Shrinking point bifurcations of resonance tongues for piecewise-smooth, continuous maps

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Abstract
Resonance tongues are mode-locking regions of parameter space in which stable periodic solutions occur; they commonly occur, for example, near Neimark–Sacker bifurcations. For piecewise-smooth, continuous maps these tongues typically have a distinctive lens-chain (or sausage) shape in two-parameter bifurcation diagrams. We give a symbolic description of a class of ‘rotational’ periodic solutions that display lens-chain structures for a general $N$-dimensional map. We then unfold the codimension-two, shrinking point bifurcation, where the tongues have zero width. A number of codimension-one bifurcation curves emanate from shrinking points and we determine those that form tongue boundaries.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Mathematical models often incorporate a nonsmooth component in order to describe a physical discontinuity or sudden change. Piecewise-smooth, continuous maps are a class of such systems. We say that a map

$$x_{n+1} = F(x_n),$$

where $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$, is piecewise-smooth continuous if $F$ is everywhere continuous and $\mathbb{R}^N$ can be partitioned into countably many regions where $F$ has a different smooth functional form. The map is nondifferentiable on codimension-one region boundaries called switching manifolds.

Piecewise-smooth, continuous maps have been used as models in many areas, for example, economics [1, 2], power electronics [3–5], and cellular neural networks [6]. Furthermore, they arise as Poincaré maps of piecewise-smooth systems of differential equations, particularly near...
sliding bifurcations [7, 8] and near so-called corner collisions [9, 10]. System parameter values often correspond to some user controllable input such as a forcing frequency or a chemical concentration.

Piecewise-smooth, continuous maps may exhibit border-collision bifurcations when a fixed point crosses a switching manifold under smooth variation of system parameters [11–14]. A fundamental characteristic of these bifurcations is that multipliers associated with the fixed point may change discontinuously at the bifurcation. In some cases, border-collision bifurcations resemble bifurcations of smooth maps (such as saddle-node and flip bifurcations); however, invariant sets created at the bifurcation generically exhibit a linear (not quadratic) growth in size under variation of system parameters. Alternatively, border-collision bifurcations may have no smooth analogue (for example, an immediate transition from a fixed point to a period-three cycle in a one-dimensional map). Furthermore, these bifurcations may be extremely complicated, producing multiple attractors and chaotic dynamics.

Resonance (or Arnold) tongues are regions in parameter space within which there is an attracting periodic solution. In some cases (e.g. circle maps), the periodic solutions in a tongue can be characterized by their rotation number. As was first observed for a one-dimensional, piecewise-linear, circle map [15], two-parameter pictures of resonance tongues for piecewise-smooth, continuous maps typically exhibit a distinctive lens-chain (or sausage) geometry [1, 16–19], as in figure 1. As we will show, for maps with many parameters generically this geometry is observed when any two parameters are chosen. The boundaries of a resonance tongue correspond either to a loss of stability of the associated periodic solution or to a collision of one point of the solution with the switching manifold. The latter case usually corresponds to the collision of the stable periodic solution with an unstable periodic solution of the same period and is known as a border-collision fold bifurcation (the word ‘fold’ is interchangeable with ‘saddle-node’). (This need not always occur, as is shown in [16].) Following [15], we call the codimension-two points at which resonance tongues have zero width, shrinking points (sometimes called ‘waist points’). The purpose of this paper is to explore the bifurcation structure that occurs near such points for a general $N$-dimensional map in the neighbourhood of a switching manifold.

A switching manifold locally divides phase space into two regions. Any nearby period-$n$ cycle with no points on the switching manifold consists of, say, $l$ points on one side and $n-l$ points on the other side of the switching manifold. We observe that $l$ differs by one for coexisting stable and unstable periodic orbits in lens-shaped resonance regions. Moreover, the number $l$ also differs by one for stable periodic orbits in adjoining lenses. For example, figure 2 shows a magnification of figure 1 near a generic shrinking point in the $2/7$-tongue. The borders of a lens near such a shrinking point correspond to border-collision bifurcations of the two orbits in the lens, and thus to the occurrence of an orbit with a point on the switching manifold.

Consequently, a shrinking point is characterized geometrically by the occurrence of a periodic solution that has two points on the switching manifold. However, in section 5 we show that there are two cases that must be treated separately: terminating and non-terminating shrinking points. It turns out to be more convenient to define non-terminating shrinking points by the singularity of certain matrices (the ‘border-collision’ matrices). Terminating shrinking points correspond to the case $l = 1$ or $n - 1$; they are defined by the occurrence of multipliers with a particular value. Throughout this paper when we say ‘shrinking point’ we mean either a non-terminating or a terminating shrinking point.

As we will describe in section 4, periodic solutions of a piecewise-affine map may be obtained by solving a linear system. When this system is nonsingular its unique solution corresponds to a period-$n$ cycle that is realized by the map and said to be admissible when
Figure 1. Resonance tongues corresponding to period-$n$ cycles up to $n = 30$ of \((2)\) with 
\[
A_1 = \begin{pmatrix}
\cos(2\pi \omega) & 1 \\
-\frac{\omega}{b} & 0
\end{pmatrix}, \quad A_R^n = \begin{pmatrix}
\frac{1}{2} \cos(2\pi \omega) & 1 \\
-\frac{\omega}{2} & 0
\end{pmatrix}, \quad b = e_1 \text{ and } \mu = 1, \text{ as shown in } [16].
\]

The parameters \(\omega\) and \(s_R\) are chosen so that the multipliers of the fixed point ‘jump’ across the unit circle at the border-collision bifurcation. The rotation number associated with each resonance tongue is indicated.

Each of the \(n\) points lies on the appropriate side of the switching manifold, and is otherwise said to be virtual.

Lemma 7, which is given in section 5, essentially states (under certain nondegeneracy conditions, see definitions 2 and 3) the following:

- The algebraic manner by which we define shrinking points implies that there exists a periodic solution with two points on the switching manifold.
- At a shrinking point, the map has an invariant polygon (that is typically nonplanar) on which the dynamics are conjugate to a rigid rotation.

Since a non-terminating shrinking point is defined by two conditions (two points on the switching manifold), its unfolding requires the variation of two parameters. In the generic two-parameter picture, a non-terminating shrinking point appears, at first glance, to lie at the intersection of two smooth curves. However, this is an illusion: there are really four distinct curves corresponding to four different border-collision fold bifurcations. Theorem 9 essentially states the following:

- Each of the four border-collision curves is quadratically tangent to one of the others and appears on only one side of the shrinking point (for instance in figure 2 the curves (0) and (5) are tangent, as are (1) and (3)).
- The extension of these curves through the shrinking point corresponds to virtual solutions.
- At each border-collision fold bifurcation one particular point of a particular period-$n$ cycle lies on the switching manifold.
- There are curves in parameter space along which each point of this cycle lies on the switching manifold, see figure 2; however, except for the four that form the resonance tongue boundaries, these correspond to virtual solutions.

The rest of this paper is organized as follows. In section 2 we introduce the general map that will be studied throughout this paper and determine its fixed points. Symbolic dynamics are
Figure 2. A magnification of figure 1 showing the 2/7-resonance tongue near a shrinking point. Sketches of four period-7 orbits are shown. The orbit with $l = 3$ (its symbol sequence is LLRRLRR, see section 3) is admissible in both grey regions. In the upper region this orbit is stable; in the lower region it is a saddle. Let $\{x_i\}$ denote the points on this orbit. On the curves labelled (i) the point $x_i$ lies on the switching manifold. By theorem 9 (see section 6), portions of the curves (0), (1), (3) and (5) form tongue boundaries. A saddle period-7 orbit with $l = 2$ is admissible in the upper region and coincides with $\{x_i\}$ on the boundaries (0) and (1). Similarly, a stable period-7 orbit with $l = 4$ is admissible in the lower region and coincides with $\{x_i\}$ on the boundaries (3) and (5). At the shrinking point there is an invariant heptagon. On the curve $\Gamma$ the linear system (24) that determines the orbit $\{x_i\}$ is singular. As any point on $\Gamma$ other than the shrinking point is approached, the amplitude of $\{x_i\}$ tends to infinity, however $\{x_i\}$ is virtual here.

Introducing the new concept of a ‘rotational symbol sequence’ and using symbolic dynamics are able to classify periodic solutions as either rotational or non-rotational. The motivation is that rotational periodic solutions correspond to lens-chain resonance regions whereas non-rotational periodic solutions may not [16]. In section 4 we describe how a periodic orbit with a particular symbol sequence may be determined by solving a linear system. In section 5 we provide formal definitions for terminating and non-terminating shrinking points. This section describes the singular nature of shrinking points and the existence of invariant, nonplanar, polygons. This mostly preparatory work sets up section 6 where we present the unfolding of the shrinking point bifurcation summarized by theorem 9. Concluding comments are given in section 7.

2. A framework for local analysis

It is well known (see for instance [12]) that dynamical behaviour of the piecewise-smooth, continuous map (1) in the neighbourhood of a nondegenerate border-collision bifurcation of a fixed point at a smooth point on a switching manifold is described by a piecewise-affine map of the form

$$x_{n+1} = f_\mu(x_n) = \begin{cases} f_\mu^L(x_n), & s \leq 0, \\ f_\mu^R(x_n), & s \geq 0, \end{cases}$$

(2)
where
\[ f^i_\mu(x) = \mu b + A_i x, \]
for \( i = L, R \). Here
\[ s = e^T_1 x, \]
denotes the first component of \( x \in \mathbb{R}^N \), \( \mu \in \mathbb{R} \) is a system parameter, \( b \in \mathbb{R}^N \), and \( A_L \) and \( A_R \) are \( N \times N \) matrices, which, by continuity of (2), must be identical in all but possibly their first columns. Nonlinear terms only affect local dynamics in degenerate cases (for example, when \( A_L \) has a multiplier on the unit circle). Also, for simplicity, we have assumed the switching manifold is analytic and therefore may be transformed to the plane \( s = 0 \).

