Gluon contribution to the structure function $g_2(x, Q^2)$

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Abstract:

We calculate the one-loop twist-3 gluon contribution to the flavor-singlet structure function $g_2(x, Q^2)$ in polarized deep-inelastic scattering and find that it is dominated by the contribution of the three-gluon operator with the lowest anomalous dimension (for each moment $N$). The similar property was observed earlier for the nonsinglet distributions, although the reason is in our case different. The result is encouraging and suggests a simple evolution pattern of $g_2(x, Q^2)$ in analogy with the conventional description of twist-2 parton distributions.
1. Introduction

Twist-three parton distributions in the nucleon are attracting increasing interest as unique probes of quark-gluon correlations in hadrons. They have clear experimental signatures and give rise to various spin asymmetries in experiments with polarized beams and targets. Quantitative studies of such asymmetries are becoming possible with the increasing precision of experimental data at SLAC and RHIC, and can constitute an important part of the future spin physics program on high-luminosity accelerators like ELFE, eRHIC, etc.

The structure function $g_2(x, Q^2)$ in polarized deep inelastic scattering presents the classical example of a twist-3 observable and received considerable attention in the past. The experimental studies at SLAC [1, 2, 3] have confirmed theoretical expectations about the shape of $g_2(x, Q^2)$ and provided first evidence on the most interesting twist-3 contribution. On the theoretical side, a lot of effort was invested to understand the physical interpretation of twist-3 distributions (see e.g. [4, 5, 6] for the review of various aspects) and their scale dependence [7, 8, 9, 10, 11, 12, 13]. Conditions for the validity of the Burhardt-Cottingham (BC) sum rule [14]

$$\int_0^1 dx g_2(x) = 0$$

were discussed in detail [4, 15, 16, 17] and the second moment $\int_0^1 dx x^2 g_2(x)$ was estimated using QCD sum rules [18] and on the lattice [19].

In spite of the significant progress that has been achieved, understanding of the scale dependence of $g_2(x, Q^2)$ still poses an outstanding theoretical problem. To explain the difficulty, we remind that to the tree-level accuracy the structure function $g_2(x, Q^2)$ or, equivalently, $g_T(x, Q^2) = g_1(x, Q^2) + g_2(x, Q^2)$ is given by the quark-antiquark light-cone correlation function in a transversely polarized nucleon (see, e.g. [15])

$$g_T(x) = \frac{1}{2M} \int_{-\infty}^{\infty} d\lambda \frac{d\lambda}{2\pi} e^{i\lambda x} \langle p, s_\perp|\bar{q}(0)\gamma_5 q(\lambda n)|p, s_\perp\rangle, \quad (1.1)$$

where $p_\mu$ and $s_\mu$ are nucleon momentum and spin vectors, respectively, and we assumed that the nucleon is moving in the $z-$direction, $p_\mu = (p_+, p_-, 0_\perp)$, $p^2 = 2p_+p_- = M^2$ and $p \cdot s = 0$. The light-like vector $n$ is given by $n_\mu = (0_+, 1/p_+, 0_\perp)$ so that $n^2 = 0$, $(pn) = 1$ and the transverse direction is defined as orthogonal to the $(p, n)$ plane. For comparison, the leading twist-2 spin structure function is written as

$$g_1(x) = \int_{-\infty}^{\infty} d\lambda \frac{d\lambda}{2\pi} e^{i\lambda x} \langle p, s_\perp|\bar{q}(0)\gamma_5 q(\lambda n)|p, s_\perp\rangle. \quad (1.2)$$

In the parton model, $g_1(x)$ measures the quark helicity distribution in a longitudinally polarized nucleon. Such an interpretation can be made because the quark helicity operator $\Sigma_\mu$ commutes with the free-quark Dirac Hamiltonian $H = \alpha_z p_z$. On the contrary, the quark spin operator projected along the transverse direction $\Sigma_\perp$ does not commute with the Hamiltonian and thus there exists no energy eigenstate $|p_\perp\rangle$ such that $\Sigma_\perp|p_\perp\rangle = s_\perp|p_\perp\rangle$. The transverse spin of the nucleon cannot, therefore, be thought of as being composed of transverse spins of individual quark (gluon) constituents. The transverse spin average of

\footnote{Throughout the paper we shall use the following definition of the light-cone components $p_\pm = (p_0 \pm p_3)/\sqrt{2}$ and $p_\perp = (p_1, p_2)$. In addition, we shall not display the gauge factors connecting the quark fields and ensuring the gauge invariance of nonlocal light-cone operators.}
quarks in (1.1) that defines $g_T(x)$ is sensitive to the dynamics of quark-gluon interactions and does not have any probabilistic interpretation in terms of one-particle quark parton densities.

One possible way to see the relation of the transverse spin to gluonic degrees of freedom is to decompose the quark field operator in “good” (+) and “bad” (−) components $q(x) = q_+(x) + q_-(x)$ where $P_+ = \frac{1}{2} \gamma_- \gamma_+$ and $P_- = \frac{1}{2} \gamma_+ \gamma_-$ are the corresponding projection operators $[20]$. It is easy to check that the correlation function in (1.2) involves only good quark components, while in (1.1) necessarily one good and one bad components are involved. In the approach of $[20]$ only good field components correspond to genuine partonic degrees of freedom, while bad components are not dynamically independent and can be eliminated through the equations of motion in favor of good components and insertions of quark masses or gluon fields. An important point is that this relation is nonlocal and involves quark and gluon fields with different positions on the light-cone $[20]$: \[ q_-(\lambda n) = -i \int \frac{d\lambda'}{2\pi} \int \frac{d\nu}{2\nu} e^{-i\nu(\lambda-\lambda')} \not{D}_\bot (\lambda' n) q_+ (\lambda' n), \]

where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative.

Eq. (1.3) states that degrees of freedom associated with the bad component of the quark field in (1.1) are, in fact, those of one quark and one gluon. The structure function $g_T(x, Q^2)$ is, therefore, naturally related to the quark-antiquark-gluon correlation function in the nucleon. More precisely, $g_T(x)$ presents by itself only one special projection of this more general three-particle distribution $D(\xi_1, \xi_2, \xi_3)$ that depends, generally, on the momentum fractions $\xi_i$ carried by three partons. It is this special projection, $g_T(x)$, that can be measured in deep inelastic scattering with a transversely polarized target. On the other hand, the scale dependence of the quark-antiquark-gluon distribution function involves the “full” function $D(\xi_1, \xi_2, \xi_3)$ in a nontrivial way $[8, 21, 22]$ and the knowledge of one particular projection $g_T(x, Q_0^2)$ at a given value of $Q_0^2$ does not allow to predict $g_T(x, Q^2)$ at different momentum transfers: a DGLAP-type evolution equation for $g_T(x, Q^2)$ in QCD does not exist or, at least, is not warranted. The reason is simply that inclusive measurements in general do not provide complete information on the relevant three-particle parton correlation function $D(\xi_1, \xi_2, \xi_3)$.

From the phenomenological point of view such situation is not satisfactory since it would mean that one cannot relate results of the measurements of $g_2(x)$ at different values of $Q^2$ to one another without model assumptions. The theoretical challenge is, therefore, to find out whether the complicated pattern of quark-gluon correlations can be reduced to a few effective degrees of freedom. One has to look for meaningful approximations to the scale dependence that introduce a minimum amount of nonperturbative parameters.

In particular, it was found by Ali, Braun and Hiller (ABH) $[10]$ that the scale-dependence of the flavor-nonsinglet contribution to $g_T(x, Q^2)$ simplifies dramatically in the limit of large number of colors $N_c \to \infty$. To explain this result, it is convenient to use the language of the Operator Product Expansion (OPE), see Sect. 2 for more details. The statement of the OPE is that odd moments $n = 3, 5, \ldots$ of the structure function $g_2(x, Q^2)$ can be expanded in contributions of multiplicatively renormalized local twist-3
quark-antiquark-gluon operators\footnote{Here we neglect the twist-2 contribution to $g_2(x)$.}

\begin{equation}
\int_0^1 dx \, x^{n-1} g_2(x, Q^2) = \sum_{k=0}^{n-3} C_{n-3}^k \left( \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right)^{\gamma_{n-3}^k/b} \langle \langle O_N^k(\mu) \rangle \rangle,
\end{equation}

where $C_{n-3}^k$ are the coefficients and $\langle \langle O_N^k(\mu) \rangle \rangle$ reduced matrix elements normalized at the scale $\mu$; $\gamma_{n-3}^k$ are the corresponding anomalous dimensions that we assume are ordered with $k$: $\gamma_{n-3}^0 < \gamma_{n-3}^1 < \ldots < \gamma_{n-3}^{n-3}$ for each $n$, and $b = 11N_c/3 - 2n_f/3$. The exact analytic expression for the lowest anomalous dimension in the spectrum, $\gamma_{n-3}^0$, has been found in \cite{10} and it was also noticed that the coefficient functions of all other operators (with higher anomalous dimensions for each $n$) are suppressed by powers of $1/N_c^2$. Thus, to the stated $\mathcal{O}(1/N_c^2)$ accuracy, each moment of $g_2(x)$ involves a single nonperturbative parameter while the complicated degrees of freedom related to quark-antiquark-gluon correlations essentially decouple. The result can be reformulated as a DGLAP-type evolution equation

\begin{equation}
Q^2 \frac{d}{dQ^2} g_{NS}^N(x, Q^2) = \frac{\alpha_s}{4\pi} \int_x^1 dz \frac{d}{dz} P_{NS}(x/z) g_{NS}^N(z, Q^2),
\end{equation}

\begin{equation}
P_{NS}(z) = \left[ \frac{4C_F}{1-z} \right]_+ + \delta(1-z) \left[ C_F + \frac{1}{N_c} \left( 2 - \frac{\pi^2}{3} \right) \right] - 2C_F,
\end{equation}

where $C_F = (N_c^2 - 1)/(2N_c)$ and we have included the $1/N_c^2$ corrections calculated in \cite{13}.

The present paper is devoted to the extension of this analysis to the flavor-singlet sector in which case twist-3 three-gluon operators have to be included. We calculate the leading one-loop $\mathcal{O}(\alpha_s)$ gluon contribution to the coefficient function and examine its properties. We find that the one-loop coefficient function is such that it mainly picks up the contribution of the twist-3 three-gluon operator with the lowest anomalous dimension for each moment $N$. The dominance of the lowest three-gluon “state” is observed both for the logarithmic contribution $\sim \ln Q^2/\mu^2$ that reflects mixing with the quark-antiquark-gluon operators, and the constant term that gives rise to a “genuine” gluon contribution.

The result is very encouraging and allows to hope for the similar pattern of a simplified evolution that mainly involves a single quark-antiquark gluon and a single three-gluon parton distribution corresponding to the “trajectories” with the lowest anomalous dimension as important degrees of freedom, although the reason for such a simplification is different. The properties of these trajectories have been recently studied in \cite{21}.

