EQUILIBRIUM AND STABILITY OF TENSEGRITY STRUCTURES: A CONVEX ANALYSIS APPROACH

FRANCO MACERI, MICHELE MARINO AND GIUSEPPE VAIRO
Department of Civil Engineering
University of Rome “Tor Vergata”
Via del Politecnico, 1 - 00133 Rome, Italy

Abstract. In this paper, tensegrity structures are modeled by introducing suitable energy convex functions. These allow to enforce both ideal and non-ideal constraints, gathering compatibility, equilibrium, and stability problems, as well as their duality relationships, in the same functional framework. Arguments of convex analysis allow to recover consistently a number of basic results, as well as to formulate new interpretations and analysis criterions.

1. Introduction. A tensegrity structure is a space truss made up of struts and guys. Truss-type structures were developed in XIX century together with metallurgic industry. In the last part of the XX century, due to advances in technology and imagination of artists (see the works by K. Snelson), architects and engineers [1, 2, 3, 4], a special truss variety, the tensegrity, was conceived and implemented. In tensegrities, struts are usually made by bars, that is structural members carrying both tension and compression (namely, bilateral members), and guys are realized by cables, that is unilateral members able to carry only tension. An essential contribution to the rigorous analysis of tensegrities has been provided by mathematicians [5, 6, 7, 8].

A tensegrity structure can be modeled as a discrete system of points in space (nodes) whose positions are restrained through frictionless constraints described by linear equalities and inequalities. When a structural member is modeled as inextensible, the corresponding constraint is said to be ideal and tensegrity models accounting only for ideal (opposite, non-ideal) members, are denoted as ideal (opposite, non-ideal). Otherwise, when both ideal and non-ideal constraints occur, a mixed-type tensegrity will be here referred to.

Although many energy-based frameworks are available for non-ideal structures (wherein possible ideal members are treated by introducing a fictitious extensibility) [6], a variational framework explicitly accounting for ideal behavior has been proposed only very recently in [9]. There, following [10] and within the mathematical framework of the convex analysis [11, 12], restrictions corresponding to ideal members are regarded as internal constraints for the variational formulation of kinematic and static structural problems. These internal constraints are defined by means of suitable free-energy contributions, depending on nodal configuration and fulfilling convexity requirements.

2010 Mathematics Subject Classification. 74G65, 74G99, 70E50, 52A41, 52A41.
Key words and phrases. Mechanics of tensegrity structures, unilateral problems, convex analysis.
In this paper, tensegrity structures modeled considering both ideal and non-ideal (herein assumed as at most linearly viscoelastic) constraints are addressed by means of a consistent and non-conventional variational approach. The internal constraints describing ideal members are enforced by introducing suitable dissipative pseudo-potential contributions, depending on nodal velocities.

Several classical results are consistently recovered by energy arguments, in the context of both kinematic and static problems of ideal, mixed-type and non-ideal tensegrities. The novel perspective herein focused allows to gather engineering and mathematical approaches. Accordingly, basic results are coupled with non-conventional energy-based physical interpretations, even in the case of ideal or mixed-type structures, wherein classical regular energy arguments cannot be applied. Following such a formulation, an operative stability criterion is introduced, enabling the stability analysis of mixed-type tensegrities.

\section{Notation} Let \( E \) be the three-dimensional Euclidean space, \( V := \{v = P - Q, \forall P, Q \in E\} \) the vector space associated with \( E \), endowed with the usual inner product \( \langle a, b \rangle \in \mathbb{R} \) between \( a, b \in V \), and let \( \| \cdot \| \) be the Euclidean norm on \( V \).

Let \((O, \xi_1, \xi_2, \xi_3)\) be a time-independent Cartesian frame in \( E \), \( \{\xi_1, \xi_2, \xi_3\} \) being an orthonormal basis for \( V \).

Given two sets \( a := \{a_1, \ldots, a_n\} \in V^n \) and \( b := \{b_1, \ldots, b_n\} \in V^n \), the following operators are defined: \( a + b := \{a_1 + b_1, \ldots, a_n + b_n\} \in V^n \), \( (a, b) := \sum_{j=1}^{n} a_j \cdot b_j \in \mathbb{R} \) and \( a \wedge b := \{a_1 \wedge b_1, \ldots, a_n \wedge b_n\} \in V^n \). Moreover, vector \([a] := [a_1 \ldots a_n]^t \in \mathbb{R}^{3n}\) collects vectors \( a_j \in V \) which define \( a \in V^n \), symbol \( \emptyset \) denotes the empty set, \( \mathcal{O}_n \in V^n \) the set made up of \( n \) null vectors, and \( \mathcal{U}_n^a := \{a_1, \ldots, a_n | a_j = a \in V, j = 1, \ldots, n\} \in V^n \).

Let \( t \in [0, +\infty) \) be the time variable and assume that all time-dependent functions are as regular as needed in time.

Let \( p_j(t) \in E \), with \( j \in \{1, \ldots, n_p\} \) and \( \mathbb{N} \ni n_p > 1 \), be the point occupied by the node \( j \) at time \( t \), and \( p_j(t) = p_j(0) \) its position vector in \( V \) at time \( t \). Let \( \mathcal{N} \) be a set of \( n_p \) nodes, such that \( p_j \neq p_k \forall j, k \in \{1, \ldots, n_p\} \) with \( j \neq k \). Denote by \( p(t) := \{p_1(t), \ldots, p_{n_p}(t)\} \in V^{n_p} \) a configuration of \( \mathcal{N} \) at time \( t \), with \( p^0 := p(0) \).

Moreover, let \( \dot{p}_j(t) \in V \) be the velocity of \( p_j \) at time \( t \) (that is, \( \dot{p}_j := \frac{dp_j}{dt} \)), \( \ddot{p}(t) := \{\dot{p}_1(t), \ldots, \dot{p}_{n_p}(t)\} \in V^{n_p} \), and \( \dddot{p}^0 := \dddot{p}(0) \).

Let now consider \( n_e \) bilateral scalar constraints for nodal positions as well as \( n_b \) bilateral (namely, corresponding to bar-type structural elements) and \( n_u \) unilateral (that is, cable-type members) constraints restricting the relative position of couples of nodes in \( \mathcal{N} \), and let define the following sets:

\[
\begin{align*}
\mathcal{E} & := \{(j, m) | \exists \text{ external scalar constraint } m \text{ on node } j\} \quad (1) \\
\mathcal{I}_b & := \{(i,j,k) | \exists \text{ bar } k \text{ between nodes } i \text{ and } j, i < j\} \quad (2) \\
\mathcal{I}_u & := \{(i,j,h) | \exists \text{ cable } h \text{ between nodes } i \text{ and } j, i < j\} \quad (3)
\end{align*}
\]

Whenever necessary, apex \( b \) (respectively, \( u \) or \( e \)) will denote in the following quantities associated to bilateral bar-type constraints (respectively, unilateral cable-type constraints or bilateral external supports).

\textbf{Definition 2.1.} The tensegrity \( T_r \) is the set of nodes collected in \( \mathcal{N} \) and of constraints identified by \( \mathcal{E} \), \( \mathcal{I}_b \) and \( \mathcal{I}_u \).
3. Tensegrity modeling and problems statement. For a given tensegrity structure $T_r$, several models could be conceived depending on the description of kinematic and static features of the involved members, as well as depending on the modeling of their joints. Structural members restrict nodal positions or velocities by means of reaction forces. These restrictions (namely, constraints) can be external or internal, depending on whether they are applied to single nodes or to couples of nodes. Kinematics and statics of constraints herein addressed are described in what follows.

3.1. Constraints modeling. For each internal constraint among node $i$ and $j$, let define the length as

$$\ell^k_{ik}(t) = \ell^k_{jk}(p(t)) := \| \mathbf{p}_j(t) - \mathbf{p}_i(t) \| \quad \text{for} \ (i,j,k) \in \mathcal{I}_b ,$$

$$\ell^m_{ij}(t) = \ell^m_{ij}(p(t)) := \| \mathbf{p}_j(t) - \mathbf{p}_i(t) \| \quad \text{for} \ (i,j,h) \in \mathcal{I}_a ,$$

and the unit vector identifying the constraint axis as

$$\beta^k_{ij}(t) = \beta^k_{ij}(p(t)) := [ \mathbf{p}_j(t) - \mathbf{p}_i(t) ]/\ell^k_{ij}(t) \quad \text{for} \ (i,j,k) \in \mathcal{I}_b ,$$

$$\gamma^m_{ij}(t) = \gamma^m_{ij}(p(t)) := [ \mathbf{p}_j(t) - \mathbf{p}_i(t) ]/\ell^m_{ij}(t) \quad \text{for} \ (i,j,h) \in \mathcal{I}_a ,$$

while $\mathbf{e}_m^o \in \mathbf{V}$ (with $\| \mathbf{e}_m^o \| = 1$) identifies the axis of the $m$th external constraint.