The map (2) is a homeomorphism if and only if \( \det(A_L) \det(A_R) > 0 \). Furthermore, (2) has the scaling symmetry
\[ f_{\lambda \mu}(\lambda x) = \lambda f_\mu(x), \quad \forall \lambda > 0, \]
and consequently every bounded invariant set of (2) collapses to the origin as \( \mu \to 0 \).

Potential fixed points of (2) are obtained by solving \( x^*^{(i)} = f^i_\mu(x^*^{(i)}) \). Whenever \( A_i \) does not have a multiplier of one, \( x^*^{(i)} \) is unique and given by
\[ x^*^{(i)} = \mu(I - A_i)^{-1} b. \]

Each \( x^*^{(i)} \), as given by (6), is an admissible fixed point of (2) whenever \( s^*^{(i)} \) (the first component of \( x^*^{(i)} \)) has the appropriate sign (negative for \( i = L \) and positive for \( i = R \)), otherwise it is virtual. We have
\[ s^*^{(i)} = \frac{\mu}{\det(I - A_i)} e^T_1 \operatorname{adj}(I - A_i)b, \]
where \( \operatorname{adj}(X) \) denotes the adjugate matrix of a matrix \( X \), \( (\operatorname{adj}(X))X = \det(X)I \). Since all columns of \( I - A_L \) and \( I - A_R \), except possibly the first, are identical, their adjugates share the same first row, which we denote by \( \varrho^T \):
\[ \varrho^T = e^T_1 \operatorname{adj}(I - A_L) = e^T_1 \operatorname{adj}(I - A_R). \]

Thus
\[ s^*^{(i)} = \frac{\mu \varrho^T b}{\det(I - A_i)}. \]

The condition \( \varrho^T b \neq 0 \) is a nondegeneracy condition for the border-collision bifurcation, guaranteeing that the distance between each \( x^*^{(i)} \) and the switching manifold varies linearly with the bifurcation parameter \( \mu \). Furthermore, if \( I - A_i \) is nonsingular and \( a_i \) denotes the number of real multipliers of \( A_i \) that are greater than one, then \( \det(I - A_i) \) is positive if and only if \( a_i \) is even. Thus, as described in [20], by (8), if \( a_L + a_R \) is even, \( x^{*^{(L)}} \) and \( x^{*^{(R)}} \) are admissible for different signs of \( \mu \), thus as \( \mu \) is varied through zero a single fixed point persists. Conversely if \( a_L + a_R \) is odd, \( x^{*^{(L)}} \) and \( x^{*^{(R)}} \) coexist for the same sign of \( \mu \). At \( \mu = 0 \) the fixed points collide and annihilate in a nonsmooth fold bifurcation.

### 3. Symbolic dynamics

We refer to any sequence, \( S \), that has elements taken from the alphabet, \( \{L, R\} \), as a symbol sequence. This paper focuses on periodic solutions; thus, we will assume \( S \) is finite and let \( n \) denote the length of \( S \). We index the elements of \( S \) from 0 to \( n - 1 \). Arithmetic on the indices
of $S$ is usually modulo $n$. For clarity, throughout this paper we omit ‘mod $n$’ where it is clear modulo arithmetic is being used.

Here we introduce notation relating to symbol sequences. Given $n \in \mathbb{N}$, the collection of all symbol sequences of length $n$ is $(L, R)^n \equiv \Sigma_2^n$. The $i$th left cyclic permutation is an operator, $\sigma_i : \Sigma_2^n \rightarrow \Sigma_2^n$, defined by $(\sigma_i S)_j = S_{j+i}$ and we use the notation, $S^{(i)} \equiv \sigma_i S$. The $i$th flip permutation is an operator, $\chi_i : \Sigma_2^n \rightarrow \Sigma_2^n$, that flips the $i$th element of $S$ (i.e. $L \rightarrow R$ and $R \rightarrow L$) and leaves all other elements unchanged. We use the notation, $S^T \equiv \chi_i S$. For example if $S = \text{LRLRR}$ then $S^3 = \text{LRLRR}$ and $S^{(2)} = \text{LLRLR}$. In general $S^{(j)} \equiv (S^T)^{(j)} \neq S^{(j)}$ because with the same example $S^{(2)} = \text{LLRLR}$ and $S^{(2)} = \text{LLRRR}$.

The $i$th multiplication permutation is an operator, $\pi_i : \Sigma_2^n \rightarrow \Sigma_2^n$, defined by $(\pi_i S)_j = S_{ij}$ (for example, if again $S = \text{LRLRR}$, then $\pi_2S = \text{LLRRR}$). Note $(\pi_i, \pi_j S)_k = (\pi_j S)_i k = S_{i+k}$, thus

$$\pi_i \pi_j = \pi_{ij}. \tag{9}$$

Consequently $\pi_i$ is an invertible operator if and only if $\gcd(i, n) = 1$ (where ‘$\gcd$’ denotes greatest common divisor) and the inverse of $\pi_i$ is $\pi_{i^{-1}}$ (where $i^{-1}$ is the multiplicative inverse of $i$ modulo $n$). Also note $(\sigma_i, \pi_j S)_k = (\pi_j S)_i k = S_{i+k} = (\sigma_i \pi_j S)_k$. Hence

$$\sigma_i \pi_j = \pi_j \sigma_i. \tag{10}$$

We let $S\hat{S}$ denote the concatenation of the symbol sequences $S$ and $\hat{S}$ and let $S^k \in \Sigma_2^{kn}$ denote the symbol sequence formed by the concatenation of $k$ copies of $S$. A symbol sequence is called primitive if it cannot be written as a power, $S^k$, for any $k > 1$. $S$ is primitive if and only if $S \neq S^{(i)}$ for all $i \neq 0$.

Of particular interest is the situation that the map (2) exhibits an invariant, topological circle that crosses the switching manifold at two points. When the restriction of the map to this circle is homeomorphic to a monotone increasing circle map we find that periodic solutions on this circle can only have certain symbol sequences. We call these sequences rotational symbol sequences.

**Definition 1 (Rotational symbol sequence).** Let $l, m, n \in \mathbb{N}$, with $l, m < n$ and $\gcd(m, n) = 1$. Let $S = S[l, m, n]$ be the symbol sequence of length $n$ defined by

$$S_d = \begin{cases} L, & i = 0, \ldots, l - 1, \\ R, & i = l, \ldots, n - 1, \end{cases} \tag{11}$$

where $d$ is the multiplicative inverse of $m$ modulo $n$, i.e. $dm \equiv 1 \mod n$. Then we say $S$ and any cyclic permutation of $S$ is a rotational symbol sequence.

Note $d$ always exists and is unique because $m/n$ is an irreducible fraction [21]. For example if $(l, m, n) = (3, 2, 7)$ then $d = 4$, hence $S_0 = S_1 = S_2 = L$ in modulo 7 arithmetic, thus $S[3, 2, 7] = \text{LLRLR}$.

A pictorial method for computing $S$ in terms of $l, m$ and $n$ is to select $n$ points on a circle, then draw a vertical line through the circle such that $l$ points lie to the left of the line, see figure 3. Label the first point to the left of the lower intersection of the circle and line, point 0. Move $m$ points clockwise from 0 and label this point 1. Continue stepping clockwise labelling every $m$th point with a number that is one greater than the previous number until all points are labelled. Then $S_i = L$ if the point $i$ lies to the left of the vertical line and $R$ otherwise.

The circle represents an invariant circle of (2) and the vertical line represents the switching manifold. Taking $m$ steps clockwise corresponds to evaluating the map (2) once. Thus
Figure 3. Illustration of the pictorial method for determining $S[3, 2, 7] = \text{LLBLRRLR}$.

$S[l, m, n]$ is the symbol sequence of a period-$n$ cycle of (2) that has $l$ points to the left of the switching manifold and with rotation number $m/n$.

If $N_n [N^\text{rot}_n]$ denotes the number of primitive symbol sequences (primitive rotational symbol sequences) of length $n$ that are distinct up to cyclic permutation, then $N_n$ grows like $e^n$, whereas $N^\text{rot}_n$ grows like $n^2 [22]$. Thus although for $2 \leq n \leq 5$ all primitive symbol sequences are rotational, for large $n$ the majority of primitive symbol sequences are non-rotational.

The following lemma states some basic properties of rotational symbol sequences which are used to prove theorem 9.

**Lemma 1.** Suppose $S[l, m, n]$ is a rotational symbol sequence.

(a) $S[l, m, n] = \pi_m S[l, 1, n]$.
(b) If $S[l, m, n]_0^{(i)} = L$ and $S[l, m, n]_{-d}^{(i)} = R$ then $i = 0$.
(c) $S[l, m, n]$ is primitive.
(d) $S[l, m, n]^{(i-1)d} = S[l, n-m, n]$.
(e) $S[l, m, n]^{(l-1)d} = S[l, m, n]^{(n-1)d}$.
(f) If $0 < m_1, m_2 < \frac{n}{2}$ are distinct integers coprime to $n$ and $l \neq 1, n - 1$, then $S[l, m_1, n]$ is not a cyclic permutation of $S[l, m_2, n]$.

**Proof.** Recall the notation $\sigma_i S = S^{(i)}$, for the $i$th left cyclic permutation and $\chi_i S = S^i$ for the flip of the $i$th element.

