Eigenvalues of Euclidean wedge domains in higher dimensions

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Abstract

In this paper, we use a weighted isoperimetric inequality to give a lower bound for the first Dirichlet eigenvalue of the Laplacian on a bounded domain inside a Euclidean cone. Our bound is sharp, in that only sectors realize it. This result generalizes a lower bound of Payne and Weinberger in two dimensions.

1 Introduction

Lower bounds for the first Dirichlet eigenvalue of the Laplacian often arise from an integral inequality relating the boundary of a domain to its interior (see, for example, [Ch, LT, P]). Moreover, these inequalities can lead one to a characterization of the optimal domains, that is, the domains for which the inequality is an equality. The Faber-Krahn inequality [FK] provides the model for such a lower bound, using the classical isoperimetric inequality to prove that, among all domains with the same volume in a simply connected space of constant curvature, geodesic balls have the least eigenvalue. In this paper we prove a similar estimate for domains inside a convex cone.

To state our main theorem, we first introduce some notation. Let \( \Omega \) be a convex domain in the upper unit hemisphere \( S_{\mathbb{R}}^{n-1} = S^{n-1} \cap \{ x_n > 0 \} \), and let

\[
W = \{(r, \theta) \mid r \geq 0, \theta \in \Omega \}
\]

be the cone over \( \Omega \). For \( r_0 > 0 \), define the sector

\[
S_{r_0} = \{(r, \theta) \mid 0 \leq r \leq r_0, \theta \in \Omega \}.
\]

If \( D \) is a bounded domain in \( W \), we denote the first Dirichlet eigenvalue of the Laplacian on \( D \) by \( \lambda_1(D) \).

Next, let \( \mu \) be the first Dirichlet eigenvalue of the Laplacian on \( \Omega \), with eigenfunction \( \psi \). Normalize \( \psi \) so that \( \int_{\Omega} \psi^2 = 1 \). Observe that

\[
w(r, \theta) = r^\alpha \psi(\theta), \quad \alpha := \frac{2-n}{2} + \sqrt{\left(\frac{2-n}{2}\right)^2 + \mu} \tag{1}
\]

is a positive harmonic function in \( W \), which is zero on the boundary of \( W \). Monotonicity, with \( \Omega \subset S_{\mathbb{R}}^{n-1} \), implies \( \mu > n - 1 \), and so \( \alpha > 1 \).

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Theorem 1. Let $D$ be a bounded domain in the cone $\mathcal{W}$, and choose $r_0$ so that
\[
\int_D w^2 dV = \int_{S_{r_0}} w^2 dV.
\]
Then $\lambda_1(D) \geq \lambda_1(S_{r_0})$, with equality if and only if $D = S_{r_0}$ almost everywhere.

This theorem extends an eigenvalue estimate of Payne and Weinberger [PW] to dimensions greater than two. Like their estimate, ours relies on a weighted isoperimetric inequality; such inequalities are sometimes called isoperimetric inequalities for measures [Ro] or isoperimetric inequalities for densities [M,RBM].

The rest of this paper proceeds as follows. First we compute the first Dirichlet eigenvalue of the Laplacian on a sector in Section 2, both in terms of the radius $r_0$ and the integral of the weight function $w$. The bulk of this paper lies in Section 3 where we prove a weighted isoperimetric inequality for domains in the cone $\mathcal{W}$. Finally, in Section 4 we use our isoperimetric inequality to estimate the Rayleigh quotient, proving our main theorem.

We conclude this introduction by briefly comparing the two-dimensional and higher-dimensional theorems. First observe that in the two-dimensional case, the link $\Omega$ is an interval $(0, \pi/\alpha)$, for some $\alpha > 1$, and the cone $\mathcal{W}$ has the form
\[
\mathcal{W} = \{(r, \theta) \mid 0 < \theta < \pi/\alpha\}
\]
in polar coordinates. In both the two-dimensional and higher-dimensional cases, the key weighted isoperimetric inequality comes from an inequality for domains in a half-space (see Lemma 4 below). In two dimensions, there is an obvious way to open the cone up to a half-plane, whereas in higher dimensions this is more subtle. The second main difference between the two proofs is a technical complication. In the two-dimensional case, the first eigenvalue of the link is $\mu = \alpha^2$, and the harmonic weight is $w(r, \theta) = r^\alpha \sin(\alpha \theta)$ (up to a constant multiple). The fact the the opening angle of the cone, the first eigenvalue of the link, and the exponent of the radial part of the harmonic weight function all agree simplifies much of the analysis.

