Hodge cycles for cubic hypersurfaces

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Abstract

We study an algebraic cycle of the form $Z_0 = r\mathbb{P}^n + \hat{r}\tilde{\mathbb{P}}^n$, $r \in \mathbb{N}, \hat{r} \in \mathbb{Z}$, $1 \leq r, |\hat{r}| \leq 10$, $\gcd(r, \hat{r}) = 1$, inside the cubic Fermat variety $X_0$ of even dimension $n \geq 4$ and with $\dim (\mathbb{P}^n \cap \tilde{\mathbb{P}}^n) = m$. We take a smooth deformation space $S$ of $X_0$ such that the triple $(X_0, \mathbb{P}^n, \tilde{\mathbb{P}}^n)$ becomes rigid. For $m = \frac{n}{2} - 2$ and for many examples of $N \in \mathbb{N}$ and $n$ we show that the $N$-th order Hodge locus attached to $Z_0$ is smooth and reduced of positive dimension if and only if $(r, \hat{r}) = (1, -1)$. In this case, the underlying algebraic cycles are conjectured to be cubic ruled cycles. For $m = \frac{n}{2} - 3$ the same happens for all choices of coefficients $r$ and $\hat{r}$ and we do not know what kind of algebraic cycles might produce such Hodge cycles. The first case gives us a conjectural description of a component of the Hodge locus, and the second case gives us strong computer assisted evidences for the existence of new Hodge cycles for cubic hypersurfaces. Whereas the well-known construction of Hodge cycles due to D. Mumford and A. Weil for CM abelian varieties, and Y. André’s motivated cycles can be described in theoretical terms, the full proof of the existence of our Hodge cycle seems to be only possible with more powerful computing machines.

1 Introduction

A. Weil starts his article [Wei77] with the following: ‘In searching for possible counterexamples to the “Hodge conjecture”, one has to look for varieties whose Hodge ring is not generated by its elements of degree 2’. The description of such elements for CM abelian varieties is fairly understood and it is due to D. Mumford and A. Weil himself, see [Mum66, page 166-167], [Wei77]. Other examples are Y. André’s motivated Hodge cycles in [And96]. None of these methods can be applied to hypersurfaces. By Lefschetz hyperplane section theorem, for hypersurfaces of even dimension $n \geq 4$ we have only a one dimensional subspace of $H^2$ generated by elements of degree 2 (the class of a hyperplane section), and producing interesting Hodge cycles in this case, for which the Hodge conjecture is not known, is extremely difficult, and hence, they might be a better candidate for a counterexample to the Hodge conjecture. Even for Fermat varieties the methods introduced by Z. Ran, T. Shioda in [Ran81], [Shi79] are not based on explicit description of Hodge cycles, and the author in his book [Mov19, Chapter 15] had to work out a computer implementable description of such cycles. This resulted, for instance, to the verification of integral Hodge conjecture for many examples of the Fermat variety, see [AMV19]. The main goal of Chapter 18 of this book is to describe a computer assisted project in order to classify components of Hodge loci passing through Fermat point, and in this way to discover new Hodge cycles by deforming the class of algebraic cycles. The main difficulty is that the parameter space of hypersurfaces are usually of huge dimensions and it is hard to carry out computations even with the modern computers of today. In November 2018, P. Deligne wrote many comments regarding this chapter, and this gave the author some force to push forward a tiny step toward one of the main goals of this book. This is namely to find a smaller parameter space for cubic hypersurfaces and describe instances, where a rigid algebraic cycle deforms into a Hodge cycle.

Let $n \geq 4$ be an even number and let us consider the projective space $\mathbb{P}^{n+1}$ with the coordi-
nate system \([x_0 : x_1 : \cdots : x_{n+1}]\) and \(\mathbb{P}^n_\mathbb{R}, \mathbb{P}^n_\mathbb{C} \subset \mathbb{P}^{n+1}\) given by:

\[
\begin{align*}
\mathbb{P}^n_\mathbb{R} & : \begin{cases}
x_0 - \zeta_6 x_1 = 0, \\
x_2 - \zeta_6 x_3 = 0, \\
x_4 - \zeta_6 x_5 = 0, \\
\vdots \\
x_{n} - \zeta_6 x_{n+1} = 0,
\end{cases} & \mathbb{P}^n_\mathbb{C} & : \begin{cases}
x_0 - \zeta_6 x_1 = 0, \\
\vdots \\
x_{2m} - \zeta_6 x_{2m+1} = 0, \\
x_{2m+2} + x_{2m+3} = 0, \\
\vdots \\
x_n + x_{n+1} = 0,
\end{cases}
\end{align*}
\]

