Factorial and Cumulant Moments in a Simple Cascade Model

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Abstract
Factorial and cumulant moments in a simple cascade models are considered. Their characteristic features are shown to be similar to those observed in QCD jets.
1 Introduction

The characteristics of multiplicity distributions in multiparticle dynamics has recently gained much attention due to a QCD analysis of generating functional of these distributions and their moments. A very clear review of this subject can be found in [1], where one can also find the relevant references, see also a recent review [2]. The main focus in the above studies was initially a striking QCD prediction on the oscillatory behavior of cumulants as functions of their rank. Actually it turned out to be more convenient to analyze a ratio of cumulants to factorial moments. The experimental data from many reactions (see [2]) confirmed this prediction. The situation seemed to be all the more interesting, that conventional multiplicity distributions such as negative binomial one, did not predict such oscillations. However, a subsequent analysis [3] has shown, that other more phenomenological models such as dual parton model [4] or quark-gluon string model [5] are also explaining the data. This brings us to a natural question. What is the real (basic) origin of the oscillations of the cumulants? To answer this question we construct a simple probabilistic cascade and look at the properties of the cumulants. We find exactly the same oscillation pattern as predicted theoretically and observed experimentally, thus confirming one of the conclusions of [3], that the basic origin of these oscillations is a cascading origin of the underlying process of particle formation.

2 The model.

Let us consider a model of cascade with discrete steps in time. Initially (at time $t = 0$) we have only one "particle". When time changes from $t$ to $t + 1$ any particle existing at a time $t$ splits with probability $p$ (a parameter of the model) into 2 particles or does not split with probability $1 - p$.

Now $P_n(t)$ is a probability that on $t$ step $n$ particles were produced. The "first split" equation reads

$$P_n(t) = (1 - p)P_n(t - 1) + p \sum_{n_1=0}^{n} P_{n_1}(t - 1)P_{n - n_1}(t - 1) , \quad P_n(0) = \delta_{n1} \quad (1)$$

Let us now consider a generating function related to the probabilities $P_n(t)$:

$$G(t, z) = \sum_{n=0}^{\infty} P_n(t)(1 + z)^n \quad (2)$$

From (1) we have

$$G(t, z) = (1 - p)G(t - 1, z) + pG^2(t - 1, z) \quad (3)$$

The boundary conditions for this relation is

1. $G(t, 0) = 1$ – the total probability conservation;
2. $G(0, z) = 1 + z$ – corresponding to $P_n(0) = \delta_{n1}$
3 Mean multiplicity and moments.

By definition a mean multiplicity \( n(t) \) is related to a generating function by

\[
n(t) = \sum_{n=0}^{\infty} n P_n(t) = \left. \frac{\partial G(t, z)}{\partial z} \right|_{z=0}.
\]

(4)

It gives for \( n(t) = (1 - p)n(t - 1) + 2pn(t - 1) = (1 + p)n(t - 1) = (1 + p)^t. \)

(5)

Last equality follows from \( G(t, 0) = 1 \) (see above) and \( n(0) = 1. \)

Factorial moments are introduced as coefficients of \( F_q(t) \) in the expansion

\[
G(t, z) = 1 + \sum_{q=1}^{\infty} \frac{z^q}{q!} F_q(t)n^q(t).
\]

(6)

After substitution into (3) we find

\[
(1 + p)^q F_q(t) = (1 - p)F_q(t - 1) + p \sum_{k=0}^{q} C_k^q F_k(t - 1)F_{q-k}(t - 1), \quad q > 1
\]

(7)

It is easy to demonstrate that the limit \( \lim_{t \to \infty} F_q(t) = F_q \) exists. Therefore from (7) we get

\[
F_q = \frac{1}{(1 + p)^{q-1}} - \frac{p}{1 + p} \sum_{k=1}^{q-1} C_k^q F_kF_{q-k}, \quad q > 1 \quad F_1 = 1
\]

(8)

Now it is possible to treat asymptotic factorial moments as factorial moments of "asymptotic" generating function.

\[
G(a) = 1 + \sum_{q=1}^{\infty} \frac{F_q}{q!} a^q.
\]

(9)

The equation on \( G(a) \) now reads

\[
G((1 + p)a) = (1 - p)G(a) + pG^2(a)
\]

(10)

with the initial conditions \( G(0) = 1 \) and \( G'(0) = 1. \)

4 Cumulants and \( H_q \)

Cumulants are introduced as coefficients \( K_q(t) \) in the expansion

\[
\ln G(t, z) = \sum_{q=1}^{\infty} \frac{z^q}{q!} K_q(t)n^q(t).
\]

(11)
The cumulant and factorial moments are actually not independent. The relation between the two can be traced from the formula \( G_z(\ln G)'_z = G'_z \). From (3) and (11) we obtain

\[
K_q = F_q - \sum_{k=1}^{q-1} C_{q-k}^k K_{q-k} \tag{12}
\]

It turns out convenient to consider the ratios \( H_q = K_q/F_q \) obtained from (3) and (12). The resulting \( H_q \) for some values of \( p \) are shown in Fig. 1. We see, that the ratio of cumulant to factorial moments \( H_q \) in our simple cascade model provides the same oscillation pattern that follows from a number of theoretical models and is observed experimentally. How good is a quantitative predictions of the model when compared to the experimental data is questionable. We restrict ourselves to illustrate this situation at Fig. 2.

