FIELDS GENERATED BY
A MOVING RELATIVISTIC POINT MASS AND
MATHEMATICAL CORRECTION TO FEYNMAN’S LAW

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This paper is dedicated to my teachers: Sir Isaac Newton, Gottfried W. Leibnitz, Rudolf O. S. Lipschitz, James C. Maxwell, Hendrik A. Lorentz, Albert Einstein, Stefan Banach, Stanislaw Mazur, Richard P. Feynman, and Stan M. Ulam, who directly or indirectly inspired and influenced this research.

Abstract. Feynman using a formula, known as Feynman’s law for a moving point charge, explained the phenomenon of synchrotron radiation and derived formulas for phenomena concerning electromagnetic radiation at large distance from the source, such as reflection, refraction, interference, diffraction, and scattering. These facts show the importance of the formula.

The formula is supposed to represent the intensities of the electric and magnetic fields in free space, that is satisfying homogeneous system of Maxwell equations.

Feynman’s law contains a mathematical inconsistency. It involves implicitly a field representing the retarded time and an ordinary derivative in place where a partial derivative should be.

In this note the author shows how to prove, using Banach’s contraction mapping theorem, the existence and uniqueness of the retarded time field for any relativistically admissible trajectory of a point mass.

The Lorentzian frame, the trajectory, and the retarded time field uniquely determine a system of fundamental fields.

By means of these fields one can represent and establish relations between the system of wave-gauge equations and Maxwell equations, and to prove that electromagnetic field represented by the amended Feynman’s formula satisfies the homogenous system of Maxwell equations and the system of wave-gauge equations.

As applications the author proves the existence and uniqueness of the solution to the n-body problem in the resulting joint gravitational and electromagnetic fields as also in any other relativistic force field representing a regular nonanticipating operator of the system of trajectories.

1. Analysis of the Original Feynman’s Formula
   for a Moving Point Charge

Importance of Maxwell’s laws for logical analysis of electromagnetic phenomena stems from Einstein’s work on special theory of relativity [8]. An asymmetry in the

1991 Mathematics Subject Classification. 35L05, 53C50, 53C80, 78A25, 78A35, 81V10, 83C50.
Key words and phrases. Maxwell equations, Feynman’s law, electrodynamics, motion of charged particles, electromagnetic interaction, electromagnetic theory, gravitation, n-body problem.
interpretation of an electromagnetic phenomena based on Lorentz force [14], acting on a charged particle, lead Einstein to the discovery of this theory.

Einstein proved that laws of physics have to be invariant under transformations discovered earlier by Lorentz [14]. In the process he established that Maxwell’s equations are invariant under such transformations [8].

Feynman derived heuristically a formula for the electromagnetic field generated by a moving point charge. In this paper we will prove that after replacing the ordinary derivative with respect to time by the partial derivative and treating all quantities appearing in Feynman’s formula as fields, the corrected formula yields electromagnetic field \((E, B)\) satisfying Maxwell equations. In the process we shall also prove that Liénard-Wiechert formulas provide an electromagnetic potential for such a field.

To see that one has to be careful when treating derivatives

\[
\frac{d}{dt} \quad \text{and} \quad \frac{\partial}{\partial t}
\]

of composed function consider the following example. Let \(u = u(x, y, z, t)\) be a differentiable function in \(\mathbb{R}^4\) representing some physical quantity as a function of position and time. Let \((x(t), y(t), z(t))\) denote a path of a particle as a function of time. Then the derivative with respect to time along the path is

\[
\frac{d}{dt} u = u_x \dot{x} + u_y \dot{y} + u_z \dot{z} + u_t.
\]

Obviously it is not the same as partial derivative on the path

\[
\frac{\partial}{\partial t} u = u_t.
\]

Any moving point mass \(m_0\) obeying Einstein’s laws of special theory of relativity generates a pair of fields \((E, B)\) satisfying Maxwell’s equations just from geometrical considerations of the Lorentzian frame. No other physical properties are required. Thus the obtained field \(E\) may as well represent for instance the illusive gravity field that Einstein was looking for [9].

To formulate briefly the main result of the paper assume that in a given Lorentzian frame we introduce fundamental fields that are uniquely determined by the trajectory of a moving point mass. By means of these fields and their partial derivatives we can represent in explicit form fields given by amended Feynman’s and Liénard-Wiechert’s formulas and prove the relations between such fields and Maxwell’s and wave equations.

The main result can be stated as follows:

**Theorem 1.1** (Bogdan-Feynman Theorem). Assume that in a Lorentzian frame is given an admissible trajectory \(t \mapsto r_2(t)\) from \(\mathbb{R}\) into \(\mathbb{R}^3\) representing a path of a point mass \(m_0\). Let \(G\) denote the set of points that do not lie on the path. Define Newton-Feynman field by the formula

\[
E(r_1, t) = u^2 e + \frac{1}{c} u^{-1} \frac{\partial}{\partial t} (u^2 e) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (e) \quad \text{for all} \quad r_1 \in \mathbb{R}^3, \ t \in \mathbb{R}
\]

and the associated field by

\[
B(r_1, t) = \frac{1}{c} e \times E \quad \text{for all} \quad r_1 \in \mathbb{R}^3, \ t \in \mathbb{R}.
\]
Every pair of such fields \((E, B)\) satisfies the homogenous system of Maxwell equations and the homogenous system of wave equations on the set \(G\). As such these fields propagate through space with velocity \(c\) of light.

The 4-vector obtained by Liénard-Wiechert formulas provides a pair of scalar-vector potentials for the pair \((E, B)\).

For the notion of an admissible trajectory see (2.1) and for definitions of the involved fields \(u\) and \(e\) see (4.1).

We shall start with analysis of Feynman’s original formulas:

The intensity of the electric field \(E\) and of the magnetic field \(B\) at any time \(t\) and any point \(r_1 \in \mathbb{R}^3\), not lying on the trajectory of a moving charge \(q\), are given by

\[
E = \frac{q}{4\pi \varepsilon_0} \left[ \frac{e}{|r|^2} + \frac{|r|}{c} \frac{d}{dt} \left( \frac{e}{|r|^2} \right) + \frac{1}{c^2} \frac{d^2}{dt^2} \right], \quad B = \frac{1}{c} e \times E,
\]

where \(r = r_1 - r_2(t')\) is a vector, starting on the trajectory \(t \mapsto r_2(t)\) of a moving point charge at the retarded time \(t' = t - |r|/c\) and ending at the point \(r_1\) at time \(t\) where the field is to be evaluated. Here \(e\) denotes the unit vector corresponding to the vector \(r\), and \(c\) the speed of light, and \(\varepsilon_0\) the electrostatic constant.

For reference see Feynman-Leighton-Sands [10], vol. 1, chapter 28, formulas (28.3) and (28.4).

Let us pose for a moment to analyze the formulas (1.1) for their mathematical content. On the left side of the formula we have the quantity \(E\) representing intensity of the electric field at point \(r_1\) at time \(t\). Thus \(E = E(r_1, t)\) represents a function of the point \((r_1, t)\). So on the right side we should also find an expression representing a function of these variables. The vector \(r = r_1 - r_2(t')\) and therefore the vector \(e = r/|r|\) involves a retarded time that must satisfy the equation

\[
t' = t - |r|/c
\]
equivalently

\[
t' = t - |r_1 - r_2(t')|/c
\]
Since the trajectory \(t \mapsto r_2(t)\) is fixed the above equation implicitly defines the retarded time as a function of \((r_1, t)\). If one can solve the above equation explicitly we will get \(t' = t'(r_1, t)\) in the form of a function. Once this function is known and we can prove that it has continuous partial derivatives up to order 2 and the trajectory \(r_2(t)\) itself has continuous derivative up to order 2, then the composed functions

\[
r(r_1, t) = r_1 - r_2(t'(r_1, t)) \quad \text{and} \quad e(r_1, t) = \frac{1}{|r(r_1, t)|} r(r_1, t)
\]
will be well defined on the set \(G = \{(r_1, t) : |r(r_1, t)| > 0\}\) and the functions will be of class \(C^2\).

Our immediate goal will be to find an explicit formula for the retarded time function and to prove that the functions appearing in Feynman’s formula indeed have the required properties.
To simplify the notation we select units of measure so that the speed of light is \( c = 1 \) and the electrostatic constant satisfies the condition \( 4\pi\varepsilon_0 = 1 \). We shall also assume that all the intensities of the involved fields are per unit of charge, that is we assume that \( q = 1 \).

After these modifications and clarifications the formula for \( E \) can be reduced to a formula

**Definition 1.2** (Newton-Feynman formula).

\[
E(r_1, t) = u^2 e + u^{-1} \frac{\partial}{\partial t}(u^2 e) + \frac{\partial^2}{\partial t^2} e \quad \text{for all} \quad (r_1, t) \in G.
\]

Notice that, when the point mass representing the charge, is in its rest frame, that is \( \dot{r}_2(t) = 0 \) for all \( t \), then the above formula reduces to a single term

\[
u^2 e = \frac{1}{|r_1 - r_2|^3}(r_1 - r_2)
\]
differing from Newton’s gravitation formula just by a constant of proportionality. Thus it is proper to call the expression (1.2) the **Newton-Feynman formula**.

2. **Considerations concerning trajectories**

In this section we shall refine the notion of the trajectory \( t \mapsto r_2(t) \) so that we could prove that the functions involved in amended Feynman’s formulas are of class \( C^3 \) that is the functions and their partial derivatives up to order 3 are continuous on the set \( G \) of points not lying on the trajectory. We will need this class of regularity in order to be able to prove that the fields generated by amended Feynman’s formula coincide with fields generated by means of Liénard-Wiechert potentials.

A trajectory of a path of a point mass can be parameterized in several different ways. It is important to understand which of these parameterizations depend on the Lorentzian frame, which are invariant under Lorentzian transformations and thus belong to Einstein’s special theory of relativity, and which can be carried over to general theory of relativity.

Following Einstein [8] we define a **Lorentzian frame** to consist of an orthogonal coordinate system in \( \mathbb{R}^3 \), having right hand orientation of axes, and equipped with a clock. A physical event in such a frame is described by a point \((r_1, t)\), where \( r_1 \) denotes a position in \( \mathbb{R}^3 \) and \( t \in \mathbb{R} \) the time of the event. Denote such a frame by \( S \). Assume that \( S' \) denotes another Lorentzian frame whose origin initially coincides with the origin of the frame \( S \). Moreover the frame \( S' \) moves as a rigid body away from the frame of \( S \) at a constant velocity. The transformation of coordinates of events from the frame \( S \) into the frame \( S' \) forms a linear transformation that preserves the quadratic form

\[
|r_1'|^2 - (t')^2 = |r_1|^2 - t^2.
\]

By **geometry of Lorentz space-time** we shall understand the product space \( \mathbb{R}^3 \times \mathbb{R} \) with the transformations of coordinates as described above. These transformations form a group with composition of transformations as a group operation. Though one could expand the group by adding affine transformations, the linear transformations are sufficient for description of dynamics in physical processes in
Einstein’s special theory of relativity. Any affine orthogonal transformation can be reduced to a linear one just by moving the origin of the coordinate system.

More general groups of transformations related to Lorentz group were studied by several authors. For generalization of such transformations and further references see Vogt [16].

Let \( \alpha \mapsto y(\alpha) \) be a mapping of an interval \( I \) into \( \mathbb{R}^4 \), of class \( C^3 \), that is having continuous derivatives up to order 3 on the entire interval \( I \). Assume that the mapping forms a parametric representation of a path of a point mass in space.

Moreover assume that the tangent vector field \( y' \) consists of time-like vectors that is

\[
y_1'(\alpha)^2 + y_2'(\alpha)^2 + y_3'(\alpha)^2 - y_4'(\alpha)^2 < 0 \quad \text{for all } \alpha \in \mathbb{R}.
\]

As a derivative of a covariant field with respect to a free parameter the tensor \( y' \) itself forms a covariant field over \( I \). Thus it is invariant under Lorentzian transformations and it can be carried over to the general theory of relativity as in Dirac [7]. The transition from covariant to contravariant tensors is given by means of transformation

\[
y^j = g^{jk} y_k
\]

where summation is with respect to index \( k = 1, 2, 3, 4 \) and the matrix \( g^{jk} \) for orthogonal axes has elements on the diagonal equal respectively 1, 1, 1, -1 and non-diagonal elements are zero.

The time along the path is given by \( t = y_4(\alpha) \), since from the relation (2.1) follows that \( \frac{dt}{d\alpha} = y_4'(\alpha) > 0 \) for all \( \alpha \in I \), the correspondence \( \alpha \mapsto t \) represents a diffeomorphism of \( I \) onto some interval \( J \) and is also of class \( C^3 \), that is both maps \( \alpha \mapsto t = y_4(\alpha) \) and its inverse \( t \mapsto \alpha \) are of class \( C^3 \).

Thus in every Lorentzian frame we can represent our path in the form

\[
y = (r_2(t), t) \quad \text{for all } t \in J,
\]

where \( t \mapsto r_2(t) \) is from some interval \( J \) into \( \mathbb{R}^3 \) and represents the position of the mass as a function of time \( t \) in that Lorentzian frame. Clearly this representation is also of class \( C^3 \) and forms another equivalent parametric representation of the path but this representation is frame dependent.

Most important parametrization of a path is with respect the proper time \( s \) of the moving mass \( m_0 \). It is unique up to an additive constant and can be found from the formula

\[
(ds)^2 = (dy_4)^2 - ((dy_1)^2 + (dy_2)^2 + (dy_3)^2) = (dt)^2 - |dr_2|^2.
\]

The condition (2.1) can be translated into

\[
|\dot{r}_2(t)| = \frac{|dr_2(t)|}{dt} < 1 = c \quad \text{for all } t \in J,
\]

that is velocity along any path of a point mass is less then the speed \( c \) of light. The proper time of a body carries over to general theory of relativity and thus it is also invariant under Lorentzian transformations.

Maxwell established that waves in electromagnetic field propagate with velocity of light \( c \). From considerations of Einstein and Rosen [9] follows that even disturbances in gravity field should propagate with velocity of light.

From results of Bogdan [1] and [2], Proposition 5.2, follows that if we consider the dynamics of \( n \) bodies interacting with each other by means of fields propagating with velocity of light, the equations of evolution are non-anticipating differential
equations and their solutions, not only depend on the initial conditions like in
Newtonian mechanics, but also on the initial trajectory of the entire system.
Assuming for instance that in a Lorentzian frame we are starting with $n$ bodies
whose initial trajectories $t \mapsto y_j(t)$, where $j = 1, \ldots, n$, are known and we intend to
observe the dynamics of evolution of the system for a period of time $t_1$, and we can
apriori estimate the bound $v$ on velocities, and the bound $A$ on the accelerations and
the initial diameter $\delta$ of the system, then the length of the interval of significance
is at most, according to Proposition 5.2 of [2],
$$a = (\delta + 2vt_1)/(c - v).$$
Thus it is sufficient to know the initial trajectories of the system on the closed
interval $[a, 0]$.
We should think about such trajectories as a postmortem record of the trajectory
of some particular body from the system. It is clear that such trajectories would
 correspond to a time interval $J$ that on the left is closed and on the right open or
closed, finite or infinite. In any case it suffices to restrict ourselves to trajectories
defined on intervals of the form $J = [a, b]$ closed on the left and open on the right.
The left end $a$ of such time interval will be called a point of significance.
Thus, if our trajectory $t \mapsto r_2(t)$ is of class $C^2$, from continuity of the velocity
$w(t) = \dot{r}_2(t)$ and of acceleration $\ddot{w}(t)$ on the closed interval $[a, t_1]$ follows that the
following two functions
$$q(t_1) = \sup \{|w(u)| : u \leq t_1, u \in J\} < c,$$
$$A(t_1) = \sup \{\|\dot{w}(u)\| : u \leq t_1, u \in J\} < \infty,$$
are well defined for all stopping times $t_1 \in J$, since the supremum of a continuous
function on a closed bounded interval is attained at some point of that interval.
For the sake of mathematical simplicity we shall consider only trajectories defined
on the entire interval $(-\infty, \infty) = \mathbb{R}$.

**Definition 2.1** (Admissible trajectory). Assume that we are given a path of a
point mass $m_0$ that in some Lorentzian frame has a representation in the form
$y = (r_2(t), t)$, where the function $r_2(t)$ is from $\mathbb{R}$ into $\mathbb{R}^3$ and it has continuous
derivatives up to order 3 and that for any stopping time $t_1 \in J$ the kinetic energy
and the acceleration $\ddot{r}_2(t)$ are bounded on the interval $(-\infty, t_1)$. We shall say that
such a function $r_2(t)$ represents an admissible trajectory.

For the sake of logical completeness we should prove that every trajectory with
a point of significance can be extended on the left to an admissible trajectory. Let
us skip this for present and concentrate on admissible trajectories.

**Proposition 2.2** (Kinetic energy bound and velocity bound). Assume that a body
having rest mass $m_0$ moves along a trajectory $r_2 : \mathbb{R} \to \mathbb{R}^3$. Let $c$ denote the speed of
light. For any nonnegative function $k : \mathbb{R} \to \mathbb{R}$ define function $q : \mathbb{R} \to \mathbb{R}$ by the
formula
$$q(t) = \sqrt{1 - \frac{1}{(1 + k(t)/(m_0c^2))^2}}$$
for all $t \in \mathbb{R}$.
Then for any $t \in \mathbb{R}$ the following two conditions are equivalent
- The kinetic energy of the body $m_0$ on the interval $(-\infty, t)$ is bounded by
  $k(t)$.
• The velocity $|v|$ of the body $m_0$ on the interval $(-\infty, t)$ is bounded by $cq(t)$.

Proof. From Einstein’s formula [8], p. 22, the kinetic energy of mass $m_0$ moving with the velocity $v$ is given by the formula
\[ m_0c^2 \left( \frac{1}{\sqrt{1 - |v|^2/c^2}} - 1 \right). \]
Thus the condition
\[ m_0c^2 \left( \frac{1}{\sqrt{1 - |v(u)|^2/c^2}} - 1 \right) \leq k(t) \quad \text{for all} \quad u \leq t, u \in \mathbb{R} \]
is equivalent to the condition
\[ |v(u)| \leq cq(t) \quad \text{for all} \quad u \leq t, u \in \mathbb{R}. \]
This completes the proof. \[ \square \]

Notice that in the above proposition the quantity $q(t) < 1$ for all $t \in \mathbb{R}$.

**Theorem 2.3** (Admissible trajectory is relativistic). The notion of an admissible trajectory does not depend on the Lorentzian frame.

Proof. Assume that we have two Lorentzian frames $S$ and $S'$. Assume that the frame $S'$ moves away from frame $S$ with constant velocity $u$. Assume that $t \mapsto r_2(t)$ represents an admissible trajectory in the frame $S$ and a body with rest mass $m_0$ is moving along the trajectory.

Without loss of generality we may assume that the frames $S$ and $S'$ are oriented so that the transformation of the coordinates $y = (r, t)$ from $S$ to $S'$ is given by the formulas
\[ y'_1 = y_1 \]
\[ y'_2 = y_2 \]
\[ y'_3 = \gamma (y_3 - uy_4) \]
\[ y'_4 = \gamma (y_4 - uy_3) \]
where $\gamma = (1 - u^2)^{-1/2}$ and $y_4$ and $y'_4$ denote time in the respective frames. As before $c = 1$.

First of all notice that the time interval $(-\infty, \infty)$ maps onto itself from frame $S$ into $S'$. Indeed we have
\[ \frac{dy'_4}{dy_4} = \gamma \frac{dy_4 - u dy_3}{dy_4} = \gamma (1 - uv) \geq \gamma (1 - |u|) > 0 \quad \text{for all} \quad t = y_4 \in \mathbb{R}. \]
Define function $g$ by the formula
\[ g(t) = y'_4(y_4) \quad \text{for all} \quad t = y_4 \in \mathbb{R}. \]
From Cauchy’s mean value theorem we have
\[ g(t) - g(0) = t g'(\theta) \geq t \gamma (1 - |u|) \quad \text{for all} \quad t > 0. \]
Thus $y_4 = g(t) \to \infty$ if $t \to \infty$. Similarly
\[ g(t) - g(0) = t g'(\theta) \leq t \gamma (1 - |u|) \quad \text{for all} \quad t < 0. \]
Thus \( y_4 = g(t) \to -\infty \) if \( t \to -\infty \). Since any continuous function maps an interval onto an interval the function \( g \) maps \( \mathbb{R} \) onto \( \mathbb{R} \).