3.1.1. Kinematic modeling. The following three types of constraints are herein considered

$$\omega^m_{ij}(p_j) := (\mathbf{p}_j - \mathbf{p}_o^j) \cdot \mathbf{e}_m^o = 0 \quad \text{with} \ (j,m) \in \mathcal{E} ,$$

$$\phi^k_{ij}(p_i,p_j) := \ell^k_{ij}(t) - b^k_{ij} = \delta^k_{ij}(t) \quad \text{with} \ (i,j,k) \in \mathcal{I}_b ,$$

$$\theta^m_{ij}(p_i,p_j) := \ell^m_{ij}(t) - e^m_{ij} \begin{cases} \leq 0 & \text{if} \ \chi^m_{ij}(t) = 0 \\ \geq 0 & \text{if} \ \chi^m_{ij}(t) > 0 \end{cases} \quad \text{with} \ (i,j,h) \in \mathcal{I}_a ,$$

where $b^k_{ij}$ and $e^m_{ij}$ are the reference lengths of the $k$th bilateral and $m$th unilateral constraint, respectively, and where $\delta^k_{ij}(t) \in [-b^k_{ij},+\infty)$, $\delta^m_{ij}(t) \in [-e^m_{ij},+\infty)$, and $\chi^m_{ij}(t) \in \mathbb{R}^+ \cup \{0\}$ depend on constraints constitutive response, which will be defined in Section 3.2.1.

3.1.2. Static modeling. Denoting as $\mathbf{r}_s^j$ the reactive force exerted by node $s$ on constraint $q$ (opposite to the reaction of $q$ on $s$), the constraints’ static behavior is assumed to be described by:

$$\mathbf{r}_s^m(t) := \nu^m_{ij}(t) \mathbf{e}_m^o \quad \text{with} \ (j,m) \in \mathcal{E} ,$$

$$\mathbf{r}_s^k(t) := -\mathbf{r}_s^k(t) := \lambda^k_{ij}(t) \beta^k_{ij} \quad \text{with} \ (i,j,k) \in \mathcal{I}_b ,$$

$$\mathbf{r}_s^m(t) := -\mathbf{r}_s^m(t) := \lambda^m_{ij}(t) \gamma^m_{ij} \quad \text{with} \ (i,j,h) \in \mathcal{I}_a ,$$

where $\nu^m_{ij}(t), \lambda^k_{ij}(t), \lambda^m_{ij}(t) \in \mathbb{R}$ have the physical meaning of reaction force values.

3.2. Tensegrity structure modeling. The structure $T_r$ is assumed to be loaded only by external forces at nodes and any inertial effect is disregarded. Among possible models for $T_r$, let $T$ be the one wherein bars (respectively, cables) are assumed to be massless and to enforce frictionless pin-jointed internal bilateral (respectively, unilateral) constraints. Internal constraints are assumed to behave as elastic, viscoelastic or ideal, while external constraints are assumed to be ideal. The ideal behavior is defined by:
Definition 3.1. A bilateral (respectively, unilateral) constraint is said to be ideal if its reaction force acting on nodes makes a zero virtual work (respectively, non-negative) for any admissible (that is, satisfying the constraint’s kinematic restriction) virtual displacement of the constrained nodes.

For $T$, let $\tilde{T}_b \subseteq \tilde{T}_b$ and $\tilde{T}_u \subseteq \tilde{T}_u$ (of cardinality $\tilde{n}_b$ and $\tilde{n}_u$, respectively) be the sets identifying the non-ideal constraints, so that the sets $\tilde{T}_b = \tilde{T}_b \setminus \tilde{T}_b$ and $\tilde{T}_u = \tilde{T}_u \setminus \tilde{T}_u$ collect all constraints modeled as ideal (with $\tilde{n}_b = n_b - \tilde{n}_b$ and $\tilde{n}_u = n_u - \tilde{n}_u$, respectively).

3.2.1. Constitutive modeling. For $(i, j, k) \in \tilde{T}_b$ and $(i, j, h) \in \tilde{T}_u$, reactive force values $\lambda^b_k$ and $\lambda^n_h$ are split in two contributes: dissipative ($\zeta^b_k$ and $\zeta^n_h$) and non-dissipative ($f^b_k$ and $f^n_h$).

Non-ideal constraints are herein modeled as linearly viscoelastic. Therefore, let $\kappa^b_k$ (respectively, $\eta^b_k$) and $\kappa^n_h$ (respectively, damping coefficients) of the corresponding structural member. Assuming $\kappa^b_k, \kappa^n_h \in \mathbb{R}^+$ and $\eta^b_k, \eta^n_h \in \mathbb{R}^+ \cup \{0\}$, the constraints constitutive behavior is assumed to be described by:

$$\delta^b_k(t) := \begin{cases} f^b_k(t)/\kappa^b_k & \text{if } (i, j, k) \in \tilde{T}_b, \\ 0 & \text{if } (i, j, k) \in \tilde{T}_b, \end{cases}$$

$$\delta^n_h(t) := \begin{cases} f^n_h(t)/\kappa^n_h & \text{if } (i, j, h) \in \tilde{T}_u, \\ 0 & \text{if } (i, j, h) \in \tilde{T}_u, \end{cases}$$

$$\zeta^b_k(t) := \eta^b_k v^b_k(p(t), \ddot{p}(t)), \quad \zeta^n_h(t) := H(t^n_h(t) - c^n_h) \eta^n_h v^n_h(p(t), \ddot{p}(t)), \quad (16)$$

where $v^b_k(p(t), \ddot{p}(t)) = [\dot{p}_j(t) - \dot{p}_i(t)] \cdot \mathcal{B}_k(t)$, $v^n_h(p(t), \ddot{p}(t)) = [\dot{p}_j(t) - \dot{p}_i(t)] \cdot \gamma_h(t)$ and $H(x - x_o)$ is the Heaviside function centered in $x_o$. It is worth pointing out that both dissipative and non-dissipative contributions of cable response are assumed to be unilateral.

Function $\chi_h(t)$, introduced in Eq. (10) and governing the kinematics of the $h^{th}$ unilateral constraint, is defined as:

$$\chi_h(t) := \begin{cases} \lambda^b_h(t) \in \mathbb{R}^+ \cup \{0\} & \text{if } (i, j, h) \in \tilde{T}_b, \\ \lambda^n_h(t) \in \mathbb{R}^+ \cup \{0\} & \text{if } (i, j, h) \in \tilde{T}_u. \end{cases} \quad (17)$$

Accordingly, $\chi_h$ has the mechanical meaning of reactive force value (respectively, non-dissipative part) of the $h^{th}$ internal unilateral ideal constraint (respectively, non-ideal).

By employing Eqs. (15) and (17), $\delta^n_h(t)$ is restricted by: $\delta^n_h(t) \in \mathbb{R}^+ \cup \{0\}$. The elastic behavior is recovered by considering in Eqs. (16) $\eta^b_k = \eta^n_h = 0$.

Since such a constitutive model, the reference lengths $b^b_k$ and $c^n_h$ acquire the physical meaning of the unstressed lengths of the $k^{th}$ bar and of the $h^{th}$ cable, respectively. Furthermore, from Definition 3.1 it follows that internal ideal constraints correspond to inextensible structural members, ensuring that the distance between the constrained nodes is always equal to (for bilateral) or not greater than (for unilateral) the unstressed length.

3.2.2. Linearized model. As customary in structural theories and useful in some applications, a linearized model can be obtained by considering statics at $t = 0$ and a first-order approximation in time around $t = 0$ of the kinematical restrictions.
From Eqs. (8), (9) and (10), and by employing Eqs. (17) and (15), the linearized kinematical model for $t \to 0^+$ results in:

$$\omega_m(p_j^o) + t\omega_m(p_j^o) = 0 \quad \Rightarrow \quad \dot{p}_j(0) \cdot e_m = 0 \quad \text{with} \quad (j, m) \in \mathcal{E}, \quad (18)$$

$$\phi_k(p_i^o, p_j^o) + t\phi_k(p_i^o, p_j^o) = \delta_k^0(0) + t\delta_k^0(0)$$

$$\Rightarrow \quad [\dot{p}_j(0) - \dot{p}_i(0)] \cdot \beta_k(0) = \delta_k^0(0) \quad \text{with} \quad (i, j, k) \in \mathcal{I}_b, \quad (19)$$

$$\theta_k(p_i^o, p_j^o) + t\theta_k(p_i^o, p_j^o) \begin{cases} \leq 0 & \text{if} \quad \chi_h(0) = \hat{\chi}_h(0) = 0 \\ = \delta_h^0(0) + t\delta_h^0(0) & \text{else} \end{cases}$$

$$\Rightarrow \quad [\dot{p}_j(0) - \dot{p}_i(0)] \cdot \gamma_h(0) \begin{cases} \leq 0 & \text{if} \quad \begin{cases} \ell_h^0(0) = c_h^0 \\ \delta_h^0(0) = 0 \end{cases} \quad \text{with} \quad (i, j, h) \in \mathcal{I}_u. \\ = \delta_h^0(0) & \text{else} \end{cases} \quad (20)$$

### 3.3. Compatibility and equilibrium problems.

#### 3.3.1. Kinematics. Denote by $v = \{v_1, \ldots, v_n\} \in \mathbb{V}^{nr}$ a virtual velocity for $\mathcal{N}$, $v_j \in \mathbb{V}$ being a virtual velocity of node $j$. Let $\mathcal{V}_r$ be the space of rigid-body virtual velocities:

$$\mathcal{V}_r := \{v \in \mathbb{V}^{nr} \mid v = \mathcal{U}_+^{nr} + (\mathcal{U}_+^{nr})^\perp \land p, \forall t, \omega \in \mathbb{V}\} \quad (21)$$

and $\mathcal{V}_{nr} := \mathbb{V}^{nr} \setminus \mathcal{V}_r$ the set of non-rigid-body velocities.

