Symplectic capacities of domains in $\mathbb{C}^2$

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1 Introduction

In his paper [3] M. Gromov proved his celebrated non-squeezing theorem. We will study domains $D$ in $\mathbb{C}^2$ with standard coordinates $(z_1, z_2)$ and projections $\pi_1$ and $\pi_2$ onto the $z_1$ and $z_2$ planes respectively. The standard symplectic form on $\mathbb{C}^2$ is $\omega = \frac{i}{2} \sum_{j=1}^{2} dz_j \wedge d\bar{z}_j$ and this restricts to a symplectic form on the balls $B(r) = \{|z_1|^2 + |z_2|^2 < r^2\}$. In this notation Gromov’s non-squeezing theorem states that if area$(\pi_1(D)) \leq C$ and there exists a symplectic embedding $B(r) \rightarrow D$ then $\pi_2 \leq C$. Nowadays this can be rephrased as saying that the Gromov width of $D$ is at most $C$. Of course this is sharp when $D$ is a cylinder $\{|z_1| < r\}$.

For general $D$ it is natural to ask whether we can estimate the Gromov width instead in terms of the cross-sectional areas area$(D \cap \{z_2 = b\})$. But for any $\epsilon > 0$ there exists a construction of F. Schlenk, [4], of a domain $D$ lying in a cylinder $\{|z_1| < 1\}$ with Gromov width at least $\pi - \epsilon$ but with all cross-sections having area less than $\epsilon$. At least if we drop the condition on the domain lying in the cylinder, the cross-sections can even be arranged to be star-shaped, see [5]. Nevertheless in this note we will obtain such an estimate in terms of the areas of the cross-sections for domains whose cross-sections are all star-shaped about the axis $\{z_1 = 0\}$.

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**Theorem 1** Let $D \subset \mathbb{C}^2$ be a domain whose cross-sections $D \cap \{z_2 = b\}$ are star-shaped about center $z_1 = 0$. Define $C = \sup_b \text{area}(\{z_2 = b\} \cap D)$. Then if $B(r) \to D$ is a symplectic embedding we have $\pi r^2 \leq C$. In other words, $D$ has Gromov width at most $C$.

In section 2 we will establish an estimate on the Gromov width for such domains $D$. This is combined with a symplectic embedding construction to obtain our result in section 3.

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## 2 Embedding estimate

Here we prove the following theorem.

**Theorem 2** Fix constants $0 < K \leq M$ and $0 < t < 1$. Let $D \subset \mathbb{C}^2$ be a domain of the form $D = \{r < c(\theta, z_2), |z_2| < M\}$ where $(r, \theta)$ are polar coordinates in the $z_1$ plane and $c(\theta, z_2)$ is a real-valued function satisfying $t \leq c(\theta, z_2) \leq 1$ and $\left| \frac{dc}{dz_2} \right| \leq \frac{1}{K}$.

Define $C = \sup_b \text{area}(\{z_2 = b\} \cap D)$. Then if $B(r) \to D$ is a symplectic embedding of the standard ball of radius $r$ in $\mathbb{C}^2$ we have $\pi r^2 < C + 3\sqrt{\frac{M}{1K^2}}$.

Its key implication for us is the following.

**Corollary 3** Let $D = \{r < c(\theta, z_2), |z_2| < M\} \subset \mathbb{C}^2$ and $C = \sup_b \text{area}(\{z_2 = b\} \cap D)$. For any $L > 0$ the domain $D$ is a symplectic manifold with symplectic form $\omega_L = \frac{1}{L}(dz_1 \wedge d\overline{z}_1 + Ldz_2 \wedge d\overline{z}_2)$. Let $r > 0$ with $\pi r^2 > C$. Then for all $L$ sufficiently large the symplectic manifold $(D, \omega_L)$ does not admit a symplectic embedding of the ball $B(r)$.

This follows by rescaling. Note above that the volume of $(D, \omega_L)$ approaches infinity as $L \to \infty$.

