Optimal Dividend Strategy for An Insurance Group with Contagious Default Risk

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Abstract

This paper studies the optimal dividend for a multi-line insurance group, in which each subsidiary runs a product line and is exposed to some external credit default risk. The external default contagion is considered in the sense that one default event can affect the default probabilities of all surviving subsidiaries. The total dividend problem is formulated for the insurance group and we reveal for the first time that the optimal singular dividend strategy is still of the barrier type. Furthermore, we show that the optimal barrier for each subsidiary is modulated by the current default state, namely how many and which subsidiaries have defaulted will determine the dividend threshold for each surviving subsidiary. These interesting conclusions are based on our analysis of the associated recursive system of Hamilton-Jacobi-Bellman variational inequalities (HJBVIs), which is new to the literature. The existence of the classical solution is established and the rigorous proof of the verification theorem is provided. For the case of two subsidiaries, the value function and optimal barriers for each subsidiary are explicitly constructed. Some numerical examples are also presented to illustrate the economic insights.

Keywords: Insurance group, external default contagion, optimal dividend, reflection control, default-state-modulated barriers, recursive system of HJBVIs.

1 Introduction

Dividend payment decision making is a fundamental topic in insurance industry and corporate finance, which represents an important financial signal about a firm’s future growth opportunities and may influence the wealth of the shareholders. [23] studies the relationship between a company’s dividend policy and the valuation of its shares. Insurance companies generally accumulate large amount of cash to pay future claims due to the nature of their products. A higher level of surplus will provide the better protection against the adverse claim volatility. However, insurers are also under increasing pressure to pay dividends to shareholders when surplus level keeps growing higher. The optimal dividend problem is first studied by [15], which solved the problem in a discrete time model by assuming that the surplus process follows a simple random walk and that the decision maker aims to maximize the expected total dividends until the financial ruin. [16] provides solutions for optimal dividend problem in both discrete and continuous models. [4] examines the problem using the theory of controlled diffusion processes. In the past decades, vast research...
has been devoted to finding optimal dividend strategies in different settings and context. See a short list of some pioneer work among [19, 7, 6, 13, 22, 29] and references therein. We refer to [2] and [5] for some comprehensive surveys of dividend control problems.

On the other hand, mergers and acquisitions are effective ways to form large insurance groups to expand their business in a market that is currently not served and seek secure profitable growth. Largest insurers generally have a wide coverage of product lines run by subsidiaries in different markets, e.g. new or existing markets, positive or negatively correlated products and markets, etc. Due to the intrinsic correlation between different product lines or subsidiaries, it is of great importance to study the operation strategy for such a large multi-line insurance group. [28] develops a financial pricing model to price insurance by line in a multi-line insurance group to overcome the drawbacks of previous models of single-line insurance company. [24] investigates the strategy of capital allocation for a multi-line insurance company. It is shown that the allocations depend on the uncertainty of each line’s losses and the marginal contribution of each line. [20] further generalizes the previous models for multi-line insurance companies and studies the insurance premiums under the assumption that losses created by insurer default are allocated following a sharing rule. On the other hand, there are recent work on dividend optimization with multiple business entities. [1] and [18] study the optimal dividend strategies for two collaborating insurance companies under compound Poisson and diffusion models, respectively. [17] extends the work in [18] and investigates a two-dimensional dividend optimization problem with a different solvency criteria. With the correlation and collaborations between multiple decision makers, the dividend control problems become multi-dimensional and more mathematically challenging.

The present paper aims to contribute to the study of cooperative dividend payment problem for the insurance group from a new perspective, namely the optimal dividend in the presence of potential systemic credit risk. Due to the multiple product lines, determining the risks born by the insurance group becomes necessary and crucial. Each product line in different market has its own risk process with very distinctive claim frequency and severities. Premium principles of each product line in different markets can also be significantly different. When a product line in a market is insolvent, it is subject to termination and will not be able to generate profit in the future, leading to a decline of total dividend payment for the whole insurance group. In addition, the correlation between different markets and product lines makes the solvency more complicated, that is, one product line’s termination may lead to the termination of another product line. As opposed to the conventional single-company dividend control problem, the systemic risk for multiple companies naturally arises within the group. Recent empirical studies find that defaults are contagious in certain cases and perform as a so-called phenomenon of default-clustering, see [14]. The contagious defaults are widely considered as correlated and depended to certain common factors. [30] studies a dependent credit risk model and analyze the contagious defaults affected by a common macroeconomic condition. [3] develops a financial network models and investigates the contagious defaults that are linked to a common macroeconomic shock. For insurance companies, insurance products are also subject to common economic and financial shocks and contagious default factors and default events within the insurance group are contagious to each other. Hence, it is important to investigate the impact of risk among different product lines within the large insurance group.

To make the mathematical model tractable, we work in the interacting intensity framework for default contagion, which allows sequential defaults and assumes that the external default event of one product line can affect all other surviving names as each credit default intensity will change afterwards. This type of default contagion has been actively studied recently in the context of risky assets management, see among [9, 8, 10, 11, 12] and many others. To the best of our knowledge, our work appears as the first one attempting to introduce the default contagion to the insurance group dividend control framework. In particular, the present paper distinguishes the ruin caused by insurance claims (i.e. the surplus process diffuses to zero) and the ruin caused by external credit default jump. Several novel and interesting observations can be drawn from our new model set up. In particular, it is shown in this paper that the optimal barrier for each
insurance product is default-state-modulated, i.e., the optimal barrier level of each surviving subsidiary will be adjusted if other subsidiaries go default due to some external credit risk. Based on different parameters of default intensities, our numerical examples illustrate that the new adjustment of the barrier for dividend payment may happen in two opposite ways: the surviving subsidiary may pay dividend immediately as the subsidiary may go default soon because of the relatively strong default contagion, low level of premium and large frequency of claims; on the other hand, the group may lift up the subsidiary’s barrier for dividend if the contagion effect is relatively weak and the surplus level is healthy so that the subsidiary can strategically accumulate more surplus to protect itself against future claims and extends its life-time of operation. These phenomena are consequent on some complicated and implicit orders of free boundary points involved in the recursive system of HJBVIs.

The main mathematical contribution of the present paper is the introduction and study of a recursive system of HJBVIs for the stochastic singular control problem, which is new to the literature. The recursion is conducted based on number of defaulted subsidiaries and the depth of recursion equals the total number of subsidiaries we consider initially. In general, it is very challenging to examine the existence and uniqueness of the classical solution to a system of variational inequalities. However, we can take the full advantage of the structure from the value function, which is rooted in the mechanism of sequential defaults as well as the risk neutral valuation of the singular control problem. Firstly, we don’t have to handle the fully coupled system of variational inequalities. Instead, we can follow the order from the case when there is only one surviving subsidiary and work inductively to the case when all subsidiaries are alive. The classical solution we establish for the step with \( k \) surviving subsidiaries will appear as variable coefficients in the step with \( k + 1 \) surviving subsidiaries, which will help to conclude the existence of classical solution for the step with \( k + 1 \) names. Secondly, to show the existence of classical solution at each step with a given number of subsidiaries, we can identify the key separation form of the value function, and split the variational inequality for the group dividend problem into a sub-system of auxiliary variational inequalities. To tackle each auxiliary variational inequality, we first obtain the existence of classical solution to the PDE part. By employing smooth-fit principle, we can deduce the existence of a constant free boundary point depending on default states of all subsidiaries and construct the classical solution to the auxiliary variational inequality using the previous solution of the PDE problem. The rigorous proof of the verification theorem is provided to characterize the value function as the unique classical solution to the associated HJBVI. Furthermore, the optimal dividend is shown to be a reflection strategy with optimal barriers modulated by default states. It will be interesting to extend our setting to incorporate the constraint that the accumulative dividend strategy is absolutely continuous with respect a Lebesgue measure, as studied before in the single insurance company model by [21, 26, 27] and [25]. Our recursive analysis for the external default contagion may also work to check the optimality of the barrier control as refraction dividend strategy with thresholds modulated by default states. Some technical efforts will be required to analyze the recursive system of HJB equations and prove the verification theorem, which will be left as one future project.

The rest of the paper is organized as follows. Section 2 introduces the model set up for the multi-line insurance group with external credit default contagion. The optimal dividend problem combining all subsidiaries is formulated and the main theorem that the optimal dividend is of barrier type is given whereafter. In Section 3, we formally derive the recursive system of HJBVIs for the case of two-line insurance group and solve the value function in a fully explicit manner. The optimal barriers for the dividend strategy, modulated by default states, are constructed using the smooth-fit principle. Section 4 generalizes the results to a multi-line insurance group. The proof of the verification theorem is given in Section 5. Section 6 reports some numerical examples of comparative statics for two subsidiaries. Some heuristic arguments to derive the associated HJBVI are presented in Appendix A.


2 Model Formulation

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space where \(\mathbb{F} := \{\mathcal{F}_t\}\) is a right-continuous, \(\mathbb{P}\)-completed filtration. We consider an insurance group that includes a total of \(N\) subsidiary business units. Each business unit is managed independently within the group. The decision maker of each subsidiary business unit collects the premiums and contributes shares of the dividend for the whole group.

In our model, we assume all subsidiaries have the same form of surplus processes with different drifts and claim distributions. We denote the pre-default surplus process \(\hat{X}_i(t)\) as the diffusion-approximation of the classical Cramér-Lundberg model:

\[
d\hat{X}_i(t) = a_i dt - b_i dW_i(t),
\]

where constants \(a_i > 0\) and \(b_i > 0\) represent the mean and the volatility of the surplus process, respectively, and \(W_i(t)\) is a standard \(\mathbb{P}\)-Brownian motion.

It is assumed in this paper that each subsidiary confronts some external credit risk, which can lead to direct default at some future time. To model the possible default jump, we choose the so-called default indicator process that is described by a \(N\)-dimensional \(\mathbb{F}\)-adapted process \((Z_1, Z_2, \ldots Z_N)\) taking values on \([0, 1]^N\). For \(i = 1, 2, \ldots N\), \(Z_i(t) = 1\) indicates that the default event corresponding to the \(i\)-th subsidiary has happened up to time \(t\), while \(Z_i(t) = 0\) indicates that the \(i\)-th subsidiary is still alive at time \(t\). The default time \(\rho_i\) for the \(i\)-th subsidiary, \(i = 1, 2, \ldots N\), is given by

\[
\rho_i = \inf\{t \geq 0; Z_i(t) = 1\}.
\]

For each \(i = 1, 2, \ldots N\), the stochastic intensity of \(\rho_i\) is modeled to be \((1 - Z_i(\cdot)) \lambda_i(Z(\cdot))\), where \(\lambda_i\) maps \([0, 1]^N\) to \(\mathbb{R}_+\), i.e., the process

\[
M_i(t) := Z_i(t) - \int_0^{\rho_i} \lambda_i(Z(s)) \, ds,
\]

follows an \(\mathbb{F}\)-martingale. Take \(N = 2\) for example and let us consider the default state \(Z(t) = (0, 0)\) at time \(t\). The values \(\lambda_1(0, 0)\) and \(\lambda_2(0, 0)\) give the default intensity of subsidiary 1 and subsidiary 2 at time \(t\) respectively. Suppose that subsidiary 1 has already defaulted before time \(t\) and only subsidiary 2 is alive, then \(\lambda_2(1, 0)\) represent the default intensity of subsidiary 2 at time \(t\). Similarly, if the subsidiary 2 has already defaulted before time \(t\) and only subsidiary 1 is alive, then \(\lambda_1(0, 1)\) represent the default intensity of subsidiary 1 at time \(t\).

