FREE PRE-LIE ALGEBRAS OF FINITE POSETS

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ABSTRACT. We first recall the construction of a twisted pre-Lie algebra structure on the species of finite connected topological spaces. Then, we construct a corresponding non-coassociative permutative (NAP) coproduct on the subspecies of finite connected $T_0$ topological spaces, i.e., finite connected posets, and we prove that the vector space generated by isomorphism classes of finite posets is a free pre-Lie algebra and also a cofree NAP coalgebra. Further, we give an explicit duality between the non-associative permutative product and the proposed NAP coproduct. Finally, we prove that the results in this paper remain true for the finite connected topological spaces.

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1. Introduction

In this paper, we introduce a twisted non-associative permutative algebra structure on the species of finite connected posets. We recall the definition of a non-associative permutative algebra (in short NAP algebra), and the dual definition of a non-coassociative permutative coalgebra (in short NAP coalgebra). A (left) NAP algebra is a vector space $V$ equipped with a bilinear product $\cdot$ satisfying the relation:

\[
a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \text{for all } a, b \text{ and } c \in V.
\]

We note also that the notion of NAP algebra has emerged from the reference [15] by M. Livernet. It has also been previously defined in [11] by A. Dzhumadil’daev, and C. Löfwall under the name of ”left commutative algebra”. Dually a (left) NAP coalgebra is a vector space $V$ equipped with

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a bilinear coproduct $\delta: V \to V \otimes V$ satisfying the relation:
\begin{equation}
(Id \otimes \delta) \delta = \tau^{12} (Id \otimes \delta) \delta.
\end{equation}
We recall the following rigidity theorem [15]: any pre-Lie algebra $(V, \triangleright)$, together with a non-associative permutative connected coproduct $\delta$ satisfying the distributive law
\begin{equation}
\delta(a \triangleright b) = a \otimes b + (a \otimes 1 + 1 \otimes a) \triangleright \delta(b),
\end{equation}
is a free pre-Lie algebra and a cofree NAP coalgebra.

We recall also in this paper the description of the free pre-Lie algebra in terms of rooted trees given in [10]. We define a new bilinear product $\triangleleft$ in the species of finite connected posets $\mathbb{U}$ by: for all $P \in \mathbb{U}_{X_1}$ and $Q \in \mathbb{U}_{X_2}$, where $X_1$ and $X_2$ are two finite sets
\[ P \triangleleft Q = \sum_{v \in \min(Q)} P \downarrow v \circ Q, \]
where $P \downarrow v \circ Q$ is obtained from the Hasse graphs $G_1$ and $G_2$ of $P$ and $Q$ by adding an (oriented) edge from $v$ in $G_2$ to any minimal vertex of $G_1$ [3].

We show that the bilinear product $\triangleleft$ verifies (1.1) in the monoidal category of species, and therefore endows the species $\mathbb{U}$ with a NAP algebra structure.

For any finite set $X$, we define the coproduct $\delta$ by:
\[ \delta : \mathbb{U}_X \longrightarrow (\mathbb{U} \otimes \mathbb{U})_X = \bigoplus_{Y \cup Z = X} \mathbb{U}_Y \otimes \mathbb{U}_Z \]
\[ P \longmapsto \frac{1}{|\min(P)|} \sum_{I \supset P} I \otimes P \setminus I, \]
where $I \otimes P$ means that $I$ is a subset of $P$ such that:
- $I$ is a singleton included in $\min(P)$,
- and $I$ is a connected component of the set $\{x \in P, I_\prec P x\}$.

with the set $I$ is equal to the space $\{x \notin I, \text{ there exists } y \in I \text{ such that } x \preceq P y\}$.

We prove that the coproduct $\delta$ is a twisted NAP coproduct, i.e., (1.2) is verified in the monoidal category of species.

M. Livernet in [15] proved the following rigidity theorem: for any pre-Lie algebra $(V, \triangleright)$ together with a NAP connected coproduct $\delta$ satisfying relation (1.3), $V$ is the free pre-Lie algebra generated by $\text{Prim}(V)$.

Later in this paper, we prove a compatibility relation between the pre-Lie $\setminus_{\triangleright}$ structure and the coproduct $\delta$ by proving the twisted version of (1.3) namely:
\[ \delta(P \setminus_{\triangleright} Q) = P \otimes Q + (P \otimes 1 + 1 \otimes P) \setminus_{\triangleright} \delta(Q), \]
for any pair $(P, Q)$ of finite connected posets, where the unit $1$ is identified to the empty poset.

Applying Aguiar-Mahajan’s bosonic Fock functor $\mathcal{K}$ [2] and M. Livernet’s rigidity theorem [15] leads to the main result of the paper: $(\mathcal{K}(\mathbb{U}), \setminus_{\triangleright})$ endowed with the coproduct $\delta$ is a free pre-Lie algebra and a cofree NAP coalgebra.
We end up Section 3 by proving that the NAP product $\circ$ and the NAP coproduct $\delta$ are dual to each other, and we show that $(\overline{K}(P), \Delta_{\downarrow})$ is a coassociative cofree coalgebra, where $\Delta_{\downarrow}$ is the coassociative coproduct defined in [3].

In the last section, we prove that the results in this paper remain true for the finite connected topological spaces, with a small change on the definition of the coproduct $\delta$.

2. Basics of finite topologies

A partial order on a set $X$ is a transitive, reflexive and antisymmetric relation on $X$. A finite poset is a finite set $X$ endowed with a partial order $\leq$. Let $P = (X, \leq_P)$ and $Q = (X, \leq_Q)$ be two posets. We say that $P$ is finer than $Q$ if:

$$x \leq_P y \implies x \leq_Q y$$

for any $x, y \in X$. The Hasse diagram of poset $P = (X, \leq_P)$ is obtained by representing any element of $X$ by a vertex, and by drawing a directed edge from $a$ to $b$ if and only if $a <_P b$, and, for any $c \in X$ such that $a \leq_P c \leq_P b$, one has $a = c$ or $b = c$. We can say that $I$ is an upper ideal of $P$ ($I \subset P$) if, for all $x, y \in P$, $(x \in I, x \leq_P y) \implies y \in I$. We denote $J(P)$ the set of upper ideals of $P$.

Recall [1,12] that a finite topological space is a finite quasi-poset and vice versa. Any topology $\mathcal{T}$ (hence any quasi-order on $X$) gives rise to an equivalence relation:

$$(2.1) \quad x \sim_\mathcal{T} y \iff (x \leq_\mathcal{T} y \text{ and } y \leq_\mathcal{T} x).$$

Equivalence classes will be called bags here. This equivalence relation is trivial if and only if the quasi-order is a (partial) order, or equivalently, if the corresponding topology is $\mathcal{T}_0$. Any topology $\mathcal{T}$ on $X$ defines a poset structure on the quotient $X/ \sim_\mathcal{T}$, corresponding to the partial order induced by the quasi-order $\leq_\mathcal{T}$. More on finite topological spaces can be found in [4,9,12,18].

