Dynamics of narrow solitons in two Ablowitz-Ladik chains with different interchain couplings

M T Primatarowa and R S Kamburova
Georgi Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria
E-mail: prima@issp.bas.bg

Abstract. We study the evolution of discrete (narrow) solitons in a system of coupled Ablowitz-Ladik (AL) chains. Two types of interchain coupling are investigated: one which admits reduction of the system to the standard (integrable) AL model and one which couples opposite sites of the chains and does not admit reduction to the AL model. The condition for a perfect soliton switching between the two chains is obtained and the characteristics of the different couplings are analyzed.

1. Introduction
The study of nonlinear waves is receiving much attention due to the potential application within different branches of physics from nonlinear optics to Bose-Einstein condensate. The interplay between discrete diffraction and nonlinearity leads to the formation of discrete solitons. They promise an efficient way to control switching of optical signals in a system of coupled waveguides. So waveguide-based devices have received considerable attention in literature and this field has been extensively explored theoretically and experimentally. A discrete coupler involving two waveguides which exchange power as a result of weak overlap of their evanescent fields is the basic realization of a waveguide switch. The rate of power swapped back and forth between waveguides depends on the strength of the coupling, the degree of similarity of the waveguides and the initial pulse energy [1-5]. Waveguide arrays are particularly interesting because of their possible applications in signal processing [6-10]. All considerations regard discrete soliton switching mainly in homogenous waveguides with constant or smoothly varying coupling between them. However in practical applications the properties of inhomogeneous waveguides are more interesting and rather inevitable for switching [11-15].

Widely investigated are the standard discrete nonlinear Schrödinger (NLS) equation, as well as the completely integrable discrete Ablowitz-Ladik (AL) equation [16-18]. Although the two equations have the same linear properties and yield the same NLS equation in the continuum limit, their nonlinear properties are different. This leads to differences in the dynamics of narrow solitons (bright or dark) for the two models. Soliton solutions in two coupled discrete nonlinear chains were found and their stability was investigated in [19-21]. In the present paper we study the soliton dynamics in two coupled Ablowitz-Ladik chains with a complicated coupling that includes linear, nonlinear and dispersive interactions between the chains.
2. The model

We shall consider two parallel chains of particles described by the following system of coupled Ablowitz-Ladik equations:

\[
\begin{aligned}
  i \frac{\partial \alpha_n}{\partial t} &= M(\alpha_{n+1} + \alpha_{n-1})(1 + \gamma|\alpha_n|^2) + [d_1(\beta_{n+1} + \beta_{n-1}) + 2d_2\beta_n](1 + \gamma|\alpha_n|^2) \\
  i \frac{\partial \beta_n}{\partial t} &= M(\beta_{n+1} + \beta_{n-1})(1 + \gamma|\beta_n|^2) + [d_1(\alpha_{n+1} + \alpha_{n-1}) + 2d_2\alpha_n](1 + \gamma|\beta_n|^2)
\end{aligned}
\]  

(1)

\(\alpha_n(t)\) [\(\beta_n(t)\)] is the amplitude of an excitation at site \(n\) of the first (second) chain. \(M\) is the coupling interaction between neighboring particles in one and the same chain. The two chains are coupled to each other through the real parameters \(d_1\) and \(d_2\). The parameter \(d_2\) governs the interchain coupling between opposite sites (nondispersive), while \(d_1\) is a coupling of the AL type and allows reduction of the system to the AL model which is completely integrable. Both interactions include linear and nonlinear terms. The parameter \(\gamma\) determines the type of the soliton solution (bright for \(\gamma > 0\) and dark for \(\gamma < 0\)) of the AL equation. In what follows we consider only bright solitons and set \(\gamma = 1\) due to the scaling property of the AL system.

We shall study the influence of the two different couplings on the soliton properties. (1) is a generalization of the models considered in [19] and [20] and can be derived from the Hamiltonian

\[
H = \sum_n [M(\alpha^*_n\alpha_{n-1} + \alpha^*_n\alpha_{n-1}) + \beta^*_n\beta_{n-1} + \beta^*_n\beta_{n-1}) + d_1(\alpha^*_n\beta_{n-1} + \alpha^*_n\beta_{n-1}) + \alpha^*_n\beta_{n-1}) + 2d_2(\alpha^*_n\beta_{n-1} + \alpha^*_n\beta_{n-1})]
\]  

(2)

using the deformed Poisson brackets [17,18]

\[
\{\alpha_n, \alpha^*_m\} = i(1 + |\alpha_n|^2)\delta_{n,m} \\
\{\alpha_n, \alpha_m\} = \{\alpha^*_n, \alpha^*_m\} = 0
\]

