Manifolds of $G_2$ Holonomy from $N = 4$ Sigma Model

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Abstract

Using two dimensional (2D) $N = 4$ sigma model, with $U(1)^r$ gauge symmetry, and introducing the ADE Cartan matrices as gauge matrix charges, we build "toric" hyper-Kahler eight real dimensional manifolds $X_8$. Dividing by one toric geometry circle action of $X_8$ manifolds, we present examples describing quotients $X_7 = \frac{X_8}{U(1)}$ of $G_2$ holonomy. In particular, for the $A_r$ Cartan matrix, the quotient space is a cone on a $S^2$ bundle over $r$ intersecting $\text{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the $A_r$ Dynkin diagram.
1 Introduction

Over the few past years, there has been an increasing interest in studying string dualities. One of the important consequences of these studies is that all superstring models are equivalent in the sense that they correspond to different limits in moduli space of the same theory, called M-theory [1, 2, 3]. The latter, which is considered nowadays as the best candidate for the unification of the weak and strong coupling sectors of superstring models, is described, at low energies, by an eleven dimensional supergravity theory.

More recently, a special interest has been given to the compactification of the M-theory on seven real manifolds \(X_7\) with non trivial holonomy. This interest is due to the fact that these manifolds provide a potential point of contact with low energy semi realistic physics from M-theory. In particular, one can obtain four dimensional theory with \(N = 1\) supersymmetry by compactifying M-theory on \(R^{1,3} \times X_7\) where \(X_7\) a seven manifold with \(G_2\) holonomy [4-11]. In this regard, the \(N = 1\) four dimensional resulting physics models depend on geometric properties of \(X_7\). For instance, if \(X_7\) is smooth, the low energy theory contains, in addition to \(N = 1\) supergravity, only abelian gauge group and neutral chiral multiplets. However, non abelian gauge symmetries with chiral fermions can be obtained by considering limits where \(X_7\) develops singularities [10,11]. For this reason, it is interesting to study M-theory on singular seven manifold with \(G_2\) holonomy. Following [11], an interesting analysis for building such spaces is to consider the quotient of conical hyper-Kahler eight manifolds \(X_8\) by a \(U(1)\) symmetry. This approach, which is called the unfolding of the singularity, guarantees the \(G_2\) holonomy group of the quotient space \(\frac{X_8}{U(1)}\). A remarkable feature of this method, which may be related to two dimensional (2D) \(N = 4\) sigma model Calabi-Yau fourfolds construction \(CY^4\), is that the \(\frac{X_8}{U(1)}\) space solutions differ by what kind of \(U(1)\) symmetry is chosen and moreover the matter fields, in four dimensions, are obtained using the techniques of the geometric engineering of quantum field theory [12-15].

The aim of this work is to contribute in this direction by considering models with \(N = 4\) 2D sigma model ADE Cartan matrix gauge charges for building \(X_8\) manifolds. This study is motivated by the following points: (1) Actually, these vector charges go beyond of the one given in the first example studied in [11], where the matrix charge of the hypermultiplets \(\phi_i\), under \(U(1)^r\) gauge symmetry, was

\[
q^a_i = -\delta^a_{i-1} + \delta^a_i, \quad a = 1, \ldots, r. \tag{1.1}
\]

\(^1\)The latter is the maximal subgroup of \(SO(7)\) which can break the eight dimensional spinorial representation of \(SO(7)\) to the seven fundamental representation of \(G_2\) plus one singlet \((8 \rightarrow 7 + 1)\).
They may give an analogue connection appearing between the toric Calabi-Yau geometries, used in string theory compactifications, and the structures of ADE Lie algebras. The latter may lead to a similar analysis of the geometric engineering method of quantum field theory embedded in string theory.

