Entanglement measure for multipartite pure states and its numerical calculation

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The quantification and classification of quantum entanglement is a very important and still open question of quantum information theory. In this paper, we describe an entanglement measure for multipartite pure states (the minimum of Shannon’s entropy of orthogonal measurements). This measure is additive, monotone under LOCC, and coincides with the reduced von Neumann entropy on bipartite states. A method for numerical calculation of this measure by genetic algorithms is also presented. Moreover, the minimization of entropy technique is extended to fermionic states.

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I. INTRODUCTION

Multipartite quantum entanglement plays significant role in whole quantum science. For example, one can easily obtain that we can’t achieve algorithmic speedup in quantum computer without quantum entanglement. Also entanglement is needed for other quantum protocols: quantum teleportation, quantum error correction, etc. What is more, the theory of quantum entanglement may help us to understand multiparticle quantum physics deeply.

It’s well known that the entanglement of bipartite pure states is fully described. Schmidt coefficients uniquely define a local unitary orbit of a bipartite quantum state $|\psi\rangle$ and unambiguously determine the class of states that can be obtained from $|\psi\rangle$ by LOCC [1].

Schmidt decomposition

$$\sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B = \sum_i \sqrt{\lambda_i} |i\rangle_A \otimes |i\rangle_B$$

is an equivalent of matrix SVD decomposition

$$\sum_{i,j} c_{ij} |i\rangle_A \langle j|_B = \sum_i \sqrt{\lambda_i} |i\rangle_A \langle i|_B,$$

$$A = USV^*.$$

It’s easy to see that there is no Schmidt decomposition even for three qubits. The $W$ state

$$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

cannot be represented as

$$\lambda_0 |\tilde{0}\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle + \lambda_1 |\tilde{1}\rangle \otimes |\tilde{1}\rangle \otimes |\tilde{1}\rangle.$$
A stronger counterexample exists for SVD decomposition: there is no lower-rank orthogonal approximation even for third-order tensors [2].

While a lot of applications use SVD: image compression (for example, [3]), LSI (Latent Semantic Indexing, for example, [4]), statistics (for example, [5]), etc., the appropriate expanding of SVD decomposition (Schmidt decomposition) to higher order tensors (to more than two subsystems) is an important task not only for quantum information theory but for fundamental (tensor algebra) and applied (statistics, machine learning, etc.) math. Some higher order analogues of SVD may be found (for example, [6]), but none of them can be directly applied to the quantum pure state case.

Whereas there is no full theory of multipartite quantum entanglement, some important results are known.

A full classification of pure states of three qubits in terms of SLOCC (stochastic LOCC) is given in [7]. A description of quantum entanglement with nilpotent polynomials is constructed in [8]. Some entanglement measures of pure and mixed states can be found, for example, in [9, 10, 11, 12, 13, 14]. Theory of quantum entanglement measures is studied in, for example, [15, 16, 17, 18, 19].

The paper is organized as follows: In the first section of this paper we present the minimum of Shannon’s entropy of orthogonal measurements and prove that this function possesses all necessary pure state entanglement measure properties. Some other features of this measure and its values for some states are also presented. Next we describe a method to compute this measure using genetic algorithms. An extension of minimal measurements entropy to fermionic states is described in the last part of the paper.

A. Notation

$H_{sh}$ is the Shannon entropy,
$H_{vN}$ is the von Neumann entropy.

II. SHANNON’S ENTROPY OF ORTHOGONAL MEASUREMENTS
MINIMUM

Consider a pure $n$-qudit quantum state

$$|\psi\rangle = \sum_{i_1,i_2,...,i_n=1}^d a_{i_1,i_2,...,i_n} |i_1i_2...i_n\rangle.$$

(1)

The measurement of this state in the computational basis gives states $|i_1i_2...i_n\rangle$ with probabilities $|a_{i_1,i_2,...,i_n}|^2$. So we can regard this measurement process as a signal generator. We shall say that Shannon’s entropy of this generator is called measurements entropy of the state $|\psi\rangle$:

$$H_{meas}(|\psi\rangle) = H_{sh}(Diag(|\psi\rangle\langle\psi|)) = \sum_{i_1,i_2,...,i_n=1}^d |a_{i_1,i_2,...,i_n}|^2.$$

But this characteristic of a quantum state is not invariant under local changes of the measurement basis. Shannon’s source coding theorem shows that, in the limit, the average length of the shortest possible representation to encode the messages in a given alphabet
is their entropy divided by the logarithm of the number of symbols in the target alphabet. That’s why the natural way to construct an invariant is the minimization of \( H_{\text{meas}}(\psi) \) over all possible local changes of the measurements basis:

\[
E_{H_{\text{min}}}(\psi) = \min_{U_1, U_2, \ldots, U_n} H_{\text{meas}}(U_1 \otimes U_2 \otimes \ldots \otimes U_n \psi);
\]

and \( E_{H_{\text{min}}} \) is a measure of entanglement of the pure state \( |\psi\rangle \).

