Research Article

A Fixed-Point Theorem for Ordered Contraction-Type Decreasing Operators in Banach Space with Lattice Structure

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In this work, we mainly improve the results in Amini-Harandi and Emami (2010). By introducing a new kind of ordered contraction-type decreasing operator in Banach space, we obtain a unique fixed point by using the iterative algorithm. An example is also presented to illustrate the theorem.

1. Introduction

In this work, we obtain a unique fixed point for a kind of ordered contraction-type decreasing operator in Banach space by using the iterative algorithm. The fixed-point study is mainly focused on two aspects. On the one hand, it is about the research of contraction-type mapping, for example, in [1–8]. On the other hand, it is about the study of monotone operators with concavity and convexity, for example, in [9–15]. There is little research on operators that only satisfy the partial-order constrictions. Applications of operator theory in fractional differential equations can be seen in [16–43].

The following generalization of Banach’s contraction principle is due to Geraghty [44].

Lemma 1. Let $(M, d)$ be a complete metric space and let $f : M \to M$ be a map. Suppose there exists $\beta \in \zeta$ such that for each $x, y \in M$,

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y),$$

where $\zeta$ denotes the class of those functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition $\beta(t_n) \to 1 \Rightarrow t_n \to 0$. Then $f$ has a unique fixed-point $z \in M$, and $\{f^n(x)\}$ converges to $z$, for each $x \in M$.

Very recently, Amini-Harandi and Emami [6] proved the following existence theorem which is a version of Lemma 1 in the context of partially ordered complete metric spaces:

Lemma 2. Let $(M, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ in $M$ such that $(M, d)$ is a complete metric space. Let $f : M \to M$ be an increasing map such that there exists an element $x_0 \in M$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in \zeta$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y), \quad x, y \in M, y \preceq x. \quad (2)$$

Assume that either $f$ is continuous or $M$ is such that if an increasing sequence $\{x_n\} \to x$ in $M$, then $x_n \preceq x, \forall n$. Besides, if for each $x, y \in M$, there exists $z \in M$, which is comparable to $x$ and $y$.

Then $f$ has a unique fixed point.

We found that in (2), the contraction is concerning a metric. But in fact, the relation of partial order does not play any role in (2). That is to say, in (1), the constriction $y \preceq x$ is not effective because $d(x, y) = d(y, x)$. A question appears naturally in the authors’ minds: “Can the contraction condition be merely about partial order so that $y \preceq x$ can be
effective?” The authors have been haunted by this question since it was found. Driven by this idea, we introduce a new kind of ordered contraction-type decreasing operator in Banach space with lattice structure and obtain a unique fixed point of the operator. Our results are helpful and meaningful for studies of fixed point. Comparing to [6], our improvements are in three aspects.

First, the contraction is merely about partial order, and the relation of partial-order y ≤ x does play an important role in the contraction condition. This has never been seen. Second, we consider the situation when the operator is decreasing. Third, we only use the iterative algorithm, and we can start the iterative process with any initial point, i.e., we do not need any assumptions of the existence of upper or lower solutions. An example is also presented to illustrate the theorem.

The outline of this paper is as follows. In the remainder of this section, we will give some preliminaries. In Section 2 of this paper, we present the existence and uniqueness theorem. In Section 3, an example is illustrated.

**Definition 3** (see [45]). Let E be a real Banach space. A non-empty convex closed set P ⊂ E is called a cone if

(i) \( x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P \)

(ii) \( x \in P, -x \in P \Rightarrow x = \theta; \theta \) is the zero element in E

In the case that P is a given cone in a real Banach space \((E, \|\|)\), a partial order “≤” can be induced on E by \( x \leq y \Leftrightarrow y - x \in P \). The cone \( P \) is called normal if there exists a constant \( N > 0 \), such that for all \( x, y \in E, 0 \leq x \leq y \) implies that \( \|x\| \leq N \|y\| \). Details about cones and fixed point of operators can be found in [45, 46].

**Definition 4** (see [47, 48]). We call a set \( X \subset E \) a lattice under the partial ordering ≤, if sup\{x, y\} and inf\{x, y\} exist for arbitrary \( x, y \in X \).

