Feature Selection with Annealing for Big Data Learning

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Abstract—Many computer vision and medical imaging problems are faced with learning from large-scale datasets, with millions of observations and features. In this paper we propose a novel efficient learning scheme which tightens a sparsity constraint by gradually removing variables based on a criterion and a schedule. The attractive fact that the problem size keeps dropping throughout the iterations makes it particularly suitable for big data learning. Our approach applies generically to the optimization of any differentiable loss function, and finds applications in regression, classification and ranking. The resultant algorithms build variable screening into estimation and are extremely simple to implement. We provide theoretical guarantees of convergence and selection consistency. In addition, one dimensional piecewise linear response functions are used to account for nonlinearity and a second order prior is imposed on these functions to avoid overfitting. Experiments on real and synthetic data show that the proposed method compares very well with other state of the art methods in regression, classification and ranking while being computationally very efficient and scalable.

Index Terms—feature selection, supervised learning, regression, classification, ranking.

1 INTRODUCTION

Feature selection is a popular and crucial technique to speed computation and to obtain parsimonious models that generalize well. Many computer vision and medical imaging problems require learning classifiers from large amounts of data, with millions of features and even more observations. Such big data pose great challenges for feature selection.

- Efficiency. Learning algorithms that are fast and scalable are attractive in large-scale computation.
- Statistical guarantee. In consideration of the inevitable noise contamination and numerous nuisance dimensions in big datasets, a trustworthy learning approach must recover genuine signals with high probability. Learning the truth is more important than obtaining a globally optimal solution (which is usually infeasible in high dimensions).
- Universality. Rather than restricting to a specific problem, a universal learning scheme can adapt to different types of problems, including, for instance, regression, classification, ranking and others.
- Implementation ease. Algorithms that are simple to implement can avoid over-fitting and ad-hoc designs. Parameters are better for robustness and ease of tuning. In some real-world applications, it is helpful to have an algorithm with customizable cost based on computing resources.

Nonlinearity. Linear combinations of explanatory variables may not suffice in learning and modeling. Incorporating nonlinearity is vital in many big data applications.

Recently, penalized methods have received a lot of attention in high-dimensional feature selection. They solve a class of optimization problems with sparsity-inducing penalties such as the $L_1$, $L_0$, and SCAD. There is a statistical guarantee that junk dimensions can be removed with high probability (even in high dimensions). But these optimization based algorithms are not scalable enough and the tuning of the penalty parameter could be time consuming on large datasets. Most of the these methods cannot adaptively capture the nonlinearity.

Boosting can also be used for feature selection when restricting each weak learner to be dependent on a single variable only. Boosting algorithms run in a progressive manner: at each iteration a weak learner is added to the current model for the sake of decreasing the value of a certain loss function. Such an algorithm design is greedy in nature because what feature will be selected in the next boosting iteration strongly depends on the subset of selected features and their current coefficients. This dependence structure makes it more difficult to obtain theoretical selection guarantees for boosting in a general way. In addition, each boosting iteration could be computationally expensive, especially since hundreds or even thousands of such iterations are usually required.

There also exist numerous ad-hoc procedures designed for feature selection in specific problems. Although many ideas in this class of methods are motivating, there is a lack of universal learning schemes that are simple to implement and can adapt to different situations.

In this paper we combine the regularization technique and the sequential algorithm design to bring forward a novel feature selection scheme that is extremely suitable...
for big data learning.

Rather than growing a model by adding one variable at a time, we consider a shrinkage estimation problem in the whole predictor space, together with the use of annealing to lessen greediness. An attractive feature is that a number of variables are removed while the model parameters are updated each time, which makes the problem size keep dropping during the iteration process. It is worth mentioning that our learning scheme is not ad-hoc and the principle of keep or kill has an exact form with theoretical guarantee of optimality and consistency.

The proposed feature selection approach can handle large datasets without being online (which might be too greedy and inaccurate). The total amount of data the algorithm needs to access for training is about 2-10 times the size of the training set, which can be orders of magnitude faster than penalization or Boosting.

2 The Feature Selection with Annealing Algorithm

Let \((x_i, y_i), i = 1, N\) be training examples with \(x_i \in \mathbb{R}^M\) and a loss function \(L(\beta)\) defined based on these examples. We formulate the feature selection problem as a constrained optimization

\[
\beta = \arg \min_{\{j: \beta_j \neq 0\} \leq k} L(\beta)
\]

where the number \(k\) of relevant features is a given parameter, and the loss function \(L(\beta)\) is differentiable with respect to \(\beta\). This constraint form facilitates parameter tuning because in comparison with penalty parameters such as \(\lambda \|\beta\|_1\), our regularization parameter \(k\) is much more intuitive and easier to specify. The experiments in Section 5.1 also demonstrate the robustness of the choice of \(k\) as long as it is within a large range. Of course, with such a nonconvex (and discrete) constraint, the optimization problem is challenging to solve for large \(M\).

2.1 Basic Algorithm Description

Our key ideas in the algorithm design are: a) using an annealing plan to lessen the greediness in reducing the dimensionality from \(M\) to \(k\), and b) gradually removing the most irrelevant variables to facilitate computation. The prototype algorithm summarized in Algorithm 1 is actually pretty simple. It starts with an initial value of the parameter vector \(\beta\), usually \(\beta = 0\), and alternates two basic steps: one step of parameter updates towards minimizing the loss \(L(\beta)\) by gradient descent

\[
\beta \leftarrow \beta - \eta \frac{\partial L(\beta)}{\partial \beta},
\]

and one variable selection step that removes some variables according to the coefficient magnitudes \(|\beta_j|, j = 1, M\).

Through the annealing schedule, the support set of the coefficient vector is gradually tightened till we reach \(|\{j: \beta_j \neq 0\}| \leq k\). Step 4 conducts an adaptive screening, which results in a nonlinear operator that increases the difficulty of the theoretical analysis. Perhaps surprisingly, the keep-or-kill rule is simply based on the magnitude of coefficients and does not involve any information of the objective function \(L\). This is in contrast to many ad-hoc backward elimination approaches. Nicely, Theorem 5.1 shows such a design always has a rigorous guarantee of computational convergence and statistical consistency.

Algorithm 1 Feature Selection with Annealing (FSA)

Input: Training examples \(\{(x_i, y_i)\}_{i=1}^N\).
Output: Trained classifier parameter vector \(\beta\).

1: Initialize \(\beta = 0\).
2: for \(e=1\) to \(N_{iter}\) do
3: \hspace{1em} Update \(\beta \leftarrow \beta - \eta \frac{\partial L(\beta)}{\partial \beta}\).
4: \hspace{1em} Keep only the \(M_e\) variables with highest \(|\beta|\) and renumber them 1, ..., \(M_e\).
5: end for

The prototype FSA algorithm is extremely simple to implement. More importantly, the problem size and thus the complexity keep dropping, owing to the removal process. With the annealing schedule, the nuisance features that are difficult to identify are handled only when we are close to an optimal solution, while those ‘apparent’ junk dimensions are eliminated at earlier stages to save the computational cost.

Figure 1 gives a demonstration of the removal and convergence process for a classification problem with \(N = 1,000\) observations and \(M = 1,000\) variables described in Section 5.2. Notice that some of the \(\beta_j\) are zeroed after each iteration. The algorithm stabilizes very quickly (about 80 steps for \(M = 1,000\)).

Fig. 1. The value of \(\beta_j, j = 1, M\) vs iteration number for simulated data with \(N = 1,000\), \(M = 1,000\), \(k = 10\) with \(\eta = 20\), \(\mu = 300\).

2.2 Some Implementation Details

In this part, we provide empirical values of the algorithmic parameters in implementation.
First, any annealing schedule \(\{M_e\}\) slow enough works well in terms of estimation and selection accuracy. But a fast decaying schedule could reduce the computational cost significantly. Our experience shows that the following inverse schedule with a parameter \(\mu\) provides a good balance between efficiency and accuracy:

\[
M_e = k + (M - k) \max(0, \frac{N_{iter}}{2e} - 2e), \quad e = \frac{1}{N_{iter}}
\]  

(3)

Figure 2 plots the schedules for six different choices of \(\mu\) with \(M = 1,000, k = 10\) and \(N_{iter} = 500\).

