Holonomic control operators in quantum completely integrable Hamiltonian systems

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Abstract.
We provide geometric quantization of a completely integrable Hamiltonian system in the action-angle variables around an invariant torus with respect to the angle polarization. The carrier space of this quantization is the pre-Hilbert space of smooth complex functions on the torus. A Hamiltonian of a completely integrable system in this carrier space has a countable spectrum. If it is degenerate, its eigenvalues are countably degenerate. We study nonadiabatic perturbations of this Hamiltonian by a term depending on classical time-dependent parameters. It is originated by a connection on the parameter space, and is linear in the temporal derivatives of parameters. One can choose it commuting with a degenerate Hamiltonian of a completely integrable system. Then the corresponding evolution operator acts in the eigenspaces of this Hamiltonian, and is an operator of parallel displacement along a curve in a parameter space.

1 Introduction

By the well-known KAM theorem \[1, 11\], there are perturbations of a Hamiltonian of a completely integrable Hamiltonian system around an invariant torus such that trajectories of a perturbed Hamiltonian system do not leave a compact neighbourhood of this torus. Here, we show that one can obtain the similar effect by means of a suitable control operator, though trajectories of the control system need not live on tori. The perturbed system that we construct depends on parameters. A Hamiltonian of such a system is a sum of a dynamic Hamiltonian of the initial completely integrable system and a perturbation term which is linear in the temporal derivatives of parameters. The key point is that integration of this term over time through a parameter function depends only on a trajectory of this function in a parameter space, but not on its parameterization by time. The corresponding evolution operator can be treated as parallel displacement with respect to some connection on a parameter space, and plays the role of a classical geometric Berry factor. One can use it as a control operator in a perturbed completely integrable Hamiltonian system. The goal is quantization of such a system.

Note that, at present, geometric factor phenomena in quantum systems with classical parameters attract special attention in connection with holonomic quantum computation
This approach to quantum computing is based on the generalization of Berry’s adiabatic phase to the non-Abelian case corresponding to adiabatically driving an $n$-fold degenerate eigenstate of a Hamiltonian over the parameter manifold \[21\]. Information is encoded in this degenerate state, while the parameter manifold plays the role of a control parameter space.

There are different approaches to quantization of autonomous completely integrable Hamiltonian systems \[8, 9\]. An advantage of geometric quantization is that it remains equivalent under symplectic isomorphisms. One has studied geometric quantization of an autonomous completely integrable Hamiltonian system around an invariant torus with respect to polarization generated by Hamiltonian vector fields of first integrals of this system \[14, 15\]. The problem is that the associated quantum algebra contains functions which are not globally defined, and that elements of its carrier space fail to be smooth sections of the quantum bundle. We choose a different polarization.

Let \((\mathbb{Z}, \Omega)\) be a \(2m\)-dimensional symplectic manifold which admits \(m\) smooth functions \(F_k, k = 1, \ldots, m\), which are pairwise in involution and whose differentials \(dF_k\) are linearly independent almost everywhere. This is called a completely integrable Hamiltonian system.

Let \(M\) be its (connected) compact manifold, i.e., all \(F_k\) are constant on \(M\). By the well-known theorem \[1, 11\], if \(dF_k \neq 0\) on \(M\), there exists a small neighbourhood of \(M\) which is isomorphic to the symplectic annulus

\[
W = V \times T^m,
\]

where \(V \subset \mathbb{R}^m\) is a nonempty (open, contractible) domain and \(T^m\) is an \(m\)-dimensional torus. The \(W\) is equipped with the action-angle coordinates \((I_k, \phi_k \mod 2\pi)\). With respect to these coordinates, the symplectic form on \(W\) reads

\[
\Omega = dI_k \wedge d\phi^k,
\]

and all \(F_k\) are functions of action coordinates \((I_k)\) only. A Hamiltonian on \(W\) is an arbitrary analytic function \(\mathcal{H}\) of action coordinates \(I_k\). The corresponding Hamilton equation reads

\[
\dot{I}_k = 0, \quad \dot{\phi}^k = \partial^k \mathcal{H}.
\]

