Convergence to the Plancherel Measure of Hecke Eigenvalues

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Abstract

We give improved uniform estimates for the rate of convergence to Plancherel measure of Hecke eigenvalues of holomorphic forms of weight 2 and level N. These are applied to determine the sharp cutoff for the non-backtracking random walk on arithmetic Ramanujan graphs and to Serre’s problem of bounding the multiplicities of modular forms whose coefficients lie in number fields of degree $d \geq 1$.

1 Introduction

It is well known that the distribution of Hecke eigenvalues of modular forms at primes $p_1, \ldots, p_r$ converges to the product of the corresponding $p$-adic Plancherel measures as one varies over certain families ([Sar87], [Ser92], [CDF92]). Our aim in this paper is to establish uniform rates on this convergence and to apply these to problems of sharp cutoff for random walks on Ramanujan graphs ([NS21]) and to the factorization of the Jacobian of the modular curve $X_0(N)$ as in [Ser97].

The Eichler-Selberg trace formula expresses the trace of the Hecke operator $T_n$ on the space $S(N) := S_2(N)$ of holomorphic cusp forms of weight 2 for $\Gamma_0(N)$ in terms of class numbers of binary quadratic forms. Using this, one can show ([Ser97], Prop. 4) that as $f$ runs over a Hecke basis $H(N)$ of such eigenforms with eigenvalues $T_nf =: \lambda_f(n)\sqrt{n} \cdot f, \ (n, N) = 1,$ we will have

$$\frac{1}{|H(N)|} \left| \sum_{f \in H(N)} \lambda_f(n) - \frac{\delta_N(n, \square)}{12} \psi(N) \right| \ll_n (n/N)^{1/2} \cdot d(N),$$

(1)

where

$$\delta_N(n, \square) := \begin{cases} 1 & \text{if } n \text{ is a square modulo } N, \\ 0 & \text{otherwise}, \end{cases}$$

$d(N) := \sum_{d|N} 1 \ll N^\varepsilon$ is the divisor function, and $\psi(N) := N \prod_{p|N} (1 + 1/p)$ is the Dedekind psi function. Murty and Sinha ([MS09]) give explicit and effective bounds in (1).

To extend the range of $n$ for which the left-hand side of (1) goes to 0 with $N$, we introduce and remove $1/L(1, \text{Sym}^2 f)$ weights into the sum. This allows us to use the Petersson trace formula, replacing class numbers with Kloosterman sums, which enjoy sharp bounds coming from the arithmetic geometry of curves ([Wei48]). This technique applied to a similar problem is outlined in [Sar02] and is used on other problems in [Iwa84] and [ILS00]. It allows us to double the exponent in the range of $n$, which is crucial for certain applications.
In what follows, our aim is to establish sharp estimates, and to simplify the analysis, we assume that \( N \) is prime. In much of what we do, this assumption can be removed. We address this further in the final section.

**Theorem 1.** Let \( \varepsilon > 0 \) and let \( m, n \) be integers coprime to \( N \). Then:

\[
\frac{1}{|H(N)|} \sum_{f \in H(N)} \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \atop d^2k=mn}} \frac{1}{k} + O_{\varepsilon}\left(\frac{(mn)^{1/8}}{N^{1/2}}(mn)^{\varepsilon}\right).
\]

With this theorem, we can formulate and prove a corresponding uniform convergence to the Plancherel measure. Let \( r \geq 1 \), and for \( \ell_1, \ldots, \ell_r \geq 0 \), let \( \mathcal{P}^{\ell_1, \ldots, \ell_r} \) denote the set of polynomials in \( x_1, \ldots, x_r \) of degrees at most \( \ell_1, \ldots, \ell_r \), respectively, that is,

\[
\mathcal{P}^{\ell_1, \ldots, \ell_r} := \left\{ \sum_{j_1=0}^{\ell_1} \cdots \sum_{j_r=0}^{\ell_r} a_{j_1, \ldots, j_r} x_1^{j_1} \cdots x_r^{j_r} : a_{j_1, \ldots, j_r} \in \mathbb{C} \right\}.
\]

For \( p \nmid N \), let \( \theta_f(p) \in [0, \pi] \) be such that

\[
\lambda_f(p) = 2\cos(\theta_f(p))
\]

(such a \( \theta_f \) exists because of self-adjointness of \( T_p \) and thanks to the Ramanujan bound \( |\lambda_f(p)| \leq 2 \) due to Eichler). Let \( \mu_p \) be the \( p \)-adic Plancherel measure:

\[
d\mu_p := \frac{2}{\pi} \cdot \frac{(p + 1) \sin^2 \theta}{(p^{1/2} + p^{-1/2})^2 - 4 \cos^2 \theta} d\theta.
\]

(2)

We have the following uniform convergence result:

**Theorem 2.** Let \( r \geq 1 \) and \( \eta > 0 \). Then uniformly for \( p_1^{\ell_1} \cdots p_r^{\ell_r} < N^{2-\eta} \) and \( P \in \mathcal{P}^{\ell_1, \ldots, \ell_r}, \)

\[
\frac{1}{|H(N)|} \sum_{f \in H(N)} |P(\cos \theta_f(p_1), \ldots, \cos \theta_f(p_r))|^2 = (1 + o(1)) \int_0^\pi \cdots \int_0^\pi |P|^2 d\mu_{p_1} \cdots d\mu_{p_r}
\]

as \( N \to \infty \).

This result with an exponent of \( N \) larger than 1 (which corresponds to \( mn \) going up to \( N^{2-\varepsilon} \) in Theorem 1) is what is needed to settle the cutoff window for the non-backtracking random walks on Ramanujan graphs (NS21). In fact, it yields the conjectured asymptotics for the variance for these walks (see end of section).

Another application of Theorem 2 is to multiplicities of \( f \)'s in a Hecke basis with given \( \lambda_f(p) \)'s for \( p \in \{p_1, \ldots, p_r\} \). Let \( s(N) := |H(N)| = \dim S(N) \), so for \( N \) prime, \( s(N) = \left\lfloor \frac{N+1}{12} \right\rfloor - 1 \) when \( N \equiv 1 \) (mod 12) and \( s(N) = \left\lfloor \frac{N+1}{12} \right\rfloor \) otherwise. For \( \varphi \in S(N) \), let

\[
M_N(y, \varphi) := \#\{ f \in H(N) : \lambda_f(p) = \lambda_\varphi(p) \text{ for } p \leq y, (p, N) = 1 \}.
\]

For a fixed \( y \), Theorem 2 implies that uniformly in \( \varphi, \)

\[
M_N(y, \varphi) \ll \frac{s(N)}{(\log N)^r},
\]

(3)

where \( r = \pi(y) \) is the number of primes up to \( y \). If \( y \) is allowed to increase with \( N \), then one can exploit that for \( f \) in the set defining \( M_N(y, \varphi) \), we also have \( \lambda_f(m) = \lambda_\varphi(m) \) for all \( y \)-smooth numbers \( m \) (which are numbers all of whose prime factors are at most \( y \)). This allows one to improve vastly.
Such an argument using the large sieve for Dirichlet characters is due to Linnik ([Lin41]). In the modular form setting, Duke and Kowalski ([DK00]) establish that the number of non-monomial newforms of square-free level up to $N$ that have prescribed eigenvalues $\lambda_f(p)$ at primes $p \leq y = (\log N)^\beta$ satisfies

$$M_{\leq N}(y, \varphi) \# \ll_{\beta} N^{10/\beta + \varepsilon},$$

which is non-trivial for $\beta > 5$. Lau and Wu ([LW08]) show that for $y = C \log N$ with $C$ a large constant, there is $c > 0$ s.t.

$$M_N(y, \varphi) \ll \exp \left( -c \log N \over \log \log N \right) s(N).$$

Our interest is in smaller $y$'s, namely $y = (\log N)^\beta$ with $0 < \beta < 1$.

