Large Field Polynomial Inflation: Parameter Space, Predictions and (Double) Eternal Nature

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Abstract. Simple monomial inflationary scenarios have been ruled out by recent observations. In this work we revisit the next simplest scenario, a single–field model where the scalar potential is a polynomial of degree four which features a concave “almost” saddle point. We focus on trans–Planckian field values. We reparametrize the potential, which greatly simplifies the procedure for finding acceptbale model parameters. This allows for the first comprehensive scan of parameter space consistent with recent Planck and BICEP/Keck 2018 measurements. Even for trans–Planckian field values the tensor–to–scalar ratio $r$ can be as small as $O(10^{-8})$, but the model can also saturate the current upper bound. In contrast to the small–field version of this model, radiative stability does not lead to strong constraints on the parameters of the inflaton potential. For very large field values the potential can be approximated by the quartic term; as well known, this allows eternal inflation even for field energy well below the reduced Planck mass $M_{\text{Pl}}$, with Hubble parameter $H \sim 10^{-2} M_{\text{Pl}}$. More interestingly, we find a region of parameter space that even supports two phases of eternal inflation. The second epoch only occurs if the slope at the would–be saddle point is very small, and has $H \sim 10^{-5} M_{\text{Pl}}$; it can only be realized if $r \sim 10^{-2}$, within the sensitivity range of next–generation CMB observations.

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1 Introduction and Motivation

Inflation, as invented in the 1980s [1–4], is an elegant paradigm of early universe physics. It not only solves the horizon, flatness and monopole problems of standard cosmology, but also generates initial seeds for structure formation arising from vacuum fluctuations [5]. In the simplest, “slow–roll” implementation of this idea, a spin\(^{-\frac{3}{2}}\) inflaton field is introduced which slowly rolls down a sufficiently flat potential; see Ref. [6] for a comprehensive review of models of inflation.

The simplest model assumes a monomial \(\phi^p\) potential; in renormalizable models whose potential is bounded from below \(p\) is either 2 or 4. However, such potentials are sufficiently flat, i.e. the first and second derivatives of the potential are sufficiently small compared to the potential itself, only at large field values. This leads to the overproduction of tensor modes, i.e. the tensor–to–scalar ratio \(r\) is predicted beyond the upper bound established by recent measurements of Cosmic Microwave Background (CMB) anisotropies [7, 8].\(^1\)

In this paper, we still assume that the inflaton is a real scalar field \(\phi\), but we allow a general, renormalizable polynomial potential. This next to simplest scenario has been analyzed many times since 1990 [11–20]. Such polynomial inflation can also be realized in string theory [21]. All these analyses have been performed before the release of the 2018 Planck and BICEP/Keck results [7, 8]. Unlike previous investigations, we aim to work out the full parameter space that agrees with the latest measurements, and derive the allowed

\(^1\)One may also consider monomials with fractional power, e.g. the monodromy inflationary model where \(V(\phi) \propto \phi^{2/3}\) [9, 10]. The BICEP/Keck 2018 results [8] then require \(p < 0.53\) at 95% c.l. if CMB scales experienced no more than 60 e–folds of inflation, in strong tension with monodromy inflation.
range of $r$. This is timely since future precise observations, for example by CORE [22], AliCPT [23], LiteBIRD [24], and CMB-S4 [25], should greatly extend the sensitivity, down to $r \sim \mathcal{O}(10^{-3})$.

As noted above, the potential should be flatter than a monomial at not too large field values. Here we achieve this by canceling several contributions, with different powers of the field, around a (near) saddle point, where both the first and second derivative of the potential become small. This is similar to the inflection point inflationary scenario [26–32]; however, we employ a purely renormalizable potential, i.e. only allow terms up to $\phi^4$. Just below the would–be saddle point the potential has a concave shape, as favored by the Planck 2018 data [7]. We rewrite the potential in terms of the location $\phi_0$ of the would–be saddle point, a quantity $\beta$ which governs the slope of the potential at $\phi_0$, and a multiplicative factor which only affects the overall normalization of the CMB anisotropies. Based on this reparametrization, we work out, for the first time, the full parameter space with predictions (power spectrum, spectral index and its running) consistent with Planck and BICEP/Keck 2018 measurements [7]. We find that the current upper bound on $r$ can be saturated, which means that part of the parameter space should be testable in the near future.

Another aim of this work is to investigate at which scale(s) eternal inflation\(^2\) might have occurred. In slow–roll inflation the classical change of the inflaton field during one Hubble time dominates over its quantum fluctuation; the inflaton field thus moves essentially deterministically downhill towards its minimum, until the end of inflation. However, in the opposite situation, where the quantum fluctuations dominate over the classical evolution, the inflaton field can move uphill rather than downhill. The Hubble patches where this happens inflate longer; in fact, in this case some such patches will inflate forever, i.e. inflation becomes eternal, although in our patch inflation obviously must have ended. The possibility that inflation can be eternal was first discussed in Ref. [34]. Later it was shown [35] that eternal inflation is in fact inevitable if the potential is a monomial with positive power, assuming only that the initial field value is sufficiently large; this occurs at energy scales well below the reduced Planck mass, $M_{Pl} \simeq 2.4 \cdot 10^{18}$ GeV. One thus may not need to worry about quantum gravity effects when describing eternal inflation [36]. “Hilltop” models can also lead to eternal inflation [37, 38].

During the eternal expansion, infinitely many independent “mini–universes” (or “pocket universes”) with different de Sitter vacua are generated via a self-reproducing process [39]. It has been speculated that this process can “populate” (or probe) the landscape of string theory [40–42]. This mechanism also naturally provides a scientific justification for the (weak) anthropic principle. From the perspective of eternal inflation, nearly everything is possible, provided only that the overall energy density is dominated by the potential energy of the inflaton field. For example, independent mini–universes may feature different types of compactification leading to different fundamental physical laws and/or different values of physical “constants” (which are field–dependent in superstring theory). Some of these laws and constants support life of our type, and clearly we (as living beings) can only observe those mini–universes where this is indeed the case [39, 43]. Finally, eternal inflation may help to relax the initial conditions problem. By this we mean the probability that some initial configuration of the inflaton field, and of the other dynamical degrees of freedom, gives rise to sufficiently long exponential expansion of the universe.\(^3\) As argued in [46], any

\(^2\)See e.g. Ref. [33] for a review.

\(^3\)For a review see e.g. Refs. [44, 45].
initial configuration that leads to eternal inflation will produce an infinite spacetime volume, making the probability of this initial configuration less significant. 

Since for large field values our potential is dominated by the $\phi^4$ term, it is not surprising that it leads to eternal inflation at sufficiently large values of $\phi$; the minimal required inflaton field energy turns out to be slightly lower than in pure $\phi^4$ inflation. More intriguingly, for sufficiently small values of $\beta$, i.e. a sufficiently flat potential near the would-be saddle point, a second period of eternal inflation can occur, at a value of the Hubble parameter smaller by more than three orders of magnitude than that during the first epoch of eternal inflation. However, this can only be realized if $\phi_0 \gtrsim 15 M_{Pl}$, with $r \gtrsim 0.01$. These scenarios can therefore be probed in the near future.

The remainder of this paper is organized as follows. In sec. 2 we describe the general setup, with emphasis on the reparametrization of the inflaton potential. In sec. 3 methods to scan the allowed parameter space are shown, and the corresponding predictions are given. The radiative stability of the potential is checked and the maximal reheating temperature is determined in sec. 4. In sec. 5 we investigate the possibility of realizing eternal inflation in our scenario, with focus on the calculation of the corresponding energy scale(s). Finally, sec. 6 summarizes this work. In this paper, we use Planckian units, i.e. we set the reduced Planck mass $M_{Pl} = \sqrt{\frac{1}{8\pi G}} \simeq 2.4 \cdot 10^{18}$ GeV to unity.

2 The Setup

In this section we first describe the inflaton potential and the resulting expressions for the parameters of inflation; in the second subsection we present simplified analytical results in some limits.

2.1 General Analysis

The action for the inflaton field in the Einstein frame is given by:

$$ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - V(\phi) \right], \quad (2.1) $$

where $g$ is the determinant of the metric; we assume it to be of the Friedmann–Robertson–Walker (FRW) type, i.e. $g_{\mu\nu} = \text{diag}(+1,-a^2,-a^2,-a^2)$ with $a$ denoting the scale factor. The corresponding Euler–Lagrange equation of motion for the classical background field is

$$ \ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + V'(\phi) = 0. \quad (2.2) $$

Here $\nabla$ denotes derivatives with respect to comoving spatial coordinates, and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter, which is determined by the Friedmann equation:

$$ H^2 = \frac{1}{3} \rho(\phi) = \frac{1}{3} \left[ \frac{1}{2} (\dot{\phi})^2 + V(\phi) + \frac{1}{2a^2} (\nabla \phi)^2 \right]. \quad (2.3) $$