For general case with \(N\) subsidiaries, the default indicator process at time \(t\) may jump from a state \((Z_1(t), \ldots , Z_{i-1}(t), Z_i(t), Z_{i+1}(t), \ldots , Z_N)\) in which the subsidiary \(i\) is alive \((Z_i(t) = 0)\) to the neighbour state \((Z_1(t), \ldots , Z_{i-1}(t), 1-Z_i(t), Z_{i+1}(t), \ldots , Z_N)\) in which the subsidiary \(i\) has defaulted at the stochastic rate \(\lambda_i(Z(t))\). We assume from this point onwards that \(Z_i, i = 1, 2, \ldots N\) will not jump simultaneously. Because the default intensity of the \(i\)-th subsidiary \(\lambda_i(Z(t))\) depends on the whole vector \(Z(t)\), the default intensity of the \(i\)-th subsidiary may change if any other subsidiary defaults and this is what we mean by default contagion. Let us denote the vector \(\lambda(z) = (\lambda_i(z); i = 1, 2, \ldots N)^T\) for the given default vector \(z \in \{0, 1\}^N\).

The real surplus process of insurance product \(i\) after the incorporation of external credit risk is denoted by \(\bar{X}_i(t)\), where \(i = 1, 2, \ldots , N\), and it is defined as

\[
\bar{X}_i(t) = (1 - Z_i(t)) \hat{X}_i(t).
\]

Given the surplus process \(\bar{X}_i(t)\) for each subsidiary \(i\), we can then introduce the dividend policy. A dividend strategy \(D(\cdot)\) is an \(\mathcal{F}_t\)-adapted process \(\{D(t) : t \geq 0\}\) corresponding to the accumulated amount of
dividends paid up to time \( t \) such that \( D(t) \) is a nonnegative and nondecreasing stochastic process that is right continuous and have left limits with \( D(0^-) = 0 \). The dividend strategy fits the type of singular control. The jump size of \( D \) at time \( t \geq 0 \) is denoted by \( \Delta D(t) := D(t) - D(t^-) \), and \( D^c(t) := D(t) - \sum_{0 \leq s \leq t} \Delta D(s) \) denotes the continuous part of \( D(t) \).

For the \( i \)-th subsidiary, the resulting surplus process in the presence of dividend payments can be written as
\[
X_i(t) = \bar{X}_i(t) - D_i(t), \quad X_i(0) = x_i \geq 0, \tag{2.5}
\]
where \( x_i \) stands for the initial surplus of the \( i \)-th subsidiary.

The objective function for the insurance group is formulated as a corporative singular control of total dividend strategy \( \mathbf{D}(t) = (D_1(t), \ldots, D_N(t)) \) under the expected value of discounted future dividend payments up to the ruin time
\[
J(x, z, \mathbf{D}(\cdot)) = \mathbb{E} \left( \sum_{i=1}^{N} \alpha_i \int_{0}^{{\tau}_i} e^{-rt} dD_i(t) \right), \tag{2.6}
\]
where the weight parameter satisfies \( \alpha_1 + \alpha_2 + \ldots + \alpha_N = 1 \), and \( r \) is the given discount factor. Note that the insurance group is the decision maker, hence the surplus process of each subsidiary is completely observable to the decision maker. Here, the ruin time \( {\tau}_i \) of the subsidiary \( i, i = 1, \ldots, N \), is defined by
\[
{\tau}_i := \inf \{ t \geq 0 : X_i(t) < 0 \}.
\]

Both \( x \) and \( z \) denote \( N \)-dimensional vectors. The initial surplus level is denoted by \( X_i(0) = x_i \) and \( X(0) = x = (x_1, \ldots, x_N) \), and the initial default state is denoted by \( Z_i(0) = z_i \) and \( Z(0) = z = (z_1, \ldots, z_N) \). It is worth noting that each admissible control \( D_i \) can not jump simultaneously with \( Z_i \), i.e., the dividend for the subsidiary \( i \) can not be paid right at the moment when the subsidiary \( i \) goes default due to external credit risk. This conclusion lies in the fact that \( D_i(t) \) is càdlàg and the default time \( \rho_i \) is totally inaccessible in view of the existence of default intensity \( \lambda_i \). The fact that \( D_i \) and \( Z_i \) will not jump simultaneously is important when we derive the associated HJBVI.

The aim of this paper is to look for the optimal dividend strategy \( \mathbf{D}^* \) such that the value function can be achieved that
\[
f(x, z) = \sup_{\mathbf{D}} J(x, z, \mathbf{D}) = J(x, z, \mathbf{D}^*). \tag{2.7}
\]
In particular, we are interested in the case that all subsidiaries are alive at the initial time, i.e., we want to characterize the value function \( f(x, 0) \) with the zero vector \( 0 = (0, \ldots, 0) \).

A barrier dividend strategy is to pay dividend whenever the surplus process excess over the barrier, which corresponds to the reflection control in stochastic control theory. Given the objective of maximizing the discounted total dividend payments until financial ruin, the optimal dividend strategy has been shown to fit this type of barrier strategy, see [5]. By focusing on a single insurance company, most of the existing work in the literature are devoted to find the optimal barriers under various classical risk models.

In our setting of insurance group with external default contagion, we can again verify that the optimal dividend fits into the barrier reflection control. Nevertheless, the optimal barrier for each subsidiary is no longer a fixed point as in the single company case. Instead, we identify that the optimal barrier is modulated by the default states, i.e. the default subsidiaries and surviving ones. This indicates that the dividend barrier will be adjusted in the observation of sequential defaults. Under some parameters, it might be the case that the more default events occur, the more likely that the surviving subsidiary will pay more dividend as its threshold drops after each default. The next theorem is the main result of the present paper.
Theorem 2.1. Let us consider the initial surplus level \( X(0) = x \in \mathbb{R}_+^N \) and the initial default state \( Z(0) = z = (z_1, \ldots, z_N) = 0 \) that all subsidiaries are alive at the initial time. The value function \( f(x, 0) \) defined in (2.7) is the unique classical solution to the variational inequalities

\[
\max_{1 \leq i \leq N} \left\{ \mathcal{L} f(x, 0) + \sum_{l=1}^{N} \lambda_l(0) f(x, z^l), \alpha_i - \partial_l f(x, 0) \right\} = 0, \tag{2.8}
\]

where the operator

\[
\mathcal{L} f(x, 0) := - \left( r + \sum_{l=1}^{N} \lambda_l(0) \right) f + \sum_{i=1}^{N} \left( a_i \partial_i f + \frac{1}{2} b_i^2 \partial_{ll} f \right),
\]

and \( \partial_l f = \frac{\partial f}{\partial x_l}, \partial_{ll} f = \frac{\partial^2 f}{\partial x_l^2} \) and \( z^l = (z_1, \ldots, z_{l-1}, 1 - z_l, z_{l+1}, \ldots, z_N) \).

Moreover, for each \( i = 1, \ldots, N \), there exists a mapping \( m_i : \{0, 1\}^N \mapsto \mathbb{R}_+ \) such that the optimal dividend \( D^*_i \) for the \( i \)-th subsidiary is given by the reflection type strategy

\[
D^*_i(t) := \{ 0, \sup_{0 \leq s \leq t} \{ \bar{X}_i(s) - m_i(Z(s)) \} \}, \quad i = 1, \ldots, N, \tag{2.9}
\]

and \( m_i(Z(t)) \) represents the optimal barrier for the \( i \)-th subsidiary modulated by the default state process \( Z(t) \) at time \( t \).

From the HJBVI (2.8), we can see the solution \( f(x, 0) \) depends on the value function \( f(x, z^l) \) with the initial default state \( z^l \) that one subsidiary has already defaulted. Therefore, to show the existence of classical solution to HJBVI (2.8) with \( z = 0 \), we have to investigate the classical solution to the system of HJBVI for all different values of \( z \in \{0, 1\}^N \). To this end, we follow a recursive scheme that is based on the default states of all subsidiaries, and the proof of the theorem above will be provided in Section 5.

3 Analysis of HJB Variational Inequalities: Two Subsidiaries

To make our recursive formulation and mathematical arguments more accessible to readers, we first present the main result for only 2 subsidiaries. As we can see in this section, the associated HJB variational inequalities can be solved explicitly for 2 initial survival subsidiaries and the optimal barriers of dividend for each subsidiary at time \( t \) can be constructed based on the default state \( Z(t) \). The recursive scheme to analyze the variational inequalities has a hierarchy feature, which is operated in a backward manner. To be more precise, we first solve a standard optimal dividend problem when only one subsidiary survives initially, and the corresponding value function will appear in the associated HJBVI as variable coefficients for the step when both subsidiaries are initially alive. We then continue to solve the HJBVI for two subsidiaries by using a conjectured separation form and the smooth-fit principle. As there are only two companies, let us set \( \alpha_1 = \alpha \) and \( \alpha_2 = 1 - \alpha \).

3.1 One Survival subsidiary

We assume in this section that there is only one survival subsidiary at the initial time and present the results for two separated cases.

Let us first consider the initial surplus level \( x_1 \geq 0 \) for the subsidiary 1 and the initial default state is \( z = (0, 1) \), namely only subsidiary 1 is alive and the subsidiary 2 has already defaulted due to the external credit risk. The associated HJBVI for the default state \( (0, 1) \) can be

\[
\max \left\{ \mathcal{L}^{(0,1)} f(x_1, (0, 1)), \alpha - \partial_1 f(x_1, (0, 1)) \right\} = 0, \tag{3.1}
\]
where the operator is defined by
\[
L^{(0,1)} f := - (r + \lambda_1(0,1)) f + \left( a_1 \frac{\partial f}{\partial x_1} + \frac{1}{2} b_1^2 \frac{\partial^2 f}{\partial x_1^2} \right).
\]

Here, we recall that \( \lambda_1(0,1) \) stands for the default intensity for subsidiary 1 given that subsidiary 2 has already defaulted.

