PHANTOM MAPS AND FIBRATIONS

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Abstract. Given pointed CW-complexes $X$ and $Y$, $\text{Ph}(X,Y)$ denotes the set of homotopy classes of phantom maps from $X$ to $Y$ and $\text{SPh}(X,Y)$ denotes the subset of $\text{Ph}(X,Y)$ consisting of homotopy classes of special phantom maps. In a preceding paper, we gave a sufficient condition such that $\text{Ph}(X,Y)$ and $\text{SPh}(X,Y)$ have natural group structures and established a formula for calculating the groups $\text{Ph}(X,Y)$ and $\text{SPh}(X,Y)$ in many cases where the groups $[X,\Omega\tilde{Y}]$ are nontrivial. In this paper, we establish a dual version of the formula, in which the target is the total space of a fibration, to calculate the groups $\text{Ph}(X,Y)$ and $\text{SPh}(X,Y)$ for pairs $(X,Y)$ to which the formula or existing methods do not apply. In particular, we calculate the groups $\text{Ph}(X,Y)$ and $\text{SPh}(X,Y)$ for pairs $(X,Y)$ such that $X$ is the classifying space $BG$ of a compact Lie group $G$ and $Y$ is a highly connected cover $Y'$ of a nilpotent finite complex $Y'$ or the quotient $G/H$ of $G=U,O$ by a compact Lie group $H$.

1. Introduction

Given two pointed CW-complexes $X$ and $Y$, a map $f : X \to Y$ is called a phantom map if for any finite complex $K$ and any map $h : K \to X$, the composite $fh$ is null homotopic. The concept of a phantom map, which is one of the most important concepts in homotopy theory, is essential to understanding maps with infinite dimensional sources ([8, 11]).

Let $\text{Ph}(X,Y)$ denote the subset of $[X,Y]$ consisting of homotopy classes of phantom maps, and let $\text{SPh}(X,Y)$ denote the subset of $\text{Ph}(X,Y)$ consisting of homotopy classes of special phantom maps, defined by the exact sequence of pointed sets

$$0 \to \text{SPh}(X,Y) \to \text{Ph}(X,Y) \xrightarrow{e_Y} \text{Ph}(X,\tilde{Y}),$$

where $e_Y : Y \to \tilde{Y} = \prod_p Y_{(p)}$ is a natural map called the local expansion (cf. [11, p. 150]). The target $\tilde{Y}$ is usually assumed to be nilpotent of finite type.

Previous calculations of $\text{Ph}(X,Y)$ had generally assumed that $[X,\Omega\tilde{Y}]$ is trivial, in which case generalizations of Miller’s theorem are directly applicable, and calculations of $\text{SPh}(X,Y)$ had rarely been reported (see [4, Section 1]). In [4], we gave a sufficient condition such that $\text{Ph}(X,Y)$ and $\text{SPh}(X,Y)$ have natural group structures, which is much weaker than the conditions obtained by Meier and McGibbon ([9, 7, Theorem 4]), and established a formula which enables us to calculate not only $\text{Ph}(X,Y)$ but also $\text{SPh}(X,Y)$ in many cases where the groups $[X,\Omega\tilde{Y}]$ are nontrivial (see Section 2.1 for these results, which are recorded as Theorems 2.1 and 2.2).

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In this paper, we establish a dual version of the formula and apply it to calculate the groups $\text{Ph}(X, Y)$ and $\text{SPh}(X, Y)$ for pairs $(X, Y)$ with $[X, \Omega Y'] \neq 0$ to which the formula or existing methods do not apply.

We state the main results of this paper more precisely.