Recall [2,14] that a linear (tensor) species is a contravariant functor from the category of finite sets $\text{Fin}$ with bijections into the category $\text{Vect}$ of vector spaces (on some field $k$). Let $\mathbb{E}$ be a linear species. We note $\overline{K}(\mathbb{E}) = \bigoplus_{n \geq 0} \mathbb{E}_n/S_n$, where $\mathbb{E}_n/S_n$ denotes the space of $S_n$-coinvariants of $\mathbb{E}_n$. The functor $\overline{K}$ from linear species to graded vector spaces is intensively studied in [2, chapter 15] under the name "bosonic Fock functor".

The species $\mathcal{T}$ of finite topological spaces is defined as follows: for any finite set $X$, $\mathcal{T}_X$ is the vector space freely generated by the topologies on $X$. For any bijection $\varphi : X \to X'$, the isomorphism $\mathcal{T}_\varphi : \mathcal{T}_X \to \mathcal{T}_{X'}$ is defined for any topology $\mathcal{T}$ on $X'$, by the following relabelling:

$$\mathcal{T}_\varphi(\mathcal{T}) = \{\varphi^{-1}(Y), Y \in \mathcal{T}\}$$

It is well-known that the species $\mathcal{T}$ of finite topological spaces is a twisted commutative Hopf algebra [12]: the product $m$ is given by disjoint union, and the coproduct $\Delta$ is a natural generalization of the Connes-Kremmer coproduct on rooted forests. We defined another twisted Hopf
algebra structure in [3] by replacing $\Delta$ with the coproduct
\[
\Delta : \mathbb{T}_X \rightarrow (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus_{Y \subseteq Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z
\]
\[
\mathfrak{T} \leftarrow \sum_{Y \subseteq \mathfrak{T}} \mathfrak{T}_Y \otimes \mathfrak{T}_{X \setminus Y}.
\]

Where $Y \subseteq \mathfrak{T}$, stands for
- $Y \in \mathfrak{T},$
- $\mathfrak{T}_Y = \mathfrak{T}_1 \ldots \mathfrak{T}_n$, such that for all $i \in \{1, \ldots, n\}$, $\mathfrak{T}_i$ is connected and $(\min \mathfrak{T}_i = (\min \mathfrak{T}) \cap \mathfrak{T}_i$, or there is a single common ancestor $x_i \in X \setminus Y$ to $\min \mathfrak{T}_i, \text{where } \overline{X \setminus Y} = (X \setminus Y)/\sim_{\mathfrak{T}\setminus Y}.$

If we apply the functor $\overline{\mathfrak{X}}$, we therefore two commutative graded connected Hopf algebras $\overline{\mathfrak{X}}(\mathbb{T}, m, \Delta)$ and $\overline{\mathfrak{X}}(\mathbb{T}, m, \Delta_{\setminus})$.

**Definition 2.1.** [8, 16] A left pre-Lie algebra over a field $k$ is a $k$-vector space $A$ with a binary composition $\triangleright$ that satisfies the left pre-Lie identity:
\[
(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z),
\]
for all $x, y, z \in A$. The left pre-Lie identity rewrites as:
\[
L_{[x,y]} = [L_x, L_y],
\]
where $L_x : A \rightarrow A$ is defined by $L_x y = x \triangleright y$, and where the bracket on the left-hand side is defined by $[x, y] = x \triangleright y - y \triangleright x$. As a consequence this bracket satisfies the Jacobi identity.

In fact, as mentioned in [3], the structure $\setminus$ on the species of connected finite topological spaces $\mathbb{V}$ defined by: for all $\mathfrak{T}_1 = (X_1, \leq_{\mathfrak{T}_1})$ and $\mathfrak{T}_2 = (X_2, \leq_{\mathfrak{T}_2})$ two finite connected topological spaces as:
\[
\mathfrak{T}_1 \setminus \mathfrak{T}_2 = \sum_{v \in \mathfrak{T}_2} \mathfrak{T}_1 \setminus_{\mathfrak{T}_1} \mathfrak{T}_2,
\]
is a pre-Lie structure, where $\mathfrak{T}_1 \setminus_{\mathfrak{T}_1} \mathfrak{T}_2$ is obtained from the Hasse graphs $G_1$ and $G_2$ of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ by adding an (oriented) edge from $v$ in $G_2$ to any minimal vertex of $G_1$.

We denote by $\mathbb{P}$ the sub-species of $\mathbb{T}$ consisting in partial order (i.e., $T_0$-topologies). The binary product $m$ (resp. $\setminus$) restricts to finite posets (resp. finite connected posets), as well as both coproducts $\Delta$ (resp. $\Delta_{\setminus}$). Hence the triple $(\mathbb{P}, m, \Delta)$ is a twisted Hopf subalgebra of $(\mathbb{T}, m, \Delta)$, the triple $(\mathbb{P}, m, \Delta_{\setminus})$ is a twisted Hopf subagebra of $(\mathbb{T}, m, \Delta_{\setminus})$ and $(\mathbb{P}, \setminus)$ is a twisted pre-Lie subalgebra of $(\mathbb{T}, \setminus)$.

### 3. Free pre-Lie algebras and cofree coalgebras

#### 3.1. Free pre-Lie algebras and rooted trees.**

Let $\mathcal{T}$ be the vector space spanned by the set of isomorphism classes of rooted trees and $H = S(\mathcal{T})$. Grafting pre-Lie algebras of rooted trees were studied for the first time by F. Chapoton and M. Livernet [10], although the use of rooted trees can be traced back to A. Cayley [5]. The grafting product is given, for all $t, s \in \mathcal{T}$, by:
\[
t \rightarrow s = \sum_{s' \text{ vertex of } s} t \rightarrow_{s'} s,
\]
where \( t \rightarrow s \) is the tree obtained by grafting the root of \( t \) on the vertex \( s' \) of \( s \). In other words, the operation \( t \rightarrow s \) consists of grafting the root of \( t \) on every vertex of \( s \) and summing up.

