ON FAR-OUTLYING CMC SPHERES IN ASYMPTOTICALLY FLAT RIEMANNIAN 3-MANIFOLDS

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Abstract. We extend the Lyapunov-Schmidt analysis of outlying stable CMC spheres in the work of S. Brendle and the second-named author [3] to the “far-off-center” regime and to include general Schwarzschild asymptotics. We obtain sharp existence and non-existence results for large stable CMC spheres that depend very delicately on the behavior of scalar curvature at infinity.

1. Introduction

We complement our recent work [5] on the characterization of the leaves of the canonical foliation as the unique large closed embedded stable constant mean curvature surfaces in strongly asymptotically flat Riemannian 3-manifolds. More precisely, we extend here the Lyapunov-Schmidt analysis of outlying stable constant mean curvature spheres that developed by S. Brendle and the second-named author in [3] to also include the far-off-center regime and general Schwarzschild asymptotics.

We begin by introducing some standard notation.

Throughout this paper, we consider complete Riemannian 3-manifolds \((M, g)\) so there are both a compact set \(K \subset M\) and a diffeomorphism \(M \setminus K \cong \{ x \in \mathbb{R}^3 : |x| > 1/2 \}\) such that in this chart at infinity, for some \(q > 1/2\) and non-negative integer \(k\),

\[ g_{ij} = \delta_{ij} + \tau_{ij} \]

where

\[ \partial_I \tau_{ij} = O(|x|^{-q-|I|}) \]

for all multi-indices \(I\) of length \(|I| \leq k\). Moreover, we require that the boundary \(\partial M\) of \(M\), if non-empty, is a minimal surface, and that the components of \(\partial M\) are the only connected closed minimal surfaces in \((M, g)\). We say that \((M, g)\) is \(C^k\)-asymptotically flat of rate \(q\).

It is convenient to denote, for \(r > 1\), by \(S_r\) the surface in \(M\) corresponding to the centered coordinate sphere \(S_r(0) = \{ x \in \mathbb{R}^3 : |x| = r \}\), and by \(B_r\) the bounded open region enclosed by \(S_r\) and \(\partial M\). Given \(A \subset M\), we let

\[ r_0(A) := \sup\{ r > 1 : B_r \subset A \}. \]
A particularly important example of an asymptotically flat Riemannian 3-manifold is Schwarzschild initial data

\[ M = \{ x \in \mathbb{R}^3 : |x| \geq m/2 \} \quad \text{and} \quad g = \left(1 + \frac{m}{2|x|}\right)^4 \sum_{i=1}^{3} dx^i \otimes dx^i \]

where \( m > 0 \) is the mass parameter.

We say that \((M, g)\) as above is \(C^k\)-asymptotic to Schwarzschild of mass \( m > 0 \), if, instead of (1), we have

\[ g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + \sigma_{ij} \]

where

\[ \partial_I \sigma_{ij} = O(|x|^{-2-|I|}) \]

for all multi-indices \( I \) of length \(|I| \leq k\).

The contributions in this paper combined with the key result in [5] lead to the following theorem.

**Theorem 1.1** ([5]). Let \((M, g)\) be a complete Riemannian 3-manifold that is \(C^6\)-asymptotic to Schwarzschild of mass \( m > 0 \) and whose scalar curvature vanishes. Every connected closed embedded stable constant mean curvature surface with sufficiently large area is a leaf of the canonical foliation.

The canonical foliation \( \{ \Sigma_H \}_0 < H < H_0 \) of \( M \setminus K \) through stable constant mean curvature spheres \( \Sigma_H \) (with respective mean curvature \( H \)) was discovered by G. Huisken and S.-T. Yau [8]. They show that, for every \( s \in (1/2, 1] \), there is \( H_s \in (0, H_0) \) such that \( \Sigma_H \) for \( H \in (0, H_s) \) is the only stable constant mean curvature sphere of mean curvature \( H \) in \((M, g)\) that encloses the ball \( \{ x \in \mathbb{R}^3 : |x| < H^{-s} \} \) in the chart at infinity. This characterization of the leaves was later refined by J. Qing and G. Tian [12]: Upon enlarging \( K \) and shrinking \( H_0 > 0 \) accordingly, if necessary, each \( \Sigma_H \) of the canonical foliation \( \{ \Sigma_H \}_0 < H < H_0 \) is the unique stable constant mean curvature sphere of mean curvature \( H \) in \((M, g)\) that encloses \( K \). In joint work with A. Carlotto [4], inspired by earlier work of J. Metzger and the second-named author [6], we have extended this characterization further under the additional assumption that the scalar curvature of \((M, g)\) is non-negative in the following way: Choose a point \( p \in M \). Every connected stable constant mean curvature sphere \( \Sigma \subset M \) that encloses \( p \) and whose area is sufficiently large is a leaf of the canonical foliation. Thus, to prove an unconditional uniqueness result along the lines of Theorem 1.1, it remains to understand large stable constant mean curvature spheres that are outlying in the sense that the region they enclose is disjoint and — in view of the results in [4] — far from \( K \). The center of mass flux integrals used in [8] [12] as a centering device vanish in this case regime; new ideas are needed. S. Brendle and the second-named author [3] have discovered a subtle relationship between scalar curvature and outlying stable constant mean curvature spheres. They give examples of divergent sequences \( \{ \Sigma_k \}_{k=1}^{\infty} \) of outlying stable constant mean curvature spheres in \((M, g)\) asymptotic to Schwarzschild with \( m > 0 \), which is the setting of [8] [12]. In fact, \( \Sigma_k \) is a perturbation of the coordinate sphere

\[ S_{\lambda_k} (\lambda_k \xi) = \{ |x - \lambda_k \xi| = \lambda_k : x \in \mathbb{R}^3 \} \]
in the chart at infinity, where $\xi \in \mathbb{R}^3$ is such that $|\xi| > 1$ and $\lambda_k \to \infty$. On the other hand, they show that no such sequences can exist in $(M, g)$ if the scalar curvature is non-negative, provided a further technical assumption on the expansion of the metric in the chart at infinity holds.

**Theorem 1.2** (S. Brendle and M. Eichmair [3]). Let $(M, g)$ be a complete Riemannian 3-manifold that is $C^4$-asymptotic to Schwarzschild with mass $m > 0$, where, in addition to (2), we also ask that

\begin{equation}
    g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + T_{ij} + o(|x|^{-2}) \quad \text{as} \quad |x| \to \infty
\end{equation}

with corresponding estimates for all partial derivatives of order $\leq 4$, and where $T_{ij}$ is homogeneous of degree $-2$. There does not exist a sequence of outlying stable constant mean curvature surfaces $\{\Sigma_k \subset M\}_{k=1}^{\infty}$ whose inner radius $r_0(\Sigma_k)$ and mean curvature $H(\Sigma_k)$ satisfy

\begin{equation}
    r_0(\Sigma_k) \to \infty \quad \text{and} \quad r_0(\Sigma_k)H(\Sigma_k) \to \eta > 0.
\end{equation}

In our recent work [5], we show that when $(M, g)$ is asymptotic to Schwarzschild with mass $m > 0$ and if the scalar curvature is non-negative, there are no sequences of embedded stable constant mean curvature spheres $\{\Sigma_k \subset M\}_{k=1}^{\infty}$ in $(M, g)$ with

\begin{align*}
    r_0(\Sigma_k) &\to \infty \quad \text{and} \quad r_0(\Sigma_k)H(\Sigma_k) \to 0.
\end{align*}

Assuming in addition that the metric has the form in Theorem 1.2 this leaves only the case of

\begin{align*}
    r_0(\Sigma_k) &\to \infty, \\
    \text{area}_g(\Sigma_k) &\to \infty, \\
    r_0(\Sigma_k)H(\Sigma_k) &\to \infty.
\end{align*}

To rule out this scenario, we revisit the Lyapunov–Schmidt reduction in [3].

Our other main goal here is to investigate whether top-order homogeneity in the expansion of the metric (3) off of Schwarzschild in Theorem 1.2 is really necessary. Neither the results [8, 12, 4] for spheres that are not outlying nor the main result of [5] require such an assumption. It turns out that Theorem 1.2 is false without additional such conditions.

**Theorem 1.3.** There is an asymptotically flat complete Riemannian 3-manifold $(M, g)$ with non-negative scalar curvature that is smoothly asymptotic to Schwarzschild of mass $m > 0$ in the sense that

\begin{equation}
    g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + \sigma_{ij}
\end{equation}

where

\begin{equation}
    \partial_I \sigma_{ij} = O(|x|^{-2-|I|})
\end{equation}

for all multi-indices $I$, which contains a sequence of outlying stable constant mean curvature spheres $\Sigma_k \subset M$ with

\begin{align*}
    r_0(\Sigma_k) &\to \infty \quad \text{and} \quad r_0(\Sigma_k)H(\Sigma_k) \to \eta > 0.
\end{align*}

It turns out that it is possible to recover a version of Theorem 1.2 without demanding homogeneity in the expansion of the metric if instead we impose a mild growth condition on the scalar curvature.
Theorem 1.4. Let \((M,g)\) be a complete Riemannian 3-manifold that is \(C^4\)-asymptotic to Schwarzschild in the sense that

\[
g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + \sigma_{ij}
\]

where \(\partial_I \sigma_{ij} = O(|x|^{-2-|I|})\) for all multi-indices \(I\) of length \(|I| \leq 4\). We also assume that either

\[
R = o(|x|^{-4}) \quad \text{as} \quad |x| \to \infty
\]

or

\[
x^i \partial_i (|x|^2 R) \leq o(|x|^{-4}) \quad \text{as} \quad |x| \to \infty.
\]

There does not exist a sequence of outlying stable constant mean curvature surfaces \(\Sigma_k \subset M\) whose inner radius \(r_0(\Sigma_k)\) and mean curvature \(H(\Sigma_k)\) satisfy

\[
r_0(\Sigma_k) \to \infty \quad \text{and} \quad r_0(\Sigma_k)H(\Sigma_k) \to \eta > 0.
\]

Note that (5) holds in either one of the following two cases.

(i) When \(R = 0\). This is for example the case when \((M,g)\) is time symmetric initial data for a vacuum spacetime.

(ii) When the metric in the chart at infinity has the special form (3) in Theorem 1.2 then

\[
R = S + o(|x|^{-4}) \quad \text{where} \quad S = \sum_{i,j=1}^3 (\partial_i \partial_j T_{ij} - \partial_i \partial_i T_{jj}).
\]

Note that \(S\) is homogeneous of degree \(-4\). Euler’s theorem gives that (5) holds if and only if \(R \geq -o(|x|^{-4})\). As such, Theorem 1.4 generalizes Theorem 1.2 to the non-homogeneous setting.

It is interesting to compare (5) to condition (H3) in S. Brendle’s version of Alexandrov’s theorem for certain warped products [1]. We remark that the example constructed in Theorem 1.3 is a warped product. We also mention that S. Ma has constructed examples of \((M,g)\) that contain large unstable constant mean curvature spheres [9]. The scalar curvature in these examples is negative in some places; see the discussion preceding the statement of Theorem 1.1 in [9] and the proof of Lemma 4.7 therein.

We now turn to the case of surfaces that are very far outlying in the sense that

\[
r_0(\Sigma_k)H(\Sigma_k) \to \infty.
\]

These surfaces are not within the scope of the Lyapunov–Schmidt reduction carried out in [3], where the case (4) is considered. The main difficulty in this regime is that the “Schwarzschild contribution” to the reduced area functional leveraged in [3] is no longer on the order of \(O(1)\), but is instead decaying. As such, it is necessary to obtain rather involved estimates for the reduced functional. To describe our results, we first recall some terminology from [3] that we will also adopt. A standard application of the implicit function theorem gives that for \(\lambda > 0\) and \(\xi \in \mathbb{R}^3\) large, we can find closed surfaces \(\Sigma_{(\xi,\lambda)}\) in the chart at infinity so that the following hold:

- \(\Sigma_{(\xi,\lambda)}\) bounds volume \(4\pi \lambda^3 / 3\) with respect to the metric \(g\).
\[ \Sigma_{(\xi, \lambda)} \text{ is the Euclidean graph of a function } u_{(\xi, \lambda)} \text{ on } S_\lambda(\lambda \xi), \text{ i.e.} \]
\[ \Sigma_{(\xi, \lambda)} = \{ \lambda \xi + y + u_{(\xi, \lambda)}(x) y/\lambda : x = \lambda \xi + y \in S_\lambda(\lambda \xi) \}, \]

where
\[ \sup_{S_\lambda(\lambda \xi)} |u_{(\xi, \lambda)}| + \lambda \sup_{S_\lambda(\lambda \xi)} |\nabla u_{(\xi, \lambda)}| + \lambda^2 \sup_{S_\lambda(\lambda \xi)} |\nabla^2 u_{(\xi, \lambda)}| = O(1/|\xi|). \]

- \( u_{(\xi, \lambda)} \) is orthogonal to the first spherical harmonics on \( S_\lambda(\lambda \xi) \) with respect to the Euclidean metric.
- The mean curvature of \( \Sigma_{(\xi, \lambda)} \) with respect to \( g \) viewed as a function on \( S_\lambda(\lambda \xi) \) is the restriction of a linear function.

