ON TIME FRACTIONAL PSEUDO-PARABOLIC EQUATIONS WITH NONLOCAL INTEGRAL CONDITIONS

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Abstract. In this paper, we study the nonlocal problem for pseudo-parabolic equation with time and space fractional derivatives. The time derivative is of Caputo type and of order $\sigma$, $0 < \sigma < 1$ and the space fractional derivative is of order $\alpha, \beta > 0$. In the first part, we obtain some results of the existence and uniqueness of our problem with suitably chosen $\alpha, \beta$. The technique uses a Sobolev embedding and is based on constructing a Mittag-Leffler operator. In the second part, we give the ill-posedness of our problem and give a regularized solution. An error estimate in $L^p$ between the regularized solution and the sought solution is obtained.

1. Introduction. Fractional differential equations (FDEs) have been extensively studied during the past two decades by many researchers because of their diverse applications in physics, electrochemistry, viscoelasticity, etc. Time-fractional PDEs have been used as a major tool for modeling various practical fields and there are a number of publications devoted to the study of time-fractional PDEs and their applications (we refer the reader to [2, 28, 18, 20]).

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In this paper, for $\alpha, \beta \in (0, 1)$, we consider the integral boundary problem for the fractional differential equation as follows:

$$\begin{cases}
\frac{\partial^\sigma_t}{\partial t^\sigma} (u + \kappa (-\Delta)\alpha u)(x, t) + (-\Delta)\beta u(x, t) = F(x, t), & \text{in } \Omega \times (0, T], \\
u = 0 & \text{in } \partial \Omega \times (0, T], \\
\rho_1 u(x, T) + \rho_2 \int_0^T u(x, t)dt = f(x), & \text{in } \Omega.
\end{cases} \quad (P)$$

Here we consider a domain $\Omega \subset \mathbb{R}^N$ with the smooth boundary $\partial \Omega$ and the constants $\rho_1, \rho_2 \geq 0$ satisfying $\rho_1^2 + \rho_2^2 > 0$. Our problem is studied with the time fractional derivative of order $\sigma \in (0, 1)$ in the sense of Caputo which is denoted by $\frac{\partial^\sigma_t}{\partial t^\sigma}$.

In the main equation of problem $(P)$, if we take $\kappa = 0$, and $\sigma = \alpha = \beta = 1$, we have the usual parabolic equation, which has been investigated by many researchers; see [15], [23], [19] and the references therein. If $\sigma = \alpha = \beta = 1$, and $\kappa > 0$, our main equation becomes the pseudo-parabolic equation. This type of FDEs can be used to model many phenomena in many fields of science. In [8], Peter J. Chen and Morton E. Gurtin presented a theory about a non-simple material for which the conductive temperature and the thermodynamic temperature do not coincide. The nonstationary processes in semiconductors in the presence of sources can be analyzed by the equation (see [31])

$$\frac{d}{dt}u - \frac{d}{dt} \Delta u - \Delta u = u^q.$$ 

The unidirectional propagation of nonlinear, dispersive, long waves is also described by the classical pseudo-parabolic (see [4]). For more applications, we refer the reader to [9], [24].

Problems with the usual Cauchy conditions such as the initial condition $u(x, 0) = \phi(x)$ or the condition at the terminal time $t = T$ are familiar. Usually, we can obtain well-posedness results for problems with initial conditions. In contrast, problems with Cauchy conditions at the terminal time are often ill-posed; we refer the reader to some recent results [17, 25] on the terminal value problem.

Our paper considers a non-local in time condition replacing the usual Cauchy conditions, that is

$$\rho_1 u(x, T) + \rho_2 \int_0^T u(x, t)dt = f(x). \quad (1)$$

In [12], [30], we can find two types of condition similar to the above such as

$$u(x, 0) = \sum_{j=1}^m \alpha_j u(x, T_j) + \int_0^T v(\tau) u(x, \tau)d\tau + \varphi(x) \quad \text{or} \quad \int_0^T u(x, t)dt = \phi(x). \quad (2)$$

M. Beshtokov [7, 5, 6] considered boundary value problem for FPPDE. In practice, some phenomena will be simulated more effectively if we investigate the problems with the non-local condition. Indeed, in some models for meteorology, the time-averaged data help us to get a more reliable long-term weather forecast (see [3]).

The problem with this type of condition can also be used when investigating radionuclides propagation in Stokes fluid, diffusion and flow in porous media ([13], [22], [26]). Compared with usual local initial/final value conditions, non-local conditions are more difficult to handle and motivated by this reason and their high application value, we work on time-fractional pseudo-parabolic equations with non-local final
conditions, and our paper provides new results for the linear source term case (to
the best of the authors’ knowledge, it seems that a problem like (P) has not really
been studied). Our paper will investigate problem (P) and the main results of this
work are as follows:

• The regularity results for the mild solution.
• The proof for the instability of the solution to the initial data recovery problem.
• The regularization of the initial data recovery problem.

This paper is organized as follows. Section 2 gives some preliminaries that are
needed throughout the paper. In section 3, we give the regularity result for the
mild solution, and an example which shows that our solution is unstable in the case
\( t = 0 \), and moreover, we give a regularized solution for the initial data recovery
problem.