Remark II.1. The definition of \( E_{H_{\text{min}}} \) for subsystems with different dimensions is the same to (2). As non of the following reasoning bears on the equality of subsystem dimensions, here and further the common \( d \) for different \( i, j \) in (1) is used only for notation simplicity.

A. Necessary properties of entanglement measure

Entanglement measure for pure quantum states must satisfy the following intuitive conditions:

(i) must be equal to zero for fully unentangled states
(ii) must be invariant under local unitary operations
(iii) must be invariant under attachment and detachment of unentangled ancilla
(iv) must not increase under LOCC.

Let’s show that \( E_{H_{\text{min}}} \) satisfies (i)-(iv). First of all we consider another one important property:

Property II.2. (Additivity of \( E_{H_{\text{min}}} \))

Let \( |\psi_1\rangle \) and \( |\psi_2\rangle \) be multipartite (may be entangled) pure states, then

\[
E_{H_{\text{min}}}(|\psi_1\rangle \otimes |\psi_2\rangle) = E_{H_{\text{min}}}(|\psi_1\rangle) + E_{H_{\text{min}}}(|\psi_2\rangle).
\]

The proof is straightforward from the definition of \( E_{H_{\text{min}}} \) and additivity of Shannon’s entropy.

The properties (i)-(ii) are trivial and (iii) is the consequence of \( E_{H_{\text{min}}} \) additivity. To prove (iv), since (ii) and (iii) are correct, we may only prove that \( E_{H_{\text{min}}} \) doesn’t increase under orthogonal measurements in the mean. To show monotonicity under orthogonal measurements in the mean we need some lemmas.

Lemma II.3. Let \( \rho_{AB} \) be a density matrix of a pure bipartite state. Then

\[
H_{\text{sh}}(\text{Diag}(\rho_{AB})) \leq H_{\text{sh}}(\text{Diag}(\rho_A)) \leq H_{\text{sh}}(\text{Diag}(\rho_A)) + H_{\text{sh}}(\text{Diag}(\rho_B)),
\]

where \( \rho_A \) and \( \rho_B \) are reduced density matrices.

Proof. The left inequality follows from generalized grouping of Shannon’s entropy:

\[
H_{\text{sh}}(p_1, \ldots, p_{\sigma_1}, p_{\sigma_1+1}, \ldots, p_{\sigma_2}, \ldots, p_{\sigma_{n-1}+1}, \ldots, p_{\sigma_n})
= H_{\text{sh}}(p_1 + \ldots + p_{\sigma_1}, p_{\sigma_1+1} + \ldots + p_{\sigma_2}, \ldots, p_{\sigma_{n-1}+1} + \ldots + p_{\sigma_n})
+ \sum_{i=1}^{n} (p_{\sigma_i+1} + \ldots + p_{\sigma_i}) H_{\text{sh}}(p_{\sigma_i+1}/\sum_{j=\sigma_i+1}^{\sigma_i} p_j, \ldots, p_{\sigma_i}/\sum_{j=\sigma_i+1}^{\sigma_i} p_j).
\]
The right part follows the sub-additivity of entropy:

\[ H_{sh}(v_{11}, v_{12}, ..., v_{1m}, v_{21}, ..., v_{2m}, ..., v_{n1}, ..., v_{nm}) \leq H_{sh}\left( \sum_{i=1}^{n} v_{i1}, \sum_{i=1}^{n} v_{i2}, ..., \sum_{i=1}^{n} v_{im} \right) + H_{sh}\left( \sum_{j=1}^{m} v_{1j}, \sum_{j=1}^{m} v_{2j}, ..., \sum_{j=1}^{m} v_{nj} \right). \]

Lemma II.4. Let

\[ |\psi\rangle = \sum_{i_1, i_2, ..., i_n=1}^{d} a_{i_1, i_2, ..., i_n} |i_1 i_2 \ldots i_n\rangle \]

be a pure \( n \)-qudit state. The states \( |j\rangle \otimes |\psi_j\rangle \) are results of the measurement of \( |\psi\rangle \) in computational basis with probabilities \( p_j \). Then

\[ H_{meas}(U_1|\psi\rangle) \geq \sum_{j=1}^{d} p_j H_{meas}(|j\rangle \otimes |\psi_j\rangle), \quad (3) \]

where \( U_1 \) is an arbitrary unitary transformation of the first qudit.

