The Existence of Quantum Entanglement Catalysts

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Abstract

Without additional resources, it is often impossible to transform one entangled quantum state into another with local quantum operations and classical communication. Jonathan and Plenio [Phys. Rev. Lett. 83, 3566(1999)] presented an interesting example showing that the presence of another state, called a catalyst, enables such a transformation without changing the catalyst. They also pointed out that in general it is very hard to find an analytical condition under which a catalyst exists. In this paper we study the existence of catalysts for two incomparable quantum states. For the simplest case of $2 \times 2$ catalysts for transformations from one $4 \times 4$ state to another, a necessary and sufficient condition for existence is found. For the general case, we give an efficient polynomial time algorithm to decide whether a $k \times k$ catalyst exists for two $n \times n$ incomparable states, where $k$ is treated as a constant.

Index Terms — Quantum information, entanglement states, entanglement transformation, entanglement catalysts.

1 Introduction

Entanglement is a fundamental quantum mechanical resource that can be shared among spatially separated parties. The possibility of having entanglement is a distinguishing feature of quantum mechanics that does not exist in classical mechanics. It plays a central role in some striking applications of quantum computation and quantum information such as quantum teleportation [1], quantum superdense coding [2] and quantum cryptography [3]. As a result, entanglement has been recognized as a useful physical resource [4]. However, many fundamental problems concerning quantum entanglement are still unsolved. An important such problem concerns the existence of entanglement transformation. Suppose that Alice and Bob each have one part of a bi-partite state. The question then is what

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other states can they transform the entangled state into? Since an entangled state is separated spatially, it is natural to require that Alice and Bob can only make use of local operations and classical communication (LOCC). Significant progress in the study of entanglement was made by Bennett, Bernstein, Popescu and Schumacher \[5\] in 1996. They proposed an entanglement concentration protocol which solved the entanglement transformation problem in the asymptotic case. In 1999, Nielsen \[6\] made another important advance. Suppose there is a bi-partite state $|\psi_1\rangle = \sum_{i=1}^{n} \sqrt{\alpha_i}|i\rangle_A|\bar{i}\rangle_B$ shared between Alice and Bob, with ordered Schmidt coefficients (OSCs for short) $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$, and they want to transform $|\psi_1\rangle$ into another bi-partite state $|\psi_2\rangle = \sum_{i=1}^{n} \sqrt{\beta_i}|i\rangle_A|\bar{i}\rangle_B$ with OSCs $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$. It was proved that $|\psi_1\rangle \rightarrow |\psi_2\rangle$ is possible under LOCC if and only if $\lambda_{\psi_1} < \lambda_{\psi_2}$, where $\lambda_{\psi_1}$ and $\lambda_{\psi_2}$ are the vectors of ordered Schmidt coefficients, i.e. $\lambda_{\psi_1} = (\alpha_1, \ldots, \alpha_n)$, $\lambda_{\psi_2} = (\beta_1, \ldots, \beta_n)$, $< \lambda \lambda$ denotes the majorization relation \[7, 8\], i.e. for $1 \leq l \leq n$,

$$\sum_{i=1}^{l} \alpha_i \leq \sum_{i=1}^{l} \beta_i,$$

with equality when $l = n$. This fundamental contribution by Nielsen provides us with an extremely useful mathematical tool for studying entanglement transformation. A simple but significant fact implied by Nielsen’s theorem is that there exist incomparable states $|\psi_1\rangle$ and $|\psi_2\rangle$ with both transformations $|\psi_1\rangle \rightarrow |\psi_2\rangle$ and $|\psi_2\rangle \rightarrow |\psi_1\rangle$ impossible. Shortly after Nielsen’s work, a quite surprising phenomenon of entanglement, namely, entanglement catalysis, was discovered by Jonathan and Plenio \[9\]. They gave an example showing that one may use another entangled state $|c\rangle$, known as a catalyst, to make an impossible transformation $|\psi\rangle \rightarrow |\phi\rangle$ possible. Furthermore, the transformation is in fact one of $|\psi\rangle \otimes |c\rangle \rightarrow |\phi\rangle \otimes |c\rangle$, so that the catalyst $|c\rangle$ is not modified in the process.