**Theorem 3.** Fix $\beta \in (0, 1)$. Then for $y = (\log N)^\beta$ and uniformly in $\varphi$,

$$M_N(y, \varphi) \leq \exp \left( -1 \over \beta \right) (\log N)^\beta + o ((\log N)^\beta) s(N)$$

as $N \to \infty$.

We apply this to a question of Serre ([Ser97]). Assume that all $f = \sum_{n \geq 1} a(n)e(nz) \in H(N)$ are normalized so $a(1) = 1$. The Fourier coefficients $a(n)$ are algebraic integers in a totally real field of degree $d(f)$. For $d \geq 1$, let $s(N)_d$ denote the number of $f$'s for which $d(f) = d$. Serre shows that for $d$ fixed, $s(N)_d = o(s(N))$ (see also [MS09], Theorem 5), and asks for stronger upper bounds. Theorem 3 implies such a bound.

**Theorem 4.** Fix $d \geq 1$ and $\beta < \frac{2}{d+1}$. Then as $N \to \infty$,

$$s(N)_d \leq \exp \left( -c(d, \beta) (\log N)^\beta + o ((\log N)^\beta) \right) s(N),$$

where $c(d, \beta) := (1 - \beta) / \beta - d/2 > 0$.

This falls short of Serre’s conjecture, which asserts that Theorem 1 holds for $\beta = 1$ and $c = c(d) > 0$ (i.e., $s(N)_d \ll s(N)^\alpha$ for some $\alpha < 1$).

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## 2 Weight Removal in the Petersson Formula

Throughout this section we assume $N$ is prime. Let $H(N)$ denote a simultaneous eigenbasis of Hecke operators $T_k$, $(k, N) = 1$, acting on the space $S(N)$ of dimension $s(N)$ of weight 2 level $N$ cusp forms for $\Gamma_0(N)$, and for $f \in H(N)$, let $a_f(n)$ and $\lambda_f(n)$ be such that

$$f(z) = \sum_{n \geq 1} a_f(n)e(nz) = \sum_{n \geq 1} \sqrt{n} \lambda_f(n)e(nz),$$

where $e(z) := e^{2\pi iz}$. Assume $f$ are normalized so $a_f(1) = 1$.

Our starting point for this section is the Petersson trace formula estimated via the Weil bound on Kloosterman sums, as presented in [ILS00], Corollary 2.2 or [IK04], Corollary 14.24:

**Petersson Formula.** With $H(N)$ as above, $(mn, N) = 1$, and $\varepsilon > 0$,

$$\sum_{f \in H(N)} h \lambda_f(m) \lambda_f(n) = \delta(m, n) + O_\varepsilon \left( \left( mn \right)^{1/4} N (mn)^{-\varepsilon} \right).$$

(4)

Here the $h$ superscript signifies adding "harmonic" weights: $\sum_{f \in H(N)} \alpha_f = \frac{1}{4\pi} \sum_{f \in H(N)} \alpha_f \left\| f \right\|^2$, where $\left\| \cdot \right\|$ denotes the Petersson norm.
We derive Theorem \[ \text{four.lf} \] from the Petersson formula by removing the harmonic weights. The Petersson norm is related to the special value of the symmetric square \( L \)-function at 1 (\cite{symmetric-square-l-results}, Lemma 2.5) via:

\[
4\pi \|f\|^2 = \frac{s(N)}{\zeta(2)} L(\text{Sym}^2 f, 1)
\]

(where \( L(\text{Sym}^2 f, s) = \zeta(2s) (1 - N^{-2s}) \sum_{n \geq 1} \lambda_f(n^2)n^{-s} := \sum_{n=1}^{\infty} \lambda_{\text{sym}^2 f}(n)n^{-s} \), so

\[
\frac{1}{s(N)} \sum_{f \in H(N)} \lambda_f(m) \lambda_f(n) = \sum_{f \in H(N)}^{h} \lambda_f(m) \lambda_f(n) \cdot \frac{4\pi \|f\|^2}{s(N)} = \sum_{f \in H(N)}^{h} \lambda_f(m) \lambda_f(n) \frac{L(\text{Sym}^2 f, 1)}{\zeta(2)},
\]

(5)

and to prove Theorem \[ \text{five.lf} \] we need to derive a suitable approximation for \( L(\text{Sym}^2 f, s) \).

Let \( \Psi(x) \geq 0 \) be a smooth function supported on \([-1, 1]\) with \( \Psi(0) = 1 \). The Mellin transform

\[
\hat{\Psi}(s) = \int_{0}^{\infty} \Psi(x)x^s \frac{dx}{x}
\]

is an analytic function of \( s = \sigma + it \) for \( \sigma > -1 \), except for a simple pole at 0 with residue 1, and decreases rapidly as \( |t| \to \infty \) for \(-1 \leq \sigma \leq 2\).

For a parameter \( x > 0 \), let

\[
A := \frac{1}{2\pi i} \int_{\sigma=2} L(\text{Sym}^2 f, s+1)x^s \hat{\Psi}(s) ds
= \frac{1}{2\pi i} \int_{\sigma=2} \sum_{\nu \geq 1} \frac{\lambda_{\text{sym}^2 f}(\nu)}{\nu^s+1} x^s \hat{\Psi}(s) ds
= \sum_{\nu \geq 1} \frac{\lambda_{\text{sym}^2 f}(\nu)}{\nu} \int_{\sigma=2} \left( \frac{\nu}{x} \right)^{-s} \hat{\Psi}(s) ds
= \sum_{\nu \leq x} \frac{\lambda_{\text{sym}^2 f}(\nu)}{\nu} \Psi \left( \frac{\nu}{x} \right)
\]

by the Mellin inversion theorem. Shifting the integral defining \( A \) to \( \text{Re}(s) = -1/2 \) picks up the simple pole of \( \hat{\Psi} \) at \( s = 0 \), so by the residue theorem,

\[
L(\text{Sym}^2 f, 1) = \sum_{\nu \leq x} \frac{\lambda_{\text{sym}^2 f}(\nu)}{\nu} \Psi \left( \frac{\nu}{x} \right) + R(f, x),
\]

(6)

where

\[
R(f, x) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} L(\text{Sym}^2 f, 1/2 + it)x^{-1/2+i} \hat{\Psi}(-1/2 + it) dt.
\]

