PERTURBATIONS

We describe the background evolution and the evolution of linear perturbations in a homogeneous and isotropic static background spacetime and show equivalence with the standard dynamics of the corresponding FRW solution in GR and its linear perturbations.

#### A. Evolution of the Background Fields

The background evolution of the scalar field (to which particle masses are proportional [1]) is described on a static maximally-symmetric background spacetime. The Einstein tensor components $G^i_{ij}$, associated with the metric $g_{ij} = \text{diag}(-1, r^2, r^2, r^2 \sin^2 \theta)$, with conformal rather than cosmic time, are $G^0_i = -3K$ and $G^1_i = -K\delta^1_i$. Here, 'i, j' indices stand for the spatial coordinates.

The energy-momentum tensor of a perfect fluid is $(T_M)_{\mu \nu} = \rho_M \cdot \text{diag}(-1, w_M, w_M, w_M)$, where $w_M = \rho_M/\rho_M$ is the equation of state (EOS) describing a matter with energy density $\rho_M$ and pressure $P_M$. In case that $L_M = -\rho_M$ (i.e., that $L_M$ does not explicitly depend on the scalar field derivatives) then it immediately follows from Eq. (6) that $\rho_M \propto |\phi|^{1-3w_M}$ where $w_M \neq 0$. The case $w_M = 0$ is treated separately below. As expected, $\rho_M$ is a quartic potential in the case $w_M = -1$, and is independent of $|\phi|$, i.e. of masses, in the case $w_M = 1/3$. In the special case of nonrelativistic (NR) fermions ($w_M = 0$) that are described by the Yukawa-like term implicitly appearing in $L_M$ [1], it readily follows from Eq. (6) in flat and static background metric, i.e. $\rho_M' = \lambda g_{ij}\psi\phi \phi^* + h.c. = \frac{1}{2}\rho_M (\phi' \phi^* + h.c.)$ where $\psi \phi^* = \text{constant}$ (since $\psi$ is proportional to the number density of NR fermionic particles in static space), that $\rho_M \propto |\phi|$. Together with the result $\rho_M \propto |\phi|^{1-3w_M}$ for $w_M \neq 0$ we conclude that $\rho_M \propto |\phi|^{1-3w_M}$ for any $w_M$. Here, $f' \equiv \frac{df}{\eta}$ is the derivative of a function $f$ with respect to conformal time $\eta$.

In the following we consider a real scalar field, i.e. $\phi = |\phi|e^{i\theta}$, with $\theta = 0$, and for notational simplicity we denote the scalar field modulus $\phi$ where it is clear that $\theta > 0$. In the full cosmological model [6] we relax this and indeed show that the phase $\theta$ plays a crucial role. Using $\rho_M \propto \phi^{1-3w_M}$ Eqs. (3) become

$$Q^2 + K = \frac{\rho_M}{\phi^2},$$  \hfill (7)

$$2Q' + Q^2 + K = -\frac{3w_M \rho_M}{\phi^2},$$  \hfill (8)

where we used $Q \equiv \phi'/\phi$. In comparison, the standard Friedmann equation reads $H^2 + K = \rho_M/\phi^2$, with $H \equiv \phi'/a$, and $\rho_M \propto a^{-3(1+w_M)}$, i.e. the right hand side of the Friedmann equation is $a^{-3-3w_M}$.

This is analogous to the term $\rho_M/\phi^2 \propto \phi^{1-3w_M}$ appearing on the right hand side of Eq. (7) when $\phi$ is replaced with $\phi(\eta)$, i.e. $H$ is replaced by $Q$ throughout. Consequently, Eqs. (7) & (8) are identical in form to the Friedmann equations in conformal time coordinates. In other words, the relation between $\eta$ and the energy-density, i.e. the distance-redshift relation, is unchanged compared to standard cosmology. We thus obtained an equivalent description to that of the standard cosmological model in conformal scalar-tensor gravity, at least at the background level.

The energy-momentum tensor of matter itself is known to be generally non-conserved in scalar-tensor theories of gravity, as is indeed evident from Eq. (4). By virtue of the formal equivalence to the Friedmann equation in the standard cosmological model, the conserved quantity in the homogeneous and isotropic background is $\rho_M/\phi^4$. Another way to see this is to consider the continuity equation, which reads $\rho_M + 3(1+w_M)H\rho_M = 0$ in FRW spacetimes. The analogous (non-) conservation equation in a static background model is Eq. (4), which yields $(\frac{\rho_M}{\phi^2})' + 3(1+w_M)Q(\frac{\rho_M}{\phi^2}) = 0$, and $\rho_M \propto \phi^{1-3w_M}$ is recovered. This is analogous to the relation obtained from the Friedmann equation, $\phi(\eta) \propto \eta^{2/(1+3w_M)}$.