**Proof of Theorem 2**
We consider the symplectic manifold $S^2 \times \mathbb{C}$ with a standard product symplectic form $\omega = \omega_1 \oplus \omega_2$ and still use coordinates $(z_1, z_2)$, where $z_1$ now extends from $\mathbb{C}$ to give a coordinate on the $S^2 = \mathbb{CP}^1$ factor. Still $\pi_1$ and $\pi_2$ denote the projections onto the coordinate planes. Let $F$ be the area of the first factor, we suppose that this is sufficiently large that the complement of $\{z_1 = \infty\}$ can be identified with a neighborhood of $\{|z_1| \leq 1\}$ in $\mathbb{C}^2$, the identification preserving the product complex and symplectic structures. In other words, from now we assume that $D \subset S^2 \times \mathbb{C} \setminus \{z_1 = \infty\}$ and satisfies the conditions on its cross-sections. Let $D^c$ denote the complement of $D$ in $S^2 \times \mathbb{C}$.

Now let $\phi : B(r) \to D$ be a symplectic embedding. Then we consider almost-complex structures $J$ on $S^2 \times \mathbb{C}$ which are tamed by $\omega$ and coincide with the standard product structure on $D^c$. By now it is well-known, see [3], that for all such $J$ the almost-complex manifold $S^2 \times \mathbb{C}$ can be foliated by $J$-holomorphic spheres. In $\{|z_2| \geq M\}$ the foliation simply consists of the $S^2$ factors.

Let $S$ denote the image of the holomorphic curve in our foliation passing through $\phi(0)$. By positivity of intersections $S$ intersects $\{z_1 = \infty\}$ in a single point, say $\{z_2 = b\}$. As above we will use polar coordinates $(r, \theta)$ in the plane $\{z_2 = b\}$. So we can write $D \cap \{z_2 = b\} = \{r \leq c(\theta, b) := c(\theta)\}$. Let $A = \text{area}(\{z_2 = b\} \cap D)$. We intend to obtain lower bounds for both $\int_{S \cap D^c} \omega_1$ and $\int_{S \cap D^c} \omega_2$.

First of all, we will suppose that $\pi_1(S \cap D^c) = \{r \geq g(\theta)\}$ for a positive function $g$ and that $S \cap D^c$ is a graph $\{z_2 = u(z_1)\}$ over this region. We explain later how essentially the same proof applies to the general case. Recall that our assumptions imply that $t \leq c(\theta), g(\theta) \leq 1$ for all $\theta$. Define $h(\theta) = |g(\theta) - c(\theta)|$.

Define a holomorphic function $f : \{r \leq \frac{1}{g(-\theta)}\} \to \{|z_2| \leq M\}$ by $f(z) = u(\frac{1}{z})$. Then $f(0) = b$ and $|f(z)| \leq M$ for all $z$. Therefore composing $f$ with a translation we can redefine $f$ as a function $f : \{r \leq \frac{1}{g(-\theta)}\} \to \{|z_2| \leq 2M\}$ with $f(0) = 0$.

As $g(\theta) \leq 1$ for all $\theta$ the map $f$ restricts to one from $\{|z| \leq 1\}$ and so by the Schwarz Lemma, if $|z| < 1$ we have $|f'(z)| \leq \frac{2M}{1-|z|}$. On the boundary of the
disk, our assumptions on the boundary of $D$ imply that $|f(\frac{1}{g(-\theta)}e^{i\theta})| \geq Kh(\theta)$.

Now we estimate

$$\int_{S \cap D^c} \omega_2 = \text{area}(\text{image}(f))$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{g(-\theta)}} r|f'(z)|^2 dr$$

$$= \int_0^{2\pi} g(-\theta) d\theta \left( \int_0^{\frac{\pi}{g(-\theta)}} r^2 |f'(z)|^2 dr \right) \left( \int_0^{\frac{\pi}{g(-\theta)}} dr \right)$$

$$\geq t \int_0^{2\pi} d\theta \left( \int_0^{\frac{1}{g}} r^2 |f'(z)|dr \right)^2.$$

