Alternative Mathematics without Actual Infinity *

Toru Tsujishita

2012.6.12

Abstract

An alternative mathematics based on qualitative plurality of finiteness is developed to make non-standard mathematics independent of infinite set theory. The vague concept “accessibility” is used coherently within finite set theory whose separation axiom is restricted to definite objective conditions. The weak equivalence relations are defined as binary relations with sorites phenomena. Continua are collection with weak equivalence relations called indistinguishability. The points of continua are the proper classes of mutually indistinguishable elements and have identities with sorites paradox. Four continua formed by huge binary words are examined as a new type of continua. Ascoli-Arzela type theorem is given as an example indicating the feasibility of treating function spaces.

The real numbers are defined to be points on linear continuum and have indefiniteness. Exponentiation is introduced by the Euler style and basic properties are established. Basic calculus is developed and the differentiability is captured by the behavior on a point. Main tools of Lebesgue measure theory is obtained in a similar way as Loeb measure.

Differences from the current mathematics are examined, such as the indefiniteness of natural numbers, qualitative plurality of finiteness, mathematical usage of vague concepts, the continuum as a primary inexhaustible entity and the hitherto disregarded aspect of “internal measurement” in mathematics.

*Thanks to Ritsumeikan University for the sabbatical leave which allowed the author to concentrate on doing research on this theme.
Contents

Abstract 1

Contents 2

0 Introduction 6
  0.1 Nonstandard Approach as a Genuine Alternative 6
  0.2 Multiple Levels of Finiteness 7
  0.3 Points of Conflicts with Modern Mathematics 9
  0.4 Background 10
    0.4.1 Qualitative Plurality of Numbers 10
    0.4.2 Properties without Extension 15
    0.4.3 Coherence of Vague Concepts 19
    0.4.4 Continuum 19
  0.5 Outline of Contents 21

1 Fundamentals 23
  1.1 Numbers 23
    1.1.1 Accessibility 23
    1.1.2 Rational Numbers 24
  1.2 Sets and Classes 25
    1.2.1 Basic Concepts 25
    1.2.2 Subclasses and Subsets 25
    1.2.3 Objective Conditions and Semisets 26
    1.2.4 Class Constructions 27
    1.2.5 Functions 27
    1.2.6 $\sigma$-finite Classes 29
  1.3 Induction Axioms 31
  1.4 Overspill Principles 32
  1.5 Concrete Sequences 33

2 Continuum 35
  2.1 Sorites Relations 35
11 Concluding Remarks

11.1 Recapitulation ........................................... 141
  11.1.1 Vague Concepts and Semisets ................. 141
  11.1.2 Treatment as Naive theory ................. 142
  11.1.3 Continua and Points ......................... 142
  11.1.4 Idealization ................................. 143
  11.1.5 Transfer Principle .......................... 144

11.2 Future Direction ........................................ 144
  11.2.1 Accessibility of Higher Order Objects .... 144
  11.2.2 Relative Accessibility ................. 145
  11.2.3 Continua of Syntactic Objects .......... 146

11.3 Philosophical Implications .......................... 148
  11.3.1 Rejection of Existence Absolutism ........ 148
  11.3.2 Internal Measurement ...................... 150

Acknowledgement ............................................. 151

References ..................................................... 152

Index .......................................................... 157
0 Introduction

0.1 Nonstandard Approach as a Genuine Alternative

Mathematics has evolved by integrating paradoxes. The Galilei paradox and the Sorites paradox represent two logical phenomena concerning infinity. The former, the inevitable paradox of actual infinity that a part is the same size as the total is incorporated in mathematics as the very definition of infinite sets. The latter, the inevitable paradox resulting from two incommensurable view points such as micro vs macro is in harmony with the infinite phenomena daily experienced by us and is taken in mathematics implicitly by nonstandard mathematics.

As a result mathematics has currently two methods of treating infinity, Cantorian set theory and nonstandard mathematics. In contrast to the former which handles infinity as a definite concept, the latter handles infinity as being “incarnated” in finiteness thus removing the inconvenient dichotomies such as infinite vs finite and continuous vs discrete. Nonstandard mathematics shows new ways of making mathematical discourse more intuitive without losing logical rigor and giving more flexible ways of constructing mathematical objects. We may say that by discriminating between “actual finiteness” and “ideal finiteness”, we obtain a better system of handling infinity than the “actual infinity” offers.

Surely the nonstandard mathematics was born and has been bred in the realm of Cantorian set theory. Various axiomatic systems for nonstandard mathematics such as IST (Internal Set Theory) \cite{Nel77} of E.Nelson, RST (Relative Set Theory) \cite{Per92} of Péraire and EST (Enlargement Set Theory) \cite{Bal93} of D. Ballard are conservative extensions of ZFC, so that the statements of usual mathematics proved in the new axiomatic systems can be proved without them. It is natural that many researchers considered the conservativeness the crucial point since the significance of nonstandard mathematics at first was able to be claimed only through its relation to current mathematics. Besides one could believe the consistency of axiomatic systems of nonstandard mathematics only through reducing it to that of the standard systems.

However as long as it remains grafted to Cantorian set theory, the nonstandard mathematics will not unveil its seminal significance as a genuine alternative to modern mathematics and its potentiality will not be fully brought to fruition. It seems high time to break the fetters and to make nonstandard mathematics independent of Cantorian set theory. In fact, after 50 years after its birth, there seems to be widespread conviction that most of modern mathematics can be rebuild more efficiently by nonstandard mathematics and that new wine must be put in new bottle, namely
the foundation of nonstandard mathematics itself should be rebuild without recourse to infinite set theory.

In fact, already in 1991, P. Vopěnka [Vop91] clearly stated such a view as follows:\[1\]

As long as this master-vassal relationship lasts, Non-standard Analysis cannot use all its potential, which lies mainly in new formalizations of various situations and not in new proofs of classical theorems. . . . It is necessary to approach the study of natural infinity directly and not through its pale reflection as found within Cantor’s Set Theory. Such a direct approach is what Alternative Set Theory attempts.

E. Nelson [Nel07] points out the importance of thinking of nonstandard analysis as a genuine alternative to modern mathematics.

Heretofore nonstandard analysis has been used primarily to simplify proofs of theorems. But it can also be used to simplify theories. There are several reasons for doing this. First and foremost is the aesthetic impulse, to create beauty. Second and very important is our obligation to the larger scientific community, to make our theories more accessible to those who need to use them. To simplify theories we need to have the courage to leave results in simple, external form —— fully to embrace nonstandard analysis as a new paradigm for mathematics. Much can be done with what may be called minimal nonstandard analysis.

0.2 Multiple Levels of Finiteness

The crucial point of the nonstandard mathematics is to afford qualitative multitudeness of finiteness. Unfortunately one must take currently a long detour to actualize the qualitative multitudeness of finiteness in modern mathematics, because of its deep belief in the qualitative uniqueness of finiteness symbolized as “the infinite set \( N \)”. However the presence of qualitatively different levels of finiteness is an undeniable state of affairs in real life and may be assumed as a fundamental principle much more secure than the belief in the \( N \)-dogma.

The importance of considering seriously the qualitatively different kind of natural numbers has been stressed repeatedly by many mathematicians from the middle of the last century. In 1952, E. Borel [Bor52] considered “inaccessible numbers” would be important. Around 1960, E. Volpins

\[1\]For more quotations of similar views, see § 0.4.1
claimed the multitude of natural number sequences and in 1971 R. Parikh \cite{Par71} pointed out various paradoxical phenomena resulting from the uniqueness of the natural number concept, e.g., construction of certain formulas which are shown to be provable but the proof is too long to be actually carried out. See §0.4.1 for more comments on these aspects.

Now there have been many trials to lay foundation of mathematics based on the multitude of finiteness. Vopěnka’s Alternative Set Theory is one of the most elaborated approaches admitting only finite sets some of which are huge containing actually all the “concrete numbers”. Similar systems are elaborated in Hyperfinite Set theory \cite{AG06} of Gordon et al..

The outstanding variance of nonstandard mathematics from the conventional mathematics is the acceptance of so called “vague concept” in mathematics\footnote{See §0.4.3 for a criticism to the Dammetts’ arguments on the incoherence of vague concepts.}. The totality of accessible objects is indefinite since the accessibility depends on the methods of access and even if a method is fixed it is not clear how far we can access. Hence one cannot consider the collection of all the accessible numbers as a set and must treat it as a proper class like the totality of sets. However this collection is contained in the finite set of numbers less than an inaccessible number, whence the notion of semisets of Vopěnka will play vital roles in this new mathematics. See §0.4.2 for more points on concepts without extension.

In educational studies of mathematics, it has been pointed out that the concept of “measuring infinity” \cite{Tal80} such as the hyperfinite numbers in nonstandard mathematics is more intuitive than that of “cardinal infinity” of Cantorian set theory \cite{TT01,BMW10}. Regrettably the usage of nonstandard mathematics in elementary levels of university education is not workable at present because of various artifacts in its usual framework resulting from the detour through infinite set theory\footnote{This is another aspect of the natural infinity in the sense of P. Vopěnka. See §0.4.}

However E. Nelson \cite{Nel87b} clearly showed that “minimal nonstandard analysis” captures directly the essence of a deep mathematical theory in an elementary way without artificial arguments when freed from the burden of the infinite sets theory. The preface states clearly his intention as follows.

This work is an attempt to lay new foundations for probability theory, using a tiny bit of nonstandard analysis. The mathematical background required is little more than that which is taught in high school, and it is my hope that it will make deep results from the modern theory of stochastic processes readily available to anyone who can add, multiply, and reason. What makes this
possible is the decision to leave the results in nonstandard form. Nonstandard analysts have a new way of thinking about mathematics, and if it is not translated back into conventional terms then it is seen to be remarkably elementary.

Mathematicians are quite rightly conservative and suspicious of new ideas. They will ask whether the results developed here are as powerful as the conventional results, and whether it is worth their while to learn nonstandard methods. These questions are addressed in an appendix, which assumes a much greater level of mathematical knowledge than does the main text. But I want to emphasize that the main text stands on its own.

Just as it took only a few decades for mathematicians to get comfortable with the cardinal infinity, it may not take long that discourse using the measuring infinity become common practice as tools more fundamental and more versatile than the cardinal infinity. But it will surely take at least a few decades and most mathematicians might hesitate to take the risk of get involved in such a long range uncertain project. But various trials to develop such a genuine alternative to modern mathematics are indispensable for healthy evolution of future mathematics in view of the strong evidence of the radical superiority of the alternative over the current mathematics. Besides already mentioned contributions [Vop79], [Nel87b] there are many proposals and trials of alternative mathematics based on similar intention such as [SLSZ], [Myc81], [Har83], [Bec80, Bec79] [Lut87], [Lut92], [Lau92], [Die92] to mention a few. I hope this another trial would play some role, however small it may be, to strengthen and quicken the movement to free nonstandard analysis from current mathematics.

0.3 Points of Conflicts with Modern Mathematics

The followings are some of the features of our approach radically different from the usual mathematics.

Sets are finite. The usual “infinite sets” such as \( \mathbb{N} \) and \( \mathbb{Q} \) are considered as proper classes so that the totality is not considered as a definite object.

Sorites Axiom. A number \( x \) is called accessible if there is a certain concrete method of obtaining it\(^5\). We postulate the existence of inaccessible numbers as the most basic axiom of our framework. The accessible numbers form an nonending number series which is closed under the

\(^5\)For example there is a concrete Peano formula \( P(u) \) such that \( x \) is the minimal number satisfying \( P(x) \).
operation \( x \mapsto x + 1 \) but differs from the total number series. Accordingly, fundamental notions such as transitivity, equivalence relation, provability, compatibility, etc. become relative to the number series chosen.

**The overspill axiom.** If an objective condition holds for all accessible numbers, then it holds also for an inaccessible number. Here a condition is called objective if it can be specified without the notion of accessibility.

**Vague conditions.** The vagueness of the accessibility prohibits us to regard the collection of accessible numbers as a set. It is a proper class contained in a finite set, called semiset in Alternative Set Theory of Vopěnka [Vop79].

**Continua are not infinite sets.** The real line is considered as the “quotient” of the proper class \( \mathbb{Q} \) by the indistinguishability relation defined by \( r \approx r' \) if and only if \( k|r - r'| < 1 \) for every accessible number \( k \). Although this “quotient” is used only as a way of speech, we can represent for example the “unit interval” \( \{ r \in \mathbb{Q} \mid 0 \leq r \leq 1 \} / \approx \) by the quotient of the finite set \( \{ \frac{i}{\Omega} \mid 0 \leq i \leq \Omega \} / \approx \) with an inaccessible number \( \Omega \). See §0.4.4 for more discussions on continuum.

**Functions not as arbitrary mappings.** A function on a proper class must be given by an explicit objective specification. However functions on sets are precisely the usual arbitrary mappings since every map has an explicit specification as a finite table. A function on a semiset can be extended to a mapping defined on a set including \( D \). For example a sequence defined on the accessible numbers is uniquely extended up to a certain inaccessible number.

0.4 Background

We augment the above position by examining key differences between Cantorian infinity and “Robinsonian infinity”.

0.4.1 Qualitative Plurality of Numbers

“The infinite set \( \mathbb{N} \)” has brought phenomenal evolution of mathematics by its boundless productivity. However it still remains a pure dogma, without any supporting mathematical phenomena. On the contrary, there have been found many mathematical observations against it such as Skolem theorem and Gödel’s incompleteness theorem signifying respectively ontological and epistemological indefiniteness of the collection of natural numbers. As a
result various disbelief in “the infinite set $\mathbb{N}$” has never vanished and quite a few mathematicians have stated strong views against it.

Perhaps one of the earliest positive criticism against it is stated by E. Borel in [Bor52] where he pointed out the potential productivity of taking accessibility into account as follows.

Il me semble que les mathématiciens, tout en conservant le droit d’élaborer des théories abstraites déduites d’axiomes arbitraires non contradictoires, ont intérêt, eux aussi, à distinguer, parmi les êtres de raison qui sont la substance de leur science, ceux qui sont véritablement accessibles, c’est-à-dire ont une individualité, une personnalité qui les distingue sans équivoque; on est ainsi conduit à définir avec précision une science de l’accessible et du réel, au delà de laquelle il reste possible de développer une science de l’imaginaire et de l’imaginé, ces deux sciences pouvant, dans certains cas, se prêter un appui mutuel.

Around 1960, E. Volpin [Vol70] stated the radical view of the multitude-ness of natural number series which has given various impetus to explore alternative mathematics freed from the dogma of “the infinite set $\mathbb{N}$”. An example is the seminal paper of R. Parikh [Par71] which showed several paradoxical consequences of the $\mathbb{N}$-dogma and suggested the importance of taking the notion of “feasibility” into account in mathematics.

Does the Bernays’ number $67^{257^{729}}$ actually belong to every set which contains 0 and is closed under the successor function? The conventional answer is yes but we have seen that there is a very large element of fantasy in conventional mathematics which one may accept if one finds it pleasant, but which one could equally sensibly (perhaps more sensibly) reject.

Another example is an outline [Ras73] by P.K. Rashevsky of radically different type of mathematical theory on numbers as follows.

What would correspond more to the spirit of physics would be a mathematical theory of the integers in which numbers, when they became very large, would acquire, in some sense, a “blurred”

---

6 “It seems mathematicians have interests in distinguishing really accessible objects, namely, those which have individuality with personality distinguishing them clearly from others, among the intellectual objects which constitute the substance of their discipline, keeping of course the right to elaborate the abstract theories deduced from arbitrary consistent axioms. Thus one can precisely define a science of accessibility and reality from which it is possible to develop a science of imagination and imagined objects, and in certain cases these two sciences can support each other.”
form and would not be strictly defined members of the sequence of natural numbers as we consider it. The existing theory is, so to speak, over-accurate: adding unity changes the number, but what does the addition of one molecule to the gas in a container change for the physicist? If we agree to accept these considerations even as a remote hint of the possibility of a new type of mathematical theory, then first and foremost, in this theory one would have to give up the idea that any term of the sequence of natural numbers is obtained by the successive addition of unity - an idea which is not, of course, formulated literally in the existing theory, but which is provoked indirectly by the principle of mathematical induction. It is probable that for “very large” numbers, the addition of unity should not, in general, change them (the objection that by successively adding unity it is possible to add on any number is not quoted, by force of what has been said above).

See [Isl80], [May00], [Saz95] for similar views.

Around the same period, although not directly connected with the above tide, A. Robinson [Rob66] created nonstandard analysis, which took advantage of a mathematical phenomenon conflicting with the N-dogma. As is often quoted, he comments on the last page of his book [Rob66]:

Returning now to the theory of this book, we observe that it is presented, naturally, within the framework of contemporary Mathematics, and thus appears to affirm the existence of all sorts of infinitary entities. However, from a formalist point of view we may look at our theory syntactically and may consider that what we have done is to introduce new deductive procedures rather than new mathematical entities. Whatever our outlook and in spite of Leibniz’ position, it appears to us today that the infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers.

Our main purpose is to give another support to the position that “the existence of all sorts of infinitary entities” is not indispensable for nonstandard mathematics. We try to show this by the strategy of developing core mathematics without infinite set theory taking the multitudeness of finiteness as the very basic axiom considered as more reliable than that of its uniqueness.

S. Yatabe observes in [Yat09] that sorites phenomena is unavoidable for models of natural numbers in set theories in a non-classical logic.
Quotations  The followings are quotations from authors who take the position that nonstandard mathematics is a genuine alternative way of handling infinity and infinitesimals.

P. Vopěnka [Vop79] wrote in 1976

Cantor set theory is responsible for this detrimental growth of mathematics; on the other hand, it imposed limits for mathematics that cannot be surpassed easily. All structures studied by mathematics are a priori completed and rigid, and the mathematician’s role is merely that of an observer describing them. This is why mathematicians are so helpless in grasping essentially inexact things such as realizability, the relation of continuous and discrete, and so on.

In 1991 [Vop91], he analyzed philosophically the Cantorian set theory and called its infinity as “classical” and introduced the concept of “natural infinity” to capture the aspect of infinity present already in huge finite sets emerging from the “horizon” which bounds our “view”, and write

Even classical mathematics then studies natural infinity; however, it does so inappropriately. Classical mathematics is restricted by the accepted limitations, mainly by those inflicted on the horizon. The acceptance of the hypothesis that the sharpening process can lead to a complete sharpening does not extend the field of our study but rather to the contrary, restricts it. The study of situations where the sharpening process itself is essential is thus completely blocked. To put it briefly, the laws that govern classical infinity are nothing more than a drastic restriction of the laws that govern natural infinity.

Incidentally the following remark in [Vop91] on the nature of the “horizon” seems helpful to understand the main idea behind the concept of semisets.

The following three characteristics of the horizon are now important for our theme. Firstly, we do not understand the horizon as the boundary of the world, but as a boundary of our view. So the world continues even beyond the horizon. Secondly, the horizon is not some line drawn and fixed in the world but it moves depending on the view in question, specifically on the degree of its sharpness. The further we manage to push the horizon, the sharper the view. Thirdly, for a phenomenon situated in front of the horizon, the closer it is to the horizon, the less definite it is.

G. Reeb [Ree81] wrote in 1981
Donc, contrairement à une légende, il ne s’agit pas du tout de compléter $\mathbb{N}$, par l’adjonction d’objets nouveaux, en un ensemble plus large $\mathbb{N}^*$; mais il s’agit de reconnaître que seulement quelques objets privilégiés de $\mathbb{N}$, en particulier $0, 1, 2, 3, 4$ etc., méritent le label standard.

In 1983, J.Harthong [Har83] wrote

Je voudrais montrer dans cette communication que ..., si on admet que les entiers naïfs ne remplissent pas $\mathbb{N}$, la seule théorie des ensembles finis suffit à rendre compte de toutes les propriétés du continu, et il est inutile de recourir à des ensembles non dénombrables.

In 1985, A.G.Dragalin [Dra85] points out the inconsistency of feasibility can be tamed by taking into account the qualitative difference of length of proofs.

We investigate theories with notions “infinitely large” and respectively “feasible” numbers of various orders. These notions are inconsistent in a certain sense, so our theories turn out to be inconsistent in an exact sense. Nevertheless, we show that by the short proofs in these theories we get true formulas.

In 1996, R. Chuaqui and P.Suppes [CS95] consider it important to ignore the standard part operation.

To reflect the features mentioned above that are characteristic of works in theoretical physics, the foundational approach we develop here has the following properties:

(i) The formulation of the axioms is essentially a free-variable one with no use of quantifiers.

(ii) We use infinitesimals in an elementary way drawn from non-standard analysis, but the account here is axiomatically self-contained and deliberately elementary in spirit.

(iii) Theorems are left only in approximate form; that is, strict equalities and inequalities are replaced by approximate equalities and inequalities. In particular, we use neither the notion of

---

8 “Therefore, contrary to the legend, it is not the question of augmenting $\mathbb{N}$ by adding new objects to a larger set $\mathbb{N}^*$ but it is only the matter of recognizing that some privileged elements of $\mathbb{N}$, in particular $0, 1, 2, 3, 4$ etc. are entitled to be labeled standard.”

9 “I would like to show in this communication that if the naive integers do not fill $\mathbb{N}$ then only the finite set theory suffices to treat all the properties of continuum and it is not to necessary to have recourse to uncountable sets.”
standard function nor the standard part function. Such approximations are not explicit in physics, but can be viewed as implicit in the way infinitesimals are used.

In 2005, Y. Pétraire [Pé05] pointed out that nonstandard analysis made it possible to express indefiniteness in mathematics.

The recent history of nonstandard mathematics is displayed so as to reveal a modification in the used language as well that in the way the referentiation of the statements is done. These changes could lead to bring the mathematical language closer to a language of communication. The profusion of constructions of sets can be limited thanks to a little richer vocabulary making it possible to express the indetermination, indiscernibility, inaccessibility . . . when it is necessary, and permit also to explore more precisely with the mathematical language, using a sort of translation of the ordinary language, some concepts about which the language of conventional mathematics is almost dumb such as concepts of point, infinity or infinitesimality.

In 2006, Hrbaček et al. [HLO10b] also recognizes the key point of nonstandard mathematics is to incorporate vague concepts with “soritic properties” into mathematics and write as follows.

There are many examples of “soritic properties” for which mathematical induction does not hold (“number of grains in a heap”, “number that can be written down with pencil and paper in decimal notation”, “macroscopic number”, ... ), but mathematicians traditionally take no account of them in their theories, with the excuse that such properties are vague. We present here a mathematically rigorous theory in which a soritic property is put to constructive use.

0.4.2 Properties without Extension

The above quotations may be said to point out the essence of nonstandard mathematics consists in the positive usage of indefiniteness in mathematics, which means the rejection of the monism of sets in modern mathematics. How is it possible to treat conditions without definite extensions?

Surely modern mathematics do not exclude conditions because it is without extension. For example the condition $x \notin x$ is not considered as nonsense even though we cannot consider its extension. In fact, from purely formalistic points of view, a “vague” concept has no difference from the
usual ones provided the rule of its usage is precisely given. In fact in the
axiomatic formulation of nonstandard mathematics such as the internal set
theory [Nel04], the rules of the usage of the word “standard” is precisely
given among which is the prohibition to consider its extension. It might
be said that we have already enough experience about reasoning coherently
with conditions without extensions at least formally.

However in order to “do mathematics” actually, purely formalistic posi-
tion is not helpful and it is beneficial if even vague concepts have certain
kind of extensions so that they have “set theoretical” meaning. It is P.
Vopěnka [Vop79] who found the notion of semisets which disclosed essential
difference of nature between sets and “external sets” often used informally.

We can not only coherently and naively develop an alternative mathe-
ematics admitting properties without extension but also enjoy its advantage
over usual mathematics since we can treat infinitary concepts and continuum
more naturally by keeping their indefiniteness. In [Vop79 Vop91], P.
Vopěnka points out that infinite sets are not necessary to treat infinitary
phenomena. He also points out the merit of his alternative set theory
which allows new kind of natural concepts which are not available in usual mathematics.

0.4.3 Coherence of Vague Concepts

We do not take the ultrafinitistic standpoint and admit the existence of
inaccessible numbers. Just as infinite sets, huge numbers are ideal objects,
but, in contrast to Cantorian infinity, huge finiteness is philosophically less
problematic and intuitively more in harmony with naive concepts of infinite
quantities.

---

10 “We shall deal with the phenomenon of infinity in accordance with our experience,
i.e., as a phenomenon involved in the observation of large, incomprehensible sets. We shall
be no means use any ideas of actually infinite sets. Let us note that by eliminating actually
infinite sets we do not deprive mathematics of the possibility of describing actually infinite
sets sufficiently well in the case that they would prove to be useful.”
11 “Our theory makes possible a natural mathematical treatment of notions that either
have not yet been defined mathematically or that have been defined in a unsatisfactory
way. As an example we have here the chapter dealing with motion.”
12 According to R. Tragesser [Tra98], ultrafinitistic aim is not to restrict mathematics
to concrete objects but to reconstruct the idealization of mathematics properly. In this
sense, our program might be called ultrafinitistic.
13 This view is supported by educational studies of university mathematics. For example
J.Monaghan [Mon01] says as follows.

Cantor’s transfinite universe became the infinite paradigm during the 20th
Century. This affected educational studies, which tended to view
children’s responses against Cantorian ideas. Robinson’s non-standard universe
(Robinson, 1966) is equally authoritative (though not as well known) and it
is a different paradigm. It offers researchers a release from a single paradigm
However the concepts of accessibility and hence that of hugeness interpreted as inaccessibility are vague. Since Frege, vague concepts have been considered as useless in mathematics because of various incoherences associated to them. For example, M. Dummett [Dum75] argues for this Frege’s position that use of vague expressions is fundamentally incoherent and concludes as follows.

Let us review the conclusions we have established so far.

(1) Where non-distinguishable difference is non-transitive, observational predicates are necessarily vague.

(2) Moreover, in this case, the use of such predicates is intrinsically inconsistent.

(3) Wang’s paradox merely reflects this inconsistency. What is in error is not the principles of reasoning involved, nor, as on our earlier diagnosis, the induction step. The induction step is correct, according to the rules of use governing vague predicates such as ‘small’: but these rules are themselves inconsistent, and hence the paradox. Our earlier model for the logic of vague expressions thus becomes useless: there can be no coherent such logic.

(4) The weakly infinite totalities which must underlie any strict finitist reconstruction of mathematics must be taken as seriously as the vague predicates of which they are defined to be the extensions. If conclusion [2], that vague predicates of this kind are fundamentally incoherent, is rejected, then the conception of a weakly infinite but weakly finite totality must be accepted as legitimate. However, on the strength of conclusion [2], weakly infinite totalities may likewise be rejected as spurious: this of course entails the repudiation of strict finitism as a viable philosophy of mathematics.

He identifies the condition of transitivity

\[ a \approx b \approx c \text{ implies } a \approx c \]

with the multiple transitivity

\[ a_1 \approx a_2 \approx a_3 \approx \cdots \approx a_n \text{ implies } a_1 \approx a_n. \quad (1) \]

This identification is based on the tacit assumption that the notion of natural number is uniquely determined, which is precisely the ultrafinitistic position and allows them to interpret children’s ideas with reference to children’s ideas instead of with reference to Cantorian ideas.
doubts. When there are two kinds of natural numbers, for example feasible and unfeasible ones, it is possible to define the weak transitive relations for which the multiple transitivity holds only for feasible $n$. Hence the conclusions and are untenable if the assumption of the qualitative uniqueness of finiteness is abandoned, which opens the possibility to use weakly transitive relations consistently. Namely, weak transitivity of non-distinguishable difference turns out to be one of the corner stone of the new approach to continuum developed here.

As for the conclusion, the key arguments against the skepticism about induction is as follows. Assume the ultrafinitistic position that a proof is legitimate only when the totality of the inferences is survayable. A number $n$ is called apodictic if a proof, without induction principle, of length less than or equal to $n$ is legitimate as a proof from ultrafinitistic standpoint. Then the condition of being apodictic is inductive in the sense that 0 is apodictic and if $n$ is apodictic then $n + 1$ is apodictic. Moreover a number less that an apodictic number is also apodictic. If a condition $F$ is inductive then $F(n)$ is true whenever $n$ is apodictic since there is the obvious proof of $F(n)$ consisting of $n$ lines of modus ponen. Now choose an apodictic number $k$ and define the condition $S(n)$ to be $n + k$ is apodictic. Then $S$ is obviously inductive. Suppose there are an apodictic number $n$ such that $n + k$ is not apodictic. Then $S(n)$ is false by definition but since $S$ is inductive $S(n)$ is true, a contradiction. Hence he concludes that the arguments against the induction principle is not tenable and also implicitly that the notion of apodictic is incoherent and hence the ultrafinitistic standpoint is incoherent.

However the contradiction comes from the assumption that there are two apodictic numbers $k, n$ such that $n + k$ is not apodictic. However this is based on the tacit assumption that there are no nontrivial inductive properties of numbers closed under the addition which ultrafinitistic position doubts. Since not only the induction remains problematic but also there is coherent usage of “non-transitive non-distinguishable difference” the conclusion is untenable.

Since conclusion is misleading, so is the conclusion. See for similar criticism against Dummett’s arguments.

Thus Dummett’s arguments against not only to ultrafinitism but also to any alternative mathematics which use vague concepts is essentially grounded on the basic assumption of modern mathematics that there is unique concept of natural numbers, which is exactly the alternative approach in this paper negates.

In fact, the secret of effectiveness of nonstandard analysis might be pin downed to the vague concept “standard” which forbids formation of the set of standard elements.
0.4.4 Continuum

The infinite sets are considered indispensable to modern mathematics since the continua are infinite sets. For example the interval, the simplest continuum, is identified with “a set of real numbers between 0 and 1” which have more elements than “the set of natural numbers”. However historically this atomic view regarding continuum as a mere aggregation of its points has been criticized repeatedly from various points of view since ancient times to today.

H. Weyl gives in 1921 an overview\(^\text{14}\) of the two opposing approach to continuum, culminating respectively to Cantorian set theoretic approach and Brouwer’s intuitionistic approach. He did not satisfied with the atomic approach to continuum of his book \([\text{Wey94}]\) published in 1917 and recognized the need to reconstruct it radically according to his philosophy, but he regrets in the “preface to the 1932 Reprint”\(^\text{15}\) that he has no time to undertake it \([\text{Wey94}]\).

Now that topology has become one of the major disciplines of mathematics, there seems to be quite a few mathematicians who, independently of the antagonism between classical logic vs intuitionistic one, consider continua as primitive objects. For example R. Thom amplifies the claim that continuum ontologically precedes discrete objects in \([\text{Tho92}]\).\(^\text{16}\)

\[\text{Ici, je voudrais m'attaquer à un mythe profondément ancré dans}\]

\(^{14}\) “An atomistic view, taking the continuum to consist of individual points, and a view that takes it to be impossible to understand the continuum flux in this manner, have been opposing each other from time immemorial. The atomistic one has a system of existing elements that can be conceptually grasped, but it is incapable of explaining motion and action. In it, all change must degenerate into appearance. The second conception was not capable, in antiquity, and up to the time of Galilei, to lift itself from the sphere of vague intuition to the one of abstract concepts that would be suitable for a rational analysis of reality. The solution that was finally achieved is the one whose mathematical systematic form is given in the differential and integral calculus. Modern criticism of analysis is destroying this solution from within, however, without being particularly conscious of the old philosophical problems, and it lead to chaos and nonsense. The two rescue attempts discussed here revive the old antithesis in a sharper and more clarified form. The previously described theory is radically atomistic([I am saying this] in full awareness of the fact that, as it is, this theory does not fully capture the intuitive continuum, the idea being that the concepts are capable of grasping only rigid existence.) Brouwer’s theory, on the other hand, undertakes to do justice to Becoming in a valid and tenable manner. \([\text{Man98}]\)"

\(^{15}\) “It seems not to be out of the question that the limitation prescribed in the present treaties– i.e., unrestricted application of the concepts “existence” and “universality” to the natural numbers, but not to sequences of natural numbers– can once again be of fundamental significance. It would not be possible, without radical rebuilding, to bring the content of this monograph into harmony with my current beliefs – and such a project would keep me from satisfying other demands on my time.”

\(^{16}\) “It might be said that such viewpoints is reflected for example in the computational approach to topology such as \([\text{RS10}]\)."
la mathématique contemporaine, à savoir que le continu s'engendre (voire se définit) à partir de la générativité de l’arithmétique, celle de la suite des entiers naturels. Je fais bien entendu allusion à la construction de Dedekind où \( \mathbb{R} \) se définit par complétion des coupures définies sur les rationnels. J’estime, au contraire, que le continu archétypique est un espace ayant la propriété d’une homogénéité qualitative parfaite. \(^{17}\)

Our intention is to develop mathematics which takes as primitive both the discrete and the continuous based on the plurality of finiteness. This might be said to conform with the viewpoint of Brouwer who stated as follows in his dissertation 1907 according to \(^{vS90}\).

Since in the Primordial Intuition the continuous and the discrete appear as inseparable complements, each with equal rights and equally clear, it is impossible to avoid one as a primitive entity and construct it from the other, posited as the independent primitive.

However, H. Weyl was disappointed with the Brouwer’s approach in which it is awkward to carry out usual mathematics as is remarked\(^{18}\) in his book \(^{Wey49}\) published in 1949.

The success of nonstandard mathematics suggests high feasibility of this approach.

We regard intervals as primitive objects and the basic operation is to divide them to subintervals which are similar in nature to the total interval. This fractal nature is the essential feature of continua. Dividing the subintervals again and again, we get many small intervals with many points which bounds them. Although we can divide only concrete number of times, the division process can be continued to huge number of times in principle. Thus we can imagine a set of huge finite number of infinitesimal intervals each of which we cannot discriminate from the neighboring ones. Moreover since the intervals obtained are so small that each interval determines uniquely

\(^{17}\)“Here I would like to attack a myth deeply anchored in modern mathematics which says that continuum is obtained from the generative feature of the arithmetics and the series of natural numbers. Of course I am referring to Dedekind construction which defines \( \mathbb{R} \) by completion using the cuts on rationals. I consider on the contrary that archetype of continuum is a space with qualitatively complete homogeneity.”

\(^{18}\) “Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the larger part of his towering edifice which he believed to be built of concrete blocks dissolve into mist before his eyes.”
a position up to indistinguishability on the interval. The subintervals are
infinitesimal but are not identified with the positions they determine. They
are themselves continua which have the same property as the initial interval.

So we arrive at an imaginary picture that intervals are composed of huge
number of infinitesimal intervals, which themselves can be divided indefi-
nitely into smaller intervals. The result of a huge number of division can
be described by the huge finite set of the rationals coding the positions of
the boundaries of the resulting infinitesimal intervals. This huge finite set,
called a representation of the interval, has an indistinguishability binary re-
lation satisfying the usual axiom of equivalence relation but the huge chain
of indistinguishable elements connects distinguishable elements and there
arises the sorites paradox. So we define a rigid mesh continuum as a huge fi-
nite set equipped with such a paradoxical equivalence relation, called sorites
relation, which exists by virtue of Axiom 2.

Thus continuum as a primitive entity should be represented as an “equiv-
alanse class” of rigid mesh continua, but we use more handy formulation,
for example, of the linear continuum $\mathbb{R}$ as the proper class $\mathbb{Q}$ of rational
numbers equipped with the indistinguishability relation. Many continua
such as the real line and various types of intervals are given as subcontinua
defined by possibly vague conditions.

Besides the Euclidean continua, a large class of continua is provided by
metric spaces with rational distance functions by defining the indistinguisha-
bility $x \approx y$ as $d(x, y) \approx 0$. Symmetric graphs with infinitesimal positive
distances given to edges form a rich subclass of metric continua. This con-
struction given for the first time by L. van den Dries and A. J. Wilkie
[vdDW84] plays vital roles in their proof of the Gromov’s theorem on groups
of polynomial growth using nonstandard method. We note that Urysohn
space [Ver98] can be regarded as a “universal continuum” which includes all
metric continua as subcontinua.

0.5 Outline of Contents

In Section 1 we explain fundamentals in a naive way to emphasize the
approach is more easily assimilated than that of infinite set theory. Then we
treat directly “continua” represented as a “quotient” of finite sets by weak
equivalent relation of indistinguishability in Section 2. Usual topological
concepts are reformulated by the indistinguishability relation in Section 3.

---

19 A rational $r$ is infinitesimal and written $r \approx 0$ if $|r| < \frac{1}{k}$ for every accessible number $k$
and two rational numbers are considered indistinguishable if their difference is infinitesimal.

20 For the real line, the condition is the finiteness, namely the absolute value is less than
an accessible number. For the open intervals such as $(0, 1)$, the condition is $0 \prec x \prec 1
where x \prec y$ means that that $y$ is visibly greater than $x$. 
The following two sections discuss concrete examples of continua. Section 4 treats the continua arising from the finite sets of the 01-words of an inaccessible fixed length endowed with various distances, which demonstrates the drastic increase of freedom of construction in the new approach. Section 5 investigates the continuum of morphisms and show the Ascoli-Arzela Theorem, with the purpose to demonstrate how our framework can treat function spaces.

As a special case of continua, we treat “real numbers” as rational numbers under the weak equivalence relation of indistinguishability in Section 6. Section 7 treats real valued functions and proves the mean value theorem and the maximum principle. The exponential functions is treated just like in the Euler’s way.

The calculus of one variable and multiple variables are treated respectively in Sections 8 and 9. A new feature is that the differentiability of a function controls its behavior only on large infinitesimals and the behavior on tiny infinitesimal neighborhood can be taken rather arbitrarily, which seems to open new freedom to represent functions. The integration is treated in a way similar to Loeb measure in Section 10.
1 Fundamentals

1.1 Numbers

We assume usual elementary arithmetic taught up to high school. For examples, we have natural numbers 0, 1, 2, 3, · · · sometimes simply called numbers, and the integers 0, ±1, ±2, ±3, · · · with the addition and multiplication satisfying the axiom of rings. We have also the rational numbers ±\frac{p}{q} with natural numbers p, q ≠ 0 with the addition and multiplication satisfying the axiom of field.

1.1.1 Accessibility

A number is called accessible if it can be actually accessed somehow. For example, numbers which can be written by some notation is accessible. Since such a naive concept of accessibility has inevitable vagueness, we use it as an undefined terminology obeying rigorously the axioms which reflects naive meaning of accessibility\[^{21}\].

Axiom 1 (Accessible Numbers) 1. The numbers 0 and 1 are accessible.

2. The sum and product of two accessible numbers are accessible.

3. Every number less than an accessible number is accessible\[^{22}\].

The following reflects the intuition that there are inaccessible numbers.

Axiom 2 (Sorites Axiom) There are numbers which are not accessible.

By Axiom 1, the number 0 is accessible and if n is accessible then n + 1 is accessible, whence every numbers are accessible if the unrestricted induction principle is applied, contradicting to Axiom 2. This is one version of the sorites paradox. We weaken in §1.3 the induction principle in order to use the concept accessibility coherently.

If a natural number n is accessible, we say n is finite and write n < ∞. If a natural number n is not accessible, we say n is huge and write n ≫ 1. An integer is called accessible if its absolute value is accessible. We call a rational number r is bounded from above and write r < ∞ if there is an

\[^{21}\] We remark that the concept of “accessible numbers” is semantically vague but just as vague as that of “numbers” and less vague than that of “infinite sets”.

\[^{22}\] Hence numbers less than a big numbers such as 10^{10} are considered to be accessible although most of them cannot be written explicitly.
accessible number \( n \) with \( r < n \) and \( r \) is \textit{bounded from below} and write \(-\infty < r \) if \(-r < \infty \). A rational number is called \textit{finite} if its absolute value is bounded from above.

We say a rational number is \textit{accessible} if it is written as \( \pm \frac{p}{q} \) with accessible \( p, q \). An accessible rational number is finite but the converse is not true. For example the rational number \( \frac{1}{N} \) with \( N \gg 1 \) is finite but is not accessible.

### 1.1.2 Rational Numbers

We call a rational number \( r \) \textit{infinitesimal} and write \( r \approx 0 \) if \( |r| < \frac{1}{k} \) for all accessible number \( k \). We say two rational numbers \( r, s \) are \textit{indistinguishable} and write \( r \approx s \) if \( r - s \) is infinitesimal.

We remark that the assertion “for all accessible number \( k \) the condition \( P(k) \) is true” means that there is a proof of the assertion \( P(k) \) with parameter \( k \) which do not use peculiarity of \( k \). See §1.2.2 for more elucidation about this.

Axiom 2 implies

**Proposition 1.1.1** There are nonzero infinitesimal rational numbers.

**Proof.** Let \( r = \frac{1}{N} \) with \( N \gg 1 \). Let \( k \) be an accessible number. Then \( k < N \) whence \( kr < 1 \). Hence \( r \) is an infinitesimal but nonzero rational number. \( \qed \)

For rational numbers \( r, s \), we write \( r < s \) and say that \( r \) is \textit{visibly smaller than} \( s \) if there is an accessible number \( k \) satisfying \( r + \frac{1}{k} < s \). We write \( r \preceq s \) if \( r < s \) or \( r \approx s \).

Note that \( r \leq s \) implies \( r \preceq s \) but the other implication is generally false and \( r < s \) implies \( r \preceq s \) but the other implication is generally false. In fact if \( \varepsilon \) is a positive infinitesimal, we have \( r \leq r - \varepsilon \) and \( r \not< r + \varepsilon \).

Obviously we have

**Proposition 1.1.2**

(1) \( r < s \) satisfies the transitivity.

(2) The conditions \( r < s, r \approx s, s \prec r \) are mutually exclusive and just one of them is valid.

(3) The relation \( \prec \) is \( \approx \)-congruent, namely, if \( r \prec s, r \approx r' \) and \( s \approx s' \) then \( r' \prec s' \).

(4) The relation \( \preceq \) is \( \approx \)-congruent, namely, if \( r \preceq s, r \approx r' \) and \( s \approx s' \) then \( r' \preceq s' \).

(5) If \( 0 < r, s \) then \( 0 \prec r + s, rs \).

(6) If \( 0 < r, s_1 \prec s_2 \) then \( rs_1 \prec rs_2 \).
1.2 Sets and Classes

1.2.1 Basic Concepts

A collection of objects is called a \textit{class} if its elements have distinctiveness, namely, given two objects \(x, y\) qualified as its elements, it is possible to determine \(x = y\) or \(x \neq y\). If an object \(x\) belongs to a class \(X\), we write \(x \in X\).

A \textit{set} is a class \(X\) with enumeration \(X = \{x_1, x_2, \cdots, x_n\}\) for some number \(n\). An enumeration of a finite set without repetition is called a \textit{tight enumeration}. The number of elements of a set \(A\) is denoted by \(\#(A)\).

We take usual naive set theory for granted with the exception of those concepts and propositions referring to infinite sets.

A class which is not a set is called a \textit{proper class}. For example, the collections \(\mathbb{N}\), \(\mathbb{Z}\) and \(\mathbb{Q}\) respectively of natural numbers, integers and rational numbers are proper classes.

Let \(X\) and \(Y\) be classes. We say they are equal and write \(X = Y\) if and only if we can prove that every object belongs to \(X\) if and only if it belongs to \(Y\). We say \(X\) is different from \(Y\) and write \(X \neq Y\) if and only if we can find an object \(x\) either satisfying \(x \in X\) and \(x \notin Y\) or satisfying \(x \notin X\) and \(x \in Y\). Hence it is not logically evident that either \(X = Y\) or \(X \neq Y\) holds. Hence classes have no distinctiveness so that the collection of classes do not form a class.

1.2.2 Subclasses and Subsets

A class \(Y\) is a \textit{subclass} of a class \(X\) written as \(Y \subset X\) if every element of \(Y\) is also an element of \(X\).

A subset of a class \(X\) is a set with elements in \(X\).

\textbf{Bounded Conditions} We say a quantification is \textit{bounded} if it is either \(\forall x \in a\) or \(\exists x \in a\) with \(a\) being a set. A condition is called \textit{definite}\footnote{Usually called \(\Delta_0\)-conditions.} if it has only bounded quantification. Since sets can be exhausted, a definite condition \(P\) has semantically definite truth value and either \(P\) is true or \(P\) is false. A condition on the class \(\mathbb{N}\) is bounded precisely when the quantifications are of the form \(\forall x \leq n\) or \(\exists x \leq n\).

If \(X\) is a proper class, the truth value of an unbounded condition such as \(\forall x \in X.P(x)\) or \(\exists x \in X.P(x)\) cannot be determined semantically, namely, by evaluating the truth value of \(P(x)\) for each \(x \in X\) since a proper class
cannot be exhausted by any procedures. So we adopt the proof-theoretic interpretation that “∀x ∈ X. P(x) is true” means that the assertion P(a) with the parameter a has a proof which is independent of the parameter a, and “it is false” means that the assumption that every x ∈ X satisfies P(x) implies a contradiction. For example if we have found an a for which P(a) is false, it is false. Similarly “∃x ∈ X. P(x) is true” means that we have constructed an object a satisfying P(a) and “it is false” means that the existence of an object a such that P(a) implies a contradiction.

