ON MEAGER FUNCTION SPACES, NETWORK CHARACTER AND MEAGER CONVERGENCE IN TOPOLOGICAL SPACES

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ABSTRACT. For a non-isolated point x of a topological space X let \( \text{nw}_X(x) \) be the smallest cardinality of a family \( \mathcal{N} \) of infinite subsets of X such that each neighborhood \( O(x) \subseteq X \) of x contains a set \( N \in \mathcal{N} \). We prove that

- each infinite compact Hausdorff space X contains a non-isolated point x with \( \text{nw}_X(x) = \aleph_0 \);
- for each point \( x \in X \) with \( \text{nw}_X(x) = \aleph_0 \) there is an injective sequence \( (x_n)_{n \in \omega} \) in X that \( F \)-converges to \( x \) for some meager filter \( F \) on \( \omega \);
- if a functionally Hausdorff space X contains an \( F \)-convergent injective sequence for some meager filter \( F \), then for every path-connected space Y that contains two non-empty open sets with disjoint closures, the function space \( C_p(X,Y) \) is meager.

Also we investigate properties of filters \( F \) admitting an injective \( F \)-convergent sequence in \( \beta \omega \).

This paper was motivated by a question of the second author who asked if the function space \( C_p(\omega^*,2) \) is meager. Here \( \omega^* = \beta \omega \setminus \omega \) is the remainder of the Stone-Cech compactification of the discrete space of finite ordinals \( \omega \) and \( 2 = \{0,1\} \) is the doubleton endowed with the discrete topology. According to Theorem 4.1 of [13] this question is tightly connected with the so-called meager convergence of sequences in \( \omega^* \).

A filter \( F \) on \( \omega \) is meager if it is meager (i.e., of the first Baire category) in the power-set \( \mathcal{P}(\omega) = 2^\omega \) endowed with the usual compact metrizable topology. By the Talagrand characterization [13], a free filter \( F \) on \( \omega \) is meager if and only if \( \xi(F) = \mathfrak{g}r \) for some finite-to-one function \( \xi : \omega \to \omega \). A function \( \xi : \omega \to \omega \) is finite-to-one if for each point \( y \in \omega \) the preimage \( \xi^{-1}(y) \) is finite and non-empty. A filter \( F \) on \( \omega \) is defined to be \( \xi \)-meager for a surjective function \( \xi : \omega \to \omega \) if \( \xi(F) = \mathfrak{g}r \).

We shall say that for a filter \( F \) on \( \omega \), a sequence \( (x_n)_{n \in \omega} \) of points of a topological space X \( F \)-converges to a point \( x_\infty \in X \) if for each neighborhood \( O(x_\infty) \subseteq X \) of \( x_\infty \) the set \( \{n \in \omega : x_n \in O(x_\infty)\} \) belongs to the filter \( F \). Observe that the usual convergence of sequences coincides with the \( \mathfrak{g}r \)-convergence for the Fréchet filter \( \mathfrak{g}r = \{A \subseteq \omega : \omega \setminus A \text{ is finite}\} \) that consists of all cofinite subsets of \( \omega \). The filter convergence of sequences has been actively studied both in Analysis [1, 4] and Topology [5]. A sequence \( (x_n)_{n \in \omega} \) will be called meager-convergent if it is \( F \)-convergent for some meager filter \( F \) on \( \omega \). A sequence \( (x_n)_{n \in \omega} \) is called injective if \( x_n \neq x_m \) for all \( n \neq m \).

We shall prove that for a zero-dimensional Hausdorff space X the function space \( C_p(X,2) \) is meager if X contains an injective meager-convergent sequence. We recall that a topological space X is functionally Hausdorff if for any distinct points \( x, y \in X \) there is a continuous function \( \lambda : X \to I \) such that \( \lambda(x) \neq \lambda(y) \). Here \( I = [0,1] \) is the unit interval. For topological spaces \( X, Y \) by \( C_p(X,Y) \) we denote the space of continuous functions endowed with the topology of pointwise convergence.

Theorem 1. Let X be a functionally Hausdorff space and Y be a topological space that contains two open non-empty subsets with disjoint closures. Assume that X is zero-dimensional or Y is path-connected. If X contains an injective meager-convergent sequence, then the function space \( C_p(X,Y) \) is meager.