[4] has already studied this stochastic singular control problem for single company. Let \( \hat{\theta}_1, -\hat{\theta}_2 \) be the positive and negative roots of the equation \( \frac{1}{2} b_1^2 x^2 + a_1 x - (r + \lambda_1(0,1)) = 0 \) respectively that
\[
\hat{\theta}_1 = -a_1 + \sqrt{a_1^2 + 2b_1^2(r + \lambda_1(0,1))}, \quad -\hat{\theta}_2 = -a_1 - \sqrt{a_1^2 + 2b_1^2(r + \lambda_1(0,1))}.
\]

According to [4], the solution to variational inequality (3.1) can be obtained as
\[
f_1(x_1, (0,1)) = \begin{cases} 
\alpha C_1(0,1)(e^{\hat{\theta}_1 x_1} - e^{-\hat{\theta}_2 x_1}), & 0 \leq x_1 \leq m_1(0,1), \\
\alpha C_1(0,1)(e^{\hat{\theta}_1 m_1(0,1)} - e^{-\hat{\theta}_2 m_1(0,1)}) + \alpha(x_1 - m_1(0,1)), & x_1 \geq m_1(0,1).
\end{cases}
\]

Here
\[
m_1(0,1) = \frac{2}{\theta_1 + \theta_2} \log \left( \frac{\hat{\theta}_2}{\hat{\theta}_1} \right) = \frac{b_1^2}{\sqrt{a_1^2 + 2b_1^2(r + \lambda_1(0,1))}} \log \frac{\sqrt{a_1^2 + 2b_1^2(r + \lambda_1(0,1)) + a_1}}{\sqrt{a_1^2 + 2b_1^2(r + \lambda_1(0,1)) - a_1}},
\]
\[
C_1(0,1) = \frac{1}{\theta_1 e^{\hat{\theta}_1 m_1(0,1)} + \theta_2 e^{-\hat{\theta}_2 m_1(0,1)}}.
\]

Similarly, let us consider the initial surplus level \( x_2 \geq 0 \) for the subsidiary 2 and initial default state \( z = (1,0) \), i.e., the subsidiary 2 is our target and the subsidiary 1 has already defaulted because of the external credit risk. The associated HJBVI for the default state \( (1,0) \) is written by
\[
\max \left\{ L^{(1,0)} f(x_2, (1,0)), (1 - \alpha) - \partial_1 f(x_2, (1,0)) \right\} = 0,
\]
where the operator is defined by
\[
L^{(1,0)} f := - (r + \lambda_2(1,0)) f + \left( a_2 \frac{\partial f}{\partial x_2} + \frac{1}{2} b_2^2 \frac{\partial^2 f}{\partial x_2^2} \right).
\]

Let \( \tilde{\theta}_1, -\tilde{\theta}_2 \) be the positive and negative roots of the equation \( \frac{1}{2} b_2^2 x^2 + a_2 x - (r + \lambda_2(1,0)) = 0 \) that
\[
\tilde{\theta}_1 = -a_2 + \sqrt{a_2^2 + 2b_2^2(r + \lambda_2(1,0))}, \quad -\tilde{\theta}_2 = -a_2 - \sqrt{a_2^2 + 2b_2^2(r + \lambda_2(1,0))}.
\]

We have that
\[
f_2(x_2, (1,0)) = \begin{cases} 
(1 - \alpha) C_2(1,0)(e^{\tilde{\theta}_1 x_2} - e^{-\tilde{\theta}_2 x_2}), & 0 \leq x_2 \leq m_2(1,0), \\
(1 - \alpha) C_2(1,0)(e^{\tilde{\theta}_1 m_2(1,0)} - e^{-\tilde{\theta}_2 m_2(1,0)}) + (1 - \alpha)(x_2 - m_2(1,0)), & x_2 \geq m_2(1,0),
\end{cases}
\]

where
\[
m_2(1,0) = \frac{2}{\theta_1 + \theta_2} \log \left( \frac{\hat{\theta}_2}{\hat{\theta}_1} \right) = \frac{b_2^2}{\sqrt{a_2^2 + 2b_2^2(r + \lambda_2(1,0))}} \log \frac{\sqrt{a_2^2 + 2b_2^2(r + \lambda_2(1,0)) + a_2}}{\sqrt{a_2^2 + 2b_2^2(r + \lambda_2(1,0)) - a_2}},
\]
\[
C_2(1,0) = \frac{1}{\theta_1 e^{\tilde{\theta}_1 m_2(1,0)} + \theta_2 e^{-\tilde{\theta}_2 m_2(1,0)}}.
\]
3.2 Auxiliary Results for Two Subsidiaries

We now continue to consider the desired scenario that both subsidiaries are alive at time \( t = 0 \) with the initial surplus level \( x = (x_1, x_2) \) and initial default state \( z = (0, 0) \). Using heuristic arguments in Appendix A, the associated HJBVI is written by

\[
\max \left\{ \mathcal{L}^{(0,0)} f(x, (0,0)), \alpha - \partial_1 f(x, (0,0)), (1 - \alpha) - \partial_2 f(x, (0,0)) \right\} = 0, \tag{3.7}
\]

with the operator

\[
\mathcal{L}^{(0,0)} f(x, (0,0)) := -(r + \lambda_1(0,0) + \lambda_2(0,0)) f(x, (0,0))
+ \left( a_1 \partial_1 f(x, (0,0)) + \frac{1}{2} b_1^2 \partial^2_{11} f(x, (0,0)) \right)
+ \left( a_2 \partial_2 f(x, (0,0)) + \frac{1}{2} b_2^2 \partial^2_{22} f(x, (0,0)) \right)
+ \lambda_1(0,0) f_2(x_2, (1,0)) + \lambda_2(0,0) f_1(x_1, (1,0)),
\]

where functions \( f_1(x_1, (0,1)) \) and \( f_2(x_2, (1,0)) \) are given explicitly in (3.2) and (3.5) respectively, and \( \partial_i f(x, (0,0)) := \frac{\partial f(x(0,0))}{\partial x_i} \) and \( \partial_{ii} f(x, (0,0)) := \frac{\partial^2 f(x(0,0))}{\partial x_i^2} \) for \( i = 1, 2 \).

To show the existence of a classical solution to variational inequalities (3.7), we conjecture that the solution \( f(x, (0,0)) \) with \( x = (x_1, x_2) \in \mathbb{R}^2_+ \) admits a key separation form that

\[
f((x_1, x_2), (0,0)) = f_1(x_1, (0,0)) + f_2(x_2, (0,0)), \quad x_1, x_2 \geq 0. \tag{3.8}
\]

for some smooth functions \( f_1 \) and \( f_2 \), i.e., functions of \( x_1 \) and \( x_2 \) can be decoupled. The rigorous proof of this separation form will be given in the next subsection.

Following the conjecture in (3.8), to solve variational inequality (3.7), we can first consider the following two auxiliary variational inequalities for one dimensional variable \( x \in \mathbb{R}_+ \) defined by

\[
\max \left\{ \mathcal{A}_1 f_1(x, (0,0)) + \lambda_2(0,0) f_1(x, (0,1)), \alpha - f'_1(x, (0,0)) \right\} = 0, \quad x \geq 0 \tag{3.9}
\]

\[
\max \left\{ \mathcal{A}_2 f_2(x, (0,0)) + \lambda_1(0,0) f_2(x, (1,0)), 1 - \alpha - f'_2(x, (0,0)) \right\} = 0, \quad x \geq 0. \tag{3.10}
\]

where the operators are defined as

\[
\mathcal{A}_i f(x, (0,0)) := \frac{1}{2} b_i^2 f''(x, (0,0)) + a_i f'(x, (0,0)) - (r + \lambda_1(0,0) + \lambda_2(0,0)) f(x, (0,0)), \quad i = 1, 2.
\]

(3.11)

Equations (3.9) and (3.10) satisfy the boundary condition \( f_i(0, (0,0)) = 0 \) respectively for \( i = 1, 2 \).

To show the existence of classical solution to variational inequalities (3.9) and (3.10), it is then enough to consider the following general form of variational inequality for one dimensional variable \( x \in \mathbb{R}_+ \) that

\[
\max \left\{ \mathcal{A} f(x) + h(x), \gamma - f'(x) \right\} = 0, \tag{3.12}
\]

where

\[
\mathcal{A} f(x) := -\mu f(x) + \nu f'(x) + \frac{1}{2} \sigma^2 f''(x), \tag{3.13}
\]

and the function \( h \) is a \( C^2 \) function satisfying \( h(0) = 0, h(x) \geq 0, h'(x) \geq 0, \) and \( h''(x) \leq 0 \) for \( x \geq 0 \).

To tackle the variational inequality, we propose to first examine the solution to the PDE part in the next lemma.
Lemma 3.1. Let us consider the PDE problem

\[ Ag(x) + h(x) = 0, \quad x \geq 0 \] (3.14)

with the constrain \( g(0) = 0 \) and the operator \( A \) is defined in (3.13). The classical solution \( g \) to (3.14) admits the form

\[ g(x) = \phi_1(x) + C\phi_2(x). \] (3.15)

where \( C \) is a parameter in \( \mathbb{R} \), and

\[ \phi_1(x) = \frac{1}{\theta_1 + \theta_2} \int_0^x h(u)(e^{\theta_1(x-u)} - e^{-\theta_2(x-u)})du, \quad x \geq 0, \]
\[ \phi_2(x) = e^{\theta_1 x} - e^{-\theta_2 x}, \quad x \geq 0. \]

Here \( \theta_1, -\theta_2 \) are the roots of the equation

\[ \frac{1}{2} \sigma^2 \theta^2 + \nu \theta - \mu = 0. \] (3.16)

Proof of Lemma 3.1. We first rewrite the PDE (3.14) in a vector form as

\[ \frac{d}{dx} \left( \begin{array}{c} g \\ g' \end{array} \right) = A \left( \begin{array}{c} g \\ g' \end{array} \right) + \beta, \] (3.17)

where

\[ A = \left( \begin{array}{cc} 0 & 1 \\ 2\sigma^{-2}\mu & -2\sigma^{-2}\nu \end{array} \right), \]
\[ \beta(x) = \left( \begin{array}{c} 0 \\ -2\sigma^{-2}h(x) \end{array} \right). \] (3.18)

One can solve it as

\[ \left( \begin{array}{c} g(x) \\ g'(x) \end{array} \right) = e^{Ax} \int_0^x e^{-Au} \beta(u)du + e^{Ax} \beta_0. \] (3.19)

The constrain \( g(0) = 0 \) then yields that \( \beta_0 = (0, g'(0))^T \) and \( e^{Ax} \beta_0 = (C(e^{\theta_1 x} - e^{-\theta_2 x}), C(\theta_1 e^{\theta_1 x} + \theta_2 e^{-\theta_2 x}))^T \) for some constant \( C \). Note also that \( \beta(x) = (0, h(x)) \), hence it yields that

\[ e^{Ax} \int_0^x e^{-Au} \beta(u)du = \int_0^x e^{A(x-u)} \left( \begin{array}{c} 0 \\ h(u) \end{array} \right) du = \int_0^x h(u)e^{A(x-u)} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) du \]

Let

\[ \left( \begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right) = e^{At} \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \] (3.20)

we get that

\[ \frac{d}{dt} \left( \begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right) = A \left( \begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right), \quad y_1(0) = 0, y_2(0) = 1. \] (3.21)
Then \( y'_1(t) = y_2(t) \) implies that \( y_1(t) = C_1 e^{\theta_1 t} + C_2 e^{-\theta_2 t} \), \( y_1(0) = 0, y'_1(0) = 1 \). We then deduce that 
\[ C_1 = -C_2 = \frac{1}{\theta_1 + \theta_2}. \]
Therefore
\[
e^{\lambda x} \int_0^x e^{-Au} \beta(u) du = \int_0^x h(u)e^{A(x-u)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} du
= \frac{1}{\theta_1 + \theta_2} \int_0^x \left( h(u)(e^{\theta_1(x-u)} - e^{\theta_2(x-u)}) \right) du.
\] (3.22)

\[
g(x, (0, 0)) = \frac{1}{\theta_1 + \theta_2} \int_0^x h(u)(e^{\theta_1(x-u)} - e^{\theta_2(x-u)}) du + C(e^{\theta_1 x} - e^{-\theta_2 x})
= \phi_1(x) + C\phi_2(x).
\] (3.23)

where \( C \) is a parameter, and
\[
\phi_1(x) = \frac{1}{\theta_1 + \theta_2} \int_0^x h(u)(e^{\theta_1(x-u)} - e^{\theta_2(x-u)}) du, \ x \geq 0
\] (3.24)
\[
\phi_2(x) = e^{\theta_1 x} - e^{-\theta_2 x}, \ x \geq 0.
\] (3.25)