Given a sequence of connected closed stable constant mean curvature surfaces \( \{\Sigma_k\}_{k=1}^\infty \) with \( r_0(\Sigma_k) \to \infty \) and \( r_0(\Sigma_k)H(\Sigma_k) \to \infty \), the same argument as in [3, p. 676] shows that \( \Sigma_k = \Sigma_{(\xi_k, \lambda_k)} \) for appropriate \( \lambda_k > 0 \) and \( \xi_k \in \mathbb{R}^3 \) when \( k \) is sufficiently large. Note that \( \lambda_k > 0 \) and \( \xi_k \in \mathbb{R}^3 \) are both large in this case. Whether \((M, g)\) admits such sequences can now be decided using the following result.

**Theorem 1.5.** Let \((M, g)\) be a complete Riemannian 3-manifold that is \(C^{5+\ell}\)-asymptotic to Schwarzschild with mass \( m = 2 \), where \( \ell \geq 0 \) is an integer. Let \( \lambda > 0 \) and \( \xi \in \mathbb{R}^3 \) be large. We have\(^1\)
\[ \text{area}_g(\Sigma_{(\xi, \lambda)}) = 4\pi \lambda^2 - \frac{2\pi}{15} \lambda^4 R(\lambda \xi) - \frac{\pi}{105} \lambda^6 (\Delta R)(\lambda \xi) - \frac{8\pi}{35} |\xi|^{-6} + O(\lambda^{-1}|\xi|^{-6}) + O(|\xi|^{-7}) \]
where \( R \) is the scalar curvature of \((M, g)\). This expansion can be differentiated \( \ell \) times with respect to \( \xi \).

As in [3], we use that for \( \ell \geq 1 \), the map
\[ \xi \mapsto \text{area}_g(\Sigma_{(\xi, \lambda)}) \]
has a critical point at \( \xi \) if, and only if, \( \Sigma_{(\xi, \lambda)} \) is a constant mean curvature sphere. If \( \ell \geq 2 \), then the critical point is stable if, and only if, \( \Sigma_{(\xi, \lambda)} \) is a stable constant mean curvature sphere. This immediately leads to the following corollary.

**Corollary 1.6.** Let \((M, g)\) be a complete Riemannian 3-manifold that is \(C^6\)-asymptotic to Schwarzschild in the sense that
\[ g_{ij} = \left( 1 + \frac{m}{2|x|} \right)^4 \delta_{ij} + \sigma_{ij} \]
where \( \partial_I \sigma_{ij} = O(|x|^{-2-|I|}) \) for all multi-indices \( I \) of length \( |I| \leq 6 \). Assume that the scalar curvature vanishes. There does not exist a sequence of connected closed stable constant mean curvature surfaces \( \{\Sigma_k\}_{k=1}^\infty \) in \((M, g)\) with
\[ r_0(\Sigma_k) \to \infty, \quad \text{area}_g(\Sigma_k) \to \infty, \quad \text{and} \quad r_0(\Sigma_k)H(\Sigma_k) \to \infty. \]

Lyapunov-Schmidt reduction has also been used by e.g. R. Ye [13], S. Nardulli [10], and F. Pacard and X. Xu in [11] to study when small geodesic spheres admit perturbations to constant

\(^1\)We may compute the Laplacian of scalar curvature either with respect to \( g \) or with respect to the Euclidean background metric in the chart at infinity. The difference may be absorbed into the error terms of the expansion.
mean curvature. S. Nardulli [10] has studied the expansion for small volumes of the isoperimetric profile of a Riemannian manifold.

The analogue of Theorem 1.4 in this setting is not so clear-cut. We have the following result.

Corollary 1.7. Let \((M,g)\) be a complete Riemannian 3-manifold that is \(C^7\)-asymptotic to Schwarzschild in the sense that
\[
g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + \sigma_{ij}
\]
where \(\partial_I \sigma_{ij} = O(|x|^{-2-|I|})\) for all multi-indices \(I\) of length \(|I| \leq 6\). We also assume that the scalar curvature \(R\) of \((M,g)\) is radially convex at infinity in the sense that
\[
x_i x_j \partial_i \partial_j R \geq 0
\]
outside of a compact set. There does not exist a sequence of connected closed stable constant mean curvature surfaces \(\{\Sigma_k\}_{k=1}^\infty\) in \((M,g)\) with
\[
r_0(\Sigma_k) \to \infty, \quad \text{area}_g(\Sigma_k) \to \infty, \quad \text{and} \quad r_0(\Sigma_k)H(\Sigma_k) \to \infty.
\]

It turns out that the hypothesis (7) is surprisingly sharp. Comparing with Theorem 1.2 or Theorem 1.4 one might be lead to conjecture that it can be weakened to
(i) assuming that \(x_i x_j \partial_i \partial_j R \geq -o(|x|^{-4})\) as \(|x| \to \infty\), or
(ii) assuming that \(\sigma_{ij} = T_{ij} + o(|x|^{-2})\) as \(|x| \to \infty\) where \(T_{ij}\) homogeneous of order \(-2\), and that the scalar curvature is non-negative.

The second alternative assumption here implies the first — by Euler’s theorem.

The following example dashes any hope of such generalizations.

\[\text{Theorem 1.8.} \] There is an asymptotically flat complete Riemannian 3-manifold \((M,g)\) with non-negative scalar curvature such that, in the chart at infinity,
\[
g_{ij} = (1 + |x|^{-1})^4 \delta_{ij} + T_{ij} + o(|x|^{-4}) \quad \text{as} \quad |x| \to \infty
\]
along with all derivatives, where \(T_{ij}\) is homogeneous of degree \(-2\), and which contains outlying stable constant mean curvature spheres \(\Sigma_k \subset M\) with
\[
r_0(\Sigma_k) \to \infty, \quad \text{area}_g(\Sigma_k) \to \infty, \quad \text{and} \quad r_0(\Sigma_k)H(\Sigma_k) \to \infty.
\]

Finally, we note that there is by now an impressive body of work on stable constant mean curvature spheres in general asymptotically flat Riemannian 3-manifolds. We refer the reader to Section 2.1 in [5] for an overview and references to results in this direction.

Acknowledgments. We thank S. Brendle for helpful conversations. M. Eichmair has been supported by the START-Project Y963-N35 of the Austrian Science Fund.

2. Proof of Theorem 1.4

The proof follows the strategy of [3], with one important difference: We do not assume here that the deviation of the metric from Schwarzschild is homogeneous of degree \(-2\) to top order. Without
loss of generality, we may assume that the mass $m$ is equal to 2. Thus,

$$g_{ij} = (1 + |x|^{-1})^4\delta_{ij} + \sigma_{ij}$$

where

$$\partial_I\sigma_{ij} = O(|x|^{-2-|I|})$$

for all multi-indices $I$ of length $|I| \leq 4$.

Let $\Omega$ be a bounded subset with compact closure in $\mathbb{R}^3 \setminus \overline{B_1(0)}$. For $\xi \in \Omega$ and $\lambda > 0$ sufficiently large, we may use the implicit function theorem to find surfaces $\Sigma(\xi, \lambda)$ as in Proposition 4 of [3]. Moreover, the surface $\Sigma(\xi, \lambda)$ is a constant mean curvature sphere (respectively, a stable constant mean curvature sphere) if, and only if, $\xi$ is a critical point (respectively, a stable critical point) for the map

$$\xi \mapsto \text{area}_g(\Sigma(\xi, \lambda)).$$

The derivation of Proposition 5 in [3] carries over to give

$$\text{area}_g(\Sigma(\xi, \lambda)) = 4\pi\lambda^2 + \frac{\pi}{2} F_0(|\xi|) + F_\sigma(\xi, \lambda) + o(1) \quad \text{as} \quad \lambda \to \infty. \quad (8)$$

The assumption that $\sigma$ is homogeneous is neither needed nor used at this point of [3]. We recall that

$$F_0(t) = -14 + 16t^2\log\frac{t^2 - 1}{t^2} + (15t - t^{-1})\log\frac{t + 1}{t - 1}$$

is the contribution from the Schwarzschild background, while

$$F_\sigma(\xi, \lambda) = \frac{1}{2} \int_{S(\xi, \lambda)} \text{tr} S(\xi, \lambda) \sigma - \frac{1}{\lambda} \int_{B(\xi, \lambda)} \text{tr} \sigma$$

is the contribution from $\sigma$.

Here and below, unless explicitly noted otherwise, all geometric operations are with respect to the Euclidean background metric in the chart at infinity.

As in [3], given $\xi \in \mathbb{R}^3$ and $\lambda > 0$, we will often write

$$S(\xi, \lambda) = S_\lambda(\lambda \xi) \quad \text{and} \quad B(\xi, \lambda) = B_\lambda(\lambda \xi).$$

2.1. Radial variation. The computation of the radial derivative of (9) in Section 3 of [3] uses the top-order homogeneity of $\sigma$ that is part of their assumption repeatedly. Here, we compute this derivative in the general case, employing several integration by parts to derive a geometric expression involving the scalar curvature on the nose.

$$(\nabla_\xi F_\sigma)(\xi, \lambda) = \frac{d}{ds}\bigg|_{s=1} F_\sigma(s \xi, \lambda)$$

$$= \frac{\lambda}{2} \int_{S(\xi, \lambda)} \text{tr} S(\xi, \lambda) \nabla_\xi \sigma - \int_{S(\xi, \lambda)} (\text{tr} \sigma)(\xi, \nu)$$

We write $\xi = \xi^\top + \langle \xi, \nu \rangle \nu$.

$$= \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) \nabla_\nu \sigma)(\xi, \nu) - \int_{S(\xi, \lambda)} (\text{tr} \sigma)(\xi, \nu)$$
We write

\[ \langle \frac{\lambda}{2} \int_{S(\xi, \lambda)} \left( \text{tr} S(\xi, \lambda) \right) \nabla \xi^\top \sigma \rangle \]

\[ = \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) \nabla \nu \sigma) \langle \xi, \nu \rangle - \int_{S(\xi, \lambda)} (\text{tr} \sigma \langle \xi, \nu \rangle)
+ \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\nabla \xi^\top \text{tr} \sigma - (\nabla \xi^\top \sigma)(\nu, \nu)\]

\[ = \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) \nabla \nu \sigma) \langle \xi, \nu \rangle - \int_{S(\xi, \lambda)} (\text{tr} \sigma \langle \xi, \nu \rangle)
+ \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\nabla \xi^\top (\text{tr} \sigma - \sigma(\nu, \nu)) + 2 \sigma(\nabla \xi^\top \nu, \nu)\]

\[ = \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) \nabla \nu \sigma) \langle \xi, \nu \rangle - \int_{S(\xi, \lambda)} (\text{tr} \sigma \langle \xi, \nu \rangle)
+ \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\nabla \xi^\top \sigma(\nu, \nu) - \sigma(\nu, \nu) \langle \xi, \nu \rangle + \sigma(\xi^\top, \nu))\]

\[ = \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) \nabla \nu \sigma) \langle \xi, \nu \rangle + \int_{S(\xi, \lambda)} (\sigma(\xi, \nu) - 2 \sigma(\nu, \nu) \langle \xi, \nu \rangle).\]

We define a vector field

\[ Y = \langle \xi, \nu \rangle \sigma(\nu, \cdot)^2\]

on \( S(\xi, \lambda) \) and compute

\[ \text{div}_{S(\xi, \lambda)} Y = \frac{1}{\lambda} \sigma(\xi, \nu) - \frac{1}{\lambda} \langle \xi, \nu \rangle \sigma(\nu, \nu) + \langle \xi, \nu \rangle \text{tr}_{S(\xi, \lambda)} (\nabla \cdot \sigma)(\nu, \cdot) + \frac{1}{\lambda} \langle \xi, \nu \rangle \text{tr}_{S(\xi, \lambda)} \sigma.\]