Note that the condition Y ⊂ X is not definite if Y is a proper class. Moreover for two proper subclasses Y_i ⊂ X (i = 1, 2), the condition Y_1 = Y_2 is not definite. Hence the collection of proper subclasses of X is not a class. It will turn out that the collection of subsets of X is a class when X is σ-finite in the sense defined in §1.2.6.

**Power Set** The collection of subsets of a set forms a set as follows. Let A be a set with a tight enumeration \{a_1, \cdots, a_n\}. An integer k ∈ [1..2^n] defines a set S_k ⊂ A by

\[ S_k := \{ a_i \mid \text{the binary expansion of } k - 1 \text{ has } 1 \text{ on the } i - 1\text{-th position} \}. \]

Conversely, for each B ⊂ A, define k = ∑_{a_i ∈ A} 2^{i-1} + 1. Then S_k = B.

Hence the subsets of a set A defines the power set pow(A) with the explicit enumeration \{ S_k \mid k ∈ [1..2^n] \}. We show in §1.2.4 the subsets of a σ-finite class form a σ-finite class.

**1.2.3 Objective Conditions and Semisets**

A condition is called **objective** if it is specified independently of the concept of accessibility. An **objective subclass** is a subclass defined by an objective definite condition.

**Remark 1.2.1** If proper subclasses A and B are not objective, then the equality condition A = B is not definite and the collection of subclasses of a class is not a class generally. However the collection of objective subclasses form a class since the equality condition of objective subclasses is definite.

A subclass of a set is called a **semiset**. A semiset which is not a set is called a **proper semiset**. We write A ⊏ x if A is a semiset included in a set x. A set including a proper semiset is called an **environment set** of it. Note that the intersection of two environment sets is also an environment set.

**Axiom 3 (Objective seperation)** An objective semiset is a set.
The class $\mathbb{N}_{\text{acc}}$ of accessible numbers is a proper class and hence is a proper semiset. Generally a proper semiset present itself only when the defining condition depends on accessibility explicitly or implicitly. Thus proper semisets play vital roles in the mathematical treatment of vague concepts such as accessibility.

The proper semisets plays in our theory the similar role as is played by the infinite sets in usual mathematics. The following is the key tool in the arguments of the proper semisets.

**Theorem 1.2.1 (General Overspill Principle)** Let $A$ be a proper semiset of a set $X$. Suppose every element of $A$ satisfies a definite objective condition $P$ on $X$. Then there is an $x \in X \setminus A$ satisfying $P$.

**Proof.** Axiom 3 implies that $B = \{x \in X \mid P(x)\}$ is a subset which includes $A$. Since $A$ is not a set, the class $B \setminus A$ must be nonempty.

### 1.2.4 Class Constructions

If $A, B$ are subclasses of $X$ defined respectively by definite conditions $P_A, P_B$, then usual Boolean operations

$$A \cap B, A \cup B, A \setminus B$$

are defined respectively by the definite conditions “$P_A$ and $P_B$”, “$P_A$ or $P_B$” and “$P_A$ but not $P_B$” and obeys usual algebraic laws of Boolean operations.

If $A_i (i \in [1..n])$ are subclasses of a class $X$, then subclasses $\bigcup_{1 \leq i \leq n} A_i$ and $\bigcap_{1 \leq i \leq n} A_i$ of $X$ are defined respectively by the definite conditions $\forall i \leq n.x \in A_i$ and $\exists i \leq n.x \in A_i$.

If $X_i (i = 1, 2)$ are classes, the product class $X_1 \times X_2$ is defined as the collection of the ordered pair $\langle x_1, x_2 \rangle$ of $x_i \in X_i (i = 1, 2)$. The coproduct class $X_1 \coprod X_2$ is defined as the collection of $\langle i, x_i \rangle$ with $x_i \in X_i (i = 1, 2)$ with the canonical inclusions $i_i : X_i \to X_1 \coprod X_2 (i = 1, 2)$ defined by $i_i(x_i) = (i, x_i) (i = 1, 2)$.

If there is a rule to define a class $A_n$ for each $n \in \mathbb{N}$ such that $A_n \subset A_{n+1}$ for all $n$, we say \{ $A_n \mid n \in \mathbb{N}$ \} is an increasing family of classes. Then the union class $\bigcup_{n \in \mathbb{N}} A_n$ is defined as the collection of the elements of some $A_n$. There is a function $\text{rank} : \bigcup_{n \in \mathbb{N}} A_n \to \mathbb{N}$ defined by $\text{rank}(x) = \min \{ k \leq m \mid x \in A_k \}$ for $x \in A_m$, which satisfies $x \in A_{\text{rank}(x)}$.

### 1.2.5 Functions

Let $X, Y$ be classes and suppose $f$ is a correspondence which assigns $x \in X$ to $f(x) \in Y$ by an objective definite rule $R_f$. Here a rule is called objective
if it is specified without recourse to the concept of accessibility and is called
definite if the specification does not involve unbounded quantification. We
say then that \( f \) is a function from \( X \) to \( Y \) and write \( f : X \rightarrow Y \).

Two functions \( f, g : A \rightarrow B \) are called equal and written \( f = g \) if
\[
\forall x \in A. f(x) = g(x)
\]
is proved. We say \( f \neq g \) if we have found an \( a \in A \) such that \( f(a) \neq g(a) \).
Since it is not logically obvious that either \( f = g \) or \( f \neq g \) is true, functions
defined on proper classes have generally no distinctiveness and do not form a
class. However Proposition 1.2.4 below shows that the collection of functions
defined on a semiset forms a class.

**Axiom 4 (Extension Axiom)** If \( f : A \rightarrow Y \) is a function from a proper
semiset \( A \) to a set \( Y \), then there is an environment set \( b \) and a function
\( g : b \rightarrow Y \) which coincides with \( f \) on \( A \).

A rationale of this axiom is as follows. Suppose \( x \) is an environment set of
\( A \). The condition on the elements of the set \( x \) that the defining condition
of \( f \) has meaning and determines an element of \( Y \) is objective and includes
\( A \), hence defines a set \( b \subset x \) such that \( A \sqsubseteq b \subset x \).

We call a class \( Y \) set-like if every function from a semiset to \( Y \) can be
extended to an environment set of \( Y \).

Two extensions coincides on an appropriate environment set.

**Proposition 1.2.2** Let \( f : A \rightarrow Y \) be a function from a proper semiset \( A \)
to a set-like class \( Y \) and \( f_i : a_i \rightarrow Y \) \( (i = 1, 2) \) be its extensions. Then
\( f_1 = f_2 \) on an environment set included in \( a_1 \cap a_2 \).

**Proof.** The objective condition \( f_1(x) = f_2(x) \) on the elements \( x \in a_1 \cap a_2 \)
is satisfied on \( A \), whence defines an environment set of \( A \).

Moreover we can choose extensions of family of functions so that their
domains coincides.

**Proposition 1.2.3** Let \( A \) be a proper semiset and
\[
f_i : A \rightarrow Y \quad i \in [1..n]
\]
a family of functions to a set-like class \( Y \). Then there is a set \( b \) and exten-
sions \( f_i \) \( (i \in [1..n]) \) with domains \( b \).

**Proof.** Just take any extensions of \( f_i \)'s and then restrict them to the intersection
of their domains.
Proposition 1.2.4 If $X_1$ is a semiset and $X_2$ is a set-like class. Then the collection of functions from $X_1$ to $X_2$ forms a class $\text{Fun}(X_1, X_2)$.

Proof. If $X_1$ is a set, the equality of two functions is obviously definite.

Suppose $X_1$ is a proper semiset. Let $f_i : X_1 \to X_2$ $(i = 1, 2)$ be functions. By Proposition 1.2.3 we can choose their extensions $\tilde{f}_i : b \to X_2$ $(i = 1, 2)$ with a common domain set $b$. Then the equality condition

$$\forall x \in X_1. f_1(x) = f_2(x),$$

which is unbounded since $X_1$ is proper, is equivalent to the bounded condition

$$\exists c \subset b. \forall x \in c. \tilde{f}_1(x) = \tilde{f}_2(x),$$

(2)

whence the indistinguishability is definite. Note that the validity of (2) is independent of the choice of the extensions.

Hence the functions from $X_1$ to $X_2$ forms a class $\text{Fun}(X_1, X_2)$.

Suppose $X, Y$ are sets with tight enumerations $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_m\}$. Each $k \in [1..m^n]$ defines a function $f_k : X \to Y$ by the rule $f_k(x_i) = y_j$ if and only if $j - 1$ is the number in the $i - 1$-th position of the $m$-ary expansion of $k - 1$. Obviously any function from $X$ to $Y$ is given as $f_k$ for some $k$, whence the collection of functions from $X$ to $Y$ forms a set $Y^X$ with explicit enumeration $\{f_1, \cdots, f_{\#(Y)^\#(X)}\}$.

Thus we can define a function from a set $X$ to a set $Y$ by choosing an arbitrary element of $Y$ for each $x \in X$. In particular we have the following choice principle.

We remark that if $f : X \to Y$ is a function between sets, then it induces functions on the power sets. Namely, if $A \subset X$, $B \subset Y$ are subsets, then

$$f(A) := \{f(a) \mid a \in A\} \subset Y,$$

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\} \subset X$$

are subsets. In particular, for $y \in Y$,

$$f^{-1}(y) := f^{-1}\{y\}$$

is a subset.

1.2.6 $\sigma$-finite Classes

The union of an increasing family of sets is called $\sigma$-finite. If the sequence $\{A_n\}$ is strictly increasing in the sense that $A_n \neq A_{n+1}$ for all $n$, then $\bigcup_{n \in \mathbb{N}} A_n$ is a proper class.
Two increasing sequences \( \{ A_n \mid n \in \mathbb{N} \} \) and \( \{ B_n \mid n \in \mathbb{N} \} \) are called equivalent if there are functions \( f, g : \mathbb{N} \to \mathbb{N} \) such that \( A_n \subset B_{f(n)} \) and \( B_n \subset A_{g(n)} \) hold for all \( n \). Obviously we have the following.

**Proposition 1.2.5** Let \( \{ A_n \mid n \in \mathbb{N} \} \) and \( \{ B_n \mid n \in \mathbb{N} \} \) by equivalent increasing sequences of sets. Then

\[
\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n.
\]

For a \( \sigma \)-finite class \( X \), an increasing sequence of sets \( \{ A_n \mid n \in \mathbb{N} \} \) with a function \( \rho : X \to \mathbb{N} \) called ranking satisfying \( x \in A_{\rho(x)} \) for all \( x \in X \) is called its representation. We write then \( X = \bigcup_{n \in \mathbb{N}} A_n \) with the implicit agreement of the existence of a ranking function.

For example, \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) are \( \sigma \)-finite classes with the following representations.

\[
\begin{align*}
\mathbb{N} &= \bigcup_{n \in \mathbb{N}} [0..n], \\
\mathbb{Z} &= \bigcup_{n \in \mathbb{N}} [-n..n], \\
\mathbb{Q} &= \bigcup_{n \in \mathbb{N}} \mathbb{Q}_n.
\end{align*}
\]

where

\[
\mathbb{Q}_n := \left\{ \frac{p}{q} \mid p, q \in [-n..n], q \neq 0 \right\} \subset \mathbb{Q}.
\]

If \( X = \bigcup_{n \in \mathbb{N}} X_n \) is a representation of a \( \sigma \)-finite class then every subset is contained in some \( X_n \). In fact if \( a \subset X \), then \( a \subset X_n \) with

\[
n := \max \{ \text{the rank of } u \mid u \in a \}.
\]

Hence, the subsets of \( X \) forms a \( \sigma \)-finite class \( \text{pow} (X) \) with a representation

\[
\text{pow} (X) = \bigcup_{n \in \mathbb{N}} \text{pow} (X_n).
\]

If \( X \) is a set and \( Y = \bigcup_{n} Y_n \) is a \( \sigma \)-finite class, then a function from \( X \) to \( Y \) is given by a map from \( X \) to \( Y_n \) for some \( n \) and hence the collection of functions from \( X \) to \( Y \) forms the \( \sigma \)-finite class \( Y^X \) with the representation \( Y^X = \bigcup_{n} Y_n^X \).

If a class \( X \) is \( \sigma \)-finite, the elements of \( X^n := X^{[1..n]} \) is called a sequence of length \( n \) in \( X \) and is written as \( (x_1, \cdots, x_n) \). Hence, we have \( \sigma \)-finite classes \( \mathbb{N}^n, \mathbb{Z}^n, \mathbb{Q}^n \) of sequences of length \( n \) for each \( n \).
Note that an objective subclass of a $\sigma$-finite class is $\sigma$-finite by Axiom 3, namely the objective separation axiom.

If $X_i$ ($i \in [1..n]$) are $\sigma$-finite classes, their union class $\bigcup_{i \in [1..n]} X_i$ is obviously $\sigma$-finite.

The product class

$$\prod_{1 \leq i \leq n} X_i$$

is defined as the collection of functions

$$f : [1..n] \to \bigcup_{1 \leq i \leq n} X_i$$

satisfying $f(i) \in X_i$ for all $i \in [1..n]$.

The coproduct class

$$\coprod_{1 \leq i \leq n} X_i$$

is defined as the collection of the pairs

$$(i, x) \in [1..n] \times \bigcup_{1 \leq j \leq n} X_j$$

such that $x \in X_i$.

Obviously the product and the coproduct of $\sigma$-finite classes are $\sigma$-finite.

**Lemma 1.2.6** If $X_i$'s are semisets, then the product $\prod_{1 \leq i \leq n} X_i$ and the coproduct $\coprod_{1 \leq i \leq n} X_i$ are semisets.

**Proof.** Suppose $X_i$ is a subclass of a set $a_i$ for $i \in [1..n]$. Then the product class is a subclass of the set $\prod_{1 \leq i \leq n} a_i$ and the coproduct class is a subclass of the set $\coprod_{1 \leq i \leq n} a_i$.

1.3 Induction Axioms

A condition $P(n)$ on numbers is called inductive if $P(0)$ is true and for all $n$ if $P(n)$ is true then $P(n + 1)$ is true. Even when $P(x)$ is an inductive condition, we do not think it is obvious that $P(a)$ is true for all $a \in \mathbb{N}$ but rather we take it as an evidence that we can refine the class $\mathbb{N}$ so that the condition $P(x)$ turns out to be true on it. In this way, we consider the class $\mathbb{N}$ open to continual refinements whenever new inductive conditions are found. With the tacit understanding that such refinements being done background automatically, we are convinced of the validity of following axiom.
Axiom 5 (Strong Induction Axiom)  If $P$ is an objective definite inductive condition on the class $\mathbb{N}$, $P(x)$ holds for every $x \in \mathbb{N}$.

A condition $P(n)$ on numbers, which may not be necessarily objective, is called weakly inductive if it satisfies that $P(0)$ is true and for every accessible number $n$, if $P(n)$ is true then $P(n + 1)$ is true.

Axiom 6 (Weak Induction Axiom)  If $P$ is a weakly inductive definite condition, then $P(i)$ holds for every accessible $i$.

Note that the condition that a primitive recursive function is totally defined is not bounded for most primitive recursive functions and hence we cannot show that they are total functions using Strong Induction Axiom.

However the condition that a primitive recursive function is defined for accessible numbers and has accessible values can be expressed by bounded formula by virtue of the overspill principle, whence the weak induction axiom shows that every primitive recursive function is defined at least on accessible numbers with accessible values. We omit the detail.

The following is used frequently.

Lemma 1.3.1  Let $f(x)$ be a primitive recursive function. Then for each huge $K$, there is a huge $M$ such that $f(M)$ is defined and satisfies $f(M) < K$.

Proof. Since the set $\{ n \mid f(n) < K \}$ contains all accessible numbers, it contains also a huge number $M$.  

For example, for every huge $K$, there is a huge $L$ with $L^L < K$.

1.4 Overspill Principles

The following is a special case of Theorem 1.2.1

Theorem 1.4.1 (Overspill Principle)  Let $P$ be a definite objective condition on natural numbers. If all the accessible numbers satisfy the condition $P$, then there is a huge number satisfying $P$.

The contraposition of the theorem for the negation of $P$ gives the following

Corollary 1.4.2  If all the inaccessible numbers satisfy a definite objective condition $P$, then an accessible number satisfy $P$. 

The following over spill principle is often used.

**Theorem 1.4.3** Let \( P \) be a definite objective condition. Then the following two conditions are equivalent:

(a) There is an accessible number above which every accessible number satisfy \( P \).

(b) There is a huge number under which every huge number satisfy \( P \).

**Proof.** Suppose (a) and every accessible number greater than \( k \) satisfies \( P \). Denote by \( Q(n) \) the condition that every accessible \( x \in [k+1..n] \) satisfies \( P \). Since \( Q \) is definite and every accessible number satisfies \( Q \), the Theorem 1.4.1 implies \( Q(K) \) for a huge number \( K \), hence every huge number \( I \leq K \) satisfies \( P(I) \) since \( I > k \), whence (b).

Conversely suppose (b) and there is an \( M \gg 1 \) such that every huge numbers \( I \leq M \) satisfies \( P(I) \). Denote by \( R(x) \) the condition that every number \( n \in [x..M] \) satisfies \( P(n) \). Since \( R \) is definite and every huge number satisfies it, the Corollary 1.4.2 implies that there is an accessible \( i \) satisfying \( R(i) \), whence (a).

Taking its contraposition for the negation of \( P \), we have

**Theorem 1.4.4** Suppose \( P \) is a definite objective condition. Then the following conditions are equivalent.

(a) For every accessible number \( n \), there is an accessible number \( x > n \) satisfying \( P(x) \).

(b) For every huge number \( N \), there is a huge number \( K \leq N \) satisfying \( P(K) \).

### 1.5 Concrete Sequences

Let \( A \) be a class. A function \( a : \mathbb{N}_{\text{acc}} \to A \) is called a concrete sequence on \( A \), which is often written as \( a = (a_1, a_2, \cdots) \). A function \([1..n] \to A \) is called a huge sequence if \( n \) is huge. The Axiom 4 implies the following extension property of concrete sequences.

**Theorem 1.5.1** A concrete sequence in a class can be extended to a huge sequence in it. More precisely, if \( a = (a_1, a_2, \cdots) \) is a concrete sequence in a class \( A \), then there is a huge \( N \) and a function \( f : [1..N] \to A \) satisfying \( f(i) = a_i \) if \( i \) is accessible.

**Proof.** By Axiom 4 there is a subset \( b \) such that \( \mathbb{N}_{\text{acc}} \subset b \subset [1..N] \) and a function \( g : b \to A \) which restricts to \( a \) on \( \mathbb{N}_{\text{acc}} \). The condition on \( n \) that \([1..n] \subset b \)
is obviously objective and definite. Moreover it is satisfied by all \( x \in \mathbb{N}_{acc} \), whence by Theorem 1.4.1, there is a huge \( N \gg 1 \) such that \([1..N] \subset b\), whence \( f = g|[1..N] \) is an extension with the desired properties.

Although extensions of a concrete sequence to huge sequences are not unique, the “germ” of the extensions is unique in the following sense.

**Proposition 1.5.2** If \( N_i \) (\( i = 1, 2 \)) are huge numbers and maps

\[ f_i : [1..N_i] \to A, \quad i = 1, 2 \]

satisfy \( f_i(k) = a_k \) for accessible \( k \) (\( i = 1, 2 \)). Then there is a huge \( K \leq \min \{ N_1, N_2 \} \) such that \( f_1(j) = f_2(j) \) for \( j \leq K \).

**Proof.** Let \( N = \min \{ N_1, N_2 \} \). Since the condition on natural numbers \( n \) that

\[ n \leq N \text{ and } f_1(n) = f_2(n) \tag{3} \]

hold for every accessible \( n \), there is a huge \( K \) such that \( (3) \) holds for any \( n \leq K \). From \( (3) \) for \( n = K \), we have \( K \leq N \).

We say a class \( A \) a quasi-set if every concrete sequence in \( A \) can be extended to a huge sequence in \( A \).

We use often the following lemma.

**Proposition 1.5.3** Suppose, for \( f : \mathbb{N}_{acc} \to \mathbb{N} \) is a function such that \( f(i) \) is huge for all \( i \). Then there is a huge number \( N \) satisfying \( N \leq f(i) \) for all accessible \( i \).

**Proof.** By Theorem 1.5.1, the concrete sequence \( f \) can be extended to a huge sequence \( \tilde{f} : [1..M] \to \mathbb{N} \) for some huge \( M \). Define a mapping \( g : [1..M] \to \mathbb{N} \) by

\[ g(i) := \min_{1 \leq j \leq i} f(j) \]

Then, for accessible \( i \), the following holds

\[ i < g(i), \tag{4} \]

\[ g(i) \leq g(j) \text{ for all } j \leq i. \tag{5} \]

Since these conditions are definite, there is a huge number \( K \) for which the conditions \( (4) \) and \( (5) \) hold for \( i = K \). Take then \( N = g(K) \).

If \( P \) is definite, Theorem 1.4.1 implies the following.

**Proposition 1.5.4** Let \( P \) be a definite objective weakly inductive condition. Then there is a huge \( M \) such that \( P(i) \) holds for all \( i \leq M \).

**Proof.** By the weak induction axiom, \( P(n) \) is true for all accessible \( n \), hence by the overspill principle of Theorem 1.4.1 there is a huge \( M \) such that \( P(n) \) is true for \( n \leq M \).
2 Continuum

2.1 Sorites Relations

A subclass $R$ of the product class $X \times X$ of a class $X$ is called a binary relation on $X$.

A basic example is the binary relation $x \approx y$ of indistinguishability between rationals $x, y$. This relation is not objective but is definite since it can be expressed by bounded quantifier

$$\forall n \leq N. \text{if } n \text{ is accessible then } |x - y| < \frac{1}{n},$$

using any huge $N$.

Usual notions of symmetry, anti-symmetry, reflexivity, transitivity have meaning for $R$, with the proviso that the validity of unbounded $\forall$-statements is understood proof-theoretically as in §1.2.2.

A sequence $(x_1, \cdots, x_N)$ is called an $R$-chain if $x_iRx_{i+1}$ for $i \in \{1..N-1\}$.

**Proposition 2.1.1** If a transitive binary relation $R$ is objective, the following condition holds

$$\text{If } (x_1, \cdots, x_n) \text{ is an } R\text{-chain on } X, \text{ then } x_1Rx_n \quad (6)$$

for all $n \in \mathbb{N}$.

**Proof.** Suppose $(x_1, \cdots, x_n)$ is an $R$-chain. Since $R$ is objective, the class $A := \{ i \mid x_1Rx_i \} \subset \{1..n\}$ is a set. Suppose its greatest element $m$ is less than $n$. Then $x_1Rx_m$ and $x_nRx_{m+1}$ but not $x_1Rx_{m+1}$, which contradicts to the transitivity of $R$. Hence $m = n$ and we have $x_1Rx_n$.

Note that if $R$ is not objective, the subclass $A$ in the proof may be proper and have no greatest element and the above arguments fail. Hence (6) might not hold for huge $n$ although it holds for accessible $n$ by the weak induction axiom.

The relation $R$ is called **strictly transitive** if (6) holds for every $n$ and an equivalence relation is called **strict** if it is strictly transitive. An equivalence relation which is not strict is called a sorites relation. An $R$-chain $(x_1, \cdots, x_N)$ without the validity of $x_1Rx_N$ is called a sorites sequence.

For example, the equivalence relation $\approx$ on the set $\{1..N\}$ defined by $i \approx j$ if and only if the rational number $\frac{i-j}{N}$ is infinitesimal is a sorites relation since $(1, 2, \cdots, N)$ is a sorites sequence. In fact $i \approx i + 1$ for $i \in \{1..N-1\}$ but $1 \not\approx N$. 
2.2 Continuum

Continuum is a pair $C = (|C|, \approx_C)$ of a class $|C|$ and an equivalence relation $\approx$ on $|C|$ which might not be strict. The class $|C|$ is called the support of the continuum $C$.

Elements of $|C|$ are called the positions of $C$. The relation $\approx_C$ is called the indistinguishability relation of $C$. We say two positions $p, q$ are indistinguishable if $p \approx_C q$.

For a position $a \in C$, the point determined by $a$, is defined as the subclass

$$[a] := \{ x \in |C| \mid x \approx a \},$$

which is proper in most cases. A point of $C$ is the point determined by some position of $C$. The notation $a \in C$ stand for the phrase that $a$ is a point of $C$. For $p \in C$, a position $x \in |C|$ such that $x \in p$ is said to represent the point $p$ and the point $p$ is represented by the position $x$.

**Remark 2.2.1** The collection of points do not form a class in most cases, since the condition of equality of proper classes is not definite. This is reasonable since if it formed a class, then we would have a paradox "$[x_1] = [x_2] = \cdots = [x_N]$ but $[x_1] \neq [x_N]$" if $(x_1, \cdots, x_N)$ is a sorites sequence.

This conforms to the view that the "points" of a continuum have inevitable indefiniteness which however is not perceived by any observation however accurate it may be.

A continuum $C$ is called a mesh continuum if $|C|$ is a semiset and a rigid mesh continuum if $|C|$ is a set.

2.2.1 Examples

**Linear continuum** A basic example is the continuum $(\mathbb{Q}, \approx)$, called the linear continuum denoted by $\mathbb{R}$.

**Metric continuum** A metric class $(X, d)$ is a class with a function $d : X \times X \to \mathbb{Q}$ satisfying the usual property of distance function. The relation $x \approx_d y$ defined by $d(x, y) \approx 0$ is an equivalence relation which might not be strict. We call $(X, \approx_d)$ the metric continuum defined by $(X, d)$. If $X$ is a semiset, $(X, d)$ is called a metric semispace and $(X, \approx_d)$ is a mesh continuum. If $X$ is a set, $(X, d)$ is called a metric space and the continuum $(X, \approx_d)$ is a rigid mesh continuum.

**Subcontinuum** If $C$ is a continuum class, a subclass $Y \subset |C|$ defines a continuum $(Y, \approx |Y|)$ called the subcontinuum of $C$ with support $Y$. 
Interval continuum  Let \( a \in \mathbb{Q} \). Then the definite conditions \( |x| < \infty \), \( a < x < \infty \), \( a \leq x < \infty \), \( -\infty < x < a \) and \( -\infty < x \leq a \) define respectively the subclasses \( (-\infty, \infty)_\mathbb{Q} \), \( (a, \infty)_\mathbb{Q} \), \( [a, \infty)_\mathbb{Q} \), \( (-\infty, a)_\mathbb{Q} \) and \( (-\infty, a]_\mathbb{Q} \).

Let \( a, b \in \mathbb{Q} \) with \( a < b \). Then the definite condition \( a < x < b \) defines the subclass \( (a, b)_\mathbb{Q} \subset \mathbb{Q} \). Similarly the conditions \( a \leq x < b \), \( a < x \leq b \) and \( a \leq x \leq b \) define respectively the objective subclasses \( (a, b)_\mathbb{Q} \), \( [a, b)_\mathbb{Q} \), \( (a, b]_\mathbb{Q} \) and \( [a, b]_\mathbb{Q} \). Note that only \( [a, b]_\mathbb{Q} \) is objective subclass.

The nine strings \( (-\infty, \infty) \), \( (a, \infty) \), \( (a, \infty] \), \( (-\infty, a) \), \( (-\infty, a] \), \( (a, b) \), \( [a, b) \), \( (a, b] \) and \( [a, b] \) are called interval symbols. The interval symbols without \( \infty \) is called finite interval symbols. If \( I \) is an interval symbol, then the subclass \( I_\mathbb{Q} \subset \mathbb{Q} \) defines a subcontinuum denoted by \( I \). For example \([0, 1]_\mathbb{Q} \) denotes the subcontinuum \( (0, 1)_\mathbb{Q}, \approx \).

Let \( r \) be a nonzero rational number. We write by \( r\mathbb{Z} \) the class of rationals which can be written as \( nr \) with \( n \in \mathbb{Z} \). For an interval symbol \( I \), we write \( I_r := I_\mathbb{Q} \cap r\mathbb{Z} \).

Lemma 2.2.1 If \( I \) is an interval symbol then \( I_r \) is a semiset. If \( I = [a, b] \), then \( I_r \) is a set.

Proof. Let \( I \) be an interval symbol. Let \( M \gg 1 \). Then \( I_r \subset [-M, M]_r \). Take \( K \) such that \( rK > M \), then

\[
I_r \subset [-M, M]_r \subset \{ x \in \mathbb{Q} \mid x \in [-K..K] \}
\]

whence \( I_r \) is a semiset.

Suppose \( I = [a, b] \). Since \( a \leq x \leq b \) means \( \frac{a}{r} \leq x \leq \frac{b}{r} \), we have \( I_r = \{ x \in \mathbb{Q} \mid x \in [a'..b'] \} \) where \( a' \) and \( b' \) are the integer parts of \( \frac{a}{r} \) and \( \frac{b}{r} \) respectively. Hence \( I_r \) is a set.

For each rational \( r \neq 0 \) and interval symbol \( I \), we obtain a subcontinuum with support \( I_r \) denoted also by \( I_r \), which are mesh continuum by the above lemma, among which \([a, b]_r \) is rigid.

Euclidean continuum  Since \( \mathbb{Q} \) is \( \sigma \)-finite, every number \( n \) defines the product \( \mathbb{Q}^n \) with the metric function \( d_\infty \) defined by

\[
d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|,
\]

where \( x_i \) is the \( i \)-th coordinate of \( x \in \mathbb{Q}^n \). The continuum \( (\mathbb{Q}^n, \approx_{d_\infty}) \) is called the \( n \)-dimensional Euclidean continuum.

The subcontinuum defined by the subclass \( [0, 1]^n_\mathbb{Q} \subset \mathbb{Q}^n \) is called the unit \( n \)-hypercube.
Product continuum  Let $C_i \ (i \in [1..N])$ be rigid mesh continua. Then the product set $\prod_{i \in [1..N]} |C_i|$ has an equivalence relation $x \approx y$ defined by $x_i \approx_{C_i} y_i$ for all $i$. We call the rigid mesh continuum $(\prod_{i \in [1..N]} |C_i|, \approx)$ the *product continuum* of the family $\{ C_i \mid i \in [1..N] \}$ and denote it by $\prod_{i \in [1..N]} C_i$.

Let $C_i \ (i \in [1..k])$ be mesh continua with $k$ a very small number so that we can write $1, 2, \cdots, k$ without the ellipsis. Then we have the product semiset $\prod_{i \in [1..k]} |C_i|$ and an equivalence relation $\approx$ defined by $x \approx y$ if and only if $x_i \approx y_i$ for all $i$.

Graphs as rigid mesh continua  Let $G = (V, E)$ be a connected symmetric graph with $V, E \subset X \times X$ being sets. Let $w : E \to \mathbb{Q}$ be a symmetric positive valued function. Define the length of a path $\gamma = (x_1, \cdots, x_n)$ by $\ell_w(\gamma) = \sum_{1 \leq i < n} w((x_i, x_{i+1}))$. Let $d_w(p, q)$ be the minimum of the length of paths connecting $p$ and $q$. Then $d_{G, w}$ is a rational valued metric function on $V$. The rigid mesh continuum defined by $(V, d_w)$ is called the *rigid mesh continuum* generated by the graph $G$ with the edge length function $w$.

**Example 2.2.2**  Fix a huge $\Omega \gg 1$ and put $V_\Omega := \prod_{i \leq \Omega} \{ 0, 1 \}$, the set of finite 01-words of length less than or equal to $\Omega$. Define

$$E = \{ \langle w, wi \rangle, \langle wi, w \rangle \mid w \in V_{\Omega-1}, i = 0, 1 \}.$$

Then the graph $G = (V, E)$ is the binary tree of depth $\Omega$. If we give uniform infinitesimal length $\frac{1}{\Omega}$ to edges, we obtain a continuum of hyperbolic type. If the length of the edge $\langle w, wi \rangle$ is given $2^{-|w|}$ where $|w|$ denotes the length of the word $w$, then the continuum induced from $(G, w)$ is the Cantor spaces. In §4.1, we study the topological properties of these continua.
2.3 Morphism

Let $C_i$ ($i = 1, 2$) be continua. A function $f : |C_1| \to |C_2|$ is called continuous if $x \approx y$ implies $f(x) \approx f(y)$.

We call two continuous functions $f, g : |C_1| \to |C_2|$ indistinguishable and write $f \approx g$ if we can prove $f(x) \approx g(x)$ for all $x \in |C_1|$.

The collection of continuous function indistinguishable from $f$ is not generally a class. So we formally introduce a symbol $[f]$ and use it as if it were a class as follows.

1. A morphism from $C_1$ to $C_2$ is a symbol of the form $[f]$ for some continuous function from $|C_1|$ to $|C_2|$.

2. If $C_i$ ($i = 1, 2$) are continua, the notation $F : C_1 \to C_2$ means that $F$ is a morphism from $C_1$ to $C_2$.

3. If $F$ is a morphism, then the expression $g \in F$ means $g \approx f$ if $F = [f]$.

   If $g \in F$, we say that the morphism $F$ is represented by $g$ and $g$ represents $F$.

4. If $F$ and $G$ are morphisms represented respectively by $f$ and $g$, then the expression $F = G$ means $f \approx g$.

5. A condition on continuous functions is called a condition on morphisms if it is $\approx$-invariant.

The identity morphism $id_{C_i}$ is represented by the identity map $id_{|C_i|}$.

The value of a morphism $F$ at a point $p$ of $C_1$ is defined to be the point $[f(t)]$ for $f \in F$ and $t \in p$. This does not depend on the choice of representations.

If $F_1 : C_1 \to F_2$ and $F_2 : C_2 \to C_3$, then the composition $F_2 \circ F_1 : C_1 \to C_3$ is the morphism, represented by $f_2 \circ f_1$, where $f_i \in [F_i]$ ($i = 1, 2$), which does not depend on the choices of $f_i$ ($i = 1, 2$). We may write the definition symbolically by

$$(F_2 \circ F_1)(p) := F_2(F_1(p))$$

for $p \in C_1$,

with proviso that the precise meaning is understood as above since a morphism cannot be defined as a correspondence which maps points to arbitrary points.

When $C_i$ ($i = 1, 2$) are rigid mesh continua, we will construct in §5.1 a continuum $C(C_1, C_2)$ whose points are precisely morphisms from $C_1$ to $C_2$.

Example 2.3.1 (Morphism defined by $\frac{1}{x}$) Taking the inverses of nonzero rationals define a function $f : (0, 1]_\mathbb{Q} \to [1, \infty)_\mathbb{Q}$. In fact, if $x \in (0, 1]_\mathbb{Q}$,
namely, \(0 < x \leq 1\), then \(1 \leq \frac{1}{x} < \infty\) and \(\frac{1}{x} \in [1, \infty)\). Moreover if \(r, s \in (0, 1]_\mathbb{Q}\) and \(r \approx s\), then since \(r, s \notin 0\), we have \(\frac{1}{r} \approx \frac{1}{s}\) as will be shown in §6.2. Thus \([f]\) is a morphism from \((0, 1]\) to \([1, \infty)\).

Example 2.3.2 (Morphism to product continuum mesh) Let \(C\) and \(C_i\) \((i \in \{1, \ldots, n\})\) be continua. An \(n\)-tupple \((F_1, \ldots, F_n)\) of morphisms \(F_i : C \to C_i\) defines a morphism \(F : C \to \prod_{1 \leq i \leq n} C_i\) which is represented by \(f\) which assigns \(a \in C\) to \(f(a) := (f_i(a))_{i \in \{1, \ldots, n\}}\), where \(f_i \in F_i\) \((i \in \{1, \ldots, n\})\).

Conversely a morphism \(F : C \to \prod_{i=1}^n C_i\) defines morphisms \(F_i = \pi_i \circ F : C \to C_i\) \((i \in \{1, \ldots, n\})\), where \(\pi_i : \prod_{i=1}^n C_i \to C_i\) is the projection morphism represented by \(\pi_i : (x_1, \ldots, x_n) \mapsto x_i\). The morphism \(F_i\) is called the \(i\)-th component of \(F\).

2.4 Equivalence

A morphism \(F : C_1 \to C_2\) is called injective and surjective if it is represented by a continuous map \(f : |C_1| \to |C_2|\) satisfying respectively

\[f(x) \approx f(y)\] implies \(x \approx y\) for all \(x, y \in C_1\), \hspace{1cm} \text{(7)}

and

for every \(x_2 \in C_2\), there is an \(x_1 \in C_1\) with \(f(x_1) \approx x_2\). \hspace{1cm} \text{(8)}

Note that unless \(C_i\) \((i = 1, 2)\) are rigid mesh continua, these conditions are not definite. Note also that the above conditions are independent of the choice of \(f \in F\).

A morphism \(F : C_1 \to C_2\) is an equivalence if there is a morphism \(G : C_2 \to C_1\) satisfying

\[G \circ F = \text{id}_{C_1}\] and \(F \circ G = \text{id}_{C_2}\).

The morphism \(G\) is uniquely determined by \(F\) and is called the inverse of \(F\) and is denoted by \(F^{-1}\).

If and equivalence \(F\) and its inverse \(F^{-1}\) are represented respectively by \(f : |C_1| \to |C_2|\) and \(g : |C_2| \to |C_1|\), then

\[g \circ f \approx \text{id}_{|C_1|}\] and \(f \circ g \approx \text{id}_{|C_2|}\).

Such \(g\) is uniquely determined by \(f\) up to indistinguishability and is called an almost inverse of \(f\).

**Proposition 2.4.1** Suppose \(C_i\) \((i = 1, 2)\) are rigid mesh continuum. Then a morphism \(F : C_1 \to C_2\) is an equivalence if and only if it is injective and surjective.
Proof. Suppose $F : C_1 \to C_2$ is an equivalence and let $f \in F$ and $g \in F^{-1}$. Then $f(x) \approx f(y)$ implies
\[ x \approx g(f(x)) \approx g(f(y)) \approx y. \]
Furthermore, for every $x_2 \in C_2$, we have $f(x_1) \approx x_2$ if we put $x_1 = g(x_2)$.

Conversely suppose $f \in F$ satisfies the conditions (7) and (8). For any $x_2 \in |C_2|$, we can choose by (8) an $x_1 \in |C_1|$ satisfying $f(x_1) \approx x_2$. Define $g(x_2) := x_1$. Then the map $g : |C_2| \to |C_1|$ is continuous by the condition (7).

By definition $f(g(x_2)) \approx x_2$ hold for $x_2 \in |C_2|$. For $x_1 \in |C_1|$, we have $f(g(f(x_1))) \approx f(x_1)$ by definition of $g$, whence by the condition (7) we obtain $g(f(x_1)) \approx x_1$. Hence $g$ is an almost inverse to $f$.

If there is an equivalence $F : C_1 \to C_2$, we say the continuum $C_1$ is equivalent to $C_2$ and write $C_1 \simeq C_2$. Then $\simeq$ satisfies the axiom of equivalence relations.

Let $C$ be a continuum and $C_i \subset C$ ($i = 1, 2$) be subcontinua. An equivalence $\alpha : C_1 \simeq C_2$ is called a quasi-identity if it satisfies $\alpha(x) = x$ for all $x \in C_1$. By definition, a quasi-identity is uniquely determined if it exists.

2.4.1 Examples

Proposition 2.4.2 Let $r$ be a nonzero infinitesimal rational number. The inclusion
\[ i_r : r\mathbb{Z} \to \mathbb{Q} \]
represents a quasi-identity, for which the function $\kappa_r : \mathbb{Q} \to r\mathbb{Z}$ defined by
\[ \kappa_r(s) := [sr^{-1}]r, \]
where $[x]$ denotes the integer part of $x$, gives an almost inverse of $i_r$.

Proof. Obviously $\kappa_r(i(nr)) = nr$ for $n \in \mathbb{Z}$.
From $x \leq [x] < x + 1$ it follows
\[ s = (sr^{-1})r < [sr^{-1}]r < (sr^{-1} + 1)r = s + r \]
whence $[sr^{-1}]r \approx s$ since $r \approx 0$. Hence
\[ i_r \circ \kappa_r \approx id. \]

Hence $\kappa_r$ is an almost inverse to $i_r$.

If $I$ is an interval symbol, the inclusion
\[ i_r|I_r : I_r \to I_{\mathbb{Q}} \]
represents a quasi-identity $(I_r, \approx) \to I$. 
Corollary 2.4.3 If $I$ is an interval symbol, then

$$(I_r, \approx) \simeq I$$

for every nonzero infinitesimal rational $r$.

The mesh continuum $(I_r, \approx)$ is called a representation of the continuum $I$. Obviously representations are unique up to equivalences.

For example, let $r, s$ be nonzero infinitesimal rationals and $I$ an interval symbol. Then the representations $(I_r, \approx)$ and $(I_s, \approx)$ of $I$ are equivalent by the morphism given by

$$h(nr) = [nr/s]s,$$

which is the restriction $h := g_s \circ \iota_r : r\mathbb{Z} \to s\mathbb{Z}$ on $I_r$.

Example 2.4.2 Let $a \prec b$ be finite rationals. Then $[a, b] \approx [0, 1]$. In fact define $f : [a, b]_\mathbb{Q} \to [0, 1]_\mathbb{Q}$ by $f(x) = \frac{x-a}{b-a}$. Then it has an inverse $g : [0, 1]_\mathbb{Q} \to [a, b]_\mathbb{Q}$ defined by $g(x) = (b-a)x + a$.

Since $(a, b, r, \approx) \simeq [a, b]$ for nonzero infinitesimal $r$, we have equivalence between rigid mesh continua $f_{rs} : [a, b]_r \approx [0, 1]_s$ for each nonzero infinitesimals $r, s$. This equivalence is given by

$$f_{rs}(nr) = \left[\frac{nr-a}{s(b-a)}\right]s.$$

2.5 Saturation

Let $C$ be a continuum. A subset $a \subset |C|$ defines a subclass $\bar{a} \subset |C|$ of such elements $y$ as satisfying the definite condition that there is an $x \in a$ with $y \approx x$.

A subset $a \subset |C|$ is called dense in $C$ if $\bar{a} = |C|$. For example $[a, b]_r \subset [a, b]_\mathbb{Q}$ is dense if $r$ is nonzero infinitesimal.

Obviously the correspondence $a \mapsto \bar{a}$ satisfies the following conditions of closure operators

Proposition 2.5.1 1. $a \subset \bar{a}$,

2. $a \subset b$ implies $\bar{a} \subset \bar{b}$,

3. If $\bar{a}$ is a set, then $\bar{a} = \bar{a}$.

Moreover the following holds.

$$a \bar{b} = \bar{a} \bar{b}$$

(9)
A subset $a \subset |C|$ is called saturated if $\bar{a} = a$. The saturated subsets are rare. In fact we have

**Proposition 2.5.2** Suppose $C$ is connected, namely, every two elements are connected by an $\approx$-chain. Then a saturated subset is either $\emptyset$ or $|C|$.

**Proof.** Suppose $a = \bar{a}$ and $a \neq \emptyset$ and $a \neq |C|$. Let $x \in a$ and $y \notin a$. Let $(x_1, \ldots, x_N)$ be a $\approx$-chain such that $x_1 = x$ and $x_N = y$. Let $i \in [1..N]$ be the minimum satisfying $x_i \notin a$. Then $x_{i-1} \in a$ and $x_i \notin a$. The former condition and $x_i \approx x_{i-1}$ implies $x_i \in \bar{a} = a$ which contradicts to the latter condition. \(\blacksquare\)
3 Topology of Continuum

3.1 Convergence of Sequences

Let $C$ be a continuum. A sequence $a = (a_1, \cdots, a_N)$ in $|C|$ is called a sequence in $C$. A sequence $a$ in $C$ converges to $c \in |C|$ if there is a huge $K \leq N$ such that for every huge $I \leq K$, $a_I \approx c$. The limit $c$ is uniquely defined up to indistinguishability.

Let $F : C_1 \to C_2$ be a morphism and $f \in F$. Then a sequence $a$ in $C_1$ defines a sequence $f(a) = (f(a_1), \cdots, f(a_N))$ in $C_2$ and if $c \in |C_1|$ is a limit of the sequence $a$ then $f(c)$ is a limit of the sequence $f(a)$.

Let $B \subset |C|$ be a subset and $c \in C$. We call $c$ an accumulation point of $B$ if there is a huge subset $B' \subset B$ whose elements are indistinguishable from $c$. We call $c \in B$ an isolated point if $c \neq b \in B$ implies $c \not\approx b$. Note that there can be an element $c \in B$ which is neither isolated nor accumulation point of $B$ since it is possible that $\{c\} \cap (B \setminus \{c\})$ contains only one element.

A continuum $C$ is called perfect if every $c \in C$ is an accumulation point of $|C| \setminus \{c\}$. Basic Euclidean continua are obviously perfect.

3.2 Compactness

A continuum $C$ is called compact if for every huge $N$ there is a dense subset $A \subset |C|$ with $\#(A) \leq N$.

**Proposition 3.2.1** For finite rationals $a, b$ with $a \prec b$, the continuum $[a, b]$ is compact.

**Proof.** For every huge $N$, the subset $[a, b]_{\frac{1}{N}} \subset [a, b]_{\mathbb{Q}}$ is dense and has $N$ elements.

A continuum equivalent to a compact continuum is compact. In fact we have the following.

