GENERALIZED RICCI FLOW I: HIGHER DERIVATIVES
ESTIMATES FOR COMPACT MANIFOLDS

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Abstract. In this paper, we consider a generalized Ricci flow with a given 3-from (it’s not necessarily closed) and establish the higher derivatives estimates for compact manifolds. As an application, we prove the compactness theorem for this generalized Ricci flow. Our work can be viewed as an extension of J. Streets [8] in which he considered a generalized Ricci flow with a closed 3-form.

1. Introduction

The 2-dimensional nonlinear sigma model is a quantum field theory of maps $X^i$ from a 2-dimensional Riemannian manifold $(\Sigma, h)$ to another Riemannian manifold $(M, g)$. We let $\epsilon$ be the volume element of $\Sigma$ and let $R(h)$ denote the scalar curvature of $\Sigma$ associated to metric $h$. Then the sigma model action is given by

$$S = -\frac{1}{\alpha'} \int_\Sigma d^2\sigma \left[ \sqrt{h} h^{\alpha\beta} g_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j - \alpha' \sqrt{h} \Phi R(h) \right]$$

where $(\sigma^1, \sigma^2)$ are coordinates on $\Sigma$, $\alpha'$ is the constant, and $g_{ij}, B_{ij} = -B_{ji}$, and $\Phi$ are the target space metric, $B$-field, and dilaton respectively.

The first order renormalization group (RG) flow [6] of this action with respect to $\alpha'$ is

$$\frac{\partial g_{ij}}{\partial t} = -\alpha' \left( R_{ij} + 2 \nabla_i \nabla_j \Phi - \frac{1}{4} H_{ikl} H_{jl} \right),$$

$$\frac{\partial H}{\partial t} = \alpha' \left( \frac{1}{2} \Delta_L B - d(H, \text{grad} \Phi) \right),$$

$$\frac{\partial \Phi}{\partial t} = -A + \alpha' \left( \frac{1}{2} \Delta \Phi - |\nabla \Phi|^2 + \frac{1}{24} |H|^2 \right),$$
where $\Delta$ is the Laplacian, $\Delta_{LB}$ is the Laplace-Beltrami operator acting on the 3-form $H := dB$, and $A$ is a constant. Under a family of diffeomorphisms, the RG flow can be rewritten as
\[
\frac{\partial g_{ij}}{\partial t} = -\alpha'(R_{ij} - \frac{1}{4}H_{ikl}H_{j}^{kl}),
\]
\[
\frac{\partial H}{\partial t} = \frac{\alpha'}{2} \Delta_{LB} H,
\]
\[
\frac{\partial \Phi}{\partial t} = -A + \alpha' \left( \frac{1}{2} \Delta \Phi + \frac{1}{24} |H|^2 \right).
\]

Therefore the Ricci flow is derived by setting $B = 0$, i.e., Ricci flow is a RG flow with zero $B$-field.

This is the first of a series papers in which we study the Ricci flow with nonzero $B$-field, called the generalized Ricci flow (GRF). Here ”generalized” refers to the flow with a nonzero 3-form.

Let $(M, g_{ij}(x))$ denote an $n$-dimensional compact Riemannian manifold with a 3-form $H = \{H_{ijk}\}$. In this paper we consider the following GRF on $M$:
\[
\begin{align*}
\frac{\partial g_{ij}(x,t)}{\partial t} &= -2R_{ij}(x,t) + \frac{1}{2}H_{ikl}(x,t)H_{j}^{kl}(x,t), \\
\frac{\partial H(x,t)}{\partial t} &= \Box_{g(x,t)} H(x,t), H(x,0) = H(x), g(x,0) = g(x).
\end{align*}
\]

Here $\Box_{g(x,t)}$ is the Laplace-Beltrami operator associated to $g(x,t)$. Finally, we consider a kind of generalized Ricci flow including this case.

We use the following definitions of curvatures. The Ricci curvature tensors, and scalar curvatures are given by
\[
R_{ij} = g^{kl}R_{ikjl}, \quad R = g^{ij}R_{ij} = g^{ij}g^{kl}R_{ikjl}
\]
respectively. For any two tensors $T := \{T_{ijkl}\}$ and $U := \{U_{ijkl}\}$ we define the inner product of $T$ and $U$,
\[
\langle T_{ijkl}, U_{ijkl} \rangle := g^{i\alpha} g^{j\beta} g^{k\gamma} g^{l\delta} T_{ijkl} U_{\alpha\beta\gamma\delta},
\]
and then the norm of $T$ is denoted by
\[
|T|^2 = |T_{ijkl}|^2 := \langle T_{ijkl}, T_{ijkl} \rangle.
\]

We also denote by $\nabla T_{ijkl}$ the covariant derivatives of $\{T_{ijkl}\}$ with respect to the metric $g$, and $\nabla^m T_{ijkl}$ the $m^{th}$ covariant derivatives of $\{T_{ijkl}\}$. The

\footnote{From the definition, the exterior derivative $d$ does no depend on the choice of metric however the dual operator $d^*$ depends on the metric. Hence we write $\Box_{g(x,t)} = -(dd^* + d^*d)$ to indicate that the Laplace-Beltrami opetator depends on the metric $g(x,t)$.}
Laplace operator of $g$ is given by $\Delta_g := g^{ij}\nabla_i\nabla_j$.

Throughout this paper we denote by $C$ the universal constants depending only on the dimension of $M$ or some other constants, which may take different values at different places.

When $H(x)$ is closed, we should consider a refined generalized Ricci flow (RGRF):

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij}(x,t) &= -2R_{ij}(x,t) + \frac{1}{2} H_{ikl}(x,t) H_j^{kl}(x,t), \\
\frac{\partial}{\partial t} H(x,t) &= -d_{g(x,t)}^* d_{g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x).
\end{align*}
\]

Here $d_{g(x,t)}^*$ is the dual operator of $d$ with respect to the metric $g(x,t)$, and actually $d_{g(x,t)}$ is the exterior derivative $d$ itself.

**Lemma 1.1.** Under RGRF, the 3-forms $H(x,t)$ are closed if the initial value $H(x)$ is closed.

**Proof.** Since the exterior derivative $d$ is independent of the metric, we have

\[
\frac{\partial}{\partial t} dH(x,t) = d \frac{\partial}{\partial t} H(x,t) = d (-d_{g(x,t)}^* d_{g(x,t)} H(x,t)) = 0.
\]

so $dH(x,t) = dH(x) = 0$. \qed

**Proposition 1.2.** If $(g(x,t), H(x,t))$ is a solution of RGRF and the initial value $H(x)$ is closed, then it is also a solution of GRF.

**Proof.** From Lemma 1.1 and the assumption we know that $H(x,t)$ are all closed. Hence $\Box_{g(x,t)} H(x,t) = -d_{g(x,t)}^* d_{g(x,t)} H(x,t)$. \qed

First we prove the short-time existence of GRF. The short-time existence for RGRF has been established in [5], moreover, the authors in [5] have also showed the short-time existence for a kind of generalized Ricci flow including our case. The proof presented here and the context of next section will be used in the forthcoming paper.

**Theorem 1.3.** There is a unique solution to GRF for a short time. More precisely, let $(M, g_{ij}(x))$ be an $n$-dimensional compact Riemannian manifold with a 3-form $H = \{H_{ijk}\}$, then there exists a constant $T = T(n) > 0$ depending only on $n$ such that the evolution system

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij}(x,t) &= -2R_{ij}(x,t) + \frac{1}{2} g^{kp}(x,t) g^{lq}(x,t) H_{ikl}(x,t) H_{jqp}(x,t), \\
\frac{\partial}{\partial t} H(x,t) &= \Box_{g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x),
\end{align*}
\]

has a unique solution $(g_{ij}(x,t), H_{ijk}(x,t))$ for a short time $0 \leq t \leq T$. 

After establishing the local existence, we are able to prove the higher derivatives estimates for GRF. Precisely, we have the following theorem

**Theorem 1.4.** Suppose that \((g(x,t), H(x,t))\) is a solution to GRF on a compact manifold \(M^n\) and \(K\) is an arbitrary given positive constant. Then for each \(\alpha > 0\) and each integer \(m \geq 1\) there exists a constant \(C_m\) depending on \(m, n, \max\{\alpha, 1\}\), and \(K\) such that if

\[
|Rm(x,t)|_{g(x,t)} \leq K, \quad |H(x)|_{g(x)} \leq K
\]

for all \(x \in M\) and \(t \in [0, \frac{\alpha K}{K}]\), then

\[
(1.6) \quad |\nabla^{m-1} Rm(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \leq \frac{C_m}{t^{m/2}}
\]

for all \(x \in M\) and \(t \in (0, \frac{\alpha K}{K}]\).

As an application, we can prove the compactness theorem for GRF.

**Theorem 1.5. (Compactness for GRF)** Let \(\{(M_k, g_k(t), H_k(t), O_k)\}_{k \in \mathbb{N}}\) be a sequence of complete pointed solutions to GRF for \(t \in [\alpha, \omega) \ni 0\) such that

(i) there is a constant \(C_0 < \infty\) independent of \(k\) such that

\[
\sup_{(x,t) \in M_k \times (\alpha, \omega)} |Rm(g_k(x,t))|_{g_k(x,t)} \leq C_0, \quad \sup_{x \in M_k} |H_k(x, \alpha)|_{g_k(x, \alpha)} \leq C_0,
\]

(ii) there exists a constant \(\iota_0 > 0\) satisfies

\[
\text{inj}_{g_k(0)}(O_k) \geq \iota_0.
\]

Then there exists a subsequence \(\{j_k\}_{k \in \mathbb{N}}\) such that

\((M_{j_k}, g_{j_k}(t), H_{j_k}(t), O_{j_k}) \longrightarrow (M_\infty, g_\infty(t), H_\infty(t), O_\infty), t \in [\alpha, \omega)\) to GRF as \(k \to \infty\).

The case of \(H\) closed is very important in physics. Recently, J. Streets [8] considered the connection Ricci flow in which \(H\) is the torsion of connection and proved all of the results. The authors in [7,8] have already proved the similar results for Ricci Yang-Mills flow. Using the similar method, the above results are easily extended to a kind of generalized Ricci flow introduced in [5]. We will discuss it in the last section.

The rest of this paper is organized as follows. In Sec.2, we prove the short-time existence and uniqueness of the GRF for any given 3-form \(H\). In Sec.3, we compute the evolution equations for the Levi-Civita connections, Riemann, Ricci, and scalar curvatures of a solution to the GRF. In Sec.4, we establish higher derivative estimates for GRF, called **Bernstein-Bando-Shi (BBS) derivative estimates** (e.g., [1] or [2]). In Sec.5, we prove the compactness theorem for GRF by using BBS estimates. In Sec.6, based on the work of [5], we prove the higher derivatives
estimates and the compactness theorem for a kind of generalized Ricci flow.

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2. Short-time existence

In this section we establish the short-time existence for GRF. Our method is standard, that is, DeTurck trick which is used in Ricci flow to prove its short-time existence. We assume that \( M \) is an \( n \)-dimensional compact Riemannian manifold with metric
\[
(2.1) 
 d\tilde{s}^2 = \tilde{g}_{ij}(x)dx^i dx^j
\]
and with Riemannian curvature tensor \( \{\tilde{R}_{ijkl}\} \). We also assume that \( \tilde{H} = \{\tilde{H}_{ijk}\} \) is a fixed 3-form on \( M \). In the following we put
\[
(2.2) 
 h_{ij} := H_{ikl}H_{j}^{kl}.
\]
Suppose the metrics
\[
(2.3) 
 ds_t^2 = \frac{1}{2}\hat{g}_{ij}(x,t)dx^i dx^j
\]
are the solutions of
\[
(2.4) 
 \frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = -2\hat{R}_{ij}(x,t) + \hat{h}_{ij}(x,t), \quad \hat{g}_{ij}(x,0) = \tilde{g}_{ij}(x)
\]
for a short time \( 0 \leq t \leq T \). Consider a family of smooth diffeomorphisms \( \varphi_t : M \to M(0 \leq t \leq T) \) of \( M \). Let
\[
(2.5) 
 ds_t^2 := \varphi_t^* ds_t^2, \quad 0 \leq t \leq T
\]
be the pull-back metrics of \( ds_t^2 \). For coordinates system \( x = \{x^1, \cdots, x^n\} \) on \( M \), let
\[
(2.6) 
 ds_t^2 = g_{ij}(x,t)dx^i dx^j
\]
and
\[
(2.7) 
 y(x,t) = \varphi_t(x) = \{y^1(x,t), \cdots, y^n(x,t)\}.
\]
Then we have
\[
(2.8) 
 g_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \hat{g}_{\alpha \beta}(y,t).
\]
\footnote{In the following computations we don’t need to use the evolution equation for \( H(x,t) \), hence we only consider the evolution equation for metrics.}
By the assumption $\tilde{g}_{\alpha\beta}(x, t)$ are the solutions of
\begin{equation}
\frac{\partial}{\partial t}\tilde{g}_{\alpha\beta}(x, t) = -2\tilde{R}_{\alpha\beta}(x, t) + \tilde{h}_{\alpha\beta}(x, t), \quad \tilde{g}_{\alpha\beta}(x, 0) = \tilde{g}_{\alpha\beta}(x).
\end{equation}