The first variation formula gives

\[ \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) (\nabla \cdot \sigma)(\nu, \cdot)) \langle \xi, \nu \rangle = \frac{1}{2} \int_{S(\xi, \lambda)} (3 \sigma(\nu, \nu) - \text{tr}_{S(\xi, \lambda)} \sigma) \langle \xi, \nu \rangle - \frac{1}{2} \int_{S(\xi, \lambda)} \sigma(\xi, \nu).\]

We insert this into the above expression, and continue.

\[ \frac{d}{ds} \Big|_{s=1} F(s \xi, \lambda) = \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) \nabla \nu \sigma - \text{tr}_{S(\xi, \lambda)} (\nabla \cdot \sigma)(\nu, \cdot)) \langle \xi, \nu \rangle
- \frac{1}{2} \int_{S(\xi, \lambda)} (\text{tr} \sigma \langle \xi, \nu \rangle - \sigma(\xi, \nu))\]

We write \( \langle \xi, \nu \rangle = -|\xi|^2 + \lambda^{-1} \langle \xi, X \rangle \) in the first integrand, where \( X \) is the position field.

\[ = \frac{\lambda}{2} \int_{S(\xi, \lambda)} (\text{tr} S(\xi, \lambda) (\nabla \cdot \sigma)(\nu, \cdot) - \text{tr}_{S(\xi, \lambda)} \nabla \nu \sigma) (|\xi|^2 - \lambda^{-1} \langle \xi, X \rangle)
- \frac{1}{2} \int_{S(\xi, \lambda)} ((\text{tr} \sigma \langle \xi, \nu \rangle - \sigma(\xi, \nu))\]
\[ F = \frac{\lambda}{2} \int_{\Sigma(\xi,\lambda)} (\text{tr} (\nabla \cdot \sigma)(\nu, \cdot) - \text{tr} \nabla \nu \sigma)(|\xi|^2 - \lambda^{-1} \langle \xi, X \rangle) \]

\[- \frac{1}{2} \int_{\Sigma(\xi,\lambda)} (\text{tr} \sigma) \langle \xi, \nu \rangle - \sigma(\xi, \nu) \]

We define a vector field \( W = \text{div} \sigma - \nabla \text{tr} \sigma. \)

\[ = \frac{1}{2} \int_{\Sigma(\xi,\lambda)} \langle \xi, \lambda \xi - X \rangle (W, \nu) \]

\[- \frac{1}{2} \int_{\Sigma(\xi,\lambda)} \langle \xi, \nu \rangle - \sigma(\xi, \nu) \]

\[ = \frac{1}{2} \int_{B(\xi,\lambda)} \text{div}(\langle \xi, \lambda \xi - X \rangle W) \]

\[- \frac{1}{2} \int_{\Sigma(\xi,\lambda)} ((\text{tr} \sigma) \langle \xi, \nu \rangle - \sigma(\xi, \nu)) \]

\[ = \frac{1}{2} \int_{B(\xi,\lambda)} (\text{div} W) \langle \xi, \lambda \xi - X \rangle \]

\[- \frac{1}{2} \int_{B(\xi,\lambda)} \langle \xi, W \rangle \]

\[- \frac{1}{2} \int_{\Sigma(\xi,\lambda)} \langle \xi, \nu \rangle - \sigma(\xi, \nu) \]

Note that \( \langle \xi, W \rangle = \text{div}(\sigma(\xi, \cdot) - (\text{tr} \sigma)\xi). \) We apply the divergence theorem.

\[ = \frac{1}{2} \int_{B(\xi,\lambda)} (\text{div} W) \langle \xi, \lambda \xi - X \rangle \]

\[- \frac{1}{2} \int_{B(\xi,\lambda)} \sigma(\xi, \nu) - (\text{tr} \sigma) \langle \xi, \nu \rangle \]

\[- \frac{1}{2} \int_{\Sigma(\xi,\lambda)} \langle \xi, \nu \rangle - \sigma(\xi, \nu) \]

\[ = \frac{1}{2} \int_{B(\xi,\lambda)} (\text{div} W) \langle \xi, \lambda \xi - X \rangle. \]

Note that

\[ \text{div} W = R + O(|x|^{-5}) \]

where \( R \) is the scalar curvature of \( g. \) In conclusion, we obtain

\[ (\nabla_\xi F_\sigma)(\xi, \lambda) = \frac{d}{ds} \bigg|_{s=1} F(s \xi, \lambda) = \frac{1}{2} \int_{B(\xi,\lambda)} \langle \xi, \lambda \xi - X \rangle R + o(1) \quad \text{as} \quad \lambda \rightarrow \infty. \]

This computation connects the radial derivative of \( F_\sigma \) with the scalar curvature \( R \) of \( g. \) We emphasize again that our derivation parallels the proof of Proposition 7 in [3], though we do not assume the top order homogeneity of \( \sigma. \)
2.2. **Radial variation in spherical coordinates.** Assume first that
\[ R \geq 0 \quad \text{and} \quad x^i \partial_i (|x|^2 R) \leq 0. \]

For definiteness, we assume that \[ \xi = |\xi| e_3 \]
where \(|\xi| > 1\). In this subsection, we compute the radial variation
\[ \int_{B_\lambda(\lambda \xi)} \langle \lambda \xi - X, \xi \rangle R \]
in spherical
\[ (\rho, \phi, \theta) \mapsto (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \]
on the complement of the z-axis. The radial line in direction \( (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \)
intersects the sphere \( B_\lambda(\lambda \xi) \) in the \( \rho \)-interval whose endpoints are the solutions
\[ \rho_{\pm} = \lambda |\xi| \left( \cos \phi \pm \sqrt{1/|\xi|^2 - \sin^2 \phi} \right) \]
of the quadratic equation
\[ \rho^2 - 2 \rho |\xi| \cos \phi + \lambda^2 (|\xi|^2 - 1) = 0. \]
The intersection is non-empty for angles \( \phi \in [0, \phi_+] \) where \( \phi_+ \in (0, \pi) \) solves
\[ \sin^2 \phi_+ = 1/|\xi|^2. \]

We then have that
\[
\int_{B_\lambda(\lambda \xi)} \langle \lambda \xi - X, \xi \rangle R \\
= \int_0^{2\pi} \int_0^{\phi_+} \int_{\rho_-}^{\rho_+} R(\rho, \phi, \theta) \left( \lambda |\xi|^2 - \rho |\xi| \cos \phi \right) \rho^2 \sin \phi d\rho d\phi d\theta \\
= |\xi| \int_0^{2\pi} \int_0^{\phi_+} \int_{\rho_-}^{\rho_+} \rho^2 R(\rho, \phi, \theta) \left( |\xi| \cos \phi \right) \sin \phi d\rho d\phi d\theta \\
\geq |\xi| \int_0^{2\pi} \int_0^{\phi_+} \left( |\xi| / \cos \phi \right)^2 R(|\xi| / \cos \phi, \phi, \theta) \left( \int_{\rho_-}^{\rho_+} (|\lambda| - \rho \cos \phi) d\rho \right) \sin \phi d\phi d\theta. \\
\]

Now, for every \( \phi \in (0, \phi_+) \),
\[
\int_{\rho_-}^{\rho_+} (|\lambda| - \rho \cos \phi) d\rho = (\rho_+ - \rho_-) |\xi| \sin^2 \phi > 0 \\
\]
so that, in conclusion,
\[ \int_{B_\lambda(\lambda \xi)} \langle \lambda \xi - X, \xi \rangle R \geq 0. \]

Arguing as in [3, p. 677] shows that \( \Sigma_{(\xi, \lambda)} \) cannot be a constant mean curvature sphere.

We now observe that the above arguments go through under the weaker assumption \( (5) \). Indeed, using that \( R = O(|x|^{-4}) \) from asymptotic flatness, we obtain upon integrating inwards from infinity
that
\[ R \geq -o(|x|^{-4}) \quad \text{as} \quad |x| \to \infty. \]

Under these assumptions, the preceding computation leads to the estimate
\[ \int_{B_{R}(\lambda \xi)} \langle \lambda \xi - X, \xi \rangle R \geq -o(1) \quad \text{as} \quad \lambda \to \infty. \]

We also mention that (5) is implied by the assumption
\[ R \geq -o(|x|^{-4}) \quad \text{and} \quad 4R + x^i \partial_i R \leq o(|x|^{-4}) \]
both as \(|x| \to \infty\). In particular, it follows from the assumptions in Theorem 1.2.

3. Proof of Theorem 1.3

Our strategy here parallels the proof of Theorem 1 in [3] in that we construct our metric to have a pulse in its scalar curvature, which in turn forces the reduced area functional \( \xi \mapsto \text{area}_g(\Sigma(\xi,\lambda)) \) to have stable critical points. Unlike in [3], our examples are spherically symmetric (which also simplifies the analysis) and, more importantly, they have non-negative scalar curvature.

Let \( S : (0, \infty) \to (-\infty, 0] \) be a smooth function with
\[ S^{(\ell)}(r) = O(r^{-4-\ell}). \]

We define a smooth function \( \varphi : (0, \infty) \to \mathbb{R} \) by
\[ \varphi(r) = \frac{1}{r} \int_r^\infty (\rho - r) \rho S(\rho) \, d\rho. \]

Note that
\[ \varphi'(r) = -\frac{1}{r^2} \int_r^\infty \rho^2 S(\rho) \, d\rho \]
so
\[ (r^2 \varphi')' = S(r). \]  

Lemma 3.1. We have that
\[ \varphi^{(\ell)}(r) = O(r^{-2-\ell}). \]

Proof. Because \( S(r) = O(r^{-4}) \), we see that
\[ \varphi(r) = O(r^{-2}) \quad \text{and} \quad \varphi'(r) = O(r^{-3}). \]

Using (11), we find
\[ \varphi''(r) + 2\varphi'(r)/r = S(r). \]

From this, the asserted decay of the higher derivatives can be verified by induction. \( \square \)

On \( \mathbb{R}^3 \setminus \{0\} \), we define a conformally flat Riemannian metric
\[ g = (1 + 1/r + \varphi(r))^4 \bar{g} = (1 + 1/r)^4 \bar{g} + O(1/r^2) \]
where \( r = |x| \). Note that \( g \) is smoothly asymptotic to Schwarzschild with mass 2. Its scalar curvature is easily computed as

\[
R = -8(1 + 1/r + \varphi(r))^{-5}(r^2 \varphi')/r^2 = -8(1 + O(1/r))S(r).
\]

In particular, it is non-negative on the complement of a compact set. We now make a particular choice for \( S \). Fix \( \chi \in C^\infty(\mathbb{R}) \) that is positive on \((3, 4)\) and supported in \([3, 4]\). Let

\[
S(r) = -A \sum_{k=0}^{\infty} 10^{-4k} \chi(10^{-k}r)
\]

where \( A > 0 \) is a large constant that we will fix later. Recall from (8) that

\[
\text{area}_g(\Sigma(\xi, \lambda)) = 4\pi\lambda^2 + 2\pi F_0(|\xi|) + \frac{1}{2\pi} F_\sigma(\xi, \lambda) + o(1) \quad \text{as} \quad \lambda \to \infty.
\]

We choose \( \xi \in \mathbb{R}^3 \) with \( 2 \leq |\xi| \leq 9 \) and \( \lambda = 10^j \) where \( j \geq 1 \) is a large integer. Using (10), we compute the radial derivative as

\[
\frac{d}{ds} \bigg|_{s=1} \text{area}_g(\Sigma(s\xi, \lambda)) = 2\pi|\xi| F_0'(|\xi|) + \frac{1}{4\pi} \int_{X \in B_\lambda(\lambda \xi)} R(X) \langle \xi, \lambda \xi - X \rangle + o(1)
\]

\[
= 2\pi|\xi| F_0'(|\xi|) - \frac{2}{\pi} \int_{X \in B_\lambda(\lambda \xi)} S(|X|) \langle \xi, \lambda \xi - X \rangle + o(1)
\]

\[
= 2\pi|\xi| F_0'(|\xi|) + \frac{2A}{\pi} \int_{Y \in B_1(\xi)} \chi(|Y|) \langle \xi, \xi - Y \rangle + o(1) \quad \text{as} \quad \lambda \to \infty.
\]

When \( |\xi| = 2\sqrt{2} \), the integral on the last line is negative. We choose \( A > 0 \) large so that the sum of the first two terms is negative. When \( |\xi| = 5 \), the second term vanishes while the first term is strictly positive. Thus, for \( j \geq 1 \) sufficiently large, the derivative

\[
\frac{d}{ds} \bigg|_{s=1} \text{area}_g(\Sigma(s\xi, \lambda))
\]

is negative when \( |\xi| = 2\sqrt{2} \) and positive when \( |\xi| = 5 \). Using that the metric \( g \) is rotationally symmetric, we see that the map

\[
\xi \mapsto \text{area}_g(\Sigma(\xi, 10^j))
\]

has a stable critical point (a local minimum) at some \( \xi_j \in \mathbb{R}^3 \) with \( |\xi_j| \in (2\sqrt{2}, 5) \). In other words, \( \Sigma(\xi_j, 10^j) \) is a “far-off-center” stable constant mean sphere for \( j \) sufficiently large.