Proof. Let \( (x_n)_{n \in \omega} \) be a sequence in X that \( F \)-converges to \( x_\infty \in X \) for some meager filter \( F \) in \( \omega \). Then there is a finite-to-one surjection \( \xi : \omega \to \omega \) such that \( \xi(F) = \mathfrak{g}r \). By our assumption, Y contains two non-empty open subsets \( W_0, W_1 \) with disjoint closures.

For every \( n \in \omega \) consider the subset

\[
C_n = \{ f \in C_p(X,Y) : \forall i \in \{0,1\} (f(x_\infty) \notin W_i) \Rightarrow \forall m \geq n \exists k \in \xi^{-1}(m) \ (f(x_k) \notin W_i) \}.
\]
The meager property of $C_p(X,Y)$ will follow as soon as we check that $C_p(X,Y) = \bigcup_{n \in \omega} C_n$ and each set $C_n$ is nowhere dense in $C_p(X,Y)$.

To show that $C_p(X,Y) = \bigcup_{n \in \omega} C_n$, fix any continuous function $f \in C_p(X,Y)$. Since $Y = (Y \setminus \overline{W_0}) \cup (Y \setminus \overline{W_1})$, there is $i \in \{0,1\}$ such that $f(x_\infty) \notin \overline{W_i}$. Since $(x_n)$ is $F$-convergent to $x_\infty$ and $f^{-1}(Y \setminus \overline{W_i})$ is an open neighborhood of $x_\infty$, the set $F = \{n \in \omega : f(x_n) \notin \overline{W_i}\}$ belongs to the filter $F$ and thus the image $\xi(F)$, being cofinite in $\omega$, contains the set $\{m \in \omega : m \geq n\}$ for some $n \in \omega$. Then $f \in C_n$ by the definition of the set $C_n$.

Next, we show that each set $C_n$ is nowhere dense in $C_p(X,Y)$. Fix any non-empty open set $U \subseteq C_p(X,Y)$. Without loss of generality, $U$ is a basic open set of the following form:

$$U = \{f \in C_p(X,Y) : \forall z \in Z \ f(z) \in U_z\}$$

for some finite set $Z \subseteq X$ and non-empty open sets $U_z \subseteq Y$, $z \in Z$. We can additionally assume that $x_\infty \in Z$. We need to find a non-empty open set $V \subseteq C_p(X,Y)$ such that $V \subseteq U \setminus C_n$. If $U \cap C_n$ is empty, then put $V = U$. So we assume that $U \cap C_n$ contains some function $f_0$. For this function we can find $i \in \{0,1\}$ such that $f_0(x_\infty) \notin \overline{W_i}$. Since $f_0(x_\infty) \in U_{x_\infty}$, we lose no generality assuming that $U_{x_\infty} \subseteq Y \setminus \overline{W_i}$.

Since the sequence $(x_n)_{n \in \omega}$ is injective, we can find $m \geq n$ such that the set $X_m = \{x_k : k \in \xi^{-1}(m)\}$ does not intersect the finite set $Z$. Choose any function $g : Z \cup X_m \to Y$ such that $g(z) = f_0(z)$ for all $z \in Z$ and $g(x) \in W_{1,i}$ for all $x \in X_m$.

We claim that the function $g$ has a continuous extension $\bar{g} : X \to Y$. By our assumption, $X$ is zero-dimensional or $Y$ path-connected. In the first case we can find a retraction $r : X \to Z \cup X_m$ and put $\bar{g} = g \circ r$. If $Y$ is path-connected, then take any injective function $\phi : g(Z \cup X_m) \to I$ and extend the function $\phi \circ g : Z \cup X_m \to I$ to a continuous map $\lambda : X \to I$ using the functional Hausdorff property of $X$. Since $Y$ is path-connected, the map $\phi^{-1} : (\phi \circ g)(Z \cup X_m) \to Y$ extends to a continuous map $\psi : I \to Y$. Then the continuous map $\bar{g} = \psi \circ \lambda : X \to Y$ is a required continuous extension of $g$.

In both cases the set

$$V = \{f \in C_p(X,Y) : \forall z \in Z \ f(z) \in U_z, \mbox{ and } \forall x \in X_m \ f(x) \in W_{1,i}\}$$

is an open neighborhood of $\bar{g}$ that lies in $U \setminus C_n$, witnessing that the set $C_n$ is nowhere dense in $C_p(X,Y)$. \qed

In light of Theorem 1 it is important to detect topological spaces that contain injective meager-convergent sequences. This will be done for spaces containing points with countable network character.