Now

$$\int_0^{\frac{1}{g}} |f'(z)| dr \geq Kh(\theta)$$

and over all such functions $|f'(z)|$ the final integral above is minimized by taking $|f'(z)|$ as large as possible for small values of $r$. We compute

$$\int_0^{1 - e^{-\frac{Kh(\theta)}{2M}}} \frac{2M}{1-r}dr = Kh(\theta)$$

when $y = 1 - e^{-\frac{Kh(\theta)}{2M}} < \frac{1}{g(-\theta)}$. Therefore putting $y = x^2$ we have

$$t \int_0^{2\pi} d\theta \left( \int_0^{\frac{1}{g(-\theta)}} r^2 |f'(z)|dr \right)^2 \geq t \int_0^{2\pi} d\theta \left( \int_0^{\frac{x^2}{1-r}} \frac{2M \sqrt{r}}{1-r}dr \right)^2$$

$$= 4M^2 t \int_0^{2\pi} d\theta \left( \left[-2\sqrt{r} + \ln \left( \frac{1 + \sqrt{r}}{1 - \sqrt{r}} \right) \right]_0^{x^2} \right)^2$$

$$= 4M^2 t \int_0^{2\pi} d\theta \left( -2x + \ln \left( \frac{1 + x}{1 - x} \right) \right)^2$$

$$\geq 4M^2 t \int_0^{2\pi} \frac{4x^6}{9} d\theta$$

for the final estimate using the fact that $0 < x < 1$.

Now

$$x^2 = 1 - e^{-\frac{Kh(\theta)}{2M}} \geq (1 - e^{-\frac{1}{2}}) \frac{Kh(\theta)}{M}$$

since $\frac{Kh(\theta)}{2M} \leq \frac{1}{2}$. 

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Therefore
\[
\int_{S \cap D^c} \omega_2 \geq 4M^2 t \int_0^{2\pi} \frac{4r^6}{9} \theta d\theta \\
\geq \frac{16}{9} (1 - e^{-\frac{k}{2}})^3 \frac{t K^3}{M} \int_0^{2\pi} h(\theta)^3 d\theta.
\]

Next we compute
\[
\int_{S \cap D^c} \omega_1 = F - \frac{1}{2} \int_0^{2\pi} g(\theta)^2 d\theta \\
= F - A - \frac{1}{2} \int_0^{2\pi} (g(\theta)^2 - c(\theta)^2) d\theta \\
\geq F - A - \frac{1}{2} \int_0^{2\pi} (g(\theta) - c(\theta))(g(\theta) + c(\theta)) d\theta \\
\geq F - A - \int_0^{2\pi} h(\theta) d\theta.
\]

Therefore writing \( k = \frac{16}{9} (1 - e^{-\frac{k}{2}})^3 \frac{t K^3}{M} \) we have
\[
\int_{S \cap D^c} \omega \geq F - A - \int_0^{2\pi} (h(\theta) - kh(\theta)^3) d\theta \\
\geq F - A - 2\pi \frac{2}{3\sqrt{3k}} \\
= F - A - \pi \left( \frac{M}{3(1 - e^{-\frac{k}{2}})^3 t K^3} \right).
\]

Thus \( S \cap D \) has symplectic area at most \( A + \pi \left( \frac{M}{3(1 - e^{-\frac{k}{2}})^3 t K^3} \right) < A + 3 \sqrt{\frac{M}{tK^3}} \), since \( S \) itself has area \( F \).

We assumed above that \( \pi_1(S \cap D^c) \) is starshaped about \( z_1 = 0 \) and that \( S \cap D^c \) is a graph over this region. If the projection \( \pi_1 : S \to \pi_1(S \cap D^c) \) is a branched cover then we can define a function \( f \) as before simply choosing a suitable branch along the rays \( \{ \theta = \text{constant} \} \). The proof then applies as before.

Now suppose that \( \pi_1(S \cap D^c) \) is not starshaped about \( z_1 = 0 \). Then we find the smallest possible starshaped set \( \{ r \leq g(\theta) \} \) containing the complement of \( \pi_1(S \cap D^c) \). The defining function \( g \) will then have discontinuities but this does not affect the proof which again proceeds as before.

Finally we choose a \( J \) which coincides with the push forward of the standard complex structure on the ball \( B(r) \) under \( \phi \) but remains standard outside \( D \).
The part of $S$ intersecting the image of $\phi$ is now a minimal surface with respect to the standard pushed forward metric on the ball and so must have area at least $\pi r^2$, giving our inequality as required.

