ON *r*th COEFFICIENT OF DIVISORS OF $x^n - 1$

SAI TEJA SOMU

**Abstract.** Let $r$, $n$ be two natural numbers and let $H(r, n)$ denote the maximal absolute value of $r$th coefficient of divisors of $x^n - 1$. In this paper, we show that $\sum_{n \leq x} H(r, n)$ is asymptotically equal to $c(r)x(\log x)^{2r-1}$ for some constant $c(r) > 0$. Furthermore, we give an explicit expression of $c(r)$ in terms of $r$.

1. Introduction

Cyclotomic polynomials are irreducible divisors of $x^n - 1$. The factorization of $x^n - 1$ is as follows,

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Applying Mobius inversion formula we get $\Phi_n(x) = \prod_{d|n}(x^d - 1)^{\mu(d)}/d$. Coefficients of $n$th cyclotomic polynomial has been the subject of study in \[1\], [2] and [3].

In [4], the study of coefficients of divisors of $x^n - 1$ has started. Recently, in [6] and [7] we proved that for every finite sequence of integers $n_1, \cdots, n_r$ there exists a natural number $n$ and a divisor $d(x)$ of $x^n - 1$ of the form $d(x) = 1 + n_1x + \cdots + n_rx^r + \sum_{j=r+1}^{\deg(d)} c_jx^j$.

Let $(f)_r$ denote $r$th coefficient of the $f(x)$ for any formal power series $f(x)$. That is, $f(x) = \sum_{i=0}^{\infty}(f)_i x^i$. In [6], the function $H(r, n) = \max\{|(d)_r| : d(x)|x^n - 1\}$ has been studied. In that paper, we worked on the maximal order of $H(r, n)$ and we showed that

$$H(r, n) \leq (1 + o(1))n^{r(\log 2 + o(1))}. $$

In this paper, we work on the average order of $H(r, n)$ for a fixed $r$. We will prove the following theorem.

**Theorem 1.1.** Let $r$ be any natural number then

$$\sum_{n \leq x} H(r, n) \sim c(r)x(\log x)^{2r-1}$$

as $x \to \infty$ where

$$c(r) = \frac{1}{(2^r - 1)!2^{r-1}} \prod_{p \text{ prime}} \left( \left( 1 + \frac{2^r - 1}{p} \right) \left( 1 - \frac{1}{p} \right)^{2^r-1} \right).$$
2. Notation

Let \( \nu(n), \tau(n) \) denote number of prime factors of \( n \) and number of divisors of \( n \) respectively. Let \( \delta(n) \) denote the function \( \delta(n) = \begin{cases} -1 & n = 1 \\ 1 & n > 1 \end{cases} \). Note that,

\[
\delta(n)\Phi_n(x) = \prod_{d|n}(1 - x^d)^{\nu(\frac{n}{d})}.
\]

We say that a formal power series \( f(x) = \sum_{n=1}^{\infty}(f_n)x^n \) is dominated by another power series \( g(x) = \sum_{n=1}^{\infty}(g_n)x^n \) if \( |(f_n)| \leq (g_n) \) for all \( n \in \mathbb{N} \). If \( f(x) \) and \( g(x) \) are two formal power series then we denote \( f(x) \equiv g(x) \mod x^{r+1} \) if the coefficients of \( x^i \) in the power series of \( f(x) \) and \( g(x) \) are equal for \( 0 \leq i \leq r \).

3. Proof of the Main Theorem

We require several lemmas in order to prove the main theorem.

**Lemma 3.1.** If \( f_1(x), \ldots, f_i(x) \) are dominated by \( g_1(x), \ldots, g_i(x) \) then \( f(x) = \prod_{j=1}^{i} f_j(x) \) is dominated by \( g(x) = \prod_{j=1}^{i} g_j(x) \).