A family $\mathcal{N}$ of subsets of a topological space $X$ is called a $\pi$-network at a point $x \in X$ if each neighborhood $O(x) \subset X$ of $x$ contains some set $N \in \mathcal{N}$. If each set $N \in \mathcal{N}$ is infinite, then $\mathcal{N}$ will be called an $\omega$-network at $x$. An $\omega$-network at $x$ exists if and only if each neighborhood of $x$ in $X$ is infinite. In this case let $\text{nw}_\chi(x;X)$ denote the smallest cardinality $|\mathcal{N}|$ of an $\omega$-network $\mathcal{N}$ at $x$. If some neighborhood of $x$ in $X$ is finite, then let $\text{nw}_\chi(x;X) = 1$. If the space $X$ is clear from the context, then we write $\text{nw}_\chi(x)$ instead of $\text{nw}_\chi(x;X)$ and call this cardinal the network character of $x$ in $X$. If $X$ is a $T_1$-space, then $\text{nw}_\chi(x) \geq \aleph_0$ if and only if the point $x$ is not isolated in $X$. The cardinal $\text{hnw}_\chi(x) = \sup\{\text{nw}_\chi(x;A) : x \in A \subset X\}$ is called the hereditary network character at $x$.

Points $x \in X$ with $\text{hnw}_\chi(x) \leq \aleph_0$ are called Pytkeev points, see [11].

**Theorem 2.** If some point $x$ of a topological space $X$ has $\text{nw}_\chi(x) = \aleph_0$, then for each finite-to-one function $\xi : \omega \to \omega$ with $\lim_{n \to \infty} |\xi^{-1}(n)| = \infty$ there is an injective sequence $(x_n)_{n \in \omega}$ in $X$ that $F$-converges to $x$ for some $\xi$-meager filter $F$.

**Proof.** Let $(N_i)_{i \in \omega}$ be a countable $\omega$-network at $x$. Since each set $N_i$ is infinite, we can choose an injective sequence $(x_k)_{k \in \omega}$ in $X$ such that for every $n \in \omega$ and $0 \leq i < |\xi^{-1}(n)|$ the set $N_i$ meets the set $\{x_k : k \in \xi^{-1}(n)\}$.

It is clear that the sequence $(x_n)_{n \in \omega}$ $F$-converges to $x$ for the filter

$$F = \{\{n \in \omega : x_n \in O(x)\} : O(x) \mbox{ is a neighborhood of } x \mbox{ in } X\}.$$  

It remains to check that the filter $F$ is $\xi$-meager. Given any neighborhood $O(x) \subset X$ of $x$ we need to find $n \in \omega$ such that for every $m \geq n$ there is $k \in \xi^{-1}(m)$ with $x_k \in O(x)$. Since $(N_i)_{i \in \omega}$ is a network at $x$, there is $i \in \omega$ such that $N_i \subset O(x)$. Taking into account that $\lim_{n \to \infty} |\xi^{-1}(n)| = \infty$, find $n \in \omega$ such that $|\xi^{-1}(m)| > i$ for all $m \geq n$. Now the choice of the sequence $(x_k)$ guarantees that for every $m \geq n$ there is $k \in \xi^{-1}(m)$ with $x_k \in N_i \subset O(x)$.

\qed

In light of Theorem 2 it is important to detect points $x$ with countable network character $\text{nw}_\chi(x)$. Let us recall that the character $\chi(x)$ (resp. the $\pi$-character $\pi\chi(x)$) of a point $x$ in a topological space $X$ is equal to
the smallest cardinality of a neighborhood base (resp. a $\pi$-base) at $x$. A $\pi$-base at $x$ is any $\pi$-network at $x$ consisting of non-empty open subsets of $X$. These definitions imply the following simple:

**Proposition 1.** For any non-isolated point $x$ of a $T_1$-space $X$,

1. $\nu_{\chi}(x) \leq \chi(x)$;
2. $\nu_{\chi}(x) \leq \pi\chi(x)$ provided that $x$ has a neighborhood containing no isolated point of $X$;
3. $\nu_{\chi}(x) = \mathbb{N}_0$ if $x$ is the limit of an injective $\mathfrak{F}$-convergent sequence in $X$.

