Teleporting Quantum Information Encoded in Fermionic Modes

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Quantum teleportation is considered a basic primitive in many quantum information processing tasks and has been experimentally confirmed in various photonic and matter-based setups. Here, we consider teleportation of quantum information encoded in modes of a fermionic field. In fermionic systems, superselection rules lead to a more differentiated picture of entanglement and teleportation. In particular, one is forced to distinguish between single-mode entanglement swapping, and qubit teleportation with or without authentication via Bell inequality violation, as we discuss here in detail. We focus on systems subject to parity superselection where the particle number is not fixed, and contrast them with systems constrained by particle number superselection which are relevant for possible practical implementations. Finally, we analyse the consequences for the operational interpretation of fermionic mode entanglement and examine the usefulness of so-called mixed maximally entangled states for teleportation.

I. INTRODUCTION

Quantum teleportation refers to the transference of quantum information encoded in the complex amplitudes of an unknown quantum state of a localised system to a remote system solely via initially entangled, local operations, and exchange of classical information. First proposed in [1], quantum teleportation was experimentally confirmed in [2], followed soon thereafter by further experiments refining various aspects of teleportation using photon polarization [3, 4], optical coherence [5], and nuclear magnetic resonance [6]. Since then, teleportation has become a conceptual cornerstone of many tasks in quantum communication and quantum information processing. Among other methods [7], teleportation can be seen as a way of detecting and certifying the usefulness of entanglement, because the latter is necessary to achieve a nontrivial teleportation fidelity. Practical teleportation protocols have been developed for photonic degrees of freedom, e.g., in the context of long-distance high-fidelity communication [8–11], chip-to-chip teleportation with applications to integrated photonic quantum technologies [12], or multi-party settings [13] relevant, e.g., for measurement-based quantum computation [14, 15]. In parallel to advances in photonic setups, much progress has been made for teleportation in matter-based systems [16–19].

Information carriers that are typically used for quantum information processing in solid-state and atomic systems (for instance, quantum dots [20], and ions in radio-frequency traps [21] or optical lattices [22]) are electrons, i.e., fermions or even the more elusive Majorana fermions [23–27]. Fermionic systems are described by anti-commuting operators and are subject to superselection rules [28–31], both of which require a careful approach to questions concerning correlations and entanglement [31–41], in particular, with regards to definition of mode subsystems [42, 43]. Nonetheless, many features known from bosonic quantum optics settings can be successfully carried over to fermions, for instance, phase-space methods for the description of Gaussian states and channels [44–51]. Research in this direction has previously mostly been confined to the domain of theoretical analysis, but impressive technological advances in the control and manipulation of individual electrons [52–56], as well as in the generation of electronic mode-entangled states [57] motivate further studies of fermionic entanglement also from a practical perspective.

A key open problem in this area concerns the assignment of clear operational meaning to fermionic entanglement. Fermionic systems with variable or indefinite numbers of particles (but subject to superselection rules) allow for different ways of quantifying entanglement, see, e.g., [30, 31]. But what do these quantifiers tell us about the usefulness of the corresponding states in practical tasks? Consider a single fermionic excitation in an equally weighted superposition of two different field modes in analogy to a single photon that is delocalized in a two-path interferometer. Formally, such a state can be seen as being maximally entangled: The state of either mode is maximally mixed. But is this type of entanglement operationally meaningful? For instance, can one use it to violate a Bell inequality? If a quantum state allows
such a violation then it can be unambiguously concluded that the state is entangled. For fermionic mode entanglement it was shown in [58] that this is indeed possible provided that two locally processed copies of a maximally entangled two-mode state (four modes in total) are used. While this can be seen as a device-independent certification of the entanglement of the state, one may nonetheless wonder to what (further) practical use this fermionic mode entanglement can be put. Here, we therefore investigate teleportation using fermionic mode entanglement as a resource.

Inspired by initial work in this direction [59], we review and more closely examine the different ways of interpreting the standard teleportation protocol [1] for the task of teleporting fermionic quantum information in the presence of superselection rules (SSRs). While we confirm that fermionic mode entanglement does allow for quantum teleportation, the parity superselection rule (P-SSR) imposes restrictions that require a more differentiated specification of what is meant by ‘teleporting quantum information’ in the first place.

For a single fermionic mode that is not entangled with any other mode(s), the parity SSR implies that the encoded information is classical. Consequently, teleporting such a state requires no shared entanglement in principle. However, when the mode in question is entangled with an another (auxiliary) mode, an entangled resource state is necessary for teleportation-based entanglement swapping. When more than one mode is considered, the equivalent of one qubit of quantum information can be directly encoded in the teleported state (e.g., dual-rail encoding in two modes). However, we find that the corresponding protocols require more resources as compared to standard qubit teleportation. To transfer the complex amplitudes of a single qubit, one can make do with sharing a single maximally entangled fermionic mode pair and two bits of classical information (a fermionic single-mode teleportation channel), but one also needs to transfer additional information about the teleported state (the state of the second mode) via a fermionic quantum channel. This channel may be realized by another fermionic single-mode teleportation channel, increasing the required resources to two copies of maximally entangled two-mode states, along with the usual two bits of classical information.

Within the framework of these variations of standard teleportation we discuss the consequences of further restrictions. In particular, we consider the potential of fermionic Gaussian states and operations for teleportation, as well as the limitations imposed by particle number superselection, which is highly relevant for potential experimental implementations (in particular, using state-of-the-art methods in electron quantum optics [60]). Finally, we apply our findings to understand the wider implications for the quantification of fermionic entanglement, especially with a view to the notion of ‘mixed maximally entangled’ (MME) fermionic states [61] and their usefulness for teleportation.

The paper is structured as follows. In Sec. II, we briefly discuss the mathematical framework of fermionic modes and their entanglement. In Sec. III, we then turn to teleportation. First, we review the standard protocol for qubit teleportation as a backdrop and discuss how fermionic teleportation deviates from this well-established paradigm in Sec. III.1. We then analyze protocols for teleporting the state of a single fermionic mode in Sec. III.2, as well as their implementation via fermionic Gaussian operations in Sec. III.3, before we turn to teleportation of states of several modes in Sec. III.4. In Sec. IV, we then discuss how the presented teleportation schemes (and potential practical implementations in electron quantum optics [60]) are influenced by the additional constraint of a SSR for the particle number. The implications on the quantification of fermionic (mode) entanglement are analyzed in Sec. V, with a special view to MME states.

II. FRAMEWORK

II.1. Fermionic modes

We consider quantum information encoded in the modes of a fermionic field\(^1\). To each mode labelled \(i\) we associate a pair of fermionic mode operators \(b_i\) and \(b_i^{\dagger}\), which satisfy

\[
\{ b_i, b_j^{\dagger}\} = \delta_{ij}, \quad \{ b_i, b_j \} = 0 \quad \forall i, j, \tag{1}
\]

where \(\{ \cdot, \cdot\}\) denotes the anticommutator. The corresponding Fock space is constructed by the action of the creation operators \(b_i^{\dagger}\) on the vacuum state \(|0\rangle\), which itself is annihilated by all annihilation operators \(b_i\), i.e., \(b_i |0\rangle = 0 \quad \forall i\). The creation operators \(b_i^{\dagger}\) populate the vacuum with single fermions, that is, \(|b_i^{\dagger}|0\rangle = |1_i\rangle\). Due to the indistinguishability of the particles the tensor product of single-particle states needs to be antisymmetrized when two or more fermions are created. Here, we use the convention

\[
|b_i^{\dagger}|0\rangle = |1_k\rangle \wedge |1_{k'}\rangle = |1_k\rangle |1_{k'}\rangle, \tag{2}
\]

where we use double-lined notation to indicate the antisymmetrized wedge product “\(^\wedge\)" between two or more single-mode state vectors with particle content (in contrast to the notation \(|\cdot\rangle|\cdot\rangle = |\cdot\rangle \otimes |\cdot\rangle\) for a tensor product), i.e., we have \(|1_k\rangle \wedge |1_{k'}\rangle = - |1_{k'}\rangle \wedge |1_k\rangle\), whereas combinations of states with and without particle content satisfy \(|0\rangle \wedge |1_k\rangle = |1_k\rangle |0\rangle = |1_k\rangle\). With this definition at hand, arbitrary pure states on the Fock space can be written as

\[
|\Psi\rangle = \gamma_0 |0\rangle + \sum_{i=1}^{n} \gamma_i |1_i\rangle + \sum_{j,k} \gamma_{jk} |1_j\rangle |1_k\rangle + \ldots \tag{3}
\]

\(^1\) For now, we impose no further constraints such as a particular (half-integer) spin, fixed mass or charge on the field excitations, but we discuss such restrictions in Sec. IV.
However, the parity superselection rule (see, e.g., [28–31]) implies that coherent superpositions of even and odd numbers of fermions cannot exist. For instance, a general pure state of two modes $A$ and $A'$ of the form

$$\left| \Psi_{A,A'} \right\rangle = \gamma_0 \left| 0 \right\rangle + \gamma_A \left| 1_A \right\rangle + \gamma_{A'} \left| 1_{A'} \right\rangle$$

$$+ \left| \gamma_{A'\gamma} \right\rangle \left| 1_A \right\rangle \left| 1_{A'} \right\rangle , \quad (4)$$

must be an even- or odd-parity state, i.e., the probability amplitudes must satisfy either

$$\gamma_A = \gamma_{A'} = 0, \quad |\gamma_0|^2 + |\gamma_{A'\gamma}|^2 = 1 \quad \text{ (even parity)},$$

or

$$\gamma_0 = \gamma_{A'\gamma} = 0, \quad |\gamma_A|^2 + |\gamma_{A''\gamma}|^2 = 1 \quad \text{ (odd parity)}.$$  

While coherent superpositions of states with different parity are thus forbidden, incoherent mixtures are still possible. In particular, this implies that any fermionic single-mode state must be of the form

$$\rho_A = p \left| 0 \right\rangle \left\langle 0 \right| + (1 - p) \left| 1_A \right\rangle \left\langle 1_A \right| , \quad (5)$$

for $0 \leq p \leq 1$.

