Autonomous generation of all Wigner functions and marginal probability densities of Landau levels

B. Demircioğlu\(^1\) and A. Verçin\(^2\)\(^\dagger\)

\(^1\) Ankara Nuclear Research and Training Center

06100, Beşevler, Ankara-Turkey

\(^2\) Department of Physics, Ankara University, Faculty of Sciences,

06100, Tandoğan-Ankara, Turkey

Abstract

Generation of Wigner functions of Landau levels and determination of their symmetries and generic properties are achieved in the autonomous framework of deformation quantization. Transformation properties of diagonal Wigner functions under space inversion, time reversal and parity transformations are specified and their invariance under a four-parameter subgroup of symplectic transformations are established. A generating function for all Wigner functions is developed and this has been identified as the phase-space coherent state for Landau levels. Integrated forms of generating function are used in generating explicit expressions of marginal probability densities on all possible two dimensional phase-space planes. Phase-space realization of unitary similarity and gauge transformations as well as some general implications for the Wigner function theory are presented.

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\(^*\)E.mail:bengu.demircioglu@taek.gov.tr

\(^\dagger\)E.mail:vercin@science.ankara.edu.tr
I. INTRODUCTION

Landau levels are equally spaced, highly degenerate energy levels of a charged particle moving on a plane under the influence of a perpendicular uniform and static magnetic field [1,2]. They play very important roles in our current understanding of two dimensional electron systems and their characteristic behaviors such as quantum Hall effect [2–4] and superconductivity [5] and in exploring electronic properties of various bound states of matter such as atoms, molecules and condensed matter in strong magnetic fields [6]. Regarding recent experiments [7,8] devised to measure Wigner function of various states of light and matter and to observe its remarkable properties, determination of Wigner functions of Landau levels and search for their salient features become very promising since they can be realized in several classes of systems.

Wigner function is quantum mechanical analogue of classical phase-space probability distribution function. It is the central concept of Weyl-Wigner-Groenewold-Moyal (WWGM) quantization which was initiated just after the formulation of quantum mechanics [9]. WWGM quantization consists of well established correspondence rules between the operator formulation of quantum mechanics and its phase-space formulation [9–11]. According to these rules Wigner function can be computed from

$$W_{\lambda \lambda'} = \int_{\mathbb{R}^D} \psi_{\lambda}(q + \frac{1}{2}y)\overline{\psi}_{\lambda'}(q - \frac{1}{2}y)e^{-i\frac{\hbar}{2}\mathbf{y} \cdot \mathbf{p}}dV(y),$$

(1.1)

where, $\hat{H}$ being the Hamiltonian operator, $\psi_{\lambda}$ is an eigenfunction of the Schrödinger equation $\hat{H}\psi_{\lambda} = E_{\lambda}\psi_{\lambda}$, $\overline{\psi}$ denotes the complex conjugation of $\psi$ and $(\mathbf{q}, \mathbf{p}) \equiv (q_1, \ldots, q_D, p_1, \ldots, p_D)$ are the phase space coordinates. $dV(y)$ stands for $D$-dimensional volume element in $y_j$ variables, $h = h/2\pi$ is the Planck’s constant, and $\mathbf{y} \cdot \mathbf{p}$ denotes the usual scalar product of vectors $\mathbf{y}$ and $\mathbf{p}$. The functions $W_{\lambda \lambda'} \equiv W_{\lambda \lambda'}(\mathbf{q}, \mathbf{p})$ are called off-diagonal Wigner functions and what is referred to as Wigner function is known also as the diagonal (or, pure state) Wigner function which corresponds taking $\psi_{\lambda} = \psi_{\lambda'}$ in (1.1), where $\lambda$ may be a multi index set of quantum numbers. Throughout this paper the pair $(q_j, p_j)$ will denote the canonically conjugate position and momentum variables for each $j = 1, 2, \ldots, D$, bold face letters will stand for vector quantities and when there is no risk of confusion arguments of functions will be suppressed.

Deformation quantization which was formulated in the pioneering work [12] in the seventies by generalizing WWGM quantization has become the third autonomous and logically complete formulation of quantum mechanics beyond the conventional ones based on operators in Hilbert space or path integral [13,14]. In this formulation quantum effects are encoded in a new operation called $\star$-product. This is a noncommutative but associative composition rule between functions defined on phase-space which can be any symplectic manifold [14,15] (see also [13,16] for recent developments). In terms of $\star$-product the spectrum and corresponding phase-space eigenfunctions of a Hamiltonian function $H$ can be determined through two-sided $\star$-eigenvalue equation [17–19]

$$H \star W_\lambda = W_\lambda \star H = E_\lambda W_\lambda.$$  

(1.2)

In that case the Wigner function of $H$ corresponding to the eigenvalue $E_\lambda$ is the $\star$-eigenfunction $W_\lambda$ of (1.1). When the underlying phase-space is $\mathbb{R}^{2D}$ with globally defined coordinates $(\mathbf{q}, \mathbf{p})$, the $\star$-product is given by
\[
\star = \exp\left[\frac{1}{2i\hbar} \sum_{j=1}^{D} \left( \partial_{q_j} \partial_{p_j} - \partial_{p_j} \partial_{q_j} \right) \right],
\]

(1.3)

where we use the abbreviation \( \partial_{x_i} \equiv \partial / \partial x_i \) and the convention that \( \partial_{\hat{\theta}} \) and \( \partial_{\hat{\phi}} \) are acting, respectively, on the left and on the right. (1.3) is known as the Moyal \( \star \)-product.

Apart from a few simple cases the use of (1.1) to obtain Wigner function is rather difficult since for a given wavefunction the resulting integrals are not easy to cope with. On the other hand, only a few \( \star \)-eigenvalue problems have been analytically handled until now. In fact, the basic physical systems such as H atom and harmonic oscillator were considered in [12] with special emphasis on determination of their spectra. Later on, some studies utilizing different techniques considered the same and some other systems in phase-space formulation [17–21]. To the best of our knowledge, the Ref. [21] is the first in dealing with Wigner function of Landau levels by using some WWGM correspondence rules. Systematic use of \( \star \)-eigenvalue equations were taken up in Ref. [19] which inspired us much.

The main goal of the present paper is to generate all Wigner functions and corresponding marginal probability densities of Landau levels on all possible phase-space planes. In doing that we shall follow an autonomous approach by working entirely within the framework of deformation quantization. All generic properties such as reality, normalization, projection and orthogonality as well as transformation and symmetry properties of these bound state Wigner functions are rigorously investigated in this framework without any reference to wavefunctions or to WWGM correspondence rules. After solving a \( \star \)-eigenvalue equation for the ground state Wigner function, Wigner functions of the excited states will be produced algebraically by successive applications of creation and annihilation functions. Then, off-diagonal Wigner functions and marginal probability densities will be obtained by means of associated generating functions.

The organization of the paper is as follows. In the next section the main points of deformation quantization, mainly those needed in subsequent investigation are briefly reviewed. Formulation of the problem and a closed algebraic form of diagonal Wigner functions of Landau levels are given in Sec. III, and their explicit functional forms are derived in Secs. IV and V. Symmetries and other generic properties are taken up in Secs. VI and VII. A generating function is introduced in Sec. VIII and all (diagonal and off-diagonal) Wigner functions are generated by means of it. In Sec. IX phase-space coherent state property of this generating function is presented and it is shown that its integrated forms serve as generating functions for marginal probability densities derived in Sec. X and presented altogether in Tables I and II. Final section is devoted to some remarks concerning justifications of some our results, implementation of unitary similarity transformations and gauge transformations in a phase space and to some general implications for Wigner function theory. In appendix we derive an operator identity used in the main text and exhibit a new finite sum formula for the generalized Laguerre polynomials.

II. STAR PRODUCT AND MOYAL BRACKET

Let us denote the linear space of complex valued, smooth (differentiable to all orders) functions defined on \( \mathbb{R}^{2D} \) by \( \mathcal{N} \). The Moyal \( \star \)-product (1.3) is a bilinear product \( \star : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \), which is, for all \( f_j \in \mathcal{N} \), associative.
\[
f_1 \ast (f_2 \ast f_3) = (f_1 \ast f_2) \ast f_3,
\]
and obeys the so-called closedness property [17,22]
\[
\int_{\mathbb{R}^{2D}} f_1 \ast f_2 dV = \int_{\mathbb{R}^{2D}} f_1 f_2 dV = \int_{\mathbb{R}^{2D}} f_2 \ast f_1 dV.
\]
This can be proved via integration by parts and it implies that integration, all over the phase-space, is a trace on the \(*\)-algebra of functions. From (1.3) immediately follows that under complex conjugation the \(*\)-product of functions satisfies the following relation
\[
\overline{(f_1 \ast f_2)} = \overline{f_2} \ast \overline{f_1}. \tag{2.3}
\]
It is also evident that \(f_1(q) \ast f_2(q) = f_1(q)f_2(q)\) and \(g_1(p) \ast g_2(p) = g_1(p)g_2(p)\), where the product on the right hand sides is the usual commutative pointwise product of functions.

In terms of \(*\)-product Moyal bracket \(\{,\}_M\) is defined, for all \(f,g \in \mathcal{N}\), as
\[
\{f,g\}_M = f \ast g - g \ast f. \tag{2.4}
\]
The properties of the \(*\)-product ensure that Moyal bracket (MB) is bilinear, antisymmetric and obey the Jacobi identity and Leibniz rule. With respect to MB, \(\mathcal{N}\) acquires a Lie algebra structure. The most important properties of \(*\)-product and MB are the following limit relations
\[
\lim_{\hbar \to 0} f \ast g = fg, \tag{2.5a}
\]
\[
\lim_{\hbar \to 0} (i\hbar)^{-1}\{f,g\}_M = \{f,g\}_P, \tag{2.5b}
\]
where
\[
\{f,g\}_P = \sum_{k=1}^{D} (\partial_{q_k} f \partial_{p_k} g - \partial_{p_k} f \partial_{q_k} g), \tag{2.6}
\]
is the usual Poisson bracket (PB) of the classical mechanics. The limit relations (2.5) reveal the fact that the associative \(*\)-algebra and Lie algebra structure of \(\mathcal{N}\) given by MB are, respectively, deformations (in the sense of Gerstanhaber [23]) of associative algebra structure of \(\mathcal{N}\) with respect to above mentioned pointwise product and of Lie algebra structure determined with respect to PB.

