ON BLOWING UP THE WEIGHTED PROJECTIVE PLANE

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Abstract. We investigate the blow-up of a weighted projective plane at a general point. We provide criteria and algorithms for testing if the result is a Mori dream surface and we compute the Cox ring in several cases. Moreover applications to the study of $\overline{M}_{0,n}$ are discussed.

1. Introduction

Let $a, b, c$ be pairwise coprime positive integers and denote by $\mathbb{P}(a, b, c)$ the associated weighted projective plane, defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. We consider the blow-up

$$\pi : X(a, b, c) \rightarrow \mathbb{P}(a, b, c)$$

at the point $[1, 1, 1] \in \mathbb{P}(a, b, c)$ and ask whether $X = X(a, b, c)$ is a Mori dream surface, i.e., has finitely generated Cox ring

$$\mathcal{R}(X) = \bigoplus_{\Gamma(X, \mathcal{O}(D))} \Gamma(X, \mathcal{O}(D)).$$

This problem has been studied by several authors and the results have been used to prove that $\overline{M}_{0,n}$ is not a Mori dream space for $n \geq 13$, see [4, 8, 9]. In fact, as we will see below, $\overline{M}_{0,n}$ is not even a Mori dream space for $n \geq 10$.

However, it still remain widely open questions, which of the $X(a, b, c)$ are Mori dream surfaces, and, if so, how does their Cox ring look like. We provide new results and computational tools. Our approach goes through the description of the Cox ring of $X = X(a, b, c)$ as a saturated Rees algebra:

$$\mathcal{R}(X) = S[I]^{\text{sat}} := \bigoplus_{\mu \in \mathbb{Z}} (I^{-\mu} : J^{\infty})t^\mu,$$

where $S$ is the Cox ring of $\mathbb{P}(a, b, c)$ and $I, J \subseteq S$ are the weighted homogeneous ideals of the points $(1, 1, 1)$ and $(0, 0, 0)$ respectively; see [10, Prop. 5.2]. We say that an element of the Cox ring $\mathcal{R}(X)$ is of Rees multiplicity $\mu$ if it belongs to the component $(I^\mu : J^{\infty})t^{-\mu}$.

Our theoretical results concern the cases that the Cox ring of $X$ is generated by elements of low Rees multiplicity. We characterize this situation in terms of $a, b, c$ and we provide generators and relations for the Cox ring of $X$, where we list the degree of a generator $T_i$ in $\text{Cl}(X) = \mathbb{Z}^2$ as the $i$-th column of the degree matrix $Q$.

Theorem 1.1. Let $X = X(a, b, c)$ be as before. Then the following statements are equivalent.

(i) The surface $X$ admits a nontrivial $\mathbb{K}^*$-action.

(ii) One of the integers $a, b, c$ lies in the monoid generated by the other two.

(iii) The Cox ring of $X$ is generated by homogeneous elements of Rees multiplicity at most one.

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If one of these conditions holds, then \( X \) is a Mori dream surface. Moreover, if \( a \) lies in the monoid generated by \( b \) and \( c \), then the Cox ring of \( X \) is given by

\[
\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_5]/(T_4 T_5 - T_1^c + T_2^d),
\]

and the Rees multiplicities of the generators \( T_1, \ldots, T_5 \) are 0, 0, 1, 1, \(-1\) respectively. In particular, \( X \) is a toric surface if and only if at least one of the three integers \( a, b, c \) equals one.

Theorem 1.2. Assume that none of \( a, b, c \) is contained in the monoid generated by the remaining two. Then, for \( X = X(a, b, c) \), the following statements are equivalent.

(i) The Cox ring of \( X \) is generated by elements of Rees multiplicity at most two.

(ii) After suitably reordering \( a, b, c \), one has \( 2a = nb + mc \) with positive integers \( n, m \) such that \( b \geq 3m \) and \( c \geq 3n \).

Moreover, if one of these conditions holds, then \( X \) is a Mori dream surface and its Cox ring is given by

\[
\mathcal{R}(X) = \mathbb{K}[x, y, z, s_1, \ldots, s_4, t]/(I_2 : t\infty),
\]

\[
Q = \begin{bmatrix}
a & b & c & 2a & \frac{b(c+m)}{2} & \frac{c(b+m)}{2} & bc & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & -2 & 1
\end{bmatrix},
\]

where the Rees multiplicities of the generators \( x, y, z, s_1, \ldots, s_4, t \) are 0, 0, 0, 1, 1, 1, \(-1\), respectively, and the ideal \( I_2 \subseteq \mathbb{K}[x, y, z, s_1, \ldots, s_4, t] \) is generated by the polynomials

\[
\begin{align*}
x^2 - y^n z^m - s_1 t, & \quad xz^{\frac{b}{2}} - y^{\frac{c+m}{2}} - s_2 t, & \quad xy^{\frac{z-m}{2}} - z^{\frac{c}{2}} - s_3 t, \\
x y^{\frac{z-3m}{2}} - s_4, & \quad z^{\frac{b-m}{2}} - s_1 - s_2 - s_3 - s_4 t, & \quad y^{\frac{b-m}{2}} z^{\frac{b-3m}{2}} s_2^2 - s_2 s_3 - x s_4, \\
y^{\frac{z-m}{2}} - s_1 - z^m s_2 - x s_3, & \quad z^{\frac{b-m}{2}} s_1 - x s_2 - y^n s_3, & \quad s_3^2 + y^{\frac{z-3m}{2}} s_1 s_2 - z^m s_4, & \quad s_2^2 + z^{\frac{b-3m}{2}} s_1 s_3 - y s_4.
\end{align*}
\]

In fact, we expect the ideal \( I_2 \) generated by the polynomials displayed in Theorem 1.2 to be prime and thus to coincide with the saturation \( I_2 : t\infty \). As we will see in Corollary 5.3, Theorem 1.2 comprises the in particular the surfaces \( X(3, b, c) \) such that none of \( b, c \) lies in the monoid generated by the remaining two.

In Section 6, we present computational tools and discuss applications to the study of \( M_{0,n} \). Algorithm 6.1 verifies a guess of generators for the Cox ring of a blow-up of an arbitrary Mori dream space. Moreover, Algorithm 6.3 implements the Mori dreamness criterion for \( X(a, b, c) \) given in Proposition 2.4. As an application, we obtain:

Theorem 1.3. Let \( a < b < c \leq 30 \) be pairwise coprime positive integers. Then \( X(a, b, c) \) is a Mori dream surface whenever the triple \( a, b, c \) does not occur in the following list.


The triples \( a, b, c \) marked with \( \star \) are known to give non Mori dream surfaces, see [8]. For the other listed \( a, b, c \), the Cox ring of \( X(a,b,c) \) needs generators of Rees multiplicities at least 15.

The fact that all \( X(a,b,c) \) with \( \min(a,b,c) \leq 6 \) are Mori dream surfaces is due to Cutkosky [7]. Besides the cases covered by Theorems 1.1 and 1.2, Theorem 1.3 yields 514 new Mori dream surfaces \( X(a,b,c) \). The question whether or not the \( X(a,b,c) \) listed without \( \star \) in Theorem 1.3 are Mori dream surfaces remains open — in fact, we expect some of them to be Mori dream surfaces, e.g. those marked with \( \bullet \).

Let us discuss the applications to the question whether or not \( \overline{M}_{0,n} \) is a Mori dream space. Recall that for \( n \leq 6 \), there is an affirmative answer [3]. For higher \( n \), the idea of Castravet and Tevelev [4] is to construct sequences

\[
\overline{M}_{0,n} \rightarrow \overline{T}_n \rightarrow X(a,b,c),
\]

where the first arrow is the canonical proper surjections onto the blow-up \( \overline{T}_n \) of the Losev-Manin space \( \overline{T}_n \) at the general point and the second one is a composition of small quasimodifications and proper surjections. This allows to conclude that if \( X(a,b,c) \) is not a Mori dream space, the same holds for \( \overline{M}_{0,n} \). Applying results from [9], Castravet and Tevelev obtain that \( \overline{M}_{0,n} \) is not a Mori dream space for \( n \geq 134 \). Gonzales and Karu [8] gave further sufficient conditions on \( X(a,b,c) \) to be not a Mori dream surface and, as a consequence, showed that \( \overline{M}_{0,n} \) is not a Mori dream space for \( n \geq 13 \). In fact, as we will see, the results of [8] even lead to the following:

**Addendum 1.4.** \( \overline{M}_{0,n} \) is not a Mori dream space for \( n \geq 10 \).

For the remaining open cases \( n = 7, 8, 9 \), our algorithms yield that all \( X(a,b,c) \) that can be reached via a surjection of any modified Losev-Manin space \( L'_n \) as in the above sequence are Mori dream surfaces. In particular, the treatment of the cases \( n = 7, 8, 9 \) needs new ideas.
2. ORTHOGONAL PAIRS I

Here we introduce our main tool to decide when a given $X = X(a, b, c)$ is a Mori dream surface. It depends on the specific situation and it allows to answer the question entirely in terms of (computable) data of $\mathbb{P}(a, b, c)$, see Proposition 2.3.

We first introduce the necessary notation and recall some background.