3 Proof of Theorem 1

For any domain $E \subset \mathbb{C}^2$ we will write $C(E) = \sup_b \text{area}(\{z_2 = b\} \cap E)$. Again we let $C = C(D)$. Arguing by contradiction suppose that $B(r) \to D$ is a symplectic embedding with $\pi r^2 > C + \epsilon$.

Let $B$ be the image of the ball of radius $r$ in $D$. We will prove Theorem 1 by finding a symplectic embedding of $B$ into $(D_1, \omega_L)$ for all sufficiently large $L$, where $D_1$ is a domain $C^0$ close to $D$ and with $C(D_1) < C(D) + \epsilon$. Such embeddings would contradict Corollary 3.

First we choose a lattice of the $z_2$ plane sufficiently fine that if we denote the gridsquares by $G_i$ then $\sup_i \text{area}(\pi_1(D \cap \pi_2^{-1}(G_i))) < C(D) + \epsilon$. Then we let $D_1 = \bigcup_i \pi_1(D \cap \pi_2^{-1}(G_i)) \times G_i$, suitably smoothed.

Let $\{b_j\}$ be the vertices of our lattice. We make the following simple observation.

Lemma 4 Suppose that $B \cap \{z_2 = b_j\} = \emptyset$ for all $j$. Then there exists a symplectic embedding of $B$ into $(D_1, \omega_L)$ for all sufficiently large $L$.

Proof It suffices to find a diffeomorphism $\psi$ of $\mathbb{C} \setminus \{b_j\}$ which preserves the $G_i$ and such that $\psi^*(L\omega_0) = \omega_0$, letting $\omega_0 = dz \wedge d\bar{z}$ be the standard symplectic form. It is not hard to construct such a map, and the product of this map on the $z_2$ plane with the identity map on the $z_1$ plane gives a suitable embedding.

Given Lemma 4, to find our embedding it remains to find a symplectic isotopy of $D_1$ such that the image of $B$ is disjoint from the planes $C_j = \{z_2 = b_j\}$. Equivalently we will find a symplectic isotopy of the union of the $C_j$, compactly supported in a neighborhood of $B$ and moving the $C_j$ away from $B$.

We may assume that the embedding of the ball of radius $r$ extends to a symplectic embedding of a ball of radius $s$ where $s$ is slightly greater than $r$. 6
Let $U$ be the image of this ball and $J_0$ the push-forward of the standard complex structure on $\mathbb{C}^2$ to $U$ under the embedding.

**Lemma 5** There exists a $C^0$ small symplectic isotopy supported near $\partial U$ which moves each $C_j$ into a $J_0$-holomorphic curve near $\partial U$.

**Proof** Let $(x + iy, u + iv)$ be local coordinates on $\mathbb{C}^2$. Let $C$ be one of our curves. We may assume that in these coordinates near to the origin $C \cap \partial U$ is the curve $\{(x, 0, 0, 0)\}$ and therefore that nearby $C$ is the graph over the $(x, y)$ plane of a function $h(x, y) = (u, v)$. So $u = v = 0$ when $y = 0$.

There exists a constant $k$ such that $|u, |v, |\frac{\partial u}{\partial x}|$ and $|\frac{\partial v}{\partial x}|$ are all bounded by $k|y|$ near $y = 0$.

Now, such a graph is symplectic provided

$$\left|\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}\right| < 1.$$ 

We can make $C$ holomorphic near $\partial U$ by replacing $h$ by $(\chi u, \chi v)$ where $\chi$ is a function of $y$, equal to 0 near $y = 0$ and 1 away from a small neighborhood. The resulting graph remains symplectic provided

$$\left|\chi \frac{\partial u}{\partial x} (\chi' \frac{\partial v}{\partial y}) - \chi \frac{\partial v}{\partial x} (\chi' u + \frac{\partial u}{\partial y})\right| < 1$$

or rewriting

$$\left|\chi^2 \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}\right) + \chi \chi'(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x})\right| < 1.$$ 

If we assume that $|\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}| < 1 - \delta$ the graph remains symplectic if $\chi$ is chosen such that

$$\left|\chi \chi'(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x})\right| < \delta$$

which is guaranteed if $\chi' < \frac{\delta}{k^2}.$

Since the integral $\int_0^t \frac{\chi}{k^2} dy$ diverges a function $\chi$ satisfying this condition while being equal to 0 near 0 and 1 away from an arbitrarily small neighborhood does indeed exist as required. The resulting surface is clearly isotopic through symplectic surfaces to the original $C$. 

