DIFFERENTIABILITY OF HAUSDORFF DIMENSION OF THE NON-WANDERING SET IN A PLANAR OPEN BILLIARD

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ABSTRACT. We consider open billiards in the plane satisfying the no-eclipse condition. We show that the points in the non-wandering set depend differentiably on deformations to the boundary of the billiard. We use Bowen’s equation to estimate the Hausdorff dimension of the non-wandering set of the billiard. Finally we show that the Hausdorff dimension depends differentiably on sufficiently smooth deformations to the boundary of the billiard, and estimate the derivative with respect to such deformations.

1. Introduction. The dimension theory of dynamical systems studies the dimensional characteristics (such as Hausdorff dimension) of the invariant sets of dynamical systems. See [20] for an introduction to the theory, or [4] for a recent review of the field. Past work has examined how dimensional characteristics of various dynamical systems can change with respect to perturbations of the system; for example the differentiability of entropy of Anosov flows [11], SRB measures in hyperbolic flows [24], and Hausdorff dimension of horseshoes [15]. However, this kind of problem has not been considered in the context of open billiard systems. Open billiards of the type described here have been investigated in [9, 18, 22, 27, 28, 14, 21]. The Hausdorff dimension of the non-wandering set has been estimated for open billiards in the plane [13] and in higher dimensions [31]. In this paper, we show that the Hausdorff dimension of the non-wandering set for an open billiard in the plane depends smoothly on perturbations to the boundary of the billiard. Specifically, if the boundary of the billiard is $C^r$-smooth and depends $C^{r'}$-smoothly on a perturbation parameter $\alpha$ (where $r \geq 4, r' \geq 2$), then the Hausdorff dimension is $C^{\min(r-3,r'-1)}$-smooth with respect to $\alpha$. Further, we find bounds for the derivative of the Hausdorff dimension with respect to $\alpha$, and we show that if the boundary is real analytic then the dimension is real analytic.

A billiard is a dynamical system in which a single pointlike particle moves at constant speed in some domain $Q \subset \mathbb{R}^n$ and reflects off the boundary $\partial Q$ according to the classical laws of optics [25]. Open billiards are a class of billiard in which the domain $Q$ is unbounded. Let $K = K_1 \cup \ldots \cup K_m$ ($m \geq 3$) be a subset of $\mathbb{R}^2$, where each $K_i$ is a compact, strictly convex, disjoint domain in $\mathbb{R}^2$ with a $C^r$ boundary ($r \geq 2$) that has strictly positive curvature. Set $Q = \overline{\mathbb{R}^2 \setminus K}$. We assume that $K$ satisfies the no-eclipse condition (H) introduced by Ikawa in [9]:

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(H): For distinct 1 ≤ i, j, k ≤ m, the convex hull of \( K_i \cup K_j \) is disjoint from \( K_k \). This condition ensures that the collision angle \( \phi \) is bounded above by a constant \( \phi_{\text{max}} < \frac{\pi}{2} \), and prevents discontinuities in the non-wandering set \( M_0 \) which consists of all bounded billiard trajectories in \( Q \).

In this paper, we consider smooth homotopies (with a parameter \( \alpha \)) between different billiards, which we call billiard deformations. Section 2 contains some preliminaries on open billiards and a precise definition of the deformations. We show that the periodic trajectories in the non-wandering set are differentiable and Lipschitz with respect to \( \alpha \). We extend this to the whole non-wandering set, and show that the curvature of the stable and unstable manifolds are also differentiable and Lipschitz.

The Hausdorff dimension of the non-wandering set was estimated in [13] by investigating convex fronts. This was later improved and extended to higher dimensional billiards in [31]. In this paper, using techniques from Pesin’s book on dimension theory in dynamical systems [20], we recover the estimates in [13] and show that they also apply to the lower and upper box dimensions. That is,

\[
\frac{2 \log(m-1)}{\log(1 + d_{\text{max}}k_{\text{max}})} \leq \dim_B M_0 = \overline{\dim}_BM_0 = \dim_H M_0 \leq \frac{2 \log(m-1)}{\log(1 + d_{\text{min}}k_{\text{min}})},
\]

where \( d_{\text{min}}, d_{\text{max}}, k_{\text{min}}, k_{\text{max}} \) are constants that depend on simple geometric characteristics of the obstacles. Furthermore we show that these dimensions depend differentiably on the boundary of the billiard obstacles. That is, if the billiard is shifted or deformed smoothly with some parameter \( \alpha \), then the function

\[
D(\alpha) = \dim_B M_0 = \overline{\dim}_BM_0 = \dim_H M_0
\]

is differentiable with respect to \( \alpha \). In fact, it is almost as smooth as the deformation, and for the first derivative we have

\[
\left| \frac{d}{d\alpha} D(\alpha) \right| \leq C,
\]

where \( C \) is a constant depending only on simple geometrical characteristics of the obstacles.

2. Open billiards. Consider the set \( Q \) described in the introduction. We describe a particle in the billiard by \( S_t x = x_t = (q_t, v_t) \) where \( q_t \in Q \) is the position of the particle and \( v_t \in \mathbb{S}^1 \) is its velocity at time \( t \). The map \( S_t \) is called the billiard flow. Then for as long as the particle stays inside \( Q \), it satisfies

\[
(q_{t+s}, v_{t+s}) = S_s(x_t) = (q_t + sv_t, v_t).
\]

Collisions with the boundary are described by

\[
v^+ = v^- - 2(v^-, n)n,
\]

where \( n \) is the normal vector (into \( Q \)) of \( \partial Q \) at the point of collision, \( v^- \) is the velocity before reflection, and \( v^+ \) is the velocity after reflection.

2.1. Non-wandering set. For \( x = (q, v) \) with \( q \in \partial K, v \in \mathbb{S}^1 \), we denote by \( n = n_K(q) \) the outward unit normal vector of \( \partial K \) at \( q \), by \( \phi(x) \) the angle between \( v \) and \( n \), by \( \phi_{\text{max}} \) the supremum of this angle over \( M_0 \), by \( \kappa(q) \) the curvature of \( \partial K \) at \( q \), and by \( t_j(x) \in [-\infty, \infty] \) for \( j \in \mathbb{Z} \) the time of the \( j \)-th reflection of \( x \) (with the convention that \( t_0(q, v) = t_1(q, -v) \) if \( q \in \text{Int}(Q) \), or \( t_0(q, v) = 0 \) if \( q \in \partial Q \)). If the forward trajectory of \( x \) does not have at least \( j \) reflections, then \( t_j(x) = \infty \), and if the backward trajectory does not have at least \( j \) reflections then
\[ t_{-j}(x) = -\infty. \] Let \( d_{j}(x) = t_{j}(x) - t_{j-1}(x) \) and abbreviate \( d(x) = d_{1}(x) \). Let \( M = \{(q,v) \in \partial K \times S^{1} : (n(q),v) \geq \cos \phi_{\text{max}} \} \), and \( M' = \{x \in M : t_{1}(x) < \infty\} \).

Let \( \pi : M \to \partial K \) be the canonical projection \( (q,v) \mapsto q \). Then define the billiard map \( B : M' \to M \) by \( Bx = S_{t_{1}(x)}(x) \). Then \( B \) is \( C^{-1} \). On the tangent space \( T_{x}M \), for \( x \in M \), we will use the norm \( ||(dq, dv)|| = \cos \phi_{\text{max}} \) (see e.g. [8]).

The set \( M \) together with an inner product inducing this norm is a Riemannian manifold. The non-wandering set of a billiard is the set of points with bounded trajectories. The non-wandering set of the billiard flow is denoted \( \Omega(S) \) or \( \Omega \), and its restriction to the boundary of \( K \) is \( M_{0} = \Omega \cap (\partial K \times S^{1}) \). Equivalently, \( M_{0} = \{x \in M : |t_{j}(x)| < \infty, \forall j \in \mathbb{Z}\} \) is the non-wandering set of the billiard map \( B \). Then \( B \) is a \( C^{-1} \) diffeomorphism on \( M_{0} \).

### 2.2. Notation for upper bounds on derivatives. We will say that a function of two variables \( f(x, y) \) is \( C^{A,B} \)-smooth or simply \( C^{A,B} \) if for every \( a \leq A, b \leq B \) the derivatives \( \frac{\partial^{a+b} f}{\partial x^{a} \partial y^{b}} \) are continuous. Frequently we will have some quantity that depends on a scalar \( \alpha \) and a vector (or sometimes a scalar) \( u \), and we will show that its derivatives are bounded by some constants. Rather than numbering these constants, we will label them with the quantity being differentiated in the subscript and the number of differentiations in the superscript. For example, if \( f \) is a function of \( \alpha \), we will say \( \frac{d^{2}f}{d\alpha^{2}} \leq C^{(2)}_{f} \). If \( g \) is a function of \( u = (u_{0}, \ldots, u_{n-1}) \) and \( \alpha \), then for each \( q, q' \geq 0 \) (but not \( q = q' = 0 \)), we will say \( \left| \frac{\partial^{q} g}{\partial \alpha^{q'}} \nabla^{q} g \right| \leq C^{(q,q')}_{g} \) for all \( u \) in its domain. These constants may depend on \( \alpha \) but not on \( u \). When there is only one variable and only the first derivative is required, we will simply write \( \left| \frac{df}{d\alpha} \right| \leq C_{f} \). It will be clear what each constant refers to each time we use this notation.

### 2.3. Billiard deformations. Here we define precisely what we mean by deformations to the boundary. We will always assume the boundaries \( \partial K_{i} \) of each obstacle are parametrized counterclockwise.