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2 First eigenvalue of a sector

In this section we compute the first Dirichlet eigenvalue of a sector $S_{r_0}$, to compare with our estimates for the Rayleigh quotient below in Section 4.

Lemma 2. Let
\[
a = \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu = \alpha + \frac{n-2}{2}}
\]
and consider the sector $S_{r_0}$ for $r_0 > 0$. Then
\[
\lambda_1(S_{r_0}) = \frac{j_\alpha^2}{r_0^2} = \left[ (2a + 2) \int_{S_{r_0}} w^2 dV \right]^{-\frac{1}{2}} j_\alpha^2,
\]
where $j_\alpha$ is the first positive zero of the Bessel function $J_\alpha$.

Notice that $a = \alpha$ in dimension two.

Proof. We write the eigenfunction for the sector as $u(r, \theta) = f(r)\psi(\theta)$, and denote eigenvalue by $\lambda$. Then we rewrite the eigenvalue equation
\[
-\lambda u = \Delta u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_{n-1} u,
\]

2
as

\[ 0 = r^2 f'' + (n-1)rf' + (\lambda r^2 - \mu) f. \]  

(2)

Look for a solution of the form \( f(r) = r^q J_a(kr) \), where \( J_a \) is the Bessel function of weight \( a \), and \( k \) and \( q \) are positive constants. Observe that

\[ k^2 r^2 J''_a(kr) + kr J'_a(kr) + (k^2 r^2 - a^2) J_a(kr) = 0. \]

Plugging this choice of \( f \) into equation (2), we get

\[
0 = r^2 f'' + (n-1)rf' + (\lambda r^2 - \mu) f \\
= q(q-1)r^q J_a + 2kqr^{q+1} J'_a + k^2 r^{q+2} J''_a + (n-1) qr^q J_a + (n-1) kr^{q+1} J'_a + (\lambda r^2 - \mu) r^q J_a(kr) \\
= r^q[k^2 r^2 J''_a + k(2q + n - 1)r J'_a + (\lambda r^2 - \mu + q(n + q - 2))J_a] \\
= r^q[k(2q + n - 2)r J'_a + ((\lambda - k^2)r^2 - \mu + q(n + q - 2) + a^2)J_a].
\]

Setting the individual terms to zero, we obtain

\[
q = \frac{2 - n}{2}, \quad k = \sqrt{\lambda}, \quad a^2 = \mu - q(n + q - 2) = \mu + \left(\frac{n-2}{2}\right)^2.
\]

(3)

The fact that the eigenfunction vanishes on the boundary of \( S_{r_0} \) forces

\[ 0 = f(r_0) = r_0^{2a+n} J_a(\sqrt{\lambda} r_0) \Rightarrow \sqrt{\lambda} r_0 = j_a. \]

In other words, the eigenvalue is

\[ \lambda = \lambda_1(S_{r_0}) = \frac{j_a^2}{r_0^2}, \quad a = \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu} \]

and the eigenfunction is

\[ u(r, \theta) = r^{\frac{2a+n}{2}} J_a \left(\frac{j_a r}{r_0}\right) \psi(\theta). \]

Finally, we rewrite our expression for \( \lambda \) in terms of the integral \( \int_{S_{r_0}} w^2 dV \). Recall

\[ \alpha = \frac{2 - n}{2} + \sqrt{\left(\frac{2 - n}{2}\right)^2 + \mu} = a - \frac{n-2}{2}. \]

By explicit computation,

\[
\int_{S_{r_0}} w^2 dV = \int_{\Omega} \psi^2 dA_{s_{\alpha-1}} \int_{0}^{r_0} r^{2a+n-1} dr = \frac{r_0^{2a+n}}{2a+n} = \frac{r_0^{2a+2}}{2a+2},
\]

or

\[ r_0 = \left[(2a+n) \int_{S_{r_0}} w^2 dV \right]^{\frac{1}{2a+n}} = \left[(2a+2) \int_{S_{r_0}} w^2 dV \right]^{\frac{1}{2a+2}}. \]

Plugging this into our formula for \( \lambda \), we see

\[ \lambda_1(S_{r_0}) = \left[(2a+2) \int_{S_{r_0}} w^2 dV \right]^{-\frac{1}{2a+2}} j_a^2. \]

□
3 Isoperimetric inequality

The goal of this section is to establish the following weighted isoperimetric inequality for bounded domains $D \subset \mathcal{W}$.

