Canonical Structure of Locally Homogeneous Systems on Compact Closed 3-Manifolds of Types $E^3$, Nil and Sol

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Abstract

In this paper we investigate the canonical structure of diffeomorphism invariant phase spaces for spatially locally homogeneous spacetimes with 3-dimensional compact closed spaces. After giving a general algorithm to express the diffeomorphism-invariant phase space and the canonical structure of a locally homogeneous system in terms of those of a homogeneous system on a covering space and a moduli space, we completely determine the canonical structures and the Hamiltonians of locally homogeneous pure gravity systems on orientable compact closed 3-spaces of the Thurston-type $E^3$, Nil and Sol for all possible space topologies and invariance groups. We point out that in many cases the canonical structure becomes degenerate in the moduli sectors, which implies that the locally homogeneous systems are not canonically closed in general in the full diffeomorphism-invariant phase space of generic spacetimes with compact closed spaces.
§1 Introduction

Locally homogeneous systems are, roughly speaking, extensions of spatially homogeneous systems on simply connected spaces to spaces with non-trivial topologies. Recent increase of interest in such systems are closely related with the investigations of quantum gravity\[1, 2, 3, 4, 5\], although early work on them was mainly concerned with applications to the observational cosmology(see Ref.\[6\] for review).

In quantum gravity spatially homogeneous systems were intensively studied in the name of minisuperspace models, because they give simple gravity systems of finite degrees of freedom. If one applies the quantization procedure to such systems, however, one encounters the problem that the Hamiltonian diverges due to the infinite spatial volume except for Bianchi IX models. Since the evolution equations for space metric do not depend on the spatial volume, this difficulty was often avoided by considering a virtual finite region of space. Though this prescription works well in the investigation of classical structure of canonical dynamics, it is not satisfactory in the quantum problem because the virtual volume of the region affects quantum behavior of the system though the Hamiltonian.

A more natural and reasonable way to resolve this difficulty is to consider spacetimes with compact spaces instead of simply connected open spaces. In this approach, however, two new problems arise. Firstly, most of compact closed 3-manifolds do not allow a Bianchi-like globally homogeneous metric, i.e., a metric whose isometry group acts simply transitively on the manifolds. This obliges us to replace the requirement of global homogeneity to a local one\[1, 2\]. Secondly, since compact Riemannian manifolds with non-trivial topologies are not globally isomorphic even if they are locally isomorphic, the structure of a locally homogeneous space is not fully described by the components of the metric with respect to an invariant basis(even if it is well-defined) as in the Bianchi models, but depends on the so-called moduli parameters. Thus the dynamical structure of a locally homogeneous system on a compact manifold differs from that of a globally homogeneous system on its simply connected covering manifold.

These problems also occur in (2 + 1)-dimensional systems and are studied well. There a natural definition of locally homogeneity is obtained from the requirement that the Ricci scalar curvature is spatially constant. The locally homogeneous closed two surfaces in this sense are always covered by a simply connected homogeneous space and is obtained as a quotient of the latter by its discrete isometry group. Though the Ricci homogeneity is too weak in the (3 + 1)-dimensional systems, the latter characterization, i.e., the requirement that its covering spacetime is globally spatially homogeneous, can be easily extended to higher dimensions. Furthermore if we adopt this as the definition of spatial local homogeneity of (3+1)-dimensional systems, we can almost completely determine all the possible topologies of compact closed spaces and their locally homogeneous structures including the moduli with the help of Thurston’s theorem.

Recently Koike, Tanimoto and Hosoya\[3\] completely determined the degree of moduli freedom of locally homogeneous 3-manifolds for all possible topologies by this approach. They further applied this result to the investigation of dynamics of spatially locally homogeneous spacetimes\[4, 5\], and gave a systematic algorithm to determine the dynamics of moduli. Their strategy was as follows. First, they defined a spatially locally homogeneous spacetime \((M, g)\) as a spacetime with compact space which is obtained as a quotient of
a spatially homogeneous spacetime \((\tilde{M}, \tilde{g})\) with a simply connected space by its discrete spacetime isometry group \(K\) contained in the spatial homogeneity subgroup. Here it is important to distinguish the isometry group \(G(t)\) of each constant time slice of \(\tilde{M}\) and its subgroup \(G_e\) whose transformations are extendible to spacetime isometries. Next they classified the covering spatially homogeneous vacuum solutions and put them into some standard form \((\tilde{M}, \tilde{g}_0)\) to fix the freedom of extensible homogeneity-preserving diffeomorphisms (EHPDs), which are defined as transformations of the covering spacetimes \((\tilde{M}, \tilde{g})\) preserving the spatial homogeneity. By this procedure one obtains spacetime diffeomorphism classes of \((M, g)\), each of which is described by the canonical form of the metric \(\tilde{g}_0\) or a evolutionary family of covering 3-metric \(\tilde{q}_0(t)\), and the conjugate class of the discrete subgroup \(K\) in \(G_e\). Since EHPDs are a subset of the transformations of each constant-time slice which preserve the spatial homogeneity of \(\tilde{q}_0(t)\), one can further reduce the freedom of \(\tilde{q}_0(t)\) on each slice by the latter transformations. This time-dependent reduction maps \(K\) to a time-dependent conjugate class \(K(t)\) with respect to \(G(t)\), which corresponds the standard moduli/Teichmüller freedom. Thus one can determine the time evolution of the 3-metric and moduli parameters of a locally homogeneous space.

This algorithm is very useful in determine the time evolution of moduli parameters in the sense defined above at least in the classical framework. However, it has some disadvantages in the investigation of canonical structures of the systems and their quantization. First, the knowledge on the structure of spacetime solutions is required in advance in their method. Though this is not an obstacle for the pure gravity systems, such information is not available for systems coupled with matter in general. Second, since they start from the spacetime solutions, the information on the momentum variable conjugate to the 3-metric or on the Hamiltonian is not directly given. It must be calculated by inserting spacetime solutions in the generic definitions of the momentums and the Hamiltonian. This procedure often fails because defining momentums requires the knowledge of the canonical structure, which is generally not available in the moduli sector in their approach. Since the information on the canonical structure is crucial in going to quantum theory, this defect is serious. In fact, we will show in this paper that the actual canonical structure is often degenerate in the moduli sector. Such information can not be obtained in their approach.

On the basis of these considerations, in the present paper, we give a slightly different scheme to determine the dynamics of locally homogeneous systems. The main point is that we directly treat the diffeomorphism-invariant phase space of locally homogeneous canonical data including momentums on a compact closed 3-manifold \(M\), and determine its canonical structure and the Hamiltonian directly from those of the diffeomorphism-invariant phase space of generic canonical data on \(M\). This enables us to discuss the off-shell behavior of the systems as well as relations of the canonical structures of locally homogeneous systems to generic systems, such as the degeneracy of the canonical structure in the locally homogeneous sector.

The paper is organized as follows. First, in the next section, we prove a theorem which enables us to express the diffeomorphism-invariant phase space of locally homogeneous data on a compact closed 3-manifold \(M\) in terms of globally homogeneous data on its universal covering manifold \(\tilde{M}\) and a moduli freedom of the embedding of the fundamental group \(\pi_1(M)\) into an invariance group of the covering data. There Thurston’s theorem on the classification and the uniqueness of maximal geometries on compact closed 3-manifolds
play an essential role. Then we explain a general algorithm to determine the canonical structure and the Hamiltonian of them. We further prove an important theorem on the dynamics of moduli parameters. In the subsequent three sections, following this algorithm, we completely determine the canonical structures and the Hamiltonians of locally homogeneous pure gravity systems on orientable compact closed 3-space of the Thurston-type $E^3$, Nil and Sol. As a result we point out that in many of them the canonical structure becomes degenerate in the moduli sector. Section 6 is devoted to summary and discussions.

§2 General Theory

2.1 Diffeomorphism-Invariant Phase Space

Let $M$ be a compact closed 3-manifold and $\Phi$ be a set of canonical variables on it. For example, for the pure gravity system, $\Phi$ consists of a pair $(q, p)$ of a three metric $q$ on $M$ and its conjugate momentum $p$. When it is coupled with matter, canonical pairs of matter fields should be included in $\Phi$.

Naively it is natural to define that canonical data $\Phi$ are locally homogeneous when around each point of $M$ there exist a set of local fields $\xi_I$ which contain at least three linearly independent fields and leave $\Phi$ invariant, i.e., $\mathcal{L}_{\xi_I}\Phi = 0$. However, this local definition is not convenient for our purpose because we must treat cases in which the base space $M$ has non-trivial topology.

If we instead require that the fields $\xi_I$ are globally defined on $M$, the analysis of the problems gets much simplified. However, this requirement is too stringent and excludes many interesting cases. For example, consider a compact Riemannian manifold $(M, q)$ constructed from the cubic region $0 \leq x, y, z \leq 1$ of Euclidean space $E^3$ by identifying the pair of faces $x = 0$ and $x = 1$ by rotating by $\pi$ and the other two pairs of faces normally. Clearly $(M, q)$ is locally isometric to $E^3$ and locally homogeneous, but translation Killing vectors parallel to the $y-z$ plane cannot be extended to the whole space $M$. Another example is given by the class B open Bianchi models which cannot be compactified keeping the global transitive symmetries\(^1\).

Since the main obstruction against the existence of global symmetries in these examples is of a topological nature, a better definition which remedies the defects of the definitions above is obtained by considering the universal covering data. Let $\tilde{M}$ be a universal covering space of $M$, and $j$ be a covering map from $\tilde{M}$ onto $M$. Then through the pullback by $j$ we obtain unique data $\tilde{\Phi}$ on $\tilde{M}$ from $\Phi$ on $M$. Clearly $\tilde{\Phi}$ is locally homogeneous in the first definition if $\Phi$ is. Further if $\tilde{\Phi}$ is real analytic and $\Phi$ contains the metric data, the local symmetries of $\tilde{\Phi}$ can always be extended to global symmetries\(^2\). Hence in this case local homogeneity of $\Phi$ in the naive definition implies global global homogeneity of the covering data $\tilde{\Phi}$. This global symmetry defines an invariance group

$$\text{InvG}(\tilde{\Phi}, \tilde{M}) := \{ f \in \text{Diff}(\tilde{M}) \mid f_\ast \tilde{\Phi} = \tilde{\Phi} \}.$$  \hspace{1cm} (2.1)

When there occurs no confusion, we often write $\text{InvG}(\tilde{\Phi}, \tilde{M})$ as $\text{InvG}(\tilde{\Phi})$.

\(^1\)There had been some misunderstanding on this point in the early literature\(^3\). It was corrected in Ref.\(^4\).
From these observations, in this paper, we define that canonical data $\Phi$ on $M$ are \textit{locally $G$-homogeneous} if there exists a universal covering space $j : \tilde{M} \to M$ such that the invariance group of the pullback $\tilde{\Phi} = j^*\Phi$, $\text{Inv}_G(\tilde{\Phi}, M)$, is isomorphic to $G$ and acts transitively on $\tilde{M}$. The reason why we have introduced the symmetry group $G$ in an abstract way is that $\text{Inv}_G(\tilde{\Phi}, \tilde{M})$ is not intrinsic to the original manifold $M$, and manifests itself only as local symmetries on $M$. We denote the set of all locally $G$-homogeneous data on $M$ by $\Gamma_{LH}(M, G)$.

This definition allows us to translate the problem on compact manifold with complicated topological structures to that on much simpler simply-connected manifolds, and is widely adopted in the recent literature on this problem\cite{6, 2, 3}. Now we closely examine the relation between these two problems.

\subsection{The freedom of the covering space $\tilde{M}$}

Our definition gives the description of locally homogeneous data on $M$ in terms of the set of its universal covering space $\tilde{M}$, a covering map $j$, and homogeneous covering data $\tilde{\Phi}$. There is clearly a redundancy in this description since all the universal covering spaces are mutually diffeomorphic. This redundancy is eliminated with the help of the following well-known fact.

\begin{proposition}
Let $j : \tilde{M} \to M$ and $j' : \tilde{M}' \to M'$ be a pair of two universal covering spaces.

1) For any diffeomorphism $f : M \to M'$, there exists a diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}'$ up to the freedom of covering transformations such that $f \circ j = j' \circ \tilde{f}$, i.e., the following diagram commutes:

\begin{equation}
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{j}} & \tilde{M}' \\
\downarrow j & & \downarrow j' \\
M & \xrightarrow{f} & M'
\end{array}
\end{equation}

2) For any diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}'$ which preserves fibers, i.e., maps each fiber $j^{-1}(x) \subset \tilde{M}$ onto some fiber $j'^{-1}(x') \subset \tilde{M}'$, there is a unique diffeomorphism $f : M \to M'$ such that $f \circ j = j' \circ \tilde{f}$.

The first part of this proposition follows from the uniqueness of the lift of curves in the covering space. For example, let $x_0$ and $y_0$ be a pair of points such that $y_0 = f(x_0)$ and pick up a point $\tilde{x}_0 \in j^{-1}(x_0)$ and another point $\tilde{y}_0 \in j'^{-1}(y_0)$. For a given point $\tilde{x} \in \tilde{M}$, take a curve $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}$. Then the curve $f \circ j(\tilde{\gamma})$ in $M'$ is uniquely lifted to a curve $\tilde{\gamma}'$ in $\tilde{M}'$ starting from $\tilde{y}_0$. Let its endpoint be $\tilde{y}$. Since any two curves connecting $\tilde{x}_0$ and $\tilde{x}$ are homotopic due to the simple-connectedness of $\tilde{M}$, $\tilde{y}$ does not depend on the choice of $\tilde{\gamma}$ and defines a map $\tilde{y} = \tilde{f}(\tilde{x})$. It is a simple task to show that $\tilde{f}$ is a diffeomorphism and the diagram commutes. The second part is trivial.

Now let $j : \tilde{M} \to M$ and $j' : \tilde{M}' \to M'$ be two universal covering spaces of $M$ defining the local homogeneity of data $\Phi$ on $M$. Then the above proposition implies that there exists a diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}'$ such that the following diagram commutes:
From this it follows that the covering data $\tilde{\Phi}$ on $\tilde{M}$ and $\Phi'$ on $\tilde{M}'$ are related by $\tilde{f}_*\tilde{\Phi} = \tilde{f}_*j_*^{-1}\Phi = j'^{-1}_*\Phi = \Phi'$. This implies that $\text{InvG}(\tilde{\Phi}, \tilde{M})$ and $\text{InvG}(\Phi', \tilde{M}')$ are connected by the external automorphism induced by $\tilde{f}$ as $\text{InvG}(\Phi', \tilde{M}') = \tilde{f}\text{InvG}(\tilde{\Phi}, \tilde{M})\tilde{f}^{-1}$. In particular, if $\text{InvG}(\tilde{\Phi}, \tilde{M}) \cong G$ then $\text{InvG}(\Phi', \tilde{M}') \cong G$, and vice versa. Hence there is a one-to-one correspondence between the description by $G$-invariant data on $\tilde{M}$ and that on $\tilde{M}'$ up to the freedom of the covering transformations. Therefore we can work in one fixed universal covering space to construct $\Gamma_{\text{LH}}(M, G)$.

2.1.2 Moduli: intrinsic representation of the freedom of covering map $j$

If we apply the argument in the last paragraph to the case corresponding to a pair of different covering maps $j$ and $j'$ by the same covering space $\tilde{M}$, we obtain an automorphism of the covering data on $\tilde{M}$ induced by the change of the covering map. This implies that we can also fix the projection map $j$ to see the correspondence of data on $\tilde{M}$ and those on $\tilde{M}'$. However, this approach is not adequate for our purpose because we cannot fix the invariance group of $\tilde{\Phi}$ isomorphic to $G$, which introduces complications into the investigation of the structure of allowed covering data.

For example, consider the 3-dimensional Euclidean space $E^3$ as the covering space and construct the 3-dimensional torus $M = T^3$ as the quotient manifold by the discrete transformation group $K_0$ generated by the standard lattice $K_0 = \{(l, m, n) \mid l, m, n \in \mathbb{Z}\}$. The standard metric on $E^3$ is represented in terms of the natural Descartes coordinate as $\tilde{q}_0 = dx^2 + dy^2 + dz^2$ and its invariance group is given by $\text{IO}(3)$. If we denote the standard projection map from $E^3$ to $T^3$ by $j_0$, $q_0 = j_0^*\tilde{q}_0$ gives locally $\text{IO}(3)$-homogeneous data on $T^3$. Next let us consider a linear transformation $x' = \tilde{f}(x) := Ax$. The metric $\tilde{q}$ obtained from $q_0$ by this transformation, $\tilde{q} = \tilde{f}_*\tilde{q}_0$, gives another data $q$ on $T^3$ through $j_0$. It is not isomorphic to $q_0$ if $\tilde{f}$ does not belong to $\text{IO}(3)$ because the lengths of closed geodesics in $T^3$ with respect to $q_0$ and $q$ are different. Clearly the invariance groups $\text{InvG}(\tilde{q}_0, E^3)$ and $\text{InvG}(\tilde{q}, E^3)$ are different, though they are connected by the external automorphism induced by $\tilde{f}$ and both give locally $\text{IO}(3)$-homogeneous data on $T^3$.

On the other hand if we consider the projection defined by $j = j_0^*\tilde{f}$, $q = j_0^*\tilde{f}_*\tilde{q}_0 = j_0^*\tilde{q}$ coincides with the data induced from the original standard metric $\tilde{q}_0$ by the new projection map $j$. Hence by changing the projection map, we can construct non-isomorphic data on $T^3$ from covering data with the same invariance group.

On the basis of this observation we do not fix the projection map, and take the approach to express its freedom in an intrinsic way. The basis of this approach is provided by the following fact.

Proposition 2.2 For a given point $\tilde{x}_0$ in $\tilde{M}$, each homotopy class of the loops with the base point $j(\tilde{x}_0)$ induces a unique covering transformation of the universal covering space $\tilde{M}$. Let us denote the corresponding monomorphic embedding of $\pi_1(M, j(\tilde{x}_0))$ into $\text{Diff}(\tilde{M})$ as $j_{\tilde{x}_0}^\sharp$. Then the image of this monomorphism, $j_{\tilde{x}_0}^\sharp(\pi_1(M, j(\tilde{x}_0)))$, does not depend on the
choice of $\tilde{x}_0$, and gives a unique discrete subgroup $\tilde{j}^*(\pi_1(M))$. $\tilde{\Phi} = j^*\Phi$ is invariant under this subgroup for any data $\Phi$ on $M$, i.e., $\tilde{j}^*(\pi_1(M)) \subseteq \text{InvG}(\tilde{\Phi}, \tilde{M})$.

The former half of the proposition is a well-known fact and is proved by considering the special case in Prop. 2.1 that $(M', j', M') = (\tilde{M}, j, M)$ and $x_0$ and $y_0$ are taken as the start point and the end point of the lift of a closed loop. To show the latter half, take two points $\tilde{x}_0$ and $\tilde{y}_0$ in $\tilde{M}$. Let $\tilde{\mu}$ be a curve from $\tilde{x}_0$ to $\tilde{y}_0$, and $\mu$ be its projection on $M$. Then from the way of construction of the covering transformation, it is easily shown that $j_{\tilde{y}_0}^\mu(\lbrack \mu \alpha \mu^{-1} \rbrack) = j_{\tilde{x}_0}^\mu(\lbrack \alpha \rbrack)$ holds for any closed loop $\alpha$ with the base point $j(\tilde{x}_0)$ where $\lbrack \alpha \rbrack$ represents the homotopy class of $\alpha$. The invariance of $\tilde{\Phi}$ under $\tilde{j}^*(\pi_1(M))$ is trivial from the definition of the pullback.

This proposition shows that the discrete subgroup $\tilde{j}^*(\pi_1(M))$ yields an intrinsic representation of the covering map. Though this discrete group does not uniquely determine the covering map $j$, this representation is very useful in our problem. To see this, suppose that two covering maps $j_1$ and $j_2$ map $\pi_1(M)$ to the same discrete group $K$. Then since a fiber containing a point $\tilde{x}$ is given by the set $\tilde{j}^*(\pi_1(M))\tilde{x}$ in general, $j_1^{-1}(x) \cap j_2^{-1}(y) \neq \emptyset$ implies $j_1^{-1}(x) = j_2^{-1}(y)$ in the present case. This shows that the identity transformation $\tilde{f} = \text{id}$ gives a fiber preserving diffeomorphism between the two covering space $(\tilde{M}, j_1, M)$ and $(\tilde{M}, j_2, M)$. Hence applying Prop. 2.1 to the following diagram,

\[
\begin{array}{ccc}
\tilde{M} \\
j_1 \searrow & f & \nwarrow j_2 \\
M & \rightarrow & M
\end{array}
\]

we obtain a diffeomorphism $f \in \text{Diff}(M)$ such that this diagram commutes. Therefore for any covering data $\tilde{\Phi}$ on $\tilde{M}$ such that $K \subset \text{InvG}(\tilde{\Phi}, \tilde{M})$, the corresponding data $\Phi_1 = j_1^*\tilde{\Phi}$ and $\Phi_2 = j_2^*\tilde{\Phi}$ on $M$ are related by the diffeomorphism $f$ as $\Phi_2 = f_*\Phi_1$. Hence the pair $(\tilde{\Phi}, K)$ uniquely determines a diffeomorphism class of locally $G$-homogeneous data $\Phi$ on $M$. This suggests that the problem of classifying the diffeomorphism classes of the data in $\Gamma_{\text{LH}}(M, G)$ can be replaced by that of the pair of data $(\tilde{\Phi}, K)$. In other words, if we denote the set of covering data whose invariance group coincides with $\tilde{G}(\subset \text{Diff}(M))$ as

\[
\Gamma_H(\tilde{M}, \tilde{G}) := \left\{ \tilde{\Phi} \mid \text{InvG}(\tilde{\Phi}, \tilde{M}) = \tilde{G} \right\},
\]

and the set of all possible images of $\pi_1(M)$ in $\tilde{G}$ by $\tilde{j}^*$ as

\[
\mathcal{M}(M, \tilde{G}) := \left\{ K : \text{a subgroup of } \tilde{G} \mid K = \tilde{j}^*(\pi_1(M)) \text{ for some } j : \tilde{M} \rightarrow M \right\},
\]

the problem is replaced by that of determining the diffeomorphism class of $\bigcup_{\tilde{G} \supseteq G} \Gamma_H(\tilde{M}, \tilde{G}) \times \mathcal{M}(M, \tilde{G})$.

2.1.3 Eliminating the freedom of $\tilde{G}$

Though the above reformulation reduces the problem to a simpler one, it still has a cumbersome large freedom associated with $\tilde{G}$. In order to eliminate this freedom, we utilize Thurston’s theorem on the geometric structures on 3-dimensional manifolds.
| Space | $G_{\text{max}}$ | $G_{\text{max}}^+$ | $(G_{\text{min}})_0$ | Bianchi type |
|-------|-----------------|------------------|----------------|-------------|
| $E^3$ | IO(3)           | ISO(3)           | $R^3$          | I           |
|       |                 |                  | $\text{VII}_0^{(\pm)}$ | VII(0)      |
| Nil   | Isom(Nil)       | Isom(Nil)        | Nil            | II          |
|       | $\cong R \times \text{IO}(2)$ |                  |                |             |
| Sol   | Isom(Sol)       | $\text{Sol} \times D_2$ | Sol           | VI(0)       |
|       | $\cong \text{Sol} \times D_4$ |                  |                |             |
| $H^3$ | Isom($H^3$)     | $\text{PSL}_2 C$ | $\text{PSL}_2 C$ | V, VII($A \neq 0$) |
|       | $\cong \text{PSL}_2 C \times \mathbb{Z}_2$ |                  |                |             |
| $\tilde{S\text{L}}_2 R$ | Isom($\tilde{S\text{L}}_2 R$) | Isom($\tilde{S\text{L}}_2 R$) | $R \times \text{PSL}_2 R$ | III, VIII |
|       | $\cong R \times \text{PSL}_2 R \times \mathbb{Z}_2$ |                  |                |             |
| $H^2 \times E^1$ | Isom($H^2 \times E^1$) | (PSL$_2 R \times R \times \mathbb{Z}_2$) | PSL$_2 R \times R$ | III       |
|       | $\cong \text{PSL}_2 R \times \mathbb{Z}_2 \times \text{IO}(1)$ |                  |                |             |
| $S^3$ | O(4)           | SO(4)            | SU(2)          | IX          |
| $S^2 \times E^1$ | O(3) $\times$ IO(1) | SO(3) $\times$ $R \times \mathbb{Z}_2$ | SO(3) $\times$ R | Kantowski-Sachs models |

Table 1: Thurston types

For each Thurston type the maximal symmetry group $G_{\text{max}}$, its orientation-preserving component $G_{\text{max}}^+$, the connected component of minimum transitive invariance groups $(G_{\text{min}})_0$, and the Bianchi types of the simply transitive subgroups are shown\cite{2,3}.

First we give a couple of basic definitions. A pair $(\tilde{M}, G_{\text{max}})$ of a connected and simply connected manifold $\tilde{M}$ and its transformation group $G_{\text{max}}$ is said to define a maximal geometry on $\tilde{M}$ if there exists a metric on $\tilde{M}$ such that its isometry group coincides with $G_{\text{max}}$ and there exists no metric with a larger isometry group. Two maximal geometries $(\tilde{M}, G_{\text{max}})$ and $(\tilde{M}', G_{\text{max}}')$ are said to be isomorphic if there exists a diffeomorphism $f: \tilde{M} \rightarrow \tilde{M}'$ such that $G_{\text{max}}' = fG_{\text{max}}f^{-1}$. Further a compact closed manifold $M$ is called a compact quotient of a maximal geometry $(\tilde{M}, G_{\text{max}})$ if $G_{\text{max}}$ has a discrete subgroup $K$ such that the quotient space $\tilde{M}/K$ is diffeomorphic to $M$. With these definitions Thurston’s theorem is stated as follows\cite{9,10,11}:

**Theorem 2.1 (Thurston)**

1) \emph{[Classification theorem]}

Any 3-dimensional maximal geometry $(\tilde{M}, G_{\text{max}})$ is isomorphic to one of 8 types listed in Table 1, provided that it allows a compact quotient.
2) [Uniqueness theorem]
For any 3-dimensional compact closed manifold \( M \), if \( M \) is a compact quotient of a maximal geometry, the corresponding maximal geometry is unique up to isomorphism.

