Probing negative dimensional integration:
two-loop covariant vertex and one-loop light-cone integrals

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Abstract

Negative dimensional integration method (NDIM) seems to be a very promising technique for evaluating massless and/or massive Feynman diagrams. It is unique in the sense that the method gives solutions in different regions of external momenta simultaneously. Moreover, it is a technique whereby the difficulties associated with performing parametric integrals in the standard approach are transferred to a simpler solving of a system of linear algebraic equations, thanks to the polynomial character of the relevant integrands. We employ this method to evaluate a scalar integral for a massless two-loop three-point vertex with all the external legs off-shell, and consider several special cases for it, yielding results, even for distinct simpler diagrams. We also consider the possibility of NDIM in non-covariant gauges such as the light-cone gauge and do some illustrative calculations, showing that for one-degree violation of covariance (i.e., one external, gauge-breaking, light-like vector $n_\mu$) the ensuing results are concordant with the ones obtained via either the usual dimensional regularization technique, or the use of principal value prescription for the gauge dependent pole, while for two-degree violation of covariance — i.e., two external, light-like vectors $n_\mu$, the gauge-breaking one, and (its dual) $n^*_\mu$ — the ensuing results are concordant with the ones obtained via causal
constraints or the use of the so-called generalized Mandelstam-Leibbrandt
prescription.
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I. INTRODUCTION.

The equivalence between fermionic integration, in positive dimensional space-time, and bosonic integration, in negative dimensional space-time, is a remarkable property [1]. Based on this fact, Halliday and co-workers suggested the use of such a property to tackle the problem of calculating Feynman integrals. The advantage to be derived in such an approach can be appreciated and understood — at least in principle — from the point of view that Grassmannian integrals are linear operators defined with few rules and properties and thus easier to solve them than the ordinary integrals.

Negative dimensional integration method (NDIM) allows us to perform massless Feynman integrals with propagators raised to arbitrary powers even at the two-loop level [2]. Box diagrams with massive propagators can also be calculated easily and several results are obtained according to the different possibilities of dimensionless ratios defined by internal mass and external momenta [3]. This work is intended to advance our testing to the next natural step at the two-loop level of three-point vertex diagrams in the framework of off-shell external momenta. Furthermore, recalling that the light-cone gauge loop integrals are notoriously more difficult to handle than their covariant counterparts in virtue of the structure of the gauge boson propagator [4], we show how NDIM can simplify things out even in this case.

So, our proposal here is to give clear-cut examples to demonstrate the beauty and power of this technique in dealing with Feynman integrals of the perturbative quantum field theories. Without loss of generality, here we restrict ourselves to massless fields. The outline for this work is given as follows: In Section II we solve a scalar integral for the diagram of Fig.1 and discuss some special cases stemming from it. In Section III we consider a scalar and a vectorial one-loop light-cone integrals pertaining to two-point self-energy diagrams and finally in the last Section we present our concluding remarks.
II. OFF-SHELL TWO-LOOP THREE-POINT VERTEX.

This computation is performed following a few simple steps outlined in [5,6]. First of all, let us consider the Gaussian-like integral,

\[
I = \int \int d^D q \, d^D r \, \exp \left[ -\alpha q^2 - \beta (q - p)^2 - \gamma r^2 - \xi (q - r - k)^2 \right]
\]

where the arguments in the exponential function of the integrand correspond to propagators in the diagram of Fig. 1 and for compactness we define \( \phi = \alpha \gamma + \alpha \xi + \beta \gamma + \beta \xi + \gamma \xi \). The arbitrary parameters \((\alpha, \beta, \gamma, \xi)\) are chosen such that their real parts are positive to make sure we have well-defined objects over the whole integration space.

Expanding \( I \) in Taylor series we obtain,

\[
I = \sum_{i,j,l,m=0}^{\infty} \frac{(-1)^i j l m}{i! j! l! m!} \alpha^i \beta^j \gamma^l \xi^m S_{NDIM},
\]

where \( S_{NDIM} \) is the relevant integral in negative \( D \), defined by

\[
S_{NDIM} = \int \int d^D q \, d^D r \, (q^2)^i (r^2)^l \left[(q - p)^2\right]^j \left[(r - q + k)^2\right]^m.
\]