In order to solve the variational inequality, we can apply the smooth-fit principle to mandate the solution to be smooth at the boundary. To be precise, for the given constant \( \gamma \), we hope to choose \((C, m)\) such that 
\( \phi'_1(m) + C\phi'_2(m) = \gamma, \phi''_1(m) + C\phi''_2(m) = 0 \). We start with some identities for derivatives. Straightforward calculations give that
\[
\phi'_1(x) = \frac{1}{\theta_1 + \theta_2} \int_0^x h(u)(\theta_1 e^{\theta_1(x-u)} + \theta_2 e^{-\theta_2(x-u)}) du \geq 0,
\] (3.26)
\[
\phi''_1(x) = \frac{1}{\theta_1 + \theta_2} \int_0^x h''(u)(\theta_1 e^{\theta_1(x-u)} + \theta_2 e^{-\theta_2(x-u)}) du \geq 0,
\] (3.27)

where the second inequality holds because \( h(0) = 0 \), and
\[
\phi''_2(x) = \frac{1}{\theta_1 + \theta_2} h(x)\phi''_2(0) + \frac{1}{\theta_1 + \theta_2} \int_0^x h(u)\phi''_2(x-u) du
= \frac{1}{\theta_1 + \theta_2} \left( h(x)\phi''_2(0) - h(0)\phi''_2(x) \right) + \frac{1}{\theta_1 + \theta_2} \int_0^x h(u)\phi''_2(x-u) du
= -\frac{1}{\theta_1 + \theta_2} \int_0^x h(u)\phi''_2(x-u) du + \frac{1}{\theta_1 + \theta_2} \int_0^x h'(u)\phi'_2(x-u) du
+ \frac{1}{\theta_1 + \theta_2} \int_0^x h(u)\phi''_2(x-u) du
= \frac{1}{\theta_1 + \theta_2} \int_0^x h'(u)\phi'_2(x-u) du.
\] (3.28)

Note that \( \phi''_2(0) = \theta_1^2 - \theta_2^2 < 0 \). Let \( \zeta := \inf\{ u : \phi''_2(u) = 0 \} \), then we have \( \zeta > 0 \) because \( \phi''_2(x) > 0 \) and we have that \( \phi''_2(x) < 0 \) for \( x \in [0, \zeta) \).

**Lemma 3.2.** There exist constants \((C, m)\) such that \( C > 0, m \in (0, \zeta) \), satisfying
\[
\begin{cases}
\phi'_1(m) + C\phi'_2(m) = \gamma, \\
\phi''_1(m) + C\phi''_2(m) = 0.
\end{cases}
\] (3.29)
Proposition 3.3. Define the function

\[ q(x) := \phi''_1(x) + \frac{\gamma - \phi'(x)}{\phi'_2(x)} \phi''_2(x) = 0. \]  

(3.30)

Define \( m := \inf \{ u : q(u) = 0 \} \), and we set \( m \) to be \(+\infty\) if \( q \) doesn’t admit any roots on \( \mathbb{R} \). As \( \phi'_1(0) = \phi''_1(0) = 0 \) (according to (3.26) and (3.27)), we obtain that \( q(0) = \frac{\gamma \phi''_2(0)}{\phi'_2(0)} < 0 \). Similarly, we have \( q(\zeta) = \phi''_1(\zeta) > 0 \), where the inequality strictly holds because \( \zeta > 0 \). Hence, there exists a constant \( m \in (0, \zeta) \), and we can choose that \( C = \frac{\gamma - \phi'_1(m)}{\phi'_2(m)} = -\frac{\phi''_2(m)}{\phi'_2(m)} > 0 \).

With the parameter \((C, m)\) obtained in Lemma 3.2, we now come to the construction of a classical solution to the desired general variational inequality.

Proposition 3.3. The variational inequality

\[ \max \{ \mathcal{A} f(x) + h(x), \gamma - f'(x) \} = 0, \quad x \geq 0 \]  

(3.31)

with the constrain \( f(0) = 0 \) admits a \( C^2 \) solution.

Proof of Proposition 3.3. Define the function

\[ f(x) := \begin{cases} 
  g(x) = \phi_1(x) + C \phi_2(x), & x \in [0, m), \\
  \phi_1(m) + C \phi_2(m) + \gamma (x - m), & x \in [m, +\infty),
\end{cases} \]  

(3.32)

where \( g(x) \) is the classical solution to PDE problem (3.14), and constants \( C \) and \( m \) are determined by function \( g(x) \) by Lemma 3.2. That is to say, the function coincides with the solution to the PDE problem in Lemma 3.2 for \( x \leq m \) and the function is a linear function for \( x > m \). We aim to prove that the function \( f \) is the desired \( C^2 \) solution to the variational inequality (3.31). In view of its definition, it is straightforward to see that \( f \) belongs to \( C^2 \). Lemma 3.1 gives that \( \mathcal{A} f(x) + h(x) = 0, \ x \in [0, m] \). Thanks to Lemma 3.2, we deduce that \( f'(m) = \gamma, f''(m) = 0 \). Therefore the proposition holds once we show that \( f'(x) = \phi'_1(x) + C \phi'_2(x) \geq \gamma \) for \( x \in [0, m] \) as well as \( \mathcal{A} f(x) + h(x) \leq 0 \) for \( x \geq m \).

Let us first verify that \( \phi'_1(x) + C \phi'_2(x) \geq \gamma \) for \( x \in [0, m] \). Define the elliptic operator

\[ Lf := -\frac{1}{2} \sigma^2 f'' - \nu f' \]  

(3.33)

We get that \( L \phi_1 = h - \mu \phi_1 \). Note that \( h \) is twice continuously differentiable, and that \( h'' \leq 0, \phi''_1 \geq 0 \). It therefore follows that \( L \phi''_1 = h'' - \mu \phi''_1 \leq 0 \).

According to the weak maximum principle, we can see that \( \phi''_1(x) \leq \max \{ \phi''_1(0), \phi''_1(m) \} \), \( x \in [0, m] \). If \( \max \{ \phi''_1(0), \phi''_1(m) \} = \phi''_1(0) \), then \( \phi''_1(x) + C \phi''_2(x) \leq \phi''_1(0) + C \phi''_2(x) = C \phi'_2(x) < 0, \ x \in [0, m] \). If \( \max \{ \phi''_1(0), \phi''_1(m) \} = \phi''_1(m) \), then we obtain from \( \phi''_1 > 0 \) that \( \phi''_1(x) + C \phi''_2(x) \leq \phi''_1(m) + C \phi''_2(m) = 0, \ x \in [0, m] \). In conclusion, we obtain that \( \phi''_1(x) + C \phi''_2(x) \leq 0, \ x \in [0, m] \). Therefore \( \phi'_1(x) + C \phi'_2(x) \geq \phi'_1(m) + C \phi'_2(m) = \gamma, \ x \in [0, m] \).

We next show that \( \mathcal{A} f'(x) + h'(x) \leq 0 \) for \( x \geq m \). In our previous argument, we have shown that \( \phi''_1(x) + C \phi''_2(x) \leq 0, \ x \in [0, m] \), i.e., \( f''(x) \leq 0, \ x \in [0, m] \). It follows that

\[ f'''(m-) = \lim_{x \to m-} \frac{f''(m) - f''(x)}{m - x} = -\lim_{x \to m-} \frac{f''(x)}{m - x} \geq 0. \]  

(3.34)
Thanks to the definition of $f$, we have that $Af'(x) + h'(x) = 0$ on $x \in [0, m)$. By sending $x \to m_-$, we get

$$Af'(m_-) + h'(m) = 0. \quad (3.35)$$

That is to say

$$-\mu \gamma + h'(m) = -\frac{1}{2}b^2 f''(m_+) \leq 0. \quad (3.36)$$

For $x > m$, we have $f''(x) = 0$, $f'(x) = \gamma$, and $h'(x) \leq h'(m)$ as $h'' \leq 0$. Hence we have

$$Af'(x) + h'(x) = -\mu f'(x) + h'(x) \leq -\mu \gamma + h'(m) \leq 0. \quad (3.37)$$

Then for $x \geq m$, we arrive at

$$Af(x) + h(x) \leq Af(m) + h(m) = 0. \quad (3.38)$$

Putting all pieces together, we can conclude that $f$ is the desired $C^2$ solution to the variational inequality (3.31).

\[\square\]

**Corollary 3.4.** There exists a solution of (3.31), which admits the form

$$f(x) = \begin{cases} 
\phi_1(x) + C\phi_2(x), & x \in [0, m], \\
\phi_1(m) + C\phi_2(m) + \gamma(x - m), & x \in [m, +\infty).
\end{cases} \quad (3.39)$$

Here $\phi_1(x)$ and $\phi_2(x)$, $x \geq 0$, are defined in (3.24) and (3.25) respectively and parameters $C$ and $m$ are determined by functions $\phi_1$ and $\phi_2$ in Lemma 3.2.

### 3.3 Main Results

After we obtain the explicit solution to the general variational inequality (3.31), by setting $A = A_1$, $h(x) = \lambda_2(0, 0)f_1(x_1, (0, 1))$, $\gamma = \alpha$, we can deduce the explicit solution $f_1(x_1, (0, 0))$ to variational inequality (3.9). Similarly, by taking $A = A_2$, $h(x) = \lambda_1(0, 0)f_2(x_2, (0, 0))$, $\gamma = 1 - \alpha$, we get the explicit solution $f_2(x_2, (0, 0))$ to variational inequality (3.10). Moreover, let us denote the constant $m$ and $C$ for variational inequality (3.9) by $m_1(0, 0)$ and $C_1(0, 0)$ as we can verify later that the constant $m_1(0, 0)$ is the optimal barrier of the dividend strategy for the subsidiary 1. Similarly, we denote $m$ and $C$ for variational inequality (3.10) by $m_2(0, 0)$ and $C_2(0, 0)$, which are associated to subsidiary 2.