where \(\zeta_6 := e^{\frac{2\pi \sqrt{-1}}{3}}\). These are linear algebraic cycles in the cubic Fermat variety \(X_0 \subset \mathbb{P}^{n+1}\) given by the homogeneous polynomial \(x_0^3 + x_1^3 + \cdots + x_{n+1}^3 = 0\), and satisfy \(\mathbb{P}^n_\mathbb{R} \cap \mathbb{P}^n_\mathbb{C} = \mathbb{P}^m\). For the main purpose of this paper we will only consider cubic Fermat varieties and \(m = \frac{n}{2} - 2, \frac{n}{2} - 3\). The other cases are fairly discussed in [Mov19, Chapter 18]. We choose a deformation of the Fermat variety

\[
X_1 : x_0^3 + x_1^3 + \cdots + x_{n+1}^3 - \sum_{\alpha \in I} t_{\alpha} x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} = 0, \quad t = (t_\alpha)_{\alpha \in I} \in S := \mathbb{C}^{|I|},
\]

where \(\alpha\) runs through a finite subset \(I\) of all three elements subsets of \(\{0, 1, \ldots, n+1\}\). This deformation is taken in such a way that that the triple \((X_0, \mathbb{P}^n_\mathbb{R}, \mathbb{P}^n_\mathbb{C})\) does not deform, see \([2]\). For \(n = 4, 6, 8, 10\) we have computed such a deformation and the corresponding monomials are listed in Table 3 and Table 4. For instance, for \(n = 4\) we have considered the deformations:

\[
\begin{align*}
X_1 & : \ x_0^3 + x_1^3 + \cdots + x_{n+1}^3 - (t_1 x_2 + t_2 x_3) x_1 x_5 \text{ the case } m = 0, \\
X_1 & : \ x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - (t_1 x_2 + t_2 x_3) x_3 x_5 \text{ the case } m = -1.
\end{align*}
\]

For any algebraic cycle \(Z_0\) of dimension \(\frac{n}{2}\) in \(X_0\) let \([Z_0] \in H_n(X_0, \mathbb{Z})\) be the homology class of \(Z_0\) which is a Hodge cycle. For a Hodge cycle \(\delta_0 \in H_n(X_0, \mathbb{Z})\), and in particular \(\delta_0\) any linear combination of \([\mathbb{P}^n_\mathbb{R}]\) and \([\mathbb{P}^n_\mathbb{C}]\) with \(\mathbb{Z}\) coefficients, and \(N \in \mathbb{N}\), we can define the Hodge locus \(V_{\delta_0}\) and \(N\)-th order infinitesimal Hodge locus \(V^{N}_{\delta_0}\), see \([3]\). The first order Hodge locus \(V^{1}_{\delta_0}\) is the tangent space \(T_{0}V_{\delta_0}\) of \(V_{\delta_0}\) at 0 and its study is mainly done under the name infinitesimal variation of Hodge structures (IVHS) introduced by P. Griffiths and his coauthors in [CCGHS3].

**Theorem 1.** Let \(\mathbb{P}^n_\mathbb{R}\) and \(\mathbb{P}^n_\mathbb{C}\) be linear cycles in \((1)\) with \(\mathbb{P}^n_\mathbb{R} \cap \mathbb{P}^n_\mathbb{C} = \mathbb{P}^m\) and \(n = 4, 6, 8, 10, 12\). For \(m = \frac{n}{2} - 2, \frac{n}{2} - 3\) consider the family of hypersurfaces \((2)\) with monomials coming from Table 3 and Table 4 respectively.

1. \((m = \frac{n}{2} - 2)\) For all \(r, \bar{r} \in \mathbb{Z}, 1 \leq |r|, |\bar{r}| \leq 10\), the infinitesimal Hodge locus \(V^{N}_{r[\mathbb{P}^n_\mathbb{R}]+\bar{r}[\mathbb{P}^n_\mathbb{C}]}\) is smooth at 0 for the cases in Table 1 with \(\check{\text{V}}\) mark, and further for \(r \neq -\bar{r}\) it is not smooth at 0 for the cases in Table 1 with \(X\) mark, and so, for all these cases the Hodge locus \(V^{N}_{r[\mathbb{P}^n_\mathbb{R}]+\bar{r}[\mathbb{P}^n_\mathbb{C}]}\) as analytic scheme is either non-reduced or its underlying analytic variety is singular at the Fermat point 0. Moreover, the infinitesimal Hodge locus \(V^{N}_{[\mathbb{P}^n_\mathbb{R}]-[\mathbb{P}^n_\mathbb{C}]}\) is smooth for all \(N\)‘s listed in the last row of Table 1.

2. \((m = \frac{n}{2} - 3)\) For all \(r, \bar{r} \in \mathbb{Z}, 1 \leq |r|, |\bar{r}| \leq 10\), the infinitesimal Hodge locus \(V^{N}_{r[\mathbb{P}^n_\mathbb{R}]+\bar{r}[\mathbb{P}^n_\mathbb{C}]}\) is smooth at 0 for all \(N\)‘s listed in Table 2. Moreover, their tangent spaces form a pencil with the origin as axis, that is, they have the same dimension and intersect each other only at the origin.