The set \(\mathcal{A}\) of phase-space functions \(f\)'s which satisfy
\[
\{f,g\}_M = i\hbar \{f,g\}_P, \tag{2.7}
\]
for all \(g \in \mathcal{N}\) plays an important role in deformation quantization. First, \(\mathcal{A}\) is a Lie subalgebra with respect to PB which is preserved under the deformation \(PB \to MB\). Second, with respect to PB, \(\mathcal{A}\) acts as a derivation on the \(*\)-algebra of functions, that is,
\[
\{f,g_1 \ast g_2\}_P = \{f,g_1\}_P \ast g_2 + g_1 \ast \{f,g_2\}_P, \tag{2.8}
\]
for all \(g_1, g_2 \in \mathcal{N}\) and \(f \in \mathcal{A}\). Finally, quantum mechanical time evolution of any element of \(\mathcal{A}\) is a classical evolution. This implies that when the Hamiltonian function belongs to \(\mathcal{A}\),
the classical and quantum time evolutions of an observable coincide. In the nomenclature of the deformation quantization $\mathcal{A}$ is called the invariance algebra of the $\star$-product and its elements are called preferred (good or distinguished) observables. As can easily be verified from (1.3) and (2.4) \( \mathcal{A} \) is spanned by

$$\{1, q_j, p_j, q_j q_{k\geq j}, p_j p_{k\geq j}, q_j p_k\}. \quad (2.9)$$

These define $2D^2 + 3D + 1$ dimensional affine symplectic algebra $w_D \oplus sp(2D)$, where $w_D$ stands for $2D + 1$ dimensional Heisenberg-Weyl algebra and $sp(2D)$ denotes $2D^2 + D$ dimensional symplectic algebra.

For $g = c_1 \cdot p + f$, where $c_1$ is a constant vector and $f = f(q)$, we have

$$g \star g^n = g^{n+1} + \sum_{j=2}^{\infty} \frac{1}{j!} \left( \frac{i\hbar}{2} \right)^j f(\sum_{k=1}^{D} \partial_{q_k} \partial_{p_k})^j g^n.$$ 

Hence, if $f$ is first order in $q_k$’s then $g \star g^n = g^{n+1}$ for any $n$, which implies that

$$(g_\star)^n = g^n, \quad \text{for} \quad g = c_0 + c_1 \cdot p + c_2 \cdot q. \quad (2.10)$$

where $c_0$ is a constant, $c_2$ is another constant vector and we have defined the $\star$-power of $g \in \mathcal{N}$ as

$$(g_\star)^n \equiv g \star g \star \cdots \star g, (n \text{ times}). \quad (2.11)$$

Finally in this section we should note that for two functions of the form $f = a_1 + h_1, g = a_2 + h_2$ such that $a_1, a_2 \in \mathcal{A}$, and $\{h_1, h_2\}_m = i\hbar \{h_1, h_2\}_p$ we have $\{f, g\}_m = i\hbar \{f, g\}_p$. As particular cases this holds for $h_j = h_j(p)$, or for $h_j = h_j(q)$.

**III. WIGNER FUNCTIONS OF LANDAU LEVELS**

The well known Landau Hamiltonian $H_L$ for a spinless particle of charge $q > 0$ and mass $m$ moving on the $q_1q_2$-plane is (in the Gaussian units)

$$H_L = \frac{1}{2m} (p - \frac{q}{c} A)^2 = \frac{1}{2} m(v_1^2 + v_2^2), \quad (3.1)$$

where $c$ is the speed of light, $A \equiv A(q)$ is the vector potential of the magnetic field

$$B = \partial_{q_1} A_2 - \partial_{q_2} A_1, \quad (3.2)$$

and $v = (p - \frac{q}{c} A)/m$ is the velocity vector whose components obey

$$\{v_1, v_2\}_m = \frac{i\hbar q}{m^2 c} B. \quad (3.3)$$

In writing (3.3) we have used the fact that the phase-space of the Landau problem is $\mathbb{R}^4$ with canonical coordinates $(q,p)$. In that case $\star$-product and MB are given by (1.3) and (2.4) for $D = 2$. We should also note that (3.3) is valid for any $B$. 

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When the magnetic field is constant the general solution of (3.2) is

$$ A = \left( -\frac{1}{2} B q_2 + \partial_{q_1} \chi, \frac{1}{2} B q_1 + \partial_{q_2} \chi \right), \quad (3.4) $$

where $\chi \equiv \chi(q)$ is an arbitrary gauge function. In such a case we have, in any gauge $\chi$, two constants of motion

$$ X_1 = m (v_2 + \omega q_1), \quad (3.5a) $$
$$ X_2 = -m (v_1 - \omega q_2), \quad (3.5b) $$

where $\omega = qB/mc$ is the cyclotron frequency. ($X_1/m\omega, X_2/m\omega$) correspond to the coordinates of cyclotron centre and satisfy the following gauge-independent relations

$$ \{X_1, X_2\}_M = -i m \hbar \omega, \quad (3.6a) $$
$$ \{v_j, X_k\}_M = 0 = \{H_L, X_k\}_M, \quad j, k = 1, 2. \quad (3.6b) $$

The relations $\{H_L, X_k\}_M = 0$ imply that $H_L, X_1$ and $X_2$ are constants of motion and since

$$ dX_1 \wedge dX_2 \wedge dH_L = m^3 \omega [(v_1 dq_2 - v_2 dq_1) \wedge dv_1 \wedge dv_2 + \frac{\omega}{m} \mathbf{v} \cdot \mathbf{dp} \wedge dq_1 \wedge dq_2] \quad (3.7) $$

they are functionally independent provided that $\mathbf{v} \neq 0$. In Eq. (3.7) $dq_j, dp_j$ denote the coordinate differentials (coordinate 1-forms) and $\wedge$ stands for the antisymmetric exterior product [15]. The above observations emphasize the quantum superintegrability of the problem. Moreover, by the remark at the end of previous section, it is easy to check that MBs in Eqs. (3.3) and (3.6) are equal to $i\hbar$ times the corresponding Poisson brackets. This implies that $\{H_L, X_1, X_2\}$ are at the same time classical constants of motion in any gauge, and hence the problem is also classically superintegrable. As a result, the Landau problem is one of the rare problems for which the classical and quantum superintegrability coincide.

For the phase-space quantization of the Landau problem we shall use $H_L$ and

$$ J = \frac{1}{2m\omega} (X^2 - m^2v^2), \quad (3.8) $$

as a complete Moyal-commuting phase-space functions. $J$ corresponds to canonical angular momentum $q_1p_2 - q_2p_1$ in the symmetric gauge $\chi = \text{constant}$. We then introduce two mutually commuting pairs of dimensionless creation and annihilation functions

$$ a = \frac{1}{\gamma \omega} (v_1 + iv_2), $$
$$ \bar{a} = \frac{1}{\gamma \omega} (v_1 - iv_2), \quad (3.9a) $$
$$ b = \frac{1}{m\gamma \omega} (X_2 + iX_1), $$
$$ \bar{b} = \frac{1}{m\gamma \omega} (X_2 - iX_1), \quad (3.9b) $$

where $\gamma = (2\hbar c/qB)^{1/2} = (2\hbar/m\omega)^{1/2}$ is the so-called magnetic length. $\gamma$ corresponds the radius of a disc from which a flux quantum $\hbar c/e$ passes for $q = e$, $e$ being the elementary charge. The nonvanishing Moyal brackets of (3.9) are
\{a, \bar{a}\}_M = 1 = \{b, \bar{b}\}_M. \quad (3.10)

In terms of real number functions defined by

\[ N_a = \bar{a} \star a, \quad N_b = \bar{b} \star b, \quad (3.11) \]

(3.1) and (3.8) take the form

\[ H_L = \hbar \omega (N_a + \frac{1}{2}), \quad (3.12a) \]
\[ J = \hbar (N_b - N_a). \quad (3.12b) \]

By making use of

\[ \{a, (\bar{a} \star a)^k\}_M = k(\bar{a} \star a)^{k-1}, \quad (3.13a) \]
\[ \{b, (\bar{b} \star b)^k\}_M = k(\bar{b} \star b)^{k-1}, \quad (3.13b) \]

and by defining the ground state Wigner function \( W_0 \) by

\[ a \star W_0 = 0 = b \star W_0, \quad (3.14) \]

it is straightforward to check that

\[ W_{nl} = \frac{1}{n!l!}(\bar{a} \star a)^n \star (\bar{b} \star b)^l \star W_0 \star (a \star a)^n \star (b \star b)^l, \quad (3.15) \]

satisfy the following “\( \star \)-ladder structure”

\[ a \star W_{nl} = W_{n-1,l} \star a, \]
\[ b \star W_{nl} = W_{n,l-1} \star b, \]
\[ \bar{a} \star W_{nl} = W_{n+1,l} \star \bar{a}, \]
\[ \bar{b} \star W_{nl} = W_{n,l+1} \star \bar{b}, \quad (3.16a) \]

where \( n, l \) are positive integers. We assume that \( W_0 \equiv W_{00} \) is a real function. Eqs. (3.16b) can also be verified by taking the complex conjugates of (3.16a) and then performing the shifts \( n \to n + 1 \) and \( l \to l + 1 \). For the number functions (3.11) we have

\[ N_a \star W_{nl} = W_{nl} \star N_a = \bar{a} \star W_{n-1,l} \star a = nW_{nl}, \quad (3.17a) \]
\[ N_b \star W_{nl} = W_{nl} \star N_b = \bar{b} \star W_{n,l-1} \star b = lW_{nl}. \quad (3.17b) \]

Hence, \( \{W_{nl} : n, l = 0, 1, \ldots\} \) is the set of simultaneous eigenfunctions of \( N_a \) and \( N_b \) whose spectra are bounded from below.

