DIAMETERS OF CAYLEY GRAPHS OF \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \)

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Abstract. We show that for integers \( k \geq 2 \) and \( n \geq 3 \), the diameter of the Cayley graph of \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \) associated to a standard two-element generating set, is at most a constant times \( n^2 \ln k \). This answers a question of A. Lubotzky concerning \( \text{SL}_n(\mathbb{F}_p) \) and is unexpected because these Cayley graphs do not form an expander family. Our proof amounts to a quick algorithm for finding short words representing elements of \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \).

1. Introduction

This paper concerns expressing elements of \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \), for integers \( k \geq 2 \) and \( n \geq 3 \), as words in the two-element generating set \( \{ \mathcal{A}_n, \mathcal{B}_n \} \), where

\[
\mathcal{A}_n := \begin{pmatrix}
1 & 1 \\
1 & 1 \\
\vdots & \ddots \\
1 & 1
\end{pmatrix}, \quad \mathcal{B}_n := \begin{pmatrix}
0 & 1 \\
0 & 1 \\
\vdots & \ddots \\
(-1)^{n-1} & 1 \\
0
\end{pmatrix}.
\]

From the point of view of word length, one might suspect this to be an inefficient generating set because the conjugates of \( \mathcal{A}_n \) by small powers of \( \mathcal{B}_n \) generate a nilpotent group, and the diameters of nilpotent groups are large \( \mathbb{1} \). However we show in this paper:

**Theorem 1.1.** For all integers \( k \geq 2 \) and \( n \geq 3 \),

\[
\text{Diam}_{\text{Cay}}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), \{ \mathcal{A}_n, \mathcal{B}_n \}) \leq 3600 n^2 \ln k.
\]

Moreover, there is an algorithm which expresses matrices in \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \) as words on \( \mathcal{A} \) and \( \mathcal{B} \) of length \( O(\ln |\text{SL}_n(\mathbb{Z}/k\mathbb{Z})|) \) in time \( O(\ln |\text{SL}_n(\mathbb{Z}/k\mathbb{Z})|) \).

The \( n^2 \ln k \) term is the best possible because a logarithm of \( |\text{SL}_n(\mathbb{Z}/k\mathbb{Z})| \sim k^{n^2-1} \) gives a lower bound on the diameter of \( \text{Cay}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), \{ \mathcal{A}_n, \mathcal{B}_n \}) \). More precise tracking of word length in our arguments would lead to an improvement of the constant from 3600 to at least 1400, but at the expense of complicating the exposition.

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Our result is better than that obtainable by known methods that use the heavy machinery of Property $T$, Kazhdan constants and expander families. For fixed $n \geq 3$, Property $T$ of $\text{SL}_n(\mathbb{Z})$ implies that

$$\{\text{Cay}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), \{A_n, B_n\}) \mid k \geq 2\}$$

is an expander family. So the diameter of $\text{Cay}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), \{A_n, B_n\})$ is at most $C(n) \ln k$, where the constant $C(n)$ is related to the Kazhdan constant $\mathcal{K}(\text{SL}_n(\mathbb{Z}), \{A_n, B_n\})$ by $C(n) < n^2 / \mathcal{K}(\text{SL}_n(\mathbb{Z}), \{A_n, B_n\})^2$. Lower bounds for Kazhdan constants are hard to come by. Using the bounds of [1] for $\mathcal{K}(\text{SL}_n(\mathbb{Z}), S)$, where $S$ is the set of all elementary matrices $e_{i,j}$, one can show $\mathcal{K}(\text{SL}_n(\mathbb{Z}), \{A_n, B_n\}) > n^{-3/2}$. This implies that $C(n) = O(n^5)$.