Now comparing both expressions for the original integral \( I \) we are led to the conclusion that in order to have equality between these expressions, the factor \( S_{NDIM} \) must be given by the multiple series,

\[
S_{NDIM} = G(i, j, l, m; D) \sum_{n_1, \ldots, n_9 = 0}^{\infty} \frac{(p^2)^{n_1} (k^2)^{n_2} (t^2)^{n_3}}{\prod_{i=1}^{9} n_i!} \frac{(n_1 + n_2 + n_3 + n_4 + n_5 + n_6)^a (n_1 + n_3 + n_4 + n_5 + n_7 + n_8)^b (n_1 + n_3 + n_4 + n_5 + n_7 + n_9)^c (n_2 + n_3 + n_4 + n_6 + n_8 + n_9)^d},
\]

where \((a, b, c, d)\) stand respectively for

\[
a = n_1 + n_2 + n_3 + n_5 + n_6 ,
\]

\[
b = n_1 + n_2 + n_4 + n_7 + n_8 ,
\]

\[
c = n_1 + n_3 + n_4 + n_5 + n_7 + n_9 ,
\]

\[
d = n_2 + n_3 + n_4 + n_6 + n_8 + n_9 ,
\]

where \(G(i, j, l, m; D)\) is a factor related to the integrals in negative \( D \).
with

\[ G(i, j, l, m; D) = (-\pi)^D \Gamma(1 + i)\Gamma(1 + j)\Gamma(1 + l)\Gamma(1 + m)\Gamma(1 - \sigma - \frac{1}{2}D), \]

and for convenience we use the definition \( \sigma = i + j + l + m + D \). The system we must solve is, therefore,

\[
\begin{align*}
  n_1 + n_2 + n_3 + n_5 + n_6 &= i \\
  n_1 + n_2 + n_4 + n_7 + n_8 &= j \\
  n_1 + n_3 + n_4 + n_5 + n_7 + n_9 &= l \\
  n_2 + n_3 + n_6 + n_8 + n_9 &= m \\
  n_1 + n_2 + n_3 + n_4 &= \sigma.
\end{align*}
\] (6)

It is an easy matter to see that this system is composed of five equations with nine “unknowns” (the sum indices), so that it cannot be solved unless it is done in terms of four arbitrary “unknowns”. These, of course, will label the four remnant summations, which means that the answer will be in terms of a fourfold summation series. From the combinatorics, it is a straightforward matter to see that there are many different ways we can choose those four indices; indeed, we can choose \( C_9^5 = 126 \) different ways. In other words, what we need to do is to solve 126 different systems. Of these, 45 are unsolvable systems, i.e., they are systems whose set solution is empty. There remains therefore 81 which have non-trivial solutions. Of course, the trivial solutions are of no interest at all. However, the non-trivial solutions generate a space of functions with different basis, characterized by their functional variable, according to the different possibilities allowed for ratios of external momenta. Each basis is a solution for the pertinent Feynman integral, which is connected by analytic continuation to all other basis defined by other set solutions. We remind ourselves that a basis that generates a given space can be composed of one or several linearly independent functions combined in what is called linear combination.

Each representation of the Feynman integral will be given by a basis of functions gen-
erated by the solutions of the systems [4]. Of course, only linearly independent and non-degenerate solutions are relevant to define a basis.

It can be easily seen that the diagram we are dealing with here is symmetric under the exchange of external momenta \( k^2 \leftrightarrow t^2 \). This symmetry is reflected by the systems we have to solve, and the solutions display this fact. Moreover, in order to save ourselves space, only those solutions which are not thus connected have been written down explicitly (see [8]). Here we further restrict ourselves to those solutions which are of present interest, that is, the ones we can compare with results obtained from positive dimensional techniques.

With these clarifying statements and with the solutions of the many systems in hands — it is an easy matter to write down a computer program to solve all the systems — and the general form of the results (4), we can start building the power series representations for the Feynman graph.