**Remark 3.2.** S. Brendle has already observed in Theorem 1.5 of [1] that, as a consequence of the work by F. Pacard and X. Xu in [11], every rotationally symmetric Riemannian manifold whose scalar curvature has a strict local extremum contains small stable constant mean curvature spheres.

### 4. Proof of Theorem 1.5

Consider

\[
g_{ij} = (1 + |x|^{-1})^4 \delta_{ij} + \sigma_{ij}
\]

with

\[
\partial_I \sigma_{ij} = O(|x|^{-2-|I|}) \quad \text{as} \quad |x| \to \infty
\]
for all multi-indices $I$ of length $|I| \leq 7$.

Our proof is guided by the Lyapunov–Schmidt reduction and the related expansion for the reduced area functional as developed in [3]. The goal is to extend these ideas to allow for $\xi \to \infty$. For a useful analysis in this regime, it is necessary to develop the expansion of the reduced area functional to a higher order than was necessary in [3], which turns out to be quite delicate. Our computations are also related and in part inspired by those for exact Schwarzschild in Appendix A of [2].

We also note that part of our expansion for the reduced area functional $\text{area}_g(\Sigma(\xi,\lambda))$ follows, upon rescaling the chart at infinity by $\lambda|\xi|$, from the work of S. Nardulli [10] or F. Pacard and X. Xu [11]. The estimate for the error term in (6) in e.g. [11] is $O(\lambda^2|\xi|^{-5})$ where we obtain $O(\lambda^{-1}|\xi|^{-6}) + O(|\xi|^{-7})$. Our stronger estimate is crucial for our applications here.

Let $\xi \in \mathbb{R}^3$ and $\lambda > 0$ large. There is $r > 1$ with $r \sim \lambda$ and a smooth function $u(\xi,\lambda)$ on the sphere $S_r(\lambda \xi)$ that is perpendicular to constants and linear functions with respect to the Euclidean metric and such that the mean curvature with respect to $g$ of the Euclidean normal graph $\Sigma(\xi,\lambda)$ of $u(\xi,\lambda)$ — as a function on $S_r(\lambda \xi)$ — is a linear combination of constants and linear functions and such that

$$\text{vol}_g(\Sigma(\xi,\lambda)) = 4\pi \lambda^3/3.$$  

Moreover,

$$\sup_{S_r(\lambda \xi)} |u(\xi,\lambda)| + \lambda \sup_{S_r(\lambda \xi)} |\nabla u(\xi,\lambda)| + \lambda^2 \sup_{S_r(\lambda \xi)} |\nabla^2 u(\xi,\lambda)| = O\left(1/|\xi|\right).$$

This is a standard consequence of the implicit function theorem and elementary analysis; cf. Proposition 4 in [3].

We will improve estimate (12) below.

It is convenient to abbreviate $a = \lambda \xi$.

We will frequently use the computations results listed in Appendix A in this section.

4.1. Estimating $\text{vol}_g(B_r(a))$. Recall the following expansion for the determinant of a matrix

$$\sqrt{\det(I + A)} = 1 + \frac{1}{2} \text{tr } A + \frac{1}{8} (\text{tr } A)^2 - \frac{1}{4} \text{tr } A^2 + O(|A|^3).$$  

Thus, we have

$$(1 + |x|^{-1})^6 \sqrt{\det(\delta_{ij} + (1 + |x|^{-1})^{-4}\sigma_{ij})}$$

$$= (1 + |x|^{-1})^6$$

$$+ \frac{1}{2} (1 + |x|^{-1})^2 \text{tr } \sigma$$
\begin{align*}
&+ \frac{1}{4} (1 + |x|^{-1})^{-2} \left( \frac{1}{2}(\text{tr } \sigma)^2 - |\sigma|^2 \right) \\
&+ O(|x|^{-6}).
\end{align*}

Repeating the computations in Proposition 17 of [2] (noting the dependence of the error on \( r \)), we find
\[
\int_{B_r(a)} (1 + |x|^{-1})^6 = \frac{4\pi}{3} r^3 (1 + |a|^{-1})^6 \left( 1 + 3(1 + |a|^{-1})^{-2} \frac{r^2}{|a|^2} + \frac{9}{7} \frac{r^4}{|a|^6} \right) + O(r^8 |a|^{-7})
\]

We now turn to the second term in the expansion of the volume form. We will write \( \sigma \) for \( \sigma \) evaluated at \( a \) (we will use the convention that if \( \sigma \) appears with a derivative, the derivative is taken and then the quantity is evaluated at \( a \)).

First, note that for \( y \in B_r(0) \) with \( x = a + y \),
\[
(1 + |x|^{-1})^2 = (1 + |a|^{-1})^2 + 2(1 + |a|^{-1})(|a + y|^{-1} - |a|^{-1}) + \underbrace{(|a + y|^{-1} - |a|^{-1})^2}_{= O(|a|^{-4}|y|^2)}
\]
as well as
\[
|y + a|^{-1} - |a|^{-1} = -\frac{\langle a, y \rangle}{|a|^3} - \frac{1}{2} \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} + O(r^3 |a|^{-4}).
\]

Finally, we have
\[
\text{tr } \sigma = \text{tr } \sigma + \nabla_y (\text{tr } \sigma) + \frac{1}{2} \nabla^2_{y,y} (\text{tr } \sigma)
\]
\[
+ \frac{1}{6} \nabla^3_{y,y,y} (\text{tr } \sigma) + \frac{1}{24} \nabla^4_{y,y,y,y} (\text{tr } \sigma)
\]
\[
+ O(|y|^5 |x|^{-7}).
\]

We will frequently consider such Taylor expansions for expressions involving \( \sigma \).

Combining the above expansions and using the expressions found in Appendix A, we have
\[
\frac{1}{2} \int_{B_r(a)} (1 + |x|^{-1})^2 \text{tr } \sigma = \frac{1}{2} (1 + |a|^{-1})^2 \int_{B_r(a)} \text{tr } \sigma
\]
\[
+ (1 + |a|^{-1}) \int_{B_r(a)} (|a + y|^{-1} - |a|^{-1}) \text{tr } \sigma
\]
\[
+ O(r^5 |a|^{-6})
\]
\[
= \frac{1}{2} (1 + |a|^{-1})^2 \int_{B_r} \text{tr } \sigma
\]
\[
+ \frac{1}{4} (1 + |a|^{-1})^2 \int_{B_r} \nabla^2_{y,y} \text{tr } \sigma
\]
\[
+ \frac{1}{48} (1 + |a|^{-1})^2 \int_{B_r} \nabla^4_{y,y,y,y} \text{tr } \sigma
\]
\[
- (1 + |a|^{-1}) |a|^{-3} \int_{B_r} \langle a, y \rangle \nabla_y \text{tr } \sigma
\]
\[
+ O(r^5 |a|^{-6}) + O(r^7 |a|^{-7})
\]
\[
= \frac{2\pi}{3} r^3 (1 + |a|^{-1})^2 \text{tr } \sigma
\]
Continuing on, we have that
\[
\frac{1}{4} \int_{B_r(a)} (1 + |x|^{-1})^{-2} \left( \frac{1}{2} (\text{tr} \sigma)^2 - |\sigma|^2 \right) = \frac{\pi}{3} r^3 (1 + |a|^{-1})^{-2} \left( \frac{1}{2} (\text{tr} \sigma)^2 - |\sigma|^2 \right) + O(r^5|a|^{-6})
\]
Now, putting these terms together, we find that
\[
\text{vol}_g(B_r(a)) = \frac{4\pi}{3} r^3 (1 + |a|^{-1})^6 \left( 1 + 3(1 + |a|^{-1})^{-2} \frac{r^2}{|a|^4} + \frac{9}{7} \frac{r^4}{|a|^6} \right)
\]
\[
+ \frac{2\pi}{3} r^3 (1 + |a|^{-1})^2 \text{tr} \sigma
\]
\[
+ \frac{\pi}{15} r^5 (1 + |a|^{-1})^2 \Delta(\text{tr} \sigma)
\]
\[
+ \frac{\pi}{420} r^7 \Delta(\Delta(\text{tr} \sigma))
\]
\[
- \frac{4\pi}{15} (1 + |a|^{-1}) r^5 |a|^{-3} \nabla_a(\text{tr} \sigma)
\]
\[
+ \frac{\pi}{3} r^3 (1 + |a|^{-1})^{-2} \left( \frac{1}{2} (\text{tr} \sigma)^2 - |\sigma|^2 \right)
\]
\[
+ O(r^5|a|^{-6}) + O(r^7|a|^{-7}).
\]

4.2. Estimating $\text{area}_g(S_r(a))$. Using the above expansion, we have that the volume form of $S_r(a)$ becomes
\[
d\mu_g = (1 + |x|^{-1})^4 \sqrt{\text{det}(\delta_{ij} + (1 + |x|^{-1})^{-4} \sigma_{ij})}
\]
\[
= (1 + |x|^{-1})^4
\]
\[
+ \frac{1}{2} \text{tr} S \sigma
\]
\[
+ \frac{1}{4} (1 + |x|^{-1})^{-4} \left( \frac{1}{2} (\text{tr} S \sigma)^2 - |\sigma|^2 \right)
\]
\[
+ O(|a|^{-6})
\]
\[
= (1 + |x|^{-1})^4
\]
\[
+ \frac{1}{2} \text{tr} \sigma - \frac{1}{2} r^{-2} \sigma(y, y)
\]
\[
+ \frac{1}{4} (1 + |x|^{-1})^{-4} \left( \frac{1}{2} (\text{tr} \sigma)^2 - r^{-2}(\text{tr} \sigma)\sigma(y, y) - |\sigma|^2 + 2r^{-2} |\sigma(y, \cdot)|^2 - \frac{1}{2} r^{-4} \sigma(y, y)^2 \right)
\]
\[
+ O(|a|^{-6}).
\]
As in Proposition 17 of [2], we have that
\[
\int_{S_r(a)} (1 + |x|^{-1})^4 = 4\pi r^2(1 + |a|^{-1})^4 \left(1 + 2(1 + |a|^{-1})^{-2} \frac{r^2}{|a|^2} + \frac{6}{5} \frac{r^4}{|a|^6}\right) + O(r^7|a|^{-7})
\]

We compute, using Appendix A,
\[
\frac{1}{2} \int_{S_r(a)} \sigma \, \text{tr} \, \sigma = \frac{1}{2} \int_{S_r} \text{tr} \, \sigma
+ \frac{1}{4} \int_{S_r} \nabla^2_{y,y} \text{tr} \, \sigma
+ \frac{1}{48} \int_{S_r} \nabla^4_{y,y,y,y} \text{tr} \, \sigma
+ O(r^6|a|^{-8})
= 2\pi r^2 \text{tr} \, \sigma
+ \frac{\pi}{3} r^4 \Delta(\text{tr} \, \sigma)
+ \frac{\pi}{60} r^6 \Delta(\Delta(\text{tr} \, \sigma))
+ O(r^8|a|^{-8})
\]

and
\[
\frac{1}{2} \int_{S_r(a)} r^{-2} \sigma(y, y) = \frac{1}{2} \int_{S_r} r^{-2} \sigma(y, y)
+ \frac{1}{4} \int_{S_r} r^{-2} \nabla^2_{y,y} \sigma(y, y)
+ \frac{1}{48} \int_{S_r} r^{-2} \nabla^4_{y,y,y,y} \sigma(y, y)
+ O(r^8|a|^{-8})
= \frac{2\pi}{3} r^2 \text{tr} \, \sigma
+ \frac{\pi}{15} r^4 \Delta(\text{tr} \, \sigma)
+ \frac{2\pi}{15} r^4 \text{div} (\text{div} \, \sigma)
+ \frac{\pi}{420} r^6 \Delta(\Delta(\text{tr} \, \sigma))
+ \frac{\pi}{105} \Delta(\text{div} (\text{div} \, \sigma))
+ O(r^8|a|^{-8}).
\]

