Braneworld Gravity under gravitational decoupling

P. León
*Departamento de Física, Universidad de Antofagasta, Aptdo 02800, Chile.

A. Sotomayor†
Departamento de Matemática, Universidad de Antofagasta, Aptdo 02800, Chile.

The main objective of this work is to study in some detail the Randall-Sundrum gravity under the gravitational decoupling through the minimal geometric deformation approach (MGD-decoupling). We show a family of new black hole solutions as well as new exact interior solutions for self-gravitating stellar systems and we discuss the corresponding matching conditions.

Keywords: BraneWorld, Gravitational Decoupling, Internal Solutions, Black Hole Solutions

I. INTRODUCTION

A problem that has caught the attention of physicists in the last decades is to find new theories to describe the gravitational interaction beyond General Relativity. This search is based on the fact that the Einstein’s theory of General Relativity, despite his great success, can not explain in a satisfactory way different fundamental aspects of the gravitational interaction like the existence of dark matter and dark energy and the hierarchy problem. Moreover, this theory breaks down at very high energies, which makes it incompatible with the Standard Model of particles.

Among all the known candidates to describe the gravitational interaction beyond General Relativity, there are for example those theories that include extra dimensions, which take it inspiration in the Superstring or in the M-theory. These theories are particularly interesting because they can explain some of the fundamental problems of the physics. In fact, one of these theories, proposed by Randall and Sundrum [1, 2], is the Braneworld, from which it is possible to explain the scale hierarchy problem.

In the most simple Braneworld scenario, our observable universe is modeled as a four dimensional hypersurface, known as the 3-brane, embedded in a five dimensional space, usually called the bulk. The novel idea of the Braneworld is that all the gauge interactions, described by the Standard Model, are confined to live in the 3-brane while the gravitational interaction can spread in to the five dimensions of the space. For this reason it is possible to explain the fact that the gravitational scale is very low, compared to the Planck scale, a consequence of the fact that only a part of gravitational interaction is in our four dimensional observable universe, while the another part is spread in a fifth dimension that can be very large. For this reason the study of the modifications of the General Relativity in the Randall-Sundrum models, due to the interaction of the 3-brane with the bulk, are very important.

Nevertheless, even when there is known a covariant formulation of the Randall-Sundrum Braneworld theory that made a little more easy the study of these models [3], there are many issues that, until now, remain unsolved [4–8]. One of the reasons for these problems is due to lack of solutions to the complete five dimensional theory, that takes into account the brane and the bulk. However, an approach that can shed some light on this problem, consists in find solutions to the effective Einstein’s field equations in four dimensions and in this way get some information about the complete geometry in five dimensions.

Now, it is a known problem that the searching of analytical solutions to Einstein’s equations in General Relativity is a very difficult task and is even more complicate when we are searching in particular interior stellar solutions. This is mostly due to the nonlinearities of the resulting field equations. In the context of the Randall-Sundrum models there are additional nonlinear contributions to effective energy momentum tensor, in four dimensions, coming from the high energy corrections [3, 9]. For this reason the task of find exact physically acceptable solutions for the effective Einstein field equations in four dimensions its seems to be almost impossible.

In the last few years, a new method was found, known as minimal geometric deformation (MGD)-decoupling method, which allows us to solve the Einstein field equations in a very systematic and simple way. The earlier version of the method was proposed [10, 11] in the context of the brane-world and then was used to obtain new black hole solutions in Refs. [12, 13] (for some initial works on MGD, see Refs. [14–17], and Refs. [18–28] for some recent applications). For the more recent applications of the MGD-decoupling method [29, 30] see for instance Refs. [31–52].

Indeed, this approach is very useful to solve the Einstein’s equations for energy-momentum tensors of the form

\[ \bar{T}_{\mu\nu} = T_{\mu\nu} + \theta_{\mu\nu}, \]

because, instead of solve the complete system of equations for the source \( \bar{T}_{\mu\nu} \), we first solve Einstein’s equa-
tions for the primary source and then we solve a system of equations, similar to Einstein’s field equations, for the second source $\theta_{\mu\nu}$. Then, by performing a combination of the two solutions we can obtain the solution for the complete system. It is the purpose of this work to show how we can use this new method to solve the novelties which appear in the Randal-Sundrum theory in this context. Indeed, we will use this algorithm to find new analyti-
physically acceptable solutions to the effective Ein-
stein’s field equations by extending every known solution of General Relativity to its braneworld versions.

We would like to emphasize that while it is true that the braneworld has a well-deserved theoretical importance, it is fair to mention that there is no experimental evidence on it. However, given that it manages to explain the problem of the hierarchy of fundamental forces in a simple and highly non-trivial way, its theoretical im-
portance continues today. Especially since it could serve as a guide to construct a new theory that cannot only explain the problem of the hierarchy of fundamental inter-
actions, but can also be tested experimentally. If this will be possible the MGD-decoupling would be an ideal approach to study the equations of movement associated with the corrections suffered by General Relativity.

This work is organized as follows: in section 2 we make a brief review of how to decouple Einstein’s field equations, using the MGD decoupling method, in the case we have a combination of two gravitational sources. Then, in section 3, we write the four dimensional effective Einstein’s equations in the five-dimensional Braneworld world. The MGD decoupling in the Braneworld domain is explained in section 4 while in section 5 we present some black holes solutions. Finally, in section 6 we show how initial internal solutions taken as a seed the Tolman IV solution of General Relativity.

II. MINIMAL GEOMETRIC DEFORMATION DECOUPLING METHOD

In this section we present a brief review of the most important results to solve Einstein’s equations using the minimal geometric deformation decoupling method for spherically symmetric and static systems.

Let us start by writing the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -k^2 T_{\mu\nu}, \tag{2}$$

where we will assume that the energy momentum ten-
sor $T_{\mu\nu}$ has contributions of two different gravitational sources, that is

$$T_{\mu\nu} = T_{\mu\nu}^0 + \theta_{\mu\nu}. \tag{3}$$

The line element for our case has the following form in

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{4}$$

from which we can see in a straightforward way, using [3], that Einstein’s field equations [2] take can be rewritten by

$$k^2 \rho = k^2 (T_0^0 + \theta_0^0) = \frac{1}{r^2} - e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right), \tag{5}$$

$$k^2 \bar{p}_r = -k^2 (T_1^1 + \theta_1^1) = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right), \tag{6}$$

$$k^2 \bar{p}_t = -k^2 (T_2^2 + \theta_2^2) = \frac{e^{-\lambda}}{4} \left( 2\nu'' + \nu' - \lambda' \nu' \right) + 2\nu'' - \lambda' \nu', \tag{7}$$

where the prime indicates derivatives respect to variable $r$ and $\rho$, $\bar{p}_r$, and $\bar{p}_t$ are defined as the effective energy den-
sity, the effective radial pressure and the effective tangen-
tial pressure, respectively.