We note \( T(n) \) the linear span of rooted trees of degree \( n \). For a vector space \( V \), we denote by \( T(V) \) the space \( \bigoplus_n T(n) \otimes V^{\otimes n} / S_n \). This is the linear span of rooted trees decorated by a basis of \( V \). Here \( S_n \) is the symmetric group on \( n \) elements. Following the notation of Connes and Kreimer [6], any tree \( t \) writes \( t := B(v, t_1, \ldots, t_n) \) that is decorated by \( v \), and \( t_1, \ldots, t_n \) are trees.

We notice that, if using the pre-Lie product \( \rightarrow \) in \( T(V) \), one has:

\[
B(v, t_1, \ldots, t_n) = t_n \rightarrow B(v, t_1, \ldots, t_{n-1}) - \sum_{0 \leq k < n} B(v, t_1, \ldots, t_k \rightarrow t_{k+1}, \ldots, t_{n-1})
\]

**Proposition 3.1.** [10] Equipped by \( \rightarrow \), the space \( T \) is the free pre-Lie algebra with one generator.

**Proposition 3.2.** [15] Let \( V \) be a vector space. Then, \( T(V) \) together with the coproduct

\[
\delta(B(v, t_1, t_2, \ldots, t_n)) = \sum_{0 \leq k < n+1} t_k \otimes B(v, t_1, \ldots, \hat{t}_k, \ldots, t_n),
\]

is the cofree NAP connected coalgebra generated by \( V \).

### 3.2. Twisted NAP algebras of finite connected posets.

Let \( P \) be a finite connected poset, and \( I \) be a subset of \( P \). We denote by \( I_\neg \) the set \( \{ x \not\in I \mid \text{there exists } y \in I \text{ such as } x \leq_P y \} \).

We note \( I \otimes P \) whenever:

- \( I_\neg \) is a singleton included in \( \min(P) \),
- and \( I \) is a connected component of the set \( \{ x \in P \mid I_\neg <_P x \} \).

**Lemma 3.1.** For any finite connected poset \( P \), and any subsets \( I, J \) of \( P \), we have:

1. if \( I \otimes P \) and \( J \otimes P \), then \( I = J \) or \( I \cap J = \emptyset \).
2. if \( I \otimes P \), then \( \min(P \setminus I) = \min(P) \).
3. if \( I \otimes P \), then \( P \setminus I \) is a connected poset.

**Proof.** 1- Let \( P \) be a finite connected poset and \( I, J \) two subsets of \( P \), such that \( I, J \otimes P \). So, we have two possible cases: either \( I_\neg \not\equiv J_\neg \) or \( I \equiv J_\neg \).
- If \( I_\neg \not\equiv J_\neg \), with \( I_\neg = \{ y \} \) and \( J_\neg = \{ z \} \), then \( I \) is a connected component of the set \( E_1 = \{ x \in P \mid y <_P x \} \) and \( J \) is a connected component of the set \( E_2 = \{ x \in P \mid z <_P x \} \). So \( I \) and \( J \) is a connected component of the set \( E_1 \sqcup E_2 \), then \( I = J \) or \( I \cap J = \emptyset \). Gold \( I \equiv J_\neg \), then \( I \equiv J \). Hence \( I \cap J = \emptyset \).
- If \( I_\neg = J_\neg = \{ y \} \), then \( I \) and \( J \) is two connected components of the set \( \{ x \in P \mid y <_P x \} \), then: \( I = J \) or \( I \cap J = \emptyset \).

2- If there exists \( x \in \min(P) \cap I \), then \( I_\neg <_P x \), which is absurd. Thus, \( \min(P) \cap I = \emptyset \). Then, \( \min(P \setminus I) = \min(P) \).

3- If \( I \otimes P \), then \( I \) is a connected component of the set \( E = \{ x \in P \mid I_\neg <_P x \} \). We have \( P \setminus I \) is a connected poset if and only if, for all \( x, y \in P \setminus I \) there exist \( a_1, \ldots, a_n \in P \setminus I \) such that \( x \mathcal{R} a_1 \ldots \mathcal{R} a_n \mathcal{R} y \), where \( \mathcal{R} \) is defined by: \( t_1 \mathcal{R} t_2 \iff (t_1 <_P t_2 \text{ or } t_2 <_P t_1) \). We assume that \( x, a_1, \ldots, a_n, y \) is the smallest chain that connects \( x \) to \( y \).

If \( P \setminus I \) is not connected, there is \( k \in [n] \) such that \( a_k \not\in I \). By choosing \( k \) appropriately we can also assume that \( a_{k-1} \) or \( a_{k+1} \) are not in \( I \), then we have four possible cases:
First case; \( a_{k-1} \leq_P a_k \leq_P a_{k+1} \). Then this chain is not the smallest chain that connects \( x \) to \( y \), which is absurd.

Second case; \( a_{k-1} \geq_P a_k \geq_P a_{k+1} \), the proof is similar.

Third case; \( a_{k-1} \geq_P a_k \leq_P a_{k+1} \). Since \( a_k \in I \), then \( I_\prec_P a_k \). Moreover \( \prec_P \) is transitive, then \( I_\prec_P a_{k-1} \), so \( a_{k-1} \in E \). Then we obtained: \( (a_{k-1}, a_k) \in E \times I \), \( I \) is a connected component of \( E \) and \( a_k \prec_P a_{k-1} \). Hence \( a_{k-1} \in I \), and similarly \( a_{k+1} \in I \). Then \( a_{k-1} \in I \) and \( a_{k+1} \in I \), which is in contradiction with the assumptions.

Fourth case; \( a_{k-1} \leq_P a_k \geq_P a_{k+1} \). In this case we have \( a_{k-1} \in I_\prec \) and \( a_{k+1} \in I_\prec \), so \( a_{k-1} = a_{k+1} \). Then this chain is not the smallest chain that connects \( x \) to \( y \), which is absurd. Hence, \( P \setminus I \) is a connected poset.

\[ \square \]

**Definition 3.1.** The bilinear product \( \bigcirc \rightarrow \) is defined in the species of finite connected posets \( \mathbb{U} \) as follows: for all \( P \in \mathbb{U}_{X_1} \) and \( Q \in \mathbb{U}_{X_2} \), where \( X_1 \) and \( X_2 \) are two finite sets:

\[ P \bigcirc \rightarrow Q = \sum_{v \in \text{min}(Q)} P \backslash_{uv} Q. \]

**Proposition 3.3.** The bilinear product \( \bigcirc \rightarrow \) endows the species \( \mathbb{U} \) with a twisted NAP algebra structure i.e., the following identity is verified:

\[ P \bigcirc \rightarrow (Q \bigcirc \rightarrow R) = Q \bigcirc \rightarrow (P \bigcirc \rightarrow R) \]

for any three finite connected posets \( P, Q \) and \( R \).