Were

$$\{\text{Cay}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), \{A_n, B_n\}) \mid k \geq 2, n \geq 3\}$$

an expander family, our $O(n^2 \ln k)$ bound would immediately follow. But this is not so: on page 105 of [11] an argument of Yael Luz is given that shows the expander constant of $\text{Cay}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), \{A_n, B_n\})$ to be at most $5/n$, which is not bounded away from 0. (In fact, in [11] the generating set considered has one additional element; none-the-less the argument there applies to $\{A_n, B_n\}$.) We remark that it follows that $\mathcal{K}(\text{SL}_n(\mathbb{Z}), \{A_n, B_n\}) \leq \sqrt{2/(n-2)}$.

Analogous results for $\text{SL}_2(\mathbb{Z}/k\mathbb{Z})$ and $\text{SL}_2(\mathbb{F}_p)$ cannot be proved using our methods. Indeed, there is no known fast algorithm which writes elements in $\text{SL}_2(\mathbb{F}_p)$ as short words in $A$ and $B$. For results in this direction see [3, 4, 9].

This article builds on methods in [13], where it is shown (Theorem 5.1) that for all $n \geq 3$, the diameter of $\text{Cay}(\text{SL}_n(\mathbb{F}_p), S)$ is at most a constant times $n^2 \ln p$, where $S$ is the set of all elementary matrices $e_{i,j}$. By expressing the elementary matrices as words in $A_n$ and $B_n$ one deduces [13, Corollary 1.1] that the diameter of $\text{Cay}(\text{SL}_n(\mathbb{F}_p), \{A_n, B_n\})$ is at most a constant times $n^3 \ln p$.

Theorem [14] affirmatively answers a question of A. Lubotzky [10] Problem 8.1.3 and improves on, and provides a constructive proof for, a result of Lubotzky, Babai and Kantor:

**Proposition 1.2** ([2, 10]). There exists $K > 0$ such that for all $n \geq 3$ and primes $p$, there is a set $\Sigma$ of three generators for $\text{SL}_n(\mathbb{F}_p)$ such that

$$\text{Diam} \ \text{Cay}(\text{SL}_n(\mathbb{F}_p), \Sigma) \leq Kn^2 \ln p.$$
subgroup, there exists \( K > 0 \) such that the diameter of any Chavalle group \( \Gamma \) over \( \mathbb{Z}/k\mathbb{Z} \), of rank at least 2, is at most \( K \ln |\Gamma| \).

A further extension shows that for every finite simple group \( \Gamma \) of Lie type and of rank at least 2, there is a 3 element generating set \( S \) such that \( \text{Diam Cay}(\Gamma, S) \leq K \ln |\Gamma| \). Combined with a similar result for the rank 1 groups from [2] and the corresponding result for the alternating/symmetric groups [10, Proposition 8.1.6], this yields

**Theorem 1.3.** There exists \( K > 0 \) such that every finite simple non-abelian group \( \Gamma \) has a 4 element generating set \( S \) such that

\[
\text{Diam Cay}(\Gamma, S) \leq K \ln |\Gamma|.
\]

Moreover, as all the proofs are suitably constructive, there is a fast algorithm which, given \( g \in \Gamma \) produces a word on \( S \) representing \( g \), provided that \( \Gamma \) is not a factor of a lattice in a rank 1 Lie group.

The following problems offer a broader context for the study of diameters of Cayley graphs of \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \) and \( \text{SL}_n(\mathbb{F}_p) \).

**Problem 1.4.** Fix \( n \geq 3 \). Does \( \text{SL}_n(\mathbb{Z}) \) enjoy uniform Property \( T \)?

**Problem 1.5.** Fix \( n \geq 2 \). Does there exist \( K(n) > 0 \) such that for all generating sets \( X \) for \( \text{SL}_n(\mathbb{Z}) \) and all primes \( p \)

\[
\text{Diam Cay}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), X) \leq K(n) \ln p.
\]

For fixed \( n \geq 3 \), an affirmative answer to Problem 1.4 would imply an affirmative answer to 1.5. More details can be found in [10] and [12].

**Problem 1.6.** What are the maximal and minimal Kazhdan constants for \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \) over generating sets of bounded size?