The conservation equation for this system, which can be obtained as a linear combination of the equations [5]-[7], is given by

$$\nabla_{\mu} T^{\mu\nu} = (\bar{p}_r)' - \frac{\nu'}{2}(\rho + \bar{p}_r) - \frac{2}{r}(\bar{p}_t - \bar{p}_r) = 0, \tag{8}$$

which in terms of the gravitational sources $T_{\mu\nu}$ and $\theta_{\mu\nu}$, takes the following form

$$(T_1^1)' - \frac{\nu'}{2} (T_0^0 - T_1^1) - \frac{2}{r} (T_2^2 - T_1^1)$$

$$+ \alpha \left( (\theta_1^1)' - \frac{\nu'}{2} (\theta_0^0 - \theta_1^1) - \frac{2}{r} (\theta_2^2 - \theta_1^1) \right) = 0. \tag{9}$$

At this point it is easy to see, from equations [5]-[7], that the combination of the two sources in the energy-momentum tensor will describe a fluid with local anisotropy on the pressures.

Now in order to solve the system of equations [5]-[7], we will apply the MGD decoupling method. The first step in this approach is to neglect the contributions of the source $\theta_{\mu\nu}$ and consider a solution of Einstein’s equations for the source $T_{\mu\nu}$, whose line element can be written as

$$ds^2 = e^{\mu(r)} dt^2 - \frac{1}{\mu(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{10}$$

were

$$\mu(r) \equiv 1 - \frac{k^2}{r} \int_0^r x^2 T_0^0 \, dx = 1 - \frac{2m(r)}{r}, \tag{11}$$

is the standard definition of the mass function in General Relativity.
The next step is to include all the contributions $\theta_{\mu\nu}$ on $T_{\mu\nu}$. This will be done by considering that the effects, induced by the gravitational source $\theta_{\mu\nu}$, are encoded in a deformation of the temporal and radial components of the metric given by

\[ \nu(r) = \xi(r) + \alpha g(r), \quad (12) \]
\[ e^{-\lambda(r)} = \mu(r) + \alpha f(r), \quad (13) \]

were $g(r)$ and $f(r)$ are two unknown functions. The minimal geometric deformation is associated with the case $g = 0$. In this particular case we will only be considering deformations in the radial component of the metric, this is

\[ \nu(r) = \xi(r), \quad (14) \]
\[ e^{-\lambda(r)} = \mu(r) + \alpha f^*(r). \quad (15) \]

Now, using (14), it is easy to show that Einstein’s field equations will be separated into two different systems of equations. The first of them are the Einstein field equations for the source $T_{\mu\nu}$, given by

\[ k^2 T_0^0 = \frac{1}{r^2} - \frac{\mu}{r^2} - \frac{\mu'}{r}, \quad (16) \]
\[ -k^2 T_1^1 = -\frac{1}{r^2} + \mu \left( \frac{1}{r^2} + \frac{\nu'}{r} \right), \quad (17) \]
\[ -k^2 T_2^2 = \frac{\mu}{4} \left( 2\nu'' + \nu'^2 + \frac{2\nu'}{r} \right) + \frac{\mu'}{4} \left( \nu' + \frac{2}{r} \right), \quad (18) \]

with the correspondent conservation equation

\[ (T_1^1)' - \frac{\nu'}{2} \left( T_0^0 - T_1^1 \right) - \frac{2}{r} \left( T_2^2 - T_1^1 \right) = 0, \quad (19) \]

while the second system is only related to the gravitational source $\theta_{\mu\nu}$ and is written as

\[ k^2 \theta_0^0 = -\alpha f^* - \frac{\alpha f'}{r}, \quad (20) \]
\[ k^2 \theta_1^1 = -\alpha f^* \left( \frac{1}{r^2} + \frac{\nu'}{r} \right), \quad (21) \]
\[ k^2 \theta_2^2 = -\alpha f^* \left( 2\nu'' + \nu'^2 + 2\frac{\nu'}{r} \right) - \alpha f^{*'} \left( \nu' + \frac{2}{r} \right), \quad (22) \]

and the conservation equation associated with the source is

\[ (\theta_1^1)' - \frac{\nu'}{2} \left( \theta_0^0 - \theta_1^1 \right) - \frac{2}{r} \left( \theta_2^2 - \theta_1^1 \right) = 0. \quad (23) \]

We notice that the system of equations (20)-(22), due to a missing term of $1/r^2$ in the first two equations, is not a Einstein system of equations for $\theta_{\mu\nu}$. However, it is always possible to redefine the components of $\theta_{\mu\nu}$ in order to include the factor of $1/r^2$ in the system (20)-(22) and obtain an Einstein system of equations for this source. Also we can notice, from equations (10), (21) and (22), that the interaction between the sources $T_{\mu\nu}$ and $\theta_{\mu\nu}$ is purely gravitational. Then we can conclude that we have decoupled the Einstein field equations.

The fact of having decoupled the Einstein field equations, for the combination of two sources, represents a huge simplification to the problem of finding solutions to the system of equations (10)-(12). This is because instead of solving the equations for the complete system, we can solve first the Einstein’s equations for the source $T_{\mu\nu}$, (10)-(12), and determine $(T_{\mu\nu}, \xi, \mu)$. Then, we can solve the system of equations (20)-(22) for the source $\theta_{\mu\nu}$ to find $(\theta_{\mu\nu}, f^*)$. Finally, the solution for the complete system can be obtained by a simple combination of these two results.

The simple and systematic approach of the MGD decoupling method to solve the Einstein field equations represent a powerful tool in the analysis of more complicated and realistic distributions of matter in the context of General Relativity. Indeed, we can find solutions to the Einstein field equations, for very complicated distributions of matter, in two different ways:

- We can choose known simple solutions of Einstein’s equations for the energy-momentum tensor $T_{\mu\nu}$.
- We can start with a very complicated expression for the energy momentum $T_{\mu\nu}$. Then in order to find a solution for this case, we can separate the energy-momentum tensor in its more simpler components, that is

\[ T_{\mu\nu} = \sum_i T_{\mu\nu}^i. \quad (24) \]

Now, we can solve the Einstein’s equations for each $T_{\mu\nu}^i$. Then, by simple combinations of these solutions, we can found the solutions of the Einstein field equations for the most general energy-momentum tensor $T_{\mu\nu}$.

III. FIELD EQUATIONS IN THE BW

The main feature of the braneworld models is to consider that our $(3 + 1)$ observable universe is confined on a 3-brane in a five dimensional space time, usually called the bulk, with $Z_2$ symmetry. This five dimensional theory induces modifications to the Einstein field equations on the brane, which can by written as

\[ G_{\mu\nu} = -g_{\mu\nu} \Lambda - k^2 T_{\mu\nu}^T, \quad (25) \]
were $k^2 = 8\pi G_N$, $\Lambda$ is the cosmological constant on the brane and $T^T_{\mu\nu}$ is a effective energy-momentum tensor given by

$$T^T_{\mu\nu} = T_{\mu\nu} + \frac{6}{\sigma} S_{\mu\nu} + \frac{1}{8\pi} \mathcal{E}_{\mu\nu} + \frac{4}{\sigma} \mathcal{F}_{\mu\nu}, \quad (26)$$

which, through the inclusion of the last three terms, take into account all the effects of the bulk onto the 3-brane. Here $\sigma$ is the brane tension.

