Equivariant Khovanov Homology of Periodic Links

Wojciech Politarczyk

April 3, 2015

Abstract

In this paper we study Khovanov homology of periodic links. We construct a variant of the Khovanov homology – the equivariant Khovanov homology – which is adapted to the equivariant setting. In the construction of the equivariant Khovanov homology we utilize the action of a finite cyclic group \( \mathbb{Z}/n \) on the Khovanov complex of a periodic link. Further we prove the invariance with respect to the equivariant Reidemeister moves, show that it is isomorphic to the Khovanov homology modulo \( n \)-torsion and express values of this homology theory on \( p \)-periodic trivial links in terms of the group cohomology of \( \mathbb{Z}/p \), if \( p \) is a prime. Additionally we construct a spectral sequence, analogous to the one from [ET12], converging to the equivariant Khovanov homology and we utilize this spectral sequence to compute the equivariant Khovanov homology of torus links \( T(n,2) \) with respect to the symmetry of order 2.

1 Introduction

One of the main themes in topology is the study of symmetries of certain objects like topological spaces or manifolds. In knot theory one is particularly interested in symmetries of knots, that is symmetries of the 3-sphere that preserve the given knot.

In this paper, which is based on the authors PhD Thesis [Pol15b], we study knots which possess certain kind of symmetry of finite order which is derived from a semi-free action of the cyclic group on the 3-sphere, i.e., for a fixed knot \( K \) we are interested in diffeomorphisms \( f: (S^3, K) \to (S^3, K) \) of finite order, with the property that \( K \) is disjoint with the fixed point set of \( f \). Due to the resolution of the Smith Conjecture in [MB84], the existence of such symmetry can be rephrased in the following way. Let \( \rho_n \) be the rotation of \( \mathbb{R}^3 \) by the \( \frac{2\pi}{n} \) angle about the \( OZ \) axis. We are interested in knots \( K \subset \mathbb{R}^3 \), which are disjoint from the \( OZ \) axis and invariant under \( \rho_n \). A knot \( K \) is \( n \)-periodic if it admits such rotational symmetry. \( n \)-periodic links are defined analogously.

The importance of periodic links stems from the fact, that according to [PS01], a 3-manifold \( M \) admits an action of the cyclic group of prime order \( p \) with the fixed point set being an unknot if, and only if, it can be obtained as a surgery on a \( p \)-periodic link. Additionally, \( M \) admits a free action of the cyclic group \( \mathbb{Z}/p \) if, and only if, it can be obtained as a surgery on a link formed by taking a \( p \)-periodic link \( L \) together with the fixed point axis \( F \). Hence, these
two classes of cyclic symmetries of 3-manifolds are determined by symmetries of their Kirby diagrams.

Another possible application of periodic links is, according to Przytycki [Prz89], to give a unified theory of skein modules for branched and unbranched coverings. Skein module of a 3-manifold \( M \) is a certain algebraic objects associated to \( M \), which serves as a generalization of a certain polynomial link invariant, like the Jones polynomial, for links in \( M \). For more details on skein modules refer to [Prz06].

There are many techniques at hand to study periodic knots. The first significant results were obtained by Trotter in [Tro61], where the author studies actions of the cyclic group on the fundamental group of the complement of the knot, to derive all possible periods of torus links. Murasugi studied periodic links with the aid of the Alexander polynomial in [Mur71], obtaining very strong criterion for detecting periodicity. In [DL91] authors consider the question whether a Laurent polynomial, which satisfies the congruence of Murasugi, can be realized as the Alexander polynomial of a periodic link.

Several authors studied the Jones polynomial of periodic links. The first result in this direction was obtained by Murasugi in [Mur88]. Besides that, [Prz89, Tra90a, Yok91b] give other criteria for detecting periodicity of knots in terms of their Jones polynomial. Several other authors [PS03, Chb02, Prz89, Tra91, Tra90b, Yok91b] studied \( SU_n \)-quantum polynomials and the HOMFLYPT polynomial of periodic links. A summary of these results can be found in [Prz04].

Khovanov in [Kho00] made a breakthrough in knot theory, by constructing certain homology theory of links, called the Khovanov homology, which categorifies the Jones polynomial, i.e., the Jones polynomial can be recovered from the Khovanov homology as an appropriately defined Euler characteristic. Hence, it is natural to ask whether this homology theory can be utilized to study periodic links. The first such attempt was made in [Chb10]. However, the author works only with \( \mathbb{Z}/2 \) coefficients due to certain technical problem with signs, which appears along the way. Nevertheless, the author obtains an invariant of periodic links and shows, via transfer argument, that his invariant is isomorphic to the submodule of fixed points of the induced action on the Khovanov homology.

The purpose of this article is to construct and study a generalized version of the invariant from [Chb10], called the equivariant Khovanov homology of periodic links. If \( p \) is an odd prime, then the equivariant Khovanov homology from [Chb10] is isomorphic to \( KH\mathbb{Z}/p - 1(D; \mathbb{Z}/2) \), see Definition 4.2. This is a consequence of the fact, that the group algebra \( \mathbb{Z}/2 \mathbb{Z}[\mathbb{Z}/p] \) is semi-simple.

In a subsequent paper [Pol15a] we define equivariant analogues of the Jones polynomial and study their properties. We prove that they satisfy certain invariant of the skein relation and use this property to derive certain periodicity criterion, which generalizes the ones from [Prz89, Tra90a]. Moreover, we study the state sum expansion of these polynomials, which leads to a proof of the congruence from [Mur88].

The paper is organized as follows. Section 2 contains a summary of some facts from representation theory and homological algebra which are essential in further considerations.

In section 3 we show that if \( D \) is an \( n \)-periodic diagram of an \( n \)-periodic link \( L \), then the Khovanov complex \( \text{CKH}(D) \) becomes a cochain complex of graded \( \mathbb{Z} [\mathbb{Z}/n] \)-modules. This enables us to study the Khovanov homology of periodic
In section 4.1, with the aid of integral representation theory of cyclic groups, we construct the equivariant Khovanov homology – denoted by \( \text{Kh}^{*,*,*}_{\mathbb{Z}/n}(D) \) – a triply graded homology theory, where the third grading is supported only in positive degrees, which divide \( n \). Further, we show that this is indeed an invariant of periodic links, utilizing machinery from [BN05].

**Theorem 4.3** Equivariant Khovanov homology groups are invariants of periodic links, that is, they are invariant under equivariant Reidemeister moves.

For the definition of equivariant Reidemeister moves see [3.12]. Intuitively, equivariant Reidemeister moves are deformation of periodic link diagrams in which we perform simultaneously \( n \) Reidemeister moves in such a way that the symmetry of the diagram is preserved at all times. Additionally, we cannot cross the fixed point axis while performing the equivariant Reidemeister moves.

Next, we investigate the relation between the equivariant Khovanov homology and the classical Khovanov homology.

**Theorem 4.4** Let \( p_1, \ldots, p_s \) be the collection of all prime divisors of \( n \). Define the ring \( R_n = \mathbb{Z}\left[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_s}\right] \). Under this assumption there exists a natural map

\[
\bigoplus_{d | r} \text{Kh}^{*,*,d}_{\mathbb{Z}/n}(L) \to \text{Kh}(L)
\]

which, when tensored with \( R_n \), becomes an isomorphism.

Hence, the equivariant Khovanov homology, after collapsing the third grading, encodes the same information as the classical Khovanov homology, modulo torsion of order dividing \( n \).

Further, we analyze the structure of this invariant for trivial links. It turns out, that the equivariant Khovanov homology of \( p^n \)-periodic trivial link, for some prime \( p \), is expressible in terms of the group cohomology of subgroups of \( \mathbb{Z}/p^n \) with coefficients in cyclotomic rings, see Proposition 4.10.

Computations of the equivariant Khovanov homology of trivial links suggests that, unlike the classical Khovanov homology, \( \text{Kh}^{1,*,d}_{\mathbb{Z}/n}(D) \) can be non-trivial for \( i > n_+(D) \), where \( n_+(D) \) stands for the number of positive crossings of the diagram \( D \). Theorem 4.3 implies, that for such values of \( i \), the abelian group \( \text{Kh}^{i,*,d}_{\mathbb{Z}/n}(D) \) is necessarily torsion. Our next goal is to use an additional algebraic structure of the equivariant Khovanov homology to extract some information about the additional torsion. This additional algebraic structure manifest itself in the fact that for any \( 0 \leq s \leq n \), \( \text{Kh}^{s,*,*}_{\mathbb{Z}/p^n}(D) \) is a graded module over the graded ring \( \text{Ext}^2_{\mathbb{Z}/p^n}[Z[\xi_{p^s}], Z[\xi_{p^s}]] \). This ring is isomorphic to certain quotient of the polynomial ring \( Z[T]/(T^s) \). Analysis of this structure yields the periodicity result, which can be thought of as an analogue of the periodicity of the cohomology groups of the cyclic groups.

**Corollary 4.13** Let \( m \in \mathbb{Z} \) be chosen such that \( \text{Kh}^{m,*}_{\mathbb{Z}/n}(D) \neq 0 \) and \( \text{Kh}^{i,*}_{\mathbb{Z}/n}(D) \) vanishes for \( i > m \). Let \( T_s \) denote the cohomology class in the ext ring

\[
T_s \in \text{Ext}^2_{\mathbb{Z}/p^n}[Z[\xi_{p^s}], Z[\xi_{p^s}]]
\]

from Proposition 2.19. Then, multiplication by \( T_s \)

\[
- \cup T_s : \text{Kh}^{i,*,p}_{\mathbb{Z}/p^n}(D) \to \text{Kh}^{i+2,*,p}_{\mathbb{Z}/p^n}(D)
\]
is an epimorphism for \( i = m \) and isomorphism for \( i > m \).

As a consequence we can obtain some information about the additional torsion appearing in the equivariant Khovanov homology.

**Corollary 4.15.** Let \( m \) be as in the previous corollary. For \( i > m \), \( \text{Kh}^{i,*,1}_{\mathbb{Z}/p^n}(D) \) is annihilated by \( p^n \), and for \( 1 \leq s \leq n \), \( \text{Kh}^{i,p^s,1}_{\mathbb{Z}/p^n}(D) \) is annihilated by \( p^{n-s+1} \).

In section 4.3 we study the structure of the rational equivariant Khovanov homology. These considerations are sufficient to compute the rational equivariant Khovanov homology of torus links \( T(n,2) \) and, if \( \gcd(n,3) = 1 \), for torus knots \( T(n,3) \), with respect to the \( \mathbb{Z}/d \)-symmetry, provided that \( d \mid n \) is odd and greater than 2. In all of these cases we have

\[
\text{Kh}^{*,*,1}_{\mathbb{Z}/d}(D; \mathbb{Q}) = \text{Kh}^{*,*}(D; \mathbb{Q}),
\]

\[
\text{Kh}^{*,*,k}_{\mathbb{Z}/d}(D; \mathbb{Q}) = 0, \quad k > 1, \quad k \mid d.
\]

Theorem 5.11, which is the main result of Section 5, yields a spectral sequence converging to the equivariant Khovanov homology of a periodic link. Since the long exact sequence of Khovanov homology, coming from two resolutions of a single crossing of \( D \), cannot be adapted to the equivariant setting, the spectral sequence is supposed to fill in this gap and provide a computational tool. Instead of resolving a single crossing, we resolve crossings from a single orbit. We take all possible resolutions of these crossings and assemble these data into a spectral sequence. In the non-equivariant setting we can recover from this construction the spectral sequence from [ET12].

The above spectral sequence is used, in Section 6, to compute the rational 2-equivariant Khovanov homology of torus links, i.e. the equivariant Khovanov homology with respect to the \( \mathbb{Z}/2 \)-symmetry. It turns out, that something analogous happen as in the case of symmetries of order \( d > 2 \). Namely, almost always the only non-trivial part is \( \text{Kh}^{*,*,1}_{\mathbb{Z}/2} \), with an exception of torus links \( T(2n,2) \) for which

\[
\text{Kh}^{i,j,2}_{\mathbb{Z}/2}(T(2n,2); \mathbb{Q}) = \begin{cases} 
\mathbb{Q}, & i = 2n, \quad j = 6n, \\
0, & \text{otherwise}.
\end{cases}
\]

**Convention** In the remainder part of this article we adopt the convention that all links and tangles are oriented unless stated otherwise.

**Acknowledgments** The author is very grateful to Józef Przytycki for bringing this problem to his attention, to Maciej Borodzik and Prof. Krzysztof Pawałowski for their suggestions and corrections which improved the presentation of the paper.

## 2 Preliminaries

### 2.1 Representation theory

Before we start, we will briefly recall some notions from integral representation theory that are essential in the remainder part of this thesis. The exposition of the material in this section is based on [CR90].
**Definition 2.1.** Let $A$ be a semi-simple and finite dimensional $\mathbb{Q}$-algebra. We say that $\Lambda \subset A$ is a $\mathbb{Z}$-order if it is a subring of $A$ which contains the unit and some $\mathbb{Q}$-basis of $A$.

An example of a $\mathbb{Z}$-order is the group ring $\mathbb{Z}[G]$, of a finite group $G$, contained in the rational group algebra $\mathbb{Q}[G]$.