Entanglement catalysis is another useful protocol that quantum mechanics provides. Therefore to exploit the full power of quantum information processing, we first have to solve the following basic problem: given a pair of incomparable states $|\psi_1\rangle$ and $|\psi_2\rangle$ with $|\psi_1\rangle \not\rightarrow |\psi_2\rangle$ and $|\psi_2\rangle \not\rightarrow |\psi_1\rangle$, determine whether there exists a catalyst $|c\rangle$ such that $|\psi_1\rangle \otimes |c\rangle \rightarrow |\psi_2\rangle \otimes |c\rangle$. According to Nielsen’s theorem, solving the problem requires determining whether there is a state $|c\rangle$ for which the majorization relation $\lambda_{\psi_1 \otimes c} < \lambda_{\psi_2 \otimes c}$ holds. As pointed out by Jonathan and Plenio \[9\], it is very difficult to find an analytical and both necessary and sufficient condition for the existence of a catalyst. The difficulty is mainly due to lack of suitable mathematical tools to deal with majorization of tensor product states, and especially the flexible ordering of the OSCs of tensor products. In \[9\], Jonathan and Plenio only gave some simple necessary conditions for the existence of catalysts, but no sufficient condition was found. Those necessary conditions enabled them to show that entanglement catalysis can happen in the transformation between two $n \times n$ states with $n \geq 4$. One of the main aims of the present paper is to give a necessary and sufficient condition for entanglement catalysis in the simplest case of entanglement transformation between $4 \times 4$ states with a $2 \times 2$ catalyst. For general case, the fact that an analytical condition under which incomparable states are catalyzable is not easy to find leads us naturally to an alternative approach; that is, to seek some efficient algorithm to decide catalyzability of entanglement transformation. Indeed, an
algorithm to decide the existence of catalysts was already presented by Bandyopadhyay and Roychowdhury [10]. Unfortunately, for two $n \times n$ incomparable states, to determine whether there exists a $k \times k$ catalyst for them, their algorithm runs in exponential time with complexity $O((nk)!^2)$, and so it is intractable in practice. The intractability of Bandyopadhyay and Roychowdhury’s algorithm stimulated us to find a more efficient algorithm for the same purpose, and this is exactly the second aim of the present paper.

This paper is organized as follows. In the second section we deal with entanglement catalysis in the simplest case of $n = 4$ and $k = 2$. A necessary and sufficient condition under which a $2 \times 2$ catalyst exists for an entanglement transformation between $4 \times 4$ states is presented. This condition is analytically expressed in terms of the OSCs of the states involved in the transformation, and thus it is easily checkable. Also, some interesting examples are given to illustrate the use of this condition. The third section considers the general case. We propose a polynomial time algorithm to decide the existence of catalysts. Suppose $|\psi_1\rangle$ and $|\psi_2\rangle$ are two given $n \times n$ incomparable states, and $k$ is any fixed natural number. With the aid of our algorithm, one can quickly find all $k \times k$ catalysts for the transformation $|\psi_1\rangle \rightarrow |\psi_2\rangle$ using only $O(n^{2k+3.5})$ time. Comparing to the time complexity $O((nk)!^2)$ of the algorithm given in [10], for constant $k$, our algorithm improves the complexity from superexponential to polynomial. We make conclusions in section 4, and some open problem are also discussed.

To simplify the presentation, in the rest of the paper, we identify the state $|\psi\rangle = \sum_{i=1}^{n} \sqrt{\gamma_i} |i\rangle$ with the vector of its Schmidt coefficients $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, the meaning will be clear from the context.

## 2 A necessary and sufficient condition of entanglement catalysis in the simplest case ($n = 4, k = 2$)

Jonathan and Plenio [9] has shown that entanglement catalysis only occurs in transformations between $n \times n$ states with $n \geq 4$. In this section, we consider the simplest case that a transformation from one $4 \times 4$ state to another possesses a $2 \times 2$ catalyst. Assume $|\psi_1\rangle = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $|\psi_2\rangle = (\beta_1, \beta_2, \beta_3, \beta_4)$ are two $4 \times 4$ states, where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq 0$, $\sum_{i=1}^{4} \alpha_i = 1$, $\beta_1 \geq \beta_2 \geq \beta_3 \geq \beta_4 \geq 0$, and $\sum_{i=1}^{4} \beta_i = 1$. The potential catalyst is supposed to be a $2 \times 2$ state, denoted by $|\phi\rangle = (c, 1-c)$, where $c \in [0.5, 1]$.

It was proved in [9] that if $|\psi_1\rangle \not\rightarrow |\psi_2\rangle$, but $|\psi_1\rangle \otimes |\phi\rangle \rightarrow |\psi_2\rangle \otimes |\phi\rangle$ then

$$\alpha_1 \leq \beta_1, \quad \alpha_1 + \alpha_2 > \beta_1 + \beta_2, \quad \alpha_1 + \alpha_2 + \alpha_3 \leq \beta_1 + \beta_2 + \beta_3,$$

or equivalently,

$$\alpha_2 + \alpha_3 + \alpha_4 \geq \beta_2 + \beta_3 + \beta_4, \quad \alpha_3 + \alpha_4 < \beta_3 + \beta_4, \quad \alpha_4 \geq \beta_4.$$

(2)

Note that $\{\alpha_i\}$ and $\{\beta_i\}$ are arranged in decreasing order, so we have

$$\beta_1 \geq \alpha_1 \geq \alpha_2 \geq \beta_2 \geq \alpha_3 \geq \alpha_4 \geq \beta_3 \geq \beta_4 \geq \beta_4$$

(3)
These inequalities are merely necessary conditions for the existence of catalyst $|\phi\rangle$, and it is easy to see that they are not sufficient. In the following theorem we give a condition which is both necessary and sufficient.

