On the number of hyperbolic 3-manifolds of a given volume

Hidetoshi Masai
(Joint work with Craig Hodgson)

Tokyo Institute of Technology DC2,
(University of Melbourne)

2012, May, 31st
Representation spaces, twisted topological invariants and geometric structures of 3-manifolds
Contents

1 Introduction
   - Hyperbolic volume
   - On small $N(\nu)$
   - On large $N(\nu)$

2 sketch of proofs
   - Main theorem 1
   - Main theorem 2

3 Open problems
Contents

1 Introduction
   - Hyperbolic volume
   - On small $N(\nu)$
   - On large $N(\nu)$

2 sketch of proofs
   - Main theorem 1
   - Main theorem 2

3 Open problems
Mostow-Prasad rigidity

Today, by the word "hyperbolic 3-manifold" (or just "manifold"), we mean a connected orientable complete hyperbolic 3-manifold of finite volume.
Mostow-Prasad rigidity

Today, by the word "hyperbolic 3-manifold" (or just "manifold"), we mean a connected orientable complete hyperbolic 3-manifold of finite volume.

**Theorem (Mostow-Prasad rigidity theorem)**

Let $M_1, M_2$ be hyperbolic 3-manifolds. Then, 

$$\pi_1(M_1) \cong \pi_1(M_2) \iff M_1 \text{ is isometric to } M_2$$

By Mostow-Prasad rigidity the volume of a given hyperbolic 3-manifold is a topological invariant. For example, for a given hyperbolic link, the volume is a link invariant.
Mostow-Prasad rigidity

Today, by the word "hyperbolic 3-manifold" (or just "manifold"), we mean a connected orientable complete hyperbolic 3-manifold of finite volume.

**Theorem (Mostow-Prasad rigidity theorem)**

\[ \pi_1(M_1) \cong \pi_1(M_2) \iff M_1 \text{ is isometric to } M_2 \]

- By Mostow-Prasad rigidity the volume of a given hyperbolic 3-manifold is a **topological invariant**.
- For example, for a given hyperbolic link, the volume is a link invariant.
Theorem (Jørgensen-Thurston)

Let \( \mathcal{H} \) be isometry classes of hyperbolic 3-manifolds. Then the volume function \( \text{vol} : \mathcal{H} \rightarrow \mathbb{R}_{>0} \) is a finite-to-one function. Further, the image \( \text{vol}(\mathcal{H}) \) is a well-ordered subset of \( \mathbb{R}_{>0} \) of order type \( \omega^\omega \).

By this theorem, for a given \( v \in \mathbb{R}_{>0} \), there exists a natural number \( N(v) := \# \text{vol}^{-1}(v) \).
What can we say about $N(v)$?

Not much is known!

Gabai-Meyerhoff-Milley proved that the Weeks manifold $W$ is the unique smallest volume manifold among all hyperbolic 3-manifolds, i.e.

$$N(v) = 1$$

and for $v < \text{vol}(W)$;

$$N(v) = 0$$

(As far as I know) prior to our work, this is the only result which gives the exact value of $N(v)$. 
$N(v)$

**Question**

What can we say about $N(v)$?

Not much is known!
Introduction

Hyperbolic volume

\[ N(\nu) \]

**Question**

What can we say about \( N(\nu) \)?

Not much is known!

- Gabai-Meyerhoff-Milley proved that the Weeks manifold \( W \) is the *unique* smallest volume manifold among all hyperbolic 3-manifolds.
  
  i.e \( N(\text{vol}(W)) = 1 \) and for \( \nu < \text{vol}(W) \), \( N(\nu) = 0 \)

- (As far as I know) prior to our work, this is the only result which gives the *exact* value of \( N(\nu) \).
There are many interesting classes of hyperbolic 3-manifolds. For example,

- $\mathcal{C}$: cusped manifolds.
- $\mathcal{A}$: arithmetic manifolds.
- $\mathcal{G}$: manifolds with geodesic boundaries.
- $\mathcal{L}$: link complements.

