Partial distinguishability and photon counting probabilities in linear multiport devices

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Probabilities of photon counts at the output of a multiport optical device are generalised for optical sources of arbitrary quantum states in partially distinguishable optical modes. For the single-mode photon sources, the generating function for the probabilities is a linear combination of the matrix permanents of positive semi-definite Hermitian matrices, where each Hermitian matrix is a Hadamard product of a submatrix of the multiport matrix and a Hermitian matrix describing partial distinguishability. For the multi-mode sources the generating function is given by an integral of the Husimi functions of the sources. When each photon source outputs exactly a Fock state, the obtained expression reduces to the probability formula derived for partially distinguishable photons, Physical Review A 91, 013844 (2015). The derived probability formula can be useful in analysing experiments with partially distinguishable sources and error bounds of experimental Boson Sampling devices.

Keywords: Quantum Optics, Photon Counts, Linear Networks

I. INTRODUCTION

The purpose of this work is to derive a formula for photon counts at the output of a linear unitary optical network with arbitrary photon sources at the input, i.e., of arbitrary input quantum states in each port with arbitrary state of partial distinguishability of the internal optical modes between the ports. Such a result is in order to study linear quantum optical networks with important applications in computational complexity [1, 2] and the rise of Boson Sampling with single photons [3] with proof-of-principle experiments of several groups [4–10]. The computational complexity of linear quantum optical networks is not
specific to single photons, as it extends to Gaussian quantum states at the input \[^{11,12}\]. Quantum optical networks for single photons are also at the core of the universal quantum computing with linear optics \[^{13}\].

For realistic photon sources at the input of a linear multiport, photon distinguishability affects “quantumness” of such a device \[^{14–18}\]. For example, quantum supremacy of the Boson Sampling requires that indistinguishability of single photons must be very small \[^{19,20}\]. In general, the probability distribution at the output of a unitary linear network with of partial distinguishable single photons (or, more generally, multi-photon input in Fock states) is now well studied \[^{19,21–25}\], nevertheless these results do not extend to the case of sources producing arbitrary quantum states in the Fock space. The formula for photon counts of an optical field can be found in many books (see for instance, Refs. \[^{26,27}\]). In principle, it can be applied to optical sources producing photons in partially distinguishable optical modes, which then are sent through a linear network, but such an application has been not considered in detail. Due to the recently realised significance of linear unitary optical networks for demonstrating quantum supremacy over digital computers, such a result is in order for study of realistic setups of Boson Sampling and quantum to classical transition in linear optical networks.

The text is organised as follows. In section II we derive a generating function for photon counting probabilities at the output of a multiport device with arbitrary quantum states in partially distinguishable (internal) optical modes at the input, with some details relegated to the appendices. In section III, the results are generalised to multi-mode independent sources. In the concluding section IV a brief statement of the results is given.

II. GENERATING FUNCTION FOR PROBABILITIES OF PHOTON COUNTS IN CASE OF SINGLE-MODE OPTICAL SOURCES

Consider a unitary linear \(M\)-port optical device with arbitrary independent \(M\) optical sources at its input (number of sources can be less than \(M\), in this case some of the sources output the vacuum state, see below). Since the sources are independent, they output quantum states in generally different set of internal optical modes (which are the degrees of freedom not affected by the multiport and not resolved by the detectors). Here we consider each source to be single-mode. The optical modes of the sources can be accounted for by
introducing a common basis of the optical modes for all $M$ sources, where there are two
indices: the first index takes care of the degrees of freedom operated on by a linear multiport
(a linear combination of which is resolved by the detectors) and the second for the degrees
of freedom invariant under the action of a multiport (the internal modes). The creation
operators of the input, $\hat{a}^\dagger_{k,s}$, and the output, $\hat{b}^\dagger_{l,s}$, basis are related by a multiport optical
device described by an unitary matrix $U$ as follows

$$\hat{b}^\dagger_{l,s} = \sum_{l=1}^{M} U_{k,l} \hat{a}^\dagger_{k,s}, \quad l = 1, \ldots, M,$$

where $s = 1, \ldots, M$ enumerates the basis of the internal optical modes. We assume that the
optical source at the input port $k$ of the device outputs an arbitrary single-mode quantum
state $\rho^{(k)}$, whose internal mode is described by the creation operator $\hat{c}^\dagger_k$. The latter can be
expanded over the input basis

$$\hat{c}^\dagger_k = \sum_{s=1}^{M} \phi_{k,s} \hat{a}^\dagger_{k,s}, \quad \sum_{s=1}^{M} |\phi_{k,s}|^2 = 1.$$  

