Brane embeddings in sphere submanifolds

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Abstract
Wrapping a D(8−p)-brane on AdS₂ times a submanifold of S⁸−p introduces point-like defects in the context of AdS/CFT correspondence for a Dp-brane background. We classify and work out the details in all possible cases with a single-embedding angular coordinate. Brane embeddings of the temperature and beta-deformed near-horizon D3-brane backgrounds are also examined. We demonstrate the relevance of our results to holographic lattices and dimers.

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(Some figures may appear in colour only in the online journal)

1. Prolegomena

When branes wrap internal manifolds, they have the tendency to shrink. However, they can be stabilized by turning on a world-volume gauge field with a quantized flux. A pioneering example of such flux stabilization of D-branes was presented in [1] for the wrapping of a probe D2-brane inside an SU(2) group manifold, and a connection with results from an exact CFT approach was also made. There have been numerous works in the literature considering brane embeddings in various backgrounds and dimensions [2–7] (and references therein). In particular, the authors of [4] considered configurations where a D(8−p)-background wraps an S⁷−p inside an S⁸−p in the background of a Dp-brane. The stabilization occurs at quantized values of the equatorial angle of the bigger sphere. This result, besides being aesthetically beautiful, is also relevant in a holographic approach to condensed matter lattices and dimers [8].

Motivated by these works we realized that for generic values of p there are more embedding possibilities even when one considers the simplest case of one embedding coordinate. Instead of S⁷−p, one could also select other submanifolds of S⁸−p whose isometry groups are essentially given by subgroups of SO(9 − p), the latter being the isometry group of S⁸−p. These submanifolds are presented in table 1 for p = 0, 1, . . . , 5. The coloring is introduced for later convenience.

This paper is organized as follows. In section 2, we minimize the action of the brane probe and calculate the semi-classical energy for each one of the aforementioned configurations. In
general, the energy depends on the ratio of the flux units \( n \) of the world-volume gauge field to the number of the Dp-branes \( N \) that we stack together to form the background. For a given value of \( p \), these energies depend on the specific submanifold that is wrapped. We have checked by using kappa symmetry that the embeddings in consideration are supersymmetric, hence ensuring stability.

In section 3, we present brane embeddings in \( \beta \)-deformed backgrounds [9]. In this case, it turns out that the \( \gamma \)-dependence of the deformation drops out completely in the probe computation. Pertaining to the \( \sigma \)-deformation, which involves an S-duality, we formulated the problem mathematically, but we were not able to find minimal configurations explicitly due to its complexity.

In section 4, we turn on the temperature and examine its effect on the stability of our constant embeddings. We conclude, by considering a small fluctuation analysis, that these are perturbatively stable, even though there is no underlying supersymmetry. In section 5, we apply our results in the context of holographic lattices and dimers. We show that the free energy and, hence, the physical behavior of the systems are sensitive in a simple manner to the different wrappings we have constructed. Finally, in section 6, we present concluding remarks and comment on future directions.

2. Brane embeddings in Ramond–Ramond backgrounds

The geometry created by a stack of \( N \) coincident Dp-branes in the near-horizon region is described by the ten-dimensional metric [10]

\[
\mathrm{d}s^2 = \left( \frac{r}{R} \right)^{\frac{7-p}{2}} (- \mathrm{d}r^2 + \mathrm{d}r^2 + \frac{R}{r^3} (\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{7-p})),
\]

where \( \mathrm{d}\Omega^2_{7-p} \) is generally the line element of a unit \( p \)-sphere and the parameter \( R \) is given by

\[
R^{7-p} = N g_s 2^{5-p} \pi^{\frac{7-p}{2}} (\alpha')^{\frac{7-p}{2}} \Gamma \left( \frac{7-p}{2} \right).
\]

The background is also supported by a dilaton \( \Phi(r) \) and a nonzero Ramond–Ramond (RR) field strength \( F_{(8-p)} \) given by

\[
e^{-\Phi(r)} = \left( \frac{R}{r} \right)^{\frac{7-p}{2}},
\]

\[
F_{(8-p)} = (7-p)R^{7-p} \text{Vol}(S^{(8-p)}) = dC_{(7-p)},
\]

where \( \text{Vol}(S^{(8-p)}) \) denotes the volume form of the unit \( p \)-sphere and \( C_{(7-p)} \) is the RR potential. We split the \( (8-p) \) spherical coordinates as \( (\theta, \phi_1, \ldots, \phi_{7-p}) \) and let \( \theta \) and \( x_i \) be the embedding coordinates of the probe brane.