**Definition 2.2.** Let $\Lambda$ be a $\mathbb{Z}$-order in a semi-simple $\mathbb{Q}$-algebra $A$. We say that $\Lambda$ is maximal if it is not contained in any other $\mathbb{Z}$-order in $A$.

**Theorem 2.3** ([CR90, Thm. 26.20]). Let $A$ be a finite dimensional semi-simple $\mathbb{Q}$-algebra.

1. Every $\mathbb{Z}$-order $\Lambda \subset A$ is contained in a some maximal order $\Lambda'$.
2. If $A$ is commutative, then it possesses a unique maximal order.
3. If we are given a Wedderburn decomposition of $A$
   $$A = A_1 \oplus \ldots \oplus A_k,$$
   then every maximal order $\Lambda' \subset A$ admits a decomposition into a direct sum of ideals
   $$\Lambda' = \Lambda'_1 \oplus \ldots \oplus \Lambda'_k,$$
   where each $\Lambda'_i \subset A_i$ is a maximal order in $A_i$ for $i = 1, \ldots, k$. In fact, if $e_i$ is a central idempotent in $A$ which acts as an identity on $A_i$ and by 0 on $A_j$ for $j \neq i$, then $\Lambda'_i = e_i \Lambda'$.

**Example 2.4.** The group algebra $\mathbb{Q} [\mathbb{Z}/n]$ admits the following Wedderburn decomposition
   $$\mathbb{Q} [\mathbb{Z}/n] = \bigoplus_{d|n} \mathbb{Q}[\xi_d].$$
   The maximal order in $\mathbb{Q}[\xi_d]$ is the ring of cyclotomic integers $\mathbb{Z}[\xi_d]$. Therefore, the unique maximal order $\Lambda' \subset \mathbb{Q}[\mathbb{Z}/n]$ is equal to the following direct sum
   $$\Lambda' = \bigoplus_{d|n} \mathbb{Z}[\xi_d].$$

**Proposition 2.5** ([CR90, Prop 27.1]). Let $\Lambda'$ be a maximal order in $\mathbb{Q}[\mathbb{Z}/n]$. Under this assumption, the following chain of inclusions is satisfied.
   $$\mathbb{Z}[\mathbb{Z}/n] \subset \Lambda' \subset \frac{1}{n} \mathbb{Z}[\mathbb{Z}/n].$$
   Therefore, there exists an exact sequence
   $$0 \to \mathbb{Z}[\mathbb{Z}/n] \to \Lambda' \to M \to 0,$$
   where $n \cdot M = 0$.

**Definition 2.6.** Let $m \leq n$ be positive integers and suppose that $L$ is a $\mathbb{Z}[\mathbb{Z}/m]$-module and $T$ is a $\mathbb{Z}[\mathbb{Z}/n]$-module. Define the induction of $L$ to be the $\mathbb{Z}[\mathbb{Z}/n]$-module
   $$\text{Ind}_{\mathbb{Z}[\mathbb{Z}/m]}^{\mathbb{Z}[\mathbb{Z}/n]} L = L \otimes_{\mathbb{Z}[\mathbb{Z}/m]} \mathbb{Z}[\mathbb{Z}/n].$$
where $\mathbb{Z}/\mathbb{Z}[n]$ is considered as a $\mathbb{Z}[\mathbb{Z}/m]$ module via the inclusion

$$\mathbb{Z}/m \hookrightarrow \mathbb{Z}/n.$$  

Define the restriction of $T$ to be the $\mathbb{Z}[\mathbb{Z}/m]$-module, whose underlying abelian group remains the same, however we restrict the action of $\mathbb{Z}/n$ to the action of its subgroup isomorphic to $\mathbb{Z}/m$.

**Proposition 2.7.** Let $p$ be a prime and $n$ a positive integer. Choose $0 \leq s, m \leq n$, then

$$\text{Res}_{\mathbb{Z}/p^n}^{\mathbb{Z}/p^m} \mathbb{Z} \left[ \mathbb{Z}/p^{n-s} \right] = \begin{cases} 
\mathbb{Z}^\phi(p^{n-s}), & m \leq s, \\
\mathbb{Z} \left[ \mathbb{Z}/p^{n-s} \right] p^{n-m}, & m > s.
\end{cases}$$

**Proof.** Since $\text{Res}_{\mathbb{Z}/p^n}^{\mathbb{Z}/p^m} \mathbb{Z} \left[ \mathbb{Z}/p^{n-s} \right]$ is the maximal order in $\text{Res}_{\mathbb{Z}/p^n}^{\mathbb{Z}/p^m} \mathbb{Q} \left[ \mathbb{Z}/p^{n-s} \right]$, it is sufficient to check this equality for the cyclotomic fields. Let $\chi_{p^{n-s}, p^n}$ denote the character of $\mathbb{Q} \left[ \mathbb{Z}/p^{n-s} \right]$ treated as an irreducible $\mathbb{Q}[\mathbb{Z}/p^n]$-module.

**Lemma 2.8.** Consider the cyclic group $\mathbb{Z}/p^n$, where $p$ is a prime. Let $0 \leq s < n$, $0 \leq j \leq p^{n-s} - 1$ and $0 \leq m \leq p^s - 1$. The characters of $\mathbb{Z}/p^n$ are given by the following formulas.

$$\begin{align*}
\chi_{1, p^n}(t^j) &= 1, \\
\chi_{p^{n-s}, p^n}(t^{j+m} p^{n-s}) &= \begin{cases} 
\phi(p^{n-s}), & j = 0, \\
-p^{n-s-1}, & j \mid p^{n-s-1}, \\
0, & \text{otherwise,}
\end{cases}
\end{align*}$$

where $0 \leq j \leq p^{n-s} - 1$.

**Proof.** This follows from [CR90] Cor. 7.19] applied to the complex characters of $\mathbb{Z}/p^n$. 

The above lemma implies that the following formulas hold.

$$\text{Res}_{\mathbb{Z}/p^n}^{\mathbb{Z}/p^m} \chi_{p^{n-s}, p^n} = \begin{cases} 
\varphi(p^{n-s}) \chi_{1, p^m}, & m \leq s, \\
p^{n-m} \chi_{p^{n-s}, p^m}, & m > s.
\end{cases}$$

Hence, the proposition follows.

### 2.2 Ext groups

The purpose of this section is to sketch the theory of the derived Hom functor. The exposition is based on [TSPA14].

To fix the notation, assume that $C^*$, $D^*$ and $E^*$ are bounded cochain complexes.

**Definition 2.9.** Let $C^*$ be a cochain complex. For $n \in \mathbb{Z}$ denote by $C[n]^*$ a new cochain complex obtained from $C^*$ by applying the following shift

$$C[n]^k = C^{k-n},$$

$$d_{C[n]}^k = (-1)^n d_C^{k-n}.$$
The category of cochain complexes can be equipped with the structure of the differential graded category i.e. morphism sets can be made into cochain complexes themselves.

**Definition 2.10.** For two cochain complexes, $C^*$ and $D^*$, define the Hom complex $\text{Hom}_R^n(C^*, D^*)$ to be

$$\text{Hom}_R^n(C^*, D^*) = \prod_{p \in \mathbb{Z}} \text{hom}_R(C^p, D^{p+n}), \quad n \in \mathbb{Z}$$

and equip it with the following differential.

$$d_{C^*, D^*}(\psi)_p = (d_D \circ \psi_p - (-1)^p \psi_{p+1} \circ d_C)_p$$

**Remark 2.11.** Notice that since $C^*$ and $D^*$ are bounded, the Hom complex $\text{Hom}_R^n(C^*, D^*)$ is also bounded.

The next proposition is a mere reformulation of the definition of the chain homotopy.

**Proposition 2.12.** The following equalities hold

$$H^n(\text{Hom}_R^n(C^*, D^*)) = [C^*, D^*[-n]],$$

where the outer square brackets denote the set of homotopy classes of chain maps.

Analogously as in the case of modules, in order to preserve the exactness of the Hom functor, we need to replace every cochain complex by its injective or projective resolution.

**Definition 2.13.** Let $C^*$ be cochain complex. Let $I^*$ be bounded below cochain complex of injective modules. We say that $I^*$ is an injective resolution of $C^*$ if there exists a quasi-isomorphism $C^* \to I^*$.

Categories of modules have enough injectives, which guarantees that every module possesses an injective resolution. It turns out that the same condition is sufficient to construct an injective resolution of a cochain complex, provided that the complex is bounded from below, see [TSPA14, Lemma 18.3]

**Definition 2.14.** Let $I^*$ be an injective resolution of a cochain complex $D^*$. Define the derived Hom complex

$$\text{RHom}_R^n(C^*, D^*) = \text{Hom}_R^n(C^*, I^*).$$

Ext groups are defined as the homology of the derived Hom complex.

$$\text{Ext}^n_R(C^*, D^*) = H^n(\text{RHom}_R^n(C^*, I^*)).$$

In other words,

$$\text{Ext}^n_R(C^*, D^*) = [C^*, I^*[-n]].$$

Properties of classical Ext groups extended to their generalized version.
Proposition 2.15. Let $C^*$ and $D^*$ be as in the previous definition.

1. If $C'^*$ and $D'^*$ are bounded cochain complexes quasi-isomorphic to $C^*$ and $D^*$, respectively, then these quasi-isomorphisms induce the following isomorphisms,

$$\operatorname{Ext}^n_R(C^*, D^*) \cong \operatorname{Ext}^n_R(C'^*, D'^*), \quad n \in \mathbb{Z}.$$ 

2. If

$$0 \to C_1^* \to C_2^* \to C_3^* \to 0$$

is a short exact sequence of bounded cochain complexes, then there exists a long exact sequence of Ext groups

$$\ldots \to \operatorname{Ext}^n_R(C_3^*, D^*) \to \operatorname{Ext}^n_R(C_2^*, D^*) \to \operatorname{Ext}^n_R(C_1^*, D^*) \to \ldots .$$

3. Analogously, if

$$0 \to D_1^* \to D_2^* \to D_3^* \to 0$$

is a short exact sequence of bounded cochain complexes, then there exists a long exact sequence of Ext groups

$$\ldots \to \operatorname{Ext}^n_R(C^*, D_1^*) \to \operatorname{Ext}^n_R(C^*, D_2^*) \to \operatorname{Ext}^n_R(C^*, D_3^*) \to \ldots .$$

4. There exist bilinear maps, induced by composition of maps on the cochain level,

$$\mu: \operatorname{Ext}^n_R(C^*, D^*) \times \operatorname{Ext}^m_R(B^*, C^*) \to \operatorname{Ext}^{n+m}_R(B^*, D^*).$$

Hence, $\operatorname{Ext}^*_R(C^*, C^*)$ and $\operatorname{Ext}^*_R(D^*, D^*)$ are graded rings. Additionally, $\operatorname{Ext}^*_R(C^*, D^*)$ is a graded $(\operatorname{Ext}^*_R(D^*, D^*), \operatorname{Ext}^*_R(C^*, C^*))$-bimodule.

Proof. See the discussion after the Definition 27.1 in [TSPA14].

Classical result of Cartan and Eilenberg provides two spectral sequences which converge to the higher derived functors of Hom.

Theorem 2.16 ([TSPA14, Lemma 21.3]). There are two spectral sequences

$$\{\tilde{I}E^p, q, d_r\}, \{\tilde{II}E^p, q, d_r\}$$

converging to $\operatorname{Ext}^*_R(C^*, D^*)$ satisfying

$$\tilde{I}E^p, q = \operatorname{Ext}^p_R(C^*, H^q(D))$$
$$\tilde{II}E^p, q = H^q(\operatorname{Ext}^p_R(C^*, D^*))$$

Part 4 of Proposition 2.15 shows that Ext groups possess a multiplicative structure. This additional structure is derived from certain bilinear maps defined on the cochain level. The bilinear map is compatible with the filtration of Cartan and Eilenberg, thus its existence is manifested in the Cartan-Eilenberg spectral sequence.
**Theorem 2.17.** For cochain complexes $B^*, C^*$ and $D^*$ there are bilinear maps of spectral sequences

\[
\mu: iE^{p,q}_r(C^*, D^*) \times iE^{p',q'}_r(B^*, C^*) \to iE^{p+p',q+q'}_r(B^*, D^*)
\]

\[
\mu: iI'E^{p,q}_r(C^*, D^*) \times iI'E^{p',q'}_r(B^*, C^*) \to iI'E^{p+p',q+q'}_r(B^*, D^*)
\]

commuting with differentials i.e.

\[
d^{B^*,D^*}_r(\mu(x,y)) = \mu(d^{C^*,D^*}_r(x), y) + (-1)^{p+q}\mu(x, d^{B^*,C^*}_r(y)),
\]

and converging to bilinear maps from Proposition 2.15.

When a cochain complex $D^*$ is equipped with a filtration, the Hom complex $\text{Hom}_R(C^*, D^*)$ becomes filtered. This filtration is defined by considering homomorphisms whose images are contained in the respective submodule of the filtration of $D^*$. Moreover, the filtration of $D^*$ induces a filtration on the derived Hom complex. This leads to a spectral sequence.