**Proof.** For all \( j \in \{0\} \cup \mathbb{N} \), we have

\[
|(f_1 \cdot \ldots \cdot f_i)_j| = | \sum_{h_1 + \ldots + h_i = j} (f_1)_{h_1} \cdots (f_i)_{h_i} |.
\]

From triangle inequality and the fact that \( f_k(x) \) are dominated by \( g_k(x) \) we have

\[
|(f_1 \cdot \ldots \cdot f_i)_j| \leq \sum_{h_1 + \ldots + h_i = j} |(f_1)_{h_1}| \cdots |(f_i)_{h_i}|
\]

\[
\leq \sum_{h_1 + \ldots + h_i = j} (g_1)_{h_1} \cdots (g_i)_{h_i}
\]

\[
= (g_1 \cdots g_i)_j.
\]

\[\square\]

**Lemma 3.2.** There exists a constant \( c_1(r) \) depending only on \( r \) such that

\[
H(r, n) \leq \frac{1}{2^rr!}2^{\nu(n)} + c_1(r)2^{(r-1)\nu(n)}
\]

for all natural numbers \( n > 1 \).

**Proof.** The proof is similar to that of the proof of Theorem 4.1 of [9]. Let \( d(x) \) be a divisor of \( x^n - 1 \). As

\[
\max\{|(d)_r| : d(x)|x^n - 1, d(x) \in \mathbb{Z}[x]\} = \max\{|(d)_r| : d(x)|x^n - 1, d(x) \in \mathbb{Z}[x], d(0) = 1\}
\]

without loss of generality we can assume \( d(0) = 1 \). Every divisor \( d(x) \) of \( x^n - 1 \) with \( d(0) = 1 \) will be of the form \( \prod_{m \in S} \delta(m)\Phi_m(x) \) where \( S \) is a subset of set of
divisors of $n$. Now,

$$d(x) = \prod_{m \in S} \prod_{d|m} (1 - x^d)^{\mu(d/k)}$$

$$\equiv \prod_{m \in S} \prod_{d|m, d \leq r} (1 - x^d)^{\mu(d/k)} \mod x^{r+1}$$

$$\equiv \prod_{d \leq r} (1 - x^d)^{\Sigma_1(d) - \Sigma_2(d)} \mod x^{r+1}$$

where

$$\Sigma_1(d) = \sum_{\mu(\frac{d}{k}) = 1, m \equiv 0 \mod d} 1$$

and

$$\Sigma_2(d) = \sum_{\mu(\frac{d}{k}) = -1, m \equiv 0 \mod d} 1.$$ 

As $\sum_1(d) = \sum_{\mu(k)=1} 1 \leq \sum_{\mu(k)=1} 1 = 2^{\nu(n)} - 1$ and $\sum_2(d) = \sum_{\mu(k)=-1} 1 \leq \sum_{\mu(k)=-1} 1 = 2^{\nu(n)} - 1$, we can conclude that $|\sum_1(d) - \sum_2(d)| \leq 2^{\nu(n)-1}$. Therefore, $(1 - x^d)^{\sum_1(d) - \sum_2(d)}$ is dominated by $(1 - x^d)^{2^{\nu(n)-1}}$. From Lemma 3.1, it follows that

$$|\langle d(x) \rangle_r| = \left| \prod_{d \leq r} (1 - x^d)^{\sum_1(d) - \sum_2(d)} \right|_r$$

$$\leq \left( \prod_{d \leq r} (1 - x^d)^{-2^{\nu(n)-1}} \right)^r$$

$$= \left( \sum_{c_1 + 2c_2 + \cdots + rc_r = r, c_i \geq 0} \binom{2^{\nu(n)}-1 + c_1}{c_1} \cdots \binom{2^{\nu(n)}-1 + c_r}{c_r} \right)$$