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We now replace the $C_j$ by their images under the isotopy from Lemma 5.

We let $J$ be an almost-complex structure on $U$ which is tamed by $\omega$, coincides with $J_0$ near $\partial U$, and such that the $C_j \cap U$ are $J$-holomorphic.

Now $(U, J)$ is an (almost-complex) Stein manifold in the sense that it admits a plurisubharmonic exhaustion function $\phi : U \to [0, R)$. In fact, work of Eliashberg, see [1] and [2], implies that such a plurisubharmonic exhaustion exists with a unique critical point, its minimum. Generically this will be disjoint from the $C_j$.

Near the boundary we can take $\phi$ to be the push-forward under the embedding of a function $\frac{|z|^N}{C}$ for some integer $N \geq 2$ (depending perhaps on $U$) and (any given) constant $C$. The definition of a plurisubharmonic function states that $\omega_\phi = -dd^c \phi$ is a symplectic form on $U$ which is compatible with $J$ (for a function $f$ we define $d^c f := df \circ J$). We can choose $C$ such that $\omega_\phi|_{\partial U} = \omega|_{\partial U}$ and thus by Moser’s lemma the symplectic manifolds $(U, \omega)$ and $(U, \omega_\phi)$ are symplectomorphic via a symplectomorphism $F$ fixing the boundary. In fact, adjusting the isotopy provided by Moser’s method we may assume that $F$ fixes the $C_j$ (since they are symplectic with respect to both $\omega$ and $\omega_\phi$). Let $V$ denote the image of $U \setminus B$ under $F$ and suppose that $\{\phi \geq R_0\} \subset V$.

It now suffices to find a symplectic isotopy of the $C_j$ in $(U, \omega_\phi)$ moving the surfaces into the region $\{\phi \geq R_0\}$. Then the preimages of these surfaces under $F$ gives a symplectic isotopy moving them away from $B$ as required.

Let $Y$ be the gradient of $\phi$ with respect to the Kähler metric associated to $\phi$. Equivalently $Y$ is defined by $Y|\omega_\phi = -d^c \phi$. Define $\chi : [0, R) \to [0, 1]$ to have compact support but satisfy $\chi(t) = 1$ for $t \leq R_0$. Then the images of the $C_j$ under the one-parameter group of diffeomorphisms generated by $X = \chi(\phi) Y$ will eventually lie in $\{\phi \geq R_0\}$. Thus we can conclude after checking that they remain symplectic during this isotopy. We recall that the $C_j$ are $J$-holomorphic and finish with the following lemma.

**Lemma 6** Let $G$ be a diffeomorphism of $U$ generated by the flow of the vector-field $X$. Then $G^* \omega_\phi(Z, JZ) > 0$ for all non-zero vectors $Z$.
Proof For any function $f$ we compute

$$
\mathcal{L}_X f(\phi) d\phi = f'(\phi) X|d\phi \wedge d\phi + f(\phi) X|dd\phi + d(f(\phi) X|d\phi)
= (f'(\phi)d\phi(X) + f(\phi)\chi(\phi))d\phi.
$$

Thus $G^*d\phi = g(\phi)d\phi$ for some function $g$ and

$$
G^*\omega_\phi = g(\phi)\omega_\phi - g'(\phi)d\phi \wedge d\phi.
$$

The function $g$ is certainly positive and so $G^*\omega_\phi$ evaluates positively on the (contact) planes $\{d\phi = d^c\phi = 0\}$. Therefore if $G^*\omega_\phi$ evaluates nonpositively on a $J$-holomorphic plane then there exists such a plane containing $Y$. But this is clearly not the case, as $G^*\omega_\phi(Y, JY) = \omega_\phi(G_*Y, G_*JY) = -kd^c\phi(G_*JY)$ for some positive constant $k$ and $-d^c\phi(G_*JY) = -G^*d^c\phi(JY) = g(\phi)d\phi(Y) > 0$.

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