New decay rates for a Cauchy thermoelastic laminated Timoshenko problem with interfacial slip under Fourier or Cattaneo laws

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The objective of the present paper is to investigate the decay of solutions for a laminated Timoshenko beam with interfacial slip in the whole space $\mathbb{R}$ subject to a thermal effect acting only on one component modeled by either Fourier or Cattaneo law. When the thermal effect is acting via the second or third component of the laminated Timoshenko beam (rotation angle displacement or dynamic of the slip), we obtain that both systems, Timoshenko–Fourier and Timoshenko–Cattaneo systems, satisfy the same polynomial stability estimates in the $L^2$-norm of the solution and its higher order derivatives with respect to the space variable. The decay rate depends on the regularity of the initial data. In addition, the presence and absence of the regularity-loss type property are determined by some relations between the parameters of systems. However, when the thermal effect is acting via the first component of the system (transversal displacement), a new stability condition is introduced for both Timoshenko–Fourier and Timoshenko–Cattaneo systems. This stability condition is in the form of threshold between polynomial stability and convergence to zero. To prove our results, we use the energy method in Fourier space combined with judicious choices of weight functions to build appropriate Lyapunov functionals.

KEYWORDS
energy method, Fourier analysis, Fourier and Cattaneo laws, heat conduction, interfacial slip, Timoshenko beam

MSC CLASSIFICATION
34B05; 34D05; 34H05

1 INTRODUCTION

In this paper, we investigate the decay properties of a thermoelastic laminated Timoshenko beam with interfacial slip in the whole space $\mathbb{R}$ where the thermal effect is modeled by Fourier law or Cattaneo law. The first system we consider is the coupling of a laminated Timoshenko system with a heat conduction described by Fourier law and given by
and the second system of interest is the coupling between a laminated Timoshenko system with a heat conduction described by Cattaneo law and given by

\[
\begin{align*}
\phi_{tt} - k_1 (\phi_x + \psi + w)_x + \tau_1 \gamma \eta_x &= 0, \\
\psi_{tt} - k_2 \psi_{xx} + k_1 (\phi_x + \psi + w) + \tau_2 \gamma \eta_x &= 0, \\
w_{tt} - k_3 w_{xx} + k_1 (\phi_x + \psi + w) + \tau_3 \gamma \eta_x &= 0, \\
\eta_t - k_4 q_x + \gamma (\tau_1 \phi_{xx} + \tau_2 \psi_{xx} + \tau_3 w_{xx}) &= 0, \\
q_t + k_5 q + k_4 \eta_x &= 0,
\end{align*}
\]  

(1.2)

where \( k_1, k_2, k_3, k_4, k_5, \gamma > 0 \), \( \phi = \phi(x, t) \), \( \psi = \psi(x, t) \), \( \eta = \eta(x, t) \), and \( q = q(x, t) \) denoting the transversal displacement and the rotation angle of the beam, the temperature, and the heat flow, respectively; \( w = w(x, t) \) is proportional to the amount of slip along the interface, so the third equation in (1.1) and (1.2) describes the dynamics of the slip, \( x \in \mathbb{R} \) and \( t > 0 \). The thermal effect \( \gamma \eta_x \) is acting only on one equation of the laminated Timoshenko system, so

\[(\tau_1, \tau_2, \tau_3) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}. \]  

(1.3)

Systems (1.1) and (1.2) are, respectively, subject to the initial conditions

\[
\begin{align*}
(\phi, \psi, w, \eta)(x, 0) &= (\phi_0, \psi_0, w_0, \eta_0)(x), \\
(\phi_t, \psi_t, w_t)(x, 0) &= (\phi_1, \psi_1, w_1)(x)
\end{align*}
\]  

(1.4)

and

\[
\begin{align*}
(\phi, \psi, w, \eta, q)(x, 0) &= (\phi_0, \psi_0, w_0, \eta_0, q_0)(x), \\
(\phi_t, \psi_t, w_t)(x, 0) &= (\phi_1, \psi_1, w_1)(x).
\end{align*}
\]  

(1.5)

The main purpose of this paper is to investigate the capacity of the dissipation, generated by the heat conduction \( \gamma \eta_x \) via only one equation of the laminated Timoshenko system, to stabilize (1.1) and (1.2), and to determine its influence on the decay rate of solutions. We will show that the two cases

\[ (\tau_1, \tau_2, \tau_3) = (1, 0, 0) \text{ and } (\tau_1, \tau_2, \tau_3) \in \{(0, 1, 0), (0, 0, 1)\} \]

are completely different in the following sense:

**Case** \((\tau_1, \tau_2, \tau_3) = (1, 0, 0)\): Systems (1.1) and (1.2) are stable if and only if

\[ k_2 \neq k_3, \]  

(1.6)

and when \( k_2 \neq k_3 \), the following polynomial stability result holds true for (1.1) and (1.2): For any \( N, \ell \in \mathbb{N} \) with \( \ell \leq N \), \( j \in \{0, \ldots, N - \ell\} \), and \( U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R}) \), there exists \( c_0 > 0 \) such that

\[ \| \partial_x^j U \|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-1/12 - j/6} \| U_0 \|_{L^1(\mathbb{R})} + c_0 (1 + t)^{-\ell/2} \| \partial_x^j \partial_t^\ell U_0 \|_{L^2(\mathbb{R})}, \quad \forall t \in \mathbb{R}_+, \]  

(1.7)

where \( \partial_x^j = \frac{\partial^j}{\partial x^j} \), and \( U \) and \( U_0 \) are defined in Section 2.

**Case** \((\tau_1, \tau_2, \tau_3) \in \{(0, 1, 0), (0, 0, 1)\} \): When the three speeds of wave propagations of the laminated Timoshenko system are equal; that is,

\[ k_1 = k_2 = k_3, \]  

(1.8)
systems (1.1) and (1.2) are stable with the following decay rate: For any $N, \ell \in \mathbb{N}$ with $\ell \leq N, j \in \{0, \ldots, N-\ell\}$ and $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$, there exists $c_0, \epsilon_0 > 0$ such that
\[
\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-\ell/4} \|U_0\|_{L^2(\mathbb{R})} + c_0 e^{-\epsilon_0 t} \|\partial_x^j U_0\|_{L^2(\mathbb{R})}, \forall t \in \mathbb{R}_+.
\] (1.9)

If (1.8) is not satisfied, then the following estimate holds true for (1.1) and (1.2):
\[
\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-\ell/4} \|U_0\|_{L^2(\mathbb{R})} + c_0 t^\epsilon \|\partial_x^j U_0\|_{L^2(\mathbb{R})}, \forall t \in \mathbb{R}_+.
\] (1.10)

It is well known in the literature that the behavior of the Fourier transform of $U$ in the low-frequency region determines the rate of decay of $U$, while its behavior in the high-frequency region imposes a regularity restriction on the initial data known as the regularity-loss property (see several studies1–6). It seems that the dissipation generated by the heat conduction is so weak in the high-frequency region that it leads to the regularity-loss property in the estimates (1.7) and (1.10). On the other hand, the restriction (1.6) and the fact that the decay rate in (1.7) is smaller than the one in (1.9) and (1.10) indicate that the effect of the heat conduction is better propagated to the whole system from the second or third equation of the laminated Timoshenko system than from the first one.

A model describing laminated Timoshenko beams with interfacial slip based on the Timoshenko theory (see, e.g., several studies7–9) is given by
\[
\begin{align*}
\varphi_t - k_1 (\varphi_x + \psi + w)_x &= 0, \\
\psi_t - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + w) &= 0, \\
w_t - k_3 \psi_{xx} + k_1 (\varphi_x + \psi + w) &= 0,
\end{align*}
\] (1.11)

and can be derived from the following more general model of Bresse type:
\[
\begin{align*}
\varphi_t - k_1 (\varphi_x + \psi + lw)_x - \bar{l} k_3 (w_x - \bar{l} \varphi) &= 0, \\
\psi_t - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + lw) &= 0, \\
w_t - k_3 (w_x - \bar{l} \varphi)_x + l k_1 (\varphi_x + \psi + lw) &= 0,
\end{align*}
\] (1.12)

where $l$ and $\bar{l}$ are positive constants. System (1.12) coincides with (1.11) when $l = 1$ and $\bar{l} = 0$. When $w = l = \bar{l} = 0$, system (1.12) is reduced to the following Timoshenko-type system:
\[
\begin{align*}
\varphi_t - k_1 (\varphi_x + \psi)_x &= 0, \\
\psi_t - k_2 \psi_{xx} + k_1 (\varphi_x + \psi) &= 0.
\end{align*}
\] (1.13)

The well-posedness as well as the stability questions for (1.11)–(1.13) have been the subject of various studies in the literature, where different controls (dampings, memories, heat conduction, ...) and/or boundary conditions (Dirichlet, Neumann, mixed, ...) have been used to force the solution to converge to zero when time $t$ goes to infinity and get information on its speed of convergence.

In case of bounded domains, we refer the reader to, for example, several studies9–23 and the references therein.

In case of unbounded domains, the stability of (1.12) and (1.13) has been also treated in the literature for the last few years. In this direction, we mention the papers3,24–28 (see also the references therein), where some polynomial stability estimates for $L^2$-norm of solutions have been proved using frictional dampings, heat conduction effects, or memory controls. In some particular cases, the optimality of the decay rate was also proved.

Our results in the present paper give extensions from the bounded to the unbounded domain case of some ones known in the literature (as in several studies9–12,16,18–20,23). The proof is based on the energy method combined with the Fourier analysis (by using the transformation in the Fourier space) and well-chosen weight functions.

The paper is organized as follows. In Section 2, we formulate (1.1) and (1.2) as first-order Cauchy systems and give some preliminaries. In Sections 3 and 4, we prove our polynomial stability estimates for (1.1) and (1.2), respectively. We end our paper by some general comments and other related issues in Section 5.
To formulate (1.1) and (1.2) in abstract first-order systems, we introduce the new variables

\[ u = \varphi_t, \quad y = \psi_t, \quad \theta = \omega_t, \quad v = \varphi_x + \psi + \omega, \quad z = \psi_x, \quad \text{and} \quad \phi = \omega_x. \]  

(2.1)

Then, systems (1.1) and (1.2) can be rewritten, respectively, in the forms

\[
\begin{align*}
&v_t - u_x - y - \theta = 0, \\
u_t - k_1 v_x + \tau_1 \gamma \eta_x = 0, \\
z_t - y_x = 0, \\
y_t - k_2 z_x + k_1 v + \tau_2 \gamma \eta_x = 0, \\
\phi_t - \theta_x = 0, \\
\theta_t - k_3 \phi_x + k_1 v + \tau_3 \gamma \eta_x = 0, \\
\eta_t - k_4 \eta_{xx} + \gamma (\tau_1 u_x + \tau_2 y_x + \tau_3 \theta_x) = 0, \\
\end{align*}
\]

and

\[
\begin{align*}
&v_t - u_x - y - \theta = 0, \\
u_t - k_1 v_x + \tau_1 \gamma \eta_x = 0, \\
z_t - y_x = 0, \\
y_t - k_2 z_x + k_1 v + \tau_2 \gamma \eta_x = 0, \\
\phi_t - \theta_x = 0, \\
\theta_t - k_3 \phi_x + k_1 v + \tau_3 \gamma \eta_x = 0, \\
\eta_t + k_4 \eta_{xx} + \gamma (\tau_1 u_x + \tau_2 y_x + \tau_3 \theta_x) = 0, \\
\end{align*}
\]

Now, we define the variable \( U \) and its initial data \( U_0 \) by

\[
U = \begin{cases}
(v, u, z, y, \phi, \theta, \eta)^T & \text{for (2.2)}, \\
(v, u, z, y, \phi, \theta, \eta, q)^T & \text{for (2.3)}
\end{cases}
\quad \text{and} \quad
U_0 = \begin{cases}
(v, u, z, y, \phi, \theta, \eta)^T (\cdot, 0) & \text{for (2.2)}, \\
(v, u, z, y, \phi, \theta, \eta, q)^T (\cdot, 0) & \text{for (2.3)}
\end{cases}
\]

Systems (2.2) and (2.3) with initial conditions (1.4) and (1.5), respectively, are equivalent to

\[
\begin{align*}
&U_t(x, t) + A_2 U_{xx}(x, t) + A_1 U_x(x, t) + A_0 U(x, t) = 0, \\
&U(x, 0) = U_0(x),
\end{align*}
\]

(2.4)

where, for (2.2),

\[
A_2 = \begin{pmatrix}
0 & -u_x & -k_1 v_x + \tau_1 \gamma \eta_x & 0 \\
0 & -y_x & 0 & 0 \\
0 & \tau_2 \gamma \eta_x & 0 & 0 \\
-k_4 \eta_{xx} & \gamma (\tau_1 u_x + \tau_2 y_x + \tau_3 \theta_x) & \tau_3 \gamma \eta_x & 0 \\
\end{pmatrix}
\]

(2.5)
and for (2.3),

\[ A_2 = 0, \quad A_1 U_x = \begin{pmatrix} -u_x \\ -k_1 v_x + \tau_1 \eta_x \\ -y_x \\ -k_2 z_x + \tau_2 y_x \\ -\theta_x \\ -k_3 \phi_x + \tau_3 \eta_x \\ k_4 q_x + \gamma (\tau_1 u_x + \tau_2 y_x + \tau_3 \theta_x) \\ k_4 q_x \end{pmatrix} \quad \text{and} \quad A_0 U = \begin{pmatrix} -y - \theta \\ 0 \\ 0 \\ k_1 v \end{pmatrix}. \quad (2.6) \]

For a function \( h : \mathbb{R} \to \mathbb{C} \), \( Re h \), \( Im h \), \( \hat{h} \), and \( \hat{\hat{h}} \) denote, respectively, the real part of \( h \), the imaginary part of \( h \), the conjugate of \( h \), and the Fourier transformation of \( h \). Using the Fourier transformation (with respect to the space variable \( x \)) to transform (2.4) in the Fourier space, we obtain the following first-order Cauchy system:

\[
\begin{align*}
\hat{U}(\xi, t) - \xi^2 A_2 \hat{U}(\xi, t) + i \xi A_1 \hat{U}(\xi, t) + A_0 \hat{U}(\xi, t) &= 0, \\
\hat{U}(\xi, 0) &= \hat{U}_0(\xi),
\end{align*}
\]

The solution of (2.7) is given by

\[
\hat{U}(\xi, t) = e^{-i \xi^2 A_2 + i \xi A_1 + A_0} \hat{U}_0(\xi).
\]

The energy \( \hat{E} \) associated with (2.7) is defined by

\[
\hat{E}(\xi, t) = \frac{1}{2} \left[ k_1 |\hat{\phi}|^2 + |\hat{\eta}|^2 + k_2 |\hat{x}|^2 + |\hat{y}|^2 + k_3 |\hat{\phi}|^2 + |\hat{\eta}|^2 + |\hat{\eta}|^2 \right]
\]

in case (2.2) and

\[
\hat{E}(\xi, t) = \frac{1}{2} \left[ k_1 |\hat{\phi}|^2 + |\hat{\eta}|^2 + k_2 |\hat{x}|^2 + |\hat{y}|^2 + k_3 |\hat{\phi}|^2 + |\hat{\theta}|^2 + |\hat{\eta}|^2 + |\hat{\eta}|^2 \right]
\]

in case (2.3).