It is shown in [8] that \( \max \{ \mathcal{K}(\text{SL}_n(\mathbb{Z}/k\mathbb{Z}), X) \mid |X| < 30 \} > 10^{-4} \), and so there are 30-element generating sets, with respect to which the Cayley graphs of \( \text{SL}_n(\mathbb{Z}/k\mathbb{Z}) \) (with both \( n \) and \( k \) allowed to vary) form an expander family.

## 2. Generating bit-row and bit-column matrices

All the computations in this and the next section are in \( \text{SL}_n(\mathbb{Z}) \) where \( n \geq 3 \). Let us fix some notation and terminology. Denote the matrix with 1’s on the diagonal and in the \( i, j \)-th place, and 0’s everywhere else by \( e_{i,j} \). Suppressing their subscripts, we denote \( A_n \) and \( B_n \) by \( A \) and \( B \). Define a row (column) matrix to be a square matrix whose diagonal entries are all 1’s and which differs from the identity only in one row (column). A bit-row (bit-column) matrix is a row (column) matrix whose entries are all in \{0, \pm 1\}. For a sequence \( m = \{m_2, m_3, m_4, \ldots, m_n\} \) define \( R_m \) to be the row matrix whose entries all agree with those of the identity matrix except for those in row 1 which is

\[
(1, m_2, m_3, m_4, \ldots, m_n).
\]
This section is devoted to proving the following proposition and an analogy concerning column matrices.

**Proposition 2.1.** Suppose $M \in \text{SL}_n(\mathbb{Z})$ is a bit-row matrix. There is a word on $A$ and $B$ that represents $M$ and, if the first row of $M$ differs from the identity, has length at most $48n$, and at most $49n$ otherwise.

**Proof.** For integers $s_2, \ldots, s_{n-1}, t_2, \ldots, t_{n-2}$ define

$$N := \begin{pmatrix}
1 & 0 & 0 \\
1 & -s_2 & -t_2 \\
1 & -s_3 & -t_3 \\
& & \ddots \\
1 & -s_4 & -t_{n-2} \\
& & & \ddots \\
& & & & \ddots \\
& & & & & -s_{n-1} \\
& & & & & & 1
\end{pmatrix}.$$  

**Lemma 2.2.** The matrix $Ne_{1,2}N^{-1}$ is equal to $R_m$, where the sequence $\{m_i\}$ is defined recursively by $m_2 = 1$, $m_3 = s_2$ and $m_i = m_{i-1}s_{i-1} + m_{i-2}t_{i-2}$ for $4 \leq i \leq n$.

**Proof.** Rows 2 to $n$ of $R_mN$ and $Ne_{1,2}$ are the same as rows 2 to $n$ of $N$. The recursion defining $\{m_i\}$ ensures that the first rows of $R_mN$ and $Ne_{1,2}$ also agree.

**Lemma 2.3.** Suppose $m_2 = 1$ and $m_3, \ldots, m_n \in \{0, \pm 1, \pm 2\}$ satisfy one of the two conditions:

(i) $m_{2i} = 1$ for all $i$,
(ii) $m_{2i+1} = 1$ for all $i$.

Then there exist $s_2, \ldots, s_{n-1} \in \{0, \pm 1, \pm 2\}$ and $t_2, \ldots, t_{n-2} \in \{0, 1\}$ satisfying the equations in Lemma 2.2.

**Proof.** There is a solution in case (i) with $s_{2i} = m_{2i+1}$, $s_{2i+1} = 0$, $t_{2i} = 1$, $t_{2i+1} = 0$ for all $i \geq 1$, and in case (ii) with $s_2 = 1$, $s_{2i+2} = 0$, $s_{2i+1} = m_{2i+2}$, $t_{2i} = 0$, $t_{2i+1} = 1$, for all $i \geq 1$.

Our next lemma is an immediate consequence of the previous two.