The term $S_{\mu\nu}$, called the high-energy corrections, in the effective energy-momentum tensor arise from the extrinsic curvature terms in the projected Einstein tensor onto the brane. This is given by

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\rho} T^\rho_{\nu} + \frac{1}{24} [3T_{\rho\lambda} T^{\rho\lambda} - T^2], \quad (27)$$

where $T$ is the trace of $T_{\mu\nu}$. The third term, $\mathcal{E}_{\mu\nu}$, is known by the name of Kaluza-Klein corrections and represents the projection of the Weyl tensor of the bulk. For the case of spherically symmetric and static distributions of matter, which is the only case that we will consider in this paper, this term can be written as

$$k^2 \mathcal{E}_{\mu\nu} = \frac{6}{\sigma} \left[ \mathcal{U} \left( u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} \right], \quad (28)$$

with

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \quad (29)$$
$$\mathcal{P}_{\mu\nu} = \mathcal{P} \left( r_\mu r_\nu - \frac{1}{3} h_{\mu\nu} \right), \quad (30)$$

where $\mathcal{U}$, $\mathcal{P}_{\mu\nu}$, $h_{\mu\nu}$, $u_\mu$ and $r_\mu$ are the bulk Weyl scalar, the anisotropic stress, the projection operator operator, the four velocity of fluid element and a radial unitary vector, respectively.

The last correction to the effective energy-momentum tensor, $\mathcal{F}_{\mu\nu}$, depends on all the stresses in the bulk apart from the cosmological constant. Thus, in general there will be an interchange of energy momentum tensor between the bulk and the brane. From now on, we will restrict our self to the case where only the cosmological constant is present in the bulk, which implies $\mathcal{F}_{\mu\nu} = 0$. In this particular case, we recover the standard conservation equation of GR

$$\nabla^\nu T_{\mu\nu} = 0. \quad (31)$$

Now in order to study the effects of the BW on perfect fluids the energy-momentum tensor $T_{\mu\nu}$ must have the following form

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu}, \quad (32)$$

where $\rho, p$ represents the energy density and the pressure of the perfect fluid, respectively, and $u_\mu = \exp -\nu/2\delta^0_0$. In this case the equilibrium equation leads to

$$p' + \nu' \frac{\rho}{2} = 0. \quad (33)$$

Finally, with all these ingredients, we are ready to write the effective Einstein’s equation, with $\Lambda = 0$, in the four dimensional 3-brane. Then, using (10) and (26)-(32), the equation (25) leads to

$$k^2 \left[ \rho + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \frac{6\mathcal{U}}{k^4} \right) \right] = \frac{1}{r^2} e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right), \quad (34)$$
$$k^2 \left[ p + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \frac{\rho p + 2\mathcal{U}}{k^4} \right) \right] = \frac{4\mathcal{P}}{k^4} = -\frac{1}{r^2} e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right), \quad (35)$$
$$k^2 \left[ p + \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \frac{\rho p + 2\mathcal{U}}{k^4} - \frac{2\mathcal{P}}{k^4} \right) \right] = \frac{e^-\lambda}{4} \left( 2\nu'' + \nu'^2 - \lambda' \nu' + 2\nu' - \frac{\lambda'}{r} \right). \quad (36)$$

Finally, from this equations, it is evident that the inclusion of the high energy corrections to the Einstein’s equations represent a huge complication with respect to General Relativity case. Furthermore, it is easy to see that the equations (34)-(36) represent an indefinite system because extra information is required, related with the geometry of the bulk, in order to solve the system. In the next sections we will show how to solve this problems using the MGD-decoupling method.

**IV. GRAVITATIONAL DECOUPLING IN THE BW**

Let us implement the gravitational decoupling to the braneworld system represented by the expressions (34)-(36). If we compare the effective energy-momentum tensor in the equation (26) with (3) we can identify the generic energy-momentum tensor $\theta_{\mu\nu}$ with that corresponding to the braneworld, namely

$$\theta_{\mu\nu} = \frac{6}{\sigma} S_{\mu\nu} + \frac{1}{8\pi} \mathcal{E}_{\mu\nu}, \quad (37)$$

where we has been used $\mathcal{F}_{\mu\nu} = 0$. Hence, comparing the system (34)-(36) with (5)-(7) we find that the energy-momentum tensor $T_{\mu\nu}$ corresponds to a perfect fluid and that $\theta_{\mu\nu}$ is given by

$$\theta^{\mu\nu} = \frac{1}{\sigma} \left( \frac{\rho^2}{2} + \frac{6\mathcal{U}}{k^4} \right), \quad (38)$$
\[
\theta_1^1 = -\frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} U \right) - \frac{4}{k^4} \frac{\mathcal{P}}{\sigma}, \quad (39)
\]
\[
\theta_2^2 = -\frac{1}{\sigma} \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} U \right) + \frac{2}{k^4} \frac{\mathcal{P}}{\sigma}. \quad (40)
\]

Therefore, the system (20)-(22) becomes
\[
k^2 \left( \frac{\rho^2}{2} + \frac{6U}{k^4} \right) = -\frac{f^*}{r^2} - \frac{f^*}{r}, \quad (41)
\]
\[
k^2 \left[ \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} U \right) + \frac{4}{k^4} \mathcal{P} \right] = f^* \left( \frac{1}{r^2} + \frac{\xi'}{r} \right), \quad (42)
\]
\[
k^2 \left[ \left( \frac{\rho^2}{2} + \rho p + \frac{2}{k^4} U \right) - \frac{2}{k^4} \mathcal{P} \right] = \frac{1}{4} \left[ f^* \left( 2 \xi'' + \xi^2 + \frac{2}{r^2} \xi' \right) + f^* \left( \xi' + \frac{2}{r} \right) \right], \quad (43)
\]

where we have identified \( \alpha = \sigma^{-1} \). On the other hand, the conservation equation (19) reads
\[
k^4 \left( \rho \rho' + \rho' \rho + \rho p \rho' \right) + 2 (U' + 2 \mathcal{P}') + \frac{\rho'}{2} \left[ k^4 \rho (\rho + p) + 4 (2U + \mathcal{P}) \right] + \frac{12}{r} \mathcal{P} = 0, \quad (44)
\]
which can be simplified by using the conservation equation for the perfect fluid (33), yielding
\[
U' + 2 \mathcal{P} + \xi' (2U + \mathcal{P}) + \frac{6}{r} \mathcal{P} = k^4 \left( \rho + p \right) \rho'. \quad (45)
\]