**Lemma 2.1.** System (2.7) is dissipative, since

\[
\frac{d}{dt} \hat{E}(\xi, t) = -k_4 |\hat{\eta}|^2 \quad (2.11)
\]

in case (2.2) and

\[
\frac{d}{dt} \hat{E}(\xi, t) = -k_5 |\hat{\eta}|^2 \quad (2.12)
\]

in case (2.3).

**Proof.** First, we remember the following two trivial identities which will be frequently used in this paper: For any two differentiable functions \( h, d : \mathbb{R} \to \mathbb{C} \), we have

\[
\frac{d}{dt} Re(h \bar{d}) = Re(h \bar{d} + d \bar{h}) \quad (2.13)
\]

and

\[
\frac{d}{dt} Re(ih \bar{d}) = Re \left[ i(h \bar{d} - d \bar{h}) \right] \quad (2.14)
\]
In case (2.5), the first equation in (2.7) is equivalent to

$$\begin{aligned}
\dot{v}_i - i\xi \ddot{u} - \ddot{\phi} - \dot{\theta} &= 0, \\
\dot{u}_i - ik_1 \ddot{v} + ir_1 \gamma \ddot{\eta} &= 0, \\
\ddot{z}_i - i\xi \ddot{y} &= 0, \\
\ddot{y}_i - ik_2 \ddot{z} + k_1 \ddot{v} + ir_2 \gamma \ddot{\eta} &= 0, \\
\ddot{\phi}_i - i\xi \ddot{\theta} &= 0, \\
\ddot{\theta}_i - ik_3 \ddot{\phi} + k_1 \ddot{v} + ir_3 \gamma \ddot{\eta} &= 0, \\
\ddot{\eta}_i + k_4 \dot{\xi}^2 \ddot{\eta} + i\gamma \xi (r_1 \ddot{u} + r_2 \ddot{v} + r_3 \ddot{\theta}) &= 0.
\end{aligned}$$

(2.15)

Multiplying the equations in (2.15) by $k_1 \ddot{u}$, $k_2 \ddot{z}$, $k_3 \ddot{\phi}$, $k_4 \ddot{\phi}$, $k_5 \ddot{\theta}$, and $k_6 \ddot{\eta}$, respectively, adding the obtained equations, taking the real part of the resulting expression, and using (2.13), we obtain (2.11). Similarly, in case (2.6), the first equation of (2.7) is reduced to

$$\begin{aligned}
\dot{v}_i - i\xi \ddot{u} - \ddot{\phi} - \dot{\theta} &= 0, \\
\dot{u}_i - ik_1 \ddot{v} + ir_1 \gamma \ddot{\eta} &= 0, \\
\ddot{z}_i - i\xi \ddot{y} &= 0, \\
\ddot{y}_i - ik_2 \ddot{z} + k_1 \ddot{v} + ir_2 \gamma \ddot{\eta} &= 0, \\
\ddot{\phi}_i - i\xi \ddot{\theta} &= 0, \\
\ddot{\theta}_i - ik_3 \ddot{\phi} + k_1 \ddot{v} + ir_3 \gamma \ddot{\eta} &= 0, \\
\ddot{\eta}_i + k_4 \dot{\xi}^2 \ddot{\eta} + i\gamma \xi (r_1 \ddot{u} + r_2 \ddot{v} + r_3 \ddot{\theta}) &= 0.
\end{aligned}$$

(2.16)

Multiplying the equations in (2.16) by $k_1 \ddot{u}$, $k_2 \ddot{z}$, $k_3 \ddot{\phi}$, $k_4 \ddot{\phi}$, $k_5 \ddot{\theta}$, and $k_6 \ddot{\eta}$, respectively, adding the obtained equations, taking the real part of the resulting expression, and using (2.13), we get (2.12).

\[\square\]

Remark 1. It is clear that the energy $\tilde{E}$ is equivalent to $|\ddot{U}|^2$ defined in case (2.15) by

$$|\ddot{U}(\xi, t)|^2 = |\ddot{v}|^2 + |\ddot{u}|^2 + |\ddot{z}|^2 + |\ddot{y}|^2 + |\ddot{\phi}|^2 + |\ddot{\theta}|^2 + |\ddot{\eta}|^2$$

and in case (2.16) by

$$|\ddot{U}(\xi, t)|^2 = |\ddot{v}|^2 + |\ddot{u}|^2 + |\ddot{z}|^2 + |\ddot{y}|^2 + |\ddot{\phi}|^2 + |\ddot{\theta}|^2 + |\ddot{\eta}|^2 + |\ddot{\phi}|^2.$$

Since, for $\alpha_1 = \frac{1}{2} \min\{k_1, k_2, k_3, 1\}$ and $\alpha_2 = \frac{1}{2} \max\{k_1, k_2, k_3, 1\}$, we have

$$\alpha_1 |\ddot{U}(\xi, t)|^2 \leq \tilde{E}(\xi, t) \leq \alpha_2 |\ddot{U}(\xi, t)|^2, \ \forall \xi \in \mathbb{R}, \ \forall t \in \mathbb{R}_+.$$ \hspace{1cm} (2.17)

We finish this section by proving two lemmas, which will be also frequently used in the proof of our stability results.

**Lemma 2.2.** Let $\sigma$, $p$, and $r$ be real numbers such that $\sigma > -1$ and $p$, $r > 0$. Then there exists $C_{\sigma,p,r} > 0$ such that

$$\int_0^1 \xi^\sigma e^{-rt\xi^p} d\xi \leq C_{\sigma,p,r} (1 + t)^{-(\sigma+1)/p}, \ \forall t \in \mathbb{R}_+.$$ \hspace{1cm} (2.18)

*Proof.* For $0 \leq t \leq 1$, (2.18) is evident, for any $C_{\sigma,p,r} \geq \frac{2^{(\sigma+1)/p}}{(p+1)^{1/p}}$. For $t > 1$, we have

$$\int_0^1 \xi^\sigma e^{-rt\xi^p} d\xi = \int_0^1 \xi^{p+1-\sigma} e^{-rt\xi^p} \xi^{\sigma-1} d\xi = \int_0^1 (\xi^{p})^{(\sigma+1-p)/p} e^{-rt\xi} \xi^{p-1} d\xi.$$
Taking $\tau = rt^{\frac{1}{p}}$, then
\[ \frac{r}{rt} = r t^{\frac{1}{p}} \text{ and } d\xi = \frac{1}{p r \tau} d\tau. \]

Substituting in the above integral, we find
\[ \int_0^1 (\xi^{p})^{(\sigma+1)/p} e^{-r t^{p}} \xi^{p-1} d\xi = \int_0^1 \left( \frac{r}{rt} \right)^{(\sigma+1)/p} e^{-\tau} \frac{1}{p r \tau} d\tau \]
\[ \leq \frac{1}{p(r t)^{\sigma+1}/p} \int_0^{+\infty} \tau^{(\sigma+1)/p} e^{-\tau} d\tau \leq \frac{2(\sigma+1)/p}{p r^{\sigma+1}/p} C_{\sigma,p} (t+1)^{-\sigma+1}/p, \]

where
\[ C_{\sigma,p} = \int_0^{+\infty} \tau^{(\sigma+1)/p} e^{-\tau} d\tau, \]
which is a convergent integral, for any $\sigma > -1$ and $p > 0$. This completes the proof of (2.18) with
\[ C_{\sigma,p} = \max \left\{ \frac{2(\sigma+1)/p}{\sigma+1}, \frac{2(\sigma+1)/p}{p r^{(\sigma+1)/p} C_{\sigma,p}} \right\}. \]

\[ \square \]

**Lemma 2.3.** For any positive real numbers $\sigma_1$, $\sigma_2$, and $\sigma_3$, we have
\[ \sup_{|\xi| \geq 1} |\xi|^{-\sigma_2} e^{-\sigma_2 t|\xi|^{-\sigma_3}} \leq (1 + \sigma_1/(\sigma_2 \sigma_3))^{\sigma_1/\sigma_3} (1 + t)^{-\sigma_1/\sigma_3}, \forall t \in \mathbb{R}_+. \] (2.19)

**Proof.** It is clear that (2.19) is satisfied for $t = 0$. Let $t > 0$ and $h(x) = x^{-\sigma_1} e^{-\sigma_2 x^{-\sigma_3}}$, for $x \geq 1$. Direct and simple computations show that
\[ h'(x) = (\sigma_2 \sigma_3 x^{-\sigma_3} - \sigma_1) x^{-\sigma_2} e^{-\sigma_2 x^{-\sigma_3}}. \]

If $t \geq \sigma_1/(\sigma_2 \sigma_3)$, then
\[ h(x) \leq h((\sigma_2 \sigma_3 t)/\sigma_1)^{1/\sigma_3} = ((\sigma_2 \sigma_3)/\sigma_1)^{-\sigma_1/\sigma_3} e^{-\sigma_1/\sigma_3} (1 + t)^{-\sigma_1/\sigma_3}, \]
\[ \leq ((\sigma_2 \sigma_3)/\sigma_1)^{-\sigma_1/\sigma_3} (1 + (\sigma_2 \sigma_3)/\sigma_1)^{\sigma_1/\sigma_3} (1 + t)^{-\sigma_1/\sigma_3} = (1 + \sigma_1/(\sigma_2 \sigma_3))^{\sigma_1/\sigma_3} (1 + t)^{-\sigma_1/\sigma_3}, \]
which gives (2.19) by taking $x = |\xi|$. If $0 < t < \sigma_1/(\sigma_2 \sigma_3)$, then
\[ h(x) \leq h(1) = e^{-\sigma_2 (1 + t)^{\sigma_1/\sigma_3} (1 + t)^{-\sigma_1/\sigma_3}} \leq (1 + \sigma_1/(\sigma_2 \sigma_3))^{\sigma_1/\sigma_3} (1 + t)^{-\sigma_1/\sigma_3}, \]
which implies (2.19), for $x = |\xi|$. \[ \square \]

### 3 STABILITY: FOURIER LAW (1.1)

This section is dedicated to the investigation of the asymptotic behavior, when time $t$ goes to infinity, of the solution $U$ of (2.4) in case of Fourier law (1.1). We will prove (1.7), (1.9), and (1.10) by showing, first, that $|\hat{U}|^2$ converges exponentially to zero with respect to time $t$. Then, we prove that solution (2.8) of (2.7) does not converge to zero when $t$ goes to infinity if $(r_1, r_2, r_3) = (1, 0, 0)$ and $k_2 = k_3$.

In this section and in the next one, $C$ denotes a generic positive constant, and $C_{\epsilon}$ denotes a generic positive constant depending on some positive constant $\epsilon$. These generic constants can be different from line to line. Before distinguishing between the three cases (1.3), we prove several identities, which will play a crucial role in the proofs.
Multiplying (2.15)\textsubscript{4} and (2.15)\textsubscript{3} by \(i\xi\ddot{\tau}\) and \(-i\xi\ddot{\gamma}\), respectively, adding the resulting equations, taking the real part, and using (2.14), we obtain

\[
\frac{d}{dt} \Re \left( i\xi \ddot{\gamma} \ddot{\tau} \right) = \xi^2 \left( |\ddot{\gamma}|^2 - k_2 |\ddot{\tau}|^2 \right) - k_1 \Re \left( i\xi \ddot{\gamma} \ddot{\tau} \right) + \tau_2 \gamma \xi^2 \Re \left( \ddot{\gamma} \ddot{\tau} \right) .
\]  

(3.1)

Multiplying (2.15)\textsubscript{2} and (2.15)\textsubscript{1} by \(i\xi\ddot{\gamma}\) and \(-i\xi\ddot{\tau}\), respectively, adding the resulting equations, taking the real part, and using (2.14), we find

\[
\frac{d}{dt} \Re \left( i\xi \ddot{\gamma} \ddot{\tau} \right) = \xi^2 \left( |\ddot{\tau}|^2 - k_1 |\ddot{\gamma}|^2 \right) - \Re \left( i\xi \ddot{\gamma} \ddot{\tau} \right) - k_1 \Re \left( i\xi \ddot{\gamma} \ddot{\tau} \right) + \tau_1 \gamma \xi^2 \Re \left( \ddot{\gamma} \ddot{\tau} \right) .
\]  

(3.2)

Multiplying (2.15)\textsubscript{6} and (2.15)\textsubscript{5} by \(i\xi\ddot{\beta}\) and \(-i\xi\ddot{\beta}\), respectively, adding the resulting equations, taking the real part, and using (2.14), we get

\[
\frac{d}{dt} \Re \left( i\xi \ddot{\phi} \ddot{\beta} \right) = \xi^2 \left( |\ddot{\beta}|^2 - k_3 |\ddot{\phi}|^2 \right) - k_1 \Re \left( i\xi \ddot{\phi} \ddot{\beta} \right) + \tau_3 \gamma \xi^2 \Re \left( \ddot{\beta} \ddot{\phi} \right) .
\]  

(3.3)

Multiplying (2.15)\textsubscript{6} and (2.15)\textsubscript{1} by \(-\xi^2 \ddot{v}\) and \(-\xi^2 \ddot{\tau}\), respectively, adding the resulting equations, taking the real part, and using (2.13), we have

\[
\frac{d}{dt} \Re \left( -\xi^2 \ddot{v} \ddot{\phi} \right) = \xi^2 \left( k_1 |\ddot{v}|^2 - |\ddot{\beta}|^2 \right) - \xi^2 \Re \left( i\xi \ddot{v} \ddot{\phi} \right) - k_3 \xi^2 \Re \left( i\xi \ddot{v} \ddot{\phi} \right) \\
- \xi^2 \Re \left( \ddot{v} \ddot{\phi} \right) + \tau_3 \gamma \xi^2 \Re \left( i\xi \ddot{v} \ddot{\phi} \right) .
\]  