**Theorem 2.1** There exists a catalyst $|\phi\rangle$ for two states $(|\psi_1\rangle, |\psi_2\rangle)$ with $|\psi_1\rangle \not\rightarrow |\psi_2\rangle$, if and only if

$$
\max \left\{ \frac{\alpha_1 + \alpha_2 - \beta_1}{\beta_2 + \beta_3}, 1 - \frac{\alpha_4 - \beta_4}{\beta_3 - \alpha_3} \right\} \leq \min \left\{ \frac{\beta_1}{\alpha_1 + \alpha_2}, \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, 1 - \frac{\beta_4}{\alpha_3 + \alpha_4} \right\}
$$

and Eq. (7) hold. In addition, for any $c \in [0.5, 1]$ such that

$$
\max \left\{ \frac{\alpha_1 + \alpha_2 - \beta_1}{\beta_2 + \beta_3}, 1 - \frac{\alpha_4 - \beta_4}{\beta_3 - \alpha_3} \right\} \leq c \leq \min \left\{ \frac{\beta_1}{\alpha_1 + \alpha_2}, \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, 1 - \frac{\beta_4}{\alpha_3 + \alpha_4} \right\}
$$

$|\phi\rangle = (c, 1 - c)$ is a catalyst for $(|\psi_1\rangle, |\psi_2\rangle)$.

**Proof:** Assume $|\psi_1\rangle \not\rightarrow |\psi_2\rangle$ but $|\psi_1\rangle \otimes |\phi\rangle \rightarrow |\psi_2\rangle \otimes |\phi\rangle$ under LOCC. From Eq. (8) in [9] we know Eq. (11) holds. So Eq. (2) and Eq. (3) hold too.

A routine calculation shows that the Schmidt coefficients of $|\psi_1\rangle|\phi\rangle$ and $|\psi_2\rangle|\phi\rangle$ are

$$
A = \{\alpha_1 c, \alpha_2 c, \alpha_3 c, \alpha_4 c; \alpha_1 (1 - c), \alpha_2 (1 - c), \alpha_3 (1 - c), \alpha_4 (1 - c)\}
$$

and

$$
B = \{\beta_1 c, \beta_2 c, \beta_3 c, \beta_4 c; \beta_1 (1 - c), \beta_2 (1 - c), \beta_3 (1 - c), \beta_4 (1 - c)\},
$$

respectively. Sort the elements in $A$ and $B$ in decreasing order and denote the resulted sequences by $a^{(1)} \geq a^{(2)} \geq \cdots \geq a^{(8)}$ and $b^{(1)} \geq b^{(2)} \geq \cdots \geq b^{(8)}$. It is clear that $a^{(1)} = \alpha_1 c$, $a^{(8)} = c$, $b^{(1)} = \beta_1 c$, and $b^{(8)} = \beta_4 (1 - c)$. Since $|\psi_1\rangle \otimes |\phi\rangle \rightarrow |\psi_2\rangle \otimes |\phi\rangle$, Nielsen’s theorem tells us that

$$
\sum_{i=1}^{l} a^{(i)} \leq \sum_{i=1}^{l} b^{(i)} \ (\forall 1 \leq l \leq 8)
$$

Since $\{\beta_i\}$ is ordered, and $c \geq 0.5$, thus

$$
\beta_1 c \geq \beta_2 c \geq \beta_3 c \geq \beta_4 c, \ \beta_1 (1 - c) \geq \beta_2 (1 - c) \geq \beta_3 (1 - c) \geq \beta_4 (1 - c), \ \beta_1 c \geq \beta_1 (1 - c) \geq \beta_2 c \geq \beta_3 c. \ \beta_4 (1 - c)
$$

Now we are going to demonstrate that

$$
\beta_1 c \geq \beta_1 (1 - c) > \beta_2 c \geq \beta_3 c \geq \beta_2 (1 - c) \geq \beta_3 (1 - c) \geq \beta_4 (1 - c) \geq \beta_4 c \geq \beta_4 (1 - c).
$$

and consequently fix the ordering of $B$. The key idea is: the sum of the biggest $l$ numbers in a set is greater than or equal to the sum of any $l$ numbers in this set.

First, by definition of $\{a^{(i)}\}$ we have $a^{(1)} + a^{(2)} \geq \alpha_1 c + \alpha_2 c$. So Nielsen’s theorem leads to $b^{(1)} + b^{(2)} \geq a^{(1)} + a^{(2)} \geq \alpha_1 c + \alpha_2 c$. From inequality (11), $a_1 + a_2 > \beta_1 + \beta_2$, so $b^{(1)} + b^{(2)} > \beta_1 c + \beta_2 c$, i.e. $b^{(2)} > \beta_2 c$. Combining this with inequality (5), we see that the only case is $b^{(2)} = \beta_1 (1 - c), b^{(3)} = \beta_2 c$ and $\beta_1 (1 - c) > \beta_2 c$.
Similarly, we have

\[ a^{(1)} + a^{(2)} + a^{(3)} + a^{(4)} \geq \alpha_1 c + \alpha_2 c + \alpha_1 (1 - c) + \alpha_2 (1 - c) = \alpha_1 + \alpha_2. \]

So it holds that

\[ b^{(1)} + b^{(2)} + b^{(3)} + b^{(4)} \geq a^{(1)} + a^{(2)} + a^{(3)} + a^{(4)} \geq \alpha_1 + \alpha_2 > \beta_1 + \beta_2. \]

This implies \( b^{(4)} > \beta_2 (1 - c) \). Then it must be that \( b^{(4)} = \beta_3 c \), and \( \beta_3 c > \beta_2 (1 - c) \).