Let pairwise coprime positive integers $a, b, c$ be given. The homogeneous coordinate ring of the weighted projective plane $\mathbb{P}(a, b, c)$ is the $\mathbb{Z}$-graded polynomial ring

$$S := \mathbb{K}[x, y, z], \quad \deg(x) := a, \quad \deg(y) := b, \quad \deg(z) := c.$$ 

For a homogeneous polynomial $f \in S_d$, we denote by $V(f)$ the associated (not necessarily reduced) curve on $\mathbb{P}(a, b, c)$. The divisor class group of $\mathbb{P}(a, b, c)$ is freely generated by $A := \eta V(x) + \zeta V(y)$, where we fix $\eta, \zeta \in \mathbb{Z}$ with $\eta a + \zeta b = 1$. We regard the Cox ring of $\mathbb{P}(a, b, c)$ as a divisorial algebra

$$\mathcal{R}(\mathbb{P}(a, b, c)) = \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}(a, b, c), \mathcal{O}(dA)).$$

Observe that the identification of this algebra with the homogeneous coordinate ring $S$ goes via

$$S_d \ni f \mapsto f x^{-\eta} y^{-\zeta} \in \Gamma(\mathbb{P}(a, b, c), \mathcal{O}(dA)).$$

As before, $X = X(a, b, c)$ is the blow-up of $\mathbb{P}(a, b, c)$ at the point $1 = [1, 1, 1]$ and the blow-up morphism is denoted by $\pi: X \to \mathbb{P}(a, b, c)$. The divisor class group $\text{Cl}(X) = \mathbb{Z}^2$ is generated by the classes of

$$H := \pi^*(A), \quad E := \pi^* \cdot (1).$$

In particular, the intersection form on $\text{Cl}_2(X)$ is determined by the intersection numbers

$$H^2 = \frac{1}{abc}, \quad H \cdot E = 0, \quad E^2 = -1.$$ 

As we did with $\mathbb{P}(a, b, c)$, we regard the Cox ring of $X = X(a, b, c)$ as a divisorial algebra. More explicitly, we write

$$\mathcal{R}(X) = \bigoplus_{(d, \mu) \in \mathbb{Z}^2} \mathcal{R}(X)_{dh + \mu E}, \quad \mathcal{R}(X)_{dh + \mu E} = \Gamma(X, \mathcal{O}(dh + \mu E)).$$

The canonical pullback homomorphism $\pi^*$ realizes the Cox ring of $\mathbb{P}(a, b, c)$ as the Veronese subalgebra of $\mathbb{Z}H \subseteq \text{Cl}(X)$ inside the Cox ring of $X$. We will make use of the fact that, as any Cox ring with torsion free grading group, $\mathcal{R}(X)$ is a unique factorization domain.
Let \( I \subseteq S \) and \( J \subseteq S \) denote the homogeneous ideals of the points \((1,1,1)\in \mathbb{K}^3\) and \((0,0,0)\in \mathbb{K}^3\), respectively. Then we have the saturated Rees algebra, graded by \( \mathbb{Z}^2 \), as follows

\[
S[I]^{\text{sat}} := \bigoplus_{\mu \in \mathbb{Z}} (I^{-\mu} : J^\infty) t^\mu = \bigoplus_{(d,\mu) \in \mathbb{Z}^2}(I^{-\mu} : J^\infty) d t^\mu.
\]

For \( f \in S[I]^{\text{sat}}_{(d,\mu)} \), we refer to \( d \) as its \textit{degree} and to \(-\mu\) as its \textit{Rees multiplicity}. We identify the saturated Rees algebra with the Cox ring \( \mathcal{R}(X) \) of \( X = X(a,b,c) \) via the explicit isomorphism

\[
S[I]^{\text{sat}}_{(d,\mu)} \ni f t^\mu \mapsto \pi^* f \in \mathcal{R}(X)_{dH+E}.
\]

see \cite{10} Prop. 5.2. Observe that \( t \in S[I]^{\text{sat}}_{(0,1)} \) is of Rees multiplicity \(-1\) and, in the Cox ring \( \mathcal{R}(X) \), it represents the canonical section of the exceptional divisor \( E \). Moreover, in terms of \( S \) and \( S[I]^{\text{sat}} \), the pullback map \( \pi^* \) between the Cox rings of \( \mathbb{P}(a,b,c) \) and \( X \) is given as

\[
S_d \ni f \mapsto f t^0 \in S[I]^{\text{sat}}_{(d,0)}.
\]

We now assign also to every homogeneous polynomial \( f \in S_d \subseteq S \) a Rees multiplicity.

**Definition 2.1.** Consider a polynomial \( f \in S_d \subseteq S \). The \textit{Rees multiplicity} of \( f \) is the maximal non-negative integer \( \mu \) such that \( f \in I^\mu : J^\infty \) holds.

**Remark 2.2.** For every \( f \in S_d \subseteq S \) and every \( \mu \in \mathbb{Z}_{\geq 0} \), the following statements are equivalent.

(i) The polynomial \( f \) is of Rees multiplicity \( \mu \).

(ii) The curve \( V(f) \subseteq \mathbb{P}(a,b,c) \) has multiplicity \( \mu \) at \( 1 \in \mathbb{P}(a,b,c) \).

(iii) The exceptional divisor \( E \) occurs with multiplicity \( \mu \) in \( \text{div}(\pi^* f) \).

If \( f \in S_d \subseteq S \) is of Rees multiplicity \( \mu \in \mathbb{Z}_{\geq 0} \), then the strict transform of the curve \( V(f) \) in \( \mathbb{P}(a,b,c) \) associated with \( f \) is given as

\[
\text{div}_{dH-\mu E}(\pi^* f) = \text{div}(\pi^* f) + dH - \mu E.
\]

In particular, the element \( f t^{-\mu} \in S[I]^{\text{sat}}_{(d,\mu)} \) is prime if and only if \( V(f) \) is a reduced irreducible curve, or equivalently \( f \in S \) is irreducible, see \cite{11} Prop. 1.5.3.5.

**Definition 2.3.** Let \( f_1 \in S_{d_1} \) and \( f_2 \in S_{d_2} \) be two non-constant homogeneous polynomials in \( S \) of Rees multiplicities \( \mu_1 \) and \( \mu_2 \) respectively. We call \( f_1, f_2 \) an \textit{orthogonal pair} if the following holds:

(i) we have \( d_1^2 \leq \mu_1^2 abc \) and there is no \( f'_1 \in S_{d_1} \) with \( d'_1 < d_1 \) satisfying this condition;

(ii) we have \( d_1 d_2 = \mu_1 \mu_2 abc \) and \( f_1 \nmid f_2 \) and there is no \( f'_2 \in S_{d_2} \) with \( d'_2 < d_2 \) satisfying these conditions.

**Proposition 2.4.** As before, let \( a, b, c \) be pairwise coprime positive integers. Then the following statements are equivalent.

(i) \( X(a,b,c) \) is a Mori dream surface.

(ii) There exists an orthogonal pair \( f_1, f_2 \in S \).

Moreover, if (ii) holds, then the two polynomials \( f_1, f_2 \in S \) are both irreducible.

**Proof.** In \( \text{Cl}_{\mathbb{Q}}(X) = \mathbb{Q}^2 \), we consider the inclusions of the (two-dimensional) cones of ample, semiample, movable, nef and effective divisor classes:

\[
\text{Ample}(X) \subseteq \text{SAmple}(X) \subseteq \text{Mov}(X) \subseteq \text{Nef}(X) \subseteq \text{Eff}(X).
\]

The ample cone is the relative interior of the nef cone. As \( H \) is semiample but not ample, it generates an extremal ray of the semiample cone and thus also of the nef cone. Moreover, the nef cone and the effective cone are dual to each other with
with respect to the intersection product. In particular, $E$ generates an extremal ray of the effective cone because we have $H \cdot E = 0$. Finally, from [11] we know that $X$ is a Mori dream surface if and only if the semiample cone equals the nef cone and is polyhedral in $\text{Cl}_2(X)$.

We prove "(i)⇒(ii)". Since $X$ is a Mori dream surface, the effective and the semiample cone are polyhedral and the semiample cone equals the moving cone. Consequently, we find non-associated prime elements

$g_1 \in \mathcal{R}(X)_C, \quad C = d_1H + \mu_1E, \quad g_2 \in \mathcal{R}(X)_D, \quad D = d_2H + \mu_2E$

such that $C$ and $E$ generate the effective cone and $D$ and $H$ the semiample cone; see [11] Prop. 3.3.2.1 and Prop. 3.3.2.3. Observe that we have $\mu_i < 0 < d_i$ in both cases. We choose the $g_i$ such that the degrees $d_i$ are minimal with respect to the above properties. Since the semiample cone also equals the nef cone, $C \cdot D = 0$ holds.

Consider the element $f_it^{\mu_i} \in S[I]_{(d_i, \mu_i)}$ corresponding to $g_i \in \mathcal{R}(X)$. We have $f_i \in S_{d_i}$. Moreover, we claim that $-\mu_i$ is the Rees multiplicity of $f_i$. Indeed, the order of $f_i$ along $E_i$ is at least $-\mu_i$. If it were bigger, then $f_it^{\mu_i}$ were divisible by $t$, which is impossible by primality of $g_i$. Thus, Remark 2.2 gives the claim.