Let us recall that the conservation equation (45) is a linear combination of the equations of motion for the braneworld gravitational sector, namely, the expressions in Eqs. (41)-(43). We see that after the decoupling, we end with three independent equations, namely, the system (41)-(43), to find three unknown functions \( \{U, \mathcal{P}, f^*\} \). This means that every perfect fluid configuration \( \{p, \rho, \mu, \xi\} \) will have a specific braneworld solution under the MGD-decoupling. In this respect, a natural question regarding the BW vacuum \( \{p = \rho = 0, U, \mathcal{P}\} \) arises: which is the exterior Schwarzschild deformed solution obtained under the MGD-decoupling? To answer it, we impose the vacuum condition \( \{p = \rho = 0\} \) on the system (41)-(43). Hence we found a first order differential equation for the deformation \( f^* \), namely,
\[
\left( \frac{\xi'}{2} + \frac{2}{r} \right) (f^*)' + \left( \xi'' + \frac{\xi'}{2} + \frac{2}{r^2} \xi' + \frac{2}{r^2} \right) f^* = 0 \quad (46)
\]
which solution is given by
\[
f^* = \frac{D}{F(r)}, \quad (47)
\]
where \( D \) is an integration constant and
\[
F(r) = \exp \int \frac{\xi'' + \frac{\xi'}{2} + \frac{2}{r^2} \xi' + \frac{2}{r^2}}{\xi'} dr. \quad (48)
\]

Using the Schwarzschild solution in Eq. (47) we obtain
\[
f^* = \frac{D}{2} \left( 1 - \frac{2M}{r} \right) \left( 1 - \frac{2M}{r} \right), \quad (49)
\]
then we can write the minimally deformed radial metric component as
\[
e^{-\lambda} = \left( 1 - \frac{2M}{r} \right) \left[ 1 + \frac{D}{2\sigma} \left( r - \frac{3M}{2} \right) \right], \quad (50)
\]
and the functions
\[
U = -\frac{4\pi DM}{3r^2(3M - 2r)^2}, \quad (51)
\]
\[
\mathcal{P} = -\frac{4\pi D(4M - 3r)}{3r^2(3M - 2r)^2}. \quad (52)
\]

This result represent the only possible deformation of the Schwarzschild exterior vacuum under the MGD-decoupling method in the BW. Nevertheless, this does not represent a new black hole solution for the BW. Indeed, this solution was first found in Ref. [8] and later on in Ref. [14].

Now, we known that there exist others black hole solutions in the BW context different from the obtained here [4, 63, 64]. Then the fact that (50) is the only possible deformation to the Schwarzschild vacuum shows the limitations of the MGD-decoupling method to obtain new black holes solutions in the BW. Therefore to find new black hole solutions in the BW we need to implement an extension of the MGD-decoupling, like in Ref. [30], which will not be studied in this paper. Instead, we will see that still it is possible to generate new black hole solutions in the BW by the MGD-decoupling when the vacuum is filled not only with the BW fields \( \{U, \mathcal{P}\} \) but also with a generic source \( \theta_{\mu\nu} \), as implemented in Ref. [33]. We will show this next.

### A. Black holes by MGD-decoupling in the BW

Let us start by identifying the energy-momentum tensor \( T_{\mu\nu} \) in equation (3) as the one representing the BW sector, hence
\[
T_{\mu\nu}^{(tot)} = \frac{6}{\sigma} S_{\mu\nu} + \frac{1}{8\pi} \mathcal{E}_{\mu\nu} + \theta_{\mu\nu}. \quad (53)
\]

Therefore after decoupling the system (35)-(7) we end with i) Einstein equations for a pure BW sector in equations (16)-(18) to determine \( \{T_{\mu\nu}, \xi, \mu\} \), and ii) field equations for the source \( \theta_{\mu\nu} \) in equations (20)-(22) to determine \( \{\theta_{\mu\nu}, f^*\} \).

Among all known BH solutions in the BW, here we will be interested in to deform the simplest one, namely,
the well-known tidally charged solution whose deformed version, under MGD-decoupling, reads

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{q}{r^2}\right) dt^2 - \frac{dr^2}{1 - 2M/r - \frac{q}{r^2} + \alpha f^* r^2} - r^2 d\Omega^2 .$$

(54)

Since the system (20)-(22) has four unknown functions to determine \(\{\theta_{\mu\nu}, f^*\}\), we need to prescribe one additional equation. Next we will demand some physically motivated restriction on the energy-momentum tensor \(\theta_{\mu\nu}\).

Let us start by considering the case of isotropic pressure, so that

$$\theta_{1} = \theta_{2} = \theta_{3} .$$

(55)

Eqs. (20) and (22) yields a differential equation for the MGD function, namely

$$f^* \left(\xi^2 + \frac{2}{r}\right) + f^* \left(2\xi'' + \xi'^2 - 2\frac{\xi'}{r} - \frac{4}{r^2}\right) = 0 .$$

(56)

Following the MGD approach, we plug the the temporal metric component \(\xi\) of the metric shown in (54) in the expression (56), whose general solution is given by

$$f^*(r) = \left(1 - \frac{2M}{r} - \frac{q}{r^2}\right) \left(\frac{r}{\ell_{iso}}\right)^2 e^{\frac{4\xi}{3M}} \left(1 - \frac{M}{r}\right)^{2 + \frac{4\xi}{M}} ,$$

(57)

where \(\ell_{iso}\) is a constant with dimensions of a length. Hence, the MGD radial component for an isotropic deformation of the tidally charged exterior becomes

$$e^{-\lambda} = e^{\xi} + \alpha f^*$$

$$= \left(1 - \frac{2M}{r} - \frac{q}{r^2}\right)$$

$$\times \left[1 + \alpha \left(\frac{r}{\ell_{iso}}\right)^2 e^{\frac{4\xi}{3M}} \left(1 - \frac{M}{r}\right)^{2 + \frac{4\xi}{M}} \right] ,$$

(58)

which is clearly not asymptotically flat for \(r \gg M\). We therefore conclude that the additional source \(\theta_{\mu\nu}\) cannot contain an isotropic pressure if we wish to preserve asymptotic flatness.

Now let us consider that the source \(\theta_{\mu\nu}\) is associated with a conformal gravitational sector. Since the energy-momentum tensor for a conformally symmetric source must be traceless, we have

$$2\theta_{2} = -\theta_{0} - \theta_{1} ,$$

(59)

so that the system (20)-(22) becomes

$$-k^2 \theta_{0} = \frac{f^*}{r^2} + \frac{f'^*}{r} ,$$

(60)

$$-k^2 \theta_{1} = f^* \left(\frac{1}{r^2} + \frac{\xi'}{r}\right) ,$$

(61)

where \(f^*\) is again MGD function and \(\xi\) the non-deformed tidally charged function shown in (54). From Eq. (59), we find the radial deformation must satisfy the differential equation

$$f^* \left(\xi^2 + \frac{2}{r}\right) + f^* \left(2\xi'' + \xi'^2 + 2\frac{\xi'}{r} + \frac{2}{r^2}\right) = 0 .$$

(62)

and it is important to highlight that the conservation equation (23) remains a linear combination of the system (59)-(61). The general solution for Eq. (62) is given by

$$f^*(r) = \left(1 - \frac{2M}{r} - \frac{q}{r^2}\right) \frac{\ell_{e}}{r^2} e^{\frac{3M Ar + T_{\mu\nu}}{\sqrt{-9M^2 - 8q}}} \left(1 - \frac{M}{r}\right)^{2 + \frac{3M Ar + T_{\mu\nu}}{\sqrt{-9M^2 - 8q}}} ,$$

(63)

with \(\ell_{e}\) a constant with units of a length. Thus the conformally deformed tidally charged becomes

$$e^{-\lambda} = \left(1 - \frac{2M}{r} - \frac{q}{r^2}\right)$$

$$\left[1 + \ell_{e} e^{\frac{3M Ar + T_{\mu\nu}}{\sqrt{-9M^2 - 8q}}} \left(1 - \frac{M}{r}\right)^{2 + \frac{3M Ar + T_{\mu\nu}}{\sqrt{-9M^2 - 8q}}} \right] ,$$

(64)

where \(\ell = \alpha \ell_{e}\). The solution in Eq. (64) represents a black hole with the same two horizons that those of the tidally charged solution (Eq. (54) with \(\alpha = 0\)), namely, \(r_{\pm} = M \pm \sqrt{M^2 + q}\).

