Abstract

Flexible estimation of heterogeneous treatment effects lies at the heart of many statistical challenges, such as personalized medicine and optimal resource allocation. In this paper, we develop a general class of two-step algorithms for heterogeneous treatment effect estimation in observational studies. We first estimate marginal effects and treatment propensities in order to form an objective function that isolates the causal component of the signal. Then, we optimize this data-adaptive objective function. Our approach has several advantages over existing methods. From a practical perspective, our method is flexible and easy to use: For both steps, we can use any loss-minimization method, e.g., penalized regression, deep neutral networks, or boosting; moreover, these methods can be fine-tuned by cross validation. Meanwhile, in the case of penalized kernel regression, we show that our method has a quasi-oracle property: Even if the pilot estimates for marginal effects and treatment propensities are not particularly accurate, we achieve the same regret bounds as an oracle who has a priori knowledge of these two nuisance components. We implement variants of our approach based on both penalized regression and boosting in a variety of simulation setups, and find promising performance relative to existing baselines.

1 Introduction

The problem of heterogeneous treatment effect estimation in observational studies arises in a wide variety application areas (Athey, 2017), ranging from personalized medicine (Obermeyer and Emanuel, 2016) to offline evaluation of bandits (Dudík et al., 2011), and is also a key component of several proposals for learning decision rules (Athey and Wager, 2017; Hirano and Porter, 2009). There has been considerable interest in developing flexible and performant methods for heterogeneous treatment effect estimation. Some notable recent advances include proposals based on the lasso (Imai and Ratkovic, 2013), recursive partitioning (Athey and Imbens, 2016; Su et al., 2009), BART (Hahn, Murray, and Carvalho, 2017; Hill, 2011), random forests (Athey, Tibshirani, and Wager, 2018; Wager and Athey, 2017), boosting (Powers et al., 2018), neural networks (Shalit, Johansson, and Sontag, 2017), etc., as well as combinations thereof (Künzel et al., 2017; Luedtke and van der Laan, 2016); see Dorie et al. (2017) for a recent survey and comparisons.

However, although this line of work has led to many promising methods, the literature does not yet provide a comprehensive answer as to how machine learning methods should be adapted for treatment effect estimation. First of all, there is no definitive guidance on
how to turn a good generic predictor into a good treatment effect estimator that is robust to confounding. The process of developing “causal” variants of machine learning methods is still a fairly labor intensive process, effectively requiring the involvement of specialized researchers. Second, with some exceptions, the above methods are mostly justified via numerical experiments, and come with no formal convergence guarantees or regret bounds proving that the methods in fact succeed in isolating causal effects.

In this paper, we discuss a new approach to estimating heterogeneous treatment effects that addresses both of these concerns. Our framework allows for fully automatic specification of heterogeneous treatment effect estimators in terms of arbitrary loss minimization procedures. Moreover, we show how the resulting methods can achieve comparable regret bounds to oracle methods that know everything about the data-generating distribution except the treatment effects.

We formalize our problem in terms of the potential outcomes framework (Neyman, 1923; Rubin, 1974). The analyst has access to independent and identically distributed examples \((X_i, Y_i, W_i), i = 1, ..., n\), where \(X_i \in \mathcal{X}\) denotes per-person features, \(Y_i \in \mathbb{R}\) is the observed outcome, and \(W_i \in \{0, 1\}\) is the treatment assignment. We posit the existence of potential outcomes \(\{Y_i(0), Y_i(1)\}\) corresponding to the outcome we would have observed given the treatment assignment \(W_i = 0\) or \(1\) respectively, such that \(Y_i = Y_i(W_i)\), and seek to estimate the conditional average treatment effect (CATE) function

\[
\tau^*(x) = \mathbb{E} \left[ Y(1) - Y(0) \mid X = x \right].
\]  

In order to identify \(\tau^*(x)\), we assume unconfoundedness, i.e., the treatment assignment is as good as random once we control for the features \(X_i\) (Rosenbaum and Rubin, 1983).

**Assumption 1.** The treatment assignment \(W_i\) is unconfounded, \(\{Y_i(0), Y_i(1)\} \perp \perp W_i \mid X_i\).

We write the treatment propensity as \(e^*(x) = \mathbb{P} [W = 1 \mid X = x]\) and the conditional response surfaces as \(\mu^*_w(x) = \mathbb{E} [Y(w) \mid X = x]\) for \(w \in \{0, 1\}\) (throughout this paper, we use \(*\)-superscripts to denote unknown population quantities). Under unconfoundedness, we can check that \(Y_i = \mu^*_0(X_i) + W_i \tau^*(X_i) + \epsilon_i\) where \(\mathbb{E} [\epsilon_i \mid X_i, W_i] = 0\). A few more lines of algebra then imply that the CATE function \(\tau^*(x)\) can be expressed as follows in terms of the propensity \(e^*(x)\) and the mean outcome \(m^*(x) = \mathbb{E} [Y \mid X = x]\):

\[
Y_i - m^*(X_i) = (W_i - e^*(X_i)) \tau^*(X_i) + \epsilon_i.
\]  

This decomposition was originally used by Robinson (1988) to estimate parametric components in partially linear models, and has regularly been discussed in both statistics and econometrics ever since. For example, Zhao, Small, and Ertefaie (2017) use Robinson’s transformation for post-selection inference on effect modification; Athey, Tibshirani, and Wager (2018) rely on it to grow a causal forest that is robust to confounding; Robins (2004) builds on it in developing G-estimation for sequential trials, and Chernozhukov et al. (2017) present it as a leading example on how machine learning methods can be put to good use in estimating nuisance components for semiparametric inference.

With the representation (2), an oracle who knew both the functions \(m^*(x)\) and \(e^*(x)\) a priori could estimate the heterogeneous treatment effect function \(\tau^*(\cdot)\) by simple loss minimization:

\[
\hat{\tau}(\cdot) = \arg\min_{\tau(\cdot)} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( (Y_i - m^*(X_i)) - (W_i - e^*(X_i)) \tau(X_i) \right)^2 + A_n (\tau(\cdot)) \right\},
\]  

(3)
where the term $\Lambda_n(\tau(\cdot))$ is interpreted as a regularizer on the complexity of the $\tau(\cdot)$ function. In practice, this regularization could be explicit as in penalized regression, or implicit, e.g., as provided by a carefully designed deep neural network.

The difficulty, however, is that we never know the weighted main effect function $m^*(x)$ and usually don’t know the treatment propensities $e^*(x)$ either, and so the estimator (3) is not feasible. In this paper, we study the following class of two-step estimators motivated by the oracle procedure (3):

1. Fit $\hat{m}(x)$ and $\hat{e}(x)$ via appropriate methods tuned for optimal predictive accuracy, then

2. Estimate treatment effects via a plug-in version of (3), where $\hat{e}(-i)(X_i)$, etc., denote held-out predictions, i.e., predictions made without using the $i$-th training example,$^1$

$$
\hat{\tau}(\cdot) = \arg\min_{\tau} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( (Y_i - \hat{m}(-i)(X_i)) - (W_i - \hat{e}(-i)(X_i)) \tau(X_i) \right)^2 + \Lambda_n(\tau(\cdot)) \right\}.
$$

In other words, the first step learns an approximation for the oracle objective, and the second step optimizes it. We refer to this approach as the $R$-learner in recognition of the work of Robinson (1988), and also to emphasize the role of residualization. We will also refer to the squared loss in (4) as the $R$-loss.

This approach has several advantages over existing, more ad hoc proposals. The steps 1 and 2 capture the conceptual aspects of heterogeneous treatment effect estimation in a general way, and so implementing new variants of these methods reduces to routine work. Moreover, the key distinctive step using (4) is simply a loss minimization problem, and so can be efficiently solved via off-the-shelf software such as glmnet for high-dimensional regression (Friedman, Hastie, and Tibshirani, 2010), XGboost for boosting (Chen and Guestrin, 2016), or TensorFlow for deep learning (Abadi et al., 2016). Finally, it is straightforward to tune the regularizer $\Lambda_n(\tau)$ by simply cross validating the squared-error loss in (4), which avoids the use of more sophisticated model-assisted cross-validation procedures as developed in Athey and Imbens (2016) or Powers et al. (2018).

This paper makes the following contributions. First, we implement variants of our method based on penalized regression and boosting. In each case, we find that the $R$-learner exhibits promising performance relative to existing proposals. Second, we prove that—in the case of penalized kernel regression—regret bounds for the feasible estimator for $\hat{\tau}(\cdot)$ asymptotically match the best available bounds for the oracle method $\tilde{\tau}(\cdot)$. Crucially, this quasi-oracle behavior holds even if the first-step predictors $\hat{e}(x)$ and $\hat{m}(x)$ are up to an order of magnitude less accurate than $\hat{\tau}(\cdot)$ itself. Both results highlight the promise of the $R$-learner as a general purpose framework for estimating heterogeneous treatment effects in observational studies.

$^1$Using hold-out prediction for nuisance components, also known as cross-fitting, is an increasingly popular approach for making machine learning methods usable in classical semiparametrics (Athey and Wager, 2017; Chernozhukov et al., 2017; Schick, 1986; van der Laan and Rose, 2011; Wager et al., 2016).
1.1 Related Work

Under unconfoundedness (Assumption 1), the CATE function can be written as

$$\tau^*(x) = \mu_{(1)}^*(x) - \mu_{(0)}^*(x), \quad \mu_{(w)}^*(x) = \mathbb{E}[Y \mid X = x, W = w]. \quad (5)$$

As a consequence of this representation, it may be tempting to first estimate $\hat{\mu}_{(w)}(x)$ separately, and then set $\hat{\tau}(x) = \hat{\mu}_{(1)}(x) - \hat{\mu}_{(0)}(x)$. This approach, however, is often not robust: Because $\hat{\mu}_{(1)}(x)$ and $\hat{\mu}_{(0)}(x)$ are not trained together, their difference may be unstable. As an example, consider fitting the lasso (Tibshirani, 1996) to estimate $\hat{\mu}_{(1)}(x)$ and $\hat{\mu}_{(1)}(x)$ in the following high-dimensional linear model,

$$Y_i(w) = X_i^T \beta^*_w + \epsilon_i(w) \quad \text{with} \quad X_i, \beta^*_w \in \mathbb{R}^d, \quad \text{and} \quad E[\epsilon_i(w) \mid X_i] = 0.$$  

A naive approach would fit two separate lassos to the treated and control samples,

$$\hat{\beta}(w) = \arg\min_{\beta(w)} \left\{ \sum_{i: W_i = w} \left( Y_i - X_i^T \beta(w) \right)^2 + \lambda(w) \| \beta(w) \|_1 \right\}, \quad (6)$$

and then use it to deduce a treatment effect function, $\hat{\tau}(x) = x(\hat{\beta}_{(1)} - \hat{\beta}_{(0)})$. However, the fact that both $\hat{\beta}_{(0)}$ and $\hat{\beta}_{(1)}$ are regularized towards 0 separately may inadvertently regularize the treatment effect estimate $\hat{\beta}_{(1)} - \hat{\beta}_{(0)}$ away from 0, even when $\tau^*(x) = 0$ everywhere. This problem is especially acute when the treated and control samples are of different sizes; see Künzel et al. (2017) for some striking examples.

The recent literature on heterogeneous treatment effect estimation has proposed several ideas on how to avoid such “regularization bias”. In particular, several recent papers have proposed structural changes to various machine learning methods aimed at focusing on accurate estimation of $\tau(\cdot)$ (Athey and Imbens, 2016; Hahn et al., 2017; Imai and Ratkovic, 2013; Powers et al., 2018; Shalit et al., 2017; Su et al., 2009; Wager and Athey, 2017). For example, with the lasso, Imai and Ratkovic (2013) advocate replacing (6) with a single lasso as follows,

$$\left( \hat{b}, \hat{\delta} \right) = \arg\min_{b, \delta} \left\{ \sum_{i=1}^n \left( Y_i - X_i^T b + (W_i - 0.5)X_i^T \delta \right)^2 + \lambda_b \| b \|_1 + \lambda_\delta \| \delta \|_1 \right\}, \quad (7)$$

where then $\hat{\tau}(x) = x^T \hat{\delta}$. This approach always correctly regularizes towards a sparse $\delta$-vector for treatment heterogeneity. The other approaches cited above present variants and improvements of similar ideas in the context of more sophisticated machine learning methods; see, for example, Figure 1 of Shalit et al. (2017) for a neural network architecture designed to highlight treatment effect heterogeneity without being affected by confounders.

Another trend in the literature, closer to our paper, has focused on meta-learning approaches that are not closely tied to any specific machine learning method. Künzel, Sekhon, Bickel, and Yu (2017) proposed two approaches to heterogeneous treatment effect estimation via generic machine learning methods. One, called the X-learner, first estimates $\mu_{(w)}(x)$ via appropriate non-parametric regression methods. Then, on the treated observations, it defines pseudo-effects $D_i = Y_i - \mu_{(w)}(X_i)$, and uses them to fit $\hat{\tau}_{(1)}(X_i)$ via a non-parametric regression. Another estimator $\hat{\tau}_{(0)}(X_i)$ is obtained analogously (see Künzel et al. (2017) for details), and the two treatment effect estimators are aggregated as

$$\hat{\tau}(x) = (1 - \hat{\epsilon}(x))\hat{\tau}_{(1)}(x) + \hat{\epsilon}(x)\hat{\tau}_{(0)}(x). \quad (8)$$
Another method, called the $U$-learner, starts by noticing that

$$E \left[ U_i \mid X_i = x \right] = \tau(x), \quad U_i = \frac{Y_i - m^*(X_i)}{W_i - e^*(X_i)},$$

and then fitting $U_i$ on $X_i$ using any off-the-shelf method. Athey and Imbens (2016) propose a “transformed outcome” approach that involves regressing $Y_i(W_i - e^*(X_i))/(e^*(X_i)(1 - e^*(X_i)))$ on $X_i$ to estimate $\tau^*(x)$.

In our experiments, we compare our method at length to those of Küenzel et al. (2017). Relative to this line of work, our main contribution is our method, the $R$-learner, which provides meaningful improvements over baselines in a variety of settings, and our analysis, which provides the first “quasi-oracle” regret bound we are aware of for non-parametric regression, i.e., where the regret of $\hat{\tau}$ may decay faster than that of $\hat{e}$ or $\hat{m}$.

Conceptually, we draw from the literature on semiparametric efficiency and constructions of orthogonal moments including Robinson (1988) and, more broadly, Belloni et al. (2017), Bickel et al. (1998), Newey (1994), Robins (2004), Robins and Rotnitzky (1995), Robins et al. (2017), Tsiatis (2007), van der Laan and Rose (2011), etc., that aim at $\sqrt{n}$-rate estimation of a target parameter in the presence of nuisance components that cannot be estimated at a $\sqrt{n}$ rate. Algorithmically, our approach has a close connection to targeted maximum likelihood estimation (Scharfstein et al., 1999; Van Der Laan and Rubin, 2006), which starts by estimating nuisance components non-parametrically, and then uses these first stage estimates to define a likelihood function that is optimized in a second step.

The main difference between this literature and our results is that existing results typically focus on estimating a single (or low-dimensional) target parameter, whereas we seek to estimate an object $\tau^*(\cdot)$ that may also be quite complicated itself. Another research direction that also use ideas from semiparametrics to estimate complex objects is centered on estimating optimal treatment allocation rules (Athey and Wager, 2017; Dudík et al., 2011; Luedtke and van der Laan, 2016; Zhang et al., 2012; Zhao et al., 2012). This problem is closely related to, but subtly different from the problem of estimating $\tau^*(\cdot)$ under squared-error loss; see Kitagawa and Tetenov (2018), Manski (2004) and Murphy (2005) for a discussion. We also note the work of van der Laan, Dudoit, and van der Vaart (2006), who consider non-parametric estimation by empirical minimization over a discrete grid.

2 A Motivating Example

To see how the $R$-learner works in practice, we consider an example motivated by Arceneaux, Gerber, and Green (2006), who studied the effect of paid get-out-the-vote calls on voter turnout. A common difficulty in comparing the accuracy of heterogeneous treatment effect estimators on real data is that we do not have access to the ground truth. From this perspective, a major advantage of this application is that Arceneaux et al. (2006) found no effect of get-out-the-vote calls on voter turnout, and so we know what the correct answer is. We then “spike” the original dataset with a synthetic treatment effect $\tau^*(\cdot)$ such as to make the task of estimating heterogeneous treatment effects non-trivial. In other words, both the baseline signal and propensity scores are from real data; however, $\tau^*(\cdot)$ is chosen by us, and so we can check whether different methods in fact succeed in recovering it.

The dataset of Arceneaux et al. (2006) has many covariates that are highly predictive of turnout, and the original study assigned people to the treatment and control condition
with variable probabilities, resulting in a non-negligible amount of confounding. A naive analysis (without correcting for variable treatment propensities) estimates the average intent to treat effect of a single get-out-the-vote call on turnout as 4%; however, an appropriate analysis finds with high confidence that any treatment effect must be smaller than 1% in absolute value. The full sample has \( n = 1,895,468 \) observations, of which \( n_1 = 59,264 \) were treated. We focus on \( p = 11 \) covariates (including state, county, age, gender, etc.). Both the outcome \( Y \) and treatment \( W \) are binary.

As discussed above, we assume that the treatment effect in the original data is 0, and spike in a synthetic treatment effect \( \tau^*(X_i) = -\text{VOTE00}_i/(2 + 100/\text{AGE}_i) \), where \( \text{VOTE00}_i \) indicates whether the \( i \)-th unit voted in the year 2000, and \( \text{AGE}_i \) is their age. Because the outcomes are binary, we add in the synthetic treatment effect by strategically flipping some outcome labels. As is typical in causal inference applications, the treatment heterogeneity here is quite subtle, with \( \text{Var}[\tau^*(X)] = 0.016 \), and so a large sample size is needed in order to reject a null hypothesis of no treatment heterogeneity. For our analysis, we focus on a subset of 148,160 samples containing all the treated units and a random subset of the controls (thus, 2/5 of our analysis sample was treated). We further divide this sample into a training set of size 100,000, a test set of size 25,000, and a holdout set with the rest.