Let the following problems be introduced:

**Problem 1.** For $\mathcal{T}$ in $p^o$ find $v \in \mathbb{V}^{nr}$ such that $v_j \cdot e_m = 0 \quad \forall \quad (j, m) \in \mathcal{E}$,

$$(v_j - v_i) \cdot \beta_k(0) = 0 \quad \forall \quad (i, j, k) \in \mathcal{I}_b, \quad \text{and} \quad (v_j - v_i) \cdot \gamma_h(0) \leq 0 \quad \text{if} \quad (i, j, h) \in \mathcal{I}_u \text{ and } \ell_h^0(0) = c_h^0.$$

**Problem 2.** Find $v \in \mathcal{V}_{nr}$ such that $v$ satisfies Eqs. (18), (19) and (20) with:

$$\delta_h^0(0) = 0 \quad \forall \quad (i, j, k) \in \mathcal{I}_b, \quad \delta_h^0(0) = 0 \quad \forall \quad (i, j, h) \in \mathcal{I}_u. \quad (22)$$

**Definition 3.2.** The velocity $\hat{v}$ is said to be kinematically admissible for $\mathcal{T}$ when $\hat{v}$ is a solution of Problem 1.

**Definition 3.3 (Rigidity).** $\mathcal{T}$ is said to be rigid when Problem 2 has no solution.

#### 3.3.2. Statics. Denote by $f_j \in \mathbb{V}$ the resultant of the active forces on node $j$, and by $f = \{f_1, \ldots, f_n\} \in \mathbb{V}^{nr}$ the set of nodal forces on $\mathcal{N}$. Moreover, let the set of the self-equilibrated nodal forces be defined as:

$$\mathcal{F}_n := \{f \in \mathbb{V}^{nr} \mid \sum_{j=1}^{n_p} f_j = 0 \text{ and } \sum_{j=1}^{n_p} p_j \land f_j = 0\}. \quad (23)$$

Denote also by $r_j \in \mathbb{V}$ the nodal resultant of a set (generally not unique if it exists) of reactive forces exerted on constraints by node $j$ according to Eqs. (11), (12) and (13), and let $r = \{r_1, \ldots, r_n\} \in \mathbb{V}^{nr}$ be a set of nodal reactions.

The equilibrium problem for model $\mathcal{T}$ is:

**Problem 3.** Given $f \in \mathbb{V}^{nr}$ acting on $\mathcal{T}$ in a given state $(p(t), \dot{p}(t))$, find $r$ such that $r = f$. 
Problem 3 can be formulated in a variational form through the application of the Principle of Virtual Powers:

\[
\text{Find } r \in V^{np} \text{ such that } \sum_{j=1}^{np} r_j \cdot v_j = \sum_{j=1}^{np} f_j \cdot v_j \quad \forall v \in V^{np}.
\]  

(24)

**Definition 3.4.** \( T \) in \((p(t), \dot{p}(t))\) is said to be in equilibrium under nodal forces \( \hat{f}(t) \) when Problem 3 has solution with \( f = \hat{f}(t) \).

**Definition 3.5.** \( T \) is said to be in steady-state equilibrium under time-independent nodal forces \( \hat{f} \) when Problem 3 has a solution at \( t = 0 \) with \( f = \hat{f} \) and \( \dot{p}^0 = \emptyset_{np} \).

**Definition 3.6.** A set \( \hat{r} \) of reactive forces is said to be self-equilibrated when \( \hat{r} \) is solution of Problem 3 with null nodal forces, that is \( f = \emptyset_{np} \).

**Definition 3.7 (Static-rigidity).** \( T \) is said to be static-rigid when it is in steady-state equilibrium for any set of nodal forces such that \( f \in \mathcal{F}_v \).

**Definition 3.8 (Pre-stressability).** \( T \) is said to be pre-stressable when \( T \) is in steady-state equilibrium under null nodal forces \( (f = \emptyset_{np}) \) and \( \{\chi_1(0), \ldots, \chi_{nu}(0)\} \in (\mathbb{R}^+)^{nu} \).

3.3.3. **Remarks.** Since previous definitions, the following remarks hold:

**Remark 1.** Rigidity, static-rigidity and pre-stressability properties do not depend on parameters \( \kappa^b_k, \kappa^u_h, \eta^b_k \) and \( \eta^u_h \).

In fact, static-rigidity and pre-stressability problems (both static), do not involve material behavior. Addressing rigidity (kinematic problem), quantities \( \eta^b_k \) and \( \eta^u_h \) do not affect the statement of Problem 2. Moreover, due to the constitutive response in Eqs. (14) and (15), request (22) ensures that stiffness constants do not intervene in Problem 2. As a consequence, it results:

**Remark 2.** Rigidity, static-rigidity and pre-stressability are properties of \( T_r \), independent on the choice of the model \( T \) (e.g., ideal, linearly elastic, linearly viscoelastic).

**Remark 3.** It should be pointed out that rigidity is a kinematic property, whereas static-rigidity is a static property. They are proved to be equivalent, as shown in [9].

3.4. **Stability problem and tangent response.** Non-ideal \((\dot{\hat{n}}_b = \dot{\hat{n}}_u = 0)\) or mixed-type (when both ideal and non-ideal internal constraints occur) tensegrities are addressed in this section.

Let the time-dependent positive-defined scalar quantity \( \mathcal{L}(t) \) be defined as:

\[
\mathcal{L}(t) := \sum_{(i,j,k) \in \hat{I}_b} \kappa^b_k [\ell^b_k(t) - b^b_k]^2 + \sum_{(i,j,h) \in \hat{I}_u} \kappa^u_h H(\ell^u_h(t) - c^u_h)[\ell^u_h(t) - c^u_h]^2,
\]  

(25)

and let introduce:

**Problem 4.** Find \( v(t) \in V_{nr} \) and \( \bar{t} > 0 \) such that \( p(t) = p(0) + \int_0^{\bar{t}} v(\tau) d\tau \) satisfies the kinematic restrictions imposed only by ideal constraints in \( T \) and \( \mathcal{L}(t) \leq \mathcal{L}(0) \) for any \( t \in [0, \bar{t}] \).

**Definition 3.9 (Stability).** \( T \) is said to be stable when it is in steady-state equilibrium under \( \hat{f} = \emptyset_{np} \) and Problem 4 has no solution.
In many engineering problems, structural equilibrium is combined with the condition that all cables have to be tensioned (namely, there does not exist a slack cable). This is equivalent to require that at a given time $t$

$$
T \text{ is in equilibrium under } f(t) \text{ and } \begin{cases} 
\ell_h^u(t) > \ell_h^o & \forall (i,j,h) \in \hat{I}_u \\
\lambda_h^u(t) > 0 & \forall (i,j,h) \in \hat{I}_u 
\end{cases} \quad (26)
$$

In this case, as an engineering task, it is frequently asked to identify the effects of small perturbations superimposed upon the equilibrium configuration of the structure. Two problems can be formulated:

**Problem 5 (Direct tangent response).** For a tensegrity model $T$ in the state $(p(t), \dot{p}(t))$ and satisfying the condition (26), find the nodal force variation $\delta f \in V_n p$ which ensures the equilibrium when perturbations $\delta p, \delta \dot{p} \in V_n p$ admissible with ideal constraints are assigned.

**Problem 6 (Inverse tangent response).** For a tensegrity model $T$ in a stable steady-state equilibrium and satisfying the condition (26) in the reference configuration (that is, for $t = 0$), find the variation $\delta p \in V_n p$ admissible with ideal constraints and which ensures the equilibrium for a given perturbation $\delta f \in V_n p$ of applied nodal forces.

Problems 5 and 6 will be solved by adopting a constructive approach, described in the following. In particular, it will be shown that under condition (26) unilateral constraints behave as bilateral in a neighborhood of the structure configuration, and thereby each of previous problems has an unique solution.

4. **An energy-based approach.** Statics and kinematics of tensegrity structures will be now recovered within a variational approach based on convex analysis arguments enabling to model both ideal and non-ideal constraints in a unique framework. The definition of free-energy and dissipative potential functions in the case of non-ideal members is coupled with a non-standard variational description of ideal constraints. This non-standard description is physically motivated by the fact that a motion along a non-admissible direction would imply the dissipation of an infinite amount of energy.

4.1. **Convex framework.** As customary in convex analysis [12], define $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, where the regular addition is completed by the rules: $a + (+\infty) = +\infty$ ($\forall a \in \mathbb{R}$) and $+\infty + (+\infty) = +\infty$, while multiplication by positive numbers is completed by $a \times (+\infty) = +\infty$ ($\forall a \in \mathbb{R}^+$). Let $I_a(x)$ be the indicator function of the zero of $\mathbb{R}$ (that is $I_a(x) = 0$ if $x = 0$ and $I_a(x) = +\infty$ elsewhere) and $I^-(x)$ be the indicator function of the non positive numbers. If $g$ is a convex function defined on a convex part $X$ of a real vector space, denote by $\nabla g(x)$ and $\partial g(x)$ a subgradient and the subdifferential set of $g$ at point $x \in X$, respectively. Omitting the proof (see [9]), the following result is recalled:

**Theorem 4.1.** Let $g$ be a convex function taking only two values (0 or $+\infty$), with $g \neq +\infty$. Function $g$ is subdifferentiable at $x$ if and only if $g(x) < +\infty$.

