Students' obstacles in understanding the properties of the closed sets in terms of the APOS theory

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Abstract. An understanding of the concepts and principles of open sets can be done through activities in the form of mathematical actions, processes, objects organized in a scheme. This was the operationalization of the APOS theory (Action, Processes, Objects, and Schema). The purpose of this study was to describe students' mistakes in understanding the properties of open sets in terms of the APOS Theory. This research was part of development research. It was the stage need assessment stage. We want to determine the characteristics of students to determine the right learning model. The subject of this study were 15 students of mathematics education at the University of Bengkulu. Students were selected based on the tasks that they do in learning Real Analysis. Data was collected from paper and pencil results and task-based interview results. Data were analyzed qualitatively using fixed comparison techniques. The results of this study were that students can only coordinate two or more sets, and the difficulty of proving a set theorem closes. One such theorem was "F was a subset of R, F closed sets if and only if F contains all the cluster points". He can only take actions separately from the characteristics of the set given. The conclusion of this study was the difficulty of students in understanding the properties of the closed sets was the inability of students to carry out interiorization from actions to processes. Also, it was not able to encapsulate processes into objects. He was unable to reach the scheme of the properties of closed sets.

1. Introduction

Real analysis was a course that was difficult for students to understand. The deductive nature of the subject requires valid logical abilities, as well as good axiomatic deductive abilities. The real number system was the main study [1]. Therefore, the ability to understand each concept, the nature and axioms was an absolute requirement for students.

Understanding of concepts, principles and axioms can be done through activities in the form of mathematical actions, processes, objects organized in a scheme. This was the operationalization of the APOS theory (Action, Processes, Objects, and Schema)[2]. The purpose of this study was to describe students' mistakes in understanding the properties of open sets in terms of APOS Theory.

APOS theory was an analytical tool used to describe the development of one's scheme [3][4]. The development of the scheme was a process that was physical, and always changing. Knowledge grows based on certain mechanisms and includes three levels (intra level, inter level, and trans level)[5]. The fixed sequence implies that the triad level was hierarchical. That was that the intra level was the lowest level, the inter level was the middle level, and the trans level was the highest level. Another characteristic of triad leveling was functional, not structural. Also, these levels can be found if someone analyzes a scheme's development [6].
In solving problems, students have their own methods. The method will be clearly seen if he presents the arguments for solving the problem comprehensively, at least he will express it in writing and be equipped verbally [7]. In order for students to express well, the cognitive process needs to be raised by using the initial stimulus that meets the conditions required [7]. One of the initial stimulus that fulfills these requirements was a problem that must involve more than the scheme [8]. These were schemes that must be cognitively integrated to be used to solve a problem.

Dubinsky treats using certain computer experience in an effort to help students carry out object-scheme-process actions. It was developed through reflective abstraction, so that a coherent scheme was built [9] [10]. APOS theory was a constructivist theory about how the possibility of achievement / learning of a mathematical concept or principle.

APOS theory can be used as a predictive tool that was about the ability of students to make certain mental constructs. Also, it can be used to explain and describe the transcript of the interview in very good details. APOS theory can also be used to try to find mathematical ideas and possibilities that exist in the form of student performance. Then try to find an explanation of the differences in terms of constructing or not building certain actions, processes, objects and / or schemes [9]. The theory attempts to explain the successes and failures of students. Therefore, a collection of mental activities carried out by students in understanding mathematical material was a strong basis for describing their mathematical understanding abilities. That was the ability that can be categorized in the domain of cognitive processes. Understanding ability can be classified into four, namely 1) factual knowledge, 2) conceptual knowledge, 3) procedural knowledge, and 4) metacognitive knowledge [11].

One of the basic understanding skills about real analysis was advanced mathematical thinking [12]. It was an axiomatic deductive thinking ability that students will meet with an entirely new construction. Therefore, axiomatic objects in the form of traits (expressed as axioms) which were the starting points and concepts that must be constructed with logical deduction [13]. That was a continuation of calculus learning. In fact, calculus was one of the difficult material for some students, so that they experience misconceptions and principles in calculus [14]. Therefore, we suspect that many students experience errors in understanding real analysis. In that we understand it, such as students' understanding that links many concepts and principles. Like closed sets understanding that was associated with the concept and principle of cluster points. Thus we were interested in exploring students' mistakes in understanding the closed sets characteristics using the APOS theory.

2. Method
This research was part of the development research. This was the need assessment stage. Our long-term research was the implementation of a research master plan for the Mathematics Education Postgraduate Program, University of Bengkulu, Indonesia. That was a development research. We were to determine the characteristics of students to determine the right learning model. That was a search that we do about the mistakes of understanding that students experience in doing assignments. The subject of this study were 15 students of mathematics education at the University of Bengkulu. Students were selected based on the tasks that they do in learning Real Analysis. The data was collected based on the paper and pencil and task-based interview from subject (namely Sh). The task was to ask students to prove the truth about a closed sets statement. One statement that must be proven was “F was a closed set in a set of real numbers if and only if F contains all of its cluster points.” Data were analyzed qualitatively and the constant comparative technique.

3. Results and Discussion
We carry out long-term research, through development research. This was part of the research. We carried it out in the regular learning process for real analytical meta-lectures. We do not interfere with the learning process of the research subject. Our researchers were members of the teaching lecturer team in real analysis. That was an activity that makes the research subject not feel researched. We apply participatory techniques. Researchers conduct in-depth interviews fairly and casually so that we obtain complete data. It was a task-based interview.
The results of the initial analysis show the frequency of students making mistakes. There were five types of mistakes we get. These five were fact errors, conceptual errors, principle errors and operating/procedural errors and random errors [15][16]. Data shows the frequency of errors as listed in Figure 1.