Putting these two expressions together, we find
\[
\frac{1}{2} \int_{S_r(a)} \text{tr} \, \sigma - \frac{1}{2} \int_{S_r(a)} r^{-2} \sigma(y, y) = 2\pi r^2 \text{tr} \, \sigma
- \frac{2\pi}{3} r^2 \text{tr} \, \sigma
+ \frac{\pi}{3} r^4 \Delta(\text{tr} \, \sigma)
- \frac{\pi}{15} r^4 \Delta(\text{tr} \, \sigma)
- \frac{2\pi}{15} r^4 \text{div} (\text{div} \, \sigma)
\]
Finally, we compute
\[
\frac{1}{4} \int_{S_r(a)} \left(1 + |x|^{-1}\right)^{-4} \left( \frac{1}{2} (\text{tr} \, \sigma)^2 - r^{-2} (\text{tr} \, \sigma) \sigma(y, y) - |\sigma|^2 + 2r^{-2} |\sigma(y, \cdot)|^2 - \frac{1}{2} r^{-4} \sigma(y, y)^2 \right) \\
= \frac{1}{8} (1 + |a|^{-1})^{-4} \int_{S_r} (\text{tr} \, \sigma)^2 \\
- \frac{1}{4} (1 + |a|^{-1})^{-4} \int_{S_r} r^{-2} (\text{tr} \, \sigma) \sigma(y, y) \\
- \frac{1}{4} (1 + |a|^{-1})^{-4} \int_{S_r} |\sigma|^2 \\
+ \frac{1}{2} (1 + |a|^{-1})^{-4} \int_{S_r} r^{-2} |\sigma(y, \cdot)|^2 \\
- \frac{1}{8} (1 + |a|^{-1})^{-4} \int_{S_r} r^{-4} \sigma(y, y)^2 \\
+ O(r^4 |a|^{-6}) \\
= \frac{\pi}{2} r^2 (1 + |a|^{-1})^{-4} (\text{tr} \, \sigma)^2 \\
- \frac{\pi}{3} r^2 (1 + |a|^{-1})^{-4} (\text{tr} \, \sigma)^2 \\
- \pi r^2 (1 + |a|^{-1})^{-4} |\sigma|^2 \\
+ \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} |\sigma|^2 \\
- \frac{\pi}{30} r^2 (1 + |a|^{-1})^{-4} (\text{tr} \, \sigma)^2 - \frac{\pi}{15} r^2 (1 + |a|^{-1})^{-4} |\sigma|^2 \\
+ O(r^4 |a|^{-6}) \\
= \frac{2\pi}{15} r^2 (1 + |a|^{-1})^{-4} (\text{tr} \, \sigma)^2 - \frac{2\pi}{5} r^2 (1 + |a|^{-1})^{-4} |\sigma|^2 + O(r^4 |a|^{-6}) \\
= -\frac{2\pi}{5} r^2 (1 + |a|^{-1})^{-4} |\sigma|^2 + O(r^4 |a|^{-6}).
\]

Thus, putting this together, we find that
\[
\text{area}_g(S_r(a)) = 4\pi r^2 (1 + |a|^{-1})^{-4} \left(1 + 2(1 + |a|^{-1})^{-2} \frac{r^2}{|a|^4} + \frac{6}{5} \frac{r^4}{|a|^6}\right)
\]
4.3. Estimating \( \mathcal{F}(S_r(a)) \). We define
\[
\mathcal{F}(S_r(a)) = \text{area}_g(S_r(a)) - 2r^{-1}(1 + |a|^{-1})^{-2} \text{vol}_g(S_r(a)).
\]
We then compute
\[
\mathcal{F}(S_r(a)) = 4\pi r^2 (1 + |a|^{-1})^4 \left( 1 + 2(1 + |a|^{-1})^{-2} \frac{r^2}{|a|^4} + \frac{6}{5} \frac{r^4}{|a|^6} \right)
\]
\[+ \frac{4\pi}{3} r^2 \text{tr} \sigma \]
\[+ \frac{4\pi}{15} r^4 \Delta(\text{tr} \sigma) - \frac{2\pi}{15} r^4 \text{div(div}(\sigma)) \]
\[+ \frac{\pi}{70} r^6 \Delta(\Delta(\text{tr} \sigma)) - \frac{\pi}{105} r^6 \Delta(\text{div(div}(\sigma)) \]
\[+ \frac{2\pi}{5} r^2 (1 + |a|^{-1})^{-4} |\tilde{a}|^2 \]
\[+ O(r^4|a|^{-6}) + O(r^7|a|^{-7})
\]
Thus, unless noted otherwise. It follows that the mean curvature of $S_r(a)$. Consider

$$
\hat{g}_{ij} = \bar{g}_{ij} + \hat{\sigma}_{ij} \quad \text{where} \quad \hat{\sigma}_{ij} = (1 + |x|^{-1})^{-4} \sigma_{ij}.
$$

By the computation in Lemma 7.4 of [7], we have

$$
\hat{H} = H - r^{-1} \text{tr}_S \hat{\sigma} + r^{-3} \hat{\sigma}(y, y) - r^{-1} \text{tr}_S (\nabla \cdot \hat{\sigma})(y, \cdot) + r^{-1} \frac{1}{2} \text{tr}_S \nabla y \hat{\sigma} + O(r^{-1}|a|^{-4})
$$

$$
= \frac{2}{r^2} \left( 1 + |x|^{-1} \right)^{-2} - \frac{4}{r} \left( 1 + |x|^{-1} \right)^{-3} \langle x, y \rangle + O(|a|^{-6})
$$

$$
= 2^r \left( 1 + |x|^{-1} \right)^{-2} \langle x, y \rangle - r^{-1} \text{tr} \sigma + 2r^{-3} \left( 1 + |x|^{-1} \right)^{-6} \sigma(y, y)
$$

$$
= 2r^{-1} \left( 1 + |x|^{-1} \right)^{-2} - 4r^{-1} \left( 1 + |x|^{-1} \right)^{-3} \langle x, y \rangle + O(|a|^{-6})
$$

and that geometric quantities are computed with respect to the Euclidean background metric $\bar{g}$ unless noted otherwise. It follows that the mean curvature of $S_r(a)$ with respect to $g$ is given by

$$
H_g = \frac{2}{r^2}\left( 1 + |x|^{-1} \right)^{-2} - 4r^{-1} \left( 1 + |x|^{-1} \right)^{-3} \langle x, y \rangle + O(|a|^{-6})
$$

Computing as in Lemma 18 of [2],

$$
2r^{-1} \left( 1 + |a|^{-1} \right)^{-2} - 4r^{-1} \left( 1 + |a|^{-1} \right)^{-3} \langle a, y \rangle + O(r^2|a|^{-4})
$$

Thus,

$$
H_g = 2r^{-1} \left( 1 + |a|^{-1} \right)^{-2} - \left( |a|^{-3} |y|^2 - 3|a|^{-5} \langle a, y \rangle^2 \right) + O(r^2|a|^{-4})
$$

$$
H_g = 2r^{-1} \left( 1 + |a|^{-1} \right)^{-2} - \left( |a|^{-3} |y|^2 - 3|a|^{-5} \langle a, y \rangle^2 \right) + O(r^2|a|^{-4})
$$

$$
= \frac{8\pi}{15} r^4 |a|^{-3} \nabla_a (\text{tr} \sigma)
$$

$$
+ O(r^4|a|^{-6}) + O(r^7|a|^{-7}).
$$
Now, we consider the (Euclidean) projection of $H_g$ to $\Lambda_2$ and $\Lambda_{>2}$ where $\Lambda_2$ is the space of second eigenfunctions on $S_r$ and $\Lambda_{>2}$ is the $L^2(S_r)$-orthogonal complement of $\Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$.

\[
\text{proj}_{\Lambda_2} H_g = \frac{2}{r} \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \\
+ 2r^{-3}(1 + |a|^{-1})^{-6} \text{proj}_{\Lambda_2} \sigma(y, y) \\
+ O(r^2 |a|^{-4}) \\
= - \frac{2}{r} \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \\
+ 2r^{-3}(1 + |a|^{-1})^{-6} \left( \sigma(y, y) - \frac{1}{3} |y|^2 \text{tr} \sigma \right) \\
+ O(r^2 |a|^{-4}).
\]

For the higher eigenspaces, we will be content with the estimate

\[
\text{proj}_{\Lambda_{>2}} H_g = O(|a|^{-3}) + O(r^2 |a|^{-4}).
\]

### 4.5. Estimates for $u$

Our goal here is to improve upon the initial estimate (12).

Let $t \in [0, 1]$. Consider the Euclidean graph over $S_r(a)$ of the function $tu$. The initial normal speed with respect to $g$ of this family can be computed as

\[
w = u g(y/r, \nu_g).
\]

Note that

\[
w = (1 + O(|x|^{-1})) u
\]

up to and including second derivatives. We will give a more precise estimate later. Thus, the second variation of area implies that

\[
\Delta_g^{S_r(a)} w + (|h_g|^2 + \text{Ric}_g(\nu_g, \nu_g)) w = H_g - H_g^\Sigma + O(\lambda^{-3} |\xi|^{-2})
\]

where, as before, $H_g$ is the mean curvature of $S_r(a)$ with respect to $g$. It follows that

\[
\Delta_g^{S_r(a)} u + 2r^{-2} u = H_g - H_g^\Sigma + O(\lambda^{-3} |\xi|^{-2}).
\]

Since

\[
\text{proj}_{\Lambda_{>1}} (H_g - H_g^\Sigma) = \text{proj}_{\Lambda_{>1}} H_g = O(\lambda^{-3} |\xi|^{-2}) + O(\lambda^{-2} |\xi|^{-3}),
\]

we obtain that

\[
\sup_{S_r(\lambda \xi)} |u(\xi, \lambda)| + \lambda \sup_{S_r(\lambda \xi)} |\nabla u(\xi, \lambda)| + \lambda^2 \sup_{S_r(\lambda \xi)} |\nabla^2 u(\xi, \lambda)| = O(\lambda^{-1} |\xi|^{-2}) + O(|\xi|^{-3}).
\]

This allows us to improve the coarse estimate above to

\[
\Delta_g^{S_r(a)} w + (|h_g|^2 + \text{Ric}_g(\nu_g, \nu_g)) w = H_g - H_g^\Sigma + O(\lambda^{-5} |\xi|^{-4}) + O(\lambda^{-3} |\xi|^{-6}).
\]
At this point, we can improve our earlier estimate for \( w \) to

\[
w = ((1 + |x|^{-1})^2 + O(|x|^{-2})) \ u
\]

up to and including second derivatives. Thus

\[
\Delta^S g_{(a)} w = (1 + |a|^{-1})^{-2} \Delta^S (a) u + O(\lambda^{-4}|\xi|^{-4}) + O(\lambda^{-3}|\xi|^{-5}).
\]

Continuing on, we have that

\[
|h_g|^2 = 2r^{-2}(1 + |a|^{-1})^{-4} + O(\lambda^{-4}|\xi|^{-2})
\]

and

\[
\text{Ric}_g(\nu_g, \nu_g) = O(\lambda^{-3}|\xi|^{-3}).
\]

Putting these estimates together, we find that

\[
(1 + |a|^{-1})^{-2} \Delta^S (a) u + 2r^{-2}(1 + |a|^{-1})^{-2} u = H_g - H^\Sigma_g + O(\lambda^{-4}|\xi|^{-4}) + O(\lambda^{-3}|\xi|^{-5}).
\]

Hence,

\[
\Delta^S (a) \text{proj}_A u + 2r^{-2} \text{proj}_A u = \text{proj}_A (\Delta^S (a) u + 2r^{-2} u)
\]

\[
= (1 + |a|^{-1})^2 \text{proj}_A H_g + O(\lambda^{-4}|\xi|^{-4}) + O(\lambda^{-3}|\xi|^{-5})
\]

\[
= \frac{2}{r^3} \frac{1}{(1 + |a|^{-1})^4} (\sigma(y,y) - \frac{1}{3}|y|^2 \text{tr} \sigma) - \frac{2}{r} \frac{|a|^2|y|^2 - 3 \langle a,y \rangle^2}{|a|^5} + O(\lambda^{-2}|\xi|^{-4}).
\]

This implies that

\[
\text{proj}_A u = -\frac{1}{2r} \frac{1}{(1 + |a|^{-1})^4} (\sigma(y,y) - \frac{1}{3}|y|^2 \text{tr} \sigma) + \frac{r}{2} \frac{|a|^2|y|^2 - 3 \langle a,y \rangle^2}{|a|^5} + O(|\xi|^{-4})
\]

together with two derivatives. Note that in particular

\[
\text{proj}_A u = O(\lambda^{-1}|\xi|^{-2}) + O(|\xi|^{-3})
\]

along with two derivatives. The above expression also implies that

\[
\text{proj}_{A^2} u = O(\lambda^{-1}|\xi|^{-3}) + O(|\xi|^{-4})
\]

with two derivatives.