**Theorem 2.18.** Suppose that $D^*$ is a bounded and filtered cochain complex. Then there exists a spectral sequence $\{E^*_r, d_r\}$ converging to $\text{Ext}^*_R(C^*, D^*)$ such that

\[
E^*_1 = \text{Ext}^*_R(C^*, \text{F}_p(D^*)/\text{F}_{p+1}(D^*)).
\]

The next lemma provides a description of Ext rings of cyclotomic rings. This description will be used later to obtain some information about the torsion of the equivariant Khovanov homology.

**Proposition 2.19.** Let $p$ be a prime and let $0 \leq s \leq n$ be integers. Denote by $\Phi_{p^n}(t)$ the $p^n$-th cyclotomic polynomial. Under these assumptions, there exists an isomorphism

\[
\text{Ext}^2_{\mathbb{Z}/p^n} (\mathbb{Z} [\xi_{p^n}], \mathbb{Z} [\xi_{p^n}]) \cong \mathbb{Z} [\xi_{p^n}] / (\Phi_{p^n}(\xi_{p^n}) T_s),
\]

where

\[
\Phi_{p^n}(t) = \frac{t^{p^n} - 1}{\Phi_p(t)}
\]

and $T_s \in \text{Ext}^2_{\mathbb{Z}/p^n} (\mathbb{Z} [\xi_{p^n}], \mathbb{Z} [\xi_{p^n}])$ is a class represented by the following Yoneda extension

\[
0 \to \mathbb{Z} [\xi_{p^n}] \to \mathbb{Z} [\mathbb{Z}/p^n] \xrightarrow{\Phi_p(t)} \mathbb{Z} [\mathbb{Z}/p^n] \to \mathbb{Z} [\xi_{p^n}] \to 0.
\]

Additionally, for any $\mathbb{Z} [\mathbb{Z}/p^n]$-module $N$, multiplication by $T_s$ is an isomorphism for $i > 0$ and epimorphism for $i = 0$. In particular

\[
\text{Ext}^2_{\mathbb{Z}/p^n} (\mathbb{Z} [\xi_{p^n}], \mathbb{Z} [\xi_{p^n}]) = \begin{cases} 
\mathbb{Z}/p^m, & i > 0, s = 0, \\
\mathbb{Z}/p^{m-s+1}, & i > 0, s > 0, \\
\mathbb{Z} [\xi_{p^n}], & i = 0, \\
0, & \text{otherwise}.
\end{cases}
\]
Proof. The first part follows from [Wal79, Lemma 1.1]. To prove the second part, notice that

\[
\Phi_{p^r,p^m}(\xi_{p^r}) = \lim_{z \to \xi_{p^r}} \frac{z^{p^m} - 1}{\Phi_{p^r}(z)} = \lim_{z \to \xi_{p^r}} (z^{p^{r-1}} - 1) \frac{z^{p^m} - 1}{z^{p^r} - 1} = p^{m-s}(\xi_{p^r} - 1).
\]

by de L’Hospital rule. Since the algebraic norm of \(\xi_{p^r} - 1\) is equal to \(p\), it follows readily that \(\mathbb{Z}[\xi_{p^r}]/(\Phi_{p^r,p^m}(\xi_{p^r})) \cong \mathbb{Z}/p^{m-s+1}\).

\[\square\]

3 Periodic links

Let us start with the recollection of the definition of a periodic link.

Definition 3.1. Let \(n\) be a positive integer, and let \(L\) be a link in \(S^3\). We say that \(L\) is \(n\)-periodic, if there exists an action of the cyclic group of order \(n\) on \(S^3\) satisfying the following conditions.

1. The fixed point set, denoted by \(F\), is the unknot.
2. \(L\) is disjoint from \(F\).
3. \(L\) is a \(\mathbb{Z}/n\)-invariant subset of \(S^3\).

Example 3.2. Borromean rings provide an example of a 3-periodic link. The symmetry is visualized on Figure 1. The dot marks the fixed point axis.
Figure 2: 4-periodic planar diagram.

Figure 3: Torus knot $T(3,4)$ as a 4-periodic knot obtained from the planar diagram from Figure 2.

**Example 3.3.** Torus links constitute an infinite family of periodic links. In fact, according to [Mur71], the torus link $T(m,n)$ is $d$-periodic if, and only if, $d$ divides either $m$ or $n$.

Periodic diagrams of periodic links can be described in terms of planar algebras. For the definition of planar algebras see [BN05]. Take an $n$-periodic planar diagram $D_n$ with $n$ input disks, like the one on Figure 2. Choose a tangle $T$ which possesses enough endpoints, and glue $n$ copies of $T$ into the input disks of $D_n$. In this way, we obtain a periodic link whose quotient is represented by an appropriate closure of $T$. See Figure 3 for an example.

Using this description of periodic links, it is possible to exhibit a cobordism which induces an action of $\mathbb{Z}/n$ on the Khovanov bracket $[D]_{Kh}$, as defined in [BN05], where $D$ is a periodic link diagram. First, notice that we can assume that $D$ represents a link in $D^2 \times I$ and the symmetry comes from a rotation of the $D^2$ factor, where $D^2$ denotes the 2-disk. In order to construct the cobordism, notice that the diffeomorphism, denote it by $f$, generating the $\mathbb{Z}/n$-symmetry of $D^2 \times I$, is isotopic to the identity. Indeed, this isotopy can be chosen in such a way that it changes the angle of rotation linearly from $0$ to $\frac{2 \pi}{n}$. Denote this
isotopy by $H$. The cobordism in question is the trace of $H$.

$$\Sigma_H = \{(H(x,t), t) \in D^2 \times I \times I : x \in L, \ t \in I\}.$$ 

Cobordism $\Sigma_H$ is diffeomorphic to the cylinder $S^1 \times I$, however it is not isotopic, rel boundary, to the cylinder, which is equal to $\Sigma_{H_0}$, where $H_0$ denotes the constant isotopy from the identity to the identity. However, $\Sigma_H$ is invertible in $\text{Kob}_h(2\ell)$, because the composition $\Sigma_H \circ \Sigma_H$, where $H(\cdot, t) = H(\cdot, 1-t)$, is isotopic to $\Sigma_{H_0}$, rel boundary.

Before proceeding further, one remark is in order. During the construction of the Khovanov’s bracket $[D]_{\text{Kh}}$ it was necessary to multiply each summand of the differential with $\pm 1$. This particular choice of signs forces us to do the same with maps between complexes.

Recall from [BN05] that we need to number all crossings of $D$ and based on that impose an ordering of the set of crossings. Consider two vector spaces – $W$ spanned by the crossings of $D$ and its exterior algebra $V = \Lambda^*W$. Each Kauffman state of $D$ can be labeled with a unique vector from the distinguished basis of $V$, consisting of vectors of the form

$$c_{i_1} \wedge c_{i_2} \wedge \ldots \wedge c_{i_k}, \ i_1 < i_2 < \ldots < i_k$$

where $c_{i_j} \in \text{Cr}D$ is a crossing of $D$. Choose a tangle $T$ and let $W_T$ be the vector space associated to $T$ and $D = D_n(T, \ldots, T)$ with $W_D$ defined analogously. Under this assumptions

$$W_D \cong W_T^n, \quad \text{and} \quad \Lambda^*W_D \cong (\Lambda^*W_T)^{\otimes n}.$$ 

Symmetry of $D$ induces an action of $\mathbb{Z}/n$ on $\Lambda^*W_D$, which permutes factors in the above decomposition. Cobordism $\Sigma_H$ discussed above induces a map

$$\Sigma_H : [D]_{\text{Kh}} \to [D]_{\text{Kh}}$$

which permutes Kauffman states of $D$. This permutation is compatible with the induced action on $\Lambda^*W_D$. Geometrically, the map $\Sigma_H|_{[D]_{\text{Kh}}}$ is induced by a “permutation” cobordisms similar to the one from Figure 4. However, additional sign is needed to assure that this map commutes with the differential. Let us define

$$\psi : (\Lambda^*W_T)^{\otimes n} \to (\Lambda^*W_T)^{\otimes n}$$

$$\psi : x_1 \otimes x_2 \otimes \ldots \otimes x_n \mapsto (-1)^{\alpha} x_2 \otimes \ldots \otimes x_n \otimes x_1, \quad x_i \in W_T,$$  \hspace{1cm} (1)
where

\[ \alpha = (n - 1)n_-(T) + \deg x_1(\deg x_2 + \deg x_3 + \ldots + \deg x_n). \]

Above, \( n_-(T) \) stands for the number of negative crossings of \( T \). Automorphism \( \psi \) maps any vector from the distinguished basis of \( \Lambda^* W_D \) to \( \pm 1 \) multiplicity of some other vector from the basis.

\[ \psi(v) = \text{sign}(\psi, v)w \]

We can utilize these signs to change the definition of \( \Sigma_H \) as follows.

\[
\Sigma_H|_{D_v} : D_v \rightarrow D_w, \\
\Sigma_H|_{D_v} = \text{sign}(\psi, v)\Sigma_{v,w},
\]

(2)

where \( \Sigma_{v,w} \) denotes the appropriate permutation cobordism. This discussion leads to the following proposition.

**Proposition 3.4.** If \( D \) is a periodic link diagram, then \( \text{CKh}(D) \) is a complex of graded \( \mathbb{Z}/\mathbb{Z}/n \)-modules.

**Remark 3.5.** This sign convention was implicitly described in [BN05].

**Proof of Prop. 3.4.** The only thing left to prove, is the commutativity of \( \Sigma_H \) and the differential of \( \text{CKh}(D) \). Geometric properties of the Khovanov bracket imply that the components of both maps commute up to sign. Hence, the only thing left to check is that all these maps really commute.

Let \( x_1, \ldots, x_n \in V \) be homogeneous vectors. Consider the following linear maps

\[
d_D : (\Lambda^* W_T)^\otimes n \rightarrow (\Lambda^* W_T)^\otimes n, \\
d_D : x_1 \otimes \ldots \otimes x_n \mapsto \sum_{i=1}^n (-1)^{\alpha_i} x_1 \otimes \ldots \otimes d_T(x_i) \otimes \ldots \otimes x_n, \\
\sigma_i : (\Lambda^* W_T)^\otimes n \rightarrow (\Lambda^* W_T)^\otimes n, \\
\sigma_i : x_1 \otimes \ldots \otimes x_n \mapsto (-1)^{\deg x_1 + \deg x_{i+1}} x_1 \otimes \ldots \otimes x_{i+1} \otimes x_i \otimes \ldots \otimes x_n, \\
\tilde{\sigma}_i = (-1)^{n_-(T)} \sigma_i.
\]

where \( 1 \leq i \leq n - 1, \alpha_i = (-1)^{\deg(x_n) + \ldots + \deg(x_{i+1})} \) and

\[
d_T(w) = \sum_{v \in \operatorname{Cr} T} w \wedge v.
\]

Notice that the map \( \psi \) from \([1]\) is expressible as the composition of the maps \( \tilde{\sigma}_1 \) in the following way.

\[
\psi = \tilde{\sigma}_{n-1} \circ \tilde{\sigma}_{n-2} \circ \ldots \circ \tilde{\sigma}_1.
\]

The map \( d_D \), on the other hand, corresponds to the differential, i.e. let

\[
d_{r-n_-(D)} : \text{CKh}^r-n_-(D) \rightarrow \text{CKh}^{r+1-n_-(D)}(D)
\]
be the differential in the Khovanov complex. It can be expressed in the following form
\[ d_{r-\lambda} (D) = \sum_{(v,w)} \text{sign}(v,w) \Sigma_{(v,w)}, \]
where the summation extends over pairs \((v,w)\) consisting of \(v \in \text{Cr } D\) and \(w\) a vector from the distinguished basis of \(\Lambda W_D\), such that \(v \wedge w \neq 0\). Also, \(\Sigma_{(v,w)}\) denotes the appropriate permutation cobordism. It is not hard to check, that the coefficient \(\text{sign}(v,w)\) is equal to the coefficient of \(v \wedge w\) in \(d_D(v)\).

Therefore, it is sufficient to check that for \(1 \leq i \leq n-1\) the following equality holds
\[ \sigma_i \circ d_D = d_D \circ \sigma_i. \]
This can be verified by an elementary calculation.

Let us now analyze the structure of the cochain complex \(\text{CKh}(D)\).

**Definition 3.6.**
1. Let \(S_r(D)\) denote the set of Kauffman states of \(D\) which were obtained by resolving exactly \(r\) crossings with the 1-smoothing.
2. For \(d \mid n\), let \(S_d(D)\) denote the set of Kauffman states which inherit a symmetry of order \(d\) from the symmetry of \(D\), that is Kauffman states of the form
\[ D_n(T_1, T_2, T_3, \ldots, T_{\frac{n}{d}}, T_1, T_2, T_3, \ldots, T_{\frac{n}{d}}), \]
where \(T_1, \ldots, T_{\frac{n}{d}}\) are distinct resolutions of \(T\).
3. For a Kauffman state \(s\), write \(\text{Iso}_D(s) = \mathbb{Z}/d\) if, and only if \(s \in S_d(D)\).
4. Define \(S_d^r(D) = S_d(D) \cap S_r(D)\).
5. Notice that there is an induced action of \(\mathbb{Z}/\frac{n}{d}\) on \(S_d^r(D)\), indeed since all Kauffman states in \(S_d^r(D)\) are fixed by \(\mathbb{Z}/d\). Define \(S_d^r(D)\) to be the quotient of \(S_d^r(D)\) by the action of \(\mathbb{Z}/\frac{n}{d}\).