Proof. If \( |j\rangle \otimes |\psi_j\rangle \) are results of \( |\psi\rangle \) measurement in computational basis with probabilities \( p_j \), then \( |\psi\rangle \) can be represented in the view

\[ |\psi\rangle = \sum_{j=1}^{d} c_j |j\rangle \otimes |\psi_j\rangle, \]

where

\[ |c_j|^2 = p_j. \]

Let \( \psi_j^k, k=1,\ldots,d^{n-1} \) be the amplitudes of \( |\psi_j\rangle \), then

\[ H_{meas}(|\psi\rangle) = H_{meas}\left( \sum_{j=1}^{d} c_j |j\rangle \otimes |\psi_j\rangle \right) = \]

\[ = H_{sh}(|c_1|^2|\psi_1|^2, \ldots, |c_1|^2|\psi_1^{d-1}|^2, \ldots, |c_d|^2|\psi_d|^2, \ldots, |c_d|^2|\psi_d^{d-1}|^2) \]

\[ = \{ \text{by the strong additivity of Shannon’s entropy} \} \]

\[ = \sum_{j=1}^{d} p_j H_{sh}(|\psi_j|^2, |\psi_j^2|^2, \ldots, |\psi_j^{d-1}|^2) + H_{sh}(p_1, p_2, \ldots, p_d) \]

\[ = \sum_{j=1}^{d} p_j H_{meas}(|j\rangle \otimes |\psi_j\rangle) + H_{sh}(p). \]

Using this we can rewrite (3) in the equivalent form

\[ H_{meas}(|\psi\rangle) \leq H_{meas}(U_1|\psi\rangle) + H_{sh}(p). \quad (4) \]
Let \( \rho = |\psi\rangle \langle \psi | \) be the density matrix of the state \( |\psi\rangle \), \( \rho_1 = Tr_{2,3,...,d} (\rho) \), \( \rho_2 = Tr_1 (\rho) \). Then by Lemma II.3 for \( \rho \) we get
\[
H_{sh}(\text{Diag}(\rho)) \leq H_{sh}(\text{Diag}(\rho_2)) + H_{sh}(\text{Diag}(\rho_1)),
\]
or equivalently
\[
H_{\text{meas}}(|\psi\rangle) \leq H_{sh}(\text{Diag}(\rho_2)) + H_{sh}(\text{Diag}(\rho_1)). \tag{5}
\]

Let \( \rho_{U_1} = U_1 |\psi\rangle \langle \psi | U_1^* \) be the density matrix of the state \( U_1 |\psi\rangle \), \( \rho_{U_1}^2 = Tr_1 (\rho_{U_1}) \). Since \( U_1 \) affects the first qudit only, \( \rho_{U_1}^2 = \rho_2 \).

By Lemma II.3
\[
H_{sh}(\text{Diag}(\rho_2)) = H_{sh}(\text{Diag}(\rho_{U_1}^2)) \leq H_{sh}(\text{Diag}(\rho_{U_1})). \tag{6}
\]

By (5) and (6) it follows that (4) is correct, consequently (3) is correct too. \( \square \)

**Theorem II.5.** (the monotonicity of \( E_{H_{\text{min}}} \) under orthogonal measurements)
Let \( |\psi\rangle \) be an \( n \)-qudit state. The states \( |\psi_j\rangle \) are results of some orthogonal measurement with probabilities \( p_j \). Then
\[
E_{H_{\text{min}}}(|\psi\rangle) \geq \sum_j p_j E_{H_{\text{min}}}(|\psi_j\rangle).
\]

**Proof.** We may assume, without loss of generality, that we measure first qudit in the basis \( \{\psi^1_1\} \) (a measurement of more than one qudit can be replaced by sequential one-qudit measurements). Then \( |\psi_j\rangle = |\psi^1_1\rangle \otimes |\psi^2_j\rangle \). Let \( |\psi_{\text{min}}\rangle \) has minimal measurements entropy over local unitary orbit of \( |\psi\rangle \), and \( |\psi_{\text{min}}\rangle = U_1 \otimes U_2 \otimes \ldots \otimes U_n |\psi\rangle \). Then
\[
\sum_j p_j E_{H_{\text{min}}}(|\psi_j\rangle) = \sum_j p_j E_{H_{\text{min}}}(|\psi^1_1\rangle \otimes |\psi^2_j\rangle) \leq \sum_j p_j H_{\text{meas}}(|j\rangle \otimes (U_2 \otimes \ldots \otimes U_n)|\psi_j\rangle)
\]
\[
\leq \{ \text{by Lemma II.4} \} \leq H_{\text{meas}}(\psi_{\text{min}}) = E_{H_{\text{min}}}(|\psi\rangle). \tag{iv}
\]

Thus, from (iv) we get (iv). \( \square \)

**B. Other properties**

**Lemma II.6.** (Klein’s lemma)
Let \( \rho \) be a density matrix, then
\[
H_{sh}(\text{Diag}(\rho)) \geq H_{vN}(\rho).
\]