It is also interesting to ask

$$N_{\mathcal{X}}(v) = \# \{ \text{vol}^{-1}(v) \cap \mathcal{X} \}$$
For particular classes some of the exact values are known.
For particular classes some of the exact values are known.

- Cao-Meyerhoff proved that m003 and m004 are the smallest cusped manifolds i.e. $N^C(\text{vol}(m003)) = 2$. 
For particular classes some of the exact values are known.

- Cao-Meyerhoff proved that m003 and m004 are the smallest cusped manifolds i.e. \( N_C(\text{vol}(m003)) = 2 \).
- Gabai-Meyerhoff-Milley detected first 10 smallest cusped manifolds.
For particular classes some of the exact values are known.

- Cao-Meyerhoff proved that m003 and m004 are the smallest cusped manifolds i.e. $N_C(\text{vol}(m003)) = 2$.

- Gabai-Meyerhoff-Milley detected first 10 smallest cusped manifolds.

- For Weeks manifold $W$ and Meyerhoff manifold $M$, Chinburg-Friedman-Jones-Reid proved $N_A(\text{vol}(W)) = 1$, $N_A(\text{vol}(M)) = 1$.
For particular classes some of the exact values are known.

- Cao-Meyerhoff proved that m003 and m004 are the smallest cusped manifolds i.e. \( N_C(\text{vol}(m003)) = 2 \).

- Gabai-Meyerhoff-Milley detected first 10 smallest cusped manifolds.

- For Weeks manifold \( W \) and Meyerhoff manifold \( M \), Chinburg-Friedman-Jones-Reid proved \( N_A(\text{vol}(W)) = 1, N_A(\text{vol}(M)) = 1 \).

- Kojima-Miyamoto detected the smallest compact manifolds with geodesic boundaries and Fujii proved that there are 8 of them. i.e. \( N_G(6.452...) = 8 \).
Computer experiments

SnapPy has many good censuses of hyperbolic manifolds.

- Orientable Cusped Census. (at most 8 ideal tetrahedra)
- Orientable Closed Census. (by Hodgson and Weeks)
SnapPy has many good censuses of hyperbolic manifolds.

- Orientable Cusped Census. (at most 8 ideal tetrahedra)
- Orientable Closed Census. (by Hodgson and Weeks)
- Census Knots. (at most 7 ideal tetrahedra)
- Link Exteriors (using Rolfsen’s notation).
- (Non) Alternating Knot Exteriors (up to 16 crossings).
- MorwenLinks(up to 14 crossings, about 180k links).
Computer experiments

SnapPy has many good censuses of hyperbolic manifolds.

- Orientable Cusped Census. (at most 8 ideal tetrahedra)
- Orientable Closed Census. (by Hodgson and Weeks)
- Census Knots. (at most 7 ideal tetrahedra)
- Link Exteriors (using Rolfsen’s notation).
- (Non) Alternating Knot Exteriors (up to 16 crossings).
- MorwenLinks (up to 14 crossings, about 180k links).
- Nonorientable Cusped (or Closed) Census.
Computer experiments

SnapPy has many good censuses of hyperbolic manifolds.

- Orientable Cusped Census. (at most 8 ideal tetrahedra)
- Orientable Closed Census. (by Hodgson and Weeks)
- Census Knots. (at most 7 ideal tetrahedra)
- Link Exteriors (using Rolfsen’s notation).
- (Non) Alternating Knot Exteriors (up to 16 crossings).
- MorwenLinks(up to 14 crossings, about 180k links).
- Nonorientable Cusped (or Closed) Census.

We used the first two censuses and compute $N_{\text{census}}(v)$'s.
Computer experiments

Closed and Cusped Manifolds

$N(v)$ vs Volume
Computer experiments

Cusped Manifolds

\[ N(v) \]

Volume
Main theorem 1

**Theorem (Unique volume Manifolds)**

There exists an infinite sequence of hyperbolic manifolds \( \{ M_i \} \) such that \( N(\text{vol}(M_i)) = 1 \).

