On Minimal Pseudo-Codewords of Tanner Graphs from Projective Planes

Pascal O. Vontobel  Roxana Smarandache
Dept. of ECE  Dept. of Mathematics
University of Wisconsin  University of Notre Dame
Madison, WI 53706, USA  Notre Dame, IN 46556, USA
vontobel@ece.wisc.edu  rsmarand@nd.edu

Abstract

We would like to better understand the fundamental cone of Tanner graphs derived from finite projective planes. Towards this goal, we discuss bounds on the AWGNC and BSC pseudo-weight of minimal pseudo-codewords of such Tanner graphs, on one hand, and study the structure of minimal pseudo-codewords, on the other.

1 Introduction

In this paper we focus solely on certain families of codes based on finite projective planes. More precisely, the codes under investigation are the families of codes that were called type-I PG-LDPC codes in [1, 2], see also [3]. They are defined as follows. Let $q \triangleq 2^s$ for some positive integer $s$ and consider a (finite) projective plane $PG(2, q)$ (see e.g. [4, 5]) with $q^2 + q + 1$ points and $q^2 + q + 1$ lines: each point lies on $q + 1$ lines and each line contains $q + 1$ points. A standard way of associating a parity-check matrix $H$ of a binary linear code to a finite geometry is to let the set of points correspond to the columns of $H$, to let the set of lines correspond to the rows of $H$, and finally to define the entries of $H$ according to the incidence structure of the finite geometry. In this way, we can associate to the projective plane $PG(2, q)$ the code $C_{PG(2, q)}$ with parity-check matrix $H \triangleq H_{PG(2, q)}$.

It turns out that this code has block length $n = q^2 + q + 1$, dimension $n - 3^s - 1$, and minimum Hamming distance $q + 2$. The parity-check matrix $H_{PG(2, q)}$ has size $n \times n$ and it has uniform column weight $w_{col} = q + 1$ and uniform row weight $w_{row} = q + 1$. Moreover, this code has the nice property that with an appropriate ordering of the columns and rows, the parity-check matrix is a circulant matrix, meaning that $C_{PG(2, q)}$ is a cyclic code. This fact can e.g. be used for efficient encoding. Such symmetries can also substantially...
simplify the analysis. Note that the automorphism group of $C_{\text{PG}(2,q)}$ contains many more automorphisms besides the cyclic-shift-automorphism implied by the cyclicity of the code.

In this paper we continue the investigations started in [3] related to these codes. Our goal is to improve our knowledge about the fundamental cone [6, 7] of the parity-check matrix $H_{\text{PG}(2,q)}$, as a better understanding of this fundamental cone yields a better understanding of linear programming (LP) decoding [7] of this code. Moreover, the connection made by Koetter and Vontobel [6, 8] between iterative decoding and LP decoding suggests that results for LP decoding have immediate implications for iterative decoding. We will use the same notations and definitions of [3] that we briefly review here. We let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ be the set of real numbers, the set of non-negative real numbers, and the set of positive real numbers, respectively.

**Definition 1 ([6, 7])**. Let $C$ be an arbitrary binary linear code that is described by a parity-check matrix $H$ of size $m \times n$. We let $J \triangleq J(H) \triangleq \{1, \ldots, m\}$ and $I \triangleq I(H) \triangleq \{1,2,\ldots,n\}$ be the set of row and column indices of $H$, respectively. For each $j \in J$, we let $I_j \triangleq I_j(H) \triangleq \{i \in I \mid h_{ji} = 1\}$ and for each $i \in I$ we let $J_i \triangleq J_i(H) \triangleq \{j \in J \mid h_{ji} = 1\}$. We define the fundamental cone $K(H)$ of $H$ to be the set of vectors $\omega \in \mathbb{R}^n$ that satisfy

$$\forall j \in J, \forall i \in I_j : \sum_{i' \in I_j \setminus \{i\}} \omega_{i'} \geq \omega_i \quad \text{and} \quad \forall i \in I : \omega_i \geq 0. \quad (1)$$

Vectors in the fundamental cone will be called pseudo-codewords. Note that two pseudo-codewords that are equal up to a positive scaling constant will be considered to be equivalent. The edges of the fundamental cone will be called minimal pseudo-codewords. It can be shown that all minimal pseudo-codewords stem from valid configurations in covers of the base Tanner graph, and that minimal pseudo-codewords that are unnormalized [9] are equal (modulo 2) to some codewords of the code $C$. $\square$

Note that the fundamental cone is a function of the parity-check matrix representing a code. Because of the equivalence of parity-check matrix and Tanner graph, the fundamental cone can also be seen as a function of the Tanner graph representing a code. Therefore, in order to emphasize the dependence of minimal pseudo-codewords on the representation of the code, we will talk about the minimal pseudo-codewords of a Tanner graph.

