Boundary regularity for parabolic systems in convex domains

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Abstract
In a cylindrical space–time domain with a convex, spatial base, we establish a local Lipschitz estimate for weak solutions to parabolic systems with Uhlenbeck structure up to the lateral boundary, provided homogeneous Dirichlet data are assumed on that part of the lateral boundary.

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1 INTRODUCTION

This paper studies boundary regularity of weak solutions \( u : \Omega_T \to \mathbb{R}^N, N \geq 1 \), to nonlinear parabolic systems of the type

\[
\partial_t u^i - \sum_{\alpha=1}^{n} \left[ a(|Du|)u^i_{x_\alpha} \right]_{x_\alpha} = b^i \quad \text{for } i = 1, \ldots, N, \tag{1.1}
\]

in a space–time cylinder \( \Omega_T = \Omega \times (0, T) \), where \( \Omega \subset \mathbb{R}^n \) is a bounded open convex set, \( n \geq 2 \) and \( T > 0 \). We assume that \( u \) satisfies a homogeneous Dirichlet boundary condition on some part of the lateral boundary \( (\partial \Omega)_T = \partial \Omega \times (0, T) \). The nonlinearity \( a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) fulfills a growth condition of the type \( a(r) + ra'(r) \approx r^{p-2} \) for some growth exponent \( p > \frac{2n}{n+2} \). As such the diffusion part in (1.1) is said to have the Uhlenbeck structure. For the right-hand side, we require \( b \in L^\sigma(\Omega_T, \mathbb{R}^N) \) for some \( \sigma > n+2 \).

The primary purpose of this paper is to establish

\[
Du \in L^\infty(\Omega_T \cap Q_\sigma(z_0), \mathbb{R}^{Nn})
\]

whenever \( u \) is a weak solution to the system (1.1) satisfying \( u \equiv 0 \) on the subset \((\partial \Omega)_T \cap Q_\sigma(z_0)\) of the lateral boundary. Here, \( Q_\sigma(z_0) : = B_\sigma(x_0) \times (t_0 - \sigma^2, t_0) \) for some \( z_0 = (x_0, t_0) \in (\partial \Omega)_T \).

We only require that \( \Omega \) is a bounded open convex set. No further regularity of \( \partial \Omega \) is assumed. The qualitative assertion is confirmed by a quantitative \( L^\infty \)-estimate for the spatial gradient \( Du \).

Regularity problems for nonlinear equations or systems of the parabolic \( p \)-Laplacian type and their stationary counterparts were very difficult to access in the past. The interior \( C^{1,\lambda} \)-regularity had been longstanding open problems. The first major breakthrough was achieved by Uraltseva [34] in 1968. She showed that solutions to \( p \)-Laplacian equations, whose model is given by

\[
- \text{div} \left( |Du|^{p-2} Du \right) = 0 \quad \text{in } \Omega,
\]
are of class $C^{1,\lambda}$ in the interior of the domain $\Omega \subset \mathbb{R}^n$. This result was generalized in 1977 by Uhlenbeck in her famous paper [33] to the $p$-Laplacian type systems (that is, elliptic version of (1.1))

$$-\sum_{\alpha=1}^{n} \left[ a(|Du|)u_{\alpha} \right]_{x_{\alpha}} = 0 \quad \text{for } i = 1, \ldots, N. \quad (1.2)$$

More general structures, replacing $|Du|^2$ by some quadratic expression $Q(Du, Du)$ and including a sufficiently regular dependence on $x \in \Omega$, have been considered by Tolksdorf [32]. Roughly speaking, in the weak formulation of (1.2) the nonlinear term $a(|Du|)Du$ is replaced by $a(x, Q(Du, Du)^{\frac{1}{2}})Q(Du, \cdot)$. Similar $C^{1,\lambda}$-regularity results have been shown for minimizers of integral functionals with $p$-growth. The degenerate case with growth exponent $p \geq 2$ goes back to Giaquinta and Modica [23], while the singular case $1 < p < 2$ was treated by Acerbi and Fusco [1]. For systems of $p$-Laplacian type as in (1.2), sharp pointwise interior gradient bounds in terms of a nonlinear potential of the right-hand side $b$ have been established in [17].

Regarding the boundary regularity for $p$-Laplacian type systems the picture is less complete. Global $C^{1,\lambda}$-regularity is known only for homogeneous Dirichlet and Neumann boundary data (see Hamburger [25]). For general boundary data, it is still an open problem. However, local $L^\infty$-gradient bounds (Lipschitz estimates at the boundary) have been established by Foss [19] for minimizers to asymptotically regular integral functionals on domains with $C^{1,\lambda}$-boundary (see also [20, 27]). Again for homogeneous Dirichlet or Neumann data, global Lipschitz estimates (in terms of the right-hand side $b$ of (1.2) and under minimal assumptions on the regularity of $\partial \Omega$ and $b$) have been proved by Cianchi and Maz'ya in [10, 11]. These results are global in nature and only valid if $u$ or its outer normal derivative $\partial_n u$ vanishes on the whole boundary $\partial \Omega$. It is noteworthy that their results hold for convex domains in particular. In contrast to these global results, Banerjee and Lewis [3] established local boundary Lipschitz estimates with homogeneous data for convex domains. Their result is of local nature as they only require the homogeneous boundary condition on a part of the boundary. Inspired by the technique introduced in [3], Marcellini, the first two, and the last author were able to establish the first local boundary Lipschitz estimate for integral functionals with non-standard $p, q$-growth (see [5]).

The interior $C^{1,\lambda}$ regularity theory for the parabolic $p$-Laplacian type systems (1.1) is a fundamental achievement by DiBenedetto and Friedman; see [14–16] and the monograph [12, Chapters VIII, IX, X]; see also Chen [8] and Wiegner [35]. For systems without Uhlenbeck structure of the type

$$\partial_t u^i - \sum_{\alpha=1}^{n} \left[ a_{\alpha}^i(Du) \right]_{x_{\alpha}} = 0 \quad \text{for } i = 1, \ldots, N,$$

with a nonlinear diffusion $a$ that behaves asymptotically like the $p$-Laplacian at the origin, that is, $s^{1-p}a(s\xi) \to |\xi|^{p-2}\xi$ in the limit $s \downarrow 0$ for any $\xi$, partial $C^{1,\lambda}$-regularity has been established by Bögelein and Duzaar and Mingione [6].

In contrast to the interior regularity theory, the boundary regularity is largely an open problem. There were two results achieved by Chen and DiBenedetto in [9] for the parabolic systems with the Uhlenbeck structure in $C^{1,\lambda}$-domains. The first was about the Hölder continuity of a solution $u$ up to the lateral boundary with any Hölder exponent in $(0,1)$, given sufficiently regular boundary data (see also [12, Chapter X, Theorem 1.1]). The second dealt with the Hölder
continuity of $Du$ up to the lateral boundary, given homogeneous boundary data (see [12, Chapter X, Theorem 1.2]). The results have been achieved by a boundary flattening procedure. This allows us, after freezing the coefficients, to reduce the problem to the interior case via reflection along the flat boundary. At this stage it is important that the transformed coefficients admit certain quantitative Hölder-regularity. In the course of the proof the authors established gradient sup-estimates for the model case of $p$-Laplacian systems with homogeneous Dirichlet data when the boundary is flat (see [9, Propositions 3.1, 3.1']). These estimates serve as reference inequalities when comparing the solution with the one to the frozen system. This is why $\partial \Omega$ and $g$ are assumed to be $C^{1,\lambda}$. This approach fails in the case of Lipschitz domains.

Boundary regularity for more general parabolic systems has been considered by the first author in [4]. The main result ensures the boundedness up to the lateral boundary of the spatial derivative of weak solutions to asymptotically regular parabolic systems. Roughly speaking, this means that the $C^1$-coefficients of the diffusion part behave like the $p$-Laplacian when $|Du|$ becomes large. The result holds true for inhomogeneous boundary values. As in [9], the proof relies on a boundary flattening procedure and comparison arguments. Therefore, $\partial \Omega$ and $g$ have to be of class $C^{1,\lambda}$.

1.1 Statement of the result

We assume that the nonlinearity $a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $C^1(\mathbb{R}_{>0}, \mathbb{R}_{>0})$ and satisfies

$$\lim_{r \downarrow 0} ra(r) = 0. \quad (1.3)$$

Moreover, $a$ fulfills a standard monotonicity and $p$-growth condition

$$m(\mu^2 + r^2)^{\frac{p-2}{2}} \leq a(r) + ra'(r) \leq M(\mu^2 + r^2)^{\frac{p-2}{2}} \quad \text{for all } r > 0, \quad (1.4)$$

with positive constants $0 < m \leq M$, some parameter $\mu \in [0, 1]$, and some growth exponent $\frac{2n}{n+2} < p < \infty$. Note that in the case $\mu > 0$ the parabolic system (1.1) is non-degenerate, while for $\mu = 0$ the diffusion part becomes either degenerate or singular at points with $|Du| = 0$. For the inhomogeneity $b = (b^1, \ldots, b^N) : \Omega_T \to \mathbb{R}^N$, we assume the integrability condition

$$b \in L^\sigma(\Omega_T, \mathbb{R}^N) \quad \text{for some } \sigma > n + 2. \quad (1.5)$$

**Definition 1.1.** Assume that the nonlinearity $a$ and the inhomogeneity $b$ satisfy the assumptions (1.3)–(1.5). A map $u : \Omega_T \to \mathbb{R}^N$ with

$$u \in C^0([0,T];L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$$

is called a weak solution to the nonlinear parabolic system (1.1) if and only if

$$\iint_{\Omega_T} [u \cdot \varphi_t - a(|Du|)Du \cdot D\varphi] \, dx \, dt = \iint_{\Omega_T} b \cdot \varphi \, dx \, dt$$

for any test function $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$. 
Throughout this article, $d$ denotes the scaling deficit given by

$$d := \begin{cases} \frac{p}{2}, & \text{if } p \geq 2, \\ \frac{2p}{p(n+2)-2n}, & \text{if } \frac{2n}{n+2} < p < 2. \end{cases}$$  \hspace{1cm} (1.6)$$

Now we can state our main result.

**Theorem 1.2** ($L^\infty$-gradient bound at the boundary). Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set, and assume that the structural assumptions (1.3)–(1.5) are in force and let $u \in C^0([0,T]; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to the parabolic system (1.1) in the sense of Definition 1.1 satisfying the homogeneous Dirichlet boundary condition

$$u \equiv 0 \text{ on } (\partial \Omega)_T \cap Q_{2\varepsilon}(z_0) \text{ in the sense of traces},$$

where $z_0 = (x_0, t_0)$ is a point with space center $x_0 \in \partial \Omega$ and time $t_0 \in (0,T)$, and $\varepsilon \in (0, \frac{1}{2} \sqrt{t_0})$. Then, we have

$$Du \in L^\infty(\Omega_T \cap Q_{\varepsilon/2}(z_0)).$$

Moreover, the following quantitative $L^\infty$-gradient bound

$$\sup_{\Omega_T \cap Q_{\varepsilon/2}(z_0)} |Du| \leq C \left[ 1 + \varepsilon^{n+2} \|b\|_{L^\infty(\Omega_T \cap Q_{\varepsilon}(z_0))} \right] \left\{ \left( \frac{n+2}{\sigma-2} \right)^{\frac{n+2}{\sigma-2}} \right\}^\frac{q}{p} \int_{\Omega_T \cap Q_{\varepsilon}(z_0)} (1 + |Du|^p) \, dx \, dt$$

holds with a constant $C$ depending on $n, N, p, \sigma, m, M$, and the geometry of the boundary.

**Remark 1.3.** The dependence of the constant $C$ on the geometry of the boundary can be quantified in terms of the expression $\Theta_{\varepsilon/2}(x_0)$ defined in Section 2.1.

