BLOW-UP SOLUTIONS AND STRONG INSTABILITY OF GROUND STATES FOR THE INHOMOGENEOUS NONLINEAR SCHröDINGER EQUATION

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Abstract. Using variational methods we study the stability and strong instability of ground states for the focusing inhomogeneous nonlinear Schrödinger equation (INLS)

\[ i\partial_t u + \Delta u + |x|^{-b}|u|^{p-1}u = 0. \]

We construct two kinds of invariant sets under the evolution flow of (INLS). Then we show that the solution of (INLS) is global and bounded in \( H^1(\mathbb{R}^N) \) in the first kind of the invariant sets, while the solution blow-up in finite time in the other invariant set. Consequently, we prove that if the nonlinearity is \( L^2 \)-supercritical, then the ground states are strongly unstable by blow-up.

1. Introduction. In this paper we consider the following inhomogeneous nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u + |x|^{-b}|u|^{p-1}u = 0, \tag{1.1} \]

where \( u = u(x,t) \) is a complex-valued function of \( (x,t) \in \mathbb{R}^N \times \mathbb{R} \), \( 1 < p < 2^* \) and \( 0 < b < \min\{2, N\} \) where \( 2^* = 1 + \frac{4}{N-2} \) if \( N \geq 3 \) or \( 2^* = \infty \) if \( N = 1, 2 \). Notice that since \( p > 1 \), we assume the technical restriction \( 0 < b < \min\{2, N\} \).

The equations of the form (1.1) arise in many physical problems, especially in non-linear optic. We refer the readers to [1] for the derivation and applications of (1.1). The nonlinearity \( |x|^{-b}|u|^{p-1}u \) describes the interaction between particles. Moreover, if \( b > 0 \), the nonlinearity is used to model an inhomogeneity, or defect, in the medium in which the wave propagates. The equation (1.1) has been extensively studied due to its application in numerous fields of mathematics and physics; see [4, 5, 13, 14, 15, 6, 2] and references therein.

It is well-known that (1.1) is locally well-posed in the energy space \( H^1(\mathbb{R}^N) \) (see [8, 3, 14] for more details). Indeed, for \( u_0 \in H^1(\mathbb{R}^N) \), there exists \( T_* = T(||u_0||_{H^1}) > 0 \) and a unique maximal solution \( u \in C([0, T_*), H^1(\mathbb{R})) \) of (1.1) satisfying \( u(0) = u_0 \). The maximality of the solution \( u(t) \) is in the sense that if \( T_* < +\infty \), then

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\[ \lim_{t \to T_*} \| \nabla u(t) \|^2_{L^2} = +\infty. \] In addition, the solution satisfies the conservation of energy and mass
\[ E(u(t)) = E(u_0) \quad \text{and} \quad M(u(t)) = M(u_0), \]
for all \( t \in [0, T_*) \), where
\[ E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1} \, dx, \quad M(u) = \int_{\mathbb{R}^N} |u|^2 \, dx. \]
We say that the solution \( u(t) \) of (1.1) blows up in positive finite time if \( T_* < +\infty \).

In this paper we discuss the variational structure and orbital stability/instability of standing waves of the form \( u(x,t) = e^{i\omega t} \phi(x) \) where \( \omega > 0 \) is a frequency and \( \phi \) is a nontrivial solution of the elliptic problem
\[ -\Delta \phi + \omega \phi - |x|^{-b}|\phi|^{p-1}\phi = 0 \quad \text{in} \quad \mathbb{R}^N. \] (1.2)
Notice that if \( 1 < p < 1 + \frac{4-2b}{N} \), then the global well-posedness of (1.1) holds in \( H^1(\mathbb{R}^N) \) by Gagliardo-Nirenberg inequality and conservation laws. Moreover, in the \( L^2 \)-critical case \( p = 1 + \frac{4-2b}{N} \), if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \) and \( u(t) \) is the corresponding solution to (1.1) with \( u(0) = u_0 \), then \( T_* = +\infty \), i.e., \( u(t) \) is global. Here \( Q(x) \) is the unique positive radial solution of the stationary problem (1.2) with \( \omega = 1 \); see [6, 13] for more details.

**Definition 1.1.** We define the following functionals of class \( C^2 \) on \( H^1(\mathbb{R}^N) \):
\[ S_\omega(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\omega}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1} \, dx, \]
\[ I_\omega(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \omega \int_{\mathbb{R}^N} |u|^2 \, dx - \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1} \, dx, \]
\[ P(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N(p-1)+2b}{2(p+1)} \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1} \, dx. \]

Notice that if \( \omega \leq 0 \), then (1.2) has no solution. Indeed, suppose that (1.2) has solution \( \varphi \in H^1(\mathbb{R}^N) \). Since \( I_\omega(\varphi) = 0 \) and \( P(\varphi) = 0 \), we infer that \( \varphi \) satisfies
\[ \frac{(N+2)-p(N-2)-2b}{2(p+1)} \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1} \, dx = \omega \int_{\mathbb{R}^N} |u|^2 \, dx. \]
Since \( p < 1 + \frac{4-2b}{N-2} \), it follows that \( \omega > 0 \).

Now, for \( \omega > 0 \), we denote the set of non-trivial solutions of (1.2) by
\[ \mathcal{A}_\omega = \{ \varphi \in H^1(\mathbb{R}^N) \setminus \{ 0 \} : S_\omega'(\varphi) = 0 \}. \]
A ground state is a nontrivial solution of the elliptic problem (1.2) with the least action \( S_\omega \). The set of ground states is denoted by \( \mathcal{G}_\omega \) and
\[ \mathcal{G}_\omega = \{ \varphi \in \mathcal{A}_\omega : S_\omega(\varphi) \leq S_\omega(v) \quad \text{for all} \quad v \in \mathcal{A}_\omega \}. \]

Next, we consider the minimization problem
\[ d(\omega) = \inf \{ S_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{ 0 \} , I_\omega(u) = 0 \}, \]
\[ \mathcal{M}_\omega = \{ \varphi \in H^1(\mathbb{R}^N) : S_\omega(\varphi) = d(\omega) \quad \text{and} \quad I_\omega(u) = 0 \}. \] (1.3)
The set \( \{ u \in H^1(\mathbb{R}^N) \setminus \{ 0 \} , I_\omega(u) = 0 \} \) is called the Nehari manifold. We now state the first result, which provides at least one minimizer for the minimization problem (1.3).
Proposition 1. Let \( N \geq 1, \omega > 0 \) and \( 1 < p < 2^* \). Then the set of ground states \( \mathcal{G}_\omega \) is not empty and \( \mathcal{G}_\omega = \mathcal{M}_\omega \). Moreover, there exists a unique real-valued, positive, and spherically symmetric function \( \phi_\omega \in H^1(\mathbb{R}^N) \) such that \( \mathcal{G}_\omega = \{ e^{i\theta} \phi_\omega; \theta \in \mathbb{R} \} \).

We remark that Proposition 1 was claimed without proof in [8]. For the sake of completeness, we give a proof in Section 2. Next, for \( \omega > 0 \), we define the following subsets in \( H^1(\mathbb{R}^N) \) as follows

\[
\mathcal{K}_\omega^+ = \{ \varphi \in H^1(\mathbb{R}^N) : S_\omega(\varphi) < S_\omega(\phi_\omega) \quad \text{and} \quad P(\varphi) \geq 0 \}, \\
\mathcal{K}_\omega^- = \{ \varphi \in H^1(\mathbb{R}^N) : S_\omega(\varphi) < S_\omega(\phi_\omega) \quad \text{and} \quad P(\varphi) < 0 \}.
\]

In the following result we study the global existence/blow-up property for the sets defined above. More specifically, we classify the global behavior of the solution whose action is less that \( d(\omega) \).

**Theorem 1.2.** Let \( \omega > 0 \) and \( 1 + \frac{4-2b}{N} < p < 2^* \). The sets \( \mathcal{K}_\omega^+ \) and \( \mathcal{K}_\omega^- \) are invariant sets under flow generated by (1.1). Moreover, let \( u_0 \in H^1(\mathbb{R}^N) \) and let \( u(t) \) be the corresponding solution to (1.1) with life time \([0,T_\ast)\). Then we have the following properties.

(i) If \( u_0 \in \mathcal{K}_{\omega}^+ \), then the corresponding solution \( u(t) \) exists globally.

(ii) If \( u_0 \in \mathcal{K}_{\omega}^- \), then one of the following two cases holds:

1. \( T_\ast < \infty \) and \( \lim_{t \rightarrow T_\ast} \| \nabla u(t) \|_{L^2}^2 = \infty \) or,
2. \( T_\ast = \infty \) and there exists a time sequence \( \{ t_n \} \) such that \( t_n \rightarrow \infty \) and \( \lim_{t_n \rightarrow \infty} \| \nabla u(t_n) \|_{L^2}^2 = \infty \).

Here \( Q(x) \) is the unique positive radial solution of (1.2) with \( \omega = 1 \).

Notice that the sharp criteria between blow-up and global well-posedness in Theorem 1.2 above is described in terms of \( S_\omega(\phi_\omega) \) and \( P(\varphi) \). Moreover, the blow-up result is proved by the method of Du-Wu-Zhang [7]. Now, as a consequence of the Theorem 1.2(ii) we have.