It was suggested in [11] that the unfolding of the singularity may be adapted to others examples of $X_8$ manifolds, in particular toric-hyper-Kahler manifolds. In this paper, we would like to present a new class of these $X_8$ manifolds, which will be called toric-hyper-Kahler eight manifolds $X_8$, with the Calabi-Yau condition in sigma model construction given by

$$\sum_i q_i = 0.$$  \hspace{1cm} (1.2)

We will refer to such manifolds as Calabi-Yau fourfolds. Then we give their quotients by a $U(1)$ group symmetry using toric geometry circle actions. Our way involves two steps:

First we introduce the ADE Cartan matrices as $2D N = 4$ $U(1)^r$ linear sigma model matrix gauge charges. Second mimicking the analysis of [11] and using toric geometry circle actions, we discuss the construction of a new class of the quotients $X_8/U(1)$ of $G_2$ holonomy group. In particular, for the $A_r$ Cartan matrix, the quotient space is a cone on a $S^2$ bundle over $r$ intersecting $WCP^2_{(1,2,1)}$ weighted projective spaces according to the $A_r$ Dynkin geometry.

The organization of this paper is as follows: In section 2, we give an overview on aspects of $2D N = 4$ linear sigma model. Then we give examples illustrating the field theoretical construction of hyper-Kahler manifolds. In section 3, we introduce the ADE Cartan matrices as matrix gauge charges in the $2D N = 4$ field theory construction of $X_8$ manifolds. For the $A_r$ Lie algebra, the moduli space of the classical theory is given by the cotangent bundle over $r$ intersecting $WPC^2_{(1,2,1)}$ weighted projective spaces according to the $A_r$ Dynkin graph, extending the $A_r$ singularity of K3 surfaces described by $N = 2$ type IIA superstring sigma model used in the geometric engineering method. In section 4, we identify the $U(1)$ symmetry group with the toric geometry circle actions of $X_8$ to present quotients $X_7 = X_8/U(1)$ of $G_2$ holonomy. For the $A_r$ Cartan matrix gauge charge, the geometry is a cone on a $S^2$ bundle over $r$ intersecting $WCP^2_{(1,2,1)}$ weighted projective spaces according to the $A_r$ Dynkin diagram. Discussion and conclusion will be given in section 5.

2 $N = 4$ sigma model approach

In this section we review the main lines of the $N = 4$ sigma model approach for building the hyper-Kahler manifolds involved in the study of superstring, M-theory and F-theory, com-
pactifications, Yang Mills small instanton singularities and more general in supersymmetric field theories with eight supercharges [16,17,18]. For this purpose, consider 2D $N = 4$ supersymmetric $U(1)^r$ gauge theory with $n$ hypermultiplets $\phi_i$ ($i = 1, \ldots, n$) of a matrix charge $q^a_i$ ($a = 1, \ldots, r$), under $U(1)^r$ gauge symmetry, and $r$ 3-vectors FI coupling $\tilde{\xi}_a$. The equations defining the hyper-Kahler moduli space of this classical gauge theory are given by the following D-terms

$$\sum_i q^i_a [\phi^\alpha_i \bar{\phi}^\beta_i + \phi^{\beta}_i \bar{\phi}^\alpha_i] = \tilde{\xi}_a \tilde{\sigma}^\alpha_\beta.$$  \hfill (2.1)

The double index $(i, \alpha)$ of $\phi^\alpha_i$’s refer to component field doublets of the $n$ hypermultiplets, and $\tilde{\sigma}^n_\alpha$ are the traceless $2 \times 2$ Pauli matrices. For later use it is interesting to note the following points:

1. Equations (2.1) have a formal analogy with the D-flatness equations of 2D $N = 2$ $U(1)^r$ toric sigma model involved in the study of type II superstring compactifications on ALE spaces with ADE singularities[13]. The latters are given by:

$$\sum_{i=1}^n q^a_i |x_i|^2 = R_a, \quad a = 1, \ldots, r$$  \hfill (2.2)

where $r$ is the rank of the ADE Lie algebras and where $q^a_i$, up some details, the minus of the corresponding Cartan matrices satisfying the Calabi-Yau condition

$$\sum_{i=1}^n q^a_i = 0.$$  \hfill (2.3)

Equation (2.2) has a nice geometrical interpretation in terms of toric geometry. This has been a beautiful interplay between 2D $N = 2$ sigma models and toric geometry. In this way, (2.2) have a toric diagram which consists of $n$ vertices $\{v_i\}$, in the standard lattice $\mathbb{Z}^{n-r}$, satisfying the following constraint equations

$$\sum_{i=1}^n q^a_i v_i = 0, \quad a = 1, \ldots, r.$$  \hfill (2.4)

For instance, if one takes $r = 1$, this space describes the $n - 1$ dimensional weighted projective with weights $q_i$.