**Proposition 3.2.2** If there is a surjective morphism $F : C_1 \to C_2$ and $C_1$ is compact then $C_2$ is also compact.

**Proof.** Let $f \in F$. Let $N$ be a huge number. Take a dense subset $A \subset |C_1|$ with $\#(A) \leq N$. Then $f(A) \subset |C_2|$ is dense since for every $x \in |C_2|$, there is an $y \in |C_1|$ with $f(y) \approx x$. Take $a \in A$ such that $a \approx y$. Then $x \approx f(y) \approx f(a) \in f(A)$. Hence $f(A)$ is dense in $|C_2|$.
Theorem 3.2.3 If a continuum $C$ is compact, then every huge subset $B \subset |C|$ has an accumulation point.

Proof. Suppose $C$ is compact and $B \subset |C|$ a huge set. Take a huge $N$ with $N^2 < \#(B)$. Let $E \subset |C|$ be a dense subset with $\#(E) \leq N$. Define a map $f : B \to E$ which carries $b \in B$ to an $e \in E$ such that $e \approx b$. If $\#(f^{-1}e) < N$ for all $e \in E$, then
\[
\#(B) = \sum_{e \in E} \#(f^{-1}e) < N \#(E) \leq N^2 < \#(B)
\]
a contradiction. Hence there is an $e \in E$ with $\#(f^{-1}e) \geq \#(E)$. Let $b \in B \cap f^{-1}e$. Then $x \in f^{-1}e$ implies $x \approx e \approx b$ whence every element of $f^{-1}e$ is indistinguishable from $b$, whence $b$ is an accumulation point of $B$.

Corollary 3.2.4 If $C$ is compact, every huge subset of $|C|$ has a pair of indistinguishable elements.

A subset $a \subset |C|$ is called discrete if $x, y \in a$ and $x \approx y$ implies $x = y$. For example $[-M..M]$ is a discrete subset of $[-M,M]_Q$.

Proposition 3.2.5 A continuum with a discrete huge subset is not compact.

Proof. Suppose a continuum $C$ is compact with a huge discrete subset $a \subset |C|$. Let $b \subset |C|$ be a dense subset with $\#(b) < \#(a)$. Define a map $f : a \to b$ by assigning $x \in a$ to an $y \in b$ such that $x \approx y$. Since $\#(b) < \#(a)$, there must be $x, y \in a$ with $f(x) = f(y)$ but $x \neq y$. Then $x \approx f(x) = f(y) \approx y$ contradicting to the discreteness of $a$.

Let $C$ be a continuum. An objective subclass $R \subset |C| \times |C|$ is called an objective discrimination of $C$ if $R(x,y)$ implies $x \neq y$. A subclass $A \subset X$ is called $R$-discrete if $x \neq y$ implies $R(x,y)$ for $x, y \in A$. For example, in $\mathbb{Q}$, if $k$ is accessible then the relation $R_k := \{(x, y) \mid |x - y| > \frac{1}{k}\}$ is an objective discrimination and the subclass $\mathbb{Z} \subset \mathbb{Q}$ is $R_2$-discrete.

Proposition 3.2.6 A continuum $C$ is not compact if it has an objective discrimination $R$ and a subset $a \subset |C|$ such that for each accessible number $k$, there is an $R$-discrete subset of $a$ of size greater than $k$.

Proof. Let $R$ be an objective discrimination of $C$ and suppose that for each accessible $k$ the condition
\[
\text{there is an } R\text{-discrete subset of } a \text{ with at least } k \text{ elements} \quad (10)
\]
is satisfied. Since the condition $\text{(10)}$ is objective and definite and satisfied by all accessible $k$, it is satisfied also by a huge $k$. Hence by Proposition 3.2.5 $C$ is not compact.
Example 3.2.1  1. If \( m - n > 0 \) is huge, the continuum \([n, m]\) is not compact since it has the huge discrete subset \([n..m]\).

2. The continuum \((-\infty, \infty)\) is not compact. In fact for any concrete \( k \), the subset \([1..k] \subset (-\infty, \infty)\) has \( k \) elements and \( R\)-discrete for the objective discrimination \( R = \{ (x, y) \mid |x - y| > \frac{1}{2} \} \).

3.3 Connectedness

Let \( C \) be a continuum. We say \( x \in C \) is connected to \( y \in C \) and write \( x \sim y \) if there is an \( \approx \)-chain connecting \( x \) and \( y \). If \( x \sim y \) for every \( x, y \in C \), the continuum is called connected. Note that this condition is not definite generally.

Obviously, if \( f : C_1 \to C_2 \) is a surjective morphism and \( C_1 \) is connected then \( C_2 \) is also connected. In particular connectedness is equivalence invariant.

Suppose \( C \) is a rigid mesh continuum. Then the binary relation \( x \sim y \) is definite and for each \( x \in |C| \), we have the equivalence class

\[
[x] := \{ y \mid x \sim y \}
\]

which is a semiset called the connected component containing \( x \). Each \( x \in |C| \) belongs to the connected component \([x]\). A rigid mesh continuum \( C \) is called totally disconnected if \([x] = \{ x \}\) for all \( x \in |C|\).

Remark 3.3.1  1. One may think that the terminology “arcwise connected” conforms with usual mathematics. However, the popular example of connected space which is not arcwise connected in the usual mathematics turns out to be connected in our sense. A continuum corresponding to it is the subcontinuum \( H \) of \([0, 1]^2\) defined by

\[
H = (0, 1)_Q \times \{ 0 \}\cup \{ 0 \} \times (0, 1)_Q \cup \left( \bigcup \left\{ \frac{1}{i} \mid i \leq N \right\} \right) \times [0, 1)_Q
\]

where \( N \) is a huge number. Then \((0, y) \approx (\frac{1}{N}, y)\) and hence \((0, y)\) can be connected by a sorites sequence to any other point.

2. Note that the usual definition of connectedness asserts that there is a nontrivial disjoint decomposition \( X = A_1 \cup A_2 \) with \( A_i \ (i = 1, 2) \) being open and closed, which cannot be used since every rigid mesh continuum \( C \) is totally disconnected with respect to the “\( S\)-topology”.

\[\]
3.4 Topology of Metric Continuum

The distance function gives refined statements on the topology of continua.

A metric continuum is a triple \((X, d, \approx)\) where \((X, d)\) is a metric class and \((X, \approx)\) is a continuum defined by \(x \approx y\) if and only if \(d(x, y) \approx 0\). A metric continuum \((X, d, \approx)\) is called metric mesh continuum if \(X\) is a semiset and rigid metric mesh continuum if \(X\) is a set.

3.4.1 Completeness

Let \((X, d, \approx)\) be a metric continuum. A concrete sequence \(a = (a_1, a_2, \cdots)\) in \(X\) converges to \(c \in X\) if for each accessible number \(k\) there is an accessible number \(\ell\) such that for every accessible \(i \geq \ell\)

\[d(a_i, c) < \frac{1}{k}.\]

A concrete sequence \(a = (a_1, a_2, \cdots)\) is a Cauchy sequence if for each accessible number \(k\) there is an accessible number \(\ell\) such that for every accessible \(i, j \geq \ell\)

\[d(a_i, a_j) < \frac{1}{k}.\]

**Proposition 3.4.1** Let \(a = (a_1, a_2, \cdots)\) be a concrete sequence on a rigid metric mesh continuum \((X, d, \approx)\). Let \(\tilde{a} = (a_1, \cdots, a_N)\) be an extension of it to a huge sequence. Then

1. For \(c \in X\), the accessible sequence \(a\) converges to \(c\) if and only if the extended \(\tilde{a}\) converges to \(c\).

2. The sequence \(a\) is a Cauchy sequence if and only if the extended \(\tilde{a}\) is convergent.

Hence every accessible sequence is convergent if and only if it is a Cauchy sequence.

**Proof.** Let \(a\) be a concrete sequence with a huge extension \(\tilde{a} = (a_1, \cdots, a_N)\). Suppose \(\tilde{a}\) converges to \(c\). There is a huge \(M\) such that \(d(a_L, c) \approx 0\) for every huge \(L \leq M\). Let \(k\) be an arbitrary accessible number. Since the objective condition

\[d(a_i, c) < \frac{1}{k}\]  \hspace{1cm} (11)

is satisfied by every huge number \(i \leq M\), there is an accessible number \(\ell\) such that every accessible \(i > \ell\) satisfies [11] by Theorem 1.4.3. Hence the concrete sequence \(a\) converges to \(c\).
Conversely suppose that the concrete sequence $a$ converges to $c$. Let $k$ be an accessible number. There is an accessible $\ell_k$ such that (11) holds for every accessible $i \geq \ell_k$, whence there is a huge $M_k$ such that (11) holds for $i \in [\ell_k..M_k]$. By Proposition 1.5.3, there is a huge $M$ satisfying $M \leq M_k$ for every accessible $k$. If $I \leq M$ is huge, then (11) holds for each accessible $k$ since $I \leq M_k$. Hence $a_I \approx c$, which means that $\tilde{a}$ converges to $c$.

Suppose now that $\tilde{a}$ converges to $c$. Then $a$ converges and hence it is a Cauchy sequence by the usual arguments.

Conversely suppose that $a$ is a Cauchy sequence. Let $k$ be an accessible number. Then there is an accessible number $\ell_k$ such that $d(a_i, a_j) < \frac{1}{k}$ holds for every accessible $i, j \geq \ell_k$. Hence for every accessible $i, j \in [\ell_k..p]$, there is a huge $M_k$ such that (12) holds for every $i,j \in [\ell_k..M_k]$. By Proposition 1.5.3, there is a huge $M$ satisfying $M \leq M_k$ for every accessible $k$. Then for every huge $I \leq M$, (12) holds for $i = I, j = M$ holds for every accessible $k$ since $I, M \in [\ell_k..M_k]$, whence $a_I \approx a_M$. Hence $\tilde{a}$ converges to $a_M$.

A metric class is called complete if every concrete Cauchy sequence converges.

By Proposition 3.4.1, we have the following.

**Theorem 3.4.2** A metric space $(X, d)$ is complete.

Moreover we have the following.

**Theorem 3.4.3** Suppose $(X, d, \approx)$ is a metric class and $A \subseteq X$ is a quasi-subset, namely, every concrete sequence of $A$ can be extended to a huge sequence in $A$. Then the metric mesh continuum $(A, d, \approx)$ is complete. In particular, for a subset $b \subseteq X$, the metric class $(b, d)$ is complete.

**Proof.** Suppose $A \subseteq X$ is a quasi-subset. Let $a_1, a_2, \ldots$ be a concrete Cauchy sequence of $A$. Extend it to a huge sequence $\tilde{a}$ in $A$. By Proposition 3.4.1, $\tilde{a}$ converges and hence $a$ converges.

Let $b$ be a subset of $X$. It suffices to show that $A := \tilde{b}$ is a quasi-subset. Let $a$ be a concrete sequence in $A$. Then there is a concrete sequence $x$ in $b$ with $a_i \approx x_i$ for all accessible $i$. Extend $a$ to huge sequence $\tilde{a} = (a_1, \cdots, a_{N_i})$ in $x$ and huge sequence $\tilde{x} = (x_1, \cdots, x_{N_2})$. Since for accessible $i$, we have $d(a_i, x_i) < \frac{1}{I}$, there is a huge $M$ with $d(a_I, x_I) < \frac{1}{I}$ for all $I \leq M$. Hence $a_I \approx x_I$ for huge $I \leq M$ and $a_I \in \tilde{b} = A$. Namely $(a_1, \cdots, a_M)$ is an extension of $a$ in $A$. 

\[\]
3.4.2 Compactness

Let \((X,d,\approx)\) be a metric continuum. For \(x \in X\) and a positive rational number \(r\), define the \(r\)-ball with center \(x\) by

\[
B_r(x) := \{ y \in X \mid d(x,y) \leq r \},
\]

which is a set if \(X\) is a set. If \(r > 0\), then \(B_r(x)\) is called a visible ball.

We say the continuum is precompact if for each accessible \(k\), there is an accessible number of points \(\{x_1, \cdots, x_\ell\}\) such that

\[
X = \bigcup_{1 \leq i \leq \ell} B_1^r(x_i).
\]

**Proposition 3.4.4** An rigid mesh metric continuum is compact if and only if it is precompact.

**Proof.** Let \((X,d,\approx)\) be a rigid mesh metric continuum.

Assume \((X,d)\) is precompact. Let \(K\) be a huge number and \(I\) the set of numbers \(n\) such that there is a subset \(Y \subset X\) satisfying \(\#(Y) \leq K\) and \(X = \bigcup_{y \in Y} B_{1/n}(y)\). By assumption \(I\) contains every accessible number and hence a huge number \(N\). Then there is a subset with \(\#(Y) \leq K\) such that \(X = \bigcup_{y \in Y} B_{1/N}(y)\). Hence \(Y = X\). This means that \((X,\approx)\) is compact.

Conversely suppose \((X,\approx)\) is compact. Let \(n\) be an accessible number. Let \(I\) be the set of numbers \(K\) such that there is a subset \(Y \subset X\) satisfying \(\#(Y) \leq K\) and \(\bigcup_{y \in Y} B_{1/n}(y) = X\). If \(K\) is huge then there is a dense subset \(Y \subset X\) with \(\#(Y) \leq K\), whence \(K \in I\). Thus \(I\) contains all huge numbers and hence an accessible number \(k\). Namely \(X\) is covered by an accessible number of balls of radius \(\frac{1}{n}\). Hence \((X,d)\) is precompact.

**Corollary 3.4.5** A rigid mesh subcontinuum of a compact rigid metric continuum is compact.

**Proof.** Suppose \(C\) is a compact rigid metric continuum and \(X \subset |C|\). Since \(C\) is precompact, the metric continuum \((X,d,\approx)\) with the restricted distance function is precompact and hence is compact.

An \(x \in X\) is an accumulation point of a huge sequence \(a = (a_1, \cdots, a_N)\) in \(X\) if either there is a huge number of \(j\) satisfying \(a_j = x\) or the support \(\{a_i \mid i \in [1..N]\}\) is huge and has \(x\) as its accumulation point.

An element \(x \in X\) is an accumulation point of a concrete sequence \((a_1, a_2, \cdots)\) if for each accessible number \(k\) there is an accessible \(i \geq k\) with \(d(x, a_i) < \frac{1}{k}\).
§3 Topology of continua

Proposition 3.4.6 A concrete sequence on a rigid mesh metric continuum \((X,d,\approx)\) has an accumulation point if and only if every huge extension of it has an accumulation point.

Proof. Let \(a = (a_1,a_2,\cdots)\) be a concrete sequence in \(X\). If \(a\) is contained in a set with accessible number of points, then the assertion is obvious. Otherwise every extended sequence has the huge support.

Suppose \(x\) is an accumulation point of the sequence \(a\) and \(\tilde{a} = (a_1,\cdots,a_N)\) is a huge sequence extending it such that \(#(\{ i \mid a_i = x \})\) is accessible. We show that \(x\) is an accumulation point of \(A := \{ a_i \mid i \in [1..N]\} \). For each accessible \(k\), the number of elements \(B_{\frac{1}{k}}(x) \cap A\) is huge hence there is a huge \(M\) such that

\[ #(B_{\frac{1}{k}}(x) \cap A) \geq M \quad (13) \]

for all accessible \(k\) by Proposition 1.5.3. Hence there is a huge \(K\) such that \((13)\) holds for \(k = K\), which means \(x\) is an accumulation point of \(A\).

Suppose every huge extension of \(a\) has an accumulation point and suppose \(a\) has no accumulation point. Then for every \(x \in X\) there is an accessible \(k_x\) such that for accessible \(i \geq k_x\)

\[ d(x,a_i) \geq \frac{1}{k_x}. \]

Put \(k = \max_{x \in X} k_x\). Then for all \(x \in X\) and for accessible \(i \geq k\),

\[ d(x,a_i) \geq \frac{1}{K}. \]

In particular, for every accessible \(i,j \geq k\) we have

\[ d(a_i,a_j) \geq \frac{1}{K}. \quad (14) \]

If \((a_1,\cdots,a_N)\) is a huge extension of \(a\), then there is a huge \(M \leq N\) such that \[(14)\] holds for every \(i,j \in [k..M]\). This means the extended sequence \((a_1,\cdots,a_M)\) has no accumulation point, a contradiction.

Hence by virtue of Theorem 3.2.3 we have

Corollary 3.4.7 A concrete sequence in a compact rigid mesh metric continuum has an accumulation point.
4 Continua of Huge Binary Words

The concept of continuum makes it possible to construct continuum directly from syntactic objects. As an illustration we examine topological properties of four metric continua of huge binary words.

Denote by \( \{ 0, 1 \}^{\leq N} \) the set of words on 0, 1 of length less than or equal to \( N \) and \( \{ 0, 1 \}^N \) the subset consisting of words of length \( N \).

The following interpretations with appropriate distance functions give four rigid mesh continua with topological properties different from one another.

1. \( \{ 0, 1 \}^{\leq N} \) is the vertex set of binary trees of depth \( N \) and \( \{ 0, 1 \}^N \) is the set of its leaves. 
2. \( \{ 0, 1 \}^N \) is the vertex set of the \( N \)-dimensional hypercube. 
3. \( \{ 0, 1 \}^N \) is the set of the characteristic functions of subsets in \([0..1]^N\). 

4.1 Binary Trees

Consider the symmetric graph \( BTree_N \) with the vertex set \( \{ 0, 1 \}^{\leq N} \) and the edges are \( \{ w, w0 \} \), \( \{ w, w1 \} \) for \( w \in \{ 0, 1 \}^{\leq N-1} \). Let \( d_0(p, q) \) be the path distance, namely the length of the shortest path joining \( p \) and \( q \) where every edges are given unit length.

Lemma 4.1.1

\[
d_0(x, y) = |x| + |y| - 2m(x, y),
\]

where

\[
m(x, y) := \min \{ i \mid x_i \neq y_i \}.
\]

Proof. Denote by \( x_i \) the \( i \)-th character of the word \( x \). Put \( x = ux', y = uy' \) with \( x'_i \neq y'_i \) if both \( x' \) and \( y' \) are not empty word. Note that this decomposition is unique. Then the shortest path joining \( x \) and \( y \) is composed the path of length \( |x'| \) from \( x \) to \( u \) and the path of length \( |y'| \) joining \( u \) to \( y \). Hence, noting \( |u| = m(x, y) \) we have

\[
d_0(x, y) = |x'| + |y'| = |x| - |u| + |y| - |u| = |x| + |y| - 2m(x, y).
\]

Lemma 4.1.2

\[
m(x, z) \geq \min \{ m(x, y), m(y, z) \}.
\]
Proof. Suppose \( m(x, y) = m(y, z) \). Then

\[
x = ux', y = uy', z = uz'
\]

with \( x_1' \neq y_1' \) if both \( x' \) and \( y' \) are nonempty words and \( y_1' \neq z_1' \) if both \( y' \) and \( z' \) are not empty words. Hence \( m(x, z) \geq |u| = m(x, y) = m(y, z) \).

Suppose \( m(x, y) \neq m(y, z) \). We may assume \( m(x, y) < m(y, z) \). Then we can write

\[
x = ux', y = uvy', z = uvz'
\]

with \( x_1' \neq v_1 \) and \( y_1' \neq z_1' \) if \( y' \) and \( z' \) are nonempty. Hence \( m(x, z) = |u| = m(x, y) \leq \min \{ m(x, y), m(y, z) \} \).

From the function \( m \) we obtain various ultrametrics.

**Lemma 4.1.3** If \( f \) is a positive descreasing function on positive rationals, then the function \( d_f \) defined by \( d_f(x, x) = 0 \) and \( d_f(x, y) := f(m(x, y)) \) if \( x \neq y \) is an ultrametric.

**Proof.** Obviously \( d_f \) is symmetric and reflexive and if \( x \neq y \) then \( m(x, y) > 0 \), whence \( d_f(x, y) > 0 \). The ultrametric triangle relation

\[
d_f(x, z) \leq \max \{ d_f(x, y), d_f(y, z) \}.
\]

follows from Lemma 4.1.2.

**Lemma 4.1.4** Suppose \( C \) is a metric continuum with an ultrametric \( d \). Then there are no sorites sequences. If \( C \) is a rigid mesh continuum, then it is totally disconnected.

**Proof.** Let \( (x_1, \cdots, x_N) \) be an \( \approx \)-chain. Put \( \varepsilon = \max \{ d(x_i, x_{i+1}) \mid i \in [1..N-1] \} \approx 0 \). Suppose \( d(x_1, x_N) > \varepsilon \). Then

\[
k := \min \{ i \mid d(x_1, d_i) > \varepsilon \} \leq N.
\]

Since \( d(x_{k-1}, x_k) \leq \varepsilon \), we have

\[
\varepsilon < d(x_1, x_k) \leq \max \{ d(x_1, x_{k-1}), d(x_{k-1}, x_k) \} \leq \varepsilon
\]

a contradiction. Hence we have \( d(x_1, x_N) \leq \varepsilon \) and hence \( x_1 \approx x_N \).

### 4.1.1 Hyperbolic Space

Define on \( \{ 0, 1 \}^{\leq \Omega} \)

\[
d_{hyp}(x, y) = \frac{|x| + |y| - 2m(x, y)}{2\Omega}.
\]
Restrired on \( \{ 0, 1 \}^\Omega \) we have
\[
d_{\text{hyp}}(x, y) = 1 - \frac{m(x, y)}{\Omega},
\]
which is an ultrametric.

The metric continuum mesh \( Hyp_\Omega := (\{ 0, 1 \}^\Omega, d_{\text{hyp}}, \approx) \) is called the hyperbolic continuum of binary words of length \( \Omega \). See the left graph of Fig. 1.

Figure 1: Binary tree with 256 leaves marked by red dots. In the left graph edges are given uniform length whereas in the right the edges connecting the \( k \)-th level vertices to its children is given the length \( 2^{-k} \). The subspace of red dots of the left graph “approximates” the hyperbolic space and that on the right the Cantor space.

A metric continuum is locally compact if for every virtual point \( x \) there is a rational \( r > 0 \) such that \( B_r(x) \) is compact.

**Proposition 4.1.5** The hyperbolic continuum mesh \( Hyp_\Omega \) is perfect but is neither connected nor locally compact.

**Proof.** Since for every \( w \in \{ 0, 1 \}^\Omega \), the ball \( B_r(w) \) with \( 0 \neq r \approx 0 \) is a huge set, the continuum \( Hyp_\Omega \) is perfect.

Since \( d_{\text{hyp}} \) is ultrametric, the continuum \( Hyp_\Omega \) is totally disconnected by Lemma 4.1.4.

To show \( Hyp_\Omega \) is not locally compact, let \( w \) be any word of length \( \Omega \). Let \( r > 0 \) be a rational. Let \( K \) be the integer part of \( r\Omega \) so that \( \frac{K}{2} < r \) and \( \frac{K}{2} \approx r \). Decompose as \( w = w_1w_2 \) with \( |w_2| = K \). Let \( L \) be the integer part of \( \frac{K}{2} + 1 \) so that \( L \geq \frac{K}{2} \). Then
\[
B_r(w) \supset B_{\frac{K}{2}}(w) = \{ w_1u \mid |u| = K \} \supset P
\]
where
\[
P := \{ w_1u_11^L \mid |u_1| = K - L \}.
\]
The set $P$ is huge with $2^{K-L}$ elements and has no accumulation points. In fact if $x_i = w_i u_i 1^L \in P$ \((i = 1, 2)\), then

$$m(x_1, x_2) \leq |w_1 u_1| = \Omega - K + (K - L) = \Omega - L,$$

whence

$$d_{hyp}(x_1, x_2) \geq \frac{L}{\Omega} \geq \frac{K}{2\Omega} \approx \frac{r}{2}.$$ 

Hence $B_r(w)$ is not compact for every $r > 0$.

4.1.2 Cantor Space

Let $d_C$ be the distance function on the graph $BTree$ when the edge of level $n$ is given the length $2^{-n}$.

**Lemma 4.1.6** The distance function $d_C$ is given by

$$d_C(x, y) := 2^{-m(x,y)+1} - 2^{-|x|} - 2^{-|y|}$$

for $x, y \in \{0, 1\}^{\leq \Omega}$ and if $x, y \in \{0, 1\}^{\Omega}$,

$$d_C(x, y) = 2(2^{-m(x,y)} - 2^{-\Omega}).$$

Hence $d_C$ on $\{0, 1\}^{\Omega}$ is an ultrametric.

**Proof.** The path which connects $x$ to the empty word is

$$\sum_{i=1}^{\lfloor x \rfloor} 2^{-i} = 1 - 2^{-|x|}.$$ 

The length of the path connecting $x = ux'$ and $y = uy'$ with $|u| = m(x,y)$ is the sum of the length of the paths from $x$ to $u$ and from $u$ to $y$, whence

$$d_C(x, y) = d_C(x, \lambda) + d_C(y, \lambda) - 2d_C(u, \lambda) = (1 - 2^{-|x|}) + (1 - 2^{-|y|}) - 2(1 - 2^{-m(x,y)}) = 2^{-m(x,y)+1} - 2^{-|x|} - 2^{-|y|}.$$ 

Hence if $|x| = |y| = \Omega$, then

$$d_C(x, y) = 2(2^{-m(x,y)} - 2^{-\Omega}).$$ 

Hence by Lemma 4.1.2 $d_C$ is an ultrametric.

The rigid mesh metric continuum $\left(\{0, 1\}^{\Omega}, d_C, \approx\right)$ is called the Cantor space. See the right graph of Fig. 1.

**Proposition 4.1.7** Let $\Omega$ be a huge number.
1. The rigid mesh continuum \( (\{0, 1\}^\Omega, d_C, \approx) \) is compact but is not perfect nor connected.

2. The Cantor space \( (\{0, 1\}^\Omega, d_C, \approx) \) is compact and perfect but is not connected.

**Proof.** Let \( N \) be a huge number and take a huge \( K < \Omega \) with \( 2^{K+1} < N \). Denote by \( Y \) the set of all the words of length \( \leq K \). Then \( \#(Y) = 2^{K+1} < N \).

Every word of length \( \leq K \) is in \( Y \). Words \( w \) of length in \([K+1..\Omega]\) is decomposed as \( w = uv \) with \( |u| = K \) and
\[
d(w, u) = 2^{-|w|+1} - 2^{-|u|} - 2^{-|u|} \leq 2^{-|u|} = 2^{-K} \approx 0,
\]
whence \( w \approx u \in Y \), namely \( w \in Y \). As a result \( Y = \{0, 1\}^\leq \Omega \). Thus we can make the sizes of dense subsets as small as possible within huge numbers. Hence \( (\{0, 1\}^\leq \Omega, d_C, \approx) \) is compact. It is not perfect nor connected since the words of accessible length are isolated points.

By Corollary 3.4.5, the Cantor space is compact since it is a rigid mesh subcontinuum of the compact rigid mesh continuum \((\{0, 1\}^\leq \Omega, d_C, \approx)\).

The Cantor space is not connected because \( d_C \) is an ultrametric. However it is perfect. In fact, let \( x \in \{0, 1\}^\Omega \). Put \( x = x_1x_2 \) with \( |x_1| = \Omega/2 \). Then for any \( w \) with \( |w| = |x_2| \), \( d(x, x_1w) = 2(2^{-|x_1|} - 2^\Omega) \approx 2^\Omega \approx 0 \). Hence \( x \) is an accumulation.

### 4.2 Power Set

Let \( C \) be a rigid mesh continuum. We can regard \( \{0, 1\}^{|C|} \) as the power set of \( |C| \) identifying \( \chi \) with the subset \( \{x \mid \chi(x) = 1\} \subset |C| \). Let \( \text{pow}^+(|C|) \) be the set of nonempty subset of \( |C| \).

Define for nonempty subsets \( A, B \subset |C| \),
\[
d_p(A, B) := \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(A, b) \right\},
\]
where \( d(a, B) := \min \{ |a - b| \mid b \in B \} \). Obviously \( d_p \) is a metric function.

**Lemma 4.2.1** \( d_p(A, B) \approx 0 \) if and only if \( \overline{A} = \overline{B} \).

**Proof.** Suppose \( d_p(A, B) \approx 0 \). Then \( d(a, B) \approx 0 \) for every \( a \in A \), which means \( a \approx b \) for some \( b \in B \). Hence \( A \subset \overline{B} \). Similarly \( B \subset \overline{A} \), whence \( \overline{A} = \overline{B} \).
Suppose $\overline{A} = \overline{B}$. Then for every $a \in A$, there is a $b \in B$ with $d(a, b) \approx 0$. Hence $d(a, B) \approx 0$ for every $a \in A$. Hence $\max_{a \in A} d(a, B) \approx 0$. Similarly $\max_{b \in B} d(A, b) \approx 0$. Hence $d_p(A, B) \approx 0$.

The rigid mesh metric continuum $\text{pow}^+(C) = (\{0, 1\}^{|C|} \setminus \{0\}, d_p, \approx)$ is called the power continuum of $C$.

**Proposition 4.2.2** The power continuum $\text{pow}^+(C)$ of a rigid mesh continuum $C$ is connected if $C$ is connected, perfect if $C$ is perfect and compact if $C$ is compact.

**Proof.** Suppose $C$ is connected. Let $A \subseteq |C|$. Put $A = \{a_1, \ldots, a_K\}$. Fix $c \in |C|$ and for each $i \in [1..K]$, let $a_i = a_i0, a_i1, \ldots, a_iL_i = c$ be a sorites sequence connecting $a_i$ to $c$. Let $L = \max\{L_i \mid i \in [1..K]\}$ and put $a_{ip} = c$ for $p > L_i$.

Define for $i \in [1..K]$,

$$A_i = \{a_{ij} \mid i \in [1..K], j \in [0..L]\}.$$ Then $A = A_0, A_1, \ldots, A_L$ is a sorites sequence. Define now

$$B_p = \{a_{ij} \mid i \in [1..K], j \in [p..L]\}.$$ Then $B_0 = A_L, B_1, \ldots, B_L = \{c\}$ is a sorites sequence. Hence every subset is connected by a sorites sequence to $\{c\}$ whence the power continuum $\text{pow}^+(C)$ is connected.

Suppose now $C$ is perfect and $A \subseteq |C|$ be nonempty and $a \in A$. Since $C$ is perfect, there is a huge subset $B$ whose elements are indistinguishable from $a$. For each $b \in B$, we have $A \approx A_b$ where

$$A_b := A \triangle \{b\},$$

$\triangle$ denoting the symmetric difference. Hence there are huge number of subsets indistinguishable from $A$. Hence the power continuum $\text{pow}^+(C)$ is perfect.

Suppose now $C$ is compact. Let $N$ be a huge number. Let $M$ be the number satisfying

$$2^M \leq N < 2^{M+1}.$$ Then $M$ is huge. Since $X$ is compact, there is a subset $A \subseteq X$ with $\#(A) \leq M$ and $\overline{A} = X$. Let $B \subseteq X$. Then for each $b \in B$, there is an $a_b \in A$ with $a_b \approx b$. Define

$$\tilde{B} := \{a_b \mid b \in B\} \subseteq A,$$ then $B \approx \tilde{B}$. Hence $\text{pow}(A) = \text{pow}(X)$, with $\#(\text{pow}(A)) = 2^\#(A) \leq 2^M \leq N$. Since $N$ is an arbitrarily huge number, the power continuum is compact.  

**4.3 Hypercube**

As in the previous subsection, we consider $\{0, 1\}^\infty$ as the power set of $X = [0, 1]_1 \setminus \{0\}$. Let $d_h$ be the distance function of the graph $\text{Hyper}_{\infty}^\infty$.  


whose nodes are subsets of $X$ and the edges are $\{A, A \cup \{b\}\}$ ($b \notin A$) with length $\frac{1}{\Omega}$. Obviously we have

$$d_h(A, B) = \frac{\#(A \Delta B)}{\Omega}.$$ 

The rigid mesh metric continuum $\left(\{0, 1\}^\Omega, d_h, \approx\right)$ is called the hypercube continuum of size $\Omega$. We remark that this continuum seems essentially the same as a metric space constructed in [CT08] from finite hypercubes by limiting process.

![Figure 2: Hypercubes on 3, 4, 5, 6, 7 nodes](image)

**Proposition 4.3.1** The hypercube continuum is connected and perfect but is not locally compact.

**Proof.** By similar arguments as in the proof of Proposition 4.2.2, the hypercube continuum is connected and perfect.

However the hypercube continuum is not compact. In fact we show that there are huge number of words with mutual distance greater than $\frac{1}{2}$. Choose a huge $M$ with $2^{M+1} < \Omega \leq 2^{M+2}$. For $i \leq M$, let $A_i$ be the set of integers less than
Continua of huge binary words

$2^{M+1}$ whose binary expansion have 1 on the $i$-th position. Then $\#(A_i) = 2^{M-1}$ and $\#(A_i \Delta A_j) = 2^{M-2}$ for $i \neq j$. Hence, since $2^{M+2} \geq \Omega$,

$$d_h(A_i, A_j) = \frac{2^{M-1}}{\Omega} \geq \frac{1}{8}.$$ 

Similar arguments show that the hypercube is not locally compact.

If we give uniform probability density on $X$, then $A \approx B$ if and only if $m(A \Delta B) \approx 0$. Hence the hypercube continuum of binary words is a special case of the continuum of the powerset of a probability space with this distance function. See §10.

**Remark 4.3.1** The metrics $d_p$ and $d_h$ on the power set $\{0, 1\}^\Omega$ are not comparable. For example, for

$$A = \left\{ \frac{i}{\Omega} \mid 2i \leq \Omega \right\}, \quad B = A \cup \{1\},$$

$A \approx_{d_h} B$ but $d_p(A, B) = \frac{1}{2}$. On the other hand, for

$$C = \left\{ \frac{2i}{\Omega} \mid 2i \leq \Omega \right\}, \quad D = \left\{ \frac{2i + 1}{\Omega} \mid 2i + 1 \leq \Omega \right\},$$

$C \approx_{d_p} D$ but $d_h(C, D) = 1$ since $C \Delta D = X$. 
§5 Continuum of Morphisms

One might think that this alternative mathematics cannot treat function spaces, for which “infinite sets” are indispensable. However every compact continua are represented by rigid mesh continua whose supports are sets and any class of morphisms between continua is represented by maps between sets, which forms a set.

In this section, we show how to formulate the continuum of morphisms between two continua and show an Ascoli-Arzela type theorem as an illustration showing the usability of our framework for “usual mathematics” involving infinite sets.

5.1 Continuum of Functions

Let $C_i$ $(i = 1, 2)$ be continua. We call two functions $f_i : |C_i| \to |C_2|$ $(i = 1, 2)$ are indistinguishable and write $f \approx g$ if $f(x) \approx g(x)$ for all $x \in C_1$.

If $C_1$ is a mesh continuum and $|C_2|$ is set-like, then we have a continuum $\text{Fun}(C_1, C_2) = (\text{Fun}(|C_1|, |C_2|), \approx)$, called the continuum of functions by Proposition 1.2.4.

Let $C_i$ $(i = 1, 2)$ be rigid mesh continua. Then the continuum $\text{Fun}(C_1, C_2)$ is a rigid mesh continuum with $\#(|C_2|)\#(|C_1|)$ virtual points.

The subclass of the continuous functions in $\text{Fun}(|C_1|, |C_2|)$ forms a mesh continuum, called the continuum of morphisms from $C_1$ to $C_2$ and written $C(C_1, C_2)$, which is a subcontinuum of $\text{Fun}(C_1, C_2)$. Note that a point of $C(C_1, C_2)$ is the collection of continuous functions indistinguishable from a fixed continuous function and hence is exactly a morphism from $C_1$ to $C_2$ introduced in §2.3.

Even if $C_i \approx C_i'$ $(i = 1, 2)$, the continuum $\text{Fun}(C_1, C_2)$ is not necessarily equivalent to $\text{Fun}(C_1', C_2')$ but the continua of morphisms are equivalent as is seen as follows.

**Proposition 5.1.1** Suppose $C_i$ $(i = 1, 2)$ are mesh continua and $C_i'$ $(i = 1, 2)$ are continua. Let $g_i : |C_i| \to |C_i'|$ $(i = 1, 2)$ be representations of equivalences with almost inverses $g_i^{-1} : |C_i'| \to |C_i|$ $(i = 1, 2)$. Define

$$\alpha : \text{Fun}(C_1, C_2) \to \text{Fun}(C_1', C_2')$$

by $\alpha(f) = g_2 \circ f \circ g_1^{-1}$, and

$$\beta : \text{Fun}(C_1', C_2') \to \text{Fun}(C_1, C_2)$$
by
\[ \beta(f') = g_2^{-1} \circ f' \circ g_1. \]

\[
\begin{array}{c}
C_1 & \xrightarrow{f} & C_2 \\
g_1^{-1} & \approx & g_2 \\
C_1' & \xrightarrow{\alpha(f)} & C_2' \\
g_1^{-1} & \approx & g_2^{-1}
\end{array}
\]

Then
\[ \alpha : |C(C_1, C_2)| \rightarrow |C(C_1', C_2')| \]
represents an equivalence with an almost inverse \( \beta \).

**Proof.** To show \( \alpha \) is continuous, suppose \( f \approx f' \). Then \( f(g_1^{-1}(x)) \approx f'(g_1^{-1}(x)) \)
whence
\[ \alpha(f)(x) = g_2(f(g_1(x))) \approx g_2(f'(g_1(x))) \approx \alpha(f')(x). \]
Similarly \( \beta \) is continuous.

To show that \( \beta \) is an almost inverse of \( \alpha \) we need the continuity of \( f \) and \( f' \).
Since \( f \) is continuous, so is \( \alpha(f) \). Hence
\[ \beta(\alpha(f)) \approx f \]
since
\[ \beta(\alpha(f))(x) = g_2^{-1}g_2(f(g_1(g_1^{-1}(x)))) \approx g_2^{-1}(g_2(f(x))) \approx f(x). \]
Similarly \( \alpha(\beta(f')) \approx f' \) if \( f' \) is continuous.

The following is the well-known lemma which plays important roles everywhere.

**Lemma 5.1.2 (Robinson)** Let \((X,d,\approx)\) be a metric continuum. Let \((a_1,\ldots,a_M),(b_1,\ldots,b_M)\)
be huge sequences in \(X\). If \(a_i \approx b_i\) for accessible \(i\), then it holds for \(i \in [1..K]\)
for some huge \(K \leq M\).

**Proof.** Since the objective condition \(d(a_i,b_i) < \frac{1}{\ell}\) holds for all accessible \(i\), it
holds for \(i \leq K\) for some huge \(K \leq M\). For huge \(I \leq K\), \(d(a_I,b_I) < \frac{1}{\ell}\)
implies \(a_I \approx b_I\).

The continuity can be rephrased by an \(\varepsilon - \delta\)-like condition.

**Proposition 5.1.3** If \(C_i = (|C_i|,d_i,\approx_i)\) \((i = 1,2)\) are rigid mesh metric continua, then a map \(f : |C_1| \rightarrow |C_2|\) is continuous if and only if for every accessible \(k\), there is an accessible \(\ell\) such that for all \(x,y \in |C_1|\), \(d(x,y) < \frac{1}{\ell}\)
implies \(d(f(x),f(y)) < \frac{1}{\ell}\).
Proof. Suppose \( f \) is continuous. Let \( k \) be an accessible number. For every huge number \( N \), the condition \( d(x, y) < \frac{1}{N} \) implies \( x \approx y \), \( f(x) \approx f(y) \) and hence \( d(f(x), f(y)) < \frac{1}{k} \). Since the condition on \( i \) that
\[
d(x, y) < \frac{1}{i} \text{ implies } d(f(x), f(y)) < \frac{1}{k}
\]
is objective and satisfied by every huge \( i \), we have an accessible \( i \) for which (15) holds.

Conversely suppose that for every accessible \( k \), there is an accessible \( \ell \) such that (15) holds for \( i = \ell \). Then, for every accessible \( k \), \( x \approx y \) implies \( d(f(x), f(y)) < \frac{1}{k} \), since \( d(x, y) < \frac{1}{k} \). Hence \( f(x) \approx f(y) \).

5.2 Ascoli-Arzela Theorem

A map \( \kappa : [1..N] \to [1..M] \) with huge \( N, M \) is called a scale of approximation if it satisfies the condition that \( i \) is accessible if and only if \( \kappa(i) \) is accessible.

Lemma 5.2.1 Suppose \( (X_i, d_i, \approx_i) \ (i = 1, 2) \) are rigid mesh metric continua, Then a map \( f : X_1 \to X_2 \) is continuous if there is a scale of approximation \( \kappa : [1..N_2] \to [1..N_1] \) and a huge number \( K \leq N_2 \) such that
\[
d(x, y) < \frac{1}{\kappa(i)} \text{ implies } d(f(x), f(y)) < \frac{1}{i}
\]
holds for every \( i \leq K \) and \( x, y \in X_1 \).

Proof. Suppose there is a scale of approximation \( \kappa \) such that (16) holds for every \( i \leq K \) with a huge \( K \). Suppose \( x \approx y \). Let \( k \) be an accessible number.

Since \( d(x, y) < \frac{1}{\kappa(k)} \), we have \( d(f(x), f(y)) < \frac{1}{k} \). Hence \( f(x) \approx f(y) \).

Let \( C_i = (X_i, d_i, \approx_i) \ (i = 1, 2) \) be rigid mesh metric continua. For a scale of approximation \( \kappa : [1..N_2] \to [1..N_1] \), denote by \( C_\kappa(C_1, C_2) \) the set of functions \( f : |C_1| \to |C_2| \) satisfying
\[
d(x, y) < \frac{1}{\kappa(i)} \text{ implies } d(f(x), f(y)) < \frac{1}{i}
\]
for all \( i \in [1..N_2] \) and \( x, y \in |C_1| \).

By Lemma 5.2.1 \( C_\kappa(C_1, C_2) \) is a subclass of \( C(C_1, C_2) \) and in fact is a subset since the condition of membership is objective.

A class \( F \) of morphisms from \( C_1 \) to \( C_2 \) is called equicontinuous if there is a scale of approximation \( \kappa \) such that \( F \subset C_\kappa(C_1, C_2) \).

The following shows that a set of morphisms from \( C_1 \) to \( C_2 \) is necessarily equicontinuous.
Proposition 5.2.2 Suppose \( C_i = (X_i, d_i, \approx_i) \) \((i = 1, 2)\) are rigid mesh metric continua. Then for every subset \( F \subseteq |C(C_1, C_2)| \), there is a scale of approximation such that \( F \subseteq C_\kappa(C_1, C_2) \).

Proof. For each number \( i \), define

\[
r_F(i) := \min \left\{ d(x, y) \mid d(f(x), f(y)) \geq \frac{1}{i} \text{ for some } f \in F \right\}.
\]

If \( d(x, y) < r_F(i) \) then \( d(f(x), f(y)) < \frac{1}{i} \) for all \( f \in F \). Define

\[
\kappa(i) := \max \left\{ \left( \frac{1}{r_F(i)} \right) + 1, i \right\}.
\]

Then \( \kappa(i) \geq i \) and \( \kappa(i) > \frac{1}{r_F(i)} \), whence \( \frac{1}{r_F(i)} < \kappa(i) \). Hence

\[
d(x, y) < \frac{1}{\kappa(i)} \text{ implies } d(f(x), f(y)) < \frac{1}{i}
\]

holds for all \( f \in F \). By Proposition 5.1.3 \( \kappa(i) \) is accessible if \( i \) is accessible and \( \kappa \) is a scale of approximation. Hence \( F \subseteq C_\kappa(C_1, C_2) \).

Theorem 5.2.3 (Ascoli-Arzela) If continua \( C_i \) \((i = 1, 2)\) are compact rigid mesh metric continua, then every subcontinuum of \( C(C_1, C_2) \) with set support is compact. In particular, if \( \kappa \) is a scale of approximation, then \( C_\kappa(C_1, C_2) \) is compact.

Proof. Let \( K \) be a huge number. Select a huge \( L \) with \( L^L \leq K \). Since \( C_i \) \((i = 1, 2)\) are compact, there are dense subsets \( A_i \subseteq |C_i| \) with \( \#(A_i) \leq L \) \((i = 1, 2)\). Let \( F \subseteq |C(C_1, C_2)| \) be a subset.

For each \( f \in F \) and \( a \in A_1 \), choose an element \( b \in A_2 \) such that \( f(a) \approx b \) and put \( f(a) := b \). Then \( f \in \text{Fun}(A_1, A_2) \). Define a map \( \alpha : F \to \text{Fun}(A_1, A_2) \) by \( \alpha(f) := f \).

Then \( \alpha \) is injective, namely, \( \alpha(f) \approx \alpha(g) \) implies \( f \approx g \). In fact, suppose \( \alpha(f) \approx \alpha(g) \) and \( x \in |C_1| \). Choose \( y \in A_1 \) such that \( x \approx y \). Then

\[
f(x) \approx f(y) \approx \alpha(f)(y) \approx \alpha(g)(y) \approx g(y) \approx g(x).
\]

Hence \( f \approx g \).