Let $\mathcal{CW}$ denote the category of pointed connected $\mathcal{CW}$-complexes and homotopy classes of maps and let $\mathcal{N}$ denote the full subcategory of $\mathcal{CW}$ consisting of nilpotent $\mathcal{CW}$-complexes of finite type. Let $\mathcal{Q}$ be the full subcategory of $\mathcal{CW}^{\text{op}} \times \mathcal{N}$ consisting of $(X, Y)$ such that for each pair $i, j > 0$, the rational cup product on $H^i(X; \mathbb{Q}) \otimes H^j(X; \mathbb{Q})$ or the rational Whitehead product on $(\pi_{i+1}(Y) \otimes \mathbb{Q}) \otimes (\pi_{j+1}(Y) \otimes \mathbb{Q})$ is trivial. Then, $\text{Ph}(X, Y)$ and $\text{SPh}(X, Y)$ have natural divisible abelian group structure for $(X, Y) \in \mathcal{Q}$ (see Theorem 2.1).

The following is the main theorem of this paper, which is a dual version of Theorem 2.2. Note that $j_X \text{Ph}(X, L)$ and $j_Y \text{SPh}(X, L)$ are subgroups of $\text{Ph}(X, Y)$ and $\text{SPh}(X, L)$ respectively (Theorem 2.1(2)). Let $\hat{\mathbb{Z}}$ denote the product $\prod_p \hat{\mathbb{Z}}_p$ of the $p$-completions of $\mathbb{Z}$, in which $\mathbb{Z}$ is diagonally contained. Similarly, let $\hat{\mathbb{Z}}$ denote the product $\prod_p \mathbb{Z}(p)$ of the $p$-localizations of $\mathbb{Z}$, in which $\mathbb{Z}$ is diagonally contained.

**Theorem 1.1.** Let $(X, Y)$ be in $\mathcal{Q}$. Suppose that there exists a fibration sequence $L \xrightarrow{j} Y \xrightarrow{q} Y'$ with $Y'$ nilpotent of finite type and $[X, \Omega Y'] = 0$. Then there exist natural split exact sequences of abelian groups given by

\[
0 \rightarrow j_X \text{Ph}(X, L) \rightarrow \text{Ph}(X, Y) \rightarrow \prod_{i>0} H^i(X; \pi_{i+1}(Y)/j_X \pi_{i+1}(L) \otimes \hat{\mathbb{Z}}/\mathbb{Z}) \rightarrow 0,
\]

\[
0 \rightarrow j_Y \text{SPh}(X, L) \rightarrow \text{SPh}(X, Y) \rightarrow \prod_{i>0} H^i(X; \pi_{i+1}(Y)/j_Y \pi_{i+1}(L) \otimes \hat{\mathbb{Z}}/\mathbb{Z}) \rightarrow 0.
\]

Let us recall the generalizations of Miller’s theorem [10] and Anderson-Hodgkin’s theorem [2] to obtain many pairs $(X, Y')$ with $[X, \Omega Y'] = 0$. A space whose $i$th homotopy group is zero for $i \leq n$ and locally finite for $i = n + 1$ is said to be $n\frac{1}{2}$-connected. Define the classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and $\mathcal{B}'$ by

- $\mathcal{A}$ = the class of $\frac{1}{2}$-connected Postnikov spaces, the classifying spaces of compact Lie groups, $\frac{1}{2}$-connected infinite loop spaces and their iterated suspensions.
- $\mathcal{B}$ = the class of nilpotent finite complexes, the classifying spaces of compact Lie groups and their iterated loop spaces.
- $\mathcal{A}'$ = the class of $1\frac{1}{2}$-connected Postnikov spaces of finite type and their iterated suspensions.
- $\mathcal{B}'$ = the class of $BU$, $BO$, $BSp$, $BSO$, $U/Sp$, $Sp/U$, $SO/U$, $U/SO$, and their iterated loop spaces.

If $(X, Y')$ is in $\mathcal{A} \times \mathcal{B}$ or $\mathcal{A}' \times \mathcal{B}'$, then $[X, \Omega Y'] = 0$ ([4, Corollary 6.4]).

We have the following corollaries to Theorem 1.1. Let $K(n)$ denote the $n$-connected cover of $K$.