Now what remains is to determine the order between \( b^{(5)} \) and \( b^{(7)} \). We consider \( b^{(7)} \) first. Nielsen’s theorem yields \( b^{(7)} + b^{(8)} \leq a^{(7)} + a^{(8)} \). By definition, we know that \( a^{(7)} + a^{(8)} \leq \alpha_3 (1 - c) + \alpha_4 (1 - c) \). Therefore,

\[ b^{(7)} + b^{(8)} \leq \alpha_3 (1 - c) + \alpha_4 (1 - c) = (\alpha_3 + \alpha_4) (1 - c) < (\beta_3 + \beta_4) (1 - c), \]

the last inequality is due to \( \Box \). Since \( b^{(8)} = \beta_4 (1 - c) \), it follows that \( b^{(7)} < \beta_3 (1 - c) \).

Furthermore, we obtain \( b^{(7)} = \beta_4 c, b^{(6)} = \beta_3 (1 - c) \), and \( \beta_3 (1 - c) > \beta_4 c \).

Finally, only \( \beta_3 (1 - c) \) leaves, so \( b^{(5)} = \beta_2 (1 - c) \). Combining the above arguments, we finish the proof of inequality \( \Box \).

Clearly, inequality \( \Box \) implies that

\[ \frac{\beta_2}{\beta_2 + \beta_3} < c < \left\{ \frac{\beta_1}{\beta_1 + \beta_2}, \frac{\beta_3}{\beta_3 + \beta_4} \right\} \]  

(7)

This is needed in the remainder of the proof.

Remembering the order of \( B \) has been found out, it enables us to calculate easily \( \sum_{i=1}^{l} b^{(i)} \) for each \( l \). The only rest problem is how to calculate \( \sum_{i=1}^{l} a^{(i)} \). To this end, we need the following simple lemma:

**Lemma 2.1** Assume \( A = \{a_1, \ldots, a_n\} \), \( B = \{b_1, \ldots, b_n\} \). Sort \( B \) in decreasing order and denote the resulted sequence by \( b^{(1)} \geq b^{(2)} \geq \cdots \geq b^{(n)} \). Then \( A \prec B \) if and only if for \( 1 \leq l \leq n \),

\[ \max_{A' \subseteq A, |A'| = l} \sum_{a_i \in A'} a_i \leq \sum_{i=1}^{l} b^{(i)} \]  

(8)

with equality when \( l = n \).

**Proof of Lemma:** The “if” part is obvious. For the “only if” part, we sort \( A \) in decreasing order and denote the resulted sequence by \( a^{(1)} \geq a^{(2)} \geq \cdots \geq a^{(n)} \). Then \( A \prec B \) if and only if for \( 1 \leq l \leq n \),

\[ \sum_{i=1}^{l} a^{(i)} \leq \sum_{i=1}^{l} b^{(i)} \]

It is easy to see that \( \sum_{i=1}^{l} a^{(i)} = \max_{A' \subseteq A, |A'| = l} \sum_{a_i \in A'} a_i \), so the lemma holds.
Proof of Theorem 2.1 (continued): Now the above lemma guarantees a quite easy way to deal with $\sum_{i=1}^{l} a^{(i)}$: enumerating simply all the possible cases. For example, $a^{(1)} + a^{(2)} = \alpha_1 c + \alpha_1 (1 - c)$ or $\alpha_1 c + \alpha_2 c$, i.e. $a^{(1)} + a^{(2)} = \max \{ \alpha_1 c + \alpha_1 (1 - c), \alpha_1 c + \alpha_2 c \}$. The treatments for $\sum_{i=1}^{3} a^{(i)}, \ldots, \sum_{i=1}^{8} a^{(i)}$ are the same. What we still need to do now is to solve systematically the inequalities of $\sum_{i=1}^{l} a^{(i)} \leq \sum_{i=1}^{l} b^{(i)} (1 \leq l \leq 8)$. We put this daunting but routine part in the Appendix.

The above theorem presents a necessary and sufficient condition when a $2 \times 2$ catalyst exists for a transformation from one $4 \times 4$ state to another. Moreover, it is also worth noting that the theorem is indeed constructive. The second part of it gives all $2 \times 2$ catalysts (if any) for such a transformation. To illustrate the utility of the above theorem, let us see some simple examples.