We use $R_{ij}, \tilde{R}_{ij}, \hat{R}_{ij}, \Gamma^k_{ij}, \hat{\Gamma}^k_{ij}, \nabla, \bar{\nabla}, \hat{h}_{ij}, \tilde{h}_{ij}, \bar{h}_{ij}$ to denote the Ricci curvatures, Christoffel symbols, covariant derivatives, and products of the 3-form $H$ with respect to $\tilde{g}_{ij}, \hat{g}_{ij}, g_{ij}$ respectively. Then
\begin{equation}
\frac{\partial}{\partial t}g_{ij}(x, t) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \left( \frac{\partial}{\partial t} \tilde{g}_{\alpha\beta}(y, t) \right) + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial x^j} \tilde{g}_{\alpha\beta}(y, t) + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial x^j} \left( \frac{\partial y^\beta}{\partial t} \right) \tilde{g}_{\alpha\beta}(y, t).
\end{equation}
From (2.9) we have
\begin{equation}
\frac{\partial}{\partial t}\tilde{g}_{\alpha\beta}(y, t) = -2\hat{R}_{\alpha\beta}(y, t) + \hat{h}_{\alpha\beta}(y, t) + \partial_{\alpha\beta} \partial y^\gamma \partial y^\gamma \partial t,
\end{equation}
and
\begin{equation}
\frac{\partial}{\partial t}g_{ij}(x, t) = -2\frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \hat{R}_{\alpha\beta}(y, t) + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \hat{h}_{\alpha\beta}(y, t) + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \partial_{\alpha\beta} \partial y^\gamma \partial y^\gamma \partial t + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \partial_{\alpha\beta} \partial y^\gamma \partial t \tilde{g}_{\alpha\beta}(y, t) + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial x^j} \left( \frac{\partial y^\beta}{\partial t} \right) \tilde{g}_{\alpha\beta}(y, t).
\end{equation}
Since
\begin{equation}
R_{ij}(x, t) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \hat{R}_{\alpha\beta}(y, t), \quad h_{ij}(x, t) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \hat{h}_{\alpha\beta}(y, t),
\end{equation}
using the equation ([7], Sec. 2, (29)), we obtain
\begin{equation}
\frac{\partial}{\partial t}g_{ij}(x, t) = -2R_{ij}(x, t) + h_{ij}(x, t)
\end{equation}
(2.10) + \nabla_i \left( \frac{\partial y^\alpha}{\partial t} \frac{\partial x^k}{\partial y^\alpha} g_{jk} \right) + \nabla_j \left( \frac{\partial y^\alpha}{\partial t} \frac{\partial x^k}{\partial y^\alpha} g_{ik} \right).
\end{equation}
According to DeTurck trick, we define $y(x, t) = \varphi_t(x)$ by the equation
\begin{equation}
\frac{\partial y^\alpha}{\partial t} = \frac{\partial y^\alpha}{\partial x^i} g^{\beta\gamma} (\Gamma^k_{\beta\gamma} - \hat{\Gamma}^k_{\beta\gamma}), \quad y^\alpha(x, 0) = x^\alpha,
\end{equation}
then (2.10) becomes
\begin{equation}
\frac{\partial}{\partial t}g_{ij}(x, t) = -2R_{ij}(x, t) + h_{ij}(x, t)
\end{equation}
(2.12) + \nabla_i V_j + \nabla_j V_i, \quad g_{ij}(x, 0) = \tilde{g}_{ij}(x)
where
\begin{equation}
V_i = g_{ik} g^{\beta\gamma} (\Gamma^k_{\beta\gamma} - \hat{\Gamma}^k_{\beta\gamma}).
\end{equation}
Lemma 2.1. The evolution equation (2.12) is a strictly parabolic system. Moreover,
\[
\frac{\partial}{\partial t} g_{ij} = g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta g_{ij} - g^{\alpha\beta} g_{ip} \tilde{g}^{pq} \tilde{R}_{jqo\beta} - g^{\alpha\beta} g_{jp} \tilde{g}^{pq} \tilde{R}_{ioq\beta} \\
+ \frac{1}{2} g^{\alpha\beta} g^{pq} (\tilde{\nabla}_p g_{\alpha q} \cdot \tilde{\nabla}_j g_{\beta i} + 2 \tilde{\nabla}_j g_{jp} \cdot \tilde{\nabla}_\alpha g_{iq} - 2 \tilde{\nabla}_\alpha g_{jp} \cdot \tilde{\nabla}_\beta g_{iq} \\
- 2 \tilde{\nabla}_j g_{lp} \cdot \tilde{\nabla}_\beta g_{iq} - 2 \tilde{\nabla}_i g_{lp} \cdot \tilde{\nabla}_\beta g_{jq}) + \frac{1}{2} g^{\alpha\beta} g^{pq} H_{\alpha p} H_{iq}.
\]

Proof. It is an immediate consequence of Lemma 2.1 of [7]. □

Now we can prove the short-time existence of GRF.

Theorem 2.2. There is a unique solution to GRF for a short time. More precisely, let \((M, g_{ij}(x))\) be an \(n\)-dimensional compact Riemannian manifold with a 3-form \(H = \{H_{ijk}\}\), then there exists a constant \(T = T(n) > 0\) depending only on \(n\) such that the evolution system
\[
\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \frac{1}{2} g^{kp}(x,t) g^{lq}(x,t) H_{ikl}(x,t) H_{jpq}(x,t), \\
\frac{\partial}{\partial t} H(x,t) = \Box g(x,t) H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x),
\]
has a unique solution \((g_{ij}(x,t), H_{ijk}(x,t))\) for a short time \(0 \leq t \leq T\).

Proof. We proved that the first evolution equation is strictly parabolic by Lemma 2.1. Form Bochner formula, we have \(\Box g(x,t) H = \Delta_g(x,t) H + \text{Rm} \ast H\) which is also strictly parabolic. Hence from the standard theory of parabolic systems, the evolution system has a unique solution. □

3. Evolution of curvatures

The evolution equation for the Riemann curvature tensors to the usual Ricci flow (e.g., [1], [2], [3]) is given by
\[
(3.1) \quad \frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + \psi_{ijkl}
\]
where
\[
\psi_{ijkl} = 2(B_{ijkl} - B_{ijlk} - B_{ijlk} + B_{ikjl}) - g^{pq}(R_{ijpq} R_{kl} + R_{ipql} R_{kj} + R_{ijpq} R_{kl} + R_{ijqp} R_{kl}),
\]
and \(B_{ijkl} = g^{pr} g^{qs} R_{ipqr} R_{ks}.\) From this we can easily deduce the evolution equation for the Riemann curvature tensors to GRF.

Let \(v_{ij}(x,t)\) be any symmetric 2-tensor, we consider the flow
\[
(3.2) \quad \frac{\partial}{\partial t} g_{ij}(x,t) = v_{ij}(x,t).
\]
Then the evolution equation for Riemann curvature tensors is
\[
\frac{\partial}{\partial t} R_{ijkl} = -\frac{1}{2} (\nabla_i \nabla_k v_{lj} - \nabla_i \nabla_l v_{jk} - \nabla_j \nabla_k v_{il} + \nabla_j \nabla_l v_{ik}) \\
+ \frac{1}{2} g^{pq} (R_{ijkl} v_{ql} + R_{ijpl} v_{qk}).
\]
Applying to our case \(v_{ij} := -2R_{ij} + \frac{1}{2} h_{ij}\) where \(h_{ij} = H_{ikl} H_{j}^{kl}\), we obtain
\[
\frac{\partial}{\partial t} R_{ijkl} = -\frac{1}{2} \left(-2 \nabla_i \nabla_k R_{jl} + \frac{1}{2} \nabla_i \nabla_k h_{jl} - 2 \nabla_j \nabla_l R_{ik} - \frac{1}{2} \nabla_j \nabla_l h_{ik}\right) \\
+ \frac{1}{2} g^{pq} \left(R_{ijkl} v_{ql} + R_{ijpl} v_{qk}\right).
\]

**Proposition 3.1.** For GRF we have
\[
\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2 \left( R_{ijkl} - R_{ijlk} - R_{ijkl} + B_{ikjl}\right) \\
- g^{pq} (R_{ijkl} R_{ql} + R_{ipkl} R_{qj} + R_{ijpl} R_{qk} + R_{ijkp} R_{ql}) \\
+ \frac{1}{4} \left[-\nabla_i \nabla_k h_{jl} + \nabla_i \nabla_l h_{jk} + \nabla_j \nabla_k h_{il} - \nabla_j \nabla_l h_{ik}\right] \\
+ \frac{1}{4} g^{pq} [R_{ijkl} h_{ql} + R_{ijpl} h_{qk}].
\]

In particular,

**Corollary 3.2.** For GRF we have
\[
(3.3) \quad \frac{\partial}{\partial t} R_{m} = \Delta R_{m} + R_{m} * R_{m} + H * H * R_{m} + \sum_{i=0}^{2} \nabla^i H * \nabla^{2-i} H
\]
Proof. From above proposition, we obtain
\[ \frac{\partial}{\partial t} R_m = \Delta R_m + R_m * R_m + \nabla^2 h + h * R_m. \]

On the other hand, \( h = H * H \) and
\[ \nabla^2 h = \nabla(\nabla(H * H)) = \nabla(\nabla H * H) = \nabla^2 H * H + \nabla H * \nabla H. \]

Combining these terms, we yield the result. \( \square \)

**Proposition 3.3.** For GRF we have
\[
\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + 2\langle R_{piqk}, R_{pq} \rangle - 2\langle R_{pi}, R_{pk} \rangle + \frac{1}{4} \left[ \langle h_{iq}, R_{iklq} \rangle + \langle R_{ip}, h_{kp} \rangle \right] + \frac{1}{4} \left[ -\nabla_i \nabla_k |H|^2 + g^{jl} \nabla_i \nabla_l h_{jk} + g^{jl} \nabla_j \nabla_k h_{il} - \Delta h_{ik} \right].
\]

Proof. Since
\[
\frac{\partial}{\partial t} R_{ik} = g^{jl} \frac{\partial}{\partial t} R_{ijkl} + 2g^{jp} \frac{\partial}{\partial t} R_{ijkl} R_{pq},
\]
and
\[
g^{ij} h_{ij} = g^{ij} H_{ipq} H^{pq} = g^{ij} g^{rs} H_{ipq} H_{jrs} = |H_{ijk}|^2 := |H|^2,
\]
it follows that
\[
g^{jl}[-\nabla_i \nabla_k h_{jl} + \nabla_i \nabla_l h_{jk} + \nabla_j \nabla_k h_{il} - \nabla_j \nabla_l |H|^2 + g^{jl} \nabla_i \nabla_l h_{jk} + g^{jl} \nabla_j \nabla_k h_{il} - \Delta h_{ik} + g^{jl} g^{pq} h_{ql} R_{ijkp} + g^{pq} h_{qk} R_{ip}.
\]

From these identities, we get the result. \( \square \)

As a consequence, we obtain the evolution equation for scalar curvature.