**Corollary 1.2.** Let $(X, Y')$ be in $\mathcal{A} \times \mathcal{B}$ or $\mathcal{A}' \times \mathcal{B}'$, and let $m$ be a positive integer. Suppose that $X$ is a $\mathcal{CW}$-complex of finite type and that $(X, Y'(m))$ is in $\mathcal{Q}$. Then
there exist natural isomorphisms of groups
\[
\text{Ph}(X, Y'; m) \cong \prod_{i > 0} H^i(X; \pi_{i+1}(Y'; m)) \otimes \hat{\mathbb{Z}}/\mathbb{Z}),
\]
\[
\text{SPh}(X, Y'; m) \cong \prod_{i > 0} H^i(X; \pi_{i+1}(Y'; m)) \otimes \hat{\mathbb{Z}}/\mathbb{Z}).
\]

Let \( L \) be a pointed \( CW \)-complex endowed with an action of a compact Lie group \( H \). Defining the homotopy quotient \( L \sslash H \) by \( L \sslash H = EH \times L \), we have the fiber bundle
\[
L \rightarrow L \sslash H \rightarrow BH,
\]
where \( EH \rightarrow BH \) is the universal principal \( H \)-bundle. If an injective homomorphism \( H \rightarrow G \) of topological groups is given, then \( G \sslash H \) is usually denoted by \( G/H \).

The following corollary is derived using a result of Atiyah-Segal [3].

**Corollary 1.3.** Let \( X \) be the classifying space \( BG \) of a compact Lie group \( G \) or its iterated suspension. Let \( G \) denote the infinite unitary group \( U \) or the infinite orthogonal group \( O \), and let \( H \) be a compact Lie group which is a topological subgroup of \( G \). Then \((X, G/H)\) is in \( Q \) and there exist natural isomorphisms of groups
\[
\text{Ph}(X, G/H) \cong \prod_{i > 0} H^i(X; \pi_{i+1}(G/H)/j\pi_{i+1}(G) \otimes \hat{\mathbb{Z}}/\mathbb{Z}),
\]
\[
\text{SPh}(X, G/H) \cong \prod_{i > 0} H^i(X; \pi_{i+1}(G/H)/j\pi_{i+1}(G) \otimes \hat{\mathbb{Z}}/\mathbb{Z}).
\]

Further applications of Theorem 1.1 are given in Section 2 (see Examples 2.7-2.8 and Remark 2.9).

**Remark 1.4.** Most calculations of \( \text{Ph}(X, Y) \) have assumed that \( Y \) is a nilpotent finite complex or its iterated loop space ([4, Theorem A and Remark 2.6]). Corollaries 1.2-1.3 and Examples 2.7-2.8 and Remark 2.10 give calculational results for \((X, Y)\) such that \( Y \) is not in \( B \) or \( B' \).

2. Proofs of main results

In this section, we prove Theorem 1.1 and Corollaries 1.2-1.3 and then give further applications of Theorem 1.1.

We begin by recalling the basic results on \( \text{Ph}(X, Y) \) and \( \text{SPh}(X, Y) \).

2.1. Groups of homotopy classes of phantom maps. In this subsection, we make a review of the main results of [4].

Recall the definition of the full subcategory \( Q \) of \( CW^{op} \times N \) from Section 1. A pair \((X, Y)\) is in \( Q \) if \( X \) is a co-\( H_0 \)-space or \( Y \) is an \( H_0 \)-space. \( Q \) contains many other pairs (see [4, Section 4.2]).

The following theorem, which is Theorem 2.3 in [1], is a fundamental result on group structures on \( \text{Ph}(X, Y) \) and \( \text{SPh}(X, Y) \).