### II.2. Entanglement of fermionic modes

The parity superselection rule also has interesting consequences for defining entanglement between fermionic mode subsystems. In principle, one can define subsystems containing complementary, non-overlapping sets of fermionic modes and consider (quantum) correlations between them, see, for instance [30, 31, 35, 38, 42, 62]. In particular, the ‘local’ operators assigned to different subsystems need not commute as one would usually assume, but they can also anticommute. In fact, there is no particular reason why the modes in question need to be spatially separated. For instance, one may consider two spatially overlapping (but orthogonal) field modes with different frequencies. In any case, particular care must be taken to deal with the definition of partial traces [42, 43] to avoid ambiguities such as those discussed in [63–65]. In other words, two-mode states like

even: $\left| \psi^e \right\rangle_{A,A'} = \alpha \left| 0 \right\rangle + \beta \left| 1_A \right\rangle \left| 1_{A'} \right\rangle , \quad (6a)$
odd: $\left| \psi^o \right\rangle_{A,A'} = \alpha \left| 1_{A'} \right\rangle + \beta \left| 1_A \right\rangle , \quad (6b)$

can be regarded as entangled (for $\alpha \beta \neq 0$). In particular, we can define a basis of **maximally entangled two-mode states** $\left| \Phi^e \right\rangle_{AB}$ and $\left| \Psi^e \right\rangle_{AB}$ given by

$$\left| \Phi^e \right\rangle_{AB} = \frac{1}{\sqrt{2}} \left( \left| 0 \right\rangle \left| 1_B \right\rangle \pm \left| 1_A \right\rangle \left| 0 \right\rangle \right) . \quad (7a)$$

$$\left| \Psi^e \right\rangle_{AB} = \frac{1}{\sqrt{2}} \left( \left| 1_B \right\rangle \left| 0 \right\rangle \pm \left| 0 \right\rangle \left| 1_A \right\rangle \right) . \quad (7b)$$

The parity superselection rule leads to some interesting differences with respect to the corresponding two-qubit states. First, one notes that measurements in any local single-mode basis other than the ‘computational’ basis $\{ \left| 0 \right\rangle, \left| 1_A \right\rangle \}$ are prevented by parity superselection. At the same time, measurements in a single product basis are not sufficient to distinguish entanglement from purely classical correlations. Consequently, two copies of each state need to be processed simultaneously to allow for the violation of a Bell inequality [58] (see also Ref. [66]). Second, the superselection rule also restricts the physically allowed pure-state decompositions for any given mixed state, which enters in convex-roof entanglement measures. For instance, consider the entanglement of formation (EOF) [67], defined as

$$E_{\text{OF}}(\rho) := \inf \mathcal{D}(\rho) \sum p_i S(\rho^{(i)}) , \quad (8)$$

where $S(\rho) = -\text{tr}(\rho \log(\rho))$ is the von Neumann entropy and the infimum is taken over all pure-state decompositions, that is, $\mathcal{D}(\rho)$ is normally taken to be the set of all sets $\{ (p_i, \left| \psi_i \right\rangle) \}$, for which $\rho = \sum p_i \left| \psi_i \right\rangle \left\langle \psi_i \right|$, with $\sum p_i = 1$ and $0 \leq p_i \leq 1$. For fermionic modes it can now be argued [61] that the set $\mathcal{D}(\rho)$ should be restricted to allow only pure state decompositions $\{ (p_i, \left| \psi_i \right\rangle) \}$, where the states $\{ \left| \psi_i \right\rangle \}$ satisfy the parity superselection rule. This assumption leads to the notion of mixed maximally entangled (MME) states [61], i.e., mixtures of maximally entangled pure states from different parity subspaces that are still as entangled (according to the value of the superselected EOF) as the individual pure states. The question that remains is, what is the operational significance of the value of the fermionic EOF?

Here, we therefore want to investigate the role of superselection rules and fermionic entanglement in teleportation protocols. More specifically, we aim to extend previous work [59] in this direction and identify if and how quantum information encoded in fermionic modes can be teleported, which resources need to be shared and which information needs to be communicated, before we return to a discussion of the implications for fermionic entanglement in Sec. V.

### III. FERMIONIC TELEPORTATION

#### III.1. Fermionic versus qubit teleportation

To set the stage for explaining fermionic teleportation scenarios, let us briefly sketch the standard protocol for teleporting a single qubit between two observers called Alice and Bob, as illustrated in Fig. 1. There, to teleport an unknown state $\left| \psi \right\rangle$ of qubit $A$, held by Alice, a maximally entangled two-qubit state $\left| \Phi \right\rangle_{AB}$ of qubits $A$ and $B$ is shared between Alice and Bob. Then, a projective measurement in a maximally entangled two-qubit basis is performed on qubits $A$ and $\tilde{A}$ by Alice. The result of the measurement, encoded in two classical bits with values $n_1$ and $n_2$ is then sent to Bob, who applies a corresponding unitary $U_{n_1n_2}$ on qubit $B$, recovering the state $\left| \psi \right\rangle_{B}$. 

$$\left| \Psi_{A,A'} \right\rangle = \gamma_0 \left| 0 \right\rangle + \gamma_A \left| 1_A \right\rangle + \gamma_{A'} \left| 1_{A'} \right\rangle$$
The basic observation to understand where teleportation of fermionic quantum information deviates from standard teleportation of qubits \[ |\psi\rangle_A \] is that parity superselection implies that single-mode states of fermionic fields are of the form of Eq. (5). On the one hand, this means that a single mode can locally only encode classical information (the equivalent of a classical bit). Consequently, teleportation of the quantum information stored solely in a single fermionic mode state is trivial: One can simply measure the state, send the result as one bit of classical information via a classical channel and prepare the corresponding state at the other end. On the other hand, the entropy of the single-mode state can arise from lack of information but also from entanglement with another mode. That is, the state \( \rho_A \) that is to be teleported and which is of the form of Eq. (5) may be the marginal of a two-mode state \( \rho_{AA'} \), for instance, as in Eq. (6a) or Eq. (6b) with \( p = |\alpha|^2 \), or even an incoherent mixture of the two. In this case, a purely classical ‘measure and prepare’ protocol would transfer classical information stored locally in mode \( A \), but would not be able to preserve entanglement with mode \( A' \).

To consider teleportation of quantum information using fermionic modes in any nontrivial way, we hence first have to decide what we mean by ‘quantum information’: If by ‘quantum information’ we mean the state of a system that itself contains only classical information but which might be entangled with another system, then we can consider teleporting the state of a single fermionic mode, as we discuss in Sec. III.2. If, on the other hand, we require the transfer of the equivalent of one qubit of quantum information, then we either have to relax the rules of the teleportation protocol (see Sec. III.2) or consider the teleportation of an entangled two-mode state from Eq. (6), since this definition implies a single fermionic mode cannot contain quantum information.

As in the teleportation using qubits, the teleportation of fermionic quantum information of any kind requires two resources:

(i) Shared entangled states: For qubits, one usually considers the number of ‘ebits’, i.e., shared maximally entangled qubit pairs. Here, we consider the number of required fermionic ebits, i.e., maximally entangled two-mode states, which we call ‘ebits’.

(ii) Sending classical information (in bits).

For qubits, the minimal amount of resources for teleportation of 1 qubit is 1 ebit and 2 bits. For the teleportation of fermionic quantum information, the minimally required resources depend on the particular scenario one considers. In the following, we will discuss these different scenarios, the corresponding resources, advantages and drawbacks.

### III.2. Fermionic single-mode teleportation

For the teleportation of fermionic quantum information, we explore a situation where Alice wishes to teleport the state of a single mode labelled \( A \) to Bob, as illustrated in Fig. 2. The mode \( A \) may (potentially) be entangled with another mode \( A' \) that is itself not necessarily teleported and whose role we discuss in more detail in Sec. III.4. For simplicity, let us for now assume that the two modes are prepared in the state \( |\psi\rangle_{AA'} \) as in Eq. (6).

We further assume that Alice and Bob share one maximally entangled fermionic two-mode state \( |\varphi\rangle_{AB} \), i.e., 1 fbit, as a resource to teleport the state of mode \( A \) from...
Alice to Bob. Alice, then performs a projective measurement with respect to the basis \( \{ \Phi^\pm \}_A, \{ \Psi^\pm \}_A \) on the modes \( \hat A \) and \( \hat A \).