(a) Since $dm = 1$,

$$\pi_m S[l, 1, n]_{id} = S[l, 1, n]_{i} = L$$

if and only if $i = 0, \ldots, l - 1$,

matching the definition of $S[l, m, n]$, (11).

(b) Let $j = im$ (equivalently, $i = jd$). Then $S[l, m, n]_0^{(i)} = S[l, m, n]_{id} = L$ implies $0 \leq j \leq l - 1$. Similarly, $S[l, m, n]_{-d}^{(i)} = S[l, m, n]_{(l-1)d} = R$ implies $l \leq j - 1 \leq n - 1$. The only value of $j$ that satisfies both inequalities is zero, hence $i = 0$.

(c) From part (b), $S[l, m, n]$ differs from any nontrivial cyclic permutation of itself in either the 0th or the $(-d)$th element. Therefore, $S[l, m, n]$ is primitive.
(d) By definition,
\[ S[l, m, n]_d = L \quad \text{if and only if } i = 0, \ldots, l - 1, \]
and
\[ S[l, n - m, n]_{(n-d)} = L \quad \text{if and only if } i = 0, \ldots, l - 1, \]

since the multiplicative inverse of \((n - m)\) is \((n - d)\). From (12), by letting \(j = i - l + 1\) we obtain
\[ S[l, m, n]_{j}^{d-l} = L \quad \text{if and only if } j = -l + 1, \ldots, 0, \]
thus
\[ S[l, m, n]_{j}^{d-l} = L \quad \text{if and only if } j = 0, \ldots, l - 1, \]
which matches (13).

(e) From (14) we obtain
\[ S[l, m, n]_{j}^{d-l} = L \quad \text{if and only if } i = -l + 1, \ldots, -1. \]

Also
\[ S[l, m, n]_{1}^{d-l} = L \quad \text{if and only if } i = 1, \ldots, l - 1, \]
thus
\[ S[l, m, n]_{j}^{d-l} = L \quad \text{if and only if } j = -l + 1, \ldots, -1, \]
where we have set \(j = i - l\).

(f) Let \(d_1\) and \(d_2\) denote the multiplicative inverses of \(m_1\) and \(m_2\) modulo \(n\), respectively. Let
\[ \hat{S} = \pi_{d_1} S[l, m_1, n], \]
\[ \hat{S} = \pi_{d_2} S[l, m_2, n]^{(k)}, \]
where \(k \in \mathbb{Z}\). Since \(\pi_{d_1}\) is an invertible operator, it remains to show that \(\hat{S} \neq \hat{S}\) for any \(k \in \mathbb{Z}\).

Using (9) and part (a) we find
\[ \hat{S} = S[l, 1, n]. \]

Re-expressing \(\hat{S}\) in the form we desire is a little more complicated but requires no more that the basic known properties concerning multiplicative permutations, \(\pi\).

\[ \hat{S} = (\pi_{d_1} S[l, m_2, n]^{(k)})^{(k,m_1)}, \quad \text{by (10)} \]
\[ = (\pi_{d_1} \pi_{m_2} S[l, 1, n]^{(k)})^{(k,m_1)}, \quad \text{by part (a)} \]
\[ = (\pi_{d_1} S[l, 1, n])^{(k,m_1)} \]
\[ = S[l, \hat{m}, n]^{(k,m_1)}, \quad \text{by part (a)}. \]

Note \(\hat{m} \neq 1\) since \(m_1 \neq m_2\). Also \(\hat{m} \neq n - 1\) since, otherwise, \(d_1m_2 = -1 \Rightarrow (n-d_1)m_2 = 1 \Rightarrow m_2 = n - m_1\) (since the multiplicative inverse of \((n - d_1)\) is \((n - m_1)\), thus \(m_1 + m_2 = n\) which is a contradiction since \(m_1, m_2 < \frac{n}{2}\) by assumption.

Using (20), by the definition of a rotational symbol sequence (11), if \(\hat{S} = L\), then \(\hat{S}_{i+d} = L\) for all but one value of \(i \in \{0, \ldots, n - 1\}\) (where \(d\hat{m} = 1\)). We now show this property of \(\hat{S}\) is not exhibited by \(S\) and hence \(\hat{S}\) and \(S\) cannot be equal.

Using (19) and remembering \(d, i \neq 1, n - 1, \) if \(d \leq i\), then \(\hat{S}_{i+d} = \hat{S}_{i-d+1} = L\) and \(\hat{S}_1 = \hat{S}_{n+1} = R\). Similarly if \(d > i\), we have \(\hat{S}_0 = \hat{S}_1 = L\) and \(\hat{S}_d = \hat{S}_{d+1} = R\). In either case
\[ \hat{S} \neq \hat{S} \quad \text{for any } k \in \mathbb{Z}. \]
\[ \square \]
4. Describing and locating periodic solutions

Each orbit of (2) can be coded by a symbol sequence that gives its itinerary relative to the switching manifold. However, instead of defining symbol sequences for orbits, we find it preferable to do the reverse. Given a point \(x = x_0 \in \mathbb{R}^N\), we denote \(x_i\) as the \(i\)th iterate of \(x\) under the maps \(f_L^\mu\) and \(f_R^\mu\) in the order determined by \(S \in \Sigma_2^n\):

\[
x_{i+1} = f_S^\mu(x_i).
\]

(22)

In general this is different from iterating \(x\) under the map (2). However, if the sequence \(\{x_i\}\) satisfies the admissibility condition:

\[
S_i = \begin{cases} L, & \text{whenever } s_i < 0, \\ R, & \text{whenever } s_i > 0 \end{cases}
\]

(23)

for every \(i\), then \(\{x_i\}\) coincides with the forward orbit of \(x\) under (2). Note if \(s_i = 0\) there is no restriction on \(S_i\). When (23) holds for every \(i\), \(\{x_i\}\) is admissible, otherwise it is virtual.

For a given symbol sequence \(S\), we are interested in finding \(x_0 \in \mathbb{R}^N\) such that \(x_0 = x_n\), because then \(\{x_0, x_1, \ldots, x_{n-1}\}\) is a period-\(n\) cycle. We call this orbit an \(S\)-cycle. \(S\)-cycles are determined by the linear system

\[
x_1 = A_{S_0}x_0 + \mu b,
\]
\[
x_2 = A_{S_1}x_1 + \mu b,
\]
\[
\vdots
\]
\[
x_0 = A_{S_n-1}x_{n-1} + \mu b.
\]

Elimination of the points \(x_1, \ldots, x_{n-1}\), gives

\[
(I - M_S)x_0 = \mu P_S b.
\]

(24)

where

\[
M_S = A_{S_{n-1}} \ldots A_{S_0},
\]

(25)

\[
P_S = I + A_{S_{n-1}} + A_{S_{n-1}}A_{S_{n-2}} + \cdots + A_{S_1} \ldots A_{S_0}.
\]

(26)

We call (24) the \(n\)-cycle solution system of \(S\). If \(I - M_S\) is nonsingular, then (24) has the unique solution

\[
x_0 = \mu(I - M_S)^{-1} P_S b.
\]

(27)

Stability of the period-\(n\) cycle is determined by \(M_S\) and for this reason we call \(M_S\) the stability matrix of \(S\). In view of lemma 5 (see below), we call \(P_S\) the border-collision matrix of \(S\).

Note \(P_S\) is independent of \(S_0\), thus

\[
P_S = P_{S_0}.
\]

(28)

Also, it is easily verified that \(M_S\) and \(M_{S_0}\) differ in only their first column. Consequently, the map that describes the \(n\)th iterate of \(x_0\) under either \(S\) or \(S_S\):

\[
x_n = \begin{cases} \mu P_S b + M_S x, & s \leq 0, \\ \mu P_S b + M_{S_0} x, & s > 0 \end{cases}
\]

(29)

is piecewise-smooth continuous and has the same form as (2). For this reason (2) may be used to investigate dynamical behaviour local to border-collision bifurcations of periodic solutions.

We now state five fundamental lemmas relating to \(n\)-cycle solution systems that we will utilize in sections 5 and 6. Lemmas 5 and 6 are generalizations of those given in [16].
Table 1. The nature of solutions to (24) when $\mu \neq 0$ and $\varrho^T b \neq 0$ as determined by lemmas 5 and 6. $M_S$ is the stability matrix of $S$, (25), and $P_S$ is the border-collision matrix of $S$, (26).

| Condition                  | Nature of Solution |
|----------------------------|--------------------|
| $\det(I - M_S) \neq 0$    | Unique solution and $s_0 \neq 0$ |
| $\det(P_S) \neq 0$        | Unique solution and $s_0 = 0$ |
| $\det(I - M_S) = 0$       | No solution         |
| $\det(P_S) = 0$           | Possibly uncountably many solutions |

Lemma 2. Suppose $x$ is a solution of the $n$-cycle solution system (24) of $S$ and $s_i = e_1^T x_i = 0$. Then $x$ also solves the $n$-cycle solution system of $S^T$.

Proof. By continuity: $A_L x_i = A_R x_i$, hence there is no restriction on the $i$th element of $S$. □

Lemma 3. Suppose $x$ and $\hat{x}$ solve the $n$-cycle solution systems of $S$ and $S^T$ respectively. Then $\det(I - M_S)x = \det(I - M_S^T)\hat{x}$.

Proof. By (24) and (28), we have $(I - M_S)x = (I - M_S^T)\hat{x}$. Since $I - M_S$ and $I - M_S^T$ are identical except in the first column, the first row of their adjugates are identical. Multiplication on the left by this row to the previous equation yields the desired result. □

Lemma 4. For any $i$, $\det(I - M_S(i)) = \det(I - M_S)$.