In order to geometrically quantize the symplectic manifold \((W, \Omega)\), we choose the angle polarization spanned by the almost-Hamiltonian vector fields \(\partial^k\) of angle variables \[3\]. The associated quantum algebra \(\mathcal{A}\) consists of functions which are affine in action variables \(I_k\). We obtain the continuum set of its nonequivalent representations by first order differential operators in the separable pre-Hilbert space \(\mathbb{C}^\infty(T^m)\) of smooth complex functions on the torus \(T^m\). This set is indexed by homomorphisms of the de Rham cohomology group \(H^1(T^m)\) of \(T^m\) to the circle group \(U(1)\), i.e., by collections \([\lambda_k]\), \(k = 1, \ldots, m\), of real numbers \(\lambda_k \in [0, 1)\). In particular, the action operators read

\[
\hat{I}_k = -i\partial_k - \lambda_k.
\]
By virtue of the multidimensional Fourier theorem \[4\], the orthonormal basis for \(C^\infty(T^m)\) consists of functions

\[
\psi_{(n_r)} = \exp[i(n_r\phi^r)], \quad (n_r) = (n_1, \ldots, n_m) \in \mathbb{Z}^m.
\]  

(5)

These are eigenvectors

\[
\hat{I}_k \psi_{(n_r)} = (n_k - \lambda_k) \psi_{(n_r)}
\]

of the action operators \[4\] and, consequently, of a Hamiltonian

\[
\hat{H} \psi_{(n_r)} = \mathcal{H}(\hat{I}_j) \psi_{(n_r)} = \mathcal{H}(n_j - \lambda_j) \psi_{(n_r)}.
\]  

(6)

In particular, if a Hamiltonian of a completely integrable system is independent on some action variables, its eigenstates are countably degenerate. In the framework of holonomic quantum computation, such a degenerate state can be utilized for encoding information. The goal is to build a holonomic control operator acting in this state. We construct it depending on classical time-dependent parameters.

## 2 Quantization of a completely integrable Hamiltonian system

We follow the standard geometric quantization procedure \[2, 20, 22\]. Since the symplectic form \(\Omega\) \[2\] is exact, the prequantum bundle is a trivial complex line bundle \(C \rightarrow W\). Let its trivialization

\[
C \cong W \times \mathbb{C}
\]  

(7)

hold fixed. Any other trivialization leads to equivalent quantization of \(W\). Given the associated bundle coordinates \((I_k, \phi^k, c), c \in \mathbb{C}\), on \(C \[6\], one can treat its sections as smooth complex functions on \(W\).

The Konstant–Souriau prequantization formula associates to each smooth real function \(f \in C^\infty(W)\) on \(W\) the first order differential operator

\[
\hat{f} = -i\nabla_\phi f + f
\]  

(8)

on sections of \(C\). Here

\[
\phi = \partial^k f \partial_k - \partial_k f \partial^k
\]

is the Hamiltonian vector field of \(f\), and \(\nabla\) is the covariant differential with respect to a \(U(1)\)-principal connection \(A\) on \(C\) whose curvature form obeys the prequantization condition \(R = i\Omega\). Such a connection reads

\[
A = A_0 + icI_k d\phi^k \otimes \partial_c,
\]  

(9)
where $A_0$ is a flat $U(1)$-principal connection on $C$. The classes of gauge conjugated flat principal connections on $C$ are indexed by the set $\mathbb{R}^m/\mathbb{Z}^m$ of homomorphisms of the de Rham cohomology group

$$H^1(W) = H^1(T^m) = \mathbb{R}^m$$

of the annulus $W$ (1) to $U(1)$ (2). Let us choose the representatives

$$A_0[\lambda_k] = dI_k \otimes \partial^k + d\phi^k \otimes (\partial_k + i\lambda_k c\partial_c), \quad \lambda_k \in [0, 1),$$

of these classes. Then the connection $A$ (3) up to gauge conjugation reads

$$A[\lambda_k] = dI_k \otimes \partial^k + d\phi^k \otimes (\partial_k + i(I_k + \lambda_k)c\partial_c). \quad (10)$$

For the sake of simplicity, let $\lambda_k$ in the expression (10) be arbitrary real numbers, but we will bear in mind that connections $A[\lambda_k]$ and $A[\lambda'_k]$ with $\lambda_k - \lambda'_k \in \mathbb{Z}$ are gauge conjugated.

Given a connection (10), the prequantization operators (8) read

$$\hat{f} = -i\theta f + (f - (I_k + \lambda_k)\partial^k f). \quad (11)$$

Since the divergence of any Hamiltonian vector field with respect to canonical coordinates vanishes, the prequantization operators $\hat{f}$ also keep their form (11) on sections of the quantum bundle $C \otimes D_{1/2}$, where $D_{1/2} \to W$ is a metalinear bundle, whose sections are half-forms on $W$.