The computation time is proportional to the area under the graph of the schedule curve and can be easily calculated. Examples of computation times are in Table 1. In reality, the overall computational complexity of FSA is linear in \(MN\) (the problem size).

| Annealing param \(\mu\) | Computation Time |
|-------------------------|------------------|
| \(\mu = 0\)             | \(125MN + kN_{iter}\) |
| \(\mu = 1\)             | \(97MN + kN_{iter}\) |
| \(\mu = 10\)            | \(41MN + kN_{iter}\) |
| \(\mu = 100\)           | \(10MN + kN_{iter}\) |
| \(\mu = 300\)           | \(5MN + kN_{iter}\) |
| \(\mu = 1000\)          | \(2MN + kN_{iter}\) |

In addition to the annealing schedule, the performance of FSA depends on two other parameters:

- Gradient learning rate \(\eta\), which can be arbitrarily small provided that the number of iterations is large enough. Of course, if \(\eta\) is too large, the coefficients \(\beta_j\) may not converge. We used \(\eta = 20\) for classification and \(\eta = 1\) for regression.
- Number of iterations \(N_{iter}\), large enough to ensure the parameters have converged to a desired tolerance. In our experiments we used \(N_{iter} = 500\).

Finally, we observe that the performance of the algorithm is rather stable for a large range of values for the parameters \(\eta, \mu, N_{iter}\) (cf. Section 5.1). This is advantageous in implementation and parameter tuning.

**Large Scale Implementation.** The FSA algorithm can be parallelized for large scale problems by subdividing the \(N \times M\) data matrix into a grid of sub-blocks that fit into the memory of the processing units. Then the per-observation response vectors can be obtained from a row-wise reduction of the partial sums computed by the units. The parameter updates are done similarly, via column-wise reduction. A GPU based implementation could offer further computation cost reductions.

### 2.3 Examples and Variants

The FSA algorithm can be used for the optimization of any differentiable loss function subject with a sparsity constraint as described in eq. (1). Some examples are given as follows in regression, classification and ranking.

**FSA for Regression.** Given training examples \((x_i, y_i) \in \mathbb{R}^M \times \mathbb{R}, i = 1, N\), we have the penalized squared-error loss

\[
L(\beta) = \sum_{i=1}^{N} (y_i - x_i^T \beta)^2 + \sum_{j=1}^{M} \rho(\beta_j)
\]  

(4)

with a differentiable prior function \(\rho\) such as \(\rho(\beta) = s\beta^2\).

**FSA for Classification.** FSA can be used for classification and feature selection. The loss functions use a set of differentiable losses on the interval \([-1, 1]\) and feature selection. The loss functions use a set of differentiable loss functions subject with a sparsity constraint as described in eq. (1). Some examples are given as follows in regression, classification and ranking.

- **The Logistic Loss** is

  \[
  L_D(\beta) = -\sum_{i=1}^{N} w_i \ln(1 + \exp(-y_i x_i^T \beta)) + \sum_{j=1}^{M} \rho(\beta_j)
  \]  

  (5)

  where \(\hat{y}_i = 2y_i - 1 \in \{-1, 1\}\).

- **The SVM Loss** we use is a differentiable approximation of the primal SVM objective function from [9]:

  \[
  L_D(\beta) = \sum_{i=1}^{N} L_h(y_i x_i^T \beta) + \sum_{j=1}^{M} \rho(\beta_j)
  \]  

  (6)

  where \(L_h : \mathbb{R} \to \mathbb{R}\) is the Huber-style differentiable approximation of the hinge loss [9]:

  \[
  L_h(x) = \begin{cases} 
  0 & \text{if } x > 1 + h \\
  (1 + h - x)^2 & \text{if } |1 - x| \leq h \\
  1 - x & \text{if } x < 1 - h
  \end{cases}
  \]  

  (7)

- **The Lorenz Loss** is

  \[
  L_D(\beta) = \sum_{i=1}^{N} L(y_i x_i^T \beta) + \sum_{j=1}^{M} \rho(\beta_j)
  \]  

  (8)

  where \(L : \mathbb{R} \to \mathbb{R}\) is the following differentiable function:

  \[
  L(x) = \begin{cases} 
  0 & \text{if } x > 1 \\
  \ln(1 + (1 - x)^2) & \text{else}
  \end{cases}
  \]  

  (9)

The Lorenz loss is differentiable everywhere, it is zero for \(x \in [1, \infty)\) and grows logarithmically with respect to \(|x|\) as \(x \to -\infty\). These properties make the Lorenz loss behave like the SVM loss in the sense that correctly classified examples that are far from the margin don’t contribute to the loss. Moreover, the Lorenz loss is more robust to label noise than the SVM and logistic losses.
because the loss values for the misclassified examples that are far from the margin is not much higher than for those that are close to the margin. This loss is not convex, but it works well in practice together with the FSA algorithm, as it will be seen in experiments.

**FSA for Ranking.** We developed an extension of FSA to deal with ranking problems. Let \( x_i \in \mathbb{R}^M \) be the training instances and \( r_{ij} \in [0,1] \) be the true rankings between observations \( x_i, x_j \), for some pairs \((i, j) \in C \subset \{1, ..., N\} \times \{1, ..., N\}\). A criterion (e.g., an error measure) can be used to compare instances \( x_i \) and \( x_j \) and generate the true rankings \( r_{ij} \in [0,1] \), which can be for example 0 if \( x_i \) is "better" than \( x_j \), 0.5 if they are "equally good" and 1 if \( x_i \) is "worse" than \( x_j \).

Training means finding a ranking function \( f_{\beta}(x) : \mathbb{R}^M \to \mathbb{R} \) specified by a parameter vector \( \beta \) such that \( f_{\beta}(x_i) - f_{\beta}(x_j) \) agrees as much as possible with the true rankings \( r_{ij} \).

There are different criteria that could be optimized to measure this degree of agreement, but we will use the differentiable criterion from [3] and \( f_{\beta}(x) = x^T \beta \)

\[
L_C(\beta) = \sum_{(i,j) \in C} \ln(1 + \exp(x_i^T \beta - x_j^T \beta)) - \sum_{(i,j) \in C} r_{ij}(x_i^T \beta - x_j^T \beta) + \sum_{j=1}^M \rho(\beta_j),
\]

(10)

where we added the prior term \( \sum_{j=1}^M \rho(\beta_j) \) that helps with generalization. More details are given in Section 2.

### 3 Convergence and Consistency Theorem

We investigate the performance of the FSA estimators in regression and classification problems. In the first case, each \( y_i \) follows a Gaussian distribution \( \mathcal{N}(x_i^T \beta^*, \sigma^2 I) \), while in the latter situation each \( y_i \) is binary following the Bernoulli distribution with mean \( x_i^T \beta^* \). For simplicity, we focus on the log-likelihood based loss (denoted by \( I \)), which is the squared-error loss from [5] and the logistic loss from [4] respectively. For clarity, we redefine them as follows:

Regression: \( F(\beta) = \frac{1}{2} \sum_{i=1}^N (y_i - x_i^T \beta)^2 \),

(11)

Classification: \( F(\beta) = \sum_{i=1}^N (-y_i x_i^T \beta + \log(1 + \exp(x_i^T \beta))) \).

(12)

The FSA applications may have \( M \) large (possibly much greater than \( N \)). In the rest of the section, we set \( N_{\text{iter}} = +\infty \) in the FSA algorithm. Let \( \beta^{(e)} \) be the value of \( \beta \) at iteration \( e \). Let \( M_e \) be a bounded and monotone annealing schedule satisfying \( M \geq M_e \geq k \), \forall e and \( M_e = k \) for sufficiently large values of \( e \). Suppose \( L = F \) (for now) in either regression or classification. Let \( ||\beta||_0 = |\{j : \beta_j \neq 0\}| \).