We check that $f_1 \in S_{d_1}$ and $f_2 \in S_{d_2}$ form the desired orthogonal pair. The inequality in 2.3(i) is due to $C^2 \leq 0$, the equation in 2.3(ii) follows from $C \cdot D = 0$. We verify the minimality condition for $d_1$. Let $f \in S_{d_1}$ be of Rees multiplicity $\mu$ and satisfy the inequality of (i). Let $g \in \mathcal{R}(X)_F$, where $F = dH - \mu E$, be the element corresponding to $ft^{\mu} \in S[I]_{(d_1, -\mu)}$. Then $F^2 \leq 0$ holds. Consider

$C_0 := \text{div}(g_1) + C = \text{div}(g_1) + d_1H - \mu_1E,$

$F_0 := \text{div}(g) + F = \text{div}(g) + dH - \mu E.$

Since $g_1 \in \mathcal{R}(X)$ is prime, $C_0$ is a reduced irreducible curve. Moreover, $F_0$ is an effective curve. The class of $C_0$ equals that of $C$ and the class of $F_0$ equals that of $F$. In particular, we have

$C_0^2 \leq 0, \quad F_0^2 \leq 0, \quad C_0 \cdot F_0 \leq 0.$

If $C_0^2 = 0$ holds, then all the above intersection numbers vanish, $F$ lies on the ray through $C$ and by the choice of $G$, we have $d_1 \leq d$. If $C_0^2 < 0$ holds, then $C_0 \cdot F_0 < 0$ holds and we conclude that $C_0$ is a component of $F_0$. This implies $d_1 \leq d$ and we obtained the minimality condition for $d_1$.

We turn to the minimality condition of $d_2$. Let $f \in S_{d_2}$ be of Rees multiplicity $\mu$ such that $f_1$ does not divide $f$ in $S$ and $f$ satisfies the equation of (ii). As before, consider the element $g \in \mathcal{R}(X)_F$ corresponding to $ft^{\mu} \in S[I]_{(d_1, -\mu)}$, where $F = dH - \mu E$. Then $F \cdot C = 0$ holds and thus $F$ defines a class on the ray through $D$. By the choice of $f_2$, this implies $d_2 \leq d$.

We prove "(ii)⇒(i)". Let $f_1, f_2$ form an orthogonal pair, denote by $d_1, d_2$ the respective degrees and by $\mu_1, \mu_2$ the Rees multiplicities. Consider $C = d_1H - \mu_1E$ and $D = d_2H - \mu_2E$ and the elements $g_1 \in \mathcal{R}(X)_C$ and $g_2 \in \mathcal{R}(X)_D$ corresponding to $f_1t^{\mu_1} \in S[I]_{(d_1, -\mu_1)}$ and $f_2t^{\mu_2} \in S[I]_{(d_2, -\mu_2)}$ respectively. By the definition of an orthogonal pair we have $C^2 \leq 0$ and $C \cdot D = 0$.

We show that $g_1 \in \mathcal{R}(X)$ and $f_1 \in S$ are prime elements. Otherwise, we have a decomposition $g_1 = g'h$ with homogeneous non-units $g', h \in \mathcal{R}(X)$. Because of the minimality of $d_1$ with respect to $C^2 \leq 0$, the corresponding decomposition of the degree $(d_1, -\mu_1)$ of $g_1$ is of the shape

$$(d_1, -\mu_1) = (d_1, -\mu'_1) + (0, k) \in \mathbb{Z}^2 = \text{Cl}(X),$$

where $\mu'_1 > \mu_1$ and $k > 0$. We conclude that $h$ is a power of $t$, the canonical section of the exceptional divisor. This contradicts the fact that $\mu_1$ is the Rees multiplicity
of \(f_1\); see Remark 2.2 and [11 Prop. 1.5.3.5]. Thus, \(g_1 \in \mathcal{R}(X)\) is prime, and, again by Remark 2.2, the polynomial \(f_1 \in S\) is prime.

We claim that \(C\) generates an extremal ray of the effective cone of \(X\). Otherwise, we find a prime element \(g \in \mathcal{R}(X)\) such that its degree \(F = dH - \mu E\), where \(d, \mu \in \mathbb{Z}_{\geq 0}\) lies outside the cone generated by \(E\) and \(C\). Similarly as earlier, we consider

\[
C_0 := \text{div}(g_1) + C = \text{div}(g_1) + d_1H - \mu_1E,
\]

\[
F_0 := \text{div}(g) + F = \text{div}(g) + dH - \mu E.
\]

Since \(g_1\) and \(g\) are prime elements in \(\mathcal{R}(X)\), these are reduced irreducible curves on \(X\). The class of \(C_0\) equals that of \(C\) and the class of \(F_0\) equals that of \(F\). In particular, we have

\[
C_0^2 \leq 0, \quad F_0^2 < 0, \quad C_0 \cdot F_0 < 0.
\]

We conclude that \(F_0\) is a component of \(C_0\) and thus \(F_0 = C_0\) holds. In particular, the class \(F\) lies in the cone generated by \(E\) and \(C\); a contradiction. We obtained that \(E\) and \(C\) generate the effective cone of \(X\).

Since \(D\) is orthogonal to \(C\), it generates an extremal ray of the nef cone of \(X\). Thus, the nef cone of \(X\) is the polyhedral cone generated by \(D\) and \(H\). Since \(f_1\) does not divide \(f_2\) in \(S\), we conclude via Remark 2.2 and [11 Prop. 1.5.3.5] that \(g_1\) does not divide \(g_2\) in \(\mathcal{R}(X)\) and thus the curve \(C_0\) is not a component of the effective curve \(D_0 := D + \text{div}(g_2)\). This, together with the fact that \(nD\) is linearly equivalent to \(rC + B\) for some \(n, r \in \mathbb{Z}_{\geq 0}\) and a very ample divisor \(B\), implies that the stable base locus of \(D\) is at most zero-dimensional. By Zariski’s theorem [14 Theorem 6.2], one concludes that \(D\) is semiample. So, the nef cone equals the semiample cone and thus \(X\) is a Mori dream surface.

We turn to the supplement. Let \(f_i \in S\) and \(g_i \in \mathcal{R}(X)\) be as in the proof of the implication “(ii)⇒(i)”. We already saw that \(f_1\) is irreducible in \(S\). To obtain irreducibility of \(f_2\) note that by [11 Prop. 3.3.2.3] there is at least one prime generator \(g \in \mathcal{R}(X)\) which is not divisible by \(g_1\) and has its degree on the ray through \(D\) bounding the semiample cone. The minimality condition of [23(ii)] yields that \(g_2\) is among these \(g\) and thus prime. \(\square\)

**Remark 2.5.** Let \(f_1 \in S_{d_1}\) and \(f_2 \in S_{d_2}\) be two homogeneous polynomials in \(S\) of Rees multiplicities \(\mu_1\) and \(\mu_2\), respectively, and assume that \(f_1, f_2\) is an orthogonal pair. From Proposition 2.4 and its proof, we infer the following:

(i) The effective cone of \(X\) is polyhedral in \(\text{Cl}_Q(X)\); one ray is generated by \(E\), the other we denote by \(g\).

(ii) The element \((d_1, -\mu_1) \in \mathbb{Z}^2 = \text{Cl}(X)\) is the class of a prime divisor \(C_1\) and it is the shortest non-zero lattice vector which lies on \(g\) and belongs to the monoid of effective divisor classes of \(X\).

(iii) The semiample cone of \(X\) is polyhedral in \(\text{Cl}_Q(X)\); one ray is generated by \(H\), the other we denote by \(\tau\); here \(g = \tau\) is possible.

(iv) The element \((d_2, -\mu_2) \in \mathbb{Z}^2 = \text{Cl}(X)\) is the class of a prime divisor \(C_2 \neq C_1\), and it is the shortest non-zero lattice vector which lies on \(\tau\) and is the class of a prime divisor \(C_2 \neq C_1\).

This means in particular that for any two orthogonal pairs \(f_1, f_2\) and \(f'_1, f'_2\), we have \(d_1 = d'_1\) and \(d_2 = d'_2\) for the respective degrees and \(\mu_1 = \mu'_1\) and \(\mu_2 = \mu'_2\) for the Rees multiplicities. Moreover, \((d_1, -\mu_1)\) and \((d_2, -\mu_2)\) occur in the set of \(\text{Cl}(X)\)-degrees of any system of homogeneous generators of the Cox ring \(\mathcal{R}(X)\).
3. Proof of Theorem 1.1

The setting and the notation are the same as in the preceding section. We begin with preparing the proof of Theorem 1.1.

Lemma 3.1. For \( i = 1, 2 \) let \( f_i \in S_{d_i} \) be irreducible of Rees multiplicity \( \mu_i \) and write \( C_i \subseteq X \) for the strict transform of \( V(f_i) \subseteq \mathbb{P}(a, b, c) \). If \( C_1 \cdot C_2 = 0 \) holds, then \( V(f_1) \cap V(f_2) \) contains only the point \( 1 \in \mathbb{P}(a, b, c) \).

Proof. We have \( C_i = \text{div}(\pi^*f_i) + d_iH - m_iE \). Thus \( C_1 \cdot C_2 = 0 \) is equivalent to \( d_1d_2 = abc\mu_1\mu_2 \). Bezout’s theorem in \( \mathbb{P}(a, b, c) \) tells us that the zero-dimensional scheme \( V(f_1, f_2) \) has degree \( \mu_1\mu_2 \). Since \( V(f_1) \) and \( V(f_2) \) intersect with multiplicity at least \( \mu_1\mu_2 \) at \( 1 \), we conclude that they intersect exactly with multiplicity \( \mu_1\mu_2 \) at \( 1 \) and nowhere else. \( \square \)

Lemma 3.2. Consider homogeneous polynomials \( f_1 \in S_{d_1} \) and \( f_2 \in S_{d_2} \), both of Rees multiplicity one, and assume that \( f_1, f_2 \) is an orthogonal pair. Then there is an orthogonal pair \( f'_1, f'_2 \) of binomials \( f'_i \in S_{d_i} \) of Rees multiplicity one.