**Proposition 3.4.** For GRF we have
\[
\frac{\partial}{\partial t} R = \Delta R + 2|R|^2 - \frac{1}{2} \Delta |H|^2 + \frac{1}{2} \langle h_{ij}, R_{ij} \rangle + \frac{1}{2} g^{ik} g^{jl} \nabla_i \nabla_j h_{kl}.
\]
Proof. From the usual evolution equation for scalar curvature under the Ricci flow, we have

\[
\frac{\partial}{\partial t} R = \Delta R + 2|R|^2 + \frac{1}{4} g^{ik} \left( [h_{lk}, R_{ik}] + \langle R_{ip}, h_{ip} \rangle \right) \\
+ \frac{1}{4} g^{ik} \left[ -\nabla_i H |h_k|^2 + g^{jl} \nabla_i h_{jk} + g^{ji} \nabla_j h_{il} - \Delta h_{ik} \right] \\
= \Delta R + 2|R|^2 + \frac{1}{4} \langle h_{ij}, R_{ij} \rangle + \frac{1}{4} \langle R_{ip}, h_{ip} \rangle \\
- \frac{1}{4} \Delta |H|^2 + \frac{1}{4} g^{ik} g^{jl} \nabla_i \nabla_l h_{jk} + \frac{1}{4} g^{ik} g^{ji} \nabla_j \nabla_k h_{il} - \frac{1}{4} \Delta |H|^2.
\]

Simplifying the terms, we obtain the required result. \( \square \)

4. Derivative estimates

In this section we are going to prove BBS estimates. At first we review several basic identities of commutators \([\Delta, \nabla]\) and \([\frac{\partial}{\partial t}, \nabla]\). If \(A = A(t)\) is a \(t\)-dependency tensor, and \(\frac{\partial A}{\partial t} = v_{ij}\), then

\[
(4.1) \quad \frac{\partial}{\partial t} \nabla A - \nabla \frac{\partial}{\partial t} A = A * \nabla v, \quad \nabla (\Delta A) - \Delta (\nabla A) = \nabla Rm * A + Rm * \nabla A.
\]

Applying above formulas on GRF, we have

\[
\frac{\partial}{\partial t} \nabla Rm = \nabla \frac{\partial}{\partial t} Rm + Rm * \nabla (Rm + H * H) \\
= \nabla (\Delta Rm + Rm * Rm + H * H * Rm) \\
+ \nabla^2 H * H + \nabla H * \nabla H \\
+ Rm * \nabla Rm + H * \nabla H * Rm \\
= \Delta (\nabla Rm) + Rm * \nabla Rm + H * \nabla H * Rm \\
+ H * H * \nabla Rm + H * \nabla^3 H + \nabla H * \nabla^2 H \\
= \Delta (\nabla Rm) + \sum_{i+j=0} \nabla^i Rm * \nabla^j Rm \\
+ \sum_{i+j+k=0} \nabla^i H * \nabla^j H * \nabla^k Rm \\
+ \sum_{i+j=0+2} \nabla^i H * \nabla^j H.
\]

More generally, we have
Proposition 4.1. For GRF and any nonnegative integer \( l \) we have

\[
\frac{\partial}{\partial t} \nabla^l Rm = \Delta (\nabla^l Rm) + \sum_{i+j=l} \nabla^i Rm \ast \nabla^j Rm \\
+ \sum_{i+j+k=l} \nabla^i H \ast \nabla^j H \ast \nabla^k Rm \\
+ \sum_{i+j=l+2} \nabla^i H \ast \nabla^j H.
\]

(4.2)

Proof. For \( l = 1 \), we have proved before the proposition. Suppose that the formula holds for \( 1, \ldots, l \). By induction to \( l \), for \( l + 1 \) we have

\[
\frac{\partial}{\partial t} \nabla^{l+1} Rm = \frac{\partial}{\partial t} \nabla (\nabla^l Rm) \\
= \nabla \frac{\partial}{\partial t} (\nabla^l Rm) + \nabla^l Rm \ast \nabla (Rm + H \ast H) \\
= \nabla \left( \Delta (\nabla^l Rm) + \sum_{i+j=l} \nabla^i Rm \ast \nabla^j Rm \\
+ \sum_{i+j+k=l} \nabla^i H \ast \nabla^j H \ast \nabla^k Rm \\
+ \sum_{i+j=l+2} \nabla^i H \ast \nabla^j H \right) \\
+ \nabla^l Rm \ast \nabla Rm + H \ast \nabla H \ast \nabla^l Rm \\
= \Delta (\nabla^{l+1} Rm) + \nabla Rm \ast \nabla^l Rm + Rm \ast \nabla^{l+1} Rm \\
+ \sum_{i+j=l} (\nabla^{i+1} Rm \ast \nabla^j Rm + \nabla^i Rm \ast \nabla^{j+1} Rm) \\
+ \sum_{i+j+k=l} (\nabla^{i+1} H \ast \nabla^j H \ast \nabla^k Rm \\
+ \nabla^i H \ast \nabla^{j+1} H \ast \nabla^k Rm + \nabla^i H \ast \nabla^j H \ast \nabla^{k+1} Rm) \\
+ \sum_{i+j=l+2} (\nabla^{i+1} H \ast \nabla^j H + \nabla^i H \ast \nabla^{j+1} H) \\
+ H \ast \nabla H \ast \nabla^l Rm.
\]

Simplifying these terms, we obtain the required result. \( \square \)

As an immediate consequence, we have an evolution inequality for \( |\nabla^l Rm|^2 \).
Corollary 4.2. For GRF and any nonnegative integer $l$ we have
\[
\frac{\partial}{\partial t}|\nabla^l Rm|^2 \leq \Delta|\nabla^l Rm|^2 - 2|\nabla^{l+1} Rm|^2
+ C \cdot \sum_{i+j=l} |\nabla^i Rm| \cdot |\nabla^j Rm| \cdot |\nabla^l Rm|
+ C \cdot \sum_{i+j+k=l} |\nabla^i Rm| \cdot |\nabla^j Rm| \cdot |\nabla^k Rm| \cdot |\nabla^l Rm|
\]
(4.3)
where $C$ are universal constants depending only on the dimension of $M$.

Next we derive the evolution equations for the covariant derivatives of $H$.

Proposition 4.3. For GRF and any positive integer $l$ we have
\[
\frac{\partial}{\partial t} \nabla^l H = \Delta(\nabla^l H) + \sum_{i+j=l} \nabla^i H \ast \nabla^j Rm + \sum_{i+j+k=l} \nabla^i H \ast \nabla^j H \ast \nabla^k H.
\]
(4.4)

Proof. From the Bochner formula, the evolution equation for $H$ can be rewritten as
\[
\frac{\partial}{\partial t} H = \Delta H + Rm \ast H.
\]
(4.5)

For $l = 1$, we have
\[
\frac{\partial}{\partial t} \nabla H = \nabla \frac{\partial}{\partial t} H + H \ast \nabla (Rm + H \ast H)
= \nabla (\Delta H + Rm \ast H) + H \ast \nabla Rm + H \ast H \ast \nabla H
= \nabla (\Delta H) + H \ast \nabla Rm + \nabla H \ast Rm + H \ast H \ast \nabla H
= \Delta(\nabla H) + \nabla Rm \ast H + \nabla H \ast Rm + H \ast H \ast \nabla H.
\]

Using (4.1) and the same argument, we can prove the evolution equation for higher covariant derivatives. □

Similarly, we have an evolution inequality for $|\nabla^l H|^2$.

Corollary 4.4. For GRF and for any positive integer $l$ we have
\[
\frac{\partial}{\partial t} |\nabla^l H|^2 \leq \Delta|\nabla^l H|^2 - 2|\nabla^{l+1} H|^2
+ C \cdot \sum_{i+j=l} |\nabla^i H| \cdot |\nabla^j Rm| \cdot |\nabla^l H|
+ C \cdot \sum_{i+j+k=l} |\nabla^i H| \cdot |\nabla^j H| \cdot |\nabla^k H| \cdot |\nabla^l H|
\]
(4.6)
while
\begin{equation}
\frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 - 2|\nabla H|^2 + C \cdot |\text{Rm}| \cdot |H|^2.
\end{equation}

**Theorem 4.5.** Suppose that \((g(x,t), H(x,t))\) is a solution to GRF on a compact manifold \(M^n\) for a short time \(0 \leq t \leq T\) and \(K_1, K_2\) are arbitrary given nonnegative constants. Then there exists a constant \(C_n\) depending only on \(n\) such that if
\[ |\text{Rm}(x,t)|_{g(x,t)} \leq K_1, \quad |H(x)|_{g(x)} \leq K_2 \]
for all \(x \in M\) and \(t \in [0, T]\), then
\begin{equation}
|H(x,t)|_{g(x,t)} \leq K_2 e^{C_n K_1 t}
\end{equation}
for all \(x \in M\) and \(t \in [0, T]\).

**Proof.** Since
\[ \frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 + C_n |\text{Rm}| \cdot |H|^2 \leq \Delta|H|^2 + C_n K_1 |H|^2, \]
using maximum principle, we obtain \(u(t) \leq u(0) e^{C_n K_1 t} , u(t) := |H|^2. \)

The main result in this section is the following estimates for higher derivatives of Riemann curvature tensors and 3-forms. Some special cases were proved in [8], [9], and [10].

**Theorem 4.6.** Suppose that \((g(x,t), H(x,t))\) is a solution to GRF on a compact manifold \(M^n\) and \(K\) is an arbitrary given positive constant. Then for each \(\alpha > 0\) and each integer \(m \geq 1\) there exists a constant \(C_m\) depending on \(m, n, \max\{\alpha, 1\}\), and \(K\) such that if
\[ |\text{Rm}(x,t)|_{g(x,t)} \leq K, \quad |H(x)|_{g(x)} \leq K \]
for all \(x \in M\) and \(t \in [0, \frac{\alpha}{K}]\), then
\begin{equation}
|\nabla^{m-1}\text{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \leq \frac{C_m}{t^{m/2}}
\end{equation}
for all \(x \in M\) and \(t \in (0, \frac{\alpha}{K}]\).

**Proof.** In the following computations we always let \(C\) be any constants depending on \(n, m, \max\{\alpha, 1\}\), and \(K\), which may take different values at different places. From the evolution equations and Theorem 4.5, we have
\[ \frac{\partial}{\partial t}|\text{Rm}|^2 \leq \Delta|\text{Rm}|^2 - 2|\nabla \text{Rm}|^2 + C + C|\nabla^2 H| + C|\nabla H|^2, \]
\[ \frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 - 2|\nabla H|^2 + C, \]
\[ \frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + C|\nabla \text{Rm}| \cdot |\nabla H| + C|\nabla H|^2. \]
Consider the function $u = t|\nabla H|^2 + \gamma |H|^2 + t|Rm|^2$. Directly computing, we obtain

$$
\frac{\partial}{\partial t}u \leq \Delta u - 2t|\nabla^2 H|^2 + Ct|\nabla^2 H| + (C - 2\gamma)|\nabla H|^2 + C + C\gamma
- 2t|Rm|^2 + Ct \cdot |\nabla Rm| \cdot |\nabla H|
\leq \Delta u + 2(C - \gamma) \cdot |\nabla H|^2 + C(1 + \gamma).
$$

If we choose $\gamma = C$, then $\frac{\partial}{\partial t}u \leq \Delta u + C$ which implies that $u \leq C$ since $u(0) \leq C$. With this estimate we are able to bound the first covariant derivative of $Rm$ and the second covariant derivative of $H$. In order to control the term $|\nabla Rm|^2$, we should use the evolution equations of $|H|^2$, $|\nabla H|^2$ and $|\nabla^2 H|^2$ to cancel with the bad terms, i.e., $|\nabla^2 Rm|^2$, $|\nabla^2 H|^2$, and $|\nabla^3 H|^2$, in the evolution equation of $|\nabla Rm|^2$:

$$
\frac{\partial}{\partial t}|\nabla Rm|^2 \leq \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + C|\nabla Rm|^2 + \frac{C}{t^{1/2}}|\nabla Rm|
+ C \cdot |\nabla Rm| \cdot |\nabla^3 H| + \frac{C}{t^{1/2}}|\nabla^2 H| \cdot |\nabla Rm|,
$$

$$
\frac{\partial}{\partial t}|\nabla^2 H|^2 \leq \Delta |\nabla^2 H|^2 - 2|\nabla^3 H|^2 + C \cdot |\nabla^2 Rm| \cdot |\nabla^2 H|
+ \frac{C}{t^{1/2}}|\nabla Rm| \cdot |\nabla^2 H| + C|\nabla^2 H|^2 + \frac{C}{t}|\nabla^2 H|.
$$

As above, we define

$$
u := t^2(|\nabla^2 H|^2 + |\nabla Rm|^2) + t\beta(|\nabla H|^2 + |Rm|^2) + \gamma |H|^2,
$$

and therefore, $\frac{\partial u}{\partial t} \leq \Delta u + C$. Motivated by cases for $m = 1$ and $m = 2$, for general $m$, we can define a function

$$
u := t^m(|\nabla^m H|^2 + |\nabla^{m-1} Rm|^2) + \sum_{i=1}^{m-1} \beta_i t^{i}(|\nabla^i H|^2 + |\nabla^{i-1} Rm|^2) + \gamma |H|^2,
$$

where $\beta_i$ and $\gamma$ are positive constants determined later. In the following, we always assume $m \geq 3$. Suppose

$$
|\nabla^{i-1} Rm| + |\nabla^i H| \leq \frac{C_{i}}{t^{i/2}}, \quad i = 1, 2, \ldots, m - 1.
$$
For such \( i \), from Corollary 4.4, we have

\[
\frac{\partial}{\partial t} |\nabla^i H|^2 \leq \Delta |\nabla^i H|^2 - 2 |\nabla^{i+1} H|^2 + C \sum_{j=0}^{i} |\nabla^j H| \cdot |\nabla^{i-j} Rm| \cdot |\nabla^i H|
\]

\[
+ C \sum_{j=0}^{i} \sum_{l=0}^{i-j} |\nabla^j H| \cdot |\nabla^{i-j-l} H| \cdot |\nabla^l H| \cdot |\nabla^i H|
\]

\[
\leq \Delta |\nabla^i H|^2 - 2 |\nabla^{i+1} H|^2 + C \cdot |\nabla^i H| \sum_{j=0}^{i} \frac{C_j}{t^{j+1}} \cdot \frac{C_{i-j+1}}{t^{i-j+1}}
\]

\[
+ C \cdot |\nabla^i H| \sum_{j=0}^{i} \sum_{l=0}^{i-j} \frac{C_j}{t^{j+1}} \cdot \frac{C_{i-j-l}}{t^{i-j-l}} \cdot \frac{C_l}{t^l}
\]

\[
\leq \Delta |\nabla^i H|^2 - 2 |\nabla^{i+1} H|^2 + C \cdot |\nabla^i H| + \frac{C_i}{t^{i+1}} |\nabla^i H|.
\]

