Temperatures of extremal black holes

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Abstract

The temperature of an extremal Reissner - Nordstrom black hole is not restricted by the requirement of absence of a conical singularity. It is demonstrated how Kruskal-like coordinates may be constructed corresponding to any temperature whatsoever. A recently discovered stringy extremal black hole which apparently has an infinite temperature is also shown to have its temperature unrestricted by conical singularity arguments.

A classical black hole has a horizon beyond which nothing can leak out. This suggests that it can be assigned a zero temperature. But the relation between the area of the horizon and the mass and other parameters like the charge indicates a close similarity with the thermodynamical laws, thus allowing the definition of a temperature. This analogy was understood as being of quantum origin and made quantitative after the discovery of Hawking radiation. The associated Hawking temperature vanishes only in the classical limit. The thermodynamics of black holes has been extensively studied since then.

Most of the studies were first made for the simplest kind of black hole, viz., the Schwarzschild spacetime. Of more recent interest is the case of the so-called extremal black

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holes which have peculiarities not always present in the corresponding non-extremal cases \cite{3,4}. For extremal Reissner - Nordstrom black holes, the na"ively defined temperature is zero, but the area, which is usually thought of as the entropy, is nonzero. For extremal dilatonic black holes, where the temperature is not zero, the area vanishes.

In this note we shall reexamine the temperature of an extremal Reissner - Nordstrom black hole, for which the definition in terms of the surface gravity leads to a zero temperature. We discuss the conical singularity approach in detail: there is no conical singularity in the extremal case, so that there is no constraint on the temperature from this point of view. Thereafter we consider Kruskal-like coordinates and show that they may be constructed for an arbitrary temperature.

We also comment on some stringy black holes. One of them has an infinite temperature in the surface gravity approach. We show however that there is no conical singularity in this case.

**REISSNER- NORDSTROM BLACK HOLES**

The metric of the Reissner - Nordstrom spacetime is given by

\[
ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{1}
\]

in general, with \(M\) and \(Q\) denoting the mass and the charge respectively. There are apparent singularities at

\[
r_{\pm} = M \pm \sqrt{M^2 - Q^2} \tag{2}
\]

provided \(M \geq Q\). Cosmic censorship dictates that this inequality holds and then there is a horizon at \(r_+\). The limiting case when \(Q = M\) and therefore \(r_+ = r_- (= M)\) is referred to as the extremal case. Whereas the case with \(Q < M\) is qualitatively similar to a Schwarzschild black hole, it is clear that the horizon in the extremal case will behave differently: the metric singularity becomes stronger here.
The best known method of calculating the temperature of a black hole is through the
relation with surface gravity. To distinguish this temperature from those arising in other
approaches, we may call it the Unruh temperature. Here

\[
T = \frac{1}{2\pi} \frac{1}{\sqrt{g_{rr}}} \left. \frac{d\sqrt{-g_{tt}}}{dr} \right|_{r=r_+} \\
= \frac{1}{2\pi} \sqrt{(1 - \frac{r_+}{r})(1 - \frac{r_+}{r})} \left. \frac{d\sqrt{(1 - \frac{r_+}{r})(1 - \frac{r_+}{r})}}{dr} \right|_{r=r_+} \\
= \frac{r_+ - r_-}{4\pi r_+^2}.
\]  

(3)

Now this expression vanishes in the extremal limit where \( r_+ = r_-. \) As the properties of
extremal black holes are qualitatively different from those of non-extremal ones, it is better
to calculate the temperature afresh in the extremal case. One finds

\[
T = \frac{1}{2\pi} \frac{1}{\sqrt{g_{rr}}} \left. \frac{d\sqrt{-g_{tt}}}{dr} \right|_{r=M} \\
= \frac{1}{2\pi} (1 - \frac{M}{r}) \left. \frac{d(1 - \frac{M}{r})}{dr} \right|_{r=M} = 0.
\]  

(4)

Thus the result is the same as is obtained through the limiting procedure. This means that
the surface gravity is continuous in this limit.