To use the R-learner, we first estimate \( \hat{e}(\cdot) \) and \( \hat{m}(\cdot) \) to form the R-loss function in (4). To do so, we fit models for the nuisance components via both boosting and the lasso (both with tuning parameters selected via cross-validation), and pick the model that minimized cross-validated error. Perhaps unsurprisingly noting the large sample size, this criterion leads us to choose boosting for both \( \hat{e}(\cdot) \) and \( \hat{m}(\cdot) \). Another option would have been to try to combine predictions from the lasso and boosting models, as advocated by Van der Laan, Polley, and Hubbard (2007).

Next, we optimize the R-loss function. We again try methods based on both the lasso and boosting. This time, the lasso achieves a slightly lower training set cross-validated R-loss than boosting, namely 0.1816 versus 0.1818. Because treatment effects are so weak (and so there is potential to overfit even in cross-validation), we have also examined R-loss on the holdout set. The lasso again comes out ahead on the holdout set, and the improvement in R-loss is stable, 0.1781 versus 0.1783. We thus choose the lasso-based \( \hat{\tau}(\cdot) \) fit as our final model for \( \tau^*(\cdot) \).

Because we know the true CATE function \( \tau^*(\cdot) \) in our semi-synthetic data generative distribution, we can evaluate the oracle test set mean-squared error, \( 1/n_{\text{test}} \sum_{i \in \text{test}} (\hat{\tau}(X_i) - \tau^*(X_i))^2 \). Here, it is clear that the lasso does substantially better than boosting, achieving a mean-squared error of \( 0.47 \times 10^{-3} \) versus \( 1.23 \times 10^{-3} \) that corresponds to a factor 2.6 improvement in efficiency. The right panel of Figure 1 compares \( \hat{\tau}(\cdot) \)
In order to illustrate formal aspects of the $R$-learner, we study a variant of (4) based on penalized kernel regression. The problem of regularized kernel learning covers a broad class of methods that have been thoroughly studied in the statistical learning literature (see, e.g., Bartlett and Mendelson, 2006; Caponnetto and De Vito, 2007; Cucker and Smale, 2002; Steinwart and Christmann, 2008; Mendelson and Neeman, 2010), and thus provides an ideal case study for highlighting the promise of the $R$-learner. Throughout this section, we compares our approach to both the single lasso approach (7), and a popular non-parametric approach to heterogeneous treatment effect estimation via BART (Hill, 2011), with the estimated propensity score added in as a feature following the recommendation of Hahn, Murray, and Carvalho (2017). The single lasso got an oracle test set error of $0.61 \times 10^{-3}$, whereas BART got $4.05 \times 10^{-3}$. It thus appears that, in this example, there is value in using a non-parametric method for estimating $\hat{\epsilon}(\cdot)$ and $\hat{m}(\cdot)$, but then using the simpler lasso for $\hat{\tau}(\cdot)$. In contrast, the single lasso approach uses linear modeling everywhere (thus leading to inefficiencies and potential confounding), whereas BART uses non-parametric modeling everywhere (thus making it difficult to obtain a stable $\tau(\cdot)$ fit). Section 4 has a more comprehensive simulation evaluation of the $R$-learner relative to several baselines, including the meta-learners of Künzel et al. (2017).

3 Asymptotic Theory

In order to illustrate formal aspects of the $R$-learner, we study a variant of (4) based on penalized kernel regression. The problem of regularized kernel learning covers a broad class of methods that have been thoroughly studied in the statistical learning literature (see, e.g., Bartlett and Mendelson, 2006; Caponnetto and De Vito, 2007; Cucker and Smale, 2002; Steinwart and Christmann, 2008; Mendelson and Neeman, 2010), and thus provides an ideal case study for highlighting the promise of the $R$-learner. Throughout this section, we

estimates from minimizing the $R$-loss using the lasso and boosting respectively. We see that the lasso is somewhat biased, but boosting is noisy, and the bias-variance trade-off favors the lasso. With a larger sample size, we’d expect boosting to prevail.

We also compared our approach to both the single lasso approach (7), and a popular non-parametric approach to heterogeneous treatment effect estimation via BART (Hill, 2011), with the estimated propensity score added in as a feature following the recommendation of Hahn, Murray, and Carvalho (2017). The single lasso got an oracle test set error of $0.61 \times 10^{-3}$, whereas BART got $4.05 \times 10^{-3}$. It thus appears that, in this example, there is value in using a non-parametric method for estimating $\hat{\epsilon}(\cdot)$ and $\hat{m}(\cdot)$, but then using the simpler lasso for $\hat{\tau}(\cdot)$. In contrast, the single lasso approach uses linear modeling everywhere (thus leading to inefficiencies and potential confounding), whereas BART uses non-parametric modeling everywhere (thus making it difficult to obtain a stable $\tau(\cdot)$ fit). Section 4 has a more comprehensive simulation evaluation of the $R$-learner relative to several baselines, including the meta-learners of Künzel et al. (2017).
will seek to derive performance guarantees for the $R$-learner that match the best available guarantees for the infeasible oracle learner (3). We conduct our analysis in the following setting.

We study $\| \cdot \|_{\mathcal{H}}$-penalized kernel regression, where $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS) with a continuous, positive semi-definite kernel function $\mathcal{K}$. Let $\mathcal{P}$ be a non-negative measure over the compact metric space $\mathcal{X} \subset \mathbb{R}^d$, and let $\mathcal{K}$ be a kernel with respect to $\mathcal{P}$. Let $T_{\mathcal{K}} : L_2(\mathcal{P}) \to L_2(\mathcal{P})$ be defined as $T_{\mathcal{K}}(f)(\cdot) = \mathbb{E} [\mathcal{K}(\cdot, X)f(X)]$. By Mercer’s theorem (Cucker and Smale, 2002), there is an orthonormal basis of eigenfunctions $(\psi_j)_{j=1}^{\infty}$ of $T_{\mathcal{K}}$ with corresponding eigenvalues $(\sigma_j)_{j=1}^{\infty}$ such that

$$\mathcal{K}(x, y) = \sum_{j=1}^{\infty} \sigma_j \psi_j(x) \psi_j(y).$$

Consider the function $\phi : \mathcal{X} \to l_2$ defined by $\phi(x) = (\sqrt{\sigma_j} \psi_j(x))_{j=1}^{\infty}$. Following Mendelson and Neeman (2010), we define the RKHS $\mathcal{H}$ to be the image of $l_2$: For every $t \in l_2$, define the corresponding element in $\mathcal{H}$ by $f_t(x) = \langle \phi(x), t \rangle$ with the induced inner product $\langle f_s, f_t \rangle_{\mathcal{H}} = \langle s, t \rangle$.

**Assumption 2.** Without loss of generality, we assume $\mathcal{K}(x, x) \leq 1$ for all $x \in \mathcal{X}$. We assume that the eigenvalues $\sigma_j$ satisfy $G = \sup_{j \geq 1} j^{1/p} \sigma_j$ for some constant $G < \infty$, and that the orthonormal eigenfunctions $\psi_j(\cdot)$ with $\|\psi_j\|_{L_2(\mathcal{P})} = 1$ are uniformly bounded, i.e., $\sup_j \|\psi_j\|_{\infty} \leq A < \infty$. Finally, we assume that the outcomes $Y_i$ are almost surely bounded, $|Y_i| \leq M$.

**Assumption 3.** The oracle CATE function $\tau^*(x) = \mathbb{E} [Y_i(1) - Y_i(0) \mid X_i = x]$ satisfies $\|T_{\mathcal{K}}(\tau^*(\cdot))\|_{\mathcal{H}} < \infty$ for some $0 < \alpha < 1/2$.

We emphasize that, above, we do not assume that $\tau^*(\cdot)$ has a finite $\mathcal{H}$-norm; rather, we only assume that we can make it have a finite $\mathcal{H}$-norm after a sufficient amount of smoothing. More concretely, with $\alpha = 0$, $T_{\mathcal{K}}^0$ would be the identity operator, and so this assumption would be equivalent to the strongest possible assumption that $\|\tau^*(\cdot)\|_{\mathcal{H}} < \infty$ itself. Then, as $\alpha$ grows, this assumption gets progressively weaker, and at $\alpha = 1/2$ it devolves to simply asking that $\tau^*(\cdot)$ belong to the space $L_2(\mathcal{P})$ of square-integrable functions.\(^5\)

Given this setup, we study oracle penalized regression rules of the following form,

$$\hat{\tau}(\cdot) = \arg\min_{\tau \in \mathcal{H}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( (Y_i - m^*(X_i)) - (W_i - e^*(X_i)) \tau(X_i) \right)^2 + \Lambda_n (\|\tau\|_{\mathcal{H}}) : \|\tau\|_{\infty} \leq 2M \right\}, \tag{10}$$

\(^5\)To see this, let $f(x) = \sum_{j=1}^{\infty} \sqrt{\sigma_j} \psi_j(x) t_j$ for some $t \in l_2$, in which case $\|f\|_{L_2(\mathcal{P})} = \sum_{j=1}^{\infty} \sigma_j t_j^2$. We also note that by taking $\phi_j(x) = f_{e_j}(x) = (\phi(x), e_j)$ where $e_j \in l_2$ is 1 at the $j$-th position and 0 otherwise, we have $\|\phi_j\|_{\mathcal{H}} = (\langle e_j, e_j \rangle)^{1/2} = (\langle e_j, e_j \rangle)^{1/2} = 1$. Then,

$$\|T_{\mathcal{K}}^{1/2}(f)\|_{\mathcal{H}} = \| \sum_{j=1}^{\infty} \sigma_j^{1/2} \sqrt{\sigma_j} \psi_j(x) t_j \|_{\mathcal{H}} = \| \sum_{j=1}^{\infty} \sigma_j^{1/2} \phi_j t_j \|_{\mathcal{H}} = \sum_{j=1}^{\infty} \sigma_j^{1/2} t_j^2,$$

and so $\|T_{\mathcal{K}}^{1/2}(f)\|_{\mathcal{H}} = \|f\|_{L_2(\mathcal{P})}$ for all $f \in L_2(\mathcal{P})$. 


as well as feasible analogues obtained by cross-fitting (Chernozhukov et al., 2017; Schick, 1986):

\[
\hat{\tau}(\cdot) = \arg\min_{\tau \in \mathcal{H}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( Y_i - \hat{m}^{(-q(i))}(X_i) \right) - \left( W_i - \hat{e}^{(-q(i))}(X_i) \right) \tau(X_i) \right\}^2 + A_n(\|\tau\|_H): \|\tau\|_\infty \leq 2M, \tag{11}\n\]

where \( q \) is a mapping from the \( i = 1, \ldots, n \) sample indices to \( Q \) evenly sized data folds, such that \( \hat{e}^{(-q(i))}(x) \) and \( \hat{m}^{(-q(i))}(x) \) are each trained without considering observations in the \( q(i) \)-th data fold; typically we set \( Q \) to 5 or 10. Adding the upper bound \( \|\tau\|_\infty \leq 2M \) (or, in fact, any finite upper bound on \( \tau \)) enables us to rule out some pathological behaviors.

We seek to characterize the accuracy of our estimator \( \hat{\tau}(\cdot) \) by bounding its regret \( R(\hat{\tau}) \),

\[
R(\tau) = L(\tau) - L(\tau^*), \quad L(\tau) = \mathbb{E} \left[ \left( Y_i - m^*(X_i) - \tau(X_i) (W_i - e^*(X_i)) \right)^2 \right]. \tag{12}
\]

Note that if we have overlap, i.e., there is an \( \eta > 0 \) such that \( \eta < e^*(x) < 1 - \eta \) for all \( x \in \mathcal{X} \), then following from \( R(\tau) = \mathbb{E}[|W_i - e^*(X_i)|^2|\tau(X_i) - \tau^*(X_i)|^2] \), we have

\[
(1 - \eta)^{-2}R(\tau) < \mathbb{E}[|\tau(X_i) - \tau^*(X_i)|^2] < \eta^{-2}R(\tau), \tag{13}
\]

meaning that regret bounds directly translate into squared-error loss bounds for \( \tau(\cdot) \), and vice-versa.

The sharpest regret bounds for (10) given Assumptions 2 and 3 are due to Mendelson and Neeman (2010) (see also Steinwart, Hush, and Scovel (2009)), and scale as

\[
R(\hat{\tau}) = \tilde{O} \left( n^{-\frac{1+2a}{p+1+2a}} \right), \tag{14}
\]

where the \( \tilde{O} \)-notation hides logarithmic factors. In the case \( \alpha = 0 \) where \( \tau^* \) is within the RKHS used for penalization, we recover the more familiar \( n^{-1/\|H\|} \) rate established by Caponnetto and De Vito (2007). Again, our goal is to establish regret bounds for our feasible estimator \( \hat{\tau} \) that match those for \( \hat{\tau} \).

### 3.1 Fast Rates and Isomorphic Coordinate Projections

In order to establish regret bounds for \( \hat{\tau} \), we first need to briefly review the proof techniques underlying (14). The argument of Mendelson and Neeman (2010) relies on the following quasi-isomorphic coordinate projection lemma of Bartlett (2008). To state this result, write

\[
\mathcal{H}_c = \{ \tau : \|\tau\|_H \leq c, \|\tau\|_\infty \leq 2M \} \tag{15}
\]

for the radius-\( c \) ball of \( \mathcal{H} \) capped by \( 2M \), let \( \tau^*_c = \arg\min \{ L(\tau) : \tau \in \mathcal{H}_c \} \) denote the best approximation to \( \tau^* \) within \( \mathcal{H}_c \), and define \( c \)-regret \( R(\tau; c) = L(\tau) - L(\tau^*_c) \) over \( \tau \in \mathcal{H}_c \). We also define the estimated and oracle \( c \)-regret functions

\[
\hat{R}_n(\tau; c) = \hat{L}_n(\tau) - \hat{L}_n(\tau^*_c), \quad \tilde{R}_n(\tau; c) = \bar{L}_n(\tau) - \bar{L}_n(\tau^*_c), \tag{16}
\]
where
\[ \tilde{L}_n(\tau) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - m^{\ast}(X_i) - \tau(X_i)(W_i - c^{\ast}(X_i)))^2 \] (17)

is the oracle loss function on the samples used for empirical minimization, and
\[ \hat{L}_n(\tau) = \frac{1}{n} \sum_{i=1}^{n} \left( Y_i - \hat{m}^{(-\eta(i))}(X_i) - \tau(X_i) \left( W_i - \hat{c}^{(-\eta(i))(X_i)} \right) \right)^2 \] (18)

is the feasible cross-fitted loss function. \( \hat{R}_n(\tau; c) \) is not actually observable in practice as it depends on \( \tau^\ast_c \); however, this does not hinder us from establishing high-probability bounds for it. The lemma below is adapted from Bartlett (2008).

**Lemma 1.** Let \( \hat{L}_n(\tau) \) be any loss function, and \( \hat{R}_n(\tau; c) = \hat{L}_n(\tau) - \hat{L}_n(\tau^\ast_c) \) be the associated regret. Let \( \rho_n(c) \) be a continuous positive function that is increasing in \( c \). Suppose that, for every \( 1 \leq c \leq C \) and some \( k > 1 \), the following inequality holds:
\[ \frac{1}{k} \hat{R}_n(\tau; c) - \rho_n(c) \leq R(\tau; c) \leq k\hat{R}_n(\tau; c) + \rho_n(c) \quad \text{for all } \tau \in \mathcal{H}_c. \] (19)

Then, writing \( \kappa_1 = 2k + \frac{1}{k} \) and \( \kappa_2 = 2k^2 + 3 \), any solution to the empirical minimization problem with regularizer \( \lambda_n(c) \geq \rho_n(c) \),
\[ \hat{\tau} \in \arg\min_{\tau \in \mathcal{H}_c} \{ L(\tau) + \kappa_1\lambda_n(\|\tau\|_{\mathcal{H}}) \}, \] (20)

also satisfies the following risk bound:
\[ L(\hat{\tau}) \leq \inf_{\tau \in \mathcal{H}_C} \{ L(\tau) + \kappa_2\lambda_n(\|\tau\|_{\mathcal{H}}) \}. \] (21)

In other words, the above lemma reduces the problem of deriving regret bounds to establishing quasi-isomorphisms as in (19) (and any with-high-probability quasi-isomorphism guarantee yields a with-high-probability regret bound). In order to prove a regret bound (14) following the approach of Mendelson and Neeman (2010), we first establish a quasi-isomorphism in terms of the oracle empirical regret,
\[ \frac{1}{k} \hat{R}_n(\tau; c) - \rho_n(c) \leq R(\tau; c) \leq k\hat{R}_n(\tau; c) + \rho_n(c), \] (22)

and then get a regret bound in terms of the regularized risk as in (21). Mendelson and Neeman (2010) then conclude their argument by noting that, for any \( 1 \leq c \leq C \) (see their Corollary 2.7 for details),
\[ \inf_{\tau \in \mathcal{H}_C} \{ L(\tau) + \kappa_2\rho_n(\|\tau\|_{\mathcal{H}}) \} \leq L(\tau^\ast) + (L(\tau^\ast_c) - L(\tau^\ast)) + \kappa_2\rho_n(c), \] (23)

and that by optimizing this bound (up to logarithmic factors) with \( C = \infty \) and a specific choice of \( \rho_n(\cdot) \), and by setting \( c \) to be \( c_n = n^{\alpha/(p+1-2\alpha)} \), we recover the regret bound (14) for the oracle learner. Their argument for choosing the optimal value of \( c_n \) to use in (23) leverages the approximation error bound
\[ \|\tau^\ast_c - \tau^\ast\|_{L_2(P)}^2 \leq \eta^{-2}c^{2\alpha-1/2} \|T^\alpha_K(\tau^\ast(\cdot))\|_{\mathcal{H}}^{1/\alpha} \] (24)

\(^6\)Corollary 2.7 in Mendelson and Neeman (2010) proves (23) when \( C = \infty \). To show (23) with the extra constraint \( c \leq C \) for \( C \leq \infty \), it directly follows from the original proof with the added constraint.
established by Smale and Zhou (2003) given Assumption 3.

