THE MANIFOLD OF FINITE RANK PROJECTIONS IN THE SPACE $\mathcal{L}(H)$.

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Abstract. Given a complex Hilbert space $H$ and the von Neumann algebra $\mathcal{L}(H)$ of all bounded linear operators in $H$, we study the Grassmann manifold $M$ of all projections in $\mathcal{L}(H)$ that have a fixed finite rank $r$. To do it we take the Jordan-Banach triple (or JB*-triple) approach which allows us to define a natural Levi-Civita connection on $M$ by using algebraic tools. We identify the geodesics and the Riemann distance and establish some properties of $M$.

0 Introduction

In this paper we are concerned with the differential geometry of the infinite-dimensional Grassmann manifold $M$ of all projections in $Z := \mathcal{L}(H)$, the space of bounded linear operators $z: H \rightarrow H$ in a complex Hilbert space $H$. Grassmann manifolds are a classical object in Differential Geometry and in recent years several authors have considered them in the Banach space setting. Besides the Grassmann structure, a Riemann and a Kähler structure has sometimes been defined even in the infinite-dimensional setting. Let us recall some aspects of the history of the topic that are relevant for our purpose.

The study of the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra was made by U. Hirzebruch in [4], who proved that such a manifold is a compact symmetric Riemann space of rank 1, and that every such a space arises in this way. Later on, Nomura in [13, 14] established similar results for the manifold of fixed finite rank projections in a topologically simple real Jordan-Hilbert algebra. On the other hand, the Grassmann manifold $M$ of all projections in the space $Z := \mathcal{L}(H)$ of bounded linear operators has been discussed by Kaup in [7] and [10]. It is therefore reasonable to ask whether a Riemann structure can always be defined in $M$ and how does it behave when it exists. It is known that $M$ has several connected components $M_r \subset M$ each of which consists of the projections in $\mathcal{L}(H)$ that have a fixed rank $r$, $1 \leq r \leq \infty$. We prove that $M_r$ admits a Riemann structure if and only if $r < \infty$ establishing a distinction between the finite and the infinite dimensional cases. We then assume $r < \infty$ and proceed to discuss the behaviour of the Riemann manifold $M_r$, which looks very much like in the finite-dimensional case. One of the novelties is that we take JB*-triple approach instead of the Jordan-algebra approach of [4] and [13]. As noted in [1] and [5], within this context the algebraic structure of JB*-triple acts as a substitute for the Jordan algebra structure and provides a local scalar product known as the Levi form [10]. Although $\mathcal{L}(H)$ is not a Hilbert space, the JB*-triple approach and the use of the Levi form allows us to define a torsion-free affine connection $\nabla$ on

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$M_e$ that is invariant under the group $\text{Aut}^\circ(Z)$ of all surjective linear isometries of $\mathcal{L}(H)$. We integrate the equation of the geodesics and define an $\text{Aut}^\circ(Z)$-invariant Riemann metric on $M_e$ with respect to which $\nabla$ is a Levi-Civita connection. We prove that any two distinct points in $M_e$ can be joined by a geodesic which (except for the case of a pair of antipodal points) is uniquely determined and is a minimizing curve for the Riemann distance, that is also computed. We prove that $M_e$ is a symmetric manifold on which $\text{Aut}^\circ(Z)$ acts transitively as a group of isometries.

1 JB*-triples and tripotents.

For a complex Banach space $Z$, denote by $\mathcal{L}(Z)$ the Banach algebra of all bounded linear operators on $Z$. A complex Banach space $Z$ with a continuous mapping $(a, b, c) \mapsto \{abc\}$ from $Z \times Z \times Z$ to $Z$ is called a JB*-triple if the following conditions are satisfied for all $a, b, c, d \in Z$, where the operator $a\Box b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[,]$ is the commutator product:

1. $\{abc\}$ is symmetric complex linear in $a, c$ and conjugate linear in $b$.
2. $[a\Box b, c\Box d] = \{abc\}\Box d - c\Box \{dab\}$.
3. $a\Box a$ is hermitian and has spectrum $\geq 0$.
4. $\|\{aaa\}\| = \|a\|^3$.

If a complex vector space $Z$ admits a JB*-triple structure, then the norm and the triple product determine each other. A derivation of a JB*-triple $Z$ is an element $\delta \in \mathcal{L}(Z)$ such that $\delta\{zzz\} = \{(\delta z)zz\} + \{z(\delta z)z\} + \{zz(\delta z)\}$ and an automorphism is a bijection $\phi \in \mathcal{L}(Z)$ such that $\phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\}$ for $z \in Z$. The latter occurs if and only if $\phi$ is a surjective linear isometry of $Z$. The group $\text{Aut}(Z)$ of automorphisms of $Z$ is a real Banach-Lie group whose Banach-Lie algebra is the set of derivations of $Z$. The connected component of the identity in $\text{Aut}(Z)$ is denoted by $\text{Aut}^\circ(Z)$. Two elements $x, y \in Z$ are orthogonal if $x\Box y = 0$. An element $e \in Z$ is called a tripotent if $\{eee\} = e$. The set $\text{Tri}(Z)$ of tripotents is endowed with the induced topology of $Z$. If $e \in \text{Tri}(Z)$, then $e\Box e \in \mathcal{L}(Z)$ has the eigenvalues $0, \frac{1}{2}, 1$ and we have the topological direct sum decomposition

$$Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e)$$

called the Peirce decomposition of $Z$. Here $Z_k(e)$ is the $k$-eigenspace and the Peirce projections are

$$P_1(e) = Q^2(e), \quad P_{1/2}(e) = 2(e\Box e - Q^2(e)), \quad P_0(e) = \text{Id} - 2e\Box e + Q^2(e),$$

