Partitioning a Symmetric Rational Relation into Two Asymmetric Rational Relations

Stavros Konstantinidis\textsuperscript{1}, Mitja Mastnak\textsuperscript{1}, and Juraj Šebej\textsuperscript{1,2}

\textsuperscript{1} Saint Mary’s University, Halifax, Nova Scotia, Canada, 
\texttt{s.konstantinidis@smu.ca, mmastnak@cs.smu.ca}

\textsuperscript{2} Institute of Computer Science, Faculty of Science, P. J. Šafárik University, Košice, Slovakia \texttt{juraj.sebej@gmail.com}

Abstract. We consider the problem of partitioning effectively a given symmetric (and irreflexive) rational relation \( R \) into two asymmetric rational relations. This problem is motivated by a recent method of embedding an \( R \)-independent language into one that is maximal \( R \)-independent, where the method requires to use an asymmetric partition of \( R \). We solve the problem when \( R \) is realized by a zero-avoiding transducer (with some bound \( k \)): if the absolute value of the input-output length discrepancy of a computation exceeds \( k \) then the length discrepancy of the computation cannot become zero. This class of relations properly contains all recognizable, all left synchronous, and all right synchronous relations.

We leave the asymmetric partition problem open when \( R \) is not realized by a zero-avoiding transducer. We also show examples of total word-orderings for which there is a relation \( R \) that cannot be partitioned into two asymmetric rational relations such that one of them is decreasing with respect to the given word-ordering.

Keywords: asymmetric relations, transducers, synchronous relations, word-orderings

1 Introduction

The abstract already serves as the first paragraph of the introduction.

The structure of the paper is as follows. The next section contains basic concepts about relations, word orderings and transducers. Section 3 contains the mathematical statement of the rational asymmetric partition problem and the motivation for considering this problem. Section 4 presents the concept of a \( C \)-copy of a transducer \( t \), which is another transducer that contains many copies of the states of \( t \) (one copy for each \( c \in C \)). A \( C \)-copy of \( t \), for appropriate \( C \), allows us to produce two transducers realizing an asymmetric partition of the relation realized by \( t \). Section 5 deals with the simple case where the transducer is letter-to-letter (Proposition 1). Section 6 introduces zero avoiding transducers \( t \) with some bound \( k \in \mathbb{N}_0 \) and shows a few basic properties: the minimum \( k \) is less than the number of states of \( t \) (Proposition 2); every left synchronous and every right
synchronous relation is realized by some zero-avoiding transducer with bound
0 (Proposition 4). Section 7 shows a construction, from a given input-altering
transducer \( s \), that produces a certain \( C \)-copy \( \alpha(s) \) of \( s \) such that \( \alpha(s) \) realizes
the set of all pairs in \( R(s) \) for which the input is greater than the output with
respect to the radix total order of words (Theorem 3). This construction solves
the rational asymmetric partition problem when the given relation is realized
by a zero-avoiding transducer. Section 8 discusses a variation of the problem,
where we have a certain fixed total word ordering \([ > ]\) (e.g., the radix one) and
we want to know whether there is a rational symmetric relation \( S \) such that not
both of \( S \cap [ > ] \) and \( S \cap [ < ] \) are rational (Proposition 5). This section also offers
as an open problem the general rational asymmetric partition problem (that is
when the given \( R \) is not realized by a zero-avoiding transducer). The last section
contains a few concluding remarks.

2 Basic Terminology and Notation

We assume the reader is familiar with basic concepts of formal languages: alphabet, words (or strings), empty word \( \lambda \), language (see e.g., [7,4]). We shall
use a totally ordered alphabet \( \Sigma \); in fact for convenience we assume that
\( \Sigma = \{0, 1, \ldots, q-1\} \), for some integer \( q > 0 \). If a word \( w \) is of the form \( w = uv \) then
\( u \) is called a prefix and \( v \) is called a suffix of \( w \). We shall use \( x/y \) to denote the
pair of words \( x \) and \( y \).

A (binary word) relation \( R \) over \( \Sigma \) is a subset of \( \Sigma^* \times \Sigma^* \), that is, \( R \subseteq \Sigma^* \times \Sigma^* \).
We shall use the
infix notation \( xRy \) to mean that \( x/y \in R \); then, \( xRy \) means \( x/y \notin R \).
The domain \( \text{dom} R \) of \( R \) is the set \( \{x \mid x/y \in R\} \). The inverse \( R^{-1} \) of \( R \) is the relation \( \{y/x \mid x/y \in R\} \).

Word orderings. The following types of, and notation about relations over \( \Sigma \)
are important in this work, where \( x, y, z \) are any words in \( \Sigma^* \).

- A relation \( R \) is called (i) irreflexive, if \( xRx \); (ii) reflexive, if \( xRx \) for all \( x \in \text{dom} R \); (iii) symmetric, if \( xRy \) implies \( yRx \); (iv) transitive, if \( xRy \) and \( yRz \)
implies \( xRz \).
- A relation \( A \) is called asymmetric, if \( xAy \) implies \( yAx \). In this case, \( A \)
must be irreflexive. Moreover we have that
\[
A \cap A^{-1} = \emptyset \quad \text{and} \quad A \subseteq (\Sigma^* \times \Sigma^*) \setminus \{w/w : w \in \Sigma^*\}.
\]

A total asymmetry is an asymmetric relation \( A \) such that either \( uAv \) or \( vAu \),
for all words \( u, v \) with \( u \neq v \). We shall use the notation \([>]\) for an arbitrary
total asymmetry, as well as the notation \([>\alpha]\) for a specific total asymmetry
where \( \alpha \) is some identifying subscript. Then, we shall write \( u > v \) to indicate
that \( u/v \in [>] \). Moreover, we shall write \([<] \) (and \([<\alpha]\) for the inverse of
\([>] \) (and \([<\alpha]\)).
A total strict ordering $[<]$ is a total asymmetry that is also transitive. Examples of total strict orderings are the radix $['<r']$ and the lexicographic $['<\ell']$ ordering. The lexicographic ordering is the standard dictionary order, for example, $12 <_r 12 <_l 3$. The radix ordering is the standard integer ordering when words are viewed as integers and no symbol of $\Sigma$ is interpreted as zero: $3 <_r 12 <_r 112$. In both of these orderings, the empty word is the smallest one.

**Pathology.** A path $P$ of a labelled (directed) graph $G = (V, E)$ is a string of consecutive edges, that is, $P \in E^*$ and is of the form

$$P = (q_0, \alpha_1, q_1)(q_1, \alpha_2, q_2) \cdots (q_{\ell-1}, \alpha_{\ell}, q_\ell),$$

for some integer $\ell \geq 0$, where each $q_i \in V$, each $\alpha_i$ is a label, and each $(q_{i-1}, \alpha_i, q_i) \in E$. We shall use the following shorthand notation for that path

$$P = (q_{i-1}, \alpha_i, q_i)_{i=1}^\ell.$$

The empty path is denoted by $\lambda$. We shall concatenate paths in the same way that we concatenate words, provided that the concatenated sequence consists of consecutive edges; thus, $PQ$ is a path of $G$ when $P, Q$ are paths and the last vertex in $P$ is equal to the first vertex in $Q$. As usual, $P\lambda = \lambda P = P$, for all paths $P$. A cycle is a path as above such that $q_0 = q_\ell$. The path $P$ contains a cycle $C$, if $C = (q_{j-1}, \alpha_j, q_j)_{j=s}^t$, for some indices $s, t$, with $1 \leq s \leq t \leq \ell$ and $q_{s-1} = q_t$. In this case, we have $P = BCD$ for some paths $C, D$. Moreover, the path $BD$ that results when we remove $C$ from $P$ is well-defined. Based on this terminology, we have the following remark.

