The Second Virial Coefficient of Spin-1/2 Anyon System

"Sahng-kyoon Yoo and D. K. Park

*Department of Physics, Seonam University, Namwon 590-711, Korea

bDepartment of Physics, Kyungnam University, Masan 631-701, Korea

(August 13, 2018)

Abstract

We calculate the second virial coefficient of spin-1/2 anyon gas in the various values of the self-adjoint extension parameter by incorporating the self-adjoint extension method into the Green’s function formalism. Especially, the completely different cusp- and discontinuity-structures from the result of previous literature are obtained when the self-adjoint extension parameter goes to infinity. This is originated from the different condition for the occurrence of irregular states.
Since the Kronig-Penny model \[1\] described successfully the band structure of energy spectrum in the solid-state physics, the point interaction problem has been applied to the various branches of physics for a long time. Recent application of the point interaction in the theoretical physics seems to be concentrated upon the subjects related to the quantum mechanical renormalization \[2,3\] and anyonic theories. \[1\]

The frequent use of the point interaction in the quantum mechanical renormalization is mainly due to its advantage of permitting the derivation of the exact solution. Hence, the comparison of an exact solution with a perturbative solution, which can be obtained by solving the Lippmann-Schwinger equation iteratively, allows us to understand the subtleties of renormalization scheme encountered frequently in the quantum field theories. It may be also very helpful to understand the highly non-trivial concepts like dimensional transmutation \[5\] and anomalies \[3\].

In spin-1/2 Aharonov-Bohm problem \[7,8\], which is directly related to the fermion-based anyonic theory, the two-dimensional point interaction is realized as a Zeeman interaction of the spin with a magnetic flux tube. The two different approaches, renormalization and self-adjoint extension \[9\], for the quantum mechanical point interaction and the coincidence of their results are presented by Jackiw \[10\]. In Ref. \[11\] one of us analyzed the incorporation of the self-adjoint extension into the Green’s function formalism by solving the Lippmann-Schwinger equation nonperturbatively.

One of the way to investigate the statistical properties of the spin-1/2 anyon system is to evaluate the second virial coefficient as a function of a statistical parameter. The second virial coefficient of the Boson-based anyonic system is firstly calculated by Arovas et al. \[12\]. The most interesting feature of their result is that the virial coefficient interpolates between the Bose and Fermi values with a periodic dependence on the flux \(\alpha\) carried by anyon. Subsequently, the second virial coefficient for a system of spin-1/2 anyon gas is computed by Blum et al. \[13\] by using the condition of the irregular solution

\[
|m + \alpha| < 1,
\]
where \( m \) is the angular momentum quantum number and \( s \) is \( \pm 1 \) for spin up and spin down, respectively. Their striking result is that there exist discontinuities in the virial coefficient at all even, nonzero values of \( \alpha \).

In this Letter we compute the second virial coefficient for a spin-1/2 anyon gas by incorporating the self-adjoint extension method into the Green’s function formalism. In this case, it is well-known that the arbitrary combination of the regular and irregular solution, which is characterized by the self-adjoint extension parameter, is derived when \( | m + \alpha | < 1 \). We expect the absence of the latter condition in Eq.(1) for the appearance of the irregular solution in the self-adjoint extension method does yield completely different cusp- and discontinuity-structure in the virial coefficient from the known in the previous literatures. The evaluation is performed at various self-adjoint extension parameter \( \lambda_m \). In particular, we find new discontinuities at all non-zero integer values of flux when \( \lambda_m \to \infty \). When, however, \( \lambda_m \) becomes finite, the discontinuities turn into cusps. This might be an existence of different kind of phase transitions from the previous cases.