Introduce a function \( f \) by the formula

\[
 f(w) = \left( \frac{1}{\sqrt{1 - w^2}} - 1 \right) \quad \text{for all} \quad w \geq 0.
\]

Notice that the function \( f \) is nondecreasing and the kinetic energy of the mass \( m_0 \) moving along the trajectory can be represented as

\[
 m_0 f(|v|)
\]

where

\[
 v = \frac{dy_3}{dy_4} = \dot{r}_2
\]

is the velocity of the body in the frame \( S \).

The velocity of the body in frame \( S' \) is given by

\[
 v' = \frac{dy'_3}{dy'_4} = \frac{dy_3 - u dy_4}{dy_4 - u dy_3} = \frac{v - u}{1 - uv}
\]

Thus we have the estimate

\[
 |v'| \leq \frac{|v| + |u|}{1 - |u|} \leq \frac{q(t) + |u|}{1 - |u|}
\]

for all times in the initial interval \(( -\infty, t)\). The quantity \( q(t) \) denotes the velocity bound on the initial interval. Thus the velocity \( v' \) is bounded on every initial interval \(( -\infty, t')\) in the frame \( S' \). Therefore its kinetic energy is bounded on every initial interval.

Now let us consider the acceleration in the frame \( S' \). It can be expressed as

\[
 \frac{dv'}{dy'_4} = \frac{\dot{v}(1 - u^2)}{\gamma(1 - uv)^3}
\]

in terms of quantities in frame \( S \). Thus on every initial interval \(( -\infty, t)\) we have the estimate

\[
 \left| \frac{dv'}{dy'_4} \right| \leq \frac{A(t)(1 - u^2)}{\gamma(1 - |u|)^3}
\]

where \( A(t) \) is the bound on the acceleration in the initial time interval \(( -\infty, t)\) in the frame \( S \). Hence the acceleration in the frame \( S' \) is bounded on every initial time interval \(( -\infty, t')\).

Therefore the trajectory in the frame \( S' \) forms an admissible trajectory. \( \square \)

### 3. Retarded Time Field

Now consider any point \(( r_1, t)\) in a fixed Lorentzian frame and let \(( r_2(\tau), \tau)\) denote a point on the path of the point mass with the property that light beam emitted from the trajectory will arrive at position \( r_1 \) at time \( t \).

The time \( \tau \) is called the **retarded time**. It must satisfy the relation

\[
 |r_1 - r_2(\tau)|^2 - (t - \tau)^2 = 0,
\]

which is preserved under Lorentzian transformations.

The following theorem establishes that the retarded time is well defined as a function of the variables \(( r_1, t) \in \mathbb{R}^3 \times \mathbb{R} \).
Theorem 3.1 (The retarded time \( \tau \) is unique and forms a continuous function). Assume that we are given in a Lorentzian frame an admissible trajectory \( t \mapsto r_2(t) \). Then for any point \( r_1 \in \mathbb{R}^3 \) and any time \( t \in \mathbb{R} \) there exists a unique number \( \tau \leq t \) such that

\[
\tau = t - |r_1 - r_2(\tau)|.
\]

Moreover the map \( (r_1, t) \mapsto \tau \) represents a locally Lipschitzian function on the space \( \mathbb{R}^3 \times \mathbb{R} \). Thus \( \tau(r_1, t) \) is continuous on \( \mathbb{R}^3 \times \mathbb{R} \).

Proof. For fixed \( r_1 \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \) introduce a function \( f \) by the formula

\[
|f(s)| = t - |r_1 - r_2(s)| \quad \text{for all } s \leq t.
\]

The function \( f \) is well defined and maps the closed interval \( (-\infty, t) \) into itself. The function represents a contraction. Indeed

\[
|f(s) - f(\tilde{s})| = |(t - |r_1 - r_2(s)|) - (t - |r_1 - r_2(\tilde{s})|)| \leq |r_2(s) - r_2(\tilde{s})|
\]

(3.1)

where \( v_1 = q(t) < c = 1 \) is the velocity bound corresponding to stopping time \( \tau \). Therefore by Banach’s contraction mapping theorem there exists one and only one solution of the equation \( \tau = f(\tau) \).

Thus the map \( (r_1, t) \mapsto \tau \) is well defined in our Lorentzian frame for all points \( (r_1, t) \in \mathbb{R}^3 \times \mathbb{R} \).

To prove that the function \( \tau \) is locally Lipschitzian it suffices to prove that it is Lipschitzian on every open set of the form \( \mathbb{R}^3 \times (-\infty, t_1) \). To this end take any two points \( (r_1, t) \) and \( (\tilde{r}_1, \tilde{t}) \) from the domain of \( \tau \) such that \( t, \tilde{t} < t_1 \). Let \( v_1 < 1 \) denote the velocity bound corresponding to our trajectory on the interval \( (-\infty, t_1) \).

To avoid unnecessarily complex notation denote by \( \tau \) and \( \tilde{\tau} \) the retarded times corresponding to the points \( (r_1, t) \) and \( (\tilde{r}_1, \tilde{t}) \) respectively. We have

\[
|\tau - \tilde{\tau}| = |f(\tau) - f(\tilde{\tau})| = |(t - |r_1 - r_2(\tau)|) - (\tilde{t} - |\tilde{r}_1 - r_2(\tilde{\tau})|)|
\]

\[
\leq |t - \tilde{t}| + |r_1 - \tilde{r}_1| + |r_2(\tau) - r_2(\tilde{\tau})|
\]

\[
= |t - \tilde{t}| + |r_1 - \tilde{r}_1| + \int_\tau^{\tilde{\tau}} |\dot{r}_2(u)| du
\]

\[
\leq |t - \tilde{t}| + |r_1 - \tilde{r}_1| + v_1|\tau - \tilde{\tau}|.
\]

Taking the last term in the above inequality onto the left side and dividing by \((1 - v_1)\) both sides of the obtained inequality we get

\[
|\tau(r_1, t) - \tau(\tilde{r}_1, \tilde{t})| \leq \frac{1}{1 - v_1}(|t - \tilde{t}| + |r_1 - \tilde{r}_1|) \quad \text{for all } (r_1, t), (\tilde{r}_1, \tilde{t}) \in \mathbb{R}^3 \times (-\infty, t_1)
\]

Thus the function \( \tau \) is continuous on the entire space \( \mathbb{R}^3 \times \mathbb{R} \). \( \square \)

For a proof of Banach’s contraction mapping theorem see, for instance, Loomis and Sternberg [13] page 229.

Theorem 3.2 (An explicit formula for the retarded time function \( \tau \)). Assume that we are given in a Lorentzian frame an admissible trajectory \( t \mapsto r_2(t) \).

Take any stopping time \( t_1 \in \mathbb{R} \) and let \( v_1 = q(t_1) < c \) denote the corresponding velocity bound for \( t \leq t_1 \).
Put $s_0(r_1, t) = 0$ and define recursively the sequence

$$s_n(r_1, t) = f(s_{n-1}(r_1, t)) \quad \text{for all } n = 1, 2, 3, \ldots; \text{ and } r_1 \in R^3, t \leq t_1,$$

where $f(s) = t - |r_1 - r_2(t - s)|$ for all $s \leq t$.

The retarded function $\tau$ is given by the formula

$$\tau(r_1, t) = \lim_n s_n(r_1, t) \quad \text{for all } r_1 \in R^3 \text{ and } t \in R.$$

Moreover we have the following convenient estimate for the rate of convergence

$$|\tau(r_1, t) - s_n(r_1, t)| \leq \frac{v_1^n}{1 - v_1^4} |t - |r_1 - r_2(t)|| \quad \text{for all } r_1 \in R^3 \text{ and } t \leq t_1.$$

Proof. The proof follows from Theorem 4.7 page 37 of Bogdan [2] or Theorem 9.1 on page 229 of Loomis and Sternberg [13].

4. THE FUNDAMENTAL FIELDS ASSOCIATED WITH AN ADMISSIBLE TRAJECTORY

Now define the delay function $T(r_1, t) = t - \tau(r_1, t)$ and notice that it satisfies the equation

$$T = |r_1 - r_2(t - T)| \quad \text{for all } t \in R \text{ and } r_1 \in R^3.$$  \hspace{1cm} (4.1)

It is important to stress that in this paper we shall use the term field as synonymous with a function defined on a set of the space $R^4$. The space $R^3 \times R = R^4$ is treated as a fixed Lorentzian frame. So we will use such expressions as scalar field, vector field, tensor field, etc., to describe the functions taking values in corresponding spaces. Thus the functions $\tau$ and $T$ represent continuous scalar fields defined on the entire space $R^4$.

Since the function $T$ as difference of two continuous functions is continuous the set

$$G = \{(r_1, t) \in R^3 \times R : T(r_1, t) > 0\} = T^{-1}(0, \infty)$$

as an inverse image of an open set by means of a continuous function is itself open. The set $G$ consists of points that do not lie on the trajectory.

By assumption the trajectory $r_2$ has continuous derivatives $\dot{r}_2(t) = w(t)$ and $\dot{w}(t)$, so we can define the vector fields

$$r_{12}(r_1, t) = r_1 - r_2(\tau(r_1, t)), \quad v(r_1, t) = \dot{r}_2(\tau(r_1, t)), \quad a(r_1, t) = \dot{w}(\tau(r_1, t))$$

for all $(r_1, t) \in R^4$. Introduce the unit vector field $e = r_{12}/T$ and fields $u$ and $z$ by the formulas

$$u = \frac{1}{T} \quad \text{and} \quad z = \frac{1}{(1 - \langle e, v \rangle)} \quad \text{on} \ G.$$  

In the above $\langle e, v \rangle$ denotes the dot product of the vectors $e$ and $v$. Since $|\langle e, v \rangle| \leq |v| < c = 1$ the vector field $z$ is well defined on the set $G$.

**Definition 4.1** (Fundamental fields). Assume that we are given in a Lorentzian frame an admissible trajectory $t \mapsto r_2(t)$.

Define the time derivative $\dot{w}(t) = \dot{r}_2(t)$. The fields given by the formulas

$$\tau, T, r_{12} = r_1 - r_2 \circ \tau, \quad v = \dot{w} \circ \tau, \quad a = \ddot{w} \circ \tau \quad \text{for all } (r_1, t) \in G$$

and

$$u = 1/T, \quad e = u r_{12}, \quad \text{and} \quad z \quad \text{for all } (r_1, t) \in G.$$
will be called the fundamental fields associated with the trajectory \( r_2(t) \). The operation \( \circ \) denotes here the composition of functions.

The fundamental fields are continuous on their respective domains. This follows from the fact that composition of continuous functions yields a continuous function. Thus all of them, for sure, are continuous on the open set \( G \) of points that do not lie on the trajectory.

Analogous fields defined by similar formulas on an open set \( G \subset \mathbb{R}^4 \) appear in the problems involving plasma flows \([3]\) and \([4]\), or more generally flows of matter.

We would like to stress here that the fundamental fields depend on the Lorentzian frame, in which we consider the trajectory. It is important to find expressions involving fundamental fields that yield fields invariant under Lorentzian transformations.

Lorentz and Einstein \([8]\), Part II, section 6, established that fields satisfying Maxwell equations are invariant under Lorentzian transformations.

Our main goal is to prove that fields given by amended Feynman formulas and fields obtained from Liénard-Wiechert potentials satisfy Maxwell equations. We shall do this by showing that these fields are representable by means of fundamental fields and using the formulas for partial derivatives of the fundamental fields prove that such fields generate fields satisfying Maxwell equations.

The following theorem represents the main pillar of the entire structure of the proof. Consider each of the formulas as bricks from which the pillar is constructed. If any one brick is rotten the whole structure of the proof will collapse. This is just a gentle warning to an impatient reader not to skip the computations involved.

Introduce operators \( D = \frac{\partial}{\partial t} \) and \( D_i = \frac{\partial}{\partial x^i} \) for \( i = 1, 2, 3 \) and \( \nabla = (D_1, D_2, D_3) \).

Observe that \( \delta_i \) in the following formulas denotes the \( i \)-th unit vector of the standard base in \( \mathbb{R}^3 \) that is \( \delta_1 = (1, 0, 0), \delta_2 = (0, 1, 0), \delta_3 = (0, 0, 1) \).

**Theorem 4.2** (Partial derivatives of fundamental fields). Assume that in some Lorentzian frame we are given an admissible trajectory \( t \mapsto r_2(t) \). Define the time derivative \( w(t) = \dot{r}_2 \circ \tau \). For partial derivatives with respect to coordinates of the vector \( r_1 \) we have the following identities on the set \( G \)

\[
\begin{align*}
(4.2) \quad D_i T &= ze_i \quad \text{where} \quad v = \dot{r}_2 \circ \tau, \\
(4.3) \quad D_i u &= -zu^2 e_i, \\
(4.4) \quad D_i v &= -e_i z a \quad \text{where} \quad a = \dot{w} \circ \tau, \\
(4.5) \quad D_i \tau &= -ze_i, \\
(4.6) \quad D_i e &= -uze_i e + u\delta_i + u ze_i v \quad \text{where} \quad \delta_i = (\delta_{ij}), \\
(4.7) \quad D_i z &= -z^3 e_i (e, a) - u z^3 e_i + uz^2 e_i + uz^2 v_i + uz^2 e_i (v, v), \\
(4.8) \quad \nabla T &= ze, \\
(4.9) \quad \nabla u &= -zu^2 e, \\
(4.10) \quad \nabla z &= -z^3 (e, a) e - uz^3 e + uz^2 e + uz^2 v + uz^3 (v, v)e.
\end{align*}
\]
and for the partial derivative with respect to time we have

\begin{align}
(4.11) & \quad DT = 1 - z, \\
(4.12) & \quad Du = zu^2 - u^2, \\
(4.13) & \quad D\tau = z, \\
(4.14) & \quad Dv = za \quad \text{where} \quad a = \dot{w} \circ \tau, \\
(4.15) & \quad De = -ue + uz e - uz v, \\
(4.16) & \quad Dz = uz - 2uz^2 + z^3(e,a) + uz^3 - uz^3\langle v,v \rangle.
\end{align}

Since the expression on the right side of each formula represents a continuous function, the fundamental fields are at least of class \( C^1 \) on the set \( G \). Moreover if the trajectory is of class \( C^{\infty} \) then also the fundamental fields are of class \( C^{\infty} \) on \( G \).

**Proof.** Proof of formula (4.2): Applying the operator \( D_i \) to both sides of equation (4.1) we get

\[
D_i T = \langle e, D_i(r_1 - r_2) \rangle = \langle e, (\delta_{ij}) \rangle - \langle e, v(\tau) \rangle D_i \tau
\]

yielding formula (4.2).

Formulas (4.3), (4.8), and (4.9) follow from formula (4.2).

**Proof of formula (4.4):**

\[
D_i v = a D_i \tau = a(-D_i T) = -e_iz a
\]

**Proof of formula (4.5):**

\[
D_i \tau = D_i(t - T) = -D_i T = -ze_i
\]

**Proof of formula (4.6):**

\[
D_i e = D_i[u(r_1 - r_2)] = [D_iu][r_{12}] + u[D_ir_1 - D_ir_2]
\]

\[
= [D_iu]u^{-1}e + u[D_ir_1 - D_ir_2]
\]

\[
= -zu^2 e_i u^{-1}e + u\delta_i - u[D_i \tau]v
\]

\[
= -uze_i e + u\delta_i - u[-ze_i]v
\]

\[
= -uze_i e + u\delta_i + uz e_i v
\]

**Proof of formula (4.7):**

\[
D_i z = D_i(1 - \langle e, v \rangle)^{-1} = (-1)z^2(-D_i\langle e, v \rangle)
\]

\[
= z^2[(e, D_i v) + (v, -uze_i e + u\delta_i + uz e_i v)]
\]

\[
= z^2[(e, -e_iz a) + (v, -uze_i e + u\delta_i + uz e_i v)]
\]

\[
= -z^3 e_i \langle e, a \rangle - uz^3 e_i \langle v, e \rangle + uz^2 e_i + uz^3 e_i \langle v, v \rangle
\]

\[
= -z^3 e_i \langle e, a \rangle - uz^3 e_i [1 - z^{-1}] + uz^2 v_i + uz^3 e_i \langle v, v \rangle
\]

\[
= -z^3 e_i \langle e, a \rangle - uz^3 e_i + uz^2 e_i + uz^2 v_i + uz^3 e_i \langle v, v \rangle
\]

Formula (4.10) follows from the formula (4.7).
Proof of formula (4.11): Applying the operator $D$ to both sides of the equation (4.1) we get

$$DT = D|r_{12}| = (e, D[r_{12}]) = -(e, Dr_2)$$
$$= -(e, v)D\tau = -(e, v)(1 - DT) = -(e, v) + (e, v)DT.$$  

The above yields

$$DT = \frac{-(e, v)}{1 - (e, v)} = 1 - z.$$  

Proof of formula (4.12):

$$Du = DT^{-1} = (-1)T^{-2}DT = (-1)u^2(1 - z) = zu^2 - u^2.$$  

Proof of formula (4.13):

$$D\tau = D(t - T) = 1 - DT = 1 - (1 - z) = z.$$  

Proof of formula (4.14):

$$Dv = aD\tau = za.$$  

Proof of formula (4.15):

$$De = D[ur_{12}] = [Du]r_{12} + u[Dr_{12}]$$
$$= [zu^2 - u^2]u^{-1}e - u[Dr_2]$$
$$= [zu - u]e - u[(D\tau)v]$$
$$= zue - ue - uzv = -ue + uze - uzv.$$  

Proof of formula (4.16):

$$Dz = D(1 - (e, v))^{-1} = (1 - (e, v))^{-2}D(e, v)$$
$$= z^2\langle De, v \rangle + z^2\langle e, De \rangle$$
$$= z^2(ue - u - uzv), v \rangle + z^2(e, za)$$
$$= uz^3\langle e, v \rangle - uz^2\langle e, v \rangle - uz^3\langle v, v \rangle + z^3\langle e, a \rangle$$
$$= uz^3[1 - z^{-1}] - uz^2[1 - z^{-1}] - uz^3\langle v, v \rangle + z^3\langle e, a \rangle$$
$$= uz^3 - uz^2 + uz - uz^3\langle v, v \rangle + z^3\langle e, a \rangle$$
$$= uz^3 - 2uz^2 + uz - uz^3\langle v, v \rangle + z^3\langle e, a \rangle$$
$$= uz - 2uz^2 + z^3\langle e, a \rangle + uz^3 - uz^3\langle v, v \rangle$$

$\Box$

5. Electromagnetic potentials satisfying wave and gauge equations induce field satisfying Maxwell equations

The following theorem represents the second main pillar of the structure. Each step one should prove in detail to see that no heuristic is introduced. The proof of this theorem one can trace back to the work of Lorentz [14].
Theorem 5.1 (Wave and gauge imply Maxwell equations). Let on an open set $G \subset \mathbb{R}^4$ be given two scalar fields $\phi$ and $S$ and two vector fields $A$ and $J$.

Assume that the fields $\phi$ and $A$ have second partial derivatives with respect to the coordinates of the point $(r_1, t) \in G$ and these derivatives are continuous on the entire set $G$.

If these fields satisfy the following wave equations

$$\nabla^2 \phi - \frac{\partial^2}{\partial t^2} \phi = -S, \quad \nabla^2 A - \frac{\partial^2}{\partial t^2} A = -J,$$

and Lorentz gauge formula

$$\nabla \cdot A + \frac{\partial}{\partial t} \phi = 0$$

on the set $G$, then the fields $E$ and $B$ defined by the formulas

$$E = -\nabla \phi - \frac{\partial}{\partial t} A \quad \text{and} \quad B = \nabla \times A$$

for all $(r_1, t) \in G$ will satisfy the following Maxwell equations

$$(a) \quad \nabla \cdot E = S, \quad (b) \quad \nabla \times E = -\frac{\partial}{\partial t} B,$$

$$(c) \quad \nabla \cdot B = 0, \quad (d) \quad \nabla \times B = \frac{\partial}{\partial t} E + J$$

(5.1)

on the entire set $G$.

Proof. In this proof we will use the usual notation for dot product of two vectors $F \cdot H = \langle F, H \rangle$ to be close to the customary notation used in textbooks. The following proof represents a reversed argument presented in Feynman-Leighton-Sands [10], vol. 2, chapter 18, section 6. Namely we show that potentials, that satisfy wave and Lorentz gauge equations, induce a pair of fields $E$ and $B$ satisfying Maxwell equations.

Since for any vector field $F$, which is twice continuously differentiable, we have $\nabla \cdot (\nabla \times F) = 0$, the vector field $B$ must satisfy Maxwell’s equation (c).