**Corollary 1.** Let \( \omega > 0 \), \( 1 + \frac{4-2b}{N} < p < 2^* \), \( s_c = \frac{N}{2} - \frac{2-b}{p-2} \), \( u_0 \in H^1(\mathbb{R}^N) \) and let \( u(t) \) be the solution of (1.1) with \( u(0) = u_0 \), satisfying

\[
E(u_0)^{s_c} M(u_0)^{1-s_c} < E(Q)^{s_c} M(Q)^{1-s_c}.
\]

(i) If

\[
\| \nabla u_0 \|_{L^2}^{s_c} \| u_0 \|_{L^2}^{1-s_c} > \| \nabla Q \|_{L^2}^{s_c} \| Q \|_{L^2}^{1-s_c},
\]

then one of the following two statements holds true:

1. \( T_\ast < \infty \) and \( \lim_{t \rightarrow T_\ast} \| \nabla u(t) \|_{L^2}^2 = \infty \).
2. \( T_\ast = \infty \) and there exists a time sequence \( \{ t_n \} \) such that \( t_n \rightarrow \infty \) and \( \lim_{t_n \rightarrow \infty} \| \nabla u(t_n) \|_{L^2}^2 = \infty \).

Corollary 1 follows from Theorem 1.2 and the fact that if \( u_0 \) satisfies (1.4) and (1.5), then there exists \( \omega_0 > 0 \) such that \( u_0 \in \mathcal{K}_{\omega_0}^- \) and therefore we have the same conclusion as in Theorem 1.2(ii). We remark that in [9], it was proved that if the initial data \( u_0 \) satisfies (1.4)-(1.5) and \( |x|u_0 \in L^2(\mathbb{R}^N) \), then the corresponding solution \( u(t) \) with \( u(0) = u_0 \) blows up in finite time.

We also have the following blow-up result, which will be important in the proof of the strong instability of standing waves.

**Corollary 2.** Let \( \omega > 0 \), \( 1 + \frac{4-2b}{N} < p < 2^* \). If \( u_0 \in \mathcal{K}_\omega^- \cap \Sigma \), then the corresponding solution \( u(t) \) blows up in finite time in \( H^1(\mathbb{R}^N) \). Here \( \Sigma = \{ u \in H^1(\mathbb{R}^N) : |x|u \in L^2(\mathbb{R}^N) \} \).
The proof of statement (ii) of Theorem 1.2 is based on the following result which is interesting in its own right.

**Theorem 1.3.** Let \(1 + \frac{4-2b}{N} < p < 2^*\), \(u_0 \in H^1(\mathbb{R}^N)\) and let \(u \in C([0,T_*), H^1(\mathbb{R}^N))\) be the corresponding maximal solution of (1.1) with \(u(0) = u_0\). If there exists \(\gamma < 0\) such that

\[
\sup_{t \in (0,T_*)} P(u(t)) \leq \gamma,
\]

then there exists no global solution \(u \in C([0,\infty), H^1(\mathbb{R}^N))\) with

\[
\sup_{t \in [0,\infty)} \|u(t,)\|_{L^q} < \infty \quad \text{for some } q > (p + 1) \frac{N}{N - b}.
\]

We remark that if \(u_0 \in \mathcal{K}_\omega\), then by Lemma 3.3 below, the corresponding solution \(u(t)\) with \(u(0) = u_0\) satisfies the hypotheses of Theorem 1.3. Roughly speaking, under the assumption of the above result, there exist no global solutions whose \(L^q\) norms are uniformly bounded for all \(t \in \mathbb{R}\). We refer the reader to Section 4 for the proof of Theorem 1.3.

Now, the basic symmetry associated to equation (1.1) is the phase-invariance. Indeed, the translation invariance does not hold due to the inhomogeneity \(|x|^{-b}\). Therefore, the definition of stability takes into account only this type of symmetry.

**Definition 1.4.** Let \(\omega > 0\).

- We say that a standing wave \(u(x,t) = e^{i\omega t}\phi(x)\) of (1.1) is stable in \(H^1(\mathbb{R}^N)\) if for any \(\epsilon > 0\) there exists \(\eta > 0\) such that if \(u_0 \in H^1(\mathbb{R}^N)\) satisfies \(\|u_0 - \phi\|_{H^1} < \eta\), then the corresponding solution \(u(t)\) to (1.1) exists globally in time and satisfies

  \[
  \sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\phi\|_{H^1} < \epsilon.
  \]

- We say that the standing wave \(e^{i\omega t}\phi(x)\) is unstable in \(H^1(\mathbb{R}^N)\) if \(e^{i\omega t}\phi(x)\) is not stable.

- We say that the standing wave \(u(x,t) = e^{i\omega t}\phi(x)\) strongly unstable in \(H^1(\mathbb{R}^N)\) if for \(\epsilon > 0\), there exists \(u_0 \in H^1(\mathbb{R}^N)\) such that \(\|u_0 - \phi\|_{H^1} < \eta\) and the corresponding solution \(u(t)\) of (1.1) with initial data \(u_0\) blows up in a finite time.

As a consequence of Proposition 1 and Corollary 2 we obtain the following stability/instability results for the standing waves of equation (1.1).

**Theorem 1.5.** Let \(N \geq 1\), \(\omega > 0\) and \(0 < b < \min\{2,N\}\).

(i) If \(1 < p < 1 + \frac{4-2b}{N}\), then the standing wave \(e^{i\omega t}\phi(x)\) of (1.1) is orbitally stable.

(ii) If \(1 + \frac{4-2b}{N} < p < 2^*\), then the standing wave \(e^{i\omega t}\phi(x)\) of (1.1) is strongly unstable.

We remark that the stability result (for \(N \geq 3\)) in Theorem 1.5(i) was obtained in [8], however we will give the proof for \(N \geq 1\) (with a different argument). Notice that the approach developed in [8] does not work in dimensions 1 or 2. In the critical case \(p = 1 + \frac{4-2b}{N}\), Genoud [13] proved the strong instability of the ground states of (1.1). To the best of our knowledge, this is the first mathematical result about strong instability of ground states of (1.1) in the \(L^2\)-supercritical case.

This present paper is organized as follows. In Section 2 we prove the existence of ground states (Proposition 1). In Section 3 we prove our global existence/blow-up
Taking the infimum, we see
\[
H \leq \min \{ |x|^{\frac{b}{p+1}} u \}^{p+1}_{L^{p+1}} \leq C_{GN} \| \nabla u \|_{L^2}^{\frac{N(p-1)}{2} + b} \| u \|_{L^2}^{p+1 - \frac{N(p-1)}{2} - b},
\]  
where \( 0 < b < \min \{2, N\} \),
\[
C_{GN} = \left( \frac{N(p-1) + 2b}{2(p+1) - N(p-1) - b} \right)^{\frac{4 - N(p-1) - 2b}{2}} \frac{2(p+1)}{(N(p-1) + 2b) \| Q \|_{L^2}^{-1}}
\]
and \( Q \) is the unique positive solution of (1.2) with \( \omega = 1 \). We have divided the proof of Proposition 1 into a sequence of lemmas. First we recall the following Gagliardo-Nirenberg inequality (see [13])
\[
\| x |^{\frac{b}{p+1}} u \|_{L^{p+1}}^{p+1} \leq C_{GN} \| \nabla u \|_{L^2}^{\frac{N(p-1)}{2} + b} \| u \|_{L^2}^{p+1 - \frac{N(p-1)}{2} - b},
\]
where \( 0 < b < \min \{2, N\} \),
\[
C_{GN} = \left( \frac{N(p-1) + 2b}{2(p+1) - N(p-1) - b} \right)^{\frac{4 - N(p-1) - 2b}{2}} \frac{2(p+1)}{(N(p-1) + 2b) \| Q \|_{L^2}^{-1}}
\]
and \( Q \) is the unique positive solution of (1.2) with \( \omega = 1 \). We have divided the proof of Proposition 1 into a sequence of lemmas. First we recall the following result (see [6]).

**Lemma 2.1.** Let \( N \geq 1 \), \( 0 < b < \min \{2, N\} \) and \( 1 < p < 2^* \). Then the embedding \( H^1(\mathbb{R}^N) \hookrightarrow L^{p+1}_b(\mathbb{R}^N) \) is compact, where
\[
L^{p+1}_b(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+1} dx < \infty \right\}
\]

**Lemma 2.2.** Let \( \omega > 0 \). Then the quantity \( d(\omega) \) is positive.

**Proof.** Let \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) be such that \( I_\omega(u) = 0 \). By Gagliardo-Nirenberg inequality (2.1), we infer that there exists \( C_1 > 0 \) such that
\[
\| x |^{\frac{b}{p+1}} u \|_{L^{p+1}}^{p+1} \leq C \| u \|_{H^1}^{p+1} \leq C_1 \left( \| x |^{\frac{b}{p+1}} u \|_{L^{p+1}}^{p+1} \right)^{\frac{p+1}{p+1}}
\]
This implies that
\[
\| x |^{\frac{b}{p+1}} u \|_{L^{p+1}}^{p+1} \geq \left( \frac{1}{C_1} \right)^{\frac{p+1}{p+1}} > 0.
\]
Therefore, if \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) with \( I_\omega(u) = 0 \), then there exists \( C_2 > 0 \) such that
\[
\| x |^{\frac{b}{p+1}} u \|_{L^{p+1}}^{p+1} \geq C_2. \]
This implies that
\[
S_\omega(u) = \frac{1}{2} I_\omega(u) + \frac{p-1}{2(p+1)} \| x |^{\frac{b}{p+1}} u \|_{L^{p+1}}^{p+1} + \frac{p-1}{2(p+1)} \| x |^{\frac{b}{p+1}} u \|_{L^{p+1}}^{p+1} \geq \frac{p-1}{2(p+1)} C.
\]
Taking the infimum, we see
\[
d(\omega) = \inf \{ S_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\} , I_\omega(u) = 0 \} \geq C > 0.
\]
This concludes the proof of lemma.