4.6. Estimating \( \mathcal{F}(\Sigma) \). We have that

\[
\mathcal{F}(\Sigma) = \mathcal{F}(S_r(a)) + \int_{S_r(a)} (H_g - 2r^{-1}(1 + |a|^{-1})^{-2}) w \ d\mu_g
\]

\[
+ \frac{1}{2} \int_{S_r(a)} H_g (H_g - 2r^{-1}(1 + |a|^{-1})^{-2}) w^2 d\mu_g
\]

\[
- \frac{1}{2} \int_{S_r(a)} (\Delta^S g_{(a)} w + |h_g|^2 + \text{Ric}_g(\nu_g, \nu_g) w) w d\mu_g
\]

\[
+ O(\lambda^{-4}|\xi|^{-6}) + O(\lambda^{-1}|\xi|^{-9})
\]
Recall that
\[ w = g(y/r, \nu_y) = (1 + |x|^{-1})^2 \left( 1 + \frac{1}{2} (1 + |x|^{-1})^{-4} r^{-2} \sigma(y, y) \right) + O(|x|^{-4}). \]

We have seen above that
\[ d\mu_g = (1 + |x|^{-1})^4 \left( 1 + \frac{1}{2} (1 + |x|^{-1})^{-4} \text{tr} \sigma - \frac{1}{2} (1 + |x|^{-1})^{-4} r^{-2} \sigma(y, y) \right) d\mu_g + O(|x|^{-4}). \]

We begin with the first term.
\[
\int_{S_{r(a)}} (H_g - 2r^{-1}(1 + |a|^{-1})^{-2}) w d\mu_g = \int_{S_{r(a)}} (H_g - 2r^{-1}(1 + |a|^{-1})^{-2}) u (1 + |x|^{-1})^6 \\
+ O(\lambda^{-4} |\xi|^{-6}) + O(\lambda^{-3} |\xi|^{-7}) \\
= (1 + |a|^{-1})^6 \int_{S_{r(a)}} (H_g - 2r^{-1}(1 + |a|^{-1})^{-2}) u \\
+ O(\lambda^{-3} |\xi|^{-6}) + O(\lambda^{-2} |\xi|^{-7}) \\
= -2r^{-1}(1 + |a|^{-1})^6 \int_{S_{r(a)}} \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) u \\
+ 2r^{-3} \int_{S_{r(a)}} u \Xi(y, y) \\
+ O(\lambda^{-2} |\xi|^{-6}) + O(\lambda^{-1} |\xi|^{-7}) \\
= - \int_{S_r} \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right)^2 \\
+ r^{-2}(1 + |a|^{-1})^2 \int_{S_r} \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) \Xi(y, y) \\
- r^{-4}(1 + |a|^{-1})^{-4} \int_{S_r} \left( \Xi(y, y) - \frac{1}{3} |y|^2 \text{tr} \Xi \right)^2 \\
+ r^{-2} \int_{S_r} \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) \Xi(y, y) \\
+ O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}) \\
= - \int_{S_r} \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right)^2 \\
+ \frac{2}{r^2} \int_{S_r} \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) \Xi(y, y) \\
\frac{1}{r^4} \int_{S_r} \left( \Xi(y, y) - \frac{1}{3} |y|^2 \text{tr} \Xi \right)^2 \\
+ O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\]

The second term satisfies
\[
\frac{1}{2} \int_{S_{r(a)}} H_g (H_g - 2r^{-1}(1 + |a|^{-1})^{-2}) u^2 d\mu_g = O(\lambda^{-4} |\xi|^{-6}) + O(\lambda^{-1} |\xi|^{-9}).
\]
Finally, the last term satisfies

\[-\frac{1}{2} \int_{S_r(a)} (\Delta^S_{\gamma(a)} w + (|h_g|^2 + \text{Ric}_g(\nu_g, \nu_g)) w) d\mu_g \]

\[= -\frac{1}{2} (1 + |a|^{-1})^4 \int_{S_r(a)} \left( \Delta^S_{\gamma} u + 2 r^{-2} u \right) u \]

\[+ O(\lambda^{-3}|\xi|^{-6}) + O(\lambda^{-1}|\xi|^{-8}) \]

\[= 2r^{-2} (1 + |a|^{-1})^4 \int_{S_r(a)} (\text{proj}_{\Lambda_2} u)^2 \]

\[+ O(\lambda^{-2}|\xi|^{-6}) + O(|\xi|^{-7}) \]

Putting this together, we find that

\[\mathcal{F}(\Sigma) = \mathcal{F}(S_r(a)) \]

\[= \int_{S_r} \left( \frac{|a|^2|y|^2 - 3 \langle a, y \rangle^2}{|a|^3} \right)^2 \]

\[+ 2r^{-2} \int_{S_r} \left( \frac{|a|^2|y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) \sigma(y, y) \]

\[+ \frac{1}{r} \int_{S_r} \left( \frac{|a|^2|y|^2 - 3 \langle a, y \rangle^2}{|a|^3} \right)^2 \sigma(y, y) \]

\[- r^{-4} (1 + |a|^{-1})^{-4} \int_{S_r} \left( \sigma(y, y) - \frac{1}{3} |y|^2 \text{tr} \sigma \right)^2 \]

\[+ \frac{1}{2} r^{-4} (1 + |a|^{-1})^{-4} \int_{S_r} \left( \sigma(y, y) - \frac{1}{3} |y|^2 \text{tr} \sigma \right)^2 \]

\[- r^{-2} \int_{S_r} \left( \frac{|a|^2|y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right)^2 \sigma(y, y) \]

\[+ \frac{1}{2} \int_{S_r} \left( \frac{|a|^2|y|^2 - 3 \langle a, y \rangle^2}{|a|^3} \right)^2 \]

\[+ O(\lambda^{-1}|\xi|^{-6}) + O(|\xi|^{-7}) \]

\[= \mathcal{F}(S_r(a)) \]

\[- \frac{1}{2} \int_{S_r} \left( \frac{|a|^2|y|^2 - 3 \langle a, y \rangle^2}{|a|^3} \right)^2 \]

\[+ r^{-2} \int_{S_r} \left( \frac{|a|^2|y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) \sigma(y, y) \]

\[- \frac{1}{2} r^{-4} (1 + |a|^{-1})^{-4} \int_{S_r} \left( \sigma(y, y) - \frac{1}{3} |y|^2 \text{tr} \sigma \right)^2 \]
We now use the expansions given in Appendix A.3.

\[= \mathcal{F}(S, (a))\]

\[- \frac{8\pi}{5} |\xi|^{-6}\]

\[+ \frac{8\pi}{15} r^4 \left( \text{tr} \sigma - 3|a|^{-1} \sigma(a, a) \right)\]

\[- \frac{4\pi}{15} r^2 (1 + |a|^{-1})^{-4} |\hat{a}|^2\]

\[+ O(\lambda^{-1}|\xi|^{-6}) + O(|\xi|^{-7})\]

\[= \frac{4\pi}{3} r^2 (1 + |a|^{-1})^4 + \frac{48\pi}{35} r^6 |a|^5\]

\[+ \frac{2\pi}{15} r^4 \Delta (\text{tr} \sigma) - \frac{2\pi}{r^4} \text{div(div(} \sigma))\]

\[+ \frac{\pi}{105} r^6 \Delta (\Delta (\text{tr} \sigma)) - \frac{\pi}{105} r^6 \Delta (\text{div(div(} \sigma))\]

\[- \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} |\hat{a}|^2\]

\[- \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} \left( \frac{1}{2} (\text{tr} \sigma)^2 - |\sigma|^2 \right)\]

\[+ \frac{8\pi}{15} r^4 |a|^{-3} \nabla_a (\text{tr} \sigma)\]

\[- \frac{8\pi}{5} |\xi|^{-6}\]

\[+ \frac{8\pi}{15} r^4 \left( \text{tr} \sigma - 3|a|^{-1} \sigma(a, a) \right)\]

\[- \frac{4\pi}{15} r^2 (1 + |a|^{-1})^{-4} |\hat{a}|^2\]

\[+ O(\lambda^{-1}|\xi|^{-6}) + O(|\xi|^{-7})\]

\[= \frac{4\pi}{3} r^2 (1 + |a|^{-1})^4 - \frac{8\pi}{35} |\xi|^{-6}\]

\[+ \frac{2\pi}{15} r^4 \Delta (\text{tr} \sigma) - \frac{2\pi}{15} r^4 \text{div(div(} \sigma))\]

\[+ \frac{\pi}{105} r^6 \Delta (\Delta (\text{tr} \sigma)) - \frac{\pi}{105} r^6 \Delta (\text{div(div(} \sigma))\]

\[- \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} |\hat{a}|^2\]

\[- \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} \left( \frac{1}{2} (\text{tr} \sigma)^2 - |\sigma|^2 \right)\]

\[+ \frac{8\pi}{15} r^4 |a|^{-3} \nabla_a (\text{tr} \sigma)\]

\[+ \frac{8\pi}{15} r^4 \left( \text{tr} \sigma - 3|a|^{-1} \sigma(a, a) \right)\]

\[+ O(\lambda^{-1}|\xi|^{-6}) + O(|\xi|^{-7}).\]
Using that \( \text{vol}_g(\Omega) = \frac{4\pi}{3} \lambda^3 \), we obtain

\[
\text{area}_g(\Sigma) = \frac{4\pi}{3} r^2 (1 + |a|^{-1})^4 + \frac{8\pi}{3} \lambda^3 r^{-1} (1 + |a|^{-1})^{-2} - \frac{8\pi}{35} |\xi|^{-6}
\]

\[
+ \frac{2\pi}{15} r^4 \Delta(\text{tr} \sigma) - \frac{2\pi}{15} r^4 \text{div}(\text{div} \sigma)
\]

\[
+ \frac{\pi}{105} r^6 \Delta(\Delta(\text{tr} \sigma)) - \frac{\pi}{105} r^6 \Delta(\text{div}(\text{div} \sigma))
\]

\[
- \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} |\sigma|^2
\]

\[
- \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} \left( \frac{1}{2} (\text{tr} \sigma)^2 - |\sigma|^2 \right)
\]

\[
+ \frac{8\pi}{15} r^4 |a|^{-3} \nabla_a (\text{tr} \sigma)
\]

\[
+ \frac{8\pi}{15} r^4 |\sigma| (\text{tr} \sigma - 3 |a|^{-2} \sigma(a,a))
\]

\[
+ O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\]

### 4.7. Estimating \( r \)

We now use the expansion

\[
\text{vol}_g(\Omega) = \text{vol}_g(B_r(a)) + \int_{S_r(a)} w \, d\mu_g + \frac{1}{2} \int_{S_r(a)} H_g w^2 \, d\mu_g
\]

\[
+ O(\lambda^{-3} |\xi|^{-6}) + O(|\xi|^{-9})
\]

to relate \( \lambda \) and \( r \). Note that because \( u \) is orthogonal to constants and to linear functions,

\[
\int_{S_r(a)} w \, d\mu_g = O(|\xi|^{-5}) + O(\lambda^{-1} |\xi|^{-6})
\]

and

\[
\frac{1}{2} \int_{S_r(a)} H_g w^2 \, d\mu_g = O(\lambda^{-1} |\xi|^{-4}) + O(\lambda |\xi|^{-6}).
\]

Hence, using the expression for \( \text{vol}_g(B_r(a)) \) obtained previously, we find that

\[
\frac{4\pi}{3} \lambda^3 = \frac{4\pi}{3} r^3 (1 + |a|^{-1})^6 + \frac{2\pi}{3} r^3 (1 + |a|^{-1})^2 \text{tr} \sigma + O(\lambda |\xi|^{-4})
\]

\[
= \frac{4\pi}{3} r^3 (1 + |a|^{-1})^6 \left( 1 + \frac{1}{2} (1 + |a|^{-1})^{-4} \text{tr} \sigma + O(\lambda^{-2} |\xi|^{-4}) \right).
\]