**Remark 1.** If $P = BAD$ is a path of $G$, where $A$ is a cycle, and the path $BD$ contains a cycle $C$, then also $P$ contains the cycle $C$.

Let $\mathcal{S}$ be a subset of the cycles contained in $P$. The first $\mathcal{S}$-cycle of $P$ is the cycle $\langle q_{j-1}, \alpha_j, q_j \rangle_{j=s}^t$ of $P$ that has the smallest index $t$ among the cycles of $P$ that are in $\mathcal{S}$. The path $P - \mathcal{S}$ is the path that results if we remove from $P$ all $\mathcal{S}$-cycles: remove the first $\mathcal{S}$-cycle of $P$ to get a path $P_1$, then the first $\mathcal{S}$-cycle of $P_1$, etc.

**Transducers.** ([1,10,8]) A transducer is a quintuple$^3$ $t = (Q, \Sigma, E, I, F)$ such that $(Q, E)$ is a labelled graph with labels of the form $x/y$, for some $x, y \in \Sigma \cup \{\lambda\}$, and $I, F \subseteq Q$ with $I \neq \emptyset$. The set of vertices $Q$ is also called the set of states of $t$. The set of edges $E$ is also called the set of transitions of $t$. In a transition $e = (p, x/y, q)$ of $t$, $p$ is called the source state of $e$, and $q$ is called the destination state of $e$. The sets $I, F$ are called the initial and final states of $t$, respectively. The label of a path $\langle q_{i-1}, x_i/y_i, q_i \rangle_{i=1}^\ell$ is the pair $x_1 \cdots x_\ell/y_1 \cdots y_\ell$. We write label($P$) to denote the label of a path $P$. In particular, label($\lambda$) = $\lambda/\lambda$.

A computation of $t$ is a path $P$ of $t$ such that, either $P$ is empty, or the first

$^3$ In general, $t$ has an input and an output alphabet, but in our context these are equal.
state of \( P \) is in \( I \). We write \( \text{Comput}(t) \) to denote the set of all computations of \( t \). The computation \( P \) is called accepting if, either \( P = \lambda \) and \( I \cap F \neq \emptyset \), or \( P \neq \lambda \) and the last state of \( P \) is in \( F \). We write \( \text{AccComput}(t) \) to denote the set of accepting computations of \( t \). The relation realized by \( t \) is the set \( R(t) = \{ \text{label}(P) \mid P \in \text{AccComput}(t) \} \).

If \( R(t) \) is irreflexive then \( t \) is called input-altering. If \( R(t) \subseteq [>], \) for some total asymmetry \([>]\), then \( t \) is called input-decreasing (with respect to \([>]\)). The size \( |t| \) of \( t \) is the number of states of \( t \) plus the number of transitions of \( t \).

3 Statement and Motivation of the Main Problem

In this section, we make precise the problem we are dealing with, and we explain its context as well as the motivation for considering it.

Let \( I \) be an irreflexive relation. An asymmetric partition of \( I \) is a partition \( \{A, B\} \) of \( I \) such that \( A, B \) are asymmetric. If \( I \) is rational, then a rational asymmetric partition of \( I \) is an asymmetric partition \( \{A, B\} \) of \( I \) such that \( A, B \) are rational.

Remark 2. If \( I \) is any irreflexive relation and \([>]\) is any total asymmetry then \( \{I \cap [>], I \cap [<]\} \) is an asymmetric partition of \( I \). As any asymmetric \( A \) is irreflexive, we also have that \( \{A \cap [>, A \cap [<]\} \) is an asymmetric partition of \( A \). If \( S \) is a symmetric and irreflexive relation and \( \{A, B\} \) is an asymmetric partition of \( S \) then \( B = A^{-1} \).

**The Rational Asymmetric Partition Problem.** Which symmetric-and-irreflexive rational relations have a rational asymmetric partition?

Remark 3. Any relation \( R \) that is not irreflexive cannot have an asymmetric partition; otherwise, \( R \) would contain a pair \( u/u \) and this cannot be an element of any asymmetric relation. We also have the following observation.

– If \( A \) is any rational asymmetric relation then \( \{A, A^{-1}\} \) is a rational asymmetric partition of \( A \cup A^{-1} \).

**Motivation for the above problem.** For a relation \( R \) and language \( L \), we say that \( L \) is \( R \)-independent, \([9,11]\), if

\[
    uRv, u \in L, v \in L \rightarrow u = v.
\]

If \( R \) is irreflexive and realized by some transducer \( t \) then, \([5]\), the above condition is equivalent to \( t(L) \cap L = \emptyset \). In any case, we have that \( L \) is \( R \)-independent, if and only if it is \((R \cup R^{-1})\)-independent; and of course the relation \((R \cup R^{-1})\) is *always*
symmetric. The concept of independence provides tools for studying code-related properties such as prefix codes and error-detecting languages (according to the relation $R$). In [6], for a given input-altering transducer $t$ and regular language $L$ that is $R(t)$-independent, the authors provide a formula for embedding $L$ into a maximal $R(t)$-independent language, provided that $t$ is input-decreasing with respect to $[>_{r}]$. Of course then, $R(t)$ is asymmetric. Thus, to embed an $S$-independent language $L$ into a maximal one, where $S$ is symmetric, it is necessary to find a transducer $t$ such that $S = R(t) \cup R(t^{-1})$ and $R(t)$ is asymmetric.

4 Multicopies of Transducers

In this section we fix a finite nonempty set $C$, whose elements are called copy labels. Let $S$ be any set and let $c \in C$. The copy $c$ of $S$ is the set $S^c = \{s^c | s \in S\}$.  

**Definition 1.** Let $t = (Q, \Sigma, T, I, F)$ be a transducer. A C-copy of $t$ is any transducer $t' = (Q', \Sigma, T', I', F')$ satisfying the following conditions.  

1. $Q' = \cup_{c \in C}Q^c$, $I' \subseteq \cup_{c \in C}I^c$, $F' \subseteq \cup_{c \in C}F^c$.  
2. $T' \subseteq \{(p^c, x/y, q^d) | c, d \in C, (p, x/y, q) \in T\}$. If $e' = (p^c, x/y, q^d) \in T'$ then the edge $e = (p, x/y, q)$ of $t$ is called the edge of $t$ corresponding to $e'$ and is denoted by $\text{corr}(e')$.

For each edge $e$ of $t$, we define the set of edges of $t'$ corresponding to $e$ to be the set 

$$\text{Corr}(e) = \{e' | e = \text{corr}(e')\}.$$ 

**Example 1.** The transducer $\alpha_0(s)$ in Fig. 1 is a $C$-copy of $s$, where $C = \{\lambda, A, R\}$. It has three copies of the states of $s$. We have that 

$$\text{Corr}(q_1, 0/1, q_2) = \{(q_1^\lambda, 0/1, q_2^R), (q_1^A, 0/1, q_2^A), (q_1^R, 0/1, q_2^R)\}.$$ 

Each edge $(p, x/y, q)$ of $s$ has corresponding edges in $\alpha_0(s)$ of the form $(p^c, x/y, q^d)$ such that the source state $p^c$ is in the copy $c$ (initially, $c = \lambda$) and the destination state $q^d$ is in the copy $d$, where possibly $d = c$. Edges of $\alpha_0(s)$ with source state in the copies $A, R$ have a destination state in the same copy. On the other, an edge of $\alpha_0(s)$, with some label $x/y$, whose source state is in the copy $\lambda$ has a destination state in the copy $\lambda$ if $x = y$; in the copy $A$ if $x >_{r} y$; and in the copy $R$ if $x <_{r} y$. As $\alpha_0(s)$ has final states only in the copy $A$, it follows that for any $u/v \in R(\alpha_0(s))$ we have that $u >_{r} v$ and $u/v \in R(s)$. This example is useful when solving the rational asymmetric partitioning problem for letter-to-letter transducers—see Section 5.