Now let us consider the source \(\theta_{\mu\nu}\) satisfy the condition of null tangential pressure, namely,

$$\theta_{2} = 0 ,$$

(65)

which according to Eq. (22) yields

$$f^* \left(\xi^2 + \frac{2}{r}\right) + f^* \left(2\xi'' + \xi'^2 + 2\frac{\xi'}{r} + \frac{2}{r^2}\right) = 0 ,$$

(66)

whose solution is given by

$$f^* = C \left(1 - \frac{2M}{r} - \frac{q}{r^2}\right) \left(1 - \frac{M}{r}\right)^{\frac{2M}{r}} e^{\frac{3M Ar + T_{\mu\nu}}{\sqrt{-9M^2 - 8q}}} ,$$

(67)

with \(C\) a constant. Hence the deformed tidally charged solution becomes

$$e^{-\lambda} = \left(1 - \frac{2M}{r} - \frac{q}{r^2}\right)$$

$$\left[1 + C \left(1 - \frac{M}{r}\right)^{\frac{2M}{r}} e^{\frac{3M Ar + T_{\mu\nu}}{\sqrt{-9M^2 - 8q}}} \right] ,$$

(68)

which is not asymptotically flat for \(r \gg M\). We therefore conclude that the source \(\theta_{\mu\nu}\) must contain a non-null tangential pressure if we wish to preserve asymptotic flatness.

**B. Interior solutions by MGD-decoupling in the BW**

Now let us find interior solutions for a self-gravitating system. The deformed interior metric under MGD-
decoupling reads
\[ ds^2 = \left( 1 - \frac{2m(r)}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2m(r)}{r} + \alpha f^* - r^2 d\Omega^2}. \] (69)

Let us remind that after the decoupling, we end with three independent equations, namely, the system \([41]-[43]\), to find three unknown functions \(\{U, P, f^*\}\). By combining Eqs. \([41]-[43]\) we find the first order differential equation for the function \(f^*\), given by
\[
\left( \frac{\xi'}{2} + \frac{2}{r} \right) (f^*)' + \left( \xi'' + \left( \frac{\xi'}{2} \right)^2 + \frac{2\xi'}{r} + \frac{2}{r^2} \right) f^* = k^2 \rho (\rho + 3p),
\] (70)
whose formal solution is
\[
f^*(r) = \frac{D}{F(r)} + \frac{2k^2}{F(r)} \int \frac{F(r) r}{r^2 \xi' + 4} \rho (\rho + 3p) dr,
\] (71)
with
\[ F(r) = \exp \left( \int \left( \xi'' + \left( \frac{\xi'}{2} \right)^2 + \frac{2\xi'}{r} + \frac{2}{r^2} \right) / \left( \frac{\xi'}{2} + \frac{2}{r} \right) dr \right)
\] (72)
where we see that in the vacuum \(\rho = p = 0\) the expression \([71]\) yields the one in \([40]\).

Now it is easy to show, from Eqs. \([41]-[43]\), that the functions \(U\) and \(P\) have the following form for any solution of RG

\[
U = - \frac{8\pi}{3(\xi' r + 4)} \left[ 2\pi \rho (8 + \xi' r) + 12p \right] - f^* \left( \xi'' + \left( \frac{\xi'}{2} \right)^2 + \frac{3\xi'}{2r} \right),
\] (73)
\[
P = \frac{4\pi}{3(\xi' r + 4)} \left[ -4\pi \rho (\rho + 3p) (\xi' r + 2) + f^* \left( \frac{6}{r} \left( \xi' + \frac{1}{r} \right) + (\xi')^2 - \xi'' \right) \right].
\] (74)

Using this expressions we can compute all the components of the source \(\theta_{\mu\nu}\) from Eqs. \([68]-[10]\) and later, from the combinations
\[
\bar{\rho} = \rho + \theta_0^0, \quad \bar{p}_r = p - \theta_1^1, \quad \bar{p}_t = p - \theta_2^2,
\] (75)
it is possible to find the expressions for the effective energy density and the effective pressures.

Then, what we show here is that given a solution of Einstein’s equations in General Relativity it is always possible to extend it to the Brane World scenario using the MGD-decoupling method. It is important to recall that we are assuming that the source \(T_{\mu\nu}\) in \([4]\) corresponds to a perfect fluid. The case where the \(T_{\mu\nu}\) represents a fluid with local anisotropy in pressure can be obtained directly following the same steps that we presented here.

In order to avoid singularities at the surface of our distribution we must impose the well known matching conditions. The exterior geometry in the Brane-Wold context is characterized by a Weyl fluid with
\[
\rho = p = 0, \quad U = U^+, \quad P = P^+,
\] (76)
whose metric can be written in a generically way as
\[
ds^2 = e^{\nu^+} dt^2 - e^{\lambda^+} dv^2 - r^2 d\Omega^2.
\] (77)

From \([66]\) it is easy to see that the effective pressures and the effective density in the outer region will be, in general, different from zero due to the contributions coming from the interaction of our universe with the bulk. On the other hand, from the Eqs. \([41]-[43]\) it is evident that the system of equations for the exterior region has more unknown functions than equations. So, in order to close the system, it is necessary to impose further conditions under \(U^+\) and \(P^+\). Then, unlike the General Relativistic case, in the BW we can have many possible static and spherically symmetric vacuum solutions, in which the Schwarzschild’s vacuum is only a particular case \((U^+ = P^+ = 0)\). It can be shown that, in the most general case, the first and second fundamental form lead to
\[
e^{\nu^+} \bigg|_{r = R} = e^{\nu^+} \bigg|_{r = R},
\] (78)
\[
\left( 1 - \frac{2m(r)}{r} + \frac{1}{\sigma} f^*(r) \right) \bigg|_{r = R} = e^{-\lambda^+} \bigg|_{r = R},
\] (79)
\[
\left( p(r) + \frac{f^*}{8\pi} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) \right) \bigg|_{r = R} = \left( 2 \frac{U^+}{k^2} + \frac{4}{k^4} \frac{P^+}{\sigma} \right) \bigg|_{r = R},
\] (80)

In the General Relativistic scenario, the second fundamental form \([80]\) lead to the condition
\[
p(R) = 0,
\] (81)
however in the BW scenario, even when the physical pressure equal to zero at the surface of the distribution, the effective radial pressure will be different from zero at \(r = R\). Now, in this paper we will consider the case when the condition \([81]\) is satisfied and also the case when only \([80]\) is fulfilled but not \([81]\).