1. Even-parity resource states

With the mentioned choice of measurement basis in mind, we can write the joint initial state \( \| \psi \rangle_{\hat A \hat B}, \| \varphi \rangle_{\hat A \hat B} \) for the specific case where \( \| \psi \rangle_{\hat A \hat A'} = \| \psi^\prime \rangle_{\hat A \hat A'} \) and \( \| \varphi \rangle_{\hat A \hat B} = \| \Phi^\prime \rangle_{\hat A \hat B} \), i.e., both states have even parity. Then, we have

\[
\| \psi^\prime \rangle_{\hat A \hat A'} \| \Phi^\prime \rangle_{\hat A \hat B} = \frac{1}{2} \left[ \| \Phi^+ \rangle_{\hat A \hat A} \left( \alpha \| 0 \rangle \pm \beta \| 1 \rangle \right) \| 1 \rangle \right] + \| \Phi^- \rangle_{\hat A \hat A} \left( \alpha \| 0 \rangle \mp \beta \| 1 \rangle \right) \| 1 \rangle \right]
\]

The measurement with respect to the basis \( \{ \Phi^\pm \}_A, \{ \Psi^\pm \}_A \) results in one of four possible outcomes corresponding to the four orthogonal basis states. Alice encodes the outcome in two classical bits, \( n_1 \) and \( n_2 \), and communicates them to Bob via a classical channel. If the outcome suggests that the modes \( \hat A \) and \( \hat A' \) have been projected onto the state \( \| \Phi^\pm \rangle_{\hat A \hat A'} \), i.e., the initially shared resource state, then the modes \( \hat B \) and \( \hat A' \) are left in the ‘correct’ state \( \| \psi^\prime \rangle_{\hat B \hat B'} = \alpha \| 0 \rangle \pm \beta \| 1 \rangle \| 1 \rangle \), without any further action. If the obtained outcome is \( \| \Phi^\pm \rangle_{\hat A \hat A'} \), i.e., an outcome in the same (even) parity sector as the resource state but with a relative phase of \( \pi \), then a phase flip transformation is required which can be represented by the unitary

\[
U_\pi = \exp(i\pi b_\hat A b_\hat B),
\]  

(10)

which maps \( \| 1 \rangle \) to \( -\| 1 \rangle \) and leaves all other modes invariant. When the outcome corresponds to a state in the opposite (odd) parity sector, i.e., \( \| \Psi^\pm \rangle_{\hat A \hat A'} \), then Bob needs to apply a unitary \( U_\pi \) to switch the parity of the state in the modes \( \hat B \) and \( \hat A' \). Due to parity superselection this is of course only possible via a parity conserving operation on a larger Hilbert space. We therefore append an auxiliary mode \( \hat C \) that is initially not populated and define the unitary \( U_\pi \) as

\[
U_\pi = \left( b_\hat C + b_\hat C^\dagger \right) \left( b_\hat B - b_\hat B^\dagger \right),
\]

(11)

such that

\[
U_\pi \| 1 \rangle \| 1 \rangle = \| 1 \rangle \| 1 \rangle, \quad (12a)
\]

\[
U_\pi \| 1 \rangle \| 1 \rangle = -\| 1 \rangle \| 1 \rangle. \quad (12b)
\]

One can confirm that the unitarity condition \( U_\pi^\dagger U_\pi = U_\pi U_\pi = 1 \) is satisfied using the anticommutation relations of (1). Bob may thus obtain the desired state

\[
\| \psi' \rangle_{\hat A \hat A'} = \| \Phi^+ \rangle_{\hat A \hat B} \| \Phi^+ \rangle_{\hat B \hat B'} + \| \Phi^- \rangle_{\hat A \hat B} \| \Phi^- \rangle_{\hat B \hat B'},
\]

\[
\| \varphi \rangle_{\hat A \hat B} = \| \Phi^+ \rangle_{\hat A \hat B} \| \Phi^- \rangle_{\hat B \hat B'} - \| \Phi^- \rangle_{\hat A \hat B} \| \Phi^+ \rangle_{\hat B \hat B'},
\]

(13)

For outcomes in the same parity sector as the resource state, the applied corrections are either trivial or correspond to \( U_\pi \) from Eq. (10). When the outcomes are in the odd-parity sector, we have to apply \( U_\pi \) in addition, which acts as

\[
U_\pi \| 0 \rangle \| 1 \rangle = -b_\hat B^\dagger b_\hat B \| 0 \rangle \| 1 \rangle = \| 1 \rangle \| 1 \rangle, \quad (14a)
\]

\[
U_\pi \| 1 \rangle \| 1 \rangle = b_\hat B^\dagger b_\hat B \| 0 \rangle \| 1 \rangle = -\| 0 \rangle \| 1 \rangle \| 1 \rangle. \quad (14b)
\]

Crucially, the combinations of outcomes and corrections, summarized in Table 1, are exactly the same for the even-parity state \( \| \psi' \rangle_{\hat A \hat A'} \), such that Bob is not required to have information about the parity of the unknown state to successfully teleport it.

| Outcome | Correction |
|---------|------------|
| \( \| \Phi^+ \rangle_{\hat A \hat A} \) | \( U_\pi \) |
| \( \| \Phi^- \rangle_{\hat A \hat A} \) | \( U_\pi \) |
| \( \| \Psi^+ \rangle_{\hat A \hat A} \) | \( U_\pi U_\pi = U_\pi \) |
| \( \| \Psi^- \rangle_{\hat A \hat A} \) | \( U_\pi U_\pi = U_\pi \) |

Table 1. Correction operations for even- and odd-parity resource states. Depending on which of the four outcomes (rows) is obtained, one of the four unitary corrections \( U_\pi \) or \( U_\pi U_\pi \) needs to be applied, depending on the resource state (columns used).

2. Odd-parity resource states

We can of course also consider the cases where the entangled resource state for the teleportation is an odd-parity state, \( \| \varphi \rangle_{\hat A \hat B} = \| \Psi^\pm \rangle_{\hat A \hat B} \). For a teleported state

\[
\| \psi' \rangle_{\hat A \hat A'} = \| \Phi^\prime \rangle_{\hat A \hat A} \| \Phi^\prime \rangle_{\hat A \hat B} + \| \Phi^- \rangle_{\hat A \hat A} \| \Phi^- \rangle_{\hat A \hat B},
\]

(15)
with even parity we then have
\[ \frac{1}{2} \left\langle \psi^\prime \left| \Phi^{+} \right| \psi^\prime \right\rangle_{AB} = \frac{1}{2} \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \right) \right) \]

\[ + \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \left( \alpha \left| 1_a \right\rangle \pm \beta \left| 1_a' \right\rangle \right) \left( \alpha \left| 1_b \right\rangle \pm \beta \left| 1_b' \right\rangle \right) \right) \]

\[ + \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \left( \alpha \left| 1_a \right\rangle \pm \beta \left| 1_a' \right\rangle \right) \left( \beta \left| 1_b \right\rangle \pm \alpha \left| 1_b' \right\rangle \right) \right) \]

\[ + \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \left( \beta \left| 1_a \right\rangle \pm \alpha \left| 1_a' \right\rangle \right) \left( \beta \left| 1_b \right\rangle \pm \alpha \left| 1_b' \right\rangle \right) \right) \]}

(15)

while an odd-parity state to be teleported results in
\[ \frac{1}{2} \left\langle \psi^\prime \left| \Phi^{+} \right| \psi^\prime \right\rangle_{AB} = \frac{1}{2} \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \right) \right) \]

\[ - \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \left( \alpha \left| 1_a \right\rangle \pm \beta \left| 1_a' \right\rangle \right) \left( \alpha \left| 1_b \right\rangle \pm \beta \left| 1_b' \right\rangle \right) \right) \]

\[ + \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \left( \beta \left| 1_a \right\rangle \pm \alpha \left| 1_a' \right\rangle \right) \left( \beta \left| 1_b \right\rangle \pm \alpha \left| 1_b' \right\rangle \right) \right) \]

\[ - \left( \frac{1}{2} \left\langle \Phi^{+} \left| \psi^\prime \right\rangle \left( \alpha \left| 1_a \right\rangle \pm \beta \left| 1_a' \right\rangle \right) \left( \beta \left| 1_b \right\rangle \pm \alpha \left| 1_b' \right\rangle \right) \right) \]}

(16)

From Eqs. (12) and (14), we see that the corresponding combinations of outcomes and corrections (for both \[ \left\langle \psi^\prime \left| \Phi^{+} \right| \psi^\prime \right\rangle \] and \[ \left\langle \psi^\prime \left| \Phi^{+} \right| \psi^\prime \right\rangle \] ) are the same regardless of the parity of the teleported state but of course depend on the specific resource state used, as summarized in Table 1.

### III.3. Implementation via fermionic Gaussian operations

It is interesting to note that the whole teleportation protocol can be implemented via fermionic Gaussian operations. For the correction operation \[ U_c \] this is easy to see since its generator \[ b_1 b_2 \] is quadratic in the mode operators (of mode \[ B \] ). For the operator \[ U_c \], this is also the case, which can be seen in the following way. First, note that \[ U_c \] is Hermitian, \[ U_c^\dagger = U_c \]. Since this implies that \[ U_c^2 = 1 \], the unitary \[ U_c \] can be considered to coincide with the Hamiltonian generating the unitary up to a global phase. That is, we can define \[ H_c := \frac{\pi}{2} (U_c - 1) \] and calculate
\[ e^{-i H_c} = \sum_{n=0}^{\infty} \frac{(-i H_c)^n}{n!} = e^{\frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-i \frac{\pi}{2} U_c)^n}{n!}} \]

\[ = i \left( \sum_{n=0}^{\infty} \frac{(-i \frac{\pi}{2} U_c)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i \frac{\pi}{2} U_c)^{2n+1}}{(2n+1)!} \right) \]

\[ = i \left( \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-i \frac{\pi}{2} U_c)^{2n}}{(2n)!} + U_c \sum_{n=0}^{\infty} \frac{(-i \frac{\pi}{2} U_c)^{2n+1}}{(2n+1)!} \right) \]

\[ = i \left( \frac{1}{2} \cos \left( \frac{\pi}{2} \right) - i U_c \sin \left( \frac{\pi}{2} \right) \right) = U_c \]}

(17)

Since both operators \[ U_c \] and \[ U_c \] are quadratic in the mode operators, each individually, and hence also their combination \[ U_c U_c \] are fermionic Gaussian operations. And since the Bell states are Gaussian \[ \{49\} \] the Bell measurement is a fermionic Gaussian operation \[ \{45\} \]. We thus see that teleportation can be carried out by Gaussian means: sharing 1 bit, performing a Gaussian measurement, sending classical information (2 bits encoding the outcome of Alice’s measurement) from Alice to Bob, and applying fermionic Gaussian corrections depending on the bit values.

### III.4. Teleporting ‘one qubit of quantum information’ – the role of mode \[ \tilde{A} \]

Let us now more carefully discuss the purpose of the explicit inclusion of the mode \[ \tilde{A} \] in our previous calculations. As we have already mentioned in Sec. III.1, using the teleportation protocol as outlined above just to transfer information about mode \[ \tilde{A} \] could be considered to be a waste of resources. Parity superselection constrains the state of mode \[ \tilde{A} \] to be of the form of Eq. (3), i.e., diagonal in the occupation number basis, and hence a classical state. However, there are two ways in which the protocol above can nonetheless be seen as transferring quantum information, both of which rely on the mode \[ \tilde{A} \].