Proof. Suppose w.l.o.g., $S_0 = L$. If $A_L$ is nonsingular, then $I - M_S(i) = A_L(I - M_S)A_L^{-1}$ which verifies the result for $i = 1$. Since nonsingular matrices are dense in the set of all matrices and the determinant of a matrix is a continuous function of its elements, the result for $i = 1$ is also true even when $A_L$ is singular. Repetition of this argument completes the result for any $i$. □

Lemma 5. Suppose $I - M_S$ is nonsingular, $\mu \neq 0$ and $\varrho^T b \neq 0$. Then the point $x_0$, given by (27), lies on the switching manifold if and only if $P_S$ is singular.

Lemma 6. Suppose $P_S$ is nonsingular, $\mu \neq 0$ and $\varrho^T b \neq 0$. Then the $n$-cycle solution system (24) has a solution if and only if $I - M_S$ is nonsingular.

Proofs for lemmas 5 and 6 are given in appendix A. An interpretation of these last two lemmas is presented in table 1. The situation $\det(I - M_S) = \det(P_S) = 0$ is generically codimension-two. When appropriate nondegeneracy conditions are satisfied it is equivalent to the occurrence of a shrinking point, see section 5.

5. Shrinking points

Roughly speaking, as in [15], we call points where lens-chain shaped resonance tongues have zero width, shrinking points. The aim of this section is to provide a rigorous foundation for the unfolding of shrinking points described in section 6. We define two classes of shrinking points: terminating and non-terminating. This categorization provides a distinction between shrinking points that lie at the end of a lens-chain (terminating) and those that lie in the middle (non-terminating) (recall figure 1). We assume associated symbol sequences are rotational (see section 3); this assumption is crucial to our analysis. Our main results are lemma 7, which states that at a shrinking point there exists an invariant, nonplanar (though planar in special...
cases) polygon and corollary 8, which shows that shrinking points are a hub for the singularity of important matrices.

Near a shrinking point, a lens-shaped resonance tongue corresponding to the existence of an admissible, stable period-$n$ cycle, $\{x_i\}$, has two boundaries. These correspond to border-collision fold bifurcations of $\{x_i\}$ with an unstable orbit of the same period. At each boundary, one point on the orbit lies on the switching manifold. The first question to address is the following: which points lie on the switching manifold at the two resonance tongue boundaries?

Suppose $\{x_i\}$ has an associated symbol sequence that is rotational, $S[l, m, n]$. We may picture the points, $x_i$, lying on a topological circle, as in figure 3. The switching manifold intersects the circle at two points. If this structure is maintained as parameters vary, then it seems reasonable that only points that lie adjacent to an intersection can collide with the switching manifold. When $l \neq 1, n - 1$, there are four such points: $x_0$, $x_{-d}$, $x_{(l-1)d}$ and $x_{ld}$.

(For example, if $[l, m, n] = [3, 2, 7]$, as in figure 3, then $d = 4$ and the four adjacent points are $x_0, x_3, x_1$ and $x_5$.)

Suppose w.l.o.g. that $x_0$ lies on the switching manifold at one resonance tongue boundary. In the interior of the tongue the symbol sequence of the corresponding unstable periodic solution then differs from $S$ in the 0th element, i.e. is $S^\delta$. If we assume that $\{x_i\}$ collides and annihilates with the same unstable periodic solution on the second boundary, it must be $x_{(l-1)d}$ that lies on the switching manifold there because the only index $i$ for which $S^\delta$ is a cyclic permutation of $S^\delta$, is $i = l - 1$. Consequently, in view of lemma 5 since $\{x_i\}$ is not always well defined, we define non-terminating shrinking points by the singularity of $P_S$ and $P_{S[0, l-1, 0]}$ (definition 2). In section 6 we will show that resonance tongue boundaries at which the remaining two points, $x_{-d}$ and $x_{ld}$, lie on the switching manifold, form a second lens-shaped resonance tongue emanating from the shrinking point.

As we will see, the cases $l = 1$ and $l = n - 1$ correspond to terminating shrinking points. It suffices to consider only one of these cases, we choose $l = n - 1$, because they are interchangeable via swapping $L$ and $R$. For the rest of this paper we assume $l \neq 1$ which greatly simplifies our analysis. In particular, we find it is always reasonable to assume that the unstable periodic solution described above always exists (though it may not be admissible). That is $I - M_S$ is nonsingular and therefore the $n$-cycle solution system, (24), of $S^\delta$ has the unique solution

$$p = p_0 = \mu (I - M_S)^{-1} P_S b.$$  \hspace{1cm} (30)

We denote the $i$th iterate of $p$ via the symbol sequence $S^\delta$ by $p_i$ and let

$$t_i = e_1^T p_i.$$  \hspace{1cm} (31)

The orbit, $\{p_i\}$, will play a pivotal role in our analysis. In particular, we will show that at a shrinking point two points of the orbit $\{p_i\}$ lie on the switching manifold (see lemma 7), whereas the orbit $\{x_i\}$ is not well defined (see corollary 8).

Loosely speaking, a shrinking point is a point in parameter space at which a resonance tongue has zero width. In order to analyse these points rigorously we give an alternative, formal, algebraic definition.

**Definition 2 (Non-terminating shrinking point).** Consider the map (2) with $N \geq 2$ and suppose that $\mu \neq 0$ and $0^T b \neq 0$. Let $S = S[l, m, n]$ be a rotational symbol sequence with $1 < l < n - 1$. Suppose

$$P_S$$ and $$P_{S[0, l-1, 0]}$$ are singular \hspace{1cm} (the singularity condition).

Let $S = S^\delta$ and $\bar{S} = S^\delta$ and assume $I - M_S$ and $I - M_{\bar{S}}$ are nonsingular. Suppose the orbit, $\{p_i\}$, of (30), is admissible. Then we say (2) is at a non-terminating shrinking point.
Definition 2 essentially characterizes non-terminating shrinking points as where the border-collision matrices $P_S$ and $P_{S_0(l-1)d}$ are simultaneously singular. Since there are two independent requirements, a non-terminating shrinking point is a codimension-two phenomena. The following definition for terminating shrinking points is quite different because the defining characteristic of these points, in our opinion, is the codimension-two requirement that one fixed point (here $x^{(L)}$ because we are assuming $l = n - 1$) is admissible and has a pair of associated multipliers on the unit circle with a particular rational angular frequency.

**Definition 3 (Terminating shrinking point).** Consider the map (2) with $N \geq 2$, suppose $I - A_L$ is nonsingular and $\mu_\rho^T b/(\det(I - A_L)) < 0$ (i.e. the fixed point $x^{(L)}$ is admissible, see (8)). Let $S = S[l, m, n]$ be a rotational symbol sequence with $l = n - 1$ and $n \geq 3$. Let $\check{S} = S^\check{S}$ and suppose $I - M_{\check{S}}$ is nonsingular. Suppose $e^\pm 2\pi i m/n$ are multipliers of $A_L$ (the singularity condition).

Then we say (2) is at a terminating shrinking point.

In two dimensions, terminating shrinking points are centre bifurcations for rational rotation numbers, $m/n$, studied in [23, 24]. Here it is shown that there exists an invariant polygon that has one side on the switching manifold, within which all points other than the fixed point belong to periodic orbits with rotation number, $m/n$. As we will show, in higher dimensions this behaviour occurs on the centre manifold of $x^{(L)}$ corresponding to the multipliers $e^\pm 2\pi i m/n$, call it $E_c$.

The following lemma concerns the orbit, $\{p_i\}$, of (30), and describes dynamical behaviour at shrinking points. The lemma is used to prove the main result, theorem 9. The reader should take care to note that if $P_{S_0(l-1)d}$ is singular then by lemma 5 $p_{td}$ (not $p_{(l-1)d}$) lies on the switching manifold because

$$P_{S_0(l-1)d} = P_{S_0l0} = P_{S_0(l-1)d0} = P_{S_0(l-1)d},$$

where the second equality is lemma 1(e) and (28) is used for the last equality.

**Lemma 7.** Suppose (2) is at a shrinking point. Then,

(a) $t = t_{ld} = 0$;

(b) if the shrinking point is non-terminating, then $t_d, t_{(l-1)d} < 0$ and $t_{(l+1)d}, t_{-d} > 0$ as in figure 4; if the shrinking point is terminating, then $t_{ld} < 0$ for all $i \neq 0, -1$.
Figure 5. The invariant, nonplanar polygon, $\mathcal{P}$, at a non-terminating shrinking point corresponding to the rotational symbol sequence $S[2, 2, 5] = LRRLR$ for the map (2) when $A_L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$, $A_R = \begin{bmatrix} -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$, $b = e_1$ and $\mu = 1$. Here $p_0 = (0, -\frac{1}{2})^T$. The switching manifold, $s = 0$, is shaded grey.

(c) $\{p_i\}$ has period $n$;
(d) the iterates of $p$ are the vertices of an invariant, nonplanar $n$-gon, $\mathcal{P}$, that is a collection of uncountably many $S$-cycles and the restriction of (2) to $\mathcal{P}$ is homeomorphic to rigid rotation with rotation number, $m/n$.

Separate proofs for non-terminating and terminating shrinking points are given in appendix B. If the shrinking point is terminating, then the polygon $\mathcal{P}$ is planar. In general, when $N > 2$, $\mathcal{P}$ is nonplanar as in figure 5. In any case $\mathcal{P}$ has two vertices on the switching manifold and may be constructed by connecting each $p_i$ to $p_{i+1}$ with a line segment.