Let \( I \subseteq [-\infty, \infty] \) be a closed interval. A deformation will be described by adding an extra variable \( \alpha \in I \) to the parametrizations \( \tilde{\varphi}_{i}, i = 1, \ldots, m \), so that any point on \( \partial K_{i} \) is described by \( \tilde{\varphi}_{i}(\tilde{u}_{i}, \alpha) \). Denote the perimeter of \( \partial K_{i}(\alpha) \) by \( L_{i}(\alpha) \). Then let

\[ R_{i} = \{(\tilde{u}_{i}, \alpha) : \alpha \in I, \tilde{u}_{i} \in [0, L_{i}(\alpha)]\}. \]

**Definition 2.1.** Let \( I \subseteq [-\infty, \infty] \) be a closed interval and let \( m \geq 3 \) be an integer. For any \( \alpha \in I \), let \( K(\alpha) \) be a subset of \( \mathbb{R}^{2} \). For integers \( r \geq 2, r' \geq 1 \), we call \( K(\alpha) \) a \( C^{r,r'} \)-billiard deformation if the following conditions hold for all \( \alpha \in I \):

1. \( K(\alpha) = \bigcup_{i=1}^{m} K_{i}(\alpha) \) satisfies the no-eclipse condition (H).
2. Each \( K_{i}(\alpha) \) is a compact, convex set, the boundary of which is \( C^{r} \) and has strictly positive curvature, with total arc length \( L_{i}(\alpha) \).
3. Each \( K_{i} \) is parametrized counterclockwise by arc length with \( C^{r,r'} \) functions \( \tilde{\varphi}_{i} : R_{i} \to \mathbb{R}^{2} \).
4. For all integers \( 0 \leq q \leq r, 0 \leq q' \leq r' \) (apart from \( q = q' = 0 \)), there exist constants \( C^{(q,q')}_{\alpha} \) depending only on \( \alpha \) and the parametrizations, such that
for all integers \(i = 1, \ldots, m\),
\[
\left\| \frac{\partial^{q+q'} \tilde{\varphi}}{\partial u^i \partial \alpha^{q'}} \right\| \leq C^{(q,q')}_{\varphi}.
\]

We call \(\alpha\) the \textit{deformation parameter}, and the \(C^{(q,q')}_{\varphi}\) \textit{deformation constants}. We assume that only one obstacle is affected by the deformation. This results in stronger estimates for the derivatives. The general case can be covered by considering several successive deformations, or by deforming several at once (see Remark 4.3 for details on this). Define a function \(\delta_i\) such that \(\delta_i = 0\) if \(K_i(\alpha) = K_i\) is constant for all \(\alpha\), and \(\delta_i = 1\) if \(K_i(\alpha)\) depends on \(\alpha\).

Since the obstacles are parametrized by arc length, we always have \(C^{(1,0)}_{\varphi} = 1\); in fact \(\frac{2\kappa}{\omega_i} = 1\). The curvature of \(\partial K(\alpha)\) is \(\left\langle n, \frac{\partial^2 \tilde{\varphi}}{\partial u^i \partial \alpha^{q'}} \right\rangle\), which is bounded below by \(\kappa_{\min}\) and above by \(\kappa_{\max} = C_{\varphi}^{(2,0)}\).

2.4. **Symbolic model.** Let
\[
\Sigma_n = \{\xi = (\xi_1, \ldots, \xi_n) : \xi_i \in \{1, \ldots, m\}, \xi_i \neq \xi_{i+1}, \xi_n \neq \xi_1\}.
\]
This is the symbol space that models \(n\)-periodic trajectories, that is, trajectories \(x\) such that \(B^n x = x\). Sequences \(\xi\) that satisfy \(\xi_i \neq \xi_{i+1}\) and \(\xi_n \neq \xi_1\) are called \textit{admissible}.

For a fixed \(\alpha\), let \(M_\alpha \subset M_0\) be the set of \(n\)-periodic trajectories. Let \((q_j, v_j) = B^n x\) and let \(u_j \in [0, L_j(\alpha)]\) such that \(\tilde{\varphi}_{\xi_i}(\alpha, u_j) = q_j\). When considering a fixed sequence \(\xi\), we will use the abbreviation \(\tilde{\varphi}_j = \tilde{\varphi}_{\xi_i}j\).

Let the \textit{two-sided subshift} \(\sigma: \Sigma_n \to \Sigma_n\) be defined by \((\sigma\xi) = \xi_{i+1}\). Then \(\sigma\) is continuous under the following metric \(d_\theta\) for any \(\theta \in (0, 1)\),
\[
d_\theta(\xi, \xi') = \begin{cases} 0, & \text{if } \xi_i = \xi'_i \text{ for all } i \in \mathbb{Z}, \\ \theta^n, & \text{where } n = \max\{j \geq 0 : \xi_i = \xi'_i \text{ for all } |i| < j\} \end{cases}
\]

For any point \(x \in M_n\), define the corresponding sequence \(\xi = (\xi_1, \ldots, \xi_n) \in \Sigma_n\) such that \(\pi B^n x = K_{\xi_j}\) for all \(j = 1, \ldots, n\). We denote \(K_\xi = K_{\xi_1} \times \ldots \times K_{\xi_n}\). Let the \textit{length function} \(F = F_\xi : K_\xi \to \mathbb{R}\) be defined by
\[
F(q_1, \ldots, q_n) = \sum_{j=1}^n ||q_j - q_{j+1}||,
\]
where we write \(q_{n+1} = q_1\). Consider the function
\[
G = G_\xi : [0, L_{\xi_1}] \times \ldots \times [0, L_{\xi_n}] \times I \to \mathbb{R},
\]
\[
G(u_1, \ldots, u_n, \alpha) = F(\varphi_{\xi_1}(u_1, \alpha), \ldots, \varphi_{\xi_n}(u_n, \alpha)).
\]
If \(K(\alpha)\) is a billiard deformation, then \(G\) also depends on \(\alpha\) and is \(C^{r,r'}\)-smooth.

**Lemma 2.2.** ([27], cf. [22]) If \(K(\alpha)\) is a billiard deformation, then for a fixed \(\xi\) and \(\alpha\), the function \(G_\xi\) has exactly one minimum at
\[
\alpha \mapsto (u_1(\alpha), \ldots, u_n(\alpha)).
\]
\(F_\xi\) has a corresponding minimum
\[
(p_1, \ldots, p_n) = (\varphi_{\xi_1}(u_1(\alpha), \alpha), \ldots, \varphi_{\xi_n}(u_n(\alpha), \alpha)).
\]
These points determine a billiard trajectory that satisfies the classical laws of optics.
This shows that the map $x \mapsto \xi$ is invertible and its inverse is $\chi \xi = (p_1, u_{12})$, where $v_{12}$ is the unit vector from $p_1$ to $p_2$, and the $p_i$'s are found by minimising the length function. For any $\theta \in (0, 1)$, $\chi$ is a homeomorphism from $M_0$ onto $(\Sigma, d_0)$, and the shift map $\sigma$ is topologically conjugate to $B$, that is $B = \chi^{-1} \circ \sigma \circ \chi$ (see e.g. [18, 28]).

For any $x$, let $\kappa_j = \kappa(\pi B^j x)$ be the curvature at $\pi B^j x$, let $\phi_j = \phi(B^j x)$ be the angle between the velocity vector and the normal vector of $B^j x$, and let $\gamma_j = \frac{2\kappa_j}{\cos \phi_j}$. Let $d_{\min}, \kappa_{\min}$, and $\gamma_{\min}$ be the respective minimum values of $d_j(x)$, $\kappa_0(x)$ and $\gamma_0(x)$ over all $x \in M_0$, and let $d_{\max}, \kappa_{\max}$, and $\gamma_{\max}$ be the respective maximum values. Note that $\gamma_{\min} = 2\kappa_{\min}$ and $\gamma_{\max} = \frac{2\kappa_{\max}}{\cos \phi_{\max}}$. Also recall that $\phi_{\max} < \frac{\pi}{2}$ is the maximum value of $\phi(x)$ over $x \in M_0$. Whenever we are considering a fixed sequence $\xi$, we will use the abbreviation $\varphi_j = \varphi_{\xi j}$.

3. Derivatives of parameters. Let $K(\alpha)$ be a $C^{r, r'}$ billiard deformation satisfying the conditions in Definition 2.1. Fix a finite admissible sequence $\xi = (\xi_0, \ldots, \xi_{n-1}) \in \Sigma_n$. Let

$$R_\xi = \{(u, \alpha) : \alpha \in I, u = (u_0, \ldots, u_{n-1}), u_j \in [0, L_{\xi_j}(\alpha)] \text{ for } j = 0, \ldots, n-1\}.$$  

For each $j = 0, \ldots, n-1$ set $\varphi_j = \varphi_{\xi j}$. By Lemma 2.2, there exist numbers $u_j(\alpha) = u_j(\xi, \alpha)$ and points $p_j(\alpha) = p_j(\xi, \alpha) = \varphi_j(u_j(\alpha), \alpha) \in \partial K_{\xi_j}$ which correspond to a billiard trajectory.

**Theorem 3.1.** Let $K(\alpha)$ be a $C^{r, r'}$ billiard deformation, $r \geq 2, r' \geq 1$. For any finite admissible sequence $\xi \in \Sigma_n$, let $p_j = \varphi_j(u_j(\alpha), \alpha)$ be the periodic points corresponding to $\xi$. Then the parameters $u_j(\alpha)$ are $C^{\min\{r-1, r'\}}$ with respect to the deformation parameter $\alpha$.

**Proof.** Fix a sequence $\xi$ with period $n$. Recall that the periodic points corresponding to $\xi$ are given by the global minimum of the length function $G = G_\xi : R_\xi \to \mathbb{R}$ defined by

$$G(u, \alpha) = \sum_{j=1}^n \| \varphi_j(u_j, \alpha) - \varphi_{j-1}(u_{j-1}, \alpha) \|.$$  

This is a $C^{r, r'}$ function of $u$ and $\alpha$. We will use the notation $I_j = \{j - 1, j + 1\}$. For each $j$, we can take the partial derivative of $G$ with respect to $u_j$ to get the equation

$$\frac{\partial G}{\partial u_j}(u, \alpha) = \sum_{\ell \in I_j} \left( \frac{\varphi_j(u_j, \alpha) - \varphi_\ell(u_i, \alpha)}{\| \varphi_j(u_j, \alpha) - \varphi_\ell(u_i, \alpha) \|} \frac{\partial \varphi_j}{\partial u_j}(u_j, \alpha) \right).$$  

By Lemma 2.2, for each $\alpha \in I$ the function $G$ has a single critical point $u = (u_1, \ldots, u_n)$, which satisfies

$$\frac{\partial G}{\partial u_j}(u(\alpha), \alpha) = 0 \text{ for all } j = 0, \ldots, n-1.$$  

Now define a function $h_j : R_j \times I \to \mathbb{R}^n$ by $h_j(u, \alpha) = \frac{\partial G}{\partial u_j}(u, \alpha)$, and let $h$ be the vector $(h_0, \ldots, h_{n-1})$. This is a $C^{r-1, r'}$ function of $u$ and $\alpha$. The Jacobian of $h$ with respect to $u$ is the Hessian matrix of $G$:

$$H_{ij} = \frac{\partial^2 G}{\partial u_i \partial u_j}.$$
This matrix is non-singular and positive definite (see [27] or [22]), so we can apply the implicit function theorem. There exists a function \( u(\alpha) \) that satisfies \( h(u(\alpha),\alpha) = 0 \), and the \( u_j(\alpha) \) are exactly the parameters that minimize \( G \). So \( \varphi_j(u_j(\alpha),\alpha) \), \( j = 0, \ldots, n-1 \) are the periodic points corresponding to \( \xi \). Furthermore, by the implicit function theorem [1], \( u_j(\alpha) \) is \( C^{\min\{r-1, r'\}} \).