(3)

and the equations of motion

\[
\frac{\partial \alpha_n}{\partial t} = \{H, \alpha_n\}, \quad \frac{\partial \beta_n}{\partial t} = \{H, \beta_n\}.
\]  

(4)

The system (1) is nonintegrable but has two integrals of motion, the Hamiltonian \(H\) and the total number of particles \(N\)

\[
N = \sum_n [\ln(1 + |\alpha_n|^2) + \ln(1 + |\beta_n|^2)].
\]  

(5)

For \(d_2 = 0\) and the symmetric reduction \(\alpha_n(t) \equiv \beta_n(t)\) the system (1) turns in an AL equation with the well known soliton solution:

\[
\alpha_n(t) = \beta_n(t) = \sinh \frac{1}{L} \sech \frac{n - vt}{L} e^{i(n\mathbf{k} - \omega t)} \\
v = -2(M + d_1)\sinh \frac{1}{L} \sin k, \quad \omega = 2(M + d_1)\cosh \frac{1}{L} \cos k
\]  

(6)

The parameters \(k\) (wavenumber) and \(L\) (width) determine the velocity \(v\) and frequency \(\omega\) of the soliton. In this case the conserved quantities have the form

\[
H = 8(M + d_1)\sinh \frac{1}{L} \cos k, \quad N = 4/L
\]  

(7)
Figure 1. Propagation of equal narrow ($L = 2$) solitons in the two chains with $M = -1$, $k = 0.1$ and different coupling constants. (a): $d_1 = 0.628$, $d_2 = 0$; (a'): $d_1 = -0.628$, $d_2 = 0$; (b): $d_1 = 0$, $d_2 = 0.628$; (b'): $d_1 = 0$, $d_2 = -0.628$. The time is in units of $1/|M|$.

and it holds $\omega = \partial H / \partial N$.

Figure 1 shows the propagation of the soliton (6) when the amplitudes in the two chains are equal for different coupling constants. As can be expected the dispersive coupling $d_1$ influences the velocity of the soliton and according to the expression in (6) $v = 0.077$ for $d_1 = 0.628$ [figure 1(a)] and $v = 0.339$ for $d_1 = -0.628$ [figure 1(a')]. The coupling between opposite sites in the two chains does not change significantly the velocity and $v = 0.208$ for positive as well as negative values of $d_2$ [figures 1(b) and (b')]. The value of $d_2$ has mainly influence on the soliton's amplitude through the factor $\sqrt{M/(M + d_2)}$.

In the continuum limit $\alpha_n(t) \rightarrow \alpha(x, t)$, $\beta_n(t) \rightarrow \beta(x, t)$ which holds for wide solitons ($L \gg 1$) and for $\alpha(x, t) \equiv \beta(x, t)$ the system (1) reduces to the standard NLS equation of the form

$$i\frac{\partial \alpha}{\partial t} = G\alpha + (M + d_1)\frac{\partial^2 \alpha}{\partial x^2} + G|\alpha|^2\alpha, \quad G = 2(M + d_1 + d_2). \quad (8)$$

3. Numerical results

We shall investigate the propagation of an AL soliton which at the initial time is launched in one of the chains

$$\alpha_n(0) = \sinh \frac{1}{L} \text{sech} \frac{n - n_0}{L} e^{ikn}, \quad \beta_n(0) = 0 \quad (9)$$

solving numerically the system (1). The simulations are carried out for 1000 sites of each chain, $n_0 = 500$ and periodic boundary conditions.

In the pure linear case the excitation will transfer from one chain to the other and back with a period $t_0 = \pi/|d|$, where $d$ is the total linear coupling.
Figure 2. Switching of a narrow soliton with $k = 0$ between two chains for $|d_1| = 0.628$, $d_2 = 0$ (a) and $d_1 = 0$, $|d_2| = 0.628$ (b). The other parameters are the same as in figure 1.

For our complicated model (which is not linear) we observe that an energy exchange between the two chains take place with nearly the same period

$$t_0 = \frac{\pi}{|d_1 + d_2|}$$

and the energy exchange rate depends on the strength of the coupling. For small values of the coupling constants the soliton is only partially transferred. When the coupling increases the transferred rate grows. This behavior is due to the nonlinear coupling terms. We obtained that a soliton can transfer (perfect soliton switching) when the simple condition

$$4|d_1 + d_2|L^2 \gg 1$$

is fulfilled. Figure 2 shows the soliton transfer when the coupling between the two chains is of type $d_1$ [figure 2(a)] or $d_2$ [figure 2(b)]. The process does not depend on the sign of the coupling. We observe that in the case of propagating solitons the soliton can be switched and its velocity remains the same as the initial one.