**Proposition 3.** Let $D \subset \mathcal{W}$ be a bounded domain with piecewise smooth boundary. Then

$$
\int_{\partial D} w^2 dA \geq \left(2a + 2\right) \int_D w^2 dV \frac{2a+2}{2a+4}, \quad a = \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu} = \alpha + \frac{n-2}{2}.
$$

Moreover, equality implies $D$ is (almost everywhere) a sector $S_{\alpha_0}$ for some $\alpha_0 > 0$.

**Remark 1.** In proving our weighted isoperimetric inequality, we will not explicitly compute the coefficient $(2a + 2)^{2a+4}$, but rather show that there is some constant $c$ depending only on $n$ and $\Omega$ such that

$$
\int_{\partial D} w^2 dA \geq c \left[ \int_D w^2 dV \right]^{\frac{2a+4}{2a+2}}.
$$

Then we recover the constant $c$ by computing the relevant integrals for a sector.

We start with the following weighted isoperimetric problem in a half-space.

**Lemma 4.** Let $\tilde{D} \subset \mathbb{R}^n_+ = \{ x_n \geq 0 \}$ be a bounded domain with piecewise smooth boundary. If we hold $\int_{\tilde{D}} x_n^2 dV$ constant, the minimizers of $\int_{\partial \tilde{D}} x_n^2 dA$ are hemispheres whose centers lie on the hyperplane $\{ x_n = 0 \}$.

**Proof.** If $n = 1$ the lemma is automatically true. Payne and Weinberger [PW] proved that in dimension two the minimizers of the variational problem are semi-circles centered on the $x_1$ axis. Indeed, if one lets $\tilde{D} = \{ (x_1, x_2) \mid 0 \leq x_2 \leq \phi(x_1) \}$, a straight-forward computation shows that the Euler-Lagrange equation is

$$
2\sqrt{1 + (\phi')^2} - \frac{2(\phi')^2}{\sqrt{1 + (\phi')^2}} - \phi \left( \frac{\phi'}{\sqrt{1 + (\phi')^2}} \right)' = \Lambda \phi,
$$

where the constant $\Lambda$ is the Lagrange multiplier. For any choice of constants $R$ and $c$, the function $\phi(x_1) = \sqrt{R^2 - (x_1 - c)^2}$ solves equation (5). By uniqueness of solutions to ODEs, semi-circles are the only critical points and thus minimize. We pause here to remark that the Euler-Lagrange equation in higher dimensions is

$$
2\sqrt{1 + |\nabla \phi|^2} - \frac{2|\nabla \phi|^2}{\sqrt{1 + |\nabla \phi|^2}} - \phi \div \left( \frac{\nabla \phi}{\sqrt{1 + |\nabla \phi|^2}} \right) = \Lambda \phi.
$$

For $n \geq 3$ we reduce the variational problem to a two-dimensional problem using Steiner symmetrization (see note A of [PS]), starting with $\{ x_n = 0 \}$ as the symmetry hyperplane. Given a point $\bar{x} = (0, x_2, \ldots, x_n)$ in the symmetry hyperplane, let $l_\bar{x}$ be the line through $\bar{x}$ perpendicular to to $\{ x_n = 0 \}$. By Sard’s theorem, for almost every $\bar{x}$ such that the intersection $\tilde{D} \cap l_\bar{x}$ is nonempty one can write

$$
\tilde{D} \cap l_\bar{x} = \cup_{k=1}^{m_\bar{x}} \{(a_k, x_2, \ldots, x_n), (b_k, x_2, \ldots, x_n)\}.
$$

The symmetrized domain $\tilde{D}_1$ is given by

$$
\tilde{D}_1 \cap l_\bar{x} = \{(-\bar{a}, x_2, \ldots, x_n), (\bar{a}, x_2, \ldots, x_n)\}, \quad 2\bar{a} = \sum_{k=1}^{m_\bar{x}} b_k - a_k.
$$