**Lemma 2.3.** The set \( \mathcal{M}_\omega \) is non-empty.
Proof. Let \( \{ u_n \} \) be a minimizing sequence of \( d(\omega) \). We claim that \( \{ u_n \} \) is bounded in \( H^1(\mathbb{R}^N) \). Indeed, since \( S_\omega(u_n) = \frac{p-1}{2(p+1)} \| x \|^{\frac{p-1}{p+1}} u_n \|_{L^{p+1}} \rightarrow d(\omega) \) as \( n \) goes to \( +\infty \), it follows that \( \| x \|^{\frac{p-1}{p+1}} u_n \|_{L^{p+1}} \) is bounded. This, since \( I_\omega(u_n) = 0 \), we infer that
\[
\| \nabla u_n \|_{L^2}^2 + \omega \| u_n \|_{L^2}^2 \leq C, 
\]
for some \( C > 0 \) and for all \( n \geq 1 \).

In particular, this implies that \( \{ u_n \} \) is bounded in \( H^1(\mathbb{R}^N) \). Therefore there exists \( \varphi \in H^1(\mathbb{R}^N) \) such that, up to a subsequence, \( u_n \rightharpoonup \varphi \) weakly in \( H^1(\mathbb{R}^N) \). From Lemma 2.1, we obtain that \( \| x \|^{\frac{p-1}{p+1}} u_n \|_{L^{p+1}} \rightarrow \| x \|^{\frac{p-1}{p+1}} \varphi \|_{L^{p+1}} \) as \( n \) goes to \( +\infty \) and
\[
I_\omega(\varphi) \leq \lim_{n \to \infty} I_\omega(u_n) = 0.
\]

By Lemma 2.2, we obtain that \( \varphi \neq 0 \). Next, suppose that \( I_\omega(\varphi) < 0 \). Set
\[
\lambda := \left( \frac{\| \nabla \varphi \|_{L^2}^2 + \omega \| \varphi \|_{L^2}^2}{\| x \|^{\frac{p-1}{p+1}} \varphi \|_{L^{p+1}}} \right)^{\frac{1}{p-1}}.
\]
Notice that \( \lambda \in (0, 1) \) and \( I_\omega(\lambda \varphi) = 0 \). Thus, by definition of \( d(\omega) \) we see that
\[
d(\omega) \leq S_\omega(\lambda \varphi) = \lambda^{p-1} \frac{(p-1)}{2(p+1)} \| x \|^{\frac{p-1}{p+1}} \varphi \|_{L^{p+1}} < d(\omega),
\]
This is a contradiction. Therefore \( I_\omega(\varphi) = 0 \) and
\[
S_\omega(\varphi) = \frac{1}{2} I_\omega(\varphi) + \frac{(p-1)}{2(p+1)} \| x \|^{\frac{p-1}{p+1}} \varphi \|_{L^{p+1}} = d(\omega),
\]
which implies, by the definition of \( d(\omega) \), that \( \varphi \in \mathcal{M}_\omega \). This concludes the proof. \( \square \)

Lemma 2.4. Let \( \omega > 0 \). Then there exists a unique real-valued, positive, and spherically symmetric function \( \phi_\omega \in H^1(\mathbb{R}^N) \) such that \( \mathcal{M}_\omega = \{ e^{i\theta} \phi_\omega; \theta \in \mathbb{R} \} \).

Proof: Let \( u \in \mathcal{M}_\omega \), then there exists a Lagrange multiplier \( \lambda \in \mathbb{R} \) such that \( S'_\omega(u) = \lambda I'_\omega(u) \). Notice that
\[
0 = I_\omega(u) = \langle S'_\omega(u), u \rangle = \lambda \langle I'_\omega(u), u \rangle, \tag{2.2}
\]
and
\[
\langle I'_\omega(u), u \rangle = -(p-1) \| x \|^{\frac{p-1}{p+1}} \varphi \|_{L^{p+1}} < 0. \tag{2.3}
\]
This implies that \( \lambda = 0 \), i.e. \( S'_\omega(u) = 0 \). So \( u \) is solution of the elliptic equation (1.2). We claim that there exists \( \theta \in \mathbb{R} \) such that \( u(x) = e^{i\theta} \varphi(x) \), where \( \varphi \) is a positive and spherically symmetric function. Indeed, it is not difficult to show that the following variational problem is equivalent to \( d(\omega) \):
\[
d_1(\omega) = \inf \left\{ \frac{p-1}{2(p+1)} \| x \|^{\frac{p-1}{p+1}} u \|_{L^{p+1}}^{p+1} : u \in H^1(\mathbb{R}^N) \setminus \{ 0 \}, I_\omega(u) \leq 0 \right\}. 
\]
Therefore, since \( \| \nabla(|u|) \|_{L^2}^2 \leq \| \nabla u \|_{L^2}^2 \), we infer that \( S_\omega(|u|) \leq S_\omega(u) \) and \( I_\omega(|u|) \leq I_\omega(u) = 0 \). This implies that \( |u| \in \mathcal{M}_\omega \) and
\[
\| \nabla |u| \|_{L^2}^2 = \| \nabla u \|_{L^2}^2. \tag{2.4}
\]
On the other hand, since \( u \) satisfies the stationary problem (1.2), by an elliptic regularity/bootsrap we obtain that \( u \in C^1(\mathbb{R}^N, \mathbb{C}) \). We remark that by strong maximum principle \( |u| > 0 \) in \( \mathbb{R}^N \). Next we set \( w(x) := \frac{u(x)}{|u(x)|} \). Since \( |w|^2 = 1 \), this implies that \( \text{Re}(\bar{w} \nabla w) = 0 \) and
\[
\nabla u = (\nabla |u|) w + |u| \nabla w = w(\nabla |u| + |u| \bar{w} \nabla w). 
\]
Then we see that $|\nabla u|^2 = |\nabla|u|^2| + |u|^2|\nabla w|^2$. We then use (2.4) to deduce that

$$
\int_{\mathbb{R}^N} |u|^2 |\nabla w|^2 dx = 0,
$$
and thus $|\nabla w| = 0$. Hence $w$ is constant with $|w| = 1$, we find $\theta \in \mathbb{R}$ such that $u = e^{i\theta} \varphi(x)$ where $\varphi(x) := |u(x)| > 0$. We now prove that $\varphi$ is necessarily radial. Indeed, denoting by $\varphi^*$ the Schwarz symmetric rearrangement of $\varphi$, it is well known that

$$
\int_{\mathbb{R}^N} |\nabla \varphi^*(x)|^2 dx \leq \int_{\mathbb{R}^N} |\nabla \varphi(x)|^2 dx
$$

$$
\int_{\mathbb{R}^N} |x|^{-b} |\varphi^*(x)|^{p+1} dx > \int_{\mathbb{R}^N} |x|^{-b} |\varphi(x)|^{p+1} dx \quad \text{unless } \varphi = \varphi^*.
$$

Therefore, if $\varphi$ is not radial, then $S_\omega(\varphi^*) < S_\omega(\varphi) = d(\omega)$ and $I_\omega(\varphi^*) < I_\omega(\varphi) = 0$. This gives a contradiction. Finally, since $\varphi$ is radially symmetric, it follows from Yanagida [17] for $N \geq 3$ (see also [8]), Genoud [12] for $N = 2$ and Toland [16] for $N = 1$, that such solution is unique. This completes of proof.

**Lemma 2.5.** Let $\omega > 0$. Then $G_\omega = M_\omega$.

**Proof.** By Lemma 2.3, $M_\omega$ is non-empty. Let $\varphi \in M_\omega$. Then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $S_\omega'(u) = \lambda I_\omega'(u)$. By (2.2) and (2.3), we infer that $\lambda = 0$, hence $\varphi \in A_\omega$. On the other hand, if $u \in A_\omega$, then $S_\omega'(u) = 0$. This implies that $I_\omega(u) = \langle S_\omega'(u), u \rangle = 0$ and, by definition of $d(\omega)$ we see that $S_\omega(\varphi) \leq S_\omega(v)$; that is

$$
S_\omega(\varphi) \leq S_\omega(v) \quad \text{for all } v \in A_\omega.
$$

Therefore, $\varphi \in G_\omega$ and $M_\omega \subset G_\omega$. In particular, we have $G_\omega$ is non-empty. Next, let $\varphi \in G_\omega$. Since $\varphi \in A_\omega$, we have $I_\omega(\varphi) = \langle S_\omega'(\varphi), \varphi \rangle = 0$. Moreover, from the fact that $M_\omega \subset G_\omega$, we infer that $S_\omega(\varphi) = d(\omega)$, and so $\varphi \in M_\omega$. The proof is complete.