For the quantum state $\rho^{(k)}$ we thus have

$$\rho^{(k)} = \sum_{n,m \geq 0} \rho^{(k)}_{n,m} \frac{\langle c^\dagger_k^n | 0 \rangle \langle 0 | c^\dagger_k^m}{\sqrt{n!m!}}.$$  

Here we do not make assumptions on the relation between optical modes of different sources
(i.e., their scalar product $\sum_{s=1}^{M} \phi^*_{k,s} \phi_{l,s}$ is arbitrary).

For each fixed $k$, one can complement the operator $\hat{c}^\dagger_k$ to a standard basis of the creation
operators $\hat{c}^\dagger_{k,s}$, with $s = 1, \ldots, M$, such that there is an unitary transformation between the
two sets of basis operators $\hat{a}^\dagger_{k,s}$ and $\hat{c}^\dagger_{k,s}$ enumerated by $s = 1, \ldots, M$. In other words, Eq.
(2) can be complemented to a unitary transformation for such an extended set of operators
$\hat{c}^\dagger_{k,s}$, $s = 1, \ldots, M$. It turns out that we do not need below the exact form of such an unitary
transformation, but only the mere fact that (for each $k$) the equation inverse to Eq.  (2) can
written as follows

$$\hat{a}^\dagger_{k,s} = \phi^*_{k,s} \hat{c}^\dagger_k + \hat{d}^\dagger_{k,s},$$

where $\hat{d}^\dagger_{k,s}$ is a linear combination of creation operators for the internal optical modes ortho-
ogonal to that of $\hat{c}_k$. Since the sources produce vacuum in the optical mode described by
$\hat{d}^\dagger_{k,s}$ for $k, s = 1, \ldots, M$, there will be no effect of $\hat{d}^\dagger_{k,s}$ in Eq.  (4) on the photon counts at the
multiport output (thanks to the normal ordering of the creation and annihilation operators in the photon counting formula, Eqs. (5)-(9) below).

The well-known general formula for photon counts (see, for instance, Refs. [26, 27]) can be conveniently described by a generating function

\[ P_0(\eta) = \left\langle N \left\{ \exp \left( - \sum_{k=1}^{M} \hat{\mathcal{I}}_k \right) \right\} \rightangle , \]  

(5)

where the detector attached at output mode \( k \) is described by the operator \( \hat{\mathcal{I}}_k = \eta_k \sum_{s=1}^{M} \hat{b}^\dagger_{l,s} \hat{b}_{l,s} \), \( 0 \leq \eta_k \leq 1 \) being its efficiency (here we take into account that the internal modes are not resolved), \( N \) stands for the normal ordering of creation and annihilation operators, and \( \langle \ldots \rangle \) stands for the averaging with an input quantum state \( \rho = \rho^{(1)} \otimes \ldots \otimes \rho^{M} \) of the photon sources. The generating function in Eq. (5) is also the probability of zero photon counts (detecting the vacuum) at the output of a multiport device. The probability of detecting \( m = (m_1, \ldots, m_M) \) photons at the output ports \( l = 1, \ldots, M \) of an \( M \)-port is given as follows [26, 27]

\[ P_m(\eta) = \left\langle N \left\{ \prod_{l=1}^{M} \frac{\hat{\mathcal{I}}_l^{m_l}}{m_l!} \exp \left( - \hat{\mathcal{I}}_l \right) \right\} \right\rangle = \prod_{l=1}^{M} \frac{\eta_l^{m_l}}{m_l!} \left( - \frac{\partial}{\partial \eta_l} \right)^{m_l} P_0(\eta). \]  

(6)

First of all, let us express the total detection operator \( \sum_{l=1}^{M} \hat{\mathcal{I}}_l \) in the \( \hat{c} \)-mode basis, using Eqs. (1) and (4) we have (dropping on the way the operators \( \hat{d}^\dagger_{k,s} \) which have no effect on the photon counts)

\[
\sum_{l=1}^{M} \hat{\mathcal{I}}_l = \sum_{l=1}^{M} \eta_l \sum_{k,j=1}^{M} U_{k,l} U^*_j l \sum_{s=1}^{M} \hat{a}^\dagger_{k,s} \hat{a}_{j,s} = \left[ \sum_{k,j=1}^{M} U_{k,l} \eta_l U^*_j \right] \sum_{s=1}^{M} \phi_{k,s}^* \phi_{j,s} \hat{c}_k^\dagger \hat{c}_j = \hat{c}^\dagger U \Lambda U^\dagger \circ \mathcal{V} \hat{c},
\]