2. Preliminary. Before we introduce the main results of our works, some prelim-
inary materials are given.

Definition 2.1. Let \( \| \cdot \|_B \) be a norm on a Banach space \( B \). Then, we define the
following spaces

- For \( 1 \leq q < \infty \)
  \[
  L^q(0, T; B) = \left\{ f : (0, T) \to B \left\| f \right\|_{L^q(0, T; B)} = \left( \int_0^T \| f(t) \|_B^q dt \right)^{\frac{1}{q}} < \infty \right\}.
  \]

- For \( q = \infty \)
  \[
  L^\infty(0, T; B) = \left\{ f : (0, T) \to B \left\| f \right\|_{L^\infty(0, T; B)} = \text{ess sup}_{t \in (0, T)} \| f(t) \|_B < \infty \right\}.
  \]

Remark 1. The spaces \( L^q(0, T; B), L^\infty(0, T; B) \) with the corresponding norms
\( \left\| \cdot \right\|_{L^q(0, T; B)} \) and \( \left\| \cdot \right\|_{L^\infty(0, T; B)} \), respectively, are Banach spaces.

Definition 2.2. Let \( (B, \| \cdot \|_B) \) be a Banach space. Then we use the notation
\( C^k([0, T]; B) \) to denote the Banach space of all \( k \in \mathbb{N} \) time’s derivative continuous
functions equipped with the norm
\[
\left\| f \right\|_{C^k([0, T]; B)} = \sum_{j=1}^n \sup_{t \in [0, T]} \| f^j(t) \|_B.
\]

Definition 2.3. (see [16]) The fractional Laplace operator of order \( \alpha \in (0, 1) \) is
defined as a Fourier multiplier with symbol \( -|\xi|^{2\alpha} \) given by
\[
\hat{\mathcal{F}}[(\Delta)^\alpha u](\xi) = -|\xi|^{2\alpha} \hat{\mathcal{F}}u(\xi),
\]\nand it is equivalent to
\[
(\Delta)^\alpha u = \mathcal{F}^{-1}(-|\xi|^{2\alpha} \mathcal{F}u),
\]\nwhere the notations \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) stand for the Fourier transform and the inverse
Fourier transform, respectively.

Next, let us recall the spectral problem for the fractional Laplace operator on
the bounded domain \( \Omega \) as follows
\[
\begin{cases}
(\Delta)^\alpha \psi_j(x) = \lambda_j^\alpha \psi_j(x), & x \in \Omega, \forall j \in \mathbb{N}, \\
\psi_j(x) = 0, & x \in \partial \Omega, \forall j \in \mathbb{N},
\end{cases}
\]
where the sequence of positive eigenvalues \( \{ \lambda_j \}_{j=1}^\infty \) satisfy
\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \nearrow \infty,
\]
whose corresponding set of real eigenfunctions \( \{ \psi_j(x) \}_{j=1}^\infty \) is orthogonal and complete.

**Remark 2.** If the inner product on \( L^2(\Omega) \) is denoted by \( (\cdot, \cdot)_{L^2(\Omega)} \), then the Fourier series of a function \( u \) in \( L^2(\Omega) \) can be formulated as
\[
u(x,t) = \sum_{j=1}^{\infty} (u(t), \psi_j)_{L^2(\Omega)} \psi_j(x).
\]

**Definition 2.4.** For any \( \eta > 0 \), we define the fractional Hilbert scale space by
\[
\mathcal{H}^\eta(\Omega) = \left\{ w \in L^2(\Omega) : \| w \|_{\mathcal{H}^\eta(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^{2\eta} (w, \psi_j)_{L^2(\Omega)}^2 < \infty \right\}.
\]

We denote the dual space \( \mathcal{H}^\eta(\Omega) \) by \( \mathcal{H}^{-\eta}(\Omega) \) provided that the dual space of \( L^2(\Omega) \) is identified with itself. The space \( \mathcal{H}^{-\eta}(\Omega) \) is a Hilbert space, equipped with the norm
\[
\| f \|_{\mathcal{H}^{-\eta}} = \left( \sum_{j=1}^{\infty} \lambda_j^{-2\eta} (w, \psi_j)_*^2 \right)^{\frac{1}{2}},
\]
for \( w \in \mathcal{H}^{-\eta} \) where \( (\cdot, \cdot)_* \) represents the dual product between \( \mathcal{H}^\eta(\Omega) \) and \( \mathcal{H}^{-\eta}(\Omega) \).

**Lemma 2.5.** (See\cite{1},\cite{10}) Let \( \Omega \subset \mathbb{R}^N, 1 \leq p < \infty \) such that \( k \geq m \geq 0 \) and \( (k-m)p < N \). Then we have
\[
\begin{align*}
& W^{k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega), \quad for \quad 1 \leq q < \frac{pN}{N - (k-m)p}, \\
& \mathcal{H}^\eta(\Omega) \hookrightarrow W^{\eta,2}, \quad for \quad \eta > 0, \\
& L^p(\Omega) \hookrightarrow \mathcal{H}^\eta(\Omega), \quad for \quad -\frac{N}{2} < \eta \leq 0, \quad p \geq \frac{2N}{N - 2\eta}, \\
& \mathcal{H}^\eta(\Omega) \hookrightarrow L^p(\Omega), \quad for \quad 0 \leq \eta < \frac{N}{2}, \quad p \leq \frac{2N}{N - 2\eta}.
\end{align*}
\]

For more details about the definition of the fractional Sobolev spaces, we refer the reader to \cite{10}.