Figure 3 demonstrates the interplay between the two types of coupling. Their action is additive. When the constants $d_1$ and $d_2$ are of the same sign [figure 3(a)] the soliton switching is similar to that for one coupling’s type but with the greater strength (figure 2). When the two couplings are of different signs the resulting coupling decreases and can be set to nearly zero as shown on figure 3(b). We observe that the soliton $\alpha_n$ remains in the first chain where it was initially excited and in the second chain $\beta_n \simeq 0$. 
Figure 3. Evolution of a soliton with \( k = 0 \) excited in one of the chains for \( d_1 = d_2 = 0.314 \) (a) and \( d_1 = -d_2 = 0.314 \) (b). The other parameters are the same as in figure 1.

Figure 4 illustrate the differences between the two types of interchain coupling. For large values of the coupling constants the soliton switching is preserve for the coupling between opposite sites of the chains \( d_2 \) [figure 4(b)]. For the dispersive coupling \( d_1 \) an energy exchange with the same period \( t_0 \) takes place but the soliton formation is destroyed [figure 4(a)]. Obviously to achieve soliton switching \( |d_1| < |M| \) has to be fulfilled. We have observed that for wide solitons \((L \gg 1)\) the dispersive character of the coupling \( d_1 \) vanish and the solitons in the two chains are stable.

4. Conclusion
We have studied the evolution of discrete solitons in a system of coupled Ablowitz-Ladik chains. Two types of interchain coupling are introduced and compared. The coupling of the AL type admits reduction of the system to an AL model. Perfect soliton switching can be obtained for large enough values of the coupling constant in both cases. In the case of dispersive coupling the soliton is destroyed when the coupling constant exceed a critical value which depends on the soliton width and the interaction between the neighboring particles in the chains.

Acknowledgments
This work is supported by the National Science Fund of Bulgaria under Grant No. DO 02-264.

References
[1] Jensen S M, 1982 *IEEE J. Quantum Electron.* 18 1580-83
[2] Trillo S, Wabnitz S and Stegeman G 1989 *IEEE J. Quantum Electron.* 25 1907-16
[3] Paré C and Florjańczyk M 1990 *Phys. Rev.* A 41 6287-95
Figure 4. Switching of a narrow soliton with $k = 0$ between two chains for $|d_1| = 1.256$, $d_2 = 0$ (a) and $d_1 = 0$, $|d_2| = 1.256$ (b). The other parameters are the same as in figure 1.

[4] Kivshar Yu S 1993 Optics Lett. 18 7-9
[5] Chu P L, Peng G D and Malomed B A 1993 Optics Lett. 18 328-30
[6] Aceves A B, De Angelis C, Peschel T, Muschall R, Lederer F, Trillo S and Wabnitz S 1996 Phys. Rev. E 53 1172-89
[7] Christodoulides D N and Eugenieva E D 2001 Phys. Rev. Lett. 87 233901(4)
[8] Peschel U, Morandotti R, Arnold J M, Aitchison J S, Eisenberg H S, Silberberg Y, Pertsch T and Lederer F 2002 J. Opt. Soc. Am. A 19 2637-44
[9] Vicencio R A, Molina M I and Kivshar Yu S 2003 Optics Lett. 28 1942-44
[10] Molina M I, Vicencio R A and Kivshar Yu S 2006 Optics Lett. 31 1693-95
[11] Królikowski W and Kivshar Yu S 1996 J. Opt. Soc. Am. B 13 876-87
[12] Longhi S 2006 Phys. Rev. E 74 026602(9)
[13] Morales-Molina L and Vicencio R A 2006 Optics Lett. 31 966-68
[14] Goodman R H and Weinstein M I 2008 Physica D 237 2731-60
[15] Belmonte-Beitia J, Pérez-García V M and Torres P J 2009 Nonlin. Science 19 437-51
[16] Ablowitz M J and Ladik J F 1976 J. Math. Phys. 17 1011-18
[17] Scharf R and Bishop A R 1991 Phys. Rev. A 43 6535-44
[18] Cai D, Bishop A R and Grønbech-Jensen N 1996 Phys. Rev. E 53 4131-36
[19] Bilow A, Hennig D and Gabriel H 1999 Phys. Rev. E 59 2380-92
[20] Malomed B A and Yang J 2002 Phys. Lett. A 302 163-70
[21] Kamburova R S and Primatarowa M T 2010 AIP Conf. Proc. 1203 261-266