To apply this theorem, first note that from the argument in §2.1.1 for any covering map \( j : \tilde{M} \to M \) and \( \tilde{f} \in \text{Diff}(\tilde{M}) \), \( j' = \tilde{f}^{-1} \circ j \) gives a new covering map, and that conversely for any two covering maps \( j \) and \( j' \) there exists a diffeomorphism \( \tilde{f} \) such that \( \tilde{f} \circ j' = j \). Further, in this case, the invariance groups of covering data \( \tilde{\Phi} \) and \( \tilde{\Phi}' \) corresponding to the same data \( \Phi \) on \( M \) are mutually conjugate by \( \tilde{f} \). On the other hand, from the above theorem, the base manifold \( M \) uniquely determines a diffeomorphism class of a maximal geometry. Let us pick up one representative pair of it, \((\tilde{M}, G_{\text{max}})\), and fix it. Then if the data \( \Phi \) contains metric data \( q \), its invariance group \( \text{Inv}_G(\tilde{\Phi}) \) is contained in some maximal symmetry group \( G'_{\text{max}} \) since \( \text{Inv}_G(\tilde{\Phi}) \subseteq \text{Inv}_G(\tilde{q}) \), and again from Thurston’s theorem there exists a diffeomorphism \( \tilde{f} \in \text{Diff}(\tilde{M}) \) such that \( G'_{\text{max}} = \tilde{f}G_{\text{max}}\tilde{f}^{-1} \). Therefore, for any given data \( \Phi \) on \( M \), we can always find a covering map \( j \) such that \( \text{Inv}_G(\tilde{\Phi}) \) is contained in the fixed transformation group \( G_{\text{max}} \).

With the help of this result we can construct a mapping from \( \Gamma_{\text{LH}}(M, G) \) to a space with a much simpler structure. First let us define \( \text{TSB}(G_{\text{max}}) \) as a set of subgroups of \( G_{\text{max}} \) such that it contains just one representative element for each conjugate class of the subgroups of \( G_{\text{max}} \) with respect to \( \text{Diff}(\tilde{M}) \). Then from the above argument, each locally \( G \)-homogeneous data \((\Phi, M)\) uniquely determines a subgroup \( \tilde{G} \in \text{TSB}(G_{\text{max}}) \) such that \( \text{Inv}_G(j^*\Phi) = \tilde{G} \) holds for some covering map \( j \). The corresponding covering map in turn determines data \((j^*\Phi, j^*(\pi_1(M))) \in \Gamma_H(\tilde{G}, M) \times \mathcal{M}(M, \tilde{G}) \). Though the covering map \( j \) here is not unique, its freedom is quite restricted. In fact, if \( \text{Inv}_G(j^*\Phi) = \text{Inv}_G(j^*\Phi) = \tilde{G} \) holds, there should exists \( \tilde{f} \in \text{Diff}(\tilde{M}) \) such that \( j' \circ \tilde{f} = j \), but this diffeomorphism must satisfy the condition \( \tilde{f}\tilde{G}\tilde{f}^{-1} = \tilde{G} \). That is, the freedom in the covering map is represented by homogeneity preserving diffeomorphisms (HPDs for brevity). Hence, by denoting the set of HPDs with respect to the symmetry group \( \tilde{G} \) as

\[
\text{HPDG}(\tilde{M}, \tilde{G}) := \left\{ \tilde{f} \in \text{Diff}(\tilde{M}) \mid \tilde{f}\tilde{G}\tilde{f}^{-1} = \tilde{G} \right\}, \tag{2.7}
\]

we naturally obtain the map

\[
F : \Gamma_{\text{LH}}(M, G) \to \bigcup_{G \cong \tilde{G} \in \text{TSB}(G_{\text{max}})} \left( \Gamma_H(\tilde{M}, \tilde{G}) \times \mathcal{M}(M, \tilde{G}) \right) / \text{HPDG}(\tilde{M}, \tilde{G}), \tag{2.8}
\]

where the action of \( \tilde{f} \in \text{HPDG}(\tilde{M}, \tilde{G}) \) is defined as

\[
\tilde{f}_*(\tilde{\Phi}, K) = (\tilde{f}_*\tilde{\Phi}, \tilde{f}_*K\tilde{f}^{-1}). \tag{2.9}
\]

For any data \((\tilde{\Phi}, K) \in \Gamma_H(\tilde{M}, \tilde{G}) \times \mathcal{M}(M, \tilde{G}) \), from the definition of \( \mathcal{M}(\pi_1(M), \tilde{G}) \), there is a covering map \( j : \tilde{M} \to M \) such that \( j^*(\pi_1(M)) = K \). Since \( K \subset \tilde{G} = \text{Inv}_G(\tilde{\Phi}) \), there exist unique locally \( G \)-homogeneous data \( \Phi \) on \( M \) whose pullback by \( j \) coincide with \( \tilde{\Phi} \). Hence the map \( F \) is surjective.
2.1.4 An expression of the diffeomorphism-invariant phase space

The map $F$ defined above induces an isomorphism between the diffeomorphism-invariant phase space of the locally $G$-homogeneous data and the target space of $F$.

To see this, we first show that $F$ induces a well-defined map from $\Gamma_{\text{LH}}(M,G)/\text{Diff}(M)$ to $\Gamma_H(M,\tilde{G}) \times \mathcal{M}(M,\tilde{G})$. Suppose that two data $\Phi_1$ and $\Phi_2$ on $M$ are connected by a diffeomorphism $f$ of $M$: $\Phi_2 = f^*\Phi_1$. From the argument in §2.1.3, for each data $\Phi_i(i = 1, 2)$, we can find a covering map $j_i$ such that $\text{InvG}(\Phi_i) \in \text{TSB}(G_{\max},\tilde{M})$ where $\Phi_i = j_i^*\Phi_i$. On the other hand, from Prop. 2.1, there exists a diffeomorphism $\tilde{f} \in \text{Diff}(M)$ such that the diagram

$$
\begin{array}{ccc}
(\tilde{\Phi}_1, \tilde{M}) & \xrightarrow{\tilde{f}} & (\tilde{\Phi}_2, \tilde{M}) \\
\downarrow j_1 & & \downarrow j_2 \\
(\Phi_1, M) & \xrightarrow{f} & (\Phi_2, M)
\end{array}
$$

(2.10)

commutes. Clearly the covering data $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are related by $\tilde{f}$ as $\tilde{\Phi}_2 = \tilde{f}_\ast \tilde{\Phi}_1$, and their invariance groups as $\text{InvG}(\tilde{\Phi}_2) = \tilde{f} \text{InvG}(\tilde{\Phi}_1) \tilde{f}^{-1}$. However from the definition of $\text{TSB}(G_{\max})$, this implies $\text{InvG}(\tilde{\Phi}_1) = \text{InvG}(\tilde{\Phi}_2) = \tilde{G}$ for some $\tilde{G} \cong G$. Thus $\tilde{G} = \tilde{f} \tilde{G} \tilde{f}^{-1}$ holds and $\tilde{f}$ must be a HPD. Further, since diffeomorphisms preserve the fundamental group, it follows from Prop. 2.2 that $j_2^\ast(\pi_1(M)) = \tilde{j}_2^\ast \pi_1(M)$. Hence we obtain $F(\Phi_1) = F(\Phi_2)$, which implies that $F$ induces the map

$$F_\ast : \Gamma_{\text{LH}}(M,G)/\text{Diff}(M) \to \bigcup_{G \cong \tilde{G} \in \text{TSB}(G_{\max})} \left( \Gamma_H(\tilde{M},\tilde{G}) \times \mathcal{M}(M,\tilde{G}) \right) / \text{HPDG}(\tilde{M},\tilde{G}).$$

(2.11)

Next we show that the map $F_\ast$ is injective. Suppose that $F(\Phi_1) = F(\Phi_2)$ holds for $\Phi_1, \Phi_2 \in \Gamma_{\text{LH}}(M,G)$. Then there exist covering maps $j_1, j_2$, $\tilde{G} \in \text{TSB}(G_{\max})$, and $\tilde{f} \in \text{HPDG}(\tilde{M},\tilde{G})$ such that $\tilde{\Phi}_2 = \tilde{j}_2^\ast \tilde{\Phi}_2 = \tilde{f}_\ast j_1^\ast \tilde{\Phi}_1 = \tilde{f}_\ast \tilde{\Phi}_1$ and $j_2^\ast(\pi_1(M)) = \tilde{j}_2^\ast(\pi_1(M)) \tilde{f}^{-1}$.

Since $j_2^\ast(\pi_1(M)) \tilde{f}(\tilde{x}) = \tilde{f}(j_2^\ast(\pi_1(M)) \tilde{x})$ holds for any point $\tilde{x} \in \tilde{M}$, $\tilde{f}$ preserves fibers. Hence from Prop. 2.1 there exists a diffeomorphism $f \in \text{Diff}(M)$ such that the diagram (2.10) commutes. For this map $f, \Phi_1 = \Phi_2$ holds. Hence $\Phi_1$ and $\Phi_2$ belong to the same diffeomorphism class in $\Gamma_{\text{LH}}(M,G)$.

Thus we obtain the following theorem:

**Theorem 2.2** For a locally $G$-homogeneous system on a compact closed 3-manifold $M$ of type $(\tilde{M},G_{\max})$, if the canonical variables contain the metric, the diffeomorphism-invariant phase space is represented as

$$\Gamma_{\text{LH}}(M,G)/\text{Diff}(M) = \bigcup_{G \cong \tilde{G} \in \text{TSB}(G_{\max})} \left( \Gamma_H(\tilde{M},\tilde{G}) \times \mathcal{M}(M,\tilde{G}) \right) / \text{HPDG}(\tilde{M},\tilde{G}).$$

(2.12)

Here note that this formula holds for $M$ of any dimension if we replace $\text{TSB}(G_{\max})$ by a set of representative subgroups for the general conjugate classes of transitive transformation groups with respect to $\text{Diff}(M)$. However, this generalization will be of little use in higher dimension because possible types of universal covering spaces and the structure of the conjugate classes of their transformation groups are not known well. On the other hand for two dimension, the theorem holds in the present form because the uniformization theorem holds. In this case the maximal geometries are classified into three types: $(E^2, \text{IO}(2))$, $(S^2, \text{O}(3))$ and $(H^2, \text{PO}(2,1))$. 


2.1.5 Note on orientation

In physics we usually consider only orientable spaces because the laws of physics are not invariant under the reversal of space orientation. When we restrict spaces to orientable ones, it is natural to restrict covering maps to orientation preserving ones by fixing orientations of the base manifold and its universal covering space. Let $\Gamma_{LH}^+(M, G)$ be the corresponding phase space of locally homogeneous data on $M$. Then by inspecting the proof, it is easy to see that Theorem 2.2 holds if we replace $\text{Diff}(M)$, $\text{Diff}(\tilde{M})$ and $\text{InvG}(\Phi)$ by the orientation preserving ones $\text{Diff}^+(M)$, $\text{Diff}^+(\tilde{M})$ and $\text{InvG}^+(\Phi) = \text{InvG}(\Phi) \cap \text{Diff}^+(M)$, $\text{TSB}(G_{\text{max}})$ by a representative system of the conjugate classes of transitive subgroups of $G_{\text{max}}$ with respect to $\text{Diff}^+(\tilde{M})$, and $\Gamma_{H}(\tilde{M}, \tilde{G})$ and $\text{HPDG}(\tilde{M}, \tilde{G})$ by $\Gamma_{H}^+(\tilde{M}, \tilde{G}) = \{ \tilde{\Phi} \mid \text{InvG}^+(\tilde{\Phi}) = \tilde{G} \}$ and $\text{HPDG}^+(\tilde{M}, \tilde{G}) = \text{HPDG}(\tilde{M}, \tilde{G}) \cap \text{Diff}^+(\tilde{M})$:

$$\Gamma_{LH}^+(M, G)/\text{Diff}(M) = \bigcup_{G \cong \tilde{G} \in \text{TSB}^+(G_{\text{max}})} \left( \Gamma_{H}^+(\tilde{M}, \tilde{G}) \times \mathcal{M}(M, \tilde{G}) \right)/\text{HPDG}^+(\tilde{M}, \tilde{G}). \quad (2.13)$$

From now on, in the arguments which apply to the orientation-preserving case just by these replacements, we indicate it by putting the suffix (+) on relevant quantities as $\Gamma_{LH}^+(M, G)$ and $\text{InvG}^+(\Phi)$.

2.2 The algorithm to determine $\Gamma_{LH}(M, G)/\text{Diff}(M)$

Theorem 2.2 gives us a systematic algorithm for the classification and the determination of the diffeomorphism-invariant phase space of locally $G$-homogeneous data on a compact closed manifold of a given Thurston-type $(\tilde{M}, G_{\text{max}})$. It is divided into three parts.

2.2.1 Determination of $\mathcal{M}(M, G_{\text{max}})$

Each compact closed 3-manifolds modeled on $(\tilde{M}, G_{\text{max}})$ is diffeomorphic to a quotient space $\tilde{M}/K$ where $K$ is some discrete subgroup of $G_{\text{max}}$ acting freely. Two quotient manifolds $\tilde{M}/K$ and $\tilde{M}/K'$ are diffeomorphic if and only if $K$ and $K'$ are conjugate with respect to $\text{Diff}(\tilde{M})$. In the three-dimensional case this condition is equivalent to the condition that $K$ and $K'$ are isomorphic as abstract groups, as in the two-dimensional case, except for the Thurston-type $\tilde{M} = S^3$ for the following reason.

First for the Thurston-types other than $S^3$ and $H^3$, $\pi_1(M)$ contain a discrete subgroup isomorphic to $\mathbb{Z}^k$ [14]. This implies that $H_1(M)$ is an infinite group, and $M$ becomes a Haken manifold [12]. Hence from Waldhausen’s theorem [13] the diffeomorphism class of $M$ is completely determined by the isomorphism class of the fundamental group $\pi_1(M)$. On the other hand for $H^3$ the same result holds from Mostow’s rigidity theorem [11].

Since all the manifolds modeled on $E^3$, Nil, $\text{SL}_2\mathbb{R}$, $H^2 \times E^1$ and $S^2 \times E^1$ allow the structure of Seifert fiber space, their abstract fundamental groups are determined by the indices of base orbifolds and the Seifert index. On the other hand compact closed manifolds of type Sol have the structure of torus bundle or Klein-bottle bundle over $S^1$, and their abstract fundamental groups are determined by the gluing map of the torus or the Klein bottle. Hence from the above fact the moduli space $\mathcal{M}(M, G_{\text{max}})$ for these types is
easily determined by examining all the possible embedding of the corresponding abstract fundamental group into $G_{\text{max}}$.

All possible fundamental groups of manifolds of type $S^3$ and their embedding into $O(4)$ are also determined with the help of the Seifert bundle structure of them. The classification of the spaces modeled on $H^3$ is not completed yet.

### 2.2.2 Classification of $G$-homogeneous data on $\tilde{M}$

This part is divided into the following three steps:

i) Classification of the conjugate classes of transitive subgroups of $G_{\text{max}} \subseteq \text{TSB}^{(+)}(G_{\text{max}})$.

ii) Determination of HPDs for each $\tilde{G} \in \text{TSB}^{(+)}(G_{\text{max}}) \Rightarrow \text{HPDG}^{(+)}(\tilde{M}, \tilde{G})$.

iii) Determination of the structure of $\tilde{G}$-invariant $\tilde{\Phi} \Rightarrow \Gamma_H^{(+)}(\tilde{M}, \tilde{G})$.

Some comments are in order. Firstly, in connection with step i), it should be noted that two subgroups of $G_{\text{max}}$ may not be conjugate with respect to $G_{\text{max}}$ even if they are with respect to $\text{Diff}^{(+)}(\tilde{M})$. However, the following useful proposition holds.

**Proposition 2.3** Let $G_{st}$ be a simply transitive subgroup of $G_{\text{max}}$. Then for $G_{st} \subseteq G_1, G_2 \subseteq G_{\text{max}}$, if $G_1$ and $G_2$ are conjugate with respect to $\text{Diff}^{(+)}(\tilde{M})$, they are with respect to $\text{HPDG}^{(+)}(\tilde{M}, G_{\text{max}}) \times \text{HPDG}^{(+)}(\tilde{M}, G_{st})$.

Let $f \in \text{Diff}^{(+)}(\tilde{M})$ be a transformation such that $G_2 = fG_1f^{-1}$. Then $fG_{st}f^{-1} \subseteq G_{\text{max}}$. Since explicit analyses show that all the simply-transitive subgroups of $G_{\text{max}}$ are mutually conjugate with respect to $\text{HPDG}^{(+)}(\tilde{M}, G_{\text{max}})$ (see §3, §4 and §5 for Isom($E^3$), Isom(Nil) and Isom(Sol)), this implies that there exists $f' \in \text{HPDG}^{(+)}(\tilde{M}, G_{\text{max}})$ such that $f'fG_{st}f^{-1}f'^{-1} = G_{st}$. From this it follows that $f'' = f'f \in \text{HPDG}^{(+)}(\tilde{M}, G_{st})$. Hence $f = f''^{-1}f'' \in \text{HPDG}^{(+)}(\tilde{M}, G_{\text{max}}) \times \text{HPDG}^{(+)}(\tilde{M}, G_{st})$.

Secondly, in step ii), since $f \in \text{HPDG}^{(+)}(\tilde{M}, \tilde{G})$ implies $f \in \text{EAut}(\tilde{G}, \tilde{M}) \subseteq \text{Aut}(\tilde{G})$ where $\text{EAut}(\tilde{G}, \tilde{M})$ is the subset of $\text{Aut}(\tilde{G})$ whose elements are induced from transformations of $\tilde{M}$, $\text{HPDG}^{(+)}(\tilde{M}, \tilde{G})$ is easily determined from the automorphism group of the Lie algebra, $\text{Aut}(\mathcal{L}(\tilde{G}))$, as follows. Let $\phi$ be an element of $\text{Aut}(\mathcal{L}(\tilde{G}))$ which is expressed in terms of a basis $\xi_I$ of $\mathcal{L}(\tilde{G})$ as $\phi(\xi_I) = \xi_J A^J_I$. Then if $\phi$ is induced from $f \in \text{Diff}^{(+)}(\tilde{M})$, $f$ should satisfy the equation

$$\phi(\xi_I) = f_* \xi_I = \xi_J A^J_I.$$  \hspace{1cm} (2.14)

In a local coordinate system $(x^\mu)$ this gives a set of differential equations

$$\xi_I^\mu(x) \partial_\nu f^\mu(x) = \xi_J^\nu(f(x)) A^J_I.$$ \hspace{1cm} (2.15)

$\text{Aut}(\mathcal{L}(\tilde{G}))$ is easily determined by simple algebraic calculations.

Thirdly, in step iii), even if $\tilde{\Phi}$ is invariant with respect to $\tilde{G}$ and $\text{InvG}^{(+)}(\tilde{\Phi}) \cap G_{\text{max}} = \tilde{G}$, $\tilde{\Phi}$ may not belong to $\Gamma_H(\tilde{M}, \tilde{G})$ because it can be invariant under some transformation $\check{f} \not\in G_{\text{max}}$. However, in the case $\tilde{G}$ contains a simply transitive subgroup $G_{st}$, we can check rather easily whether $\tilde{\Phi}$ belongs to $\Gamma_H(\tilde{M}, \tilde{G})$ with help of the following proposition.
Proposition 2.4 If \( \text{Inv}^{(+)}(\tilde{\Phi}) \) contains a simply transitive subgroup \( G_{st} \), there exists \( f \in \text{HPDG}^{(+)}(\tilde{M}, G_{st}) \) such that \( \text{Inv}^{(+)}(f_{st}\Phi) \subseteq G_{max} \). In particular if \( \text{Inv}^{(+)}(f_{st}\Phi) \cap G_{max} = \tilde{G} \) for any \( f \in \text{HPDG}^{(+)}(\tilde{M}, G_{st}) \), then \( \text{Inv}^{(+)}(\tilde{\Phi}) = G \).

The proof is almost the same as that of Prop. 2.3. First from Thurston’s theorem there exists \( f' \in \text{Diff}^{(+)}(\tilde{M}) \) such that \( f'\text{Inv}^{(+)}(\tilde{\Phi})f'^{-1} \subseteq G_{max} \). From the assumption this implies \( f'G_{st}f'^{-1} \subseteq G_{max} \). Then, since any subgroup of \( G_{max} \) is conjugate to \( G_{st} \) with respect to \( \text{HPDG}^{(+)}(\tilde{M}, G_{max}) \) if it is with respect to \( \text{Diff}^{(+)}(\tilde{M}) \), there exists \( f'' \in \text{HPDG}^{(+)}(\tilde{M}, G_{max}) \) such that \( f''fG_{st}f''^{-1}G_{max} = G_{st} \). This implies \( f = f''f' \in \text{HPDG}^{(+)}(\tilde{M}, G_{st}) \), and \( \text{Inv}^{(+)}(f_{st}\Phi) = f\text{Inv}^{(+)}(f'_{st}\Phi)f^{-1} \subseteq f''G_{max}f''^{-1} = G_{max} \).

From this proposition it follows that even if \( \text{Inv}^{(+)}(\tilde{\Phi}) \cap G_{max} = \tilde{G}, \tilde{\Phi} \) should not be included in \( \Gamma_{H}^{(+)}(\tilde{M}, \tilde{G}) \) if there exists \( f \in \text{HPDG}^{(+)}(\tilde{M}, G_{st}) \) such that \( \text{Inv}^{(+)}(f_{st}\Phi) \supset \tilde{G} \) and \( \neq \tilde{G} \). For example, as we will show in §3, for the locally homogeneous system \( \Phi = (q, p) \) of the type \( E^{3} \), the smallest group \( \tilde{G} \) for which \( \tilde{G} \supset \tilde{R}^{3} \) and \( \Gamma_{H}^{(+)}(\tilde{M}, \tilde{G}) \neq \emptyset \) is given not by \( \tilde{R}^{3} \) but by \( \tilde{R}^{3} \times D_{2} \) where \( D_{2} \) is the dihedral group of order 4.

In practice we can show by direct analyses that all the transitive connected subgroups in \( G_{max}^{(+)} \) are conjugate with respect to \( \text{HPDG}^{(+)}(\tilde{M}, G_{max}) \) if they are in \( \text{Diff}^{(+)}(\tilde{M}) \). Hence the proposition still holds even if \( G_{st} \) is replaced by the connected component of \( \tilde{G} \) containing identity.

Finally we comment on the diffeomorphism constraint. When \( \Gamma^{(+)}_{\text{LH}}(M, G)/\text{Diff}^{(+)}(M) \) is regarded as a subspace of the diffeomorphism-invariant phase space \( \Gamma^{(+)}(M)/\text{Diff}^{(+)}(M) \) of generic data on \( M \), the canonical structure of the latter induces that of the former. Let \( \Gamma^{(+)}_{D}(M) \) be the set of data that satisfy the diffeomorphism constraint. Then the canonical structure of the generic diffeomorphism-invariant phase space becomes non-degenerate only in the subspace \( \Gamma^{(+)}_{D}(M)/\text{Diff}^{(+)}(M) \). Hence in the study of the canonical structure of the phase space of locally homogeneous data we must also restrict consideration to the subspace satisfying the diffeomorphism constraint.

Let us denote this subspace by
\[
\Gamma^{(+)}_{\text{LH,inv}}(M, G) := \left( \Gamma^{(+)}_{\text{LH}} \cap \Gamma^{(+)}_{D} \right)/\text{Diff}^{(+)}(M).
\] (2.16)

Then, since the diffeomorphism constraint on \( \Phi \in \Gamma^{(+)}_{\text{LH}}(M, G) \) is represented as that on the covering data \( \tilde{\Phi} \in \Gamma^{(+)}_{H}(\tilde{M}, \tilde{G}) \) and does not depend on the moduli freedom, Eq.(2.12) and Eq.(2.13) give
\[
\Gamma^{(+)}_{\text{LH,inv}}(M, G) = \bigcup_{G \supset \tilde{G} \in \text{TSB}^{(+)}(G_{max})} \left( \Gamma^{(+)}_{H, D}(\tilde{M}, \tilde{G}) \times \mathcal{M}(M, \tilde{G}) \right)/\text{HPDG}^{(+)}(\tilde{M}, \tilde{G}),
\] (2.17)
where
\[
\Gamma^{(+)}_{H, D}(\tilde{M}, \tilde{G}) := \Gamma^{(+)}_{H}(\tilde{M}, \tilde{G}) \cap \Gamma^{(+)}_{D}(\tilde{M}).
\] (2.18)
This restriction to the subspace generally selects data with higher symmetry. This point should be taken care of in the determination of \( \Gamma^{(+)}_{\text{LH,inv}}(M, G) \).

2.2.3 Parametrization of \( \Gamma^{(+)}_{\text{LH,inv}}(M, G) \)

In order to express the canonical structure and the hamiltonian constraint explicitly in terms of independent canonical variables, we must parameterize the phase space \( \Gamma^{(+)}_{\text{LH,inv}}(M, G) \)
by picking out representative points of the HPDG-orbits in $\Gamma_{H,D}^{(+)}(\tilde{M}, \tilde{G}) \times \mathcal{M}(M, \tilde{G})$. In this procedure the following proposition plays a crucial role.