**Definition 2.6.** For \( \alpha > 0 \), and an arbitrary constant \( \beta \in \mathbb{R} \), the Mittag-Leffler function can be defined by (see\cite{14})
\[
E_{\alpha,\beta}(z) = \sum_{j=1}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad z \in \mathbb{C},
\]
\( 6 \)
where $\Gamma$ is the usual Gamma function.

**Lemma 2.7.** (see [14]) For $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha \in [\alpha_1, \alpha_2]$, there exist positive constants $m_\alpha$ and $M_\alpha$, depending only on $\alpha$ such that

(i) $E_{\alpha,1}(-z) > 0$, for $z > 0$.

(ii) $\frac{m_\alpha}{1+z} \leq E_{\alpha,\beta}(-z) \leq \frac{M_\alpha}{1+z}$, for $\beta \in \mathbb{R}$, $z > 0$.

3. Main results. Using the Laplace transform method, we can find the formula of the solution to the first equation of (P) as follows

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\begin{align*}
u(x,t) = \sum_{j=1}^{\infty} E_{\sigma,1} \left( \frac{-\lambda_j^\sigma t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) (u(0), \psi_j)_{L^2(\Omega)} \psi_j(x) \\
+ \sum_{j=1}^{\infty} \int_0^t \frac{(t-s)^{\sigma-1}}{1 + \kappa \lambda_j^\alpha} E_{\sigma,\sigma} \left( \frac{-\lambda_j^\beta (t-s)^\sigma}{1 + \kappa \lambda_j^\alpha} \right) (F(s), \psi_j)_{L^2(\Omega)} ds \psi_j(x). \quad (7)
\end{align*}
\]

To find the formula of the mild solution to problem (P), we need to find the representation of the initial data $u(x,0)$. Using our non-local final condition, we have

\[
(f, \psi_j)_{L^2(\Omega)} = (u(0), \psi_j)_{L^2(\Omega)} \left[ \rho_1 E_{\alpha,1} \left( \frac{-\lambda_j^\sigma T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_1 \int_0^T E_{\alpha,1} \left( \frac{-\lambda_j^\sigma t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) dt \right]
\]

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\end{align*}
\]

Therefore, the formula of the mild solution to the problem (P) can be given by

\[
\begin{align*}
u(x,t) = \sum_{j=1}^{\infty} E_{\sigma,1} \left( \frac{-\lambda_j^\sigma t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) (f, \psi_j)_{L^2(\Omega)} \psi_j(x) \\
- \sum_{j=1}^{\infty} \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\sigma T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \int_0^T \frac{(T-s)^{\sigma-1}}{1 + \kappa \lambda_j^\alpha} E_{\sigma,\sigma} \left( \frac{-\lambda_j^\beta (T-s)^\sigma}{1 + \kappa \lambda_j^\alpha} \right) (F(s), \psi_j)_{L^2(\Omega)} ds \psi_j(x) \\
- \sum_{j=1}^{\infty} \rho_2 E_{\sigma,1} \left( \frac{-\lambda_j^\sigma t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \int_0^T \frac{(T-s)^{\sigma-1}}{1 + \kappa \lambda_j^\alpha} E_{\sigma,\sigma} \left( \frac{-\lambda_j^\beta (T-s)^\sigma}{1 + \kappa \lambda_j^\alpha} \right) (F(s), \psi_j)_{L^2(\Omega)} dtds \psi_j(x) \\
+ \sum_{j=1}^{\infty} \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\sigma T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \int_0^T \frac{(T-s)^{\sigma-1}}{1 + \kappa \lambda_j^\alpha} E_{\sigma,\sigma} \left( \frac{-\lambda_j^\beta (T-s)^\sigma}{1 + \kappa \lambda_j^\alpha} \right) (F(s), \psi_j)_{L^2(\Omega)} ds \psi_j(x) \\
:= Q_1(x,t) + Q_2(x,t) + Q_3(x,t) + Q_4(x,t).
\end{align*}
\]

Next we introduce the structure for this section.

**Part 1:** Regularity of the mild solution.

**Part 2:** The ill-posedness of the initial data recovery problem.
Part 3: Regularization and $L^p$ error estimate for the initial data recovery problem.

3.1. Regularity result.

Lemma 3.1. Let $0 < \alpha, \beta < 1$. Then we can find a constant $C_0 > 0$ such that

(i) If $0 < \beta \leq \frac{\alpha}{2}$, we have

$$\left[ \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) dt \right]^{-1} \leq \frac{1}{C_0}. \quad (10)$$

(ii) If $\frac{\alpha}{2} < \beta < 1$, we have

$$\left[ \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) dt \right]^{-1} \leq \frac{\lambda_j^{\beta - \frac{\alpha}{2}}}{C_0}. \quad (11)$$

Proof. First, we use the Cauchy inequality to get

$$\frac{\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \leq \frac{t^\sigma}{2\sqrt{\kappa}} \lambda_j^{\beta - \frac{\alpha}{2}}. \quad (12)$$

Using Lemma 2.7, it follows that

$$E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \geq \frac{m_\sigma}{1 + \frac{\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha}} \geq \frac{m_\sigma}{1 + \frac{\sigma^2}{2\sqrt{\kappa}} \lambda_j^{\beta - \frac{\alpha}{2}}}. \quad (13)$$