2. For each $U(1)$ factor, there are three real constraint equations transforming as an iso-triplet of $SU(2)$ R-symmetry ($SU(2)_R$) acting on the hyper-Kahler structures.

3. Using the $SU(2)_R$ transformations

$$\phi^\alpha = \varepsilon^{\alpha\beta} \phi_\beta, \quad \varepsilon_{12} = \varepsilon^{21} = 1$$

$$\overline{(\phi^\alpha)} = \bar{\phi}_\alpha.$$  \hfill (2.5)
and replacing the Pauli matrices by their expressions, the identities (2.1) can be split as follows:

\[ \sum_{i=1}^{n} q_i^a (|\phi^1_i|^2 - |\phi^2_i|^2) = \xi_a^3 \] (2.6)

\[ \sum_{i=1}^{n} q_i^a \phi^1_i \bar{\phi}_i = \xi_a^1 + i\xi_a^2 \] (2.7)

\[ \sum_{i=1}^{n} q_i^a \phi^2_i \bar{\phi}_i = \xi_a^1 - i\xi_a^2. \] (2.8)

Note that these equations have similar features of the description of [16] leaving only half the supersymmetry of the gauge model.

(4) After dividing the moduli space of zero energy states of the classical gauge theory (2.1) by the action of the \( U(1) \) gauge symmetry, we find precisely a toric-hyper-Kahler variety \( X_{4(k-r)} \) of \( 4(k-r) \) real dimensions. This construction is called the hyper-Kahler quotient extending the Kahler one involved in 2D N = 2 toric sigma model [16,17,18].

(5) The solutions of eqs (2.1) depend on the values of the FI couplings. For the case where \( \xi^1 = \xi^2 = 0 \) and \( \xi^3 > 0 \), it is not difficult to see that eqs (2.1) describe the cotangent bundle over a toric variety defined by

\[ \sum_{i=1}^{n} q_i^a |\phi^1_i|^2 = \xi_a^3. \] (2.9)

Indeed, if we set all \( \phi^2_i = 0 \), the \( \phi^1_i \)'s, modulo the complexified \( U(1) \) gauge group, determine a toric variety \( \frac{\mathbb{C}^m}{\mathbb{C}^*} \) of \( 2(n-r) \) real dimensions, see equations (2.2). The equations (2.6-7) mean that the \( \phi^1_i \)'s define the cotangent fiber directions over the toric variety given by (2.9). To see this feature, we assume that \( \xi^1_a = \xi^2_a = 0, \ a = 1 \) and we set \( q_i^a = q_i = 1 \), so we have

\[ \sum_{i=1}^{n} (|\phi^1_i|^2 - |\phi^2_i|^2) = \xi^3 \] (2.10)

and

\[ \sum_{i=1}^{n} \phi^1_i \bar{\phi}^2_i = 0. \] (2.11)

Equation (2.10), for \( \phi^2_i = 0 \), defines the \( \mathbb{CP}^{n-1} \) projective space while equation (2.11) means that \( \phi^1_i \) parameterizes the cotangent directions over it. In what follows, we give two extra examples illustrating this analysis and reconsidering the example given in [11]. In the first example, we consider a 2D \( N = 4 \) \( U(1) \) linear sigma model with two hypermultiplets of a vector charge \( (1,-1) \). The D-flatness conditions of this model read as

\[ (|\phi^1_1|^2 - |\phi^1_2|^2) - (|\phi^2_1|^2 - |\phi^2_2|^2) = \xi^3 \] (2.12)

\[ \phi^1_1 \bar{\phi}^2_1 - \phi^2_1 \bar{\phi}^2_2 = 0 \] (2.13)

\[ \phi^2_1 \bar{\phi}^2_1 - \phi^2_2 \bar{\phi}^2_2 = 0. \] (2.14)
Permuting the role of $\varphi_1$ and $\bar{\varphi}_2$, and making the following field change $\varphi_1 = \phi_1^1 \varphi_2 = -\bar{\varphi}_2^2$, $\psi_1 = \phi_1^2$ and $\psi_2 = \bar{\phi}_2^1$, the constraint equations (2.12-14) become

