Modeling Co-location in Multi-Operator mmWave Networks with Spectrum Sharing

Rebal S. Jurdi, Abhishek K. Gupta, Jeffrey G. Andrews, and Robert W. Heath, Jr.

Abstract

Competing cellular operators aggressively share infrastructure in many major US markets. If operators also were to share spectrum in next-generation mmWave networks, intra-cellular interference will become correlated with inter-cellular interference. We propose a mathematical framework to model a multi-operator mmWave cellular network with co-located base-stations. We then characterize the SINR distribution for an arbitrary network and derive its coverage probability. To understand how varying the spatial correlation between different networks affects coverage probability, we derive special results for the two-operator scenario, where we construct the operators’ individual networks from a single network via probabilistic coupling. For external validation, we devise a method to quantify and estimate spatial correlation from actual base-station deployments. We compare our two-operator model against an actual macro-cell-dominated network and an actual DAS-node-dominated network of different scales. Using the actual deployment data to set the parameters of our model, we observe that coverage probabilities for the model and actual deployments not only compare very well to each other, but also match nearly perfectly for the case of the DAS-node-dominated deployment. Another interesting observation is that spectrum and infrastructure sharing has a lower rate coverage probability for lower thresholds, which would make it less suitable for low-rate applications.

I. INTRODUCTION

Millimeter-wave communication will be central to delivering the anticipated performance of next-generation cellular networks [1]–[4]. A key feature of mmWave systems is directional communication [5], [6], which reduces the effect of out-of-cell interference as compared to communication at UHF frequencies [7], and opens up the possibility of sharing spectrum licenses with no coordination.
Fig. 1. The percentage of sites shared with one or more cellular operators. Sites considered include macro-cellular towers, rooftops, and DAS nodes. Figures are given for four major US cellular operators in three major cellular market areas (CMAs). In CMA III, Operator A shares about 11% of the sites it occupies with competing operators. In CMA I, three out of four operators share over 66% of their sites.

between network operators [8]–[10]. While spectrum sharing is a future possibility, infrastructure sharing is already a reality, and there has been a progression in the cellular operator industry towards sharing network infrastructure such as the network core, backhaul, and cell towers as a means of expanding coverage at a reduced cost [11], [12]. When multiple closed-access cellular networks share spectrum, inter-network interference adds to intra-network interference. When cellular networks also share infrastructure, inter-network interference becomes coupled to intra-network interference because many of the base-stations (BSs) of the different networks are stationed at the exact same location. In this paper, we propose a mathematical framework that accurately models the co-location of BSs of multiple operators that share spectrum licenses, and suggest how to estimate model parameters from actual deployments.

A. Background and Related Work

Cellular operators have already been sharing network infrastructure through a variety of business models to increase their coverage and capacity while reducing capital and operational expenditures [13]. We surveyed three geographically diverse US cellular market areas and tallied the sites that are occupied by a single operator and those that are shared by two or more operators. These sites included macro-cellular towers, rooftops, and DAS nodes. Figure 1 shows the percentage of sites shared with one or more competing operators. For every market, the bar plot shows the sharing ratio per operator, i.e. the percentage of the operator’s BSs that are co-located with those of at least one other. Sharing
ratios range from 11% to 78%, with some markets displaying more aggressive site sharing than other markets. This trend is expected to continue in next generation cellular networks through a dense overlay of multi-operator and virtual-host small cells targeting enterprise and entertainment venues, on top of the existing layer of macro towers housing BSs of competing operators [14]–[17].

Spectrum is a valuable asset that can also be shared by competing cellular networks [18]–[20]. Studies have suggested that cognitive radios are able to efficiently utilize existing sparse, sporadically used spectrum and facilitate spectrum sharing between different networks [10], [21]–[23]. Other studies have called for Licensed Shared Access (LSA) to enable 5G small cell to access mmWave spectrum bands that are already in use [24]. Combining spectrum and infrastructure sharing between operators couples inter-network interference with intra-network interference. It is unclear how this affects the coverage of the involved networks. The objective of this paper is to model infrastructure sharing between networks that have access to the same spectrum, and to study the effect of varying the extent of sharing on the downlink rate achieved by network subscribers.

In recent work, the performance of a number of mmWave cellular systems spanning different combinations of spectrum and access sharing methods was analyzed in a stochastic geometry framework [20]. Two particular systems were studied: a two-operator system with closed access and full spectrum sharing, and a two-operator system where all the BSs of the two operators are housed on the same towers. The work in [20], however, did not consider a system with partial base station co-location. The same authors also established the feasibility of secondary licensing in mmWave licensed, yet the model they used to represent the locations of the primary and secondary BSs did not generalize to scenarios where the primary and secondary networks share infrastructure [25]. In other work related to infrastructure sharing, a statistical approach to model multi-operator networks with shared deployment patterns was presented in [26], but performance of the models therein was evaluated only through simulation. In a subsequent paper by the same authors [27], the impact of spatial clustering, network density, and spectrum access coordination on network coverage in a multi-operator system was studied analytically. Finally, in [28], different configurations of infrastructure and spectrum sharing were considered, and corresponding SINR and rate coverage probabilities were compared and evaluated against different channel, antenna, and BS patterns. Prior work such as [20], [25]–[28] lacks a model that reproduces any extent of co-location between the BSs of any set of operators, and that allows straightforward analysis of key performance metrics like SINR and rate coverage probabilities.
B. Contributions

In this paper, we propose an analytical framework to model a multi-operator mmWave system with any desired co-location extent of any set of operators. We use independent Poisson point processes (PPPs) of different densities to describe the co-location pattern of any group of operators. This pattern denotes locations of sites that undergo dual BS co-location between every pair of operators, triple BS co-location between every group of three operators, and so on. Hence, BS locations of any operator can be represented by superposing the appropriate point processes. To evaluate the performance of a participating cellular network, we derive the SINR probability of coverage of a typical user of this network.

We then focus on the more tractable case of two operators to understand how varying the spatial correlation between the two networks affects coverage probability. We begin with a mother point processes representing the BS locations of the two operators combined, and then derive two child point processes while forcing a desired degree of spatial correlation by probabilistically coupling the resultant processes on the same probability space (here, spatial correlation denotes the linear correlation of the number of BSs of the two operators in an area of space). We use the notion of second moment measure to quantify spatial correlation which we capture by the overlap coefficient. The two-operator system allows us to analyze the impact of correlation on the performance of the individual networks. To evaluate and compare the how the SINR probability of coverage varies as a function of overlap, we consider two perspectives of infrastructure sharing: fixed individual densities (FID), and fixed combined density (FCD). In FID, the densities of the individual networks’ BSs are fixed with varying overlap. In FCD, these densities increase as overlap increases, but the density of the total BSs is fixed. In a real deployment, FID corresponds to relocating BSs to sites that are already occupied by a competitor, while FCD corresponds to expanding into such sites.

Finally, we compare the probability of rate coverage and median rate for different systems under FID and FCD. To measure how accurately our model reflects the performance of an actual two-operator system, we compare the SINR probability of coverage obtained for those two systems. Since there are no mmWave networks currently deployed, we suggest ways of extracting model parameters, namely, overlap and individual densities, from actual deployments. We consider both macro-tower-dominated deployments and deployments predominantly comprised of distributed antenna system nodes (DAS nodes) in major US cellular markets. Our results show that coverage probabilities for the PPP model and actual deployments compare very well, and they are even almost identical in the case of the DAS-node-dominated deployment.
The rest of the paper is organized as follows. Section II describes the channel and multi-operator system model. Section III gives the expressions of the probability of SINR and rate coverage. In Section IV, we consider the two-operator model from a different angle which allows us to quantify the spatial correlation between two BS deployments, and estimate its value from actual deployments. Section V presents numerical results and provides some insights. Finally, we conclude in Section VI.

II. SYSTEM MODEL

We consider a mmWave cellular network of $M$ cellular operators that share spectrum licenses as well as infrastructure. We combine independent point processes representing BS locations to produce any amount of co-location of any subset of the $M$ operators. Our model reduces to the two extremes of co-location described in [20] of full independence, where BSs of each operator are represented by their own point process, and full overlap, where BSs of all operators are at the exact same locations and thus represented by a single point process.