Now, by Cauchy-Schwarz,

\[
|R(f, x)|^2 = \frac{1}{4\pi^2 x} \left| \int_{-\infty}^{\infty} L(\text{Sym}^2 f, 1/2 + it)x^it \hat{\Psi}(-1/2 + it) dt \right|^2
\ll x^{-1} \int_{-\infty}^{\infty} \left| L(\text{Sym}^2 f, 1/2 + it) \right|^2 \cdot \left| \hat{\Psi}(-1/2 + it) \right| dt,
\]

so

\[
\sum_{f \in H(N)}^{h} |R(f, x)|^2 \ll x^{-1} \int_{-\infty}^{\infty} \left| \hat{\Psi}(-1/2 + it) \right| \sum_{f \in H(N)}^{h} |L(1/2 + it, \text{Sym}^2 f)|^2 dt.
\]
According to the Lindelöf on average result due to Iwaniec and Michel for this family of \( L \)-functions ([IM01]),
\[
\sum_{f \in \mathcal{H}(N)} |L(1/2 + it, \text{Sym}^2 f)|^2 \ll_{\varepsilon} N^\varepsilon(|t| + 1)^8,
\]
so
\[
\sum_{f \in \mathcal{H}(N)} |R(f, x)|^2 \ll_{\varepsilon} x^{-1} N^\varepsilon. \tag{7}
\]
Substituting (7) and (6) into (5),
\[
\frac{1}{s(N)} \sum_{f \in \mathcal{H}(N)} \lambda_f(m)\lambda_f(n) = \frac{1}{\zeta(2)} \sum_{f \in \mathcal{H}(N)} \lambda_f(m)\lambda_f(n) \left( \sum_{\nu \leq x} \frac{\lambda_{\text{Sym}^2 f}(\nu)}{\nu} \Psi \left( \frac{\nu}{x} \right) + R(f, x) \right)
= \frac{1}{\zeta(2)} (I + II), \tag{8}
\]
where
\[
I = \sum_{\nu \leq x} \frac{\lambda_{\text{Sym}^2 f}(\nu)}{\nu} \Psi \left( \frac{\nu}{x} \right) \sum_{f \in \mathcal{H}(N)} \lambda_f(m)\lambda_f(n)
\]
and
\[
|II| \leq \sum_{f \in \mathcal{H}(N)} |\lambda_f(m)\lambda_f(n)||R(f, x)| \ll_{\varepsilon} N^\varepsilon \left( \sum_{f \in \mathcal{H}(N)} (mn)^\varepsilon \right)^{1/2} \left( \sum_{f \in \mathcal{H}(N)} |R(f, x)|^2 \right)^{1/2} \ll_{\varepsilon} (mnN)^\varepsilon, \]
where we used Cauchy-Schwartz and (7).
To estimate \( I \), we use Hecke relations
\[
\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f \left( \frac{mn}{d^2} \right)
\]
and the formula
\[
\lambda_{\text{Sym}^2 f}(\nu) = \sum_{t^2 k = \nu \atop (t,N) = 1} \lambda_f(k^2).
\]
From this,
\[
I = \sum_{t^2 k \leq x} \frac{\Psi \left( \frac{t^2 k}{x} \right)}{t^2 k} \sum_{d|(m,n)} \sum_{f \in \mathcal{H}(N)} \lambda_f \left( \frac{mn}{d^2} \right) \lambda_f \left( k^2 \right),
\]
so by the Petersson formula,
\[
I = \sum_{t^2 k \leq x} \frac{\Psi \left( \frac{t^2 k}{x} \right)}{t^2 k} \sum_{d|(m,n)} \sum_{m^2 = d^2 k^2} 1 + O_{\varepsilon} \left( \sum_{t^2 k \leq x} \frac{(mn)^\varepsilon (mnk^2)^{1/4}}{N} (Nmn)^\varepsilon \right)
= \left( \sum_{d|(m,n)} \frac{1}{k} \right) \sum_{t^2 k \leq x/k} \frac{\Psi \left( \frac{t^2 k}{x} \right)}{t^2} + O_{\varepsilon} \left( \frac{x^{1/2} (mn)^{1/4}}{N} (mnN)^\varepsilon \right)
= \zeta(2) \left( \sum_{d|(m,n)} \frac{1}{k} \right) \sum_{m^2 = d^2 k^2} 1 + O_{\varepsilon} \left( \frac{x^{1/2} (mn)^{1/4}}{N} (mnN)^\varepsilon \right).
\]

Combining estimates of I and II with (8),

\[
\frac{1}{s(N)} \sum_{f \in H(N)} \lambda_f(m) \lambda_f(n) = \sum_{d | (mn)} \frac{1}{k} + O_\varepsilon((mnN)^\varepsilon/x^{1/2}) + O_\varepsilon\left(\frac{x^{1/2}(mnN)^{1/4}}{N}\right).
\]

Choosing \( x = \frac{N}{(mn)^{1/4}} \), we finish the proof of Theorem 1.

3 Convergence to the Plancherel Product Measure

In this section we address Theorem 2. Fix an integer \( r > 0 \), let \( \ell_1, \ldots, \ell_r > 0 \), and let \( p_1, \ldots, p_r \) be distinct primes with \( (p_j, N) = 1 \).

Consider a polynomial \( P(x_1, \ldots, x_r) \in \mathbb{C}[x_1, \ldots, x_r] \) of degree at most \( \ell_i \) in \( x_i \) for \( 1 \leq i \leq r \). For \( n \geq 0 \), let

\[
U_n(\cos \theta) := \frac{\sin((n+1)\theta)}{\sin \theta}
\]

be the \( n \)th Chebyshev polynomial of the second kind. \( U_n \) is a degree \( n \) polynomial in \( \cos \theta \) with real coefficients, so we can find \( a_{t_1, \ldots, t_r} \in \mathbb{C} \) such that

\[
P(x_1, \ldots, x_r) = \sum_{t_1=0}^{\ell_1} \cdots \sum_{t_r=0}^{\ell_r} a_{t_1, \ldots, t_r} U_{t_1}(x_1) \cdots U_{t_r}(x_r).
\]

Moreover, Hecke relations imply that for \( (p, N) = 1 \), \( U_n(\lambda(p)) = \lambda(p^n) \). From this,

\[
|P(\theta_f(p_1), \ldots, \theta_f(p_r))|^2 = \sum_{t_1, s_1=0}^{\ell_1} \cdots \sum_{t_r, s_r=0}^{\ell_r} a_{t_1, \ldots, t_r} u_{s_1, \ldots, s_r} U_{t_1}(\theta_f(p_1)) \overline{U_{s_1}(\theta_f(p_1))} \cdots U_{t_r}(\theta_f(p_r)) \overline{U_{s_r}(\theta_f(p_r))} =
\]

\[
= \sum_{t_1, s_1} a_{t_1, \ldots, t_r} u_{s_1, \ldots, s_r} \lambda_f(p_1^{t_1}) \overline{\lambda_f(p_1^{s_1})} \cdots \lambda_f(p_r^{t_r}) \overline{\lambda_f(p_r^{s_r})},
\]

so by Theorem 1

\[
\frac{1}{|H(N)|} \sum_{f \in H(N)} |P(\theta_f(p_1), \ldots, \theta_f(p_r))|^2 = \sum_{t_1, s_1} a_{t_1, \ldots, t_r} u_{s_1, \ldots, s_r} \frac{1}{|H(N)|} \sum_{f \in H(N)} \lambda_f(p_1^{t_1}) \cdots \lambda_f(p_r^{t_r}) \overline{\lambda_f(p_1^{s_1}) \cdots \lambda_f(p_r^{s_r})} = I + II,
\]

where

\[
I = \sum_{s_1, \ldots, s_r} a_{s_1, \ldots, s_r} \sum_{m \sim p_1^{s_1} \cdots p_r^{s_r}} 1/k,
\]

and

6
Applying Cauchy-Schwartz and summing the geometric series,

\[ (*) \leq \left( \frac{p_1^{2t_1} \cdots p_r^{2t_r}}{\sqrt{N}} \right) \sum_{0 \leq t_i \leq \ell_i} p_1^{t_1/4} \cdots p_r^{t_r/4} \left( \sum_{0 \leq t_i \leq \ell_i} |a_{t_1, \ldots, t_r}| \right)^2 \leq \left( \frac{p_1^{2t_1} \cdots p_r^{2t_r} \mathcal{N}}{N^2} \right)^{1/2} \sum_{0 \leq t_i \leq \ell_i} |a_{t_1, \ldots, t_r}|^2. \]