**Remark 3.5.** We present the explicit form of functions $f_1(x_1, (0, 0))$ and $f_2(x_2, (0, 0))$ as below.

Let $h_1(x) = \lambda_2(0, 0)f_1(x_1, (0, 1))$ and $K_1 = \alpha C_1(0, 1)(e^{x_1+1} - e^{x_1-1}) - \alpha m_1(0, 1)$. We can construct the explicit solution of the variation inequality (3.9). Let us denote $\theta_{11}, -\theta_{12}$ as the positive and negative roots of the equation $\frac{1}{2}b_1^2 \theta^2 + a_1 \theta - (r + \lambda_1(0, 0) + \lambda_2(0, 0)) = 0$ respectively that

$$\theta_{11} = \frac{-a_1 + \sqrt{a_1^2 + 2b_1^2(r + \lambda_1(0, 0) + \lambda_2(0, 0))}}{b_1^2}, \quad -\theta_{12} = \frac{-a_1 - \sqrt{a_1^2 + 2b_1^2(r + \lambda_1(0, 0) + \lambda_2(0, 0))}}{b_1^2}.$$
Let us first define for the variable \( x \geq 0 \) that

\[
f_{11}(x) := \frac{\alpha \lambda_2(0,0) C_1(0,1)}{\theta_{11} + \theta_{12}} \\
\times \left[ \left( \frac{1}{\theta_1 - \theta_{11}} - \frac{1}{\theta_1 + \theta_{12}} \right) e^{\hat{\theta}_1 x} + \left( \frac{1}{\theta_2 + \theta_{11}} + \frac{1}{-\theta_2 + \theta_{12}} \right) e^{-\hat{\theta}_2 x} \right] \\
- \left( \frac{1}{\theta_1 - \theta_{11}} + \frac{1}{\theta_2 + \theta_{11}} \right) e^{\theta_{11} x} + \left( \frac{1}{\theta_1 + \theta_{12}} - \frac{1}{-\hat{\theta}_2 + \theta_{12}} \right) e^{-\theta_{12} x} \right], \quad 0 \leq x \leq m_1(0,1),
\]

\[
f_{112}(x) := \frac{\alpha \lambda_2(0,0) C_1(0,1)}{\theta_{11} + \theta_{12}} \\
\times \left[ \frac{1}{\theta_1 - \theta_{11}} e^{\theta_{11} x} \left( e^{(\hat{\theta}_1 - \theta_{11}) m_1(0,1)} - 1 \right) + \frac{1}{\theta_1 + \theta_{12}} e^{-\theta_{12} x} \left( - e^{(\hat{\theta}_1 + \theta_{12}) m_1(0,1)} + 1 \right) \right] \\
+ \frac{1}{\theta_2 + \theta_{11}} e^{\theta_{11} x} \left( e^{(\hat{\theta}_2 + \theta_{11}) m_1(0,1)} - 1 \right) + \frac{1}{-\theta_2 + \theta_{12}} e^{-\theta_{12} x} \left( e^{(\hat{\theta}_2 + \theta_{12}) m_1(0,1)} - 1 \right) \right] \\
+ \frac{K_1 \lambda_2(0,0)}{\theta_{11} + \theta_{12}} \left[ \frac{1}{\theta_{11}} \left( e^{\theta_{11} x - \theta_{11} m_1(0,1)} - 1 \right) + \frac{1}{\theta_{12}} \left( e^{-\theta_{12} x + \theta_{12} m_1(0,1)} - 1 \right) \right] \\
+ \frac{\alpha \lambda_2(0,0)}{\theta_{11} + \theta_{12}} \left[ \frac{1}{(\theta_{12})^2} \left( - \theta_{12} x - 1 + \theta_{11} m_1(0,1) + 1 \right) e^{\theta_{11} x - \theta_{11} m_1(0,1)} \right] \\
+ \frac{1}{\theta_{12}^2} \left( - \theta_{12} x + 1 + \theta_{11} m_1(0,1) - 1 \right) e^{-\theta_{12} x + \theta_{12} m_1(0,1)} \right], \quad m_1(0,1) \leq x,
\]

(3.40)

\[
f_{12}(x) = e^{\theta_{11} x} - e^{-\theta_{12} x}, \quad x \geq 0.
\]

(3.41)

In view of Lemma 3.2 and Corollary 3.4, we can define constant

\[
m_1(0,0) := \inf \{ s : q_1(s) = 0 \},
\]

where

\[
q_1(x) := f_{11}''(x, (0,0)) + \frac{\alpha - f_{11}'(x, (0,0))}{f_{12}'(x, (0,0))} f_{12}''(x, (0,0)).
\]

We also define \( C_1(0,0) := \frac{\alpha - f_{11}'(m_1(0,0))}{f_{12}'(m_1(0,0))} \).

Based on the values of \( m_1(0,0) \) and \( m_1(0,1) \), we can separate two cases for the presentation of solution to (3.9):

**Case I:** If \( m_1(0,0) \geq m_1(0,1) \).

Then for \( 0 \leq x_1 \leq m_1(0,0) \), the solution of variational inequality (3.9) satisfies

\[
f_1(x_1, (0,0)) = f_{11}(x_1, (0,0)) + C_1(0,0) f_{12}(x_1, (0,0))
\]

\[
= \begin{cases} 
  f_{111}(x_1) + C_1(0,0) f_{12}(x_1, (0,0)), & 0 \leq x_1 < m_1(0,1), \\
  f_{112}(x_1) + C_1(0,0) f_{12}(x_1, (0,0)), & m_1(0,1) \leq x_1 \leq m_1(0,0).
\end{cases}
\]

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For $x_1 > m_1(0,0)$, the solution to (3.9) is

$$f_1(x_1, (0,0)) = f_{112}(m_1(0,0)) + C_1(0,0)f_{12}(m_1(0,0), (0,0)) + \alpha(x_1 - m_1(0,0)).$$

**Case II:** If $m_1(0,0) < m_1(0,1)$.

For $0 \leq x_1 \leq m_1(0,0)$, the solution of (3.9) can be simply written as

$$f_1(x_1, (0,0)) = f_{111}(x_1) + C_1(0,0)f_{12}(x_1, (0,0))$$

For $x_1 > m_1(0,0)$, the solution of (3.9) is written as

$$f_1(x_1, (0,0)) = f_{111}(m_1(0,0)) + C_1(0,0)f_{12}(m_1(0,0), (0,0)) + \alpha(x_1 - m_1(0,0)).$$

Similarly, let us take $h_2(x) = \lambda_1(0,0)f_2(x_2, (1,0))$ and

$$K_2 = (1 - \alpha)C_2(1,0)(e^{\theta_1m_2(1,0)} - e^{-\theta_2m_2(1,0)}) - (1 - \alpha)m_2(1,0).$$

We can obtain the solution of variational inequality (3.10). We denote $\theta_{21}, -\theta_{22}$ as the positive and negative roots of the equation $\frac{1}{2}b_2^2\theta^2 + a_2\theta - (r + \lambda_1(0,0) + \lambda_2(0,0)) = 0$ that

$$\theta_{21} = -a_2 + \sqrt{a_2^2 + 2b_2^2(r + \lambda_1(0,0) + \lambda_2(0,0))}, \quad \theta_{22} = -a_2 - \sqrt{a_2^2 + 2b_2^2(r + \lambda_1(0,0) + \lambda_2(0,0))}.$$

Let us define functions for the variable $x \geq 0$ that

\[
\begin{align*}
 f_{211}(x) & := \frac{(1 - \alpha)\lambda_1(0,0)C_2(1,0)}{\theta_{21} + \theta_{22}} \\
 & \times \left[ \left( \frac{1}{\theta_1 - \theta_{21}} - \frac{1}{\theta_1 + \theta_{22}} \right)e^{\theta_1x} + \left( \frac{1}{\theta_2 + \theta_{21}} + \frac{1}{\theta_2 - \theta_{22}} \right)e^{-\theta_2x} \\
 & - \left( \frac{1}{\theta_1 - \theta_{21}} + \frac{1}{\theta_2 + \theta_{21}} \right)e^{\theta_{21}x} + \left( \frac{1}{\theta_1 + \theta_{22}} - \frac{1}{\theta_2 - \theta_{22}} \right)e^{-\theta_{22}x} \right], \quad 0 \leq x \leq m_2(1,0),
\end{align*}
\]

\[
\begin{align*}
 f_{212}(x) & := \frac{(1 - \alpha)\lambda_1(0,0)C_2(1,0)}{\theta_{21} + \theta_{22}} \\
 & \times \left[ \frac{1}{\theta_1 - \theta_{21}} e^{\theta_{21}x} \left( e^{(\theta_1 - \theta_{21})m_2(1,0)} - 1 \right) + \frac{1}{\theta_1 + \theta_{22}} e^{-\theta_{22}x} \left( -e^{(\theta_1 + \theta_{22})m_2(1,0)} + 1 \right) \\
 & + \frac{1}{\theta_2 + \theta_{21}} e^{\theta_{21}x} \left( e^{(\theta_2 + \theta_{21})m_2(1,0)} - 1 \right) + \frac{1}{\theta_2 - \theta_{22}} e^{-\theta_{22}x} \left( e^{-(\theta_2 + \theta_{22})m_2(1,0)} - 1 \right) \right] \\
 & + \frac{K_2\lambda_1(0,0)}{\theta_{21} + \theta_{22}} \left[ \frac{1}{\theta_{21}} \left( e^{\theta_{21}x - \theta_{21}m_2(1,0)} - 1 \right) + \frac{1}{\theta_{22}} \left( e^{-\theta_{22}x + \theta_{22}m_2(1,0)} - 1 \right) \right] \\
 & + \frac{(1 - \alpha)\lambda_1(0,0)}{\theta_{21} + \theta_{22}} \left[ \frac{1}{(\theta_{21})^2} \left( -\theta_{21}x - 1 + (\theta_{21}m_2(1,0) + 1)e^{\theta_{21}x - \theta_{21}m_2(1,0)} \right) \\
 & + \frac{1}{(\theta_{22})^2} \left( -\theta_{22}x + 1 + (\theta_{22}m_2(1,0) - 1)e^{-\theta_{22}x + \theta_{22}m_2(1,0)} \right) \right], \quad m_2(1,0) \leq x,
\end{align*}
\]
For Case I

We also define

\[ C(x, (0, 0)) = e^{\theta_{21} x} - e^{-\theta_{22} x}, \quad x \geq 0. \]  

(3.43)

In view of Lemma 3.2 and Corollary 3.4, we can define constant

\[ m_2(0, 0) := \inf\{ s : q_2(s) = 0 \}, \]

where

\[ q_2(x) := f''_{21}(x, (0, 0)) + \frac{1 - \alpha - f'_{21}(x, (0, 0))}{f''_{22}(x, (0, 0))} f''_{22}(x, (0, 0)). \]

We also define \( C_2(0, 0) := \frac{1 - \alpha - f'_{21}(m_2(0, 0))}{f''_{22}(m_2(0, 0))}. \)

Again, we need to separate two cases based on values of \( m_2(0, 0) \) and \( m_2(1, 0) \):

**Case I:** If \( m_2(0, 0) \geq m_2(1, 0) \).

For \( 0 \leq x_2 \leq m_2(0, 0) \), the solution of the variational inequalities (3.10) satisfies

\[
\begin{align*}
f_2(x_2, (0, 0)) &= f_{21}(x_2, (0, 0)) + C_2(0, 0) f_{22}(x_2, (0, 0)) \\
&= \begin{cases} 
    f_{211}(x_2) + C_2(0, 0) f_{22}(x_2, (0, 0)), & 0 \leq x_2 < m_2(1, 0), \\
    f_{212}(x_2) + C_2(0, 0) f_{22}(x_2, (0, 0)), & m_2(1, 0) \leq x_2 \leq m_2(0, 0).
\end{cases}
\end{align*}
\]

For \( x_2 > m_2(0, 0) \), the solution to (3.10) is

\[
f_2(x_2, (0, 0)) = f_{212}(m_2(0, 0)) + C_2(0, 0) f_{22}(m_2(0, 0), (0, 0)) + (1 - \alpha)(x_2 - m_2(0, 0)).
\]

**Case II:** If \( m_2(0, 0) < m_2(1, 0) \).

For \( 0 \leq x_2 \leq m_2(0, 0) \), the solution of variational inequalities (3.10) can be simply written as

\[
f_2(x_2, (0, 0)) = f_{211}(x_2) + C_2(0, 0) f_{22}(x_2, (0, 0))
\]

For \( x_2 > m_2(0, 0) \), the solution of (3.10) is written as

\[
f_2(x_2, (0, 0)) = f_{211}(m_2(0, 0)) + C_2(0, 0) f_{22}(m_2(0, 0), (0, 0)) + (1 - \alpha)(x_2 - m_2(0, 0)).
\]

We can now verify our conjecture \( f(x, (0, 0)) = f_1(x_1, (0, 0)) + f_2(x_2, (0, 0)) \) in (3.8) and prove the existence of a classical solution to variational inequality (3.7) in the next main theorem.

