Large-scale structures predicted by linear models of wall-bounded turbulence

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Large-scale structures predicted by linear models of wall-bounded turbulence

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Abstract.
The objective of this article is to determine for which scales stochastic forcing of the linearized Navier-Stokes equations, recast as the resolvent operator, is sufficient to reproduce second-order statistics in turbulent channel flow. Our focus is on the large scales at a friction Reynolds number of $Re_f = 2003$. We consider a molecular resolvent operator, where only the kinematic viscosity appears, and an eddy resolvent operator, where the kinematic viscosity is augmented with an eddy viscosity profile. The molecular resolvent operator is able to identify the wall-normal height where the maximum energy of a structure is located, but it fails to predict the most energetic wave speed. It also overestimates the streamwise velocity component and underestimates the spatial support of the structures in the wall-normal direction. When the eddy resolvent operator identifies the most energetic wave speed, it also predicts the correct statistics for a given spatial scale. For spatial scales where this criterion is not met, the eddy viscosity overdamps the linear response. As a result, it predicts energetic wave speeds which are too low and velocity structures which are too energetic close to the wall. We conclude that eddy viscosity works best for structures which are most energetic in the wake region while its performance deteriorates for structures that are active in the log region.

1. Introduction
The Navier-Stokes equations linearized around the mean flow have successfully identified coherent structures in a variety of flows where linear mechanisms are important. In particular, the resolvent-based approach of McKeon & Sharma [1], where any terms nonlinear in the perturbations are treated as an unknown forcing, has reproduced key features of wall-bounded turbulent flows [2, 3]. Recent work has suggested that adding an eddy viscosity to the linearized Navier-Stokes equations leads to improved predictions of coherent structures in wall-bounded turbulence. While the use of an eddy viscosity dates back to the work of Reynolds & Hussain [4], it has subsequently been used by many studies to investigate transient growth [5, 6, 7], energy amplification due to stochastic and harmonic forcing [8], and the impulse response [9].

The performance of the linearized Navier-Stokes equations has also been studied in the context of estimation by Illingworth et al. [10]. It was observed that the linear estimator without an eddy viscosity tended to overestimate the streamwise velocity. This behavior is a consequence of component-type non-normality [11, 12], where the resolvent favors disturbances that maximize lift-up due to the presence of mean shear. The most amplified output is aligned in the streamwise direction while its corresponding input is in the streamwise-perpendicular plane. To improve the estimation, Illingworth et al. [10] added an eddy viscosity to the linear model and were
able to quantify its accuracy. The linear model was also able to predict the range of spatial wavenumbers where the estimation performed well.

An alternative estimation technique was developed in Madhusudanan et al. [13], who used spectral linear stochastic estimation to build linear estimators from the linearized Navier-Stokes equations. Similar to Illingworth et al. [10], one of the main conclusions was that adding eddy viscosity to the linear operator improved its ability to predict the wall-normal coherence of structures. In terms of estimation of the fluctuating velocity field, however, it overestimated the intensity of fluctuations at the wall. The molecular viscosity case, on the other hand, failed to capture the wall-normal coherence of structures due to the critical layer mechanism [1], which localizes the perturbations at a wall-normal height where the disturbance travels at a wave speed equal to the mean velocity. This behavior was also remarked upon by Morra et al. [14], who demonstrated that the eddy viscosity correctly predicted second-order velocity statistics for both a scale representative of the near wall streaks and a large scale in the outer region.

In this paper, we are interested in understanding the spatial scales and temporal frequencies for which stochastic forcing is appropriate to identify the correct statistics and structures. The predictions from resolvent analysis without and with eddy viscosity are compared to data from a direct numerical simulation (DNS) of channel flow at a friction Reynolds number of $Re_τ = 2003$. In Section 2, we provide a brief overview of resolvent analysis and make clear the differences between the molecular and eddy viscosity cases. In Section 3, we describe how the predictions from stochastic forcing of the operators are compared to DNS using second-order velocity statistics and spectral proper orthogonal decomposition (SPOD). The DNS dataset is described in Section 4 and in Section 5, we compare the performance of the two resolvent operators across all temporal frequencies for the largest scales. Finally, we conclude in Section 6 and suggest future directions for resolvent-based predictions of wall-bounded shear flows.