Observe that $\tilde{D}_1$ is now symmetric about the $\{ x_1 = 0 \}$ hyperplane, and has the same volume as $\tilde{D}$. Also, because $x_n$ is constant along the line $l_\bar{x}$ and the length of $\tilde{D} \cap l_\bar{x}$ is the same as the length of $\tilde{D}_1 \cap l_\bar{x}$, Fubini’s theorem implies

$$
\int_{\tilde{D}_1} x_n^2 dV = \int_{\tilde{D}} x_n^2 dV.
$$
Next we show examine the boundary integral. Let $G$ be the projection of $\partial D$ onto the $\{x_1 = 0\}$ hyperplane, and notice that $G$ is also the orthogonal projection of $\bar{D}$ onto $\{x_1 = 0\}$. Moreover, our representations of $D$ and $\bar{D}$ as graphs over the $\{x_1 = 0\}$ hyperplane in equations (6) and (7) are smooth almost everywhere, and so the integral identities below are valid. Now rewrite the boundary integral as

$$\int_{\partial\bar{D}} x_n^2 dA = \int_0^\infty \int_{\partial D \cap \{x_n = t\}} x_n^2 dA$$

$$= \int_0^\infty t^2 \int_{G \cap \{x_n = 1\}} \left[1 + \sum_{k=1}^m \sum_{j=2}^n \left(\frac{\partial a_k}{\partial x_j}\right)^2 + \left(\frac{\partial b_k}{\partial x_j}\right)^2 \right] dx_2 \cdots dx_{n-1} dt$$

$$\geq \int_0^\infty t^2 \int_{G \cap \{x_n = 1\}} \left[1 + 2 \sum_{j=2}^n \left(\frac{\partial a}{\partial x_j}\right)^2 \right] dx_2 \cdots dx_{n-1} dt$$

$$= \int_{\partial\bar{D}_1} x_n^2 dA.$$  

Here we have used the fact that $2\bar{a} = \sum_{k=1}^m b_k - a_k$ and Minkowski’s inequality.

The lemma now follows by induction.  

**Remark 2.** One can adapt this proof to show that for any $q > -1$ the minimizers of $\int_{\partial\bar{D}} x^q dA$, with $\int_{\bar{D}} x^q dV$ held constant, are also hemispheres whose centers lie on the $\{x_1 = 0\}$ hyperplane. (The Euler-Lagrange equation is almost the same for the case of an arbitrary power $q$.) The case of $q = 0$ is the classical isoperimetric inequality. The condition $q > -1$ ensures that the relevant integrals converge.

Let

$$\omega_k = \text{Vol}(S^{k-1} \subset \mathbb{R}^k) = \frac{k\pi^{k/2}}{1!k/2!}.$$  

We combine an explicit computation for hemispheres with Lemma 1 to see that

$$\int_{\partial\bar{D}} x_n^2 dA \geq \omega_{n-1} \left[\int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx \right]^{1/2} \cdot \left[\int_{\bar{D}} x_n^2 dV \right]^{1/2}$$

$$= c(n) \left[\int_{\bar{D}} x_n^2 dV \right]^{1/2}$$

for all bounded domains $\bar{D} \subset \mathbb{R}^n_+$.

Next we open the wedge $W$ up to the upper half-space $\mathbb{R}^n_+$. To describe this map, recall that the first eigenvalue of $\Omega$ is $\mu$, with eigenfunction $\psi$, and that our harmonic weight function is $w(r, \theta) = r^n \psi^2(\theta)$, with $\alpha$ given by \[.\] Now let

$$\Psi: W \rightarrow \mathbb{R}^n_+, \quad \Psi(r, \theta) = r^{\alpha n - 1} \omega_\alpha (\Pi(\nabla \psi), \psi) = (x_1, \ldots, x_n).$$

Here we consider $\nabla \psi$ as a vector in $T_S S^{n-1} \subset \mathbb{R}^n$, and let $\Pi$ be orthogonal projection onto the horizontal hyperplane $\{x_n = 0\}$. Note that, because $\Omega$ is contained in the open upper hemisphere, $\Pi: T_S S^{n-1} \rightarrow \mathbb{R}^{n-1} = \{x_n = 0\}$ is a linear isomorphism for each $\theta \in \Omega$, and so $\Pi(\nabla \psi) = 0$ only if $\nabla \psi = 0$. Also, $\Omega$ is convex, so a theorem of Brascamp and Lieb [42] (also see [43]) implies the first eigenfunction $\psi$ is log concave. In particular, the only critical point of $\psi$ is its maximum, and this is a nondegenerate critical point. We conclude $\Psi$ is indeed a diffeomorphism.