**Theorem 3.6.** There exists a \( C^2 \) solution to HJBVI (3.7).

*Proof.* Thanks to Proposition 3.3, equations (3.9) and (3.10) both admit \( C^2 \) solutions. Let \( f_1, f_2 \) be the solutions to (3.9) and (3.10) respectively. Set \( f(x, (0, 0)) := f_1(x_1, (0, 0)) + f_2(x_2, (0, 0)) \), then we have

\[
\mathcal{L}^{(0, 0)} f(x, (0, 0)) := -r f_1(x_1, (0, 0)) - r f_2(x_2, (0, 0))
\]

\[ + \left( a_1 \partial_1 f_1(x_1, (0, 0)) + \frac{1}{2} b_1^2 \partial^2_{11} f_1(x_1, (0, 0)) \right) \]

\[ + \left( a_2 \partial_2 f_2(x_2, (0, 0)) + \frac{1}{2} b_2^2 \partial^2_{22} f_2(x_2, (0, 0)) \right) \]

\[ + \lambda_1(0, 0) (f_2(x_2, (1, 0)) - f_1(x_1, (0, 0))) \]

\[ + \lambda_2(0, 0) (f_1(x_1, (0, 1)) - f_2(x_2, (0, 0))). \]
It readily yields that
\[
\mathcal{L}^{(0,0)} f(x, (0, 0)) = A_1 f_1(x_1, (0, 0)) + \lambda_2(0, 0) f_1(x_1, (0, 1)) + A_2 f_2(x_2, (0, 0)) + \lambda_1(0, 0) f_2(x_2, (1, 0)),
\]
\[
\alpha - \partial_1 f(x, (0, 0)) = \alpha - f'_1(x_1, (0, 0)) \quad \text{and} \quad 1 - \alpha - \partial_2 f(x, (0, 0)) = 1 - \alpha - f'_2(x_2, (0, 0)).
\] (3.44)

As \( f_1, f_2 \) solves HJBVI (3.9) and HJBVI (3.10) respectively, we have that
\[
\max \left\{ \mathcal{L}^{(0,0)} f(x, (0, 0)), \alpha - \partial_1 f(x, (0, 0)), 1 - \alpha - \partial_2 f(x, (0, 0)) \right\} \leq 0.
\] (3.45)

Moreover, if \( \mathcal{L}^{(0,0)} f(x, (0, 0)) < 0 \), then \( A_1 f_1(x_1, (0, 0)) + \lambda_2(0, 0) f_1(x_1, (0, 1)) < 0 \) or \( A_2 f_2(x_2, (0, 0)) + \lambda_1(0, 0) f_2(x_2, (1, 0)) < 0 \). Without loss of generality, we assume that \( A_1 f_1(x_1, (0, 0)) + \lambda_1(0, 0) f_1(x_1, (0, 0)) < 0 \), then by (3.9) we have that \( \alpha - \partial_1 f(x, (0, 0)) = \alpha - f'_1(x_1, (0, 0)) = 0 \), hence
\[
\max \left\{ \mathcal{L}^{(0,0)} f(x, (0, 0)), \alpha - \partial_1 f(x, (0, 0)), 1 - \alpha - \partial_2 f(x, (0, 0)) \right\} = 0.
\]

and it is proved that \( f(x, (0, 0)) \) is the solution to HJBVI (3.7). \( \square \)

### 4 Analysis of HJB Variational Inequalities: Multiple Subsidiaries

This section generalizes our previous arguments and results to the case with multiple subsidiaries. In order to solve the original variational inequality (2.8) for \( N \) subsidiaries using the backward recursive scheme, we can employ the mathematical induction argument. To this end, we will start to focus on the case that there are \( k \leq N \) subsidiaries defaulted at the initial time and show the existence of classical solution to the associated variational inequality. The final proof for \( N \) survival subsidiaries and the verification of the optimal reflection dividend strategy will be given in the next section.

We start by introducing some notations. For \( 0 \leq k \leq N \), let us consider the initial default state \( z = 0^{j_1, \ldots, j_k} \) that \( k \) subsidiaries have defaulted, which denotes the \( N \) dimensional vector that \( j_1, \ldots, j_k \) components are \( 1 \) and all other components are \( 0 \). It is then sufficient to consider the initial surplus as \((N - k)\)-dimensional vector \( x = (x_{j_{k+1}}, \ldots, x_{j_N}) \), where \( \{j_{k+1}, \ldots, j_N\} = \{0, 1\}^N \setminus \{j_1, \ldots, j_k\} \). We denote the partial differential operator \( \partial_I f := \frac{\partial f}{\partial x_I} \) and \( \partial_{II} f = \frac{\partial^2 f}{\partial x_I \partial x_I} \). Let us also define the linear operator \( \mathcal{L}^z \)
\[
\mathcal{L}^z f(x, z) := - \left( r + \sum_{l=k+1}^{N} \lambda_l(z) \right) f + \sum_{l=k+1}^{N} \left( a_l \partial_l f(x, z) + \frac{1}{2} b_{ll}^2 \partial_{II}^2 f(x, z) \right).
\]

The HJBVI becomes
\[
\max_{k+1 \leq l \leq N} \left\{ \mathcal{L}^z f(x, z) + \sum_{l=k+1}^{N} \lambda_l(z) f(x^j_l, z^j_l), \alpha_i - \partial_i f(x, z) \right\} = 0,
\] (4.1)

where we denote \( z^j_l = 0^{j_1, \ldots, j_k, j_l} \) and \( x^j_l = (x_{j_{k+1}}, x_{j_{l-1}}, x_{j_{l+1}}, x_{j_N}) \).

Without any loss of generality, we may assume that \( \{j_1, \ldots, j_k\} = \{1, \ldots, k\} \) to simplify the presentation, then \( z = 0^{1 \ldots k}, x = (x_{k+1}, \ldots, x_N) \), and
\[
\mathcal{L}^z f(x, z) := - \left( r + \sum_{i=k+1}^{N} \lambda_i(z) \right) f(x, z) + \sum_{i=k+1}^{N} \left( a_i \partial_i f(x, z) + \frac{1}{2} b_{ii}^2 \partial_{II}^2 f(x, z) \right).
\]
The HJB equation thus changes accordingly to

\[
\max_{k+1 \leq i \leq N} \left\{ \mathcal{L}^N f(x, z) + \sum_{l=k+1}^{N} \lambda_l(z) f(x^l, z^l), \alpha_i - \partial_i f(x, z) \right\} = 0. \tag{4.2}
\]

Similar to the previous section, we seek for the solution that admits the form \( f(x, z) = \sum_{i=k+1}^{N} f_i(x_i, z), \) where we define for \( x \geq 0 \)

\[
f_i(x, z) = \begin{cases} 
    f_{i,1}(x, z) + C_i(z)f_{i,2}(x, z), & 0 \leq x \leq m_i(z), \\
    f_{i,1}(m_i(z), z) + C_i(z)f_{i,2}(m_i(z), z) + \alpha_i(x - m_i(z)), & x \geq m_i(z). 
\end{cases} \tag{4.3}
\]

We suppose that for each \( 1 \leq n \leq N \), \( N - n \leq k \leq N \) and \( z = 0^{1\ldots k} \), there exists a solution \( f \) to (4.2), where \( f \) admits the form \( f(x, z) = \sum_{i=k+1}^{N} f_i(x_i, z) \), satisfying (4.3) and \( f_i(0, z) = 0 \), \( f_i \geq 0 \), \( f''_i \geq 0 \), \( f''_i \leq 0 \). The previous section has shown that the statement holds true for \( n = 1, 2 \).

Now we show by induction that the statement is also true for \( n \geq 3 \). For any given \( n \), suppose that the statement is true for \( 1, \ldots, n-1 \). Then for \( k = N-n \), equation (4.2) turns out to be

\[
\max_{k+1 \leq i \leq N} \left\{ \mathcal{L}^N f(x, z) + \sum_{i=k+1}^{N} \left( \sum_{l \neq i}^{N} \lambda_l(z) f_l(x, z^l) \right), \alpha_i - \partial_i f(x, z) \right\} = 0. \tag{4.4}
\]

In the same fashion of the previous section with two subsidiaries, we consider the following auxiliary equation for \( k+1 \leq i \leq N \) and one dimensional variable \( x \geq 0 \) that

\[
\max \left\{ \mathcal{A}^{z,i} f_i(x, z) + \left( \sum_{l \neq i} \lambda_l(z) f_l(x, z^l) \right), \alpha_i - f_i'(x, z) \right\} = 0. \tag{4.6}
\]

Here we define the operator

\[
\mathcal{A}^{z,i} f := - \left( r + \tilde{\lambda}(z) \right) f + a_i f' + \frac{1}{2} b_i^2 f'', \tag{4.7}
\]

where \( \tilde{\lambda}(z) := \sum_{i=k+1}^{N} \lambda_i(z) \).