**Proposition 2.5** Let $\tilde{H}(K)$ be the isotropy group at $K$ of the action of HPDG$(\tilde{M}, \tilde{G})$ on $\mathcal{M}(M, \tilde{G})$. If the diffeomorphism constraint becomes trivial on some $\tilde{G}$-homogeneous system on $\tilde{M}$, i.e., $\Gamma_{H}^{(+)}(\tilde{M}, \tilde{G}) = \Gamma_{H,D}^{(+)}(\tilde{M}, \tilde{G})$, the maximal connected subgroup of $\tilde{H}(K)$ is contained in $\tilde{G}$.

To prove this proposition, for a given $K \in \mathcal{M}(M, \tilde{G})$, take a covering map $j$ such that $K = j^{\#}(\pi_{1}(M))$ and fix it. Suppose that there exists a diffeomorphism $\tilde{f} \in \text{HPDG}^{(+)}(\tilde{M}, \tilde{G})$ such that $\tilde{f}K\tilde{f}^{-1} = K$. Then $\tilde{f}$ is a fiber-preserving transformation of the covering space $j : \tilde{M} \to M$. Hence from Prop. 2.1 it induces a diffeomorphism $f \in \text{Diff}^{(+)}(M)$ such that $j \circ \tilde{f} = f \circ j$. In particular, if $\tilde{H}(K)_{0}$ (the connected component containing the identity) is non-trivial, there is a vector field $\xi$ on $M$ such that its pullback $\tilde{\xi} = j^{*}\xi$ gives an infinitesimal HPD on $\tilde{M}$.

Let $\tilde{\Phi} = (\tilde{Q}, \tilde{P})$ be a $\tilde{G}$-homogeneous canonical system on $\tilde{M}$ satisfying the assumption, and $D_{K}$ be a fundamental region of the action of $K$ on $\tilde{M}$. Then, since $K \subset \tilde{G}$, there exist canonical data $\Phi = (Q, P)$ on $M$ whose pullback by $j$ coincide with $\tilde{\Phi}$. Clearly for these fields and the above vector fields, the following equality holds:

$$\int_{D_{K}} \mathcal{L}_{\tilde{\xi}}\tilde{Q} \cdot \tilde{P} = \int_{M} \mathcal{L}_{\xi}Q \cdot P, \quad (2.19)$$

where $\cdot$ denotes natural inner products of $Q$ and $P$ and of $\tilde{Q}$ and $\tilde{P}$ as tensor fields. However, from the definition the right-hand side of this equation is written in terms of the diffeomorphism constraint $C_{j}(Q, P)$ as

$$\int_{M} \mathcal{L}_{\xi}Q \cdot P = \int_{M} \xi^{i}C_{j}(Q, P) = \int_{D_{K}} \tilde{\xi}^{i}C_{j}(\tilde{Q}, \tilde{P}) \quad (2.20)$$

which vanishes identically for $\tilde{\Phi} \in \Gamma_{H,D}^{(+)}(\tilde{M}, \tilde{G})$ from the assumption. On the other hand, since HPDs preserve $\tilde{G}$-invariance of fields, $\mathcal{L}_{\xi}\tilde{Q} \cdot \tilde{P}/\sqrt{\tilde{q}}$ becomes constant on $\tilde{M}$ where $\tilde{q}$ is some reference $\tilde{G}$-invariant metric on $\tilde{M}$. Hence $\mathcal{L}_{\xi}\tilde{Q} \cdot \tilde{P}$ vanishes identically, which implies

$$\mathcal{L}_{\xi}\tilde{Q} = 0 \quad (2.21)$$

because the inner product $\cdot$ is non-degenerate. By the similar argument and the identity

$$\int_{M} \mathcal{L}_{\xi}Q \cdot P = -\int_{M} Q \cdot \mathcal{L}_{\xi}P, \quad (2.22)$$

it follows that

$$\mathcal{L}_{\xi}\tilde{P} = 0. \quad (2.23)$$

However, since the isotropy group at $\tilde{\Phi}$ of the action HPDG$^{(+)}(\tilde{M}, \tilde{G})$ on $\Gamma_{H,D}^{(+)}(\tilde{M}, \tilde{G})$ coincides with $\tilde{G}$, Eqs.(2.21)-(2.23) implies that $\tilde{\xi}$ belongs to the Lie algebra of $\tilde{G}$. This proves the proposition.
As will be shown in the following sections, the assumption of the proposition is satisfied for the locally homogeneous pure gravity systems of types $E^3$, Nil and Sol. Hence one natural method of parametrizing $\Gamma_{\text{LH,inv}}^{(+)}(M, \mathcal{G})$ is obtained by picking up a subset $\mathcal{M}_0(M, \mathcal{G})$ of $\mathcal{M}(M, \mathcal{G})$ which is transversal to the HPD-orbits in $\mathcal{M}(M, \mathcal{G})$. If the assumption of the proposition is satisfied, this leads to

$$\Gamma_{\text{LH,inv}}^{(+)}(M, \mathcal{G}) \cong \bigcup_{\tilde{\mathcal{G}}} \left( \Gamma_{\text{H}}^{(+)}(\tilde{M}, \tilde{\mathcal{G}}) \times \mathcal{M}_0(M, \tilde{\mathcal{G}}) \right) / (\text{discrete HPDs}). \tag{2.24}$$

Hence natural parameterizations of $\tilde{\mathcal{G}}$-homogeneous data and the reduced moduli space $\mathcal{M}_0$ give canonical coordinates of the phase space. In the case in which the assumption is not satisfied, $\Gamma_{\text{H,D}}^{(+)}(\tilde{M}, \tilde{\mathcal{G}})$ should be further reduced to a subspace to fix residual HPD freedoms. As will be illustrated in the following sections, this parameterization is quite useful in the study of the canonical structure.

Of course other methods of parameterization can be adopted depending on the purpose. For example, if we first pick up a representative point in each HPD-orbit in $\Gamma_{\text{H,D}}^{(+)}(\tilde{M}, \tilde{\mathcal{G}})$ such that the metric data acquire the highest possible symmetry, the gauge fixing of the residual HPD freedom reduces $\mathcal{M}$ to a larger subspace, which corresponds to the standard moduli space of locally homogeneous compact Riemann manifolds [3].

### 2.3 Canonical Structure and Dynamics

With the help of the representation theorem proved in the previous subsection, we can determine the canonical structure of locally homogeneous systems rather easily. We next describe its procedure.

#### 2.3.1 General system

First we briefly summarize basic features of the canonical structure and dynamics of general Einstein-gravity systems. Let $\Gamma(M)$ be the phase space of general canonical data $\Phi = (Q, P)$ on a 3-manifold $M$. Then the canonical 1-form of $\Gamma(M)$ is given by

$$\Theta = \int_M d^3 x \delta Q \cdot P, \tag{2.25}$$

where $\delta$ denotes the exterior derivative in the phase space as a functional space. Then its exterior derivative $\omega = \delta \Theta$ gives the symplectic form, which defines the canonical structure of $\Gamma(M)$ by the following procedure. First for a functional $F$ on $\Gamma(M)$, a vector field $X_F$ corresponding to the infinitesimal canonical transformation is defined by

$$\delta F = -I_{X_F} \omega, \tag{2.26}$$

where $I_X$ is the inner product operator on differential forms. Then the Poisson bracket of two functionals $F$ and $G$ on $\Gamma(M)$ is defined by

$$\{F, G\} = -X_F G. \tag{2.27}$$

Since infinitesimal diffeomorphisms are expressed as infinitesimal canonical transformations generated by the diffeomorphism constraint functionals $C_D$, diffeomorphism-invariant
functionals are characterized as those which have vanishing Poisson brackets with $C_D$. From this and the above definition, it follows that diffeomorphism-invariant functionals makes a closed Poisson algebra. Though this algebra is in general degenerate, it becomes non-degenerate when restricted to diffeomorphism-invariant functionals on the subset $\Gamma_D$ of $\Gamma$ satisfying the diffeomorphism constraints. Hence a well-defined canonical structure is defined on the diffeomorphism-invariant phase space $\Gamma_{\text{inv}}(M) := \Gamma_D(M)/\text{Diff}(M)$.

Though the hamiltonian constraint functional

$$C_H(\Phi; N) := \int_M d^3x N(x) \mathcal{H}_\perp(x; \Phi)$$ (2.28)

is not diffeomorphism-invariant if $N(x)$ is given as an explicit function on $M$. However, it becomes diffeomorphism-invariant if we take as $N(x)$ a functional $\hat{N}(x; \Phi)$ on the phase space which takes value in the space of functions on $M$ and transforms covariantly under diffeomorphism:

$$f^*\hat{N}(\cdot, \Phi) = \hat{N}(\cdot, f^*\Phi) \quad \forall f \in \text{Diff}(M).$$ (2.29)

Hence $C_H(\Phi; \hat{N})$ gives a hamiltonian constraint functional on $\Gamma_{\text{inv}}(M)$.

Let $\Gamma_h$ be the subspace defined by

$$\Gamma_h := \left\{ \Phi \in \Gamma_{\text{inv}} \mid C_H(\Phi; \hat{N}) = 0 \quad \forall \hat{N} \right\},$$ (2.30)

where we have denoted a diffeomorphism-invariant class and its representative data by the same symbol. Let $Y_{\hat{N}}$ be the infinitesimal canonical transformation defined by

$$Y_{\hat{N}} := X_{C_H(\hat{N})}. \quad (2.31)$$

Then, since these vector fields commute on $\Gamma_h$, it defines an involutive system there and foliates $\Gamma_h$ into leaves of integration subspaces. It is shown that each leaf is in one-to-one correspondence with a spacetime-diffeomorphism class of the solutions to the Einstein equations in the sense that two solutions connected by a spacetime diffeomorphism are mapped into curves in the same leaf, and any non-degenerate curve in a given leaf corresponds to a solution in the same spacetime-diffeomorphism class.

Let $\gamma(t)$ be a non-degenerate curve contained in a leaf. Then, since its tangent vector is linear combination of $Y$-vectors, there exists a one-parameter family of $\hat{N}$-type functionals, $\hat{N}_t$ such that $\gamma_s \partial_s = Y_{\hat{N}_t}$. By operating it on a functional $F$ on $\Gamma_h$, we obtain the canonical equation of motion,

$$\dot{F} = \{F, C_H(\hat{N}_t)\}, \quad (2.32)$$

which is equivalent to the Einstein equations. This canonical equation coincides with the Euler equation for the Lagrangian

$$L = \int_M d^3x \dot{Q} \cdot \mathcal{P} - C_H(Q, \mathcal{P}; \hat{N}_t),$$ (2.33)

which is obtained from the Einstein-Hilbert action by some gauge-fixing.
2.3.2 Reduction to the locally homogeneous system

Since the diffeomorphism-invariant phase space $\Gamma_{\text{LH},\text{inv}}(M,G)$ of locally homogeneous data on $M$ is a finite-dimensional subspace of $\Gamma_{\text{inv}}(M)$, $\Theta$ on the latter uniquely defines a one form on the former. Let us denote it by the same symbol. Then its exterior derivative $\omega = d\Theta$ defines a canonical structure of $\Gamma_{\text{LH},\text{inv}}$. One potential subtle problem in this procedure is the possibility that the reduced $\omega$ becomes degenerate. As we will show explicitly in the following sections, this pathology occurs frequently. In such cases, the canonical structure of $\Gamma_{\text{LH},\text{inv}}$ is ill-defined, which implies that the system is not canonically closed in the general phase space $\Gamma_{\text{inv}}$. Note that even in such cases the restriction of $L$ defined by Eq.(2.33) to $\Gamma_{\text{LH},\text{inv}}$ gives a well-defined canonical Lagrangian for the locally homogeneous system because

$$\int_M d^3x \dot{Q} \cdot \dot{P} \text{ is nothing but the value of } \Theta \text{ on the tangent vector of the curve, } \gamma_t, \text{ which is well-defined even if the canonical structure is degenerate.}$$

From now on we use this kinetic term to express the canonical 1-form and denote it by the same symbol $\Theta$:

$$\Theta = \int_M d^3x \dot{Q} \cdot \dot{P}. \quad (2.34)$$

Now we describe how to obtain explicit expressions for $\Theta$ and the hamiltonian $H$ when a parameterization of $\Gamma_{\text{LH},\text{inv}}(M,G)$ is given. In this subsection we omit the symbol (+) on the orientation for simplicity.

Let us denote a representative point of each HPD-orbit in $\Gamma_{\text{D}}(M,G) \times M$ by $(\tilde{\Phi},K(\lambda))$ where $\lambda = (\lambda_j)$ is a set of variables parametrizing the reduced moduli space $M_0(M,G)$. We do not introduce an explicit parameterization for $\tilde{\Phi} = (\tilde{Q}, \tilde{P})$ here.

First we pick up a point $K_0$ as the base point in $M(M,\tilde{G})$ and identify $M$ with $\tilde{M}/K_0$. Let us denote the corresponding natural covering map by $j_0$:

$$j_0 : \tilde{M} \to M = \tilde{M}/K_0. \quad (2.35)$$

Then it defines an embedding of the phase space $\Gamma_{\text{inv}}(M)$ into $\Gamma_{\text{inv}}(D_0)$ through $\tilde{\Phi} = j_0^* \Phi$ where $D_0$ is a fundamental region of the action of $K_0$ on $\tilde{M}$. This embedding is isomorphic in the sense that it preserves the canonical structure:

$$\Theta = \int_M d^3x \dot{Q} \cdot P = \int_{D_0} d^3x \dot{\tilde{Q}} \cdot \tilde{P}. \quad (2.36)$$

Further, for each $K(\lambda)$ there exists a covering map $j_\lambda$ and a transformation $f_\lambda$ of $\tilde{M}$ such that

$$j_\lambda = j_0 \circ f^{-1}_\lambda, \quad (2.37)$$

$$j_\lambda^*(\pi_1(M)) = K(\lambda), \quad (2.38)$$

$$K(\lambda) = f_\lambda K_0 f^{-1}_\lambda \iff K(\lambda) = (f_\lambda)_* K_0. \quad (2.39)$$

From the second condition $\tilde{\Phi}$ defines canonical data $\Phi$ on $M$ such that $\tilde{\Phi} = j_\lambda^* \Phi$. Its embedding by $j_0$ into $\Gamma_{\text{inv}}(D_0)$ is represented by the data $j_0^* \Phi = f_\lambda^* \tilde{\Phi}$ from the first condition. Hence from Eq.(2.36) the canonical 1-form of $\Gamma_{\text{LH,inv}}(M,G)$ is expressed in terms of the coordinates $(\tilde{Q}, \tilde{P}, \lambda)$ as

$$\Theta = \int_{D_0} d^3x f_\lambda^* (\dot{\tilde{Q}} \cdot \tilde{P} + \dot{\lambda}^j \mathcal{L}_{\zeta_j(\lambda)} \tilde{Q} \cdot \tilde{P}), \quad (2.40)$$

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where $\zeta_j(\lambda)$ is the vector field defined by

$$(\zeta_j(\lambda))_x := (\partial_{\lambda_j} f_\lambda)(f_\lambda^{-1}(x)).$$

From now on we call the transformation $f_\lambda$ a deformation map from the base point $K_0$ to $K(\lambda)$.

In terms of the components with respect to an invariant basis this expression can be rewritten in a more useful form. Let $G_{st}$ be a simply transitive subgroup of $\tilde{G}$, and $\tau^a$ and $\tau_b$ be bases of $G_{st}$-invariant tensors of types $\tilde{Q}$ and $\tilde{P}$ which are normalized as

$$\tau^a \cdot \tau_b = \delta^a_b.$$  (2.42)

Then, since $\tilde{P}$ is a tensor density, $\tilde{Q}$ and $\tilde{P}$ are expressed in terms of them as

$$\tilde{Q} = Q_a \tau^a, \quad \tilde{P} = \sqrt{|\tilde{q}|} P^b \tau_b,$$  (2.43)

where $|\tilde{q}|$ is the determinant of the metric data in $\tilde{Q}$, and $Q_a$ and $P^b$ are spatially constant components.

Let us write the pullback of $\tau^a$ by $f_\lambda$ as

$$f^*_\lambda \tau^a = F^a_b \tau^b.$$  (2.44)

Then from Eq.(2.42) the pullback of $\tau_a$ is written as

$$f^*_\lambda \tau_a = \tau_b (F^{-1})^a_b.$$  (2.45)

Hence the components $Q_a$ and $P^b$ transform as

$$\begin{align*}
(f^*_\lambda \tilde{Q})_a &= Q_b F^b_a, \\
(f^*_\lambda \tilde{P})^a &= (F^{-1})^a_b P^b. 
\end{align*}$$  (2.46) (2.47)

Inserting these expressions into Eq.(2.36), we obtain

$$\Theta = \int_{D_0} d^3 x (f^*_\lambda \sqrt{|\tilde{q}|}) (\hat{Q}_a P^a + (\hat{F} F^{-1})^a_b Q_a P^b)$$

$$= V(Q, \lambda) \hat{Q}_a P^a + \dot{\lambda}^j C^a_{jb} Q_a P^b,$$  (2.48)

where

$$V(Q, \lambda) := \int_{D_0} d^3 x f^*_\lambda \sqrt{|\tilde{q}|},$$  (2.49)

$$C^a_{jb} = \int_{D_0} d^3 x (f^*_\lambda \sqrt{|\tilde{q}|}) (\partial_{\lambda_j} F F^{-1})^a_b.$$  (2.50)

Here note that the matrix $F$ depends on the position as well as on $\lambda$ in general because $f_\lambda$ maps $\tau_a$ to invariant tensors if and only if $f_\lambda$ is in $\text{HPDG}(\tilde{M}, G_{st})$.

Similarly the hamiltonian gets a simple expression in terms of the components $Q_a$ and $P^b$. Since $\mathcal{H}_\perp(Q, P)$ behaves as a scalar density, its pullback to $\tilde{M}$ by $j_\lambda$ is written in terms of the metric and a function $H_\perp(Q_a, P^b)$ as

$$j^*_\lambda \mathcal{H}_\perp(Q, P) = \sqrt{|\tilde{q}|} H_\perp(Q_a, P^b).$$  (2.51)
Further the pullback of the lapse function $\hat{N}$ should be represented by a spatially constant function $N$ on $\tilde{M}$ in the locally homogeneous sector. Hence the Hamiltonian is written as

$$H = \int_M d^3x \hat{N} \mathcal{H}_\perp(Q, P)$$

$$= \int_{D_0} d^3x N(f^*_\chi |\tilde{q}|) H_\perp(Q_a, P_b) = NV(Q, \lambda) H_\perp(Q_a, P_b). \quad (2.52)$$

In particular for the pure gravity system in which $\Phi = (q, p)$, $\tilde{q}$ and $\tilde{p}$ are expressed in terms of the basis of invariant vectors, $X_I$, and its dual basis $\chi^I$ as

$$\tilde{q} = Q_{IJ} \chi^I \otimes \chi^J, \quad \tilde{p} = \sqrt{|\tilde{q}|} P_{IJ} X_I \otimes X_J, \quad (2.53)$$

where $|\tilde{q}|$ is now written as

$$|\tilde{q}| = |Q||\chi|^2. \quad (2.54)$$

For the pullback of the basis

$$f^*_\chi \chi^I = F^I J \chi^J, \quad (2.55)$$

the transformation of the components is written in the matrix form as

$$f^*_\chi Q = t F Q F, \quad (2.56)$$

$$f^*_\chi P = F^{-1} Q t F^{-1}, \quad (2.57)$$

$$f^*_\chi |\tilde{q}| = |F||\chi|\sqrt{|Q|}. \quad (2.58)$$

Hence the canonical 1-form and the Hamiltonian are expressed as

$$\Theta = \Omega(\lambda) \sqrt{|Q|} \left( \text{Tr}(\dot{Q} P) + 2 \dot{\lambda}^j \text{Tr}(C_j(\lambda) Q P) \right), \quad (2.59)$$

$$H = 2\kappa^2 N \Omega(\lambda) \sqrt{|Q|} \left[ \text{Tr}(Q P Q P) - \frac{1}{2} \left( \text{Tr}(Q P) \right)^2 - \frac{1}{4\kappa^2} R(Q) \right], \quad (2.60)$$

where

$$\Omega(\lambda) := \int_{D_0} d^3x |\chi||F|, \quad (2.61)$$

$$C_j(\lambda) := \frac{1}{\Omega(\lambda)} \int_{D_0} d^3x |\chi||F| \partial_{\lambda_j} F F^{-1}, \quad (2.62)$$

and $R(Q)$ is the Ricci scalar curvature.

Finally note that $(Q, P)$ must satisfy the diffeomorphism constraint. This constraint is expressed in terms of these components with respect to the invariant basis as

$$H_I := 2(c^K_{IJ} Q^P_{KJ} + c^K_{JL} Q_{KL} P^L) = 0, \quad (2.63)$$

where $c^K_{JL}$ is the structure constant of $G_{st},$

$$d\chi^I = -\frac{1}{2} c^K_{JK} \chi^J \wedge \chi^K, \quad (2.64)$$

and $c_I := c^I_{JJ}.$
2.3.3 Dynamics of the moduli parameters

As touched upon in the previous subsection, the canonical structure of $\Gamma_{\text{LH,inv}}(M,G)$ often gets degenerate. In such cases the time evolution of moduli parameters is not fully determined by the canonical equation of motion. However, it does not mean that their time evolution is really uncertain, but it just means that the locally homogeneous system is not canonically closed in the generic phase space as the following theorem shows.

**Theorem 2.3** For a locally $G$-homogeneous system $(\Phi, M)$ of type $(\tilde{M}, G_{\max})$, let $(\tilde{\Phi}(t), K(t))$ be a representative trajectory in $\Gamma^{(+)}(\tilde{M}, \tilde{G}) \times \mathcal{M}(M, \tilde{G})$ corresponding to a locally $G$-homogeneous solution $\Phi(t)$ on $M$ to the Einstein equations. Then $K(t)$ is expressed in terms of a family of transformations $f_t \in \text{HPDG}^{(+)}(\tilde{M}, \tilde{G})$ as

$$K(t) = f_t K(t_0) f_t^{-1}. \quad (2.65)$$

As in the previous subsection, let us identify $M$ with $\tilde{M}/K(t_0)$ and let the corresponding covering map be $j_0 : \tilde{M} \to M$. Then there exists a family of transformations $f_t \in \text{Diff}^{(+)}(\tilde{M})$ such that

$$K(t) = f_t K(t_0) f_t^{-1}(\subset \tilde{G}), \quad (2.66)$$

$$\tilde{\Phi}(t) = (f_t)_* j_0^* \Phi(t). \quad (2.67)$$

Hence the Einstein equation

$$\dot{\Phi} = \{\Phi, C_H(\Phi; N)\} \quad (2.68)$$

is written in terms of $\tilde{\Phi}$ and $f_t$ as

$$\partial_t (f_t^* \tilde{\Phi}) = \{f_t^* \tilde{\Phi}, C_H(f_t^* \tilde{\Phi}; N)\}. \quad (2.69)$$

Since the Poisson bracket operation commutes with diffeomorphism and $N$ is spatially constant, this equation is rewritten as

$$\partial_t \tilde{\Phi} + \mathcal{L}_{\eta(t)} \tilde{\Phi} = \{\tilde{\Phi}, C_H(\tilde{\Phi}, N)\}, \quad (2.70)$$

where $\eta(t)$ is the vector field defined by

$$\eta(t)_x = (\partial_t f_t)(f_t^{-1}(x)). \quad (2.71)$$

Since $\partial_t \tilde{\Phi}$ and $\{\tilde{\Phi}, C_H(\tilde{\Phi}, N)\}$ are $\tilde{G}$-homogeneous, it follows from this equation that $\mathcal{L}_{\eta} \tilde{\Phi}$ is also $\tilde{G}$-homogeneous. Hence for any $g \in \tilde{G}$, $\mathcal{L}_g \tilde{\Phi} = g_* \mathcal{L}_{\eta} \tilde{\Phi} = \mathcal{L}_{g_* \eta} \tilde{\Phi}$. Since $\text{Inv}G^{(+)}(\tilde{\Phi}) = \tilde{G}$, this is equivalent to the condition

$$g_* \eta(t) - \eta(t) \in \mathcal{L}(\tilde{G}). \quad (2.72)$$

From this it immediately follows that

$$[\xi, \eta(t)] \in \mathcal{L}(\tilde{G}) \quad \forall \xi \in \mathcal{L}(\tilde{G}). \quad (2.73)$$

This equation implies that $\eta(t)$ induces an automorphism of $\mathcal{L}(\tilde{G}) = \mathcal{L}(\tilde{G}_0)$ where $\tilde{G}_0$ is the connected component of $\tilde{G}$ containing the identity transformation.
We first show that \( f_t \) belongs to \( \text{HPDG}^+(\tilde{M}, \tilde{G}_0) \). For \( \xi \in \mathcal{L}(\tilde{G}) \), let \( \xi(t) \) be a vector field defined by

\[
\xi(t) := \text{Ad}(f_t) \xi = (f_t)_* \xi.
\] (2.74)

By differentiating this equation with respect to \( t \), we obtain

\[
\frac{d}{dt} \xi(t) = -[\eta(t), \xi(t)].
\] (2.75)

Let us introduce a basis \( \xi_I, \xi'_\alpha \) of \( \mathcal{L}(\text{Diff}(\tilde{M})) \) such that \( \xi_I \) gives a basis of \( \mathcal{L}(\tilde{G}_0) \), and expand \( \xi(t) \) in terms of them as

\[
\xi(t) = c^I(t) \xi_I + d^\alpha(t) \xi'_\alpha.
\] (2.76)

Since \( \xi(t) \) belongs to a finite dimensional subspace of \( \mathcal{L}(\text{Diff}(\tilde{M})) \) due to Eq.(2.73), the right-hand side of this equation has well-defined meaning. Then since \( [\eta(t), \xi_I] \in \mathcal{L}(\tilde{G}_0) \), \( d^\alpha(t) \) satisfies a closed evolution equation of the form

\[
\dot{d}^\alpha = f^\alpha_\beta d^\beta.
\] (2.77)

However, since \( d^\alpha(0) = 0 \), we obtain \( d^\alpha(t) = 0 \). Hence \( \text{Ad}(f_t) \xi \in \mathcal{L}(\tilde{G}_0) \) and \( f_t \in \text{HPDG}^+(\tilde{M}, \tilde{G}_0) \).