**Theorem II.7.** \( E_{H_{\text{min}}} \) coincides with the reduced von Neumann entropy for bipartite states. I.e., let \( |\psi\rangle \) be a pure bipartite state; \( \rho = |\psi\rangle \langle \psi | \) is its density matrix, \( \rho_A = Tr_B (\rho) \) and \( \rho_B = Tr_A (\rho) \) are reduced density matrices of subsystems. Then \( E_{H_{\text{min}}}(|\psi\rangle) = H_{vN}(\rho_A) = H_{vN}(\rho_B) \).
Proof. Consider the Schmidt decomposition of $|\psi\rangle$:

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle.$$ 

Then

$$E_{H_{\min}}(|\psi\rangle) = \min_{U_A, U_B} H_{\text{meas}}(U_A \otimes U_B |\psi\rangle) \leq H_{sh}(\lambda_i) = H_{vN}(\rho_A) = H_{vN}(\rho_B).$$

From Lemma II.3 we get

$$H_{\text{meas}}(|\psi\rangle) = H_{sh}(\text{Diag}(\rho)) \geq H_{sh}(\text{Diag}(\rho_A)).$$

Further, by Klein’s lemma we have

$$H_{sh}(\text{Diag}(\rho_A)) \geq H_{vN}(\rho_A) = H_{vN}(\rho_B).$$

Thus

$$E_{H_{\min}}(|\psi\rangle) = H_{vN}(\rho_A) = H_{vN}(\rho_B).$$

Property II.8. ($E_{H_{\min}}$ of generalized GHZ states) Consider a generalized GHZ-state:

$$|GHZ\rangle = \sum_{i=1}^d a_i |i\rangle_1 \otimes |\tilde{i}\rangle_2 \otimes \ldots \otimes |\tilde{i}\rangle_n,$$

where $a_i \in \mathbb{R}$, $\sum_{i=1}^d |a_i|^2 = 1$.

Then

$$H_{\text{meas}}(|GHZ\rangle) = \min_{U_1, U_2, \ldots, U_n} H_{\text{meas}}(U_1 \otimes U_2 \otimes \ldots \otimes U_n |GHZ\rangle).$$

Proof. We can consider the space of GHZ state as bipartite space of the first qudit and the remaining qudits. Then

$$|GHZ\rangle = \sum_{i=1}^d a_i |i\rangle_1 \otimes |\tilde{i}\rangle.$$ 

From Theorem II.7 and Remark II.1 we have that Schmidt decomposition has minimal measurements entropy. From this we get

$$H_{\text{meas}}(|GHZ\rangle) = H_{\text{meas}}(\sum_{i=1}^d a_i |i\rangle_1 \otimes |\tilde{i}\rangle) \leq H_{\text{meas}}(U_1 \otimes \tilde{U} |GHZ\rangle),$$

where $U_1$ is a unitary transformation of the first qudit and $\tilde{U}$ is an arbitrary unitary transformation of $2, 3, \ldots, n$ qudits. We can take $U_2 \otimes U_3 \otimes \ldots \otimes U_n$ as $\tilde{U}$, then

$$H_{\text{meas}}(|GHZ\rangle) \leq H_{\text{meas}}(U_1 \otimes U_2 \otimes \ldots \otimes U_n |GHZ\rangle),$$

this finishes the proof.

So we have $E_{H_{\min}}(|GHZ\rangle) = - \sum_{i=1}^d |a_i|^2 \ln |a_i|^2.$
C. Numerical properties

The following properties were obtained numerically by genetic algorithms.

**Numerical Result II.9.** \((E_{H_{\text{min}}} \text{ of generalized W states})\)

Consider a generalized W state

\[ |W\rangle = a_1|0\ldots01\rangle + a_2|0\ldots10\rangle + a_n|1\ldots00\rangle, \]

where \(\sum_{i=1}^{n}|a_i|^2 = 1\). Then

\[ E_{H_{\text{min}}}(W) = H_{\text{meas}}(W) = \sum_{i=1}^{n}|a_i|^2 \ln |a_i|^2. \]

**Numerical Result II.10.** Let

\[ |\psi\rangle = \sum_{i_1,i_2,\ldots,i_n=1}^{d} a_{i_1,i_2,\ldots,i_n}|i_1i_2\ldots i_n\rangle, \]

\[ |\varphi\rangle = \sum_{i_1,i_2,\ldots,i_n=1}^{d} b_{i_1,i_2,\ldots,i_n}|i_1i_2\ldots i_n\rangle, \]

and \(H_{\text{meas}}(|\psi\rangle) = H_{\text{meas}}(|\varphi\rangle) = E_{H_{\text{min}}}(|\psi\rangle) = E_{H_{\text{min}}}(|\varphi\rangle)\) (in other words \(|\psi\rangle\) and \(|\varphi\rangle\) have equal \(E_{H_{\text{min}}}\) and they are in minimal entropy representation), then

\[ |a_{i_1,i_2,\ldots,i_n}|^2 = |b_{i_1,i_2,\ldots,i_n}|^2 \]

to within local permutations of basis vectors.