**Lemma 4.1.** Equation (4.6) with the constrain \( f(0, z) = 0 \) admits a \( C^2 \) solution \( f_i(x, z) \), where

\[
f_i(x, z) = \begin{cases} 
    f_{i,1}(x, z) + C_i(z)f_{i,2}(x, z), & 0 \leq x \leq m_i(z), \\
    f_{i,1}(m_i(z), z) + C_i(z)f_{i,2}(m_i(z), z) + \alpha_i(x - m_i(z)), & x \geq m_i(z). 
\end{cases} \tag{4.8}
\]

**Proof.** Our induction assumption gives the boundary condition \( \sum_{l \neq i} \lambda_l(z) f_l(0, z^l) = 0 \) as well as the results

\[
\sum_{l \neq i} \lambda_l(z) f_l(x, z^l) \geq 0, \quad \left( \sum_{l \neq i} \lambda_l(z) f_l(x, z^l) \right)' \geq 0, \quad \left( \sum_{l \neq i} \lambda_l(z) f_l(x, z^l) \right)'' \leq 0.
\]

Therefore, we conclude the existence of solution following the same argument of Proposition 3.3 and Corollary 3.4. \( \square \)
Proof. For \(k+1 \leq i \leq N\), let \(f_i(x, z)\) be the solution to the HJBVI (4.6), and set \(f(x, z) := \sum_{i=k+1}^{N} f_i(x_i, z)\). It is then obvious that \(f\) is \(C^2\). In view of (4.6),

\[
L^z f(x, z) + \sum_{i=k+1}^{N} \left( \sum_{l \neq i} \lambda_l(z) f_i(x_i, z^l) \right) = \sum_{i=k+1}^{N} A^{z,i} f_i(x_i, z) + \sum_{i=k+1}^{N} \left( \sum_{l \neq i} \lambda_l(z) f_i(x_i, z^l) \right) = 0.
\]

Furthermore, \(\alpha_i - \partial_i f(x, z) = \alpha_i - f_i(x_i, z) \leq 0, i = k+1, \ldots, N\). Then it follows that

\[
\max_{k+1 \leq i \leq N} \left\{ L^z f(x, z) + \sum_{i=k+1}^{N} \left( \sum_{l \neq i} \lambda_l(z) f_i(x_i, z^l) \right), \alpha_i - \partial_i f(x, z) \right\} \leq 0. \tag{4.10}
\]

Now we claim that

\[
\max_{k+1 \leq i \leq N} \left\{ L^z f(x, z) + \sum_{i=k+1}^{N} \left( \sum_{l \neq i} \lambda_l(z) f_i(x_i, z^l) \right), \alpha_i - \partial_i f(x, z) \right\} = 0. \tag{4.11}
\]

Fix \(x \geq 0\) and \(z \in \{0,1\}^N\). If \(L^z f(x, z) + \sum_{i=k+1}^{N} \left( \sum_{l \neq i} \lambda_l(z) f_i(x_i, z^l) \right) = 0\), then the equality trivially holds. If \(L^z f(x, z) + \sum_{i=k+1}^{N} \left( \sum_{l \neq i} \lambda_l(z) f_i(x_i, z^l) \right) < 0\), it follows that \(A^{z,i} f_i(x_i, z) + \left( \sum_{l \neq i} \lambda_l(z) f_i(x_i, z^l) \right) < 0\) for some \(i\). As \(f_i\) is chosen to solve (4.6), it holds that \(\alpha_i - \partial_i f(x, z) = \alpha_i - f_i(x_i, z) = 0\). By combining (4.10), our claim is verified, which completes the proof. \(\square\)

Thanks to Theorem 4.2, we have completed the mathematical induction.

5 Proof of Verification Theorem

In this section, we construct the optimal strategy with the \(C^2\) solution to equation (4.1). For \(0 \leq k \leq N\), let \(z = (z_{j_1}, \ldots, z_{j_k})\), and \(x = (x_{j_{k+1}}, \ldots, x_{j_N})\). In view of Proposition 3.3 and the induction in section 4, the solution \(f\) admits the form \(f(x, z) = \sum_{i=k+1}^{N} f_i(x_i, z)\), where we define for \(x \geq 0\) that

\[
f_i(x, z) = \begin{cases} 
    f_{i,1}(x, z) + C_i(z) f_{i,2}(x, z), & 0 \leq x \leq m_i(z), \\
    f_{i,1}(m_i(z), z) + C_i(z) f_{i,2}(m_i(z), z) + \alpha_i(x - m_i(z)), & x \geq m_i(z).
\end{cases} \tag{5.1}
\]

Based on previous analysis of the recursive system of HJB variational inequalities, we can present the final proof of Theorem 2.1.

Proof of Theorem 2.1.

First, because of Theorem 4.2 and the explicit classical solution for \(k = 1\) and \(k = 2\), we can conclude that variational inequality (2.8) for \(k = N\) also admits the \(C^2\) solution. Moreover, the existence of mapping \(m_i : \{0,1\}^N \mapsto \mathbb{R}_+\) can be obtained in the similar fashion of Lemma 3.2 for each recursion step \(k\).
By using Itô’s formula, we first get

\[
\sum_{i=1}^{N} \alpha_i \int_0^\tau e^{-rs} dD_i(s) + e^{-r} f(X(\tau), Z(\tau)) - f(x, z)
\]

\[
= \int_0^\tau e^{-rs} \left[ \mathcal{L}Z(s) f(X(s), Z(s)) + \sum_{i=k+1}^{N} \lambda_i(Z(s)) f(X^{\bar{i}}(s), Z^{\bar{i}}(s)) \right] \, ds
\]

\[
+ \sum_{i=1}^{N} \int_0^\tau e^{-rs} [\alpha_i - \partial_t f(X(s), Z(s))] \, dD_i(s)
\]

\[
+ \sum_{0<s\leq\tau, \Delta Z(s)\neq 0} e^{-rs} \sum_{j=1}^{N} \Delta Z_j(s) \left[ f(X^{\bar{j}}(s) - \Delta D^{\bar{j}}(s), Z^{\bar{j}}(s)) - f(X^{\bar{j}}(s), Z^{\bar{j}}(s)) \right]
\]

\[
+ \sum_{i=1}^{N} \alpha_i \Delta D_i(s)
\]

\[
+ \sum_{0<s\leq\tau, \Delta Z(s)=0} e^{-rs} \left[ f(X(s) - \Delta D(s), Z(s)) - f(X(s), Z(s)) + \sum_{i=1}^{N} \alpha_i \Delta D_i(s) \right] + \mathcal{M}_\tau
\]

\[
= I + II + III + IV + \mathcal{M}_\tau. \tag{5.2}
\]

Here for any vector \(\xi \in \mathbb{R}^N\), we have used the notation \(\xi^j := (\xi_1, \ldots, \xi_{j-1}, 0, \xi_{j+1}, \ldots, \xi_N)\). As \(f\) solves (4.1), we have that \(I, II, IV \leq 0\). Moreover, by noting that \(f(x, z^j)\) also solves (4.1), we deduce that \(III \leq 0\). Note that \(\mathcal{M}_{\lambda \wedge \mathcal{T}}\) is a local martingale. There exists a sequence of stopping times \(\{T_n\}_{n=1}^\infty\) satisfying \(T_n \uparrow \infty\), and

\[
\mathbb{E} \left[ \sum_{i=1}^{N} \alpha_i \int_0^{\tau \wedge T_n} e^{-rs} dD_i(s) \right]
\]

\[
\leq \lim_{n \to \infty} \mathbb{E} \left[ \sum_{i=1}^{N} \alpha_i \int_0^{\tau \wedge T_n} e^{-rs} dD_i(s) + e^{-r(\tau \wedge T_n)} f(X(\tau \wedge T_n), Z(\tau \wedge T_n)) \right]
\]

\[
\leq f(x, z) + \lim_{n \to \infty} \mathbb{E}[\mathcal{M}_{\lambda \wedge T_n}] = f(x, z).
\]

Let us consider the càdlàg strategy

\[
D^*_i(t) := \max \left\{ 0, \sup_{0 \leq s \leq t} \left\{ \bar{X}_i(s) - m_i(Z(s)) \right\} \right\},
\]

\[
X^*_i(t) = \bar{X}_i(t) - D^*_i(t). \tag{5.3}
\]

We set

\[
A_i(t) := 1_{\{D^*_i(t) = \bar{X}_i(t) - m_i(Z(t))\}}. \tag{5.4}
\]

It follows that

\[
X^*_i(t) = \bar{X}_i(t) - D^*_i(t) \leq m_i(Z(t)), \tag{5.5}
\]

\[
dD^*_i(t) = A_i(t) dD_i(t). \tag{5.6}
\]

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On \( \{D_i^+(t) = \bar{X}_i(t) - m_i(Z(t))\} \), we have that \( X_i^*(t) = \bar{X}_i(t) - D_i^+(t) = m_i(Z(t)) \) and vise versa. Hence

\[
dD_i^+(t) = A_i(t)dD_i^+(t) = 1_{\{X_i^*(t) = m_i(Z(t))\}}dD_i^+(t). \tag{5.7}
\]

Furthermore, we have on \( \{X_i^*(t) = m_i(Z(t))\} \) that

\[
X_i^*(t-) = X_i^*(t) + \Delta D_i^+(t) \geq X_i^*(t) = m_i(Z(t)). \tag{5.8}
\]

In view of (5.5), (3.39), we have that

\[
\mathcal{L}^{Z(s)}f(X^*(s), Z(s)) + \sum_{l=k+1}^{N} \lambda_l(Z(s))f((X^*)^l(s), Z^l(s)) = 0. \tag{5.9}
\]

Note that for \( x_i \geq m_i(z) \), \( \partial_i f(x, z) = f_i'(x_i, z) = \alpha_i \). Hence on \( \{X_i^*(t) = m_i(Z(t))\} \), it holds that \( \partial_i f(X^*(s), Z(s)) = \alpha_i \). Then it follows that

\[
\sum_{i=1}^{N} \int_{0}^{\tau} e^{-rs} [\alpha_i - \partial_i f(X^*(s), Z(s))] (D_i^+)^c(s) \tag{5.10}
\]

\[
= \sum_{i=1}^{N} \int_{0}^{\tau} e^{-rs} [\alpha_i - \partial_i f(X^*(s), Z(s))] 1_{\{X_i^*(t) = m_i(Z(t))\}} d(D_i^+)^c(s) = 0. \tag{5.11}
\]

By virtue of (5.8), we can see that whenever \( \Delta D_i^+(s) \neq 0 \), it holds that \( X_i^*(s-) > X_i^*(s-) - \Delta D_i^+(s) = X_i^+(s) = m_i(Z(s)) \). By using the fact that \( \partial_i f(x, z) = f_i'(x_i, z) = \alpha_i \) for \( x_i \geq m_i(z) \) again, we obtain that

\[
\sum_{j=1}^{N} \Delta Z_j(s) \left[ f((X^*)^j(s-), Z^j(s-), Z^j(s-)) - f((X^*)^j(s-), Z^j(s-)) \right]
\]

\[
= \sum_{j=1}^{N} \alpha_j \Delta (D^*)^j_i(s) \tag{5.12}
\]

Similarly, we attain the equality that

\[
\sum_{0<s \leq \tau, \Delta Z(s)=0} e^{-rs} \left[ f(X^*(s-), Z(s-)) - f(X^*(s-), Z(s-)) + \sum_{i=1}^{N} \alpha_i \Delta D_i^+(s) \right]
\]

\[
= \sum_{0<s \leq \tau, \Delta Z(s)=0} e^{-rs} \left[ f(X^*(s-), Z(s-)) - f(X^*(s-), Z(s-)) + \sum_{i=1}^{N} \alpha_i \Delta D_i^+(s) \right] = 0. \tag{5.13}
\]
Putting all pieces together, we conclude from (5.2) and (5.9) \(\sim\) (5.13) that

\[
\sum_{i=1}^{N} \alpha_i \int_{0}^{\tau} e^{-rs} dD_i^*(s) + e^{-r} f (X^*(\tau), Z(\tau)) - f(x, z) = \mathcal{M}_r,
\]

(5.14)

where \(\mathcal{M}_{r\wedge t}\) is a local martingale. Hence there exists a sequence of stopping times \(\{T_n\}_{n=1}^{\infty}\) satisfying \(T_n \uparrow \infty\), and

\[
\mathbb{E} \left[ \sum_{i=1}^{N} \alpha_i \int_{0}^{\tau \wedge T_n} e^{-rs} dD_i^*(s) + e^{-r \wedge T_n} f (X^*(\tau \wedge T_n), Z(\tau \wedge T_n)) \right] - f(x, z) = \mathbb{E} [\mathcal{M}_{r\wedge T_n}] = 0.
\]

(5.15)

Note that \(0 \leq X_i^*(t) \leq m_i (Z(t)) \leq C_i(Z(t))\). Thus \(f (X^*(T_n), Z(T_n))\) is bounded, and

\[
\lim_{n \to \infty} e^{-r \wedge T_n} f (X^*(\tau \wedge T_n), Z(\tau \wedge T_n)) = 0 \quad a.s.
\]

(5.16)

Passing \(n\) to infinity in (5.15), we arrive at

\[
\mathbb{E} \left[ \sum_{i=1}^{N} \alpha_i \int_{0}^{\tau} e^{-rs} dD_i^*(s) \right] - f(x, z) = 0,
\]

(5.17)

which completes the proof of verification. \(\square\)

### 6 Numerical Examples

This section presents some numerical examples of the value function and optimal barriers for two subsidiaries. In particular, we aim to present some distinctive cases of model parameters and show how each subsidiary adjust its optimal barrier for dividend payment after the other subsidiary defaults.