Let us choose the manifested angle polarization. It is the vertical tangent bundle $V\pi$ of the fiber bundle

$$\pi : V \times T^m \to T^m$$

spanned by the vectors $\partial^k$. The corresponding quantum algebra $\mathcal{A} \subset C^\infty(W)$ consists of affine functions

$$f = a^k(\phi^r)I_k + b(\phi^r) \quad (12)$$

of action coordinates $I_k$. The carrier space $\mathcal{E}$ of its representation (11) consists of sections $\rho$ of the quantum bundle $C \otimes D_{1/2} \to W$ of compact support which obey the condition $\nabla_\theta \rho = 0$ for any Hamiltonian vector field $\theta$ subordinate to the polarization $V\pi$. This condition reads

$$\partial_k f \partial^k \rho = 0, \quad \forall f \in C^\infty(W).$$

It follows that elements of $\mathcal{E}$ are independent of action variables and, consequently, fail to be of compact support, unless $\rho = 0$, i.e., $\mathcal{E}$ reduces to $\rho = 0$. 

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Therefore, let us modify the standard quantization procedure as follows [7]. Given an imbedding
\[ i_T : T^m \to V \times T^m, \]
let \( C_T = i_T^* C \) be the pull-back of the prequantum bundle \( C \) onto the torus \( T^m \). It is a trivial complex line bundle \( C_T = T^m \times \mathbb{C} \) whose sections are smooth complex functions on \( T^m \). Let
\[ \overline{A}[\lambda_k] = i_T^* A[\lambda_k] = d\phi^k \otimes (\partial_k + i(I_k + \lambda_k)c\partial_c) \]
be the pull-back of the connection \( A[\lambda_k] \) onto \( C_T \), and let \( \nabla \) denote the corresponding covariant differential. Let \( \mathcal{D}_T \) be a metahermitian bundle of complex half-forms on the torus \( T^m \). It admits the canonical lift of any vector field \( \tau \) on \( T^m \), and the corresponding Lie derivative of its sections reads
\[ \mathbf{L}_\tau = \tau^k \partial_k + \frac{1}{2} \partial_k \tau^k. \]
Let us consider the tensor product
\[ Y = C_T \otimes \mathcal{D}_T \to T^m. \] (13)
Since the Hamiltonian vector fields
\[ \vartheta_f = a^k \partial_k - (I_\tau \partial_\tau a^\tau + \partial_\tau b)\partial^k \]
of functions \( f \) are projectable onto \( T^m \), one can associate to each \( f \in \mathcal{A} \) the first order differential operator
\[ \hat{f} = (-i\nabla_{\pi \vartheta_f} + f) \otimes \text{Id} + \text{Id} \otimes \mathbf{L}_{\pi \vartheta_f} = -ia^k \partial_k - i\frac{1}{2} \partial_k a^k - a^k \lambda_k + b \] (14)
on sections of \( Y \). A direct computation shows that the operators (14) obey the Dirac condition
\[ [\hat{f}, \hat{f}'] = -i\{f, f'\}. \]
Sections \( s \) of \( Y \to T^m \) constitute a pre-Hilbert space \( \mathcal{E}_T \) with respect to the nondegenerate Hermitian form
\[ \langle s | s' \rangle = \left( \frac{1}{2\pi} \right)^m \int_{T^m} s s', \quad s, s' \in \mathcal{E}_T. \]
Then \( \hat{f} \) are Hermitian operators in \( \mathcal{E}_T \). They provide the desired geometric quantization of a completely integrable Hamiltonian system on the annulus \( W \).
Of course, this quantization depends on the choice of a connection $A[\lambda_k]$ (10) and a metalinear bundle $D_T$, which need not be trivial.

If $D_T$ is trivial, sections of the quantum bundle $Y \to T^m$ (13) obey the transformation rule

$$s(\phi^k + 2\pi) = s(\phi^k)$$

for all indices $k$. They are naturally complex smooth functions on $T^m$, i.e., the carrier space $\mathcal{E}_T$ coincides with the pre-Hilbert space $\mathbb{C}^\infty(T^m)$ of smooth complex functions on $T^m$. Its orthonormal basis consists of the functions $\psi_{(n_r)}$ (8). The action operators $\hat{I}_k$ (14) in this space take the form (4). Other elements of the algebra $A$ are decomposed into the pull-back functions $\pi^*\psi_{(n_r)}$ on $W$ which act on $\mathbb{C}^\infty(T^m)$ by multiplications

$$\pi^*\psi_{(n_r)}\psi_{(n'_r)} = \psi_{(n_r)}\psi_{(n'_r)} = \psi_{(n_r+n'_r)}.$$ (15)