**Lemma 2.4.** If $m_2 = 0$ and $m_3, \ldots, m_n \in \{0, \pm 1\}$ then there exist two matrices $N_1, N_2$ of the same form as $N$, having all entries in $\{0, \pm 1, \pm 2\}$, and satisfying $R_m = N_1e_{12}N_1^{-1}N_2e_{12}^{-1}N_2^{-1}$. 
Lemma 2.5. If \( s_2, \ldots, s_{n-1} \in \{0, \pm 1, \pm 2\} \) and \( t_2, \ldots, t_{n-2} \in \{0, \pm 1\} \) then the word
\[
w := B^{-(n-2)}Q_{n-1}BQ_{n-2}B \ldots Q_2B,
\]
where \( Q_i := A^{-s_i}[A, B^{-1}AB]^{-t_i} \) and \( t_{n-1} := 0 \), represents \( N \) and has length at most \( 12n - 24 \) as a word on \( A \) and \( B \).

Proof. This result follows from the observations that \( w \) equals
\[
(B^{-(n-2)}Q_{n-1}B^{n-2})(B^{-(n-3)}Q_{n-2}B^{n-3}) \ldots (B^{-1}Q_2B),
\]
in \( \text{SL}_n(\mathbb{Z}) \), and \( (B^{-(i-1)}Q_iB^{i-1}) \) equals the row matrix whose entries in the \( i \)-th row agree with those of \( N \) and whose remaining entries agree with the identity matrix.

To complete the proof of Proposition 2.1 in the row matrix case, first suppose \( M = R_m \) where \( m_2 = 0 \) and \( m_3, \ldots, m_n \in \{0, \pm 1\} \). Lemmas 2.4 and 2.5 supply a word \( w_m \) on \( A \) and \( B \) that represents \( M \) and has length at most \( 4(12n - 24) + 2 \). We can change \( m_2 \) to \( \pm 1 \) by right-multiplying by \( e_{1,2}^{\pm 1} = A^{\pm 1} \). This proves that \( \ell(w_m) < 48n \), as claimed.

Conjugating a matrix by a power of \( B \) moves its entries diagonally (changing some of their signs if \( n \) is even). So any given bit-row matrix \( M \) equals \( B^l R_m B^{-l} \) for some \( m = \{m_i\}_{i=2}^n \) with \( m_i \in \{0, \pm 1\} \), and some \( -n/2 \leq l \leq n/2 \). The cost to word length of conjugating is at most \( n \) and so \( M \) can be written as a word on \( A \) and \( B \) of length at most \( 49n \). \( \square \)

There is a natural analog of Proposition 2.1 for bit-column matrices:

Proposition 2.6. If \( M \in \text{SL}_n(\mathbb{Z}) \) is a bit-column matrix then there is a word on \( A \) and \( B \) that represents \( M \) and has length at most \( 48n \) if the final column differs from the identity, and at most \( 49n \) otherwise.

Proof. As in Proposition 2.2 if we define
\[
\bar{N} := \begin{pmatrix}
1 & -s_1 & -t_1 & & & \\
1 & -s_2 & -t_2 & & & \\
& & \ddots & \ddots & & \\
1 & -s_{n-3} & -t_{n-3} & & & \\
1 & -s_{n-2} & 0 & & & \\
1 & 0 & & & & \\
1 & & & & & &
\end{pmatrix},
\]
we find \( \bar{N}^{-1} e_{n-1,n} \bar{N} \) is a column matrix \( C^t_m \) whose final column is
\[
(m'_1, \ldots, m'_{n-1}, 1)^t,
\]
where \( m'_{n-1} = 1, m'_{n-2} = s_{n-2} \) and \( m'_i = m'_{i+1} s_i + m'_{i+2} t_i \) for \( 1 \leq i \leq n-3 \).