2. Resolvent analysis

The linear operators for resolvent analysis are derived from the incompressible Navier-Stokes equations, which, for turbulent channel flow, are non-dimensionalized by the channel half height $h$ and the friction velocity $u_τ = \sqrt{τ_w/ρ}$:

$$\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} = -\nabla \tilde{p} + Re_τ^{-1} \nabla^2 \tilde{u},$$

$$\nabla \cdot \tilde{u} = 0.$$ (1b)

Here, $\tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w})$ and $\tilde{p}$ are the velocity and pressure, respectively, $ρ$ is the density, and $τ_w$ is the wall shear stress. The friction Reynolds number is $Re_τ = h u_τ/ν$ and $ν$ is the kinematic viscosity. The flow is periodic in the streamwise and spanwise directions, denoted as $x$ and $y$. The wall-normal coordinate $z$ extends from lower to upper walls at $z/h = −1$ and $z/h = 1$, respectively, where no slip and no penetration boundary conditions are enforced.

Eq. 1 is Reynolds-decomposed into a mean and fluctuating component

$$\tilde{u}(x, y, z, t) = U(z) + u(x, y, z, t),$$ (2)

where $U(z)$ is the mean velocity profile that varies in the wall-normal direction only. Due to homogeneity in the streamwise and spanwise directions and in time, the fluctuations are expressed as

$$\tilde{u}(k_x, k_y, \omega; z) = \int_{−∞}^{∞} \int_{−∞}^{∞} \int_{−∞}^{∞} u(x, y, z, t) e^{−i(k_x x + k_y y − ω t)} dx dy dt,$$ (3)

where $k_x$ is the streamwise wavenumber, $k_y$ is the spanwise wavenumber, and $ω$ is the angular frequency. After elimination of the pressure, the equations for the wall-normal velocity and
vorticity ($\tilde{\eta} = ik_y \tilde{u} - ik_x \tilde{v}$) fluctuations become

$$-i\omega \left( \begin{array}{c} \tilde{w} \\ \tilde{\eta} \end{array} \right) + \left( \begin{array}{cc} \Delta & 0 \\ 0 & L \end{array} \right)^{-1} \left( \begin{array}{cc} \mathcal{L}_{OS} & 0 \\ ik_y U' & \mathcal{L}_{SQ} \end{array} \right) \left( \begin{array}{c} \tilde{w} \\ \tilde{\eta} \end{array} \right) = B \tilde{f},$$

where $k^2 = k_x^2 + k_y^2$, $\mathcal{D}$ and $'$ represent wall-normal differentiation, $\Delta = \mathcal{D}^2 - k^2$, $\tilde{f}$ represents all nonlinear terms, and

$$B = \left( \begin{array}{cc} \Delta & 0 \\ 0 & 1 \end{array} \right)^{-1} \left( \begin{array}{ccc} -ik_x \mathcal{D} & -ik_y \mathcal{D} & -k^2 \\ ik_y & -ik_x & 0 \end{array} \right).$$

The Orr-Sommerfeld $\mathcal{L}_{OS}$ and Squire $\mathcal{L}_{SQ}$ operators are

$$\mathcal{L}_{OS} = -ik_x U \Delta + ik_x U'' + Re^{-1} \Delta^2,$$

$$\mathcal{L}_{SQ} = -ik_x U + Re^{-1} \Delta.$$  

Eq. 4 can be arranged into the following input-output form

$$\hat{\mathcal{u}} = \mathcal{H}(k_x, k_y, \omega) \tilde{f},$$

where

$$\mathcal{H}_m(k_x, k_y, \omega) = C(-i\omega + L)^{-1} B,$$

is the molecular resolvent operator, henceforth referred to as linearized Navier-Stokes (LNS), as denoted by the subscript $m$. The output matrix $C$ is defined as

$$C = \frac{1}{k^2} \left( \begin{array}{cc} ik_x \mathcal{D} & -ik_y \\ ik_y \mathcal{D} & ik_x \\ \mathcal{L} & 0 \end{array} \right),$$

and $L = \mathcal{M} \mathcal{L}$.