If $D_T$ is a nontrivial metalinear bundle, sections of the quantum bundle $Y \to T^m$ (13) obey the transformation rule

$$\rho_T(\phi^j + 2\pi) = -\rho_T(\phi^j)$$ (16)

for some indices $j$. In this case, the orthonormal basis for the pre-Hilbert space $\mathcal{E}_T$ can be represented by double-valued complex functions

$$\psi_{(n_i,n_j)} = \exp[i(n_i\phi^i + (n_j + \frac{1}{2})\phi^j)]$$ (17)

on $T^m$. They are eigenvectors

$$\hat{I}_i\psi_{(n_i,n_j)} = (n_i - \lambda_i)\psi_{(n_i,n_j)}, \quad \hat{I}_j\psi_{(n_i,n_j)} = (n_j - \lambda_j + \frac{1}{2})\psi_{(n_i,n_j)}$$

of the operators $\hat{I}_k$ (14), while the pull-back functions $\pi^*\psi_{(n_r)}$ act on the basis (14) by the above law (15). It follows that the representation of the quantum algebra $A$ determined by the connection $A[\lambda_k]$ (10) in the space of sections (16) of a nontrivial quantum bundle $Y$ (13) is equivalent to its representation determined by the connection $A[\lambda_i, \lambda_j - \frac{1}{2}]$ in the space $\mathbb{C}^\infty(T^m)$ of smooth complex functions on $T^m$. Therefore, one can restrict the study of representations of the quantum algebra $A$ to its representations in $\mathbb{C}^\infty(T^m)$ defined by different connections (10). These representations are nonequivalent, unless $\lambda_k - \lambda'_k \in \mathbb{Z}$ for all indices $k$.

Given the representation (14) of the quantum algebra $A$ in $\mathbb{C}^\infty(T^m)$, any polynomial Hamiltonian $\mathcal{H}(I_k)$ of a completely integrable system is uniquely quantized as a Hermitian element $\hat{\mathcal{H}}(I_k) = \mathcal{H}(\hat{I}_k)$ of the enveloping algebra $\overline{A}$ of $A$. It has the countable spectrum (8). Note that, since $I_k$ are diagonal operators, one can also quantize the Hamiltonians which are analytic functions on $\mathbb{R}^m$.  

6
3 The classical controllable completely integrable system

A generic momentum phase space of a Hamiltonian system with time-dependent parameters is a composite fiber bundle

\[ \Pi \to \Sigma \to \mathbb{R}, \]

where \( \Pi \to \Sigma \) is a symplectic bundle, \( \Sigma \to \mathbb{R} \) is a parameter bundle whose sections are parameter functions, and \( \mathbb{R} \) is the time axis \([5, 12, 19]\). Here, we assume that all bundles are trivial and, moreover, their trivializations hold fixed. Then the momentum phase space of a completely integrable Hamiltonian system on the symplectic annulus \( W \) is the product

\[ \Pi = \mathbb{R} \times S \times W, \]

(18)
equipped with the coordinates \((t, \sigma^\lambda, I_k, \phi)\). It is convenient to suppose for a time that parameters are dynamic variables. The momentum phase space of such a system is

\[ \Pi' = \mathbb{R} \times T^* S \times W, \]

(19)
coordinated by \((t, \sigma^\lambda, p_{\lambda}, I_k, \phi)\).

The dynamics of a time-dependent mechanical system on the momentum phase space \( \Pi' \) is characterized by a Hamiltonian form

\[ H_\Sigma = p_{\lambda} d\sigma^\lambda + I_k d\phi^k - H_\Sigma(t, \sigma^\mu, p_{\mu}, I_j, \phi^j) dt \]

(20)
For any Hamiltonian form \( H_\Sigma \), there exists a unique vector field \( \gamma_H \) on \( \Pi' \) such that

\[ \gamma_H^* dt = 1, \quad \gamma_H^* dH_\Sigma = 0. \]

It defines the first order differential Hamilton equation on \( \Pi' \).