Similarly, from Corollary 4.2 we also have

\[
\frac{\partial}{\partial t} |\nabla^{i-1} Rm|^2 \leq \Delta |\nabla^{i-1} Rm|^2 - 2 |\nabla^i Rm|^2
\]

\[
+ C \sum_{j=0}^{i-1} |\nabla^j Rm| \cdot |\nabla^{i-1-j} Rm| \cdot |\nabla^{i-1} Rm|
\]

\[
+ C \sum_{j=0}^{i-1} \sum_{l=0}^{i-1-j} |\nabla^j H| \cdot |\nabla^{i-1-j-l} H| \cdot |\nabla^l Rm| \cdot |\nabla^{i-1} Rm|
\]

\[
+ C \sum_{j=0}^{i-1} |\nabla^j H| \cdot |\nabla^{i-1+j} H| \cdot |\nabla^{i-1} Rm|
\]

\[
\leq \Delta |\nabla^{i-1} Rm|^2 - 2 |\nabla^i Rm|^2
\]

\[
+ C \cdot |\nabla^{i-1} Rm| \sum_{j=0}^{i-1} \frac{C_{j+1}}{t^{j+1}} \cdot \frac{C_{i-j}}{t^{i-j}}
\]

\[
+ C \cdot |\nabla^{i-1} Rm| \sum_{j=0}^{i-1} \sum_{l=0}^{i-1-j} \frac{C_j}{t^{j+1}} \cdot \frac{C_{i-1-j-l}}{t^{i-1-j-l}} \cdot \frac{C_{l+1}}{t^{l+1}}
\]

\[
+ C \cdot |\nabla^{i-1} Rm| \sum_{j=1}^{i} \frac{C_j}{t^j} \cdot \frac{C_{i+1-j}}{t^{i+1-j}} + C \cdot |\nabla^{i+1} H| \cdot \frac{C_i}{t^i}
\]

\[
\leq \Delta |\nabla^{i-1} Rm|^2 - 2 |\nabla^i Rm|^2
\]

\[
+ \frac{C_i}{t^{i+1}} |\nabla^{i-1} Rm| + \frac{C_i}{t^i} |\nabla^{i+1} H| + \frac{C_i}{t^{i+1}} |\nabla^{i-1} Rm|.
\]
The evolution inequality for $u$ now is given by
\[
\frac{\partial u}{\partial t} \leq ml^{m-1}(|\nabla^m H|^2 + |\nabla^{m-1} Rm|^2)
\]
\[
+ \sum_{i=1}^{m-1} i \beta_i t^{i-1}(|\nabla^i H|^2 + |\nabla^{i-1} Rm|^2)
\]
\[
+ t^m \left( \frac{\partial}{\partial t} |\nabla^m H|^2 + \frac{\partial}{\partial t} |\nabla^{m-1} Rm|^2 \right)
\]
\[
+ \sum_{i=1}^{m-1} \beta_i t^i \left( \frac{\partial}{\partial t} |\nabla^i H|^2 + \frac{\partial}{\partial t} |\nabla^{i-1} Rm|^2 \right)
\]
\[
+ \gamma \cdot \frac{\partial}{\partial t} |H|^2.
\]
It’s easy to see that the second term is bounded by
\[
\sum_{i=1}^{m-1} i \beta_i t^{i-1} C_i = \sum_{i=1}^{m-1} i \beta_i C_i t^{-1},
\]
but this bound depends on $t$ and approaches to infinity when $t$ goes to zero. Hence we use the last second term to control this bad term. The evolution inequality for the third term is the combination of the following two inequalities
\[
\frac{\partial}{\partial t} |\nabla^m H|^2 \leq \Delta |\nabla^m H|^2 - 2|\nabla^{m-1} H|^2
\]
\[
+ C \sum_{i=0}^{m} |\nabla^i H| \cdot |\nabla^{m-i} Rm| \cdot |\nabla^m H|
\]
\[
+ C \sum_{i=0}^{m} \sum_{j=0}^{m-i} |\nabla^j H| \cdot |\nabla^{m-i-j} H| \cdot |\nabla^i H| \cdot |\nabla^m H|
\]
\[
\leq \Delta |\nabla^m H|^2 - 2|\nabla^{m+1} H|^2
\]
\[
+ C \sum_{i=1}^{m-1} \frac{C_i}{t^2} \cdot \frac{C_{m-i+1}}{t^{\frac{m-i}{2}}} \cdot |\nabla^m H|
\]
\[
+ C \cdot |\nabla^m Rm| \cdot |\nabla^m H| + C |\nabla^m H|^2
\]
\[
+ C \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} \frac{C_i}{t^2} \cdot \frac{C_{m-i-j}}{t^{\frac{m-i-j}{2}}} \cdot |\nabla^m H|
\]
\[
\leq \Delta |\nabla^m H|^2 - 2|\nabla^{m+1} H|^2 + C |\nabla^m H|^2
\]
\[
+ C \cdot |\nabla^m Rm| \cdot |\nabla^m H|
\]
\[
+ \frac{C_m}{t^{\frac{m+1}{2}}} |\nabla^m H| + \frac{C_m}{t^{\frac{m}{2}}} |\nabla^m H|,
\]
and
\[
\frac{\partial}{\partial t} |\nabla^{m-1}Rm|^2 \leq \Delta |\nabla^{m-1}Rm|^2 - 2|\nabla^m Rm|^2 \\
+ C \sum_{i=0}^{m-1} |\nabla^i Rm| \cdot |\nabla^{m-1-i} Rm| \cdot |\nabla^{m-1} Rm| \\
+ C \sum_{i=0}^{m-1} \sum_{j=0}^{m-1-i} |\nabla^j H| \cdot |\nabla^{m-1-i-j} H| \\
\cdot |\nabla^i Rm| \cdot |\nabla^{m-1} Rm| \\
+ C \sum_{i=0}^{m+1} |\nabla^i H| \cdot |\nabla^{m+1-i} H| \cdot |\nabla^{m-1} Rm| \\
\leq \Delta |\nabla^{m-1} Rm|^2 - 2|\nabla^m Rm|^2 \\
+ C \sum_{i=1}^{m-2} \frac{C_{i+1}}{t^{\frac{i+1}{2}}} \cdot \frac{C_{m-i}}{t^{\frac{m-i}{2}}} |\nabla^{m-1} Rm| + C |\nabla^{m-1} Rm|^2 \\
+ C \sum_{i=0}^{m-2} \sum_{j=0}^{m-1-i} \frac{C_{j}}{t^{\frac{j}{2}}} \cdot \frac{C_{m-1-i-j}}{t^{\frac{m-1-i-j}{2}}} \cdot \frac{C_{i+1}}{t^{\frac{i+1}{2}}} |\nabla^{m-1} Rm| \\
+ C \sum_{i=0}^{m-1} \frac{C_{i}}{t^{\frac{i}{2}}} \cdot \frac{C_{m+1-i}}{t^{\frac{m+1-i}{2}}} \cdot |\nabla^{m-1} Rm| \\
+ C |\nabla^m H| \cdot |\nabla H| \cdot |\nabla^{m-1} Rm| \\
+ C |\nabla^{m+1} H| \cdot |\nabla^{m-1} Rm| \\
\leq \Delta |\nabla^{m-1} Rm|^2 - 2|\nabla^m Rm|^2 \\
+ C |\nabla^{m-1} Rm|^2 + \frac{C}{t^2} \cdot |\nabla^m H| \cdot |\nabla^{m-1} Rm| \\
+ C \cdot |\nabla^{m+1} H| \cdot |\nabla^{m-1} Rm| \\
+ \frac{C_m}{t^{m+1}} |\nabla^{m-1} Rm| + \frac{C_m}{t^2} |\nabla^{m-1} Rm|.
Therefore we have
\[
\frac{\partial u}{\partial t} \leq mt^{m-1}(|\nabla^m H|^2 + |\nabla^{m-1} Rm|^2)
+ \sum_{i=1}^{m-1} i \beta_i t^{i-1}(|\nabla^i H|^2 + |\nabla^{i-1} Rm|^2)
+ t^m \left( \Delta |\nabla^m H|^2 - 2 |\nabla^{m+1} H|^2 + \frac{C}{t^{m-1}} |\nabla^m H| + C |\nabla^m H|^2 \right)
+ C |\nabla^m Rm| \cdot |\nabla^m H| + \Delta |\nabla^{m-1} Rm|^2
- 2 |\nabla^m Rm|^2 + \frac{C}{t^{m-1}} |\nabla^{m-1} Rm| + C |\nabla^{m-1} Rm|^2
+ \frac{C}{t^{1/2}} |\nabla^m H| \cdot |\nabla^{m-1} Rm| + C |\nabla^{m+1} H| \cdot |\nabla^{m-1} Rm|
+ \sum_{i=1}^{m-1} \beta_i t^i \left( \frac{C_i}{t^{i-1}} |\nabla^{i-1} Rm| + \Delta |\nabla^i H|^2 - 2 |\nabla^{i+1} H|^2 \right)
+ \Delta |\nabla^{i-1} Rm|^2 + \frac{C_i}{t^{i-1}} |\nabla^i H| + \frac{C_i}{t^i} |\nabla^{i+1} H| - 2 |\nabla^i Rm|^2 \right)
+ \gamma (\Delta |H|^2 - 2 |\nabla H|^2 + C)
\leq \Delta u - 2t^m |\nabla^{m+1} H|^2 + Ct^m |\nabla^{m+1} H| \cdot |\nabla^{m-1} Rm|
- 2t^m |\nabla^m Rm|^2 + Ct^m |\nabla^m Rm| \cdot |\nabla^m H|
+ \sum_{i=0}^{m-2} (i + 1) \beta_{i+1} t^i (|\nabla^{i+1} H|^2 + |\nabla^i Rm|^2)
- 2 \sum_{i=1}^{m-1} \beta_i t^i (|\nabla^{i+1} H|^2 + |\nabla^i Rm|^2) - 2gamma |\nabla H|^2 + gamma C
+ Ct^{m-1} |\nabla^m H|^2 + Ct^{m-1} |\nabla^{m-1} Rm|^2
+ Ct^{m-1} |\nabla^m H| + Ct^{m-1} |\nabla^{m-1} Rm|
+ Ct^{m-1} |\nabla^m H| \cdot |\nabla^{m-1} Rm| + Ct^m |\nabla^{m+1} H| \cdot |\nabla^{m-1} Rm|
+ \sum_{i=1}^{m-1} \beta_i C_i t^{i/2} |\nabla^{i+1} H| + \sum_{i=1}^{m-1} \beta_i C_i t^{i-1} (|\nabla^i H|^2 + |\nabla^{i-1} Rm|).
\]

Choosing
\[
(i + 1) \beta_{i+1} = \beta_i, \quad \beta_i = \frac{A}{i!}, \quad i \geq 0,
\]
where $A$ is constant which is determined later, and noting that
\[
\sum_{i=1}^{m-1} \beta_i C_i t^{i/2} |\nabla^{i+1} H| \leq \frac{1}{2} \sum_{i=1}^{m-1} \beta_i t^i |\nabla^{i+1} H|^2 + \frac{1}{2} \sum_{i=1}^{m-1} \beta_i C_i^2,
\]
It yields that

\[
\sum_{i=1}^{m-1} \beta_i C_i t^{i-1} (|\nabla^i H| + |\nabla^{i-1} R_m|) \leq \beta_1 C_1 (|\nabla H| + |R_m|) + \sum_{i=1}^{m-2} \beta_{i+1} C_{i+1} t^i (|\nabla^{i+1} H| + |\nabla^i R_m|)
\]

\[
\leq \beta_1 C_1 (|\nabla H| + |R_m|) + \sum_{i=1}^{m-2} \beta_{i+1} C_{i+1} t^i \left(\frac{t^i |\nabla^{i+1} H|^2}{2\beta_{i+1} C_{i+1}} + \frac{t^i |\nabla^i R_m|^2}{2\beta_{i+1} C_{i+1}} + \frac{\beta_{i+1} C_{i+1}}{\beta_i}\right)
\]

\[
\leq \beta_1 C_1 (|\nabla H| + |R_m|) + \frac{1}{2} \sum_{i=1}^{m-2} \beta_i t^i (|\nabla^{i+1} H|^2 + |\nabla^i R_m|^2) + \sum_{i=1}^{m-2} \frac{\beta_i C_i t^i}{\beta_i}
\]