**Remark 4.** The below observations follow from the above definitions and are simple and helpful facts to use when proving statements about $C$-copies of transducers. Let $t$ be a transducer and let $t'$ be a $C$-copy of $t$.  

1. The set of edges of $t'$ is equal to $\bigcup_{e \in T} \text{Corr}(e)$, where $T$ is the set of edges of $t$. Thus, to define the edges of a $C$-copy of $t$, it is sufficient to specify, for all edges $e$ of $t$, the sets $\text{Corr}(e)$.
2. If \( t' \) has a state that is both initial and final then so does \( t \). Thus, if \( \lambda \in \text{AccComput}(t') \) then we have that \( \lambda \in \text{AccComput}(t) \) and \( \lambda/\lambda \in R(t) \).

**Definition 2.** Let \( t' \) be a \( C \)-copy of a transducer \( t \), and let \( P' = \langle e'_i \rangle_{i=1}^\ell \in \text{Path}(t') - \{ \lambda \} \). For each edge \( e'_i \) of \( P' \), let \( e_i = \text{corr}(e'_i) \). Then the string \( \langle e_i \rangle_{i=1}^\ell \) is a path of \( t \) and is called the (unique) path of \( t \) corresponding to \( P' \) and is denoted by \( \text{corr}(P') \).

Conversely, if \( P = \langle e_i \rangle_{i=1}^\ell \) is a path of \( t \), then we define the set of paths of \( t' \) corresponding to \( P \) to be the set of all paths of \( t' \) of the form \( \langle e'_i \rangle_{i=1}^\ell \), where each \( e'_i \in \text{Corr}(e_i) \); this set is denoted by \( \text{Corr}(P) \).

We also define \( \text{corr}(\lambda) = \lambda \) and \( \text{Corr}(\lambda) = \{ \lambda \} \).

**Remark 5.** Let \( t \) be a transducer and let \( t' \) be a \( C \)-copy of \( t \). Let \( P \in \text{Path}(t) \) and \( P' \in \text{Path}(t') \).

1. We have that \( P' \in \text{Corr}(P) \) if and only if \( P = \text{corr}(P') \).
2. If \( e' \) is an edge of \( t' \) and \( P'e' \in \text{Path}(t') \), then \( \text{corr}(P'e') = \text{corr}(P') \text{corr}(e') \).

**Lemma 1.** If \( t' \) is a \( C \)-copy of a transducer \( t \), then \( R(t') \subseteq R(t) \).

**Proof.** Let \( u/v \in R(t') \). We show that \( u/v \in R(t) \). There is \( P' \in \text{AccComput}(t') \) with \( \text{label}(P') = u/v \). If \( P' = \lambda \) then \( u/v = \lambda/\lambda \) and the statement follows from Remark 4. Now suppose that \( P' = \langle q_i/v_i, q_i^c \rangle_{i=1}^\ell \) with \( \ell > 0 \). Then, \( P = \text{corr}(P') \) is a path of \( t \). Moreover, as \( q_0^b \) is initial in \( t' \) and \( q_0^e \) is final in \( t' \), the state \( q_0 \) is initial in \( t \) and \( q_\ell \) is final in \( t \); hence \( P \in \text{AccComput}(t) \). Also as \( P = \langle q_i/v_i, q_i^c \rangle_{i=1}^\ell \), we have \( \text{label}(P) = u/v \). Hence, \( u/v \in R(t) \).

5 **Asymmetric Partition of Letter-to-letter Transducers**

A transducer \( t \) is called letter-to-letter, [8], if all its transition labels are of the form \( \sigma/\tau \), where \( \sigma, \tau \in \Sigma \). Here we provide a solution to the asymmetric partition problem for letter-to-letter transducers in Proposition 1, which is based on Construction 1 below. We note that this construction is a special case of the more general construction for zero-avoiding transducers in Section 7, but we present it separately here as it is simpler than the general one.

**Construction 1** Let \( s = (Q, \Sigma, T, I, F) \) be a letter-to-letter transducer. Let \( C = \{ \lambda, A, R \} \). We construct a transducer \( \alpha_0(s) = (Q', \Sigma, T', I', F') \), which is a \( C \)-copy of \( s \), as follows.

First, \( Q' = Q^\lambda \cup Q^A \cup Q^R \), \( I' = I^\lambda \) and \( F' = F^A \).
Then, $T'$ is defined as follows.

$$T' = \{(p^c, \sigma/\tau, q^c) \mid (p, \sigma/\tau, q) \in T, c \in \{A, R\}\}$$

$$\cup \{(p^\lambda, \sigma/\sigma, q^\lambda) \mid (p, \sigma/\sigma, q) \in T\}$$

$$\cup \{(p^\lambda, \sigma/\tau, q^A) \mid (p, \sigma/\tau, q) \in T, \sigma > \tau\}$$

$$\cup \{(p^\lambda, \sigma/\tau, q^R) \mid (p, \sigma/\tau, q) \in T, \sigma < \tau\}.$$

**Explanation.** The constructed transducer $\alpha_0(s)$ contains two exact copies of $s$: a copy whose states are the $A$ copies of $Q$, and a copy whose states are the $R$ copies of $Q$; it also contains a sub-copy of $s$ which contains a $\lambda$ copy of $Q$ and only transitions with labels of the form $\sigma/\sigma$. Any computation $P'$ of $\alpha_0(s)$ starts at an initial state $i^\lambda$ and continues with states in the $\lambda$ copy of $Q$ as long as transition labels are of the form $\sigma/\sigma$. If a transition label is $\sigma/\tau$ with $\sigma > \tau$ then the computation $P'$ continues in the $A$ copy and never leaves that copy. As final states are only in the $A$ copy, we have that $P'$ is accepting if and only if $\text{corr}(P')$ is accepting and label($P'$) = $u/v$ such that $u$ is of the form $x\sigma y_1$ and $v$ of the form $x\tau y_2$ with $\sigma > \tau$ and $|u| = |v|$. Note that, in the computation $P'$, if a transition label is $\sigma/\tau$ with $\sigma < \tau$ and the current state is in the $\lambda$ copy, then $P'$ would continue in the $R$ copy of $\alpha_0(s)$, which has no final states, so $P'$ would not be accepting.

**Remark 6.** In fact the $R$ copy of $s$ is not necessary as it has no final states. It was included to make the construction a little more intuitive.

Using Lemma 1 and based on the above explanation, we have the following lemma—it is a special case of the lemma for zero-avoiding transducers in Section 7 where a more rigorous proof is given.

**Lemma 2.** Referring to Construction 1, we have that

$$R(\alpha_0(s)) = R(s) \cap \{u/v : u > v\}.$$

**Proposition 1.** Let $s$ be any input-altering letter-to-letter transducer. Let $t_1 = \alpha_0(s)$ and $t_2 = (\alpha_0(s^{-1}))^{-1}$, where $\alpha_0(s)$ is the transducer produced in Construction 1. The following statements hold true.