Note also that the fundamental cone is independent of the specific memoryless binary-input channel through which we are transmitting; however, the influence of a pseudo-codeword on the LP decoding behavior is measured by a channel-dependent pseudo-weight. For the binary-input additive white Gaussian noise channel, the AWGNC-pseudo-weight turns out to be $w_p(\omega) \triangleq w_p^{\text{AWGNC}}(\omega) \triangleq \frac{\|\omega\|^2}{\|\omega\|^2}$ if $\omega \in \mathbb{R}_+^n \setminus \{0\}$ and $w_p(\omega) \triangleq w_p^{\text{AWGNC}}(\omega) \triangleq 0$ if $\omega = 0$ [10, 11, 6]; the formula for the binary symmetric channel (BSC) pseudo-weight $w_p^{\text{BSC}}(\omega)$ can be found in [11]. Finally, for the binary erasure channel, the BEC-pseudo-weight is $w_p^{\text{BEC}}(\omega) \triangleq |\text{supp}(\omega)|$ [11].

Let $w_p^{\min}(H)$ be the minimum AWGNC pseudo-weight of a parity-check matrix $H$. One can show that $w_p^{\min}(H_{\text{PG}(2,q)}) \geq q+2$ (e.g. using Th. 1 in [12]) and because this lower bound matches the minimum Hamming weight, we actually know that $w_p^{\min}(H_{\text{PG}(2,q)}) = q+2$. Similarly, one can show that $w_p^{\text{BSC},\min}(H_{\text{PG}(2,q)}) = q+2$, and that $w_p^{\text{BEC},\min}(H_{\text{PG}(2,q)}) = q+2$.

\footnote{Because of space reasons we omit the rather lengthy definition of $w_p^{\text{BSC}}(\omega)$; however, in Sec. 8 we will discuss some of the consequences of the $w_p^{\text{BSC}}(\omega)$ definition.}
Example 2. Consider the parity-check matrix $H_{\text{PG}(2,q)}$ for $q = 4$ and its associated Tanner graph. Fig. 11 shows the histograms of the AWGNC, BSC, and BEC pseudo-weight of minimal pseudo-codewords of this Tanner graph.

Without going into any details, it is apparent from Fig. 11 that the influence of minimal pseudo-codewords can vary depending on the channel that is used. (For related observations about varying influences of minimal pseudo-codewords, see the discussion in [13].)

It is well-known that the support set of any pseudo-codeword is a stopping set [14] and that for any stopping set there exists a pseudo-codeword whose support set equals that stopping set. Therefore, the BEC pseudo-weight of a pseudo-codeword equals the size of a certain stopping set and so the work by Kashyap and Vardy [15] on (minimal) stopping sets for finite-geometry-based codes has implications for our setup, in particular when studying the BEC pseudo-weight.

**Definition 3.** Let $\omega \in \mathbb{R}_+^n$. We call $t \triangleq t(\omega) = (t_\ell(\omega))_{\ell \in \mathbb{R}_+}$ the type of $\omega$, where $t_\ell \triangleq t_\ell(\omega)$ is the number of components of the vector $\omega$ that are equal to $\ell$. (Note that in the following we do not assume that $\ell$ is a non-negative integer, only that it is a non-negative real number.)

It follows from this definition that only finitely many $t_\ell$’s are non-zero and that $\sum \ell \ t_\ell = |I| = n$ for any $\omega \in \mathbb{R}_+^n$. Moreover, because $\|\omega\|_1 = \sum \ell \ t_\ell$, $\|\omega\|_2^2 = \sum \ell^2 t_\ell$, and $|\text{supp}(\omega)| = \sum_{\ell>0} t_\ell$ we have

$$w_p(\omega) = \frac{(\sum \ell t_\ell)^2}{\sum \ell^2 t_\ell} \quad \text{and} \quad w_{p\text{BEC}}(\omega) = \sum_{\ell>0} t_\ell.$$

If $\tilde{\omega} = \alpha \cdot \omega$ for some $\alpha \in \mathbb{R}_+$ then its type $\tilde{t} \triangleq \tilde{t}(\omega)$ is such that $\tilde{t}_\ell = t_\ell$ for all $\ell$.