where $Q(e)z = \{zee\}$ for $z \in Z$. We will use the Peirce rules $\{Z_i(e)Z_j(e)Z_k(e)\} \subset Z_{i-j+k}(e)$ where $Z_l(e) = \{0\}$ for $l \neq 0, 1/2, 1$. We note that $Z_1(e)$ is a complex unital JB*-algebra in the product $a \circ b := \{aeb\}$ and involution $a^\#: = \{eae\}$. Let

$$A(e) := \{z \in Z_1(e) : z^\# = z\}.$$ 

Then we have $Z_1(e) = A(e) \oplus iA(e)$. The Peirce spaces of $Z$ with respect to a an orthogonal family of tripotents $\mathcal{E} = (e_i)_{i \in I}$ are defined by

$$Z_{ii} := Z_1(e_i)$$
$$Z_{ij} := Z_{1/2}(e_i) \cap Z_{1/2}(e_j), \quad i \neq j$$
$$Z_{i0} := Z_{0i} := Z_{1/2}(e_i) \bigcap_{j \neq i} Z_0(e_j)$$
$$Z_{00} := \bigcap_{i \in I} Z_0(e_i)$$
The Peirce sum \( P(\mathcal{E}) := \bigoplus_{i,j \in I} Z_{ij} \) relative to the family \( \mathcal{E} \) is direct and we have \( Z = P(\mathcal{E}) \) whenever \( \mathcal{E} \) is a finite set. Every \( \mathcal{E} \)-Peirce space is a JB*-subtriple of \( Z \) and the Peirce rules

\[
\{ Z_{ij} Z_{jk} Z_{kl} \} \subset Z_{il}
\]

hold for all \( i, j, k, l \in I \).

A tripotent \( e \) in a JB*-triple \( Z \) is said to be minimal if \( P_{1}(e)Z = Ce \), and we let \( \text{Min}(Z) \) be the set of them. Clearly \( e = 0 \) lies in \( \text{Min}(Z) \) and is an isolated point there. If \( e \in \text{Min}(Z) \) and \( e \neq 0 \) then \( \| e \| = 1 \) and by the Peirce multiplication rules we have \( \{ euv \} \in Z_{1}(e) = Ce \) for all \( u, v \in Z_{1/2}(e) \). Therefore we can define a sesquilinear form, called the Levi form, \( \langle \cdot, \cdot \rangle_{e}: Z_{1/2}(e) \times Z_{1/2}(e) \to \mathbb{C} \) by

\[
\{ euv \} = \langle v, u \rangle_{e}, \quad u, v \in Z_{1/2}(e).
\]

It is known [10] that \( \langle \cdot, \cdot \rangle_{e} \) is positive definite hence it defines a scalar product in \( Z_{1/2}(e) \) whose norm, called the Levi norm and denoted by \( | \cdot |_{e} \), satisfies

\[
| u |_{e}^{2} \leq \| u \|^{2}, \quad u \in Z_{1/2}(e)
\]

that is, we have the continuous inclusion \( (Z_{1/2}(e), \| \cdot \|) \hookrightarrow (Z_{1/2}(e), | \cdot |_{e}) \). To simplify the notation, we shall omit the subindex \( e \) in both the Levi form and the Levi norm if no confusion is likely to occur.

JB*-triples include C*-algebras and JB*-algebras. A C*-algebra is a JB*-triple with respect to the triple product \( 2\{ abc \} := (ab^{*}c + cb^{*}a) \). Every JB*-algebra with Jordan product \( (a, b) \mapsto a \circ b \) and involution \( a \mapsto a^{*} \) is a JB*-triple with triple product \( \{ abc \} = (a \circ b^{*}) \circ c - (c \circ a) \circ b^{*} + (b^{*} \circ c) \circ a \).

We refer to [8,9,10,12] for the background of JB*-triples theory.

### 2 The manifold \( M \) of minimal projections

Let \( Z := \mathcal{L}(H) \), where \( H \) is a complex Hilbert space, and let \( M \subset \text{Tri}(Z) \) denote the set of all projections in \( Z \) endowed with its topology as subspace of \( Z \). Fix any non zero projection \( e_{0} \in M \) and denote by \( M \) the connected component of \( e_{0} \) in \( M \). Then all elements in \( M \) have the same rank as \( e_{0} \) and \( \text{Aut}^{0}(Z) \) acts transitively on \( M \) which is an \( \text{Aut}^{0}(Z) \)-invariant real analytic manifold whose tangent space at a point \( e \in M \) is

\[
T_{e}M = Z_{1/2}(e)_{s},
\]

the selfadjoint part of the \( 1/2 \)-eigenspace of \( e \). If we set \( k_{u} := 2(u \square e - e \square u) \), then by [1, th. 3.3] a local chart of \( M \) in a suitable neighbourhood \( U \) of \( 0 \) in \( Z_{1/2}(e)_{s} \) is given by

\[
u \mapsto f(u) := \exp k_{u}(e).
\]

Let \( \mathfrak{D}(M) \) be the Lie algebra of all real analytic vector fields on \( M \), and as in [1], define an affine connection \( \nabla \) on \( M \) by

\[
(\nabla_{X} Y)_{e} := P_{1/2}(e)Y'_{e}X_{e}, \quad e \in M, \quad X, Y \in \mathfrak{D}(M)
\]

(1).