In general, the second virial coefficient can be calculated by making an energy spectrum discrete. Therefore, we introduce the simple harmonic oscillator potential in the Hamiltonian

\[
H = H_0 + v\delta(r),
\]

where

\[
H_0 = \frac{1}{2M}(\mathbf{p} - e\mathbf{A})^2 + \frac{M\omega^2}{2}r^2.
\]

The energy-dependent Green’s function \( \hat{G}[\mathbf{r}_1, \mathbf{r}_2 : E] \) for \( H \) can be derived by solving the Lippmann-Schwinger equation nonperturbatively. Following Ref. \[\square\] one can arrive at

\[
\hat{A}[\mathbf{r}_2, \mathbf{r}_1 : E] = \frac{\hat{G}_0[\mathbf{r}_2, \epsilon_1 : E][\hat{G}_0[\mathbf{r}_2, \mathbf{r}_1 : E]]}{1 + \lim_{\epsilon_2 \to \epsilon_1} + \hat{G}_0[\epsilon_2, \epsilon_1 : E]},
\]

where

\[
\hat{A}[\mathbf{r}_2, \mathbf{r}_1 : E] \equiv \hat{G}[\mathbf{r}_2, \mathbf{r}_1 : E] - \hat{G}_0[\mathbf{r}_2, \mathbf{r}_1 : E]
\]
and $\hat{G}_0[\mathbf{r}_2, \mathbf{r}_1 : E]$ is energy-dependent Green’s function for $H_0$. Since $\hat{G}[\mathbf{r}_2, \mathbf{r}_1 : E] = \sum_n \phi_n^*(\mathbf{r}_2)\phi_n(\mathbf{r}_1)/(E + E_n)$, $\hat{G}$ and $\phi$ must obey the same boundary condition, which is given by self-adjoint extension method at the origin. In Ref. [11] bound state energy is derived by imposing the boundary condition to $\hat{A}[\mathbf{r}_2, \mathbf{r}_1 : E]$ instead of $\hat{G}[\mathbf{r}_2, \mathbf{r}_1 : E]$. In this case the bound state spectrum can be obtained by solving the equation

$$-\frac{1}{\lambda_m} = (M\omega)^{|m+\alpha|} \frac{\Gamma(1 - |m + \alpha |)}{\Gamma(1 + |m + \alpha |)} \frac{\Gamma \left( \frac{1+|m+\alpha|+E/\omega}{2} \right)}{\Gamma \left( \frac{1-|m+\alpha|+E/\omega}{2} \right)},$$

when $|m+\alpha| < 1$. Here, $\lambda_m$ is the self-adjoint extension parameter. If $\hat{G}_0[\mathbf{r}_2, \mathbf{r}_1 : E]$ does not have any pole, the imposition of the boundary condition to $\hat{A}[\mathbf{r}_2, \mathbf{r}_1 : E]$ is physically relevant because the most contribution to $\hat{G}[\mathbf{r}_2, \mathbf{r}_1 : E]$ is $\hat{A}[\mathbf{r}_2, \mathbf{r}_1 : E]$ at bound state energies. If, however, $\hat{G}_0[\mathbf{r}_2, \mathbf{r}_1 : E]$ does have poles, we can not conclude a priori the relevance of the above procedure. In order to get the credibility in this case, the poles in $\hat{G}_0[\mathbf{r}_2, \mathbf{r}_1 : E]$ must not contributed to the poles in $\hat{G}[\mathbf{r}_2, \mathbf{r}_1 : E]$ at the final stage. We confirmed by explicit calculation that this is indeed the case in our model. Furthermore, imposing the boundary condition to $\hat{G}[\mathbf{r}_2, \mathbf{r}_1 : E]$, one arrives at the same result.

When $|m+\alpha| > 1$, no irregular solution occurs, and therefore, the bound state energies are given by

$$\epsilon_{n,m} = (2n + 1 + |m + \alpha|)\omega,$$

where $n$ is a non-negative integer and $m$ is an integer. We first consider the two special cases in Eq.(6): $\lambda_m = 0$ and $\lambda_m = \infty$. In both cases, we can obtain the analytic solutions for the bound state spectrum. For the former case, the bound state energies coincide with Eq.(7), which means that there is no irregular states even at $|m + \alpha| < 1$. (hard-core repulsion potential exists.) For the latter case, however, the bound state energies are given by

$$\epsilon_{n,m} = (2n + 1 - |m + \alpha|)\omega.$$
self-adjoint extension method. We plot the energy \( \epsilon_{0,0} \) for \( \lambda_m = \infty \) as a function of \( \alpha \) in Fig. 1. In contrast to Ref. [13], there are two discontinuities at \( \alpha = -1 \) and +1, respectively, and one cusp at \( \alpha = 0 \). This will probably imply the different behavior of second virial coefficient from that of Blum et al.