**Theorem (Unique volume Cusped manifolds)**

There exists an infinite sequence of cusped hyperbolic manifolds \( \{ M_i^C \} \) such that \( N_C(\text{vol}(M_i^C)) = 1 \).

- These manifolds are obtained by Dehn filling on \( m004 \) and \( m129 \) respectively.

\( m004 = \) complement of figure eight knot, \( m129 = \) complement of Whitehead link
Growth rate

In the above theorem, we discussed the case $N(v)$ is small.

**Question**

How large can $N(v)$ be?

- Wielenberg: For all $n \in \mathbb{N}$, there exists $v \in \mathbb{R}_{>0}$ such that $N(v) > n$.
- Zimmerman: $N_{\text{closed}}(v) > n$.

**Theorem (Chesebro-DeBlois, 2012)**

$C(v)$ can be arbitrary large. Where $C(v)$ is the number of commensurability classes that contain manifolds of volume $v$. 
Computer experiments

Closed and Cusped Manifolds

\[ N(v) \]

Volume
Computer experiments

Cusped Manifolds

\[ N(v) \]

Volume
Growth rate

Question

How fast can $N(v)$ grow?

In other words, what can we say about $G(V)$? Where

$$\max_{v \leq V} N(v) \preceq G(V)$$
**Known results**

**Theorem (Belolipetsky, Gelander, Lubotzky, Shalev, 2010)**

There exists constants $a, b > 0$ such that for $x \gg 0$,

$$x^{ax} < \max_{x_i \leq x} N_A(x_i) < x^{bx}$$

**Theorem (Frigerio, Martelli and Petronio, 2003)**

There exists a constant $c > 0$ such that for $x \gg 0$,

$$N_G(x) > x^{cx}$$

($A$: arithmetic manifolds  
$G$: manifolds with geodesic boundaries)
Main theorem 2

**Theorem (Hodgson-M)**

*There exists* $c > 1$ *such that*

$$N_{\mathcal{L}}(x) > c^x$$

($\mathcal{L}$: Link complements)
1 Introduction
- Hyperbolic volume
- On small $N(\nu)$
- On large $N(\nu)$

2 sketch of proofs
- Main theorem 1
- Main theorem 2

3 Open problems
Main theorem 1

Theorem (Unique volume Manifolds)

There exists an infinite sequence of hyperbolic manifolds \( \{ M_i \} \) such that \( N(\text{vol}(M_i)) = 1 \).

Theorem (Unique volume Cusped manifolds)

There exists an infinite sequence of cusped hyperbolic manifolds \( \{ M_i^C \} \) such that \( N_C(\text{vol}(M_i^C)) = 1 \).

- These manifolds are obtained by Dehn filling on m004 and m129 respectively.

(m004 = complement of figure eight knot, m129 = complement of Whitehead link)
Hyperbolic Dehn surgery theorem

- $M$: hyperbolic 3-manifold with a cusp $T$.
- $M(a, b)$: manifold after Dehn filling $T$ along slope $(a, b)$.
- $L(a, b)$: length of the slope $(a, b)$ on $T$.

Theorem (Thurston)

Then there exist a constant $C_1 = C_1(M)$ such that the Dehn filling $M(a, b)$ is hyperbolic whenever $L(a, b) > C_1$, and $\text{vol}(M(a, b)) \to \text{vol}(M)$ as $L(a, b) \to \infty$. 
Neuman-Zagier asymptotic formula

- $A$: area of the horotori
- $Q(a, b) = \frac{L(a, b)^2}{A}$
- $\Delta(a, b) = \text{vol}(M) - \text{vol}(M(a, b)) > 0$

**Theorem (Neumann-Zagier)**

There exist a constant $C_2 = C_2(M) > 0$ such that,

$$\left| \frac{\pi^2}{\Delta(a, b)} - Q(a, b) \right| < C_2$$
Key idea

\[(a_0, b_0)\] : pair of relatively prime integers such that

(i) \(Q(a, b) = Q(a_0, b_0) = Q_0\) has few integer solutions,

(ii) there is large enough 2-sided gap around \(Q_0\) in the set of possible value of \(Q(x, y)\) for \((x, y)\) relatively prime integers.