The resolvent operator can be augmented with an eddy viscosity profile [15, 4]

$$\nu_T(z) = \frac{\nu}{2} \left( 1 + \frac{\kappa}{3} (1 - z^2) (1 + 2z^2) \left( 1 - e^{\left(1 - \frac{1}{2} Re \right)} \right)^2 \right)^{1/2} + \frac{\nu}{2},$$

where $\kappa = 0.426$ and $A = 25.4$ are chosen based on a least-squares fit to experimentally obtained mean velocity profiles at $Re = 2000$ [5]. The addition of eddy viscosity modifies $\mathcal{L}_{OS}$ and $\mathcal{L}_{SQ}$ to

$$\mathcal{L}_{OS} = -ik_x U \Delta + ik_x U'' + \nu_T \Delta^2 + 2\nu_T \mathcal{D} \Delta + \nu_T'' (\mathcal{D}^2 + k^2),$$

$$\mathcal{L}_{SQ} = -ik_x U + \nu_T \Delta + \nu_T' \mathcal{D}.$$  

Consequently, the resolvent operator $\mathcal{H}_e(k_x, k_y, \omega)$ can also be constructed with an eddy viscosity profile and we refer to this as eLNS. To extract coherent structures, the resolvent operators $\mathcal{H}_m$ and $\mathcal{H}_e$ are decomposed using the singular value decomposition (SVD)

$$\mathcal{H} = \Psi \Sigma \Phi^*.$$

The matrix $\Psi$ consists of velocity structures $\hat{\psi}_j$, which are ranked in terms of their kinetic energy gain $\sigma_j$ as indicated by the diagonal matrix $\Sigma$. The matrix $\Phi$ contains structures $\hat{\phi}_j$ which are ranked by how susceptible they are to linear amplification. Under the assumption of stochastic forcing, there is no preferred projection of $\tilde{f}$ onto any given $\hat{\phi}_j$, so the most amplified velocity structures $\hat{\psi}_j$ should correspond to those observed in the flow.
3. Methods

3.1. Second-order velocity statistics

We analyze the predictive capabilities of the two models by computing second-order velocity statistics for an arbitrary $k$. These are obtained by multiplying Eq. 7 by its conjugate transpose

$$
\hat{u}^{*} = H \hat{f} \hat{f}^{*} H^{*},
$$

(13)

where $S_{uu}$ and $S_{ff}$ are the cross-spectral density tensors of the velocity fluctuations and nonlinear terms, respectively. Our objective is to determine for which scales stochastic forcing is sufficient, so Eq. 13 reduces to

$$
S_{uu} = HH^{*}.
$$

(14)

If the true nonlinear forcing is white, then LNS will correctly reproduce the velocity statistics. If it is not, i.e. $S_{ff} \neq I$, then the eddy viscosity needs to model $S_{ff}$ in order for eLNS to reproduce the correct statistics with stochastic forcing.

3.2. Spectral proper orthogonal decomposition

We can also compare the flow structures observed in DNS to those predicted from LNS and eLNS. The velocity structures are extracted from the DNS data using spectral proper orthogonal decomposition (SPOD) [16]. The data-computed $S_{uu}$ is expressed in terms of its eigenvalue decomposition, i.e.

$$
S_{uu} = V \Lambda V^{-1},
$$

(15)

where $V$ are the SPOD modes which are ranked in terms of their kinetic energy in the diagonal matrix $\Lambda$. The cross spectral density tensor can also be expressed in terms of resolvent modes by rewriting the resolvent operators in Eq. 14 in terms of their singular value decompositions

$$
S_{uu} = \Psi \Sigma \Phi^{*} \Phi \Sigma \Psi^{*} = \Psi \Sigma^{2} \Psi^{*}.
$$

(16)

If the nonlinear forcing is white, then the right-hand sides of Eq. 15 and 16 can be equated

$$
V \Lambda V^{-1} = \Psi \Sigma^{2} \Psi^{*}.
$$

(17)

It follows from Eq. 17 that (i) SPOD and resolvent modes are equivalent [17] and (ii) the kinetic energy of each SPOD mode is equal to the amplification factor squared of each resolvent mode, i.e. $\lambda_{j} = \sigma_{j}^{2}$.