We concentrate first to the cases corresponding to the entries of the table 1 that involve solely spheres. For these cases, the metric of the compact space will have the form

\[
\mathrm{d}\Omega^2_{8-p} = \mathrm{d}\theta^2 + \cos^2 \theta \, \mathrm{d}\Omega^2_q + \sin^2 \theta \, \mathrm{d}\Omega^2_{7-p-q}, \quad q = 0, 1, \ldots, \left[ \frac{7-p}{2} \right].
\]

Table 1. Submanifolds of \( S^{8-p} \).

| \( p \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) |
|---|---|---|---|---|---|
| 0 | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) |
| 1 | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) |
| 2 | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) |
| 3 | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) |
| 4 | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) |
| 5 | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) | \( S^7 \times S^3 \times S^4 \) |
This parametrization of the metric corresponds to splitting the \(9 - p\) representation of the symmetry group \(SO(9 - p)\) under the subgroup \(SO(q + 1) \times SO(8 - p - q)\) as \((9 - p) \rightarrow (q + 1, 1) \oplus (1, 8 - p - q)\). The variable \(\theta \in [0, \pi]\), unless \(q = 0\) in which case \(\theta \in [0, \pi/2]\). Consequently, the RR potential will be written as
\[
 C_{(7-p)} = R^{7-p} f(\theta) \omega_{(7-p)},
\]
where
\[
 \omega_{(7-p)} = d\text{Vol}(\mathbb{S}^q) \wedge d\text{Vol}(\mathbb{S}^{7-p-q}) = \sqrt{\tau} d\phi_1 \wedge d\phi_2 \ldots d\phi_{7-p}.
\]
The corresponding volume is
\[
 \mathcal{V}_M = \int_{\mathcal{M}} d^7\tau \sqrt{\hat{h}} = \frac{4\pi^{3/2}}{\Gamma(\frac{1+q}{2})\Gamma(\frac{8-p-q}{2})}.
\]
For \(q = 0\), we should divide this formula by 2 since the general expression for \(\text{Vol}(\mathbb{S}^q)\) gives 2 for \(q = 0\). The function \(f(\theta)\) is given for \(q \neq 0\) by
\[
f(\theta) = \frac{7 - p}{8 - p - q} \sin^q \theta F_2 \left( \frac{1 - q}{2}, \frac{4 - p + q}{2}, \frac{5 - p + q}{2}, \sin^2 \theta \right),
\]
and for \(q = 0\) by
\[
f(\theta) = 2^{8-p} \frac{7 - p}{8 - p} \sin^q \theta F_2 \left( \frac{p}{2}, \frac{3 - p}{2}, \frac{5 - p}{2}, \sin^2 \theta \right).
\]
This difference originates from the two different ranges that the angular variable \(\theta\) takes, as mentioned above. Note also that this is not the most general form for the RR potential, but it is the only one consistent for the particular embedding that we will consider in this paper.

The D(8−p)-brane probe is described by the sum of a Dirac–Born–Infeld (DBI) and a Wess–Zumino (WZ) term
\[
 S = - T_{8-p} \int d^9\tau \sigma e^{-\Phi} \sqrt{-\det(\hat{g} + F)} + T_{8-p} \int C_{(7-p)} \wedge F,
\]
where \(\hat{g}\) is the induced metric on the brane, \(F\) is an Abelian gauge field strength living on the world volume of the brane and
\[
 T_{8-p} = (2\pi)^{p-8} (\alpha')^{\frac{p-8}{2}} (g_s)^{-1}
\]
is the tension of the probe brane. The integration is performed over the world-volume coordinates of the brane, which are taken to be \(\sigma = (t, r, \phi_1, \ldots, \phi_{7-p})\). In general, the embedding coordinate may depend on any world-volume coordinate. Here, we shall restrict ourselves to the case where \(\theta\) depends only on the radial coordinate, which is also consistent with the form of the RR potential (5). Since the WZ term acts as a source term for the Abelian gauge field strength \(F\), the latter one is constrained to be \(F = F_{\tau r} \wedge dr\). We also set the spacelike world-volume coordinates \(x_{\parallel}\) to constants, which is consistent with their equations of motion.