By the implicit function theorem, we have the following system of equations:

\[
\frac{\partial^2 G}{\partial \alpha \partial u_j}(u(\alpha),\alpha) + \sum_{i=1}^{n} \frac{\partial u_i}{\partial \alpha} \frac{\partial^2 G}{\partial u_i \partial u_j}(u(\alpha),\alpha) = 0,
\]

which we can write as a matrix equation:

\[
H \frac{\partial u}{\partial \alpha} = - \frac{\partial}{\partial \alpha} \nabla G.
\]  

Theorem 3.2. For any \( \xi \in \Sigma_n \), the derivatives of the parameters satisfy

\[
\left| \frac{\partial u_j}{\partial \alpha} \right| \leq \frac{1}{\cos \phi_j} \frac{C^{(0,1)}_\varphi + C^{(1,1)}_\varphi \delta_{\min} d_{\min}}{\kappa_{\min} d_{\min}}.
\]

Proof. The two following sections cover the proof of this theorem. We use the notation \( a_{ij} = 1/\|p_i - p_j\|, v_{ij} = a_{ij}(p_i - p_j) \). Denote by \( n_j \) the normal vector to \( \partial K \) at \( p_j \), by \( \kappa_j \) the curvature at \( p_j \), and by \( \phi_j = \phi_j(p_j, v_{jj+1}) \) the collision angle. We will use the following vector identity several times.

Proposition 3.3. If \( u, v, w \) are unit vectors in the plane, then

\[
\langle u, w \rangle - \langle v, u \rangle \langle v, w \rangle = \langle v, u^\perp \rangle \langle v, w^\perp \rangle,
\]  

where \( v^\perp \) is a positive (counterclockwise) rotation by a right angle.

3.1. Estimating \( -\frac{\partial}{\partial \alpha} \nabla G \). Note that \( v_{jj-1} + v_{jj+1} = -2 \cos \phi_j n_j \), where \( \phi_j \) is the collision angle at \( (p_j, v_{jj+1}) \) and \( n_j \) is the normal vector of \( K_\xi \) at \( \phi_j \). We also use the vector identity (2).

\[
\frac{\partial^2 G}{\partial \alpha \partial u_j} = \frac{\partial}{\partial \alpha} \sum_{i \in I_j} \left\langle \varphi_j - \varphi_i, \frac{\partial \varphi_j}{\partial u_j} \right\rangle
\]

\[
= \sum_{i \in I_j} \left\langle v_{ji}, \frac{\partial^2 \varphi_j}{\partial \alpha \partial u_j} \right\rangle + \sum_{i \in I_j} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial \alpha} - \frac{\partial \varphi_i}{\partial \alpha}, \frac{\partial \varphi_j}{\partial u_j} \right\rangle
\]

\[
- \sum_{i \in I_j} a_{ji} \left\langle v_{ji}, \frac{\partial \varphi_j}{\partial \alpha} - \frac{\partial \varphi_i}{\partial \alpha}, \frac{\partial \varphi_j}{\partial u_j} \right\rangle
\]

\[
= \sum_{i \in I_j} a_{ji} \left\langle v_{ji}, \frac{\partial \varphi_j}{\partial \alpha} - \frac{\partial \varphi_i}{\partial \alpha}, \frac{\partial \varphi_j}{\partial u_j} \right\rangle - 2 \cos \phi_j \left\langle n_j, \frac{\partial \varphi_j}{\partial \alpha \partial u_j} \right\rangle.
\]

We have \( \left\langle v_{ji}, \frac{\partial \varphi_j}{\partial \alpha} \right\rangle = \cos \phi_j \), \( \left\langle v_{ji}, \frac{\partial \varphi_j}{\partial \alpha} \right\rangle = \delta_j C^{(0,1)}_\varphi, \) and \( \left\langle n_j, \frac{\partial \varphi_j}{\partial \alpha} \right\rangle \leq \delta_j C^{(1,1)}_\varphi \). We get the inequality

\[
\left| \frac{1}{\cos \phi_j} \frac{\partial^2 G}{\partial \alpha \partial u_j} \right| \leq C^{(0,1)}_\varphi (a_{jj-1} \delta_{j-1} + (a_{jj-1} + a_{jj+1}) \delta_j + a_{jj+1} \delta_{j+1}) + 2 \delta_j C^{(1,1)}_\varphi.
\]  

(3)
Let \( b_j = \frac{1}{\cos \phi_j} \frac{\partial^2 G}{\partial \phi_i \partial \phi_j} \). Since only one obstacle is deformed, either \( \delta_j = 0 \) or both \( \delta_{j-1} \) and \( \delta_{j+1} = 0 \), so let \( b_{\text{max}} = 2C_{(0,1)}^{(1,0)} + 2C_{(1,1)}^{(1,1)} \) and note that \( |b_j| \leq b_{\text{max}} \) for all \( j \).

3.2. The Hessian matrix. The Hessian of \( G \) is a matrix composed of the derivatives \( \frac{\partial^2 G}{\partial \phi_i \partial \phi_j} \). This section follows [27] and Section 2.2 of [22]. The first derivatives of \( G \) can be written

\[
\frac{\partial G}{\partial u_j} = \sum_{i \in I_j} \left\langle \frac{\varphi_j - \varphi_i}{\| \varphi_j - \varphi_i \|}, \frac{\partial \varphi_j}{\partial u_j} \right\rangle.
\]

If \( i \in I_j \), we can use (2) to get

\[
\frac{\partial^2 G}{\partial u_j \partial u_i} = -a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j}, \frac{\partial \varphi_i}{\partial u_i} \right\rangle + a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j}, \frac{\partial \varphi_i}{\partial u_i} \right\rangle = -a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j}, \frac{\partial \varphi_i}{\partial u_i} \right\rangle = a_{ji} \cos \phi_j \cos \phi_i.
\]

Along the diagonal \( i = j \) we have

\[
\frac{\partial^2 G}{\partial u_j^2} = \sum_{i \in I_j} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j}, \frac{\partial \varphi_j}{\partial u_j} \right\rangle - \sum_{i \in I_j} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j}, \frac{\partial \varphi_j}{\partial u_i} \right\rangle^2 - \sum_{i \in I_j} \left\langle \frac{\partial \varphi_j}{\partial u_j}, \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\rangle.
\]

Recall that \( v_{jj-1} + v_{jj+1} = -(2 \cos \phi_j) n_j \), where \( n_j \) is the outward unit normal vector. Also recall that \( \kappa_j = \left\langle n_j, \frac{\partial \varphi_j}{\partial u_j} \right\rangle \). So we have \( \sum_{i \in I_j} \left\langle v_{ji}, \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\rangle = 2 \kappa_j \cos \phi_j \).

Using the vector identity (2) we get

\[
\frac{\partial^2 G}{\partial u_j^2} = (a_{jj-1} + a_{jj+1}) \cos^2 \phi_j + 2 \kappa_j \cos \phi_j.
\]

Finally, if \( i \notin I_j \cup \{ j \} \), then \( \frac{\partial^2 G}{\partial u_i \partial u_i} = 0 \). We will now show the derivatives \( \frac{\partial u_i}{\partial \alpha} \) are bounded.

4. Solving the cyclic tridiagonal system. From (1) and the results of Section 3.2, we now have the following system of equations:

\[
\frac{\partial^2 G}{\partial \alpha \partial u_j} = -\sum_{i=1}^{n} \frac{\partial u_i}{\partial \alpha} \frac{\partial^2 G}{\partial \phi_i \partial \phi_j} \frac{\partial u_j^{-1}}{\partial \alpha}
= a_{jj-1} \cos \phi_j \cos \phi_{j-1} \frac{\partial u_{j-1}}{\partial \alpha} + \left( (a_{jj-1} + a_{jj+1}) \cos^2 \phi_j + 2 \kappa_j \cos \phi_j \right) \frac{\partial u_j}{\partial \alpha} + a_{jj+1} \cos \phi_j \cos \phi_{j+1} \frac{\partial u_{j+1}}{\partial \alpha}.
\]

For each \( j \), make the substitutions \( y_j = \frac{\partial u_j}{\partial \alpha} \cos \phi_j \), \( \gamma_j = \frac{2 \kappa_j}{\cos \phi_j} \) and \( a_j = a_{jj-1} \), \( a_1 = a_{n1} \). Divide through by \( \cos \phi_j \), then we can rearrange the system to

\[
\frac{1}{\cos \phi_j} \frac{\partial^2 G}{\partial \alpha \partial u_j} = a_j y_{j-1} + (a_j + a_{j+1} + \gamma_j) y_j + a_j y_{j+1}.
\]

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We can write this as a matrix equation $Ay = b$, where $y = (y_1, \ldots, y_n)^\top$, $b = (b_1, \ldots, b_n)^\top$, and $A$ is a matrix.

$$
\begin{pmatrix}
  a_1 + a_2 + \gamma_1 & a_2 & 0 & a_1 \\
  a_2 & a_2 + a_3 + \gamma_2 & a_3 & 0 \\
  a_3 & \ddots & \ddots & \ddots \\
  0 & \ddots & \ddots & a_n \\
  a_1 & 0 & a_n & a_n + a_1 + \gamma_n
\end{pmatrix}
\begin{pmatrix}
y_1 \\
\vdots \\
y_n
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
\vdots \\
b_n
\end{pmatrix}.
$$

A tridiagonal matrix only has non-zero elements in the main diagonal and the first diagonals above and below the main diagonal. $A$ is cyclic tridiagonal, meaning it can have two more non-zero elements in the corners. It is also diagonally dominant by rows since $a_j + a_{j+1} + \gamma_j > a_j + a_{j+1} > 0$. The problem now is to estimate the solutions $y_j$ of this equation. We could estimate

$$
\|y\|_2 \leq \frac{\|b\|_2}{\|A^{-1}\|_\infty} \leq \sqrt{n} \frac{b_{\max}}{\|A^{-1}\|_\infty}.
$$

This may seem to be the obvious approach to take. However since $\sqrt{n}$ is unbounded we cannot use this to find constant bounds on $y_j$ that hold for all $n$. Instead we use the following theorem of Varah [29]. Let $\|\cdot\|_\infty$ denote the matrix norm induced by the infinity norm.

**Theorem 4.1.** [29] Let $A = (A_{ij})_{i,j=1}^n$ be a diagonally dominant matrix. Then

$$
\|A^{-1}\|_\infty \leq \frac{1}{h},
$$

where

$$
h = \min_i \left( |A_{ii}| - \sum_{j \neq i} |A_{ij}| \right).
$$

For our matrix, we have

$$
h = \min_i \left( (a_i + a_{i+1} + \gamma_i) - (a_i + a_{i+1}) \right) = \min_i \gamma_i \geq 2\kappa_{\min}.
$$

Returning to the system $Ay = b$, we have $\|y\|_\infty \leq \|A^{-1}\|_\infty \|b\|_\infty$, so $|y_j| \leq \frac{b_{\max}}{2h_{\kappa_{\min}}}$.