We point out that the gradient bound from the preceding theorem is the exact analogue of the interior gradient bounds in [12, Chapter VIII, Theorems. 5.1, 5.2] for the case $b = 0$.

**1.2 \hspace{1cm} Strategy of the proof**

The usual boundary flattening procedure via a local Lipschitz representation of $\partial \Omega$ leads to a non-linearity depending on the gradient of the Lipschitz graph. Due to the limited regularity of $\partial \Omega$, the transformed nonlinearity admits only a measurable dependence on the spatial variables. This prevents the reduction of the problem to the interior by freezing, comparing and reflection arguments. Therefore, we pursue a different strategy, which is inspired by ideas from Banerjee and Lewis [3]; see also [5] for the corresponding boundary estimate for minimizers to integral functionals with non-standard $p, q$ growth. The present paper represents in some sense the parabolic counterpart of [3].
We establish the sup-estimate of $Du$ in Theorem 1.2 as the limit of similar estimates for more regular approximating problems. More precisely, we approximate the convex domain $\Omega$ in Hausdorff distance from outside by smooth convex domains $\Omega_{\varepsilon}$, regularize the nonlinearity $a$ into $a_{\varepsilon}$, extend $u$ and $b$ by zero outside of $\Omega_T$, and mollify them properly into $g_{\varepsilon}$ and $b_{\varepsilon}$. Then we solve in $(\Omega_{\varepsilon})_T \cap Q_{\varepsilon}(x_o,t_o)$ the Cauchy–Dirichlet problem associated to $a_{\varepsilon}$ and $b_{\varepsilon}$, with boundary values $g_{\varepsilon}$ on the parabolic boundary of $(\Omega_{\varepsilon})_T \cap Q_{\varepsilon}(x_o,t_o)$. The unique solution $u_{\varepsilon}$ — which exists by standard methods — fulfills the Dirichlet condition $u_{\varepsilon} = 0$ on $(\partial \Omega_{\varepsilon})_T \cap Q_{\varepsilon}(x_o,t_o)$ by construction. Since the domain of $u_{\varepsilon}$ is smooth, we may use a reflection argument together with the interior $C^{1,\lambda}$-regularity results and the Schauder estimates for linear parabolic systems to show that these solutions are smooth up to the boundary (see Appendix B).

Next, we prove a quantitative sup-estimate for $Du_{\varepsilon}$, which is uniform in the parameter $\varepsilon$. Its proof consists of two steps. In the first step, we derive an energy estimate for the second-order derivatives (see Proposition 3.2). The key ingredient is a differential geometric identity from [24] (see Lemma 2.1). This identity allows us to exploit the convexity of $\partial \Omega$, in the sense that the boundary integral, which cannot be controlled by integrals over $(\Omega_{\varepsilon})_T \cap Q_{\varepsilon}(x_o,t_o)$, admits a sign and can be discarded in the estimate. Based on the energy estimate, we then perform a Moser iteration, which leads to the sup-estimate for $Du_{\varepsilon}$ in Proposition 3.1.

Finally, we pass to the limit $\varepsilon \downarrow 0$, which can be achieved by certain compactness arguments. Decisive for this argument are the uniform (in $\varepsilon$) energy estimates for the solutions to the regularized problems. The main obstruction at this stage is that testing the original parabolic system by the difference $u - u_{\varepsilon}$ is not allowed, since $u_{\varepsilon}$ does not admit zero boundary values on $(\partial \Omega)_T \cap Q_{\varepsilon}(x_o,t_o)$. Moreover, $u$ is not sufficiently regular in time, that is, the extension of $u$ by zero outside of $\Omega_T$ does not necessarily admit a distributional time derivative on $(\Omega_{\varepsilon})_T \cap Q_{\varepsilon}(x_o,t_o)$. This is why we will not choose the zero extension of $u$ as boundary values for $u_{\varepsilon}$, but the modified version $g_{\varepsilon} := \eta_{\varepsilon}(x)u$. The cutoff function $\eta_{\varepsilon}$ is chosen to vanish on the set $\{ x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \}$. With the aid of Hardy’s inequality, one checks that $g_{\varepsilon}$ admits a time derivative in the dual space on the domain $(\Omega_{\varepsilon})_T \cap Q_{\varepsilon}(x_o,t_o)$. Note that this choice of the boundary values does not affect the sup-estimate for $Du_{\varepsilon}$. On the other hand, the choice allows us to derive an appropriate uniform energy estimate for $u_{\varepsilon}$. Thus, we can pass to a weakly convergent subsequence with the weak limit $\tilde{u}$. Since the sup-estimate for $Du_{\varepsilon}$ is uniform in $\varepsilon$, it can be transferred to $D\tilde{u}$. To conclude, it is left to show $\tilde{u} = u$. This, however, follows from the uniqueness.

2 | PRELIMINARIES

2.1 | A remark on convex domains

The dependence of the constant from Theorem 1.2 on the domain is given by the quantity

$$\Theta_\varphi(x_o) := \frac{\varphi^n}{|\Omega \cap B_\varphi(x_o)|}, \quad \text{for } x_o \in \partial \Omega \text{ and } \varphi > 0. \quad (2.1)$$

Since every bounded convex set satisfies a uniform cone condition, $\Theta_\varphi(x_o)$ can be bounded independently of $x_o \in \partial \Omega$ and $\varphi > 0$ by a constant only depending on the domain $\Omega$. For a more detailed discussion, we refer to [5, Section 2.1].
2.2 A differential geometric identity

For a $C^2$-domain $\Omega \subset \mathbb{R}^n$, the second fundamental form of $\partial \Omega$ is defined by

$$B_x(\xi, \eta) := -\partial_x \nu(x) \cdot \eta$$

for any $x \in \partial \Omega$ and all tangential vectors $\xi, \eta \in T_x(\partial \Omega)$, where $\nu \in C^1(\partial \Omega, \mathbb{R}^n)$ denotes the outer unit normal vector field on $\partial \Omega$.

We will use the following differential geometric identity due to Grisvard [24, Equation (3.1.1.8)].

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^2$-domain and $w \in C^1(\overline{\Omega}, \mathbb{R}^n)$ a vector field. Then we have the following identity on $\partial \Omega$:

$$(w \cdot \nu) \text{div} w - \partial_w w \cdot \nu = \text{div}_T((w \cdot \nu)w_T) - (\text{trace } B)(w \cdot \nu)^2 - B(w_T, w_T) - 2w_T \cdot \nabla_T(w \cdot \nu),$$

where $w_T := w - (w \cdot \nu)\nu$ denotes the tangential component of $w$ and $\nabla_T$, $\text{div}_T$ the gradient and the divergence, respectively, with regard to the tangential directions.

Note that for a convex domain $\Omega$, our sign convention for the second fundamental form implies

$$B_x(\eta, \eta) \leq 0 \quad \text{for any } \eta \in T_x(\partial \Omega).$$

2.3 Properties of the coefficients $a(r)$

Keeping in mind assumption (1.3), we observe

$$a(r) = \frac{1}{r} \int_0^r \frac{d}{ds}[sa(s)]ds = \int_0^1 [a(r\sigma) + r\sigma a'(r\sigma)]d\sigma \quad \text{for any } r > 0.$$ 

Therefore, assumption (1.4) and standard estimates (cf. [1, Lemma 2.1; 22, Lemma 2.1]) imply

$$c^{-1}m\left(\mu^2 + r^2\right)^{\frac{p-2}{2}} \leq a(r) \leq cM\left(\mu^2 + r^2\right)^{\frac{p-2}{2}} \quad \text{(2.2)}$$

for all $r > 0$ and a constant $c = c(p)$. For the derivatives of the coefficients $a(|\xi|)\xi^i_\alpha$, we compute

$$\partial_{\xi^j_\beta} [a(|\xi|)\xi^i_\alpha] = a(|\xi|)\delta_{\alpha\beta}\delta^i_j + \frac{a'(|\xi|)}{|\xi|} \xi^i_\alpha \xi^j_\beta$$

for any $\xi \in \mathbb{R}^{Nn}$ with $\xi \neq 0$. This implies the monotonicity and growth property

$$h(|\xi|)|\lambda|^2 \leq \sum_{i,j=1}^N \sum_{\alpha, \beta=1}^n \partial_{\xi^i_\beta} [a(|\xi|)\xi^i_\alpha] \lambda^i_\alpha \lambda^j_\beta \leq H(|\xi|)|\lambda|^2, \quad \text{(2.3)}$$
for any $\xi, \lambda \in \mathbb{R}^n$ with $\xi \neq 0$, where we abbreviated
\begin{equation}
\begin{aligned}
&\{ h(r) := \min\{a(r), a(r) + ra'(r)\} \geq c^{-1}m(\mu^2 + r^2)^{\frac{p-2}{2}}, \\
&H(r) := \max\{a(r), a(r) + ra'(r)\} \leq cM(\mu^2 + r^2)^{\frac{p-2}{2}}. \}
\end{aligned}
\end{equation}

The estimates follow from (2.2) and (1.4) by distinguishing the cases $a'(r) \geq 0$ and $a'(r) < 0$.

### 2.4 Sobolev’s constant on convex domains

To determine the dependencies of the constants in the Moser iteration, we rely on the following version of Sobolev’s embedding valid for convex domains (cf. [5, Lemma 2.3; 13, Chapter 10, Theorem 8.1]).

**Lemma 2.2.** Let $K \subset \mathbb{R}^n$ be a bounded open convex set and $1 \leq p < n$. Then, for any $w \in W^{1,p}(K)$ we have
\[
\left[ \int_K |w|^{p^*} \, dx \right]^\frac{1}{p^*} \leq c(n, p) \frac{(\text{diam} K)^n}{|K|} \left[ \int_K |Dw|^p \, dx \right]^\frac{1}{p} + \left[ \int_K |w|^p \, dx \right]^\frac{1}{p},
\]
with the Sobolev exponent $p^* = \frac{np}{n-p}$.

### 2.5 Auxiliary lemma

The following elementary assertions will be used in the Moser iteration.

**Lemma 2.3.** Let $A > 1$, $\theta > 1$, $\gamma > 0$ and $k \in \mathbb{N}$. Then, we have
\begin{equation}
\prod_{j=1}^k A^{\frac{j-1}{\theta(\theta^k - 1)}} = A^{\frac{\theta}{\theta(\theta - 1)}}
\end{equation}
and
\begin{equation}
\prod_{j=1}^k A^{\frac{j-1}{\theta(\theta^k - 1)}} \leq A^{\frac{\theta^2}{\theta(\theta - 1)^2}}
\end{equation}

**Proof.** For the first product, we compute
\[
\prod_{j=1}^k A^{\frac{j-1}{\theta(\theta^k - 1)}} = \exp \left[ \log A \sum_{j=1}^k \frac{\theta^{j-1}}{\gamma(\theta^k - 1)} \right] = \exp \left[ \frac{\log A}{\gamma(1 - \theta^{-k})} \sum_{j=1}^k \theta^{-j+1} \right] = \exp \left[ \frac{\log A}{\gamma(1 - \theta^{-1})} \right] = A^{\frac{\theta}{\theta(\theta - 1)}}.
\]
Similarly, we re-write the second product in the form
\[
\prod_{j=1}^{k} A^{j \partial_{j}^{k+1}} = \exp \left[ \log A \sum_{j=1}^{k} j \partial_{j}^{k+1} \right] = \exp \left[ \frac{\log A}{\gamma(1 - \theta^{-k})} \sum_{j=1}^{k} j \theta^{-j+1} \right].
\]

To estimate the right-hand side further, we observe that for any \( t \in (0, 1) \) we have
\[
\frac{1}{1 - t^k} \sum_{j=1}^{k} j t^{j-1} = \frac{1}{1 - t^k} \frac{d}{dt} \sum_{j=0}^{k} t^j = \frac{1}{1 - t^k} \frac{d}{dt} \frac{1 - t^{k+1}}{1 - t} \leq \frac{1}{(1 - t)^2}.
\]

We use this estimate with the choice \( t = \theta^{-1} \in (0, 1) \) and obtain
\[
\prod_{j=1}^{k} A^{j \partial_{j}^{k+1}} \leq \exp \left[ \frac{\log A}{\gamma(1 - \theta^{-1})^2} \right] = A^{\frac{\partial^2}{\gamma(1 - \theta^{-1})^2}}.
\]

This proves the claim. \( \square \)

### 3 A PRIORI ESTIMATES FOR SMOOTH SOLUTIONS

We begin by proving the desired gradient bound in the case of regular data. More precisely, we additionally assume that the boundary \( \partial \Omega \) is of class \( C^2 \) and that the solution is of class \( C^3 \). Moreover, we consider parabolic systems that are non-degenerate, that is, \( \mu > 0 \), and inhomogeneities with \( \text{spt} b \subset \Omega \times \mathbb{R} \). The precise statement reads as follows.