I.e. modulus squares of a minimal entropy representation of the local unitary orbit are unique.

D. \(E_{H_{\text{min}}}\) is substantially multipartite

Numerical computations using genetic algorithms show that two equivalent under bipartite entanglement states of three qubits \(|\varphi\rangle\) and \(|\psi\rangle\) may have different \(E_{H_{\text{min}}}\). This means that \(E_{H_{\text{min}}}\) is substantially multipartite. Equivalence under bipartite entanglement means

\[ \exists U_1^1, U_2^1, U_3^1, U_2^2, U_3^2, U_3^3, U_{12}^3 : \]

\[ |\varphi\rangle = U_1^1 \otimes U_2^1 \otimes U_3^1 |\psi\rangle, \]

\[ |\varphi\rangle = U_2^2 \otimes U_3^2 |\psi\rangle, \]

\[ |\varphi\rangle = U_3^3 \otimes U_1^3 |\psi\rangle, \]

where \(U_i^k\) are unitary evolutions of \(i\)-th qubit, and \(U_{ij}^k\) are unitary evolutions (may be entangled) of \(i\)-th and \(j\)-th qubit.

From \(E_{H_{\text{min}}}(|\psi\rangle) \neq E_{H_{\text{min}}}(|\varphi\rangle)\) we have that \(|\varphi\rangle\) and \(|\psi\rangle\) are not equivalent under local unitary transformations, i.e

\[ \exists U_1, U_2, U_3 : |\varphi\rangle = U_1 \otimes U_2 \otimes U_3 |\psi\rangle. \]
This property of $E_{H\text{min}}$ gives an answer to the problem of equivalence of bipartite and multipartite entanglement proposed in [20]: the bipartite and multipartite entanglements are not equivalent. This also means that entanglement measures based only on Schmidt coefficients of different decompositions of a state are not good for quantifying multipartite entanglement.

III. CALCULATION OF $E_{H\text{min}}$ USING GENETIC ALGORITHMS

A. Formalization of the optimization problem

Consider the $n$-qudit state

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_n = 1}^d a_{i_1, i_2, \ldots, i_n} |i_1 i_2 \ldots i_n\rangle.$$

Then

$$E_{H\text{min}}(|\psi\rangle) = \min_{U_1, U_2, \ldots, U_n} H_{\text{meas}}(U_1 \otimes U_2 \otimes \ldots \otimes U_n |\psi\rangle),$$

where $U_i$ is a $d$-dimension unitary operator on $i$-th qudit,

$$H_{\text{meas}}(|\psi\rangle) = \sum_{i_1, i_2, \ldots, i_n = 1}^d |a_{i_1, i_2, \ldots, i_n}|^2 \ln |a_{i_1, i_2, \ldots, i_n}|^2.$$

To solve this optimization problem we need to parameterize unitary matrices $U_i$. When $d = 2$ this parametrization is well-known:

$$U_i(\beta_i, \delta_i, \gamma_i) = \begin{pmatrix} e^{i(-\beta_i-\delta_i)} \cos \gamma_i & -e^{i(-\beta_i+\delta_i)} \sin \gamma_i \\ e^{i(\beta_i-\delta_i)} \sin \gamma_i & e^{i(\beta_i+\delta_i)} \cos \gamma_i \end{pmatrix},$$

where $\beta_i, \delta_i, \gamma_i$ are real numbers.

Parametrization of unitary matrices for $d > 2$ is a still open and interesting question. The best known parametrization is proposed in [21, 22], but it is very slow, so we have chosen another one. Let’s parameterize hermitian matrix $H$ by $d^2$ real numbers (the diagonal has $d$ real numbers, and $d(d-1)/2$ complex numbers above the diagonal need $d(d-1)$ real numbers to be parameterized). Then we take $U = e^{iH}$ as a unitary matrix.

In such a way the calculation of $E_{H\text{min}}$ is an optimization task of $3n$ real parameters for qubits and $nd^2$ real parameters for qudits ($d > 2$). To calculate $E_{H\text{min}}(|\psi\rangle)$ we need to minimize the function

$$f^\psi(x_1, \ldots, x_{n,k}) = H_{\text{meas}}(U(x_1, \ldots, x_k) \otimes U(x_{k+1}, \ldots, x_{2k}) \otimes \ldots \otimes U(x_{(n-1)k+1}, \ldots, x_{nk}) |\psi\rangle),$$

where $U(x_1, \ldots, x_k)$ is a parametrization of $d \times d$ unitary matrix by $k$ real parameters.

In the general case, functions $f^\psi(x_1, \ldots, x_{n,k})$ may be multimodal (have many local minimums), because of this fact we can’t use gradient-based optimization methods. Thus genetic algorithm has been chosen.
B. Genetic algorithm (GA)

There are a lot of publications about GA, but for a brief overview and references we recommend Wikipedia [23].