Example 2.1 This example is exactly the original example that Jonathan and Plenio [9] used to demonstrate entanglement catalysis. Let $|\psi_1\rangle = (0.4, 0.4, 0.1, 0.1)$ and $|\psi_2\rangle = (0.5, 0.25, 0.25, 0)$. Then

$$\max \left\{ \frac{\alpha_1 + \alpha_2 - \beta_1}{\beta_2 + \beta_3}, 1 - \frac{\alpha_4 - \beta_4}{\beta_3 - \alpha_3} \right\} = \max \{0.6, 1 - 2/3\} = 0.6,$$

$$\min \left\{ \frac{\beta_1 - \beta_1 - \alpha_1}{\alpha_1 + \alpha_2}, \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, 1 - \frac{\beta_4}{\alpha_3 + \alpha_4} \right\} = \min \{5/8, 2/3, 1 - 0\} = 0.625.$$

Since 0.6 < 0.625, Theorem 2.1 gives us a continuous spectrum $|\phi\rangle = (c, 1 - c)$ of catalysts for $|\psi_1\rangle$ and $|\psi_2\rangle$, where $c$ ranges over the interval [0.6, 0.625]. Especially, when choosing $c = 0.6$, we get the catalyst $|\phi\rangle = (0.6, 0.4)$, which is the one given in [9].

Example 2.2 We also consider the example in [10]. Let $|\psi_1\rangle = (0.4, 0.36, 0.14, 0.1)$ and $|\psi_2\rangle = (0.5, 0.25, 0.25, 0)$. The catalyst for $|\psi_1\rangle$ and $|\psi_2\rangle$ given there is $\phi = (0.65, 0.35)$. Note that

$$\max \left\{ \frac{\alpha_1 + \alpha_2 - \beta_1}{\beta_2 + \beta_3}, 1 - \frac{\alpha_4 - \beta_4}{\beta_3 - \alpha_3} \right\} = \max \{0.52, 1 - 10/11\} = 0.52,$$

$$\min \left\{ \frac{\beta_1 - \beta_1 - \alpha_1}{\alpha_1 + \alpha_2}, \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, 1 - \frac{\beta_4}{\alpha_3 + \alpha_4} \right\} = \min \{25/38, 10/11, 1 - 0\} = 25/38,$$

and 0.52 < 0.65 < 25/38, Theorem 2.1 guarantees that $|\phi\rangle$ is really a catalyst; and it allows us to find much more catalysts $|\phi\rangle = (c, 1 - c)$ with $c \in [0.52, 25/38]$.

3 An efficient algorithm for deciding existence of catalysts

In the last section, we was able to give a necessary and sufficient condition under which a $2 \times 2$ catalyst exists for an transformation between $4 \times 4$ states. The key idea enabling us to obtain such a condition is that the order among the Schmidt coefficients of the tensor product of the catalyst and the target state in the transformation is uniquely determined by Nielsen’s Theorem. However, the same idea does not work when we deal with higher
dimensional states, and it seems very hard to find an analytical condition for existence of catalyst in the case of higher dimension. On the other hand, existence of catalysts is a dominant problem in exploiting the power of entanglement catalysis in quantum information processing. Such a dilemma forces us to explore alternatively the possibility of finding an efficient algorithm for deciding existence of catalysts. The main purpose is to give a polynomial time algorithm to decide whether there is a $k \times k$ catalyst for two incomparable $n \times n$ states $|\psi_1\rangle, |\psi_2\rangle$, where $k \geq 2$ is a fixed natural number.

To explain the intuition behind our algorithm more clearly, we first cope with the case of $k = 2$. Assume $|\psi_1\rangle = (\alpha_1, \ldots, \alpha_n)$, and $|\psi_2\rangle = (\beta_1, \ldots, \beta_n)$ are two $n \times n$ states, and assume that the potential catalyst for them is a $2 \times 2$ state $\phi = (x, 1-x)$. The Schmidt coefficients of $|\psi_1\rangle|\phi\rangle$ and $|\psi_2\rangle|\phi\rangle$ are then given as

$$A_x = \{\alpha_1 x, \alpha_2 x, \ldots, \alpha_n x; \alpha_1 (1-x), \ldots, \alpha_n (1-x)\}$$

and

$$B_x = \{\beta_1 x, \beta_2 x, \ldots, \beta_n x; \beta_1 (1-x), \ldots, \beta_n (1-x)\},$$

respectively. Sort them in decreasing order and denote the resulting sequences by $a^{(1)}(x) \geq a^{(2)}(x) \geq \cdots \geq a^{(2n)}(x)$ and $b^{(1)}(x) \geq b^{(2)}(x) \geq \cdots \geq b^{(2n)}(x)$. By Nielsen’s theorem we know that a necessary and sufficient condition for $|\psi_1\rangle|\phi\rangle \rightarrow |\psi_2\rangle|\phi\rangle$ is

$$\sum_{i=1}^{l} a^{(i)}(x) \leq \sum_{i=1}^{l} b^{(i)}(x) \quad (l = 1, \ldots, 2n).$$