The rest of this paper is structured as follows. Whereas in Sec. 2 we will discuss bounds on the AWGNC pseudo-weight, in Sec. 3 we will investigate the so-called effectiveness of minimal pseudo-codewords, and in Sec. 4 we will study the structure of minimal pseudo-codewords. Finally, in Sec. 5 we offer some conclusions.

## 2 Bounds on the AWGNC Pseudo-Weight

In this section we present some bounds on the AWGNC pseudo-weight, in particular we present bounds that depend only on the type of a pseudo-codeword.

**Lemma 4.** Let $\omega \in \mathbb{R}_+^n$ be a vector. If its type $t = t(\omega)$ is such that only $t_0$, $t_1$, and $t_2$ are non-zero, then

$$w_p(\omega) \geq \max \left\{ \frac{15}{16} t_1 + \frac{12}{16} t_2, \frac{3}{4} t_1 + t_2 \right\}.$$

**Proof:** Using the well-known bound $\sqrt{\frac{1}{16} t_1 t_2} \leq \frac{t_1 + t_2}{2}$, i.e. $\frac{t_1 t_2}{t_1 + 4t_2} \leq \frac{1}{16} (t_1 + 4t_2)$, we obtain $w_p(\omega) = \frac{(t_1 + 2t_2)^2}{t_1 + 4t_2} = t_1 + t_2 - \frac{t_1 t_2}{t_1 + 4t_2} \geq t_1 + t_2 - \frac{t_1 t_2}{t_1 + 4t_2} = \frac{15t_1}{16} + \frac{19t_2}{16}$. For the second inequality we have $w_p(\omega) = \frac{(t_1 + 2t_2)^2}{t_1 + 4t_2} = t_2 + \frac{t_1(t_1 + 3t_2)}{t_1 + 4t_2} \geq \frac{3}{4} t_1 + t_2$.  

$\square$
Figure 1: Histograms of the AWGNC, BSC, and BEC pseudo-weight of minimal pseudo-codewords of the PG(2, 4)-based code, see also [3]. (Note that the y-axis is logarithmic.)

Lemma 5. Let $\omega \in \mathbb{R}^n_+$ and let $\eta \neq 0$ be some arbitrary real number. Then

$$w_p(\omega) \geq \frac{2\eta\|\omega\|_1 - \|\omega\|_2^2}{\eta^2} = \sum_{i=1}^{n} \omega_i(2\eta - \omega_i)$$

with equality if and only if $\omega = 0$ or $\eta = \|\omega\|_2^2/\|\omega\|_1$.

Proof: If $\omega = 0$ then the statement is certainly true, so let us assume that $\omega \neq 0$. The square of any real number is non-negative, therefore

$$(\eta\|\omega\|_1 - \|\omega\|_2^2)^2 \geq 0,$$

with equality if and only if $\eta = \|\omega\|_2^2/\|\omega\|_1$. Multiplying out and rearranging we obtain

$$\eta^2\|\omega\|_1^2 \geq 2\eta\|\omega\|_1\|\omega\|_2^2 - \|\omega\|_2^4.$$ 

Finally, dividing by $\eta^2\|\omega\|_2^2$ and using the definition of $w_p(\omega)$, we obtain the desired result. \qed

Corollary 6. Let $\omega \in \mathbb{R}^n_+$, let $t \triangleq t(\omega)$ be the type of $\omega$, and let $\eta \neq 0$ be some arbitrary real number. Then

$$w_p(\omega) \geq \sum_{\ell} \beta_\ell t_\ell \quad \text{with} \quad \beta_\ell = \frac{\ell(2\eta - \ell)}{\eta^2} = 1 - \left(1 - \frac{\ell}{\eta}\right)^2.$$
Proof: The result follows immediately from Th. 5.

Note that choosing \( \eta = 4/3 \) in Cor. 6 yields \( \beta_0 = 0, \beta_1 = 15/16, \) and \( \beta_2 = 12/16, \) and that choosing \( \eta = 2 \) in Cor. 6 yields \( \beta_0 = 0, \beta_1 = 3/4, \) and \( \beta_2 = 1. \) This recovers Lemma 4.

