Multimagic Squares

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1. INTRODUCTION. Suppose that $M$ is an $m \times m$ matrix whose entries are natural numbers. Then $M$ is called a magic square (of order $m$) if the sum of all elements in each column, each row, and each of the two main diagonals gives the same number, the so-called magic number. If, in addition, its matrix elements consist of the consecutive integers $1, 2, \ldots, m^2$ the matrix $M$ is called a pure (or normal) magic square. Pure magic squares have been studied for more than four thousand years. During this long period of time many subclasses have been introduced, such as panmagic squares, which have the property that their broken diagonals also add up to the same magic number (a broken diagonal is a diagonal parallel to one of the two main diagonals), and multimagic squares, the subject of this article. A most-perfect magic square is a normal magic square of order $n$, a multiple of four, with the following two additional properties: (1) each $2 \times 2$ subsquare sums to $2s$, where $s = n^2 + 1$, and (2) all pairs of integers along any diagonal and at distance $n/2$ from each other sum to $s$. Recently some exciting new results have been found concerning these squares. For instance, the first method of constructing all most-perfect magic squares and their enumeration appeared in [10] (see also [14]). Another highlight is the verification (by computer) of the nonexistence of an $8 \times 8$ magic knight tour in [13] (i.e., it is impossible for a knight to make a tour on a checkerboard starting somewhere with the number 1, then putting 2 after one jump, and so forth, and after completing its tour to obtain a magic square). Finally, we would like to mention that in [2] a new method is found to construct and enumerate all Franklin squares. Franklin squares have the following properties: (i) the sum of the numbers in half a row (respectively half a column) is constant, (ii) each $2 \times 2$ subsquare has the same sum, and (iii) each of the 4 ‘bent diagonals’ has the same sum. Here a ‘bent diagonal’ is formed by joining half of one diagonal to half of another diagonal at a $90^\circ$ angle. However, there are still many unsolved problems (see [1] and [12]) or, to put it in the words of Clifford Pickover ([12, p. 26]) “the field of magic square study is wide open.”

In this paper we concentrate on one of the aforementioned open problems, namely, the existence and construction of so-called multimagic squares. To describe this class of magic squares, let $M$ be an $m \times m$ matrix and $d$ a positive integer. Then $M^{*d}$ denotes the $m \times m$ matrix obtained by raising each entry of $M$ to the power $d$. Now let $n$ be a positive integer. Then $M$ is called an $n$-multimagic square if $M$ is a normal magic square and $M^{*2}, M^{*3}, \ldots, M^{*n}$ are all magic. The first 2-multimagic square was found by Pfeffermann in 1891. It is the following square of order 8:

$$
\begin{array}{cccccccc}
56 & 34 & 8 & 57 & 18 & 47 & 9 & 31 \\
33 & 20 & 54 & 48 & 7 & 29 & 59 & 10 \\
26 & 43 & 13 & 23 & 64 & 38 & 4 & 49 \\
19 & 5 & 35 & 30 & 53 & 12 & 46 & 60 \\
15 & 25 & 63 & 2 & 41 & 24 & 50 & 40 \\
6 & 55 & 17 & 11 & 36 & 58 & 32 & 45 \\
61 & 16 & 42 & 52 & 27 & 1 & 39 & 22 \\
44 & 62 & 28 & 37 & 14 & 51 & 21 & 3
\end{array}
$$

In 1905 the first 3-multimagic square, a square of order 128, was constructed by Tarry. In 2001 both a 4- and a 5-multimagic squares were constructed by Boyer and Viricel,
squares of order 512 and 1024, respectively (see [4] and [5], where a nice history of the subject is given). The record up to now has been a 6-multimagic square of order 4096 constructed by Pan Fengchu in October 2003 [8]. However, the following question remained open: Do there exist $n$-multimagic squares when $n \geq 7$?

In this paper we give an affirmative answer to this question by providing an explicit construction for each $n$ greater than 2. In particular, we give the first proof of the existence of 7-multimagic squares (ours has order 13^3), the first 8-multimagic squares (order 17^b), etc. Our construction is based on elementary facts from linear algebra over finite fields and can easily be extended to construct $n$-multimagic cubes and hypercubes. At the end of this paper we comment on various generalizations of our construction.

2. CONSTRUCTION OF MULTIMAGIC SQUARES. Looking at the examples of multimagic squares described in the introduction, one observes that they all are matrices whose orders are powers of prime numbers. Not all multimagic squares need have prime power order. For example, Trump recently constructed the smallest 3-multimagic square, which has order 12 [15]. Nevertheless, each $n$-multimagic square we are going to construct will have order $p^n$, where $p$ is an odd prime number (at the end of this paper we indicate how to construct $n$-multimagic squares of order $q^n$, where $q$ can be any number greater than 2).