Now the difficulty arises from the fact that we do not know the exact order of elements in $A$ and $B$. Let us now consider this problem in a different way. If we fix $x$ to some constant $x_0$, we can calculate the elements in $A, B$ and sort them. Then if we moves $x$ slightly from $x_0$ to $x_0 + \epsilon$, the order of the elements in $A$ (or $B$) does not change, except the case that $x$ goes through a point $x^*$ with $\alpha_i (1-x^*) = \alpha_j x^*$ (or $\beta_i (1-x^*) = \beta_j x^*$), i.e. $x^* = \frac{\alpha_i}{\alpha_i + \alpha_j}$ (resp. $x^* = \frac{\beta_i}{\beta_i + \beta_j}$) for some $i < j$. This observation leads us to the following algorithm:
Algorithm 1

1. \( \rho_{i,j} \leftarrow \alpha_i \alpha_j \delta_{i,j} \leftarrow \beta_i \beta_j \), \( 1 \leq i < j \leq n \)
2. Sort \( \{\rho_{i,j}\} \cup \{\delta_{i,j}\} \) in nondecreasing order, the resulted sequence is denoted by \( \gamma^{(1)} \leq \gamma^{(2)} \leq \ldots \leq \gamma^{(n^2-n)} \)
3. \( \gamma^{(0)} \leftarrow 0.5, \gamma^{(n^2-n+1)} \leftarrow 1 \)
4. For \( i = 0 \) to \( n^2 - n \) do
5. \( c \leftarrow \frac{\gamma^{(i)} + \gamma^{(i+1)}}{2} \)
6. Determine the order of elements in \( A_c \) and \( B_c \), respectively
7. Solve the system of inequalities:
\[
\left\{ \begin{array}{l}
\sum_{i=1}^{l} a^{(i)}(x) \leq \sum_{i=1}^{l} b^{(i)}(x) \\
\gamma^{(i)} \leq x \leq \gamma^{(i+1)}
\end{array} \right. \quad (l = 1, \ldots, 2n)
\]
8. OUTPUT: Catalysts do not exist, if for all \( i \in \{0,1,\ldots,n^2-n\} \), the solution set of the above inequalities is empty; catalyst \((x, 1-x)\), if for some \( i \) the inequalities has solution.

It is easy to see that this algorithm runs in \( O(n^3) \) time. In [10], an algorithm for the same purpose was also given, but it runs in \( O(n!) \) time.

By generalizing the idea explained above to the case of \( k \times k \) catalyst, we obtain:

**Theorem 3.1** For any two \( n \times n \) states \( |\psi_1\rangle = (\alpha_1, \ldots, \alpha_n) \) and \( |\psi_2\rangle = (\beta_1, \ldots, \beta_n) \), the problem whether there exists a \( k \times k \) catalyst \( |\phi\rangle = (x_1, \ldots, x_k) \) for them can be decided in polynomial time about \( n \). Furthermore, if there exists a \( k \times k \) catalyst, our algorithm can find all the catalysts in \( O(n^{2k+3.5}) \) time.

**Proof.** The algorithm is similar to the one for the case \( k = 2 \). Now the Schmidt coefficients of \( |\psi_1\rangle|\phi\rangle \) and \( |\psi_2\rangle|\phi\rangle \) are

\[
A_x = \{\alpha_1 x_1, \ldots, \alpha_n x_1, \alpha_1 x_2, \ldots, \alpha_n x_2, \ldots, \alpha_n x_k\}
\]
and

\[
B_x = \{\beta_1 x_1, \ldots, \beta_n x_1, \beta_1 x_2, \ldots, \beta_n x_2, \ldots, \beta_n x_k\}.
\]

If we move \( x \) in the \( k \)-dimensional space \( \mathbb{R}^k \), the order of the elements in \( A_x \) (or \( B_x \)) will change if and only if \( x \) goes through a hyperplane \( \alpha_{i_1} x_{i_2} = \alpha_{j_1} x_{j_2} = \beta_{i_1} x_{i_2} = \beta_{j_1} x_{j_2} \) for some \( i_1 < j_1 \) and \( i_2 > j_2 \). (Indeed, the area that \( x \) ranges over should be \( (k-1) \)-dimensional because we have a constrain of \( \sum_{i=1}^{k} x_i = 1 \).) So first we can write down all the equations of these hyperplanes

\[
\Gamma = \{\alpha_{i_1} x_{i_2} = \alpha_{j_1} x_{j_2} | i_1 < j_1, i_2 > j_2\} \cup \{\beta_{i_1} x_{i_2} = \beta_{j_1} x_{j_2} | i_1 < j_1, i_2 > j_2\},
\]