Since the differential operators $\nabla$ and $D$ commute, due to the assumption that the fields $\phi$ and $A$ are of class $C^2$, and since from the Lorentz gauge formula follows that $\nabla \cdot A = -D\phi$, we must have

$$\nabla \cdot E = \nabla \cdot (-\nabla \phi - DA) = -\nabla^2 \phi - D\nabla \cdot A = -\nabla^2 \phi + D^2 \phi = S.$$

Therefore the equation (a) is satisfied.

Since $\nabla \times (\nabla h) = 0$ for any scalar function, that is twice continuously differentiable on the set $G$, and for any such vector function $F$ defined on the open set $G$ we have $\nabla \times DF = D\nabla \times F$, we get

$$\nabla \times E = \nabla \times (-\nabla \phi - DA) = -D\nabla \times A = -DB$$

and this means that Maxwell’s equation (b) is satisfied.

Finally for the last equation, Maxwell’s equation (d), from the algebraic identity

$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$$

and Lorentz gauge equation and the wave equation

$$\nabla \cdot A + D\phi = 0, \quad \nabla^2 A - D^2 A = -J$$
we get
\[ \nabla \times B - D E = \nabla \times (\nabla \times A) - D(-\nabla \phi - D A) \]
\[ = \nabla(\nabla \cdot A) - \nabla^2 A + D\nabla \phi + D^2 A \]
\[ = \nabla[(\nabla \cdot A) + D\phi] - (\nabla^2 A - D^2 A) = J. \]

\[ \square \]

Notice that in the above theorem the field \( S \) corresponds to the field \( \rho \) of charge density and \( J \) to the field \( j \) of intensity of charge flow. Since we redefined the electrostatic constant \( \epsilon_0 \) to satisfy the condition \( 4\pi \epsilon_0 = 1 \) our new unit of charge became equal to \( 4\pi \) [coulomb].

It is worthwhile to mention also the following proposition.

**Proposition 5.2** (Equation of continuity for flow of charge). *If the electromagnetic potentials \( \phi \) and \( A \) have continuous partial derivatives of order 3 on an open set \( G \subset \mathbb{R}^4 \), then from the Lorentz gauge equation*

\[ D\phi + \nabla \cdot A = 0 \quad \text{for all} \quad (r_1, t) \in G \]

*follows the equation*

\[ DS + \nabla \cdot J = 0 \quad \text{for all} \quad (r_1, t) \in G \]

*of continuity for charge flow.*

**Proof.** Indeed introduce the D’Alembertian operator

\[ \square^2 = D^2 - \nabla^2. \]

From our assumptions follows the commutativity of the differential operators

\[ D, \quad \nabla, \quad \square^2 \]

when acted on fields \( \phi \) and \( A \), respectively. Hence we have

\[ \square^2 D = D \square^2 \quad \text{and} \quad \square^2 \nabla = \nabla \square^2. \]

The above implies

\[ DS + \nabla \cdot J = D \square^2 \phi + \nabla \cdot \square^2 A = \square^2(D\phi + \nabla \cdot A) = 0 \]

and this is the equation of continuity of flow of charge. \[ \square \]
6. LIÉNARD-WIECHERT POTENTIALS AND WAVE AND GAUGE EQUATIONS

In this section we will deal only with Liénard-Wiechert potentials

\[ \phi = uz \quad \text{and} \quad A = uzv. \]

The author in the proof of the following theorem used DOS based interactive computer software called DERIVE allowing to input user data files and perform on them interactive operations. More details about DERIVE and actual programs used in computation we shall present in the last section of the paper.

Any program, that can symbolically manipulate polynomials of an arbitrary number of variables, can be used to verify the proof. For a reader, who would like personally to verify the proof by hand or another software system, we present here just the key intermediate formulas as computational mile stones.

In the following formulas the vector field \( \hat{a} \) denotes the composition of the function \( \frac{\partial}{\partial \tau}f_2(t) \) with the retarded time field \( \tau \).

**Theorem 6.1** (Homogenous wave and gauge equations are satisfied). For any admissible trajectory \( r_2(t) \) of class \( C^3 \) the two homogenous wave equations

\[ (\nabla^2 - D^2)[uz] = 0, \quad (\nabla^2 - D^2)[uzv] = 0, \]

and Lorentz gauge equation

\[ \nabla \cdot [uzv] + D[uz] = 0 \]

are satisfied on the entire set \( G = \{(r_1, t) \in R^4 : T(r_1, t) > 0\} \).

**Proof.** We have the following identities

\[ \nabla^2[uz] = D^2[uz] = +uz^4\dot{e} + u^2z^3(e, a) + 3uz^5(e, a)^2 + 3u^3z^2 + 3u^3z^5 + 3u^3z^5(v, v)^2 + 6u^2z^5(e, a) + 6u^3z^4(v, v) \]
\[ - 3u^2z^4(v, a) - 6u^2z^3(e, a) - 6u^3z^5(v, v)(e, a) - 6u^3z^4 + 6u^3z^5(v, v) - u^3z^3(v, v) \]

Thus the first wave equation takes the form

\[ \nabla^2[u] - D^2[u] = 0. \]

Similarly for the second wave equation we have the formulas

\[ \nabla^2[uzv] = D^2[uzv] = +uz^4\dot{e} + u^2z^3(e, a) + u^2z^2a + u^2z^3(e, a)v + 3uz^5(e, a)a + 3u^2z^4a + 3u^3z^3v \]
\[ + 3u^3z^5(v, v)^2 + 3u^3z^5v + 6u^2z^5(e, a)v + 6u^3z^4(v, v)v \]
\[ - u^2z^4(v, v)v - 3u^2z^4(e, a)v + 3u^3z^4(v, v)v + 4u^3z^3a \]
\[ - 6u^2z^3(e, a)v - 6u^2z^3(v, v)(e, a)v - 6u^3z^4v - 6u^3z^5(v, v)v. \]

Computing terms of Lorentz gauge equation we get

\[ \nabla \cdot [uzv] = \langle \nabla, uzv \rangle = +u^2z^3(v, v) - u^2z^3 + u^2z^2 - uz^3(e, a) \]
and
\[ D[uz] = -u^2z^3(v,v) + u^2z^3 - u^2z^2 + uz^3(e,a). \]
Hence
\[ \langle \nabla, u z v \rangle + D[uz] = 0. \]
\[ \Box \]

Liénard-Wiechert potentials generate a solution to a homogenous system of Maxwell equations. Indeed from previous theorems we get the following corollary.

**Corollary 6.2** (Liénard-Wiechert potentials and Maxwell equations). Let \( r_2(t) \) be any admissible trajectory and let \( G \) denote the open set of points that do not lie on the trajectory, that is
\[ G = \{ (r_1,t) \in R^4 : T(r_1,t) > 0 \}. \]
Define the fields by formulas
\[ \phi = uz, \quad A = u z v, \quad S = 0, \quad J = 0 \quad \text{for all} \quad (r_1,t) \in G. \]
Then the fields \( E \) and \( B \) defined by the equalities
\[ E = -\nabla \phi - \frac{\partial}{\partial t} A \quad \text{and} \quad B = \nabla \times A \quad \text{for all} \quad (r_1,t) \in G \]
will satisfy the following homogenous Maxwell equations
\[ \nabla \cdot E = 0, \quad \nabla \times E + \frac{\partial}{\partial t} B = 0, \quad \nabla \cdot B = 0, \quad \nabla \times B - \frac{\partial}{\partial t} E = 0 \]
on the set \( G \).

7. Proof of the amended Feynman’s formulas

As before let \( r_2(t) \) be any admissible trajectory of class \( C^3 \) in some Lorentzian frame and let
\[ G = \{ (r_1,t) \in R^4 : T(r_1,t) > 0 \} \]
be the set of points not lying on the trajectory. Let \( \phi = uz \) and \( A = u z v \) denote the Liénard-Wiechert potentials corresponding to the trajectory \( r_2 \).

We remind the reader that all intensities of the considered fields are per unit of charge, thus if we are dealing with a charge \( q \) the corresponding potentials will take the form \( \phi = quz \) and \( A = qu z v \) and similar adjustments should be made with the fields \( E, B \).

Feynman using his formula for the electromagnetic potentials, see vol. 2, Feynman-Leighton-Sands [10], chapter 15, page 15.15, had derived on page 21.9, of chapter 21, a formula for electromagnetic potential of a moving point charge. A formula, which did not agree with the Liénard-Wiechert formulas.

Nevertheless on page 21.11 in the last sentence he suggested that perhaps the reader may take his word for it, that Liénard-Wiechert potentials yield the same field as his formulas in the case of a moving point charge. In the footnote he mentioned that he has done that, but it took him a lot of time and paper.
This little footnote provided the author with the main impetus to try to verify the computations. Feynman did not leave any hints concerning possible simplifications of the computations or that quantities involved should be considered as fields.

The author taking Feynman’s suggestion performed by hand, step by step, computations presented in the following proof. These computations have provided an essential hint how to develop and to structure the proofs of the theorems presented in this paper.

Thus the credit has to be given with gratitude to Feynman for discovery of this gem stone of Einstein’s Special Theory of Relativity and consequently of Maxwell’s Theory of Electrodynamics.

To verify in a more efficient way that the electric fields $E$ and $E_f$ coincide, we provide a computer program in the last section of this paper. However to show that the magnetic fields $B$ and $B_f$ are equal a direct computation seems to be more efficient.

To Feynman are due immense thanks and gratitude beyond words! Here are the computations as they might have looked under Feynman’s hand.

**Theorem 7.1** (Equality of amended Feynman and Liénard-Wiechert fields). Assume that in some Lorentzian frame we are given an admissible trajectory $r_2(t)$ of class $C^3$.

Let

$$G = \{(r_1, t) \in R^4 : T(r_1, t) > 0\}$$

denote the set of points not lying on the trajectory.

Let $u$ and $e$ denote the fundamental fields (4.1) generated by the trajectory $r_2(t)$.

Define fields

$$E = -\nabla \phi - DA \quad \text{and} \quad B = \nabla \times A,$$

$$E_f = u^2 e + u^{-1} D(u^2 e) + D^2 e \quad \text{and} \quad B_f = e \times E_f,$$

Then we have the equalities $E = E_f$ and $B = B_f$ on the set $G$.

**Proof.** From the formulas (4.10) and (4.9) we get

$$\nabla \phi = \nabla (zu) = u \nabla z + z \nabla u$$

$$= u[-z^3 \langle e, a \rangle e - u z^3 e + u z^2 v + u z^3 \langle v, v \rangle e] + z[-u^2 e]$$

$$= [-u z^3 \langle e, a \rangle e - u^2 z^3 e + u^2 z^2 v + u^2 z^3 \langle v, v \rangle e] - u^2 z^2 e$$

$$= -u z^3 \langle e, a \rangle e + u^2 z^2 v - u^2 z^3 e + u^2 z^3 \langle v, v \rangle e$$

and

$$D[uv] = [Du]v + u Dv = [zu^2 - u^2]v + uza$$

$$= u z v - u^2 v + uza$$

$$= uza - u^2 v + u^2 z v$$
Now consider the second term of the expression $E$. For the third term of the expression $Dzuv = [Dz]uv + zD[uv]$

$$= [uz - 2uz^2 + z^3\langle e, a \rangle + uz^3 - uz^3\langle v, v \rangle]uv + z[uz - u^2v + u^2zv]$$

$$= u^2zv - 2u^2z^2v + uz^3\langle e, a \rangle v + u^2z^3v$$

$$- u^2z^3\langle v, v \rangle v + uz^2a - u^2zv + u^2z^2v$$

$$= u^2a - u^2z^2v + uz^3\langle e, a \rangle v + u^2z^3v - u^2z^3\langle v, v \rangle v$$

thus

$$E = -\nabla\phi - DA$$

$$= -[-uz^3\langle e, a \rangle e + u^2z^2v - u^2z^3e + u^2z^3\langle v, v \rangle e]$$

$$- [uz^2a - u^2zv + uz^3\langle e, a \rangle v + u^2z^3v - u^2z^3\langle v, v \rangle v]$$

$$= -[uz^2a - u^2z^2v + uz^3\langle e, a \rangle v$$

$$- u^2z^3e + u^2z^3\langle v, v \rangle e + u^2z^3v - u^2z^3\langle v, v \rangle v]$$

Now consider the second term of the expression $E_f$. We have

$$u^{-1}D[u^2e] = u^{-1}[D[u^2]]e + uDe$$

$$= u^{-1}2u[Du]e + u[-ue + uze - uze]$$

$$= 2[u^2z - u^2e - u^2zv]$$

$$= 2u^2ze - 2u^2e - u^2ze - u^2zv$$

$$= 3u^2ze - 3u^2e - u^2zv$$

$$= 3u^2ze - 3u^2e - u^2zv$$

For the third term of the expression $E_f$ we get

$$D^2e = D[De] = D[-ue + uze - uze] = D\{u[z(e - v) - e]\}$$

$$= \{Du\}[z(e - v) - e] + uD[z(e - v) - e]$$

$$= \{u^2z - u^2\}[ze - zv - e]$$

$$+ u[(Dz)(e - v) + zD(e - v) - De]$$

$$= 2u^2e - 3u^2ze + u^2zv - uz^2a + uz^3(e, a) - uz^3(e, a)v$$

$$+ u^2z^3e - u^2z^3\langle v, v \rangle e - u^2z^3v + u^2z^3\langle v, v \rangle v$$
Compute the field $E_f$

$$E_f = u^2 e + u^{-1} D(u^2 e) + D^2 e$$

$$= u^2 e + [-3u^2 e + 3u^2 ze - u^2 zv]$$

$$+ [2u^2 e - 3u^2 ze + u^2 zv - uz^3 e, a] e - uz^3 e, a v$$

$$+ u^2 z^3 e - u^2 z^3 (v, v)e - u^2 z^3 v + u^2 z^3 (v, v)v$$

$$= - uz^2 a + uz^3 (e, a)e - uz^3 (e, a)v$$

$$+ u^2 z^3 e - u^2 z^3 (v, v)e - u^2 z^3 v + u^2 z^3 (v, v)v$$

and compare with

$$E = -[uz^2 a - uz^3 (e, a)e + uz^3 (e, a)v$$

$$- u^2 z^3 e + u^2 z^3 (v, v)e + u^2 z^3 v - u^2 z^3 (v, v)v].$$

Now the magnetic field from Feynman’s formula is

$$B_f = e \times E_f = e \times [-uz^2 a - uz^3 (e, a)v - u^2 z^3 v + u^2 z^3 (v, v)v]$$

$$= - uz^2 e \times a - uz^3 (e, a) e \times v - u^2 z^3 e \times v + u^2 z^3 (v, v) e \times v$$

and since

$$\nabla [uz] = - uz^3 (e, a)e + u^2 z^3 v - u^2 z^3 e + u^2 z^3 (v, v)e,$$

$$\nabla \times v = - z e \times a,$$

we have from Liénard-Wiechert potentials that

$$B = \nabla \times A = \nabla \times [uzv] = [uz] \nabla \times v + \nabla [uz] \times v$$

$$= [uz](- z e \times a) + (- uz^3 (e, a)e + u^2 z^3 v)$$

$$- u^2 z^3 e + u^2 z^3 (v, v)e \times v$$

$$= - uz^2 e \times a - uz^3 (e, a)e \times v - u^2 z^3 e \times v + u^2 z^3 (v, v)e \times v$$

Clearly we have the equality for both fields

$$E = E_f \quad \text{and} \quad B = B_f.$$

**Eureka! A salute is due to Feynman!**

Now let us collect the formulas representing the fields $E$ and $B$ by means of the fundamental fields (111) into one convenient statement. These formulas can be very useful for computations especially with the help of a digital computer.

Notice that all the fundamental fields generated by the admissible trajectory are defined by means of algebraic operations and operation of composition of functions, with the exception of the retarded time field $\tau$, but Banach’s fixed point theorem (3.2) provides a fast converging algorithm for computing this function.

**Proposition 7.2** (Representation of amended Feynman’s fields by fundamental fields). Assume that in some Lorentzian frame we are given an admissible trajectory $r_z(t)$ of class $C^1$ of a moving point mass $m_0$. Let

$$G = \{(r_1, t) \in R^4 : T(r_1, t) > 0\}$$

de note the set of points not lying on the trajectory. Define fields

$$E = u^2 e + u^{-1} D(u^2 e) + D^2 e \quad \text{and} \quad B = e \times E,$$
Then the fields $E$ and $B$ have the following representation in terms of the fundamental fields (4.1) generated by the trajectory $r_2(t)$.

\[
E = -uz^2a + uz^3(e,a)e - uz^3(e,a)v \\
+ u^2z^3e - u^2z^3(v,v)e - u^2z^3v + u^2z^3(v,v)v,
\]

\[
B = -uz^3 e \times a - uz^3(e,a)e \times v - u^2z^3 e \times v + u^2z^3(v,v)e \times v.
\]

8. Modified theory of generalized functions

Now let us prove that any pair $(E, B)$ of fields of class $C^2$ on an open set $G$ in $\mathbb{R}^4$ that satisfies the homogenous system of Maxwell equations must satisfy the homogenous wave equations on $G$.

It will be convenient here to use the theory of generalized functions, called also distributions, originally introduced heuristically by Dirac in his works on Quantum Mechanics and put on precise mathematical footing by L. Schwartz [15] and Gelfand and Shilov [11]. We do not introduce here any notions from the theory of Lebesgue-Bochner integral. We base the proofs only on elementary properties of continuous and differentiable functions that can be found in any book on Calculus.

We will need to make a few modifications for the sake of our problem. We need distributions defined over any open set $G \subset \mathbb{R}^4$ and having their values in the vector space $\mathbb{R}^k$.

Let $\mathcal{D}_k$ denote the space $C^\infty_0(G, \mathbb{R}^k)$ of infinitely differentiable functions having compact supports contained in $G$ and values in $\mathbb{R}^k$.

On the space $\mathcal{D}_k$ we introduce the usual sequential topology. A sequence of functions $g_n \in \mathcal{D}_k$ converges to a function $g \in \mathcal{D}_k$ if it converges uniformly together with all its partial derivatives $D^\alpha g$, on every compact subset $K$ of $G$, to the function $g$, that is for every $\alpha$ the following sequence converges uniformly on $K$

\[
D^\alpha g_n(x) \to D^\alpha g(x)
\]

where $\alpha = (k_1, \ldots, k_4)$ denotes the multi-index of a partial derivative

\[
D^\alpha = D_{k_1}^1 \cdots D_{k_4}^4
\]

with $k_j = 0, 1, 2, \ldots$ and $D_j = \frac{\partial}{\partial x_j}$ where $j = 1, 2, 3, 4$.

**Definition 8.1** (Generalized vector valued functions). Let $\mathcal{D}_k'$ denote the space of all linear continuous real functionals on $\mathcal{D}_k$. Convergence in $\mathcal{D}_k'$ will be understood as pointwise convergence. The space $\mathcal{D}_k'$ will be called the space of **generalized vector valued functions** or the space of vector distributions.

We shall use the following equivalent notation for such a functional

\[
(8.1) \quad f(g) = \langle f, g \rangle = \int_G f(x) \cdot g(x) \, dx \quad \text{for all} \quad g \in \mathcal{D}_k.
\]

In the above $y \cdot x$ denotes the dot product, also called the scalar product, of two vectors $y, x \in \mathbb{R}^k$.

Any continuous function $f$ on the set $G$ generates by means of the above integral formula a vector distribution.
Notice that the space $D'_1$ is linearly and topologically isomorphic with the Cartesian product of $k$ copies of the space $D'_1$

$$D'_1 \times \cdots \times D'_1 = (D'_1)^k.$$ 

Indeed, if for every $m = 1, \ldots, k$ we define maps $P_m : R \to R^k$ by the condition

$$P_m(t) = x \Leftrightarrow \{x_m = t \text{ and } x_j = 0 \text{ if } j \neq m\},$$

where $x = (x_1, \ldots, x_k) \in R^k$, then the functionals

$$f_m(g) = \int_G f(x) \cdot P_m(g(x)) \, dx \quad \text{for all } g \in D_1$$

are well defined and represent elements of the space $D'_1$. Thus the element

$$(f_1, \cdots, f_k) \in (D'_1)^k.$$ 

Conversely define maps $P'_m : R^k \to R$ for $m = 1, \ldots, k$ by the formula

$$P'_m(x) = x_m \quad \text{for all } x \in R^k, \ m = 1, \ldots, k.$$ 

Then the formula

$$f(g) = \sum_m \int_G f_m(g(x)) \, dx \quad \text{for all } g \in D_k$$

yields a linear continuous functional on the space $D_k$. It is easy to verify that the transformation $Q : D'_k \to (D'_1)^k$ defined by

$$f \mapsto (f_1, \ldots, f_k)$$

is indeed a linear and topological isomorphism of the two spaces.