We make the usual assumption that the classical background field $\phi$ is homogeneous, i.e. $\phi \equiv \phi(t)$ is a function of the cosmic time $t$ only, so that all gradient terms for the background field vanish. The potential we are considering is the general renormalizable one$^4$:

$$ V(\phi) = b \phi^2 + c \phi^3 + d \phi^4. \quad (2.4) $$

$^4$A linear term can be removed through a shift of $\phi$. We neglect the tiny cosmological constant term, which would be generated by a constant term in the potential.
We need $d > 0$ for the potential to be bound from below, and we consider $b > 0$ so that the minimum of the potential is at $\phi = \phi_{\text{min}} = 0$, with $V(\phi_{\text{min}}) = 0$. Since the potential is symmetric under the transformation $c \to -c$, $\phi \to -\phi$, we set $c \leq 0$ without loss of generality, so that inflation occurs at positive field values.\footnote{The simpler case with $c = 0$, so that the potential has only two terms, has been investigated in Ref. \[47\]; our analysis shows that this scenario is no longer viable. In Ref. \[48\], the two-term scenario with radiative corrections is investigated.} The first and second derivatives of the potential are given by:

$$V'(\phi) = 2b \phi + 3c \phi^2 + 4d \phi^3; \quad V''(\phi) = 2b + 6c \phi + 12d \phi^2.$$  \hspace{1cm} (2.5)

We need the potential to be very flat over some range of field values. Suppose first that the potential features an exact saddle point at $\phi = \phi_0$, i.e. $V'(\phi_0) = V''(\phi_0) = 0$, which requires

$$\phi_0 = -\frac{3c}{8d}; \quad b = \frac{9c^2}{32d},$$  \hspace{1cm} (2.6)

from which we learn that the ratio $c/d$ determines the position of the saddle point. Allowing for a finite slope even at $\phi_0$, one can reparametrize the potential as

$$V(\phi) = d \left[ \phi^4 + \frac{c}{d} (1 - \beta) \phi^3 + \frac{9}{32} \left( \frac{c}{d} \right)^2 \phi^2 \right] = \frac{1}{d} \left[ \phi^4 + A (1 - \beta) \phi^3 + \frac{9}{32} A^2 \phi^2 \right].$$  \hspace{1cm} (2.7)

Here $A \equiv \xi d \equiv -\frac{8}{3} \phi_0$ determines the location of the flat region of the potential. Note that the potential (2.7) still contains three free parameters, $d$, $A$ and $\beta$, i.e. it is a genuine reparametrization of the general ansatz (2.4). However, the form (2.7) is far more convenient, since the overall multiplicative factor $d$ only affects the overall normalization of the density perturbations, while $\beta$ directly controls the slope near $\phi_0$. For $\beta < 0$ the potential has a second minimum at $\phi > \phi_0$ where the inflaton field may get stuck, in which case there would be no hot Big Bang. We therefore require $\beta \geq 0$; recall that for $\beta = 0$ the potential has an exact saddle point at $\phi_0$. In this paper, we focus on the large field inflation scenario where $\phi_0 \geq 1$; a detailed analysis of the small field case can be found in Ref. \[49\].

The traditional potential slow–roll (SR) parameters \[50\] are:

$$\epsilon_V \equiv \frac{1}{2} \left( \frac{V'}{V} \right)^2 = \frac{2}{\phi^2} \left[ \frac{9A^2 - 48A(\beta - 1)\phi + 64\phi^2}{9A^2 - 32A(\beta - 1)\phi + 32\phi^2} \right]^2;$$

$$\eta_V \equiv \frac{V''}{V} = \frac{6}{\phi^2} \left[ \frac{3A^2 - 32A(\beta - 1)\phi + 64\phi^2}{9A^2 - 32A(\beta - 1)\phi + 32\phi^2} \right];$$

$$\xi_V^2 \equiv \frac{V'V''}{V^2} = -\frac{384[A(\beta - 1) - 4\phi] (9A^2 - 48A(\beta - 1)\phi + 64\phi^2)}{\phi^3 (9A^2 - 32A(\beta - 1)\phi + 32\phi^2)^2}.$$  \hspace{1cm} (2.8)

These quantities do not depend on $d$. During SR inflation, all these parameters must be small, $\epsilon_V$, $|\eta_V|$ and $|\xi_V^2| \ll 1$.

Inflation ends at a field value $\phi_{\text{end}}$ where $\epsilon_V(\phi_{\text{end}}) = 1$. Since for $\phi \ll \phi_0$ the term $9A^2 = 64\phi_0^2$ dominates over other terms in eq. (2.8), $\epsilon_V \approx 2/\phi^2 \approx \eta_V$ and hence $\phi_{\text{end}} \approx 1.41$
if $\phi_0 \gg 1$. For smaller $\phi_0$, $\phi_{\text{end}}$ is closer to $\phi_0$ so that the cubic and quartic terms need to be included in its determination.

Another important quantity is the total number $N_{\text{CMB}}$ of e–folds of inflation that occurred after the CMB pivot scale $k_\star = 0.05\text{Mpc}^{-1}$ first crossed out the horizon; in our model it can be computed analytically (within the SR approximation). The full result is given in the Appendix; for $\beta \ll 1$ it reduces to

$$N_{\text{CMB}} = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{\sqrt{2\epsilon_V}} d\phi \approx \frac{1}{24} \left\{ 3\phi^2 - 4\phi_0^2 + 15\phi_0^2 \sqrt{\frac{2}{\beta}} \arctan \left( \frac{\phi_0 - \phi}{\sqrt{2\beta}\phi_0} \right) - \phi_0^2 \ln \left( (\phi_0 - \phi)^2 \right) \right\} \bigg|_{\phi_{\text{end}}}^{\phi_{\text{CMB}}}.$$  

Here $\phi_{\text{CMB}}$ denotes value of the field when $k_\star$ crossed out of the horizon. In order to solve the flatness and horizon problems, $\sim 50$ e–folds of inflation are needed.

During SR inflation with a quasi de Sitter spacetime, Gaussian curvature perturbations are generated, with power

$$P_\zeta = \frac{V}{24\pi^2 \epsilon_V}.$$  

The spectral index $n_s$ and its running $\alpha$ are given by

$$n_s = 1 - 6\epsilon_V + 2\eta_V; \quad \alpha = 16\epsilon_V\eta_V - 24\epsilon_V^2 - 2\xi_V^2;$$  

measurements of these quantities can be used to constrain the model parameters $\beta$ and $A$.

The final observable of interest is the tensor–to–scalar ratio $r$, which is given by

$$r = 16\epsilon_V = \frac{32}{\phi^2} \left[ \frac{9A^2 - 48A(\beta - 1)\phi + 64\phi^2}{9A^2 - 32A(\beta - 1)\phi + 32\phi^2} \right]^2.$$  

The final Planck 2018 measurements at the pivot scale $k_\star = 0.05\text{Mpc}^{-1}$, including their own measurements plus results on baryonic acoustic oscillations (BAO), give in a 7 parameter cosmological model (baseline $\Lambda$CDM plus running $n_s$) [7]:

$$P_\zeta = (2.1 \pm 0.1) \times 10^{-9}; \quad n_s = 0.9659 \pm 0.0040; \quad \alpha = -0.0041 \pm 0.0067.$$  

So far no evidence for a non–vanishing tensor–to–scalar ratio $r$ has been found. The most recent upper bound, from BICEP/Keck 2018 results [8], is

$$r_{0.05} < 0.035$$  

at 95% C.L., after extrapolation to our pivot scale.\footnote{Note that in the experimental literature (e.g. the $r - n_s$ plots of Planck [7] or BICEP/Keck [8]) the bound on $r$ is usually quoted at scale $k = 0.002\text{Mpc}^{-1}$, denoted by $r_{0.002}$. For our choice of the pivot scale $k_\star = 0.05\text{Mpc}^{-1}$, one has $r_{0.05} \approx r_{0.002} (0.05/0.002)^2 = r_{0.002} (0.002/0.05)^2$, where the (small) running of the tensor spectral index $n_T \approx -r/8$ has been neglected.}

The combined constraint on $r$ and $n_s$ (adapted from Ref. [8]) is shown in Fig. 1. We also present predictions for three sets of free parameters of the potential (2.7), chosen such that $\mathcal{O}(10^{-3}) \lesssim r \lesssim \mathcal{O}(10^{-2})$, which should be testable in the near future [22–25].
Figure 1: The blue shaded region is the currently allowed part of the $r-n_s$ plane; it has been adapted from the recent BICEP/Keck 2018 results [8]. The red lines show predictions for our model, defined in eq.(2.7), where the parameters have been chosen such that $\mathcal{O}(10^{-3}) \lesssim r \lesssim \mathcal{O}(10^{-2})$. The small and big red dots correspond to $N_{\text{CMB}} = 50$ and 60, respectively.

2.2 Approximations

Eqs.(2.8) to (2.12) allow a fully analytical calculation of all SR parameters in terms of the free parameters of the potential plus the value of $\phi_{\text{CMB}}$, which is also a free parameter. However, these equations are too complicated to be solved analytically for the free parameters, for given $n_s$, $\alpha$ and $N_{\text{CMB}}$. In this section, we therefore present simplified analytical expressions, which work well in some limits.

1. $\phi \approx \phi_0$:

In this regime, analytical results for the inflationary predictions can be obtained by rewriting the field as [49]

$$\phi = \phi_0 (1 - \delta).$$

(2.15)

Decreasing $\phi$ corresponds to increasing $\delta$. Note that fluctuations at scales probed by observations of the CMB must have been created at $\phi < \phi_0$, where the inflaton potential is concave, i.e. $\eta_V < 0$, so that $n_s < 1$ can be reproduced. Since both $\delta$ and $\beta$ are rather small (as we will see, $\delta \lesssim \mathcal{O}(\sqrt{\beta})$ is needed, so that $\beta \ll \delta \ll 1$), it will be sufficient to keep only terms linear in $\beta$ and up to quadratic in $\delta$ in the analysis.