**Case I:** Let us choose the following model parameters: \(a_1 = 1, b_1 = 0.1, a_2 = 1.5, b_2 = 0.2, \lambda_1(0, 0) = 0.2, \lambda_1(0, 1) = 0.4, \lambda_2(0, 0) = 0.3, \lambda_2(1, 0) = 0.7, r = 0.1\) and \(\alpha = 0.4\). We recall that the value function of the insurance group can be written as \(f((x_1, x_2), (0, 0)) = f_1(x_1, (0, 0)) + f_2(x_2, (0, 0))\). We first graph the three dimensional surface of the value function \(f((x_1, x_2), (0, 0))\) in Figure 1 and then graph both functions \(f_1(x, (0, 0))\) and \(f_2(x, (0, 0))\) in Figure 2 for the purpose of presentation and comparison. Moreover, note that \(m_1(0, 0)\) and \(m_1(0, 1)\) stand for the optimal barriers for subsidiary 1 before and after the subsidiary 2 defaults. From Figure 2, we obtain the order of the optimal barriers that \(m_1(0, 0) \leq m_1(0, 1)\), which implies that the subsidiary 1 increases its optimal barrier after learning that the subsidiary 2 defaults. On the other hand, as we have the opposite order \(m_2(0, 0) > m_2(1, 0)\) for the subsidiary 2, the subsidiary 2 drops its optimal barrier for dividend payment after learning that the subsidiary 1 defaults. Comparing with parameters for two subsidiaries, the default contagion effect \(\lambda_1(0, 1) - \lambda_1(0, 0) = 0.2\) to the subsidiary 1 is relatively small, while the default contagion effect \(\lambda_2(1, 0) - \lambda_2(0, 0) = 0.4\) to subsidiary 2 is relatively large. The numerical results in Figure 2 illustrate that if the contagion impact is low, the subsidiary will strategically raise its barrier to pay dividend after the observation of default events so that the subsidiary can accumulate sufficient surplus to protect itself against the future claims and extend its lifetime of business and pay more future dividends. On the other hand, if the contagion impact is relatively high, the subsidiary predicts that itself will go default very soon due to the default events of others. As the default time is totally inaccessible, the subsidiary will prefer to pay the dividend as soon as possible, before its own default occurs. Therefore, it is reasonable that the survival subsidiary will lower its barrier to make the dividend payment in this case, which matches with the real life situations.
Case II: We choose the following model parameters: $a_1 = 1$, $b_1 = 0.1$, $a_2 = 1.5$, $b_2 = 0.2$, $\lambda_1(0,0) = 0.02$, $\lambda_1(0,1) = 0.4$, $\lambda_2(0,0) = 0.3$, $\lambda_2(1,0) = 0.7$, $r = 0.1$ and $\alpha = 0.4$. Comparing with Case I, we only reduce the value of $\lambda_1(0,0)$ to $\lambda_1(0,0) = 0.02$ and all other parameters remain the same. In this case, we can see that both default contagion effects $\lambda_1(0,1) - \lambda_1(0,0) = 0.38$ and $\lambda_2(1,0) - \lambda_2(0,0) = 0.4$ for two subsidiaries are relatively large. Figure 3 shows the comparison results that $m_1(0,0) > m_1(0,1)$ and $m_2(0,0) > m_2(1,0)$, which imply that both subsidiaries will drop the optimal barriers for dividend payment after the other subsidiary defaults. These observations are again consistent with our intuition that as the default contagion impact is high, the survival subsidiary takes into account that itself will go default very soon with high probability and therefore prefers to pay dividend as soon as possible before the unexpected default happens.
Case III: We choose the following model parameters: $a_1 = 1, b_1 = 0.1, a_2 = 1.5, b_2 = 0.2, \lambda_1(0,0) = 0.2, \lambda_1(0,1) = 0.4, \lambda_2(0,0) = 0.3, \lambda_2(1,0) = 0.4, r = 0.1$ and $\alpha = 0.4$. Comparing with Case I, we only change the value of $\lambda_2(1,0)$ to $\lambda_2(1,0) = 0.4$ and retain all other parameters. In this case, we can see that the default contagion effects $\lambda_1(0,1) - \lambda_1(0,0) = 0.2$ and $\lambda_2(1,0) - \lambda_2(0,0) = 0.1$ to both subsidiaries are relatively small. Figure 4 shows the order of the optimal barriers that $m_1(0,0) < m_1(0,1)$ and $m_2(0,0) < m_2(1,0)$, which imply that both subsidiary 1 and subsidiary 2 will lift up the optimal barriers for dividend payment after the other subsidiary defaults. These observations are consistent with our intuition that as the default contagion impact is low, the dominating effect of the survival subsidiary is to maintain the surplus at a reasonable level to protect itself instead of to fear the coming external default. The optimal strategy for the survival name is then to set higher barrier to pay dividend so that it can keep sufficient surplus for a longer term business of the whole insurance group.

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A Appendix

A.1 Derivation of (3.7)

For the default process starting from $Z(0) = (z_1, z_2) := (0, 0)$, we present here our heuristic argument to derive the associated HJBVI using the Itô’s formula. For a given function $\psi(\cdot, z) \in C^2(\mathbb{R}^2)$ for each $z$, let us rewrite

$$\alpha \int_0^\tau e^{-rs}dD_1(s) + (1-\alpha) \int_0^\tau e^{-rs}dD_2(s) + e^{-r}\psi(X(\tau), Z(\tau)) - \psi(x, z)$$

where

$$\alpha \int_0^\tau e^{-rs}d\tilde{D}^{(0)}(s)ds + \int_0^\tau e^{-rs}[\alpha - \partial_1\psi(s)]dD_1^c(s) + \int_0^\tau e^{-rs}[(1-\alpha) - \partial_2\psi(s)]dD_2^c(s)$$

+ $\alpha \int_0^\tau e^{-rs}d\tilde{D}^1_1(s) + (1-\alpha) \int_0^\tau e^{-rs}d\tilde{D}^1_2(s)$

+ $\sum_{0<s\leq \tau, \Delta Z(s)\neq 0} e^{-rs}[\psi(X(s), Z(s)) - \psi(X(s), Z(s-))]$  

+ $\sum_{0<s\leq \tau, \Delta Z(s)=0} e^{-rs}[\psi(X(s) + \Delta D(s), Z(s-)) - \psi(X(s), Z(s-))] + M_\tau$

$$= \int_0^\tau e^{-rs}d\tilde{D}^{(0)}(s)ds + \int_0^\tau e^{-rs}[\alpha - \partial_1\psi(s)]dD_1^c(s) + \int_0^\tau e^{-rs}[(1-\alpha) - \partial_2\psi(s)]dD_2^c(s)$$

+ $\sum_{0<s\leq \tau, \Delta Z(s)\neq 0} e^{-rs}\Delta Z_1(s) [f_2(X_2(s) - \Delta D_2(s), (1, 0)) - f_2(X_2(s), (1, 0))] + (1-\alpha)\Delta D_2(s)]$

+ $\sum_{0<s\leq \tau, \Delta Z(s)\neq 0} e^{-rs}\Delta Z_2(s) [f_1(X_1(s) - \Delta D_1(s), (0, 1)) - f_1(X_1(s), (0, 1))] + \alpha\Delta D_1(s)]$

+ $\sum_{0<s\leq \tau, \Delta Z(s)=0} e^{-rs}[\psi(X(s) - \Delta D(s), Z(s-)) - \psi(X(s), Z(s-))] + \alpha\Delta D_1(s) + (1-\alpha)\Delta D_2(s)]$

+ $M_\tau$,  \hspace{1cm} (A.1)

where $f_1(x_1, (0, 1))$ and $f_2(x_2, (1, 0))$ are given in (3.2) and (3.5) respectively.

Let us focus on the jump part. According to the assumption on simultaneous jumps, it follows that

$$\Delta Z_1(s)\Delta D_1(s) = \Delta Z_2(s)\Delta D_2(s) = \Delta Z_1(s)\Delta Z_2(s) = 0.$$  \hspace{1cm} (A.2)

On $\{\Delta Z(s) \neq 0\}$, consider $Z(s-) = (0, 0)$.

\begin{align*}
e^{-rs}[\psi(X(s), Z(s)) - \psi(X(s), Z(s-))]
&= e^{-rs}\Delta Z_1(s) [\psi((0, X_2(s-) - \Delta D_2(s)), (1, 0)) - \psi(X(s), (0, 0))] \\
&+ e^{-rs}\Delta Z_2(s) [\psi((X_1(s) - \Delta D_1(s), 0), (0, 1)) - \psi(X(s), (0, 0))].  \hspace{1cm} (A.3)
\end{align*}
Moreover,
\[
e^{-rs} \Delta Z_1(s) [\psi((0, X_2(s) - \Delta D_2(s)), (1, 0)) - \psi(X(s), (0, 0))] \\
= e^{-rs} \Delta Z_1(s) [\psi((0, X_2(s) - \Delta D_2(s)), (1, 0)) - \psi(X(s), (1, 0))] \\
+ e^{-rs} \Delta Z_1(s) [\psi((0, X_2(s)), (1, 0)) - \psi(X(s), (0, 0))].
\] (A.4)

Similarly,
\[
e^{-rs} \Delta Z_2(s) [\psi((X_1(s) - \Delta D_1(s), 0), (0, 1)) - \psi(X(s), (0, 0))] \\
= e^{-rs} \Delta Z_2(s) [\psi((X_1(s) - \Delta D_1(s), 0), (0, 1)) - \psi(X(s), (0, 1))] \\
+ e^{-rs} \Delta Z_2(s) [\psi((X_1(s)), (0, 1)) - \psi(X(s), (0, 0))].
\] (A.5)

On \(\{\Delta Z(s) = 0\}\),
\[
e^{-rs} [\psi(X(s), Z(s)) - \psi(X(s), Z(s))] \\
= e^{-rs} [\psi(X(s) - \Delta D(s), Z(s)) - \psi(X(s), Z(s))].
\] (A.6)

\[
\alpha \int_0^\tau e^{-rs} dD_1^j(s) = \sum_{0 < s \leq \tau, \Delta Z_2(s) \neq 0} \alpha e^{-rs} \Delta D_1^j(s) + \sum_{0 < s \leq \tau, \Delta Z_2(s) = 0} \alpha e^{-rs} \Delta D_1^j(s) \tag{A.7}
\]

In conclusion, we arrive at the HJBVI (3.7).

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