**Theorem 3.1:** The following convergence and consistency results hold under \( 0 < \eta < 4/\|X\|_2^2 \) for classification and \( 0 < \eta < 1/\|X\|_2^2 \) for regression, respectively, where \( \|X\|_2 \) stands for the spectral norm of the design:

(i) The algorithm converges in the sense that \( F(\beta^{(e)}) \) for sufficiently large values of \( e \) decreases monotonically to a limit.

(ii) In regression (cf. (11)), \( \lim_{e \to \infty} \beta^{(e)} \) always exists; in classification (cf. (12)), under the overlap condition in the appendix, the same conclusion holds. Moreover, the limit point is a locally optimal solution to \( \min_{\|\beta\|_2 \leq k} F(\beta) \).

(iii) Suppose, asymptotically, \( N \to +\infty \) and the limit of the scaled Fisher information matrix exists, i.e., the design \( X(N) \) and true coefficient \( \beta^* \) satisfy \( \lim \|X^T X/N \to I^* \) in regression, or \( \lim X^T \text{diag} \left\{ \frac{\sigma^2}{1 + e^{(\beta_j^*/\sigma^2)}^2} \right\} X/N \to I^* \) in classification, for some \( I^* \) positive definite. Let \( k \geq ||\beta^*||_0 \). Then, there exists a slow enough schedule \( \{M_e\} \) such that any \( \beta^{(e)} \) for \( e \) sufficiently large is a consistent estimator of \( \beta^* \), and \( \{j : \beta_j^* \neq 0\} \subset \{j : \beta_j^{(e)} \neq 0\} \) occurs with probability tending to 1.

The proof details are given in the supplementary material. The theorem holds more generally for smoothly penalized loss criteria. For example, when \( L = F + \frac{1}{2} \lambda \|\beta\|_2^2 \), (i) and (ii) are true for any \( \lambda > 0 \), with no need of the overlap assumption in classification.

The convergence results regardless of how large \( k \) or \( M \) can be (or even \( k > N \)) are reassuring in computation. They also imply that in implementation, we may adopt a universal choice of the stepsize at any iteration, as long as it is properly small. Moreover, in view of (iii), there is no need to evaluate a global minimum (or even a local minimum). To attain good accuracy, the cooling schedule has to be slow enough. Although coming up with an adaptive schedule that is theoretically sound is tempting, our current results seem to be way too slow in practice. Based on our empirical experience we recommend using an inverse function (3) to attain a good balance of accuracy and efficiency. The optimal cooling schedule is left to further theoretical/empirical investigations in the future.

### 4 Some Related Works

Penalized loss algorithms add a sparsity inducing penalty such as the \( L_1 \) [6, 12, 24, 44], SCAD [15], MCP [42] and the \( L_0 + L_2 \) [33, 36] and optimize a non-differentiable objective function loss in various ways. The proposed method is different from the penalized methods because variable selection is not obtained by imposing a sparsity prior on the variables, but by a successive optimization and reduction of the \( L_0 \) constrained loss function. The sparsity parameter \( k \) in FSA is more intuitive than penalty parameters and provides direct cardinality control of the obtained model.

FSA does not introduce any undesired bias on the coefficients. In contrast, the bias introduced by the \( L_1 \) penalty for a certain sparsity level might be too large
and it can lead to poor classification performance [5], [13], [15]. This is why it is a common practice when using the $L_1$ penalty to fit the penalized model only for variable selection and to refit an unpenalized model on the selected variables afterwards. Such a two-step procedure is not necessary in the approach proposed in this paper.

FSA shares some similarity to the Recursive Feature Elimination [20] (RFE) procedure, which alternates training an SVM classifier on the current feature set and removing a percentage of the features based on the magnitude of the variable coefficients. However, our approach has the following significant differences:

1) It removes numerous junk variables long before the parameters $\beta$ have converged, thus it is much faster than the RFE approach where all coefficients are fully trained at each iteration.

2) It can be applied to any loss function, not necessarily the SVM loss and we present applications in regression, classification and ranking.

3) It offers rigorous theoretical guarantees of variable selection and parameter consistency.

It would also be interesting to compare FSA with boosting. Boosting algorithms – such as Adaboost [34], Logitboost [18], Floatboost [31], Robust Logitboost [30] to cite only a few – optimize a loss function in a greedy manner in $k$ iterations, at each iteration adding a weak learner that decreases the loss most. Boosting algorithms do not explicitly enforce sparsity but can be used for feature selection by making the weak learners depend on a single variable (feature). What feature will be selected in the next boosting iteration depends on what features have already been selected and their current coefficients. This dependence structure makes it difficult to obtain a general theoretical variable selection guarantee for boosting.

The approach introduced in this paper is different from boosting because it starts with all the variables and gradually removes variables, according to an elimination schedule. Indeed, its top-down design is opposite to that of boosting, but seems to be less greedy in feature selection based on our experiments in Section 5 and Section 8.

FSA can be viewed as a backward elimination method [19]. But its variable elimination is built into the optimization process. Although there are numerous ways for variable removal and model update, our algorithm design by combining the optimization update and progressive killing is unique to the best of our knowledge. These principles enjoy theoretical guarantees of convergence, variable selection and parameter consistency.

Another related class of methods are based on Stochastic Gradient Descent, such as [28], [39], [40], [43]. However, they still use a sparsity inducing penalty to obtain feature selection, which makes it difficult to optimize and can be slow in practice. We will present in Section 5.2 an evaluation of an implementation of [59] and see that it lags behind our method in computation time, feature selection accuracy and prediction power.

5 Synthetic Data Experiments

We first present simulations on synthetic data to evaluate feature selection and prediction performance and compare it with other state of the art feature selection methods. In this section we focus on FSA in classification and regressions problems. More data applications are reported in Section 9.

The data for simulations has correlated predictors sampled from a multivariate normal $x \sim \mathcal{N}(0, \Sigma)$ where $\Sigma_{ij} = \delta_{(i,j)}$ and $\delta = 0.9$.

For classification, the label $y$ for a data point $x \in \mathbb{R}^M$ is

$$y = \begin{cases} 1 & \text{if } \sum_{i=1}^{k^*} x_{10i} > 0 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

Thus only the variables with index $10i, i = 1, k$ are relevant. We will also use a version of the data with noisy labels, where 10% of the examples had random labels, thus about 5% of the examples have incorrect labels.

All experiments were performed on a six core Intel Core I7-980 machine at 3.3GHz with 24Gb RAM.

5.1 Stability of All Algorithmic Parameters

In this experiment, we evaluate the stability of the FSA Algorithm 1 with respect to its tuning parameters: the learning rate $\eta$, the annealing rate $\mu$ and the number of iterations $N^{\text{iter}}$. The experiment was conducted on the linearly separable data with $M=N=1000$, $k=k^*=10$.

In Figure 4 are shown the dependence of the average area under the ROC curve (AUC) with respect to $\eta$ (left), $\mu$ (middle) and $N^{\text{iter}}$ (right). For the left plot, we had $\mu = 300$, $N^{\text{iter}} = 500$, for the middle plot $\eta = \mu/10$, $N^{\text{iter}} =$
500 and for the right plot μ = 300, η = 20. The obtained curves are the averages of 10 runs.

One can see that all three parameters have a large range of values that yield optimal prediction performance. This robustness property is in contrast to the sensitivity issue of penalty parameters in $L_1$ or $L_0$ like methods. It greatly facilitates parameter tuning and reduces ad-hocness.

### 5.2 Classification Experiments

In this experiment, we compare the variable selection and the prediction performance of the FSA algorithm with the Logitboost algorithm and various sparsity-inducing penalties that are popular in the literature. In calling Logitboost for feature selection, we require each weak learner depends on only one variable.