Given the above conditions, the probe brane action assumes in general the form
\[
 S = \int_{\mathcal{M}} d^7\tau \phi \int dt \sqrt{\mathcal{L}(\theta, F)},
\]
where the Lagrangian density is computed to be
\[
 \mathcal{L}(\theta, F) = - T_{8-p} R^{7-p} \sqrt{\hat{h}} \left[ \frac{f(\theta)}{7-p} \sqrt{1 - F_{\tau r}^2 + r^2 \theta'^2} - f(\theta) F_{\tau r} \right].
\]
By varying the Lagrangian density with respect to the world-volume gauge potentials $A_{ir}$ and $A_r$, one observes that

$$\frac{\partial L}{\partial F_{ir}} = \text{const.} \quad (14)$$

Then, it turns out that the gauge field assumes the form

$$F_{ir} = \frac{f''(\theta)}{\sqrt{(7 - p)^2(f'(\theta))^2 + (f''(\theta))^2}} \quad (15)$$

In order to attribute physical meaning to this constant, we consider the coupling of our system to fundamental strings [4]. This is achieved by replacing $F$ with $F - B$ in $L$, where $B$ is the Kalb–Ramond field. By expanding, at first order in $B$, we pick out a term of the form

$$\int_{\mathcal{M}} d^7r \phi \int dt dr \frac{\partial L}{\partial F_{ir}} B_{tr}. \quad (16)$$

We can interpret the coefficient in front of $B_{tr}$ as a charge ($n$ units of $T_f$) that multiplies the Kalb–Ramond potential of the fundamental string. Therefore, the fundamental strings ‘feel’ a potential in this background, whose strength is proportional to their number $n$, and their tension $T_f = 1/(2\pi\alpha')$. Consequently, one writes

$$\int_{\mathcal{M}} d^7r \phi \frac{\partial L}{\partial F_{ir}} B_{tr} = nT_f, \quad n \in \mathbb{Z}. \quad (17)$$

In order to find semi-classical minima of the embeddings, solving the equations of motion arising from the Lagrangian density would suffice. However, since we are also interested in computing the energies of our configurations, we will obtain the minima through the Hamiltonian procedure. By performing a Legendre transformation, which actually removes the WZ part, the Hamiltonian of the system is given by

$$H = \int_{\mathcal{M}} d^7r \phi \int dt dr \left[ \frac{\partial L}{\partial F_{ir}} F_{tr} - L \right]. \quad (18)$$

Using the explicit form of the Lagrangian (13) and the quantization condition (17), the Hamiltonian becomes

$$H = \lambda NT_f \int dt dr \left[ \frac{1}{1 + r^2 \dot{\theta}^2} \sqrt{\left(\frac{f'(\theta)}{7 - p}\right)^2 + \left(\nu \lambda^{-1} - f(\theta)\right)^2} \right], \quad (19)$$

where we have defined

$$\nu = \frac{n}{N}, \quad \lambda = \frac{\Gamma\left(\frac{7-p}{2}\right)}{\Gamma\left(\frac{8-p-q}{2}\right) \Gamma\left(\frac{1+q}{2}\right)}. \quad (20)$$

Since the origin of the constant $\lambda$ is $\gamma_{\lambda}$, it turns out that, for reasons explained below (7), for $q = 0$, we should divide the above formula by $2$. It is obvious from the expression for $H$ that it is consistent to look for constant $\theta$ configurations, since in this case the $r$-dependence drops out. Setting $\theta' = 0$ and requiring $\partial H/\partial \theta = 0$ gives the condition

$$f''(\theta) = (7 - p)^2 \left(\nu \lambda^{-1} - f(\theta)\right). \quad (21)$$

As one can see in table 2, in some cases, depending on the specific values for $p$ and $q$, this equation admits an exact solution $\theta(\lambda)$, but in general it can only be solved numerically. In the rest of the paper, in order to avoid a plethora of symbols, we will denote by $\theta$ the solution of (21). The energy density is defined by

$$H = \int dr \mathcal{E}. \quad (22)$$
For general values of $p$ and $q$, it is given by

$$E_{p,q} = \lambda NT_f \sqrt{(\cos \theta)^{2q} (\sin \theta)^{2(p-1)q} + (v\lambda^{-1} - f(\theta))^2}. \quad (23)$$

For the case where a one-cycle is manifest, i.e. $q = 1$, the above formula, as well as the one for the minima, has a much simpler form given by

$$\sin \theta = \left(\frac{7 - p}{6 - p}\right)^{1/2}, \quad E_{p,1} = NT_f \sqrt{v^2 + (\sin \theta)^{12-2p-2} - 2v(\sin \theta)^{3-p}}. \quad (24)$$

Noting that $f(0) = 0$ and $f\left(\frac{\pi}{2}\right) = \lambda^{-1}$, we find the limiting behaviors

$$E_{p,q} = nT_f + O(v^2), \quad E_{p,q} = (N-n)T_f + O(1-v)^2. \quad (25)$$

The results of our computations regarding the minima and the corresponding energies are summarized in the table 2. In all cases, the angle $\theta$ ranges from 0 to $\pi/2$. We have also included two more cases, apart from the products of spheres, which arise for odd values of $p$, by writing the $S^{8-p}$ as a $U(1)$ bundle over $CP^{4-p}_1$. We use the conventions of [11] and [12] for the $CP^{2}$ and $CP^{3}$, respectively. The normalizations for the metrics are such that $R_{\mu\nu} = \frac{16}{p+1}g_{\mu\nu}$ (for the values $p = 1$ and $p = 3$ that are of interest to us).