GA has been already used for quantum entanglement calculation (to calculate relative entropy of entanglement of mixed bipartite states) in [24].

Now we describe GA that was used for $E_{H_{min}}$ calculations.

Every parameter $x$ of $f$ is being encoded by $n_{gen}$ real value genes $g_j, j = 1, n_{gen}$ by the rule $x = \sum_{j=1}^{n_{gen}} 10^{1-j}g_j$. So, if we have $k$ parameters, then a chromosome is a vector $\{g_i\}$ of length $n_{gen} \cdot k$. (A full tuple of parameters of $f$ is encoded by a chromosome.)

A mutation of a chromosome $\{g_i\}$ is determined by the following probabilities:

$$P(g_i^m = g_i) = (1 - p_{mut}), \quad P(g_i^m = g_i + \xi) = p_{mut},$$

where $\{g_i^m\}$ is a chromosome after mutation, $p_{mut}$ is a mutation probability, $\xi$ is a random variable that has uniform distribution on $[-m_{mut}, m_{mut}]$.

Crossover of chromosomes $\{g_i^1\}$ and $\{g_i^2\}$ is a chromosome $\{g_i^r\}$, where

$$P(g_i^r = g_i^1) = P(g_i^r = g_i^2) = \frac{1}{2}.$$ 

A fitness function is $-f$.

The algorithm.

1. Initialize the first population of $n_{population}$ random chromosomes with uniformly distributed on $[-m_{init}, m_{init}]$ genes.

2. Wait an epoch (this step will be described further). After the epoch we have a new population.

3. If one of termination conditions is satisfied, algorithm stops and the fitness of the best chromosome of the last population is taken as a result, else we repeat from Step 2.

Termination conditions:

We can fix the maximal number of epochs ($n_{epochs}$), the precision $\varepsilon$, and the maximal number of nonchanging epoches $n_{term}$. Then termination conditions will be:

– We reach $n_{epochs}$ epoch.
– Examine the best chromosome from each of the last $n_{term}$ epoches. If the fitness function values of these chromosomes differ from each other by less than $\varepsilon$, then this termination condition is true.
The epoch.

1. Two random pairs of chromosomes are chosen from the best \( n_{\text{population}} - n_{\text{bad}} \) chromosomes of the population, \( n_{\text{bad}} \) is a number of the weakest chromosomes of the population, which cannot be used for reproduction. From each pair we take a chromosome with the best fitness. After that we put a crossover result of the two selected chromosomes to the new population.

2. Repeat Step 1 while the new population is lesser than \( n_{\text{population}} \).

3. Mutate every chromosome of the new population.

While \( f^\psi \) for |\( \psi \rangle \) from local orbits of "easy" states like GHZ or W has no "difficult" local minimums, the situation is reverse for states with random amplitudes. For example, one 7-qubit state has a local minimum 4.0220, while another one (probably global) minimum is 3.968. To solve this problem we use GA with "islands": we form \( n_{\text{islands}} \) islands with rather small equal populations. Epoches at these islands are independent and only infrequent migrations are allowed. The increase of islands count is an equivalent of the the whole GA repeating, so the probability of an error decreases exponentially with the number of islands. Increase of the probability of migration speedups the convergence, but it also increases the probability of local minimums results.

Remark (about software implementation)

One of the advantages of GA is its parallelism, so the software realization of \( E_{H_{\text{min}}} \) calculation was multithreading. Moreover, the implementation of \( E_{H_{\text{min}}} \) calculation for qubits was realized using nVidia CUDA [25] technology. Using cheap personal GPU, 7x speedup against the best 4-core CPU implementation has been achieved. Software complex can easily calculate \( E_{H_{\text{min}}} \) for up to 17 qubits inclusively.

IV. \( E_{H_{\text{min}}} \) FOR FERMIONIC STATES

Consider a pure state of \( n \) fermions in \( p \)-dimensional Hilbert space with basis \( f_1, f_2, \ldots, f_n \):

\[
|f\rangle = \sum_{i_1, i_2, \ldots, i_n = 1, i_1 < i_2 < \ldots < i_n} \lambda_{i_1 i_2 \ldots i_n} |i_1 i_2 \ldots i_n\rangle,
\]

where

\[
\sum_{i_1, i_2, \ldots, i_n = 1, i_1 < i_2 < \ldots < i_n} |\lambda_{i_1 i_2 \ldots i_n}|^2 = 1,
\]

\[
|i_1 i_2 \ldots i_n\rangle = \frac{1}{\sqrt{n!}} \begin{vmatrix} f_{i_1}(r_1) & \cdots & f_{i_1}(r_n) \\ \vdots & \ddots & \vdots \\ f_{i_n}(r_1) & \cdots & f_{i_n}(r_n) \end{vmatrix}
\]

are Slater determinants.