Next we estimate the effect of $\Psi$ on the arclength and area elements. Let $d\theta$ be the standard arclength element on $S^{n-1}$, so that $ds = dr^2 + r^2 d\theta^2$ is the arclength element in $D$. We denote the arclength element in the image domain $\bar{D}$ by $d\bar{s}^2 = dx_1^2 + \cdots + dx_n^2$. We also denote the corresponding area elements by $dA$ (on $\partial D$) and $d\bar{A}$ (on $\partial\bar{D}$). A quick computation gives

$$dx_1^2 + \cdots + dx_n^2 \leq \left(\frac{2\alpha + n - 1}{n + 1}\right)^2 r^{4\alpha n - 4} \left(\psi^2 + |\nabla \psi|^2\right) dr^2 + r \frac{4\alpha + n - 2}{n + 1} \left(|\nabla \psi|^2 + |D^2 \psi|^2\right) d\theta^2$$

$$\leq c\|\psi\|_2^2 r^{4\alpha n - 4} (dr^2 + r^2 d\theta^2).$$
Here \(|D^2\psi|\) is the \(L^2\)-norm of \(D^2\psi\), considered as a quadratic form. We have also used the fact that \(\Pi\) is norm-decreasing, so (for instance) \(|\Pi(\nabla\psi)| \leq |\nabla\psi|\). Combine the computation above with estimates for the eigenfunction \(\psi\) (see \([L]\)) to conclude that there is a constant \(\hat{c} = \hat{c}(n, \Omega)\) such that
\[
dx_1^2 + \cdots + dx_n^2 \leq \hat{c}(n, \Omega) r^{\frac{4(n-1)}{n+1}} (dr^2 + r^2 d\theta^2).
\]
Therefore, for some constant \(\hat{c}\) depending on \(n\) and \(\Omega\),
\[
x_a^2 d\tilde{A} \leq \hat{c} r^{\frac{4n+2n-2}{n+1}} \psi^2 \left(r^{\frac{2n+2}{n+1}}\right)^{n-1} dA = \hat{c} r^{2n} \psi^2 dA = \hat{c} w^2 dA.
\]

In order to prove the inequality \((11)\) we need one final ingredient, which is an inequality of Szegő \([Sz]\) (also see Lemma 2 of \([RT]\)). If \(f \geq 0\), \(F' = f\), \(g\) is nondecreasing, and \(G' = g\), then for any bounded measurable set \(E \subset \mathbb{R}\),
\[
G \left(\int_E f(x) dx\right) \leq \int_E g(F(x)) f(x) dx,
\]
with equality if and only if \(E\) is almost everywhere an interval of the form \([0, R]\).

We now prove our weighted isoperimetric inequality.

**Proof.** Let \(D\) be a bounded domain in the wedge \(W\), and \(\tilde{D} = \Psi(D)\) its image under the change of variables. We apply inequality \((10)\) to the isoperimetric inequality \((8)\), and conclude
\[
\int_{\partial D} w^2 dA \geq c \int_{\partial \tilde{D}} x_a^2 d\tilde{A} \geq c \left[\int_{\tilde{D}} x_a^2 d\tilde{V}\right]^{\frac{n+1}{n+1}}
\]
\[
= c \left[\int_{\tilde{D}} r^{\frac{4n+2n-2}{n+1}} \psi^2 |\det(D\Psi)| drdA_{g_{n-1}}\right]^{\frac{n+1}{n+1}}
\]
\[
\geq c \left[\int_{\tilde{D}} r^{\frac{4n+2n-2}{n+1}} \psi^2 r^{\frac{2n}{n+1}} \left(r^{\frac{2n}{n+1}}\right)^{n-1} drdA_{g_{n-1}}\right]^{\frac{n+1}{n+1}}
\]
\[
= c \left[\int_{\tilde{D}} r^{\frac{2n+2n-2}{n+1}} \psi^2 drdA_{g_{n-1}}\right]^{\frac{n+1}{n+1}}.
\]

Here \(c\) is a generic constant depending only on \(n\) and \(\Omega\) which we allow to change line to line in the computation. The estimate of \(|\det D\Psi|\) arises from
\[
D\Psi = \begin{bmatrix}
\frac{r^{\frac{2n+2}{n+1}}}{r^{\frac{2n}{n+1}} \Pi(\nabla\psi)} & \left(\frac{2n+2}{n+1}\right) r^{\frac{2n}{n+1}} \Pi(\nabla\psi)
\end{bmatrix}.
\]
Restricted to \(\{r = 1\}\), the map \(\Psi\) is a diffeomorphism between compact domains, and so there are positive constants \(c_1 \leq c_2\) such that
\[
c_1 \leq |\det D\Psi|_{|r=1|} \leq c_2.
\]