We should clarify two subcases of the above table. First, the results for the $CP^2$ and the $S^1 \times S^3$ submanifolds coincide. This happens because the wrapping in the first case involves the $U(1)$ fiber with group structure $S^1$ and a submanifold inside $CP^2$, which has a similar structure with $S^3$. The same happens with the results for the $CP^3$ and the $S^1 \times S^3$ submanifolds. Second, for $p = 5$, we have the solutions $\theta = 0$ and $\theta = \frac{\pi}{2}$, which correspond to the collapse of the D-brane at the poles of the 3-sphere, thus rendering them singular. We also note that we omit the respective equations that give the minima and energies for the submanifolds $S^{7-p} \subset S^{8-p}$, which can be found in [4].

1 Our embedding coordinate $\theta(r)$ in these cases is identified with the coordinates $\chi$ and $\mu$ in equations (5) and (4.1) in [11] and [12], respectively.
Having obtained the energies for the various values for $p$, we plot them together with the energies found in [4] in figures 1–5. The colors (black, blue, purple and red) correspond to the entries with the same colors in table 1. The energies are plotted as the functions of the ratio $\nu$, in units of $NT_f$. Curves with the same value for $p$, but a different one for $q$, might intersect. We also use the obvious notation $(q \perp q', \nu)$.

We observe, from figures 1–5, that for a given value of $p$, the maximally symmetric submanifolds corresponding to $q = 0$ have the lowest energy. When the submanifold in consideration includes an $S^1$ or a $\mathbb{CP}$ space, the ratio $\nu$ cannot exceed the value $\frac{6 - p}{7 - p}$ found from (24). At this maximum value, the corresponding value of the energy density is $\frac{6 - p}{7 - p}$.

2.1. Kappa symmetry

We now turn our attention to studying the portion of supersymmetry preserved by our embeddings, by examining the kappa symmetry. We will briefly present results for the D3-brane background case; the others follow in a similar manner. In order to have
supersymmetric configurations of a Dp-brane probe in a given background, the following condition [13–15]:

$$\Gamma_{\kappa} \epsilon = \epsilon$$

should be satisfied. Here, $\epsilon$ is the Killing spinor of the background which, for the maximally supersymmetric case of the $\text{AdS}_5 \times \mathbb{S}^5$ background, is unconstrained. We shall also put the two Majorana–Weyl spinors of type-IIB theory into a doublet of chiral spinors, which transforms as a vector under $SO(2)$:

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}.$$  

(27)

The kappa symmetry operator $\Gamma_{\kappa}$ for the case of type-IIB theory is defined by

$$\Gamma_{\kappa} = \frac{1}{\sqrt{-\det(g + F)}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \gamma^{m_1 n_1 \cdots m_n n_n} F_{m_1 n_1} \cdots F_{m_n n_n}$$

$$\times \ (\sigma_3)^{n+1} (i\sigma_2) \otimes \Gamma_{(0)}.$$  

(28)
Here, $\hat{g}$ is the induced metric on the Dp-brane and
\[
\mathcal{F} = F - P[B],
\]
where $F$ is the world-volume gauge field and $P[B]$ is the pullback of the Kalb–Ramond $B$ field. $\gamma_m$ denotes the induced world-volume gamma matrices defined as $\gamma_m = \partial_m \sigma^a e_a^A \Gamma_A$, with $(m, n)$ and $A$ being curved and flat indices, respectively, while $\Gamma_A$ are the ten-dimensional flat Dirac matrices. Finally, $\Gamma_0$ is defined by
\[
\Gamma_0 = \frac{1}{(p + 1)!} \epsilon^{a_1 \ldots a_{p+1}} \gamma_{a_1 \ldots a_{p+1}},
\]
and $\sigma_i$ are the Pauli matrices, which act in the usual way on the doublet of chiral spinors.