The following simple example shows that the usual convergence of the injective sequence in Proposition 1(3) cannot be replaced by the meager convergence. It also shows that Theorem 2 cannot be reversed.

**Example 1.** Let $F$ be the meager filter on $\omega$ consisting of the sets $F \subset \omega$ such that

$$\lim_{n \to \infty} \frac{|F \cap [2^n, 2^{n+1})|}{2^n} = 1.$$ 

On the space $X = \omega \cup \{\infty\}$ consider the topology in which all points $n \in \omega$ and isolated while the sets $F \cup \{\infty\}$, $F \in F$, are neighborhoods of $\infty$. It is clear that the sequence $x_n = n, n \in \omega, F$-converges to $\infty$ in $X$. On the other hand, a simple diagonal argument shows that $\nu_{\chi} (\omega; X) > \mathbb{N}_0$.

**Theorem 3.** Each infinite compact Hausdorff space $X$ contains a point $x \in X$ with $\nu_{\chi}(x) = \mathbb{N}_0$.

**Proof.** Theorem trivially holds if $X$ contains no non-trivial convergent sequence. So we assume that $X$ contains no non-trivial convergent sequence. Then $X$ contains a closed subset $C \subset X$ that admits a continuous map $g: C \to \mathbb{I}$ onto the unit interval $\mathbb{I} = [0, 1]$, see [1], p.172. Replacing $C$ by a smaller subset, we can assume that the map $g: C \to \mathbb{I}$ is irreducible, which means that $g(C') \neq \mathbb{I}$ for any proper closed subset $C' \subset C$. Fix any countable base $B$ of the topology of $\mathbb{I}$. The irreducibility of the map $g: C \to \mathbb{I}$ implies that the space $C$ has no isolated points. Also the irreducibility of $g$ implies that the countable family $\mathcal{N} = \{g^{-1}(U) \cup B\}$ of open infinite subsets of $C$ is an $i$-network at each point $x \in C$. Consequently, $\nu_{\chi}(x) = \mathbb{N}_0$ for each point $x \in C$. \hfill $\square$

Theorems 1, 2, and 3 imply:

**Corollary 1.** For each infinite zero-dimensional compact Hausdorff space $X$ and each topological space $Y$ containing two non-empty open sets with disjoint closures the function space $C_p(X,Y)$ is meager. In particular, the function space $C_p(\omega^*, 2)$ is meager.

Also Theorems 2 and 3 imply:

**Corollary 2.** Let $\xi: \omega \to \omega$ be a finite-to-one function with $\lim_{n \to \infty} |\xi^{-1}(n)| = \infty$. Each infinite compact Hausdorff space $X$ contains an injective $\mathcal{F}$-convergent sequence for some $\xi$-meager filter $\mathcal{F}$ on $\omega$.

In fact, the condition $\lim_{n \to \infty} |\xi^{-1}(n)| = \infty$ in Corollary 2 can not be weakened.

Let us recall that an infinite subset $A$ is called a pseudointersection of a family of sets $\mathcal{F}$ if $A \subseteq F$ for all $F \in \mathcal{F}$ where $A \subseteq F$ means that $A \setminus F$ is finite. If a sequence $(x_n)_{n \in \omega}$ in a topological space $\mathcal{F}$-converges to a point $x_\infty$ for some filter $\mathcal{F}$ with infinite pseudointersection $A \subseteq \omega$ then the subsequence $(x_k)_{k \in A}$ converges to $x_\infty$ in the standard sense.

**Lemma 1.** Let $I$ be a countable set and $C = \bigcup_{i \in I} C_i$, where the sets $C_i$ are nonempty and mutually disjoint, and $\sup_{i \in I} |C_i| < \omega$. If $\mathcal{H}$ is a filter on $C$ all of whose elements intersect all but finitely many $C_i$'s, then $\mathcal{H}$ has an infinite pseudointersection.