We begin our analysis of the solutions for the systems by looking at the simpler ones, i.e., those having two variables, defined by ratios of external momenta. Of course, there are in fact fours sums but two of them have unity argument, making it possible for us to actually sum the pertinent series as we shall shortly see. The variables are \((x, y), (z, y^{-1}), (x^{-1}, z^{-1})\) where we define the dimensionless ratios

\[
x = \frac{p^2}{k^2}, \quad y = \frac{t^2}{k^2}, \quad z = \frac{p^2}{t^2}.
\]  

(7)

Each of the three pairs above appears twelve times among the total of 81 systems with non-trivial solutions. Yet, each of these is twelve-fold degenerate just as in the on-shell case calculated in [4]. A possible way of expressing the first solution in positive \(D\) — the analytic continuation to this region is carefully explained in [3, 4] — is:

\[
S_{1AC}^{AC} = \pi^D(p^2)^{\sigma} P_1^{AC} \\
\times F_4(-\sigma, -l - m - \frac{1}{2}D; 1 + j - \sigma, 1 + i - \sigma \mid x^{-1}, z^{-1}),
\]  

(8)

with \(P_1^{AC}\) being given by
\[ P_1^{AC} = (-i|\sigma)(-j|\sigma)(-l| - m - \frac{1}{2}D)(-m|l + m + \frac{1}{2}D) \]
\[ \times (l + m + D| - l - \frac{1}{2}D)(\sigma + \frac{1}{2}D| - 2\sigma - \frac{1}{2}D), \quad (9) \]

and where we use the Pochhammer symbol \((a|k) \equiv (a)_k = \Gamma(a + k)/\Gamma(a)\), together with one of its properties, \((a|b + c) = (a + b|c)(a|b)\) as well as the well-known Gauss’ summation formula \([9]\) for the hypergeometric function \(_2F_1\) with unit argument,

\[ _2F_1(a, b; c|1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad (10) \]

so that two of the four series above are summed up. The remaining two series are by definition an Appel hypergeometric function \(F_4\) with two variables \([4]\). This function, in the special case of \(i = j = l = m = -1\) further reduces to Gauss’ hypergeometric function \(_2F_1\) \([4]\).

In a manner similar to the previous results, there are solutions which have three remaining variables, meaning that one of the series with unit argument is summed out. There are six sets of these, determined by their variables, each appearing four times, i.e., a fourfold degeneracy. Just to keep our accounting straight, \(4 \times 6 = 24\) systems yielding solutions with three variables. These, added to the 36 of the previous two-variables results, give us 60 from the total of 81 non-trivial systems.

The solutions within this category have functional dependencies given by \((x, x, y), (z, z, y^{-1}), (x, y, y), (z, y^{-1}, y^{-1}), (x^{-1}, x^{-1}, z^{-1}), (z^{-1}, z^{-1}, x^{-1})\).

Note that the above triplets are conveniently arranged into pairs connected by the symmetry \(k^2 \leftrightarrow t^2\).

Here we focus on the triple power series representation of interest, given by

\[ S_2^{AC} = P_2^{AC} \sum_{n_1, n_2, n_3=0}^{\infty} \frac{(z)_n_1+n_2(y^{-1})_n_3}{n_1!n_2!n_3!} \frac{(m + \frac{1}{2}D|n_1)}{(1 - i - j - \frac{1}{2}D|n_1 + n_2)} \]
\[ \times \frac{(l + \frac{1}{2}D|n_2)(-i|n_1 + n_2 + n_3)(-\sigma|n_1 + n_2 + n_3)}{(l + m + D|n_1 + n_2)(1 - i - l - m - D|n_3)}, \quad (11) \]

where
\[ P_{2}^{AC} = \pi^{2}(t^{2})^{\sigma}(-j|\sigma)(-l|l + m + 1_{2}D)(-m|l + m + 1_{2}D) \]
\[ \times (l + m + D|i + j - l - m - 1_{2}D)(\sigma + 1_{2}D| - 2\sigma - 1_{2}D). \]

Lastly, we consider solutions of systems of linear algebraic equations leading to four variables, i.e., fourfold summation series with four variables. There are 21 of these, which completes the total of 81 non-trivial solutions for the systems. Again the functional variables are given paired with their corresponding symmetries, as follows: \((z^{-1}, z^{-1}, x^{-1}, x^{-1})\), \((x, y, z, y^{-1})\), \((y, y^{-1}, x, y)\), \((x, y, x^{-1}, z^{-1})\), \((z, y^{-1}, z^{-1}, x^{-1})\), \((z, z, y^{-1}, y^{-1})\), \((z, z, y^{-1}, x^{-1})\), \((z, y^{-1}, z^{-1}, z^{-1})\), \((x, y, x^{-1}, x^{-1})\).