**Corollary 7.** Let \( \omega \in \mathbb{R}_+^n \) and let \( t \triangleq t(\omega). \) Moreover, let \( r \) be the ratio of the largest positive \( \ell \) such that \( t_\ell \) is non-zero and the smallest positive \( \ell \) such that \( t_\ell \) is non-zero. Then we have the lower bound

\[
w_p(\omega) \geq \frac{4r}{(r + 1)^2} \cdot |\text{supp}(\omega)|.
\]

(This bound was also obtained by Wauer [16] using a different derivation.)

**Proof:** Let \( m \) be the largest positive \( \ell \) such that \( t_\ell \) is non-zero and let \( m' \) be the smallest positive \( \ell \) such that \( t_\ell \) is non-zero. These definitions obviously yield \( r = m/m'. \) Consider Cor. 6 with \( \eta = \frac{m + m'}{2}. \) We obtain \( w_p(\omega) \geq \sum_\ell \beta_\ell t_\ell \) with \( \beta_\ell = 4\ell(m + m') - 1 - (1 - \frac{2\ell}{m + m'})^2. \) We observe that \( \beta_m' = \beta_\omega = \frac{2^m m' - 4}{(m + m')^2} = \frac{4r}{(r + 1)^2}. \) Since \( \beta_\ell \) is strictly concave in \( \ell \) we must have \( \beta_\ell > \beta_m' = \beta_\omega = \frac{4r}{(r + 1)^2} \) for all \( m' < \ell < m. \)

Choosing \( \{\beta_\ell'\} \) such that \( \beta_\ell' \leq \beta_m' \) for all \( \ell, \) the above lower bound in (\( \ast \)) can be turned into the lower bound \( w_p(\omega) \geq \sum_\ell \beta_\ell' t_\ell \) because \( t_\ell \geq 0 \) for all \( \ell. \) We choose \( \beta_\ell' \triangleq \frac{4r}{(r + 1)^2} \) for all \( m' \leq \ell \leq m \) and \( \beta_\ell' \triangleq 0 \) otherwise. The observations in the previous paragraph show that these are valid choices and we finish the proof by noting that

\[
w_p(\omega) \geq \sum_\ell \beta_\ell' t_\ell = \sum_\ell \frac{4r}{(r + 1)^2} t_\ell = \frac{4r}{(r + 1)^2} \sum_\ell t_\ell \geq \frac{4r}{(r + 1)^2} \cdot |\text{supp}(\omega)|.
\]

**Theorem 8.** Let \( H \triangleq H_{PG(2,q)} \) and let \( \omega \in \mathcal{K}(H) \) be of type \( t \) with both \( t_0 \) non-negative, \( t_1 \geq q + 2, t_2 \) positive, and \( t_\ell = 0 \) otherwise. Then

\[
w_p(\omega) \geq \frac{4}{3} (q + 2).
\]

**Proof:** For any \( i \in I \) we must have \( \sum_{i' \in I \setminus \{i\}} \omega_{i'} = (q + 1)\omega_i \) where at step (\( \ast \)) we used the fact that all variable nodes are at graph distance two from each other in the Tanner graph associated to \( H, \) and where at step (\( \ast \ast \)) we used the inequalities in (1). Adding \( \omega_i \) to both sides we obtain \( \sum_{i' \in I} \omega_{i'} \geq (q + 2)\omega_i. \) Note, fix an \( i \in I \) for which \( \omega_i = 2 \) holds and express \( \sum_{i' \in I} \omega_{i'} \) in terms of \( t \): it must hold that \( t_1 + 2t_2 \geq 2(q + 2), \) or, equivalently, \( t_2 \geq q + 2 - t_1 / 2. \) For any \( \eta \neq 0 \) we obtain

\[
w_p(\omega) \overset{(\ast)}{\geq} \frac{(2\eta - 1)t_1 + (4\eta - 4)t_2}{\eta^2} \overset{(\ast \ast)}{\geq} \frac{(2\eta - 1)t_1 + (4\eta - 4)(q + 2 - t_1 / 2)}{\eta^2} = \frac{t_1 + (4\eta - 4)(q + 2)}{\eta^2},
\]