Proof. Since \( f_i \) is of Rees multiplicity one, we have \( 1 \in V(f_i) \). In particular, there are at least two monomials of degree \( d_i \) occurring with non-zero coefficients in \( f_i \). We consider binomials \( f'_i \) which are the difference of two monomials of \( f_i \). Each such \( f'_i \) is of degree \( d_i \). Moreover, \( V(f'_i) \) has multiplicity one at \( 1 \in \mathbb{P}(a, b, c) \) and thus Remark 2.2 tells us that \( f'_i \) is of Rees multiplicity one. Observe that all binomials \( f'_i \) are prime due to the minimality condition on the degree \( d_i \). Every pair \( f'_1, f'_2 \) fulfills obviously all conditions of an orthogonal pair, except \( f'_1 \parallel f'_2 \). In fact, this condition needs not be satisfied automatically. We show how to achieve it.

If \( \dim_{\mathbb{K}}(S_{d_i}) > 2 \) holds, then we have at least three different choices for the binomial \( f'_i \). As the binomial \( f'_2 \) has at most one prime factor vanishing at the point \( (1, 1, 1) \), we find a pair \( f'_1, f'_2 \) with \( f'_1 \parallel f'_2 \). We treat the case \( \dim_{\mathbb{K}}(S_{d_i}) = 2 \).

Then \( f'_1 \) is a scalar multiple of \( f_1 \). Consider the list \( f'_2, \ldots, f'_{2, r} \) of all possible binomials made from monomials of \( f_2 \). Because of \( f_2(1, 1, 1) = 0 \), the coefficients of the monomials of \( f_2 \) sum up to zero and thus \( f_2 \) is a linear combination of the binomials \( f'_{2, j} \). Since \( f_1 \) does not divide \( f_2 \), there must be a binomial \( f'_2 = f'_{2, j} \) which is not divisible by \( f'_1 \). \( \square \)

Lemma 3.3. Let \( f_1 \in S_{d_1} \) and \( f_2 \in S_{d_2} \) be binomials of Rees multiplicity one. If \( f_1, f_2 \) is an orthogonal pair, then one of the numbers \( a, b, c \) lies in the monoid generated by the remaining two.

Proof. Proposition 2.3 tells us that \( f_1 \) and \( f_2 \) are both irreducible. According to Lemma 3.1 the zero loci of \( f_1 \) and \( f_2 \) intersect only at the point \( 1 \in \mathbb{P}(a, b, c) \). Thus, reordering \( a, b, c \) suitably, we may assume

\[
\begin{align*}
f_1 &= x^{p_1} - y^{p_2} z^{p_3}, \\
f_2 &= y^{q_1} - x^{q_2} z^{q_3}.
\end{align*}
\]

The homogeneity of the two binomials and the orthogonality condition give us the following equations:

\[
\begin{align*}
ap_1 &= bp_2 + cp_3, & b q_1 &= aq_2 + cq_3, & p_1q_1 &= c.
\end{align*}
\]

Substituting \( c = p_1q_1 \) in the first equation and using the coprimality of \( b \) and \( c \) we obtain \( p_2 = p_1p'_2 \) with \( p'_2 \in \mathbb{Z}_{\geq 1} \). Similarly one shows that \( q_2 = q_1q'_2 \) with \( q'_2 \in \mathbb{Z}_{\geq 1} \). Consider the case \( p_2q_2 \neq 0 \). Then, from the first two equations, we deduce

\[
\begin{align*}
a &= bp'_2 + q_1p_3, & b &= aq'_2 + p_1q_3.
\end{align*}
\]

In particular \( a \geq b \geq a \), so that \( a = b \), and thus \( a \) is in the monoid generated by \( b \) and \( c \). We now treat the case \( p_2q_2 = 0 \). We may assume \( q_2 = 0 \). Then from
Proof of Theorem 1.1] We prove “(i)⇒(ii)”. If $X$ has a non-trivial $K^*$-action, then this action stabilizes the exceptional curve $E \subseteq X$ and thus $\mathbb{P}(a,b,c)$ inherits a non-trivial $K^*$-action having $[1,1,1]$ as a fixed point. According to [3], this means that $\operatorname{Aut}(\mathbb{P}(a,b,c))$ must contain a root subgroup, i.e., there must be a monomial in two variables in $K[x,y,z]$ of degree $a$, $b$, or $c$. This is only possible, if one of $a, b, c$ lies in the monoid generated by the remaining two.

We show that (ii) implies (i), (iii) and the supplement. We may assume that $a = nb + nc$ holds with non-negative integers $m$ and $n$. Then the morphism

$$\varphi: \mathbb{P}(a,b,c) \to \mathbb{P}(a,b,c), \quad [z_1, z_2, z_3] \mapsto [z_1 - z_2^m z_3^n, z_2, z_3]$$

sends $[1,1,1]$ to $[0,1,1]$. The blowing-up of $\mathbb{P}(a,b,c)$ at $[0,1,1]$ obviously admits a $K^*$-action. To obtain the Cox ring, observe first

$$\mathbb{P}(a,b,c) \cong V(T_a - T_1 + T_1) \subseteq \mathbb{P}(b,c,a, bc).$$

The Cox ring of $X(a,b,c)$ is now computed via a toric ambient modification, see [1, Sec. 4.1.3]: We blow up $\mathbb{P}(b,c,a, bc)$ at $[1,1,0,0]$. Then $X = X(a,b,c)$ is isomorphic to the strict transform $V(T_a - T_1 + T_1)$ and its Cox ring is as claimed. Observe that the degree matrix $Q$ is given with respect to the basis $H, E$ of $\operatorname{Cl}(X) = \mathbb{Z}^2$. The last column in the degree matrix is the class of $E$ and thus we see that the Rees multiplicities of the generators are as in the assertion. In particular, we obtain (iii).

We prove “(iii)⇒(ii)”. By assumption, $X$ is a Mori dream surface. Take homogeneous non-associated prime generators $g_1 \in \mathcal{R}(X)_C$ and $g_2 \in \mathcal{R}(X)_D$ as in the proof of “(i)⇒(ii)” of Proposition 2.3. Then the effective cone of $X$ is generated by $C$ and $E$ and the semiample cone by $D$ and $H$. Moreover, $g_1$ and $g_2$ occur (up to scalars) in any system of homogeneous generators of $\mathcal{R}(X)$. Thus, since $g_1$ and $g_2$ are of positive Rees multiplicity, the assumption says that they are of Rees multiplicity one. Let $f_i \in S_d$, denote the polynomial such that $g_i$ corresponds to $\pi^*(f) i^{-1} \in S/T_{[d, -1]}$. By primality of the $g_i$, the $f_i$ are of Rees multiplicity one. Moreover, they are non-associated primes forming an orthogonal pair, which means in particular $d_1 d_2 = abc$. According to Lemma 3.3 we may assume that $f_1, f_2 \in I$ are binomials. Then Lemma 3.3 gives condition (ii).

4. Orthogonal pairs II

Remark 3.4. Assume that we have $c = ma + nb$ with non-negative integers $m$ and $n$. Then the describing matrix $P$ of $X(a,b,c)$ in the sense of [1] is of the form

$$P = \begin{bmatrix} -c & b & 0 & 0 & 0 \\ -c & 0 & 1 & 1 & 0 \\ -m & -n & 0 & 1 & 1 \end{bmatrix}.$$
where $\mathbb{T}^k = (\mathbb{K}^*)^k$ denotes the standard $k$-torus. The restriction $\kappa: \mathbb{T}^3 \to \mathbb{T}^2$ of the quotient map is a homomorphism of tori and thus given by monomials. Let $f_0$ be any monomial of $f$. Then we have

$$\frac{f}{f_0} = \kappa^*(h)$$

with a unique $h \in \mathbb{K}[u^{\pm 1}, v^{\pm 1}]$ on $\mathbb{T}^2$. The Laurent polynomial $h$ generates the defining ideal of $V(f)$ on $\mathbb{T}^2$. Thus, the multiplicity of $V(f)$ at $1 \in \mathbb{P}(a,b,c)$ equals the multiplicity of $h$ at $(1,1) \in \mathbb{T}^2$.

**Lemma 4.2.** Let $\alpha, \beta, \gamma \in \mathbb{K}$ with $\alpha + \beta + \gamma = 0$ and $k, n_1, n_2, m_1, m_2 \in \mathbb{Z}_{\geq 0}$ such that we obtain a non-constant homogeneous polynomial

$$f := \alpha z^k + \beta x^{n_1} y^{n_2} + \gamma x^{m_1} y^{m_2} \in \mathbb{K}[x, y, z].$$

Assume that $ck/l \notin (a,b)$ holds whenever $l \in \mathbb{Z}_{>1}$ is a common divisor of $k, n_1, n_2$ or of $k, m_1, m_2$. Then the multiplicity of $V(f)$ at $1 \in \mathbb{P}(a,b,c)$ is at most one.