In view of the above relations it is now easy to show that \( W_{nl} \) satisfy

\[ H_L \star W_{nl} = W_{nl} \star H_L = E_n W_{nl}, \quad (3.18a) \]
\[ J \star W_{nl} = W_{nl} \star J = J_n W_{nl}, \quad (3.18b) \]

with \( \star \)-eigenvalues
$$E_n = \hbar \omega (n + \frac{1}{2}), \quad (3.19a)$$

$$J_{nl} = \hbar (l - n), \quad n, l = 0, 1, 2, ... \quad (3.19b)$$

$E_n$’s are the well-known infinitely degenerate (since every $E_n$ is independent from the quantum number $l$) Landau levels. The corresponding Wigner functions are given by (3.15) (or, by Eq. (3.22) below) in a closed algebraic way and in a gauge-independent manner. We should note that the degeneracy of a Landau level is a reflection of the fact that energy is independent of the location of the guiding center of gyrating charge. It is infinite when the motion takes place on the whole $\mathbb{R}^2$ plane and otherwise it is finite.

In the next section we will choose the symmetric gauge $\chi =$ constant, that is

$$(A_1, A_2) = \frac{B}{2} (-q_2, q_1), \quad (3.20)$$

and discuss the Wigner functions in any gauge in section XI. Before doing that we should note that in a gauge which is at most quadratic function of $q_j$’s $a, \bar{a}, b, \bar{b}$ are some linear functions of $q_j$’s and $p_j$’s. Hence, by virtue of (2.10) we get

$$(\bar{a}_s)^k = \bar{a}^k, \quad (a_s)^k = a^k, \quad (3.21a)$$

$$(\bar{b}_s)^k = \bar{b}^k, \quad (b_s)^k = b^k, \quad (3.21b)$$

and rewrite (3.15) as

$$W_{nl} = \frac{1}{n!l!} a^n \star \bar{b}^l \star W_0 \star a^n \star b^l. \quad (3.22)$$

Note that the $\star$-product between $a$’s and $b$’s can be removed for $\bar{a} \star \bar{b} = \bar{a} \bar{b}$ and $a \star b = ab$.

### IV. THE GROUND STATE WIGNER FUNCTION

Let us introduce the complex coordinates

$$z = \frac{1}{\sqrt{2}} (q_1 + iq_2), \quad p = \frac{1}{\sqrt{2}} (p_1 - ip_2), \quad (4.1a)$$

$$\bar{z} = \frac{1}{\sqrt{2}} (q_1 - iq_2), \quad \bar{p} = \frac{1}{\sqrt{2}} (p_1 + ip_2), \quad (4.1b)$$

which imply

$$\partial_{q_1} = \frac{1}{\sqrt{2}} (\partial_z + \partial_{\bar{z}}), \quad \partial_{p_1} = \frac{1}{\sqrt{2}} (\partial_p + \partial_{\bar{p}}), \quad (4.2a)$$

$$\partial_{q_2} = \frac{i}{\sqrt{2}} (\partial_z - \partial_{\bar{z}}), \quad \partial_{p_2} = \frac{i}{\sqrt{2}} (\partial_p - \partial_{\bar{p}}). \quad (4.2b)$$

In terms of new coordinates the $\star$-product (1.3) (for $D = 2$) transforms to

$$\star = \exp \left[ \frac{i}{\hbar} \left( \partial_z \partial_p + \partial_{\bar{z}} \partial_{\bar{p}} - \partial_p \partial_{\bar{z}} - \partial_{\bar{p}} \partial_z \right) \right], \quad (4.3)$$
and the annihilation functions take the form

\[ a = \frac{\sqrt{2}}{m\gamma\omega} (\bar{p} - \frac{im\omega}{2} \bar{z}), \quad (4.4a) \]

\[ b = \frac{\sqrt{2}}{m\gamma\omega} (-p + \frac{im\omega}{2} z). \quad (4.4b) \]

In that case the defining relations (3.14) of \( W_0 \) are as follow

\[ \bar{p} \star W_0 - \frac{im\omega}{2} z \star W_0 = 0, \quad (4.5a) \]

\[ p \star W_0 - \frac{im\omega}{2} \bar{z} \star W_0 = 0. \quad (4.5b) \]

Combining the complex conjugate of (4.5a) with (4.5b) we get

\[ \partial_z W_0 = -\frac{m\omega}{\hbar} \bar{z} W_0, \quad \partial_{\bar{p}} W_0 = -\frac{4}{m\omega\hbar} p W_0. \quad (4.6) \]

By ignoring a possible factor \( f(\bar{z}, p) \) which destroys the reality of \( W_0 \), and by defining

\[ H_0 = \frac{1}{m} p\bar{p} + \frac{1}{4} m\omega^2 \bar{z} \bar{z} = \frac{p^2}{2m} + \frac{m\omega^2}{8} q^2, \quad (4.7) \]

which corresponds the Hamiltonian function of a two-dimensional isotropic harmonic oscillator, the general real solution of Eqs. (4.6) is

\[ W_0 = 4e^{-\frac{\hbar}{m\omega} H_0}. \quad (4.8) \]

We have normalized \( W_0 \) as \( \int_{\mathbb{R}^4} W_0 \, dV = \hbar^2 \), where the volume form \( dV \) is

\[ dV = dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2 = dz \wedge d\bar{z} \wedge dp \wedge d\bar{p}. \quad (4.9) \]

We note that the coordinate transformation given by (4.1) is a canonical transformation since the symplectic 2-form \( \Omega \) remains invariant:

\[ \Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = dz \wedge dp + d\bar{z} \wedge d\bar{p}. \quad (4.10) \]

V. WIGNER FUNCTIONS OF HIGHER STATES

Observing that the \( \star \)-product (4.3) can be expressed in terms of \( a, \bar{a}, b, \bar{b} \), explicit expressions of the Wigner functions and other related computations can be carried out in an easier and more suggestive way. To this end from Eqs. (4.4) and their complex conjugates we obtain

\[ z = \frac{i\gamma}{\sqrt{2}} (a + \bar{b}), \quad p = \frac{m\omega\gamma}{2\sqrt{2}} (\bar{a} - b), \quad (5.1a) \]

\[ \bar{z} = -\frac{i\gamma}{\sqrt{2}} (\bar{a} + b), \quad \bar{p} = \frac{m\omega\gamma}{2\sqrt{2}} (a - \bar{b}), \quad (5.1b) \]
\[ \partial z = -\frac{i}{\gamma\sqrt{2}}(\partial_a + \partial_b), \quad \partial_p = \frac{\sqrt{2}}{m\omega\gamma}(\partial_a - \partial_b), \quad (5.2a) \]
\[ \partial \bar{z} = \frac{i}{\gamma\sqrt{2}}(\partial_a + \partial_b), \quad \partial \bar{p} = \frac{\sqrt{2}}{m\omega\gamma}(\partial_a - \partial_b). \quad (5.2b) \]

Then (4.3) and (4.7) can be written as
\[ \ast = \exp\left(\frac{1}{2}(\partial_a \partial_a + \partial_b \partial_b - \partial_a \partial_a - \partial_b \partial_a)\right), \quad (5.3a) \]
\[ H_0 = \frac{1}{2}\hbar\omega(\bar{a}a + \bar{b}b). \quad (5.3b) \]

By inserting (5.3b) and (4.8) into (3.22) and then by defining
\[ w_n = \frac{1}{n!}\bar{a}^n \ast e^{-2\bar{a}} \ast a^n, \quad (5.4a) \]
\[ w_l = \frac{1}{l!}\bar{b}^l \ast e^{-2\bar{b}} \ast b^l, \quad (5.4b) \]

\( W_{nl} \) can be cast to the following factorized form
\[ W_{nl} = 4w_n(a, \bar{a})w_l(b, \bar{b}). \quad (5.5) \]

By virtue of (5.3a) one can easily verify that
\[ e^{-2\bar{a}} \ast a^n = \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \frac{1}{k!}(\partial_a^k e^{-2\bar{a}})(\partial_a^k a^n) = 2^n a^n e^{-2\bar{a}}, \quad (5.6) \]
where \((\binom{n}{k})\) being a binomial number we have used the identity \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\). On substituting (5.6) into (5.4a) we obtain
\[ w_n = \frac{2^n}{n!}\bar{a}^n \ast (a^n e^{-2\bar{a}}) \]
\[ = \frac{2^n}{n!}(\bar{a} - \frac{1}{2}\partial_a)^n(a^n e^{-2\bar{a}}) = e^{-2\bar{a}}P_n(2\bar{a}), \quad (5.7) \]

where we have defined
\[ P_n(u) = \frac{1}{n!}e^u(1 - \frac{d}{du})^n(u^n e^{-u}). \quad (5.8) \]

In the appendix we will prove the following operator identity
\[ (1 - \frac{d}{du})^n(u^n e^{-u}) = (-1)^n e^u(\frac{d}{du})^n(u^n e^{-2u}). \quad (5.9) \]

On multiplying both sides of these identity by \(e^u/n!\) and by recalling the Rodriguez formula of the Laguerre polynomials (see the appendix)
\[ L_n(2u) = e^{2u} \frac{d^n}{du^n}(u^n e^{-2u}), \]  
(5.10)

we see that

\[ P_n(u) = (-1)^n L_n(2u). \]  
(5.11)

Therefore, from (5.7) and (5.11) the explicit functional form of diagonal Wigner functions (5.5) are obtained as follows

\[ W_{nl} = (-1)^{n+l} 4 L_n(4\bar{a}\bar{a}) L_l(4\bar{b}\bar{b}) e^{-2(\bar{a}\bar{a} + \bar{b}\bar{b})}. \]  
(5.12)