Now notice the following fact.

**Proposition 8.2** (Imbedding of continuous functions into $D'_k$ is one-to-one). Given two continuous functions $f_1$ and $f_2$ on the open set $G$ such that they generate the same vector distribution, that is

$$\int_G f_1(x)g(x) \, dx = \int_G f_2(x)g(x) \, dx \quad \text{for all } g \in D_k,$$

then they coincide

$$f_1(x) = f_2(x) \quad \text{for all } x \in G.$$ 

**Proof.** Indeed, from linearity of the map $f \mapsto \langle f, g \rangle$ and the previous isomorphism it is sufficient to prove that for any real continuous function $f$ such that

$$\int_G f(x)g(x) \, dx = 0 \quad \text{for all } g \in D_1$$

follows that $f = 0$ on $G$.

Assuming that this is not true then at some point $x_0 \in G$ we have $f(x_0) \neq 0$. We may assume without loss of generality that $2\delta = f(x_0) > 0$ otherwise we would consider the function $-f$. From continuity of $f$ follows that there is a rectangular neighborhood $V \subset G$ of $x_0$ such that

$$f(x) \geq \delta \quad \text{for all } x \in V.$$
There exists a nonnegative function \( g \) of class \( C^\infty \) with support in the set \( V \) with integral \( \int_G g(x) \, dx = 1 \). Thus for such a function we would get
\[
\int_G f(x)g(x) \, dx = \int_V f(x)g(x) \, dx \geq \int_V \delta g(x) \, dx = \delta > 0.
\]
A contradiction. So all we have to do is to show that there exists a function \( g \) having the above properties. To this end consider a nonnegative function \( g_0(t) \) defined by the formula
\[
g_0(t) = \alpha e^{-1/(1-t^2)} \quad \text{if} \quad |t| < 1; \quad \text{and} \quad g_0(t) = 0 \quad \text{if} \quad |t| \geq 1,
\]
where the constant is selected so that \( \int_{-1}^1 g_0(t) \, dt = 1 \).

It follows from the above formula that the function \( g_0 \) is infinitely differentiable at every point except perhaps at \( t = +1 \) or \( t = -1 \). One can prove that at these points the one-sided derivatives exist and they are equal. Thus the function \( g_0 \) is of class \( C^\infty \).

Now consider the function \( g_1(x) = g_0(x_1)g_0(x_3)g_0(x_2)g_0(x_4) \) for all \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \) with the integral, over the cube in \( \mathbb{R}^4 \) representing its support, equal to 1. For the fixed \( x_0 \in G \) and sufficiently large \( n \) we see that the function given by the formula
\[
g_n(x) = n^4 g(n(x - x_0)) \quad \text{for all} \quad x \in \mathbb{R}^4
\]
will satisfy our requirements.

Any linear continuous operator \( H : \mathcal{D}_k \rightarrow \mathcal{D}_k \) generates a linear continuous dual operator \( H' : \mathcal{D}'_k \rightarrow \mathcal{D}'_k \) by the formula
\[
\langle H' f, g \rangle = \langle f, H g \rangle \quad \text{for all} \quad g \in \mathcal{D}_k.
\]

The dual operator corresponding to scalar multiplication \( g \mapsto \lambda g \) is scalar multiplication \( f \mapsto \lambda f \) as follows from the above definition.

**Definition 8.3 (Generalized differential operator).** By a **generalized partial derivative** \( D_i = \frac{\partial}{\partial x_i} \), acting onto the m-th component of \( f \in \mathcal{D}'_k \)
\[
D_{i,m}(f_1, \ldots, f_k) = (f_1, \ldots, D_i f_m, \ldots, f_k)
\]
we shall understand the dual operator to the operator acting onto the m-th component of \( g \in \mathcal{D}_k \) by the formula
\[
(-1)D_{i,m}(g_1, \ldots, g_k) = (g_1, \ldots, (-1)D_i g_m, \ldots, g_k).
\]

In the case when the set \( G \) represents a Cartesian product of open bounded intervals and the vector function \( f \) is continuous together with \( D_i f_m \), the above formula can be easily verified through iterated integral and integration by parts. In this case the ordinary partial derivative will produce the derivative in the sense of distributions. So it is natural to extend this property to vector distributions.

**Definition 8.4 (Weak and strong partial derivatives).** Now every distribution and in particular every continuous function is differentiable in the sense of distributions. If it happens that the distributional partial derivative is representable by means of a continuous function, such a function is called a **weak derivative**. If a vector function has a continuous partial derivative such a derivative is called a **strong derivative**.
It is not obvious that weak and strong derivatives coincide in the general case of an arbitrary open set $G$ in $\mathbb{R}^4$. This calls for the following theorem.

**Theorem 8.5 (Weak and strong partial derivatives coincide).** Assume that $f = (f_1, \ldots, f_k)$ is a vector-valued function on an open set $G \subset \mathbb{R}^4$ and that its $m$-th component has a continuous partial derivative $D_if_m$.

Then this derivative coincides with the derivative in the sense of distribution, that is the weak derivative coincides with the strong one.

**Proof.** To prove this fact for general open sets in $\mathbb{R}^4$ notice that for every point $x \in G$ there exists a neighborhood $V(x) \subset G$ in the form of the Cartesian product of open intervals. Restricting our test functions to functions with support in $V(x)$ will yield that on such a domain the weak and strong partial derivatives coincide.

Since every open set in $\mathbb{R}^4$ is a union of a countable number of compact sets, from the collection $V(x)$ ($x \in G$) one can extract a locally finite countable cover of the set $G$.

Now using partition of unity theorem, (for reference concerning this theorem see for instance Gelfand and Shilov [11], vol. 1, Appendix to Chapter 1, Section 2,) we can prove this theorem for any open set $G \subset \mathbb{R}^4$. □

Now let $A$ denote the algebra of all linear transformations $H : D'_k \to D'_k$. The multiplication of two transformations $H$ and $S$ in the algebra $A$ is defined as composition $H \circ S$.

Since any two partial derivatives commute and they commute with scalar multiplication operators, the above definition permits one to represent any polynomial of the operators $D_{i,m}$ as a dual of some operator acting in the space $D_k$. It follows from the above definition that any operator $H$, generated by a homogenous quadratic polynomial $p$, is formally self dual

$\langle Hf, g \rangle = \langle f, Hg \rangle$ for all $g \in D_k$.

Now let us consider a particular space of interest $E = D'_3 \times D'_3$.

Let $A_0$ denote the algebra of all linear transformations from $D'_3$ into itself with multiplication defined again as a composition of transformations.

**Proposition 8.6 (Matrix representation of operators on $E$).** A transformation $H$ belongs to the algebra $A$ if and only if there exists a $2 \times 2$ matrix such that

$$H(f) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ for all } f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in E,$$

where transformations $H_{ij}$ belong to algebra $A_0$. This correspondence establishes isomorphism between the algebra $A$ and the algebra of all $2 \times 2$ matrices with elements in the algebra $A_0$.

**Proof.** The proof is evident and we leave it to the reader. □

A transformation $H$ with matrix of the form $a_{ij}I$, where $a_{ij} \in R$ and $I$ is the identity map in $A_0$, will be called scalar transformation. A scalar transformation $H$ can be simply represented as

$$H(f) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ for all } f \in E.$$
Notice that the identity transformation, the unit element $I$ in the algebra $\mathcal{A}$, can be represented as a **scalar transformation** with matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Of a particular interest are the following two scalar transformations that are inverse to each other

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad J^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -J.$$

A transformation $H \in \mathcal{A}$ is called **simple** if its matrix representation is of the form

$$(8.3) \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \quad \text{where} \quad h \in \mathcal{A}_0.$$ 

We shall use a shorthand notation when dealing with an action by a simple transformation. We shall write $h(f)$ or just $hf$ instead of

$$(8.4) \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{for all} \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f \in \mathcal{E}.$$ 

**Proposition 8.7** (Commutativity of $J$ with any simple transformation). Every simple transformation $H \in \mathcal{A}$ commutes with transformation $J$ that is

$$H \circ J = J \circ H.$$ 

In shorthand notation we can write $hJ = Jh$ if no confusion is possible.

**Proof.** The proof is obvious and we leave it to the reader. \qed

Compare the above constructions with the arguments presented in Kaiser [12], page 207.

**Theorem 8.8** (Homogenous Maxwell equations imply homogenous wave equation). Let $E$ and $H$ be two vector distributions from the space $\mathcal{D}'_3$. If they satisfy the homogenous system of Maxwell equations

$$\nabla \times E = -DB, \quad \nabla \cdot E = 0,$$

$$\nabla \times B = +DE, \quad \nabla \cdot B = 0,$$

then they satisfy the wave equations

$$(\nabla^2 - D^2)E = 0,$$

$$(\nabla^2 - D^2)B = 0.$$ 

**Proof.** Let

$$f = \begin{bmatrix} E \\ B \end{bmatrix}.$$ 

Then Maxwell’s equations can be written as

$$(a) \quad (\nabla \times )f = DJf, \quad (b) \quad (\nabla \cdot )f = 0.$$ 

Applying operator $J^{-1}D = -JD$ to both sides of equation $(a)$ we get

$$(8.5) \quad -JD(\nabla \times )f = D^2 f.$$
On the other hand applying the simple operator $(\nabla \times)$ to both sides of equation (a) and using the commutativity of the operators $J, D, (\nabla \times)$ we get

\[(\nabla \times)^2 f = (\nabla \times) J D f = JD(\nabla \times) f.\]

Since it is easy to verify that

\[(\nabla \times)^2 f = \nabla((\nabla \cdot f) - \nabla^2 f) \quad \text{for all} \quad f \in \mathcal{E},\]

observing that the terms on the right side represent differential operators generated by quadratic homogeneous polynomials and the above identity holds for any test function from $\mathcal{D}_3$, by Maxwell’s equation (b) we get that the solution $f$ of the system of Maxwell equations must satisfy the equation

\[(\nabla \times)^2 f = -\nabla^2 f.\]

Thus from equations (8.6) and (8.7) follows that the solution $f$ of the system of Maxwell equations must satisfy the wave equation

\[\nabla^2 f - D^2 f = 0\]

and this completes the proof. \hfill \Box

**Proposition 8.9.** Assume that $E$ and $B$ are two, twice continuously differentiable, vector fields from an open set $G$ in the space $\mathbb{R}^4$ into $\mathbb{R}^3$ satisfying the homogenous system of Maxwell’s equations

\[\begin{align*}
\nabla \times E &= -DB, \quad \nabla \cdot E = 0, \\
\nabla \times B &= +DE, \quad \nabla \cdot B = 0,
\end{align*}\]

Then these fields satisfy the homogenous wave equations

\[\begin{align*}
\nabla^2 E - D^2 E &= 0, \\
\nabla^2 B - D^2 B &= 0.
\end{align*}\]

**Proof.** The proof follows from the previous theorem and from the fact that for continuous functions with continuous partials their partial derivatives in the sense of generalized functions coincide with the corresponding ordinary partial derivatives that is the weak derivatives coincides with the strong derivatives, see Theorem 8.5. \hfill \Box

9. **Maxwell’s mathematical experiment**

At the beginning of the 19-th century the magnetic and electric phenomena were considered as independent. Ampere’s law that electric current can produce a magnetic field was the first breakthrough in this field.

Around 1830 M. Faraday confirmed validity of Ampere’s law. Further he discovered that changing magnetic flux can induce an electric current. He established the principle of electric induction leading to transformers of alternating current.

In 1846 Faraday wrote a paper *Thoughts on ray vibrations*. In his imagination he saw interaction of gravitational, electric, and magnetic forces along the force lines, which connected particles and masses together, contrary to prevailing Newtonian view of actions at a distance. Faraday in his work suggested that light itself is some kind of vibration in the lines of force. He postulated that vibrating charge may produce vibrations in the lines of force.
J. C. Maxwell was inspired by Faraday’s work and after augmenting Ampère’s law he discovered the specific velocity $c$ associated with electromagnetism that is now known as the speed of light and proved that electromagnetic waves in free space propagate with velocity $c$.

We can perform this experiment now in a more general setting and a greater generality. Maxwell had done it for harmonic waves in electromagnetic field.

Let us assume that we are working in some Lorentzian frame and we have some physical quantity $u$, it could be a component of the electric or magnetic field, gravity field, or any other quantity, that satisfies the homogeneous wave equation in some open set $G \in \mathbb{R}^4$. We can select a rectangular subset of $G$, so let us assume that $G$ is in rectangular form, that is, it is representable as a Cartesian product of 4 intervals. The wave equation in standard g-m-s units for quantity $u$ will look as follows

$$
\nabla^2 u - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u = 0 \quad \text{for all} \quad (r, t) \in G.
$$

We may assume without loss of generality that the origin of our Lorentzian frame is in the center of the 4-dimensional cube $G$. Take any function $f(t)$ of class $C^2$ on $R$ and consider a wave that this function generates by means of the formula

$$
u(y_1, y_2, y_3, t) = f(y_1 - vt) \quad \text{for all} \quad (y_1, y_2, y_3, t) \in G.
$$

At time $t = 0$ the shape of the wave has graph represented by function the $t \mapsto f(t)$. To visualize this wave we may assume that the wave looks like tsunami wave with the crest at $t = 0$. At any time $t$ the crest is at position $y_1 = vt$. Thus the wave moves with velocity $v$.

We want to find when such a wave will satisfy our wave equation. Notice that $u$ does not depend on $y_2$ and $y_3$ thus

$$
\nabla^2 u = f''(y_1 - vt) \quad \text{for all} \quad (y_1, y_2, y_3, t) \in G.
$$

and

$$
\frac{\partial^2}{\partial t^2} u = v^2 f''(y_1 - vt) \quad \text{for all} \quad (y_1, y_2, y_3, t) \in G.
$$

The wave equation (9.1) will be satisfied for any function of class $C^2$ if and only if $c^2 = v^2$ that is $v = +c$ or $v = -c$. Thus such waves propagate with velocity $c$. This is the essence of Maxwell’s mathematical experiment. Hence we have the following corollary

**Corollary 9.1** (Maxwell’s experiment). Assume that in a Lorentzian frame we have some physical quantity $u$ satisfying the homogenous wave equation (9.1), then waves of that quantity propagate in free space with velocity of light $c$.

---

**10. Einstein’s Formulas for Transformation of Electromagnetic Field**

Lorentz [14] and Einstein [8], Part 2, section 6, established that a pair of fields

$$
E = (E_1, E_2, E_3) \quad \text{and} \quad B = (B_1, B_2, B_3)
$$
that satisfy homogenous system of Maxwell equations in an open set $G$ transforms as a part of an antisymmetric tensor of second rank in Lorentzian space-time $R^3 \times R$.

The matrix of this tensor looks as follows

\[
\begin{bmatrix}
0 & +E_1 & +E_2 & +E_3 \\
-E_1 & 0 & +B_3 & -B_2 \\
-E_2 & -B_3 & 0 & +B_1 \\
-E_3 & +B_2 & -B_1 & 0 \\
\end{bmatrix}
\]

For formulas for transformation of electromagnetic field under Lorentzian change of coordinates see Feynman-Leighton-Sands [10], vol. 2, chapter 26. In particular formulas for Lorentz transformations of the pair of fields $E$ and $B$ on page 26-9.

Following [10] assume that our orthogonal frames of reference are $S$ and $S'$ and the frame $S'$ moves along the first coordinate axis of $S$ with velocity $w = (w_1, 0, 0)$. Then the transformation of the components of the vector fields

\[E = (E_1, E_2, E_3)\quad \text{and} \quad B = (B_1, B_2, B_3)\]

will look as follows

\[
\begin{align*}
E'_1 &= E_1 \\
E'_2 &= \gamma(E + w \times B)_2 \\
E'_3 &= \gamma(E + w \times B)_3 \\
B'_1 &= B_1 \\
B'_2 &= \gamma(B - w \times E)_2 \\
B'_3 &= \gamma(B - w \times E)_3 \\
\end{align*}
\]

where $\gamma = 1/\sqrt{1 - |w|^2}$.

Since

\[w \times B = w \times (e \times E) = (w \cdot E)e - (w \cdot e)E\]

we get final formula involving just the vector fields $E$ and $e$

\[
\begin{align*}
E'_1 &= E_1 \\
E'_2 &= \gamma(E_2 + (w \cdot E)e_2 - (w \cdot e)E_2) = \gamma(E_2 + w_1 E_1 e_2 - w_1 e_1 E_2) \\
E'_3 &= \gamma(E_3 + (w \cdot E)e_3 - (w \cdot e)E_3) = \gamma(E_3 + w_1 E_1 e_3 - w_1 e_1 E_3) \\
\end{align*}
\]

11. **Bogdan-Feynman Theorem for a moving point mass**

From the theorems of the previous sections we can conclude that the following theorem is true. Please notice that the physical nature of the fields is completely irrelevant.

The important fact is that we are working in a Lorentzian frame, in which we are given an admissible trajectory, see definition (2.1). This trajectory alone generates in a unique way the system of the fundamental fields by means of which we are able to construct fields that are preserved by Lorentzian transformations.

As we have proved earlier the notion of an admissible trajectory is invariant under any Lorentzian transformation from one frame to any other frame moving with a constant velocity. This fact is one of the consequences of Einstein’s formula on kinetic energy of a point mass.

We remind the reader that we are working in a fixed Lorentzian frame. The position vector is denoted by $r_1 \in R^3$ and time by $t \in R$. The position axes are oriented so as to form right screw orientation. The units are selected so that the speed of light $c = 1$. 
Partial derivatives with respect to coordinates of $r_1$ are denoted by $D_1$, $D_2$, $D_3$ and with respect to time just by $D$. The gradient differential operator is denoted by $\nabla = (D_1, D_2, D_3)$ and the D’Alembertian operator by $\Box^2 = \nabla^2 - D^2$.

**Theorem 11.1 (Bogdan-Feynman Theorem).** Assume that in a given Lorentzian frame the map $t \mapsto r_2(t)$ from $\mathbb{R}$ to $\mathbb{R}^3$ represents an admissible trajectory of class $C^3$. Assume that $G$ denotes the open set of points that do not lie on the trajectory. All the following field equations are satisfied on the entire set $G$.

Consider the pair of fields $E$ and $B$ over the set $G$ given by the formulas

$$E = u^2 e + u^{-1} D(u^2 e) + D^2 e \quad \text{and} \quad B = e \times E$$

where $u$ and $e$ represent fundamental fields (4.1) associated with the trajectory $r_2(t)$.

Then this pair of fields will satisfy the following homogenous system of Maxwell equations

$$\nabla \times E = -DB, \quad \nabla \cdot E = 0,$$
$$\nabla \times B = +DE, \quad \nabla \cdot B = 0,$$

and the homogenous wave equations

$$\Box^2 E = 0, \quad \Box^2 B = 0.$$

Moreover Liénard-Wiechert potentials, expressed in terms of the fundamental fields as $A = uzv$ and $\phi = uz$, satisfy the homogenous system of wave equations with Lorentz gauge formula

$$\Box^2 A = 0, \quad \Box^2 \phi = 0, \quad \nabla \cdot A + D\phi = 0$$

and generate the fields $E$ and $B$ by the formulas

$$E = -\nabla \phi - DA \quad \text{and} \quad B = \nabla \times A,$$

Finally we have the following explicit formula for the field $E$ in terms of the fundamental fields

$$E = -u z^2 a + u z^3 (e,a)e - u z^3 (e,a)v$$
$$+ u^2 z^3 e - u^2 z^3 (v,v)e - u^2 z^3 v + u^2 z^3 (v,v)v.$$ 

As a consequence of the above theorem the components of the quantities $E$, $B$, $A$, and $\phi$ propagate in the Lorentzian frame with velocity of light $c = 1$.

12. **Einstein’s illusive gravity field**

Matter in the universe is distributed in chunks like atoms, particles, molecules, or stellar objects, like planets, stars, galaxies, etc. So it is natural to consider a system on $n$ bodies carrying charges $q_j$ and having rest masses $m_0$.

Einstein using general theory of relativity proved that waves in gravity fields should also propagate in space with velocity $c$ like electromagnetic waves. For reference see Einstein and Rosen [9]. We shall assume that the field $E$ given by Newton-Feynman formula has this property. We want to find out how to develop dynamics in such fields.

Let us assume that we are working in a Lorentzian frame with units selected so that the speed of light $c = 1$ the electrostatic constant in free space satisfies
the condition \(4\pi\varepsilon_0 = 1\) and the unit of mass is selected so that the gravitational constant \(G = 1\).