Let \( \beta : \alpha(F) \to |C(C_1, C_2)| \) be a right inverse of \( \alpha \), namely, \( \alpha(\beta(g)) \equiv g \). Then \( \text{Im}(\beta) \) is dense in \( F \). In fact, for each \( f \in F \), \( \alpha(\beta(\alpha(f))) \equiv \alpha(f) \) implies \( f \approx \beta(\alpha(f)) \).

Since

\[
\#(\text{Im}(\beta)) \leq \#(\alpha(F)) \leq \#(\text{Fun}(A_1, A_2)) \leq L^L < K,
\]

we have a dense subset of \( F \) with the number of elements less than \( K \). Hence \( F \) is compact. The latter assertion is the special case of the former since \( C_\kappa(C_1, C_2) \) is a set.
Corollary 5.2.4 (Ascoli-Arzela) An equicontinuous concrete sequence of morphisms between compact rigid mesh metric continua has an accumulation point.

Proof. Suppose rigid mesh metric continua $C_i$ ($i = 1, 2$) are compact. Let $f = (f_1, f_2, \cdots)$ be an equicontinuous concrete sequence of morphisms from $C_1$ to $C_2$. Then $f$ is a concrete sequence in the continuum $C_\kappa(C_1, C_2)$ for some scale $\kappa$ of approximation, which is compact by Theorem 5.2.3 whence has an accumulation point by Corollary 3.4.7.
6 Real Numbers

6.1 Real Numbers

A point of the continuum \( \mathbb{R} = (\mathbb{Q}, \approx) \) is called a real number, namely a real number is a class

\[
[r] := \{ s \in \mathbb{Q} \mid s \approx r \}
\]

for some rational number \( r \in \mathbb{Q} \). A real number \( a \) is said to be represented by a rational number \( r \) if \( a = [r] \).

By Proposition 2.4.1, we have the following.

Lemma 6.1.1 If \( \varepsilon > 0 \) is an infinitesimal, we have \( r \approx [\frac{r}{\varepsilon}]\varepsilon \) for every \( r \in \mathbb{Q} \). In particular, every real number is represented by a rational of the form \( m\varepsilon \) with \( m \in \mathbb{Z} \).

A representation of a real number by a rational in \( \varepsilon \mathbb{Z} \) is called \( \varepsilon \)-separate.

Since each real number is a proper class, the equality of real numbers is not a definite condition and the collection of real numbers do not form a class. In other words, the “a real number” should not be regarded as a definite object. They have uneliminable indefiniteness indicated by the sorites paradox that \( x_1 = x_2 = \cdots = x_N \) but \( x_1 \neq x_N \) if the equality of real numbers had definite meaning.

We defined the relations \( r \approx s, r < s \) and \( r \leq s \) for rationals \( r, s \) in §1.1.2 which induce relations of real numbers \( p = q, p < q \) and \( p \leq q \) respectively owing to Proposition 1.1.2. A real number \( p \) is called positive and negative respectively when \( p > 0 \) and \( p < 0 \).

The absolute value \( |p| \) of a real number \( p = [r] \) is defined by \( |p| := |[r]| \).

The following is obvious but shows that there are no nonzero infinitesimal reals.

Proposition 6.1.2 If a real number \( p \) satisfies \( |p| \leq \frac{1}{k} \) for every accessible number \( k \), then \( p = 0 \).

We also defined the notion of finiteness of rationals in §1.1.1 which induces finiteness of real numbers. A real number is called commensurable if it is represented by an accessible rational number. Since for accessible rational numbers \( r, s \), the indistinguishability implies equality, every commensurable real number is represented by a unique accessible rational number. We identify each accessible rational number with the commensurable real number represented by it. For example, the accessible rational number \( \frac{1}{2} \) denotes also the commensurable real number \( \lfloor \frac{1}{2} \rfloor \).
In the following, we define following operations and functions of real numbers.

1. Addition and multiplication of finite real numbers,

2. For accessible number $n$, the $n$-power of finite real numbers, and $n$-th root of finite non-negative real numbers.

3. Exponentiation of finite real numbers and logarithm of positive real numbers,

4. Power of finite positive real numbers to finite real numbers.

### 6.2 Arithmetic Operations

The arithmetic operations on rational numbers induce those on real continuum.

**Lemma 6.2.1** Suppose $r, s, r_i, s_i \in \mathbb{Q}$ satisfy $r \approx s$ and $r_i \approx s_i$ ($i = 1, 2$).

1. $r_1 + r_2 \approx s_1 + s_2$.

2. If $r_i$ ($i = 1, 2$) are finite then $s_i$ ($i = 1, 2$) are finite and
   
   $r_1 r_2 \approx s_1 s_2$.

3. If $r$ is not infinitesimal, then $s$ is neither infinitesimal and both $\frac{1}{r}$ and $\frac{1}{s}$ are finite and satisfies
   
   \[ \frac{1}{r} \approx \frac{1}{s}. \]

4. If $r$ is finite and $n$ is an accessible number then
   
   $r^n \approx s^n$.

**Proof.** Suppose $r_i$ ($i = 1, 2$) are finite. Since a rational indistinguishable from a finite rational is finite, $s_i$ ($i = 1, 2$) are finite. From

\[
|r_1 r_2 - s_1 s_2| \leq |r_1||r_2 - s_2| + |s_2||r_1 - r_2| \leq k(|r_2 - s_2| + |r_1 - r_2|)
\]

where $k = \max \{|r_1|, |s_2|\} < \infty$ it follows $r_1 r_2 \approx s_1 s_2$.

Let $\text{inv} : \mathbb{Q} \setminus \{0\} \to \mathbb{Q}$ be the function $\text{inv}(r) = \frac{1}{r}$. If $r \neq 0$, then $s \neq 0$ hence there is an accessible $k$ such that $|r|, |s| > \frac{1}{k}$. Hence

\[
\left| \frac{1}{r} - \frac{1}{s} \right| = \frac{|r - s|}{|rs|} < k|r - s| \approx 0.
\]
On the other hand the condition $|r|, |s| < \infty$ implies that $|\frac{1}{r}|, |\frac{1}{s}| \neq 0$. Hence $inv$ represents a morphism from $(-\infty, 0) \cup (0, \infty)$ to itself.

Put $K = n(\max \{ |r|, |s| \})^n$, then $K$ is finite and

$$|r^n - s^n| \leq K|r - s|,$$

hence $r^n \approx s^n$.

Hence we have the following morphisms

**Theorem 6.2.2**

1. The addition defines a morphism

   $$+ : \mathbb{R}^2 \to \mathbb{R}.$$

2. The multiplication defines a morphism

   $$\times : (-\infty, \infty)^2 \to (-\infty, \infty).$$

3. The inverse defines an equivalence

   $$(-)^{-1} : (-\infty, 0) \cup (0, \infty) \to (-\infty, 0) \cup (0, \infty).$$

4. The powers $r \mapsto r^n$ defines morphisms

   $$\text{pow}_n : (-\infty, \infty) \to (-\infty, \infty)$$

   for each accessible $n$.

We express this symbolically by the following point wise “definition” on real numbers.

**Definition 6.2.1** Let $p = [r]$ and $q = [s]$.

1. $p + q := [r + s],$
2. $pq := [rs], \text{ when } p, q \text{ are finite},$
3. $\frac{1}{p} = [\frac{1}{r}], \text{ when } p \neq 0,$
4. $\frac{p}{q} := p\frac{1}{q}, \text{ when } q \neq 0,$
5. $p^n := [r^n], \text{ when } p \text{ is finite and } n \text{ is accessible}.$
It should be noted that since the real numbers are vague objects without definite identity, precise meaning of this definition is given by the above Theorem 6.2.2.

The usual axiom of field is satisfied by these operations. Let 0 and 1 denotes the commensurable real numbers represented by the rational 0 and 1 respectively. Define \(-[r] := [-r]\).

**Proposition 6.2.3** Let \(p, q, r\) be finite real numbers. Then

1. The addition and multiplication are associative and commutative.
2. \(0 + p = p, 1 \times p = p\),
3. \(p + (-p) = 0\),
4. if \(p \neq 0\) then \(p \times \frac{1}{p} = 1\),
5. \(p \times (q + r) = p \times q + p \times r\).

This implies for example the following.

**Lemma 6.2.4** If \(a, b\) are finite rationals and \(a\) is not infinitesimal, then \(a \approx b\) if and only if \(a^{-b} \approx 1\).

**Proof.** Put \(p = [a]\) and \(q = [b]\). Then the statement means \(p = q\) if and only if \(\frac{p}{q} = 1\), which follows from Proposition 6.2.3.

Let \(\varepsilon > 0\) be an infinitesimal. Then the above operations can be realized by those on the \(\varepsilon\)-separate representations. These representations of operations of reals are considered fundamental in the computational treatments of real numbers. See [RR96, CWF+09] for example.

**Proposition 6.2.5** (1) The addition defines a morphism
\[+ : (\varepsilon\mathbb{Z}, \approx)^2 \rightarrow (\varepsilon\mathbb{Z}, \approx)\]

(2) The multiplication is represented by the morphism
\[\text{mult}_\varepsilon : ((-\infty, \infty)_\varepsilon, \approx)^2 \rightarrow ((-\infty, \infty)_\varepsilon, \approx)\]
defined by
\[\text{mult}_\varepsilon(n\varepsilon, m\varepsilon) := [mn\varepsilon]\varepsilon \quad m, n \in \mathbb{Z}\]

(3) The inverse is represented by the morphism
\[\text{inv}_\varepsilon : ((-\infty, 0)_\varepsilon \cup (0, \infty)_\varepsilon, \approx) \rightarrow ((-\infty, 0)_\varepsilon \cup (0, \infty)_\varepsilon, \approx)\]
defined by
\[\text{inv}_\varepsilon(n\varepsilon) = \left\lfloor \frac{1}{n\varepsilon^2} \right\rfloor \varepsilon\]
(4) If \( n \) is an accessible number, the power morphism \( r \mapsto r^n \) is represented by the morphism

\[
\text{pow}_{n,\varepsilon} : ((-\infty, \infty), \varepsilon) \to ((-\infty, \infty), \varepsilon)
\]

defined by

\[
\text{pow}_{n,\varepsilon}(k\varepsilon) := \left[ k^{n-1} \varepsilon \right].
\]

As for the root operation, we can define it only through representations.

**Lemma 6.2.6** Let \( k \) be an accessible number and \( \varepsilon \) a positive infinitesimal. Then for each finite positive rational number \( x \),

\[
\text{root}_{k,\varepsilon}(x)^k \approx x,
\]

where

\[
\text{root}_{k,\varepsilon}(x) := \varepsilon \max \left\{ m \in \mathbb{N} \mid (m\varepsilon)^k \leq x \right\}.
\]

Moreover \( \text{root}_{k,\varepsilon}(x) = 0 \) if and only if \( x = 0 \).

**Proof.** Put \( m := \max \left\{ m \in \mathbb{N} \mid (m\varepsilon)^k \leq x \right\} \). Then

\[
(m\varepsilon)^k \leq x < ((m + 1)\varepsilon)^k.
\]

Since \( m\varepsilon \approx (m + 1)\varepsilon \), we have \( (m\varepsilon)^k \approx ((m + 1)\varepsilon)^k \) by the accessibility of \( k \) and hence \( x \approx (m\varepsilon)^k \).

**Lemma 6.2.7** If finite nonnegative rationals \( u, v \) satisfy \( u^k \approx v^k \) for an accessible \( k \), then \( u \approx v \). In particular, the function \( \text{root}_{k,\varepsilon} \) is continuous for infinitesimal \( \varepsilon > 0 \).

**Proof.** Suppose \( u \neq v \) but \( u^k \approx v^k \). We may assume \( u \prec v \). Then \( u + \frac{1}{k} < v \) for some accessible \( k \). We may assume \( 0 \prec u \). Then \( \frac{nu^{k-1}}{k} \succ 0 \), whence

\[
u^n < u^n + \frac{nu^{k-1}}{k} \leq (u + \frac{1}{k})^n < v^n
\]

which contradicts \( u^n \approx x \approx y \approx v^n \). Hence \( u \approx v \).

Suppose rational numbers \( a, b \) satisfy \( a \approx b \). Then

\[
\left( \text{root}_{k,\varepsilon}(a) \right)^k \approx a \approx b \approx \left( \text{root}_{k,\varepsilon}(b) \right)^k
\]

and the above conclusion implies \( \text{root}_{k,\varepsilon}(a) \approx \text{root}_{k,\varepsilon}(b) \).

Hence we have proved
Theorem 6.2.8 If $k$ is accessible then the power operator
\[ \text{pow}_k : [0, \infty) \rightarrow [0, \infty) \]
represents an equivalence. For each nonzero infinitesimal $\varepsilon$, the function
\[ x \mapsto \text{root}_{k,\varepsilon}([x/\varepsilon]\varepsilon) \]
is an almost inverse of the power operator $\text{pow}_k$.

From this we define the $k$-th root $p^{\frac{1}{k}}$ of a finite nonnegative real number $p = [r]$ by
\[ p^{\frac{1}{k}} := \text{root}_{k,\varepsilon}(r), \]
where $\varepsilon > 0$ is an infinitesimal. The above theorem shows that this does not depend on the choice of $r$ and $\varepsilon$ and the following holds:
\[ (p^{\frac{1}{k}})^k = p, (p^{k})^{\frac{1}{k}} = p. \]

If $s = \frac{\ell}{k}$ is accessible, namely, $\ell, k$ are accessible numbers, we define for a finite positive real number $p$
\[ p^{\frac{\ell}{k}} := (p^{\frac{1}{k}})^{\ell}. \]

Remark 6.2.1 The exponentiation $x^y$ for general finite $x, y$ will be defined as $\exp(y \log x)$ after defining the exponentiation $\exp(x)$ and the logarithm function $\log$ as the inverse of $\exp$.

6.3 Sequence

In §3.4 we defined convergence of concrete sequences and Cauchy sequences on on metric spaces.

Two concrete sequence of rational numbers $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ are indistinguishable if $a_i \approx b_i$ for all $i$. A concrete sequence of real numbers is the collection of the concrete sequence of rational numbers indistinguishable with one such $(a_1, a_2, \cdots)$. This is not a class but we use the symbol $[a] = ([a_1], [a_2], \cdots)$ to denote this collection.

If $p = (p_1, p_2, \cdots)$ is a concrete sequence of real numbers, then a concrete sequence of rational numbers $a = (a_1, a_2, \cdots)$ is said to represent $p$ if $a_i \in p_i$ for all $i$. Note that we cannot form a representation by arbitrarily choosing elements of each $p_i$. 
A concrete sequence of real numbers \((p_1, p_2, \cdots)\) converges to a real number \(q\) if for each accessible number \(k\) there is an accessible number \(\ell\) such that for every accessible \(i \geq \ell\) we have
\[
|p_i - q| < \frac{1}{k}.
\]
By Proposition \[6.1.2\], such \(q\) is uniquely determined and is called the limit of the sequence \(p\) and is denoted by \(\lim_{i \to \infty} p_i\).

We say that a concrete sequence of real numbers \(p = (p_1, p_2, \cdots)\) is a \emph{Cauchy sequence} if for every accessible \(k\), there is an accessible \(\ell\) such that for every accessible \(i, j \geq \ell\) we have
\[
|p_i - p_j| < \frac{1}{k}.
\]
This means that \(p\) is represented by a Cauchy concrete sequence of rational numbers.

By Proposition \[3.4.1\], a concrete sequence of rational numbers \(a\) converges if and only if it is a Cauchy sequence whence we have the following “completeness” of the metric continuum \(\mathbb{R}\).

**Theorem 6.3.1** Concrete Cauchy sequences of real numbers converge.

We have also

**Theorem 6.3.2** An increasing concrete sequence of real numbers bounded from above converges.

**Proof.** Let \(p = (p_1, p_2, \cdots)\) be a concrete sequence of real numbers such that \(p_i \leq p_{i+1}\) for all \(i\) and, for some some accessible number \(k\), \(p_i \leq k\) for all \(i\).

Let \(a_1, a_2, \cdots\) be a concrete sequence of rational numbers representing \(p\). Then \(a_i \preceq a_{i+1}\) and \(a_i \preceq k\) for all \(i\).

Since the continuum \([a_1, k]\) is compact by Proposition \[3.2.1\], the sequence has an accumulation point \(c\). Hence for every accessible \(\ell\), the numbers \(i \geq \ell\) satisfying
\[
|a_i - c| < \frac{1}{\ell}.
\]  
(19)
is not finite. Let \(i_0\) be one such number. If there is an accessible \(j > i_0\) with \(c + \frac{1}{\ell} \leq a_j\), then \(j < m\) implies
\[
c + \frac{1}{2\ell} < c + \frac{1}{\ell} \leq a_j \leq a_m
\]
and hence the number of \(i\) satisfying \[19\] with \(\ell\) replaced by \(2\ell\) is less than or equal to \(j\), a contradiction. Hence \(i_0 < i\) implies \[19\], which means that \((a_1, a_2, \cdots)\) converges to \(c\).
6.4 Series

The addition of rationals can be extended to a function

\[ \mathbb{Q}^N \ni (a_1, \cdots, a_N) \mapsto \sum_{i \in [1..N]} a_i \in \mathbb{Q}. \]

However this does not define a morphism \( \mathbb{R}^N \to \mathbb{R} \) since generally \( \sum_{i=1}^N a_i \neq \sum_{i=1}^N b_i \) even if \( a_i \approx b_i \) (\( i \in [1..N] \)).

We call that the sum \( \sum_{i=1}^N a_i \) converges if the following holds.

\[ \sum_{i=1}^N a_i \approx 0 \text{ for every huge } I \leq N. \tag{20} \]

Similarly the huge sum \( \sum_{i=1}^N a_i \) converges absolutely if \( \sum_{i=1}^N |a_i| \approx 0 \) for every huge \( I \leq N \).

Note that if we define

\[ S_k := \sum_{i=1}^k a_i, \quad k \in [1..N], \]

the condition (20) is equivalent to the convergence of the sequence \((S_1, \cdots, S_N)\).

The following can be easily proved.

**Lemma 6.4.1** If \( a_i \approx b_i \) for \( i \in [1..K] \) and \( \sum_{i=1}^{[1..K]} a_i \) converges then for some huge \( L \leq K \), \( \sum_{i=1}^{[1..L]} b_i \) converges and their limits coincide up to indistinguishability.

**Proof.** Put \( \varepsilon = \max \{ |a_i - b_i| \mid i \in [1..K] \} \approx 0 \). Since \( n \varepsilon \approx 0 \) for every accessible \( n \), we can choose a huge \( L \) such that \( L \varepsilon \approx 0 \). Then for huge \( I \leq L \),

\[
\sum_{i \in [I..L]} |b_i| \leq \sum_{i \in [I..L]} |b_i - a_i| + \sum_{i \in [I..L]} |a_i| \\
\leq \varepsilon L + \sum_{i \in [I..L]} |a_i| \approx 0.
\]

hence the sum \( \sum_{1 \leq i \leq L} b_i \) converges. Moreover

\[
\left| \sum_{i \in [1..L]} a_i - \sum_{i \in [1..L]} b_i \right| \leq \sum_{i \in [1..L]} |a_i - b_i| \leq \varepsilon L \approx 0.
\]

A point of \( \mathbb{R}^N \) is called a sequence of real numbers and is denoted by \( p = (p_1, \cdots, p_N) \). It is represented by a sequence of rational number
a = (a_1, \cdots, a_N) \in \mathbb{Q}^N. We say that p converges if its representation converges and define the sum \( \sum_i p_i = [\sum_i a_i] \). By Lemma 6.4.1, the condition of convergence and the value of the sum are independent of the choice of representations.
7 Real Functions on Continua

7.1 Real Functions

Let $C$ be a continuum. A morphism from $C$ to $\mathbb{R}$ is called a real function on $C$. Recall it is a formal symbol $[f]$ where $f$ is a rational valued continuous function on $|C|$. See §2.3.

The value of a real function $F$ at a point $p$ of $C$ is defined to be the real number $[f(t)]$ for $f \in F$ and $t \in p$. This does not depend on the choice of representations.

We saw in §5.1 that if $C$ is a mesh continuum, the continuous rational valued functions form a subcontinuum $C(C, \mathbb{R}) \subset \text{Fun}(|C|, \mathbb{Q})$ and the indistinguishability condition is definite. Hence in this case, the symbol $[f]$ can be interpreted by the class $\{ g \in C(C, \mathbb{R}) \mid g \approx f \}$ and the above definition of the symbol of real function conforms to this interpretation.

Suppose $\alpha : C_1 \to C_2$ is a morphism between continua $C_i$ ($i = 1, 2$).

If $F$ is a real function on $C_2$ represented by $f$, then the real function $[f \circ \alpha]$ does not depend on $f$ since $f \approx g$ implies $f \circ \alpha \approx g \circ \alpha$. The real function $[f \circ \alpha]$ on $C_1$ is called the pull back of $F$ by $\alpha$ and denoted by $F \circ \alpha$. Note that if $\alpha \approx \alpha'$, then $F \circ \alpha = F \circ \alpha'$ since $f \circ \alpha \approx f \circ \alpha'$.

A representation of a real function $F$ on $C$ is defined to be a pair $(f, \alpha)$, where $\alpha : C \to C_1$ is an equivalence of continua and $f$ is a rational valued continuous function on $C_1$ such that $f \circ \alpha \in F$. Obviously we have the following.

**Proposition 7.1.1** Suppose $\alpha : C \to C_1$ is an equivalence of continua. The assignment $F \mapsto F \circ \alpha$ defines a one-to-one correspondence from the collection of real functions on $C_1$ onto those on $C$. In particular, every real function is represented as $(f, \alpha)$ for some rational valued continuous function $f$ on $C_1$.

Let $D$ be a subcontinuum of the linear continuum $\mathbb{R}$. A continuous rational valued function $f$ on $C$ is called $D$-valued if $f(x) \in D$ for all $x \in C$.

**Proposition 7.1.2** If $C$ is a mesh continuum, the condition of being $D$-valued is definite. In particular, the $D$-valued continuous rational valued functions on $C$ forms a subcontinuum $C(C, D) \subset C(C, \mathbb{R})$. 
Proof. Let $f$ be a real function on $C$. Let $\tilde{f} : b \to \mathbb{Q}$ be an extension of it. Then $f$ is $D$-valued if and only if it satisfies the bounded condition
$$\exists b' \subset b \ \forall x \in b' \ [x \in |C| \text{ implies } \tilde{f}(x) \in D].$$

Note that $D$-valuedness is not objective condition in general but if $D$ is an objective subclass and $C$ is rigid then it is objective.

Let $I$ be an interval symbol defined in §2.2.1 and $C$ is a mesh continuum then $I$-valued continuous rational valued function on $C$ defines a subcontinuum denoted by $C(C, I) \subset C(C, \mathbb{R})$. A continuous rational valued function $f$ is called finite if $f$ is $(-\infty, \infty)$-valued.

Let $C$ be a continuum. A real function $F$ on $C$ is called $D$-valued if it is represented by a $D$-valued continuous rational valued function. In particular $F$ is called finite if it is represented by a finite continuous rational valued function.

Let $C$ be a mesh continuum. Let $D_i \ (i = 1, 2)$ be subcontinua of $\mathbb{R}$ and $\beta : D_1 \to D_2$ be a quasi-identity in the sense explained in §2.4. If $f$ is a $D_1$-valued continuous rational valued function, then $\beta \circ f$ is $D_2$-valued and if $f_1 \approx f_2$, then $\beta \circ f_1 \approx \beta \circ f_2$ whence $\beta$ induces a morphism
$$\beta_* : C(C, D_1) \to C(C, D_2)$$
which is an equivalence since $\gamma_\epsilon$ is an almost inverse whenever $\gamma$ is an almost inverse of $\beta$.

Note that a real function $F$ is $D_1$-valued if and only if $D_2$-valued, since if $f \in F$ is $D_1$-valued then $f \approx \beta \circ f$ is $D_2$-valued and hence $F$ is also represented by $D_2$-valued function.

For example if $\epsilon > 0$ is infinitesimal, the inclusion function
$$\iota_\epsilon : \epsilon \mathbb{Z} \to \mathbb{Q}$$
defines an equivalence
$$\iota_{\epsilon*} : C(C, \epsilon \mathbb{Z}) \to C(C, \mathbb{R})$$
with the almost inverse given by $\kappa_{\epsilon*}$ where $\kappa_{\epsilon} : \mathbb{Q} \to \epsilon \mathbb{Z}$ is the almost inverse defined in Proposition 2.4.2.

Similarly, for every interval symbol $I$ and an infinitesimal $\epsilon > 0$, we have an equivalence
$$\iota_{\epsilon*} : C(C, I_{\epsilon}) \to C(C, I).$$

Note that even if $C$ is not a mesh continuum, every real function $F = [f]$ on $C$ is represented by an $\epsilon \mathbb{Z}$-valued continuous function such as $\kappa_{\epsilon} \circ f$. Similarly every $I$-valued real function $F = [f]$ on $C$ is represented by an $I_{\epsilon}$-valued function $\kappa_{\epsilon} \circ f$. 
Proposition 7.1.3 Suppose a continuum $C$ has a dense subcontinuum $M$ with an almost inverse $\kappa$ for the inclusion morphism $i : M \to C$. Then the assignments $i^* : F \mapsto F \circ i$ and $\kappa^* : G \mapsto G \circ \kappa$ are inverse to one another and defines a one-to-one correspondence between the real functions on $C$ and those on $M$.

Proof. Since $\kappa \circ i = id_M$ and $i \circ \kappa \approx id_C$, $F \circ i \circ \kappa = F$ and $G \circ \kappa \circ i = G$. 

Thus if $C$ and $D \subset \mathbb{R}$ are continua and there are dense mesh subcontinua $i : C_0 \subset C$ and $j : D_0 \subset D$ whose inclusions morphisms have almost inverses

$$\lambda : C \to C_0 \text{ and } \kappa : D \to D_0,$$

then the correspondence $F \leftrightarrow \kappa \circ F \circ i$ defines one-to-one correspondence between $D$-valued real functions on $C$ and $D_0$-valued real functions on $C_0$. Hence although there is no such continuum as “$C(D)$”, we can treat $D$-valued real functions on $C$ via mesh continua such as $C(C_0, D_0)$.

Composition Suppose $C$ is a mesh continuum and $D \subset \mathbb{R}$. Let $F$ be a $D$-valued real function on $C$ and $G$ be a real function on $D$. Then a real function $G \circ F$ is defined by

$$G \circ F := [g \circ f]$$

with $f \in F$ and $g \in G$. This is well-defined since $f \approx f'$ and $g \approx g'$ implies $g \circ f \approx g' \circ f'$.

However this “point wise definition” cannot be given precise meaning as a morphism

$$C(C, D) \times C(D', \mathbb{R}) \to C(C, \mathbb{R})$$

since there are no such continuum as “$C(D, \mathbb{R})$”. However if $D' \subset D$ is a dense mesh subcontinuum with an almost inverse $\kappa : D \to D'$. Then the real functions on $D$ corresponds to those on $D'$ in bijective way and we can take $C(D', \mathbb{R})$ as one realization of the phantom “$C(D, \mathbb{R})$”.

Then the composition $(F, G) \mapsto G \circ F$ is realized by the morphism

$$\gamma : C(C, D) \times C(D', \mathbb{R}) \to C(C, \mathbb{R})$$

defined by $(f, g) \mapsto g \circ \kappa \circ f$.

This does not depend on the choice of $D'$ in the sense that the following diagram commutes up to indistinguishability, whenever $\kappa_i : D \to D'_i$ ($i = 1, 2$) are almost inverse of the inclusions and $\beta$ is the restriction of $\kappa_2$ on $D'_2$. 
7.2 Examples

7.2.1 Polynomial Functions

Let \( f(x) \) be a polynomial

\[
    f(x) = \sum_{i=0}^{n} a_i x^i
\]

with \( n \) accessible and \( a_i \in (-\infty, \infty)_\mathbb{Q} \). Then the function \( r \mapsto f(r) \) is continuous on \((-\infty, \infty)_\mathbb{Q}\) and defines a real function \( \lambda x. P(x) \) on \((-\infty, \infty)\), called the polynomial functions defined by \( f \). It is denoted by

\[
    F(x) := \sum_{i=0}^{n} p_i x^i,
\]

where \( p_i = [a_i] \) and is called the real polynomial of degree \( n \) if \( p_n \neq 0 \). For a finite real number \( t \), its value is

\[
    F(t) := \sum_{i=0}^{n} p_i t^i.
\]

7.2.2 Exponential

For huge \( T \) and rational \( r \), define a rational number by

\[
    \exp(r, T) := \sum_{i=0}^{T} \frac{r^i}{i!}.
\]

**Proposition 7.2.1** If \( r \in (-\infty, \infty)_\mathbb{Q} \), the sum \( \exp(r, T) \) converges. In particular, if \( T, S \) are huge, then

\[
    \exp(r, T) \approx \exp(r, S).
\]

The proposition follows directly from the following Lemmas 7.2.2, 7.2.3.
Lemma 7.2.2 If \( r > 0 \) is a finite positive rational, and \( T \) a huge number, then \( \sum_{i=0}^{T} \frac{r^i}{i!} \) is finite.

**Proof.** Take an accessible number \( k \) satisfying \( 2r < k \). The sum \( \sum_{i=0}^{k-1} \frac{r^i}{i!} \) is finite being the sum of an accessible number of bounded rational. Hence it suffices to show that

\[
\sum_{i=k}^{T} \frac{r^i}{i!}
\]

is bounded.

\[
\sum_{i=k}^{T} \frac{r^i}{i!} = \frac{r^k}{k!} \left( 1 + \sum_{i=1}^{T-k} \frac{r^i}{(k+i)(k+i-1) \cdots (k+1)} \right) < \frac{r^k}{k!} \left( 1 + \sum_{i=1}^{T-k} \left( \frac{r}{k} \right)^i \right) < \frac{r^k}{k!} \frac{1 - \left( \frac{r}{k} \right)^{T-k}}{1 - \frac{r}{k}} < 2 \frac{k^k}{2k!}
\]

\( \blacksquare \)

Lemma 7.2.3 If \( r > 0 \) is a finite rational, and \( T, N \) are huge numbers, then

\[
\sum_{T \leq i \leq T+N} \frac{r^i}{i!} \approx 0.
\]

**Proof.**

\[
\sum_{i=T}^{T+N} \frac{r^i}{i!} = \frac{r^T}{T!} \left( 1 + \sum_{i=1}^{N} \frac{r^i}{(T+1)(T+2) \cdots (T+i)} \right) \leq \frac{r^T}{T!} \sum_{i=0}^{N} \frac{r^i}{i!}
\]

Take an accessible \( k \) with \( r < k \). Since \( \frac{r}{T} < 1 \) for \( i > k \), we have

\[
\frac{r^T}{k!} \frac{r^{T-k}}{(k+1)(k+2) \cdots T} = \frac{r^k}{k!} \frac{r}{k+1} \frac{r}{k+2} \cdots \frac{r}{T} \leq \frac{r^k}{k!} \approx 0.
\]

By Lemma 7.2.2 \( \sum_{i=0}^{N} \frac{r^i}{i!} \) is finite, whence

\[
\frac{r^T}{T!} \sum_{i=0}^{N} \frac{r^i}{i!} \approx 0.
\]
The following is a more primitive expression for the exponential function. This gives an example of huge number of product of rationals indistinguishable from 1 gives a number $\succ 1$.

**Proposition 7.2.4** If $r$ is finite and $T$ is huge, then

$$\exp(r, T) \approx \left(1 + \frac{r}{T}\right)^T.$$ 

**Proof.**

$$\left(1 + \frac{r}{T}\right)^T = \sum_{i=0}^{T} \binom{T}{i} \frac{r^i}{T^i} = \sum_{i=0}^{T} a_i r^i,$$

where

$$a_i := \left(1 - \frac{1}{T}\right) \left(1 - \frac{2}{T}\right) \ldots \left(1 - \frac{i-1}{T}\right).$$

If $i$ is accessible $a_i \approx 1$ hence

$$\sum_{i=0}^{n} \frac{a_i x^i}{i!} \approx \sum_{i=1}^{n} \frac{x^i}{i!} \quad (21)$$

holds for accessible $n$. Hence by Robinson’s lemma, there is a huge $N$ such that $(21)$ holds for $n \leq N$. If $T \leq N$ we have nothing more to show. Suppose $T > N$.

$$\left(1 + \frac{r}{T}\right)^T \approx \sum_{i=0}^{N} r^i \frac{1}{i!} + \sum_{i=N+1}^{T} a_i r^i \frac{1}{i!}.$$  

Since $a_i < 1$, the second term is infinitesimal by Lemma 7.2.3.

**Proposition 7.2.5** If $p$ is a finite real number and $r \in p$ and $T \gg 1$. Then the real number $[\exp(r, T)]$ does not depend on the representation $r$ and $T$.

**Proof.** If $r, r' \in p$ then

$$\sum_{i=0}^{n} \frac{r^i}{i!} \approx \sum_{i=0}^{n} \frac{r'^i}{i!}, \quad (22)$$

for accessible $n$. Hence by the Robinson’s lemma [p.12], there is a huge $N$ such that $(22)$ holds for $n \leq N$. Hence, by Proposition 7.2.1

$$\exp(r, T) \approx \exp(r, N) \approx \exp(r', N) \approx \exp(r', T').$$
Hence the function
\[ \exp_T : (-\infty, \infty)_\mathbb{Q} \to (0, \infty)_\mathbb{Q} \]
defined by \( \exp_T(r) := \exp(r, T) \) is continuous by Proposition 7.2.5 and represents a real function, called the exponential function:
\[ \exp : (-\infty, \infty) \to (0, \infty), \]
which is independent of \( T \). Its value at a real number \( p = [r] \) is given by
\[ \exp(p) := [\exp(r, T)] \]
by any \( T \gg 1 \).

**Proposition 7.2.6** If \( x_i \ (i = 1, 2) \) are finite real numbers, then
\[ \exp(x_1 + x_2) = \exp(x_1) \exp(x_2). \]
In other words, if \( r_i \ (i = 1, 2) \) are finite rationals then \( \exp(r_1 + r_2, T) \simeq \exp(r_1, T) \exp(r_2, T) \).

**Proof.** If \( N \) is huge,
\[
\exp(r_1, N) \exp(r_2, N) = \left( \sum_{i=0}^{N} \frac{r_1^i}{i!} \right) \left( \sum_{j=0}^{N} \frac{r_2^j}{j!} \right) = \sum_{k=0}^{2N} \sum_{i+j=k, i \leq N, j \leq N} \frac{r_1^i r_2^j}{i! j!} = \sum_{k=0}^{N} \frac{(r_1 + r_2)^k}{k!} + U = \exp(r_1 + r_2, N) + U,
\]
where
\[ U := \sum_{k=N+1}^{2N} \sum_{i+j=k, i \leq N, j \leq N} \frac{r_1^i r_2^j}{i! j!}. \]

By Lemma 7.2.3
\[
|U| \leq \sum_{k=N+1}^{2N} \sum_{i+j=k, i \leq N, j \leq N} \frac{|r_1|^i |r_2|^j}{i! j!} \leq \sum_{k=N+1}^{2N} \sum_{i+j=k} \frac{|r_1|^i |r_2|^j}{i! j!} = \sum_{k=N+1}^{2N} \frac{(|r_1| + |r_2|)^k}{k!} \approx 0 \quad (24)
\]
**Corollary 7.2.7** If $x$ is a finite real number then $\exp(x)\exp(-x) = 1$. In other words, for a finite rational $r$ and huge $T$,

$$\exp(r, T) \exp(-r, T) \approx 1.$$  

**Lemma 7.2.8** If $x$ is a nonzero finite real number, then

$$\exp(x) > 1 + x.$$

In other words, if $r \in (-\infty, \infty) \mathbb{Q}$ satisfies $r \neq 0$ then

$$\exp(r, T) > 1 + r.$$  \hfill{(25)}

**Proof.**

We may assume $r \geq -1$ since otherwise the right hand is non positive. Suppose $0 \prec r$. Then

$$\exp(r, T) = 1 + r + \sum_{i=2}^{T} \frac{r^i}{i!} \geq 1 + r + \frac{r^2}{2} > 1 + r,$$

Suppose now $-1 < r < 0$. Then $-r > 0$ and

$$\exp(-r, T) = 1 + (-r) + \sum_{i=2}^{T} \frac{(-r)^i}{i!} \leq -\frac{r^2}{2} + \sum_{i=0}^{T} (-r)^i \leq \frac{1 - (-r)^{T+1}}{1 + r} \leq \frac{1}{1 + r}.$$  

Hence $\exp(-r, T) \prec \frac{1}{1 + r}$ namely $\exp(r, T) > 1 + r$ and \ref{25} holds also for $-1 < r < 0$.

**Corollary 7.2.9** If $T, S \gg 1$ then $\exp(-T, S) \approx 0$ and $\exp(T, S) \gg 1$.

**Proof.** By Lemma \ref{7.2.8} for accessible $n$,

$$\exp(n, S) > 1 + n,$$

whence for some huge $T$,

$$\exp(T, S) > S + T \gg 1.$$  

If $T_1 > T$, then

$$\exp(T_1, S) > \exp(T, S) \gg 1.$$  

By Proposition \ref{7.2.3} and Lemma \ref{7.2.8} for accessible $n$,

$$\exp(-n, S) \approx \frac{1}{\exp(n, S)} < \frac{1}{1 + n}.$$  

whence
\[ \exp(-n, S) < \frac{1}{1+n} \]
for all accessible \( n \), whence there is a huge \( M \) satisfying such that for all \( T \leq M \)
\[ \exp(-T, S) < \frac{1}{1+T} \approx 0, \]
whence \( \exp(-T, S) \approx 0 \). If \( T_1 > T \), then
\[ \exp(-T_1, S) \approx 1 \exp(T_1, S) \leq \exp(T, S) \approx \exp(-T, S) \approx 0. \]

**Proposition 7.2.10** The exponential is injective and order preserving. Namely, for finite real numbers \( p, q \), \( \exp(p) = \exp(q) \) if and only if \( p = q \) and \( \exp(p) < \exp(q) \) if and only if \( p < q \). In other words, for finite rationals \( x, y \) and huge \( N \), the condition \( \exp(x, N) \approx \exp(y, N) \) implies \( x \approx y \), and the condition \( \exp(x, N) \prec \exp(y, N) \) implies \( x \prec y \).

**Proof.** If \( p > 0 \), then by Lemma 7.2.8, \( \exp(p) > 1 + p > 1 \). Hence if \( p < q \), then \( \exp(q - p) > 1 \), which implies \( \exp(p) < \exp(q) \) when multiplied by \( \exp(p) \). The other assertions follow from observing that the mutually disjoint and exhausting conditions
\[ \exp(p) < \exp(q), \quad \exp(p) = \exp(q), \quad \exp(p) > \exp(q) \]
hold according respectively to the mutually disjoint and exhausting conditions \( p < q, p = q, q > p \).

### 7.2.3 Logarithm

**Lemma 7.2.11** Let \( r \in (0, \infty)_\mathbb{Q} \) and \( T \gg 1 \). Define
\[ \log(r, T) := \frac{1}{T} \max \left\{ k \in [-T^2..T^2] \mid \exp\left(\frac{k}{T}, T\right) \leq r \right\}. \]
Then for \( s \in (-\infty, \infty)_\mathbb{Q}, r, r' \in (0, \infty)_\mathbb{Q} \)
\[ (1) \ \exp(\log(r, T), T) \approx r, \]
\[ (2) \ \log(\exp(s, T), T) \approx s, \]
(3) \( \log(r, T) \approx s \) if and only if \( r \approx \exp(s, T) \),

(4) \( \log(rr', T) \approx \log(r, T) + \log(r', T) \),

(5) \( \log(r, T) \) is monotone increasing,

(6) \( \log(r, T) > 0, \log(r, T) \approx 0, \) and \( \log(r, T) < 0 \) according respectively to \( r > 1, r \approx 1, \) and \( r < 1 \).

(7) If \( r \approx r' \), then \( \log(r, T) \approx \log(r', T) \).

Hence the function \( r \mapsto \log(r, T) \) defines a morphism

\[ \log_T : (0, \infty) \to (-\infty, \infty), \]

which is an almost inverse of \( \exp_T \). If \( T' \) is another huge number then the morphisms \( \log_T \) and \( \log_{T'} \) are indistinguishable.

**Proof.** First, we show that \( \log(r, T) \) is well-defined. By Corollary 7.2.9, \( \exp(-T, T) \approx 0 \) whence the set \( \{ t \in [-T, T]_T \mid \exp(t, T) \leq r \} \) is not empty and its maximum \( t_0 \) is defined, which is finite. In fact if \( r \geq 1 \) then \( t_0 < r - 1 \) and \( t_0 \geq 0 \) since \( \exp(0, T) = 1 \) Suppose \( r < 1 \). Obviously \( t_0 < 0 \). If \( t_1 := \frac{T}{2} - 1 \) then

\[ \exp(-t_1, T) \approx \frac{1}{\exp(t_1, T)} \leq \frac{1}{1 + t_1} = \frac{r}{2} \times r, \]

whence \( t_0 > -t_1 > -\infty \).

Then

\[ \exp\left(\frac{k}{T}, T\right) \leq r < \exp\left(\frac{k + 1}{T}, T\right), \]

Hence \( \log(r, T) = \frac{k}{T} \) satisfies

\[ \exp(\log(r, T), T) \approx r. \]

Let \( s \in (-\infty, \infty)_Q \). Then \( \exp\left(\frac{k}{T}, T\right) \leq \exp(s, T) \) is equivalent to \( \frac{k}{T} \leq s \), hence

\[ \log(\exp(s, T), T) = \frac{[sT]}{T} \approx s. \]

The third assertions follow from the first and Proposition 7.2.10.

The fourth assertion can be verified as follows. Let \( r, r' \in (0, \infty)_Q \). Then

\[ \exp(\log(rr', T), T) \approx rr' = \exp(\log(r, T), T) \exp(\log(r', T), T) \approx \exp(\log(r, T) + \log(r', T), T), \]

whence the assertion follows by Proposition 7.2.10.

The real function \( \log := [\log_T] \) on the continuum \((0, \infty)\) is called the **natural logarithm function**. Lemma 7.2.11 can be rephrased as follows.
Proposition 7.2.12 Let $x, x' > 0$ and $y$ be finite real numbers. Then

1. $\exp(\log(x)) = x,$
2. $\log(\exp(y)) = y,$
3. $\log(x) = y$ if and only if $x = \exp(y),$ 
4. $\log(xx') = \log(x) + \log(x'),$
5. $\log(x, T)$ is monotone increasing,
6. $\log(x) > 0, \log(x) = 0, \text{ and } \log(x) < 0$ according respectively to $x > 1,\ x = 1,\ \text{and } x < 1.$

For finite real numbers $x, y$ with $x > 0$, we define 

$$x^y := \exp(y \log(x)).$$

It is easily verified the following.

Proposition 7.2.13 (1) If $y$ is commensurable, then $x^y$ coincides with the $x^y$ defined in § 6.2.

2. $x^{y+z} = x^y x^z,$
3. $x^{yz} = (x^y)^z.$

7.3 Mean Value Theorem

Theorem 7.3.1 (Mean value theorem) Let $p, q$ be real numbers with $p < q$ represented respectively by $a, b \in \mathbb{Q}$. If a real function $F$ on $[a, b]$ satisfies $F(p) < t < F(q)$ for a real number $t$, then there is a real number $s$ satisfying $F(s) = r$ and $p < s < q$.

In other words, let $a, b \in \mathbb{Q}$ with $a < b$. If a continuous rational valued function $f$ on $[a, b]$ satisfies $f(a) < c < f(b)$ for a rational number $c$, then there is an $r \in [a, b]_\mathbb{Q}$ satisfying $f(r) \approx c$.

Proof. Let $\varepsilon > 0$ be an infinitesimal and put $m_1 := [a/\varepsilon] + 1$ and $m_2 := [b/\varepsilon]$. Then $m_1 \varepsilon, m_2 \varepsilon \in [a, b]_\varepsilon$ and 

$$f(m_1 \varepsilon) \approx f(a) < c < f(b) \approx f(m_2 \varepsilon).$$

Hence we can define 

$$x := \varepsilon \min \{ i \in [m_1..m_2] \mid f(i \varepsilon) \geq c \},$$

which satisfies $f(x - \varepsilon) < c \leq f(x).$ Since $f(x - \varepsilon) \approx f(x)$, we have $f(x) \approx c.$

Note that in the theorem the real numbers $p$ or $q$ may be infinite.
7.4 Maximum Principle

Theorem 7.4.1 (Maximum principle) Let $F$ be a real function on a continuum $C$ with a dense subset $A$. Then there are points $p, q$ of $C$ such that

$$F(p) \leq F(x) \leq F(q)$$

for all points $x$ of $C$. In other words, if $f \in F$ is a continuous rational valued function on $C$, then there are positions $x_m, x_M \in A$ such that

$$f(x_m) \leq f(x) \leq f(x_M)$$

(26)

for all $x \in X$.

Proof. Let $f(x_m) = \min_{x \in A} f(x)$ and $f(x_M) = \max_{x \in A} f(x)$. Then for all $x \in A$

$$f(x_m) \leq f(x) \leq f(x_M).$$

(27)

Let $y \in C$. Then there is a $z \in A$ with $y \approx z$.