(3.4)

Multiplying (2.15)\textsubscript{4} and (2.15)\textsubscript{1} by \(-\xi^2 \ddot{v}\) and \(-\xi^2 \ddot{\gamma}\), respectively, adding the resulting equations, taking the real part, and using (2.13), we infer that

\[
\frac{d}{dt} \Re \left( -\xi^2 \ddot{v} \ddot{\gamma} \right) = \xi^2 \left( k_1 |\ddot{v}|^2 - |\ddot{\gamma}|^2 \right) - \xi^2 \Re \left( i\xi \ddot{v} \ddot{\gamma} \right) - k_2 \xi^2 \Re \left( i\xi \ddot{v} \ddot{\gamma} \right) \\
- \xi^2 \Re \left( \ddot{v} \ddot{\gamma} \right) + \tau_2 \gamma \xi^2 \Re \left( i\xi \ddot{v} \ddot{\gamma} \right) .
\]  

(3.5)

Multiplying (2.15)\textsubscript{3} and (2.15)\textsubscript{5} by \(i\xi\ddot{\beta}\) and \(-i\xi\ddot{\gamma}\), respectively, adding the resulting equations, taking the real part, and using (2.14), we entail

\[
\frac{d}{dt} \Re \left( i\xi \ddot{\gamma} \ddot{\tau} \right) = -\xi^2 \Re \left( \ddot{\gamma} \ddot{\tau} \right) + k_3 \xi^2 \Re \left( \ddot{\beta} \ddot{\phi} \right) + k_1 \Re \left( i\xi \ddot{v} \ddot{\gamma} \right) - \tau_3 \gamma \xi^2 \Re \left( \ddot{\gamma} \ddot{\tau} \right) .
\]  

(3.6)

Multiplying (2.15)\textsubscript{5} and (2.15)\textsubscript{4} by \(i\xi\ddot{\gamma}\) and \(-i\xi\ddot{\phi}\), respectively, adding the resulting equations, taking the real part, and using (2.14), we arrive at

\[
\frac{d}{dt} \Re \left( i\xi \ddot{\phi} \ddot{v} \right) = -\xi^2 \Re \left( \ddot{\phi} \ddot{v} \right) + \tau_2 \xi^2 \Re \left( i\xi \ddot{\phi} \ddot{v} \right) + k_1 \Re \left( i\xi \ddot{v} \ddot{\phi} \right) - \tau_2 \gamma \xi^2 \Re \left( \ddot{\phi} \ddot{v} \right) .
\]  

(3.7)

Multiplying (2.15)\textsubscript{2} and (2.15)\textsubscript{3} by \(-\ddot{v}\) and \(-\ddot{\tau}\), respectively, adding the resulting equations, taking the real part, and using (2.13), it follows that

\[
\frac{d}{dt} \Re \left( -\ddot{v} \ddot{\tau} \right) = -k_1 \Re \left( i\xi \ddot{v} \ddot{\tau} \right) - \Re \left( i\xi \ddot{v} \ddot{\tau} \right) + \tau_1 \gamma \Re \left( i\xi \ddot{v} \ddot{\tau} \right) .
\]  

(3.8)

Finally, multiplying (2.15)\textsubscript{2} and (2.15)\textsubscript{3} by \(-\ddot{\phi}\) and \(-\ddot{\tau}\), respectively, adding the resulting equations, taking the real part, and using (2.13), it appears that

\[
\frac{d}{dt} \Re \left( -\ddot{\phi} \ddot{\tau} \right) = -k_1 \Re \left( i\xi \ddot{\phi} \ddot{\tau} \right) - \Re \left( i\xi \ddot{\phi} \ddot{\tau} \right) + \tau_1 \gamma \Re \left( i\xi \ddot{\phi} \ddot{\tau} \right) .
\]  

(3.9)
3.1 | Case 1: \((\tau_1, \tau_2, \tau_3) = (1, 0, 0)\)

We start by presenting the exponential stability result for (2.7) in the next lemma.

**Lemma 3.1.** Assume that \(k_2 \neq k_3\). Let \(\hat{U}\) be solution (2.8) of (2.7). Then there exist \(c, \hat{c} > 0\) such that

\[
|\hat{U}(\xi, t)|^2 \leq \hat{c} e^{-c/f(\xi)}|\hat{U}_0(\xi)|^2, \ \forall \xi \in \mathbb{R}, \ \forall t \in \mathbb{R}_+.
\]

(3.10)

where

\[
f(\xi) = \frac{\xi^6}{1 + \xi^8}.
\]

(3.11)

**Proof.** Multiplying (2.15)\(_2\) and (2.15)\(_7\) by \(i\xi \bar{\eta}\) and \(-i\xi \bar{u}\), respectively, adding the resulting equations, taking the real part, and using (2.14), we get

\[
\frac{d}{dt} \operatorname{Re} \left( i\xi \bar{\eta} \hat{\theta} \right) = \gamma \xi^2 (|\bar{\eta}|^2 - |\bar{u}|^2) + k_4 \xi^2 \operatorname{Re} \left( i\xi \bar{\eta} \bar{u} \right) - k_1 \xi^2 \operatorname{Re} \left( \hat{\nu} \bar{\eta} \right).
\]

(3.12)

Similarly, multiplying (2.15)\(_6\) and (2.15)\(_7\) by \(\bar{\eta}\) and \(\bar{\theta}\), respectively, adding the resulting equations, taking the real part, and using (2.13), we find

\[
\frac{d}{dt} \operatorname{Re} \left( \bar{\eta} \hat{\theta} \right) = \gamma \operatorname{Re} \left( i\xi \bar{\theta} \bar{u} \right) - k_4 \xi^2 \operatorname{Re} \left( \bar{\eta} \bar{\theta} \right) + k_3 \operatorname{Re} \left( i\xi \bar{\phi} \bar{\eta} \right) - k_1 \operatorname{Re} \left( \hat{\nu} \bar{\eta} \right).
\]

(3.13)

Also, multiplying (2.15)\(_4\) and (2.15)\(_7\) by \(\bar{\eta}\) and \(\bar{\gamma}\), respectively, adding the resulting equations, taking the real part, and using (2.13), we obtain

\[
\frac{d}{dt} \operatorname{Re} \left( \bar{\eta} \bar{\gamma} \right) = -\gamma \operatorname{Re} \left( i\xi \bar{\gamma} \bar{u} \right) - k_4 \xi^2 \operatorname{Re} \left( \bar{\eta} \bar{\gamma} \right) + k_2 \operatorname{Re} \left( i\xi \bar{\phi} \bar{\eta} \right) - k_1 \operatorname{Re} \left( \hat{\nu} \bar{\eta} \right).
\]

(3.14)

We define the functional \(F_0\) as follows:

\[
F_0(\xi, t) = \operatorname{Re} \left[ i\xi \left( \lambda_1 \bar{\gamma} \bar{\xi} + \lambda_3 \bar{\theta} \bar{\phi} + i\lambda_4 \xi \bar{\theta} \bar{v} - \frac{(\lambda_4 + 1)k_2}{k_2 - k_3} \bar{\xi} \bar{\theta} + \frac{(\lambda_4 + 1)k_3}{k_2 - k_3} \bar{\phi} \bar{v} \right) - \xi^2 \bar{\gamma} \bar{\xi} \bar{v} + i\lambda_2 \xi \bar{\phi} \bar{v} \right),
\]

(3.15)

where \(\lambda_1, \lambda_2, \lambda_3,\) and \(\lambda_4\) are positive constants to be defined later (\(F_0\) is well defined since \(k_2 \neq k_3\)). By multiplying (3.1)–(3.4), (3.6), and (3.7) by \(\lambda_1, \lambda_2, \lambda_3, -\frac{(\lambda_4 + 1)k_2}{k_2 - k_3}\), and \(\frac{(\lambda_4 + 1)k_3}{k_2 - k_3}\), respectively, adding the obtained equations, and adding (3.5), we deduce that

\[
\frac{d}{dt} F_0(\xi, t) = -\xi^2 \left( k_2 \lambda_1 |\bar{\xi}|^2 + k_3 \lambda_3 |\bar{\phi}|^2 + (1 - \lambda_1) |\bar{\gamma}|^2 + (\lambda_4 - \lambda_3) |\bar{\theta}|^2 + (k_1 \lambda_2 - k_1 \lambda_4 - k_1) |\bar{v}|^2 \right)
\]

\[
+ I_1 \operatorname{Re}(i\xi \bar{\gamma} \bar{v}) + I_2 \operatorname{Re}(i\xi \bar{v} \bar{\phi}) + \xi^2 \left[ \lambda_3 |\bar{u}|^2 - \operatorname{Re} \left( i\xi \left( \lambda_4 \bar{u} \bar{\phi} + \bar{u} \bar{\gamma} \right) \right) \right]
\]

\[
- \lambda_2 \operatorname{Re} \left[ i\xi \left( \bar{\gamma} \bar{\xi} + \bar{\theta} \bar{\phi} + i\gamma \xi \bar{\phi} \right) \right],
\]

(3.16)

where

\[
I_1 = k_2 \xi^2 - k_1 \lambda_1 - \frac{(\lambda_4 + 1)k_1 k_2}{k_2 - k_3} \text{ and } I_2 = k_3 \lambda_4 \xi^2 - k_1 \lambda_3 + \frac{(\lambda_4 + 1)k_3 k_3}{k_2 - k_3}.
\]

We put

\[
F_1(\xi, t) = \xi^4 \left[ F_0(\xi, t) - \frac{1}{k_1} \operatorname{Re} \left( I_1 \bar{\xi} \bar{\phi} + I_2 \bar{\xi} \bar{\phi} \right) \right].
\]

(3.17)
Multiplying (3.8) and (3.9) by \( \frac{I_1}{k_1} \) and \( \frac{I_2}{k_2} \), respectively, adding the obtained equations, adding (3.16), and multiplying the resulting formula by \( \xi^4 \), we arrive at

\[
\frac{d}{dt} F_1(\xi, t) = -\xi^6 \left( k_2 \lambda_1 |\hat{z}|^2 + k_3 \lambda_3 |\hat{p}|^2 + (1 - \lambda_1) |\hat{y}|^2 + (\lambda_4 - \lambda_3) |\hat{\phi}|^2 + (k_1 \lambda_2 - k_1 \lambda_4 - k_1) |\hat{v}|^2 \right) 
+ \lambda_2 \xi^6 |\hat{u}|^2 + \gamma \xi^4 \text{Re} \left[ \xi^2 \left( \frac{I_1}{k_1} \hat{z} \hat{\phi} + \frac{I_2}{k_1} \hat{\phi} \hat{\phi} \right) \right] + \lambda_2 \xi^2 \hat{y} \hat{\hat{\phi}} + \xi^4 \text{Re} \left( i I_2 \xi \hat{u} \hat{\phi} + i I_4 \xi \hat{u} \hat{\phi} \right),
\]

(3.18)

where

\[ I_3 = \lambda_2 + \frac{1}{k_1} I_2 - \lambda_4 \xi^2 \quad \text{and} \quad I_4 = \lambda_2 + \frac{1}{k_1} I_1 - \xi^2. \]

Let \( \lambda_5 > 0 \) and

\[
F(\xi, t) = F_1(\xi, t) + \lambda_5 \xi^4 \text{Re} \left( i \xi \hat{u} \hat{\hat{\phi}} \right) + \frac{1}{\gamma} I_3 \xi^4 \text{Re} \left( \hat{\theta} \hat{\theta} \right) + \frac{1}{\gamma} I_4 \xi^4 \text{Re} \left( \hat{\hat{\phi}} \hat{\hat{\phi}} \right).
\]

(3.19)

Multiplying (3.12)–(3.14) by \( \lambda_5 \xi^4 \), \( \frac{1}{\gamma} I_3 \xi^4 \), and \( \frac{1}{\gamma} I_4 \xi^4 \), respectively, adding the obtained equations, and adding (3.18), we see that

\[
\frac{d}{dt} F(\xi, t) = -\xi^6 \left( k_2 \lambda_1 |\hat{z}|^2 + k_3 \lambda_3 |\hat{p}|^2 + (1 - \lambda_1) |\hat{y}|^2 + (\lambda_4 - \lambda_3) |\hat{\phi}|^2 + (k_1 \lambda_2 - k_1 \lambda_4 - k_1) |\hat{v}|^2 \right) 
- (\gamma \lambda_5 - \lambda_2) \xi^6 |\hat{u}|^2 + \gamma \lambda_5 \xi^6 |\hat{\hat{\phi}}|^2 + \lambda_5 \xi^6 \text{Re} \left( i k_4 \xi \hat{\hat{u}} \hat{\hat{\phi}} - k_1 \hat{v} \hat{\hat{\phi}} \right) 
+ \frac{1}{\gamma} I_3 \xi^4 \text{Re} \left( ik_3 \xi \hat{\hat{\phi}} \hat{\hat{\phi}} - k_3 \xi^2 \hat{\hat{\phi}} \hat{\hat{\phi}} - k_1 \hat{v} \hat{\hat{\phi}} \right) + \frac{1}{\gamma} I_4 \xi^4 \text{Re} \left( ik_3 \xi \hat{\hat{\phi}} \hat{\hat{\phi}} - k_3 \xi^2 \hat{\hat{\phi}} \hat{\hat{\phi}} - k_1 \hat{v} \hat{\hat{\phi}} \right) 
+ \gamma \xi^4 \text{Re} \left[ \xi^2 \left( \frac{I_1}{k_1} \hat{\hat{\phi}} \hat{\phi} + \frac{I_2}{k_1} \hat{\phi} \hat{\phi} \right) \right] + \lambda_2 \xi^2 \hat{\hat{\phi}} \hat{\hat{\phi}}.
\]

(3.20)

Applying Young’s inequality for the terms depending on \( \hat{\hat{\phi}} \) in (3.20) and using the following evident inequality:

\[
|\xi|^m_2 \leq |\xi|^m_1 + |\xi|^m_3, \quad \forall \xi \in \mathbb{R}, \quad \forall 0 \leq m_1 \leq m_2 \leq m_3,
\]