This means that if we keep meter as our unit of length the unit of time is approximately 3.3 nanoseconds and the unit of mass is about 3.871 metric tons and the unit of charge is 4\(\pi\) of coulombs.

Let us assume that we have a system consisting of \(n\) bodies indexed by \(j, k = 1, 2, \ldots, n\) with rest masses \(m_{0j}\) and charges \(q_j\). We want to find the equations of motion for a time period starting at time \(t = s\) and ending at time \(t = e\). Assume that the bodies during this time period \([s, e]\) move without collision.

We shall follow here the constructions developed in Bogdan [1]. Now consider the trajectories. Assume that we know the formulas for the trajectories of the bodies up to time \(t = s\). We can setup the equations of motion of these bodies in their own force fields. To this end let \(r_j(t)\) be the trajectory of the \(j\)-th body. Let \(t_{jk}(t)\) be the retarded time for a wave travelling at the speed of light to reach from the \(k\)-th trajectory the \(j\)-th trajectory at the point \(r_j(t)\) at time \(t\).

This function must satisfy the equation
\[
t - t_{jk}(t) = |r_j(t) - r_k(t_{jk}(t))| \quad \text{for all} \quad t \leq e.
\]
As follows from Theorem (3.1) on uniqueness of the retarded times, the function \(t_{jk}(t)\) is uniquely determined by the above condition.

Put \(T_{jk} = t - t_{jk}\) and call this function the delayed time. Introduce a vector function \(r_{jk}\) by the formula
\[
r_{jk}(t) = r_j(t) - r_k(t_{jk}(t)) \quad \text{for all} \quad t \leq e, \ j \neq k.
\]
and the unit vector function \(e_{jk}\) by
\[
e_{jk}(t) = r_{jk}(t)/|r_{jk}(t)| \quad \text{for all} \quad t \leq e, \ j \neq k.
\]

Since \(T_{jk} = |r_j(t) - r_k(t_{jk}(t))|\) and by assumption the bodies do not collide for any \(t \in I\) the unit vector \(e_{jk}(t)\) is well defined. Let \(v_j(t)\) be the velocity of the \(j\)-th body and let \(y_j(t)\) denote the state vector \((r_j(t), v_j(t))\) and let \(y(t)\) be the list of all state vectors \(y_j(t)\) for \(j = 1, \ldots, n\).

Thus the vector function \(y(t)\) has its values in the space \(R^{6n}\). By the norm of the state vector \(y \in R^{6n}\) of the system \((r_j, v_j)\), where \(j = 1, \ldots, n\), we shall understand
\[
|y| = \max \{|r_j|, |v_j| : j = 1, \ldots, n\},
\]
where \(|r_j|\) and \(|v_j|\) denote the usual Euclidian norm in \(R^3\). Since the vector \(r_{jk}\), and also the vector \(e_{jk}\), is directed from the \(k\)-th body towards the \(j\)-th body the Lorentz force induced by the \(k\)-th body is given by the formula
\[
q_j[E_{jk} + v_j \times (e_{jk} \times E_{jk})]
\]
and Newton’s force is given by
\[
-m_{0k}E_{jk}
\]
from the fact that force of action and force of reaction are equal in magnitude but have opposite directions. Thus the total force \(F_{jk}\) acting onto the \(j\)-th body by the \(k\)-th body is given by the formula
\[
F_{jk} = q_j[E_{jk} + v_j \times (e_{jk} \times E_{jk})] - m_{0k}E_{jk}
\]
Introduce an operator $H$ by the formula
\begin{equation}
H_{jk}(h) = q_j \{h + v_j \times (e_{jk} \times h)\} - m_0 h \quad \text{for all } h \in \mathbb{R}^3.
\end{equation}

For fixed $y$ and $t$ and $j \neq k$ the operator $H_{jk}$ is linear with respect to $h$. Denote by $U$ the algebra of all linear transformations from the space $\mathbb{R}^3$ into itself.

The total force acting onto the $j$-th body now can be represented in the form
\begin{equation}
F_j = \sum_{k=1; k \neq j}^{k=n} H_{jk}(E_{jk}) + F_{j0},
\end{equation}
where $F_{j0}$ represents external forces due to some external fields. It could be a force field due to self action or due to loss of energy through the electromagnetic radiation. We shall want to find conditions that the fields should satisfy in order to have uniqueness of the solution of the system of evolution equations.

Now for a body of rest mass $m_0$ moving under the influence of a force $F$ from Newton-Einstein formula, time derivative of the momentum is equal to force, we must have
\[
\dot{p} = F
\]
where $\gamma = (1 - |v|^2)^{-1/2}$ and $p = m_0 \gamma v$ is the relativistic momentum. Computing the derivative of $p$ with respect to time we get
\[
\dot{p} = m_0 \gamma \dot{v} + m_0 \gamma v = m_0 \gamma \dot{v} + m_0 \gamma^3 \langle v, \dot{v} \rangle v
= m_0 \gamma (\dot{v} + \gamma^2 \langle v, \dot{v} \rangle v) = m(\dot{v} + \gamma^2 \langle v, \dot{v} \rangle v)
= m \Gamma(v)(\dot{v}),
\]
where $m = m_0 \gamma$ is the relativistic mass and $\Gamma(v)$ is the linear transformation of $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the formula
\[
\Gamma(v)(h) = h + \gamma^2 \langle v, h \rangle v \quad \text{for all } h \in \mathbb{R}^3.
\]
Notice that for fixed velocity $v$ the transformation $\Gamma(v)$ represents a symmetric, positive definite transformation with eigenvalues equal respectively to
\[
(1 + \gamma^2 |v|^2), \quad 1, \quad 1.
\]
Thus the inverse transformation $\hat{\Gamma}(v)$ exists and is also symmetric. Since its eigenvalues are
\[
(1 + \gamma^2 |v|^2)^{-1}, \quad 1, \quad 1
\]
and the norm of positive definite symmetric transformation is its maximal eigenvalue, we must have
\[
|\hat{\Gamma}(v)| = 1 \quad \text{for all } v \in \mathbb{R}^3
\]
and we have also
\[
\hat{\Gamma}(v)\Gamma(v) = e \quad \text{for all } v \in \mathbb{R}^3
\]
where $e$ denotes here the identity transformation in the algebra Lin$(\mathbb{R}^3, \mathbb{R}^3)$ that is $e(h) = h$ for all $h \in \mathbb{R}^3$.

We shall call the function $v \mapsto \hat{\Gamma}$ the \textbf{reciprocal to the function} $\Gamma(v)$, since
\[
\hat{\Gamma}(v) = [\Gamma(v)]^{-1} \quad \text{for all } v \in \mathbb{R}^3.
\]
Thus the Newton-Einstein equation for the $j$-th body can be written as

$$\Gamma(v_j)(\dot{v}_j) = \frac{1}{m_j} F_j \quad \text{for all} \quad j = 1, \ldots, n$$

(12.3)

Since trajectories $y_j = (r_j, v_j)$ are admissible \([2.1]\), for stopping time $t_1 = e$ we must have

\[
\begin{align*}
\sup \{ |v_j(t)| : t \leq e, j = 1, 2, \ldots, n \} &= q < 1, \\
\sup \{ |\dot{v}_j(t)| : t \leq e, j = 1, 2, \ldots, n \} &= A < \infty
\end{align*}
\]

(12.4)

Now put $x(t) = y(t)$ for $t \leq s$. Such trajectory will be called an initial trajectory. Clearly $x$ is of class $\mathcal{C}^1$. We shall follow here the notation used in Bogdan \[1\].

**Definition 12.1** (Initial domain). For a given initial trajectory $x$ and nonnegative numbers $q < 1$ and $A < \infty$ and an interval $I = [s, e]$ define the set

$$D = D(x, q, A, I)$$

of functions $y(t) = (r(t), v(t))$, where $v(t) = \dot{r}(t)$, extending the trajectory $x$ to the interval $I$ and satisfying the inequalities

$$|r_j(t) - r_j(\tilde{t})| \leq q|t - \tilde{t}| \quad \text{for all} \quad t, \tilde{t} \leq e.$$  

(12.5)

$$|v_j(t) - v_j(\tilde{t})| \leq A|t - \tilde{t}| \quad \text{for all} \quad t, \tilde{t} \leq e.$$  

Such a set $D$ will be called an initial domain generated by the parameters $x, q, A, I$.

**Definition 12.2** (Space $\mathcal{C}(I, Y)$ of continuous functions). For any interval $I$ and any Banach space $Y$ let $\mathcal{C}(I, Y)$ denote the set of all continuous functions $f$ from $I$ into $Y$.

In the case when the interval $I$ is closed and bounded the space $\mathcal{C}(I, Y)$ with norm $||f|| = \sup \{|f(t)| : t \in I\}$ forms a Banach space.

The set $D(x, q, A, I)$ can be considered as a subset of the Banach space $\mathcal{C}(I, \mathbb{R}^{6n})$. It is clear that the original trajectory $y$ of the entire system of bodies belongs to the initial domain $D(x, q, A, I)$ for parameters $q$ and $A$ as defined in \(12.4\).

**Definition 12.3** (Space $\mathcal{T}$ of admissible trajectories). For fixed $n$ and an interval $I$ of the form $I = (-\infty, s]$ denote by $\mathcal{T}(I)$ the set of all admissible trajectories $x$ of the form

$$x_j(t) = (r_j(t), v_j(t)) \quad \text{for} \quad t \in I \text{ and } j = 1, \ldots, n.$$  

Trajectories $x$ of this form will be called admissible trajectories on interval $I$ and the set $\mathcal{T}(I)$ the space of admissible trajectories on $I$.

The union $\mathcal{T}$ of all such spaces $\mathcal{T}(I)$ will be called the space of admissible trajectories, that is

$$\mathcal{T} = \bigcup_I \mathcal{T}(I).$$

Notice that every initial domain $D(x, q, A, [s, e])$ forms a subset of the space

$$\mathcal{T}((-\infty, e]) \subset \mathcal{T}.$$
Definition 12.4 (Space \( \mathcal{U}(Y) \) of nonanticipating operators). For fixed \( n \) and a fixed Banach space \( Y \) let \( P \) denote an operator on the space \( \mathcal{T} \) mapping an admissible trajectory \( x \) on an interval \( I = (-\infty, d] \) into a continuous function from the space \( C(I, Y) \).

Such an operator is well defined if it has the following property: for every two functions \( y, z \) and every number \( d \in \mathbb{R} \) the condition that \( y(t) = z(t) \) for all \( t \leq d \), implies that the images \( P(y)(t) \) and \( P(z)(t) \) coincide for all \( t \leq d \). Every such operator will be called a **nonanticipating operator** and the space \( \mathcal{U}(Y) \) the **space of nonanticipating operators** with values in the space \( Y \).

The following theorem represents Th. 4.6 of Bogdan [2].

Theorem 12.5 (Initial domain is compact and convex). Every initial domain \( D(x, q, A, I) \) is compact and convex in the Banach space \( C = C(I, \mathbb{R}^{6n}) \).

Thus every initial domain forms a closed set in the space \( C(I, \mathbb{R}^{6n}) \).

Definition 12.6 (Section \( D_d \) of Initial Domain). Let \( D = D(x, q, A, I) \) be an initial domain with \( I = [s, e] \). For any \( d \in [s, e] \) denote by \( I_d \) the interval \([s, e]\) and by \( D_d \) the initial domain \( D(x, q, A, I_d) \). Any such initial domain \( D_d \) will be called a **section of the initial domain** \( D \).

It is plain from the definition that every nonanticipating operator \( P \) on an initial domain \( D \) induces a nonanticipating operator \( P_d \) on any section \( D_d \) of the initial domain \( D \).

Definition 12.7 (Uniformly Lipschitzian Operator). A nonanticipating operator \( P \) from an initial domain \( D = D(x, q, A, I) \) into the space \( C(I, U) \), of continuous functions from the interval \( I \) into a Banach space \( U \), will be called **uniformly Lipschitzian** if there exists a constant \( M \) such that for every \( d \in I \) we have

\[
\|P_d(y) - P_d(y^-)\|_d \leq M \|y - y^-\|_d \quad \text{for all } y, y^- \in D_d,
\]

where \( \| \|_d \) denotes the norm in the space \( C(I_d, U) \).

Obviously the operators \( y \rightarrow r_j \) and \( y \rightarrow v_j \) are nonanticipating and uniformly Lipschitzian on any initial domain \( D \).

Definition 12.8 (The space \( \text{Lip}(D, U) \)). Let \( D = D(x, q, A, I) \) denote an initial domain and \( U \) a Banach space. Let \( \text{Lip}(D, U) \) denote the space of all nonanticipating uniformly Lipschitzian operators from the domain \( D \) into the Banach space \( C(I, U) \).

If \( P \) is an operator in \( \text{Lip}(D, U) \), let

\[
\|P\| = \sup \{\|P(y)\| : y \in D\},
\]

\[
l(P) = \inf \{M : \|P(y) - P(y^-)\|_d \leq M \|y - y^-\|_d \quad \text{for all } y, y^- \in D, \ d \in I\}.
\]

Clearly since \( D \) is compact, this norm \( \|P\| \) is well defined. The norm in the expression \( \|P(y)\| \) is understood as the supremum norm in the space \( C(I, U) \). Define

\[
\|P\|_l = \|P\| + l(P)\quad \text{for all } P \in \text{Lip}(D, U).
\]
The space $\text{Lip}(D,U)$ with norm $\| \cdot \|$ will be called the space of nonanticipating uniformly Lipschitzian operators.

13. Nonanticipating differential equations

Let $D = D(x,q,A,I)$ be an initial domain and $\Lambda \in \text{Lip}(D,R^{3n})$. Consider the system of differential equations

\[(13.1) \quad \dot{v}_j(t) = \Lambda_j(y)(t) \quad \text{for all} \quad t \in I, \ j = 1, \ldots, n.\]

Introduce the operator $X$ whose component $X_j$ corresponding to the $j$-th body is defined by the formulas

$$X_j(y)(t) = (v_j(t), \Lambda_j(y)(t)) \quad \text{for all} \quad y = (r,v) \in D \text{ and } t \in I.$$  

The right side of the equation (13.1) represents a continuous function of the variable $t$, so derivative on the left side also represents a continuous function. Thus we can apply the integral to both sides to get an equivalent equation

$$v_j(t) = v_j(s) + \int_s^t \Lambda_j(y)(u) \, du \quad \text{for all} \quad t \in I, \ j = 1, \ldots, n.$$  

We also have

$$r_j(t) = r_j(s) + \int_s^t v_j(u) \, du \quad \text{for all} \quad t \in I, \ j = 1, \ldots, n.$$  

Now if we define the operator $\Omega_j$ for all $y \in D$ by the formula

$$(r_j(s) + \int_s^t v_j(u) \, du, \ v_j(s) + \int_s^t \Lambda_j(y)(u) \, du)$$

we can rewrite the previous two equations in the form

$$y_j(t) = (r_j(t), v_j(t)) = \Omega_j(y)(t) \quad \text{for all} \quad t \in I, \ j = 1, \ldots, n.$$  

or even in a shorter form as

$$y = \Omega(y)$$

where $\Omega$ represents the list of operators $\Omega_j$, $(j = 1, \ldots, n)$. This operator belongs to the space $\text{Lip}(D,R^{3n})$. In this way we reduced the problem to a fixed point problem.

Now we are ready to prove that the fixed point problem has a unique local solution. Compare the following theorem with Th. 7 of Bogdan [1].

**Theorem 13.1** (Local existence and uniqueness). Assume that the operator $\Lambda$ is nonanticipating and for every value of $A > 0$ there exists an interval $I = [s,e]$ such that $\Lambda$ restricted to the initial domain $D = D(x,q,A,I)$ yields an operator belonging to $\text{Lip}(D,R^{3n})$.

Let $A_0 > |\Lambda(x)(s)|$. Let $A > 2A_0$. Compactness of the initial domain $D = D(x,q,A,I)$ implies the compactness of its image $\Lambda(D)$. Thus there exists a non-decreasing function $\omega(\delta)$, called modulus of continuity, such that $\omega(\delta) \to 0$ when $\delta \to 0$ and

$$|\Lambda(y)(t) - \Lambda(y)(t')| \leq \omega(|t - t'|) \quad \text{for all} \quad y \in D, \ t,t' \in I.$$  

Let $\beta$ be such that $\omega(\beta) \leq A_0$ and $\Lambda > \max \{ l(A_j) : j = 1, \ldots, n \}$, where $l(A_j)$ denotes the Lipschitz constant of the operator $A_j \in \text{Lip}(D,R^3)$.  

Let \( J = [s, s + \delta] \) where \( \delta \) is such that
\[
0 < \delta < \min \{1, \beta, \lambda^{-1}\}.
\]
Then the transformation \( \Omega \) maps the domain \( D = D(x, q, A, J) \) into \( D \) and forms a contraction map. Thus for the initial trajectory \( x \) of class \( C^1 \) there exists a unique local solution in the domain \( D \) to the evolution equation of the system of \( n \) bodies
\[
\dot{y}(t) = X(y)(t) \quad \text{for all} \quad t \in I.
\]

**Proof.** First let us establish that the operator \( \Omega \) maps the initial domain \( D \) into itself. Take any \( y = (r, v) \in D \) and let \( \tilde{y} = (\tilde{r}, \tilde{v}) = \Omega(y) \). For the position component we have
\[
\tilde{r}(t) = r(s) + \int_s^t v(u) \, du \quad \text{for all} \quad t \leq s + \delta.
\]
and for the velocity component
\[
\tilde{v}(t) = v(s) + \int_s^t \dot{r}(u) \, du \quad \text{for all} \quad t \leq s
\]
and
\[
\tilde{v}(t) = v(s) + \int_s^t \Lambda(y)(u) \, du \quad \text{for all} \quad s \leq t \leq s + \delta.
\]
Notice that the above formulas represent a trajectory \( \tilde{y} \) extending the initial trajectory \( x \).

To check the Lipschitz conditions notice that we have for the position component
\[
|\tilde{r}(t) - \tilde{r}(t^\sim)| = |\int_t^{t^\sim} v(u) \, du| \leq q|t - t^\sim| \quad \text{for all} \quad t, t^\sim \in I.
\]
For the velocity component we have
\[
|\tilde{v}(t) - \tilde{v}(t^\sim)| = \left| \int_{t^\sim}^{t} \Lambda(y)(u) \, du \right|
\]
\[
\leq \int_{t^\sim}^{t} \Lambda(y)(s) \, ds + \left| \int_{t}^{t^\sim} \Lambda(y)(s) - \int_{t}^{t^\sim} \Lambda(y)(u) \, du \right|
\]
\[
\leq \int_{t^\sim}^{t} \Lambda(y)(s) \, ds + \left| \int_{t}^{t^\sim} (\Lambda(y)(s) - \Lambda(y)(u)) \, du \right|
\]
\[
\leq |\Lambda(x)(s)||t - t^\sim| + \omega(\delta)|t - t^\sim|
\]
\[
\leq (A_0 + \omega(\delta))|t - t^\sim| \leq (A_0 + \omega(\beta))|t - t^\sim|
\]
\[
\leq 2A_0|t - t^\sim| \leq A|t - t^\sim| \quad \text{for all} \quad t, t^\sim \in I,
\]
Thus we have that \( \tilde{y} \in D \).

The transformation \( \Omega \) is a contraction. Indeed, take any \( y, y^\sim \in D \) and consider their images \( z = \Omega(y) \) and \( z^\sim = \Omega(y^\sim) \). Introducing the notation for components of \( z = (r_1, v_1) \) and \( z^\sim = (r_1^\sim, v_1^\sim) \) we get
\[
|r_1(t) - r_1^\sim(t)| = \left| \int_s^t (v(u) - v^\sim(u)) \, du \right| = \left| \int_s^t v - v^\sim \, du \right|
\]
\[
\leq \delta \|y - y^\sim\| \quad \text{for all} \quad y, y^\sim \in D, \text{ and } t \in I
\]
and for velocity component of $z, z^\sim$
\[
|v_1(t) - v^\sim_1(t)| = \left| \int_t^s (\Lambda(y)(u) - \Lambda(y^\sim)(u)) \, du \right| \\
\leq \delta \lambda \|y - y^\sim\| \quad \text{for all} \quad y, y^\sim \in D, \text{and} \ t \in I
\]
The above inequalities imply
\[
\|z - z^\sim\| \leq \max\{\delta, \delta \lambda\} \|y - y^\sim\| \quad \text{for all} \quad y, y^\sim \in D,
\]
or equivalently
\[
\|\Omega(y) - \Omega(y^\sim)\| \leq q \|y - y^\sim\| \quad \text{for all} \quad y, y^\sim \in D,
\]
where $q = \max\{\delta, \delta \lambda\}$. Since it follows from the assumptions of the theorem that the constant $q$ is less than 1, the transformation $\Omega$ represents a contraction mapping.