Then \( \nabla \) is a torsion-free \( \text{Aut}^{0}(Z) \)-invariant affine connection on \( M \). For each \( e \in M \) and \( u \in Z_{1/2}(e)_{s} \) we let \( \gamma_{e,u}: \mathbb{R} \to M \) denote the curve \( \gamma_{e,u}(t) := \exp tk_{u}(e) \). Clearly we have \( \gamma_{e,u}(0) = e \) and \( \gamma_{e,u}(0) = u \in T_{e}M \). By [1, th. 2.7], \( \gamma_{e,u} \) is a \( \nabla \)-geodesic of \( M \). Let us introduce a binary product in \( Z \) by \( x \circ y := \{ xey \} \). Then \( (Z, \circ) \) is a complex Jordan algebra where, as usual, \( x^{(n)} \)
denotes the $n$-th power of $x$ in $(Z, \circ)$ for $n \in \mathbb{N}$. For $u \in Z_{1/2}(e)$, the real Jordan subalgebra of $(Z, \circ)$ generated by the pair $(e, u)$ is denoted by $J[e, u]$ and we have $\gamma_{e,u} (\mathbb{R}) \subset J[e,u]$.

To make a more detailed study of the manifold $M$, we shall assume that $e_0$ is minimal. In such a case $J[e, u]$ coincides with the closed real linear span of the set $\{e, u, u^{(2)}\}$, in particular $\dim J[e, u] \leq 3$ and

$$\gamma_{e,u}(t) = (\cos^2 t\theta) e + \left(\frac{1}{2\theta}\sin 2t\theta\right) u + \left(\frac{1}{\theta^2}\sin^2 t\theta\right) u^{(2)}, \quad t \in \mathbb{R} \quad (2)$$

for some angle $0 \leq \theta < \pi$. If $a, b$ are two distinct minimal projections and they are not orthogonal (that is, if the Peirce projection $P_1(a)b$ is invertible in the JB*-algebra $Z_1(a)$) then there is an unique geodesic $\gamma_{a,u}(t)$ joining $a$ with $b$ in $M$. Moreover, due to the minimality of $e$ the tangent space $Z_{1/2}(e) \approx \{e\}^\perp$ appears naturally endowed with the Levi form $(\cdot, e)$ and it turns out that the Levi norm $| \cdot |_e$ and the operator norm $\| \cdot \|$ are equivalent in $Z_{1/2}(e)$ (see \[6, th.5.1\]). Thus $(Z_{1/2}(e), | \cdot |_e)$ is a Hilbert space and an $\text{Aut}^0$-invariant Riemann structure can be defined in $M$ by

$$g_e(X, Y) := \langle X_e, Y_e \rangle_e, \quad X, Y \in \mathcal{D}(M) \quad (3)$$

where $V_e \in Z_{1/2}(e)$ denotes the value taken by the vector field $V$ at the point $e \in M$. By \[1\] $g$ satisfies

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X, Y, Z \in \mathcal{D}(M) \quad (4)$$

Therefore $\nabla$ is the only Levi-Civita affine connection on $M$, and the geodesics are minimizing curves for the Riemann distance in $M$, which is given by the formula

$$d(a, b) = \cos^{-1} \left(\|P_1(a)b\|_e^2\right) = \theta.$$  

$M$ is symmetric Riemann manifold on which $\text{Aut}^0(Z)$ acts transitively as a group of isometries and there is a real analytic diffeomorphism of $M$ onto the projective space $\mathbb{P}(H)$ over $H$, endowed with the Fubini-Study metric. We refer to \[1,5,6,13\] for proofs and background about these facts.

3 The manifold of finite rank projections in $\mathcal{L}(H)$.

In what follows we let $M$ and $M_r$ be the set of all projections in $Z$ and the set of all projections that have a fixed finite rank $r$, respectively. If $a \in M_r$ then a frame for $a$ is any family $(a_1, \ldots, a_r)$ of pairwise orthogonal minimal projections in $Z$ such that $a = \Sigma a_k$. Note that then the $a_k$ have the form $a_k = (\cdot, \alpha_k)_{a_k}$ where $(\alpha_k)$ is an orthonormal family of vectors in the range $a(H)$.

3.1 Proposition. For every projection $a \in M$ the following conditions are equivalent:

1. The rank of $a$ is finite.
2. The Banach space $Z_{1/2}(a)$ is linearly homeomorphic to a Hilbert space.

Proof. Let us choose an orthonormal basis $(\alpha_i)_{i \in I}$ in the range $a(H) \subset H$ of $a$. Then $a_i := (\cdot, \alpha_i)_{a_i}, i \in I$, is a family of pairwise orthogonal minimal projections that satisfy

$$a = \Sigma_{i \in I} a_i \quad \text{strong operator convergence in } Z$$  

(5)

The space $Z_{1/2}(a)_s$ consists of the operators $u \in Z$ such that $2\{aau\} = u$ and using (5) it is easy to check that $u$ can be represented in the form

$$u = \Sigma_{i \in I} (\cdot, \xi_i)_{a_i} + (\cdot, \alpha_i)_{\xi_i} \quad \text{strong operator convergence in } Z$$

where $\xi_i := u(\alpha_i)$ are vectors in $H$ that satisfy $\xi_i \in a(H)^\perp$. By (4) each $u \in Z_{1/2}(a)_s$ is determined by the family $(\xi_i)_{i \in I}$. To simplify the notation, set $K := a(H)^\perp$ and $L := \ell_\infty(I, K)$ for the
Banach space of the families \((\xi_i)_{i \in I} \subset K\) with the norm of the supremum \(\| (\xi_i) \| = \sup_{i \in I} \| \xi_i \|\).