In order to show how the MGD-decoupling method works let us begin with the Tolman IV perfect fluid solution given by
\[ e^\xi = B^2 \left(1 + \frac{r^2}{A^2}\right), \quad (82) \]
\[ e^{-\lambda} = \frac{\left(1 - \frac{r^2}{A^2}\right) \left(1 + \frac{r^2}{A^2}\right)}{(1 + \frac{2r}{\sqrt{A^2}})}, \quad (83) \]
\[ \rho(r) = \frac{3A^4 + A^2(3C^2 + 7r^2) + 2r^2(C^2 + 3r^2)}{8\pi C^2(A^2 + 2r^2)^2}, \quad (84) \]
\[ \rho(r) = \frac{C^2 - A^2 - 3r^2}{8\pi C^2(A^2 + 2r^2)}, \quad (85) \]

where \( A, B, \) and \( C \) are constants that must be determined by the matching conditions. This perfect fluid configuration \( \{p, \rho, \mu, \xi\} \) is a solution of the system \([16][18],[18]_1\), and will have a specific frameworld solution under the MGDecoupling. Then, plugging the expressions in Eqs. \([82]-[85]\) in Eq. \((71)\), we found that

\[
f^*(r) = -\frac{(A^2 + r^2)}{384\pi C^4 r (2A^2 + 3r^2)^{3/2}} \left( r \sqrt{2A^2 + 3r^2} \right.
\]
\[
\times \left( -\frac{9 (A^4 - 4C^4)}{A^2 + 2r^2} + \frac{4A^4 (A^2 + 2C^2)^2}{(A^2 + 2r^2)^3} \right.
\]
\[
+ \frac{17 (A^3 + 2AC^2)^2}{(A^2 + 2r^2)^2} - 36C^2 + 54r^2 \right)
\]
\[
- 24 (A^2 + 2C^2)^2 \tan^{-1} \left( \frac{r}{\sqrt{2A^2 + 3r^2}} \right)
\]
\[
- 48\sqrt{3}C^4 \log \left( \sqrt{6A^2 + 9r^2 + 3r} \right)
\]
\[
+ \frac{D(A^2 + r^2)}{r (2A^2 + 3r^2)^{3/2}}. \quad (86) \]

from which follows that \( f^* \) will be finite at the center of the distribution only if \( D = 0 \) and \( A^2 = 1/6 \). This last condition can be seen easily if we expand \( f^*(r) \) in a power series around \( r = 0 \), which yields

\[
f^*(r) = \frac{\sqrt{3}}{2} \log \left( \sqrt{6}\sqrt{A^2} \right)
\]
\[
- \frac{5r \left( \sqrt{3}\sqrt{A^2} \log \left( \sqrt{6}\sqrt{A^2} \right) \right)}{16\pi \sqrt{A^2} r}
\]
\[
- \frac{3r^2 \left( A^2 + C^2 \right)}{8\pi A^4 C^2} + O(r^3), \quad (87) \]

where it is clear that \( A^2 = 1/6 \) in order to get an expression for the deformation function finite at the origin. Then \( f^*(r) \) can be written as

\[
f^*(r) = -\frac{(6r^2 + 1)}{384\sqrt{3}C^4 r (9r^2 + 1)^{3/2} (12r^2 + 1)^3}
\]
\[
\times \left( \sqrt{3} \left( r \sqrt{9r^2 + 1} (72C^4 (216r^4 + 70r^2 + 5)
\right.
\]
\[
- 24C^2 (1296r^6 + 324r^4 + 10r^2 - 1)
\]
\[
+ 46656r^8 + 11664r^6 + 864r^4 + 26r^2 + 1)
\]
\[
- 72C^4 (12r^2 + 1)^3 \log \left( \sqrt{9r^2 + 1 + 3r} \right)
\]
\[
- (12C^2 + 1)^2 (12r^2 + 1)^3 \tan^{-1} \left( \frac{r}{\sqrt{3r^2 + 5}} \right) \right) \quad (88) \]

Now, using \([88]\) we can found that the functions \( \mathcal{U} \) and \( \mathcal{P} \) can be written as

\[
\mathcal{U} = -\frac{4\pi f^*(r) (18r^2 + 5)}{54r^4 + 15r^2 + 1}
\]
\[
- \frac{3(6C^2 (4r^2 + 1) + 72r^4 + 14r^2 + 1)G_1(r)}{4 (9r^2 + 1)(12Cr^2 + C)^4}, \quad (89) \]
\[
\mathcal{P} = \frac{2pf^*(r) (144r^4 + 22r^2 + 1)}{54r^6 + 15r^4 + r^2}
\]
\[
+ \frac{3rH_1(r) (6C^2 (4r^2 + 1) + 72r^4 + 14r^2 + 1)}{2C^4 (9r^2 + 1)(12r^2 + 1)^3}. \quad (90) \]

where

\[
G_1(r) = 6C^2 (132r^4 + 41r^2 + 3) - 216r^6
\]
\[
- 42r^4 + 72r^2 + 1, \quad (91) \]
\[
H_1(r) = (4 (9r^4 + r^2) - 3C^2 (8r^2 + 1)). \quad (92) \]

1. The case with \( p(R) = 0 \)

Assuming that Eq. \([81]\) is satisfied, then using \([81]\) it can be shown that

\[
C^2 = A^2 + 3R^2. \quad (93) \]

Then we can compute the value of the radial pressure at \( r = R \), which yields

\[
P(R) = \frac{3(I_1(R) + I_2(R) + I_3(R))}{256\pi^2 \sigma R^3 (9R^2 + 1)^{3/2} (216R^4 + 30R^2 + 1)}, \quad (94) \]

where
\[
I_1(R) \equiv 6 - 3R\sqrt{9R^2 + 1} (216R^4 + 66R^2 + 5), \quad (95)
\]
\[
I_2(R) \equiv 3\sqrt{3} (12R^2 + 1)^3 \tan^{-1}\left(\frac{R}{\sqrt{3R^2 + \frac{3}{4}}}\right), \quad (96)
\]
\[
I_3(R) \equiv 2 (12R^2 + 1) (18R^2 + 1)^2 \\
\times \log \left(\sqrt{9R^2 + 1} + 3R\right). \quad (97)
\]