(3.21)

it follows that, for any \( \varepsilon_0 > 0 \),

\[
\frac{d}{dt} F(\xi, t) \leq - (k_2 \lambda_1 - \varepsilon_0) \xi^6 |\hat{z}|^2 - (k_3 \lambda_3 - \varepsilon_0) \xi^6 |\hat{p}|^2 - (1 - \lambda_1 - \varepsilon_0) \xi^6 |\hat{y}|^2 
- (\lambda_4 - \lambda_3 - \varepsilon_0) \xi^6 |\hat{\phi}|^2 - (k_1 \lambda_2 - k_1 \lambda_4 - k_1 - \varepsilon_0) \xi^6 |\hat{v}|^2 
+ C_{\varepsilon_0, \lambda_1, \ldots, \lambda_5} (1 + \xi^8) \xi^2 |\hat{\hat{\phi}}|^2.
\]

(3.22)

We choose \( 0 < \lambda_1 < 1, \lambda_2 > 1, \lambda_5 > \frac{1}{\gamma} \lambda_2, 0 < \lambda_3 < \lambda_4 < \lambda_2 - 1, \) and

\[
0 < \varepsilon_0 < \min \left\{ k_2 \lambda_1, k_3 \lambda_3, 1 - \lambda_1, \lambda_4 - \lambda_3, k_1 \lambda_2 - k_1 \lambda_4 - k_1, \gamma \lambda_5 - \lambda_2 \right\}.
\]

Hence, using the definition (2.9) of \( \hat{E} \), (3.22) leads to, for some positive constant \( c_1 \),

\[
\frac{d}{dt} F(\xi, t) \leq -c_1 \xi^6 \hat{E}(\xi, t) + C \left( 1 + \xi^8 \right) \xi^2 |\hat{\hat{\phi}}|^2.
\]

(3.23)

Now, we introduce the perturbed energy \( L \) as follows:

\[
L(\xi, t) = \lambda \hat{E}(\xi, t) + \frac{1}{1 + \xi^8} F(\xi, t),
\]

(3.24)

where \( \lambda \) is a positive constant to be fixed later. Then from (2.11), (3.23), and (3.24), we have

\[
\frac{d}{dt} L(\xi, t) \leq -c_1 f(\xi) \hat{E}(\xi, t) - (k_4 \lambda - C) \xi^2 |\hat{\hat{\phi}}|^2,
\]

(3.25)
where \( f \) is defined in (3.11). Moreover, using (3.21) and the definitions (2.9), (3.19), and (3.24) of \( \hat{E}, F, \) and \( L, \) respectively, we get, for some \( c_2 > 0 \) (not depending on \( \lambda \)),

\[
|L(\xi, t) - \lambda \hat{E}(\xi, t)| \leq \frac{c_2 (\xi^4 + \xi^6)}{1 + \xi^8} \hat{E}(\xi, t) \leq 2c_2 \hat{E}(\xi, t). \tag{3.26}
\]

Therefore, for \( \lambda \) large enough so that \( \lambda > \max \left\{ \frac{c}{k_4}, 2c_2 \right\} \), we deduce from (3.25) and (3.26) that

\[
\frac{d}{dt} L(\xi, t) + c_1 f(\xi) \hat{E}(\xi, t) \leq 0 \tag{3.27}
\]

and

\[
c_3 \hat{E}(\xi, t) \leq L(\xi, t) \leq c_4 \hat{E}(\xi, t), \tag{3.28}
\]

where \( c_3 = \lambda - 2c_2 > 0 \) and \( c_4 = \lambda + 2c_2 > 0. \) Consequently, a combination of (3.27) and the second inequality in (3.28) leads to, for \( c = \frac{c_3}{c_4}, \)

\[
\frac{d}{dt} L(\xi, t) + c f(\xi) L(\xi, t) \leq 0. \tag{3.29}
\]

Finally, by integration (3.29) with respect to time \( t \) and using (2.17) and (3.28), (3.10) follows with \( \tilde{c} = \frac{c_2 c_4}{c_2 c_4}. \)

**Theorem 3.2.** Assume that \( k_2 \neq k_3. \) Let \( N, \ell' \in \mathbb{N} \) such that \( \ell' \leq N, \)

\[
U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R}),
\]

and \( U \) be the solution of (2.4). Then, for any \( j \in \{0, \ldots, N - \ell'\}, \) there exists \( c_0 > 0 \) such that

\[
\| \partial_x^j U \|_{L_t^2(\mathbb{R})} \leq c_0 (1 + t)^{-1/12-j/6} \| U_0 \|_{L^1(\mathbb{R})} + c_0 (1 + t)^{-\ell'/2} \| \partial_x^{j+\ell} U_0 \|_{L_t^1(\mathbb{R})}, \forall t \in \mathbb{R}_+.
\tag{3.30}
\]

**Proof.** From (3.11), we have (low and high frequencies)

\[
f(\xi) \geq \begin{cases} \frac{1}{2} \xi^6 & \text{if } |\xi| \leq 1, \\ \\
\frac{1}{2} \xi^{-2} & \text{if } |\xi| > 1. \end{cases} \tag{3.31}
\]

Applying Plancherel's theorem and (3.10), we entail

\[
\| \partial_x^j U \|_{L_t^2(\mathbb{R})} = \left\| \widehat{\partial_x^j U}(\xi, t) \right\|_{L_t^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \xi^{2j} |\hat{U}(\xi, t)|^2 d\xi
\leq \tilde{c} \int_{|\xi| \leq 1} \xi^{2j} e^{-c f(\xi)t} |\hat{U}_0(\xi)|^2 d\xi
\leq \tilde{c} \int_{|\xi| \leq 1} \xi^{2j} e^{-c f(\xi)} |\hat{U}_0(\xi)|^2 d\xi + \tilde{c} \int_{|\xi| > 1} \xi^{2j} e^{-c f(\xi)} |\hat{U}_0(\xi)|^2 d\xi := J_1 + J_2.
\]

Using (2.18) (with \( \sigma = 2j, r = \frac{\xi}{2}, \) and \( p = 6) \) and (3.31), it follows, for the low-frequency region,

\[
J_1 \leq C \| \hat{U}_0 \|_{L^\infty(\mathbb{R})}^2 \int_{|\xi| \leq 1} \xi^{2j} e^{-\frac{\xi^4}{2} t} d\xi \leq C(1 + t)^{-\frac{1}{6}(1+2j)} \| U_0 \|_{L^1(\mathbb{R})}^2.
\tag{3.33}
\]
For the high-frequency region, using (3.31), we observe that

\[
J_2 \leq C \int_{|\xi| > 1} |\xi|^2 e^{-\xi t} e^{-\xi^2 t} \left| \hat{U}(\xi, 0) \right|^2 d\xi
\]

\[
\leq C \sup_{|\xi| > 1} \left\{ |\xi|^{-2} e^{-\xi t} \right\} \int_{\mathbb{R}} |\xi|^2 (j+\xi) \left| \hat{U}(\xi, 0) \right|^2 d\xi,
\]

and then using (2.19) (with $\sigma_1 = 2l$, $\sigma_2 = \frac{c}{2}$ and $\sigma_3 = 2$),

\[
J_2 \leq C(1+t)^{-\epsilon} \| \partial^j_{\xi^\epsilon} U_0 \|_{L^2(\mathbb{R})}^2,
\]

(3.34)

and so, by combining (3.32)–(3.34), we get (3.30).

We finish this subsection by proving that the condition $k_2 \neq k_3$ is necessary for the stability of (2.7) in case (2.5) with $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$.

**Theorem 3.3.** Assume that $k_2 = k_3$. Then $|\hat{U}(\xi, t)|$ doesn't converge to zero when time $t$ goes to infinity.

**Proof.** We show that, for any $\xi \in \mathbb{R}$, the matrix

\[
A := (-\xi^2 A_2 + i\xi A_1 + A_0)
\]

(3.35)

has at least a pure imaginary eigenvalue; that is,

\[
\forall \xi \in \mathbb{R}, \exists \lambda \in \mathbb{C} : Re(\lambda) = 0, \text{ Im}(\lambda) \neq 0, \text{ and } det(\lambda I - A) = 0,
\]

where $I$ denotes the identity matrix. From (2.5) with $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$ and $k_2 = k_3$, we have

\[
\lambda I - A = \begin{pmatrix}
\lambda & -i\xi & 0 & -1 & 0 & -1 & 0 \\
-i\xi & \lambda & 0 & 0 & 0 & 0 & i\gamma \xi \\
0 & 0 & \lambda & -i\xi & 0 & 0 & 0 \\
k_1 & 0 & -ik_2 \xi & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & -i\xi & 0 \\
k_1 & 0 & 0 & 0 & -ik_2 \xi & \lambda & 0 \\
0 & i\gamma \xi & 0 & 0 & 0 & 0 & k_4 \xi^2 + \lambda
\end{pmatrix}
\]

(3.36)

A direct computation shows that

\[
det(\lambda I - A) = 2k_1 \lambda (\lambda^2 + k_2 \xi^2) \left[ \lambda (\lambda + k_4 \xi^2) + \gamma^2 \xi^2 \right] + (\lambda^2 + k_2 \xi^2)^2 \left[ \lambda (\lambda + k_4 \xi^2) + \gamma^2 \xi^2 \right] + k_1 \xi^2 (\lambda + k_4 \xi^2).
\]

(3.37)

It is clear that, if $\xi \neq 0$, then $\lambda = i\sqrt{k_2} \xi$ is a pure imaginary eigenvalue of $A$. If $\xi = 0$, then $\lambda = i\sqrt{2k_1}$ is a pure imaginary eigenvalue of $A$. Consequently, according to (2.8) (see Teschl\cite{19}), the solution of (2.7) doesn't converge to zero when time $t$ goes to infinity.

**3.2  | Case 2: $(\tau_1, \tau_2, \tau_3) = (0, 1, 0)$**

We present, first, our exponential stability result for (2.7).

**Lemma 3.4.** Let $\hat{U}$ be the solution (2.8) of (2.7). Then there exist $c, \hat{c} > 0$ such that (3.10) is satisfied with

\[
f(\xi) = \begin{cases}
\frac{\xi^4}{1+\xi^4} & \text{if } k_1 = k_2 = k_3, \\
\frac{1}{1+\xi^2} & \text{if not.}
\end{cases}
\]

(3.38)
Proof. Multiplying (2.15)$_4$ and (2.15)$_7$ by $i\xi\bar{\eta}$ and $-i\xi\bar{\eta}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we get

$$
\frac{d}{dt} \text{Re} \left( i\xi\bar{\eta} \right) = \gamma \epsilon^2 \left( |\eta|^2 - |\bar{\eta}|^2 \right) + k_4 \epsilon^2 \text{Re} \left( i\xi\bar{\eta} \right) - k_2 \epsilon^2 \text{Re} \left( \bar{\eta} \right) - k_1 \text{Re} \left( i\xi\bar{\eta} \right). 
$$

(3.39)

Similarly, multiplying (2.15)$_2$ and (2.15)$_7$ by $\bar{\eta}$ and $\bar{u}$, respectively, adding the resulting equations, taking the real part, and using (2.13), we find

$$
\frac{d}{dt} \text{Re} \left( \bar{u} \right) = -\gamma \text{Re} \left( i\xi\bar{\eta} \right) - k_4 \epsilon^2 \text{Re} \left( \bar{u} \right) + k_1 \text{Re} \left( i\xi\bar{\eta} \right).
$$

(3.40)

Also, multiplying (2.15)$_7$ and (2.15)$_6$ by $i\xi\bar{\theta}$ and $-i\xi\bar{\eta}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we obtain

$$
\frac{d}{dt} \text{Re} \left( i\xi\bar{\eta} \right) = \gamma \epsilon^2 \text{Re} \left( \bar{\eta} \right) - k_4 \epsilon^2 \text{Re} \left( i\xi\bar{\eta} \right) + k_3 \epsilon^2 \text{Re} \left( \phi\bar{\eta} \right) + k_1 \text{Re} \left( i\xi\bar{\eta} \right).
$$

(3.41)

Let us define the functionals

$$
F_0(\xi, t) = \text{Re} \left[ i\xi \left( \lambda_1 \bar{\eta} - \lambda_2 \bar{u} \bar{v} + \lambda_3 \bar{\phi} \right) - \lambda_4 \epsilon^2 \bar{\theta} \bar{v} + \xi^2 \bar{\eta} \right].
$$

(3.42)

$$
F_1(\xi, t) = \left( \frac{k_2}{k_1} \epsilon^2 + \lambda_1 \right) \text{Re} \left( \bar{u} \right),
$$

(3.43)

$$
F_2(\xi, t) = \frac{k_2}{k_1 k_3} \left( k_3 \lambda_4 \epsilon^2 - k_1 \epsilon \right) \text{Re} \left( i\xi\bar{\theta} - \bar{u} \bar{z} - i k_3 \xi \bar{\phi} \bar{y} \right),
$$

(3.44)

and

$$
F_3(\xi, t) = -\frac{k_2}{k_3} \left( \lambda_4 \epsilon^2 + \lambda_2 \right) \text{Re} \left( i\xi\bar{\theta} - \bar{u} \bar{z} - i k_3 \xi \bar{\phi} \bar{y} + \lambda_1 \epsilon^2 \bar{\eta} \right),
$$

(3.45)

where $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ are positive constants to be fixed later. Multiplying (3.1)–(3.5) by $\lambda_1$, $-\lambda_2$, $\lambda_3$, $\lambda_4$, and $-1$, respectively, and adding the obtained equations, it follows that

$$
\frac{d}{dt} F_0(\xi, t) = (\lambda_1 + 1) \epsilon^2 |\bar{\eta}|^2 + \text{Re} \left( (1 - \lambda_4) \epsilon^2 \bar{\theta} \bar{v} + (\epsilon^2 - \lambda_2) i\xi \bar{u} \bar{y} - i\eta \epsilon^3 \bar{\eta} \bar{v} + \gamma \lambda_1 \epsilon^2 \bar{\eta} \right)
$$

(3.46)

$$
- \epsilon^2 \left( (k_1 - k_2) \lambda_2 - k_1 \lambda_4) |\bar{\eta}|^2 + k_2 \lambda_4 |\bar{\eta}|^2 + (\lambda_4 - \lambda_2) |\bar{\theta}|^2 + \lambda_2 |\bar{u}|^2 + k_3 \lambda_3 |\bar{\phi}|^2 \right)
$$

$$
+ (k_2 \epsilon^2 + k_1 \lambda_1) \text{Re} \left( i\xi \bar{\theta} \bar{v} \right) + (k_1 \lambda_4 \epsilon^2 - k_1 \lambda_3) \text{Re} \left( i\xi \bar{\phi} \bar{y} \right) + \left( \lambda_4 \epsilon^2 + \lambda_2 \right) \text{Re} \left( i\xi \bar{\theta} \bar{u} \right).
$$