Then the mapping

\[ L \to Z_{1/2}(a)_s, \quad (\xi_i) \mapsto u_\xi = \sum_{i \in I} [\langle \cdot, \alpha_i \rangle \xi_i + \langle \cdot, \xi_i \rangle \alpha_i] \]

is a continuous real linear vector space isomorphism, hence a homeomorphism. Thus if the operator norm in \(Z_{1/2}(a)_s\) is equivalent to a Hilbert space norm the same must occur with \(\ell_\infty(I, K)\), hence \(I\) must be a finite set which means that \(a = \Sigma a_i\) has finite rank. The converse is easy. \(\square\)

### 3.2 Lemma

Let \(a, b \in M_r\) with \(a = \Sigma a_k\) where the \((a_k)\) is a frame for \(a\), and let \(Q(a_k)b = \lambda_k a_k\), \((k = 1, \ldots, r)\). If \(P_1(a)\) is invertible in the JB*-algebra \(Z_1(a)\), then \(\lambda_k \neq 0\) for all \(k\). The set of all elements \(b \in M_r\) for which \(P_1(a) b\) is invertible in \(Z_1(a)\) is dense in \(M_r\).

**Proof.** Suppose that \(a_k = (\cdot, \alpha_k)\alpha_k\) and \(b_j = (\cdot, \beta_j)\beta_j\) are frames for \(a\) and \(b\) respectively. Then for each fixed \(k\) we have

\[ Q(a_k)b = \{a_k b a_k\} = (\Sigma_j (|\alpha_k, \beta_j|)^2) a_k = \lambda_k a_k \]

where \(\lambda_k \geq 0\). Moreover \(\lambda_k = 0\) if and only if \(\alpha_k \in \{\beta_1, \ldots, \beta_r\}^\perp\) which is equivalent to \(a_k \perp b\).

But in such a case \(\text{range}(a_k) \subset \ker\{a_k b a_k\} = \ker P_1(a) b\) which contradicts the invertibility of \(P_1(a) b\). To simplify the notation set \(K := a(H) \subset H\) and note that \(\dim K = \text{rank} a = r < \infty\). The operators in \(Z_1(a) = aZa\) can be viewed as operators in \(L(K)\), therefore the determinant function is defined in \(Z_1(a)\) and an element \(z \in Z_1(a)\) is invertible if and only if \(\det(z) \neq 0\). Thus the set of the operators \(b \in Z\) for which \(P_1(a) b\) is invertible in \(Z_1(a)\) is an open dense subset of \(M_r\). \(\square\)

### 3.3 Lemma

If \(a, p\) and \(q\) are projections in \(M_r\) and \(P_{1/2}(a)p = P_{1/2}(a)q\), then \(p = q\).

**Proof.** Take frames for \(a, p, q\), compute \(P_{1/2}(a)p = 2(D(a \square a) - Q(a)^2)p\) and proceed similarly with \(q\). An elementary exercise of linear algebra yields \(\text{range}(p) = \text{range}(q)\), hence \(p = q\). \(\square\)

Let \(a \in M_r\) and choose any frame \((a_1, a_2, \ldots, a_r)\) for \(a\). As above \(Z_{1/2}(a)_s\) consists of the operators \(u = \Sigma (\cdot, \xi_k)\alpha_k + (\cdot, \alpha_k)\xi_k\) where \(\xi_k := u(\alpha_k)\) are vectors in \(H\) that satisfy \(\xi_k \in a(H)^\perp\).

Write \(u_k := (\cdot, \xi_k)\alpha_k + (\cdot, \alpha_k)\xi_k\). Then we have \(u = \Sigma u_k\) where the \(u_k\) are selfadjoint operators in \(Z = L(H)\) (in fact \(u_k \in Z_{1/2}(a)_s\)) that satisfy

\[ u_j \square a_k = a_k \square u_j = 0, \quad j \neq k, \quad (j, k = 1, 2, \ldots, r) \tag{6} \]

The above properties of the \(a_k, u_k\) hold whatever is the frame \((a_1, a_2, \ldots, a_r)\). There are many families in those conditions and we are going to prove that, by making an appropriate choice of the \(a_k\) (a choice in which the tangent vector \(u \in Z_{1/2}(a)\) is also involved) we can additionally have

\[ u_k \square u_j = u_j \square u_k = 0, \quad j \neq k, \quad (j, k = 1, 2, \ldots, r) \tag{7} \]

This will simplify considerably the calculations in the sequel. We need some material.

### 3.4 Lemma

With the above notation the set of minimal tripotents in \(Z_{1/2}(a)\) is

\[ \{ (\cdot, \alpha)\xi + (\cdot, \xi)\alpha : \alpha \in a(H), \xi \in a(H)^\perp, \|\alpha\| = 1 = \|\xi\| \} \]

**Proof.** Let \(x \in Z\) be of the form \(x = (\cdot, \alpha)\xi + (\cdot, \xi)\alpha\) where \(\alpha, \xi \in H\) satisfy the above conditions. It is a matter of routine calculation to see that then \(2\{aax\} = x\) hence \(x \in Z_{1/2}(a)_s\). Moreover \(\{xx\} = x\) so that \(x\) is a tripotent and we can easily see that \(\{xZ_{1/2}(a)x\} \subset Cx\) which proves the minimality of \(x\) in \(Z_{1/2}(a)\). The converse is similar. \(\square\)

The following result should be compared to [14, prop. 3.4]
3.5 Lemma. Two minimal tripotents \( x = (\cdot, \alpha)\xi + (\cdot, \xi)\alpha \) and \( y = (\cdot, \beta)\eta + (\cdot, \eta)\beta \) in \( Z_{1/2}(a) \), are orthogonal if and only if \( \alpha \perp \beta \) and \( \xi \perp \eta \). In particular \( Z_{1/2}(a) \) has rank \( r \) for all \( a \in M \).