(7)

where “\( \circ \)” stands for the by-element (Hadamard) product of two matrices, we have introduced a row-vector of operators \( \hat{c}^\dagger = (\hat{c}_1^\dagger, \ldots, \hat{c}_M^\dagger) \), a diagonal matrix \( \Lambda = \text{diag}(\eta_1, \ldots, \eta_M) \), and a positive semi-definite Hermitian matrix

\[
\mathcal{V}_{k,l} \equiv \sum_{s=1}^{M} \phi_{k,s}^* \phi_{l,s} = \phi_{k,l}^\dagger \phi_k
\]

(8)

with \( \phi_k \equiv (\phi_{k,1}, \ldots, \phi_{k,M})^T \) being the column-vector of the internal optical mode of source \( k \) in the common basis \( \hat{a}^\dagger_{k,s} \).
A convenient way to obtain an explicit expression for the probability in Eq. (6) is to use the Husimi functions and averaging in the coherent basis. For the latter we can convert Eq. (5) to an equivalent form but involving the anti-normal ordering of boson operators. A general formula for such a conversion (for some  \( \hat{c} = (\hat{c}_1, \ldots, \hat{c}_M)^T \) ) reads (see appendix A)

\[
\mathcal{N} \{ \exp (-\hat{c}^\dagger (I_M - H) \hat{c}) \} = \frac{A \{ \exp (-\hat{c}^\dagger (H^{-1} - I_M) \hat{c}) \}}{\det(H)},
\]

where \( H \) is an \( M \)-dimensional positive-definite Hermitian matrix with the eigenvalues bounded by 1 (\( I_M = \text{diag}(1, \ldots, 1) \)). In our case we have

\[
H = U(I_M - \Lambda)U^\dagger \circ V = I_M - U\Lambda U^\dagger \circ V
\]

and the generating function becomes

\[
P_0(\eta) = \langle \mathcal{N} \{ \exp (-\hat{c}^\dagger (I_M - H) \hat{c}) \} \rangle = \langle \det(H^{-1})A \{ \exp (-\hat{c}^\dagger (H^{-1} - I_M) \hat{c}) \} \rangle.
\]

Introducing a Husimi function for the quantum state of each optical source, using the coherent state for \( \hat{c}_k \)-mode, \( \hat{c}_k \alpha; \phi_k) = \alpha \alpha; \phi_k \rangle \),

\[
Q^{(k)}(\alpha) = \frac{1}{\pi}\langle \alpha; \phi_k | \rho^{(k)} | \alpha; \phi_k \rangle,
\]

we obtain the generating function in the form of a multi-mode integral

\[
P_0(\eta) = \int \left[ \prod_{k=1}^M d\alpha_k Q^{(k)}(\alpha_k) \right] \frac{\exp \{-\alpha^\dagger (H^{-1} - I_M) \alpha\}}{\det(H)},
\]

where \( \alpha \equiv (\alpha_1, \ldots, \alpha_M)^T \).

In case of \( N < M \) photon sources (i.e., the rest \( M - N \) input ports receive the optical vacuum), one can easily integrate the vacuum inputs out and reduce the integration in Eq. (13) to the \( N \) non-vacuum sources only. In this case, the matrix \( H \) of Eq. (10) is replaced by a reduced one, built using \( (N \times M) \)-submatrix of a multiport matrix \( U \) and the \( N \)-dimensional submatrix of \( V \) the non-vacuum sources. Indeed, when some of the sources output the vacuum state, each such source has \( Q(\alpha) = e^{-|\alpha|^2}/\pi \). As the internal optical mode for the vacuum state can be chosen arbitrarily, we take the internal modes for the vacuum Husimi functions to be orthogonal to each other and to the internal modes of the rest of the sources. This results in a block-matrix structure for \( V \) and, hence, for \( H \) from Eq. (10):

\[
V = \begin{pmatrix} V^{(I)} & 0 \\ 0 & V^{(II)} \end{pmatrix}, \quad H = \begin{pmatrix} H^{(I)} & 0 \\ 0 & H^{(II)} \end{pmatrix},
\]