Note that the Hamiltonian \( H_\Sigma \) takes the form

\[ \mathcal{H}_\Sigma = p_{\lambda} \Gamma^\lambda + I_k (\Lambda^k + \Gamma^\lambda \Lambda^k_{\lambda}) + \mathcal{H}, \]

(21)
where

\[ \Lambda \circ \Gamma = dt \otimes (\partial_t + \Gamma^\lambda \partial_{\lambda} + (\Lambda^k + \Gamma^\lambda \Lambda^k_{\lambda}) \partial_k) \]
is a connection on \( \mathbb{R} \times S \times T^m \to \mathbb{R} \) which is the composition of a connection

\[ \Gamma = dt \otimes (\partial_t + \Gamma^\lambda(t, \sigma^\mu) \partial_{\lambda}) \]

(22)
on the parameter bundle \( \mathbb{R} \times S \to \mathbb{R} \) and of a connection

\[ \Lambda = dt \otimes (\partial_t + \Lambda^k(t, \sigma^\mu, \phi^j) \partial_k) + d\sigma^\lambda \otimes (\partial_{\lambda} + \Lambda^k(t, \sigma^\mu, \phi^j) \partial_k) \]

(23)
on \( R \times S \times T^m \to R \times S \) \cite{12, 13}.

Bearing in mind that \( \sigma^\lambda \) are parameters, one should choose the Hamiltonian \( H_\Sigma \) (21) affine in momenta \( p_\lambda \). Furthermore, in order to describe a Hamiltonian system with a fixed parameter function \( \sigma^\lambda = \xi^\lambda(t) \), one defines the connection \( \Gamma \) (22) such that
\[
\nabla^\Gamma \xi = 0, \quad \Gamma^\lambda(t,\xi^\mu(t)) = \partial_t \xi^\lambda.
\]

Then the pull-back
\[
H = \xi^* H_\Sigma = I_k d\phi^k - (I_k\Lambda^k(t,\xi^\mu,\phi^j) + \Lambda^k_\lambda(t,\xi^\mu,\phi^j)\partial_t \xi^\lambda) + \tilde{H}(t,\xi^\mu, I_j, \phi^j) \, dt
\]
is a Hamiltonian form on \( R \times W \). Let us put
\[
\tilde{H} = H - I_k \Lambda^k.
\]

Then the Hamiltonian form
\[
H = I_k d\phi^k - H' \, dt = I_k d\phi^k - [I_k\Lambda^k(t,\xi^\mu,\phi^j)\partial_t \xi^\lambda + H(I_j)] \, dt
\]
describes a time-dependent perturbed completely integrable Hamiltonian system on \( R \times W \). The corresponding Hamilton equation reads
\[
\partial_t I_k = -\partial_k \Lambda^k_\lambda I_j \partial_t \xi^\lambda, \quad \partial_t \phi^k = \partial^k H + \Lambda^k_\lambda \partial_t \xi^\lambda.
\]

In order to make the term
\[
\Delta = I_k \Lambda^k_\lambda \partial_t \xi^\lambda
\]
in the perturbed Hamiltonian \( H' \) (24) to a control operator, let us assume that the coefficients \( \Lambda^k_\lambda \) of the connection (23) are independent of time. Then, in view of the trivialization (18), the second term
\[
d\sigma^\lambda \otimes \left( \partial_\lambda + \Lambda^k_\lambda(\sigma^\mu,\phi^j)\partial_k \right)
\]
of the connection (23) can be seen as a connection on the fiber bundle \( S \times T^m \to S \). Let the dynamic Hamiltonian \( \mathcal{H} \) be independent of action variables with some \( l \) indices \( a, b, c, \ldots \), and let the perturbation term \( \Delta \) (26) be independent of the action and angle variables with the rest indices \( i, j, k, \ldots \). Then the Hamilton equation (25) falls into the two independent equations
\[
\begin{align*}
\partial_t I_k &= 0, \quad \partial_t \phi^k = \partial^k \mathcal{H}, \\
(a) \quad \partial_t I_a &= -I_b \partial_a \Lambda^k_\lambda \partial_t \xi^\lambda, \\
(b) \quad \partial_t \phi^a &= \Lambda^k_\lambda \partial_t \xi^\lambda.
\end{align*}
\]