For our purposes, the upshot is that if we can match the strength of the quasi-isomorphism bounds (22) with our feasible loss function, then we can also match the rate of any regret bounds proved using Lemma 1. We do so via the following result.

Lemma 2. Given the conditions in Lemma 1, suppose that the propensity estimate \( \hat{e}(x) \) is uniformly consistent,

\[
\xi_n := \sup_{x \in X} |\hat{e}(x) - e^*(x)| \rightarrow_p 0, \tag{25}
\]

and the \( L_2 \) errors converge at rate

\[
\mathbb{E} \left[ (\hat{m}(X) - m^*(X))^2 \right], \mathbb{E} \left[ (\hat{e}(X) - e^*(X))^2 \right] = \mathcal{O}(a_n^2) \tag{26}
\]

for some sequence \( a_n \) such that

\[
a_n = \mathcal{O}(n^{-\kappa}) \text{ with } \kappa > \frac{1}{4}. \tag{27}
\]

Suppose, moreover, that we have overlap, i.e., \( \eta < e^*(x) < 1 - \eta \) for some \( \eta > 0 \), and that Assumptions 2 and 3 hold. Then, for any \( \varepsilon > 0 \), there is a constant \( U(\varepsilon) \) such that the regret functions induced by the oracle learner (10) and the feasible learner (11) are coupled as

\[
\left| \hat{R}_n(\tau; c) - \tilde{R}_n(\tau; c) \right| \\
\leq U(\varepsilon) \left( c^p R(\tau; c)^{1-\frac{2}{\alpha}} a_n^2 + c^{2p} R(\tau; c)^{1-p} \frac{1}{\sqrt{n}} \log(n) + c^{2p} R(\tau; c)^{1-p} \frac{1}{n} \log \left( \frac{cn^{1-p}}{R(\tau; c)} \right) \right) \\
+ c^{p} R(\tau; c)^{1-\frac{2}{\alpha}} \frac{1}{\sqrt{n}} \log \left( \frac{cn^{1-p}}{R(\tau; c)} \right) + c^p R(\tau; c)^{1-\frac{2}{\alpha}} a_n \frac{1}{\sqrt{n}} \log \left( \frac{cn^{1-p}}{R(\tau; c)} \right) \\
+ \xi_n R(\tau; c), \tag{28}
\]

simultaneously for all \( 1 \leq c \leq c_n \log(n) \) with \( c_n = n^{-\frac{1}{1-2\alpha}} \) and \( \tau \in \mathcal{H}_c \), with probability at least \( 1 - \varepsilon \).

This result implies that we can turn any quasi-isomorphism for the oracle learner (22) with error \( \rho_n(c) \) into a quasi-isomorphism bound for \( \hat{R}(\tau) \) with error inflated by (28). Thus, given any regret bound for the oracle learner built using Lemma 1, we can also get an analogous regret bound for the feasible learner provided we regularize just a little bit more. The following result makes this formal.

Theorem 3. Given the conditions of Lemma 2 and that \( 2\alpha < 1 - p \), suppose that we obtain \( \hat{\tau}(\cdot) \) via a penalized kernel regression variant of the \( \mathcal{R} \)-learner (11), with a properly chosen penalty of the form \( \Lambda_n(||\hat{\tau}||_H) \) specified in the proof. Then \( \hat{\tau}(\cdot) \) satisfies the same regret bound (14) as \( \tilde{\tau}(\cdot) \), i.e.,

\[
R(\hat{\tau}) = \tilde{\mathcal{O}}_p \left( n^{-(1-2\alpha)/(p+(1-2\alpha))} \right). \tag{29}
\]
In other words, we have found that with penalized kernel regression, the R-learner can match the best available performance guarantees available for the oracle learner (10) that knows everything about the data generating distribution except the true treatment effect function—both the feasible and the oracle learner satisfy

\[ R(\hat{\tau}), R(\tilde{\tau}) = \tilde{O}(r_n^2), \quad \text{with} \quad r_n = n^{-\frac{1}{2} \frac{1-2\alpha}{p+1-2\alpha}}. \]  \hspace{1cm} (30)

As we approach the semiparametric case, i.e., \( \alpha, p \to 0 \), we recover the well-known result from the semiparametric inference literature that, in order to get \( 1/\sqrt{n} \)-consistent inference for a single target parameter, we need 4-th root consistent nuisance parameter estimates (see Robinson (1988), Chernozhukov et al. (2017), and references therein).

We emphasize that our quasi-oracle result depends on a local robustness property of the \( R \)-loss function, and does not hold for general meta-learners; for example, it does not hold for the X-learner of Künnzel et al. (2017). To see this, pick \( \varepsilon > 0 \) such that \( 0.25 + \varepsilon < (1-2\alpha)/(2(p+(1-2\alpha)) \), and modify the nuisance components used to form the X-learner in (8) such that \( \hat{\mu}_0(x) \leftarrow \hat{\mu}_0(x) - c/n^{0.25+\varepsilon} \) and \( \hat{\mu}_1(x) \leftarrow \hat{\mu}_1(x) + c/n^{0.25+\varepsilon} \). Recall that the X-learner fits \( \hat{\tau}_1(x) \) by minimizing \( n^{-1} \sum_{i=1}^{n} (Y_i - \hat{\mu}_0(x)_i - \tau_1(x)_i)^2 \), and fits \( \hat{\tau}_0(x) \) by solving an analogous problem on the controls. Combining the \( \hat{\tau}_1(x) \) estimates from these two loss functions, we see by inspection that its final estimate of the treatment effect is also shifted by \( \hat{\tau}(x) \leftarrow \hat{\tau}(x) + c/n^{0.25+\varepsilon} \). The perturbations \( c/n^{0.25+\varepsilon} \) are vanishingly small on the \( n^{-1/4} \) scale, and so would not affect conditions analogous to those of Theorem 3; yet they have a big enough effect on \( \hat{\tau}(x) \) to break any convergence results on the scale of (30).

In other words, while the R-learner can achieve the oracle regret bound with its nuisance parameters \( \hat{\epsilon} \) and \( \hat{\mu}_0 \) learned at a slower rate (by Theorem 3), the X-learner’s regret bound is capped at the rate at which its nuisance parameters \( \hat{\mu}_1 \) and \( \hat{\mu}_0 \) are estimated, and thus does not exhibit the same kind of quasi-oracle property.\(^7\)

4 Simulation Experiments

As discussed several times already, our approach to heterogeneous treatment effect estimation via learning objectives can be implemented using any method that is framed as a loss minimization problem, such as boosting, decision trees, etc. In this section, we focus on simulation experiments using the \textbf{R-learner}, a direct implementation of (4) based on both the lasso and boosting.

We consider the following methods for heterogeneous treatment effect estimation as baselines. The \textbf{S-learner} fits a single model for \( f(x, w) = \mathbb{E}[Y | X = x, W = w] \), and then estimates \( \hat{\tau}(x) = \hat{f}(x, 1) - \hat{f}(x, 0) \); the \textbf{T-learner} fits the functions \( \mu^*_w(x) = \mathbb{E}[Y | X = x, W = w] \) separately for \( w \in \{0, 1\} \), and then estimates \( \hat{\tau}(x) = \hat{\mu}_1(x) - \hat{\mu}_0(x) \); the \textbf{X-learner} and \textbf{U-learner} are as described in Section 1.1.\(^8\)

\(^7\)Künnzel et al. (2017) do have some quasi-oracle type results; however, they only focus on the case where the number of control units \( |\{W_i = 0\}| \) grows much faster than the number of treated units \( |\{W_i = 1\}| \). In this case, they show that the X-learner performs as well as an oracle who already knew the mean response function for the controls, \( \mu^*_{0}(x) = \mathbb{E}[Y_i(0) | X_i = x] \). Intriguingly, in this special case, we have \( m^*(x) \approx \hat{\mu}_0^*(x) \) and \( e^*(x) \approx 0 \), and so the R-learner as in (11) is roughly equivalent to the X-learner procedure (8). Thus, at least qualitatively, we can interpret the result of Künnzel et al. (2017) as a special case of our result in the case where the number of controls dominates the number of treated units (or vice-versa).

\(^8\)The \( S^*, T^*, X^*, \) and \( U^* \)-learners are named following the nomenclature of Künnzel et al. (2017).
In addition, for the boosting-based experiments, we consider the causal boosting algorithm (denoted by CB in Section 4.2) proposed by Powers et al. (2018).

Finally, for the lasso-based experiments, we consider an additional variant of our method, the RS-learner, that combines the spirit of R- and S-learners by adding an additional term in the loss function: using \( \hat{\tau}(x) = x^\top \delta \) with

\[
\left( \hat{b}, \hat{\delta} \right) = \arg\min_{b, \delta} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( Y_i - \hat{m}^{(-i)}(X_i) - X_i^\top b \right. \\
- \left. \left( W_i - \hat{e}^{(-i)}(X_i) \right) X_i^\top \delta \right) + \lambda (\|b\|_1 + \|\delta\|_1) \right\}. \tag{31}
\]

Heuristically, one may hope that the RS-learner may be more robust, as it has a “second chance” to eliminate confounders.

In all simulations, we generate data as follows: for different choices of \( X \)-distribution \( P_d \) indexed by dimension \( d \), noise level \( \sigma \), propensity function \( e^*(\cdot) \), baseline main effect \( b^*(\cdot) \), and treatment effect function \( \tau^*(\cdot) \):

\[
\begin{align*}
X_i & \sim P_d. \quad W_i \mid X_i \sim \text{Bernoulli}(e^*(X_i)), \quad \varepsilon_i \mid X_i \sim \mathcal{N}(0, 1), \\
Y_i & = b^*(X_i) + (W_i - 0.5)\tau^*(X_i) + \sigma \varepsilon_i. \tag{32}
\end{align*}
\]

We consider the following specific setups.

**Setup A** Difficult nuisance components and an easy treatment effect function. We use the scaled Friedman (1991) function for the baseline main effect \( b^*(X_i) = \sin(\pi X_{i1} X_{i2}) + 2(X_{i3} - 0.5)^2 + X_{i4} + 0.5 X_{i5} \), along with \( X_i \sim \text{Unif}(0, 1)^d \), \( e^*(X_i) = \text{trim}_{0.1}(\sin(\pi X_{i1} X_{i2})) \) and \( \tau^*(X_i) = (X_{i1} + X_{i2})/2 \), where \( \text{trim}_\eta(x) = \max\{\eta, \min(x, 1 - \eta)\} \).

**Setup B** Randomized trial. Here, \( e^*(x) = 1/2 \) for all \( x \in \mathbb{R}^d \), so it is possible to be accurate without explicitly controlling for confounding. We take \( X_i \sim \mathcal{N}(0, I_{d\times d}) \), \( \tau^*(X_i) = X_{i1} + \log(1 + e^{X_{i2}}) \), and \( b^*(X_i) = \max\{X_{i1} + X_{i2}, X_{i3}, 0\} + \max\{X_{i4} + X_{i5}, 0\} \).

**Setup C** Easy propensity score and a difficult baseline. In this setup, there is strong confounding, but the propensity score is much easier to estimate than the baseline: \( X_i \sim \mathcal{N}(0, I_{d\times d}) \), \( e^*(X_i) = 1/(1 + e^{-X_{i1} + X_{i2}}) \), \( b^*(X_i) = 2\log(1 + e^{X_{i1} + X_{i2} + X_{i5}}) \), and the treatment effect is constant, \( \tau^*(X_i) = 1 \).

**Setup D** Unrelated treatment and control arms, with data generated as \( X_i \sim \mathcal{N}(0, I_{d\times d}) \), \( e^*(X_i) = 1/(1 + e^{-X_{i1} + e^{-X_{i2}}}) \), \( \tau^*(X_i) = \max\{X_{i1} + X_{i2} + X_{i3}, 0\} - \max\{X_{i4} + X_{i5}, 0\} \), and \( b^*(X_i) = \max\{X_{i1} + X_{i2} + X_{i3}, 0\} + \max\{X_{i4} + X_{i5}, 0\} \).

Here, \( \mu^*_0(X) \) and \( \mu^*_1(X) \) are uncorrelated, and so there is no upside to learning them jointly.

### 4.1 Lasso-based experiments

In this section, we compare \( S-, T-, X-, U- \), and our \( R- \) and \( RS- \) learners implemented via the lasso on simulated designs. For the \( S \)-learner, we follow Inai and Ratkovic (2013) using (7),
while for the $T$-learner, we use (6). For the $X_-$, $R_-$, and RS-learners, we use $L_1$-penalized logistic regression to estimate propensity $\hat{e}$, and the lasso for all other regression estimates.

For all estimators, we run the lasso on the pairwise interactions of a natural spline basis expansion with 7 degrees of freedom on $X_i$. We generate $n$ data points as the training set and generate a separate test set also with $n$ data points, and the reported mean-squared error is on the test set. All lasso regularization penalties are chosen from 10-fold cross validation. For the $R_-$ and RS-learners, we use 10-fold cross-fitting on $\hat{e}$ and $\hat{m}$ in (4). All methods are implemented via glmnet (Friedman et al., 2010).\footnote{The $U$-learner suffers from high variance and instability due to dividing by the propensity estimates. Therefore, we set a cutoff for the propensity estimate to be at level 0.05. We have also found empirically $U$-learner achieves much lower estimation error if we choose to use the largest regularization parameter that achieves 1 standard error away from the minimum in the cross validation step. Therefore, the $U$-learner uses lambda.ise as its cross validation parameter, while all other learners use lambda.min in glmnet.}

In Figure 2, we compare the performance of our 6 considered methods to an oracle that runs the lasso on (3), for different values of sample size $n$, dimension $d$, and noise level $\sigma$. As is clear from these illustrations, the considered simulation settings differ vastly in difficulty, both in terms of the accuracy of the oracle, and in terms of the ability of feasible methods to approach the oracle. A full list of specifications considered along with all numbers depicted in Figure 2 is available in Appendix B.

In Setups $A$ and $C$, where there is complicated confounding that needs to be overcome before we can estimate a simple treatment effect function $\tau^*(\cdot)$, the $R_-$ and RS-learners stand out. All methods do reasonably well in the randomized trial (Setup $B$) where it was not necessary to adjust for confounding (the $X_-$, $S_-$, and $R$-learners do best). Finally, having completely disjoint functions for the treated and control arms is unusual in practice. However, we consider this possibility in Setup $D$, where there is no reason to model $\mu^*_{(0)}(x)$ and $\mu^*_{(1)}(x)$ jointly, and find that the $T$-learner—which in fact models them separately—performs well.

Overall, the $R_-$ and RS-learner consistently achieve good performance and, in most simulation specifications, essentially match the performance of the oracle (3) in terms of the mean-squared error. The $U$-learner suffers from high loss due to its instability.

### 4.2 Gradient boosting-based experiments

We move on to compare $S_-$, $T_-$, $X_-$, $U_-$, and $R$-learners implemented via gradient boosting, as well as the causal boosting (CB) algorithm. We use the causallr R package for CB, while all other methods are implemented via XGboost (Chen and Guestrin, 2016). For fitting the objective in each subroutine in all methods, we draw a random set of 10 combinations of hyperparameters from the following grid: subsample= [0.5, 0.75, 1], colsample_bytree= [0.6, 0.8, 1], eta= [5e-3, 1e-2, 1.5e-2, 2.5e-2, 5e-2, 8e-2, 1e-1, 2e-1], max_depth= [3, ..., 20], gamma=Uniform(0, 0.2), min_child_weight= [1, ..., 20], max_delta_step= [1, ..., 10], and cross validate on the number of boosted trees for each combination with an early stopping of 10 iterations. We experiment on the same set of setups and parameter variations (including variations on sample size $n$, dimension $d$, and noise level $\sigma$) as in Section 4.1, and include all numbers depicted in Figure 3 in Appendix B.

In Figure 3, we observe again that $R$-learner stands out in Setup $A$ and $C$, with all methods performing reasonably well in the randomized control setting of Setup $B$; in Setup $D$, $T$-learner performs best since the the treated and control arms are generated from completely different functions.
Figure 2: Performance of lasso-based $S$, $T$, $X$, $U$, $RS$- and $R$-learners, relative to a lasso-based oracle learner (3), across simulation setups described in Section 4. All mean-squared error numbers are aggregated over 500 runs and reported on an independent test set, and are plotted on the logarithmic scale.
Figure 3: Performance of boosting-based $S$-, $T$-, $X$-, $U$-, $R$-learners as well as causal boosting ($CB$), relative to a boosting-based oracle learner (3), across simulation setups described in Section 4. All mean-squared error numbers are aggregated over 200 runs and reported on an independent test set, and are plotted on the logarithmic scale.
Before we conclude this section, we note that in both sets of the experiments, for simplicity of illustration, we have used lasso and boosting respectively to learn \( \hat{m}(\cdot) \) and \( \hat{e}(\cdot) \). In practice, we recommend cross validating on a variety of black-box learners (lasso, random forests, neural networks, etc.) that are tuned for prediction accuracy to learn these two pilot quantities. All simulation results above can be replicated using the rlearner R package.\(^{10}\)

5 Discussion

We introduced the \( R \)-learner, a method for heterogeneous treatment effect estimation in observational studies whose performance guarantees are robust to mild inaccuracies in estimated treatment propensities \( \hat{e}(\cdot) \) and baseline effects \( \hat{m}(\cdot) \). The \( R \)-learner starts by forming a data-adaptive \( R \)-loss function based on nuisance parameter estimates, and then optimizes this loss function with appropriate regularization. Our approach is motivated by the transformation of Robinson (1988), and draws more broadly from the literature on semiparametric inference and constructions of orthogonal moments (Athey and Wager, 2017; Belloni et al., 2017; Bickel et al., 1998; Luedtke and van der Laan, 2016; Newey, 1994; Robins, 2004; Robins and Rotnitzky, 1995; Robins et al., 2017; Scharfstein et al., 1999; Tsiatis, 2007; van der Laan and Rose, 2011). Our main result establishes that, in the case of penalized kernel regression, the \( R \)-learner achieves the same regret bounds as an oracle who knew \( e^*(\cdot) \) \( m^*(\cdot) \) a priori, even if \( \hat{e}(\cdot) \) and \( \hat{m}(\cdot) \) may converge an order of magnitude slower than the target rate for \( \hat{\tau}(\cdot) \).