Next, in order to show that \( f_t \) preserves \( \tilde{G} \), let us define \( \Psi(t) \) by

\[
\Psi(t) := (f_t)_* \Psi_0
\] (2.78)

where \( \Psi_0 \) is arbitrary \( \tilde{G} \)-invariant data. Then for any \( g \in \tilde{G} \), we obtain

\[
\partial_t (g_* \Psi - \Psi) = \mathcal{L}_\eta (g_* \Psi - \Psi) + \mathcal{L}_{g_* \eta - \eta} g_* \Psi.
\] (2.79)

Since \( f_t \) maps \( \tilde{G}_0 \)-invariant data to data with the same invariance and \( \tilde{G}_0 \) is a normal subgroup of \( \tilde{G} \), \( g_* \Psi \) is \( \tilde{G}_0 \)-invariant. Hence from Eq.(2.72), the second term on the right-hand side of this equation vanishes, and we get a closed evolution equation for \( g_* \Psi - \Psi \). If we expand this quantity in terms of a basis \( \sigma_a \) of the \( \tilde{G}_0 \)-invariant fields as \( g_* \Psi - \Psi = \psi^a \sigma_a \), then this equation is written as

\[
\partial_t \psi^a \sigma_a = \psi^a \mathcal{L}_\eta \sigma_a.
\] (2.80)

Since \( \mathcal{L}_\eta \sigma_a \) is again a \( \tilde{G}_0 \)-invariant field, and written as a linear combination of \( \sigma_a \), it gives a closed linear evolution equation for \( \psi^a \). Hence from \( g_* \Psi(0) - \Psi(0) = 0 \), we obtain \( g_* \Psi(t) - \Psi(t) = 0 \), which implies that \( f_t \) maps \( \tilde{G} \)-invariant fields to themselves. Therefore \( f_t \) must be HPDs with respect to \( \tilde{G} \).

This theorem states that the dynamics of the moduli parameters is essentially frozen as pointed out by Tanimoto, Koike and Hosoya[4]. In particular, in the parameterization in which the continuous HPD-freedom with respect to \( \tilde{G} \) is completely removed, the moduli parameters become constants of motion. We will confirm this in the following sections explicitly.
§3 LHS of type $E^3$

In this section we completely determine the canonical structure of locally homogeneous pure gravity systems on compact closed orientable manifolds of type $E^3$.

3.1 Basic properties

3.1.1 $G_{\text{max}}^+$ and HPDG\(^+\)(\(G_{\text{max}}^+\))

The maximal symmetry group of $E^3$ is given by $G_{\text{max}} = \text{Isom}(E^3) = \text{IO}(3)$ and its orientation-preserving subgroup by $G_{\text{max}}^+ = \text{Isom}^+(E^3) = \text{ISO}(3)$. We denote each element of ISO(3),

$$f(x) = Ax + a; \quad A \in \text{SO}(3), \ a \in \mathbb{R}^3$$

by the pair $(a, A)$. In this notation the product of two elements are written as

$$(a, A)(b, B) = (a + Ab, AB). \quad (3.2)$$

The structure of the Lie algebra $\mathcal{L}(\text{ISO}(3))$ of ISO(3) is expressed in terms of the basis

$$T_I = \partial_I, \quad J_I = \epsilon_{IJK} x^J \partial_K$$

as

$$[T_I, T_J] = 0, \quad [J_I, T_J] = -\epsilon_{IJK} T_K, \quad [J_I, J_J] = -\epsilon_{IJK} J_K. \quad (3.4)$$

In terms of this basis a generic element $\phi$ of the automorphism group $\text{Aut}(\mathcal{L}(\text{ISO}(3)))$ is written in the matrix form as

$$\phi \left( \begin{array}{cccc} T_1 & T_2 & T_3 & J_1 & J_2 & J_3 \end{array} \right) = \left( \begin{array}{cccc} T_1 & T_2 & T_3 & J_1 & J_2 & J_3 \end{array} \right) \left( \begin{array}{cc} kR & RV \\ 0 & R \end{array} \right), \quad (3.5)$$

where $k$ is a non-zero constant, $R \in \text{SO}(3)$, and $V$ is a 2-dimensional anti-symmetric matrix. From Eq.(2.15) this automorphism belongs to $\text{EAut}(\mathcal{L}(\text{ISO}(3)))$ if and only if $V$ is written in terms of a vector $c = (c^1, c^2, c^3)$ as

$$V_{IJ} = -\epsilon_{IJK} (R^{-1})^K L c^L, \quad (3.6)$$

and if $k > 0$, it is induced from $f \in \text{HPDG}^+(E^3, \text{ISO}(3))$ given by

$$f(x) = kRx + c. \quad (3.7)$$

3.1.2 Transitive subgroups of $G_{\text{max}}^+$

Conjugate classes of connected transitive subgroups of $G_{\text{max}}^+$ are determined as follows. For a connected transitive subgroup $\tilde{G}$, let $\xi_a = A'_a T_I + B'_a J_I$ be a basis of its Lie algebra. Since $\tilde{G}$ is transitive, the rank of the matrix $(A'_a)$ must be equal to 3. Further from the exact sequence

$$0 \to \mathbb{R}^3 \xrightarrow{j} \text{ISO}(3) \xrightarrow{p} \text{SO}(3) \to 1, \quad (3.8)$$
\( p(\tilde{G}) \), the isotropy group at \( x = 0 \), must be a connected Lie subgroup of \( \text{SO}(3) \), which is either 1 or \( \text{SO}(2) \) or \( \text{SO}(3) \).

First, in the case \( p(\tilde{G}) = 1 \), \( \tilde{G} \) coincides with the normal subgroup \( \mathbb{R}^3 \), which is simply transitive and corresponds to the Bianchi-type I group. Its invariant basis is given by

\[
\tilde{G} = \mathbb{R}^3 : \quad \chi^1 = dx^1, \quad \chi^2 = dx^2, \quad \chi^3 = dx^3. \tag{3.9}
\]

Second, in the case \( p(\tilde{G}) = \text{SO}(2) \), its generator can be put to \( J_3 \) by conjugate transformation in \( \text{ISO}(3) \). Then \( \tilde{G} \cap \mathbb{R}^3 \) must be an invariant subspace of \( \mathbb{R}^3 \) for rotations around \( x^3 \)-axis, and its dimension is larger than one.

If the dimension is two, \( \tilde{G} \cap \mathbb{R}^3 \) should coincide with the subgroup generated by \( T_1 \) and \( T_2 \). Since \( \tilde{G} \) becomes a simply transitive group, by a linear transformation if necessary, \( \xi_I \) can be put in the form \( \xi_1 = T_1, \xi_1 = T_2, \xi_3 = cT_3 - J_3 \). Further the constant \( c \) can be transformed to \( \pm 1 \) by a scaling of the \( x^3 \)-coordinate by a positive constant which is in \( \text{HPDG}^+(E^3, \text{ISO}(3)) \). Thus in this case \( \tilde{G} \) is conjugate to a group generated by

\[
\xi_1 = T_1, \quad \xi_2 = T_2, \quad \xi_3 = -\epsilon T_3 - J_3, \tag{3.10}
\]

where \( \epsilon = \pm 1 \). Its algebra is given by

\[
[\xi_1, \xi_2] = 0, \quad [\xi_3, \xi_1] = \xi_2, \quad [\xi_3, \xi_2] = -\xi_1. \tag{3.11}
\]

and a generic element of this group is written as \((a, R_3(\epsilon a^3))\). Since this group is isomorphic to the Bianchi-type VII(0) group, we denote it as \( \text{VII}^\epsilon(0) \). The corresponding invariant basis is given by

\[
G = \text{VII}^\epsilon(0) : \quad \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} = R(-\epsilon x^3) \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}, \quad \chi^3 = dx^3. \tag{3.12}
\]

Note that if we allow orientation-reversing transformations, \( \text{VII}^+(0) \) and \( \text{VII}^-(0) \) become conjugate with each other.

On the other hand if the dimension of \( \tilde{G} \cap \mathbb{R}^3 \) is three, we obtain a four-dimensional transitive subgroup isomorphic to \( \mathbb{R}^3 \times \text{SO}(2) \). In terms of the basis

\[
\xi_1 = T_1, \quad \xi_2 = T_2, \quad \xi_3 = T_3, \quad \xi_4 = J_3, \tag{3.13}
\]

the structure of its Lie algebra is given by

\[
[\xi_I, \xi_J] = 0 \quad (I, J = 1, 2, 3),
[\xi_4, \xi_1] = -\xi_2, \quad [\xi_4, \xi_2] = \xi_1, \quad [\xi_4, \xi_3] = 0. \tag{3.14}
\]

Note that this subgroup contains both of the simply transitive groups \( \mathbb{R}^3 \) and \( \text{VII}^\pm(0) \).

Finally in the case \( p(\tilde{G}) = \text{SO}(3) \), \( \tilde{G} \) obviously coincides with \( \text{ISO}(3) \).

Next we determine invariance groups with disconnected components. The maximal connected subgroup of each transitive invariance group must be isomorphic to one of the connected subgroups determined above. Hence after an appropriate conjugate transformation, \( \tilde{G} \) is decomposed as

\[
0 \rightarrow \tilde{G}_0 \overset{i}{\rightarrow} \tilde{G} \overset{p}{\rightarrow} H \rightarrow 1 \quad \text{(exact)}, \tag{3.15}
\]

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where $H$ is a discrete subgroup of $\text{SO}(3)$. Each element of $H$ is expressed as a rotation around some vector $n$, $R_n(\theta)$. It is easily shown that if covering data $(\tilde{q}, \tilde{p})$ on $E^3$ is invariant by the rotation $\theta \neq \pi (\text{mod} 2\pi)$, it is invariant under $R_n(\theta)$ with arbitrary value of $\theta$. Hence $\theta$ should be equal to $\pi$, which implies that $H$ is isomorphic to a subgroup of the dihedral group $D_2 = \{1, R_1(\pi), R_2(\pi), R_3(\pi)\}$, where $R_j(\pi)(j = 1, 2, 3)$ is the rotation matrix of angle $\pi$ around the $x^j$-axis.

In the case $\tilde{G}_0 = R^3$, it follows from this that $H$ is transformed to either 1, $\{1, R_3(\pi)\}$ or $D_2$ by a $\text{SO}(3)$-transformation, which leaves $R^3$ invariant. On the other hand, in the case $\tilde{G}_0 = \text{VII}^{\pm}(0)$, $R_n(\pi)$ leaves $\tilde{G}_0$ invariant only when $n$ is parallel to or orthogonal to the $x^3$-axis. Hence by a rotation around the $x^3$-axis, which leaves $\tilde{G}_0$ invariant, $H$ is transformed to either 1, $\{1, R_1(\pi)\}$, $\{1, R_3(\pi)\}$ or $D_2$. Similarly in the case $\tilde{G}_0 = \text{R}^3 \times \text{SO}(2)$, $n$ must be parallel to the $x^3$-axis, and $H$ is transformed to either 1 or $\{1, R_1(\pi)\}$ by a rotation around the $x^3$-axis. Thus every transitive invariance group contained in $\text{ISO}(3)$ is conjugate to one of the following groups with respect to $\text{HPDG}^+(E^3, \text{ISO}(3))$: $\text{R}^3$, $\text{R}^3 \times \{1, R_1(\pi)\}$, $\text{R}^3 \times D_2$, $\text{VII}^{\prime}(0)$, $\text{VII}^{\prime}(0) \times \{1, R_1(\pi)\}$, $\text{VII}^{\prime}(0) \times D_2$, $\text{R}^3 \times \text{SO}(2)$, $\text{R}^3 \times \text{SO}(2) \times \{1, R_1(\pi)\}$, and $\text{ISO}(3)$.

### 3.1.3 Topology of orientable compact quotients

All the compact closed manifolds modeled on $(E^3, \text{IO}(3))$ admit Seifert fibration with $\chi = 0$ and $e = 0$, where $\chi$ is the Euler number of the base orbifold and $e$ is the Euler number of the Seifert bundle $[1]$. From this it is shown that they are all covered by three torus $T^3$. In particular the orientable ones are diffeomorphic to either $T^3$, $T^3/\mathbb{Z}_2$, $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, $T^3/\mathbb{Z}_3$, $T^3/\mathbb{Z}_4$, and $T^3/\mathbb{Z}_6$, whose fundamental groups and their representation in $\text{IO}(3)$ are listed in Table $[4]$.

We need the volume $\Omega_K$ of the fundamental region $D_K$ for the action of each $K = j^\sharp(\pi_1(M))$ in writing down the expression for the canonical 1-form and the Hamiltonian (see Eqs. (2.39) - (2.40)). It is determined as follows.

First for $M = T^3$, $D_K$ is given by $x = ua + vb + wc$ with $0 \leq u, v, w \leq 1$. Its volume is given by $\Omega_K = |(a \times b) \cdot c|$.

Second for $M = T^3/\mathbb{Z}_k(k = 2, 3, 4, 6)$, since $\alpha$, $\beta$ and $\gamma$ generate the fundamental group of the covering $T^3$, $D_K$ is given by $x = ua + vb + wc(0 \leq u, v, w \leq 1)$ where $c = (1 + R + \cdots + R^{k-1})c/k$. Hence its volume is again given by $\Omega_K = |(a \times b) \cdot c|$.

Finally for $M = T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, $\alpha^2$, $\beta^2$ and $\gamma^2$ generate $\pi_1(T^3)$, and the quotient group $\pi_1(M)/\pi_1(T^3)$ is isomorphic to the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the cosets $[\alpha]$ and $[\beta]$. Hence the volume of $D_K$ is given by the fourth of that of the covering torus. In terms of $\tilde{a} = (1 + R_\alpha)a/2$, $\tilde{b} = (1 + R_\beta)b/2$ and $\tilde{c} = (1 + R_\gamma)c/2$, it is expressed as $\Omega_K = 2|\tilde{a}| |\tilde{b}| |\tilde{c}|$.

### 3.2 $\tilde{G}_0 = R^3$

The automorphism group of $R^3$ coincides with $\text{GL}(3, R)$, and its action on the Lie algebra is expressed as

$$\phi(\xi) = \xi_1 A^I I, \quad A \in \text{GL}(3, R).$$  \hspace{1cm} (3.16)

For $\det A > 0$ it is induced from $f \in \text{HPDG}^+(E^3, R^3)$ given by

$$f(x) = Ax + a \quad a \in R^3.$$  \hspace{1cm} (3.17)
Space Fundamental group and representation

\( T^3 \)  
\(< \alpha, \beta, \gamma | [\alpha, \beta] = 1, [\beta, \gamma] = 1, [\gamma, \alpha] = 1 >\)  
\( \alpha = (a, 1), \beta = (b, 1), \gamma = (c, 1); \)  
\((a, b, c) \in \text{GL}(3, \mathbb{R}).\)

\( T^3 / \mathbb{Z}_2 \)  
\(< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} \alpha = 1, \gamma \beta \gamma^{-1} \beta = 1 >\)  
\( \alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma); \)  
\( R_\gamma = R_{axb}(\pi), \)  
\((a, b, c) \in \text{GL}(3, \mathbb{R}).\)

\( T^3 / \mathbb{Z}_2 \times \mathbb{Z}_2 \)  
\(< \alpha, \beta, \gamma | \alpha \beta \gamma = 1, \alpha \beta^2 \alpha^{-1} \beta^2 = 1, \beta \alpha^2 \beta^{-1} \alpha^2 = 1 >\)  
\( \alpha = (a, R_\alpha), \beta = (b, R_\beta), \gamma = (c, R_\gamma); \)  
\((a + R_\alpha a, b + R_\beta b, c + R_\gamma c) \in \text{GL}(3, \mathbb{R}).\)  
\( R_\alpha, R_\beta \) and \( R_\gamma \) are rotations of the angle \( \pi \) around mutually orthogonal axes.  
\( R_\beta a + R_\alpha b + R_\alpha c = 0.\)

\( T^3 / \mathbb{Z}_3 \)  
\(< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \alpha^{-1} \beta^{-1} >\)  
\( \alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma); \)  
\( R_\gamma = R_{axb}(\frac{2\pi}{3}), \)  
\( b = R_\gamma a, (a, b, c) \in \text{GL}(3, \mathbb{R}).\)

\( T^3 / \mathbb{Z}_4 \)  
\(< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta^{-1}, \gamma \beta \gamma^{-1} = \alpha >\)  
\( \alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma); \)  
\( R_\gamma = R_{axb}(\frac{\pi}{2}), \)  
\( b = R_\gamma a, (a, b, c) \in \text{GL}(3, \mathbb{R}).\)

\( T^3 / \mathbb{Z}_6 \)  
\(< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \alpha^{-1} \beta >\)  
\( \alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma); \)  
\( R_\gamma = R_{axb}(\frac{\pi}{3}), \)  
\( b = R_\gamma a, (a, b, c) \in \text{GL}(3, \mathbb{R}).\)

Table 2: Fundamental groups and their representation in \( \text{Isom}(E^3) \) of compact closed orientable 3-manifolds of type \( E^3 \)
Since the invariant basis \((3.9)\) transforms by \(f\) as
\[
f^* \chi^I = A^I J \chi^J,
\]
the components of \(R^3\)-invariant data \(\tilde{\Phi} = (\tilde{q}, \tilde{p})\) with respect to the invariant basis, \(Q = (Q_{IJ})\) and \(P = (P^I J)\), transforms as
\[
(f^* Q) = t A Q A, \quad (f^* P) = A^{-1} P^I A^{-1}.
\]

With the helps of these formula we can show that \(R^3\)-invariant data always have higher symmetry. First from Eq.\((3.19)\), \(Q\) can be always transformed to the unit matrix \(I_3\) by a HPD in HPDG\(^+\)(\(E^3, R^3\)). This leaves residual HPDs given by \(A \in SO(3)\), which can be used to diagonalize \(P\) as \([P^1, P^2, P^3]\). If all of these eigenvalues of \(P\) are distinct with each other, \(\text{InvG}^+(\tilde{\Phi}) \cap G_{\text{max}} = R^3 \times D_2\). Since it is clear that there does not exist \(f \in \text{HPDG}^+(E^3, R^3)\) such that \(\text{InvG}^+(f, \tilde{\Phi}) \cap G_{\text{max}}\) increases, from Prop.\((2.3)\) it follows that \(\text{InvG}^+(\tilde{\Phi}) = R^3 \times D_2\). On the other hand, if two of the eigenvalues coincide, \(\text{InvG}^+(\tilde{\Phi})\) is obviously equal to or larger than a group isomorphic to \(R^3 \times SO(2)\). Thus the smallest invariant group containing \(R^3\) is conjugate to \(R^3 \times D_2\).

The HPDs for \(R^3 \times D_2\) is easily determined from HPDG\(^+\)(\(E^3, R^3\)) by finding \(f\) such that \(f D_2 f^{-1} \subset R^3 \times D_2\). The result is
\[
f \in \text{HPDG}^+(E^3, R^3 \times D_2) \Leftrightarrow A = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix} B
\]
where \(p, q,\) and \(r\) are positive constants and \(B\) is an element of the octahedral group. That is, \(A\) must be a regular matrix with positive determinant whose components vanish except for three entries.

If \(Q\) and \(P\) are invariant under \(R^3 \times D_2\), they must be given by the diagonal matrices, \(Q = [Q_1, Q_2, Q_3]\) and \(P = [P^1, P^2, P^3]\). By a transformation in HPDG\(^+\)(\(E^3, R^3 \times D_2\)) they can be put in the form \(Q = I_3\) and \(P = [Q_1 P^1, Q_2 P^2, Q_3 P^3]\). Hence by the same argument as above, we obtain
\[
\Gamma_\Pi(E^3, R^3 \times D_2) = \{ Q = [Q_1, Q_2, Q_3], P = [P^1, P^2, P^3] \mid Q_I P^I \neq Q_J P^J (I \neq J) \}. \tag{3.21}
\]

From the general formula \((2.60)\) we can easily write down the Hamiltonian for this system. Let us define the variable \(\alpha\) by
\[
\epsilon^{3\alpha} := |F|(Q_1 Q_2 Q_3)^{1/2}, \tag{3.22}
\]
where \(F\) is the matrix defined by Eq.\((2.54)\), which turns out to be always spatially constant in the present case. Then, since the metric is flat (\(R(Q) = 0\)) and \(|\chi| = 1\), the Hamiltonian is written as
\[
H = \frac{\kappa^2}{12 \Omega} N \epsilon^{-3\alpha} (-p^2_\alpha + p^2_+ + p^2_-), \tag{3.23}
\]
where \(\Omega\) is the coordinate volume of a standard fundamental region to be defined later, and \(p_\alpha\) and \(p_\pm\) are the momentum variables defined by
\[
p_\alpha := 2 \Omega \epsilon^{3\alpha} (Q_1 P^1 + Q_2 P^2 + Q_3 P^3), \tag{3.24}
p_- := 2 \sqrt{3} \Omega \epsilon^{3\alpha} (Q_1 P^1 - Q_2 P^2), \tag{3.25}
p_+ := 2 \Omega \epsilon^{3\alpha} (Q_1 P^1 + Q_2 P^2 - 2 Q_3 P^3). \tag{3.26}
\]
The difference of topology affects the Hamiltonian only through the value of Ω.

On the other hand the structure of the canonical 1-form is sensitive to topology. Next we determine it for each topology of M. Note that K = j^2π1(M) is contained in \( \tilde{G} = R^3 \times D_2 \) only when M is either \( T^3 \), \( T^3/Z_2 \) or \( T^3/Z_2 \times Z_2 \) because \( R_a(2\pi/k)(k = 3, 4, 6) \) is not contained in it(see Table 2).

### 3.2.1 \( T^3 \)

Following the procedure explained in §2.2.3, we fix the freedom of continuous HPDs by reducing the moduli space in order to obtain the expression for \( \Gamma_{\text{LH,inv}}^+(T^3, R^3 \times D_2) \) of the form (2.24). From Table 2 the moduli space \( M(T^3, R^3 \times D_2) \) is parameterized by a matrix \( \begin{pmatrix} a & b & c \end{pmatrix} \in GL(3, R) \) apart from the discrete \( GL(3, Z) \) modular transformations. By HPDG\(^+\)(\( E^3, R^3 \times D_2 \)) it can be always put in the form

\[
F = \begin{pmatrix} 1 & X & Y \\ 0 & 1 & Z \\ 0 & 0 & 1 \end{pmatrix} R(\phi, \theta, \psi),
\]

where \( X, Y, \) and \( Z \) are arbitrary real constants, and \( R(\phi, \theta, \psi) = R_3(\psi)R_1(\theta)R_3(\phi) \) is a SO(3) matrix parameterized by the Euler angles. By this gauge fixing the HPDs reduced to a discrete subgroup. Note that it does not fix the discrete modular transformations.

Let us take \( K_0 \cong \{ (l, m, n) \mid l, m, n \in Z \} \) corresponding to the parameters \( X = Y = Z = 0 \) and \( R(\theta, \phi, \psi) = 1 \) as the base point in the moduli space, and identify \( T^3 \) with \( E^3/K_0 \). Then the deformation map is given by \( f_\lambda(x) = Fx \), and the volume Ω of the fundamental region \( D_0 \) is unity. Hence from Eq.(2.25) the canonical 1-form is expressed as

\[
\Theta = e^{3\alpha}[\dot{Q}_1P^1 + \dot{Q}_2P^2 + \dot{Q}_3P^3 + 2\text{Tr}(\dot{F}F^{-1}QP)].
\]

From the expression for \( F \) given above, after a short calculation, we find that it is put in the canonical form

\[
\Theta = \dot{\alpha}p_\alpha + \dot{\beta}_+p_+ + \dot{\beta}_-p_- + \dot{\theta}p_\theta + \dot{\phi}p_\phi + \dot{\psi}p_\psi,
\]

if we change the variables \( Q_1, Q_2 \) and \( Q_3 \) to \( \alpha \) and \( \beta_\pm \) defined by

\[
Q_1 = e^{2(\alpha + \beta_+ + \sqrt{3}\beta_-)}, \quad Q_2 = e^{2(\alpha + \beta_+ - \sqrt{3}\beta_-)}, \quad Q_3 = e^{2(\alpha - 2\beta_+)},
\]

and introduce the momentum variables conjugate to \( \phi, \theta \) and \( \psi \) by

\[
\begin{pmatrix} p_\phi \\ p_\theta \\ p_\psi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta \cos \phi & \sin \theta \sin \phi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}}p_-X \\ -\frac{1}{2\sqrt{3}}p_-(Y + XZ) - \frac{1}{2}p_+(Y - XZ) \\ -\frac{1}{2\sqrt{3}}p_- + \frac{1}{2}p_+ \end{pmatrix}.
\]
3.2.2 $T^3/Z_2$

For $T^3/Z_2$, as given in Table 2, the generator $\gamma$ is represented by a glide rotation. In order for this to be contained in $R^3 \times D_2$, the corresponding rotation matrix $R_\gamma$ coincides with either $R_1(\pi)$, $R_2(\pi)$ or $R_3(\pi)$. These three cases are connected with each other by HPDs because HPDG$^+\left(E^3, R^3 \times D_2\right)$ contains rotations exchanging two of $x^1$, $x^2$ and $x^3$. Hence we can choose the gauge such that $\alpha = (a, R_1(\pi))$, $\beta = (b, R_2(\pi))$ and $\gamma = (c, R_3(\pi))$ where $a^3 = b^3 = 0$. Further, since $\gamma$ changes by a translation $(d, 1)$ as

\[(d, 1)(c, R_3(\pi))(d, 1)^{-1} = (c + d - R_3(\pi)d, R_3(\pi))\]

we can put $c^1$ and $c^2$ to zero. Finally by automorphisms (i.e., modular transformation) $\alpha \leftrightarrow \beta$ and $\gamma \rightarrow \gamma^{-1}$ if necessary, det $\left( \begin{array}{ccc} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right)$ and $c^3$ can be made positive.