To characterize the co-location of deployments, we first consider the set $O = \{1, 2, \ldots, M\}$ of operators and the power set $\mathcal{P}(O)$ of $O$. Next, we consider the collection $\{\Phi_S\}$, indexed by a set $S \in \mathcal{P}(O)$, of independent homogeneous PPPs containing the locations of the sites undergoing dual co-location, triple co-location, and all the way up to $M$-tuple co-location of the operators. We will refer to elements (point processes) of this collection as blocks. The superposition of independent PPPs is another PPP [29]. Moreover, given the Poisson nature of these point processes, any collection thereof shares no points in common almost surely; hence “double counting” of sites is not a concern. These blocks, independent and disjoint, constitute the most elementary components of the model. Additionally, we consider the collection $\{\Phi_m\}$, $m \in O$, where $\Phi_m$ describes the locations of all sites housing BSs of operator $m$, both shared and non-shared, and is given by

$$\Phi_m = \bigcup_{S : m \in S} \Phi_S.$$  \hfill (1)

It is important to note the difference between $\Phi_S$, $S \in \mathcal{P}(O)$, and $\Phi_m$, $m \in O$. The first denotes a block and is the main building block of the model. The second characterizes the BS locations of an operator and is the result of combining different blocks. Consider, for example, the two-operator scenario ($M = 2$). The locations of BSs are represented by $\Phi_{\{1\}}$, $\Phi_{\{2\}}$, and $\Phi_{\{1,2\}}$, which in turn are subsets of $\Phi_1$ or $\Phi_2$. We make the simplifying assumption that all operators own licenses of an equal amount of spectrum, and leave the generalization to future work.

We make the following assumptions about the blockage and channel models.
**Blocking model:** We assume the independent blocking model where the link established between the typical user and a BS located at a distance $r$ away can either be LOS (denoted by $L$) with probability $p_L(r)$ or NLOS (denoted by $N$) with a probability $p_N(r) = 1 - p_L(r)$. We adopt the exponential blocking model introduced in [7], where $p_L(r) = \exp(-\beta r)$. Hence, conditioned on the typical user, each system block $\Phi_S$ of density $\lambda_S$ is divided into two independent non-homogeneous PPP as a direct result of the independent thinning theorem [29], and we obtain the two sub-blocks:

- $\mathcal{L}_S$ containing all BSs with LOS links to the user. It has density $\lambda_{S,L}(r) = p_L(r)\lambda_S$ and measure $\Lambda_{S,L}$.
- $\mathcal{N}_S$ containing all BS with NLOS links. It has density $\lambda_{S,N}(r) = p_N(r)\lambda_S$ and measure $\Lambda_{S,N}$.

Then, it follows that the average number of BSs in the sub-blocks $\mathcal{L}_S$ and $\mathcal{N}_S$ in the Euclidean ball $B_0(r)$ centered at the origin and of radius $r$ is

$$\Lambda_{S,L}(B_0(r)) = 2\pi\lambda_S \int_0^r p_L(t)tdt = \frac{2\pi\lambda_S}{\beta^2} \gamma(2, \beta r),\quad (2)$$

$$\Lambda_{S,N}(B_0(r)) = 2\pi\lambda_S \int_0^r p_N(t)tdt = \pi\lambda_S \left( r^2 - \frac{2}{\beta^2} \gamma(2, \beta r) \right),\quad (3)$$

where $\gamma(\ldots)$ is the lower incomplete gamma function.

**Transmit and noise power:** We assume that all BSs transmit at a fixed power $P_t$. We consider a noise power spectral density $N_0$ and a total bandwidth $B$.

**Path loss:** We consider the power-law path loss functions for LOS and NLOS links:

\[ \ell_L(r) = c_L r^{-\alpha_L}, \quad \text{and} \quad \ell_N(r) = c_N r^{-\alpha_N}, \]

where $c_L$ and $c_N$ correspond to the power attenuation at $r = 1$ for LOS and NLOS links.

**Directionality gain:** Similar to [20], base stations are equipped with steerable antennas characterized by a main-lobe gain $G$ and side-lobe gain $g$, while user mobile devices have a single omni-directional antenna. Hence, all points $\{X_k\}$ representing BS locations are endowed by marks $\{G_k\}$ which are IID Bernoulli distributed with PMF

\[ G_k = \begin{cases} G & \text{w.p. } \theta_b/\pi \\ g & \text{w.p. } (\pi - \theta_b)/\pi, \end{cases} \quad (4) \]

where $\theta_b$ is the half beamwidth and assumed to be identical across all BSs. Since signals received from co-located BSs are transmitted from antenna arrays pointed in different directions to serve different users, we can assume that directionality gains are independent. In reality, actual array patterns can be different from those produced by this model because of scattering and dispersion [30]. Nevertheless, we use this model for analytical tractability.
**Association rule:** We consider a closed-access system where the users can only connect to the base stations of their parent network. Moreover, the typical user associates to the BS that corresponds to the smallest path loss, or equivalently, the BS providing the maximum received signal averaged over fading. Once the BS is chosen and a link is established, the BS antenna array aligns its beam with the user to ensure maximum signal gain. The typical user could form either a LOS or a NLOS link with the serving station.

**Small-scale fading:** We assume that the channel undergoes flat Rayleigh fading. Equivalently, the fading power $H_k$ of the signal received from the BS at $X_k$ is exponentially distributed with unit mean. We verify in Section V that the relative performance remains unchanged when Nakagami fading and lognormal shadowing are used. Despite the fact that the large-scale propagation losses of co-located transmitters of opposite networks are equal at any distance, we assume that signals received from these transmitters at any point undergo independent fades. This is reasonable given the different locations of BS antennas on the tower are typically further than the (vertical) coherence distance of the channel.

III. **Coverage Analysis**

We use the SINR probability of coverage as the system performance metric, which is defined as the value of the SINR CCDF at a threshold $T$

$$P_c(T) = P(\text{SINR} > T).$$

(5)

Suppose that the typical user associates with $b$th BS of the $n$th network at a distance $R$ via a link of type $\tau(b,n)$ which can be LOS or NLOS. We define $c_{\tau(b,n)}$ and $\alpha_{\tau(b,n)}$ to be the path loss constant and exponent corresponding to $\tau(b,n)$, and $\sigma^2$ to be the thermal noise power normalized by the transmit power, i.e. $\sigma^2 = N_0B/P_t$. We also define $I$ to be the interference from all blocks and is expressed as

$$I = \sum_{i: X_i \in \Phi_n} c_{\tau(i,n)} H_{i,n} G_{i,n} ||X_{i,n}||^{-\alpha_{\tau(i,n)}}$$

(6)

$$+ \sum_{m \in O} \sum_{j: X_j \in \Phi_m} c_{\tau(j,m)} H_{j,m} G_{j,m} ||X_{j,m}||^{-\alpha_{\tau(j,m)}}.$$ 

(7)

Therefore, the SINR of the typical user is

$$\text{SINR} = \frac{c_{\tau(b,n)} H_{b,n} G R^{-\alpha_{\tau(b,n)}}}{\sigma^2 + I}.$$ 

(8)

The first term of the sum accounts for the interference from BSs of the same operator, while the second term describes the interference from all BSs of different operators. Note that $X_{j,m}$ and $X_{j',m'}$,
the locations of BS $j$ of network $m$ and BS $j'$ of network $m'$, need not be distinct. If $m, m' \in \mathcal{S}$, then $\Phi_m$ and $\Phi_{m'}$ share points in common, as they are both derived from block $\Phi_S$.

In the remainder of this section, we analyze the coverage probability of a typical user of Network 1 since the coverage analysis of all networks is mathematically identical. The networks could have different coverage due to the various parameter values. However, this does change the analysis. We first investigate the association of the typical user of Network 1 to any of its BSs. We then compute its SINR probability of coverage and derive the rate probability of coverage which is a tangible metric in quantifying user experience.
A. Association Criterion

The probability that a typical user of Network 1 is covered depends on what block they are associated with. Association could take place through any of the $2^{M-1}$ blocks of $\{ \Phi_S \}$, $1 \in S$, and any of their sub-blocks. Since these blocks and sub-blocks are independent, the events of associating to distinct blocks are disjoint. Hence, we can compute the total probability of coverage by adding the joint probabilities of coverage and association.