The experiments are performed on the linearly separable data and its noisy version described above. The algorithms being compared are:

- **FSA** - The FSA Algorithm [1] for the logistic loss [5] with the μ = 300 annealing schedule, η = 20.
- **FSV, FSL** - The FSA Algorithm [1] for the SVM loss [6] and Lorenz loss [5] respectively, with the μ = 300 annealing schedule, η = 1.
- **L1** - The interior point method [2] for $L_1$-penalized Logistic Regression using the implementation from [http://www.stanford.edu/~boyd/l1_logreg/](http://www.stanford.edu/~boyd/l1_logreg/). To obtain a given number $k$ of variables, the value of the $L_1$ penalty coefficient $λ$ is found using the bisection method [7]. The bisection procedure calls the interior point training routine about 9 times until a $λ$ is found that gives exactly $k$ nonzero coefficients. Then an unpenalized model was fitted on the selected variables.
- **EL** - Elastic net on the Logistic loss with $L_1 + L_2$ penalty using the stochastic gradient descent algorithm. We used the Python implementation sklearn.linear_model.SGDClassifier of [39], 1000 epochs for convergence, and the bisection method for finding the appropriate $L_1$ penalty coefficient. After feature selection, the model was refit on the selected variables with only the $L_2$ penalty $α = 0.001$.
- **L2** - SVM using the Python implementation sklearn.linear_model.SGDClassifier with 1000 epochs, and choosing the $L_2$ penalty coefficient $α ∈ \{10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$ that gave the best result.
- **QTP** - The quantile TISP algorithm with 10 thresholding iterations and 500 more iterations on the selected variables for convergence.
- **MCP, SCD** - Logistic regression using MCP (Minimax Concave Penalty) [12] and SCAD penalty respectively. Two implementations were evaluated: the ncvreg R package based on the coordinate descent algorithm [1] and the cvlogistic R package based on the Majorization-Minimization by Coordinate Descent (MMCD) algorithm [22]. The cvlogistic package obtained better results, which are reported in this paper.
- **LB** - Logitboost using univariate linear regressors as weak learners. In this version, all $M$ linear regressors (one for each variable) are trained at each boosting iteration and the best one is added to the classifier.
- **LB1** - Similar to LB, but only 10% of the learners were selected at random and trained at each boosting iteration and the best one was added to the classifier.

In Tables 2 and 3 are shown the all-variable detection rate (DR) and the average percent of correctly detected variables (PCD) obtained from 100 independent runs. The PCD is the average value of $|\{j, β_j \neq 0\} \cap \{j, \beta^*_j \neq 0\}|/k^* \cdot 100$. A more stringent criterion is the DR which is the percentage of times when all $k^*$ variables were correctly found i.e. $\{j, β_j \neq 0\} = \{j, β^*_j \neq 0\}$. The average area under the ROC curve on unseen data of same size as the training data, and the average training times for...
### 5.3 Regression Experiments

Similar to the classification simulations, the observations are sampled from a multivariate normal $x \sim \mathcal{N}(0, \Sigma)$ where $\Sigma_{ij} = \delta|i-j|$ and $\delta = 0.9$. Given $x$, the dependent variable $y$ is obtained as

$$y = \sum_{i=1}^{k^*} x_{10i} + \epsilon, \epsilon \sim \mathcal{N}(0, 1)$$

We experimented with different data sizes and number $k^*$ of relevant variables. The results of the experiments, averaged over 100 runs, are given in Table 4.

The following algorithms were evaluated:

1. FSA - The FSA Algorithm 1 with the $\mu = 300$ annealing schedule, $\eta = 20$.
2. L1 - The built in lasso function from Matlab. The model was refit on the selected variables by least squares.
3. EL - Elastic net with the built in lasso function from Matlab with mixing coefficient 0.99. The model was refit on the selected variables by least squares with shrinkage penalty 0.01.
4. L2 - ordinary least squares with the shrinkage ($L_2$ penalty) coefficient $\alpha \in \{0, 0.1, 0.01, 0.001, 0.0001\}$ that gave the best result.
5. QTP - The quantile TISP algorithm with 10 thresholding iterations and 500 more iterations on the selected variables for convergence.
6. MCP, SCD. The MCP and SCAD penalized regression using coordinate descent 4. The ncvreg C++ implementation was used.

One could see from Table 4 that the FSA algorithm consistently finds the true variables more often than the other methods and obtains better predictions in terms of root mean square error (RMSE) than the other methods. The other methods need at least ten times more data to obtain a similar performance to the FSA method.

We also observe that the FSA algorithm scales quite well to large data sizes, in fact it scales as $O(MN)$ where $N$ is the number of observations and $M$ is the number of variables.

### 6 Capturing Nonlinearity in Regression, Classification and Ranking

In this section we present methods based on the FSA technique to capture nonlinearity and structural information in the feature space in conjunction with feature selection.

We will use a type of nonlinearity that is compatible with feature selection, obtained by replacing $x^2/\beta$ with...
a nonlinear response function that is a sum of a number of univariate functions
\[ f_{\beta}(x) = \sum_{j=1}^{M} f_{\beta_j}(x_j), \quad (14) \]
where \( \beta_j \) is a parameter vector characterizing the response function on variable \( j \).

The univariate functions we will use are piecewise linear, as described in the next section.

### 6.1 Piecewise Linear Learners

A piecewise linear (PL) learner \( f_{\beta}(x) : \mathbb{R} \rightarrow \mathbb{R} \) is a piecewise function that only depends on one variable \( x \) of the instance \( x \in \Omega \). It is defined based on the range \([x_{\text{min}}, x_{\text{max}}]\) of that variable and a predefined number \( B \) of bins.

Let \( b = (x_{\text{max}} - x_{\text{min}})/B \) be the bin length. For each value \( x \), the learner finds the bin index \( j(x) = [(x - x_{\text{min}})/b] \in \{0, ..., B-1\} \) and the relative position in the bin \( \alpha(x) = (x - x_{\text{min}})/b - j(x) \in [0, 1) \) and returns
\[ f_{\beta}(x) = \beta_{j(x)}(1 - \alpha(x)) + \beta_{j(x)+1} \alpha(x) \]

Let
\[ u_i(x) = \begin{cases} 
1 - \alpha(x) & \text{if } i = j(x) \\
\alpha(x) & \text{if } i = j(x) + 1 \\
0 & \text{else}
\end{cases} \]

for \( i \in \{0, ..., B\} \) be a set of \( B + 1 \) piecewise linear basis functions. Then \( f_{\beta}(x) \) can be written as a linear combination:
\[ f_{\beta}(x) = \sum_{i=0}^{B} \beta_i u_i(x) = u^T(x) \beta \]

where \( u(x) = (u_0(x), ..., u_B(x))^T \) is the vector of responses of the basis functions and \( \beta = (\beta_0, ..., \beta_B)^T \in \mathbb{R}^{B+1} \) is the parameter vector.

Some recent works \[21, 22\] use nonlinear additive models that depend on the variables through one-dimensional smooth functions. In \[22\] it was proved that cubic B-splines optimize a smoothness criterion on these 1D functions. Variable selection was obtained by a group lasso penalty. A similar model is presented in \[23\] where a coordinate descent soft thresholding algorithm is used for optimizing an \( L_1 \) group-penalized loss function. Our work differs from these works by imposing constraints on the coefficients instead of biasing them with the \( L_1 \) penalty. Moreover, our optimization is achieved by a novel gradual variable selection algorithm that works well in practice and is computationally efficient.

**Fig. 5.** Piecewise linear response functions \( f_{\beta_j}(x_j) = u_j^T(x_j) \beta_j \) obtained on an eye detection problem using the second order prior \[15\].

**Nonlinear Response Regularization.** Aside from the shrinkage penalty \( \rho(\beta_j) = \lambda \| \beta_j \|^2 \), we will experiment with the second order prior
\[ \rho(\beta_j) = \lambda \| \beta_j \|^2 + c \sum_{k=1}^{B-1} \beta_{j+k+1} + \beta_{j+k-1} - 2 \beta_{jk} \]  \( (15) \)

that favors “smooth” feature response functions \( h_j(x_j) \), as shown in Figure 5.

Other priors could be used, such as differentiable versions of the total variation regularization
\[ \rho(\beta_j) = q \sum_{k=1}^{B} h(\beta_{j,k} - \beta_{j,k-1}) \]  \( (16) \)

where \( h : \mathbb{R} \rightarrow \mathbb{R} \) could be for example the Huber approximation of the \( L_1 \) norm.