Thus the matrix $\left( \begin{array}{ccc} a & b & c \end{array} \right)$ parametrizing $M(T^3/Z_2, R^3 \times D_2)$ is reduced by HPDs to $M_0(T^3/Z_2, R^3 \times D_2)$ parameterized by

\[F = \left( \begin{array}{ccc} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) R_3(\phi),\]

where $X$ and $\phi$ are arbitrary real numbers. The residual HPDs form a discrete group. This gauge fixing reduces the freedom of modular transformations to $SL(2, \mathbb{Z})$.

If we take $K_0$ with $X = 0$ and $\phi = 0$ as the base point of the reduced moduli space, $f_\lambda(x) = Fx$ gives the deformation map from $K_0$ to $K$ corresponding to the above moduli matrix $F$. Further from the argument in §3.1.3, the fundamental region $D_0$ and its volume coincide with those in the case $M = T^3$. Hence the canonical 1-form is obtained from that in the previous case by putting $Y = Z = 0$ and $\theta = \psi = 0$:

\[\Theta = \dot{\alpha} p_{\alpha} + \dot{\beta}_+ p_+ + \dot{\beta}_- p_- + \dot{\phi} p_\phi,\]

where

\[p_\phi = \frac{1}{\sqrt{3}} p_- X.\]

3.2.3 $T^3/Z_2 \times Z_2$

In this case, from Table 2 all the generators are represented by glide rotations, whose axes are mutually orthogonal. Since the corresponding rotation matrices must be taken from $D_2$, by the HPDs producing permutations in $D_2$, the generators can be put in the form

\[\alpha = (a, R_1(\pi)), \quad \beta = (b, R_2(\pi)), \quad \gamma = (c, R_3(\pi)).\]

Then the constraint $R_\beta a + R_\gamma b + R_\alpha c = 0$ gives

\[c^1 = a^1 + b^1, \quad a^2 = b^2 + c^2, \quad b^3 = a^3 + c^3.\]
Hence, if we make $b^1$, $c^2$ and $a^3$ vanish by the transformation corresponding to a translation $(d, 1)$,

\[
(a^1, a^2, a^3) \rightarrow (a^1, a^2 + 2d^2, a^3 + 2d^3),
\]

\[
(b^1, b^2, b^3) \rightarrow (b^1 + 2d^1, b^2, b^3 + 2d^3),
\]

\[
(c^1, c^2, c^3) \rightarrow (c^1 + 2d^1, c^2 + 2d^2, c^3),
\]

we obtain $a^1 = c^1$, $a^2 = b^2$ and $b^3 = c^3$. Since we can make $b^2c^3$ positive by the modular transformation $\beta^{-1} \rightarrow \beta$ and $\beta^2\gamma \rightarrow \gamma$, these constants can be put to 1 by rescalings of $x^1$, $x^2$ and $x^3$ in $\text{HPDG}^+(E^3, R^3 \times D_2)$. Thus the moduli space is reduced to a single point corresponding to

\[
\begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}.
\]

(3.43)

There exists no residual HPD or modular transformation.

From the argument in §3.1.3, the volume of the fundamental region is given by $\Omega = 2$. Hence by putting $F = I_3$ in Eq.(3.28), we obtain

\[
\Theta = \dot{\alpha}p_\alpha + \dot{\beta}p_\beta + \dot{\gamma}p_\gamma,
\]

(3.44)

where $\alpha$ and $\beta_\pm$ are defined by the same relations Eqs.(3.30)-(3.32) as in the previous case.

3.3 $\tilde{G}_0 = \text{VII}^\pm(0)$

The automorphism group of $\text{VII}^0(0)$ is isomorphic to $R_+ \tilde{\times} \text{IO}(2)$ and its element $\phi$ is represented as

\[
\phi(\xi_I) = \xi_J T^J_I, \quad T = \{1, R_1(\pi)\} \times \begin{pmatrix}
kR(\theta) & -c^2 \\
0 & c^1
\end{pmatrix},
\]

(3.45)

where $k > 0$, and $\theta$, $c^1$ and $c^2$ are arbitrary real numbers. It is induced from $f \in \text{HPDG}^+(E^3, \text{VII}^0(0))$ given by

\[
f(\mathbf{x}) = \{1, R_1(\pi)\} \left[ \begin{pmatrix}
kR(\theta) & 0 \\
0 & 0
\end{pmatrix} \mathbf{x} + \begin{pmatrix}
c^1 \\
c^2 \\
c^3
\end{pmatrix} + R_3(e^{-c^3}) \begin{pmatrix}
d^1 \\
d^2 \\
d^3
\end{pmatrix} \right].
\]

(3.46)

By this HPD the invariant basis (3.12) transforms as

\[
f^* \chi^I = A^I_J \chi^J,
\]

(3.47)

where

\[
A = \{1, R_1(\pi)\} \times R_3(-e^{-c^3}) \begin{pmatrix}
kR(\theta) & -ev^2 \\
0 & ev^1
\end{pmatrix}.
\]

(3.48)

As in the case $\tilde{G}_0 = R^3$, we can show with the helps of these formula that $\text{VII}(0)$-homogeneous data always have higher symmetries. To see this, first note that we can
transform the matrix $Q$ representing the components of the metric data $\tilde{q}$ with respect to the invariant basis to the diagonal matrix $[Q_1, 1/Q_1, Q_3]$ by the above HPDs. After this transformation, the diffeomorphism constraint $[\ref{2.63}]$ is expressed in terms of the coefficient matrix $P$ of the momentum data $\tilde{p}$ as

$$P^{13} = P^{23} = 0, \quad (Q_1 - Q_1^{-1})P^{12} = 0. \quad (3.49)$$

If $Q_1 \neq 0$, this implies that $P$ is diagonal. Hence $\text{InvG}(\tilde{\Phi})$ contains $\text{VII}^{(e)}(0)\tilde{\times}D_2$ at least. On the other hand, if $Q_1 = 1$, by the residual HPDs of the form $A = R_3(-\epsilon c^2)$, we can put $P$ diagonal. Hence the same conclusion holds also in this case.

As in the case of $\tilde{G}_0 = R^3$, HPDG$^+(E^3, \text{VII}^{(e)}\tilde{\times}D_2)$ is obtained from $f \in \text{HPDG}^+(E^3, \text{VII}^{(e)})$ such that $fD_2f^{-1} \subset \text{VII}^{(e)}(0)\tilde{\times}D_2$. Its explicit form is given by

$$f(x) = \{1, R_1(\pi)\} \begin{pmatrix} kR(\epsilon c^3 + \frac{l}{2}) & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix}, \quad (3.50)$$

where $l$ is an arbitrary integer. This induces the following transformation of the invariant basis:

$$f^*\chi^l = A^l j\chi^j; \quad A = \{1, R_1(\pi)\} \begin{pmatrix} kR(l\pi/2) & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.51)$$

Covering data invariant under $\text{VII}^{(e)}(0)\tilde{\times}D_2$ are represented by diagonal matrices $Q = [Q_1, Q_2, Q_3]$ and $P = [P^1, P^2, P^3]$. From Prop.$\ref{2.24}$ it is easily shown that the data has higher symmetries if and only if $Q_1 = Q_2$ and $P^1 = P^2$. Hence $\Gamma^+_{\text{H.D}}$ for $\tilde{G} = \text{VII}^{(e)}(0)\tilde{\times}D_2$ is given by

$$\Gamma^+_{\text{H.D}}(E^3, \text{VII}^{(e)}(0)\tilde{\times}D_2) = \{ Q = [Q_1, Q_2, Q_3], P = [P^1, P^2, P^3] \mid Q_1 \neq Q_2 \quad \text{or} \quad P^1 \neq P^2 \}. \quad (3.52)$$

Now we determine explicit expressions for the canonical 1-form and the Hamiltonian of $\Gamma^+_{\text{H.inv}}(M, \text{VII}^{(e)}(0)\tilde{\times}D_2)$.

### 3.3.1 $T^3$

All elements of $\pi_1(T^3)$ are represented by translations. However, since each element of $\tilde{G} = \text{VII}^{(e)}(0)\tilde{\times}D_2$ is expressed in the form $(a, R_3(\epsilon a^3))$ or its product with $R_1(\pi)$, a translation $(a, 1)$ is contained in $\tilde{G}$ if and only if $a^3$ is an integer multiple of $\pi$. Hence the matrix

$$\begin{pmatrix} a & b & c \end{pmatrix}$$

parametrizing the moduli space must take the form

$$\begin{pmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ l\pi & m\pi & n\pi \end{pmatrix}, \quad (3.53)$$

where $l, m, n \in \mathbb{Z}$. However, since for any integer vector $(l, m, n)$ there exists $C \in \text{GL}(3, \mathbb{Z})$ such that $(l, m, n)C = (0, 0, k)$ where $k > 0$ is the greatest common divisor of $l, m$ and
n, we can always put \( l = m = 0 \) and \( n > 0 \) by modular transformations. Further by the modular transformation \( \alpha \leftrightarrow \beta \) if necessary, we can make \( \det \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \) positive. Hence by applying HPDs of the form (3.50), we can finally put \( \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \) in the form

\[
B = \begin{pmatrix} X & Y & Z \\ 0 & X^{-1} & W \\ 0 & 0 & n\pi \end{pmatrix},
\]

where \( X > 0 \), \( n \) is a positive integer, and \( Y, Z \) and \( W \) are arbitrary real numbers. This completely fixes the HPD freedom of the moduli parameters except in the subspace \( Y = 0 \), which is invariant by the residual HPDs \( R_3(\pm \pi/2) \). These residual HPD transformations produce a singularity in the invariant phase space with the structure of corned lens space. On the other hand the freedom of modular transformations represented by upper triangle matrix in \( SL(3, \mathbb{Z}) \) still remains. Since these residual modular transformations do not change the value of \( n \), the invariant phase space \( \Gamma \text{ inv}(T^3, \text{VII}(0) \sim D_2) \) consists of two families (VII\( \pm \)(0)) each of which has countably infinite number of connected components.

To find the expression for \( \Theta \), let us take \( K_0 \) corresponding to \( X = 1 \) and \( Y = Z = W = 0 \) as the base point of the moduli space. Then \( f_\lambda \) defined by

\[
f_\lambda(x) = Ax; \quad A = \begin{pmatrix} X & Y & Z \\ 0 & X^{-1} & W \\ 0 & 0 & 1 \end{pmatrix}
\]

gives the deformation map from \( K_0 \) to \( K \) corresponding to the matrix \( B \). The fundamental region \( D_0 \) is given by \( 0 \leq x^1, x^2 \leq 1 \) and \( 0 \leq x^3 \leq n\pi \), and its volume by \( \Omega = n\pi \).

Since \( f_\lambda \) transforms the invariant basis as

\[
f^*_\lambda \chi^I = F^I_J \chi^J; \quad F = R_3(-\varepsilon x^3)AR_3(\varepsilon x^3),
\]

the last term in Eq.(2.59) is expressed as

\[
2Tr(F^{-1}PQ) = \left[ \frac{\dot{X}}{X} \cos(2\varepsilon x^3) + (XY - Y\dot{X}) \sin(2\varepsilon x^3) \right] (Q_1P^1 - Q_2P^2).
\]

Though it is not non-zero, its integration over \( D_0 \) vanishes. Hence, noting that \( |\chi| = 1 \) and \( |F| = 1 \), we obtain from Eq.(2.59) the following expression for \( \Theta \):

\[
\Theta = \Omega \sqrt{Q_1Q_2Q_3(\dot{Q}_1P^1 + \dot{Q}_2P^2 + \dot{Q}_3P^3)},
\]

which is diagonalized by the change of variables \( (3.24)-(3.25) \) and \( (3.30)-(3.32) \) as

\[
\Theta = \dot{\alpha}p_\alpha + \dot{\beta}_+ p_+ + \dot{\beta}_- p_-.
\]

Thus the canonical structure is completely degenerate in the 4-dimensional moduli sector.

For the present system the Ricci scalar curvature of the metric \( \tilde{q} \) is given by

\[
R = -\frac{1}{2Q_3} \left( \frac{Q_1}{Q_2} + \frac{Q_2}{Q_1} - 2 \right) = -2e^{-2\alpha+4\beta_+} \sinh^2(2\sqrt{3}\beta_-).
\]

Hence from Eq.(2.60) the Hamiltonian is expressed as

\[
H = \frac{\kappa^2}{12\Omega} Ne^{-3\alpha} \left[ -p_\alpha^2 + p_+^2 + p_-^2 + \frac{12Q^2}{\kappa^4} e^{4(\alpha+\beta_+)} \sinh^2(2\sqrt{3}\beta_-) \right]
\]
3.3.2 $T^3/Z_2$

Since each transformation in $\text{VII}^e(0)$ is a glide rotation around the $x^3$-axis, the rotation matrix $R_\gamma$ associated with the generator $\gamma$ in $\text{VII}^e(0) \times D_2$ must be of the form $R_3(\theta) R_1(\pi) (I = 1, 2) \text{ or } R_3(\pi)$. The former can be always transformed to $R_2(\pi)$ by some transformation in $\text{VII}^e(0)$ because they can be written as $R_3(\theta/2) R_1(\pi) R_3(-\theta/2)$, and $R_2(\pi) = R_3(\pi/2) R_1(\pi) R_3(-\pi/2)$. On the other hand, there exists no transformation in $\text{HPDG}^+(E^3, \text{VII}^e(0) \times D_2)$ which transforms $R_2(\pi)$ to $R_3(\pi)$. Hence in this case the moduli space $\mathcal{M}(T^3/Z_2, \text{VII}^e(0) \times D_2)$ is divided into two disconnected families.

\textbf{a) $R_\gamma = R_3(\pi)$:} In this case $a^3 = b^3 = 0$, and $c^3$ must be written as $n\pi$ with non-vanishing integer $n$ as in the previous case. On the other hand $c^1$ and $c^2$ can be put to zero by translations along $x^1 - x^2$ plane. Further by modular transformations, $\alpha \leftrightarrow \beta$ and $\gamma \rightarrow \gamma^{-1}$, det $\begin{pmatrix} a & b & c \\ \end{pmatrix}$ and $n$ can be made positive. Hence by HPDs the moduli matrix $\begin{pmatrix} a & b & c \end{pmatrix}$ can be put into the canonical form

$$ B = \begin{pmatrix} X & Y & 0 \\ 0 & X^{-1} & 0 \\ 0 & 0 & n\pi \end{pmatrix}, \quad (3.62) $$

where $X > 0$, $n$ is a positive integer and $Y$ is an arbitrary real number. There remains no residual freedom of HPDs except in the subspace $Y = 0$, while there remains residual modular transformations corresponding $\alpha^p \beta \rightarrow \beta$ with $p \in Z$.

This moduli matrix is the special case $Z = W = 0$ of that in the previous case. Hence the canonical structure is degenerate in the 2-dimensional moduli sector, and the the value of $\Omega$ and the forms of $\Theta$ and $H$ are exactly the same as in the previous case.

\textbf{b) $R_\gamma = R_2(\pi)$:} In this case the moduli matrix $\begin{pmatrix} a & b & c \end{pmatrix}$ takes the form

$$ \begin{pmatrix} a^1 & b^1 & c^1 \\ 0 & 0 & c^2 \\ l\pi & n\pi & m\pi \end{pmatrix}, \quad (3.63) $$

where $l, m$ and $n$ are integers. $l$ can be put to zero by unimodular transformations among $\alpha$ and $\beta$, and $m$ to 0 or 1 by a transformation in $\text{VII}^e(0) \times D_2$;

$$ (d, 1) \begin{pmatrix} (c^1, c^2, m\pi), R_2(\pi) \end{pmatrix} (d, 1)^{-1} = \begin{pmatrix} (c^1, c^2, m\pi + k\pi), R_2(\pi) \end{pmatrix}, \quad (3.64) $$

where $d = (0, 0, k\pi)$. Further by modular transformations $\alpha \rightarrow \alpha^{-1}$, $\beta \rightarrow \beta^{-1}$ and $\gamma \rightarrow \gamma^{-1}$ if necessary, $a^1$ and $c^2$ and $n$ can be made positive. Finally by translation along the $x^1$-axis and HPDs, $\begin{pmatrix} a & b & c \end{pmatrix}$ is transformed to

$$ B = \begin{pmatrix} X & Y & \frac{m Y}{n} \\ 0 & 0 & X^{-1} \\ 0 & n\pi & m\pi \end{pmatrix}, \quad (3.65) $$

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where $X > 0$, $n$ is a positive integer, $m = 0, 1$ and $Y$ is an arbitrary real number. As in the case a), this gauge fixing leaves only the residual modular transformations isomorphic to $\mathbb{Z}$.

Let us take $K_0$ with $X = 1$ and $Y = 0$ as the base point for each of the connected component of the reduced moduli space $\mathcal{M}_0(T^3/\mathbb{Z}_2, \text{VII}^{(e)}(0) \times D_2)$. Then $f_\lambda$ defined by

$$f(x) = Ax; \quad A = \begin{pmatrix}
X & 0 & Y/n\pi \\
0 & 1/X & 0 \\
0 & 0 & 1
\end{pmatrix}$$

maps $K_0$ to $K$ corresponding to the generic matrix $B$. This is again the special case obtained from that for $M = T^3$ by putting $Y \rightarrow 0$, $Z \rightarrow Y/n\pi$ and $W \rightarrow 0$. Hence the canonical structure is completely degenerate in the 2-dimensional moduli sector. $\Omega$, $\Theta$ and $H$ are the same as those in the previous case.

### 3.3.3 $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$

As in the case of $\tilde{G} = \mathbb{R}^3 \times D_2$, three generators can be put into the form

$$\alpha = (a, R_1(\pi)), \quad \beta = (b, R_2(\pi)), \quad \gamma = (c, R_3(\pi)), \quad (3.67)$$

where $(a \ b \ c)$ is expressed in the present case as

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} a^1 & b^1 & c^1 \\
               a^2 & b^2 & c^2 \\
l\pi & m\pi & n\pi \end{pmatrix}$$

with integers $l$, $m$ and $n$. As in §3.2.3, $b^1$ and $c^2$ can be put to zero by translations parallel to the $x^1 - x^2$ plane, which entails $a^1 = c^1$ and $a^2 = b^2$. Further $l$ can be put to 0 or 1 by translation along the $x^3$-axis. Hence taking account of the constraint $m = l + n$, $(a \ b \ c)$ is finally put by HPDs into the form

$$B = \begin{pmatrix}
X & 0 & X \\
X^{-1} & X^{-1} & 0 \\
l\pi & (l + n)\pi & n\pi
\end{pmatrix},$$

where $X > 0$, $l = 0, 1$ and $n$ is a non-zero integer. Thus the invariant phase space consists of two families ($l = 0, 1$) of countably infinite number of connected components.

Let us take $K_0$ with $X = 0$ as the base point of each connected component. Then $f_\lambda$ defined by

$$f_\lambda(x) = Ax; \quad A = \begin{pmatrix}
X & 0 & 0 \\
0 & 1/X & 0 \\
0 & 0 & 1
\end{pmatrix}$$

yields the deformation map to the generic point corresponding to $B$. This is a special case of Eq.(3.55) with $Y = Z = W = 0$. Hence the canonical structure is degenerate in the 1-dimensional moduli sector, and $\Theta$ and $H$ are again given by Eq.(3.59) and Eq.(3.61). The only difference is the value of $\Omega$, which is given by $2|n|\pi$ in the present case.
3.3.4 $T^3/Z_k(k = 3, 4, 6)$

In order for $K \subset \text{VII}^{(c)}(0) \times \mathcal{D}_2$, from Table 2, its generator should be of the form

$$\alpha = (a, 1), \quad \beta = (b, 1), \quad \gamma = (c, R_3(\pm 2\pi/k)), \quad \text{(3.71)}$$

where the translation part \( \begin{pmatrix} a & b & c \end{pmatrix} \) is given by

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ 0 & 0 & c^3 \end{pmatrix}, \quad \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = R(\pm \frac{2\pi}{k}) \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}, \quad \text{(3.72)}$$

and $c^3$ is a discrete number of the form $\pm 2\pi/k + n\pi(n \in \mathbb{Z})$. By the rotation $R_1(\pi) \in \text{HPDG}^+(E^3, \text{VII}^{(c)} \times \mathcal{D}_2)$, $R_3$ can be put to $R_3(2\pi/k)$ and $a^1b^2 - a^2b^1$ can be made positive. Further by translations along the $x^1 - x^2$ plane we can put $c^1$ and $c^2$ to zero, and $a^2$ to zero by a translation along $x^3$ in $\text{VII}^{(0)} \times \mathcal{D}_2$. Hence by HPDs the moduli matrix is finally transformed to

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} 1 \cos(2\pi/k) & 0 & 0 \\ 0 \sin(2\pi/k) & 0 & 0 \\ 0 & 0 & 2\pi\epsilon/k + n\pi \end{pmatrix}. \quad \text{(3.73)}$$

Thus the reduced moduli space $\mathcal{M}_0$ consists of countably infinite number of discrete points. The canonical 1-form and the Hamiltonian are given by the same expressions as for $T^3$ except that $\Omega = \sin(2\pi/k)|2\pi\epsilon/k + n\pi|$ in the present case.

3.4 $\tilde{G}_0 = R^3 \times \text{SO}(2)$

An automorphism of $R^3 \times \text{SO}(2)$ is expressed as

$$\phi(\xi_a) = \xi_b T^b_a; \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \begin{pmatrix} k_1 R(\theta) & 0 & c^2 \\ 0 & 0 & -c^1 \\ 0 & k_2 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(3.74)}$$

where $k_1 > 0$. When $d = 0$ and $k_2 > 0$, it is induced from $f \in \text{HPDG}^+(E^3, R^3 \times \text{SO}(2))$ given by

$$f(\mathbf{x}) = \{1, R_1(\pi)\} \left[ \begin{pmatrix} k_1 R(\theta) & 0 \\ 0 & 0 & k_2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} \right], \quad \text{(3.75)}$$

where $c^3$ is an integration constant. By this HPD the invariant basis Eq.(3.9) transforms as

$$f^* \chi_I = A^I J \chi^J; \quad A = \{1, R_1(\pi)\} \begin{pmatrix} k_1 R(\theta) & 0 \\ 0 & 0 & k_2 \end{pmatrix}. \quad \text{(3.76)}$$

From the symmetry the components $Q$ and $P$ of covering data $\bar{\Phi} = (\bar{q}, \bar{p})$ with respect to the invariant basis Eq.(3.9) must be represented by the diagonal matrices with constant
| Symmetry | Topology | Moduli | $\Omega$ | $\Theta$ |
|----------|----------|--------|----------|----------|
| $R^3 \times D_2$ | $T^3$ | \( \begin{pmatrix} 1 & X & Y \\ 0 & 1 & Z \\ 0 & 0 & 1 \end{pmatrix} \) | \( R(\theta, \phi, \psi) \) | 1 | $\dot{\alpha} p_\alpha + \dot{\beta} p_+ + \dot{\beta} p_- + \dot{\theta} p_\theta + \dot{\phi} p_\phi + \dot{\psi} p_\psi$ |
| $T^3 / Z_2$ | | \( \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) | \( R_3(\phi) \) | 1 | $\dot{\alpha} p_\alpha + \dot{\beta} p_+ + \dot{\beta} p_- + \dot{\phi} p_\phi$ |
| $T^3 / Z_2 \times Z_2$ | | \( \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \) | \( R_\alpha = R_1(\pi), R_\beta = R_2(\pi), R_\gamma = R_3(\pi) \) | 2 | $\dot{\alpha} p_\alpha + \dot{\beta} p_+ + \dot{\beta} p_-$ |
| $\text{VII}^{(e)} \times D_2$ | $T^3$ | \( \begin{pmatrix} X & Y & Z \\ 0 & X^{-1} & W \\ 0 & 0 & n\pi \end{pmatrix} \) | \( n\pi \) | $\dot{\alpha} p_\alpha + \dot{\beta} p_+ + \dot{\beta} p_-$ |
| $T^3 / Z_2$ | | \( \begin{pmatrix} X & Y & 0 \\ 0 & X^{-1} & 0 \\ 0 & 0 & n\pi \end{pmatrix} \) | \( n\pi \) | $\dot{\alpha} p_\alpha + \dot{\beta} p_+ + \dot{\beta} p_-$ |
| $T^3 / Z_2 \times Z_2$ | | \( \begin{pmatrix} X & 0 & X \\ X^{-1} & X^{-1} & 0 \\ l\pi & (l + n)\pi & n\pi \end{pmatrix} \) | \( 2|n|\pi \) | $\dot{\alpha} p_\alpha + \dot{\beta} p_+ + \dot{\beta} p_-$ |
| $T^3 / Z_k$ | | \( \begin{pmatrix} 1 & \cos \frac{2\pi}{k} & 0 \\ 0 & \sin \frac{2\pi}{k} & 0 \\ 0 & 0 & \sin \frac{2\pi}{k} \end{pmatrix} \) | \( |\frac{2\pi}{k} + n| \times \sin \frac{2\pi}{k} \) | $\dot{\alpha} p_\alpha + \dot{\beta} p_+ + \dot{\beta} p_-$ |

Table 3: Canonical Structure of compact orientable closed 3-manifold of type $E^3$.  