From the above estimate, we find that

- If $0 < \beta \leq \frac{\alpha}{2}$, we have

$$\rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) dt \geq \frac{m_\sigma (\rho_1 + \rho_2 T)}{1 + \frac{\sigma^2}{2\sqrt{\kappa}} \lambda_j^{\beta - \frac{\alpha}{2}}}. \quad (14)$$

- If $\frac{\alpha}{2} < \beta < 1$, we have

$$\rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) dt \geq \frac{m_\sigma (\rho_1 + \rho_2 T)}{\lambda_j^{\beta - \frac{\alpha}{2}} \left( \frac{1}{\lambda_j^{\beta - \frac{\alpha}{2}}} + \frac{\sigma^2}{2\sqrt{\kappa}} \right)}. \quad (15)$$

From the above estimates, our lemma is proved. \qed

Theorem 3.2. We assume that the constants $\sigma, \alpha, \beta, \theta, \eta, p, k, m$ satisfy

$$1/2 < \sigma < 1, \quad \alpha/2 < \beta < 1, \quad p < 1/\alpha, \quad 0 < \theta < \frac{2\sigma - 1}{2\sigma}, \quad 0 < \eta < 2, \quad k \leq \eta, \quad 1 \leq m \leq \frac{2N}{N + 2k - 2\eta}. \quad \text{Then the mild solution } u \text{ of the Problem (P) will belong to } L^p(0,T;W^{k,m}(\Omega)) \text{ and the following holds}

$$\|u\|_{L^p(0,T;W^{k,m}(\Omega))} \lesssim \|f\|_{H^{\eta+,\frac{\sigma}{2}}(\Omega)} + \|F\|_{L^\infty(0,T;H^{\eta-,\beta+\frac{\sigma}{2}}(\Omega))} + \|F\|_{L^2(0,T;H^{\eta-,\beta+\frac{\sigma}{2}}(\Omega))}, \quad (16)$$

where we use the notation $a \lesssim b$ if we can find a positive constant $K$ such that $a \leq Kb$. \quad \text{Theorem 3.2 is proved.}
Thanks to Lemma 3.1 and Lemma 2.7, we obtain

\[ \|u(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)} \leq \|Q_1(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)} + \|Q_2(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)} + \|Q_3(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)} + \|Q_4(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)}. \] (17)

Hence, we need to estimate the four terms on the right-hand side of the above to obtain the regularity results for our mild solution.

- **Estimate of the first term.** Parseval’s identity gives us

\[ \|Q_1(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)}^2 = \sum_{j=1}^{\infty} \lambda_j^{2\eta} \left[ \frac{E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{t^\sigma}{1+\kappa \lambda_j^\sigma} \right)}{\rho_1 E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{T^\sigma}{1+\kappa \lambda_j^\sigma} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{t^\sigma}{1+\kappa \lambda_j^\sigma} \right) dt} \right]^2. \] (18)

Thanks to Lemma 3.1 and Lemma 2.7, we obtain

\[ \lambda_j^{2\eta} \left[ \frac{E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{t^\sigma}{1+\kappa \lambda_j^\sigma} \right)}{\rho_1 E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{T^\sigma}{1+\kappa \lambda_j^\sigma} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{t^\sigma}{1+\kappa \lambda_j^\sigma} \right) dt} \right]^2 \leq t^{-2\alpha} \sum_{j=1}^{\infty} \lambda_j^{2\eta+2\beta-\alpha} \left( 1 + \kappa \lambda_j^\sigma \right)^2 \left( f, \psi_j \right)_{L^2(\Omega)}^2 \]

\[ \lesssim t^{-2\alpha} \sum_{j=1}^{\infty} \left( \lambda_j^{2\eta-\alpha} + \kappa \lambda_j^{2\eta+\alpha} \right) \left( f, \psi_j \right)_{L^2(\Omega)}^2 \]

\[ \lesssim t^{-2\alpha} \left( \|f\|_{\mathcal{H}^\sigma(\Omega)}^2 + \|f\|_{\mathcal{H}^\sigma(\Omega)}^2 \right). \] (19)

The Sobolev embedding \( \mathcal{H}^\sigma \rightarrow \mathcal{H}^{\sigma-\frac{2}{n}} \) enables us to get

\[ \|Q_1(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)} \leq t^{-\sigma} \|f\|_{\mathcal{H}^{\sigma-\frac{2}{n}}(\Omega)}. \] (20)

- **Estimate of the second term.**

\[ \|Q_2(\cdot, t)\|_{\mathcal{H}^\sigma(\Omega)}^2 \leq \sum_{j=1}^{\infty} \lambda_j^{2\eta} \left[ \frac{\rho_1 E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{t^\sigma}{1+\kappa \lambda_j^\sigma} \right) \int_0^T (T-s)^{\sigma-1} E_{\sigma,\sigma} \left( -\lambda_j^{\sigma} (T-s)^\sigma \frac{1+\kappa \lambda_j^\sigma}{1+\kappa \lambda_j^\sigma} \right) (F(s), \psi_j)_{L^2(\Omega)} ds}{\rho_1 E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{T^\sigma}{1+\kappa \lambda_j^\sigma} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{t^\sigma}{1+\kappa \lambda_j^\sigma} \right) dt} \right]^2. \] (21)