The SR parameters defined in (2.8) can then be approximated as [49]:

$$\epsilon_V \simeq \frac{72}{\phi_0^2} \left(2\beta + \delta^2\right)^2; \quad \eta_V \simeq \frac{24}{\phi_0^2} \left(2\beta - \delta\right); \quad \xi_V^2 \simeq \frac{288}{\phi_0^4} \left(2\beta + \delta^2\right).$$

(2.16)
Using the simplified result for $\epsilon_V$, the number of e–folds becomes \[49\]

$$
N_{\text{CMB}} = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{\sqrt{2\epsilon_V}} d\phi = \frac{\phi_0^2}{12} \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{d\delta}{(2\beta + \delta^2)} \\
\simeq \frac{\phi_0^2}{12\sqrt{2}\beta} \left[ \frac{\pi}{2} - \arctan \left( \frac{\delta_{\text{CMB}}}{\sqrt{2}\beta} \right) \right]. \quad (2.17)
$$

The normalization of the power spectrum, its spectral index and the running of the spectral index defined in (2.11), simplify to \[49\]

$$
P_\zeta \simeq \frac{d\phi^6_0}{5184\pi^2(\delta^2 + 2\beta)^2}; \quad (2.18)$$

$$
n_s \simeq 1 - \frac{48\delta}{\phi_0^2}; \quad (2.19)$$

$$
\alpha \simeq -\frac{576(2\beta + \delta^2)}{\phi_0^4}. \quad (2.20)
$$

Finally, the tensor–to–scalar ratio $r$ defined in eq.(2.12) reduces to \[49\]

$$
r \simeq \frac{1152(2\beta + \delta^2)^2}{\phi_0^2}. \quad (2.21)
$$

Evidently eq.(2.19) immediately determines $\delta$, i.e. $\phi_{\text{CMB}}$. $\beta$ can then be fixed using eq.(2.17): the argument of the arctan needs to be $O(1)$, giving $\beta \sim O(\delta^2)$, as claimed above. Moreover, $|\eta_V| \gg \epsilon_V$ in this scenario. The SR conditions thus cease to be satisfied at $\phi_{\text{end}} \simeq \phi_0 \left( 1 - \phi_0^2/24 \right)$, where $\eta_V$ crosses $-1$ \[49\]. However, exponential expansion continues until $\epsilon_V \simeq 1$, which is satisfied for $\phi \simeq \phi_0 \left( 1 - \sqrt{\phi_0/(6\sqrt{2})} \right)$.

In ref.[49] it was shown that this approximation works very well for $\phi_0 \leq 1$. Here we find that it still works quite well even for $\phi_0 \lesssim 5$.

2. $\phi \gg \phi_0$:

Here the potential is dominated by the quartic term:

$$
V \simeq d\phi^4. \quad (2.22)
$$

The corresponding SR parameters are:

$$
\epsilon_V = \frac{8}{\phi_0^2}; \quad \eta_V = \frac{12}{\phi_0^2}, \quad (2.23)
$$

while the spectral index is given by

$$
n_s = 1 - 6\epsilon_V + 2\eta_V = 1 - \frac{24}{\phi_0^2}. \quad (2.24)
$$
The Planck central value $n_s = 0.9659$ (cf. eq.(2.13)) thus requires $\phi_{\text{CMB}} \approx 27$, which implies $r = \frac{128}{9\phi_0^2} \approx 0.18$. This is well above the current upper bound on $r$, see fig. 1. Moreover, $\phi_{\text{CMB}} = 27$ with a $\phi^4$ potential yields $N_{\text{CMB}} \approx 90$, which is also well above the allowed range. This re–derives the by now quite well–known result that simple $\phi^4$ inflation is excluded; as we noted in the Introduction, this holds for any monomial inflaton potential, $V(\phi) \propto \phi^p$ with $p \geq 2$.

3. $\phi \gtrsim \phi_0$:

If $\phi$ is very close to $\phi_0$ the first approximation can again be used, but now with $\delta < 0$. Eq.(2.19) shows that then $n_s > 1$ if $\beta \ll 1$, in conflict with observation. Solutions with $\phi_{\text{CMB}} > \phi_0$ require larger values of $\beta$ than those with $\phi_{\text{CMB}} < \phi_0$ in order to keep $N_{\text{CMB}}$ within the acceptable range. In fact, for $\beta > 0.1$, $n_s < 1$ even for $\phi_{\text{CMB}} = \phi_0$, since $6\epsilon_V > 2\eta_V$ then. However, for $\phi_{\text{CMB}} = \phi_0$ and $\beta = 0.1$ one has $r = 128/(9\phi_0^2)$. $r < 0.035$ then requires $\phi_0 > 20$, which in turn yields $N_{\text{CMB}} > 150$. In order to obtain a value of $n_s$ close to its upper bound (2.13) at $\phi_{\text{CMB}} = \phi_0$ one even needs $\beta \gtrsim 0.2$; $r < 0.035$ then requires $\phi_0 > 28$ and $N_{\text{CMB}} > 245$, about a factor of 4 above the desired range.

In the regime with $\phi > \phi_0$ but neither $\phi \sim \phi_0$ nor $\phi \gg \phi_0$, the potential remains convex, i.e. $\eta_V > 0$. As for the case $\phi_{\text{CMB}} \approx \phi_0$ one can still get $n_s < 1$, since for sufficiently large $\beta$ and/or $\phi_{\text{CMB}} \epsilon_V$ becomes larger than $\eta_V/3$. However, once again this leads to too large values for $N_{\text{CMB}}$ or $r$ (or both). Therefore no viable solution with $\phi_{\text{CMB}} \geq \phi_0$ exists.

4. $\phi \ll \phi_0$:

If $\phi \ll \phi_0$, the quadratic term in the potential (2.7) dominates:

$$V(\phi) \simeq b\phi^2.$$  \hspace{1cm} (2.25)

The corresponding SR parameters are

$$\epsilon_V = \frac{2}{\phi^2} = \eta_V,$$  \hspace{1cm} (2.26)

giving a spectral index

$$n_s = 1 - 6\epsilon_V + 2\eta_V = 1 - \frac{8}{\phi^2}.$$  \hspace{1cm} (2.27)

The central value (2.13) of $n_s$ measured by Planck 2018 is reproduced for $\phi_{\text{CMB}} \approx 15$, which in turn leads to $r \simeq 0.14$, well above its upper bound. ($N_{\text{CMB}} \approx 56$ comes out correctly in this case.)

5. $\phi < \phi_0$:

In the regime with $\phi < \phi_0$ but with $\phi_{\text{CMB}}$ neither close to $\phi_0$ nor $\phi_{\text{CMB}} \ll \phi_0$, the potential can maintain a small and negative curvature due to the negative contribution from the cubic term slightly overcompensating the positive contributions from the quadratic and quartic terms. In fact, for $\beta \ll 1$ the potential remains concave for

\begin{footnote}{For such large values of $\beta$ eq.(2.9) is no longer valid, and the full expression given in the Appendix should be used.}

\end{footnote}
\(\phi_0(1-2\beta) > \phi > \phi_0(1+2\beta)/3\). If \(\phi_0 \lesssim 1\), \(|\eta V|\) begins to exceed 1, signaling the end of SR inflation, already at a value of \(\phi\) close to \(\phi_0\), see eq.(2.16). However, for \(\phi_0 > 1\) SR inflation can extend to field values well below \(\phi_0\). In fact for very large \(\phi_0\), the potential is effectively approaching a quadratic one again, since the last \(\lesssim 65\) e-folds of inflation happen at \(\phi \ll \phi_0\); this leads back to the case discussed in the previous paragraph, which is excluded by the upper bound on \(r\). This argument shows that there must be an upper bound on \(\phi_0\) in our model. In the remainder of this paper we will explore the parameter space with \(\phi_0 > 1\) and \(\phi_{\text{CMB}} < \phi_0\) in detail.

3 Model Parameters and Predictions

In this section, we first describe our methods to search for acceptable model parameters and then scan over the full parameter space that is consistent with the latest CMB observations (2.13) and (2.14) at the 2\(\sigma\) level.

3.1 Method to Find Model Parameters and Examples

We have learned that the location of the plateau, i.e. \(\phi_0\), is determined by the parameter \(A\) of the rewritten potential (2.7); we treat it as a free parameter. The slope of the plateau is determined by \(\beta\), i.e. for given field value (not too far from \(\phi_0\)) the SR parameter \(\epsilon V\) will become larger when \(\beta\) is increased. Of course, the SR parameters, as well as \(N_{\text{CMB}}\) and \(r\), also depend on \(\phi_{\text{CMB}}\). On the other hand, the overall coupling \(d\) in eq.(2.7) only affects the normalization of the power spectrum, see eq.(2.10).

In practice we first fix \(\phi_0\). The parameters \(\phi_{\text{CMB}}\) and \(\beta\) should then be chosen such that \(n_s\) and \(N_{\text{CMB}}\) have the desired values. As argued in the previous subsection, viable solutions only exist for \(\phi_{\text{CMB}} < \phi_0\). Reducing \(\phi_{\text{CMB}}\) for given \(\beta\) means that one is moving away from the flattest part of the potential (apart from the region near the minimum, which cannot lead to inflation); this increases \(\epsilon V\) and reduces \(\eta V\) (often making it more negative). This means that \(1-n_s\) and \(r\) both increase when \(\phi_{\text{CMB}}\) is reduced, but \(N_{\text{CMB}}\) becomes smaller. Reducing \(\beta\) for fixed \(\phi_{\text{CMB}}\) has the opposite effect: the potential becomes flatter, which increases \(N_{\text{CMB}}\) but decreases \(r\) and usually also \(1-n_s\).