**Lemma 5** (see [45]). A cone P is normal if and only if there exists a norm \( \|\| \) on E which is equivalent to \( \|\| \) such that for any \( 0 \leq x \leq y \), \( \|x\| \leq \|y\| \), i.e., \( \|\| \) is monotone. The equivalence of \( \|\| \) and \( \|\| \) means that there exist \( M > m > 0 \) such that \( m \|\| \leq \|\| \leq M \|\| \).

**Lemma 6** (see [45]). Let P be a normal cone in a real Banach space E. Suppose that \( \{x_n\} \) is a monotone sequence which has a subsequence \( \{x_{n_k}\} \) converging to \( x^* \), then \( \{x_n\} \) also converges to \( x^* \). Moreover, if \( \{x_n\} \) is an increasing sequence, then \( \{x_n\} \leq x^* (n = 1, 2, 3, \cdots) \); if \( \{x_n\} \) is a decreasing sequence, then \( x^* \leq \{x_n\} (n = 1, 2, 3, \cdots) \).

2. The Main Results

We suppose that E is a partially ordered Banach space. P is a normal cone. The partial-order “≤” on E is induced by the cone P.

Let \( \zeta \) denote the class of those functionals \( \beta : P \rightarrow [0, 1] \) which satisfy the condition

\[
\beta(w_n) \rightarrow 1 \Rightarrow w_n \rightarrow \theta\text{ for any monotonic sequence }\{w_n\} \subset P.
\]

**Theorem 7** (main theorem). Suppose that \( X \subset E \) is a closed subset, \( P \subset X \) is a lattice. \( A : X \rightarrow X \) is a decreasing operator and satisfies the following ordered contraction condition:

(H) Suppose that there exists \( \beta \in \zeta \) such that

\[
Au - Av \leq \beta(v - u)(v - u), \quad \forall u, v \in X, u \leq v.
\]

Then \( A \) has unique fixed-point \( u_* \in X \). Moreover, constructing successively sequence

\[
u_n = Au_{n-1} (n = 1, 2, \cdots),
\]

for any initial \( u_0 \in X \), we have

\[
\lim_{n \to \infty} u_n = u_*.
\]

**Remark 8.** Here, we study the decreasing operator while most of the contractions are about metric. The contraction condition (4) is merely about the partial order, while most of the contractions are about metric.

**Remark 9.** Two elements \( x \) and \( y \) in an ordered set \((X, \leq)\) are said to be comparable if either \( x \leq y \) or \( y \leq x \), and we denote it as \( x \sim y \).

**Proof.** Let \( u_0 \in X \), we have \( Au_0 \in X \). So we have the following two cases.

Case 1. When \( u_0 \) is comparable with \( Au_0 \). Firstly, without loss of generality, we suppose that

\[
u_0 \leq Au_0.
\]

If \( u_0 = Au_0 \), then the proof is finished. Suppose that \( u_0 < Au_0 \). Since \( A \) is decreasing, we obtain \( Au_0 \geq A^2 u_0 \), and it is easy to prove that \( A^2 \) is increasing. Using the contractive condition (4), we have

\[
Au_0 - A^2 u_0 \leq \beta(Au_0 - u_0)(Au_0 - u_0) \leq Au_0 - u_0,
\]

hence,

\[
u_0 \leq A^2 u_0.
\]

From (4), we have

\[
A^2 v - A^2 u \leq \beta(Au - Av)(Av - Av)
\]

\[
\leq \beta(Au - Av)(v - u)(v - u),
\]

\[
\leq \beta(v - u)(v - u), \quad \forall u, v \in X, u \leq v.
\]
Let
\[ Bu = A^2 u, \quad \forall u \in X. \] (11)

From (9) and (10), we have the following two conclusions:

(a) There exists a functional \( \beta \in \zeta \) such that for \( u, v \in X \) with \( u \leq v \)
\[ Bv - Bu \leq \beta(v - u)(v - u) \] (12)

(b) There exists \( u_0 \in X \) such that \( u_0 \leq Bu_0 \)

We assert that the operator \( B \) has unique fixed point in \( X \). And the unique fixed point of \( B \) is also the unique fixed point of \( A \). In order to be clear, we divide the process of proof into three steps.