Since (27) is valid for $x = z$ and $f(y) \approx f(z)$, it is valid also for $x = y$ by Proposition 1.1.2.

Corollary 7.4.2 Every real function on a compact continuum has maximum and minimum values.

7.5 Behavior of Real Functions on a Point

We introduce a method of describing the infinitesimal behavior of functions on a point which will be used extensively in the treatment of calculus.

In the following, $C = (|C|, d, \approx_d)$ denotes a metric continuum and $f(x), g(x)$ are rational valued functions on $|C|$.

Definition 7.5.1 We write for $a \in |C|$ and accessible $n$

$$f(x) \equiv_n 0 \text{ if } x \approx a$$

(28)

if

$$\frac{|f(x)|}{d(x, a)^n} \approx 0$$

(29)

whenever $x \neq a$ and $x \approx a$. 

This relation is not indistinguishability-invariant, namely, \( f \approx 0 \) does not necessarily imply (28). For example, let \( \varepsilon > 0 \) be an infinitesimal and define a rational valued function \( f \) on \([-1,1]_\mathbb{Q}\) by

\[
f(x) := \begin{cases} 
|x|^n & \text{for } |x| \leq \varepsilon \\
\varepsilon^n & \text{for } |x| > \varepsilon
\end{cases}
\]

Then \( f \approx 0 \) but \( f(x) \approx n \) for \( |x| \leq \varepsilon \) and hence it is not the case that \( f(x) \equiv_n 0 \) if \( x \approx 0 \).

However the following weaker condition is indistinguishability-invariant.

**Definition 7.5.2** We write, for \( a \in |C| \) and accessible \( n \)

\[
f(x) \approx_n 0 \text{ if } x \approx a
\]

if there is an infinitesimal \( \varepsilon > 0 \) such that (29) holds whenever \( d(x,a) \geq \varepsilon \) and \( x \approx a \)

The following two propositions give basic properties of this condition.

**Proposition 7.5.1** If \( f, g \) are rational valued functions on \(|C|\) such that \( f \approx g \) and \( a, b \in |C| \) are indistinguishable, then

\[
f(x) \approx_n 0 \text{ if } x \approx a \iff g(x) \approx_n 0 \text{ if } x \approx b .
\]

**Proof.** Let \( n \) be an accessible number and \( a \in C \).

First we show that if \( f \approx 0 \), then \( f(x) \approx_n 0 \) if \( x \approx a \). Put

\[
\eta := \max \{|f(x)| \mid x \in |C|\} \approx 0 .
\]

Let \( \varepsilon \) be an infinitesimal such that \( \eta < \varepsilon^{n+1} \). For example take a huge \( N \) and the minimal number \( k \) such that \( \eta < \left( \frac{k}{N} \right)^{n+1} \) and put \( \varepsilon = \frac{k}{N} \). Then \( \eta < \varepsilon^{n+1} \).

Moreover \( \left( \frac{k-1}{N} \right)^{n+1} \leq \eta \) implies \( \varepsilon^{n+1} \approx \eta \approx 0 \) whence \( \varepsilon \approx 0 \) by Lemma 6.2.7.

If \( \varepsilon < d(x,a) \), then

\[
\left| \frac{f(x)}{d(x,a)^n} \right| < \frac{\eta}{\varepsilon^n} < \varepsilon \approx 0 ,
\]

whence \( f(x) \approx_n 0 \) if \( x \approx a \).

Suppose now \( a \approx b \) and \( f(x) \approx_n 0 \) if \( x \approx a \). Then there is an infinitesimal \( \varepsilon > 0 \) such that (29) holds when \( x \approx a \) and \( d(x,a) > \varepsilon \). Put \( \varepsilon_1 := \max \{ \varepsilon + d(a,b), 3d(a,b) \} \approx 0 \) and suppose \( 0 \approx d(x,b) > \varepsilon_1 \). Then

\[
d(x,a) > d(x,b) - d(b,a) \geq \varepsilon ,
\]
whence (29) holds. On the other hand
\[ d(x,a) \geq d(x,b) - d(a,b) \geq 3d(a,b) - d(a,b) = 2d(a,b), \]
whence
\[ d(x,b) \geq d(x,a) - d(a,b) \geq \frac{1}{2}d(x,a). \]
This implies
\[ |f(x)| d(x,b)^n \leq 2^n |f(x)| d(x,a)^n \approx 0 \]
whence \( f(x) \approx_n 0 \) if \( x \approx b \).

**Proposition 7.5.2** Suppose \( C_i \ (i = 1, 2) \) are metric continua and \( \alpha : |C_1| \to |C_2| \) a function satisfying, for \( x,y \in |C_1| \),
\[ K_1 \ d_1(x,y) < d_2(\alpha(x),\alpha(y)) < K_2 \ d_1(x,y) \]
with finite rational numbers \( K_1, K_2 \) and an infinitesimal \( \eta > 0 \). Suppose \( f \) is a continuous rational valued function on \( |C_2| \) and \( a \in |C_1| \). Then
\[ f(y) \approx_n 0 \text{ if } y \approx \alpha(a) \text{ on } C_2, \] (32)
implies
\[ f(\alpha(x)) \approx_n 0 \text{ if } x \approx a \text{ on } C_1. \] (33)

**Proof.** Suppose (32) holds. Then there is an infinitesimal \( \varepsilon > 0 \) such that if \( \varepsilon < d_2(y,\alpha(a)) \approx 0 \), then
\[ \frac{|f(y)|}{d(y,\alpha(a))^n} \approx 0. \]
If \( d_1(x,a) > \max \left\{ \eta, \frac{\varepsilon}{K_1} \right\} \), then \( d_2(\alpha(x),\alpha(a)) > K_1 d_1(x,a) > \varepsilon \), whence
\[ \frac{|f(\alpha(x))|}{d_1(x,a)^n} \leq K_2^{-n} \frac{|f(\alpha(x))|}{d_2(\alpha(x),\alpha(a))^n} \approx 0. \]
Hence (33) holds.

The following shows that the relation \( \approx_n \) is invariant under quasi-identities.

**Proposition 7.5.3** Let \( \varepsilon, \eta > 0 \) be infinitesimal and \( \alpha : [0,1]^n \to [0,1]^n \) represents a quasi-identity and \( f \) a continuous rational valued function on \( [0,1]^n \). Then, for accessible \( n \) and \( a \in X \),
\[ f(y) \approx_n 0 \text{ if } y \approx \alpha(a) \] (34)
if and only if
\[ f(\alpha(x)) \approx_n 0 \text{ if } x \approx a. \] (35)
§7 real functions on continua

Proof. Let \( \beta \) be an almost inverse of \( \alpha \). Then there is an infinitesimal \( \delta > 0 \) such that
\[
d(x, \alpha(x)), d(y, \beta(y)) < \delta
\]
holds for all \( x \in [0, 1]^n \) and \( y \in [0, 1]^n \). Then
\[
d(x, y) - 2\delta < d(\alpha(x), \alpha(y)) < d(x, y) + 2\delta.
\]
Hence if \( d(x, y) > 4\delta \), then
\[
\frac{1}{2} d(x, y) < d(\alpha(x), \alpha(y)) < \frac{3}{2} d(x, y).
\]
Hence by Proposition 7.5.2 the condition (34) implies (35).

Assume now (35). Put \( b = \alpha(a) \). Since \( a \approx \beta(b) \), we have
\[
f(\alpha(x)) \approx_n 0 \text{ if } x \approx \beta(b).
\]
Since \( \frac{1}{2} d(x, y) < d(\beta(x), \beta(y)) < \frac{3}{2} d(x, y) \)
whenever \( d(x, y) > 4\delta \), we infer from Proposition 7.5.2 that (36) implies
\[
f(\alpha(\beta(y))) \approx_n 0 \text{ if } y \approx b.
\]
Since \( f(\alpha(\beta(y))) \approx f(y) \) and \( b \approx \alpha(a) \), we obtain (34) by Proposition 7.5.1.

Although the relation \( \approx_n \) is weaker than \( \equiv_n \), the following proposition shows that the former implies the latter if the continuum is replaced with a suitable equivalent continuum.

Proposition 7.5.4 Let \( \varepsilon > 0 \) be an infinitesimal. Suppose \( C = [r, s]^n \) is given the metric \( d_{\infty} \) and a continuous rational valued function \( f \) on \( |C| \times |C| \) satisfies
\[
f(x, a) \approx_n 0 \text{ if } x \approx a
\]
for all \( a \in C \). Then there is an integer \( L > 0 \) such that \( L\varepsilon \approx 0 \) and
\[
g(x, b) \equiv_n 0 \text{ if } x \approx b,
\]
for all \( b \in [r, s]_{L\varepsilon} \) where \( g = f|X \times X \).

Proof. For each \( a \in [r, s]^n \) let \( \delta_a > 0 \) be an infinitesimal such that
\[
\frac{|f(x, a)|}{d(x, a)^n} \approx 0
\]
holds if \( x \) satisfies \( 0 \approx d(x, a) > \delta_a \). Let \( \delta = \max \{ \delta_a \mid a \in [r, s]^n \} \approx 0 \). Then (37) holds for all \( x, a \in [r, s]^n \) with \( d(x, a) > \delta \). Let \( L \) be the least integer greater than \( \frac{2}{\varepsilon} \). Then \( \delta < L\varepsilon \approx 0 \) and different \( y, z \in [0, 1]_{L\varepsilon} \) satisfies \( d(y, z) > L\varepsilon > \delta \). Hence (37) holds for different \( x, a \in [0, 1]_{L\varepsilon} \), whence \( g(y, b) \approx_n 0 \) if \( y \approx b \) for each \( b \in [r, s]_{L\varepsilon} \).

The following proposition shows that the condition (31) is characterized by the behavior of \( f \) for \( x \not\approx a \).
Proposition 7.5.5 Let $C$ be a metric continuum and $f$ is a rational valued function on $|C|$. Then for accessible $n$ and $a \in |C|$, the following conditions are equivalent.

(A) $f(x) \approx_n 0$ if $x \approx a$,

(B) for each accessible number $k$, there is an accessible number $\ell$, such that if $0 \neq d(x,a) < \frac{1}{\ell}$ then $|f(x)| \leq \frac{1}{k} d(x,a)^n$.

Proof. Suppose the condition (A) is satisfied and there is a positive $\varepsilon \approx 0$ such that \(^{(29)}\) if $0 \approx d(x,a) > \varepsilon$. Let $k$ be an accessible number. Then for all huge $i$, the objective condition

$$\varepsilon < d(x,a) < \frac{1}{i} \implies \left| \frac{f(x)}{d(x,a)^n} \right| < \frac{1}{k}$$

holds, whence it holds also for an accessible $i = \ell$, whence (B).

Conversely suppose the condition (B) holds. Let $k$ be an accessible number. Then there is an accessible $\ell$ such that for all accessible $p$, the objective condition

$$\frac{1}{p} < d(x,a) < \frac{1}{\ell} \implies \left| \frac{f(x)}{d(x,a)^n} \right| < \frac{1}{k}.$$  \hspace{1cm} (39)

holds. Hence for some huge $M_k$, the condition \(^{(39)}\) holds for $p = M_k$. By Proposition 1.5.3 there is a huge $M$ such that $M \leq M_k$ for all $k$. Then if $\frac{1}{M} < d(x,a) \approx 0$ then $\left| \frac{f(x)}{d(x,a)^n} \right| < \frac{1}{k}$ for all accessible $k$, whence \(^{(29)}\).

Proposition 7.5.6 Let $n$ be an accessible number and $a \in [0,1]_\varepsilon$. If a rational valued continuous function $f$ on $[0,1]_\varepsilon$ satisfies

$$f(x) \equiv_n 0 \text{ if } x \approx a$$

then

$$g(x) \equiv_{n+1} 0 \text{ if } x \approx a$$

where $g$ is defined by

$$g(x) = \begin{cases} \sum_{a \leq u < x} f(u) \Delta x & \text{for } x \geq a \\ \sum_{x \leq u < a} f(u) \Delta x & \text{for } x < a \end{cases}$$

with $\Delta x := \varepsilon$.

Proof. By hypothesis if $0 \approx |x-a| \neq 0$ then

$$c := \max_{x \approx a, x \neq a} \frac{|f(x)|}{|x-a|^n} \approx 0.$$
If $x > a$

$$\left| \sum_{a \leq u < x} f(u) \Delta x \right| \leq \frac{1}{|x - a|^{n+1}} \sum_{a \leq u < x} \left| \frac{f(u)}{|x - a|^{n}} \right| \Delta x \leq \frac{c}{|x - a|} \sum_{a \leq u < x} \Delta x = c \approx 0$$

Together with similar arguments for $x < a$ we conclude $g(x) \equiv_{n+1} 0$ if $x \approx a$.

The following lemmas show that the monomials are linearly independent even within a point.

**Lemma 7.5.7** Let $k$ be an accessible number and $a_0, \ldots, a_k$ finite rational numbers. Suppose $\sum_{i=0}^{k} a_i x^i \approx_k 0$ if $x \approx 0$. Then $a_i \approx 0$ for $i \in [0..k]$.

**Proof.** By hypothesis, there is an infinitesimal $\varepsilon > 0$ such that $0 \approx |x| \geq \varepsilon$, then

$$\sum_{i=0}^{k} a_i x^i \approx 0.$$ 

Let $N$ be a huge number greater than $\frac{1}{\varepsilon}$, then $\varepsilon < \frac{1}{N} \approx 0$ for $j \in [1..k+1]$. Hence

$$\sum_{i=0}^{k} a_i N^{k-i} j^{i-k} \approx 0,$$

for $j = 1, \ldots, k+1$.

Define $(k+1, k+1)$ matrix $B$ and $k+1$ vector $a$ by

$$B_{j,i} = ((j+1)^{i-k}), a_i := (a_i N^{k-i}),$$

$(0 \leq j, i \leq k)$ then $Ba \approx 0$. Since the Van der Monde matrix $B$ has an inverse whose components are bounded, we have

$$a \approx 0.$$ 

Hence for each $i$, $a_i N^{k-i} \approx 0$, whence $a_i \approx 0$.

For multi-index $J = (j_1, \ldots, j_n)$, put $|J| := j_1 + \cdots + j_n$ and $x^J := x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$.

**Lemma 7.5.8** Let $k$ and $n$ be accessible numbers. Suppose for each multi-index $J$ with $|J| \leq n$ a finite rational number $a_J$ is given and satisfies

$$\sum_{|J| \leq n} a_J x^J \approx_k 0 \text{ if } x \approx 0$$

on $[0,1]^n$. Then $a_J \approx 0$ for all $J$.

**Proof.** By assumption, there is an infinitesimal $\varepsilon > 0$ such that $0 \approx d(0, x) \geq \varepsilon$ implies

$$\sum_{|J| \leq k} a_J x^J \approx d(0, x)^k \approx 0.$$  

(40)
Let $N = \left[ \frac{1}{2} \right]$. If components of $\alpha \in \mathbb{Z}^n$ are accessible, then

$$d(0, \frac{\alpha}{N}) = \frac{d(0, \alpha)}{N} > \varepsilon,$$

whence by substituting $x = \frac{\alpha}{N}$ with $\alpha \in [0..k]^n$ in (40), we obtain

$$\sum_{|J| \leq k} \alpha^I(a_J N^{k-|J|}) \approx 0, \quad \alpha \in [0..k]^n. \quad (41)$$

Since the vectors

$$\{ (\alpha^I)_{|J| \leq k} \mid \alpha \in [0..k]^n \}$$

are linearly independent, there is a subset $T \subset [0..k]^n$ such that $A = (\alpha^I)_{|J| \leq k, \alpha \in T}$ is regular and the inverse matrix has bounded components. Hence (41) implies $a_J N^{k-|J|} \approx 0$ for all $|J| \leq k$, whence $a_J \approx 0$. \hfill \blacksquare
8 Differentiation: Single variable

It turns out that the differentiability of real functions can be defined as the possibility of choosing good representations with continuous difference quotients. One might be puzzled that every real function seems to become differentiable according to this definition since its representations can be arbitrarily specified within a point. However the fringe of a point is not separated from that of other “neighboring points” and the nearer we move to the “boundary” of a point, the less freedom we have for the specification of the behaviour of representations of a real function. This feature might be understood by the fact that for every infinitesimal $\varepsilon$, there is an huge number $L$ such that $L\varepsilon$ is still infinitesimal, which means that given any two places within a point, we can magnify the point to have visible extent without breaking their indistinguishability.

8.1 Difference Quotient

In this section $\varepsilon > 0$ is a fixed infinitesimal and $C = ([0, 1]_\varepsilon, \approx)$ the rigid mesh continuum representing $[0, 1]$.\footnote{We consider only the unit interval continuum $[0, 1]$ but everything can be directly generalized to general interval $[a, b]$.}

Denote by $[0, 1]_\varepsilon^-$ the subset of $[0, 1]_\varepsilon$ obtained by removing the greatest element, namely, $[\frac{1}{\varepsilon}]_\varepsilon$. and denote the next larger element of $x$ of $[0, 1]_\varepsilon^-$ by $x^+$, namely, $x^+ = x + \Delta x$, where $\Delta x = \varepsilon$. Note that $([0, 1]_\varepsilon^-, \approx)$ is also a rigid mesh continuum representing $[0, 1]$.

**Definition 8.1.1 (Difference operator)** For a rational valued function $f$ on $[0, 1]_\varepsilon$, define its difference $\Delta f$ defined for $x \in [0, 1]_\varepsilon^-$ by

$$\Delta f(x) := f(x^+) - f(x).$$

The quotient

$$\frac{\Delta f}{\Delta x}(x) := \frac{f(x^+) - f(x)}{\Delta x} \in \mathbb{Q}$$

is called the difference quotient of $f$ at $x$ and the rational valued function $\frac{\Delta f}{\Delta x}$ is called the difference quotient function of $f$.

For a rational valued function $f$ on $[0, 1]_\varepsilon$ and $a, b \in [0, 1]_\varepsilon$ with $a < b$, we define

$$\sum_{a}^{b} f := \sum_{u \in [a, b]_\varepsilon} f(u) = \sum_{i=0}^{b-a} f(a + i\varepsilon)$$
Proposition 8.1.1 If $f$ is a rational valued function on $[0, 1]_{\varepsilon}$ and $a, b \in [0, 1]_{\varepsilon}^{-}$ with $a < b$, then
\[
\sum_{a}^{b} \frac{\Delta f}{\Delta x} \Delta x = f(b^+) - f(a).
\] (42)

Proof. Since \( \frac{\Delta f}{\Delta x}(x) \Delta x = \Delta f(x) \), we have
\[
\sum_{a}^{b} \frac{\Delta f}{\Delta x} \Delta x = \sum_{x \in [a, b]_{\varepsilon}} (f(x^+) - f(x)) = f(b^+) - f(a).
\]

Proposition 8.1.2 If rational valued functions $f, g$ on $[0, 1]_{\varepsilon}$ satisfy $f \approx g$, then
\[
\sum_{a}^{b} f \Delta x \approx \sum_{a}^{b} g \Delta x,
\]
for $a, b \in [0, 1]_{\varepsilon}^{-}$ with $a < b$.

Proof. Put $c := \max \left\{ |f(x) - g(x)| \mid x \in [0, 1]_{\varepsilon} \right\}$. Then $c \approx 0$ and
\[
\left| \sum_{a}^{b} f - \sum_{a}^{b} g \right| \leq \sum_{a}^{b} |f - g| \leq \sum_{a}^{b} c \Delta x = c(b^+ - a) \leq c \approx 0.
\]

Basic relation between $f$ and its difference quotient $\frac{\Delta f}{\Delta x}$ is as follows:

Proposition 8.1.3 Suppose a rational valued function $f$ on $[0, 1]_{\varepsilon}$ is continuous and the difference quotient function $\frac{\Delta f}{\Delta x}$ is finite on $[0, 1]_{\varepsilon}^{-}$. Then, for $x, y \in [0, 1]_{\varepsilon}$ with $x < y$,

(1) \( f(y) \leq f(x) + M(y - x) \), where
\[
M = \max \left\{ \frac{\Delta f}{\Delta x}(u) \mid u \in [0, 1]_{\varepsilon}, x \leq u < y \right\},
\]

(2) \( f(y) \geq f(x) + m(y - x) \), where
\[
m = \min \left\{ \frac{\Delta f}{\Delta x}(u) \mid u \in [0, 1]_{\varepsilon}, x \leq u < y \right\},
\]
\[ |f(y) - f(x)| \leq M_1 |y - x|, \text{ where} \]
\[ M_1 = \max \left\{ \left| \frac{\Delta f}{\Delta x}(u) \right| \mid u \in [0,1], x \leq u < y \right\}. \]

**Proof.**

\[ f(y) - f(x) = \sum_{x}^{y-\varepsilon} \frac{\Delta f}{\Delta x} \Delta x \leq \sum_{x}^{y-\varepsilon} M \Delta x = M(y - x). \]

The other assertions can be shown similarly. \[ \]
8.2 Differentiability

Let \( \varepsilon \) be a positive infinitesimal. We say a real function \( F \) on \([0,1]\) is represented by \((f, [0,1]_\varepsilon)\), sometimes simply represented by \( f \), if \( f \) is a continuous rational-valued function \( f \) on \([0,1]_\varepsilon \) and \( f \circ \kappa_\varepsilon \) represents \( F \) where \( \kappa_\varepsilon \) is the map defined in Proposition 2.4.2.

**Definition 8.2.1** A real function \( F \) on \([0,1]\) is called differentiable if it is represented by \((f, [0,1]_\varepsilon)\) with \( \varepsilon \) a positive infinitesimal whose difference quotient \( \frac{\Delta F}{\Delta x} \) is continuous. We say \((f, [0,1]_\varepsilon)\) is a representation of \( F \) with continuous difference quotient.

The real function on \([0,1]\) represented by \((\frac{\Delta f_i}{\Delta x}, [0,1]_\varepsilon)\) does not depend on the choice of the representation \((f, [0,1]_\varepsilon, \alpha)\) by Proposition 8.2.1. It is denoted by \( F' \) and is called the derivative of \( F \). It is also denoted by \( \frac{dF}{dx} \).

**Proposition 8.2.1** Let \( F \) be a real function on \([0,1]\). Let \((f_i, [0,1]_\varepsilon)\) \((i = 1, 2)\) be representations of \( F \) such that the difference quotients of \( f_1, f_2 \) are continuous. Then the real functions on \([0,1]\) represented by the difference quotients \((\frac{\Delta f_i}{\Delta x}, [0,1]_\varepsilon)\) \((i = 1, 2)\) coincides.

**Proof.** Put \( \alpha_i = \kappa_\varepsilon \mid_{[0,1]} \) \((i = 1, 2)\). By Theorem 8.3.1 of the next section, we have for \( a \in [0,1] \)

\[
f_i(y) \approx f_i(\alpha_i(a)) + \frac{\Delta f_i}{\Delta x}(y - \alpha_i(a)) \quad \text{if} \quad y \approx \alpha_i(a)
\]
on \([0,1]_\varepsilon \), for \( i = 1, 2 \).

For \( i = 1, 2 \), by Proposition 7.5.3

\[
f_i(\alpha_i(x)) \approx f_i(\alpha_i(a)) + \frac{\Delta f_i}{\Delta x}(\alpha_i(a)(x - \alpha_i(a)) \quad \text{if} \quad x \approx a
\]
and, since \( \alpha_i(x) - \alpha_i(a) \approx x - a \), Proposition 7.5.1 implies

\[
f_i(\alpha_i(x)) \approx f_i(\alpha_i(a)) + \frac{\Delta f_i}{\Delta x}(\alpha_i(a)(x - a) \quad \text{if} \quad x \approx a.
\]

Since \( f_1 \circ \alpha_1 \approx f_2 \circ \alpha_2 \), Proposition 7.5.1 implies

\[
\frac{\Delta f_1}{\Delta x}(\alpha_1(a)(x - a) \approx f_1(\alpha_1(a)(x - a) \quad \text{if} \quad x \approx a.
\]

Hence by Lemma 7.5.7 we have

\[
\frac{\Delta f_1}{\Delta x}(\alpha_1(a)) \approx \frac{\Delta f_2}{\Delta x}(\alpha_2(a)).
\]

---

\[25 \text{This is equivalent to the condition that } F \text{ is represented by } (f, [0,1]_\varepsilon, \kappa_\varepsilon \mid_{[0,1]} \text{ in the terminology of §7.1} \]
Note that in the special case when $\varepsilon_1 = \varepsilon_2$, the independence can be proved directly as follows.

**Lemma 8.2.2** Suppose $f, g$ are continuous rational valued functions on $X = [0, 1]_\varepsilon$ with continuous difference quotients. If $f \approx g$, then

$$\frac{\Delta f}{\Delta x} \approx \frac{\Delta g}{\Delta x}.$$

**Proof.** It suffices to show that if $f \approx 0$ and $\frac{\Delta f}{\Delta x}$ is continuous then $\frac{\Delta f}{\Delta x} \approx 0$.

Suppose $\frac{\Delta f}{\Delta x} \not\approx 0$. We may suppose that the maximum of $\frac{\Delta f}{\Delta x}$ is positive finite rational number. Let $\frac{\Delta f}{\Delta x}(a) = r$ be one of the maxima. If $x \approx a$ then $\frac{\Delta f}{\Delta x}(x) \approx r$, whence $\frac{\Delta f}{\Delta x}(x) > \frac{r}{2}$. Therefore, if $k$ is huge

$$|x - a| < \frac{1}{k} \text{ implies } \frac{\Delta f}{\Delta x}(x) > \frac{r}{2}. \quad (43)$$

Hence there is an accessible $n$ such that $(43)$ holds for $k = n$. Let $x_1$ and $x_2$ be respectively the minimum and the maximum of $[a - \frac{1}{n}, a + \frac{1}{n}]_\varepsilon$. Then

$$x_1 \approx a - \frac{1}{n} \quad \text{and} \quad x_2 \approx a + \frac{1}{n},$$

whence by Proposition 8.1.3

$$f(x_2) - f(x_1) \geq \frac{r}{2}(x_2 - x_1) \approx \frac{r}{2} \frac{2}{n} = \frac{r}{n} > 0,$$

which contradicts $f \approx 0$.

### 8.3 Infinitesimal Taylor Formula

We fix a positive infinitesimal $\varepsilon$ in this section.

**Theorem 8.3.1 (First order Infinitesimal Taylor formula)** If $f$ is a function on $[0, 1]_\varepsilon$ with continuous difference quotients, then for $a \in [0, 1]_\varepsilon$,

$$f(x) \equiv_1 f(a) + \frac{\Delta f}{\Delta x}(a)(x - a) \text{ if } x \approx a. \quad (44)$$

In particular

$$f(x) \approx_1 f(a) + \frac{\Delta f}{\Delta x}(a)(x - a) \text{ if } x \approx a.$$
§8 Differentiation: Single variable

For $x \in [0, 1]$, $x \approx a$ and $a < x$, then

\[ f(x) - f(a) = \sum_{a \leq u < x} \frac{\Delta f}{\Delta x}(u)(u^+ - u) \]

\[ = \sum_{a \leq u < x} \frac{\Delta f}{\Delta x}(a)(u^+ - u) + \sum_{\varepsilon < u < x} \left( \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right)(u^+ - u) \]

\[ = \frac{\Delta f}{\Delta x}(a)(x - a) + \sum_{\varepsilon < u < x} \left( \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right)(u^+ - u) \]

If we put

\[ c := \max_{a \leq u < x} \left| \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right|, \]

then

\[ |f(x) - f(a) - \frac{\Delta f}{\Delta x}(a)(x - a)| \leq \sum_{\varepsilon < u < x} c(u^+ - u) = c|x - a|. \]

By the continuity of $\frac{\Delta f}{\Delta x}$, we have $c \approx 0$, which implies (44).

The proof for the case $x < a$ is similar.

**Theorem 8.3.2** If a real function $F$ on $[0, 1]$ is differentiable then for every representation $(f, [0, 1])$ of $F$ there is a continuous rational valued function $g$ on $[0, 1]$ satisfying, for each $a \in [0, 1]$, $f(x) \approx f(a) + g(a)(x - a)$ if $x \approx a$. \hfill (45)

Conversely if a real function $F$ on $[0, 1]$ has a representation $(f, [0, 1])$ with a continuous rational valued function $g$ on $[0, 1]$ satisfying (45) for each $a \in [0, 1]$ then $F$ is differentiable.

**Proof.** Suppose $F$ is differentiable and let $(f_1, [0, 1])$ be a representation of $F$ such that the difference $\frac{\Delta f_1}{\Delta x}$ is continuous. By Theorem 8.3.1, we have (45) for $a \in [0, 1]$ with $g = \frac{\Delta f_1}{\Delta x}$. Let $b \in [0, 1]$. Then

\[ f_1(x) \equiv_1 f_1(a_1(b)) + \frac{\Delta f_1}{\Delta x}(a_1(b))(x - a_1(b)) \text{ if } x \approx a_1(b), \] \hfill (46)

whence by Proposition 7.5.1

\[ f_1(a_1(x)) \equiv_1 f_1(a_1(b)) + \frac{\Delta f_1}{\Delta x}(a_1(b))(x - b) \text{ if } x \approx b. \]

Since $f_1 \circ a_1 \in F$, the relation (45) holds for $f = f_1 \circ a_1$ and $g = \frac{\Delta f_1}{\Delta x} \circ a_1$.

Now let $(f_2, [0, 1])$ be an arbitrary representation of $F$. By Proposition 7.1.1 there is a continuous rational valued function $g_2$ on $[0, 1]$ such that

\[ g_2 \circ a_2 \approx \frac{\Delta f_1}{\Delta x} \circ a_1, \]
where \( \alpha_i = \kappa \varepsilon_i \) \((i = 1, 2)\). Since \( f_2 \circ \alpha_2 \approx f_1 \circ \alpha_1 \), we have

\[
f_2(\alpha_2(x)) \approx f_2(\alpha_2(b)) + g_2((\alpha_2(b))(x - b) \text{ if } x \approx b.
\]

Hence by Proposition 7.5.3, the condition (45) holds for \( f = f_2 \) and \( g = g_2 \).

Conversely suppose \((f, [0, 1]_\varepsilon)\) is a representation of \( F \) and there is a continuous rational valued function \( g \) on \([0, 1]_\varepsilon\) satisfying (45) for each \( a \in [0, 1]_\varepsilon \). By Proposition 7.5.4, there is a subcontinuum \([0, 1]_{\varepsilon'}\) of \([0, 1]_\varepsilon\) such that

\[
f_1^1 (x) \equiv f_1(a) + g_1(a)(y - a) \text{ if } y \approx a.
\]

(47) for each \( a \in [0, 1]_{\varepsilon'} \), where \( f_1 := f\mid[0, 1]_{\varepsilon'} \) and \( g_1 := g\mid[0, 1]_{\varepsilon'} \). Hence

\[
\frac{\Delta f_1}{\Delta x}(a) - g_1(a) = \frac{(f_1(a + \varepsilon') - f_1(a) - g_1(a)\varepsilon')}{\varepsilon'} \approx 0
\]

for each \( a \in [0, 1]_{\varepsilon'} \) and \( \frac{\Delta f_1}{\Delta x} \approx g_1 \) is continuous. Then \((f_1, [0, 1]_{\varepsilon'})\) represents \( F \) and have continuous difference quotients. Hence \( F \) is differentiable.

The last part of the proof shows the following Corollary which asserts that an arbitrary representation of a differentiable function has continuous difference quotients when restricted on a coarser but dense rigid mesh subcontinuum.

**Corollary 8.3.3** Suppose a differentiable function \( F \) on \([0, 1] \) is represented by \((f, [0, 1]_\varepsilon)\). Then there is an infinitesimal \( \varepsilon' \in [0, 1]_\varepsilon \) and a continuous rational valued function \( g \) on \([0, 1]_{\varepsilon'} \) such that for \( a \in [0, 1]_{\varepsilon'} \)

\[
f(x) \equiv f(a) + g(a)(x - a) \text{ if } x \approx a.
\]

on \([0, 1]_{\varepsilon'} \). In particular the difference quotient of \( f\mid[0, 1]_{\varepsilon'} \) is continuous.

This can be rephrased as follows.

**Corollary 8.3.4** If a differentiable function \( F \) on \([0, 1] \) is represented by \((f, [0, 1]_\varepsilon)\), then there is an infinitesimal \( \varepsilon' \in [0, 1]_\varepsilon \) such that the rational valued function

\[
g(x) := \frac{f(x + \varepsilon') - f(x)}{\varepsilon'}
\]

on \([0, 1]_{\varepsilon'} \) is continuous and represents \( F' \).

**Proposition 8.3.5** Let \( F_1, \ldots, F_k \) be an accessible number of differentiable functions on \([0, 1] \) and \( \alpha : [0, 1] \xrightarrow{\sim} [0, 1]_\varepsilon \) be a quasi-identity. Then each \( F_i \) is represented by a continuous rational valued function on \([0, 1]_\varepsilon \) whose difference quotient is continuous.
§8 Differentiation: Single variable

Proof. First we represent each \( F_i \) by a rational valued continuous function on \([0, 1]_\varepsilon\). Using Corollary 8.3.3 accessible number of times, we obtain a dense subset \([0, 1]_\varepsilon' \subset [0, 1]_\varepsilon\) for which the exact Taylor formula holds for each \( f_i \). Then we extend them to functions on \([0, 1]_\varepsilon\) by linear interpolation. Details are omitted.

The next lemma is used in the section of inverse function theorem.

Lemma 8.3.6 Let \( f \) be a rational valued function on \([0, 1]_\varepsilon\) with continuous difference quotient. If \( \frac{\Delta f}{\Delta x}(a) \neq 0 \), then for some rationals \( K_1, K_2, c > 0 \)

\[
K_1|x - a| \leq |f(x) - f(a)| \leq K_2|x - a|
\]

holds for \( |x - a| < c \).

Proof. By Theorem 8.3.3 we have

\[
f(x) - f(a) \equiv f'(a)(x - a) \quad \text{if} \quad x \approx a,
\]

namely if \( 0 < |x - a| \approx 0 \) then

\[
\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| \approx 0.
\]

Hence if \( K_1 := \frac{|f'(a)|}{4} > 0 \), then for every \( N \gg 1 \) the condition \( 0 < |x - a| < \frac{1}{N} \) implies

\[
\left| \frac{f(x) - f(a)}{x - a} \right| > 2K_1 \gg K_1 \tag{48}
\]

which, by the overspill principle, holds also if \( |x - a| < c := \frac{1}{n} \) for some accessible \( n \). Hence if \( |x - a| < c \) then

\[
|f(x) - f(a)| \gg K_1|x - a|.
\]

On the other hand Proposition 8.1.3 implies

\[
|f(x) - f(a)| < K_2|x - a| \quad \text{if} \quad |x - a| < c
\]

if \( K_2 = \max \left\{ \left| \frac{\Delta f}{\Delta x}(x) \right| \mid |x - a| < \frac{1}{n} \right\} \).

8.4 Chain Rule

Theorem 8.4.1 Let \( F_i \ (i = 1, 2) \) be differentiable real functions on \([0, 1]\) such that \( F_1 \) is \([0, 1]_\varepsilon\)-valued. Then the composition \( F_2 \circ F_1 \) is differentiable and satisfies

\[
(F_2 \circ F_1)' = (F_2' \circ F_1)F_1'.
\]

Proof. First we represent \( F_2 \) by \((f_2, [0, 1]_{\varepsilon_2})\) with continuous difference quotient.
Let \( (g, [0, 1]_\eta) \) represents \( F \). Then \( (g_1 := \kappa_{\varepsilon_2} \circ g, [0, 1]_{\eta}) \) also represents \( F \) and it is \([0, 1]_{\varepsilon_2\varepsilon_1} \)-valued. By Corollary 8.3.3, there is a positive infinitesimal \( \varepsilon_1 \in [0, 1]_\eta \) such that \( (f_1 := g_1|_{[0, 1]_\varepsilon_1}, [0, 1]_{\varepsilon_1}) \) represents \( F \) and has the continuous difference quotient.

The composition \( F_2 \circ F_1 \) is represented by \( f_2 \circ f_1 \). Put \( \Delta_i x = \varepsilon_i \) \((i = 1, 2) \). Suppose \( f_1(x_1) < f_1(x_1 + \Delta_1 x) \). Then \( h = f_2 \circ f_1 \) satisfies

\[
\Delta h(a) = h(a + \Delta_1 x) - h(a) = f_2(f_1(a + \Delta_1 x)) - f_2(f_1(a)) = \sum_{f_1(a) \leq u < f_1(a + \Delta_1 x)} \frac{\Delta f_2}{\Delta x}(u)(u^+ - u) = A + \frac{\Delta f_2}{\Delta x}(f_1(a))(\Delta f_1(a)),
\]

where

\[
A = \sum_{f_1(a) \leq u < f_1(a + \Delta_1 x)} \left( \frac{\Delta f_2}{\Delta x}(u) - \frac{\Delta f_2}{\Delta x}(f_1(a)) \right)(u^+ - u).
\]

Since \( \frac{\Delta f_2}{\Delta x}(u) \) is continuous, \( \frac{\Delta f_2}{\Delta x}(u) - \frac{\Delta f_2}{\Delta x}(f_1(a)) \approx 0 \) hence by Proposition 7.5.6, \( A \equiv 0 \) if \( x \approx a \). Hence

\[
\frac{\Delta h}{\Delta_1 x}(a) = A + \frac{\Delta f_2}{\Delta x}(f_1(a)) \frac{\Delta f_1}{\Delta_1 x}(a) \approx \frac{\Delta f_2}{\Delta x}(f_1(a)) \frac{\Delta f_1}{\Delta_1 x}(a),
\]

which implies that \( \frac{\Delta h}{\Delta_1 x}(a) \) is continuous and the derivative of \( F_2 \circ F_1 \) is represented by \( (\frac{\Delta f_2}{\Delta x} \circ f_1)(\frac{\Delta f_1}{\Delta_1 x}) \), namely \( (F_2' \circ F_1)F_1' \).

### 8.5 Inverse Function Theorem

**Theorem 8.5.1 (Inverse Function)** Let \( F \) be a differentiable real function on \([-1, 1]\) such that \( F(0) = 0 \) and \( F'(0) \neq 0 \). Then there is a rational \( c > 0 \) and a differential real function on \([-c, c]\) with values in \([-1, 1]\) such that

- \( F(G(y)) = y \) if \( y \in [-c, c] \),
- \( G(F(x)) = x \) if \( F(x) \in [-c, c] \),
- \( G' = \frac{1}{F' \circ G} \).

It suffices to show the following.

**Lemma 8.5.2** Let \( f \) be a rational valued continuous function on \([-1, 1]_\varepsilon\) such that \( f(0) = 0 \) and have the continuous difference quotient and \( \frac{\Delta f}{\Delta x}(0) \neq 0 \). Then there is a rational valued function \( g \) on \([-c, c]_\varepsilon\) for some rational \( c > 0 \) such that
such that if $|x| < a_1$ then $\frac{\Delta f}{\Delta x}(x) > \frac{a}{2}$. Hence $f$ is strictly increasing on $[-a_1, a_1]$. Then if $x \in [-c, c]$ satisfies $f(x) \in [-c, c]$, then by Proposition 8.1.3, $f(a_1) > c$, $f(-a_1) < -c$, where $c = \frac{a_1}{2}$.

For $y \in [-c, c]$, define

$$g(y) := \max \{ x \in [-1, 1] \mid f(x) \leq y \}$$

Then if $y \in [-c, c]$, $f(g(y)) \leq y < f(g(y) + \Delta x) \approx f(g(y))$, whence

$$f(g(y)) \approx y.$$ 

Suppose $y_1, y_2 \in [-c, c]$ satisfies $y_1 \approx y_2$. Then $f(g(y_1)) \approx y_1 \approx y_2 \approx f(g(y_2))$ with $g(y_1), g(y_2) \in [-a_1, a_1]$, whence by Corollary 8.1.4 $g(y_1) \approx g(y_2)$. Thus $g$ is continuous.

On the other hand, since $f$ is strictly increasing on $[-a_1, a_1]$, if $x \in [-a_1, a_1]$ satisfies $f(x) \in [-c, c]$ then

$$f(g(f(x))) \leq f(x) \leq f(g(f(x)) + \Delta x)$$

implies

$$g(f(x)) \leq x \leq g(f(x)) + \Delta x,$$

whence

$$g(f(x)) \approx x.$$ 

Let $b_1 \in [-c, c]$. By Lemma 8.3.6 there is a rational $K_1, K_2, c_1 > 0$, such that

$$K_1 |x - b_1| \leq |f(x) - f(b_1)| \leq K_2 |x - b_1|$$

if $|x - b_1| < c_1$. If we put $y := f(x)$ and $b := f(b_1)$, then $x \approx g(y)$ and $b_1 \approx g(b)$, whence

$$\frac{1}{K_2} |y - b| \leq |g(y) - g(b)| \leq \frac{1}{K_1} |y - b|.$$
Then
\[ f(x) - f(b_1) \equiv_1 \frac{\Delta f}{\Delta x}(b_1)(x - b_1) \quad \text{if } x \approx b_1 \]
implies by virtue of Proposition 7.5.3 that
\[ f(g(y)) - f(g(b)) \approx_1 \frac{\Delta f}{\Delta x}(g(b))(g(y) - g(b)) \quad \text{if } y \approx b, \]
whence (49) since \( y - b \approx f(g(y)) - f(g(b)) \).

### 8.6 Second Order Differentiability

**Definition 8.6.1** A real valued function \( F \) on \([0, 1]\) is called differentiable up to second order if it is differentiable and its derivative is also differentiable. The derivative of \( F' \) is denoted by \( F'' \). We often write it also by \( F^{(2)} \) or by \( \frac{d^2 F(x)}{dx^2} \).

**Theorem 8.6.1** If a real valued function \( F \) on \([0, 1]\) is differentiable up to second order, then for every positive infinitesimal \( \varepsilon \), \( F \) has a representation \((f, [0, 1]_{\varepsilon})\) with continuous difference quotients up to second order, namely, not only \( \frac{\Delta f}{\Delta x} \) but also its difference quotient \( \frac{\Delta^2 f}{\Delta x^2} := \frac{\Delta (\frac{\Delta f}{\Delta x})}{\Delta x} \) is continuous and \((\frac{\Delta^2 f}{\Delta x^2}, \alpha)\) represents \( F'' \).

**Proof.** By Proposition 8.3.5, the differentiable functions \( F \) and \( F' \) have representatives \((f, [0, 1]_{\varepsilon})\) and \((g, [0, 1]_{\varepsilon})\). Put
\[
\hat{f}(x) = f(0) + \sum_{0 \leq u < x, u \in X} g(u) \Delta x
\]
where \( \Delta x = \varepsilon \). Then \( \frac{\Delta f}{\Delta x} \approx g \) implies
\[
\hat{f}(x) \approx f(0) + \sum_{0 \leq u < x} \frac{\Delta f}{\Delta x}(u) \Delta x = f(x),
\]
whence \( \hat{f} \approx f \). Then \( \hat{f} \) is a representation such that both \( \frac{\Delta f}{\Delta x} = g \) and \( \frac{\Delta^2 f}{\Delta x^2} = \frac{\Delta g}{\Delta x} \)
are continuous. The last statement is obvious.

**Theorem 8.6.2** Let \( F \) be a real valued function on \([0, 1]\) differentiable up to second order. Let \((f, [0, 1]_{\varepsilon})\) be a representation of \( F \) with continuous \( \frac{\Delta f}{\Delta x} \) and \( \frac{\Delta^2 f}{\Delta x^2} \). Then for \( a \in [0, 1]_{\varepsilon} \)
\[
f(x) \approx_2 f(a) + \frac{\Delta f}{\Delta x}(a)(x - a) + \frac{\Delta^2 f}{\Delta x^2}(a)\frac{(x - a)^2}{2} \quad \text{if } x \approx a. \quad (50)
\]
**Proof.** By Theorem 8.3.1 we have for $a \in [0,1]_\varepsilon$,

$$\frac{\Delta f}{\Delta x}(u) \equiv_1 \frac{\Delta f}{\Delta x}(a) + \frac{\Delta^2 f}{\Delta x^2}(a)(u-a) \quad \text{if } u \approx a$$  \hfill (51)

Assume $x > a$. By substituting this into

$$f(x) = f(a) + \sum_{a \leq u < x} \frac{\Delta f}{\Delta x}(u) \Delta x,$$

with $\Delta x = \varepsilon$, we obtain by Proposition 7.5.6

$$f(x) \equiv_2 f(a) + \frac{\Delta f}{\Delta x}(a)(x-a) + \frac{\Delta^2 f}{\Delta x^2}(a) \sum_{a \leq u < x} (u-a) \Delta x \quad \text{if } x \approx a \text{ and } x > a$$

since $|x-a| \geq |u-a|$ if $a \leq u < x$

Put $M := \frac{|x-a|}{\Delta x}$. Then $u \in [0,1]_\varepsilon$ with $a \leq u < x$ is written as $u = a + i\Delta x$ with $i = \frac{u-a}{\Delta x} \in [0..M]$. Hence

$$\sum_{a \leq u < x} (u-a) \Delta x = \sum_{0 \leq i \leq M} i(\Delta x)^2 = \frac{M(M+1)}{2}(\Delta x)^2 = \frac{(x-a)^2}{2} + |x-a|\frac{\Delta x}{2}.$$  \hfill (52)

Suppose $0 \approx |x-a| \geq \sqrt{\Delta x}$. Then

$$\frac{|x-a|\Delta x}{|x-a|^2} \leq \frac{1}{2}\sqrt{\Delta x} \approx 0.$$  \hfill (53)

Hence

$$f(x) \approx_2 f(a) + \frac{\Delta f}{\Delta x}(a)(x-a) + \frac{\Delta^2 f}{\Delta x^2}(a)\frac{(x-a)^2}{2} \quad \text{if } x \approx a \text{ and } x > a.$$  \hfill (54)

The proof in the case when $x < a$ is similar.