Using Lemma 2.7 we have

\[ \left( \rho_1 E_{\sigma,1} \left( -\lambda_j^{\sigma} \frac{t^\sigma}{1+\kappa \lambda_j^\sigma} \right) \int_0^T (T-s)^{\sigma-1} E_{\sigma,\sigma} \left( -\lambda_j^{\sigma} (T-s)^\sigma \frac{1+\kappa \lambda_j^\sigma}{1+\kappa \lambda_j^\sigma} \right) (F(s), \psi_j)_{L^2(\Omega)} ds \right)^2 \leq \rho_1^2 \frac{M^2_{\sigma}}{1 + \lambda_j^{\sigma} (T-s)^\sigma \frac{1+\kappa \lambda_j^\sigma}{1+\kappa \lambda_j^\sigma}} \left( \int_0^T (T-s)^{\sigma-1} (F(s), \psi_j)_{L^2(\Omega)} ds \right)^2. \] (22)
In the same way as in the previous step, we obtain

\[ \leq \left( 1 + \kappa \lambda^2 \right)^2 \left( \int_0^T (T - s)^{\sigma - \sigma \theta - 1} (F(s), \psi_j)_{L^2(\Omega)} ds \right)^2. \]

Now, the Hölder inequality will be applied to get the following estimate

\[ \left( \int_0^T (T - s)^{\sigma - \sigma \theta - 1} (G(s), \psi_j)_{L^2(\Omega)} ds \right)^2 \leq T^{\sigma - \sigma \theta} \left( \int_0^T (T - s)^{\sigma - \sigma \theta - 1} (G(s), \psi_j)_{L^2(\Omega)} ds \right)^2. \]

Combining these and noting that \( \lambda_j^{2\eta - 2\beta \theta - \alpha} \leq \lambda_j^{2\eta - 2\beta \theta + \alpha}, \forall j \in \mathbb{N} \), we deduce that

\[ \| Q_2(\cdot, t) \|_{H^\eta(\Omega)}^2 \leq t^{-2\sigma} \sum_{j=1}^\infty \lambda_j^{2\eta - 2\beta \theta + \alpha} \left( \int_0^T (T - s)^{\sigma - \sigma \theta - 1} (F(s), \psi_j)_{L^2(\Omega)} ds \right)^2 \]

\[ = t^{-2\sigma} \int_0^T (T - s)^{\sigma - \sigma \theta - 1} \| F(s) \|_{H^{\eta - \beta \theta + \frac{\alpha}{2}}(\Omega)}^2 ds. \]

\[ \leq t^{-2\sigma} \| F \|^2_{L^2(0, T; H^{\eta - \beta \theta + \frac{\alpha}{2}}(\Omega))}. \]

- **Estimate of the third term.**

In the same way as in the previous step, we obtain

\[ \left( \rho \lambda^2 \frac{\lambda^2}{1 + \kappa \lambda^2} \right) \left( \int_0^t (t - s)^{\sigma - 1} 1 + \kappa \lambda^2 \frac{\lambda^2}{1 + \kappa \lambda^2} \right) (F(s), \psi_j)_{L^2(\Omega)} ds dt \]

\[ \leq \left( 1 + \kappa \lambda^2 \right)^2 \left( \int_0^t (t - s)^{\sigma - \sigma \theta - 1} (F(s), \psi_j)_{L^2(\Omega)} ds \right)^2 \]

\[ \leq \left( 1 + \kappa \lambda^2 \right)^2 \left( \int_0^t (F(s), \psi_j)_{L^2(\Omega)} ds \right). \]

Note that, we get the estimate above by using the Hölder inequality as follows

\[ \left( \int_0^t (t - s)^{\sigma - \sigma \theta - 1} (F(s), \psi_j)_{L^2(\Omega)} ds \right)^2 \leq \left[ \int_0^t \left( \int_0^t (t - s)^{\sigma - 2\sigma \theta - 2} ds \right) \left( \int_0^t (F(s), \psi_j)_{L^2(\Omega)} ds \right)^2 dt \right]^2. \]

\[ \leq \left( \int_0^t \left[ \frac{T^{2\sigma - 2\sigma \theta - 1}}{2\sigma - 2\sigma \theta - 1} \right] dt \right)^2 \left( \int_0^T (F(s), \psi_j)_{L^2(\Omega)}^2 ds \right). \]

We thus have

\[ \| Q_3(\cdot, t) \|^2_{H^\eta(\Omega)} \leq t^{-2\sigma} \sum_{j=1}^\infty \lambda_j^{2\eta - 2\beta \theta + \alpha} \left( \int_0^T (F(s), \psi_j)_{L^2(\Omega)}^2 ds \right) \]

\[ = t^{-2\sigma} \| F \|^2_{L^2(0, T; H^{\eta - \beta \theta + \frac{\alpha}{2}}(\Omega))}. \]
• **Estimate of the fourth term.** This term can be treated more simply than the above two terms. Indeed, thanks to Lemma 2.7 and the Hölder inequality, we have

\[ \|Q_A(t)\|_{H^\alpha(\Omega)}^2 = \sum_{j=1}^\infty \lambda_j^{2\eta} \left( \int_0^t \frac{(t-s)^{\sigma-1}}{1 + \kappa \lambda_j^\sigma} E_{\sigma,\sigma} \left( \frac{-\lambda_j^\beta (t-s)^\sigma}{1 + \kappa \lambda_j^\sigma} \right) (F(s, \psi_j)_{L^2(\Omega)}) \, ds \right)^2 \]

\[ \lesssim \sum_{j=1}^\infty \lambda_j^{2\eta - \beta \theta} \left( \int_0^t (t-s)^{\sigma - \sigma \theta - 1} (F(s, \psi_j)_{L^2(\Omega)}) \, ds \right)^2 \]

\[ \lesssim \|F\|^2_{L^\infty(0,T,H^{\alpha - \beta \theta}(\Omega))}. \]