**Proof.** The proposition will be proved by induction on $n = \sup_{i \in I} |C_i|$. If $n = 1$ there is nothing to prove. Suppose that it is true for all $k < n$ and let $I$, $\{C_i : i \in I\}$, $\mathcal{H}$ be as above with $\max\{|C_i| : i \in I\} = n$. If for every $H \in \mathcal{H}$ the set $\{i \in I : |C_i \cap H| > n\}$ is finite, then $C$ itself is a pseudointersection of $\mathcal{H}$. So suppose that $J = \{i \in I : |C_i \cap H_0| < n\}$ is infinite for some $H_0 \in \mathcal{H}$. In this case we may use our inductive hypothesis for $J$, $\{C_i \cap H_0 : i \in J\}$, $\mathcal{G} = \mathcal{H} \setminus (\bigcup_{i \in J} C_i \cap H_0)$, and $n-1$. Thus $\mathcal{G}$ has an infinite pseudointersection, and hence so does $\mathcal{H}$. \hfill $\square$

**Proposition 2.** If $\mathcal{F}$ is a $\xi$-meager filter on $\omega$ for some surjective function $\xi: \omega \to \omega$ with $\lim_{n \to \infty} |\xi^{-1}(n)| < \infty$, then any sequence $(x_n)_{n \in \omega}$ in a topological space $X$ that $\mathcal{F}$-converges to a point $x_\infty \in X$ contains a subsequence $(x_{n_k})_{k \in \omega}$ that converges to $x_\infty$. 

Proof. Choose infinite set \(I \subseteq \omega\) such that \(\sup_{i \in I} |\xi^{-1}(i)| < \omega\). Let \(C_i = \xi^{-1}(i)\) for every \(i \in I\), \(C = \bigcup_{i \in I} C_i\) and \(\mathcal{H} = \{F \cap C : F \in \mathcal{F}\}\). According to Lemma [1] there exists an infinite set \(D \subseteq C\) such that \(D \subseteq^* H\) for every \(H \in \mathcal{H}\). Then the subsequence \((x_i)_{i \in D}\) converges to \(x_\infty\).

Now let us compare two facts:

1. Assume that \(\beta\omega\) contains no injective \(\mathfrak{S}_r\)-convergent sequences;
2. each infinite compact Hausdorff space \(X\) contains an injective \(\mathcal{F}\)-convergent sequence for some meager filter \(\mathcal{F}\).

These two facts suggest a problem of finding the borderline between filters \(\mathcal{F}\) that admit an injective \(\mathcal{F}\)-convergent sequence in \(\beta\omega\) and filters that admit no such sequences. We hope that this borderline passes near analytic filters. Let us recall the definitions of some properties of filters.

A filter \(\mathcal{F}\) is analytic (resp. an \(F_\sigma\)-filter, \(F_{\sigma\delta}\)-filter) if \(\mathcal{F}\) is an analytic (resp. \(F_\sigma\)-subset, \(F_{\sigma\delta}\)-subset) of the power-set \(\mathcal{P}(\omega) = 2^\omega\) endowed with the natural compact metrizable topology.

A filter \(\mathcal{F}\) is measurable (resp. null) if it is measurable (resp. has measure zero) with respect to the Haar measure on the Cantor cube \(2^\omega\) considered as the countable product of 2-element groups. It is well-known that a filter is measurable if and only if it is null. The relations between meager and null filters are not trivial and were investigated in [18] and [2]. Since each analytic filter is meager and null we get the following chain of properties of filters:

\[ F_\sigma \Rightarrow \text{analytic} \Rightarrow \text{meager \& null}. \]

We are going to show that some meager and null filter \(\mathcal{F}\) admits an injective \(\mathcal{F}\)-convergent sequence in \(\beta\omega\) while no \(F_\sigma\)-filter \(\mathcal{F}\) admits such a sequence. The latter fact holds more generally for analytic \(P^+\)-filters.

A filter \(\mathcal{F}\) on \(\omega\) is called a \(P\)-filter (resp. a \(P^+\)-filter) if each countable subfamily \(\mathcal{C} \subseteq \mathcal{F}\) has a pseudointersection \(A\) that belongs to \(\mathcal{F}\) (resp. to \(\mathcal{F}^+\)). Here

\[ \mathcal{F}^+ = \{ A \subseteq \omega : \forall F \in \mathcal{F}, A \cap F \neq \emptyset \} \]

coincides with the union of all filters that contain \(\mathcal{F}\). It is clear that each \(P\)-filter is a \(P^+\)-filter. In particular, the Fréchet filter \(\mathcal{F}\) is both a \(P\)-filter and \(P^+\)-filter.