On the one hand, the fermionic single-mode teleportation protocol can be considered as entanglement swapping from the modes \[ \tilde{A} \] and \[ \tilde{A}' \] to the modes \[ B \] and \[ A' \], regardless of who is controlling mode \[ \tilde{A}' \]. If the modes \[ \tilde{A} \] and \[ \tilde{A}' \] are initially in an entangled state, then the modes \[ B \] and \[ A' \] are in that very same entangled state after the teleportation protocol. More generally, this is true for any arbitrary state \[ \rho_{\tilde{A}\tilde{A}'} \] of these modes, since the details of the protocol (for fixed resource state) do not depend on the parity of the teleported state, and any state \[ \rho_{\tilde{A}\tilde{A}'} \] must be a convex mixture of even- and odd-parity states of \[ \tilde{A} \] and \[ \tilde{A}' \]. A fully classical information transfer whereby the mode \[ \tilde{A} \] is measured and the result is sent to Bob via a classical channel cannot achieve this, despite the fact that such a procedure would be able to transmit all locally available information about the mode \[ \tilde{A} \]. The described fermionic entanglement swapping protocol can thus be considered to transfer the equivalent of one qubit of quantum information in the sense of being able to transfer one half of a mode pair in an arbitrary (unknown and potentially entangled) state. The resources for this transfer are 1 bit of shared fermionic entanglement and communicating 2 bits of classical information.

On the other hand, one may argue that the equivalent of one qubit of quantum information should be defined in terms of the ability to encode the same complex amplitudes \[ \alpha \] and \[ \beta \] (with \[ |\alpha|^2 + |\beta|^2 = 1 \] as in a single-qubit state \[ \alpha |0 \rangle + \beta |1 \rangle \]). Clearly, a single fermionic mode does not provide this ability, but two modes do. Therefore, one can realize the above single-mode protocol on both modes \[ \tilde{A} \] and \[ \tilde{A}' \] at once, in the way that ‘one qubit of quantum information’ can be combined into full-fledged two-mode teleportation by the iteration of the initially described entanglement swapping protocol. That is, by using 2 fbits entanglement, transferring 4 bits of classical information and performing Gaussian operations (as discussed in Sec. III.3), one may teleport the modes \[ \tilde{A} \] and
Figure 3. Fermionic two-mode teleportation. In this scenario, teleporting quantum information encoded in the two-mode state $|\psi\rangle_{AB}$, from modes $A$ and $\tilde{A}'$ to the modes $B$ and $B'$ requires two maximally entangled states $|\varphi\rangle_{AB}$ and $|\varphi\rangle_{A'B'}$ (2 fbits) and communicating four bits of classical information with values $n_1$, $n_2$, $n_3$, and $n_4$, where the bit pairs $(n_1, n_2)$ and $(n_3, n_4)$ encode the outcomes of the (independent) measurements on the mode pairs $(\tilde{A}, A)$ and $(\tilde{A}', A')$, respectively. To complete the protocol a unitary operation $U_{n_1n_2n_3n_4}$ that depends on the bit values $n_i$ for $i = 1, 2, 3, 4$ is applied to the modes $B$ and $B'$, and to an auxiliary mode $C$. This may be realized as two consecutive operations $U_{n_1n_2}$ and $U_{n_3n_4}$ acting on the mode pairs $\{B, C\}$ and $\{B', C\}$, respectively, and the state of mode $C$ does not need to be reset inbetween. The output state of the auxiliary mode $C$ is denoted as $|\chi_C\rangle_C$, and depends on the local unitary operation $U_{n_1n_2n_3n_4}$ but remains separable from the other modes. The number of classical bits communicated from Alice to Bob can be reduced from 4 to 2, if non-Gaussian operations are used.

$\tilde{A}'$ as illustrated in Fig. 3. In this way, the complex amplitudes $\alpha$ and $\beta$ of any unknown two-mode state $|\psi\rangle_{\tilde{A}A'}$ (or, likewise, single-qubit state $|\psi\rangle_{A\tilde{A}}$, in a dual-rail encoding) can be transferred.

Indeed, one can even perform the two-mode teleportation protocol sharing only 2 bits of classical information, if non-Gaussian operation are allowed, as Alice and Bob each locally perform a projective measurement of the parity of the resource state. For instance, if the resource state is $|\Phi^+\rangle_{AB} |\Phi^+\rangle_{A'B'}$, then an ‘even’ outcome projects into $\frac{1}{\sqrt{2}} (|0\rangle + |1_A, 1_B, 1_{A'}, 1_{B'}\rangle)$, whereas an ‘odd’ outcome results in $\frac{1}{\sqrt{2}} (|1_A, 1_B\rangle + |1_{A'}, 1_{B'}\rangle)$. In either case, Alice may then perform teleportation with a Bell measurement adapted to the measured parity and sending the usual 2 classical bits (see, e.g., the example in Sec. IV.3 for comparison), while Bob learns the relevant parity from his local measurement. Thus it may seem as if an fbit is only half as powerful as an ebit, since two are needed to teleport a single qubit. However, this difference (almost) disappears if one allows to teleport many fermionic modes at once. Then the even-parity sector of $n$ modes spans a $2^{n-1}$-dimensional Hilbert space uninhibited by P-SSR in which $n-1$ qubits can be encoded, and which can be faithfully teleported using $n$ fbits (whereas $n-1$ ebits would suffice without SSR). The resource costs of all three variants are summarized in Table 2.

### IV. FERMIONIC TELEPORTATION SUBJECT TO PARTICLE NUMBER SUPERSELECTION

#### IV.1. Non-fundamental superselection rules

This far, we have viewed the task of fermionic teleportation as a fundamental problem, i.e., we have taken into account parity superselection but no other limitations. However, in practice, other restrictions such as non-fundamental SSRs typically do apply. In particular, we now want to discuss the influence of particle number superselection (N-SSR).

Let us begin by noting that it is less clear than with P-SSR (see, e.g., the discussion in [28]), if N-SSR is a fundamental restriction of Nature or not. On the one hand, we note that superpositions of different fermion numbers are not ruled out by charge conservation, much like superpositions of different energy eigenstates are not excluded by energy conservation. Instead, this can be viewed as an issue of not having available an appropriate reference frame, see, e.g., the discussion in Ref. [68, Sec. IV] or the argument by Aharonov and Susskind [69]. At the same time, there does not appear to exist any process (to the best of our knowledge) that could result in a superposition of different electric charges. An example for a state with indefinite particle number sometimes referred to in this context is the BCS ground state [70]. However, the BCS ground state with indefinite electron number can be understood as convenient approximation of the actual physical state with fixed electron number [71]. Here, we therefore cannot conclusively answer the question if superpositions of different charges exist or not.

However, even if one were to adopt charge superselection axiomatically [72], one may of course consider species of uncharged fermions (both composite and fundamental). There, the question of the existence of pure states with indefinite particle number is tied to the question of the existence (or not) of Majorana fermions as fundamental objects in Nature. Although we cannot directly answer this question either, admittedly, the prospects of creating coherent superpositions of different numbers of fermions useful for quantum information processing are nevertheless daunting (to say the least) either way. For practical purposes, particle number superselection is hence a sensible restriction for practical implementations of fermionic teleportation such as in Ref. [60].
To begin, it is interesting to put into perspective the usefulness of fbits as resource states for teleportation when constraints due to N-SSR apply. In the context of the single-mode teleportation protocol discussed in Sec. III.2, particle number superselection implies that it is not possible to create or project into even-parity states of two fermionic modes $A$ and $A$ other than $|0\rangle$ and $|1_A\rangle\langle 1_A|$. More specifically, this means that the outset of the single-mode teleportation protocol is the restriction to the odd-parity state $|\psi\rangle_{\tilde{A}A'}$, as the state to be teleported, and $|\Psi^+\rangle_{\tilde{A}A}$ as the shared resource state to achieve this. In addition, let us assume that the Bell measurement carried out by Alice can only result in states with definite particle number, i.e., the state of modes $\tilde{A}$ and $A$ will be projected into either $|\Psi^+\rangle_{\tilde{A}A'}$, $|0\rangle_{\tilde{A}A}$, or $|1_A\rangle\langle 1_A|$. Consequently, it is instructive to write the initial joint state w.r.t. this choice of basis as

$$\begin{align*}
|\psi\rangle_{\tilde{A}A'}\langle\psi|_{\tilde{A}A} &= \frac{1}{2}\left[|0\rangle_{\tilde{A}A} \sqrt{2} \alpha \langle 1_A| \right. \\
&\pm |1_A\rangle\langle 1_A| \sqrt{2} \beta \langle 0|_{\tilde{A}A'} \\
&\mp |\Psi^+\rangle_{\tilde{A}A} \left( \alpha \langle 1_A'\right| + \beta \langle 1_A| \right) \\
&\mp |\Psi^-\rangle_{\tilde{A}A} \left( \alpha \langle 1_A\right| - \beta \langle 1_A| \right)
\end{align*}$$

As one can clearly see from this decomposition, single-mode teleportation can in this case only be successful if either $|\Psi^+\rangle_{\tilde{A}A}$ or $|\Psi^-\rangle_{\tilde{A}A}$ is obtained as outcome on Alice’s side, resulting in an average single-mode teleportation fidelity that is reduced by 50% with respect to the case where no N-SSR applies. In principle, one may consider a more general scenario, where a three-element POVM $\{P_{\Psi^+}, P_{\Psi^-}, P_{\text{even parity}}\}$ is performed. In case of the third outcome, the quantum information might still be present. However, it is delocalized between Alice and Bob, and we are not aware of any way to complete the transfer if particle number superselection applies.