3. The codimension of the Zariski tangent space of \(V^{N}_{r[\mathbb{P}^n_\mathbb{R}]+\bar{r}[\mathbb{P}^n_\mathbb{C}]}\) at 0 in both cases as above is listed in the third row of Table 3 and Table 4 respectively.
4. The Fermat variety $X_0$ together with its algebraic cycles $\mathbb{P}^2_\mathbb{F}_r$, $\mathbb{P}^{2,\mathbb{F}_r}$ with $m = \frac{n}{2} - 2$ (resp. $m = \frac{n}{2} - 3$) is rigid inside the family \cite{Mov19} with monomials coming from Table 2 (resp. 3).

Theorem 1 and similar computation in \cite{Mov19, Chapter 18} for the full family of hypersurfaces lead us to speculate many conjectures. The following might be the most evident one.

**Conjecture 1.**

1. For the full family of cubic hypersurfaces the Hodge locus $V_{\mathbb{P}^2_\mathbb{F}_r - [\mathbb{F}_r]}$ with $\mathbb{P}^2_\mathbb{F} \cap \mathbb{P}^{2,\mathbb{F}_r} = \mathbb{P}^{2,\mathbb{F}_r - 2}$ is smooth and it is larger than the deformation space of the triple $(X_0, \mathbb{P}^2_\mathbb{F}, \mathbb{P}^{2,\mathbb{F}_r})$.

2. For the full family of cubic hypersurfaces the Hodge locus $V_{r[\mathbb{P}^2_\mathbb{F} - r\mathbb{F}]}$ with $\mathbb{P}^2_\mathbb{F} \cap \mathbb{P}^{2,\mathbb{F}_r} = \mathbb{P}^{2,\mathbb{F}_r - 3}$ and $|r|, |\bar{r}| \in \mathbb{N}$, is smooth and it is larger than the deformation space of the triple $(X_0, \mathbb{P}^2_\mathbb{F}, \mathbb{P}^{2,\mathbb{F}_r})$. The difference of dimensions in this case is 1.

Conjecture 1 is true for cubic fourfolds for trivial reasons. In this case $h^{40} = 0$, $h^{31} = 1$ and so any Hodge locus $V_{\delta_0}$, $\delta_0 \in H_n(X_0, \mathbb{Z})$ primitive non-zero Hodge cycle, is given by one equation, which turns out that it has non-zero linear part when we consider the full parameter space of cubic hypersurfaces, and hence it is always smooth. Both the integral and rational Hodge conjecture are proved in this case, see \cite{Zuc77} and \cite{Voi13, Theorem 2.11 and the comments thereafter}. In this case, an effective construction of algebraic cycles for hypersurfaces parameterized by $V_{r[\mathbb{P}^2_\mathbb{F} + r[\mathbb{P}^2]}$ might give some hint to do a similar verification in higher dimensions. For a partial verification of Conjecture 1 part 1 see \cite{Zuc77}.

For the proof of Theorem 1 the author has written many procedures which are collected in the library foliation.lib of SINGULAR, see \cite{GPS01}. In order to check the computations of the present paper, we first get this library from the author’s web page.\footnote{http://w3.impa.br/~hossein/foliation-allversions/foliation.lib} Then we run the example session of the procedure. For instance, for the procedure InterTang used in \cite{Zuc77} we run

```
LIB foliation.lib;
example InterTang;
```

Modifying, the code in the example session (for instance changing the dimension $n$ or degree $d$ of the hypersurface) we get all the claimed statements.
Table 3: Monomials of a deformation: $m = \frac{n}{2} - 2$

| $n = 4$ | $x_1 x_2 x_5$, $x_1 x_3 x_5$ |
| $n = 6$ | $x_1 x_3 x_4$, $x_1 x_3 x_5$, $x_1 x_3 x_6$, $x_1 x_3 x_7$, $x_1 x_4 x_7$, $x_3 x_4 x_7$, $x_1 x_5 x_7$, $x_3 x_5 x_7$ |
| $n = 8$ | $x_1 x_3 x_5$, $x_1 x_3 x_6$, $x_1 x_3 x_7$, $x_1 x_4 x_7$, $x_3 x_4 x_7$, $x_1 x_5 x_7$, $x_3 x_5 x_7$, $x_1 x_7 x_7$, $x_3 x_7 x_7$, $x_1 x_9 x_9$, $x_3 x_9 x_9$, $x_1 x_9 x_9$, $x_3 x_9 x_9$ |
| $n = 10$ | $x_1 x_3 x_5$, $x_1 x_3 x_6$, $x_1 x_3 x_7$, $x_1 x_4 x_7$, $x_3 x_4 x_7$, $x_1 x_5 x_7$, $x_3 x_5 x_7$, $x_1 x_7 x_7$, $x_3 x_7 x_7$, $x_1 x_9 x_9$, $x_3 x_9 x_9$, $x_1 x_9 x_9$, $x_3 x_9 x_9$ |
| $n = 12$ | $x_0 x_3 x_5$, $x_1 x_3 x_5$ |