Proof. By \([2, \text{p. 18}]\) \( x \) and \( y \) are orthogonal if and only if the conditions \( xy^* = 0 = y^*x \) hold. Now it is elementary to complete the proof of the first statement. For the second part, let \( (u_i)_{i \in I} \) be a family of pairwise minimal orthogonal tripotents in \( Z_{1/2}(a) \). Then \( u_i = (\cdot, \alpha_i)\xi_i + (\cdot, \xi_i)\alpha_i \), where \( (\alpha_i) \subset a(H) \) and \( \xi_i \subset a(H)^+ \) are orthonormal families of vectors in \( H \). In particular \( a_i = (\cdot, \alpha_i) \alpha_i \) is a family of pairwise orthogonal projections with \( \Sigma a_i \leq a \). Since \( \text{rank}(a) = r \), we have cardinal \( (I) \leq r \). The converse is easy. \( \square \)

Let \( a \in M \) be a fixed projection and take any tangent vector \( u \in Z_{1/2}(a) \) to \( M \) at \( a \). By lemma 3.2 \( Z_{1/2}(a) \) has finite rank, hence \([9, \text{cor. 4.5}]\) \( u \) has a spectral decomposition in the JB*-triple \( Z_{1/2}(a) \) of the form

\[
\begin{aligned}
&u = \rho_1 u_1 + \cdots + \rho_s u_s, \\
&0 \leq \rho_1 \leq \cdots \leq \rho_s = \|u\|, \\
&1 \leq s \leq r
\end{aligned}
\]    

where the \( u_k \) are pairwise orthogonal minimal tripotents in \( Z_{1/2}(a) \). Therefore

\[
u_k = (\cdot, \alpha_k)\xi_k + (\cdot, \xi_k)\alpha_k, \quad \alpha_k \in a(H), \quad \xi_k \in a(H)^+, \]

\[
\|\alpha_k\| = 1 = \|\xi_k\|, \quad \alpha_j \perp \alpha_k, \quad \xi_j \perp \xi_k, \quad j \neq k
\]

Then \( a_k = (\cdot, \alpha_k)\alpha_k \) are pairwise orthogonal minimal projections in \( Z \) and \( \Sigma a_k \leq a \). In case \( s < r \), which occurs if some of the \( \rho_k = 0 \), we pick additional minimal orthogonal projections \( a_{s+1}, \ldots, a_r \) so as to have \( a = \Sigma a_k \). For the family \( (a_1, \ldots, a_r) \) so constructed, called a frame associated to the pair \( (a, u) \), both properties (6) and (7) hold. Remark that this frame needs not be unique, it depends on \( a \) and on \( u \) as well, and it is invariant under the group \( \text{Aut}^+(Z) \). In fact some more properties are valid now.

In accordance with section §1, each pair \( (a_k, u_k) \) gives rise to a real Jordan algebra \( J_k = J[a_k, u_k] \) with the product \( x \circ k y = \{xa_k y\} \). We have \( \dim(J_k) = 3 \) and \( \{a_k, u_k, u_k^{(2)}\} \) is a basis of \( J_k \). Moreover, \( J_k \) is invariant under the operator \( g_k = 2(a_k \Box a_k - u_k \square a_k) \) where triple products are computed in \( Z = \mathcal{L}(H) \). In case \( s < \text{rank}(a) \) we set \( J_n = \mathbb{R}a_n \) as real Jordan algebras.

3.6 Lemma. The Jordan algebras \( J_k \) and \( J_l \) with \( k \neq l \), \( (k, l = 1, \ldots, r) \) are orthogonal in the JB*-triple sense in \( Z \), that is \( \{J_kJ_l\} = 0 \).

Proof. For \( n \in \{k, l\} \subset \{1, \ldots, s\} \) with \( k \neq l \), let \( z_n \) be any element in the basis \( \{a_n, u_n, u_n^{(2)}\} \) of \( J_n \). Clearly it suffices to show that \( z_k z_l = 0 = z_l z_k \). As an example, we shall prove that \( u_k^{(2)} u_l^{(2)} = 0 \). It is a routine to check that \( u_k u_l = 0 \). Then

\[
u_k^{(2)} u_l^{(2)} = \{u_k a_k u_k\} \{u_l a_l u_l\} = (u_k a_k u_k)(u_l a_l u_l) = u_k a_k (u_k u_l) a_l u_l = 0
\]

as we wanted to see. \( \square \)

Consider now the vector space direct sum \( J := \bigoplus J_k \), and define a product \( z \circ w := \{zaw\} \) in \( J \) by

\[
z \circ w := \{zaw\} = \frac{1}{2} (zaw + waz) = \frac{1}{2} \sum_{1}^{r} (z_k a_k w_k + w_k a_k z_k) = \sum_{1}^{r} z_k \circ_k w_k
\]

where \( z_k, w_k \) are respectively the \( J_k \)-component of \( z \) and \( w \). It is now clear that \( J \) is a real Jordan algebra, that the product in \( J \) induces in each \( J_k \) its own product \( z \circ_k w = \{zaw\} \) and that the \( J_k \) are orthogonal as Jordan subalgebras of \( J \). It is also clear that \( J \) coincides with the closed real linear span of the set \( \bigcup_{1}^{r} \{a_k, u_k, u_k^{(2)}\} \), in particular \( \dim J \leq 3r < \infty \). Finally \( J[a, u] \subseteq J \) and we conjecture that the equality holds (see \([14, \text{prop. 3.5} \& \text{th. 3.6}]\)
4 Geodesics and the exponential mapping.