**IV.3. Two-mode teleportation & particle number superselection**

Let us now consider two-mode teleportation in the presence of particle superselection. The state to be teleported in this scenario is a two-mode state containing a single fermion, which can be considered as dual-rail encoding of a qubit. If we combine two fbits in the odd-parity sector (two single-fermion states as in the implementations proposed in Ref. [60] and as discussed in Sec. III.4) as resource states and naively perform the teleportation for each mode separately as before, then we see that the teleportation fidelity is further reduced to 25%, since the teleportation of either mode is only successful half the time (on average). However, as we shall see shortly, particle superselection does not intrinsically limit the fidelity in this way. Using the same resource state (a pair of two single-fermion fbits), the fidelity can be increased to 50%, and for other resource states (subject to particle superselection) one may even achieve 100% teleportation fidelity.

To see this, let us consider a different resource state for the modes $A$, $B$, $A'$, and $B'$ in a setup subject to particle superselection. Take, for instance, the state

$$|\Psi_R^+\rangle_{ABA'B'} = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle_{AA'A'} \right. \\
\pm |1_A\rangle\langle 1_A| + |1_B\rangle\langle 1_B| + |1_A'\rangle\langle 1_A'| + |1_B'\rangle\langle 1_B'| \right), \tag{19}$$

and let Alice carry out a projective measurement on the modes $\tilde{A}$, $A'$, $A$, and $A'$ in the ‘basis’ given by the four states

$$|\Phi_R^+\rangle_{AA'A'} = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle_{AA'A'} \right. \\
\pm |1_A\rangle\langle 1_A| + |1_A'\rangle\langle 1_A'| \right), \tag{20a}$$

$$|\Phi_R^-\rangle_{AA'A'} = \frac{1}{\sqrt{2}}\left(|\Psi^-\rangle_{AA'A'} \right. \\
\pm |1_A\rangle\langle 1_A| + |1_A'\rangle\langle 1_A'| \right). \tag{20b}$$

Here, we have put ‘basis’ in quotation marks, since these four states form a basis only of that subspace of the 2-fermion subspace of the four modes in question where there is exactly 1 fermion in the modes $A$ and $A'$. Consequently, when a single-fermion state $|\psi\rangle_{\tilde{A}A'} = |\psi\rangle_{\tilde{A}A'}$ is prepared for the states $\tilde{A}$, $A'$, we can write

$$|\psi\rangle_{\tilde{A}A'}\langle\psi|_{\tilde{A}A'} = \frac{1}{2}\left[|\Psi^+\rangle_{\tilde{A}A'} \left( \alpha |1_B\rangle - \beta |1_B'| \right) \\
- |\Psi^-\rangle_{\tilde{A}A'} \left( \alpha |1_B\rangle + \beta |1_B'| \right) \\
- |\Phi^+\rangle_{\tilde{A}A'} \left( \alpha |1_B\rangle + \beta |1_B'| \right) \\
+ |\Phi^-\rangle_{\tilde{A}A'} \left( \alpha |1_B\rangle - \beta |1_B'| \right)
\right].$$

We thus see that the encoding of the teleported state, the preparation of the resource state, the Bell measurements, as well as any correction operations required on the modes $B$ and $B'$ can all in principle be carried out while respecting particle number (and charge) superselection, both globally and locally (with respect to the partition $AA'[AA'BB']$, achieving a teleportation fidelity of 100%.

However, we observe that the resource (state) for this teleportation is not a pair of fbits anymore. This can be understood in a simple way: Although both resource states, $|\Psi_R^+\rangle_{ABA'B'}$ and $|\Psi_R^-\rangle_{AB}$, are pure states with the same particle content (2 fermions), and can hence be transformed into each other by global (on $A$, $B$, $A'$, $B'$) particle-number conserving unitaries, this cannot be achieved by unitaries acting locally with respect to the bipartition $AA'[BB']$. To see this, simply note that the reduced states of the modes $A$ and $A'$ have different rank for the different resource states. That is, both states are entangled w.r.t. this cut, but (it seems) not

---

2 The 2-particle sector of the Fock space of 4 fermionic modes is 6-dimensional, but for two of these states, $|1_A\rangle\langle 1_A|$, $|1_A\rangle\langle 1_A|$, and $|1_A\rangle\langle 1_A|$, the particle content of the subspace of modes $A$ and $A'$ is different from 1.
equally strongly (w.r.t. to an entanglement measure suitable to the applicable SSR). Nevertheless, if two fbits \( \| \Psi^+ \rangle \rangle_{AB} \otimes \| \Psi^+ \rangle \rangle_{A'B'} \) are used as a resource, one may still achieve 50% fidelity. If Alice performs a projective measurement of the total particle number in modes \( A \) and \( A' \) before performing the Bell measurement, this will result in the state \( \| \Psi_R \rangle \rangle_{ABA'B'} \) in half the cases (when there is 1 particle in the modes \( A \) and \( A' \)), and in separable states \( \| 1_A, 1_{A'} \rangle \rangle \) (when there are no particles in the modes \( A \) and \( A' \)) and \( \| 1_A, 1_{A'} \rangle \rangle \) (two particles in the modes \( A \) and \( A' \)) otherwise.

In other words, problems arise from using resource states whose marginals have support in different superselection sectors. All restrictions disappear, of course, if all logical qubits are locally supported in a subspace of fixed particle number. Then N-SSR does not restrict any logical operations, the projection on each of the four logical Bell states is permitted, and standard teleportation (of logical qubits) works as usual. The limitation of the fidelity due to particle number superselection in potential experimental settings [60] is hence more a practical (but nonetheless very challenging) problem of determining ways to prepare states like \( \| \Psi_R \rangle \rangle_{ABA'B'} \) from Eq. (19) directly (rather than by post-selection after preparing two fbits).

V. IMPLICATIONS FOR FERMIONIC ENTANGLEMENT

In this section, we want to relate our previous observations about fermionic teleportation with the quantification of entanglement subject to SSRs. In particular, we aim here to contrast the notion of superselected entanglement of formation (EOF, as discussed in Sec. II.2) with a state’s usefulness for teleportation.

For pure states, i.e., 1 or 2 fbits, this appears to be rather straightforward. The superselected EOF of \( n \) (pure) fbits is equal to \( n \), and we can refer to Table 2 for the corresponding resources for different tasks. However, the superselected EOF allows for the notion of ‘mixed maximally entangled’ (MME) fermionic states [61]. Take, for instance, the MME state

\[
\rho_{\text{MME}}^{AB} = \frac{1}{2} \| \Phi^+ \rangle \langle \Phi^+ \| + \frac{1}{2} \| \Psi^+ \rangle \langle \Psi^+ \|. \tag{21}
\]

Because the two parity subspaces do not mix, one fbit is required per copy to create \( \rho_{\text{MME}}^{AB} \) and the (parity) superselected EOF evaluates\(^3\) to \( \xi_{\text{EOF}}(\rho_{\text{MME}}^{AB}) = \log(2) = 1 \). The entanglement of \( \rho_{\text{MME}}^{AB} \) can thus be considered to be maximal in this sense. But are MME states useful for teleportation? In the following, we will discuss this question in more detail for P-SSR and N-SSR.

\[\text{V.1. Teleportation using mixed maximally entangled states for P-SSR}\]

Let us consider fermionic teleportation using the state \( \rho_{\text{MME}}^{AB} \) as a resource state for teleporting the state of mode \( A \) (which may be entangled with another mode \( A' \)) as illustrated in Fig. 2. If the state of modes \( A \) and \( A' \) has even parity and is given by \( \| \psi^\prime \rangle \rangle_{A'A'} \), and the outcome of the Bell-measurement in the basis \( \{ \| \Phi^+ \rangle \rangle_{AA}, \| \Psi^+ \rangle \rangle_{AA} \} \) gives the outcome \( \| \Phi^+ \rangle \rangle_{AA} \). Eqs. (9) and (15) allow us to conclude that the state of modes \( B \) and \( A' \) prior to any corrections is an equally weighted mixture of \( \alpha \| 0 \rangle \rangle 1_B \rangle \rangle + \beta \| 1_B \rangle \rangle 1_{A'} \rangle \rangle \) and \( \alpha \| 1_B \rangle \rangle - \beta \| 1_{A'} \rangle \rangle \). In particular, this means that the reduced state of mode \( B \) is given by \( \frac{1}{2} \| 0 \rangle \| 0 \rangle + \frac{1}{2} \| 1_B \rangle \rangle \), and hence maximally mixed. This means, no information whatsoever about the teleported state is locally available in mode \( B \). However, if we consider the joint state of modes \( B \) and \( A' \), we see that all information about the teleported state is still available. That is, a joint parity measurement on both modes projects either into the state \( \| \psi^\prime \rangle \rangle_{A'A'} \) or into the state \( \| \Phi^+ \rangle \rangle_{1_A1_B} \rangle \rangle - \| \Phi^+ \rangle \rangle_{1_A1_{A'}} \rangle \rangle \). In the former case, one has already retrieved the desired state. In the latter case, one applies the unitary correction \( U_{\text{c}} \) to complete the teleportation. As this example illustrates (and as can easily be confirmed for other combinations of resource states and teleported states), MME states can indeed be useful for ‘teleportation’ in this sense, but this has some caveats.