1. $|\alpha_0(s)| = \Theta(|s|)$.
2. $\{R(t_1), R(t_2)\}$ is a rational asymmetric partition of $R(s)$.

**Proof.** It follows from the construction and the lemma.

## 6 Discrepancies of Computations and Zero-avoiding Transducers

Here we introduce the concept of a zero-avoiding transducer with some bound $k \in \mathbb{N}_0$, which relates to length discrepancies of the computations of the transducer. We show that the bound $k$ is always less than the number of states of the
transducer. We also show that the class of relations realized by zero-avoiding transducers (of any bound) is a proper superset of all left and right synchronous relations. Thus, they also include all recognizable relations and all relations of bounded length discrepancy.

**Definition 3.** Let $u, v \in \Sigma^*$, let $t$ be a transducer and let $P = \langle q_{i-1}, x_i/y_i, q_i \rangle_{i=1}^\ell \in \text{Path}(t)$.

1. **The length discrepancy of the pair $u/v$** is the integer $d(u/v) = |u| - |v|$.
2. **The length discrepancy of $P$** is the integer
   \[ d(P) = d(x_1x_2\cdots x_{\ell}/y_1y_2\cdots y_{\ell}). \]
3. **The maximum absolute length discrepancy of $P$** is the integer
   \[ d_{\text{max}}(P) = \max_{Q \in \text{Prefix}(P)} \{|d(Q)|\}. \]

**Remark 7.** We have that $d(\lambda/\lambda) = 0$ and $d_{\text{max}}(\lambda) = 0$. Moreover,
- if $P_1P_2$ is a path of $t$, then $d(P_1P_2) = d(P_1) + d(P_2)$. 

Fig. 1: Example of Construction 1 applied to transducer $s$ to get transducer $\alpha_0(s)$. 

\[ \begin{array}{c}
\text{s:} \\
\xrightarrow{1/0} q_1 \\
\xrightarrow{0/1} q_2 \\
\xrightarrow{0/0} q_3 \\
\xrightarrow{1/0} q_4 \\
\xrightarrow{1/1, 1/0, 0/1} q_5 \\
\end{array} \]
Definition 4. A transducer $t$ is called zero-avoiding, if there is an integer $k \geq 0$ such that the following condition is satisfied:

$$\text{for any } P \in \text{Comput}(t), \text{ if } d_{\max}(P) > k \text{ then } d(P) \neq 0.$$ 

In this case, $t$ is called zero-avoiding with bound $k$. It is called zero-avoiding with minimum bound $k$, if it is zero-avoiding with bound $k$ and not zero-avoiding with bound $k - 1$.\(^4\)

Remark 8. In the above definition of a zero-avoiding transducer, if a computation $P$ has length discrepancy $> k$, or $<-k$, then any continuation of $P$ cannot have zero as its length discrepancy.

Remark 9. Let $t = (Q, \Sigma, T, I, F)$ be a transducer. For any path $P$ of $t$ there is a unique path $P^{-1}$ of $t^{-1}$ whose labels are the inverses of the labels in $P$. Thus, $d(P^{-1}) = -d(P)$ and $|d(P^{-1})| = |d(P)|$. This implies that

- if $t$ is zero-avoiding with some bound $k$ then also $t^{-1}$ is zero-avoiding with bound $k$.

Remark 10. Let $s = (Q, \Sigma, T, I, F)$ be a transducer and let $t = (Q', \Sigma, T', I', F')$ be a $C$-copy of $s$. Let $P'$ be a computation of $s$, and let $P = \text{corr}(P')$; then $P$ and $P'$ have exactly the same sequence of labels in their transitions. Thus the following statements hold.

1. $d_{\max}(P') = d_{\max}(P)$.
2. $d(Q) = d(Q')$ for any prefixes $Q, Q'$ of $P, P'$ of the same length.
3. If $s$ is zero-avoiding with some bound $k$ then also $t$ is zero-avoiding with the same bound $k$.

Lemma 3. Let $t$ be an $n$-state transducer, for some integer $n > 0$.

1. If $t$ has a path $P$ with $d(P) \geq n$ (resp., $d(P) \leq -n$) then $P$ contains a cycle $B$ with $d(B) > 0$ (resp., $d(B) < 0$).
2. If $t$ has a computation $P$ with $d_{\max}(P) \geq n$ and $d(P) = 0$, then $P = BC_1AC_2D$ such that $C_1, C_2$ are cycles with $d(C_1)d(C_2) < 0$.

Proof. The proofs of the statements make use of the path terminology in Section 2. For the first statement, we show only the case $d(P) \geq n$, as the other case is symmetric. As every transition in $P$ changes the discrepancy by at most one, $P$ contains at least $n$ transitions, so there are at least $n + 1$ states in $P$, which implies that $P$ contains a cycle. Let $S$ be the set of cycles $C$ of $P$ with $d(C) \leq 0$. Then, $d(P - S) \geq d(P)$ and, as $d(P - S) \geq n$, $P - S$ must contain a cycle $B$, which must have $d(B) > 0$. Then, the cycle $B$ is also contained in $P$.

For the second statement, we split the computation $P$ into two paths $P = P_1P_2$ such that $P_1$ is the shortest prefix of $P$ with $|d(P_1)| = n$. First we consider the case where $d(P_1) = n$. Then, $P_1$ contains a cycle $C_1$ with $d(C_1) > 0$. Then, $d(P_2) = -n$, which implies that $P_2$ contains a cycle $C_2$ with $d(C_2) < 0$. The case where $d(P_1) = -n$ is analogous.

\(^4\) This is well-defined: if $t$ is zero-avoiding with bound $k$ then it is also zero-avoiding with bound $k'$ for all $k' > k$. 
Proposition 2. Let \( t \) be an \( n \)-state transducer, for some integer \( n \geq 1 \). If \( t \) is zero-avoiding with minimum bound \( k \) then \( k < n \).

**Proof.** The statement holds trivially if \( k = 0 \). So suppose \( k \geq 1 \) and assume for the sake of contradiction that \( k \geq n \). As \( t \) is not zero-avoiding with bound \( k-1 \), there is a computation \( P \) of \( t \) such that \( d_{\text{max}}(P) = k \) and \( d(P) = 0 \). Then, Lemma 3 implies that \( P = BC_1AC_2D \) for some cycles \( C_1, C_2 \) with \( d(C_1)d(C_2) < 0 \). We can use the cycles \( C_1, C_2 \) to make a new computation \( Q \) of \( t \) such that \( d_{\text{max}}(Q) > k \) and \( d(Q) = 0 \). Without loss of generality, assume that \( d(C_1) > 0 \) and \( d(C_2) < 0 \). The required computation of \( t \) is

\[
Q = BC_1C_1^{[d(C_2)|k(1+|d(B)|)]}AC_2C_2^{d(C_1)[1+|d(B)|]}D.
\]

Then we have that \( d(Q) = d(P) + d(C_1)d(C_2)[k(1+|d(B)|)] + d(C_2)d(C_1)[1 + |d(B)|] = 0 \) and

\[
d_{\text{max}}(Q) \geq d_{\text{max}}(BC_1C_1^{[d(C_2)|k(1+|d(B)|)]}) \geq d(B) + d(C_1) + d(C_2)[k(1 + |d(B)|)] > k
\]

which contradicts the assumption that \( t \) is zero-avoiding with bound \( k \).

Proposition 3. Let \( t \) be an \( n \)-state transducer, for some integer \( n \geq 1 \). The following statements are equivalent.