Let \( \mu_\infty(\theta) : \frac{2}{\pi} \sin^2 \theta d\theta \) be the Sato-Tate measure on \([0, \pi]\). From the definition of \( d\mu_p \), it follows that

\[ d\mu_\infty \cdots d\mu_\infty \ll_r d\mu_{p_1} \cdots d\mu_{p_r}. \]

Hence, from (9) and the orthonormality of \( U_n \) with respect to \( d\mu_\infty \), we have

\[ \sum_{0 \leq t_i \leq \ell_i} |a_{t_1, \ldots, t_r}|^2 \ll \| P \|^2 \| \mu_{p_1, \ldots, p_r} \|. \]

We conclude that

\[ \Pi \ll \varepsilon \left( \frac{p_1^{t_1} \cdots p_r^{t_r} \mathcal{N}}{N^2} \right)^{1/2} \| P \|^2 \| \mu_{p_1, \ldots, p_r} \|. \]

It remains to evaluate \( I \). To interpret \( I \) as the integral against the Plancherel measure, we need the following observation:

**Proposition 3.1.** Let \( m, n \geq 0 \), and let \( d\mu_p \) be the \( p \)-adic Plancherel measure. Then:

\[
\int_0^\pi U_n(\theta) U_m(\theta) d\mu_p = \begin{cases} 
\frac{p}{p-1} \left( \frac{1}{p^{[m-1]/2}} - \frac{1}{p^{m+1/2}} \right) & \text{if } m \equiv n \pmod{2} \\
0 & \text{otherwise}
\end{cases} \quad (11)
\]

We leave the proof until the end of the section. From (9),

\[
\int_{[0, \pi]^n} |P(\theta_1, \ldots, \theta_r)|^2 d\mu_{p_1} \cdots d\mu_{p_r} = \sum_{t_i, s_i \leq \ell_i} a_{t_1 s_1} \cdots a_{t_r s_r} \int_{[0, \pi]^n} U_{t_1}(\theta_1) U_{s_1}(\theta_1) \cdots U_{t_r}(\theta_r) U_{s_r}(\theta_r) d\mu_{p_1} \cdots d\mu_{p_r},
\]

\[
= \sum_{t_i, s_i \leq \ell_i} a_{t_1 s_1} \cdots a_{t_r s_r} \prod_i \int_0^\pi U_i(\theta) U_i(\theta) d\mu_{p_i}. \quad (12)
\]

We substitute the inner product (11) into (12). Observe that

\[
\frac{p}{p-1} \left( \frac{1}{p^{[m-1]/2}} - \frac{1}{p^{m+1/2}} \right) = \frac{1}{p^{[m-n]/2}} + \frac{1}{p^{m+n+1/2}} + \cdots + \frac{1}{p^{m+n/2}}
\]

\[
= \sum_{m \equiv n \pmod{2}} \frac{1}{p^{m+n/2 -\ell}} P^{\frac{1}{2}}
\]

\[
= \sum_{d \mid p^m, p^n} \frac{1}{k^d},
\]

where

\[ d \mid p^m, p^n \implies d^2 k^2 = p^m p^n. \]
\[
\|P\|_{\mu_{p_1}, \ldots, \mu_{p_r}}^2 = \sum_{s_i, t_i \leq \ell_i, s_i \equiv t_i \mod 2} a_{t_i} \overline{a}_{s_i} \cdots a_{t_i} \overline{a}_{s_i} \cdot \prod_i \sum_{d_i(p_i^{t_i} - p_i^{s_i})} \frac{1}{k_i} = \sum_{s_i, t_i \leq \ell_i, s_i \equiv t_i \mod 2} a_{t_i} \overline{a}_{s_i} \cdots a_{t_i} \overline{a}_{s_i} \cdot \sum_{d_i(p_i^{t_i} - p_i^{s_i})} \frac{1}{k_i} = I.
\]

Combining I and II,

\[
\frac{1}{|H(N)|} \sum_{f \in H(N)} |P(\theta_f(p_1), \ldots, \theta_f(p_r))|^2 = \|P\|_{\mu_{p_1}, \ldots, \mu_{p_r}}^2 \left(1 + O(N^{6e} \cdot N^{-n/4})\right) = \|P\|_{\mu_{p_1}, \ldots, \mu_{p_r}}^2 \left(1 + o(1)\right)
\]

for small enough \(\varepsilon\). This concludes the proof of Theorem 2.

Proof of Proposition 3.7: From the trigonometric identity for the product of sines,

\[
\int_0^\pi U_n(\theta)U_m(\theta)d\mu_p = \frac{p + 1}{2\pi} \int_0^\pi \sin((n + 1)\theta)\sin((m + 1)\theta)d\theta
\]

\[
= \frac{p + 1}{4\pi} \int_0^\pi \cos((m - n)\theta) - \cos((m + n + 2)\theta)d\theta
\]

\[
= \mathcal{I}(m - n) - \mathcal{I}(m + n + 2),
\]

where \(\mathcal{I}(k) := \frac{p + 1}{4\pi} \int_0^\pi \frac{\cos(k\theta)}{4\pi^2 + \sin^2 \theta}d\theta\).

Since \(\sin^2(\theta) = \sin^2(\pi - \theta)\) and \(\cos(k(\pi - x)) = -\cos kx\) for odd \(k\), \(\mathcal{I}(k) = 0\) for odd \(k\), so the integral is 0 when \(m, n\) and \(p\) have different parity. To prove the proposition, it suffices to show that for all integers \(T \geq 0\),

\[
\mathcal{I}(2T) = \frac{(p + 1)}{4\pi} \int_0^\pi \frac{\cos(2T\theta)}{4\pi^2 + \sin^2 \theta}d\theta = \frac{p}{(p - 1)p^T}.
\]

We prove this statement by induction. Let \(\zeta := e^{ix}, c := \frac{(p - 1)^2}{4p}\), and let \(\alpha := 2 + 4c = p + 1/p\). Then:

\[
\int_0^\pi \frac{\cos(2T\theta)}{4\pi^2 + \sin^2 \theta}d\theta = \frac{1}{2} \int_0^{2\pi} \frac{(\zeta^{2T} + \zeta^{-2T})/2}{c + ((\zeta - \zeta^{-1})/2)^2}d\zeta =
\]

\[
= \frac{1}{2} \int_0^{2\pi} \frac{(\zeta^{2T} + \zeta^{-2T})}{-4c + (\zeta - \zeta^{-1})}d\zeta
\]

\[
= \frac{1}{2} \int_{\zeta^{2T-1}} \frac{(\zeta^{2T} + 1)}{(\zeta^4 - \alpha \zeta^2 + 1)}d\zeta
\]

\[
= \frac{1}{2} \int_{\zeta^{2T-1}} (\zeta^{2T} + 1)/(\zeta^2 - p)(\zeta^2 - 1/p)d\zeta.
\]