Fix an odd prime $p$. The starting point of our construction is the following observation: a square matrix $M$ of order $p^n$ whose entries comprise all integers from 1 to $(p^n)^2$ can be viewed as a bijection from the set of pairs $(i, j)$ with $1 \leq i, j \leq p^n$ to the set of consecutive integers from 1 to $(p^n)^2$. To find such bijections we first consider the bijection $N: \mathbb{Z}/p^2 \mathbb{Z} \to \{0, 1, \ldots, p-1\}$ given by $N(\overline{i}) = i (0 \leq i \leq p-1)$, where as usual $\overline{i}$ denotes the class of $i$ modulo $p$. We use $k$ to denote the field $\mathbb{Z}/p^2 \mathbb{Z}$. It is easy to verify that $N(a) + N(-a - 1) = p - 1$ for all $a$ in $k$.

More generally, for each positive integer $m$ we obtain a bijection $N_m: k^m \to \{1, 2, \ldots, p^m\}$ defined by means of the formula

$$
N_m((a_1, \ldots, a_m)) = 1 + \sum_{j=1}^{m} p^{j-1} N(a_j) \quad ((a_1, \ldots, a_m) \in k^m).
$$

Using the relation $N(a) + N(-a - 1) = p - 1$ one quickly deduces that

$$
N_m(a) + N_m(-a+c) = p^m + 1 \quad (1)
$$

for all $a$ in $k^m$, where $c = (-1, -1, \ldots, -1)$.

Next we choose a bijection $F: (k^n)^2 \to (k^n)^2$. The composition

$$
M := N_{2n} \circ F \circ (N_{n}^{-1} \times N_{n}^{-1}) \quad (2)
$$

gives a bijection from $\{1, 2, \ldots, p^n\}^2$ to $\{1, 2, \ldots, p^{2n}\}$. Clearly, not every choice of $F$ gives rise to an $n$-multimagic square. To obtain such squares we first choose $F$ to be an affine map given by an invertible $2n \times 2n$ matrix $A$ over $k$ and a vector $t$ in $k^{2n}$ (i.e.,

$$
F((a, b)) = A \begin{pmatrix} a \\ b \end{pmatrix} + t \quad (3)
$$

for $(a, b)$ in $(k^n)^2$). Since $A$ is invertible, $F$ is bijective, hence so is $M$. In order to ensure that $M$ is $n$-multimagic we impose on $A$ the following condition: all $n \times n$
minors of $P$, $Q$, $P + Q$, and $P - Q$ are units in $k$, where $P$ (respectively, $Q$) is the $2n \times n$ matrix formed by the first (respectively, last) $n$ columns of $A$. (Of course, we could have simply said that the minors are nonzero. However, since we will later work with arbitrary commutative rings rather than fields, we chose to write “units in $k$” instead of “nonzero elements in $k$”). Such a matrix $A$ is called an $n$-multimagic generator matrix. In the next section we will give explicit formulas for these matrices when $n \geq 3$ and $p \geq 2n - 1$. Now we are ready to state the main result of this paper.

**Theorem 2.1.** Let $F$ be as in (3), where $A$ is an $n$-multimagic generator matrix. Then the matrix $M$ defined by (2) is an $n$-multimagic square.

**Proof.** Since, as observed, (2) is a bijection, the entries of the matrix $M$ include all integers from 1 to $p^{2n}$. Thus $M$ is normal.

(i) We write $A = (P|Q)$ as indicated, and let $d$ be an integer with $1 \leq d \leq n$. We show that each column-sum of $M^{ad}$ is the same constant that depends only on $p$ and $n$. Therefore we fix an element $b$ in $k^n$. To compute the sum of the $N_n(b)$th column of $M^{ad}$, which we denote by $S_d(b)$, we have to add the elements $M^{ad}_{N_n(a), N_n(b)}$, where $a$ runs through $k^n$ (remember that $N_n$ is a bijection). It follows from the definition of $M$ that

$$S_d(b) = \sum_{a \in k^n} \left(1 + \sum_{j=1}^{2n} C_j(a, b)\right)^d,$$

where

$$C_j(a, b) = p^{j-1}N(P_{(j)}a + Q_{(j)}b + t_j) \quad (1 \leq j \leq 2n)$$

and $P_{(j)}$ (respectively, $Q_{(j)}$) denotes the $j$th row of $P$ (respectively, $Q$). Now observe that if $x_1, x_2, \ldots, x_{2n}$ are variables, then $(1 + x_1 + \cdots + x_{2n})^d$ is equal to $1 + g$, where $g$ is a sum of terms of the form $\alpha x_{j_1}^{e_1} \cdots x_{j_s}^{e_s}$. Here $\alpha$ is a positive integer, $1 \leq j_1 < j_2 < \cdots < j_s \leq 2n$, $e_1, \ldots, e_s \geq 1$, and $e_1 + \cdots + e_s \leq d$. In particular $s \leq d \leq n$.