Now compute \(\det D\Psi\) in general by expanding along the last column. Each entry in the last column of \(D\Psi\) has a factor of \(r^{\frac{2n}{n+1}}\), while all the other entries have a factor of \(r^{\frac{2n+2}{n+1}}\). Thus
\[
|\det D\Psi| \geq c_1 r^{\frac{2n}{n+1}} \left(r^{\frac{2n}{n+1}}\right)^{n-1} = c_1 r^{\frac{2n}{n+1} - 2n - 1}.
\]

Denote the inner integral of \((12)\) by
\[
I = \int_{\tilde{D}} r^{\frac{2n+2n-2+2n-3}{n+1}} \psi^2 drdA_{g_{n-1}}.
\]

For a given \(\theta \in \Omega\), we let \(L_\theta\) be the radial slice
\[
L_\theta = \{r \in [0, \infty) \mid (r, \theta) \in D\}.
\]
and rewrite the integral $I$ as

$$I = \int_{\Omega} \int_{L_\theta} r^{\beta+1} dA_{\gamma} = \int_{\Omega} r^\beta (\beta+1)^\gamma dr \psi^2(\theta) dA_{\gamma}.$$

Here $\beta$ and $\gamma$ are positive parameters we will choose later.

We let $f(r) = (\beta+1)r^\beta, \quad g(x) = x^\gamma,$ and apply inequality (11), yielding

$$I \geq \frac{1}{\beta + 1} \int_{\Omega} \int_{L_\theta} f(r)g(F(r)) dr \psi^2 dA_{\gamma}.$$

Moreover, equality only occurs if $L_\theta$ is an interval of the form $(0, R(\theta))$ for almost every $\theta$.

Define the measure $\nu$ on $\Omega$ by $d\nu = \psi^2 dA_{\gamma}$, and notice it has total measure $\nu(\Omega) = 1$. Applying Hölder’s inequality with exponents $\gamma + 1$ and $\frac{\gamma+1}{\gamma}$, to the functions $h(\theta) = \int_{L_\theta} r^\beta dr$ and 1, gives us

$$\left[ \int_{\Omega} \left( \int_{L_\theta} r^\beta dr \right)^{\gamma+1} d\nu \right]^{\frac{\gamma}{\gamma+1}} = \left[ \int_{\Omega} \left( \int_{L_\theta} r^\beta dr \right)^{\gamma+1} d\nu \right]^{\frac{\gamma}{\gamma+1}} \left[ \int_{\Omega} d\nu \right]^{\frac{\gamma}{\gamma+1}} \geq \int_{\Omega} \int_{L_\theta} r^\beta dr d\nu.$$

Raise each side of this inequality to the power $\gamma + 1$, and plug the result into inequality (13) to obtain

$$I \geq \frac{1}{(\beta+1)\gamma+1} \left( \int_{\Omega} \int_{L_\theta} r^\beta dr \psi^2(\theta) dA_{\gamma} \right)^{\frac{\gamma+1}{\gamma}}. \quad (14)$$

Moreover, equality only occurs if $\int_{L_\theta} r^\beta dr$ is constant in $\theta$.

Plug the inequality (14) into (12) to get

$$\int_{\partial D} w^2 dA \geq c \left( \int_{\Omega} \int_{L_\theta} r^\beta dr \psi^2 dA_{\gamma} \right)^{\frac{(\gamma+1)(\gamma+1)}{\gamma+2}}. \quad (15)$$

Finally, we choose the parameters $\beta$ and $\gamma$. In order for the integral to be $\int_D w^2 dV$, we need $\beta = 2\alpha + n - 1$. Thus

$$r^{2\alpha+2\alpha^2+n-3} \gamma \Rightarrow \gamma = \frac{2\alpha - 2}{2\alpha + n^2 + 2\alpha + n} = \frac{2\alpha - 2}{(2\alpha + n)(n+1)}.$$

Plugging these choices of $\beta$ and $\gamma$ into inequality (15) gives us

$$\int_{\partial D} w^2 dA \geq c \left( \int_D w^2 dV \right)^{\frac{2\gamma+1}{\gamma+2}} = c \left( \int_D w^2 dV \right)^{\frac{2\alpha+1}{\gamma+2}},$$

as claimed.