The outline of the paper is as follows. Sect. 2 is introductory and reviews existing results on the OPE of the antisymmetric part of the T-product of two electromagnetic currents to twist-3 accuracy. The calculation of the coefficient function of twist-3 three-gluon operators is presented in Sect. 3 and its structure is elaborated upon in Sect. 4; in Sect. 5 we summarize. Technical details on the twist separation in gluon operators are presented in the Appendix.
2. The Operator Product Expansion

As well known, the hadronic tensor which appears in the description of deep inelastic scattering of polarized leptons on polarized nucleons, involves two structure functions

\[
W_{\mu\nu}^{(A)} = \frac{1}{p \cdot q} \varepsilon_{\mu\alpha\beta\gamma} q^\beta \left\{ s^\gamma g_1(x_B, Q^2) + \left[ s^\gamma - \frac{s \cdot q}{p \cdot q} \right] g_2(x_B, Q^2) \right\},
\]

where the nucleon spin vector is defined as \( s^\gamma = \tilde{u}(p, s) \gamma^\gamma u(p, s) \) is the nucleon spinor \( (\tilde{u}(p, s) u(p, s) = 2M, s^2 = -4M^2) \), and is related to the imaginary part of the Fourier-transform of the T-product of two electromagnetic currents, antisymmetrized over the Lorentz indices:

\[
W_{\mu\nu}^{(A)} = \frac{1}{\pi} \text{Im} T_{\mu\nu}^{(A)},
\]

\[
i T_{\mu\nu}^{(A)} = \frac{i}{2} \int d^4 x \ e^{iqx} \langle p, s | T \{ j_\mu(x/2) j_\nu(-x/2) - j_\nu(x/2) j_\mu(-x/2) \} | p, s \rangle.
\]

We are going to examine the light-cone expansion of (2.2) at \( x^2 \to 0 \) and write down the answer in terms of nonlocal light-cone operators of increasing twist, schematically

\[
\frac{i \varepsilon_{\mu\alpha\beta}}{16\pi^2} \frac{\partial}{\partial x^\alpha} \left\{ [C * O_\beta]^{tw-2} + [C * O_\beta]^{tw-3} + \text{(higher twists)} \right\},
\]

where \( C * O_\beta \) stands for the product (convolution) of the coefficient functions and operators of the corresponding twist. This expression is explicitly \( U(1) \)-gauge invariant, i.e. \( \partial/\partial x_\mu T \{ j_\mu(x) j_\nu(-x) \} = 0 \).

2.1. Leading-order results

To the leading order in the strong coupling, the OPE of the antisymmetric part of the T-product in (2.3) can be written in a compact form as

\[
[C * O_\beta]^{tw-2} = \frac{x_\beta}{x^4} \sum_{q=u,d,s,...} e_q^2 \int_0^1 du \ \bar{q}(ux) \not{\gamma} \gamma_5 q(-ux) + (x \to -x),
\]

\[
[C * O_\beta]^{tw-3} = \frac{i}{2x^2} \sum_{q=u,d,s,...} e_q^2 \int_0^1 du \int_{-u}^u \left[ (u + v) S_\beta^+(u, v, -u) + (u - v) S_\beta^-(u, v, -u) \right] \not{\gamma} q(-ux) + (x \to -x),
\]

where notation was introduced for light-cone nonlocal quark-gluon operators

\[
S_\mu^+(a, b, c) = \frac{1}{2} \tilde{q}(ax)[ig \tilde{G}_{\mu\nu}(bx) \pm g G_{\mu\nu}(bx) \gamma_5] x^\nu \not{q}(cx).
\]

The dual gluon strength tensor is defined as \( \tilde{G}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} G^{\alpha\beta} \) and we use the conventions \( \gamma_5 = i\gamma_0 \gamma_2 \gamma_3 \) and \( \varepsilon^{0123} = 1 \), see [24]. To save space, here and below we do not show the gauge factors connecting the quark (gluon) fields:

\[
[a, bx] = P \exp \left[ ig \int_{a}^{b} du \ x_\mu A^\mu(ux) \right].
\]
Figure 1: Leading-order twist-3 contributions to the OPE of the T-product of electromagnetic currents

The full twist-3 contribution (2.4) to (2.3) is assembled from the Feynman diagrams shown in Fig. 1. The individual contributions of the two diagrams in Fig. 1 correspond to the two possibilities to apply the derivative in (2.3): $\partial/\partial x^\alpha (S^\pm/\beta x^2) = -2x_\alpha S^\pm/\beta x^4 + 1/x^2 \partial S^\pm/\beta x^\alpha$. The term $\sim 1/x^4$ presents the contribution of the diagram in Fig. 1a, rewritten in terms of quark-gluon operators using equations of motion. The contribution $\sim 1/x^2$ corresponds to the diagram with gluon emission from the hard propagator in Fig. 1b and is necessitated by gauge invariance of (2.3).

Going over to matrix elements, we introduce the usual quark helicity distributions $\Delta q(x_B) = q^\uparrow(x_B) - q^\downarrow(x_B)$

$$\langle p, s | \bar{q}(x) \not{x} \gamma_5 q(-x) | p, s \rangle = (s x) \int_{-1}^1 d\xi e^{2i\xi p x} \Delta q(\xi, \mu^2),$$

where $\mu$ refers to the normalization scale of the operator in the l.h.s. and positive (negative) $\xi$ correspond to the contribution of quarks (antiquarks) $\Delta q(x_B) = q^\uparrow(x_B) - q^\downarrow(x_B)$, $\Delta \bar{q}(x_B) \equiv \Delta \bar{\bar{q}}(x_B) = \bar{q}^\downarrow(x_B) - \bar{q}^\uparrow(x_B)$, respectively.

Similarly, we define the twist-3 quark-antiquark-gluon parton correlation functions as (cf. [25])

$$\langle p, s | S^\pm_\mu(u, v, -u) | p, s \rangle = 2i(px) | s_\mu(px) - p_\mu(sx) | \int_{-1}^1 D\xi e^{ipx[\xi_1 u + \xi_2 v - \xi_3 u]} D^\pm_q(\xi_1, \xi_2, \xi_3)$$

where

$$\int_{-1}^1 D\xi \equiv \int_{-1}^1 d\xi_1 d\xi_2 d\xi_3 \delta(\xi_1 + \xi_2 + \xi_3).$$

The correlation functions $D^\pm_q(\xi_i)$ have the following symmetry property:

$$D^\pm_q(\xi_1, \xi_2, \xi_3) = (D^\mp_q)^*(-\xi_3, -\xi_2, -\xi_1)$$

They are in general complex functions, but the imaginary parts do not contribute to the structure functions and can be omitted [25].

We always imply that $x^2$ can be put to zero in the operator matrix elements. Under this condition, the nonlocal operators $S^\pm_\beta$ still contain a superposition of twist-3 and twist-4 terms [26], with the twist-4 terms being explicitly proportional to $x_\beta$. The easiest way to separate the genuine twist-3 contribution is to take the transverse projection $S^\pm_\beta \to S^\perp_\beta$, see also [3, 26] for explicit expressions.
For further use, it is convenient to introduce a separate notation for the nucleon matrix element of the specific combination of quark-antiquark-gluon operators entering the OPE in (2.4):

\[
\int_{-u}^{u} dv \langle p, s | [(u + v) S^+_{\mu}(u, v, -u) + (u - v) S^-_{\mu}(u, v, -u)] | p, s \rangle =
\]

\[
= -i \left[ s_{\mu} - p_{\mu} \left( \frac{sx}{px} \right) \right] \int_{-1}^{1} d\xi e^{2i\xi u_{\mu}} \Delta q_{T}(\xi, \mu^2). \tag{2.11}
\]

The function \( \Delta q_{T}(x_B, Q^2) \) will play an important rôle in what follows. It describes the momentum fraction distribution inside the nucleon of the transverse spin and has the same support property as the parton distribution in (2.4). However, in contrast with (2.7), it does not have any probabilistic interpretation but can rather be expressed through the more general three parton correlation functions \( D_q^{ij}(\xi_1, \xi_2, \xi_3) \), integrating out the dependence on the gluon momentum fraction \( \xi_3 \). Making a Fourier transformation of (2.5), taking imaginary part and comparing with the definition of structure functions in (2.4) one obtains

\[
g_1(x_B, Q^2) = \frac{1}{2} \sum_q e_q^2 \left[ \Delta q(x_B, \mu^2 = Q^2) + \Delta q(-x_B, \mu^2 = Q^2) \right], \tag{2.12}
\]

where \( x_B = Q^2/(2pq) \) is the Bjorken variable, and \( 2 \leq q \leq 10 \)

\[
g_2(x_B, Q^2) = g_{2WW}(x_B, Q^2) + \frac{1}{2} \sum_q e_q^2 \int_{x_B}^{1} \frac{dy}{y} \left[ \Delta q_T(y, Q^2) + \Delta q_T(-y, Q^2) \right] \tag{2.13}
\]

where

\[
g_{2WW}(x_B, Q^2) = -g_1(x_B, Q^2) + \int_{x_B}^{1} \frac{dy}{y} g_1(y, Q^2) \tag{2.14}
\]

is the the familiar Wandzura-Wilczek contribution \( [27] \) and will not be considered from now on.

Going over to the moments in (2.13) we obtain

\[
\int_0^1 dx_B x_B^{n-1} g_2(x_B, Q^2) = \frac{1}{2n} \sum_q e_q^2 \int_0^1 dx_B x_B^{n-1} \left[ \Delta q_T(x_B, Q^2) + \Delta q_T(-x_B, Q^2) \right]. \tag{2.15}
\]

For odd \( n \geq 3 \) the relevant integrals of the transverse spin quark distributions \( \Delta q_T \) are given by the reduced matrix elements of local twist-3 operators that arise via the Tailor-expansion of the nonlocal operators (2.5) at short distances \( x_\mu \to 0 \):

\[
[S^\pm_\mu]^k_N = \frac{1}{2} \tilde{q}(\vec{D} \cdot x)^k [ig\tilde{G}_{\mu\nu} \pm gG_{\mu\nu}\gamma_5] \not{x}^\nu (\vec{D} \cdot x)^{N-k}q. \tag{2.16}
\]

According to (2.8), the reduced matrix elements \( \langle \ldots \rangle \) of these operators

\[
\langle p, s | [S^\pm_\mu]^k_N | p, s \rangle = 2(ipx)^{N+1} \left[ s_{\mu}(px) - p_{\mu}(sx) \right] \langle [S^\pm_\mu]^k_N \rangle \tag{2.17}
\]

are equal to moments of the quark-antiquark-gluon three-particle distribution amplitude

\[
\langle [S^\pm_\mu]^k_N \rangle = \int_{-1}^{1} d\xi \xi^k \xi^{N-k} D_q^\pm(\xi_i). \tag{2.18}
\]
The symmetry relation (2.10) implies
\[ \langle [S^\pm]_N^k \rangle^* = (-1)^N \langle [S^\mp]_N^{-k} \rangle. \] (2.19)

Expanding Eq. (2.11) at short distances we obtain
\[ \int_{-1}^{1} d\xi \xi^{N+2} \Delta q_T(\xi) = 2 \sum_{k=0}^{N} (-1)^{N-k} [(k+1) \langle [S^-]_N^k \rangle] + (N - k + 1) \langle [S^+]_N^k \rangle] \]
\[ = 4 \sum_{k=0}^{N} (-1)^{N-k} (k+1) \text{Re} \langle [S^-]_N^k \rangle \]
\[ = 4 \int_{-1}^{1} D\xi \text{Re} D_q^\rightarrow (\xi) \frac{\partial}{\partial \xi_1} \frac{1}{\xi_1 + \xi_3} \left[ \xi_1^{N+2} - (-\xi_3)^{N+2} \right]. \] (2.20)

The last equality can also be obtained directly from the definition in (2.8), (2.11). The following comments are in order.