8
We evaluate the integral using the residue theorem. For $T = 0$, the poles are at $\pm \sqrt{1/p}$, and both residues are equal to 
\[
\frac{1/\sqrt{p}}{(p-1/p)(2/\sqrt{p})} = \frac{1}{p-1/p},
\]
so 
\[
\mathcal{I}(0) = (p+1) \cdot 2\pi (2p/(p^2 - 1))/4\pi = p/(p-1).
\]
For $T = 1$, the pole at 0 has residue $-1$ and the poles at $\pm 1/\sqrt{p}$ have residues 
\[
\frac{1/p^2+1}{(1/\sqrt{p})(p-1/p)(2/\sqrt{p})} = \frac{p^2+1}{2(p^2-1)},
\]
so 
\[
\mathcal{I}(2) = (p+1) \cdot 2\pi (1 + (p^2 + 1)/(p^2 - 1))/4\pi = 1/(p-1).
\]
Assume now $T \geq 2$. The rational function 
\[
-\frac{x^{2T} + 1}{x^{2T-1}(x^2 - p)(x^2 - 1/p)}
\]
has three poles inside the unit circle: $0$, $\omega = 1/\sqrt{p}$ and $-\omega$, and the two latter ones have the same residue. Let $A(T)$, $B(T)$ be the residues at 0 and $\omega$ respectively. Then:
\[
B(T) = -\frac{(\omega^2)^{2T} + 1}{2(\omega^2)^2(\omega^2 - p)} = \frac{1}{2(p-1/p)}(p^T + 1/p^T),
\]
and $A(T)$ is the coefficient of $x^{2T-2}$ in $1/(1 - \alpha x^2 + x^4) = \sum_{r \geq 0} x^{4r - \alpha x^2}$. Notice that both $A(T)$ and $B(T)$ satisfy the recurrence relation
\[
F(T + 2) - \alpha F(T + 1) + F(T) = 0.
\]
It remains to notice 
\[
\frac{p}{(p-1)p^T}
\]
satisfies the same recurrence relation, which proves \[14\].

To end this section, we apply Theorem 2 to the question of sharp cutoff of random walks on certain Ramanujan graphs. Let $\ell$ be a fixed prime and $p$ a large prime, $p \equiv 1 \pmod{12}$ (the notation here is made to conform with \[CGL09\]). The Brandt- Ihara-Pizer "super singular isogeny graphs," $G(p, \ell)$, are $d := \ell + 1$ regular graphs on $n := \frac{p^2 - 1}{12} + 1$ vertices (see \[CGL09\], page 4, for a description). The non-trivial eigenvalues of $G(p, \ell)$ are the numbers $2\sqrt{\ell} \cos(\theta_f(\ell))$ for $f \in H(N)$ ($N = p$ in our notation). The $G(p, \ell)$'s are $d$-regular Ramanujan graphs on $n$ vertices. The $L^2$-variance, $W_2(t)$, for the $t$-step non-backtracking random walk on $G(p, \ell)$ is given by (see \[NS21\], page 13)
\[
W_2(t) = \frac{\ell^t}{n} \sum_{f \in H(N)} |R_\ell(\cos \theta_f(\ell))|^2,
\]
where $R_\ell$ is the $t$th orthogonal polynomial on $[0, \pi]$ with respect to $d\mu_\ell$, normalized so that
\[
\int_0^\pi |R_\ell(\theta)|^2 d\mu_\ell(\theta) = \frac{\ell + 1}{\ell}.
\]
Applying Theorem 2 with $r = 1$, $p_1 = \ell$, and $\ell_1 = t$ yields that uniformly for $t < (2 - \eta) \log_\ell n$, 
\[
W_2(t) \sim (\ell + 1)\ell^{t-1} \text{ as } n \to \infty.
\]
Note that $N(t)$, the number of non-backtracking walks of length $t$, is $(\ell + 1)\ell^{t-1}$, so that
\[
W_2(t) \sim N(t) \text{ for } t < (2 - \eta) \log_\ell n.
\]
This proves conjecture 1.8 in \[NS21\] for graphs $G(p, \ell)$. For the application to bounded window cutoff one needs $t$ to be as large as $(1 + \varepsilon) \log_\ell n$, which is provided by the key doubling of the degree of $P$ in Theorem 2. In order to prove Conjecture 1.8 in \[NS21\] for the more general Ramanujan graphs constructed using modular forms, one would need to identify the images of division algebras in $H(N)$ under the Jacquet-Langlands correspondence and restrict the sums in Theorem 2 to those forms.
4 Multiplicity of Eigenvalue Tuples

Recall that for a fixed prime level $N$ and $\varphi \in S(N)$ a weight 2 holomorphic cusp form for $\Gamma_0(N)$, we let $M_N(y, \varphi)$ be the multiplicity of the tuple of eigenvalues of $\varphi$ at primes up to $y$ in a Hecke basis $H(N)$, i.e.

$$M_N(y, \varphi) := \# \{|f \in H(N) : \lambda_f(p) = \lambda_\varphi(p) \text{ for } p \leq y, (p, N) = 1\}|.$$ 

In this section we bound $M_N(y, \varphi)$ uniformly in $\varphi$ in the range $y = (\log N)^\beta$ for a fixed $\beta \in (0, 1)$. Specifically, we prove Theorem 3 via the large sieve and smooth number estimates.

From now on, we assume $y = o(\log N)$. We let $p_1, \ldots, p_r$ denote the first $r$ prime numbers, where $r = \pi(y)$ is the number of primes up to $y$.

An integer $m$ is called $y$-smooth if all primes $p|m$ satisfy $p \leq y$. The set of $y$-smooth numbers is denoted with $\mathcal{S}_y$, and the de Bruijn function $\Psi(y, M)$ is the counting function for $y$-smooth numbers up to $M$:

$$\Psi(y, M) := \# \{|m \in \mathcal{S}_y, m \leq M\}|.$$ 

We use the large sieve inequality as in [IK04], Theorem 7.26 (the inequality is stated in [IK03] for weight $k > 2$ but holds for $k = 2$ as well – see comment after the proof):

**Large Sieve Inequality.** Let $\mathcal{F}$ be an orthonormal basis of $S(N)$, $f(z) := \sum \rho_f(n) e(nz)$ for $f \in \mathcal{F}$. Then for any complex numbers $\{c_n\}$ we have

$$\sum_{f \in \mathcal{F}} \left| \sum_{n \leq M} c_n \rho_f(n) \frac{1}{\sqrt{n}} \right|^2 \ll (1 + M/N) \|c\|^2, \quad (15)$$

where $\|c\|^2 = \sum_{n \leq M} |c_n|^2$ and the implied constant is absolute.