$\mathcal{P}$ is comprised of many $S$-cycles, in other words the $n$-cycle solution system of $S$ has more than one distinct solution. Therefore at a shrinking point the matrix $I - M_S$ is singular. Furthermore, for each $i$, by lemma 4, $I - M_{S^{(i)}}$ is singular. However, there exist solutions to the $n$-cycle solution system of $S^{(i)}$ (such as $p_i$), so to avoid a contradiction with lemma 6, $P_{S^{(i)}}$ must be singular. Consequently we have the following:

**Corollary 8.** Suppose (2) is at a shrinking point. Then,
(a) $(I - M_S)$ is singular,
(b) $P_{S^{(i)}}$ is singular, for all $i$.

The singularity of a matrix is a codimension-one phenomenon. Thus by corollary 8, we expect there will be curves in two-dimensional parameter space passing through shrinking points along which $I - M_S$ and each $P_{S^{(i)}}$ is singular. Some of these curves form resonance tongue boundaries, and it is these that form the focus of the following section.

6. Unfolding shrinking points

This section presents an unfolding of the dynamics near terminating and non-terminating shrinking points. We begin by assuming (2) is parameter dependent and perform a coordinate
transformation such that, locally, in two dimensions, two tongue boundaries lie on the positive coordinate axes.

Suppose \((2)\) varies with a set of parameters \(\xi \in \mathbb{R}^K \) \((K \geq 2)\) and is \(C^{k+1} \) \((k \geq 2)\). Then \(A_L\) and \(A_R\) are \(C^k\) functions of \(\xi\). Suppose \((2)\) is at a shrinking point when \(\xi = 0\), for some fixed \(\mu \neq 0\). Then, \(\det(P_S(\xi))\) and \(\det(P_{S(i\rightarrow l)}(\xi))\) are \(C^k\) scalar functions that have the value zero when \(\xi = 0\). If the \(2 \times 2\) Jacobian \((\partial(\det(P_S), \det(P_{S(i\rightarrow l)}))/\partial(\xi_i, \xi_j)\) is nonsingular for some \(i\)th and \(j\)th components of \(\xi\), we may utilize the implicit function theorem to obtain a coordinate transformation of \(\xi\), such that \(\det(P_S) = 0\) when \(\eta = \xi = 0\) and \(\det(P_{S(i\rightarrow l)}) = 0\) when \(\nu = \xi_j = 0\). In what follows below we omit dependence of \((2)\) on the remaining \((K - 2)\) components of \(\xi\).

Let \(\tilde{x}(\eta, \nu)\) denote the unique solution to the \(n\)-cycle solution system, \((24)\), of \(\tilde{S} = S^\eta\) for small values of \(\eta\) and \(\nu\). Let \(p = \tilde{x}(0, 0)\) (to coincide with \((30)\)). If the shrinking point is non-terminating, let \(\tilde{x}(\eta, \nu)\) denote the unique solution to the \(n\)-cycle solution system of \(\tilde{S} = S^\eta\) for small values of \(\eta\) and \(\nu\). Let \(\tilde{x}_i\) and \(\tilde{x}_s\) denote the first components of \(\tilde{x}_i\) and \(\tilde{x}_s\), respectively (then \(\tilde{x}_i(0, 0) = t_i\)). From the above assumptions on \(\eta\) and \(\nu\), by lemma 5 we have \(\tilde{x}(0, 0) = 0\) and \(\tilde{x}_d(\eta, 0) = 0\), (see \((32)\)). If \((\partial/\partial \eta)\tilde{x}(0, 0)\) and \((\partial/\partial \nu)\tilde{x}_d(0, 0)\) are both nonzero, we may redefine \(\eta\) and \(\nu\) via a nonlinear scaling so that

\[
\tilde{s}(\eta, \nu) = \eta(1 + O(1)),
\]

\[
\tilde{x}_d(\eta, \nu) = \nu(1 + O(1)).
\]

Note, we use \(O(k)\) (and \(o(k)\)) to denote terms that are order \(k\) or larger (larger than order \(k\)) in all variables and parameters.

**Theorem 9.** Suppose the map \((2)\) is a \(C^{k+1}\) \((k \geq 2)\) function of parameters \(\eta\) and \(\nu\), has a shrinking point when \(\eta = \nu = 0\) and we have \((33)\) and \((34)\). Write

\[
\det(I - M_S(\eta, \nu)) = k_1 \eta + k_2 \nu + O(2),
\]

and assume \(k_1, k_2 \neq 0\). As usual we assume \(N \geq 2\).

Then for small \(\eta\) and \(\nu\), the \(S\) and \(\tilde{S}\)-cycles are admissible in the first quadrant of \((\eta, \nu)\)-space (which we will call \(\Psi_1\)) and collide in border-collision fold bifurcations at the boundaries. If the shrinking point is non-terminating, the \(S\) and \(\tilde{S}\)-cycles are admissible in a subset, \(\Psi_2\), of the third quadrant of \((\eta, \nu)\)-space and collide in border-collision fold bifurcations at the boundaries of \(\Psi_2\). Furthermore, the boundaries of \(\Psi_2\) are \(C^k\), intersect at the origin tangent to the coordinate axes, as in figure 6.

Near non-terminating shrinking points, theorem 9 predicts the bifurcation set sketched in figure 6. The four resonance tongue boundaries that emanate from the shrinking point correspond to the collision of the points of the \(S\)-cycle: \(x_0, x_{-d}, x_{l_{-1}d}, x_{ld}\), with the switching manifold. As mentioned in section 5 these are the four points adjacent to the switching manifold in the interior of the tongue. By applying the implicit function theorem to \((35)\) there exists a \(C^k\) function

\[
h(\eta) = -\frac{k_1}{k_2} \eta + O(\eta^2),
\]

such that \(\det(I - M_S(\eta, h(\eta))) = 0\) (labelled \(\Gamma\) in figure 6, as in figure 2). But by theorem 9, \(S\)-cycles exist throughout the first quadrant of parameter space; therefore, the curve \(\nu = h(\eta)\)
Figure 6. A schematic of the generic bifurcation set in a neighbourhood of a non-terminating shrinking point.

does not enter this quadrant. Hence, since $h'(0) = -k_1/k_2$,

$$k_1 k_2 < 0,$$

as in figure 6.

By corollary 8, the border-collision matrix of every cyclic permutation of $S$ is singular at a shrinking point. Theorem 9 describes curves along which some of these matrices remain singular. We now describe all other such curves. If $p_0$ and $p_{ld}$ are the only points of the period-$n$ cycle, $\{p_i\}$, that lie on the switching manifold (as is generically the case), then for each $i \neq 0, l - 1, l, -1$ if the shrinking point is non-terminating and for each $i \neq 0, -2, -1$ if the shrinking point is terminating, there exists a $C^k$ function

$$q_i(\eta) = -\frac{k_1 t(i+1) d}{k_2 t_{ld}} \eta + O(\eta^2),$$

(38)

that satisfies $\det(P_{S(id)}(\eta, q_i(\eta))) = 0$. In each case $t(i+1) d$ and $t_{ld}$ have the same sign, thus by (37) we have $q_i'(0) < 0$. Although the border-collision matrix, $P_{S(id)}$, is singular along $v = q_i(\eta)$, these curves are not seen in resonance tongue diagrams because the related periodic solutions are virtual.

So far we have described curves passing through the shrinking point along which $P_{S(id)}$ is singular for each $i$, except for $i = -1$ when the shrinking point is terminating. Here $S^{(-d)} = RL^{n-1}$, hence $P_{S^{(-d)}} = I + A_1 + \cdots + A_{n-1}^{-1} = (I - A_1)^{-1}(I - A_n)$. The matrix $A_n$ has a multiplier 1 with an algebraic multiplicity of at least two, therefore $\det(P_{S^{(-d)}}(\eta, \nu)) = O(2)$. Thus in this case, for small $\eta$ and $\nu$, generically $P_{S^{(-d)}}$ is singular only when $\eta = \nu = 0$.

In addition, for terminating shrinking points there generically exists a curve in parameter space passing through the shrinking point along which the matrix, $A_1$, has a pair of complex multipliers on the unit circle. As one moves along the curve, the angular frequency of multipliers changes. Whenever the angular frequency is rational there may be additional terminating shrinking points. Such curves are described in [16, 19, 23, 24].

The border-collision bifurcations occurring on resonance tongue boundaries are described by piecewise-affine maps of the form (29). Theorem 9 states that these bifurcations are nonsmooth folds, thus we may use Feigen’s results [20], summarized in section 2, to determine the relative stability of periodic solutions near shrinking points. Let $a_i[S]$ denote the number of real multipliers of $M_S$ that are greater than one in the interior of $\Psi_i$. Then $a_1[S] + a_1[\hat{S}]$
and \(a_2[S] + a_2[\hat{S}]\) are odd. Also, det\((I - M_S(\eta, v)) = O(1), (35)\), thus \(|a_1[S] - a_2[S]| = 1\). Consequently, if \(S\)-cycles are stable in the interior of \(\Psi_1\), then \(\hat{S}\)-cycles are unstable and \(S\)-cycles are unstable in the interior of \(\Psi_2\).

Finally we turn to the proof of theorem 9.