A natural generalization of our setup arises when, in some applications, we need to work with multiple treatment options. For example, in medicine, we may want to compare a control condition to multiple different experimental treatments. If there are \( k \) different treatments (along with a control arm), we can encode \( W \in \{0,1\}^k \), and note that a multivariate version of Robinson’s transformation suggests the following estimator,

\[
\hat{\tau}(\cdot) = \arg\min_{\tau} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \left( Y_i - \hat{m}(-i)(X_i) \right) - \left( W_i - \hat{e}(-i)(X_i), \tau(X_i) \right) \right)^2 + \Lambda_n(\tau(\cdot)) \right\},
\]

where the angle brackets indicate an inner product, \( e(x) = \mathbb{E}[W \mid X = x] \in \mathbb{R}^k \) is a vector, and \( \tau_l(x) \) measures the conditional average treatment effect of the \( l \)-th treatment arm at \( X_i = x \), for \( l = 1, \ldots, k \). When implementing variants of this approach in practice, different choices of \( \Lambda_n(\tau(\cdot)) \) may be needed to reflect relationships between the treatment effects of different arms (for example, whether there is a natural ordering of treatment arms, or if there are some arms that we believe a priori to have similar effects).

It would also be interesting to consider extensions of the \( R \)-learner to cases where the treatment assignment \( W_i \) is not unconfounded, and we need to rely on an instrument to identify causal effects. Chernozhukov et al. (2017) discusses how Robinson’s approach to the partially linear model generalizes naturally to this case, and Athey, Tibshirani, and Wager (2018) adapt their causal forest to work with instruments. The underlying estimating equations, however, cannot be interpreted as loss functions as easily as (3), especially in the case where instruments may be weak, and so we leave this extension of the \( R \)-learner to future work.

\(^{10}\)https://github.com/xnie/rlearner
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A Proofs

A.1 Preliminaries

A.1.1 A useful inequality relating function norms in RKHS

Before beginning our proof, we present an inequality that we will use frequently. Under Assumption 2, directly following from Lemma 5.1 of Mendelson and Neeman (2010), there is a constant $B$ depending on $A$, $p$ and $G$ such that for all $\tau \in \mathcal{H}$,

$$\|\tau\|_\infty \leq B \|\tau\|_\mathcal{H}^p \|\tau\|_{L^2(P)}^{1-p}.$$  \hfill (34)

If $\eta < e^*(x) < 1 - \eta$ for some $\eta > 0$, a consequence of the above inequality is as follows: for $\tau \in \mathcal{H}_c$,

$$\|\tau - \tau^*_c\|_\infty \leq B \|\tau - \tau^*_c\|_\mathcal{H}^p \|\tau - \tau^*_c\|_{L^2(P)}^{1-p} \leq B 2^p \eta^{-(1-p)} c^p R(\tau; c)^{1-p}. $$ \hfill (35)

We note that the second inequality in (35) follows from combining (13) with the fact that for $\tau \in \mathcal{H}_c$, $\|\tau - \tau^*_c\|_\mathcal{H} \leq 2c$ by the triangle inequality.

A.1.2 Talagrand’s Inequalities

Below we state Talagrand’s Concentration Inequality for an empirical process indexed by a class of uniformly bounded functions (Talagrand, 1994, 1996). The version of the inequality we shall use here is due to Massart (2000).

Let $\mathcal{F}$ be a class of functions defined on $(\Omega, \mathcal{P})$ such that for every $f \in \mathcal{F}$, $\|f\|_\infty \leq b$, and $E[f] = 0$. Let $X_1, \ldots, X_n$ be independent random variables distributed according to $\mathcal{P}$ and set $\sigma^2 = \sup_{f \in \mathcal{F}} E[f^2]$. Define

$$Z = \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} f(X_i) \quad \text{and} \quad \bar{Z} = \sup_{f \in \mathcal{F}} \frac{1}{n} \left| \sum_{i=1}^{n} \epsilon_i f(X_i) \right|.$$ 

Then, there exists an absolute constant $C$ such that for every $t > 0$, and every $\rho > 0$,

$$\mathbb{P} \left[ Z > (1 + \rho)E[Z] + \frac{\sigma}{\sqrt{n}} \sqrt{Ct} + \frac{C}{n} (1 + \rho^{-1})bt \right] \leq e^{-t},\hfill (36)$$

$$\mathbb{P} \left[ Z < (1 - \rho)E[Z] - \frac{\sigma}{\sqrt{n}} \sqrt{Ct} - \frac{C}{n} (1 + \rho^{-1})bt \right] \leq e^{-t},$$ \hfill (37)

and the same inequalities holds for $\bar{Z}$.

We will also make use of the following bound given by Talagrand (Corollary 3.4 in Talagrand (1994)) below:

$$E \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f^2(X_i) \right] \leq n\sigma^2 + 8b^2 E \left[ \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} \epsilon_i f(X_i) \right| \right], \hfill (38)$$

where $\epsilon_i$ are independent Rademacher variables independent of the variables $X_i$. 

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A.2 Technical definitions and lemmas

Before we proceed with the proof of Lemma 2, it is helpful to prove the following results.

Proof of Lemma 1. First, we note that for $1 \leq c \leq C$, $H_c$ defined as $\{\tau \in H, ||\tau||_H \leq c, ||\tau||_\infty \leq 2M\}$ is an ordered set, i.e. $H_c \subseteq H_{c'}$ for $c \leq c'$. Without loss of generality, it suffices to consider $\Lambda_n(\cdot) = \rho_n(\cdot)$ because if (19) holds with $\rho_n(c)$, it also holds with $\rho_n(c)$ replaced by $\Lambda_n(c) \geq \rho_n(c)$. Define $\tau^*_c = \arg\min_{\tau \in H_c} L(\tau)$, $\hat{\tau}_c = \arg\min_{\tau \in H_c} \hat{L}(\tau)$, $\hat{m} = ||\hat{\tau}||_H$.

Following the proof of Theorem 4 in Bartlett (2008), first check the following facts:

- In the event that $c \geq \hat{m}$ (see Lemma 5 of Bartlett (2008)),
  $$L(\hat{\tau}) \leq L(\tau^*_c) + \max\{k\kappa_1 + 2, 3\} \rho_n(c).$$

- In the event that $c \leq \hat{m}$ (see Lemma 6 of Bartlett (2008)),
  $$L(\tau^*_c) \leq L(\tau^*_c) + \left(\frac{1}{k^2} - \frac{\kappa_1}{k} + 1\right) \rho_n(\hat{m}) + \frac{\kappa_1 \rho_n(c)}{k}.$$

- In the event that $c \leq \hat{m}$ (see Lemma 7 of Bartlett (2008)),
  $$L(\hat{\tau}) \leq L(\tau^*_c) + \left(\frac{1}{k^2} - \frac{\kappa_1}{k} + 2\right) \rho_n(\hat{m}) + \frac{\kappa_1 \rho_n(c)}{k}.$$

Now, choosing $\kappa_1 = \frac{1}{k} + 2k$ shows that

$$L(\hat{\tau}) \leq L(\tau^*_c) + \frac{\kappa_1 \rho_n(c)}{k}.$$

Let $\kappa_2 = 2k^2 + 3$, combining the above,

$$L(\hat{\tau}) \leq \inf_{1 \leq c \leq C} L(\tau^*_c) + \kappa_2 \rho_n(c).$$

Finally, for any $\tau = \arg\min_{\tau \in H_c} \{L(\tau) + \kappa_2 \Lambda_n(\tau)\}$, $\tau = \tau^*_c \tau_{\tau||H}$. Suppose not, then $L(\tau) + \kappa_2 \Lambda_n(\tau) > L(\tau^*_c) + \kappa_2 \Lambda_n(\tau_{\tau||H})$, which is a contradiction. Thus, the claim follows.

Definition 1 (Definition 2.4 from Mendelson and Neeman (2010)). Given a class of functions $F$, we say that $\{F_c : c \geq 1\}$ is an ordered, parameterized hierarchy of $F$ if the following conditions are satisfied:

- $\{F_c : c \geq 1\}$ is monotone;
- for every $c \geq 1$, there exists a unique element $f^*_c \in F_c$ such that $L(f^*_c) = \inf_{f \in F_c} L(f)$;
- the map $c \to L(f^*_c)$ is continuous;
Proof. Our proof proceeds by generic chaining. Defining random variables

\[ H \cdot \text{Lemma } 5 \cdot \text{chaining) } \]

\[ \inf_{S} \]

Proof. Here, \( \gamma \)

Lemma 4. \( \mathcal{H}_c := \{ \tau \in \mathcal{H}, \| \tau \|_\mathcal{H} \leq c, \| \tau \|_\infty \leq 2M \} \) is an ordered, parameterized hierarchy of \( \mathcal{H} \).

Proof. First, we show that \( \mathcal{H}_1 \) is compact. Let \( (\tau_n)_n \) be a sequence in \( \mathcal{H}_1 \). Following from the fact that \( B_1 = \{ \tau \in \mathcal{H}, \| \tau \|_\mathcal{H} \leq 1 \} \) is compact with respect to \( L_2 \)-norm, \( \tau_n \) has a converging subsequence \( (\tau_{n_k})_k \) with a limit \( \tau \in B_1 \). For any \( \varepsilon > 0 \), there exists \( K \) such that for all \( k > K, \| \tau_{n_k} - \tau \|_{L_2(\mathcal{P})} \leq \varepsilon \). Suppose \( \| \tau \|_\infty > 2M \), then take \( \tau'(x) = \min(\max(\tau(x), -2M), 2M) \), we see that \( \| \tau_{n_k}(x) - \tau'(x) \|_{L_2(\mathcal{P})} \leq \| \tau_{n_k}(x) - \tau(x) \|_{L_2(\mathcal{P})} \) for all \( k \geq K \). So the limit \( \tau(x) = \tau'(x) \). Thus the subsequence converges to a limit in \( \mathcal{H}_1 \), and so \( \mathcal{H}_1 \) is compact. The proof now follows exactly the proof of Lemma 3.6 in Mendelson and Neeman (2010).

Lemma 5 (chaining). Let \( \mathcal{H} \) be an RKHS with kernel \( K \) satisfying Assumption 2, let \( X_1, ..., X_n \) be \( n \) independent draws from the measure \( \mathcal{P} \), and let \( Z_1, ..., Z_n \) be independent mean-zero sub-Gaussian random variables with variance proxy \( M^2 \), conditionally on the \( X_i \). Then, there is a constant \( B \) such that, for any (potentially random) weighting function \( \omega(x) \),

\[
\mathbb{E} \left[ \sup_{h \in \mathcal{H}_{c, \delta}} \left\{ \frac{1}{n} \sum_{i=1}^{n} Z_i \omega(X_i) h(X_i) \right\} \right] \leq B M c^p \delta^{-p} \mathbb{E} \left[ \omega^2(X) \right]^{\frac{1}{2}} \frac{\log(n)}{\sqrt{n}}, \tag{39}
\]

where \( \mathcal{H}_{c, \delta} := \{ h \in \mathcal{H} : \| h \|_\mathcal{H} \leq c, \| h \|_{L_2(\mathcal{P})} \leq \delta \} \).

Proof. Our proof proceeds by generic chaining. Defining random variables

\[ Q_h = \frac{1}{n} \sum_{i=1}^{n} Z_i \omega(X_i) h(X_i), \]

the basic generic chaining result of Talagrand (2006) (Theorem 1.2.6) states that if \( \{Q_h\}_{h \in \mathcal{H}_{c, \delta}} \) is a sub-Gaussian process relative to some metric \( d \), i.e., for every \( h_1, h_2 \in \mathcal{H}_{c, \delta} \) and every \( u \geq 1 \),

\[
The \mathbb{P} |Q_{h_1} - Q_{h_2}| \geq ud(h_1, h_2) | \leq 2e^{-\frac{u^2}{2}}, \tag{40}
\]

then for some universal constant \( B \) (not the same as in (39)),

\[
\mathbb{E} \left[ \sup_{h \in \mathcal{H}_{c, \delta}} Q_h \right] \leq B \gamma_2(\mathcal{H}_{c, \delta}, d). \tag{41}
\]

Here, \( \gamma_2 \) is a measure of the complexity of the space \( \mathcal{H}_{c, \delta} \) in terms of the metric \( d \): writing \( \mathcal{S}_j, j = 1, 2, ..., \) for a sequence of collections of elements form \( \mathcal{H}_{c, \delta} \),

\[
\gamma_2(\mathcal{H}_{c, \delta}, d) = \inf_{(\mathcal{S}_j)} \left\{ \sup_{h \in \mathcal{H}_{c, \delta}} \left\{ \sum_{j=0}^{\infty} 2^j d(h, \mathcal{S}_j) \right\} : |\mathcal{S}_0| = 1, |\mathcal{S}_j| = 2^j \text{ for } j > 0 \right\}, \tag{42}
\]

where the infimum is with respect to all sequences of collections \( (\mathcal{S}_j)_{j=0}^{\infty} \), and \( d(h, \mathcal{S}) = \inf_{g \in \mathcal{S}} d(h, g) \).
To establish (39), we start by applying generic chaining conditionally on \(X_1, \ldots, X_n\): given a (possibly random) distance measure \(d\) such that (40) holds conditionally on the \(X_i\), then (41) also provides a uniform bound conditionally on the \(X_i\). To this end, we study the following metric:

\[
d(h_1, h_2) = \frac{1}{n} Md_{\infty,n}(h_1, h_2) \sqrt{\sum_{i=1}^{n} \omega^2(X_i)},
\]

(43)

\[
d_{\infty,n}(h_1, h_2) = \sup \{ |h_1(X_i) - h_2(X_i)| : i = 1, \ldots, n \}.
\]

(44)

Conditionally on the \(X_i\), \(Q_{h_1} - Q_{h_2}\) is a sum of \(n\) independent mean-zero sub-Gaussian random variables, the \(i\)-th of which has its sub-Gaussian variance proxy

\[
n^{-2}M^2 d_{\infty,n}^2(h_1, h_2) \omega^2(X_i),
\]

so (40) holds by elementary properties of sub-Gaussian random variables. Finally, noting that \(d(\cdot, \cdot)\) is a constant multiple of \(d_{\infty,n}(\cdot, \cdot)\) conditionally on \(X_1, \ldots, X_n\), the definition of \(\gamma_2\) implies that

\[
\gamma_2(H_c, \delta, d) = \frac{1}{n} M \sqrt{\sum_{i=1}^{n} \omega^2(X_i) \gamma_2(H_c, \delta, d_{\infty,n})}.
\]

Our argument so far implies that

\[
\mathbb{E} \left[ \sup_{H_c, \delta} \left\{ \frac{1}{n} \sum_{i=1}^{n} Z_i \omega(X_i) h(X_i) \right\} \bigg| X_1, \ldots, X_n \right] \leq \frac{BM}{n} \sqrt{\sum_{i=1}^{n} \omega^2(X_i) \gamma_2(H_c, \delta, d_{\infty,n})}.
\]

(45)

It now remains to bound moments of \(\gamma_2(H_c, \delta, d_{\infty,n})\).