Multiplying (3.8) by $-\left( \frac{k_2}{k_1} \epsilon^2 + \lambda_1 \right)$, we entail

$$
\frac{d}{dt} F_1(\xi, t) = - (k_2 \epsilon^2 + k_1 \lambda_1) \text{Re} \left( i\xi \bar{\theta} \bar{v} \right) + \left( \frac{k_2}{k_1} \epsilon^2 + \lambda_1 \right) \text{Re} \left( i\xi \bar{\theta} \bar{u} \right).
$$

(3.47)

Multiplying (3.7) by $-\frac{k_2}{k_1}$, adding (3.6) and (3.8), and multiplying the obtained equation by

$$
\frac{k_2}{k_1 k_3} \left( k_3 \lambda_4 \epsilon^2 - k_1 \lambda_3 \right),
$$
we arrive at
\[
\frac{d}{dt} F_3(\xi, t) = \frac{k_2}{k_1 k_3} (k_3 \lambda_4 \xi^2 - k_1 \lambda_3) \Re \left[ \frac{(k_3}{k_2} - 1) \bar{\xi} \bar{\gamma} \bar{\phi} - i \xi \bar{\gamma} \bar{u} + \frac{\gamma k_3}{k_2} \xi^2 \bar{\eta} \phi \right] \\

- (k_3 \lambda_4 \xi^2 - k_1 \lambda_3) \Re \left( i \xi v \phi \right). 
\] (3.48)

Similarly, adding (3.7) and (3.9), multiplying by $-\frac{k_2}{k_3}$, adding (3.6) and (3.8), and multiplying the obtained formula by $-\frac{k_2}{k_3} (\lambda_4 \xi^2 + \lambda_2)$, we deduce that
\[
\frac{d}{dt} F_3(\xi, t) = \frac{k_2}{k_3} \left( \lambda_4 \xi^2 + \lambda_2 \right) \Re \left[ \left( 1 - \frac{k_3}{k_2} \right) \xi^2 \bar{\gamma} \bar{\phi} + \frac{\gamma k_3}{k_2} \xi^2 \bar{\eta} \phi \right] \\

- (\lambda_4 \xi^2 + \lambda_2) \Re \left( i \xi \bar{\theta} \bar{u} \right). 
\] (3.49)

Now, let us introduce the functional $F_4$
\[
F_4(\xi, t) = F_0(\xi, t) + F_1(\xi, t) + F_2(\xi, t) + F_3(\xi, t).
\] (3.50)

By combining (3.46)–(3.49), we deduce that
\[
\frac{d}{dt} F_4(\xi, t) = -\xi^2 \left( (k_1 - k_1 \lambda_2 - k_1 \lambda_4) \bar{\gamma}^2 + k_2 k_1 [\bar{\gamma}]^2 + (\lambda_4 - \lambda_3) [\bar{\theta}]^2 + \lambda_2 [\bar{u}]^2 + k_3 \lambda_3 [\bar{\phi}]^2 \right) \\

+ F_3(\xi, t), 
\] (3.51)

where
\[
F_3(\xi, t) = \Re \left( I_1 \xi^2 \bar{\gamma} \bar{\phi} + i I_2 \xi^2 \bar{\eta} \phi - I_3 \xi^2 \bar{\eta} \phi + i \gamma \xi^2 \bar{\eta} \phi + \gamma (\lambda_4 \xi^2 \bar{\gamma} \bar{\phi}) \right) + (\lambda_1 + 1) \xi^2 |\bar{\gamma}|^2, 
\] (3.52)

\[
I_1 = 1 - \lambda_4 + \frac{k_2}{k_1 k_3} \left( \frac{k_3}{k_2} - 1 \right) (k_1 \lambda_4 \xi^2 - k_1 \lambda_3) + \frac{k_2}{k_3 - 1} \lambda_4 \xi^2 + \left( \frac{k_2}{k_3 - 1} \right) \lambda_2, 
\]

\[
I_2 = \left( \frac{k_2}{k_3 - k_1} \right) \lambda_4 + \frac{k_2}{k_1 - 1} \xi^2 + \lambda_1 + \left( \frac{k_2}{k_3 + 1} \right) \lambda_2 + \frac{k_2}{k_3} \lambda_3, 
\]

and
\[
I_3 = \gamma \left( 1 - \frac{k_3}{k_1} \right) \lambda_4 \xi^2 + \gamma (\lambda_2 + \lambda_3). 
\]

Let $\lambda$ and $\lambda_5$ be positive constants, and $F$ and $L$ be the functionals
\[
F(\xi, t) = F_4(\xi, t) + \lambda_5 \Re \left( i \xi \bar{\gamma} \bar{\eta} \phi \right) - \frac{1}{\gamma} I_1 \Re \left( i \xi \bar{\gamma} \bar{\phi} \right) + \frac{1}{\gamma} I_2 \Re \left( i \bar{\theta} \bar{u} \phi \right) 
\] (3.53)

and
\[
L(\xi, t) = \lambda \bar{E}(\xi, t) + \frac{\xi^2}{f(\xi)} F(\xi, t), 
\] (3.54)

where
\[
f(\xi) = \begin{cases} 
1 + \xi^4 & \text{if } k_1 = k_2 = k_3, \\
1 + \xi^2 & \text{if not}.
\end{cases} 
\] (3.55)

Multiplying (3.39)–(3.41) by $\lambda_5$, $\frac{1}{\gamma} I_2$, and $-\frac{1}{\gamma} I_1$, respectively, adding the obtained equations, and adding (3.51), it appears that
\[
\frac{d}{dt} F(\xi, t) = -\xi^2 \left( (k_1 - k_1 \lambda_2 - k_1 \lambda_4) [\bar{\gamma}]^2 + k_2 \lambda_1 [\bar{\gamma}]^2 + (\lambda_4 - \lambda_3) [\bar{\theta}]^2 + \lambda_2 [\bar{u}]^2 + k_3 \lambda_3 [\bar{\phi}]^2 \right) \\

- \xi^2 (\gamma \lambda_5 - (\lambda_1 + 1)) |\bar{\gamma}|^2 + F_0(\xi, t). 
\] (3.56)
where
\[
F_6(\xi, t) = \gamma \lambda_5 \xi^2 |\tilde{\eta}|^2 + \xi^2\Re \left( \gamma \lambda_1 \tilde{\eta} \tilde{\zeta} - i \gamma \xi \tilde{\eta} \tilde{\omega} - i \gamma \tilde{\zeta} \tilde{\phi} + ik_4 \lambda_5 \xi \tilde{\eta} \tilde{\zeta} - k_2 \lambda_5 \tilde{\eta} \tilde{\zeta} \right) - k_1 \lambda_5 \Re \left( i \xi \tilde{\eta} \tilde{\omega} \right)
- \frac{1}{\gamma} I_1 \Re \left( k_3 \xi^2 \tilde{\eta} \tilde{\phi} + ik_4 \xi \tilde{\zeta} \tilde{\phi} - ik_4 \xi^2 \tilde{\eta} \tilde{\phi} \right) + \frac{1}{\gamma} I_2 \Re \left( ik_1 \xi \tilde{\eta} \tilde{\omega} - k_4 \xi^2 \tilde{\eta} \tilde{u} \right).
\] (3.57)

Noticing that, if \( k_1 = k_2 = k_3 \), then \( I_1, I_2, \) and \( I_3 \) are constants. Otherwise, \( I_1, I_2, \) and \( I_3 \) are of the form \( \text{const} \xi^2 + \text{const} \). Then, by applying Young’s inequality and using (3.21), we see that, for any \( \epsilon_0 > 0 \), we have
\[
F_6(\xi, t) \leq \epsilon_0 \xi^2 \left( |\tilde{\eta}|^2 + |\tilde{\theta}|^2 + |\tilde{\theta}|^2 + |\tilde{\zeta}|^2 + |\tilde{\zeta}|^2 \right) + C_{\epsilon_0,k_1,\ldots,k_4} \int f(\xi) |\tilde{\eta}|^2.
\] (3.58)

Therefore, we conclude from (3.56) and (3.58) that
\[
\frac{d}{dt} F(\xi, t) \leq -c_1 \xi^2 \tilde{E}(\xi, t) + C f(\xi) |\tilde{\eta}|^2.
\] (3.59)

We choose \( 0 < \lambda_1, 0 < \lambda_2 < 1, 0 < \lambda_3 < \lambda_4 < 1 - \lambda_2,\lambda_5 > \frac{1}{\gamma} (\lambda_1 + 1), \) and
\[
0 < \epsilon_0 < \min \{ k_1 - k_2 \lambda_2 - k_4 \lambda_4, k_2 \lambda_1 - \epsilon_0 \lambda_4, (\lambda_4 - \lambda_3 - \epsilon_0) |\tilde{\theta}|^2 + (\lambda_2 - \epsilon_0) |\tilde{\theta}|^2 + (k_2 \lambda_3 - \epsilon_0) |\tilde{\phi}|^2 \}.
\]

Thus, using definition (2.9) of \( \tilde{E} \), (3.59) leads to, for some positive constant \( c_1 \),
\[
\frac{d}{dt} F(\xi, t) \leq -c_1 \xi^2 \tilde{E}(\xi, t) + C f(\xi) |\tilde{\eta}|^2.
\] (3.60)

Then, from (2.11), (3.54), and (3.60), we infer that
\[
\frac{d}{dt} L(\xi, t) \leq -c_1 f(\xi) \tilde{E}(\xi, t) - (k_4 \lambda - C) \xi^2 |\tilde{\eta}|^2,
\] (3.61)

where \( f \) is defined in (3.38). On the other hand, definitions (2.9), (3.53), and (3.54) of \( \tilde{E}, F, L \), respectively, imply that there exists \( c_2 > 0 \) (not depending on \( \lambda \)) such that
\[
|L(\xi, t) - \lambda \tilde{E}(\xi, t)| \leq c_2 \frac{\xi^2 + |\xi|^5}{f(\xi)} \tilde{E}(\xi, t) \leq 2c_2 \tilde{E}(\xi, t).
\]

So, we choose \( \lambda = \max \left\{ \frac{C}{k_5}, 2c_2 \right\} \), and we get (3.27) and (3.28) with \( c_3 = \lambda - 2c_2 > 0 \) and \( c_4 = \lambda + 2c_2 > 0 \). The proof can be ended as for Lemma 3.1.

**Theorem 3.5.** Let \( N, \ell \in \mathbb{N} \) such that \( \ell \leq N \),
\[
U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R}),
\]
and \( U \) be the solution of (2.4). Then, for any \( j \in \{0, \ldots, N - \ell\} \), there exist \( c_0, \bar{c}_0 > 0 \) such that, for any \( t \in \mathbb{R}_+ \),
\[
\| \partial_x^j U \|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-1/8 - j/4} \| U_0 \|_{L^1(\mathbb{R})} + c_0 e^{-\bar{c}_0 t} \| \partial_x^j U_0 \|_{L^2(\mathbb{R})} \text{ if } k_1 = k_2 = k_3,
\] (3.62)

and
\[
\| \partial_x^j U \|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-1/8 - j/4} \| U_0 \|_{L^1(\mathbb{R})} + c_0 (1 + t)^{-\ell/4} \| \partial_x^{j+\ell} U_0 \|_{L^2(\mathbb{R})} \text{ if not.}
\] (3.63)
Proof. From (3.38), we have (low and high frequencies)

$$
f(\xi) \geq \begin{cases} 
\frac{1}{2} \xi^4 & \text{if } |\xi| \leq 1, \\
\frac{1}{2} & \text{if } |\xi| > 1
\end{cases}
$$

(3.64)

and

$$
f(\xi) \geq \begin{cases} 
\frac{1}{2} \xi^4 & \text{if } |\xi| \leq 1, \\
\frac{1}{2} e^{-4} & \text{if } |\xi| > 1
\end{cases}
$$

(3.65)

The proof of (3.63) is identical to the one of Theorem 3.2 by using (3.65) and applying (2.18) (with $\sigma = 2j, r = \frac{c}{2}$, and $p = 4$) and (2.19) (with $\sigma_1 = 2l, \sigma_2 = \frac{c}{2}$, and $\sigma_3 = 4$). To get (3.62), notice that the low frequencies can be treated as for (3.63). For the high frequencies, we have just to remark that (3.64) implies that

$$
\int_{|\xi| > 1} |\xi|^{2j} e^{-c(\xi_0)} |\tilde{U}(\xi, 0)|^2 d\xi \leq \int_{|\xi| > 1} |\xi|^{2j} e^{-\frac{c}{2}} |\tilde{U}(\xi, 0)|^2 d\xi \leq e^{-\frac{c}{2}} \int_{\mathbb{R}} |\xi|^{2j} |\tilde{U}(\xi, 0)|^2 d\xi \leq e^{-\frac{c}{2}} \|\partial_x^2 U_0\|_{L^2(\mathbb{R})}^2,
$$

so (3.62) holds true with $\tilde{c}_0 = \frac{c}{4}$.

3.3 | Case 3: $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$

In this case, we prove the same stability results for (2.7) and (2.4) that are given in Section 3.2, and moreover, the proofs are very similar.

Lemma 3.6. The result of Lemma 3.4 holds true also when $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$.