\(\Rightarrow\) There are few Dehn fillings \(M(a, b)\) with the same volume as \(M(a_0, b_0)\)
m004 : the figure eight knot complement

- m004: the figure eight knot complement.
- Then $Q(a, b) = a^2 + 12b^2$ for suitably chosen basis on the cusp of m004.
m004 : the figure eight knot complement

- m004: the figure eight knot complement.
- Then \( Q(a, b) = a^2 + 12b^2 \) for suitably chosen basis on the cusp of m004.

\[ \Rightarrow \text{some number theory proves the existence of a sequence } \{(a_i, b_i)\} \text{ such that} \]

(i) \( (Q_0 :=) Q(a, b) = Q(a_i, b_i) \Rightarrow (a, b) = (a_i, b_i) \)

(ii) there is large enough 2-sided gap around \( Q_0 \) in the set of possible value of \( Q(x, y) \) for \((x, y)\) relatively prime integers.

\[ \Rightarrow M(a_i, b_i) \text{ is a unique volume manifold among all Dehn fillings of m004 and m003}. \]
Smallest cusped manifolds

By the work of Cao-Meyerhoff, m003 and m004 are the smallest volume cusped manifolds.
Smallest cusped manifolds

By the work of Cao-Meyerhoff, m003 and m004 are the smallest volume cusped manifolds.

⇒ (+ Hyperbolic Dehn surgery theorem)

∃ε > 0 such that for all manifolds N with

\[ 2.029... - \epsilon < \text{vol}(N) < 2.029... \]

can be obtained from m004 or m003 by a Dehn filling.
Smallest cusped manifolds

By the work of Cao-Meyerhoff, m003 and m004 are the smallest volume cusped manifolds.
⇒ (+ Hyperbolic Dehn surgery theorem)
∃ε > 0 such that for all manifolds N with

$$2.029... - \varepsilon < \text{vol}(N) < 2.029...$$

can be obtained from m004 or m003 by a Dehn filling.
⇒ If $M(a, b)$ is a unique volume manifold among all Dehn fillings of m004 or m003, then $M(a, b)$ is unique among all hyperbolic 3-manifolds.
(m004: the figure eight knot complement
m003: the sister of m004)
Main theorem 1

Theorem (Unique volume Manifolds)

There is an infinite sequence of hyperbolic manifolds \( \{M_i\} \) such that \( N(\text{vol}(M_i)) = 1 \).

Theorem (Unique volume Cusped manifolds)

There is an infinite sequence of hyperbolic manifolds \( \{M_i^c\} \) such that \( N_C(\text{vol}(M_i^c)) = 1 \).

- These manifolds are obtained by Dehn filling on m004 and m129 respectively.

(m004 = complement of figure eight knot, m129 = complement of Whitehead link)
Main theorem 2

Theorem (Hodgson-M)

There exists $c > 1$ such that

$$N_{\mathcal{L}}(x) > c^x$$

($\mathcal{L}$: Link complements)
Hyperbolic graph

- **$G$:** trivalent spatial graph in $S^3$.
- **$V$:** the set of vertices of $G$. 

Define $N_G$ by

$$N_G := m_{N(G)} \cup \bigcup_{v \in V} N(v)$$

where $N(v)$ is an open regular neighborhood of $v$. Then $N_G$ is a manifold with 3-punctured sphere boundaries, one corresponds to each vertex of $G$. 

---

**Main theorem 2**

**Introduction**

**Sketch of proofs**

**Open problems**
Hyperbolic graph

- $G$: trivalent spatial graph in $S^3$.
- $V$: the set of vertices of $G$.

Define $N_G$ by

$$N_G := M \setminus G \setminus \bigcup_{v \in V} N(v)$$

where $N(v)$ is an open regular neighborhood of $v$. 
Hyperbolic graph

- $G$: trivalent spatial graph in $S^3$.
- $V$: the set of vertices of $G$.