This states correspond to normalized elements of the exterior algebra \( \Lambda^n \mathbb{C}^k \). Let’s take states that correspond to separable elements \( x_1 \wedge x_2 \wedge \ldots \wedge x_n \) of \( \Lambda^n \mathbb{C}^k \) as unentangled, i.e. the states that are represented as a single Slater determinant in some basis of one-particle
space are unentangled. Thus, we can take unitary changes of basis as unentangling transforms.

This is more mathematical than physical approach. For example, if we have two electrons in two quantum dots their one-particle space must at the least include spin and position coordinates. In the simplest case the whole space will be $H_{\text{spin}} \otimes H_{\text{position}}$ and its basis will be $|\uparrow\rangle \otimes |1\rangle$, $|\uparrow\rangle \otimes |2\rangle$, $|\downarrow\rangle \otimes |1\rangle$, $|\downarrow\rangle \otimes |2\rangle$. As we can see, unitary transformations of this basis are “entangled” in an intuitive physical way. So, the construction of a physical formalism of unentangled local transformations of identical particles is an important direction for future research.

But mathematical approach is nevertheless very important and is used in many papers devoted to the entanglement of indistinguishable particles. For example, the Slater decomposition (an analogue of the Schmidt decomposition) of two fermions is constructed in [26], and we will discuss it further. Another interesting result is a classification of $\Lambda^3 \mathbb{C}^6$ fermionic states in terms of SLOCC [27].

Now let’s define $E_{H_{\text{min}}}$ for fermionic states.

The change of basis is determined by the unitary matrix $U$ and in the new basis the state becomes

$$U \circ |f\rangle = \sum_{i_1, i_2, ..., i_n = 1, i_1 < i_2 < ... < i_n}^p \lambda'_{i_1 i_2 ... i_n} |i_1 i_2 ... i_n\rangle,$$

$$\lambda'_{j_1, j_2, ..., j_n} = \sum_{i_1, i_2, ..., i_n = 1, i_1 < i_2 < ... < i_p}^p \lambda_{j_1, j_2, ..., j_n} M^{j_1, j_2, ..., j_n}_{i_1, i_2, ..., i_n},$$

where $M^{j_1, j_2, ..., j_n}_{i_1, i_2, ..., i_n}$ is a determinant of the matrix that is constructed by the crossing of $j_1, j_2, ..., j_n$ columns and $i_1, i_2, ..., i_n$ rows of $U$.

Measurements entropy of $|f\rangle$ is

$$H_{\text{meas}}(|f\rangle) = \sum_{i_1, i_2, ..., i_n = 1, i_1 < i_2 < ... < i_n}^p |\lambda_{i_1 i_2 ... i_n}|^2 \log |\lambda_{i_1 i_2 ... i_n}|^2.$$ 

And

$$E_{H_{\text{min}}}(|f\rangle) = \min_U H_{\text{meas}}(U \circ |f\rangle).$$

The parametrization of $U$ is described in Section III A. To get $U \circ |f\rangle$ amplitudes we need to calculate all minors of $U$, this can be done recursively using determinant expansion by minors.

### A. Slater decomposition

Now consider a state of 2 fermions in $2p$-dimensional space:

$$|f_2\rangle = \sum_{i_1, i_2 = 1, i_1 < i_2}^{2p} \lambda_{i_1 i_2} |i_1 i_2\rangle.$$

Then this state can be represented as a linear combination of only $p$ Slater determinants using a unitary change of basis [26]:

$$U_{\text{slater}} \circ |f_2\rangle = \sum_{i=1}^p z_i |2i, 2i + 1\rangle.$$
This representation is called \textit{Slater decomposition}. This fact follows from the existence of the symplectic basis for antisymmetric matrices \cite{28}, full proof may be found in \cite{26}.

An important \textit{numerical} result has been achieved for Slater decomposition and $E_{H_{min}}$

\textbf{Numerical Result IV.1.} Let $|f_2\rangle$ be the state of 2 fermions with $2p$-dimensional one-particle space. Then the following two conditions are equivalent:

1. The state $|f_2\rangle$ is in the Slater decomposition representation:

$$|f_2\rangle = \sum_{i=1}^{p} z_i |2i, 2i + 1\rangle.$$ 

2. $E_{H_{min}}(|f_2\rangle) = H_{meas}(|f_2\rangle).$

(I.e. Slater decomposition, like Schmidt decomposition, has minimal measurements entropy, and the process of finding $E_{H_{min}}$ for a state of two fermions is equal to finding its Slater decomposition.)

\section{V. CONCLUSION}

Entanglement measure $E_{H_{min}}$ for multipartite pure states was presented. Also we proved that this measure is additive, satisfies all necessary entanglement measure conditions, and coincides with the reduced von Neumann entropy for bipartite states. The method of numerical calculation of $E_{H_{min}}$ by genetic algorithm was presented and tested on up to 17 qubits inclusively. Moreover, $E_{H_{min}}$ was generalized to fermionic states, and this generalization corresponds to Slater decomposition for two-fermions states.