The first equation (27) keeps the form of the Hamilton equation (3) of an autonomous completely integrable system, while the second one is the control equation as follows.
Let us rewrite the equation (28b) as the countable system of equations
\[
\partial_t \psi_{(n_a)} = i\psi_{(n_a)} n_a \partial_t \phi^a = i\psi_{(n_a)} n_a \Lambda^a_\lambda \partial_t \xi^\lambda
\] (29)
for functions \(\psi_{(n_a)}\). The left hand-side of these equations is a multidimensional Fourier series with time-dependent coefficients. Therefore, the equations (29) for all collections of \(l\) integers \((n_a)\) make up a countable system of ordinary linear differential equations with time-dependent coefficients:
\[
\partial_t \psi_{(n_a)} = iM_{\lambda(n_a)}(\xi^\mu) \partial_t \xi^\lambda \psi_{(n_b)}.
\]
Its solution with the initial dates \(\phi^a(0)\) can be written formally as the time-ordered exponential
\[
\psi_{(n_a)}(t) = U(t)_{(n_b)}^{(n_a)} \psi_{(n_b)}(0),
\]
\[
U(t) = T \exp \left[ i \int_0^t M_\lambda(\xi^m(t')) \partial_t \xi^\lambda dt' \right] = T \exp \left[ i \int_{\xi([0,t])} M_\lambda(\sigma^\mu) d\sigma^\lambda \right],
\] (30)
\[\text{[10, 16].}\] A glance at the evolution operator \(U(t)\) (30) shows that solutions \(\psi_{(n_a)}(t)\) of the equations (29) are functions of a point \(\sigma\) of the curve \(\xi : \mathbb{R} \to S\) in the parameter space \(S\).

Substituting this solution into the equation (28a), we obtain the system of \(l\) ordinary linear differential equations with time-dependent coefficients:
\[
\partial_t I_a = -L^b_{\lambda a}(\psi_{(n_a)}(t), \xi^\mu(t)) I_b \partial_t \xi^\lambda, \quad L^b_{\lambda a} = \partial_a \Lambda^b_\lambda.
\]
Its solution is given by the time-ordered exponential
\[
I_a(t) = U(t)^b_a I_b(0),
\]
\[
U(t) = T \exp \left[ - \int_0^t L_\lambda(\psi_{(n_a)}(t'), \xi^\mu(t')) \partial_t \xi^\lambda dt' \right] = T \exp \left[ - \int_{\xi([0,t])} L_\lambda(\psi_{(n_a)}(\sigma), \sigma^\mu) d\sigma^\lambda \right].
\]
This solution is also a functions of a point \(\sigma\) of the curve \(\xi : \mathbb{R} \to S\) in the parameter space \(S\).

Consequently, the equation (28a) can be regarded a control equation. In particular, one can choose a parameter function \(\xi(t)\) such that the trajectory of the perturbed system does not leave a compact neighbourhood of the invariant torus \(T^m\) of the initial completely integrable Hamiltonian system.
4 The quantum controllable completely integrable system

Let us quantize the classical controllable completely integrable system on the momentum phase space $\mathbb{R} \times W$ in previous Section.

The manifold $W' = \mathbb{R} \times W$, coordinated by $(t, I_k, \phi^k)$, is provided with the Poisson structure

$$\{f, f'\} = \partial_k f \partial_k f' - \partial_k f \partial_k f', \quad f, f' \in C^\infty(W').$$

This is the direct product of the symplectic structure $\Omega$ on the symplectic annulus $W$ and of the zero Poisson structure on the time axis $\mathbb{R}$. In particular, the Poisson algebra $(C^\infty(W'), \{,\})$ of smooth real functions on $W'$ is the Lie algebra over the ring $C^\infty(\mathbb{R})$ of smooth real functions on $\mathbb{R}$. In order to quantize the Poisson manifold $(W', \{,\})$, we therefore can essentially simplify the general procedure of instantwise geometric quantization of time-dependent Hamiltonian systems in Ref. [7]. Namely, we repeat geometric quantization of the symplectic manifold $(W, \Omega)$ in Section 2, but replace functions on $T_m$ with those on $\mathbb{R} \times T_m$.