We first define some notation. Let $\nu_{S,\tau} = \Lambda_{S,\tau} \circ B_0$, where $\Lambda_{S,\tau}$ is the measure for the appropriate sub-block of $\mathcal{T}_S$, and the subscript $\tau$ denotes an arbitrary link type. The operator $\circ$ denotes composition, i.e., $\nu_{S,\tau}(r) = \Lambda_{S,\tau}(B_0(r))$. Additionally, let $D_L$ be the exclusion function of LOS transmitters of Network 1 when the user is associated with a NLOS transmitter of the same network. Similarly, let $D_N$ be the exclusion function of NLOS transmitters when the user is associated with a LOS transmitter. An exclusion function gives the radius of the region around the tagged BS within which no other BSs in the same or different blocks exist. These functions are given in [7] as

$$D_L(r) = \left( \frac{c_L}{c_N} \right) \frac{1}{\alpha_L} \frac{r^{\frac{\alpha_N}{\alpha_L}}}{r^{\frac{\alpha_N}{\alpha_L}}}, \text{ and } D_N(r) = \left( \frac{c_N}{c_L} \right) \frac{1}{\alpha_N} \frac{r^{\frac{\alpha_L}{\alpha_N}}}{r^{\frac{\alpha_L}{\alpha_N}}}.$$ 

Moreover, let $\mathcal{A}_{T_S}$ be the event of association with sub-block $\mathcal{T}_S$. Define $P_c(T; \Phi_S)$ and $P_c(T; \mathcal{T}_S)$ to be the probabilities that the user is in coverage for a threshold $T$ and that the user is associated to a BS in any block $\Phi_S$ and sub-block $\mathcal{T}_S$ thereof:

$$P_c(T; \mathcal{T}_S) = P \left( \text{SINR} > T \cap \mathcal{A}_{T_S} \right), \quad (9)$$

These probabilities can be obtained by integrating the ccdf of the appropriate SINR given by (8), (6), and weighted by the PDF $f_R(\cdot; \mathcal{T}_S)$ of the length $R$ of the established link with a BS of $\mathcal{T}_S$ as

$$P_c(T; \mathcal{T}_S) = \int_{r \geq 0} P \left( \text{SINR} > T \cap \mathcal{A}_{T_S} \right) f_R(r; \mathcal{T}_S) \, dr. \quad (10)$$

What remains to be derived is the PDF $f_R(\cdot; \mathcal{T}_S)$ for an arbitrary sub-block $\mathcal{T}_S$. To accomplish this, we follow the derivation in [20]. We draw an analogy between the sub-blocks that a typical user of Network 1 can associate with, defined in this paper, and the tiers as defined in [20]. In [20], a typical user (of Network 1, let’s say) is permitted to access Network 1, their home network, and every other network that is in the access class of Network 1. In a closed-access system, the access class is Network 1, and in an open-access system, the access class is all networks. The key to computing $f_R(r; \mathcal{T}_S)$ is computing the probability $f_R^0(r; \mathcal{T}_S)$ that all BSs of every other sub-block $\mathcal{T}_T$, $\mathcal{T} \in \mathcal{P}(\mathcal{O}) \setminus S$, $1 \in S$ that are accessible by the typical user are outside the exclusion radius $r$. 

For an arbitrary sub-block \( T \), this is given by the void probability \( \mu_T(r) = P(\mathcal{T}(B_0(r)) = 0) \). Since all sub-blocks are mutually independent, \( f^o_R(r; \mathcal{L}_S) \) is given as the product of void probabilities
\[
f^o_R(r; \mathcal{L}_S) = \mu_{\mathcal{L}_S}(D_L(r)) \cdot \prod_{T \in \mathcal{P}(O) \setminus S} \mu_{\mathcal{L}_T}(r) \mu_{\mathcal{N}_T}(D_L(r)).
\] (11)

The density \( f_R(\cdot; \mathcal{L}) \) is obtained according to [31, Section V-C] and [20, Equation (9)] as
\[
f_R(r; \mathcal{L}_S) = \frac{\partial}{\partial r} (\mu_{\mathcal{L}_S}(r)) f^o_R(r; \mathcal{L}_S) = 2\pi \lambda_S p_L(r) e^{-\nu_{S,L}(r)} \cdot \prod_{T \in \mathcal{P}(O) \setminus S} e^{-\nu_{T,L}(r)}.
\] (12)

Since the events \( \{A_T\} \) of association with different sub-blocks \( \{T\} \) are disjoint [20], the SINR coverage probability \( P_c(T) \) for the typical user is obtained by adding these individual block coverage probabilities over all accessible block of Network 1:
\[
P_c(T) = \sum_{S \in \mathcal{P}(O) \setminus 1 \in S} P_c(T; \Phi_S) = \sum_{S \in \mathcal{P}(O) \setminus 1 \in S} P_c(T; \mathcal{L}_S) + P_c(T; \mathcal{N}_S).
\] (13)

Next, we derive the expression for \( P_c(T; \mathcal{T}_S) \) for an arbitrary sub-block.

**B. Interference Characterization**

Computing the probability of coverage under Rayleigh fading can be readily reduced to finding the Laplace transform of interference. To illustrate this, (5) can be expanded as
\[
P(\text{SINR} > T | R) = P(c_T G H R^{-\alpha_T} > T (\sigma^2 + I) | R) = E\left[ F_H\left(\frac{R^{\alpha_T}T}{c_T G} (\sigma^2 + I) | R, I\right)\right],
\] (14)
where \( F_H \) is the CCDF of the fading encountered by the signal emitted from the tagged BS which can be expressed as a single exponential. Noting that \( F_H(u) = e^{-u} \), (14) becomes
\[
P(\text{SINR} > T | R) = \exp\left(-\frac{\sigma^2}{c_T G} \frac{R^{\alpha_T}T}{c_T G}\right) \cdot \mathcal{L}_{1|R}\left(\frac{R^{\alpha_T}T}{c_T G}\right).
\] (15)

Since the BSs of Network 1 are spatially co-located with those of other networks, inter-network interference is no longer independent and \( \mathcal{L}_{1|R} \) is not the product of Laplace transforms of inter-network and intra-network interference. To resolve this, we reformulate the interference expression in (6), which is given with respect to the correlated point processes of \( \{\Phi_m\}_{m \in O} \), as a sum over the uncorrelated block of \( \{\Phi_S\}_{S \in \mathcal{P}(O)} \). We give the resulting expression in the following proposition.
Proposition 1. Given that the typical user associates to a BS at a distance \( r \) in the LOS sub-block \( \mathcal{L}_\mathcal{T}, \mathcal{T} \in \mathcal{P}(O) \), then the Laplace transform of the interference random variable is given as

\[
\mathcal{L}_{I_L}(s) = u_L(s,r)^{|T|-1} \prod_{S: 1 \notin S} \exp \left( -2\pi \lambda_S \int_{t \geq 0} \left( 1 - u_L(s,t)^{|S|} \right) p_L(t) dt \right) \cdot \prod_{S': 1 \in S'} \exp \left( -2\pi \lambda_{S'} \int_{t \geq r} \left( 1 - u_L(s,t)^{|S'|} \right) p_L(t) dt \right) \cdot \prod_{S: 1 \notin S} \exp \left( -2\pi \lambda_S \int_{t \geq 0} \left( 1 - u_N(s,t)^{|S|} \right) p_N(t) dt \right) \cdot \prod_{S': 1 \in S'} \exp \left( -2\pi \lambda_{S'} \int_{t \geq D_L(r)} \left( 1 - u_N(s,t)^{|S'|} \right) p_N(t) dt \right),
\] (16)

where \( u_r(s,t) = \mathbb{E}_G [\mathcal{L}_{H|G}(s_GGt^{-ar})] \). Moreover, if fading power is exponentially distributed with unit mean and antenna gain follows a Bernoulli distribution, \( \mathcal{L}_{I_L}(s) \) is given as

\[
u_r(s,t) = \frac{\theta_b/\pi}{1 + s_GGt^{-ar}} + \frac{(\pi - \theta_b)/\pi}{1 + s_Ggt^{-ar}}.
\] (17)

Proof: The proof is detailed in Appendix A.

The Laplace transform is the product of Laplace transforms of independent random variables corresponding to different classes of BSs, grouped according to the network they belong to as well as the type of potential link they can establish with the user. The first term of (16) represents the contribution of the LOS BSs of different operators co-located with the typical user’s tagged BS at a distance \( r \) away, since there are a total of \(|T|\) co-located BSs in sub-block \( \mathcal{L}_\mathcal{T} \). The second term gives the LOS interference from all the sites where no BS of Network 1 is deployed. The third term gives the LOS interference from all the sites housing BSs of Network 1 averaged outside the exclusion ball associated with the tagged BS. The remaining terms are almost identical to the second and third terms, with the only difference being that they account for NLOS interference.

The probability of coverage is expressed in terms of the interference Laplace transform (see (10) and (15)). Now that we have determined the Laplace transform, we give the coverage probability in the next proposition.

Corollary 1. The SINR probability of coverage of a typical user of Network 1 in a multi-operator system is given by

\[
P_c(T) = \int_{r \geq 0} \sum_{S \in \mathcal{P}(O)} \left( e^{-\sigma^2 s_L} \mathcal{L}[I; \mathcal{L}_S](s_L) f_R(r; \mathcal{L}_S) + e^{-\sigma^2 s_N} \mathcal{L}[I; \mathcal{N}_S](s_N) f_R(r; \mathcal{N}_S) \right) dr,
\] (18)
where $\mathcal{L}[I; T]$ is the Laplace transform of interference given the event $A_T$, and $s_L$ and $s_N$ are given by

$$s_L = \frac{Tr_1}{\alpha c_L G}, \quad s_N = \frac{Tr_N}{\alpha c_N G}$$

**Proof:** The result follows by substituting the expression of the Laplace transform of the interference random variable given from Proposition 1 in (15).

