Conformal isometry of the Reissner-Nordström-de Sitter black hole

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Abstract

It was pointed out by Couch and Torrence that the extreme Reissner-Nordström solution possesses a discrete conformal isometry. Using results of Romans, it is shown that such a symmetry also exists when a non-zero cosmological constant is allowed.

Introduction

In \cite{1}, Couch and Torrence found a conformal isometry that interchanges the event horizon and null infinity $J$ of an extremal Reissner-Nordström black hole. Related but distinct ideas have also appeared in a string theory context \cite{2}. Here we will show that such a conformal isometry exists also for the case of a positive cosmological constant, provided that the surface gravities of the two horizons are equal. We will leave global issues aside and refer the reader to \cite{3} for those matters. See also \cite{4} for a discussion of conserved quantities in asymptotically de Sitter spacetimes from a Hamiltonian point of view.

The calculation

In the Couch-Torrence case the conformal isometry switches the roles of the event horizon and infinity. With a positive cosmological constant, there is another geometrically distinguished object: the cosmological horizon. As $\Lambda \to 0$, this horizon approaches infinity. Let us therefore provisionally assume that the putative conformal isometry interchanges the black hole horizon and the cosmological horizon since this produces the correct limiting behavior; this assumption will later turn out to be correct.

Assume that the BH horizon is at $r = a$ and the cosmological horizon at $r = b$. It is convenient to introduce the coordinate $x$ defined by

$$x = \frac{r - a}{b - r} \frac{b}{a}$$

The strategy here is to first re-express the above metric in terms of the $x$ coordinate. In such a coordinate system, the metric will be manifestly conformally invariant under the inversion $x \mapsto 1/x$ as we will now show (note that the

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surface gravities of the two horizons will remain equal after the inversion since the surface gravity $\kappa$, suitably defined, is a conformal invariant [5].

For a general spherically symmetric black hole, the metric in the standard coordinates is

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2d\Omega^2$$  \hspace{1cm} (1)$$

where $V(r)$ is given by

$$V(r) = \frac{1 - 2m}{r} + \frac{Z^2}{r^2} - \frac{1}{3}\Lambda r^2$$  \hspace{1cm} (2)$$

A calculation shows that $\kappa = |V'|/2$. For the particular case of so-called “lukewarm black holes”, where the surface gravities of the two horizons are equal, Romans has shown [6] that the parameters take the values

$$m = \frac{ab}{a + b} = \pm Z \quad \Lambda = \frac{3}{(a + b)^2}$$

Note that the mass remains equal to the charge when a cosmological constant is introduced, so in this sense the black hole is still extremal. Inserting those parameter values into $V(r)$, one obtains

$$V(r) \propto \frac{1}{r^2}(r - a)(r - b)(r^2 + (a + b)r - ab)$$

In terms of $x$, the right-hand side of (2) becomes

$$W(x) = \frac{x}{(1 + x)^2(b + ax)^2} \left(2ab(1 + x^2) + x(a + b)^2\right)$$  \hspace{1cm} (3)$$

We are now in a position to verify that the map $x \mapsto 1/x$ is indeed a conformal isometry. The condition for this to hold is that $ds^2(x) = f^2ds^2(1/x)$ where $f^2$ is the conformal factor. The easiest way to verify this is to notice that the $d\Omega^2$ term in the metric only involves $x$ through the $r^2$, so we immediately find that the conformal factor $f^2$ has to satisfy

$$f^2 = \left(\frac{a + bx}{b + ax}\right)^2$$  \hspace{1cm} (4)$$

One can check that the conformal isometry conditions for the $dt^2$ and the $dr^2$ terms both lead to

$$\frac{W(x)}{W(1/x)} = f^2$$

Since $f^2$ is known from (4) and $W(x)$ is given by (3), one can simply calculate the left-hand side of the above expression and find that the equality does hold, meaning that the inversion is indeed a conformal isometry.
Optical geometry

All this can be nicely illustrated in terms of optical geometry\(^2\). The optical geometry is obtained as the spatial part of the conformally rescaled metric

\[
\frac{1}{V(r)} ds^2 = -dt^2 + \frac{dr^2}{V(r)^2} + \frac{r^2}{V(r)} d\Omega^2
\]

A null geodesic in the full spacetime projects down to a geodesic in the optical geometry, hence its name.

The optical geometry looks roughly like figure 1. As outlined above, we would expect the conformal inversion to interchange the two horizons; in this picture, this would amount to swapping the two spheres at infinity (they appear as circles in the figure). The neck between the spheres is where the optical geometry has its minimal radius; just like in the Schwarzschild case, this occurs at \(r = 2m\), i.e. \(r = 2ab/(a+b)\). One can verify that this is a fixed point of the conformal inversion \(x \mapsto 1/x\).

A calculation reveals the curvature scalar of the optical geometry associated with 1 to be

\[
R_{opt} = \frac{1}{2r^2} \left(4V - 4V^2 - 3r^2V'^2 + 4rVV' + 4r^2VV''\right)
\]

This means that \(R_{opt} \to -3V'^2/2 = -6\kappa^2\) as \(r \to a\). If we restrict the definition of a Couch-Torrence symmetry to be an isometry of the optical geometry, one obtains in this way a necessary condition for such a symmetry to exist: since \(R_{opt} \to 0\) when one approaches infinity for an asymptotically flat black hole, \(\kappa\) must equal zero at the horizon in that case.

![Figure 1](optical_geometry.png)

Figure 1: Optical geometry of the R-N-dS black hole with one dimension suppressed, embedded in the Poincaré ball. The dashed line marks the location of the neck at \(r = 2m\).

\(^2\)For a more complete exposition of optical geometry for Reissner-Nordström black holes, see [7]. For optical geometry in general, see [8] or [9].
Conclusions and open ends

One may ask the question whether there are other black-hole spacetimes for which a conformal isometry of the Couch-Torrence type exists. This seems to be tricky to answer; since the isometry is discrete, infinitesimal methods using Killing vectors are not likely to be very useful. One could hope for the existence of a sufficient condition something along the lines of the curvature scalar condition referred to above. This remains to be investigated.

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