It is instructive to look more attentively at two extreme limits of our cascade model.

1. \( p \to 0 \)

Keeping the leading power of \( p \) in (10) we get

\[
p a G'(a) = pG(a)(G(a) - 1) \tag{13}
\]

The solution of this equation that satisfies boundary conditions is

\[
G(a) = \frac{1}{1 - a} \tag{14}
\]

so \( F_q^{(0)} = q! \), \( K_q^{(0)} = (q - 1)! \) and \( H_q^{(0)} = 1/q \). It is worth noting that this result is valid only for \( pq \ll 1 \) (not for \( p \ll 1 \)).

2. \( p = 1 \)

Although in the context of our study this limit looks quite peculiar, let us proceed and perform an explicit calculation, which gives

\[
G(2a) = G^2(a) \tag{15}
\]

The obvious solution of this equation is

\[
G(a) = e^a \tag{16}
\]

so \( F_q^{(1)} = 1 \), \( K_q^{(1)} = \delta_{q1} \) and \( H_q^{(1)} = \delta_{q1} \). Such moments are produced by Poisson distribution.

5 The continuous limit.

In the context of our model investigations it is useful to determine continuous limit of (3). This allows to calculate not only asymptotic shape of generating function and moments but also their dependence on mean multiplicity \( n \).
It is convenient to give (3) the form

\[ G(t, z) - G(t - 1, z) = pG(t - 1, z)(G(t - 1, z) - 1). \]  

(17)

If \( p \) is considered to be a small one can replace the difference in l.h.s. of this equation by derivative (the change at one step becomes infinitesimal)

\[ \frac{\partial G(t, z)}{\partial t} = pG(t, z)(G(t, z) - 1) \]

(18)

(we have also changed \( t - 1 \) to \( t \)). In this limit the equation on mean multiplicity is (see [4])

\[ \frac{dn(t)}{dt} = pn(t) \]

(19)

that is

\[ n(t) = e^{pt}n(0) = e^{pt} \]

(20)

according to the initial condition \( P_n(0) = \delta_{n1} \). Solution of (18) is clear

\[ G(t, z) = \left(1 - \frac{n(t)z}{1 + z}\right)^{-1}, \]

(21)

where initial conditions and (20) have been used. From this we obtain factorial moments and cumulants

\[ F_q(t) = q! \left(1 - \frac{1}{n(t)}\right)^{q-1} \]

(22)

\[ K_q(t) = (q - 1)! \left(1 - \frac{1}{n(t)}\right)^q - \frac{(-1)^q}{n^q(t)} \]

(23)

so

\[ H_q(t) = \frac{1}{q} \left(1 - \frac{1}{n(t)}\right) \left(1 - \frac{(-1)^q}{n(t)^q}\right). \]

(24)

In the limit \( n \to \infty \) this result becomes the same as in (14).

Here it is appropriate to discuss some relations between our model (in continuous limit) and a PB (pure birth) one considered in [6]. That model is formulated as an equation on probability to obtain \( n \) particles at the moment of (continuous!) time \( t \)

\[ \frac{\partial P(n, t)}{\partial t} = \lambda(n - 1)P(n - 1, t) - \lambda nP(n, t). \]

(25)

For this equation on \( P \)'s the equation on generating function (which we denote by \( \hat{G} \)) is

\[ \frac{\partial \hat{G}}{\partial t} = \lambda z(1 + z) \frac{\partial \hat{G}}{\partial z} \]

(26)
Mean multiplicity $\tilde{n}$ in this case satisfies
\[ \frac{\partial \tilde{n}(t)}{\partial t} = \lambda \tilde{n}(t) = e^{\lambda t} \tilde{n}(0). \] (27)

It is easy to see that the equation (27) has a solution
\[ \tilde{G}(t, z) = \tilde{G} \left( \frac{e^{\lambda t} z}{1 + z} \right) = f \left( \tilde{n}(t) \frac{z}{1 + z} \right). \] (28)

Here $f$ is a function determined by initial ($t = 0$) distribution. Universal conditions on $f$ are only $f(0) = 1$ (from $\tilde{G}(t, z = 0) = 1$) and $f'(0) = 1$ (from $\partial \tilde{G}(t, z = 0)/\partial z = \tilde{n}(t)$). So in this model the asymptotic ($\tilde{n} \rightarrow \infty$) behavior of factorial moments and cumulants is determined by initial distribution while the dependence on $\tilde{n}$ is described by differential equations (25) and (26).

It is clear that in the continuous limit our model turns into PB if one specifies some initial conditions to (3).

6 Conclusions

From the study of the simple cascade model of particle multiplication we see, that it correctly reproduces the oscillation pattern predicted by a number of theoretical schemes and observed experimentally. We conclude that the basic origin of such oscillations is a cascade structure of underlying dynamic picture. This cascade could in principle originate both from the quark-gluon phase of the evolution of hadronic system and from the nonperturbative hadronization stage.

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Fig. 1 Cumulants to factorial moments ratio in the model for 3 values of p.
NA22 $\pi + $Al

- experiment
- - - - model $p = 0.583$

$H_q$

$q$

Fig. 2