To get the accounting straight, let us again mention that the first three appear just once while the remaining nine appear twice, totalling the needed 21 of this category. Among them, there is a solution we want to pinpoint here, namely,

\[ S_{3}^{AC} = P_{3}^{AC} \sum_{\{n_{i}\} = 0}^{\infty} \frac{(z)^{n_{1}+n_{2}}(y^{-1})^{n_{7}+n_{8}}(-j|n_{1} + n_{2} + n_{7} + n_{8})(l + 1_{2}D|n_{2} + n_{8})}{n_{1}!n_{2}!n_{7}!n_{8}!} \frac{l + 1_{2}D|n_{2} + n_{8}}{(1 - j + \sigma|n_{7} + n_{8})} \times \frac{(m + 1_{2}D|n_{1} + n_{7})}{(1 - i - j - 1_{2}D|n_{1} + n_{2})}, \]

where

\[ P_{3}^{AC} = (k^{2})^{\sigma} y^{i} (-i|l - m - D)(-l|l + m + 1_{2}D)(-m|l + m + 1_{2}D) \]
\[ \times (\sigma + 1_{2}D|i + j - \sigma). \]

\textbf{A. On-Shell Limit.}

Of course, particular cases of on-shell external legs must be contained in the set of off-shell solutions. To check on this, let us take two legs on-shell, namely, let \(k^{2} = t^{2} = 0\). Not all off-shell solutions \(S^{AC}\) are suitable for taking this particular limit, because some of them either vanish or are not defined in this regime. It is easy to see that such a suitable solution
is given by Eq.(8), because in this limit only the first term in the $F_4$ series is non-zero while all the others vanish, leaving us with

$$S_{AC}^1(k^2 = t^2 = 0) = \pi^D (p^2)^\sigma (-i|\sigma) (-j|\sigma)(-l| - m - \frac{1}{2}D)(-m|l + m + \frac{1}{2}D)$$

$$\times (l + m + D| - l - \frac{1}{2}D)(\sigma + \frac{1}{2}D) - 2\sigma - \frac{1}{2}D). \quad (14)$$

This result is valid for arbitrary $D$ and negative exponents of propagators. In order to confront this result with the known one we still need to go further in specializing to the case where $i = j = l = m = -1$. Then, what we get is the very result obtained by Hathrell using standard procedures for calculating Feynman diagrams in positive $D$. Of course, a more straightforward way of getting this result using NDIM is to put the corresponding legs on-shell from the very beginning, in Eq.(11), and what we get then is twelve systems to solve with non-trivial solutions, exactly the number we have for the solution in question: a twelvefold degeneracy giving the same result.

We can also consider another special case, namely, the one where $p^2 = k^2 = 0$. This one is interesting because it contributes to another two-loop three-point diagram if we apply the integration by parts technique. The general result, for arbitrary (negative) exponents of propagators and positive dimension can be read off from the solution $S_2$, Eq.(11),

$$S_{AC}^2(p^2 = k^2 = 0) = \pi^D (t^2)^\sigma (-j|\sigma)(-l|l + m + \frac{1}{2}D)(-m|l + m + \frac{1}{2}D)$$

$$\times (l + m + D|i + j - l - m - \frac{1}{2}D)(\sigma + \frac{1}{2}D) - 2\sigma - \frac{1}{2}D). \quad (15)$$

Taking the same particular case of Kramer et al., i.e., $l = m = -2$ and $i = j = -1$, we obtain the well-known result in Euclidean space.

Two other simpler special cases can be read off from this graph, namely, when $p = 0$, $k^\nu = t^\nu$ (see Fig.2) and $k = 0$, $p^\nu = -t^\nu$, as follows:

The solution $S_2$ gives us the first one,

$$S_{AC}^2(p = 0, k^\nu = t^\nu) = (-i - j|i)(i + l + m + D| - i)P_{AC}^2. \quad (16)$$
Note that in the $n_1$ and $n_2$ series only the first term contributes and the $n_3$ series reduces to a Gauss hypergeometric function with unit argument, thus a summable one.

The second case, $k = 0$, $p' = -t'$, can be read off from $S_1$,

$$S_{1}^{AC}(k = 0, p' = -t') = \pi^D(p^2)^\sigma P_1^{AC}(-i + \sigma| - \sigma)\left(\frac{1}{2}D + j - \sigma|\sigma\right).$$  \hspace{1cm} (17)

This result can be used to study two self-energy two-loop graphs and agrees with our previously published results \cite{3}.