4.2. **Free-energy and pseudo-potential of dissipation.** Take the configuration $p$ as the set of state variables for $T$ and $\dot{p}$ as the variables describing the evolution.
The internal forces $\mathbf{r}_j$ are split in dissipative $\mathbf{r}_{jd}^d$ and non dissipative $\mathbf{r}_{jd}^{nd}$ components, defined through the following constitutive laws:

$$\mathbf{r}_{jd}^{nd} := \frac{\partial \Psi}{\partial \mathbf{p}_j}, \quad \mathbf{r}_{jd}^d := \frac{\partial \Phi}{\partial \mathbf{p}_j},$$

(27)

where $\Psi(p)$ is the free-energy (gathering all the time-independent properties of the system) and $\Phi$ is the pseudo-potential of dissipation of $T$. As defined by Jean Jacques Moreau [12], pseudo-potential of dissipation function $\Phi$ is a positive convex system) and $\Phi$ is the pseudo-potential of dissipation of $\dot{\mathbf{p}}$, with value 0 for $\dot{\mathbf{p}} = 0$. In this paper, function $\Phi(p, \dot{\mathbf{p}})$ is chosen as dependent also on $p$ in order to take into account the influence of the actual state on such a dissipation.

Denoting by $\Psi^\varepsilon$ the strain-energy function associated with the actual configuration, by $\Phi^\varepsilon$ (respectively, $\hat{\Phi}$) the dissipative pseudo-potential associated with non-ideal (respectively, ideal) constraints, the constitutive laws are defined by:

$$\Psi(p) := \Psi^\varepsilon(p), \quad \Phi(p, \dot{\mathbf{p}}) := \Phi^\varepsilon(p, \dot{\mathbf{p}}) + \hat{\Phi}(p, \dot{\mathbf{p}}).$$

(28)

In turn, by employing the previously-introduced apex rule, $\Psi^\varepsilon$ is herein defined as:

$$\Psi^\varepsilon(p) := \sum_{(i,j,k) \in \mathcal{I}_b} \Psi^\varepsilon_k(p) + \sum_{(i,j,k) \in \mathcal{I}_a} \Psi^\varepsilon_k(p),$$

(29)

where ideal constraints do not contribute to $\Psi$.

Moreover, $\Phi^\varepsilon$ is chosen as:

$$\Phi^\varepsilon(p, \dot{\mathbf{p}}) := \sum_{(i,j,k) \in \mathcal{I}_b} \Phi^\varepsilon_k(p, \dot{\mathbf{p}}) + \sum_{(i,j,k) \in \mathcal{I}_a} \Phi^\varepsilon_k(p, \dot{\mathbf{p}}),$$

(30)

and $\hat{\Phi}$ as:

$$\hat{\Phi}(p, \dot{\mathbf{p}}) := \sum_{(i,j,k) \in \mathcal{I}_b} \hat{\Phi}^b_k(p, \dot{\mathbf{p}}) + \sum_{(i,j,k) \in \mathcal{I}_a} \hat{\Phi}^a_k(p, \dot{\mathbf{p}}) + \sum_{(i,j,m) \in \mathcal{E}} \hat{\Phi}^e_m(p, \dot{\mathbf{p}}).$$

(31)

4.3. Ideal constraints. For $(i,j,k) \in \mathcal{I}_b$, the kinematic constraint in Eq. (9) combined with Eq. (14) reads also as:

$$\frac{d}{dt} \| \mathbf{p}_j(t) - \mathbf{p}_i(t) \|^2 = 0 \iff (\mathbf{p}_j - \mathbf{p}_i) \cdot (\dot{\mathbf{p}}_j - \dot{\mathbf{p}}_i) = 0.$$  

(32)

For a given configuration $p$, the set of velocities $v \in \mathcal{V}^{np}$ satisfying $(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{v}_j - \mathbf{v}_i) = 0$ is a vector space. Therefore, the function

$$\mathcal{V}^{np} \ni v \mapsto \mathbf{\hat{\Phi}}^b_k(p, v) := L_b((\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{v}_j - \mathbf{v}_i))$$

(33)

is convex in $\mathcal{V}^{np}$ and endowed with generalized derivative:

$$\frac{\partial \mathbf{\hat{\Phi}}^b_k}{\partial \mathbf{v}_s} = \begin{cases} 0 & \text{if } s \neq i, j \\ \nabla \mathbf{\hat{\Phi}}^b_k[\mathbf{p}_j - \mathbf{p}_i] & \text{if } s = j \\ -\nabla \mathbf{\hat{\Phi}}^b_k[\mathbf{p}_j - \mathbf{p}_i] & \text{if } s = i \end{cases},$$

(34)

where

$$\partial \mathbf{\hat{\Phi}}^b_k \ni \nabla \mathbf{\hat{\Phi}}^b_k = \begin{cases} \alpha \in \mathbb{R} & \text{if } (\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{v}_j - \mathbf{v}_i) = 0 \\ \# & \text{else} \end{cases}.$$  

(35)

Thereby, the definition set $\mathcal{D}$ of $\partial \mathbf{\hat{\Phi}}^b_k(p, v)$ is:

$$\mathcal{D}(\partial \mathbf{\hat{\Phi}}^b_k(p, v)) := \{ v \in \mathcal{V}^{np} \mid (\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{v}_j - \mathbf{v}_i) = 0 \},$$

(36)

and, from Eqs. (27), the generalized derivative of $\mathbf{\hat{\Phi}}^b_k$ corresponds to $\mathbf{r}_s^k$ for $s = i, j$. 


Similarly, for \((i, j, h) \in \tilde{\mathcal{I}}_{u}\), the kinematic constraint in Eq. (10) combined with Eq. (15) reads as:
\[
(p_j - p_i) \cdot (p_j - p_i) \begin{cases} 
\text{any} & \text{if } \|p_j(t) - p_i(t)\| < \epsilon_h^p \\
\leq 0 & \text{if } \|p_j(t) - p_i(t)\| = \epsilon_h^p.
\end{cases}
\]
(37)

For a given configuration \(p(t)\) and introducing
\[
H(p_i, p_j) := \begin{cases} 
0 & \text{if } \|p_j - p_i\| < \epsilon_h^p \\
1 & \text{if } \|p_j - p_i\| = \epsilon_h^p,
\end{cases}
\]
the function
\[
\mathcal{V}^u \ni v \mapsto \hat{\Phi}^u_h(p, v) \in \mathbb{R}, \quad \hat{\Phi}^u_h(p, v) := I^{-1}(H(p_i, p_j)[(p_j - p_i) \cdot (v_j - v_i)])
\]
(39)
is convex in \(\mathcal{V}^u\) and its generalized derivative results in
\[
\frac{\partial \hat{\Phi}^u_h}{\partial v_s} = \begin{cases} 
0 & \text{if } s \neq i, j \\
H(p_i, p_j)\nabla \hat{\Phi}^u_h[p_j - p_i] & \text{if } s = j \\
-H(p_i, p_j)\nabla \hat{\Phi}^u_h[p_j - p_i] & \text{if } s = i
\end{cases}
\]
(40)

with
\[
\partial \hat{\Phi}^u_h \ni \nabla \hat{\Phi}^u_h = \begin{cases} 
\alpha \in \mathbb{R}^+ \cup \{0\} & \text{if } H(p_i, p_j)(p_j - p_i) \cdot (v_j - v_i) = 0 \\
0 & \text{if } H(p_i, p_j)(p_j - p_i) \cdot (v_j - v_i) < 0 \\
\# & \text{else}
\end{cases}
\]
(41)

Thereby, the definition set of \(\partial \hat{\Phi}^u_h\) is:
\[
\mathcal{D}(\partial \hat{\Phi}^u_h(p, v)) := \begin{cases} 
\mathcal{V}^u & \text{if } \|p_j - p_i\| < \epsilon_h^p \\
\{v \in \mathcal{V}^u \mid (p_j - p_i) \cdot (v_j - v_i) \leq 0\} & \text{if } \|p_j - p_i\| = \epsilon_h^p,
\end{cases}
\]
(42)

and, from Eqs. (27), the generalized derivative of \(\hat{\Phi}^u_h\) corresponds to \(r^h_s\) for \(s = i, j\).

**Remark 4.** If \(\|\partial \hat{\Phi}^u_h / \partial v_j\| > 0\) then \((p_j - p_i) \cdot (v_j - v_i) = 0\). Therefore, an unilateral ideal member with a non-trivial reaction force kinematically behaves as a bilateral ideal one in the neighborhood of \(p(t)\).