For \(\phi_0 \gtrsim 5\) the approximation described by eqs.(2.15)–(2.21) still works fairly well. As already noted in the corresponding discussion, in this case one can use the spectral slope \(n_s\) to determine \(\phi_{\text{CMB}}\), and then chose \(\beta\) such that \(N_{\text{CMB}}\) is reproduced. For this range of parameters \(r\) is still very small, well below the present bound.

This procedure yields \(1-\phi_{\text{CMB}}/\phi_0 \propto \phi_0^2\), so the approximation \(\phi_0 - \phi_{\text{CMB}} \ll \phi_0\) begins to break down for \(\phi_0 \gtrsim 5\). The free parameters \(\phi_{\text{CMB}}\) and \(\beta\) then need to be determined together. We find that this can still be done iteratively. One starts with a guess for \(\beta\), e.g. the small–\(\phi_0\) value \(\approx 10^{-6}\phi_0^4\). For this value of \(\beta\), \(\phi_{\text{CMB}}\) is selected such that the spectral index \(n_s\) comes out as desired. One then fixes \(\phi_{\text{CMB}}\) and varies \(\beta\) until \(N_{\text{CMB}}\) takes the desired value. With this new value of \(\beta\), a new value of \(\phi_{\text{CMB}}\) can be computed using \(n_s\), and so on. This iteration usually converges fairly quickly. At the end, the overall coupling strength \(d\) is determined using eq.(2.10) with \(\mathcal{P}_\zeta = 2.1 \cdot 10^{-9}\).

Of course, one should also check that \(r\) and the running of the spectral index \(\alpha\) have acceptable values. We find that \(\alpha\) is always negative, and lies within the currently allowed range given in (2.13). On the other hand, for large \(\phi_0\) the tensor–to–scalar ratio \(r\) may come out too large. Moreover, while for sufficiently small \(\phi_0\) the desired values of \(n_s\) and \(N_{\text{CMB}}\) can always be attained, this is not necessarily true for larger values of \(\phi_0\).
Figure 2: The values of $\phi_{\text{CMB}}$ (black, divided by 100) and $\beta$ (red) that lead to $n_s = 0.9659$ and $N_{\text{CMB}} = 60$, as function of $\phi_0$. The blue curve shows the resulting prediction for the tensor–to–scalar ratio $r$.

This procedure is illustrated in Fig. 2, for $\phi_0 \geq 5$ where the deviation from the small–$\phi_0$ solution begins to be sizable. Here we have chosen $N_{\text{CMB}} = 60$ and $n_s = 0.9659$, the current central value. We see that for $\phi_0 \lesssim 10$, $\phi_{\text{CMB}}$ (shown in black) has to remain quite close to $\phi_0$. For fixed ratio $\phi_{\text{CMB}}/\phi_0$ an increase of $\phi_0$ reduces the SR parameters due to the overall $1/\phi^2$ factors in eqs.(2.8), moving $n_s$ closer to 1. This has to be compensated by decreasing $\phi_{\text{CMB}}/\phi_0$. On the other hand, $N_{\text{CMB}}$ depends not only on the SR parameter $\epsilon_V$, but also on the range of field values over which the integral in eq.(2.9) has to be evaluated. An increase of $\phi_0$ thus has to be compensated by an increase in $\beta$, shown by the red curve, in order to leave $N_{\text{CMB}}$ unchanged. This leads to a rapid increase of $\epsilon_V$, and hence of $r$ (shown in blue); however, in this region of parameter space we still have $\epsilon_V \ll |\eta_V|$, i.e. the spectral index $n_s$ is essentially determined by $\eta_V$.

For $\phi_0 > 10$ the curve for $\phi_{\text{CMB}}$ flattens out. Recall that even for a purely quadratic potential $\phi_{\text{CMB}} \simeq 15$, see eq.(2.27), and our potential is significantly flatter, hence requiring smaller $\phi_{\text{CMB}}$ in order to give the correct $N_{\text{CMB}}$. At the same time $\epsilon_V$ keeps increasing, so that its contribution to $n_s$ becomes significant. This flattens the increase of $\beta$, which reaches
a maximum at $\phi_0 \approx 17$. For yet larger values of $\phi_0$, $\phi_{\text{CMB}}$ becomes almost independent of $\phi_0$, i.e. inflation now occurs further and further away from the saddle point. Keeping the potential sufficiently flat then requires a reduction of $\beta$. For the given choice of $n_s$ and $N_{\text{CMB}}$ no solution can be found for $\phi_0 \geq 25.4$; the red curve drops very steeply at the end, since the potential at $\phi_{\text{CMB}} \sim 0.55\phi_0$ depends only weakly on the slope at $\phi_0$.

Note that $r$ keeps increasing even when $\beta$ is decreasing; for the chosen parameters, it exceeds the bound of 0.035 for $\phi_0 \geq 17.5$. However, for other choices of $N_{\text{CMB}} \in [50, 65]$ and $n_s \in [0.9579, 0.9739]$ (the 2σ range) the solution terminates before the bound on $r$ is saturated. Notice also that $\beta$ remains quite small throughout, i.e. the potential indeed needs to feature a near–inflection point.

In Table 1 we explore a wider range of $N_{\text{CMB}}$ and $n_s$, for $\phi_0$ between 1 and 20. The overall trends are as in Fig. 2: $\phi_{\text{CMB}}$ remains close to $\phi_0$ for $\phi_0 \lesssim 10$, but increases only slowly once $\phi_0 > 15$; and $\beta$ at first increases quickly, but reaches a maximum at $\phi_0$ around 15 and then quickly diminishes again. We also see that the model can saturate the upper bound on $r$ for $\phi_0 > 14$.

Moreover, the running $\alpha$ of the spectral index is always negative, and well within the currently allowed range given in (2.13). It is essentially independent of $\phi_0$ for $\phi_0 \lesssim 5$, but becomes smaller in magnitude for $\phi_0 \geq 8$. In fact, $\xi^2_\beta$ changes sign at $\phi_{\text{CMB}} = 2\phi_0(1 - \beta)/3$; for the parameters used in Fig. 2 this happens at $\phi_0 \approx 18$. However, by then $\epsilon_V$ has become so large that the first two terms in the expression for $\alpha$ in eq.(2.11) dominate; these terms are negative since $\eta_V < 0$.

Table 1 also lists the values of $d$ needed to reproduce the observed normalization of the spectrum of CMB anisotropies, computed from eq.(2.10). At small $\phi_0$ the coupling scales like $d \propto \phi_0^2$, since $V(\phi_{\text{CMB}}) \propto d\phi_0^4$ and $\epsilon_V \propto \phi_0^6$. However, for $\phi_0 > 5$ the growth of $\epsilon_V$ with increasing $\phi_0$ slows down. As a result, $d$ reaches a maximum value near $10^{-13}$ at $\phi_0 \sim 10$, and decreases again for yet larger values of $\phi_0$. As a result, the physical inflaton mass $m_\phi = 2\sqrt{d}\phi_0$ increases $\propto \phi_0^2$ for $\phi_0 \lesssim 5$, but depends only weakly on $\phi_0$ for $\phi_0 > 10$.

### 3.2 Complete Scan of Parameter Space

In order to explore the full parameter space, we scan over the parameters $\phi_0$, $\phi_{\text{CMB}}$ and $\beta$; $d$ has again been fixed such that $P_k \approx 2.1 \cdot 10^{-9}$. We accept all combinations of parameters that yield $50 \leq N_{\text{CMB}} \leq 65$ and satisfy the constraints on $n_s$ and $r$ shown in Fig. 1. We see from Fig. 3 that this scan in fact fills the entire presently allowed region of the $(n_s, r)$ plane.

The allowed ranges of the model parameters and the resulting predictions for $r$, $n_s$ and $N_{\text{CMB}}$ are further shown as function of $\phi_0$ in Fig. 4.

The upper left panel shows $\phi_{\text{CMB}}$. For $\phi_0 \lesssim 5$, $\phi_{\text{CMB}} \approx \phi_0$ can be analytically obtained by [49]

$$\phi_{\text{CMB}} = \phi_0(1 - \delta_{\text{CMB}}) \approx \phi_0 \left(1 - 7.10 \cdot 10^{-4} \phi_0^2\right);$$  \hspace{1cm} (3.1)

here we have used $\delta_{\text{CMB}} \approx 1 - \frac{\phi_0^2}{28}(1 - n_s)$ (cf. Eq.(2.19)) with $n_s = 0.9659$. This analytical approximation is shown by the red line, which describes the numerical results very well for

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As long as only field values below $\phi_0$ are considered, one could find additional solutions with larger $\phi_0$ and negative $\beta$. Recall, however, that in this case the potential features a second minimum above $\phi_0$, in which the inflaton would get stuck if it came from yet larger field values; in particular, in scenarios featuring a phase of eternal inflation, as discussed below. This is why we restrict ourselves to $\beta \geq 0$.