**Proof.** If $f$ is a monomial, then $V(f)$ is of multiplicity zero at $1$. If $f$ is a binomial, then it is of multiplicity one at $1$. So, we may assume that $\alpha, \beta, \gamma$ all differ from zero. We follow Remark [11]. Consider the homomorphism of tori $\kappa: \mathbb{T}^3 \to \mathbb{T}^2$ and let $u, v$ be coordinates on $\mathbb{T}^2$. Then there are monomials $u^{p_1} v^{p_2}$ and $u^{q_1} v^{q_2}$ with

$$\kappa^*(u^{p_1} v^{p_2}) = \frac{x^{n_1} y^{n_2}}{z^k}, \quad \kappa^*(u^{q_1} v^{q_2}) = \frac{x^{m_1} y^{m_2}}{z^2}.$$

We have $f = z^k \kappa^*(h)$ for $h := \alpha + \beta u^{p_1} v^{p_2} + \gamma u^{q_1} v^{q_2}$ and the multiplicity of $f$ at $1$ equals the multiplicity of $h$ at $(1,1)$. Assume that latter is at least two. Then $h$ and its derivatives $\partial h/\partial u$ and $\partial h/\partial v$ vanish simultaneously at $(1,1)$. This means

$$\alpha + \beta + \gamma = 0, \quad \beta p_1 + \gamma q_1 = 0, \quad \beta p_2 + \gamma q_2 = 0,$$

which implies that $(p_1, p_2)$ and $(q_1, q_2)$ are proportional and thus $u^{p_1} v^{p_2}$ and $u^{q_1} v^{q_2}$ are powers of a monomial $g = u^{w_1} v^{w_2}$ with coprime exponents $w_1, w_2$. The pullback monomials are thus of the form

$$\frac{x^{n_1} y^{n_2}}{z^k} = \kappa^*(g)^{l_1}, \quad \frac{x^{m_1} y^{m_2}}{z^k} = \kappa^*(g)^{l_2}.$$ 

Observe that $l_1$ divides $k, n_1, n_2$. Thus we have $ck/l_1 = an_1/l_1 + bn_2/l_1 \in (a,b)$. By the assumption, this means $l_1 = 1$. Analogously, $l_2$ divides $k, m_1, m_2$ and we conclude $l_2 = 1$. Thus, $f$ is a binomial and vanishes of order one at $1$. Consequently, $h$ cannot vanish of order at least two at $(1,1)$. \hfill $\Box$

**Lemma 4.3.** Assume that none of $a, b, c$ lies in the monoid generated by the other two and that $2c$ lies in the monoid generated by $a$ and $b$. Then any $0 \neq f \in S_{2c}$ vanishes with multiplicity at most one at $1 \in \mathbb{P}(a,b,c)$.

**Proof.** A monomial $x^{n_1} y^{n_2} z^{n_3} \in S$ is of degree $2c$ if and only if it equals $z^2$ or is of the shape $x^{n_1} y^{n_2}$. Indeed, we must have $n_3 \leq 2$ and $n_3 = 1$ is impossible, because this means $2c = a + n_1 + b n_2$, contradicting $c \notin (a,b)$. We obtain

$$g = \alpha z^2 + \beta x^{n_1} y^{n_2} + \gamma x^{m_1} y^{m_2},$$

with coefficients $\alpha, \beta, \gamma \in \mathbb{K}$, as [13, 4.4, p. 80] tells us that there are at most two monomials of degree $2c$ only depending on $x$ and $y$. If $1 \in V(f)$ holds, then we have $\alpha + \beta + \gamma = 0$ and Lemma 4.2 gives the assertion. \hfill $\Box$
Proposition 4.4. Let \( f_1 \in S_{d_1} \) and \( f_2 \in S_{d_2} \) be an orthogonal pair. If one of the \( f_i \) is of Rees multiplicity two, then the other is not.

Proof. If one of \( a,b,c \) lies in the monoid generated by the other two, then Theorems \( \star \star \) and Remark \( \star \star \star \) give the assertion. So, we only have to consider the case, where none of \( a,b,c \) lies in the monoid generated by the other two.

Assume that both members \( f_1, f_2 \) of the orthogonal pair are of Rees multiplicity two. Then, by Proposition \( \star \star \), each \( V(f_i) \subseteq \mathbb{P}(a,b,c) \) is an irreducible curve and the strict transforms \( C_i \subseteq X \) satisfy \( C_1 \cdot C_2 = 0 \). Thus, Lemma \( \star \star \star \) says that 1 is the only intersection point of \( V(f_1) \) and \( V(f_2) \).

In a first step we show that each \( V(f_i) \) contains at least one of the toric fixed points \([1,0,0], [0,1,0] \) and \([0,0,1] \). Assume that one \( V(f_i) \) does not. Then \( f_i \) is of the shape

\[
\alpha x^n + \beta y^p + \gamma z^m + f_i',
\]

where \( \alpha, \beta, \gamma \in \mathbb{K}^* \) and the monomials of \( f_i' \in S_{d_i} \) are all in two or three variables. Since \( f_i \) is homogeneous of degree \( d_i \), we obtain

\[
d_i = p_1 a = p_2 b = p_3 c.
\]

As \( a,b,c \) are pairwise coprime, \( d_i = abc \) holds with \( n \in \mathbb{Z}_{\geq 1} \). The orthogonality condition \( d_1 d_2 = 4 abc \) gives \( n d_j = 4 \) for the \( j \neq i \). This implies \( a,b,c \leq 4 \). Then one of \( a,b,c \) lies in the monoid generated by the other two; a contradiction.

Thus, we saw that each of the curves \( V(f_i) \) contains at least one toric fixed point and no toric fixed point is contained in both of them. After suitably reordering \( a,b,c \), we are left with the following three cases.

Case 1. Each of the curves \( V(f_1) \) and \( V(f_2) \) contains exactly one toric fixed point, namely \([1,0,0] \) and \([0,1,0] \) respectively. Then \( f_1 \) and \( f_2 \) are of the shape

\[
f_1 = \beta_1 y^p + \gamma_1 z^m + f_1', \quad f_2 = \alpha_1 x^n + \gamma_2 z^m + f_2',
\]

where \( \alpha_1, \beta_1, \gamma_1 \in \mathbb{K}^* \) and the monomials of \( f_i' \in S_{d_i} \) are all in two or three variables. The homogeneity of the \( f_i \) implies

\[
d_1 = p_1 a = p_2 b = p_3 c.
\]

Pairwise coprimality of \( a,b,c \) gives \( d_1 = b c n \) and \( d_2 = a c m \) with \( n,m \in \mathbb{Z}_{\geq 1} \). The orthogonality condition \( d_1 d_2 = 4 abc \) implies \( c n m = 4 \). We conclude \( c = 4 \) and \( n = m = 1 \), because \( c \leq 2 \) would imply \( a \in \langle b,c \rangle \) or \( b \in \langle a,c \rangle \). Thus, \( d_1 = 4 b \) holds.

Now we use Condition \( \star \star \star \):

\[
d_1^2 \leq 4abc \implies 16b^2 \leq 16ab \implies b \leq a \implies b < a,
\]

where the last conclusion is due to \( a \notin \langle b,c \rangle \). On the other hand, \( a \notin \langle b,c \rangle \) implies that \( a \) is less or equal to the Frobenius number of the monoid \( \langle b,c \rangle \). This means

\[
a \leq (c-1)(b-1) - 1 = (4-1)(b-1) - 1 = 3b - 4.
\]

Moreover, \( a - b \) and \( 2b \) are even but not divisible by \( c = 4 \). Consequently, \( 3b - a \) is divisible by 4. We claim

\[
f_1 = \beta_1 y^b + \gamma_1 z^m + \delta x y z^{3b-4}, \quad \beta_1, \gamma_1, \delta \in \mathbb{K}^*.
\]

Note that we need at least three terms, because binomials are of Rees multiplicity one. The task is to show that there are no further monomials of degree \( 4b \) than the ones above. Each monomial \( x^n y^m z^l \) of degree \( 4b \) gives an equation

\[
an + bm + 4l = 4b, \quad n,m \in \mathbb{Z}_{\geq 0}.
\]

Clearly, \( m \leq 4 \) holds. Because of \( a > b \), we have \( n \leq 3 \). As \( an + bm \) is divisible by 4, the only possibilities for \((n,m)\) are \((0,4), (0,0) \) and \((1,1) \). Having verified
the special shape for $f_1$, we can compute the multiplicity of $V(f_1)$ according to Remark [4.1]. The quotient map is given on the tori as
\[
\kappa : \mathbb{T}^3 \to \mathbb{T}^2, \quad (x, y, z) \mapsto \left(\frac{z^b}{y^4}, \frac{x^2 - \beta y^2}{y^4}\right).
\]
We have $f_1 = y^4\kappa^*(h)$ with $h := \beta_1 + \gamma_1 u + \delta v$. The polynomial $h$ has multiplicity one at $(1, 1)$; a contradiction.