**Corollary 8.6.3** Let $F$ be a real valued function on $[0,1]$ differentiable up to second order. Let $(g,[0,1]_\varepsilon)$ be its representation. Let $g_1$ and $g_2$ be continuous rational valued function on $[0,1]_\varepsilon$ representing $F'$ and $F''$. Then for $a \in [0,1]_\varepsilon$

$$g(x) \approx_2 g(a) + g_1(a)(x-a) + g_2(a)\frac{(x-a)^2}{2} \quad \text{if } x \approx a.$$  \hfill (55)

**Proof.** By Theorem 8.6.1 there is a representation $(f,[0,1]_\varepsilon)$ of $F$ with continuous $\frac{\Delta f}{\Delta x}$ and $\frac{\Delta^2 f}{\Delta x^2}$. By Theorem 8.6.2 we have (50) on $[0,1]_\varepsilon$. Put $\gamma := \kappa_\delta[0,1]_\varepsilon$ which is a quasi-identity from $[0,1]_\varepsilon$ to $[0,1]_\varepsilon$. Then by Proposition 7.5.3 we have

$$f(\gamma(x)) \approx_2 f(\gamma(a)) + \frac{\Delta f}{\Delta x}(\gamma(a))(\gamma(x)-\gamma(a)) + \frac{\Delta^2 f}{\Delta x^2}(\gamma(a))\frac{(\gamma(x)-\gamma(a))^2}{2} \quad \text{if } x \approx a.$$  \hfill (56)
Since $f \circ \gamma \approx g_1$, $\frac{\Delta f}{\Delta x} \circ \gamma \approx g_1$, and $\frac{\Delta^2 f}{\Delta x^2} \circ \gamma \approx g_2$, we obtain (53) by Proposition 7.5.1.

**Proposition 8.6.4** Suppose a real valued function $F$ has representations $(f_i, [0, 1]_{\varepsilon_i})$ $(i = 1, 2)$ with continuous difference quotients up to second order. Then $\frac{\Delta^2 f_1}{\Delta x^2}$ and $\frac{\Delta^2 f_2}{\Delta x^2}$ represent one and the same real function. In particular, the second derivative $F''$ is represented by $\frac{\Delta^2 f}{\Delta x^2}$ of any representation $(f, [0, 1]_{\varepsilon})$ of $F$ with continuous difference quotients up to second order.

**Proof.** By Theorem 8.6.2

$$f_i(x) - f_i(a) - \frac{\Delta f_i}{\Delta x}(a)(x-a) - \frac{\Delta^2 f_i}{\Delta x^2}(a)\frac{(x-a)^2}{2} \approx 0 \text{ if } x \approx a \quad (55)$$
on $[0,1]_{\varepsilon_i}$, $(i = 1, 2)$.

Let $\beta := \kappa_{\varepsilon_2}|[0,1]_{\varepsilon_1}$ be the quasi-identity from $[0,1]_{\varepsilon_1}$ to $[0,1]_{\varepsilon_2}$. Then by Propositions 7.5.1 and Proposition 7.5.3

$$f_2(\beta(x)) - f_2(\beta(a)) - \frac{\Delta f_2}{\Delta x}(\beta(a))(x-a) - \frac{\Delta^2 f_2}{\Delta x^2}(\beta(a))\frac{(x-a)^2}{2} \approx 0 \text{ if } x \approx a \quad (56)$$

Since $f_2 \circ \beta \approx f_1$, Propositions 7.5.1 implies

$$\left(\frac{\Delta f_2}{\Delta x}(\beta(a)) - \frac{\Delta f_1}{\Delta x}(a)\right)(x-a) + \left(\frac{\Delta^2 f_2}{\Delta x^2}(\beta(a)) - \frac{\Delta^2 f_1}{\Delta x^2}(a)\right)\frac{(x-a)^2}{2} \approx 0 \text{ if } x \approx a. \quad (57)$$

Hence by Lemma 7.5.7

$$\frac{\Delta^2 f_2}{\Delta x^2}(\beta(a)) \approx \frac{\Delta^2 f_1}{\Delta x^2}(a).$$

Since $\beta \circ \kappa_{\varepsilon_1} \approx \kappa_{\varepsilon_2}$,

$$\frac{\Delta^2 f_2}{\Delta x^2} \circ \kappa_{\varepsilon_2} \approx \frac{\Delta^2 f_2}{\Delta x^2} \circ \beta \circ \kappa_{\varepsilon_1} \approx \frac{\Delta^2 f_1}{\Delta x^2} \circ \kappa_{\varepsilon_1}.$$ 

**Theorem 8.6.5** (Characterization of second order differentiability)

A real function $F$ on $[0, 1]$ is differentiable up to second order if it has a representation $(f, [0, 1]_{\varepsilon})$ and there are continuous rational valued functions $g_1$ and $g_2$ on $X$ satisfying for each $a \in [0, 1]_{\varepsilon}$

$$f(x) \approx g_1(a)(x-a) + g_2(a)\frac{(x-a)^2}{2} \text{ if } x \approx a. \quad (58)$$

Moreover $F'$ and $F''$ are represented respectively by $g_1$ and $g_2$. 


Proof. By Lemma 7.5.4 there is a positive infinitesimal $\delta \in [0, 1]_\epsilon$ such that on $[0, 1]_\delta$,
\[
f(x) \equiv_2 f(a) + g_1(a)(x - a) + g_2(a)\frac{(x - a)^2}{2} \quad \text{if } x \approx a \quad (59)
\]
Substituting $x = a + \delta$ we have
\[
\frac{\Delta f}{\Delta x}(a) \approx g_1(a).
\]
Substituting $x = a + 2\delta$ and using
\[
\frac{\Delta^2 f}{\Delta x^2}(a) = \frac{f(a + 2\delta) - 2f(a + \delta) + f(a)}{\delta^2},
\]
we obtain
\[
\frac{\Delta^2 f}{\Delta x^2}(a) \approx g_2(a).
\]

8.7 Higher Order Differentiability

Definition 8.7.1 Let $k \geq 3$ be an accessible number. A real function $F$ on $[0, 1]$ is differentiable up to order $k$ if it is differentiable up to order $k - 1$ and its $k - 1$-th derivative is differentiable. The derivative of its $k - 1$-th derivative is called its $k$-th derivative and is denoted by $F^{(k)}$ and $\frac{d^k F(x)}{dx^k}$.

Theorem 8.6.1 extends to general order.

Theorem 8.7.1 Let $\epsilon$ be a positive infinitesimal and $k$ an accessible number. If a real function $F$ on $[0, 1]$ is differentiable up to $k$-th order, then $F$ has a representation $(f, [0, 1]_\epsilon)$ with continuous difference quotients up to $k$-th order, namely, the higher order difference quotients defined inductively by
\[
\frac{\Delta^i f}{\Delta x^i} := \frac{\Delta(\frac{\Delta^{i-1} f}{\Delta x^{i-1}})}{\Delta x}
\]
is continuous and represents $F^{(i)}$ for $i \leq k$.

Proof. By Proposition 8.3.5, the real functions $F^{(i)}$ have representatives $(g_i, [0, 1]_\epsilon)$ for $i \in [0..k]$. Define $\hat{g}_{k-i}$ for $i \in [0..k]$ inductively by $\hat{g}_k = g_k$ and, for $i \geq 1$,
\[
\hat{g}_{k-i}(x) = g_{k-i}(0) + \sum_{0 \leq u < x} \hat{g}_{k-i+1}(u) \Delta x.
\]
Then $\hat{g}_i \approx g_i$ for $i \in [0..k]$ and if $i < k$
\[
\frac{\Delta \hat{g}_i}{\Delta x} = \hat{g}_{i+1},
\]
whence $\frac{\Delta^i \hat{g}_i}{\Delta x^i} = \hat{g}_i \approx g_i$ is continuous.
Theorem 8.7.2 (Taylor formula) Let $F$ be a real function on $[0, 1]$ differentiable up to $k$-th order with accessible $k$. Let $(f, [0, 1])$ be a representation with continuous $\frac{\Delta^i f}{\Delta x^i}$ for $i \in [1..k]$. Then for $a \in X$

\[ f(x) \approx_k f(a) + \sum_{i=1}^{k} \frac{\Delta^i f}{\Delta x^i}(a) (x - a)^i \quad \text{if } x \approx a. \]  \hspace{1cm} (60)

**Proof.** By induction on $\ell \in [1..k]$, we can show

\[ f(x) - f(a) \approx_k \sum_{i=1}^{\ell} \frac{\Delta^i f}{\Delta x^i}(a) \frac{(x - a)^i}{i!} + \sum_{a \leq u_1 \leq \cdots \leq u_2 \leq x} \frac{\Delta^i f}{\Delta x^i}(u_t)(\Delta x)^t \quad \text{if } x \approx a. \]  \hspace{1cm} (61)

In fact, for $\ell = 1$, this is essentially (42). Suppose (61) holds for $\ell \leq t$.

\[ \sum_{a \leq u_1 \leq \cdots \leq u_2 \leq x} \frac{\Delta^i f}{\Delta x^i}(u_t)(\Delta x)^t = \sum_{a \leq u_1 \leq \cdots \leq u_2 \leq x} \left( \frac{\Delta^i f}{\Delta x^i}(a) + \sum_{a \leq u_{t+1} \leq u_t} \frac{\Delta^{t+1} f}{\Delta x^{t+1}}(u_{t+1})(\Delta x)^{t+1} \right) (\Delta x)^t \]

\[ = \frac{\Delta^i f}{\Delta x^i}(a) \sum_{a \leq u_1 \leq \cdots \leq u_2 \leq x} (\Delta x)^t + \sum_{a \leq u_{t+1} \leq u_t \leq \cdots \leq u_2 \leq x} \frac{\Delta^{t+1} f}{\Delta x^{t+1}}(u_{t+1})(\Delta x)^{t+1} \]

\[ \approx_k \frac{\Delta^i f}{\Delta x^i}(a) \frac{(x - a)^i}{i!} + \sum_{a \leq u_{t+1} \leq u_t \leq \cdots \leq u_2 \leq x} \frac{\Delta^{t+1} f}{\Delta x^{t+1}}(u_{t+1})(\Delta x)^{t+1}, \]

by Lemma 8.7.3 below. Hence (61) holds for $\ell = t + 1$.

Finally we calculate the last term of (61) with $\ell = k$. Since $\frac{\Delta^k f}{\Delta x^k}$ is continuous,

\[ \sum_{a \leq u_k \leq \cdots \leq u_2 \leq x} \frac{\Delta^k f}{\Delta x^k}(u_k)(\Delta x)^k \approx_k \sum_{a \leq u_k \leq \cdots \leq u_2 \leq x} \frac{\Delta^k f}{\Delta x^k}(a)(\Delta x)^k \approx_k \frac{\Delta^k f}{\Delta x^k}(a) \frac{(x - a)^k}{k!} \]

by Lemma 8.7.3 below. Hence by Proposition 7.5.1

\[ \sum_{a \leq u_k \leq \cdots \leq u_2 \leq x} \frac{\Delta^k f}{\Delta x^k}(u_k)(\Delta x)^k \approx_k \frac{\Delta^k f}{\Delta x^k}(a) \frac{(x - a)^k}{k!} \quad \text{if } x \approx a \]
Lemma 8.7.3 Let $\ell$ be an accessible number and $a, x \in [0, 1]$ with $a \approx x$. If $\frac{|a-x|}{\varepsilon}$ is huge then

$$
\sum_{a \leq u_1 \leq \cdots \leq u_2 \leq u_1 < x} (\Delta x)^{\ell} \approx \frac{(x-a)^{\ell}}{\ell!}
$$

where $\Delta x = \varepsilon$. In particular, by Proposition 7.5.1, for any accessible number $k$

$$
\sum_{a \leq u_1 \leq \cdots \leq u_2 \leq u_1 < x} (\Delta x)^{\ell} \approx_k \frac{(x-a)^{\ell}}{\ell!} \text{ if } x \approx a,
$$

Proof. Put $L = \frac{x-a}{\Delta x}$ and assume $L \gg 1$. Then

$$
\sum_{a \leq u_1 \leq \cdots \leq u_2 \leq u_1 < x} (\Delta x)^{\ell} = \# \{ (i_1, \ldots, i_{\ell}) | 0 \leq i_1 \leq \cdots \leq i_{\ell} < L \} (\Delta x)^{\ell} = \binom{L}{\ell} (\Delta x)^{\ell} = \frac{L^\ell}{\ell!} \left( 1 - \frac{1}{L} \right) \left( 1 - \frac{2}{L} \right) \left( 1 - \frac{\ell - 1}{L} \right) (\Delta x)^{\ell} \approx \frac{L^\ell}{\ell!} (\Delta x)^{\ell} = \frac{(x-a)^{\ell}}{\ell!}.
$$

Remark 8.7.1 The infinitesimal Taylor formula gives usual one by virtue of Proposition 7.5.5.

Corollary 8.7.4 extends to general order. The proof is similar and omitted.

Corollary 8.7.4 Let $F$ be a real function on $[0, 1]$ differentiable up to $k$-th order with a representation $(g, [0, 1]_e)$. Let $g_i$ $(1 \leq i \leq k)$ be continuous rational valued functions on $[0, 1]_e$ representing $\frac{d^iF}{dx^i}$. Then for $a \in [0, 1]_e$

$$
g(x) \approx_k g(a) + \sum_{i=1}^{k} g_i(a) \frac{(x-a)^i}{i!} \text{ if } x \approx a.
$$

Theorem 8.7.5 (Characterizaion of higher order differentiability) A real function $F$ on $[0, 1]$ is differentiable up to $k$-th order if it has a representation $(f, [0, 1]_e)$ and there are continuous rational valued functions $g_i$ $(i \in [1..k])$ on $[0, 1]_e$ satisfying (64) for each $a \in [0, 1]_e$. Then $g_i$ represents $\frac{d^iF}{dx^i}$ for $i \in [1..k]$. 


8.8 Fundamental Theorem of Calculus

Definition 8.8.1 Let $F$ be a real function on $[0, 1]$ with a representation $(f, [0, 1]_\epsilon)$. The rational valued function $\Sigma f \Delta x$ on $[0, 1]_\epsilon$ defined by

$$(\Sigma f \Delta x)(u) := \sum_{0}^{u} f \Delta x = \sum_{0 \leq x < a} f(x) \Delta x,$$

where $\Delta x = \epsilon$, is continuous and finite. The real function represented by $\Sigma f \Delta x$ is called the indefinite integral of $F$ and is written as $\int_{0}^{t} F(x)dx$. \hfill \blacksquare$

This definition is legitimate since $\Sigma f \Delta x$ is continuous and finite by Proposition 8.8.1 and $\int_{0}^{t} F(x)dx$ does not depend on the representation of $F$ by Proposition 8.8.3.

Proposition 8.8.1 If $f$ is a continuous rational valued function on $[0, 1]_\epsilon$, then the rational valued function $\Sigma f \Delta x$ on $[0, 1]_\epsilon$ is continuous and finite.

Proof. Let $M$ be the maximum of $|f|$, which is finite by assumption. If $a, b \in [0, 1]_\epsilon$ and $a < b$, then

$$|(\Sigma f \Delta x)(b) - (\Sigma f \Delta x)(a)| \leq \sum_{a \leq x < b} |f(x)| \Delta x \leq M \sum_{a \leq x < b} \Delta x = M|b - a|.$$

Hence $\Sigma f \Delta x$ is continuous. It is finite since

$$|(\Sigma f \Delta x)(u)| \leq M \sum_{0}^{u} \Delta x = Mu.$$

Lemma 8.8.2 Suppose $\epsilon_i$ ($i = 1, 2$) are positive infinitesimals such that $\epsilon_1 \in [0, 1]_{\epsilon_2}$. For a continuous rational valued function $g$ on $[0, 1]_{\epsilon_1}$, define $\tilde{g} := g \circ \kappa_{\epsilon_1}$ on $[0, 1]_{\epsilon_2}$, namely, put

$$\tilde{g}(x) := g([x/\epsilon_1] \epsilon_1).$$

Then $\tilde{g}$ is continuous and on $[0, 1]_{\epsilon_2}$

$$\Sigma \tilde{g} \Delta x \approx (\Sigma g \Delta x) \circ \kappa_{\epsilon_1}. \quad (65)$$

In particular $\Sigma g \Delta x$ and $\Sigma \tilde{g} \Delta x$ represent one and the same real function on $[0, 1]$. 

Proof. The continuity of \( \tilde{g} \) is obvious.

For \( u \in [0, 1]_{\varepsilon_2} \), put \( u_- := \kappa_{\varepsilon_1}(u) = [u/\varepsilon_1]\varepsilon_1 \). Then

\[
(S\tilde{g}\Delta x)(u) = \sum_{0 \leq t < u, t \in [0, 1]_{\varepsilon_2}} \tilde{g}(t)\varepsilon_2 
= \sum_{0 \leq t < u_-} \tilde{g}(t)\varepsilon_2 + \sum_{u_- < t < u} \tilde{g}(t)\varepsilon_2 
= \sum_{0 \leq t < u_-} g(t)\varepsilon_1 + \sum_{u_- < t < u} \tilde{g}(t)\varepsilon_2 \approx (Sg\Delta x)(u_-),
\]

whence \((65)\).

**Proposition 8.8.3** If \((f_i, [0, 1]_{\varepsilon_i}) \ (i = 1, 2)\) represent a real function \(F\) on \([0, 1]\), then \(Sf_1\Delta x\) and \(Sf_2\Delta x\) represent one and the same real function on \([0, 1]\).

Proof. Let \(\varepsilon\) be an infinitesimal such that \(\varepsilon_i \in [0, 1]_{\varepsilon} \ (i = 1, 2)\). For example, if \(\varepsilon_i = \frac{p_i}{q_i}\), then one may take \(\varepsilon = \frac{1}{q_1q_2}\).

Then \(F\) is represented by \(\tilde{f}_i := f_i \circ \kappa_{\varepsilon_i} \circ [0, 1]_{\varepsilon} \ (i = 1, 2)\) and by Proposition 8.1.2

\[
S\tilde{f}_1\Delta x \approx S\tilde{f}_2\Delta x,
\]
and by Lemma 8.8.2 for \(i = 1, 2\),

\[
S\tilde{f}_i\Delta x \approx (Sf_i\Delta x) \circ \kappa_{\varepsilon_i}.
\]

Since \(\kappa_{\varepsilon_i} \circ \kappa_{\varepsilon} = \kappa_{\varepsilon_i} \ (i = 1, 2)\) , for \(i = 1, 2\) we have

\[
Sf_1\Delta x \circ \kappa_{\varepsilon_1} = Sf_1\Delta x \circ \kappa_{\varepsilon_1} \circ \kappa_{\varepsilon} \approx S\tilde{f}_1\Delta x \circ \kappa_{\varepsilon}.
\]

Hence by \((66)\),

\[
Sf_1\Delta x \circ \kappa_{\varepsilon_1} \approx Sf_2\Delta x \circ \kappa_{\varepsilon_2},
\]

namely \(Sf_i\Delta x \ (i = 1, 2)\) represent one and the same real function on \([0, 1]_{\varepsilon}\).

Generally, the indefinite integral \(\int_a^t F(x)dx\) is defined for a real function \(F\) on general interval \([a, b]\).

**Proposition 8.8.4** Suppose \(F\) is a real function on \([0, 1]\). Then the real function \(\int_0^t F(x)dx\) on \([0, 1]\) is differentiable and its derivative is \(F\).

Proof. The indefinite integral \(\int_0^t F(x)dx\) is represented by \(Sf\Delta x\) using a representation \((f, [0, 1]_{\varepsilon})\) of \(F\). Its difference quotient is

\[
\frac{\Delta(Sf\Delta x)(x)}{\Delta x} = \frac{(Sf\Delta x)(x^+) - (Sf\Delta x)(x)}{\Delta x} = f(x),
\]
where $\Delta x = \varepsilon$. Thus the indefinite integral has the representation $\Sigma f \Delta x$ with the continuous difference quotients $f$, whence is differentiable and its derivative is $F$. 

### 8.9 Ordinary Differential Equation

**Theorem 8.9.1** Let $K$ be a finite positive real number and $F$ a real function on $A = [0, 1] \times [-2K, 2K]$ satisfying

$$|F(x, y)| < K$$

for all $(x, y) \in A$. Further suppose that there is a finite real number $L$ such that

$$|F(x, y_1) - F(x, y_2)| \leq L|y_1 - y_2|$$

holds for all $x \in [0, 1]$ and $y_i \in [-2K, 2K]$ $(i = 1, 2)$. Then there is a unique real function $G(x, a)$ on $[0, 1] \times [-K, K]$ satisfying

$$G(0, a) = a, \quad \frac{dG(x, a)}{dx} = F(x, G(x, a)).$$

This is equivalent to the following statement.

**Proposition 8.9.2** Let $\varepsilon > 0$ be an infinitesimal and $K < 0$ a finite rational number. If $f$ is a rational valued continuous function on $X = [0, 1]_\varepsilon \times [-2K, 2K]_\varepsilon$ satisfying

$$|f(x, y)| < K,$$

and there is a finite rational number $L$ such that

$$|F(x, y_1) - F(x, y_2)| \leq L|y_1 - y_2|$$

for all $x \in [0, 1]_\varepsilon$ and $y_i \in [-2K, 2K]_\varepsilon$ $(i = 1, 2)$.

Then there is a rational valued continuous function $\varphi$ on $Y = [0, 1]_\varepsilon \times [-K, K]_\varepsilon$ satisfying

$$\varphi(0, a) \approx a$$

for all $a \in [-K, K]_\varepsilon$ and

$$\frac{\varphi(x + \Delta x, a) - \varphi(x, a)}{\Delta x} \approx f(x, \varphi(x, a)) \quad (68)$$

for all $(x, a) \in [0, 1]_\varepsilon \times [-K, K]_\varepsilon$.

Furthermore if another continuous rational valued function $\psi$ on $[0, 1]_\varepsilon \times [-K, K]_\varepsilon$ satisfies $[68]$ with $\varphi$ replaced by $\psi$, then $\varphi \approx \psi$. 
\section*{Proof.} For $x \in [0, 1]_\varepsilon$ and $a \in [-K, K]_\varepsilon$, we can define $\varphi(x, a)$ by “induction on $x$” as follows.

\[ \varphi(0, a) := a, \quad \varphi(x + \Delta x, a) := \varphi(x, a) + f(x, \varphi(x, a))\Delta x, \]

where $\Delta x = \varepsilon$.

In fact, suppose we have defined $\varphi(x, a)$ for $x \leq b$ for some $b < 1$. Then for $0 \leq x \leq b$

\[ |\Delta_x \varphi(x, a)| \leq |f(x, \varphi(x, a))| < K, \]

where $\Delta_x \varphi(x, a) = \varphi(x + \Delta x, a) - \varphi(x, a)$. Hence by Proposition 8.1.3

\[ |\varphi(b, a) - \varphi(0, a)| < bK < K. \]

Hence $|\varphi(b, a)| < K + a < 2K$ and $f(b, \varphi(b, a))$ has value and $\varphi(b + \Delta x, a)$ is defined.

From $|\Delta_x \varphi| < K$, it follows

\[ |\varphi(x, a) - \varphi(y, a)| < K|x - y|. \]

Hence $\varphi$ is continuous with respect to the first variable and the partial difference quotient with respect to $x$

\[ \frac{\Delta_x \varphi(x, a)}{\Delta x} = f(x, \varphi(x, a)) \]

is also continuous with respect to $x$.

To show the continuity of $\varphi$ with respect to the second variable, put $h(x) := \varphi(x, a) - \varphi(x, b)$. Then

\[ \left| \frac{h(x + \Delta x) - h(x)}{\Delta x} \right| = \left| f(x, \varphi(x, a)) - f(x, \varphi(x, b)) \right| \leq L|h(x)|. \]

Hence by Lemma 8.9.3 below,

\[ h(x) \leq h(0) \exp(Lx) = |a - b| \exp(Lx). \]

Hence $a = b$ implies $\varphi(x, a) \approx \varphi(x, b)$. Thus it is verified that $\varphi$ is continuous.

Let $\psi$ be a rational valued function on $[0, 1]_\varepsilon \times [-K, K]_\varepsilon$ satisfying (68) with $\varphi$ replaced by $\psi$.

Fix $a \in [-K, K]_\varepsilon$ and put $h(x) = \varphi(x, a) - \psi(x, a)$. Then we can show similarly

\[ \left| \frac{\Delta h(x)}{\Delta x} \right| \leq L|h(x)| \quad \text{for all } x \in [0, 1]_\varepsilon, \tag{69} \]

whence by the following Lemma 8.9.3

\[ |h(x)| \leq |h(0)| \exp(Lx). \tag{70} \]

Since $|h(0)| \approx 0$ and $\exp(xL)$ is bounded, we obtain $|h(x)| \approx 0$ for all $x \in [0, 1]_\varepsilon$. \null\null\null
Lemma 8.9.3 If a rational valued continuous function $h$ on $[0, 1]$ satisfies (69), then (70) holds.

Proof. Put
$$\delta := \max \left\{ 0, \text{max} \left\{ \left| \frac{\Delta h(x)}{\Delta x} \right|, x \in [0, 1] \right\} \right\} \approx 0.$$ Then
$$\left| \frac{h(x + \Delta x) - h(x)}{\Delta x} \right| \leq L|h(x)| + \delta.$$ Hence
$$|h(x + \Delta x)| \leq (1 + L\Delta x)|h(x)| + \delta \Delta x,$$ which is equivalent to
$$|h(x + \Delta x)| + \frac{\delta}{L} \leq \left( |h(x)| + \frac{\delta}{L} \right) (1 + L\Delta x).$$ Hence
$$|h(x)| + \frac{\delta}{L} \leq \left( h(0) + \frac{\delta}{L} \right) (1 + L\Delta x) \approx x \exp(xL).$$ If we put $T = \frac{x}{\Delta x}$, then $\frac{1}{T} \approx 0$ whenever $x \neq 0$ and by Proposition 7.2.4 we have
$$(1 + L\Delta x) \exp(xL) = \left( 1 + \frac{xL}{T} \right)^T \approx \exp(xL).$$ Hence
$$|h(x)| \approx |h(x)| + \frac{\delta}{L} \leq \left( |h(0)| + \frac{\delta}{L} \right) \exp(xL) \approx |h(0)| \exp(xL)$$ when $x > 0$, which implies (70) by Proposition 1.1.2. \qed
9 Differentiation: Multiple Variables

Let $n$ be an accessible number. We consider real functions only on the continuum $[0,1]^n$ for simplicity but nothing changes essentially for general continuum of the form $\prod_i [a_i, b_i]$.

We say a real function $F$ on $[0,1]^n$ is represented by a continuous rational valued function $f : [0,1]^n \to \mathbb{Q}$ if $f \circ \kappa \in F$. We say then that $(f, [0,1]^n)$ is a representation of $F$.

9.1 Partial Difference Quotients

Let $f$ be a rational valued function on $X = [0,1]^n$ with accessible $n$. Define the partial differences of $f$ by

$$\Delta_if(x) := f(x + \Delta x e_i) - f(x)$$

for $x_i + \Delta x \leq 1$ and $\Delta_if(x) := \Delta_i f(x - \Delta x e_i)$ otherwise. Here $\Delta x := \varepsilon$ and $e_i$ the is the $n$-vector with the $i$-th component 1 and other components 0. The quotient $D_if(a) := \Delta_if(a)/\Delta x$ is called the $i$-th partial difference quotient at $a$ and $D_if$ is a rational valued function on $X$.

**Lemma 9.1.1** If $f$ is a rational valued function on $X = [0,1]^n$ and $a \in X$, then

$$f(x) - f(a) = \varepsilon \sum_{i=1}^n \sum_{u \in I_i(a,x)} \text{sgn}(x_i - a_i) \Delta_i f(x[i-1] + u e_i),$$

where

$$x[i] = (x_1, \ldots, x_i, a_{i+1}, \ldots, a_n)$$

with $x[0] = a$,

$$I_i(a,x) = \left\{ \begin{array}{ll} \{ u & | a_i \leq u < x_i \} & \text{if } a_i \leq x_i \\ \{ u & | x_i \leq u < a_i \} & \text{if } x_i < a_i \end{array} \right.$$ 

and $\text{sgn}$ is the signature function defined by

$$\text{sgn}(r) = \left\{ \begin{array}{ll} \frac{r}{|r|} & \text{for } r \neq 0 \\ 0 & \text{otherwise} \end{array} \right.$$ 

**Proof.** Obviously we have

$$f(x) - f(a) = \sum_{i=1}^n (f(x[i]) - f(x[i-1])).$$
Put \( g(t) = f(x_1, \ldots, x_{i-1}, t, a_{i+1}, \ldots, a_n) \). Then \( g(a_i) = f(x[i - 1]) \) and \( g(x_i) = f(x[i]) \) and \( \frac{\Delta g}{\Delta x_i}(u) = D_i f(x[i - 1] + u e_i) \). If \( a_i \leq x_i \)

\[
g(x_i) = g(a_i) + \sum_{a_i \leq u < x_i} \frac{\Delta g}{\Delta x_i}(u) \varepsilon.
\]

If \( x_i < a_i \), then

\[
g(a_i) = g(x_i) + \sum_{x_i \leq u < a_i} \frac{\Delta g}{\Delta x_i}(u) \varepsilon,
\]

whence

\[
g(x_i) = g(a_i) + \sum_{u \in I_i(a,x)} sgn(x_i - a_i) \frac{\Delta g}{\Delta x_i}(u) \varepsilon.
\]

### Proposition 9.1.2

If \( f \) is a rational valued function on \( X = [0, 1]^n \) and

\[ |D_i f(x)| < M \]

for all \( i \in [1..n] \) and \( x \in X \). Then \( |f(x) - f(a)| \leq nMd(x,a) \) for all \( x, a \in X \).

**Proof.** By Lemma 9.1.1 we have

\[
|f(x) - f(a)| \leq \sum_{i=1}^{n} \sum_{u \in I_i(a,x)} M = M \sum_{i=1}^{n} |x_i - a_i| \leq nMd(x,a).
\]

### 9.2 Differentiability

**Definition 9.2.1** A real function \( F \) on \( [0, 1]^n \) is called differentiable if it has a representation whose partial difference quotients are continuous, namely, \( F \) has a representation \((f, [0, 1]^n)\) such that the partial difference quotients \( D_i f \) are continuous. By Corollary 9.2.2 below, this definition does not depend on the choice of representations.

The real function on \([0, 1]^n\) represented by \((D_i f, [0, 1]^n)\) is called the \( i \)-th partial derivative and is denoted by \( \frac{\partial F}{\partial x_i} \). Sometimes we write it as \( \partial_i F \) for brevity. These functions are independent not only of the choice of \( f \) as is seen by the arguments in Lemma 8.2.2 but also of the choice of \( \varepsilon \) by Corollary 9.2.2.
Theorem 9.2.1 (Infinitesimal Taylor formula of first order) Let $f$ be a continuous rational valued function on $[0, 1]^n$ with continuous partial difference quotients $D_i f$ ($i \in [1..n]$). Then

$$f(x) \equiv_1 f(a) + \sum_{i=1}^{n} D_i f(a)(x_i - a_i) \quad \text{if } x \approx a.$$  \hfill (71)

In particular the following also holds.

$$f(x) \approx_1 f(a) + \sum_{i=1}^{n} D_i f(a)(x_i - a_i) \quad \text{if } x \approx a \quad (72)$$

Proof. For simplicity we consider the case $a_i \leq x_i$.

By the continuity of $D_i f$,

$$f(x[i]) - f(x[i - 1]) = \sum_{a_i \leq u < x_i} D_i f(x[i - 1] + u e_i) \Delta x \equiv_1 D_i f(a) \sum_{a_i \leq u < x_i} \Delta x = D_i f(a)(x_i - a_i) \quad \text{if } x \approx a.$$  

by Proposition 7.5.6.

Corollary 9.2.2 If $F$ is a differentiable function on $[0, 1]^n$, then for any representation $(f, [0, 1]^n_\varepsilon, \alpha)$, there are continuous functions $g_1, \ldots, g_n$ such that

$$f(x) \approx_1 f(a) + \sum_{i=1}^{n} g_i(a)(x_i - a_i) \quad \text{if } x \approx a.$$  \hfill (73)

and $(g_i, [0, 1]^n_\varepsilon, \alpha)$ represents $\partial_i F$.

Proof. Suppose $(h, [0, 1]_\varepsilon)$ is a representation of $F$ with continuous partial difference quotients so that

$$h(y) \approx_1 h(b) + \sum_{i=1}^{n} D_i h(b)(y_i - b_i) \quad \text{if } y \approx b.$$  \hfill (74)

Let $\alpha : [0, 1]^n_\varepsilon \rightarrow [0, 1]^n_\delta$ be the restriction of $k_\varepsilon^\alpha$ and put $g_i(x) = D_i h(\alpha(x))$ for $x \in [0, 1]^n_\varepsilon$. Since $\alpha$ is a quasi-identity, (74) implies (73) by Proposition 7.5.3.

Since $g_i \simeq D_i h$, the continuous function $g_i$ represents $\partial_i F$.

Proposition 9.2.3 (Characterization of differentiability) Suppose a real function $F$ on $[0, 1]^n$ has a representation $(f, [0, 1]^n_\varepsilon)$ with continuous rational valued functions $g_1, \ldots, g_n$ such that

$$f(x) \approx_1 f(a) + \sum_{i=1}^{n} g_i(a)(x_i - a_i) \quad \text{if } x \approx a.$$  \hfill (75)

Then $F$ is differentiable.
Proof. By Proposition 5.5.4 there is a huge number $L$ such that $L \epsilon \approx 0$ and for $a \in [0, 1]_L^n$

$$f(x) \equiv f(a) + \sum_{i=1}^n g_i(a)(x_i - a_i) \quad \text{if } x \approx a,$$

(76) on $[0, 1]^n_L$. Then

$$D_i f(a) = \frac{f(a + L \epsilon e_i) - f(a)}{L \epsilon} = g_i(a)$$

for $a \in [0, 1]^n_L$ hence the partial difference quotients of $f$ restricted on $[0, 1]^n_L$ are continuous.

9.3 Chain Rule

Let $F : [0, 1]^{n_1} \to [0, 1]^{n_2}$ be a morphism. Then $F = (F_1, \cdots, F_n)$ with real functions $F_i$ on $[0, 1]^{n_1}$. We call $F$ a differentiable morphism if each $F_i$ is differentiable. Let $G$ be a real function on $[0, 1]^{n_2}$. The composition $G \circ F$ is a real function on $[0, 1]^{n_1}$.

Theorem 9.3.1 If $F$ and $G$ are differentiable, then the composition $G \circ F$ is differentiable and for $i \in [1..n]$

$$\frac{\partial (G \circ F)}{\partial x_i} = \sum_{j=1}^{n_2} \left( \frac{\partial G}{\partial x_j} \circ F \right) \frac{\partial F_i}{\partial x_i}.$$  

(77)

Proof. Let $(g, [0, 1]^{n_2}_L)$ be a representation of $G$ with continuous difference quotients. By [0.2.1] for $b \in [0, 1]^{n_2}_L$

$$g(y) \equiv_1 g(b) + \sum_{i=1}^{n_2} D_i g(b)(y_i - b_i) \quad \text{if } y \approx b \quad (78)$$

Let $F$ be represented by $f : [0, 1]^{n_1}_L \to [0, 1]^{n_2}_{1Q}$. Then the composition $G \circ F$ is represented by $g \circ \kappa^n_\epsilon \circ f$

$$[0, 1]^{n_1}_L \xrightarrow{f} [0, 1]^{n_2}_{1Q} \xrightarrow{\kappa^n_\epsilon} [0, 1]^{n_2}_L$$

$$\xrightarrow{g} \mathbb{Q}$$

By Corollary 9.2.2 we have for $i \in [1..n_1]$ and $a \in [0, 1]^{n}_L$

$$f_i(x) - \left( f_i(a) + \sum_{j=1}^{n_1} D_j f_i(a)(x_j - a_j) \right) \approx_1 0 \quad \text{if } x \approx a.$$  

(79)
Since $\kappa_\varepsilon$ is a quasi-identity, this implies

$$
\kappa_\varepsilon(f(x)) = \left( \kappa_\varepsilon(f(a)) + \sum_{j=1}^{n_1}(D_j f_i(a))(x_j - a_j) \right) \approx 1 \quad \text{if } x \approx a, \quad (80)
$$

by Proposition 7.5.2. Put $\alpha = (\kappa_\varepsilon, \cdots, \kappa_\varepsilon) : [0,1]^{n_1} \to [0,1]_{\varepsilon}$. If $x \approx a$ then $\alpha(f(x)) - \alpha(f(a)) \approx 0$, whence by substituting $y = \alpha(f(x)), y_i = \kappa_\varepsilon(f_i(x))$ and $b = \alpha(f(a))$ in (78) we obtain

$$
g(\alpha(f(x))) \equiv_1 g(\alpha(f(a))) + \sum_{i=1}^{n_2} D_i g(\alpha(f(a)))(\kappa_\varepsilon(f_i(x)) - \kappa_\varepsilon(a)) \quad \text{if } x \approx a \quad (81)
$$

Hence by (80), we obtain for $a \in [0,1]_{\varepsilon}^{n_1}$,

$$
g(\alpha(f(x))) \approx_1 g(\alpha(f(a))) + \sum_{i=1}^{n_2} D_i g(\alpha(f(a)))(\sum_{j=1}^{n_1}(D_j f_i(a))(x_j - a_j))
$$

$$
= g(\alpha(F(a))) + \sum_{j=1}^{n_1} g_j(a)(x_j - a_j)
$$

with

$$
g_j(a) = \sum_{i=1}^{n_2} D_i g(\alpha(F(a)))(D_j f_i(a)).
$$

Hence by Proposition 9.2.3 $G \circ F$ is differentiable. Moreover, since $D_i g(\alpha(f(a))$ represents $\frac{\partial G}{\partial x_i} \circ F$, we have (77).

### 9.4 Implicit Function Theorem

The following implicit function theorem is in essence the inverse function theorem 8.3.1 of one variable with parameters and is proved by similar arguments.

**Theorem 9.4.1 (Implicit Function Theorem)** Let $F$ be a differentiable function on $[-1,1]^n$ such that

$$
F(0) = 0 \quad , \frac{\partial F}{\partial x_n}(0) \neq 0.
$$

Then there is a differentiable function $G$ on $[-c,c]^n$ with some $c > 0$ such that for $y \in [-c,c]$ and $x' = (x_1, \cdots, x_{n-1}) \in [-c,c]^{n-1}$,

$$
F(x', G(x', y)) = y,
$$

and if $x \in [-c,c]^n$ satisfies $|F(x', x_n)| \leq c$, then

$$
G(x', F(x', x_n)) = x_n.
$$
The partial derivatives are given by
\[
\frac{\partial G(x', y)}{\partial x_i} = \begin{cases} 
\frac{-\partial F(x', G(x', y))}{\partial x_i} & \text{for } i \in \{1..n-1\} \\
\frac{-\partial F(x', G(x', y))}{\partial x_n} & \text{for } i = n
\end{cases}
\]

This follows from the following lemma.

**Lemma 9.4.2** Let \( f \) be a rational valued function on \([-1, 1]^n \) with continuous partial difference quotients and suppose
\[
f(0) = 0, \quad \mathcal{D}_n f(0) \neq 0.
\]

Then there is a continuous rational valued function \( g \) on \([-c, c]^n \) with some \( c > 0 \) such that for \( y \in [-c, c] \) and \( x' \in [-c, c]^{n-1} \)
\[
f(x', g(x', y)) \approx y.
\]

Moreover if \( (x', x_n) \in [-c, c]^{n-1} \) satisfies \( f(x', x_n) \in [-c, c]), then
\[
g(x', x_n) \approx x_n.
\]

The difference quotients of \( g \) at \( (x', y) \in [-c, c]^n \) is given by
\[
\mathcal{D}_i g(x', y) \approx \begin{cases} 
\frac{-\mathcal{D}_i f(x', g(x', y))}{\mathcal{D}_n f(x', g(x', y))} & \text{for } i \in \{1..n-1\} \\
\frac{-\mathcal{D}_n f(x', g(x', y))}{\mathcal{D}_n f(x', g(x', y))} & \text{for } i = n
\end{cases}
\]
and are continuous.

**Proof.** We may assume \( \mathcal{D}_n f(0) > 0 \). Then there are \( \alpha, c_1 > 0 \) such that \( \mathcal{D}_n f(x) \geq \alpha \) if \( x \in [-c_1, c_1]^n \). By Proposition 8.1.3
\[
f(0, c_1) \prec -\frac{\alpha}{2}c_1, \quad \frac{\alpha}{2}c_1 \prec f(0, c_1),
\]
whence there is \( 0 < c_2 < c_1 \) such that if \( x' \in [-c_2, c_2]^{n-1} \) then
\[
f(x', -c_1) \prec -\frac{\alpha}{2}c_1, \quad \frac{\alpha}{2}c_1 \prec f(x', c_1).
\]
Put \( c = \min \{ c_2, \frac{\alpha c_1}{4} \} \), then \( (x', y) \in [-c, c]^n \) implies
\[
f(x', -c_1) \prec y \prec f(x', c_1),
\]
since \( f(x', c_1) > \frac{\alpha}{2}c_1 \geq 2c > c > y \) for the first equality. Define
\[
g(x', y) = \max \{ u \in [-c_1, c_1] | f(x', u) \leq y \}.
\]
Then
\[
f(x', g(x', y)) \leq y < f(x', g(x', y) + \Delta x),
\]
which implies
\[ f(x', g(x', y)) \approx y. \]  \hspace{1cm} (82)

By Corollary 8.1.4
\[ \text{if } (x', u), (x', v) \in [-c_1, c_1]^2 \text{ satisfies } f(x', u) \approx f(x', v) \text{ then } u \approx v. \]  \hspace{1cm} (83)

If $(x', u) \in [-c, c]^2$ satisfies $f(x', u) \in [-c, c]$, then (82) with $y = f(x', u)$ implies
\[ f(x', g(x', \kappa_\varepsilon(f(x', u)))) \approx \kappa_\varepsilon(f(x', u)) \approx f(x', u), \]
whence by (83)
\[ u \approx g(x', \kappa_\varepsilon(f(x', u))), \]
since the values of $g$ is in $[-c_1, c_1]^2$.

If $(x'_i, y_i) \in [-c, c]^2$ (i = 1, 2) satisfies $x'_i \approx x'_2, y_1 \approx y_2$, then
\[ f(x'_1, g(x'_1, y_1)) \approx y_1 \approx y_2 \approx f(x'_2, g(x'_2, y_2)) \approx f(x'_1, g(x'_2, y_2)), \]
whence, by (83) again, $g(x'_1, y_1) \approx g(x'_2, y_2)$. Thus $g$ is continuous.

Suppose $f(x', x_n) = y$ and $u' \approx u_n = w$, which imply respectively $x_n \approx g(x', y)$ and $u_n \approx g(u', w)$. By Theorem 9.2.1,
\[ f(x', x_n) \equiv f(u', u_n) + \sum_{i=1}^{n-1} D_if(u', u_n)(x_i - u_i) + D_nf(u', u_n)(x_n - u_n) \]
if $(x', x_n) \approx (u', u_n)$.

Hence, from Lemma 8.3.6 below and Proposition 7.5.2 it follows
\[ y \approx w + \sum_{i=1}^{n-1} D_i f(u', u_n)(x_i - u_i) + D_n f(u', u_n)(g(x', y) - g(u', w)) \text{ if } (x', y) \approx (u', w). \]

Since $D_nf(u', u_n) \neq 0$, by Proposition 7.5.2 we have
\[ g(x', y) \equiv g(u', w) - \sum_{i=1}^{n-1} \frac{D_i f(u', g(u', w))(x_i - u_i)}{D_n f(u', g(u', w))} + \frac{1}{D_n f(u', g(u', w))}(y - w). \]

Since
\[ h_i(u', w) := \begin{cases} \frac{-D_i f(u', g(u', w))}{D_n f(u', g(u', w))} & \text{for } i \leq n - 1, \\ \frac{1}{D_n f(u', g(u', w))} & \text{for } i = n \end{cases} \]
are continuous, Proposition 9.2.3 implies $g$ represents a differentiable function whose $i$-th partial derivative is represented by $h_i$.