We next check the kappa symmetry for each one of the embeddings that we considered in the previous section. We consider first the kappa symmetry for the $S^2 \times S^2$ case. Note that $F$ here is just the world-volume field strength, since there is no Kalb–Ramond field in our background. Hence, the infinite sum in (28) has only two terms, since only the $F_{tr}$ components of the field strength have been turned on. We compute that
\[
\Gamma_0 = \frac{1}{\sqrt{1 - F_{tr}^2 + r^2 \theta'^2}} (\sigma_1 \otimes P_1 P_2 + i F_{tr} \sigma_2 \otimes P_2),
\]
where we have defined the commuting operators
\[
P_1 = \Gamma_{01} + r \theta' \Gamma_{05}, \quad P_2 = \Gamma_{6789}.
\]
Consistency requires that $\Gamma_0 \Gamma_0 = 1$, which is easily proven. Breaking into components, one arrives at the following consistent algebraic system:
\[
(P_1 P_2 + F_{tr} P_2) \epsilon_2 = \sqrt{1 - F_{tr}^2 + r^2 \theta'^2} \epsilon_1,
\]
\[
(P_1 P_2 - F_{tr} P_2) \epsilon_1 = \sqrt{1 - F_{tr}^2 + r^2 \theta'^2} \epsilon_2.
\]
We conclude that half of the components of the Killing spinor are related to the other half, and hence, the configuration we considered here retains $1/2$ of the original supersymmetry.

The cases of the $S^1 \times S^3$ and $\mathbb{CP}^2$ wrappings are similar to the one above, and hence, the results will not be presented here. They are also found to be $1/2$ BPS configurations and similarly for other cases with $p = 0, 1, 2, 4$. 

Figure 5. Submanifolds for $p = 4$. 

Here, $\hat{g}$ is the induced metric on the Dp-brane and
\[
\mathcal{F} = F - P[B],
\]
3. Brane embeddings in deformed backgrounds

In this section, we consider brane embeddings inside $\gamma$-deformed background solutions of type-IIB supergravity \[9\]. We begin with the $\gamma$-deformation of the $\text{AdS}_5 \times S^5$ background for which the $\text{AdS}_5$ part of the metric remains the same, while the metric of the $\gamma$-deformed 5-sphere is written as

$$d\Omega_{5,\gamma}^2 = \sum_{i=1}^{3} (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + GR^2 \gamma^2 \mu_1^2 \mu_2^2 \mu_3^2 \left(\sum_{i=1}^{3} d\phi_i\right)^2,$$

(34)

where

$$G^{-1} = 1 + R^2 \gamma^2 (\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2)$$

(35)

and $(\mu_1, \mu_2, \mu_3) \equiv (\cos \theta, \sin \theta \cos \psi, \sin \theta \sin \psi)$. The NS sector of the background includes a dilaton and a Kalb–Ramond two-form, given by

$$e^{2\phi} = Ge^{2\phi_0},$$

$$B_{\text{NS}} = \gamma R^4 G (\mu_1^2 \mu_2^2 d\phi_1 \wedge d\phi_2 + \text{cyclic})$$

(36)

and the RR potential and field strengths

$$C_2 = -4\gamma R^4 w_1 \wedge (d\phi_1 + d\phi_2 + d\phi_3),$$

$$C_4 = 4R^2 (w_4 + G w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3),$$

$$\text{Vol}(\text{AdS}_5) = dw_4,$$

(37)

$$F_5 = 4R^4 (\text{Vol}(\text{AdS}_5) + G \text{Vol}(S^5)), $$

$$\text{Vol}(S^5) = dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3.$$

We consider D5-brane embeddings in this deformed background. The brane will wrap the four angles of the deformed sphere so that the world-volume coordinates will be $(t, r, \psi, \phi)$ and the embedding coordinates are taken as $\vec{x}_i = \text{const}$ and $\theta = \theta(r)$. As before, we also turn on an Abelian world-volume gauge field strength $F_\gamma$. The action of the probe brane is given by a sum of a DBI and a WZ term. Some extra care is needed since there are new terms arising from the induced Kalb–Ramond field and the RR potentials. The action assumes the generic form

$$S = -T_5 \int_{D5} e^{-\Phi} \sqrt{g} [\mathcal{P} + F + T_5 \sum_p C_p \wedge e^{F}],$$

(38)

where $\mathcal{P}$ is given in (29). After performing the computation, the action for the D5-brane reads

$$S = -\frac{T_5 R^4}{2} \int d\psi d^3\phi \sin 2\psi \int dr \left(\cos \theta \sin^3 \theta \sqrt{1 - F_\gamma^2 + r^2 \theta^2} - F_\gamma \sin^4 \theta\right).$$

(39)

The entire $\gamma$-dependence has dropped out completely due to non-trivial cancellations in the DBI and WZ terms, separately. In fact, this action is exactly the same as that computed for the $p = 3$ and $q = 1$ case in which the D5-brane wraps the $S^1 \times S^4$ submanifold of $S^5$. Indeed, one may check that the above Lagrangian falls into the generic family (13) with $f(\theta) = \sin^4 \theta$, which is the correct function appearing in the RR potential for the aforementioned case.