We apply this with $M = N$ and

$$c_n := \begin{cases} \lambda_\varphi(n) & \text{if } n \in \mathcal{S}_y \\ 0 & \text{otherwise.} \end{cases}$$

Using that $\rho_f(n) = \sqrt{n} \cdot \lambda_f(n) \rho_f(1)$ and the definition of $\mathcal{S}_y$, we have

$$M_N(y, \varphi) |\rho_\varphi^2(1)| \left| \sum_{n \leq N} \lambda_\varphi(n)^2 \right|^2 \leq \sum_{f \in \mathcal{F}} \rho_f(1) \left| \sum_{n \leq N} \lambda_\varphi(n)^2 \right|^2 \leq \sum_{f \in \mathcal{F}} \left| \rho_f(1) \lambda_\varphi(n) \lambda_\varphi(n) \right|^2 \ll \sum_{n \leq N} \lambda_\varphi(n)^2,$$

which, combined with the Hoffstein-Lockhart estimate $|\rho_\varphi^2(1)| \gg N^{-1} (\log N)^{-2}$ ([HL94]) yields

$$M_N(y, \varphi) \ll N(\log N)^2 \sum_{n \leq N} \lambda_\varphi(n)^2. \quad (16)$$

To prove Theorem 3 we bound $\sum_{n \leq N} |\lambda_\varphi(n)|^2$ away from 0 uniformly in $\varphi$. We use the following fact:

**Proposition 4.1.** Let $k, x \in \mathbb{R}$. Then:

$$\max \{|\sin kx/\sin x|, |\sin((k + 1)x)/\sin x|\} \geq 1/2. \quad (17)$$

(where the functions are extended continuously to the $x$ with $\sin x = 0$).
Proof. If $\sin x = 0$, (17) is true since $\max\{k, k + 1\} \geq 1/2$, so assume $\sin x \neq 0$. Let $\frac{\sin^2 kx}{\sin^2 x} = \varepsilon^2$, where $0 \leq \varepsilon < 1$ (since $\varepsilon \geq 1$, (17) clearly holds). Then:

$$|\cos kx| = \sqrt{1 - \varepsilon^2\sin^2 x} = \sqrt{1 - \varepsilon^2 + \varepsilon^2\cos^2 x},$$

so by the trigonometric identity for sine of a sum,

$$\left|\frac{\sin(kx + x)}{\sin x}\right| = \left|\frac{\sin kx}{\sin x}\cos x + \cos kx\right| \geq |\cos kx| - \varepsilon|\cos x| = \sqrt{1 - \varepsilon^2 + \varepsilon^2|\cos x|^2} - \varepsilon|\cos x|.$$

It remains to minimize $f_\varepsilon(t) := \sqrt{1 - \varepsilon^2 + \varepsilon^2t^2} - \varepsilon t$ for $t \in [0, 1]$. The function $f_\varepsilon(t)$ has a non-vanishing derivative in this interval when $\varepsilon < 1$, so

$$\left|\frac{\sin((k + 1)x)}{\sin x}\right| \geq \min\{f_\varepsilon(0), f_\varepsilon(1)\} = \min\{\sqrt{1 - \varepsilon^2}, 1 - \varepsilon\} = 1 - \varepsilon,$$

and so

$$\max\left\{\left|\frac{\sin kx}{\sin x}\right|, \left|\frac{\sin((k + 1)x)}{\sin x}\right|\right\} \geq \max\{\varepsilon, 1 - \varepsilon\} \geq 1/2.$$

Using the Hecke relation

$$\lambda_\varphi(p^k) = \frac{\sin((k + 1)\theta_\varphi(p))}{\sin(\theta_\varphi(p))},$$

(17) implies that

$$\max\{|\lambda_\varphi(p^k)|, |\lambda_\varphi(p^{k+1})|\} \geq 1/2$$

for all $\varphi$ and $k \geq 0$. Since $\lambda_\varphi(n)$ is multiplicative, for any $r$-tuple $(\alpha_1, \ldots, \alpha_r)$ of non-negative integers, there are $(\delta_1, \ldots, \delta_r) \subseteq \{0, 1\}^r$ such that

$$|\lambda_\varphi(p_1^{2\alpha_1 + \delta_1} \cdots p_r^{2\alpha_r + \delta_r})|^2 \geq 4^{-r}.$$  (18)

The set $S_y$ of all $y$-smooth numbers is a disjoint union of sets

$$\mathcal{E}_{\alpha_1, \ldots, \alpha_r} := \{p_1^{2\alpha_1 + \delta_1} \cdots p_r^{2\alpha_r + \delta_r} | \delta_i \in \{0, 1\}\}$$

of size $2^r$, and (18) implies that each $\mathcal{E}_{\alpha_1, \ldots, \alpha_r}$ contains an element $t$ with $|\lambda_\varphi(t)|^2 \geq 4^{-r}$. Moreover, for every $s \in S_y \cap [0, N/(p_1 \cdots p_r)]$, the set $\mathcal{E}_s$ containing $s$ is fully contained in $S_y \cap [0, N]$. Hence, there are at least $\Psi[y, N/(p_1 \cdots p_r)]/2^r$ sets $\mathcal{E}_s$ fully contained in $S_y \cap [0, N]$, so

$$\sum_{\substack{n \leq N \\ n \in S_y}} |\lambda_\varphi(n)|^2 \geq 8^{-r}\Psi\left(y, \frac{N}{\prod_{p \leq y} p}\right) = 8^{-r}\Psi\left(y, \frac{N}{N^{1+\log(N)/\log y}}\right),$$

where we used the prime number theorem in the second step. We use a result of Hildebrand and Tenenbaum on the size of $\Psi(y, X)$ in the range $y = o(\log X)$:

**Theorem (HTT86, Corollary 1).**

Let $y = o(\log X)$ such that $y \to 0$ as $X \to \infty$. Let

$$\alpha(X, y) := (1 + o(1))\frac{y}{\log X \log y},$$

and

$$\zeta(\alpha, y) := \prod_{p \leq y} (1 - p^{-\alpha})^{-1}.$$

Then:

$$\Psi(X, y) = (1 + o(1))X^\alpha \zeta(\alpha, y) \sqrt{\log y/(2\pi y)}.$$  (20)
From (20),
\[ \log \Psi(X, y) = \alpha \log X + \log \zeta(\alpha, y) + O(\log y) = \log \zeta(\alpha, y) + O(y/\log y), \] (21)
and using the Taylor series expansion,
\[
\log \zeta(\alpha, y) = -\sum_{p \leq y} \log \left(1 - e^{-\alpha \log p}\right) \\
= -\sum_{p \leq y} \log(\alpha \log p(1 + O(\alpha \log p))) \\
= -\pi(y) \log \alpha - \sum_{p \leq y} \log \log p + O(\alpha \sum_{p \leq y} \log p). \tag{22}
\]
Using partial summation,
\[
\left| \sum_{p \leq y} \log \log p - \pi(Y) \log \log Y \right| \ll \int_{2}^{Y} \frac{dt}{\log^2 t} = \left(\text{li}(t) - \frac{t}{\log t}\right) \Big|_{2}^{Y} \ll \frac{Y}{\log^2 Y},
\]
and by the prime number theorem, \( \sum_{p \leq y} \log p = (1 + o(1))y \). Hence,
\[
\text{(22)} \Rightarrow \pi(y) \left(- \log y + \log \log X + \log \log y + O(1) - \log \log y\right) + O(y^2/(\log y \log X)) \\
= (1 + o(1)) \frac{y}{\log y} \log \left(\frac{\log X}{y}\right),
\]
and so (21) gives
\[
\log \Psi(X, y) = (1 + o(1)) \frac{y}{\log y} \log \left(\frac{\log X}{y}\right). \tag{23}
\]
Finally, we combine the results above. Let \( X = N^{1-(1+o(1))\frac{y}{\log N}} \), so
\[
\log X = \log N \left(1 - (1 + o(1)) \frac{y}{\log N}\right) = \log N(1 + o(1))
\]
when \( y = o(\log N) \). Clearly, for such \( y \) we also have \( y = o(\log X) \), so it follows from Theorem 4 that (23) holds for such \( y \) and \( X \). Finally, combining (23) with (16) and (19) and using that \( r = \pi(y) = o(y) \) (i.e. \( \log(8^r) = o(y) \)), we see that
\[
\log(M_N(y, \varphi)/N) \leq O(\log \log N) + o(y) + (1 + o(1)) \frac{y}{\log y} \log \left(\frac{\log N}{y}\right)
\]
as long as \( y = o(\log N) \) and \( y \to \infty \) as \( N \to \infty \). In particular, when \( y = (\log N)^\beta \) for \( 0 < \beta < 1 \),
\[
\log \left(\frac{\log N}{y}\right) = \frac{1 - \beta}{\beta} \log y,
\]
so
\[
\log(M_N(y, \varphi)/N) \leq (1 + o(1)) \frac{1 - \beta}{\beta} y.
\]
Since \( s(N) \approx N \), this proves Theorem 3.
5 Number of Forms with Degree $d$ Hecke Fields

For a prime level $N$, let $H(N)_d \subseteq H(N)$ denote Hecke forms whose Hecke eigenvalues span a number field of degree exactly $d$. We bound the size of $H(N)_d$ using the multiplicity bound from the previous section.