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| Symmetry  | Topology | Moduli | Ω | Θ |
|----------|----------|--------|---|---|
| $R^3 \times O(2)$ | $T^3$ | \[
\begin{pmatrix}
X & Y & Z \\
0 & X^{-1} & W \\
0 & 0 & 1
\end{pmatrix} R(\theta, \phi)
\] | 1 | $\dot{\alpha}_p + \dot{\beta}_+ p_+ + \dot{\theta} \phi + \dot{\phi} p_\phi$ |
|  |  | $(X > 0)$ |  |  |
|  | $T^3 / Z_2$ | \[
\begin{pmatrix}
0 & 0 & X^{-1} \\
X & Y & 0 \\
0 & 1 & 0
\end{pmatrix} R_3(\phi)
\] | 1 | $\dot{\alpha}_p + \dot{\beta}_+ p_+ + \dot{\phi} p_\phi$ |
|  |  | $(X > 0, R_\gamma = R_2(\pi))$ |  |  |
|  | $T^3 / Z_2 \times Z_2$ | \[
\begin{pmatrix}
X & 0 & X \\
X^{-1} & X^{-1} & 0 \\
0 & 1 & 1
\end{pmatrix}
\] | 2 | $\ddot{\theta} \phi$ |
|  |  | $(X > 0, R_\alpha = R_1(\pi), R_\beta = R_2(\pi), R_\gamma = R_3(\pi))$ |  |  |
|  | $T^3 / Z_k$ | \[
\begin{pmatrix}
1 & \cos \frac{2\pi}{k} & 0 \\
0 & \sin \frac{2\pi}{k} & 0 \\
0 & 0 & \pm 1
\end{pmatrix}
\] | $\sin \frac{2\pi}{k}$ | $\ddot{\theta} \phi$ |
|  |  | $(k = 3, 4, 6, R = R_3(2\pi/k))$ |  |  |
| ISO(3) | $T^3$ | \[
\begin{pmatrix}
X & Z & W \\
0 & Y & U \\
0 & 0 & 1/XY
\end{pmatrix}
\] | 1 | $\dot{\alpha}_p$ |
|  |  | $(X, Y > 0)$ |  |  |
|  | $T^3 / Z_2$ | \[
\begin{pmatrix}
X & Z & 0 \\
0 & Y & 0 \\
0 & 0 & 1/XY
\end{pmatrix}
\] | 1 | $\ddot{\theta} \phi$ |
|  |  | $(X, Y > 0, R_\gamma = R_3(\pi))$ |  |  |
|  | $T^3 / Z_2 \times Z_2$ | \[
\begin{pmatrix}
X & 0 & X \\
Y & Y & 0 \\
0 & 1/XY & 1/XY
\end{pmatrix}
\] | 2 | $\ddot{\theta} \phi$ |
|  |  | $(X, Y > 0, R_\alpha = R_1(\pi), R_\beta = R_2(\pi), R_\gamma = R_3(\pi))$ |  |  |
|  | $T^3 / Z_k$ | \[
\begin{pmatrix}
0 & X \cos \frac{2\pi}{k} & 0 \\
0 & X \sin \frac{2\pi}{k} & 0 \\
0 & 0 & \pm 1/X^2
\end{pmatrix}
\] | $\sin \frac{2\pi}{k}$ | $\ddot{\theta} \phi$ |
|  |  | $(k = 3, 4, 6, R_\gamma = R_3(2\pi/k))$ |  |  |

Table 3 (continued).
from Eq.(2.59) and Eq.(2.60), after a short calculation, we obtain the reduced moduli space, \( f \). The translation vectors generating the moduli \( K \) connected. As in the corresponding case with \( \tilde{G} = \text{VII}^{(\pm 0)} \) the rotation matrix \( R \) associated with the generator \( \gamma \) can be put to \( R_3(\pi) \) or \( R_1(\pi) \). Accordingly the invariant phase space consists of two connected components.

3.4.1 \( T^3 \)

The translation vectors generating the moduli \( K \) can be put into the form

\[
( \begin{array}{ccc} a & b & c \\ \end{array} ) = F = \begin{pmatrix} X & Y & Z \\ 0 & X^{-1} & W \\ 0 & 0 & 1 \end{pmatrix} R(\theta, \phi)
\]

by modular transformations \( \alpha \leftrightarrow \alpha^{-1}, \beta \leftrightarrow \beta^{-1} \) and \( \gamma \leftrightarrow \gamma^{-1} \), and HPDs, where \( X > 0 \) and \( R(\theta, \phi) := R_3(\phi - \pi/2)R_1(\theta)R_3(\pi/2 - \phi) \) is a matrix which rotates a vector with the angular direction \( (\theta, \phi) \) to a vector parallel to the \( x^3 \)-axis. This fixes the gauge freedom up to residual discrete HPDs and modular transformations. The invariant phase space is connected.

If we take \( K_0 \) with \( X = 1 \) and \( Y = Z = W = \theta = \phi = 0 \) as the base point of the reduced moduli space, \( f_\lambda(x) = Fx \) maps \( K_0 \) to a generic \( K \) corresponding to \( F \). Hence from Eq.(2.59) and Eq.(2.60), after a short calculation, we obtain

\[
\Theta = \dot{\alpha}p_\alpha + \dot{\beta}_+p_+ + \dot{\theta}p_\theta + \dot{\phi}p_\phi,
\]

\[
H = \frac{\kappa^2}{12\Omega} Ne^{-3\alpha}(-p_\alpha^2 + p_\phi^2).
\]

Here \( \alpha \) and \( \beta_+ \) are related to \( Q_1 \) and \( Q_2 \) by the equations obtained from Eq.(3.30) and Eq.(3.32) by putting \( \beta_- = 0 \), \( p_\alpha \) and \( p_+ \) are defined by Eq.(3.24) and Eq.(3.26) with \( Q_1 P^1 = Q_2 P^2 \), and \( p_\phi \) and \( p_\theta \) are given by

\[
p_\phi := \frac{1}{2}p_+ \left[ (YW - Z/X) \sin \phi + XW \cos \phi \right] \sin \theta,
\]

\[
p_\theta := \frac{1}{2}p_+ \left[ - (YW - Z/X) \cos \phi + XW \sin \phi \right].
\]

Of course \( \Omega = 1 \) is understood in the present case. The canonical structure is partially degenerate in the moduli sector.

3.4.2 \( T^3/Z_2 \)

As in the corresponding case with \( \tilde{G} = \text{VII}(\pm 0) \) the rotation matrix \( R \) associated with the generator \( \gamma \) can be put to \( R_3(\pi) \) or \( R_1(\pi) \). Accordingly the invariant phase space consists of two connected components.
a) \( R_{3} = R_{3}(\pi) \): By a procedure similar to that in §3.3.2, the moduli matrix \( \begin{pmatrix} a & b & c \end{pmatrix} \) can be put in the form

\[
F = \begin{pmatrix} X & Y & 0 \\ 0 & X^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\] (3.83)

where \( X > 0 \). This gauge fixing eliminates the freedom of HPDs except in the subspace \( Y = 0 \), while it leaves modular transformations isomorphic to \( \mathbb{Z} \). By taking \( K_0 \) with \( X = 1 \) and \( Y = 0 \) as the base point, the deformation map \( f_\lambda \) is given by \( f_\lambda(x) = F x \). This is obtained from the corresponding map for \( T^3 \) by putting \( Z = W = 0 \) and \( R(\theta, \phi) = 1 \). Hence \( \Theta \) is given by

\[
\Theta = \dot{\alpha} p_\alpha + \dot{\beta}_+ p_+,
\] (3.84)

and \( H \) by Eq.(3.80) with \( \Omega = 1 \). Thus the canonical structure is completely degenerate in the 2-dimensional moduli sector.

b) \( R_{1} = R_{1}(\pi) \): In this case the moduli matrix has the form

\[
\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix}.
\] (3.85)

c^2 and \( c^3 \) can be put to zero by translations along the \( x^2 - x^3 \) plane. Hence by modular transformations and HPDs it can be put in the form

\[
B = \begin{pmatrix} 0 & 0 & X^{-1} \\ X & Y & 0 \\ 0 & 1 & 0 \end{pmatrix} R_3(\phi),
\] (3.86)

where \( X > 0 \). There remains residual discrete HPDs as well as modular transformations isomorphic to \( \text{SL}(2, \mathbb{Z}) \).

If we take \( K_0 \) with \( X = 1 \) and \( Y = 0 \) and \( \phi = 0 \) as the base point, \( f_\lambda \) given by

\[
f_\lambda(x) = F x; \quad F = \begin{pmatrix} 1/X & 0 & 0 \\ 0 & X & Y \\ 0 & 0 & 1 \end{pmatrix} R_1(\phi)
\] (3.87)

maps \( K_0 \) to \( K \) corresponding to \( B \). Inserting this into Eq.(2.59) with \( \Omega = 1 \), we obtain

\[
\Theta = \dot{\alpha} p_\alpha + \dot{\beta}_+ p_+ + \dot{\phi} p_\phi,
\] (3.88)

where

\[
p_\phi := \frac{Y}{2X} p_+.
\] (3.89)

The canonical structure is again completely degenerate in the moduli sector.
3.4.3 \( T^3 / \mathbb{Z}_2 \times \mathbb{Z}_2 \)

As in §3.3.3, the generator of \( K \) can be transformed to the same form as Eq.(3.67). Further by HPDs and modular transformations, the moduli matrix can be put into the form

\[
\begin{pmatrix}
a & b & c \\
\end{pmatrix} =
\begin{pmatrix}
X & 0 & X \\
1/X & 1/X & 0 \\
0 & 1 & 1
\end{pmatrix},
\]

(3.90)

where \( X > 0 \). No freedom of HPD and modular transformation remains. Hence the phase space \( \Gamma_{\text{LH,inv}}(T^3 / \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{R}^3 \times \text{O}(2)) \) is connected.

For the base point \( K_0 \) with \( X = 1 \) and \( \Omega = 2 \), the deformation map \( f_\lambda \) is given by

\[
f_\lambda(x) = Fx; \quad F = \begin{pmatrix} X & 0 & 0 \\
0 & 1/X & 0 \\
0 & 0 & 1 \end{pmatrix}.
\]

(3.91)

This is the special case of \( f_\lambda \) in §3.4.2-a) with \( Y = 0 \). Hence \( \Theta \) is given by the same equation as Eq.(3.84), and the canonical structure is degenerate in the 1-dimensional moduli sector.

3.4.4 \( T^3 / \mathbb{Z}_k \) \((k = 3, 4, 6)\)

By an argument similar to that in §3.3.4, the generators of \( K \) can be put in the form

\[
\alpha = (a, 1), \quad \beta = (b, 1), \quad \gamma = (c, R_3(2\pi/k)),
\]

(3.92)

with

\[
\begin{pmatrix}
a & b & c \\
\end{pmatrix} = \begin{pmatrix} 1 & \cos(2\pi/k) & 0 \\
0 & \sin(2\pi/k) & 0 \\
0 & 0 & \pm 1 \end{pmatrix}.
\]

(3.93)

Thus the reduced moduli space consists of two points which are connected by orientation-reversing HPDs. The volume of the fundamental region is given by \( \Omega = \sin(2\pi/k) \), and \( \Theta \) and \( H \) by Eq.(3.84) and Eq.(3.80), respectively.

3.5 \( \tilde{G} = \text{ISO}(3) \)

For \( \tilde{G} = \text{ISO}(3) \), the phase space of homogeneous covering data is obviously given by

\[
\Gamma_{\text{H,D}}(E^3, \text{ISO}(3)) = \left\{ Q = [Q_1, Q_1, Q_1], P = [P^1, P^1, P^1] \right\}.
\]

(3.94)

The arguments to determine \( \Theta \) and \( H \) for each spaces are almost the same as those in the previous case. So I just give outlines of derivations and results.

3.5.1 \( T^3 \)

By HPDs (3.7) and modular transformations the modular matrix can be put in the form

\[
\begin{pmatrix}
a & b & c \\
\end{pmatrix} = F = \begin{pmatrix} X & Z & W \\
0 & Y & U \\
0 & 0 & 1/XY \end{pmatrix},
\]

(3.95)
| Symmetry Space | Space | Q | P | N<sub>m</sub> | N<sub>c</sub> | N<sub>d</sub> | N<sub>cc</sub> | HPD | Modular |
|---------------|-------|---|---|-------------|-------------|-------------|-------------|--------|----------|
| \( \mathbb{R}^3 \times D_2 \) | \( T^3 \) | 3 | 3 | 6 | 12 | 12 | 0 | 1 | □ | □ |
| \( T^3/\mathbb{Z}_2 \) | | 3 | 3 | 2 | 8 | 8 | 0 | 1 | □ | □ |
| \( T^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) | | 3 | 3 | 0 | 6 | 6 | 0 | 1 | × | × |
| \( \text{VII(0)} \tilde{\times} D_2 \) | \( T^3 \) | 3 | 3 | 4 | 10 | 6 | 6 | 1 | ∞ | △ | □ |
| \( T^3/\mathbb{Z}_2 \) | | 3 | 3 | 2 | 8 | 6 | 2 | ∞ | △ | □ |
| \( T^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) | | 3 | 3 | 1 | 7 | 6 | 1 | ∞ | × | × |
| \( T^3/\mathbb{Z}_k(k = 3, 4, 6) \) | | 3 | 3 | 0 | 6 | 6 | 0 | ∞ | × | × |
| \( \mathbb{R}^3 \tilde{\times} \text{O}(2) \) | \( T^3 \) | 2 | 2 | 6 | 10 | 8 | 2 | 1 | □ | □ |
| \( T^3/\mathbb{Z}_2 \) | | 2 | 2 | 2 | 6 | 4 | 2 | 1 | △ | □ |
| \( T^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) | | 2 | 2 | 3 | 7 | 6 | 1 | 1 | □ | □ |
| \( T^3/\mathbb{Z}_k(k = 3, 4, 6) \) | | 2 | 2 | 0 | 4 | 4 | 0 | 2 | × | × |
| \( \text{ISO}(3) \) | \( T^3 \) | 1 | 1 | 5 | 7 | 2 | 5 | 1 | △ | □ |
| \( T^3/\mathbb{Z}_2 \) | | 1 | 1 | 3 | 5 | 2 | 3 | 1 | △ | □ |
| \( T^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) | | 1 | 1 | 2 | 4 | 2 | 2 | 1 | × | × |
| \( T^3/\mathbb{Z}_k(k = 3, 4, 6) \) | | 1 | 1 | 1 | 3 | 2 | 1 | 2 | × | × |

Table 4: Canonical and degenerate degrees of freedom for LHS of type \( E^3 \)

The last two columns show whether there exist residual discrete HPDs and modular transformations. □ indicates the existence, × the non-existence, and △ the existence at lower-dimensional subspaces.
where $X, Y > 0$. There exists no residual HPD except in the subspace $ZUX = 0$, but the freedom of modular transformations isomorphic to $SL(2, \mathbb{Z})$ remains.

The deformation map $f_\lambda$ is given by the linear transformation by $F$. Inserting it into Eqs. (2.59)-(2.60), we find that the moduli parameters do not appear in the canonical 1-form, which is simply written as

$$\Theta = \dot{\alpha}p_\alpha.$$  \hspace{1cm} (3.96)

The Hamiltonian is given by

$$H = -\frac{\kappa^2}{12\Omega} Ne^{-3\alpha}p_\alpha^2,$$ \hspace{1cm} (3.97)

where $\Omega = 1$.

3.5.2 $T^3/\mathbb{Z}_2$

By $SO(3)$ transformation $\gamma$ can be put in the form $\gamma = (c, R_3(\pi))$. The freedom of HPDs is fixed by putting the moduli matrix to the form

$$\begin{pmatrix} a & b & c \\ X & Z & 0 \\ 0 & Y & 0 \\ 0 & 0 & 1/XY \end{pmatrix} = F = \begin{pmatrix} X & Z & 0 \\ 0 & Y & 0 \\ 0 & 0 & 1/XY \end{pmatrix},$$ \hspace{1cm} (3.98)

except at the subspace $Z = 0$ where $X, Y > 0$. The residual freedom of modular transformations is isomorphic to $\mathbb{Z}$. The phase space is connected and the canonical structure is completely degenerate in the moduli sector as in the previous case. $\Omega$, $\Theta$ and $H$ are the same as those for $T^3$.

3.5.3 $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$

The canonical form of the moduli matrix is given by

$$\begin{pmatrix} a & b & c \\ X & 0 & X \\ Y & Y & 0 \\ 0 & 1/XY & 1/XY \end{pmatrix},$$ \hspace{1cm} (3.99)

where $X, Y > 0$. For the base point $K_0$ with $X = Y = 1$, the deformation map is given by

$$f_\lambda(\mathbf{x}) = F\mathbf{x}; \quad F = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & 1/XY \end{pmatrix},$$ \hspace{1cm} (3.100)

and the volume of the fundamental region by $\Omega = 2$. The phase space is connected, and its canonical structure becomes completely degenerate in the 2-dimensional moduli sector. Hence $\Theta$ and $H$ are given by Eqs. (3.96)-(3.97) with $\Omega = 2$. 

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3.5.4 $T^3/Z_k(k = 3, 4, 6)$

The moduli matrix for the reduced moduli space is given by

$$
(a \ b \ c) = F = \begin{pmatrix}
X & X \cos(2\pi/k) & 0 \\
0 & X \sin(2\pi/k) & 0 \\
0 & 0 & \pm 1/X^2
\end{pmatrix}, \tag{3.101}
$$

and the deformation map by $f_\lambda(x) = Fx$. There remains no freedom of residual HPDs and modular transformations. Thus the phase space has two connected components, and $\Theta$ and $H$ are given by Eqs. (3.96)-(3.5.1) with $\Omega = \sin(2\pi/k)$.

3.6 Degeneracy of the canonical structure

The reduced form of the moduli matrix, the corresponding canonical 1-form $\Theta$ and the volume $\Omega$ of the standard fundamental region obtained so far are summarized in Table 3.

As we have seen, the canonical 1-form often becomes degenerate in the sector of moduli parameters. The number of degenerate degrees of freedom, $N_d$, for each space and symmetry are listed in Table 3 with the independent degrees of freedom of $Q$ and $P$, the dimension $N_m$ of moduli space, the total dimension $N$ of the invariant phase space, the number $N_c$ of the non-degenerate canonical degrees of freedom and the number of connected components $N_{cc}$.

From this table we see that the degrees of degeneracy increases as the covering data have higher symmetries in general. However, it is difficult to regard this to be simply brought about by the decrease of the freedom of homogeneous data due to symmetry because invariant phase spaces corresponding to different invariance groups have no simple relations with each other. For example the moduli sector of the invariance phase spaces for $G \cong R^3 \times D_2$ are all compact due to the modular transformations, while their counter parts for $G \cong R^3 \times O(2)$ are all non-compact.

Furthermore the degeneracy for the systems with $G = \text{VII}(0)$ cannot be a result of the existence of isotropy groups because the invariance group is simply transitive. The calculation of the canonical 1-form in §3.3.1 rather shows that the degeneracy arises partly because the couplings among variables vanishes by integration over the fundamental region. This implies that canonical variables describing the locally homogeneous sector has non-trivial couplings with locally inhomogeneous degrees of freedom.

In any case, when the canonical structure is degenerate, the canonical equations of motion alone do not determine the time evolution of locally homogeneous systems completely. As touched upon in §2.3.3, however, this does not implies that the degenerate moduli freedom is non-dynamical, because, from Theorem 2.3, the full Einstein equations completely determine the time evolution of the parameters describing the reduced moduli space obtained by eliminating the HPD freedom. They are actually constants of motion. For the locally homogeneous systems of type $E^3$ (as well as other types considered in the present paper), it is directly confirmed that the moduli parameters in the non-degenerate sectors are conserved from the structures of the canonical 1-form and the Hamiltonian. Thus the degeneracy in the canonical structure of locally homogeneous systems means that they are not canonically closed in the full diffeomorphism-invariant phase space.
§4 LHS of type Nil

In this section we investigate the canonical structure of locally homogeneous pure gravity systems of type Nil.

4.1 Basic properties

4.1.1 $G_{\text{max}}$ and HPDG$^+(G_{\text{max}})$

The space Nil has the structure of the Heisenberg group, and classified as type II in the Bianchi scheme. As a subgroup of $\text{GL}(3, R)$, Nil is expressed as

$$\text{Nil} \cong \left\{ \begin{pmatrix} 1 & x & z + xy/2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in R \right\}.$$  \hspace{1cm} (4.1)

Here we have adopted a parameterization which is different from the standard one. This parameterization is more convenient for our purpose because it enables us to express all the HPDs by linear transformations. In this section we denote each element of Nil by the coordinate $(x, y, z)$. In this notation the product of two elements is expressed as

$$(a, b, c)(x, y, z) = (a + x, b + y, c + z + \frac{ay - bx}{2}).$$ \hspace{1cm} (4.2)

In terms of the generators

$$\xi_1 = \partial_x + \frac{1}{2}y\partial_z, \quad \xi_2 = \partial_y - \frac{1}{2}x\partial_z, \quad \xi_3 = \partial_z,$$ \hspace{1cm} (4.3)

the structure of the Lie algebra $\mathcal{L}(\text{Nil})$ is expressed as

$$[\xi_1, \xi_2] = -\xi_3, \quad [\xi_3, \xi_1] = 0, \quad [\xi_3, \xi_2] = 0.$$ \hspace{1cm} (4.4)

The corresponding invariant basis is given by

$$\chi^1 = dx, \quad \chi^2 = dy, \quad \chi^3 = dz + \frac{1}{2}(ydx - xdy),$$ \hspace{1cm} (4.5)

$$d\chi^1 = 0, \quad d\chi^2 = 0, \quad d\chi^3 = -\chi^1 \wedge \chi^2.$$ \hspace{1cm} (4.6)

As in the case of $E^3$, $|\chi| = 1$ holds for this basis.

The maximally symmetric metric on Nil is given by

$$ds^2 = Q_1 \left( (\chi^1)^2 + (\chi^2)^2 \right) + Q_3(\chi^3)^2 = Q_1(dx^2 + dy^2) + Q_3[dz + \frac{1}{2}(ydx - xdy)]^2,$$ \hspace{1cm} (4.7)

where $Q_1$ and $Q_3$ are positive constants. From this we find that the maximal symmetry group of Nil is 4-dimensional and has following decomposition:

$$0 \to \text{Nil} \to \text{Isom(Nil)} \to \text{O}(2) \to 1 \quad \text{(exact)},$$ \hspace{1cm} (4.8)
where $O(2)$ is the group generated by rotations around the $z$-axis, $R_z(\theta)$, and the rotation of angle $\pi$ around the $x$-axis, $R_x(\pi)$. Hence a generic transformation $f$ in $\text{Isom}(\text{Nil})$ is expressed as

$$f(\mathbf{x}) = \{1, R_x(\pi)\} \times R_z(\theta) \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}d^2 & \frac{1}{2}d^1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} d^1 \\ d^2 \\ d^3 \end{pmatrix} \right]. \quad (4.9)$$

The above invariant basis transforms by $f$ as

$$f^* \chi^I = A^I J \chi^J; \quad A = \{1, R_x(\pi)\} \times R_z(\theta). \quad (4.10)$$

$\text{Isom}(\text{Nil})$ has two connected components, $\text{Isom}_0(\text{Nil})$ and $R_x(\pi) \times \text{Isom}_0(\text{Nil})$, both of which preserve orientation.

The generators of $L(\text{Isom}(\text{Nil}))$ are given by $\xi_I (I = 1, 2, 3)$ and

$$\xi_4 = -y \partial_x + x \partial_y, \quad (4.11)$$

whose commutation relations with $\xi_I$ are

$$[\xi_4, \xi_1] = -\xi_2, \quad [\xi_4, \xi_2] = \xi_1, \quad [\xi_1, \xi_3] = 0. \quad (4.12)$$

From these commutation relations an automorphism of $L(\text{Isom}(\text{Nil}))$ is represented as

$$\phi(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_1, \xi_2, \xi_3, \xi_4) \left( \begin{pmatrix} k & 0 & 0 \\ 0 & \pm k & 0 \\ a^3 & b^3 & \pm k^2 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} R(\theta) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \pm k^2 & d \\ 0 & 0 \end{pmatrix} \right), \quad (4.13)$$

where $k > 0$ and $R(\theta) \in \text{SO}(2)$. This automorphism is induced from $f$ in $\text{HPDG}^+(\text{Isom}_0(\text{Nil}))$ if $a^3, b^3$ and $d$ are written as

$$(a^3, b^3) = \pm k(-d^2, d^1), \quad d = \pm \frac{1}{2} \left((d^1)^2 + (d^2)^2\right). \quad (4.14)$$

The explicit form of $f$ are

$$f(\mathbf{x}) = \{1, R_x(\pi)\} \times R_z(\theta) \left[ \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ -\frac{k}{2}d_2 & \frac{k}{2}d_1 & k^2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} d^1 \\ d^2 \\ d^3 \end{pmatrix} \right], \quad (4.15)$$

where $d^3$ is an integration constant. This is a product of a scaling represented by a diagonal matrix $[k, k, k^2]$ and an element of $\text{Isom}(\text{Nil})$.

### 4.1.2 Transitive subgroups of $\text{Isom}(\text{Nil})$

Let $\tilde{G}$ be a connected transitive subgroup of $\text{Isom}(\text{Nil})$. Then from Eq.(4.8) its projection on $O(2)$ is either 1 or $\text{SO}(2)$. In the former case $\tilde{G}$ should be equal to $\text{Nil}$. In the latter case, $\tilde{G} \cap \text{Nil}$ must have a dimension greater than one. If the dimension is two, from Eq.(4.14),
it should be generated by $\xi_3$ and a linear combination of $\xi_1$ and $\xi_2$, $a\xi_1 + b\xi_2$. As the third
generator of $\tilde{G}$ we can take one with the form $\xi'_1 = \xi_4 + c^I\xi_I$. However, since $[\xi'_4, a\xi_1 + b\xi_2]
is written as $-b\xi_1 + a\xi_2 + c\xi_3$ with some constant $c$, these three generators do not form
a closed algebra. Hence $\tilde{G}$ must coincide with $\text{Isom}_0(\text{Nil})$. Thus the connected transitive subgroups of $\text{Isom}(\text{Nil})$ are given by $\text{Nil}$ and $\text{Isom}_0(\text{Nil})$.