Writing \(\sigma_j\) for the eigenvalues of \(K\) and \(A\) for the uniform bound on the eigenfunctions as in Assumption 2, Mendelson and Neeman (2010) show that for another universal constant \(B\), (Theorem 4.7)

\[
\mathbb{E} \left[ \gamma_2^2(H_c, \delta, d_{\infty,n}) \right]^{\frac{1}{2}} \leq AB \log(n) \sqrt{\sum_{j=1}^{\infty} \min \{ \delta^2, \sigma_j c^2/4 \}},
\]

(46)

and that for yet another universal constant \(B_p\) depending only on \(p\), (Lemma 3.4)

\[
\sum_{j=1}^{\infty} \min \{ \delta^2, \sigma_j c^2/4 \} \leq B_p \delta^{2(1-p)} c^{2p} G,
\]

(47)

where \(G = \sup_{j \geq 1} j^{\frac{3}{2}} \sigma_j\) as defined in Assumption 2. Thus, by Cauchy-Schwartz,

\[
\mathbb{E} \left[ \gamma_2(H_c, \delta, d_{\infty,n}) \right] \leq \mathbb{E} \left[ \gamma_2^2(H_c, \delta, d_{\infty,n}) \right]^{\frac{1}{2}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \omega^2(X_i) \right]^{\frac{1}{2}} \leq B \delta^{1-p} c^p \mathbb{E} \left[ \omega^2(X) \right]^{\frac{1}{2}},
\]

where \(B\) is a (different) constant. The desired result then follows. \[\square\]
Lemma 6. Suppose we have overlap, i.e., \( \eta < c^* (x) < 1 - \eta \) for some \( \eta > 0 \), for \( 1 < c < c' \). Then, the following holds:

\[
\| \tau_c^*(X_i) - \tau_{c'}^*(X_i) \|_{L_2(\mathcal{P})} \leq \frac{1}{\eta} \left( 1 - \frac{c}{c'} \right) \| \tau_c^* \|_{L_2(\mathcal{P})} .
\]  

(48)

Proof. First, we note that following a similar derivation behind (13), we have for any \( \tau, \tau' \in \mathcal{H} \),

\[
\eta^2 \| \tau(X_i) - \tau'(X_i) \|^2_{L_2(\mathcal{P})} \leq |L(\tau) - L(\tau')| \leq (1 - \eta)^2 \| \tau(X_i) - \tau'(X_i) \|^2_{L_2(\mathcal{P})} .
\]  

(49)

Then, we have

\[
\left\| \tau_c^*(X_i) - \frac{c}{c'} \tau_{c'}^*(X_i) \right\|^2_{L_2(\mathcal{P})} \leq \eta^{-2} \left( L \left( \frac{c}{c'} \tau_{c'}^* \right) - L(\tau_c^*) \right) \\
\leq \eta^{-2} \left( L \left( \frac{c}{c'} \tau_{c'}^* \right) - L(\tau_c^*) \right) \\
\leq \left( \frac{1 - \eta}{\eta^2} \right)^2 \left\| \tau_c^* \right\|^2_{L_2(\mathcal{P})} \left( 1 - \frac{c}{c'} \right)^2 .
\]  

(50)

Finally, by triangle inequality,

\[
\left\| \tau_c^*(X_i) - \tau_{c'}^*(X_i) \right\|_{L_2(\mathcal{P})} \leq \left\| \tau_c^*(X_i) - \frac{c}{c'} \tau_{c'}^*(X_i) \right\|_{L_2(\mathcal{P})} + \left\| \tau_c^*(X_i) - \frac{c}{c'} \tau_{c'}^*(X_i) \right\|_{L_2(\mathcal{P})} \\
\leq \left( 1 - \frac{c}{c'} \right) \| \tau_c^* \|_{L_2(\mathcal{P})} + \frac{1 - \eta}{\eta} \| \tau_c^* \|_{L_2(\mathcal{P})} \left( 1 - \frac{c}{c'} \right) \\
= \frac{1}{\eta} \left( 1 - \frac{c}{c'} \right) \| \tau_c^* \|_{L_2(\mathcal{P})} ,
\]

where the second inequality follows from (50). \( \square \)

Lemma 7. Simultaneously for all \( \tau \in \mathcal{H}_c, c \geq 1, \delta \leq 4M \) where \( \| \tau - \tau_c^* \|_{L_2(\mathcal{P})} \leq \delta \), we have

\[
\frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau_c^*(X_i))^2 \\
= \mathcal{O}_p \left( \delta^2 + c^2 p^2 (1 - p) \log(n) \right) + c^2 p^2 (1 - p) \left( \frac{1}{n} \log \left( \frac{cn^p \delta^2}{\delta^2} \right) \right) + c^2 p^2 (1 - p) \left( \frac{1}{n} \log \left( \frac{cn^p \delta^2}{\delta^2} \right) \right)
\]  

(51)

Proof. We proceed by a localization argument by bounding the quantity of interest over sets indexed by \( c \) and \( \delta \) such that \( \| \tau - \tau_c^* \|_{L_2(\mathcal{P})} \leq \delta \), i.e. we bound

\[
\sup_{\tau \in \mathcal{H}_c} \left\{ \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau_c^*(X_i))^2 : \| \tau - \tau_c^* \|_{L_2(\mathcal{P})} \leq \delta \right\}.
\]
First we bound the expectation. Let \( \varepsilon_i \) be i.i.d. Rademacher random variables.

\[
\mathbb{E} \left[ \sup_{\tau \in \mathcal{H}_c} \left\{ \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau_c^*(X_i))^2 : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\} \right] \\
(a) \leq \sup_{\tau \in \mathcal{H}_c} \left\{ \|\tau - \tau_c^*\|_{L_2(P)}^2 : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\} + 8 \sup_{\tau \in \mathcal{H}_c} \left\{ \|\tau - \tau_c^*\|_{\infty} : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\}.
\]

\[
= \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (\tau(X_i) - \tau_c^*(X_i))^2 : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right] \right\}
\]

\[
\leq B \left( \delta^2 + Bc^2\delta^2(1-p) \frac{\log(n)}{\sqrt{n}} \right),
\]

where (a) follows from (38), (b) follows from the fact that \( \varepsilon_i \) are symmetrical around 0, (c) follows from (39), and \( B \) is an absolute constant.

Let \( f_{\tau,c}(X_i) = (\tau(X_i) - \tau_c^*(X_i))^2 - \mathbb{E} \left[ (\tau(X_i) - \tau_c^*(X_i))^2 \right] \). Let \( G = \sup_{\tau \in \mathcal{H}_c} \left\{ \frac{1}{n} \sum_{i=1}^{n} f_{\tau,c}(X_i) : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\} \). Note that for a different constant \( B \),

\[
\mathbb{E} \left[ G \right] \leq \mathbb{E} \left[ \sup_{\tau \in \mathcal{H}_c} \left\{ \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau_c^*(X_i))^2 : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\} \right]
\]

\[
+ \sup_{\tau \in \mathcal{H}_c} \left\{ \mathbb{E} \left[ (\tau(X_i) - \tau_c^*(X_i))^2 \right] : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\}
\]

\[
\leq B \left( \delta^2 + c^2\delta^2(1-p) \frac{\log(n)}{\sqrt{n}} \right),
\]

where we note that bounding the first summand on the right-hand side of the first inequality above follows immediately from (52).

We also note that by (34),

\[
\sup_{\tau \in \mathcal{H}_c} \left\{ \|f_{\tau,c}\|_{\infty} : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\} \leq B \|\tau - \tau_c^*\|_{\infty}^2 \leq Bc^2\delta^2(1-p)
\]

for another different constant \( B \), and that

\[
\sup_{\tau \in \mathcal{H}_c} \left\{ \mathbb{E} \left[ f_{\tau,c}^2 \right] : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\} \leq \sup_{\tau \in \mathcal{H}_c} \left\{ \mathbb{E} \left[ (\tau(X_i) - \tau_c^*(X_i))^4 \right] : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\}
\]

\[
\leq \sup_{\tau \in \mathcal{H}_c} \left\{ \|\tau - \tau_c^*\|_{L_2(P)}^2 : \|\tau - \tau_c^*\|_{L_2(P)} \leq \delta \right\} \leq c^2\delta^2(1-p)+2.
\]
By Talagrand’s concentration inequality (36), for a fixed $c$ and $\delta$, we have that with probability $1 - \varepsilon$,

$$G \leq B \left( \delta^2 + c^2 \delta^2(1-p) \frac{\log(n)}{\sqrt{n}} + c^2 \delta^2 - p \frac{1}{\sqrt{n}} \log \left( \frac{1}{\varepsilon} \right) + \frac{1}{n} c^2 \delta^2(1-p) \log \left( \frac{1}{\varepsilon} \right) \right). \quad (53)$$

We conclude that for a fixed $c$ and $\delta$, we have that with probability $1 - \varepsilon$, for a different constant $B$,

$$\sup_{\tau \in \mathcal{H}_c} \left\{ \frac{1}{n} (\tau(X_i) - \tau^*_c(X_i))^2 : \|\tau - \tau^*_c\|_{L_2(P)} \leq \delta \right\}$$

$$\leq G + \sup_{\tau \in \mathcal{H}_c} \left\{ \mathbb{E} \left[ (\tau(X_i) - \tau^*_c(X_i))^2 \right] : \|\tau - \tau^*_c\|_{L_2(P)} \leq \delta \right\}$$

$$\leq B \left( \delta^2 + c^2 \delta^2(1-p) \frac{\log(n)}{\sqrt{n}} + c^2 \delta^2 - p \frac{1}{\sqrt{n}} \log \left( \frac{1}{\varepsilon} \right) + \frac{1}{n} c^2 \delta^2(1-p) \log \left( \frac{1}{\varepsilon} \right) \right).$$

We proceed with bounding the above for all values of $c$ and $\delta$ simultaneously. For a fixed $k = 0, 1, 2, \cdots$, define $C^{k,\delta} := \{2^k + jn^{-1/\theta} \delta 2^k, j = 0, 1, 2, \cdots, \lceil \frac{1}{\theta} (n^{1/\theta} - 1) \rceil \}$. For a fixed $\delta$, and for any $c \geq 1$, let $u(c, \delta) = \min \{d : d > c, d \in C^{2^k,\delta} \}$. Recall that $\|\tau^*_c\|_{L_2(P)} \leq 2M$ by definition, and so by Lemma 6, there is a constant $D$ such that

$$\left\| \tau^*_c - \tau^*_{u(c,\delta)} \right\|_{L_2(P)} \leq Dn^{-\frac{1}{2\theta}} \delta. \quad (54)$$

Thus, for any $c \geq 1$,

$$\sup_{\tau \in \mathcal{H}_c} \left\{ \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau^*_c(X_i))^2 : \|\tau - \tau^*_c\|_{L_2(P)} \leq \delta \right\}$$

$$\leq \sup_{\tau \in \mathcal{H}_{u(c,\delta)}} \left\{ \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau^*_c(X_i))^2 : \|\tau - \tau^*_{u(c,\delta)}\|_{L_2(P)} \leq \delta + Dn^{-\frac{1}{2\theta}} \delta \right\}$$

$$\leq \sup_{\tau \in \mathcal{H}_{u(c,\delta)}} \left\{ \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau^*_{u(c,\delta)}(X_i))^2 : \|\tau - \tau^*_{u(c,\delta)}\|_{L_2(P)} \leq \delta + Dn^{-\frac{1}{2\theta}} \delta \right\}$$

$$+ \sup_{\tau \in \mathcal{H}_{u(c,\delta)}} \left\{ \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau^*_c(X_i))^2 - \frac{1}{n} \sum_{i=1}^{n} (\tau(X_i) - \tau^*_{u(c,\delta)}(X_i))^2 : \|\tau - \tau^*_{u(c,\delta)}\|_{L_2(P)} \leq \delta + Dn^{-\frac{1}{2\theta}} \delta \right\}.$$

Let the two summands be $Z_{1,c,\delta}, Z_{2,c,\delta}$ respectively. Starting with the former, we note that for all $c, \delta > 0$,

$$Z_{1,c,\delta} \leq Z_{1,u(c,\delta),2^{|k_{\delta}(\delta)|}},$$

and so it suffices to bound this quantity on a set with $c \in C^{k_{c,\delta}}$, with $\delta = 4M \cdot 2^{-k_{\delta}}$ for $k_{c, \delta} = 0, 1, 2, \ldots$. Applying (53) unconditionally with probability threshold $\varepsilon \propto$
Suppose that the propensity estimate $\hat{\epsilon}(x)$ is uniformly consistent,

$$
\xi_n := \sup_{x \in X} |\hat{\epsilon}(x) - e^*(x)| \to_p 0,
$$

and the $L_2$ errors converge at rate

$$
E \left( \left( \hat{m}(X) - m^*(X) \right)^2 \right), \ E \left( \left( \hat{\epsilon}(X) - e^*(X) \right)^2 \right) = \mathcal{O} \left( a_n^2 \right)
$$

for some sequence $a_n \to 0$. Suppose, moreover, that we have overlap, i.e., $\eta < e^*(x) < 1 - \eta$ for some $\eta > 0$, and that Assumptions 2 and 3 hold. Then, for any $\varepsilon > 0$, there is a constant
$U(\varepsilon)$ such that the regret functions induced by the oracle learner (10) and the feasible learner (11) are coupled as

$$
|\hat{R}_n(\tau; c) - \bar{R}_n(\tau; c)| \\
\leq U(\varepsilon) \left( c^p \mathcal{R}(\tau; c)^{1-p} a_n^2 + \frac{1}{n} c^p \mathcal{R}(\tau; c)^{1-p} \log \left( \frac{cn^{1-p}}{\delta^2} \right) \right) \\
+ \frac{1}{n} a_n c^p \mathcal{R}(\tau; c)^{1-p} a_n \mathcal{R}(\tau; c)^{1-p} \frac{1}{\sqrt{n}} + c^p \mathcal{R}(\tau; c)^{1-p} \frac{1}{n} \\
+ \frac{a_n c^p \mathcal{R}(\tau; c)^{1-p}}{\sqrt{n}} \sqrt{\log \left( \frac{cn^{1-p}}{\delta^2} \right)} + \frac{1}{n} c^p \mathcal{R}(\tau; c)^{1-p} \log \left( \frac{cn^{1-p}}{\delta^2} \right) \\
+ \xi_n \mathcal{R}(\tau; c) + c^p \mathcal{R}(\tau; c)^{1-p} \frac{\log(n)}{\sqrt{n}} + c^p \mathcal{R}(\tau; c)^{1-p} \frac{1}{\sqrt{n}} \sqrt{\log \left( \frac{cn^{1-p}}{\delta^2} \right)}
$$

(61)

simultaneously for all $c \geq 1$ and $\tau \in \mathcal{H}_c$, with probability at least $1 - \varepsilon$.

**Proof.** We start by decomposing the feasible loss function $\hat{L}(\tau)$ as follows:

$$
\hat{L}(\tau) = \frac{1}{n} \sum_{i=1}^{n} \left( (Y_i - \hat{m}^{(-q(i))}(X_i)) - \tau(X_i) \left( W_i - \hat{e}^{(-q(i))}(X_i) \right) \right)^2 \\
= \frac{1}{n} \sum_{i=1}^{n} \left( (Y_i - m^*(X_i)) + (m^*(X_i) - \hat{m}^{(-q(i))}(X_i)) \right) \\
- \tau(X_i) (W_i - e^*(X_i)) - \tau(X_i) (e^*(X_i) - \hat{e}^{(-q(i))}(X_i)) \right)^2 \\
= \frac{1}{n} \sum_{i=1}^{n} \left( (Y_i - m^*(X_i)) - \tau(X_i) (W_i - e^*(X_i)) \right)^2 \\
+ \frac{1}{n} \sum_{i=1}^{n} \left( (m^*(X_i) - \hat{m}^{(-q(i))}(X_i)) - \tau(X_i) (e^*(X_i) - \hat{e}^{(-q(i))}(X_i)) \right)^2 \\
+ \frac{1}{n} \sum_{i=1}^{n} 2 (Y_i - m^*(X_i)) \left( m^*(X_i) - \hat{m}^{(-q(i))}(X_i) \right) \\
- \frac{1}{n} \sum_{i=1}^{n} 2 (Y_i - m^*(X_i)) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) \tau(X_i) \\
- \frac{1}{n} \sum_{i=1}^{n} 2 (W_i - e^*(X_i)) \left( m^*(X_i) - \hat{m}^{(-q(i))}(X_i) \right) \tau(X_i) \\
+ \frac{1}{n} \sum_{i=1}^{n} 2 (W_i - e^*(X_i)) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) \tau(X_i)^2.
$$

Furthermore, we can verify that some terms cancel out when we restrict attention to our main object of interest $\hat{R}(\tau; c) - R(\tau; c) = \hat{L}(\tau) - \hat{L}(\tau^*) - \bar{L}(\tau) + \bar{L}(\tau^*)$; in particular, note
that the first summand above is exactly $\bar{L}(\tau)$:

\[
\bar{R}(\tau; c) - \bar{R}(\tau; c)
= -\frac{2}{n} \sum_{i=1}^{n} \left( m^*(X_i) - \hat{m}^{(-q(i))}(X_i) \right) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) \left( \tau(X_i) - \tau_c^*(X_i) \right)
+ \frac{1}{n} \sum_{i=1}^{n} \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right)^2 \left( \tau(X_i)^2 - \tau_c^*(X_i)^2 \right)
- \frac{1}{n} \sum_{i=1}^{n} 2 (Y_i - m^*(X_i)) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) \left( \tau(X_i) - \tau_c^*(X_i) \right)
- \frac{1}{n} \sum_{i=1}^{n} 2 (W_i - e^*(X_i)) \left( m^*(X_i) - \hat{m}^{(-q(i))}(X_i) \right) \left( \tau(X_i) - \tau_c^*(X_i) \right)
+ \frac{1}{n} \sum_{i=1}^{n} 2 (W_i - e^*(X_i)) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) \left( \tau(X_i)^2 - \tau_c^*(X_i)^2 \right).
\]

Let $A_1^e(\tau), A_2^e(\tau), B_1^e(\tau), B_2^e(\tau)$ and $B_3^e(\tau)$ denote these 5 summands respectively. We now proceed to bound them, each on their own.

Starting with $A_1^e(\tau)$, by Cauchy-Schwarz,

\[
|A_1^e(\tau)| \leq 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( m^*(X_i) - \hat{m}^{(-q(i))}(X_i) \right)^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right)^2} \|\tau - \tau_c^*\|_\infty.
\]

This inequality is deterministic, and so trivially holds simultaneously across all $\tau \in \mathcal{H}_c$. Now, the two square-root terms denote the mean-squared errors of the $m$- and $e$-models respectively, and decay at rate $O_P(a_n)$ by Assumption 3 and a direct application of Markov's inequality. Thus, applying (35) to bound the infinity-norm discrepancy between $\tau$ and $\tau_c^*$, we find that simultaneously for all $c \geq 1$,

\[
\sup \left\{ e^{-p R(\tau; c)} \mathbb{I}_{|A_1^e(\tau)|} : \tau \in \mathcal{H}_c, c \geq 1 \right\} = O_P \left( a_n^2 \right).
\]

To bound $A_2^e(\tau)$, note that

\[
\tau^2(X_i) - \tau_c^*(X_i)^2 = 2 \tau_c^*(X_i)[\tau(X_i) - \tau_c^*(X_i)] + (\tau(X_i) - \tau_c^*(X_i))^2
\]

and so,

\[
|A_2^e(\tau)| \leq 2 \|\tau - \tau_c^*\|_\infty \|\tau_c^*\|_\infty \frac{1}{n} \sum_{i=1}^{n} \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right)^2
+ \|\tau - \tau_c^*\|_\infty^2 \frac{1}{n} \sum_{i=1}^{n} \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right)^2
= A_{2,1}^e(\tau) + A_{2,3}^e(\tau).
\]

To bound the two terms above, we can use a similar argument to the one used to bound $A_1^e(\tau)$. Specifically, $\frac{1}{n} \sum_{i=1}^{n} \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right)^2$ is bounded with high probability and
does not depend on $c$ or $\tau$, whereas terms that depend on $c$ or $\tau$ are deterministically bounded via (35); also, recall that $\|\tau^*_c\|_\infty \leq 2M$ by (15). We thus find that
\[
\sup \left\{ e^{q_p R(\tau; c) - \frac{1}{2} c} |A_{2,\tau}(\tau)| : \tau \in \mathcal{H}_c, c \geq 1 \right\} = O_P \left( a_n^2 \right),
\]
\[
\sup \left\{ e^{q_p R(\tau; c) - (1-p) \frac{1}{2} c} |A_{2,\tau}(\tau)| : \tau \in \mathcal{H}_c, c \geq 1 \right\} = O_P \left( a_n^2 \right),
\]
which all in fact decay at the desired rate.