Now consider the case of equality. To achieve equality, we must have equality in (13), which forces $L_\theta = \{ r \in [0, \infty) \mid (r, \theta) \in D \} = (0, R(\theta))$ almost everywhere. Moreover, equality in our use of Hölder’s inequality forces $\int_{L_\theta} r^\beta dr$ to be constant in $\theta$. If $L_\theta = (0, R(\theta))$, this latter integral is $\frac{1}{\beta+1} R^\beta(\theta)$, which can only be constant if the function $R(\theta)$ is constant. The combination of these two properties forces $D$ to be a sector.
At this point, we can recover the constant \( c \) by examining the case of the sector. If \( D = \mathcal{S}_{r_0} \), then
\[
\int_{\mathcal{S}_{r_0}} w^2 dV = \frac{1}{2a + 2} r_0^{2a+2}, \quad \int_{\partial \mathcal{S}_{r_0}} w^2 dA = r_0^{2a+1}.
\]
Solving for \( r_0 \) and rearranging, we get
\[
\int_{\partial \mathcal{S}_{r_0}} w^2 dA = \left(2a + 2\right) \int_{\mathcal{S}_{r_0}} w^2 dV \]
and so
\[
c = (2a + 2)^{2a+1}
\]
as we claimed.

4 Estimating the Rayleigh quotient

In this section we prove our main theorem by estimating the Rayleigh quotient
\[
\frac{\int_D |\nabla u|^2 dV}{\int_D u^2 dV}
\]
for an appropriate test function. We proceed as in [PW] and [RT].

Proof. Let \( u \geq 0 \), and write \( u(r, \theta) = w(r, \theta)v(r, \theta) \) for some \( v \in C^2_0(W) \). Using the fact that \( \Delta w = 0 \) and the divergence theorem, we see
\[
\int_D vw(\nabla v, \nabla w) dV = -\int_D v \text{div}(vw\nabla w) dV = -\int_D v (vw\Delta w + v|\nabla w|^2 + w\langle \nabla v, \nabla w \rangle) dV
\]
Rearranging yields
\[
2 \int_D vw(\nabla v, \nabla w) dV = -\int_D v^2 |\nabla w|^2 dV,
\]
and so
\[
\int_D |\nabla u|^2 dV = \int_D v^2 |\nabla w|^2 + 2vw(\nabla v, \nabla w) + w^2 |\nabla v|^2 dV = \int_D w^2 |\nabla v|^2 dV.
\]
For \( 0 \leq t \leq \bar{v} = \max(v) \), let \( D_t := v^{-1}((t, \bar{v})) \) and
\[
\zeta(t) := \int_{D_t} w^2 dV.
\]
By the coarea formula,
\[
\frac{d\zeta}{dt} = -\int_{\partial D_t} \frac{w^2}{|\nabla v|} dA < 0,
\]
and so \( \zeta \) is a decreasing function and has an inverse \( t = t(\zeta) \). Then
\[
\int_D w^2 v^2 dV = \int_0^\bar{v} t^2 \int_{\partial D_t} \frac{w^2}{|\nabla v|} dAdt = -\int_0^\bar{v} t^2 \frac{d\zeta}{dt}(t)dt = \int_0^\bar{v} t^2 d\zeta.
\]
Here \( \bar{\zeta} = \zeta(0) = \max(\zeta) \).

Next we estimate the Dirichlet energy. For this computation, it will be convenient to define
\[
p = \frac{2a + 1}{2a + 2} = \frac{2a + n - 1}{2a + n}.
\]
Note that, applying the Cauchy-Schwartz inequality to \( w^2 = (|\nabla v|^{-1/2})(|\nabla v|^{1/2}) \), we have
\[
\int_{\partial D_t} w^2 dA \leq \left( \int_{\partial D_t} \frac{w^2}{|\nabla v|} dA \right)^{1/2} \left( \int_{\partial D_t} w^2 |\nabla v| dA \right)^{1/2} \Rightarrow \int_{\partial D_t} w^2 |\nabla v| dA \geq \frac{\left( \int_{\partial D_t} w^2 dA \right)^2}{\int_{\partial D_t} \frac{w^2}{|\nabla v|} dA}
\]
Now use the coarea formula, the Cauchy-Schwartz inequality as above, and our inequality (14) to conclude
\[
\int_D |\nabla u|^2 dV = \int_D w^2 |\nabla v|^2 dV = \int_0^1 \int_{\partial D_t} w^2 |\nabla v| dAdt \tag{17}
\]
\[
\geq \int_0^1 \left( \int_{\partial D_t} w^2 dA \right)^2 dt
\]
\[
\geq (2a + 2) \frac{2a+1}{3} \int_0^1 \left( \frac{\int_{\partial D_t} w^2 dV}{-\frac{d}{dt}} \right)^{2p} dt
\]
\[
= (2a + 2) \frac{2a+1}{3} \int_0^1 \left( \frac{\frac{d}{dt} t^2}{d\zeta} \right)^{2p} d\zeta.
\]