The ultimate metric for evaluating the performance of a cellular network is the per-user downlink rate distribution since it reflects an aspect of service quality experienced by the user. We can transform the SINR coverage probability into a rate coverage probability with a few assumptions. The amount of bandwidth resources allotted to a typical user is a function of the total number of users served by the associated BS, as well as the total available bandwidth $B$. We assume a fair resource allocation algorithm where the BS scheduler divides bandwidth resources equally among each of the $N_U$ users of spatial density $\lambda_U$. Due to the closed-access nature of our multi-operator system, users of a particular operator can only connect to their operator’s home network. Hence, the mean number of connected users in a cell can be given based on the approximate load model in [20], [32], [33] as $N_U = 1 + 1.28 \left( \frac{\lambda_U}{\alpha} \right)$. Finally, the probability that a typical user of Network 1 experiences a rate of at least $R$ bps is $P(\text{Rate} > R) = P_c \left( 2^{RN_U/B} - 1 \right)$.

**IV. THE TWO-OPERATOR CASE**

In this section, we analyze the probability of coverage for the two-operator case. This scenario is an important special case because it allows to parameterize the system using only three quantities: the densities of the two networks and the extent of overlap between them. In addition, it is difficult to simplify the general case because one has to sweep the densities $\{\lambda_S\}$ of all the underlying blocks $\{\Phi_S\}$, and consider all possible ways these blocks could be combined to construct the point processes $\{\Phi_m\}$ describing the BS locations of every operator.

In this section, we describe how to construct the operator point processes when there are only two operators, and analyze the probability of coverage for Network 1. Next, we describe how to estimate the parameters of our model from actual two-operator deployments. Then, we consider two perspectives to study how the coverage probability changes as a function of the overlap between the two networks.

**A. Two-Operator Model**

The two-operator cases allows for a more natural construction of the operator point processes that allows to describe the model using a few parameters: the densities of the two networks and the extent
of overlap between them. We start with a mother point process $\Phi$ of density $\lambda$, and extract the two child point processes from it. We identify $\Phi$ as sites managed by a network infrastructure provider, and $\Phi_1$ and $\Phi_2$ as sites leased to two independent operators. We build our model by capturing the density of the first network (Network 1) and the density of the second (Network 2), denoted by $\lambda_1$ and $\lambda_2$. We introduce the overlap coefficient $\rho(\cdot, \cdot)$ as a measure of spatial correlation between Network 1 and Network 2 over two sets. For any two given sets $A$ and $B$, $\rho(A, B)$ is a function of the covariance between the number of Network 1 sites in $A$ and that of Network 2 in $B$ as

$$
\rho(A, B) \triangleq \frac{\text{Cov}(\Phi_1(A), \Phi_2(B))}{\mathbb{E}[\Phi(A \cap B)]}.
$$

Notice that the numerator of the overlap coefficient is a function of the cross-moment $\mathbb{E}[\Phi_1(A)\Phi_2(B)]$ which describes the interaction between two point processes $\Phi_1$ and $\Phi_2$. The normalization by the total density $\lambda$ is necessary so that one can compare the spatial correlation of two networks across different markets of distinct sizes. Since we are interested in the correlation of these two point processes over the entire geographical window $W$, we use $\rho$ as a shorthand notation to $\rho(W W)$. Proposition 2 shows that $\rho$ is in fact directly proportional to $\lambda$, which is a direct result of extracting the child processes from a mother process that is Poisson. If the mother process is not Poisson, the overlap coefficient will not necessarily be proportional to the total density $\lambda$.

We now explain how to mathematically construct $\Phi_1$ and $\Phi_2$ from $\Phi$. The key to construct $\Phi_1$ and $\Phi_2$ from $\Phi$ is the coupling technique, where we enforce that the derived point processes have some points $\{X_k\}$ of $\Phi$ in common by coupling them on the same probability space. We begin by marking the points of $\Phi$ with independent random variables $\{U_k\}_{k \geq 0}$ uniformly distributed between 0 and 1. We next consider two parameters $a$ and $b$, where $0 \leq b \leq a \leq 1$, and the retention functions $q_1(X_k) = a$ and $q_2(X_k) = 1 - b$. A retention function assigns to every point of a point process a probability of being retained, or alternatively, discarded [29]. Here the probability that a point $X_k$ in $\Phi$ is retained by $\Phi_1$ and $\Phi_2$ is $a$ and $1 - b$, respectively. As a result of thinning $\Phi$ separately with $q_1$ and $q_2$, the following two child processes can be obtained

$$
\Phi_1(\omega) = \{ X_k(\omega) \mid U_k(\omega) \leq a \}, \quad \Phi_2(\omega) = \{ X_k(\omega) \mid U_k(\omega) > b \},
$$

where $\omega \in \Omega$, and $\Omega$ is the common sample space. Let $\Phi_{12} = \Phi_{\{1,2\}} = \Phi_1 \cap \Phi_2$ be the point process describing the locations of shared sites, and let $\lambda_{12}$ be its density. We now give a proposition that validates the above construction of the individual networks from a greater one.
Proposition 2. Given $\lambda_1$, $\lambda_2$ and an overlap coefficient $\rho$ on a common geographical window $W \in \mathbb{R}^2$, the thinning of $\Phi$ with $a = \frac{\lambda_1}{\lambda}$ and $b = 1 - \frac{\lambda_2}{\lambda}$ yields $\Phi_1$ with density $\lambda_1$, $\Phi_2$ with density $\lambda_2$, and $\lambda = \frac{\lambda_1 + \lambda_2}{\rho}$. Furthermore, $\Phi_1$, $\Phi_2$, and $\Phi_{12}$ are PPPs.

Proof: See Appendix B.

Since $\rho$ turns out to be the fraction of co-located BSs, and $0 \leq \rho \leq 1$, we use it as a proxy for $\lambda_{12}$ to obtain $\lambda_1 + \lambda_2 - \rho \lambda = \lambda$. In the next proposition, we give the expression for the interference Laplace transform, which is the stepping stone towards the coverage probability expression.

Proposition 3. Given that the typical user establishes a LOS link of length $r$ with a BS of Network 1 that is not co-located with a BS of Network 2, the Laplace transform of the interference random variable is

\[
\mathcal{L}_{I_L}(s) = \exp \left( -2\pi \lambda \int_{0}^{r} (1 - u_L(s, t)) (1 - a) p_L(t) t \, dt \right) \cdot \exp \left( -2\pi \lambda \int_{r}^{+\infty} (1 - u_L(s, t)) (1 + \rho u_L(s, t)) p_L(t) t \, dt \right) \cdot \exp \left( -2\pi \lambda \int_{0}^{D_L(r)} (1 - u_N(s, t)) (1 - a) p_N(t) t \, dt \right) \cdot \exp \left( -2\pi \lambda \int_{D_L(r)}^{+\infty} (1 - u_N(s, t)) (1 + \rho u_N(s, t)) p_N(t) t \, dt \right). \tag{20}
\]

If, otherwise, the tagged BS of Network 1 is co-located with a BS of Network 2, then the Laplace transform of interference is given as

\[
\mathcal{L}_{I_L}^\prime(s) = u_L(s, r) \mathcal{L}_{I_L}(s). \tag{21}
\]

Proof: This follows directly from Proposition 1 with $M = 2$, $\mathcal{S} \in \{\{1\}, \{1, 2\}\}$, and $\mathcal{S}' = \{2\}$.

We examine the extreme cases of infrastructure sharing, namely full spatial independence and full spatial co-location, through (20). These two cases mimic the closed access with full spectrum sharing and co-located BSs with closed access and full spectrum license sharing systems studied in [20]. In particular, values of $\rho = 1$ (and $a = 1$) fold the Laplace transform expression in (20) into a product of the second and the fourth terms. These two factors account for the interference contribution from all BSs of the two networks that are located in the LOS and NLOS exclusion balls $B_0(r)$ and $B_0(D_L(r))$ centered at the typical user. Note that the Laplace transform expression for NLOS interference follows the same derivation, so it is excluded.
Before giving the final proposition, we introduce some notation. Let \( \Phi_1 = \Phi_1 \setminus \Phi_2 \) and \( \Phi_2 = \Phi_2 \setminus \Phi_1 \) be two point processes with respective densities \( \lambda_1' \) and \( \lambda_2' \). Now, we give a corollary that relates the probability of coverage of a typical subscriber of Network 1 in a two-operator system to the different system parameters.