\footnote{Let \( C \) be the code defined by \( H. \) If a pseudo-codeword is an unscaled pseudo-codeword \([9]\) then it is equal (modulo 2) to a codeword of \( C. \) Therefore, the number of odd components of an unscaled pseudo-codeword must either be zero or at least equal to the minimum Hamming weight of the code. So, if we actually know that \( \omega \) in the theorem statement is an unscaled pseudo-codeword then the requirement \( t_1 \geq q + 2 \) is equal to the requirement \( t_1 \geq 1. \)}
Figure 2: (a)-(d): Codewords and pseudo-codewords used in Ex. 9. (e): Part of PG(2, 4)

discussed in Ex. 14 where at step (∗) we used Cor. 6 and at step (∗∗) we used the inequality on
\( t_2 \) that we just found above. Using the assumption that \( t_1 \geq q_0 + 2 \) from the theorem statement we
get \( w_p(\omega) \geq \frac{(4r-3)(q+2)}{q^2} \). The right-hand side of this expression is maximized by \( \eta^* = \frac{3}{2} \); inserting this value yields the lower bound in the theorem statement.

A possible goal for future research is to weaken the assumptions about \( t_1 \) in the theorem statement without weakening the lower bound on the AWGNC pseudo-weight of pseudo-codewords that are not (multiples of) codewords: in light of Footnote 3 it would be desirable to prove that the same lower bound holds also if \( \omega \) is an unscaled pseudo-codeword with \( t_1 = 0 \) and which is not a multiple of a codeword.

Note that the above theorem can be generalized to the setup where \( \omega \in \mathcal{K}(H) \) has type \( t \) with \( t_0 \) non-negative, \( t_m \) positive for some integer \( m \geq 2 \), \( t_\ell \) non-negative for \( 1 \leq \ell \leq m-1 \), \( t_\ell = 0 \) for \( \ell \geq m+1 \), and \( \sum_{\text{odd } \ell} t_\ell \geq q+2 \). Then \( w_p(\omega) \geq \frac{m^2}{m^2-m+1}(q+2) \).

Example 9. One can exhibit minimal pseudo-codewords whose AWGNC pseudo-weight matches the leading-term behavior of the lower bound in Th. 8 (when \( q \) grows). Consider first the case \( q = 2 \). The projective plane for \( q = 2 \) is shown in Fig. 2 (a): it has 7 points and 7 lines and we consider the points to be variables and the lines to be checks. Fig. 2 (a and b) shows two codewords of weight \( q+2 = 4 \); note that their supports overlap in \( \frac{q+2}{2} = 2 \) positions. Adding these two codewords together yields the pseudo-codeword shown in Fig. 2 (c). Switching the zero value into a two results in the pseudo-codeword in Fig. 2 (d); it can be checked that this pseudo-codeword is actually a minimal pseudo-codeword. It has AWGNC pseudo-weight 6.25, whereas the lower bound in Th. 8 is 5.33.

Similarly, in the case of \( q = 4 \) it is possible to start with two codewords of weight \( q+2 = 6 \) whose supports overlap in \( \frac{q+2}{2} = 3 \) positions. After adding them and switching two zeros (that are specifically chosen and lie on the same line) into two twos, one gets a minimal pseudo-codeword of AWGNC pseudo-weight 9.85, whereas the lower bound in Th. 8 is 8.00.

In general, we conjecture that for any \( q = 2^s \), where \( s \) is a positive integer, it is possible to construct a minimal pseudo-codeword of type \( t \) with \( t_1 = q+2 \) and \( t_2 = \frac{q}{2} + s + 1 \) and \( t_\ell = 0 \) for \( \ell \notin \{0, 1, 2\} \): take two codewords of weight \( q+2 \) whose supports overlap in \( \frac{q+2}{2} = \frac{q}{2} + 1 \) positions and switch \( s \) zeros (that are specifically chosen) into \( s \) twos. The points corresponding to these \( s \) twos (together with the lines through them) should then form a simplex. These pseudo-codewords have weight

\[
w_p(\omega) = \frac{\|\omega\|_2^2}{\|\omega\|_2^2} = \frac{4}{3} \cdot (q+2) \cdot \frac{1 + f(q)}{1 + \frac{f(q)}{3(1+f(q))}},
\]

where \( f(q) = \frac{\log_2(q)}{q+2} \). (We wrote the last term on the right-hand side such that it is readily apparent that it is not smaller than 1, i.e. the bound in Th. 8 is clearly satisfied.)
3 Effective Minimal Pseudo-Codewords