It is convenient to write

\[
\lambda^3 = r^3 (1 + |a|^{-1})^6 (1 + \psi)
\]

for

\[
\psi = \frac{1}{2} (1 + |a|^{-1})^{-4} \text{tr} \sigma + O(\lambda^{-2} |\xi|^{-4}) = O(\lambda^{-2} |\xi|^{-2})
\]

We now estimate the first line in the expansion for \( \text{area}_g(\Sigma) \) obtained above.

\[
\frac{4\pi}{3} r^2 (1 + |a|^{-1})^4 + \frac{8\pi}{3} \lambda^3 r^{-1} (1 + |a|^{-1})^{-2}
\]

\[
= \frac{4\pi}{3} r^2 (1 + |a|^{-1})^4 + \frac{8\pi}{3} r^2 (1 + |a|^{-1})^4 (1 + \psi)
\]

\[
= 4\pi r^2 (1 + |a|^{-1})^4 \left( 1 + \frac{2}{3} \psi \right)
\]
\[ = 4\pi r^2 (1 + |a|^{-1})^4 (1 + \psi)^\frac{3}{2} + \frac{4\pi}{9} r^2 (1 + |a|^{-1})^4 \psi^2 + O(r^2 \psi^3) \]

\[ = 4\pi \lambda^2 + \frac{\pi}{9} r^2 (1 + |a|^{-1})^{-4} (\text{tr} \, \sigma)^2 + O(\lambda^{-2} |\xi|^{-6}). \]

4.8. **Concluding the estimate for** \( \text{area}_g(\Sigma) \). Combining the previous two subsections, we conclude that

\[
\text{area}_g(\Sigma) = 4\pi \lambda^2 - \frac{8\pi}{35} |\xi|^{-6}
\]

\[ + \frac{\pi}{9} r^2 (1 + |a|^{-1})^{-4} (\text{tr} \, \sigma)^2
\]

\[ + \frac{2\pi}{15} r^4 \Delta (\text{tr} \, \sigma) - \frac{2\pi}{15} r^4 \text{div}(\text{div}(\sigma))
\]

\[ + \frac{\pi}{105} r^6 \Delta (\Delta (\text{tr} \, \sigma)) - \frac{\pi}{105} r^6 \Delta (\text{div}(\text{div}(\sigma)))
\]

\[ - \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} |\sigma|^2
\]

\[ - \frac{2\pi}{3} r^2 (1 + |a|^{-1})^{-4} \left( \frac{1}{2} (\text{tr} \, \sigma)^2 - |\sigma|^2 \right)
\]

\[ + \frac{8\pi}{15} r^4 |a|^{-3} \nabla_a (\text{tr} \, \sigma)
\]

\[ + \frac{8\pi}{15} |a|^3 (\text{tr} \, \sigma - 3 |a|^{-2} \sigma(a, a))
\]

\[ + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7})
\]

\[ = 4\pi \lambda^2 - \frac{8\pi}{35} |\xi|^{-6}
\]

\[ + \frac{2\pi}{15} r^4 \Delta (\text{tr} \, \sigma) - \frac{2\pi}{15} r^4 \text{div}(\text{div}(\sigma))
\]

\[ + \frac{\pi}{105} r^6 \Delta (\Delta (\text{tr} \, \sigma)) - \frac{\pi}{105} r^6 \Delta (\text{div}(\text{div}(\sigma)))
\]

\[ + \frac{8\pi}{15} r^4 |a|^{-3} \nabla_a (\text{tr} \, \sigma)
\]

\[ + \frac{8\pi}{15} |a|^3 (\text{tr} \, \sigma - 3 |a|^{-2} \sigma(a, a))
\]

\[ + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7})
\]

\[ = 4\pi \lambda^2 - \frac{8\pi}{35} |\xi|^{-6}
\]

\[ + \frac{2\pi}{15} \lambda^4 (1 + |a|^{-1})^{-8} (\Delta (\text{tr} \, \sigma) - \text{div}(\text{div}(\sigma)))
\]

\[ + \frac{\pi}{105} \lambda^6 (\Delta (\Delta (\text{tr} \, \sigma)) - \Delta (\text{div}(\text{div}(\sigma))))
\]

\[ + \frac{8\pi}{15} \lambda^4 |a|^{-3} \nabla_a (\text{tr} \, \sigma)
\]

\[ + \frac{8\pi}{15} |a|^3 (\text{tr} \, \sigma - 3 |a|^{-2} \sigma(a, a))
\]

\[ + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\]
4.9. **Estimating \( R \) and \( \Delta_g R \).** We now relate the previous expression to the scalar curvature \( R \) of \((M, g)\). As with mean curvature, we first consider

\[
\hat{g}_{ij} = \bar{g}_{ij} + \hat{\sigma}_{ij} \quad \text{where} \quad \hat{\sigma}_{ij} = (1 + |x|^{-1})^{-4} \sigma_{ij}.
\]

Then,

\[
R_{\hat{g}} = \text{div div } \hat{\sigma} - \Delta \text{tr } \hat{\sigma} + O(|x|^{-6}).
\]

Note that

\[
\text{div } \hat{\sigma} = (1 + |x|^{-1})^{-4} \text{div } \sigma + 4(1 + |x|^{-1})^{-5} |x|^{-3} \sigma(x, \cdot).
\]

Thus, we find that

\[
\text{div div } \hat{\sigma} = (1 + |x|^{-1})^{-4} \text{div div } \sigma + 4(1 + |x|^{-1})^{-5} |x|^{-3} \sigma(x)
\]

\[
+ 20(1 + |x|^{-1})^{-6} |x|^{-6} \sigma(x, x) - 12(1 + |x|^{-1})^{-5} |x|^{-5} \sigma(x, x)
\]

\[
+ 4(1 + |x|^{-1})^{-5} |x|^{-3} \text{div } \sigma(x) + 4(1 + |x|^{-1})^{-5} |x|^{-3} \text{tr } \sigma
\]

\[
= (1 + |x|^{-1})^{-4} \text{div div } \sigma + 8 \text{div } \sigma(x)
\]

\[
+ 4(1 + |x|^{-1})^{-5} |x|^{-3} (\text{tr } \sigma - 3 |x|^{-2} \sigma(x, x))
\]

\[
+ O(|x|^{-6}).
\]

Similarly,

\[
\Delta \text{tr } \hat{\sigma} = \Delta \left( (1 + |x|^{-1})^{-4} \text{tr } \sigma \right)
\]

\[
= (1 + |x|^{-1})^{-4} \Delta \text{tr } \sigma
\]

\[
+ 8(1 + |x|^{-1})^{-5} |x|^{-3} \nabla_x \text{tr } \sigma
\]

\[
+ (\text{tr } \sigma) \Delta (1 + |x|^{-1})^{-4}
\]

\[
= (1 + |x|^{-1})^{-4} \Delta \text{tr } \sigma
\]

\[
+ 8 |x|^{-3} \nabla_x \text{tr } \sigma
\]

\[
+ O(|x|^{-6}).
\]

Thus, we find that

\[
R_{\hat{g}} = (1 + |x|^{-1})^{-4} \left( \text{div div } \sigma - \Delta \text{tr } \sigma \right)
\]

\[
+ 4 |x|^{-3} (\text{tr } \sigma - 3 |x|^{-2} \sigma(x, x))
\]

\[
+ 8 |x|^{-3} \text{div}(\sigma)(x)
\]

\[
- 8 |x|^{-3} \nabla_x \text{tr } \sigma
\]

\[
+ O(|x|^{-6}).
\]

It follows that

\[
R = -8(1 + |x|^{-1})^{-5} \Delta_{\hat{g}} |x|^{-1} + (1 + |x|^{-1})^{-4} R_{\hat{g}}
\]

\[
= -8(1 + |x|^{-1})^{-5} \Delta_{\hat{g}} |x|^{-1}
\]

\[
+ (1 + |x|^{-1})^{-8} (\text{div}(\sigma)) - \Delta \text{tr } \sigma
\]
Thus, it remains to estimate $\Delta_{\hat{g}}|x|^{-1}$. We have that

$$\sqrt{\det \hat{g}} = \sqrt{\det(\delta_{ij} + \hat{\sigma}_{ij})} = 1 + \frac{1}{2} \text{tr} \hat{\sigma} + O(|x|^{-4})$$

and

$$\hat{g}^{ij} = \delta^{ij} - \hat{\sigma}^{ij} + O(|x|^{-4}).$$

Thus,

$$\Delta_{\hat{g}}|x|^{-1} = -3|x|^{-5} \sigma(x, x) + |x|^{-3} \text{tr} \sigma + |x|^{-3} \text{div} \sigma(x) - \frac{1}{2} |x|^{-3} \nabla_x \text{tr} \sigma + O(|x|^{-6}).$$

Thus, we find that

$$R = (1 + |x|^{-1})^{-8} (\text{div}(\text{div}(\sigma)) - \Delta \text{tr} \sigma) - 4|x|^{-3} (\text{tr} \sigma - 3|x|^{-2} \sigma(x, x)) - 4|x|^{-3} \nabla_x \text{tr} \sigma + O(|x|^{-6}).$$

Similarly,

$$\Delta R = \Delta (\text{div}(\text{div}(\sigma)) - \Delta \text{tr} \sigma) + O(|x|^{-7}).$$

4.10. Reduced area-functional. We finally obtain that, for $\xi \in \mathbb{R}^3$ and $\lambda > 0$ large,

$$(\xi, \lambda) \mapsto \text{area}_{\hat{g}}(\Sigma_{(\xi, \lambda)}) = 4\pi \lambda^2 - \frac{2\pi}{15} \lambda^4 R - \frac{\pi}{105} \lambda^6 \Delta R - \frac{8\pi}{35} |\xi|^{-6} + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7})$$

where $R$ is the scalar curvature of $(M, g)$, and where

$$R = R(\lambda \xi) \quad \text{and} \quad \Delta R = (\Delta R)(\lambda \xi).$$

The Laplacian is computed with respect to the Euclidean background metric. This is $\Box$.

We also record here the first radial derivative

$$\frac{d}{ds} \bigg|_{s=1} \text{area}_g(\Sigma_{(s \xi, \lambda)}) = -\frac{2\pi}{15} \lambda^5 |\xi| \partial_s R - \frac{\pi}{105} \lambda^7 |\xi| \partial_s \Delta R + \frac{48\pi}{35} |\xi|^{-6}$$

$$+ O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).$$
where the underscore indicates evaluation at $\lambda \xi$ after all derivatives are taken.

This completes the proof of Theorem 1.5.