Finally, let us check up on a solution that has four series with four variables. Let $j = 0$, so that we get the graph of Fig.3. The solution that allows us to consider this limit is $S_3$,

$$S_{3}^{AC}(j = 0) = \pi^D(k^2)^\sigma P_3^{AC}(j = 0).$$  \hspace{1cm} (18)

Observe that there is no sum in the result. This is due to the factor $(-j|n_1 + n_2 + n_7 + n_8)$ which leads to only one non-vanishing term, i.e., when $n_1 = n_2 = n_7 = n_8 = 0$ in the series.

### III. LIGHT-CONE GAUGE LOOP INTEGRALS.

For the vector gauge fields in the light-cone gauge \cite{4} ghosts decouple from the physical fields and for this reason the number of vertices we have in the theory may be considerably reduced — and consequently the number of Feynman diagrams to deal with — but the price we have to pay is that we are left with a more complicated gauge boson propagator. This complexity manifests itself in the form of a gauge dependent pole of the form $(k \cdot n)^{-1}$ where $n^\mu$ is a light-like vector which defines the gauge. It is a well-known fact admit the light-cone experts that the use of Cauchy principal value (CPV or PV for short) prescription to treat such a pole is plagued with pathologies such as the impossibility of Wick rotation, the emergence of double pole singularities at the one-loop level, and the incorrect exponentiation of the Wilson loop \cite{15}, to name a few. To circumvent these difficulties, by the middle of '80s, Mandelstam \cite{14} and Leibbrandt \cite{13} independently suggested prescriptions to treat the so-called “spurious” singularities generated by the light-cone propagator. Soon after it
has been shown that both prescriptions were in fact equivalent and became known as the ML prescription. Still later on the prescription has been generalized to deal with generic non-covariant axial gauges and sometimes it has been referred as the generalized Mandelstam-Leibbrandt prescription \[16\] in this context. In parallel, it became clear that all those pathologies do arise because the PV prescription violates causality and once causality is carefully taken into account, no prescription is in fact needed \[17\].

After these few words of elucidation to introduce the light-cone gauge to those unfamiliar with it, we are in position to propose applying the NDIM technology to see what happens in this case. We can anticipate, of course, some subtle intrinsic properties.

Just to make things easier, we follow the usual notation for the light-cone that can be found in the majority of the specific literature on it.

First of all, let us consider the simplest of the one-loop integrals,

\[
L_1 = \int \frac{d^2 \omega k}{k^2(k-b)^2(k\cdot n)},
\]

which can be seen as the limiting case, i.e., \(a \to 0\), of the more general one,

\[
L_2 = \int \frac{d^2 \omega k}{(k-a)^2(k-b)^2(k\cdot n)}.
\]

Although one can easily compute this last integral — which arises in the computation of one-loop four-point functions — with the help of dimensional regularization technique, the result of this particular integral is not tabulated in the literature as far as we know it. Therefore it is a suitable object to do our lab testing in NDIM, since the previous \(L_1\) integral is by far the most well-known integral in light-cone gauge.

Our starting point is again the Gaussian integral,

\[
\mathcal{A} = \int d^2 \omega k \exp \left[ -\alpha(k-a)^2 - \beta(k-b)^2 - \gamma(k\cdot n) \right] = \pi^\omega \lambda \exp \left\{ -\frac{1}{\lambda} \left[ \alpha\beta(a-b)^2 - \alpha\gamma(a\cdot n) - \beta\gamma(b\cdot n) \right] \right\} = \sum_{i,j,l=0}^\infty \frac{(-1)^{i+j+l}}{i!j!l!} \mathcal{N}(i,j,l),
\]

where \(\lambda = \alpha + \beta\) and

\[
\mathcal{N}(i,j,l).
\]
\[ N(i, j, l) = \int d^2k \ (k - a)^2i(k - b)^2j(k \cdot n)^l. \] (22)

Note that in the middle line of Eq.(21) we are left with no factor proportional to \( n^2 \) in the argument of the exponential, since it is zero in the light-cone gauge (by choice the \( n^\mu \) vector is a light-like one.) Also note that in NDIM scheme one has generic values of \((i, j, l)\) and not just the specific quantum field theory values \( i = j = l = -1 \).