Let us choose the angle polarization spanned by the vectors $\partial^i$. The corresponding quantum algebra $\mathcal{A} \subset C^\infty(W')$ consists of affine functions

$$f = a^i(t, \phi^j)I_i + b(t, \phi^j)$$

of action coordinates $I_i$. It is represented by the first order differential operators

$$\hat{f} = -i a^i \partial_t - \frac{i}{2} \partial_t a^i - a^i \lambda_i + b, \quad \lambda_i \in \mathbb{R},$$

in the space $C^\infty(\mathbb{R} \times T_m)$ of smooth complex functions on $\mathbb{R} \times T_m$. Given different collections of real numbers $(\lambda_i)$ and $(\lambda'_i)$, the representations (31) are nonequivalent, unless $\lambda_i - \lambda'_i \in \mathbb{Z}$ for all indices $i$. The carrier space $C^\infty(\mathbb{R} \times T_m)$ is provided with the structure of the pre-Hilbert $C^\infty(\mathbb{R})$-module with respect to the nondegenerate $C^\infty(\mathbb{R})$-bilinear form

$$\langle \psi | \psi' \rangle = \left(\frac{1}{2\pi}\right)^m \int_{T_m} \psi \overline{\psi'}, \quad \psi, \psi' \in C^\infty(\mathbb{R} \times T_m).$$

The basis for this pre-Hilbert module is made up by the pull-backs onto $\mathbb{R} \times T_m$ of functions $\psi_{(n,)}$ (5) on $T_m$. They are the eigenvectors of the action operators

$$\hat{I}_k = -i \partial_k - \lambda_k, \quad \hat{I}_k \psi_{(n,)} = (n_k - \lambda_k) \psi_{(n,)}.$$

As in previous Section, let us assume that the dynamic Hamiltonian $\mathcal{H}$ is independent of action variables with some $l$ indices $a, b, c, \ldots$, and let the perturbation term $\Delta$ (26) be
indepedent of the time and the action-angle variables with the rest indices \( i, j, k, \ldots \), i.e., the perturbed Hamiltonian \( \mathcal{H}' \) \( (24) \) takes the form

\[
\mathcal{H}' = \Lambda^a_\lambda(\xi^\mu, \phi^b) \partial_i \xi^\lambda I_a + \mathcal{H}(I_j).
\]

The perturbation term

\[
\Delta = \Lambda^a_\lambda(\xi^\mu, \phi^b) \partial_i \xi^\lambda I_a
\]

of this Hamiltonian is an element of the quantum algebra \( \mathcal{A} \), and is quantized by the operator

\[
\hat{\Delta} = -i\Lambda^a_\beta \partial_i \xi^\beta \partial_a - \frac{i}{2} \partial_a (\Lambda^a_\beta) \partial_i \xi^\beta - \lambda_a \Lambda^a_\beta \partial_i \xi^\beta.
\]

The (polynomial or analytic) dynamic Hamiltonian \( \mathcal{H}(I_j) \) is quantized as in Section 2, i.e.,

\[
\hat{\mathcal{H}} = \mathcal{H}(\hat{I}_j).
\]

Since the operators \( \hat{\Delta} \) and \( \hat{\mathcal{H}} \) mutually commute, the corresponding quantum evolution operator reduces to the product

\[
T \exp \left[ -i \int_0^t \hat{\mathcal{H}} dt' \right] = U_1(t) \circ U_2(t) = T \exp \left[ -i \int_0^t \hat{\mathcal{H}} dt' \right] \circ T \exp \left[ -i \int_0^t \hat{\Delta} dt' \right]. \tag{32}
\]

The first operator in this product is the dynamic evolution operator of the quantum completely integrable Hamiltonian system. It reads

\[
U_1(t) \psi_{(n_j)} = \exp[-i\mathcal{H}(n_j - \lambda_j) t] \psi_{(n_j)}. \tag{33}
\]

Its eigenvalues are countably degenerate. Recall that the operator \( (33) \) acts in the \( \mathbb{C}^\infty(\mathbb{R}) \)-module \( \mathbb{C}^\infty(\mathbb{R} \times T^m) \). Its eigenvalues are smooth complex functions on \( \mathbb{R} \), and its eigenspaces are \( \mathbb{C}^\infty(\mathbb{R}) \)-submodules of \( \mathbb{C}^\infty(\mathbb{R} \times T^m) \) of countable rank.

The second multiplier in the product \( (32) \) is

\[
U_2(t) = T \exp \left[ \int_0^t \{ -\Lambda^a_\beta(\phi^b, \xi^\mu) \partial_a - \frac{1}{2} \partial_a \Lambda^a_\beta(\phi^b, \xi^\mu(t')) + i\lambda_a \Lambda^a_\beta(\phi^b, \xi^\mu(t')) \} \partial_i \xi^\beta dt' \right]
\[
= T \exp \left[ \int_{\xi([0,t])} \{ -\Lambda^a_\beta(\phi^b, \sigma^\mu) \partial_a - \frac{1}{2} \partial_a \Lambda^a_\beta(\phi^b, \sigma^\mu) + i\lambda_a \Lambda^a_\beta(\phi^b, \sigma^\mu) \} d\sigma^\beta \right]. \tag{34}
\]