We now move to bounding $B_1(\tau)$. To do so, first define
\[
B_{1,q}(\tau) = \sum_{\{i : q(i) = q\}} 2(Y_i - m^*(X_i)) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) (\tau(X_i) - \tau^*_c(X_i)),
\]
and note that $|B_{1,q}(\tau)| \leq \sum_{q=1}^Q |B_{1,q}(\tau)|$. We first bound $\sup_{\tau \in \mathcal{H}_c} B_{1,q}(\tau)$. To proceed, we bound this quantity over sets indexed by $c$ and $\delta$ such that $\|\tau - \tau^*_c\|_{L_2(P)} \leq \delta$, i.e., we bound
\[
\sup_{\tau \in \mathcal{H}_c} \left\{ B_{1,q}(\tau) : \|\tau - \tau^*_c\|_{L_2(P)} \leq \delta \right\}.
\]
Let $\mathcal{I}^{(-q)} = \{X_i, W_i, Y_i : q(i) \neq q\}$. By cross-fitting,
\[
\mathbb{E} \left[ B_{1,q}^{(-q)}(\tau) \right] = \sum_{\{i : q(i) = q\}} \mathbb{E} \left[ \frac{2(Y_i - m^*(X_i)) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) (\tau(X_i) - \tau^*_c(X_i))}{|\{i : q(i) = q\}|} \right] \mathbb{I}^{(-q)}
\]
\[
= \sum_{\{i : q(i) = q\}} \mathbb{E} \left[ \frac{2(Y_i - m^*(X_i)) \left( e^*(X_i) - \hat{e}^{(-q(i))}(X_i) \right) (\tau(X_i) - \tau^*_c(X_i))}{|\{i : q(i) = q\}|} \right] \mathbb{E} \left[ (Y_i - m^*(X_i)) \mid X_i \right] \mathbb{I}^{(-q)}
\]
\[
= 0,
\]
where the last equation follows because $\mathbb{E} \left[ (Y_i - m^*(X_i)) \mid X_i \right] = 0$ by definition. Moreover, by conditioning on $\mathcal{I}^{(-q)}$, the summands in $B_{1,q}^{(-q)}(\tau)$ become independent, as $\hat{e}^{(-q(i))}(X_i)$ is now only random in $X_i$. By Lemma 5 and (60), we can bound the expectation of the supremum of this term as
\[
\mathbb{E} \left[ \sup_{\tau \in \mathcal{H}_c} \left\{ B_{1,q}^{(-q)}(\tau) : \|\tau - \tau^*_c\|_{L_2(P)} \leq \delta \right\} \mid \mathcal{I}^{(-q)} \right] = O \left( e^{\beta^1 - p \log(n)} \right),
\]
and so, in particular,
\[
\mathbb{E} \left[ \sup_{\tau \in \mathcal{H}_c} \left\{ B_{1,q}^{(-q)}(\tau) : \|\tau - \tau^*_c\|_{L_2(P)} \leq \delta \right\} \mid \mathcal{I}^{(-q)} \right] = O_P \left( a_n e^{\beta^1 - p \log(n)} \right). \tag{65}
\]
It now remains to bound stochastic fluctuations of this supremum; and we do so using Talagrand’s concentration inequality (36). To proceed, first note that for an absolute constant
for $k_c$ and so it suffices to bound this quantity on a set with $c, \delta > 0$, such that for all $c \in \mathbb{Z}$, \[ \sup_{\tau \in \mathcal{H}_c} \left\{ \mathbb{E} \left[ \left( 2(Y_i - m^*(\cdot))(e^*(\cdot) - \hat{c}(\cdot))(\tau(\cdot) - \tau^*_c(\cdot)) \right)^2 \right] : \| \tau - \tau^*_c \|_{L^2(P)} \leq \delta \right\} \leq Bc^2\delta^{1-p}, \] and for a different constant $B$, \[ \sup_{\tau \in \mathcal{H}_c} \left\{ \mathbb{E} \left[ \left( 2(Y_i - m^*(X_i)) \left( e^*(X_i) - \hat{c}(\cdot)(X_i) \right) (\tau(X_i) - \tau^*_c(X_i)) \right)^2 \right] : \| \tau - \tau^*_c \|_{L^2(P)} \leq \delta \right\} \leq Bc^2\delta^{1(p-a_n^2)}. \]

Following from (36) and (65), for any fixed $c, \delta, \varepsilon > 0$, there exists an (again, different) absolute constant $B$ such that, with probability at least $1 - \varepsilon$, \[ \sup_{\tau \in \mathcal{H}_c} \left\{ B_{1,q}^c(\tau) \left| \mathcal{T}^{(-q)} : \| \tau - \tau^*_c \|_{L^2(P)} \leq \delta \right\} \right\} < B \left( c^p\delta^{1-p} a_n \frac{\log(n)}{\sqrt{n}} + \frac{c^p\delta^{1-p} a_n}{\sqrt{n}} \sqrt{\log \left( \frac{1}{\delta} \right)} + \frac{1}{n} c^p\delta^{1-p} \log \left( \frac{1}{\varepsilon} \right) \right) \tag{66} \]

Because the right-hand side does not depend on $\mathcal{T}^{(-q)}$, this bound also holds unconditionally. Our next step is to establish a bound that holds for all values of $c$ and $\delta$ simultaneously, as opposed to single values only as in (66). For $k = 0, 1, 2, \cdots$, define \[ C^{k,\delta} := \left\{ 2^k + jn^{-\frac{1}{1-p}} \delta 2^k, j = 0, 1, 2, \cdots, \left\lceil \left( n^{\frac{1}{1-p}} - 1 \right) / \delta \right\rceil \right\}. \]

For any $c \geq 1$, let $u(c, \delta) = \min\{ d : d > c, d \in C^{k_c,\delta} \}$. Recall that $\| \tau^*_c \|_{L^2(P)} \leq 2M$ by definition (15), and so by Lemma 6, there is a constant $D$ such that \[ \| \tau^*_c - \tau^*_u(c, \delta) \|_{L^2(P)} \leq Dn^{-\frac{1}{1-p}} \delta. \tag{67} \]

Thus, for any $c \geq 1$, \[ \sup_{\tau \in \mathcal{H}_c} \left\{ B_{1,q}^c(\tau) : \| \tau - \tau^*_c \|_{L^2(P)} \leq \delta \right\} \leq \sup_{\tau \in \mathcal{H}_{u(c, \delta)}} \left\{ B_{1,q}^c(\tau) : \| \tau - \tau^*_u(c, \delta) \|_{L^2(P)} \leq \delta + Dn^{-\frac{1}{1-p}} \delta \right\} \leq \sup_{\tau \in \mathcal{H}_{u(c, \delta)}} \left\{ B_{1,q}^u(c, \delta) (\tau) : \| \tau - \tau^*_u(c, \delta) \|_{L^2(P)} \leq \delta + Dn^{-\frac{1}{1-p}} \delta \right\} \right\} + \sup_{\tau \in \mathcal{H}_{u(c, \delta)}} \left\{ B_{1,q}^c(\tau) - B_{1,q}^u(c, \delta) (\tau) : \| \tau - \tau^*_u(c, \delta) \|_{L^2(P)} \leq \delta + Dn^{-\frac{1}{1-p}} \delta \right\}. \]

Let the two summands be $Z_{1,c,\delta}^B$ and $Z_{2,c,\delta}^B$ respectively. Starting with the former, we note that for all $c, \delta > 0$, \[ Z_{1,c,\delta}^B \leq Z_{1,u(c, \delta), 2^k, \delta}^B, \]

and so it suffices to bound this quantity on a set with $c \in C^{k_c,\delta}$ with $\delta = 4M \cdot 2^{-k_\delta}$, for $k_c, k_\delta = 0, 1, 2, \cdots$. Applying (66) unconditionally with probability threshold $\varepsilon \propto
2^{-k_e-k_s}n^{-\frac{1}{1-p}}2^{-k_s} = 2^{-k_e}2^{-2k_s}n^{-\frac{1}{1-p}}, we can use a union bound to check that

\[ Z_{1,c,\delta}^{B_1} = \mathcal{O}_p \left( e^p (\delta + Dn^{-\frac{1}{1-p}}\delta) a_n \log(n) \sqrt{n} + \frac{e^p (\delta + Dn^{-\frac{1}{1-p}}\delta)^{1-p}a_n}{\sqrt{n}} \right) \sqrt{\log \left( \frac{cn^\frac{1}{1-p}}{\delta^2} \right)} \]

\[ + \frac{1}{n} e^p (\delta + Dn^{-\frac{1}{1-p}}\delta)^{1-p} \log \left( \frac{cn^\frac{1}{1-p}}{\delta^2} \right) \]

\[ = \mathcal{O}_p \left( e^p \delta^{1-p}a_n \log(n) \right) \sqrt{\log \left( \frac{cn^\frac{1}{1-p}}{\delta^2} \right)} + \frac{1}{n} e^p \delta^{1-p} \log \left( \frac{cn^\frac{1}{1-p}}{\delta^2} \right) \]

simultaneously for all \( c > 1 \) and \( \delta \leq 4M \). Next, to bound \( Z_{2,c,\delta}^{B_1} \), we use Cauchy-Schwartz to check that

\[ \sum_{(i,q(i)=q)} 2 (Y_i - m^e(X_i)) \left( e^e(X_i) - \hat{e}^{(-q(i))}(X_i) \right) \left( \tau_{u(c,\delta)}^e(X_i) - \tau_c^e(X_i) \right) \]

\[ \leq D \sqrt{ \sum_{(i,q(i)=q)} \left( e^e(X_i) - \hat{e}^{(-q(i))}(X_i) \right)^2 } \sqrt{ \sum_{(i,q(i)=q)} \left( \tau_{u(c,\delta)}^e(X_i) - \tau_c^e(X_i) \right)^2 } \]

\[ \leq D \left\| \tau_{u(c,\delta)}^e - \tau_c^e \right\|_{\infty} \]

\[ = \mathcal{O}_p \left( \frac{a_n e^p \delta^{1-p}}{n} \right). \]

where the last equality follows from (67), (34) and (60) with a direct application of Markov’s inequality. Note that the term that depends on \( c \) is deterministically bounded, so the above bound holds for all \( c \geq 1 \). We can similarly bound \(-B_1^c(\tau)\). For any \( T, \)

\[ P \left[ \sup_{\tau \in \mathcal{H}_c} |B_1^c(\tau)| \geq T \right] \leq P \left[ \sup_{\tau \in \mathcal{H}_c} B_1^c(\tau) \geq T \right] + P \left[ \sup_{\tau \in \mathcal{H}_c} -B_1^c(\tau) \geq T \right], \]

the desired result then follows. Similar arguments apply to bounding \( B_2^c(\tau) \) as well, and the same bound (up to constants) suffices.

Now moving to bounding \( B_3^c(\tau) \), note that by (63),

\[ B_3^c(\tau) \leq 4 \sum_{n=1}^{n} (W_i - e^e(X_i)) \left( e^e(X_i) - \hat{e}^{(-q(i))}(X_i) \right) (\tau(X_i) - \tau_c^e(X_i)) \tau_c^e(X_i) \]

\[ + 2 \sum_{n=1}^{n} (W_i - e^e(X_i)) \left( e^e(X_i) - \hat{e}^{(-q(i))}(X_i) \right) (\tau(X_i) - \tau_c^e(X_i))^2. \]

Denote the two summands by \( D_2^{B_3,c}(\tau) \) and \( D_2^{B_3,c}(\tau) \) respectively. Note that since \( \| \tau_c^e \|_{\infty} \leq 2M \), we can use a similar argument to the one used to bound \( \sup B_3^c(\tau) \), and the same bound (up to constants) suffices.

We now proceed to bound \( D_2^{B_3,c} \). First, we note that

\[ D_2^{B_3,c} \leq \sum_{i=1}^{n} 2 \| Y_i - m^e(\cdot) \|_{\infty} \| e^e(\cdot) - \hat{e}^{(-q(i))}(\cdot) \|_{\infty} (\tau(X_i) - \tau_c^e(X_i))^2 \]

\[ \leq B \| e^e(\cdot) - \hat{e}^{(-q(i))}(\cdot) \|_{\infty} \sum_{i=1}^{n} (\tau(X_i) - \tau_c^e(X_i))^2, \]
where $B$ is an absolute constant. By Lemma 7, uniformly for all $\tau \in \mathcal{H}_c, c \geq 1, \delta \leq 4M$ where $\|\tau - \tau^*_c\|_{L_2(P)} \leq \delta$, we have

$$D_{2}^{B_{\delta,c}} = O_P \left( \xi_n \delta^2 + c^2 \delta^2 \log(n) \right) + \frac{1}{\sqrt{n}} c^2 \delta^2 \log\left( \frac{c n \delta^2}{\delta^2} \right) + \frac{1}{\sqrt{n}} c^2 \delta^2 \log\left( \frac{c n \delta^2}{\delta^2} \right) + \frac{1}{\sqrt{n}} c^2 \delta^2 \log\left( \frac{c n \delta^2}{\delta^2} \right),$$

where $\xi_n = \|\varphi^{*}(-\varphi^{(i)})\|_{\infty} = o(1)$. Finally, recalling that from (13), $R(\tau; c)$ is within a constant factor of $\|\tau - \tau^*_c\|_{L_2(P)}^2$ given overlap, we obtain our desired result.

A.3 Proof of Lemma 2

Proof. Comparing (61) with (28), we note that given the conditions, all other terms that are omitted in (28) are on lower order to the first leading term in (28).

A.4 Proof of Theorem 3

As discussed earlier, the arguments of Mendelson and Neeman (2010) can be used to get regret bounds for the oracle learner. In order to extend their results, we first review their analysis briefly. Their results imply the following facts (details see Theorem A and the proof of Theorem 2.5 in the Appendix section in Mendelson and Neeman (2010)). For any $\varepsilon > 0$, there is a constant $U(\varepsilon)$ such that

$$\rho_n(c) = U(\varepsilon) \left( 1 + \log(n) + \log \log \left( c + e^1 \right) \right) \left( \frac{c + 1}{\sqrt{n}} \log(n) \right) \tau^2 \rho \quad (71)$$

satisfies, for large enough $n$ with probability at least $1 - \varepsilon$, simultaneously for all $c \geq 1$, the condition

$$0.5 \bar{R}_n(\tau; c) - \rho_n(c) \leq R(\tau; c) \leq 2 \bar{R}_n(\tau; c) + \rho_n(c). \quad (72)$$

Thus, thanks to Lemma 1 and (23), we know that

$$R(\tilde{\tau}) \leq O_P \left( (L(\tau^*_c) - L(\tau^*)) + \rho_n(c_n) \right) \text{ with } c_n = n^{-\frac{\alpha}{2(1-2\alpha)}}, \quad (73)$$

and then pairing (24) with the form of $\rho_n(c)$ in (71), we conclude that

$$R(\tilde{\tau}) \leq O \left( \max \left\{ \left( L(\tau^*_c) - L(\tau^*) \right), \rho_n(c_n) \right\} = \tilde{O} \left( n^{-\frac{1+2\alpha}{2(1-2\alpha)}} \right). \quad (74)$$

Our present goal is to extend this argument to get a bound for $R(\tilde{\tau})$.\footnote{We note that the $R-$learning objective can be written as a weighted regression problem: $\tilde{\tau}(x) = \arg\min_{c \in \mathcal{H}_c} \frac{1}{n} \sum_{i=1}^{n} \left( W_i e^{-n^{(1)}(X_i)} \right)^2 \left( Y_i - e^{-n^{(1)}(X_i)} - \tau(X_i) \right)^2$. To adapt the setting in Mendelson and Neeman (2010) to our setting, note that we weight the data generating distribution of $\{X_i, Y_i, W_i\}$ by the weights $(W_i e^{-n^{(1)}(X_i)})^2$. In addition, by Lemma 4, the class of functions we consider $\mathcal{H}_c$ with capped infinity norm is also an ordered, parameterized hierarchy, thus their results follow.}
Towards this end, first we copy from Lemma 2 that under the conditions from Lemma 2,
\[
\left| \tilde{R}_n(\tau; c) - \tilde{R}_n(\tau; c) \right|
\leq U(\varepsilon) \left( c^p R(\tau; c) \frac{1}{\sqrt{n}} a_n^2 + c^2 p R(\tau; c) \log(n) + c^2 p R(\tau; c) \right) \frac{1}{\sqrt{n}} \log\left( \frac{c n^{1-p}}{R(\tau; c)} \right)
\leq c^p R(\tau; c) \frac{1}{\sqrt{n}} \log\left( \frac{c n^{1-p}}{R(\tau; c)} \right) + c^2 p R(\tau; c) \frac{1}{\sqrt{n}} \log\left( \frac{c n^{1-p}}{R(\tau; c)} \right)
\leq \xi_n R(\tau; c)
\]
with probability at least $1 - \varepsilon$, for all $\tau \in \mathcal{H}_c, 1 \leq c \leq c_n \log(n)$ with $c_n = \frac{n^{1-p}}{1-p}$.