### 6.2 Example: Nonlinear FSA Classifier

We are interested in binary classification in an instance space \( \Omega \subset \mathbb{R}^M \). To introduce nonlinearity, we will aggregate a number of piecewise linear learners described in Section 6.1. Other ways to introduce nonlinearity could be used (e.g. splines) and are subject to further exploration.

Using the piecewise linear learners, we obtain the following logistic regression classifier:
\[ p(x, \beta) = \frac{1}{1 + \exp(-\beta_0 - \sum_{j=1}^{M} u_j^T(x_j) \beta_j)} \]  \( (17) \)

where the parameter vector \( \beta = (\beta_0, \beta_1, ..., \beta_M) \) contains the intercept \( \beta_0 \) and the per-variable parameter vectors \( \beta_j \in \mathbb{R}^{B+1}, j = 1, M \).
The logistic loss function with the piecewise linear learners is
\[
L(\beta) = -\sum_{n=1}^{N} w_n \ln(1+e^{-\tilde{y}_n(\beta_0+\sum_{j=1}^{M} u_j^T(x_n)\beta_j)}) + \sum_{j=1}^{M} \rho(\beta_j)
\]  
(18)

6.3 Example: Nonlinear FSA For Ranking

Using the notations from Section 6.1, we can use the nonlinear ranking function without intercept
\[
f(\beta(x)) = \sum_{k=1}^{M} u_k^T(x_k)\beta_k,
\]  
(19)

where \( u_k(x_k) \) is the basis response vector and \( \beta_k \in \mathbb{R}^{B+1} \) is the coefficient vector of variable \( k \).

The loss function (1) in this case has the partial derivatives
\[
\frac{\partial L}{\partial \beta_k} = \sum_{i,j \in C} \frac{1}{1+e^{f(\beta(x_i))-f(\beta(x_j))}} - r_{ij}(u_k(x_{ik}) - u_k(x_{jk}))
\]
\[
+ \frac{\partial \rho(\beta_k)}{\partial \beta_k}
\]

where \( x_{ik} \) is variable \( k \) of observation \( x_i \).

For ranking we use the shrinkage prior for each coefficient vector \( \beta_k \in \mathbb{R}^{B+1} \)
\[
\rho(\beta_k) = \lambda \| \beta_k \|^2,
\]  
(20)

which discourages large values of the coefficients.

The FSA-Rank method will be used in the next section to compare motion segmentations and choose the best one from a set of segmentations with different parameters.

7 Ranking for Motion Segmentation

Sparse motion segmentation is the problem of grouping a given set of trajectories of feature points (that were tracked through the frames of an image sequence) into a number of groups according to their common, usually rigid, motion. A popular method for sparse motion segmentation is spectral clustering [29], where the feature point trajectories are projected to a lower dimensional space where spectral clustering is performed according to an affinity measure.

A major difficulty in this approach is that a rigid motion lies in a low dimensional space that does not have a fixed dimension. As a result, when there are several motions present in the same video sequence, it is hard to determine the best projection dimension for spectral clustering. Consequently, some segmentation methods [11], [29] propose to project to a number of spaces of different dimensions and find the best results according to some measure.

However, it is hard to find a versatile measure that consistently finds the best dimension in all scenarios. Moreover, segmentation algorithms always have one or more parameters, such as the noise level, the separability of the affinity measure, etc, that need to be tuned according to different scenarios. It is also hard to expect there exists a set of parameters that work well for all problems.

Furthermore, many motion segmentation algorithms have been published in recent years, each with their own strength and weaknesses. It would be of practical importance to segment one sequence by many different algorithms and find an automatic way to select the best segmentation.

In this work, we address the problem of choosing the best segmentation from a larger set of segmentations that are generated by different algorithms or one algorithm with different parameters. We formalize it as a ranking problem and solve it using supervised learning with the FSA-Rank algorithm.

7.1 Segmentation by Spectral Clustering

The candidate segmentation results are generated by the velocity clustering (VC) algorithm [11], which we briefly describe below to make the paper self-contained.

A trajectory \( t = [(x_1, y_1), ..., (x_F, y_F)] \) is transformed into a velocity vector
\[
v(t) = [x^1 - x^2, y^1 - y^2, ..., x^{F-1} - x^F, y^{F-1} - y^F, x^F, y^F]^T
\]
(21)

where \( F \) is the number of frames of the image sequence. Then the velocity vectors are projected to spaces of different dimensions in range \([2K, 4K]\) by truncated SVD, where \( K \) is the number of motions. The range contains the possible dimensions of spaces containing \( K \) mixed rigid motions. At last, spectral clustering is applied to obtain the segmentation using the angular affinity
\[
A_{ij} = \left( \frac{v_i^T v_j}{\|v_i\| \|v_j\|} \right)^{2\alpha}, i, j \in \{1, ..., P\}
\]  
(22)

where \( t_i \) and \( t_j \) are two projected trajectories, and \( \alpha \) is a tuning parameter to improve inter-cluster separability.

In this paper the value \( \alpha \) is set to 2, as in VC [11]. Please refer to [11] for more details.

After removing possible repetitive segmentations, around \( 2K + 1 \) segmentations would be generated for each sequence. While the VC method proposes an error measure to select the best segmentation, this paper solves the same problem by learning.

7.2 Likelihood and Prior Based Features

A motion segmentation can be described by a labeling \( L : \{1, ..., P\} \rightarrow \{1, ..., K\} \). We will use two types of features that can characterize the ranking of a motion segmentation \( L \): likelihood features and prior features.

Under the orthographic camera assumptions, the point trajectories of each rigid motion should lie in a 3 dimensional affine subspace.

For a segmentation the likelihood features are used to measure how far are the point trajectories of the same label from lying in a 3D linear subspace.
For both the original trajectory vectors and the points obtained by projection to space of dimension \(d\), where \(d\) is a parameter, we fit in a least squares sense 3-D affine subspaces \(S_i\) through the points of motion label \(l \in \{1, \ldots, K\}\). Denote \(L(i)\) as the label of trajectory \(t_i\) and let \(D(t, S)\) be the euclidean distance of point \(t\) to plane \(S\). Let \(N\) be the total number of trajectories.

We use three types of likelihood features:
- The average distance \(\frac{1}{N} \sum_{i=1}^{N} D(t_i, S_{L(i)})\)
- The average squared distance \(\frac{1}{N} \sum_{i=1}^{N} D^2(t_i, S_{L(i)})\)
- The average thresholded distance
  \[
  \frac{1}{N} \sum_{i=1}^{N} I(D(t_i, S_{L(i)}) \geq \tau),
  \]
where \(I(\cdot)\) is the indicator function taking on value 1 if its argument is true or 0 otherwise, and \(\tau\) is a threshold.

Inspired by VC, the first and second types of features obtained in all dimensions \(d \in [2K, 4K]\) are sorted and the smallest 4 values are used as features.

By changing the threshold \(\tau\) and dimension \(d\) a number of features of the third type can be obtained.

The prior features measure the compactness of the partition over different graphs.

For a given \(k\), the \(k\)-nearest neighbor (kNN) graph is constructed using a distance measure in a space of a given dimension \(d\). The distance could be either the Euclidean distance or the angular distance defined in eq. (22).

By changing the dimension \(d\), number of neighbors \(k\) and distance measure a number of different graphs and features are obtained.

On the kNN graph \(G = (V, E)\) the prior feature is the proportion of the edges that connect vertices with different labels
\[
F_G = \frac{|(i,j) \in E, L(i) \neq L(j)|}{|E|},
\]
where \(L(i)\) is the segmentation label of vertex \(i \in V\).

In total, the features described in this section result in more than 2000 features for each segmentation.

### 7.3 Training the Ranking Function

The performance of a segmentation is characterized by the misclassification error

\[
\text{Misclassification Error} = \frac{\# \text{ misclassified points}}{\text{total \# of points}}, \tag{23}
\]
which could be easily calculated by comparison to the ground truth segmentation.

The true rankings \(r_{ij}, (i,j) \in C\) are constructed based on the relative misclassification errors of the segmentations. Since at test time only the segmentations belonging to the same sequence will be compared, the set \(C\) contains only pairs of segmentations obtained from the same sequence.