We note the function \( \Delta q_T(\xi) \) takes real values. The expression in the last line of (2.20) can be used for an analytic continuation to \( N \to -2 \) and remains finite provided that the corresponding integral of the \( D^- \) function converges. This convergence, thus, presents a necessary condition for the validity of the BC sum rule at \( N = -2 \).

At \( N = -1 \) an absence of a local twist-3 operator (2.16) with dimension four implies the constraint
\[ \int_{-1}^{1} d\xi \xi \Delta q_T(\xi) = \int_{0}^{1} dx_B x_B [\Delta q_T(x_B) - \Delta q_T(-x_B)] = 0. \] (2.21)

This relation should be compared with the first moment of \( g_2(x) \) given by (2.13) that involves the combination of the same distributions but with a different C-parity
\[ \int_{0}^{1} dx_B x_B g_2(x_B, Q^2) = \frac{1}{4} \sum_q e_q^2 \int_{0}^{1} dx_B x_B [\Delta q_T(x_B) + \Delta q_T(-x_B)]. \] (2.22)

Vanishing of this integral (known as Efremov-Leader-Teryaev sum rule [28, 3]) is, therefore, not warranted by the OPE, although its numerical value can be small since the r.h.s. does not receive contribution from the valence quarks.

### 2.2. The scale dependence

The dependence of the structure function \( g_2(x, Q^2) \) on \( Q^2 \) is driven by the scale dependence of the distribution functions \( \Delta q_T(x, \mu^2) \). Going over to moments (2.22), it corresponds to the renormalization-group scale dependence of the local operators \( [S^\pm]_N^k \). Similar to the familiar case of the helicity distributions \( \Delta q(x_B) \), one has to distinguish between the components with different flavor symmetry as they have a different scaling behavior. For instance, the flavor decomposition of the \( u \)-quark distribution looks like
\[
\Delta u_T(x_B) = \frac{1}{2} (\Delta u_T - \Delta d_T)(x_B) + \frac{1}{6} (\Delta u_T + \Delta d_T - 2\Delta s_T)(x_B) \\
+ \frac{1}{3} (\Delta u_T + \Delta d_T + \Delta s_T)(x_B). \] (2.23)
Renormalization of flavor-nonsinglet contributions, $\Delta q^\text{NS}$, given by either $\Delta u - \Delta d$, or $\Delta u + \Delta d - 2\Delta s$ is simpler since they do not mix with gluons. Still, the number of contributing operators rises linearly with $N$. By an explicit calculation one obtains, for the two lowest moments \[ 3, 8, 10, 11 \]

\[
\frac{1}{4} \int_{-1}^{1} d\xi \xi^2 \Delta q_T^\text{NS}(\xi, Q^2) = L^{\gamma_0^0/b} \langle \langle S_0^0(\mu^2) \rangle \rangle,
\]

\[
\frac{1}{4} \int_{-1}^{1} d\xi \xi^4 \Delta q_T^\text{NS}(\xi, Q^2) = L^{\gamma_2^0/b} \left[ 0.807 \langle \langle S_0^0(\mu^2) \rangle \rangle - 2.320 \langle \langle S_1^1(\mu^2) \rangle \rangle + 2.894 \langle \langle S_2^2(\mu^2) \rangle \rangle \right]
+ L^{\gamma_2^1/b} \left[ 0.028 \langle \langle S_0^0(\mu^2) \rangle \rangle - 0.014 \langle \langle S_1^1(\mu^2) \rangle \rangle - 0.026 \langle \langle S_2^2(\mu^2) \rangle \rangle \right]
+ L^{\gamma_2^2/b} \left[ 0.165 \langle \langle S_0^0(\mu^2) \rangle \rangle + 0.334 \langle \langle S_1^1(\mu^2) \rangle \rangle + 0.132 \langle \langle S_2^2(\mu^2) \rangle \rangle \right]
\]

where we used $\langle \langle S_N^k(\mu^2) \rangle \rangle \equiv \text{Re} \langle \langle S_N^{-1}_k(\mu^2) \rangle \rangle$ as a shorthand and $L = \alpha_s(Q^2)/\alpha_s(\mu^2)$. Anomalous dimensions are equal to \[ 10 \]

\[
\gamma_0^0 = 8.5, \quad \gamma_2^0 = 10.89, \quad \gamma_2^1 = 13.71, \quad \gamma_2^2 = 16.15.
\]

Eq. (2.24) illustrates the main difficulty: since the number of contributing operators proliferates with $N$, so does the number of independent nonperturbative parameters $\langle \langle S_N^k(\mu^2) \rangle \rangle$. On the other hand, Eq. (2.24) reveals a remarkable pattern: coefficients in front of the operators with higher anomalous dimensions are much smaller than those with the lowest anomalous dimension. This structure is not accidental, but related to a dramatic simplification of the renormalization-group evolution of flavor-nonsinglet operators in the large–$N_c$ limit. As was found in \[ 10 \], the small coefficients in (2.24) are in fact suppressed by powers of $1/N_c^2$ and one obtains in the limit $N_c \to \infty$

\[
\int_{-1}^{1} d\xi \xi^2 \Delta q_T^\text{NS}(\xi, Q^2) \overset{N_c \to \infty}{=} L^{\gamma_2^0/b} \left[ \langle \langle S_0^0(\mu^2) \rangle \rangle - 2\langle \langle S_1^1(\mu^2) \rangle \rangle + 3\langle \langle S_2^2(\mu^2) \rangle \rangle \right]
= L^{\gamma_2^0/b} \int_{-1}^{1} d\xi \xi^2 \Delta q_T^\text{NS}(\xi, \mu^2),
\]

so that the scale evolution of the second moment of $\Delta q_T^\text{NS}$ involves the same moment. The similar phenomenon takes place for arbitrary $N$:

\[
\int_{-1}^{1} d\xi \xi^N \Delta q_T^\text{NS}(\xi, Q^2) \overset{N_c \to \infty}{=} L^{\gamma_N^0/b} \int_{-1}^{1} d\xi \xi^N \Delta q_T^\text{NS}(\xi, \mu^2).
\]

Here $\gamma_N^\text{NS}$ is the lowest anomalous dimension in the spectrum of flavor-nonsinglet twist-3 operators. It is known analytically in the large–$N_c$ limit \[ 10 \] and the $1/N_c^2$ corrections have been recently calculated using the large–$N$ expansion in \[ 13 \]:

\[
\gamma_N^\text{NS} = N_c \left( 2\psi(N + 3) + 2\gamma_E + \frac{1}{N + 3} - \frac{1}{2} \right)
- \frac{2}{N_c} \left( \ln(N + 3) + \gamma_E + \frac{3}{4} - \frac{\pi^2}{6} + \mathcal{O}(1/N^2) \right) + \mathcal{O}\left( \frac{1}{N_c^4} \right).
\]
Figure 2: Leading-order twist-3 gluon contribution to the OPE of the T-product of electromagnetic currents.

Here $\psi(x) = d \ln \Gamma(x)/dx$ stands for the Euler $\psi$-function. Eq. (2.27) together with (2.28) are equivalent to the DGLAP evolution equation in (1.5).

The scale dependence of the flavor-singlet distribution $\Delta q^S_T = \Delta u_T + \Delta d_T + \Delta s_T$ differs from the above in three aspects. First, the mixing matrices of the relevant quark-antiquark-gluon operators receive extra terms related to the possibility of quark-antiquark annihilation. Second, they mix in addition with an entirely new and equally big set of three-gluon operators. Third, the three-gluon operators themselves contribute to the OPE of the T-product of the electromagnetic currents starting order $\alpha_s$. Concerns have been raised (see e.g. [29]) that in particular the last contribution does not have a simple structure and will spoil any ABH–type approximation in the singlet case. We begin, therefore, with the corresponding calculation.

3. The gluon contribution to the structure functions

The leading-order gluon contribution to the T-product of two electromagnetic currents in (2.2) is described by the box diagram shown in Fig. 2. Its calculation can be easily done using the background field approach of Ref. [9]. Namely, considering outgoing gluons as classical background Yang-Mills fields we calculate the box diagram replacing the free quark propagators by propagators in an external fields. Then, the antisymmetric part of the T-product in (2.2) is given by

$$\Pi^{(A)}_{\mu\nu}(x,-x) = \frac{1}{2} T \{ j_\mu(x) j_\nu(-x) \} - (\mu \leftrightarrow \nu)$$

$$= \frac{1}{2} \sum_{q=u,d,s,...} e_q^2 \text{Tr} \left[ \gamma_\mu S(x,-x) \gamma_\nu S(-x,x) \right] - (\mu \leftrightarrow \nu) ,$$

(3.1)
and $k$ second is singular in the limit of $x^2 \to 0$ the propagator $S(x, -x)$ exhibits light-cone singularities that one handles using the dimensional regularization with $d = 4 - 2\epsilon$ and $\epsilon < 0$. Then, expanding $S(x, -x)$ in powers of the deviation from the light-cone and retaining contributions of gluon operators up to twist-3 we find
\[
(4\pi)^{d/2} S(x, -x) = -\frac{\Gamma(d/2)}{(-x^2)^{d/2}} S_0 - \frac{\Gamma(d/2 - 1)}{(-x^2)^{d/2 - 1}} S_1 - \frac{\Gamma(d/2 - 2)}{(-x^2)^{d/2 - 2}} S_2 + \ldots
\]  
(3.3)

First two terms in the light-cone expansion of the propagator are known
\[
S_0(x, -x) = \not{x},
\]
\[
S_1(x, -x) = \frac{1}{2} \int_{-1}^{1} du \left\{ [x, ux] \left[ g\tilde{G}_{\alpha\beta}(ux)\gamma_\alpha\gamma_5 - iugG_{\alpha\beta}(ux)\gamma_\alpha \right][ux, -x] + \not{u} \int_{-1}^{u} dv (1 - u)(1 + v)[x, ux]gG_{\alpha\beta}(ux)[ux, vx]gG_{\alpha\beta}(vx)[vx, -x] \right\}
\]  
(3.4)

and the further ones can be calculated using the technique described in Appendix A of [3]. Here and below we use a shorthand notation $G_{\alpha\beta} = G^a_{\alpha\beta}x^a t^a$ with $t^a$ being the generators of fundamental (quark) representation of the $SU(N_c)$.

