Diffeomorphism invariant Colombeau algebras.
Part I: Local theory

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Abstract

This contribution is the first in a series of three: it reports on the construction of (a fine sheaf of) diffeomorphism invariant Colombeau algebras $\mathcal{G}^d(\Omega) = E^M(\Omega)/N(\Omega)$ on open sets of Euclidean space ([Gro01]), which completes earlier approaches ([Col94, Jel99]). Part II and III will show, among others, the way to an intrinsic definition of Colombeau algebras on manifolds which, locally, reproduces the algebra(s) $\mathcal{G}^d(\Omega)$.

Key words. Algebras of generalized functions, Colombeau algebras, diffeomorphism invariance

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1 Introduction

Since its introduction in [Col84] the question of the functor property of Colombeau’s construction was at hand as a crucial one: let $\mu : \tilde{\Omega} \to \Omega$ be a diffeomorphism between open subsets of $\mathbb{R}^s$, is it possible to extend the operation $\mu^* : f \mapsto f \circ \mu$ on smooth distributions on $\Omega$ to an operation $\hat{\mu}$ on the Colombeau algebra such that $(\mu \circ \nu)^* = \hat{\nu} \circ \hat{\mu}$ and $(\text{id})^* = \text{id}$ are satisfied or—to phrase it differently—is it possible to achieve a diffeomorphism invariant construction of full Colombeau algebras? (In this work we shall focus on full algebras distinguished by the existence of a canonical embedding of distributions and henceforth omit the term “full”. By contrast the so called special (or simplified) algebras with their elements basically depending on a real regularization parameter $\varepsilon \in (0, 1]$ do not allow for a canonical embedding of distributions. However, by their relative ease of construction and the fact that diffeomorphism
invariance of the basic definitions is automatically satisfied they provide a flexible tool to model singularities in an nonlinear context: global analysis in this setting has been investigated in [Kun01].

To begin with we briefly recall Colombeau’s construction as given in [Col85] and shall refer to the corresponding algebra as the “elementary” one. Let $\Omega \subseteq \mathbb{R}^s$ and for $\varphi \in \mathcal{D}(\mathbb{R}^s)$ set $\varphi_\varepsilon(x) = (1/\varepsilon)^n \varphi(x/\varepsilon)$. Then we define:

\[
\mathcal{A}_0(\mathbb{R}^s) := \{ \varphi \in \mathcal{D}(\mathbb{R}^s) : \int \varphi(x) \, dx = 1 \}
\]
\[
\mathcal{A}_q(\mathbb{R}^s) := \{ \varphi \in \mathcal{A}_0(\mathbb{R}^s) : \int \varphi(x) x^\alpha \, dx = 0, \, 1 \leq |\alpha| \leq q \} \quad (q \in \mathbb{N})
\]
\[
U(\Omega) := \{(\varphi, x) \in \mathcal{A}_0(\mathbb{R}^s) \times \Omega : \text{supp}(\varphi) \subseteq \Omega - x\}
\]
\[
\mathcal{E}^e(\Omega) := \{ R : U(\Omega) \to \mathbb{C} : \forall (\varphi, x) \in U(\Omega) \text{ the map } y \mapsto R(\varphi, y) \text{ is smooth in a neighborhood of } x \}\n\]
\[
\mathcal{E}^e_M(\Omega) := \{ R \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega \forall \alpha \in \mathbb{N}_0^s \exists N \in \mathbb{N}_0 \forall \varphi \in \mathcal{A}_p(\mathbb{R}^s) : \sup_{x \in K} |\partial_\alpha R(\varphi_x, x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0 \}
\]
\[
\mathcal{N}^e(\Omega) := \{ R \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega \forall \alpha \in \mathbb{N}_0^s \forall n \in \mathbb{N}_0 \exists q \in \mathbb{N}_0 \forall \varphi \in \mathcal{A}_q(\mathbb{R}^s) : \sup_{x \in K} |\partial_\alpha R(\varphi_x, x)| = O(\varepsilon^n) \text{ as } \varepsilon \to 0 \}.
\]

The “elementary” Colombeau algebra of generalized functions on $\Omega$ finally is defined as the quotient space

\[
\mathcal{G}^e(\Omega) := \mathcal{E}^e_M(\Omega) / \mathcal{N}^e(\Omega).
\]