The following was used in the proof of Lemma 9.4.2

**Lemma 9.4.3** Let $f$ be a rational valued function on $[0, 1]^n$ with continuous difference quotient. Define a function $F : [0, 1]^n \to [0, 1]^n$ by
\[ F(x) = (x_1, \ldots, x_{n-1}, f(x)). \]
If $\mathcal{D}_n f(a) \neq 0$, then for some rationals $K_1, K_2, c > 0$

\[ K_1 d(x, a) \leq d(F(x), F(a)) \leq K_2 d(x, a) \]

holds for if $d(x, a) < c$.

**Proof.** Put $M := \max \{ |\mathcal{D}_i f(x)| : i \in [1..n], x \in [0,1]^n \}$. Then by Proposition 9.1.2

\[ |f(x) - f(a)| \leq M \sum_{i \in [1..n]} |x_i - a_i| \leq n M d(x, a) \]

for $x, a \in [0,1]^n$. Put $x' = (x_1, \ldots, x_{n-1})$ and $a' = (a_1, \ldots, a_{n-1})$. Then

\[ d(F(x), F(a)) = \max \{ d(x', a'), |f(x) - f(a)| \}. \]

Since $d(x', a') \leq d(x, a)$, if we put $K_2 = \max \{ 1, n M \}$, then

\[ d(F(x), F(a)) \leq K_2 d(x, a). \]

Put $b = \mathcal{D}_n f(a)$. We may assume $b > 0$. Define

\[ b_2 := \max_{1 \leq i \leq n-1} \frac{b}{|\mathcal{D}_i f(a)|} \]

Then we can choose $c_1 > 0$ and $c_2 > 0$ such that $d(x, a) < c_1$ implies $\mathcal{D}_n f(x) > \frac{b}{2}$

and $|\mathcal{D}_i f(x)| < 2 b_2 \mathcal{D}_n f(x)$ for $i \leq n - 1$.

Put $L = \min \left\{ 2, \frac{b}{4}, \frac{b}{8 b_2 M} \right\}$ and suppose $d(x', a') < L|x_n - a_n|$. Then by Lemma 9.1.1

\[ |f(x) - f(a)| \geq \frac{b}{2} |x_n - a_n| - 2 b_2 \sum_{i=1}^{n-1} \sum_{u \in I_i(a,x)} |\mathcal{D}_n f(x[i - 1] + u \epsilon_i)| |\epsilon| \]

\[ \geq \frac{b}{2} |x_n - a_n| - 2 b_2 \sum_{i=1}^{n-1} \sum_{u \in I_i(a,x)} |\mathcal{D}_n f(x[i - 1] + u \epsilon_i)| |\epsilon| \]

\[ \geq \frac{b}{2} |x_n - a_n| - 2 b_2 M \sum_{i=1}^{n-1} |x_i - a_i| \]

\[ \geq \frac{b}{2} - 2 b_2 M L |x_n - a_n| \geq \frac{b}{4} |x_n - a_n| \]

Since

\[ d(x, a) = \max \{ d(x', a'), |x_n - a_n| \} \leq \max \{ L|x_n - a_n|, |x_n - a_n| \} = L|x_n - a_n| \]

we conclude that $d(x', a') < L|x_n - a_n|$ implies

\[ |f(x) - f(a)| > \frac{b}{4 L} d(x, a). \]

Hence

\[ d(F(x), F(a)) = \max \{ d(x', a'), |f(x) - f(a)| \} > \max \left\{ d(x', a'), \frac{b}{4 L} d(x, a) \right\} = \frac{b}{4 L} d(x, a), \]
since $\frac{b}{4L} \geq 1$.

Suppose $d(x', a') \geq L|x_n - a_n|$. Then

$$d(x, a) = \max \left\{ d(x', a'), |x_n - a_n| \right\} \leq \max \left\{ d(x', a'), \frac{1}{L}d(x', a') \right\} = L_2d(x', a'),$$

where $L_2 := \max \left\{ 1, \frac{1}{L} \right\}$. Hence

$$d(F(x), F(a)) \geq d(x', a') \geq \frac{1}{L_2}d(x, a).$$

So if we put $K_1 = \min \left\{ \frac{b}{4L}, \frac{1}{L_2} \right\}$, then $d(x, a) < c_1$ implies

$$K_1d(x, a) \leq d(F(x), F(a)).$$

### 9.5 Inverse Mapping Theorem

Let $F = (F_1, \cdots, F_m)$ be a differentiable morphism from $[-1, 1]^n$ to $(-\infty, \infty)^m$. The matrix

$$dF(p) := (\partial_j f_i(p))_{i\in[1..n], j\in[1..m]}$$

is called the Jacobian of $F$ at a point $p \in (-1, 1)^n$.

A differentiable map $F : [-1, 1]^n \to [-1, 1]^n$ with $F(0) = 0$ is called a local diffeomorphism at the point 0 if there is a $c > 0$ and a differentiable morphism

$$G : [-c, c]^n \to [-1, 1]^n$$

such that for $y \in [-c, c]^n$

$$F(G(y)) = y$$

and

$$G(F(x)) = x,$$

for $x \in [-c, c]^n$ such that $F(x) \in [-c, c]^n$. $G$ is called a local inverse of $F$.

**Theorem 9.5.1** Let $F$ be a differentiable mapping $[-1, 1]^n$ to $(-\infty, \infty)^n$ such that $F(0) = 0$ with invertible Jacobian at 0. Then $F$ is a local diffeomorphism at 0.

Theorem 9.4.1 implies the following special case.

**Lemma 9.5.2** Let $F_n$ be a differentiable function on $[-1, 1]^n$ such that $F_n(0) = 0$, $\partial_n F_n(0) = 1$ and $\partial_i F_n(0) = 0$ for $i < n$. Then the differential morphism $F : [-1, 1]^n \to (-\infty, \infty)^n$ defined by $F(x) = (x_1, \cdots, x_{n-1}, F_n(x))$ is a local diffeomorphism at $[0]$. Moreover the Jacobian matrix of every local inverses of $F$ at 0 is the identity matrix.
Proof of Lemma 9.5.2 Let \( f, [-1, 1]^n] \) represents \( F_n \) so that \( D_n f(0) \approx 1 \) and \( D_j f(0) \approx 0 \) for \( i \in [1..n - 1] \). By Theorem 9.4.1 there is a differentiable function \( g \) on \([-c, c] \) with \( c > 0 \) such that for \( x \in [-c, c] \)

\[
f(x', g(x)) \approx x_n
\]

where \( x' := (x_1, \ldots, x_{n-1}) \) and if \( f(x) \in [-c, c] \) then

\[
g(x', f(x)) \approx x_n.
\]

Put, for \( x \in [-c, c]^n \), \( G(x) := (x', g(x)) \). Then it is obvious that \( G \) satisfies the conditions of local inverse. By Theorem 9.4.1

\[
D_i g(0) \approx \begin{cases} -\frac{D_i f(0)}{D_j f(0)} & \text{for } i < n \\ -\frac{D_n f(0)}{D_j f(0)} & \text{for } i = n 
\end{cases}
\]

Hence the Jacobian matrix of \( G \) at 0 is the identity matrix.

Proof of Theorem 9.5.1 By applying the inverse of \( dF(0) \) to \( F \), we may suppose \( dF(0) \) is the identity matrix \( I_n \) of size \( n \).

Put \( \Phi_n(x) := (x_1, \ldots, x_{n-1}, F_n(x)) \). Applying Lemma 9.5.2 for \( F_n \), we obtain differential morphism \( \Gamma_n : [-c_n, c_n]^n \rightarrow \mathbb{R}^n \) such that for \( x \in [-c_n, c_n]^n, \)

\[
\Phi_n(\Gamma_n(x)) = x
\]

and hence

\[
F_n(\Gamma_n(x)) = x_n,
\]

and if \( \Phi_n(x) \in [-c_n, c_n]^n \) then

\[
\Gamma_n(\Phi_n(x)) = x.
\]

We define inductively local diffeomorphisms \( \Gamma_{n-1}, \Gamma_{n-1}, \ldots, \Gamma_1 \) satisfying, for each \( j, \)

\[
d\Gamma_j(0) = I_n \quad (84)
\]

and

\[
F_k(\Gamma_j(x)) = x_k \text{ for } k \in [j..n],
\]

(85) with \( \hat{\Gamma}_j = \Gamma_n \circ \Gamma_{n-1} \circ \cdots \circ \Gamma_j \). Then \( \hat{\Gamma}_1 \) is the required local diffeomorphism.

Suppose we have constructed \( \Gamma_n, \ldots, \Gamma_{i+1} \) satisfying the conditions (84) and (85) for \( j \geq i + 1 \). Define

\[
\hat{F}_i(x) := F_i(\hat{\Gamma}_{i+1}(x)).
\]

Since \( d\hat{\Gamma}_{i+1}(0) = I_n \), we have \( \partial_j \hat{F}_i(0) = \partial_j F_i(0) = \delta_{ij} \). Hence by Lemma 9.5.2 there is a local diffeomorphism \( \Gamma_i \) such that \( d\Gamma_i(0) = I_n \), \( \hat{F}_i(\Gamma_i(x)) = x_i \) and, for \( j \neq i \), the \( j \)-th component of \( \Gamma_i(x) \) is \( x_j \). Then for \( k > i \)

\[
F_k(\hat{\Gamma}_i(x)) = F_k(\hat{\Gamma}_{i+1}(\Gamma_i(x))) = (\Gamma_i(x))_k = x_k
\]

and

\[
F_i(\hat{\Gamma}_i(x)) = F_i(\hat{\Gamma}_{i+1}(\Gamma_i(x))) = \hat{F}_i(\Gamma_i(x)) = x_i,
\]

whence (85) holds for \( j \geq i \).
9.6 Second Order Differentiability

**Definition 9.6.1** A real function $F$ on $[0,1]^n$ is differentiable up to second order if it is differentiable and its partial derivatives are differentiable. The partial derivatives $\partial_j \partial_i F$ are called second order partial derivatives. It will be shown that it is symmetric with respect to $i,j$, namely, $\partial_j \partial_i F = \partial_i \partial_j F$.

**Theorem 9.6.1** Suppose a function $F$ on $[0,1]^n$ is differentiable up to second order. Let $g_i$ and $g_{ij}$ are rational valued functions on $[0,1]^n$ representing respectively $F$, $\partial_i F$ and $\partial_i \partial_j F$. Then there is an infinitesimal $\delta \in \varepsilon \mathbb{Z}$ such that if $a \in [0,1]^n$, then

\[
 f(x) \equiv_2 f(a) + \sum_{i=1}^{n} g_i(a)(x_i-a_i) + \sum_{1 \leq j < i \leq n} g_{ji}(a)(x_i-a_i)(x_j-a_j) + \frac{1}{2} \sum_{i} g_{ii}(a)(x_i-a_i)^2 \text{ if } x \approx a.
\]

(86)

**Proof.** By Theorem 9.2.1, we have

\[
 f(x) \approx_1 f(a) + \sum_{i=1}^{n} g_i(a)(x_i-a_i) \text{ if } x \approx a
\]

(87)

\[
 g_i(x) \approx_1 g_i(a) + \sum_{j=1}^{n} g_{ji}(a)(x_j-a_j) \text{ if } x \approx a
\]

(88)

Hence by Corollary 8.3.3 there is a number $L$ with $\varepsilon' := L\varepsilon \approx 0$ such that if $a \in X := [0,1]^n$, then

\[
 f(x) \equiv_1 f(a) + \sum_{i=1}^{n} g_i(a)(x_i-a_i) \text{ if } x \approx a
\]

(89)

\[
 g_i(x) \equiv_1 g_i(a) + \sum_{j=1}^{n} g_{ji}(a)(x_j-a_j) \text{ if } x \approx a
\]

(90)

Assume $x \approx a \in X$ and $a_i \leq x_i$ for all $i$. In the other cases, the assertion can be proved similarly. Put $\Delta x = \delta$. Using the notation in Lemma 9.1.1, (89) implies

\[
 f(x) - f(a) = \sum_{i=1}^{n} (f(x[i]) - f(x[i-1])).
\]

\[
 \approx \sum_{i=1}^{n} \left( \sum_{0 \leq u < \frac{x_i-a_i}{\Delta x}} g_i(x[i-1] + u\Delta x) \Delta x \right)
\]

\[
 = \sum_{i=1}^{n} g_i(x_i-a_i) + \sum_{i=1}^{n} \left( \sum_{0 \leq u < \frac{x_i-a_i}{\Delta x}} A_i(u) \Delta x \right),
\]
where

\[ A_i(u) := g_i(x[i - 1] + u\Delta e_i) - g_i(x) \]

\[ = g_i(x[i - 1] + u\Delta e_i) - g_i(x[i - 1]) + \sum_{j=1}^{i-1} g_i(x[j]) - g_i(x[j - 1]). \]

By (90) and Proposition 7.5.6,

\[ A_i(u) \approx \sum_{0 \leq w < u} g_{ii}(x_{i-1} + \Delta x_e) \Delta x + \sum_{0 \leq w < \frac{x_{i+1} - a_i}{\Delta x}} g_{ji}(x_j) \Delta x \]

if \( x \approx a \)

\[ \approx g_i(x) u \Delta x + \sum_{j=1}^{i-1} g_{ji}(x_j - a_j) \]

Hence

\[ \sum_{i=1}^{n} \sum_{0 \leq u < x_{i+1} - a_i} A_i(u) \Delta x \]

\[ \approx 2 \sum_{i=1}^{n} \left( g_{ii}(x) \sum_{0 \leq u < \frac{x_{i+1} - a_i}{\Delta x}} u(\Delta x)^2 \right) + \sum_{i=1}^{n} \sum_{j=1}^{i-1} g_{ji}(x_j - a_j) \sum_{0 \leq u < \frac{x_{j+1} - a_j}{\Delta x}} \Delta x \]

\[ = \sum_{i=1}^{n} \left( g_{ii}(x) \frac{1}{2}(x_i - a_i)(x_i - a_i + \Delta x) \right) + \sum_{j<i} g_{ji}(x_j - a_j)(x_i - a_i) \]

\[ \approx 2 \sum_{i=1}^{n} \frac{1}{2} g_{ii}(x)(x_i - a_i)^2 + \sum_{j<i} g_{ji}(x_j - a_j)(x_i - a_i) \]

Hence, we have (86) with \( \equiv_2 \) replaced with \( \approx_2 \).

By Proposition 7.5.4, there is an integer \( L' > 0 \) with \( \delta := L'e' \approx 0 \) such that (86) holds for \( a \in [0, 1]_s \).

Let \( f \) be a rational valued function on \( X = [0, 1]^n \) with accessible \( n \). The second order partial difference quotients are defined by

\[ D_{ij}f := D_i(D_jf). \]

More explicitly we have the following.

Lemma 9.6.2

\[ D_{ii}f := \frac{f(x + 2\Delta e_i) - 2f(x + \Delta e_i) + f(x)}{(\Delta x)^2}, \]
\[ D_{ij} f := \frac{f(x + \Delta x e_i + \Delta x e_j) - f(x + \Delta x e_i) - f(x + \Delta x e_j) + f(x)}{(\Delta x)^2} \text{ if } i \neq j. \]

In particular, \( D_{ij} f = D_{ji} f \).

**Proposition 9.6.3** Suppose a rational valued function \( f \) on \([0,1]^n\) with an infinitesimal positive rational \( r \) satisfies (86) with continuous \( g_i \) and \( g_{ij} \). Then
\[ D_i f \approx g_i, \]
and if \( j < i \)
\[ D_{ji} f \approx g_{ji}. \]

**Proof.** Put \( \Delta x = r \). Substituting \( x = a + \Delta x e_i \) and \( x = a + 2\Delta x e_i \) in (86) we obtain
\[ f(a + \Delta x e_i) \equiv_2 f(a) + g_i(a)\Delta x + \frac{1}{2} g_{ii}(a)(\Delta x)^2 \]  
(93)
\[ f(a + 2\Delta x e_i) \equiv_2 f(a) + 2g_i(a)\Delta x + 2g_{ii}(a)(\Delta x)^2. \]  
(94)
Hence (93) implies \( D_i f(a) \approx g_i(a) \).

By (91), (93) and (94), we have
\[ D_i f(a) \approx g_i(a). \]

Putting \( x = a + \Delta x e_j + \Delta x e_i \) we have if \( j < i \)
\[ f(a + \Delta x e_i + \Delta x e_j) \equiv_2 f(a) + (g_i(a) + g_j(a))\Delta x + \frac{1}{2} g_{ji}(a)(\Delta x)^2. \]

Hence by (92) if \( j < i \), \( D_{ji} f(a) \approx g_{ji}(a) \).

By Theorem 9.6.1 and Proposition 9.6.3 we have the following.

**Corollary 9.6.4** If a real function \( F \) is differentiable up to second order, then \( F \) is represented by a rational valued function \( f \) with continuous partial difference quotients up to second order and \( \partial_i \partial_j F \) is represented by \( D_{ij} f \).

By Lemma 9.6.2 and Corollary 9.6.4 we have proved the following.

**Theorem 9.6.5** Let \( F \) be a function on \([0,1]^n\) differentiable up to second order. Then
\[ \partial_i \partial_j F = \partial_j \partial_i F. \]  
(95)
Corollary 9.6.6 Suppose a function $F$ on $[0,1]^n$ is differentiable up to second order. Let $f, g_i$ and $g_{ij}$ are rational valued functions on $[0,1]^n$ representing respectively $F$, $\partial_i F$ and $\partial_i \partial_j F$. Then $g_{ij} \approx g_{ji}$ and there is an integer $L$ with $r = L \varepsilon \approx 0$ such that if $a \in [0,1]^n$ then

$$f(x) \equiv_2 f(a) + \sum_{i=1}^n g_i(a)(x_i - a_i) + \frac{1}{2} \sum_{1 \leq i,j \leq n} g_{ji}(a)(x_i - a_i)(x_j - a_j) \quad \text{if } x \approx a.$$  

(96)
10 Measure

In this section, we show how to obtain the basic tools of Lebesgue integration in our framework. We start with a set \( X \) with positive probability density \( p \), which give measure \( m(A) \) of subsets of \( A \subset X \). A condition is said to be true almost everywhere if there are subsets with arbitrary small measure outside of which it holds. The integral of a rational valued function \( f \) on \( X \) is defined by
\[
E(f) := \sum_{x \in X} p(x) f(x).
\]
A function \( f \) is called \( L^1 \) function if \( E(f) \approx E(f^a) \) for all huge \( a \) where \( f^a \) denotes the function obtained by modifying \( f(x) \) to zero when \( f(x) > a \). Note that this concept get meaning since we have functions with huge values. Then the \( L^1 \) functions form a complete metric space with respect to the distance function \( d_1(f,g) = E(|f - g|) \).

A concrete sequence of \( L^1 \) functions converges with respect to \( d_1 \) then a subsequence converges pointwise almost everywhere.

10.1 Probability Density

Let \( X \) be a set and \( p \) be a probability density function, namely, a rational valued function on \( X \) satisfying
\[
\begin{align*}
0 &\leq p(x) \leq 1 \\
\sum_{x \in X} p(x) &= 1.
\end{align*}
\]
For a subset \( A \subset X \), define its measure \( m(A) := \sum_{a \in A} p(a) \). Then for \( A, B \subset X \), we have obviously
\[
m(A \cup B) = m(A) + m(B) - m(A \cap B).
\]
If \( A_1, \cdots, A_N \) is a huge sequence of mutually disjoint subsets, then we have also
\[
m\left( \bigcup_{i=1}^{N} A_i \right) = \sum_{i=1}^{N} m(A_i).
\]

10.2 Null Semisets

We call a subsemiset \( P \subset X \) null semiset and write \( P^{\approx a.e.} \emptyset \) if for each accessible \( k \), there is a subset \( A \subset X \) satisfying \( P \subset A \) and \( m(A) < \frac{1}{k} \). Note that a subset \( A \) is a null semiset if and only if \( m(A) \approx 0 \).

Obviously intersection and union of two null semisets are null.

We call a condition \( Q \), not necessarily objective, holds almost everywhere (a.e. for short) if the subsemiset defined by \( \neg Q \) is a null semiset. Obviously
Q holds a.e. if and only if for every accessible \( k \) there is a subset \( B \subset X \) such that \( m(B) > 1 - \frac{1}{k} \) and \( Q(x) \) holds for all \( x \in B \).

A typical condition we encounter is \( f(x) \approx 0 \) for a rational valued function \( f \) on \( X \).

**Lemma 10.2.1** Let \( f \) be a rational valued function on \( X \). Then \( f(x) \approx 0 \) a.e. if and only if there is a subset \( A \subset X \) with \( m(A) \approx 0 \) such that \( f(x) \approx 0 \) for all \( x \notin A \).

**Proof.** Suppose \( f(x) \approx 0 \) a.e.. For accessible \( k \), there is a subset \( A_k \subset X \) such that \( m(A_k) < \frac{1}{k} \) and \( f(x) \approx 0 \) for \( x \notin A_k \). Then the set of numbers

\[
\{ \ k \mid \text{there is a subset } A \subset X \text{ such that } m(A) < \frac{1}{k} \text{ and } |f(x)| < \frac{1}{k} \text{ for all } x \notin A \} 
\]

contains all accessible numbers and hence also a huge number \( K \). Hence there is a subset \( A \subset X \) such that \( m(A) < \frac{1}{K} \) and \( |f(x)| < \frac{1}{K} \) for \( x \notin A \). Then \( m(A) \approx 0 \) and \( f(x) \approx 0 \) for \( x \notin A \).

The converse is obvious. \( \square \)

The following properties hold obviously.

**Proposition 10.2.2**

- If \( P \) is a null semiset and \( Q \subseteq P \), then \( Q \) is also a null semiset.
- If \( P_1 \approx \emptyset (i = 1, 2) \) are null semisets, then their union is also a null semiset.

The infinite union of null semisets is also a null semiset.

**Theorem 10.2.3** Suppose \( P_1, P_2, \cdots \) is a concrete sequence of null semisets. Then their union

\[
\bigcup_i P_i
\]

is also a null semiset.

**Proof.** Let \( k \) be an accessible number. Since \( P_i \) is null, for each accessible \( i \), there is a set \( A_i \supset P_i \) with

\[
m(A_i) < \frac{1}{k2^i}.
\]

(97)

By the over-spill axiom, the concrete sequence \( A_1, A_2, \cdots \) can be extended to a huge sequence \( (A_1, \cdots, A_N) \) which satisfy (97). Put

\[
A = \bigcup_{i=1}^{M} A_i.
\]
Then obviously $A \supseteq \bigcup_{i=1}^{\infty} P_i$ and

$$m(A) \leq \sum_i m(A_i) \leq \frac{1}{k} \left( 1 - \left( \frac{1}{2} \right)^{M+1} \right) < \frac{1}{k}.$$  

For subsemisets $P_i \subseteq X$ $(i = 1, 2)$, we write $P_1 \approx^0 P_2$ if $P_1 \Delta P_2 \approx^0 \emptyset$.

**Lemma 10.2.4** The relation $\approx^0$ is an equivalence relation.

**Proof.** The reflexivity and symmetricity are obvious. For the transitivity, it suffices to show

$$P_1 \Delta P_3 \subseteq (P_1 \Delta P_2) \cup (P_2 \Delta P_3).$$  

(98)

Suppose $x \in P_1 \setminus P_3$. If $x \in P_2$ then $x \in P_2 \setminus P_3 \subseteq P_2 \Delta P_3$, whereas if $x \notin P_2$ then $x \in P_1 \setminus P_2 \subseteq P_1 \Delta P_2$. Hence $P_1 \setminus P_3$ is included in the right hand side of (98). Similarly it can be shown that $P_3 \setminus P_1$ is included in the right hand side of (98).

### §10 Measure

10.3 Measurable Semisets

A subsemiset $P \subseteq X$ is called **measurable** if there is a subset $A \subseteq X$ with null subsemiset $A \Delta P$. We define then $\overline{m}(P) := [m(A)]$, which is independent of the choice of $A$ by the following lemma 10.3.1 hence uniquely defined as a real number.

Note that if $P$ is measurable, then $\overline{m}(P) = 0$ means that $P$ is null semiset.

**Lemma 10.3.1** If $P \subseteq X$ and $A_1, A_2 \subseteq X$ satisfies $A_i \Delta P \approx^0 \emptyset$ $(i = 1, 2)$, then $m(A_1) \approx m(A_2)$.

**Proof.** First we show that $m(A_1 \Delta A_2) \approx 0$, for which it suffices to show

$$A_1 \Delta A_2 \subseteq (A_1 \Delta P) \bigcup (A_2 \Delta P),$$

since there are $B_i \subseteq X$ such that $A_i \Delta P \subseteq B_i$ and $m(B_i) \approx 0$ $(i = 1, 2)$ . Suppose $x \in A_1 \setminus A_2$. If $x \in P$ then $x \in P \Delta A_2$ and if $x \notin P$ then $x \in P \Delta A_1$, whence

$$x \in (A_1 \Delta P) \bigcup (A_2 \Delta P).$$  

(99)

Similarly it can be shown that (99) holds for $x \in A_2 \setminus A_1$.

Define $C = A_1 \cap A_2$, $A'_i = A_i \setminus C$ $(i = 1, 2)$. Then from $A_1 \Delta A_2 = A'_1 \cup A'_2$ it follows $m(A'_i) \approx 0$ $(i = 1, 2)$. Hence

$$m(A_1) = m(A'_1) + m(C) \approx m(A'_2) + m(C) = m(A_2).$$

Measurability of $P$ can be rephrased as follows.
Proposition 10.3.2 A subsemiset $P \sqsubseteq X$ is measurable if and only if for each accessible number $k$, there exist subsets $A, B \subset X$ satisfying
\[ A \setminus B \sqsubseteq P \sqsubseteq A \cup B, \quad (100) \]
and $m(B) < \frac{1}{k}$.

Proof. Suppose $P$ is measurable. Then there is a subset $B \subset X$ with $m(B) < \frac{1}{k}$ satisfying $P \Delta A \sqsubseteq B$. Put
\[ C = P \cap A \sqsubseteq X, \quad P' = P \setminus C, \quad A' = A \setminus C. \quad (101) \]
Then
\[ A', P' \sqsubseteq A' \cup P' = P \Delta A \sqsubseteq B. \]
Hence
\[ A \setminus B \subset A \setminus A' = C \sqsubseteq P = P' \cup C \subset B \sqsubseteq C \subset B \cup A. \]

Conversely suppose for each accessible $k$ there is a subset $A, B$ with $m(B) < \frac{1}{k}$ satisfying $P \Delta A \sqsubseteq B$. In fact, if we define $C, A', P'$ by (101), then it suffices to show $A' \sqsubseteq P' \sqsubseteq B$. Since $A' \cap P = \emptyset$,
\[ A' \setminus B \subset A \setminus B \subset P \]
implies $A' \setminus B = \emptyset$, namely, $A' \subset B$. On the other hand
\[ P' = P \setminus A \sqsubseteq (A \cup B) \setminus A \subset B. \]

Proposition 10.3.3 If $k$ is accessible and $P_i \sqsubseteq X$ ($i \in [1..k]$) are measurable, then their intersection and union are measurable.

Proof. There are subsets $A_i$ with $P_i \Delta A_i \approx \emptyset$ for $i \in [1..k]$. It suffices to show the following.
\[ \bigcap_{i=1}^{k} P_i \Delta \left( \bigcup_{i=1}^{k} A_i \right) \sqsubseteq \bigcup_{i=1}^{k} (P_i \Delta A_i) \quad (102) \]
\[ \bigcup_{i=1}^{k} P_i \Delta \left( \bigcap_{i=1}^{k} A_i \right) \sqsubseteq \bigcap_{i=1}^{k} (P_i \Delta A_i) \quad (103) \]

To show (102), let $x$ be an element of the left hand side. If $x \in \bigcap P_i$ and $x \notin \bigcap A_i$, then there is a $j$ such that $x \notin A_j$, whence $x \in P_j \Delta A_j$ and $x$ belongs to the right hand side, which holds also in the case $x \notin \bigcap P_i$ and $x \in \bigcap A_i$ by similar arguments.

To show (103), let $x$ be an element of the left hand side. If $x \in \bigcup P_i$ and $x \notin \bigcup A_i$, then there is a $j$ with $x \in P_j$ and $x \notin A_j$, whence $x \in P_j \Delta A_j$ and $x$ belongs to the right hand side, which holds also in the case $x \notin \bigcup P_i$ and $x \in \bigcup A_i$. \hfill \[\blacksquare\]
Proposition 10.3.4 If $k$ is accessible and $P_i \subseteq X$ ($i \in [1..k]$) are measurable and mutually almost disjoint in the sense that $P_i \cap P_j$ is null for $i \neq j$, then
\[ m(\bigcup_i P_i) = \sum_i m(P_i). \]

Proof. For $i \in [1..k]$, let $A_i \subset X$ be a set satisfying $P_i \Delta A_i \approx \emptyset$. Define
\[ B_i = P_i \cap A_i, \quad P_i' = P_i \setminus B_i, \quad A_i' = A_i \setminus B_i. \]
Since
\[ P_i' \cup A_i' = P_i \Delta A_i \approx \emptyset \]
$P_i', A_i'$ are null semisets. Since
\[ B_i \cap B_j \subset P_i \cap P_j \approx \emptyset \]
if $i \neq j$, we have
\[ A_i \cap A_j \subset (B_i \cap B_j) \cup (A_i' \cup A_j') \approx \emptyset. \]
Hence, if we put
\[ C_i := A_i \setminus \bigcup_{j \geq i} (A_i \cap A_j), \]
then $C_i$’s are mutually disjoint and $\bigcup_i C_i = \bigcup_i A_i$. Since
\[ m(C_i) \leq m(A_i) \leq m(C_i) + \sum_{j \geq i} m(A_j \cap A_i) \approx m(C_i), \]
we have $m(A_i) \approx m(C_i)$. Hence
\[ m(\bigcup_i A_i) = m(\bigcup_i C_i) = \sum_i m(C_i) \approx \sum_i m(A_i) \approx \sum_i m(P_i). \]
Hence, by (103) and Proposition 10.2.2
\[ m(\bigcup_i P_i) \approx m(\bigcup_i A_i) \approx \sum_i m(P_i). \]

Theorem 10.3.5 If $P_1, P_2, \ldots$ is a concrete sequence of measurable sub-semisets and mutually almost disjoint, then the sub-semiset
\[ \bigcup_{i=1}^\infty P_i \]
is also measurable and
\[ m(\bigcup_{i=1}^\infty P_i) = \sum_{i=1}^\infty m(P_i). \]
\textbf{Proof.} For accessible $i$, choose $A_i \subset X$ satisfying $$P_i \Delta A_i \overset{a.c.}{\approx} \emptyset.$$ Since $A_i$ is a set, we extend the concrete sequence $(A_1, A_2, \cdots)$ to a huge sequence of subsets $(A_1, \cdots, A_{M_0})$.

Put $b_p := m(\bigcup_{1 \leq i \leq p} A_i)$ for $p \in [1..M_0]$. By Lemma 6.3.2 the increasing sequence of rationals $(b_1, \cdots, b_{M_0})$ has an upper bound 1, whence it converges and there is an $M_1 \leq M_0$ such that $\lim_p b_p \approx b_K$ for all huge $K \leq M_1$.

If $i$ is accessible, it follows from $P_i \overset{a.c.}{\approx} A_i$ and (102) $$(A_i \cap A_j) \Delta (P_i \cap P_j) \subset (A_i \Delta P_i) \cup (A_j \Delta P_j) \overset{a.c.}{\approx} \emptyset,$$ whence if $i \neq j$ then $A_i \cap A_j \overset{a.c.}{\approx} P_i \cap P_j \overset{a.c.}{\approx} \emptyset$. By Lemma 10.2.4 we conclude $A_i \cap A_j \overset{a.c.}{\approx} \emptyset$, for accessible $i \neq j$. By Proposition 10.3.4, for accessible $k$,

$$m\left(\bigcup_{i \in [1..L]} A_i\right) \approx \sum_{i \in [1..L]} m(A_i),$$

whence for some huge $M_2 \leq M_1$, for every huge $L \leq M_2$

$$m\left(\bigcup_{i \in [1..M]} A_i\right) \approx \sum_{i \in [1..M]} m(A_i).$$

Since

$$(\bigcup_{i \in [1..p-1]} A_i) \cap (\bigcup_{i \in [p..M]} A_i) \subset \bigcup_{i \in [1..p-1], j \in [p..M]} (A_i \cap A_j) \overset{a.c.}{\approx} \emptyset,$$ (104)

holds for accessible $\ell$, there is a huge $M_3 \leq M_2$ such that (104) holds also for huge $\ell \leq M_3$. Thus if $M \leq M_3$ is huge we have

$$m\left(\bigcup_{i \in [1..M]} A_i\right) = m\left(\bigcup_{i \in [1..p-1]} A_i\right) + m\left(\bigcup_{i \in [p..M]} A_i\right) - m\left(\bigcap_{i \in [1..p-1]} A_i \cap \bigcup_{i \in [p..M]} A_i\right) \approx m\left(\bigcup_{i \in [1..p-1]} A_i\right) + m\left(\bigcup_{i \in [p..M]} A_i\right).$$

Hence

$$\lim_p m\left(\bigcup_{i \in [p..M]} A_i\right) = m\left(\bigcup_{i \in [1..M]} A_i\right) - \lim_p m\left(\bigcup_{i \in [1..p-1]} A_i\right) = b_M - \lim_p (b_p) \approx 0, \quad (105)$$

Put $A = \bigcup_{i=1}^{M_3} A_i$. We show

$$\left(\bigcup_{i=1}^{\infty} P_i\right) \Delta A \overset{a.c.}{\approx} \emptyset.$$
Now, using \(103\), we have
\[
\left( \bigcup_{i=1}^{\infty} P_i \right) \Delta A = \left( \bigcup_{i=1}^{\ell} P_i \right) \Delta \left( \bigcup_{i=1}^{\infty} P_i \right) \Delta \left( \bigcup_{i=1}^{M} A_i \right)
\]

since for each \(i\),
\[
P_i \setminus A \subseteq P_i \setminus A_i \subseteq P_i \Delta A_i \quad \text{a.e.}
\]
the first and the second term in \(106\) are null semisets. The measure of the last term converges to zero when \(\ell \to \infty\) by \(105\). Hence the left hand side \(\left( \bigcup_{i=1}^{\infty} P_i \right) \Delta A\) is a null semiset. Thus \(\bigcup_{i=1}^{\infty} P_i\) is measurable. By Proposition 10.3.4,
\[
\overline{m}\left( \bigcup_{i=1}^{\ell} P_i \right) = [m(A)] = [b_{M_\ell}] = \lim_{p} b_{p} = \lim_{p} \overline{m}\left( \bigcup_{i=1}^{p} A_i \right)
\]
\[
= \lim_{p} \overline{m}\left( \bigcup_{i=1}^{p} P_i \right) = \lim_{p} \sum_{i=1}^{p} \overline{m}(P_i) = \sum_{i=1}^{\infty} \overline{m}(P_i).
\]

**Theorem 10.3.6** If \((P_1, P_2, \cdots)\) is an increasing concrete sequence of measurable subsemisets, then \(\bigcup_{i=1}^{\infty} P_i\) is also measurable and
\[
\overline{m}\left( \bigcup_{i=1}^{\infty} P_i \right) = \lim_{i} \overline{m}(P_i).
\]

**Proof.** Define \(Q_1 = P_1\) and \(Q_i = P_i \setminus P_{i-1}\) for \(i > 1\). Since \((Q_1, Q_2, \cdots)\) is a concrete sequence of subsemisets which is mutually disjoint, Theorem 10.3.6 implies that
\[
\bigcup_{i=1}^{\infty} P_i = \bigcup_{i=1}^{\infty} Q_i,
\]
is measurable and
\[
\overline{m}\left( \bigcup_{i=1}^{\infty} P_i \right) = \overline{m}\left( \bigcup_{i=1}^{\infty} Q_i \right) = \lim_{p} \sum_{i=1}^{p} \overline{m}(Q_i) = \lim_{p} \overline{m}\left( \bigcup_{i=1}^{p} Q_i \right) = \lim_{p} \overline{m}(P_p).
\]

**Theorem 10.3.7** If \((P_1, P_2, \cdots)\) is a concrete sequence of measurable subsemisets, then \(\bigcup_{i=1}^{\infty} P_i\) is measurable and
\[
\overline{m}\left( \bigcup_{i=1}^{\infty} P_i \right) \leq \sum_{i=1}^{\infty} \overline{m}(P_i).
\]
Proof. Put $Q_p = \bigcup_{i=1}^{p} P_i$. Then $(Q_1, Q_2, \cdots)$ is increasing and
\[ \bigcup_{i=1}^{\infty} P_i = \bigcup_{i=1}^{\infty} Q_i, \]
is measurable and
\[ m\left( \bigcup_{i=1}^{\infty} P_i \right) = m\left( \bigcup_{i=1}^{\infty} Q_i \right) = \lim_{p} m(Q_p) = \lim_{p} m\left( \bigcup_{i=1}^{p} P_i \right). \]

Put $P'_1 = P_1$ and $P'_i = P_i \setminus \bigcup_{j \in [1, \ldots, i-1]} P_j$ for $i \geq 2$. Then
\[ m\left( \bigcup_{i=1}^{p} P_i \right) = m\left( \bigcup_{i=1}^{p} P'_i \right) = \sum_{i=1}^{p} m(P'_i) \leq \sum_{i=1}^{p} m(P_i). \]
Taking the limit, we obtain
\[ m\left( \bigcup_{i=1}^{\infty} P_i \right) = \lim_{p} m\left( \bigcup_{i=1}^{p} P_i \right) \leq \lim_{p} \sum_{i=1}^{p} m(P_i) = \sum_{i=1}^{\infty} m(P_i). \]

10.4 Integration

Let $(X, p)$ be a set $X$ with a probability density. For a huge number $M$, the collection of functions $F_M(X) := \text{Fun}(X, [-M, M]_{\mathbb{R}})$ is a set with $(M(2M + 1))^{X}$ elements.

For $f \in F_M(X)$ define its integration by
\[ E(f) := \sum_{x \in X} p(x)f(x) \]
and for $f, g \in F_M$ define
\[ d_1(f, g) := E(|f - g|). \]
If $d_1(f, 0) < \infty$, we say $f$ is integrable. The class of integrable elements of $F_M(X)$ is denoted by $F_M^{\text{int}}(X)$.

Proposition 10.4.1 (1) If $f, f' \in F_M(X)$ satisfies $f \approx f'$ and $f$ is integrable, then $f'$ is also integrable and $E(f) \approx E(f')$.

(2) If $f_i \approx f'_i$ $(i = 1, 2)$ then
\[ d_1(f_1, f_2) \approx d_1(f'_1, f'_2). \]
(3) If $f_1, f_2 \in F^{\text{int}}_M(X)$, then $d_1(f_1, f_2)$ is finite.

**Proof.** Put $\varepsilon := \max_{x \in X} |f(x) - f'(x)| \approx 0$. Then

$$E(|f - f'|) \leq E(\varepsilon) = \varepsilon.$$  

Hence $E(f) \approx E(f')$ and if $E(f) < \infty$ then $E(f') < \infty$.

Since $|f_1 - f_2| \approx |f'_1 - f'_2|$, we have

$$d_1(f_1, f_2) \approx d_1(f'_1, f'_2).$$

If $f_1, f_2 \in F^{\text{int}}_M(X)$, then $d_1(f_1, f_2) = E(|f_1 - f_2|) \leq E(|f_1|) + E(|f_2|)$ is finite.  

**Proposition 10.4.2**

$$E(f) = \sum_{\lambda} \lambda m(f^{-1}(\lambda))$$

**Proof.** Since $X = \bigsqcup_{\lambda} f^{-1}(\lambda)$, we have

$$E(f) = \sum_{\lambda} \sum_{f(x) = \lambda} f(x)p(x) = \sum_{\lambda} \lambda \sum_{f(x) = \lambda} p(x) = \sum_{\lambda} \lambda m(f^{-1}(\lambda)).$$

**Proposition 10.4.3 (Chebishev Inequality)** Suppose $f$ is integrable, $f \geq 0$ and $c$ is a positive rational. Then

$$m(\{ x \mid f(x) \geq c \}) \leq \frac{E(f)}{c}.$$  

**Proof.**

$$E(f) \geq \sum_{\lambda \geq c} \lambda m(f^{-1}(\lambda)) \geq c \sum_{\lambda \geq c} m(f^{-1}(\lambda)) = c m(\{ x \mid f(x) \geq c \}).$$

**Theorem 10.4.4** If $f, g \in F^{\text{int}}_M(X)$ satisfies $d_1(f, g) \approx 0$, then $f \approx g$ a.e..

**Proof.** It suffices to show that if $E(h) \approx 0$ then $h \approx 0$ a.e..

Suppose $E(h) \approx 0$. For accessible $k$,

$$m \left( \left\{ x \mid |h(x)| > \frac{1}{k} \right\} \right) \leq kE(h) \approx 0.$$
Hence the set
\[
\left\{ k \mid m \left( \left\{ x \mid |h(x)| > \frac{1}{k} \right\} \right) \leq \frac{1}{k} \right\}
\]
includes all accessible numbers and hence also a huge number \( K \). Put \( A := \left\{ x \mid |h(x)| > \frac{1}{K} \right\} \). Then \( m(A) \leq \frac{1}{K} \approx 0 \). Moreover if \( x / \notin A \) then \(|h(x)| \leq \frac{1}{K} \approx 0 \).
Hence \( h \approx 0 \) a.e.

Remark 10.4.1 The converse does not hold, namely, \( E(f) \approx E(g) \) does not hold always even if \( f \approx g \) a.e.. For example, suppose \( p \) is uniform distribution, namely, \( p(x) = \frac{1}{|X|} \). Define
\[
f(x) := \begin{cases} |X| & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}
\]
Then \( f \approx 0 \) a.e. but \( 1 = E(f) \neq E(0) = 0 \).
If \( f, g \) are \( L^1 \) functions defined in the next subsection, then \( d_1(f, g) \approx 0 \) if and only if \( f \approx g \) a.e..

10.5 \( L^1 \)-functions

For \( f \in F_M(X) \) and a rational \( a \in [-M, M] \frac{1}{M} \), define
\[
f^a(x) := \begin{cases} 0 & \text{if } |f(x)| > a \\ f(x) & \text{otherwise} \end{cases}
\]

Definition 10.5.1 An \( f \in F_M(X) \) is called an \( L^1 \) function if \( E(|f - f^a|) \approx 0 \) for huge rationals \( a \).

Lemma 10.5.1 \( f \in F_M(X) \) is an \( L^1 \) function if and only if for huge rationals \( a \)
\[
\sum_{|f(x)| > a} |f(x)|p(x) \approx 0,
\]
if and only if
\[
\sum_{\lambda > a} \lambda m(|f|^{-1}\lambda) \approx 0
\]

Proof. Obvious since
\[
E(|f - f^a|) = \sum_{|f(x)| > a} |f(x)|p(x) = \sum_{\lambda > a} \sum_{|f(x)| = \lambda} \lambda p(x) = \sum_{\lambda > a} \lambda m(|f|^{-1}\lambda).
\]
Proposition 10.5.2  An \( f \in F_M(X) \) is an \( L^1 \) function, if and only if the following conditions are satisfied.

1. \( f \in F_M^{\text{int}} \),
2. \( E(|f|\chi_A) \approx 0 \) if \( m(A) \approx 0 \), where \( \chi_A \) denotes the characteristic function of \( A \subset X \).

Proof. Suppose \( f \) is an \( L^1 \) function. Then for every huge \( K \),
\[
E(|f|) \leq E(|f - f^a|) + E(|f^a|) \approx E(|f^a|) \leq K.
\]
Hence there is an accessible \( k \) with \( E(|f|) \leq k \).

Suppose a subset \( A \subset X \) satisfies \( m(A) \approx 0 \). Then
\[
E(|f|\chi_A) \leq E(|f - f^a|\chi_A) + E(|f^a|\chi_A) \leq E(|f - f^a|) + a m(A) \approx 0.
\]
Conversely suppose the conditions 1, 2 are satisfied. Suppose \( E(|f - f^a|) \succ 0 \) for some huge \( a \). Then the subset
\[
A := \{ x \mid f(x) \geq a \}
\]
satisfies, by Proposition 10.4.3,
\[
m(A) \leq \frac{E(f)}{a} \approx 0.
\]
On the other hand, since \( f^a = 0 \) on \( A \) and \( f - f^a = 0 \) on \( A^c \), we have
\[
E(|f|\chi_A) = E(|f - f^a|\chi_A) = E(|f - f^a|) \succ 0.
\]
This contradicts the latter assumption.

Corollary 10.5.3  
- If \( f, g \) are \( L^1 \) functions, then \( f + g \) is also an \( L^1 \) function.
- If \( f \) is an \( L^1 \) function and \( g \) is finite, then \( fg \) is also an \( L^1 \) function.
- If \( f \) is an \( L^1 \) function and \( |g| \leq |f| \), then \( g \) is also an \( L^1 \) function.