$$|\varphi_1|^2 + |\varphi_2|^2 = \xi^3 \quad (2.15)$$

$$\varphi_1 \bar{\psi}_1 + \varphi_2 \bar{\psi}_2 = 0 \quad (2.16)$$

$$\bar{\varphi}_1 \psi_1 + \bar{\varphi}_2 \psi_2 = 0 \quad (2.17)$$

and describe a cotangent bundle over a $\mathbb{CP}^1$ projective. In this way, the $\mathbb{CP}^1$ is defined by the following equation:

$$|\varphi_1|^2 + |\varphi_2|^2 = \xi^3. \quad (2.18)$$

Recall, in passing, that the cotangent bundle over $\mathbb{CP}^1$, which is known by the resolved $A_1$ singularity of K3 surfaces, is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$ and plays a crucial role in the study of the non perturbative limit of type II superstring dynamics in six and four dimensions [13,14,15].

The second example we want to consider deals with the generalization of the first one. This concerns a $2D N = 4 U(1)^r$ linear sigma model with $(r+1)$ hypermultiplets of a matrix charge satisfying (1.1). Using the same procedure, the D-flatness conditions (2.1) become:

$$|\varphi_{a-1}|^2 + |\varphi_a|^2 = \xi_a^3 \quad (2.19)$$

$$\bar{\psi}_{a-1} \varphi_{a-1} + \varphi_a \bar{\psi}_a = 0 \quad (2.20)$$

$$\bar{\varphi}_{a-1} \bar{\psi}_{a-1} + \psi_a \varphi_a = 0. \quad (2.21)$$

The solution of these equations describes the cotangent bundle over $r$ intersecting complex curves $\mathbb{CP}^1$. In the limit when all $\xi_a^3$ go to zero, the $CP^1$'s shrink and one ends with the $A_r$ singularity of local K3 surfaces. Note that this example has been used in [11] to construct seven real dimensional manifolds $X_7$ of $G_2$ holonomy group from the quotient of $X_8$ hyper-Kahler eight real dimensional manifolds by an $U(1)$ group symmetry. These eight dimensional spaces are obtained using $2D N = 4 U(1)^{(r-1)}$ linear sigma model with $(r+1)$ hypermultiplets, where the missing $U(1)$ invariance is explored to get the quotient $X_8/U(1)$ of $G_2$ holonomy group [11].

In what follows we want to give a new class of $X_8$ manifolds, which will be called toric-hyper-Kahler Calabi-Yau fourfolds ($CY^4 = X_8$) by introducing the $ADE$ Cartan matrices instead of the gauge matrix charge given in equation (1.1).
3 Toric-Hyper-Kahler eight manifolds with Calabi-Yau condition

We start this section by recalling that complex Calabi-Yau manifolds are the best ingredients for obtaining semi-realistic models of superstrings/M/F-theory [18,19,20], with minimal supercharges in lower dimensions. In particular, for later use, Calabi-Yau fourfolds, compact, non compact, singular or non-singular, are considered as ways for getting \( N = 1 \) supersymmetric models in four dimensions from the F-theory compactifications [20,21]. In M-theory context, compatifications on manifolds of \( G_2 \) holonomy can be effectively described by four dimensional \( N = 1 \) supersymmetry. Furthermore, from supersymmetry breaking viewpoint, the above geometries, which preserve both the same supercharges in particular \( \frac{1}{8} \) of initial ones of the uncompactified theory, have a similar role in superstrings and M-theory compactifications. From this physical argument and the string duality results, connecting type IIA and type IIB strings, we think that there are, at least, two natural questions. The latters are as follows: (1) Exist there a four dimensional duality connecting M-theory on manifolds of \( G_2 \) holonomy and F-theory on Calabi-Yau fourfolds?. (2) Or exist there a link between the corresponding geometries,(manifolds of \( G_2 \) holonomy and Calabi-Yau fourfolds)?. These questions, which are quite similar to the link between M-theory on manifolds of \( G_2 \) holonomy and heterotic strings on Calabi-Yau threefolds, need deeper study. However, here we try to give a modest comment on the the second one; while the first one will be dealt with in future work. This comment is based on the following known points:

(i) Manifolds with \( G_2 \) holonomy can be constructed as \( U(1) \) quotients of eight manifolds.
(ii) The maximal group of automorphisms in eight dimensions is \( SO(8) \). Using Dynkin geometries, this group, including the SU(4) group, can give the \( G_2 \) group.
(iii) Eight manifolds can have hyper-Kahler constructions in terms of \( N = 4 \) sigma model.