3.3. Role of the nonlinear forcing

Although the nonlinear forcing is treated as white in space and time in this article, it is not white in space or time for an actual turbulent flow. The coloring in space has an impact on the structure of velocity fluctuations and the coloring in time plays a role in setting their amplitude. If the linear model (LNS or eLNS) can reproduce the correct structure for a given $k$, then stochastic forcing in space is appropriate. Stochastic forcing in time is suitable when the linear model can reproduce the correct temporal power spectrum for a given wavenumber pair $(k_{x}, k_{y})$.

4. DNS dataset

We compare velocity statistics and structures for incompressible turbulent channel flow at $Re_{tau} = 2003$. The DNS has been performed by the Universidad Politécnica de Madrid [18] and the range of wavenumbers provided are $0.25 \leq |k_{x}| \leq 8$ and $0.67 \leq k_{y} \leq 21.0$, where the channel half-width is assumed to be $h = 1$ (see Madhusudanan et al. [13] for more details).
1146 time-resolved snapshots are available at a time resolution of $\Delta t = 0.27$. The cross-spectral densities are estimated using Welch’s method [19]. To this end, the data are divided into overlapping blocks containing 256 snapshots of the flow with an overlap of 75% resulting in 14 blocks. In order to improve convergence for the largest scales, the number of blocks is doubled by adding $k_x = -k_x$ snapshots. Since the signs for $k_x$ and $\omega$ are opposite in Eq. 3, the temporal frequency spectrum must be reflected, i.e. bins corresponding to $\omega$ now correspond to $-\omega$, for the $-k_x$ snapshots. While 28 blocks is slightly lower than values used by Muralidhar et al. [20] and Morra et al. [14], they are sufficient to obtain convergence for the quantities of interest in this study, namely the second-order velocity statistics and leading SPOD modes for any given $k$.

5. Results

5.1. Predictive capabilities of LNS and eLNS

We now investigate the predictive capabilities of LNS and eLNS with stochastic forcing in space and time. First, the second-order velocity statistics are computed using Eq. 14 and are directly compared to DNS. In order to quantify their performance against the DNS, an error $E$ is computed for a $(k_x, k_y)$ wavenumber pair

$$E(k_x, k_y) = \int \omega W(\omega) \frac{\text{trace} [\mathcal{H}(\omega)\mathcal{H}^*(\omega) - S_{uu}^{DNS}(\omega)]}{\text{trace} [S_{uu}^{DNS}(\omega)]} d\omega.$$  (18)

Due to the limited number of snapshots available, $E$ is integrated across all relevant temporal frequencies for smoother results. The linear model does not provide the amplitude of the energy, so the second-order statistics computed from the model as well as the DNS are normalized in Eq. 18 such that their maximum is unity. The values of $E(k_x, k_y)$, therefore, assume values between zero and one, i.e. $0 \leq E(k_x, k_y) \leq 1$. An additional weighting factor $W(\omega)$ is introduced to weight each $\omega$ by its contribution to the overall kinetic energy for the $(k_x, k_y)$ selected. This allows us to discount non-energetic wave speeds where the linear models are not expected to provide good predictions.

Figure 1 maps the error for both models in wavenumber space. For LNS, we see in Figure 1(a) that the error for almost all the scales is close to one, indicating that LNS is not correctly predicting any scales. For eLNS, on the other hand, we see from Figure 1(b) that the error depends primarily on the choice of $k_y$. The smallest errors are obtained for $k_y \approx 2$ and the error increases when moving away from this trough. We select two wavenumber pairs for further analysis in the remainder of the paper. The first is $(k_x = 1, k_y = 2.67)$, where the error reaches a local minimum for eLNS as seen in Figure 1(d). It is a good choice, therefore, to illustrate when eLNS produces good predictions. The second is $(k_x = 1, k_y = 6)$, which has an error approximately three times as large as its $k_y = 2.67$ counterpart for eLNS. As such, it is capable of illustrating the shortcomings of eLNS. Additionally, the advantage of keeping $k_x = 1$ constant between the two scales is that the resolution in terms of wave speed, which is defined as $c = \omega/k_x$, is the same.