1. \( t \) is not zero-avoiding
2. \( t \) has a computation \( P \) with \( d_{\text{max}}(P) \geq n \) and \( d(P) = 0 \).
3. \( t \) has a computation \( P \) of the form \( P = BC_1AC_2D \) such that \( C_1, C_2 \) are cycles with \( d(C_1)d(C_2) < 0 \).

**Proof.** That the first statement is equivalent to the second one follows logically from Definition 4 and the above lemma. Now we show that the second and third statements are equivalent. The second statement implies the third one, by Lemma 3. Now suppose that the third statement holds, that is, there is a computation \( P = BC_1AC_2D \) of \( t \) such that \( d(C_1)d(C_2) < 0 \). As in the proof of the above lemma, we can use the cycles \( C_1, C_2 \) to make a computation \( Q \) of \( t \) with \( d_{\text{max}}(Q) \geq n \) and \( d(Q) = 0 \).

**Relating left (right) synchronous and zero-avoiding relations.** A natural question that arises is how zero-avoiding relations are related to the well-known left (or right) synchronous relations. A relation \( R \) is called left synchronous if the relation \( \overrightarrow{R} \) can be realized by a letter-to-letter transducer over the alphabet \( \Sigma \cup \{\#\} \) with \( \# \notin \Sigma \) [2,3]. Here we use the notation

\[
\overrightarrow{u/v} = (u/v^{[u]-[v]}) \text{ if } |u| \geq |v|; \text{ and } \overrightarrow{u/v} = (u^{[v]-[u]}/v) \text{ if } |u| < |v|.
\]

Then, \( \overrightarrow{R} = \{\overrightarrow{u/v} : u/v \in R\} \). An equivalent definition is given in [8]: \( R \) is left synchronous if it is a finite union of relations, each of the form \( S(A \times \{\lambda\}) \) or
$S(\{\lambda\} \times A)$, where $A$ is a regular language and $S$ is realized by a letter-to-letter transducer. The concept of a right synchronous relation is symmetric: we can define it either via

\[ \widehat{u/v} = (u/\#|u|-|v|v) \text{ if } |u| \geq |v|; \]  

\[ \widehat{u/v} = (\#|v|-|u|u/v) \text{ if } |u| < |v|, \]

or via finite unions of relations of the form $(A \times \{\lambda\})S$ or $(\{\lambda\} \times A)S$.

**Proposition 4.** The classes of left synchronous relations and right synchronous relations are proper subsets of the class of zero-avoiding relations with bound 0.

**Proof.** Consider any left synchronous relation $R = S_1C_1 \cup \cdots \cup S_mC_m$ such that each $S_i$ is realized by a letter-to-letter transducer $t_i$, and each $C_i$ is of the form $(A_i \times \{\lambda\})$ or $((\{\lambda\} \times A_i)$, where $A_i$ is a regular language. Then, each $C_i$ is realized by a transducer $s_i$, whose transition labels are either all in $\Sigma \times \{\lambda\}$, or all in $\{\lambda\} \times \Sigma$. The zero-avoiding transducer realizing $R$ is the transducer $t = \bigvee_{i=1}^m t_is_i$. It is zero-avoiding with bound 0 because each computation $P$ of $t$ is in one component $t_is_i$ and, once a prefix $Q$ of $P$ has $d(Q) > 0$ (resp., $d(Q) < 0$) then also $d(P) > 0$ (resp., $d(P) < 0$). Similarly, one has that every right synchronous relation is also zero-avoiding with bound 0.

The rational relation $R = (00/0)^*$ is zero-avoiding but not left synchronous. To see that $R = (00/0)^*$ is zero-avoiding we use the transducer in Fig. 2 realizing $R$ and satisfying the zero-avoiding condition with bound 0. To show that $R$ is not left synchronous, we use contradiction. Assume that it is, then the relation $\overline{R} = \{(0^i0^i/0^i\#^i) : i \in \mathbb{N}_0\}$ would be rational, which implies that the language $\{0^i\#^i : i \in \mathbb{N}_0\}$ would be regular; a contradiction. Similarly, we have that $R$ is not right synchronous.

**Zero-avoiding relations are not closed under intersection.** To see this consider the zero-avoiding relations

\[ (0/0)^*(\lambda/1)^* \quad \text{and} \quad (0/\lambda)^*(1/0)^*. \]

Their intersection is $\{0^i1^i/0^i : i \in \mathbb{N}_0\}$, which is non-rational (if it were rational then the language $\{0^i1^i : i \in \mathbb{N}_0\}$ would be regular).
7 Asymmetric Partition of Zero-avoiding Transducers

We present next a solution to the asymmetric partition for any relation realized by some zero-avoiding transducer $s$—see Construction 2 and Theorem 3. The required asymmetric relation is realized by a transducer $\alpha(s)$ which is a $C$-copy of $s$, where $C$ is shown in (1) further below. In fact $R(\alpha(s)) = (R(s) \cap \{ > \})$; thus, $u/v \in R(\alpha(s))$ implies $u >_r v$. The set of states of $\alpha(s)$ is $Q' = \cup_{c \in C} Q^c$.

The reason why all these copies of $Q$ are needed is to know at any point during a computation $P^{'}$ of $\alpha(s)$ whether $d_{\text{max}}(P^{'})$ has exceeded $k$, where $k$ is either the known bound of zero-avoidance, or the number of states of $s$.

Meaning of states of $\alpha(s)$ in Construction 2. A state $q^c$ of $\alpha(s)$ has the following meaning. Let $P^{'}$ be any computation of $\alpha(s)$ ending with $q^c$ and having some label $w_{\text{in}}/w_{\text{out}}$. Then, state $q^c$ specifies which one of the following mutually exclusive facts about $P^{'}$ holds.

- $q^c = q^A$ means: $w_{\text{in}} = w_{\text{out}}$.
- $q^c = q^+u$ means: $w_{\text{in}} = w_{\text{out}}u$, for some word $u$ with $1 \leq |u| \leq k$, so $w_{\text{in}} >_r w_{\text{out}}$.
- $q^c = q^-u$ means: $w_{\text{out}} = w_{\text{in}}u$, for some word $u$ with $1 \leq |u| \leq k$, so $w_{\text{in}} <_r w_{\text{out}}$.
- $q^c = q^A^c$ means: $w_{\text{in}} = x\sigma y$, $w_{\text{out}} = x\tau z$, $\sigma >_r \tau$, $\ell = |y| - |z| = d(P^{'})$, and $-k \leq \ell \leq k$. Note that the $A$ in $q^A^c$ is a reminder of $\sigma >_r \tau$ and indicates that $P^{'}$ could be the prefix of an Accepting computation $Q'$ having $d(Q') \geq 0$, in which case $w_{\text{in}} >_r w_{\text{out}}$ where $w_{\text{in}}^{'}/w_{\text{out}}^{'}$ is the label of $Q'$.
- $q^c = q^R^c$ means: $w_{\text{in}} = x\sigma y$, $w_{\text{out}} = x\tau z$, $\sigma <_r \tau$, $\ell = |y| - |z| = d(P^{'})$, and $-k \leq \ell \leq k$. Note that the $R$ in $q^R^c$ is a reminder of $\sigma <_r \tau$ and indicates that $P^{'}$ could be the prefix of a Rejecting computation $Q'$ having $d(Q') \leq 0$, in which case $w_{\text{in}} <_r w_{\text{out}}$ where $w_{\text{in}}^{'}/w_{\text{out}}^{'}$ is the label of $Q'$.
- $q^c = q^R$ means: $d_{\text{max}}(P^{'}) > k$ and $d(P^{'}) = |w_{\text{in}}| - |w_{\text{out}}| > 0$.
- $q^c = q^A$ means: $d_{\text{max}}(P^{'}) < k$ and $d(P^{'}) = |w_{\text{in}}| - |w_{\text{out}}| < 0$.