Table 4: Monomials of a deformation for $m = \frac{n}{2} - 3$
The present work would not have been possible without the attention of two great mathematicians: thanks go S.-T. Yau for all his effort to create lovely ambients to do mathematics, from CMSA to TSIMF which I enjoyed both institutes during the preparation of this text, and to P. Deligne for all his enlightening emails and comments to the author’s book and this article. I would also like to thank D. van Straten for his help in § 5.

2 Smaller deformation space

Let $T$ be the (full) parameter space of smooth cubic hypersurfaces of dimension $n$ and $X/T$ be the corresponding family. Let also $X_0$, $0 \in T$ be a smooth hypersurface given by the zero set of a homogeneous polynomial of the form

$$f = f_1 f_{s+1} + f_2 f_{s+2} + \cdots + f_s f_{2s}, \ f_i \in \mathbb{C}[x]_{d_i}, \ f_{s+i} \in \mathbb{C}[x]_{d-d_i}, \ s := \frac{n}{2} + 1. \tag{5}$$

We call $Z_0 : f_1 = f_2 = \cdots = f_s = 0$ a complete intersection algebraic cycle. Let also $V_{Z_0} \subset (T, 0)$ be the analytic variety parameterizing deformations of $(X_0, Z_0)$. This corresponds to variation of polynomials $f_i$ as above. This is a branch of the algebraic set $T_d \subset T$ which parameterizes all hypersurfaces given by $f$ of the form 5. For $d_1 = d_2 = \cdots = d_s = 1$, $T_d$ has $N := 1 \cdot 3 \cdot (n-1)(n+1)d^{n+1}$ branches near the Fermat point $0 \in T$, see [Mov19, §17.4]. If $f_1, f_2, \cdots, f_{2s}$ have not common zeros in $\mathbb{P}^{n+1}$ then they form a regular sequence and a Koszul complex argument tells us that $V_Z$ is smooth at 0 and its tangent space at this point is given by the degree $d$ part of the homogeneous ideal $\langle f_1, f_2, \cdots, f_{2s} \rangle$, see [Mov19, Proposition 17.5]. In particular, this is the case for two linear cycles

$$\mathbb{P}^\frac{n}{2} : f_1 = f_2 = \cdots = f_s = 0,$$

$$\mathbb{P}^\frac{n}{2} : f_1 = f_2 = \cdots = f_s = 0,$$

inside the Fermat variety $X_0$ and given in [11]. Moreover, the intersection $V_{\mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2}} := V_{\mathbb{P}^\frac{n}{2}} \cap V_{\mathbb{P}^\frac{n}{2}}$ is smooth at 0 and its tangent space at this point is given by the intersection of tangent spaces of $V_{\mathbb{P}^\frac{n}{2}}$ and $V_{\mathbb{P}^\frac{n}{2}}$ at 0, [Mov19, Proposition 17.8]. We conclude that

$$T_0 V_{\mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2}} = I_d, \ \mathcal{I} := \langle f_1, f_2, \ldots, f_{2s} \rangle \cap \langle \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{2s} \rangle.$$

We make the identification of $T$ with a Zariski open subset of $\mathbb{C}[x]$ such that $f \in T$ parametrizes the hypersurface given by $x_0^d + x_1^d + \cdots + x_{2s+1}^d + f = 0$. In this way, $T_0 T = \mathbb{C}[x]$. We choose a monomial basis $x^\alpha, \ \alpha \in I$ of degree $d$ piece of the quotient $\mathbb{C}[x]/\mathcal{I}$ and define $S$ to be a linear subspace of $\mathbb{C}[x]$ generated by these monomials. By definition it is perpendicular to $\mathcal{I}_d$, and this is all what we need. The monomials in Table 3 and Table 4 are obtained in this way. This has been implemented in the procedure InterTang.
3 Infinitesimal Hodge loci