Combining these points with the Calabi-Yau condtnion, \( \sum, q_i = 0 \), in sigma model approach, one may say that seven real dimensional manifolds of \( G_2 \) holonomy group may be constructed from hyper-Kahler eight manifolds with the Calabi-Yau condition. In what follows we refer to such manifolds as Calabi-Yau fourfords geometries. In this way, the \( G_2 \) manifolds can be obtained using quotients by one finite circle, preserving the supercharges. In this present study, using similar ideas of [11], we would like to discuss the construction of seven dimensional manifolds with \( G_2 \) holonomy group from Calabi-Yau fourfolds geometry physics data, but with a different realization of the \( U(1) \) group symmetry for obtaining the quotient. This study involves two steps. First we will introduce, in the field theoretical construction of Calabi-Yau
fourfolds $X_8$, the ADE Cartan matrices as $2D N = 4$ linear sigma model matrix gauge charges. 

Second, mimicking the method of [11] and using toric geometry circle actions, we will discuss quotients $\frac{X_8}{U(1)}$ of $G_2$ holonomy group which will be given in the next section. Roughly speaking the toric-hyper- Calabi-Yau fourfolds $CY^4 = X_8$ may be viewed as the moduli space of $2D N = 4$ supersymmetric $U(1)^r$ gauge theory with $(r + 2) \phi^a_i$ hyper-multiplets $(4(r + 2 - r) = 8)$ with a matrix charge $q^a_i$ with the Calabi-Yau condition (1.2). In what follows, we will consider a matrix charge going beyond the equation (1.1). Our choice will be given by ADE Cartan matrices. For simplicity, we first consider the $A_r$ Lie algebra where the Cartan matrix is given by

$$q^a_i = -2\delta^a_i + \delta^a_{i-1} + \delta^a_{i+1}, \quad a = 1, \ldots, r,$$

(3.1)
satisfying naturally the Calabi-Yau condition $\sum_i q^a_i = 0$. Putting these equations into the D-flatness equations (2.1), one gets the following system of $3r$ equations:

\begin{align*}
\left( |\phi^1_{a-1}|^2 + |\phi^2_{a+1}|^2 - 2|\phi^1_{a-1}|^2 \right) - \left( |\phi^2_{a-1}|^2 + |\phi^2_{a+1}|^2 - 2|\phi^2_{a-1}|^2 \right) &= \xi_a \\
\phi^1_{a-1}\phi^2_{a-1} + \phi^1_{a+1}\phi^2_{a+1} - 2\phi^1_{a}\phi^2_{a} &= 0 \\
\phi^2_{a-1}\phi^2_{a-1} + \phi^2_{a+1}\phi^2_{a+1} - 2\phi^2_{a}\phi^2_{a} &= 0.
\end{align*}

(3.2) (3.3) (3.4)

We first solve these equations for the simple example of $U(1)$ gauge theory. Then we will give the result for the $U(1)^r$ gauge model. For $r = 1$, the above equations reduce to

\begin{align*}
\left( |\phi^1_0|^2 + |\phi^1_1|^2 - 2|\phi^1_0|^2 \right) - \left( |\phi^2_0|^2 + |\phi^2_1|^2 - 2|\phi^2_1|^2 \right) &= \xi \\
\phi^1_0\phi^2_0 + \phi^1_1\phi^2_1 - 2\phi^1_1\phi^2_1 &= 0 \\
\phi^2_0\phi^2_1 + \phi^2_1\phi^2_1 - 2\phi^2_1\phi^2_1 &= 0.
\end{align*}

(3.5) (3.6) (3.7)