Generically $N_{\text{CMB}} < 65$ unless there is an exotic reheating phase following the end of inflation [51, 52]. For our case, inflation ends with a usual quadratic potential, hence we adopt $50 \leq N_{\text{CMB}} \leq 65$. 

small $\phi_0$. As we already saw in Fig. 2, this approximation breaks down for $\phi_0 > 5$, where the difference between $\phi_0$ and $\phi_{\text{CMB}}$ increases quickly. Eventually $\phi_{\text{CMB}}$ becomes nearly independent of $\phi_0$, taking values around 12; larger $\phi_{\text{CMB}}$ would require larger $\epsilon_V$ in order to keep $N_{\text{CMB}}$ within the acceptable range, in conflict with the upper bound on $r$.

The middle left panel of Fig. 4 gives $\beta$ as function of $\phi_0$. Here the analytical small $\phi_0$ approximation gives [49]

$$\beta \simeq 1.65 \cdot 10^{-6} \phi_0^4, \quad (3.2)$$

| $\phi_0$ | $d/10^{-14}$ | $\beta/10^{-3}$ | $\phi_{\text{CMB}}$ | $n_s$ | $r/10^{-3}$ | $\alpha/10^{-3}$ | $N_{\text{CMB}}$ |
|---------|--------------|-----------------|---------------------|-------|-------------|-----------------|---------------|
| 1       | 0.00636      | 0.00009         | 0.999205            | 0.9619| 6.81 \cdot 10^{-6} | -1.40           | 64.6          |
| 1       | 0.164        | 0.0017          | 0.999285            | 0.9659| 1.76 \cdot 10^{-5}  | -2.24           | 54.6          |
| 1       | 0.268        | 0.0023          | 0.999368            | 0.9699| 2.87 \cdot 10^{-5}  | -2.87           | 50.2          |
| 2       | 0.27         | 0.015           | 1.99356             | 0.9619| 4.64 \cdot 10^{-4}  | -1.43           | 64.3          |
| 2       | 0.65         | 0.027           | 1.99418             | 0.9659| 1.11 \cdot 10^{-3}  | -2.22           | 55.2          |
| 2       | 1.03         | 0.036           | 1.99482             | 0.9699| 1.77 \cdot 10^{-3}  | -2.80           | 51.1          |
| 3       | 0.66         | 0.08            | 2.97783             | 0.9619| 5.74 \cdot 10^{-3}  | -1.47           | 63.8          |
| 3       | 1.52         | 0.14            | 2.97976             | 0.9659| 1.32 \cdot 10^{-2}  | -2.24           | 55.2          |
| 3       | 2.49         | 0.19            | 2.98175             | 0.9699| 2.17 \cdot 10^{-2}  | -2.87           | 50.7          |
| 4       | 1.16         | 0.25            | 3.9461              | 0.9619| 3.18 \cdot 10^{-2}  | -1.43           | 64.4          |
| 4       | 2.77         | 0.45            | 3.9499              | 0.9659| 7.62 \cdot 10^{-2}  | -2.22           | 55.2          |
| 4       | 4.39         | 0.60            | 3.9542              | 0.9699| 1.21 \cdot 10^{-1}  | -2.81           | 51.0          |
| 5       | 2.23         | 0.7             | 4.8899              | 0.9619| 0.15            | -1.55           | 61.9          |
| 5       | 4.29         | 1.1             | 4.8968              | 0.9659| 0.29            | -2.16           | 55.2          |
| 5       | 7.02         | 1.5             | 4.9037              | 0.9699| 0.48            | -2.77           | 50.6          |
| 8       | 4.54         | 4.0             | 7.4815              | 0.9619| 2.04            | -1.23           | 62.3          |
| 8       | 8.76         | 6.7             | 7.4879              | 0.9659| 4.01            | -1.72           | 55.0          |
| 8       | 13.05        | 9.0             | 7.5048              | 0.9699| 6.07            | -2.12           | 51.2          |
| 11      | 5.19         | 10              | 9.4906              | 0.9619| 8.5             | -0.84           | 62.1          |
| 11      | 9.43         | 19              | 9.4772              | 0.9659| 16.1            | -1.18           | 54.7          |
| 11      | 11.99        | 25              | 9.5482              | 0.9699| 21.1            | -1.35           | 52.7          |
| 14      | 6.93         | 26              | 10.6090             | 0.9619| 29.8            | -0.82           | 53.7          |
| 14      | 7.05         | 33              | 10.8235             | 0.9659| 31.5            | -0.83           | 55.3          |
| 14      | 7.36         | 41              | 11.0439             | 0.9699| 34.2            | -0.86           | 56.6          |
| 17      | 3.28         | 8               | 11.6109             | 0.9619| 27.1            | -0.49           | 61.1          |
| 17      | 3.43         | 22              | 11.8913             | 0.9659| 30.0            | -0.51           | 62.6          |
| 17      | 3.73         | 39              | 12.1859             | 0.9699| 34.9            | -0.55           | 63.6          |
| 20      | 2.25         | 3               | 12.4500             | 0.9651| 32.9            | -0.42           | 64.3          |
| 20      | 2.28         | 8               | 12.5183             | 0.9659| 33.9            | -0.42           | 64.6          |
| 20      | 2.30         | 13              | 12.6000             | 0.9668| 34.7            | -0.43           | 65.0          |

Table 1: Examples of model parameters and corresponding predictions. The overall coupling strength $d$ has been chosen to reproduce the central value of power spectrum, i.e. $P_\zeta \simeq 2.1 \cdot 10^{-9}$. The predictions for $n_s$ and $\alpha$ are consistent with Planck 2018 results (2.13) at the 1$\sigma$ level. Predictions for the tensor–to–scalar ratio $r$ satisfy the current bound $r < 0.035$ (from the recent BICEP/Keck 2018 [8]) and range from $O(10^{-8})$ to $O(10^{-2})$. 


for our choices $N_{\text{CMB}} = 55$ and $n_s = 0.9659$; again, this reproduces the numerical results for $\phi_0 \lesssim 5$. On the other hand, for $\phi_0 \geq 15$, a relatively wide range of values of $\beta$ is allowed, since for $\beta \ll 1$ the potential at $\phi \lesssim \phi_{\text{CMB}}$ only weakly depends on $\beta$; recall from the top–left frame that $\phi_{\text{CMB}} \leq 0.7\phi_0$ lies well below the near–inflection point for these large values of $\phi_0$.

The lower left panel of Fig. 4 shows $d$ as function of $\phi_0$. In this case the small $\phi_0$ approximation yields \[^49\]

$$d \simeq 1.55 \cdot 10^{-15}\phi_0^2,$$

(3.3)

for $P_\zeta = 2.1 \cdot 10^{-9}$ and our default values of $n_s$ and $N_{\text{CMB}}$.\[^{10}\] On the other hand, we already saw in the discussion of Table 1 that for $\phi_0 > 10$, a smaller coupling $d$ is required in order to obtain the correct power spectrum.

The upper right panel of Fig. 4 depicts $r$ as function of $\phi_0$. For $\phi_0 \lesssim 5$ this is again quite well described by the analytical approximation \[^{49}\]

$$r \simeq 1.66 \cdot 10^{-8}\phi_0^6.$$ 

(3.4)

For $\phi_0 > 5$ $r$ increases slightly less quickly with increasing $\phi_0$; nevertheless for $\phi_0 \gtrsim 12$ solutions can be found that saturate the current upper bound on $r$. In fact, for $\phi_0 > 22$ all solutions that give $N_{\text{CMB}} \leq 65$ predict too large a value of $r$.

\[^{10}\]The larger scatter of the blue points around the red curve in this frame, compared to the top left and middle left frames, is largely a plotting artifact. The $y$–axis in the latter spans 6 orders of magnitude compared to “only” 4 orders of magnitude in the lower left frame, making the blue “bars” appear correspondingly shorter. Moreover, for $\phi_0 \lesssim 5$ the dynamics is more usefully described by the scaled difference $\delta$ introduced in eq.(2.15); the scatter in $\delta_{\text{CMB}}$ for $\phi_0 \lesssim 5$ is similar to that in $\beta$ and $d$. 

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**Figure 3:** The numerical scan over the free parameters $\phi_0$, $\phi_{\text{CMB}}$ and $\beta$, requiring $50 \leq N_{\text{CMB}} \leq 65$, fills the entire allowed region of the $(n_s, r)$ plane.
Figure 4: Blue dots represent allowed sets of model parameters (left frames) and the corresponding predictions (right frames) using the full expressions. The red lines depict the analytical approximation valid for $\phi_0 - \phi_{\text{CMB}} \ll \phi_0$, for fixed $n_s = 0.9659$ and $N_{\text{CMB}} = 55$, as described in the text.

The middle right panel of Fig. 4 gives $n_s$ as function of $\phi_0$. For $\phi_0 \leq 13$ essentially the entire currently allowed range can be covered by our model. For larger values of $\phi_0$ the parameter space begins to get squeezed by the conflicting constraints $N_{\text{CMB}} \leq 65$ and $r \leq 0.035$. Probably coincidentally the prediction of $n_s$ for the largest allowed $\phi_0$ is quite close to the present central value.

Finally, the lower right panel of Fig. 4 gives results for $N_{\text{CMB}}$. Again for $\phi_0 \leq 13$ all values between our chosen limits of 50 and 65 can be reproduced. For larger $\phi_0$ the upper bound on $r$ leads to a lower bound on $N_{\text{CMB}}$, since a flatter potential reduces the former but increases the latter. As we saw above, the two constraints become incompatible for $\phi_0 > 22$.