**Remark 3.7.** If \(S_d^r(D)\) is non-empty, then \(d \mid \gcd(n,r)\).

**Definition 3.8.** Let \(\mathbb{Z}_{\leq 0}\) be the following \(\mathbb{Z}[\mathbb{Z}/n]\)-module.
\[ \mathbb{Z}_{\leq 0} = \begin{cases} \mathbb{Z} [\mathbb{Z}], & 2 \mid n, \\ \mathbb{Z}, & 2 \nmid n. \end{cases} \]
In other words, if \(n\) is even, the generator of the cyclic group \(\mathbb{Z}/n\) acts on \(\mathbb{Z}_{\leq 0}\) by multiplication by \(-1\), otherwise it is the trivial module.

**Lemma 3.9.** Let \(T\) be any TQFT functor whose target is the category of \(R\)-modules, for \(R\) a commutative ring with unit. If \(s_1, \ldots, s_n \in S_d^r(D)\), for \(d \mid \gcd(n,r)\) and \(d \geq 1\), are Kauffman states constituting a single orbit, then
\[ \bigoplus_{i=1}^{\frac{n}{d}} T([s_i]_{\text{Kh}}) \cong \text{Ind}_{\mathbb{Z}/d}^{\mathbb{Z}/n} \left( T([s_1]_{\text{Kh}}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\leq 0}^{\otimes (n,r,d)} \right). \]
as $R[\mathbb{Z}/n]$-modules, where

$$s(n, r, d) = \frac{(n - 1)n_-(D) + r(d - 1)}{d}$$

Proof. We will prove that if $s \in S^d_r(D)$, then $\Sigma_D^\mathbb{Z}_n(s) = (-1)^{s(n,r,d)}s$. The orbit of $s$ consists of $\frac{n}{d}$ Kauffman states which are permuted by the action of $\mathbb{Z}/n$. Hence, the lemma will follow from Proposition [CR90, Prop. 10.5] once we determine the induced action of $\mathbb{Z}/d$ on $T([s_1]_{Kh})$. Since $T([s_1]_{Kh})$ possesses a natural action of $\mathbb{Z}/d$, the induced action will differ from this one by a certain sign. Appearance of this sign is a consequence of our sign convention.

The Kauffman state $s_1$ corresponds to a vector of the form $w = \underbrace{v \otimes v \otimes \ldots \otimes v}_d$, where $v = v_1 \otimes v_2 \otimes \ldots \otimes v_\mathbb{Z}$ and $v_1, \ldots, v_\mathbb{Z} \in \Lambda^* W_T$ belong to the distinguished basis. Consequently, according to the adopted sign convention,

$$\psi^\mathbb{Z}_n(w) = (-1)^{k(d-1)+\frac{n_-(T)n(n-1)}{d}}w = (-1)^{r(d-1)+\frac{n_-(D)n(n-1)}{d}}w,$$

where $k = \deg v_1 + \deg v_2 + \ldots + \deg v_\mathbb{Z}$. □

Corollary 3.10. If $T$ is as in the previous lemma and $0 \leq r \leq n_+ + n_-$, then

$$T([D]_{Kh}^{r-n_-}) = \bigoplus_{d|\gcd(n, r)} \bigoplus_{s \in S^d_r} \text{Ind}_{\mathbb{Z}/d}^{\mathbb{Z}} \left(T([s]_{Kh}) \otimes \mathbb{Z}^{\otimes s(n,r,d)} \right) \{r + n_+(D) - n_-(D)\}.$$
Theorem 3.13. Any Reidemeister move induces a map,

\[ R: [D]_{\text{Kh}} \to [D']_{\text{Kh}}, \]

which is an isomorphism in \( \text{Kob}_{\text{Kh}}(0) \). In particular, this map induces a chain homotopy equivalence after application of any TQFT functor.

However, in the equivariant setting we obtain a considerably weaker invariance result.

Theorem 3.14. If \( D \) and \( D' \) are as above and \( T \) is a TQFT functor whose target is the category of \( R \)-modules, then the map

\[ R: [D]_{\text{Kh}} \to [D']_{\text{Kh}}, \]

induced by an equivariant Reidemeister move, yields a quasi-isomorphism

\[ T(R): T([D]_{\text{Kh}}) \to T([D']_{\text{Kh}}) \]

in the category of cochain complexes of \( R[\mathbb{Z}/n] \)-modules.

Proof. It is sufficient to prove, that \( T(R) \) is a morphism in the category of \( R[\mathbb{Z}/n] \)-modules, because Theorem 3.13 implies that it automatically induces an isomorphism on homology.

To check this condition, refer to the proof of [BN05, Thm. 2]. The bracket \([D]_{\text{Kh}}\) is constructed along the lines of the formal tensor product of copies of the complex \([T]_{\text{Kh}}\). Each collection of morphisms

\[ f_i: [T]_{\text{Kh}} \to [T']_{\text{Kh}}, \]

for \( i = 1, \ldots, n \), yield a morphism

\[ f_1 \otimes \ldots \otimes f_n: [D]_{\text{Kh}} \to [D']_{\text{Kh}}. \]

Taking into account the symmetry of \( D \) and \( D' \), we obtain the following commutative diagram

\[
\begin{array}{ccc}
[D]_{\text{Kh}} & \xrightarrow{f_1 \otimes \ldots \otimes f_n} & [D']_{\text{Kh}} \\
\Sigma_D & \downarrow & \downarrow \Sigma_{D'} \\
[D]_{\text{Kh}} & \xrightarrow{f_2 \otimes \ldots \otimes f_n \otimes f_1} & [D']_{\text{Kh}}
\end{array}
\]

where \( \Sigma_D \) and \( \Sigma_{D'} \) denote the automorphisms of complexes induced by the action of \( \mathbb{Z}/n \). Since \( R \) is of the form \( R = R' \otimes \ldots \otimes R' \), where

\[ R': [T]_{\text{Kh}} \to [T']_{\text{Kh}} \]

is induced by a single Reidemeister move, it follows that \( T(R) \) is a morphism in the category of \( R[\mathbb{Z}/n] \)-modules. \( \square \)
4 Equivariant Khovanov homology

4.1 Integral equivariant Khovanov homology

Let $L$ be an $n$-periodic link. It was shown earlier, that under this assumption, the Khovanov complex $\text{CKh}(D)$, where $D$ is an $n$-periodic diagram of $L$, admits the structure of a cochain complex of graded $\mathbb{Z}/n$-modules. We could try to obtain an invariant of $L$ by defining the equivariant Khovanov homology to be

$$H^{*,*} (\text{Hom}_{\mathbb{Z}/n} (M, \text{CKh}(D))),$$

for some $\mathbb{Z}/n$-module $M$. Unfortunately, Theorem 3.14 indicates that this approach might not work. This is indeed the case, which is illustrated by the following example.

**Example 4.1.** Consider the 2-periodic diagram $D$ from Figure 5. We will show that due to the lack of exactness of the Hom functor, the ordinary cohomology with coefficients depends on the chosen diagram. Since

$$\text{Hom}_{\mathbb{Z}/2} (\mathbb{Z}_-, M) = \{ x \in M : t \cdot x = -x \},$$

we obtain

$$\text{Hom}_{\mathbb{Z}/2} (\mathbb{Z}_-, \text{CKh}^{1,*}(D)) = \langle \begin{bmatrix} 1 \otimes 1 \\ -1 \otimes 1 \end{bmatrix}, \begin{bmatrix} 1 \otimes X \\ -X \otimes 1 \end{bmatrix}, \begin{bmatrix} X \otimes 1 \\ -1 \otimes X \end{bmatrix}, \begin{bmatrix} X \otimes X \\ -X \otimes X \end{bmatrix} \rangle$$

$$\text{Hom}_{\mathbb{Z}/2} (\mathbb{Z}_-, \text{CKh}^{2,*}(D)) = (1, X).$$
Inspection of the differential $d: \text{CKh}^1(D) \to \text{CKh}^2(D)$ yields

$$d: \left[ \begin{array}{cc} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{array} \right] \mapsto -2 \cdot \mathbb{1},$$

$$d: \left[ \begin{array}{cc} \mathbb{1} & X \\ -X & \mathbb{1} \end{array} \right] \mapsto -2 \cdot X,$$

therefore

$$H^{2,*}\left(\text{Hom}_{\mathbb{Z}/2}(\mathbb{Z}^{-}, \text{CKh}(D))\right) = \mathbb{Z}/2\{5\} \oplus \mathbb{Z}/2\{3\}.$$

On the other hand,

$$H^{2,*}\left(\text{Hom}_{\mathbb{Z}/2}(\mathbb{Z}^{-}, \text{CKh}(U))\right) = 0,$$

where $U$ denotes the crossingless diagram of the unknot.

The above example shows the necessity of considering the derived functors $\text{Hom}^*_{\mathbb{Z}/n}(M, -)$ and their homology. This is due to the fact, that the Khovanov complex of a periodic link is built from permutation modules, which are in general neither projective nor injective. This causes the discrepancy visible in the previous example.

**Definition 4.2.** Define the equivariant Khovanov homology of an $n$-periodic diagram $D$ to be the following triply-graded module, for which the third grading is supported only for $d | n$.

$$\text{Kh}^{*,*,d}_{\mathbb{Z}/n}(L) = \text{Ext}^{*,*}_{\mathbb{Z}[[\xi_d]]}(\mathbb{Z}[[\xi_d]], \text{CKh}(D)).$$

It is worth to notice, that since $\text{CKh}(D)$ is a complex of graded modules, $\text{Ext}$ groups become also naturally graded, provided that we regard $\mathbb{Z}[[\xi_d]]$ as a graded module concentrated in degree 0.

To check the equivariant Khovanov homology is independent of the choice of periodic diagram, it is sufficient to check its invariance under the equivariant Reidemeister moves, see Definition 3.12.

**Theorem 4.3.** Equivariant Khovanov homology is an invariant of a periodic link, that it is invariant under the equivariant Reidemeister moves.

**Proof.** Theorem 3.14 implies that a single application of an equivariant Reidemeister move to a periodic link diagram, yields a quasi-isomorphism of the corresponding Khovanov complexes. Proposition 2.15 implies, that this quasi-isomorphism induces an isomorphism of the equivariant Khovanov homology.

One of the first questions, regarding the properties of the equivariant Khovanov homology, we can ask, is the question about its relation to the classical Khovanov homology. The answer is given in the following theorem.

**Theorem 4.4.** Let $p_1, \ldots, p_s$ be the collection of all prime divisors of $n$. Define the ring $R_n = \mathbb{Z}\left[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_s}\right]$. There exists a natural map

$$\bigoplus_{d|n} \text{Kh}^{*,*,d}_{\mathbb{Z}/n}(L) \to \text{Kh}(L),$$

which, when tensored with $R_n$, becomes an isomorphism.
Proof. From Theorem 2.16 it follows that
\[ \text{Ext}^r_{\mathbb{Z}/n} (\mathbb{Z}/n, \text{CKh}(D)) \cong \text{Kh}^r(D). \]
Indeed, entries in the $E_2$ page of the Cartan-Eileberg spectral sequence are equal to
\[ \text{Ext}^r_{\mathbb{Z}/n} (\mathbb{Z}/n, \text{Kh}^q(D)) = \begin{cases} \text{Hom}_{\mathbb{Z}/n} (\mathbb{Z}/n, \text{Kh}^q(D)), & p = 0, \\ 0, & p > 0. \end{cases} \]
Hence, the spectral sequence collapses at $E_2$. Since
\[ \text{Hom}_{\mathbb{Z}/n} (\mathbb{Z}/n, \text{Kh}^q(D)) \cong \text{Kh}^q(D), \]
we obtain the desired conclusion.

The short exact sequence from Proposition 2.5 implies that
\[ \Lambda' \otimes_{\mathbb{Z}} R_n = \mathbb{Z}/n \otimes_{\mathbb{Z}} R_n = R_n [\mathbb{Z}/n], \]
where $\Lambda'$ denotes the maximal order in $\mathbb{Q}[\mathbb{Z}/n]$. Therefore,
\[ \text{Ext}^r_{\mathbb{Z}/n} (\Lambda', \text{CKh}(D)) \otimes_{\mathbb{Z}} R_n \cong \text{Ext}^r_{R_n[\mathbb{Z}/n]} (\Lambda' \otimes_{\mathbb{Z}} R_n, \text{CKh}(D) \otimes_{\mathbb{Z}} R_n) \cong \text{Ext}^r_{R_n[\mathbb{Z}/n]} (R_n [\mathbb{Z}/n], \text{CKh}(D) \otimes_{\mathbb{Z}} R_n) \cong \text{Kh}^r(D) \otimes_{\mathbb{Z}} R_n, \]
because $R_n$ is flat over $\mathbb{Z}$. The last step of the proof consist of noticing, that since $\Lambda' = \bigoplus_{d|n} \mathbb{Z} [\xi_d]$, hence
\[ \text{Ext}^r_{\mathbb{Z}/n} (\Lambda', \text{CKh}(D)) = \bigoplus_{d|r} \text{Kh}^r(G_d)^* \text{Kh}^*_d(L). \]

Until the end of this section we will restrict our attention to the case of $p^n$-periodic links, where $p$ is an odd prime. This restriction is needed to perform the computations of the Khovanov homology of trivial links.