It will follow from (4) that $S_d(b)$ depends only on $p$ and $n$ if we can show that for each set of exponents $e_1, \ldots, e_s$ and indices $j_1, \ldots, j_s$ the sum

$$\sum_{a \in k^n} C_{j_1}(a, b)^{e_1} \cdots C_{j_s}(a, b)^{e_s}$$

depends only on $p$ and $n$ (and, of course, on the $e_i$ and $j_i$). To see this, put $J = (j_1, \ldots, j_s)$ and consider the affine map $L : k^n \rightarrow k^s$ given by the formula

$$L(a) = P_{(j)}a + Q_{(j)}b + t_{(j)},$$

where $P_{(j)}$ (respectively, $Q_{(j)}$) is the $s \times n$ matrix formed by the rows $P_{(j_1)}, \ldots, P_{(j_s)}$ (respectively, $Q_{(j_1)}, \ldots, Q_{(j_s)}$) and $t_{(j)}$ is the column of length $s$ with components $t_{j_1}, \ldots, t_{j_s}$. Since $s \leq n$ and all $n \times n$ minors of $P$ are units in $k$, we infer that $L$ is surjective. On the basis of (5) we know that $C_{j_i}(a, b) = p^{b-1}N(L(a)_i)$ for $i = 1, 2, \ldots, s$. Accordingly, the expression (6) is equal to

$$p^{e_1(j_1-1) + \cdots + e_s(j_s-1)} \sum_{a \in k^n} N(L(a)_1)^{e_1} \cdots N(L(a)_s)^{e_s}.$$
By Lemma 2.2, which we will establish shortly, the surjectivity of $L$ implies that this expression equals

$$p^{n-s} p^{e_1(j_1-1)+\ldots+e_s(j_s-1)} \left( \sum_{i=0}^{p-1} i^{e_1} \right) \ldots \left( \sum_{i=0}^{p-1} i^{e_s} \right),$$

which indeed depends only on $p$ and $n$, as desired. Interchanging the roles of $a$ and $b$ in the argument just given we find that all row sums of $M^{sd}$ are also equal to the same constant.

(ii) To compute the sum of the main diagonal elements of $M^{sd}$ we have to add all elements $M^d_{N^a(a),N^a(a)}$, where $a$ runs through $k^n$. For this we repeat the preceding arguments with $b = a$ and define $L_1 : k^n \rightarrow k^s$ by the formula

$$L_1(a) = P_{(j)}a + Q_{(j)}a + t_{(j)} = (P + Q)_{(j)}a + t_{(j)}.$$  

Then $L_1$ is surjective since all $n \times n$ minors of $P + Q$ are units of $k$. The rest of the proof is the same as that given in (i).

(iii) Finally, to compute the sum of all elements of the “second” main diagonal of $M^{sd}$ we have to add the elements $M^d_{N^a(a),p^n+1-N^a(a)}$. Using (1) we can repeat the arguments in (i) with $b = -a + c$ and define $L_2 : k^n \rightarrow k^s$ by

$$L_2(a) = P_{(j)}a + Q_{(j)}(-a + c) + t_{(j)} = (P - Q)_{(j)} + (Q_{(j)}c + t_{(j)}).$$

Again $L_2$ is surjective because all $n \times n$ minors of $P - Q$ are units in $k$. The rest of the proof mimics that in (i).

To complete the proof of Theorem 2.1 it remains to fill in the following missing piece:

**Lemma 2.2.** If $L : k^n \rightarrow k^s$ is a surjective affine map and $e_1, \ldots, e_s$ are nonnegative integers, then

$$\sum_{a \in k^n} N(L(a))^{e_1} \ldots N(L(a))^{e_s} = p^{n-s} \left( \sum_{i=0}^{p-1} i^{e_1} \right) \ldots \left( \sum_{i=0}^{p-1} i^{e_s} \right).$$

**Proof.** Put $V(a) = N(L(a))^{e_1} \ldots N(L(a))^{e_s}$ for each $a$ in $k^n$. Observe that $k^n$ is a disjoint union of the fibers $L^{-1}(y)$, where $y = (y_1, \ldots, y_s)$ runs through $k^s$, and that $V$ is constant (equal to $N(y_1)^{e_1} \ldots N(y_s)^{e_s}$) on each fiber $L^{-1}(y)$. Since by the surjectivity and linearity of $L$ each fiber has $p^{n-s}$ elements, we see that

$$\sum_{a \in k^n} V(a) = p^{n-s} \sum_{y \in k^s} N(y_1)^{e_1} \ldots N(y_s)^{e_s} = p^{n-s} \left( \sum_{y_1 \in k} N(y_1)^{e_1} \right) \ldots \left( \sum_{y_s \in k} N(y_s)^{e_s} \right).$$

Because $N : k \rightarrow \{0, 1, \ldots, p - 1\}$ is a bijection, the lemma follows.