Proof. Suppose \( f, g \) are \( L^1 \) functions. From \( E(|f + g|) \leq E(|f|) + E(|g|) \), it follows \( f + g \) is integrable. On the other hand if \( m(A) \approx 0 \) then \( E(|f + g|\chi_A) \leq E(|f|\chi_A) + E(|g|\chi_A) \approx 0 \). Hence \( f + g \) is an \( L^1 \) function.

Suppose \( f \) is an \( L^1 \) function and \( g \) is finite. Let \( k \) be an accessible number such that \( |g(x)| < k \) for all \( x \in X \). Then \( E(|fg|) \leq k E(|f|) \), hence \( fg \) is integrable. On the other hand if \( m(A) \approx 0 \) then
\[
E(|fg|\chi_A) \leq k E(|f|\chi_A) \approx 0.
\]
Hence $fg$ is an $L^1$ function.

Suppose $f$ is an $L^1$ function and $|g| \leq |f|$. Then $E(|g|) \leq E(|f|) < \infty$. If $m(A) \approx 0$, then

$$E(|g|\chi_A) \leq E(|f|\chi_A) \approx 0.$$ 

**Proposition 10.5.4** If $f$ is an $L^1$ function and $d_1(f, g) \approx 0$, then $g$ is also an $L^1$ function.

**Proof.** Suppose $f$ is an $L^1$ function and $d_1(f, g) \approx 0$. Then $g$ is integrable since $f$ is integrable. Moreover if $m(A) \approx 0$ then

$$E(|g|\chi_A) \leq E(|f-g|\chi_A) + E(|f|\chi_A) \approx 0.$$ 

The class of $L^1$ functions forms a subclass of the set $F_M(X, d_1)$, which defines a continuum denoted by $L^1_M(X)$.

**Theorem 10.5.5** If $f, g \in L^1_M(X)$, then $d_1(f, g) \approx 0$ if $f \approx g$ a.e..

**Proof.** Suppose $h \in L^1_M(X)$ satisfies $h \approx 0$ a.e.. By Lemma 10.2.1, there is a subset $A \subset X$ such that $m(A) \approx 0$ and $h(x) \approx 0$ if $x \not\in A$. Then

$$E(|h|) = E(|h|\chi_A) + E(|h|\chi_A^c) \approx E(|h|\chi_A) \approx 0.$$ 

**Theorem 10.5.6** If $(f_1, f_2, \cdots)$ is a concrete sequence of $L^1$ functions and converges to an $L^1$ function $f$ pointwise a.e., then

$$\lim_i E(|f_i - f|) \approx 0.$$ 

In particular

$$\lim_i E(f_i) \approx E(f).$$ 

**Proof.** Suppose $f_i$ converges a.e. to $f$ pointwise. By Lemma 10.5.8 below, there is a subset $A \subset X$ such that $m(A) \approx 0$ and if $x \not\in A$ then $f_i(x)$ converges to $f(x)$. Then by Lemma 10.5.7, $f_i$ converges uniformly to $f$ on $A^c$. Hence for each accessible $k$, there is an accessible $n$ such that if $i$ is accessible with $i > n$ then, 

$$|f - f_i| < \frac{1}{k} \text{ on } A^c$$

and we have

$$E(|f - f_i|) = E(|f - f_i|\chi_A) + E(|f - f_i|\chi_A^c) \leq \frac{1}{k} + E(|f - f_i|\chi_A).$$
Since $|f - f_i|$ is an $L^1$ function, by Proposition 10.5.2 we have

$$E(|f - f_i|\chi_A) \approx 0.$$  

Thus for accessible $k$, there is an accessible $n$ such that for every accessible $i \geq n$, we have $E(|f - f_i|) \leq \frac{2}{k}$, namely, the concrete sequence $f_1, f_2, \cdots$ converges to $f$ with respect to the metric $d_1$.

**Lemma 10.5.7** If a concrete sequence $f_1, f_2, \cdots$ of functions on a set $X$ converges pointwise to $f$ everywhere, then it converges uniformly to $f$.

**Proof.** Suppose $f_i$ converges to $f$ pointwise. For each accessible $k$ and $x \in X$, there is an accessible $\ell_x$ such that if $n > \ell_x$ then

$$|f_n(x) - f(x)| < \frac{1}{k}.$$  

(107)

If we put $\ell = \max \{ \ell_x \mid x \in X \}$, then for $n > \ell$ conditions (107) holds for every $x \in X$. Namely the sequence $f_i$ converges to $f$ uniformly.

**Lemma 10.5.8** If a concrete sequence $f_1, f_2, \cdots$ of functions on $X$ converges pointwise to $f$ almost everywhere, then there is a subset $A \subset X$ such that $m(A) \approx 0$ and the concrete sequence of rationals $f_i(x)$ converges to $f(x)$ for $x \notin A$.

**Proof.** Suppose a concrete sequence $f_i$ converges almost everywhere to $f$.

Then for each accessible $k$, there is a subset $A_k \subset X$ satisfying

1. $m(A_k) < \frac{1}{k}$
2. If $x \notin A_k$ then $(f_1(x), f_2(x), \cdots)$ converges to $f(x)$.

We may suppose $A_k$ is decreasing since $A'_k = A_1 \cap A_2 \cap \cdots \cap A_k$ satisfies the same conditions.

Extend the concrete sequence $(f_1, f_2, \cdots)$ to a sequence $(f_1, \cdots, f_N)$ in $F_M(X)$. If $x \notin A_k$, then there is a huge $M_x$ such that for all huge $i \leq M_x$,

$$f_i(x) \approx f(x)$$  

(108)

holds. Put $M_k = \min_{x \notin A_k} M_x$. Then for huge $i \leq M_k$, (108) holds for all $x \notin A_k$. Since

$$|f_i(x) - f(x)| < \frac{1}{k}$$  

(109)

holds for all huge $i \leq M_k$, there is an accessible $m_k$ such that (109) holds for all $i$ with $m_k \leq i \leq M_k$. Note that we may take $m_k > k$.

Thus for every accessible $k$, there are an accessible number $m_k$, a huge number $M_k$, and subset $A_k \subset X$ satisfying the following conditions.
1. $k < m_k$,
2. $m(A_k) < \frac{1}{k}$,
3. the condition \[109\] holds for all $i$ satisfying $m_k \leq i \leq M_k$,
4. if $i < j \leq k$ then $m_i \leq m_j < M_j \leq M_i, A_j \subset A_i$.

Since these conditions on $k$ are objective, we have a huge $K$ such that there are numbers $m_K, M_K$ and a subset $A_K \subset X$ satisfying the above four conditions for $k = K$.

Then if $x \notin A_K$, for every huge $i \leq M_K$ and accessible $k$, we have $i \leq M_K \leq M_k$ and $A_K \subset A_k$, whence \[109\] holds whence \[108\] holds. Thus if $x \notin A_K$ then $\lim_i f_i(x) \approx f(x)$.

We note that if a concrete sequence of $L^1$ functions converges everywhere to a function $f$ then the following can be proved easily.

**Proposition 10.5.9** Let $(f_1, f_2, \cdots)$ be a concrete sequence of $L^1$ functions converging to $f$ pointwise everywhere. Then $f$ is an $L^1$ function and

$$\lim_i E(|f - f_i|) \approx 0.$$  

**Proof.** By Lemma \[105.7\] $f_i$ converges uniformly to $f$. In particular, there is an $L^1$ function $g = f_i$ such that $|f(x) - g(x)| < 1$. Hence

$$|f(x)| \leq 1 + |g(x)|$$

for all $x$ and $f$ is an $L^1$ function. Since $f_i$ converges uniformly to $f$, it is obvious that

$$\lim_i E(|f - f_i|) \approx 0.$$  

**Theorem 10.5.10** Suppose a concrete sequence $(f_1, f_2, \cdots)$ of $L^1$ functions converges to $g \in F_M(X)$ with respect to the distance $d_1$. Then $g$ is also an $L^1$ function and a subsequence of $f_1, f_2, \cdots$ converges to $g$ pointwise a.e..

**Proof.** For each accessible $k$, there is an accessible $n_k$ such that

$$E(|f_i - g|) \leq \frac{1}{4^k} \text{ for accessible } i \geq n_k. \quad (110)$$

The function $g$ is integrable since

$$E(|g|) \leq E(|g - f_{n_i}|) + E(|f_{n_i}|) \leq \frac{1}{4^i} + E(|f_{n_i}|).$$
Furthermore, if $C$ is any subset of $X$ with $m(C) \approx 0$, then since $f_{n_i}$ is an $L^1$ function, we have $E(|f_{n_i}| \chi_C) \approx 0$, and for every accessible $i$

\[ E(|g| \chi_C) \leq E(|g - f_{n_i}| \chi_C) + E(|f_{n_i}| \chi_C) \approx E(|g - f_{n_i}|) \leq \frac{1}{4^i} \]

Hence

\[ E(|g| \chi_C) \approx 0. \]

By Proposition 10.5.2, $g$ is also an $L^1$ function.

Put $A_k := \{ x \in X \mid |f_{n_k}(x) - g(x)| \geq \frac{1}{2^k} \}$, then by Proposition 10.4.3

\[ m(A_k) \leq \frac{1}{2^k} = \frac{1}{2^k}. \]

Extend the concrete sequence $A_1, A_2, \cdots$ to a huge sequence $A_1, \cdots, A_K$ such that for all $i \leq K$, $m(A_i) \leq \frac{1}{2^i}$.

Put $B_i = \bigcup_{j \leq i} A_j$. Then

\[ m(B_i) \leq \sum_{i \leq j \leq K} \frac{1}{2^j} \approx \frac{1}{2^{i-1}}. \]  

(111)

Put $g_i := f_{n_i}$. Suppose $x \notin B_i$. Since for all concrete $j \geq i$, we have $x \notin A_j$, whence

\[ |g_j(x) - g(x)| < \frac{1}{2^i} \leq \frac{1}{2^i}, \]

which implies that $(g_1(x), g_2(x), \cdots)$ converges to $g(x)$. Hence by (111), the subsequence $(f_{n_1}, f_{n_2}, \cdots)$ converges pointwise to $g$ almost everywhere.

\[ \blacksquare \]

Corollary 10.5.11 (Completeness of $L^1_M(X)$) The metric space $L^1(X)$ is complete, namely, every concrete Cauchy sequence of $L^1$ functions with respect to $d_1$ converges to an $L^1$ function.

Proof. Suppose $f_1, f_2, \cdots$ is a concrete Cauchy sequence of $L^1$ functions with respect to $d_1$. Extend to a huge sequence $(f_1, \cdots, f_N)$ in $F_M(X)$, which is convergent by Proposition 3.4.1. Hence $(f_1, \cdots, f_N)$ converges to a $g \in F_M(X)$ with respect to $d_1$. Then the concrete sequence $f_1, f_2, \cdots$ converges to $g$ by the same proposition. By Theorem 10.5.10, $g$ is an $L^1$ function. Hence $f_1, f_2, \cdots$ converges in $L^1_M(X)$.

Remark 10.5.1 If $L^1_M(X)$ were a set, then completeness follows from Proposition 3.4.1 directly. Since $L^1_M(X)$ is a proper semiset which cannot be defined by objective conditions, the extension of a concrete sequence to huge sequence is not possible in $L^1_M(X)$.

\[ \blacksquare \]
11 Concluding Remarks

11.1 Recapitulation

We showed that basic mathematical concepts with infinitary aspects such as real number, calculus, topology, measure can be developed by replacing the infinite axiom by the “sorites axiom” giving qualitative plurality of finiteness. We gave the terminology “standard” a semantical meaning of accessibility in order to make the “validity” of basic axioms obvious. We hope by this strategy the “over-technicality” of the traditional axiomatic foundation of nonstandard mathematics is reduced considerably.

11.1.1 Vague Concepts and Semisets

The crucial point in order to actualize directly the qualitative plurality of finiteness is to use vague conditions such as “accessibility” side by side with the usual mathematical conditions. The essential difference between these types of conditions give rises to the so-called “overspill phenomena”, which turn out to be one of the basic principles in nonstandard mathematics.

However, logical usage of vague concepts needs drastic change of basic concepts and principles of mathematics, although it should be stressed that the change is of such a kind as to reduce unnecessary complication of current mathematics resulting from not discriminating between theoretical possibility and actual possibility.

The most radical change is the introduction of “proper semisets” which are proper classes included in a huge finite set. A typical example of a proper semiset is the collection of accessible natural numbers. We have seen that semisets play the role of the infinite sets but in a more appropriate way since inaccessible numbers are not separated from the accessible numbers by virtue of the overspill principle.

On this account we introduced three types of collection, namely, proper classes, semisets, and sets. Furthermore only finite sets are entitled to be sets but finite sets are ramified to huge sets and concrete ones which have actual enumeration. Parenthetically we note that we used the word “collection” only informally.

Furthermore we introduced two attributes of conditions, objective and definite. A condition is definite if it can be stated without unbounded quantifications and objective if it can be sated without using accessibility. We restricted the separation axiom only to objective conditions, which implied the overspill principle.

Another deviation from the current mathematics is the understanding
of functions. We require functions to have explicit objective definite specification since the usual notion of function as mapping have no definite semantical meaning for proper classes. We checked that this restriction is void for functions defined on sets but that functions defined on semisets are extended uniquely to a mapping on surrounding sets, which is revealed to be another basic principle in addition to the overspill principle.

All these changes might be appear unnecessary complication at first sight but they reflect important aspects of our way of understanding the world and as a result they enrich mathematics with more intuitive ways of arguments hitherto considered as merely informal ones.

11.1.2 Treatment as Naive theory

We did not and do not intend to present our new framework as a formal theory. The reasons are as follows.

Firstly the so called “formal theory” itself depends on the current mathematics with the doctrine of “the $\mathbb{N}$”. For example, even syntactic concept as the provability is ramified in our alternative mathematics with multiple concepts of finiteness. Hence “formal theory” itself is not reliable from the point of view of the alternative approach presented here.

But the more decisive reason is that our intention is to present an alternative approach to mathematics without technical artifacts so that freshmen could follow without specialized training of specific area such as mathematical logic. We intend to grow an alternative mathematics for “doing mathematics” just as current mathematics are done mostly by naive set theory without exact knowledge of axiomatic set theory.

11.1.3 Continua and Points

A continuum is usually identified with the infinite set consisting of its points and the topology is captured as an additional structure given by metric or topological structure. In contrast we captured a continuum as a collection, often a huge finite set, endowed with indistinguishability relation. Here distinguishability is understood from practical point of view and discriminated from theoretical distinguishability. As a result the indistinguishability is a vague relation, and the identity of a point on a continuum, defined as the collection of elements indistinguishable from a fixed element, has persistent indefiniteness, which is embodied in the sorites paradox that both $x_1 \approx x_2 \approx \cdots \approx x_N$ and $x_1 \not\approx x_N$ can hold if $N$ is not accessible. In addition to the indefiniteness, a point of a continuum itself has a structure of a continuum if the indistinguishability is properly sharpened, which reflects the fractal nature of continuum.
11 Concluding Remarks

We did not however try to define general continua as primitive entity but only defined the linear continuum $\mathbb{R}$ as the proper class of rational numbers with the indistinguishability relation. Subcontinua of $\mathbb{R}$ such as the unit interval $[0, 1]$ can be represented by finite sets with indistinguishability.

Since points have nontrivial extensions, a morphism between continua cannot be determined as a correspondence of points. Besides indistinguishable elements must correspond to indistinguishable ones, whence morphisms are represented by continuous maps in the usual sense. Discontinuity simply means ill-definedness.

We may say that nonstandard mathematics, by embracing indefiniteness via "standardness", has given alternative approach to continuum more appropriate not only than current mathematics but also than the invaluable intuitionistic mathematics.

We note in passing that there is constructive approaches to nonstandard mathematics, for example [Pa95], based on intuitionistic type theory [ML90], which however appear to be rather too formal to be relevant to the above mentioned intention to develop alternatives mathematics at the same naive level as the usual one, based on intuitively clear simple semantics.

11.1.4 Idealization

We postulated that a number less than an accessible number is also accessible. However, for example, most of the numbers less than $10^{10^{10}}$, accessible by the exponentiation, cannot be described concretely in any fixed notational system. Thus our accessibility is too idealized to have something to do with actual accessibility from the radical ultrafinistic point of view.

Moreover the existence of the huge numbers inaccessible by any concrete methods might seem similar idealization as in the introduction of the infinite sets. However the character of idealization is utterly different. The idealization of infinity as infinite sets is to regard essentially indefinite objects as definite ones whereas idealization of infinite as huge sets keeps the indefiniteness so that it has potentially vast superiority over the infinite sets. For example it is intuitively more acceptable and more importantly it is safer from contradiction. This allows us not to consider the coherence problem so seriously.

On might think that the usage of vague concepts might be a new potential source of incoherency. In this respect, the usual coherent usage of the terminology "standard" which is vague, in the sense that it has no extension, gives us psychological assurance of our treatment of accessibility, since ours are in a sense a tiny portion of most axiomatic systems of nonstandard mathematics currently used.
11.1.5 Transfer Principle

We did not mention “the transfer principle” usually considered as the key point of nonstandard mathematics. However we found that its importance comes only from the requirement for nonstandard mathematics to be conservative extension of current mathematics and transfer principle is not necessary in developing mathematics itself.

However we briefly show that the transfer principle for definite objective conditions is a trivial consequence of the concept of accessibility if it is interpreted as the possibility of explicit specification.

Suppose \( P(x) \) is a definite objective condition with all the parameters accessible\(^{26}\) and is satisfied by all accessible numbers. If some inaccessible numbers does not satisfy it, then the minimal numbers which do not satisfy \( P \) is accessible by definition, which contradacts to the assumption.

However we did not restrict the meaning of accessibility in order only to get the transfer principle, which is not necessary if we do not insist on the conservativeness of the nonstandard mathematics.

11.2 Future Direction

We remark on some of the important aspects not touched here and some of the future promising directions.

11.2.1 Accessibility of Higher Order Objects

Many arguments of nonstandard mathematics are carried over to our framework except for those dependent on the standard-part operation. For example, the compactness is usually defined by the condition that every element is near-standard, that is, indistinguishable from a standard one. However we have more intuitive characterization of compactness, as a sort of “pigeon principle”, namely a continuum is compact if every huge subset has at least two mutually indistinguishable elements.

Since not only “standard-part operation” but also the concept standardness itself applied to higher order objects such as sets and functions seems to result in undesirable technicalities in usual nonstandard mathematics.

We guess that the counterpart of the “standard part arguments” in our framework is given by a sort of constructivity as is seen in the following examples.

\(^{26}\)Note that if \( \Omega \) is a huge number then the definite condition “\( x < \Omega \)” is satisfied by all accessible number but is not by \( \Omega \).
Suppose an increasing family of sets $X(n)$ parametrized by accessible numbers $n$ are constructed by a method independent of the specificity of the number $n$. By extension principle, we can substitute a huge number $\Omega$ to obtain a huge set $X(\Omega)$. Such huge sets are considered to be constructed by the series $\{X(n)\}$.

Two different constructions of a set can be considered as its different structures. For example, the huge set $[1..2^\Omega]$ have two constructions, one by substituting $2^\Omega$ to $n$ in $[1..n]$ and the other by substituting $\Omega$ in $n$ of $[1..2^n]$. The former series is constructed by adding $n + 1$ to $[1..n]$. The latter is constructed first by regarding $[1..2^n]$ as the set of infinitesimal intervals of width $2^{-n}$ and the step from $[1..2^n]$ to $[1..2^{n+1}]$ is done by halving all the intervals.

The vectors in $\mathbb{Q}^{[1..\Omega]}$ are sequences of rationals $(a_1, \cdots, a_\Omega)$ of length $\Omega$. The meaning of their accessibility depends on to the way the huge set $[1..\Omega]$ is constructed. For example, if $[1..\Omega]$ is constructed by the series $\{[1..n]\}$ as above, then a vector $(a_1, \cdots, a_\Omega)$ is accessible if its essential is captured by the subsequences $(a_1, \cdots, a_n)$. These vectors form the $L^1$ space of convergent sequences.

If $[1..2^\Omega]$ is the collection of infinitesimal intervals obtained by the halving processes as above, then the vector $(a_1, \cdots, a_\Omega)$, considered as a function which is constant on the intervals of width $2^{-L}$, is accessible if its essential part is captured by the functions constant on the intervals of width $2^{-n}$ with accessible $n$. These vectors form the space of measurable functions on $[0, 1]$.

The above sort of “accessible” sequences might be considered as typical ones of those usually called near-standard. We guess that standard-part operation can be captured by incorporating the constructivity of huge finite sets as above. In this way, it seems that we arrive at a mathematics which have much in common with the constructive nature of the intuitionistic approach.

### 11.2.2 Relative Accessibility

There are now various frameworks which relativize standardness such as RST (Relative Set Theory) of Péraire [Pér92], EST (Enlargement Set Theory) of D. Ballard [Bal94], relative arithmetic of S. Sandars [San10] to mention a few.

Similarly we can relativize accessibility as follows. Define a binary relation “number $y$ is accessible from a number $x$ or simply $x$-accessible” if there is some method of reaching $y$ using $x$ and the numbers less than $x$. A number $y$ is called $x$-inaccessible and written $y \gg x$ if it is not accessible from $x$. The 1-accessible numbers are accessible numbers of $\S1.1.1$.
The axioms are relativized as follows. A rational number $x$ is called \textit{infinitesimal at the level} $x$, or simply $x$-\textit{infinitesimal} if $|x| < \frac{1}{k}$ for all numbers $k$ accessible from $x$. Two rational numbers $y, z$ is called $x$-\textit{indistinguishable} and written $y \approx x z$ if $y - z$ is $x$-infinitesimal.

We postulate that for every number $x$, there are $x$-inaccessible numbers. A condition is called \textit{objective} if it is defined without using the binary relation $x \gg y$. We postulate the separation axiom for objective conditions so that the collection of elements in a set satisfying an objective condition is a set. This implies the general overspill principle to the effect that if an objective condition is satisfied by all the $x$-accessible numbers then it is satisfied also by an $x$-inaccessible number.

\subsection{Continua of Syntactic Objects}

One of the innovative aspects of our approach is the possibility of using huge syntactic objects, such as huge words, huge terms. In contrast to the "infinite words", every operations on finite words carry over to them and it is expected the mathematical world of huge syntactic objects has new phenomena with both aspects of finite and infinite and give new insights into the mysteries of the "complex systems" for whose understanding the dichotomy between finite and infinite is a severe barrier.

For example we consider it one of the main innovative points that continua can be directly constructed from huge syntactic objects. As a simplest example, we studied in §4 topological properties of the continua formed by huge binary words with respect to a few distance functions. Similarly the Cayley graphs of infinite groups define directly complete metric spaces whose topological properties are closely related to the algebraic properties of the groups. The investigation of these relations has been one of hot topics since 1980s as is exposed in \cite{Gro99} and \cite{EPC+92}. Fig 3 shows parts of the Cayley graph of the free group with two generators.

Another example are associahedra. The terms of one binary symbol $b$ and one variable $x$ with $N$ occurrences of $b$ form a connected graph when a term is connected to another if one is obtained from the other directly by the associativity rule. The number of nodes is the $N$-th Catalan number $c_N = \frac{1}{N+1} \binom{2N}{N}$.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
$N$ & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
$c_N$ & 5 & 14 & 42 & 132 & 429 & 1430 & 4862 & 16796 & 58786 & 208012 \\
\hline
\end{tabular}
\end{center}

These graphs are called associahedra for accessible $N$. See Fig 4. We do not know yet much about the topological property of this huge associahedron except that the diameter is $N$ since the shortest path between the farthest
Figure 3: Cayley Graphs of Free Group on two generators. Parts formed by words of length $\leq n$ ($n = 3, 4, 5, 6, 7$)
pair \( t := r^N_x x \) and \( s := l^N_x x \) is of length \( N \), where \( r_x w = b(w, x) \) and \( l_x w = b(x, w) \). There are many paths connecting \( t \) and \( s \) but not so many compared with the hypercube treated in §4.3.

The continua of huge syntactic objects seem to have potentiality to become topics of productive investigation.

11.3 Philosophical Implications

11.3.1 Rejection of Existence Absolutism

Although modern mathematics has acquired a sort of autonomy in human intellectual activities and seems to continue flourishing without end, it has a problem deeply embedded in its core which seems not only to diminish its cultural value but also to endanger its very existence in future. The problem is that modern mathematics allows no indefiniteness whatsoever and consid-
ers even infinity as a definite entity. For example, although “real numbers”
can not be determined by actual calculations and have essential epistemo-
logical indefiniteness, they are regarded as entities ontologically determined
exactly. The “current state” of the universe is conceptually conceived as
a definite element of a huge mathematical space and the ideal of scientists is
to find a law which determines the future state from the current state. In a
sense, the scientific determinism usually referred to by the name “Laplace’s
demon” is implicitly intertwined to the basic way of thinking of modern
people. This determinism cannot be formulated without the determinism of
modern mathematics.

The world is the physical universe with complex phenomena, some of
which has the appearance of living entities and among them the human
beings appear to possess mind and free will, which is nothing but associated
phenomena of the physical activity of the brain. This naive reductionism
joined with the above scientific determinism seems to be deeply embedded
in the quotidian view of the world.

For example, there are many informal talks on the possibility of “artificial
reality” and of immortality of a person in artificial reality transferring his
complete data into computer. Even admitting the possibility of describing
the universe completely by mathematics, such arguments lose sense if math-
ematics considers it non-sense to think of an entity as determinate if it has
potentially infinite details.

This radical determinism of modern mathematics seems not only to nar-
row its influence on other disciplines such as biology and sociology but also
underestimate the potentiality of human being. Moreover, the determin-
ism makes mathematics more difficult to be learned than necessary. This
difficulty is caused by the conflicting position treating indefinite entities as
definite ones.

The mathematical determinism results in the sharp dichotomy between

\[11\] In [Vop91], P. Vopěnka emphasizes the close connection between “natural infinity”
and indefiniteness is broken by the “classical infinity” of modern mathematics. He claims
that the study of natural infinity via alternative set theory is also a possible foundation
for the science of indefiniteness.

\[27\] This determinism is connected with what H. Weyl calls the existence absolutism:
"Before God", or "in itself", everything is determined into the last detail.
This existence absolutism is governed by a belief similar to the one that a
process in the external world that we experience does not, in itself, carry any
vagueness, even though our intuition can always only pick out spatial points
and qualities in an approximate manner, and never delimit them with abso-
lute sharpness. (“The Current Epistemological Situation in Mathematics
[Man98, p128])

\[28\] It may be objected that indeterminacy can be covered by probability theory, which
however is based on the definiteness of the probability densities.
finite and infinite, which undervalue finiteness as trivial compared with “unfathomable infinity”. This dichotomy mismatches the naive insight that infinity is an aspect of huge finiteness recognized under the limitation of our cognitive ability. Similarly the mathematical determinism results in the sharp dichotomy between discrete and continuous which also mismatches the naive insight that real continuum arises from the congregation of finite but huge number of discrete objects seen under the limitation of the granularity of our cognitive organs.

Most actual things are both finite and infinite depending on our standpoints. In everyday life, incessant switching of viewpoints is indispensable to comprehend the world well. Similarly, by incorporating two incommensurable standpoints in mathematics, infinity and continuity emerge as the phenomena resulting from the interplay between two viewpoints. This is what is achieved by incorporating sorites paradox in mathematics.

### 11.3.2 Internal Measurement

The indefiniteness thus introduced in mathematics reflects appropriately the aspect so called internal measurement [Gun04, Mat95] of our cognition of the world. Internal measurement refers to the stance taking seriously into account the inevitable temporality and incompleteness of the interaction between the observer and the observed behind the cognition. The “objects” are not fixed entities independent of observation but are phenomena acquiring more clear features through observation.

In mathematics, axiomatics correspond to the way of observing mathematical objects and each theorem may be regarded as an observation. In contrast to the usual view that the objects are definite and immutable, we think that a new theorem deforms the essence of the object. Such a view is seen in the predicative reformulation of mathematics proposed by E. Nelson.

Let $C$ be an inductive formula. ... We can replace our concept of number (any $x$) by a more refined concept of number (any $x$ such that $C^\exists[x]$). We can read $C^\exists[x]$ as “$x$ is a number“ (leaving open the possibility of formalizing an even more refined concept of number at some time in future) [Nel87a, p14].

Here is a new view that mathematical investigation change nature of the objects by proving theorems. The induction inference is not considered as an axiom but as the decision of adding new axioms which refines the concept of numbers by a condition proven to be inductive. This might be said to take account of internal measurement in mathematical investigation and the indefiniteness of the totality of the objects under study is crucial to support such view.
Acknowledgement  I would like to express my deep gratitude to Shuichiro Tsunoda for showing the crucial and ubiquitous roles played by indefiniteness in mathematics, to Yukio-Pegio Gunji and Kouichiro Matuno for showing essential aspects of the viewpoint of internal measurement convincing me its importance in mathematics, to Taichi Haruna for carefully reading the manuscript and for pointing out many errors both mathematical and typographical, which improved the manuscript considerably. I am indebted to Ichiro Tsuda, Kunihiko Kaneko and Takashi Ikegami whose stimulant unprecedented activities on complex systems led me to notice the great blind spot of modern mathematics supposedly arising from the peace and easiness in the ”Cantor’s paradise” where infinities are treated as handy definite entities thereby closing the way and making indifference to understanding essential aspects of living being where indefiniteness plays vital roles. I am also indebted to Yoshitsugu Oono, Gen Kuroki and Toshio Sunada for their various strong critisims on the alternative view of mathematical science expressed in [Tsu98] which emphasized the importance of indefiniteness in mathematics based on the internal measurement point of view. Their criticism helped me to pursue the alternative way more concretely. I am thankful to many mathematicians for stimulative discussions, especially to Shinsuke Shimogawa, Yohe Yamasaki, Akihiko Gyoja, Yoshifumi Takeda, Makoto Kikuchi, Shunsuke Yatabe. I am also greatfull for many researchers whose concern, sympathy and encouragement has been supporting my research activity in this isolated direction, especially to Yoshihiro Fukumoto, Kazuyuki Tanaka, Yoshinori Shiozawa, Hideo Mori, Kazufumi Nakajima, Isao Naruki, Etsuro Date, Noriaki Kawanaka, Toshio Mikami. Finally I would like to express my deep gratitude to Shunichi Tanaka who turned my attention to ”complex systems”, to the late Kunihiko Kodaira who encouraged me to proceed to new domain of research, and to Koji Shiga who constantly showed me, from my student days, the open-mindedness to mathematics and invariable passionate concern with the mystery of infinity, which formed in me the courage to pursue freed from the past without worry whatever topics I judge important.
References

[AG06] P.V. Andreev and E.I. Gordon, *A theory of hyperfinite sets*, Annals of Pure and Applied Logic 143 (2006), no. 1-3, 3–19.

[Bal94] D. Ballard, *Foundational aspects of "non" standard mathematics*, vol. 176, Amer Mathematical Society, 1994.

[Bec79] Jon M. Beck, *Simplicial sets and the foundations of analysis*, Proceedings of Conference on Sheaf Theory, Durham, England (July 1977), Lecture Notes in Mathematics, vol. 753, Springer, Berlin, 1979, pp. 113–124.

[Bec80] _____, *On the relationship between algebra and analysis*, Journal of Pure and Applied Algebra 19 (1980), 43–60.

[BMW10] A. Brown, M.A. McDonald, and K. Weller, *Step by step: Infinite iterative processes and actual infinity*, Research in collegiate mathematics education 7 (2010), 115.

[Bor52] Emil Borel, *Les nombres inaccessibles*, Gauthier-Villars, 1952.

[CS95] R. Chuaqui and P. Suppes, *Free-variable axiomatic foundations of infinitesimal analysis: a fragment with finitary consistency proof*, The Journal of Symbolic Logic 60 (1995), no. 1, 122–159.

[CT08] Peter J. Cameron and Sam Tarzi, *Limits of cubes*, Topology Appl. 155 (2008), no. 14, 1454–1461. MR 2435141 (2010c:54040)

[CWF+09] A. Chollet, G. Wallet, L. Fuchs, G. Largeteau-Skapin, and E. Andres, *Insight in discrete geometry and computational content of a discrete model of the continuum*, Pattern recognition 42 (2009), no. 10, 2220–2228.

[Die92] M. Diener, *Application du calcul de harthong-reeb aux routines graphiques*, Le Labyrinthe du Continu (1992), 424–435.

[Dra85] A. Dragalin, *Correctness of inconsistent theories with notions of feasibility*, Computation theory (1985), 58–79.

[Dum75] M.E. Dummett, *Wang's paradox*, Synthese 30 (1975), 301–324.

[EPC+92] David B. A. Epstein, M. S. Paterson, G. W. Camon, D. F. Holt, S. V. Levy, and W. P. Thurston, *Word processing in groups*, A. K. Peters, Ltd., Natick, MA, USA, 1992.

[Gro99] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152,
References

Birkhäuser Boston Inc., Boston, MA, 1999, Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR 1699320 (2000d:53065)

[Gun04] Yukio-Pegio Gunji, protocomputing and ontological measurement, in japanese, University of Tokyo Press, 2004, ISBN 4130100971.

[Har83] J. Harthong, Éléments pour une théorie du continu, Astérisque 109 (1983), no. 110, 235–244.

[HLO10a] K. Hrbacek, O. Lessmann, and R. O’Donovan, Analysis with ultrasmall numbers, American Mathematical Monthly 117 (2010), no. 9, 801–816.

[HLO10b] Karel Hrbacek, Olivier Lessmann, and Richard O’Donovan, Analysis with ultrasmall numbers, Amer. Math. Monthly 117 (2010), no. 9, 801–816. MR 2760381 (2011j:26047)

[Isi80] David Isles, Remarks on the notion of standard non-isomorphic natural number series, Constructive Mathematics, Proceedings of the New Mexico State University Conference Held at Las Cruces, New Mexico, August 11–15, 1980 (F. Richman, ed.), Lecture Notes in Mathematics, vol. 873, Springer, 1980, pp. 111–134.

[Lau92] Detlef Laugwitz, Leibniz’ principle and omega calculus, Le Labyrinthe du Continu. Paris: Springer France (1992), 144–154.

[Lut87] Robert Lutz, Réveries infinitésimales, Gaz. Math. (1987), no. 34, 79–87. MR 918184 (89f:03065)

[Lut92] R. Lutz, La force des théories infinitésimales faibles, le labyrinthe du continu, Springer France, Paris (1992), 414–423.

[Mag07] O. Magidor, Strict finitism refuted?, Proceedings of the Aristotelian Society (Hardback), vol. 107, Wiley Online Library, 2007, pp. 403–411.

[Man98] Paolo Mancosu (ed.), From Brouwer to Hilbert. The debate on the foundations of mathematics in the 1920s, Oxford University Press, Oxford and New York, 1998.

[Mat95] K. Matsuno, Quantum and biological computation, BioSystems 35 (1995), no. 2-3, 209–212.
[May00] J. P. Mayberry, The foundations of mathematics in the theory of sets, Encyclopedia of Mathematics and its Applications, no. 82, Cambridge University Press, Cambridge, 2000, ISBN 0-521-77034-3.

[ML90] P. Martin-Löf, Mathematics of infinity, COLOG-88, Springer, 1990, pp. 146–197.

[Mon01] J. Monaghan, Young peoples’ ideas of infinity, Educational Studies in Mathematics 48 (2001), no. 2, 239–257.

[Myc81] Jan Mycielski, Analysis without actual infinity, J. Symbolic Logic 46 (1981), 625–633.

[Nel77] E. Nelson, Internal set theory: a new approach to nonstandard analysis, Bulletin of the American Mathematical Society 83 (1977), 1165–1198.

[Nel87a] ______, Predicative arithmetic, Princeton University Press, 1987, ISBN 0-691-08455-6.

[Nel87b] ______, Radically elementary probability theory, Annals of Mathematics Studies, no. 117, Princeton University Press, 1987, ISBN 0-691-08455-6.

[Nel04] Edward Nelson, Bookreview:gnomes in the fog: The reception of brouwer’s intuitionism in the 1920s, by denis e. hesseling, science networks –historical studies, vol.28 birkhauer,base,2003.xxviii + 447 pp.,isbn 3-7643-6536-6, Bulletin of the AMS (2004).

[Nel07] ______, The virtue of simplicity, The strength of nonstandard analysis, SpringerWienNewYork, Vienna, 2007, pp. 27–32. MR 2341412

[Pal95] Erik Palmgren, A constructive approach to nonstandard analysis, Ann. Pure Appl. Logic 73 (1995), no. 3, 297–325. MR 1336645 (96c:03123)

[Par71] R. Parikh, Existence and feasibility in arithmetic, Journal of Symbolic Logic 36 (1971), 494–508.

[Pér92] Y. Péraire, Théorie relative des ensembles internes, Osaka J. Math 29 (1992), no. 2, 267–297.

[Pér05] ______, Le replacement du référent dans les pratiques de l’analyse issues de e. nelson et de g. reeb, Philosophia Scientiae. Travaux d’histoire et de philosophie des sciences (2005), no. CS 5, 257–273.
[Ras73] P.K. Rashevskii, *On the dogma of the natural numbers*, Russian Mathematical Surveys 28 (1973), no. 4, 143–148.

[Ree81] G. Reeb, *Mathematique non standard (essai de vulgarisation)*, Bulletin APMEP 328 (1981), 259–273.

[Rob66] A. Robinson, *Non-standard analysis*, North-Holland Publishing Company, Amsterdam, 1966.

[RR96] Jean-Pierre Reveills and Denis Richard, *Back and forth between continuous and discrete for the working computer scientist*, Annals of Mathematics and Artificial Intelligence 16 (1996), no. 1, 89–152.

[RS10] Ana Romero and Francis Sergeraert, *Discrete vector fields and fundamental algebraic topology*, CoRR abs/1005.5685 (2010).

[San10] S. Sanders, *Relative arithmetic*, Mathematical Logic Quarterly 56 (2010), no. 6, 564–572.

[Saz95] Vladimir Yu. Sazonov, *On feasible numbers*, Logic and computational complexity (Leviant D, ed.), Lecture Notes in computer science, vol. 960, Springer, 1995, pp. 30–51.

[SLSZ] Antonín Sochor, Alistair Lachlan, Marian Srebrny, and Andrzej Zarach, *Differential calculus in the alternative set theory*, pp. 273–284, Springer Berlin / Heidelberg.

[Tal80] D. Tall, *The notion of infinite measuring number and its relevance in the intuition of infinity*, Educational Studies in Mathematics 11 (1980), no. 3, 271–284.

[Tho92] René Thom, *L'antériorité ontologique du continu sur le discret*, Le labyrinthe du continu (Cerisy-la-Salle, 1990), Springer, Paris, 1992, pp. 137–143. MR 1413523

[Tra98] Robert Tragesser, *Part i:ultrafinitism,naturalism,vagueness*, http://www.cs.nyu.edu/pipermail/fom/1998-April/001825.html 4 1998.

[Tsu98] Toru Tsujishita, , *life and complex systems, in japanese*, Science of Complex Systems and Modern Thought, pp. 75–225, Seidosha, 1998, ISBN 4-7917-9145-2.

[TT01] D. Tall and D. Tirosh, *Infinity—the never-ending struggle*, Educational studies in Mathematics 48 (2001), no. 2, 129–136.
References

[vdDW84] L. van den Dries and A. J. Wilkie, *Gromov’s theorem on groups of polynomial growth and elementary logic*, J. Algebra **89** (1984), no. 2, 349–374. MR 751150 (85k:20101)

[Ver98] A. M. Vershik, *The universal Uryson space, Gromov’s metric triples, and random metrics on the series of natural numbers*, Uspekhi Mat. Nauk **53** (1998), no. 5(323), 57–64. MR 1691182 (2000b:53055)

[Vol70] A.S. Essenin Volpin, *The ultra-intuitionistic criticism and the anti-traditional programme for foundations of mathematics*, Intuitionism and Proof Theory (J. Myhill A. Kino and R.E. Vesley, eds.), North-Holland, Amsterdam, 1970, pp. 3–45.

[Vop79] Petr Vopěnka, *Mathematics in the alternative set theory*, Teubner, 1979.

[Vop91] P. Vopěnka, *The philosophical foundations of alternative set theory*, International Journal Of General System **20** (1991), no. 1, 115–126.

[vS90] Walter P. van Stigt, *Brouwer’s intuitionism*, Studies in the History and Philosophy of Mathematics, vol. 2, North-Holland Publishing Co., Amsterdam, 1990. MR 1075018 (92d:01054)

[Wey49] Hermann Weyl, *Philosophy of Mathematics and Natural Science. Revised and Augmented English Edition Based on a Translation by Olaf Helmer*, Princeton University Press, Princeton, N. J., 1949. MR 0029851 (10,670c)

[Wey94] ———, *The continuum*, Dover Publications Inc., New York, 1994, A critical examination of the foundation of analysis, Translated from the German by Stephen Pollard and Thomas Bole, With a foreword by John Archibald Wheeler and an introduction by Pollard, Corrected reprint of the 1987 translation [Thomas Jefferson Univ. Press, Kirksville, MO; MR1040831 (91h:01105)]. MR 1280464

[Yat09] Shunsuke Yatabe, *Comprehension contradicts to the induction within lukasiewicz predicate logic*, Arch. Math. Logic **48** (2009), no. 3-4, 265–268. MR 2500986 (2010f:03025)
## Index

\[ \frac{dF}{dx}, F', \] 94  
\#(A), number of elements, 25  
\[ n, \] 42  
\[ m(P), \] 128  
\[ Fun(C_1, C_2), \] 59  
\[ P \approx \emptyset, \] 126  
p \approx q on continuum, 36  
r \approx s, 24  
r \prec s, 24  
r \preceq s, 24  
a.e., 126  
accumulation point of sequences, 49  
accumulation point of subset, 44  
auxiliary, inverse, 40  
Axiom, accessible numbers, 23  
Axiom, sorites, 23  
binary tree, 51  
binary tree, Cantor continuum, 54  
binary tree, hyperbolic distance, 52  
binary tree, hamming distance, 56  
binary tree, power set continuum, 55  
Cantor continuum, 54  
\[ \sigma - \text{finite class}, \] 30  
\[ \sigma - \text{finite class}, \] ranking, 30  
\[ \alpha, \] 42  
\[ \Gamma, \] 91  
\[ \delta, \] 94  
\[ \Delta, \] 94  
\[ \gamma, \] 94  
\[ \lambda, \] 94  
condition, definite, 25  
condition, objective, 26  
condition, respective, 26  
connected component, 46  
continuous function, 39  
continuum, 38  
subcontinuum, 36  
continuum generated by a graph, 38  
continuum morphism, 39  
continuum of functions, 59  
continuum of morphisms, 59  
continuum, n-dimensional Euclidean, 37  
continuum, compact, 44  
continuum, connected, 46  
continuum, equivalent, 40  
continuum, Euclidean, 37  
continuum, hyperbolic, 57  
continuum, indistinguishability relation, 36  
continuum, linear, 36  
continuum, locally compact metric, 53  
continuum, mesh, 36  
continuum, metric, 36  
continuum, point, 36  
continuum, position, 36  
continuum, power, 56  
continuum, product, 38  
continuum, representation, 12  
continuum, rigid mesh, 36  
continuum, support, 36  
convergence, finite sequences, 44  
convergence of series, 71  
convergence of series, absolute, 71  
dense, 42  
derivative, 94  
difference operator, 91  
difference quotient, 91  
differentiable, second order, 122  
differentiable, up to order \( k \), 104  
discrete subset, 45  
enumeration, tight, 25  
equicontinuous, 61  
equivalence relation, strong, 35  
equivalence relation, weak, 35  
exponential morphism, 76  
indistinguishable functions, 59  
big number, 23
hypercube, 56

indifinite integreal, 107

induction for definite conditions, 32

induction for vague conditions, 32

infitesimal, 24

interval symbols, 37

interval symbols, finite, 37

local diffeomorphism, 120

logarithm, 81

measure, $m(A)$, 126

measure, subset, 126

metric class, 36

metric semispace, 36

metric space, 36

mutually almost disjoint, 130

objective discrimination, 45

perfect, 44

precompact, 49

probability density, 126

quantification, bounded, 25

quasi-identity, 41

quasi-set, 34

quasi-subset, 48

$R$-chain, 35

rational number, accessible, 23

rational number, finite, 23

rational number, huge, 23

$D$-valued real function, 74

finite real function, 74

real function, representation, 73

real number, 64

real number, finite, 64

real number, $\varepsilon$-separate representation, 64

real number, commensurable, 64

Robinson’s lemma, 60

saturation, 42

scale of approximation, 61

semiset, environment set, 26

semiset, mesurable, 128

semiset, null, 126

semiset, proper, 26

sigma additivity, 130, 132

sorites relation, 35

sorites sequence, 35

subclass, objective, 26

totally disconnected, 46

transitive, strongly, 35

transitive, weakly, 34

unit $n$-hypercube, 37

visible ball, 47

visibly, 24
Toru Tsujishita
Department of Mathematics, Ritsumeikan University, Shiga 525-8577, JAPAN
email: tjst@se.ritsumei.ac.jp