**Proof.** Let \( P, Q \) and \( R \) be three finite connected posets, we have:

\[
P \bigcirc \rightarrow (Q \bigcirc \rightarrow R) = \sum_{v \in \text{min}(R)} P \bigcirc \rightarrow (Q \backslash_{uv} R)
\]

\[
= \sum_{v \in \text{min}(Q \backslash_{uv} R)} \sum_{u \in \text{min}(R)} P \backslash_{uu} (Q \backslash_{uv} R),
\]

and since \( \text{min}(Q \backslash_{uv} R) = \text{min}(R) \), for all \( v \in \text{min}(R) \) then:

\[
P \bigcirc \rightarrow (Q \bigcirc \rightarrow R) = \sum_{v, u \in \text{min}(R)} P \backslash_{uu} (Q \backslash_{uv} R),
\]

Which is symmetric on \( P \) and \( Q \). Correspondingly, we obtain:

\[
P \bigcirc \rightarrow (Q \bigcirc \rightarrow R) = Q \bigcirc \rightarrow (P \bigcirc \rightarrow R).
\]

\[ \square \]

**Corollary 3.1.** [2, Chapter 17] Applying the functor \( \mathcal{K} \) gives that \( (\mathcal{K}(\mathbb{U}), \bigcirc \rightarrow) \) is a NAP algebra.

**Definition 3.2.** [15] Let \( (V, \delta) \) be a coalgebra, i.e. be a vector space \( V \) together with linear map \( \delta : V \rightarrow V \otimes V \). The following defines a filtration on \( V \):

- \( \text{Prim}(V) = V_1 = \{ x \in V, \delta(x) = 0 \} \),
- \( V_n = \{ x \in V, \delta(x) \in \bigoplus_{0<k<n} V_k \otimes V_{n-k} \} \).

The coalgebra \( (V, \delta) \) is said to be connected if \( V = \bigcup_{k>0} V_k \).
Proposition 3.4. The coproduct $\delta : U \rightarrow U \otimes U$ defined for any finite set $X$ by

$$\delta : U_X \rightarrow (U \otimes U)_X = \bigoplus_{Y \cup Z = X} U_Y \otimes U_Z$$

$$P \mapsto \frac{1}{|\min(P)|} \sum_{I \text{ such } I \cup J = P} I \otimes P \setminus I,$$

is a twisted connected NAP coproduct, i.e. the following identity is verified

$$(Id \otimes \delta) \delta = \tau_{12} (Id \otimes \delta) \delta,$$

where $\tau_{12} = \tau \otimes Id$, $\tau$ is the flip.

Proof. Let $P$ be a finite connected poset. We have

$$(Id \otimes \delta) \delta(P) = \frac{1}{|\min(P)|} \sum_{I \text{ such } I \cup J = P} I \otimes \delta(P \setminus I)$$

$$= \frac{1}{|\min(P)|} \sum_{I \text{ such } I \cup J = P} \frac{1}{|\min(P \setminus I)|} \sum_{J \text{ such } I \cup J = P} I \otimes J \otimes ((P \setminus I) \setminus J)$$

$$= \frac{1}{|\min(P)|^2} \sum_{I \text{ such } I \cup J = P} \sum_{J \text{ such } I \cup J = P} I \otimes J \otimes (P \setminus (I \sqcup J))$$

$$= \tau_{12} (Id \otimes \delta) \delta(P).$$

By the Definition 2.6 in [15] and the definition of the coproduct $\delta$, it is clear that the coalgebra $(U, \delta)$ is connected. Hence, $(U, \delta)$ is a twisted connected NAP coalgebra. \hfill $\square$

Example 3.1. $\delta(\bigwedge) = 0$  \hspace{1cm} $\delta(\bigvee) = 2 \cdot 1$

$\delta(\bigcap) = \frac{1}{2} \cdot \bigwedge$  \hspace{1cm} $\delta(\bigtriangleup) = \bigwedge \otimes \cdot$

Corollary 3.2. [2, Chapter 17] Applying the functor $K$ gives that $(\mathcal{K}(U), \delta)$ is a NAP coalgebra.

3.3. Free twisted pre-Lie algebras of finite posets. In this subsection, we prove the compatibility relation [15, Theorem] between the pre-Lie product $\cdot$ and the NAP coproduct $\delta$.

Theorem 3.1. Let $P, Q$ be two finite connected posets. We have the following identity

$$\delta(P \setminus Q) = P \otimes Q + (P \otimes 1 + 1 \otimes P) \setminus \delta(Q),$$

where the unit $1$ is identified to the empty poset.
Proof. Let \( P = (X_1, \leq_P), Q = (X_2, \leq_Q) \) be two finite connected posets, we have:

\[
\delta(P \setminus Q) = \sum_{v \in X_2} \delta(P \setminus_v Q)
\]

\[
= \sum_{v \in X_2, \min(P \setminus_v Q)} 1 \sum_{\min(P \setminus_v Q)} I \otimes ((P \setminus_v Q) \setminus I)
\]

\[
= \sum_{v \in X_2, \min(Q)} 1 \sum_{\min(Q)} I \otimes ((P \setminus Q) \setminus I)
\]

\[
= \frac{1}{\min(Q)} \left( \sum_{v \in X_2, \min(Q)} I \otimes ((P \setminus Q) \setminus I) + \sum_{v \in X_2, \min(Q)} I \otimes ((P \setminus Q) \setminus I) \right),
\]

we notice that

- if \( v \in \min(Q) \), then \( I, I \otimes (P \setminus_v Q) \) = \( \{P\} \cup \{I, I \otimes Q\} \),

and

- if \( v \notin \min(Q) \), then \( I, I \otimes (P \setminus_v Q) \) = \( \{P \setminus_v J, J \otimes Q, v \in J\} \cup \{J, J \otimes Q, v \notin J\} \).