Specifically, let $y > 0$, $r = \pi(y)$, and for $f \in H(N)_d$, and let $a_f(p) = \lambda_f(p)\sqrt{p}$ be the $p^{th}$ Hecke operator eigenvalue of $f$. To prove Theorem we combine the multiplicity bound with an upper bound on the set

$$T_N(y)_d := \{ (a_f(p_1), \ldots, a_f(p_r)) | f \in H(N)_d \}$$

of possible tuples of eigenvalues of a Hecke form at the first $r$ primes. We get this bound by exploiting that $a_f(p)$ is a totally real algebraic integers whose conjugates are bounded by $2\sqrt{p}$ in size.

Proposition 5.1.

$$\# |T_N(y)_d| \leq \exp(yd/2 + o_d(y)). \quad (24)$$

Lemma 5.2. For $f \in H(N)_d$, let $K_{f,r} := \mathbb{Q}(a_f(p_1), \ldots, a_f(p_r))$. Then:

$$\# \{ K_{f,r} | f \in H(N)_d \} \ll_d y^\kappa$$

where $\kappa = \kappa(d)$ is a constant depending on $d$.

Proof. Let $K = K_{f,r}$ for some $f \in H(N)_d$. Let $K_i := \mathbb{Q}(a_f(p_i))$ be of degree $d_i \leq d$ with discriminant $\Delta_i$, and let $P_i(x) = \prod (x - \beta_j)$ be the minimal polynomial of $a_f(p_i)$. Then

$$|\Delta_i| = \frac{|\text{disc}(P_i)|}{|\mathcal{O}_{K_i} : \mathbb{Z}[a_f(p_i)]|^2} \leq \prod_{i \neq j} |(\beta_i - \beta_j)| \leq (4\sqrt{p})^{d_i(d_i - 1)} \ll_d y^{d^2/2}.$$

Since $K$ has degree at most $d$, it can be expressed as a composition of at most $\log_2 d$ fields $K_i$, so the discriminant $\Delta$ of $K$ satisfies

$$|\Delta| \ll_d y^k$$

for some constant $k$ depending only on $d$. This implies a bound of the same form on the number of possibilities for $K$ by the Theorem of Schmidt (Sch93).

Lemma 5.3. Let $K$ be a totally real number field of degree $d \leq d$. Then for $M > 1$, the number of $\alpha \in \mathcal{O}_K$ such that all the Galois conjugates of $\alpha$ are bounded by $M$ is at most $C(d)M^d$ for some constant $C(d)$ which does not depend on $K$.

Proof. Consider that standard embedding $\iota : K \hookrightarrow \mathbb{R}^d$. For $\alpha \in \mathcal{O}_K$, the coordinates of $\iota(\alpha)$ are the Galois conjugates of $\alpha$; their product is a non-zero integer, so the non-zero vectors in the lattice formed by the image of $\mathcal{O}_K$ under $\iota$ have length $\geq 1$. From this, sphere packing bounds imply immediately that the number of lattice points in the box $[-M,M]^d$ is bounded by $O_d(1)M^d$ (this can be seen, for example, by placing (disjoint) balls of diameter 1 at each lattice point in the box and comparing volumes).

Proof of Proposition 5.1. From Lemma 5.3 we see that for a fixed degree $K$ number field, the number of possible tuples $(a_f(p_1), \ldots, a_f(p_r))$ with $a_f(p_i) \in K$ is at most

$$\prod_{p \leq y} C(d)(2\sqrt{p})^d = (2C(d))^r \exp \left( \frac{(d/2)}{y} \sum_{p \leq y} \log p \right) = \exp(dy/2 + o_d(y)),$$

where the last step uses the prime number theorem. On the other hand, from Lemma 5.2 the number of choices for $K$ is $\exp(O_d(y)) = \exp(o_d(y))$, so multiplying the two proves the proposition statement.
Combining this Proposition with Theorem \[3\]

\[
\log s(N)_d/s(N) \leq - \left( \frac{1-\beta}{\beta} - \frac{d}{2} \right) y + o_d(y),
\]

which concludes the proof of Theorem \[4\].

Note that for the coefficient of \(y\) to be negative, we have to choose \(\beta\) small, which is why we dealt with multiplicity bounds in Theorem 3 only for \(0 \leq \beta \leq 1\).

### 6 Composite Level

The discussion up to this point was restricted to weight \(k = 2\) and level \(N\) being prime. Theorems \[1\] and \[2\] can be extended without much change to allow varying weight and general \(N\), as long as the relatively prime conditions \((mN, N) = 1\) and \((p_j, N) = 1\) are maintained. Indeed, the starting point, which is an application of the Petersson formula [Petersson 1900, Corollary 2.2] or [IK04, Corollary 14.24], gives the desired uniformity in the "harmonic" weighted form. To remove the weights, one has to take care with new and old forms and the Atkin-Lehner involutions in relating \(\|f\|^2\) and \(L(1, \text{Sym}^2 f)\).

On the other hand, in the proof of the multiplicity bounds in Theorem \[3\], we used \(y\)-smooth numbers and the assumption that \((p, N) = 1\) for \(p \leq y < \log N\). As we show in Theorem \[5\] below, similar bounds can be proved for \(N\)'s that do not have an abnormal number of small prime factors. For "super-smooth" numbers, such as \(N = \prod_{p \leq y} p\), we cannot make use of the approach to the Plancherel measure of the Hecke eigenvalues for small primes, and our bounds in Theorem \[3\] and \[4\] don't apply.

In what follows, we restrict ourselves to the \(s^*(N)\)-dimensional space \(S^*(N)\) of weight 2 level \(N\) newforms, which admits a simultaneous eigenbasis \(H^*(N)\) with respect to Hecke operators \(T_n\) with \((n, N) = 1\) (we assume these forms are normalized to have constant Fourier coefficient 1).