**Proof of Proposition 1.** Proposition 1 follows from Lemmas 2.4 and 2.5.

3. **Global existence and blow-up in finite time.** In this section, we prove Theorem 1.2. We have divided the proof of theorem into a sequence of lemmas. Firstly we give

**Lemma 3.1.** Let $\omega > 0$ and $1 + \frac{4-2b}{N} < p < 2^*$. If $P(u) \geq 0$, then

$$
S_\omega(u) \leq ||u||^2_{L^2} + \omega ||u||^2_{L^2} \leq 2 \left( \frac{N(p-1) + 2b}{N(p-1) + 2b - 4} \right) S_\omega(u).
$$

**Proof.** It is clear that $S_\omega(u) \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx + \omega \int_{\mathbb{R}^N} |u|^2 dx$. On the other hand, if $P(u) \geq 0$ we infer that

$$
0 \leq ||u||^2_{L^2} - \frac{N(p-1) + 2b}{2(p+1)} ||x||_{L^{p+1}} u_{L^{p+1}}
$$

$$
= \frac{4 - (N(p-1) + 2b)}{4} ||\nabla u||^2_{L^2} + \frac{N(p-1) + 2b}{2} E(u).
$$

This implies that

$$
\frac{(N(p-1) + 2b) - 4}{4} ||\nabla u||^2_{L^2} \leq \frac{(N(p-1) + 2b) - 4}{4} \omega ||u||^2_{L^2} \leq \frac{2}{4} \left( E(u) + \frac{\omega}{2} ||u||^2_{L^2} \right).
$$
Since \((N(p - 1) + 2b) - 4 > 0\), by definition of \(S_\omega(u)\), we see that
\[
\|\nabla u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 \leq 2 \left( \frac{N(p - 1) + 2b}{N(p - 1) + 2b - 4} \right) S_\omega(u).
\]
This completes the proof. \(\square\)

**Lemma 3.2.** Let \(\omega > 0\). Then
\[
S_\omega(\phi_\omega) = \inf \{ S_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\}, P(u) = 0 \}.
\]

**Proof.** Let \(n(\omega) := \inf \{ S_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\}, P(u) = 0 \}\). Since \(P(\phi_\omega) = 0\), it follows that \(n(\omega) \leq S_\omega(\phi_\omega)\). Now we show that \(n(\omega) \geq S_\omega(\phi_\omega)\). Let \(u \in H^1(\mathbb{R}^N)\) be such that \(P(u) = 0\). Notice that if \(L_\omega(u) = 0\), then by Proposition we infer that \(1 S_\omega(u) \geq S_\omega(\phi_\omega)\). Assume that \(I_\omega(u) \neq 0\). For \(\lambda \in \mathbb{R}\) we set \(u_\lambda(x) = e^{\lambda x} u(e^{\lambda} x)\).

From straightforward calculations we see that
\[
\|\nabla u_\lambda\|_{L^2}^2 = e^{2\lambda}\|\nabla u\|_{L^2}^2, \quad \left\| \frac{\lambda}{p+1} u_\lambda \right\|_{L^{p+1}} = e^{\lambda \frac{N(p-1)+2b}{2}} \left\| \frac{\lambda}{p+1} u \right\|_{L^{p+1}}^{p+1}.
\]
Thus,
\[
I_\omega(u_\lambda) = e^{2\lambda}\|\nabla u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 - \frac{e^{\lambda \frac{N(p-1)+2b}{2}}}{p+1} \left\| \frac{\lambda}{p+1} u \right\|_{L^{p+1}}^{p+1}.
\]

Since \(\lim_{\lambda \to -\infty} I_\omega(u_\lambda) = \omega\|u\|_{L^2}^2 > 0\) and \(\lim_{\lambda \to \infty} I_\omega(u_\lambda) = -\infty\), we infer that there exists \(\lambda_0\) such that \(I_\omega(u_{\lambda_0}) = 0\). Now we observed that
\[
\partial_\lambda S_\omega(u_{\lambda_0}) = e^{2\lambda} \left\{ \|\nabla u\|_{L^2}^2 - \left( \frac{N(p-1)+2b}{2} \right) \frac{e^{\lambda \frac{N(p-1)+2b}{2}}}{p+1} \left\| \frac{\lambda}{p+1} u \right\|_{L^{p+1}}^{p+1} \right\}.
\]

Since \(N(p-1)+2b-4 > 0\) and \(P(u) = 0\), it is not difficult to show that \(\lambda = 0\) is the only zero of the equation \(\partial_\lambda S_\omega(u_{\lambda_0}) = 0\), and consequently
\[
\partial_\lambda S_\omega(u_{\lambda_0}) > 0 \quad \text{if } \lambda < 0
\]
\[
\partial_\lambda S_\omega(u_{\lambda_0}) < 0 \quad \text{if } \lambda > 0.
\]

This implies that \(S_\omega(u_{\lambda_0}) < S_\omega(u)\) for all \(\lambda \neq 0\). In particular, \(S_\omega(u_{\lambda_0}) < S_\omega(u)\). Now since \(I_\omega(u_{\lambda_0}) = 0\), from Proposition 1, we get \(S_\omega(\phi_\omega) \leq S_\omega(u_{\lambda_0}) < S_\omega(u)\). Thus taking the infimum, we get \(n(\omega) \geq S_\omega(\phi_\omega)\), which finishes the proof. \(\square\)

**Lemma 3.3.** Let \(\omega > 0\) and \(1 + \frac{4-2b}{N} < p < 2^*.\) If \(u \in K_\omega\), then
\[
P(u) \leq -(S_\omega(\phi_\omega) - S_\omega(u)).
\]

**Proof.** Let \(I(\lambda) = S_\omega(u_{\lambda})\), where \(u_{\lambda}(x) = e^{\lambda x} u(e^{\lambda} x)\) for \(\lambda \in \mathbb{R}\). By (3.1), we infer that
\[
I(\lambda) = \frac{e^{2\lambda}}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda}{2} \|u\|_{L^2}^2 - \frac{e^{\lambda \frac{N(p-1)+2b}{2}}}{p+1} \left\| \frac{\lambda}{p+1} u \right\|_{L^{p+1}}^{p+1},
\]
\[
I'(\lambda) = e^{2\lambda} \|\nabla u\|_{L^2}^2 - \left( \frac{N(p-1)+2b}{2} \right) \frac{e^{\lambda \frac{N(p-1)+2b}{2}}}{p+1} \left\| \frac{\lambda}{p+1} u \right\|_{L^{p+1}}^{p+1},
\]
\[
I''(\lambda) = 2e^{2\lambda} \|\nabla u\|_{L^2}^2 - \left( \frac{N(p-1)+2b}{2} \right)^2 \frac{e^{\lambda \frac{N(p-1)+2b}{2}}}{p+1} \left\| \frac{\lambda}{p+1} u \right\|_{L^{p+1}}^{p+1}.
\]

Now we remark that
\[
I''(\lambda) - 2I'(\lambda)
\]
Thus, we get globally in $[0, \infty)$ because $l'(0) = P(u) < 0$, we infer that $l'(\lambda) < 0$ for sufficiently small $\lambda$. Moreover, there exists $\lambda_0 < 0$ such that $l'(\lambda) < 0$ for $\lambda \in (\lambda_0, 0]$ and $l'(\lambda_0) = 0$. Integrating (3.2) on $[\lambda_0, 0]$ we get

$$l'(0) - l'(\lambda_0) \leq (l(0) - l(\lambda_0)).$$

Thus, since $P(u_{\lambda_0}) = l'(\lambda_0) = 0$, from Lemma 3.2 we obtain

$$P(u) \leq -(S_\omega(\phi_\omega) - S_\omega(u)),$$
Proof of Corollary 2. Let \( u_0 \in \mathcal{K}_0 \cap \Sigma \) and \( u(t) \) be the solution of (1.1) with \( u(0) = u_0 \). By Theorem 1.2, we infer that \( u(t) \in \mathcal{K}_0 \cap \Sigma \) for all \( t \in [0, T_\ast) \). Let \( \delta := S_\omega(\phi_\omega) - S_\omega(u_0) > 0 \). By Lemma 3.3, we see that \( P(u(t)) \leq -\delta \) for all \( t \in [0, T_\ast) \). Now since \( u_0 \in \Sigma \), by the virial identity (see [9]) we have
\[
\partial_t^2 \int_{\mathbb{R}^N} |x|^2 |u(x,t)|^2 dx = 8P(u(t)) \leq -8\delta \quad \text{for all } t \in [0, T_\ast).
\] (3.3)
It is clear that \( \int_{\mathbb{R}^N} |x|^2 |u(t)|^2 dx \) can not verify (3.3) for all time \( t \). Therefore, it must be the case that \( T_\ast < \infty \), thus, the solution \( u(t) \) blows up in finite time. \( \square \)

4. Blow-up criterion for INLS. We start the section, proving the Corollary 1. We show that if \( u_0 \in H^1 \) satisfies the conditions (1.4) and (1.5), then there exists \( \omega > 0 \) such that \( u_0 \in \mathcal{K}_0 \). Thereby, by Theorem 1.2 (ii) we obtain the result.