For any two segmentations \(i, j\) obtained from the same sequence, the ranking \(r_{ij}\) is based on the misclassification errors of the two segmentations, with value 1 if \(i\) is better than \(j\), 0.5 if they have the same error and 0 if \(j\) is better than \(i\).

These ground truth rankings and the feature vectors for each segmentation are used in the FSA-Rank Algorithm to obtain the parameter vector \(\beta\) that generates the nonlinear ranking function without intercept
\[
f_\beta(x) = \sum_{k=1}^{M} u_k^T(x_k)\beta_k, \tag{24}
\]
where \(u_k(x_k)\) is the basis response vector and \(\beta_k \in \mathbb{R}^{B+1}\) is the coefficient vector of variable \(k\) using the notations from Section 6.1.

### 7.4 Motion Segmentation Algorithm

Given a new sequence, the learned parameter vector \(\beta\) is used to select the best segmentation for that sequence. The whole procedure is described in Algorithm 2.

**Algorithm 2 Motion Segmentation using Ranking**

**Input:** The measurement matrix \(W = [t_1, t_2, \ldots, t_P] \in \mathbb{R}^{2P \times P}\) whose columns are point trajectories, and the number of clusters \(K\).

**Preprocessing:** Build the velocity measurement matrix \(W' = (v(t_1), \ldots, v(t_P))\) where \(v(t)\) is given in eq. (21).

```plaintext
for \(d = d_{\text{min}}\) to \(d_{\text{max}}\) do
  1. Perform SVD: \(W' = U \Sigma V^T\)
  2. Obtain \(P\) projected points as the columns of the \(d \times P\) matrix \(X_d = [v_1, \ldots, v_d]^T\)
  where \(v_i\) is the \(i\)-th column of \(V\).
  3. Apply spectral clustering to the \(P\) points of \(X_d\) using the affinity measure \(\mathcal{D}\), obtaining segmentation \(L_d\).
  4. Extract feature vector \(x_d\) from segmentation \(L_d\) as described in Section 7.2.
  5. Compute the ranking

\[
f_\beta(L_d) = \sum_{k=1}^{M} u_k^T(x_{dk})\beta_k
\]
end for

**Output:** The segmentation result \(L_d\) with the largest value of \(f_\beta(L_d)\).

### 8 Experiments on Real Data

We present FSA experiments on face keypoint detection using classification and regression and on motion segmentation using ranking.

#### 8.1 Face Keypoint Detection Experiments

As this feature selection method is intended to be used in computer vision, we present experiments on detecting face keypoints from color images. The face keypoints such as eye centers, nose sides, mouth corners, chin, bottom of ears, are represented as 2D points \((x, y)\).

**AFLW.** The dataset used for training and testing is the AFLW dataset [25], which has 21123 images containing 24386 faces annotated with 21 points. Of them, 16207 images were found to contain one face per image and 999 of them were selected for training (AFLWT). There
were 2164 images containing at least 2 annotated faces. By visual inspection, 1555 of them were found to have all the faces annotated and were used as the test dataset AFLWMF. These 1555 images contain 3861 faces.

**Feature pool.** All classifiers were trained using a feature pool consisting of \(288 \times 3 = 864\) Histograms of Oriented Gradients (HOG) features and 61000 Haar features extracted from the RGB channels in a \(24 \times 24\) pixel window centered at the point of interest \((x, y)\) in one of the images of a Gaussian pyramid with 4 scales per octave (i.e. resized by powers of \(2^{1/4}\)).

**Training examples.** The training examples are points on the Gaussian pyramid, with the positives within one pixel from the keypoint annotation on the images of the pyramid where the inter-eye distance is in the \([20, 40]\) pixel range. The negatives are all points at least 4 pixels from the keypoint annotation. In total the 999 AFLWT training images contain about 1 billion negatives. All the negatives were used for training the classifiers through a negative mining procedure similar to [16], with the difference that about 20,000 hard negatives were added to the training set at each iteration, thus the set of training images contain about 1 billion negatives. All the faces annotated and were used as the test dataset.

**Detection criteria.** The following criteria were used for evaluating detection performance. The visible face keypoint is considered detected in an image if a detection is found at most 5% of the IED (inter-eye distance, computed by fitting a rigid 3D face model) away in one of the images of the pyramid. A detected point \(p\) in one of the images of the pyramid is a false positive if it is at least 10% of the IED away from the face part being evaluated (visible or not) of any face of the image.

### 8.1.1 Regression Based Object Detection

Applying the sliding window classifier to all locations in the image pyramid can be computationally expensive. A faster alternative is based on the fact that many times the object of interest is surrounded by context, for example the face keypoint is part of the face. The context can be used to predict the object location. This idea has been used in [45] for detecting and segmenting the left ventricle of the heart in ultrasound images.

#### Algorithm 3 Regression Based Keypoint Detection

**Input:** Image \(I\).

- Set \(D = \emptyset\)
- for \(I_s\) on a Gaussian pyramid of \(I\) do
  - Construct a grid \(G_s\) of equally spaced points in \(I_s\)
  - for \((x,y) \in G_s\) do
    - Predict \((dx, dy) = f(I_s, x, y)\) using regressor \(f\)
    - Obtain \((x_1, y_1) = (x + dx, y + dy)\)
    - if \(c(I_s, x_1, y_1) > \tau\) then
      - Set \(D = D \cup \{(x_1, y_1, s)\}\)
    - end if
  - end for
- end for

**Output:** Set of detected points \(D\).

In this paper we apply the same idea, with the difference that we used a regular grid instead of a random set of points. The keypoint detection algorithm proceeds as described in Algorithm 3. The algorithm uses the image based regressor \(f(I, x, y)\) and the classifier \(c(I, x, y)\). The algorithm is also illustrated in Figure 6.

#### Regressor training. The regressor has 1000 weak learners, each being a 2D function on one variable \(f_{\beta_j}(x) = (u^T(x))\beta_{j1}, u^T(x)\beta_{j2}\).

### 8.1.2 Results

**Algorithms.** We compared the following learning algorithms:

1) FSA - The FSA method on the Logistic loss \(\hat{\beta}\) with piecewise linear learners, \(\mu = 300, N^{iter} = 500\).
2) FSA-SVM - The FSA method on the SVM loss \(\hat{\beta}\) with piecewise linear learners, \(\mu = 300, N^{iter} = 500\).
3) FSA-Lorenz - The FSA method on the Lorenz loss [8] with piecewise linear learners, $\mu = 300$, $N_{\text{iter}} = 500$.

4) FSA-Lorenz R - The regression-based detection method from section 8.1.1 using the FSA-Lorenz classifier above for verification.

5) LB - Logitboost using univariate piecewise constant regressors as weak learners. For speed reasons, only 10% of the learners were selected at random and trained at each boosting iteration and the best one was added to the classifier.

6) SVM-PL HOG - The SVM algorithm with piecewise linear response on each variable. The variables were the 864 HOG features.

In Figure 7 are shown the precision-recall curves for detecting nine keypoints on the AFLWMF data. One can see that the FSA-SVM and FSA-Lorenz perform similarly and slightly outperform the FSA on the logistic loss. All three FSA versions outperform Logitboost and greatly outperform the piecewise linear SVM on the HOG features. At the same time, the FSA algorithm is about 8 times faster than the LB algorithm, which is 10 times faster than the full LB version that trains all weak learners at each boosting iteration. The regression-based FSA-Lorenz method is at least as good as the sliding window classifiers, while being about 4 times faster.