For a positive integer \(N\), the number of distinct prime divisors of \(N\) is at most

\[(1 + o(1)) \frac{\log N}{\log \log N} =: P.
\]

Let \(y = y(N) = o(\log N)\) be a parameter going to infinity with \(N\), and let \(r := \pi(y) \sim y/\log y\). Let \(q_1, \ldots, q_r\) be the first \(r\) primes which don’t divide \(N\). Since \(q_k\) is at most the \((k + P)\)th prime and \(k \leq r = o(P)\), we can conclude via the prime number theorem that

\[q_k \leq (1 + o(1))(P \log P).\]

(26)

In the spirit of section \[4\] we want to give a lower bound for the function

\[\Phi(q_1, \ldots, q_r, X) := \#\{(\alpha_1, \ldots, \alpha_r) : q_1^{\alpha_1} \cdots q_r^{\alpha_r} \leq X\}\]

for \(X\) (to be chosen later) satisfying

\[\log X = (1 + o(1)) \log N.\]

(27)

By (26),

\[
\Phi(q_1, \ldots, q_r, X) = \#\{(\alpha_1, \ldots, \alpha_r) : q_1^{\alpha_1} \cdots q_r^{\alpha_r} \leq X\} \geq
\]

\[
\#\{(\alpha_1, \ldots, \alpha_r) : (1 + o(1))P \log P)^{\alpha_1 + \cdots + \alpha_r} \leq X\} \geq
\]

\[
\#\{(\alpha_1, \ldots, \alpha_r) : \alpha_1 + \cdots + \alpha_r \leq \log X/ \log ((1 + o(1))(P \log P))\} \geq
\]

\[
\#\{(\alpha_1, \ldots, \alpha_r) : \alpha_1 + \cdots + \alpha_r \leq (1 + o(1)) \log X/ \log \log X\}.
\]

(28)
The number of non-negative integer solutions to $x_1 + \ldots + x_A \leq B$ is
\[
\binom{A + B}{A} \geq \left( \frac{B}{A} \right)^A,
\]
so (28) implies
\[
\log \Phi(q_1, \ldots, q_r, X) \geq r \log \frac{(1 + o(1)) \log X}{r \log \log X}. \tag{29}
\]

For $\varphi \in S^\ast(N)$ a weight 2 holomorphic cusp newform for $\Gamma_0(N)$, we let $M_N^\varphi(q_1, \ldots, q_r, \varphi)$ be the multiplicity of the tuple of eigenvalues of $\varphi$ at primes $q_i$, i.e.
\[
M_N^\varphi(q_1, \ldots, q_r, \varphi) := \# \{ f \in H^\ast(N) : \lambda_f(q_i) = \lambda_\varphi(q_i) \text{ for all } i \leq r \}.
\]

We bound $M_N^\varphi(q_1, \ldots, q_r, \varphi)$ for a fixed $\varphi$ via the large sieve inequality identically to section (4). Taking $c_n := \lambda_\varphi(n)$ if $n = q_1^{a_1} \cdots q_r^{a_r} \leq X$ otherwise,

we get
\[
M_N^\varphi(q_1, \ldots, q_r, \varphi) \ll N(\log N)^2 / \sum_{n=q_1^{a_1}\cdots q_r^{a_r}} |\lambda_\varphi(n)|^2. \tag{30}
\]

Recreating the proof in section (4) we can see that
\[
\sum_{n \leq N \in \mathcal{S}_{q_1, \ldots, q_r}} |\lambda_\varphi(n)|^2 \gg 8^{-r} \Phi \left( q_1, \ldots, q_r, \frac{N}{q_1 \cdots q_r} \right) \geq 8^{-r} \Phi \left( q_1, \ldots, q_r, \frac{N}{P^{2r}} \right). \tag{31}
\]

Let $X := N/P^{2r}$. Recall that $r = \pi(o(\log N)) = o(\log N / \log \log N)$, so
\[
\log X = \log N - 2r(1 + o(1)) \log \log N = \log N(1 + o(1)),
\]
which means this choice of $X$ satisfies (27) and hence also satisfies (29). Let $0 < \beta < 1$ and let
\[
y := (\log N)^\beta,
\]
so
\[
r = (1 + o(1))(\log X)^\beta/\beta \log \log X.
\]

Then (29) becomes
\[
\log \Phi(q_1, \ldots, q_r, X) \geq (1 + o(1)) \frac{(\log X)^\beta}{\beta \log \log X} \log \frac{\beta \log X}{(\log X)^\beta} = (1 + o(1)) \frac{1 - \beta}{\beta} (\log X)^\beta,
\]
i.e.,
\[
\log \Phi(q_1, \ldots, q_r, N) \geq (1 + o(1)) \frac{1 - \beta}{\beta} (\log N)^\beta.
\]
Finally, by (30) and (31),
\[
\log M_N^\varphi(q_1, \ldots, q_r, \varphi)/N \leq \log \log N + r \log 8 - (1 + o(1)) \frac{1 - \beta}{\beta} (\log N)^\beta = -(1 + o(1)) \frac{1 - \beta}{\beta} (\log N)^\beta \tag{32}
\]
(note that this bound is identical to the one in section \[\text{section 4}\] which is sharp).

We apply (32) to extend Theorem 4 to more general \(N\)'s. For \(T \geq 1\) fixed and for some \(y = y(N)\) with \(\log \log N \ll y \ll \log N\), we say that a large \(N\) is \(T\)-super-smooth if

\[
\frac{\pi(y^T; N)}{\pi(y)} = o(1),
\]

where \(\pi(z; N) = \#\{p \leq z : (p, N) = 1\}\) is the number of primes up to \(z\) that don't divide \(N\). Clearly, very few numbers are \(T\)-super-smooth for all \(T\).

Let \(H^*(N)_d := \{f \in H^*(N) : \lambda_1(f) = d\}\) be the set of Hecke newforms whose Fourier coefficients span a number field of degree \(d\), \(s^*(N)_d = |H^*(N)_d|\). The following theorem extends Theorem 4 to non-\(T\)-super-smooth numbers.

**Theorem 5.** Let \(0 \leq \beta \leq 1\), \(y = (\log N)^\beta\), \(T \geq 1\), and \(d \geq 1\). Then for \(N\) not \(T\)-super-smooth,

\[
s^*(N)_d \leq \exp\left(-\left(1 - \frac{\beta}{\beta} - \frac{dT}{2} \right) y + o_{T,d}(y)\right) s^*(N)
\]

as \(N \to \infty\).

**Proof.** The proof emulates that of section \[\text{section 5}\] for \(f \in H^*(N)_d\), let \(a_f(q_i) = \lambda_f(q_i)\sqrt{q_i}\) be the eigenvalue of \(f\) for the Hecke operator \(T_{q_i}\), and let

\[
T^*_N(q_1, \ldots, q_r)_d := \{(a_f(q_1), a_f(q_2), \ldots, a_f(q_r)) | f \in H^*(N)_d\}
\]

denote the set of possible eigenvalue tuples of a form in \(H^*(N)_d\) at the first \(r\) primes not dividing \(N\). Repeating verbatim the proof of Lemma 5.2 there are \(\ll_d y^{T_\pi(d)}\) possible number fields of the form \(\mathbb{Q}(a_f(q_1), \ldots, a_f(q_r))\). Tautologically, for \(N\) as in the statement of the theorem, the first \(r\) primes \(q_1, \ldots, q_r\) not dividing \(N\) satisfy \(q_i \leq y^T\), so using Lemma 5.3 we get

\[
\#T^*_N(q_1, \ldots, q_r)_d \leq y^{T_\pi(d)} \prod_{i \leq r} C(d) y^{T_d/2} \leq \exp((d/2)T \log y + o_{T,d}(y)) \leq \exp((d/2)T y + o_{d,T}(y)).
\]

Combined with (32), this gives

\[
\log s(N)_d / N \leq -\left(1 - \frac{\beta}{\beta} - \frac{dT}{2} \right) y + o_{T,d}(y).
\]

It remains to note that \(s^*(N) \asymp \varphi(N)\), the Euler's totient function, and \(\log \varphi(N) = \log N + O(\log \log \log N)\), which completes the proof.

\[
\square
\]

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