### 3. FINDING $n$-MULTIMAGIC GENERATOR MATRICES

In order to construct $n$-multimagic squares using Theorem 2.1, we need to show how to find $n$-multimagic generator matrices $A$ over $k$, where as earlier $k$ denotes the field of $p$ elements for an odd prime $p$. Therefore we will assume that $n \geq 3$ and $p \geq 2n - 1$. Then when $0 \leq j < i \leq 2n - 2$ the differences $i - j$ are units in $k$. By appealing to Vandermonde
determinants we conclude that all \( n \times n \) minors of the following \( 2n \times n \) matrix \( P \) are units in \( k \):

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 1 & 1 & \ldots & 1 \\
1 & 2 & 4 & \ldots & 2^{n-1} \\
1 & 3 & 9 & \ldots & 3^{n-1} \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
1 & (2n-2) & (2n-2)^2 & \ldots & (2n-2)^{2n-1} \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}.
\]

Using this matrix it is easy to make an \( n \)-multimagic generator matrix \( A \) as follows: write \( P = (P_1 \mid P_2) \), where \( P_1 \) (respectively, \( P_2 \)) is the \( n \times n \) matrix formed by the first (respectively, last) \( n \) rows of \( P \) and set \( Q = (\frac{1}{2}P_1 \mid -2P_2) \). Then define \( A = (P\mid Q) \).

Using elementary column operations one can reduce

\[
A = \begin{pmatrix}
P_1 \\
P_2
\end{pmatrix} = \begin{pmatrix}
2P_1 \\
-2P_2
\end{pmatrix}
\]

to the matrix

\[
\begin{pmatrix}
P_1 & 0 \\
P_2 & -4P_2
\end{pmatrix},
\]

which is clearly invertible because both \( \det P_1 \) and \( \det(-4P_2) \) are units in \( k \). It follows without difficulty that \( A \) is an \( n \)-multimagic generator matrix.

4. THE ORDERS OF MULTIMAGIC SQUARES. Now that we know that \( n \)-multimagic squares exist for each positive integer \( n \) one might wonder whether for each such \( n \)-multimagic squares exist of order \( m \), say if \( m \) is sufficiently large. The next result shows that this is not the case. For example, taking \( e = 1 \) and \( p = 2 \) in Theorem 4.1 shows that 3-multimagic squares of order \( m \) such that \( m \equiv 2 \pmod{4} \) cannot exist. A very simple proof of this was communicated to us by Christian Boyer. It goes as follows: Let \( M \) be a 3-magic square of order \( m \). Then its magic sum \( S_1 \) and the magic sum \( S_3 \) of \( M^{*3} \) are equivalent modulo 2, for every integer is equivalent to its cube modulo 2. On the other hand, it is well known that \( S_1 = m(m^2 + 1)/2 \) and \( S_3 = mS_1^2 \). Thus if \( m \equiv 2 \pmod{4} \), then \( S_3 \) is even, whereas \( S_1 \) is odd!

**Theorem 4.1.** Let \( M \) be an \( n \)-multimagic square of order \( m \). If the prime number \( p \) divides \( m \) with multiplicity \( e \geq 1 \), then \( n \leq p^{e+1} - 2 \).

**Proof.** Consider the polynomial

\[
f(x) = \binom{x-1}{n}
\]

in \( \mathbb{Q}[x] \). Then \( f \) has degree \( n \). Since \( M \) is \( n \)-multimagic it follows that the matrix \( f(M) \) obtained by applying \( f \) to each entry of \( M \) is a magic square consisting of (nonnegative) integers. In particular, the sum \( f(1) + f(2) + \cdots + f(m^2) \) is \( m \) times the magic sum. The sum

\[
\binom{0}{n} + \binom{1}{n} + \cdots + \binom{m^2-1}{n},
\]

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which is equal to \( \binom{m^2}{n+1} \), is thus divisible by \( m \). Hence the (rational) number

\[
q := \frac{m(m^2 - 1)(m^2 - 2) \ldots (m^2 - n)}{(n + 1)!}
\]

is an integer. Since an \( n \)-multimagic square of order \( m \) cannot exist if an \((n - 1)\)-multimagic square of order \( m \) cannot exist, it suffices to show that the existence of such a square in the case \( n = p^{e+1} - 1 \) leads to a contradiction. Let \( n = p^{e+1} - 1 \).