\(\mathcal{G}^e(\Omega)\) is a fine sheaf of differential algebras. Smooth functions are embedded into \(\mathcal{G}^e\) by the “constant” embedding \(\sigma\), i.e., \(\sigma(f)(\varphi, x) = f(x)\), turning \(\mathcal{C}^\infty\) into a faithful subalgebra. Distributions, on the other hand, are embedded via (anti)convolution with the \(\varphi\)’s, i.e. \(\iota : \mathcal{D}'(\Omega) \to \mathcal{G}^e(\Omega)\) with

\[
\iota(u)(\varphi, x) = \langle u, \varphi(-, x) \rangle.
\]  \hfill (1)

It is one of the fundamental properties of Colombeau algebras that \(\iota|_{\mathcal{C}^\infty} = \sigma\).

Now, let \(\mu : \Omega \to \Omega\) denote a diffeomorphism as above. Accepting \(\langle \mu^* u, \varphi \rangle := \langle u, (\varphi \circ \mu^{-1}) \cdot |\text{det } D\mu^{-1}| \rangle\) and \(\hat{\hat{\varphi}}\) as the explicit forms of \(\mu^* : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)\) resp. \(\iota : \mathcal{D}' \to \mathcal{E}\), the basic requirement \(\hat{\hat{\mu}} \circ \iota = \iota \circ \mu^*\) amounts to

\[
\langle \hat{\hat{\mu}}(u) \rangle(\hat{\varphi}, \hat{x}) = \langle \mu u \rangle((\hat{\varphi}(\mu^{-1}(-, \mu \hat{x}) - \hat{x}) \cdot |\text{det } D\mu^{-1}(-, \mu \hat{x})|, \mu \hat{x})
\]

where \((\hat{\varphi}, \hat{x}) \in U(\hat{\hat{\Omega}})\), which in turn, enforce

\[
\langle \hat{\hat{\mu}}R \rangle(\hat{\varphi}, \hat{x}) := R((\hat{\varphi}(\mu^{-1}(-, \mu \hat{x}) - \hat{x}) \cdot |\text{det } D\mu^{-1}(-, \mu \hat{x})|, \mu \hat{x})
\]

as the definition of the action of the diffeomorphism on a representative \(R\) of a generalized function in a Colombeau algebra on \(\Omega\).
2 Towards diffeomorphism invariance

Colombeau and Meril in their paper [Col94] (using earlier ideas of [Col84]) made the first decisive steps to incorporate formula (2) into the construction of a Colombeau algebra which they claimed to be diffeomorphism invariant. Before discussing their work in some more detail let us introduce some terminology (cf. [Gro01], Section 9) which eases the understanding of the definitions to be given below.

Every Colombeau algebra is constructed as a quotient space of moderate modulo negligible sequences (nets) of (smooth) functions \( R \) belonging to some basic space usually denoted by \( E \) (plus some superscript to distinguish the algebras to be constructed). The respective properties of moderateness and negligibility are then defined by inserting scaled test objects (e.g. \( \varphi_{\varepsilon} \) with \( \varphi \in A_0 \) resp. \( A_q \) as above) into \( R \) and analyzing the asymptotic behavior of the latter on these “paths” as the scaling parameter \( \varepsilon \) tends to zero (and consequently \( \varphi_{\varepsilon} \to \delta \) weakly): we shall refer to the respective processes as testing for moderateness resp. negligibility. In this terminology diffeomorphism invariance of a Colombeau algebra is ensured by diffeomorphism invariance of the respective tests, of course including diffeomorphism invariance of the respective class of (scaled) test objects. As opposed to this testing procedure the elements of the algebra themselves do not depend in any way from \( \varepsilon \). We regard this distinction as fundamental clarifying several misinterpretations in the literature and call it the policy of “separating definitions from testing.”

In [Gro01], Section 3 there was given a blueprint collecting all the definitions and theorems necessary for the construction of a Colombeau algebra. In the following we shall use this collection as a guiding line in discussing the various variants of the algebra proposed in the literature beginning with the one of Colombeau and Meril [Col94].

There are basically three modifications introduced by the authors of [Col94] distinguishing their construction—which we call \( G^1 \)—from \( G^c \), namely:

(i) Smooth dependence of \( R \) on \( \varphi \) in place of arbitrary dependence.

(ii) Dependence of test objects on \( \varepsilon \), i.e., bounded paths \( \varepsilon \mapsto \phi(\varepsilon) \in D(\Omega) \).