Components of periodic trivial links can be divided into two categories. The first category consists of components, which are invariant under the $\mathbb{Z}/n$ action, i.e. they link non-trivially with the fixed point axis $F$ and the generator of the cyclic groups rotates them by the angle $\frac{2\pi}{p^n}$. The second category consists of components, which are freely permuted by $\mathbb{Z}/n$. Let $T_{k,p^n+f}$ denote a crossingless $p^n$-periodic diagram of the trivial link consisting of $f$ fixed circles, i.e. components from the first category, and $k$ orbits of free components.

In order to express the equivariant Khovanov homology of trivial links, let us define the following family of polynomials.
Definition 4.5. Define a sequence of Laurent polynomials i.e. elements of the ring $\mathbb{Z}[q, q^{-1}]$.

$$P_0(q) = q + q^{-1}$$

$$P_n(q) = \frac{1}{p^n} \sum_{1 \leq k \leq p^{n-1}} \binom{p^n}{k} q^{2k-p^n} + \frac{1}{p^n} \sum_{1 \leq s < n} \sum_{1 \leq k \leq p^{n-1}} \left( \binom{p^n}{k} - \binom{p^n-s}{k'} \right) q^{2k-p^n},$$

where $k' = k/p^n$ and $n \geq 1$.

Definition 4.6. If $M^*$ is a graded and free $\mathbb{Z}$-module, then define its quantum dimension as in $[\text{Tur08}]$.

$$\text{qdim } M^* = \sum_i q^i \dim M^i \otimes \mathbb{Q}.$$ 

Proposition 4.7. The Khovanov complex $\text{CKh}(T_{kp^n})$ can be decomposed into a direct sum of permutation modules in the following way

$$\text{CKh}^0_*(T_{kp^n}) = \bigoplus_{s=0}^n \text{Ind}_{\mathbb{Z}/p^n^{n-s}}^{\mathbb{Z}/p^n} M^k_s,$$

where $M^k_s$ is a free abelian group and a trivial $\mathbb{Z}/p^n^{n-s}$-module satisfying

$$\text{qdim } M^k_s = \sum_{\ell=1}^k p^{s}(\ell-1)\mathcal{P}_s(q^{p^n-s}) \ell \prod_{0 \leq i_0, \ldots, i_{s-1} \leq k} (p^{i_j}\mathcal{P}_j(q^{p^n-i_j}))^{i_j}.$$ 

Proof. Let

$$\mathcal{A} = \mathbb{Z}[X] / (X^2),$$

where

$$\deg 1 = 1,$$

$$\deg X = -1.$$ 

Since the induced action on

$$\text{CKh}^0_*(T_{kp^n}) = \text{CKh}^0_*(T_{kp^n}) = \mathcal{A}^{\otimes k} \otimes \mathcal{A}^{\otimes k} \otimes \ldots \otimes \mathcal{A}^{\otimes k}$$

permutes the factors in the tensor product above, thus it is sufficient to consider the restriction of this action to the basis of $\mathcal{A}^{\otimes k}$ consisting of tensor products of vectors $1$ and $X$.

Let us start with the case $k = 1$. Denote by $\ell$ the following map

$$\ell: \mathcal{A}^{\otimes p^n} \rightarrow \mathcal{A}^{\otimes p^n},$$

$$\ell: x_1 \otimes x_2 \otimes \ldots \otimes x_{p^n} \mapsto x_2 \otimes \ldots \otimes x_{p^n} \otimes x_1.$$
In order to obtain the desired decomposition of $\text{CHK}_{\bar{0},*}(T_{\bar{p}^s})$ it is sufficient to decompose the basis of $\mathcal{A}^{\otimes p^n}$ into disjoint union of orbits. Observe that if $v = x_1 \otimes \ldots \otimes x_{p^n}$ satisfies $\theta^{p^s}(v) = v$, for some $s \leq n$, then $v$ is completely determined by its first $p^s$ factors $x_1, \ldots, x_{p^s}$.

**Lemma 4.8.** The set of basis vectors satisfying the following conditions, for fixed $s \leq n$,

1. $\text{Iso}(v) = \mathbb{Z}/p^{n-s}$,
2. $\deg v = p^{n-s}(2k - p^s)$, for $1 \leq k \leq p^s - 1$

has cardinality

$$\left\{ \begin{array}{ll}
(p_{k}^{s}), & \gcd(k, p^s) = 1, \\
(p_{k}^{s}) - (p_{k'}^{s-1}), & \gcd(k, p^s) = p^u, \\
\end{array} \right.$$  

where $k' = k/p^u$. In particular, $\text{qdim } M_1^s = P_s(p^{n-s})$.

**Proof.** Notice first, that if a vector $v$ is fixed by $\theta^{p^s}$, then necessarily $p^{n-s} | \deg v$. If $k$ is not divisible by $p$, then $v$ automatically satisfies the first condition and cardinality of the set of such vectors is equal to $\binom{p^s}{k}$.

On the other hand, when $\gcd(k, p^s) = p^u$, there are vectors $v$ such that $\deg v = p^{n-s}(2k - p^s)$ and $\text{Iso}(v)$ contains properly $\mathbb{Z}/p^{n-s}$. There are exactly $\binom{p^s}{k/p^u}$ such vectors. Therefore, the overall cardinality in this case is equal to $\binom{p^s}{k} - \binom{p^s}{k/p^u}$.  

To perform the inductive step, notice that if $v \in A^{(k-1)p^n}$ and $w \in A^{p^n}$ satisfy $\text{Iso}(v) = \mathbb{Z}/p^{n-s}$ and $\text{Iso}(w) = \mathbb{Z}/p^{n-s'}$, then $\text{Iso}(v \otimes w) = \mathbb{Z}/p^{\min(n-s, n-s')}$. When $\text{Iso}(v) = \mathbb{Z}/p^{n-s}$, then the orbit of $v$ can be identified with $\mathbb{Z}/p^s$, with the action coming from the following quotient map

$$\mathbb{Z}/p^n \to \mathbb{Z}/p^s.$$  

The product of two orbits $\mathbb{Z}/p^s$ and $\mathbb{Z}/p^{s'}$, with the diagonal action, consists of several orbits. This decomposition is given below.

$$\mathbb{Z}/p^s \times \mathbb{Z}/p^{s'} = \bigoplus_{i=1}^{p^{s'}} \mathbb{Z}/p^s, \quad (3)$$

if $s \geq s'$. Thus,

$$M_s^k = \bigoplus_{0 \leq s' < s} \Big( (M_s^{k-1} \otimes_{\mathbb{Z}} M_1^1)^{p^{s'}} \oplus (M_s^{k-1} \otimes_{\mathbb{Z}} M_1^1)^{p^{s'}} \Big) \oplus (M_s^{k-1} \otimes_{\mathbb{Z}} M_1^1)^{p^s}.$$  

Consequently,

$$\text{qdim } M_s^k = \sum_{0 \leq s' < s} p^{s'}(\text{qdim } M_s^{k-1} \cdot \text{qdim } M_1^1) + \text{qdim } M_s^{k-1} \cdot \text{qdim } M_1^1 + \text{qdim } M_s^{k-1} \cdot \text{qdim } M_1^1.$$  

Expanding this inductive formula yields the desired result.  

![](image-url)
Corollary 4.9. The Khovanov complex \( \text{CKh}(T_{kp^n+f}) \) decomposes in the following way.

\[
\text{CKh}^{0,\ast}(T_{kp^n+f}) = \bigoplus_{s=0}^n \text{Ind}_{\mathbb{Z}_{p^n}}^{\mathbb{Z}_{p^n-s}} M_s^{k,f},
\]

where \( M_s^{k,f} \) is a trivial \( \mathbb{Z}_{p^n-s} \)-module such that

\[
\text{qdim} M_s^{k,f} = (q+q^{-1}) \text{qdim} M_s^k.
\]

Proof. Corollary follows from Proposition 4.7. Indeed, because

\[
\text{CKh}^{0,\ast}(T_{kp^n+f}) = \text{CKh}^{0,\ast}(T_{kp^n}) \otimes_{\mathbb{Z}} \text{CKh}^{0,\ast}(T_f).
\]

Arguing as in the proof of Proposition 4.7 we can consider orbits of vectors which are tensor products of 1 and \( X \). Every such orbit is a tensor product of an orbit from \( \text{CKh}^{0,\ast}(T_{kp^n}) \) and a trivial orbit from \( \text{CKh}^{0,\ast}(T_f) \). This additional trivial orbit from \( \text{CKh}^{0,\ast}(T_f) \) preserves the isotropy group of the orbit from \( \text{CKh}^{0,\ast}(T_{kp^n}) \), therefore

\[
M_s^{k,f} = M_s^k \otimes_{\mathbb{Z}} A^\otimes f,
\]

which concludes the proof. \( \square \)

Analysis of the Khovanov complex of the trivial link yields the following result giving explicit formulas for the equivariant Khovanov homology of trivial links.

Proposition 4.10. The equivariant Khovanov homology of the trivial link \( T_{kp^n+f} \) is given below.

\[
\text{Kh}^{0,\ast}_{\mathbb{Z}/p^n}(T_{kp^n+f}) = \bigoplus_{s=0}^n W_{s,u}^{k,f},
\]

where

\[
W_{s,u}^{k,f} = \left\{ \begin{array}{ll}
\bigoplus_i H^i(\mathbb{Z}/p^n-s, \mathbb{Z}d_i \cdot p^{n-s}) \{i\}, & n-s \leq u, \\
\bigoplus_i H^i(\mathbb{Z}/p^n-s, \mathbb{Z}/p \cdot p^{n-s}) \{i\}, & n-s > u.
\end{array} \right.
\]

and

\[
\text{qdim} M_s^{k,f} = \sum_i d_i q^i,
\]

for some non-negative integers \( d_i \).

Proof. From corollary 4.9 one obtains the following equality

\[
\text{Ext}_{\mathbb{Z}/p^n}^{\ast} \left( \mathbb{Z} \left[ \mathbb{Z} \right]_{p^n-s} \right), \text{Ind}_{\mathbb{Z}/p^n}^{\mathbb{Z}_{p^n-s}} M_s^{k,f} \right) =
\]

\[
= \bigoplus_{s=0}^n \text{Ext}_{\mathbb{Z}/p^n}^{\ast} \left( \mathbb{Z} \left[ \mathbb{Z} \right]_{p^n-s} \right), \text{Ind}_{\mathbb{Z}/p^n}^{\mathbb{Z}_{p^n-s}} M_s^{k,f} \right).
\]

From Shapiro’s Lemma and Proposition 2.7

\[
\text{Ext}_{\mathbb{Z}/p^n}^{\ast} \left( \mathbb{Z} \left[ \mathbb{Z} \right]_{p^n-s} \right), \text{Ind}_{\mathbb{Z}/p^n}^{\mathbb{Z}_{p^n-s}} M_s^{k,f} \right) \cong
\]

\[
\cong \text{Ext}_{\mathbb{Z}/p^n}^{\ast} \left( \text{Res} \mathbb{Z} \left[ \mathbb{Z} \right]_{p^n-s} \mathbb{Z} \left[ \mathbb{Z} \right]_{p^n-s} \right), \text{Ind}_{\mathbb{Z}/p^n}^{\mathbb{Z}_{p^n-s}} M_s^{k,f} \right) \cong
\]

\[
\cong \left\{ \begin{array}{ll}
\text{Ext}_{\mathbb{Z}/p^n}^{\ast} \left( \mathbb{Z} \left[ \mathbb{Z} \right]_{p^n-s} \right), \text{Ind}_{\mathbb{Z}/p^n}^{\mathbb{Z}_{p^n-s}} M_s^{k,f} \right), & n-s \leq u, \\
\text{Ext}_{\mathbb{Z}/p^n}^{\ast} \left( \mathbb{Z} \left[ \mathbb{Z} \right]_{p^n-s} \right), \text{Ind}_{\mathbb{Z}/p^n}^{\mathbb{Z}_{p^n-s}} M_s^{k,f} \right), & n-s > u.
\end{array} \right.
\]
Corollary 4.11. When \( k = 0 \), then the following equality holds.

\[
\text{Kh}_{Z/p^n}^{*, *}(T_f) = \bigoplus_{i=0}^f H^*(\mathbb{Z}/p^n, \mathbb{Z}[[\xi]])\{i\} \{2i-f\}.
\]

4.2 Torsion in the equivariant Khovanov homology

Theorem 4.4 together with Proposition 4.7 imply that the equivariant Khovanov homology contains an abundance of torsion not present in the classical Khovanov homology. However, we can use the additional algebraic structure of the equivariant Khovanov homology to obtain some information about this extra torsion.