For any $\gamma_n, \zeta_n > 0$, and $0 \leq \nu_\gamma, \nu_\zeta < 1 - p$, by concavity,
\[
R(\tau; c) \leq \gamma_n \frac{1-p-\nu_\gamma}{\nu_\gamma} + \frac{1-p-\nu_\gamma}{\nu_\gamma} \gamma_n \frac{1}{\sqrt{n}} \log\left( \frac{c n^{1-p}}{R(\tau; c) - \gamma_n} \right)
= \frac{1}{1-p-\nu_\gamma} \frac{1}{\sqrt{n}} \log\left( \frac{c n^{1-p}}{R(\tau; c) - \gamma_n} \right)
\]
\[
R(\tau; c) \leq \gamma_n \frac{1-p-\nu_\zeta}{\nu_\zeta} + \frac{1-p-\nu_\zeta}{\nu_\zeta} \gamma_n \frac{1}{\sqrt{n}} \log\left( \frac{c n^{1-p}}{R(\tau; c) - \zeta_n} \right)
= \frac{1}{1-p-\nu_\zeta} \frac{1}{\sqrt{n}} \log\left( \frac{c n^{1-p}}{R(\tau; c) - \zeta_n} \right)
\]
We then apply the above bounds with choices of $\gamma_n, \zeta_n$ that make the linear coefficients of $R(\tau; c)$ in (75) small, and show that the remaining terms are lower order to $\rho_n(c)$ for all $1 \leq c \leq c_n \log(n)$. More formally, it suffices to show that
\[
\left| \tilde{R}_n(\tau; c) - \tilde{R}_n(\tau; c) \right| \leq 0.125 R(\tau; c) + o(\rho_n(c)),
\]
with probability at least $1 - \varepsilon$, for all $\tau \in \mathcal{H}_c, 1 \leq c \leq c_n \log(n)$ with $c_n = \frac{n^{1-p}}{1-p}$ for large enough $n$. The above would imply that
\[
R(\tau; c) \leq 2 \tilde{R}_n(\tau; c) + \rho_n(c)
\leq 2 \tilde{R}_n(\tau; c) + 0.25 R(\tau; c) + 2 \rho_n(c),
\]
which implies that
\[
R(\tau; c) \leq \frac{2}{0.75} \tilde{R}_n(\tau; c) + 2 \rho_n(c)
\leq 3 \tilde{R}_n(\tau; c) + 2 \rho_n(c)
\]
for large $n$ for all $1 \leq c \leq c_n \log(n)$, with probability at least $1 - 2\varepsilon$. Following a symmetrical argument, (78) would imply that
\[
\frac{1}{3} \tilde{R}_n(\tau; c) - 2 \rho_n(c) \leq R(\tau; c) \leq 3 \tilde{R}_n(\tau; c) + 2 \rho_n(c)
\]
(81)
for $n$ large enough for all $1 \leq c \leq c_n \log(n)$ with probability at least $1 - 4\varepsilon$.

We now proceed to show (78) holds. First, following from (13), $R(\tau; c) < (1 - \eta)2M^2 = O(1)$. Let $J$ be a constant such that $R(\tau; c) < J$. Now we bound each term in (75) as follows:

To bound the terms $c^p R(\tau; c)\frac{1 - p}{2} a_n^2$, let $\gamma_n = (U(\varepsilon))^{\frac{1}{1+\varepsilon}} \left(\frac{1 - p}{1+\varepsilon}\right)^{\frac{2p}{1+\varepsilon}} c^{\frac{2p}{1+\varepsilon}} a_n^{\frac{2p}{1+\varepsilon}}$. Note that since $a_n = o(n^{-\frac{1}{4}})$, $\gamma_n = o(\rho_n(c))$ for all $c \geq 1$.

Following from (76),
\[
\frac{c^p R(\tau; c)\frac{1 - p}{2} a_n^2}{U(\varepsilon)} = \frac{1}{0.02 R(\tau; c) + o(\rho_n(c))}.
\]

(82)

To bound the term $c^{2p} R(\tau; c)^{1-p} \frac{1}{\sqrt{n}} \log(n)$, let $\zeta_n = U(\varepsilon)^{\frac{1}{2}} \left(\frac{1 - p}{0.02}\right)^{\frac{1}{2}} c^{2p} n^{-\frac{1}{\varepsilon}} \log(n)^{\frac{1}{2}}$. When $c = c_n \log(n)$,
\[
\zeta_n = \frac{c_n^{2p}}{U(\varepsilon)^{\frac{1}{2}}} \left(\frac{1 - p}{0.02}\right)^{\frac{1}{2}} c_n^{2p} \log(n)^{\frac{1}{2}} n^{-\frac{1}{\varepsilon}} \log(n)^{\frac{1}{2}}
= \mathcal{O} \left(n^{\frac{2p}{1+2p(1+\varepsilon)} - \frac{1}{2p\varepsilon}} \log(n)^{\frac{1}{2}}\right)
= \mathcal{O} \left(n^{\frac{2p}{1+2p(1+\varepsilon)} - \frac{1}{2p\varepsilon}} \log(n)^{\frac{1}{2}}\right)
= \mathcal{O} \left(c_n^{2p} n^{-\frac{1}{2p\varepsilon}} \log(n)^{\frac{1}{2}}\right)
= o(\rho_n(c_n \log(n))),
\]
where $(a)$ follows from a few lines of algebra and the assumption that $2\alpha < 1 - p$. Since the exponent on $c$ in $\zeta_n$ is greater than that in $\rho_n(c)$, we can verify that for any $c \leq c_n \log(n)$,
\[
\frac{\zeta_n(c)}{\rho_n(c)} \leq \frac{\zeta_n(c_n \log(n))}{\rho_n(c_n \log(n))} = o(1).
\]

Following from (77),
\[
c^{2p} R(\tau; c)^{1-p} \frac{1}{\sqrt{n}} \log(n) \leq \frac{1}{U(\varepsilon)} \left(0.02 R(\tau; c) + o(\rho_n(c))\right).
\]

To bound the term $c^{2p} R(\tau; c)^{1-p} \frac{1}{\sqrt{n}} \log(n)$, let $\zeta_n = U(\varepsilon)^{\frac{1}{2}} \left(\frac{1 - p}{0.02}\right)^{\frac{1}{2}} c^{2p} n^{-\frac{1}{\varepsilon}} \log(n)^{\frac{1}{2}}$.
\[
\zeta_n = \frac{c_n^{2p}}{U(\varepsilon)^{\frac{1}{2}}} \left(\frac{1 - p}{0.02}\right)^{\frac{1}{2}} c_n^{2p} \log(n)^{\frac{1}{2}} n^{-\frac{1}{\varepsilon}} \log(n)^{\frac{1}{2}}
= \mathcal{O} \left(n^{\frac{2p}{1+2p(1+\varepsilon)} - \frac{1}{2p\varepsilon}} \log(n)^{\frac{1}{2}}\right)
= \mathcal{O} \left(n^{\frac{2p}{1+2p(1+\varepsilon)} - \frac{1}{2p\varepsilon}} \log(n)^{\frac{1}{2}}\right)
= \mathcal{O} \left(c_n^{2p} n^{-\frac{1}{2p\varepsilon}} \log(n)^{\frac{1}{2}}\right)
= o(\rho_n(c_n \log(n))),
\]
where $(a)$ follows from $2\alpha < 1$. Since the exponent on $c$ in $\zeta_n$ is greater than that in $\rho_n(c)$, we can verify that for any $c \leq c_n \log(n)$,
\[
\frac{\zeta_n(c)}{\rho_n(c)} \leq \frac{\zeta_n(c_n \log(n))}{\rho_n(c_n \log(n))} = o(1).
\]

Following from (77),
\[
c^{2p} R(\tau; c)^{1-p} \frac{1}{n} \log(n) \log(c) \leq \frac{1 - p}{2J^n U(\varepsilon)} \left(0.02 R(\tau; c) + o(\rho_n(c))\right).
\]

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Thus,
\[
c^{2p} R(\tau; c)^{1-p} \frac{1}{n} \log \left( \frac{c n^{\frac{1}{1-p}}}{R(\tau; c)} \right) \leq \frac{1}{U(\varepsilon)} (0.02 R(\tau; c) + o(\rho_n(c))).
\]

To bound the terms \(c^p R(\tau; c)^{1-p} \frac{1}{n} \sqrt{\log \left( \frac{c n^{\frac{1}{1-p}}}{R(\tau; c)} \right)}\), since \(R(\tau; c)^{\frac{1}{2}} \sqrt{\log(1/R(\tau; c))} < \), let \(a\) different \(\gamma\) such that 0

\[
R(\tau; c)^{\frac{1}{2}} < J^{\frac{1}{2}} = O(1), \quad \text{and} \quad \sqrt{\log(c n^{\frac{1}{1-p}})} < \sqrt{\log(c)} + \frac{1}{\sqrt{1-p}} \sqrt{\log(n)} < \frac{2}{\sqrt{1-p}} \sqrt{\log(n)} \sqrt{\log(c)}
\]

it is sufficient to bound \(c^p R(\tau; c)^{1-p} \frac{1}{n} \sqrt{\log(n)} \sqrt{\log(c)}\). To proceed, let a different \(\gamma_n = \left( \frac{2}{\sqrt{1-p}} J^{\frac{1}{2}} U(\varepsilon) \right)^{\frac{2}{1-p}} \left( \frac{1-p}{2} \right)^{\frac{2}{1-p}} c^{\frac{2}{1-p}} n^{\frac{1}{1-p}} (\log(n))^{\frac{1}{1-p}} (\log(c))^{\frac{1}{1-p}}\).

Note that for \(1 \leq c \leq c_n \log(n)\), \((\log(c))^{\frac{1}{1-p}} \leq (\log(c_n \log(n)))^{\frac{1}{1-p}}\). For a different constant \(D\) and \(D'\),

\[
\gamma_n \leq D c^{\frac{2}{1-p}} n^{\frac{1}{1-p}} (\log(n))^{\frac{1}{1-p}} (\log(c_n \log(n)))^{\frac{1}{1-p}}
\]

\[
\leq D' c^{\frac{2}{1-p}} n^{\frac{1}{1-p}} (\log(n))^{\frac{1}{1-p}}
\]

\[
= o(\rho_n(c)).
\]

Following from (76),
\[
c^p R(\tau; c)^{1-p} \frac{1}{n} \sqrt{\log(n)} \sqrt{\log(c)} \leq \frac{\sqrt{1-p}}{2 J^{\frac{1}{2}}} U(\varepsilon) (0.02 R(\tau; c) + o(\rho_n(c))).
\]

Thus,
\[
c^p R(\tau; c)^{1-p} \frac{1}{n} \sqrt{\log \left( \frac{c n^{\frac{1}{1-p}}}{R(\tau; c)} \right)} \leq \frac{1}{U(\varepsilon)} (0.02 R(\tau; c) + o(\rho_n(c))).
\]

To bound the term \(c^p R(\tau; c)^{1-p} a_n \frac{1}{\sqrt{\log \left( \frac{c n^{\frac{1}{1-p}}}{R(\tau; c)} \right)}}\), since

\[
R(\tau; c)^{\frac{1}{2}} \sqrt{\log(1/R(\tau; c))} < R(\tau; c)^{\frac{1}{n}}
\]

\[
< J^{\frac{1}{2}} = O(1),
\]

and

\[
\sqrt{\log(c n^{\frac{1}{1-p}})} = \sqrt{\log(c) + \frac{1}{1-p} \log(n)}
\]

\[
< \sqrt{\log(c)} + \frac{1}{\sqrt{1-p}} \sqrt{\log(n)}
\]

\[
< \frac{2}{\sqrt{1-p}} \sqrt{\log(c)} \sqrt{\log(n)},
\]

and \(a_n = o(n^{-\frac{1}{2}})\), it is sufficient to bound \(c^p R(\tau; c)^{1-p-\nu} n^{-\frac{3}{2}} \sqrt{\log(n)} \sqrt{\log(c)}\) for some \(\nu\), such that 0 < \(\nu < 1 - p\). Let a different \(\gamma_n = \left( \frac{2}{\sqrt{1-p}} J^{\frac{1}{2}} U(\varepsilon) \right)^{\frac{2}{1-p}} \left( \frac{1-p-\nu}{0.04} \right)^{\frac{2}{1-p}} c^{\frac{2}{1-p}} n^{\frac{1}{1-p}} (\log(n) \log(c))^{\frac{1}{1-p}}\).
Let $\nu_c = \frac{1-p}{2}$, it is straightforward to check that $\gamma_n = o(\rho_n(c))$ for all $c \geq 1$. Following from (76),
\[
e^p R(\tau; c) \geq \frac{1-p}{2} n^{-\frac{1}{2}} \sqrt{\log(n)} \sqrt{\log(c)} \leq \frac{1-p}{2 J^2} U(\varepsilon) (0.02 R(\tau; c) + o(\rho_n(c))).
\]
Thus,
\[
e^p R(\tau; c) \geq \frac{n a}{\sqrt{n}} \log\left(\frac{c \nu_c}{R(n; c)}\right) \leq \frac{1}{U(\varepsilon)} (0.02 R(\tau; c) + o(\rho_n(c))).
\]
Finally, to bound the term $\xi_n R(\tau; c)$, note that since $\xi_n \to 0$, for $n$ large enough, $\xi_n R(\tau; c) \leq \frac{1}{\nu_c} 0.025 R(\tau; c)$.

Given the above derivations, (78) is now immediate. Thus, with probability $1 - 4\varepsilon$, (81) holds for all $1 \leq c \leq c_n \log(n)$. Then applying the same argument as above, we use Lemma 1 to check that the constrained estimator defined as
\[
\hat{\tau} \in \arg\min_{\tau \in \mathcal{H}} \left\{ \hat{L}_n(\tau) + 2k_1 \rho_n(\|\tau\|_\mathcal{H}) : \|\tau\|_\mathcal{H} \leq \log(n)c_n, \|\tau\|_\infty \leq 2M \right\}
\]
\[
\subseteq \arg\min_{\tau \in \mathcal{H}} \left\{ \hat{R}_n(\tau) + 2k_1 \rho_n(\|\tau\|_\mathcal{H}) : \|\tau\|_\mathcal{H} \leq \log(n)c_n, \|\tau\|_\infty \leq 2M \right\}
\]
has regret bounded on the order of
\[
L(\hat{\tau}) - L(\tau^*) \leq P \left( (L(\tau^*_n) - L(\tau^*)) + \rho_n(c_n) \right) \leq \rho_n(c_n),
\]
where we note that $\hat{L}_n(\tau) = \hat{R}_n(\tau) + \hat{L}_n(\tau^*)$. We see that for some constant $B$ and $B'$,
\[
\min_{\tau \in \mathcal{H}} \left\{ \hat{R}_n(\tau) + 2k_1 \rho_n(\|\tau\|_\mathcal{H}) : \|\tau\|_\mathcal{H} \leq \log(n)c_n, \|\tau\|_\infty \leq 2M \right\}
\]
\[
\leq \hat{R}_n(\tau^*_n) + 2k_1 \rho_n(c_n)
\]
\[
\leq 3 R(\tau^*_n) + (2k_1 + 6) \rho_n(c_n) \quad \text{w.p.} \ 1 - 4\varepsilon
\]
\[
\leq B c_n^{\frac{2k_1}{2}} + (2k_1 + 6) \rho_n(c_n)
\]
\[
\leq B' \rho_n(c_n).
\]
where $(a)$ follows from (81), $(b)$ follows from (13) and (24), and $(c)$ follows from (73) and (74). In addition, we see that
\[
\inf_{\tau \in \mathcal{H}} \left\{ \hat{R}_n(\tau) + 2k_1 \rho_n(\|\tau\|_\mathcal{H}) : \|\hat{\tau}\|_\mathcal{H} = \log(n)c_n, \|\tau\|_\infty \leq 2M \right\} \geq P \rho_n(c_n \log(n))
\]
which, combined with (90), implies that the optimum of the problem (87) occurs in the interior of its domain (i.e., the constraint is not active). Thus, the solution $\hat{\tau}$ to the unconstrained problem matches $\hat{\tau}$, and so $\hat{\tau}$ also satisfies (88) and hence the regret bound (74).