Now we define \( \lambda_\ast \) to be the least eigenvalue of
\[
\frac{d}{d\zeta} \left( \zeta^{2p} \frac{d}{d\zeta} \right) + \lambda_\ast t = 0, \quad t(\zeta) = 0 = \lim_{\zeta \to 0^+} \left( \zeta^{2p} \frac{dt}{d\zeta} \right).
\]

Observe that, by the boundary conditions in equation (13),
\[
\int_0^\zeta \zeta^{2p} \left( \frac{dt}{d\zeta} \right)^2 d\zeta = -\int_0^\zeta t \frac{d}{d\zeta} \left( \zeta^{2p} \frac{dt}{d\zeta} \right) d\zeta.
\]

Thus we combine inequalities (16) and (17) with the eigenvalue equation (18) to get
\[
(2a + 2) \frac{2a+1}{3} \lambda_\ast = (2a + 2) \frac{2a+1}{3} \inf \left[ \frac{\int_0^\zeta \zeta^{2p} \left( \frac{dt}{d\zeta} \right)^2 d\zeta}{\int_0^1 t^2 d\zeta} \right] \leq \inf \left[ \frac{\int_D |\nabla u|^2 dV}{\int_D w^2 dV} \right] = \lambda_1(D). \tag{19}
\]

We can write the solution to (13) in terms of Bessel functions. We try
\[
t(\zeta) = \zeta^q J_b \left( \frac{\sqrt{\lambda_\ast}}{m} \zeta^m \right),
\]
where \( J_b \) is the Bessel function of weight \( b \). Taking a derivative, we see
\[
t' = q \zeta^{q-1} J_b + \sqrt{\lambda_\ast} \zeta^{q+m-1} J'_b,
\]
and so equation (13) becomes
\[
-\lambda_\ast \zeta^q J_b = \frac{d}{d\zeta} \left( \zeta^{2p} t' (\zeta) \right) = \frac{d}{d\zeta} \left( q \zeta^{2p+q-1} J_b + \sqrt{\lambda_\ast} \zeta^{2p+q+m-1} J'_b \right)
\]
\[
= (2pq + q - q) \zeta^{2p+q-2} J_b + \sqrt{\lambda_\ast} (2p + 2q + m - 1) \zeta^{2p+q+m-2} J'_b + \lambda_\ast \zeta^{2p+q+m-2} J''_b
\]
\[
= (b^2 m^2 + 2pq + q - q) \zeta^{2p+q-2} J_b + \lambda_\ast \zeta^{2p+q+m-2} J_b + \sqrt{\lambda_\ast} (2p + 2q - 1) \zeta^{2p+q+m-2} J'_b,
\]
where we have used Bessel's identity in the last equality. We force the last term to be zero by choosing
\[
q = \frac{1 - 2p}{2} = \frac{2 - 2\alpha - n}{4a + n} = -\frac{2a}{4a - n + 4}.
\]
We force the middle term to be equal to the left hand side \(-\lambda_\ast t = -\lambda_\ast \zeta^q J_b\) by choosing
\[
m = 1 - p = \frac{1}{2a + n} = \frac{1}{2a + 2}.
\]
Finally, we force the last term to be zero by choosing
\[
b^2 = \left( \frac{2p - 1}{2p - 2} \right)^2.
\]
Notice that we have a choice in the sign of the square root, and we choose
\[ b = -\frac{2p - 1}{2p - 2} - \frac{1 - 2p}{2p - 2} = \frac{1 - \frac{2a + 1}{a + 1}}{2} = \frac{a + 1 - 2a - 1}{2a + 1 - 2a - 2} = a. \]

Putting this all together, we have
\[ t(\zeta) = \zeta^{-\frac{1}{2a + 1}} J_a \left( (2a + 2) \sqrt{\lambda_0 \zeta} \right). \]

Next we use the boundary condition \( t(\bar{\zeta}) = 0 \) to find \( \lambda_* \). If \( j_a \) is the first positive zero of \( J_a \), we must have
\[ j_a = (2a + 2) \sqrt{\lambda_0 \zeta}, \]
and so
\[ \lambda_* = \frac{\bar{\zeta}^{-\frac{1}{2a + 1}} j_a^2}{2} = \frac{1}{(2a + 2)^2} \left[ \int_D w^2 dV \right] \frac{1}{2} \lambda_0. \]

Using the explicit computation of \( \lambda_* \) above to compare inequality (19) with Lemma 2 completes the proof of our main theorem. \( \square \)

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