(14)
where the superscripts \((I)\) and \((II)\) stand for the non-vacuum and the vacuum sources, respectively, with the \((II)\)-matrices being diagonal. These properties allow one to integrate over the \(\alpha\)-variables corresponding to the vacuum sources (which is a Gaussian integral given in appendix [3]), where, taking into account that \(\det(H) = \det(H^{(I)}) \det(H^{(II)})\), we obtain the final result in exactly the same form as Eq. \((13)\) except that now the integration is only over the \(\alpha\)-variables of the non-vacuum sources and \(H = H^{(I)}\).

Now, given explicit expressions for Husimi functions of all optical sources, one can obtain all the needed output probabilities. Below derive other equivalent expressions for the generating function of the output probabilities, making connection with the previous results.

One can integrate in the expression for the generating function \((13)\) by performing the series expansion of the Husimi functions, with the coefficients being proportional to the terms in the Fock-space expansion of the quantum state \(\rho \equiv \rho^{(1)} \otimes \ldots \otimes \rho^{(M)}\) of the sources:

\[
Q^{(k)}(\alpha) = \frac{1}{\pi} e^{-|\alpha|^2} \sum_{n,m \geq 0} \rho_{n,m}^{(k)} \frac{\alpha^m}{\sqrt{n!m!}} \equiv \frac{1}{\pi} e^{-|\alpha|^2} G^{(k)}(\alpha^*, \alpha). \tag{15}
\]

Substituting this expression into Eq. \((13)\) and observing that the infinite series \(G^{(k)}(\alpha^*_k, \alpha_k)\) can be obtained by application of derivatives over some complex dummy variables \(\lambda^\dagger = (\lambda^*_1, \ldots, \lambda^*_M)\) in the exponent, we obtain (see appendix [4])

\[
P_0(\eta) = \frac{1}{\det(H)} \int \left[ \prod_{k=1}^M \frac{d^2 \alpha_k}{\pi} G^{(k)} \left( \frac{\partial}{\partial \lambda_k}, \frac{\partial}{\partial \lambda^*_k} \right) \right] \exp \left\{ \alpha^\dagger H^{-1} \alpha + \alpha^\dagger \lambda + \lambda^\dagger \alpha \right\} \bigg|_{\lambda_k=0} \tag{16}
\]

Eq. \((16)\) leads to an explicit form of the generating function given in terms of the matrix permanents of some Hermitian matrices (with repeated rows and columns, in general) built using rows and columns of the Hermitian matrix \(H\), defined in Eq. \((10)\). Indeed, expanding the exponent in Eq. \((16)\) into the Taylor series, we obtain

\[
\prod_{k=1}^M \left( \frac{\partial}{\partial \lambda_k} \right)^{n_k} \left( \frac{\partial}{\partial \lambda^*_k} \right)^{m_k} \bigg|_{\lambda=0} \frac{(\lambda^\dagger H \lambda)^p}{p!} = \delta_{|m|,|p|} \delta_{|n|,p} \sum_{\sigma \in S_p} \prod_{i=1}^p H_{l_i, k_{\sigma(i)}} \operatorname{per}(H[m, n]), \tag{17}
\]

where \(H[m, n]\) is a matrix built from \(H\) on rows \(l_1, \ldots, l_p\) and columns \(k_1, \ldots, k_p\) with repetitions given by the \(m = (m_1, \ldots, m_M)\) and \(n = (n_1, \ldots, n_M)\), respectively, \(S_p\) is the
group of permutations of \( p \) objects, \(|n| \equiv \sum_{i=1}^{M} n_i\), and \( \text{per}(\ldots) \) is the matrix permanent \([28]\). Hence, by the definition of \( G^{(k)} \) as an infinite series in Eq. (13), Eq. (16) becomes

\[
P_0(\eta) = \sum_{p \geq 0} \sum_{|n|,|m| = p} \frac{\text{per}(H[n,m])}{\sqrt{m!n!}} \prod_{k=1}^{M} \rho_{nk,mk}^{(k)},
\]

(18)

where \( n! \equiv n_1! \ldots n_M! \).