Hence there exists a unique trajectory $y \in D$ such that $y = \Omega(y)$ or equivalently that
\[
\dot{y}(t) = X(y)(t) \quad \text{for all} \quad t \in I.
\]
This completes the proof of the theorem. □

14. Existence and Uniqueness of Global Solutions

Definition 14.1 (Operator with local uniqueness property). Assume that we have a nonanticipating operator $X : T \mapsto \mathbb{R}^{6n}$ with the property that for every admissible initial trajectory $x \in T$ there exists a unique local solution of the equation
\[
\dot{y}(t) = X(y)(t) \quad \text{for all} \quad t \in I
\]
extending the trajectory $x$. We shall say that such operator has the local uniqueness property.

Assume that we found a solution $y$ of the differential equation in an initial domain $D(x, q, A, I)$, where $I = [s, e]$. Taking the endpoint $e$ as our new starting point $s$ and the trajectory $y$ as our new initial trajectory $x$ one can extend the solution onto a larger interval. The object is to show that between all possible extensions there exists a maximal extension. Moreover that the maximal extension is unique. Here we will make the first step in this direction.

Theorem 14.2 (Uniqueness of extensions of solutions). Assume that a nonanticipating operator $X : T \mapsto \mathbb{R}^{6n}$ has a local uniqueness property for any starting time $s$ and any initial trajectory $x \in T((-\infty, s])$. Let $I$ be an interval starting at the point $s$. On the right the interval may be either open or closed, either bounded or unbounded.

If $y$ and $y^\sim$ are two solutions of the differential equation
\[
\dot{y}(t) = X(y)(t) \quad \text{for all} \quad t \in I,
\]
extending the initial trajectory $x$ onto the interval $I$, then these trajectories are identical on the entire interval $I$ that is
\[
y(t) = y^\sim(t) \quad \text{for all} \quad t \in I.
\]
Proof. Let $I$ be an interval starting at the point $s$. On the right the interval may be either open or closed, either bounded or unbounded.

Since difference of two continuous functions on the set $I$ yields a continuous function, the function $\phi = y(t) - y^\sim(t)$ is continuous on $I$. Consider the set $J$ defined by

$$J = \{ t \in I : \phi(u) = 0 \text{ for all } u \leq t \}.$$

Clearly $s \in J$ so the set $J$ is nonempty.

Since $I$ is an interval and since the set $J$ with any two points $t_1, t_2 \in J$ such that $t_1 < t_2$ contains all points $u$ such that $t_1 < u < t_2$, the set $J$ itself forms an interval.

If the interval $J$ is, either unbounded on the right, or is bounded on the right but the least upper bound $\sup(J)$ coincides with the right end of the interval $I$, we immediately see that $I = J$.

So let us consider the case when $J$ is bounded on the right but $s_0 = \sup(J)$ does not coincide with the right end of the interval $I$. We shall prove that this leads to a contradiction.

It follows from the continuity of the solutions $y_1$ and $y_2$ that they coincide up to time $t = s_0$ including $t = s_0$ and are of class $C^1$. From our assumption about the point $s_0$ follows that it must lie inside of the interval $I$. Taking the part of the trajectories up to time $t = s_0$ as a new initial trajectory $x$, we can construct two unique local solutions defined on some intervals $I_1, I_2 \subset I$ to the right of $s$. We may assume that $I_1 = I_2$, otherwise we would take their intersection $I^\sim = I_1 \cap I_2$ as a new interval.

Let $\delta > 0$ denote the length of the interval $I^\sim$. Since the solution of the equation is unique, we would have that

$$y(t) = y^\sim(t) \text{ for all } t \in I_0 = I \cap [s_0, s_0 + \delta],$$

which would yield that $s_0 + \delta \leq s_0$. A contradiction. Thus the interval $J$ must coincide with the entire interval $I$. $\square$

**Definition 14.3** (Maximal global solution). Assume that $X$ is a nonanticipating operator with uniqueness property. If between all possible solutions extending an initial trajectory $x$, there exists a solution with longest possible interval, the such solution is called maximal global solution.

**Theorem 14.4** (Existence and uniqueness of the maximal global solution). Assume that $X : T \mapsto \mathbb{R}^{6n}$ is a nonanticipating operator with uniqueness property.

Then for every initial trajectory $x \in T$, between all possible solutions $y$ of the differential equation

$$\dot{y} = X(y),$$

which extend $x$, there exists a maximal global solution. Moreover this solution is unique.
Proof. To prove the theorem let \( \tilde{y} \) denote a trajectory, extending the initial trajectory \( x \), and defined on some interval \( \tilde{I} = [s, \tilde{e}] \). Assume that \( \tilde{y} \) represents a solution of the differential equation

\[
\dot{y} = X(y).
\]

Take all such solutions with their intervals \( \tilde{I} \) and let \( I = \bigcup \tilde{I} \) be the union of all such intervals. If a point \( t \) belongs to \( I \) it belongs to some interval \( \tilde{I} \) being the domain of a solution \( \tilde{y} \). Put

\[
y(t) = \tilde{y}(t).
\]

From the theorem on uniqueness of extensions follows that the value \( y(t) \) is well defined, that is it does not depend on the choice of the function \( \tilde{y} \).

Indeed if \( t \) belongs to any two intervals \( \tilde{I}_1 \) and \( \tilde{I}_2 \) then on the intersection \( J \) of these intervals the corresponding solutions \( \tilde{y}_1 \) and \( \tilde{y}_2 \) must coincide in accord with the Theorem 14.2. Thus we must have that \( \tilde{y}_1(t) = \tilde{y}_2(t) \).

Moreover the obtained trajectory \( y \) is the solution of our differential equation on the entire interval \( I \). Since the graph of the function \( y \) contains in it the graph of any other solution of our equation, the solution \( y \) represents a maximal global solution. Again from the Theorem 14.2 follows that this solution is unique. \( \Box \)

15. Spaces of nonanticipating uniformly Lipschitzian operators

We shall remind the reader the definition 12.8 of the space \( \text{Lip}(D, U) \) of uniformly Lipschitzian operators on the initial domain \( D \).

Let \( D = D(x, q, A, I) \) denote an initial domain and \( U \) a Banach space. Let \( \text{Lip}(D, U) \) denote the space of all nonanticipating uniformly Lipschitzian operators from the domain \( D \) into the Banach space \( C(I, U) \), of all continuous functions from the interval \( I \) into the space \( U \). If \( P \) is an operator in \( \text{Lip}(D, U) \), let

\[
\|P\| = \sup \{\|P(y)\| : y \in D \},
\]

\[
l(P) = \inf \{M : \|P(y) - P(y^\sim)\|_d \leq M \|y - y^\sim\|_d \text{ for all } y, y^\sim \in D_d, d \in I \}.
\]

Clearly, since \( D \) is compact, this norm \( \|P\| \) is well defined. The norm in the expression \( \|P(y)\| \) is understood as the supremum norm in the space \( C(I, U) \). Define

\[
\|P\|_l = \|P\| + l(P) \text{ for all } P \in \text{Lip}(D, U).
\]

The space \( \text{Lip}(D, U) \) with norm \( \| \|_l \) is called the space of nonanticipating uniformly Lipschitzian operators.

**Proposition 15.1** (The pair \( (\text{Lip}(D, U), \| \|_l) \) forms a Banach space). For every initial domain \( D \) and for every Banach space \( U \) the space \( \text{Lip}(D, U) \), of nonanticipating uniformly Lipschitzian operators, equipped with the norm \( \| \|_l \) forms a Banach space.

*Proof.* The proof follows from Th. 2.15 of Bogdan [2]. \( \Box \)
It follows from the above theorem that the space $\text{Lip}(D,U)$ is closed under addition and scalar multiplication operations induced by respective operations in the Banach space $U$. In the sequel we will extend this class of operations to include all Lipschitzian functions.

The following is Th. 6.10 of Bogdan [2].

**Theorem 15.2** (Joint Continuity). If $P \in \text{Lip}(D,U)$ is a nonanticipating uniformly Lipschitzian operator then the map

$$(t,y) \rightarrow P(y)(t)$$

is jointly continuous from the product $I \times D$ into the Banach space $U$, that is for every point $(t_0,y_0) \in I \times D$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|P(y)(t) - P(y_0)(t_0)| \leq \varepsilon \quad \text{when} \quad |t - t_0| \leq \delta \quad \text{and} \quad \|y - y_0\| \leq \delta \quad \text{and} \quad (t,y) \in I \times D.$$

**Corollary 15.3** (Uniform continuity of $P(y)(t)$). Assume that $P \in \text{Lip}(D,U)$. Since the set $I \times D$, as product of compact sets, is itself compact. Thus the function $(t,y) \rightarrow P(y)(t)$ is uniformly continuous on the set $I \times D$ into the Banach space $U$.

**Definition 15.4** (The set $\text{ran}(P)$ is the range of operator $P$). Assume that $P \in \text{Lip}(D,U)$. For shorthand we shall write $\text{ran}(P)$ to denote the set

$$P(D)(I) = \{P(y)(t) : y \in D, t \in I\}.$$

**Corollary 15.5** (Compactness of $\text{ran}(P)$). Assume that $P \in \text{Lip}(D,U)$. Since the set $I \times D$ is compact and the function $(t,y) \rightarrow P(y)(t)$ is continuous on it, the set $\text{ran}(P)$, as an image of a compact set by means of a continuous map, forms a compact set in the Banach space $U$.

**Definition 15.6** (Lipschitzian Operators over Cartesian Products). Let

$$U_1, \ldots, U_n; Z$$

be some Banach spaces. We shall say that an operator $P$, from a subset $G$ of the cartesian product $U_1 \times \ldots \times U_n$ into the Banach space $Z$, is Lipschitzian on $G$, if for some constant $M$ we have

$$\|P(u_1, \ldots, u_n) - P(\tilde{u}_1, \ldots, \tilde{u}_n)\| \leq M(\|u_1 - \tilde{u}_1\| + \cdots + \|u_n - \tilde{u}_n\|)$$

for all $(u_1, \ldots, u_n), (\tilde{u}_1, \ldots, \tilde{u}_n) \in G$.

The following is Th. 6.15 of Bogdan [2].
Theorem 15.7 \((\text{Lip}(D, U) \text{ spaces are closed under composition with Lipschitzian operators})\). Let \(D\) be an initial domain and \(U_1, \ldots, U_n; Z\) be some Banach spaces. Assume that \(P_j \in \text{Lip}(D, U_j)\) for \(j = 1, \ldots, n\) and \(Q\) is a Lipschitzian operator from the product \(\text{ran}(P_1) \times \cdots \times \text{ran}(P_n)\) into the space \(Z\). Define the composed operator \(P\) by the formula
\[
P(y)(t) = Q(P_1(y)(t), \ldots, P_n(y)(t)) \quad \text{for all} \quad y \in D \text{ and } t \in I.
\]
Then the operator \(P\) belongs to the space \(\text{Lip}(D, Z)\).

Definition 15.8 \((n\text{-linear bounded operators})\). Let \(U_1, \ldots, U_n; Z\) be some Banach spaces. We shall say that an operator \(P\), from the Cartesian product \(U_1 \times \cdots \times U_n\) into the Banach space \(Z\), is \(n\)-linear if for every variable \(u_j\), where \(j = 1, \ldots, n\), when other variables are fixed the map
\[
u_j \rightarrow P(u_1, \ldots, u_j, \ldots, u_n)
\]
is linear from the space \(U_j\) into the space \(Z\). Such an operator is said to be bounded if for some constant \(M\) we have
\[
|P(u_1, \ldots, u_n)| \leq M |u_1| |u_2|, \ldots, |u_n| \quad \text{for all} \quad u_1 \in U_1, \ldots, u_n \in U_n.
\]

Proposition 15.9 \((n\text{-linear bounded operator is Lipschitzian on bounded sets})\). If \(P\) is an \(n\)-linear bounded operator from a Cartesian product \(U_1 \times \cdots \times U_n\) of Banach spaces into a Banach space \(Z\), then it is Lipschitzian on every set of the form
\[
B_\delta = \{(u_1, \ldots, u_n) \in U_1 \times \cdots \times U_n : |u_j| \leq \delta, \; j = 1, \ldots, n\}
\]

Proof. The proof is straightforward and we leave it to the reader. \(\square\)

Corollary 15.10 \((\text{Compositions with } n\text{-linear operators})\). If \(Q\) is an \(n\)-linear bounded operator from a Cartesian product \(U_1 \times \cdots \times U_n\) of Banach spaces into a Banach space \(Z\), and \(P_j\) are in \(\text{Lip}(D, U_j)\) for \(j = 1, \ldots, n\), then the composition \(Q \circ (P_1, \ldots, P_n)\) belongs to the space \(\text{Lip}(D, Z)\).

Proof. Since the real-valued functions \(u_j \rightarrow |u_j|\) are Lipschitzian on \(U_j\), they are continuous. Thus on the compact set \(\text{ran}(P_j)\) they attain their maximum \(\delta_j\). Put \(\delta = \max \{\delta_j : \; j = 1, \ldots, n\}\).

Then the operator \(Q\) is Lipschitzian on the bounded set \(B_\delta\) and the operator representing the composition of operators
\[
P = Q \circ (P_1, \ldots, P_n)
\]
is well defined and we have \(P \in \text{Lip}(D, Z)\). \(\square\)
**Definition 15.11 (Differentiable operators).** Let \( f \) be an operator from an open set \( G \) in a Banach space \( U \) into a Banach space \( Z \). We shall say that the operator \( f \) is differentiable at a point \( a \in G \) if there is a ball

\[
B(a, r) = \{ u \in U : |u - a| < r \} \subset G
\]

and a linear bounded operator \( g \in \text{Lin}(U, Z) \) such that

\[
|f(x) - f(a) - g(x - a)| \leq o(|x - a|) \quad \text{for all} \quad x \in B(a, r),
\]

where \( o(h) \) denotes some function of the variable \( h \geq 0 \) such that

\[
o(h)/h \to 0 \quad \text{when} \quad h \to 0.
\]

Such an operator \( g \) is unique, and is called the derivative of \( f \) at the point \( a \), and it is denoted by \( f'(a) = g \).

It follows from the above definition that if the map \( f \) has a derivative at a point \( a \in G \) then \( f \) is continuous at \( a \). Notice also that the space \( \text{Lin}(R, R) \) is isomorphic and isometric with the space \( R \) of reals.

The theorem, that follows, can be found in Cartan [6, page 27, Th. 2.2.1].

**Theorem 15.12 (Chain Rule).** Assume that \( U, V, W \) are Banach spaces and \( G \subset U \) and \( H \subset V \) are open sets. Let \( f : G \to H \) and \( g : H \to W \) be differentiable on their domains. The composed function \( h = f \circ g \) defined by

\[
h(u) = f \circ g(u) = f(g(u)) \quad \text{for all} \quad u \in G
\]

is differentiable and

\[
h'(u) = f'(g(u))g'(u) \quad \text{for all} \quad u \in G.
\]

The following theorem can be found in Cartan [6, page 41, Th. 3.3.2].

**Theorem 15.13 (Maps with bounded derivative on convex sets are Lipschitzian).** Let \( f \) be an operator from an open set \( G \) in a Banach space \( U \) into a Banach space \( Z \). Assume that the derivative \( f'(u) \) exists at every point \( u \in G \). Assume that for some convex set \( W \subset G \) and a constant \( M \) we have

\[
|f'(u)| \leq M \quad \text{for all} \quad u \in W.
\]

Then

\[
|f(u) - f(u^\sim)| \leq M |u - u^\sim| \quad \text{for all} \quad u, u^\sim \in W.
\]

The space \( \text{Lin}(U, U) \) of linear bounded operators from a Banach space \( U \) into itself beside being closed under addition and scalar multiplication is also closed under the composition of operators: \( P \circ Q \in \text{Lin}(U, U) \) for all \( P, Q \in \text{Lin}(U, U) \). This operation has the property that \( |P \circ Q| \leq |P||Q| \).
Definition 15.14 (Banach Algebras). A Banach space $U$ is called a Banach algebra, if it is equipped with a bilinear operation $(u, w) \rightarrow uw$, from the product $U \times U$ into $U$, that is associative: $(uw)z = u(wz)$, and such that $|uw| \leq |u||w|$ for all $u, w \in U$.

If in addition there is an element $e$ in $U$ such that $eu = u = ue$ for all $u \in U$, then such an algebra is called a Banach algebra with unit. In such a case an element $u$ is called invertible if for some $w \in U$, called the inverse of $u$, we have $uw = wu = e$.

The unit element and the inverse elements are unique. We denote the inverse of $u$ by $u^{-1}$.

The following proposition is very useful in numerical computations of the inverse transformations. It represents a simple application of the Banach fixed point theorem and it is essential in the development of the theory presented in this paper.

It represents Pro. 6.25 of Bogdan [2].

Proposition 15.15 (Inverse $(e - v)^{-1}$ exist if $|v| < 1$). Let $U$ be a Banach algebra with unit. Then for every element $v \in U$ such that $|v| < 1$ the inverse $w = (e - v)^{-1}$ exists. It is a fixed point of the operator $f$ defined by the following formula

$$f(u) = e + vu \quad \text{for all } u \in U.$$ 

Moreover the sequence $w_n = f(w_{n-1})$ where $w_0 = 0$ is of the form

$$w_n = e + v + v^2 + \cdots + v^n = \sum_{0 \leq k \leq n} v^k \quad \text{for all } n > 0$$

and we have the following estimate for the distance of the fixed point $w$ and the approximation $w_n$

$$|w - w_n| \leq \frac{|v|^n}{1 - |v|} \quad \text{for all } n > 0,$$

and we also have an explicit formula for the inverse element $w$ as the sum of an absolutely convergent series

$$w = e + v + v^2 + \cdots = \sum_{n \geq 0} v^n.$$

Notice that in any Banach algebra $Lin(U, U)$ the identity map: $e(u) = u$ for all $u \in U$, is the unit element. It is easy to prove that the norm of the unit element $e$ in any Banach algebra is equal to 1, $|e| = 1$. Therefore for every invertible element $u$ we have $1 = |u^{-1}w| \leq |u||u^{-1}|$, thus $|u^{-1}| > 0$.

The following is Pro. 6.26 of Bogdan [2].

Proposition 15.16 (When the inverse map $u \mapsto u^{-1}$ is Lipschitzian?). Assume that $U$ is a Banach algebra with unit.

(1) If for some $u_0$ the inverse $u_0^{-1}$ exists then every element in the ball $B(u_0, r) = \{ u \in U : |u - u_0| < r \}$,

where $r = 1/|u_0^{-1}|$, has an inverse.
(2) The domain $G$ of existence of the inverse $u^{-1}$ is an open set and the function $f(u) = u^{-1}$ is continuous on $G$.

(3) The function $f(u) = u^{-1}$ is differentiable on $G$ and

$$f'(u)(h) = -u^{-1}h u^{-1} \quad \text{for all } h \in U \text{ and } u \in G.$$  

(4) If $W$ is a compact convex set such that $W \subset G$, then $f(u)$ is Lipschitzian on $W$.

**Theorem 15.17** (If reciprocal $g$ of $f \in \text{Lip}(D,U)$ exists then $g \in \text{Lip}(D,U)$). Let $D$ be any initial domain and $U$ a Banach algebra with unit $e$. Then if $f$ is a uniformly Lipschitzian function from the domain $D$ into the algebra $U$ and

$$[g(y)(t)] [f(y)(t)] = e \quad \text{for all } y \in D \text{ and } t \in I$$

then $g \in \text{Lip}(D,U)$.

**Proof.** By continuity of the functions $(y,t) \mapsto f(y)(t)$, and $x \mapsto x^{-1}$, and the norm $y \mapsto |y|$ and compactness of the set $D \times I$ we get the existence of a constant $M$ such that

$$|[f(y)(t)]^{-1}| \leq M \quad \text{for all } y \in D \text{ and } t \in I.$$  

From the algebraic identity

$$x_1^{-1} - x_2^{-1} = x_1^{-1}(x_2 - x_1)x_2^{-1}$$

we get

$$|g(y_1)(t) - g(y_2)(t)| = |[f(y_1)(t)]^{-1}[f(y_2)(t) - f(y_1)(t)][f(y_2)(t)]^{-1}|$$

$$\leq M^2 l(f) \|y_1 - y_2\|_d \quad \text{for all } y_1, y_2 \in D_d \text{ and } t \in I_d, d \in I$$

Therefore

$$l(g) \leq M^2 l(f)$$

and this implies that $g \in \text{Lip}(D,U)$. \hfill \square

**Theorem 15.18** $(f \in \text{Lip}(D,R^k) \iff f_j \in \text{Lip}(D,R) \forall j = 1, \ldots, k)$. Let $f$ be a map from an initial domain $D$ into the space $R^k$. Assume that $f = (f_1, \ldots, f_k)$ where $f_j$ is from $D$ into the reals $R$ for all $j = 1, \ldots, k$.