**Proof of theorem 9.** Write

\[
\det(I - M_{\hat{S}}(\eta, v)) = k_3 + O(1),
\]

(39)

\[
\det(I - M_{\hat{S}}(\eta, v)) = k_4 + O(1),
\]

(40)

where \(k_3, k_4 \neq 0\) by assumption. Let

\[
x(\eta, v) = \mu(I - M_S(\eta, v))^{-1} P_S(\eta, v)b,
\]

(41)

whenever \(v \neq h(\eta)\), (36). For small \(v \neq 0\), if \(\eta = 0\) then \(x\) is defined and coincides with \(\hat{x}\) because \(\det(P_S) = 0\). Therefore \(x_{id}(0, v) = \hat{x}_{id}(0, v)\), for all \(i\). In particular,

\[
s_{id}(0, v) = t_{id} + O(v),
\]

(42)

so that by (33) and (34)

\[
s(0, v) = 0,
\]

\[
s_{id}(0, v) = v + O(v^2).
\]

(43)

Similarly for small \(\eta \neq 0\), if \(v = 0\) then \(x\) is defined and \(\det(P_{S(l-1)d}) = 0\), thus \(x\) solves the \(n\)-cycle solution system of \(S^{(l-1)d} = \hat{S}^{(l)d}\). Therefore \(x_{id}(\eta, 0) = \hat{x}_{id}(\eta, 0)\), for all \(i\). In particular,

\[
s_{id}(\eta, 0) = t_{(i+1)d} + O(\eta),
\]

(44)

so that

\[
s_{-d}(\eta, 0) = \eta + O(\eta^2),
\]

\[
s_{l-1,d}(\eta, 0) = 0.
\]

We now compute lowest order terms in the Taylor series of \(\hat{s}_{id}(\eta, v)\). An application of lemma 3 to the symbol sequence \(S^{(l)d}\), using also lemma 4, produces

\[
\hat{s}_{id}(\eta, 0) = k_2 t_{(i+1)d} + O(\eta^2),
\]

(45)

\[
\hat{s}_{id}(\eta, 0) = k_1 t_{id} + O(v^2) + O(\eta^2).
\]

Using (35), (40), (43) and (44) we obtain

\[
\hat{s}_{id}(0, v) = \frac{k_2}{k_4} v^2 + O(v^3),
\]

\[
\hat{s}_{id}(\eta, 0) = \frac{k_1 t_{id}}{k_4} \eta + O(\eta^2).
\]

By the implicit function theorem, there exists a unique \(C^k\) function \(g_1 : \mathbb{R} \to \mathbb{R}\) such that for small \(v\), \(\hat{s}_{id}(g_1(v), v) = 0\) and

\[
g_1(v) = -\frac{k_2}{k_1 t_{id+1}} v^2 + O(v^3).
\]

(46)

In a similar fashion, by applying lemma 3 to the symbol sequence \(S^{(-d)}\) and noting \(S^{(-d)\hat{P}} = \hat{S}\), we obtain

\[
\hat{s}(0, v) = \frac{k_2 t_{-d}}{k_4} v^2 + O(v^3),
\]

(47)

\[
\hat{s}(\eta, 0) = \frac{k_1}{k_4} \eta^2 + O(\eta^3),
\]
so that the implicit function theorem gives the function
\[ g_2(\eta) = -\frac{k_1}{k_{2t-d}}\eta^2 + O(\eta^3), \] (48)
which satisfies \( \hat{s}(\eta, g_2(\eta)) = 0. \)

We now prove \( k_1k_2 > 0. \) Let \( \Sigma_i \subset \mathbb{R}^2 \) denote the intersection of the interior of the \( i \)-th quadrant with a sufficiently small neighbourhood of the origin \((i = 1, \ldots, 4)\). Suppose for a contradiction, \( k_1k_2 < 0. \) Then, by (35), \( I - M_S \) is nonsingular throughout \( \Sigma_2 \) and hence \( s \) is continuous throughout \( \Sigma_2. \) Note \( s = 0 \) only when \( \eta = 0, \) thus the sign of \( s \) is constant in \( \Sigma_2. \) When \( v = 0, \) by admissibility and using (44) we have \( s(\eta, 0) = t_d + O(1) < 0, \) thus \( s \) is negative throughout \( \Sigma_2. \) In particular, \( s \) is negative when \( v = k_1\eta/k_2 \) for small \( \eta < 0. \) By applying lemma 3 to \( S, \) we obtain
\[
\det(I - M_S(\eta, v))s(\eta, v) = \det(I - M_S(\eta, v))\hat{s}(\eta, v) = 0. \quad (49)
\]
Substituting \( v = k_1\eta/k_2 \) and using (33), (35) and (39) gives
\[
(2k_1 + O(\eta^2))s\left(\frac{\eta}{k_2}\right) = (k_3 + O(\eta))s(\eta, v) = k_3 + O(\eta),
\]
\[
\Rightarrow k_1k_3 < 0. \quad (50)
\]
Via a similar argument with \( s_{(l-1)d}, \) we find \( k_2k_3 < 0. \) This inequality provides a contradiction with (50), thus \( k_1k_2 > 0. \) Hence \( s \) and \( s_{(l-1)d} \) are in fact continuous and negative throughout \( \Sigma_1, \) \( \Sigma_2 \), and \( (1, k_3) < 0. \) Via similar arguments we find \( k_1k_4 > 0. \) Consequently we have
\[
\sgn(k_1) = \sgn(k_2) = -\sgn(k_3) = \sgn(k_4) \quad (51)
\]
Furthermore, from (46) and (48) we have
\[
g''_1(0), g''_2(0) < 0, \quad (52)
\]
since \( t_{(l+1)d}, t_{-d} > 0. \)

The regions \( \Psi_1 \) and \( \Psi_2 \) are then
\[
\Psi_1 = \{(\eta, v) \mid \eta, v \geq 0\},
\]
\[
\Psi_2 = \{(\eta, v) \mid \eta \leq g_1(v), v \leq g_2(\eta)\}.
\]
Indeed, near \( (\eta, v) = (0, 0), \) since \( \{p_i\} \) is admissible, \( \hat{S}-\)cycles are admissible if and only if \( \hat{s}, \hat{s}_{td} \geq 0, \) thus only in \( \Psi_1 \) by (33) and (34). Similarly \( \hat{S}-\)cycles are admissible only if \( \hat{s}, \hat{s}_{td} \leq 0. \) By (47), \( (\partial \hat{s}/\partial v)(0, 0) = (k_{2t-d}/k_4) \) is positive by (51) and admissibility. Thus \( \hat{s}(\eta, v) \leq 0 \) when \( v \leq h_2(\eta). \) Similarly, by (45), \( \hat{s}_{td}(\eta, v) \leq 0 \) when \( \eta \leq h_1(v). \) Therefore \( \hat{S}-\)cycles are admissible in \( \Psi_2. \) The result for \( S-\)cycles follows by looking at the signs of each \( s_i \) on the boundaries of \( \Psi_1 \) and \( \Psi_2. \)

7. Discussion

Dynamics local to a nondegenerate border-collision bifurcation of a fixed point of a piecewise-smooth, continuous map are described by a piecewise-affine approximation of the form (2). The border-collision bifurcation occurs when the parameter, \( \mu, \) is zero. As \( \mu \) is varied from zero, invariant sets such as periodic solutions and topological circles may emanate from the
bifurcation. The focus of this paper has been to fix $\mu$ at some nonzero value, assume (2) varies with other independent parameters, and investigate the nature of regions in parameter space where periodic solutions exist. In two-dimensional parameter space these resonance tongues commonly display a lens-chain structure. In this paper we have given a rigorous unfolding about points for which resonance tongues have zero width-shrinking points. Since the general map (1) is conjugate to the form (2), the unfolding is valid when there are two generic parameters that can be made to satisfy (33) and (34).

Curves corresponding to four different border-collision fold bifurcations form resonance tongue boundaries near non-terminating shrinking points. If $\{x_i\}$ denotes the corresponding $S$-cycle, each boundary corresponds to the collision of one of the points $x_0, x_{-d}, x_{(-1)d}$ and $x_{ld}$ with the switching manifold. By lemma 5, at each boundary the corresponding border-collision matrix (one of $P_S, P_{S(d)}, P_{S(1-d)}$ and $P_{S(ld)}$) is singular.

At a shrinking point there exists an invariant polygon, $\mathcal{P}$. If $N = 2$, or the shrinking point is terminating, $\mathcal{P}$ is planar, but in general $\mathcal{P}$ is nonplanar. $\mathcal{P}$ comprises uncountably many periodic solutions that have a given rotational symbol sequence, $S$. Under variation of parameters, $\mathcal{P}$ may persist as an invariant topological circle. In this case there exists curves along which one intersection point of the circle with the switching manifold maps to the other intersection point in a fixed, finite number of map iterations. We hypothesize that shrinking points generically occur densely along such curves, see [16].

The two coexisting periodic solutions for parameter values in a lens-shaped resonance region have associated rotational symbol sequences with identical values for $m$ and $n$ but with values of $l$ that differ by one. By adding or subtracting one from the value of $l$ of these symbol sequences, one is able to obtain the symbol sequences of the periodic solutions in an adjoining lens. In short, throughout a lens-chain, the rotational number $m/n$ is constant and $l$ differs by one between lenses. Numerically we have observed that as $n \to \infty$, the overall width of the lens-chain tends to zero and the number of individual lenses increases. We speculate that in the limit, a resonance tongue for a quasiperiodic solution of fixed frequency is a curve along which the fraction of points of the solution that lie in each half plane changes continuously.

Shrinking points require solutions to intersect the switching manifold at two distinct points. For this reason shrinking points do not occur for (2) in one dimension because here the switching manifold is a single point. The one-dimensional, piecewise-linear, circle map studied in [15] is able to exhibit lens-chain structures because it has two switching manifolds.