Also shown are the supervised descent method [41] and the CNN based face point detection method [37] on the eye and mouth, which were the keypoints that were in common with the keypoints we evaluated.

These two methods outperform the classification and regression-based FSA detectors. However, we must point out that the two face alignment methods are top-down methods that rely on the face being detected first by a face detector, which in the case of the CNN method was trained with about 100k faces. In contrast, our point detectors are bottom-up detectors that were trained with 999 faces to directly detect the keypoints without the intermediary step of finding the face. If we involve our own 3D-model based face detector [3] that uses all nine FSA-Lorenz R keypoint detectors to detect the face and its 3D pose, we obtain the curve denoted as FSA-Lor RFace. These results were obtained using a top-down pruning step that keeps only the keypoint detections that are within 0.5 IED (Inter-Eye Distance) from the predicted locations from the 3D pose. We see that using the top-down information we obtain results comparable to the CNN method [37] and slightly better than the supervised descent method [41].
8.2 Ranking Experiments

The FSA Rank based method for motion segmentation was evaluated on the Hopkins 155 dataset [38]. The Hopkins 155 Dataset has been created with the goal of providing an extensive benchmark for testing sparse motion segmentation algorithms. It contains 155 sets of trajectories of 2 or 3 motions from 50 videos, along with the corresponding ground truth. Based on the content of the video and the type of motion, the 155 sequences can be categorized into three main groups: checkerboard, traffic and articulated checkerboard. Figure 8 shows sample frames from three videos of the Hopkins 155 database with the feature points superimposed.

8.3 The RankBoost Algorithm

The RankBoost algorithm [17] is used in this paper as a baseline method to compare performance in learning the ranking function. At iteration $t$, RankBoost selects the best weak ranker $h_t$ among a pool of candidate weak rankers, and adds $\alpha_t h_t$ to the ranking function $f_{t-1}(x)$. The learning algorithm tries to find a ranking function $H : \mathbb{R}^M \rightarrow \mathbb{R}$ that minimizes the weighted sum of wrong orderings:

$$\text{loss}_D = \sum_{(i,j) \in C} D_{ij} I(H(x_i) \leq H(x_j))$$

where again $I(\pi)$ is 1 if predicate $\pi$ holds and 0 otherwise. The ranking function $H(x)$ is a weighted sum of weak rankers that are selected iteratively

$$H(x) = \sum_{t=1}^T \alpha_t h_t(x).$$

At iteration $t$, RankBoost selects the best weak ranker $h_t$ along with its weighted ranking score $\alpha_t$ from the pool of candidate weak rankers, and adds $\alpha_t h_t(x)$ to the ranking function $f_{t-1}(x)$.

We used threshold-based weak rankers

$$h(x) = \begin{cases} 1 & \text{if } x_i > \theta \\ 0 & \text{if } x_i \leq \theta \end{cases}$$

that depend on the threshold $\theta \in \mathbb{R}$ and the variable index $i$. The pool of weak rankers is generated using all variables $i = 1, M$ and $B = 64$ equally spaced thresholds on the range of each feature.

Parameter Settings. The parameters for RankBoost were the following: the number of thresholds $B = 64$, and the number of boosting iterations was set to 100. The parameters for our FSA-Rank method were: number of bins $B = 4$, the number of selected features $k = 40$. The other parameters are $N^{iter} = 300, \eta = 0.5, \mu = 300, \lambda = 0.01$.

Ten Fold Cross Validation. The Hopkins 155 dataset contains sequences from 50 videos. The 50 videos were divided at random into 10 subsets, each subset containing 5 videos. The 155 Hopkins sequences were also divided into 10 subsets, each subset containing all sequences corresponding to one of the 10 subsets of 5 videos. The reason for separating the videos first and then the sequences is fairness. Some 2 motion sequences are subsets of 3 motion sequences, and it is possible that the segmentation from 2 motions is a subset of that of 3 motions. If this happens, then it would be unfair to have a 3-motion sequence in the training set and a 2-motion subset from the same sequence for testing.

At round $k$ of the cross validation, we select the $k$–th of the 10 subsets of sequences as the test set and form the training set from the remaining 9 subsets. After training, we apply the obtained ranking function to rank the motion segmentations for each sequence. The best one is picked as the final result to calculate the misclassification rate.

8.3.1 Misclassification Error

Ranking Accuracy. Each sequence would be selected in the training set 9 times and in the test set once. In Table 5 are shown the average misclassification errors over all sequences when they were in the training set and when they were in the test set. Other methods are compared, such as randomized voting (RV) [23], spectral clustering (SC) [29], sparse spectral clustering (SSC) [14] and velocity clustering (VC) [11].

Our method outperforms the RankBoost algorithm in every category on both training and test sets, even though Rankboost uses 100 boosting iterations (thus about 100 features) while FSA-Rank uses only 40 features.
Also the difference in misclassification rate between the training set and test set is very small for FSA-Rank, especially for 2-motion sequences. In comparison, the average misclassification rate of 3 motions on test set of RankBoost is about 50% larger than that on training set, while these two misclassification rates are quite close on our method. This is probably due to the small number of features selected and the shrinkage prior \(15\), which together helped obtain a small training error and good generalization.

Compared to VC \(11\) which uses a fixed measure to select best segmentation, our method works better on all categories. Moreover, the average misclassification rates of our method on both 2 motions and 3 motions are almost half of those from SC \(29\).

From the cumulative distributions shown in Figure \(9\) we see that for 2 motions our method performs much better than the other methods compared, while for 3 motions our method is comparable to the best (VC). Nevertheless, our method outperforms RankBoost in both situations.

### 9 Conclusion and Future Work

This paper presented a novel learning scheme for feature selection in high dimensional data applications. It gradually identifies and removes some irrelevant variables and proceeds according to an annealing schedule. We showed that it solves a constrained optimization problem and has a performance guarantee in both estimation and selection.

As opposed to the \(L_1\) penalized method, the proposed method runs much more efficiently and does not introduce any undesired bias in estimation. It kills variables in terms of their importance progressively, which is opposite to the model growing process of boosting, but usually brings improvement in variable selection and prediction.

The algorithm is extremely suitable for big data computation due to its simplicity and ability to reduce the problem size throughout the iteration. The total amount of data the algorithm needs to access for training is only about 2-10 times the size of the training set, which makes it amenable for large scale problems. Hence in computation, FSA has similar advantages as an online algorithm (that accesses each training observation once) while being much more accurate. Our approach applies generically to many types of problems, including regression, classification and ranking for instance. Extensive experiments on both synthetic data and real data support FSA as a competitive alternative to many up-to-date feature selection methods.