8
where $|\Gamma| = \binom{k}{2} \binom{2}{2} = O(n^2)$. In the $k$-dimensional space $\mathbb{R}^k$, these $O(n^2)$ hyperplanes can at most divide the whole space into $O(O(n^2)^k) = O(n^{2k})$ different parts. Note the number of parts generated by these hyperplanes is a polynomial of $n$. Now we enumerate all these possible parts. In each part, for different $x$, the elements in $A_x$ (or $B_x$) has the same order. Then we can solve the inequalities
\[
\sum_{i=1}^{l} a^{(i)}(x) \leq \sum_{i=1}^{l} b^{(i)}(x) \quad (1 \leq l \leq nk)
\]
and check the order constrains by linear programming. Following the well-known result that linear programming is solvable in $O(n^{3.5})$ time, our algorithm runs in $O(n^{2k+3.5})$ time, it is a polynomial time of $n$ whenever $k$ is a given constant. □

Indeed, Theorem 3.1 is constructive too, and its proof gives an algorithm which is able not only to decide whether a catalyst of a given dimension exists but also to find all such catalysts when they do exist. The algorithm before this theorem is just a more explicit presentation of the proof for the case of $k = 2$.

4 Conclusion and discussion

In this paper, we investigate the problem concerning existence of catalysts for entanglement transformations. It is solved for the simplest case in an analytical way. We give a necessary and sufficient condition for the existence of a $2 \times 2$ catalyst for a pair of two incomparable $4 \times 4$ states. For the general case ($k \times k$ catalysts for $n \times n$ states), although we fail to give an analytical condition, an efficient polynomial time algorithm is found when $k$ is treated as a constant. However, if $k$ is a variable, ranging over all positive integers, the problem of determining the existence of catalysts still remains open. We believe it is NP-hard, since the set $A_x = \{\alpha_1 x_1, \ldots, \alpha_n x_1; \alpha_1 x_2, \ldots, \alpha_n x_2; \ldots; \alpha_n x_k\}$ in the proof of Theorem 3.1 potentially has exponential kind of different orders.

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References

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[2] C. H. Bennett and S. J. Wiesner, “Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states”, Phys. Rev. Lett., vol. 69, pp. 2881–2884, Nov. 1992.
5 Appendix: Proof of Theorem 2.1

Proof of Theorem 2.1 (remaining part): We need to solve the system of inequalities \( \sum_{i=1}^{l} a^{(i)} \leq \sum_{i=1}^{l} b^{(i)} \) \((1 \leq l \leq 8)\). This is carried out by the following items:

1. First, we have:
   \[ a^{(1)} \leq b^{(1)} \iff \alpha_1 c \leq \beta_1 c \iff \alpha_1 \leq \beta_1. \]  
   (9)

2. The inequality \( a^{(1)} + a^{(2)} \leq b^{(1)} + b^{(2)} \) may be rewritten as
   \[ \max\{\alpha_1 c + \alpha_1 (1 - c), \alpha_1 c + \alpha_2 c\} \leq \beta_1 c + \beta_1 (1 - c) \iff \]
   \[ c \leq \frac{\beta_1}{\alpha_1 + \alpha_2}, \quad \alpha_1 \leq \beta_1. \]  
   (10)
   (11)

3. We now consider \( a^{(1)} + a^{(2)} + a^{(3)} \leq b^{(1)} + b^{(2)} + b^{(3)} \). It is equivalent to
   \[ \max\{\alpha_1 c + \alpha_1 (1 - c) + \alpha_2 c, \alpha_1 c + \alpha_2 c + \alpha_3 c\} \leq \beta_1 c + \beta_1 (1 - c) + \beta_2 c \iff \]
   \[ c \leq \left\{ \frac{\beta_1}{\alpha_1 + \alpha_2 + \alpha_3 - \beta_2}, \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} \right\}. \]  
   (12)
(4) It holds that
\[ a^{(1)} + a^{(2)} + a^{(3)} + a^{(4)} \leq b^{(1)} + b^{(2)} + b^{(3)} + b^{(4)} \iff \max\{\alpha_1 c + \alpha_1 (1 - c) + \alpha_2 c + \alpha_2 (1 - c), \alpha_1 c + \alpha_2 c + \alpha_3 c + \alpha_1 (1 - c),
\alpha_1 c + \alpha_2 c + \alpha_3 c + \alpha_4 c\} \leq \beta_1 c + \beta_1 (1 - c) + \beta_2 c + \beta_3 c \iff \frac{\alpha_1 + \alpha_2 - \beta_1}{\beta_2 + \beta_3} \leq c \leq \left\{ \frac{\beta_1}{1 - \beta_2 - \beta_3}, \frac{\beta_1 - \alpha_1}{\alpha_2 + \alpha_3 - \beta_2 - \beta_3} \right\} \] (13)

(5)
\[ a^{(1)} + a^{(2)} + a^{(3)} + a^{(4)} + a^{(6)} \leq b^{(1)} + b^{(2)} + b^{(3)} + b^{(4)} + b^{(5)} \iff a^{(6)} + a^{(7)} + a^{(8)} \geq b^{(6)} + b^{(7)} + b^{(8)} \iff \min\{\alpha_2 (1 - c) + \alpha_3 (1 - c) + \alpha_4 (1 - c), \alpha_3 (1 - c) + \alpha_4 c + \alpha_4 (1 - c)\} \geq \beta_3 (1 - c) + \beta_4 c + \beta_4 (1 - c) \iff 1 - \frac{\alpha_4 - \beta_4}{\beta_3 - \alpha_3} \leq c \leq 1 - \frac{\beta_4}{\alpha_2 + \alpha_3 + \alpha_4 - \beta_3} \] (14)