VI. SYMMETRY PROPERTIES

Some transformation and symmetry properties of Wigner functions immediately follow from their factorized forms given by (5.12) and from following relations

\[ \bar{a}a = \frac{H_L}{\hbar \omega} = \frac{1}{2\hbar \kappa} |p + i\kappa \bar{z}|^2 \]
\[ = \frac{1}{4\hbar \kappa} [(p_1 + \kappa q_2)^2 + (p_2 - \kappa q_1)^2], \]  
(6.1a)

\[ \bar{b}b = \frac{J}{\hbar} + \frac{H_L}{\hbar \omega} = \frac{1}{2\hbar \kappa} |p - i\kappa \bar{z}|^2 \]
\[ = \frac{1}{4\hbar \kappa} [(p_1 - \kappa q_2)^2 + (p_2 + \kappa q_1)^2], \]  
(6.1b)

where \( \kappa = m \omega / 2 \). As is apparent from the first equalities of (6.1a) and (6.1b), the values of \( W_{nl} \) depend on the level sets of \( H_L / \hbar \omega \) and of \( J / \hbar \) which are classical constants of motion and define some surfaces in \( \mathbb{R}^4 \). Rewriting (5.12) as

\[ W_{nl} = (-1)^{n+l} 4 L_n(4\bar{a}\bar{a}) L_l(4\bar{b}\bar{b}) e^{-2\bar{a}\bar{a} - 4\bar{b}\bar{b}}. \]  
(6.2)

makes it possible to read out the values of \( W_{nl} \)’s on various level sets very easily. In particular, since \( L_n(0) = 1 \), at the phase-space origin which corresponds to \( H_L = 0 = J \) we have

\[ W_{nl}(0, 0) = 4(-1)^{n+l}. \]  
(6.3)

The following relations now easily follow from (5.12) and (6.1),

\[ W_{nl}(-q, p) = W_{ln}(q, p) = W_{ln}(q, -p), \]  
(6.4a)

\[ W_{nl}(-q, -p) = W_{nl}(q, p), \]  
(6.4b)

\[ W_{nl}(q_2, q_1, p_2, p_1) = W_{ln}(q, p). \]  
(6.4c)

(6.4a) shows that under the space inversion \( (q \rightarrow -q) \) and time reversal \( (p \rightarrow -p) \) transformations all Wigner functions transform in the same way. In particular, Wigner functions for
which $n = l$ possess both space inversion and time reversal symmetries separately and they are positive valued. Eq. (6.4b) directly results from (6.4a) and it implies that all Wigner functions are invariant under the phase-space parity transformation $(q, p) \rightarrow (-q, -p)$. The transformation property given by (6.4c) will be helpful in deriving marginal probability densities (see Sec. X). We should note that although (6.1a) and (6.1b) have some translational invariance properties they do not lead to a translational symmetry of $W_{nl}$. Namely, the following translational invariance of $H_L$:

$$H_L(q + c, p - \kappa c^*) = H_L(q, p),$$

where $c = (c_1, c_2)$, $c^* = (c_2, -c_1)$, is not compatible with that of (6.1b).

Due to the fact that $W_{nl}$ depend on the products $a\bar{a}, b\bar{b}$, a larger class of symmetries can immediately be identified. Indeed, it is evident that under transformations given by

$$(a, b, \bar{a}, \bar{b}) \rightarrow (ue^{i\xi}a, ve^{i\eta}b, u^{-1}e^{-i\xi}\bar{a}, v^{-1}e^{-i\eta}\bar{b}),$$

where $\xi, \eta$ and $u \neq 0, v \neq 0$ are some real parameters, all $W_{nl}$ remain invariant. Obviously, this transformation also leaves $H_L$ and $J$ invariant. To express this transformation in terms of $(q, p)$ coordinates, we define two column matrices $x$ and $y$ by

$$x^T = (a, b, \bar{a}, \bar{b}),$$

$$y^T = (q_1, q_2, p_1, p_2),$$

where $x^T$ denotes the transpose of $x$. We then introduce two nonsingular matrices

$$A = \begin{pmatrix}
ue^{i\xi} & 0 & 0 & 0 \\
0 & ve^{i\eta} & 0 & 0 \\
0 & 0 & u^{-1}e^{-i\xi} & 0 \\
0 & 0 & 0 & v^{-1}e^{-i\eta}
\end{pmatrix},$$

$$B = \gamma^2 \gamma^2 \begin{pmatrix}
i & -i & -i & i \\
1 & 1 & 1 & 1 \\
\kappa & -\kappa & \kappa & -\kappa \\
-\kappa & -\kappa & \kappa & i\kappa
\end{pmatrix},$$

having the determinants $detA = 1, detB = -\gamma^4\kappa^2 = -\hbar^2$. In this matrix notation the transformation (6.5) and relation between $y$ and $x$ simply read as

$$x' = Ax, \quad y = Bx.$$  

In writing $y = Bx$ we made use of Eqs. (4.1) and (5.1). Since $y' = Bx' = BABx$ we deduce, from (6.7) that the transformation (6.5) can be written, in the canonical coordinates, as

$$y' = Cy,$$

$$C = BAB^{-1} = \frac{1}{4} \begin{pmatrix}
M & -\kappa^{-1}N \\
\kappa N & M
\end{pmatrix},$$

where $y'^T = (q'_1, q'_2, p'_1, p'_2)$ and

$$M = \begin{pmatrix}
c_+ & -s_- \\
s_- & c_+
\end{pmatrix}, \quad N = \begin{pmatrix}
s_+ & c_- \\
-c_- & s_+
\end{pmatrix},$$

$$c_\pm = u_+ \cos \xi \pm v_+ \cos \eta + i(u_\pm \sin \xi \pm v_\pm \sin \eta),$$

$$s_\pm = u_+ \sin \xi \pm v_+ \sin \eta - i(u_\pm \cos \xi \pm v_\pm \cos \eta),$$

$$u_\pm = u \pm \frac{1}{u_\mp}, \quad v_\pm = v \pm \frac{1}{v_\mp}.$$
In view of $\det C = \det A = 1$ and

\begin{align}
NN^T + MM^T &= 161, \\
NM^T - MN^T &= 0,
\end{align}

where $1$ and $0$ denote, respectively, $2 \times 2$ unit and zero matrices, it is straightforward to check that $C$ is a symplectic matrix. That is,

\[ J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

being the standard symplectic matrix, $C$ satisfies the symplectic condition $CJ_0C^T = J_0$. Hence $C$ belongs to the symplectic group $Sp(4, \mathbb{C})$. Evidently, $W_{nl}(C_1y) = W_{nl}(y) = W_{nl}(C_2y)$ imply that $W_{nl}(C_1C_2^{-1}y) = W_{nl}(y)$. In other words, if $C_1$ and $C_2$ matrices leave $W_{nl}(y)$ invariant so does $C_1C_2^{-1}$. This means that the transformations (6.8) form a four parameter subgroup of $Sp(4, \mathbb{C})$.

Finally in this section we note that the volume form and symplectic 2-form in $(a, b, \bar{a}, \bar{b})$ coordinates take the form

\begin{align}
dV &= dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2 = \hbar^2 da \wedge d\bar{a} \wedge db \wedge d\bar{b}, \\
\Omega &= dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = i\hbar (da \wedge d\bar{a} + db \wedge d\bar{b}).
\end{align}

Then, the symplectic property of the transformation (6.5) can directly be observed from the second equality of (6.12b).

**VII. GENERIC PROPERTIES**

In this section we shall exhibit some generic properties of diagonal Wigner functions, such as the reality, $\star$-projection, normalization, and orthogonality properties, within autonomous framework of deformation quantization. Marginal probability densities and their basic properties will be taken up in section X. As is apparent from (2.3), (3.15) (or (3.22)) and (4.8) the reality condition $W_{nl} = W_{nl}$ of diagonal Wigner functions is guaranteed by construction from the beginning. On the other hand, the following form of the so-called $\star$-projection property

\[ W_{nl} \star W_{n'l'} \propto \delta_{nn'}\delta_{ll'} W_{nl} \]

(7.1)

can be easily inferred from (3.18) by taking the $\star$-multiplication of them from the left and then from the right by $W_{n'l'}$. We start by deriving the exact form of this relation for the Wigner function of Landau levels.

Noting that, by (5.3a)

\[ (a\bar{a})^{k+1} = (a\bar{a})^k \star a\bar{a} + \frac{k^2}{4} (a\bar{a})^{k-1} \]

(7.2)

one can verify $(a\bar{a})^k \star W_0 = (k!/2^k)W_0$ by induction and
\[ e^{-2ta\bar{a}} \ast W_0 = \frac{1}{1 + t} W_0. \] (7.3)

This is a special case of \[ e^{-2ta\bar{a}} \ast e^{-2s\bar{a}} = (1 + ts)^{-1} \exp[-2(t + s)(1 + ts)^{-1}a\bar{a}] \] which can be proved by other means (see Eq. 34 in [19]). Hence, we obtain the \( \ast \)-projection property of \( W_0 \) as follows\(^1\)

\[ W_0 \ast W_0 = 4e^{-2a\bar{a}} \ast e^{-2b\bar{b}} \ast W_0 = W_0. \] (7.4)

By virtue of (3.13) we have

\[ a^n \ast a^{n'} = \prod_{j=n+1}^{n+1}(N_a + j), \quad \text{for} \quad n' > n, \]
\[ a^{n-1} \ast a^{n'-1} \ast (N_a + n') = a^{n-2} \ast a^{n'-2} \ast (N_a + n' - 1) \ast (N_a + n'). \] (7.5)

This leads us to, by induction

\[ a^n \ast a^{n'} = \begin{cases} 
\prod_{j=n+1}^{n+1}(N_a + j), & \text{for} \quad n' > n, \\
\prod_{j=1}^{n-1}(N_a + j), & \text{for} \quad n' < n, \\
\prod_{j=1}^{n}(N_a + j), & \text{for} \quad n' = n,
\end{cases} \] (7.6)

where

\[ \prod_{j=i}^{n'}(N_a + j) \equiv (N_a + i) \ast (N_a + i + 1) \ast \cdots \ast (N_a + n'). \] (7.7)