Proof of Corollary 1. For \( \omega > 0 \) and \( \varphi \in H^1 \), consider the function
\[
f(\omega) = S_\omega(\varphi_\omega) - S_\omega(\varphi),
\]
where \( \phi_\omega \) is the radial positive solution to elliptic equation (1.2). From direct computations, we have that \( S_\omega(\varphi_\omega) = \omega^{1-s_c}S_1(Q) \) where \( s_c = \frac{p}{2} - \frac{2b}{p-1} \) and \( Q \) is the solution to elliptic equation (1.2) with \( \omega = 1 \).

First, we will prove that the condition
\[
E(u_0)^{s_c}M(u_0)^{1-s_c} < E(Q)^{s_c}M(Q)^{1-s_c}.
\] (4.1)
is sufficient to the existence of \( \omega > 0 \) so that \( S_\omega(u_0) < S_\omega(\phi_\omega) \), i.e., \( f(\omega) > 0 \).

Indeed, note that to study the existence of \( \omega \) such that \( f(\omega) > 0 \) is equivalent to show that \( \sup_{\omega > 0} f(\omega) > 0 \). The function \( f \) has a maximal point at
\[
\omega_0 = \left( \frac{M(u_0)}{2(1-s_c)S_1(Q)} \right)^{-\frac{1}{s_c}} > 0
\]
and
\[
f(\omega_0) = \omega_0^{1-s_c}S_1(Q) - \frac{\omega_0}{2} ||u_0||_L^2 - \frac{1}{2} \|\nabla u_0\|_L^2 + \frac{1}{p+1} \int |x|^{-b} |u|^{p+1} dx
\]
\[
= \left( \frac{M(u_0)}{2(1-s_c)S_1(Q)} \right)^{-\frac{1}{s_c}} S_1(Q) - \frac{1}{2} \left( \frac{M(u_0)}{2(1-s_c)S_1(Q)} \right)^{-\frac{1}{s_c}} M(u_0) - E(u_0)
\]
\[= s_c [2(1-s_c)]^{1-s_c} \frac{S_1(Q)^{\frac{1}{s_c}}}{M(u_0)^{\frac{1}{s_c}}} - E(u_0).
\] (4.2)
Furthermore, since
\[
||Q||_L^2 = \frac{(p-1)(1-s_c)}{p-1} \|\nabla Q\|^2_{L^2} = \frac{(p-1)(1-s_c)}{p+1} \int |x|^{-b} |Q|^{p+1} dx,
\] (4.3)
we have
\[
S_1(Q) = \frac{1}{2(1-s_c)} ||Q||_{L^2}^2 \quad \text{and} \quad E(Q) = \frac{s_c}{2(1-s_c)} ||Q||_{L^2}^2,
\]
and consequently,
\[
s_c [2(1-s_c)]^{1-s_c} S_1(Q) = E(Q)M(Q)^{\frac{1-s_c}{s_c}}.
\] (4.4)
Thereby, by (4.1), (4.2) and (4.4), we obtain
\[ f(\omega) = \frac{E(Q)M(Q)^{\frac{1-s_c}{4}}}{M(u_0)^{\frac{1-s_c}{4}}} - E(u_0) > 0. \]

Now, we claim that the conditions (4.1) and
\[ \|\nabla u_0\|_{L^2}^s \|u_0\|_{L^2}^{1-s_c} > \|\nabla Q\|_{L^2}^s \|Q\|_{L^2}^{1-s_c} \]  
(4.5)

imply that there exists \( \gamma < 0 \) such that
\[ P(u(t)) < \gamma < 0 \quad \text{for all} \quad t \in (-T, T). \]  
(4.6)

For this end, by definition of \( P \), mass and energy conservation laws, (4.1), (4.3) and (4.5), we get
\[ P(u(t)) = \|\nabla u(t)\|_{L^2}^2 - \frac{N(p-1) + 2b}{2(p+1)} \int |x|^{-b} |u(t)|^{p+1} \, dx \]
\[ = \frac{N(p-1) + 2b}{2} E(u_0) - \left( \frac{N(p-1) + 2b}{4} - 1 \right) \|\nabla u(t)\|_{L^2}^2 \]
\[ < \frac{N(p-1) + 2b}{2} M(Q)^{\frac{1-s_c}{4}} E(Q) \]
\[ - \left( \frac{N(p-1) + 2b}{4} - 1 \right) \|\nabla Q\|_{L^2}^s \|Q\|_{L^2}^{1-s_c} \frac{\|u_0\|_{L^2}^{\frac{1-s_c}{4}}}{\|u_0\|_{L^2}^{\frac{1-s_c}{4}}} = 0, \]

that is,
\[ P(u(t)) < 0 \quad \text{for all} \quad t \in (-T, T). \]  
(4.7)

From the Gagliardo-Nirenberg inequality (2.1) and (4.7)
\[ \frac{2(p+1)}{N(p-1) + 2b} \|\nabla u(t)\|_{L^2}^2 < \||x|^{-\frac{b}{p+1}} u(t)\|_{L^{p+1}}^{p+1} \]
\[ \leq C_{GN} \|\nabla u(t)\|_{L^2}^{N(p-1) + 2b} \|u(t)\|_{L^2}^{\frac{p+1-N(p-1)+2b}{2}}, \]

and thus, since \( \frac{N(p-1) + 2b}{2} > 2 \), there exists \( \varepsilon_0 > 0 \) such that \( \|\nabla u(t)\|_{L^2} > \varepsilon_0 \), for all \( t \in (-T, T) \).

There exists \( \delta_0 > 0 \) such that
\[ P(u(t)) < -\delta_0 \|\nabla u(t)\|_{L^2}^2. \]

Suppose by contradiction that for all \( \delta_n \) there exists \( t_n \in (-T, T) \) such that
\[ -\delta_n \left( \frac{N(p-1) + 2b}{4} - 1 \right) \|\nabla u(t_n)\|_{L^2}^2 < P(u(t_n)) < 0, \]

where \( \delta_n \to 0 \) as \( n \to \infty \). So, from the definition of \( P \), we have
\[ \frac{1}{2} \|\nabla u(t_n)\|_{L^2}^2 < (1 - \delta_n) \left( \frac{1}{2} - \frac{2}{N(p-1) + 2b} \right) \|\nabla u(t_n)\|_{L^2}^2 < E(u(t_n)). \]

Moreover, by (4.1), (4.3) and (4.5), we obtain
\[ E(u_n)^{s_c} M(u_n)^{1-s_c} > (1 - \delta_n)^{s_c} E(Q)^{s_c} M(Q)^{1-s_c}. \]

Using the mass and energy conservation laws and making \( n \to \infty \)
\[ E(u_0)^{s_c} M(u_0)^{1-s_c} \geq E(Q)^{s_c} M(Q)^{1-s_c}, \]

which is a contradiction with (4.5). Thus, (4.6) holds taking \( \gamma = -\delta_0 \varepsilon_0 \).  \( \square \)
Before of the proof of Proposition 1.3 we prove the following lemma on the boundedness of the weighted $L^{p+1}$-norm.

**Lemma 4.1.** Let $N \geq 1$, $0 < b < \min\{N, 2\}$, $1 < p$ and $q > (p+1)\frac{N}{N-b}$. Then, $L^q(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is continuously embedded in $L^{p+1}_b(\mathbb{R}^N)$, i.e.,

$$L^q(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \hookrightarrow L^{p+1}_b(\mathbb{R}^N).$$

**Proof.** Let $u \in L^q \cap L^2$. We divide the integral $\|| \cdot |^{-\frac{N}{N-b}} u \|_{L^{p+1}}^{p+1}$ of the following way

$$\int |x|^{-b} |u|^{p+1} \, dx = \int_{|x| \leq 1} |x|^{-b} |u|^{p+1} \, dx + \int_{|x| > 1} |x|^{-b} |u|^{p+1} \, dx.$$  

Since $q > (p+1)\frac{N}{N-b} > p + 1 > 2$, by Hölder’s inequality, there exists $\theta \in (0, 1)$ such that

$$\int_{|x| > 1} |x|^{-b} |u|^{p+1} \, dx < \int_{|x| > 1} |u|^{p+1} \, dx \leq c\|u\|_{L^q}^{1-\theta} \|u\|_{L^2}^\theta.$$

On the other hand, note that if $\alpha < N$, then

$$\int_{|x| \leq 1} |x|^{-\alpha} \, dx < \infty. \quad (4.8)$$

Recalling the condition $(p+1)\frac{N}{N-b} < q$, we get $\frac{N-b}{N} - \frac{p+1}{q} > 0$. Let $0 < \varepsilon < \frac{N-b}{N} - \frac{p+1}{q}$ and $r' = \frac{b}{N} + \varepsilon$ such that $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{N}{N-b} < (p+1)r < q$. Thus, by Hölder’s inequality, Sobolev embedding and (4.8)

$$\int_{|x| \leq 1} |x|^{-b} |u|^{p+1} \, dx \leq \left( \int_{|x| \leq 1} |x|^{-br'} \, dx \right)^{\frac{1}{r'}} \left( \int_{|x| \leq 1} |u|^{(p+1)r} \, dx \right)^{\frac{1}{r}} \leq c\|u\|_{L^q}^{\frac{1}{r}},$$

which concludes the proof. 