To be more precise, Proposition 2.15 implies that the equivariant Khovanov homology \( \text{Kh}_{Z/p^n}^{*, *}(D) \) is a module over the Ext ring \( \text{Ext}^*_{Z[[\xi]]}(Z[[\xi]], Z) \).

This is the key to understand the additional torsion.

Proposition 4.12. For any \( r > 0 \), the ring \( \text{Ext}^*_{Z[[\xi]]}(Z[[\xi]], Z) \) acts on \( E^r_{i,j} \) page this action agrees with the natural action of the Ext ring on the module \( \text{Ext}^*_{Z/p^n}(Z[[\xi]], Z[[\xi]]) \).

Proof. It is a direct consequence of Theorem 2.17, because the appropriate version of the Cartan-Eilenberg spectral sequence for \( \text{Ext}^*_{Z[[\xi]]}(Z[[\xi]], Z) \) collapses at the \( E^r_1 \) page.

Corollary 4.13. Let \( m \in \mathbb{Z} \) be chosen such that \( \text{Kh}_{Z/p^n}^{m,*}(D) \neq 0 \) and \( \text{Kh}_{Z/p^n}^{i,*}(D) \) vanishes for \( i > m \). Let \( T_\ast \) denote the cohomology class in the ext ring

\[
T_\ast \in \text{Ext}^*_{Z[[\xi]]}(Z[[\xi]], Z[[\xi]])
\]

from Proposition 2.19. Then, multiplication by \( T_\ast \)

\[
- \cup T_\ast : \text{Kh}_{Z/p^n}^{i,*}(D) \to \text{Kh}_{Z/p^n}^{i+2,*}(D)
\]

is an epimorphism for \( i = m \) and isomorphism for \( i > m \).

Remark 4.14. This is an analogue of [Wal79] Lemma 1.1.
Proof. The action of the class $T_s$ can be described in terms of the dimension shifting, see \[ML95\]. Since the class $T_s$ is represented by the following Yoneda extension
\[
0 \to \mathbb{Z} [\xi_{p^r}] \to \mathbb{Z} [\mathbb{Z}/p^n] \to \mathbb{Z} [\mathbb{Z} / p^n] \to 0,
\]
we can split this exact sequence into two short exact sequences
\[
0 \to \mathbb{Z} [\xi_{p^r}] \to \mathbb{Z} [\mathbb{Z}/p^n] \to M \to 0,
\]
\[
0 \to M \to \mathbb{Z} [\mathbb{Z}/p^n] \to \mathbb{Z} [\xi_{p^r}] \to 0,
\]
where $M$ is a submodule of $\mathbb{Z} [\mathbb{Z}/p^n]$. Let
\[
\delta_1 : \text{Ext}_{\mathbb{Z}/p^n}^{i+1} (M, \text{CKh}(D)) \to \text{Ext}_{\mathbb{Z}/p^n}^{i+2} (M, \text{CKh}(D)),
\]
\[
\delta_2 : \text{Ext}_{\mathbb{Z}/p^n}^{i} (M, \text{CKh}(D)) \to \text{Ext}_{\mathbb{Z}/p^n}^{i+1} (M, \text{CKh}(D)),
\]
be the connecting homomorphisms from the long exact cohomology sequences derived from the short exact sequences \ref{4} and \ref{5}, respectively. The map
\[
- \cup T_s : \text{Ext}_{\mathbb{Z}/p^n}^{i+1} (M, \text{CKh}(D)) \to \text{Ext}_{\mathbb{Z}/p^n}^{i+2} (M, \text{CKh}(D))
\]
is equal to the composition $\delta_2 \circ \delta_1$.

Since $\text{Ext}_{\mathbb{Z}/p^n}^{i}(\mathbb{Z} [\mathbb{Z}/p^n], \text{CKh}(D)) \cong \text{Kh}^{i,*}(D)$, see the proof of \ref{4.3}, we deduce that $\delta_1$ and $\delta_2$ are epimorphisms for $i = m$ and isomorphisms for $i > m$, because $\text{Kh}^{i,*}(D) = 0$ for $i > m$. Hence the corollary follows.

\begin{corollary}
Let $m$ be as in the previous corollary. For $i > m$, $\text{Kh}_{\mathbb{Z}/p^n}^{i+1} (D)$ is annihilated by $p^n$, and for $1 \leq s \leq n$, $\text{Kh}_{\mathbb{Z}/p^n}^{i+2} (D)$ is annihilated by $p^{n-s+1}$.
\end{corollary}

Proof. This follows from previous corollary and the fact that $p^n T_0 = 0$ and $p^{n-s+1} T_s = 0$, for $1 \leq s \leq n$.

\section{4.3 Rational equivariant Khovanov homology}
Since the rational group algebra $\mathbb{Q} [\mathbb{Z}/n]$ is semi-simple and artinian, the algebraic structure of the equivariant Khovanov homology with rational coefficients simplifies considerably.

\begin{proposition}
If $D$ is an $n$-periodic link diagram, then
\[
\text{CKh}_{\mathbb{Z}/n}^{*,*}(D; \mathbb{Q}) \cong \bigoplus_{d|n} \text{CKh}_{\mathbb{Z}/n}^{*,d}(D; \mathbb{Q}),
\]
where
\[
\text{CKh}_{\mathbb{Z}/n}^{*,d}(D; \mathbb{Q}) = \text{Hom}_{\mathbb{Q}[\mathbb{Z}/n]} (\mathbb{Q} [\xi_d], \text{CKh}_{\mathbb{Z}/n}^{*,*}(D; \mathbb{Q})).
\]
Moreover,
\[
\text{CKh}_{\mathbb{Z}/n}^{*,d}(D; \mathbb{Q}) = \text{CKh}_{\mathbb{Z}/n}^{*,d}(D; \mathbb{Q}) \cdot e_d,
\]
and
\[
\text{Kh}_{\mathbb{Z}/n}^{*,d}(D; \mathbb{Q}) = \text{Kh}_{\mathbb{Z}/n}^{*,d}(D; \mathbb{Q}) \cdot e_d = H^{*,*}(\text{CKh}(D; \mathbb{Q} \cdot e_d),
\]
where $e_d$ is a central idempotent from $\mathbb{Q} [\mathbb{Z}/n]$, which annihilates $\mathbb{Q} [\xi_d]$, for $d' \mid n$ and $d \neq d'$, and acts as an identity on $\mathbb{Q} [\xi_d]$.

\end{proposition}
Proof. The proposition is a consequence of the Wedderburn decomposition and Schur’s Lemma. For more details refer to [CR90, Chap. 1].

Example 4.17. The rational Khovanov homology of the trivial link $T_{kp^n+f}$, for some prime $p$, which possesses $k$ free orbits of components and $f$ fixed circles, is given by the following formula

$$
\text{Kh}_{Z_p}^{0,*}(T_{kp^n+f}; \mathbb{Q}) = \bigoplus_{s=n-u}^n (M^k,f_s^{(p^n-u)}) \otimes \mathbb{Q},
$$

where $M^k,f_s$ was defined in Proposition 4.7.

The above proposition has the following corollary.

Corollary 4.18. Suppose that $D$ is an $n$-periodic diagram of a link. Choose $d \mid n$. If for any $i,j$ we have $\dim \mathbb{Q} \text{Kh}^{i,j}(D; \mathbb{Q}) < \varphi(d)$, where $\varphi$ denotes the Euler’s totient function, then $\text{Kh}_{Z_n}^{*,*,d}(D; \mathbb{Q}) = 0$.

Proof. Indeed, since $\text{Kh}_{Z_n}^{*,*,d}(D; \mathbb{Q})$ is a $\mathbb{Q}[\xi_d]$ vector space, it follows readily that $\dim \mathbb{Q} \text{Kh}_{Z_n}^{*,*,d}(D; \mathbb{Q})$ is divisible by $\dim \mathbb{Q} [\xi_d] = \varphi(d)$.

The above corollary can be used to compute the equivariant Khovanov homology in some cases.

Corollary 4.19. Let $T(n,2)$ be the torus link. Let $d > 2$ be a divisor of $n$. According to Example 3.3, $T(n,2)$ is $d$-periodic. Let $d' > 2$ and $d' \mid d$.

$$
\text{Kh}_{Z_n}^{*,*,d'}(T(n,2); \mathbb{Q}) = 0.
$$

Proof. Indeed, because according to [Kho00, Prop. 35] for all $i, j$ we have

$$
\dim \mathbb{Q} \text{Kh}^{i,j}(T(n,2); \mathbb{Q}) \leq 1
$$

and $\varphi(d') > 1$ if $d' > 2$.

Corollary 4.20. Let $\gcd(3, n) = 1$. The 3-equivariant Khovanov homology $\text{Kh}_{Z_3}^{*,*,d}(T(n,3); \mathbb{Q})$ of the torus knot $T(n,3)$ vanishes.

If $d > 2$ divides $n$, $d' > 2$ and $d' \mid d$, then $\text{Kh}_{Z_3}^{*,*,d'}(T(n,3); \mathbb{Q}) = 0$.

Proof. Indeed, because [Tur08, Thm. 3.1] implies that for all $i, j$ we have

$$
\dim \mathbb{Q} \text{Kh}^{i,j}(T(n,3); \mathbb{Q}) \leq 1,
$$

provided that $\gcd(3, n) = 1$.

5 The spectral sequence

Computation of the classical Khovanov homology is usually done with the aid of the long exact sequence of Khovanov homology, applied to a chosen link crossing, which is then resolved in two different ways. However, resolution of a single crossing kills the symmetry of a periodic link diagram in question. Instead, we need to resolve a whole orbit of crossings to obtain another periodic diagram.
But this is not sufficient to recover the equivariant Khovanov homology. We have to take into account all possible resolutions of a chosen orbit of crossings. These data are organized into a spectral sequence, the construction of which is the main goal of this chapter. During the construction, we obtain a filtration which yields the spectral sequence from [ET12]. The spectral sequence that we obtain is applied to the computation of the 2-equivariant Khovanov homology of torus links $T(n, 2)$.

Start with a link $L$ and its $n$-periodic diagram $D$. Choose a subset of crossings $X \subseteq \text{Cr}(D)$.

**Definition 5.1.** Let $\alpha: \text{Cr}(D) \to \{0, 1, x\}$ be a map.

1. If $i \in \{0, 1, x\}$ define $|\alpha|_i = \#\alpha^{-1}(i)$.
2. Define the support of $\alpha$ to be $\text{supp}\alpha = \alpha^{-1}(\{0, 1\})$.
3. Define also the following family of maps
   $$B_k(X) = \{\alpha: \text{Cr}(D) \to \{0, 1, x\} | \text{supp}\alpha = X, \ |\alpha|_1 = k\}.$$  
4. Denote by $D_\alpha$ the diagram obtained from $D$ by resolving crossings from $\alpha^{-1}(0)$ by 0-smoothing and from $\alpha^{-1}(1)$ by 1-smoothing.

First, we will work with $\overline{\text{CKh}}(D)$, which is defined in the following ways

$$\overline{\text{CKh}}(D) = \text{CKh}(D)[n-(D)]\{2n-(D) - n+(D)\},$$

where $n_+(D)$ and $n_-(D)$ denote the number of positive and negative crossings of $D$, respectively. We will construct a filtration of this complex. The filtration on $\text{CKh}(D)$ will be obtained from this one by an appropriate shift in degree.