**Case 2.** The curve $V(f_1)$ contains $[1, 0, 0]$ and $[0, 1, 0]$ and $V(f_2)$ contains $[0, 0, 1]$. Then $f_1$ and $f_2$ are of the shape
\[
f_1 = \gamma_1 z^p + f'_1, \quad f_2 = \alpha_2 x^q + \beta_2 y^r + f'_2,
\]
where $\alpha, \beta, \gamma \in \mathbb{K}^*$, and the polynomials $f'_i \in S_d$ have only monomials in two or three variables. By homogeneity of the $f_i$ we have
\[
d_1 = cp, \quad d_2 = aq_1 = bq_2 = ab,
\]
where $n$ is a positive integer. The orthogonality condition $d_1d_2 = 4abc$ provides us with $np = 4$. The case $n = p = 2$ is impossible: we would have $2c \in \langle a, b \rangle$ and, by Lemma [4.3], the multiplicity of $f_1$ at $1 \in \mathbb{P}(a, b, c)$ would be one. We end up with $p = 4$ and $n = 1$. This means $d_1 = 4c$ and $d_2 = ab$. Condition [2.3] (i) gives
\[
d_1^2 \leq 4abc \implies 16c^2 \leq 4abc \implies c \leq \frac{ab}{4}.
\]
This implies $2c \notin \langle a, b \rangle$, because otherwise we find a binomial $g = z^2 - x^ny^m$ of degree $2c$ and Rees multiplicity $1$ which satisfies Condition [2.3] (i), contradicting the minimality of the degree of $f_1$. We determine $f_1$ more explicitly. Each monomial $x^ny^mz^l$ of degree $4c$ gives an equation
\[
an + bm + lc = 4c, \quad n, m, l \in \mathbb{Z}_{\geq 0}.
\]
Here, $l = 2, 3$ are excluded because of $2c \notin \langle a, b \rangle$ and $c \notin \langle a, b \rangle$. Thus, we have $l \leq 1$. If $4c < ab$ holds, then we can apply [13, 4.4, p. 80] and obtain that there is at most one monomial of the form $zx^{n_1}y^{n_2}$ and at most one of the form $x^{n_1}y^{n_2}$ in degree $4c$. Thus, we have
\[
f_1 = \alpha z^4 + \beta zx^{n_1}y^{n_2} + \gamma x^{n_1}y^{n_2}
\]
and Lemma [4.2] tells us that $V(f_1)$ is multiplicity one at $1$; a contradiction. We are left with discussing the case $4c = ab$. By coprimality of $a$ and $b$, we obtain $a = 4a'$ or $b = 4b'$. Thus, $c = a'b$ and $c = b'a$, both contradicting $c \notin \langle a, b \rangle$. Thus, Case 2 cannot occur.

**Case 3.** The curve $V(f_1)$ contains $[1, 0, 0]$ and $V(f_2)$ contains $[0, 1, 0]$ and $[0, 0, 1]$. Then $f_1$ and $f_2$ are of the shape
\[
f_1 = \beta_1 y^p + \gamma_1 z^p + f'_1, \quad f_2 = \alpha_2 x^q + f'_2,
\]
where $\alpha, \beta, \gamma \in \mathbb{K}^*$, and the polynomials $f'_i \in S_d$ have only monomials in two or three variables. By homogeneity of the $f_i$ we have
\[
d_1 = bp_1 = cp_2 = bcn, \quad d_2 = aq,
\]
where $n$ is a positive integer. The orthogonality condition $d_1d_2 = 4abc$ gives $nq = 4$. We obtain $q = 4$ and $n = 1$, because $q = 1$ is excluded by $a \notin \langle b, c \rangle$ and $q = 2$ is impossible due to Lemma [4.3]. Thus, we have $d_1 = bc$ and $d_2 = 4a$. Condition [2.3] (i) gives
\[
d_1^2 \leq 4abc \implies b^2c^2 \leq 4abc \implies bc \leq 4a.
\]
We have $2a \notin \langle b, c \rangle$, because otherwise, there is a binomial $f'_2 = x^2 - \gamma_1 z^{n_2}$ of degree $d'_2 = 2a$ and Rees multiplicity $\mu'_2 = 1$ satisfying the orthogonality condition;
Moreover, in this case, the Cox ring $R$. In other words, there are at most three monomials in the algebra to $I$. Proposition 5.1. In the above situation, let $E$ of the $R$. Then $\xi$ is generated as a $R$-algebra by $t$. We search for solutions with $l \leq 3$. The cases $l = 3, 2$ are excluded because of $a \not\in \langle b, c \rangle$ and $2a \not\in \langle b, c \rangle$. Thus, we look for pairs $m, n \in \mathbb{Z}_{\geq 0}$ satisfying one of the equations

$$mb + nc = 4a, \quad mb + nc = 3a.$$ 

Consider the case $b \not\in \{2, 3, 4\}$. Then $b$ does not divide $ka$ for $k = 1, 2, 3, 4$. Fix positive integers $u, v$ with $ub - vc = 1$. Then [2, Corollary 1.6] says that the number $\xi_{b, c}(ka)$ of pairs $(m, n) \in \mathbb{Z}_{\geq 0}^2$ satisfying $mb + nc = ka$ is given as

$$\xi_{b, c}(ka) = \left\lceil \frac{uka}{c} \right\rceil - \left\lfloor \frac{uka}{b} \right\rfloor, \quad \text{for } k = 1, 2, 3, 4.$$ 

As just seen, we have $\xi_{b, c}(a) = 0$ and $\xi_{b, c}(2a) = 0$. The first equality implies that the two numbers $ua/c$ and $va/b$ lie in some open interval $[s, s + 1]$, where $s \in \mathbb{Z}$. The second equality implies that both numbers even lie either in $[s, s + 1/2]$ or in $[s + 1/2, s + 1]$. We obtain

$$\xi_{b, c}(3a) \leq 1, \quad \xi_{b, c}(4a) \leq 1. \quad \text{In other words, there are at most three monomials in } S_{4a}, \text{ namely } x^4, xy^{m_1}z^{n_2} \text{ and } y^{m_2}z^{n_3}. \text{ Lemma 1.2 says that } V(f_2) \text{ is of multiplicity at most one at } 1; \text{ a contradiction. Analogously, the case } c \not\in \{2, 3, 4\} \text{ is excluded. Thus, we are left with } b, c \in \{2, 3, 4\}. \text{ But this impossible due to } a \not\in \langle b, c \rangle \text{ and } 2a \not\in \langle b, c \rangle. \quad \square$$

5. Proof of Theorem 1.2

We will use the following general criterion for verifying Cox ring generators. Consider an arbitrary Mori dream space $X_1$ and the blow-up $X_2$ of an irreducible subvariety $C \subseteq X_1$ contained in the smooth locus of $X_1$. We will denote by $I \subseteq R_1 := \mathcal{R}(X_1)$ the homogeneous ideal corresponding to $C \subseteq X_1$ and by $J \subseteq R_1$ the irrelevant ideal. The morphism $X_2 \to X_1$ defines a canonical pull back map $R_1 \to R_2 := \mathcal{R}(X_2)$ of Cox rings. We ask if for a given choice of homogeneous generators $f_1, \ldots, f_k \in I$ for $I$, the canonical section $t \in R_2$ of the exceptional divisor $E \subseteq X_2$ together with $f_1t^{-m_1}, \ldots, f_kt^{-m_k}$, where $i = 1, \ldots, k$ and $m_i$ is the Rees multiplicity, generate the Cox ring $R_2$ of $X_2$ as an $R_1$-algebra.

**Proposition 5.1.** *In the above situation, let $g_1, \ldots, g_m$ be homogeneous generators of the $\mathbb{K}$-algebra $R_1$ and let $f$ be the product over all $g_j$ not belonging to $I$. Set

$$B_0 := \{t^m, s_i - f_i; i = 1, \ldots, k\} \subseteq R_1[s_1, \ldots, s_k, t].$$

Then $R_2$ is generated as a $\mathbb{K}$-algebra by $t$, the $f_1t^{-m_1}, \ldots, f_kt^{-m_k}$, where $i = 1, \ldots, k$, and the $g_j$ not belonging to $I$, provided that there is a finite set $B_0 \subseteq B \subseteq \langle B_0 \rangle : \langle t \rangle^\infty$ with

$$\dim(R_1) = \dim(\langle B \cup \{t\} \rangle) > \dim(\langle B \cup \{t, f\} \rangle).$$

Moreover, in this case, the Cox ring $R_2$ of $X_2$ is isomorphic as a $\text{Cl}(X_2)$-graded algebra to

$$R_1[s_1, \ldots, s_k, t]/(\langle B \rangle : \langle t \rangle^\infty).$$*
Proof. Recall that the Cox ring $R_2$ of $X_2$ is the saturated Rees algebra $R_1[I]^{sat}$. As before, let $m_i$ be the Rees multiplicity of $f_i$ for $i = 1, \ldots, k$. The kernel of the $\Cl(X_2)$-graded homomorphism

$$\psi: R_1[s_1, \ldots, s_k, t] \to R_1[I]^{sat}, \quad s_i \mapsto f_i t^{−m_i}, \quad t \mapsto t$$

is the saturation $I_2 := I_2^+ : (t)^\infty$, where we set $I_2^+ := (B)$. Observe that the dimension of $R_1$ equals that of $I_2 + (t)$. Thus, by our assumption, we have

$$\dim(I_2 + (t)) = \dim(R_1) = \dim(I_2^+ + (t)) > \dim(I_2^+ + (t, f)) \geq \dim(I_2 + (t, f)).$$

Consequently, we meet the condition of [10, Algorithm 5.4] which guarantees that the homomorphism $\psi$ is surjective. The assertion follows.

Proof of Theorem 1.3 We show that (i) implies (ii). First note that the Cox ring of $X$ finitely generated. Indeed, $\mathcal{R}(X)$ is the saturated Rees algebra $S[I]^{sat}$ which, under the assumption (i), is generated by $t^{-1}$, the Cox ring generators $x, y, z$ of $\mathbb{P}(a, b, c)$, the elements $g_i t^{-1}$, where the $g_i$ generate the ideal $I : J^{\infty}$, and the $h_i t^{-2}$, where the $h_i$ generate the ideal $I^2 : J^{\infty}$. Thus, Proposition 2.4 provides us with an orthogonal pair $f_1, f_2$, where $f_i \in S_{d_i}$ is of Rees multiplicity $\mu_i$. Remark 2.5 says that $(d_1, -\mu_1)$ and $(d_2, -\mu_2)$ occur in the set of $\Cl(X)$-degrees of any system of generators of the Cox ring $\mathcal{R}(X)$. Thus, by assumption, we have $\mu_i \leq 2$.