**Proposition 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain with \( C^2 \)-boundary, \( B_{\varphi}(x_0) \) a ball with \( \frac{\varphi^n}{2^n |\Omega \cap B_{\varphi/2}(x_0)|} \leq \Theta \) for some constant \( \Theta > 0 \), and \( (t_0 - \varphi^2, t_0) \subset (0, T) \). Moreover, we assume that \( u \in C^3(\Omega_T \cap Q_{\varphi}(z_0), \mathbb{R}^N) \) is a solution to the parabolic system
\[
\begin{align*}
\partial_t u^i - \sum_{\alpha=1}^{n} \left[ a(|Du|) u^i_{x_\alpha} \right]_{x_\alpha} &= b^i \quad \text{in } \Omega_T \cap Q_{\varphi}(z_0),
\end{align*}
\]
for \( i = 1, \ldots, N \), where \( a \) and \( b \) satisfy assumptions (1.3)–(1.5) with \( \mu \in (0, 1] \) and
\[
\text{spt } b \subset \Omega \times \mathbb{R}.
\]

Moreover, \( u \equiv 0 \) on \( (\partial \Omega)_T \cap Q_{2\varphi}(z_0) \). Then we have the gradient sup-estimate
\[
\sup_{\Omega_T \cap Q_{\varphi/2}(z_0)} |Du| \leq C \left( 1 + \varphi^{n+2} \|b\|_{L^\infty(\Omega_T \cap Q_{\varphi}(z_0))} \right) \int_{\Omega_T \cap Q_{\varphi/4}(z_0)} (1 + |Du|^p) \, dx \, dt \right)^{\frac{d}{p}},
\]
for some constant \( C \) that depends at most on \( n, N, p, \sigma, M, \) and \( \Theta \) and where \( d \) denotes the scaling deficit defined in (1.6).

The proof is given in the following subsections.
3.1 | **Energy estimates for second-order derivatives**

The first step in the proof of Proposition 3.1 is the derivation of an energy estimate for smooth solutions to the parabolic system (3.1).

**Proposition 3.2** (Energy estimate for second derivatives). *Suppose the hypotheses in Proposition 3.1 hold. Then, for any non-negative increasing $C^1$-function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, any cutoff function $\phi \in C_0^\infty(\mathbb{B}_2(x_0, \mathbb{R}_{\geq 0})$ and any non-negative Lipschitz continuous function $\chi : [t_0 - \varphi^2, t_0] \to \mathbb{R}_{\geq 0}$, we have the estimate*

$$
\iint_{\Omega_T \cap Q_\varphi} \chi \phi^2 \left[ \partial_t [\Psi(|Du|)] + \frac{1}{2} \Phi(|Du|) \sum_{\alpha, \beta, \gamma = 1}^n \sum_{i, j = 1}^N b^{ij}_{\alpha \beta} u^i_{\alpha x} u^j_{\beta x} \right] dx dt 
\leq 2 \iint_{\Omega_T \cap Q_\varphi} \chi \Phi(|Du|) \sum_{\alpha, \beta, \gamma = 1}^n \sum_{i, j = 1}^N b^{ij}_{\alpha \beta} \phi_x u^i_{\alpha x} \phi_x u^j_{\beta x} dx dt 
+ \iint_{\Omega_T \cap Q_\varphi} \chi \phi^2 \Psi(|Du|) Du \cdot Db dx dt,
$$

*where we abbreviated*

$$
\Psi(s) := \int_0^s \Phi(\tau) d\tau
$$

*and*

$$
b^{ij}_{\alpha \beta} := a(|Du|) \delta_{\alpha \beta} \delta^{ij} + \frac{a'(|Du|)}{|Du|} u^i_{\alpha x} u^j_{\beta x}
$$

*for $\alpha, \beta = 1, \ldots, n$ and $i, j = 1, \ldots, N$.*

**Remark 3.3.** We note that the monotonicity conditions (1.4) and (2.4) imply the ellipticity and growth estimates

$$
c^{-1} m (\mu^2 + |Du|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \sum_{\alpha, \beta = 1}^n \sum_{i, j = 1}^N b^{ij}_{\alpha \beta} \lambda^i_{\alpha} \lambda^j_{\beta} \leq cM (\mu^2 + |Du|^2)^{\frac{p-2}{2}} |\lambda|^2
$$

*for any $\lambda \in \mathbb{R}^n$ (cf. (2.3)). Therefore, the preceding proposition yields an energy estimate of the form*

$$
\iint_{\Omega_T \cap Q_\varphi} \chi \phi^2 \left[ \partial_t [\Psi(|Du|)] + \frac{m}{c} \Phi(|Du|)(\mu^2 + |Du|^2)^\frac{p-2}{2} |D^2 u|^2 \right] dx dt 
\leq CM \iint_{\Omega_T \cap Q_\varphi} \chi \Phi(|Du|)(\mu^2 + |Du|^2)^\frac{p}{2} |D\phi|^2 dx dt 
+ \iint_{\Omega_T \cap Q_\varphi} \chi \phi^2 \Phi(|Du|) Du \cdot Db dx dt,
$$

(3.6)
with a constant $C = C(p) \geq 1$. This will be the starting point for the Moser iteration.

**Remark 3.4.** It is crucial that Proposition 3.2 holds for cylinders with arbitrary centers $z_o$, not only for points in the lateral boundary. This allows us to apply it on regularized domains $\Omega_\varepsilon \supset \Omega$, with the choice $z_o \in \partial \Omega \times (0, T)$ independently of $\varepsilon > 0$.

**Proof of Proposition 3.2.** For the sake of convenience, we omit the reference to the center $z_o$ in our notation. We write $v_e$ for the directional derivative of a function $v$ in the direction $e \in \mathbb{R}^n$. We start by differentiating (3.1) in the direction $e$. In view of the identities

$$
\frac{|Dv|}{|Du|} = \sum_{j=1}^N \sum_{\beta=1}^n u^j_{x\beta} u^j_{x\beta e},
$$

and

$$
[a(|Du|)]_e = \frac{a'(|Du|)}{|Du|} \sum_{j=1}^N \sum_{\beta=1}^n u^j_{x\beta} u^j_{x\beta e},
$$

we obtain for $i = 1, \ldots, N$ that

$$
(b^i)_e = \partial_t u^i_e - \sum_{\alpha=1}^n \left[ a(|Du|) u^i_{x\alpha} \right]_{x\alpha e} = \partial_t u^i_e - \sum_{\alpha, \beta=1}^n \sum_{j=1}^N b^{ij}_{\alpha\beta}(x,t) u^j_{x\alpha e}. \quad (3.8)
$$

In the last line, we used the abbreviation introduced in (3.4). Next, we compute

$$
I := \sum_{\alpha, \beta, \gamma=1}^n \sum_{i,j=1}^N \left[ a(|Du|) u^i_{x\alpha} u^j_{x\beta} u^j_{x\alpha x\beta} u^j_{x\alpha x\beta x\gamma} \right]_{x\alpha x\beta x\gamma}
$$

$$
= \sum_{\alpha, \beta, \gamma=1}^n \sum_{i,j=1}^N a'(|Du|) u^i_{x\alpha} u^j_{x\alpha x\gamma} u^j_{x\beta x\gamma}
$$

$$
+ \sum_{\alpha, \beta, \gamma=1}^n \sum_{i,j=1}^N u^i_{x\gamma} \left[ a'(|Du|) u^i_{x\alpha} u^j_{x\beta} u^j_{x\alpha x\gamma} \right]_{x\alpha x\beta x\gamma}
$$

$$
= \sum_{\alpha, \beta, \gamma=1}^n \sum_{i,j=1}^N \left[ b^{ij}_{\alpha\beta} u^i_{x\alpha x\gamma} u^j_{x\beta x\gamma} + u^i_{x\gamma} \left[ b^{ij}_{\alpha\beta} u^j_{x\beta x\gamma} \right]_{x\alpha x\beta x\gamma} \right]
$$

$$
- \sum_{\alpha, \gamma=1}^n \sum_{i=1}^N \left[ a(|Du|) u^i_{x\alpha x\gamma} u^i_{x\alpha x\gamma} + u^i_{x\gamma} \left[ a(|Du|) u^i_{x\alpha x\gamma} \right]_{x\alpha x\gamma} \right].
$$

We note that the last term on the right-hand side of the preceding identity is equal to $\Delta [g(|Du|)]$, where $\Delta$ stands for the Laplacian, and $g$ is defined by

$$
g(s) := \int_0^s r a(r) dr \quad \text{for any } s > 0.
$$

Using the differentiated system (3.8) with $e = e_\gamma$, we thus obtain

$$
I = \sum_{\alpha, \beta, \gamma=1}^n \sum_{i,j=1}^N b^{ij}_{\alpha\beta} u^i_{x\alpha x\gamma} u^j_{x\beta x\gamma} - \Delta [g(|Du|)] + \frac{1}{2} \partial_t |Du|^2 - Du \cdot Db. \quad (3.9)
$$
On the other hand, a direct calculation gives
\[
I + \Delta [g(|Du|)] = L[g(|Du|)],
\]
where \(L\) denotes the second-order elliptic differential operator defined by
\[
L[v] := \sum_{\alpha,\gamma=1}^{n} \left[ c_{\alpha\gamma} v_{\gamma x} \right]_{x_{\alpha}},
\]
with coefficients
\[
c_{\alpha\gamma}(x) := \frac{1}{a(|Du|)} \left[ a(|Du|) \delta_{\alpha\gamma} + a'(|Du|) \sum_{i=1}^{N} u_{i x_{\alpha}} u_{i x_{\gamma}} \right]
\]
and
\[
a'(|Du|) \sum_{i=1}^{N} u_{i x_{\alpha}} u_{i x_{\gamma}}.
\]

Joining (3.9) and (3.10), we find
\[
\frac{1}{2} \partial_{t} |Du|^{2} + \sum_{\alpha,\beta,y=1}^{n} \sum_{i,j=1}^{N} b_{\alpha\beta}^{ij} u_{i x_{\alpha} x_{\gamma}} u_{j x_{\beta} x_{\gamma}} = L[g(|Du|)] + Du \cdot Db.
\]