**Corollary 2.** The SINR probability of coverage of a typical user of Network 1 in a two-operator system is given by

\[
P_c(T) = \int_{r \geq 0} \sum_{S} \left( e^{-\sigma^2 s_L} L[I ; L_S](s_L) f_R(r ; L_S) + e^{-\sigma^2 s_N} L[I ; N_S](s_N) f_R(r ; N_S) \right) dr.
\]

**(22)**

*Proof:* This is a special case of Proposition 1 with \( M = 2 \) and \( S \in \{ \{1\}, \{1,2\} \} \).

As in Corollary 1, the inner summation in (22) is across all collections \( S \) of operators in which operator 1 is present. Every block \( \Phi_S \) is divided into two sub-blocks \( L_S \) and \( N_S \), and their contributions to the total probability of coverage are weighted by the pdfs \( f_R(\cdot ; L_S) \) and \( f_R(\cdot ; N_S) \).

**B. Statistics of Actual Deployments**

To assess how well our model matches with an actual two-operator deployment of comparable network densities and overlap, we first need to estimate these parameters from the actual deployment. Unfortunately, there is one realization of the point process we seek to model per market, so the first order (the densities) and second order statistics (the overlap) need to be estimated from this single realization. Hence, we conduct our statistical analysis on a single observation and assume that the underlying point process \( \Phi \) is a PPP which is stationary and ergodic [34]. The key quantities to estimate given a bounded window \( W \) are \( \lambda, \lambda_1, \lambda_2, \) and \( \rho \). A general unbiased estimator of the total density \( \lambda \) according to [34] is

\[
\hat{\lambda} = \frac{\Phi(W)}{\text{vol}(W)},
\]

(23)

and similar expression for \( \lambda_1 \) and \( \lambda_2 \) follow.

We also provide two ways to estimate the overlap coefficient given in (19), which involves a cross-moment of two point processes. Second-order moments of point processes and methods of their estimation are well studied in [35]–[38] and the references therein, yet there is very little in the stochastic geometry literature on cross moments [35], [39] and their estimation. Therefore, we devise estimating the overlap coefficient in one of two ways:

**a) Indirectly through estimating \( \lambda_{12} \):** This uses the fact that the overlap coefficient \( \rho \) is directly proportional to the density of the mother process \( \Phi \) when the latter is Poisson, with the proportionality constant being \( \lambda_{12} \). Hence, we compute \( \hat{\lambda}_{12} \) as in (23), and then set \( \hat{\rho} = \frac{\hat{\lambda}_{12}}{\hat{\lambda}} \).
b) Directly through a naive estimator: This extends the notion of sample covariance to point processes to estimate the overlap $\rho$ between two point processes. We first apply a uniform partition $\mathcal{W}^{(n)}$ of size $n$ to the observation window $W$. Finally, we compute $\hat{\rho}$ as:

$$\hat{\rho}_n = \frac{\sum_{w \in \mathcal{W}^{(n)}} \Phi_1(w)\Phi_2(w) - \hat{\lambda}_1\hat{\lambda}_2|W|}{\hat{\lambda}|W|},$$

and $\hat{\rho} = \lim_{n \to \infty} \hat{\rho}_n$. (24)

Note that the expression of $\hat{\rho}$ as given by (24) does not necessarily guarantee that $\hat{\lambda}_1 + \hat{\lambda}_2 - \hat{\rho}\hat{\lambda} = \hat{\lambda}$, which is the case when the actual deployment patterns are not Poisson.

C. System Comparison

We consider two perspectives to study how coverage probability changes as a function of overlap between two mmWave cellular networks. For instance, increasing overlap does not imply that individual densities need to increase as well. To make the comparison simpler, we assume that the two networks have equal densities.

a) Fixed individual network densities (FID): The operators can opt to share sites with one another while maintaining the densities of their overall deployments. In a practical sense, this might occur when decreasing operational expenditures gets a higher priority densification. For example, consider two operators with fixed densities $\lambda_1 = \lambda_2 = \lambda_0$. To achieve an arbitrary overlap of $\rho$, each of the networks relocates a number of BSs accounting for $\frac{\rho}{1+\rho}$ of its density into the same number of sites of the competing network. As a result, $\lambda'_1 = \lambda'_2 = \left(\frac{1-\rho}{1+\rho}\right)\lambda_0$ and $\lambda_{12} = \left(\frac{2\rho}{1+\rho}\right)\lambda_0$. We will refer to $\lambda_{12}/\lambda_0$ as the sharing ratio.

b) Fixed combined network density (FCD): Each operator can share sites with the competing operator by means of expanding into the sites owned by the latter; i.e., each operator retains the sites that it started out with. In a practical sense, this might occur as strategic action to extend an operator’s reach in a market. For example, consider as above two operators with base densities $\lambda_1 = \lambda_2 = \lambda_0$. To achieve an overlap of $\rho$, each operator expands into $\rho$ of the competitor’s network. In this case, $\lambda'_1 = \lambda'_2 = (1-\rho)\lambda_0$ and $\lambda_{12} = 2\rho\lambda_0$. Note that the highest density achieved by either operator is the sum of the individual starting densities of individual networks; i.e. $2\lambda_0$.

The main takeaway here is that increasing $\rho$ under FID increases overlap but not individual densities. In contrast, increasing $\rho$ under FCD increases overlap and individual densities.
V. Numerical Results

In this section, we present numerical results that validate our coverage analysis, evaluate the accuracy of our model, and compare performance metrics across a range of overlap extents. We give all numerical results for the two-operator mmWave system.

For these results, we assume that the combined bandwidth of the mmWave systems operated by the networks is 200 MHz, and that the operating frequency is 28 GHz as in [20]. The values of the rest of the parameters are set to those of their counterparts in [7], [20], [25]. For the power law path loss model, we consider $c_L = -60$ dB, $c_N = -70$ dB, $\alpha_L = 2$, and $\alpha_N = 4$. Additionally, we consider a transmit power of 26 dBm, main-lobe gain of 18 dB, side-lobe gain of $-2$ dB, half beam-width of $10^\circ$, and a thermal noise power spectral density of $-174$ dBm/Hz with a noise figure of 10 dB. As for the densities of the two networks, we consider a reference density of $\lambda_0 = 30$ per km$^2$ which is equivalent to a cell radius of 103 m. Each network has an associated active user density of 200 per km$^2$. For the mmWave exponential blockage model, we consider $\beta = 0.007$ corresponding to an average LOS region of 144 m.

A. Validation of SINR Coverage Analysis

![Graph](image)

Fig. 2. Probability of SINR coverage vs. SINR threshold (dB) for different sharing schemes with individual network densities fixed at $\lambda_0$. Solid curves correspond to analytical results and marked curves correspond to simulation results.

We validate the analytical expressions for the probability of SINR coverage that were obtained in Section IV-A for a two-operator mmWave system with closed access and full spectrum sharing. We numerically evaluate the probability of coverage expression in (22) for a range of SINR thresholds, and...
we plot this against the empirical probability obtained through Monte Carlo simulation; the results are shown in Fig. 2. We consider FID with different overlap coefficient values, $\rho = 0$ or no infrastructure sharing, $\rho = 0.4$ or 57% sharing, and $\rho = 1$ or full sharing (by letting $\rho$ grow large, the two networks share more sites in common but their individual densities remains constant throughout). The first thing we observe, for all considered degrees of sharing, is that plots obtained through simulation match the ones obtained through analysis; which further validates the correctness of our analysis. Moreover, we observe that increasing overlap increases coverage probability at higher SINR thresholds, yet decreases coverage at lower SINR thresholds. The reason is that, in full sharing, there are no interfering BSs closer to the user than the associated BS, which has a positive impact on coverage at low SINR thresholds. As for high SINR, the anticipated signal is received at a much higher power than that of the interfering signals combined, but this is not the case in full sharing; the associated BS is co-located with another BS, which adds yet a source of interference that is just as powerful as the intended source.

B. Comparison of Estimators for the overlap $\rho$

We compare the direct and indirect estimators for $\rho$ proposed in IV-B. Since there are no current mmWave BS deployments, we have obtained the coordinates of current BS locations of legacy networks (2G to 4G) for four major operators in three major US cellular market areas (CMAs) of different geographic and demographic characteristics. The Hasse diagram in Fig. 3 shows the number of macro towers and other structures (rooftops, stealth, and DAS nodes) that house BS antennas of the different operators in every market. While there appears to be little infrastructure sharing in CMA III, CMAs I and II display strong instances of sharing.