Final states in Construction 2. Based on the meaning of the states and the requirement that the label $w_{\text{in}}/w_{\text{out}}$ of an accepting computation $P^{'}$ of $\alpha(s)$ satisfies $w_{\text{in}} >_r w_{\text{out}}$, the final states of $\alpha(s)$ are shown in (2) further below. Let $f$ be any final state of $s$. State $f^A$ of $\alpha(s)$ is final because, if $P^{'A}$ ends in $f^A$, we have $d(P^{'A}) > 0$, which implies $w_{\text{in}} >_r w_{\text{out}}$. On the other hand, state $f^R$ is not final because any computation $P^{'}$ of $\alpha(s)$ ending in $f^A$ has $d(P^{'}) < 0$, which implies $w_{\text{in}} <_r w_{\text{out}}$. State $f^R^c$, with $\ell > 0$, is final because any computation $P^{'}$ of $\alpha(s)$ ending in $f^R^c$ has $|w_{\text{in}}| - |w_{\text{out}}| = \ell > 0$, so $w_{\text{in}} >_r w_{\text{out}}$. On the other hand, state $f^R^c$, with $\ell \leq 0$, is not final because any computation $P^{'}$ of $\alpha(s)$ ending in $f^R^c$ has $|w_{\text{in}}| - |w_{\text{out}}| = \ell \leq 0$, and $w_{\text{in}} <_r w_{\text{out}}$.

Example 2. The transducer $\alpha(s)$ consists of several modified copies of $s$ (see Fig. 3a) such that, for any computation $P$ of $s$ with label $w_{\text{in}}/w_{\text{out}}$ there is at least one corresponding computation of $\alpha(s)$ with the same label $w_{\text{in}}/w_{\text{out}}$ which goes through copies of the same states appearing in $P$. The initial states of $\alpha(s)$
are in the copy $Q^\lambda$, where any computation involving only states in $Q^\lambda$ has equal input and output labels. For a transition $e = (p, 1/\lambda, q)$ of $s$, we have that 
$e' = (p^\lambda, 1/\lambda, q^{+1})$ is one of the transitions of $\alpha(s)$ corresponding to $e$, where the transition $e'$ starts at the copy $Q^\lambda$ of $Q$ and goes to the copy $Q^{+1}$ of $Q$ (see Fig. 3b). In a computation of $\alpha(s)$ that ends in the copy $Q^{+1}$, the input label is of the form $x_1$ and the output label is of the form $x$. Then, Fig. 3c shows all possible transitions from state $p^{+1}$ to other states of $\alpha(s)$, which could be in the same or different copies of $Q$.

A $\lambda/\lambda$-free transducer is a transducer that has no label $\lambda/\lambda$. Using tools from automata theory, we have that every transducer realizes the same relation as one of a $\lambda/\lambda$-free transducer.

Construction 2 Let $s = (Q, \Sigma, T, I, F)$ be a $\lambda/\lambda$-free and zero-avoiding transducer with some bound $k$. The transducer $\alpha(s) = (Q', \Sigma, T', I', F')$ is a $C$-copy of $s$ as follows.

$$C = \{\lambda, A, R\} \cup \{+u, -u | u \in \Sigma^*, 1 \leq |u| \leq k\} \cup \{A\ell, R\ell | \ell \in \mathbb{Z}, -k \leq \ell \leq k\}.$$  

We have $Q' = \bigcup_{c \in C} Q^c$, $I' = I^\lambda$,

$$F' = F^A \cup F^{A0} \cup \{\bigcup_{1\leq|u|\leq k} F^{u}\} \cup \{\bigcup_{k=1}^{C} (F^A \cup F^{R})\}.$$  

The set $T'$ of transitions is defined next. More specifically, for each transition $(p, x/y, q) \in T$, with $x/y \in \{\sigma/\tau, \sigma/\lambda, \lambda/\tau | \sigma, \tau \in \Sigma\}$, we define the set $\text{Corr}(p, x/y, q)$. For each state $p^c \in Q'$ the transition $(p^c, x/y, q^d)$ is in $\text{Corr}(p, x/y, q)$, where $q^d$ depends on $p^c$ and $x/y$ as follows.

If $p^c = p^\lambda$:
- if $x/y = \sigma/\sigma$ then $q^d = q^\lambda$;
- if $x/y = \sigma/\tau$ and $\sigma \geq \tau$ then $q^d = q^{A0}$;
- if $x/y = \sigma/\tau$ and $\sigma \leq \tau$ then $q^d = q^{R0}$;
- if $x/y = \sigma/\lambda$, then $q^d = q^{+\sigma}$ if $k > 0$, and $q^d = q^A$ if $k = 0$;
- if $x/y = \lambda/\tau$, then $q^d = q^{-\tau}$ if $k > 0$, and $q^d = q^R$ if $k = 0$.

If $p^c = p^u$:
- if $x/y = \sigma/\lambda$ and $|u| < k$ then $q^d = q^{u+\tau}$;
- if $x/y = \sigma/\lambda$ and $|u| = k$ then $q^d = q^A$;
- if $x/y = \lambda/\tau$ and $u[0] = \tau$ then $q^d = q^{u[1..]}$;
- if $x/y = \lambda/\tau$ and $u[0] \geq \tau$ then $q^d = q^{A\ell}$ where $\ell = |u[1..]|$;
- if $x/y = \lambda/\tau$ and $u[0] < \tau$ then $q^d = q^{R\ell}$ where $\ell = |u[1..]|$;
- if $x/y = \sigma/\tau$ and $u[0] = \tau$ then $q^d = q^{u[1..]+\tau}$;
- if $x/y = \sigma/\tau$ and $u[0] \geq \tau$ then $q^d = q^{A\ell}$ where $\ell = |u|$;
- if $x/y = \sigma/\tau$ and $u[0] < \tau$ then $q^d = q^{R\ell}$ where $\ell = |u|$.

If $p^c = p^{-u}$:
- if $x/y = \sigma/\lambda$ and $u[0] = \sigma$ then $q^d = q^{u[1..]}$;
- if $x/y = \sigma/\lambda$ and $u[0] \geq \sigma$ then $q^d = q^{R\ell}$ where $\ell = |u[1..]|$;
(a) Overview of copies of states of transducer \(s\) in transducer \(\alpha(s)\). The figure also depicts some chosen transitions involving \(\sigma, \tau \in \Sigma\). The symbol \(\otimes\) represents that the copy contains some final states; initial states are only in copy \(Q^A\).