Then

\[
\delta(P \setminus Q) = \frac{1}{\min(Q)} \left( \sum_{v \in X_2, \min(Q)} P \otimes Q + \sum_{v \in X_2, \min(Q)} I \otimes (P \setminus_v (Q \setminus J)) \right)
\]

\[
+ \frac{1}{\min(Q)} \left( \sum_{v \in X_2, \min(Q)} (P \setminus_v J) \otimes Q \setminus J + \sum_{v \in X_2, \min(Q)} J \otimes (P \setminus_v (Q \setminus J)) \right),
\]

we notice that

\[
\sum_{v \in X_2, \min(Q), v \notin J} (P \setminus_v J) \otimes Q \setminus J = \sum_{J \in Q} (P \setminus 1) \setminus (J \otimes Q \setminus J),
\]

and

\[
\sum_{v \in X_2, \min(Q), v \notin J} J \otimes (P \setminus_v (Q \setminus J)) = \sum_{J \in Q} J \otimes (P \setminus_v (Q \setminus J)),
\]

then

\[
\sum_{v \in X_2, \min(Q), v \notin J} I \otimes (P \setminus_v (Q \setminus J)) + \sum_{v \in X_2, \min(Q), v \notin J} J \otimes (P \setminus_v (Q \setminus J)) = \sum_{J \in Q} (1 \otimes P) \setminus (J \otimes Q \setminus J),
\]

accordingly

\[
\delta(P \setminus Q) = \frac{1}{\min(Q)} \left( \sum_{v \in X_2, \min(Q)} P \otimes Q + \sum_{J \in Q} (P \otimes 1) \setminus (J \otimes Q \setminus J) + \sum_{J \in Q} (1 \otimes P) \setminus (J \otimes Q \setminus J) \right).
\]

Hence,

\[
\delta(P \setminus Q) = P \otimes Q + (P \otimes 1) \setminus \delta(Q) + (1 \otimes P) \setminus \delta(Q).
\]

□
Corollary 3.3. Applying the functor $\overline{K}$ gives that, $(\overline{K}(\mathcal{U}), \bigtriangleup)$ endowed with the coproduct $\delta$ is a free pre-Lie algebra and a cofree NAP coalgebra.

**Proof.** This is a direct consequence of Theorem 3.1 and M. Livernet’s rigidity theorem [15, Theorem].

Corollary 3.4. $\overline{K}(\mathcal{U})$ is generated as a pre-Lie algebra by $\text{Prim}(\overline{K}(\mathcal{U}))$.

**Proof.** Let $\overline{K}(\mathcal{U}) = \text{Prim}(\overline{K}(\mathcal{U})) = \{P \in \overline{K}(\mathcal{U}), \delta(P) = 0\}$, and let $\overline{K}(\mathcal{U})_n = \{P \in \overline{K}(\mathcal{U}), \delta(P) \in \sum_{0 \leq k \leq n} \overline{K}(\mathcal{U})_k \otimes \overline{K}(\mathcal{U})_{n-k}\}$, we notice that $\overline{K}(\mathcal{U}) = \bigcup_{k \geq 0} \overline{K}(\mathcal{U})_k$, i.e. the vector space $(\overline{K}(\mathcal{U}), \delta)$ is connected. By applying Corollary 3.9 in [15], we obtain that $\overline{K}(\mathcal{U})$ is generated as a pre-Lie algebra by $\text{Prim}(\overline{K}(\mathcal{U}))$. □

Example 3.2. Here are the posets in $\text{Prim}(\overline{K}(\mathcal{U}))$ up to four vertices:

\[., \wedge, \wedge, \land, \uparrow, \updownarrow, \downarrow, \]

3.4. Duality relation between $\odot$ and $\delta$. In this subsection, we prove that the NAP product $\odot$ and the NAP coproduct $\delta$ are dual to each other modulo a symmetry factor.

**Definition 3.3.** Let $G$ be a group acting on $X$. For every $x \in X$:
- we denote by $G \cdot x := \{g \cdot x, g \in G\}$ the orbit of $x$,
- we denote by $G_x := \{g \in G, g \cdot x = x\}$ the stabilizer subgroup of $G$ with respect to $x$.

**Proposition 3.5.** Let $G$ be a group acting on $X$, if $G$ and $X$ is finite, then the orbit-stabilizer theorem, together with Lagrange’s theorem [19, Theorem 3.9], gives:

$$|G \cdot x| = |G_x : G| = \frac{|G|}{|G_x|}.$$  

In particular, the cardinal of the orbit is a divisor of the group order.

**Definition 3.4.** For any poset $P$ on a finite set $X$, we denote by $\text{Aut}(P)$ the subgroup of permutations of $X$ which are homeomorphisms with respect to $P$. The symmetry factor is defined by $\sigma(P) = |\text{Aut}(P)|$. Let $X_1, X_2$ two finite sets, we define the linear map $(,) : U_{X_1} \otimes U_{X_2} \rightarrow \mathbb{K}$ by:

$$(Q, R) = \begin{cases} \sigma(Q) & \text{if } Q \approx R, \\ 0 & \text{otherwise}. \end{cases}$$

In other terms, $(Q, R)$ is the number of isomorphisms between $Q$ and $R$.

**Theorem 3.2.** Let $P, Q$ and $R$ three finite connected posets. We have the following identity

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\text{min}(P)|} \langle P, Q \odot R \rangle.$$

**Proof.** Let $P, Q$ and $R$ three finite connected posets.

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\text{min}(P)|} \sum_{\substack{I \subseteq P \mid I \approx Q, \sigma_1 : I \sim Q, \sigma_2 : P \setminus I \sim R}} |A|$$

with

$$A = \{(I, \sigma_1, \sigma_2) \mid I \subseteq P, \sigma_1 : I \sim Q, \sigma_2 : P \setminus I \sim R\}.$$
and:

\[ \langle P, Q \circ \to R \rangle = \sum_{v \in \min(R)}^{v \in \min(R)} \sigma(P) = |B| \]

with

\[ B = \{(v, \sigma) \mid v \in \min(R), \sigma : P \to Q \setminus v R\} \].

Let us define now a map \( \phi : A \to B \). Let \( (I, \sigma_1, \sigma_2) \in A \). We put \( w = I \). As \( I \otimes P, w \in \min(P \setminus I) \), so \( v = \sigma_2(w) \in \min(R) \). As \( I \otimes P, P = I \setminus w (P \setminus I) \), so we obtain an isomorphism \( \sigma : P \to Q \setminus w R \) by taking

\[ \sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in I, \\ \sigma_2(x) & \text{otherwise}. \end{cases} \]

We then put \( \phi(I, \sigma_1, \sigma_2) = (v, \sigma) \).

Now, we define a map \( \psi : B \to A \). If \( (v, \sigma) \in B \), we put \( I = \sigma^{-1}(Q) \). As \( Q \otimes Q \setminus v R, I \otimes P \). Moreover, \( \sigma_1 = \sigma_1 \) is a graph isomorphism from \( I \) to \( Q \) and \( \sigma_2 = \sigma_{|P,I} \) is a graph isomorphism from \( P \setminus I \) to \( R \). We put \( \psi(v, \sigma) = (I, \sigma_1, \sigma_2) \).