To evaluate the accuracy of our two-operator PPP model, we compare the probability of SINR coverage between the PPP model and actual two-operator networks in different markets. Since mmWave systems have not been deployed yet, we “down-scale” the abscissas and ordinates of the BS locations so that the individual network densities compare to that of a typical mmWave network. We set that to 60 BSs/km$^2$. This operation is known as pressing, and is essentially an affine transformation in the plane which maintains the Poisson property of the original, full-scale point process. Note that pressing a network of two operators maintains their overlap.

For each market, we first estimate the densities and overlap using (23) and (24). Then, we generate a number of realizations of our two-operator PPP model with the obtained estimates. Finally we re-estimate the densities and overlap of these realizations. The reason for this procedure is two-fold. First, mismatching estimates for $\rho$ suggest that BS locations of actual deployments are not really PPP realizations. Second, we can verify that (24) is in fact an estimator for $\rho$ at least for a PPP. We plot
Fig. 3. Hasse Diagrams showing the number of macro towers and other structures that are shared by four major US operators (A–D) in three select CMAs (I–III). Nodes in these diagrams are marked by elements in the power set of \( \{A, B, C, D\} \). Each of these elements/subsets denote the identity of operators whose BSs are co-located with one another, and the number inside the corresponding node refers to the number of these shared structures. As an example, the corner nodes, marked with letters corresponding to the sets \( \{A\}, \{B\}, \{C\} \) and \( \{D\} \), contain the number of towers occupied by BS antennas of one and only one operator. Moving inwards, edges represent set membership; equivalently, moving outwards, edges represent set inclusion. As another example, in the leftmost diagram, \( |\{A\}| = 74, |\{D\}| = 216, |\{C\}| = 27, |\{A, D\}| = 23, |\{C, D\}| = 7 \), and finally \( |\{A, D, C\}| = 5 \); where the final term represents the towers with BSs of operators A, C, D but not B.

Our findings in Figure 4. We filter the direct estimate of \( \rho \)–which is a quantity that evolves with the number of bins–with a simple moving average filter to highlight the general trend it follows. The first observation we make is that the two estimates are different in both datasets: the two estimates are off by 0.2 in the first dataset and by 0.1 in the second. This suggests that the BS locations are not quite PPP. The second observation is that difference of the two estimates for CMA II is less than the error for CMA I. This can be explained by the fact that CMA II is a dense, DAS-node-dominated network that extends over 1% the area covered by CMA I which is a macro-tower-dominated network; and DAS node locations are more random than those of cell towers.

C. Validation of Model with Actual BS Deployments

To assess how accurately our model reflects the performance of an actual two-operator system, we compare the probability of coverage of the two systems while constraining the two networks to have the same individual densities and overlap. We do this experiment for two sets of actual deployments.

We first estimate \( \lambda_1, \lambda_2 \) and \( \lambda_{12} \) as described in Section IV-B. Figure 5 shows the views of base station deployments in CMAs I and II. The figure on the left shows base stations of networks A and B, and the one on the right shows those of operators B and C. Figure 6 shows the SINR probability of coverage attained by a typical user connecting to Operator A (left) and Operator C (right). Note that the overlap coefficient estimates used in evaluating the SINR coverage probability formula were
obtained indirectly. We can see that the SINR coverage plots for our two-operator PPP model match reasonably well with those for the actual deployments. Even though our model produces an almost identical coverage curve for CMA II, it produces a coverage probability curve that slightly deviates from the actual deployment at a wide range of SINR.

D. Comparison of Sharing Schemes

We compare the probability of rate coverage between three infrastructure sharing scenarios as well as the single operator system under FID and FCD. A single operator system comprises of the network of a single operator that shares neither infrastructure nor spectrum. In the sharing scenarios, we consider $\rho = 0, 0.4, 1$.

Figure 7 compares the probability of rate coverage between the different systems under FCD. A single system is made of the network of a single operator that shares neither infrastructure nor spectrum. In the sharing scenarios, we consider $\rho = 0, 0.4, 1$. We observe that by fixing the total density to a base value $\lambda_0$ and increasing the overlap between the base stations of the two operators, the probability of coverage increases for all rate thresholds. At low rate thresholds, the performance of a single-operator network with half the bandwidth and half as many BSs excels over shared networks with $\rho = 0.4$ at the least. This is explained by having no interfering BSs closer to the user than the associated BS. Figure 7 also shows that if two operators share only spectrum, their networks will provide up to 500%
Fig. 5. Actual deployment of three major US operators in two major markets. Operator A ○, Operator B △ and Operator C □. CMA I extends over 100× the area of CMA II. Corner vacancies in the first figure correspond to outskirts of the market. Co-located BSs are represented by overlapping shapes.

increase in rate coverage. If each network further expand into 40% of the competitors’ network, i.e. an overlap coefficient of 0.4, their networks will provide up to 800% increase in rate coverage.

Additionally, we see that these trends remain the same regardless of the types of small-scale and large-scale fading considered. This strengthens our earlier assumption that signals undergo Rayleigh fading.

Figure 8 compares the probability of coverage between the same five structure sharing scenarios and the single operators case under FID and two important observations can be made. First of all, the probability of rate coverage does not change appreciably with varying overlap. This can be explained by the fact that each of the networks maintain its individual density. Of course, higher coverage at low rate thresholds for deployments with more prominent sharing and the opposite trend at high thresholds follows the same reasoning we made before in relation to Figure 2. Second, the single user system outperforms systems with sharing for low rate thresholds, which is also be observed in Figure 7 for the FCD sharing strategy. This directly affects planning for deployments that communicate with devices through very low rates, such us IoT devices that wake up infrequently to send status updates or small measurements.
Figure 9 shows the trajectory of the median rate of the two sharing strategy and the reference strategy with the increase of either of the overlap coefficient (FID), or individual operator density (single operator), or both (FCD). For FID, unlike FCD, increasing the overlap between the two operators does not increase individual densities. For the single operator case, the operator shares neither spectrum nor infrastructure with other competitors. We observe that FCD gives a higher rate than the two strategies. In particular, the median rate an operator achieves under FCD is almost double the median rate achievable by a single operator at any individual density. Even though the interference in FCD is higher due to the presence of transmitting base stations in equal density for the second operator, the used spectrum is double. Moreover, sharing gives an almost steady median rate for FID. The reason is that the number of interfering BSs of the opposite network remains the same with varying the overlap coefficient. The only difference that changing overlap makes is whether the interferers of the opposite network are co-located with those of the home network.

VI. CONCLUSION

In this paper, we proposed a novel mathematical framework for modeling BS locations of a multi-operator cellular mmWave system, and provided analytical expressions for the SINR coverage probability as a performance metric for an arbitrary network. For a tractable evaluation of system performance, we narrowed the scope to a two-operator scenario. For the two-operator scenario, we provided a method
Fig. 7. Probability of rate coverage vs. downlink rate threshold (×100 Mbps) for \( \rho = 0, 0.4, 1 \) under FCD sharing strategy, and the single operator system. Combined network density is fixed at \( \lambda_0 = 60 \) BSs/km\(^2\). Coverage probabilities were computed experimentally with Rayleigh fading (Left), and Nakagami fading with parameters \( m = 2, 3 \) for LOS and NLOS, and log-normal shadowing with power \( \sigma_{\text{dB}} = 5.2, 7.6 \) for LOS and NLOS (Right). The trends in the two figures are identical which justifies using Rayleigh fading assumption in our analysis.

Fig. 8. Probability of rate coverage vs. downlink rate threshold (×100 Mbps) for different values of \( \rho \) under FID with individual network densities fixed at \( \lambda_0 = 30 \) per km\(^2\).

to fit actual deployments with our model and estimate its necessary parameters, and validated the model with realistic deployments of cellular systems. Additionally, we suggested different schemes that describe increased infrastructure sharing: FID and FCD. We showed that the median rate in FID does not change appreciably as the overlap between the networks varies, unlike in FCD that witnesses a
Fig. 9. Comparison of median rate: FID strategy, FCD strategy, and single operator system. For a single operator, median rate is a function of the network’s density. For FID, individual network densities is fixed at 30 km²; hence, median rate is a function of the overlap coefficient. For FCD, individual network densities grows with overlap. Hence median rate varies with both overlap and individual densities.

steadily increasing median rate in parallel to increasing overlap (or equivalently, increasing the density of the individual networks). We saw that the FCD strategy outperforms the single operator system for the same BS density, which is possible due to the availability of double the spectrum resources. We also observed that, for both FID and FCD, the single operator system with half the total bandwidth excels in the low rate threshold regime. That is, it is able to provide low data rates to more users than some of the two-operator systems under FID or FCD.