![Bar chart showing frequency of student mistakes](image)

**Figure 1.** The frequency of students making mistakes

Based on Figure 1, describing the frequency of students making mistakes in completing the tasks that we provide. There were students who have errors of more than one type. The picture shows that 9 students had conceptual errors, 8 students had a principle error, 5 people had the wrong facts, 4 people had a wrong procedure and 1 person experienced a random error. In accordance with the data, there were 1 student who has the most types. The student was **Sh**. Therefore, we will explain the results of our interview with **Sh**. Figure 1 was the work of **Sh** in proving a statement that was true. The statement proved was "F was a subset of R and F closed sets if and only if each cluster point of F was contained in F."

![Image of Sh's proof](image)

**Figure 2.** Proof of the statement by **Sh**
Interview snippet between researcher (= Q) and subject (= Sh):

Q: Try to explain what the statement means?
Sh.01: ... yes ... all cluster points were members of the F set ... that's the result of F closed sets.

Q: Why was that?
Sh.02: ... that's what I understand from the statement given ...
Q: was the statement true or false?
Sh.03: I am sure that was the correct statement ... then I will explain the proof ...
Q: Why does F have members?
Sh.04: ... first, because F subset of R means having members ... then F closed sets, meaning all members of F were also cluster points of F, ...
Q: Okay ... please explain now!
Sh.05: ... yes of course ... because the R members were infinite ... there must be members included in F ...

Q: What was your argument?
Sh.06: ... 0 was definitely a member of F ... and that's a minimum ...
Q: How was your explanation continued?
Sh.07: I can also prove the opposite ... because of the x cluster points of F then x and everything in the set F ... like 0 must have been a member of F ... that was also the cluster point of F.
Q: What were your arguments from these statements?
Sh.08: ... Yes ... all points will gather at 0 ... because 0 was in the middle of a real number ... that's a sign that F was closed sets ...

Based on the interview footage with Sh, it was revealed that Sh had a wrong understanding. He misunderstood the statement given. Sh said that "all cluster points were members of the F set ... that was the result of F closed sets" (see Sh.01). It was a statement that only uses intuition, and not conceptual understanding. The actions were carried out separately. He did not understand the properties of closed sets. Also, do not understand the properties of cluster points. These two things show that Sh experienced a principle error. Sh has a procedural misunderstanding (for more details on reading comprehension domains [11]).

The principle mistake Sh made was also revealed on Sh.04. He stated that "because F subset of R means it must have members ... then F closed sets, meaning all members of F were also cluster points of F." Also, shows the scheme that was premature in its memory. He expressed the wrong argument. That was "yes surely ... because the R members were infinite ... there must be members included in F" (see Sh.05).

Fatal errors occur in cognitive processes (see Sh.06), they were statements that do not exist in the deductive structure of real analysis. It was making up and reveals a misleading understanding. He stated that "0 was definitely a member of F ... and that was minimal." It all signals that Sh only has low mental activity. Her actions were done intuitively separately. He interiorized it into a wrong process. A procedure Sh try to do using misguided arguments. The achievement of objects about cluster points and closed sets was structured from wrong understanding. As a result, Sh doesn't have a mature scheme about it. The basic qualities he ignored. He also revealed the misguided argument to declare that 0 has always been the computer point of a closed set. See Sh.8: "Yes ... all points will gather at 0 ... because 0 was in the middle of real numbers ... that was a sign that F was closed sets." This cognitive process falls into the category of intra level (Piaget & Garcia's theory [5], and see Widada [7]). Sh was experiencing weakness in the ability to understand facts, concepts. Also, Sh failed procedural understanding [11].

Sh's inability to be factual, conceptually and procedurally was the fundamental weakness he does. According to Anderson, et. al., That factual knowledge contains basic elements in which students must know if they learn or solve a problem. There were two subtypes of factual knowledge, namely 1) knowledge of terminology, and 2) knowledge of specific details and elements. Conceptual knowledge includes knowledge of categories, classifications and relationships between the two things were more
complex, forms of knowledge were organized. Conceptual knowledge includes explicit, implicit schemes, mental models, or theories in different cognitive psychology models. Procedural knowledge was knowledge about how to do things. This knowledge was often in the form of rows or lines of stages, which include skills, algorithms, techniques, methods, and procedures [11].

Based on the paper and pencil of 15 students, there was only 1 that has a unique obstacle. Therefore, we were only in depth interview of 1 subject. The results of this study were that students can only coordinate two or more set properties, and the difficulty of proving a closed sets theorem. One of these theorems was "F the set of parts of R, the F number of closed if and only if F contains all the cluster points". He can only take actions separately from the characteristics of the set given. Finally, we conclude that students' difficulties in understanding the characteristics of closed sets were the inability of students to carry out the interiorization from actions to processes. Also, it was not able to encapsulate processes into objects. He was unable to reach the scheme of closed sets properties.

4. Conclusion
We conclude that as long as students complete the real analysis task there were five errors that occur, namely errors of facts, concepts, principles, operations or procedures, and random errors. These mistakes were more due to the inability to conceptually understand the objects of real analysis. This results in a weak ability of students' logic in verifying the truth of a statement. They fail logically to connect the concepts and properties of the object to become a valid argument in a proof. Finally, he was unable to reach a mature scheme.

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