The analogue of Lemma 2.3 implies that if \( m'_1, \ldots, m'_{n-2} \in \{0, \pm 1, \pm 2\} \) satisfies \( m'_{2i} = 1 \) for all \( i \), or \( m'_{2i+1} = 1 \) for all \( i \), then \( C^t_m \) can be written as a short word on \( A \) and \( B \). And, as in Lemma 2.4 two such column
Remark 2.7. This construction yields an algorithm with running time $O(n)$ which produces a short word on $A$ and $B$ representing any given bit-row or bit-column matrix in $\text{SL}_n(\mathbb{Z})$.

Remark 2.8. If we allow $s_i \in \{1, 2, 4\}$ in $N$ and we set all the $t_i = 0$ (so $N$ has one super-diagonal, not two) then it is possible to find shorter words representing certain row and column matrices in $\text{SL}_n(\mathbb{Z})$ whose entries are particular powers of 2. If $k$ is odd we can use these row and column matrices to obtain a better upper bound than that of Theorem 1.1. This breaks down when $k$ is even because there are insufficient invertible elements in $\mathbb{F}_2$.

3. Generating row and column matrices

Before we come to the main result of this section we give a lemma which is essentially [13, Lemma 2.2]. It concerns expressing matrices $e_{i,j}F_{2t}$ and $e_{i,j}F_{2t+1}$, where the powers are Fibonacci numbers (defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{i+2} = F_{i+1} + F_i$), as short words on $\{e_{i,j} \mid i \neq j\}$. This lemma will be superseded by Lemma 3.3 but the detailed calculation we give in the proof of this simpler case is key to understanding the proof of the more general result.

Lemma 3.1. For non-negative integers $l$, the words

$$e_{1,2}^2(e_{2,1}e_{1,2})^{l}e_{1,3}(e_{2,1}e_{1,2})^{-l}e_{1,2}^{-l}e_{2,1}e_{1,2}^{-l}e_{1,3}^{-1}e_{2,1}e_{1,2}^{-l}e_{1,2}^{-1}, \text{ and}$$

$$e_{1,2}^2(e_{2,1}e_{1,2})^{l}e_{2,3}(e_{2,1}e_{1,2})^{-l}e_{1,2}^{-l}e_{2,1}e_{1,2}^{-l}e_{2,3}^{-1}e_{2,1}e_{1,2}^{-l}e_{1,2}^{-1}$$

equal $e_{1,3}F_{2t}$ and $e_{1,3}F_{2t+1}$, respectively, in $\text{SL}_3(\mathbb{Z})$.

Proof. We multiply out the first of these words from left to right as follows.

The calculation for the second is similar. The notation for each step shown is $S \xrightarrow{T} ST$. 
from which we derive the bound on $L$.

\[ \text{Lemma 3.3.} \]

Suppose $m \in \text{SL}_n(\mathbb{Z})$ is a row or column matrix with entries in $\{-K+1, \ldots, 0, \ldots, K-1\}$, where $K \geq 1$. Then there is a word on $\mathcal{A}$ and $\mathcal{B}$, representing $M$, that has length at most $1200 n \ln K + 400 n$.

**Proof.** The proof in the row matrix case generalizes Lemma 3.1 – instead of using $e_{1,3}$ and $e_{2,3}$ we use general bit-row matrices; they allow the simultaneous construction of sums of Fibonacci numbers in entries 3, \ldots, $n$ of the first row. These sums of Fibonacci numbers are as per Zeckendorf’s Theorem \cite{Zeckendorf1972}, which states that every nonzero integer $m$ can be expressed in a unique way as

\[
m = \pm(F_{l_1} + F_{l_2} + \cdots + F_{l_r}),
\]

with $l_1 \geq 2$ and $l_{j+1} - l_j \geq 2$ for all $1 \leq j < r$. This result can be proved by an easy induction argument and, in fact, $F_{l_r}$ is the largest Fibonacci number no bigger than $|m|$, and $F_{l_{r-1}}$ is the largest no bigger than $|m| - F_{l_r}$, and so on. Since $F_s = (\tau^s - (-\tau)^{-s})/\sqrt{5}$ for all $s \in \mathbb{N}$, where $\tau := (1 + \sqrt{5})/2$, we get $F_s \geq (\tau^s - 1)/\sqrt{5}$. Thus, as $F_{l_r} \leq |m|$, we find

\[
l_r \leq \log_\tau(1 + |m|/\sqrt{5}) < 2 + 3 \ln |m|,
\]

from which we derive the bound on $L$ in the following lemma.