We now multiply this identity by \(\phi^{2}(x)\), where \(\phi \in C_{0}^{\infty}(B_{r} \setminus \mathbb{R} \geq 0)\) is a smooth cutoff function. In the resulting equation we examine the diffusion term on the right-hand side. We start by noting that
\[
\sum_{\gamma=1}^{n} c_{\alpha\gamma} g(|Du|)_{x_{\gamma}} = \sum_{\beta,y=1}^{n} a(|Du|) c_{\alpha\gamma} u_{j x_{\beta} x_{\gamma}} = \sum_{\beta,y=1}^{n} b_{\alpha\beta}^{ij} u_{i x_{\alpha} x_{\gamma}} u_{j x_{\beta} x_{\gamma}}
\]
for \(\alpha = 1, \ldots, n\), where we used first (3.7) with \(e = e_{\beta}\) and then the definition (3.11) for the coefficients \(c_{\alpha\beta}\) and (3.4). This allows us to compute
\[
\phi^{2} L[g(|Du|)] = \sum_{\alpha,\gamma=1}^{n} \left[ \phi^{2} c_{\alpha\gamma} g(|Du|)_{x_{\gamma}} \right]_{x_{\alpha}} - 2\phi \sum_{\alpha,\gamma=1}^{n} \phi x_{\alpha} c_{\alpha\gamma} g(|Du|)_{x_{\gamma}}
\]
\[
= \Pi - 2\phi \sum_{\alpha,\gamma=1}^{n} \sum_{i,j=1}^{N} b_{\alpha\beta}^{ij} \phi x_{\alpha} u_{i x_{\gamma}} u_{j x_{\beta} x_{\gamma}},
\]
with the obvious meaning of \(\Pi\). Inserting this identity into (3.12) multiplied by \(\phi^{2}\) as described above, we deduce
\[
\phi^{2} \left[ \frac{1}{2} \partial_{t} |Du|^{2} + \sum_{\alpha,\beta,y=1}^{n} \sum_{i,j=1}^{N} b_{\alpha\beta}^{ij} u_{i x_{\alpha} x_{\gamma}} u_{j x_{\beta} x_{\gamma}} \right] = \Pi - 2\phi \sum_{\alpha,\gamma=1}^{n} \sum_{i,j=1}^{N} b_{\alpha\beta}^{ij} \phi x_{\alpha} u_{i x_{\gamma}} u_{j x_{\beta} x_{\gamma}} + \phi^{2} Du \cdot Db.
\]
Next, we note that due to (3.5), the matrix \((b^{ij}_{\alpha\beta})\) defines a positive definite bilinear form on \(\mathbb{R}^{Nn}\), which grants a Young type inequality for quadratic forms. That is,

\[
2\phi \left| \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\alpha\beta} \phi_{x^\alpha} u^i_{x^\gamma} u^j_{x^\gamma} \right| 
\leq \frac{1}{2} \phi^2 \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\alpha\beta} u^i_{x^\alpha} u^j_{x^\gamma} + 2 \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\alpha\beta} \phi_{x^\alpha} u^i_{x^\gamma} \phi_{x^\beta} u^j_{x^\gamma},
\]

for any \(\gamma = 1, \ldots, n\). Using this estimate in the identity above and re-absorbing the term containing the second derivatives of \(u\) into the left-hand side yields

\[
\frac{1}{2} \phi^2 \left[ \partial_t |Du|^2 + \sum_{\alpha,\beta',\gamma=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\alpha\beta} u^i_{x^\alpha} u^j_{x^\gamma} \right] 
\leq \mathbf{I} + 2 \sum_{\alpha,\beta',\gamma=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\alpha\beta} \phi_{x^\alpha} u^i_{x^\gamma} \phi_{x^\beta} u^j_{x^\gamma} + \phi^2 Du \cdot Db.
\]

Next, we multiply this identity by \(\chi(t)\Phi(|Du|)\), where \(\chi : [t_o - g^2, t_o] \to \mathbb{R}_{\geq 0}\) is a non-negative Lipschitz continuous function and \(\Phi \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})\) is increasing. For the term involving the time derivative, we compute

\[
\frac{1}{2} \chi \Phi(|Du|) \partial_t |Du|^2 = \Phi(|Du|)|Du| \partial_t |Du| = \partial_t \left[ \int_0^{|Du|} \Phi(\tau) d\tau \right] = \partial_t \left[ \Psi(|Du|) \right],
\]

with the function \(\Psi\) defined in (3.3) and obtain

\[
\chi \phi^2 \left[ \partial_t [\Psi(|Du|)] + \frac{1}{2} \Phi(|Du|) \sum_{\alpha,\beta',\gamma=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\alpha\beta} u^i_{x^\alpha} u^j_{x^\gamma} \right] 
\leq \chi \Phi(|Du|) \left[ \mathbf{I} + 2 \sum_{\alpha,\beta',\gamma=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\alpha\beta} \phi_{x^\alpha} u^i_{x^\gamma} \phi_{x^\beta} u^j_{x^\gamma} + \phi^2 Du \cdot Db \right]. \tag{3.14}
\]

Next, we analyze the term containing \(\mathbf{I}\), which will result in a boundary term. Indeed,

\[
\Phi(|Du|) \mathbf{I} = \Phi(|Du|) \sum_{\alpha,\gamma=1}^{n} \left[ \phi^2 c_{\gamma y} g(|Du|)_{x^\gamma} \right]_{x^\alpha}
\leq \sum_{\alpha,\gamma=1}^{n} \left[ \phi^2 \Phi(|Du|) c_{\gamma y} g(|Du|)_{x^\gamma} \right]_{x^\alpha} - \phi^2 \sum_{\alpha,\gamma=1}^{n} c_{\alpha\gamma} \Phi(|Du|)_{x^\alpha} g(|Du|)_{x^\gamma}
\leq \sum_{\alpha,\gamma=1}^{n} \left[ \phi^2 \Phi(|Du|) c_{\gamma y} g(|Du|)_{x^\gamma} \right]_{x^\alpha}.
\]
In the last line, we used
\[
\sum_{\alpha,\gamma=1}^{n} c_{\gamma \alpha} \Phi(|Du|)_{\gamma \alpha} g(|Du|) = \Phi'(|Du|) g'(|Du|) \sum_{\alpha,\gamma=1}^{n} c_{\gamma \alpha} |Du|_{\gamma \alpha} |Du|_{\gamma \alpha} \geq 0,
\]
since \(\Phi\) and \(g\) are both increasing and that the coefficients \(c_{\gamma \alpha}\) are positive definite.

We now integrate \(\chi \Phi(|Du|) II\) over \(\Omega_T \cap Q_\varphi\) and perform an integration by parts. This leads to a boundary integral. More precisely, denoting by \(\nu\) the outward unit normal vector on \(\partial \Omega\), we have
\[
\int_{\Omega_T \cap Q_\varphi} \chi \Phi(|Du|) II \, dx \, dt \leq \int_{(\partial \Omega)_T \cap Q_\varphi} \chi \Phi^2(|Du|) \sum_{\alpha,\gamma=1}^{n} c_{\gamma \alpha} |Du|_{\gamma \alpha} \nu_{\alpha} \, dH^{n-1} \, dt
\]
\[
= \int_{(\partial \Omega)_T \cap Q_\varphi} \chi \Phi^2(|Du|) \sum_{\alpha,\beta,\gamma=1}^{n} b^{ij}_{\gamma \alpha} u^i_{\gamma \alpha} u^j_{\gamma \beta} \nu_{\alpha} \, dH^{n-1} \, dt. \tag{3.15}
\]
The last step follows from (3.13). Now, we analyze the integrand on the right-hand side by recalling the explicit form of the coefficients. In view of (3.4), we obtain
\[
\sum_{\alpha,\beta,\gamma=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\gamma \alpha} u^i_{\gamma \alpha} u^j_{\beta \gamma} \nu_{\alpha} \leq \sum_{i=1}^{n} u^i_{\gamma \alpha} \sum_{\gamma=1}^{n} \sum_{j=1}^{N} \left[ a(|Du|) \delta_{\gamma \beta} \delta^{ij} + \frac{a'(|Du|)}{|Du|} u^i_{\gamma \alpha} u^j_{\gamma \beta} \right] u^j_{\gamma \beta} \nu_{\alpha}
\]
\[
= \sum_{i=1}^{n} u^i_{\gamma \alpha} \sum_{\gamma=1}^{n} \sum_{j=1}^{N} b^{ij}_{\gamma \alpha} u^j_{\gamma \beta} \nu_{\alpha} \leq \sum_{\gamma=1}^{n} \sum_{i=1}^{n} \sum_{\gamma=1}^{n} b^{ij}_{\gamma \alpha} u^i_{\gamma \alpha} \nu_{\alpha} = \sum_{\gamma=1}^{n} u^i_{\gamma \alpha} \sum_{i=1}^{n} \left[ a(|Du|) u^i_{\gamma \alpha} \right] \nu_{\gamma \alpha}
\]
\[
= u_{\gamma \alpha} \cdot \partial_t u - u_{\gamma \alpha} \cdot b = 0.
\]
At this point, we apply the differential geometric identity in Lemma 2.1 to the vector fields \(w := \nabla u^i, i = 1, \ldots, N\). In our case, the tangential components of \(w\) vanish since \(u \equiv 0\) on \((\partial \Omega)_T \cap Q_\varphi\). Hence, Lemma 2.1 yields the identity
\[
\Delta u^i u^i_{\gamma \alpha} = \sum_{\alpha,\gamma=1}^{n} u^i_{\gamma \alpha} \nu_{\alpha} = -\text{trace } B (u^i_{\gamma \alpha})^2 \quad \text{on } (\partial \Omega)_T \cap Q_\varphi
\]
for \(i = 1, \ldots, N\). Here, \(B\) denotes the second fundamental form of \(\partial \Omega\). Note that trace \(B \leq 0\), since \(\Omega\) is convex. Therefore, in the above identity the right-hand side is non-negative. This allows us to continue in estimating the boundary integral above. More precisely, on \((\partial \Omega)_T \cap Q_\varphi\) we obtain
\[
\sum_{\alpha,\beta,\gamma=1}^{n} \sum_{i,j=1}^{N} b^{ij}_{\gamma \alpha} u^i_{\gamma \alpha} u^j_{\beta \gamma} \nu_{\alpha} \leq \sum_{i=1}^{n} u^i_{\gamma \alpha} \sum_{\gamma=1}^{n} \sum_{j=1}^{N} \left[ a(|Du|) \delta_{\gamma \beta} \delta^{ij} + \frac{a'(|Du|)}{|Du|} u^i_{\gamma \alpha} u^j_{\gamma \beta} \right] u^j_{\gamma \beta} \nu_{\gamma \alpha}
\]
\[
= \sum_{i=1}^{n} u^i_{\gamma \alpha} \sum_{\gamma=1}^{n} \sum_{j=1}^{N} b^{ij}_{\gamma \alpha} u^j_{\gamma \beta} \nu_{\gamma \alpha} = \sum_{\gamma=1}^{n} u^i_{\gamma \alpha} \sum_{i=1}^{n} \left[ a(|Du|) u^i_{\gamma \alpha} \right] \nu_{\gamma \alpha}
\]
\[
= u_{\alpha \gamma} \cdot \partial_t u - u_{\alpha \gamma} \cdot b = 0.
\]
In the last line, we used the parabolic system (3.1), the fact \( \partial_t u \equiv 0 \) on \( (\partial \Omega)_T \cap Q_\varphi \) and \( \text{spt } b \in \Omega_T \). Recalling (3.15), we deduce that

\[
\int_{\Omega_T \cap Q_\varphi} \chi \Phi(|Du|) \Pi \, dx \, dt \leq 0.
\]

Therefore, integrating (3.14) over \( \Omega_T \cap Q_\varphi \) we obtain

\[
\int_{\Omega_T \cap Q_\varphi} \chi \Phi^2 \left[ \partial_t [\Psi(|Du|)] + \frac{1}{2} \Phi(|Du|) \sum_{\alpha, \beta, \gamma=1}^n \sum_{i=1}^N b^{ij}_{\alpha \beta} u^i_{x_\alpha x_\gamma} u^j_{x_\beta x_\gamma} \right] \, dx \, dt
\leq \int_{\Omega_T \cap Q_\varphi} \chi \Phi(|Du|) \left[ 2 \sum_{\alpha, \beta, \gamma=1}^n \sum_{i=1}^N b^{ij}_{\alpha \beta} \phi_{x_\alpha} u^i_{x_\gamma} \phi_{x_\beta} u^j_{x_\gamma} + \phi^2 Du \cdot Db \right] \, dx \, dt.
\]