It yields that

\[
\frac{\partial}{\partial t} u \leq \Delta u - 2t^m |\nabla^{m+1} H|^2 + C t^m |\nabla^{m+1} H| : |\nabla^{m-1} R_m|
\]

\[
- 2t^m |\nabla^m R_m|^2 + C t^m |\nabla^m H| : |\nabla^m R_m|
\]

\[
+ C t^{m-1} |\nabla^m H|^2 + C t^{m-1} |\nabla^{m-1} R_m|^2
\]

\[
+ C t^{m-\frac{3}{2}} (|\nabla^m R_m|^2 + |\nabla^{m-1} R_m|^2) + \beta_0 (|\nabla H|^2 + |R_m|^2)
\]

\[
- \sum_{i=1}^{m-1} \beta_i t^i (|\nabla^{i+1} H|^2 + |\nabla^i R_m|^2)
\]

\[
+ \sum_{i=1}^{m-2} \beta_i t^i (|\nabla^{i+1} H|^2 + |\nabla^i R_m|^2) + \frac{1}{2} \beta_{m-1} t^{m-1} |\nabla^m H|^2
\]

\[
+ \beta_1 C_1 |\nabla H| - 2\gamma |\nabla H|^2 + C + C\gamma
\]

\[
\leq \Delta u + C t^{m-1} |\nabla^{m-1} R_m|^2 + C t^{m-1} |\nabla^m H|^2
\]

\[
+ C t^{m-\frac{3}{2}} (|\nabla^m H|^2 + |\nabla^{m-1} R_m|^2) + \beta_0 |\nabla H|^2
\]

\[
+ \beta_1 C_1 |\nabla H| - 2\gamma |\nabla H|^2 + C + C\gamma
\]

\[
- \frac{1}{2} \beta_{m-1} t^{m-1} |\nabla^m R_m|^2
\]

\[
\leq \Delta u + \frac{1}{2} (C \sqrt{t} + C - \beta_{m-1}) t^{m-1} (|\nabla^{m-1} R_m|^2 + |\nabla^m H|^2)
\]

\[
+ (\beta_0 + \beta_1 C_1 - 2\gamma) |\nabla H|^2 + C + C\gamma + \beta_1 C_1.
\]

When we chose \( A \) and \( \gamma \) sufficiently large, we obtain \( \frac{\partial u}{\partial t} \leq \Delta u + C \) which implies that \( u(t) \leq C \) since \( u(0) \) is bounded. \( \square \)

Finally we give an estimate which plays a crucial role in the next section.
Corollary 4.7. Let \((g(x, t), H(x, t))\) be a solution of the generalized Ricci flow on a compact manifold \(M\). If there are \(\beta > 0\) and \(K > 0\) such that
\[
|\text{Rm}(x, t)|_{g(x,t)} \leq K, \quad |H(x)|_{g(x)} \leq K
\]
for all \(x \in M\) and \(t \in [0, T]\), where \(T > \frac{\beta}{K}\), then there exists for each \(m \in \mathbb{N}\) a constant \(C_m\) depending on \(m, n, \min\{\beta, 1\}\), and \(K\) such that
\[
|\nabla^{m-1}\text{Rm}(x, t)|_{g(x,t)} + |\nabla^m H(x, t)|_{g(x,t)} \leq C_m K^{m/2}
\]
for all \(x \in M\) and \(t \in \left[\frac{\min\{\beta, 1\}}{K}, T\right]\).

**Proof.** The proof is the same as in [2], we just copy it here. Let \(\beta_1 := \min\{\beta, 1\}\). For any fixed point \(t_0 \in \left[\frac{\beta_1}{K}, T\right]\) we set \(T_0 := t_0 - \frac{\beta_1}{K}\). For \(t := t - T_0\) we let \(\overline{g}(t)\) be the solution of the system
\[
\frac{\partial}{\partial t} = -2\text{Rc} + \frac{1}{2}h, \quad \frac{\partial}{\partial t} = \square_{\overline{g}} H, \quad \overline{g}(0) = g(T_0), \quad \overline{H}(0) = H(T_0).
\]
The uniqueness of solution implies that \(\overline{g}(t) = g(t + T_0) = g(t)\) for \(t \in [0, \frac{\beta_1}{K}]\). By the assumption we have
\[
|\text{Rm}(x, t)|_{\overline{g}(x,t)} \leq K, \quad |\overline{H}(x)|_{\overline{g}(x)} \leq K
\]
for all \(x \in M\) and \(t \in \left[0, \frac{\beta_1}{K}\right]\). Applying Theorem 4.5 with \(\alpha = \beta_1\), we have
\[
|\nabla^{m-1}\text{Rm}(x, t)|_{\overline{g}(x,t)} + |\nabla^m H(x, t)|_{\overline{g}(x,t)} \leq \frac{C_m}{t^{m/2}}
\]
for all \(x \in M\) and \(t \in (0, \frac{\beta_1}{K}]\). We have \(t^{m/2} \geq \beta_1^{m/2} 2^{-m/2} K^{-m/2}\) if \(t \in \left[\frac{\beta_1}{2K}, \frac{\beta_1}{K}\right]\). Taking \(t = \frac{\beta_1}{K}\), we obtain
\[
|\nabla^{m-1}\text{Rm}(x, t_0)|_{g(x,t_0)} + |\nabla^m H(x, t_0)|_{g(x,t_0)} \leq \frac{2^{m/2} C_m K^{m/2}}{\beta^{m/2} 2^{1/2}}
\]
for all \(x \in M\). Since \(t_0 \in \left[\frac{\beta}{K}, T\right]\) was arbitrary, the result follows. \(\square\)

5. **COMPACTNESS THEOREM**

In this section we prove the compactness theorem for our generalized Ricci flow. We follow Hamilton’s method [4] on the compactness theorem for the usual Ricci flow.

We review several definitions from [2, Chapter 3]. Throughout this section, all Riemannian manifolds are smooth manifolds with dimensions \(n\). The covariant derivative with respect to a metric \(g\) will be denoted by \(g\nabla\).
Definition 5.1. Let $K \subset M$ be a compact set and let $\{g_k\}_{k \in \mathbb{N}}, g_\infty$, and $g$ be Riemannian metrics on $M$. For $p \in \{0\} \cup \mathbb{N}$ we say that $g_k$ converges in $C^p$ to $g_\infty$ uniformly on $K$ with respect to $g$ if for every $\epsilon > 0$ there exists $k_0 = k_0(\epsilon) > 0$ such that for $k \geq k_0$,

$$
\|g_k - g_\infty\|_{C^p;K,g} := \sup_{0 \leq \alpha \leq p, x \in K} |g^{\alpha}(g_k - g_\infty)(x)|_g < \epsilon.
$$

Since we consider a compact set, the choice of background metric $g$ does no change the convergence. Hence we can choose $g = g_\infty$ for practice.

Definition 5.2. Suppose $\{U_k\}_{k \in \mathbb{N}}$ is an exhaustion\(^3\) of a smooth manifold $M$ by open sets and $g_k$ are Riemannian metrics on $U_k$. We say that $(U_k, g_k)$ converges in $C^\infty$ to $(M, g_\infty)$ uniformly on compact sets in $M$ if for any compact set $K \subset M$ and any $p > 0$ there exists $k_0 = k_0(K, p)$ such that $\{g_k\}_{k \geq k_0}$ converges in $C^p$ to $g_\infty$ uniformly on $K$.

A pointed Riemannian manifold is a 3-tuple $(M, g, O)$, where $(M, g)$ is a Riemannian manifold and $O \in M$ is a basepoint. If the metric $g$ is complete, the 3-tuple is called a complete pointed Riemannian manifold. We say that $(M, g(t), H(t), O), t \in (\alpha, \omega)$, is a pointed solution to the generalized Ricci flow if $(M, g(t), H(t))$ is a solution to the generalized Ricci flow.

The so-called Cheeger-Gromov convergence in $C^\infty$ is defined by

Definition 5.3. (Cheeger-Gromov convergence) A given sequence $\{(M_k, g_k, O_k)\}_{k \in \mathbb{N}}$ of complete pointed Riemannian manifolds converges to a complete pointed Riemannian manifold $(M_\infty, g_\infty, O_\infty)$ if there exist

(i) an exhaustion $\{U_k\}_{k \in \mathbb{N}}$ of $M_\infty$ by open sets with $O_\infty \in U_k$,

(ii) a sequence of diffeomorphisms $\Phi_k : M_\infty \ni U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$

such that $(U_k, \Phi_k^*(g_k|_{V_k}))$ converges in $C^\infty$ to $(M_\infty, g_\infty)$ uniformly on compact sets in $M_\infty$.

The corresponding convergence for the generalized Ricci flow is similar to the convergence for the usual Ricci flow introduced by Hamilton\cite{4}.

Definition 5.4. A given sequence $\{(M_k, g_k(t), H_k(t), O_k)\}_{k \in \mathbb{N}}$ of complete pointed solutions to the GRF converges to a complete pointed solution to the GRF $(M_\infty, g_\infty(t), H_\infty(t), O_\infty), t \in (\alpha, \omega)$ if there exist

(i) an exhaustion $\{U_k\}_{k \in \mathbb{N}}$ of $M_\infty$ by open sets with $O_\infty \in U_k$,

(ii) a sequence of diffeomorphisms $\Phi_k : M_\infty \ni U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$

---

\(^3\)If for any compact set $K \subset M$ there exists $k_0 \in \mathbb{N}$ such that $U_k \supset K$ for all $k \geq k_0$.
such that \((U_k \times (\alpha, \omega), \Phi^*_k(g_k(t)|_{V_k}) + dt^2, \Phi^*_k(H_k(t)|_{V_k}))\) converges in \(C^\infty\) to \((M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2, H_\infty(t))\) uniformly on compact sets in \(M_\infty \times (\alpha, \omega)\). Here we denote by \(dt^2\) the standard metric on \((\alpha, \omega)\).

Let \(\text{inj}_g(O)\) be the injectivity radius of the metric \(g\) at the point \(O\). The following compactness theorem due to Hamilton does not depend on any flow.

**Theorem 5.1. (Compactness for metrics (Hamilton, [4]))** Let \(\{(M_k, g_k, O_k)\}_{k \in \mathbb{N}}\) be a sequence of complete pointed Riemannian manifolds satisfying

(i) for all \(p \geq 0\) and \(k \in \mathbb{N}\), there is a sequence of constants \(C_p < \infty\) independent of \(k\) such that

\[|^{g_k} \nabla^p Rm(g_k)|_{g_k} \leq C_p\]

on \(M_k\),

(ii) there exists some constant \(\iota_0 > 0\) such that

\[\text{inj}_{g_k}(O_k) \geq \iota_0\]

for all \(k \in \mathbb{N}\).

Then there exists a subsequence \(\{j_k\}_{k \in \mathbb{N}}\) such that \(\{(M_{j_k}, g_{j_k}, O_{j_k})\}_{k \in \mathbb{N}}\) converges to a complete pointed Riemannian manifold \((M_\infty, g_\infty, O_\infty)\) as \(k \to \infty\).

As a consequence of Hamilton’s compactness for metrics, we state our compactness theorem for GRF

**Theorem 5.2. (Compactness for GRF)** Let \(\{(M_k, g_k(t), H_k(t), O_k)\}_{k \in \mathbb{N}}\) be a sequence of complete pointed solutions to GRF for \(t \in [\alpha, \omega) \ni 0\) such that

(i) there is a constant \(C_0 < \infty\) independent of \(k\) such that

\[
\sup_{(x,t) \in M_k \times (\alpha, \omega)} |Rm(g_k(x,t))|_{g_k(x,t)} \leq C_0, \quad \sup_{x \in M_k} |H_k(x, \alpha)|_{g_k(x, \alpha)} \leq C_0,
\]

(ii) there exists a constant \(\iota_0 > 0\) satisfies

\[\text{inj}_{g_k(0)}(O_k) \geq \iota_0.\]

Then there exists a subsequence \(\{j_k\}_{k \in \mathbb{N}}\) such that

\((M_{j_k}, g_{j_k}(t), H_{j_k}(t), O_{j_k}) \longrightarrow (M_\infty, g_\infty(t), H_\infty(t), O_\infty),\)

converges to a complete pointed solution \((M_\infty, g_\infty(t), H_\infty(t), O_\infty), t \in [\alpha, \omega)\) to GRF as \(k \to \infty\).