**Lemma 3.3.** Suppose $m := \{m_i\}_{i=3}^n$ is a sequence of integers, such that $|m_i| < K$ for all $i$. As per Zeckendorf’s Theorem, write

\[
m_i = \sum_{j=1}^L (c_{ij} F_{2j} + d_{ij} F_{2j+1})
\]
where \( c_{ij}, d_{ij} \in \{0, \pm 1\} \) and \( L \leq (2 + 3 \ln K)/2 - 1/2 \). Let \( u_m \) be the word
\[
(e_{2,1}e_{1,2})a_1b_1(e_{2,1}e_{1,2})a_2b_2(e_{2,1}e_{1,2}) \cdots (e_{2,1}e_{1,2})a_Lb_L
\]
in which \( a_j \) is the row matrix with first row \((1, 0, c_{3j}, \ldots, c_{nj})\) and \( b_j \) is the row matrix with second row \((0, 1, d_{3j}, \ldots, d_{nj})\). Let \( v_m \) be the word obtained from \( u_m \) by replacing each \( a_j \) and \( b_j \) by its inverse. Define
\[
w_m := e_{1,2}^2 u_m (e_{2,1}e_{1,2})^{-L} e_{1,2}^{-1} v_m (e_{2,1}e_{1,2})^{-L} e_{1,2}^{-1}.
\]
Then in \( \text{SL}_n(\mathbb{Z}) \) the row matrix with first row \((1, 0, m_3, \ldots, m_n)\) is represented by \( w_m \).

Proof. Lemma 3.1 gives the special cases of this lemma in which \( n = 3 \) and \( m_3 \) is \( F_{2l} \) or \( F_{2l+1} \). Below we multiply out \( w_m \) from left to right, using a more general and concise version of the calculation used to prove Lemma 3.1.

We display the top two rows only; all others agree with the identity matrix throughout the calculation. All the summations range over \( j = 1, \ldots, L \).

\[
\begin{pmatrix}
1 & 2 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0
\end{pmatrix}
\]

\[
\downarrow \quad u_m
\]

\[
\begin{pmatrix}
F_{2L+2} & F_{2L+3} & \sum (c_{3j}F_{2j+2} + d_{3j}F_{2j+3}) & \cdots & \sum (c_{nj}F_{2j+2} + d_{nj}F_{2j+3}) \\
F_{2L} & F_{2L+1} & \sum (c_{3j}F_{2j} + d_{3j}F_{2j+1}) & \cdots & \sum (c_{nj}F_{2j} + d_{nj}F_{2j+1})
\end{pmatrix}
\]

\[
\downarrow (e_{2,1}e_{1,2})^{-L}
\]

\[
\begin{pmatrix}
1 & 2 & \sum (c_{3j}F_{2j+2} + d_{3j}F_{2j+3}) & \cdots & \sum (c_{nj}F_{2j+2} + d_{nj}F_{2j+3}) \\
0 & 1 & \sum (c_{3j}F_{2j} + d_{3j}F_{2j+1}) & \cdots & \sum (c_{nj}F_{2j} + d_{nj}F_{2j+1})
\end{pmatrix}
\]

\[
\downarrow e_{1,2}^{-1} v_m
\]

\[
\begin{pmatrix}
F_{2L+1} & F_{2L+2} & \sum (c_{3j}F_{2j} + d_{3j}F_{2j+1}) & \cdots & \sum (c_{nj}F_{2j} + d_{nj}F_{2j+1}) \\
F_{2L} & F_{2L+1} & 0 & \cdots & 0
\end{pmatrix}
\]

\[
\downarrow (e_{2,1}e_{1,2})^{-L} e_{1,2}^{-1}
\]

\[
\begin{pmatrix}
1 & 0 & \sum (c_{3j}F_{2j} + d_{3j}F_{2j+1}) & \cdots & \sum (c_{nj}F_{2j} + d_{nj}F_{2j+1}) \\
0 & 1 & 0 & \cdots & 0
\end{pmatrix}
\]