Omitting the disconnected (gluon field independent) contribution $\sim 1/(-x^2)^d$, one obtains
\[
\Pi^{(A)}_{\mu\nu}(x, -x) = \sum_{q=u,d,s,...} e_q^2 \frac{\Gamma^2(d/2 - 1)}{4\pi^d(-x^2)^{d-2}} \left[ A_{\mu\nu}(x, -x) + \frac{d - 2}{d - 4} B_{\mu\nu}(x, -x) \right] + \mathcal{O}\left(\frac{1}{(-x^2)^{d-3}}\right),
\]  
(3.5)

where
\[
A_{\mu\nu} = \frac{1}{2} \text{Tr} \left[ \gamma_\mu S_1(x, -x)\gamma_\nu S_1(-x, x) \right] + (\mu \leftrightarrow \nu),
\]
\[
B_{\mu\nu} = \frac{1}{2} \left\{ \text{Tr} \left[ \gamma_\mu S_2(x, -x)\gamma_\nu S_0(-x, x) \right] + \text{Tr} \left[ \gamma_\mu S_0(x, -x)\gamma_\nu S_2(-x, x) \right] \right\} - (\mu \leftrightarrow \nu).
\]  
(3.6)

Several comments are in order. First, the expression in (3.3) defines the most singular, $\sim 1/x^4$ as $d \to 4$, contribution to the light-cone expansion of the T-product (2.3). This contribution alone suffices to determine the coefficient functions of the twist-2 and twist-3 gluon operators since less singular contributions of the same twists can be uniquely restored from matching (3.3) into the general $U(1)$-gauge invariant expression (2.3), cf. the discussion in Sect. 2.1. Second, notice that the first term in (3.3) is analytic and the second is singular in the limit $d \to 4$. These two terms correspond to the two distinct integration regions in the quark momentum in the loop (see Fig. 2) $k^2 \sim 1/(-x^2) \sim Q^2$ and $k^2 \ll Q^2$, respectively. The first term, coming from large momenta, determines the one-loop $\mathcal{O}(\alpha_s)$ coefficient function of gluon operators at a hard scale of order $Q$, while
the second term will be interpreted as a tree-level quark coefficient function times the one-loop evolution (mixing) into gluons. Finally, one can convince oneself that the traces in (3.3) can be calculated in dimension $d = 4$. This is obvious for the first term, and can be shown for the second, the reason being that calculation of diagrams of the type shown in Fig. 2 does not involve contraction of Lorentz indices of $\gamma$-matrices.

Calculating (3.5) we shall assume the translation invariance of $\Pi^{(A)}(x, -x)$ along the light-cone. In addition, we shall impose the equations of motion for gluon fields, $[D^\mu, G_{\mu\nu}(x)] = 0$, which amounts to putting external gluons in the box diagram (see Fig. 2) on their mass-shell. Then, using the explicit expression for $S_1(x, -x)$ given in (3.4) one obtains

$$A_{\mu\nu}(x, -x) = 8g^2 x_\mu \int_0^1 du u \bar{u} \text{Tr} \left\{ G_{x\xi}(ux)G_{\nu\zeta}(-ux) - G_{x\xi}(-ux)G_{\nu\zeta}(ux) \right\}$$

$$+ 4ig^2 x_\mu \int_0^1 du \bar{u}^2 \int_{-u}^u dv \left[ \left( 1 + v - \frac{\bar{u}}{3} \right) O_\nu(u, -u, v) + \left( 1 - v - \frac{\bar{u}}{3} \right) O_\nu(v, u, -u) \right] + \frac{2}{3} \bar{u}O_\nu(u, v, -u) - (\mu \leftrightarrow \nu),$$

(3.7)

where we used $\bar{u} \equiv 1 - u$ etc. and introduced a notation

$$O_\nu(u, v, t) = g \text{Tr} \left\{ G_{x\alpha}(ux), G_{x\beta}(vx) \right\} G_{x\gamma}(tx) = \frac{ig}{2} f^{abc} G_{x\alpha}^a(ux) G_{x\beta}^b(vx) G_{x\gamma}^c(tx).$$

(3.8)

In the last relation the gauge factors between the gluon fields in the adjoint representation are implied.

The calculation of the second, singular contribution in (3.5) is more tedious. After considerable algebra we obtain, however, an equally simple expression

$$B_{\mu\nu}(x, -x) = 16g^2 x_\mu \int_0^1 du u \bar{u} \text{Tr} \left\{ G_{x\mu}(ux)G_{x\nu}(-ux) \right\}$$

$$- 4ig^2 x_\mu \int_0^1 du \bar{u}^2 \int_{-u}^u dv \left[ \left( 3 + v - \frac{5}{3} \bar{u} \right) O_\nu(u, -u, v) + \left( 3 - v - \frac{5}{3} \bar{u} \right) O_\nu(v, u, -u) \right] + \frac{2}{3} (2 + u) O_\nu(u, v, -u) - (\mu \leftrightarrow \nu).$$

(3.9)

Substituting Eqs. (3.8) and (3.9) into (3.5) we finally obtain the leading order expression for the antisymmetric part of the T-product of electromagnetic currents that takes into account both twist-2 and twist-3 contributions. However, in order to match (3.6) into the general OPE form (2.3) we have to separate the different twists.

### 3.1. Twist separation

The two-gluon operators in (3.7) and (3.9) contain, generically, contributions of both twist-2 and twist-3. The separation of twists corresponds in this case to the separation of the terms of different symmetry and can be done using the trick described in [9]. The

\[
\text{That is, neglect contributions of the operators containing total derivatives that have vanishing forward matrix elements.}
\]
Substitution of this relation into (3.7) yields

$$\left[ \text{Tr} \left\{ G_{x \xi}(x) G_{\nu \xi}(-x) \right\} \right]^{tw-2} = \int_0^1 du \frac{\partial}{\partial x_{\nu}} \text{Tr} \left\{ G_{x \xi}(ux) G_{x \xi}(-ux) \right\}. \quad (3.10)$$

Similarly, neglecting the irrelevant operators proportional to the equations of motion and total derivatives, one obtains the twist-3 contribution as

$$\left[ \text{Tr} \left\{ G_{x \xi}(x) G_{\nu \xi}(-x) \right\} \right]^{tw-3} = \int_0^1 du \int_{-u}^u dv \text{Tr} \left\{ (1 + u^2) [G_{x \xi}(ux), G_{x \xi}(vx)] G_{x \xi}(-ux) \right\}.$$ 

Applying (3.10) we find that the twist-2 two-gluon contribution to (3.7) in fact cancels out, and the remaining twist-3 part can be rewritten using (3.11) as

$$\left[ \text{Tr} \left\{ G_{x \xi}(ux) G_{\nu \xi}(-ux) - G_{x \xi}(-ux) G_{\nu \xi}(ux) \right\} \right]^{tw-3} = -2i \int_0^u ds \int_{-u}^u dv \left\{ O_{\nu}(s, -s, t) + O_{\nu}(s, t, -s) + O_{\nu}(t, s, -s) \right\}. \quad (3.12)$$

Substitution of this relation into (3.7) yields

$$[A_{\mu \nu}]^{tw-2} = 0,$$

$$[A_{\mu \nu}]^{tw-3} = -4ig^2 x_{\mu} \int_0^1 du \tilde{u}^2 \int_{-u}^u dv \left\{ (u - v) O_{\nu}(u, -u, v) + 2u O_{\nu}(u, v, -u) + (u + v) O_{\nu}(v, u, -u) \right\} - (\mu \leftrightarrow \nu). \quad (3.13)$$

The separation of twists in (3.9) is equally simple. To this end we note that (in dimension $d = 4$)

$$G_{x \mu}(ux) G_{x \nu}(-ux) - G_{x \nu}(ux) G_{x \mu}(-ux) = \varepsilon_{\mu \rho \sigma \alpha} x^\sigma G_{x \alpha}(ux) \tilde{G}_{\rho \alpha}(-ux) \quad (3.14)$$

and the relations (3.10) – (3.11) remain true to the claimed accuracy if one of the gluon strength-tensors is substituted by its dual counterpart. We obtain

$$[B_{\mu \nu}]^{tw-2} = -16g^2 \varepsilon_{\mu \rho \sigma \alpha} x^\sigma \frac{\partial}{\partial x_{\rho}} \int_0^1 du \left( u \ln u + u\tilde{u} \right) \text{Tr} \left\{ G_{x \xi}(ux) \tilde{G}_{x \xi}(-ux) \right\},$$

$$[B_{\mu \nu}]^{tw-3} = 16ig^2 x_{\mu} \int_0^1 du \int_{-u}^u dv \left\{ (\tilde{u}u + \frac{1}{4} \tilde{u}^2 + u \ln u) \left[ v O_{\nu}(v, u, -u) + u O_{\nu}(u, v, -u) - v O_{\nu}(u, -u, v) \right] - \frac{1}{12} \tilde{u}^2 (u + 2) \left[ O_{\nu}(u, -u, v) + O_{\nu}(u, v, -u) + O_{\nu}(v, u, -u) \right] \right\} - (\mu \leftrightarrow \nu). \quad (3.15)$$
3.2. Twist-2: Results

Substituting (3.15), (3.13) into (3.5), subtracting the (collinear) singularities in the $\overline{\text{MS}}$ scheme\(^5\) and combining with the leading-order result in (2.4), we obtain the twist-2 contribution

\[
[C \ast O_\beta]^{\text{tw}-2} = \frac{x_\beta}{x^4} \sum_{q=u,d,s,...} e_q^2 \int_0^1 du \left\{ \tilde{q}(ux) \hat{x} \gamma_5 q(-ux) + (x \to -x) \right\} \mu_{\overline{\text{MS}}}^2
\]

\[
+ \frac{4\alpha_s}{\pi} \left( \ln(-x^2 \mu_{\overline{\text{MS}}}^2) + 2\gamma_E \right) (u \ln u + u\bar{u}) \text{Tr} \left\{ G_{x\xi}(ux) \tilde{G}_{x\xi}(-ux) \right\},
\]

where the subscript $[\ldots]_{\mu^2}$ indicates the normalization scale of the operator. Note simplicity of the answer: the entire gluon contribution can be eliminated by choosing the proper scale of the quark operator $\mu_{\overline{\text{MS}}}^2 = 1/(-x^2 e^{2\gamma_E})$. This property is lost in the momentum space since after the Fourier transformation contributions of different light-cone separations get mixed.

Going over to the matrix elements, we introduce the usual gluon helicity distribution \(30, 31\)

\[
\langle p, s | \text{Tr} \left\{ G_{x\alpha}(x) \tilde{G}_{x\alpha}(-x) \right\} | p, s \rangle = \frac{i}{4} (sx)(px) \int_{-1}^{1} d\xi e^{2i\xi px} \xi \Delta g(\xi, \mu^2).
\]

Note that $\Delta g(\xi) = \Delta g(-\xi)$.