Recall that $b_{\max} = \frac{2C^{(0,1)}_{\phi}}{d_{\min}} + 2C^{(1,1)}_{\phi}$ and $y_j = \frac{\partial u_j}{\partial \alpha} \cos \phi_j$. Then we get

$$
\left| \frac{\partial u_j}{\partial \alpha} \right| \leq \frac{1}{\cos \phi_j} \frac{C^{(0,1)}_{\phi} + C^{(1,1)}_{\phi} d_{\min}}{\kappa_{\min} d_{\min}}.
$$

\[\square\]

**Corollary 4.2.** Recall that the periodic points $p_0, \ldots, p_{n-1}$ are given by $p_j = \varphi_j(u_j(\alpha), \alpha)$. So each $p_j$ is differentiable with respect to $\alpha$ and we have

$$
\frac{dp_j}{d\alpha} = \frac{\partial \varphi_j}{\partial u_j} \frac{\partial u_j}{d\alpha} + \frac{\partial \varphi_j}{\partial \alpha},
$$

$$
\left| \frac{dp_j}{d\alpha} \right| \leq \frac{1}{\cos \phi_j} \frac{C^{(0,1)}_{\varphi} + C^{(1,1)}_{\varphi} d_{\min}}{\kappa_{\min} d_{\min}} + \delta_{\xi_j} C^{(0,1)}_{\varphi}
$$

$$
\leq \frac{1}{\cos \phi_{\max}} \frac{C^{(0,1)}_{\varphi} + C^{(1,1)}_{\varphi} d_{\min}}{\kappa_{\min} d_{\min}} + C^{(0,1)}_{\varphi},
$$

where $\delta_{\xi_j} = \frac{C^{(1,1)}_{\varphi}}{\kappa_{\min} d_{\min}} + C^{(0,1)}_{\varphi}$. 


where $\delta_i = 1$ if $K_i$ is affected by the deformation and 0 otherwise.

Following the notation in Section 2.2, let

\[
C_u^{(1)} = \frac{1}{\cos \phi_{\max}} \frac{d}{d\phi_{\max}} C_{\phi}^{(0,1)} + C_{\phi}^{(1,1)} d_{\min},
\]

\[
C_{P}^{(1)} = \frac{1}{\cos \phi_{\max}} \frac{d}{d\phi_{\max}} C_{\phi}^{(0,1)} + C_{\phi}^{(1,1)} d_{\min} + C_{\phi}^{(0,1)}.
\]

**Remark 4.3.** Theorem 3.1 assumes that only one obstacle is being deformed. Without this assumption, we have $b_{\max} = \frac{4C_{\phi}^{(0,1)}}{d_{\min}} + 2C_{\phi}^{(1,1)}$ instead of $\frac{2C_{\phi}^{(0,1)}}{d_{\min}} + 2C_{\phi}^{(1,1)}$. So the theorem still holds for deformations of multiple obstacles if we simply replace $\frac{C_{\phi}^{(0,1)} + C_{\phi}^{(1,1)} d_{\min}}{\kappa_{\min} d_{\min}}$ with $\frac{2C_{\phi}^{(0,1)} + C_{\phi}^{(1,1)} d_{\min}}{\kappa_{\min} d_{\min}}$.

5. **Higher derivatives of parameters.** Let $K(\alpha)$ be a $C^{r,r'}$ billiard deformation, and let $k \leq \min\{r, r', \}$. The function $h_j = \frac{\partial C}{\partial u_j}$ depends only on $\{\varphi_j\}_j$, so all of its derivatives can be estimated as

\[
\left| \frac{\partial^q}{\partial \alpha^q} \nabla^q h_j \right| \leq C_h^{(q,q')},
\]

where $C_h^{(q,q')}$ is a constant that depends on the constants $\{C_{\phi}^{(k,k')}\}_{k,k' \leq q}$ but does not depend on $n$.

**Lemma 5.1.** Let $K(\alpha)$ be a $C^{r,r'}$ billiard deformation and fix a finite admissible sequence $\xi \in \Sigma_n$. For every $1 \leq q \leq \min\{r-1, r'\}$, there exists a constant $C_u^{(q)}$ such that

\[
\left\| \frac{\partial^q u}{\partial \alpha^q} \right\|_{\infty} \leq C_u^{(q)}
\]

Proof: We already have the initial case $q = 1$. For a proof by induction, suppose that for some $q \leq \min\{r-1, r'\}$, there exist constants $C_u^{(1)}, \ldots, C_u^{(q)}$ such that

\[
\left\| \frac{\partial^k u}{\partial \alpha^k} \right\|_{\infty} \leq C_u^{(k)},
\]

for all $k = 1, \ldots, q$. We show the same is true for $q + 1$. We need to take the $q$th total derivative of $h_j(u(\alpha), \alpha)$ with respect to $\alpha$. There is a formula known as “Faà di Bruno’s formula” [2], which applies the chain rule for scalar functions an arbitrary number of times. In this case, we require a generalization of Faà di Bruno’s formula which applies to a scalar function of a vector function of a scalar.

First, write $x(\alpha) = (u_0(\alpha), \ldots, u_{n-1}(\alpha), \alpha)$. Then the formula from [17] can be applied to $h_j(x(\alpha))$. It is a long formula involving sums over integer partitions, but essentially the total derivative can be written as

\[
\frac{d^{q+1} h_j}{d \alpha^{q+1}}(x(\alpha)) = \sum_{j=0}^{n-1} \frac{\partial^2 G}{\partial u_j \partial u_i} \frac{d^{q+1} u_j}{d \alpha^{q+1}} + P \left\{ \begin{array}{c} \frac{\partial^k u}{\partial \alpha^k} \nabla^k h_j : k + k' \leq q + 1 \\ \frac{\partial^l u}{\partial \alpha^l} : l \leq q \end{array} \right\},
\]

where $P$ is a polynomial over the elements of the two sets. All of the arguments of $P$ can be estimated by the known constants $C_h^{(k,k')}$ and $C_u^{(l)}$ (with $k + k' \leq q, l \leq q$),
Theorem 6.3. Let $H$ be a constant such that
\[
\frac{\partial^{q+1} u}{\partial q^{q+1}} \leq C_u^{(q+1)}.
\]
So by induction, these estimates exist up to the min\{r - 1, r'\}'th derivative.

6. Extension to aperiodic trajectories. We now consider trajectories in the
non-wandering set that are not periodic. Define the symbol space for the whole
non-wandering set by
\[
\Sigma = \{ \xi = (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots) : \xi_i \in \{1, \ldots, m\}, \xi_i \neq \xi_{i+1} \},
\]
\[
\Sigma^+ = \{ \xi = (\xi_0, \xi_1, \ldots) : \xi_i \in \{1, \ldots, m\}, \xi_i \neq \xi_{i+1} \}.
\]
The two-sided and one-sided subshifts $\sigma : \Sigma \to \Sigma$ and $\sigma : \Sigma^+ \to \Sigma^+$ are defined by
\[(\sigma \xi)_i = \xi_{i+1}.\]
The no-eclipse condition ensures that sequences in $\Sigma$ correspond 1-1
with trajectories in the non-wandering set. The periodic sequences are dense in $\Sigma$,
and the periodic points are dense in $M_0$. This follows from [22, Lemma 10.2.1].

Let $\xi \in \Sigma$, and define a sequence of periodic admissible sequences \{\xi^{(n)}\}_{n=1} by
\[
\xi^{(n)}_j = \xi_{(j \mod n)},
\]
with the following modification to make $\xi^{(n)}$ always admissible:
if $\xi_{n-1} = \xi_0$, then let $\xi^{(n)}_{n-1} \in \{1, \ldots, m\}\{\xi_0, \xi_{n-2}\}$. This is always possible since
$m \geq 3$. Note that $\xi^{(n)}$ is equivalent to a string in $\Sigma_n$, and is on the same $(n - 1)$-
cylinder as $\xi$, meaning that the first $n - 1$ elements are the same.

**Proposition 6.1.** The following limit exists:
\[
\chi(\xi) = \lim_{n \to \infty} \chi(\xi^{(n)}),
\]
and $\chi : \Sigma \to M_0$ is the inverse of $\xi : M_0 \to \Sigma, x \mapsto \xi$.

**Proof.** From [26, Appendix B] or [22, Lemma 10.2.1], we have
\[
\|\chi \xi - \chi^{(n)}\| \leq 2C\delta^{n-1} \to 0,
\]
for constants $C > 0, \delta \in (0, 1)$. So $\chi^{(n)}$ absolutely converges to $\chi \xi$.

For any $\xi \in \Sigma$ and any $j \in \mathbb{Z}$, let $u_j(\xi, \alpha)$ be the parameter such that $p_j(\xi, \alpha) = \varphi_j(u_j(\xi, \alpha), \alpha)$ is the point $\pi B^\ast \chi(\xi)$. We show that these aperiodic trajectories
satisfy the same derivative estimates as the periodic orbits.

We will use the following proposition about uniformly convergent sequences.

**Proposition 6.2.** Let $m$ be a positive integer, let $I$ be an interval and let $X \subset \mathbb{R}^D$.
Then for any sequence of $C^m$ functions $f_n : I \to X$, if $f_n$ converges pointwise to $f$ and the $k$'th derivative $f_n^{(k)}$ converges uniformly to a function $g_k$ for all $k \leq m$, then $g$ is differentiable and $f^{(m)} = g_m$.

**Proof.** The case $m = 1$ is well known and can be found in [23], and the rest can be shown by induction.

**Theorem 6.3.** Let $K(\alpha)$ be a $C^{(r, r')}$ billiard deformation with $r \geq 2, r' \geq 2$, and
let $\xi \in \Sigma$. Then $u_j(\xi, \alpha)$ is $C^1$ with respect to $\alpha$ and
\[
\left| \frac{du_j(\xi, \alpha)}{d\alpha} \right| \leq C^{(1)}.
\]
Proof. Let $\xi \in \Sigma$ and define a sequence of finite admissible sequences $\{\xi^{(n)}\}_n$ with $\xi^{(n)} \in \Sigma_n$, such that the following limit is uniform:

$$u_j(\xi^{(n)}_n, \alpha) \to u_j(\xi, \alpha) \text{ as } n \to \infty.$$ 

Let $f_n(\alpha) = \frac{d}{d\alpha} u_j(\xi^{(n)}, \alpha)$. This sequence is uniformly bounded for $\alpha \in I$, and so are its derivatives (by Lemma 5.1), so it is equicontinuous. So by the Arzelà-Ascoli theorem [3], it has a uniformly (for $\alpha \in I$) convergent subsequence

$$f_n(\alpha) = \frac{d}{d\alpha} u_j(\xi^{(n)}, \alpha).$$

Let $f_n(\alpha) \to f(\alpha)$ as $k \to \infty$. By Proposition 6.2, $u_j(\xi, \alpha)$ is differentiable with respect to $\alpha$, and

$$\frac{d}{d\alpha} u_j(\xi, \alpha) = \lim_{k \to \infty} \frac{d}{d\alpha} u_j(\xi^{(n)}_k, \alpha).$$

Then by Theorem 3.2, we have

$$\left| \frac{du_j(\xi, \alpha)}{d\alpha} \right| \leq C_u^{(1)}.$$

Corollary 6.4. Let $K(\alpha)$ be a $C^{(r,r')}$ billiard deformation with $r, r' \geq 2$, and let $\xi \in \Sigma$. Then $u_j(\xi, \alpha)$ is $C^{\min\{r-1, r'-1\}}$ with respect to $\alpha$, and all of its derivatives are bounded by the same constants $C_u^{(q)}$ for the periodic trajectories.