Fix a crossing $c \in \text{Cr}(D)$ and consider three maps

$$\alpha_0, \alpha_1, \alpha_x: \text{Cr}(D) \to \{0, 1, x\},$$

which attain different value at $c$, i.e. $\alpha_1(c) = 1$, $\alpha_x(c) = x$ and $\alpha_0(c) = 0$, and are identical otherwise. These data yield the following short exact sequence of complexes.

$$0 \to \overline{\text{CKh}}(D_{\alpha_1})[1]\{1\} \to \overline{\text{CKh}}(D_{\alpha_0}) \to \overline{\text{CKh}}(D_{\alpha_x}) \to 0,$$

as in [Tur08]. However, there exists a chain map

$$\delta_c: \text{CKh}(D_{\alpha_0}) \to \overline{\text{CKh}}(D_{\alpha_1})\{1\},$$

such that $\overline{\text{CKh}}(D_{\alpha_x}) = \text{Cone}(\delta_c)$, where $\text{Cone}(\delta_c)$ denotes the algebraic mapping cone of $\delta_c$, and (6) is the corresponding short exact sequence of complexes. The map $\delta_c$ is obtained as follows. We identify $\text{CKh}(D_{\alpha_0})$ and $\overline{\text{CKh}}(D_{\alpha_1})$ with submodules of $\overline{\text{CKh}}(D_{\alpha_x})$ “generated” by Kauffman states with $c$ resolved by 0- or 1-smoothing, respectively. As a graded module $\overline{\text{CKh}}(D_{\alpha_x})$ splits in the following way

$$\overline{\text{CKh}}(D_{\alpha_x}) = \text{CKh}(D_{\alpha_0}) \oplus \overline{\text{CKh}}(D_{\alpha_1})\{1\}[1],$$

with $\overline{\text{CKh}}(D_{\alpha_1})$ being a subcomplex. If $\pi_1$ denotes the projection of $\overline{\text{CKh}}(D_{\alpha_0})$ onto $\overline{\text{CKh}}(D_{\alpha_1})$ and $i_0$ denotes the inclusion of $\text{CKh}(D_{\alpha_0})$, then

$$\delta_c = \pi_1 \circ d \circ i_0.$$

When we consider two crossings $c$ and $c'$, we obtain the following bicomplex
where $\alpha_{00}, \alpha_{10}, \alpha_{01}, \alpha_{11}$ differ only at $c$ or $c'$ and

$$\begin{align*}
\alpha_{00}(c) &= \alpha(c') = 0, \\
\alpha_{10}(c) &= 1, \quad \alpha_{10}(c') = 0, \\
\alpha_{01}(c) &= 0, \quad \alpha_{01}(c') = 1, \\
\alpha_{11}(c) &= \alpha_{11}(c') = 1.
\end{align*}$$

The horizontal maps are defined analogously as in the previous case. The total complex of the above bicomplex is equal to the shifted Khovanov complex of $D_{\alpha_{xx}}$, where $\alpha_{xx}$ agrees with $\alpha_{00}, \alpha_{10}, \alpha_{01}$ and $\alpha_{11}$ outside $c$ and $c'$ and

$$\alpha_{xx}(c) = \alpha_{xx}(c') = x.$$

Continuing this procedure we obtain the following bicomplex

$$N^{i,j,k} = \begin{cases} 
\bigoplus_{\alpha \in B(X)} \text{CKh}^{j,k}(D_{\alpha})\{i\}, & 0 \leq i \leq \#X, \\
0, & \text{otherwise}
\end{cases}$$

where $c(D_{\alpha}) = n -(D_{\alpha}) - n-(D)$. Vertical differentials

$$M^{i,j,*} \rightarrow M^{i,j+1,*}$$

are sums of $\pm 1$ multiplicities of differentials in the respective Khovanov complexes. Horizontal differentials

$$M^{i,j,*} \rightarrow M^{i+1,j,*},$$

on the other hand, are induced from the appropriate horizontal maps in the bicomplex $N^{*,*,*}$.

**Proposition 5.3.** The total complex of $M^{*,*,*}$ is equal to the Khovanov complex $\text{CKh}(D)$. 

27
Proof. Since the total complex of $N^{*,*,*}$ is equal to $\mathcal{CKh}(D)$ and

$$\mathcal{CKh}(D) = \mathcal{CKh}(D)[{-n_-(D)}][n_+(D) - 2n_-(D)],$$

we only need to check, that the application of the appropriate shift to $N$ results in $M$.

$$N^{i,j,k}[{-n_-(D)}][n_+(D) - 2n_-(D)] =$$

$$= \bigoplus_{\alpha \in B_i(X)} \mathcal{CKh}^{i,k}(D_\alpha)[-n_-(D)][i + n_+(D) - 2n_-(D)] =$$

$$= \bigoplus_{\alpha \in B_i(X)} \mathcal{CKh}^{i,k}(D_\alpha)[c(D_\alpha)][i + 3c(D_\alpha)\#X] = M^{i,j,k},$$

because

$$n_+(D) - n_+(D_\beta) =$$

$$= \# \text{Cr}(D) - n_-(D) - (\# \text{Cr}(D) - \#X - n_-(D_\beta)) =$$

$$= c(D_\beta) + \#X.$$

Definition 5.4. Let

$$F_i(X) = \text{Tot}(\bigoplus_{j \geq i} M^{j,*,*}),$$

for $0 \leq i \leq \#X$. The family $\{F_i(D)\}_i$ is a filtration of the Khovanov complex. This filtration is the column filtration of the bicomplex $M^{*,*,*}$, see [McC01, Thm. 2.15].

The following theorem was first proved in [ET12].

Theorem 5.5. Let $D$ be a link diagram and let $X \subset \text{Cr}(D)$. The pair $(D,X)$ determines a spectral sequence

$$\{E_r^{*,*,*}, d_r\}$$

of graded modules converging to $\text{Kh}^{*,*}(D)$ such that

$$E_1^{i,j,*} = \bigoplus_{\beta \in B_i(X)} \mathcal{Kh}^{j,*}(D_\beta)[c(D_\beta)][i + 3c(D_\beta) + \#X],$$

where $c(D_\beta) = n_-(D_\beta) - n_-(D)$.

Proof. This is the spectral sequence associated to the column filtration of the bicomplex $M$ as in [McC01] Thm. 2.15.

Suppose now, that $D$ is an $n$-periodic link diagram. If $X \subset \text{Cr}D$ is invariant, under the action of $\mathbb{Z}/n$, then for any $0 \leq k \leq \#X$ there exists an induced action on $B_k(X)$. Hence, each member $F_k(X)$, for $0 \leq k \leq \#X$, of the filtration is invariant under the action of $\mathbb{Z}/n$. This discussion leads to the following conclusion.
Proposition 5.6. If $X \subset \text{Cr} D$ is an invariant subset, then every member of the filtration, from Definition 5.4, is a $\mathbb{Z}/[\mathbb{Z}/n]$-subcomplex of $\text{CKh}(D)$.

From now on, we will assume that $X$ consists of a single orbit of crossings. We will perform analysis of the quotients $F_i(X)/F_{i+1}(X)$ to determine their structure as $\mathbb{Z}/[\mathbb{Z}/n]$-modules.

Definition 5.7. Let $0 \leq i \leq n$ and $d \mid \gcd(n,i)$. Analogously as in Definition 3.6, denote by $B^d_i(X)$ the subset of $B_i(X)$ consisting of maps satisfying $\text{Iso}(\alpha) = \mathbb{Z}/d$. Also denote by $\overline{B}^d_i(X)$ the quotient of $B^d_i(X)$ by $\mathbb{Z}/n$.

Lemma 5.8. If $\alpha \in B^d_i(X)$, then $D_\alpha$ is $\mathbb{Z}/d$-periodic.

Proof. The Lemma follows readily, because such diagrams have similar structure as the Kauffman states belonging to $\text{CKh}(D)$.

Proposition 5.9. Suppose, that $n$ is odd, then for $0 \leq i \leq n$

\[ F_i(X)/F_{i+1}(X) = \bigoplus_{d \mid \gcd(n,i)} \bigoplus_{\alpha \in \overline{B}^d_i(X)} \text{Ind}_{\mathbb{Z}/d}^{\mathbb{Z}/n} (\text{CKh}(D_\alpha)[t(\alpha)][q(\alpha)]), \]

where

\[ t(\alpha) = c(D_\alpha) + i, \]

\[ q(\alpha) = i + 3c(D_\alpha) + n. \]

Proof. To proof of the Lemma uses an adaptation of the argument from the proof of Lemma 3.9 and Corollary 3.10.

Proposition 5.10. Let $n = 2$. Under this assumption we have

\[ F_0(X)/F_1(X) = \text{CKh}(D_{\alpha_{00}})[t(\alpha_{00})][q(\alpha_{00})] \otimes_{\mathbb{Z}} \mathbb{Z}^{\otimes \frac{(\text{Ind}(\alpha_{00}))}{2}}, \]

\[ F_1(X)/F_2(X) = \text{CKh}(D_{\alpha_{10}})[t(\alpha_{10})][q(\alpha_{10})] \otimes_{\mathbb{Z}} \mathbb{Z}/2, \]

\[ F_2(X) = \text{CKh}(D_{\alpha_{11}})[t(\alpha_{11})][q(\alpha_{11})] \otimes_{\mathbb{Z}} \mathbb{Z}^{\otimes \frac{(\text{Ind}(\alpha_{11}))}{2} + 1}. \]

Proof. The middle equality follows easily, because complexes corresponding to $D_{\alpha_{10}}$ and $D_{\alpha_{00}}$ are freely permuted by $\mathbb{Z}/2$.

To check the first and third inequality, recall from Equation 1 from Section 3 the sign convention. We have two distinguished crossings $c_1, c_2 \in \text{supp}\alpha_{00}$. Let us denote by $T$, as in Section 3, the tangle from which $D$ was constructed. Let $x_1, x_2 \in \Lambda^* W_T$ be such that $x_1 \otimes x_2$ corresponds to certain Kauffman state of $D_{\alpha_{00}}$. According to our convention, the permutation map acts on $\text{CKh}(D_{\alpha_{00}})$ as follows

\[ x_1 \otimes x_2 \mapsto (-1)^{\frac{n-(\text{Ind}(\alpha_{00}))}{2} + \deg x_1 \deg x_2} x_2 \otimes x_1, \]

whereas on $\text{CKh}(D)$ it acts as follows.

\[ x_1 \otimes x_2 \mapsto (-1)^{\frac{n-(\text{Ind}(D))}{2} + \deg x_1 \deg x_2} x_2 \otimes x_1. \]

Comparison of the two coefficients yields, that we need to twist the action of $\mathbb{Z}/2$ on $\text{CKh}(D_{\alpha_{00}})$ by $\mathbb{Z}^{\otimes \frac{(\text{Ind}(\alpha_{00}))}{2}}$.

Analogous argument applied to $D_{11}$ yields the third equality.
Let $p$ be an odd prime and $n > 0$ an integer. We state the next theorem only for 2-periodic and $p^n$-periodic links, since these cases will be of importance in the remainder part of this paper and in the subsequent paper [Pol15a]. The statement in other cases can be analogously derived, however we omit it due to its technical complication, which dims the whole idea of the spectral sequence.

**Theorem 5.11.** Let $L$ be a $p^n$-periodic link, where $p$ is an odd prime, and let $X \subset \operatorname{Cr} D$ consists of a single orbit. Under this assumption, for any $0 \leq s \leq n$ there exists a spectral sequence $\{ p^{s-} E_2^{r,s}, d_r \}$ of graded modules converging to $\operatorname{Kh}_{Z/p^n} \{ p^{s-} \}$ \((D)\) with

\[
\begin{align*}
\pi^{-s} E_1^{0,j} &= \operatorname{Kh}_{Z/p^n}^{p^{s-}}(D_{\alpha_0})[c(D_{\alpha_0})][q(\alpha_0)], \\
\pi^{-s} E_1^{n,j} &= \operatorname{Kh}_{Z/p^n}^{p^{s-n}}(D_{\alpha_1})[c(D_{\alpha_1})][q(\alpha_1)], \\
\pi^{-s} E_1^{n,j} &= \bigoplus_{0 \leq v \leq s} \bigoplus_{\alpha \in \operatorname{E}_1} \operatorname{Kh}_{\mathcal{C}/p^n}^{x,k,v}(D_{\alpha})[c(D_{\alpha})][q(\alpha)](v,s)\] \end{align*}
\]

for $0 < i < p^n$. Above we used the following notation $i = p^s g$, where $\gcd(p, g) = 1$ and $\alpha_0, \alpha_1$ are the unique elements of $\mathcal{B}_0(X)$ and $\mathcal{B}_{p^n}(X)$, respectively, and

\[
q(\alpha) = i + 3c(D_{\alpha}) + p^n, \\
k(s, v) = \begin{cases} 1, & v \leq s, \\ p^{s-v}, & v > s, \end{cases} \\
\ell(s, v) = \begin{cases} \varphi(p^{s-v}), & v \leq s, \\ p^{s-v}, & v > s, \end{cases}
\]

The $E_1$ pages of the respective spectral sequences for 2-periodic links are given below.

\[
\begin{align*}
\ell_1^{0,j} &= \operatorname{Kh}_{Z/2}^{p^{s}(D_{\alpha_0})}(D_{00})[3c(D_{00}) + 2], \\
\ell_1^{1,j} &= \operatorname{Kh}_{Z/2}^{p^{s}}(D_{01})[3c(D_{01}) + 3], \\
\ell_1^{2,j} &= \operatorname{Kh}_{Z/2}^{p^{s-3}(D_{11})}(D_{11})[3c(D_{11}) + 4], \\
\ell_1^{0,j} &= \operatorname{Kh}_{Z/2}^{p^{s-3}(D_{00})}(D_{00})[3c(D_{00}) + 2], \\
\ell_1^{1,j} &= \operatorname{Kh}_{Z/2}^{p^{s-3}}(D_{01})[3c(D_{01}) + 3], \\
\ell_1^{2,j} &= \operatorname{Kh}_{Z/2}^{p^{s}}(D_{11})[3c(D_{11}) + 4], \end{align*}
\]

where $s \in \{1, 2\}$ and

\[s(D_{\alpha}) \equiv \frac{c(D_{\alpha})}{2} \pmod{2}\]

**Proof.** In the odd case apply Theorem 2.18 to the filtration $F_*(X)$. Use Proposition 5.9 and Eckamann-Shapiro Lemma to compute the entries in the $E_1$ page as in the proof of Proposition 4.7.

In the even case apply Theorem 2.18 and Proposition 5.10. \(\square\)
6 Sample computations

The purpose of this section is to compute the rational 2-equivariant Khovanov homology of torus links \( T(n, 2) \). Before we start, however, let us define the equivariant Khovanov and Jones polynomials. Although we defer the study of their properties to \([Pol15a]\), we define them here to simplify the statements of the results presented in this section.