So, in general the radial pressure will be different from zero in the surface of the distribution. Therefore we will not be able to match our internal solution to the Schwarzschild vacuum \((U^+ = P^+ = 0)\) but it is possible to use the exterior solution found at the beginning of this section, which is a deformation of the Schwarzschild metric. In this case the matching conditions \((78)-(80)\), yield

\[
B^2 (6R^2 + 1) = \left(1 - \frac{2M}{R}\right), \quad (98)
\]
\[
\frac{6R^2 + 1}{18R^2 + 1} + \frac{1}{\sigma} f^*(R) = \left(1 - \frac{2M}{R}\right) \\
\times \left[1 + \frac{D}{2\sigma(R - \frac{3}{2}M)}\right], \quad (99)
\]
\[
\frac{f^*(R)}{8\pi\sigma} \left(\frac{12}{6R^2 + 1} + \frac{1}{R^2}\right) = \frac{D}{16\pi\sigma R^2 (R - \frac{3M}{2})}. \quad (100)
\]
From equations \((99)\) and \((100)\) we can find that \(M\) is given by

\[
M = \frac{6R^3}{18R^2 + 1}. \quad (101)
\]

Thus, for a given value of \(R\) and \(\sigma\) we can obtain the value of \(M\). Then it is possible to compute \(D\) and \(B\) from equations \((99)\) and \((98)\) which leads to

\[
D = \frac{(3M - 2R)}{(2M - 1)(18R^2 + 1)} \left(36aR^2M + 2aM - 12aR^3 \\
+ 18R^3 f^*(R) + Rf^*(R)\right), \quad (102)
\]
\[
B^2 = \frac{R - 2M}{R(1 + 6R^2)}. \quad (103)
\]
In order to give an example of the qualitative behavior of the distribution we choose \(R = 0.1\) and \(\sigma = 5\). The results are shown in the figures \((1)-(5)\).

2. The case with \(p(R) \neq 0\).

Now, if we drop the condition \((81)\) it is possible to match our solution we the exterior Schwarzschild vacuum. In this case the matching condition yields

\[
FIG. 1. \text{Qualitative comparison of the pressure for a distribution of } R = 0.1 \text{ in the brane world model (continuous curve) with the general relativistic case (dashed curve) when } p(R) = 0.
\]

\[
FIG. 2. \text{Qualitative comparison of the energy density for a distribution of } R = 0.1 \text{ in the brane world model (continuous curve) with the general relativistic case (dashed curve) when } p(R) = 0.
\]

\[
FIG. 3. \text{Qualitative behavior of the radial pressure (continuous curve) and tangential pressure (dashed curve) } R = 0.1 \text{ when } p(R) = 0.
\]
Then, using Eqs. (106) we can obtain

\[ K_1(R) = 216 (18R^2 + 1) (12R^2 + 1)^3 \times \log \left( \frac{\sqrt{9R^2 + 1} + 3R}{\sqrt{9R^2 + 1}} \right) \]
\[ + 144\sqrt{3} (18R^2 + 1) (12R^2 + 1)^3 \times \frac{R}{\sqrt{3R^2 + \frac{1}{3}}} \]
\[ + 216R (41472\pi \sigma R^8 + 144(80\pi \sigma - 27)R^6 + 12(88\pi \sigma - 123)R^4 \]
\[ + 32(\pi \sigma - 5)R^2 - 5) \sqrt{9R^2 + 1}, \]
\[ K_2(R) = 24\sqrt{3} (12R^2 + 1)^3 (18R^2 + 1) \]
\[ \times \frac{R}{\sqrt{3R^2 + \frac{1}{3}}} - 72R \sqrt{9R^2 + 1} (18R^2 + 1) \left( 20736\pi \sigma R^8 + 144(40\pi \sigma - 9)R^6 + 12(44\pi \sigma - 27)R^4 \right) \]
\[ + 2(8\pi \sigma - 5)R^2 + 1), \]
\[ K_3(R) = -3\sqrt{9R^2 + 1} R + \sqrt{3} (12R^2 + 1)^3 \]
\[ \times (18R^2 + 1) \tan^{-1} \left( \frac{R}{\sqrt{3R^2 + \frac{1}{3}}} \right) - 2519424R \sqrt{9R^2 + 1} R^{11} - 769824R \sqrt{9R^2 + 1} R^9 \]
\[ - 81648R \sqrt{9R^2 + 1} R^7 - 3996R \sqrt{9R^2 + 1} R^5 \]
\[ - 132\sqrt{9R^2 + 1} R^3. \]

Thus, for a given value of \( R \), it is possible to find a value for \( F \) and then with Eqs. (104)-(105) we can find an expression for \( M \) and \( B \)

\[ M = \frac{R^3 (6F^2 + 6R^2 + 1)}{2F^2 (12R^2 + 1) - \frac{R}{2} f^*(R)}, \]

\[ B^2 = \frac{R - 2M}{R(1 + 6R^2)}. \]

Now in order to give an example of the qualitative behavior in this case we choose the values \( R = 0.1 \) and \( \sigma = 5 \). The results are presented in the figures.

V. CONCLUSIONS

In this paper we presented the general formalism to obtain analytical solutions of the effective Einstein equations in the brane world model, using the MGD-decoupling method. In particular, we used this approach to study new black holes solutions in two different ways.
The first one was by deforming the exterior Schwarzschild vacuum, in this case we obtained a solution that was reported in Ref. [8]. Nevertheless, being the only solution we can find deforming the Schwarzschild vacuum, this shows the limitations of the MGD decoupling method in this case. Then, in order to obtain more black holes solutions starting with the Schwarzschild solution it is necessary to use the extended case of the MGD decoupling method. The second approach that we used to study the BH solutions, was based starting with the known tidal charge black hole solution of the brane world. Then, using the MGD-decoupling with different conditions over the source $\theta_{\mu\nu}$ we have been able to find three new black holes solutions. However, by requiring that the new solutions be asymptotically flat, then we concluded that the source $\theta_{\mu\nu}$ can not have isotropic pressures ($\theta^1_1 = \theta^3_1 = \theta^3_3$) or tangential null pressure ($\theta^2_2 = 0$).

We also presented how to obtain internal analytical solutions of the effective Einstein’s field equations. In that sense we showed that every internal solution of the field equations in General Relativity can be extended to the brane world by using the MGD-decoupling. In order to give an example on how the method work, we used the well known Tolman IV internal solution, and using the steps presented in this work it was possible to find a new internal solution in the brane world scenario. Furthermore, we discussed the matching conditions for the obtained solution, for which we found two different possibilities. The first one was assuming that the physical pressure is zero at the surface of the distribution ($p(R) = 0$). In this case we showed that our internal solution could not be coupled with the Schwarzschild’s vacuum and instead of that we uses the external solution that was found at the beginning of the section 4. The second case was based on requiring that the effective radial pressure be zero at the surface and then we was able to found how to match our solution with the Schwarzschild’s vacuum. Now in order to obtain more physically acceptable solutions in the brane world we can choose another solution of the Einstein’s equation in General Relativity and follow the procedure that we presented here or we can also take the same initial solution of General Relativity and then use the extend version of the MGD-decoupling method.
VI. ACKNOWLEDGEMENTS

P. L. wants to say thanks for financial support received by the CONICYT PFCHA/DOCTORADO BECAS CHILE/2019-2119051 and by the Project ANT1856 of the Universidad de Antofagasta. A. S. was partially supported by Project Fondecyt 1161192 (Chile) and also by the MINEDUC-UA project, code ANT 1855.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.
M. Sharif and A. Waseem. Gravitational decoupled anisotropies in compact stars. *Eur. Phys. J.*, C78(5):370, 2018. [arXiv:1802.08000 [gr-qc]].