Proof. Fix some $j \in \mathbb{Z}$. We prove by induction that for all $1 \leq q \leq \min\{r-1, r'-1\}$, the function $u_j(\xi, \alpha)$ is $C^q$, and there exist subsequences

$$\{n_{q,k}\}_k \subseteq \{n_{2,k}\}_k \subset \{n_k\}_k,$$

such that the following limit is uniform:

$$\lim_{k \to \infty} \frac{d^q}{d\alpha^q} u(\xi^{(n_{q,k})}, \alpha) = \frac{d^q}{d\alpha^q} u(\xi, \alpha).$$

The initial case $q = 1$ is already proven. Suppose it is true for some $q \leq \min\{r-2, r'-2\}$. Then the sequence

$$\frac{d^{q+1}}{d\alpha^{q+1}} u(\xi^{(n_{q,k})}, \alpha)$$

is uniformly bounded and equicontinuous for all $k$ (by Lemma 5.1), so by the Arzelà-Ascoli theorem it has a uniformly convergent subsequence

$$\frac{d^{q+1}}{d\alpha^{q+1}} u(\xi^{(n_{q+1,k})}, \alpha).$$

So by Proposition 6.2, $u(\xi, \alpha)$ is $C^{q+1}$ and

$$\frac{d^{q+1}}{d\alpha^{q+1}} u(\xi, \alpha) = \lim_{k \to \infty} \frac{d^{q+1}}{d\alpha^{q+1}} u(\xi^{(n_{q+1,k})}, \alpha).$$

Furthermore,

$$\left| \frac{\partial^{q+1} u_j(\xi, \alpha)}{d\alpha^{q+1}} \right| \leq C_u^{(q+1)}.$$

So by induction, $u$ is at least $C^{\min\{r-1, r'-1\}}$. 

Remark 6.5. The only reason $u_j(\xi, \alpha)$ is only $C^{\min\{r-1, r'-1\}}$ and not necessarily $C^{\min\{r-1, r'-1\}}$ is the equicontinuity requirement for the Arzelà-Ascoli theorem. It may be possible to show that $u_j(\xi, \alpha)$ is $C^{\min\{r-1, r'\}}$ with another method.
Corollary 6.6. For all \( \xi \in \Sigma \), the periodic points \( p(\xi, \alpha) \) are at least \( \mathcal{C}^{\min(r-1, r'-1)} \) with respect to \( \alpha \), and

\[
\left| \frac{dp(\xi, \alpha)}{d\alpha} \right| \leq C_p.
\]

7. Derivatives of other billiard characteristics.

7.1. Estimating derivatives of distances, curvatures and collision angles.

From here on, let \( K(\alpha) \) be a \( C^{(r, r')} \) billiard deformation with \( r \geq 4, r' \geq 2 \). We can use the upper bound on \( \frac{\partial u_j}{\partial \alpha} \) to estimate the derivatives of other characteristics of billiard trajectories, specifically the distances \( d_j \), curvatures \( \kappa_j \) and angles \( \phi_j \). We will not estimate the higher derivatives of these functions. Fix a sequence \( \xi \in \Sigma \). Then the distance between successive points \( d_j = d_j(\chi(\xi)) \) is \( \mathcal{C}^{\min(r-1, r'-1)} \) with respect to \( \alpha \) and we have

\[
d_j = |\varphi_j(u_j(\alpha), \alpha) - \varphi_{j-1}(u_{j-1}(\alpha), \alpha)|
\]

\[
\left| \frac{\partial d_j}{\partial \alpha} \right| \leq \left| \frac{\partial \varphi_j}{\partial u_j} \frac{\partial u_j}{\partial \alpha} + \frac{\partial \varphi_{j-1}}{\partial u_{j-1}} \frac{\partial u_{j-1}}{\partial \alpha} + \frac{\partial \varphi_j}{\partial \alpha} - \frac{\partial \varphi_{j-1}}{\partial \alpha} \right|
\]

\[
\leq 2C_u^{(1)} + C^{(0,1)} = C_d.
\]

The derivative of \( \kappa_j \) can also be bounded using the following billiard constant. Recall that \( \frac{\partial^3 \varphi_j}{\partial u_j^3} \) and \( \frac{\partial^3 \varphi_j}{\partial u_j^2 \partial \alpha} \) are bounded above by \( C^{(3,0)} \) and \( C^{(2,1)} \) respectively. Then \( \kappa_j = \mathcal{C}^{\min(r-3, r'-1)} \) and

\[
\left| \frac{\partial d_j}{\partial \alpha} \right| \leq C^{(3,0)}C_u^{(1)} + C^{(2,1)} = C_k.
\]

The collision angle \( \phi_j \) satisfies \( \cos 2\phi_j = \frac{(p_{j+1} - p_j) \cdot (p_{j+1} - p_j)}{|p_{j+1} - p_j||p_{j+1} - p_j|} \). Hence, each \( \phi_j \) is \( \mathcal{C}^{\min(r-1, r'-1)} \) and we have

\[
\left| \frac{d \cos 2\phi_j}{d\alpha} \right| = \left| \frac{p_{j+1} - p_j}{p_{j+1} - p_j} \cdot \frac{\partial}{\partial \alpha} \frac{p_{j+1} - p_j}{|p_{j+1} - p_j|} + \frac{p_{j+1} - p_j}{p_{j+1} - p_j} \cdot \frac{\partial}{\partial \alpha} \frac{p_{j+1} - p_j}{|p_{j+1} - p_j|} \right|
\]

\[
\leq 2C_u^{(1)} + C^{(0,1)} \delta_{\xi_{j+1}} + C^{(0,1)} \delta_{\xi_j} + 2C_u^{(1)} + C^{(0,1)} \delta_{\xi_{j-1}} + C^{(0,1)} \delta_{\xi_j}
\]

\[
\leq \frac{8C_u^{(1)} + (4\delta_{\xi_j} + 2\delta_{\xi_{j+1}} + 2\delta_{\xi_{j-1}})C^{(0,1)}}{d_{\min}},
\]

\[
\cos \phi_j = \sqrt{\frac{\cos 2\phi_j + 1}{2}},
\]

\[
\left| \frac{d \cos \phi_j}{d\alpha} \right| \leq \frac{4C_u^{(1)} + (2\delta_{\xi_j} + \delta_{\xi_{j+1}} + \delta_{\xi_{j-1}})C^{(0,1)}}{2d_{\min} \cos \phi_j}.
\]

Denote this upper bound by \( C_\phi \) (this is a slight departure from the notation defined in Section 2.2). We will also use the expression \( \gamma_j = \frac{2\kappa_j}{\cos \phi_j} \). This is \( \mathcal{C}^{\min(r-3, r'-1)} \)
and we have
\[
\left| \frac{\partial \gamma}{\partial \alpha} \right| \leq \frac{2(C_{\varphi}^{(3,0)} C_{u}^{(1)} + C_{\varphi}^{(2,1)})}{\cos \phi_j} + \frac{2\kappa_{j}}{\cos \phi_j} + \frac{4C_{u}^{(1)} + (2\delta_{j} + \delta_{j-1} + \delta_{j+1})C_{\varphi}(0,1)}{\cos \phi_j} \leq \frac{2(C_{\varphi}^{(3,0)} C_{u}^{(1)} + C_{\varphi}^{(2,1)})}{\cos \phi_{\max}} + \frac{2\kappa_{\max}}{\cos \phi_{\max}} \frac{2C_{u}^{(1)} + C_{\varphi}(0,1)}{d_{\min} \cos^2 \phi_{\max}}.
\]

Denote this upper bound by \( C \).

### 7.2. Stable and unstable manifolds

With the no-eclipse condition (H), the billiard map \( B \) and the flow \( S_t \) are examples of an Axiom A diffeomorphism and an Axiom A flow respectively [14]. That is, the non-wandering set is hyperbolic, and the periodic points are dense. It is well known (see e.g. [10]) that for any point \( x \in M_0 \) there exist stable and unstable subspaces \( E(x) \) and \( E(x) \), and local stable and unstable manifolds \( W^s(x), W^u(x) \subset M \). The stable manifold is simply the time reversal of the unstable manifold, that is \( W^s(x) = \text{Refl} W^u(x) \), where \( \text{Refl} \) is a bi-Lipschitz involution given by

\[
\text{Refl}(q, v) = \begin{cases} (q, -v), & \text{for } q \in \text{Int } Q \\ (q, 2(n_K(q), v)n_K(q) - v), & \text{for } q \in \partial K. \end{cases}
\]

**Definition 7.1.** [20] An Axiom A diffeomorphism \( f \) with a hyperbolic set \( \Lambda \) is called \( u \)-conformal (respectively, \( s \)-conformal) if there exists a continuous function \( a^{(u)}(x) \) (respectively, \( a^{(s)}(x) \)) on \( \Lambda \) such that \( d_x f|_{E^{(u)}(x)} = a^{(u)}(x) \text{Isom}_x \) for all \( x \in \Lambda \) (respectively, \( d_x f|_{E^{(s)}(x)} = a^{(s)}(x) \text{Isom}_x \) for all \( x \in \Lambda \)), where \( \text{Isom}_x \) is an isometry of \( E^{(u)} \) or \( E^{(s)} \). Then \( f \) is called conformal if it is both \( u \)-conformal and \( s \)-conformal.

Since the stable and unstable subspaces are each one dimensional, the billiard map in the plane is trivially conformal. For higher dimensional billiards, \( B \) is not conformal in general. We now define the convex fronts used to calculate and differentiate the functions \( a^{(u)} \) and \( a^{(s)} \).

**Definition 7.2.** Let \( z = (q, v) \in \Omega \) and let \( z_0 \) be the unique point on \( M_0 \) such that \( S_t z_0 = z \) for some \( t \geq 0 \). Let \( X = X(z) \subset Q \) be the unique convex curve containing \( q \) such that for any \( x_0 \in W^{(u)}(z_0) \), there exist \( t \geq 0 \) and \( x \in X \) such that \( S_t(x_0) = (x, v_X(x)) \). Then \( X \) is called a convex front. Then for any \( x \in X(z) \), define \( k_z(x) \) to be the curvature of \( X(z) \) at \( x \). If \( z \in M_0 \) and \( x \in W^{(u)}(z) \), then

\[
k_z(x) = \lim_{t \downarrow 0} k_{S_t z}(S_t x).
\]

\( X(z) \) is a \( C^r \) curve and the map \( z \mapsto X(z) \) is at least \( C^1 \) in general [25, 8]. However if \( y \in X(z) \) then the curve \( X(y) \) overlaps with \( X(z) \), so \( z \mapsto X(z) \) is \( C^r \) when restricted to these curves. For a fixed \( z \), the map \( x \mapsto k_z(x) \) is \( C^{r-2} \) since curvature involves the second derivative.