Next we consider the question of a conical singularity on passing to imaginary time. The
metric

\[
ds^2 = dr^2 + r^2 d\theta^2,
\]  

(5)

which describes the flat Euclidean metric in polar variables, can be supposed to describe
distances on the surface of a cone. The cone has a singularity at its tip \( r = 0, \) except in the
limiting case when the cone opens out as a plane. In this situation \( \theta \) has a periodicity \( 2\pi, \)
so one may say that the conical singularity is avoided by making \( \theta \) an angular variable with
this period. This is relevant for black holes because such a singularity tends to arise in the
Schwarzschild and in the non-extremal cases. In the latter case, one passes to imaginary
time and writes the metric for a constant \( \theta, \phi \) surface as
\[ ds^2 = (1 - \frac{r^+}{r})(1 - \frac{r^-}{r})dt^2 + \frac{dr^2}{(1 - \frac{r^+}{r})(1 - \frac{r^-}{r})} \]

\[ = \Omega(\rho)(d\rho^2 + \rho^2 d\tau^2), \quad (6) \]

where \( \tau = \alpha t \) with the constant \( \alpha \) so chosen as to make the conformal factor \( \Omega \) finite at the horizon. For consistency, one requires

\[ \rho = e^{\alpha r_*}, \quad (7) \]

where \( r_* \) is defined by

\[ dr_* = \frac{dr}{(1 - \frac{r^+}{r})(1 - \frac{r^-}{r})}. \quad (8) \]

Near the horizon in the non-extremal case,

\[ r_* \approx \frac{r^2}{r^+ - r^-} \log(r - r^+), \quad (9) \]

so that

\[ \rho \approx (r - r^+)^{\alpha r_*^2/(r^+ - r^-)}, \quad (10) \]

which implies that \( \rho \) vanishes at the horizon, and

\[ \Omega = \frac{(1 - \frac{r^+}{r})(1 - \frac{r^-}{r})}{\alpha^2 \rho^2} \quad (11) \]

can be made finite at the horizon by making \( \rho^2 \) vanish linearly as \( r \to r^+ \), i.e., by choosing \( \alpha \) to satisfy

\[ \frac{\alpha r_*^2}{r^+ - r^-} = \frac{1}{2}. \quad (12) \]

Now for the conical singularity to be avoided, one must have a periodicity of \( 2\pi \) for \( \tau \), i.e., a periodicity for \( t \) given by \( \frac{2\pi}{\alpha} \). This corresponds to a temperature

\[ T = \frac{\alpha}{2\pi} = \frac{r^+ - r^-}{4\pi r^+_+}, \quad (13) \]

which is the standard non-extremal result given above. Thus for a non-extremal black hole, what may be called the conical temperature agrees with the Unruh temperature.
In the extremal case, things are very different. Near the horizon, now,

$$r_\ast \approx -\frac{M^2}{r - M},$$  

(14)

so that $\rho$ has an essential singularity as $r \to r_+$, and there is no value of the constant $\alpha$ which can make the conformal factor $\Omega$ regular at the horizon. This simply means that the extremal metric is not of the form (3) and there is no question of a conical singularity. Consequently, the derivation of a conical temperature fails. As far as this approach is concerned, one may say that the temperature has no specific constraint of periodicity to satisfy, and is therefore arbitrary in this extremal case [3,4].

We come finally to the question of an analogue of Kruskal coordinates for these black holes: the new coordinates have to be such that the metric components are nonsingular at the horizon. This topic may seem unconnected to our main interest, which is the temperature, but the Kruskal vacuum plays an important rôle in the theory of Hawking radiation and hence in the idea of the Hawking temperature [2]. One first writes the metric of a surface with constant $\theta, \phi$ in the double null form

$$ds^2 = -(1 - \frac{r_+}{r})(1 - \frac{r_+}{r})(dt^2 - dr_\ast^2)$$

$$= -(1 - \frac{r_+}{r})(1 - \frac{r}{r})dv dw$$  

(15)

with

$$v = t - r_\ast, \quad w = t + r_\ast.$$  

(16)

Here $r$ is understood to be implicitly defined by using $w - v = 2r_\ast$ and the relation between $r_\ast$ and $r$. One passes to new null coordinates $\bar{v}$ and $\bar{w}$ defined by

$$v = f(\bar{v}), \quad w = g(\bar{w}),$$  

(17)

with appropriate functions $f, g$. The metric becomes

$$ds^2 = -(1 - \frac{r_+}{r})(1 - \frac{r}{r})\frac{df}{d\bar{v}}\frac{dg}{d\bar{w}}d\bar{v}d\bar{w}$$  

(18)
and $r$ is understood to be determined implicitly by
\[ g(\bar{w}) - f(\bar{v}) = 2r_*. \] (19)

The functions $f, g$ are to be chosen in such a way that the coefficient of $d\bar{v}d\bar{w}$ in the right hand side of (18) is regular at the horizon.