The $\gamma$-deformed background has $\mathcal{N} = 1$ superconformal symmetry. Since for our embedding the action is actually $\gamma$-independent, we expect that the probe brane breaks one half of it, as it was shown before for the maximally supersymmetric cases. To demonstrate this explicitly, one has to work out (26) for our background using the fact that the corresponding Killing spinor is no longer unconstrained, as in the maximally supersymmetric case, but instead it is subject to two projections that reduce supersymmetry.
3.1. Embeddings in the $\sigma$-deformed background

One may also consider a more general deformation of the background, by performing an S-duality in the theory [9]. Apart from $\gamma$, the resulting background depends also on $\sigma$ which is an additional scaleless parameter. Searching for D5-brane embeddings, we choose the embedding coordinates $\vec{x} = \text{const}$ and $\theta = \theta(\psi)$. As opposed to the previous cases, here $\theta$ should depend on $\psi$, since the latter enters in the computations in a non-trivial way. Actually, as we shall explain later, this is related to the chosen embedding. The Hamiltonian of the system turns out to be

$$H = T_5 R^4 \sqrt{\mathcal{H}} \sqrt{P^2 Q + (P \sin^2 \theta - f(\psi))^2},$$

with

$$P = \frac{1}{2\mathcal{H}} \sin^2 \theta \sin 2\psi, \quad Q = \mathcal{H} \sin^2 \theta - \sin^4 \theta + \mathcal{H}\theta' \cos^2 \theta,$$

$$f(\psi) = \frac{\nu^2}{2} \sin 2\psi, \quad \mathcal{H} = 1 + \sigma^2 R^4 (\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2),$$

and $\mu_1$ as defined in the previous section. As before, the parameter $\gamma$ does not appear at all, but $\sigma$ does. It should be obvious that an attempt to find constant minima, namely $\psi$-independent solutions, is inconsistent. Varying the Hamiltonian with respect to $\theta$ gives a complicated nonlinear differential equation that one has to solve in order to find configurations that minimize the energy. We were unable to find solutions of this differential equation.

This increased level of complexity occurs due to the particular embedding that we considered. Had we chosen a similar embedding $\theta = \theta(\psi)$ for the undeformed background, i.e. the $p = 3$ case with manifest $\mathbb{S}^1 \times \mathbb{S}^3$, it would have also resulted to a similarly complicated differential equation. The two different embeddings in that case are depicted in figures 6 and 7.

For non-constant embeddings $\theta = \theta(\psi)$ in the undeformed case, it is obviously possible to rotate the north pole in a way to obtain the first configuration. In practice, this is done by performing an $SO(6)$ transformation. In the $\sigma$-deformed case, it is not obvious what the corresponding transformation would be, given that the isometry group has been reduced. The deformed sphere has the same Euler characteristic with the undeformed one so that their topology is the same. It makes sense, then, to assume that such a transformation exists, although we were not able to find it.
4. Turning on temperature

It is natural to extend the discussion to asymptotically AdS spacetimes, which are relevant to a holographic approach to dimers in condensed matter systems, as pursued in [8]. We will briefly discuss the near-horizon geometry of black D3-branes in such a way that the submanifold $\mathbb{S}^1 \times \mathbb{S}^3$ of $\mathbb{S}^5$ appears explicitly. The metric of the background is

\[
\text{d}s^2 = - f(r) \text{d}t^2 + \frac{\text{d}r^2}{f(r)} + R^2 (\text{d}\theta^2 + \cos^2 \theta \text{d}\Omega_1^2 + \sin^2 \theta \text{d}\Omega_3^2),
\]

\[
f(r) = r^4 - \mu^4 R^2 r^2,
\]

(42)

and the RR-potential changes also accordingly. The Hawking temperature is simply proportional to the parameter $\mu$. We consider a D5-brane probe with the same embedding coordinates as before, i.e., $\vec{x}_|| = \text{const}$ and $\theta = \theta(r)$. It is a straightforward task to show that the minima of the particular configuration remain the same as with the zero-temperature case, this being true for every $p$. This would not be the case for more general $r$-dependent solutions.

Supersymmetry is broken in all of these cases, due to the non-vanishing temperature. Then, one cannot use kappa symmetry arguments to ensure the stability of the configurations. Nonetheless, we can consider small fluctuations around the minima. Let

\[
\theta = \bar{\theta} + \xi, \quad F_{\mu\nu} = \bar{F}_{\mu\nu} + \chi,
\]