(iii) Asymptotically vanishing moments (see below) of test objects as compared to the stronger condition \( \phi(\varepsilon) \in A_q(R^s) \) for all \( \varepsilon \) (which is the naive analog of \( \varphi \in A_q(R^s) \) in the case of \( G^c \)).

Condition (i) is necessary to guarantee smoothness of \( \hat{\mu}R \) with respect to \( \hat{x} \) (cf. transformation (2)). However, the technical prize to pay here is the use of calculus in infinite dimensional spaces: Colombeau and Meril in particular used the concept of Silva-differentiability [Col82]. However, instead of giving the proofs they rather “invited the reader to admit” ([Col94], p. 362) the respective smoothness properties.

Change (ii) together with (iii) obviously was introduced to obtain a diffeomorphism invariant analog of the vanishing moment conditions defined above.
More precisely, define $\tilde{A}_q(\mathbb{R}^s)$ to be the set of all smooth, bounded paths $\varepsilon \mapsto \phi(\varepsilon)$ satisfying
\[
\int \phi(\varepsilon)(\xi) \, d\xi = 1 \quad \forall \varepsilon \in (0, 1] \quad \text{and} \quad \int x^\alpha \phi(\varepsilon)(\xi) \, d\xi = O(\varepsilon^q) \quad \forall \alpha \in \mathbb{N}_0^s \text{ with } 1 < |\alpha| \leq q.
\]
(3)

It may now be shown ([Col94], §3) that these moment conditions indeed are invariant under the action of a diffeomorphism. Colombeau and Meril chose their basic space to be $E_1(\Omega) := \{ R : \tilde{A}_0(\mathbb{R}^s) \times \Omega \to C(0, 1] \}$. Note that this definition is not in accordance with the policy of “separating definitions from testing” as propagated above. Moreover, their definition of the objects constituting the Colombeau algebra was not unambiguous. However, following the interpretations of [Jel99] and [Gro01], the testing process in [Col94] is defined by inserting test objects of the form $S_\varepsilon \phi(\varepsilon) := (1/\varepsilon)^s \phi(\varepsilon)/\varepsilon$ into the first slot of $R$. More precisely,
\[
E^1_M(\Omega) := \{ R \in E^1(\Omega) : \forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_0^s \exists N \in \mathbb{N} \forall \phi \in \tilde{A}_p : \sup_{x \in K} |\partial^\alpha R(S_\varepsilon \phi(\varepsilon), x)| = O(\varepsilon^{-N}) \text{ as } (\varepsilon \to 0) \}
\]
\[
N^1(\Omega) := \{ R \in E^1(\Omega) : \forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_0^s \forall n \in \mathbb{N} \exists q \forall \phi \in \tilde{A}_q : \sup_{x \in K} |\partial^\alpha R(S_\varepsilon \phi(\varepsilon), x)| = O(\varepsilon^{n}) \}
\]
Finally the Colombeau-Meril algebra on $\Omega$ is defined as the quotient space
\[
G^1(\Omega) := E^1_M(\Omega) / N^1(\Omega).
\]
Using these definitions, all the main properties of $G^\ast(\Omega)$ carry over to $G^1(\Omega)$, with almost identical proofs. Indeed, boundedness of the paths $\phi(\varepsilon)$ in $D(\mathbb{R}^s)$ assures similar estimates as in the case of single functions $\varphi$.

Unfortunately, in addition to the ambiguities mentioned above the class of test objects as defined by Colombeau and Meril still is not preserved under the action of a diffeomorphism. Nevertheless, despite these defects (which, apparently, went unnoticed by nearly all workers in the field) their construction was quoted and used many times (among others [Kun96], [Vic99], [Ned98], [Kun99]). It was only in 1998 that J. Jelinek in [Jel99] pointed out the error in [Col94] by giving a (rather simple) counter example which we shall discuss in a moment. In the same paper, he presented another version of the theory which avoided (some of) the shortcomings of [Col94] and has to be considered as the second decisive step towards a diffeomorphism invariant version of a Colombeau algebra.

Taking a closer look on the nature of test objects as used by Colombeau and Meril, from (2) we see that the action of a diffeomorphism $\mu$ introduces an additional $x$-dependence in the first slot of $R$. This in turn may be exploited by giving an example of a function in $E^1_M$ which is constant in $x$ (hence the
estimates of the derivatives follow trivially) but whose \( \mu \)-transform depending on \( x \) fails to be moderate. More precisely, set \( R(\phi, x) := \exp(i \exp(\int |\phi(\xi)|^2 \, d\xi)) \).