As

$$\left( \binom{2^{\nu(n)}-1 + c_1}{c_1} \cdots \binom{2^{\nu(n)}-1 + c_r}{c_r} \right) = \frac{2^{\nu(n)}(c_1 + \cdots + c_r)!}{c_1! \cdots c_r!} + O(2^{\nu(n)}(c_1 + \cdots + c_r - 1)!)$$

and $c_1 + 2c_2 + \cdots + rc_r = r, c_1 \geq 0, c_1 \neq r$ implies $c_1 + \cdots + c_r \leq r - 1$ we can conclude that

$$|\langle d(x) \rangle_r| \leq \binom{2^{\nu(n)}-1 + r}{r} + \sum_{c_1 + 2c_2 + \cdots + rc_r = r, c_i \geq 0, c_1 \neq r} \binom{2^{\nu(n)}-1 + c_1}{c_1} \cdots \binom{2^{\nu(n)}-1 + c_r}{c_r}$$

$$= \frac{2^{\nu(n)}r!}{2^{\nu(n)}r!} + O(2^{\nu(n)}(r-1)!)$$

Hence there exists a constant $c_1(r)$ depending only on $r$ such that

$$|\langle d(x) \rangle_r| \leq \frac{1}{2^{\nu(n)}r!} 2^{\nu(n)} + c_1(r) 2^{(r-1)\nu(n)}.$$
Therefore,
\[ H(r, n) \leq \frac{1}{2r!} 2^{r\nu(n)} + c_1(r) 2^{(r-1)\nu(n)}. \]

\[ \square \]

**Lemma 3.3.** There exists a constant \( c_2(r) \) depending only on \( r \) such that
\[ H(r, n) \geq \frac{1}{2r!} 2^{r\nu(n)} - c_2(r) 2^{(r-1)\nu(n)} \]
for all natural numbers \( n > 1 \).

**Proof.** Consider the following divisor \( d(x) \) of \( x^n - 1 \),
\[ d(x) = \prod_{\mu(m)=-1} \delta(m) \Phi_m(x) \]
\[ = \prod_{\mu(m)=-1} \prod_{d|m} (1 - x^d)\mu(\frac{m}{d}) \]
\[ = \prod_{d \leq r} (1 - x^d)^{k(d)} \text{ mod } x^{r+1}, \]
where \( k(d) = \sum_{m|n, \mu(m)=-1} \mu(\frac{m}{d}) \).

Note that, \( k(1) = -2^{\nu(n)-1} \) and \( |k(d)| \leq 2^{\nu(n)-1} \). Therefore, (2)
\[ |(1 - x^d)^{k(d)}| \leq \left( \frac{2^{\nu(n)-1} + c_d}{c_d} \right). \]

We have,
\[ (d(x))_r = (\prod_{d \leq r} (1 - x^d)^{k(d)})_r \]
\[ = \sum_{c_1+c_2+\cdots+c_r=r, c_i \geq 0} \prod_{d=1}^r (1 - x^d)^{k(d)} \text{ mod } x^{r+1}, \]
\[ \geq \left( \frac{2^{\nu(n)-1} + r}{r} \right) - \sum_{c_1 \neq r, c_1+c_2+\cdots+c_r=r} \prod_{d=1}^r (1 - x^d)^{k(d)} \text{ mod } x^{r+1}. \]

From (2), we can conclude that
\[ (d(x))_r \geq \left( \frac{2^{\nu(n)-1} + r}{r} \right) - \sum_{c_1 \neq r, c_1+c_2+\cdots+c_r=r} \prod_{d=1}^r \left( \frac{2^{\nu(n)-1} + c_d}{c_d} \right) \]
\[ = \frac{2^{r\nu(n)}}{2r!} + O(2^{(r-1)\nu(n)}) \]

Hence there exists a constant \( c_2(r) \) such that
\[ H(r, n) \geq (d(x))_r \geq \frac{2^{r\nu(n)}}{2r!} - c_2(r) 2^{(r-1)\nu(n)}. \]

\[ \square \]

**Lemma 3.4.** (See [4], Problem 4.4.17) Let \( f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \), with \( a_n = O(n^\epsilon) \). Suppose that \( f(s) = \zeta(s)^k g(s) \), where \( k \) is a natural number and \( g(s) \) is a Dirichlet
series absolutely convergent in $\text{Re}(s) > 1 - \delta$ for some $0 < \delta < 1$. We have

$$
\sum_{n \leq x} \alpha_n \sim \frac{g(1)x(\log x)^{k-1}}{(k-1)!}
$$

as $x \to \infty$.