In the non-extremal case, the choice of the new coordinates is essentially the same as in the Schwarzschild case. The horizon corresponds to $r_\ast \to -\infty$ and therefore either $v$ or $-w$ has to be infinite. The new coordinates are defined by
\[ v = f(\bar{v}) = -\frac{1}{\alpha} \log \bar{v}, \quad w = g(\bar{w}) = \frac{1}{\alpha} \log \bar{w}, \] (20)
with the constant $\alpha$ to be determined. This definition has the result that one of the new coordinates has to vanish at the horizon. If we consider a point where $\bar{w}$ vanishes, we see that the factor $\frac{da}{d\bar{w}} \propto \frac{1}{\bar{w}}$ in (18) becomes infinite and may make the product with $(1 - \frac{r_\ast}{r})$ finite. For this to happen, $\bar{w}$ must vanish linearly as $r \to r_\ast$. Now, in the non-extremal case, (18) indicates that
\[ \frac{1}{\alpha} \log \bar{w} \approx \frac{2r_+^2}{r_+ - r_-} \log(r - r_+), \] (21)
so that the condition for linearity of $\bar{w}$ is the same as (12). This fixation of $\alpha$ completes the definition of the new coordinates. We have arranged the regularity of the metric components at vanishing $\bar{w}$, but with this choice one can also check that there is no problem in the region of vanishing $\bar{v}$.

Now we have to find the temperature. A time coordinate can be defined in terms of the new null variables and a vacuum can be defined in terms of this time. Green functions corresponding to this vacuum involve the new coordinates $\bar{v}, \bar{w}$ (and $\theta, \phi$). If these Green functions can be shown to have a periodicity in the original time $t$ after rotation to the imaginary direction, the vacuum can be asserted to have a thermal character [8]. As $\bar{v} = e^{-\alpha t}, \bar{w} = e^{\alpha t}$ involve $\alpha t$, it is clear that there is an imaginary period of length $\frac{2\pi}{\alpha}$, which corresponds to the temperature found above. In other words, for a non-extremal
Reissner - Nordstrom black hole, the standard Kruskal temperature is the same as the Unruh and conical temperatures. This is not surprising because the form of the metric near the horizon of such a black hole is very similar to the Schwarzschild form.

As in the conical approach, the case of the extremal black hole is very different from the non-extremal cases. The behaviour (14) of \( r_* \) near the horizon is linear here and the coefficient in (15) has an extra power of \( (r - r_+) \). So the logarithmic transformation does not lead to metric components regular at the horizon. A transformation that does has been known for a long time [9]:

\[
v = f(\bar{v}) = M \tan \bar{v}, \quad w = g(\bar{w}) = M \cot \bar{w}.
\] (22)

It is not difficult to check the regularity of the coefficient in (18). But the new coordinates do not exhibit any periodicity in imaginary time, so that the vacuum corresponding to the time defined by the above coordinates is not thermal.

One way of having a thermal vacuum would be to use coordinates as in (20) but with \( \alpha \) undetermined. This parameter cannot indeed be determined, as no choice can make the metric components regular across the horizon. There is an incompatibility between the twin requirements of regularity and thermality. But a compromise can be made. Note that the question of regularity concerns the region near the horizon. We have found that Green functions cannot be arranged to be thermal in this region. But it should be enough to have thermal behaviour at infinity. Instead of requiring that the Green functions defined in the new vacua should have an imaginary periodicity in \( t \) everywhere, we shall impose this requirement only for the region away from the horizon.

Thus, we require our new null coordinates \( \bar{v}, \bar{w} \) to be of the form (22) near the horizon \( (v \to \infty \text{ or } w \to -\infty) \) and of the form (20) far away. These two forms can be smoothly joined in the intermediate region in many different ways, for instance by the equations

\[
\bar{v} = \tan^{-1}[\frac{v(1 + e^{u-\sigma v})}{M}], \quad \bar{w} = \cot^{-1}[\frac{w(1 + e^{u+\sigma w})}{M}],
\] (23)

where \( \sigma \) is a positive constant akin to \( \alpha \) in (20) and \( u \) is a large positive constant. It is clear that for large positive \( v \) or large negative \( w \), i.e., near the horizon, the new coordinates are
very close to those given in (22), but for negative $v$ and positive $w$, the new coordinates have the exponential dependence on $v, w$ which is characteristic of thermal behaviour. The periodicity in imaginary time is $\frac{2\pi}{\sigma}$, so that the temperature is $\frac{\sigma}{2\pi}$. This is of course arbitrary as the parameter $\sigma$ is free.

In conclusion, it may be repeated that the extremal Reissner - Nordstrom black hole does not have a unique temperature. The vacuum constructed using the time coordinate corresponding to the smooth coordinates introduced in [9] is not thermal. What we have shown is that alternative coordinates can be chosen for which the vacuum is indeed thermal and the temperature involved is completely arbitrary.