When \( 0 < \lambda_m < \infty \), the bound state spectrum for \( |m + \alpha| < 1 \) can be found by graphical analysis. The splitting between the adjacent levels is not \( 2\omega \) unlike the two above-mentioned cases, but depends on \( \lambda_m \) and \( \alpha \). Using the energy eigenvalues obtained numerically, we now calculate the second virial coefficient

\[
B_2(T) - B_2^0(T) = -2\lambda_T^2 \sum_{n,m} \left[ e^{-\beta\epsilon_{n,m}} - e^{-\beta\epsilon_{n,m}(\alpha=0)} \right],
\]

(9)

where \( B_2^0(T) \) is the second virial coefficient of free Fermion (Boson) which is given by \( \frac{1}{4} \lambda_T^2 \). \( \lambda_T \) is the thermal de Broglie wavelength and \( \beta = 1/kT \).

First let us consider the Boson-based case. We perform the summation only over the even \( m \), and obtain

\[
B_2^B(T) - B_2^{0,B}(T) = -2\lambda_T^2 \left[ \frac{e^{-\beta\omega}}{2\sinh \beta\omega} \left[ \cosh \beta\delta\omega - 1 \right] \right.

+ \sum_{n,|m+\alpha|<1} \left[ e^{-\beta\epsilon_{n,-N}} - e^{-\beta\epsilon_{n,-N}(\alpha=0)} \right], \quad N = \text{even},

\left. \frac{e^{-\beta\omega}}{2\sinh \beta\omega} \left[ \cosh \beta(1-\delta)\omega - \cosh \beta\omega \right] \right]

+ \sum_{n,|m+\alpha|<1} \left[ e^{-\beta\epsilon_{n,-N-1}} - e^{-\beta\epsilon_{n,-N-1}(\alpha=0)} \right], \quad N = \text{odd},
\]

(10)

where \( \alpha = N + \delta \), \( N \) is an integer and \( 0 < \delta < 1 \). In the \( \omega \to 0 \) limit, the result for \( B_2^B(T)/\lambda_T^2 \) at different self-adjoint extension parameters are shown in Fig. 2. The simple numerical calculations are needed to perform the summations when \( |m + \alpha| < 1 \). We add the \( e^{-\beta\epsilon_{n,m}} \) factors up to \( n = 5 \), for which the errors are less than 1%. When \( \lambda_m = 0 \), the
result is exactly same as that of Arovas et al. [12] When \( \lambda_m \) begins to be finite, the cusps at the Fermion points begin to appear and become more deeper with increasing \( \lambda_m \). This fact agrees with the recent calculation of second virial coefficient of anyons without hardcore [16]. The appearance of cusps at the Fermion point is interpreted in Ref. [17] as follows: as the possibility of the overlap between particles is introduced, the particles becomes more Boson-like. The cusps at the Bose points still remain, at variance with the result of Ref. [16] which is unbelievable on account of the abrupt appearance of the inflection point at \( \epsilon = 10 \). As the self-adjoint extension parameters increase, the values at all points except the Bose point become smaller than the hard-core values. This can be considered as follows. The introduction of the irregular states at the origin increases the possibility of overlap between particles, so that it makes the system compressed. Increasing the portion of the irregular solution in wave function, the augmentation of the self-adjoint extension parameter decreases the second virial coefficient.

The Fermion-based case can be calculated by summing odd \( m \)'s. We consider only the unpolarized Fermion system, which is obtained by averaging over the four spin states (triplet + singlet). The virial coefficient \( B_2^F(T)/\lambda_T^2 \) in the \( \omega \rightarrow 0 \) is plotted in Fig. 3 for \( \lambda_m = 0, 1/5, 1 \) and \( \infty \). When \( \lambda_m = 0 \), the result without irregular solution in Ref. [13] is recovered. But for finite \( \lambda_m \), the value at Bose point jumps down suddenly to the negative value and the second virial coefficient decreases with increasing \( \lambda_m \) as in the Boson-based case. As \( \lambda_m \rightarrow \infty \) (the case of Blum et al [13]), there exist two kinds of discontinuity at both Boson and Fermion points, which shows the different behavior from Ref. [13]. This means that it is some different phase transitions from those of Blum et al. The origin of this difference comes from the different condition for the occurrence of irregular states. As \( \lambda_m \) decreases, discontinuities turn into cusps at integer points and as \( \lambda_m \rightarrow 0 \), the result is reduced to the case in which the irregular solution is ignored.