Then according to 2 we have
\[
\hat{\mu} R(S_{\varepsilon} \hat{\phi}, \hat{x}) = R(\hat{\mu}(S_{\varepsilon} \hat{\phi}, \hat{x})) = \exp(i \exp(\int |\hat{\phi}(\frac{\mu^{-1}(\varepsilon \xi + \mu \hat{x}) - \hat{x}}{\varepsilon}) \det D\mu^{-1}(\varepsilon \xi + \mu \hat{x})|^2 \, d\xi)) .
\]

We next discuss in some detail the algebra proposed by J. Jelínek in \[Jel99\] which we shall call \( G^d \). Analogously to the previous construction we start by listing the main features distinguishing \( G^d \) from \( G^1 \).

(i) (Smooth) dependence of test objects also on \( x \in \Omega \).

(ii) In testing for moderateness test objects may take arbitrary values in \( \mathcal{A}_0(\mathbb{R}^s) \), independently of any moment condition.

While in the light of Jelínek’s counterexample (i) is compelling there seems to be no apparent necessity for (ii). Apparently (ii) widens the range of test objects which in turn reduces \( \mathcal{E}_M^d \) resp. \( \mathcal{N}_d \) in size. Yet it has to be admitted that by this reduction no generalized function of interest, neither for the further development of the theory nor in applications is lost. For the construction of a diffeomorphism invariant Colombeau algebra omitting (ii) see Part II resp. \[Gro01\]. Here, however, we focus on \( G^d \) which we regard as the standard diffeomorphism invariant algebra.

While Colombeau in his “Elementary Introduction” \[Col85\] chose to embed distributions via convolution with a mollifier, i.e., (cf. also (1) above)
\[
i^C(w)(\varphi, x) := \langle w, \varphi(\cdot, - x) \rangle ,
\]
Jelínek (following in fact earlier ideas of Colombeau presented in \[Col84\]) decided to embed distributions by letting them act on the test function, i.e.,
\[
i^J(w)(\varphi, x) := \langle w, \varphi \rangle .
\]

Since both embeddings are simply related by a translation, i.e., \( i^C = T^* i^J \) with
\[
T : \mathcal{D}(\mathbb{R}^s) \times \mathbb{R}^s \rightarrow \mathcal{D}(\mathbb{R}^s) \times \mathbb{R}^s \hspace{1cm} (\varphi, x) \mapsto (T_x \varphi, x) := (\varphi(\cdot, - x), x) ,
\]
they give rise to equivalent descriptions of virtually every Colombeau algebra, which we call \( C \)- resp. \( J \)-formalism. In \[Gro04\], Section 5 a translation formalism allowing to change from one setting to the other at any place of the construction was established and used in turn to clarify subtle questions of infinite-dimensional calculus. Before giving the actual definitions of \( G^d \) we

\[\text{5}\]
briefly comment on these issues. Jelínek uses [Yam74] as main reference while
the presentation of [Gro01] and [Gro99] is based upon the more convenient
calculus of [Kri97]. The basic idea of the latter is that a map $f : E \to F$ be-
tween locally convex spaces is smooth if it transports smooth curves in $E$ to
smooth curves in $F$, where the notion of smooth curves is straightforward (via
limits of difference quotients). This notion of smoothness in general is weaker
than Silva-differentiability but coincides with the latter on all the spaces used
in the construction of Colombeau algebras. Moreover, it displays the following
decisive advantage in applications to partial differential equations: if one
is to construct a generalized solution to a nonlinear singular equation this is
done componentwise, i.e., for fixed $\varphi$. Smoothness of the respective solution in
$\varphi$ is then guaranteed already by classical theorems on smooth dependence
of solutions on parameters.

3 The algebra $\mathcal{G}^d(\Omega)$

We now give a brief description of Jelínek’s algebra $\mathcal{G}^d$: contrary to the original
presentation using the $C$-formalism for its better familiarity (however, omitting
the superscript $C$ from now on). For a comparison of the respective features
of the two formalisms we refer to the table in [Gro01], Section 5. Apart from
closing a gap in the construction of [Jel99] the presentation in [Gro01] supplies
those parts of the resp. arguments which have not been included in [Jel99]. This
applies, in particular, to the questions of smoothness and stability of $\mathcal{E}_M, \mathcal{N}$
w.r.t. differentiation and the fact that transformed test objects are not defined
on the whole of $(0,1] \times \Omega$ in general.