Since every invariance subgroup of $\text{Isom}(\text{Nil})$ contains $\text{Nil}$, homogeneous covering data
$\tilde{\Phi} = (\tilde{q}, \tilde{p})$ is expressed by their coefficient matrices $Q$ and $P$ with respect to the invariant basis (4.5). From the exact sequence (4.8), if $\tilde{G}_0 = \text{Nil}$, $\tilde{G}$ should be a product of $\text{Nil}$ and a subgroup of $D_2 = \{1, R_1(\pi), R_2(\pi), R_3(\pi)\}$. On the other hand, if $\tilde{G}_0 = \text{Isom}_0(\text{Nil})$, $\tilde{G}$ is either $\text{Isom}_0(\text{Nil})$ or $\text{Isom}(\text{Nil})$.

4.1.3 Topology of orientable compact quotients

Compact closed 3-manifolds modeled on $(\text{Nil}, \text{Isom}(\text{Nil}))$ allow Seifert fibering with $\chi = 0$ and $e \neq 0$. Hence we can determine all possible fundamental groups from the fundamental groups of base orbifolds with $\chi = 0$ and Seifert index. They are classified into 7 types as listed in Table 5, where their general embeddings into $\text{Isom}(\text{Nil})$ are also given.

The fundamental groups have similar structure to those of type $E^3$ except that each has a parameter $n$ of positive integer. Six of them have counterparts in those of type $E^3$, and five of them are covered by the simplest one denoted as $T^3(n)$. Their covering transformation groups coincide with the corresponding ones for $E^3$ as will be explicitly shown later.

4.2 $\tilde{G}_0 = \text{Nil}$

The automorphism group of $\text{Nil}$ consists of elements whose action on $L(\text{Nil})$ is expressed as

$$\phi(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3) \left( \begin{array}{ccc} \hat{A} & 0 \\ 0 & 0 \\ a^3 & b^3 & \Delta \end{array} \right); \quad \Delta := \det \hat{A}. \quad (4.16)$$

Every automorphism $\phi$ is induced from $f \in \text{HPDG}^+(\text{Nil}, \text{Nil})$ of the form

$$f(x) = \left( \begin{array}{ccc} \hat{A} & 0 \\ 0 & 0 \\ (a^3, b^3) + \frac{1}{2}(d^2, -d^1)\hat{A} & \Delta \end{array} \right) x + \left( \begin{array}{c} d^1 \\ d^2 \\ d^3 \end{array} \right). \quad (4.17)$$

The invariant basis (4.3) transforms by $f$ as

$$f^*\chi^J = A^I J \chi^J; \quad A = \left( \begin{array}{ccc} \hat{A} & 0 \\ 0 & 0 \\ (a^3, b^3) + (d^2, -d^1)\hat{A} & \Delta \end{array} \right). \quad (4.18)$$

By these HPDs the metric matrix $Q$ can be put in the diagonal form $[1, 1, Q_3]$. For this metric the diffeomorphism constraint Eq.(2.63) is expressed as

$$P^{13} = 0, \quad P^{23} = 0. \quad (4.19)$$
| Space | Fundamental group and representation |
|-------|-------------------------------------|
| $T^3(n)$ | $< \alpha, \beta, \gamma | [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^n >$ \hspace{1cm} ($n \in \mathbb{N}$)  
$\alpha = a, \beta = b, \gamma = (0, 0, \Delta(a, b)/n); \Delta(a, b) \neq 0$ |
| $K^3(n)$ | $< \alpha, \beta, \gamma | [\alpha, \gamma] = 1, \beta \gamma^{-1} \gamma^{-1} = 1, [\alpha, \beta] = \gamma^2n >$ \hspace{1cm} ($n \in \mathbb{N}$)  
$\alpha = a, \beta = R_3(\pi) b, \gamma = (0, 0, \Delta(a, b)/n); \Delta(a, b) \neq 0$ |
| $T^3(n)/\mathbb{Z}_2$ | $< \alpha, \beta, \gamma | [\alpha, \gamma] = 1, \gamma \beta^{-1} \beta^{-1} = 1, [\alpha, \beta] = \gamma^{-2n} >$ \hspace{1cm} ($n \in \mathbb{N}$)  
$\alpha = R_3(\pi) a, \gamma = R_3(\pi) c; 2na^3 = -(a^2, a^1)R(\theta) \left( \begin{array}{c} a^1 - c^1 \\ a^2 - c^2 \end{array} \right) + \Delta(a, c)$ |
| $T^3(n)/\mathbb{Z}_3$ | $< \alpha, \beta, \gamma | [\alpha, \gamma] = 1, \gamma \beta^{-1} \beta^{-1} = 1, [\alpha, \beta] = \gamma^{3n} >$ \hspace{1cm} ($n \in \mathbb{N}$)  
$\alpha = a, \beta = b, \gamma = R_3(\pm \frac{2\pi}{3}) c;  
\left( \begin{array}{c} b^1 \\ b^2 \end{array} \right) = R \left( \begin{array}{c} a^1 \\ a^2 \end{array} \right);  
\left( \begin{array}{c} c^1 \\ c^2 \end{array} \right) = \left( \begin{array}{c} \frac{a^3-b^1}{\Delta(a,b)} R + \frac{1}{2} - \frac{a^3+b^1}{\Delta(a,b)} \end{array} \right) \left( \begin{array}{c} a^1 \\ a^2 \end{array} \right);  
c^3 = \frac{\Delta(a,b)}{3n} + \frac{1}{6}(c^1, c^2) R \left( \begin{array}{c} -c^2 \\ c^1 \end{array} \right) $ |
| $T^3(n)/\mathbb{Z}_4$ | $< \alpha, \beta, \gamma | [\alpha, \gamma] = 1, \beta^{-1} \gamma^{-1} = 1, [\alpha, \beta] = \gamma^{4n} >$ \hspace{1cm} ($n \in \mathbb{N}$)  
$\alpha = (a^1, a^2, 0), \beta = (b^1, b^2, 0), \gamma = R_3(\pm \frac{\pi}{2}) (0, 0, \Delta(a, b)/4n);  
\left( \begin{array}{c} b^1 \\ b^2 \end{array} \right) = R(\pm \frac{\pi}{2}) \left( \begin{array}{c} a^1 \\ a^2 \end{array} \right)$ |
| $T^3(n)/\mathbb{Z}_6$ | $< \alpha, \beta, \gamma | [\alpha, \gamma] = 1, \beta \gamma^{-1} = 1, [\alpha, \beta] = \gamma^{6n} >$ \hspace{1cm} ($n \in \mathbb{N}$)  
$\alpha = a, \beta = b, \gamma = R_3(\pm \frac{\pi}{6}) c;  
\left( \begin{array}{c} b^1 \\ b^2 \end{array} \right) = R \left( \begin{array}{c} a^1 \\ a^2 \end{array} \right);  
\left( \begin{array}{c} c^1 \\ c^2 \end{array} \right) = \left( \begin{array}{c} \frac{a^3-b^1}{\Delta(a,b)} R - \frac{3}{2} + \frac{a^3-b^3}{\Delta(a,b)} R \end{array} \right) \left( \begin{array}{c} a^1 \\ a^2 \end{array} \right);  
c^3 = \frac{\Delta(a,b)}{6n} + \frac{1}{3}(c^1, c^2) R \left( \begin{array}{c} -c^2 \\ c^1 \end{array} \right) $ |

Table 5: Fundamental groups and their representation in $\text{Isom}^+(\text{Nil})$ of compact closed orientable 3-manifolds of type Nil  
In this table $\Delta(a, b) = a^1 b^2 - a^2 b^1$.  

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Hence the momentum matrix can be diagonalized as $P = [P^1, P^2, P^3]$ by residual HPDs keeping the form of $Q$. Hence the invariance group of covering data $\Phi$ always contains $\text{Nil} \tilde{\times} D_2$.

For $f$ given by Eq. (4.17), the condition $f D_2 f^{-1} \subset \text{Nil} \tilde{\times} D_2$ holds if and only if the matrix $\tilde{A}$ is of the form

$$\tilde{A} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

and $(a^3, b^3) = (-d^2, d^4)\tilde{A}$. Hence $f \in \text{HPDG}^+(\text{Nil}, \text{Nil} \tilde{\times} D_2)$ is expressed as

$$f(\mathbf{x}) = \begin{pmatrix} \tilde{A} & 0 \\ \frac{1}{2}(-d^2, d^4)\tilde{A} & \Delta \end{pmatrix} \mathbf{x} + \begin{pmatrix} d^1 \\ d^2 \\ d^3 \end{pmatrix},$$

by which the invariant basis transforms as

$$f^* \chi^I = A^I J \chi^J; \quad A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.22)$$

Covering data invariant under $\text{Nil} \tilde{\times} D_2$ are represented by diagonal matrices $Q = [Q_1, Q_2, Q_3]$ and $P = [P^1, P^2, P^3]$. If we transform $Q$ to the Isom($\text{Nil}$)-invariant form, $[1, 1, Q_3/Q_1 Q_2]$, by HPDs of $\text{Nil} \tilde{\times} D_2$, $P$ transforms to $[Q_1 P^1, Q_2 P^2, Q_1 Q_2 P^3]$, which becomes Isom$_0(\text{Nil})$-invariant only when $Q_1 P^1 = Q_2 P^2$. Hence from Prop. 2.4, we obtain

$$\Gamma^+_\text{h.d}(\text{Nil}, \text{Nil} \tilde{\times} D_2) = \left\{ Q = [Q_1, Q_2, Q_3], P = [P^1, P^2, P^3] \mid Q_1 P^1 \neq Q_2 P^2 \right\}. \quad (4.23)$$

For the covering metric $\tilde{q}$ in this phase space, the Ricci scalar curvature is given by

$$R = -\frac{Q_3}{2Q_1 Q_2}. \quad (4.24)$$

Hence in terms of $\alpha$ and $\beta_\pm$ defined by Eqs. (3.30)-(3.31), and $p_\alpha$ and $p_\pm$ defined by Eqs. (3.24)-(3.26), the Hamiltonian is written as

$$H = \frac{\kappa^2}{12\Omega} Ne^{-3\alpha} \left[ -p_\alpha^2 + p_+^2 + p_-^2 + \frac{3\Omega^2}{\kappa^2} e^{4(\alpha-2\beta_+)} \right], \quad (4.25)$$

where $\Omega$ is the volume of the canonical fundamental region to be determined below.

Next we determine the canonical 1-form for each topology. Since the rotation matrices of angle $\pm 2\pi/k$ do not belong to $\text{Nil} \tilde{\times} D_2$ for $k = 3, 4, 6$, $T^3(n)/\mathbb{Z}_k$ $(k = 3, 4, 6)$ do not allow locally $\text{Nil} \tilde{\times} D_2$-invariant data as in the case of $E^3$.

### 4.2.1 $T^3(n)$

For this space the fundamental group is embedded in Nil, as is shown in Table 5, and the moduli is determined by the two generators $\alpha = (a^1, a^2, a^3)$ and $\beta = (b^1, b^2, b^3)$. Since $(p, q, r) \in \text{Nil}$ is transformed by HPDs (4.21) as

$$f(p, q, r)^{-1} = \begin{pmatrix} p & q \\ q & p \end{pmatrix}^T \tilde{A}, \Delta r + \begin{pmatrix} p & q \end{pmatrix}^T \tilde{A} \begin{pmatrix} -d^2 \\ d^1 \end{pmatrix}, \quad (4.26)$$
\[ a^3 \text{ and } b^3 \text{ can be put to zero by HPDs with } \hat{A} = 1. \text{ Further by using the freedom of } \hat{A}, \text{ we can transform the moduli matrix } \begin{pmatrix} a & b & c \end{pmatrix} \text{ to} \]

\[
B = \begin{pmatrix}
1 & X & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/n
\end{pmatrix} R_3(\phi). \tag{4.27}
\]

Then the HPD freedom is eliminated except for residual discrete ones. There remains the freedom of modular transformations among \( \alpha \) and \( \beta \) represented by \( \text{SL}(2,\mathbb{Z}) \) matrix as well.

Taking \( K_0 \) with \( X = 0 \) and \( \phi = 0 \) as the base point, a deformation map \( f_\lambda \) from \( K_0 \) to a generic moduli \( K \) is given by

\[
f_\lambda(x) = Fx; \quad F = \begin{pmatrix}
1 & X & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} R_3(\phi). \tag{4.28}
\]

For the moduli \( K_0 \) its generic element is written as

\[
a^p \beta^q \gamma^r = (p, q, r), \tag{4.29}
\]

where \( p, q, r \in \mathbb{Z} \). Hence the fundamental region \( D_0 \) of \( K_0 \) is given by \( 0 \leq x, y \leq 1 \) and \( 0 \leq z \leq 1/n \), and its volume by \( \Omega = 1/n \).

Inserting the above \( F \) into Eq.(2.59), we obtain

\[
\Theta = \Omega \sqrt{Q_1 Q_2 Q_3} \left[ \text{Tr}(\dot{Q}P) + 2\dot{\phi}X(Q_1 P_1 - Q_2 P_2) \right], \tag{4.30}
\]

which is written in terms of \( \alpha, \beta_\pm, p_\alpha, p_\pm \) and \( p_\phi \) defined by

\[
p_\phi := \frac{1}{\sqrt{3}} p_- X \tag{4.31}
\]

as

\[
\Theta = \dot{\alpha} p_\alpha + \dot{\beta}_+ p_+ + \dot{\beta}_- p_- + \dot{\phi} p_\phi. \tag{4.32}
\]

Thus no degeneracy occurs in the canonical structure.

### 4.2.2 \( K^3(n) \)

From Table 3, the generator \( \beta \) is a product of a transformation \( b \) in \( \text{Nil} \) and \( R_\beta = R_1(\pi)R_3(\theta) \), which is a rotation of angle \( \pi \) around an axis in the \( x - y \) plane. Hence, in order for \( \beta \) to be in \( \text{Nil} \times D_2 \), \( R_\beta \) should coincides with \( R_1(\pi) \) or \( R_2(\pi) \). Since these two cases are connected by the HPD with \( \hat{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), we can put \( R_\beta = R_2(\pi) \), i.e. \( \theta = \pi \), which implies \( a^2 = 0 \) from the definition of \( R(\theta) \) in Table 3. By the HPD given by Eq.(4.21) with

\[
\begin{pmatrix}
1/a^1 \\
0 \\
1/b^2
\end{pmatrix}, \tag{4.33}
\]

\[
(d^1, d^2, d^3) = \left( \frac{b_1}{2a^1}, \frac{a^3}{a^1 b^2}, \frac{b^3}{2a^1 b^2} - \frac{a^3 b^1}{4(a^1)^2 b^2} \right) \tag{4.34}
\]

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the corresponding moduli is transformed into one with the moduli matrix
\[
\begin{pmatrix}
a & b & c \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/n
\end{pmatrix}.
\] (4.35)

Hence the reduced moduli space consists of one point.

Transformations in the discrete group $K_0$ are expressed in one of the following two forms:
\[
\begin{align*}
\alpha^p \beta^{2q} \gamma^r &= (p, 2q, \frac{r}{n} + pq), \\
\alpha^p \beta^{2p+1} \gamma^r &= (p, 2q + 1, -\frac{r}{n} + \frac{1}{2}p(2q + 1)) R_2(\pi),
\end{align*}
\] (4.36) (4.37)

where $p, q, r \in \mathbb{Z}$. By these transformation, any point in Nil can be moved into the region $0 \leq x, y \leq 1$. All the transformations in $K_0$ that leave this region invariant are written in the form
\[
\gamma^r(x, y, z) = (x, y, z + \frac{r}{n}).
\] (4.38)

Hence the fundamental region $D_0$ of $K_0$ is given by $0 \leq x, y \leq 1/2$ and $0 \leq z \leq 1/n$, and its volume by $\Omega = 1/n$. Since there exists no moduli freedom, the canonical 1-form is given by
\[
\Theta = \dot{\alpha}p_\alpha + \dot{\beta}_+ p_+ + \dot{\beta}_- p_-.
\] (4.39)

### 4.2.3 $T^3(n)/\mathbb{Z}_2$

In this case all the generators of the moduli are always contained in $\text{Nil} \times D_2$. It is easy to see that by a similar argument in the case $T^3(n)$, HPDs transform a generic moduli matrix to the form (1.27) with $n$ replaced by $2n$. Hence the deformation map is given by Eq.(4.28) with the same replacement.

Transformations in $K_0$ are written as
\[
\alpha^p \beta^q \gamma^r = (p, q, \frac{r}{2n} + \frac{1}{2}pq) R_3(\pi r).
\] (4.40)

Hence by the transformations of the forms $\alpha^p \beta^q$, we can move any point in Nil to the region $|x|, |y| \leq 1/2$. This region is left invariant only by the transformations of the form $\gamma^r$. It is easy to see that we can move any point to $0 \leq z \leq 1/2n$ by these transformations. Hence the fundamental region $D_0$ of $K_0$ is given by $|x|, |y| \leq 1/2$ and $0 \leq z \leq 1/2n$, and its volume by $\Omega = 1/2n$. The canonical 1-form and the Hamiltonian are given by the same expressions for $T^3(n)$.

Finally note that $\alpha$, $\beta$ and $\gamma^2$ satisfy the same relations as those among $\alpha$, $\beta$ and $\gamma$ for $T^3(n)$. Hence we obtain the exact sequence for the fundamental group of the present manifold $M$,
\[
1 \to \pi_1(T^3(n)) \to \pi_1(M) \to \mathbb{Z}_2 \to 1,
\] (4.41)

which implies that $M \approx T^3(n)/\mathbb{Z}_2$. 

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4.2.4 \(T^3(2n)/\mathbb{Z}_2 \times \mathbb{Z}_2\)

In this case, in order for \(\alpha\) to be in \(\text{Nil} \times D_2\), \(R_\alpha = R_1(\pi)R_3(\theta)\) should be \(R_1(\pi)\) or \(R_2(\pi)\). Hence, as in the previous case, we can put \(R_\alpha = R_1(\pi)\) by a HPD. Further by the freedom \(d^1, d^2\) and \(d^3\) of HPDs, we can put \(a^3, c^1\) and \(c^2\) to zero. Then the HPD with \(\hat{A} = \begin{pmatrix} 1/a^1 & 0 \\ 0 & -1/a^2 \end{pmatrix}\) brings the generators to canonical forms

\[
\alpha = R_1(\pi)(1, -1, 0), \quad \gamma = R_3(\pi)(0, 0, 1/n).
\]

Hence the moduli space is reduced to a single point.

To determine the fundamental region of this moduli \(K_0\), let us introduce the transformation \(\beta\) defined by

\[
\beta := \alpha^{-1}\gamma = R_2(\pi)(1, 1, 1/n).
\]

Then we obtain

\[
[\alpha^2, \gamma^2] = 1, \quad [\beta^2, \gamma^2] = 1, \quad [\alpha^2, \beta^2] = (\gamma^2)^{2n}.
\]

From these relations we easily see that any transformation \(g\) in \(K_0\) is expressed in the form

\[
g = \alpha^{2p}\beta^{2q}\gamma^{2r} \times \{1, \alpha, \beta, \gamma\}.
\]

In particular, since \(\{1, \alpha, \beta, \gamma\}\) forms the Klein group \(\text{mod}(\alpha^2, \beta^2, \gamma^2)\), we obtain the following exact sequence for the fundamental group of the present manifold \(M\):

\[
1 \to \pi_1(T^3(2n)) \to \pi_1(M) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1,
\]

which justifies our notation for \(M\).

The fundamental region \(D_0\) is determined as follows. First by transformations of the form \(\alpha^{2p}\beta^{2q} \times \{1, \alpha\}\), we can move any point to the region \(0 \leq x \leq 1\), and by those of \(\{1, \alpha^2\} \times \beta^{2q}\) to \(0 \leq y \leq 1\). Only transformations which leave the region \(0 \leq x, y \leq 1\) invariant are \(\gamma^{2r} = (0, 0, 2r/n)\). Hence \(D_0\) is given by \(0 \leq x, y \leq 1\) and \(0 \leq z \leq 2/n\), and its volume by \(\Omega = 2/n\). Since there exists no moduli freedom, the canonical 1-form is given by the same expression as Eq.(4.39) for \(K^3(n)\).

4.3 \(\tilde{G}_0 = \text{Isom}_0(\text{Nil})\)

From the arguments in §4.1, \(\text{Isom}_0(\text{Nil})\)-invariant data are always invariant by \(\text{Isom}(\text{Nil})\), and the corresponding phase space is given by

\[
\Gamma^\circ_{\text{H}, \text{D}}(\text{Nil}, \text{Isom}(\text{Nil})) = \{Q = [Q_1, Q_1, Q_3], P = [P^1, P^2, P^3]\}.
\]

HPDs (4.15) for \(\text{Isom}(\text{Nil})\) transform the invariant basis (4.3) as

\[
f^*\chi^I = A^I_J \chi^J; \quad A = \{1, R_1(\pi)\} \begin{pmatrix} kR & 0 \\ 0 & 0 \end{pmatrix}.
\]
The moduli matrix is reduced by HPDs to the canonical form

\[
\begin{pmatrix}
1 & X & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/n
\end{pmatrix}
\]

\(R_3(\phi)\)

(4.50)

There exists no freedom of residual HPDs except in the subspace \(Y = 0\) for which the transformation \(X \rightarrow X^{-1}\) gives the residual HPD. There remain modular transformations isomorphic to \(Z\). For the base point \(K_0\) with \(X = 1\) and \(Y = 0\), the deformation map is given by

\[
f_\lambda(\mathbf{x}) = F\mathbf{x}; \quad F = \begin{pmatrix}
X & Y & 0 \\
0 & X^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
| Symmetry | Topology | Moduli | \( \Omega \) | \( \Theta \) |
|----------|----------|--------|--------------|--------------|
| Isom(Nil) | \( T^3(n) \) | \[
\begin{pmatrix}
X & Y & 0 \\
0 & X^{-1} & 0 \\
0 & 0 & 1/n \\
\end{pmatrix}
\] (\( X > 0 \)) | \( \frac{1}{n} \) | \( \hat{\alpha}p_\alpha + \hat{\beta}p_+ \) |
| \( K^3(n) \) | | \[
\begin{pmatrix}
X & 0 & 0 \\
0 & X^{-1} & 0 \\
0 & 0 & 1/n \\
\end{pmatrix}
\] (\( X > 0, R_\beta = R_3(\pi) \)) | \( \frac{1}{n} \) | " |
| \( T^3(n) / \mathbb{Z}_2 \) | | \[
\begin{pmatrix}
X & Y & 0 \\
0 & X^{-1} & 0 \\
0 & 0 & 1/2n \\
\end{pmatrix}
\] (\( X > 0, R_\gamma = R_3(\pi) \)) | \( \frac{1}{2n} \) | " |
| \( T^3(2n) / \mathbb{Z}_2 \times \mathbb{Z}_2 \) | | \[
\begin{pmatrix}
X & X & 0 \\
-X^{-1} & X^{-1} & 0 \\
0 & 1/n & 1/n \\
\end{pmatrix}
\] (\( X > 0, R_\alpha = R_1(\pi), R_\beta = R_2(\pi), R_\gamma = R_3(\pi) \)) | \( \frac{2}{n} \) | " |
| \( T^3(n) / \mathbb{Z}_3 \) | | \[
\begin{pmatrix}
0 & \sqrt{3}/2 & 0 \\
0 & 0 & \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{12n} \\
\end{pmatrix}
\] (\( R_\gamma = R_3(2\pi/3) \)) | \( \frac{1}{4n} \) | " |
| \( T^3(n) / \mathbb{Z}_4 \) | | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/4n \\
\end{pmatrix}
\] (\( R_\gamma = R_3(\pi/2) \)) | \( \frac{1}{4n} \) | " |
| \( T^3(n) / \mathbb{Z}_6 \) | | \[
\begin{pmatrix}
1 & 1/2 & -1/2 \\
0 & \sqrt{3}/2 & 0 \\
0 & 0 & \frac{\sqrt{3}}{12n} - \frac{\sqrt{3}}{16} \\
\end{pmatrix}
\] (\( R_\gamma = R_3(\pi/3) \)) | \( \frac{1}{8n} \) | " |

Table 6 (continued)
For this transformation $\text{Tr} \hat{F} F^{-1} Q P$ vanishes. Hence the canonical structure is degenerate in the moduli sector, and the canonical 1-form is given by

$$\Theta = \dot{\alpha} p_+ + \dot{\beta} p_+.$$  \hspace{1cm} (4.52)

The volume of the fundamental region is $\Omega = 1/n$.

### 4.3.2 $K^3(n)$

By the rotation $R_3(-\pi/2 + \theta/2)$ in HPDG$^+$(Nil, Isom(Nil)), $\alpha$ and $\beta$ transform to generators of the forms $(a^1, 0, a^3)$ and $R_3(\pi)(b^1, b^2, b^3)$, respectively. Further by a transformation $d \in \text{Nil}$, they can be put into the form $(a^1, 0, 0)$ and $R_2(\pi)(0, b^2, 0)$, and by rotations $R_1(\pi)$ and $R_2(\pi)$ if necessary, $a^1$ and $b^2$ can be made positive. Hence the reduced moduli matrix is given by

$$\begin{pmatrix} a & b & c \\ X & 0 & 0 \\ 0 & X^{-1} & 0 \\ 0 & 0 & 1/n \end{pmatrix}.$$  \hspace{1cm} (4.53)

There exists no residual freedom of modular transformations, but the discrete HPD $X \rightarrow X^{-1}$ remains.