The sums in this final matrix are, by definition, equal to \( m_3, \ldots, m_n \) and so the lemma is proved.
Returning to the proof of Proposition 3.2, note that a conjugate of $M$ by a power of $B$ is a row matrix $R_m$ in which the first row is $(1, m_2, m_3, \ldots, m_n)$. On the alphabet $A$ and $B$, we find $e_{1,2} = A$ and so has length 1, and $e_{2,1}, a_j, b_j$ are all bit-row matrices and so, by Proposition 3.1, can be expressed as words of length at most $48n$. So the word $w_m$ of Lemma 3.3 can be re-expressed as a word on $A$ and $B$ of length $390nL$, where the contributions to this estimate are

\[
\begin{align*}
4 + 4L \times 1 & \text{ from } e_{1,2} \\
4L \times 48n & \text{ from } e_{2,1} \\
2L \times 48n & \text{ from } a_j \\
2L \times 48n & \text{ from } b_j.
\end{align*}
\]

A revised version of Lemma 3.3 in which we build up Fibonacci numbers in columns 1 and $n$ using $e_{1,n}$ and $e_{n,1}$ rather than in columns 1 and 2 using $e_{1,2}$ and $e_{2,1}$, produces a word on $A$ and $B$ that represents the row matrix with first row $(1, m_2, 0, \ldots, 0)$. Mildly revising the estimates above, we check that the length of this word is at most $390nL$. Multiplying the two words together gives a word of length at most $780nL$ that represents $R_m$. Conjugating by a power of $B$ recovers $M$ at a further expense to word length of at most $n$. Then, using the bound on $L$ in Lemma 3.3, we learn that $M$ can be represented by a word on $A$ and $B$ of length at most $1200n \ln K + 400n$.

Obtain the same bound in the column matrix case by transposing and using Proposition 2.0 in place of 2.1 reverse the orders of the terms in $w_m, u_m$ and $v_m$, interchange the $e_{1,2}$’s and $e_{2,1}$’s, and make the $a_i$ and $b_i$ bit-column matrices rather than bit-row matrices. □

Remark 3.4. It follows from the construction above that there is an algorithm with running time $O(n \ln K)$ which produces a short word in $A$ and $B$ that represents any given row matrix in $\text{SL}_n(\mathbb{Z}/k\mathbb{Z})$ with entries in $\{-K+1, \ldots, 0, \ldots, K-1\}$.

4. The Diameter of $\text{SL}_n(\mathbb{Z}/k\mathbb{Z})$.

Proof of Theorem 1.1. All row matrices in $\text{SL}_n(\mathbb{Z}/k\mathbb{Z})$ come from row matrices in $\text{SL}_n(\mathbb{Z})$ with entries of absolute value less than $k/2$ and so can be represented by short words on $A$ and $B$ as per Proposition 3.2. So Lemma 4.3 below completes the proof of the bound in Theorem 1.1.

Our proof is constructive and amounts to an algorithm for expressing matrices in $\text{SL}_n(\mathbb{Z}/k\mathbb{Z})$ as words on $A_n$ and $B_n$ with running time

\[O(n^2 \ln k) = O(\ln |\text{SL}_n(\mathbb{Z}/k\mathbb{Z})|),\]

provided that $k$ is decomposed as a product of prime numbers. □

We start with a technical lemma which is also valid for rings satisfying the Bass stable range condition – see [14].
Lemma 4.1. Let \(a, b \in \mathbb{Z}/k\mathbb{Z}\). Then there exists \(s \in \mathbb{Z}/k\mathbb{Z}\) such that the ideal generated by \(a\) and \(b\) is the same as the ideal generated by \(a + sb\).