**Theorem 2.1.** Let \((X, Y)\) be an object of \( Q \).

1. \( \text{Ph}(X, Y) \) and \( \text{SPh}(X, Y) \) have natural divisible abelian group structures, for which \( \text{SPh}(X, Y) \) is a subgroup of \( \text{Ph}(X, Y) \).
(2) Let \((f^{op}, g) : (K, L) \longrightarrow (X, Y)\) be a morphism of \(CW^{op} \times N\). Then, the images \(\text{Im Ph}(f, g)\) and \(\text{Im SPh}(f, g)\) are divisible abelian subgroups of \(\text{Ph}(X, Y)\) and \(\text{SPh}(X, Y)\) respectively.

(3) If \(X\) is a co-\(H\)-space or \(Y\) is an \(H\)-space, the group structures on \(\text{Ph}(X, Y)\) and \(\text{SPh}(X, Y)\) are compatible with the multiplicative structure on \([X, Y]\).

The following theorem, which is Theorem 2.7 in [4], presents a powerful method for calculating the groups \(\text{Ph}(X, Y)\) and \(\text{SPh}(X, Y)\) for \((X, Y) \in Q\) with \([X, \Omega \hat{Y}] \neq 0\).

Note that in the theorem, \(p^\# \text{Ph}(K, Y)\) and \(p^\# \text{SPh}(K, Y)\) are the subgroups of \(\text{Ph}(X, Y)\) and \(\text{SPh}(X, Y)\) (see Theorem 2.1(2)).

Theorem 2.2. Let \((X, Y)\) be in \(Q\). Let \(X' \xrightarrow{i} X \xrightarrow{p} K\) be a cofibration sequence with \([X', \Omega \hat{Y}] = 0\), or a fibration sequence with weakly contractible maps \((X', \Omega \hat{Y})\). Then there exist natural split exact sequences of abelian groups given by

\[
0 \longrightarrow p^\# \text{Ph}(K, Y) \longrightarrow \text{Ph}(X, Y) \longrightarrow \prod_{i > 0} H^i(X; \pi_{i+1}(Y) \otimes \hat{Z}/\mathbb{Z}) / p^\# H^i(K; \pi_{i+1}(Y) \otimes \hat{Z}/\mathbb{Z}) \longrightarrow 0,
\]

\[
0 \longrightarrow p^\# \text{SPh}(K, Y) \longrightarrow \text{SPh}(X, Y) \longrightarrow \prod_{i > 0} H^i(X; \pi_{i+1}(Y) \otimes \hat{Z}/\mathbb{Z}) / p^\# H^i(K; \pi_{i+1}(Y) \otimes \hat{Z}/\mathbb{Z}) \longrightarrow 0.
\]

See [4, Corollaries 2.8-2.10 and Example 6.6] for the applications.

2.2. Proofs of Theorem 1.1 and Corollary 1.2. For a nilpotent space \(Y\) of finite type, the profinite completion \(\hat{Y}\) and the local expansion \(\hat{Y}\) are defined by \(\hat{Y} = \prod_p \hat{Y}_p\) and \(\hat{Y} = \prod_p Y(p)\), respectively, where \(\hat{Y}_p\) and \(Y(p)\) are the \(p\)-profinite completion and the \(p\)-localization of \(Y\) respectively ([12]). Thus, we can establish a commutative diagram of natural transformations

\[
\begin{array}{ccc}
Y & \xrightarrow{e_Y} & \hat{Y} \\
\downarrow & \downarrow & \downarrow \\
\hat{Y} & \xrightarrow{c_Y} & \hat{Y}.
\end{array}
\]

Let \(F_Y\) denote the homotopy fiber of \(c_Y\).

Proof of Theorem 1.1. Since the proof of Theorem 1.1 is similar to that of Theorem 2.2, our proof is sketchy; the details are found in the proof of [4, Theorem 2.7]. By replacing \(Y\) and \(Y'\) with their universal covers, we may assume that \(L\) is in \(N\) (see [4, Remark 5.6] and [6, Proposition 4.4.1]).