Forced by the choice of the embedding (1) we define the basic space to be

$$\mathcal{E}^d(\Omega) := C^\infty(U(\Omega)).$$

Partial derivatives on $\mathcal{E}^d(\Omega)$—which will become the derivatives in the algebra—
in the $C$-formalism are simply defined by

$$D_i^d : \mathcal{E}^d(\Omega) \to \mathcal{E}^d(\Omega) \quad D_i^d = \partial_i. \quad (6)$$

Recall that test objects have to depend on $\varepsilon$ and $x$, in particular are chosen
to be smooth, bounded paths $\phi : (0,1] \times \Omega \to \mathcal{A}_0(\mathbb{R}^s)$ (resp. $\mathcal{A}_q(\mathbb{R}^s)$). Denoting
their space by $C^\infty_b((0,1] \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ we are able to formulate the tests for
moderateness and negligibility.

3.1 Definition.

(i) $R \in \mathcal{E}^d(\Omega)$ is called moderate if
$$\forall K \subset \subset \Omega \quad \forall \alpha \in \mathbb{N}_0^d \quad \exists N \in \mathbb{N} \quad \forall \phi \in C^\infty_b((0,1] \times \Omega, \mathcal{A}_0(\mathbb{R}^s)) :$$

$$\sup_{x \in K} |\partial^\alpha(R(S_{\varepsilon}\phi(\varepsilon,x),x))| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0.$$

The set of all moderate elements $R \in \mathcal{E}^d(\Omega)$ will be denoted by $\mathcal{E}_M^d(\Omega)$. 

Finally to prove that $\partial \hat{\mu} R$ sued. Suppose we want to prove moderateness of $\hat{\mu}$. We present a heuristical calculation which clearly shows which path has to be pursued. We only formulate the respective test for moderateness (the case of negligibility is moderate resp. negligible; this becomes a peculiar issue due to the additional $x$-dependence of $\phi$) are several equivalent formulations of the tests given above.

To settle the question of stability w.r.t. differentiation Jelínek introduced an alternate, yet equivalent form of tests involving differentials of $R$ with respect to the test function-slot denoted by $d_1$. ([Jel99], Th. 17, resp. Th. 18, $(2^o) \Leftrightarrow (3^o)$). We only formulate the respective test for moderateness (the case of negligibility being analogous) and refer to the original for the ingenious proofs. We presume that the author was completely aware of the role Ths. 17 and 18 had to play in this respect yet for some reasons he decided not to address this issue.

3.2 Theorem. $R \in \mathcal{E}^d(\Omega)$ is a member of $\mathcal{E}_{s}^d(\Omega)$ if and only if the following condition is satisfied:

$$\forall K \subset \subset \Omega \forall n \in \mathbb{N}_0 \forall n \in \mathbb{N} \exists q \in \mathbb{N} \forall \phi \in C^\infty((0,1] \times \Omega, \mathcal{A}_q(\mathbb{R}^s)) :$$

$$\sup_{x \in K} |\partial^n R(S_{\varepsilon}\phi(\varepsilon, x), x)| = O(\varepsilon^n) \text{ as } \varepsilon \to 0.$$

The set of all negligible elements $R \in \mathcal{E}^d(\Omega)$ will be denoted by $N^d(\Omega)$.

The key ingredients in proving diffeomorphism invariance as well as stability with respect to derivatives (i.e., that the $x$-derivative of a moderate resp. negligible function again is moderate resp. negligible; this becomes a peculiar issue due to the additional $x$-dependence of $\phi$) are several equivalent formulations of the tests given above.