In order to prove the main theorem in the section, we extend a lemma for Ricci flow to GRF. After establishing this lemma, the proof of Theorem 5.6 is similar as in [2, Theorem 3.10].
Lemma 5.3. Let \((M,g)\) be a Riemannian manifold with a background metric \(g\), let \(K\) be a compact subset of \(M\), and let \((g_k(x,t), H_k(x,t))\) be a collection of solutions to the generalized Ricci flow defined on neighborhoods of \(K \times [\beta, \psi]\), where \(t_0 \in [\beta, \psi]\) is a fixed time. Suppose that

(i) the metrics \(g_k(x,t_0)\) are all uniformly equivalent to \(g(x)\) on \(K\), i.e., for all \(V \in T_xM, k, \) and \(x \in K,\)

\[
C^{-1}g(x)(V,V) \leq g_k(x,t_0)(V,V) \leq C g(x)(V,V),
\]

where \(C < \infty\) is a constant independent of \(V, k,\) and \(x,\)

(ii) the covariant derivatives of the metrics \(g_k(x,t_0)\) with respect to the metric \(g(x)\) are all uniformly bounded on \(K\), i.e., for all \(k\) and \(p \geq 1,\)

\[
|\partial^p g_k(x,t_0)|_{g(x)} + |\partial^p H_k(x,t_0)|_{g(x)} \leq C_\partial
\]

where \(C_\partial < \infty\) is a sequence of constants independent of \(k,\)

(iii) the covariant derivatives of the curvature tensors \(Rm(g_k(x,t))\) and of the forms \(H_k(x,t)\) are uniformly bounded with respect to the metric \(g_k(x,t)\) on \(K \times [\beta, \psi]\), i.e., for all \(k\) and \(p \geq 0,\)

\[
|g_k \nabla^p \text{Rm}(g_k(x,t))|_{g_k(x,t)} + |g_k \nabla^p H_k(x,t)|_{g_k(x,t)} \leq C'_\partial
\]

where \(C'_\partial\) is a sequence of constants independent of \(k,\)

Then the metrics \(g_k(x,t)\) are uniformly equivalent to \(g(x)\) on \(K \times [\beta, \psi]\), i.e.,

\[
B(t,t_0)^{-1}g(x)(V,V) \leq g_k(x,t)(V,V) \leq B(t,t_0)g(x)(V,V),
\]

where \(B(t,t_0) = C e^{|t-t_0|}\) (Here the constant \(C'_\partial\) may not be equal to the previous one), and the time-derivatives and covariant derivatives of the metrics \(g_k(x,t)\) with respect to the metric \(g(x)\) are uniformly bounded on \(K \times [\beta, \psi]\), i.e., for each \((p, q)\) there is a constant \(\tilde{C}_{p,q}\) independent of \(k\) such that

\[
\left|\frac{\partial}{\partial t^q} g \nabla^p g_k(x,t)\right|_{g(x)} + \left|\frac{\partial}{\partial t^q} g \nabla^p H_k(x,t)\right|_{g(x)} \leq \tilde{C}_{p,q}
\]

for all \(k,\)

Proof. Before proving the lemma, we quote a fact from [2, Lemma 3.13, P133]: Suppose that the metrics \(g_1\) and \(g_2\) are equivalent, i.e., \(C^{-1}g_1 \leq g_2 \leq C g_1\). Then for any \((p, q)\)-tensor \(T\) we have \(|T|_{g_2} \leq C^{(p+q)/2}|T|_{g_1}\). We denote by \(h\) the tensor \(h_{ij} := g^{kp}g^{jq}H_{ikljlpq}.\) In the following we denote by \(C\) a constant depending only on \(n, \beta, \) and \(\psi\) which may take different values at different places. For any tangent vector \(V \in T_xM\) we have

\[
\frac{\partial}{\partial t} g_k(x,t)(V,V) = -2\text{Rc}(g_k(x,t))(V,V) + \frac{1}{2}h_k(x,t)(V,V),
\]
and therefore
\[
\left| \frac{\partial}{\partial t} \log g_k(x, t)(V, V) \right| = \left| \frac{-2 \text{Rc}(g_k(x, t))(V, V) + \frac{1}{2} h_k(x, t)(V, V)}{g_k(x, t)(V, V)} \right|
\]
\[
\leq C'_0 + C |H_k(x, t)|_{g_k(x, t)}^2
\]
\[
\leq C'_0 + CC'_0^2 := C,
\]
since
\[
|\text{Rc}(g_k(x, t))(V, V)| \leq C'_0 g_k(x, t)(V, V),
\]
\[
|h_k(x, t)(V, V)| \leq C |H_k(x, t)|_{g_k(x, t)}^2 g_k(x, t)(V, V).
\]
Integrating on both sides, we have
\[
|t_1 - t_0| \geq \int_{t_0}^{t_1} \left| \frac{\partial}{\partial t} \log g_k(x, t)(V, V) \right| dt
\]
\[
\geq \left| \int_{t_0}^{t_1} \frac{\partial}{\partial t} \log g_k(t)(V, V) dt \right|
\]
\[
= \left| \log \frac{g_k(x, t_1)(V, V)}{g_k(x, t_0)(V, V)} \right|
\]
and hence we conclude that
\[
e^{-C|t_1 - t_0|} g_k(x, t_0)(V, V) \leq g_k(x, t_1)(V, V) \leq e^{C|t_1 - t_0|} g_k(x, t_0)(V, V).
\]
From the assumption (i), it immediately deduces from above that
\[
C^{-1} e^{-C|t_1 - t_0|} g(x)(V, V) \leq g_k(x, t_1)(V, V) \leq C e^{C|t_1 - t_0|} g(x)(V, V).
\]
Since \( t_1 \) was arbitrary, the first part is proved. From the definition (or refer [2], eq. (37), P134), we have
\[
(g_k)^{ec} (g \nabla a(g_k))_{bc} + g \nabla_b (g_k)_{ac} - g \nabla_c (g_k)_{ab} = 2 (g_k \Gamma)^{e}_{ab} - 2 (\gamma \Gamma)^{e}_{ab}.
\]
Thus \( |g_k \Gamma(x, t) - \gamma \Gamma(x)|_{g(x)} \leq C |g \nabla g_k(x, t)|_{g_k(x)} \). On the other hand,
\[
g \nabla a(g_k)_{bc} = (g_k)_{eb} [(g_k \Gamma)^{e}_{ac} - (\gamma \Gamma)^{e}_{ac}] + (g_k)_{ec} [(g_k \Gamma)^{e}_{ab} - (\gamma \Gamma)^{e}_{ab}]
\]
it follows that \( |g \nabla g_k(x, t)|_{g_k(x, t)} \leq C |g_k \Gamma(x, t) - \gamma \Gamma(x)|_{g_k(x, t)} \) and therefore \( g \nabla g_k \) is equivalent to \( g_k \Gamma - \gamma \Gamma = g_k \nabla - \gamma \nabla \). The evolution equation for \( \gamma \Gamma \) is
\[
\frac{\partial}{\partial t} (g_k \Gamma)^{ad}_{ab} = -(g_k)^{cd} [(g_k \nabla)_{a} (\text{Rc}(g_k))_{bd} + (g_k \nabla)_{b} (\text{Rc}(g_k))_{ad}]
\]
\[
- (g_k \nabla)_{d} (\text{Rc}(g_k))_{ab}
\]
\[
+ \frac{1}{4} (g_k)^{cd} [(g_k \nabla)_{a} (h_k)_{bd} + (g_k \nabla)_{b} (h_k)_{ad} - (g_k \nabla)_{d} (h_k)_{ab}].
\]
Since $^g\Gamma$ does not depend on $t$, it follows from the assumptions that
\[
\left| \frac{\partial}{\partial t}(^g\Gamma - ^g\Gamma) \right|_{^g\Gamma} \leq C|^{^g\nabla}(\text{Rc}(^g\Gamma))|_{^g\Gamma} + C|^{^g\nabla}(^g\Gamma)|_{^g\Gamma} \\
\leq CC_1' + C|^{^g\nabla}H_k|_{^g\Gamma} \cdot |H_k|_{^g\Gamma} \leq C_{1}' .
\]
Integrating on both sides,
\[
C_1'|t_1 - t_0| \geq \left| \int_{t_0}^{t_1} \frac{\partial}{\partial t} (^g\Gamma(t) - ^g\Gamma) dt \right|_{^g\Gamma} \\
\geq |^{^g\nabla}(^g\Gamma(t_1) - ^g\Gamma(t_0)) - ^g\Gamma|_{^g\Gamma}.
\]
Hence we obtain
\[
|^{^g\nabla}(^g\Gamma(t) - ^g\Gamma)|_{^g\Gamma} \leq C_1'|t_1 - t_0| + |^{^g\nabla}(^g\Gamma(t_0)) - ^g\Gamma|_{^g\Gamma} \\
\leq C_1'|t_1 - t_0| + C|^{^g\nabla}g_k(t_0)|_{^g\Gamma} \\
\leq C_1'|t - t_0| + C|^{^g\nabla}g_k(t_0)|_{^g\Gamma} \\
\leq C_1'|t - t_0| + C_1.
\]

The equivalency of metrics tells us that
\[
|^{^g\nabla}g_k(t)|_{^g\Gamma} \leq B(t, t_0)^{3/2}|^{^g\nabla}g_k(t)|_{^g\Gamma} \leq B(t, t_0)^{3/2} \cdot C|^{^g\nabla}(^g\Gamma(t) - ^g\Gamma)|_{^g\Gamma} \\
\leq B(t, t_0)^{3/2}(C_1'|t - t_0| + C').
\]
Since $|t - t_0| \leq \psi - \beta$, it follows that $|^{^g\nabla}g_k(t)|_{^g\Gamma} \leq \tilde{C}_{1,0}$ for some constant $\tilde{C}_{1,0}$. But $^g\Gamma$ and $g_k$ are equivalent, we have
\[
|H_k(t)|_{^g\Gamma} \leq C|H_k(t)|_{^g\Gamma} \leq CC_1' = \tilde{C}_{1,0}.
\]
From the assumptions, we also have
\[
|^{^g\nabla}H_k|_{^g\Gamma} \leq |(^g\nabla - ^g\nabla)H_k + ^g\nabla H_k|_{^g\Gamma} \\
\leq C|^{^g\nabla}g_k|_{^g\Gamma} \cdot |H_k|_{^g\Gamma} + C|^{^g\nabla}H_k|_{^g\Gamma} \\
\leq CC_1' + C\tilde{C}_{1,0}\tilde{C}_{1,0} := \tilde{C}_{1,0}.
\]
Moreover,
\[
\frac{\partial}{\partial t}^{^g\nabla}H_k = ^g\nabla(\Delta_{g_k} H_k + \text{Rm}(g_k) \ast H_k) \\
= (^g\nabla - ^g\nabla)\Delta_{g_k} H_k + ^g\nabla\Delta_{g_k} H_k \\
+ ^g\nabla\text{Rm}(g_k) \ast H_k + \text{Rm}(g_k) \ast ^g\nabla H_k
\]
where $\Delta_{g_k}$ is the Laplace operator associated to $g_k$. Hence
\[
\left| \frac{\partial}{\partial t}^{^g\nabla}H_k \right|_{^g\Gamma} \leq C|^{^g\nabla}g_k|_{^g\Gamma} \cdot |\Delta_{g_k} H_k|_{^g\Gamma} + C|^{^g\nabla}\Delta_{g_k} H_k|_{^g\Gamma} \\
+ C|^{^g\nabla}\text{Rm}(g_k)|_{^g\Gamma} \cdot |H_k|_{^g\Gamma} + C|\text{Rm}(g_k)|_{^g\Gamma} \cdot |^{^g\nabla}H_k|_{^g\Gamma} \\
\leq \tilde{C}_{2,1}.
\]
For higher derivatives we claim that

\[(5.2) \ |g^p \nabla^p \text{Rc}(g_k)|_g \leq C'_p |g^p g_k|_g + C'''_p \cdot |g^p g_k|_g + |g^p \nabla^{p-1} H_k|_g \leq \tilde{C}_{p,0},\]

for all \( p \geq 1 \), where \( C'_p, C'''_p \), and \( \tilde{C}_{p,0} \) are constants independent of \( k \). For \( p = 1 \), we have proved the second inequality, so we suffice to prove the first one with \( p = 1 \). Indeed,

\[
|g \nabla \text{Rc}(g_k)|_g \leq C |(g \nabla - g_k \nabla) \text{Rc}(g_k) + g_k \text{Rc}(g_k)|_{g_k} \\
\leq C |g \nabla - g_k \Gamma|_g \cdot |\text{Rc}(g_k)|_{g_k} + C |g^k \nabla \text{Rc}(g_k)|_{g_k} \\
\leq C'_1 |g \nabla g_k|_g + C'''_1.
\]