Let \( (v, \sigma) \in B \). We put \( \psi(v, \sigma) = (I, \sigma_1, \sigma_2) \) and \( \phi \circ \psi(v, \sigma) = (v', \sigma') \). Then, \( I = \sigma^{-1}(Q) \) and \( w' = I = w, \) so \( v' = \sigma(w) = v \). Moreover,

\[ \begin{align*} \\
\sigma_1' &= \sigma_1 = \sigma_I, \\
\sigma_2' &= \sigma_{|P,I}. \\
\end{align*} \]

so \( \sigma' = \sigma \). Hence, \( \phi \circ \psi = \text{id}_B \).

Let \( (I, \sigma_1, \sigma_2) \in A \). We put \( \phi(I, \sigma_1, \sigma_2) = (v, \sigma) \) and \( \psi \circ \phi(I, \sigma_1, \sigma_2) = (I', \sigma_1', \sigma_2') \). Then \( I' = \sigma^{-1}(Q) = I \), by construction of \( \sigma \). Even more,

\[ \begin{align*} \\
\sigma_1' &= \sigma_I = \sigma_1, \\
\sigma_2' &= \sigma_{|P,I} = \sigma_2. \\
\end{align*} \]

So \( \psi \circ \phi = \text{id}_A \). Finally, \( A \) and \( B \) are in bijection and we obtain:

\[ \langle \delta(P), Q \otimes R \rangle = \frac{1}{\min(P)} \langle P, Q \circ \to R \rangle. \]

\[ \square \]

**Remark 3.1.** As mentioned in the paper [3], the binary product \( \circ \) on the species \( \mathbb{U} \) of finite connected posets, defined by \( P \circ Q := j(j(P) \setminus j(Q)) \) for any pair \( (P, Q) \) of finite connected posets, is a pre-Lie product, where \( j \) is the involution which transforms \( \leq \) into \( \geq \). Similarly,

\[ P \circ Q := j(j(P) \circ j(Q)), \quad \text{resp.} \quad \delta = (j \otimes j) \delta \circ j \]

endows the species \( \mathbb{U} \) with a twisted NAP algebra (resp. twisted connected NAP coalgebra) structure.

**Notation.** Let \( E \) be any finite set, and let \( \pi, \rho \) be two partitions of \( E \). We denote by \( j(\pi, \rho) \) the rational number

\[ j(\pi, \rho) := \frac{1}{|E|} \sum_{\alpha \text{ is a block of } \pi}^{\beta \text{ is a block of } \rho} |\alpha \cap \beta| \frac{|\beta|}{|\alpha|}. \]
Proposition 3.6. Let $P, Q$ and $R$ three finite connected posets. We denote by $E = \{v, v \in \text{min}(R) \mid P \approx Q \downarrow v R\}$. Let $\pi$ be the partition of $E$ into $\text{Aut}(P)$-orbits, and let $\rho$ be the partition of $E$ into $\text{Aut}(R)$-orbits. We have the following identity
\[ f(\pi, \rho) = 1. \]

Proof. Let $P, Q$ and $R$ be three finite connected posets. In fact: firstly,
\[ \langle P, Q \circ R \rangle = \sum_{v \in \text{min}(R)} \langle P, Q \downarrow v R \rangle \]
\[ = \sum_{v \in \text{min}(R)} \sigma(P) \]
\[ = \sigma(P) |E|. \]

On the other hand
\[ \langle \delta(P), Q \otimes R \rangle = \frac{1}{|\text{min}(P)|} \sum_{I \in P} \langle I, Q \rangle \langle P \setminus I, R \rangle \]
\[ = \frac{1}{|\text{min}(P)|} \sum_{I \in P} \sigma(Q) \sigma(R) \]
\[ = \frac{1}{|\text{min}(R)|} \sum_{I \in Q, P \setminus J \in R} \sigma(Q) \sigma(R) \]
\[ = \frac{1}{|\text{min}(R)|} \sum_{v \in \text{min}(R)} \sum_{I \in \text{min}(P) \setminus I} N_v(P, Q, R) \sigma(Q) \sigma(R), \]

where $N_v(P, Q, R) :=$ the number of branches $I$ of $P$ above $v$ isomorphic to $Q$ such that $P \setminus I$ isomorphic to $R$.

We notice that, for all $v \in \text{min}(R)$:
- if $P \neq Q \downarrow v R$, then $N_v(P, Q, R) = 0$,
- if $P = Q \downarrow v R$, then $N_v(P, Q, R) = N_v(Q, R) :=$ the number of branches of $Q \downarrow v R$ above $v$ isomorphic to $Q$.

Accordingly
\[ \langle \delta(P), Q \otimes R \rangle = \frac{1}{|\text{min}(R)|} \left( \sum_{v \in \text{min}(R)} N_v(P, Q, R) \sigma(Q) \sigma(R) + \sum_{v \in \text{min}(R)} N_v(P, Q, R) \sigma(Q) \sigma(R) \right) \]
\[ = \frac{1}{|\text{min}(R)|} \sum_{P \in \text{min}(R)} N_v(P, Q, R) \sigma(Q) \sigma(R) \]
\[ = \frac{1}{|\text{min}(R)|} \sum_{P \in \text{min}(R)} N_v(Q, R) \sigma(Q) \sigma(R). \]
By using the orbit-stabilizer theorem, as well as Lagrange’s theorem for the group \( (\text{Aut}(R), \circ) \) endowed with the law of composition, we therefore have:

\[
|\text{Aut}(R) \cdot v| = \frac{|\text{Aut}(R)|}{|\text{Aut}(v)|}, \quad \text{i.e.} \quad |\text{Aut}(R) \cdot v| = \frac{\sigma(R)}{\sigma_v(R)}.
\]

We notice that, for all \( v \in \min(R) \):

\[
\sigma_v(Q \downarrow_v R) = \sigma(Q)\sigma_v(R)N_v(Q, R).
\]

Then

\[
\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(R)|} \sum_{v \in E} \sigma(Q)\sigma_v(R)|\text{Aut}(R) \cdot v|N_v(Q, R)
\]

\[
= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma(Q)|\text{Aut}(R) \cdot v|
\]

\[
= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma(Q)|\text{Aut}(P) \cdot v|
\]

\[
= \frac{1}{|\min(R)|} \sigma(Q)S,
\]

where

\[
S = \sum_{v \in E} \frac{|\text{Aut}(R) \cdot v|}{|\text{Aut}(P) \cdot v|}.
\]

Let \( \pi \) be the partition of \( E \) into \( \text{Aut}(P) \)-orbits, and let \( \rho \) be the partition of \( E \) into \( \text{Aut}(R) \)-orbits. We denote by \( \alpha \in \pi \) (respectively, \( \beta \in \rho \)), if \( \alpha \) is a block of \( \pi \) (respectively, if \( \beta \) is a block of \( \rho \)).