5.2. Correctly predicted scales

The second-order statistics for $(k_x = 1, k_y = 2.67)$ are computed from DNS, LNS, and eLNS for the most energetic wave speed $c^+ = 22$. The auto and cross correlations for all three velocity components are compared in Figure 2. The statistics from the DNS in Figure 2(a) illustrate a tall structure for which the streamwise velocity component is the most energetic. The LNS predictions in Figure 2(b) are highly localized structures which are dominated by the streamwise velocity. The wall-normal component tends to have the lowest energy of the three velocity components and the model’s prediction significantly underestimates its magnitude. In
Figure 1. The error between the kinetic energy of the DNS and that predicted by (a) LNS and (b) eLNS with stochastic forcing. The error for a slice along $k_x = 1$ is plotted in (c) and (d) for LNS and eLNS, respectively. In all four panels red dots denote the wavenumber pairs $(k_x = 1, k_y = 2.67)$ and $(k_x = 1, k_y = 6)$ chosen for further analysis.

Figure 2. $S_{uu}(k_x = 1, k_y = 2.67, c^+ = 22)$ for the lower channel half. All nine components have been plotted for (a) DNS, (b) LNS, and (c) eLNS. Since linear analysis does not yield the amplitude of the statistics, each panel has been normalized such that its maximum value is one.

Figure 2(b), the streamwise component is so strong that no energy is visible for the wall-normal component. The eLNS predictions in Figure 2(c), meanwhile, are a good match for all three velocity components with respect to their relative magnitudes and wall-normal support.

As discussed in Section 3, the results can be probed further by comparing the most amplified modes from resolvent analysis to the most energetic modes obtained from SPOD. This allows us to directly compare structures rather than statistics. We expect to see good agreement between the most energetic SPOD modes and eLNS modes for $(k_x = 1, k_y = 2.67)$ at the most energetic wave speed since we already obtained good agreement for the second-order velocity statistics. We compare the leading pair of SPOD modes to the leading pair of resolvent modes for LNS and eLNS in Figure 3.
Figure 3. The most energetic pair of SPOD modes for \((k_x = 1, k_y = 2.67, c^+ = 22)\) compared to the most amplified pair of resolvent modes from LNS and eLNS. The absolute value of all three velocity components has been plotted and the modes are normalized such that the maximum streamwise velocity is unity.

The streamwise and wall-normal velocity components of the LNS modes peak at approximately the correct wall-normal height when compared to the SPOD modes. However, there is poor agreement for the spanwise component. Similar to the velocity statistics in Figure 2(b), the LNS modes are confined to the critical layer and the wall-normal component is significantly underestimated. The SPOD modes are not confined to the critical layer and the wall-normal velocity is not weak with respect to the other components. The lack of agreement signifies that the nonlinear terms are structured and, therefore, stochastic forcing of the molecular operator is not appropriate.

The eLNS modes, on the other hand, are in good agreement with the SPOD modes for all three velocity components. This is consistent with the close match observed for the velocity statistics computed from DNS and eLNS in Figure 2(c). There are some discrepancies between the SPOD modes and eLNS modes, particularly near the walls for the streamwise velocity. Nevertheless, the overall agreement suggests stochastic forcing is effective for eLNS at this scale. The effects of the neglected nonlinear terms, therefore, have been correctly subsumed into the eddy resolvent operator for this particular scale.

So far, we have concentrated only on the most energetic wave speed. We can now look across various wave speeds for the \((k_x = 1, k_y = 2.67)\) scale in order to isolate those wave speeds for which we can obtain good predictions from the linear models. In Figure 4(a), the contribution of each wave speed to the variance \(u' u'\) is calculated for the DNS. The most energetic wave speed contributes significantly more to the variance than all other wave speeds, reinforcing the fact
Figure 4. The contribution of the most energetic wave speeds to the streamwise variance for $(k_x = 1, k_y = 2.67)$ computed from (a) DNS, (b) LNS, and (c) eLNS. Each curve is normalized by the peak contribution from the $c^+=22$ mode.

that the kinetic energy and wall-normal profiles for a particular $(k_x, k_y)$ can be recovered from a sparse number of temporal frequencies [21]. Figure 4(b) reveals that LNS is capable of predicting the most energetic wave speed, which for this scale is $c^+=22$, but it overestimates its importance relative to the other wave speeds. LNS is, nonetheless, capable of identifying the correct wall-normal height where the structures peak even though they are highly localized at their respective critical layers.