Note that \( q \) is nonzero since by hypothesis \( p^e \) divides \( m \), implying that \( m^2 \geq p^{2e} > p^{e+1} - 1 = n \) (\( e \) is positive). To obtain a contradiction we compute the multiplicity of \( p \) in \( q \). To this end, for any nonzero integer \( r \) we denote the multiplicity of \( p \) in \( r \) by \( v_p(r) \) and for any nonzero rational number \( r/s \) we put \( v_p(r/s) = v_p(r) - v_p(s) \). It is easy to verify that \( v_p(q_1q_2) = v_p(q_1) + v_p(q_2) \) for any pair of nonzero rational numbers \( q_1 \) and \( q_2 \). Observe that \( q \) is a product of \( m/(n + 1) \) and the factors \( (m^2 - i)/i \), for \( i = 1, \ldots, n \). It follows that \( v_p(q) = v_p(m/(n + 1)) \), for \( v_p((m^2 - i)/i) = 0 \) when \( 1 \leq i \leq n = p^{e+1} - 1 \). Recalling that \( n = p^{e+1} - 1 \), we obtain

\[
v_p(q) = v_p(m) - v_p(p^{e+1}) = e - (e + 1) = -1,
\]

which contradicts the fact that \( q \) is an integer.

\[\blacksquare\]

5. MULTIMAGIC CUBES AND HYPERCUBES. In this section we indicate how the method described earlier can be extended to construct multimagic cubes and hypercubes. Also, special subclasses of these magic cubes (hypercubes) can be constructed using the same method. We illustrate this for the class of “perfect” multimagic cubes.

We start with multimagic cubes. We remark at the outset that there is no consensus on the definition of multimagic cubes (or hypercubes or higher dimensional analogues) in the literature. The choice we make is the following (see [9] or [16]): an \( m \times m \times m \) cube of natural numbers (respectively, the consecutive numbers 1, 2, \ldots, \( m^2 \)) is called normal magic (respectively, normal magic) if the sum of all elements in each row, column, and pillar is the same and is equal to the sum of all elements of each of the four space diagonals. Furthermore, if \( n \geq 1 \), such a cube is called \( n \)-multimagic if the cube obtained by raising each of its entries to the \( d \)th power \( (1 \leq d \leq n) \) is magic.

The construction of an \( n \)-multimagic cube goes as follows: Analogously to the construction of \( n \)-multimagic squares in (2) and (3), we define a \( p^n \times p^n \times p^n \) cube \( M \) (where the prime number \( p \) has to be chosen appropriately) by the formula

\[
M = N_{3n} \circ F \circ (N_n^{-1} \times N_n^{-1} \times N_n^{-1}),
\]

where \( F \) is an affine map given by an invertible \( 3n \times 3n \) matrix \( A \) over \( k \) (the field of \( p \) elements) and a vector \( t \) in \( k^{3n} \) (i.e.,

\[
F((a, b, c)) = A \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t
\]

for \((a, b, c)\) in \((k^3)^3\). In order to make \( M \) \( n \)-multimagic we choose \( A \) to be a so-called \( n \)-multimagic 3-generator matrix. This means that \( A = (P|Q|R) \), where \( P, Q, \) and \( R \) are \( 3n \times n \) matrices over \( k \) that exhibit the following three properties:

1. \( A \) is invertible over \( k \) (which guarantees that all the natural numbers 1, 2, \ldots, \( p^{3n} \) appear as entries in \( M \));
2. all \( n \times n \) minors of the matrices \( P, Q, \) and \( R \) are units in \( k \) (which guarantees that when \( 1 \leq d \leq n \) the sum of all elements in each column, row, and pillar of \( M^{ed} \) is the same, hence equal to the magic sum);

3. all \( n \times n \) minors of the matrices \( P + Q + R, -P + Q + R, P - Q + R, \) and \( P + Q - R \) are units in \( k \) (which guarantees that when \( 1 \leq d \leq n \) the sum of all elements of each of the four space diagonals of \( M^{ed} \) is equal to the magic sum).

If we want to construct \( n \)-multimagic cubes having additional properties, we can achieve this by imposing extra conditions on the matrices \( P, Q, \) and \( R \).

To illustrate this approach we indicate how to construct so-called \( n \)-multimagic perfect cubes. A magic cube is perfect if the diagonals of each orthogonal slice have the magic sum property. Furthermore, such a cube is \( n \)-multimagic perfect if when \( 1 \leq d \leq n \) the cube obtained by raising each of its entries to the \( d \)th power is again perfect. To guarantee that an \( n \)-multimagic cube \( M \) arrived at via the foregoing construction is also \( n \)-multimagic perfect, we impose on the matrices \( P, Q, \) and \( R \) a fourth condition:

4. all \( n \times n \) minors of the matrices \( P + Q, P - Q, P + R, P - R, Q + R, \) and \( Q - R \) are units in \( k \).