Moments of the structure functions (2.3) are obtained by the expansion of the T-product (2.2) in momentum space in powers of $\omega = -2(pq)/q^2, Q^2 = -q^2$ in the unphysical region $\omega \to 0$, and matching to the corresponding expansion in terms of structure functions:

\[
T_{\mu\nu}^{(A)} = -4\varepsilon_{\mu\nu\alpha\beta} \frac{q_\alpha}{q^2} \sum_{n=1,3,...} \omega^{n-1} \left\{ s^\beta g_1(n, Q^2) + \left[ s^\beta - \frac{\hat{s} \cdot q}{p \cdot q} p^\beta \right] g_2(n, Q^2) \right\},
\]

where the moments are defined as (for any function $f$)

\[
f(n, Q^2) = \int_0^1 dx_B x_B^{n-1} f(x_B, Q^2).
\]

Taking the Fourier transform of (2.3) by using (3.16) and matching the obtained expression into (3.18) we obtain after some algebra

\[
g_1(n, Q^2) = \frac{1}{2} \sum_{q=u,d,s,...} e_q^2 \left\{ \Delta q(n, \mu_{\overline{\text{MS}}}^2) + \Delta \bar{q}(n, \mu_{\overline{\text{MS}}}^2) \right\}
\]

\[
+ \frac{\alpha_s}{2\pi} \frac{n-1}{n(n+1)} \Delta q(n, \mu^2) \left[ \ln \frac{Q^2}{\mu_{\overline{\text{MS}}}^2} - \psi(n) - 1 - \gamma_E \right],
\]

\[
[g_2(n, Q^2)]^{\text{tw}-2} = -\frac{n-1}{n} g_1(n, Q^2).
\]

\(^5\)Note that the (non-renormalized) tree-level coefficient function in coordinate space contains a not trivial $d$–dependence corresponding to the quark propagator $\Gamma(d/2) \hat{x}/(-4\pi x^2)^{d/2}$. Making the $\overline{\text{MS}}$–subtraction in coordinate space one has to keep this factor in dimension $d$ in the counter-term. Alternatively, one can make the Fourier transform first, and then subtract the divergencies in the usual way. One can check that the renormalized coefficient functions obtained in both ways are indeed related to each other by a 4-dimensional Fourier transformation.
in accord with the Wandzura-Wilczek relation (2.14). Comparing the first expression in (3.20) with the general expression
\[
g_1(n, Q^2) = \frac{1}{2} \sum_{q=u,d,s,...} e_q^2 \left\{ \Delta g(n, \mu^2) + \Delta \bar{g}(n, \mu^2) + \left[ \gamma_{gg}(n) \ln \frac{Q^2}{\mu^2} + C_g(n) \right] \Delta g(n; \mu^2) + \ldots \right\}
\]
(3.21)
we find the anomalous dimension and the gluon coefficient function as
\[
\gamma_{gg}(n) = \frac{\alpha_s}{2 \pi} \frac{n - 1}{n(n + 1)},
\]
\[
C_g(n) = -\frac{\alpha_s}{2 \pi} \frac{n - 1}{n(n + 1)} \left[ \psi(n) + 1 + \gamma_E \right].
\]
(3.22)
Going over from the moments to the momentum fraction representation, \(\gamma_{gg}(n) = \int_0^1 dx x^{n-1} P_{gg}(x)\) and \(C_g(n) = \int_0^1 dx x^{n-1} C_g(x)\), we get
\[
P_{gg}(x) = \frac{\alpha_s}{2 \pi} (2x - 1),
\]
\[
C_g(x) = -\frac{\alpha_s}{2 \pi} \left[ (2x - 1) \ln \frac{x}{1 - x} + 4x - 3 \right].
\]
(3.23)
Expressions in (3.22) and (3.23) are in agreement with the well-known results, see e.g. [32].

3.3. Twist-3: Results

The twist-3 contribution to the T-product (3.3) comes from (3.13) and (3.15). To cast it into the \(U(1)\)-gauge invariant form (2.3) we notice that
\[
x_\mu O_\nu(u, v, t) - x_\nu O_\mu(u, v, t) = \varepsilon_{\mu \rho \sigma \beta} x^\rho \tilde{O}_\beta(u, v, t) + \mathcal{O}(x^2),
\]
(3.24)
where the operators \(O_\nu\) were defined in (3.8) and
\[
\tilde{O}_\beta(u, v, t) = \frac{i g}{2 \pi} \int \frac{d^4k}{(2\pi)^4} G_{x \alpha}^a(k) \tilde{G}_{x \beta}^b(k) G^c_{x \alpha}(k) .
\]
(3.25)
Then, combining together Eqs. (3.13) and (3.15), subtracting the collinear singularities in the \(\overline{\text{MS}}\) scheme (see previous footnote) and comparing with the general structure of the OPE in (2.3) we obtain:
\[
[C \ast O_\beta]^{\text{tw-3}} = \frac{i}{2x^2} \sum_{q=u,d,s,...} e_q^2 \int_0^1 du \int_{-u}^u dv \left\{ (u + v) S_\beta(u, v, -u) + (u - v) S_\beta(-u, v, u) + u^2 \alpha_s \frac{2}{\pi} \left[ (u + v) \tilde{O}_\beta(u, v, -u) + 2u \tilde{O}_\beta(u, v, -u) + (u - v) \tilde{O}_\beta(u, -u, v) \right] \right. \]
\[
+ \frac{4\alpha_s}{\pi} \left( \ln(-x^2 \mu^2_{\overline{\text{MS}}}) + 2\gamma_E + 1 \right) \left[ (\bar{u} u + \frac{1}{2} \bar{u}^2 + u \ln u) \left[ v \tilde{O}_\beta(v, u, -u) + u \tilde{O}_\beta(u, v, -u) - v \tilde{O}_\beta(u, -u, v) \right] - \frac{1}{12} \bar{u}^2 (u + 2) \left[ \tilde{O}_\beta(u, -u, v) + \tilde{O}_\beta(u, v, -u) + \tilde{O}_\beta(v, u, -u) \right] \right) \]\n\[
\left. \right\}_{\mu^2_{\overline{\text{MS}}}}.
\]
where the subscript $\mu^2_{\text{MS}}$ indicates the normalization point of nonlocal quark and gluon operators. Here we introduced the C-even quark-gluon operator

$$S_\mu(u, v, -u) = S_\mu^+(u, v, -u) + S_\mu^-(u, v, u)$$  \hspace{1cm} (3.27)

so that

$$S_\mu(-u, v, u) = S_\mu^+(u, v, -u) + S_\mu^+(u, v, u) .$$  \hspace{1cm} (3.28)

The following comments are in order.

The twist-3 gluon contribution in (3.26) has two parts. The lengthy expression in the last two lines can in fact be eliminated by the choice of scale in the quark operator in the first line: $\mu^2_{\text{MS}} = 1/(-x^2e^{2\gamma_E+1})$. As a nontrivial check of our calculation, we have verified that our answer (3.26) is in agreement with the renormalization group equation for the twist-3 operator $S_\mu(u, v, -u)$ [1, 12]

$$[S_\beta(u, v, -u)]_{\mu^2} = [S_\beta(u, v, -u)]_{\mu_1^2} - \frac{\alpha_s}{2\pi} \mu^2_{\text{MS}} \int_{-u}^{u} ds \int_{-u}^{s} dt (2u)^{-3}$$  
\[ \times \{ [2u(s-t)+4(u-s)(t+s)]\tilde{O}_\beta(s, v, t) - 2|u|(s-t)[\tilde{O}_\beta(s, t, v) - \tilde{O}_\beta(v, s, t)] \} . \hspace{1cm} (3.29) \]

In addition, the expression in the second line in (3.26) defines a 'genuine' twist-3 gluon coefficient function that cannot be eliminated by the scale choice in the quark operator. This expression is surprisingly simple and can be cast in the form similar to that of the tree-level contribution of the quark operators, Eq. (2.4):

$$[C \ast \tilde{O}_\beta]_{\text{gluon}}^{tw-3} = \frac{i}{2x^2} \sum_{q=u,d,s,...} \int_0^1 du \int_{-u}^{u} dv \left[ (u+v) \mathcal{G}_\beta(u, v, -u) - (u-v)\mathcal{G}_\beta(-u, v, u) \right] ,$$  \hspace{1cm} (3.30)

where

$$\mathcal{G}_\beta(u, v, -u) = -\frac{\alpha_s}{\pi} u^2 \left[ \tilde{O}_\beta(v, u, -u) + \tilde{O}_\beta(u, v, u) \right] .$$  \hspace{1cm} (3.31)

The nucleon matrix element of (3.26) defines the twist-3 gluon correlation function similar to (2.8)]

$$\langle p, s | \tilde{O}_\mu(u, v, -u) | p, s \rangle = -2(px)^2 [s_\mu(px) - p_\mu(sx)] \int_{-1}^{1} D\xi e^{ipx[\xi_1u + \xi_2v - \xi_3u]} D_\beta(\xi_1, \xi_2, \xi_3)$$  \hspace{1cm} (3.32)

with the integration measure given by (2.9). Since $\tilde{O}_\mu(u, v, -u) = -\tilde{O}_\mu(-u, v, u)$ according to the definition (3.23), the correlation function $D_\beta(\xi_i)$ is antisymmetric to the interchange of the first and the third argument:

$$D_\beta(\xi_1, \xi_2, \xi_3) = -D_\beta(\xi_3, \xi_2, \xi_1) .$$  \hspace{1cm} (3.33)

Expanding the nonlocal operator (3.26) over local twist-3 gluon operators

$$[G_{\mu}]^k_{\lambda\nu} = \frac{i}{2gf} \int \hat{O}_{abc} G_{\alpha \beta}^a \left( \hat{\sigma} \cdot x \right)^k \hat{G}_{\beta \lambda} \left( \hat{\sigma} \cdot x \right)^{N-1-k} C_{\alpha \nu}^c$$  \hspace{1cm} (3.34)

*cf. the footnote to (2.8)*
and defining the reduced matrix elements $\langle \ldots \rangle$ of these operators

$$
\langle p, s | [G_{\mu}]_N^k | p, s \rangle = 2(i p x)^{N+1} [s_\mu(p x) - p_\mu(s x)] \langle [G]_N^k \rangle
$$

(3.35)

we obtain the moments of the gluon three-particle distribution amplitude $\langle \ldots \rangle$ as

$$
\langle [G]_N^k \rangle = \int_{-1}^1 d\xi \xi^k \xi^{N-1-k} D_g(\xi)
$$

(3.36)

with $0 \leq k \leq N/2 - 1$. The symmetry (3.33) implies that $\langle [G]_N^k \rangle = -\langle [G]_N^{N-1-k} \rangle$ and, therefore, the number of independent gluon matrix elements is equal to $N/2$.