Resonance tongues corresponding to periodic solutions with non-rotational symbol sequences need not exhibit a lens-chain structure [16]. But we believe that the collection of all symbol sequences that may generically exhibit a lens-chain structure is some class of which rotational symbol sequences are only the simplest type. For instance a period-$n$ cycle of (2) may undergo a Neimark–Sacker-like bifurcation when a complex conjugate pair of associated multipliers crosses the unit circle. If this crossing occurs at $\lambda = e^{\frac{2\pi i p}{q}}$, where $p, q \in \mathbb{Z}$ are coprime, then a period-$nq$ orbit may arise in a manner akin to a terminating shrinking point. Numerically we have observed that such an orbit may exhibit a lens-chain shaped resonance tongue, even though its corresponding symbol sequence is non-rotational. The tongue boundaries are seen to intersect transversely as in the usual case. This scenario corresponds to an $n$th iterate map of the same form as (2) exhibiting lens-chains.

The analysis presented in this paper applies to periodic solutions near the border-collision bifurcation of a fixed point. In general a periodic solution may result from global dynamics and enter regions of phase space bounded by many distinct switching manifolds. A study of the codimension-two points resulting from such a periodic solution simultaneously colliding with different switching manifolds is left for future investigations.
Proof of lemma 5. We first show that $I - A_L$ and $I - A_R$ cannot both be singular. Suppose otherwise. Then $\mathbf{q}^T (I - A_L) = \det(I - A_L) e_1^T = 0$ and $\mathbf{q}^T (I - A_R) = \det(I - A_R) e_1^T = 0$. Thus $\mathbf{q}^T A_L = \mathbf{q}^T A_R = \mathbf{q} ^ T$. Therefore $\mathbf{q}^T M_S = \mathbf{q}^T$, i.e. $M_S$ has an eigenvalue, $\lambda = 1$, which contradicts the assumption that $I - M_S$ is nonsingular. Thus at least one of $I - A_L$ and $I - A_R$ is nonsingular. Suppose w.l.o.g. that $I - A_L$ is nonsingular.

For any $\mathbf{b} \in \mathbb{R}^N$, let $k = \mathbf{q}^T b / \mathbf{q}^T b$, let $c = b - kb$ and let $y = \mu (I - A_L)^{-1} c$. Note $\mathbf{q}^T c = 0$, thus the first component of $y$ is zero, hence $y = \mu c + A_L y = \mu c + A_R y$. Thus $y = \mu P_S c + M_S y$ and therefore $y = \mu (I - M_S)^{-1} P_S c$.

Let $\hat{x}_0 = k x_0 + y$. Multiplication on the left by $e_1^T$ yields $\hat{x}_0 = k x_0$. Using above results we have $\hat{x}_0 = \mu (I - M_S)^{-1} P_S b$. Thus if $\hat{b} \in \text{null}(P_S)$, then $\hat{x}_0 = 0$. Alternatively if $\hat{b}$ is chosen such that $\mathbf{q}^T \hat{b} = 0$, then $\hat{b} = c$ and therefore $\hat{x}_0 = y$. Hence if $\hat{b} \in \text{null}(P_S)$ and $\mathbf{q}^T \hat{b} = 0$, then $y = 0$ and therefore $c = \hat{b} = 0$.

Now suppose $P_S$ is singular. Then there exists $\hat{b} \in \text{null}(P_S)$ with $\hat{b} \neq 0$ and by the previous result, $\mathbf{q}^T \hat{b} \neq 0$. Thus we have $\hat{x}_0 = 0$ and $k \neq 0$, therefore $s_0 = 0$, i.e. $x_0$ lies on the switching manifold.

Conversely suppose $s_0 = 0$. Then $\hat{x}_0 = \mu e_1^T (I - M_S)^{-1} P_S \hat{b} = 0$ for any $\hat{b} \in \mathbb{R}^N$. Hence $P_S$ is singular.

Proof of lemma 6. Clearly if $I - M_S$ is nonsingular, (24) has the unique solution (27). To prove the converse, suppose now that $I - M_S$ is singular. We will show that (24) has no solution. Let $K = \text{Span}(\mathbf{q})$ (the collection of all vectors orthogonal to $\mathbf{q}$). Choose any $c \in K$. We will show that the linear system, $(I - A_L)x = \mu c$, always has a solution, $x$, with $s = e_1^T x = 0$. This is clear if $I - A_L$ is nonsingular for then $x = \mu (I - A_L)/c$ and $s = (\mu \mathbf{q}^T c / \det(I - A_L)) = 0$.

Instead, suppose $I - A_L$ is singular. We now characterize the nullspace of $I - A_L$.

We first show that if $v_0 \in \text{null}(I - A_L)$ and $v_0 \neq 0$, then $e_1^T v_0 \neq 0$. Suppose for a contradiction $e_1^T v_0 = 0$. Denote the $(N - 1) \times (N - 1)$ matrix formed by removing the $i$th row and $j$th column from $I - A_L$. Then $\bar{v} \in \mathbb{R}^{N-1}$ denote the last $N - 1$ elements of $v_0$. Then $\bar{v} \neq 0$ and $B_{ij} \bar{v} = 0$ for each $i$, thus $\det(B_{ij}) = 0$ for each $i$. In other words the first column of the cofactor matrix of $I - A_L$ is zero, equivalently $\mathbf{q}^T = 0$, which contradicts the assumption: $\mathbf{q}^T b \neq 0$. Hence $e_1^T v_0 \neq 0$.

Let $u, v \in \text{null}(I - A_L)$. Then the linear combination, $ue_1^T v - ve_1^T u$, is an element of the null space of $(I - A_L)$ and the first element of this vector is zero. By the previous argument this vector must be the zero vector, thus $u$ and $v$ are linearly dependent. Hence null$(I - A_L)$ is one-dimensional, therefore null$(I - A_L) = \text{Span}(w_0)$.

We have $\mathbf{q}^T (I - A_L) = \det(I - A_L) e_1^T = 0$, thus null$(I - A_L)^T = \text{Span}(\mathbf{q}) = K^T$. By the fundamental theorem of linear algebra, range$(I - A_L) = \text{null}((I - A_L)^T)^\perp = K$. Note $\mu c \in K$, therefore there exists $w \in \mathbb{R}^N$ such that $(I - A_L)w = \mu c$. Let $x = w - (e_1^T w / e_1^T v_0)v_0$. Then $(I - A_L)x = \mu c$ and $s = 0$. Summarizing, regardless of whether $I - A_L$ is singular or nonsingular, we have found a vector, $x$, with zero first component, such that $(I - A_L)x = \mu c$.

The vector $x$ satisfies $x = \mu c + A_L x = \mu c + A_R x$, therefore $x = \mu P_S c + M_S x$ and hence $\mu P_S c \in \text{range}(I - M_S)$. But $c$ is arbitrary, hence $P_S K \subset \text{range}(I - M_S)$. $K$ is $(N - 1)$-dimensional thus $P_S K = \text{range}(I - M_S)$. By assumption, $b \neq K$, thus (24) has no solution.

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Appendix B. Proof of lemma 7

Proof. We begin with the case of non-terminating shrinking points.

(a) First, \( P_{t} = P_{S} \) is singular by assumption, thus by lemma 5, \( t = 0 \). Second, \( P_{\bar{S}(t)} = P_{\bar{S}(0)} = P_{\bar{S}(t-1)}, \) is singular by assumption, where we have utilized lemma 1(e) for the second equality and (28) for the last equality. Thus, by lemma 5, \( t_{ld} = 0 \).

(b) Suppose for a contradiction, \( t_{d} = 0 \). Since also \( t_{ld} = 0 \) from (a), by a double application of lemma 2, \( p \) solves the \( n \)-cycle solution system of \( \bar{S}(t) \). From the definition, (11), it is seen that \( \bar{S}(t) = \bar{S}(\cdot, t) \). Therefore, \( p_{d} \) solves the \( n \)-cycle solution system of \( S \), hence \( p = p_{d} \). The point \( p \) is admissible by assumption, therefore \( p = p_{kd} \) for any integer \( k \). Putting \( k = m \) we obtain \( p = p_{1} \), hence \( p \) is a fixed point of (2) and lies on the switching manifold.

However (2) cannot have a fixed point on the switching manifold because by assumption \( \mu \neq 0 \) and \( \theta^{T} h \neq 0 \), so we have a contradiction. Thus \( t_{d} \neq 0 \). The remaining three points may be proven nonzero in a similar fashion. The given signs follow immediately from admissibility.

(c) Suppose for a contradiction, \( \{ p_{i} \} \) is of period \( n \). Clearly \( n \) divides \( n \). Then \( p_{n} \) lies left of \( S \). Thus \( S_{n} = S \). Hence in view of (11), we must have \( \mu \) divides \( n \), so we have a contradiction.

(d) From part (a), \( t = 0 \), thus by lemma 2, \( p \) solves the \( n \)-cycle solution system of \( S(t) = S \). Similarly, since \( t_{ld} = 0 \), \( p \) solves the \( n \)-cycle solution system of \( S(t) = S(t-d) \) by (11) and therefore \( p_{d} \) solves the \( n \)-cycle solution system of \( S \). Let \( w(\tau) = \tau p + (1 - \tau) p_{d} \). Since \( \{ p_{i} \} \) is assumed to be admissible, \( \{ w(\tau) \} \) is an admissible \( S \)-cycle whenever \( 0 < \tau < 1 \).