Define $N_G$ by

$$N_G := M \setminus G \setminus \left( \bigcup_{v \in V} \mathcal{N}(v) \right)$$

where $\mathcal{N}(v)$ is an open regular neighborhood of $v$. Then $N_G$ is a manifold with 3-punctured sphere boundaries, one corresponds to each vertex of $G$. 

Definition

A spacial graph $G$ is hyperbolic if $N_G$ admits complete hyperbolic structure (with parabolic meridians) of finite volume with totally geodesic boundaries.

Example (Intuitive picture) of a hyperbolic graph.
Volume preserving moves

Lemma

The following moves on hyperbolic graphs in $S^3$ are volume preserving.

This lemma relates hyperbolic graphs with hyperbolic links.
Example.

Two complements have the same volume.
We apply one of the moves to the following graph.

Then we get **possibly distinct** $2^n$ links of a same volume.
We apply one of the moves to the following graph.

Then we get possibly distinct $2^n$ links of a same volume.

- We distinguish these manifold by computing the moduli of cusps and edges of canonical decomposition.
Our graph

The graph comes from a planar graph
The complements of planar hyperbolic graphs admit useful polyhedral decompositions.

Remark.
This decomposition is same as the decomposition of a fully augmented link found by Agol-Thurston.
Since each dihedral angle is $\pi/2$, this decomposition gives a circle packing on $\partial \mathbb{H}^3 \cong S^2$.

- This circle packing enables us to compute modulus of each cusp.
Our graph

Our graph, its polyhedral decomposition and corresponding circle packing.

\[ n \]
Moduli of cusps

For our graph, there are 3 different types of annuli cusps.

By gluing annuli cusps together we get a torus cusp.

\[\Rightarrow\]

We can compute cusp moduli.
Example.

This graph has 3 types of tori cusps and their shapes are
For each link that we obtain after gluing the 3-punctured sphere of

we can assign (horoball) volume to each cusp in terms of the moduli.
It gives us a way to fix a canonical decomposition.
(SnapPy demo.)
1 Introduction
   - Hyperbolic volume
   - On small $N(\nu)$
   - On large $N(\nu)$

2 sketch of proofs
   - Main theorem 1
   - Main theorem 2

3 Open problems
Open problems

1. Find additional exact values or upper bounds on $N(v)$. 
Open problems

1. Find additional exact values or upper bounds on \( N(v) \).
2. (Gromov, 1979) Is \( N(v) \) locally bounded?
Open problems

1. Find additional exact values or upper bounds on $N(v)$.
2. (Gromov, 1979) Is $N(v)$ locally bounded?
3. Are all hyperbolic Dehn fillings on m004 (or m003) determined by their volumes, amongst Dehn fillings on m004 (or m003)?
Open problems

1. Find additional exact values or upper bounds on $N(v)$.
2. (Gromov, 1979) Is $N(v)$ locally bounded?
3. Are all hyperbolic Dehn fillings on m004 (or m003) determined by their volumes, amongst Dehn fillings on m004 (or m003)?
4. What is the largest volume $v < v_\omega = 2.029883...$ of a closed hyperbolic 3-manifold which does not arise from Dehn filling of m004 or m003? (This would allow us to make the above results explicit.)
   Guess: $v = 2.02885309... = \text{vol}(\text{m006}(-5, 2))$. 
Open problems

1. Find additional exact values or upper bounds on $N(v)$.
2. (Gromov, 1979) Is $N(v)$ locally bounded?
3. Are all hyperbolic Dehn fillings on m004 (or m003) determined by their volumes, amongst Dehn fillings on m004 (or m003)?
4. What is the largest volume $v < v_\omega = 2.029883...$ of a closed hyperbolic 3-manifold which does not arise from Dehn filling of m004 or m003? (This would allow us to make the above results explicit.)
   Guess: $v = 2.02885309.. = \text{vol}(m006(-5, 2))$.
5. Does there exist $C > 0$ such that $N_\mathcal{L}(x) > x^{Cx}$?
Thank you for your attention.

Great Ocean Road (Australia)