After P. Griffiths’ work [Gri69] we know that for cubic hypersurfaces $X_t$ with $t$ in a Zariski neighborhood of $0 \in \mathbb{T}$, the primitive de Rham cohomology $H^0_{\text{dR}}(X_t)_0$ has a basis of the form

$$\omega_\beta := \text{Resi} \left( \frac{x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{k-n-2}} \Omega}{f^k} \right),$$

where $\text{Resi} : H^{n+1}_{\text{dR}}(\mathbb{P}^{n+1} - X_t) \to H^n_{\text{dR}}(X_t)_0$ is the residue map, $\beta_i$'s are distinct and the subindex $0$ refers to primitive cohomology. Moreover, this basis is compatible with the Hodge filtration, that is, $F^{n+1-a}H^0_{\text{dR}}(X_t)_0$ is generated by $\omega_\beta$ with $k \leq a$. For the definition of a Hodge locus we need $a = \frac{n}{2}$. For a Hodge cycle $\delta_0 \in H_n(X_0, \mathbb{Z})$, the Hodge locus $V_{\delta_0} \subset (\mathbb{T}, 0)$ is an analytic scheme given by the ideal

$$\left( \int_{\delta_t} \omega_\beta \mid \forall \beta \subset \{0, 1, 2, \ldots, n + 1\} \text{ with }, \# \beta = 3k - n - 2, \ k \leq \frac{n}{2} \right) \subset \mathcal{O}_{T,0}.$$

Let $\mathcal{M}_{T,0}$ be the maximal ideal of $\mathcal{O}_{T,0}$, that is, the set of germs of holomorphic functions in $(\mathbb{T}, 0)$ vanishing at $0$. The $N$-th order infinitesimal scheme $V^N_{\delta_0}$ is the induced scheme by (6) in the infinitesimal scheme $T^N := \text{Spec}(\mathcal{O}_{T,0}/\mathcal{M}_{T,0}^{N+1})$. We denote by $X^N/T^N$ the $N$-th order infinitesimal deformation of $X_0$ induced by $X/T$. Let $\text{cl}(Z_0) \in H^n_{\text{dR}}(X_0)$ be the class of an algebraic cycle $Z_0$ of codimension $\frac{n}{2}$ in $X_0$. Let us consider the Gauss-Manin connection

$$\nabla : H^n_{\text{dR}}(X/T) \to \Omega^1_T \otimes_{\mathcal{O}_T} H^n_{\text{dR}}(X/T).$$

It induces a connection in $H^n_{\text{dR}}(X^N/T^N)$ which we call it again the Gauss-Manin connection. There is a unique section $s$ of $H^n_{\text{dR}}(X^N/T^N)$ such that $\nabla(s) = 0$ and $s_0 = \text{cl}(Z_0)$. This is called the horizontal extension of $\text{cl}(Z_0)$ or a flat section of the cohomology bundle. An equivalent definition for $V^N_{[Z_0]}$ is as follows.

**Definition 1.** The Hodge locus $V^N_{[Z_0]}$ is a subscheme of $T^N$ given by the conditions

$$\nabla(s) = 0,$$

$$s \in F^a_{\mathbb{T}} H^n_{\text{dR}}(X^N/T^N),$$

$$s_0 = \text{cl}(Z_0).$$

4 Proof of Theorem [1]

The main ingredient of the proof is a closed formula for the Taylor expansion of the holomorphic functions $\int_{\delta_t} \omega_\beta$. Since we have only used a computer implementation of this formula, we do not reproduce it here and refer the reader to [Mov19 §18.5, Chapter 19]. The proof of Item 1 and Item 2 are the same as the proof of [Mov19 Theorem 18.2, Theorem 18.3]. For Item 3 note that these numbers are the dimension of the C-vector space generated by linear parts of $\int_{\delta_t} \omega_\beta$ in (6). Item 4 is just the consequence of our construction of $S$ in (2). Note that in a neighborhood of $0$ in $S$, $S$ intersects $V^{\mathbb{F}_2 + \mathbb{F}_2}$ only at $0$. The following computer code has been used for the proof of statements in Theorem 4 Item 1 and Item 2 involving arbitrary coefficients. A simple modification of it can be used for the case $(r, \bar{r}) = (1, -1)$. 

6
The tangent spaces of $V_{[P^2]} = V_{[P^2]}$ and $V_{[P^2]} = V_{[P^2]}$ are of the form $\ker(A)$ and $\ker(\bar{A})$, where $A$ and $\bar{A}$ are two $\dim(S) \times \frac{1}{\chi} + 1, \frac{1}{\chi} - 1$ matrices with entries in $\mathbb{Q}(\zeta_6)$ which can be constructed from the periods of the underlying algebraic cycles, see [Mov19] §16.5. The tangent space of $V_{[P^2]} = V_{[P^2]}$, $x \in \mathbb{Q}$ at 0 is given by $\ker(A + x\bar{A})$. By our construction of $S$, $\ker(A) \cap \ker(\bar{A}) = \{0\}$ which is the tangent space of the deformation space of $(X_0, \mathbb{P}^n, \mathbb{P}^n)$. For two distinct $x_1, x_2 \in \mathbb{Q}$, we have $\ker(A + x_1\bar{A}) \cap \ker(A + x_2\bar{A}) = \ker(A) \cap \ker(\bar{A}) = \{0\}$ and the last affirmation in Item 2 follows. Note that in all the cases considered in Table 2, $\ker(A + x_1\bar{A})$ is a one dimensional space which is a byproduct of our computations above.