By comparing the two expressions arising from the expansions of \( A \) we obtain an expression in terms of multiple power series for the integral in negative dimension, i.e.,

\[
N(i, j, l) = \pi^\omega (-1)^{i+j+l} \Gamma(1+i) \Gamma(1+j) \Gamma(1+l) \times \\
\sum_{n_1=0}^{\infty} (-1)^{n_1+n_2+n_3} \frac{(-n_1-n_2-n_3-\omega)!}{n_1!n_2!n_3!n_4!n_5!} (a-b)^{n_1}(a \cdot n)^{n_2}(b \cdot n)^{n_3} \\
\times \delta_{n_1+n_2+n_4,i} \delta_{n_1+n_3+n_5,j} \delta_{n_2+n_3,l} \delta_{n_4+n_5,-(n_1+n_2+n_3+\omega)}. \] (23)

The system here is far simpler than the former one in the two-loop covariant case. There are altogether 5 possible solutions but one of them is trivial. Thus using the procedure outlined in the previous section and also in \[5\] we construct two power series representations for the Feynman integral in question. The solutions in which the sum indices \( n_2 \) and \( n_5 \) are left undetermined give us, after a suitable analytic continuation to allow for negative values of \((i, j, l)\) and positive dimension,

\[
N(i, j, l) = \pi^\omega [(a-b)^2]^{\rho-l} \left\{ (b \cdot n)^l Q_1^{AC} \ {}_2F_1(-l, \omega+j; 1+j-\rho|z) \\
+ (a \cdot n)^{\rho-j} (b \cdot n)^{-i-\omega} Q_2^{AC} \ {}_2F_1(\omega+i, \rho+\omega; 1+\rho-j|z) \right\}, \] (24)

where

\[
Q_1^{AC} = (-j|\rho)(-i|\rho+j+\omega)(\rho+\omega| - 2\rho - \omega + l), \] (25)

and

\[
Q_2^{AC} = (-j|i+j+\omega)(-i|i+l-\rho)(-l|j+l-\rho), \] (26)

with

\[
z = \frac{a \cdot n}{b \cdot n},
\]
and $\rho = i + j + l + \omega$. Letting $a = 0$ Eq.(24) gives the well-known result for $L_1$, see e.g. [18]. If we want to write the result for $L_2$ in a more compact form we must rewrite the hypergeometric functions $2F_1$ using a transformation formula [9] of the type $2F_1(...) \rightarrow 2F_1(...|1 - x)^{-1})$. Considering the special case where $i = j = l = -1$, we identify the sum as a single hypergeometric function, in Euclidean space,

$$\mathcal{N}(-1, -1, -1) = \pi^{\omega} [(a - b)^2]^{\omega - 2} \frac{1}{(a.n)} \Gamma(2 - \omega) B(\omega - 1, \omega - 1) \ 2F_1(1, \omega - 1; 2\omega - 2|u), \quad (27)$$

where

$$u = \frac{(a - b) \cdot n}{b \cdot n}.$$ 

The remaining two other solutions for the system also lead to two Gaussian hypergeometric functions but with $z^{-1}$ as variable. It is obtained from the former result by making the simultaneous exchanges $a \leftrightarrow b$ and $i \leftrightarrow j$ (which is an inherent symmetry of the referred integral.)

However, the above results for $L_1$ and $L_2$ are conspicuously pathological in the sense that they violate causality, as discussed earlier on. In other words, they are concordant with those results obtained in the PV prescription scheme in positive dimension. This means that for each integral thus calculated one has to subtract out the zero-mode contribution from it [17]. Yet we would like to have causality preserving results without much ado. Is that possible in the NDIM scheme? The answer is yes — at least in principle — with some inherent subtleties as we shall shortly see.

Consider again the simplest of the one-loop light-cone integrals in the case we have a two-degree violation of covariance, i.e., the tadpole-like integral with tensorial structure,

$$T = \int \frac{d^{2\omega}k}{(k - p)^2} \frac{(k \cdot n^*)}{(k \cdot n)}. \quad (28)$$

Before we go on, a word of caution must be given just here. Note that the structure of the integrand is important. In the light-cone gauge Feynman-integrals, those factors bearing the external dual light-like vector $n_\mu^*$ do always appear in the numerator of the integrands.