Finally, substituting (3.26) into (2.3) and taking the Fourier transform (2.2), we match the result into the expansion (3.18) to obtain the moments of the structure function for $n = 1, 3, \ldots$

$$
[g_2(n, Q^2)]^{\text{tw-3}} = \frac{1}{2} \sum_{q=u,d,s,\ldots} e_q^2 \frac{4}{n} \int_{-1}^1 d\xi \left\{ D_q(\xi_i, \mu^2_{\text{MS}}) \Phi^q_n(\xi_1, \xi_3)ight. \right.

\left. + \frac{\alpha_s}{4\pi} \frac{D_g(\xi_i, \mu^2_{\text{MS}})}{n+1} \left[ \Phi^g_n(\xi_i) + \Omega^{gg}_n(\xi_i) \left( \ln \frac{Q^2}{\mu^2_{\text{MS}}} - \psi(n) - \gamma_E - 1 \right) \right] \right\},
$$

(3.37)

where, as usual, applicability to the lowest moments relies on the extra assumption about the high-energy asymptotics of the cross section, alias the assumption that the corresponding dispersion relation does not involve extra subtraction constants. Here $D_q(\xi_i)$ is the distribution function corresponding to the $C$–even quark-gluon operator (3.24)

$$
D_q(\xi_i, \mu^2_{\text{MS}}) = \text{Re} D_q^{-}(\xi_i; \mu^2_{\text{MS}}),
$$

(3.38)

the quark coefficient function is defined as

$$
\Phi^q_n(\xi_1, \xi_3) = \frac{\partial}{\partial \xi_1} \frac{\xi_1^{n-1} - (-\xi_3)^{n-1}}{\xi_1 + \xi_3}
$$

(3.39)

and it has already appeared in (2.20). The gluon coefficient functions can be expressed in terms of $\Phi^g_n$ as

$$
\Phi^g_n(\xi_i) = [\Phi^g_{n-1}(\xi_1, \xi_3) + \Phi^g_{n-1}(\xi_1, -\xi_1 - \xi_3)] + (\xi_1 \leftrightarrow -\xi_3),
$$

(3.40)

$$
\Omega^{gg}_n(\xi_i) = \left( 1 + \frac{2}{n(n-2)} \right) \Phi^g_n(\xi_i) + \frac{2(n-1)}{n(n-2)} \left[ \Phi^g_{n-1}(-\xi_1 - \xi_3, \xi_3) + \Phi^g_{n-1}(\xi_1 + \xi_3, -\xi_1) \right].
$$

Explicit expressions for a few first moments read:

$$
\Phi^g_3 = \Omega^{gg}_3 = 0,
$$

$$
\Phi^g_5 = 5 (\xi_1 - \xi_3), \quad \Omega^{gg}_5 = \frac{31}{5} (\xi_1 - \xi_3),
$$

$$
\Phi^g_7 = 14 (\xi_1^3 - \xi_3^3), \quad \Omega^{gg}_7 = \frac{2}{7} \left( 59 \xi_1^3 - 6 \xi_1^2 \xi_3 - 6 \xi_1 \xi_3^2 - 59 \xi_3^3 \right).
$$

(3.41)

(3.42)

The expressions (3.26), (3.30) and (3.37) for the one-loop gluon contribution to the antisymmetric part of the $T$-product of two electromagnetic currents (2.3) and the structure function $g_2(x_B, Q^2)$ present the main result of this section.
4. Properties of the twist-3 contribution

According to (3.37), the moments of the structure function are given by integrals of the quark-gluon and three-gluon distribution functions, \(D_q(\xi_i)\) and \(D_g(\xi_i)\), respectively, over momentum fractions of partons with the weights defined by the coefficient functions \(\Phi^q_n(\xi_i)\), \(\Phi^g_n(\xi_i)\) and \(\Omega_{gg}^n(\xi_i)\).

The quark coefficient function (3.39) vanishes at \(n = 1\) and for higher moments \(\Phi^q_n\) is a homogenous polynomial in momentum fractions \(\xi_i\) of degree \(n - 3\). For the gluon coefficient functions, Eqs. (3.40), one finds that \(\Omega_{gg}^n(\xi_i) = \Phi^g_{n=3}(\xi_i) = 0\) and, therefore, the gluon contribution to the first moment of \(g_2(x, Q^2)\) vanishes, in agreement with the BC sum rule, provided that the three-gluon distribution function \(D_g(\xi_i)\) does not have additional singularities. Moreover, in contrast with the quark coefficient function, \(\Omega_{gg}^{n=3}(\xi_i) = \Phi^g_{n=3}(\xi_i) = 0\), and gluon contribute to the structure function starting from \(n = 5\)th moment. For \(n \geq 5\) the coefficient functions \(\Omega_{gg}^n\) and \(\Phi^g_n\) are homogenous polynomials in \(\xi_i\) of degree \(n - 4\).

The explicit expressions for the lowest moments (3.37) look as follows

\[
\int_0^1 dx \, x^2 \, g_2^{tw-3}(x, Q^2) = \frac{2}{3} \sum_q e_q^2 \int_{-1}^1 d\xi \, D_q(\xi_i; Q^2), \tag{4.1}
\]

\[
\int_0^1 dx \, x^4 \, g_2^{tw-3}(x, Q^2) = \frac{2}{5} \sum_q e_q^2 \int_{-1}^1 d\xi \Phi^g(\xi_i; Q^2) \times \left[ (3\xi_1^2 - 2\xi_1\xi_3 + \xi_3^2)D_q(\xi_i; Q^2) - \frac{\alpha_s}{\pi} \frac{847}{1440} (\xi_1 - \xi_2) D_g(\xi_i; Q^2) \right], \tag{4.2}
\]

\[
\int_0^1 dx \, x^6 \, g_2^{tw-3}(x, Q^2) = \frac{2}{7} \sum_q e_q^2 \int_{-1}^1 d\xi \Phi^g(\xi_i; Q^2) \times \left[ (5\xi_1^4 - 4\xi_1^3\xi_3 + 3\xi_1^2\xi_3^2 - 2\xi_1\xi_3^3 + \xi_3^4)D_q(\xi_i; Q^2) \right. \\
\left. - \frac{\alpha_s}{\pi} \left( \frac{3091}{2240} (\xi_1^3 - \xi_3^3) + \frac{207}{1120} (\xi_1^2\xi_3 - \xi_2\xi_3^2) \right) D_g(\xi_i; Q^2) \right]. \tag{4.3}
\]

Using (2.18) and (3.36) we express the moments in terms of reduced quark and gluon matrix elements

\[
\int_0^1 dx \, x^2 \, g_2^{tw-3}(x, Q^2) = \frac{2}{3} \sum_q e_q^2 \langle S_0^0(Q^2) \rangle, \tag{4.4}
\]

\[
\int_0^1 dx \, x^4 \, g_2^{tw-3}(x, Q^2) = \frac{2}{5} \sum_q e_q^2 \left[ \langle S_0^0(Q^2) \rangle - 2\langle S_1^0(Q^2) \rangle + 3\langle S_2^0(Q^2) \rangle \right] + \frac{\alpha_s}{\pi} \frac{847}{720} \langle G_0^0(Q^2) \rangle, \tag{4.5}
\]

\[
\int_0^1 dx \, x^6 \, g_2^{tw-3}(x, Q^2) = \frac{2}{7} \sum_q e_q^2 \left[ \langle S_0^0(Q^2) \rangle - 2\langle S_1^0(Q^2) \rangle + 3\langle S_2^0(Q^2) \rangle - 4\langle S_3^0(Q^2) \rangle + 5\langle S_4^0(Q^2) \rangle \right]. \tag{4.6}
\]
\[ + \frac{\alpha_s}{\pi} \left( \frac{3091}{1120} \langle G_0^q(Q^2) \rangle + \frac{207}{560} \langle G_1^q(Q^2) \rangle \right). \]

We would like to stress that the relations (3.37), (4.1) and (4.4) take into account the leading order contribution of the three-gluon operators and they do not include \( \mathcal{O}(\alpha_s) \)-corrections to the coefficient functions of quark-antiquark-gluon operators. The latter corrections have been recently calculated in [33] using a different operator basis.

We can make one further step and define the momentum fraction distribution of the transverse spin carried by gluons in the nucleon by the expression similar to (2.11):

\[
\int_{-u}^{u} dv \langle p, s | \left[ (u + v) \tilde{O}_\mu(v, u, -u) + 2u \tilde{O}_\mu(u, v, -u) + (u - v) \tilde{O}_\beta(u, -u, v) \right] | p, s \rangle = 2 \left[ s_\mu(px) - p_\mu(sx) \right] \int_{-1}^{1} d\xi \, e^{2i\xi upx} \xi \Delta g_T(\xi; \mu^2),
\]

so that (cf. (2.20))

\[
\int_{-1}^{1} d\xi \, \xi^{n-1} \Delta g_T(\xi) = 2 \int_{-1}^{1} \mathcal{D}\xi \, \Phi^g_{\eta}(\xi_i) D_g(\xi_i)
\]

and \( \Delta g_T(\xi) = \Delta g_T(-\xi) \). It is easy to see that the first contribution in the square brackets in (3.37) can be easily rewritten in terms of \( \Delta g_T(n, Q^2) = \int_{0}^{1} d\xi \, \xi^{n-1} \Delta g_T(\xi) \). Unfortunately, the two coefficient functions \( \Phi^g_{\eta}(\xi_i) \) and \( \Omega^g_{\eta}(\xi_i) \) do not coincide and, therefore, the full gluon contribution to (3.37) cannot be, strictly speaking, reduced to the contribution of \( \Delta g_T(\xi) \) alone, as a yet another manifestation of the fact that we are dealing with a three-particle problem.

Our main observation is that such a reduction can, nevertheless, provide a reasonable approximation to the moments \( g_2(n, Q^2) \)

\[
\int_{-1}^{1} \mathcal{D}\xi \, D_g(\xi; Q^2) \Omega^g_{\eta}(\xi_i) \approx c(n) \int_{-1}^{1} \mathcal{D}\xi \, D_g(\xi; Q^2) \Phi^g_{\eta}(\xi_i),
\]

with the coefficient of proportionality \( c(n) \) that is independent on the momentum transfer \( Q^2 \). In this case, moments of the structure function (3.37) can be expressed in terms of the quark and gluon distributions as

\[
\int_{0}^{1} dx_B \, x_B^{n-1} g_2(x_B, Q^2) \approx \frac{1}{2n} \sum_q e_q^2 \int_{0}^{1} dx_B \, x_B^{n-1} \left[ \Delta q_T(x_B, Q^2) + \Delta q_T(-x_B, Q^2) + C_{gw}^{n-3}(n) \Delta g_T(x_B, Q^2) \right].
\]

where \( \Delta q_T \) and \( \Delta g_T \) were defined in [2.11] and (4.8), respectively, and

\[
C_{gw}^{n-3}(n) = -\frac{\alpha_s}{2\pi} \frac{1}{n + 1} \left[ (\psi(n) + \gamma_E + 1) \, c(n) - 1 \right].
\]

The reason for (4.9) to hold is that, as we shall argue below, the both projections of the three-gluon distribution defined in the l.h.s and the r.h.s of Eq. (1.9) can be identified to a good numerical accuracy with the contribution of the three-gluon multiplicatively
renormalizable operator with the lowest anomalous dimension. To be more precise, this statement refers to the “purely gluonic” operator, defined without taking into account the mixing with the quark-antiquark-gluon sector. We have checked that the mixing between three-gluon and quark-antiquark-gluon operators does not modify the result significantly; a detailed analysis will be presented elsewhere [34].