Similar relations hold for \( b^l \ast b^{l'} \). On the other hand since

\[ W_0 \ast N_a \ast W_0 = 0 = W_0 \ast N_b \ast W_0, \]
\[ W_0 \ast a^k \ast W_0 = 0 = W_0 \ast \bar{a}^k \ast W_0, \]
\[ W_0 \ast b^k \ast W_0 = 0 = W_0 \ast \bar{b}^k \ast W_0, \]

we see that

\[ I_{nn',ll'} \equiv W_0 \ast a^n \ast a^{n'} \ast b^l \ast b^{l'} \ast W_0 \]
\[ = n!! \delta_{nn'} \delta_{ll'} W_0 \ast W_0 = n!! \delta_{nn'} \delta_{ll'} W_{nl}. \] (7.8)

Then, from (3.22) and (7.8) we obtain

\[ W_{nl} \ast W_{n'l'} = \frac{1}{n!!l!!l'!!} \bar{a}^n \ast \bar{b}^l \ast I_{nn',ll'} \ast a^{n'} \ast b^{l'} = \delta_{nn'} \delta_{ll'} W_{nl}, \] (7.9)

for all diagonal Wigner functions of Landau levels

By virtue of (2.2), (3.22), (4.9) and (7.6) we have the following normalization property

\(^1\)In [24] a misleading factor \( 4/e^2 \) that appears at the end of Eq. (7.4) and in Eqs. (7.8), (7.9), (7.11) should be removed.
\[
\int_{\mathbb{R}^4} W_{nl}dV = \frac{1}{n!!} \int_{\mathbb{R}^4} \bar{a}^n \bar{b}^l \star W_0 \star a^n \star b^l dV,
\]
\[
= \frac{1}{n!!} \int_{\mathbb{R}^4} a^n \bar{a}^n \star b^l \star \bar{b}^l \star W_0 dV,
\]
\[
= \int_{\mathbb{R}^4} W_0 dV = \hbar^2.
\] (7.10)

Finally, the \(\star\)-orthogonality directly follows from (2.2), (7.9) and (7.10):
\[
\int_{\mathbb{R}^4} W_{nl} \star W_{n'l'}dV = \int_{\mathbb{R}^4} W_{nl} W_{n'l'}dV,
\]
\[
= \delta_{nn'}\delta_{ll'} \int_{\mathbb{R}^4} W_{nl} dV = \hbar^2 \delta_{nn'}\delta_{ll'}. \]
(7.11)

As is apparent, many properties of \(W_{nl}\) are guaranteed by the corresponding properties of \(W_0\).

**VIII. GENERATING FUNCTION FOR ALL WIGNER FUNCTIONS**

Let us define the phase-space functions

\[
G_1(a, \bar{a}) = e^{a_1\bar{a}} \star e^{-2\bar{a}a} \star e^{\beta_1a},
\]
\[
G_2(b, \bar{b}) = e^{a_2\bar{b}} \star e^{-2\bar{b}b} \star e^{\beta_2b},
\]
(8.1a, 8.1b)

where \(\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)\) are some parameters. Using the shorthands \(G_1 = G_1(a, \bar{a})\) and \(G_2 = G_2(b, \bar{b})\) we define and compute \(w_{n_1n_2}(a, \bar{a}), w_{l_1l_2}(b, \bar{b})\) as follows

\[
w_{n_1n_2}(a, \bar{a}) = \sqrt{1/n_1!n_2!} \bar{a}^{n_1} \partial_{\alpha_1}^{n_1} \partial_{\beta_1}^{n_2} G_1|_{\alpha_1=0=\beta_1},
\]
\[
= \sqrt{1/n_1!n_2!} \bar{a}^{n_1} \star e^{-2\bar{a}a} \star a^{n_2},
\] (8.2a)

\[
w_{l_1l_2}(b, \bar{b}) = \sqrt{1/l_1!l_2!} \bar{b}^{l_1} \partial_{\alpha_2}^{l_1} \partial_{\beta_2}^{l_2} G_2|_{\alpha_2=0=\beta_2},
\]
\[
= \sqrt{1/l_1!l_2!} \bar{b}^{l_1} \star e^{-2\bar{b}b} \star b^{l_2}.
\] (8.2b)

We then define the generating function

\[
G = G_1(a, \bar{a})G_2(b, \bar{b}),
\]
(8.3)

from which all (diagonal and off-diagonal) Wigner functions can be generated as

\[
W_{n_1n_2l_1l_2} = \frac{4}{\sqrt{n_1!n_2!l_1!l_2!}} \partial_{\alpha_1}^{n_1} \partial_{\beta_1}^{n_2} \partial_{\alpha_2}^{l_1} \partial_{\beta_2}^{l_2} G|_{\alpha=0=\beta},
\]
\[
= 4w_{n_1n_2}(a, \bar{a})w_{l_1l_2}(b, \bar{b}),
\] (8.4)

In the case of \(n_1 = n_2 = n, l_1 = l_2 = l\), (8.2) and (8.4) coincide with (5.4) and (5.5).

Making use of

\[
e^{a_1\bar{a}} \star e^{-2\bar{a}a} = e^{a_1\bar{a}}e^{-\frac{i}{\hbar}\partial_{\bar{a}}\partial_a} e^{-2\bar{a}a} = e^{2a(a_1-a)}
\] (8.5)
one can easily verify that
\[ G_1 = e^{-\alpha_1 \beta_1} e^{2(\alpha_1 \bar{a} + \beta_1 a)} e^{-2\bar{a}}, \]  
(8.6)
and a similar relation for \( G_2 \). Hence
\[ G = e^{-\alpha \beta} e^{2(\alpha \bar{a} + \beta_1 a + \alpha_2 b + \beta_2 b)} e^{-\frac{4j\omega}{\pi z}}, \]  
(8.7)
where \( \alpha \cdot \beta = \alpha_1 \beta_1 + \alpha_2 \beta_2 \) and we have used (5.3b).

Now let us consider the following expansion
\[ e^{-\alpha_1 \beta_1} e^{2(\alpha_1 \bar{a} + \beta_1 a)} = \sum_{k=0}^{\infty} \frac{\alpha_1^k}{k!} (2\bar{a})^k (1 - \frac{\beta_1}{2\bar{a}})^k e^{2\beta_1 a}, \]  
(8.8)
Making use of the generating function of the generalized Laguerre polynomials [28,29]
\[ (1 + y)^k e^{-xy} = \sum_{n=0}^{\infty} L_n^{k-n}(x)y^n, \]  
(8.9)
which holds for \( |y| < 1 \), we rewrite (8.8) as
\[ e^{-\alpha_1 \beta_1} e^{2(\alpha_1 \bar{a} + \beta_1 a)} = \sum_{k,n=0}^{\infty} \frac{\alpha_1^k}{k!} (2\bar{a})^{k-n} (-\beta_1)^n L_n^{k-n}(4a\bar{a}). \]  
(8.10)
The condition \( |y| < 1 \) impose the constraint \( 2|\bar{a}| > |\beta_1| \) for Eq. (8.10). Therefore, from (8.6) we have
\[ G_1 = e^{-2a\bar{a}} \sum_{k,n=0}^{\infty} \frac{\alpha_1^k}{k!} (2\bar{a})^{k-n} (-\beta_1)^n L_n^{k-n}(4a\bar{a}), \]  
(8.11a)
\[ G_2 = e^{-2b\bar{b}} \sum_{k,n=0}^{\infty} \frac{\alpha_2^k}{k!} (2\bar{b})^{k-n} (-\beta_2)^n L_n^{k-n}(4b\bar{b}), \]  
(8.11b)
and, by (8.2)
\[ w_{n_1 n_2}(a, \bar{a}) = \sqrt{\frac{n_2!}{n_1!}} (-1)^{n_2} (2\bar{a})^{n_1-n_2} L_{n_2}^{n_1-n_2}(4a\bar{a}) e^{-2a\bar{a}}, \]  
(8.12a)
\[ w_{l_1 l_2}(b, \bar{b}) = \sqrt{\frac{l_2!}{l_1!}} (-1)^{l_2} (2\bar{b})^{l_1-l_2} L_{l_2}^{l_1-l_2}(4b\bar{b}) e^{-2b\bar{b}}. \]  
(8.12b)
On substituting (8.12) into (8.4) we get
\[ W_{n_1 n_2 l_1 l_2} = 4 \sqrt{\frac{n_2! l_2!}{n_1! l_1!}} (-1)^{n_2+l_2} (2\bar{a})^{n_1-n_2} (2\bar{b})^{l_1-l_2} L_{n_2}^{n_1-n_2}(4a\bar{a}) L_{l_2}^{l_1-l_2}(4b\bar{b}) e^{-\frac{4j\omega}{\pi z}}. \]  
(8.13)
We should note that when superscript of a Laguerre polynomials is a negative integer the following formula must be used
\[ L_n^{k}(x) = (-x)^k \frac{(n - k)!}{n!} L_n^{k}(x), \]  
(8.14)
where \( k \) is positive integer. In such a case the resulting expression can also be obtained by starting with the following equivalent expansion of the left hand side of (8.8):
\[ e^{-\alpha_1 \beta_1} e^{2(\alpha_1 \bar{a} + \beta_1 a)} = \sum_{k=0}^{\infty} \frac{\beta_1^k}{k!} (2\bar{a})^{k} (1 - \frac{\alpha_1}{2\bar{a}})^k e^{2\alpha_1 \bar{a}}. \]  
In any case (8.13) coincides with (5.12) for \( n_1 = n_2 = n \) and \( l_1 = l_2 = l \).
IX. PHASE-SPACE COHERENT STATES AND GENERATING FUNCTIONS
FOR MARGINAL PROBABILITY DENSITIES

In this section we shall exhibit two important properties of the generating function $G$. The first is that $G$ can be interpreted as a phase-space coherent state of Landau levels. The second property is that the integrated forms of $G$ over some phase-space planes serve as generating functions for marginal probability densities (distributions) on these planes.