This finishes the proof of the proposition. \( \square \)

### 3.2 A reverse Hölder type inequality

Here, we work in the setting of Proposition 3.1. Again, we omit the reference to the center \( z_0 \) in our notation. By \( \zeta \in C^1(\mathbb{R}_{\geq 0}, [0, 1]) \), we denote a cutoff function with respect to the time variable that satisfies \( \zeta \equiv 0 \) on \([0, \frac{1}{2}]\), \( \zeta \equiv 1 \) on \([1, \infty)\), and \( 0 \leq \zeta' \leq 3 \) on \([\frac{1}{2}, 1]\). Moreover, we consider a cutoff function \( \phi \in C^\infty_0(B_{\bar{r}}, [0, 1]) \) with respect to the spatial variables. In the energy estimate (3.6) we choose the non-negative increasing function \( \Psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) in the form

\[
\Psi(|Du|) := \int_0^{|Du|} \Phi(s) \, ds
\]

\[
= \int_0^{H(Du)} \zeta^2(\tau) \tau^{\alpha+1} \, d\tau \leq \frac{1}{2+2\alpha} \mu^2(H(Du))^{2+2\alpha}.
\]

We could omit the cutoff function \( \zeta \) in the case \( \mu = 1 \). For the sake of a unified approach, we proceed using \( \zeta \) in any case. In the sequel we use the abbreviation

\[
H(\xi) := \sqrt{\mu^2 + |\xi|^2} \quad \text{for } \xi \in \mathbb{R}^{Nn},
\]

so that \( \Phi(|\xi|) = \Phi(H(\xi)) \). With this notation, we have

\[
\Psi(|Du|) \geq \frac{1}{2+2\alpha} \mu \mu^2(H(Du))^{2+2\alpha} - \frac{1}{2+2\alpha}. \quad (3.17)
\]

Now, we start with estimating the second integral on the right-hand side of (3.6). Since \( \text{spt } b(\cdot, t) \subseteq \Omega \) for any \( t \in [0, T] \) and since \( \phi \in C^\infty_0(B_{\bar{r}}) \), we are allowed to integrate by parts (with respect to \( x \)
in the integral containing $b$ and obtain

$$
\sum_{i=1}^{N} \sum_{\alpha=1}^{n} \int_{\Omega_T \cap Q_T} \chi \phi^2 \Phi(|Du|) u^i_{x_\alpha} b^i_{x_\alpha} \, dx \, dt
$$

$$
= - \sum_{i=1}^{N} \sum_{\alpha=1}^{n} \int_{\Omega_T \cap Q_T} \chi b^i \left[ \phi^2 \Phi(|Du|) u^i_{x_\alpha} \right] \, dx \, dt
$$

$$
= - \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \int_{\Omega_T \cap Q_T} \chi \phi^2 b \left[ \Phi(|Du|) \delta_{\alpha\beta} \delta^i_j + \Phi'(|Du|) \frac{u^i_{x_\alpha} u^j_{x_\beta}}{|Du|} \right] \, dx \, dt
$$

$$
- 2 \sum_{i=1}^{N} \sum_{\alpha=1}^{n} \int_{\Omega_T \cap Q_T} \chi b^i \Phi(|Du|) \phi^i_{x_\alpha} \, dx \, dt
$$

$$
=: \text{I} + \text{II},
$$

with the obvious meaning of $\text{I}$ and $\text{II}$. The first integral can be estimated as

$$
\text{I} \leq c(n,N) \int_{\Omega_T \cap Q_T} \chi \phi^2 |b| \left[ \Phi(|Du|) + |Du| \Phi'(|Du|) \right] |D^2 u| \, dx \, dt.
$$

To estimate the term in brackets, we note that $|Du| \leq \mathcal{H}(Du)$, $\zeta \leq 1$ and $|Du| \leq \mathcal{H}(Du) \leq 1$ whenever $\zeta'(\mathcal{H}(Du)) \neq 0$ and finally $\mathcal{H}(Du) \geq \frac{1}{2}$ whenever $\zeta(\mathcal{H}(Du)) \neq 0$. Therefore, we obtain

$$
\Phi(|Du|) + |Du| \Phi'(|Du|)
$$

$$
\leq \zeta(\mathcal{H}(Du)) \mathcal{H}(Du)^{2\alpha} \left[ (1 + 2\alpha) \zeta(\mathcal{H}(Du)) + 2 \zeta'(\mathcal{H}(Du)) |Du| \right]
$$

$$
\leq c (1 + 2\alpha) \zeta(\mathcal{H}(Du)) \mathcal{H}(Du)^{2\alpha}
$$

$$
\leq c (1 + 2\alpha) \zeta(\mathcal{H}(Du)) \mathcal{H}(Du)^{2\alpha + \frac{p-2-2\alpha-p}{2}}.
$$

Inserting this above yields

$$
\text{I} \leq c (1 + 2\alpha) \int_{\Omega_T \cap Q_T} \chi \phi^2 |b| \zeta(\mathcal{H}(Du)) \mathcal{H}(Du)^{2\alpha + \frac{p-2-2\alpha-p}{2}} |D^2 u| \, dx \, dt
$$

$$
\leq \frac{m}{2C} \int_{\Omega_T \cap Q_T} \chi \phi^2 \zeta^2(\mathcal{H}(Du)) \mathcal{H}(Du)^{p+2\alpha-2} |D^2 u|^2 \, dx \, dt
$$

$$
+ \frac{c (1 + 2\alpha)^2}{m} \int_{\Omega_T \cap Q_T} \chi \phi^2 \mathcal{H}(Du)^{2\alpha + (2-p)^+} |b|^2 \, dx \, dt
$$

$$
=: \frac{m}{2C} I_1 + \frac{c (1 + 2\alpha)^2}{m} I_2,
$$
for a constant \( c = c(n, N, p) \). In the last estimate, we used Young’s inequality. The constant \( C \) is from the energy estimate (3.6) and depends only on \( p \). The second term is bounded by

\[
\mathbf{I} \leq 2 \int_{\Omega_T \cap Q_p} \chi \phi |b| |\Phi(Du)| |D\phi| |Du| \, dx \, dt
\]

\[
\leq 2 \int_{\Omega_T \cap Q_p} \chi \phi |b| \xi^2(\mathbf{H}(Du)) \mathbf{H}(Du)^{2\alpha+1} |D\phi| \, dx \, dt
\]

\[
\leq c \int_{\Omega_T \cap Q_p} \chi \phi |b| \mathbf{H}(Du)^{2\alpha+\frac{p(2-p)}{2}} |D\phi| \, dx \, dt
\]

\[
\leq c M \int_{\Omega_T \cap Q_p} \chi \mathbf{H}(Du)^{p+2\alpha} |D\phi|^2 \, dx \, dt + \frac{1}{M} I_2,
\]

where, in the second-to-last step, we again used the fact \( \mathbf{H}(Du) \geq \frac{1}{2} \) on the support of \( \xi(\mathbf{H}(Du)) \), and in the last step we applied Young’s inequality. Using the above estimates for \( \mathbf{I} \) and \( \mathbf{II} \) in (3.6) and re-absorbing the term \( \frac{m}{2C} I_1 \) into the left-hand side, we arrive at

\[
\int_{\Omega_T \cap Q_p} \chi^2 \phi^2 \left[ \partial_t [\Psi(|Du|)] + \frac{m}{2C} \xi^2(\mathbf{H}(Du)) \mathbf{H}(Du)^{p+2\alpha-2} |D^2u|^2 \right] \, dx \, dt
\]

\[
\leq c \int_{\Omega_T \cap Q_p} \chi \mathbf{H}(Du)^{p+2\alpha} |D\phi|^2 \, dx \, dt + c(1 + 2\alpha)^2 I_2, \quad (3.18)
\]

where \( c = c(m, M, n, N, p) \). In (3.18), we choose \( \chi \) in the form of a product of two functions \( \chi \) and \( \tilde{\chi} \). We choose the first function \( \chi \in W^{1,\infty}([t_0 - \varrho^2, t_0]) \) to satisfy \( 0 \leq \chi \leq 1, \chi(t_0 - \varrho^2) = 0 \), and \( \partial_t \chi \geq 0 \), while the second one is defined by

\[
\tilde{\chi}(t) := \begin{cases} 
1, & t \in [t_0 - \varrho^2, \tau], \\
1 - \frac{t - \tau}{\delta}, & t \in (\tau, \tau + \delta), \\
0, & t \in [\tau + \delta, t_0],
\end{cases}
\]

where \( \delta > 0 \) and \( t_0 - \varrho^2 < \tau < \tau + \delta < t_0 \). With this specification of \( \chi \) we consider the first integral on the left-hand side. We perform an integration by parts with respect to time and obtain (observe that no boundary terms occur due to the choice of \( \chi \) and \( \tilde{\chi} \))

\[
\int_{\Omega_T \cap Q_p} \phi^2 \chi \tilde{\chi} \partial_t [\Psi(|Du|)] \, dx \, dt
\]

\[
= - \int_{\Omega_T \cap Q_p} \phi^2 \partial_t [\chi(t) \tilde{\chi}(t)] \Psi(|Du|) \, dx \, dt
\]

\[
= - \int_{\Omega_T \cap Q_p} \phi^2 [\tilde{\chi} \partial_t \chi + \chi \partial_t \tilde{\chi}] \Psi(|Du|) \, dx \, dt
\]

\[
= - \int_{\Omega_T \cap Q_p} \tilde{\chi} \phi^2 \partial_t \chi \Psi(|Du|) \, dx \, dt + \frac{1}{\delta} \int_{\Omega_T \cap B_p \times (\tau + \delta)} \chi \phi^2 \Psi(|Du|) \, dx \, dt.
\]
We insert this into (3.18) and pass to the limit $\delta \downarrow 0$. For $\tau \in (t_o - \varphi^2, t_o)$ we obtain

$$\int_{\Omega \cap B_\varphi \times \{\tau\}} \chi \phi^2 \Psi(|Du|) \, dx + \int_{\Omega \cap B_\varphi \times (t_o - \varphi^2, \tau)} \chi \phi^2 \xi^2 (H(Du)) H(Du)^{p+2\alpha - 2} |D^2 u|^2 \, dx \, dt$$

$$\leq c \left[ \int_{\Omega \cap Q_\varphi} \chi H(Du)^{p+2\alpha} |D\phi|^2 \, dx \, dt + (1 + 2\alpha)^2 I_2 + I_3 \right],$$

where $I_3$ is defined by

$$I_3 := \frac{1}{2(1 + \alpha)} \int_{\Omega \cap Q_\varphi} \hat{\phi} \hat{\xi} H(Du)^{2+2\alpha} \, dx \, dt.$$

In the estimate leading to $I_3$, we used (3.16) and the fact that $\hat{\xi}_t H \geq 0$. Note that $I_3$ is non-negative.