**Proof of Theorem 1.3.** Suppose by contradiction that there exists a global solution $u$ to (1.1) such that

$$A_0 := \sup_{t \in (0, \infty)} \|u(t)\|_{L^q} < \infty, \quad \text{for some } q > (p+1)\frac{N}{N-b}.$$  

From Lemma 4.1, the energy conservation law and the assumption above, we can define

$$B_0 := \sup_{t \in (0, \infty)} \|\nabla u(t)\|_{L^2}.$$  

Consider the function

$$I(t) = \int \psi(x) |u(x,t)|^2 \, dx \quad (4.9)$$

defined in $[0, \infty)$.

From direct computations, we obtain

$$I'(t) = 2\text{Im} \int \nabla \psi(x) \cdot \nabla u(x) \overline{\psi}(x) \, dx \quad (4.10)$$

and

$$I''(t) = 4\text{Re} \sum_{i,j=1}^{N} \partial_i u(x) \partial_j \overline{\psi}(x) \, dx - \int |u(x)|^2 \Delta^2 \psi(x).$$
If we suppose $\psi$ radial, then

$$I'(t) = 2 \text{Im} \int \partial_r \psi \frac{x}{r} \cdot \nabla u dx$$

and

$$I''(t) = 4 \int \frac{\partial_r \psi}{r} |\nabla u|^2 dx + 4 \int \left( \frac{\partial^2 \psi}{r^2} - \frac{\partial_r \psi}{r^3} \right) |x| \cdot |\nabla u|^2 dx$$

$$- \frac{2(p-1)}{p+1} \int \left[ \frac{\partial^2 \psi}{r^2} + \left( N - 1 + \frac{2b}{p-1} \right) \frac{\partial_r \psi}{r} \right] |x|^{-b} |u|^{p+1} dx + \int \Delta^2 \psi |u|^2 dx,$$

where the notation $\partial_r$ denote the derivative in relation to $r = |x|$.

For $R > 0$, consider $\psi(x) = f_1(x)$ in (4.9), where $f_1$ is a radial function such that

$$f_1(x) = \begin{cases} 
0, & \text{if } 0 \leq |x| \leq \frac{R}{2} \\
1, & \text{if } |x| \geq R,
\end{cases}$$

$0 \leq f_1(x) \leq 1$ and $\partial_r f_1(x) \leq 4$ for all $x \in \mathbb{R}^N$.

Thus, defining $m_0 = \|u_0\|_{L^2}$ and using the Hölder’s inequality, definition of $f_1$ and (4.10), we have

$$I(t) = I(0) + \int I'(s) ds \leq I(0) + t \|\partial_r f_1\|_{L^\infty} \|u\|_{L^2} \|\nabla u\|_{L^2}$$

$$\leq \int_{|x| \geq \frac{R}{2}} |u_0|^2 dx + \frac{4}{R} m_0 B_0 t.$$

Now, note that

$$\int_{|x| \geq \frac{R}{2}} |u_0(x)|^2 dx = o_R(1) \quad \text{and} \quad \int_{|x| \geq R} |u(x,t)|^2 dx \leq I(t).$$

Therefore, fixed $\eta_0 > 0$, for each $t \leq \frac{\eta_0 R}{4m_0 B_0}$, we have

$$\int_{|x| \geq R} |u(x,t)|^2 dx \leq \eta_0 + o_R(1). \tag{4.12}$$

On the other hand, we can rewrite $I''(t)$ in (4.11) as

$$I''(t) = 8P(u(t)) + K_1 + K_2 + K_3,$$

where

$$K_1 = 4 \int \left( \frac{\partial_r \psi}{r} - 2 \right) |\nabla u|^2 dx + 4 \int \left( \frac{\partial^2 \psi}{r^2} - \frac{\partial_r \psi}{r^3} \right) |x| \cdot |\nabla u|^2 dx,$$

$$K_2 = - \frac{2(p-1)}{p+1} \int \left[ \frac{\partial^2 \psi}{r^2} + \left( N + 1 - \frac{2b}{p-1} \right) \frac{\partial_r \psi}{r} \right] |x|^{-b} |u|^{p+1}$$

$$+ \frac{4(N(p-1) + 2b)}{p+1} \int |x|^{-b} |u|^{p+1} dx,$$

$$K_3 = - \int \Delta^2 \psi |u|^2 dx.$$
Making ψ(x) = f_2(x) in (4.9), where f_2 is a radial function such that
\[ f_2(x) = \begin{cases} 
|x|^2, & \text{if } 0 \leq |x| \leq R 
0, & \text{if } |x| \geq 2R,
\end{cases} \]
0 ≤ f_2(x) ≤ |x|^2, ∂_r f_2(x) ≤ 2 and ∂^2_r f_2(x) ≤ \frac{4}{R^2} for all x ∈ \mathbb{R}^N, we claim that there exist two constants C_0 = C_0(p, N, m_0, A_0) > 0 and θ_q > 0 such that
\[ I''(t) ≤ 8P(u(t)) + C_0\|u(t)\|_{L^2(|x|>R)}^{θ_q}. \quad (4.13) \]
Indeed, we first show that K_1 ≤ 0. For this end, we divide the \mathbb{R}^N in two regions
\[ \Omega = \left\{ x \in \mathbb{R}^N; \frac{∂^2_r f_2(x)}{r^2} - \frac{∂_r f_2(x)}{r^3} ≤ 0 \right\} \]
and
\[ \mathbb{R}^N \setminus \Omega = \left\{ x \in \mathbb{R}^N; \frac{∂^2_r f_2(x)}{r^2} - \frac{∂_r f_2(x)}{r^3} > 0 \right\}. \]
Since ∂^2_r f_2 ≤ 2 follows ∂_r f_2(x) ≤ 2|x| for all x ∈ \mathbb{R}^N. Thus, using the Cauchy-Schwartz inequality, we get
\[ K_1 = 4 \int \left( \frac{∂_r f_2}{r} - 2 \right) |\nabla u|^2 \, dx + 4 \int \left( \frac{∂^2_r f_2}{r^2} - \frac{∂_r f_2}{r^3} \right) |x \cdot \nabla u|^2 \, dx \]
\[ = 4 \int \left( \frac{∂_r f_2}{r} - 2 \right) |\nabla u|^2 \, dx + 4 \int \left( \frac{∂^2_r f_2}{r^2} - \frac{∂_r f_2}{r^3} \right) |x \cdot \nabla u|^2 \, dx \]
\[ + 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{∂_r f_2}{r} - 2 \right) |\nabla u|^2 \, dx + 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{∂^2_r f_2}{r^2} - \frac{∂_r f_2}{r^3} \right) |x \cdot \nabla u|^2 \, dx \]
\[ ≤ 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{∂_r f_2}{r} - 2 \right) |\nabla u|^2 \, dx + 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{∂^2_r f_2}{r^2} - \frac{∂_r f_2}{r^3} \right) |x|^2 |\nabla u|^2 \, dx \]
\[ = 4 \int \left( \frac{∂^2_r f_2}{r} - 2 \right) |\nabla u|^2 \, dx ≤ 0. \quad (4.14) \]
To estimate K_2, note that if 0 ≤ |x| ≤ R, then
\[ ∂_r f_2(x) = 2|x| = 2r, \quad ∂^2_r f_2(x) = 2, \]
and thus,
\[ ∂^2_r f_2 + \left( N - 1 + \frac{2b}{p - 1} \right) \frac{∂_r f_2}{r} - 2N - \frac{4b}{p - 1} = 0. \]
Hence,
\[ \text{supp} \left[ ∂^2_r f_2 + \left( N - 1 + \frac{2b}{p - 1} \right) \frac{∂_r f_2}{r} - 2N - \frac{4b}{p - 1} \right] \subset (R, \infty), \]
and by interpolation there exists 0 < θ_q ≤ 1 such that
\[ K_2 ≤ c \int_{|x|>R} |x|^{-b} |u|^{p+1} \, dx \leq cR^{-b} \|u\|_{L^∞(|x|>R)}^{1-θ_q} \|u\|_{L^2(|x|>R)}^{θ_q} \]
\[ ≤ cA_0^{-θ_q} \|u\|_{L^2(|x|>R)}^{θ_q}, \quad (4.15) \]
where c > 0 only dependent of p, N. Moreover, by definition of f_2
\[ K_3 ≤ cR^{-2} \|u\|_{L^2(|x|>R)}^{2}. \quad (4.16) \]
Thus, from (4.14), (4.15) and (4.16), for R > 1 one obtains (4.13).
For $R>1$ to be chosen later, applying (4.12) in (4.13), we get for any $t \leq T := \frac{n_R}{4m_0B_0}$, \[ I''(t) \leq 8P(u(t)) + C_0(\eta_0^\delta + o_R(1)). \] Now, integrating the inequality above from 0 to $s$ and then from 0 to $T$, we have \[
 I(T) \leq I(0) + I'(0)T + \int_0^T \int_0^s \left( 8P(u(t)) + C_0(\eta_0^\delta + o_R(1)) \right) \, dt \, ds \]
\[ \leq I(0) + I'(0)T + 8s \gamma + C_0\eta_0^\delta + o_R(1) \frac{1}{2} T^2. \]
Choosing $\eta_0$ such that
\[ C_0\eta_0^\delta = -\gamma \]
and taking $R$ large enough, we obtain
\[ I(T) \leq I(0) + I'(0)\frac{n_R R}{4m_0B_0} + \alpha_0 R^2, \quad (4.17) \]
where
\[ \alpha_0 = \frac{\gamma \eta^2}{(4m_0B_0)^2} < 0. \]
Note that $\alpha_0$ independent of $R$. Moreover, by definition $f_2$, we have
\[ I(0) = \int f_2(x)|u_0|^2 \, dx \leq \int_{|x| \leq \sqrt{R}} |x|^2 |u_0|^2 \, dx + \int_{\sqrt{R} < |x| < 2R} |x|^2 |u_0|^2 \, dx \]
\[ \leq Rm_0^2 + R^2 \int_{|x| > \sqrt{R}} |u_0|^2 \, dx \leq Rm_0^2 + R^2 \o_R(1) \leq R^2 \o_R(1). \quad (4.18) \]
Similarly, since $\partial f_2(x) \leq 2|x|$ for all $x \in \mathbb{R}^N$,
\[ I'(0) \leq 2 \text{Im} \int \partial_r f_2(x) \frac{x \cdot \nabla u_0 \bar{u}_0}{r} \, dx \leq c \int_{0 \leq |x| < 2R} |x||\nabla u_0||u_0| \, dx \]
\[ \leq c \int_{0 \leq |x| < 2R} |x||\nabla u_0|^2 \, dx + c \int_{0 \leq |x| < 2R} |x||u_0|^2 \, dx \leq R \o_R(1). \quad (4.19) \]
Thereby, from (4.17), (4.18), (4.19) and choosing $R$ large enough, we obtain
\[ I(T) \leq o_R(1)R^2 + \alpha_0 R^2 \leq \frac{1}{2} \alpha_0 R^2, \]
and thus, since $\alpha_0 < 0$ we have
\[ I(T) < 0, \]
which is a contradiction with definition (4.9) of $I$. \[
\square \]