5 Finding algebraic cycles

After the first draft of this paper was written, there was many email exchanges in January 2019 with P. Deligne in order to verify the Hodge conjecture for the Hodge cycles in Theorem 1 part 2. Most of the content of this section is the result of this joint effort. We were not able to give similar descriptions for Theorem 1 part 2. Let

\begin{equation}
\mathbb{P}^n_{a_1, a_2} : \begin{cases}
x_0 - \zeta_6 x_1 = 0, \\
x_{n-4} - \zeta_6 x_{n-3} = 0, \\
x_{n-2} - \zeta_6 x_{n-3} = 0, \\
x_n - 6 x_{n+1} = 0.
\end{cases}
\end{equation}

Using this notation we have $\mathbb{P}^n_{0,0} = \mathbb{P}^n_{1,1} = \mathbb{P}^n_{3,1}$. It turns out that

\begin{equation}
\mathbb{P}^n_{0,0} = \mathbb{P}^n_{1,1} = \mathbb{P}^n_{0,0} + \mathbb{P}^n_{0,1} + \mathbb{P}^n_{2,1}
\end{equation}

which is written modulo $\mathbb{P}^{n+1}$ slices of the Fermat $X_0^n$, and hence, both sides of (11) induce the same element in the primitive cohomology and have the same Hodge locus. Let $C_0$ be the algebraic cycle in the right hand side of (11). We write the three cycles in $C_0$ as

$\mathbb{P}^n_{0,0} : g_1 = g_2 = \cdots = g_{n-1} = f_{11} = f_{21} = 0,$

$\mathbb{P}^n_{0,1} : g_1 = g_2 = \cdots = g_{n-1} = f_{11} = f_{32} = 0,$

$\mathbb{P}^n_{2,1} : g_1 = g_2 = \cdots = g_{n-1} = f_{22} = f_{32} = 0,$

where $g_i$ and $f_{ij}$'s are homogeneous linear polynomials. Now, we can easily see that this algebraic cycle deforms into:

\begin{equation}
C : g_1 = g_2 = \cdots = g_{n-1} = 0, \quad \text{rank} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix} \leq 1,
\end{equation}

where $g_i$'s and $f_{ij}$'s are deformed homogeneous linear polynomials, and for simplicity, we have not introduced new notation for deformed polynomials. The cycle $C$ deforms to $C_0$ by setting $f_{12}, f_{31}$.
equal to zero. In a personal communication Duco van Straten pointed out the determinantal structure of $C$ and the fact that for $\frac{d}{2} = 2$, $C$ is the cubic ruled surface/Hirzebruch surface $F_1$. It is isomorphic to $\mathbb{P}^2$ blown up in a single point, embedded by the linear system of quadrics through the point. For this reason, we call $C$ a cubic ruled cycle of dimension $\frac{d}{2}$.

The ideal given by the determinantal variety in (14) is radical, and hence, if a smooth hypersurface of degree $d$ and dimension $n$ given by the homogeneous polynomial $f$ contains $C$ then

$$f = g_1 *_1 + g_2 *_2 + \cdots + g_{\frac{d}{2} - 1} *_{\frac{d}{2} - 1} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix},$$

where $*_i$’s are homogeneous polynomials of degree $d - \deg(g_i)$ and the first two columns consist of linear homogeneous polynomials and the last column of degree $d - 2$ homogeneous polynomials. Let $\bar{T} \subset T$ be the space of smooth hypersurfaces of the form (13) and $V_{C_0} \subset \bar{T}$ be the branch of $T$ corresponding to deformations of $(X_0, C_0)$. It is now natural to conjecture that $V_{[C_0]} = V_{C_0}$.

For this it is enough to prove that the codimension of $\bar{T}$ in $T$ is the same as the codimension of $T_0V_{[C_0]}$ in $T_0\bar{T}$ which is computed in Table 1. The author was not able to find a closed formula for the codimension of $\bar{T}$. Instead, it is possible to take random points in $\bar{T}$ and compute the tangent space of $\bar{T}$ at such points for all examples of $n$ in Table 1. It turned out that one gets the desired codimension. For this computer assisted verification, we have used CodRuledCubic.

It might be possible to prove that $\bar{T}$ is a component of the Hodge locus, that is, for generic $X_t, t \in \bar{T}$ with $C \subset X$ as above, the homology class of $C$ cannot be deformed into a Hodge cycle in $X_t, t \in T \setminus \bar{T}$. For sum of two linear cycles such statements are proved in [MV19] using computer calculations and in [Vil18] using long theoretical methods.