The BSC can be seen as a binary-input AWGNC where the values at the output are quantized to $+1$ or $-1$. It follows that the components of the log-likelihood vector $\lambda$ can only take on two values, namely $+L$ and $-L$, where $L$ is a positive constant that depends on the bit flipping probability of the BSC. Because of this quantization, there are certain effects that happen for the BSC that cannot happen for the AWGNC. Before continuing, it is worthwhile to recall what the meaning of the BSC pseudo-weight $w_B^{\text{BSC}}(\omega)$ of a pseudo-codeword $\omega$ is: $\lfloor w_B^{\text{BSC}}(\omega)/2 \rfloor$ is the minimum number of bit flips required (upon sending the zero codeword) to make a decoding error to $\omega$; moreover, these bit flips must happen at appropriate positions.

**Definition 10.** Fix a memoryless binary-input channel and let $L^{(n)} \subseteq (\mathbb{R} \cup \{\pm \infty\})^n$ be the set of all possible log-likelihood ratio vectors. Moreover, let us fix a parity-check matrix $H$ and let $M_p(K(H))$ be the set of minimal pseudo-codewords. A minimal pseudo-codeword $\omega \in M_p(K(H))$ is called *effective of the first kind* for that particular channel if there exists a $\lambda \in L^{(n)}$ such that $\langle \omega, \lambda \rangle < 0$ and $\langle \omega', \lambda \rangle \geq 0$ for all $\omega' \in M_p(K(H)) \setminus \{\omega\}$. A minimal pseudo-codeword $\omega \in M_p(K(H))$ is called *effective of the second kind* for that particular channel if there exists a $\lambda \in L^{(n)}$ such that $\langle \omega, \lambda \rangle \leq 0$ and $\langle \omega', \lambda \rangle \geq 0$ for all $\omega' \in M_p(K(H)) \setminus \{\omega\}$. (Obviously, a minimal pseudo-codeword that is effective of the first kind is also effective of the second kind.)

Let $L_0^{(n)} \subseteq L^{(n)}$ be the set where LP decoding decides in favor of the codeword 0. From the above definition it follows that a minimal pseudo-codeword shapes the set $L_0^{(n)}$ if and only if it is an effective minimal pseudo-codeword. More precisely, in the case where a minimal pseudo-codeword $\omega$ is effective of the first kind then there exists at least one $\lambda \in L^{(n)}$ where $\omega$ wins against all other minimal pseudo-codewords (and the zero codeword). However, in the case where $\omega$ is effective of the second kind we are guaranteed that there is at least one $\lambda \in L^{(n)}$ were $\omega$ is involved in a tie; if and how often $\omega$ wins against all other minimal pseudo-codewords (and the zero codeword) depends on how ties are resolved.

**Theorem 11.** For the binary-input AWGNC and any parity-check matrix $H$ all minimal pseudo-codewords of $K(H)$ are effective of the first kind.

*Proof:* This follows from some simple geometric considerations.

As the following observations show, for channels other than the AWGNC not all minimal pseudo-codeword need to be effective of the first or second kind.

**Theorem 12.** Consider data transmission over a BSC using the code defined by $H \triangleq H_{\text{PG}(2,q)}$. LP decoding can correct any pattern of $\frac{q}{2}$ bit flips and no pattern of more than $q$ bit flips.

*Proof:* It can be shown that the BSC pseudo-weight of any pseudo-codeword in $K(H)$ is at least $q + 2$. Therefore LP decoding can correct at least $\lfloor \frac{q+2-1}{2} \rfloor = \frac{q}{2}$ bit flips.

Let us now show that LP decoding can correct at most $q$ bit flips. Remember that a necessary condition for LP decoding to decode a received log-likelihood vector $\lambda$ to the

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4For the AWGNC we have $L^{(n)} = \mathbb{R}^n$, for the BSC we have $L^{(n)} = \{\pm L\}^n$ for some $L \geq 0$, and for the BEC we have $L^{(n)} = \{-\infty, 0, +\infty\}^n$. 