Recall that the billiard ball map is an Axiom A diffeomorphism, and there exist functions \( a^{(s)}, a^{(u)} : M_0 \to \mathbb{R} \), such that \( (d_x B)v = a^{(s)}(x) \text{Isom}_x v \) for all \( v \in E^{(s)} \) and \( (d_x B)v = a^{(u)}(x) \text{Isom}_x v \) for all \( v \in E^{(u)} \).

**Proposition 7.3.** Let \( K \) be a planar open billiard. Then the billiard map \( B \) is conformal on its stable and unstable manifolds, and

\[
a^{(u)}(x) = 1 + d(x)k_z(x), \quad a^{(s)}(x) = \frac{1}{1 + d(x)k_z(x)}.
\]
Proof. See e.g. [25, Lemma 2.1] or [8, (3.40)].

Note that $|a^{(\alpha)}(x)| < 1$ and $|a^{(\omega)}(x)| > 1$ for all $x \in M_0$.

7.3. Curvature of unstable manifolds.

**Definition 7.4.** For a fixed $\alpha \in I$ and $\zeta \in \Sigma$, we have a point $z = \chi(\zeta)$ and a convex front $X(z)$. Any point $x \in X$ has the same “past” as $z$, in the sense that $B^{-j}z$ and $B^{-j}x$ are on the same obstacle for all $j \geq 0$. For any $\xi \in \Sigma^+$, let $\chi_{a,\zeta}(\xi)$ be the unique point $x$ on $X(z)$ satisfying $B^j x \in K_{\xi_j}$ for all $j \geq 0$.

Fix a sequence $\zeta \in \Sigma$ and let $z = \chi(\zeta)$. Then for any point $x \in X(z)$ (sufficiently close to $z$), let $k_j(x) = k_{B^j(x)}(B^j z)$ be the curvature of the convex front $X(B^j z)$ at $B^j x$. We have the following well-known recurrence relation for $k_j$ (see e.g. [25, 8]).

$$k_{j+1} = \frac{k_j}{1 + d_j k_j} + \gamma_j.$$ 

This equation is smooth for all $d_j, k_j > 0$. Since $\gamma_j$ is $C^{\min(r-3,r'-1)}$ with respect to $\alpha$, $k_j$ is $C^{\min(r-3,r'-1)}$. If $x$ is periodic with period $n$, then $k_n = k_0$ and it is possible to solve these equations for $k_0$. We can bound $k_{\min} \leq k_j \leq k_{\max}$, where $k_{\min}, k_{\max}$ are constants calculated in [13] and [31].

Now by writing $x = \chi_{a,\zeta}(\xi)$, we can differentiate with respect to $\alpha$ to get

$$\frac{dk_j}{d\alpha} = \frac{d^2 k_j}{d\alpha^2} (1 + d_j k_j) - k_j \frac{dk_j}{d\alpha} (1 + d_j k_j)^2 + d^2 \gamma_j \frac{dk_j}{d\alpha}.$$

Let $\beta_j = \frac{1}{(1 + d_j k_j)^2}$ and $\eta_j = \frac{d\gamma_j}{d\alpha} \frac{k_j^2}{(1 + d_j k_j)^2}$. We have $\beta_j \leq \beta_{\max} = \frac{1}{(1 + d_{\min} k_{\max})^2}$.

and $\eta_j \leq \eta_{\max} = C_{\gamma} + \frac{k_{\max}^2 C_d}{(1 + d_{\min} k_{\max})^2}$. Then

$$\frac{dk_0}{d\alpha} = \frac{dk_n}{d\alpha} = \eta_{n-1} + \beta_{n-1} \frac{dk_{n-1}}{d\alpha}$$

$$= \eta_{n-1} + \beta_0 \eta_{n-2} + \ldots + \beta_1 \ldots \beta_{n-1} \eta_0 + \beta_0 \ldots \beta_{n-1} \frac{dk_0}{d\alpha}$$

$$= \frac{1}{1 - \beta_0 \ldots \beta_{n-1}} (\eta_{n-1} + \beta_0 \eta_{n-2} + \ldots + \beta_1 \ldots \beta_{n-1} \eta_0)$$

$$\leq \frac{1}{1 - \beta_{\max}} \left( 1 + \beta_{\max} + \ldots + \beta_{n-1} \right) \max_j \eta_j$$

$$\leq \frac{1}{1 - \beta_{\max}} \left( C_{\gamma} + \frac{k_{\max}^2 C_d}{(1 + d_{\min} k_{\max})^2} \right)$$

$$\leq C_k.$$

**Definition 7.5.** Fix a sequence $\zeta \in \Sigma$. Define functions $\psi_{a,\zeta}(\xi), \psi_{a,\zeta}(s) : \Sigma^+ \to \mathbb{R}$ as follows:

$$\psi_{a,\zeta}^{(u)}(\xi) = \log(1 + d(\chi_{a,\zeta}^+(\xi))k(\chi_{a,\zeta}^+(\xi)))$$

$$\psi_{a,\zeta}^{(s)}(\xi) = -\log(1 + d(\chi_{a,\zeta}^+(\xi))k(\chi_{a,\zeta}^+(\xi)))$$

where $d(x) = d_1(x)$. 

For a fixed $\xi \in \Sigma^+$, these functions are $C^{\min\{r-3,r'-1\}}$ with respect to $\alpha$, and we have

$$\left| \frac{d \psi^{(u)}(\alpha)}{d \alpha} \right| = \left| \frac{d \psi^{(u)}(\alpha)}{d \alpha} \right| = \frac{d \psi^{(u)}(\alpha)}{d \alpha} \left( k(x) \right) \leq \frac{C_d k(x) + C_k d(x)}{1 + d(x)k(x)}.$$

The expression $\frac{C_d k(x) + C_k d(x)}{1 + d(x)k(x)}$ as a function of $d, k$ reaches its maximum at one of the four corners of the rectangle $[d_{\min}, d_{\max}] \times [k_{\min}, k_{\max}]$. Denote this maximum by $C_\psi$. Using the reflection property, it is easy to see that

$$\left| \frac{d \psi^{(u)}(\alpha)}{d \alpha} \right| \leq C_\psi$$

for all $\xi \in \Sigma^+$.

8. Topological pressure and Bowen’s equation.

8.1. Entropy and pressure. Let $X$ be a compact metric space, $f : X \to X$ a continuous map, $\Lambda \subset X$ a hyperbolic $f$-invariant subset, and $\psi : X \to \mathbb{R}$ a continuous function. Denote by $\mathcal{M}(X)$ the set of all $f$-invariant Borel ergodic measures on $X$. Let $h_\mu(f)$ denote the topological entropy with respect to a measure $\mu \in \mathcal{M}(X)$, and let $P(\psi) = P_\Lambda(\psi)$ denote the topological pressure on $\Lambda$, as defined in [20] or [30]. The variational principle is

$$P_\Lambda(\psi) = \sup_{\mu \in \mathcal{M}(X)} \left( h_\mu(f) + \int_\Lambda \psi d\mu \right).$$

There is a unique equilibrium measure $\mu = \mu(\psi)$ corresponding to $\psi$ that satisfies $P_\Lambda(\psi) = h_\mu(f) + \int_\Lambda \psi d\mu$ [20].

**Proposition 8.1.** For an open billiard, the entropy of the billiard map $B$ is given by $h(B) = \log(m - 1)$ where $m$ is the number of obstacles.

**Proof.** From [11, 6], we have

$$h_{\text{top}} = \lim_{n \to \infty} \frac{1}{n} T(m, n),$$

where $T(m, n)$ is the number of $n$-periodic trajectories with $m$ obstacles. From [27], this is equal to $\log(m - 1)$. 

8.2. Pressure on the symbol space. Pressure and entropy can also be defined using operators on the symbol space $\Sigma^+$ (see [19]). For a Lipschitz function $\psi \in C(\Sigma^+)$, that is $\psi : \Sigma^+ \to \mathbb{R}$, define the Ruelle operator $L_\psi : C(\Sigma^+) \to C(\Sigma^+)$ by

$$(L_\psi w)(x) = \sum_{\sigma \in \Sigma^+} e^{\psi(\sigma)} w(\sigma').$$

Then $L_\psi$ is a bounded linear operator. The Ruelle-Perron-Frobenius theorem guarantees a simple maximum positive eigenvalue $\beta$ for $L_\psi$. We define the pressure on the symbol space by $P_{\Sigma^+}(\psi) = \log \beta$. The topological entropy can then be defined as $h_{\text{top}}(f) = P(0)$. There is a unique probability measure $\bar{\mu} = \bar{\mu}(\psi)$ such that

$$\int_{\Sigma^+} L_\psi v d\bar{\mu} = \beta \int_{\Sigma^+} v d\bar{\mu}.$$
Proposition 8.2. These definitions of pressure are equivalent in the following sense. Let $M$ be a smooth compact manifold with a continuous map $f : M \to M$ and a hyperbolic set $\Lambda$ coded by symbol spaces $\Sigma$ and $\Sigma^+$. For a fixed $\zeta \in \Sigma$ we have a homeomorphism $\chi^+_\zeta : \Lambda \to \Sigma$. Then

$$P_{\Sigma^+}(\psi \circ \chi^+_\zeta) = P_{\Lambda}(f, \psi, \Lambda).$$

Proof. This follows from combining the variational principle for $P_{\Lambda}$ with the variational principle for $P_{\Sigma^+}$ in [19, Theorem 3.5].