5. **Stability and strong instability of ground states.** In this section, we study the stability/instability of standing waves for (1.1).

**Stability.** For $\phi_\omega \in \mathcal{G}_\omega$ and $\delta > 0$, we set
\[ U_\delta(\phi_\omega) = \{ v \in H^1(\mathbb{R}^N) : \inf_{\theta \in \mathbb{R}} \| v - e^{i\theta} \phi_\omega \|_{H^1} < \delta \}. \]
We introduce the $C^1$ map $\omega(\cdot) : U_\delta(\phi_\omega) \to \mathbb{R}$ defined by
\[ \omega(v) = d^{-1} \left( \frac{p - 1}{2(p + 1)} \| x \|^{p+1} \| v \|_{L^{p+1}}^{p+1} \right). \]
Next, we need to prove the following lemma, from which the Theorem 1.5(i) is deduced.
Lemma 5.1. Let $1 < p < 1 + \frac{4 - 2b}{\lambda}$ and $\omega_0 > 0$. Then there exists $\delta = \delta(\omega_0) > 0$ such that for all $u \in U_\delta(\phi_{\omega_0})$
\[ E(u) - E(\phi_{\omega_0}) + \frac{\omega(u)}{2} \{ M(u) - M(\phi_{\omega_0}) \} \geq \frac{1}{4} d'(\omega_0)(\omega(u) - \omega_0)^2, \] (5.1)
where $M(u) = \|u\|_{L^2}^2$.

Proof. We adapt here a proof given in [11, Proposition 1] (see also Fibich and Wang [10]). Let $\omega > 0$. First we claim that
\[ S_\omega(\phi) = d_\delta(\omega) := \inf \left\{ S_\omega(u) : \|x|\frac{\partial}{\partial x} u\|_{L^{p+1}}^p = \|x|\frac{\partial}{\partial x} \phi\|_{L^{p+1}}^p \right\}. \] (5.2)
Indeed, it is clear that $d_\delta(\omega) \leq S_\omega(\phi)$. On the other hand, it is not difficult to show that
\[ \|x|\frac{\partial}{\partial x} \phi\|_{L^{p+1}}^p = \inf \left\{ \|x|\frac{\partial}{\partial x} u\|_{L^{p+1}}^p : u \in H^1(\mathbb{R}^N) \setminus \{0\}, I_u(u) = 0 \right\}. \] (5.2)
Let $v \in H^1(\mathbb{R}^N)$, such that $\|x|\frac{\partial}{\partial x} v\|_{L^{p+1}}^p = \|x|\frac{\partial}{\partial x} \phi\|_{L^{p+1}}^p$. Now we show that $I_\omega(v) \geq 0$. Assume that $I_\omega(v) < 0$. Then there exists $\lambda \in (0,1)$ such that $I_\omega(\lambda v) = 0$. By (5.2), this implies that
\[ \|x|\frac{\partial}{\partial x} \phi\|_{L^{p+1}}^p \leq \|x|\frac{\partial}{\partial x} \lambda v\|_{L^{p+1}}^p = \|x|\frac{\partial}{\partial x} v\|_{L^{p+1}}^p, \]
which is a contradiction. Therefore $I_\omega(v) \geq 0$, and
\[ S_\omega(v) \geq \frac{p - 1}{2(p + 1)} \|x|\frac{\partial}{\partial x} v\|_{L^{p+1}}^p = \frac{p - 1}{2(p + 1)} \|x|\frac{\partial}{\partial x} \phi\|_{L^{p+1}}^p = S_\omega(\phi). \]
This shows the claim, that is, $S_\omega(\phi) = d_\delta(\omega)$. On the other hand, it is easy to see that if $\phi_1$ is the ground states of equation (1.2) with $\omega = 1$, then $\phi_\omega(x) = \omega^{\frac{2 - b}{p - b}} \phi_1(\omega^{\frac{1}{2}} x)$ is the ground states of equation (1.2) for $\omega > 0$. In particular, this implies that the function $d(\omega) = S_\omega(\phi_\omega)$ is the class $C^2((0, \infty))$. Moreover, since $1 < p < 1 + \frac{4 - 2b}{\lambda}$, we have $d'(\omega) = \partial_\omega \|\phi_\omega\|_{L^2}^2 > 0$ for all $\omega > 0$. By Taylor’s expansion Theorem, we infer that there exists $\delta := \delta(\omega_0) > 0$ such that for any $\omega$ with $|\omega - \omega_0| < \delta$,
\[ d(\omega) \geq d(\omega_0) + d'(\omega_0)(\omega - \omega_0) + \frac{1}{4} d''(\omega_0)(\omega - \omega_0)^2. \] (5.3)
Now, let $u \in U_\delta(\phi_{\omega_0})$. We infer that
\[ \frac{p - 1}{2(p + 1)} \|x|\frac{\partial}{\partial x} u\|_{L^{p+1}}^p = d(\omega(u)) = \frac{p - 1}{2(p + 1)} \|x|\frac{\partial}{\partial x} \phi_{\omega(u)}\|_{L^{p+1}}^p. \] From (5.2), we see that $S_\omega(u) \geq S_\omega(\phi_{\omega(u)})$. Since $d'(\omega_0) = \frac{1}{2} M(\phi_{\omega_0})$, by (5.3), it follows that
\[ E(u) + \frac{\omega(u)}{2} M(u) \geq S_\omega(u)(\phi_{\omega(u)}) = d(\omega(u)) \geq d(\omega_0) + d'(\omega_0)(\omega(u) - \omega_0) + \frac{1}{4} d''(\omega_0)(\omega(u) - \omega_0)^2 \]
\[ = E(\phi_{\omega_0}) + \frac{1}{2} \omega_0 M(\phi_{\omega_0}) + \frac{1}{2} M(\phi_{\omega_0})(\omega(u) - \omega_0) \]
\[ + \frac{1}{4} d''(\omega_0)(\omega(u) - \omega_0)^2, \]
which implies (5.1). This completes the proof. \qed
Now, in view of the previous preparatory work, we can state the proof of Theorem 1.5 (i).