Observe also that the right-hand side of the preceding inequality is independent of $\tau$. Therefore, we can pass to the limit $\tau \uparrow t_o$ in the second integral on the left-hand side, while in the first integral we can take the supremum over $\tau \in (t_o - \varphi^2, t_o)$. This implies

$$\sup_{t_o - \varphi^2 < \tau < t_o} \int_{\Omega \cap B_\varphi \times \{\tau\}} \chi \phi^2 \Psi(|Du|) \, dx + \int_{\Omega \cap Q_\varphi} \chi \phi^2 \xi^2 (H(Du)) H(Du)^{p+2\alpha - 2} |D^2 u|^2 \, dx \, dt$$

$$\leq c \left[ \int_{\Omega \cap Q_\varphi} \chi H(Du)^{p+2\alpha} |D\phi|^2 \, dx \, dt + (1 + 2\alpha)^2 I_2 + I_3 \right],$$

with a constant $c = c(n, N, m, M, p)$. To bound the sup-term from below, we use (3.17) and multiply the resulting inequality by $(p + 2\alpha)$, from which we deduce

$$\sup_{t_o - \varphi^2 < \tau < t_o} \int_{\Omega \cap B_\varphi \times \{\tau\}} \chi \phi^2 H(Du)^{2+2\alpha} \, dx$$

$$+ (p + 2\alpha) \int_{\Omega \cap Q_\varphi} \chi \phi^2 \xi^2 (H(Du)) H(Du)^{p+2\alpha - 2} |D^2 u|^2 \, dx \, dt$$

$$\leq c (p + 2\alpha) \int_{\Omega \cap Q_\varphi} \|D\phi\|_{L^\infty}^2 H(Du)^{p+2\alpha} + \|\hat{\xi}_t H\|_{L^\infty} H(Du)^{2+2\alpha} \, dx \, dt$$

$$+ c (p + 2\alpha)^3 \int_{\Omega \cap Q_\varphi} \chi \phi^2 H(Du)^{2\alpha + (2-p)+} |b|^2 \, dx \, dt + |\Omega \cap B_\varphi|,$$

for a constant $c$ depending on $n, N, m, M, p$. After taking means, the preceding estimate takes the form

$$\sup_{t_o - \varphi^2 < \tau < t_o} \int_{\Omega \cap B_\varphi \times \{\tau\}} \chi \phi^2 H(Du)^{2+2\alpha} \, dx$$

$$+ (p + 2\alpha) \varphi^2 \int_{\Omega \cap Q_\varphi} \chi \phi^2 \xi^2 (H(Du)) H(Du)^{p+2\alpha - 2} |D^2 u|^2 \, dx \, dt$$

$$\leq c (p + 2\alpha) R_1 + c (p + 2\alpha)^3 R_2 + 1 =: R,$$

(3.19)
for a constant $c = c(n, N, m, M, p) \geq 1$ and with the abbreviations

$$R_1 := \varphi^2 \iint_{\Omega_T \cap \Gamma} \left[ \|D\varphi\|_{L^\infty}^2 H(Du)^{p+2\alpha} + \|\partial_t \varphi\|_{L^\infty} H(Du)^{2+2\alpha} \right] \, dx \, dt$$

and

$$R_2 := \varphi^2 \iint_{\Omega_T \cap \Gamma} \chi \varphi^2 H(Du)^{2\alpha + (2-p) + |b|} \, dx \, dt.$$

Observe that we kept the cutoff function $\chi \varphi^2$ in the integrand of the last integral. The reason for that will become clear later.

The next step is to perform an interpolation argument of Gagliardo–Nirenberg type. For the parameter $\delta := \frac{2(1+\alpha)}{n} > 0$, we compute

$$\left| D \left[ \phi^{1+\frac{2}{n}} \varphi^2 (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} \right] \right| \leq \frac{p+2\alpha+2\delta}{2} \phi^{1+\frac{2}{n}} \varphi^2 (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} |D^2 u|$$

$$+ 2 \phi^{1+\frac{2}{n}} \varphi (H(Du)) \zeta' (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} |D^2 u|$$

$$+ \frac{n+2}{n} \phi^{\frac{2}{n}} |D\phi| \varphi^2 (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2}$$

$$\leq c (p + 2\alpha) \phi^{1+\frac{2}{n}} \varphi (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} |D^2 u|$$

$$+ c \|D\phi\|_{L^\infty} \phi^2 \varphi (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} \quad (3.20)$$

for a constant $c = c(n)$. In the last step, we used the fact that $H(Du) \leq 1$ on the support of $\zeta'(H(Du))$, as well as the bounds $\zeta \leq 1$ and $\zeta' \leq 3$. On a fixed time slice we apply Sobolev’s embedding in Lemma 2.2 with $p = \frac{2n}{n+2}$ and then inequality (3.20), with the result

$$\int_{\Omega \cap B_y} \phi^{2+\frac{4}{n}} \varphi^4 (H(Du)) H(Du)^{p+2\alpha+2\delta} \, dx$$

$$\leq 2 C_{Sob}^2 \phi^2 \left[ \int_{\Omega \cap B_y} \left| D \left[ \phi^{1+\frac{2}{n}} \varphi^2 (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} \right] \right| \frac{2n}{n+2} \right]^{\frac{n+2}{n}}$$

$$+ 2 \left[ \int_{\Omega \cap B_y} \phi^{1+\frac{2}{n}} \varphi (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} \frac{2n}{n+2} \right]^{\frac{n+2}{n}}$$

$$\leq c C_{Sob}^2 \phi^2 (p + 2\alpha)^2 \left[ \int_{\Omega \cap B_y} \left[ \phi^{1+\frac{2}{n}} \varphi (H(Du)) H(Du) \frac{p+2\alpha+2\delta}{2} + \delta |D^2 u| \right] \frac{2n}{n+2} \right]^{\frac{n+2}{n}}$$

$$+ c C_{Sob}^2 \|D\phi\|_{L^\infty}^2 \left[ \int_{\Omega \cap B_y} \phi^2 H(Du) \frac{p+2\alpha+2\delta}{2} \frac{2n}{n+2} \right]^{\frac{n+2}{n}}.$$
In the last line, we used the estimate \( \| \phi \|_{L^\infty} \leq \varepsilon \| D \phi \|_{L^\infty} \), which holds true since \( \phi \) has compact support in \( B_{\varepsilon} \), and we assumed that \( C_{\text{Sob}} \geq 1 \). According to Lemma 2.2, we can choose the Sobolev constant \( C_{\text{Sob}} \) only depending on \( n \) and \( \Theta \). Next, we estimate both integrals on the right-hand side by means of Hölder’s inequality with exponents \( \frac{n+2}{n} \) and \( \frac{n+2}{2} \), which leads to
\[
\int_{\Omega \cap B_{\varepsilon}} \phi^{2+\frac{4}{n}} \zeta^4 (H(Du)) H(Du)^{p+2\alpha+2\delta} \, dx \leq c \, C_{\text{Sob}}^2 \phi^2 \, \mathbb{I}_1 \cdot \mathbb{I}_2^\frac{2}{n},
\]
where
\[
\mathbb{I}_1 := (p + 2\alpha)^2 \int_{\Omega \cap B_{\varepsilon}} \phi^2 \zeta^2 (H(Du)) H(Du)^{p+2\alpha-2} |D^2 u|^2 \, dx
\]
\[
+ \| D \phi \|_{L^\infty}^2 \int_{\Omega \cap B_{\varepsilon}} H(Du)^{p+2\alpha} \, dx
\]
and
\[
\mathbb{I}_2 := \int_{\Omega \cap B_{\varepsilon}} \phi^2 H(Du)^{n\delta} \, dx.
\]
At this point, the reason for our choice of \( \delta \) becomes clear. In fact, we have chosen \( \delta \) in such a way that \( n\delta = 2 + 2\alpha \) coincides with the integrability exponent in the sup-term of the energy inequality (3.19). We multiply the preceding inequality by \( \chi(t)^{1+\frac{2}{n}} \) and take the mean with respect to \( t \) over the interval \((t_0 - \varepsilon^2, t_0)\). In this way, we obtain
\[
\iint_{\Omega_T \cap Q_{\varepsilon}} [\chi \Phi^2]^{1+\frac{2}{n}} \zeta^4 (H(Du)) H(Du)^{p+2\alpha+2\delta} \, dx \, dt
\]
\[
\leq c \, C_{\text{Sob}}^2 \phi^2 \int_{(t_0 - \varepsilon^2, t_0)} \chi(t) \mathbb{I}_1 \cdot \left[ \int_{\Omega \cap B_{\varepsilon}} \chi \Phi^2 H(Du)^{2+2\alpha} \, dx \right]^\frac{2}{n} \, dt
\]
\[
\leq c \, C_{\text{Sob}}^2 \phi^2 \mathbb{R}^2 \int_{(t_0 - \varepsilon^2, t_0)} \chi(t) \mathbb{I}_1 \, dt . \quad (3.21)
\]
In the last line, we used the energy inequality (3.19). Recall that \( \mathbb{R} \) denotes the right-hand side of (3.19). For the estimate of the last integral, we again apply (3.19) and use the definition of \( \mathbb{R} \), with the result
\[
\int_{(t_0 - \varepsilon^2, t_0)} \chi(t) \mathbb{I}_1 \, dt \leq (p + 2\alpha)^2 \iint_{\Omega_T \cap Q_{\varepsilon}} \chi \Phi^2 \zeta^2 (H(Du)) H(Du)^{p+2\alpha-2} |D^2 u|^2 \, dx
\]
\[
+ \| D \phi \|_{L^\infty}^2 \iint_{\Omega_T \cap Q_{\varepsilon}} H(Du)^{p+2\alpha} \, dx \, dt
\]
\[
\leq c \, (p + 2\alpha) \frac{1}{\varepsilon^2} \mathbb{R},
\]
where the constant \( c \) depends on \( n, N, m, M, p, \) and \( \Theta \). Joining this with (3.21) yields

\[
\left( \iint_{\Omega_T \cap Q_\delta} [\chi \Phi^2]^{1 + \frac{2}{n+2}} \mathcal{H}(Du)^{p+2\alpha+2\delta} \, dx \, dt \right)^{\frac{n}{n+2}} \leq 1 + \left( \iint_{\Omega_T \cap Q_\delta} [\chi \Phi^2]^{1 + \frac{2}{n+2}} \mathcal{H}(Du)^{p+2\alpha+2\delta} \, dx \, dt \right)^{\frac{n}{n+2}} \leq c(p + 2\alpha) R. \tag{3.22}
\]

Our next goal is to estimate \( R_2 \). In view of the integrability assumption (1.5), that is, \( b \in L^\sigma(\Omega_T, \mathbb{R}^N) \) with \( \sigma > n+2 \), we can use Hölder’s inequality to estimate

\[
R_2 \leq \Theta^\frac{2}{\sigma} \int_{\sigma, \sigma, \Omega_T \cap Q_\delta} \left( \iint_{\Omega_T \cap Q_\delta} [\chi \Phi^2 \mathcal{H}(Du)^{2\alpha+(2-p)_+}]^{\frac{\sigma}{\sigma-2}} \, dx \, dt \right)^{\frac{\sigma-2}{\sigma}}, \tag{3.23}
\]

where we defined

\[
[b]_{\sigma, \Omega_T \cap Q_\delta} := \left[ \epsilon^{\sigma-n-2} \iint_{\Omega_T \cap Q_\delta} |b|^\sigma \, dx \, dt \right]^{\frac{1}{\sigma}}.
\]

Here we used the fact that \( \epsilon^{n+2}/|\Omega_T \cap Q_\delta| \) is bounded by \( \Theta \). To estimate the integral on the right-hand side of (3.23) further, we interpolate the \( L^{\frac{\sigma}{\sigma-2}} \)-norm between the \( L^1 \)-norm and the \( L^{\frac{n+2}{n+2}} \)-norm, which is possible since \( \sigma > n + 2 \). For every \( \kappa > 0 \), this yields the bound

\[
R_2 \leq \Theta^\frac{2}{\sigma} \int_{\sigma, \Omega_T \cap Q_\delta} \left[ \kappa \left( \iint_{\Omega_T \cap Q_\delta} [\chi \Phi^2 \mathcal{H}(Du)^{2\alpha+(2-p)_+}]^{\frac{n+2}{n+2}} \, dx \, dt \right)^{\frac{n}{n+2}} \right] + \kappa^{-\frac{n+2}{\sigma-n-2}} \iint_{\Omega_T \cap Q_\delta} \chi \Phi^2 \mathcal{H}(Du)^{2\alpha+(2-p)_+} \, dx \, dt
\]

\[
\leq \Theta^\frac{2}{\sigma} \int_{\sigma, \Omega_T \cap Q_\delta} \left[ \kappa \left( \iint_{\Omega_T \cap Q_\delta} [\chi \Phi^2]^{n+2} \left[ \mathcal{H}(Du)^{p+2\alpha+2\delta} + 1 \right] \, dx \, dt \right)^{\frac{n}{n+2}} \right] + \kappa^{-\frac{n+2}{\sigma-n-2}} \iint_{\Omega_T \cap Q_\delta} \mathcal{H}(Du)^{p+2\alpha} + 1 \, dx \, dt. \tag{3.24}
\]