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zero codeword is that $\langle \omega, \lambda \rangle \geq 0$ for all $\omega \in \mathcal{K}(H)$. Assume that we are transmitting the zero codeword and that $e$ bit flips happened. Hence $e$ components of $\lambda$ are equal to $-L$ and $n-e$ components of $\lambda$ are equal $+L$. It can easily be checked that the following $\omega$ is in $\mathcal{K}(H)$: let $\omega_i \triangleq 1$ if $\lambda_i = -L$ and $\omega_i \triangleq 1/q$ otherwise. For this $\omega$, the condition $\langle \omega, \lambda \rangle \geq 0$ translates into $e(-L) + (n-e)(1/q)(+L) \geq 0$, i.e. $e \leq \frac{n}{q+1} = \frac{q^2 + q + 1}{q+1} = q + \frac{1}{q+1}$. Rounding down we obtain $|e| = \lfloor q + \frac{1}{q+1} \rfloor = q$.

**Corollary 13.** Consider the code defined by $H \triangleq H_{PG(2,q)}$. For the BSC, a necessary condition for a minimal pseudo-codeword $\omega$ of $\mathcal{K}(H)$ to be effective of the second kind is that $q + 2 \leq w_{p}^{BSC}(\omega) \leq 2q + 2$.

For $q = 4$ it turns out that $\mathcal{K}(H_{PG(2,4)})$ has minimal pseudo-codewords with BSC pseudo-weight equal to 12. (These minimal pseudo-codewords have type $t$ with $t_2 = 1$, $t_1 = 12$, $t_0 = 8$, and $t_1 = 0$ otherwise.) Cor. clearly shows that these cannot be effective of the second kind for the BSC, since, for $q = 4$, any effective minimal pseudo-codeword of the second kind must fulfill $6 \leq w_{p}^{BSC}(\omega) \leq 10$.

Judging from Fig. it also seems — as far as AWGNC and BSC pseudo-weight are comparable — that soft information is quite helpful for the LP decoder when decoding the code $C_{PG(2,4)}$ defined by $H_{PG(2,4)}$.

One can also make interesting statements about the effectiveness of minimal pseudo-codewords for the BEC; however, we postpone this discussion to a longer version of the present paper.

4 The Structure of Minimal Pseudo-Codewords

In this section we discuss the geometry of minimal pseudo-codewords. The minimum weight of $C_{PG(2,q)}$, $q$ a prime power, is $q + 2$ and codewords that achieve this minimum weight correspond to point-line configurations in the projective plane that have been studied by several authors. Let us introduce some notation and results from finite geometries, cf. e.g. [4]. A $k$-arc in $PG(2, q)$ is a set of $k$ points no three of which are collinear. A $k$-arc is complete if it is not contained in a $(k+1)$-arc. The maximum number of points that a $k$-arc can have is denoted by $m(2, q)$, and a $k$-arc with this number of points is called an oval (in the case where $q$ is even this is sometimes also called a hyper-oval). One can show that $m(2, q) = q + 2$ for $q$ even and $m(2, q) = q + 1$ for $q$ odd. One can make the following two interesting observations for the case $q$ even. Firstly, if two ovals have more than half their points in common, then these two ovals coincide. Secondly, if a $q$-arc is contained in an oval then the number of such ovals is one if $q > 2$ and two if $q = 2$.

It turns out that in the case $q$ even, the codewords with minimal weight are $q + 2$-arcs and therefore ovals. However, whereas the classification of ovals for odd $q$ is simple (they all correspond to conics), the ovals for even $q$ are not classified that easily. For even $q$, one says that an oval is regular if it comprises the points of a conic and its nucleus; one can show that for $q = 2^s$, irregular ovals exist for $s = 5$ and $s \geq 7$. It turns out

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5Note that this is usually not a sufficient condition for correct decoding, e.g. in the case where ties are resolved randomly.

6This can be seen as a generalization of the so-called canonical completion [6], however instead of assigning values according to the graph distance with respect to a single node, we assign values according to the graph distance with respect to the set of nodes where $\lambda_i$ is negative. Note that special property of the Tanner graph of $H$: all variable nodes are at graph distance 2 from each other.
that the classification for irregular ovals is highly non-trivial. So, given that even the classification of the codewords of minimal weight is difficult, it is probably hopeless to obtain a complete classification of the minimal codewords and minimal pseudo-codewords of codes defined by $H_{PG(2,q)}$, however it is an interesting goal to try to understand as much as possible about the structure of these codewords and pseudo-codewords.

From now on, $q$ will always be even, i.e. a power of two. Before we state our conjecture about the structure of minimal pseudo-codewords, let us first look at an example.