For the first property we compute

$$a \star G = (a + \frac{1}{2} \partial_{\bar{a}})G = \alpha_1 G,$$

$$b \star G = (b + \frac{1}{2} \partial_{\bar{b}})G = \alpha_2 G,$$

which imply that $G$ is a left (complex) $\star$-eigenfunction of the annihilation functions. On the other hand, since

$$G \star \bar{a} = (\bar{a} + \frac{1}{2} \partial_{a})G = \beta_1 G,$$

$$G \star \bar{b} = (\bar{b} + \frac{1}{2} \partial_{b})G = \beta_2 G,$$

$G$ is at the same time the right $\star$-eigenfunction of the creation functions. For these reasons $G$ behaves as a left/right coherent state. Moreover

$$\bar{a} \star G = (\bar{a} - \frac{1}{2} \partial_{\bar{a}})G = (2\bar{a} - \beta_1)G = \partial_{\alpha_1} G,$$

$$\bar{b} \star G = (\bar{b} - \frac{1}{2} \partial_{\bar{b}})G = (2\bar{b} - \beta_2)G = \partial_{\alpha_2} G,$$

$$G \star a = (a - \frac{1}{2} \partial_{\bar{a}})G = (2a - \alpha_1)G = \partial_{\beta_1} G,$$

$$G \star b = (b - \frac{1}{2} \partial_{\bar{b}})G = (2b - \alpha_2)G = \partial_{\beta_2} G,$$

provide us with the phase-space Bopp realization of left (right) $\star$-actions of $\bar{a}, \bar{b}$ ($a, b$). To make $G$ real it is sufficient to take $\bar{\alpha} = \beta$. In such a case $G$ corresponds to the Glauber-Perelomov standard coherent state [25].

To exhibit the second property of $G$, we define the generating function for the marginal probability densities in the $q_1q_2$-plane as follows

$$M_{\alpha\beta}(q_1, q_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Gdp_1dp_2.$$

Then the marginal probability density $P_{nl}(q_1, q_2)$ in the $q_1q_2$-plane can be generated as

$$P_{nl}(q_1, q_2) = \frac{4}{n!!}(\partial_{\alpha_1} \partial_{\beta_1})^{n}(\partial_{\alpha_2} \partial_{\beta_2})^{l}M_{\alpha\beta}(q_1, q_2)|_{\alpha=0=\beta}.$$
\[ P_{nl}(q_1, q_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{nl} dp_1 dp_2, \]  
(9.6)

provided that we can change the order of derivatives with respect to parameters in integration on phase-space coordinates. The relation (9.6) justifies our naming \( M_{\alpha\beta} \) as the generating functions for marginal probability densities (see also (11.2)).

Similarly, the generating function for the marginal probability density in the \( p_1p_2 \)-plane can be defined as
\[ M_{\alpha\beta}(p_1, p_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G dq_1 dq_2, \]  
(9.7)

and the marginal probability density \( P_{nl}(p_1, p_2) \) in the \( p_1p_2 \)-plane can be generated as
\[ P_{nl}(p_1, p_2) = \frac{4}{n!} (\partial_\alpha \partial_\beta_1)^n (\partial_\alpha_2 \partial_\beta_2)^l M_{\alpha\beta}(p_1, p_2)|_{\alpha=0=\beta}. \]  
(9.8)

Generalizing these observations we can define generating functions \( M_{\alpha\beta}(q_j, p_k); j, k = 1, 2 \) in order to generate the marginal densities \( P_{nl}(q_j, p_k) \) on the \( q_jp_k \)-planes.

**X. MARGINAL PROBABILITY DENSITIES OF LANDAU LEVELS**

To ease the calculations of this section we shall use the dimensionless coordinates
\[ Z = \frac{1}{\gamma} (q_1 + iq_2), \quad \bar{Z} = \frac{1}{\gamma} (q_1 - iq_2), \]  
(10.1a)
\[ P = \frac{1}{m\omega\gamma} (p_1 + ip_2), \quad \bar{P} = \frac{1}{m\omega\gamma} (p_1 - ip_2), \]  
(10.1b)

and define the dimensionless quantities
\[ \rho^2 = Z \bar{Z} = \frac{1}{\gamma^2} (q_1^2 + q_2^2), \]  
(10.2a)
\[ \zeta^2 = P \bar{P} = \frac{\gamma^2}{4\hbar^2} (p_1^2 + p_2^2), \]  
(10.2b)
\[ I_q = (\alpha_1 + \beta_2) \bar{Z} - (\alpha_2 + \beta_1) Z, \]  
(10.2c)
\[ I_p = (\alpha_2 - \beta_1) P - (\alpha_1 - \beta_2) \bar{P}. \]  
(10.2d)

We then rewrite (8.7) as
\[ G = e^{-\alpha \cdot \beta} e^{-\rho^2 + iI_q} e^{-4\zeta^2 - 2I_p}. \]  
(10.3)

Substituting (10.3) into (9.4) we obtain
\[ M_{\alpha\beta}(q_1, q_2) = e^{-\alpha \cdot \beta} e^{-\rho^2 + iI_q} K_1 K_2 = \pi \left( \frac{\hbar}{\gamma} \right)^2 e^{-\rho^2} Q_\alpha Q_\beta, \]  
(10.4)

where
are obtained by the standard complex integration methods and

$$Q_\alpha = e^{-\alpha_1\alpha_2 + i(\alpha_1Z - \alpha_2Z)}, \quad Q_\beta = e^{-\beta_1\beta_2 - i(\beta_1Z - \beta_2Z)}.$$  

(10.5) When Eq. (10.4) is substituted into (9.5), in terms of

$$J_{nl}^1 = \partial_{\alpha_1}^n \partial_{\alpha_2}^l Q_\alpha|_{\alpha_1=0=\alpha_2}, \quad J_{nl}^2 = \partial_{\beta_1}^n \partial_{\beta_2}^l Q_\beta|_{\beta_1=0=\beta_2},$$

(10.6) we obtain

$$P_{nl}(q_1, q_2) = \frac{4\pi}{n!l!} (\frac{\hbar}{\gamma})^2 e^{-\rho^2} J_{nl}^1 J_{nl}^2.$$  

(10.7)

Similar to (8.8) and (8.10), we can expand (10.5) as follows

$$Q_\alpha = \sum_{j=0}^\infty \frac{(i\alpha_1Z)^j}{j!} (1 + i \frac{\alpha_2}{Z})^j e^{-i\alpha_2Z}$$

$$= \sum_{j,k=0}^\infty \frac{i^{j+k}}{j!} Z^{j-k} \alpha_1^j \alpha_2^k L_k^{j-k}(\rho^2),$$  

(10.8a)

$$Q_\beta = \sum_{j=0}^\infty \frac{(-i\beta_1Z)^j}{j!} (1 - i \frac{\beta_1}{Z})^j e^{i\beta_2Z}$$

$$= \sum_{j,k=0}^\infty \frac{(-i)^{j+k}}{j!} Z^{j-k} \beta_1^j \beta_2^k L_k^{j-k}(\rho^2).$$  

(10.8b)

Then, by using these in (10.6) we have

$$J_{nl}^1 = i^{n+l} l! Z^{n-l} L_l^{n-l}(\rho^2), \quad J_{nl}^2 = (-i)^{n+l} l! Z^{n-l} L_l^{n-l}(\rho^2),$$  

(10.9) and, by (10.7)

$$P_{nl}(q_1, q_2) = 4\pi \frac{l!}{n!} (\frac{\hbar}{\gamma})^2 \rho^{2(n-l)} e^{-\rho^2} [L_l^{n-l}(\rho^2)]^2.$$  

(10.10)

The generating functions $M_{\alpha\beta}(p_1, p_2), M_{\alpha\beta}(q_1, p_1), M_{\alpha\beta}(q_2, p_2)$ and the corresponding marginal probability densities can be calculated in a similar way. These are all calculated and presented together in the Tables I and II. But the calculations for the $q_1 p_2$ and $q_2 p_1$-planes are to be presented as they are a bit different.

From (10.3) we have
\[
M_{\alpha\beta}(q_1, p_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Gdqdp, \\
= \pi \hbar e^{\frac{1}{2}(r_1^2 + r_2^2)} e^{\frac{1}{2}[(\alpha_1 + i\tau) - (\beta_1 - i\tau)]^2 + (\alpha_2 - i\tau)^2 + (\beta_2 + i\tau)^2}, \tag{10.11}
\]

where \( \tau_{\pm} = (m\omega q_1 \pm 2p_2)/m\omega\). We now consider the identity
\[
\partial_{\alpha_1}^n e^{\frac{1}{2}[(\alpha_1 + i\tau) - (\beta_1 - i\tau)]^2}|_{\alpha_1=0} = (-i)^{n} \partial_{\tau_{\pm}}^n e^{\frac{1}{2}[(\alpha_1 + i\tau - (\beta_1 - i\tau)]^2}|_{\alpha_1=0}, \\
= (-i)^n \partial_{\tau_{\pm}}^n e^{-\frac{1}{2}\tau_{\pm}^2} = \left(\frac{i}{\sqrt{2}}\right)^n e^{-\frac{1}{2}\tau_{\pm}^2} H_n(\frac{\tau_{\pm}}{\sqrt{2}}). \tag{10.12}
\]

In passing to the last equality we have used the definition of the Hermite polynomials
\[
H_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2}. \tag{10.13}
\]

Then by using (10.12), (10.13) and the other three relations obtained for derivatives with respect to \( \alpha_2, \beta_1 \) and \( \beta_2 \) in
\[
P_{nl}(q_1, p_2) = \frac{4}{n!} (\partial_{\alpha_1} \partial_{\beta_1})^n (\partial_{\alpha_2} \partial_{\beta_2})^l M_{\alpha\beta}(q_1, p_2)|_{\alpha=0=\beta}, \tag{10.14}
\]
we arrive at
\[
P_{nl}(q_1, p_2) = \frac{4\pi \hbar}{n!} e^{-\frac{1}{4}(r_1^2 + r_2^2)} [H_n(\frac{\tau_{-}}{\sqrt{2}})H_l(\frac{\tau_{+}}{\sqrt{2}})]^2. \tag{10.15}
\]

As a common property, all marginal probability densities of Landau levels are positive on the corresponding phase-space plane. They determine the wavefunctions on these planes, up to some unitary phase factors. Axial symmetry on the respective planes of the first four marginal probability densities are obvious from their explicit expressions given in Table II. Also in functional dependence \( P_{nl}(q_1, p_2) \) and \( P_{nl}(q_2, p_1) \) differ from others. Marginal probability densities along \( q_j \), or \( p_j \), or along some oblique phase-space directions can be computed as well. Finally we note that \( P_{nl}(q_2, p_1) \) presented in the last row of the Table II easily follows from (10.15) by virtue of (6.4c).