Eqs. (15)-(16) and (18) constitute the main result. They can be used for derivation of the specific formulae for photon counting probabilities with general quantum sources and non-ideal detectors \((\eta_k \neq 1)\) by using the prescription in Eq. (6). Numerical computations of the generating function can turn out to be hard with increase of \( N \) and \( M \), at least in some cases, as the example considered below, due to hardness of the matrix permanent of positive definite Hermitian matrices \([29]\). The same applies to the formulae for the photon counting probabilities, computational hardness of even approximate calculation of which is at the core of the computational advantage of the Boson Sampling \([1, 3]\).

Let us show, for instance, that the expression in Eq. (18) reduces to the previously derived probability formula \([19, 21]\) for a fixed number of photons launched in each input port of a linear multiport, where photons in different input ports are partially distinguishable. In this case we have

\[
\rho^{(k)} = \frac{(\hat{c}_k^\dagger)^{n_k}|0\rangle\langle 0|(\hat{c}_k)^{n_k}}{n_k!} = |n_k; \phi_k\rangle\langle n_k; \phi_k|,
\]

(19)
i.e., the Fock state in an internal optical mode given by the vector \( \phi_k \). The generating function in this case becomes

\[
P_0(\eta) = \frac{\text{per}(H[n,n])}{n!}.
\]

(20)

Assuming that we detect all the input photons, \(|n| = N\) (the total number of photons at input), and using Eqs. (6), (10), and (20) gives

\[
P_m(\eta) = \frac{1}{m!} \prod_{l=1}^{M} \eta_l^{m_l} \left( -\frac{\partial}{\partial \eta_l^{m_l}} \right)^{m_l} \text{per}(H[n,n]) = \frac{1}{m!n!} \prod_{l=1}^{M} \eta_l^{m_l} \left( -\frac{\partial}{\partial \eta_l^{m_l}} \right)^{m_l} \sum_{\sigma \in S_N} \prod_{i=1}^{N} H_{k_i,k_{\sigma(i)}}
\]

\[
= \frac{1}{m!n!} \sum_{\sigma,\tau \in S_N} \prod_{i=1}^{N} \eta_{l_i}^{k_i,k_{\sigma(i)}} U_{l_i,k_{\sigma(i)}} U_{k_{\sigma(i)},l_i}^* \sum_{\sigma_1,\sigma_2 \in S_N} J(\sigma_2\sigma_1^{-1}) \prod_{i=1}^{N} U_{k_{\sigma_2(i)},l_i} U_{k_{\sigma_1(i)},l_i}^*;
\]

(21)

where \( k_1, \ldots, k_N \) and \( l_1, \ldots, l_N \) are the input and output ports of a multiport (generally, with repetitions) corresponding to the occupations \( n \) and \( m \), respectively, \( \eta_m = \eta_1^{m_1} \ldots \eta_M^{m_M} \).
\[ \sigma_1 = \tau^{-1}, \sigma_2 = \sigma \tau^{-1}, \text{ and} \]

\[ J(\sigma) \equiv \prod_{i=1}^{N} \mathcal{V}_{k_i, k_{s(i)}} = \prod_{i=1}^{N} \phi_{k_{s(i)}}^{\dagger} \phi_{k_{s(i)}} = \Phi^\dagger P_{\sigma^{-1}} \Phi \]  

(22)

where we have introduced the tensor-product of the internal states of photons

\[ \Phi \equiv \phi_{k_1} \otimes \ldots \otimes \phi_{k_N} = (\phi_1)^{\otimes \text{n}_1} \otimes \ldots \otimes (\phi_M)^{\otimes \text{n}_M}. \]  

(23)

and the operator representation \( P_{\sigma} \) of a permutation \( \sigma \). The result in Eqs. (21) and (22) reproduces that of Ref. [21] when \( \eta_k = 1 \) (to make a comparison clearer, note that \( U \) here is equivalent to \( U^\dagger \) in Ref. [21], and that \( \Phi \) is the vector of expansion coefficients in a basis of internal states of photons, therefore \( P_{\sigma^{-1}} \) in Eq. (22) corresponds to \( P_{\sigma} \) in the definition of \( J(\sigma) \) of Ref. [21]).