In the future we plan to apply the variable selection method to challenging object detection problems.

### References

[1] A. Agresti. *Categorical Data Analysis*. Wiley, 2012.
Fig. 9. The cumulative distribution of the misclassification rate for two and three motions in the Hopkins 155 database.
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\textbf{CONVERGENCE AND CONSISTENCY PROOFS}

\textit{Proof}. We focus on the classification case. The proof for regression is similar (yet simpler). To characterize the iteration, we define the \textit{quantile thresholding} $\Theta^\#(\cdot; k, \lambda)$, as a variant of the \textit{hard-ridge thresholding} \cite{9,13}. Given $1 \leq k \leq M$ and $\lambda \geq 0$, $\Theta^\#(\beta; k, \lambda): \mathbb{R}^M \rightarrow \mathbb{R}^M$ is defined for any $\beta \in \mathbb{R}^p$ such that the $q$ largest components of $\beta$ (in absolute value) are shrunk by a factor of $(1 + \lambda)$ and the remaining components are all set to zero. In the case of ties, a random tie breaking rule is used. We write $\Theta^\#(\beta ; k)$ for $\Theta^\#(\beta; k, 0)$. Steps 3-5 in Algorithm 1 can be characterized by

$$
\beta^{(e+1)} = \Theta^\#(\beta^{(e)} + \eta \mathbf{X}^T (y - \frac{1}{1 + e^{-\mathbf{x}^T \beta^{(e)}}}) ; M) \quad (26)
$$

where $\left[ \frac{1}{1 + e^{-\mathbf{x}^T \beta^{(e)}}} \right]$ is an $N \times 1$ vector with all operations componentwise except for the matrix-vector multiplication $\mathbf{X} \beta^{(e)}$. For notational simplicity, introduce $\mu(t) = 1/(1 + \exp(-t))$ and $\mu(\xi) = \xi$ defined componentwise for any $\xi \in \mathbb{R}^n$. The Fisher information matrix at $\beta$ is denoted by $I(X, \beta) = \mathbf{X}^T \mathbf{X} \eta (\mu(\mathbf{x}_i^T \beta)(1 - \mu(\mathbf{x}_i^T \beta)))_{i=1}^N \mathbf{X}$.

Part (i): We show that the following property holds for any $e$ large enough:

$$
F(\beta^{(e)}) \geq F(\beta^{(e+1)}), \quad \text{and} \quad \|\beta^{(e)}\|_0 \leq k. \quad \text{(30)}
$$

This indicates that the design of the algorithm is to solve $\min_{\beta} F(\beta)$ s.t. $\|\beta\|_0 \leq k$.

We begin by analyzing the sequence of iterates defined by

$$
\beta^{(e+1)} = \Theta^\#\left( \beta^{(e)} + \eta \mathbf{X}^T \left( y - \mu(\mathbf{X} \beta^{(e)}) \right) \right) ; k \quad (31)
$$

where $\eta^{(e)}$ stand for the step sizes. Here, $M_e = k \leq M$, but $M$ is possibly larger than $N$. Recall that $F(\beta) = \sum_i (-y_i x_i^T \beta + \log(1 + \exp(x_i^T \beta)))$ for classification.

\textbf{Lemma 9.1:} Suppose $0 < \eta^{(e)} \leq 4/\|\mathbf{X}\|_2^2$, then for the iterates defined by (31), we have $F(\beta^{(e)}) - F(\beta^{(e+1)}) \geq -\eta^{(e)} \|\mathbf{X}\|_2^2 \|\beta^{(e+1)} - \beta^{(e)}\|_2^2$ and $\beta^{(e)}$ obeys $\|\beta^{(e)}\|_0 \leq k$.

\textbf{Proof.} Define a function as follows

$$
G(\beta, \gamma; \omega) = \sum_i \{ -y_i x_i^T \gamma + \log(1 + e^{x_i^T \gamma}) \} + \frac{\omega}{2} \| \gamma - \beta \|_2^2
$$

\[ - \sum_i \{ \log(1 + e^{x_i^T \beta}) + \log(1 + e^{x_i^T \beta}) \} + \frac{x_i^T \gamma - x_i^T \beta}{1 + e^{x_i^T \beta}}. \]

Given $\beta$, minimizing $G$ over $\{ \gamma : \|\gamma\|_0 \leq k \}$ is equivalent to $\min_{\gamma} \frac{\omega}{2} \| \gamma - \beta \|_2^2 - \omega \|\mathbf{X}^T y + \frac{1}{\omega} \mathbf{X}^T \left[ \frac{1}{1 + e^{-x^T \gamma}} \right] \|_2^2$ s.t. $\|\gamma\|_0 \leq k$.

\textit{Proof.} It is easy to verify that for such a problem (though nonconvex) a globally optimal solution is given by the quantile thresholding \cite{13}

$$
\gamma = \Theta^\# \left( \beta + \frac{1}{\omega} \mathbf{X}^T (y - \mu(\mathbf{X} \beta)) ; k \right).
$$

On the other hand, Taylor expansion gives

$$
\sum_i \{ \log(1 + e^{x_i^T \gamma}) - \log(1 + e^{x_i^T \beta}) \} \cdots \quad \text{(32)}
$$

with the existence guaranteed under the overlap assumption (cf. \cite{1} for details). Let the SVD of $\mathbf{X}$ be $\mathbf{X} = \mathbf{U} \mathbf{D} \mathbf{V}^T$ where $\mathbf{D}$ is a square matrix with all diagonal entries positive. Let $\mathbf{V}_\perp$ be an orthogonal complement to $\mathbf{V}$. Define $\beta^\ast = \mathbf{V}^T \beta^\ast + \mathbf{V}_\perp \mathbf{V}_\perp^T \beta^{(0)}$ which still satisfies (32). Without loss of generality, we write $\beta^\ast$ as $\beta^\ast$.

With a bit algebra and from Taylor expansion, $\beta^{(e+1)} - \beta^\ast = (I - \eta \mathbf{X}^T (1 - \mu(x_i^T \beta))) \mathbf{X}$ may be singular (and thus the iteration mapping is not a contraction), introduce $\gamma = D^T \beta^\ast$, $\gamma^{(e)} = D^T \beta^{(e)}$, and $\gamma^{(e+1)} = D^T \beta^{(e+1)}$. Then $\gamma^{(e+1)} - \gamma^\ast = (I - \eta D^T U \mathbf{V}^T (1 - \mu(x_i^T \beta))) \mathbf{U} \gamma^{(e)} - \gamma^\ast$. Under the $\eta$-assumption, $\gamma^{(e)}$ converges (geometrically fast). By construction, $\mathbf{V}^T (\beta^{(e)} - \beta^\ast) = (\beta^{(e)} - \beta^\ast)$ for
any $e$, and thus $\beta^{(e)} - \beta^* = V D^{-1} (\gamma^{(e)} - \gamma^*)$. This indicates that the sequence of $\beta^{(e)}$ strictly converges and the limit point, denoted by $\beta_o(N)$, satisfies (32). It is not difficult to prove its local optimality, noticing because the perturbation cannot change the support of $\beta_o(N)$ in the constrained optimization.

**Part (iii):** We construct a two-stage cooling schedule to show the consistency. Recall the assumption $\mathcal{I}(X, \beta^*)/N \to \mathcal{I}^*$. Let $\beta_o(N)$ be a solution to (32). From the central limit theorem, $\sqrt{N} (\beta_o(N) - \beta^*) \Rightarrow N(0, \mathcal{I}^{-1})$. This indicates that with probability tending to 1, $\beta_o(N)$ is the unique solution.

Stage 1. Set $M_e = M (1 \leq e \leq E_0)$ for some $E_0$. Accordingly, squeezing operations (27)-(29) do not take any essential effect. To bound the estimation error, we write $\|\beta^{(e)} - \beta^*\|_{\infty} \leq \|\beta_o(N) - \beta^*\|_{\infty} + \|\beta^{(e)} - \beta_o(N)\|_{\infty}$. Based on the central limit theorem, the first term on the right hand side is $O_p(\frac{1}{\sqrt{N}})$ and thus $o_p(1)$. Define $S^* = \{j : \beta^*_j \neq 0\}$ and $\min |\beta_{E^*}^*| = \min_j \beta_{S^*}^*$. Then $\|\beta_o(N) - \beta^*\|_{\infty} \leq \frac{1}{\sqrt{N}} \min |\beta_{S^*}^*|$ with probability tending to 1 as $N \to \infty$.

On the other hand, from the algorithmic convergence of $\beta^{(e)}$ established above, there exists $E_1$ (or $E_1(N)$, as a matter of fact) large enough such that $\|\beta^{(e)} - \beta_o(N)\|_{\infty} \leq \frac{1}{8} \min |\beta_{E^*}^*|$ and $\|X^T \mu(X \beta^{(e)}) - X^T y\|_{\infty} \leq \frac{1}{80} \min |\beta_{E^*}^*|$, $\forall e \geq E_1$. Therefore, if we choose $E_0 = E_1$, $\|\beta^{(e)} - \beta^*\|_{\infty} \leq \frac{1}{4} \min |\beta_{E^*}^*|$ and $\|X^T \mu(X \beta^{(e)}) - X^T y\|_{\infty} \leq \frac{1}{80} \min |\beta_{E^*}^*|$, for $\forall e : E_1 \leq e \leq E_0$ occur with probability tending to 1.

Stage 2. Now set $M_e = k \geq \|\beta^*\|_0$, $\forall e > E_0$. At $e = E_0 + 1$, letting $\xi = \beta^{(e)} + \eta X^T (y - \mu(X \beta^{(e)}))$, we have $\min |\xi_{E^*}| \geq \frac{1}{8} \min |\beta_{E^*}^*|$. Hence $\beta^{(e)}(e+1) \neq 0$, $\forall j \in S^*$. After Steps (27)-(29), all relevant predictors are kept with probability tending to 1. Repeating the argument in Stage 1 gives the consistency of $\beta^{(e)}$ for any $e$ sufficiently large.

The proof in the Gaussian case follows the same lines. But in Part (ii) we do not have to assume the overlap condition (or require $N \geq M$), due to Landweber’s classical convergence result [27].