(6)
\[ \sum_{i=1}^{6} a^{(i)} \leq \sum_{i=1}^{6} b^{(i)} \iff a^{(7)} + a^{(8)} \geq b^{(7)} + b^{(8)} \iff \min\{\alpha_3 (1 - c) + \alpha_4 (1 - c), \alpha_4 c + \alpha_4 (1 - c)\} \geq \beta_4 c + \beta_4 (1 - c) \iff c \leq 1 - \frac{\beta_4}{\alpha_3 + \alpha_4}, \quad \alpha_4 \geq \beta_4 \] (15)

(7) We have
\[ \sum_{i=1}^{7} a^{(i)} \leq \sum_{i=1}^{7} b^{(i)} \iff a^{(8)} \geq b^{(8)} \iff \alpha_4 \geq \beta_4 \] (16)

Combining Eq. 1(13, 14, 15) we obtain
\[ c \leq \left\{ \frac{\beta_1}{\beta_1 + \beta_2}, \frac{\beta_3}{\beta_3 + \beta_4}, \frac{\beta_1}{\alpha_1 + \alpha_2}, \frac{\beta_1}{\alpha_1 + \alpha_2 + \alpha_3 - \beta_2}, \frac{\beta_1}{\alpha_2 - \beta_2}, \frac{\beta_1}{1 - \beta_2 - \beta_3}, \frac{\beta_1 - \alpha_1}{\alpha_2 + \alpha_3 - \beta_2 - \beta_3}, 1 - \frac{\beta_4}{\alpha_2 + \alpha_3 + \alpha_4 - \beta_3}, \frac{\beta_1 - \alpha_1}{\alpha_2 + \alpha_3 - \beta_2 - \beta_3}, \frac{\beta_1 - \alpha_1}{\alpha_2 + \alpha_3 - \beta_2 - \beta_3}, 1 - \frac{\beta_4}{\alpha_3 + \alpha_4} \right\} \] (17)

and
\[ c \geq \left\{ \frac{\alpha_1 + \alpha_2 - \beta_1}{\beta_2 + \beta_3}, 1 - \frac{\alpha_4 - \beta_4}{\beta_3 - \alpha_3} \right\} \] (18)

Since
\[ \beta_1 \geq \alpha_1 \geq \alpha_2 > \beta_2 \geq \beta_3 > \alpha_3 \geq \alpha_4 \geq \beta_4, \alpha_1 + \alpha_2 > \beta_1 + \beta_2, \]
\[ ^{1}\text{if } \alpha_2 + \alpha_3 - \beta_2 - \beta_3 \leq 0, \text{ this term is useless.} \]
it follows that
\[
\frac{\beta_1}{\beta_1 + \beta_2} > \frac{\beta_1}{\alpha_1 + \alpha_2},
\]
\[
\frac{\beta_3}{\beta_3 + \beta_4} = 1 - \frac{\beta_4}{\beta_3 + \beta_4} > 1 - \frac{\beta_4}{\alpha_3 + \alpha_4},
\]
\[
\frac{\beta_1}{\alpha_1 + \alpha_2} < \frac{\beta_1}{\alpha_1 + \alpha_2 + (\alpha_3 - \beta_2)},
\]
\[
1 - \frac{\beta_4}{\alpha_2 + \alpha_3 + \alpha_4 - \beta_3} > 1 - \frac{\beta_4}{\alpha_3 + \alpha_4},
\]
and
\[
\frac{\beta_1 - \alpha_1}{\alpha_2 + \alpha_3 - \beta_2 - \beta_3} \geq \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}.
\]

This indicates that there are six useless terms in Eq. (17), so we can omit them. Now we get
\[
\max \left\{ \frac{\alpha_1 + \alpha_2 - \beta_1}{\beta_2 + \beta_3}, 1 - \frac{\alpha_4 - \beta_4}{\beta_3 - \alpha_3} \right\} \leq c \leq \min \left\{ \frac{\beta_1}{\alpha_1 + \alpha_2}, \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, 1 - \frac{\beta_4}{\alpha_3 + \alpha_4} \right\}.
\]

Therefore, Eq. (4) is a necessary condition for the existence of catalyst.

On the other hand, we claim that Eq. (1) and Eq. (4) are the sufficient conditions. Indeed, if we choose a \( c \) satisfies Eq. (2.1), then \( c \) satisfies Eq. (17) and (18). From Eq. (9-16) we know that \( \sum_{i=1}^{k} a^{(i)} \leq \sum_{i=1}^{k} b^{(i)} \), i.e. \( |\psi_1\rangle|\phi\rangle \rightarrow |\psi_2\rangle|\phi\rangle \) under LOCC. This completes the proof.