At this point it should be clear to the reader how to proceed in higher dimensions (i.e., how to construct hypercubes and hypercubes with additional properties). It remains only to show how to select a suitable prime number \( p \) and how to construct an \( n \)-multimagic 3-generator matrix \( A \) or, more generally, how to construct for each \( d \) (\( \geq 2 \)) an \( n \)-multimagic \( d \)-generator matrix \( A \) (i.e., an invertible matrix \( A = (A_1|\cdots|A_d) \) over \( k \) such that each \( A_j \) is a \( dn \times n \) matrix with the property that for arbitrary \( \delta_1, \ldots, \delta_n \) in \( \{-1, 0, 1\} \), not all zero, all \( n \times n \) minors of each combination \( \delta_1A_1 + \cdots + \delta_nA_d \) are units in \( k \)). Before we indicate the construction of such matrices we recall the following well-known fact (which is easy to prove by induction on \( m \)):

If \( f \) is a nonzero polynomial in \( m \) variables over the integers, then there exist integers \( a_1, \ldots, a_m \) such that \( f(a_1, \ldots, a_m) \) is nonzero.

Now let \( n \geq 1 \) and \( d \geq 2 \) be given. To avoid complicating an easy matter we demonstrate only how to find \( n \)-multimagic \( d \)-generator matrices for the case \( d = 2 \). For other \( d \) the procedure is similar.

Let \( A_u = (A_{ij}) \) and \( B_u = (B_{ij}) \) be two universal \( 2n \times n \) matrices (i.e., the entries \( A_{ij} \) and \( B_{ij} \) are distinct variables). Then each \( n \times n \) minor of \( A_u, B_u, A_u + B_u, \) and \( A_u - B_u \) is a nonzero polynomial with integer coefficients in the \( 4n^2 \) variables \( A_{ij} \) and \( B_{ij} \). Let \( f \) denote the polynomial obtained by taking the product of all these minors and the polynomial \( \det(A_u|B_u) \). By (8) we can find integers \( a_{ij} \) and \( b_{ij} \) such that \( f(a_{ij}, b_{ij}) \) is a nonzero integer. Finally, let \( p \) be a prime number that does not divide this integer. It turns out that over \( k = \mathbb{Z}/p\mathbb{Z} \) the matrix \( A = (A_1|A_2) \), where \( A_1 = (a_{ij}) \) and \( A_2 = (b_{ij}) \), leads to an \( n \)-multimagic 2-generator matrix when the preceding construction is carried out.

6. GENERALIZATIONS. In the previous sections we described how to construct multimagic squares, three-dimensional cubes, and cubes of arbitrary dimension of order \( p^n \). A crucial ingredient was the field \( \mathbb{Z}/p\mathbb{Z} \) of \( p \) elements. In this section we indicate how this construction can be generalized to more general situations. More
precisely, for each integer \( q \) (\( \geq 2 \)) we show how to construct multimagic squares and their higher dimensional counterparts of order \( q^n \). Furthermore, our generalization even allows us to obtain more \( n \)-multimagic squares of order \( p^n \). The generalization involves two variations on the earlier theme: first, instead of a field of \( p \) elements we consider arbitrary commutative rings \( R \) having precisely \( q \) elements; second, in place of the bijection \( N : \mathbb{Z}/p\mathbb{Z} \to \{0, 1, \ldots, p-1\} \) we exploit more general bijections between \( R \) and \( \{0, 1, \ldots, q-1\} \). A key observation is that the bijection \( N \) has the property that \( N(a) + N(-a-1) = p-1 \) for all \( a \in \mathbb{Z}/p\mathbb{Z} \). In the general setting we consider bijections \( N : R \to \{0, 1, \ldots, q-1\} \) with the property that there exists an element \( c \) in \( R \) such that \( N(a) + N(-a+c) = q-1 \) for all \( a \) in \( R \). (In Lemma 6.1 we show that for each commutative ring \( R \) with \( q \) elements such a bijection exists!) For each positive integer \( m \) we obtain a bijection \( N_m : R^m \to \{1, 2, \ldots, q^n\} \):

\[
N_m((a_1, \ldots, a_m)) = 1 + \sum_{j=1}^{m} q^{j-1} N(a_j).
\]

Again, as in (1), we have a relation

\[
N_n(a) + N_n(-a+c') = q^n + 1
\]

for all \( a \) in \( R^n \), where \( c' = (c, c, \ldots, c) \) belongs to \( R^m \).