Suppose the claim holds for all \( p < N \) \((N \geq 2)\), we shall show that it also holds for \( p = N \). From

\[
|g \nabla^N \text{Rc}(g_k)|_g = \left| \sum_{i=1}^{N} g \nabla^{N-i} (g \nabla - g_k \nabla) g_k \nabla^{i-1} \text{Rc}(g_k) + g_k \nabla^N \text{Rc}(g_k) \right|_g \\
\leq \sum_{i=1}^{N} |g \nabla^{N-i} (g \nabla - g_k \nabla) g_k \nabla^{i-1} \text{Rc}(g_k)|_g + |g_k \nabla^N \text{Rc}(g_k)|_g
\]

we estimate each term. For \( i = 1 \), by induction and assumptions we have

\[
|g \nabla^{N-1} (g \nabla - g_k \nabla) \text{Rc}(g_k)|_g \\
\leq C |g \nabla^{N-1} (g \nabla g_k \cdot \text{Rc}(g_k))|_g \\
\leq C \left| \sum_{j=0}^{N-1} \binom{N-1}{j} g \nabla^{N-1-j} (g \nabla g_k) \cdot g \nabla^j (\text{Rc}(g_k)) \right|_g \\
\leq C \sum_{j=0}^{N-1} \binom{N-1}{j} |g \nabla^{N-j} g_k|_g \cdot |g \nabla^j \text{Rc}(g_k)|_g \\
\leq C \sum_{j=0}^{N-1} \binom{N-1}{j} (C''_j |g \nabla^j g_k|_g + C'''_j) |g \nabla^{N-j} g_k|_g \\
\leq C \sum_{j=0}^{N-1} \binom{N-1}{j} (C''_j C_{j,0} + C'''_j) |g \nabla^{N-j} g_k|_g \\
= C (N-1) (C''_0 C_{j,0} + C'''_0) |g \nabla^N g_k|_g \\
+ C \sum_{j=1}^{N-1} \binom{N-1}{j} (C''_j C_{j,0} + C'''_j) C_{N-j,0} \\
\leq C''_N |g \nabla^N g_k|_g + C'''_N.
\]
For $i \geq 2$, we have
\[
|g\nabla^{N-i}(g\nabla - g_k\nabla)g_k\nabla^{i-1}\text{Rc}(g_k)|_g \\
\leq C\left|g\nabla^{N-i}(g\nabla g_k \cdot g_k\nabla^{i-1}\text{Rc}(g_k))\right|_g \\
\leq C\sum_{j=0}^{N-i} \left(\begin{array}{c} N-i \\ j \end{array}\right) |g\nabla^{N-i-j}g_k|_g \cdot |g\nabla^j \cdot g_k\nabla^{i-1}\text{Rc}(g_k)|_g.
\]

If $j = 0$, then
\[
|g\nabla^{i-1}\text{Rc}(g_k)|_g \leq C''_{i-1}|g\nabla^{i-1}g_k|_g + C'''_{i-1} \leq C''_{i-1}\tilde{C}_{i-1,0} + C'''_{i-1}.
\]

Suppose in the following that $j \geq 1$. Hence
\[
|g\nabla^j \cdot g_k \nabla^{i-1}\text{Rc}(g_k)|_g \\
= \left|\left((g\nabla - g_k\nabla)^j + g_k\nabla^{i-1}\text{Rc}(g_k)\right)^j \cdot g_k\nabla^{i-1}\text{Rc}(g_k)\right|_g \\
\leq C\sum_{l=0}^{j} \left(\begin{array}{c} j \\ l \end{array}\right) |g\nabla^l g_k|_g \cdot |g_k\nabla^{j-l+i-1}\text{Rc}(g_k)|_g \\
\leq C\sum_{l=0}^{j} \left(\begin{array}{c} j \\ l \end{array}\right) \tilde{C}_{l,0}(C''_{j-l+i-1}\tilde{C}_{j-l+i-1,0} + C'''_{j-l+i-1}).
\]

Combining these inequalities, we get
\[
|g\nabla^N\text{Rc}(g_k)|_g \leq C''''_N|g\nabla^Ng_k|_g + C'''''_N.
\]

Similarly, we have
\[
|g\nabla^N h_k|_g \leq C''''_N|g\nabla^Ng_k|_g + C'''''_N.
\]

Since $\frac{\partial}{\partial t}g_k = -2\text{Rc}(g_k) + \frac{1}{2}h_k$, it follows that
\[
\frac{\partial}{\partial t}|g\nabla^N g_k|^2_g \\
\leq \frac{\partial}{\partial t}|g\nabla^N g_k|^2_g + |g\nabla^N g_k|^2_g \\
\leq 8|g\nabla\text{Rc}(g_k)|^2_g + \frac{1}{2}|g\nabla^N h_k|^2_g + |g\nabla^N g_k|^2_g \\
\leq (1 + 18(C''''_N)^2)|g\nabla^N g_k|^2_g + 18(C'''''_N)^2.
\]

Integrating the above inequality, we get $|g\nabla g_k|_g \leq \tilde{C}_{N,0}$ and therefore $|g\nabla^N h_k|_g \leq \tilde{C}_{N+1,0}$. We have proved lemma for $q = 0$. When $g \geq 1$, then
\[
\frac{\partial}{\partial t} g\nabla^p g_k(t) = g\nabla^p \frac{\partial^{p-1}}{\partial t^{p-1}} \left(-2\text{Rc}(g_k(t)) + \frac{1}{2}h_k(t)\right).
\]

Using the evolution equations for $\text{Rm}(g_k(t))$ and $h_k(t)$, combining the induction to $q$ and using the above method, we have $|\frac{\partial}{\partial t} g\nabla^p g_k(t)|_g + |\frac{\partial}{\partial t} g\nabla^{p-1} h_k(t)|_g \leq \tilde{C}_{p,q}$. \qed
6. Generalization

In this section, we generalize the main results in Sec. 4 and Sec. 5 to a kind of generalized Ricci flow in which the local existence has been established [5].

Let \((M, g_{ij}(x))\) be an \(n\)-dimensional compact Riemannian manifold and let \(A = \{A_i\}\) and \(B = \{B_{ij}\}\) denote a 1-form and 2-form respectively. Let \(F = dA\) and \(H = dB\). The authors in [5] proved that there exists a constant \(T > 0\) such that the evolution equations

\[
\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \frac{1}{2}h_{ij}(x,t) + 2f_{jk}(x,t), \quad g_{ij}(x,0) = g_{ij}(x),
\]

\[
\frac{\partial}{\partial t} A_i(x,t) = -\nabla_k F^k_i(x,t), \quad A_i(x,0) = A_i(x),
\]

\[
\frac{\partial}{\partial t} B_{ij}(x,t) = \nabla_k H^k_{ij}(x,t), \quad B_{ij}(x,0) = B_{ij}(x)
\]

has a unique smooth solution on \(m \times [0, T)\), where \(h_{ij} = H_{ikl}H^k_{jl}\) and \(f_{ij} = F^k_iF_{jk}\). We also call it as a GRF. Given a \(p\)-form \(\alpha\) on a manifold \((M, g)\) we recall the definition of adjoint operator \(d^*\) of \(d\) with respect to a metric \(g\) in terms of local coordinates:

\[
(d^* \alpha)_{i_1 \cdots i_{p-1}} = -g^{ik} \nabla_j \alpha_{ki_1 \cdots i_{p-1}}.
\]

In particular,

\[
(d^* F)_i = \nabla_k F^k_i, \quad (d^* H)_{ij} = -\nabla_k H^k_{ij}
\]

and hence

\[
\frac{\partial}{\partial t} F(x,t) = -dd^*_g(x,t)F = \Box_g(x,t)F = \Delta F + Rm * F,
\]

\[
\frac{\partial}{\partial t} H(x,t) = -dd^*_g(x,t)H = \Box_g(x,t)H = \Delta H + Rm * H.
\]

They also derived the evolution equations of curvatures:

\[
\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})
\]

\[
- g^{pq}(R_{pjk}R_{qj} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql})
\]

\[
+ \frac{1}{4} \nabla_i \nabla_l (H_{kpq}H^p_{jq}) - \nabla_i \nabla_k (H_{jpq}H^p_{iq})
\]

\[
- \nabla_j \nabla_l (H_{kpq}H^p_{iq}) + \nabla_j \nabla_k (H_{jpq}H^p_{iq})
\]

\[
+ \frac{1}{4} g^{mn}(H_{kpq}H^p_{inl} + H_{mpq}H^p_{inl}R_{ijkl})
\]

\[
+ \nabla_i \nabla_l (F^p_{kj}F_{jp}) - \nabla_i \nabla_k (F^p_{pj}F_{jp})
\]

\[
- \nabla_j \nabla_l (F^p_{ijp}F_{jp}) + \nabla_j \nabla_k (F^p_{ijp}F_{jp})
\]

\[
+ g^{mn}(F^p_{kj}F^m_{lp}R_{ijnl} + F^m_{ijp}F^p_{ljn}R_{ijkl}).
\]
Under our notation, it can be rewritten as
\[
\frac{\partial}{\partial t} R^m = \Delta R^m + \sum_{i+j=0} \nabla^i R^m \ast \nabla^j R^m + \sum_{i+j=0+2} \nabla^i H \ast \nabla^j H \\
+ \sum_{i+j=0+2} \nabla^i F \ast \nabla^j F + \sum_{i+j+k=0} \nabla^i H \ast \nabla^j H \ast \nabla^k R^m \\
+ \sum_{i+j+k=0} \nabla^i F \ast \nabla^j F \ast \nabla^k R^m.
\]

As before, we have

**Proposition 6.1.** For GRF and any nonnegative integer \( l \) we have
\[
\frac{\partial}{\partial t} \nabla^l R^m = \Delta (\nabla^l R^m) + \sum_{i+j=l} \nabla^i R^m \ast \nabla^j R^m + \sum_{i+j=l+2} \nabla^i H \ast \nabla^j H \\
+ \sum_{i+j=l+2} \nabla^i F \ast \nabla^j F + \sum_{i+j+k=l} \nabla^i H \ast \nabla^j H \ast \nabla^k R^m \\
+ \sum_{i+j+k=l} \nabla^i F \ast \nabla^j F \ast \nabla^k R^m.
\]

In particular,
\[
\frac{\partial}{\partial t} |\nabla^l R^m|^2 \leq \Delta |\nabla^l R^m|^2 - 2|\nabla^{l+1} R^m|^2 \\
+ C \cdot \sum_{i+j=l} |\nabla^i R^m| \cdot |\nabla^j R^m| \cdot |\nabla^l R^m| \\
+ C \cdot \sum_{i+j=l+2} |\nabla^i H| \cdot |\nabla^j H| \cdot |\nabla^l R^m| \\
+ C \cdot \sum_{i+j=l+2} |\nabla^i F| \cdot |\nabla^j F| \cdot |\nabla^l R^m| \\
+ C \cdot \sum_{i+j+k=l} |\nabla^i H| \cdot |\nabla^j H| \cdot |\nabla^k R^m| \cdot |\nabla^l R^m|.
\]

Since \( \frac{\partial}{\partial t} F = \Delta F + R^m \ast F \) it follows that
\[
\frac{\partial}{\partial t} \nabla F = \nabla \frac{\partial}{\partial t} F + F \ast \nabla (R^m + H \ast H + F \ast F) \\
= \nabla (\Delta F + R^m \ast F) + F \ast \nabla R^m \\
+ F \ast H \ast \nabla H + F \ast F \ast \nabla F \\
= \Delta (\nabla F) + \nabla R^m \ast F + R^m \ast \nabla F \\
+ F \ast H \ast \nabla H + F \ast F \ast \nabla F.
It can be expressed as
\[
\frac{\partial}{\partial t} \nabla F = \Delta (\nabla F) + \sum_{i+j=1} \nabla^i F \ast \nabla^j R_m \\
+ \sum_{i+j+k=1} \nabla^i F \ast \nabla^j F \ast \nabla^k F \\
+ \sum_{i=0}^{l-1} \sum_{j=0}^{l-i} \nabla^i F \ast \nabla^j H \ast \nabla^{l-i-j} H.
\]

More generally, we can show that

**Proposition 6.2.** For GRF and any positive integer \( l \) we have
\[
\frac{\partial}{\partial t} \nabla^l F = \Delta (\nabla^l F) + \sum_{i+j=l} \nabla^i F \ast \nabla^j R_m \\
+ \sum_{i+j+k=l} \nabla^i F \ast \nabla^j F \ast \nabla^k F \\
+ \sum_{i=0}^{l-1} \sum_{j=0}^{l-i} \nabla^i F \ast \nabla^j H \ast \nabla^{l-i-j} H.
\]

In particular,
\[
\frac{\partial}{\partial t} |\nabla^l F|^2 \leq \Delta |\nabla^l F|^2 - 2|\nabla^{l+1} F|^2 + \sum_{i+j=l} |\nabla^i F| \cdot |\nabla^j R_m| \cdot |\nabla^l F| \\
+ \sum_{i+j+k=l} |\nabla^i F| \cdot |\nabla^j F| \cdot |\nabla^k F| \cdot |\nabla^l F| \\
+ \sum_{i=0}^{l-1} \sum_{j=0}^{l-i} |\nabla^i F| \cdot |\nabla^j H| \cdot |\nabla^{l-i-j} H| \cdot |\nabla^l F|.
\]