As regards external constraints identified by \((j, m) \in \mathcal{E}\), the function
\[
\mathcal{V}^e \ni v \mapsto \hat{\Phi}^e_m(p, v) \in \mathbb{R}, \quad \hat{\Phi}^e_m(p, v) := I_o(e^o_m \cdot v_j),
\]
(43)
is convex in \(\mathcal{V}^e\) and its generalized derivative is
\[
\frac{\partial \hat{\Phi}^e_m}{\partial v_j} = \nabla \hat{\Phi}^e_m e^o_m,
\]
(44)

with
\[
\partial \hat{\Phi}^e_m \ni \nabla \hat{\Phi}^e_m = \begin{cases} 
\alpha \in \mathbb{R} & \text{if } e^o_m \cdot v_j = 0 \\
\# & \text{else}
\end{cases}
\]
(45)

Therefore, the definition set of \(\partial \hat{\Phi}^e_m\) is:
\[
\mathcal{D}(\partial \hat{\Phi}^e_m(p, v)) := \{v \in \mathcal{V}^e \mid e^o_m \cdot v_j = 0\}
\]
(46)

and, from Eqs. (27), the generalized derivative of \(\hat{\Phi}^e_m\) corresponds to \(r^e_j\).
4.4. **Non-ideal constraints.** The free-energy and dissipative pseudo-potential contributions for a bilateral non-ideal constraint \((i,j,k) \in \mathcal{K}_b\) are respectively:

\[
\Psi^b_k(p) := \frac{1}{2} \kappa^b_k (\ell^b_k(p) - b^o_k)^2, \quad \Phi^b_k(p, \dot{p}) := \frac{1}{2} \eta^b_k |v^b_k(p, \dot{p})|^2 .
\]  
(47)

The constraint reaction is obtained by differentiation of Eq. (47) as:

\[
r^b_j(p, \dot{p}) = -r^b_j(p, \dot{p}) = \{ \kappa^b_k (\ell^b_k(p) - b^o_k) + \eta^b_k (v^b_k(p, \dot{p})) \} \beta_k(p) = \lambda^b_k(p, \dot{p}) \beta_k(p) .
\]  
(48)

Denoting by \(\nabla_j\) and \(\tilde{\nabla}_j\) the gradient operators with respect to the components of \(p_j\) and \(\hat{p}_j\), respectively, simple algebra allows to prove that

\[
\nabla_j \ell^b_k = \tilde{\nabla}_j v^b_k = \beta_k, \quad \nabla_j \beta_k = (\mathbf{I} - \beta_k \otimes \beta_k) / \ell^b_k ,
\]  
(49)

where \(\mathbf{I}\) is the second-order identity tensor and symbol \(\otimes\) indicates the dyadic product.

Accordingly, at time \(t\), the variation of \(r^b_j\) with respect to \(p_j\) and \(\hat{p}_j\) leads to

\[
\delta r^b_j = \kappa^b_k \delta p^b_j + (\lambda^b_k / \ell^b_k) \delta \hat{p}^b_j + \eta^b_k \delta |p|^2 ,
\]  
(50)

where \(\delta p^b_j = [\beta_k \otimes \beta_k] \delta \hat{p}_j\) (analogously, \(\delta \hat{p}^b_j\) is the component of \(\delta p_j\) parallel to \(\beta_k\) and \(\delta \hat{p}^b_j = [\mathbf{I} - \beta_k \otimes \beta_k] \delta \hat{p}_j\) the one orthogonal to \(\beta_k\).

For a bilateral non-ideal member, the tangent stiffness matrix \(K^b_k\) and the tangent damping matrix \(D^b_k\) can be defined as:

\[
K^b_k(p, \dot{p}) := \kappa^b_k [\beta_k \otimes \beta_k] + (\lambda^b_k / \ell^b_k) \mathbf{I} - \beta_k \otimes \beta_k ,
\]  
(51)

\[
D^b_k(p) := \eta^b_k [\beta_k \otimes \beta_k] .
\]  
(52)

From Eq. (51), two contributions at the tangent elastic stiffness can be identified: the material stiffness \(K^{bM}_k := \kappa^b_k [\beta_k \otimes \beta_k]\) referred to the reference configuration, and the geometric stiffness \(K^{bG}_k := (\lambda^b_k / \ell^b_k) (\mathbf{I} - \beta_k \otimes \beta_k)\) accounting for the reorientation of stressed members.

Alternatively, tangent stiffness matrix can be arranged as

\[
K^b_k = \frac{\lambda^b_k}{\ell^b_k} \mathbf{I} + \left[ \kappa^b_k - \frac{\lambda^b_k}{\ell^b_k} \right] (\beta_k \otimes \beta_k) = \frac{\lambda^b_k}{\ell^b_k} \mathbf{I} + \frac{\kappa^b_k - \eta^b_k v^b_k}{\ell^b_k} \beta_k \otimes \beta_k ,
\]  
(53)

from which well-established results for the purely elastic case [13] can be simply recovered.

If the linearization of the equilibrium problem is performed, disregarding orthogonal contributions to \(\beta_k\), then \(K^b_k = K^{bM}_k\).

The free-energy and dissipative pseudo-potential contributions for the unilateral non-ideal constraint \((i,j,k) \in \mathcal{K}_u\) are respectively

\[
\Psi^u_k(p) := \frac{1}{2} \kappa^u_k H (\ell^u_k(p) - c^o_k)^2 , \quad \Phi^u_k(p, \dot{p}) := \frac{1}{2} \eta^u_k H (\ell^u_k(p) - c^o_k |v^u_k(p, \dot{p})|)^2 .
\]  
(54)

\[
\Phi^u_k(p, \dot{p}) := \frac{1}{2} \eta^u_k H (\ell^u_k(p) - c^o_k |v^u_k(p, \dot{p})|)^2 .
\]  
(55)

Therefore, by differentiation of Eqs. (54) and (55), the corresponding constraint reaction results in

\[
r^u_j(p, \dot{p}) = -r^u_j(p, \dot{p}) = H (\ell^u_k(p) - c^o_k) \{ \kappa^u_k |\ell^u_k(p) - c^o_k| + \eta^u_k v^u_k(p, \dot{p}) \} \gamma_k(p) ,
\]  
(56)
and tangent stiffness and damping matrices are:
\[
\mathbb{K}_h^\nu(p, \dot{p}) := \alpha \{ \kappa_h^\nu \gamma_h \otimes \gamma_h + (\lambda_h^\nu / \ell_h^\nu)[I - \gamma_h \otimes \gamma_h] \}, \tag{57}
\]
\[
\mathbb{D}_h^\nu(p) := \alpha \eta_h^\nu [\gamma_h \otimes \gamma_h], \tag{58}
\]
where the scalar quantity \( \alpha \) results from:
\[
\ell_h^\nu(p) < c_h^\nu \Rightarrow \alpha = 0, \tag{59}
\]
\[
\ell_h^\nu(p) = c_h^\nu \Rightarrow \alpha \in [0, 1], \tag{60}
\]
\[
\ell_h^\nu(p) > c_h^\nu \Rightarrow \alpha = 1. \tag{61}
\]

Remark 5. When \( \ell_h^\nu(p) > c_h^\nu \), unilateral non-ideal member behaves as a bilateral member in the neighborhood of \( p(t) \).

4.5. Tensegrity statics and kinematics. Let the following subdifferential sets be introduced as:
\[
\partial \Psi(p) := \left\{ G_\Psi \in \mathbb{V}^{n_r} \mid G_\Psi = \left\{ \frac{\partial \Psi}{\partial \mathbf{p}_1} \bigg|_p, \ldots, \frac{\partial \Psi}{\partial \mathbf{p}_{n_p}} \bigg|_p \right\} \right\}, \tag{62a}
\]
\[
\partial \Phi(p, \dot{p}) := \left\{ G_\Phi \in \mathbb{V}^{n_r} \mid G_\Phi = \left\{ \frac{\partial \Phi}{\partial \mathbf{p}_1} \bigg|_{p, \dot{p}}, \ldots, \frac{\partial \Phi}{\partial \mathbf{p}_{n_p}} \bigg|_{p, \dot{p}} \right\} \right\}, \tag{62b}
\]
\[
\partial \Phi^\dot{e}(p, \dot{p}) := \left\{ G_\Phi \in \mathbb{V}^{n_r} \mid G_\Phi = \left\{ \frac{\partial \Phi^e}{\partial \mathbf{p}_1} \bigg|_{p, \dot{p}}, \ldots, \frac{\partial \Phi^e}{\partial \mathbf{p}_{n_p}} \bigg|_{p, \dot{p}} \right\} \right\}, \tag{62c}
\]
and
\[
\partial \Phi(p, \dot{p}) := \{ G_\Phi \in \mathbb{V}^{n_r} \mid G_\Phi = G_o + G_\dot{e}, \text{ with } G_o \in \partial \Phi(p, \dot{p}), G_\dot{e} \in \partial \Phi^e(p, \dot{p}) \}, \tag{63}
\]
\[
\partial (\Psi(p) + \Phi(p, \dot{p})) := \{ G \in \mathbb{V}^{n_r} \mid G = G_\Psi + G_o + G_\dot{e}, \text{ with } G_\Psi \in \partial \Psi(p), G_o \in \partial \Phi(p, \dot{p}), G_\dot{e} \in \partial \Phi^e(p, \dot{p}) \}. \tag{64}
\]

Therefore, the definition domains of \( \partial \Phi(p, \dot{p}) \) and \( \partial \Phi^e(p, \dot{p}) \) are:
\[
\mathcal{D}(\partial \Phi(p, \dot{p})) = \left\{ \bigcap_{k=1}^{n} \mathcal{D}(\partial \Phi_k^e(p, \dot{p})) \right\} \cap \left\{ \bigcap_{h=1}^{n} \mathcal{D}(\partial \Phi_h^e(p, \dot{p})) \right\} \subseteq \mathbb{V}^{n_r}, \tag{65}
\]
\[
\mathcal{D}(\partial \Phi^e(p, \dot{p})) = \mathbb{V}^{n_r}. \tag{66}
\]

Since the constitutive laws in Eqs. (27), statics is fully described by generalized derivatives, whereas kinematic restrictions by their definition domains. Accordingly, the following remark holds:

Remark 6. Definitions 3.2, 3.4 and 3.5 can be formulated as:
- The virtual velocity \( \dot{v} \in \mathbb{V}^{n_r} \) is kinematically admissible for \( T \) if and only if \( \dot{v} \in \mathcal{D}(\partial \Phi(p^o, v)) \).
- \( T \) is in equilibrium under nodal forces \( \hat{f}(t) \) if and only if \( \hat{f}(t) \in \partial(\Psi(p(t)) + \Phi(p(t), \varnothing_{n_p})) \).
- \( T \) is in steady-state equilibrium under time-independent nodal forces \( \hat{f} \) if and only if \( \hat{f} \in \partial(\Psi(p^o) + \Phi(p^o, \varnothing_{n_p})) \).
4.6. Ideal model: Rigidity, static-rigidity and pre-stressability. As a consequence of Remark 1, rigidity, static-rigidity and pre-stressability can be addressed by assuming an ideal behavior for each structural member. In this case, from Eqs. (28), the model for $T_r$ is characterized by:

$$
\Psi^\text{id}(p) := \Psi(p) = 0, \quad \Phi^\text{id}(p, \dot{p}) := \Phi(p, \dot{p}) = \Phi(p, \dot{p}),
$$

with $\Phi(p, \dot{p})$ defined as in Eq. (31) when $I_b = I_b$ and $I_u = I_u$.