The first difference to using pure maximally entangled states manifests in the amount of information that is available locally about the teleported state. That is, the teleportation protocol using MME states becomes useful only if one has access also to the second mode \( A' \), and can perform a joint (and non-destructive as well as non-particle number resolving) parity measurement on the modes \( A' \) and \( B \). This is in principle (not considering practical experimental feasibility of doing so) the case when the state of mode \( A' \) is transferred to Bob via a fermionic channel as discussed in Sec. III.4. In particular, when this channel is a teleportation channel using a pure fbit, one may regard the resulting protocol as a teleportation protocol transferring the state of 2 modes (1 qubit) using the resource state \( \rho_{\text{MME}}^{AB} \otimes \| \varphi \rangle \rangle_{A''B''} \), where \( \| \varphi \rangle \rangle_{A''B''} \) is a pure fbit. The latter resource state is itself an MME state of 4 modes, i.e., a mixture of 2 two-fbit states in the two different parity sectors, and it is exactly as useful for teleportation as two pure fbits in terms of the amount of quantum information (the number of qubits) that can be transferred. To see this, note that by each performing a local parity measurement, Alice and Bob can project their four-mode MME state to a pure state of four modes with known local parity and each maximally entangled in a 2 × 2-dimensional subspace, i.e., each allowing for teleportation of one logical qubit.

The remarkable aspect here is not that mixed states

\(^3\) Here, we choose the logarithm to base 2 in the von Neumann entropy.
can be used for (imperfect) teleportation. Indeed, a 4-qubit mixed state corresponding to $\rho_{AB}^{\text{MME}} \wedge \varphi \otimes \varphi |A'\rangle$ (and not subject to any SSRs) could be used in a similar fashion (by including appropriate projections) to teleport 1 qubit perfectly. The crucial difference to the situation without SSRs is that the pure maximally entangled 2-qubit state is solely responsible for the ability to teleport 1 qubit, whereas the additional mixed 2-qubit state does not increase the capacity for teleportation. Moreover, a pure (maximally entangled) 4-qubit state could be used to perfectly teleport 2 qubits. In the presence of parity superselection, however, there is no pure state of 4 fermionic modes that is more useful for teleportation in terms of the number of transferred qubits than the 4-mode MME state. In addition, the pure 2-mode state $|\varphi\rangle \otimes |A'\rangle$ alone is not enough to transfer 1 qubit of quantum information, but it is in conjunction with $\rho_{AB}^{\text{MME}}$.

However, note that the state of two copies of $\rho_{AB}^{\text{MME}}$; i.e., $\rho_{AB}^{\text{MME}} \wedge \rho_{AB}^{\text{MME}}$ is not useful at all for teleportation, and is also not an MME state, since the states in both parity sectors are mixed and hence not maximally entangled.

The second difference between pure and mixed maximally entangled fermionic states lies in the security of the teleportation. That is, two (pure) fbits allow for violating a Bell inequality [58], and hence for authentication, whereas any number of copies of 2-mode MME states as in Eq. (21) alone do not. To see this more clearly, note that the two-qubit equivalent $\tilde{\rho}_{AB}$ (not subject to any SSRs) of $\rho_{AB}^{\text{MME}}$ is separable (which can easily be checked via the Peres-Horodecki criterion [73, 74]), and therefore so are two copies of $\tilde{\rho}_{AB}$. Therefore, no Bell inequality can be violated by $\tilde{\rho}_{AB}$, or by any number of copies of $\tilde{\rho}_{AB}$. This is so because SSRs further restrict the measurable operators that may appear in a Bell inequality. Consequently, the superselected state $\rho_{AB}^{\text{MME}}$ (or two copies of it) can also not violate a Bell inequality.

Nevertheless, authentication is possible (albeit, at a higher price) if one uses the 4-mode MME state for teleportation, one simply has to sacrifice twice as many (as compared to the situation using 2 fbits per teleported qubit) of the resource states for authentication to retrieve the same number of fbit pairs. Just recall that, also with pure states one has to collect statistics on measurements of sufficiently many entangled resource states to violate a Bell inequality. The choice between pure and mixed maximally entangled states hence comes down to a matter of efficiency of the authentication.

A comparison of the usefulness of MME states and fbits is shown in Table 3. In summary, we can say that MME states seem to have some usefulness comparable with fbits, and this is reflected in the matching values of EOF. At the same time, the difference in the potential to violate Bell inequalities is not captured by the (superselected) EOF. Finally, let us briefly discuss the extension of the ideas of MME states and MME-based teleportation to other SSRs.

| 1 fbit | 2-mode MME | 2 fbits | 4-mode MME | 1 ebit |
|-------|-------------|--------|------------|-------|
| EOF   | 1           | 1      | 2          | 2     | 1     |
| swapping | 1 mode     | 1 mode | 2 modes   | 2 modes | 1 qubit |
| Bell inequality violation | Yes* | No | Yes | Yes* | Yes |

Table 3. Comparison of ebits (maximally entangled 2-qubit states), fbits, and MME states in terms of entanglement of formation (EOF), teleportation fidelity in terms of the number of modes/qubits whose state can be swapped (‘entanglement swapping’ via single-mode teleportation), number of qubits (complex amplitudes) that can be teleported, and potential for Bell inequality violation (given sufficiently many copies).

V.2. Teleportation using mixed maximally entangled states for N-SSR

An obvious question that arises then concerns the usefulness (and existence) of MME states for other SSRs, in particular, for particle numbers superselection. The four-mode MME states encountered in the previous section allow for a 100% teleportation fidelity when only parity superselection applies. The crucial element is a final projective measurement of the system’s parity. However, when N-SSR is in place, the state $\rho_{AB}^{\text{MME}}$ is no longer maximally entangled in any sense. It becomes an incoherent mixture of two separable states, $|0\rangle$ and $|1\rangle |\Psi\rangle_{AB}$, with one fbit given by $|\Psi\rangle_{AB}$. This mixture is neither maximally entangled, nor useful for teleportation. Nevertheless, this does not mean one cannot consider other states that correspond to MME states in the presence of N-SSR (or, indeed, any SSR).

Let us consider a scenario where particle number superselection applies and we wish to teleport the state of 1 qutrit encoded in three fermionic modes labelled $A$, $A'$ and $A''$. This can be done by encoding the qutrit in the single-particle sector, spanned by the vectors $|1_{\Psi}\rangle$, $|1'_{\Psi}\rangle$, and $|1''_{\Psi}\rangle$, or in the two-particle sector, spanned by the vectors $|1_{\Psi}1'_{\Psi}\rangle$, $|1'_{\Psi}1''_{\Psi}\rangle$, and $|1_{\Psi}1''_{\Psi}\rangle$. As a resource for teleportation we can then use any 6-mode state (say, of modes $A, B, A', B', A''$, and $B''$) whose particle number is fixed both globally (to 2, 3, or 4 particles) and locally (to either 1 or 2 particles).

For instance, let us adopt the notation $|n; j, k\rangle$ for a state of $n$ particles of which $j$ particles are in the subspace of modes $A, A', A''$ and $k$ particles in the subspace of the modes $B, B', B''$. Then, for instance, one of the following...
states can be used for teleportation:

\[
\| 2; 1, 1 \rangle = \frac{1}{\sqrt{3}} \left( \| 1_A, 1_B \rangle + \| 1_A', 1_B' \rangle + \| 1_A'', 1_B'' \rangle \right), \\
\| 3; 1, 2 \rangle = \frac{1}{\sqrt{3}} \left( \| 1_A, 1_B, 1_B' \rangle + \| 1_A', 1_B, 1_B'' \rangle + \| 1_A'', 1_B', 1_B'' \rangle \right), \\
\| 3; 2, 1 \rangle = \frac{1}{\sqrt{3}} \left( \| 1_A, 1_A', 1_B \rangle + \| 1_A, 1_A', 1_B' \rangle + \| 1_A, 1_A'', 1_B' \rangle \right), \\
\| 4; 2, 2 \rangle = \frac{1}{\sqrt{3}} \left( \| 1_A, 1_A', 1_B, 1_B' \rangle + \| 1_A, 1_A', 1_B, 1_B'' \rangle + \| 1_A, 1_A'', 1_B, 1_B'' \rangle \right),
\]

(22a)

(22b)

(22c)

(22d)

While any of these states can be used to teleport 1 qudit, we can also consider an arbitrary incoherent mixture of any of these four states as an MME state for teleportation. Such a teleportation protocol works in the following way: Alice and Bob share the MME resource state, Alice receives the \( "A" \)-modes, and Bob receives the \( "B" \)-modes. Alice then performs a projective measurement of the particle number on the \( "A" \)-modes before performing an appropriate Bell measurement on the 3x3-dimensional subspace of the \( "A" \) and \( "A" \)-modes corresponding to the particle number of her encoded state and the result of the initial projective measurement on the \( "A" \)-modes. The result of the Bell measurement is communicated to Bob, who makes a similar projective measurement of the particle number on the \( "B" \)-modes and applies a correction depending on the classically communicated outcome of the Bell measurement.

As before for the parity SSR, the remarkable aspect lies not in the fact that mixed states can be used for teleportation in this way, but in the fact that there is no pure state of 6 modes subject to particle number superselection that could do better than teleporting a single qutrit (or log₂3 qubits) with unit fidelity (whereas a 6-qubit state could be used to teleport 3 qubits with the same fidelity). The maximum of log₂3 qubits is simply the maximum dimension \( d_{\text{SSR}}^{\max} \) of any subspace of 3 modes with fixed particle number. In general, the subspace dimension corresponding to \( k \) particles in \( n \) modes is \( \binom{n}{k} \) and hence

\[
d_{\text{SSR}}^{\max} = \begin{cases} 
\binom{n}{n/2} & \text{if } n \text{ even} \\
\binom{n}{(n-1)/2} & \text{if } n \text{ odd}
\end{cases}
\]

(23)

We thus see also for N-SSR that there exist MME states, and that these can also be useful for teleportation in terms of the number of transferred qubits, albeit with a reduced ability to violate Bell inequalities, as discussed in Sec. V.1. Moreover, analogous arguments can be made for any SSR. It is further interesting to remark that for \( n \) even and large, \( \log_2(d_{\text{SSR}}^{\max}) \) approaches \( n \) (by Stirling’s formula), i.e., \( n \) modes allow for \( n \) not-SSR-inhibited qubits asymptotically (up to \( \log(n) \) corrections).