To handle these D-terms equations, it should be interesting to note that they are quite similar to equations (2.10-11), and also (2.12-14). After permuting the role of $\phi^2_1$ and $\phi^2_2$, equations may be rewritten as

\begin{align*}
\left( |\phi^1_0|^2 + |\phi^1_1|^2 - 2|\phi^1_0|^2 \right) - \left( |\phi^2_0|^2 + |\phi^2_1|^2 - 2|\phi^2_1|^2 \right) &= \xi \\
\phi^1_0\phi^2_0 + \phi^1_1\phi^2_1 - 2\phi^1_1\phi^2_1 &= 0 \\
\phi^2_0\phi^2_1 + \phi^2_1\phi^2_1 - 2(\phi^2_1)^2 &= 0.
\end{align*}

(3.8) (3.9) (3.10)

Making the following field changes

$$
\begin{align*}
\phi^1_0 &= \varphi_1, & \phi^2_0 &= \psi_1 \\
\phi^1_1 &= \varphi_2, & \phi^2_1 &= \psi_2 \\
-\phi^1_2 &= \varphi_3, & \phi^2_1 &= \psi_1,
\end{align*}
$$

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the above equations become

\[(|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2) - (|\psi_1|^2 + |\psi_3|^2 + 2|\psi_2|^2) = \xi^3 \quad (3.11)\]

\[
\varphi_1\psi_1 + \varphi_3\psi_3 + 2\varphi_2\psi_2 = 0 \quad (3.12)
\]

\[
\overline{\varphi_1}\psi_1 + \overline{\varphi_3}\psi_3 + 2\overline{\varphi_2}\psi_2 = 0. \quad (3.13)
\]

Using similar analysis of the previous section, one sees that the above equations describe a cotangent bundle over \( \mathbf{WCP}^2_{1,2,1} \) weighted projective space. A way to see this feature is to use the link between \( N = 2 \) sigma model and toric geometry technics. Indeed, taking \( \psi_1 = \psi_2 = \psi_3 = 0 \), equations (3.11-13) reduce to

\[|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2 = \xi^3 \quad (3.14)\]

which can be encoded in a toric diagram. In this diagram, one has three vectors \( v_1, v_2 \) and \( v_3 \) in \( Z^2 \) lattice such that

\[v_1 + v_3 + 2v_2 = 0 \quad (3.15)\]

where the coefficients of \( v_i \) are exactly the ones of \( |\varphi_i|^2 \) in (3.14). Note that equation (3.15) describes a particular geometry of the one given in (2.4). Using the toric geometry language, equation (3.15) defines naturally a \( \mathbf{WCP}^2_{1,2,1} \) weighted projective space, where \( \xi^3 \) is a Kahler real parameter controlling its size. The equation (3.11-13), for generic value of \( \psi_i \), can be interpreted to mean that \( \psi_i \) parameterizes the fiber cotangent directions over \( \mathbf{WCP}^2_{1,2,1} \). Since the subset of (3.11) with \( \psi_i = 0 \) is a \( \mathbf{WCP}^2_{1,2,1} \) weighted projective space and \( \varphi_1\psi_1 + \varphi_3\psi_3 + 2\varphi_2\psi_2 = 0 \) is the analogue of equation (2.11), thus the space of solutions of (3.11-13) is isomorphic to the cotangent space over \( \mathbf{WCP}^2_{1,2,1}, T^*\mathbf{WCP}^2_{1,2,1} \). In the general case corresponding to the \( U(1)^r \) gauge theory, if we take the all \( \xi_a \)'s are no zero, it not too difficult to see that equations (3.2-4) describe the cotangent bundle over \( r \) intersecting \( \mathbf{WCP}^2_{1,2,1} \) weighted projective spaces. This means that the base geometry, of the cotangent bundle, consists of \( r \) intersecting \( \mathbf{WCP}^2_{1,2,1} \) according to the \( A_r \) Dynkin diagram, instead of one projective space in the case of \( U(1) \) gauge theory. In the limit that some \( \xi_a \)'s are zero, we obtain a singular geometry. Actually, this geometry may be used to extend the intersecting \( \mathbb{CP}^1 \) projective spaces of ALE spaces involved in the geometric engineering method of the quantum field theory [13,14,15]. We will conclude this section by noting that this analysis of the \( A_r \) Lie algebra may be extended to the others DE Lie algebras. However, these algebras contain trivalent vertex Dynkin geometries which complicates the computation. Recall that the trivalent Dynkin geometry involves a central node intersecting three other nodes once; moreover this geometry has been used in the geometric engineering of quantum field theories,
in particular in the introduction of fundamental matters in a chain of $SU$ product gauge group with $N = 2$ bifundamental matters. In toric sigma model approach, the corresponding vector charge, up the Calabi-Yau condition, is given by