It is worth remarking that, starting from $p^o$, the motions admissible with all the kinematic restrictions of the ideal model for $T_r$ are described by the closed convex cone $D(\partial \Phi^\text{id}(p^o, v))$.

**Remark 7.** $T_r$ is rigid if $\Phi(p, \dot{p})$ is convex.

In order to embed the rigidity concept in a variational framework, let the following problem be considered:

**Problem 7.** Find $\hat{v} \in \mathcal{V}_{nr} := \mathcal{V}_{nr} \cup \mathcal{O}_{n_p}$ such that $\Phi^\text{id}(p^o, \hat{v}) = \inf_{v \in \mathcal{V}_{nr}} \{\Phi^\text{id}(p^o, v)\}$.

Problem 7 is a minimization problem over the set $\mathcal{V}_{nr}$, which is non convex. This is a pitfall for finding a solution. In order to skip this drawback, let $\mathcal{E}^+$ be the set collecting fictitious linearly independent external ideal constraints which prevent rigid-body motions of the structure and let $\Phi^+_e(p, \dot{p})$ the corresponding contribution to the pseudo-potential of dissipation. An explicit definition of $\mathcal{E}^+$ (and then of $\Phi^+_e$) cannot be provided because it depends on the external constraints in $\mathcal{E}$. Nevertheless, if the tensegrity structure is a free-body in the space (i.e., $n_e = 0$ and $\mathcal{E} = \emptyset$), as in the case of tensegrity moduli) with $n_p \geq 3$, a possible choice for $\Phi^+_e$ is:

$$
\Phi^+_e(p, \dot{p}) := \sum_{k=1}^{3} I_o(\xi_k \cdot \dot{p}_J) + \sum_{k=1}^{2} I_o(\xi_k \cdot \dot{p}_J) + I_o(\xi_N \cdot \dot{p}_H),
$$

being $I, J, H \in \mathcal{N}$ three not-aligned nodes, $\xi_N = n/\|n\|$ and $n = (p_J - p_I) \wedge (p_J - p_H)$.

By considering a new form of the pseudo-potential function as:

$$
\hat{\Phi}(p, \dot{p}) := \Phi^\text{id}(p, \dot{p}) + \Phi^+_e(p, \dot{p}),
$$

it is possible to introduce:

**Problem 8.** Find $\hat{v} \in \mathcal{V}^{n_p}$ such that $\hat{\Phi}(p^o, \hat{v}) = \inf_{v \in \mathcal{V}^{n_p}} \{\hat{\Phi}(p^o, v)\}$.

From the convexity of $\mathcal{V}^{n_p}$ and of $\hat{\Phi}(p^o, v)$ over $\mathcal{V}^{n_p}$, it immediately follows:

**Proposition 1.** The set $\{\hat{v} \in \mathcal{V}^{n_p} | \hat{\Phi}(p^o, \hat{v}) = \inf_{v \in \mathcal{V}^{n_p}} (\hat{\Phi}(p^o, v))\}$ is convex.

Moreover, let introduce:

**Lemma 4.2.** Problems 7 and 8 are equivalent.

**Proof.** If $v \in \mathcal{V}^{n_p} \setminus \mathcal{V}_{nr}$, then $\hat{\Phi}(p^o, v) = +\infty$ and it is not a solution for both Problem 7 and 8. Moreover, if $v \in \mathcal{V}_{nr}$, then $\hat{\Phi}^+_e(p^o, v) = 0$. Therefore, if $v \in \mathcal{V}_{nr}$ is a solution for Problem 7, it is also solution for Problem 8 and viceversa.

**Remark 8.** $v = \mathcal{O}_{n_p}$ is a solution of Problem 8.

**Remark 9.** $T_r$ is rigid if $\mathcal{D}(\partial \hat{\Phi}(p^o, v)) = \mathcal{O}_{n_p}$.
On the basis of Proposition 1 and Lemma 4.2, as well as accounting for Remarks 8 and 9, it is immediate to prove:

**Theorem 4.3.**

\[ \mathcal{T}_r \text{ is rigid } \iff \text{Problem 8 has a unique solution.} \]

**Proof.** Since Remark 9, if a tensegrity is rigid then \( \hat{v} \notin \mathcal{D}(\partial \Phi(p^o, v)) \) for any \( \hat{v} \neq \emptyset \). Accordingly, by employing Theorem 4.1, \( \Phi(p^o, \hat{v}) = +\infty \) for any \( \hat{v} \neq \emptyset \). Thereby, \( \hat{v} = \emptyset \) (with \( \Phi(p^o, \hat{v}) = 0 \)) is the unique solution for Problem 8.

Conversely, if Problem 8 has a unique solution then \( \Phi(p^o, \hat{v}) = +\infty \) for any \( \hat{v} \neq \emptyset \) (see Remark 8). Equivalently, \( \hat{v} \notin \mathcal{D}(\partial \Phi(p^o, v)) \) for any \( \hat{v} \neq \emptyset \) (see Theorem 4.1), that is the tensegrity is rigid.

Since Remark 6 and Eq. (67), the following remark holds:

**Remark 10.** \( \mathcal{T}_r \) is static-rigid \( \iff f \in \partial \Phi^{id}(p^o, \emptyset_{n_p}), \forall f \in \mathcal{F}_o. \)

If \( \Phi^+ = 0 \), Remark 10 reads as:

\[ \mathcal{T}_r \text{ is static-rigid } \iff f \in \partial \Phi^{id}(p^o, \emptyset_{n_p}), \forall f \in \mathcal{V}^{np}. \]

Let the set \( \mathcal{R} := \{\nabla \Phi^1, \ldots, \nabla \Phi^n\} \in (\mathbb{R}^+ \cup \{0\})^n \) be introduced. Observing that \( \nabla \Phi^h \) is proportional to the reaction force of the \( h \)th cable and that the condition \( \emptyset_{n_p} \in \partial \Phi^{id}(p^o, \emptyset_{n_p}) \) means that there exists a reactive set \( r \) in a steady-state self-equilibrium, then

**Remark 11.**

\( \mathcal{T}_r \) is pre-stressable \( \iff \{\mathcal{R} | \mathcal{R} \in (\mathbb{R}^+)^n \} \neq \emptyset. \)

Duality between kinematic and static arguments in tensegrities is a well-established result [5]. It has been recently recovered through an energy-based approach developed in the context of the convex analysis [9]. For the sake of completeness and omitting proofs (see [5, 9]), the following results are recalled:

**Theorem 4.4.**

\( \mathcal{T}_r \) is rigid \( \iff \mathcal{T}_r \) is static-rigid

**Theorem 4.5.** Denoting by \( \mathcal{T} \) the bar-truss model obtained from \( \mathcal{T}_r \) by assuming as bilateral all the unilateral constraints, then:

\( \mathcal{T}_r \) is rigid \( \iff \mathcal{T} \) is rigid and \( \mathcal{T}_r \) is pre-stressable

The rigidity of the space truss \( \mathcal{T} \) can be classically determined by means of simple linear algebra, while the pre-stressability problem of \( \mathcal{T}_r \) requires to find admissible solutions of a system of \( 3n_p \) equilibrium equations. The latter problem can be faced by means of a quadratic optimization approach [9]. Therefore, Theorem 4.5 allows to move towards the development of effective algorithms enabling the solution of the rigidity problem.

**4.7. Mixed-type models: Tangent response and stability.** Consider in the following a mixed-type tensegrity model with free-energy and pseudo-potential of dissipation given in Eqs. (28), (29), (30) and (31).

Owing to the definition of non-ideal constraints, function \( \Psi(p) \) is twice differentiable in a neighborhood of \( p \). Moreover, assume that condition (26) is satisfied, which is equivalent to prescribe pre-stressability if \( t = 0 \) and \( \dot{p}^o = \emptyset_{n_p} \). Since
Remarks 4 and 5, both ideal and non-ideal unilateral members behave as bilateral in the neighborhood of $p$. Thereby, since the kinematic restrictions of unilateral constraints reduce to linear equalities, the space $\tilde{V}(p)$ of non-rigid body motions admissible with ideal constraints, that is

$$\tilde{V}(p) := D(\partial\Phi(p,v)) \cap D(\partial\Phi^+(p,v)) \subseteq V_{nr} := V_{nr} \cup \varnothing_{nr},$$

(70)
can be obtained by solving the homogeneous linear system

$$\hat{C}(p)[v] = 0,$$

(71)
where $[v] \in \mathbb{R}^{3n_p}$ and $\hat{C}(p)$ is the compatibility matrix at $p$, obtained considering only ideal constraints (i.e., identified by the sets $\mathcal{E}, \mathcal{E}^+, \mathcal{I}_b$ and $\mathcal{I}_n$) expressed by a classical bilateral format [15]. It is immediate to prove that $[\tilde{V}(p)] = \ker[\hat{C}(p)]$, where $[\tilde{V}] := \{[v] \in \mathbb{R}^{3n_p} | v \in V\}$.