VI. DISCUSSION

We have reviewed quantum teleportation in a setting where quantum information is encoded in the modes of a fermionic quantum field. As we have discussed, differences to standard qubit-based teleportation arise due to parity superselection, which influences both the encoding of quantum information in the state space, as well as the allowed operations on given quantum states. In particular, we have focused on understanding the usefulness of pure entangled states of two modes ("fbits"), which are known to allow for Bell inequality violation only when at least two copies can be jointly processed [58]. Here, we find that single copies of such states can be useful for swapping the state of a single mode via teleportation. However, this procedure in itself is only useful (beyond classical notions of state transfer) when the latter mode is part of an entangled two-mode state itself. Once two fbits are available as a shared resource, one may teleport the entire two-mode state encoding the complex probability amplitudes usually encoded in a single qubit.

We have further considered how these teleportation protocols are influenced by particle-number superselection. Although not a fundamental restriction of Nature, it is of practical relevance for many applications, see, e.g., Ref. [60]. Here, we conclude that also this stronger superselection rule allows for teleportation with unit fidelity, when an appropriate four-mode resource state is shared, but using two fbits instead reduces the fidelity by 50%, provided that no other practical restrictions limit the setup. Finally, we have discussed the peculiar notion of mixed maximally entangled states in the context of teleportation. Interestingly, such states are not merely an artefact of evaluating convex-roof entanglement measures under the restriction of superselection rules, but they do have limited usefulness for teleportation as well.

In comparison to usual qubit-based quantum information processing, fermionic systems hence provide a more differentiated picture of entanglement, non-locality, and teleportation. In this context, it may be of future interest to identify a suitably diverse set of entanglement quantifiers that can capture these different notions of useful entanglement. Such developments may further motivate revisiting previous observations about the energetic costs of creating correlations and entanglement [75, 76]. Moreover, one may even go as far as to speculate whether further differentiation of fermionic entanglement and correlations could become relevant to account for other applications, e.g., entanglement as a resource for fermionic measurement-based computation [77] or correlations relevant in molecular problems [78].

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[1] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters, *Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels*, Phys. Rev. Lett. **70**, 1895–1899 (1993).

[2] Dik Bouwmeester, Jian-Wei Pan, Klaus Mattle, Manfred Eibl, Harald Weinfurter, and Anton Zeilinger, *Experimental Quantum Teleportation*, Nature **390**, 575–579 (1997), arXiv:1901.11004.

[3] D. Boschi, S. Branca, Francesco De Martini, Lucien Hardy, and Sandu Popescu, *Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels*, Phys. Rev. Lett. **80**, 1121–1125 (1998), arXiv:quant-ph/9710013.

[4] Y. H. Kim, S. P. Kulik, and Y. Shih, *Quantum Teleportation of a Polarization State with a complete Bell State Measurement*, Phys. Rev. Lett. **86**, 1370–1373 (2001).

[5] Akira Furusawa, Jens L. Sørensen, Samuel L. Braunstein, Christopher A. Fuchs, H. Jeff Kimble, and Eugene S. Polzik, *Unconditional Quantum Teleportation*, Science **282**, 706–709 (1998).

[6] Michael A. Nielsen, Emanuel Knill, and Raymond Laflamme, *Complete Quantum Teleportation Using Nuclear Magnetic Resonance*, Nature **396**, 52–55 (1998).

[7] Nicolai Friis, Giuseppe Vitagliano, Mehul Malik, and Marcus Huber, *Entanglement Certification From Theory to Experiment*, Nat. Rev. Phys. **1**, 72 (2019), arXiv:1906.10929.

[8] Jian-Wei Pan, Matthew Daniell, Sara Gasparoni, Gregor Weihs, and Anton Zeilinger, *Experimental Demonstration of Four-Photon Entanglement and High-Fidelity Teleportation*, Phys. Rev. Lett. **86**, 4435 (2001), arXiv:quant-ph/0104047.

[9] I. Marcikic, H. de Riedmatten, W. Tittel, H. Zbinden, and N. Gisin, *Long-Distance Teleportation of Qubits at Telecommunication Wavelengths*, Nature **421**, 509–513 (2003).

[10] R. Ursin, T. Jennewein, M. Aspelmeyer, R. Kaltenbaek, M. Lindenthal, P. Walther, and Anton Zeilinger, *Quantum Teleportation Across the Danube*, Nature **430**, 849–849 (2004).

[11] X.-S. Ma, T. Herbst, T. Scheidl, D. Wang, S. Kropatschek, W. Naylor, A. Mech, B. Wittmann, J. Kofler, E. Anisimova, V. Makarov, T. Jennewein, R. Ursin, and Anton Zeilinger, *Quantum Teleportation over 143 Kilometers Using Active Feed-Forward*, Nature **480**, 269–273 (2012).

[12] Daniel Llewellyn, Yunhong Ding, Imad I. Faruque, Stefano Paesani, Davide Bacco, Raffaele Santagati, Yan-Jun Qian, Yan Li, Yun-Feng Xiao, Marcus Huber, Mehul Malik, Gary F. Sinclair, Xiaoqi Zhou, Karsten Rottwitt, Jeremy L. O’Brien, John G. Rarity, Qihuang Gong, Leif K. Oxenlowe, Jianwei Wang, and Mark G. Thompson, *Chip-to-chip quantum teleportation and multi-photon entanglement in silicon*, Nat. Phys. (2019), 10.1038/s41567-019-0727-x, arXiv:1911.07839.

[13] Zhi Zhao, Yu-Ao Chen, An-Ning Zhang, Tao Yang, Hans J. Briegel, and Jian-Wei Pan, *Experimental demonstration of five-photon entanglement and open-destination teleportation*, Nature **430**, 54 (2004), arXiv:quant-ph/0402096.

[14] Robert Raussendorf and Hans J. Briegel, *A One-Way Quantum Computer*, Phys. Rev. Lett. **86**, 5188 (2001), arXiv:quant-ph/0010033.

[15] Hans J. Briegel, Dan E. Browne, Wolfgang Dür, Robert Raussendorf, and Maarten Van den Nest, *Measurement-based quantum computation*, Nat. Phys. **5**, 19 (2009), arXiv:0910.1116.

[16] M. Riebe, H. Häffner, C. F. Roos, W. Hänsel, M. Ruth, J. Benhelm, G. P. T. Lancaster, T. W. Körber, C. Becher, F. Schmidt-Kaler, D. F. V. James, and R. Blatt, *Deterministic Quantum Teleportation with Atoms*, Nature **429**, 734–737 (2004).

[17] S. Olmschenk, D. N. Matsukevich, P. Maunz, D. Hayes, L. M. Duan, and C. Monroe, *Quantum Teleportation between Distant Matter Qubits*, Science **323**, 486–489 (2009).

[18] Christian Nölleke, Andreas Neuzner, Andreas Reiserer, Carolin Hahn, Gerhard Rempe, and Stephan Ritter, *Efficient Teleportation between Remote Single-Atom Quantum Memories*, Phys. Rev. Lett. **110**, 140403 (2013), arXiv:1212.3127.

[19] Wolfgang Pfaff, Bas Hensen, Hannes Bernien, Suzanne B. van Dam, Machiel S. Blok, Tim H. Taminiau, Marijn J. Tiggelman, Raymond N. Schouten, Matthew Markham, Daniel J. Twitchen, and Ronald Hanson, *Unconditional quantum teleportation between distant solid-state qubits*, Science **345**, 532 (2014), arXiv:1404.4369.

[20] R. Hanson, L. P. Kouwenhoven, J. R. Petta, S. Tarucha, and L. M. K. Vandersypen, *Spins in few-electron quantum dots*, Rev. Mod. Phys. **79**, 1217 (2007), arXiv:cond-mat/0610433.

[21] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, *Quantum dynamics of single trapped ions*, Rev. Mod. Phys. **75**, 281 (2003).

[22] Tobias Schaeetz, *Trapping ions and atoms optically*, J. Phys. B: At. Mol. Opt. Phys. **50**, 102001 (2017).

[23] Alexei Kitaev, *Unpaired Majorana fermions in quantum wires*, Phys. Usp. **44**, 131 (2001), arXiv:0010440.

[24] Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma, *Non-Abelian Anyons and Topological Quantum Computation*, Rev. Mod. Phys. **80**, 1083 (2008), arXiv:0707.1889.

[25] Sankar Das Sarma, Michael Freedman, and Chetan Nayak, *Majorana zero modes and topological quantum computation*, npj Quantum Information **1**, 15001 (2015), arXiv:1501.02813.

[26] Roman M. Lutchyn, Erik P. A. M. Bakkers, Leo P. Kouwenhoven, Peter Krogstrup, Charles M. Marcus, and...
Entanglement in fermionic systems, using witness operators, Phys. Rev. A 89, 022319 (2014), arXiv:1312.6583.

[28] Nicolai Friis, Reasonable fermionic quantum information theories require relativity, New J. Phys. 18, 033014 (2016), arXiv:1502.04476.

[29] Grigori G. Amosov and Sergey N. Filippov, Spectral properties of reduced fermionic density operators and parity superselection rule, Quantum Inf. Process. 16, 2 (2017), arXiv:1512.01828.

[30] Pawel Caban, Krzysztof Podlaski, Jakub Rembieliński, Kordian A. Smoliński, and Zbigniew Walczak, Entanglement and tensor product decomposition for two fermions, J. Phys. A: Math. Gen. 38, L79 (2005), arXiv:quant-ph/0405108.

[31] Mari-Carmen Bañuls, J. Ignacio Cirac, and Michael M. Wolf, Entanglement in fermionic systems, Phys. Rev. A 76, 022311 (2007), arXiv:0705.1103.

[32] John Schliemann, J. Ignacio Cirac, Marek Kuś, Maciej Lewenstein, and Daniel Loss, Quantum correlations in two-fermion systems, Phys. Rev. A 64, 022303 (2001), arXiv:quant-ph/0012094.