Combining these estimates with (17), we have

\[ \|u(\cdot, t)\|_{H^\alpha(\Omega)} \lesssim t^{-\sigma} \|f\|_{H^{\gamma + \frac{\alpha}{2}}(\Omega)} + t^{-\sigma} \|F\|_{L^\infty(0,T,H^{\alpha - \beta + \frac{\beta}{2}}(\Omega))} + t^{-\sigma} \|F\|_{L^2(0,T,H^{\alpha - \beta + \frac{\beta}{2}}(\Omega))} + \|F\|_{L^\infty(0,T,H^{\alpha - \beta \theta}(\Omega))}. \]

We apply the following Sobolev embeddings

\[
\begin{align*}
H^{\gamma - \beta + \frac{\beta}{2}}(\Omega) &\hookrightarrow H^{\alpha - \beta \theta}(\Omega), \\
H^\alpha(\Omega) &\hookrightarrow W^{\gamma,2}(\Omega), \\
W^{\gamma,2}(\Omega) &\hookrightarrow W^{k,m}(\Omega),
\end{align*}
\]

to obtain the important estimate

\[ \|u\|_{L^p(0,T;W^{k,m}(\Omega))} = \left( \int_0^T \|u(s)\|_{W^{k,m}(\Omega)}^p \, ds \right)^{\frac{1}{p}} \]

\[ \lesssim \left[ \|f\|_{H^{\gamma + \frac{\alpha}{2}}(\Omega)} + \|F\|_{L^\infty(0,T,H^{\alpha - \beta + \frac{\beta}{2}}(\Omega))} + \|F\|_{L^2(0,T,H^{\alpha - \beta + \frac{\beta}{2}}(\Omega))} \right] \left( \int_0^T t^{-\sigma p} \, dt \right)^{\frac{1}{p}} \]

\[ + \|F\|_{L^\infty(0,T,H^{\alpha - \beta + \frac{\beta}{2}}(\Omega))} \left( \int_0^T \, dt \right)^{\frac{1}{p}}. \]

Let us note that the integral \( \left( \int_0^T t^{-\sigma p} \, dt \right)^{\frac{1}{p}} \) is convergent for \( p < \frac{1}{\sigma} \). Hence, we can assert that

\[ \|u\|_{L^p(0,T;W^{k,m}(\Omega))} \lesssim \|f\|_{H^{\gamma + \frac{\alpha}{2}}(\Omega)} + \|F\|_{L^\infty(0,T,H^{\alpha - \beta + \frac{\beta}{2}}(\Omega))} + \|F\|_{L^2(0,T,H^{\alpha - \beta + \frac{\beta}{2}}(\Omega))}. \]

This show that \( u \) belongs to \( L^p(0,T;W^{k,m}(\Omega)) \) and the proof is complete. \( \square \)

3.2. **The ill-posedness of the initial data recovery problem.** From now on, we will only consider our problem in the homogeneous case i.e. when \( F = 0 \). Furthermore, we also assume that \( \beta > \alpha \) throughout the rest of the paper.

**Lemma 3.3.** Let \( \sigma \in (0,1) \) and \( 0 < \alpha < \beta < 1 \). Then, we get the following estimate

\[ \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\sigma} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\sigma} \right) \, dt \lesssim \frac{1}{\lambda_j^\beta} + \frac{1}{\lambda_j^{\beta - \sigma}}. \]
Proof. Lemma 2.7 gives us
\[
\rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \, dt \\
\leq M_\sigma \left[ \frac{\rho_1}{1 + \lambda_j^\beta t^\sigma} + \frac{\rho_2}{1 + \lambda_j^\beta T^\sigma} \right] \\
\leq \frac{1 + \kappa \lambda_j^\alpha}{\lambda_j^\alpha T^\sigma} + \int_0^T \frac{1 + \kappa \lambda_j^\alpha}{\lambda_j^\beta t^\sigma} \, dt \lesssim \frac{1}{\lambda_j^\beta} + \frac{1}{\lambda_j^{3-\alpha}}. \tag{33}
\]

In the latter inequality, we note that the integral \( \int_0^T t^{-\sigma} \, dt \) is convergent for \( \sigma \in (0, 1) \).

\[\Box\]

**Theorem 3.4.** If \( t = 0 \), the solution of the problem \((P)\) is unstable in the sense of the \( L^2(\Omega) \) norm.