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\begin{thebibliography}{9}
\bibitem{1} MA Nielsen and I.L. Chuang. Quantum computing and quantum information, 2000.
\bibitem{2} T.G. Kolda. A counterexample to the possibility of an extension of the Eckart–Young low-rank approximation theorem for the orthogonal rank tensor decomposition. \textit{Anal. Appl.}, 23:243–355, 2001.
\bibitem{3} P. Waldemar and TA Ramstad. Hybrid KLT-SVD image compression. In \textit{1997 IEEE International Conference on Acoustics, Speech, and Signal Processing, 1997. ICASSP-97.}, volume 4, 1997.
\bibitem{4} S. Deerwester, S.T. Dumais, G.W. Furnas, T.K. Landauer, and R. Harshman. Indexing by latent semantic analysis. \textit{Journal of the American society for information science}, 41(6):391–407, 1990.
\end{thebibliography}
[5] S. Hammarling. The singular value decomposition in multivariate statistics. *ACM Signum Newsletter*, 20(3):2–25, 1985.

[6] L.D. Lathauwer, B.D. Moor, and J. Vandewalle. A multilinear singular value decomposition. *SIAM J. Matrix Anal. Appl.* 1995.

[7] W. Dur, G. Vidal, and JI Cirac. Three qubits can be entangled in two inequivalent ways. *Arxiv preprint quant-ph/0005115*, 2000.

[8] A. Mandilara, VM Akulin, AV Smilga, and L. Viola. Description of quantum entanglement with nilpotent polynomials. *Arxiv preprint quant-ph/0508234*, 2005.

[9] MB Plenio. Logarithmic negativity: A full entanglement Monotone that is not Convex. *Physical review letters*, 95(9):90503–90503, 2005.

[10] G. Vidal and RF Werner. Quantum Physics Title: A computable measure of entanglement. *Journal reference: Phys. Rev. A*, 65:032314, 2002.

[11] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *Arxiv preprint quant-ph/0702225*, 2007.

[12] A. Wong and N. Christensen. A potential multipartite entanglement measure. *Arxiv preprint quant-ph/0010052*, 2000.

[13] G.K. Brennen. An observable measure of entanglement for pure states of multi-qubit systems. *Arxiv preprint quant-ph/0305094*, 2003.

[14] V. Vedral and MB Plenio. Entanglement measures and purification procedures. *Arxiv preprint quant-ph/9707035*, 1997.

[15] M.B. Plenio and S. Virmani. An introduction to entanglement measures. *Arxiv preprint quant-ph/0504163*, 2005.

[16] D. Bouwmeester, A.K. Ekert, and A. Zeilinger. *The physics of quantum information*. Springer Berlin, 2000.

[17] D. Bruß. Characterizing entanglement. *Journal of Mathematical Physics*, 43:4237, 2002.

[18] G. Vidal. Entanglement monotones. *Journal of Modern Optics*, 47, 2(3):355–376, 2000.

[19] L. Amico, R. Fazio, A. Osterloh, and V. Vedral. Entanglement in many-body systems. *Arxiv preprint quant-ph/0703044*, 2007.

[20] A. Burkov, A. Chernyavskiy, and Y. Ozhigov. Algorithmic approach to quantum theory 3: bipartite entanglement dynamics in systems with random unitary transformations. In *Proceedings of SPIE*, volume 6264, page 62640B, 2006.

[21] T. Tilma and ECG Sudarshan. Generalized Euler angle parameterization for U (N) with applications to SU (N) coset volume measures. *Journal of Geometry and Physics*, 52(3):263–283, 2004.

[22] T. Tilma and ECG Sudarshan. Generalized Euler angle parametrization for SU (N). *Journal of Physics A-Mathematical and General*, 35(48):10467, 2002.

[23] Wikipedia, the free encyclopedia. 

http://en.wikipedia.org/wiki/Genetic_algorithm

[24] R.V. Ramos and R.F. Souza. Calculation of the quantum entanglement measure of bipartite states, based on relative entropy, using genetic algorithms. *Journal of Computational Physics*, 175(2):576–583, 2002.

[25] nVidia corp. Compute unified device architecture.

http://www.nvidia.com/cuda

[26] J. Schliemann, J.I. Cirac, M. Kus, M. Lewenstein, and D. Loss. Quantum correlations in two-fermion systems. *Arxiv preprint quant-ph/0012094*, 2000.

[27] Peter Levay and Peter Vrana. Three fermions with six single particle states can be entangled
in two inequivalent ways. *Physical Review A*, 78:022329, 2008.

[28] É.B. Vinberg. *A course in algebra*. American Mathematical Society, 2003.

[29] R. Bhatia. *Matrix analysis*. Springer, 1997.