$$q_i = (0, \ldots, -2, 1, 1, 0, \ldots, 0, -1),$$

instead of the bivalent geometry (3.1). A priori there are different ways one may follow to overcome this problem. A naive way is to delete these trivalent vertices. In this case, the D-flatness constraint equations have similar solutions of the $A_n$ Lie algebra. However a tricky method is to leave and use the trivalent geometry results involved in the elliptic fibrations singularities over the complex plane. In this way, the base geometry of $X_8$ may be given by three chains of intersecting $\text{WCP}^2_{1,2,1}$ according to the trivalent geometry.

In what follows we would like to discuss the corresponding seven real dimensional manifolds of $G_2$ holonomy group using $U(1)$ quotients. Similarly to the ideas of [11], we should look for a $U(1)$ group symmetry acting on $X_8$. As mentioned before, there are many ways one may follow to choosing the $U(1)$ group action of $X_8$. In this regard, the solutions differ by what kind of $U(1)$ symmetry is chosen. Two kinds of solutions are given in [11]. But here we will consider another way. The latter is inspired from the toric geometry circle actions.

4 On the quotient space $X_7 = \frac{X_8}{U(1)}$ of $G_2$ holonomy

Having constructed toric-hyper-Kahler Calabi-Yau fourfolds $X_8$ associated to ADE Cartan matrices sigma model gauge charges, we are now in the position to carry out quotient spaces $X_7 = \frac{X_8}{U(1)}$ of $G_2$ holonomy group using circle actions involved in toric varieties. Before doing this, let us tell some things about toric geometry. The letter is a powerful tool for studying $n$-dimensional complex manifolds exhibiting toric circle actions $U(1)^n$ which allow to encode the geometric properties of the complex spaces in terms of simple combinatorial data of polytopes $\Delta$ of the $R^n$ real space [22,23,24,25]. The simple example of toric varieties is the complex plane $\mathbb{C}$. The latter admits a $U(1)$ toric action

$$z \rightarrow z e^{i\theta},$$

which has a fixed point at $z = 0$. Thus the toric geometry of $\mathbb{C}$ can be viewed as a circle fibered on a half line parameterized by $|z|$. The circle, which determined by the action of $\theta$, shrinks at $z = 0$. This realization can be generalized easily to $\mathbb{C}^n$ space where we have a $T^n$ fibration, parameterized by the angular coordinates $\theta_i$, over a $n$-dimensional real base
parameterized by $|z_i^2|$. The second example we want to give is the $\text{CP}^1$ projective space. This space has also a $U(1)$ toric action having two fixed points describing respectively north and south poles of the two sphere $S^2 \sim \text{CP}^1$. Thus the toric geometry of $\text{CP}^1$ is given by an interval fibered by $S^1$ with zero size at the endpoints of the interval. Using these ideas, the cotangent bundle over $\text{CP}^1$ can be also viewed as a toric space. In this way, we have two circle actions on this space. The first one is the one corresponding to the action on the $\text{CP}^1$ base space and the other circle acts on the fiber cotangent direction. Our next example will be the two complex dimensional projective space $\text{CP}^2$. The latter has a $U(1)^2$ toric action exhibiting three fixed points defining a triangle in the $\mathbb{R}^2$ real space. The toric geometry of this manifold is described by a triangle of $\mathbb{R}^2$ fibered by a two real dimensional torus $T^2$ which degenerates to a $S^1$ circle on the three edges and shrinks to a point on the endpoints. The cotangent bundle over $\text{CP}^2$ is a 4-dimensional (eight real) local toric geometry, where we have two extra circle actions coming from the fiber cotangent directions. Note that this analysis is similar for the $\text{WCP}^2$, in particular $\text{WCP}^2_{(1,2,1)}$, and can be extended easily to higher dimensional (weighted) projective spaces. In what follows we will consider the above toric geometry circle actions to identify the $U(1)$ group symmetry of quotient spaces $X_7 = \frac{X_8}{U(1)}$.