For the base point $K_0$ with $X = 1$, the deformation map is given by

$$f_\lambda(x) = Fx; \quad F = \begin{pmatrix} X & 0 & 0 \\ 0 & X^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (4.54)

which is a special form of that in the previous case. Hence the canonical structure becomes degenerate in the moduli sector again, and the canonical 1-form is given by Eq.(4.52). $\Omega$ is equal to $1/n$.

### 4.3.3 $T^3(n)/\mathbb{Z}_2$

By the freedom of Nil-translation $d$ of HPDs, $c^1$ and $c^2$ can be put to zero. Hence by rotations around the $z$-axis and scalings in HPDG$^+$(Nil, Isom(Nil)), the moduli matrix is put in the form

$$\begin{pmatrix} a & b & c \\ X & Y & 0 \\ 0 & X^{-1} & 0 \\ 0 & 0 & 1/2n \end{pmatrix}.$$  \hspace{1cm} (4.55)

This is obtained from that for $T^3(n)$ by replacing $n$ by $2n$, and the same deformation map can be used. Hence the results for $T^3(n)$ apply to the present case by the same replacement.

### 4.3.4 $T^3(2n)/\mathbb{Z}_2 \times \mathbb{Z}_2$

By a translation parallel to the $x - y$ plane and a rotation around the $z$-axis, $\alpha$ and $\gamma$ is put into the forms $R_1(\pi)(a^1, a^2, 0)$ and $R_3(\pi)(0, 0, -a^1a^2/n)$, respectively. By the rotation $R_1(\pi)$ and scaling they are further transformed to $R_1(\pi)(X, -1/X, 0)$ and $R_3(\pi)(0, 0, 1/n)$. Hence by introducing $\beta$ defined by

$$\beta := \alpha^{-1}\gamma = R_2(\pi)(X, 1/X, 1/n),$$  \hspace{1cm} (4.56)
we obtain the following canonical form of the moduli matrix:

\[
\begin{pmatrix}
a & b & c \\
x & X & 0 \\
x^{-1} & X^{-1} & 0 \\
0 & 1/n & 1/n
\end{pmatrix}
\]

The deformation map coincides with that for \( K^3(n) \). Therefore the canonical 1-form is given by Eq.\( (4.52) \), and \( \Omega \) by \( 2/n \).

4.3.5 \( T^3(n)/\mathbb{Z}_k (k = 3, 4, 6) \)

By translations, scalings and rotations in HPDG\(^+\)(Nil, Isom(Nil)), we can completely eliminate the continuous freedom of the moduli. Further by \( R_1(\pi) \) we can transform \( R_{\gamma} = R_3(-2\pi/k) \) to \( R_3(2\pi/k) \). Hence the moduli space is reduced to a single point whose moduli matrix is given in Table 6.

The fundamental region is determined as follows. First from the commutation relations in Table 5 we obtain

\[
[\gamma^k, \alpha] = 1, \quad [\gamma^k, \beta] = 1, \quad [\alpha, \beta] = (\gamma^k)^n,
\]

which leads to the exact sequence

\[
1 \to \pi_1(T^3(n)) \to \pi_1(M) \to \{1, \gamma, \ldots, \gamma^{k-1}\} \to 1.
\]

Since \( \{1, \gamma, \ldots, \gamma^{k-1}\} \) is isomorphic to \( \mathbb{Z}_k \) mod \( \gamma^k \), we obtain \( M \approx T^3(n)/\mathbb{Z}_k \).

In the case \( k = 3 \), transformations in \( K_0 \) corresponding to the above moduli matrix are given by one of the following three:

\[
\alpha^p \beta^q \gamma^{3r} = \left( p - \frac{q}{2}, \frac{\sqrt{3}}{2} q, \frac{\sqrt{3}^2}{4} pq + \frac{\sqrt{3}}{2n} r \right),
\]

\[
\alpha^p \beta^q \gamma^{3r+1} = \left( p - \frac{q}{2} - \frac{1}{4} \sqrt{3} \frac{2}{2} q + \frac{\sqrt{3}^3}{4} \frac{2}{4} pq + \frac{\sqrt{3}^3}{2n} (r + \frac{1}{3}) + \frac{\sqrt{3}^3}{8} - \frac{\sqrt{3}^2}{48} \right)
\]

\[
\times R_3 \left( \frac{2\pi}{3} \right),
\]

\[
\alpha^p \beta^q \gamma^{3r+2} = \left( p - \frac{q}{2} - \frac{1}{2} \frac{\sqrt{3}^2}{2} q, \frac{\sqrt{3}^3}{2n} \right)
\]

\[
\times R_3 \left( -\frac{2\pi}{3} \right).
\]

From this we find that the hexagon with the vertices

\[
\begin{pmatrix}
x_j \\
y_j
\end{pmatrix} = R \left( \frac{j\pi}{3} \right) \begin{pmatrix} 0 \\ 1/\sqrt{3} \end{pmatrix} + \begin{pmatrix} -1/4 \\ \sqrt{3}/4 \end{pmatrix} \quad (j = 0, \ldots, 5)
\]

gives a fundamental region in the \( x - y \) plane. Further \( \gamma^r \) leaves this region invariant. Hence noting that \( \gamma^3 \) maps \( z = 0 \) plane to \( z = \sqrt{3}/2n \) plane, we find that the volume of the fundamental region is given by \( \Omega = 1/4n \).
Next in the case $k = 4$, since the generators are simply given by
\[ \alpha = (1, 0, 0), \quad \beta = (0, 1, 0), \quad \gamma = R_{3}((\pi/2)(0, 0, 1/4n)), \] (4.64)
the fundamental region is given by $0 \leq x, y \leq 1, 0 \leq z \leq 1/4n$, and its volume by $\Omega = 1/4n$.

Finally in the case $k = 6$, we find by an argument similar to the case $k = 3$ that the hexagon with vertices
\[ \left( \begin{array}{c} x_j \\ y_j \end{array} \right) = R \left( \frac{j\pi}{3} \right) \left( \begin{array}{cc} 1/\sqrt{3} \\ 0 \end{array} \right) + \left( \begin{array}{c} 1 \\ -\sqrt{3} \end{array} \right) \quad (j = 0, \cdots, 5) \] (4.65)
gives a fundamental region in the $x - y$ plane, and $\gamma^{6}$ maps $z = 0$ plane to $z = \sqrt{3}/2n$ plane. Hence the volume of the fundamental region is give by $\Omega = 1/8n$.

In all the cases $\Theta$ and $H$ are give by the same expressions as those for $T^{3}(n)$.

4.4 Summary

As a summary of the results, the form of the reduced moduli matrix, the volume $\Omega$ of the fundamental region for the base moduli point and the canonical 1-form $\Theta$ are listed for each invariance group and topology in Table 6. Further the degrees of freedom of the covering data $Q$ and $P$, the dimension $N_{m}$ of the reduced moduli space, the total dimension $N$ of the invariant phase space, the non-degenerate and the degenerate degrees of freedom of the canonical variables, $N_{c}$ and $N_{d}$, and the number of connected components of the invariant phase space are listed in Table 7.

From these tables we see that the canonical structure of $\Gamma_{LH, inv}^{+}$(Nil, Nil $\times D_{2}$) is similar to that of $\Gamma_{LH, inv}^{+}(E^{3}, R^{3} \times D_{2})$, and no degeneracy occurs. On the other hand, for $\Gamma_{LH, inv}^{+}$(Nil, Isom(Nil)), the canonical structure becomes completely degenerate in the moduli sector. Though the dimensions of the phase space and the reduced moduli are different, the degrees of degeneracy coincide with those for $\Gamma_{LH, inv}^{+}(E^{3}, R^{2} \times SO(2) \times D_{2})$. This suggests that in these cases the degeneracy of the canonical structure is closely related with the
structure of the isotropy groups of the invariance groups, although the $\Gamma_{LH, inv}^+(\text{Nil, Isom(\text{Nil})})$ has no simple relation to $\Gamma_{LH, inv}^+(\text{Nil, Nil} \times D_2)$ as in the case of $E^3$.

§5 LHS of type Sol

In this section we determine the canonical structure of locally homogeneous pure gravity systems on compact closed orientable manifolds of type Sol.

5.1 Basic properties

5.1.1 $G_{\text{max}}$ and $\text{HPDG}^+(G_{\text{max}})$

Sol is a simply connected 3-dimensional group classified as type VI(0) in the Bianchi scheme, and homeomorphic to $R^3$. We adopt the parameterization of the group such that the product of two elements are written as

$$(a, b, c)(x, y, z) = (a + e^{-c}x, b + e^cy, c + z). \quad (5.1)$$

In terms of the generators of its Lie algebra,

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z - x\partial_x + y\partial_y, \quad (5.2)$$

its structure is expressed as

$$[\xi_1, \xi_2] = 0, \quad [\xi_3, \xi_1] = \xi_1, \quad [\xi_3, \xi_2] = -\xi_2. \quad (5.3)$$

As the invariant basis we adopt

$$\chi^1 = e^zdx + e^{-z}dy, \quad \chi^2 = e^zdx - e^{-z}dy, \quad \chi^3 = dz, \quad (5.4)$$

$$d\chi^1 = \chi^2 \wedge \chi^2, \quad d\chi^2 = \chi^3 \wedge \chi^1, \quad d\chi^3 = 0, \quad (5.5)$$

which is obtained by rotation of angle $\pi/4$ around the $z$-axis from the natural invariant basis.

The maximally symmetric metric on Sol is given by

$$ds^2 = Q_1[(\chi^1)^2 + (\chi^2)^2] + Q_3(\chi^3)^2 = 2Q_1(e^{2z}dx^2 + e^{-2z}dy^2) + Q_3dz^2, \quad (5.6)$$

where $Q_1$ and $Q_3$ are arbitrary positive constants. From this we see that Isom(Sol) has 8 connected components, and Sol is its maximal connected subgroup. The other discrete components are represented by the product of Sol and elements of the discrete subgroup of rank 8 generated by $R_3(\pi), -R_1(\pi)$ and $J$ defined by

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (5.7)$$
which is the rotation of angle $\pi$ around the line $x = y, z = 0$. Among these 8 components, four of them consist of orientation-reversing transformations, while the other four give the maximal orientation-preserving symmetry group $\text{Isom}^+(\text{Sol}) = \text{Sol} \times \{1, R_3(\pi), J, JR_3(\pi)\} \cong \text{Sol} \times D_2$. $f \in \text{Isom}^+(\text{Sol})$ is written as

$$f(x) = \{1, J\} \times \{1, R_3(\pi)\} \times \left[ \begin{pmatrix} e^{-c^3} & 0 & 0 \\ 0 & e^{c^3} & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} \right], \quad (5.8)$$

by which the invariant basis transforms as

$$f^* \chi^I = A^I J \chi^J; \quad A = \{1, R_1(\pi)\} \times \{1, R_3(\pi)\}. \quad (5.9)$$

The automorphism group of Sol consists of elements which transform the generators of the Lie algebra as

$$\phi(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3) \{1, J\} \times \begin{pmatrix} k_1 & 0 & c^1 \\ 0 & k_2 & -c^2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.10)$$

They are induced from $f \in \text{HPDG}^+(\text{Sol}, \text{Sol})$ given by

$$f(x) = \{1, J\} \times \left[ \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} d^1 - c^1 e^{-z} \\ d^2 + c^2 e^z \\ d^3 \end{pmatrix} \right]. \quad (5.11)$$

The corresponding transformation of the invariant basis is given by

$$f^* \chi^I = A^I J \chi^J; \quad A = \{1, R_1(\pi)\} \times \begin{pmatrix} k_+ & k_- & c^+ \\ k_- & k_+ & c^- \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.12)$$

where $k_\pm$ and $c_\pm$ are

$$k_\pm := \frac{1}{2}(k_1 e^{d^3} \pm k_2 e^{-d^3}), \quad c_\pm := c^1 e^{d^3} \pm c^2 e^{-d^3}. \quad (5.13)$$

The subset $\text{HPDG}^+(\text{Sol}, \text{Isom}^+(\text{Sol}))$ is easily obtained from these HPDs. Its elements are expressed as

$$f(x) = \{1, J\} \left[ \begin{pmatrix} k e^{-d^3} & 0 & 0 \\ 0 & k e^{d^3} & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} d^1 \\ d^2 \\ d^3 \end{pmatrix} \right], \quad (5.14)$$

which transforms the invariant basis as

$$f^* \chi^I = A^I J \chi^J; \quad A = \{1, R_1(\pi)\} \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.15)$$
Space

Fundamental group and representation

\[ \text{Sol}(n; \omega_1, \omega_2) \quad [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \alpha^p \beta^q, \gamma \beta \gamma^{-1} = \alpha^r \beta^s; \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \]

\[ n = p + s, \omega_1 = \frac{p-s+\sqrt{n^2-4}}{2}, \omega_2 = \frac{p-s-\sqrt{n^2-4}}{2} \]

\[ n > 2 : \quad \alpha = (b^1 \omega_1, b^2 \omega_2, 0) \]
\[ \beta = (b^1, b^2, 0) \quad (b^1 b^2 \neq 0) \]
\[ \gamma = (c^1, c^2, c^3); \quad e^{c^3} = \frac{n+\sqrt{n^2-4}}{2} \]

\[ n < -2 : \quad \alpha = (b^1 \omega_1, b^2 \omega_2, 0) \]
\[ \beta = (b^1, b^2, 0) \quad (b^1 b^2 \neq 0) \]
\[ \gamma = R_3(\pi)(c^1, c^2, c^3); \quad e^{c^3} = \frac{|n|+\sqrt{n^2-4}}{2} \]

Table 8: Fundamental groups and their representation in \( \text{Isom}^+(\text{Sol}) \) of compact closed orientable 3-manifolds of type \( \text{Sol} \)

5.1.2 Phase space of homogeneous covering data

It is easy to see that by HPDs (5.11) the components \( Q_{IJ} \) of the covering homogeneous metric \( \tilde{q} \) with respect to the invariant basis are put to the diagonal form \( Q = [Q_1, 1/Q_1, Q_3] \). For this metric the diffeomorphism constraint (2.63) is written as

\[ P^{12} = P^{23} = P^{13} = 0. \quad (5.16) \]

This implies that the \( \text{Sol} \)-invariant covering data satisfying the diffeomorphism constraint is always invariant by \( \text{Isom}^+(\text{Sol}) \). Hence the only non-empty phase space of homogeneous covering data is

\[ \Gamma^+_{A,D}(\text{Sol}, \text{Isom}^+(\text{Sol})) = \left\{ Q = [Q_1, Q_2, Q_3], P = [P^1, P^2, P^3] \right\}. \quad (5.17) \]

Thus, when expressed by the components with respect to the invariant basis, the covering data have the same structure as those for \( E^3 \) and \( \text{Nil} \). In particular the same parameterization of them (3.30)-(3.32) and (3.24)-(3.26) can be used to diagonalize \( \text{Tr} \tilde{Q} P \) in \( \Theta \). Further, since the Ricci scalar curvature of the metric \( \tilde{q} \) is given by

\[ R = -\frac{(Q_1 + Q_2)^2}{2Q_1 Q_2 Q_3} = -2e^{-2\alpha+4\beta_+} \cosh^2(2\sqrt{3}\beta_-), \quad (5.18) \]

the Hamiltonian is given by

\[ H = \frac{\kappa^2}{12\Omega} Ne^{-3\alpha} \left[ -p^2_\alpha + p^2_- + p^2_+ + \frac{12\Omega}{\kappa^2} e^{4(\alpha+\beta_+)} \cosh^2(2\sqrt{3}\beta_-) \right]. \quad (5.19) \]
Table 9: Canonical structure of compact orientable closed 3-manifold of type Sol

5.2 Topology and Moduli space

All the orientable compact closed manifolds of type Sol are torus bundles over $S^1$ with hyperbolic gluing maps. Conversely any torus bundle over $S^1$ defined by a hyperbolic gluing map admit a locally homogeneous structure modeled on Sol. Hence in terms of a matrix $Z = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, the fundamental group $\pi_1(M)$ is described by the relations among the generators of the fiber torus, $\alpha$ and $\beta$, and the generator $\gamma$ for the base space $S^1$ as in Table 8. The condition that the gluing map is hyperbolic is expressed in terms of $n = p + s$ as $|n| > 2$.

Abstract groups defined by $Z$ and $Z'$ in $\text{SL}(2, \mathbb{Z})$ are isomorphic if and only if there exist an integer matrix $V \in \text{GL}(2, \mathbb{Z})$ such that

$$Z' = VZV^{-1}. \quad (5.20)$$

Since this modular transformation preserves the trace of $Z$, fundamental groups with different values of $n$ are obviously non-isomorphic. However, the isomorphism class is not classified only by $n$. In fact, since the two roots of the quadratic equation

$$x = \frac{px + q}{rx + s} \quad (5.21)$$

transform as

$$x' = \frac{ax + b}{cx + d}; \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (5.22)$$

it can be easily shown that the isomorphism classes of the fundamental group are in one-to-one correspondence with the equivalence classes of the roots of Eq. (5.21) under the modular transformation (5.22) for given $n$. From the theorem of the continued fraction, the latter are determined by the divisor of $n^2 - 4$ with the form $k^2$ and the equivalence classes of the reduced quadratic irrationals associated with the determinant $D = (n^2 - 4)/k^2$, whose count is finite. Thus for each $n$ there exist a finite number of different isomorphism classes. For example, for $n = 8$, there exist two isomorphism classes represented by matrices $\begin{pmatrix} 7 & 6 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$.

In Ref. [3] it is stated that the isomorphism class is in one-to-one correspondence with the value of $n$, but it is not correct.
The nature of the embedding of the fundamental group differs for \( n > 2 \) and \( n < -2 \); in the former case \( \pi_1(M) \) is embedded into Sol, while in the latter case \( \gamma \) is contained in the component \( R_3(\pi) \times \text{Sol} \), as shown in Table 8. Now we determine their canonical structure.

### 5.3 Canonical Structure

First we consider the case \( n > 2 \). In this case we can put \( c^1 = c^2 = 0 \) by a HPD of the form \((d^1, d^2, 0) \in \text{Sol}\). Then the HPD with \( d^1 = d^2 = 0 \) keeps the form of \( \gamma \) and transforms the \((x, y)\) components of \( \alpha \) and \( \beta \) as

\[
\begin{pmatrix}
\omega_1 & b^1 \\
\omega_2 & b^2
\end{pmatrix} =
\begin{pmatrix}
ke^{-d^3}b^1 & 0 \\
0 & ke^{d^3}b^2
\end{pmatrix}
\begin{pmatrix}
\omega_1 & 1 \\
\omega_2 & 1
\end{pmatrix}.
\]

(5.23)

From the theorem of the continued fraction we can always find \( \omega_1 \) from each equivalence class such that \( \omega_1 > 1 \) and \(-1 < \omega_2 < 0\) corresponding to the reduced quadratic irrationals with the helps of the combination of the modular transformation \( \gamma \rightarrow \gamma^{-1} \) and the HPD \( J \) if necessary. For this choice \((b^1, b^2)\) can be transformed to \((1, 1)\) for \( b^1b^2 > 0 \) and to \((1, -1)\) for \( b^1b^2 < 0 \), respectively. Hence the moduli matrix is reduced to

\[
\begin{pmatrix}
a & b & c
\end{pmatrix} =
\begin{pmatrix}
\omega_1 & 1 & 0 \\
\pm\omega_2 & \pm1 & 0 \\
0 & 0 & e^{c^3}
\end{pmatrix},
\]

(5.24)

where \( e^{c^3} = (n + \sqrt{n^2 - 4})/2 \). Thus the moduli space \( \mathcal{M}(M, \text{Isom}^+(\text{Sol})) \) reduces to two points. These two points are connected only by orientation-reversing transformations. The canonical 1-form is simply given by

\[
\Theta = \dot{\alpha}p_{\alpha} + \dot{\beta}_+p_+ + \dot{\beta}_-p_-.
\]

(5.25)

For the moduli \( K_0 \) corresponding to Eq. (5.24), a generic transformation belonging to it is simply expressed as

\[
\alpha^u\beta^v\gamma^w = (u\omega_1 + v, \pm(u\omega_2 + v), we^{c^3}).
\]

(5.26)

Hence the fundamental region is given by a parallelepiped and its volume by

\[
\Omega = |\omega_1 - \omega_2|e^{c^3} = \frac{n^2 - 4 + n\sqrt{n^2 - 4}}{2|r|}.
\]

(5.27)

The Hamiltonian for the system is obtained just by inserting this into Eq. (5.19).

The argument for the case \( n < -2 \) is quite similar. Though \( \gamma \) now contains a factor \( R_3(\pi) \), it does not affect the above derivation, and we get the same result except that \( n \) should be replaced by \(|n|\) in the expression for \( \Omega \).

Thus for the locally homogeneous systems of type Sol, there exists no freedom of moduli and there occurs no degeneracy in the canonical structure.
§6 Summary and Discussion

In this paper we have developed a general algorithm to determine the diffeomorphism-invariant phase space and its canonical structure of locally homogeneous system on a compact closed 3-manifold, and have applied it to locally homogeneous pure gravity systems of the Thurston type $E^3$, Nil and Sol.

The main difference of our formulation from the conventional ones such as that adopted in Ref.[3, 4, 5] consists in the point that we have classified the diffeomorphism classes of canonical data which contain both the configuration variables and their conjugate momentums. Though this is a simple extension of the conventional approach, it has enabled us to obtain directly the canonical structure of the phase space of a locally homogeneous system from that of a generic system on a compact manifold. Further, together with a neat treatment of the invariance group of the canonical data, it has given us a natural decomposition of the diffeomorphism invariant phase space into the sector describing the local structure of locally homogeneous data and the reduced moduli sector describing the global structure of the data. This moduli sector is in general smaller than that obtained by just considering the structure of metric data. As we have shown, the dynamics of this moduli sector is frozen. Though this point was already clearly stated in Ref.[4] and was used in a crucial way in their formulation, the statement and the proof in our paper is more exact in the treatment of possible discrete components of the invariance group of data, to which the HPD freedom and the corresponding gauge-fixing in the moduli sector is very sensitive.

All the results of our analysis of the locally homogeneous pure gravity systems are contained in tables 3, 4, 6, 7 and 9. As these tables show, the canonical structure and the dynamics of locally homogeneous pure gravity systems are quite simple in our decomposition of the phase space variables to the local sector and the moduli sector, except for the degeneracy in the canonical structure. This degeneracy implies that locally homogeneous systems are not canonically closed in the full diffeomorphism-invariant phase space of generic data. In particular it is meaningless to discuss the quantum dynamics of a locally VII(0)-homogeneous system. It will be an interesting problem to find a canonically closed minimal sector which contains the locally VII(0)-system.

As we touched upon in the analysis of the locally homogeneous pure gravity systems, the reduction of the moduli space by gauge-fixing often leaves freedoms of discrete HPDs and modular transformations. These residual discrete transformations introduce non-trivial topological structures as well as conic singularities into the phase space. The influence of these topological structures of the classical phase space on their quantum theory may be an interesting problem.

Finally we comment on the other locally homogeneous pure gravity systems. In general locally homogeneous systems are covered by Bianchi models or the Kantowski-Sachs model. The former is further classified into two subclasses, class A and class B. In this classification our analysis only covers four of the class A Bianchi models, type I, II, VI(0) and VII(0). However, these models exhaust all the interesting cases except one as far as the canonical structure is concerned. First locally homogeneous systems modeled on $S^3$ (type IX) have no moduli freedom. Hence its canonical structure is essentially determined by their homogeneous covers. Second, among the class B Bianchi models, type V and type VII($A \neq 0$) belong to the same Thurston type, $H^3$, hence they do not have contin-
uous moduli freedom either (see Table 1). Among the remaining two Bianchi models with compact quotients, VIII and III, III can be embedded into two Thurston types, $H^2 \times E^1$ and $\tilde{SL}_2\mathbb{R}$. In the former case, the canonical structure is essentially determined by that in the 2-dimensional systems on $H^2$. On the other hand, in the latter case, the systems become locally VIII-homogeneous at the same time. Thus they have both class A and class B symmetries. This is the only important case we have not analyzed. Since our analysis shows that the canonical structure becomes in general degenerate when the invariance group has non-trivial isotropy groups, the analysis of these locally homogeneous systems of type $\tilde{SL}_2\mathbb{R}$ will give useful information on the origin of degeneracy of the canonical structure as well as the dynamics of local homogeneous systems with class B symmetry (cf. Ref. [5]). This problem will be pursued in a future work.

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References

[1] Ashtekar, A. and Samuel, J., Class. Quantum Grav. 8, 2129–2215 (1991).
[2] Fujiwara, Y. , Kodama, H. , and Ishihara, H., Class. Quantum Grav. 10, 859–868 (1993).
[3] Koike, T. , Tanimoto, M. , and Hosoya, A., J. Math. Phys. 35, 4855–4888 (1994).
[4] Tanimoto, M., Koike, T., and Hosoya, A., J. Math. Phys. 38, 350–368 (1997).
[5] Tanimoto, M., Koike, T., and Hosoya, A., Preprint YITP-97-24, 1–27 (1997).
[6] Lachi`eze-Rey, M. and Luminet, J.-P., Phys. Rep. C 254, 135–214 (1995).
[7] Fagundes, H.V., Gen. Rel. Grav. 24, 199–217 (1992).
[8] Kobayashi, S. and Nomizu, K., Foundations of Differential Geometry I, II, Interscience Pub. (1963).
[9] Thurston, W., The geometry and topology of 3-manifolds, Princeton Univ. Press (1979).
[10] Thurston, W.P., Bull. Amer. Math. Soc. 6, 357–381 (1982).
[11] Scott, P., Bull. London Math. Soc. 15, 401–487 (1983).
[12] Hempel, J., 3-Manifolds, Princeton Univ. Press (1976).
[13] Waldhausen, F., Ann. of Math. 87, 56–88 (1968).
[14] Mostow, G.D., *Strong rigidity of locally symmetric spaces*, Princeton Univ. Press (1968).

[15] Kodama, H., Prog. Theor. Phys. **94**, 475501 (1995).