Consider $M_r$ endowed with the affine connection $\nabla$ given by (1). To discuss its geodesics, let us define an operator $g \in Z = \mathcal{L}(H)$ by

$$g_a = g_{a,u} : = 2(u \square a - a \square u) = 2\Sigma \rho_k(u_k \square a_k - a_k \square u_k) = \Sigma \rho_k g_{a_k, u_k}$$

where $u = \Sigma \rho_k u_k$ is the spectral decomposition of $u \in Z_{1/2}(a)$, the $a_k$ is any frame associated to the pair $(a, u)$ and $g_k : = g_{a_k, u_k}$ is defined in a obvious manner. If the spectral decomposition of $u$ (see (8)) has $s \leq r$ non zero summands then we define $g_n : = 0$ for $n = s + 1, \ldots, r$. Then $g_k$ is a commutative family of operators in $Z$, more precisely we have $g_k(J_l) = \{0\}$, $g_k g_l = g_l g_k = 0$ for all $k \neq l$, $(k, l = 1, \ldots, r)$ and $g$ leaves invariant all the spaces $J$ and $J_k$. Thus

$$\gamma_{a,u}(t) : = \exp t g(a) = \Sigma \exp t g_k(a_k), \quad t \in \mathbb{R}$$

By section §1 this curve is a geodesic in $M_r$ and $\gamma_{a,u}(\mathbb{R}) \subset J[a, u] \subset J$. We can collect now the above discussion in the following statement (see [14, prop. 5.1 & 5.4]

4.1 Theorem. Suppose that we are given a point $a \in M_r$ and a tangent vector $u \in Z_{1/2}(a)$ to $M_r$ at $a$. Then the geodesic of $M_r$ that passes through $a$ with velocity $u$ is the curve

$$\gamma_{a,u}(t) = \Sigma \gamma_{a_k, u_k}(t), \quad t \in \mathbb{R},$$

where $\gamma_k : = \gamma_{a_k, u_k}$ is given by

$$\gamma_k(t) : = \gamma_{a_k, u_k}(t) = (\cos^2 \theta_k t) a_k + (\frac{1}{2\theta_k} \sin 2\theta_k t) u_k + (\frac{1}{\theta_k^2} \sin^2 \theta_k t) u_k^{(2)} \quad (G)$$

Here $u = \Sigma \rho_k u_k$ is the spectral decomposition of $u$ in $Z_{1/2}(a)$, the $a_k$ form a frame associated to the pair $(a, u)$ and the numbers $\theta_k$ are given by $\cos^2 \theta_k : = \rho_k$ with $0 \leq \theta_k < \frac{\pi}{2}$.

Now we are in a position to define the exponential mapping. Suppose the tangent vector $u$ lies in the unit ball $B_1(a) \subset Z_{1/2}(a)$, i.e. $\|u\| < 1$. For $t = 1$ the expression (G) yields

$$\gamma(1) = \Sigma (\cos^2 \theta_k) a_k + \Sigma (\frac{1}{2\theta_k} \sin 2\theta_k) u_k + \Sigma (\frac{1}{\theta_k^2} \sin^2 \theta_k) u_k^{(2)} \quad (E)$$

and a real analytic mapping form the unit ball $B_1(0) \subset Z_{1/2}(a)$ to the manifold $M$ can be defined by

$$\text{Exp}_a(u) : = \gamma_{a,u}(1)$$

An inspection of (E) yields that the Peirce decomposition of $\gamma_{a,u}(1)$ relative to $a$ is

$$P_1(a)\gamma_{a,u}(1) = \Sigma (\cos^2 \theta_k) a_k, \quad P_{1/2}(a)\gamma_{a,u}(1) = \Sigma (\frac{1}{2\theta_k} \sin 2\theta_k) u_k$$

$$P_0(a)\gamma_{a,u}(1) = \Sigma (\frac{1}{\theta_k^2} \sin^2 \theta_k) u_k^{(2)}$$

Remark that $0 < \cos^2 \theta_k \leq 1$, hence in particular $P_1(a)\gamma_{a,u}(1)$ lies in the set of all $\mathcal{N}_a$ of all invertible elements in the JB$^*$-algebra $Z_1(a)$. Clearly $\mathcal{N}_a$ is an open neighbourhood of $a$ in $Z_1(a)$. Remark also that $0 \leq \frac{1}{2\theta_k} \sin^2 \theta_k = \rho_k \leq \|u\| < 1$, hence $\Sigma (\frac{1}{2\theta_k} \sin^2 \theta_k) u_k$ is the spectral decomposition of $P_{1/2}(a)\gamma_{a,u}(1)$ in $Z_{1/2}(a)$. Thus $\text{Exp}_a B_1(a) \subset \mathcal{N}_a \subset M$. We refer to $\text{Exp}_a$ as the exponential mapping.
5 Geodesics connecting two given points. The logarithm mapping.