APPENDIX A

PROOF OF PROPOSITION 1

The user connects to Network 1 through \( L_T \). The Laplace transform of \( I_L \) and \( I_N \), the interference from the LOS and NLOS brackets, is given as follows:
\[ I_L = \sum_{S \in \mathcal{P}(O)} \sum_{1 \in S} \sum_{l \in S} c_L H_{l,i} G_{i,l} ||X_i||^{-\alpha_L} \]
\[ + \sum_{S' \in \mathcal{P}(O)} \sum_{1 \in S'} \sum_{m \in S'} c_L H_{j,m} G_{j,m} ||X_j||^{-\alpha_L} 1_{X_j \in B_0^c(r)} \]  
\[ + \sum_{n \in T} c_L H_{0,n} G_{0,n} r^{-\alpha_L} \]

\[ I_N = \sum_{S \in \mathcal{P}(O)} \sum_{1 \in S} \sum_{l \in S} c_N H_{l,i} G_{i,l} ||X_i||^{-\alpha_N} \]
\[ + \sum_{S' \in \mathcal{P}(O)} \sum_{1 \in S'} \sum_{m \in S'} c_N H_{j,m} G_{j,m} ||X_j||^{-\alpha_N} 1_{X_j \in B_0^c(D_L(r))} \]  

We derive the expression for \( \mathcal{L}_{I_L} \) (\( \mathcal{L}_{I_N} \) follows similarly). Since the point processes representing the different brackets are independent by construction,

\[ \mathcal{L}_{I_L}(s) = \prod_{n \in T} \mathbb{E} \left[ \exp \left( -s c_L H_{0,n} G_{0,n} r^{-\alpha_L} \right) \right] \cdot \prod_{S \in \mathcal{P}(O)} \mathbb{E} \left[ \exp \left( -s \sum_{1 \in S} \sum_{l \in S} c_L H_{l,i} G_{i,l} ||X_i||^{-\alpha_L} \right) \right] \cdot \prod_{S' \in \mathcal{P}(O)} \mathbb{E} \left[ \exp \left( -s \sum_{1 \in S'} \sum_{m \in S'} c_N H_{j,m} G_{j,m} ||X_j||^{-\alpha_L} 1_{X_j \in B_0^c(r)} \right) \right] . \]

Taking the Laplace transform of PPPs with respect to the function \( c_L H_{l,i} G_{i,l} ||X_i||^{-\alpha_N} \), and noting that independence of the fading and directionality gain RVs,

\[ \mathcal{L}_{I_L}(s) = \prod_{n \in T} \mathbb{E}_{G_n} \left[ \mathcal{L}_{H_n|G_n} \left( s c_L G_n r^{-\alpha_L} \right) \right] \cdot \prod_{S \in \mathcal{P}(O)} \exp \left( -2\pi \lambda_S \int_{t \geq 0} \left( 1 - \mathbb{E} \prod_{l \in S} e^{-st - \alpha_L c_L H_{l,i}} \right) t p_L(t) dt \right) \]
\[ \cdot \prod_{S' \in \mathcal{P}(O)} \exp \left( -2\pi \lambda_{S'} \int_{t \geq r} \left( 1 - \mathbb{E} \prod_{m \in S'} e^{-st - \alpha_L c_N H_{j,m}} \right) t p_L(t) dt \right) . \]
Moreover, since the independent marks \( \{ H_k \} \) and \( \{ G_k \} \) are assumed to be identically distributed, the Laplace transform simplifies as

\[
\mathcal{L}_{I_L}(s) = u_L(s, r)^{|T|-1} \cdot \prod_{S: 1 \in S} \exp \left( -2\pi \lambda_S \int_{t \geq 0} (1 - u_L(s, t)^{|S|}) p_L(t) t dt \right) \cdot \prod_{S': 1 \in S'} \exp \left( -2\pi \lambda_{S'} \int_{t \geq r} (1 - u_L(s, t)^{|S'|}) p_L(t) t dt \right).
\]

Finally, since the LOS and NLOS blocks are independent, we have that \( \mathcal{L}_{I_{L+I_N}}(s) = \mathcal{L}_{I_L}(s) \cdot \mathcal{L}_{I_L}(s) \).

**APPENDIX B**

**PROOF OF PROPOSITION 2**

According to the thinning theorem for a PPP [29], \( \Phi_1 \) and \( \Phi_2 \) are PPPs (Note that they are not independent) with respective densities \( a\lambda \) and \( (1 - b)\lambda \). Moreover, we claim that \( \rho = a - b \). To see this, consider the independently-marked point process \( \tilde{\Phi} = \sum_{k \geq 0} \delta_{(X_k, U_k)} \), where \( \delta \) is the Dirac measure, and \( \{ U_k \} \) are the iid marks, and \( U_k \sim U(0, 1), \forall k \). Given sets \( A \) and \( B \) in the Euclidean plane, and \( I, J \subset [0, 1] \), the intensity measure and the second moment measure of \( \tilde{\Phi} \) are given in [29] as

\[
\tilde{\Lambda}(A \times I) = \int_A \int_I dud\Lambda(x) = \lambda \text{vol}(A) \text{vol}(I),
\]

\[
\tilde{\mu}^{(2)}((A \times I) \times (B \times J)) = \tilde{\Lambda}(A \times I) \tilde{\Lambda}(B \times J) + \tilde{\Lambda}((A \cap B) \times (I \cap J)),
\]

where \( \text{vol}(\cdot) \) is the Lebesgue measure of a set taken with respect to the appropriate number of dimensions. Note that (28) holds true: the ground point process \( \Phi \) is a PPP on \( \mathbb{R}^2 \times [0, 1] \), then the independently marked point process \( \tilde{\Phi} \) is also a PPP [29]. Now let \( I = [0, a] \) and \( J = [b, 1] \). The
correlation between the number of sites of Network 1 in a set $A$ and Network 2 in a set $B$ is

\[
E[\Phi_1(A)\Phi_2(B)] = E \sum_{X_k \in \Phi \cap A} 1_{U_k \leq a} \sum_{X_l \in \Phi \cap B} 1_{U_l > b}
\]

\[
= E \sum_{X_k \in \Phi} 1_{X_k \in A} 1_{U_k \leq a} \sum_{X_l \in \Phi} 1_{X_l \in B} 1_{U_l > b}
\]

\[
= E \sum_{X_k \in \Phi} \sum_{X_l \in \Phi} 1_{X_k \in A} 1_{U_k \leq a} 1_{X_l \in B} 1_{U_l > b}
\]

\[
= E \int_{A \times I \times B \times J} \tilde{\Phi}^{(2)}(d(x_1, u_1, x_2, u_2))
\]

\[
= \int_{A \times I \times B \times J} \tilde{\mu}^{(2)}(d(x_1, u_1, x_2, u_2))
\]

\[
= a(1 - b)\lambda^2 \text{vol}(A) \text{vol}(B) + (a - b)\lambda \text{vol}(A \cap B),
\]

where $\tilde{\Phi}^{(2)} = \sum_{k,l \geq 0} \delta(x_k, u_k, x_l, u_l)$ is the second power of $\tilde{\Phi}$ [29]. Step (29) follows from Campbell’s mean value formula for the second power of a point process, while step (30) is the result of integrating with respect to the second moment measure given in (28). Hence, $\rho$ reduces to

\[
\rho = \frac{E[\Phi_1(A)\Phi_2(B)] - E[\Phi_1(A)]E[\Phi_2(B)]}{E[\Phi(A \cap B)]}
\]

\[
= \frac{(a - b)\lambda \text{vol}(A \cap B)}{\lambda \text{vol}(A \cap B)}
\]

\[
= a - b.
\]

REFERENCES

[1] T. S. Rappaport and S. Sun and R. Mayzus and H. Zhao and Y. Azar and K. Wang and G. N. Wong and J. K. Schulz and M. Samimi and F. Gutierrez, “Millimeter wave mobile communications for 5G cellular: It will work!” IEEE Access, vol. 1, pp. 335–349, 2013.

[2] S. Rangan, T. S. Rappaport, and E. Erkip, “Millimeter-wave cellular wireless networks: Potentials and challenges,” in Proc. IEEE, vol. 102, no. 3, pp. 366–385, March 2014.

[3] F. Boccardi, R. W. Heath, A. Lozano, T. L. Marzetta, and P. Popovski, “Five disruptive technology directions for 5G,” IEEE Commun. Mag., vol. 52, no. 2, pp. 74–80, February 2014.

[4] Z. Pi and F. Khan, “An introduction to millimeter-wave mobile broadband systems,” IEEE Commun. Mag., vol. 49, no. 6, pp. 101–107, June 2011.