For a filter \(\mathcal{F}\) on \(\omega\) by \(\chi(\mathcal{F})\) we denote its character. It is equal to the smallest cardinality \(|\mathcal{B}|\) of the base \(\mathcal{B} \subseteq \mathcal{F}\) that generates \(\mathcal{F}\) in the sense that \(\mathcal{F} = \{ F \subseteq \omega : \exists B \in \mathcal{B}, B \subseteq F \}\). It is well-known that the character of each free ultrafilter on \(\omega\) is uncountable. The uncountable cardinal \(u = \min \{ \chi(\mathcal{U}) : \mathcal{U} \in \beta\omega \setminus \omega \}\) is called the ultrafilter number, see [3], [19]. The dominating number \(\frak{d}\) is the smallest cardinality \(|\mathcal{D}|\) of a cofinal subset \(\mathcal{D}\) in the partially ordered set \((\omega^{\omega^2}, \leq)\), see [3], [19]. By Ketonen’s Theorem [10], each filter \(\mathcal{F}\) on \(\omega\) with character \(\chi(\mathcal{F}) < \frak{d}\) is a \(P^+\)-filter.

Now we can exhibit some properties of filters \(\mathcal{F}\) admitting injective \(\mathcal{F}\)-convergent sequences in \(\beta\omega\).

**Theorem 4.** Assume that a filter \(\mathcal{F}\) admits an injective \(\mathcal{F}\)-convergent sequence \((x_n)_{n \in \omega}\) in \(\beta\omega\).

1. If \(\mathcal{F}\) is a \(P^+\)-filter, then for some set \(A \in \mathcal{F}^+\) the filter \(\mathcal{F}|A = \{ F \cap A : F \in \mathcal{F}\}\) on \(A\) is an ultrafilter.
2. \(\chi(\mathcal{F}) \geq \min\{\frak{d}, u\}\); 
3. \(\mathcal{F}\) is not an analytic \(P^+\)-filter; 
4. \(\mathcal{F}\) is not an \(F_\sigma\)-filter.

**Proof.**

1. Assume that \(\mathcal{F}\) is a \(P^+\)-filter. Let \(x_\infty\) be the \(\mathcal{F}\)-limit of the \(\mathcal{F}\)-convergent sequence \((x_n)_{n \in \omega}\) in \(\beta\omega\). Since the sequence \((x_n)\) is injective, there is \(m \in \omega\) such that for every \(n \geq m\) \(x_n \neq x_\infty\) and hence we can fix a neighborhood \(U_n\) of \(x_\infty\) whose closure does not contain the point \(x_n\). Since the sequence \((x_k)\) \(\mathcal{F}\)-converges to \(x_\infty\), for every \(n \geq m\) the set \(F_n = \{ k \in \omega : x_k \in U_n \}\) belongs to the filter \(\mathcal{F}\). Since \(\mathcal{F}\) is a \(P^+\)-filter, the sequence \((F_n)_{n \geq m}\) has a pseudointersection \(A\) in \(\mathcal{F}^+\). It follows from the choice of the neighborhoods \(U_n\) that the set \(\{x_n\}_{n \in A}\) is discrete in \(\beta\omega\) and the sequence \((x_n)_{n \in A}\) is \(\mathcal{F}|A\)-convergent to \(x_\infty\). By Rudin’s Theorem [10], the map \(f : A \to \beta\omega, f : n \mapsto x_n\), has injective Stone-Čech extension \(\beta f : \beta A \to \beta\omega\), which implies that the filter \(\mathcal{F}|A\) is an ultrafilter.

2. If \(\chi(\mathcal{F}) < \min\{\frak{d}, u\}\), then \(\chi(\mathcal{F}) < \frak{d}\) and by the Ketonen’s Theorem [10] \(\mathcal{F}\) is a \(P^+\)-filter. By the preceding statement, \(\mathcal{F}|A\) is an ultrafilter for some set \(A \in \mathcal{F}^+\). Consequently,

\[ u \leq \chi(\mathcal{F}|A) \leq \chi(\mathcal{F}) < u \]

and this is a desired contradiction.
3. If $\mathcal{F}$ is an analytic $P^+$-filter, then by the first statement, $\mathcal{F}|A$ is an ultrafilter for some subset $A \in \mathcal{F}^+$. On the other hand, the filter $\mathcal{F}|A$ is analytic being a continuous image of the analytic filter $\mathcal{F}$. So, $\mathcal{F}|A$ cannot be an ultrafilter.

4. Assume that $\mathcal{F}$ is an $\mathcal{F}_\sigma$-filter. In order to apply the preceding statement, it suffices to show that $\mathcal{F}$ is a $P^+$-filter. This is done in the following lemma.