Proof. See page 301 of [4] for the proof of the lemma. \qed

Lemma 3.5. For $r \in \mathbb{N}$, we have

$$
\sum_{n \leq x} 2^{\nu(n)r} \sim \prod_{p \text{ prime}} \left( 1 + \frac{2^r - 1}{p} \right) \left( 1 - \frac{1}{p^s} \right)^{2^r-1} x(\log x)^{2^r-1}.
$$

Proof. Let

$$
f(s) = \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^s} = \prod_{p \text{ prime}} \left( 1 + \frac{2^r - 1}{p} \right) \left( 1 - \frac{1}{p^s} \right)^{2^r-1}.
$$

As $2^{\nu(n)} \leq (\tau(n))^r = O(n^r)$ we have $2^{\nu(n)} = O(n^r)$. Let

$$
g(s) = \prod_{p \text{ prime}} \left( 1 + \frac{2^r - 1}{p^s} \right) \left( 1 - \frac{1}{p^s} \right)^{2^r-1} = \prod_{p \text{ prime}} \left( 1 + \frac{d_2}{p^{2^s}} + \frac{d_3}{p^{3^s}} + \cdots + \frac{d_{2^r-1}}{p^{(2^r-1)^s}} \right)
$$

for some constants $d_2, d_3, \ldots, d_{2^r-1}$. Notice that Dirichlet’s series of $g(s)$ converges absolutely for $\text{Re}(s) > 1/2$. Observe that $f(s) = \zeta(s)^{2^r} g(s)$. Applying Lemma 3.4, we have

$$
\sum_{n \leq x} 2^{\nu(n)r} \sim g(1) \frac{x(\log x)^{2^r-1}}{(2^r - 1)!}
$$

which completes the proof of the lemma. \qed

Now, we are ready to prove our main theorem.

Proof. From lemmas 3.2 and 3.3 we have,

$$
\frac{1}{2^rr!} \sum_{1 < n \leq x} 2^{\nu(n)r} - c_2(r) \sum_{1 < n \leq x} 2^{\nu(n)(r-1)} \leq \sum_{1 < n \leq x} H(r, n) \leq \frac{1}{2^rr!} \sum_{1 < n \leq x} 2^{\nu(n)r} + c_1(r) \sum_{1 < n \leq x} 2^{\nu(n)(r-1)}.
$$

Hence from Lemma 3.5, we get

$$
\sum_{n \leq x} H(r, n) \sim c(r)x(\log x)^{2^r-1},
$$

where

$$
c(r) = \frac{1}{(2^r - 1)!2^rr!} \prod_{p \text{ prime}} \left( 1 + \frac{2^r - 1}{p} \right) \left( 1 - \frac{1}{p} \right)^{2^r-1}.
$$

\qed
References

[1] G. Bachman, *On the coefficients of cyclotomic polynomials*, Mem. Amer. Math. Soc., 106, no. 510 (1993).

[2] P. T. Bateman, C. Pomerance, and R. C. Vaughan, *On the size of the coefficients of the cyclotomic polynomial*, Topics in classical number theory, Vol. I, II (Budapest, 1981), Colloq. Math. Soc. Janos Bolyai, vol. 34, North-Holland, Amsterdam, 1984, pp. 171-202.

[3] D. M. Bloom, *On the coefficients of the cyclotomic polynomials*, Amer. Math. Monthly, 75, 372-377 (1968).

[4] M.R. Murthy, *Problems in Analytic Number Theory*, Graduate Texts in Mathematics, Springer, 2001.

[5] C. Pomerance, N.C. Ryan, *Maximal height of divisors of $x^n - 1$*, Illinois J. Math. 51 (2007) 597-604.

[6] S.T. Somu, *On the coefficients of divisors of $x^n - 1$*, Journal of Number Theory, Volume 167, October 2016, 284-293.

[7] S.T. Somu, *On the distribution of numbers related to the coefficients of divisors of $x^n - 1$*, Journal of Number Theory, Volume 170, January 2017, 3-9.

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India 400005