Obtaining the dynamic frequency equation for the plate calculation by the Finite Element Method in the form of a classical mixed method

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Abstract. This paper studies the Algorithm of Free Frequency and Corresponding Forms of Free Oscillations Finding on the basis of the Finite Element Method in the form of Classical Mixed Method (FEM CMM). This form of FEM is alternative, with respect to the traditional travel and can be used to verify the results obtained from other numerical methods.

1. Introduction
Currently, the FEM in displacements is the most widely used method for structures’ dynamic analysis. Different approaches [1-5] that generate a consistent or diagonal matrix are used to generate the finite element mass matrix for dynamic FEA. Papers [6, 7, etc.] note that using a diagonal mass matrix in order to simplify the computation of rod systems requires increasing the number of finite elements in the system if an acceptable accuracy is to be achieved. In most cases, using a diagonal mass matrix will produce lower-than-accurate results, while a consistent matrix will return excessive values.

The convention of FEA literature is to use the so-called dynamic stiffness matrix (an FE stiffness matrix combined with a mass matrix) [8]. It is however ignored that the elements of this matrix depend on alterations in the kinematic parameters of the nodes, i.e. the finite-element node forces depend on the motion in the same nodes (FE eigen-oscillation shapes and corresponding eigen-frequencies). Attempts to take this factor into account to clarify the values of dynamic stiffness matrix elements are described in some papers [7–12].

As of lately, there has been proposed a number of new shape functions to solve the problems of dynamics [2, 3]. All of them are focused seek to reduce the number of degrees of freedom in the structure by reducing the number of finite elements, the ensemble whereof represents such structure.

The efficiency of shape functions depends on how accurately frequency-dependent real oscillation shapes of elements are described [13–21]. When using an accurate shape function, computation results are accurate as well, i.e. the error of finding eigen-frequencies, for instance, depends only on which frequency equation solution method is used. In terms of mathematics, it depends on the methods used to solve the algebraic problem of eigenvalues and eigenvectors; finding the oscillation eigenfrequencies and solving the structure stability problem are both reduced to that algebraic problem.

Papers [6, 10, 22, 23] present comparative analysis of using accurate shape function as well as FEA with static shape functions for stability problems.
Paper [6] presents such analysis for dynamic problems. The paper, while giving no reference to comparative analysis, states that such approximating shape functions are less efficient for dynamics than for static problems.

For dynamic analysis of structures, computational accuracy of using conventional FEA can be improved in two ways: by a more accurate description of the dynamic behavior of the FE model selected for such computation; or by increasing the number of finite elements.

Increasing the free-oscillation tone increases the number of finite elements necessary to obtain a solution of desired accuracy. Computation intensity increases proportionally to the value \( n^2 \), where \( n \) is the number of finite elements.

Obtaining a dynamic matrix of masses by accurately integrating the differential dynamics equations of the basic finite element, in theory, gives an opportunity to increase the size of the finite element, i.e., to reduce the number of finite elements and degrees of freedom in the finite-element structure model for a lesser computation intensity.

However, obtaining such a matrix is associated with considerable mathematical difficulties and more intensive computation. This is due to the fact that the frequency equations obtained when using dynamic mass matrices are transcendent and non-linear, and there are no programs for finding their eigenvalues.

In practice, this computation option based on dynamic mass matrices has been implemented only for flat rod systems [6–10, 23].

It should also be borne in mind that due to the complexity of algorithms using accurate (transcendent) shape functions depending on the eigen-oscillation frequencies, as well as due to the numerical implementation of such algorithms, their theoretical advantages over ordinary FEA discretization are quite insignificant in terms of computation accuracy.

This is why it is more conventional to use algorithms based on pre-determination of location in the frequency spectrum of the desired frequency [7, 9, 10].

In some cases, one can build a rather simple yet efficient algorithm for numerical solution of dynamic problems, which will use the same approximating functions for dynamic and static problems.

This research is the first to use FEA implemented as a classical combined method to develop a mathematical model of dynamic response matrix. Using this model has enabled the researcher to build an algorithm for generating and solving frequency equations, which is more efficient than FEA in motions.

2. Obtaining a dynamic frequency equation for the plate calculation

Consider the obtaining of a dynamic frequency equation for the plate calculation.

The frequency equation for the problem under consideration consists of a static response matrix and a dynamic mass matrix:

\[
\left( [D] - \lambda [m] \right) \{q\} = 0,
\]

where \( D \) is the static response matrix, \( [m] \) is the dynamic mass matrix, \( \{q\} \) - vector of the main unknowns in the basic system of EM in the form of a mixed method ( \( q \) -kinematic unknowns, \( q \) -force), \( \lambda = \omega^2 \), \( \omega \) - frequency of free oscillations.