Proof. Multiplying (2.15)$_6$ and (2.15)$_7$ by $i\xi \tilde{\eta}$ and $-i\xi \tilde{\theta}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we get

$$
\frac{d}{dt} \text{Re} \left( i\xi \tilde{\theta} \tilde{\eta} \right) = \gamma \xi^2 \left( |\tilde{\eta}|^2 - |\tilde{\theta}|^2 \right) + k_4 \xi^2 \text{Re} \left( i\xi \tilde{\eta} \tilde{\theta} \right) - k_3 \xi^2 \text{Re} \left( \tilde{\phi} \tilde{\eta} \right) - k_1 \text{Re} \left( i\xi \tilde{\theta} \tilde{\eta} \right).
$$

(3.66)

Similarly, multiplying (2.15)$_2$ and (2.15)$_7$ by $\tilde{\eta}$ and $\tilde{u}$, respectively, adding the resulting equations, taking the real part, and using (2.13), we find

$$
\frac{d}{dt} \text{Re} \left( \tilde{u} \tilde{\eta} \right) = -\gamma \text{Re} \left( i\xi \tilde{\theta} \tilde{u} \right) - k_4 \xi^2 \text{Re} \left( \tilde{u} \tilde{\eta} \right) + k_1 \text{Re} \left( i\xi \tilde{\theta} \tilde{u} \right).
$$

(3.67)

Also, multiplying (2.15)$_7$ and (2.15)$_4$ by $i\xi \tilde{y}$ and $-i\xi \tilde{\eta}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we obtain

$$
\frac{d}{dt} \text{Re} \left( i\xi \tilde{\eta} \tilde{y} \right) = \gamma \xi^2 \text{Re} \left( \tilde{\theta} \tilde{y} \right) - k_4 \xi^2 \text{Re} \left( i\xi \tilde{\eta} \tilde{y} \right) + k_3 \xi^2 \text{Re} \left( \tilde{\zeta} \tilde{\eta} \right) + k_1 \text{Re} \left( i\xi \tilde{\theta} \tilde{y} \right).
$$

(3.68)

After, we define the functionals

$$
F_0(\xi, t) = \text{Re} \left[ i\xi \left( \lambda_1 \tilde{y} \tilde{\zeta} - \lambda_2 \tilde{u} \tilde{\phi} + \lambda_3 \tilde{\theta} \tilde{\phi} + \lambda_4 \xi^2 \tilde{\theta} \tilde{v} - \xi^2 \tilde{\phi} \tilde{v} \right) \right].
$$

(3.69)
\[ F_1(\xi, t) = \left( \frac{k_1}{k_1} \lambda_4 \xi^2 + \lambda_3 \right) \operatorname{Re} \left( \overline{\hat{u} \hat{\phi}} \right), \]  
(3.70)

\[ F_2(\xi, t) = -\frac{1}{k_1} \left( k_2 \xi^2 - k_1 \lambda_4 \right) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \hat{\phi} - i \frac{k_3}{k_2} \xi \hat{\phi} \hat{\gamma} + \frac{k_3}{k_2} \overline{\hat{\phi} \hat{\phi}} \right), \]  
(3.71)

and

\[ F_3(\xi, t) = (\xi^2 + \lambda_2) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} - i \frac{k_3}{k_2} \xi \hat{\phi} \hat{\gamma} + \frac{k_3}{k_2} \overline{\hat{\phi} \hat{\phi}} - \overline{\hat{\phi} \hat{\phi}} \right). \]  
(3.72)

where \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \) are positive constants to be fixed later. Multiplying (3.1)–(3.4) by \( \lambda_1, -\lambda_2, \lambda_3, \) and \( -\lambda_4, \) respectively, adding the obtained equations, and adding (3.5), we infer that

\[ \frac{d}{dt} F_0(\xi, t) = \operatorname{Re} \left[ i \left( \lambda_2 - \lambda_4 \xi^2 \right) \hat{\xi} \hat{\xi} \hat{\phi} - i \xi \lambda_3 \hat{\phi} \right] + (\lambda_4 - 1) \xi^2 \hat{\xi} \hat{\phi} - i \gamma \lambda_4 \xi^2 \hat{\theta} \hat{\theta} \]  
(3.73)

\[ + (\lambda_3 + \lambda_4) \xi^2 |\hat{\theta}|^2 - \xi^2 \left( (k_1 \lambda_4 - k_1 \lambda_2 - k_1) \xi |\hat{\xi}|^2 + k_2 \lambda_1 |\hat{\xi}|^2 + (1 - \lambda_1) |\hat{\gamma}|^2 + \lambda_2 |\hat{\alpha}|^2 + k_3 \lambda_3 \hat{\phi} \right)^2 \]  

\[ + (k_2 \xi^2 - k_1 \lambda_1) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \right) + (k_3 \lambda_4 \xi^2 + k_1 \lambda_3) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \right) + (\xi^2 + \lambda_2) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \right). \]  

Multiplying (3.9) by \(-\frac{k_2}{k_1} \lambda_4 \xi^2 + \lambda_3\), we arrive at

\[ \frac{d}{dt} F_1(\xi, t) = \left( \frac{k_3}{k_2} \lambda_4 \xi^2 + \lambda_3 \right) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \right) - (k_3 \lambda_4 \xi^2 + k_1 \lambda_3) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \right). \]  
(3.74)

Adding (3.7) and (3.9), multiplying the obtained equation by \(-\frac{k_2}{k_1} \), adding (3.6), and multiplying the resulting equation by \(-\left( \frac{k_2}{k_1} \xi^2 - \lambda_1 \right)\), it follows that

\[ \frac{d}{dt} F_2(\xi, t) = \left( \frac{k_2}{k_1} \xi^2 - \lambda_1 \right) \operatorname{Re} \left[ i \xi \left( -\frac{k_3}{k_2} \hat{\xi} \hat{\phi} - i \gamma \xi \hat{\phi} \hat{\gamma} \right) + \left( 1 - \frac{k_1}{k_2} \right) \xi^2 \hat{\theta} \hat{\theta} \right] \]  
(3.75)

\[ - (k_2 \xi^2 - k_1 \lambda_1) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \right). \]

Similarly, adding (3.7) and (3.9), multiplying the obtained equation by \(-\frac{k_2}{k_1} \), adding (3.6) and (3.8), and multiplying the resulting equation by \(\xi^2 + \lambda_2\), we entail

\[ \frac{d}{dt} F_3(\xi, t) = (\xi^2 + \lambda_2) \operatorname{Re} \left[ i \xi \left( \frac{k_3}{k_2} \hat{\xi} \hat{\phi} + i \gamma \xi \hat{\phi} \hat{\gamma} \right) + \left( \frac{k_3}{k_2} - 1 \right) \xi^2 \hat{\theta} \hat{\theta} \right] \]  
(3.76)

\[ - (\xi^2 + \lambda_2) \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} \right). \]

Let \( F_4 \) be the functional defined by (3.50). A combination of (3.73)–(3.76) implies that

\[ \frac{d}{dt} F_4(\xi, t) = -\xi^2 \left( (k_1 \lambda_4 - k_1 \lambda_2 - k_1) |\hat{\xi}|^2 + k_2 \lambda_1 |\hat{\xi}|^2 + (1 - \lambda_1) |\hat{\gamma}|^2 + \lambda_2 |\hat{\alpha}|^2 + k_3 \lambda_3 \hat{\phi} \right)^2 \]  
(3.77)

where

\[ F_5(\xi, t) = \operatorname{Re} \left( i \xi \hat{\xi} \hat{\phi} - i \frac{k_3}{k_2} \xi \hat{\phi} \hat{\gamma} + i \gamma \lambda_4 \xi^2 \hat{\phi} \hat{\phi} + \gamma \lambda_3 \xi^2 \hat{\phi} \hat{\phi} \right) + (\lambda_3 + \lambda_4) \xi^2 |\hat{\theta}|^2, \]

\[ I_1 = \left[ \left( \frac{k_3}{k_2} - 1 \right) \lambda_4 + \frac{k_2}{k_1} \lambda_1 \right] \xi^2 + \frac{k_3}{k_2} \lambda_1 + \left( \frac{k_3}{k_2} + 1 \right) \lambda_2 + \lambda_3. \]

\[ I_2 = \gamma \left( 1 - \frac{k_2}{k_1} \right) \xi^2 + \lambda_1 + \lambda_2. \]
and
\[
I_3 = \left( \frac{k_3}{k_2} - 1 \right) (\xi^2 + \lambda_2) + \left( 1 - \frac{k_1}{k_2} \right) \left( \frac{k_2}{k_1} \xi^2 - \lambda_1 \right) + \lambda_4 - 1.
\]
Let $\lambda$ and $\lambda_5$ be positive constants and $L$ be the functional defined by (3.54), where $\tilde{f}$ is defined by (3.55) and $F$ is given by
\[
F(\xi, t) = F_4(\xi, t) + \lambda_3 \text{Re} \left( i\xi^2 \tilde{\eta} \right) + \frac{1}{\gamma} I_3 \text{Re} \left( \tilde{u} \tilde{\eta} \right) - \frac{1}{\gamma} I_3 \text{Re} \left( i\xi \tilde{\eta} \tilde{\gamma} \right).
\] (3.79)

Multiplying (3.66)–(3.68) by $\lambda_5 \frac{1}{\gamma} I_1$, and $-\frac{1}{\gamma} I_3$, respectively, adding the obtained equations, and adding (3.77), we find
\[
\frac{d}{dt} F(\xi, t) = -\xi^2 \left( (k_1 \lambda_4 - k_1 \lambda_2 - k_1) |\tilde{\eta}|^2 + k_2 \lambda_1 |\tilde{z}|^2 + (1 - \lambda_1) |\tilde{\eta}|^2 + \lambda_2 |\tilde{\eta}|^2 + k_3 \lambda_4 |\tilde{\gamma}|^2 \right)
- (\gamma \lambda_5 - \lambda_3 - \lambda_4) \xi^2 |\tilde{\eta}|^2 + F_6(\xi, t),
\] (3.80)
where
\[
F_6(\xi, t) = \gamma \lambda_5 |\tilde{\eta}|^2 - \xi^2 \left( i\gamma \lambda_4 \tilde{\eta} \tilde{z} + i\gamma \lambda_4 \xi \tilde{\eta} \tilde{\gamma} - i\gamma \lambda_4 \xi \tilde{\gamma} \tilde{\eta} + ik_4 \lambda_5 \xi \tilde{\eta} \tilde{\gamma} + k_3 \lambda_4 \tilde{\gamma} \tilde{\eta} + k_1 \lambda_5 \text{Re} \left( i\xi \tilde{\gamma} \tilde{\eta} \right) \right)
- \frac{1}{\gamma} I_3 \text{Re} \left( ik_4 \xi \tilde{\gamma} \tilde{\eta} - k_2 \xi^2 \tilde{\eta} \tilde{z} - ik_1 \xi \tilde{\eta} \tilde{\gamma} \right).
\] (3.81)

We remark that, if $k_1 = k_2 = k_3$, then $I_1$, $I_2$, and $I_3$ are constants. Otherwise, $I_1$, $I_2$, and $I_3$ are of the form $\text{const} \xi^2 + \text{const}$. Then, by applying Young’s inequality and using (3.21), we see that, for any $\varepsilon_0 > 0$, we have
\[
F_6(\xi, t) \leq \varepsilon_0 \xi^2 \left( |\tilde{\eta}|^2 + |\tilde{\gamma}|^2 + |\tilde{\eta}|^2 + |\tilde{\gamma}|^2 \right) + C_{\varepsilon_0, \lambda_4, \ldots, \lambda_5} \tilde{f}(\xi) |\tilde{\eta}|^2,
\] (3.82)
where $\tilde{f}$ is defined in (3.55). Therefore, we conclude from (3.80) and (3.82) that
\[
\frac{d}{dt} F(\xi, t) \leq C_{\varepsilon_0, d_1, \ldots, d_5} \tilde{f}(\xi) |\tilde{\eta}|^2 - (\gamma \lambda_5 - \lambda_3 - \lambda_4 - \varepsilon_0) \xi^2 |\tilde{\eta}|^2
\] (3.83)
\[-\xi^2 \left( (k_1 \lambda_4 - k_1 \lambda_2 - k_1 - \varepsilon_0) |\tilde{\eta}|^2 + (k_2 \lambda_1 - \varepsilon_0) |\tilde{z}|^2 + (1 - \lambda_1 - \varepsilon_0) |\tilde{\eta}|^2 + (\lambda_2 - \varepsilon_0) |\tilde{\eta}|^2 + (k_3 \lambda_4 - \varepsilon_0) |\tilde{\gamma}|^2 \right).\]

We choose $0 < \lambda_3$, $0 < \lambda_1 < 1$, $\lambda_4 > 1$, $0 < \lambda_2 < \lambda_4 - 1$, $\lambda_5 > \frac{1}{\gamma} (\lambda_3 + \lambda_4)$, and
\[
0 < \varepsilon_0 < \min \{ k_3 \lambda_3, \lambda_2, 1 - \lambda_1, k_2 \lambda_1, k_1 \lambda_4 - k_1 \lambda_2 - k_1, \gamma \lambda_5 - \lambda_3 - \lambda_4 \}.
\]

Then, using definition (2.9) of $\tilde{E}$, (3.80) implies (3.60), and then (3.61) holds true. Consequently, the proof can be ended as for Lemma 3.4.

**Theorem 3.7.** The stability result given in Theorem 3.5 is satisfied when $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$.

*Proof.* The proof is identical to the one of Theorem 3.5. \qed

4 | **STABILITY: CATTEANEO LAW (1.2)**

This section concerns the stability of (2.4) in case of Cattaneo law (1.2). We will prove (1.7), (1.9), and (1.10). Moreover, we prove that (2.7) is not stable when $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$ and $k_2 = k_3$.

First, observe that (2.15)1–(2.15)6 are identical to (2.16)1–(2.16)6, and (2.15)7 with $k_4 \xi^2 \tilde{\eta}$ replaced by $ik_4 \xi \tilde{\gamma}$ is equal to (2.16)7. So (3.1)–(3.9) are still valid. Moreover, (3.12)–(3.14), (3.39)–(3.41), and (3.66)–(3.68) are satisfied with $ik_4 \xi \tilde{\gamma}$ instead of $k_4 \xi^2 \tilde{\eta}$. On the other hand, we prove the next expressions, which take in consideration the last equation in (2.16).