We now turn to the central issue of diffeomorphism invariance. First we present a heuristical calculation which clearly shows which path has to be pursued. Suppose we want to prove moderateness of $\hat{\mu} R$. Given $\phi \in C^\infty((0,1] \times \tilde{\Omega}, \mathcal{A}_0(\mathbb{R}^s))$ then we would have to estimate

$$\hat{\mu}(R)(S_{\varepsilon}\phi(\varepsilon, \tilde{x}), \tilde{x}) = R(\hat{\mu}(S_{\varepsilon}\phi(\varepsilon, \tilde{x})), \tilde{x}) = R(\hat{\mu} S^{(\varepsilon)}(\phi(\varepsilon, \tilde{x})), \tilde{x})$$

$$= R(S^{(\varepsilon)}(S^{(\varepsilon)})^{-1} \hat{\mu} S^{(\varepsilon)}(\phi(\varepsilon, \tilde{x})), \tilde{x}).$$
Hence we would need $R$ to pass a test for moderateness w.r.t. test objects of the form (denoting by $pr_1$ the projection to the fist component)

$$\phi(\varepsilon, x) = pr_1(S^{(e)})^{-1}\overline{\mu}S^{(e)}(\xi, x)$$

$$= \xi S_{\mu}^{-1}\left(\frac{\mu^{-1}(\varepsilon\xi + x) - \mu^{-1}x}{\varepsilon}\right) \cdot |\det D\mu^{-1}(\varepsilon\xi + x)|.$$ 

But unfortunately $\phi(\varepsilon, x) \not\in C^\infty((0, 1] \times \Omega, A_0(\mathbb{R}^s))$ since it is only defined if $\xi \in \frac{\Omega - x}{\varepsilon}$, whereas we want $\xi \mapsto \phi(\varepsilon, x)(\xi)$ to be a test function on the whole of $\mathbb{R}^s$.

However, $\phi(\varepsilon, x)$ belongs to a class of test objects providing an apparently weaker, yet, as it finally turns out, equivalent test. More precisely, from [Gro01], Th. 10.5 we have that $R \in E^d(\Omega)$ is moderate if and only if it fulfills the following condition (Z)

$$\forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_0^s \exists N \in \mathbb{N} \forall \phi : D \to A_0(\mathbb{R}^s) \quad (D, \phi \text{ as described below})$$

$$\exists C > 0 \exists \eta > 0 \forall \varepsilon (0 < \varepsilon < \eta) \forall x \in K : \langle (\varepsilon, x) \rangle \in D$$

$$\left|\partial^\alpha (R(S_{\mu} \phi(\varepsilon, x), x))\right| \leq C \varepsilon^{-N},$$

where $D \subseteq (0, 1] \times \Omega$ and for $D, \phi$ the following holds: For each $L \subset \subset \Omega$ there exists $\varepsilon_0$ and a subset $U$ of $D$ which is open in $(0, 1] \times \Omega$ such that

1. $(0, \varepsilon_0] \times L \subseteq U(\subseteq D)$ and $\phi$ is smooth on $U$, and

2. for all $\beta \in \mathbb{N}_0^s$, $\left\{\partial^\beta \phi(\varepsilon, x) \mid 0 < \varepsilon \leq \varepsilon_0, \ x \in L\right\}$ is bounded in $D(\mathbb{R}^s)$.

Now diffeomorphism invariance of the notion of moderateness is established by the following

3.3 Theorem. Let $\mu : \tilde{\Omega} \to \Omega$ be a diffeomorphism and $\tilde{\phi} \in C^\infty_b((0, 1] \times \tilde{\Omega}, A_0(\mathbb{R}^s))$. Define $D(\subseteq (0, 1] \times \tilde{\Omega})$ by

$$D := \{(\varepsilon, x) \in (0, 1] \times \Omega \mid (\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x) \in U_x(\tilde{\Omega})\}.$$ 

For $(\varepsilon, x) \in D$, set

$$\phi(\varepsilon, x)(\xi) := \tilde{\phi}(\varepsilon, \mu^{-1}x) \left(\frac{\mu^{-1}(\varepsilon\xi + x) - \mu^{-1}x}{\varepsilon}\right) \cdot |\det D\mu^{-1}(\varepsilon\xi + x)|.$$ 

Then $\phi$ satisfies the requirements specified for test objects in condition (Z).
In some more detail assume $R$ to be moderate. We show that $\hat{\mu}R$ passes the test used in Definition 3.1 (i). Indeed given $\tilde{K} \subset \subset \tilde{\Omega}$, $\alpha \in \mathbb{N}_0$ and $\phi \in C^\infty_b((0,1] \times \tilde{\Omega}, A_0(\mathbb{R}^s))$ define $\phi$ as in the preceding theorem. Then by the chain rule

$$\partial_{x}^\alpha (\hat{\mu}R)(S_\varepsilon \phi(\varepsilon, \hat{x}, \tilde{x})) = \partial_{x}^\alpha (R(S_\varepsilon \phi(\varepsilon, \mu \tilde{x}), \mu \tilde{x})) = \sum_{|\beta| \leq |\alpha|} \partial_{x}^\beta (R(S_\varepsilon \phi(\varepsilon, x), x)|_{x=\mu \tilde{x}} \cdot g_\beta(\tilde{x}),$$

where each $g_\beta$ is bounded on $\tilde{K}$. Since $R$ satisfies condition (Z) the claim follows.