2.1. The formation algorithm of a dynamic frequency equation for the plate calculation is based on FEM in the form of a classical mixed method

The algorithm under consideration includes several operations:

- the values of the nodal masses (the consistent mass matrix of the EM) and the corresponding inertial loads [9, 10] are found by the same algorithm used to provide the load distributed over the area of the EM to the nodal load;
- a static mass matrix is formed for each finite element [22];
– to obtain the dynamic mass matrix all nodal inertial forces are expressed through the same vector of main unknowns as the response matrices of the finite element [10, 24];
– the mass matrix formation of the entire structure is performed by the same algorithm as the formation of a static matrix for this construction [25].

2.2. The dynamic mass matrix construction for a bent rectangular element with 12 degrees of freedom
Let us consider the dynamic mass matrix construction used for the dynamic frequency equation for the thin plate calculation in more detail, using the example of a bent rectangular element with 12 degrees of freedom. The FEM CMM basic system and the numbering of main unknowns at the finite element nodes are shown in Figure 1 and 2.

The EM deflection function is approximated by an incomplete bicubic polynomial:
\[ w(x, y) = \alpha_1 + \alpha_2 x + ... + \alpha_{12} x y^3 = \begin{bmatrix} \Phi_1(x, y) \end{bmatrix} \begin{bmatrix} \alpha \end{bmatrix}, \]
where \[ \begin{bmatrix} \Phi_1(x, y) \end{bmatrix} = \begin{bmatrix} 1, x, y, x^2, y^2, xy, x^2y, x^3, y^3, x^3y, xy^3 \end{bmatrix}, \] \[ \{\alpha\} = \begin{bmatrix} \alpha_1, \alpha_2, ..., \alpha_{12} \end{bmatrix}^T. \]

Next, consider the algorithm for mass matrix formation of the finite element as a vector function of the mixed method’s main unknowns.
The Figure 3 shows the plate elementary volume.
The elementary mass of this volume \( dm = (\bar{m})(dx dy) \), where \( \bar{m} \) - is mass per volume unit.

Let's introduce the following notation:
\( I_{xy}^{(s)} \) is the inertia moment of a cross section of the elementary finite element \( dv \) about the y-axis (Figure 4), \( I_{xy}^{(s)} \) is the inertia moment of a cross section about the x-axis (Figure 5), \( (\rho I_{xy}^{(s)}) dx dy \) is the elementary mass moment of inertia, creating a rotation around the y axis and the xoz in-plane bending, \( (\rho I_{xy}^{(s)}) dx dy \) is the elementary mass moment of inertia, creating a rotation about the x axis and y o z in-plane bending, \( \rho \) is the plate material density.

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Figure 4. The inertia moment of a cross-section of the elementary finite element $dv$ about the $y$ axis.

Figure 5. The inertia moment of a cross-section of the elementary finite element $dv$ about the $x$-axis.

The nodal mass values (consistent mass matrix) will be obtained by the same algorithm, which is used to provide the load distributed over the area of the EM nodal load, i.e. equating the possible work of distributed loads and possible work of node inertial loads.

In considering the plate free oscillations, let's represent the function of possible vertical travel as a product of two functions, one of which depends only on the coordinates $x$, $y$ and is approximated by the same function as in the static bending of the EM, and the other depends only on time:

$$w(x, y, t) = w(x, y) f(t).$$

(1)

In this case, the elementary inertial force in the direction of the axis $z$

$$dF_z = -\left(\bar{m} \cdot dx \cdot dy\right) w(x, y) \frac{d^2 f(t)}{dt^2}.$$  

(2)

Similarly

$$dF_{p,x} = -\left(\rho I_x^{(i)} dx dy\right) \varphi_x^{(i)}(x, y) \frac{d^2 f(t)}{dt^2},$$

$$dF_{p,y} = -\left(\rho I_y^{(i)} dx dy\right) \varphi_y^{(i)}(x, y) \frac{d^2 f(t)}{dt^2}.$$  

Then the possible work of these elementary inertial forces over the entire area of the finite element:

$$\delta A_{pump} = -\int_0^a \int_0^b \left( \bar{m} \cdot w^2(x, y) + \rho I_x^{(i)} \left(\varphi_x^{(i)}(x, y)\right)^2 + \rho I_y^{(i)} \left(\varphi_y^{(i)}(x, y)\right)^2 \right) dx dy \frac{d^2 f(t)}{dt^2}.$$ 

(3)

Substituting in this equation of the $w(x, y)$, $\varphi_x(x, y)$, $\varphi_y(x, y)$ functions we get:

$$\delta A_{pump} = -\left\{\delta q\right\}^T \int_0^a \int_0^b \begin{bmatrix} \bar{m} & 0 & 0 \\ 0 & \rho I_x & 0 \\ 0 & 0 & \rho I_y \end{bmatrix} \left[\Phi\right]^T \left[\Phi\right] \cdot \left\{q\right\} dx dy \frac{d^2 f(t)}{dt^2},$$  

(4)
where \( \Phi = \begin{bmatrix} \Phi_1(x,y) & 0 & 0 \\ 0 & \frac{\partial}{\partial x} \Phi_1(x,y) & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \Phi_1(x,y) \end{bmatrix} \).