Multiplying (2.16)7 and (2.16)8 by $i\xi \tilde{\gamma}$ and $-i\xi \tilde{\gamma}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we find
\[
\frac{d}{dt} \text{Re} \left( i\xi \tilde{\gamma} \tilde{\gamma} \right) = k_4 \xi^2 \left( |\tilde{\gamma}|^2 - |\tilde{\eta}|^2 \right) + k_5 \text{Re} \left( i\xi \tilde{\gamma} \tilde{\eta} \right) + \gamma \xi^2 \text{Re} \left( \tau_1 \tilde{u} + \tau_2 \tilde{\gamma} + \tau_3 \tilde{\theta} \tilde{q} \right).
\] (4.1)
Multiplying (2.16)_1 and (2.16)_8 by $i\xi \tilde{q}$ and $-i\xi \tilde{v}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we get
\[
\frac{d}{dt} \text{Re} \left( i\xi \tilde{v} \tilde{q} \right) = -\xi^2 \text{Re} \left( \tilde{u} \tilde{q} \right) + \text{Re} \left( i\xi \tilde{y} \tilde{q} + i\xi \tilde{\theta} \tilde{q} + ik_5 \xi \tilde{q} \tilde{v} \right) - k_4 \xi^2 \text{Re} \left( \tilde{v} \tilde{\eta} \right).
\] (4.2)

Similarly, using the multipliers $\tilde{q}$ and $\tilde{v}$ instead of $i\xi \tilde{q}$ and $-i\xi \tilde{v}$, respectively, we obtain
\[
\frac{d}{dt} \text{Re} \left( \tilde{v} \tilde{q} \right) = \text{Re} \left( i\xi \tilde{u} \tilde{q} \right) + \text{Re} \left( \tilde{y} \tilde{q} + \tilde{\theta} \tilde{q} + k_5 \tilde{q} \tilde{v} \right) + k_4 \text{Re} \left( i\xi \tilde{v} \tilde{\eta} \right).
\] (4.3)

Multiplying (2.16)_5 and (2.16)_8 by $i\xi \tilde{q}$ and $-i\xi \tilde{\phi}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we arrive at
\[
\frac{d}{dt} \text{Re} \left( i\xi \tilde{\phi} \tilde{q} \right) = -\xi^2 \text{Re} \left( \tilde{\theta} \tilde{q} \right) + k_5 \text{Re} \left( i\xi \tilde{q} \tilde{\phi} \right) - k_4 \xi^2 \text{Re} \left( \tilde{\phi} \tilde{\eta} \right).
\] (4.4)

Similarly, using the multipliers $\tilde{q}$ and $\tilde{\phi}$ instead of $i\xi \tilde{q}$ and $-i\xi \tilde{\phi}$, respectively, we entail
\[
\frac{d}{dt} \text{Re} \left( \tilde{\phi} \tilde{q} \right) = \text{Re} \left( i\xi \tilde{\theta} \tilde{q} \right) + k_5 \text{Re} \left( \tilde{q} \tilde{\phi} \right) + k_4 \text{Re} \left( i\xi \tilde{\phi} \tilde{\eta} \right).
\] (4.5)

Also, multiplying (2.16)_3 and (2.16)_8 by $i\xi \tilde{q}$ and $-i\xi \tilde{z}$, respectively, adding the resulting equations, taking the real part, and using (2.14), we infer that
\[
\frac{d}{dt} \text{Re} \left( i\xi \tilde{z} \tilde{q} \right) = -\xi^2 \text{Re} \left( \tilde{y} \tilde{q} \right) + k_5 \text{Re} \left( i\xi \tilde{q} \tilde{z} \right) - k_4 \xi^2 \text{Re} \left( \tilde{z} \tilde{\eta} \right).
\] (4.6)

Similarly, using the multipliers $\tilde{q}$ and $\tilde{z}$ instead of $i\xi \tilde{q}$ and $-i\xi \tilde{z}$, respectively, it appears that
\[
\frac{d}{dt} \text{Re} \left( \tilde{z} \tilde{q} \right) = \text{Re} \left( i\xi \tilde{y} \tilde{q} \right) + k_5 \text{Re} \left( \tilde{q} \tilde{z} \right) + k_4 \text{Re} \left( i\xi \tilde{z} \tilde{\eta} \right).
\] (4.7)

4.1 | Case 1: $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$

As in Section 3.1, we start by presenting the exponential stability result of (2.7) in the next lemma.

**Lemma 4.1.** The result of Lemma 3.1 is satisfied in case (2.16) when $k_2 \neq k_3$ and $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$.

**Proof.** We use the arguments used in Section 3.1. We define the functional $F_0$ by (3.15), and we get (3.16) (because we used only the first six equations in (2.15) which are the same in (2.16)). We consider $F_1$ and $F$ defined by (3.17) and (3.19), and we find (3.20) with $k_4 \tilde{\eta}$ replaced by $ik_5 \tilde{q}$. We put, for $\lambda_0 > 0$,
\[
\tilde{F}(\xi, t) = F(\xi, t) + \lambda_0 \xi^2 \text{Re} \left( i\xi \tilde{\eta} \tilde{q} \right) + \frac{1}{k_4} I_5 \xi^2 \text{Re} \left( i\xi \tilde{v} \tilde{q} \right) + \frac{1}{k_4} I_6 \xi^4 \text{Re} \left( \tilde{\phi} \tilde{q} \right) + \frac{1}{k_4} I_7 \xi^4 \text{Re} \left( \tilde{z} \tilde{q} \right),
\]

where
\[
I_5 = (\gamma \lambda - k_1 \lambda) \xi^2 - \frac{k_1}{\gamma} (I_3 + I_4), \quad I_6 = \frac{\gamma}{k_1} I_2 - \frac{k_1}{\gamma} I_3, \quad \text{and} \quad I_7 = \frac{\gamma}{k_1} I_1 - \frac{k_2}{\gamma} I_4.
Multiplying (4.1), (4.2), (4.5), and (4.7) by $\lambda_6 \xi^4$, $\frac{1}{k_4} I_6 \xi^2$, $\frac{1}{k_4} I_6 \xi^4$, and $\frac{1}{k_4} I_7 \xi^4$, respectively, adding the obtained equations, and using (3.21), we find, for any $\varepsilon_0 > 0$,

$$\frac{d}{dt} \hat{F}(\xi, t) \leq - (k_2 \lambda_1 - \varepsilon_0) \xi^6 |\hat{\gamma}|^2 - (k_1 \lambda_3 - \varepsilon_0) \xi^6 |\hat{\nu}|^2 - (1 - \lambda_1 - \varepsilon_0) \xi^6 |\hat{\eta}|^2$$

$$- (\lambda_4 - (\lambda_3 - \varepsilon_0) \xi^6 |\hat{\gamma}|^2 - (k_1 \lambda_2 - k_1 \lambda_4 - k_1 - \varepsilon_0) \xi^6 |\hat{\nu}|^2 - (\gamma \lambda_5 - \lambda_2 - \varepsilon_0) \xi^6 |\hat{\eta}|^2$$

$$- (k_4 \lambda_6 - \gamma \lambda_5 - \varepsilon_0) \xi^6 |\hat{q}|^2 + C_{\varepsilon_0, k_1, \ldots, k_6} (1 + \xi^2 + \xi^4 + \xi^6 + \xi^8) |\hat{q}|^2.$$  

(4.8)

We choose $0 < \lambda_1 < 1$, $\lambda_2 > 1$, $\lambda_5 > \frac{1}{k_4}$, $\lambda_6 > \frac{\xi}{k_4}$, $0 < \lambda_3 < \lambda_4 < \lambda_2 - 1$, and

$$0 < \varepsilon_0 < \min \{k_2 \lambda_1, k_3 \lambda_3, 1 - \lambda_1, \lambda_4 - \lambda_3, k_1 \lambda_2 - k_1 \lambda_4 - k_1 - \gamma \lambda_5 - \lambda_2, k_4 \lambda_6 - \gamma \lambda_5\}.$$  

Hence, using definition (2.10) of $\hat{E}$, (4.8) leads to, for some positive constant $c_1$,

$$\frac{d}{dt} \hat{F}(\xi, t) \leq - c_1 \xi^6 \hat{E}(\xi, t) + C \left(1 + \xi^2 + \xi^4 + \xi^6 + \xi^8 \right) |\hat{q}|^2.$$  

(4.9)

So we consider $L$ given by (3.24), with $F$ instead of $\hat{F}$, and use (2.12) to find

$$\frac{d}{dt} L(\xi, t) \leq - c_1 f(\xi) \hat{E}(\xi, t) - (k_5 \lambda - C) |\hat{q}|^2,$$  

(4.10)

where $f$ is defined by (3.11). Finally, the proof can be finished exactly as in the proof of (3.10).

\[\square\]

**Theorem 4.2.** The result of Theorem 3.2 is satisfied in case (2.16) when $k_2 \neq k_3$ and $(r_1, r_2, r_3) = (1, 0, 0)$.

\[\square\]

The third result of this subsection says that (2.7) is not stable if $k_2 = k_3$.

**Theorem 4.3.** Assume that $k_2 = k_3$. Then $|\hat{U}(\xi, t)|$ doesn’t converge to zero when time $t$ goes to infinity.

\[\square\]

**Proof.** As in Section 3.1, we show that, for any $\xi \in \mathbb{R}$, matrix (3.35) has at least a pure imaginary eigenvalue. From (2.6) with $(r_1, r_2, r_3) = (1, 0, 0)$ and $k_2 = k_3$, we have

$$\lambda I - A = \begin{pmatrix}
\lambda & -i\xi & 0 & -1 & 0 & -1 & 0 & 0 \\
-ik_4 \xi & \lambda & 0 & 0 & 0 & 0 & i\xi & 0 \\
0 & 0 & \lambda & -i\xi & 0 & 0 & 0 & 0 \\
k_1 & 0 & -ik_2 \xi & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \lambda & -i\xi & 0 & 0 \\
k_1 & 0 & 0 & 0 & -ik_2 \xi & \lambda & 0 & 0 \\
0 & i\xi & 0 & 0 & 0 & 0 & \lambda & ik_4 \xi \\
0 & 0 & 0 & 0 & 0 & ik_4 \xi & k_5 + \lambda & 0
\end{pmatrix}.$$  

(4.11)

A direct computation shows that

$$\text{det}(\lambda I - A) = 2k_1 \lambda (\lambda + k_5) (\lambda^2 + k_2 \xi^2) (\lambda^2 + \gamma^2 \xi^2) + \lambda (\lambda + k_2)(\lambda^2 + k_2 \xi^2)^2 (\lambda^2 + (k_1 + \gamma^2) \xi^2)$$

$$- ik_4 \xi (\lambda^2 + k_2 \xi^2) [ik_1 k_2 \xi^3 (\lambda^2 + k_2 \xi^2) + ik_2^2 \xi^4 (\lambda + k_1) + ik_2 k_4 \lambda^2 \xi^3 + ik_4 k_4 \lambda \xi^2].$$  

(4.12)

We see that, if $\xi \neq 0$, then $\lambda = i\sqrt{2k_1} \xi$ is a pure imaginary eigenvalue of $A$. If $\xi = 0$, then $\lambda = i\sqrt{2k_1}$ is a pure imaginary eigenvalue of $A$. Consequently (see Teschl(20)), solution (2.8) of (2.7) doesn’t converge to zero when time $t$ goes to infinity.
4.2 Case 2: \((\tau_1, \tau_2, \tau_3) = (0, 1, 0)\)

We present, first, our exponential stability result for (2.7).

**Lemma 4.4.** The result of Lemma 3.4 is satisfied in case (2.16) with \((\tau_1, \tau_2, \tau_3) = (0, 1, 0)\).

**Proof.** We adapt the arguments used in Section 3.2. We define \(F_0-F_4\) and \(F\) as in Section 3.2, and we get (3.56), where \(F_6\) is defined by (3.57) with \(k_4\xi\eta\) replaced by \(ik_4\eta\). Let \(\lambda_6 > 0\) and

\[
F(\xi, t) = \xi^2 F(\xi, t) + \lambda_6 \xi^2 \text{Re} \left( i\xi \bar{\eta} \tilde{q} \right) + \frac{1}{k_4} \left( \frac{k_1}{\gamma} (I_1 - I_2) + (k_1 \lambda_5 - \gamma \lambda_4) \xi^2 \right) \xi^2 \text{Re} \left( \tilde{v} \tilde{q} \right) - \frac{1}{k_4} \left( I_3 + \frac{k_3}{\gamma} I_1 \right) \xi^2 \text{Re} \left( i\xi \bar{\eta} \tilde{g} \right) + \frac{1}{k_4} (k_2 \lambda_5 - \gamma \lambda_1) \xi^2 \text{Re} \left( i\xi \bar{\eta} \tilde{q} \right).
\]

Multiplying (4.1), (4.3), (4.4), (4.6), and (3.56) by \(\lambda_6 \xi^2\), \(\frac{1}{k_4} \left( \frac{k_1}{\gamma} (I_1 - I_2) + k_1 \lambda_5 - \gamma \right) \xi^2\), \(-\frac{1}{k_4} \left( I_3 + \frac{k_3}{\gamma} I_1 \right) \xi^2\), \(\frac{1}{k_4} (\gamma \lambda_1 - k_2 \lambda_5) \xi^2\), and \(\xi^2\), respectively, adding the obtained equations, applying Young's inequality for the terms depending on \(\tilde{q}\), and using (3.21), we find, for any \(\varepsilon_0 > 0\),

\[
\frac{d}{dt} \tilde{F}(\xi, t) \leq C_{\varepsilon_0, \lambda_5, \ldots, \lambda_6} \tilde{f}(\xi) \tilde{q}^2 - (\gamma \lambda_5 - \lambda_1 - 1 - \varepsilon_0) \xi^4 |\tilde{n}|^2 - (k_4 \lambda_6 - \gamma \lambda_5 - \varepsilon_0) \xi^4 |\tilde{n}|^2
\]

\[
-\xi^4 \left( (k_1 - k_1 \lambda_2 - k_1 \lambda_4 - \varepsilon_0) |\tilde{n}|^2 + (k_2 \lambda_1 - \varepsilon_0) |\tilde{v}|^2 + (\lambda_4 - \lambda_3 - \varepsilon_0) |\tilde{u}|^2 + (\lambda_2 - \varepsilon_0) |\tilde{u}|^2 + (k_3 \lambda_3 - \varepsilon_0) |\tilde{u}|^2 \right),
\]

where \(\tilde{f}\) is defined in (3.55). We choose \(0 < \lambda_1, 0 < \lambda_2 < 1, 0 < \lambda_3 < \lambda_4 < 1 - \lambda_2, \lambda_5 > \frac{1}{\gamma} (\lambda_1 + 1), \lambda_6 > \frac{\varepsilon_0}{k_4} \lambda_5\), and

\[
0 < \varepsilon_0 < \min \{k_1 - k_1 \lambda_2 - k_1 \lambda_4, k_2 \lambda_1, \lambda_4 - \lambda_3, \lambda_2, k_3 \lambda_3, \gamma \lambda_5 - \lambda_1 - 1, k_4 \lambda_6 - \gamma \lambda_5\}.
\]

Thus, using the definition of \(\hat{E}\), (4.13) implies that, for some positive constant \(c_1\),

\[
\frac{d}{dt} \tilde{F}(\xi, t) \leq -c_1 \xi^4 \hat{E}(\xi, t) + C \tilde{f}(\xi) \tilde{q}^2.
\]

Therefore, we introduce the functional

\[
L(\xi, t) = \lambda \hat{E}(\xi, t) + \frac{1}{f(\xi)} \hat{F}(\xi, t),
\]

and we deduce that, using (2.12) and (4.14),

\[
\frac{d}{dt} L(\xi, t) \leq -c_1 f(\xi) \hat{E}(\xi, t) - (k_5 \lambda - C) |\tilde{q}|^2,
\]

where \(f\) is defined in (3.38). The proof can be ended as for Lemma 3.1.