Therefore, our starting point to solve it in the NDIM scheme is,
\[ B = \int d^2\omega k \exp \left[ -\alpha(k - p)^2 - \beta(k \cdot n) - \gamma(k \cdot n^*) \right] \]
\[ = e^{-\beta(p \cdot n) - \gamma(p \cdot n^*)} \int d^2\omega k e^{-\alpha k^2 - \beta(k \cdot n) - \gamma(k \cdot n^*)} \]
\[ = \left( \frac{\pi}{\alpha} \right)^\omega \exp \left\{ -\beta(p \cdot n) - \gamma(p \cdot n^*) + \frac{\beta \gamma(n \cdot n^*)}{2\alpha} \right\} \]
\[ = \sum_{i,j,l=0}^{\infty} (-1)^{i+j+l} \frac{\alpha^i \beta^j \gamma^l}{i! j! l!} \mathcal{T}. \quad (29) \]

where \( \mathcal{T} \) is the negative dimensional integral

\[ \mathcal{T} = \int d^2\omega k \left[ (k - p)^2 \right]^i (k \cdot n)^j (k \cdot n^*)^l. \quad (30) \]

From the above equality, it is easy to get the result

\[ \mathcal{T} = (-\pi)^\omega \left( \frac{-2 p \cdot n p \cdot n^*}{n \cdot n^*} \right)^{1+i+\omega} (p \cdot n)^j (p \cdot n^*)^l \frac{(1 - i - \omega |2i + \omega)}{(1 + j|i + \omega)(1 + l|i + \omega)}. \quad (31) \]

Now comes the crucial point. As mentioned earlier, the peculiarity of the light-cone gauge is that the exponent \( l \geq 0 \) always. That means that the Pochhammer’s symbol containing it, namely \( (1 + l|i + \omega) \) must never be analytic continued to allow for negative values of \( l \). Bearing in mind this restriction and subtlety, we proceed in the same manner as usual, to get

\[ \mathcal{T}^{AC} = \pi^\omega \left( \frac{2 p \cdot n p \cdot n^*}{n \cdot n^*} \right)^{1+i+\omega} (p \cdot n)^j (p \cdot n^*)^l \frac{(-j - i - \omega)}{(i + \omega - 2i - \omega)(1 + l|i + \omega)}. \quad (32) \]

This is exactly the result we get through causal considerations (or by using the ML prescription.) It is really quite an amazing result, since no prescription has been called upon to deal with the so-called “unphysical” singularities characteristic of this gauge. The only consideration was that we used the two-degree violation of covariance where both external vectors \( n_\mu \) and its dual \( n^*_\mu \) were treated in the same footing. It seems that the outstanding property of translational invariance displayed by negative dimensional integrals takes care of the causality principle naturally, a principle that is required by ad hoc devised prescriptions.
IV. CONCLUSION.

We have shown in this paper how we can work out a two-loop vertex diagram with all external legs off-shell using the NDIM technique to solve a pertinent scalar Feynman integral. Altogether, twenty-one distinct results are obtained for the considered integral. These are expressed in terms of power series that can be identified as hypergeometric functions. The simples ones are Appel’s $F_4$ hypergeometric function with two variables, which for the particular cases where all the exponents of propagators are set to minus one, can be transformed into even simpler ones of the Gaussian $\text{}_{2}F_{1}$ hypergeometric function type. The technique also gives us several analytic continuation formulas between different results, because they arise from the same Feynman integral ($\mathbb{3}$).

The more outstanding results of this work, however, stems from our using NDIM technology to evaluate the simplest one-loop light-cone integrals, where two distinct conclusions can be drawn: (i) That when we consider light-cone gauge with one-degree violation of covariance, i.e., consider only the gauge-breaking external vector $n_{\mu}$, the results we get are concordant with the usual PV prescription results, while (ii) when we consider the light-cone gauge with two-degree violation of covariance, where both vectors, $n_{\mu}$ and its dual $n^{*}_{\mu}$ are treated on the same footing, the result we get is concordant with that obtained via causal considerations (or equivalently the use of the ML prescription.) A more outstanding conclusion we draw from this last result is that NDIM, somehow, in a manner that we still do not understand clearly, takes care naturally of the causality principle that should constrain all transition amplitudes. Moreover, that which in times past required an ad hoc prescription to deal with gauge dependent poles is with much finesse avoided by NDIM.

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Fig. 1: Off-shell two-loop three-point vertex calculated with NDIM.

Fig. 2: A special case ( $p = 0$, $k^\mu = t^\mu$ ) of the diagram of fig. 1.

Fig. 3: A special case ( $j = 0$ ) of the diagram of fig. 1.