**proof of Theorem 1.5(i).** We verify the statement of Theorem 1.5(i) by contradiction. Assume that \( e^{i\omega t} \phi_{\omega_0} \) is not stable in \( H^1(\mathbb{R}^N) \). Then we have \( \epsilon > 0 \), a sequence \((u_n,0)_{n \in \mathbb{N}} \in U_{1/n}(\phi_{\omega_0})\) such that

\[
\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \| u_n(t) - e^{i\theta} \phi_\omega \|_{H^1} \geq \epsilon,
\]

and a time sequence \((t_n)_{n \in \mathbb{N}}\) such that

\[
\inf_{\theta \in \mathbb{R}} \| u_n(t_n) - e^{i\theta} \phi_\omega \|_{H^1} = \frac{\epsilon}{2}, \tag{5.4}
\]

where \( u_n \) is the corresponding solution to (1.1) with \( u_n(0) = u_n,0 \). Set \( v_n(x) := u_n(x,t_n) \). By conservation laws, as \( n \to \infty \), we have

\[
M(v_n) = M(u_n(t_n)) = M(u_n,0) \to M(\phi_\omega) \tag{5.5}
\]

\[
E(v_n) = E(u_n(t_n)) = E(u_n,0) \to E(\phi_\omega). \tag{5.6}
\]

Notice that \( \| v(t_n) \|_{H^1} \leq C \) uniformly in \( n \). Since \( \omega(\cdot) \) is continuous, it follows that the sequence \( \omega_n := \omega(v(t_n)) \) is bounded. Taking \( n \) large enough, by Lemma 5.1 we infer that

\[
E(v_n) - E(\phi_{\omega_0}) + \frac{\omega(v_n)}{2} \{ M(v_n) - M(\phi_{\omega_0}) \} \geq \frac{1}{4} d''(\omega_0)(\omega(v_n) - \omega_0)^2.
\]

By using \( d''(\omega_0) > 0 \), from (5.5) and (5.6) we obtain that \( \omega(v_n) \to \omega_0 \) as \( n \) goes to \( \infty \). Moreover, since \( I_{\phi_\omega}(\phi_{\omega_0}) = 0 \), we get

\[
\lim_{n \to \infty} \frac{p-1}{2(p+1)} \| x |^{\frac{p-1}{2} \omega} v_n \|_{L^{p+1}}^{p+1} = \lim_{n \to \infty} d(\omega_n) = d(\omega_0) = \| x |^{\frac{p-1}{2} \omega} \phi_{\omega_0} \|_{L^{p+1}},
\]

and by (5.5)-(5.6), we have

\[
\lim_{n \to \infty} S_{\omega_n}(v_n) = d(\omega_0), \tag{5.7}
\]

as \( n \) goes to \( \infty \). Now, from (5.5), (5.6) and (5.7), it is easy to show that there exists \( l_n > 0 \) such that \( I_{\phi_\omega}(l_nv_n) = 0 \) and \( l_n \to 1 \) as \( n \to \infty \). Set \( w_n := l_nv_n \). Then we have \( \| v_n - w_n \|_{H^1} \to 0 \) and \( S_{\omega_n}(w_n) \to d(\omega_0) \), as \( n \) goes to \( \infty \); that is, \( \{ w_n \} \) is a minimizing sequence for \( d(\omega) \). By Proposition 1, there exists \( \theta \in \mathbb{R} \) such that \( \| w_n - e^{i\theta} \phi_{\omega_0} \|_{H^1} \to 0 \). This implies that \( \| u_n(t_n) - e^{i\theta} \phi_{\omega_0} \|_{H^1} \to 0 \), which is a contradiction with (5.4). This completes the proof of Theorem 1.5(i). \( \square \)

**Strong instability.** The following is the key lemma for our proof.

**Lemma 5.2.** Let \( 1 + \frac{4b}{N} < p < 2^* \). If \( \lambda > 0 \), then \( \phi_\omega^\lambda \in K_{\omega} \cap \Sigma \). Here \( \phi_\omega^\lambda(x) := e^{\frac{\lambda N}{2} \phi_\omega(e^\lambda x)} \).

**Proof.** Let \( \lambda \in \mathbb{R} \). Since \( \phi_\omega \in \Sigma \), it follows that \( \phi_\omega^\lambda \in \Sigma \). On the other hand, direct computations show that

\[
S_\omega(\phi_\omega^\lambda) = \frac{\lambda^2}{2} \| \nabla \phi_\omega \|_{L^2}^2 + \frac{\omega}{2} \| \phi_\omega \|_{L^2}^2 - \frac{\lambda^{\frac{N(p-1)+2b}{2}}}{p+1} \| x |^{\frac{p-1}{2} \omega} \phi_\omega \|_{L^{p+1}}^{p+1},
\]

\[
\partial_\lambda S_\omega(\phi_\omega^\lambda) = \frac{\lambda e^{\frac{\lambda N}{2} \phi_\omega}}{2} \left( N(p-1)+2b \right) \frac{\lambda^{\frac{N(p-1)+2b}{2}}}{p+1} \| x |^{\frac{p-1}{2} \omega} \phi_\omega \|_{L^{p+1}}^{p+1}.
\]
Since \( P(\phi_\lambda) = 0 \), a simple calculation shows that \( \partial_\lambda S_\omega(\phi_\lambda^\lambda) = 0 \) has a unique zero point \( \lambda = 0 \), which implies
\[
\partial_\lambda S_\lambda^\lambda(\phi_\omega) > 0 \quad \text{if } \lambda < 0,
\partial_\lambda S_\lambda^\lambda(\phi_\omega) < 0 \quad \text{if } \lambda > 0.
\]
Then we see that \( S_\omega(\phi_\lambda^\lambda) < S_\omega(\phi_\omega) \) for all \( \lambda \neq 0 \). Note that \( \partial_\lambda S_\lambda^\lambda(\phi_\omega) = \partial_\lambda^\lambda(\phi_\lambda^\omega) \).

This implies that
\[
P(\phi_\lambda^\lambda) > 0 \quad \text{if } \lambda < 0,
P(\phi_\lambda^\lambda) < 0 \quad \text{if } \lambda > 0.
\]
In particular, \( S_\omega(\phi_\lambda^\lambda) < S_\omega(\phi_\omega) \) and \( P(\phi_\lambda^\lambda) < 0 \) for all \( \lambda > 0 \); that is, \( \phi_\lambda^\lambda \in K^- \cap \Sigma \), which completes the proof.

Now we are able to prove Theorem 1.5 (ii).

**Proof of Theorem 1.5 (ii).** Now we prove (ii) of of Theorem 1.5. Let \( \epsilon > 0 \). Since \( \|\phi_\lambda^\lambda - \phi_\omega\|_{H^1} \to 0 \) as \( \lambda \to 0 \), there exists \( \lambda_0 > 0 \) such that \( \|\phi_\lambda^\lambda - \phi_\omega\|_{H^1} < \epsilon \). From Lemma 5.2, we infer that \( \phi_\lambda^\lambda \in K^- \cap \Sigma \). Moreover, by Corollary 2, the corresponding solution \( u(t) \) of (1.1) with \( u(0) = \phi_\lambda^\lambda \) blows up in finite time, which implies that the standing wave \( u(x, t) = e^{i\omega t} \phi_\omega(x) \) is strongly unstable. This completes the proof of Theorem 1.5 (ii). \( \square \)

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**REFERENCES**

[1] G. P. Agrawal, *Nonlinear Fiber Optics*. Academic Press, 2007.
[2] A. H. Ardila and V. D. Dinh, Some qualitative studies of the focusing inhomogeneous Gross-Pitaevskii equation, *Z. Angew. Math. Phys.*, 71 (2020), 24pp.
[3] T. Cazenave, *Semilinear Schrödinger Equations*, American Mathematical Society, Courant Institute of Mathematical Sciences, 2003.
[4] J. Chen, On a class of nonlinear inhomogeneous Schrödinger equations, *J. Appl. Math. Comput.*, 32 (2010), 237–253.
[5] J. Chen and B. Guo, Sharp global existence and blowing up results for inhomogeneous Schrödinger equations, *Discrete Contin. Dyn. Syst. Ser. B*, 8 (2007), 357–367.
[6] V. Combet and F. Genoud, Classification of minimal mass blow-up solutions for an \( L^2 \) critical inhomogeneous NLS, *J. Evol. Equ.*, 16 (2016), 483–500.
[7] D. Du, Y. Wu and K. Zhang, On blow-up criterion for the nonlinear Schrödinger equation, *Discrete Contin. Dyn. Sys.*, 36 (2016), 3639–3650.
[8] A. de Bouard and R. Fukuizumi, Stability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, *Ann. Henri Poincaré*, 6 (2015), 1157–1177.
[9] L. G. Farah, Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation, *J. Evol. Equ.*, 16 (2016), 193–208.
[10] G. Fibich and X. P. Wang, Equations with inhomogeneous nonlinearities, *Physica D*, 175 (2003), 96–108.
[11] R. Fukuizumi, Equations with critical power nonlinearity and potentials. *Adv. Differ. Equ.*, 10 (2005), 259–276.
[12] F. Genoud, Bifurcation and stability of travelling waves in self-focusing planar waveguides, *Adv. Nonlinear Stu.*, 10 (2010), 357–400.
[13] F. Genoud, An inhomogeneous, \( L^2 \)-critical, nonlinear Schrödinger equation, *Z. Anal. Anwend.*, 31 (2012), 283–290.
[14] F. Genoud and C. Stuart, Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves, *Discrete Contin. Dyn. Syst.*, 21 (2008), 137–186.
[15] T. Saanouni, Remarks on the inhomogeneous fractional nonlinear Schrödinger equation, *J. Math Phys.*, 57 (2016) 081503.

[16] J. Toland, Uniqueness of positive solutions of some semilinear Sturm-Liouville problems on the half line, *Proc. Roy. Soc. Edinburgh Sect. A*, 97 (1984), 259–263.

[17] E. Yanagida, Uniqueness of positive radial solutions of $\delta u + g(r)u + h(r)u^p = 0$ in $\mathbb{R}^N$, *Arch. Rat. Mech. Anal.*, 115 (1991), 257–274.

[18] S. Zhu, Blow-up solutions for the inhomogeneous Schrödinger equation with $L^2$ supercritical nonlinearity, *J. Math. Anal. Appl.*, 409 (2014), 760–776.

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