Fig. 3: Sketch of examples of a transducer \(\alpha(s)\) which is the result of Construction 2 on some transducer \(s\). The examples use \(\Sigma = \{0, 1\}\) and \(k = 2\). Notice that 0 in \(Q^{+0}\) is the string 0 and 0 in \(Q^{A0}\) is the number zero (length discrepancy).

\[
\begin{align*}
&\text{if } x/y = \sigma/\lambda \text{ and } u[0] <_{r} \sigma \text{ then } q^d = q^{A\ell} \text{ where } \ell = |u[1..]|; \\
&\text{if } x/y = \lambda/\tau \text{ and } |u| < k \text{ then } q^d = q^{-u}\tau; \\
&\text{if } x/y = \lambda/\tau \text{ and } |u| = k \text{ then } q^d = q^{R}; \\
&\text{if } x/y = \sigma/\tau \text{ and } u[0] = \sigma \text{ then } q^d = q^{-u[1..]}\tau; \\
&\text{if } x/y = \sigma/\tau \text{ and } u[0] >_{r} \sigma \text{ then } q^d = q^{R\ell} \text{ where } \ell = |u|; \\
&\text{if } x/y = \sigma/\tau \text{ and } u[0] <_{r} \sigma \text{ then } q^d = q^{A\ell} \text{ where } \ell = |u|. \\
\end{align*}
\]

If \(p^c = p^{X\ell}\) with \(X \in \{A, R\}:
\[
\begin{align*}
&\text{if } x/y = \sigma/\lambda \text{ and } \ell < k \text{ then } q^d = q^{X(\ell+1)}; \\
&\text{if } x/y = \lambda/\tau \text{ and } |u| < k \text{ then } q^d = q^{-u[1..]|}; \\
&\text{if } x/y = \lambda/\tau \text{ and } |u| = k \text{ then } q^d = q^{R}; \\
&\text{if } x/y = \sigma/\tau \text{ and } u[0] = \sigma \text{ then } q^d = q^{-u[1..]}\tau; \\
&\text{if } x/y = \sigma/\tau \text{ and } u[0] >_{r} \sigma \text{ then } q^d = q^{R\ell} \text{ where } \ell = |u|; \\
&\text{if } x/y = \sigma/\tau \text{ and } u[0] <_{r} \sigma \text{ then } q^d = q^{A\ell} \text{ where } \ell = |u|. \\
\end{align*}
\]
for all cases of \( \alpha \). For every computation \( P \) of \( \alpha \), the set \( \text{Corr}(P) \) is nonempty.

**Lemma 4.** For every computation \( P \) of \( \alpha \), the set \( \text{Corr}(P) \) is nonempty.

**Proof.** The statement follows from the definition of the set of transitions \( T' \) of \( \alpha \). More specifically, let \( P = \langle q_{i-1}, x_{i}/y_{i}, q_{i} \rangle_{i=1}^{\ell} \), and for any \( m \) with \( 1 \leq m \leq \ell \), let \( P_{m} \) be the prefix of \( P \) consisting of the first \( m \) transitions. Using induction, we show that \( \text{Corr}(P_{m}) \neq \emptyset \) for all \( m \). For \( m = 1 \), we have that \( \text{Corr}(P_{1}) \) is nonempty, as \( q_{0}^{m} \) is a defined state of \( \alpha \) and also a transition \( (q_{0}^{m}, x_{1}/y_{1}, q_{1}^{m}) \) of \( \alpha \) corresponding to \( (q_{0}, x_{1}/y_{1}, q_{1}) \) is defined in Construction 2 for all cases of \( x_{1}/y_{1} \). Now for some \( m < \ell \) suppose that there is \( P_{m}^{\prime} \in \text{Corr}(P_{m}) \), so we have

\[
P_{m}^{\prime} = \langle q_{i-1}^{m}, x_{i}/y_{i}, q_{i}^{m} \rangle_{i=1}^{m}.
\]

As \( q_{m}^{m} \) is defined and there is a transition \( (q_{m}^{m}, x_{m+1}/y_{m+1}, q_{m+1}^{m} \rangle) \) defined in Construction 2 for all cases of \( x_{m+1}/y_{m+1} \), we have that also the path

\[
\langle q_{i-1}^{m}, x_{i}/y_{i}, q_{i}^{m} \rangle_{i=1}^{m+1}
\]

is in \( \text{Corr}(P_{m+1}) \).

**Lemma 5.** Let \( \alpha \) be a \( \lambda/\lambda \)-free and zero-avoiding transducer. The transducer \( \alpha(\alpha) \) in Construction 2 is such that

\[
\text{R}(\alpha(\alpha)) = \text{R}(\alpha) \cap \{ u/v : u >_{R} v \}.
\]

**Proof.** First we show that, for every \( w_{\text{in}}/w_{\text{out}} \in \text{R}(\alpha(\alpha)) \), it is \( w_{\text{in}} >_{R} w_{\text{out}} \) and \( w_{\text{in}}/w_{\text{out}} \in \text{R}(\alpha) \). That \( w_{\text{in}}/w_{\text{out}} \in \text{R}(\alpha) \) follows from Lemma 1. The part \( w_{\text{in}} >_{R} w_{\text{out}} \) follows from the meaning of the states of \( \alpha(\alpha) \) and the definition of its final states. More specifically, if \( w_{\text{in}}/w_{\text{out}} \in \text{R}(\alpha(\alpha)) \) and \( P' \in \text{AccComput}(\alpha(\alpha)) \) with \( \text{label}(P') = w_{\text{in}}/w_{\text{out}} \) one shows that \( w_{\text{in}} >_{R} w_{\text{out}} \) by considering the possible cases for the last state of \( P \). For example, if \( P \) ends at a state \( f^{R} \), with \( \ell > 0 \), then this means that \( w_{\text{in}} = x\sigma y, w_{\text{out}} = x\tau z, \sigma <_{R} \tau \) and \( |y| - |z| = \ell \), which implies \( |w_{\text{in}}| > |w_{\text{out}}| \); hence \( w_{\text{in}} >_{R} w_{\text{out}} \).

Now we show that, if \( w_{\text{in}}/w_{\text{out}} \in \text{R}(\alpha(\alpha)) \) and \( w_{\text{in}} >_{R} w_{\text{out}} \), then \( w_{\text{in}}/w_{\text{out}} \in \text{R}(\alpha(\alpha)) \). Let \( P \) be any accepting computation of \( \alpha \) with label \( w_{\text{in}}/w_{\text{out}} \). By
Lemma 4, there is a computation $P' \in \text{Corr}(P)$ of $\alpha(s)$ that ends at some state $f^c$ with $f \in F$ and $c \in C$. If $f^c \in F'$ then $P'$ is accepting and, therefore, the label $w_{\text{in}}/w_{\text{out}}$ of $P'$ is in $R(\alpha(s))$. If $f^c \notin F'$ then $c \in \{\lambda, R, R0\} \cup \{-u \mid u \in \Sigma^*, 1 \leq |u| \leq k\} \cup \{A\ell, R\ell \mid \ell \in \mathbb{Z}, -k \leq \ell < 0\}$.

Then, we get a contradiction by showing that $w_{\text{in}} < w_{\text{out}}$ for any $c$ as above. For example, if $c = A\ell$ or $c = R\ell$, with $\ell < 0$, the meaning of the states implies that $|w_{\text{in}}| < |w_{\text{out}}|$.

The below theorem solves the rational asymmetric partition problem for every irreflexive relation realized by some zero-avoiding transducer.

**Theorem 3.** Let $s$ be any input-altering and zero-avoiding transducer with bound $k \in \mathbb{N}_0$. Let $t_1 = \alpha(s)$ and let $t_2 = (\alpha(s^{-1}))^{-1}$, where $\alpha(s)$ is the transducer produced in Construction 2. The following statements hold true.

1. $|\alpha(s)| = \Theta(|s||\Sigma|^k)$.
2. $\{R(t_1), R(t_2)\}$ is a rational asymmetric partition of $R(s)$.