**Proof.** We begin the proof by setting \( u(x, 0) = u_0(x) \) and defining a mapping \( \mathcal{T} \) from \( L^2(\Omega) \) to \( L^2(\Omega) \) as follows
\[
\mathcal{T} u_0(x) = \int_\Omega \varphi(x, z) u_0(z) \, dz, \tag{34}
\]
where
\[
\varphi(x, z) = \sum_{j=1}^\infty \left[ \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \, dt \right] \psi_j(x) \psi_j(z). \tag{35}
\]

It’s a simple matter to check that \( \varphi(x, z) = \varphi(z, x) \), and \( \mathcal{T} \) is a self-adjoint operator. Let us consider the following finite rank operator
\[
\mathcal{T}_M u_0(x) = \sum_{j=1}^M \left[ \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \, dt \right] \langle u_0, \psi_j \rangle_{L^2(\Omega)} \psi_j(x). \tag{36}
\]

From Lemma 3.3, it is clear that
\[
\| \mathcal{T} u_0(x) - \mathcal{T}_M u_0(x) \|^2_{L^2(\Omega)} \\
= \sum_{j=M+1}^\infty \left[ \rho_1 E_{\sigma,1} \left( \frac{-\lambda_j^\beta T^\sigma}{1 + \kappa \lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1} \left( \frac{-\lambda_j^\beta t^\sigma}{1 + \kappa \lambda_j^\alpha} \right) \, dt \right] \langle u_0, \psi_j \rangle_{L^2(\Omega)}^2 \tag{37}
\]
\[
\lesssim \left( \frac{1}{\lambda_j^{3-\alpha}} \right)^2 \sum_{j=M+1}^\infty \langle u_0, \psi_j \rangle_{L^2(\Omega)}^2 L^2(\Omega) \leq \left( \frac{1}{\lambda_j^{3-\alpha}} \right)^2 \| u_0 \|^2_{L^2(\Omega)}. \]

It follows immediately that \( \| \mathcal{T} - \mathcal{T}_M \|_{(L^2(\Omega); L^2(\Omega))} \rightarrow 0 \) as \( M \rightarrow \infty \). We also can prove that \( \mathcal{T} \) is compact. Moreover, we have
\[
\mathcal{T} u_0(x) = f(x), \tag{38}
\]
and then, we can conclude that our problem is ill-posed. Let us give an example to illustrate the ill-posedness of the problem. Taking the input data \( f_k(x) = \)
\[
\sqrt{\frac{1}{\lambda_k^\beta} + \frac{1}{\lambda_k^{\beta - \alpha}}} \psi_j(x), \text{ we can see at once that}
\]
\[
\|f_k\|_{L^2(\Omega)} = \sqrt{\frac{1}{\lambda_k^\beta} + \frac{1}{\lambda_k^{\beta - \alpha}}} \xrightarrow{k \to \infty} 0. \tag{39}
\]

However, the initial data \(u_{0,k}\) corresponding to the final data \(f_k\), is given by
\[
\begin{align*}
\lambda_k \in H^\nu(\Omega) \quad & \text{and define the supporting series as follows} \\
\end{align*}
\]
and it follows that
\[
\|u_{0,k}\|_{L^2(\Omega)} = \left\| \sqrt{\frac{1}{\lambda_k^\beta} + \frac{1}{\lambda_k^{\beta - \alpha}}} \right\|_{L^2(\Omega)} \\
\leq \sqrt{\frac{\lambda_k^{2\beta - \alpha}}{\lambda_k^\beta + \lambda_k^{\beta - \alpha}}}, \tag{41}
\]

The above estimate helps us to get the limit below
\[
\|u_{0,k}\|_{L^2(\Omega)} \xrightarrow{k \to \infty} \infty. \tag{42}
\]

From (39) and (42), we deduce that the solution to problem (P) is unstable in \(L^2(\Omega)\). \hfill \Box

### 3.3. Regularization and \(L^p\) error estimate.

**Theorem 3.5.** Let \(f_\varepsilon\) be noisy data satisfying \(\|f_\varepsilon - f\|_{L^p(\Omega)} \leq \varepsilon\), for \(p \geq 1\), and \(u_0 \in H^\nu(\Omega)\) for \(\nu > 0\). Then, we can find a regularized solution \(u_{0,\varepsilon}\) such that
\[
\|u_{0,\varepsilon} - u_0\|_{L^\frac{2N}{N-2\gamma}(\Omega)} \lesssim C(p, \eta)\varepsilon^\theta + \varepsilon^{\frac{\left(N - 1\right)(2\gamma - 1)}{2p(\gamma - 1)}} \|u_0\|_{H^\nu(\Omega)}, \quad (0 < \theta < 1), \tag{43}
\]

where
\[
\frac{-N}{2} < \eta \leq \min \left\{ 0, \frac{(p - 2)N}{2p} \right\}, \text{ and } 0 \leq \gamma < \min \left\{ \frac{N}{2}, \nu \right\} \tag{44}
\]