In the last line, we used the fact that

\[
\left[ 2\alpha + (2-p)_+ \right] \frac{n+2}{n} \leq p + 2\alpha + 2\delta \quad \text{and} \quad 2\alpha + (2-p)_+ \leq p + 2\alpha.
\]
The latter hold true for any $\alpha \geq 0$ and $p \geq 1$. Joining estimates (3.22) and (3.24), we arrive at

$$[1 - c (p + 2\alpha)^3 \Theta \frac{2}{3} [b]_{\sigma, \Omega_T \cap Q_{\varphi}}^2 \kappa] \left[ \iint_{\Omega_T \cap Q_{\varphi}} [\chi \phi^2]^{1 + \frac{2}{p}} \mathcal{H}(Du)^{p + 2\alpha + 2\delta} \, dx \, dt \right]^{\frac{n}{n+2}}$$

$$\leq c (p + 2\alpha) R_1 + c \Theta \frac{2}{3} (p + 2\alpha)^3 [b]_{\sigma, \Omega_T \cap Q_{\varphi}}^2 \kappa + 1$$

$$+ c \Theta \frac{2}{3} (p + 2\alpha)^3 [b]_{\sigma, \Omega_T \cap Q_{\varphi}}^2 \kappa \left[ \iint_{\Omega_T \cap Q_{\varphi}} \mathcal{H}(Du)^{p + 2\alpha + 1} \, dx \, dt \right].$$

At this stage, we choose the parameter $\kappa > 0$ so small that

$$c (p + 2\alpha)^3 \Theta \frac{2}{3} [b]_{\sigma, \Omega_T \cap Q_{\varphi}}^2 \kappa = \frac{1}{2}.$$

This implies in particular that the second term on the right-hand side equals $\frac{1}{2}$. On the other hand, the coefficient in front of the last term on the right equals

$$\frac{1}{2} \kappa^{-\frac{n+2}{\sigma-n-2}} = \frac{1}{2} \kappa^{-\frac{\sigma}{\sigma-n-2}} = \left[ c (p + 2\alpha)^3 \Theta \frac{2}{3} [b]_{\sigma, \Omega_T \cap Q_{\varphi}}^2 \kappa \right]^{\frac{\sigma}{\sigma-n-2}}.$$

This turns the preceding estimate into

$$\left[ \iint_{\Omega_T \cap Q_{\varphi}} [\chi \phi^2]^{1 + \frac{2}{p}} \mathcal{H}(Du)^{p + 2\alpha + 2\delta} \, dx \, dt \right]^{\frac{n}{n+2}}$$

$$\leq c (p + 2\alpha)^{\frac{3\sigma}{\sigma-n-2}} \left[ 1 + R_1 + [b]_{\sigma, \Omega_T \cap Q_{\varphi}}^{\frac{2\sigma}{\sigma-n-2}} \left[ \iint_{\Omega_T \cap Q_{\varphi}} \mathcal{H}(Du)^{p + 2\alpha + 1} \, dx \, dt \right] \right],$$

for a constant $c$ depending only on $n, N, m, M, p, \sigma,$ and $\Theta$. By an application of Young’s inequality, we can rewrite the above inequality and obtain the reverse Hölder type estimate

$$\left[ \iint_{\Omega_T \cap Q_{\varphi}} [\chi \phi^2]^{1 + \frac{2}{p}} \mathcal{H}(Du)^{p + 2\alpha + 2\delta} \, dx \, dt \right]^{\frac{n}{n+2}}$$

$$\leq c (p + 2\alpha)^{\frac{3\sigma}{\sigma-n-2}} \gamma_o \left[ \iint_{\Omega_T \cap Q_{\varphi}} \mathcal{H}(Du)^{q + 2\alpha + 1} \, dx \, dt \right],$$

(3.25)

where we defined $q := \max\{p, 2\}$ and moreover abbreviated

$$\gamma_o := 1 + \varepsilon^2 \|D\phi\|_{L^\infty}^2 + \varepsilon^2 \|\delta_\chi\|_{L^\infty}^2 + [b]_{\sigma, \Omega_T \cap Q_{\varphi}}^{\frac{2\sigma}{\sigma-n-2}}.$$

For the constant $c$ above, we have the dependencies $c(n, N, m, M, p, \sigma, \Theta)$. For the Moser iteration scheme, we need to compare the exponents on both sides of (3.25). We have

$$p + 2\alpha + 2\delta = q + 2\alpha + \frac{4}{n} (1 + \alpha) - (q - p) > q + 2\alpha,$$
since \( p > \frac{2n}{n+2} > 2 - \frac{4}{n} \).

### 3.3 The iteration scheme

We fix radii \( r, s \) with \( \frac{\varphi}{2} \leq r < s \leq \varphi \) and define

\[
\varphi_k := r + \frac{1}{2^k}(s - r) \quad \text{and} \quad Q_k := Q_{\varphi_k}(x_0, t_0)
\]

for \( k \in \mathbb{N}_0 \). We choose cutoff functions \( \phi_k \in C_0^\infty(B_{\varphi_k}(x_0), [0, 1]) \) such that \( \phi_k \equiv 1 \) on \( B_{\varphi_{k+1}}(x_0) \) and \( |D\phi_k| \leq \frac{2^{k+2}}{s-r} \) and \( \chi_k \in W^{1,\infty}((t_o - \varphi^2_k, t_o), [0, 1]) \) such that \( \chi_k(t_o - \varphi^2_k) = 0, \chi_k \equiv 1 \) on \( (t_o - \varphi^2_{k+1}, t_o) \), and \( 0 \leq \partial_t \chi_k \leq \frac{22(k+2)}{(s-r)^2} \). With these specifications inequality (3.25) yields

\[
\left[ \frac{1}{\Omega_T \cap Q_{k-1}} \iint_{\Omega_T \cap Q_k} \mathcal{H}(Du)^{q + 2\alpha(1 + \frac{2}{n} - (q-p) + \frac{4}{n})} \, dx \, dt \right]^\frac{n}{n+2} \leq C_o 4^{k} \varphi^2 \frac{(p + 2\alpha)^{\frac{3\alpha}{2-n-2}}}{(s-r)^2} \iint_{\Omega_T \cap Q_{k-1}} \mathcal{H}(Du)^{q + 2\alpha + 1} \, dx \, dt, \tag{3.26}
\]

for a constant \( C_o \) of the type

\[
C_o := C \left( 1 + [b]_{\varphi, \Omega_T \cap Q_\varphi} \right). \tag{3.27}
\]

Here \( C \) denotes a universal constant depending on \( n, N, m, M, p, \sigma, \) and \( \Theta \). To bound the left-hand side of (3.26), we use the fact

\[
\frac{|\Omega_T \cap Q_k|}{|\Omega_T \cap Q_{k-1}|} \geq \frac{|\Omega_T \cap Q_{\varphi/2}|}{|Q_{\varphi}|} \geq \frac{|\Omega \cap B_{\varphi/2}|}{4|B_1|\varphi^n} \geq \frac{1}{c(n)\Theta}.
\]

We use this in (3.26) and obtain

\[
\left[ \iint_{\Omega_T \cap Q_k} \mathcal{H}(Du)^{q + 2\alpha(1 + \frac{2}{n} - (q-p) + \frac{4}{n})} \, dx \, dt \right]^\frac{n}{n+2} \leq C_o 4^{k} \varphi^2 \frac{(p + 2\alpha)^{\frac{3\alpha}{2-n-2}}}{(s-r)^2} \iint_{\Omega_T \cap Q_{k-1}} \mathcal{H}(Du)^{q + 2\alpha + 1} \, dx \, dt, \tag{3.28}
\]

for some constant \( C_o \) with the same structure as the one in (3.27). We now define recursively a sequence \((\beta_k)_{k \in \mathbb{N}_0}\) by \( \beta_0 := 0 \) and

\[
2\beta_k := 2\beta_{k-1} \left( 1 + \frac{2}{n} \right) - (q - p) + \frac{4}{n}.
\]
Induction leads to
\[
\beta_k = \frac{4 - n(q - p)}{4} \left[ \left( 1 + \frac{2}{n} \right)^k - 1 \right].
\]
(3.29)

The choice \( \alpha = \beta_{k-1} \) turns (3.28) into
\[
\left\llbracket \mathcal{H}(Du)^{q+2\beta_k} \right\rrbracket_{\Omega_T \cap Q_k}^{\frac{n}{n+2}} \leq \frac{C_o 4^k \varphi^2}{(s - r)^2} \,(1 + \beta_k)^\frac{3\sigma}{\sigma - n - 2} \left\llbracket \mathcal{H}(Du)^{q+2\beta_{k-1}} \right\rrbracket_{\Omega_T \cap Q_{k-1}}^{\frac{n}{n+2}} + 1.
\]
(3.30)

In the last line, we used \( \beta_{k-1} < \beta_k \) to replace \( p + 2\beta_{k-1} \) by \( 2p(1 + \beta_k) \). The constant \( C_o \) in the above estimate is up to a multiplicative factor the same as the one from (3.27). This, however, does not change the dependencies in \( C_o \). To proceed further, let
\[
A_k := \left\llbracket \mathcal{H}(Du)^{q+2\beta_k} \right\rrbracket_{\Omega_T \cap Q_k}^{\frac{n}{n+2}}
\]
In terms of \( A_k \) the reverse Hölder inequality (3.30) leads to a recursion formula
\[
A_k \leq \left[ C_o 4^k \varphi^2 \,(1 + \beta_k)^\frac{3\sigma}{\sigma - n - 2} \right]^{1 + \frac{2}{n}} (A_{k-1} + 1)^{1 + \frac{2}{n}} \quad \forall k \in \mathbb{N}.
\]
Iteration of this inequality gives
\[
A_k \leq \prod_{j=1}^k \left[ C_o 4^j \varphi^2 \,(1 + \beta_j)^\frac{3\sigma}{\sigma - n - 2} \right]^{1 + \frac{2}{n}} (A_0 + 1)^{1 + \frac{2}{n}}^k
\]
for any \( k \in \mathbb{N} \). Here we enlarged \( C_o \) by a factor 2. We take this inequality to the power \( \frac{1}{q+2\beta_k} \) and obtain
\[
A_k^{\frac{1}{q+2\beta_k}} \leq \prod_{j=1}^k \left[ C_o 4^j \varphi^2 \,(1 + \beta_j)^\frac{3\sigma}{\sigma - n - 2} \right]^{(1 + \frac{2}{n})^{k-j+1}} \frac{(A_0 + 1)^{1 + \frac{2}{n} k}}{(q+2\beta_k)^{\frac{1}{q+2\beta_k}}}.
\]
(3.31)

Note that
\[
\lim_{k \to \infty} \frac{(1 + \frac{2}{n})^k}{q + 2\beta_k} = \frac{2}{4 - n(q - p)}.
\]

Therefore, we have
\[
\lim_{k \to \infty} (A_0 + 1)^{\frac{(1 + \frac{2}{n})^k}{q+2\beta_k}} = \left[ \mathcal{H}(Du)^q \right]_{\Omega_T \cap Q_s}^{\frac{2}{4 - n(q - p)}} + 1.
\]
(3.32)
With the abbreviation
\[ \gamma := \frac{4 - n(q - p)}{2} \in (0, 2], \]
formula (3.29) takes the form
\[ \beta_k = \frac{\gamma}{2} \left[ \left( 1 + \frac{2}{n} \right)^k - 1 \right] \leq \left( 1 + \frac{2}{n} \right)^k - 1. \]