Let us end this section by summarizing the parameter space for the polynomial inflation model (2.7):

$$6 \cdot 10^{-16} \lesssim d \lesssim 2 \cdot 10^{-13}; \quad 0 < \beta \lesssim 4 \cdot 10^{-2}; \quad \phi_0 \lesssim 21.5.$$  \hspace{1cm} (3.5)

The upper bound on $\phi_0$ immediately yields a lower bound on the cubic potential parameter, $A \gtrsim -57$. Recall that these limits have been derived from the observed value of $n_s$, the constraint on $N_{\text{CMB}}$, and the upper bound on $r$. It might be worth mentioning that in our
model the constraint on $n_s$ suffices to derive a theoretical upper bound on the tensor–to–
scalar ratio, $r \lesssim 0.06$, if $N_{\text{CMB}} \gtrsim 60$; however, for $N_{\text{CMB}} \lesssim 55$ significantly larger values of $r$ could have been reproduced. The recent bound on $r$ therefore significantly reduces the
allowed parameter space of our model. The above discussion also shows that a tighter upper
bound on $r$ would further limit the parameter space; in particular, the upper bound on $\phi_0$
would become smaller if next–generation experiments fail to detect CMB tensor modes.

4 Radiative Stability and Reheating

So far our analysis has been based on the tree–level potential (2.7). This is only justified
if loop corrections to the potential are small. In order to check this one can compute the
1–loop Coleman–Weinberg (CW) corrections to the potential [53] and require that they are
subdominant compared to the tree–level potential. Here we follow the procedure outlined in
Ref. [49]. In particular, we focus on the potential at $\phi_0$. Here the first and second derivatives
of the tree–level potential are suppressed by the small parameter $\beta$, which increases the
relative importance of loop corrections.

The bound on the inflaton self–coupling can be computed by requiring the second deriva-
tive of the 1–loop potential at $\phi = \phi_0$ to be comparable to the tree–level value. This gives

$$\left| \frac{d^2 \ln(16\beta)}{\pi^2} \right| < 8d\beta.$$  (4.1)

Since for $\phi_0 \geq 1$ we typically have $\beta \gtrsim 10^{-6}$ while $d \lesssim 10^{-13}$, this inequality is nearly always satisfied. As shown in Fig. 4, for $\phi_0 \gtrsim 14.5$, $\beta$ can in principle become arbitrarily small. However, the inflationary parameters basically do not change when $\beta$ is increased from, say, $10^{-15}$ to a loop corrected value of $10^{-13}$; our predictions therefore remain stable in this part of parameter space even if the inequality (4.1) is violated. This therefore does not lead to
an additional theoretical constraint on the parameter space, in contrast to the small–field
version of this model [49].

4.1 Reheating

In a complete model of very early universe cosmology the inflaton field has to couple to
external particles for reheating [50, 54]. Here we assume $\phi$ couples to daughter particles with
trilinear couplings in order to fully drain the inflaton energy so that a radiation dominated
epoch is reproduced after reheating. For a scalar field $\phi'$, e.g. the standard model Higgs
field, we introduce a term $g\phi|\phi'|^2$ in the Lagrangian; for a fermionic field $\chi$, e.g. right–
handed neutrino, the corresponding term is $y\phi\bar{\chi}\chi$. The radiative stability conditions [49]

\textit{The non–perturbative preheating effect in our model is negligible. In the bosonic case, i.e. preheating with Higgs production, the trilinear coupling induce tachyonic instabilities [55], which tends to make preheating efficient. However the Higgs self–coupling gives rise to a positive effective mass $\propto \lambda\phi^2$ (where $\lambda \sim \mathcal{O}(0.1)$ is the Higgs self-coupling and $\langle \phi^2 \rangle$ denotes the variance of the produced Higgs field), which quickly dominates over the (possibly negative) contribution $\propto g\phi$ (where the small trilinear coupling $g$ is bounded by Eq. (4.5)), thereby blocking further non–perturbative $\phi \rightarrow \phi'$ energy transfer [49, 56]. In the fermionic case, Pauli blocking implies that only a small fraction of the energy stored in the inflaton field can be non–perturbatively transferred to $\chi$ particles, unless the daughter particles decay very fast [57].}
then read:

\[ \left| \frac{y^4 - 3y^4 \ln(y^2)}{4\pi^2} \right| < 16d\beta ; \]  
\[ \frac{1}{8\pi^2} \left( \frac{g}{\phi_0} \right)^2 \left| \ln \left( \frac{g}{\phi_0} \right) - 1 \right| < 8d\beta . \]

For typical value of \( d \sim 10^{-14} \) and \( \beta \lesssim \beta_{\text{max}} \simeq 4 \cdot 10^{-2} \) (cf. Eq. (3.5)), one has

\[ y \lesssim y_{\text{max}} \simeq 2.7 \cdot 10^{-4} ; \]  
\[ \left( \frac{g}{\phi_0} \right) \lesssim \left( \frac{g}{\phi_0} \right)_{\text{max}} \simeq 1.2 \cdot 10^{-7} . \]

These upper bounds on the inflaton couplings immediately lead to upper bounds on the corresponding partial widths for \( \phi \to \phi' \phi' \) and \( \phi \to \bar{\chi}\chi \) decays, which in turn imply upper bounds on the post–inflationary reheating temperature. In the instantaneous decay approximation the latter is given by [54]

\[ T_{\text{rh}} \simeq 1.41g_{*}^{-1/4}T_{\phi}^{1/2} . \]  

For the fermionic reheating channel, this becomes

\[ T_{\text{rh}}^{\chi} \simeq 1.41g_{*}^{-1/4} \left( \frac{2\phi_0 g^2}{8\pi \sqrt{d}} \right)^{1/2} \lesssim 1.1 \cdot 10^{11} \text{ GeV} , \]  
where we have considered \( \phi_0 \lesssim 20, g_{*} = 106.75 \) and used Eq. (4.4) for the upper bound on \( y \).

For bosonic reheating, the analogous calculation allows even higher reheat temperatures,

\[ T_{\text{rh}}^{\phi'} \simeq 1.41g_{*}^{-1/4} \left( \frac{g^2}{8\pi 2\phi_0 \sqrt{d}} \right)^{1/2} \lesssim 2.5 \cdot 10^{14} \text{ GeV} , \]  
where the maximum value \( g_{\text{max}} \) reported in Eq. (4.5) has been utilized.

Both bounds are saturated at the largest allowed value of \( \phi_0 \), where the physical mass of the inflaton \( m_{\phi} = 2\sqrt{d}\phi_0 \simeq 10^{13} \text{ GeV} \). This is comfortably above the bound (4.7), which can thus be saturated using simple perturbative \( \phi \to \chi\bar{\chi} \) decays. On the other hand, \( m_{\phi} \) is an order of magnitude below the bound (4.8). This bound can therefore only be saturated if one can turn an ensemble of \( \phi' \) particles with energy \( \simeq m_{\phi}/2 \sim 5 \cdot 10^{12} \text{ GeV} \) into a thermal bath with much higher temperature, which can only happen via scattering reactions that reduce the number of particles, in particular \( 3 \to 2 \) scattering reactions. It is not clear whether the rate of such reactions is sufficiently high that the bound (4.8) can be saturated; in the absence of such reactions it would have to be replaced by \( T_{\text{rh}}^{\phi'} \lesssim m_{\phi}/2 \). The same remark holds for the maximal temperature of the radiation bath, which is typically attained well before reheating is completed and can be significantly larger than \( T_{\text{rh}} \) [54]: in the absence of fast \( 3 \to 2 \) reactions it is also bounded by \( m_{\phi}/2 \).

### 5 Eternal Polynomial Inflation

In the previous sections, we have worked out the parameter space consistent with Planck 2018 (2.13) and BICEP/Keck 2018 (2.14) and investigated the radiative stability of the inflaton potential as well as perturbative reheating for the polynomial inflation model. All this happened at field values (well) below \( \phi_0 \). In this section we analyze the situation at \( \phi \gtrsim \phi_0 \). In particular, we are interested in the energy scale(s) at which “eternal” inflation could have occurred within the allowed parameter space.
5.1 Eternal Phase I

During the SR phase, one can neglect the acceleration term in eq. (2.2), so that the classical inflaton field evolves as

$$\dot{\phi} \approx -\frac{V'(\phi)}{3H}.$$  \hfill (5.1)

This predicts a classical field excursion per Hubble time

$$\Delta \phi_{cl} = \frac{|\dot{\phi}|}{H} \approx \frac{|V'(\phi)|}{3H^2} \approx \frac{|V'(\phi)|}{V} = \sqrt{2\epsilon_V}.$$  \hfill (5.2)

On the other hand, since the inflaton field is very weakly coupled its quantum fluctuations follow a Gaussian probability distribution. In the quasi de Sitter background during SR inflation, the typical size of these quantum fluctuation over one Hubble time is given by $\delta \phi_{qu} = \frac{H}{2\pi}$. [58, 59].

Assume that the quantum fluctuation dominates over the classical field excursion, i.e. $\delta \phi_{qu} > \Delta \phi_{cl}$. In this case the inflaton field in a given Hubble volume is almost as likely to move “uphill”, towards larger values, as it is to move towards the minimum of the potential. Since after a small number of e–folds of inflation the inflaton field fills a great many Hubble volumes which henceforth evolve independently, it is virtually guaranteed that in some of these volumes the field does indeed move uphill. Since these regions subsequently will expand faster (owing to the larger Hubble parameter), in this picture “most” of space will continue inflating forever, even though inflation clearly must have ended in our own Hubble patch. This is known as eternal inflation.