In conclusion, using the self-adjoint extension method and imposing the boundary conditions at the origin, we calculate the second virial coefficients for the Boson-based and Fermion-based anyon systems. Both cases show the decreasing behaviors when the self-
adjoint extension parameter $\lambda_m$ increases. This tells us that the values of parameters determine the magnitude of repulsion between two particles. Although, for the Boson-based case, the decreasing behavior also shows up in Ref. [16], the cusp structure at Bose point is completely different. It is also shown that the Fermion-based case has a different kind of discontinuity from that of Blum et al. [13], which is due to the absence of the second condition in Eq. (1). This might exhibit another kind of phase transition.
REFERENCES

[1] R. de L.Kronig and W.G.Penny, Proc. R. Soc. A130, 499 (1931).

[2] S.K.Adhikari, T.Frederico and I.D.Goldman, Phys. Rev. Lett. 74, 487 (1995); S.K.Adhikari and T.Frederico, Phys. Rev. Lett. 74, 4572 (1995).

[3] D.R.Phillips, S.R.Beane and T.D.Cohen, hep-th/9706070.

[4] Y.H.Chen, F.Wilczek, E.Witten and B.I.Helperin, Int. J. Mod. Phys. B3, 1001 (1989).

[5] C.Thorn, Phys. Rev. D19, 639 (1979).

[6] R.Jackiw, in Current Algebra and Anomalies, edited by S.B.Treiman, R.Jackiw, B.Zumino and E.Witten (World Scientific, Singapore, 1985).

[7] Ph. de Sousa Gerbert, Phys. Rev. D40, 1346 (1989).

[8] C.R.Hagen, Int. J. Mod. Phys., A6, 3119 (1991).

[9] S.Albeverio, F.Gesztesy, R.Hoegh-Krohn and H.Holden, Solvable Models in Quantum Mechanics (Springer-Verlag, Berlin, 1988).

[10] R.Jackiw, in M.A.B.Beg Memorial Volume, edited by A.Ali and P.Hoodbhoy (World Scientific, Singapore, 1991).

[11] D.K.Park, J. Math. Phys. 36, 5493 (1995).

[12] D.P.Arovas, R.Schrieffer, F.Wilczek and A.Zee, Nucl. Phys. B251, 117 (1985).

[13] T.Blum, C.R.Hagen and S.Ramaswamy, Phys. Rev. Lett. 64, 709 (1990).

[14] D.Peak and A.Inomata, J. Math. Phys. 10, 1422 (1969).

[15] The explicit boundary condition is given in Eq.(52) of Ref. [11].

[16] C.Kim, hep-th/9706184.

[17] D.Loss and Y.Fu, Phys. Rev. Lett. 67, 294 (1991).
FIGURES

FIG. 1. The bound state energies $\epsilon_{0,0}$ as a function of $\alpha$ for $\lambda_m = \infty$ case.

FIG. 2. $B^B_2(T)/\lambda_T^2$ for Boson-based anyons at various self-adjoint extension parameters $\lambda_m$.

FIG. 3. $B^F_2(T)/\lambda_T^2$ for unpolarized Fermion-based anyons at various self-adjoint extension parameters $\lambda_m$. 
\[ B_2(\alpha) / \lambda^2 \]

\[ \lambda = \infty \]
\[ \lambda = 1 \]
\[ \lambda = 1/5 \]
\[ \lambda = 1/50 \]
\[ \lambda = 0 \]
\[ B_2(\alpha) / \lambda^2 \]

Graph showing the function \( B_2(\alpha) / \lambda^2 \) for different values of \( \lambda \):
- \( \lambda = \infty \)
- \( \lambda = 1 \)
- \( \lambda = 1/5 \)
- \( \lambda = 0 \)