The case of \(\text{Ph}(X, Y)\). By [4, Corollary 5.3] and the comment before [4, Lemma 3.5], we have the morphism of exact sequences of pointed sets

\[
\begin{array}{ccc}
[X, \Omega \hat{L}] & \longrightarrow & [X, F_L] \longrightarrow \text{Ph}(X, L) \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
[X, \Omega \hat{Y}] & \longrightarrow & [X, F_Y] \longrightarrow \text{Ph}(X, Y) \longrightarrow 0.
\end{array}
\]

Note that since \([X, \Omega \hat{Y}] = 0\), the left vertical arrow is surjective. Next, consider the induced morphism of exact sequences of pointed sets.
where \([X, \Omega Y]\) denotes the image of \([X, \Omega \hat{Y}]\) and \(\alpha\) denotes the map induced by the natural quotient map \([X, F_L] \to \text{Ph}(X, L)\). Note that this is a morphism of exact sequences of abelian groups (see [4, Theorem 2.3 and its proof]) and that since \(\Omega \hat{j}_\sharp : [X, \Omega L] \to [X, \Omega Y]\) is surjective, \(\beta\) is also surjective. Then, we have
\[
\text{Ph}(X, Y)/j_\sharp \text{Ph}(X, L) \cong [X, F_Y]/F_j[X, F_L],
\]
from which we obtain the desired sequence (see [4, Proposition 5.10 and the proof of Theorem 2.3(2)]).

**The case of** \(S\text{Ph}(X, Y)\). As mentioned in the introduction of Section 6.1 of [4], \(\text{Ph}(X, Y)\) and \(\text{Ph}(X, \hat{Y})\) generate analogous results. Thus, there exists a morphism of exact sequences of abelian groups
\[
0 \longrightarrow j_\sharp \text{Ph}(X, L) \longrightarrow \text{Ph}(X, L) \longrightarrow \prod_{i>0} H^i(X; \pi_{i+1}(Y)/j_\sharp \pi_{i+1}(L) \otimes \hat{Z}/\hat{Z}) \longrightarrow 0
\]
and if \(Y\) is nilpotent of finite type, then there also exists a natural isomorphism
\[
S\text{Ph}(K \wedge X, Y) \cong S\text{Ph}(X, \text{map}_*(K, Y))
\]

**Proof of Corollary 1.2.** We have the fibration sequence
\[
\Omega Y'(m) \longrightarrow Y'(m) \longrightarrow Y',
\]
where \(Y'(m)\) is the Postnikov \(m\)-stage of \(Y'\). Since \(X\) is a \(CW\)-complex with finite skeleta, \(\text{Ph}(X, \Omega Y'(m))\), and hence \(S\text{Ph}(X, \Omega Y'(m))\) vanishes. Thus, the result follows from Theorem [14].

**2.3. Proof of Corollary 1.3 and further applications.** For the proof of Corollary [13] we prove the following two lemmas, which are interesting in their own right.

**Lemma 2.3.** **Let** \(X\) **and** \(Y\) **be connected** \(CW\)-**complexes and** \(K\) **be a finite complex. Then there exists a natural isomorphism**
\[
\text{Ph}(K \wedge X, Y) \cong \text{Ph}(X, \text{map}_*(K, Y))
\]
**and if** \(Y\) **is nilpotent of finite type, then there also exists a natural isomorphism**
\[
S\text{Ph}(K \wedge X, Y) \cong S\text{Ph}(X, \text{map}_*(K, Y)).
\]
Proof. Note that \( f : K \wedge X \to Y \) is phantom if and only if \( f \mid_{K \wedge X_\alpha} \) is null homotopic for any finite subcomplex \( X_\alpha \) of \( X \). Then the natural isomorphism \([K \wedge X, Y] = [X, map_*(K, Y)]\) is clearly restricted to the first natural isomorphism. If \( Y \) is nilpotent of finite type, we obtain the isomorphism
\[
\text{Ph}(K \wedge X, \tilde{Y}) \cong \text{Ph}(X, map_*(K, \tilde{Y}))
\]
(see the introduction of Section 6.1 of [4]), which also implies the second isomorphism by [3, Theorem 6.3.2] and the definition of \( \text{SPh}(X, Y) \).
\[\square\]

Lemma 2.4. Let \( G \) be a compact Lie group and \( X \) a finite \( G \)-CW-complex. Then,
\[
\text{Ph}(X/\!/G, \Omega^lU) = 0 \quad \text{and} \quad \text{Ph}(X/\!/G, \Omega^lO) = 0
\]
hold for \( l \geq 0 \).