5. Proof of Corollary 1.7

We assume that $(M, g)$ is $C^6$-asymptotically Schwarzschild in the sense that

$$g_{ij} = (1 + |x|^{-1})^4 \delta_{ij} + \sigma_{ij},$$

where $\partial_I \sigma_{ij} = O(|x|^{-2-|I|})$ for all multi-indices $I$ of length $|I| \leq 6$. We also assume that

$$x^i x^j \partial_i \partial_j R \geq 0$$

outside of a compact set. This condition integrates to yield

$$x^i \partial_i R \leq 0 \quad \text{and} \quad R \geq 0$$

We now consider a sequence of connected closed stable constant mean curvature surfaces $\Sigma_k$ with

$$r_0(\Sigma_k) \to \infty, \quad \text{area}_g(\Sigma_k) \to \infty, \quad \text{and} \quad r_0(\Sigma_k) H(\Sigma_k) \to \infty.$$  

For $k$ large, we may find $\lambda > 0$ and $\xi \in \mathbb{R}^3$ large so that $\Sigma_k = \Sigma(\xi, \lambda)$. By Theorem 1.5,

$$\frac{d}{ds} \bigg|_{s=1} \text{area}_g(\Sigma(s\xi, \lambda)) = 0$$

so that, by (13),

$$0 = \frac{\pi}{105} \left( -14 \lambda^5 |\xi| \partial_r R - \lambda^7 |\xi| \partial_r \Delta R + 144 |\xi|^{-6} \right) + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).$$

It follows that

$$\partial_r R = O(\lambda^{-5} |\xi|^{-7}) = o(\lambda^{-5} |\xi|^{-5}).$$

Using this and (7), we may integrate in the in the radial direction to find that for $t \geq 0$,

$$(\partial_r R)((1 + t)\lambda \xi) \geq \partial_r R = o(\lambda^{-5} |\xi|^{-5}).$$

Integrating this again, we find that

$$R \leq o(\lambda^{-4} |\xi|^{-4}) t + R((1 + t) \lambda \xi) \leq O(\lambda^{-4} |\xi|^{-4}) (o(1)t + (1 + t)^{-4}).$$

Choosing $t$ judiciously we arrange for the term in parenthesis to be $o(1)$. We have proven that

$$R = o(\lambda^{-4} |\xi|^{-4}).$$

Now, considering the first variation of $\text{area}_g(\Sigma(\xi, \lambda))$ in directions orthogonal to $\xi$ as above. We obtain that the full derivative satisfies

$$D_R = O(\lambda^{-5} |\xi|^{-7}).$$

On the other hand, because $\partial_r R = o(\lambda^{-5} |\xi|^{-5})$, Taylor’s theorem combined with $\partial_r R \leq 0$ yields

$$\partial^2 R = o(\lambda^{-6} |\xi|^{-6}).$$
Combining this with (7), we obtain
\[ \partial^3_r R = o(\lambda^{-7}|\xi|^{-7}). \]
Similarly, combining the facts \( R \geq 0 \), \( R = o(\lambda^{-4}|\xi|^{-4}) \), and \( DR = o(\lambda^{-5}|\xi|^{-5}) \) with Taylor’s theorem yields
\[ D^2 R \geq -o(\lambda^{-6}|\xi|^{-6}). \]
Similarly, we find that
\[ D^2 \partial_r R \leq o(\lambda^{-7}|\xi|^{-7}). \]
Finally, since
\[ \partial_r \Delta R = \Delta \partial_r R - 2 |\xi|^{-1} \Delta R + 2 \lambda^{-1} |\xi|^{-1} \partial^2_r R + 2 \lambda^{-2} |\xi|^{-2} \partial_r R, \]
we see that
\[ \partial_r \Delta R \leq o(\lambda^{-7}|\xi|^{-7}). \]
Returning to the radial first variation, we see that
\[ 0 \geq 14 \lambda^5 |\xi| \partial_r R \geq 144 |\xi|^{-6} + \lambda^{-1} O(|\xi|^{-6}). \]
This contradiction completes the proof.

6. Proof of Theorem 1.8

As in the proof of Theorem 1.3, our strategy is parallel to the proof of Theorem 1 in [3], except that here we also exploit that the various terms in the reduced area functional \( \xi \mapsto \text{area}_g(\Sigma(\xi,\lambda)) \) have different orders in the regime where \( \xi \to \infty \).

Consider \( S : (0, \infty) \to (-\infty, 0] \) a smooth function with
\[ S^{(\ell)} = O(r^{-5-\ell}) \]
where \( S^{(\ell)} \) is the \( \ell \)-th derivative. We define a smooth function \( \varphi : (0, \infty) \to \mathbb{R} \) by
\[ \varphi(r) = \frac{1}{r} \int_r^{\infty} (\rho - r) \rho S(\rho) \, d\rho. \]
Arguing as in Lemma 3.1 we find that
\[ \varphi^{(\ell)}(r) = O(r^{-5-\ell}). \]
On the complement of a compact subset of \( \mathbb{R}^3 \) we define a conformally flat Riemannian metric
\[ g = (1 + 1/r + \varphi(r))^4 \bar{g} = (1 + 1/r)^4 \bar{g} + O(1/r^3). \]
Note that we can write
\[ g = (1 + 1/r)^4 \bar{g} + T_{ij} + o(1/r^2) \]
for \( T_{ij} = 0 \), so this is indeed of the form asserted in Theorem 1.8. The scalar curvature satisfies
\[ R = -8 (1 + O(1/r)) S(r). \]
Now, fix \( \chi \in C^\infty(\mathbb{R}) \) with support in \([4, 6]\) that is positive on \((4, 6)\). Assume that \( \chi'(5) = -1 \). Define
\[
S(r) = -\sum_{k=0}^{\infty} 10^{-5k} \chi(10^{-k} r).
\]
Note that \( S^{(\ell)}(r) = O(r^{-5-\ell}) \), as above.

Consider \( \xi \in \mathbb{R}^3 \) with \( |\xi| = 10^k t \) for \( t \in [3, 7] \). Then, taking \( \lambda = 10^k \), we have that
\[
\text{area}_g(\Sigma_{(10^k, \xi)}) = 4\pi \lambda^2 - \frac{2\pi}{15} 10^{4k} R(\lambda^{20k} \xi) - \frac{\pi}{105} 10^{6k} (\Delta R)(\lambda^{20k} \xi) - \frac{8\pi^2}{35} |\xi|^{-6} + O(10^{-7k})
\]
\[
= 4\pi \lambda^2 + \frac{2\pi}{15} 10^{-5k} \chi(t) - \frac{8\pi^2}{35} 10^{-6k} t^{-6} + O(10^{-7k}).
\]
Thus,
\[
\left. \frac{d}{ds} \right|_{s=1} \text{area}_g(\Sigma_{(10^k, s\xi)}) = \frac{2\pi}{15} 10^{-6k} \chi'(t) + \frac{48\pi}{35} 10^{-7k} t^{-6} + O(10^{-8k}).
\]
For \( t = 7 \), we have
\[
\left. \frac{d}{ds} \right|_{s=1} \text{area}_g(\Sigma_{(10^k, s\xi)}) = \frac{48\pi}{35} 10^{-7k} t^{-6} + O(10^{-8k}) > 10^{-5-7k}
\]
for sufficiently large \( k \). On the other hand, for \( t = 5 \), we have
\[
\left. \frac{d}{ds} \right|_{s=1} \text{area}_g(\Sigma_{(10^k, s\xi)}) = \frac{2\pi}{15} 10^{-6k} + \frac{48\pi}{35} 10^{-7k} t^{-6} + O(10^{-8k}) < 10^{-1-6k}.
\]
It follows that for some \( t_k \in (5, 7) \) and any \( \xi_k \in \mathbb{R}^3 \) with \( |\xi_k| = 10^k t_k \), the surface \( \Sigma_{(10^k, \xi_k)} \) is a stable constant mean curvature sphere. This completes the proof.

Appendix A. Some integral expressions

In this appendix, we recall several standard identities that are used in the proof of Theorem 1.5.

A.1. Integrals over \( B_r(0) \). Note that
\[
\int_{B_r(0)} (y^i)^2 = \frac{1}{3} \int_{B_r(0)} |y|^2 = \frac{4\pi}{15} r^5 \quad \text{for all } i = 1, 2, 3.
\]
Thus, for a symmetric tensor \( A_{ij} \) on \( \mathbb{R}^3 \), we have
\[
\sum_{i,j} \int_{B_r(0)} A_{ij} y^i y^j = \frac{4\pi}{15} r^5 \text{ tr } A.
\]
Similarly,
\[
\int_{B_r(0)} (y^i)^4 = \frac{4\pi}{35} r^7
\]
and for \( i \neq j \),
\[
\int_{B_r(0)} (y^i)^2(y^j)^2 = \frac{4\pi}{105} r^7
\]
For a totally symmetric tensor \( B_{ijkl} \) on \( \mathbb{R}^3 \), we have that
\[
\sum_{i,j,k,l} \int_{B_r(0)} B_{ijkl} y^i y^j y^k y^l = \sum_i B_{iii} \int_{B_r(0)} (y^i)^4 + 3 \sum_{i \neq j} B_{iijj} \int_{B_r(0)} (y^i)^2(y^j)^2
\]
\[= \frac{4\pi}{35} r^7 \left( \sum_i B_{iiii} + \sum_{i \neq j} B_{ijij} \right)\]
\[= \frac{4\pi}{35} r^7 \sum_{i,j} B_{ijij}.\]

A.2. Integrals over \( S_r(0) \). Note that
\[
\int_{S_r(0)} (y^i)^2 = \frac{4\pi}{3} r^4.
\]
It follows that, for a symmetric tensor \( A_{ij} \) on \( \mathbb{R}^3 \),
\[
\sum_{i,j} \int_{S_r(0)} A_{ij} y^i y^j = \frac{4\pi}{3} r^4 \text{ tr } A.
\]
Similarly,
\[
\int_{S_r(0)} (y^i)^4 = \frac{4\pi}{5} r^6 \quad \text{ for all } i = 1, 2, 3,
\]
\[
\int_{S_r(0)} (y^i)^2 (y^j)^2 = \frac{4\pi}{15} r^6 \quad \text{ for all } i \neq j.
\]
Thus, for a totally symmetric tensor \( B_{ijkl} \) on \( \mathbb{R}^3 \), we have
\[
\int_{S_r(0)} B_{ijkl} y^i y^j y^k y^l = \frac{4\pi}{5} r^6 B_{ijij}
\]
If \( B_{ijkl} \) is symmetric in the first two slots and in the second two slots separately, we obtain
\[
\sum_{i,j,k,l} \int_{S_r(0)} B_{ijkl} y^i y^j y^k y^l = \sum_i B_{iiii} \int_{S_r(0)} (y^i)^4 + \sum_{i,j} B_{ijij} \int_{S_r(0)} (y^i)^2 (y^j)^2
\]
\[
\quad + 2 \sum_{i,j \neq i} B_{ijij} \int_{S_r(0)} (y^i)^2 (y^j)^2
\]
\[
= \frac{4\pi}{15} r^6 \left( 3 \sum_i B_{iiii} + \sum_{i,j} B_{ijij} + 2 \sum_{i,j} B_{ijij} \right)
\]
\[
= \frac{4\pi}{15} r^6 \left( \sum_{i,j} B_{ijij} + 2 \sum_{i,j} B_{ijij} \right).
\]
Finally, we have
\[
\int_{S_r(0)} (y^i)^6 = \frac{4\pi}{7} r^8 \quad \text{ for all } i = 1, 2, 3,
\]
\[
\int_{S_r(0)} (y^i)^4 (y^j)^2 = \frac{4\pi}{35} r^8 \quad \text{ when } i \neq j,
\]
\[
\int_{S_r(0)} (y^i)^2 (y^j)^2 (y^k)^2 = \frac{4\pi}{105} r^8.
\]
Assume now that the tensor $C_{ijklmn}$ on $\mathbb{R}^3$ is symmetric in the first four indices and, separately, in the last two indices. Then,
\begin{align*}
\sum_{i,j,k,l,m,n} \int_{S_r(0)} C_{ijklmn} y^i y^j y^k y^m y^n &= \sum_i C_{iiiiii} \int_{S_r(0)} (y^i)^6 \\
&+ 6 \sum_{i,j \text{ distinct}} C_{iijjjj} \int_{S_r(0)} (y^i)^2 (y^j)^4 \\
&+ \sum_{i,j \text{ distinct}} C_{iiijjj} \int_{S_r(0)} (y^i)^4 (y^j)^2 \\
&+ 3 \sum_{i,j,k \text{ distinct}} C_{iijjkk} \int_{S_r(0)} (y^i)^2 (y^j)^2 (y^k)^2 \\
&+ 4 \sum_{i,j \text{ distinct}} C_{iiijij} \int_{S_r(0)} (y^i)^4 (y^j)^2 \\
&+ 12 \sum_{i,j,k \text{ distinct}} C_{iijkkk} \int_{S_r(0)} (y^i)^2 (y^j)^2 (y^k)^2 \\
= \frac{4\pi}{35} r^8 \left( \sum_{i,j,k} C_{iijjkk} + 4 \sum_{i,j,k} C_{iijkkk} \right).
\end{align*}

A.3. Some useful integrals. The following computations needed in the proof of Theorem 1 are readily verified using the identities from the previous subsection.
\begin{align*}
\int_{S_r(0)} \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right)^2 &= \frac{16\pi}{5} r^6 \\
\int_{S_r(0)} \left( \sigma(y, y) - \frac{1}{3} |y|^2 \text{tr} \sigma \right) \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) &= \int_{S_r(0)} \sigma(y, y) \left( \frac{|a|^2 |y|^2 - 3 \langle a, y \rangle^2}{|a|^5} \right) \\
&= \frac{8\pi}{15} r^6 \left( \text{tr} \sigma - 3 |a|^{-2} \sigma(a, a) \right) \\
\int_{S_r(0)} \left( \sigma(y, y) - \frac{1}{3} |y|^2 \text{tr} \sigma \right)^2 &= \frac{8\pi}{45} r^6 \left( 3 |\sigma|^2 - (\text{tr} \sigma)^2 \right) = \frac{8\pi}{15} r^6 |\sigma|^2.
\end{align*}

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