With respect to eLNS, there are two important observations to note from Figure 4(c). The first is that the variance profiles are accurate for the most energetic wave speed only. While the agreement is acceptable for $c^+=19.8$, it degrades for lower $c^+$ as eLNS predicts structures whose energy peaks too close to the wall. The second observation is that the relative contributions from each wave speed to the total variance are only approximately predicted by eLNS since the less dominant wave speeds are too energetic. As suggested by Morra et al. [14], one possible method for addressing this disagreement is to replace white in time forcing with the power spectrum, i.e.

$$\int \omega S_{uu}(k_x, k_y) d\omega = \int \omega a(\omega) \mathcal{H}_e \mathcal{H}_e^* d\omega,$$

where $a(\omega)$ is the power spectrum associated with the wavenumber pair $(k_x, k_y)$. Eq. 19 would increase the weight of $c^+=22$ with respect to the other wave speeds and lead to closer agreement with DNS.

From these observations, we can conclude that eLNS provides a reasonable approximation of the power spectrum, or coloring in time (e.g. [22]), of scales around $k_y \approx 2$, which was the trough identified in Figure 1. To reinforce this conclusion, we compute the power spectra in Figure 5 for various $k_x$ while holding $k_y = 2.67$ constant. The power spectra for the DNS in Figure 5(a) are compared to the sum of the singular values squared of LNS and eLNS in Figure 5(b) and 5(c), respectively. All scales are normalized by the maximum of the $k_x=1$ case. LNS predicts the most energetic wave speed $c^+_{\text{max}}$ to be slightly higher than that of the DNS. Furthermore, it predicts the energy to be highly concentrated around $c^+_{\text{max}}$, which is not representative of the true power spectrum. By contrast, eLNS selects a $c^+_{\text{max}}$ in better agreement with DNS and approximates the power spectrum for each choice of $k_x$. It is also able to predict the relative amplitude of each $k_x$ scale.

The scales chosen for analysis in Figure 5 roughly coincide with the $k_y \approx 2$ trough in Figure 1. Furthermore, the most energetic wave speed for each scale identified in Figure 5(a) is approximately $c^+ \approx 22$. Since most of the energy is contained in the most energetic wave speed, then eLNS must correctly predict the structure for this wave speed in order for the error to be low. We conclude that the trough exists in Figure 1 for $k_y \approx 2$ because eLNS
Figure 5. Power spectra for various $k_x$ and fixed $k_y = 2.67$ from the (a) DNS compared to the sum of the singular values squared of (b) LNS and (c) eLNS. All curves are normalized by the maximum value which appears in each plot.

is successfully identifying structures which travel at or near a wave speed of $c^+ = 22$. If the predictions were not good for this wave speed, then the error would be much higher than the values which appear in Figure 1. We also know that wave speed is indicative of the wall-normal height where a particular structure is more energetic. The scales for which we obtain a good prediction from the eddy operator, therefore, are primarily in the wake region at a wall-normal height of $z/h \approx 0.5$.

5.3. Overdamped scales
In this section, we analyze the $(k_x = 1, k_y = 6)$ scale, where the eLNS predictions are a factor of three worse than they are for $(k_x = 1, k_y = 2.67)$ as seen in Figure 1(d). The second-order velocity statistics are illustrated in Figure 6 for the most energetic wave speed $c^+ = 19.8$. The DNS statistics in Figure 6(a) illustrate that this structure is much closer to the wall than the previous scale. We also remark that the streamwise velocity component is significantly more energetic than the other two velocity components.