Now we discuss the possibility of joining two given projections \( a \) and \( b \) such that \( P_1(a)b \) is invertible in the Jordan algebra \( Z_1(a) \), by means of a geodesic in \( M \). The remarks in the precedent section show how to proceed. First we compute the spectral decomposition of \( u := P_{1/2}(a)b \) in the JB*-triple \( Z_{1/2}(a) \). Assume it to be

\[
u = P_{1/2}(a)b = \sum_{k=1}^r \rho_k u_k, \quad 0 \leq \rho_1 \leq \cdots \leq \rho_r = \|u\| < 1, \quad 1 \leq k \leq r \]

where the \( u_k \) are pairwise orthogonal minimal tripotents in \( Z_{1/2}(a) \). Hence By lemma 3.4 the \( u_k \) have the form \( u_k = (\cdot, \alpha_k)\xi_k + (\cdot, \xi_k)\alpha_k \) for some orthonormal families of vectors \((\alpha_k) \subset a(H) \) and \((\xi_k) \subset a(H)\) by lemma 3.2 \( Q(a_k)b = \{a_kba_k\} = \lambda_k \) where \( \lambda_k \neq 0 \) since \( P_1(a)b \) is invertible in \( Z_1(a) \). Also \( |\lambda_k| = \|\{a_kba_k\}\| \leq 1 \). Thus \( 0 < \lambda_k \leq 1 \) and a unique angle \( 0 < \theta_k < \frac{\pi}{2} \) is determined by \( \cos^2 \theta_k = \lambda_k \). In this way we have got all the elements appearing in \((E)\). Let us define \( \tilde{\gamma}(t) := \Sigma \tilde{\gamma}_k(t) \) for \( t \in \mathbb{R} \) where

\[
\tilde{\gamma}_k(t) := (\cos^2 t \theta_k) a_k + \left( \frac{1}{2\theta_k} \sin 2t \theta_k \right) u_k + \left( \frac{1}{\theta_k^2} \sin^2 t \theta_k \right) u_k^{(2)}
\]

By section §1, each \( \tilde{\gamma}_k(t) \) is a geodesic in the manifold \( M_1 \) of all rank 1 projections. By the previous discussion \( \tilde{\gamma}_j(t) \) and \( \tilde{\gamma}_k(t) \) are orthogonal whenever \( j \neq k \), \( t \in \mathbb{R} \), hence \( \tilde{\gamma}_j(t) = \Sigma \tilde{\gamma}_k(t) \), \( t \in \mathbb{R} \), is a curve in the manifold \( M \) of projections of rank \( r \). Clearly \( \tilde{\gamma}(0) = \Sigma \tilde{\gamma}_k(0) = \Sigma a_k = a \) and we shall now show that \( \tilde{b} = \gamma(1) \) coincides with \( b \). As above \( P_{1/2}(a) = \tilde{b} = \Sigma (\frac{1}{2} \sin 2\theta_k) u_k = \Sigma \rho_k u_k \) is the spectral decomposition of \( P_{1/2}(a)b \) in \( Z_{1/2}(a) \), which by construction is the spectral decomposition of \( P_{1/2}(a)b \). Hence by lemma 3.3, \( \tilde{b} = \tilde{\gamma}(1) = b \). This gives a geodesic \( \gamma(t) \) that connects \( a \) with \( b \) in the manifold \( M \), and passes through the point \( a \) with the velocity \( u := P_{1/2}(a)b \). It is uniquely determined by the data \( a, b \) and the property \( \gamma_{a,b}(1) = b \).

Now we are in a position to define the logarithm mapping. Fix a point \( a \in M \) and let \( \mathcal{N}_a \subset M \) be the set of all projections \( b \in M \) such that \( P_1(a)b \) is invertible in the JB*-algebra \( Z_1(a) \). Define a mapping \( \log_\mathcal{N} \) from \( \mathcal{N}_a \subset M \) to the unit ball \( B_1(a) \subset Z_{1/2}(a) \) by declaring \( \log_\mathcal{N}(b) \) to be the velocity at \( t = 0 \) of the unique geodesic \( \gamma_{a,b}(t) \) that joins \( a \) with \( b \) in \( M \) and \( \gamma_{a,b}(1) = b \), in other words \( \log_\mathcal{N}(b) := P_{1/2}(a)b \). We refer to \( \log_\mathcal{N} \) as the logarithm mapping. Clearly \( \log_\mathcal{N} \) and \( \exp_\mathcal{N} \) are real analytic inverse mappings. In particular, the family \( \{\mathcal{N}_a, \log_\mathcal{N} : a \in M\} \) is an atlas of \( M \). We remark the fact that \( \gamma_{a,u}(0) = 0 \) for all \( u \in B_1(a) \) which shall be needed later on to apply the Gauss lemma [11, 1.9] and summarize the above discussion in the statement (see [14, th. 5.7 & prop. 5.8])

5.1 Theorem. Let \( a \) and \( b \) be two given projections in \( M \), and assume that \( P_1(a)b \) is invertible in the Jordan algebra \( Z_1(a) \). Then there is exactly one geodesic \( \gamma_{a,b}(t) \) that joins \( a \) with \( b \) in \( M \) and \( \gamma_{a,b}(1) = b \).

6 The Riemann structure on \( M \).

Let \( a \in M_r \) and choose any frame \( (a_k) \) for \( a \). By section §1 we have vector space direct sum decomposition

\[
Z_{1/2}(a) = \bigoplus_{k=1}^r Z_{1/2}(a_k) \tag{9}
\]

which suggests to define a scalar product in \( Z_{1/2}(a) \) by

\[
\langle u, v \rangle := \frac{1}{\sqrt{r}} \sum_k \langle u_k, v_k \rangle_{a_k} \tag{10}
\]

where \( \langle \cdot, \cdot \rangle_{a_k} \) stands for the Levi form on \( Z_{1/2}(a_k) \). First we prove
6.1 Lemma. With the above notation, (9) defines an \( Aut^o \)-invariant scalar product on \( Z_{1/2}(a) \) that does not depend of the frame \( a = \Sigma_k \) and converts \( Z_{1/2}(a) \) into a Hilbert space.