Proposition 4.4 yields that $\mu_1 = 2$ holds at most once.

For both $f_i$, their degree $d_i$ is positive and thus also their Rees multiplicity $\mu_i$ is positive. Since we assume none of $a, b, c$ to lie in the monoid generated by the other two, the case $\mu_1 = \mu_2$ is excluded by Lemmas 3.2 and 3.3. We now consider the case $\mu_1 = 1$ and $\mu_2 = 2$. Then we may assume

$$f_1 = x^{p_1} - y^{p_2} z^{p_3}, \quad f_2 = \alpha y^{q_1} + \beta z^{q_2} + f'_2,$$

where $\alpha, \beta \in \mathbb{K}^*$ and $f'_2 \in S_{d_2}$ has only monomials in two or three variables. Indeed, Lemma 3.2 say that we may assume $f_1$ to be a binomial. By Proposition 2.4, the binomial $f_1$ is prime, and thus we may assume it to be of the displayed shape. In particular, the points $[0, 1, 0]$ and $[0, 0, 1]$ are contained in $V(f_1)$. Lemma 3.4 tells us that none of these two points lies in $V(f_2)$ and thus, $f_2$ must be of the above shape. Homogeneity of $f_1, f_2$ and the orthogonality condition (2.3) (ii) lead to the equations

$$ap_1 = bp_2 + cp_3, \quad bq_1 = cq_2, \quad p_1 q_1 = 2c.$$

Since $b$ and $c$ are coprime, the second equation shows that $q_1 = lc$ holds with $l \in \mathbb{Z}_{\geq 1}$. Substituting this in the last equation gives $lp_1 = 2$. Because of $a \not\in \langle b, c \rangle$, we have $p_1 \neq 1$ and thus obtain $p_1 = 2$ and $l = 1$. Consequently, $q_1 = c$ and $q_2 = b$ hold. With $n := p_2$ and $m := p_3$, the first equation thus becomes

$$2a = nb + mc.$$

We now describe the polynomial $f'_2$ in more detail. First, we determine the monomials $x^k y^p z^q$ of degree $d_2 = bc$. This means to look at the equation $ka + bp + cq = bc$, which implies

$$b(km + 2p) + c(km + 2q) = 2bc.$$

In particular, $km + 2p = rc$ holds for some integer $r \geq 1$. Substituting this in the displayed equation, we obtain $km + 2q = (2 - r)b$. This implies $r \leq 1$ and thus $r = 1$. Thus, we arrive at

$$p = \frac{c - km}{2}, \quad q = \frac{b - km}{2}.$$

In particular, we see that $k$ must be odd, as $b$ and $c$ are coprime. Up to now, we are able to express the possible monomials of degree $d_2 = bc$ in terms of $k, m, n$ and
\(b,c\). We are going to apply Remark [4] As a homomorphism of tori we take
\[
\kappa: \mathbb{T}^3 \rightarrow \mathbb{T}^2, \quad (x, y, z) \mapsto \left( \frac{y^c}{z^b}, \frac{xy + z \cdot \frac{k}{a}}{z^b} \right).
\]

Then, with the coordinates \(u, v\) on \(\mathbb{T}^2\) and \(l := (k-1)/2\), we can write the general monomial of degree \(d_2 = bc\) as
\[
x^k y^{\frac{c \cdot m}{2}} z^{\frac{k \cdot m}{2}} = z^b \kappa^*(\frac{u^k}{z^b}) = z^b \kappa^*(\frac{v^{2l+1}}{u^l}).
\]

Consequently, with suitable coefficients \(\gamma_l \in \mathbb{K}\), we can write \(f_2 = z^b h\) with a Laurent polynomial
\[
h = \alpha + \beta u + \sum_{l=0}^{s} \gamma_l \frac{u^{2l+1}}{u^l}.
\]

The fact that \(f_2\) is of Rees multiplicity two implies that \(h\) as well as its first order partial derivatives \(\partial h/\partial u\) and \(\partial h/\partial v\) vanish at \((1, 1)\). This leads to the conditions
\[
\alpha + \beta + \gamma_0 + \ldots + \gamma_s = 0,
\]
\[
\beta - \gamma_0 - \ldots - (s+1)\gamma_s = 0,
\]
\[
\gamma_0 + 3\gamma_1 + \ldots + (2s+1)\gamma_s = 0.
\]

In particular, we see that the polynomial \(f_2\) must have at least four terms. In fact, we can choose it to be
\[
f_2 = y^c + z^b - 3xy^m z^{\frac{k \cdot m}{2}} + x^3 y^{\frac{c \cdot m}{2}} z^{\frac{k \cdot m}{2}}.
\]

As all exponents are non-negative, we see that in the equation \(2a = bm + nc\) we have \(b \geq 3m\) and \(c \geq 3n\). Thus, we verified the conditions of (ii) in the case \(\mu_1 = 1\) and \(\mu_2 = 2\). If \(\mu_2 = 1\) and \(\mu_1 = 2\) holds, then we may proceed exactly the same way; observe that we only made use of the orthogonality condition [2,3] (ii).

We show that (ii) implies the supplement. First we claim that the ideal \(I \subseteq S\) is generated by the binomials
\[
f_1 = x^2 - y^n z^m, \quad f_2 := x^2 \frac{b \cdot m}{2} - y \frac{b \cdot m}{2},
\]
\[
f_3 := xy \frac{c \cdot m}{2} - z \frac{c \cdot m}{2} = x^{-1}(y \frac{c \cdot m}{2} f_1 - z \frac{c \cdot m}{2}).
\]

Indeed, from [12] Lemma 7.6] we infer that \(I\) equals the saturation \(\langle f_1, f_2 \rangle : \langle xyz \rangle\). Now, \(f_3\) lies in the saturation and \(\langle f_1, f_2, f_3 \rangle\) is prime, which gives the claim. Observe that we have
\[
f_4 := xy \frac{c \cdot m}{2} - z \frac{c \cdot m}{2} f_1 - y \frac{c \cdot m}{2} f_2 - z \frac{c \cdot m}{2} f_3 \in \langle t^2 \rangle : \langle t^\infty \rangle \setminus \langle t^2 \rangle.
\]

We want to show that the Cox ring of \(X\) is generated by the canonical section \(t\) of the exceptional divisor, the pull back sections \(x, y, z\), the sections \(s_i := f_i t^{-1}\) for \(i = 1, 2, 3\) and \(s_4 := f_4 t^{-2}\). This is equivalent to saying that the Cox ring of \(X\) is isomorphic to \(\mathbb{K}[x, y, z, s_1, \ldots, s_4, t]/I_2\), where
\[
I_2 := \langle s_1 t - f_1, s_2 t - f_2 t, s_3 t - f_3, s_4 t^2 - f_4 \rangle : t^\infty.
\]

The localization \((I_2)_t \subseteq \mathbb{K}[x, y, z, s_1, \ldots, s_4, t]_t\) is a prime ideal of dimension four and thus \(I_2\) is a prime ideal of dimension four. Moreover, the ideal \(I_2\) contains the
ideal $I'_2$ generated by the following polynomials
\[
\begin{align*}
&f_1 - s_1 t, \quad f_2 - s_2 t, \quad f_3 - s_3 t, \\
y - s_1 - z^m s_2 - x s_3, & \quad z - s_1 - x s_2 - y^n s_3, \\
xy - z - s_1 - y m s_2 - z - s_1 s_3 - s_4 t, & \quad s_2^2 + c - m s_2 - z^m s_4, \\
& s_2^2 + z - s_1 s_3 - y s_4, \\
y - z m s_2 - s_1 s_2 - x s_3.
\end{align*}
\]

Let $I''_2 \subseteq \mathbb{K}[x, y, z, s_1, \ldots, s_4]$ be the ideal generated by the polynomials obtained from the above ones by setting $t := 0$. Then the first three generators of $I''_2$ are $f_1, f_2, f_3$. We take a look at the zero set $V(I''_2) \subseteq \mathbb{K}^7$. First consider the area $W_0 \subseteq V(I''_2)$ cut out by $xyz = 0$. By the nature of $f_1, f_2, f_3$, each of $x, y, z$ vanishes identically on $V(I''_2)$ and we see that $W_0 = V(x, y, z, s_2, s_3)$ is of dimension two. Now consider the set of points $W_1 \subseteq V(I''_2)$ satisfying $xyz \neq 0$. We have a finite surjection
\[
\mathbb{K}^* \times \mathbb{K}^4 \to V(f_1, f_2, f_3), \quad (\xi, s_1, s_2, s_3, s_4) \mapsto (\xi^a, \xi^b, \xi^c, s_1, s_2, s_3, s_4).
\]
The image contains $W_1$ and the pullback of the generators number 4, 5 and 6 of $I''_2$ are multiples of
\[
\xi^b - s_1 - \xi^c m a s_2 - s_3 \in \mathbb{K}[\xi^a, s_1, s_2, s_3, s_4].
\]
Now, we eliminate $s_3$ by means of this relation and see that turns the pullbacks of the remaining three generators of $I''_2$ are multiples of a common polynomial, depending on $s_4$. We conclude that $W_1$ is of dimension three. Altogether, we verified that $I''_2 + \langle t \rangle$ has a three-dimensional zero set and $I''_2 + \langle t, xyz \rangle$ a two-dimensional one. Thus, we can apply Proposition 5.1 to see that the Cox ring of $X$ is as claimed.