(43)

where the bars denote the minima and $\chi = \partial_\mu \alpha - \partial_\nu \alpha$. It should be stressed out that this is consistent as long as one considers only the zero mode in the spherical harmonic expansion on $\mathbb{S}^5$. In order to find the complete spectrum, one should also turn on fluctuations of the field strength in every possible direction (see also [16] for a prime example). However, here, we are only interested in demonstrating perturbative stability at nonzero temperature, and for that, restricting to the zero-mode suffices. The effective Lagrangian for quadratic fluctuations is found to be

\[
\mathcal{L} = \frac{1}{2} \sqrt{R^4 T_3} \cos \bar{\theta} \sin^3 \bar{\theta} \frac{1}{\sqrt{1 - \bar{F}_{\mu\nu}^2}} \left[ R^2 (f(r)^{-1}(\partial_\mu \xi)^2 - f(r)(\partial_\nu \xi)^2) \right.
\]

\[
+ \frac{\chi^2}{1 - \bar{F}_{\mu\nu}^2} + A \xi^2 + B \chi \xi \right].
\]

(44)

The minima and the gauge field are given by

\[
\sin \bar{\theta} = \frac{4}{3} \nu, \quad \bar{F}_{\mu\nu} = \frac{9 - 16 \nu}{\sqrt{81 - 96 \nu}},
\]

(45)
and we have defined the constants

\[ A = 4 + \frac{36}{27 - 32\nu}, \quad B = \sqrt{\frac{81 - 96\nu}{3\nu - 4\nu^2}}. \]  

(46)

We obtain the equations of motion by varying \( \chi \) and \( \xi \). After combining them and concentrating on a Fourier mode of the form \( \xi = e^{i\omega t}\Psi_1(r) \), we obtain

\[
\frac{d}{dr} \left( f(r) \frac{d\Psi}{dr} \right) + \left( \frac{\omega^2}{f(r)} - \frac{C}{2R^2} \right) \Psi(r) = 0, \quad C \equiv 24 + \frac{72}{32\nu - 27},
\]  

(47)

defined for \( r \geq \mu \). We transform this into a Schrödinger equation for \( \Psi \), by appropriately changing to a new variable \( z = \int_{\infty}^{r} dr' f^{-1}(r') \), with \( z \in [0, \infty) \) as \( r \in (\infty, \mu] \). The associated potential, which can be written explicitly only in terms of \( r \), is

\[
V = \frac{C f(r)}{2R^2}.
\]  

(48)

Substituting the value for \( \theta \) in \( C \) from (45), one sees that \( C \) is non-negative. Hence, the zero mode of the configuration is always positive. In fact, \( C \) vanishes for the critical value \( \nu = \frac{3}{4} \).

In conclusion, the configuration that we considered is stable. Similar arguments also hold for the other submanifolds and for the cases \( p = 0, 1, 2, 4 \) as well.

5. Application on holographic dimers

It is interesting to investigate how the results of the previous sections affect the holographic description of dimers. The main idea was pioneered in [8]. There, the authors considered lattices of D5-branes embedded in a D3 black brane background in order to model a finite temperature system. The chosen embedding is such that each probe brane wraps an \( S^4 \subset S^5 \).

By generalizing the arguments presented there, we will show in this section that the less symmetric embeddings that we found in section 2 are more favorable in the aforementioned context.

The metric of the background under consideration is (42), alongside with the parametrization (4) for the compact manifold restricted to the case with \( p = 3 \). Hence, the embedded branes wrap an \( \text{AdS}_2 \) space times a four-dimensional submanifold. The free energy of the single D5-brane (or an anti-D5-brane) is computed by integrating the on-shell action [17]. In order to do this, one performs a Wick rotation to the Euclidean metric where time is identified as a periodic variable, which is the temperature. The free energy of a single D5-brane that goes straight down to the D3-horizon is given then by

\[
F_{D5} = -\lambda \mu N T f \mathcal{E}^3_{1, q}.
\]  

(49)

We observe that the result is proportional to the energies that we computed in section 2. The computation is very similar to that performed for the \( q = 1 \) case in [17] so that we omit the details.

As in [8], we will consider a lattice of D5- and an anti-D5-brane pairs. Each pair is essentially constituted by a D5-brane which dives into the bulk and returns with an opposite orientation, thus regarded as an anti-D5-brane. There exist two configurations then, for each pair, depending on the value of the temperature. In the disconnected configuration, the D5- and the anti-D5-brane are separated and do not interact\(^2\). The total free energy of the pair is

\(^2\) It turns out that all the essential details of analyzing this problem are similar to those in the holographic computation of the Wilson loop related to the binding energy of a quark–antiquark pair [18, 19].
just the sum of the individual free energies, which is simply equal to

\[ F_{\text{disconnected}} = 2F_{D5}. \]  

(50)

As will become transparent below, this configuration dominates at high temperature.