\section*{XI. REMARKS}

In this section some remarks that remain outside of our main goals are collected together.

\textbf{A. Comparison with Definitions}

For justification we would like to compare (4.8) and (10.10) with expressions obtainable from the definition of the Wigner functions. In the symmetric gauge (3.20) and in terms of plane polar coordinates \( (r, \theta) \) the normalized wavefunctions and the corresponding energy eigenvalues of the Landau levels are given by
\[
\psi_{n, j}(r, \theta) = \frac{1}{\sqrt{\pi(n_r + |j|)!}} \rho^{|j|} e^{ij\theta} e^{-\rho^2/2} L_{\frac{n_r}{2}}(|\rho^2|), \\
E_{n, j} = \hbar \omega (n_r + \frac{1}{2} + \frac{|j| - j}{2}), \tag{11.1}
\]
\[n_r = 0, 1, 2, ..., \quad j = 0, \pm 1, \pm 2, ...\]
where \( r = \gamma \rho = (q_1^2 + q_2^2)^{1/2} \) and \( n_r, j \) stand for the radial and angular momentum quantum numbers. Since in (10.10) it is supposed that \( n \geq l \), we can make the identifications \(|j| = n - l\) and \( n_r + |j| = n \), which imply \( n_r = l \). We then have, from (10.10) and (11.1)

\[
P_{nl}(q_1, q_2) = h^2 |\psi_{nrj}(r, \theta)|^2.
\] (11.2)

This expected result confirms Eq. (10.10) and our definition of marginal probability densities in section IX. On the other hand, by using the normalized ground state wavefunction \( \psi_0(q) = \psi_{00}(r, \theta) = \gamma \pi^{-1/2} \exp(-\rho^2/2) \) in the definition

\[
W_0 = \int_{\mathbb{R}^2} \psi_0(q_1 + \frac{1}{2}y) \overline{\psi_0}(q_1 - \frac{1}{2}y) e^{-iy\cdot p} dy_1 dy_2,
\] (11.3)

we get

\[
W_0 = e^{-4H_0/\gamma^2 \pi} \int_{-\infty}^{\infty} e^{-\frac{1}{4\gamma^2}(y_1+i\frac{2\gamma^2}{h}p_1)^2} dy_1 \int_{-\infty}^{\infty} e^{-\frac{1}{4\gamma^2}(y_2+i\frac{2\gamma^2}{h}p_2)^2} dy_2.
\] (11.4)

Noting that, each integral in (11.4) is equal to \( 2\gamma \pi^{1/2} \) we obtain

\[
W_0 = 4 e^{-4H_0/\gamma^2 \pi} \gamma^2 \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{4\gamma^2}(y_1+i\frac{2\gamma^2}{h}p_1)^2} dy_1 e^{-\frac{1}{4\gamma^2}(y_2+i\frac{2\gamma^2}{h}p_2)^2} dy_2.
\] (11.5)

B. Unitary Transformations in a Phase-Space

Let us consider a unitary similarity transformation generated by the phase-space function

\[
U \equiv U(q, p) \text{ and its } \ast-\text{inverse } U^{-1} = U:
\]

\[
U \ast U^{-1} = U^{-1} \ast U = 1.
\] (11.5)

When such a transformation applied to a \( \ast \)-eigenvalue equation \( H \ast W_\lambda = W_\lambda \ast H = E_\lambda W_\lambda \), the transformed functions

\[
H' \equiv U \ast H \ast U^{-1}, \quad W'_\lambda \equiv U \ast W_\lambda \ast U^{-1},
\] (11.6)

obey the same \( \ast \)-eigenvalue equation

\[
H' \ast W'_\lambda = W'_\lambda \ast H' = E_\lambda W'_\lambda.
\] (11.7)

In view of (2.3), (7.9) and (11.5) it is easy to verify that

\[
\overline{W}'_\lambda = U \ast \overline{W}_\lambda \ast U^{-1},
\] (11.8a)

\[
W'_\lambda \ast W'_\lambda - kW'_\lambda \delta_{\lambda\lambda'} = U \ast (W_\lambda \ast W_{\lambda'} - kW_\lambda \delta_{\lambda\lambda'}) \ast U^{-1},
\] (11.8b)

where \( k \) is a constant. That is, the reality and \( \ast \)-projection properties of \( W_\lambda \) are preserved under the unitary similarity transformations. Moreover, as is evident from

\[
\int_{\mathbb{R}^{2D}} W'_\lambda dV = \int_{\mathbb{R}^{2D}} U \ast W_\lambda \ast U^{-1} dV
\]

\[
= \int_{\mathbb{R}^{2D}} W_\lambda \ast (U^{-1} \ast U) dV = \int_{\mathbb{R}^{2D}} W_\lambda dV,
\] (11.9)

normalization properties of \( W_\lambda \) is also preserved. In (11.9) we have used (2.2) and (11.5).
C. Gauge Transformations in a Phase-Space

Now let us consider

\[ U_q = e^{i \frac{\bar{\hbar}}{c} \chi}, \quad U_q^{-1} = e^{-i \frac{\bar{\hbar}}{c} \chi}, \]  

(11.10)

where \( U_q \equiv U(q) \) and \( \chi \equiv \chi(q) \) is a real valued function of the coordinates. Obviously \( U_q \ast U_q^{-1} = U_q U_q^{-1} = 1 \). By using (1.3) with \( D = 2 \) we also have

\[ U_q \ast p \ast U_q^{-1} = p - \frac{q}{c} \nabla_q \chi, \]  

(11.11)

which implies, for the Landau Hamiltonian (3.1)

\[ H'_L = \frac{1}{2m} \left[ U_q \ast (p - \frac{q}{c} A) \ast U_q^{-1} \right] \cdot \left[ U_q \ast (p - \frac{q}{c} A) \ast U_q^{-1} \right] \]

\[ = \frac{1}{2m} [p - \frac{q}{c} (A + \nabla \chi)]^2, \]  

(11.12)

where \( H'_L = U_q \ast H_L \ast U_q^{-1} \) and \( \nabla_q \) represents two dimensional gradient operator in \( q_1, q_2 \) variables. Eqs. (11.11) and (11.12) show that the unitary transformation by \( U_q \) amounts to the so called gauge transformations of the second kind in \( p \) and \( H_L \). Hence, without changing their generic properties the Wigner functions of Landau levels in any gauge \( \chi \) can be computed directly from

\[ W'_{n_1 n_2 l_1 l_2} = U_q \ast W_{n_1 n_2 l_1 l_2} \ast U_q^{-1}, \]  

(11.13)

where \( W_{n_1 n_2 l_1 l_2} \) are given by (8.13). In that case the transformation in (11.13) may be considered as a first kind gauge transformation. We should emphasize that the two forms of \( \ast \)-product given by (1.3) and (5.3a) can equally well be used in (11.13) provided that the arguments of functions are properly written.

D. Implications for Wigner Functions

By using (1.3) we compute the following useful relation in an arbitrary dimension

\[ f_1(q) \ast e^{-\frac{\bar{\hbar}}{c} \chi} \ast f_2(q) = [e^{-\frac{\bar{\hbar}}{c} \chi} \ast (p + i \frac{\bar{\hbar}}{c} \nabla_q)] f_1(q) \ast f_2(q), \]

\[ = f_1(q + \frac{1}{2} y) [e^{-\frac{\bar{\hbar}}{c} \chi} \ast f_2(q)], \]

\[ = f_1(q + \frac{1}{2} y) f_2(q - \frac{1}{2} y) e^{-\frac{\bar{\hbar}}{c} \chi} \ast f_2(q), \]  

(11.14)

where \( \nabla_q \) denotes \( D \)-dimensional gradient operator in \( q_i \) variables. Taking \( f_1 = U_q \) and \( f_2 = U_q^{-1} \) in (11.14) yields

\[ U_q \ast e^{-\frac{\bar{\hbar}}{c} \chi} \ast U_q^{-1} = e^{i \frac{\bar{\hbar}}{c} \chi(q + \frac{1}{2} y)} e^{-i \frac{\bar{\hbar}}{c} \chi(q - \frac{1}{2} y)} e^{-\frac{\bar{\hbar}}{c} \chi} \]

(11.15)

This implies, for \( W'_{\lambda_1 \lambda_2} = U_q \ast W_{\lambda_1 \lambda_2} \ast U_q^{-1} \), that

\[ = 22 \]
\begin{align*}
W'_{\lambda_1 \lambda_2} &= \int_{\mathbb{R}^D} \psi_{\lambda_1}(q + \frac{1}{2}y) \overline{\psi}_{\lambda_2}(q - \frac{1}{2}y) U_q \ast e^{-\frac{i}{\hbar}y\cdot p} \ast U_q^{-1} dV(y), \\
&= \int_{\mathbb{R}^D} \psi'_{\lambda_1}(q + \frac{1}{2}y) \overline{\psi'}_{\lambda_2}(q - \frac{1}{2}y) e^{-\frac{i}{\hbar}y\cdot p} dV(y),
\end{align*}

(11.16)

where $W_{\lambda_1 \lambda_2}$ is an off-diagonal Wigner function defined by $1.1$ and $\psi'_{\lambda_1}(q) = e^{\frac{ic}{\hbar} \chi(q)} \psi_{\lambda_1}(q)$. If we change our notation as $W_{\psi_1 \psi_2} = W_{\lambda_1 \lambda_2}$, Eq. (11.16) implies the following important relation for gauge transformation of the Wigner function

\begin{align*}
W'_{\psi_1 \psi_2} &= U \ast W_{\psi_1 \psi_2} \ast U^{-1} = W_{\psi'_1 \psi'_2} = W_{U\psi_1, U\psi_2}.
\end{align*}

(11.17)

Finally we should note that, for $f_1 = \psi_1$ and $f_2 = \overline{\psi}_2$ Eq. (11.14) suggests the following redefinition of Wigner function

\begin{align*}
W_{\psi_1, \psi_2} &= \int_{\mathbb{R}^D} \psi_1(q) \ast e^{-\frac{i}{\hbar}y\cdot p} \ast \overline{\psi}_2(q) dV(y),
\end{align*}

(11.18)

which enables us to interpret the map $W$ as a “sesquilinear $\ast$-Fourier transformation”.