III. MULTI-MODE INDEPENDENT SOURCES

When photon sources output multi-mode quantum states, one has to employ a basis for the internal modes. We can choose any basis, since, in general, no preferred basis exists (only in the case of single-mode sources, one can select a special basis for each source, as in Eq. (3)). Therefore, we assume that there is a common basis, \( \hat{a}_{k,s}^{\dagger} \), of the internal modes and that it is of a finite dimension \( s = 1, \ldots, d \). Eq. (3) is replaced in this case by

\[ \rho^{(k)} = \sum_{n,m} \rho_{n,m}^{(k)} \prod_{s=1}^{d} (\hat{a}_{k,s}^{\dagger})^{n_s} |0\rangle \langle 0| \prod_{s=1}^{d} \hat{a}_{k,s}^{m_s} / \sqrt{n!m!}, \]  

(24)

where \( \mathbf{n} = (n_1, \ldots, n_d) \) and \( \mathbf{n}! = n_1! \cdots n_d! \). One can proceed now in a manner similar to that of section II with the two differences. First, source \( k \) is now described by a generalised Husimi function \( Q^{(k)}(\alpha^{(k)}) \) of \( d \) complex variables \( \alpha^{(k)} \equiv (\alpha_1, \ldots, \alpha_d)^T \), where

\[ Q^{(k)} = \frac{1}{\pi^d} \langle \alpha^{(k)} | \rho^{(k)} | \alpha^{(k)} \rangle, \quad | \alpha^{(k)} \rangle \equiv | \alpha_1^{(k)} \rangle \otimes \ldots \otimes | \alpha_d^{(k)} \rangle, \]  

(25)

where \( \hat{a}_{k,s} | \alpha_s^{(k)} \rangle = \alpha_s^{(k)} | \alpha_s^{(k)} \rangle \). Second, since we use a common basis, the \( M \)-dimensional Hermitian matrix \( H \) of Eq. (10) is replaced by the \( M \times d \)-dimensional one (note the tensor product below, and not the Hadamard product as in Eq. (3))

\[ \mathcal{H} = U (I_M - \Lambda) U^\dagger \otimes I_d. \]  

(26)
The rest of the derivation is just a mere repetition of the steps made in section II. We obtain the following result for the generating function of the output probability distribution

\[ P_0(\eta) = \frac{1}{\prod_{k=1}^{M} (1 - \eta_k)^d} \int \left[ \prod_{k=1}^{M} d^2 \alpha^{(k)} Q^{(k)}(\alpha^{(k)}) \right] \exp \left\{ -\alpha^{\dagger} (\mathcal{H}^{-1} - I_{M \times d}) \alpha \right\} \]  (27)

where \( \alpha^T \equiv (\alpha^{(1)}, \ldots, \alpha^{(M)})^T \) and we have used that \( \det(\mathcal{H}) = \prod_{k=1}^{M} (1 - \eta_k)^d \). Similarly as for the single-mode case in section II for \( N < M \) sources (with \( M - N \) input ports receiving the optical vacuum), one can easily integrate the vacuum inputs out and reduce the integration in Eq. (27) to \( N \) of \( \alpha^{(k)} \) corresponding to non-vacuum sources only, in quite a similar fashion.

Note that the internal-mode configuration of the sources is now contained in the Husimi functions themselves and not in the Hermitian matrix \( \mathcal{H} \), in contrast to the single-mode case, Eq. (13). Therefore, though one can integrate in Eq. (27), similar as it was done in Eqs. (15)-(18) of section II, the expression will be quite cumbersome, in general. This is the main difference between the single-mode and the multi-mode sources. The reason for this is the non-existence of a preferred basis: since, in general, any \( M \times d \)-dimensional basis of boson operators can be used in the multi-mode case, there is no point in introducing the operators \( \hat{c}_{k,s} \) (as a generalisation of \( \hat{c}_k \) of Eq. (2)) specific for each source in the multi-mode case.

Finally, recalling the example of single-photon sources, analysed in section II, there is but a marginal additional generality coming from Eq. (27) in derivation of the previous result of Ref. [21]. Indeed, since the number of photons per input is fixed, one can always expand the corresponding density matrix \( \rho^{(k)} \) of Eq. (24) as a positive combination, where each term is a product of single-mode ones of Eq. (3) (with different internal modes), and use the approach of section II. In this case, Eq. (22) is replaced by the most general one involving the “internal density matrix” instead of the scalar product of the internal modes (see, for details, Ref. [21]).