From eq.(5.2) and $\delta \phi_{qu} = \frac{H}{2\pi}$, the condition for eternal inflation is

$$\frac{H}{2\pi} > \sqrt{2\epsilon_V} \iff \frac{H^2}{8\pi^2\epsilon_V} > 1.$$  \hfill (5.3)

Hence the condition for eternal inflation is satisfied if the leading order prediction of the amplitude of curvature perturbations $P_\zeta$ exceeds unity [38].

As already emphasized in sec. 2.2, for $\phi \gg \phi_0$ effectively our model behaves like quartic inflation, i.e. the inflaton potential can be simplified to $V = d\phi^4$, so that

$$H = \sqrt{\frac{d}{3}\phi^2},$$  \hfill (5.4)

and

$$\frac{H^2}{8\pi^2\epsilon_V} = \frac{d\phi^6}{192\pi^2}.$$  \hfill (5.5)

From condition (5.3) eternal inflation then requires

$$\phi^2 > \left(\frac{192\pi^2}{d}\right)^{1/3}.$$  \hfill (5.6)

This in turn leads to a lower bound on the Hubble parameter:

$$H > H_{Ei}^\text{El} = 4\pi^{2/3} \left(\frac{d}{3}\right)^{1/6}.$$  \hfill (5.7)
which only depends on \(d\). Once \(H > H_{\text{EI}}\), an eternal inflationary phase can occur.

The usual monomial chaotic \(\lambda \phi^4\) inflation model requires \(\lambda \sim 10^{-12}\) \[^{[33]}\] in order to match the normalization of the power spectrum; eq. (5.7) then yields \(H_{\text{EI}}^c \approx 0.07\). In the last section we saw that in our polynomial scenario the quartic coupling \(d\) needs to be somewhat smaller, \(6 \cdot 10^{-16} \lesssim d \lesssim 2 \cdot 10^{-13}\). The corresponding threshold value of the inflaton field is (in Planckian units):

\[
460 \lesssim \phi_c \lesssim 1211, \tag{5.8}
\]

which evidently is indeed well above \(\phi_0\). One can further work out the threshold of the corresponding inflationary scale (again in Planckian units):

\[
0.02 \lesssim H_{\text{EI}}^c \lesssim 0.05. \tag{5.9}
\]

Evidently the corresponding energy scale is well below the Planck scale (and also somewhat below the scale required for eternal inflation in pure quartic inflation); hence our semi–classical treatment, which ignores “quantum gravity” effects, may be valid.

We thus conclude that our model does allow for an epoch of “eternal” while reproducing all present measurements of inflationary parameters.

### 5.2 Eternal Phase II

The discussion in the previous section shows that eternal inflation should occur in the polynomial model if the inflaton field and the Hubble parameter ever exceeded the critical values (5.8) and (5.9), respectively. Here we show that, at least in part of the allowed parameter space, a second, later epoch of eternal inflation will occur.

We have seen in Fig. 4 that the parameter \(\beta\) can be arbitrarily small if \(\phi_0 \gtrsim 15\). Recall that a very small \(\beta\) implies that the potential at \(\phi_0\) is very flat. Hence it is expected that a (second) eternal phase can occur. Since this eternal phase appears when the inflaton is near the saddle point \(\phi_0\), which is much smaller than \(\phi_c\) analyzed above, one can expect that the corresponding Hubble scale should be much lower than \(H_{\text{EI}}^c\) of eq.(5.9).\(^{12}\)

In order to obtain the maximum value of \(\beta\) that allows a second phase of eternal inflation, we again use condition Eq. (5.3), identical to the leading order prediction of the power spectrum being larger than unity as mentioned earlier. Since the potential is flattest at \(\phi = \phi_0\), we can use eq.(2.18) with \(\delta = 0\) to derive the condition for the existence of a second epoch of eternal inflation:

\[
\beta < \frac{\sqrt{d\phi_0^3}}{144\pi} \approx 10^{-6}; \tag{5.10}
\]

For the numerical value we have used \(d \sim 10^{-14}\) and \(\phi_0 \sim 20\).

The potential at \(\phi_0\) is given by \(d\phi_0^4/3\), up to corrections of relative order \(\beta\) which are evidently completely negligible here. This epoch of eternal inflation would thus have a Hubble parameter

\[
H_{\text{EI}} = \frac{\sqrt{d}}{3} \phi_0^2 \sim 10^{-5}; \tag{5.11}
\]

\(^{12}\)This has some similarity to eternal hilltop inflation investigated in \[^{[38]}\]. However, in our case the inflaton first rolls down to a plateau around the saddle point; in hilltop inflation, one has to impose as initial condition that the inflaton starts near a (local) maximum of the potential, which can also be very flat.
this is at least three orders of magnitude smaller than the one given in eq.(5.9).

The width of this second region in field space that allows eternal inflation is also important. Again from eq.(2.18) we see that the evolution of the inflaton field will be dominated by quantum fluctuations as long as

\[ \delta < \delta_c = \frac{d^{1/4}\phi_0^{3/2}}{6\sqrt{2}\pi}. \]  

(5.12)

In units of the size \( H_{\text{EI}}/(2\pi) \) of random walk steps the half width of the region in field space allowing eternal inflation is thus

\[ \frac{\delta_c\phi_0}{H_{\text{EI}}/(2\pi)} = \sqrt{\frac{\pi}{2}} \frac{\phi_0^{1/2}}{d^{1/4}} \simeq 1.8 \cdot 10^4. \]  

(5.13)

Given that the average excursion in a random walk is proportional to the square of the number of steps taken times the (typical) step size, starting from \( \phi = \phi_0 \) the field would thus typically need more than \( 10^8 \) steps to leave the region where quantum fluctuations dominate the dynamics. Moreover, half the time the random walk would end at \( \phi > \phi_0 (1 + \delta_c) \), in which case the classical field evolution would bring the field back into the range where quantum effects dominate. These arguments indicate that in our model indeed “most of” space would inflate eternally if \( \phi \) ever reached the region very close to \( \phi_0 \) and \( \beta < 10^{-6} \).

A typical inflationary trajectory starting at very large field values \( \phi > \phi_c \) of eq.(5.8) is shown in Fig. 5, which plots the Hubble parameter as function of the classical prediction \( N_{\text{cla}} \) of the number of e–folds that occur after the inflaton field had a certain value, given by eq.(2.9); larger \( N_{\text{cla}} \) correspond to larger \( \phi \). Of course, for the field ranges allowing eternal inflation, i.e. for \( \phi > \phi_c \) and for \( \phi \in [\phi_0(1 - \delta_c), \phi_0(1 + \delta_c)] \), the actual number of e–folds by which our Hubble patch expanded was likely very much larger than the classical prediction.

For field values below \( \phi_c \) the Hubble parameter seems to drop very steeply. However, if \( \phi_0 > 4 \) the SR conditions are satisfied for all \( \phi > \phi_0 \); essentially deterministic inflation therefore lasts from the end of the first, high–energy stage of eternal inflation to the onset of the second epoch of eternal inflation where \( \phi \simeq \phi_0 \). This is in contrast to the small–field version of this model, where the SR conditions are violated for some range of field values above \( \phi_0 \), and inflation around \( \phi_0 \) is always deterministic [49]. Finally, once \( \phi < \phi_0 (1 - \delta_c) \) inflation is deterministic again, including the last \( \sim 65 \) e–folds of inflation with \( \phi \leq \phi_{\text{CMB}} \).

Recalling the results of Fig. 4, we conclude that model parameters (a subset of (3.5)) with:

\[ 15 \lesssim \phi_0 \lesssim 21.5; \quad 0 < \beta \lesssim \mathcal{O}(10^{-6}); \quad 2 \cdot 10^{-14} \lesssim d \lesssim 6 \cdot 10^{-14}, \]  

(5.14)

satisfy all observational constraints and allow a second eternal inflationary phase. Note that the second eternal phase can only occur if \( \phi_0 \) is rather large, which implies \( r \sim \mathcal{O}(10^{-2}) \). The part of parameter space allowing the (unusual) low scale second eternal phase can thus be tested by the next generation CMB experiments, e.g. CORE [22], AliCPT [23], LiteBIRD [24] and CMB-S4 [25], with expected sensitivity down to \( r \sim \mathcal{O}(10^{-3}) \).

6 Summary and Conclusions

In this paper we revisited large field inflation with a single inflaton. We investigated a model where the inflaton potential is a polynomial of degree four. Current observations then
Figure 5: Evolution of the Hubble parameter as function of the classical prediction for the number of e–folds \( N_{\text{cla}} \) (depending on \( \phi \) via Eq. (2.9)), for \( \phi_0 = 20 \), \( \beta = 10^{-7} \) and \( d \simeq 2 \cdot 10^{-14} \). Inflation can be eternal in the gray shaded region where \( H > H_{\text{EI}}^c \simeq 0.04 \, M_{\text{Pl}} \). For \( H < H_{\text{EI}}^c \) a period of the usual SR inflation follows. For the given choice of a very small \( \beta \) a second epoch of eternal inflation occurs (between the dashed vertical lines), with Hubble parameter as low as \( O(10^{-5}) \, M_{\text{Pl}} \).

require that the potential features a near saddle point at \( \phi_0 \), making the potential concave at \( \phi \lesssim \phi_{\text{CMB}} < \phi_0 \) as required by the Planck 2018 data.