**Proof.** First, for \(\theta \in (0, 1)\), we set \(M_\varepsilon = \varepsilon^{\frac{\theta - 1}{2\gamma}}\). Then, we choose the following regularized solution
\[
\begin{align*}
u_{0,\varepsilon}(x) = \sum_{j=1}^{\lambda_j \leq M_\varepsilon} \frac{(f_\varepsilon, \psi_j)}{\rho_1 E_{\sigma, 1}} \left( -\frac{\lambda_j^\beta}{1 + \kappa \lambda_j^\beta} + \rho_2 \int_0^T E_{\sigma, 1} \left( -\frac{\lambda_j^\beta}{1 + \kappa \lambda_j^\beta} \right) dt \right) \psi_j(x), \tag{45}
\end{align*}
\]
and define the supporting series as follows
\[
\begin{align*}
u_{0,\varepsilon}^*(x) = \sum_{j=1}^{\lambda_j \leq M_\varepsilon} \frac{(f, \psi_j)}{\rho_1 E_{\sigma, 1}} \left( -\frac{\lambda_j^\beta}{1 + \kappa \lambda_j^\beta} + \rho_2 \int_0^T E_{\sigma, 1} \left( -\frac{\lambda_j^\beta}{1 + \kappa \lambda_j^\beta} \right) dt \right) \psi_j(x). \tag{46}
\end{align*}
\]
For the purpose of obtaining an error estimate between \( u_{0,\varepsilon} \) and \( u_0 \), we need two important estimates below.

- For any \( \gamma < \min \left\{ \frac{N}{2}, p \right\} \), thanks to Lemma 3.1, we have

\[
\left\| u_{0,\varepsilon} - u_{0,\varepsilon}^* \right\|_{\mathcal{H}^\gamma(\Omega)}^2 = \sum_{j=1}^{\lambda_j \leq M_\varepsilon} \lambda_j^{2\gamma} (f_\varepsilon - f, \psi_j)^2_{L^2(\Omega)}.
\]

\[
\sum_{j=1}^{\lambda_j \leq M_\varepsilon} \lambda_j^{2\gamma - 2\eta - \alpha} \lambda_j^{2\eta} (f_\varepsilon - f, \psi_j)^2_{L^2(\Omega)}
\]

\[
\leq \sum_{\lambda_j > M_\varepsilon} \lambda_j^{2\gamma} (u_0, \psi_j)^2_{L^2(\Omega)} = \sum_{\lambda_j > M_\varepsilon} \lambda_j^{2\gamma - 2\nu} \lambda_j^{2\nu} (u_0, \psi_j)^2_{L^2(\Omega)}
\]

\[
\leq M_\varepsilon^{2\gamma - 2\nu} \left\| u_0 \right\|_{\mathcal{H}^\gamma(\Omega)}^2.
\]

By applying the Sobolev embedding \( L^p(\Omega) \hookrightarrow \mathcal{H}^{\eta}(\Omega) \), we get

\[
\left\| u_{0,\varepsilon} - u_{0,\varepsilon}^* \right\|_{\mathcal{H}^{\eta}(\Omega)} \lesssim C(p, \eta) M_\varepsilon^{\gamma + \beta} \left\| f_\varepsilon - f \right\|_{L^p(\Omega)} \lesssim C(p, \eta) M_\varepsilon^{\gamma + \beta} \varepsilon.
\]

- From the formula of \( u_0 \), we deduce that

\[
\left\| u_{0,\varepsilon} - u_0 \right\|_{\mathcal{H}^{\gamma}(\Omega)}^2 = \sum_{\lambda_j > M_\varepsilon} \lambda_j^{2\gamma} (u_0, \psi_j)^2_{L^2(\Omega)}
\]

\[
= \sum_{\lambda_j > M_\varepsilon} \lambda_j^{2\gamma} (u_0, \psi_j)^2_{L^2(\Omega)} = \sum_{\lambda_j > M_\varepsilon} \lambda_j^{2\gamma - 2\nu} \lambda_j^{2\nu} (u_0, \psi_j)^2_{L^2(\Omega)}
\]

\[
\leq M_\varepsilon^{2\gamma - 2\nu} \left\| u_0 \right\|_{\mathcal{H}^{\gamma}(\Omega)}^2.
\]

Now, by combining these estimates with the triangle inequality and using the Sobolev embedding \( \mathcal{H}^{\gamma}(\Omega) \hookrightarrow L^{\frac{2N}{N - 2\gamma}}(\Omega) \), we can assert that

\[
\left\| u_{0,\varepsilon} - u_0 \right\|_{L^{\frac{2N}{N - 2\gamma}}(\Omega)} \lesssim C(p, \eta) M_\varepsilon^{\gamma + \beta} \varepsilon + M_\varepsilon^{\gamma - \nu} \left\| u_0 \right\|_{\mathcal{H}^{\gamma}(\Omega)}
\]

\[
= C(p, \eta) \varepsilon^\theta + \varepsilon^{\left(\frac{(p-1)(\gamma - \nu)}{\gamma}ight)} \left\| u_0 \right\|_{\mathcal{H}^{\gamma}(\Omega)}.
\]

It is easily seen that when \( \varepsilon \to 0 \), we have

\[
\begin{aligned}
\left\| u_{0,\varepsilon} - u_0 \right\|_{L^{\frac{2N}{N - 2\gamma}}(\Omega)} &\to 0, \\
M_\varepsilon &\to \infty,
\end{aligned}
\]

and this finishes our proof. \( \square \)

4. **Conclusion.** This paper investigates time fractional pseudo-parabolic equations with nonlocal integral conditions. The results of this work are divided into two main parts:

**Part I:** The Regularity result of the mild solution for Problem (P) is given.

**Part II:** We show that the initial data recovery problem is ill-posed. We also give the regularized solution and estimate error in \( L^p \) between the regularized solution and the sought solution.

This paper only considers the linear case, and in the future we hope to expand Problem (P) to the case of nonlinear source terms.
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