To conclude the whole proof, it suffices to show that the supplement implies (i). But this is obvious. \hfill \Box

**Remark 5.2.** Observe that in the proof of Theorem 1.2 the fourth and fifth generators of the ideal $I'_2$ come from the following syzygies of the lattice ideal $\langle f_1, f_2, f_3 \rangle$ as found by the methods from 12 Chap. 9:

Corollary 5.3. Consider a triple $(3, b, c)$ such that none of the entries lies in the monoid generated by the other two. Then the Cox ring of $X(3, b, c)$ is as in Theorem 1.2.

**Proof.** It suffices to show that $(3, b, c)$ satisfies condition (ii). To see this, observe that if $b < c$ then also $c < 2b$ holds; otherwise, $c$ would be in the semigroup $(3, b)$ as it is bigger that the Frobenius number $2(b - 1) + 1$ of the semigroup. Also observe that the equation $b + c \equiv 0 \pmod{3}$ must hold, since otherwise $c$ would belong to the semigroup $(3, b)$. We deduce that there exists a positive integer $n$ such that $2b = 3n + c$. Moreover, from $c + 3n = 2b < 2c$, we deduce $c > 3n$. \hfill \Box
6. Algorithms and applications

Our first algorithm applies to blow ups of arbitrary Mori dream spaces. We work in the setting of Proposition 5.4. Based on the criterion given there, we are able to avoid the (involved) computation of saturations performed in the related [10] Algorithm 5.6.

Algorithm 6.1 (Verify generators). Input: homogeneous generators \( g_1, \ldots, g_m \) for the Cox ring \( R_1 \) of a Mori dream space \( X_1 \) and homogeneous generators \( f_1, \ldots, f_k \in R_1 \) of the ideal \( I \) of an irreducible subvariety \( C \subseteq X_1 \) contained in the smooth locus.

- For each \( f_i \), compute the maximal \( m_i \in\mathbb{Z}_{\geq 0} \) with \( f_i \in I^{m_i} : J^\infty \).
- Let \( f \) be the product of all the generators \( g_i \) which do not vanish along \( C \).
- Set \( \mathcal{B} := \{l^{m_i} s_i - f_i; i = 1, \ldots, k \} \subseteq R_1[s_1, \ldots, s_k, t] \).
- Repeat
  - if \( \mathcal{B}' := (\mathcal{B}) : t^\infty \setminus (\mathcal{B}) \) is nonempty, enlarge \( \mathcal{B} \) by an element of \( \mathcal{B}' \),
  - if \( \dim(R_1) = \dim((\mathcal{B} \cup \{t\})) \) and \( \dim((\mathcal{B} \cup \{t\})) > \dim((\mathcal{B} \cup \{t, f\})) \)
    then return true.
- until \( \mathcal{B} : t^\infty = (\mathcal{B}) \).
- Return false.

Output: true is returned if and only if the Cox ring \( R_2 \) of the blow-up \( X_2 \) of \( X_1 \) along \( C \) is generated by \( t \) and \( f_1 t^{-m_1}, \ldots, f_k t^{-m_k} \) as an \( R_1 \)-algebra.

Proof. If the algorithm returns “true” that Proposition 5.4 guarantees that \( R_2 \) is generated by \( t \) and \( f_i t^{-m_i}, \ldots, f_k t^{-m_k} \) as an \( R_1 \)-algebra. For \( f \) is generated by \( t \) and \( f_1 t^{-m_1}, \ldots, f_k t^{-m_k} \) as an \( R_1 \)-algebra. Then the list of all \( g_j, f_1 t^{-m_1}, \ldots, f_k t^{-m_k} \) comprises a system of pairwise \( \text{Cl}(X) \)-coprime generators for \( R_2 \) and thus, the dimension conditions are fulfilled if \( \langle \mathcal{B} \rangle \) equals the defining ideal of \( R_2 \) which in turn is given as \( \langle \mathcal{B} \rangle : t^\infty \). Consequently, the algorithm returns true. \( \square \)

Remark 6.2. In the fifth line of Algorithm 6.1 as in Remark 5.2, elements of \( \mathcal{B}' \) can be obtained by determining syzygies among (products of) the \( f_i \).

The next algorithm implements Proposition 2.4 and provides a Mori dreamness test in our concrete setting, i.e., the blow-up \( X = X(a, b, c) \) of the point \([1, 1, 1] \in \mathbb{P}(a, b, c) \). As before, \( I \subseteq S \) is the ideal of \([1, 1, 1] \) in the Cox ring \( S = \mathbb{K}[x, y, z] \) of \( \mathbb{P}(a, b, c) \).

Algorithm 6.3 (Mori dreamness test). Input: pairwise coprime positive integers \((a, b, c)\).

- Compute a system \( \mathcal{B} \) of homogeneous generators of the ideal \( I \subseteq S = \mathbb{K}[x, y, z] \) of \([1, 1, 1] \in \mathbb{P}(a, b, c) \).
- For \( m = 2, 3, \ldots \) do
  - Compute the normal form of a basis of \( A_m := I^m : J^\infty \) with respect to \( A_{<m} := A_1 A_{m-1} + \cdots + A_m A_{<m} \), select the elements of minimal degree and add them to \( \mathcal{B} \).
  - If \( \mathcal{B} \) contains an orthogonal pair \( f_1, f_2 \in S \) as in Definition 2.3 then return true.

Output: true; this is returned if and only if the algorithm terminates and in this case, \( X(a, b, c) \) is a Mori dream surface.

Proof. If the algorithm terminates, then it returns “true” and thus there is an orthogonal pair in \( S \). Proposition 2.4 then yields that \( X(a, b, c) \) is a Mori dream surface. If \( X(a, b, c) \) is a Mori dream surface, then \( \mathcal{B} \) will give rise to a system of homogeneous generators for the Cox ring at some point and Remark 2.5 ensures that there is an orthogonal pair in \( \mathcal{B} \).

Finally, we discuss the applications to the investigation of the Mori dream space property for \( \overline{M}_{0,n} \). Recall the following from [4] and [8] Theorem 4.1.
Method 6.4 (Castravet/Tevelev). Given \( n \in \mathbb{Z}_{\geq 0} \), let \( N' \subseteq N \) be a saturated sublattice of \( N := \mathbb{Z}^{n-3} \) of rank \( n - 5 \) generated by subsets of \( M := \{ \pm p; p \in \{0, 1\}^{n-3}\} \subseteq N \) such that for the quotient map \( \pi: N \to N' \), there are \( v_1, v_2, v_3 \in M \) with
\[
\langle \pi(v_1), \pi(v_2), \pi(v_3) \rangle = N/N'.
\]
Further assume that there are pairwise coprime positive integers \( a, b, c \) with \( av_1 + bv_2 + cv_3 \equiv 0 \pmod{N'} \). If the blow up \( X(a, b, c) \) of \( \mathbb{P}(a, b, c) \) in the point \( [1, 1, 1] \) is not a Mori dream space, then also \( \overline{M}_{0,n} \) is not a Mori dream space.

Proof of Addendum 14. There are proper surjective morphisms \( \overline{M}_{0,n} \to \overline{M}_{0,n-1} \). Consequently, if \( \overline{M}_{0,n} \) is a Mori dream space, then also \( \overline{M}_{0,n-1} \) is one. Thus, it suffices to show that \( \overline{M}_{0,10} \) is not a Mori dream space.

The defining fan of the Losev-Manin space \( \mathcal{L}_{10} \) lives in \( N := \mathbb{Z}^7 \) and its rays are the cones generated by the vectors having either all their coordinates in \( \{0, 1\} \) or in \( \{0, -1\} \). Consider the linear map \( \mathbb{Z}^7 \to \mathbb{Z}^2 \) given by the matrix
\[
P := \begin{bmatrix} 1 & 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & -1 & -3 & -2 & 2 & 1 \end{bmatrix}
\]
A \( \mathbb{Z} \)-basis for the kernel \( N' \subseteq N \) of \( P \) is given by the following five primitive generators of the fan of \( \mathcal{L}_{10} \):
\[
e_1 + e_2 + e_4 + e_6, \quad e_1 + e_2 + e_5 + e_7, \quad -(e_1 + e_4 + e_6 + e_7),
\]
\[
e_5 + e_6, \quad -(e_2 + e_3 + e_4 + e_6 + e_7).
\]
Moreover, the primitive generators \(-(e_4 + e_5), -(e_1 + e_3 + e_6)\) and \(e_1 + e_3 + e_4 + e_5\) are mapped to the columns of
\[
\begin{bmatrix} 3 & -3 & -1 \\ 5 & -1 & -6 \end{bmatrix}
\]
which in turn generate the fan of \( \mathbb{P}(17, 13, 12) \). In particular, we have a rational toric morphism from \( \mathcal{L}_{10} \) to \( \mathbb{P}(17, 13, 12) \). By [8, Theorem 1.5], the surface \( X(17, 13, 12) \) is not Mori dream. Thus, Method 6.4 gives the assertion.

Remark 6.5. Method 6.3 fails for \( \overline{M}_{0,n} \), where \( n = 7, 8, 9 \). In these cases, for all possible projections \( \pi \) and the possible associated \( X(a, b, c) \), Algorithm 6.3 is feasible and shows that the \( X(a, b, c) \) are Mori dream surfaces. So, it remains open whether \( \overline{M}_{0,n} \) is a Mori dream space for \( n = 7, 8, 9 \).

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