**Definition 6.1.** Let \( L \) be an \( n \)-periodic link. For \( d \mid n \), define the \( d \)-th equivariant Khovanov polynomial of \( L \) as follows

\[
KhP_{n,d}(L)(t,q) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}[\xi_d]} \Kh_{i,j,d}(L; \mathbb{Q})
\]

and the \( d \)-th equivariant Jones polynomial of \( L \) as

\[
J_{n,d}(L)(q) = KhP_{n,d}(L)(-1, q).
\]

Note that above we take the dimension of the respective vector space over the cyclotomic field \( \mathbb{Q}[\xi_d] \).

Let us start with \( T(2, 2) \), which serves as a basis for further calculations.

**Example 6.2.** Consider the Hopf link as depicted on Figure 7 for \( n = 2 \). Its Khovanov bracket is depicted on Figure 8. Figures 9 and 10 depict \( \text{CKh}_{\mathbb{Z}/2}^*(T(2, 2)) \) and \( \text{CKh}_{\mathbb{Z}/2}^{*,2}(T(2, 2)) \), respectively. Equivariant Khovanov and Jones polynomials of \( T(2, 2) \) are given below.

\[
\begin{align*}
KhP_{2,1}(T(2, 2))(t, q) &= 1 + q^2 + t^2 q^4 \\
KhP_{2,2}(T(2, 2))(t, q) &= t^2 q^6 \\
J_{2,1}(T(2, 2))(q) &= 1 + q^2 + q^4 \\
J_{2,2}(T(2, 2))(q) &= q^6
\end{align*}
\]
Figure 7: Computation of $\text{Kh}_{\mathbb{Z}/2}^*,*,1(T(2, 2); \mathbb{Q})$.

|   | 1 $\otimes$ 1 | (1, 1) | 1 $\otimes$ $X$ $-$ $X$ $\otimes$ 1 |
|---|----------------|--------|-------------------------------------|
| 2 | 1 $\otimes$ $X$ $+$ $X$ $\otimes$ 1 | (X, X) | 1 $\otimes$ $X$ $-$ $X$ $\otimes$ 1 |
| 0 | $X$ $\otimes$ $X$ |                                    |

Figure 8: Computation of $\text{Kh}_{\mathbb{Z}/2}^*,*,2(T(2, 2); \mathbb{Q})$.

|   | 1 $\otimes$ 1 |                                    |
|---|----------------|-------------------------------------|
| 4 | $\frac{1}{2}$(-1, 1) | 1 $\otimes$ $X$ $+$ $X$ $\otimes$ 1 |
| 2 | $\frac{1}{2}$(-X, X) | $X$ $\otimes$ $X$ |

32
Let us also state the following proposition from [Kho00], which describes the Khovanov homology of torus links $T(n, 2)$.

**Proposition 6.3.** The Khovanov polynomial of $\text{Kh}(T(n, 2))$ is equal to

$$\text{KhP}(T(2k, 2)) = q^{2k-2} + q^{2k} + t^2 q^{2k+2} (1 + t q^4) \sum_{j=0}^{k-2} t^{2j} q^{4j} + t^{2k} q^{6k-2} + t^{2k} q^{6k},$$

$$\text{KhP}(T(2k + 1, 2)) = q^{2k-1} + q^{2k+1} + t^2 q^{2k+3} (1 + t q^4) \sum_{j=0}^{k-1} t^{2j} q^{4j},$$

for $k > 1$.

**Theorem 6.4.** Khovanov polynomials of the 2-equivariant Khovanov homology of torus links $T(n, 2)$ are given below.

$$\text{KhP}_{2,1}(T(2n + 1, 2)) = \text{KhP}(T(2n + 1, 2))$$

$$\text{KhP}_{2,2}(T(2n + 1, 2)) = 0$$

$$\text{KhP}_{2,1}(T(2n, 2)) = \text{KhP}(T(2n, 2)) - t^{2n} q^{6n}$$

$$\text{KhP}_{2,2}(T(2n, 2)) = t^{2n} q^{6n}$$

Consider the 2-periodic diagram $D$ of $T(n, 2)$ from Figure 9 with the chosen orbit marked with red circles. Orient the diagram so that all crossings are positive. The associated bicomplex in $\text{Kob}(0)$ is depicted on Figure 10.

**Lemma 6.5.** The zeroth column of the $E_1$ page of the spectral sequence from
Figure 10: Bicomplex associated to the 2-periodic diagram of $T(n, 2)$ from figure 9.

Figure 11: Diagram $D'$ isotopic to the diagram of the $D_{01}$.
Theorem 5.11 applied to the 2-periodic diagram \( D \), has the following form.

\[ 1E_1^{0,j,k} = \operatorname{Kh}_{\mathbb{Z}/2}^{j,k-1,1}(T(n-1,2)) \oplus \operatorname{Kh}_{\mathbb{Z}/2}^{j,k-3,1}(T(n-1,2)), \]
\[ 2E_1^{0,j,k} = \operatorname{Kh}_{\mathbb{Z}/2}^{j,k-1,2}(T(n-1,2)) \oplus \operatorname{Kh}_{\mathbb{Z}/2}^{j,k-3,2}(T(n-1,2)). \]

Proof. From figure 10 it is not hard to see, that the diagram \( D_{00} \) represents the split sum \( T(n-1,2) \sqcup U \), where \( U \) denotes the unknot. Additionally, \( D_{00} \) inherits orientation from \( D \), therefore \( c(D_{00}) = 0 \), because \( D \) was oriented so that all crossings are positive. This concludes the proof.

Lemma 6.6. The second column of the \( E_1 \) page of the spectral sequence from theorem 5.11 applied to the 2-periodic diagram \( D \), has the following form.

\[ 1E_1^{2,j,k} = \operatorname{Kh}_{\mathbb{Z}/2}^{j,k-4,2}(T(n-2,2)) \]
\[ 2E_1^{2,j,k} = \operatorname{Kh}_{\mathbb{Z}/2}^{j,k-4,1}(T(n-2,2)) \]

Proof. From figure 10 it follows that \( D_{11} = T(n-2,2) \). It is not hard to check that we can orient \( D_{11} \) in such a way that all crossings are positive, therefore \( c(D_{11}) = 0 \). This finishes the proof.

Lemma 6.7. The first column of the \( E_1 \) page of the spectral sequence from theorem 5.11 applied to the 2-periodic diagram \( D \) has the following form.

\[ 1E_1^{1,j,k} = 2E_1^{1,j,k} = \begin{cases} 
\operatorname{Kh}^{j,k-4}(T(2k-2,2)), & j < 2k - 2, \\
\mathbb{Q}[6k-4], & j = 2k - 2, \\
\mathbb{Q}[6k], & j = 2k - 1, \\
0, & j > 2k - 1,
\end{cases} \]
if $n = 2k$ and

$$1E_{1}^{1,j,k} = 2E_{1}^{1,j,k} = \begin{cases} 
\text{Kh}^{j,k-4}(T(2k - 1, 2)), & j < 2k, \\
\mathbb{Q}\{6k + 1\} \oplus \mathbb{Q}\{6k + 3\}, & j = 2k, \\
0, & j > 2k,
\end{cases}$$

if $n = 2k + 1$.

Proof. Let us denote by $D_{01}$ one of the diagrams in the middle column of Figure 10. First, let us compute $c(D_{01})$. It is not hard to see, that $D_{01}$ can be oriented in such a way that all crossings are positive. Therefore $c(D_{01}) = 0$.

Let us denote by $D'$ the diagram from Figure 11. Orient it, so that all crossings are positive. Quick inspection shows that $D_{01}$ and $D'$ are isotopic. In order to prove the lemma, let us compute Kh($D'$). To do this we will use the long exact sequence of Khovanov homology with respect to the crossing marked as on Figure 12. The 0-smoothing is the torus link $T(n - 2, 2)$ with $c(D_{0}) = 0$. On the other hand, $D'$ is a diagram of the unknot with $c(D') = n - 2$.

Consider first the case $n = 2k$, for $k > 1$. In the respective long exact sequence almost all terms corresponding to Kh($D'$) vanish. There are only two non-vanishing terms. Further inspection of the long exact sequence yields that there can be only one possibly non-vanishing morphism in this sequence.

$$\mathbb{Q} = \text{Kh}^{k-2,6k-6}(T(2k - 2, 2)) \rightarrow \text{Kh}^{0,-1}(U) = \mathbb{Q}$$

Suppose that $s = 0$ and notice that if $n = 2k$, then $D'$ represents a knot. It is not hard to see, that the Khovanov homology of this knot is concentrated only on two diagonals $j = 2k - 3, 2k - 1$, regardless of whether $s$ vanishes or not. Further, if $s$ vanishes, then

$$\text{KhP}(D') = q\text{KhP}(T(2k - 2, 2)) + t^{2k-1}q^{6k-7} + t^{2k-1}q^{6k-5}. \tag{7}$$

On the other hand, [Lee05, Thm. 4.4] and [Ras10, Prop. 3.3] imply that

$$\text{KhP}(D') \equiv q^{s(D')}(q + q^{-1}) \pmod{(1 + tq^4)}, \tag{8}$$

for some integer $s(D')$. However, from (7) it follows that

$$\text{KhP}(D') \equiv q^{2k-3} + q^{2k-1} + q^{-5k+8}(q^{6k-7} + q^{6k-5})(1 - q^{-4}) \pmod{(1 + tq^4)},$$

which contradicts (8). Thus, $s$ must be non-trivial.

If $n = 2k + 1$, there is also only one case to consider. Namely

$$Q = \text{Kh}^{k-1,6k-3}(T(2k - 1, 2)) \rightarrow \text{Kh}^{0,-1}(U) = Q.$$

Notice that $D'$ represents a 2-component link, whose Khovanov homology is concentrated on two diagonals. Therefore, analogously as in the previous case, [Lee05, Thm. 4.4] and [Ras10, Prop. 3.3] imply that

$$\text{KhP}(D') \equiv q^{s}(q + q^{-1}) + t^{\ell}q^{s}(q + q^{-1}) \pmod{(1 + tq^4)}, \tag{9}$$

where $\ell$ denotes the linking number of the components of $D'$. Argument analogous as in the even case yields that now $s$ must vanish. \qed
Proof of Thm. 6.4. The proof is inductive. The first case was done in Example 6.2.

Consider first $T(2n + 1, 2)$. From lemmas 6.5, 6.6 and 6.7 we can derive the $E_1$ page spectral sequence $2E_1^{*,*,*}$, which is depicted on Figures 13. In order to finish the computation we need to apply Proposition 4.16 and Proposition 6.3. Since

$$\text{Kh}^{2n+1,*}(T(2n + 1, 2)) = \mathbb{Q}\{6n + 3\},$$
$$\text{Kh}^{2n,*}(T(2n + 1, 2)) = \mathbb{Q}\{6n - 1\},$$

it follows easily from Proposition 6.3 that the differential

$$d^{0,2n}_1 : 2E_1^{0,2n,*} \to 2E_1^{1,2n,*}$$

is an isomorphism. Analogous comparisons of grading of $2E_1^{1,k}$ and $\text{Kh}^{k+1,*}(T(2n + 1, 2))$ yield that $2E_2^{*,*,*}$ is zero. Thus,

$$\text{Kh}_{Z/2}^{*,*,1}(T(2n + 1, 2)) = 0,$$

and consequently

$$\text{Kh}_{Z/2}^{*,*,2}(T(2n + 1, 2)) = \text{Kh}^{*,*}(T(2n + 1, 2)).$$

Consider now $T(2n, 2)$. The $E_1$ page of the spectral sequence is presented on Figure 13. Comparison of gradings of $2E_1^{*,*,*}$ and gradings of $\text{Kh}^{*,*}(T(2n - 2, 2))$ yields that the only non-zero entry of $2E_2^{*,*,*}$ is

$$2E_2^{1,2n-1,6n} = \mathbb{Q}.$$

Therefore,

$$\text{Kh}_{Z/2}^{*,*,2}(T(2n, 2)) = \mathbb{Q}[2n]\{6n\}.$$

References

[BN05] D. Bar-Natan, Khovanov’s homology for tangles and cobordisms, Geom. Topol. 9 (2005), 1443–1499. MR2174270 (2006g:57017)

[Bro94] K. S. Brown, Cohomology of groups, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339 (96a:20072)

[Chb02] N. Chbili, The quantum $SU(3)$ invariant of links and Murasugi’s congruence, Topology Appl. 122 (2002), no. 3, 479–485. MR1911695 (2003d:57022)

[Chb10] ______, Equivalent Khovanov homology associated with symmetric links, Kobe J. Math. 27 (2010), no. 1-2, 73–89. MR2779238 (2012d:57015)

[CR90] C. W. Curtis and I. Reiner, Methods of representation theory. Vol. I, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1990. With applications to finite groups and orders, Reprint of the 1981 original, A Wiley-Interscience Publication. MR1038525 (90k:20001)

[DL91] J. F. Davis and C. Livingston, Alexander polynomials of periodic knots, Topology 30 (1991), no. 4, 551–564. MR1133872 (92k:57008)

37
Figure 13: \(2E_1^{**} \) of \(T(2n + 1, 2)\).

Figure 14: \(2E_1^{**} \) of \(T(2n, 2)\).