The proof of Theorem 2.1 can be copied completely, simply by replacing \( k \) with \( R \) and \( p \) with \( q \). Furthermore the construction of an \( n \)-multimagic generator matrix at the end of section 3 also works if we assume that all the elements \( 1, 2, \ldots, 2n-2 \) are units in \( R \). In case \( n = 2 \) we need to assume additionally that \( 3 \) is a unit in \( R \). Finally, for the construction of \( n \)-multimagic \( d \)-generator matrices we take for \( R \) the ring \( \mathbb{Z}/q\mathbb{Z} \) and repeat all the arguments at the end of section 5.

It remains only to indicate why for every commutative ring \( R \) having \( q \) elements there exists a bijection \( N : R \to \{0, 1, \ldots, q-1\} \) with the property that, for some \( c \) in \( R \), \( N(a) + N(-a+c) = q-1 \) holds for all \( a \) in \( R \). We call such a bijection a bijection of type \( c \).

**Lemma 6.1.** If \( 2 \) is a unit in \( R \), then for each \( c \) in \( R \) there exists a bijection \( N \) of type \( c \). If \( 2 \) is not a unit in \( R \), then for each unit \( c \) in \( R \) there exists a bijection \( N \) of type \( c \).

**Proof.** For \( c \) in \( R \) define \( \varphi = \varphi_c : R \to R \) by \( \varphi(a) = -a + c \). Then \( \varphi^2 \) is equal to the identity, so all orbits of \( \varphi \) have length one or two and an element \( a \) is a fixed point of \( \varphi \) if and only if \( 2a = c \). For \( a \) in \( R \) denote its orbit under \( \varphi \) by \( O(a) \).

(i) Let \( 2 \) be a unit in \( R \). Then \( \varphi \) has exactly one fixed point, namely, \( a_0 := 2^{-1}c \). The orbit \( O(a_0) \) of \( a_0 \) consists of one element. Let \( O(a_1), \ldots , O(a_s) \) be the other orbits. Each consists of two elements. In particular, \( q = 2s+1 \). Define \( N(a_0) = s, N(a_i) = i-1 \), and \( N(\varphi(a_i)) = q-1-N(a_i) (= q-i) \) for \( i = 1, 2, \ldots, s \). Then \( N \) is the desired bijection.

(ii) Let \( 2 \) be a nonunit in \( R \). Then for \( c \) in \( R^* \), the group of units in \( R \), the map \( \varphi = \varphi_c \) has no fixed point: indeed, if \( \varphi(a) = a \) for some \( a \), then \( 2a = c \). This implies that \( 2 \) is a unit in \( R \), a contradiction. Consequently, we can write \( R \) as the disjoint union of orbits \( O(a_1), \ldots , O(a_s) \), each of which consists of the two elements. In particular, \( q = 2s \). Finally define \( N(a_i) = i-1 \) and \( N(\varphi(a_i)) = q-1-N(a_i) (= q-i) \) for all \( i \). Again \( N \) is a bijection of type \( c \).
7. EXAMPLES To conclude this paper we describe some examples of bimagic (i.e., 2-multimagic) squares obtained by the construction in section 3 and its generalizations in section 6. Recall that a magic square of order \( m \) is associative if the sum of any pair of its entries that are symmetric with respect to the center of the square is \( m^2 + 1 \).

Example 1 (A family of associative bimagic squares of order 16). Take

\[ R = \mathbb{F}_2[x]/(x^2 + x + 1) \]

and the 2-multimagic generator matrix

\[
A = \begin{pmatrix}
x & 0 & 1 & x \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
x & 1 & 0 & x
\end{pmatrix}.
\]

Let \( N : R \to \{0, 1, 2, 3\} \) be the bijection given by \( N(0) = 0, N(1) = 2, N(x) = 1, \) and \( N(x + 1) = 3 \). Then \( N \) is of type \( x + 1 \). For each \( t \in R^4 \) we obtain an associative bimagic square of order 16. Taking \( t = (0, 1, 1, 0)^T \) yields the following example from this family:

\[
\begin{array}{cccccccccccccccc}
41 & 252 & 74 & 155 & 125 & 176 & 30 & 207 & 129 & 84 & 226 & 51 & 213 & 8 & 182 & 103 \\
62 & 239 & 93 & 144 & 106 & 187 & 9 & 220 & 150 & 71 & 245 & 40 & 194 & 19 & 161 & 116 \\
3 & 210 & 100 & 177 & 87 & 134 & 56 & 229 & 171 & 122 & 204 & 25 & 255 & 46 & 160 & 77 \\
24 & 197 & 119 & 166 & 68 & 145 & 35 & 242 & 192 & 109 & 223 & 14 & 236 & 57 & 139 & 90 \\
240 & 61 & 143 & 94 & 188 & 105 & 219 & 10 & 72 & 149 & 39 & 246 & 20 & 193 & 115 & 162 \\
251 & 42 & 156 & 73 & 175 & 126 & 208 & 29 & 83 & 130 & 52 & 225 & 7 & 214 & 104 & 181 \\
198 & 23 & 165 & 120 & 146 & 67 & 241 & 36 & 110 & 191 & 13 & 224 & 58 & 235 & 89 & 140 \\
209 & 4 & 178 & 99 & 133 & 88 & 230 & 55 & 121 & 172 & 26 & 203 & 45 & 256 & 78 & 159 \\
98 & 179 & 1 & 212 & 54 & 231 & 85 & 136 & 202 & 27 & 169 & 124 & 158 & 79 & 253 & 48 \\
117 & 168 & 22 & 199 & 33 & 244 & 66 & 147 & 221 & 16 & 190 & 111 & 137 & 92 & 234 & 59 \\
76 & 153 & 43 & 250 & 32 & 205 & 127 & 174 & 228 & 49 & 131 & 82 & 184 & 101 & 215 & 6 \\
95 & 142 & 64 & 237 & 11 & 218 & 108 & 185 & 247 & 38 & 152 & 69 & 163 & 114 & 196 & 17 \\
167 & 118 & 200 & 21 & 243 & 34 & 148 & 65 & 15 & 222 & 112 & 189 & 91 & 138 & 60 & 233 \\
180 & 97 & 211 & 2 & 232 & 53 & 135 & 86 & 28 & 201 & 123 & 170 & 80 & 157 & 47 & 254 \\
141 & 96 & 238 & 63 & 217 & 12 & 186 & 107 & 37 & 248 & 70 & 151 & 113 & 164 & 18 & 195 \\
154 & 75 & 249 & 44 & 206 & 31 & 173 & 128 & 50 & 227 & 81 & 132 & 102 & 183 & 5 & 216
\end{array}
\]

Example 2 (A family of bimagic squares of odd order). Take \( R = \mathbb{Z}/q\mathbb{Z}, \) where \( q \geq 3 \) and \( q \) is odd,

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 \\
2 & 0 & 0 & 1 \\
1 & 1 & 2 & 1 \\
2 & 1 & 2 & 2
\end{pmatrix},
\]

and \( t \) an arbitrary member of \( R^4 \). Let \( N : R \to \{0, 1, \ldots, q - 1\} \) be the standard bijection: \( N(i) = i \ (0 \leq i \leq q - 1) \). Then \( N \) is of type \(-1\). Since \( A \) is a 2-multimagic generator matrix, our construction gives a family of bimagic squares of odd order. If we choose \( q = 3 \) and \( t \) the zero vector, we find the following square:
Furthermore, if we take \( q = 3 \) and \( t = (2, 1, 0, 2)^T \) we recover the associative bimagic square of order nine constructed by R. V. Heath in 1933 (see [3, p. 212]).

Before we turn to our last example, we recall that a magic square is called pandiagonal if also the sum of the elements on each broken diagonal is equal to the magic sum.

Example 3 (An associative, pandiagonal, bimagic, magic square of order 25). Take \( R = \mathbb{F}_5 \), let \( N : R \to \{0, 1, 2, 3, 4\} \) be the standard bijection (of type \(-1\)), and set

\[
X = \begin{pmatrix}
1 & 1 & 2 & 2 & 1 \\
1 & 2 & 2 & 4 & 1 \\
1 & 3 & 3 & 3 & 1 \\
1 & 4 & 3 & 2 & 1
\end{pmatrix}, \quad t = \begin{pmatrix}
0 \\
4 \\
0 \\
2
\end{pmatrix}.
\]

Then the corresponding \( 25 \times 25 \) matrix is associative, pandiagonal, and bimagic, and it has the following properties:

(i) each of the twenty-five standard \( 5 \times 5 \) submatrices is pandiagonal (with the same magic sum);

(ii) for each pair \((i, j)\) \((1 \leq i, j \leq 25)\) the \( 5 \times 5 \) matrix obtained by deleting each row with row number not equivalent to \( i \) modulo 5 and each column with column number not equivalent to \( j \) modulo 5 is pandiagonal!

\[
\begin{pmatrix}
103 & 350 & 567 & 164 & 381 \\
291 & 513 & 235 & 452 & 74 \\
584 & 176 & 423 & 20 & 362 \\
147 & 494 & 86 & 308 & 530 \\
440 & 32 & 254 & 621 & 218
\end{pmatrix}
\]

\[
\begin{pmatrix}
584 & 176 & 423 & 20 & 362 \\
147 & 494 & 86 & 308 & 530 \\
440 & 32 & 254 & 621 & 218 \\
103 & 350 & 567 & 164 & 381 \\
134 & 476 & 98 & 320 & 537
\end{pmatrix}
\]
More research into different properties and various examples can be found in the thesis of the second author [6]. The reader is also referred to the website [7].

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