**Lemma 2.** Each $\mathcal{F}_\sigma$-filter $\mathcal{F}$ on $\omega$ is a $P^+$-filter.

**Proof.** According to a result of Mazur [12] (see also [17]), for the $\mathcal{F}_\sigma$-filter $\mathcal{F}$ there exists a lower semi-continuous submeasure $\phi$ on $\mathcal{P}(\omega)$ such that $\mathcal{F} = \{A \subset \omega : \phi(\omega \setminus A) < \infty\}$. Since $\mathcal{F} \neq \mathcal{P}(\omega)$, $\phi(\omega) = \infty$ and the subadditivity of $\phi$ implies that $\phi(F) = \infty$ for all $F \in \mathcal{F}$. It follows from $\mathcal{F} = \{A \subset \omega : \phi(\omega \setminus A) < \infty\}$ that a set $A \subset \omega$ belongs to $\mathcal{F}^+$ if and only if $\phi(A) = \infty$.

To show that $\mathcal{F}$ is a $P^+$-filter, fix any decreasing sequence of sets $(A_k)_{k \in \omega}$ in $\mathcal{F}$. Let $n_0 = 0$ and by induction construct an increasing sequence of positive integers $(n_k)_{k \in \omega}$ such that $\phi([n_k, n_{k+1}) \cap A_k) > k$ for every $k \in \omega$. Then the set $A = \bigcup_{k \in \omega} [n_k, n_{k+1}) \cap A_k$ is a pseudointersection of $(A_k)_{k \in \omega}$ and belongs to the family $\mathcal{F}^+$ as $\phi(A) = \infty$.

Let us remark that Lemma 2 cannot be generalized to $\mathcal{F}_\sigma\delta$-filters. The following example was suggested to the authors by Jonathan Verner.

**Example 2.** The filter $\mathcal{F}_\delta \otimes \mathcal{F}_\delta = \{A \subset \omega \times \omega : \{n \in \omega : \{m \in \omega : (n, m) \in A\} \in \mathcal{F}_\delta\} \in \mathcal{F}_\delta\}$ on $\omega \times \omega$ is an $\mathcal{F}_\sigma\delta$ but not $P^+$.

In light of Theorem 4 it is natural to ask the following question.

**Question 1.** Does $\beta \omega$ contain an injective $\mathcal{F}$-convergent sequence for some analytic filter $\mathcal{F}$?

On the other hand, we have the following fact:

**Theorem 5.** Each infinite compact Hausdorff space $X$ contains an injective $\mathcal{F}$-convergent sequence for some meager and null filter $\mathcal{F}$.

**Proof.** Choose any finite-to-one function $\xi : \omega \to \omega$ such that

$$\lim_{n \to \infty} |\xi^{-1}(n)| = \infty \quad \text{and} \quad \prod_{n \in \omega} (1 - 2^{-|\xi^{-1}(n)|}) = 0.$$

By Corollary 2 any infinite compact Hausdorff space $X$ contains an injective $\mathcal{F}$-convergent sequence for some $\xi$-meager filter $\mathcal{F}$. It is clear that $\mathcal{F}$ is meager. It remains to check that $\mathcal{F}$ is null. The filter $\mathcal{F}$, being $\xi$-meager, lies in the union $\bigcup_{n \in \omega} \mathcal{F}_n$ where $\mathcal{F}_n = \{A \subset \omega : \forall k \geq n \ A \cap \xi^{-1}(k) \neq \emptyset\}$. It suffices to prove that each set $\mathcal{F}_n$ has Haar measure zero. Observe that the set $\mathcal{F}_n$ can be identified with the product $\prod_{k \geq n} (\mathcal{P}(\varphi^{-1}(k)) \setminus \emptyset)$, which has Haar measure

$$\prod_{k \geq n} \frac{2|\varphi^{-1}(k)| - 1}{2|\varphi^{-1}(k)|} = \prod_{k \geq n} (1 - 2^{-|\varphi^{-1}(k)|}) = 0.$$

**Remark 1.** After writing this paper the authors learned from V. Tkachuk that the meager property of the function space $C_p(\omega^*, 2)$ was also established by E.G. Pytkeev in his Dissertation [15, 3.24]. Game characterizations of topological spaces $X$ with Baire function space $C_p(X, \mathbb{R})$ were given in [9] and [14].

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