Then $f$ belongs to the space $\text{Lip}(D,R^k)$ if and only if each component $f_j$ belongs to the space $\text{Lip}(D,R)$.

**Proof.** The proof is obvious and we leave it to the reader. \hfill \square

**Theorem 15.19** $(f \in C^1(R^k,R^m)$ and $g \in \text{Lip}(D,R^k) \Rightarrow f \circ g \in \text{Lip}(D,R^m))$. Assume that $f$ is a continuous function from the space $R^k$ into the space $R^m$ having continuous partial derivatives of order 1.

Then if $g$ is a uniformly Lipschitzian function from an initial domain $D$ into the space $R^k$ then the composed function $f \circ g$ is uniformly Lipschitzian from $D$ into the space $R^m$. 


Proof. By assumption the derivative $f'(x)$ exists and is continuous from $\mathbb{R}^k$ into the space $\text{Lin}(\mathbb{R}^k, \mathbb{R}^m)$ of linear transformations from $\mathbb{R}^k$ into the space $\mathbb{R}^m$. Thus on every closed bounded convex set the map $x \mapsto |f'(x)|$ is bounded. Thus from Th. 3.3.2, page 41, of Cartan [6] follows that $f$ is Lipschitzian on every such set.

Therefore from Th. 15.7 follows that $f \circ g$ is uniformly Lipschitzian on the initial domain $D$. □

As a consequence of the above theorem we can conclude that for any polynomial $p$ the following is true.

Corollary 15.20 $(x_j \in \text{Lip}(D, \mathbb{R})) \forall j \Rightarrow p(x_1, \ldots, x_k) \in \text{Lip}(D, \mathbb{R})$. Assume that $D$ is an initial domain. Assume that $p(x)$ is a polynomial of coordinates $x_j$ of the vector $x$. If components $x_j = P_j \in \text{Lip}(D, \mathbb{R})$ then $p \circ (x_1, \ldots, x_k)$ is in the space $\text{Lip}(D, \mathbb{R})$.

16. Initial Domains with Positive Separation

In this section we will derive some estimates involving the speed $c$ of light, so for a while we will explicitly use this constant. Only in later sections it will be more convenient to change the units. Compare the following developments with constructions in Bogdan [1] in particular with results in Th. 10 and Th. 11.

Let $D = D(x, q, A, I)$ be an initial domain. In this section we will develop tools to study further the properties of the spaces $\text{Lip}(D, U)$ that are need for analysis of operators generated by moving bodies in gravitational and electromagnetic fields.

Definition 16.1 (Lattice Operations). Assume that $R$ as before denotes the field of real numbers. Introduce operations $\vee$ and $\wedge$ from $R \times R$ into $R$ by the formulas

$$p \vee r = \max\{p, r\} = \frac{1}{2}(p + r + |p - r|), \quad p \wedge r = \min\{p, r\} = \frac{1}{2}(p + r - |p - r|).$$

Any linear space $U$ of real-valued functions closed under composition with these two operations will be called a linear lattice.

Notice that if $f \in U$ then the absolute value $|f|$ of the the function $f$ is also in $U$, since $|f| = (-f) \vee f$. Now, if in addition the space $U$ is a Banach space with a norm $\|\|$, satisfying the following implication

$$g = |f| \implies \|g\| \leq \|f\| \quad \text{for all} \quad f \in U,$$

then $U$ is called a Banach linear lattice. It is easy to see that the last condition guarantees that the lattice operations are continuous in the norm topology.

The operations $\vee$ and $\wedge$ are commutative, associative, and Lipschitzian, since the unary operation $p \rightarrow |p|$ is Lipschitzian. The following theorem is useful in getting estimates of Lipschitz constant for operators in various spaces $\text{Lip}(D, U)$.

The following represents Th. 7.2 of Bogdan [2].
Theorem 16.2 \((\text{Lip}(D, R)\) is a Banach linear lattice and a Banach algebra). For every initial domain \(D\) the Banach space \(\text{Lip}(D, R)\), with the norm defined by
\[
\|P\|_I = \|P\| + k(P) \quad \text{for all } P \in \text{Lip}(D, R),
\]
where
\[
\|P\| = \sup \{\|P(y)\| : y \in D\},
\]
\[
k(P) = \inf \{M : \|P(y) - P(y^\sim)\|_d \leq M \|y - y^\sim\|_d \quad \text{for all } y, y^\sim \in D_d, d \in I\},
\]
forms a Banach linear lattice and a Banach algebra with unit.

In the sequel we will need a separation and minimal time delay operators.

Definition 16.3 (Separation and Minimal Time Delay Operators). Assume that 
\[D = D(x, q, A, I)\]
represents an initial domain with \(I = [s, e]\). Define the operator \(\text{sep}\), called the separation operator, by the formula
\[
\text{sep}(y)(t) = \min \{|r_j(t) - r_k(t)| : j, k = 1, \ldots, n, j \neq k\} \quad \text{for all } y \in D, t \leq e,
\]
and the operator \(\text{mtd}\), called the minimal time delay operator, by the formula
\[
\text{mtd}(y)(t) = \min \{T_{jk}(t) : j, k = 1, \ldots, n, j \neq k\} \quad \text{for all } y \in D, t \leq e.
\]

The following represents Pro. 7.4 of Bogdan [2].

Proposition 16.4 (Operators \(\text{sep}, \text{mtd}\) belong to \(\text{Lip}(D, R)\)). The separation operators \(\text{sep}\) and \(\text{mtd}\), when the trajectories \(y \in D\) are restricted to interval \(I\), belong to the space \(\text{Lip}(D, R)\) of nonanticipating uniformly Lipschitzian operators.

Proof. The proof follows from theorem [16.2].

Definition 16.5 (Time Delay Bound and Separation Bound). Let \(D = D(x, q, A, I)\) be an initial domain. The time delay bound \(t_b\) for the domain \(D\) is defined by
\[
t_b = \inf \{\text{mtd}(y)(t) : y \in D, t \in I\}.
\]
Similarly we define the separation bound \(s_b\) for the domain \(D\) by
\[
s_b = \inf \{\text{sep}(y)(t) : y \in D, t \in I\}.
\]
Notice since ranges of the operators \(\text{mtd}\) and \(\text{sep}\) are compact the constants \(t_b\) and \(s_b\) are well defined and are finite.

Notice that \(t_b = 0\) if and only if \(s_b = 0\). Indeed since the map \((y, t) \to \text{mtd}(y)(t)\) is continuous and the set \(D \times I\) is compact, there exist a trajectory \(y \in D\) and a point \(t \in I\) and a pair of indexes \(j, k\) such that for some \(j, k\) we have
\[
0 = t_b = T_{jk}(t) = \frac{1}{c}|r_j(t) - r_k(t - T_{jk}(t))| = \frac{1}{c}|r_j(t) - r_k(t)| \geq \frac{1}{c}\text{sep}(y)(t) \geq \frac{1}{c}s_b \geq 0,
\]
thus \(s_b = 0\). The converse that \(s_b = 0\) implies \(t_b = 0\) can be proved similarly.

Definition 16.6 (Domains with Positive Separation). Any initial domain \(D\) such that \(s_b > 0\) will be called a domain with positive separation.

The following represents Pro. 7.7 of Bogdan [2].
Proposition 16.7. If $D = D(x, q, A, I)$ is an initial domain with positive separation and $I = [s, e]$, then for every $d$ such that $0 < d - s < t_b$ and $d \leq e$ the emission time operators $t_{jk}$ satisfy the inequality
\[ t_{jk}(t) = t - T_{jk}(t) \leq s \quad \text{for all} \quad y \in D_d \text{ and } t \in [s, d]. \]

Proof. Notice that for the section $D_d$ of the domain $D$ the time interval is $[s, d]$. Thus for any $t \in [s, d]$ and any trajectory $y \in D_d$ we have
\[ t_{jk}(t) = t - T_{jk}(t) \leq d - t_b \leq d - (d - s) = s. \]

This completes the proof. \qed

The operators $y \rightarrow r_k(t_{jk})$, and $y \rightarrow v_k(t_{jk})$, and $y \rightarrow \dot{v}_k(t_{jk})$, are constant on the domain $D_d$ and they are uniquely determined by the initial trajectory $x$, since $t_{jk}(t) \leq s$ and by definition the initial trajectory $x$ has a continuous derivative. Therefore each, of the above operators, represents a continuous function of the variable $t$. Thus all the above operators belong to the space $Lip(D_d, R^3)$.

Notice that the time delay bound $t_b$ for the domain $D$ and the time delay bound $t'_b$ for its section $D_d$ satisfy the inequality $t_b \leq t'_b$. Thus taking eventually the section $D_d$ as our new domain we observe that the following corollary must be true.

Corollary 16.8 (Operators $r_k(t_{jk})$, $v_k(t_{jk})$, $\dot{v}_k(t_{jk})$, $r_{jk}$ are in $Lip(D, R^3)$). Assume that $D = D(x, q, A, I)$ is an initial domain with positive separation, and with $I = [s, e]$. If we have that $0 < e - s < t_b$, then each of the following operators
\[ r_k(t_{jk}), \ v_k(t_{jk}), \ \dot{v}_k(t_{jk}), \ r_{jk} = (r_j - r_k(t_{jk})), \ (v_j - v_k(t_{jk})), \]
where $j, k = 1, \ldots, n, j \neq k$, belongs to the space $Lip(D, R^3)$.

The following represents Th. 7.9 of Bogdan [2].

Theorem 16.9 ($s_b \leq c(1 + q)t_b$). Assume that $D = D(x, q, A, I)$ is an initial domain, and $s_b$ is the separation bound for the domain $D$, and $t_b$ is the time delay bound for the domain $D$. Then
\[ s_b \leq c(1 + q)t_b. \]

The following represents Th. 7.10 of Bogdan [2].

Theorem 16.10 (Lower estimate of separation bound $s_b$). Let $D = D(x, q, A, I)$ be an initial domain with $I = [s, e]$ and with positive initial separation $sep(x)(s) > 0$. If the length of the interval $I = e - s$ is such that
\[ e - s \leq \frac{sep(x)(s)}{3c}, \]
then we have the following lower estimate for the separation bound $s_b$
\[ \frac{sep(x)(s)}{3} \leq s_b. \]

The following represents Th. 7.11 of Bogdan [2].
**Theorem 16.11** (Negative powers of time delays are in Lip($D, R$)). Let $D = D(x, q, A, I)$ be an initial domain with positive separation. Then for every pair of indexes $j \neq k$ the operator $y \to T_{jk}^{-n}$, where $n > 0$, is in Lip($D, R$). Its Lipschitz constant can be estimated by

$$l(T_{jk}^{-n}) \leq nt_b^{n-1}l(T_{jk}) \leq \frac{2n}{c(1-q)t_b^{n+1}}.$$  

The following represents Cor. 7.12 of Bogdan [2].

**Corollary 16.12** (Operators $|r_{jk}|$, $|r_{jk}|^{-n}$, $e_{jk}$). Assume that the initial domain $D = D(x, q, A, I)$ is with positive separation. If the length of the interval $I = [s, e]$ is such that $e - s < t_b$, then the operators

$$y \to |r_{jk}|, \quad y \to |r_{jk}|^{-n}, (n > 0) \quad \text{for all} \quad j \neq k$$

are in Lip($D, R$), and the unit operator

$$y \to e_{jk} = |r_{jk}|^{-1}r_{jk} \quad \text{for all} \quad j \neq k$$

is in Lip($D, R^3$).

17. Lip($D, U$) spaces over regular initial domains

**Definition 17.1** (Regular initial domain). An initial domain $D = D(x, q, A, I)$, where $I = [s, e]$, with positive initial separation $\text{sep}(x)(s) > 0$ and such that

$$0 < e - s < \frac{\text{sep}(x)(s)}{3(1 + q)c}$$

will be called a regular initial domain.

Since on every regular initial domain $D$ we have the inequalities for the time delay bound

$$\frac{\text{sep}(x)(s)}{3(1 + q)c} \leq t_b = \inf \{\text{tdb}(y)(t) : y \in D, t \in I\}$$

and for the separation bound

$$\frac{\text{sep}(x)(s)}{3} \leq s_b = \inf \{\text{sep}(y)(t) : y \in D, t \in I\},$$

the bounds $t_b$ and $s_b$ are positive. It will be convenient from now on to assume, if not specified otherwise, that the domain $D$ is regular.

From now on we will assume that $c = 1$. We have already proved various parts of the following theorem but to be sure that we did not overlooked any particular fundamental field we collect them all into one theorem and go over the proofs again now that we have more tools to do that.

**Theorem 17.2** (Fundamental fields due to $k$-th body acting onto $j$-th body are in Lip($D, R^m$) for $m = 1$ or $m = 3$). Assume that $D = D(x, q, A, I)$ is a regular initial domain. Consider the fields acting onto the $j$-th body by the $k$-th body corresponding to the fundamental fields of the system of the trajectories.

Then the scalar fields

$$y \mapsto T_{jk}, \quad u_{jk}, \quad t_{jk}, \quad z_{jk}$$
all belong to the space $\text{Lip}(D, R)$ and the vector fields

$$y \mapsto r_j, \quad v_j, \quad r_{jk} = r_j - r_k(t_{jk}), \quad e_{jk} = u_{jk} r_{jk}, \quad r_j(t_{jk}), \quad v_j(t_{jk}), \quad a_j(t_{jk})$$

all belong to the space $\text{Lip}(D, \mathbb{R}^3)$.

**Proof.** It is obvious that operators $y \mapsto r_j$ and $y \mapsto v_j$ are in $\text{Lip}(D, \mathbb{R}^3)$. The remote operators, as follows from (16.8),

$$y \mapsto r_j(t_{jk}), \quad v_j(t_{jk}), \quad a_j(t_{jk})$$

are also in $\text{Lip}(D, \mathbb{R}^3)$. From linearity of $\text{Lip}(D, \mathbb{R}^3)$ follows that operator

$$y \mapsto r_{jk} = r_j - r_k(t_{jk})$$

is in $\text{Lip}(D, \mathbb{R}^3)$. The operator $y \mapsto T = |r_j - r_k(t_{jk})|$ as a composition of Lipschitzian function $r \mapsto |r|$ with $r_{jk}$ is in $\text{Lip}(D, R)$.

Since on a regular domain we have the estimate

$$T_{jk}(y)(t) \geq t_b > 0 \quad \text{for all} \quad y \in D \quad \text{and} \quad t \in I$$

the operator $u_{jk}$ as the reciprocal operator to $T_{jk}$ is in $\text{Lip}(D, R)$. Thus the operator $e_{jk} = u_{jk} r_{jk}$ considered as composition of operators $u_{jk} \in \text{Lip}(D, R)$ and operator $r_{jk} \in \text{Lip}(D, \mathbb{R}^3)$ with bilinear bounded operator $(\lambda, r) \mapsto \lambda r$ is in $\text{Lip}(D, \mathbb{R}^3)$.

Similarly we can prove that operators $t_{jk} = t - T_{jk}$ and $z_{jk} = (1 - \langle e_{jk}, v_{jk} \rangle)^{-1}$ are in $\text{Lip}(D, R)$.

**Theorem 17.3** (Newton-Feynman fields $E_{jk}$ are in $\text{Lip}(D, \mathbb{R}^3)$). Assume that $D = D(x, q, A, I)$ is a regular initial domain. Consider the Newton-Feynman field $E_{jk}$ acting onto the $j$-th body by the $k$-th body of the system $y \in D$ of the trajectories. Then for every pair of indexes $j \neq k$ the field $E_{jk} \in \text{Lip}(D, \mathbb{R}^3)$.

**Proof.** According to Bogdan-Feynman Theorem (11.1) the Newton-Feynman field $E$ generated by a single trajectory and expressed in terms of the fundamental fields is given by the formula

$$E = -u z^2 a + u z^3 (e, a)e - u z^3 (e, a)v + u^2 z^3 e - u^2 z^3 \langle v, v \rangle e - u^2 z^3 v + u^2 z^3 \langle v, v \rangle v.$$  \hspace{1cm} (17.1)

Thus the components of $E = (E_1, E_2, E_3)$ are representable as polynomials of the components of the fundamental fields. Hence $E_{jk}$ is representable as a polynomial of the coordinates of all the fields appearing in the previous theorem. So each coordinate of the field $E_{jk}$ is in $\text{Lip}(D, R)$ so the field itself $E_{jk}$ is in $\text{Lip}(D, \mathbb{R}^3)$. \hfill \Box
18. Existence of solutions to the evolution equations in force fields of special theory of relativity

We remind the reader that the Newton-Einstein equation for the $j$-th body can be written as

$$\Gamma(v_j)(\dot{v}_j) = \frac{1}{m_j} F_j$$

for all $j = 1, \ldots, n$
or equivalently as

$$\dot{v}_j = \frac{1}{m_j} \hat{\Gamma}(v_j)(F_j)$$

for all $j = 1, \ldots, n$

**Definition 18.1** (Regular nonanticipating operator). By a regular nonanticipating operator we shall understand an operator $F : T \rightarrow R^{3n}$ such that for every regular initial domain $D = D(x, q, A, I)$ the restriction of $F$ to $D$ represents a uniformly Lipschitzian operator, that is, it represents an element of the space $Lip(D, R^{3n})$.

Notice that in the following theorem the nature of the force operator $F$ is not important. Important is just the mathematical assumption that it represents a regular nonanticipating operator. This is the property we were looking for.

**Theorem 18.2** (Maximal solutions of equations of special theory of relativity). Assume that the operator $F : T \rightarrow R^{3n}$ representing the force field forms a regular nonanticipating operator.

Then for every admissible initial trajectory $x$ of the system of $n$ bodies such that at the initial time $t = s$ the separation operator satisfies the condition

$$\text{sep}(x)(s) > 0,$$

there exists a unique maximal solution $y$ to the Newton-Einstein system of equations

$$\dot{v}_j = \frac{1}{m_j} \hat{\Gamma}(v_j)(F_j)$$

for all $j = 1, \ldots, n$
satisfying the condition

$$\text{sep}(y)(t) > 0$$

for all $t \in I$
and extending the initial trajectory $x$ onto the time interval $I$.

**Proof.** Let $U$ denote the algebra of all linear transformations from $R^3$ into $R^3$. The map

$$g(r, u, x) = r u(x)$$

for all $r \in R$, $u \in U$, $x \in R^3$
is trilinear and bounded.

Take any regular initial domain $D = D(x, q, A, I)$. Consider first the map

$$y \mapsto m_j^{-1} = m_0^{-1} \sqrt{(1 - |v_j|^2)}.$$

Notice that $p_j(y) = (1 - |v_j|^2)$ represents a polynomial of the coordinates of the vector operator $v_j \in Lip(D, R^3)$. Thus $p_j \in Lip(D, R)$. We also have the estimate

$$a = (1 - q^2) \leq (1 - |v_j|^2) \leq 1.$$ Since the square root function is differentiable on
the closed interval \([a, 1]\) and has continuous derivative, it is Lipschitzian. Thus the operator \(y \mapsto m_j^{-1}\) is in \(\text{Lip}(D, R)\) for every \(D\).

Consider the operator \(y \mapsto \Gamma(v_j)\). By definition
\[
\Gamma(v_j)(h) = h + \gamma_j^2 \langle v_j, h \rangle v_j \quad \text{for all} \quad h \in R^3.
\]
Notice \(\gamma_j^2 = p_j^{-1}\). Since we established before that \(p \in \text{Lip}(D, R)\) its reciprocal \(\gamma^2\) is in \(\text{Lip}(D, R)\).