Figure 6. $S_{uu}(k_x = 1, k_y = 6, c^+ = 19.8)$ for the lower channel half. All nine components have been plotted for (a) DNS, (b) LNS, and (c) eLNS. Since linear analysis does not yield the amplitude of the statistics, each panel has been normalized such that its maximum value is one.

The LNS predictions are presented in Figure 6(b). The peak location of the streamwise velocity component is accurately predicted and, in comparison to the previous scale, the streamwise velocity component is more pronounced. It is so strong, however, that it completely overshadows the other velocity components. The structure is also highly localized at the critical layer, which results in relatively poor agreement with the DNS. The eLNS predictions in Figure 6(c) are not localized and accurately predict the relative magnitudes of each velocity component.
with respect to one another. Nonetheless, the wall-normal height at which the structure is most energetic is not accurately predicted. The streamwise velocity component is energetic too close to the wall when compared to DNS, so it can be concluded that stochastic forcing of eLNS for this scale is less appropriate.

These trends become clearer when comparing the most energetic pair of SPOD modes to the most amplified pair of resolvent modes from LNS and eLNS in Figure 7. The peak of the streamwise component for the SPOD modes and LNS modes are co-located around \( z/h = \pm 0.85 \). The LNS mode shapes are very narrow compared to the SPOD modes and heavily favor the streamwise velocity component. The eLNS mode shapes, by contrast, are not as biased in the streamwise direction and are not concentrated at a critical layer. In fact, they predict the spanwise and wall-normal velocity components reasonably well in comparison to LNS. Nevertheless, the streamwise velocity component peaks very close to the wall at a wall-normal height of \( z/h = \pm 0.95 \), which results in poorer agreement between the DNS and eLNS predictions for this scale when compared to the previous scale in Section 5.2.

![Figure 7](image_url)

**Figure 7.** The most energetic pair of SPOD modes for \( (k_x = 1, k_y = 6, c^+ = 19.8) \) compared to the most amplified pair of resolvent modes from LNS and eLNS. The absolute value of all three velocity components has been plotted and the modes are normalized such that the maximum streamwise velocity is unity.

Similar to the \( (k_x = 1, k_y = 2.67) \) wavenumber pair, we plot the contribution of each wave speed to the streamwise variance in Figure 8. The DNS results in Figure 8(a) highlight that the most energetic wave speed is \( c^+ = 19.8 \). LNS in Figure 8(b) overestimates the peak wave speed to be \( c^+ = 22 \) and predicts structures which are very narrow in the wall-normal direction. Despite the poor agreement between the mode shapes, they do peak at the correct wall-normal height. eLNS in Figure 8(c) underestimates the peak wave speed to be \( c^+ = 17.6 \).
Figure 8. The contribution of the most energetic wave speeds to the streamwise variance for \((k_x = 1, k_y = 6)\) computed from (a) DNS, (b) LNS, and (c) eLNS. Each curve is normalized by the peak contribution from the \(c^+ = 19.8\) mode.

Figure 9. Power spectra for various \(k_y\) and fixed \(k_x = 1\) from the (a) DNS compared to the sum of the singular values squared of (b) LNS and (c) eLNS. All curves are normalized by the maximum value which appears in each plot.

In addition to incorrectly weighting the wave speeds, the shapes and wall-normal height of the structures predicted by eLNS are incorrect. For nearly all the wave speeds which appear in Figure 8(c), the structures are energetic too close to the wall. The tendency for eLNS to concentrate energy near the wall helps to explain the overprediction of velocity fluctuations near the wall in Madhusudanan et al. [13]. It also suggests that the ‘coloring in time’ approach in Morra et al. [14] would need to be modified in order to address the incorrect structures predicted by eLNS.

Instead of comparing the power spectra across \(k_x\) as we did for \(k_y = 2.67\), we now calculate the power spectra across \(k_y\) holding \(k_x = 1\) fixed. These are displayed in Figure 9(a) for the DNS. As \(k_y\) increases, the structures gradually become slower and less energetic. The power spectra are compared to the sum of the singular values squared of LNS in Figure 9(b). Unlike the DNS, the most energetic wave speed remains fixed at \(c^+ = 23\) and is independent of \(k_y\). This is consistent with the earlier result that LNS overestimated the peak wave speed for \((k_x = 1, k_y = 6)\).