From part (c), the \( n \) points \( \{ p_{i} \} \) are distinct. We now show that the union of all the \( \{ w(\tau) \} \), call it \( \mathcal{P} \), has no self-intersections. Suppose for a contradiction that \( w_{i}(t_{1}) = w_{j}(t_{2}) \) for some \( i \neq j \) and \( 0 < t_{1}, t_{2} < 1 \). Let \( x = w_{0}(t_{1}) \), then \( \{ x_{i} \} \) is an admissible \( \mathcal{S} \)-cycle and an admissible \( \mathcal{S}(l_{i-1}) \)-cycle. Since \( t_{1} \neq 0, 1 \), by part (b), \( x_{0} < 0 \) and \( x_{1} > 0 \). Therefore \( S_{i} = S_{-i-l} = S \). By lemma 1(b), \( t_{1} \) is a contradiction. Hence \( \mathcal{P} \) has no self-intersections and is therefore an invariant, nonplanar \( n \)-gon.

To relate (2) to a map on the unit circle we describe a bijection between \( S \) and \( \mathcal{P} \). The angular coordinate, \( \theta \in [0, 2\pi] \), uniquely describes a point on \( S \). Let \( j(\theta) = \frac{n\theta}{2\pi} \) and \( \tau(\theta) = n\theta/2\pi - j(\theta) \). Let \( z : S \to \mathcal{P} \) be defined by \( z(\theta) = w_{j(\theta)}(\tau(\theta)) \). It is easily verified the function \( z \) is a bijection. The induced map on \( S \) is \( \theta = \theta + (2\pi m/n, i.e. rigid rotation with rotation number \( m/n \). \)

Proof. We now consider the case of terminating shrinking points.

Let \( v = y + iz \) be an eigenvector corresponding to the multiplier \( \lambda = e^{\frac{2\pi i m}{n}} \) for the matrix \( A_{L} \). Let

\[
D = \begin{bmatrix}
\cos \left( \frac{2\pi m}{n} \right) & \sin \left( \frac{2\pi m}{n} \right) \\
-\sin \left( \frac{2\pi m}{n} \right) & \cos \left( \frac{2\pi m}{n} \right)
\end{bmatrix}.
\]
Then
\[
D^j = \begin{bmatrix}
\cos \left( \frac{2\pi jm}{n} \right) & \sin \left( \frac{2\pi jm}{n} \right) \\
-\sin \left( \frac{2\pi jm}{n} \right) & \cos \left( \frac{2\pi jm}{n} \right)
\end{bmatrix}
\]
and \( A'_j \{ y \} = [y \{ y \}] D^j \). Note also \( D^n = I \). The first component of \( v \) must be nonzero because otherwise \( A_R \{ y \} = [y \{ y \}] D \) and hence \( M_3 \{ y \} = [y \{ y \}] D^\prime = [y \{ y \}] \), violating the assumption that \( I - M_3 \) is nonsingular. Thus \( E_c \) intersects the switching manifold. Furthermore, \( E_c \) is two-dimensional, because otherwise there exists another eigenvector for \( \lambda = e^{i \frac{2\pi}{n}} \), call it \( \hat{v} \), linearly independent to \( v \), and the linear combination, \( \nu \hat{v} - \hat{v} v^T \), is an eigenvector with zero first component contradicting the previous argument.

Let \( x = x_0 = \alpha y + \beta z + x^{\text{SI}} \) for some as yet undetermined scalars \( \alpha \) and \( \beta \). Let \( x_i \) denote the ith iterate of \( x \) via the symbol sequence \( S = L^k \). Then \( x_i = [y \{ y \}] D^i \{ \alpha \beta \} + x^{\text{SI}} \), and hence \( (x_i) \) is a period-n cycle. We wish to choose \( \alpha \) and \( \beta \) such that \( s_0 = e^T_1 [y \{ y \}] [\alpha \beta] + x^{\text{SI}} = 0 \) and \( s_{-d} = e^T_1 [y \{ y \}] D^{-d} [\alpha \beta] + x^{\text{SI}} = 0 \). Combining these two equations yields the linear system, \( X[\alpha \beta]^T = -s^{\text{SI}} X^{-1} [1 1]^T \) where
\[
X = \begin{bmatrix}
e^T_1 y & e^T_1 z \\
e^T_1 y \sin \left( \frac{2\pi}{n} \right) & -e^T_1 y \sin \left( \frac{2\pi}{n} \right) + e^T_1 z \cos \left( \frac{2\pi}{n} \right)
\end{bmatrix}
\]
has determinant, \( \det(X) = -(e^T_1 y)^2 + (e^T_1 z)^2 \sin (2\pi/n) \neq 0 \). Thus we let \( [\alpha \beta]^T = -s^{\text{SI}} X^{-1} [1 1]^T \). It follows that
\[
s_{i,d} = s^{\text{SI}} \left( 1 - \frac{\cos \left( \frac{2\pi (i + \frac{1}{2})}{n} \right)}{\cos \left( \frac{\pi}{n} \right)} \right) . \quad \text{(B1)}
\]

Thus the period-\( n \) cycle, \( \{ x_i \} \), is admissible and solves the \( n \)-cycle solution system of \( S \). Therefore \( x = p \). Thus \( \{ p_i \} \) has period \( n \) which verifies part (c) of the lemma. Also \( t_i = s_i \), for each \( i \), thus by (B1) we have parts (a) and (b). To verify part (d), note that as in the non-terminating case, \( p \) and \( p_d \) are admissible solutions to the \( n \)-cycle solution system of \( S \).

Thus \( w(\tau) = \tau p + (1 - \tau) p_d \) is an admissible solution to the \( n \)-cycle solution system of \( S \) for all \( \tau \in [0, 1] \). The union of all such cycles is an invariant, \( P \), which lies on \( E_c \), so is planar. The points, \( p_i \), lie on an ellipse and ordered so that \( P \) has no self-intersections and is an \( n \)-gon. As before, a bijection between \( S^1 \) and \( P \) may be constructed to show that the restriction of (2) to \( P \) is homeomorphic to rigid rotation with rotation number \( m/n \).

References

[1] Puu T and Sushko I (ed) 2006 Business Cycle Dynamics: Models and Tools (New York: Springer)
[2] Laugesen J and Mosekilde E 2006 Border-collision bifurcations in a dynamic management game Comput. Oper. Res. 33 464–78
[3] Zhoushaliev Z T and Mosekilde E 2003 Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems (Singapore: World Scientific)
[4] Banerjee S and Verghese G C (ed) 2001 Nonlinear Phenomena in Power Electronics (New York: IEEE Press)
[5] Tse C K 2003 Complex Behavior of Switching Power Converters (Boca Raton, FL: CRC Press)
[6] Chang H and Juang J 2004 Piecewise two-dimensional maps and applications to cellular neural networks *Int. J. Bifurcation Chaos* **14** 2223–8

[7] di Bernardo M, Kowalczyk P and Nordmark A 2002 Bifurcations of dynamical systems with sliding: derivation of normal-form mappings *Physica D* **170** 175–205

[8] Kowalczyk P 2005 Robust chaos and border-collision bifurcations in non-invertible piecewise-linear maps *Nonlinearity* **18** 485–504

[9] di Bernardo M, Budd CJ and Champneys AR 2001 Corner collision implies border-collision bifurcation *Physica D* **154** 171–94

[10] Osorio G, di Bernardo M and Santini S 2008 Corner-impact bifurcations: a novel class of discontinuity-induced bifurcations in cam-follower systems *SIAM J. Appl. Dyn. Syst.* **7** 18–38

[11] Nusse HE and Yorke JA 1992 Border-collision bifurcations including ‘period two to period three’ for piecewise smooth systems. *Physica D* **57** 39–57

[12] di Bernardo M, Budd CJ, Champneys AR and Kowalczyk P 2008 *Piecewise-Smooth Dynamical Systems. Theory and Applications* (New York: Springer)

[13] Leine RI and Nijmeijer H 2004 *Dynamics and Bifurcations of Non-smooth Mechanical systems. Lecture Notes in Applied and Computational Mathematics* vol 18 (Berlin: Springer)

[14] Banerjee S and Grebogi C 1999 Border collision bifurcations in two-dimensional piecewise smooth maps *Phys. Rev. E* **59** 4052–61

[15] Yang W-M and Hao B-L 1987 How the Arnol’d tongues become sausages in a piecewise linear circle map *Commun. Theor. Phys.* **8** 1–15

[16] Simpson DJW and Meiss JD 2008 Neimark–Sacker bifurcations in planar, piecewise-smooth, continuous maps *SIAM J. Appl. Dyn. Syst.* **7** 795–824

[17] Sushko I, Gardini L and Puu T 2004 Tongues of periodicity in a family of two-dimensional discontinuous maps of real Möbius type *Chaos Solitons Fractals* **21** 403–12

[18] Zhusubaliyev ZT, Mosekilde E, Maity S, Mohanan S and Banerjee S 2006 Border collision route to quasiperiodicity: numerical investigation and experimental confirmation *Chaos* **16** 023122

[19] Zhusubaliyev ZT and Mosekilde E 2008 Equilibrium-torus bifurcation in nonsmooth systems. *Physica D* **237** 930–6

[20] di Bernardo M, Feigin MI, Hogan SJ and Homer ME 1999 Local analysis of C-bifurcations in n-dimensional piecewise-smooth dynamical systems *Chaos Solitons Fractals* **10** 1881–908

[21] Gallian JA 1998 *Contemporary Abstract Algebra* (Boston, MA: Houghton Mifflin)

[22] Simpson DJW 2008 Bifurcations in Piecewise-Smooth, Continuous Systems PhD Thesis The University of Colorado

[23] Sushko I and Gardini L 2008 Center bifurcation for two-dimensional border-collision normal form *Int. J. Bifurcation Chaos* **18** 1029–50

[24] Sushko I and Gardini L 2006 Center bifurcation for a two-dimensional piecewise linear map *Business Cycle Dynamics: Models and Tools* ed T Puu and I Sushko (New York: Springer) pp 49–78