**B Detailed Simulation Results**

For completeness, we include the mean-squared error numbers behind Figure 2 for the lasso- and boosting-based simulations in Section 4.
| n   | d | \( \sigma \) | S  | T  | X  | U  | R  | RS | oracle |
|-----|---|------------|----|----|----|----|----|----|--------|
| 500 | 6 | 0.5        | 0.13 | 0.19 | 0.10 | 0.12 | **0.06** | **0.06** | 0.05   |
| 500 | 6 | 1          | 0.21 | 0.27 | 0.16 | 0.37 | 0.10 | **0.07** | 0.07   |
| 500 | 6 | 2          | 0.27 | 0.35 | 0.25 | 1.25 | 0.21 | **0.12** | 0.19   |
| 500 | 6 | 4          | 0.51 | 0.66 | 0.41 | 1.95 | 0.55 | **0.26** | 0.61   |
| 500 | 12| 0.5        | 0.15 | 0.20 | 0.12 | 0.17 | 0.07 | **0.06** | 0.05   |
| 500 | 12| 1          | 0.22 | 0.26 | 0.18 | 0.46 | 0.11 | **0.09** | 0.08   |
| 500 | 12| 2          | 0.30 | 0.35 | 0.26 | 1.18 | 0.23 | **0.14** | 0.23   |
| 500 | 12| 4          | 0.47 | 0.56 | 0.43 | 1.98 | 0.59 | **0.28** | 0.63   |
| 1000| 6 | 0.5        | 0.09 | 0.13 | 0.06 | 0.06 | **0.04** | 0.05 | 0.04   |
| 1000| 6 | 1          | 0.15 | 0.21 | 0.11 | 0.25 | 0.07 | **0.06** | 0.06   |
| 1000| 6 | 2          | 0.23 | 0.29 | 0.20 | 0.85 | 0.13 | **0.08** | 0.11   |
| 1000| 6 | 4          | 0.34 | 0.43 | 0.31 | 2.40 | 0.34 | **0.16** | 0.32   |
| 1000| 12| 0.5        | 0.11 | 0.14 | 0.08 | 0.11 | **0.05** | 0.05 | 0.04   |
| 1000| 12| 1          | 0.18 | 0.22 | 0.14 | 0.34 | 0.08 | **0.07** | 0.06   |
| 1000| 12| 2          | 0.25 | 0.30 | 0.21 | 0.94 | 0.14 | **0.09** | 0.12   |
| 1000| 12| 4          | 0.33 | 0.40 | 0.29 | 1.95 | 0.35 | **0.18** | 0.33   |

Table 1: Mean-squared error running lasso from Setup A. Results are averaged across 500 runs, rounded to two decimal places, and reported on an independent test set of size \( n \).

| n   | d | \( \sigma \) | S  | T  | X  | U  | R  | RS | oracle |
|-----|---|------------|----|----|----|----|----|----|--------|
| 500 | 6 | 0.5        | 0.26 | 0.43 | **0.22** | 0.46 | 0.28 | 0.29 | 0.16   |
| 500 | 6 | 1          | 0.44 | 0.66 | **0.38** | 0.83 | 0.43 | 0.72 | 0.33   |
| 500 | 6 | 2          | 0.84 | 1.12 | **0.71** | 1.27 | 0.85 | 1.26 | 0.75   |
| 500 | 6 | 4          | 1.52 | 1.73 | **1.29** | 1.40 | 1.51 | 1.41 | 1.46   |
| 500 | 12| 0.5        | 0.30 | 0.46 | **0.25** | 0.54 | 0.33 | 0.41 | 0.18   |
| 500 | 12| 1          | 0.52 | 0.71 | **0.43** | 0.90 | 0.50 | 0.95 | 0.38   |
| 500 | 12| 2          | 0.93 | 1.12 | **0.78** | 1.28 | 0.96 | 1.31 | 0.84   |
| 500 | 12| 4          | 1.62 | 1.77 | **1.33** | 1.42 | 1.55 | 1.40 | 1.54   |
| 1000| 6 | 0.5        | 0.14 | 0.24 | **0.13** | 0.24 | 0.15 | 0.15 | 0.10   |
| 1000| 6 | 1          | 0.27 | 0.43 | **0.23** | 0.46 | 0.25 | 0.36 | 0.20   |
| 1000| 6 | 2          | 0.54 | 0.73 | **0.45** | 1.12 | 0.52 | 0.92 | 0.47   |
| 1000| 6 | 4          | 1.06 | 1.31 | **0.92** | 1.34 | 1.07 | 1.34 | 1.06   |
| 1000| 12| 0.5        | 0.17 | 0.28 | **0.15** | 0.29 | 0.18 | 0.18 | 0.11   |
| 1000| 12| 1          | 0.30 | 0.45 | **0.26** | 0.55 | 0.30 | 0.52 | 0.23   |
| 1000| 12| 2          | 0.61 | 0.76 | **0.50** | 1.19 | 0.59 | 1.14 | 0.54   |
| 1000| 12| 4          | 1.15 | 1.30 | **1.01** | 1.33 | 1.19 | 1.34 | 1.13   |

Table 2: Mean-squared error running lasso from Setup B. Results are averaged across 500 runs, rounded to two decimal places, and reported on an independent test set of size \( n \).
Table 3: Mean-squared error running lasso from Setup C. Results are averaged across 500 runs, rounded to two decimal places, and reported on an independent test set of size $n$.

| n    | d  | $\sigma$ | S   | T   | X   | U   | R   | RS  | oracle |
|------|----|----------|-----|-----|-----|-----|-----|-----|--------|
| 500  | 6  | 0.5      | 0.18| 0.80| 0.18| 0.53| 0.05| 0.02| 0.01   |
| 500  | 6  | 1        | 0.33| 1.18| 0.29| 0.66| 0.10| 0.03| 0.03   |
| 500  | 6  | 2        | 0.75| 1.95| 0.58| 1.42| 0.21| 0.09| 0.12   |
| 500  | 6  | 4        | 1.68| 3.13| 1.24| 3.56| 0.64| 0.26| 0.51   |
| 500  | 12 | 0.5      | 0.18| 0.88| 0.19| 0.55| 0.08| 0.03| 0.01   |
| 500  | 12 | 1        | 0.34| 1.29| 0.31| 0.86| 0.12| 0.06| 0.04   |
| 500  | 12 | 2        | 0.81| 2.08| 0.65| 1.82| 0.24| 0.13| 0.14   |
| 500  | 12 | 4        | 1.79| 3.28| 1.43| 4.02| 0.62| 0.33| 0.58   |
| 1000 | 6  | 0.5      | 0.10| 0.49| 0.10| 0.23| 0.02| 0.01| 0.00   |
| 1000 | 6  | 1        | 0.19| 0.73| 0.17| 0.34| 0.03| 0.01| 0.01   |
| 1000 | 6  | 2        | 0.41| 1.29| 0.35| 0.82| 0.08| 0.04| 0.07   |
| 1000 | 6  | 4        | 0.97| 2.38| 0.82| 2.31| 0.27| 0.11| 0.22   |
| 1000 | 12 | 0.5      | 0.09| 0.58| 0.10| 0.41| 0.03| 0.01| 0.00   |
| 1000 | 12 | 1        | 0.19| 0.82| 0.18| 0.54| 0.04| 0.02| 0.01   |
| 1000 | 12 | 2        | 0.43| 1.40| 0.37| 1.21| 0.11| 0.05| 0.05   |
| 1000 | 12 | 4        | 1.10| 2.43| 0.87| 3.20| 0.29| 0.14| 0.21   |

Table 4: Mean-squared error running lasso from Setup D. Results are averaged across 500 runs, rounded to two decimal places, and reported on an independent test set of size $n$.

| n    | d  | $\sigma$ | S   | T   | X   | U   | R   | RS  | oracle |
|------|----|----------|-----|-----|-----|-----|-----|-----|--------|
| 500  | 6  | 0.5      | 0.46|     | 0.37| 0.45| 1.20| 0.72| 0.47   |
| 500  | 6  | 1        | 0.77|     | 0.66| 0.75| 1.68| 0.81| 1.57   |
| 500  | 6  | 2        | 1.32|     | 1.23| 1.29| 1.81| 1.43| 1.79   |
| 500  | 6  | 4        | 2.02|     | 2.20| 2.20| 1.97| 2.10| 2.20   |
| 500  | 12 | 0.5      | 0.59|     | 0.44| 0.56| 1.19| 0.63| 1.08   |
| 500  | 12 | 1        | 0.94|     | 0.77| 0.88| 1.70| 0.96| 1.74   |
| 500  | 12 | 2        | 1.47|     | 1.38| 1.45| 1.84| 1.59| 1.81   |
| 500  | 12 | 4        | 2.06|     | 2.21| 1.98| 2.12| 2.28| 1.94   |
| 1000 | 6  | 0.5      | 0.27|     | 0.21| 0.27| 0.74| 0.30| 0.41   |
| 1000 | 6  | 1        | 0.50|     | 0.41| 0.48| 1.57| 0.54| 0.87   |
| 1000 | 6  | 2        | 0.93|     | 0.79| 0.91| 1.76| 0.97| 1.74   |
| 1000 | 6  | 4        | 1.61|     | 1.58| 1.56| 1.95| 1.73| 1.83   |
| 1000 | 12 | 0.5      | 0.35|     | 0.26| 0.34| 0.76| 0.38| 0.55   |
| 1000 | 12 | 1        | 0.61|     | 0.48| 0.57| 1.54| 0.63| 1.28   |
| 1000 | 12 | 2        | 1.10|     | 0.93| 1.05| 1.78| 1.11| 1.76   |
| 1000 | 12 | 4        | 1.76|     | 1.73| 1.68| 1.94| 1.82| 1.83   |

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### Table 5: Mean-squared error running boosting from Setup A. Results are averaged across 200 runs, rounded to two decimal places, and reported on an independent test set of size \( n \).

| \( n \) | \( d \) | \( \sigma \) | \( S \) | \( T \) | \( U \) | \( X \) | \( CB \) | \( R \) | \( \text{oracle} \) |
|---|---|---|---|---|---|---|---|---|---|
| 500 | 6 | 0.5 | 0.06 | 0.10 | 0.04 | 0.05 | 0.04 | **0.03** | 0.04 |
| 500 | 6 | 1 | 0.12 | 0.20 | 0.08 | 0.11 | 0.09 | **0.06** | 0.06 |
| 500 | 6 | 2 | 0.26 | 0.44 | 0.16 | 0.20 | 0.21 | **0.13** | 0.11 |
| 500 | 6 | 4 | 0.53 | 0.90 | **0.32** | 1.04 | 0.33 | 0.35 | 0.32 |
| 500 | 12 | 0.5 | 0.07 | 0.11 | **0.04** | 0.05 | 0.05 | **0.04** | 0.04 |
| 500 | 12 | 1 | 0.13 | 0.23 | 0.08 | 0.12 | 0.10 | **0.06** | 0.05 |
| 500 | 12 | 2 | 0.27 | 0.49 | 0.17 | 0.38 | 0.21 | **0.13** | 0.11 |
| 500 | 12 | 4 | 0.48 | 0.88 | 0.34 | 1.21 | 0.34 | **0.33** | 0.32 |
| 1000 | 6 | 0.5 | 0.05 | 0.07 | **0.02** | 0.05 | 0.03 | **0.02** | 0.03 |
| 1000 | 6 | 1 | 0.09 | 0.15 | **0.05** | 0.07 | 0.06 | **0.05** | 0.04 |
| 1000 | 6 | 2 | 0.20 | 0.36 | 0.11 | 0.20 | 0.16 | **0.09** | 0.08 |
| 1000 | 6 | 4 | 0.38 | 0.68 | 0.23 | 0.50 | 0.27 | **0.19** | 0.19 |
| 1000 | 12 | 0.5 | 0.05 | 0.08 | **0.03** | 0.05 | 0.03 | **0.03** | 0.03 |
| 1000 | 12 | 1 | 0.09 | 0.16 | **0.05** | 0.10 | 0.06 | **0.05** | 0.05 |
| 1000 | 12 | 2 | 0.21 | 0.36 | 0.11 | 0.21 | 0.15 | **0.08** | 0.08 |
| 1000 | 12 | 4 | 0.41 | 0.72 | 0.24 | 0.60 | 0.29 | **0.22** | 0.24 |

### Table 6: Mean-squared error running boosting from Setup B. Results are averaged across 200 runs, rounded to two decimal places, and reported on an independent test set of size \( n \).

| \( n \) | \( d \) | \( \sigma \) | \( S \) | \( T \) | \( X \) | \( U \) | \( CB \) | \( R \) | \( \text{oracle} \) |
|---|---|---|---|---|---|---|---|---|---|
| 500 | 6 | 0.5 | 0.19 | 0.28 | **0.14** | 0.20 | 0.28 | 0.20 | 0.14 |
| 500 | 6 | 1 | 0.33 | 0.48 | **0.27** | 0.41 | 0.37 | 0.33 | 0.28 |
| 500 | 6 | 2 | 0.67 | 0.89 | **0.56** | 0.84 | 0.67 | 0.68 | 0.62 |
| 500 | 6 | 4 | 1.40 | 1.76 | **1.10** | 1.50 | 1.33 | 1.20 | 1.19 |
| 500 | 12 | 0.5 | 0.22 | 0.30 | **0.15** | 0.22 | 0.35 | 0.21 | 0.15 |
| 500 | 12 | 1 | 0.37 | 0.50 | **0.29** | 0.43 | 0.46 | 0.37 | 0.31 |
| 500 | 12 | 2 | 0.77 | 0.95 | **0.58** | 0.89 | 0.79 | 0.74 | 0.68 |
| 500 | 12 | 4 | 1.63 | 1.87 | **1.10** | 1.56 | 1.41 | 1.41 | 1.27 |
| 1000 | 6 | 0.5 | 0.13 | 0.19 | **0.08** | 0.13 | 0.18 | 0.11 | 0.09 |
| 1000 | 6 | 1 | 0.21 | 0.33 | **0.17** | 0.25 | 0.24 | 0.22 | 0.19 |
| 1000 | 6 | 2 | 0.45 | 0.65 | **0.39** | 0.58 | 0.43 | 0.46 | 0.43 |
| 1000 | 6 | 4 | 1.01 | 1.34 | **0.82** | 1.20 | 0.89 | 1.01 | 1.00 |
| 1000 | 12 | 0.5 | 0.14 | 0.21 | **0.09** | 0.13 | 0.20 | 0.13 | 0.09 |
| 1000 | 12 | 1 | 0.25 | 0.34 | **0.18** | 0.26 | 0.28 | 0.24 | 0.21 |
| 1000 | 12 | 2 | 0.50 | 0.69 | **0.41** | 0.63 | 0.51 | 0.52 | 0.49 |
| 1000 | 12 | 4 | 1.16 | 1.33 | **0.84** | 1.24 | 1.01 | 1.12 | 1.08 |

Table 6: Mean-squared error running boosting from Setup B. Results are averaged across 200 runs, rounded to two decimal places, and reported on an independent test set of size \( n \).
Table 7: Mean-squared error running boosting from Setup C. Results are averaged across 200 runs, rounded to two decimal places, and reported on an independent test set of size $n$.

| n   | d  | $\sigma$ | S  | T   | X  | U  | CB | R   | oracle |
|-----|----|----------|----|-----|----|----|----|-----|--------|
| 500 | 6  | 0.5      | 0.30 | 0.65 | 0.13 | 0.97 | 0.65 | **0.08** | 0.03 |
| 500 | 6  | 1        | 0.46 | 0.97 | 0.23 | 0.73 | 0.70 | **0.15** | 0.08 |
| 500 | 6  | 2        | 0.90 | 1.73 | 0.44 | 0.86 | 0.82 | **0.26** | 0.26 |
| 500 | 6  | 4        | 1.65 | 2.91 | 0.91 | 1.74 | 0.96 | **0.57** | 0.43 |
| 500 | 12 | 0.5      | 0.32 | 0.68 | 0.15 | 0.90 | 0.69 | **0.09** | 0.03 |
| 500 | 12 | 1        | 0.53 | 1.02 | 0.25 | 0.93 | 0.72 | **0.17** | 0.10 |
| 500 | 12 | 2        | 0.98 | 1.83 | 0.47 | 0.95 | 0.84 | **0.29** | 0.23 |
| 1000| 6  | 0.5      | 0.20 | 0.43 | 0.08 | 0.90 | 0.35 | **0.05** | 0.02 |
| 1000| 6  | 1        | 0.31 | 0.67 | 0.14 | 0.82 | 0.41 | **0.11** | 0.07 |
| 1000| 6  | 2        | 0.65 | 1.20 | 0.29 | 0.65 | 0.54 | **0.20** | 0.19 |
| 1000| 6  | 4        | 1.28 | 2.33 | 0.63 | 1.09 | 0.79 | **0.42** | 0.38 |
| 1000| 12 | 0.5      | 0.21 | 0.46 | 0.09 | 1.02 | 0.38 | **0.06** | 0.03 |
| 1000| 12 | 1        | 0.36 | 0.70 | 0.15 | 0.86 | 0.42 | **0.12** | 0.07 |
| 1000| 12 | 2        | 0.74 | 1.28 | 0.31 | 0.84 | 0.61 | **0.23** | 0.20 |
| 1000| 12 | 4        | 1.38 | 2.45 | 0.65 | 1.31 | 0.82 | **0.40** | 0.37 |

Table 8: Mean-squared error running boosting from Setup D. Results are averaged across 200 runs, rounded to two decimal places, and reported on an independent test set of size $n$.

| n   | d  | $\sigma$ | S  | T   | X  | U  | CB | R   | oracle |
|-----|----|----------|----|-----|----|----|----|-----|--------|
| 500 | 6  | 0.5      | 0.36 | **0.30** | 0.37 | 0.57 | 0.50 | 0.43 | 0.39 |
| 500 | 6  | 1        | 0.55 | **0.53** | 0.57 | 0.96 | 0.76 | 0.66 | 0.65 |
| 500 | 6  | 2        | **0.92** | 0.99 | 1.02 | 1.60 | 1.21 | 1.12 | 1.13 |
| 500 | 6  | 4        | **1.48** | 1.86 | 1.60 | 2.36 | 1.60 | 1.81 | 1.71 |
| 500 | 12 | 0.5      | 0.44 | **0.34** | 0.43 | 0.63 | 0.55 | 0.48 | 0.43 |
| 500 | 12 | 1        | 0.65 | **0.57** | 0.64 | 1.04 | 0.84 | 0.74 | 0.74 |
| 500 | 12 | 2        | **1.05** | 1.06 | 1.10 | 1.66 | 1.35 | 1.24 | 1.26 |
| 500 | 12 | 4        | **1.66** | 1.88 | 1.67 | 2.29 | 1.68 | 1.88 | 1.91 |
| 1000| 6  | 0.5      | 0.24 | **0.20** | 0.25 | 0.42 | 0.41 | 0.29 | 0.26 |
| 1000| 6  | 1        | 0.39 | **0.36** | 0.40 | 0.73 | 0.56 | 0.46 | 0.45 |
| 1000| 6  | 2        | **0.68** | 0.71 | 0.73 | 1.33 | 0.94 | 0.81 | 0.83 |
| 1000| 6  | 4        | **1.23** | 1.45 | 1.34 | 1.98 | 1.41 | 1.44 | 1.51 |
| 1000| 12 | 0.5      | 0.29 | **0.22** | 0.28 | 0.47 | 0.41 | 0.32 | 0.30 |
| 1000| 12 | 1        | 0.45 | **0.38** | 0.45 | 0.78 | 0.62 | 0.52 | 0.51 |
| 1000| 12 | 2        | 0.80 | **0.77** | 0.83 | 1.46 | 1.08 | 0.94 | 0.93 |
| 1000| 12 | 4        | **1.38** | 1.53 | 1.43 | 1.99 | 1.53 | 1.65 | 1.62 |