J. Ovalle, R. Casadio, R. da Rocha, A. Sotomayor, and Z. Stuchlík. Black holes by gravitational decoupling. *Eur. Phys. J.*, C78(11):960, 2018. [arXiv:1804.03468 [gr-qc]].

M. Sharif and S. Sadiq. Gravitational Decoupled Charged Anisotropic Spherical Solutions. *Eur. Phys. J.*, C78(5):410, 2018. [arXiv:1804.09616 [gr-qc]].

E. Contreras. Minimal Geometric Deformation: the inverse problem. *Eur. Phys. J.*, C78(8):678, 2018. [arXiv:1807.03252 [gr-qc]].

E. Contreras and P. Bargueño. Minimal geometric deformation decoupling in 2 + 1 dimensional spacetimes. *Eur. Phys. J.*, C78(7):558, 2018. [arXiv:1805.10565 [gr-qc]].

E. Morales and Francisco Tello-Ortiz. A new family of analytical solutions of Einstein’s equations in the anisotropical domain. *Fortsch. Phys.*, 66(7):1800036, 2018. [arXiv:1804.06874 [gr-qc]].

G. Panotopoulos and A. Rincón. Minimal Geometric Deformation in a cloud of strings. *Eur. Phys. J.*, C78(10):851, 2018. [arXiv:1810.08830 [gr-qc]].

M. Sharif and S. Saba. Gravitational decoupled anisotropic solutions in f(G) gravity. *Eur. Phys. J.*, C78(11):921, 2018. [arXiv:1811.08112 [gr-qc]].

E. Contreras, A. Rincón, and P. Bargueño. A general interior anisotropic solution for a BTZ vacuum in the context of the Minimal Geometric Deformation decoupling approach. *Eur. Phys. J.*, C79(3):216, 2019. [arXiv:1902.02033 [gr-qc]].

S. K. Maurya and F. Tello-Ortiz. Generalized relativistic anisotropic compact star models by gravitational decoupling. *Eur. Phys. J.*, C79(1):85, 2019.

E. Contreras. Gravitational decoupling in 2 + 1 dimensional space-times with cosmological term. *Class. Quant. Grav.*, 36(9):095004, 2019. [arXiv:1901.00231 [gr-qc]].

J. Ovalle, R. Casadio, R. da Rocha, A. Sotomayor, and Z. Stuchlík. Einstein-Klein-Gordon system by gravitational decoupling. *EPL*, 124(2):20004, 2018. [arXiv:1811.08559 [gr-qc]].

M. Sharif and A. Waseem. Anisotropic quark stars in f(R,T) gravity. *Eur. Phys. J.*, C78(10):868, 2018.

E. Morales and Francisco Tello-Ortiz. Compact Anisotropic Models in General Relativity by Gravitational Decoupling. *Eur. Phys. J.*, C78(10):841, 2018. [arXiv:1808.01699 [gr-qc]].

M. Sharif and Sobia Sadiq. Gravitational decoupled anisotropic solutions for cylindrical geometry. *Eur. Phys. J. Plus*, 133(6):245, 2018.

J. Ovalle and A. Sotomayor. A simple method to generate exact physically acceptable anisotropic solutions in general relativity. *Eur. Phys. J. Plus*, 133(10):428, 2018. [arXiv:1811.01300 [gr-qc]].

L. Gabbanelli, J. Ovalle, A. Sotomayor, Z. Stuchlík, and R. Casadio. A causal Schwarzschild-de Sitter interior solution by gravitational decoupling. *Eur. Phys. J.*, C79(6):486, 2019. [arXiv:1905.10162 [gr-qc]].

J. Ovalle, C. Posada, and Z. Stuchlík. Anisotropic ultracompact Schwarzschild star by gravitational decoupling, 2019. [arXiv:11905.12452 [gr-qc]].

P. Cedeño, X. Linares, and E. Contreras. Gravitational Decoupling in Cosmology, 2019. [arXiv:1907.04892 [gr-qc]].

S. Hensh and Z. Stuchlík. Anisotropic Tolman VII solution by gravitational decoupling, 2019. [arXiv:1906.08368 [gr-qc]].

M. Sharif and A. Waseem. Effects of charge on gravitational decoupled anisotropic solutions in f(R) gravity. *Chin. J. Phys.*, 60:426–439, 2019.

S. K. Maurya and Francisco Tello-Ortiz. Charged anisotropic compact star in f(R,T) gravity: A minimal geometric deformation gravitational decoupling approach, 2019. [arXiv:1905.13519 [gr-qc]].

M. Estrada. The extended Gravitational Decoupling method in Pure Lovelock gravity, 2019. [arXiv:1905.12129 [gr-qc]].

R. Da Rocha and Anderson A. Tomaz. Holographic entanglement entropy under the minimal geometrical deformation and extensions, 2019. [arXiv:1905.01548 [gr-qc]].

E. Contreras and P. Bargueño. Extended gravitational decoupling in 2 + 1 dimensional space-times, 2019. [arXiv:1902.09495 [gr-qc]].

E. Contreras and Pedro Bargueño. Minimal Geometric Deformation in asymptotically (A)-dS space-times and the isotropic sector for a polytropic black hole. *Eur. Phys. J.*, C78(12):985, 2018. [arXiv:1809.09820 [gr-qc]].

M. Estrada and R. Prado. The Gravitational decoupling method: the higher dimensional case to find new analytic solutions. *Eur. Phys. J. Plus*, 134(4):168, 2019. [arXiv:1809.03539 [gr-qc]].

R. Pérez Graterol. A new anisotropic solution by MGD gravitational decoupling. *Eur. Phys. J. Plus*, 133(6):244, 2018.

M. Estrada and F. Tello-Ortiz. A new family of analytical anisotropic solutions by gravitational decoupling. *Eur. Phys. J. Plus*, 133(11):453, 2018. [arXiv:1803.02344 [gr-qc]].

C. Las Heras and P. León. New algorithms to obtain analytical solutions of Einstein’s equations in isotropic coordinates, 2019. [arXiv:1905.02380 [gr-qc]].

Naresh Dadhich, Roy Maartens, Philippos Papadopoulos, and Vibhav Rezania. Black holes on the brane. *Phys. Rev. D*, 91(8):084041, 2015. [arXiv:1412.0057 [gr-qc]].

Gian Luigi Alberghi, Roberto Casadio, Octavian Micu, and Alessio Orlandi. Brane-world black holes and (R,T) gravity: A mini- geometric deformation decoupling in asymptotically (A-)dS space-times and the isotropic sector for a polytropic black hole. *Eur. Phys. J.*, C78(12):985, 2018. [arXiv:1809.09820 [gr-qc]].

Jose Ignacio Cembranos and Francisco Tello-Ortiz. Anisotropic compact Schwarzschild star by gravitational decoupling, 2019. [arXiv:11905.12452 [gr-qc]].