**Step 1.** We will use the method of iteration to construct a fixed point of \( B \). In fact, consider the iterative sequence
\[ u_{n+1} = Bu_n, \quad n = 0, 1, 2, \ldots. \] (13)

Since \( u_0 \leq Bu_0 \) and the operator \( B \) is increasing, we have
\[ u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots. \] (14)

This means that \( \{u_n\} \) is an increasing sequence. It follows from (12) that
\[ \theta \leq u_{n+1} - u_n = Bu_n - Bu_{n-1} \leq \beta(u_n - u_{n-1})(u_n - u_{n-1}). \] (15)

Since \( P \) is normal, from the equivalence of \( \|\cdot\| \) and \( \|\cdot\|_1 \) in Lemma 5 we have
\[ \|u_{n+1} - u_n\|_1 = \|Bu_n - Bu_{n-1}\|_1 \leq \beta(u_n - u_{n-1})\|u_n - u_{n-1}\|_1. \] (16)

Then, \( \{\|u_{n+1} - u_n\|_1\} \) is a decreasing sequence and bounded as follows. So
\[ \lim_{n \to \infty} \|u_{n+1} - u_n\|_1 = r \geq 0. \] (17)

Assume \( r > 0 \). Then, from (16), we have
\[ \frac{\|u_{n+1} - u_n\|_1}{\|u_n - u_{n-1}\|_1} \leq \beta(u_n - u_{n-1}). \] (18)

The above inequality yields
\[ \lim_{n \to \infty} \beta(u_n - u_{n-1}) = 1. \] (19)

And \( \beta \in \zeta \) implies \( \lim_{n \to \infty} (u_n - u_{n-1}) = 0 \). Then,
\[ \lim_{n \to \infty} \|u_n - u_{n-1}\|_1 = 0 \] (20)
and \( r = 0 \).

Now we show that \( \{u_n\} \) is a Cauchy sequence. On the contrary, assume that
\[ \lim_{m,n \to \infty} \sup \|u_n - u_m\|_1 > 0. \] (21)

For any fixed natural number \( n, m \), from (16), by the triangle inequality \( \|u_n - u_m\|_1 \leq |u_n - u_{n+1}|_1 + |u_{n+1} - u_m|_1 + \beta(u_n - u_m)\|u_n - u_{n+1}\|_1 + \|u_{n+1} - u_m\|_1 \).

Hence, we have \( \|u_n - u_m\|_1 \leq (1 - \beta(u_n - u_m))^{-1}(\|u_n - u_{n+1}\|_1 + \|u_{n+1} - u_m\|_1) \).

Since \( \lim_{m,n \to \infty} \|u_n - u_m\|_1 > 0 \) and \( \lim_{n \to \infty} \|u_n - u_{n+1}\|_1 = 0 \), then
\[ \lim_{m,n \to \infty} \sup \|u_n - u_m\|_1 = \infty, \] (22)
from which we obtain
\[ \lim_{m,n \to \infty} \beta(u_n - u_m) = 1. \] (23)

But since \( \beta \in \zeta \), we get
\[ \lim_{m,n \to \infty} \|u_n - u_m\|_1 = 0. \] (24)

This contradicts (21) and shows that \( \{u_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is closed, we can suppose that there exists a \( u_* \in X \) such that
\[ u_n \longrightarrow u_* \]. (25)

Since \( P \) is normal, (14) together with Lemma 5 implies that
\[ u_n \leq u_* \] (26)

(26), together with (12) and the equivalence of \( \|\cdot\|_1 \) and \( \|\cdot\| \), implies that
\[ \|Bu_* - Bu_n\|_1 \leq \beta(u_* - u_n)\|u_* - u_n\|_1. \] (27)

So \( \|u_* - Bu_*\|_1 \leq \|u_* - u_{n+1}\|_1 + \|Bu_* - Bu_{n+1}\|_1 \leq \|u_* - u_{n+1}\|_1 + \beta(u_* - u_n)\|u_* - u_{n+1}\|_1 \).