Now consider the 4-linear bounded map
\[
S(r, v_1, v_2, h) = r \langle v_1, h \rangle v_2 \quad \text{for all} \quad r \in R; \ v_1, v_2, h \in R^3
\]
and define map
\[
\hat{S}(r, v_1, v_2)(h) = r \langle v_1, h \rangle v_2 \quad \text{for all} \quad r \in R; \ v_1, v_2, h \in R^3
\]
Clearly the map \(\hat{S}\) is trilinear and bounded and maps the triple \((r, v_1, v_2)\) into an element of the algebra \(U\). Thus we have \(\hat{S}(\gamma_j^2, v_j, v_j) \in \text{Lip}(D, U)\). Since the unit element \(e\) of the algebra \(U\) as a constant operator belongs to \(\text{Lip}(D, U)\) and
\[
\Gamma(v_j) = e + \hat{S}(\gamma_j^2, v_j, v_j)
\]
we must have \(\Gamma(v_j) \in \text{Lip}(D, U)\). Therefore the reciprocal operator \(y \mapsto \hat{\Gamma}(v_j)\) must belong to \(\text{Lip}(D, U)\) for every regular initial domain \(D\).

Hence the operator \(\frac{1}{m_j} \hat{\Gamma}(v_j)(F_j)\) belongs to the space \(\text{Lip}(D, R^3)\) for every regular initial domain \(D\). Therefore the operator \(y \mapsto \frac{1}{m_j} \hat{\Gamma}(v_j)(F_j)\) forms a regular nonanticipating operator.

The above on the basis of the theorem on global maximal solution leads to the conclusion stated in the theorem: There is a unique maximal solution satisfying the separation condition and extending the initial trajectory.

\[\Box\]

19. Existence of solutions to equations of gravitational electrodynamics with external force fields

We have introduced in (12.1) operators \(H_{jk}\) by the the formula
\[
(19.1) \quad H_{jk}(h) = q_j \{h + v_j \times (e_{jk} \times h)\} - m_0 h \quad \text{for all} \quad h \in R^3.
\]
These operators have the following property.

\textbf{Proposition 19.1} (Operators \(H_{jk} \in \text{Lip}(D, U) \forall \ j \neq k\). The operators \(H_{jk}\) for every pair of indexes \(j \neq k\) are well defined on any regular initial domain \(D = D(x, q, A, I)\) and belong to the space \(\text{Lip}(D, U)\) where \(U\) denotes the algebra of all linear operators from \(R^3\) into \(R^3\). Thus the operators \(H_{jk}\) are regular nonanticipating operators.

\textit{Proof.} The proof is similar to the proof concerning the operator \(\hat{\Gamma}(v_j)\) and we leave it to the reader. \[\Box\]
Definition 19.2 (Lorentz-Newton force operator). By the Lorentz-Newton force operator we shall understand the system of the operators, for \( j = 1, \ldots, n \), given by the formulas

\[
F_j = \sum_{k=1; k \neq j}^{k=n} H_{jk}(E_{jk}) \quad \text{for all } j = 1, \ldots, n,
\]

where

\[
H_{jk}(h) = q_j \{ h + v_j \times (e_{jk} \times h) \} - m_0 k_h \quad \text{for all } h \in \mathbb{R}^3
\]

and \( E_{jk} \) represents the Newton-Feynman field acting onto the \( j \)-th body from the trajectory of the \( k \)-th body of the system.

Theorem 19.3 (Equations of gravitational electrodynamics with external force field). Assume that \( F_{j,0} : T \rightarrow \mathbb{R}^3 \) are regular nonanticipating operators representing external force fields. Assume that \( F_j (j = 1, \ldots, n) \) denotes the Lorentz-Newton force operator.

Then for every admissible initial trajectory \( x \), defined on the time interval \( (-\infty, s] \), with positive separation \( \text{sep}(x)(s) \) at time \( t = s \), there exist an interval \( I \) starting at \( s \), such that the Newton-Einstein system of differential equations

\[
\dot{p}_j(t) = F_j(t) + F_{j,0}(t) \quad \text{for all } t \in I \text{ and } j = 1, \ldots, n
\]

has a unique maximal solution having positive separation

\[
\text{sep}(y)(t) > 0 \quad \text{for all } t \in I.
\]

Proof. Let \( U \) denote the algebra of all linear operators from \( \mathbb{R}^3 \) into \( \mathbb{R}^3 \). Notice that Lorentz-Newton force operator forms a regular nonanticipating operator. Indeed, for fixed \( j \neq k \) the operator \( H_{jk} \in \text{Lip}(D, U) \), where \( D = D(x, q, A, I) \).

The operator \( E_{jk} \in \text{Lip}(D, \mathbb{R}^3) \). Since the map \( S(u, r) = u(r) \) considered on the product \( U \times \mathbb{R}^3 \) into \( \mathbb{R}^3 \) is bilinear and bounded we have that \( H_{jk}(E_{jk}) \in \text{Lip}(D, \mathbb{R}^3) \).

Now from linearity of the space \( \text{Lip}(D, \mathbb{R}^3) \) follows that the Lorentz-Newton force \( F_j \in \text{Lip}(D, \mathbb{R}^3) \). Since the force \( F_{j,0} \) represents a regular nonanticipating operator the sum \( A_j = F_j + F_{j,0} \) belongs to \( \text{Lip}(D, \mathbb{R}^3) \). Thus the equation (19.3) is equivalent to equation

\[
\dot{v}_j(t) = \frac{1}{m_j} A_j(t) \quad \text{for all } t \in I \text{ and } j = 1, \ldots, n.
\]

We have proved in the preceding section that such system of equations has a unique maximal solution satisfying the condition that separation between the bodies is positive at any time \( t \in I \). \( \square \)

20. Computer programs for DERIVE software system

The results presented in this paper were obtained with the help of an interactive, symbolic, DOS based, computer program DERIVE, version 2.06, developed by Soft Warehouse Inc. in Honolulu, Hawaii.

Since the differential formulas obtained in this paper essentially yield polynomials of the involved field variables, any program that can handle operations on polynomials of an arbitrary number of variables can be used.

For a reader who would be interested in verifying the formulas we enclose the header and the actual programs used in the computations.
To understand these programs we need to introduce rudimentary elements of grammar used by DERIVE.

\(<\text{variable}\>\) consists of a string of lower case letters a-z, digits 0-9, and underscore character \(_\). A variable must start with a letter.

\(<\text{algebraic expression}\>\) consists of variables joint by usual operations: addition +, subtraction -, multiplication *, division /, and exponentiation ^. Parenthesis ( ) can be used to indicate the order of operations.

\(<\text{assignment statement}\>\) has the form

\(<\text{variable}> := <\text{algebraic expression}> \text{<cr}>\)

where \(<\text{cr}>\) denotes the carriage return character.

\(<\text{comment}\>\) consists of any text enclosed into double quotes "<text>".

Using tilde ~ as the continuation character one can break long assignment statements or comments between lines. On output DERIVE breaks line after every 80 characters.

DERIVE accepts input files containing programs. For illustration assume that we prepared using a plain ASCII text editor a file \(\text{hh.mth}\) containing the text as in the program below representing the header with basic formulas. Notice that the sequence in which the formulas are entered in the program is important. One should not use things that where not defined in previous formulas.

To make a test to see that DERIVE understands the program run DERIVE. On entering derive a window appears with menu commands at the bottom of the window. Notice that each command has one letter in upper case. To select such a command just press the corresponding letter on the keyboard.

To load the program select from consecutive menus: Transfer, Load, Derive, hh. Press \(<\text{cr}>\) that is \(<\text{Enter}>\) on most keyboards. DERIVE will load the file \(\text{hh.mth}\) and the last statement in the file will be highlighted. To browse through the statements use directional keys \(<\text{up}>\) and \(<\text{down}>\) on the keyboard. The selected statement will be highlighted. To enter a statement to examine its components use directional keys \(<\text{right}>\) and \(<\text{left}>\) and to enter or exit a subcomponent use the keys \(<\text{up}>\) and \(<\text{down}>\).

On a highlighted component one can perform any operation from the menu. We need just one operation: Expand. It permits one to expand a compound component into a polynomial of variables that are involved in the component.

Expanded components are appended at the bottom of the file \(\text{hh.mth}\). To save the computations select from the consecutive menus commands: Transfer, Save, Derive, \(<\text{cr}>\). The file \(\text{hh.mth}\) will be saved to its original location overwriting the previous version. To exit DERIVE select from the menu command Quit.

Using your favorite ASCII editor update the file by new statements and comments. One can repeat this process until the goal of the computations is met.

In the following programs the names of the variables were preserved, if possible, as they appear in the paper with proper adjustments to lower case letters.

To represent the dot product \(\langle e, v \rangle\) of vectors \(e\) and \(v\), we use variable \(\text{edv}\). To represent a component \(v_i\) of the vector \(v\) we use \(v_i\).

The prefix \(d_\) in front of a variable represents the partial derivative \(D\) and the prefix \(d_\text{i}\) the partial derivative \(D_i\). Thus \(De\) and \(D_i e\) are translated as \(d_\text{e}\) and \(d_\text{i e}\), respectively.

Header used in the proofs of the results of the paper
This header was used in front of all the programs that follow.

"Header for electromagnetism"
"
"Define basic identities for dot product:
ede:=1
"To avoid double entrees for dot product variables, use
"notation that follows order of vector variables as in the list:
"e, v, a, dot_a; thus use vda and not adv for dot product of
"vector a with vector v. So edv is ok but not vde.
edv:=1-1/z
"
"Derivatives with respect to D_i"
d_it:=z*e_i
d_itau:=-z*e_i
d_iu:=-u^2*z*e_i
"
"Meaning of variables in the following expressions"
" un_ii=1; ee_ii=e_i*e_i"
" ev_ii=v_i*e_i=e_i*v_i"
d_ie_i:=u*un_ii-u*z*ee_ii+u*z*ev_ii
" ea_i=e_i*a_i"
d_iv_i:=-z*ea_i
"
"Notice that edot_a_ii=e_i*a'_i"
d_ia_i:=-z*edot_a_ii
"
"un_i denotes here the i-th unit vector of standard base"
d_ie:=+u*un_i-u*z*e_i*e+u*z*e_i*v
d_iv:=-z*e_i*a

d_ia:=-z*e_i*dot_a

d_iz:=-u^2*z*v_i-u^2z^2*e_i+u^2z^2*3*e_i+u^2z^2*3*e_i+u^2z^2*3*vdv+u^2z^2*3*vdv
"D_iz:=+uz^2v_i -uz^3e_i +uz^2e_i +uz^3(v,v)e_i -z^3(e,a)e_i"
"
"Derivatives of dot products:

d_iede:=0
d_iedv:=-u*z*ede*e_i+v_i+u*z*vdv+u*z*ede*e_i

d_ieda:=u*z*eda*e_i+u*z*eda*e_i+u*z*eda*e_i

d_iiva:=-2*z*vdv+u*vdv

"Derivatives with respect to time"
d_tau:=z
d_t:=1-z
d_u:=u^2*z-u^2

d_z:=u*z-2+u*z^2+z^2*3*eda+u*z^2+3-u*z^2*3*vdv
"
"Time derivatives of vector fields"
d_v:=z*a
d_e:=-u*e+u*z*e-u*z*v
d_a:=z*dot_a
"Time derivatives for components of the vector fields"

\[
d_{v_i} := z a_i \\
\]

\[
d_{e_i} := -u e_i + u z e_i - u z v_i \\
\]

\[
d_{a_i} := z \dot{a}_i \\
\]

"--"

"Time derivatives of dot products:"

"Remember the ordering list: e, v, a, dot_a"

\[
d_{ed} := 0 \\
\]

\[
d_{edv} := -u \dot{e}v + u z \dot{e}v + u z \dot{v}v + z \dot{e}a \\
\]

\[
d_{eda} := -u \dot{e}a + u z \dot{e}a - u z \dot{v}a + z \dot{e} \ddot{a} \\
\]

\[
d_{vd} := 2z \dot{v}a \\
\]

\[
d_{vda} := 2z \ddot{v}a \\
\]

\[
d_{a} := 2z \dot{a} \\
\]

"End of header"

Check that scalar potential \( \phi = uz \) satisfies the wave equation

Include the header at the beginning of this file to process the program.

"Checking that scalar potential \( \Phi = uz \) satisfies the wave equation"

"--"

"Compute time derivative \( D(uz) \)"

\[
d_{\phi} := z d_{u} + u d_{z} \\
\]

"Expanding right side of above yields"

\[
d_{\phi} := -u^2 z^3 vdv + u^2 z^3 - u^2 z^2 + u z^3 \dot{e}a \\
\]

"--"

"Compute derivative \( D \) of each term of the above sum"

\[
tt_{1} := -(2u d_{u} z^3 vdv + u^2 z^3 e_i - u^2 z^3 - u^2 z^2 + u z^3 \dot{e}a) \\
\]

\[
tt_{2} := 2u d_{u} z^3 + u^2 z^3 \dot{e}a \\
\]

\[
tt_{3} := -(2u d_{u} z^3 u^2 z^3 + u^2 z^3 e_i) \\
\]

\[
tt_{4} := d_{u} z^3 \dot{e}a + u z^3 d_{e} + d_{u} z^3 e_i \\
\]

"--"

"Thus the second derivative \( D^2 \) of \( uz \) is"

\[
d_{d_{\phi}} := tt_{1} + tt_{2} + tt_{3} + tt_{4} \\
\]

"--"

"\n\n\n\n\nNow compute partials \( D_i \) of \( u*v \) \n\n\n\n\nPartial derivative \( D_i \) of each term of the above sum is"

\[
u_{u1} := 2u d_{u} d_{i} z^3 vdv + u^2 z^3 e_i + u^2 z^3 d_{i} d_{v} + u^2 z^3 d_{i} e_i \\
\]

\[
u_{u2} := 2u d_{u} d_{i} z^3 e_i - u^2 z^3 d_{i} e_i + u^2 z^3 d_{i} d_{v} \\
\]

\[
u_{u3} := 2u d_{u} d_{i} z^3 v_i + u^2 z^3 d_{i} v_i + u^2 z^3 d_{i} e_i \\
\]

\[
u_{u4} := d_{u} z^3 \dot{e} a e_i - u z^3 d_{i} d_{e} e_i - u z^3 d_{i} e a e_i - u z^3 e_i d_{e} d_{i} e_i \\
\]

"--"

"Thus second partial \( (D_i)^2 \) of \( uz \) is"

\[
d_{d_{i\phi}} := u_{u1} + u_{u2} + u_{u3} + u_{u4} \\
\]

"Expanding right side yields"
"After summation over index i we get the Laplacian of Phi=u*z"

lp_phi:=+3*u^3*z^5*vdv^2*ede-6*u^3*z^5*vdv*ede+3*u^3*z^5*ede-u^3*z^5*ede*vdv-u^3*" 
3*z^5*ede*e_i^2*ede-6*u^3*z^5*ede*ede+3*u^3*z^5*ede*ede-u^3*z^5*ede*ede+3*u^3*" 
3*z^5*ede*ede+3*u^3*z^5*ede*ede-u^3*z^5*ede*ede+3*u^3*z^5*ede*ede-u^3*z^5*ede*ede" 
"Check if wave equation is satisfied"

wave:=lp_phi-dd_phi

"Check that vector potential A=u*z*v satisfies the wave equation"

Check that vector potential A=u*z*v satisfies wave equation

"Check that vector potential aa=u*z*v satisfies wave equation"
To compute Laplacian(aa) we need to take sum over index i
"remembering that \(\sum (e_i \cdot e_i) = \text{ede}; \sum (e_i \cdot v_i) = \text{edv}; \sum (ee_{ii}) = \text{ede};\)
"\(\sum (un_{ii}) = 3; \sum (v_i^2) = \text{vdv} \text{ etc.}\)
""To this end edit previous expression to make proper changes"

Thus Laplacian(aa) is

\[
lp_{aa} := 3z^5\text{ede}\text{eda}^2u\text{vdv} - 6z^5\text{ede}\text{eda}u^2\text{vdv} + 6z^5\text{ede}\text{eda}u^2\text{v} - 3z^5\text{ede}\text{eda}u^2\text{v} - 3z^5\text{ede}\text{eda}u^2\text{v}
\]

"Now compute time derivative \(D_{aa}\) of \([u*v\)]"

\[
d_{aa} := +d_u z v + u d_z v + u z d_v
\]

"It expands to"

\[
d_{aa} := +u^2 z^3 v (1 - \text{vdv}) - u^2 z^2 v + \text{eda} u z^3 v + a u z^2 z
\]

"Compute derivative of each term of above expression"

\[
pp1 := +2 u d_u z^3 v (1 - \text{vdv}) + u^2 z^3 d_v (1 - \text{vdv}) - u^2 z^3 v d_v d_v
\]
\[
pp2 := -2 u d_u z^2 v - u^2 z^2 d_z v - u^2 z^2 d_v
\]
\[
pp3 := +d_eda u z^3 v + u d_eda u z^3 v + u^2 z^3 d_z v + u^2 z^3 d_v
\]
\[
pp4 := +d_a u z^2 + u^2 z^2 d_z
\]

"The second derivative of aa with respect to time is"

\[
dd_{aa} := pp1 + pp2 + pp3 + pp4
\]

"It expands to"

\[
dd_{aa} := 3u^3 z^5 v^2 - 6u^3 z^5 v + 3u^3 z^5 v d_v v + 3u^3 z^5 v d_v d_v - 3u^3 z^5 v d_v v - 3u^3 z^5 v d_v d_v
\]

"D'Alembertian of aa is"

\[
wave_{aa} := lp_{aa} - dd_{aa}
\]

"Expanding right side of above yields:"

\[
wave_{aa} = 0
\]

"Thus the wave equation is satisfied!"

"For the record expand the right side of statement defining variable dd_{aa}"

\[
dd_{aa} := 3u^3 z^5 v^2 - 6u^3 z^5 v + 3u^3 z^5 v d_v v + 3u^3 z^5 v d_v d_v - 3u^3 z^5 v d_v v - 3u^3 z^5 v d_v d_v
\]
Check that Lorentz gauge formula $\nabla \cdot A + D\phi = 0$ is satisfied.

"Check Lorentz gauge formula $\nabla \cdot A + D\phi = 0$"
"--"
Computing $D_i A_i = D_i [u z v_i + u d_i z v_i + u z d_i v_i]$
"--"
Expanding the right side of above statement yields"
$d_i A_i = u^2 z^3 v d v_i + v_i - u^2 z^3 e_i v_i + u^2 z^2 v_i^2 - u z^3 e d a e_i v_i - u^2 z^2 e_d a_i$
"--"
Thus the divergence of the vector field $A$ is"
$\text{div}_A = u^2 z^3 v d v e_i + u^2 z^3 e_i v_i + u^2 z^2 e_d a e_i v_i - u z^3 e_d a e_i v_i - u z^2 e_d a_i$
"--"
Time derivative of $\phi$ is"
$d_\phi = d_u z + u d_z$
"--"
Lorentz gauge is"
$\text{gauge} = \text{div}_A + d_\phi$
"--"
Expanding the right side of above yields"
$\text{gauge} = 0$
"--"
For the record expand the expression defining variable $d_\phi$"
$d_\phi = -u^2 z^3 v d v + u - u^2 z^3 e_i v_i + u^2 z^3 e_i v_i + u z^2 e_d a$

Compare Feynman’s field with Liénard-Wiechert’s field

"Compare Feynman’s electric field $f f = u^2 e + u \cdot (-1) D(u^2 e) + D^2 e$"
"with Liénard-Wiechert electric field $e e = -\text{grad}(u z) - D(u z v)$"
"--"
First term of the sum defining $f f$ is"
$tt_1 = u^2 e$
The second term is"
$tt_2 = u \cdot (-1) (2 u \cdot d_u e + u^2 d_e)$
To compute the third term notice first that"
$d_e = u z (e - v) - e u$
Thus for the third term we have the formula"
$tt_3 = d_u z (e - v) + u d_z (e - v) + u z (d_e - d_v) - d_e u - e d_u$
"--"
Thus we have for Feynman field $f f$"
$ff = tt_1 + tt_2 + tt_3$
"--"
Now compute components $e e_i$ of the electric field $e e$"
where $e e = -\text{grad}(u z) - D(u z v)$"
$e e_i = - (d_i u z + u d_i z) - (d_u z v_i + u d_z v_i + u z d_v_i)$
The above expands to"
$e e_i = -u^2 z^3 v d v e_i + u^2 z^3 v d v e_i + u^2 z^3 e_i v_i + u^2 z^3 v_i^2 + u z^3 e d a e_i v_i - u^2 z^3 e d a e_i v_i - u z^2 e_d a_i$
"--"
Thus in vector notation $e e$ is"
$ee = -u^2 z^3 v d v e + u^2 z^3 v d v e + u^2 z^3 e_i v_i + u^2 z^3 e_i v_i + u z^2 e_d a$
"--"
difference:=ff-ee
"Expanding in the above statement the expression on the right side yields"
difference:=0
"So the electric fields coincide!"
"--"
"Formulas for magnetic fields involve cross product, programming of which is"
"cumbersome. It is easier to do it by a direct computation as in the text."

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