So

\[
S = \sum_{v \in E} \frac{|\text{Aut}(R) \cdot v|}{|\text{Aut}(P) \cdot v|}
\]

\[
= \sum_{v \in E} \sum_{\beta \text{ is a } \text{Aut}(R)-\text{orbit of } v} \frac{|\beta|}{|\alpha|}
\]

\[
= \sum_{\alpha \text{ is a } \text{Aut}(P)-\text{orbit}} \sum_{\beta \text{ is a } \text{Aut}(R)-\text{orbit}} \frac{|\beta|}{|\alpha|}
\]

\[
= \sum_{\alpha \text{ is a } \text{Aut}(P)-\text{orbit}} \sum_{\beta \text{ is a } \text{Aut}(R)-\text{orbit}} |\alpha \cap \beta| \frac{|\beta|}{|\alpha|}
\]

then

\[
\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(R)|} \sigma(P) j(\pi, \rho).
\]

Correspondingly,

\[
\langle \delta(P), Q \otimes R \rangle = \langle P, Q \circrightarrow R \rangle \frac{j_{P,R}(\pi, \rho)}{|\min(P)|}.
\]

By applying Theorem 3.2, we obtain:

\[
\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(P)|} \langle P, Q \circrightarrow R \rangle \frac{j_{P,R}(\pi, \rho)}{|\min(P)|}.
\]

Conclusion: for \( P, Q \) and \( R \) three finite connected posets, if we denote by \( E = \{v, v \in \min(R) \mid P \approx Q \downarrow_v R \} \), and for \( \pi \) (resp. \( \rho \)) the partition of \( E \) into \( \text{Aut}(P) \)-orbits (resp. the partition of \( E \) into \( \text{Aut}(R) \)-orbits). Then we obtain

\[
j(\pi, \rho) = 1.
\]
Remark 3.2. We do not have \( j(\pi, \rho) = 1 \) in general for two partitions of a finite set \( E \). Indeed: let \( E \) be the finite set \( \{1, 2, 3, 4\} \), \( \pi = \{\{1, 2\}, \{3, 4\}\} \) and \( \rho = \{\{1, 2, 3\}, \{4\}\} \). Then we obtain \( j(\pi, \rho) = \frac{1}{2} \) and \( j(\rho, \pi) = 1 \). This interesting index does not seem to be present in the literature.

3.5. Free Lie algebras of finite posets. F. Chapoton in [7] showed that free pre-Lie algebras are free as Lie algebras i.e., if \((V, \triangleright)\) is a free pre-Lie algebra. Then it is a free Lie algebra for the Lie bracket \([\cdot , \cdot]\) defined by \([a, b] = a \triangleright b - b \triangleright a\). This result has been obtained before with different methods by L. Foissy [13, Theorem 8.4].

Proposition 3.7. [7, 13] For any vector space \( V \), the free pre-Lie algebra on \( V \) is isomorphic as a Lie algebra to the free Lie algebra on some set \( X(V) \) of generators.

Application [17] Let \( H \) be a Hopf algebra and \( H^0 \) the graded dual of \( H \). The primitive element algebra of the graded dual \( H^0 \) with the bracket \([\cdot , \cdot]\) is a Lie algebra. We denote by \( \star \) the Grossman-Larson product on the dual of \( H \) [3]. This product restricted to \( H^0 \) is the graded dual of the coproduct \( \Delta_{\triangleright} \) [3]. Applying the general setting above to the case \( H = \overline{\mathcal{K}(\mathcal{P})} \) we are studying, we obtain:

\[
(\mathcal{K}(\mathcal{U}), \triangleright) = (\text{Prim}(H^0), \triangleright)
\]

is a free pre-Lie algebra, then it is also a free Lie algebra, therefore \( U(\text{Prim}H^0) = (S(\text{Prim}H^0), \star) \) is a free associative algebra. Then, \( (S(\text{Prim}H^0), \star)^0 = (H, \Delta_{\triangleright}) \) is a coassociative cofree coalgebra.

4. Generalization to finite topological spaces

All the results in this paper remain true for the finite connected topological spaces, with a small change on the definition of the coproduct \( \delta \). Indeed: let \( \mathcal{I} = (X, \leq_{\mathcal{I}}) \) be a finite connected topological space. We define

\[
\delta(\mathcal{I}) = \frac{1}{|\text{min}(\mathcal{I})|} \sum_{Y \in \mathcal{I}} \mathcal{I}_Y \otimes \mathcal{I}_{|X \setminus Y},
\]

where \( Y \otimes \mathcal{I} \) means that \( Y \in \mathcal{I} \) such that:

- \( \overline{Y} \) is a singleton included in \( \text{min}(\overline{\mathcal{I}}) \),
- and \( Y \) is a branch of the set \( \{x \in X \mid \text{such that for all } y \in Y, \text{we have } y \leq_{\mathcal{I}} x \text{ and } y \not\geq_{\mathcal{I}} x\} \).

Here, the set \( Y \) equal to the space \( \{x \notin Y \mid \text{there exists } y \in Y \text{ such that } x \leq_{\mathcal{I}} y\} \), \( \overline{\mathcal{I}} = (X/ \sim_{\mathcal{I}}, \leq) \) is the poset defined on \( X/ \sim_{\mathcal{I}} \) corresponding to the partial order \( \leq \) induced by the quasi-order \( \leq_{\mathcal{I}} \), and \( \overline{Y} = Y/ \sim_{\mathcal{I}} \). Let \( \text{Aut}(\mathcal{I}) \) be the subgroup of permutations of \( X \) which are homeomorphisms with respect to \( \mathcal{I} \), and let \( G = \{g \in \text{Aut}(\mathcal{I}) \mid g \cdot \overline{Y} = \overline{Y} \text{ for all } v \in X\} \). We notice that \( G \subseteq \text{Aut}(\mathcal{I}) \), \( G \) is a normal subgroup and \( \overline{\text{Aut}(\mathcal{I})} = \text{Aut}(\overline{\mathcal{I}})/G \).

We notice that the results obtained in Lemma 3.1 also work for \( Y, Z \otimes \mathcal{I} \). From \( Y, Z \otimes \mathcal{I} \) and Lemma 3.1, we get that both \( \overline{Y}, \overline{Z} \otimes \overline{\mathcal{I}} \), then \( (\overline{Y} = \overline{Z} \text{ or } \overline{Y} \cap \overline{Z} = \emptyset) \), and \( \text{min}(\overline{\mathcal{I}}_{|X \setminus Y}) = \text{min}(\overline{\mathcal{I}}) \). Then \( Y \cap Z = \emptyset \) and \( \text{min}(\overline{\mathcal{I}}_{|X \setminus Y}) = \text{min}(\overline{\mathcal{I}}) \).