Let \( n \to \infty \), we obtain \( \|u_* - Bu_*\|_1 = 0 \). So
\[ u_* = Bu_* \).
This proves that \( u_* \) is a fixed point of \( B \) in \( X \) and
\[ u_* = \lim_{n \to \infty} Bu_0. \] (28)
Step 2. We will obtain the uniqueness of the fixed point of $B$. On the contrary, if $\tilde{u}$ is another fixed point of $B$, we get $\tilde{u} = u_*$.

In fact, the first case, when $\tilde{u}$ is comparable with $u_0$, without loss of generality, we suppose that $\tilde{u} \leq u_0$. Since $B$ is increasing,

$$\tilde{u} = B^0 \tilde{u} \leq B^0 u_0.$$  \hspace{1cm} (29)

Moreover, by (12)

$$B^n u_0 - \tilde{u} = B^n u_0 - B^n \tilde{u} \leq \beta(B^{n-1} u_0 - B^{n-1} \tilde{u}) (B^{n-1} u_0 - B^{n-1} \tilde{u}),$$  \hspace{1cm} (30)

and so

$$\|B^n u_0 - \tilde{u}\| \leq \beta \|B^{n-1} u_0 - \tilde{u}\| \|B^{n-1} u_0 - \tilde{u}\|. \hspace{1cm} (31)$$

Consequently, the sequence

$$y_n = \|B^n u_0 - \tilde{u}\|, \hspace{1cm} (32)$$

is nonnegative and decreasing, and so $\lim_{n \to \infty} y_n = \gamma \geq 0$. Now we show that $\gamma = 0$.

On the contrary, assume that $\gamma > 0$. By passing to subsequences, if necessary, we may assume that $\lim_{n \to \infty} \beta(B^n u_0 - \tilde{u}) = \lambda$ exists. From (31), we obtain $\gamma \leq \lambda y$, and so $\lambda = 1$. Since $\beta \in \mathcal{I}$, then $\lim_{n \to \infty} (B^n u_0 - \tilde{u}) = \theta$, and $\gamma = \lim_{n \to \infty} \|B^n u_0 - \tilde{u}\| = 0$.

This contradiction proves $\gamma = 0$.

It can be obtained that

$$\tilde{u} = \lim_{n \to \infty} B^n u_0 = u_*.$$

The second case, when $\tilde{u}$ cannot compare with $u_0$. From $X$ which is a lattice, we obtain

$$y_1 = \inf \{\tilde{u}, u_0\} \in X,$$  \hspace{1cm} (34)

satisfying

$$y_1 \leq \tilde{u}, \hspace{0.5cm} y_1 \leq u_0,$$ \hspace{1cm} (35)

i.e., $\tilde{u}$ is comparable with $y_1$ and $u_0$ is comparable with $y_1$. Since $B$ is increasing, we know

$$B^n y_1 \leq B^n \tilde{u}, \hspace{0.5cm} B^n y_1 \leq B^n u_0, \hspace{0.5cm} n = 1, 2, \ldots.$$ \hspace{1cm} (36)

Moreover, by (12)

$$B^n u_0 - B^n y_1 \leq \beta(B^{n-1} u_0 - B^{n-1} y_1) (B^{n-1} u_0 - B^{n-1} y_1). \hspace{1cm} (37)$$

So we have

$$\|B^n u_0 - B^n y_1\| \leq \beta \|B^{n-1} u_0 - B^{n-1} y_1\| \|B^{n-1} u_0 - B^{n-1} y_1\|,$$

$$\leq \|B^{n-1} u_0 - B^{n-1} y_1\|. \hspace{1cm} (38)$$

Similar to the process of (31), (32), and (33),

$$\lim_{n \to \infty} B^n y_1 = \lim_{n \to \infty} B^n u_0 = u_*.$$  \hspace{1cm} (39)

From (36), we have

$$\lim_{n \to \infty} B^n y_1 = \lim_{n \to \infty} B^n \tilde{u} = \tilde{u}.$$ \hspace{1cm} (40)

So from (39) and (40), we get

$$\tilde{u} = u_*.$$ \hspace{1cm} (41)

(33) together with (41) implies that $u_*$ is unique fixed point of $B$.