However, matters become more complicated in the case of negligibility. First note that the resp. test objects take values in $A_q (q > 0)$ which is not a diffeomorphism invariant property. The way out is provided by the re-introduction of asymptotically vanishing moments (cf. 3) into the theory by building up (another) equivalent test using this notion. Indeed Jelínek ([Jel99], 18 (4°)) has formulated such a condition which, however, unfortunately is not equivalent to the notion of negligibility as defined above. Moreover, this condition is so strong that even $\iota(x^2) \neq (\iota(x))^2$, hence the property of $\iota$ being an algebra homomorphism on $C^\infty$ is lost. However, in [Gro01], Section 7 this flaw was removed, namely by demanding also all derivatives of the test objects to have asymptotically vanishing moments. More precisely we say that a test object $\phi \in C^\infty_b((0,1] \times \Omega, A_0(\Omega))$ is of type $[A_\infty^\infty]_{K,q}$ if on (a given) $K \subset \subset \Omega$

$$\forall \beta \in \mathbb{N}_0^s 1 \leq |\beta| \leq q \forall \gamma \in \mathbb{N}_0^s \sup_{x \in K} \int \xi^\beta \partial^\gamma \phi(\varepsilon, x)(\xi) dx = O(\varepsilon^q).$$

Then we have

**3.4 Theorem.** $R \in \mathcal{E}^d(\Omega)$ is negligible if and only if $\forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_0^s \forall n \in \mathbb{N} \exists q$ such that $\forall \phi$ of type $[A_\infty^\infty]_{K,q}$:

$$\sup_{x \in K} |\partial^\alpha R(S_\varepsilon \phi(\varepsilon, x), x)| = O(\varepsilon^n).$$

Diffeomorphism invariance of the notion of negligibility is now established by the above theorem in conjunction with an analog of Theorem 3.3 as well as an analog of condition (Z) above. Finally we define our main object of desire.

**3.5 Definition.** The diffeomorphism invariant Colombeau algebra on $\Omega$ is defined as the quotient

$$\mathcal{G}^d(\Omega) := \mathcal{E}^d_M(\Omega) / \mathcal{N}^d(\Omega).$$

Summing up we have constructed a differential algebra $\mathcal{G}^d(\Omega)$ in a diffeomorphism invariant way, in particular allowing for a diffeomorphism invariant embedding of distributions. Moreover, $\mathcal{G}^d(\Omega)$ (as usual) is a fine sheaf of differential algebras ([Gro01], Section 8).

We finally turn to the issue of commutativity of the embedding with partial derivatives in the algebra. This will guarantee that $\mathcal{G}^d(\Omega)$ indeed possesses all
the favorable properties of a Colombeau algebra. To this end it is useful to change to the J-formalism. Recall from (6) that derivatives in the C-formalism are just given by partial derivatives. Using the translation formalism of \[\text{Gro01}\] we derive that in the J-formalism, i.e., on \(\mathcal{E}'(\Omega) := (T^{-1})^*(\mathcal{E}'(\Omega))\) we have

\[
D^J_i = (T^{-1})^* \circ \partial_i \circ T^* \quad \text{i.e.,} \quad (D^J_i R)(\varphi, x) = -(d_1 R(\varphi, x))(\partial_i \varphi) + (\partial_i R)(\varphi, x). \tag{7}
\]

We now see immediately that if \(F \in \mathcal{D}'(\Omega)\) then \(i^J(F)\) (cf. \[\text{Gro99}\]) is independent of \(x\) hence the second term in (7) vanishes. Moreover, since \(i^J(F)\) is linear in \(\varphi\), 

\[
-d_1(i^J(F))(\varphi, x)(\partial_i \varphi) = \langle F, -\partial_i \varphi \rangle \quad \text{which is exactly the } i^J\text{-image of } \partial_i F.
\]

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