[5] W. Roh, J. Y. Seol, J. Park, B. Lee, J. Lee, Y. Kim, J. Cho, K. Cheun, and F. Aryanfar, “Millimeter-wave beamforming as an enabling technology for 5G cellular communications: theoretical feasibility and prototype results,” IEEE Commun. Mag., vol. 52, no. 2, pp. 106–113, February 2014.
[6] T. S. Rappaport, F. Gutierrez, E. Ben-Dor, J. N. Murdock, Y. Qiao, and J. I. Tamir, “Broadband millimeter-wave propagation measurements and models using adaptive-beam antennas for outdoor urban cellular communications,” IEEE Trans. Antennas Propag., vol. 61, no. 4, pp. 1850–1859, April 2013.

[7] T. Bai and R. W. Heath, “Coverage and rate analysis for millimeter-wave cellular networks,” IEEE Trans. Wireless Commun., vol. 14, no. 2, pp. 1100–1114, Feb 2015.

[8] H. Shokri-Ghadikolaei, F. Boccardi, C. Fischione, G. Fodor, and M. Zorzi, “Spectrum sharing in mmWave cellular networks via cell association, coordination, and beamforming,” IEEE J. Sel. Areas Commun., vol. 34, no. 11, pp. 2902–2917, Nov 2016.

[9] L. Doyle, J. Kibilda, T. K. Forde, and L. DaSilva, “Spectrum without bounds, networks without borders,” in Proc. IEEE, vol. 102, no. 3, pp. 351–365, March 2014.

[10] M. Matinmikko, H. Okkonen, M. Palola, S. Yrjola, P. Ahokangas, and M. Mustonen, “Spectrum sharing using licensed shared access: the concept and its workflow for LTE-advanced networks,” IEEE Wireless Commun., vol. 21, no. 2, pp. 72–79, April 2014.

[11] D.-E. Meddour, T. Rasheed, and Y. Gourhant, “On the role of infrastructure sharing for mobile network operators in emerging markets,” Computer Networks, vol. 55, no. 7, pp. 1576–1591, 2011.

[12] T. Frisanco, P. Tafertshofer, P. Lurin, and R. Ang, “Infrastructure sharing and shared operations for mobile network operators from a deployment and operations view,” in Proc. IEEE NOMS, April 2008, pp. 129–136.

[13] A. Khan, W. Kellerer, K. Kozu, and M. Yabusaki, “Network sharing in the next mobile network: TCO reduction, management flexibility, and operational independence,” IEEE Commun. Mag., vol. 49, no. 10, pp. 134–142, Oct 2011.

[14] I. Giannoulakis, J. O. Fajardo, J. G. Lloreda, P. S. Khodashenas, C. Ruiz, A. Betzler, E. Kafetzakis, J. Prez-Romero, A. Albanese, M. Paolino, L. Goratti, and R. Riggio, “Enabling technologies and benefits of multi-tenant multi-service 5G small cells,” in Proc. EuCNC, June 2016, pp. 42–46.

[15] S. Okasaka, R. J. Weiler, W. Keusgen, A. Pudeyev, A. Maltsev, I. Karls, and K. Sakaguchi, “Proof-of-concept of a millimeter-wave integrated heterogeneous network for 5G cellular,” Sensors, vol. 16, no. 9, August 2016.

[16] A. Ghanbari, J. Markendahl, and A. A. Widaa, “Cooperation patterns in small cell networks: Risks and opportunities to distinguish the win-win model,” in Proc. European Regional ITS Conference, 2013.

[17] R. Behrends, L. K. Dillon, S. D. Fleming, and R. E. K. Stirewalt, “Multi-operator and neutral host small cells,” 5G Americas and Small Cells Forum, Tech. Rep. 191.08.02, December 2016. [Online]. Available: http://www.5gamericas.org/files/4914/8193/1104/SCF191_Multi-operator_neutral_host_small_cells.pdf

[18] F. Fund, S. Shahsavari, S. Panwar, E. Erkip, and S. Rangan, “Resource sharing among mmwave cellular service providers in a vertically differentiated duopoly,” in Proc. IEEE ICC, Paris, France, May 2017.

[19] H. Kamal, M. Coupechoux, and P. Godlewski, “Inter-operator spectrum sharing for cellular networks using game theory,” in Proc. IEEE PIMRC, Sept 2009, pp. 425–429.

[20] A. K. Gupta, J. G. Andrews, and R. W. Heath, “On the feasibility of sharing spectrum licenses in mmWave cellular systems,” IEEE Trans. Commun., vol. 64, no. 9, pp. 3981–3995, Sept 2016.

[21] M. Song, C. Xin, Y. Zhao, and X. Cheng, “Dynamic spectrum access: from cognitive radio to network radio,” IEEE Wireless Commun., vol. 19, no. 1, pp. 23–29, February 2012.

[22] I. F. Akyildiz, W.-Y. Lee, M. C. Vuran, and S. Mohanty, “NeXt generation/dynamic spectrum access/cognitive radio wireless networks: A survey,” Computer Networks, vol. 50, no. 13, pp. 2127 – 2159, 2006.

[23] X. Kang, Y. C. Liang, H. K. Garg, and L. Zhang, “Sensing-based spectrum sharing in cognitive radio networks,” IEEE Trans. Veh. Technol., vol. 58, no. 8, pp. 4649–4654, Oct 2009.

[24] M. D. Mueck, I. Karls, R. Arefi, T. Haustein, and W. Keusgen, “Licensed shared access for wave cellular broadband communications,” in Proc. IEEE CCS, Sept 2014, pp. 1–5.

[25] A. K. Gupta, A. Alkhateeb, J. G. Andrews, and R. W. Heath, “Gains of restricted secondary licensing in millimeter wave cellular systems,” IEEE J. Sel. Areas Commun., vol. 34, no. 11, pp. 2935–2950, Nov 2016.
[26] J. Kibilda, B. Galkin, and L. A. DaSilva, “Modelling multi-operator base station deployment patterns in cellular networks,” IEEE Trans. Mobile Comput., vol. 15, no. 12, pp. 3087–3099, Dec 2015.

[27] J. Kibilda, N. J. Kaminski, and L. A. DaSilva, “Radio access network and spectrum sharing in mobile networks: A stochastic geometry perspective,” IEEE Trans. Wireless Commun., vol. 16, no. 4, pp. 2562–2575, April 2017.

[28] M. Rebato, M. Mezzavilla, S. Rangan, and M. Zorzi, “Resource sharing in 5G mmWave cellular networks,” in Proc. IEEE INFOCOM WKSHPS, April 2016, pp. 271–276.

[29] F. Baccelli and B. Blaszczyszyn, Stochastic Geometry and Wireless Networks, Volume I – Theory. NoW Publishers, 2009.

[30] R. Valenzuela, “5G technologies: Opportunities and challenges,” in Proc. IEEE CTW, June 2017.

[31] J. G. Andrews, A. K. Gupta, and H. S. Dhillon. (2016) A primer on cellular network analysis using stochastic geometry. [Online]. Available: arXiv:1604.03183

[32] S. Singh, H. S. Dhillon, and J. G. Andrews, “Offloading in heterogeneous networks: Modeling, analysis, and design insights,” IEEE Trans. Wireless Commun., vol. 12, no. 5, pp. 2484–2497, May 2013.

[33] S. Singh, M. N. Kulkarni, A. Ghosh, and J. G. Andrews, “Tractable model for rate in self-backhauled millimeter wave cellular networks,” IEEE J. Sel. Areas Commun., vol. 33, no. 10, pp. 2196–2211, Oct 2015.

[34] D. Stoyan, W. S. Kendall, and J. Mecke, Stochastic Geometry and its Applications, 2nd ed. John Wiley and Sons, 2008.

[35] B. D. Ripley, “The second-order analysis of stationary point processes,” Journal of Applied Probability, vol. 13, no. 2, pp. 255–266, Jun 1976.

[36] ———, “Modeling spatial patterns,” Journal of the Royal Statistical Society, vol. 39, no. 4, pp. 172–212, 1977.

[37] D. S. J. Osher, “On the second-order and orientation analysis of planar stationary point processes,” Biomedical Journal, vol. 23, no. 6, pp. 523–533, 1981.

[38] J. Riihijarvi, P. Mahonen, and M. Rubsamen, “Characterizing wireless networks by spatial correlations,” IEEE Commun. Lett., vol. 11, no. 1, pp. 37–39, Jan 2007.

[39] D. H. Perkel, G. L. Gerstein, and G. P. Moore, “Neuronal spike trains and stochastic point processes: I. The single spike train,” Biomedical Journal, vol. 7, no. 4, pp. 391–418, 1967.