Step 3. We will point that the unique fixed point of $B$ is also the unique fixed point of $A$. Since

$$A^2 u_* = Bu_* = u_*.$$ \hspace{1cm} (42)

Thus,

$$A^2 (Au_*) = A(A^2 u_*) = Au_*,$$ \hspace{1cm} (43)

i.e., $B(Au_*) = Au_*$. From the uniqueness of the fixed point of $B$, we know

$$Au_* = u_*.$$ \hspace{1cm} (44)

So $u_*$ is the unique fixed point of $A$ in $X$.

Case 2. Another case, when $u_0$ is not comparable to $Au_0$. From $X$ which is a lattice, we know there exists $v_0 \in E$ such that $\inf \{Au_0, u_0\} = v_0$. That is, $v_0 \leq Au_0, v_0 \leq u_0$. Since $A$ is a decreasing operator, we have

$$Au_0 \leq Av_0.$$ \hspace{1cm} (45)

This shows that

$$v_0 \leq Av_0.$$ \hspace{1cm} (46)

For any initial $v_0 \in X$, constructing successively sequence

$$v_n = Av_{n-1}.$$ \hspace{1cm} (47)

From $B = A^2$, we can get

$$v_{2n} = A^{2n} v_0 = B^n v_0, \hspace{0.5cm} n = 1, 2, \ldots.$$ \hspace{1cm} (48)
From (28), we know
\[ \lim_{n \to \infty} v_{2n} = \lim_{n \to \infty} B^n v_0 = u_*. \] (49)
Since \( v_{2n+1} = A^{2n+1} v_0 = B^n (A v_0) \) and from the arbitrary of \( v_0 \) in (28), we obtain
\[ \lim_{n \to \infty} v_{2n+1} = \lim_{n \to \infty} B^n (A v_0) = u_*. \] (50)
(49) and (50) imply that
\[ \lim_{n \to \infty} v_n = u_*, \] (51)
holds. Similarly to the proof of Step 2 and Step 3 in case 1, we get that \( u_* \) is the unique fixed point of \( A \).

3. An Example

Let \( E = \mathbb{R} \), equipped with usual normal \( \| \cdot \| = | \cdot | \) and usual partial order \( \leq \). \( X = [0, \infty) \), \( P = [0, \infty) \). Then, \( P \subset X \subset E \) is a normal cone in \( X \). \( (X, \| \cdot \|, \leq) \) is a partially ordered Banach space. And \( X \) is a lattice under the partial order \( \leq \) induced by the cone \( P \).

Then, \( X = [0, \infty) \), \( P = [0, \infty) \) satisfying the assumptions of Theorem 7. Define the mapping \( A : X \to X \) by \( A x = (m + 1 + mx)/1 + x \), \( x \in X \) where \( m \) is a fixed real number. Then, \( A \) is nonincreasing. Define \( \beta : P \to [0, \infty) \) by \( \beta(w) = 1/(1 + w) \), then \( \beta \in \mathcal{C} \). Now, for all \( x, y \in X \) with \( x \leq y \), we have
\[ Ax - Ay = \frac{m + 1 + mx}{1 + x} - \frac{m + 1 + my}{1 + y} = \frac{y - x}{1 + x + y + xy} \leq \frac{y - x}{1 + y - x} \]
\[ = \beta(y - x)(y - x), \]
so that \( A \) and \( \beta \) satisfy the assumption of Theorem 7. Observing that all the other conditions of Theorem 7 are also satisfied, \( A \) has a unique fixed-point \( x_* > 0 \). Moreover, constructing successively sequence
\[ x_{n+1} = \frac{m + 1 + mx_n}{1 + x_n} \quad (n = 0, 1, 2, \cdots), \] (53)
for any initial \( x_0 \geq 0 \), we have
\[ \lim_{x \to \infty} x_n = x_. \] (54)

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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