Proof. First, we show the first vanishing result. Since \( \Omega^lU \) is an \( H \)-space, we have only to consider the homotopy classes of maps from \( X/\!/G \) to the identity component of \( \Omega^lU \) in unbased context ([6, Proposition 1.4.3]). Note that \( K^*_G(X) \) is finite over \( R(G) \). Then, the result follows from the proof of [3, Proposition 4.2].

The second vanishing result can be similarly proved using the \( KO \)-versions of the results of [3], which are established in [1] (see the comment after Theorem 1.1 in [1] and Remark 2.5(2)).
\[\square\]

Remark 2.5. (1) See [1, Remark 5.1] for other \( G \)-spaces for which the vanishing results in Lemma 2.4 hold.
(2) The article [3] deals with not \( KO \) but \( KR \) as the real case.

Proof of Corollary 1.3. By the assumption, \( X = \Sigma^lBG \) for \( l \geq 0 \). To prove that \((X, G/H)\) is in \( Q \), we have only to show that \((BG, G/H)\) is in \( Q \), which is easily seen from [4, Example 4.6(2)]. Consider the fibration sequence
\[
G \xrightarrow{j} G/H \to BH
\]
and note that \((X, BH)\) is in \( A \times B \). Since the identities
\[
\text{Ph}(X, G) = \text{Ph}(BG, \Omega^lG) = 0
\]
hold (Lemmas 2.3 and 2.4), we obtain the result by Theorem 1.1.
\[\square\]

We give further applications of Theorem 1.1. For this, we show the following lemma, which is useful to find many pairs \((X, L)\) with \( \text{Ph}(X, L) = 0 \).

A space whose \( i \)-th \( k \)-invariant vanishes for all but finite \( i \) is called a generalized Postnikov space.

Lemma 2.6. (1) If \( \{i > 0 \mid H^i(A; \mathbb{Q}) \neq 0\} \cap \{j > 0 \mid \pi_{j+1}(B) \otimes \mathbb{Q} \neq 0\} = \emptyset, \) then \( \text{Ph}(A, B) = 0 \).
(2) If \( A \) is a CW-complex of finite type and \( B \) is a generalized Postnikov space, then \( \text{Ph}(A, B) = 0 \).

Proof. (1) The result follows from [4, Propositions 5.7 and 4.1].
(2) By the finite type assumption on \( A \), all elements of \( \text{Ph}(A, B) \) are skeletally phantom and \( B \) is homotopy equivalent to the product of the Postnikov \( n \)-stage \( B^{(n)} \) and \( \prod_{i > n} K(\pi_i(B), i) \), for sufficiently large \( n \) (see [4, Remark 3.3]). Therefore, \( \text{Ph}(A, B) = \text{Ph}(A, B^{(n)}) \times \prod_{i > n} \text{Ph}(A, K(\pi_i(B), i)) \) vanishes.
\[\square\]
For a connected CW-complex $K$, $Q(K)$ denotes the infinite loop space defined by $Q(K) = \varinjlim_n \Omega^n \Sigma^n K$.

**Example 2.7.** Let $G$ and $H$ be as in Corollary 1.3. Noticing that $Q(S^{2n+1})$ is rationally equivalent to $S^{2n+1}$, we calculate the groups $\text{Ph}(Q(S^{2n+1}), G/H)$ and $\text{SPh}(Q(S^{2n+1}), G/H)$ for $n \geq 1$.