In the second configuration in which the D5- and the anti-D5-brane are connected with each other, the two membranes are separated by \( \Delta x \) and are located at \( r = \infty \) with \( \vec{x} = (\pm \frac{\Delta x}{2}, 0, 0) \). One considers then embeddings with \( \theta(r) = \theta_x \) and \( x = x(r) \). The turning point of the D5-brane is computed by \( \frac{dr}{dx'} = 0 \) and has the same form for a generic wrapping. We scale the turning point by the temperature and we define the dimensionless parameter \( z_0 \equiv \frac{\mu m}{T} \). The turning point is associated with the spacing between the branes and the temperature by the following relation:

\[ \frac{\mu}{R^2} \Delta x = \left[ 2\left(x_0^4 - 1\right)^{1/2} \int_{z_0}^{\infty} dz \sqrt{\frac{1}{(x^4 - 1)(x^4 - x_0^4)}} \right]. \]  

(51)

Noting the similarity with the holographic computation of the binding energy of a quark–antiquark pair that we mentioned above, we perform the integration obtaining [20]

\[ \frac{\mu}{R^2} \Delta x = \frac{1}{2}B(3/4, 1/2) \left( x_0^4 - 1 \right) \frac{1}{z_0^4} \frac{1}{z_0^4} \frac{1}{z_0^4} \tilde{F}_{\text{connected}}, \]  

(52)

As seen in figure 8, for the fixed lattice spacing \( \Delta x \), a solution of this type exists only for low enough temperatures. For higher temperatures, the disconnected configuration is the only available solution. The critical temperature beyond which the latter dominates is not however given by the maximum value of the temperature, since the disconnected configuration already acquires a lower free energy at a lower temperature. To see that we compute the free energy of the connected configuration which is found to be

\[ F_{\text{connected}} = 2\lambda \mu NTf_{\text{3,q}} \int_{-z_0}^{\infty} dz \left( \sqrt{x^4 - 1} - 1 \right). \]  

(53)

where we have defined the function

\[ \tilde{F}_{\text{connected}} = \frac{z_0^4}{4} B(-1/4, 1/2) \frac{1}{z_0^4} \frac{1}{z_0^4} \tilde{F}_{\text{connected}}, \]  

(54)
and we have computed the integral. This measures the deviation of the free energy from that of the disconnected configuration. To proceed with the analysis, we note from figure 8 that for the same temperature there exist two values of $z_0$. This multivalueness is also manifest in the plot of the free energy for the connected configuration in figure 10. Based on the experience with a general analysis for the quark–antiquark binding energy performed in [21], we expect that a similar analysis here will indicate that the upper branch is unstable under small perturbations so that we disregard it completely. Note also that in figures 8–10, the black-, red- and blue-colored branches correspond to the unstable, meta-stable and stable branches, respectively.

Next, we compare the free energy of the connected configuration as a function of $z_0$ with that of the disconnected configuration. We see from figure 9 that for large values of $z_0$, equivalently for small temperatures, the connected configuration is more favorable. There exist a critical value of $z_0$, numerically equal to $z_{0,c} \simeq 1.52$, below which the disconnected
configuration is favorable. Therefore, there exists a critical temperature at which the system undergoes a phase transition. This phase transition is of first order, since the first derivative possesses a discontinuity at the critical value $z_{0,C}$.

Eventually, in order to make contact with the results of section 2, we first observe that (54) is the same for all wrappings. Thus, the only difference in (53) for different wrappings origins simply from the constant factor in front, which is essentially the energy of the wrapping.

Since $\tilde{F}_{\text{connected}}$ is always negative, the wrapping with the highest energy density, which is less symmetric, has the minimal free energy, becoming more favorable in this context. By considering lattices of pairs, one constructs dimers in a holographic way, along the lines presented in [8].

6. Concluding remarks

We classified and energetically compared all possible cases, with a single-embedding angular coordinate, in which a D($8-p$) brane can wrap AdS$_2$ times a submanifold of $S_{8-p}$ in a D$p$-brane background, thus producing a point-like defect. We worked out the details in all different cases that arise, performing also comparisons between them. We examined similar constructions in the presence of temperature and in $\beta$-deformed backgrounds. We demonstrated stability either by supersymmetry arguments or by a small fluctuation analysis around the minima.

It would be interesting to investigate and search for running solutions of the embedding coordinate, i.e. $\theta = \theta(r)$. This involves the classical equation of motion for the Hamiltonian (19). This is a highly nonlinear equation but it should be possible to analyze it numerically. Of particular interest would be solutions connecting minima corresponding to different values of $n$, especially when they correspond to the same energy.

Moreover, it would be very useful to extend our results beyond the probe approximation, by considering the backreaction of the probe branes on the background. This is significant when their number is comparable to the color number. In addition, these backreaction effects would also influence the dimerization analysis presented here, in case where many branes are located at each lattice site.

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