To justify this, we note that in a “purely gluonic” sector an arbitrary multiplicatively renormalizable twist-3 three-gluon operator $O_{N,\alpha}$ can be characterized by the coefficients in its expansion over the basis of operators $[G_{\mu}]^k_N$ defined in (3.34)

$$O_{N,\alpha} = \sum_{k=0}^{[N/2]-1} w^k_{N,\alpha} [G_{\mu}]^k_N = \frac{1}{2} \sum_{k=0}^{[N/2]-1} w^k_{N,\alpha} \left( [G_{\mu}]^k_N - [G_{\mu}]^{N-1-k}_N \right)$$

(4.12)

or, equivalently, by a characteristic polynomial

$$W_{N,\alpha}(\xi_1, \xi_3) = \frac{1}{2} \sum_{k=0}^{[N/2]-1} w^k_{N,\alpha} \left( \xi_1^{k+1} - \xi_3^{k+1} \right).$$

(4.13)

The subscript $0 \leq \alpha \leq [N/2] - 1$ enumerates the operators and we assume, for definiteness, that the operators $O_{N,\alpha}$ are ordered in such a way that a smaller $\alpha$ corresponds to a lower anomalous dimension. With this definition, it follows from (3.30) that the reduced matrix element of a multiplicatively renormalizable operator $O_{N,\alpha}$ is given by a weighted integral of the three-particle gluon distribution:

$$\langle \langle O_{N,\alpha} \rangle \rangle = \int_{-1}^{1} d\xi_1 W_{N,\alpha}(\xi_1) D_g(\xi_1).$$

(4.14)

For the purpose of our discussion, we assume that the expansion coefficients $w^k_{N,\alpha}$ in (4.12) are calculated by an explicit diagonalization of the mixing matrix given in [8] so that the polynomials $W_{N,\alpha}(\xi_1)$ are known functions.

To prove our assertion, we have to show that the coefficient functions $\Phi_n^g(\xi_1)$ and $\Omega_n^{gg}(\xi_1)$ are numerically close to $W_{N,\alpha=0}(\xi_1)$, at least for sufficiently large values of $n = N + 3$, or, equivalently, the both sides of (1.13) receive a dominant contribution from $\langle \langle O_{N,\alpha=0} \rangle \rangle$. This is not straightforward since the characteristic polynomials $W_{N,\alpha}(\xi_1)$ for different $\alpha$ are not mutually orthogonal with respect to any simple weight function, the reason being that the mixing matrices [8] are not symmetric.

In order to make a meaningful comparison we use the conformal symmetry that allows to rewrite the mixing matrices in a different basis such that they become hermitian (see [13, 35] for details). In the present context, the idea is that the conformal symmetry allows for a unique analytic continuation of the functions $W_{N,\alpha}(\xi_1), \Phi_n^g(\xi_1)$ and $\Omega_n^{gg}(\xi_1)$ defined on the plane $\xi_1 + \xi_2 + \xi_3 = 0$ to the full three-dimensional space of the momentum fraction variables $\xi_i$

$$W_{N,\alpha}(\xi_1, \xi_3) \Rightarrow \tilde{W}_{N,\alpha}(\xi_1, \xi_2, \xi_3)$$

(4.15)

e tc., such that $W_{N,\alpha}(\xi_1, \xi_3) = \tilde{W}_{N,\alpha}(\xi_1)$ for $\xi_1 + \xi_2 + \xi_3 = 0$ and the functions $\tilde{W}_{N,\alpha}(\xi_1)$ corresponding to different multiplicatively renormalizable operators are mutually orthogonal with respect to the conformal scalar product

$$\langle \tilde{W}_{N,\alpha} \tilde{W}_{N,\beta} \rangle = \int_{0}^{1} [d\xi] \xi_1^2 \xi_2^2 \xi_3^2 \tilde{W}_{N,\alpha}(\xi_1) \tilde{W}_{N,\beta}(\xi_1) \sim \delta_{\alpha\beta},$$

(4.16)
Figure 3: Conformal projection (see text) of the coefficient functions $\Phi^g_n(\xi_i)$ (crosses) and $\Omega^{qq}_n(\xi_i)$ (diamonds) on the contributions of multiplicately renormalizable three-gluon operators enumerated by an integer variable $\alpha$ in the order of the increasing anomalous dimension. Mixing with quark-antiquark-gluon operators is neglected. The two plots correspond to the moments $n = 13$ (right) and $n = 33$ (left), respectively.

where the integration measure is defined as
\[ [d\xi] = d\xi_1 d\xi_2 d\xi_3 \delta(\xi_1 + \xi_2 + \xi_3 - 1). \] (4.17)

Using this method, we can expand $\tilde{\Phi}^g_n(\xi_i)$ and $\tilde{\Omega}^{qq}_n(\xi_i)$ over the set of orthogonal polynomials $\tilde{W}_{N,\alpha}(\xi_i)$ to arrive after reduction to $\xi_1 + \xi_2 + \xi_3 = 0$ at
\[ \Phi^g_n(\xi_i) = \sum_{\alpha=0}^{[N/2]-1} \frac{\langle \tilde{\Phi}^g_n | \tilde{W}_{N,\alpha} \rangle}{\| \tilde{W}_{N,\alpha} \|^2} W_{N,\alpha}(\xi_i) = \| \tilde{\Phi}^g_n \| \sum_{\alpha=0}^{[N/2]-1} \phi_{n,\alpha} \frac{W_{N,\alpha}(\xi_i)}{\| \tilde{W}_{N,\alpha} \|} \] (4.18)

with $n = N + 3$ and similarly for $\Omega^{qq}_n(\xi_i)$. Here, the norm is defined in a natural way as $\| \Psi \|^2 = \langle \Psi | \Psi \rangle$.

The results of our calculations of the (normalized) expansion coefficients
\[ \phi_{n,\alpha} = \frac{\langle \tilde{\Phi}^g_n | \tilde{W}_{N,\alpha} \rangle}{\| \tilde{\Phi}^g_n \| \cdot \| \tilde{W}_{N,\alpha} \|}, \quad \omega_{n,\alpha} = \frac{\langle \tilde{\Omega}^{qq}_n | \tilde{W}_{N,\alpha} \rangle}{\| \tilde{\Omega}^{qq}_n \| \cdot \| \tilde{W}_{N,\alpha} \|} \] (4.19)

corresponding to the functions $\Phi^g_n$ and $\Omega^{qq}_n$, respectively, are shown in Fig. 3 for $n = 13$ and $n = 33$. It is clearly seen that the sum (4.18) is dominated by the single contribution $\alpha = 0$ of the three-gluon operator with the lowest anomalous dimension. The quality of such approximation is improving for larger moments $n$. Moreover, we find that, in accord with (4.19), the two functions $\Phi^g_n(\xi_i)$ and $\Omega^{qq}_n(\xi_i)$ are, in fact, close to each other with the coefficients $c(n)$ in (4.9) and (4.11) given by
\[ c(n) = \frac{\langle \tilde{\Omega}^{qq}_n | \tilde{W}_{N,0} \rangle}{\langle \tilde{\Phi}^g_n | \tilde{W}_{N,0} \rangle}. \] (4.20)

Since the contribution of $\Omega^{qq}_n(\xi_i)$ to (3.37) reflects the mixing of three-gluon with quark-antiquark gluon operators, this is an indication that the observed dominance of the lowest three-gluon state is not obstructed by this mixing.
From Eq. (3.40) it is easy to see that $c(n) \to 1$ for large moments $n$ whereas from (3.41) it follows that $c(5) = 31/25$. The values of $c(n)$ for other (odd) moments $n < 25$ are shown in Fig. 4. The $n$-dependence is very smooth (for odd $n$) and can be approximated as

$$c(n) = 1 + 11.11 \frac{1}{n^2} - 25.74 \frac{1}{n^3}. \tag{4.21}$$

Analytic expressions for $c(n)$ can be worked out in the large-$N_c$ limit and will be presented in [34].

5. Conclusions

We have presented a detailed calculation of the three-gluon twist-3 contribution to the flavor-singlet structure function $g_2(x, Q^2)$ to the one-loop accuracy. The result is encouraging as it indicates that the gluon coefficient function is close to that of the three-gluon operator with the lowest anomalous dimension, at least for large moments. This allows to hope for a simplified description of the scale dependence of $g_2(x, Q^2)$ in terms of DGLAP equations, similar as for the structure functions of leading twist. The construction of this approximation requires a more detailed analysis of the evolution equations for the flavor-singlet twist-3 operators and will be given in a forthcoming publication [34].

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Note added

When this work was in preparation, the work [36] appeared with the calculation of the singlet twist-3 coefficient functions using a different approach. The relation of this
calculation to our result is not obvious because of different operator basis. It appears that the answer for the \( n = 5 \) moment of \( g_2(x) \) given in (4.1) is in agreement with the appropriate projection of the coefficient function calculated in [36]. We thank A. Belitsky for the correspondence on this topic.

A Appendix

The short-distance expansion of the nonlocal two-gluon operator looks like

\[
G_{x\alpha}(x)G_{\nu\alpha}(-x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x_{\mu_1} x_{\mu_2} \cdots x_{\mu_n} G_{\mu_1 \alpha} \hat{\rightarrow} \cdots \hat{\rightarrow} G_{\mu_n \alpha} G_{\nu\alpha}(0) \quad (A.1)
\]

The twist-2 contribution to the r.h.s. originates from the local operators completely symmetric and traceless with respect to their Lorentz indices

\[
[G_{x\alpha}(x)G_{\nu\alpha}(-x)]_{\text{tw-2}} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!(n+1)} \partial_{\rho} \left( x_{\mu_1} x_{\mu_2} \cdots x_{\mu_n} x_{\mu_{n+1}} \right) G_{\mu_1 \alpha} \hat{\rightarrow} \cdots \hat{\rightarrow} G_{\mu_n \alpha} G_{\mu_{n+1} \alpha}(0), \quad (A.2)
\]

where \( \hat{\rightarrow} \) is the difference operator. It is straightforward to verify that the same expression can be obtained through the following integral representation

\[
[G_{x\alpha}(x)G_{\nu\alpha}(-x)]_{\text{tw-2}} = \int_0^1 du \frac{\partial}{\partial x_\nu} G_{x\alpha}(ux) G_{x\alpha}(-ux). \quad (A.3)
\]

Subtracting the twist-2 contribution from the nonlocal two-gluon operator we obtain the relation

\[
G_{x\alpha}(x)G_{\nu\alpha}(-x) - \int_0^1 du \frac{\partial}{\partial x_\nu} G_{x\alpha}(ux) G_{x\alpha}(-ux) =
\]

\[
= \int_0^1 du x_\rho \left[ \frac{\partial}{\partial x_\rho} G_{x\alpha}(ux) G_{\nu\alpha}(-ux) - \frac{\partial}{\partial x_\nu} G_{x\alpha}(ux) G_{\rho\alpha}(-ux) \right]
\]

\[
+ \int_0^1 du \left[ G_{x\alpha}(ux) G_{\nu\alpha}(-ux) - G_{\nu\alpha}(ux) G_{x\alpha}(-ux) \right], \quad (A.4)
\]

which can be expressed in terms of three-gluon operators using the QCD equations of motion as follows.