Let’s represent the consistent mass matrix in accordance with the accepted numbering of the main unknowns at the nodes of the finite element for the response matrix (Figure 2):

\[
[m] = \begin{bmatrix}
m_{11} & m_{12} & \cdots & m_{1,12} \\
m_{21} & m_{22} & \cdots & m_{2,12} \\
& & \cdots & \cdots \\
m_{12,1} & m_{12,2} & \cdots & m_{12,12}
\end{bmatrix},
\]

The linear displacements and node rotation angles of a finite element correspond to these masses:

\[
\begin{align*}
\bar{q}_1 &= \sum_{i=1}^{12} w_{i,k}^{(k)}(0,0) \\
\bar{q}_2 &= \sum_{i=2,3} w_{i,k}^{(k)}(0,0) \\
\bar{q}_3 &= \sum_{i=4,5,6} w_{i,k}^{(k)}(0,0) \\
\cdots & \cdots \\
\bar{q}_{10} &= \sum_{i=11,12} w_{i,k}^{(k)}(0,b) \\
\bar{q}_{11} &= \sum_{i=2,3} w_{i,k}^{(k)}(0,b) \\
\bar{q}_{12} &= \sum_{i=4,5,6} w_{i,k}^{(k)}(0,b) \\
\bar{q}_4 &= \sum_{i=7,8,9} w_{i,k}^{(k)}(0,b) \\
\bar{q}_5 &= \sum_{i=10,11,12} w_{i,k}^{(k)}(0,b) \\
\end{align*}
\]

\[
\begin{align*}
\Phi_2(x,y) &= -\left[ D_\kappa \cdot \frac{\partial^2}{\partial x^2} \Phi_1(x,y) + D_\mu \cdot \frac{\partial^2}{\partial y^2} \Phi_1(x,y) \right] \\
\Phi_3(x,y) &= -\left[ D_\kappa \cdot \frac{\partial^2}{\partial x^2} \Phi_1(x,y) + D_\mu \cdot \frac{\partial^2}{\partial y^2} \Phi_1(x,y) \right].
\end{align*}
\]

At this point, the index \( (k) \) is the unknown number in the basic system of the finite element, from the effect, which the \( w_{i,k}^{(k)}, \phi_{x,k}^{(k)}, \phi_{y,k}^{(k)} \) responses occur in all the FE nodes.

The possible work of nodal inertial forces on the corresponding nodal travel (4.34) is determined by the equation:

\[
\delta [A_{\text{in}}] = -[F] \cdot [q] \cdot \frac{d^2 f(t)}{dt^2},
\]

\[
[F] = \delta [q]^T [m].
\]
\[
[q] = \begin{bmatrix}
q_{1,1} & q_{1,2} & \cdots & q_{1,12} \\
q_{2,1} & q_{2,2} & \cdots & q_{2,12} \\
\vdots & \vdots & \ddots & \vdots \\
q_{12,1} & q_{12,2} & \cdots & q_{12,12}
\end{bmatrix}.
\]

Here \( q_{i,j} \) is displacement in the direction 1 (z axis) resulting from the effects of the main unknowns with the number \( i: 1 - w_z, (0,0), 2 - w_z(a,0), \ldots, 5 - M_z, (0,0), 6 - M_y, (0,0) \), and so on.

Substituting in (7) the equation (8), we obtain:

\[
\delta[A_{yy}] = - \delta[q]^T [m] [q] \frac{d^2 f(t)}{dt^2}. \tag{9}
\]

Comparing the equations (9) and (6), we find:

\[
[m] = \int_0^b \int_0^a \begin{bmatrix}
\bar{m} & 0 & 0 \\
0 & \rho I_x & 0 \\
0 & 0 & \rho I_y
\end{bmatrix} \phi^T \phi \, dx \, dy. \tag{10}
\]

The mass matrix assembly \([m]\) of the whole structure is performed by the same algorithm as the response matrix assembly \([D]\) of the entire structure.

3. Conclusions
The high efficiency of the realized algorithm is shown by the performed comparative calculations.

Almost complete coincidence was demonstrated by comparing the results of numerical calculations using the FEM algorithm in the form of a classical mixed method with the results of the FEM analysis of the displacements using a software application "Lira".

The algorithm advantage is especially convincing when calculating forced oscillations, where the most interesting are the forces entering into the resolving equation system as unknowns.

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