The eLNS results in Figure 9(c) are qualitatively similar to DNS in that the structures become slower and less energetic as \(k_y\) increases. Quantitatively, the rate at which they slow down and become less energetic is too fast compared to DNS. These trends are consistent with the error map in Figure 1(c) where the eLNS predictions deteriorate as the choice of \(k_y\) moves away from the trough at \(k_y \approx 2.0\). eLNS, thus, can be interpreted as being overdamped by the eddy viscosity as \(k_y\) increases.
5.4. Scale comparison

Before concluding, we summarize the predictive capabilities of stochastically-forced LNS and eLNS in terms of predicting the statistics and structures for energetic wave speeds, the wall-normal height where the structures are most energetic \( z_{\text{peak}} \), and the most energetic wave speed \( c^+_{\text{max}} \). The first class of scales considered are those where \( k_y = 2.67 \) and \( k_x \) is a free parameter. Figures 2 and 3 demonstrate for \( k_x = 1 \) that LNS cannot predict the velocity statistics or structures for energetic wave speeds while eLNS can. Both models provide good predictions for \( z_{\text{peak}} \), as seen in Figure 4, but Figure 5 shows that only eLNS can predict \( c^+_{\text{max}} \) which, for these structures, occurs around \( c^+ \approx 22 \). Therefore, eLNS provides good predictions for structures which are energetic in the wake region.

The second class of scales consist of those where \( k_x = 1 \) and \( k_y \) is a free parameter. Figures 6 and 7 show that neither LNS nor eLNS is capable of correctly identifying the statistics and structures for \( k_y = 6 \) at the most energetic wave speed \( c^+ = 19.8 \). Even though the LNS structures are located at the correct wall-normal height for various wave speeds, as seen in Figure 8, they favor the streamwise component and are too concentrated at the critical layer. The eLNS structures, meanwhile, have a \( z_{\text{peak}} \) which is too close to the wall for all the energetic wave speeds. Finally, Figure 9 reveals that neither LNS nor eLNS correctly identify \( c^+_{\text{max}} \). Unlike the first class of scales where \( c^+_{\text{max}} \) is roughly constant for DNS and eLNS, \( c^+_{\text{max}} \) gradually declines as \( k_y \) increases. LNS predicts no variation of \( c^+_{\text{max}} \) while eLNS overdamps scales in the log region, resulting in \( c^+_{\text{max}} \) predictions which are below those of DNS.

6. Conclusions and future work

We have assessed the predictive capabilities of resolvent analysis with stochastic forcing in time and space for the large scales in turbulent channel flow at \( Re_{\tau} = 2003 \). We began by quantifying the error between the second-order statistics computed from the DNS and the linear models over a range of length scales. Regardless of which spatial scale was investigated, LNS was able to identify the wall-normal height at which the structures were most energetic. These structures, however, were very localized and heavily biased towards the streamwise velocity component.

The only feature that LNS could predict with stochastic forcing was the wall-normal height at which structures were most energetic. The structure of the nonlinear terms, therefore, cannot be ignored in order to predict the correct statistics. The spatial scales which were successfully predicted by eLNS could be characterized by a spanwise wavenumber of \( k_y \approx 2 \). Even though the predictions deteriorated for lower wave speeds \( (c^+ < 19) \), the agreement between the most energetic pair of eLNS modes and SPOD modes computed from data was good. eLNS also provided a reasonable approximation of the power spectrum. For scales where the error was large, eLNS underestimated the most energetic wave speed and predicted structures too close to the wall.

The analysis where either \( k_x \) or \( k_y \) was varied while keeping the other held constant suggests two avenues for future work. First, it would be worthwhile to examine the accuracy of eLNS for large scales in the wake region at other Reynolds numbers. The large scales analyzed by Morra et al. [14] at \( Re_{\tau} = 1007 \) also peaked near \( z/h = 0.5 \), so it is possible that the observations made here regarding wake vs. logarithmic structures are true for any Reynolds number. Second, it would be useful to examine whether an eddy viscosity, which could depend on the spatial scales or wave speed, can be derived to improve predictions in the log region.

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