IV. CONCLUSION

The main result of this work is a generating function for the photon counts at the output of a linear optical multiport with arbitrary photon sources at the input. For the single-mode optical sources, the latter is a linear combination of matrix permanents of positive
semi-definite Hermitian matrices, where each Hermitian matrix is given as the Hadamard product of a submatrix of the unitary matrix of a multiport device and a Hermitian matrix describing partial distinguishability of the internal optical modes of photon sources. For the case of multi-mode sources we have, in general, the generating function as an integral of the (multidimensional) Husimi functions of the sources and a Gaussian exponent describing the action of a linear multiport and photon detection stage (without resolving the internal optical modes). When each photon source outputs exactly a Fock state, the obtained expression reduces to the probability formula derived before for partially distinguishable photons. The results can be useful in analysing the interference experiments with general optical sources of photons with controlled distinguishability and for derivation of error bounds for the experimental Boson Sampling devices.

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Appendix A: Relation between the normal and the anti-normal ordering of an exponent of a quadratic form in boson operators

Let us first show the following relation

\[ \mathcal{N} \left\{ \exp \left( -\xi \hat{b}^\dagger \hat{b} \right) \right\} = A \left\{ \exp \left( -\lambda \hat{b}^\dagger \hat{b} \right) \right\} \frac{1}{1 - \xi}, \]

where \( 0 < \xi < 1 \) and \( \lambda = \xi / (1 - \xi) \). To this end we observe that for two Fock states \( |n\rangle \) and \( |m\rangle \)

\[ \langle m| \mathcal{N} \left\{ \exp \left( -\xi \hat{b}^\dagger \hat{b} \right) \right\} |n\rangle = \sum_{k=0}^{\infty} \frac{(-\xi)^k}{k!} \langle m| \hat{b}^\dagger \hat{b}^k |n\rangle \]

\[ = \delta_{m,n} \sum_{k=0}^{n} \frac{(-\xi)^k n!}{(n-k)!k!} = \delta_{m,n} (1 - \xi)^n, \]

(A2)
similarly
\[
\langle m | A \left\{ \exp \left( -\lambda \hat{b}^\dagger \hat{b} \right) \right\} | n \rangle = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \langle m | \hat{b}^k (\hat{b}^\dagger)^k | n \rangle = \delta_{m,n} \langle 1 + \lambda \rangle^{-n-1}, \tag{A3}
\]
where we have used that
\[
\sum_{k=0}^{\infty} \frac{\lambda^k (k + n)!}{n! k!} = (1 + \lambda)^{-n-1}. \tag{A4}
\]
Eqs. (A2) and (A3) immediately give Eq. (A1).

Consider now Eq. (9) of section II. Let \( H^\dagger = H \) and \( 0 < H < I \). We can diagonalize the positive-definite Hermitian matrix \( H = V^\dagger D V \), where \( D = \text{diag}(1 - \xi_1, \ldots, 1 - \xi_M) \), \( 0 < \xi_k < 1 \), and \( V^\dagger V = I \). Using \( V \) for a canonical transformation of the operators \( \hat{c}_k \) to a new basis \( \hat{b}_k \), \( k = 1, \ldots, M \),
\[
\hat{b}_k = \sum_{l=1}^{M} V_{k,l} \hat{c}_l, \tag{A5}
\]
and observing that \( \det(H) = \prod_{k=1}^{M} (1 - \xi_k) \), we reduce Eq. (9) to an operator product form, where each element in the product is a relation given by Eq. (A1). This finishes the proof of Eq. (9).

Appendix B: A Gaussian integral

Here we give for a reference a Gaussian integral which is used in the main text
\[
\int \prod_{k=1}^{M} \frac{d^2 \alpha_k}{\pi} \exp \left( -\alpha^\dagger A \alpha + \lambda^\dagger \alpha + \alpha^\dagger \mu \right) = \frac{\exp \left( \lambda^\dagger A^{-1} \mu \right)}{\det(A)} \tag{B1}
\]
where \( A \) is a positive definite Hermitian matrix. Eq. (B1) is derived by first diagonalizing the matrix \( A = U^\dagger \text{diag}(a_1, \ldots, a_M) U \), making a change of integration variable \( z = U \alpha \), and invoking the following result
\[
\int \frac{d^2 z}{\pi} \exp \left( -a |z|^2 + \lambda^* z + z^* \mu \right) = \frac{1}{a} \exp \left( \frac{\lambda^* \mu}{a} \right). \tag{B2}
\]
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