[33] Paolo Zanardi, Quantum entanglement in fermionic lattices, Phys. Rev. A 65, 042101 (2002), arXiv:quant-ph/0104114.

[34] GianCarlo Ghirardi and Luca Marinatto, General criterion for the entanglement of two indistinguishable particles, Phys. Rev. A 70, 012109 (2004), arXiv:0401065.

[35] Lin Chen, Dragomir Ž. Đoković, Markus Grassl, and Bei Zeng, Four-qubit pure states as fermionic states, Phys. Rev. A 88, 052309 (2013), arXiv:1309.0791.

[36] Fernando Iemini, Thiago O. Maciel, Tiago Debarba, and Reinaldo O. Vianna, Quantifying correlations in fermionic systems using witness operators, Quantum Inf. Process. 12, 733 (2013), arXiv:quant-ph/1206.0024.

[37] Fernando Iemini, Tiago Debarba, and Reinaldo O. Vianna, Quantumness of correlations in indistinguishable particles, Phys. Rev. A 89, 032324 (2014), arXiv:quant-ph/1312.0839.

[38] Giacomo Mauro D’Ariano, Franco Manessi, Paolo Perinotti, and Alessandro Tosini, The Feynman problem and Fermionic entanglement: Fermionic theory versus qubit theory, Int. J. Mod. Phys. A 29, 1430025 (2014), arXiv:1403.2674.

[39] Nicolás Gigena and Raúl Rossignoli, Entanglement in fermion systems, Phys. Rev. A 92, 042326 (2015), arXiv:1509.05970.

[40] Fábio Benatti, Roberto Floreanini, and Ugo Marzolino, Entanglement in fermion systems and quantum metrology, Phys. Rev. A 89, 032326 (2014), arXiv:1403.1144.

[41] Antônio C. Lourenço, Tiago Debarba, and Eduardo I. Duazion, Entanglement of indistinguishable particles: A comparative study, Phys. Rev. A 99, 012341 (2019), arXiv:quant-ph/1905.05883.

[42] Nicolai Friis, Antony R. Lee, and David Edward Bruschi, Fermionic mode entanglement in quantum information, Phys. Rev. A 87, 022338 (2013), arXiv:1211.7217.

[43] Aiyalam Parameswaran Balachandran, Thupil R. Govindarajan, Amilcar R. de Queiroz, and Andrés Fernando Reyes-Lega, Entanglement and Particle Identity: A Unifying Approach, Phys. Rev. Lett. 110, 080503 (2013), arXiv:1303.0688.

[44] Alonso Botero and Benni Reznik, BCS-like modewise entanglement of fermion Gaussian states, Phys. Lett. A 331, 39 (2004), arXiv:quant-ph/0404176.

[45] Sergey Bravyi, Lagrangian representation for fermionic linear optics, Quantum Inf. Comput. 5, 216 (2005), arXiv:0404180.

[46] Joel F. Corney and Peter D. Drummond, Gaussian phase-space representations for fermions, Phys. Rev. B 73, 125112 (2006), arXiv:cond-mat/0411712.

[47] Victor Eisler and Zoltán Zimborás, On the partial transpose of fermionic Gaussian states, New J. Phys. 17, 053048 (2015), arXiv:1502.01369.

[48] Eliska Greplová and Géza Giedke, Degradability of Fermionic Gaussian Channels, Phys. Rev. Lett. 121, 200501 (2018), arXiv:1604.01954.

[49] Cornelia Spee, Katharina Schweiger, Géza Giedke, and Barbara Kraus, Mode-entanglement of Gaussian fermionic states, Phys. Rev. A 97, 042325 (2018), arXiv:1712.07560.

[50] Marvellous Onuma-Kalu, Daniel Grimmer, Robert B. Mann, and Eduardo Martín-Martínez, A classification of open fermionic Gaussian dynamics, J. Phys. A: Math. Theor. 52, 435302 (2019), arXiv:1902.02239.

[51] Martin Hebenstreit, Richard Jozsa, Barbara Kraus, Sergii Strelchuk, and Mathura Yoganathan, All pure fermionic non-gaussian states are magic states for matchgate computations, Phys. Rev. Lett. 123, 080503 (2019), arXiv:1905.08584.

[52] R. P. G. McNeil, M. Kataoka, C. J. B. Ford, C. H. W. Barnes, D. Anderson, G. A. C. Jones, I. Farrer, and D. A. Ritchie, On-demand single-electron transfer between distant quantum dots, Nature 477, 439 (2011), https://arXiv.org/abs/1107.3886.

[53] Sylvain Hermelin, Shintaro Takada, Michihisa Yamamoto, Seigo Tarucha, Andreas D. Wieck, Laurent Saminadayar, Christopher Bäuerle, and Tristan Meunier, Electrons surfing on a sound wave as a platform for quantum optics with flying electrons, Nature 477, 435 (2011), arXiv:1107.4759.

[54] B. Bertrand, S. Hermelin, S. Takada, M. Yamamoto, S. Tarucha, A. Ludwig, A. D. Wieck, C. Bäuerle, and T. Meunier, Fast spin information transfer between distant quantum dots using individual electrons, Nat. Nanotechnol. 11, 672 (2016), arXiv:1508.04307.

[55] Christopher J. B. Ford, Transporting and manipulating single electrons in surface-acoustic-wave minima, Phys. Status Solidi B 254, 1600658 (2017), arXiv:1702.06628.

[56] Takaumi Fujita, Timothy Alexander Baart, Christian Reichl, Werner Wegscheider, and Lieven Mark Koenraad Vandersypen, Coherent shuttle of electron-spin states, Nature Nanotechnology 3(2), 22 (2017), arXiv:1701.00815.

[57] Patrick R. Hofer, David Dasenbrook, and Christian Flindt, On-demand entanglement generation using dynamical single-electron sources, Phys. Status Solidi B 254, 1600582 (2017), arXiv:1608.06455.

[58] David Dasenbrook, Joseph Bowles, Jonatan Bohr Brask, Patrick R. Hofer, Christian Flindt, and Nicolas Brunner, Single-electron entanglement and nonlocality, New J. Phys. 18, 043036 (2016), arXiv:1511.04450.

[59] O. Morgenshtern, Benni Reznik, and I. Zalzberg, Quantum information with single fermions: teleportation
and fermion-boson entanglement conversion, (2008), arXiv:0807.0850.

[60] Edvin Olofsson, Peter Samuelsson, Nicolas Brunner, and Patrick P. Potts, Quantum teleportation of single-electron states, (2020), in preparation.

[61] Giacomo Mauro D’Ariano, Franco Manessi, Paolo Perinotti, and Alessandro Tosini, Fermionic computation is non-local tomographic and violates monogamy of entanglement, Europhys. Lett. 107, 2009 (2014), arXiv:1307.7902.

[62] Tiago Debarba, Reinaldo O. Vianna, and Fernando Iemini, Quantumness of correlations in fermionic systems, Phys. Rev. A 95, 022325 (2017), arXiv:1611.01473.

[63] Miguel Montero and Eduardo Martín-Martínez, Fermionic entanglement ambiguity in noninertial frames, Phys. Rev. A 83, 062323 (2011), arXiv:1104.2307.

[64] Kamil Brádler and Rocio Jáuregui, Comment on "Fermionic entanglement ambiguity in noninertial frames", Phys. Rev. A 85, 016301 (2012), arXiv:1201.1045.

[65] Miguel Montero and Eduardo Martín-Martínez, Reply to "Comment on 'Fermionic entanglement ambiguity in noninertial frames'", Phys. Rev. A 85, 016302 (2012), arXiv:1108.6074.

[66] Nicolai Friis, Unlocking fermionic mode entanglement, New J. Phys. 18, 061001 (2016).

[67] Charles H. Bennett, D. P. Di Vincenzo, J. A. Smolin, and William K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A 54, 3824 (1996), arXiv:quant-ph/9604024.

[68] Stephen D. Bartlett, Terry Rudolph, and Robert W. Spekkens, Reference frames, superselection rules, and quantum information, Rev. Mod. Phys. 79, 555 (2007), arXiv:quant-ph/0610030.

[69] Yakir Aharonov and Leonard Susskind, Charge Superselection Rule, Phys. Rev. 155, 1428 (1967).

[70] John Bardeen, Leon N. Cooper, and John Robert Schrieffer, Theory of Superconductivity, Phys. Rev. 108, 1175 (1957).

[71] Christina V. Kraus, Michael M. Wolf, J. Ignacio Cirac, and Géza Giedke, Pairing in fermionic systems: A quantum-information perspective, Phys. Rev. A 79, 012306 (2009), arXiv:0810.4772.

[72] Franco Strocchi and Arthur S. Wightman, Proof of the charge superselection rule in local relativistic quantum field theory, J. Math. Phys. 15, 2198 (1974).

[73] Asher Peres, Separability Criterion for Density Matrices, Phys. Rev. Lett. 77, 1413 (1996), arXiv:quant-ph/9604005.

[74] Michal Horodecki, Paweł Horodecki, and Ryszard Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A 223, 25 (1996), arXiv:quant-ph/9605038.

[75] David E. Bruschi, Martí Perarnau-Llobet, Nicolai Friis, Karen V. Hovhannisyan, and Marcus Huber, The thermodynamics of creating correlations: Limitations and optimal protocols, Phys. Rev. E 91, 032118 (2015), arXiv:1409.4647.

[76] Nicolai Friis, Marcus Huber, and Martí Perarnau-Llobet, Energetics of correlations in interacting systems, Phys. Rev. E 93, 042135 (2016), arXiv:1511.08654.

[77] Yu-Ju Chiu, Xie Chen, and Isaac L. Chuang, Fermionic measurement-based quantum computation, Phys. Rev. A 87, 012305 (2013), arXiv:1207.5846.

[78] Lexin Ding and Christian Schilling, Correlation Paradox of the Dissociation Limit: Formal Discussion and Quantitative Resolution based on Quantum Information Theory, (2020), arXiv:2001.04858