The model was described in Sec. 2. The potential contains three free parameters: an overall (quartic) coupling strength \( d \); the location \( \phi_0 \) of the almost saddle point; and \( \beta \ll 1 \) to determine the deviation from a true saddle point, with smaller \( \beta \) making the potential flatter for \( \phi \sim \phi_0 \). The parameters \( \phi_0 \) and \( \beta \) thus determine the shape of the potential, while the overall normalization, given by \( d \), can be fixed from the normalization of the power spectrum of curvature perturbation \( P_\zeta \). The value \( \phi_{\text{CMB}} \) of the inflaton field when the cosmic microwave background (CMB) scales first left the horizon is another important free parameter. It allows us to choose \( \phi_0 \) within a rather wide range, and then use the measured spectral index \( n_s \) as well as the number \( N_{\text{CMB}} \) of e–folds of inflation generated after CMB scales first left the horizon to determine, or constrain, \( \phi_{\text{CMB}} \) and \( \beta \). We also discuss various analytical approximations. In particular, for \( \phi_0 \lesssim 5 \) (in Planckian units, which we use throughout) \( \phi_0 - \phi_{\text{CMB}} \ll \phi_0 \) and the fully analytical treatment developed for the small field version of this model [49] still holds to good approximation.

In Sec. 3 a full scan of the parameter space consistent with most resent Planck and BICEP/Keck 2018 observations at the 2\( \sigma \) level (cf. (2.13) and (2.14)) is described. The final result is summarized in (3.5). The predictions for \( r \) range from unobservably small, \( O(10^{-8}) \), to the current upper bound. In fact, the current bound \( r \leq 0.035 \) together with the constraint
$N_{\text{CMB}} \leq 65$ leads to the upper bound $\phi_0 \lesssim 21.5$. Moreover, we predict negative running of the spectral index, $\alpha \sim -\mathcal{O}(10^{-3})$, which might be testable in the near future [60]. A large set of examples are listed in Table 1. To our knowledge, this is the first such comprehensive scan of parameter space of polynomial inflation taking into account the most recent CMB data. Of course, the currently allowed region described by eqs.(3.5), in particular the upper bound on $\phi_0$, should be further constrained once more precise CMB experiments are performed, such as CORE [22], AliCPT [23], LiteBIRD [24] and CMB-S4 [25], which could probe all $r \gtrsim \mathcal{O}(10^{-3})$.

In Sec. 4 we showed that radiative stability of the inflaton potential near the inflection-point $\phi_0$ leads to relatively mild constraints for our large field model, in sharp contrast to its small field version [49]. In particular, the one–loop Coleman–Weinberg (CW) corrections to the potential due to the self–interactions of the inflaton are always harmless. Moreover, large reheat temperatures, up to $10^{11}$ (2·$10^{14}$) GeV are in principle possible for perturbative inflaton decay into fermionic (bosonic) final states. However, we remind the reader that temperatures above half the inflaton mass $m_\phi$ can only be reached if the rate for reactions that reduce the number of particles (e.g. $3 \rightarrow 2$) is sufficiently high; note that $m_\phi \lesssim 10^{13}$ GeV in this model.

In Sec. 5 the possibility of eternal inflation is discussed. This is generally expected to occur in models of polynomial inflation, assuming the field value or, equivalently, the inflationary Hubble parameter reached sufficiently high values. In our case the critical Hubble parameter only depends on the size of the quartic coupling $d$, and lies in the range $0.02 \lesssim H_{\text{EI}} \lesssim 0.05$; this is somewhat below the value of 0.07 needed in the monomial $\lambda \phi^4$ model (which is in any case excluded by the upper bound on $r$).

More interestingly, we find that there exists another possibility to realize eternal inflation in our scenario. This occurs when the potential is very flat around $\phi_0$: for $\beta \lesssim 10^{-6}$ quantum fluctuations can dominate over the classical evolution already at $\phi \simeq \phi_0$, compared to $\phi \gtrsim \mathcal{O}(10^3)$ in the first epoch of eternal inflation. This second epoch of eternal inflation features a much smaller Hubble parameter, $H \sim \mathcal{O}(10^{-5})$, which is of the same order of magnitude as that when the CMB pivot scale $k_\star = 0.05$ Mpc$^{-1}$ first crossed out of the horizon. Nevertheless there will be many $e$–folds of deterministic inflation between the end of the second epoch of eternal inflation and the era when the CMB scales first crossed out of the horizon. We are therefore not aware of any immediate observational consequences of this “late” epoch of eternal inflation. However, since $\beta \lesssim 10^{-6}$ is possible only for $\phi_0 \gtrsim 15$, which in turn implies $r \gtrsim 0.01$, at least in our model this possibility can be tested by the next round of CMB experiments.

This novel scenario featuring two epochs of eternal inflation, as depicted in fig. 5, might also be conceptually interesting. Eternal inflation is the only known mechanism that might be able to populate the “landscape” of superstring theory [40–42]. During eternal inflation not only the inflaton field undergoes a random walk in field space, but so does every field whose mass is below the inflationary Hubble parameter. This might include many of the scalar fields (from the four–dimensional perspective) that determine the sizes of physical couplings in string theory. During the first, high scale epoch of eternal inflation these fields will be sampled with a typical step size $H/(2\pi) \gtrsim 5 \cdot 10^{-3}$. Since the second epoch of eternal inflation has a thousand times smaller Hubble parameter, the step size of the random walk in field space is also thousand times smaller during this second epoch. This might allow to much more efficiently “home in” on relatively small features of the landscape.

In summary, we have presented a successful large field polynomial model, worked out
the complete allowed parameter space (3.5) and offered the corresponding inflationary predictions. Combined with the earlier analysis of the small field version of this model [49] this offers the most complete analysis of the polynomial inflation model after Planck and BICEP/Keck 2018. We also pointed out for the first time that in our model the early history of the universe might feature two epochs of eternal inflation, at quite different energy scales.
A General Expression for the Number of e-Folds

\[ N_{\text{CMB}} = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{\sqrt{2\pi}} d\phi \]

\[ = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{2} \left[ \frac{32\phi^2 - 32A(\beta - 1)\phi + 9A^2}{64\phi^2 - 48A(\beta - 1)\phi + 9A^2} \right] d\phi \]

\[ = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{2} \left[ 1 + \frac{-32\phi^2 + 16A(\beta - 1)\phi}{64\phi^2 - 48A(\beta - 1)\phi + 9A^2} \right] d\phi \]

\[ = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{2} \left[ 1 + \frac{-32\phi^2 + 16A(\beta - 1)\phi}{64[(\beta - 1)^2 - 9A^2(\beta^2 - 2\beta)]} \right] d\phi \]

\[ = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{2} \left[ 1 - \frac{1}{2} \frac{\phi^2}{[\phi + \phi_0(\beta - 1)]^2 - \phi_0^2(\beta^2 - 2\beta)} - \frac{2}{3} \frac{\phi_0(\beta - 1)\phi}{[\phi + \phi_0(\beta - 1)]^2 - \phi_0^2(\beta^2 - 2\beta)} \right] d\phi . \] (A.1)

The three integrals can be evaluated analytically:

\[ \int d\phi \frac{\phi^2}{2} = \frac{\phi^2}{4} ; \] (A.2)

\[ \int -\frac{1}{4} \frac{\phi^3}{[\phi - m]^2 + n^2} d\phi = \frac{1}{8} \left\{ \frac{5m^2 - 4m\phi - \phi^2 + \frac{2(m^3 - 3mn^2)}{n}}{n} \text{arctan} \left[ \frac{m - \phi}{n} \right] \right. \]

\[ \left. - (3m^2 - n^2) \log \left[ n^2 + (m - \phi)^2 \right] \right\} ; \] (A.3)

and

\[ \int \frac{d\phi m\phi^2}{3[\phi - m]^2 + n^2} = \frac{m}{3} \left\{ \frac{\phi - (m^2 - n^2)}{n} \text{arctan} \left[ \frac{m - \phi}{n} \right] + m \log \left[ n^2 + (m - \phi)^2 \right] \right\} , \] (A.4)

with \( m = \phi_0(1 - \beta) \) and \( n^2 = \phi_0^2(2\beta - \beta^2) \). Combining these results, we obtain

\[ N_{\text{CMB}} = \left\{ -\frac{m^2 + 5mn^2}{12n} \text{arctan} \left[ \frac{m - \phi}{n} \right] + \frac{5m^2}{8} - \frac{m\phi}{6} + \frac{\phi^2}{8} \right. \]

\[ - \frac{(m^2 - 3mn^2)}{24} \ln \left[ n^2 + (m - \phi)^2 \right] \left|_{\phi_{\text{CMB}}}^{\phi_{\text{end}}} \right. \] (A.5)

\[ \approx \frac{1}{24} \left\{ 3\phi^2 - 4\phi\phi_0 + 15\phi_0^2 - \phi_0^2 \sqrt{\frac{2}{\beta}} \text{arctan} \left( \frac{\phi_0 - \phi}{\sqrt{2/\beta}\phi_0} \right) - \phi_0^2 \ln \left[ (\phi_0 - \phi)^2 \right] \right|_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} . \]

In the last step we have assumed \( \beta \ll 1 \), which is true in the allowed parameter space, but cannot be assumed a priori.
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