**STRINGY BLACK HOLES**

Here we consider extremal limits of some electrically charged non-rotating black hole solutions of heterotic string theory compactified on a 6-dimensional torus [10]. These black holes are characterized by the mass and a 28-dimensional charge vector (22 coming from the left hand sector and the other 6 from the right hand sector of the theory). Among the various extremal limits of these solutions we shall be interested only in those which are space-time supersymmetric and hence saturate the Bogomol’nyi mass bound. There are two such black holes which are described generically by the metric

$$ds^2 = -R^{-1/2}r^2dt^2 + R^{1/2}r^{-2}dr^2 + R^{1/2}(d\theta^2 + \sin^2\theta d\phi^2)$$

(24)

and the dilaton

$$e^\Phi = r^2/R^{1/2}$$

(25)

where $R$ is a function of $r$ only. The solutions for the 28 gauge fields are not of relevance in the following discussion. The two cases are as follows.

**a.** The first is characterized by the mass and charges

$$M = \frac{m_0}{2} \cosh \alpha, \quad \vec{Q}_L = \frac{m_0}{\sqrt{2}} \sinh \alpha \, \vec{n}, \quad \vec{Q}_R = \frac{m_0}{\sqrt{2}} \cosh \alpha \, \vec{p},$$

(26)
which saturate the mass bound

\[ M^2 = \frac{1}{2} \bar{Q}_R^2. \]  \hfill (27)

Here \( m_0 \) and \( \alpha \) are two real parameters while \( \vec{n} \) and \( \vec{p} \) are two unit vectors with 22 and 6 components respectively. The function \( R \) is given by \( R = r^2(r^2 + 2m_0 \cosh \alpha + m_0^2) \). In this case we can find a coordinate transformation which maps the line element, after passage to imaginary time and modulo the angular elements, to the form \( \text{(3)} \). As in the non-extremal Reissner-Nordstrom case there is a conical singularity at the tip of the cone which lies on the horizon \( r = 0 \). Regularity requires a particular periodicity in the time coordinate which fixes the conical temperature to be the same as the Unruh temperature of the black hole.

b. The second case is more interesting because the Unruh temperature of this black hole blows up. Here

\[ M = \frac{m_0}{2}, \quad \bar{Q}_L = \frac{m_0}{\sqrt{2}} \vec{n}, \quad \bar{Q}_R = \frac{m_0}{\sqrt{2}} \vec{p} \quad \hfill (28) \]

with

\[ M^2 = \frac{1}{2} \bar{Q}_R^2 = \frac{1}{2} \bar{Q}_L^2 \quad \hfill (29) \]

and \( R = r^2(r^2 + 2m_0r) \). One can rewrite the line element, after passage to imaginary time and with the angular part removed, in the form \( \text{(3)} \):

\[ ds^2 = R^{-1/2}r^2 dt^2 + R^{1/2}r^{-2} dr^2 \]
\[ = \Omega(\rho)(d\rho^2 + \rho^2 d\tau^2), \quad \hfill (30) \]

with \( \tau = \alpha t \) as before and \( \rho = \exp(\alpha r_\ast) \). In this case, \( r_\ast \) vanishes on the horizon (with an appropriate choice of the integration constant involved), so that \( \rho \) becomes unity, \( i.e. \), the horizon does not correspond to the tip of the cone. Furthermore, \( \Omega \) is given by

\[ \Omega^2 = \frac{r^2}{\alpha^2 r^2 \sqrt{r^2 + 2m_0 r^2}}, \quad \hfill (31) \]

which vanishes on the horizon. Thus the conical singularity does not arise here. Consequently, there is no constraint that can fix the parameter \( \alpha \) and \( t \) can have any periodicity.
As in the case of the extremal Reissner - Nordstrom black hole, there is no definite conical temperature.

We considered Reissner - Nordstrom black holes as well as some stringy ones. In cases where the naively defined temperature is finite, we found the conical singularity argument to lead to the same value. On the other hand, when the temperature defined from the surface gravity vanishes, as in the case of the extremal Reissner - Nordstrom black hole, or blows up, as in the case of the second extremal limit of the stringy black hole considered by us, there is no genuine conical singularity, and consequently no restriction on the temperature. In the case of the extremal Reissner - Nordstrom black hole, we went on to construct Kruskal-like coordinates corresponding to an arbitrary temperature, but in the case of the stringy black hole considered above, the only coordinate singularity is at $r = 0$, and there is no question of defining new coordinates.

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