In the redshifting universe $\phi$ is a monotonically increasing function of conformal time for any $w_M > -1/3$. Thus, unlike in the standard FRW spacetime, the observed cosmological redshift is not due to stretching of photon wavelengths in an expanding background, but rather due to evolving particle masses, e.g., due to the evolving Rydberg ‘constant’ on a static background space. What is normally considered a static object in standard cosmology, e.g. a galaxy, star, etc., is replaced in the present work by a stationary solution. More specifically, Weyl transformation is applied to any static solution $\phi(r) \& g_{\mu\nu}(r)$ with $\Omega(\eta) \propto \eta^{2/(1+3w_M)}$ so as to guarantee that particle masses transform continuously between the background and objects residing in it, e.g. galaxies, stars, etc, exactly as (fixed) masses (trivially) transform continuously in standard cosmology between expanding background space and compact objects; mass is fixed in standard cosmology by construction. Thus, the ratio between inertial and metric length scales, $\phi^{-1}/\sqrt{g_{\mu\nu}}$, is still static. From this perspective, observers in shrinking galaxies and stars residing in a static background would observe a redshifting universe.
B. Linear Perturbation Theory

As in section III.A, we assume an effective single fluid with matter density \( \rho_M \) and a (generally time-dependent) \( w_M = w_M(\eta) \). In the following we show that the peculiar velocity of matter, \( v \), and the effective \( \alpha, \varphi, \delta_{\rho_M}, \) and \( \delta_{P_M} \), evolve in the same fashion as in standard cosmology. Here, \( \varphi \) and \( \alpha \) are the Newtonian and curvature gravitational potentials, respectively, appearing in the perturbed FRW line element \( ds^2 = \varphi^2 [(1 + 2\alpha)d\eta^2 + (1 + 2\varphi)\gamma_{ij}(dx^idx^j)] \) where \( \gamma_{ij} \equiv diag(1/(1-Kr^2), r^2, r^2 \sin^2 \theta) \), and Latin indices here run over space coordinates. The energy density and pressure perturbations in energy density units are \( \delta_{\rho_M} \equiv \delta\rho_M/\rho_M \) and \( \delta_{P_M} \equiv \delta P_M/\rho_M \), respectively. The anisotropic stress is also redefined, \( \pi(s) \to \phi^2 \pi(s) \).

Next we write down the dynamical equations governing the evolution of metric and matter perturbations in the shear-free gauge. The linear order scalar perturbation equations are summarized in [4]. Defining the ‘shifted’ \( \rho \) and \( \phi \) of \([4]\) we obtain that
\[
\rho \to \rho - \delta\rho_M \quad \phi \to \phi - \delta\phi.
\]
We thus see not only that the background field equations obtained in the framework described by Eq. \((1)\) and applied to a maximally-symmetric spacetime are equivalent to the FRW dynamics of standard cosmology, but also that their perturbations are. This establishes the equivalence between the two approaches.

Vector and tensor perturbations are described by Eqs. \((53)-(55)\) and \((58)\) of \([4]\), respectively, and the equivalence between the models is straightforward to show in this case, provided that \( a \to \phi \) and \( \rho_M \to \rho_M/\phi^4 \) and \( \pi(s) \to \pi(s)/\phi^4 \), where \( \pi(s) \) and \( \pi(t) \) are the stresses associated with vector and tensor mode perturbations, respectively.

The Newtonian limit of Conformal scalar-tensor Gravity is obtained from Eqs. \((9)-(14)\) by setting \( Q, K, \) and \( w_M \) to 0. In particular, Eqs. \((9), (13) \& (14)\) are the Poisson, continuity, and Euler equations, respectively.

IV. SUMMARY

The background field equations and their linear perturbations in maximally-symmetric spacetimes are described within the framework of conformal scalar-tensor gravity, and are found to be equivalent in form to the Friedmann equation and linear perturbation equations as obtained in the standard cosmological model. The only difference between the two formulations is that in the former \( Q = \phi/\alpha \) replaces \( H = d'/a \phi \) of the latter, and shifted perturbation quantities involving fractional perturbations of the scalar field replace the standard perturbation quantities. In particular, this implies that space expansion is replaced by the scalar field evolution (i.e. mass evolution of fundamental particles) on a maximally-symmetric static background spacetime. We emphasize that \( G \), the hallmark of gravity, is absent from the background field equations Eqs. \((7)-(8)\). This is consistent with the fact that the background spacetime is characterized by a vanishing Weyl tensor and thus the background evolution on cosmological scales is an inertial rather than gravitational phenomenon \([1]\).

Our reformulation of the background field equations and linear perturbations reproduces the standard results for the observables associated with cosmic evolution, e.g. cosmological redshift, cosmological distances, linear metric perturbations; likewise, the evolution of gravitationally bound objects (dubbed ‘halos’ in the cosmological context), is identical to that of standard cosmology, but on a static rather than evolving background space (and with evolving particle masses). Allowing for particle masses to evolve generally results in energy-momentum non-conservation, yet we reproduce the continuity equation and its perturbation \((14)\) for the rescaled energy density \( \rho_M/\phi^4 \). In particular, the observed cosmological redshift is explained on a static background spacetime (with non-evolving wavelength) by evolving masses, i.e. Rydberg ‘constant’. The initial curvature singularity af-
flicting the standard expanding FRW spacetime at $\eta = 0$ is replaced by vanishing particle masses at that time. which were invaluable for this work. This work has been supported by a grant from the the JCF (San Diego, CA).

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