For the sake of notation, let $M := \dim[\ker[\hat{C}]]$, denote by $\{a_1(p), \ldots, a_M(p)\}$ an orthonormal basis for $\ker[\hat{C}(p)]$ and let

$$A(p) := [a_1(p) \ldots a_M(p)] \in \mathbb{R}^{3n_p \times M}.$$

(72)
The special case $\tilde{V}(p) = \varnothing_{nr}$ leads to a number of trivial results and it will be herein not addressed.

4.7.1. Direct tangent response. In order to provide a solution for Problem 5, let $\delta p$, $\delta \dot{p} \in \tilde{V}$ be small perturbations of $p$ and $\dot{p}$, respectively. The corresponding nodal force variation $\delta f(p, \dot{p})$ can be written in the form

$$[\delta f(p, \dot{p})] = K(p, \dot{p})[\delta p] + D(p)[\delta \dot{p}], \quad \delta p, \delta \dot{p} \in \tilde{V},$$

(73)
where $K$ is the tangent stiffness matrix

$$K(p, \dot{p}) = \begin{bmatrix} K_{11} & \cdots & K_{1n_p} \\ \vdots & \ddots & \vdots \\ K_{n_p1} & \cdots & K_{nn_p} \end{bmatrix}$$

and $D$ is the tangent damping matrix

$$D(p) = \begin{bmatrix} D_{11} & \cdots & D_{1n_p} \\ \vdots & \ddots & \vdots \\ D_{n_p1} & \cdots & D_{nn_p} \end{bmatrix}$$

(74)
with $K_{ij} = \frac{\partial^2 \Phi}{\partial p_i \partial p_j} \bigg|_{p, \dot{p}}$ and $D_{ij} = \frac{\partial^2 \Phi}{\partial \dot{p}_i \partial \dot{p}_j} \bigg|_{p, \dot{p}}$.

(75)
Matrices $K$ and $D$ are symmetric and can be assembled starting from the tangent stiffness matrices of each bar and cable, by embedding matrices in Eqs. (51) and (57) within a global reference system for the whole structure [13].

When $\delta \dot{p} = \varnothing_{nr}$, that is when only a perturbation with respect to nodal positions is considered, the nodal force variation results in

$$[\delta f(p, \dot{p})] = K(p, \dot{p})[\delta p], \quad \delta p \in \tilde{V},$$

(76)
and $[\delta f(p^{\mu}, \varnothing_{nr})]$ is usually denoted as the geometric load vector [14, 15].
4.7.2. Stability problem. Denote by $\mathbb{H}_\Psi(p)$ the Hessian matrix associated to the function $\Psi(p)$ at configuration $p$ and note that:

$$\mathcal{L}(t) = 2\Psi(p(t)), \quad \frac{\partial \mathcal{L}}{\partial t} \bigg|_t = 2\langle G_\Psi(p), v \rangle, \quad \frac{\partial^2 \mathcal{L}}{\partial t^2} \bigg|_t = 2\mathbb{H}_\Psi(p)[v] \cdot [v].$$

(77)

The following result holds:

**Lemma 4.6.** A tensegrity $T$ is in a steady-state equilibrium under nodal forces $f = \emptyset_{n_p}$ if and only if $\langle G_\Psi(p^o), \dot{v} \rangle = 0$ for any $\dot{v} \in \mathcal{D}(\partial\Phi(p^o, v))$, or equivalently for any $\dot{v} \in \dot{\mathcal{V}}(p^o)$.

**Proof.** Since Eqs. (24) and (62), and since the mechanical meaning of the generalized derivatives of free-energy and pseudo-potential functions, the application of the Principle of Virtual Powers with $f = \emptyset_{n_p}$ leads to:

$$\langle G_\Psi(p^o), \dot{v} \rangle + (G_\epsilon(p^o, \tilde{p}^o), \dot{v}) + (G_o(p^o, \tilde{p}^o), \dot{v}) = 0 \quad \forall \dot{v} \in \dot{\mathcal{V}}_{n_p}.$$  

(78)

Ideal constraints are workless by hypothesis and then $\langle G_o(p^o, \tilde{p}^o), \dot{v} \rangle = 0$ for any $\dot{v} \in \mathcal{D}(\partial\Phi(p^o, v))$. Moreover, since $\tilde{p}^o = \emptyset_{n_p}$ then $\Phi^i(p^o, \emptyset_{n_p}) = 0$ and $G_\epsilon(p^o, \emptyset_{n_p}) = \emptyset_{n_p}$. Therefore, if $T$ is in steady-state equilibrium under $f = \emptyset_{n_p}$ then

$$\langle G_\Psi(p^o), \dot{v} \rangle = 0 \quad \forall \dot{v} \in \mathcal{D}(\partial\Phi(p^o, v))$$

(79)

and vice versa. Finally, when only internal constraints are involved, Eq. (79) holds equivalently for any $\dot{v} \in \dot{\mathcal{V}}(p^o)$, rigid-body motions being excluded. $\square$

The Definition 3.9 of stability can be formulated within the variational framework herein developed by proving that:

**Lemma 4.7.**

$T$ is stable $\iff$ \begin{align} \langle G_\Psi(p^o), \dot{v} \rangle = 0 & \quad \forall \dot{v} \in \dot{\mathcal{V}}(p^o) \setminus \{\emptyset_{n_p}\}, \\
\mathbb{H}_\Psi(p^o)[\dot{v}] \cdot [\dot{v}] > 0 & \quad \forall \dot{v} \in \dot{\mathcal{V}}(p^o) \setminus \{\emptyset_{n_p}\}. \end{align}

**Proof.** Note that any $\dot{v} \in \dot{\mathcal{V}}(p^o)$ satisfies Eqs. (18), (19), (20) and belongs to $\mathcal{V}_{n^o}$. Moreover, the condition

$$\langle G_\Psi(p^o), \dot{v} \rangle = 0 \quad \forall \dot{v} \in \dot{\mathcal{V}}(p^o),$$

(80)

is equivalent to prescribe that

$$\mathcal{L}(t) - \mathcal{L}(0) = \mathbb{H}_\Psi(p^o)[\dot{v}] \cdot [\dot{v}] t^2 + o(t^2) \quad \forall \dot{v} \in \dot{\mathcal{V}}(p^o), \quad t > 0.$$  

(81)

Accordingly, if $T$ is stable then $T$ is in steady-state equilibrium under a set of null nodal forces (and Eq. (80) holds for Lemma 4.6), and

$$\mathbb{H}_\Psi(p^o)[\dot{v}] \cdot [\dot{v}] > 0 \quad \forall \dot{v} \in \dot{\mathcal{V}}(p^o) \setminus \{\emptyset_{n_p}\}.$$  

(82)

Conversely, if Eqs. (80) and (82) hold then $T$ is stable. $\square$

In what follows, prescribe that $T$ is in steady-state equilibrium under a set of null nodal forces ($\dot{f} = \emptyset_{n_p}$).

The following sufficient condition for stability reads as:

**Lemma 4.8.** $\mathbb{K}(p^o, \emptyset_{n_p})$ is positive definite $\Rightarrow T$ is stable.
Proof. Within a first-order approximation with respect to nodal positions and letting:
$$[\delta p] = [\delta \hat{t}] \forall \delta \in \mathcal{V}(p^o) \text{ and } t > 0,$$

it holds:
$$\mathbb{H}_p(p^o)[\delta \hat{t}] \cdot [\delta \hat{t}] t^2 = [\delta \hat{t}] t^2 \mathbb{H}_p(p^o, \mathcal{O}_{n_p})[\delta \hat{t}] = [\delta f(p^o, \mathcal{O}_{n_p})] \cdot [\delta \hat{t}] .$$

(83)

Accordingly, stability condition can be stated as:

$$\mathcal{T} \text{ is stable } \iff [\delta \hat{t}] \mathbb{H}_p(p^o, \mathcal{O}_{n_p})[\delta \hat{t}] > 0, \forall \delta \hat{t} \in \mathcal{V}(p^o) \setminus \{\mathcal{O}_{n_p}\},$$

and the positive definiteness of $\mathbb{H}_p(p^o, \mathcal{O}_{n_p})$ ensures stability. \hfill \Box

Since $\mathbb{H}_p(p) = \mathbb{H}_p(p, \mathcal{O}_{n_p})$, Lemma 4.8 is equivalent to

**Lemma 4.9.** $\Psi(p)$ is strictly convex in $p^o \Rightarrow \mathcal{T}$ is stable.

Proof. If $\Psi(p)$ is strictly convex in $p^o$ then $\mathbb{H}_p(p^o)[v] \cdot [v] > 0$ for any $v \in \mathcal{V}_{n_p}$. Thereby, $\mathcal{T}$ is stable. \hfill \Box

In the specialized literature, such a stability notion is often referred to as pre-stress stability [15]. This is related to the decomposition of $\mathbb{K}(p^o, \mathcal{O}_{n_p})$ in its material and geometric parts ($\mathbb{K}^M$ and $\mathbb{K}^G$). For conventional materials, $\mathbb{K}^M$ is at least positive semidefinite (strictly positive only for suitable arrangement of structural members). On the contrary, geometric stiffness matrix $\mathbb{K}^G$ can have negative eigenvalues depending on pre-stress. Thereby, $\mathbb{K}(p^o, \mathcal{O}_{n_p})$ can be positive definite or not depending on the ratios between material coefficients and pre-stress.