**Example 14.** Let $q = 4$. Then we can find a minimal pseudo-codeword $\omega$ whose type $t$ is $t_0 = 8$, $t_1 = 8$, $t_2 = 5$, and $t_\ell = 0$ otherwise. This pseudo-codeword can be obtained using a procedure similar to the one used in Ex. 9. Firstly, one has to add two vectors $x^{(1)}$ and $x^{(2)}$ of weight 6 whose supports overlap in two positions. This yields a pseudo-codeword $\tilde{\omega}$ of type $\tilde{t}$ with $\tilde{t}_0 = 11$, $\tilde{t}_1 = 8$, $\tilde{t}_2 = 2$, and $\tilde{t}_\ell = 0$ otherwise. Secondly, one has to switch three zeros (that were appropriately chosen) into three twos.

Let us analyze this procedure. Since a minimal pseudo-codeword corresponds to an edge of the fundamental cone, it is clear that the inequalities in (1) that are fulfilled with equality must form a system of linear equations whose rank is $21 - 1 = 20$. We start with two minimal codewords $x^{(1)}$ and $x^{(2)}$ that each yield a system of linear equations whose rank is $21 - 1 = 20$. These two codewords have been chosen such that their sum $\tilde{\omega}$ yields a system of linear equations whose rank is $21 - 2 = 19$.

To find the three zeros that we have to switch, we proceed as follows. It turns out that in the projective plane $PG(2,4)$ there are two lines, say $L_1$ and $L_2$, such that all the entries of $\tilde{\omega}$ that correspond to the points on these two lines are zero. Let $P_0$ be the intersection point of these two lines, cf. Fig. 2 (e). There exists a point $P_1$ on $L_1$ and a point $P_2$ on $L_2$ such that modifying $\tilde{\omega}$ by assigning them the same value $\alpha \geq 0$ yields a vector in the fundamental cone, as long as $\alpha$ is not too large. In fact, for $\alpha > 2$ the vector is outside the fundamental cone, and for $\alpha = 2$ it yields a vector that is a pseudo-codeword and that yields a system of equations of rank $21 - 1 = 20$, i.e. it is a minimal pseudo-codeword. \hfill \square

**Conjecture 15.** For the Tanner graph defined by $H_{PG(2,q)}$ every minimal pseudo-codeword is a sum of a few minimal pseudo-codewords with a change of one or two low-value components such that they become the large components in the equations associated to the lines that pass through them.

Hence, to find minimal pseudo-codewords, we have to take sums of two minimal pseudo-codewords that give rank $n - 2$ (if possible, lower otherwise) and change one component that is not significant into a significant one. We call a component significant if it is the sum of the other components that belong to a line passing through the point, for most of such lines.

Answering positively the following conjecture would result in a much better understanding of the minimal pseudo-codewords in general and of the so-called AWGNC pseudo-weight spectrum gap [3], in particular.

**Conjecture 16.** Let $H \triangleq H_{PG(2,q)}$ and consider the pseudo-codewords that have minimal AWGNC pseudo-weight among all minimal pseudo-codewords that are not multiples of minimal codewords. We conjecture that the type $t$ of these pseudo-codewords is such that $t_0$ is non-negative, $t_1$ is positive, $t_2$ is positive, and $t_\ell = 0$ otherwise. (If this conjecture is not true, find the the smallest $\hat{\ell}$ such that these pseudo-codewords have type $t$ with $t_\ell \geq 0$ for $\ell \in \{0, 1, \ldots, \hat{\ell}\}$ and $t_\ell = 0$ otherwise.)
5 Conclusions

In this paper we have gathered some new facts about minimal pseudo-codewords of codes derived from finite projective planes. We have obtained a clearer picture about the structure of these minimal pseudo-codewords, nevertheless more work is required to get a sufficiently tight characterization of them. Interestingly, in Sec. 3 we were able to use the canonical completion, a tool that so far has been very useful for characterizing families of \((j, k)\)-regular LDPC codes, with \(j, k\) bounded when the block length goes to infinity, i.e. code families where the Tanner graph diameter grows with the block length.

In addition, because the AWGNC pseudo-weight spectrum gap seems to be large for the codes considered in this paper, reflecting the fact that LP decoding performs closely to ML decoding, LP decoding might be an interesting starting point for obtaining a complete decoder for these codes, i.e. a decoder that finds the optimal codeword (or near-optimal codeword) with high probability when \(\lambda\) is drawn according to the Gaussian distribution \(\mathcal{N}(0, \sigma^2)\), for some \(\sigma^2\).

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