It is early to claim that the conjectural Hodge cycles introduced in Theorem 1 part 2 are counterexamples to the Hodge conjecture, as this needs more time and effort of other people who might try to verify the Hodge conjecture for such cycles. However, it seems to the author that they are better candidates for this than the well-known ones. Components of low codimension of the Noether-Lefschetz loci (Hodge loci for $n = 2$) parameterize surfaces with rather simple algebraic cycles such as lines or quadrics, see for instance [Gre89, Vor89]. The minimal codimension of components of the Hodge loci for cubic hypersurfaces in a Zariski neighborhood of the Fermat point $0 \in T$ is $(\frac{2d}{3} + 3) - (\frac{d}{2} + 1)^2 = (\frac{2d}{3} + 1)$ which is obtained by $T_1$ introduced in [2] see [Mov17 Theorem 2]. The next admissible codimension seems to be of $\bar{T}$ introduced in this section. Note that for cubic hypersurfaces all the components $T_{\frac{d}{2}}$ are the same as $\bar{T}_\frac{d}{2}$. After this, we have conjecturally an infinite number of components $V_{\mathbb{P}^2 + \mathbb{P}^2 + \mathbb{P}^2}, m = \frac{n}{2} - 3$ of the same codimension. For the convenience of the reader we have also computed Table 6 which contains the dimension of the full moduli, Hodge numbers and the range of codimensions of Hodge loci for cubic hypersurfaces. Finally, note that it follows from [DMO92 Theorem 2.12, Principle B] that the Hodge cycles of the present text are absolute.

6 Learning from cubic fourfolds

Despite the fact that the integral Hodge conjecture for cubic fourfolds is well-known, effective construction of algebraic cycles in this case might help us in a better understanding of the Hodge cycles in Theorem 1. For cubic fourfolds, Hodge loci is a union of codimension one irreducible subvarieties $C_D, D \equiv_6 0, 2, D \geq 8$ of $T$, see [Has00]. Here, $D$ is the discriminant of the saturated lattice generated by $[Z]$ and $[Z_\infty]$ in $H_4(X, \mathbb{Z})$ (in [Has00] notation $[Z_\infty] = h^2$), where $Z$ is an algebraic cycle $Z \subset X, X \in C_D$ whose homology class together $[Z_\infty]$ form a rank two lattice. The loci of cubic fourfolds containing a plane (resp. cubic ruled surface) is $T_{1,1,1} = C_8$ (resp.
The case \( m = -1 \) is our main interest and it is always \( \equiv_6 0, 2 \), however, the corresponding lattice might not be saturated. The first values of \( D \) above are 14, 18, 36 which are obtained by \( (r, \tilde{r}) = (1, 1), (1, -1), (2, 1) \), respectively. Below, we will describe generalizations of \( C_{14} \) and \( C_{20} \) for cubic \( n \)-folds. They seems to be far from any possible algebraic cycle for the verification of Theorem 1 part 2.

The loci \( C_{14} \) parametrizes cubic fourfolds with a quartic scroll. This is the image of

\[
P^1 \times P^1 \hookrightarrow P^5, \quad (x, y) \mapsto [f_1 g_1 : f_1 g_2 : f_1 g_3 : f_2 g_1 : f_2 g_2 : f_2 g_3],
\]

where \( f_i \)'s (resp. \( g_i \)'s) form a basis of \( \mathbb{C}[x, y]_1 \) (resp. \( \mathbb{C}[x, y]_2 \)). Putting \( f_1 = x, \ f_2 = y, \ g_1 = x^2, \ g_2 = xy, \ g_3 = y^2 \), the ideal of a quartic scroll is given by

\[
C : f_{21} f_{22} - f_{11} f_{32} = 0, \quad f_{21}^2 - f_{11} f_{31} = 0, \quad f_{22} - f_{12} f_{32} = 0 \quad \text{rank} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix} \leq 1,
\]

where \( f_{ij} \)'s are linear equations. In a similar way the image of the Veronese embedding \( P^2 \hookrightarrow P^5 \) by degree 2 monomials is given by

\[
C : f_{11} f_{21} - f_{32}^2 = f_{11} f_{31} - f_{22}^2 = f_{21} f_{31} - f_{12}^2 = 0,
\]

\[
f_{12} f_{22} - f_{31} f_{32} = f_{12} f_{32} - f_{21} f_{22} = f_{22} f_{32} - f_{11} f_{12} = 0.
\]

The loci of cubic fourfolds with a Veronese surface as above is \( C_{20} \).