Let us consider the simple example of the $U(1)$ gauge theory with three hypermultiplets. In this case the geometry $X_8$ can be viewed as $\mathbb{C}^2$ bundle over a $\text{WCP}^2_{(1,2,1)}$. This manifold has four toric geometry circle actions $U(1)^2_{\text{base}} \times U(1)^2_{\text{fiber}}$; two of them correspond to the $\text{WCP}^2_{(1,2,1)}$’s base space denoted by $U(1)^2_{\text{base}}$ while the remaining ones $U(1)^2_{\text{fiber}}$ act on the fiber cotangent directions. In what follows, we want to divide by one finite circle toric geometry action for obtaining seven real manifolds. Mimicking the analysis of [11] and identifying the $U(1)$ group symmetry of the quotient with one finite fiber circle action

$$X_7 = \frac{X_8}{U(1)_{\text{fiber}}}$$ \hspace{1cm} (4.2)$$

we can obtain a 7-dimensional geometry. Since $\frac{\mathbb{C}^2}{U(1)} = \mathbb{R}^+ \times \mathbb{C}$, the quotient space is now a $\mathbb{R}^+ \times \mathbb{C}$ bundle over a $\text{WCP}^2_{(1,2,1)}$. By compactifying the $\mathbb{C}$ complex plane, which can be done by adding a point at infinity, this space will be a $\mathbb{R}^+ \times S^2$ bundle over $\text{WCP}^2_{(1,2,1)}$. Similarly to [11], this geometry is a cone on a $S^2$ bundle over $\text{WCP}^2_{(1,2,1)}$ of $G_2$ holonomy. More generally, if we consider the $U(1)^r$ gauge theory with the $A_r$ Cartan matrix gauge charges and $(r + 2)$ hypermultiplets, then the quotient space is a cone on a $S^2$ bundle over $r$ intersecting $\text{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the $A_r$ Dynkin diagram.

Finally, a naive way to study the singularities of these $X_7$ manifolds is to consider the identification structure of the weighted projective spaces. The latters are not generally smooth
because non trivial fixed points under the variable identifications lead to singularities. To see this feature, consider the identification structure of $\text{WCP}_{1,2,1}^2$ defined by introducing three homogeneous complex coordinates $z_1, z_2, z_3$ not all of them simultaneously zero with a projective relation:

$$ (z_1, z_2, z_3) \equiv (\lambda z_1, \lambda^2 z_2, \lambda z_3). \quad (4.3) $$

Note, in passing, that these $(z_1, z_2, z_3)$ homogeneous complex coordinates can be related respectively to $\psi_1, \psi_2$ and $\psi_3$ fields of the sigma model construction. Finally, it is not hard to show that this space is singular. Indeed, if we take $\lambda = -1$, equation (4.3) reduces to

$$ (z_1, z_2, z_3) \equiv (-z_1, z_2, -z_3), \quad (4.4) $$

and so we have a $Z_2$ orbifold singularity at $(z_1, z_2, z_3) = (0, z_2, 0)$.

5 Discussion and Conclusion

In this paper, we have contributed in the M-theory compactifications to four dimensions. This involves the compactification on seven manifolds of $G_2$ holonomy group, leading to $N = 1$ four dimensional supersymmetric models. In particular, we have constructed a new class of toric-hyper-Kahler eight manifolds giving $G_2$ holonomy spaces after dividing by one finite toric geometry circle action. This building has been proceeded in two steps. We have first introduced the ADE Cartan matrices as matrix gauge charges in the $N = 4$ 2D field theoretical construction of toric-hyper-Kahler eight manifolds $X_8$. In particular, the solution for the $A_r$ Lie algebra is described by the cotangent bundle over $r$ intersecting $\text{WCP}_{1,2,1}^2$ weighted projective spaces according to the $A_r$ Dynkin diagram. Actually these spaces may extend the geometry of $A_r$ ALE space, described by 2D $N = 2$ type IIA superstring sigma model used in the geometric engineering method. Second we have used the toric geometry circle actions of $X_8$ to build quotients $X_7 = \frac{X_8}{U(1)}$ of $G_2$ holonomy group.

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