Consider the fibration sequence

$$G \xrightarrow{j} G/H \rightarrow BH.$$

Since $(Q(S^{2n+1}), BH)$ is in $A \times B$ and $\text{Ph}(Q(S^{2n+1}), G) = 0$ (Lemma 2.6(1)), we have the isomorphisms of abelian groups

$$\text{Ph}(Q(S^{2n+1}), G/H) \cong \pi_{2n+2}(G/H)/j_1\pi_{2n+2}(G) \otimes \hat{Z}/\mathbb{Z},$$

$$\text{SPh}(Q(S^{2n+1}), G/H) \cong \pi_{2n+2}(G/H)/j_1\pi_{2n+2}(G) \otimes \hat{Z}/\mathbb{Z}$$

by Theorem 1.1.

Recall that for a CW-complex $L$ with an $H$-action, we have the fiber bundle

$$L \xrightarrow{j} L/H \rightarrow BH$$

(see Section 1).

**Example 2.8.** Let $X$ be a CW-complex of finite type which is in $A$ and let $L$ be a generalized Postnikov space with an action of a compact Lie group $H$. Suppose that $(X, L/H)$ is in $Q$. Then there exist natural isomorphisms of groups

$$\text{Ph}(X, L/H) \cong \prod_{i>0} H^i(X; \pi_{i+1}(L/H)/j_2\pi_{i+1}(L) \otimes \hat{Z}/\mathbb{Z}),$$

$$\text{SPh}(X, L/H) \cong \prod_{i>0} H^i(X; \pi_{i+1}(L/H)/j_2\pi_{i+1}(L) \otimes \hat{Z}/\mathbb{Z}).$$

(Theorem 1.1 and Lemma 2.6(2).)

Let $L$ be the infinite symmetric product $SP(M)$ of a CW-complex $M$ endowed with an action of a compact Lie group $H$. Since $L$ is weak equivalent to the product of Eilenberg-MacLane complexes, this result is applicable to $L = SP(M)$.

This result is also applicable to the case where $L$ is an Eilenberg-MacLane $H$-space ([5, p. 21]).

**Remark 2.9.** Under the assumptions of Theorem 1.1 there exist (noncanonical) isomorphisms of abelian groups

$$\text{Ph}(X, Y) \cong j_2\text{Ph}(X, L) \oplus \prod_{i>0} H^i(X; \pi_{i+1}(Y)/j_2\pi_{i+1}(L) \otimes \hat{Z}/\mathbb{Z}),$$

$$\text{SPh}(X, Y) \cong j_2\text{SPh}(X, L) \oplus \prod_{i>0} H^i(X; \pi_{i+1}(Y)/j_2\pi_{i+1}(L) \otimes \hat{Z}/\mathbb{Z}).$$

Thus, we can obtain nontriviality results of $\text{Ph}(X, Y)$ and $\text{SPh}(X, Y)$ via rational homotopical computations, even if we do not know whether $j_2\text{Ph}(X, L)$ is nontrivial.

**Remark 2.10.** In this remark, we consider situations similar to those described in Corollary 1.3 and Example 2.8 in which the groups $\text{Ph}(X, Y)$ and $\text{SPh}(X, Y)$ are calculated using not Theorem 1.1 but [4] Proposition 6.1.

1. Let $X$ be a space in $A'$ and let $G$ and $H$ be as in Corollary 1.3. Suppose that $(X, G/H)$ is in $Q$. Then, we can calculate the groups $\text{Ph}(X, G/H)$ and $\text{SPh}(X, G/H)$ by [4] Proposition 6.1.
Let $X$ be a space in $\mathcal{A}$ and $L$ be a nilpotent finite complex endowed with an action of a compact Lie group $H$. Suppose that $(X, L \sslash H)$ is in $Q$. Then, we can calculate the groups $\text{Ph}(X, L \sslash H)$ and $\text{SPh}(X, L \sslash H)$ by [4, Proposition 6.1].

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