Proposition 8.3. (See e.g. [19, Proposition 4.10]) Let $f, g : \Sigma^+ \to \mathbb{R}$ and let $\tilde{\mu} = \tilde{\mu}(\psi)$. Then

$$\frac{d}{ds} P_{\Sigma^+}(f + sg) \bigg|_{s=0} = \int_{\Sigma^+} g d\tilde{\mu}.$$ 

Corollary 8.4. Let $I$ be an interval in $\mathbb{R}$. Let $f_s : \Sigma^+ \to \mathbb{R}$ be a function such that for any fixed $\xi \in \Sigma^+$, $f_s(\xi)$ is $C^2$ with respect to $s$. For any $s_0 \in I$, let $\mu_0$ be the equilibrium measure for $f_{s_0}$. Then

$$\frac{d}{ds} P_{\Sigma^+}(f_s) \bigg|_{s=s_0} = \int_{\Sigma^+} \frac{df_s}{ds} \bigg|_{s=s_0} d\mu_0.$$ 

Proof. For $t \in I$, define a function

$$\gamma(t, \xi) = \begin{cases} \frac{\partial f_s(\xi)}{\partial t} \bigg|_{t=s_0}, & t = s_0 \\ 0, & \text{otherwise} \end{cases},$$

Since $f_s$ is at least $C^2$ with respect to $t$, $\gamma$ is at least $C^1$. Define a function

$$f_{s,t}(\xi) = f_{s_0}(\xi) + (s - s_0)\gamma(t, \xi).$$

For a fixed $\xi$ this is a $C^1$ function of two variables. Clearly $f_{s,s}(\xi) = f_s(\xi)$. At $s = s_0$ we have $f_{s_0,t} = f_{s_0}(\xi)$ for all $t \in I$. By Proposition 8.3, we have

$$\frac{\partial P_{\Sigma^+}(f_{s,t})}{\partial s} \bigg|_{s=s_0} = \int_{\Sigma^+} \gamma(t, \xi)d\mu_0,$$

and

$$\frac{\partial P_{\Sigma^+}(f_{s,t})}{\partial s} \bigg|_{s=t=s_0} = \int_{\Sigma^+} \gamma(s_0, \xi)d\mu_0 = \int_{\Sigma^+} \frac{\partial f_s}{\partial s}(\xi) \bigg|_{s=s_0} d\mu_0.$$ 

It remains to show that

$$\frac{dP_{\Sigma^+}(f_s)}{ds} \bigg|_{s=s_0} = \frac{dP_{\Sigma^+}(f_{s,t})}{ds} \bigg|_{s=t=s_0}.$$ 

Define a function $p(s, t) = P_{\Sigma^+}(f_{s,t}(\xi))$. Since $P_{\Sigma^+}$ is analytic, this is a $C^1$ function of two variables. Consider $t$ as a function $t = t(s) = s$. Then by the chain rule, we have

$$\frac{dp}{ds} \bigg|_{s=s_0} = \frac{dp}{ds} (s, t(s)) \bigg|_{s=s_0} = \frac{\partial p}{\partial s}(s_0, s_0) + \frac{\partial t}{\partial s} \frac{\partial p}{\partial t}(s_0, s_0).$$

The derivative $\frac{\partial t}{\partial s}(s, t)$ is unknown for $s \neq s_0$, but since $p(s_0, t) = P_{\Sigma^+}(f_{s_0,t}) = P_{\Sigma^+}(f_{s_0})$ is constant with respect to $s$, we have $\frac{dp}{\partial t}(s_0, t) = 0$ for any $t$. So

$$\frac{dp}{ds} \bigg|_{s=s_0} = \frac{\partial p}{\partial s}(s_0, s_0) + 0,$$

and

$$\frac{dP_{\Sigma^+}(f_s)}{ds} \bigg|_{s=s_0} = \frac{dP_{\Sigma^+}(f_{s,t})}{ds} \bigg|_{s=t=s_0} = \int_{\Sigma^+} \frac{\partial f_s}{\partial s} \bigg|_{s=s_0} d\mu_0.$$
as required.

8.3. Bowen’s equation. Bowen’s equation can refer to any of a number of equations of the form \( P(-s\psi) = 0 \), where \( s \) is a dimension, \( P \) is a kind of topological pressure, and \( \psi \) is a function related to the dynamical system. The first use of the equation was Bowen’s paper on quasi-circles [7]. Manning and McCluskey used Bowen’s equation to calculate the Hausdorff dimension of Smale horseshoes in [16]. It is used by Barreira and Pesin in [5, 20] to calculate the Hausdorff dimension of hyperbolic sets.

**Theorem 8.5.** (Theorem 22.1 of [20]). Let \( \Lambda \) be the non-wandering set for a conformal Axiom A diffeomorphism \( f \) on a Riemannian manifold \( M \). Let \( D^{(u)}, D^{(s)} \) be the unique roots of Bowen’s equation,

\[
P_{\Lambda}(-D^{(u)} \log|a^{(u)}(x)|) = 0 \quad \text{and} \quad P_{\Lambda}(D^{(s)} \log|a^{(s)}(x)|) = 0.
\]

Let \( \kappa^{(u)} \) and \( \kappa^{(s)} \) be the unique equilibrium measures corresponding to the functions \(-D^{(u)} \log|a^{(u)}(x)|\) and \(D^{(s)} \log|a^{(s)}(x)|\).

1. For any \( z \in \Lambda \) and any open set \( U \subseteq W^{(u)}(z) \) such that \( U \cap \Lambda \neq \emptyset \),

\[
\dim_H(U \cap \Lambda) = \dim_B(U \cap \Lambda) = \dim_{B}(U \cap \Lambda) = D^{(u)}.
\]

2. For any \( z \in \Lambda \) and any open set \( S \subseteq W^{(s)}(z) \) such that \( S \cap \Lambda \neq \emptyset \),

\[
\dim_H(S \cap \Lambda) = \dim_B(S \cap \Lambda) = \dim_{B}(S \cap \Lambda) = D^{(s)}.
\]

3. The dimensions of the non-wandering set \( \Lambda \) are given by

\[
\dim_H(\Lambda) = \dim_B(\Lambda) = \dim_{B}(\Lambda) = D^{(u)} + D^{(s)}.
\]

4. The numbers \( D^{(u)} \) and \( D^{(s)} \) satisfy

\[
D^{(u)} = \frac{h_{\kappa^{(u)}}(f)}{\int_{\Lambda} \log |a^{(u)}(x)| d\kappa^{(u)}} \quad \text{and} \quad D^{(s)} = \frac{h_{\kappa^{(s)}}(f)}{\int_{\Lambda} \log |a^{(s)}(x)| d\kappa^{(s)}},
\]

where \( h_{\mu}(f) \) is the entropy of \( f \) with respect to \( \mu \).

9. Bounds on Hausdorff dimension. Let \( D^{(u)}, D^{(s)} \) be the solution to Bowen’s equations

\[
P_{\Lambda}(-D^{(u)} \log|a^{(u)}|) = 0 \quad \text{and} \quad P_{\Lambda}(D^{(s)} \log|a^{(s)}|) = 0.
\]

Let \( \kappa^{(u)}, \kappa^{(s)} \) be the unique equilibrium measures corresponding to the functions \(-D^{(u)} \log|a^{(u)}|\), \(D^{(s)} \log|a^{(s)}|\).

**Theorem 9.1.** Let \( \mu_{0} \) be the measure such that \( h_{\mu_{0}} = h_{\mu_{0}} \). Then the dimension of the unstable manifold satisfies

\[
\frac{\log(m-1)}{\int_{M_{0}} \log |a^{(u)}| d\mu_{0}} \leq D^{(u)} = \frac{h_{\kappa^{(u)}}}{\int_{M_{0}} \log |a^{(u)}| d\kappa^{(u)}} \leq \frac{\log(m-1)}{\int_{M_{0}} \log |a^{(u)}| d\kappa^{(u)}}.
\]

**Proof.** The topological entropy is

\[
\sup_{\mu} h_{\mu} = h_{\text{top}} = \log(m-1).
\]

Since \( \kappa^{(u)} \) is the equilibrium measure for \(-D^{(u)} \log|a^{(u)}|\), it satisfies

\[
h_{\kappa^{(u)}} - D^{(u)} \int_{M_{0}} \log |a^{(u)}| d\kappa^{(u)} = \sup_{\mu} \left( h_{\mu} - D^{(u)} \int_{M_{0}} \log |a^{(u)}| d\mu \right).
\]
Theorem 10.1. Let 

\[ h_{\kappa(u)} - D(u) \int_{M_0} \log |a^{(u)}|d\kappa(u) \geq h_{\mu_0} - D(u) \int_{M_0} \log |a^{(u)}|d\mu_0 \]

\[ 0 \geq h_{\text{top}} - D(u) \int_{M_0} \log |a^{(u)}|d\mu_0 \]

\[ D(u) \geq \frac{\log(m-1)}{\int_{M_0} \log |a^{(u)}|d\mu_0}. \]

The other inequality is trivial since \( h_{\kappa(u)} \leq h_{\text{top}} = \log(m-1). \)

\[ \square \]

Theorem 9.2. Let \( \mu_0 \) be the measure such that \( h_{\text{top}} = h_{\mu_0}. \) Then the dimension of the stable manifold satisfies

\[ \frac{\log(m-1)}{\int_{M_0} \log |a^{(s)}|d\mu_0} \leq D(s) = \frac{h_{\kappa(s)}}{\int_{M_0} \log |a^{(s)}|d\kappa(s)} \leq \frac{\log(m-1)}{\int_{M_0} \log |a^{(s)}|d\kappa(s)}. \]

Proof. The proof is very similar to the previous theorem. \( \square \)

In [13] and [31], estimates are found for the Hausdorff dimension using different methods. The latter gives stronger estimates in some cases and applies to higher dimensions.

Corollary 9.3. These estimates for the dimension of the non-wandering set are effectively the same as the estimates given in [31].

Proof. Using the estimates for \( d \) and \( k \) in [31], we have

\[ \log(1 + d_{\text{min}}k_{\text{min}}) \leq \int_{M_0} \log(1 + d(x)k(x))d\kappa(u) = \int_{M_0} \log |a^{(u)}|d\kappa(u), \]

\[ \int_{M_0} \log |a^{(u)}|d\mu_0 = \int_{M_0} \log(1 + d(x)k(x))d\mu_0 \leq \log(1 + d_{\text{max}}k_{\text{max}}). \]

The same holds for the stable manifolds, so we have

\[ \frac{2\log(m-1)}{\log(1 + d_{\text{max}}k_{\text{max}})} \leq D(u) + D(s) = \dim_H M_0 \leq \frac{2\log(m-1)}{\log(1 + d_{\text{min}}k_{\text{min}})}, \]

which is the estimate in [31] (Theorem 2.1 (i)). \( \square \)

10. Derivative of Hausdorff dimension. In this section we show that the Hausdorff dimension of the non-wandering set is differentiable with respect to \( \alpha \) and that its derivative is bounded by a constant depending only on the deformation. We use the definition of topological pressure \( P_{\Sigma^+} \) based on the Ruelle operator from Section 8.2. Recall that for any function \( \psi : M_0 \to \mathbb{R} \), this is related to the classical topological pressure by

\[ P_{\Sigma^+}(\psi \circ \chi_{\alpha,\zeta}^+) = P_{M_0}(B, \psi). \]

We can rewrite Bowen’s equation using this definition of pressure. The Hausdorff dimensions \( D(u) \) and \( D(s) \) are given by

\[ P_{\Sigma^+}(D(u)\psi^{(u)}_{\alpha,\zeta}) = P_{\Sigma^+}(D(u)\psi^{(s)}_{\alpha,\zeta}) = 0 \]

We will focus on the unstable manifolds first.