**Proof.** For the first statement, equation (1) implies that $\alpha(s)$ has

$$|C| = 3 + 4k + 2^{|\Sigma|^{k+1} - 1}/|\Sigma| - 1$$

copies of the states of $s$, which is of magnitude $O(|\Sigma|^k)$. If $s$ has $n$ states then $\alpha(s)$ has $O(n|\Sigma|^k)$ states, and if $s$ has $m$ transitions then $\alpha(s)$ has $O(m|\Sigma|^k)$ transitions.

The second statement follows when we note that

$$R(t_2) = R(s) \cap \{u/v : u <_r v\}$$

which is a consequence of Lemma 5 using standard logical arguments on the sets involved; and that $\{R(s) \cap \{u/v : u >_r v\}, R(s) \cap \{u/v : u <_r v\}\}$ is a partition of $R(s)$ because $s$ is input-altering.

Now we have the following consequence of Theorem 3 and Proposition 4.

**Corollary 1.** Every left synchronous and every right synchronous irreflexive rational relation has a rational asymmetric partition.

### 8 An Unsolved Case and a Variation of the Problem

Recall that Theorem 3 solves the rational asymmetric partition problem for any irreflexive relation realized by some zero-avoiding transducer. The main open question is the following.

**Open Question.** Does there exist any rational irreflexive relation that has no rational asymmetric partition?
We also offer a more specific open question. Consider the rational symmetric relation: $R = R_1 \cup R_1^{-1} \cup R_2 \cup R_2^{-1}$, where

$$R_1 = \{(0^a1^i0^j1^b \mid a, b, c, d, i, j \in \mathbb{N}\} \quad \text{and} \quad R_2 = \{(0^a1^i0^j1^b, 0^a1^i0^j1^b) \mid a, b, c, d, i, j \in \mathbb{N}\}.$$

$R$ is rational because $R_1$ and $R_2$ are rational—see for example the transducer $t_1$ realizing $R_1$ in Fig. 4. The more specific question is the following.

**Does there exist a rational asymmetric relation $A$ such that $A \cup A^{-1} = R$?**

We note the following facts about $R$:

- $R_1 \cap R_2 = \{(0^a1^i0^j1^b, 0^a1^i0^j1^b) \mid a, b, i, j \in \mathbb{N}\}$ is not rational.
- Also non rational are: $R_1 \cap R_1^{-1}$, $R_1 \cap R_1^{-1}$, $R_1 \cap R_1^{-1}$, $R_1 \cap R_1^{-1}$, $R_1 \cap R_1^{-1}$.
- Also non rational is the intersection $R_1 \cap R_1^{-1} \cap R_2 \cap R_2^{-1}$.

The fact that $R$ is realized by some transducer that is not zero-avoiding does not imply that $R$ is realized by no zero-avoiding transducer.

Fig. 4: Transducer $t_1$ realizing the relation $R_1 = \{(0^a1^i0^j1^b, 0^a1^i0^j1^b) \mid a, b, c, d, i, j \in \mathbb{N}\}$.

The proof of Theorem 3 implies that every zero-avoiding and irreflexive-and-symmetric rational relation $S$ has a rational partition according to the radix order $[> r]$; that is, $\{S \cap [> r], S \cap [< r]\}$ is a rational partition of $S$. A question that arises here is whether there are examples of irreflexive-and-symmetric rational $S$ for which at least one of $S \cap [> r], S \cap [< r]$ is not rational. This question can be generalized slightly by using any total asymmetry $[>]$ in place of $[> r]$. The question would be answered if we find an asymmetric rational $A$ such that at least one of $S \cap [> r], S \cap [< r]$ is not rational, where $S = A \cup A^{-1}$.

**The Rational Non-Partition Problem for a Fixed Asymmetry.** Let $[>]$ be a fixed total asymmetry. Is there an asymmetric rational relation $A$ such that at least one of $(A \cup A^{-1}) \cap [> r]$ and $(A \cup A^{-1}) \cap [< r]$ is not rational? If the answer is yes, then $A$ is called a rational non-partition witness for $[> r]$; else, $A$ is called a rational partition witness for $[> r]$.

Next we show that the answer to the above problem is positive for certain fixed total asymmetries $[> r]$. We use the following lemma. The notation $\text{pr}_1 R$ is for the language $\{u : u/v \in R\}$; and the notation $\text{pr}_2 R$ is for the language $\{v : u/v \in R\}$.
Lemma 6. Let $[>]$ be a total asymmetry and $A$ be a rational asymmetric relation. If $\text{pr}_1 A \cap \text{pr}_2 A = \emptyset$ and exactly one of the languages $\text{pr}_1(A \cap [>] ), \text{pr}_2(A \cap [<])$ is regular then $A$ is a rational non-partition witness for $[>]$.

Proof. For the sake of contradiction, assume $(A \cup A^{-1}) \cap [>]$ is rational. Then, the language

$$\text{pr}_1(A \cap [>] ) \cup \text{pr}_1(A^{-1} \cap [>])$$

must be regular. This is impossible however, because $\text{pr}_1(A^{-1} \cap [>] ) = \text{pr}_2(A \cap [<])$, and the languages $\text{pr}_1(A \cap [>] ), \text{pr}_2(A \cap [<])$ are disjoint and exactly one of the two is regular.

Proposition 5. Consider the following asymmetric rational relations.

$$A = \{1(00)^j / 00^i1^i \mid i, j \in \mathbb{N}_0\} \quad (3)$$

$$B = \{0^{2i+2}101 / 0^{i+1}0^i110 \mid i, j \in \mathbb{N}_0\} \quad (4)$$

We have that $A$ is a rational non-partition witness for $[>], r$ and a rational partition witness for $[>], l$; and $B$ is a rational non-partition witness for both $[>], r$ and $[>], l$.

Proof. We use the above lemma. First for $A$, we have that

$$A \cap [>], r = \{1(00)^j / 00^i1^i \mid i, j \in \mathbb{N}_0, j \geq i\} \quad \text{and}$$

$$A \cap [<], r = \{1(00)^j / 00^i1^i \mid i, j \in \mathbb{N}_0, j < i\}.$$ 

As, $\text{pr}_1(A \cap [>], r)$ is regular and $\text{pr}_2(A \cap [<], r)$ is not regular, we have that $A$ is a rational non-partition witness for $[>], r$. On the other hand, as $A \subseteq [>], l$ and $A \cap [<], l = \emptyset$, we have that

$$(A \cup A^{-1}) \cap [>], l = A \quad \text{and} \quad (A \cup A^{-1}) \cap [<], l = A^{-1},$$

which implies that $A$ is a rational partition witness for $[>], l$. Now for $B$, we have that

$$B \cap [>], l = B \cap [>], r = \{0^{2i+2}101 / 0^{i+1}0^i110 \mid i, j \in \mathbb{N}_0 i + 1 > j\}$$

and $B \cap [<], l = B \cap [<], r$ = the above set with $i + 1 \leq j$. The statement follows now when we note that $\text{pr}_1(B \cap [>], l)$ is regular and $\text{pr}_2(B \cap [<], l)$ is not regular.

9 Conclusions and Acknowledgement

Motivated by the embedding problem for rationally independent languages, we have introduced the rational asymmetric partition problem. Our aim was to find the largest class of rational relations that have a rational asymmetric partition. In doing so we introduced zero-avoiding transducers. These define a class of rational relations that properly contain the left and right synchronous relations and admit rational asymmetric partitions. Whether all rational relations admit such partitions remains open. We thank Jacques Sakarovitch for looking at this open problem and offering the opinion that it indeed appears to be non-trivial.
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