It is now natural to generalize the quartic scroll (14) and Veronese (15) by adding \( \frac{n}{2} - 2 \) linear equations into the defining equation of \( C \) and obtain \( \frac{n}{2} \)-dimensional subvarieties of \( P^{m+1} \). We call them a quartic scroll and Veronese cycle, respectively; carrying in mind that they are \( \frac{n}{2} \)-dimensional. Let \( T_{QS} \) (resp. \( T_V \)) be the space of smooth degree \( d \) hypersurfaces in \( P^{m+1} \) containing a quadric scroll (resp. Veronese) cycle. The ideal of both cycles are radical and written in the standard basis (groebner basis) and so the defining equation \( f \) of a hypersurface containing such a cycle can be written as \( f = \sum_{i=1}^s f_i g_i \), where \( f_i \)'s are the polynomials in (14) (resp. (15)) and \( g_i \)'s are other homogeneous polynomials such that \( \deg(f_i g_i) = d \). We have computed the codimension of the image of the derivation of the maps parameterizing \( T_{QS} \) and \( T_V \) at many random points for cubic hypersurfaces. These are conjecturally the codimensions of \( T_{QS} \) and \( T_V \) and they are listed in Table 6. It would not be hard to prove that these are actual codimensions and in the following we are going to assume this. We have used the procedures CodQuarticScroll and CodVeronese for this codimension computations. In this table \( L, CS, QS, V \) means Linear cycle \( P^{\frac{m}{2}} \), cubic scroll/ruled, quadric scroll and Veronese, respectively, and the numbers below them are the codimension of the corresponding sub loci of \( T \). The column under \( M \) is the codimension of the mysterious components of the Hodge loci that we get in Theorem 1 part 2.

Remark 1. One loci in Table 6 is definitely not a component of the Hodge loci. This is namely, the loci of cubic six folds containing a Veronese cycle. Its codimension is 10, whereas the upper
bound for components of the Hodge loci in this case is 8. Assuming the Hodge conjecture in this case, it means that the Veronese cycle is homologous to another algebraic cycle in primitive homology such that the new cycle has bigger deformation space. The same reasoning implies that the loci of cubic six folds containing a Quartic scroll is a component of the Hodge loci as its codimension is the maximal one $h^{51} = 8$.

**Remark 2.** A natural generalization of $C_D$ for arbitrary cubic $n$-folds seems to have increasing codimension with respect to $D$. If this is the case, even constructing algebraic cycles in the cubic fourfold case might not help to understand the Hodge loci of Theorem 1 part 2 whose codimension seems to be below codim(T\text{QS}).

**Remark 3.** For quartic scroll in (14), by setting $f_{12} = f_{31} = 0$, degenerates into a sum of two planes and a line $C_0 : \mathbb{P}^2 + \mathbb{P}^2 + \mathbb{P}^1$:

\[
\begin{align*}
\mathbb{P}^2_1 : & \quad f_{21} = f_{22} = f_{11} = 0, \\
\mathbb{P}^2_2 : & \quad f_{21} = f_{22} = f_{32} = 0, \\
\mathbb{P}^1_3 : & \quad f_{21} = f_{22} = f_{21}f_{22} = f_{11} = f_{32} = 0.
\end{align*}
\]

Note that $\mathbb{P}^2_1$’s are non-reduced, the intersection $\mathbb{P}^2_1 \cap \mathbb{P}^2_2$ is reduced and it is equal to the underlying reduced line of $\mathbb{P}^1_3$. It is not clear whether the Fermat fourfold contains a quartic scroll or its degeneration $C_0$. Note that if $C_0$ is inside a cubic fourfold $X$, using topological arguments it defines a class $[C_0] \in H^n_{\text{DR}}(X)$, however, it is not clear how to define it algebraically.

| dim($X_0$) | dim($T$) | range of codimensions | LCS | M | QS | V | Hodge numbers |
|------|---------|-----------------------|-----|---|----|---|----------------|
| 4    | 20      | 1, 1                  | 1   | 1 | 1  | 1 | 1              |
| 6    | 56      | 4, 8                  | 4   | 6 | 7  | 8 | 10             |
| 8    | 120     | 10, 45                | 10  | 16| 19 | 23| 25             |
| 10   | 220     | 20, 220               | 20  | 32| 38 | 45| 47             |
| 12   | 364     | 35, 364               | 35  | 55| 65 | 75| 77             |

Table 5: Codimensions of the components of the Hodge/special loci for cubic hypersurfaces

We started to prepare Table 1 and Table 2 with a computer with processor Intel Core i7-7700, 16 GB Memory plus 16 GB swap memory and the operating system Ubuntu 16.04. It turned out that for many cases such as $(n, m, N) = (12, 3, 3)$ in Table 2 we get the ‘Memory Full’ error. Therefore, we had to increase the swap memory up to 170 GB. Despite the low speed of the swap which slowed down the computation, the computer was able to use the data and give us the desired output. The computation for this example took more than 21 days. We only know that at least 18 GB of the swap were used. Other time consuming computations are the cases $(n, m, N) = (10, 2, 4), (12, 4, 3)$.

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