It acts in the eigenspaces of the operator \( U_1(t) \). For instance, such a space is exemplified by the pre-Hilbert \( \mathbb{C}^\infty(\mathbb{R}) \)-submodule \( E_0 \subset \mathbb{C}^\infty(\mathbb{R} \times T^m) \) whose orthonormal basis is made up by functions \( \psi_{(n_a)} \) for all collections of integers \( (n_a) \). Written with respect to this basis, the operator \( U_2(t) \) acts on \( E_0 \) as a matrix of countable rank.
A glance on the expression (34) shows that, in fact, the operator \( U_2(t) \) depends on the curve \( \xi([0,1]) \subset S \) in the parameter space \( S \). One can treat it as an operator of parallel displacement with respect to a connection in the \( \mathbb{C}^\infty(\Sigma) \)-module of smooth complex functions on \( \Sigma \times T^m \) along the curve \( \xi \) \[3, 13, 19\]. For instance, if \( \xi([0,1]) \) is a loop in \( S \), the operator \( U_2 \) (34) is the geometric Berry factor. In this case, one can think of \( U_2 \) as being the holonomic control operator.

It should be emphasized that the adiabatic assumption has never been involved.

References

[1] V.Arnold (Ed.), *Dynamical Systems III* (Springer-Verlag, Berlin, 1988).

[2] A.Echeverría-Enríquez, M.Muñoz-Lecanda, N.Román-Roy and C.Victoria-Monge, Mathematical foundations of geometric quantization, *Extracta Math.* 13, 135 (1988).

[3] K.Fujii, Note on coherent states and adiabatic connections, curvatures, *J. Math. Phys.* 41, 4406 (2000).

[4] G.Gallavotti, *The Elements of Mechanics* (Springer-Verlag, Berlin, 1983).

[5] G.Giachetta, L.Mangiarotti and G.Sardanashvily, E-print arXiv: quant-ph/0112011.

[6] G.Giachetta, L.Mangiarotti and G.Sardanashvily, E-print arXiv: quant-ph/0112083.

[7] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Covariant geometric quantization of nonrelativistic time-dependent mechanics, *J. Math. Phys.* 43, 56 (2002).

[8] M. de Gosson, The symplectic camel and phase space quantization, *J. Phys. A* 34, 10085 (2001).

[9] M.Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, Berlin, 1990).

[10] C.Lam, Decomposition of time-ordered products and path-ordered exponentials, *J. Math. Phys.* 39, 5543 (1998).

[11] V.Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions* (Springer-Verlag, Berlin, 1993).

[12] L.Mangiarotti and G.Sardanashvily, *Gauge Mechanics* (World Scientific, Singapore, 1998).
[13] L. Mangiarotti and G. Sardanashvily, Connections in Classical and Quantum Field Theory (World Scientific, Singapore, 2000).

[14] I. Mykytiuk and A. Prykarpatsky, Quantization of completely integrable Hamiltonian systems. Geometric aspect, Nuovo Cimento B109, 1185 (1994).

[15] I. Mykytiuk, A. Prykarpatsky, R. Andrushkiw and V. Samoilenko, Geometric quantization of Newmann-type completely integrable Hamiltonian systems, J. Math. Phys. 35, 1532 (1994).

[16] J. Oteo and J. Ros, From time-ordered products to Magnus expansion, J. Math. Phys. 41, 3268 (2000).

[17] J. Pachos and P. Zanardi, Quantum holonomies for quantum computing, Int. J. Mod. Phys. B15, 1257 (2001).

[18] G. Sardanashvily, Hamiltonian time-dependent mechanics, J. Math. Phys. 39, 2714 (1998).

[19] G. Sardanashvily, Classical and quantum mechanics with time-dependent parameters, J. Math. Phys. 41, 5245 (2000).

[20] J. Śniatycki, Geometric Quantization and Quantum Mechanics (Springer-Verlag, Berlin, 1980).

[21] F. Wilczek and A. Zee, Appearance of gauge structure in simple dynamical systems, Phys. Rev. Lett. 52, 2111 (1984).

[22] N. Woodhouse, Geometric Quantization (Clarendon Press, Oxford, 1992).

[23] P. Zanardi and M. Rasetti, Holonomic quantum computation, Phys. Lett. A264, 94 (1999).