Theorem 10.1. Let \( K(\alpha) \) be a \( C^{r,r'} \) billiard deformation with \( r \geq 4, r' \geq 2. \) Then the Hausdorff dimension \( D^{(u)} \) is at least \( C_{\text{min}}^{r-3,r'-1} \) with respect to \( \alpha. \)
Rearranging this we get
\[-D_0 \psi_{\alpha, \zeta}^{(u)} = 0.\]

Since \( \alpha \in \mathbb{I} \), and let \( D_0 \in \mathbb{R}^+ \) be such that \( P_{\Sigma^+}(D_0 \psi_{\alpha, \zeta}^{(u)}) = 0 \). Note that \( D_0 \neq 0 \) because the entropy is nonzero. Let \( \kappa_0^{(u)} \) be the unique equilibrium measure corresponding to \(-D_0 \psi_{\alpha, \zeta}^{(u)}\). First we show that

\[ \frac{\partial}{\partial D} Q(D, \alpha) \bigg|_{D_0, \alpha_0} \neq 0. \]

By Proposition 8.3, we have

\[ \frac{\partial}{\partial D} P_{\Sigma^+} \left( D_0 \psi_{\alpha, \zeta}^{(u)} + (D - D_0) \psi_{\alpha, \zeta}^{(u)} \right) \bigg|_{D = D_0} = \int_{\Sigma^+} \psi_{\alpha, \zeta}^{(u)} d\kappa_0^{(u)}. \]

By Bowen’s equation and the variational principle we have

\[ 0 = P_{\Sigma^+}(D_0 \psi_{\alpha, \zeta}^{(u)}) = h_{\kappa_0^{(u)}} + \int_{\Sigma^+} D_0 \psi_{\alpha, \zeta}^{(u)} d\kappa_0^{(u)}. \]

Rearranging this we get

\[ \frac{\partial}{\partial D} Q(D, \alpha) \bigg|_{D_0, \alpha_0} = \int_{\Sigma^+} \psi_{\alpha, \zeta}^{(u)} d\kappa_0^{(u)} = -\frac{h_{\kappa_0^{(u)}}}{D_0}. \]

Since the entropy is never zero in our model, this is non-zero. Since \( P \) is analytic, the map \( \alpha \mapsto P_{\Sigma^+}(D_\psi_{\alpha, \zeta}^{(u)}) \) is \( C^{\text{min} \{ r-3, r' - 1 \}} \) and the map \( D \mapsto P_{\Sigma^+}(D_\psi_{\alpha, \zeta}^{(u)}) \) is analytic. So the implicit function theorem is applicable. There exists a \( C^{\text{min} \{ r-3, r' - 1 \}} \) function \( D^{(u)}(\alpha) \), such that \( P_{\Sigma^+}(D^{(u)}(\alpha) \psi_{\alpha, \zeta}^{(u)}) = 0 \) for all \( \alpha \in I \), and furthermore

\[ \frac{\partial Q}{\partial \alpha}(D^{(u)}(\alpha), \alpha) + \frac{\partial D^{(u)}}{\partial \alpha} \frac{\partial Q}{\partial D}(D^{(u)}(\alpha), \alpha) = 0. \]

(6)

Since \( P_{\Sigma^+}(D^{(u)}(\alpha) \psi_{\alpha, \zeta}^{(u)}) = 0 \), the function \( D^{(u)}(\alpha) \) is precisely the Hausdorff dimension of \( M_0 \cap U \) for any \( U \subset W^{(u)}(z) \) with \( M_0 \cap U \neq \emptyset \), for any \( z = \chi \in M_0 \) and any \( \alpha \in I \). The dimension \( D^{(u)} \) does not depend on the choice of \( \zeta \). \( \square \)

**Theorem 10.2.** Let \( K(\alpha) \) be a \( C^{(r, r')} \) billiard deformation with \( r \geq 4, r' \geq 3 \). Then \( \frac{\partial D^{(u)}}{\partial \alpha} \) is bounded.

**Proof.** Since \( r' \geq 3 \), \( \psi_{\alpha, \zeta}^{(u)} \) is at least \( C^2 \). So by Corollary 8.4, we have

\[ \frac{\partial}{\partial \alpha} P_{\Sigma^+}(D_0 \psi_{\alpha, \zeta}^{(u)}) \bigg|_{\alpha = \alpha_0} = D_0 \int_{\Sigma^+} \frac{\partial \psi_{\alpha, \zeta}^{(u)}}{\partial \alpha} \bigg|_{\alpha = \alpha_0} d\kappa_0^{(u)}. \]

Then (6) becomes

\[ 0 = \frac{\partial Q}{\partial \alpha} + \frac{\partial D^{(u)}}{\partial \alpha} \frac{\partial Q}{\partial D} \bigg|_{\alpha = \alpha_0} = D^{(u)} \int_{\Sigma^+} \frac{\partial \psi_{\alpha, \zeta}^{(u)}}{\partial \alpha} d\kappa_0^{(u)} - \frac{\partial D^{(u)}}{\partial \alpha} \frac{h_{\kappa_0^{(u)}}}{D^{(u)}} \bigg|_{\alpha = \alpha_0}. \]

Since \( \alpha_0 \) was chosen arbitrarily, for any \( \alpha \in I \) we have

\[ \frac{\partial D^{(u)}}{\partial \alpha} = \frac{D^{(u)}(\alpha)^2}{h_{\kappa_0^{(u)}}} \int_{\Sigma^+} \frac{\partial \psi_{\alpha, \zeta}^{(u)}}{\partial \alpha} d\kappa_0^{(u)}. \]

(7)
The integrand is bounded by \( C_\psi \), and \( \tilde{k}^{(u)} \) is a probability measure, so we have

\[
\left| \frac{\partial D^{(u)}}{\partial \alpha} \right| \leq \frac{D^{(u)}(\alpha)^2}{h_{\kappa(s)}} C_\psi.
\]

\[ \square \]

**Theorem 10.3.** Let \( K(\alpha) \) be a \( C^{r',r} \) billiard deformation with \( r \geq 4, r' \geq 3 \). For any sequence \( \zeta \), the Hausdorff dimension \( D^{(s)} \) of \( M_0 \cap W^{(s)}(\chi z) \) is at least \( C_{\min\{r-3,r'-1\}} \)

with respect to \( \alpha \), and its first derivative is bounded by

\[
\frac{D^{(s)}(\alpha)^2 C_\psi}{h_{\kappa(s)}}.
\]

**Proof.** By the reflection property \( W_{\varepsilon(s)}(z) = \text{Refl}(W_{\varepsilon(s)}(\text{Refl}(z))) \), we have

\[
\psi^{(s)}_{\alpha,\zeta} = -\psi^{(u)}_{\alpha,\zeta} \text{ and } D^{(s)} = D^{(u)}.
\]

So the proof is very similar to the previous theorem. \[ \square \]

**Theorem 10.4.** Let \( K(\alpha) \) be a \( C^{r',r} \) billiard deformation with \( r \geq 4, r' \geq 3 \). The Hausdorff dimension \( D(\alpha) = D^{(u)} + D^{(s)} \) of the non-wandering set is \( C_{\min\{r-3,r'-1\}} \)

with respect to \( \alpha \), and its derivative is bounded by the following.

\[
\left| \frac{dD}{d\alpha} \right| \leq \frac{C_\psi D}{\log(1 + d_{\min}k_{\min})}.
\]

**Proof.** Since \( B \) is conformal, the dimension satisfies \( D(\alpha) = D^{(s)}(\alpha) + D^{(u)}(\alpha) \). So

\[
\frac{dD}{d\alpha} \leq \frac{D^{(s)}(\alpha)^2 C_\psi}{h_{\kappa(s)}} + \frac{D^{(u)}(\alpha)^2 C_\psi}{h_{\kappa(s)}} \leq \frac{C_\psi D^{(s)}(\alpha)}{\log(1 + d_{\min}k_{\min})} + \frac{C_\psi D^{(u)}(\alpha)}{\log(1 + d_{\min}k_{\min})} \leq \frac{C_\psi D(\alpha)}{\log(1 + d_{\min}k_{\min})}.
\]

This follows from the final part of Theorem 8.5. \[ \square \]

**Remark 10.5.** Note that \( r' \geq 3 \) is only required for estimating the first derivative. To find the differentiability class of the Hausdorff dimension only \( r' \geq 2 \) is required.

**Corollary 10.6.** Let \( K(\alpha) \) be a real analytic billiard deformation (i.e. real analytic in both \( \alpha \) and the parameter \( u \)). Then \( D(\alpha) \) is a real analytic function of \( \alpha \).

**Proof.** Let \( K(\alpha) \) be a real analytic billiard deformation. Then clearly the functions \( F, G \) and \( h \) are all real analytic. By the implicit function theorem for analytic functions [12], \( u_j(\xi,\alpha) \) is real analytic for any periodic admissible sequence \( \xi \). To extend this to aperiodic sequences, we show that \( u_j(\xi,\alpha) \) is equal to its Taylor series at any point \( \alpha_0 \) and for any \( \xi \in \Sigma \). Recall from the proof of Corollary 6.6.2 that for each \( q \) there exists a sequence \( (\xi^{(n,q,k)})_k \) such that \( \lim_{k \to \infty} \xi^{(n,q,k)} = \xi \)

\[
\frac{d^n}{d\alpha^n} u_j(\xi,\alpha) = \lim_{k \to \infty} \frac{d^n}{d\alpha^n} u_j(\xi^{(n,q,k)},\alpha).
\]
Now using the definition of real analyticity, we have
\[ u_j(\xi, \alpha) = \lim_{k \to \infty} u_j(\xi^{(n_q,k)}, \alpha) \]
\[ = \lim_{k \to \infty} \sum_{q=0}^{\infty} \frac{(\alpha - \alpha_0)^q}{q!} \left( \frac{d^q}{d\alpha^q} u_j(\xi^{(n_q,k)}, \alpha) \bigg|_{\alpha=\alpha_0} \right) \]
\[ = \sum_{q=0}^{\infty} \frac{(\alpha - \alpha_0)^q}{q!} \left( \lim_{k \to \infty} \frac{d^q}{d\alpha^q} u_j(\xi^{(n_q,k)}, \alpha) \bigg|_{\alpha=\alpha_0} \right) \]
\[ = \sum_{q=0}^{\infty} \frac{(\alpha - \alpha_0)^q}{q!} \left( \frac{d^q}{d\alpha^q} u_j(\xi, \alpha) \bigg|_{\alpha=\alpha_0} \right). \]

Therefore \( u_j(\xi, \alpha) \) is real analytic in \( \alpha \). Now the quantities \( p, d, \kappa, \phi \) and \( \gamma \) are all easily shown to be real analytic. The recurrence relation for \( k_j \) can be used to show that \( k_j \) is real analytic, and therefore \( \psi_{\alpha, \zeta}(\xi) \) is real analytic. Since the pressure is a real analytic operator, it follows that \( D(\alpha) \) is real analytic.

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