Similarly, we obtain

**Proposition 6.3.** For GRF and any positive integer \( l \) we have
\[
\frac{\partial}{\partial t} \nabla^l H = \Delta (\nabla^l H) + \sum_{i+j=l} \nabla^i H \ast \nabla^j R_m \\
+ \sum_{i+j+k=l} \nabla^i H \ast \nabla^j H \ast \nabla^k H \\
+ \sum_{i=0}^{l-1} \sum_{j=0}^{l-i} \nabla^i H \ast \nabla^j F \ast \nabla^{l-i-j} F.
\]
In particular,
\[
\frac{\partial}{\partial t} |\nabla^l H|^2 \leq \Delta |\nabla^l H|^2 - 2 |\nabla^{l+1} H|^2 + C \cdot \sum_{i+j=l} |\nabla^i H| \cdot |\nabla^j \text{Rm}| \cdot |\nabla^l H|
\]
\[
+ C \cdot \sum_{i+j+k=l} |\nabla^i H| \cdot |\nabla^j H| \cdot |\nabla^k H| \cdot |\nabla^l H|
\]
\[
+ C \cdot \sum_{i=0}^{l-1} \sum_{j=0}^{l-i} |\nabla^i H| \cdot |\nabla^j \text{F}| \cdot |\nabla^{l-i-j} \text{F}| \cdot |\nabla^l H|.
\]

From the evolution inequalities
\[
\frac{\partial}{\partial t} |H|^2 \leq \Delta |H|^2 - 2 |\nabla^l \text{H}|^2 + C \cdot |\text{Rm}| \cdot |H|^2,
\]
\[
\frac{\partial}{\partial t} |F|^2 \leq \Delta |F|^2 - 2 |\nabla^l \text{F}|^2 + C \cdot |\text{Rm}| \cdot |F|^2,
\]
the following theorem is obvious.

**Theorem 6.4.** Suppose that \((g(x,t), H(x,t), F(x,t))\) is a solution to GRF on a compact manifold \(M^n\) for a short time \(0 \leq t \leq T\) and \(K_1, K_2, K_3\) are arbitrary given nonnegative constants. Then there exists a constant \(C_n\) depending only on \(n\) such that if
\[
|\text{Rm}(x,t)|_{g(x,t)} \leq K_1, \quad |H(x)|_{g(x)} \leq K_2, \quad |F(x)|_{g(x)} \leq K_3
\]
for all \(x \in M\) and \(t \in [0, T]\), then
\[
(6.5) \quad |H(x,t)|_{g(x,t)} \leq K_2 e^{C_n K_1 t}, \quad |F(x,t)|_{g(x,t)} \leq K_3 e^{C_n K_1 t}
\]
for all \(x \in M\) and \(t \in [0, T]\).

Paralleling to Theorem 4.6, we can prove

**Theorem 6.5.** Suppose that \((g(x,t), H(x,t), F(x,t))\) is a solution to GRF on a compact manifold \(M^n\) and \(K\) is an arbitrary given positive constant. Then for each \(\alpha > 0\) and each integer \(m \geq 1\) there exists a constant \(C_m\) depending on \(m, n, \max\{\alpha, 1\}\), and \(K\) such that if
\[
|\text{Rm}(x,t)|_{g(x,t)} \leq K, \quad |H(x)|_{g(x)} \leq K, \quad |F(x)|_{g(x)} \leq K
\]
for all \(x \in M\) and \(t \in [0, \frac{\alpha}{K}]\), then
\[
(6.6) \quad |\nabla^{m-1} \text{Rm}(x,t)|_{g(x,t)} + |\nabla^m \text{H}(x,t)|_{g(x,t)} + |\nabla^m \text{F}(x,t)|_{g(x,t)} \leq C_m \frac{1}{t^2}
\]
for all \(x \in M\) and \(t \in (0, \frac{\alpha}{K}]\).

**Proof.** The proof is very similar to the proof of Theorem 4.6, but there exists cross terms involving \(|\nabla^l F| \cdot |\nabla^j H|\). In order to see how it works,
we first prove the theorem for $m = 1$. It's easy to see that
\[
\frac{\partial}{\partial t} |Rm|^2 \leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C
\]
\[+ C|\nabla^2 Rm|^2 + C|\nabla^2 F|^2 + C|\nabla F|^2,
\]
\[
\frac{\partial}{\partial t} |H|^2 \leq \Delta |H|^2 - 2|\nabla H|^2 + C,
\]
\[
\frac{\partial}{\partial t} |F|^2 \leq \Delta |F|^2 - 2|\nabla F|^2 + C,
\]
\[
\frac{\partial}{\partial t} |\nabla F|^2 \leq \Delta |\nabla F|^2 - 2|\nabla^2 F|^2 + C|\nabla F|^2
\]
\[+ C|\nabla Rm| \cdot |\nabla F| + C|\nabla H| \cdot |\nabla F|,
\]
\[
\frac{\partial}{\partial t} |\nabla H|^2 \leq \Delta |\nabla H|^2 - 2|\nabla^2 H|^2 + C|\nabla H|^2
\]
\[+ C|\nabla Rm| \cdot |\nabla H| + C|\nabla F| \cdot |\nabla H|.
\]
Setting
\[u := t(|\nabla H|^2 + |\nabla F|^2 + |Rm|^2) + \beta |H|^2 + \gamma |F|^2
\]
we yield that
\[
\frac{\partial}{\partial t} u \leq \Delta u - 2t|\nabla^2 H|^2 + Ct|\nabla^2 H|^2 - 2t|\nabla^2 F|^2 + Ct|\nabla^2 F|
\]
\[+ (C - 2\beta)|\nabla H|^2 + (C - 2\gamma)|\nabla F|^2
\]
\[+ 2t|\nabla Rm|^2 + Ct|\nabla Rm| \cdot |\nabla H|
\]
\[+ C|\nabla F| \cdot |\nabla H| + Ct|\nabla Rm| \cdot |\nabla F|
\]
\[\leq \Delta u + (C - 2\beta)|\nabla H|^2 + (C - 2\gamma)|\nabla F|^2 + C(1 + \beta + \gamma)
\]
\[+ 2t|\nabla Rm|^2 + Ct|\nabla Rm| \cdot |\nabla H|
\]
\[+ C|\nabla F| \cdot |\nabla H| + Ct|\nabla Rm| \cdot |\nabla F|
\]
\[\leq \Delta u + (C - 2\beta)|\nabla H|^2 + (C - 2\gamma)|\nabla F|^2 + C(1 + \beta + \gamma)
\]
\[+ 2t|\nabla Rm|^2 + Ct \left( \frac{|\nabla Rm|^2}{C} + \frac{C^2}{4} |\nabla H|^2 \right)
\]
\[+ C|\nabla F| \cdot |\nabla H| + Ct \left( \frac{|\nabla Rm|^2}{C} + \frac{C^2}{4} |\nabla F|^2 \right)
\]
\[\leq \Delta u + 2(C - \beta)|\nabla H|^2 + 2(C - \gamma)|\nabla F|^2 + C(1 + \beta + \gamma).
\]
Choosing $\beta = \gamma = C$, we get $\frac{\partial}{\partial t} u \leq \Delta u + C$ which implies that $u \leq C$ since $u(0)$ is bounded from above. The case $m = 2$ can be proved in the same way. In the following we assume that $m \geq 3$. We define a function
\[u := t^m(|\nabla^m H|^2 + |\nabla^m F|^2 + |\nabla^{m-1} Rm|^2)
\]
\[+ \sum_{i=1}^{m-1} \beta_i t^i (|\nabla^i H|^2 + |\nabla^i F|^2 + |\nabla^{i-1} Rm|^2) + \gamma |H|^2 + \alpha |F|^2
\]
where $\beta, \gamma,$ and $\alpha$ are positive constants determined later. Suppose

$$|\nabla^{i-1}Rm| + |\nabla^i H| + |\nabla^i F| \leq \frac{C_i}{t^{\frac{2}{i}}} , \quad i = 1, 2, \cdots, m - 1.$$  

We are going to prove the above inequality also holds for $i = m$. For such $i$, we have

$$\frac{\partial}{\partial t}|\nabla^i H|^2 \leq \Delta |\nabla^i H|^2 - 2|\nabla^{i+1} H|^2 + C \cdot \sum_{j=0}^{i-1} \frac{C_j}{t^{\frac{2}{j}}} |\nabla^j H|^2 + C \cdot \frac{C_i}{t^{\frac{2}{i}}} |\nabla^{i+1} H|^2.$$  

Similarly, we have

$$\frac{\partial}{\partial t}|\nabla^i F|^2 \leq \Delta |\nabla^i F|^2 - 2|\nabla^{i+1} F|^2 + C \cdot \frac{C_i}{t^{\frac{2}{i}}} |\nabla^{i+1} F|^2.$$  

From Proposition 6.1 we obtain

$$\frac{\partial}{\partial t}|\nabla^{i-1}Rm|^2 \leq \Delta |\nabla^{i-1}Rm|^2 - 2|\nabla^i Rm|^2 + \frac{C_i}{t^{\frac{2}{i}}} |\nabla^i Rm|^2.$$  

The evolution inequality for $u$ is

$$\frac{\partial}{\partial t} \leq \frac{mt^{m-1}(|\nabla^m H|^2 + |\nabla^m F|^2 + |\nabla^{m-1}Rm|^2)}{m} + \sum_{i=1}^{m-1} \beta_i t^{i-1}(|\nabla^i H|^2 + |\nabla^i F|^2 + |\nabla^{i-1}Rm|^2) + \gamma \cdot \frac{\partial}{\partial t}|H|^2 + \alpha \cdot \frac{\partial}{\partial t}|F|^2.$$  

$$+ \frac{C_i}{t^{\frac{2}{i}}} |\nabla^{i+1} H|^2 + C \cdot \frac{C_i}{t^{\frac{2}{i}}} |\nabla^{i+1} F|^2.$$  

$$+ \frac{C_i}{t^{\frac{2}{i}}} |\nabla^{i+1} H|^2 + C \cdot \frac{C_i}{t^{\frac{2}{i}}} |\nabla^{i+1} F|^2.$$  

$$+ \gamma \cdot \frac{\partial}{\partial t}|H|^2 + \alpha \cdot \frac{\partial}{\partial t}|F|^2.$$
The third term is given by
\[
\frac{\partial}{\partial t} |\nabla^m H|^2 \leq \Delta |\nabla^m H|^2 - 2|\nabla^{m+1} H|^2 + C|\nabla^m H|^2 \\
+ C \cdot |\nabla^m Rm| \cdot |\nabla^m H| + C \cdot |\nabla^m F| \cdot |\nabla^m H| \\
+ \frac{C_m}{t^{m+1}} |\nabla^m H| + \frac{C_m}{t^2} |\nabla^m H|,
\]
\[
\frac{\partial}{\partial t} |\nabla^m F|^2 \leq \Delta |\nabla^m F|^2 - 2|\nabla^{m+1} F|^2 + C|\nabla^m F|^2 \\
+ C \cdot |\nabla^m Rm| \cdot |\nabla^m F| + C \cdot |\nabla^m H| \cdot |\nabla^m F| \\
+ \frac{C_m}{t^{m+1}} |\nabla^m F| + \frac{C_m}{t^2} |\nabla^m F|,
\]
\[
\frac{\partial}{\partial t} |\nabla^{m-1} Rm|^2 \leq \Delta |\nabla^{m-1} Rm|^2 - 2|\nabla^m Rm|^2 + C|\nabla^{m-1} Rm|^2 \\
+ \frac{C}{t^{1/2}} |\nabla^m H| \cdot |\nabla^{m-1} Rm| + \frac{C}{t^{1/2}} |\nabla^m F| \cdot |\nabla^{m-1} Rm| \\
+ C \cdot |\nabla^{m+1} H| \cdot |\nabla^{m-1} Rm| + C \cdot |\nabla^{m+1} F| \cdot |\nabla^{m-1} Rm| \\
+ \frac{C}{t^{m+1}} |\nabla^{m-1} Rm| + \frac{C_m}{t^2} |\nabla^{m-1} Rm|.
\]

By the same computation as in Theorem 4.6, there exists a constant $C$ such that $\frac{\partial u}{\partial t} \leq \Delta u + C$ and therefore $u(t) \leq C$. $\square$

We can also establish the corresponding compactness theorem for this kind of generalized Ricci flow. We omit the detail since the proof is closed to the proof in Sec. 5. In the forthcoming paper, we will consider the BBS estimates for complete noncompact Riemannian manifolds.

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