Hence to show that the proposition 3.4 also works in topological cases, it suffices to change \( I, J \otimes P \) by \( Y, Z \otimes \mathcal{I} \) in the proof of proposition 3.6. To show that the result of Theorem 3.1 also works in connected finite topological spaces, we change \( I \otimes P \) by \( Y \otimes \mathcal{I} \) in the Proof 3.3.
Finally we prove the existence of a relation between $\delta$ and $\circ\rightarrow$ in the topological case. Exactly: for $\mathcal{T}, \mathcal{T}'$ and $\mathcal{T}''$ three finite connected topological spaces, we have

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{n} \sum_{\gamma \in \min(\mathcal{T})} \langle \mathcal{T}, \mathcal{T}' \circ \rightarrow \mathcal{T}'' \rangle,$$

where $n$ is the number of items in any bag $\overline{v} \in \min(\mathcal{T}'')$, such as $\mathcal{T} \approx \mathcal{T}' \circ \overline{v}, \mathcal{T}''$.

The proof is similar to the proof of Theorem 3.2. Indeed:

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{\min(\mathcal{T})} \sum_{\mathcal{T}' \in \mathcal{T}', \mathcal{T}'' \in \mathcal{T}''} \sigma(\mathcal{T}') \sigma(\mathcal{T}'') = \frac{1}{\min(\mathcal{T})} |A|$$

with

$$A = \{(Y, \sigma_1, \sigma_2) \mid Y \otimes \mathcal{T}, \sigma_1 : \mathcal{T}|_{Y} \rightarrow \mathcal{T}', \sigma_2 : \mathcal{T}|_{Y \setminus v} \rightarrow \mathcal{T}''\}.$$

On the other hand

$$\langle \mathcal{T}, \mathcal{T}' \circ \rightarrow \mathcal{T}'' \rangle = \sum_{v \in \min(\mathcal{T}'')} \langle \mathcal{T}, \mathcal{T}' \circ \rightarrow \mathcal{T}'' \rangle = \sum_{v \in \min(\mathcal{T}'')} \sigma(\mathcal{T}) = |B|,$$

with

$$B = \{(v, \sigma) \mid v \in \min(\mathcal{T}''), \sigma : \mathcal{T} \rightarrow \mathcal{T}' \circ \rightarrow \mathcal{T}''\}.$$

Let

$$C = \{(u, \sigma) \mid u = \overline{v}, v \in \min(\mathcal{T}''), \sigma : \mathcal{T} \rightarrow \mathcal{T}' \circ \overline{v}, \mathcal{T}''\}.$$

We notice that for all $u, u' \in C$, we have $|u| = |u'|$. Hence $|\overline{v}|$ is the number of items in any bag $\overline{v} \in \min(\mathcal{T}'')$, such as $\mathcal{T} \approx \mathcal{T}' \circ \overline{v}, \mathcal{T}''$. We define

$$f : A \rightarrow C$$

$$(Y, \sigma_1, \sigma_2) \mapsto (u, \sigma),$$

where $u = \overline{v} = \sigma_2(Y)$ and $\sigma$ is a graph isomorphism from $\mathcal{T}$ to $\mathcal{T}' \circ \overline{v}, \mathcal{T}''$ defined by

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

Now, we define a map

$$g : C \rightarrow A$$

$$(u, \sigma) \mapsto (Y, \sigma_1, \sigma_2),$$

where $Y = X_{\sigma^{-1}(\mathcal{T})}, \sigma_1 = \sigma_{\beta|Y}$ is a graph isomorphism from $\mathcal{T}|_{Y} \mathcal{T}'$ and $\sigma_2 = \sigma_{\beta|Y}$ is a graph isomorphism from $\mathcal{T}|_{Y \setminus v} \mathcal{T}''$.

Let $(u, \sigma) \in C$, we put $g(u, \sigma) = (Y, \sigma_1, \sigma_2)$, where $Y = X_{\sigma^{-1}(\mathcal{T})}, \sigma_1 = \sigma_{\beta'},$ and $\sigma_2 = \sigma_{\beta''}$. Then $f \circ g(u, \sigma) = (u', \sigma')$, with $u' = \overline{w} = \sigma_2(Y)$ and $\sigma'$ is a graph isomorphism from $\mathcal{T}$ to $\mathcal{T}' \circ \overline{w}, \mathcal{T}''$ defined by

$$\sigma'(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$
Then, \( u' = \sigma_2(Y) - \) u and \( \sigma' = \sigma \). Hence \( f \circ g = \text{Id}_C \).

Let \((Y, \sigma_1, \sigma_2) \in A\), we put \( f(Y, \sigma_1, \sigma_2) = (u, \sigma) \), with \( u = \bar{\nu} = \sigma_2(Y) \) and \( \sigma \) is a graph isomorphism from \( T \) to \( T' \backslash_{\nu} T'' \) defined by

\[
\sigma(x) = \begin{cases} 
\sigma_1(x) & \text{if } x \in Y, \\
\sigma_2(x) & \text{otherwise}.
\end{cases}
\]

Then, \( g \circ f(Y, \sigma_1, \sigma_2) = (Y', \sigma_1', \sigma_2') \), with \( Y' = X_{\nu^{-1}(T')} \), \( \sigma_1' = \sigma_{T'Y'} \) is a graph isomorphism from \( T'_{Y'} \rightarrow T' \) and \( \sigma_2' = \sigma_{T'X_{Y'}} \) is a graph isomorphism from \( T'_{X_{Y'}} \rightarrow T'' \).

Then, \( Y' = X_{\nu^{-1}(T')} = Y \) and

\[
\begin{cases}
\sigma_1' = \sigma_{T'Y'} = \sigma_1, \\
\sigma_2' = \sigma_{T'X_{Y'}} = \sigma_2.
\end{cases}
\]

So \( g \circ f = \text{Id}_A \).

Finally, \( A \) and \( C \) are in bijection. Hence

\[
\langle \delta(T), T' \otimes T'' \rangle = \frac{1}{|\min(T)|} |A| = \frac{1}{|\min(T')|} |C| = \frac{1}{|\min(T)|} |C| \frac{|B|}{|B||\min(T)|} \langle T, T' \otimes T'' \rangle.
\]

**Remark.** All the results in this paper remain true for the posets (resp. topological spaces) decorated by \( F \), that is to say pairs \((A, f)\), where \( A = (A, \leq_A) \) is a poset (resp. topology) and \( f : A \rightarrow F \) is a map.

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