Proof. If \(k = \prod p_i^{m_i}\) then

\[
\mathbb{Z}/k\mathbb{Z} \cong \prod \mathbb{Z}/p_i^{m_i}\mathbb{Z}
\]

by the Chinese Remainder Theorem. Let \(a_i\) and \(b_i\) be the components of \(a\) and \(b\) in \(\mathbb{Z}/p_i^{m_i}\mathbb{Z}\). Define \(s_i := 0\) if the ideal generated by \(a_i\) in \(\mathbb{Z}/p_i^{m_i}\mathbb{Z}\) contains \(b_i\) and \(s_i := 1\) otherwise. Let \(s\) be the element in \(\mathbb{Z}/k\mathbb{Z}\) with components \(s_i\). By construction, the components of \(a + sb\) are \(a_i + s_i b_i\) and in the ring \(\mathbb{Z}/p_i^{m_i}\mathbb{Z}\) the ideal generated by \(a_i + s_i b_i\) is the same as the ideal generated by \(a_i\) and \(b_i\). \(\square\)

Corollary 4.2. Suppose \(\{a_i\}_{i=1}^l\) are elements of \(\mathbb{Z}/k\mathbb{Z}\) such that the ideal they generate is the whole ring. Then there exist \(\{t_i\}_{i=2}^l\) such that

\[
a_1 + t_2 a_2 + t_3 a_3 + \cdots + t_l a_l
\]

is invertible in \(\mathbb{Z}/k\mathbb{Z}\).

In fact, (given the decomposition of \(k\) into prime factors) we can write a fast algorithm to find these coefficients. This is because \(a_i\) and \(b_i\) of Lemma 4.1 can be found quickly, being \(a_i \mod p_i^{m_i}\) and \(b_i \mod p_i^{m_i}\), respectively. The maximal power of \(p_i\) dividing \(a_i\) and \(b_i\) determines \(s_i\). And in the proof of Lemma 4.1 we can use \(k \sum s_i/p_i^{m_i}\), which is easier to compute.

Lemma 4.3. If \(M \in \text{SL}_n(\mathbb{Z}/k\mathbb{Z})\) then the matrix \(M\) can be written as a product of \(n\) row matrices, \(n\) column matrices and \(n\) elementary matrices.

Proof. We use a version of Gauss-Jordan elimination to prove by induction on \(r = 0, \ldots, n\) that \(M\) can be transformed to a matrix in which the top \(r\) rows agree with the identity matrix by left- and right-multiplying by a total of \(3r\) row, column and elementary matrices.

The base step \(r = 0\) holds vacuously. For the induction step assume \(r < n\) and the top \(r\) rows agree with the identity matrix. If the final entry on the \((r+1)\)-st row is not invertible in \(\mathbb{Z}/k\mathbb{Z}\) then, using Corollary 4.2, it can be made invertible by right-multiplying by some column matrix, because the ideal generated by the \((r+1)\)-st to \(n\)-th entries in row \(r+1\) is the whole ring \(\mathbb{Z}/k\mathbb{Z}\). Then make the \(r+1, r+1\)-entry 1 by right-multiplying by the appropriate power of \(e_{n,r+1}\). Then clear all the off-diagonal entries in row \(r+1\) by right-multiplying by the appropriate row matrix. \(\square\)

Remark 4.4. The constructions in this paper can be used to express matrices \(M \in \text{SL}_n(\mathbb{Z})\) as short words on \(A\) and \(B\) (cf. [13, Theorem 4.1]). However, the resulting upper bounds on word length are not very good because if we express \(M\) as a product of row matrices \(R_i\) then the absolute values of the entries in the \(R_i\) may be significantly larger than the absolute values of the entries in \(M\).
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