**XII. CONCLUSION**

In this paper we have presented a problem that can, with all of its features, thoroughly be considered in the framework of deformation quantization without making use of other quantization schemes. As a summary, we have obtained Wigner functions of Landau levels by solving a two-sided $\ast$-eigenvalue equation, specified a large class of its transformation and symmetry properties as well as established its generic properties in the same framework. Off-diagonal Wigner functions and marginal probability densities are generated by means of generating functions.

With its distinguishable and intriguing properties, Wigner function provides the most complete description and offers convenient interpretation of the result of several recent experiments in atomic optics, molecular physics and signal processing [7,8,26]. A basic aim of all these experiments is to probe the fundamental structure and predictions of quantum mechanics in a new way, by observing non-classical behaviors of Wigner function [27]. For this purpose marginal probability densities play equally important role since Wigner function appear from measured densities along various phase-space directions. Now explicit expressions of Wigner functions and all marginal probability densities of Landau levels at hand we do expect that the results of this paper will shed more light on future experiments in which Landau levels are realized and on their interpretations.

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APPENDIX A: THE OPERATOR IDENTITY (5.9)

In this appendix we will prove the following operator identity (see Eq.(5.9)) by induction:

\[(1 - T)^n(x^ne^{-x}) = (-1)^ne^xT^n(x^ne^{-2x}),\]  
(A1)

where \(T = d/dx\) and \(n\) is a positive integer. Then, we will present a finite sum formula for the generalized Laguerre polynomials that follows from (A1).

Eq. (A1) holds for \(n = 1\) (and obviously for \(n = 0\)). Now we suppose (A1) for \(n - 1\) and try to prove it for \(n\). Defining \(I_k = (1 - T)k(x^ke^{-x})\), the inductive hypothesis is

\[I_{n-1} = (-1)^{n-1}e^xT^{n-1}(x^{n-1}e^{-2x}).\]  
(A2)

Then we have

\[I_n = (1 - T)^n[x(x^{n-1}e^{-x})] = [x(1 - T) - n]I_{n-1},\]  
\[= (-1)^ne^x[XT^n + nT^{n-1}](x^{n-1}e^{-2x}),\]  
(A3)

\[= (-1)^ne^xT^n(x^ne^{-2x}).\]

In passing to the second equality we made use of \([1 - T]^n, x] = -n(1 - T)^{n-1}\), where \([,]\) stands for the usual commutator. The second line of (A3) easily follows from (A2) and in passing to the last line we have used

\[(xT^n + nT^{n-1})f = T^n(xf),\]  
(A4)

where \(f\) is an arbitrary differentiable function of \(x\). Eq. (A4) easily results from the commutation relation \([T^k, x] = kT^{k-1}\), or equivalently from the Leibniz rule \(T^n(xf) = \sum_{k=0}^n \binom{n}{k}(T^kx)T^{n-k}f\). We should note that (A1) can be generalized as

\[(1 - T)^n(f^e^x) = (-1)^ne^xT^n f,\]  
(A5)

which can be proved more easily, again, by induction on \(n\). (A1) corresponds to choice \(f = x^ne^{-2x}\) in (A5).

It is worth mentioning that, in view of the binomial expansion

\[(1 - T)^n = \sum_{j=0}^n \binom{n}{j}(-1)^{n-j}T^{n-j},\]  
(A6)

and by recalling the Rodrigues formula for generalized Laguerre polynomials ( [28], p. 241)

\[L_\alpha^n(x) = \frac{e^x}{n!}x^{-\alpha}T^n(x^{n+\alpha}e^{-x}),\]  
(A7)

from (A1) and (A7) we obtain the following finite sum formula

\[L_n(2x) = \sum_{j=0}^n \frac{(-x)^j}{j!}L_{n-j}^j(x) = \sum_{j=0}^n L_{n-j}^j(x)L_{n-j}(x),\]  
(A8)

where \(L_{n-j}^j(x) = (-x)^j/j!\). Since \(L_n = L_0^n\), from (A7) we get (5.10) for \(\alpha = 0\) and \(x = 2u\). We could not find (A8) in the related classical references [28,29].
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TABLE I. Generating functions for the marginal probability densities for Wigner functions in various phase-space planes.

\[
M_{\alpha\beta}(q_1, q_2) = \pi \frac{\hbar^2}{\gamma^2} e^{-\frac{1}{2} \rho^2} e^{-a_1 q_1 + i(a_1 Z - a_2 Z)} e^{-b_1 q_2 - i(b_1 Z - b_2 Z)}, \quad Z = \frac{q_1 + iq_2}{\gamma} \\
M_{\alpha\beta}(p_1, p_2) = \pi \gamma^2 e^{-4 \frac{c^2}{2}} e^{a_1 p_1 + 2(b_1 P - b_2 P)} e^{b_1 p_2 + 2(b_1 P - b_2 P)}, \quad P = \frac{p_1 + ip_2}{m\omega\gamma} \\
M_{\alpha\beta}(q_1, p_1) = \pi \hbar e^{-\tau_{\pm}^2} e^{a_1 q_1 + i(a_1 \tau_{\pm} + b_1 \tau_{\pm})} e^{b_1 p_1 - i(b_1 \tau_{\pm} - a_2 \tau_{\pm})}, \quad \tau = \frac{q_1}{\gamma} + \frac{2i\gamma p_1}{\hbar} \\
M_{\alpha\beta}(q_2, p_2) = \pi \hbar e^{-\eta_{\pm}^2} e^{-a_2 q_2 - (a_2 \eta_{\pm} + b_2 \eta_{\pm})} e^{b_2 p_2 + (b_2 \eta_{\pm} + a_1 \eta_{\pm})}, \quad \eta = \frac{q_2}{\gamma} + \frac{2i\gamma p_2}{\hbar} \\
M_{\alpha\beta}(q_1, p_2) = \pi \hbar e^{\frac{1}{2} (\tau_{\pm}^2 + \tau_{\mp}^2)} e^{\frac{1}{2} [(a_1 + i\tau_{\pm})^2 + (b_1 - i\tau_{\pm})^2 + (a_2 - i\tau_{\mp})^2 + (b_2 + i\tau_{\mp})^2]}, \quad \tau_{\pm,\mp} = \frac{q_1}{\gamma} \pm \frac{2\gamma p_2}{\hbar} \\
M_{\alpha\beta}(q_2, p_1) = \pi \hbar e^{\frac{1}{2} (\tau_{\pm}^2 + \tau_{\mp}^2)} e^{\frac{1}{2} [(a_1 - i\tau_{\pm})^2 + (b_1 + i\tau_{\pm})^2 + (a_2 - i\tau_{\mp})^2 + (b_2 + i\tau_{\mp})^2]}, \quad \tau_{\pm,\mp} = \frac{q_2}{\gamma} \pm \frac{2\gamma p_1}{\hbar}
\]
TABLE II. Marginal probability densities for Wigner functions in various phase-space planes. In this table we use the abbreviations $N_{nl} = 4\pi l!/n!$ and $N'_{nl} = 4\pi/n!2^{n+l}$.

|  |  |
|---|---|
| $P_{nl}(q_1, q_2) = N_{nl}(\frac{\hbar}{\gamma})^2 \rho^2 (n-l)! e^{-\rho^2 [L_i^{n-l}((\rho^2)])^2}$, & $\rho^2 = \frac{q_1^2 + q_2^2}{\gamma^2}$ |
| $P_{nl}(p_1, p_2) = N_{nl}\gamma^2 (2\zeta)^2 (n-l)! e^{-4\zeta^2 [L_i^{n-l}(4\zeta^2)]^2}$, & $\zeta^2 = \frac{\gamma^2 (p_1^2 + p_2^2)}{4\hbar^2}$ |
| $P_{nl}(q_1, p_1) = N_{nl}\hbar\mu_1^{2(n-l)} e^{-\mu_1^2 [L_i^{n-l}(\mu_1^2)]^2}$, & $\mu_1^2 = \frac{q_1^2 + \gamma^2 p_1^2}{\hbar^2}$ |
| $P_{nl}(q_2, p_2) = N_{nl}\hbar\mu_2^{2(n-l)} e^{-\mu_2^2 [L_i^{n-l}(\mu_2^2)]^2}$, & $\mu_2^2 = \frac{q_2^2 + \gamma^2 p_2^2}{\hbar^2}$ |
| $P_{nl}(q_1, p_2) = N'_{nl}\hbar e^{-\frac{1}{2}(\tau_+^2 + \tau_-^2)} [H_n(\frac{\tau_+}{\sqrt{2}})H_l(\frac{\tau_-}{\sqrt{2}})]^2$, & $\tau_{\pm} = \frac{q_2}{\gamma} \pm \frac{2\mu_2}{\hbar}$ |
| $P_{nl}(q_2, p_1) = N'_{nl}\hbar e^{-\frac{1}{2}(\tau'_+^2 + \tau'_-^2)} [H_n(\frac{\tau'_+}{\sqrt{2}})H_l(\frac{\tau'_-}{\sqrt{2}})]^2$, & $\tau'_{\pm} = \frac{q_2}{\gamma} \pm \frac{2\mu_1}{\hbar}$ |

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