Proof. Let \( \Sigma a_k \) and \( \Sigma a'_k \) denote two frames for \( a \) where \( a_k = (\cdot, \alpha_k)\alpha_k \) and \( a'_k = (\cdot, \alpha'_k)\alpha'_k \) for some orthonormal families \( (\alpha_k), (\alpha'_k) \subset a(H) \). Extend them to two orthonormal bases of \( H \) and let \( u \in \mathcal{L}(H) \) be the unitary operator that exchanges these bases. Then \( u \) induces an isometry \( U \in Aut^o(Z) \) by \( Uz = uzu^{-1} \) that satisfies \( Ua_k = a_k \). The invariance of the Levi form together with (10) yields part of the result. The remainder is trivial. \( \square \)

A Riemann structure can now be defined in \( M_r \) in the following way. Let \( X, Y \in \mathfrak{D}(M) \) vector fields on \( M_r \), and for \( a \in M_r \) take any frame \( a = \Sigma a_k \). Then (9) gives representation \( X = \Sigma X_k, Y = \Sigma Y_k \) with \( X_k, Y_k \in Z_{1/2}(a_k) \) and we set

\[
g_a(X, Y) = \langle X, Y \rangle = \frac{1}{\sqrt{r}} \Sigma \langle X_k, Y_k \rangle a_k = \frac{1}{\sqrt{r}} \Sigma g_{a_k}(X_k, Y_k)
\]

This is a well defined \( Aut^o \)-invariant Riemann structure on \( M_r \). By section §1 each \( g_{a_k} \) has property (4) and a routine argument gives the same property for \( g \). Thus \( g \) is the only Levi-Civita connection in \( M_r \) and we can apply the Gauss lemma [11, 1.9] to conclude that the \( \nabla \)-geodesics are minimizing curves for the Riemann distance.

Recall that for a tripotent \( a \in Z \), the mapping \( \sigma_a: x_1 + x_1/2 + x_0 \mapsto x_1 - x_1/2 + x_0 \), where \( x \in Z \) and \( x_1 + x_1/2 + x_0 \) is the Peirce decomposition of \( x \) with respect to \( a \), called the Peirce symmetry of \( Z \) with center \( a \), is an involutory automorphism of \( Z \) that induces an isometric symmetry of \( M_r \) (see [6, th. 5.1]). We let \( \text{Isom} M_r \) and \( \mathfrak{S} \) denote the group of all isometries of the Riemann manifold \( M_r \) and the subgroup generated by the set \( S = \{ \sigma_a : a \in M_r \} \), respectively.

6.2 Proposition. With the above notation, \( M_r \) is a symmetric Riemann manifold in which the group \( \mathfrak{S} \) acts transitively.

Proof. Let \( a, b \in M_r \) be such that \( b \in \mathcal{N}_a \). Then \( a \) and \( b \) can be joined in \( M_r \) by a unique geodesic with \( \gamma(0) = a, \gamma(1) = b \). If \( c = \gamma(\frac{1}{2}) \), then \( \sigma_c \) is a symmetry of \( M_r \) such that \( \sigma_c(a) = b \). Thus the set \( S \) is transitive in \( \mathcal{N}_a \) and \( S \) is locally transitive in \( M_r \). Consider now the case \( b \not\in \mathcal{N}_a \).

Since \( M_r \) is pathwise connected, we can join \( a \) with \( b \) by a curve \( \Gamma \) in \( M_r \) and by a standard compactness argument there exists a finite set \( \{ b_0, \ldots, b_s \} \subset \Gamma[0, 1] \) such that \( b_0 = a, b_s = b \) and \( b_{k+1} \in \mathcal{N}_{b_k} \) for \( k = 1, \ldots, s \). An application of the above argument to each pair of consecutive points gives the result. \( \square \)

We now compute the Riemann distance in \( M_r \). Consider first the case of two points \( a, b \in M_r \) with \( b \in \mathcal{N}_a \). Let \( \gamma_{a,b}(t) \) be the unique geodesic that joins \( a \) with \( b \) in \( M_r \) and satisfies \( b = \gamma_{a,b}(1) \). Since \( Aut^o(Z) \) is transitive in \( \mathcal{N}_a \) and the Levi norm is \( Aut^o(Z) \)-invariant, we have

\[
|\gamma_{a,b}(t)|_{\gamma_{a,b}(t)} = |\gamma_{a,b}(0)|_{\gamma_{a,b}(0)} = |u|_a
\]

On the other hand, since the Levi norm in \( Z_{1/2}(a) \) is the direct hilbertian sum of the Levi norms in the \( Z_{1/2}(a_k) \), we have by section §1

\[
|u|^2_a = \frac{1}{r} \Sigma |u_k|^2_{a_k} = \frac{1}{r} \Sigma \theta_k^2
\]

where \( u = \Sigma \rho_k u_k \) is the spectral decomposition of \( u \) in \( Z_{1/2}(a) \), \( (a_k) \) is the frame associated to the pair \( (a, u) \) and \( \cos^2 \theta_k = \rho_k \). Therefore

\[
d(a, b) = \int_0^1 |\gamma_{a,b}(t)|_{\gamma_{a,b}(t)} \, dt = \int_0^1 |u|_a \, dt = |u|_a = \frac{1}{\sqrt{r}} \left( \Sigma \theta_k^2 \right)^{1/2}
\]
Consider now the case $b \notin \mathcal{N}_a$. By lemma 3.2 we can take a sequence $(b_n)_{n \in \mathbb{N}}$ in $\mathcal{N}_a$ such that $b = \lim_{n \to \infty} b_n$. Since (D) holds for all $b_n$ and the Riemann distance is continuous, we get the validity (D) for all $a, b \in M_r$. □

Note that expression (D) is a generalization of the classical formula for the Fubini-Study metric in the projective space $\mathbb{P}(H)$.

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