**Theorem 4.5.** The result of Theorem 3.5 is satisfied in case (1.2) with \((\tau_1, \tau_2, \tau_3) = (0, 1, 0)\).

**Proof.** The proof is identical to the one of Theorem 3.5.
4.3 Case 3: \((τ_1, τ_2, τ_3) = (0, 0, 1)\)

In this case, we prove the same stability results for (2.7) and (2.4) that are given in Section 4.2, and moreover, the proofs are very similar.

**Lemma 4.6.** The result of Lemma 3.4 holds true also in case (2.16) with \((τ_1, τ_2, τ_3) = (0, 0, 1)\).

**Proof.** We define \(F_0-F_4\) and \(F\) as in Section 3.3, and we obtain (3.80), where \(F_6\) is defined in (3.81) with \(k_4ξη\) replaced by \(ik_4q\). Let \(λ_6 > 0\) and

\[
\tilde{F}(ξ, t) = ξ^2 F(ξ, t) + \lambda_6 ξ^2 Re\left(iξ̃q̃\tilde{q}\right) + \frac{1}{k_4} \left(\frac{1}{γ}(k_1 I_3 - I_1) + (k_1 λ_5 - γ λ_4) \tilde{q}^2\right) ξ^2 Re\left(\tilde{v} q\right) + \frac{1}{k_4} (γ λ_3 - k_3 λ_5) \tilde{q}^2 Re\left(iξ̃φ\tilde{q}\right) - \frac{1}{k_4} \left(I_2 + \frac{k_2}{γ} I_1\right) ξ^2 Re\left(iξ̃z\tilde{q}\right).
\]

Multiplying (4.1), (4.3), (4.4), (4.6), and (3.80) by

\[
\lambda_6 ξ^2, \frac{1}{k_4} \left(\frac{k_1}{γ}(I_3 - I_1) - (k_1 λ_5 + γ λ_4) \tilde{q}^2\right) ξ^2, \frac{1}{k_4} (γ λ_3 - k_3 λ_5) ξ^2, \frac{1}{k_4} \left(I_2 + \frac{k_2}{γ} I_1\right) ξ^2, \text{ and } ξ^2,
\]

respectively, adding the obtained equations, applying Young’s inequality for the terms depending on \(\tilde{q}\), and using (3.21), we find, for any \(ε_0 > 0\),

\[
\frac{d}{dt} F(ξ, t) ≤ C_{ε_0, λ_3, ..., λ_n} \tilde{f}(ξ) |q|^2 - ξ^4 \left(γ λ_5 - λ_3 - λ_4 - ε_0\right) |\tilde{q}|^2 - ξ^4 \left(k_4 λ_6 - γ λ_5 - ε_0\right) |\tilde{η}|^2 - ξ^4 \left((k_1 λ_4 - k_1 λ_2 - k_1 - ε_0)|\tilde{η}|^2 + (k_2 λ_4 - ε_0)|\tilde{η}|^2 + (1 - λ_1 - ε_0)|\tilde{η}|^2 + (1 - ε_0)|\tilde{η}|^2 + (k_3 λ_3 - ε_0)|\tilde{η}|^2\right)
\]

\(f\) is defined in (3.55). We choose \(0 < λ_3, 0 < λ_4 < 1, 0 < λ_2 < λ_4 - 1, λ_5 > \frac{1}{γ}(λ_3 + λ_4), λ_6 > \frac{1}{k_4} λ_5\), and

\[0 < ε_0 < \min\{k_3 λ_3, λ_2, 1 - λ_1, k_2 λ_4, k_1 λ_4 - k_1 λ_2 - k_1, γ λ_5 - λ_3 - λ_4, k_4 λ_6 - γ λ_5\}.
\]

Then, using the definition of \(\tilde{E}\), (4.17) implies (4.14), and then (4.16) holds true. Consequently, the proof can be ended as for Lemma 4.4.

**Theorem 4.7.** The stability result given in Theorem 4.5 is satisfied when \((τ_1, τ_2, τ_3) = (0, 0, 1)\).

**Proof.** The proof is identical to the one of Theorem 4.5.

5 Comments and Issues

1. The optimality of the obtained decay rates on \(||∂_ξ^j U||_{L^2(\mathbb{R})}\) is an interesting open question. This question will be the focus of our attention in a future work.
2. When \((τ_1, τ_2, τ_3) ∈ \{(0, 1, 0), (0, 0, 1)\}\) and \(k_1 = k_2 = k_3\), the function \(f\) tends to 1 when \(ξ\) goes to infinity, which avoid the regularity-loss property; that is, (1.9) with \(j = ε = 0\) gives the stability of (1.1) and (1.2) with a decay rate of \(||U||_{L^2(\mathbb{R})}\) depending only on \(||U_0||_{L^2(\mathbb{R})}\) and \(||U_0||_{L^2(\mathbb{R})}\). However, in the other cases, \(f\) tends to 0 when \(ξ\) goes to infinity; this means that the dissipation is very weak in the high-frequency region, which imposes the regularity-loss property in the estimates because (1.7) and (1.10) with \(j = ε = 0\) imply only the boundedness of \(||U||_{L^2(\mathbb{R})}\).
3. Estimate (1.9) leads to a faster speed of convergence to zero of \(||∂_ξ^j U||_{L^2(\mathbb{R})}\) than the one guaranteed by (1.7) and (1.10). This can be explained by the fact that the Cattaneo law generates dissipation stronger than the one generated by the Fourier law. On the other hand, for both laws with \((τ_1, τ_2, τ_3) ∈ \{(0, 1, 0), (0, 0, 1)\}\), the situation is more favorable when \(k_1 = k_2 = k_3\) than in the opposite case.
4. From the mathematical point of view, one can take \( \gamma \in \mathbb{R}^* \) in case (1.1) and \( \gamma, k_4 \in \mathbb{R}^* \) in case (1.2) (instead of \( \gamma, k_4 > 0 \)). The unique needed modifications of proofs when \( \gamma, k_4 < 0 \) are multiplying (3.12), (3.39), (3.66), and (4.1) by \(-1\) and using the obtained identities instead of (3.12), (3.39), (3.66), and (4.1).

5. The coupling terms
\[
\tau_j \gamma \eta_x \quad \text{and} \quad \gamma (\tau_1 \varphi_x + \tau_2 \psi_x + \tau_3 w_x)
\]
(5.1)
in (1.1) and (1.2) are of order 1 with respect to \( x \). Mathematically, these coupling terms can be replaced by (order 0 with respect to \( x \))
\[
\tau_j \gamma \eta_x \quad \text{and} \quad -\gamma (\tau_1 \varphi_t + \tau_2 \psi_t + \tau_3 w_t),
\]
(5.2)
respectively, with \( \gamma \in \mathbb{R}^* \). In this case, the terms \( i \tau_j \gamma \xi \eta \) and \( i \gamma \xi (\tau_1 \hat{u} + \tau_2 \hat{y} + \tau_3 \hat{\theta}) \) in (2.15) and (2.16) are replaced by \( \tau_j \gamma \hat{\eta} \) and \(-\gamma (\tau_1 \hat{u} + \tau_2 \hat{y} + \tau_3 \hat{\theta})\), respectively. On the other hand, (3.10) holds true with
\[
f(\xi) = \frac{\xi^8}{1 + \xi^{10}}
\]
instead of (3.11), and
\[
f(\xi) = \begin{cases} \frac{\xi^6}{1 + \xi^6} & \text{if } k_1 = k_2 = k_3, \\ \frac{\xi^8}{1 + \xi^{10}} & \text{if not} \end{cases}
\]
instead of (3.38), and so we get the stability estimates
\[
\| \partial_x^j U \|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-1/16 - j/8} \| U_0 \|_{L^2(\mathbb{R})} + c_0 (1 + t)^{-\varepsilon/2} \| \partial_x^j \gamma \|_{L^2(\mathbb{R})}, \quad \forall t \in \mathbb{R}^+
\]
instead of (3.30),
\[
\| \partial_x^j U \|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-1/12 - j/6} \| U_0 \|_{L^2(\mathbb{R})} + c_0 e^{-\varepsilon t} \| \partial_x^j U_0 \|_{L^2(\mathbb{R})} \quad \text{if } k_1 = k_2 = k_3
\]
instead of (3.62), and
\[
\| \partial_x^j U \|_{L^2(\mathbb{R})} \leq c_0 (1 + t)^{-1/12 - j/6} \| U_0 \|_{L^2(\mathbb{R})} + c_0 (1 + t)^{-\varepsilon/4} \| \partial_x^j U_0 \|_{L^2(\mathbb{R})} \quad \text{if not}
\]
instead of (3.63). These stability estimates show that the decay rates in case (5.2) are smaller than the ones obtained in case (5.1). Moreover, the nonstability result when \( k_2 = k_3 \) and \( (\tau_1, \tau_2, \tau_3) = (1, 0, 0) \) is still valid using the same arguments of proof, since we get (3.37) and (4.12) with \( \gamma^2 \) instead of \( \gamma^2 \xi^2 \).

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CONFLICT OF INTEREST
This work does not have any conflicts of interest.

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REFERENCES
1. Fatori LH, Monteiro RN, Fernández Sare HD. The Timoshenko system with history and Cattaneo law. Appl Math Comput. 2014;228:128-140.
2. Ide K, Haramoto K, Kawashima S. Decay property of regularity-loss type for dissipative Timoshenko system. *Math Mod Meth Appl Sci*. 2008;18:647-667.
3. Khader M, Said-Houari B. Decay rate of solutions to Timoshenko system with past history in unbounded domains. *Appl Math Optim*. 2017;75:403-428.
4. Rahali R, Said-Houari B. Asymptotic behavior of the Cauchy problem of the Timoshenko system in thermoelasticity of type III. *Evol Equ Control Theory*. 2013;2:423-440.
5. Santos ML, Almeida DS, Muñoz Rivera JE. The stability number of the Timoshenko system with second sound. *J Diff Equa*. 2012;253:2715-2733.
6. Soufyane A, Said-Houari B. The effect of the wave speeds and the frictional damping terms on the decay rate of the Bresse system. *Evol Equ Control Theory*. 2014;3:713-738.
7. Hansen SW. Control and estimation of distributed parameter systems: non-linear phenomena. *Int Ser Numer Anal*. 1994;118:143-170.
8. Hansen SW, Spies R. Structural damping in laminated beams due to interfacial slip. *J Sound Vib*. 1997;204:183-202.
9. Lo A, Tatar N-E. Stabilization of laminated beams with interfacial slip. *Elec J Diff Equa*. 2015;2015:1-14.
10. Alves MS, Gamboa P, Gorain GC, Rambaud A, Vera O. Asymptotic behavior of a flexible structure with Cattaneo type of thermal effect. *Indag Math*. 2016;27:821-834.
11. Beards CF, Imam IMA. The damping of plate vibration by interfacial slip between layers. *Int J Mach Tool Des Res*. 1978;18:131-137.
12. Cao XG, Liu DY, Xu GQ. Easy test for stability of laminated beams with structural damping and boundary feedback controls. *J Dyn Control Syst*. 2007;13:313-336.
13. Cavalcanti MM, Domingos Cavalcanti VN, Falcao Nascimento FA, Lasiiecka I, Rodrigues JH. Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping. *Z Angew Math Phys*. 2014;65:1189-1206.
14. Guesmia A. Asymptotic stability of Bresse system with one infinite memory in the longitudinal displacements. *Medi J Math*. 2017;14:19.
15. Guesmia A. Non-exponential and polynomial stability results of a Bresse system with one infinite memory in the vertical displacement. *Nonauton Dyn Syst*. 2017;4:78-97.
16. Guesmia A. Well-posedness and stability results for laminated Timoshenko beams with interfacial slip and infinite memory. *IMA J Math Cont Info*. 2020;37:300-350.
17. Guesmia A, Messaoudi S, Soufyane A. Stabilization of a linear Timoshenko system with infinite history and applications to the Timoshenko-heat systems. *Elec J Diff Equa*. 2012;2012:1-45.
18. Liu W, Zhao W. Exponential and polynomial decay for a laminated beam with Fourier’s type heat conduction. 2017:2017020058. https://doi.org/10.20944/preprints201702.0058.v1
19. Lo A, Tatar NE. Uniform stability of a laminated beam with structural memory. *Qual Theory Dyn Syst*. 2016;15:517-540.
20. Lo A, Tatar NE. Exponential stabilization of a structure with interfacial slip. *Discrete Contin Dyn Syst*. 2016;36:6285-6306.
21. Raposo CA. Exponential stability for a structure with interfacial slip and frictional damping. *Appl Math Lett*. 2016;53:85-91.
22. Raposo CA, Villagrán OV, Muñoz Rivera JE, Alves MS. Hybrid laminated Timoshenko beam. *J Math Phys*. 2017;58:11.
23. Wang JM, Xu GQ, Yung SP. Exponential stabilization of laminated beams with structural damping and boundary feedback controls. *SIAM J Control Optim*. 2005;44:1575-1597.
24. Djouamai L, Said-Houari B. A new stability number of the Bresse-Cattaneo system. *Math Meth Appl Sci*. 2018;41:2827-2847.
25. Ghoul TE, Khenissi M, Said-Houari B. On the stability of the Bresse system with frictional damping. *J Math Anal Appl*. 2017;455:1870-1898.
26. Khader M, Said-Houari B. Optimal decay rate of solutions to Timoshenko system with past history in unbounded domains. *Z Anal Anwend*. 2018;37:435-459.
27. Racke R, Said-Houari B. Decay rates and global existence for semilinear dissipative Timoshenko systems. *Quart Appl Math*. 2013;71:229-266.
28. Said-Houari B, Soufyane A. The effect of frictional damping terms on the decay rate of the Bresse system. *Evol Equa Cont Theory*. 2014;3:713-738.
29. Teschl G. *Ordinary Differential Equations and Dynamical Systems*, Vol. 140: American Mathematical Society; 2012.

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