We begin with the expression in the second line of (A.4):

\[
\frac{\partial}{\partial x_\rho} G_{\mu\alpha}(ux) G_{\nu\alpha}(-ux) - \frac{\partial}{\partial x_\nu} G_{\nu\alpha}(ux) G_{\rho\alpha}(-ux)
\]

\[
= G_{\mu\alpha} \left[ u \hat{\rightarrow} D_\rho - u \hat{\rightarrow} D_\rho - i \int_{-u}^u dv G_{x\rho}(vx) \right] G_{\nu\alpha}(-ux) - (\rho \leftrightarrow \nu). \quad (A.5)
\]

Throughout this Appendix we assume that gauge factors in the adjoint representation are inserted in between the gluon fields.
Following [9], we introduce a derivative over the total translation
\[ \partial_\rho G_{\mu\alpha}(ux)G_{\nu\alpha}(-ux) \equiv \frac{\partial}{\partial y_\rho} \left[ G_{\mu\alpha}(ux + y)G_{\nu\alpha}(-ux + y) \right] y \to 0 \]
\[ = G_{\mu\alpha}(ux) \left[ \overrightarrow{D}_\rho + \overrightarrow{D}_\rho - i \int_{-u}^u dv \ G_{x\rho}(vx) \right] G_{\nu\alpha}(-ux) \quad (A.6) \]
so that
\[ x_\rho x_\nu \left[ \frac{\partial}{\partial x_\rho} G_{\mu\alpha}(ux)G_{\nu\alpha}(-ux) - u \partial_\rho G_{\mu\alpha}(ux)G_{\nu\alpha}(-ux) \right] - (\rho \leftrightarrow \nu) = \]
\[ = -2uG_{x\alpha}(ux)\overrightarrow{D}_\nu G_{x\alpha}(-ux) - i \int_{-u}^u (u - v) dv \ G_{x\alpha}(ux)G_{x\nu}(vx)G_{x\alpha}(-ux) \]
\[ = 2u\partial_\alpha G_{x\alpha}(ux)G_{x\nu}(-ux) - 2uG_{x\alpha}(ux)\overrightarrow{D}_\nu G_{x\alpha}(-ux) + 2i \int_{-u}^u dv \ G_{x\alpha}(ux)G_{x\alpha}(vx)G_{x\nu}(-ux) \]
\[ - i \int_{-u}^u (u - v) dv \ G_{x\alpha}(ux)G_{x\nu}(vx)G_{x\alpha}(-ux) \quad (A.7) \]

where we used the Bianchi identity \( \overrightarrow{D}_\alpha G_{\rho\alpha} - \overrightarrow{D}_\rho G_{\nu\alpha} = \overrightarrow{D}_\nu G_{\rho\rho} \) to arrive at the expression in the second line. Neglecting total derivatives and terms \( \sim D_\alpha G_{\nu\alpha} \) this gives
\[ \int_0^1 du x_\rho \left[ \frac{\partial}{\partial x_\rho} G_{\mu\alpha}(ux)G_{\nu\alpha}(-ux) - \frac{\partial}{\partial x_\nu} G_{\mu\alpha}(ux)G_{\rho\alpha}(-ux) \right] = \quad (A.8) \]
\[ = i \int_0^1 du \int_{-u}^u \left[ 2uG_{x\alpha}(ux)G_{x\alpha}(vx)G_{x\nu}(-ux) + (v - u)G_{x\alpha}(ux)G_{x\nu}(vx)G_{x\alpha}(-ux) \right] . \]

Next, we consider the antisymmetric combination of gluon fields in the last line in (A.4) and rewrite it as
\[ G_{\mu\alpha}(x)G_{\nu\alpha}(-x) - (\mu \leftrightarrow \nu) = \]
\[ = \int_0^1 du x_\lambda \frac{\partial}{\partial x_\lambda} [G_{\mu\alpha}(ux)G_{\nu\alpha}(-ux) - (\mu \leftrightarrow \nu)] \]
\[ = \int_0^1 du x_\lambda \left[ G_{\mu\alpha}(ux)\overrightarrow{D}_\lambda G_{\nu\alpha}(-ux) - G_{\mu\alpha}(ux)\overrightarrow{D}_\lambda G_{\nu\alpha}(-ux) \right] - (\mu \leftrightarrow \nu) . \quad (A.9) \]

By a repeated application of the the Bianchi identity and separating (and then neglecting) contributions of total translation and equation of motion terms \( \sim D_\alpha G_{\nu\alpha} \), one arrives after some algebra at the following expression:
\[ x_\mu \left[ G_{\mu\alpha}(x)G_{\nu\alpha}(-x) - (\mu \leftrightarrow \nu) \right] = 2i \int_0^1 du \int_{-u}^u \left[ G_{x\alpha}(ux)G_{x\alpha}(vx)G_{x\nu}(-ux) \right. \]
\[ \left. - G_{x\alpha}(ux)G_{x\nu}(vx)G_{x\alpha}(-ux) + G_{x\nu}(ux)G_{x\alpha}(vx)G_{x\alpha}(-ux) \right] \quad (A.10) \]

that yields
\[ \int_0^1 du \left[ G_{x\alpha}(ux)G_{\nu\alpha}(-ux) - G_{\nu\alpha}(ux)G_{x\alpha}(-ux) \right] = \]
\[
\int_0^1 du (1-u^2) \int_{-u}^u \left[ G_{x\alpha}(ux)G_{x\alpha}(vx)G_{x\nu}(-ux) - G_{x\alpha}(ux)G_{x\nu}(vx)G_{x\alpha}(-ux) 
+ G_{x\nu}(ux)G_{x\alpha}(vx)G_{x\alpha}(-ux) \right].
\]  
(A.11)

Taking the sum of the expressions in (A.8), (A.11) and the color trace, we obtain the result (3.11) quoted in the text.

References

[1] K. Abe et al. [E143 Collaboration], Phys. Rev. Lett. 76 (1996) 587.
[2] K. Abe et al. [E154 Collaboration], Phys. Lett. B404 (1997) 377.
[3] P. L. Anthony et al. [E155 Collaboration], Phys. Lett. B458 (1999) 529; G. S. Mitchell [E155 Collaboration], hep-ex/9903055.
[4] B. L. Ioffe, V. A. Khoze and L. N. Lipatov, “Hard Processes. Vol. 1: Phenomenology, Quark Parton Model”, Amsterdam, Netherlands: North-Holland (1984).
[5] M. Anselmino, A. Efremov and E. Leader, Phys. Rept. 261 (1995) 1.
[6] J. Kodaira and K. Tanaka, Prog. Theor. Phys. 101 (1999) 191.
[7] E. V. Shuryak and A. I. Vainshtein, Nucl. Phys. B201 (1982) 141.
[8] A.P. Bukhvostov, E.A. Kuraev and L.N. Lipatov, Sov. Phys. JETP 60 (1982) 22;
[9] I. I. Balitsky and V. M. Braun, Nucl. Phys. B311 (1989) 541.
[10] A. Ali, V.M. Braun and G. Hiller, Phys. Lett. B266 (1991) 117.
[11] J. Kodaira, Y. Yasui and T. Uematsu, Phys. Lett. B344 (1995) 348; J. Kodaira et al., Phys. Lett. B387 (1996) 855; Prog. Theor. Phys. 99 (1998) 315.
[12] B. Geyer, D. Muller and D. Robaschik, Nucl. Phys. Proc. Suppl. 51C (1996) 106; D. Muller, Phys. Lett. B407 (1997) 314.
[13] V. M. Braun, G. P. Korchemsky and A. N. Manashov, Phys. Lett. B476 (2000) 455.
[14] H. Burkhardt and W. N. Cottingham, Annals Phys. 56 (1970) 453.
[15] R. L. Jaffe and X. Ji, Phys. Rev. D43 (1991) 724.
[16] J. Kodaira et al. Phys. Lett. B345 (1995) 527.
[17] I. P. Ivanov et al. Phys. Lett. B457 (1999) 218; Phys. Rept. 320 (1999) 175.
[18] I. I. Balitsky, V. M. Braun and A. V. Kolesnichenko, Phys. Lett. B242 (1990) 245; B318 (1993) 648 (Erratum);
E. Stein et al., Phys. Lett. B343 (1995) 369.
[19] M. Göckeler, et al., hep-ph/9909253.

[20] J. B. Kogut and D. E. Soper, Phys. Rev. D1 (1970) 2901.

[21] A. V. Belitsky, Nucl. Phys. B574 (2000) 407.

[22] S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, Nucl. Phys. B566 (2000) 203.

[23] I.I. Balitsky and V.M. Braun, in: Proc. of the XXV LNPI Winter School, pp. 105–153, Leningrad, 1990.

[24] L.B. Okun, “Leptons and Quarks”, Amsterdam, Netherlands: North-Holland (1982).

[25] R.L. Jaffe, Nucl. Phys. B229 (1983) 205.

[26] B. Geyer, M. Lazar and D. Robaschik, Nucl. Phys. B559 (1999) 339; B. Geyer and M. Lazar, Nucl. Phys. B581 (2000) 341.

[27] S. Wandzura and F. Wilczek, Phys. Lett. B72 (1977) 195.

[28] O. V. Teryaev, in: Proc. of the 8th Intern. Symposium on Polarization Phenomena in Nuclear Physics (SPIN 94), Bloomington, Indiana, 15-22 Sep 1994, pp. 467-471; A. V. Efremov, O. V. Teryaev and E. Leader, Phys. Rev. D55 (1997) 4307.

[29] X. Ji and J. Osborne, Eur. Phys. J. C9 (1999) 487.

[30] A. V. Manohar, Phys. Rev. Lett. 66 (1991) 289.

[31] I. I. Balitsky and V. M. Braun, Phys. Lett. B267 (1991) 405.

[32] G. T. Bodwin and J. Qiu, Phys. Rev. D41 (1990) 2755; R. Mertig and W. L. van Neerven, Z. Phys. C70 (1996) 637.

[33] X. Ji, W. Lu, J. Osborne and X. Song, Phys. Rev. D62, 094016 (2000) hep-ph/0006121.

[34] V.M. Braun, G.P. Korchemsky and A.N. Manashov, paper in preparation.

[35] V.M. Braun, S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, Nucl. Phys. B553 (1999) 355.

[36] A. Belitsky, X. Ji, W. Lu and J. Osborne, hep-ph/0007305.