There exists a special class of tensegrities which is stable whatever the material choice and pre-stress level. This occurs when $\mathbb{K}(p^o, \mathcal{O}_{n_p})$ is positive definite for any value of pre-stress, that is when $\mathbb{K}^M$ and $\mathbb{K}^G$ are both positive definite.

To verify stability when $\mathbb{K}(p^o, \mathcal{O}_{n_p})$ is not positive definite (or, equivalently, when $\Psi(p)$ is not strictly convex in $p^o$), necessary conditions are required [16]. To this aim, consider the following items.

**Definition 4.10.** Let the projector operators $\mathbb{P}(p)$ and $\mathbb{J}(p)$ be defined as $\mathbb{P}(p) := \Lambda(p)\Lambda(p)^t$ and $\mathbb{J}(p) := \mathbb{I} - \mathbb{P}(p)$, with $\Lambda(p)$ as introduced in Eq. (72) and $\mathbb{I}$ the $3n_p \times 3n_p$ identity matrix.

It is worth pointing out that $\mathbb{P}$ projects vectors in $\mathbb{R}^{3n_p}$ on $\ker[\hat{C}(p)]$ (that is, on the space of non-rigid body motions, admissible with ideal constraints), and $\mathbb{J}$ on $\mathbb{R}^{3n_p} \setminus \ker[\hat{C}(p)]$. Accordingly, Eq. (76) can be equivalently stated as

$$[\delta f(p, \dot{p})] = \mathbb{K}(p, \dot{p})\mathbb{P}(p)[\delta p], \text{ with } [\delta p] \in \mathbb{R}^{3n_p},$$

and Eq. (84) is the same as

**Lemma 4.11.** $\mathcal{T}$ is stable $\iff [\delta p]^t \mathbb{P}(p^o)\mathbb{K}(p^o, \mathcal{O}_{n_p})\mathbb{P}(p^o)[\delta p] > 0, \forall \mathbb{P}(p^o)[\delta p] \neq [\mathcal{O}_{n_p}].$

Introducing $\hat{\mathbb{K}}(p, \dot{p}) := \mathbb{J}(p) + \mathbb{P}(p)\mathbb{K}(p, \dot{p})\mathbb{P}(p)$, referred to as the augmented stiffness of $\mathcal{T}$, the following necessary and sufficient stability condition arises as an operative result:

**Theorem 4.12.**

$\mathcal{T}$ is stable $\iff \hat{\mathbb{K}}(p^o, \mathcal{O}_{n_p})$ is positive definite.
**Proof.** First of all, note that \( P, J \) and \( K \) are symmetric, and that \( PP = P, JJ = J \).

If \( T \) is stable then, for any \( \delta p \neq 0 \), two cases may occur:

1. \( P(p^o)[\delta p] \neq \{0\} \) and then \( [\delta p]^T PP(p^o)K(p^o, \{0\})P(p^o)[\delta p] > 0 \) by hypothesis and \( [\delta p]^T J(p^o)[\delta p] = [\delta p]^T J(p^o)J(p^o)[\delta p] \geq 0 \);

2. \( P(p^o)[\delta p] = \{0\} \) and then \( [\delta p]^T PP(p^o)K(p^o, \{0\})P(p^o)[\delta p] = 0 \) and \( [\delta p]^T J(p^o)[\delta p] = [\delta p]^T J(p^o)J(p^o)[\delta p] > 0 \).

Accordingly,
\[
[\delta p]^T [J(p^o) + PP(p^o)K(p^o, \{0\})P(p^o)] [\delta p] > 0, \quad \forall \, \delta p \neq 0,
\]
that is \( \hat{K}(p^o, \{0\}) \) is positive definite.

Conversely, if \( \hat{K}(p^o, \{0\}) \) is positive definite, that is if
\[
[\delta p]^T \hat{K}(p^o, \{0\})[\delta p] > 0, \quad \forall \, [\delta p] \in \mathbb{R}^{3n_p},
\]
then, since \( J(p^o)P(p^o)[\delta p] = \{0\} \),
\[
[\delta p]^T \hat{K}(p^o, \{0\})[\delta p] = [\delta p]^T [J(p^o) + PP(p^o)K(p^o, \{0\})P(p^o)] [\delta p] =
\]
\[
[\delta p]^T PP(p^o)K(p^o, \{0\})P(p^o)[\delta p] > 0, \quad \forall \, [\delta p] = P(p^o)[\delta p] \neq \{0\},
\]
that is, by Lemma 4.11, \( T \) is stable. \( \square \)

### 4.7.3. Inverse tangent response

Let the tensegrity structure be in a stable steady-state and satisfying condition (26) at \( t = 0 \). The tangent stiffness matrix \( K \) occurring in Eq. (76) is square but not always invertible (resulting generally positive semidefinite). Therefore, \( K \) cannot be used for solving Problem 6. On the contrary, the augmented tangent stiffness \( \hat{K}(p^o, \{0\}) \) is invertible and the variation \( P(p^o)[\delta p] \) of nodal positions, admissible with ideal constraints and related to a given small perturbation \( [\delta f] \) of external nodal forces, results in
\[
P(p^o)[\delta p] = P(p^o)\hat{K}^{-1}(p^o, \{0\})P(p^o)[\delta f], \quad \forall \, [\delta f] \in \mathbb{R}^{3n_p}.
\]
(89)

In fact, starting from Eq. (76), since \( [\delta p] = \{0\} \), and due to the symmetry of \( P, K \) and \( \hat{K} \), the following equivalences hold:
\[
0 = P[\delta f] - PKP[\delta p] = PP[\delta f] - PPKP[\delta p] + J[\delta p] =
\]
\[
P[\delta f]^T P - [\delta p]^T \hat{K} = P[\delta f]^T P\hat{K}^{-1} - [\delta p]^T P\hat{K}^{-1} P[\delta f] - [\delta p]^T \hat{K}.
\]
(90)

Accordingly, Eq. (89) provides an algebraic solution of Problem 6.

### 5. Conclusions

In this paper, a convex analysis approach for the analysis of both ideal and non-ideal or mixed-type tensegrities is proposed. A complete framework has been developed able to describe statics, kinematics and stability of such a kind of structures. The general framework promises an evolution towards the development of analysis and design algorithms based on linear algebra, quadratic programming and linear matrix inequality methods.

**Acknowledgments.** This work was developed within the framework of the Lagrange Laboratory, a European research group comprising CNRS, CNR, the Universities of Rome “Tor Vergata”, Calabria, Cassino, Pavia and Salerno, Ecole Polytechnique, University of Montpellier II, ENPC, LCPC and ENTPE.
REFERENCES

[1] S. Pellegrino, Analysis of prestressed mechanisms, International Journal of Solids and Structures, 26 (1990), 1329–1350.
[2] C. R. Calladine and S. Pellegrino, First-order infinitesimal mechanisms, International Journal of Solids and Structures, 27 (1991), 505–515.
[3] C. Sultan, M. Corless and R. E. Skelton, The prestressability problem of tensegrity structures: some analytical solutions, International Journal of Solids and Structures, 38 (2001), 5223–5252.
[4] R. Motro, “Tensegrity: Structural Systems for the Future,” Kogan Page Science, 2003.
[5] B. Roth and W. Whiteley, Tensegrity frameworks, Transactions of the American Mathematical Society, 265 (1981), 419–446.
[6] R. Connelly, Rigidity and energy, Inventiones Mathematicae, 66 (1982), 11–33.
[7] R. Connelly and W. Whiteley, Second-order rigidity and prestress stability for tensegrity frameworks, Journal on Discrete Mathematics, 9 (1996), 453–491.
[8] R. Connelly and A. Back, Mathematics and tensegrity, American Scientists, 86 (1998), 142–151.
[9] F. Maceri, M. Marino and G. Vairo, Convex analysis and ideal tensegrities, Comptes Rendus Mecanique, 339 (2011), 683–691.
[10] M. Frémond, “Non-Smooth Thermomechanics,” Springer-Verlag Berlin, 2001.
[11] P. D. Panagiotopoulos, Convex analysis and unilateral static problems, Archive of Applied Mechanics, 45 (1976), 55–68.
[12] J. J. Moreau, Fonctionnelles convexes, Editions of Department of Civil Engineering, University of Rome Tor Vergata, ISBN 978886290014, Roma, 2003.
[13] S. Guest, The stiffness of prestressed frameworks: A unifying approach, International Journal of Solids and Structures, 43 (2006), 842–854.
[14] A. Micheletti and W. O. Williams, A marching procedure for form-finding for tensegrity structures, Journal of Mechanics of Materials and Structures, 2 (2007), 857–882.
[15] W. O. Williams, A primer on the mechanics of tensegrity structures, preprint, (2007).
[16] J. Y. Zhang and M. Ohsaki, Stability conditions for tensegrity structures, International Journal of Solids and Structures, 44 (2007), 3875–3886.

Received August 2011; revised March 2012.

E-mail address: franco.maceri@lagrange.it
E-mail address: m.marino@ing.uniroma2.it
E-mail address: vairo@ing.uniroma2.it