Therefore, for any \( j \in \mathbb{N} \) we have the estimate
\[ \frac{C_o A^j g^2}{(s - r)^2} (1 + \beta_j)^{3\sigma} \lesssim \tilde{C}_o K^j \]
with the abbreviations
\[ \tilde{C}_o := \frac{C_o g^2}{(s - r)^2} = \frac{C g^2}{(s - r)^2} \left( 1 + [b]^{\frac{2\sigma}{\sigma - n - 2}}_{\sigma, \Omega_c \cap Q_p} \right) \]
and
\[ K := 4 \left( 1 + \frac{2}{n} \right)^{\frac{3\sigma}{\sigma - n - 2}} \geq 1. \]

We use this to bound the product appearing in (3.31), with the result
\[
\prod_{j=1}^{k} \left[ \frac{C_o A^j g^2}{(s - r)^2} (1 + \beta_j)^{3\sigma} \right]^{\frac{(1 + \frac{2}{n})^{k-j+1}}{q^{2\sigma} k}} \lesssim \prod_{j=1}^{k} \tilde{C}_o^{\frac{(1 + \frac{2}{n})^{k-j+1}}{q^{2\sigma} k}} \prod_{j=1}^{k} K^{\frac{(1 + \frac{2}{n})^{k-j+1}}{q^{2\sigma} k}} \prod_{j=1}^{k} \tilde{C}_o^{\frac{(1 + \frac{2}{n})^{k-j+1}}{q^{2\sigma} k}} \prod_{j=1}^{k} K^{\frac{(1 + \frac{2}{n})^{k-j+1}}{q^{2\sigma} k}}. \]

The first product on the right-hand side can be computed with the help of (2.5) from Lemma 2.3 applied with \( A = \tilde{C}_o \) and \( \theta = 1 + \frac{2}{n} \). We obtain
\[ \prod_{j=1}^{k} \tilde{C}_o^{\frac{(1 + \frac{2}{n})^{k-j+1}}{q^{2\sigma} k}} = \tilde{C}_o^{\frac{n+2}{2q}} = \tilde{C}_o^{\frac{n+2}{4n(q-p)}}. \]

Similarly, the second product can be bounded with the help of (2.6) from Lemma 2.3 applied with \( A = K \) and \( \theta = 1 + \frac{2}{n} \). This yields
\[ \prod_{j=1}^{k} K^{\frac{(1 + \frac{2}{n})^{k-j+1}}{q^{2\sigma} k}} \leq K^{\frac{(n+2)^2}{2q}} = K^{\frac{(n+2)^2}{24n(q-p)}}. \]
Inserting this above, we obtain

\[
\prod_{j=1}^{k} \left[ \frac{C_0 4^j \varphi^2}{(s-r)^2 (1 + \beta_j)} \right]^{\frac{3\sigma}{2n-1-j}} \leq \left( K \frac{n+2}{2} C_0 \right)^{\frac{n+2}{4-n(q-p)}}
\]

where \( K \) depends only on \( n \) and \( \sigma \). In particular, the right-hand side is independent of \( k \in \mathbb{N} \). This allows us to pass to the limit \( k \to \infty \) in (3.31). In view of (3.32), this yields

\[
\limsup_{k \to \infty} A_k^{\frac{1}{q+2\beta_k}} \leq \left( K \frac{n+2}{2} C_0 \right)^{\frac{n+2}{4-n(q-p)}} \left[ \iint_{\Omega_T \cap Q_s} |D(Hu)|^q \, dx \, dt + 1 \right]^{\frac{2}{4-n(q-p)}} \leq C \left[ s^{-n+2} \left( 1 + \frac{(n+2)\sigma}{s-n-2} \right) \iint_{\Omega_T \cap Q_s} \left( 1 + |Du|^2 \right)^{\frac{q}{2}} \, dx \, dt \right]
\]

In the last line we used (3.27), that is, the special form of \( C_0 \), and the fact that \( \mu \leq 1 \). Since \( \varphi_k \to r \) and \( \beta_k \to \infty \), the last estimate implies the following sup-estimate for the gradient

\[
\sup_{\Omega_T \cap Q_r} |Du| = \lim_{k \to \infty} \left[ \iint_{\Omega_T \cap Q_s} |D(Hu)|^{q+2\beta_k} \, dx \, dt \right]^{\frac{1}{q+2\beta_k}} \leq \limsup_{k \to \infty} A_k \leq \left[ s^{-n+2} \left( 1 + \frac{(n+2)\sigma}{s-n-2} \right) \iint_{\Omega_T \cap Q_s} \left( 1 + |Du|^2 \right)^{\frac{q}{2}} \, dx \, dt \right]^{\frac{1}{q+2\beta_k}} \leq C \left[ s^{-n+2} \left( 1 + \frac{(n+2)\sigma}{s-n-2} \right) \iint_{\Omega_T \cap Q_s} \left( 1 + |Du|^2 \right)^{\frac{q}{2}} \, dx \, dt \right]^{\frac{1}{q+2\beta_k}}
\]

for a constant \( C \) that depends on \( n, m, M, p, \sigma, \) and \( \Theta \). In the case \( p \geq 2 \), we have \( q = p \), and therefore (3.33) simplifies to

\[
\sup_{\Omega_T \cap Q_r} |Du| \leq C \left[ s^{-n+2} \left( 1 + \frac{(n+2)\sigma}{s-n-2} \right) \iint_{\Omega_T \cap Q_s} \left( 1 + |Du|^2 \right)^{\frac{p}{2}} \, dx \, dt \right]^{\frac{d}{p}}
\]

where \( d = \frac{1}{2} p \) is the scaling deficit from (1.6). With the choice \( r = \frac{\varphi}{2} \) and \( s = \frac{3\rho}{4} \), this yields the asserted sup-estimate for the gradient (3.2) in the case \( p \geq 2 \). Note that this is in perfect accordance with the interior estimate [12, Chapter VIII, Theorem 5.1].
### 3.4 Interpolation in the case \( \frac{2n}{n+2} < p < 2 \)

To reduce the integrability exponent in the sup-estimate from \( q = 2 \) to \( p \) in the singular case we need an additional interpolation argument. To this end, we apply (3.33) with arbitrary radii \( r, s \) satisfying \( \frac{r}{s} \leq r < s \leq \frac{5s}{4} \). On the right-hand side of the estimate, we bound a part of the integrand by its supremum and then apply Young’s inequality with exponents \( \frac{4-n(2-p)}{2(2-p)} \) and \( \frac{4-n(2-p)}{p(n+2)-2n} \). Note that this is possible if and only if \( \frac{2n}{n+2} < p < 2 \). This procedure leads us to

\[
\sup_{\Omega_T \cap Q_r} (1 + |Du|^2)^{\frac{1}{2}} \leq C \left[ \int_{\Omega_T \cap Q_s} \left( 1 + |Du|^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{4-n(2-p)}} \cdot \left[ \int_{\Omega_T \cap Q_s} \left( 1 + |Du|^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{4-n(2-p)}} \cdot \left[ \int_{\Omega_T \cap Q_s} \left( 1 + |Du|^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{4-n(2-p)}}.
\]

By a standard iteration argument (cf. [21, Chapter V, Lemma 3.1]), this implies

\[
\sup_{\Omega_T \cap Q_{s/2}} (1 + |Du|^2)^{\frac{1}{2}} \leq C \left[ \int_{\Omega_T \cap Q_{s/4}} \left( 1 + |Du|^2 \right)^{\frac{p}{2}} \right]^{\frac{d}{p}}, \quad (3.34)
\]

where \( d = \frac{2p}{p(n+2)-2n} \) is the scaling deficit (cf. (1.6)). This is exactly the claimed bound (3.2) in the singular range \( \frac{2n}{n+2} < p < 2 \), and completes the proof of the sup-estimate from Proposition 3.1. Note also that the sup-gradient estimate (3.34) is again in perfect accordance with the corresponding interior estimate [12, Chapter VIII, Theorem 5.2'].
4 | REGULARIZATION

In this section, we describe the regularization procedure that will allow us to extend the a priori estimate to the general case. We consider the situation stated in Theorem 1.2, that is, we let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and suppose that $u \in L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ is a solution to (1.1), where (1.3)–(1.5) are in force. Moreover, we assume that for some $x_o \in \partial \Omega$ and $\varphi > 0$ we have $u \equiv 0$ on $(\partial \Omega)_T \cap Q_{2\varphi}(x_0)$ in the sense of traces.

4.1 | Approximation of the domain

For any $\varepsilon \in (0, 1]$, we consider the parallel set $\tilde{\Omega}_\varepsilon := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \frac{3}{2}\varepsilon\}$. Note that $\tilde{\Omega}_\varepsilon$ is convex as $\Omega$ is convex. By a well-known result from convex analysis (see, for example, [28, Section XIII.2, Satz 2]), the domains $\tilde{\Omega}_\varepsilon$ can be approximated in Hausdorff distance by smooth convex sets $\Omega_\varepsilon$ with

$$\text{dist}_H (\Omega_\varepsilon, \tilde{\Omega}_\varepsilon) < \frac{1}{2}\varepsilon.$$

In particular, the regularized sets $\Omega_\varepsilon$ satisfy

$$\left\{ x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \varepsilon \right\} \subset \Omega_\varepsilon \subset \left\{ x \in \mathbb{R}^n : \text{dist}(x, \Omega) < 2\varepsilon \right\}.$$  (4.1)

Since the domains $\Omega_\varepsilon$ approximate $\Omega$ from the outside, we obtain

$$\sup_{\varepsilon \in (0, 1]} \frac{\varphi^n}{|\Omega_\varepsilon \cap B_{\varphi/2}(x_0)|} \leq \frac{\varphi^n}{|\Omega \cap B_{\varphi/2}(x_0)|} = 2^n \Theta_{\varphi/2}(x_o)$$  (4.2)

for every $x_o \in \partial \Omega$ and $\varphi > 0$, with the constant $\Theta_{\varphi/2}(x_o)$ introduced in (2.1). As a result, the constants in the a priori estimate will be independent of $\varepsilon \in (0, 1]$.

4.2 | Regularization of the coefficients

We regularize the coefficients by means of a mollifier $\phi \in C^\infty_0(\mathbb{R}, [0, \infty))$ with spt $\phi \subset (-1, 1)$ and $\int_{\mathbb{R}} \phi \, dx = 1$. For $\varepsilon \in (0, 1]$, we let $\phi_\varepsilon(x) := \varepsilon^{-1}\phi(\frac{x}{\varepsilon})$ and

$$c_\varepsilon(s) := (\phi_\varepsilon \ast c)(s), \quad \text{where} \quad c(s) := \begin{cases} a(e^s), & \text{if } \mu > 0, \\ a\left(\varepsilon^2 + e^{2s}\right), & \text{if } \mu = 0, \end{cases}$$

for any $s \in \mathbb{R}$. The regularized coefficients $a_\varepsilon$ are defined by

$$a_\varepsilon(r) := c_\varepsilon(\log r), \quad \text{for } r > 0.$$

Similarly as in [5, Section 4.2], we obtain the following ellipticity and growth conditions for $a_\varepsilon$ (see also Appendix A for the proof). For any $r > 0$, we have
\[
\begin{align*}
\frac{m}{\varepsilon} (\lambda^2 + r^2)^{\frac{p-2}{2}} & \leq a_\varepsilon(r) \leq cM (\lambda^2 + r^2)^{\frac{p-2}{2}}, \\
\frac{m}{\varepsilon} (\lambda^2 + r^2)^{\frac{p-2}{2}} & \leq a'_\varepsilon(r)r + a_\varepsilon(r) \leq cM (\lambda^2 + r^2)^{\frac{p-2}{2}}, \\
|a''_\varepsilon(r)r^2| & \leq cM (\lambda^2 + r^2)^{\frac{p-2}{2}},
\end{align*}
\] (4.3)

with a constant \( c = c(n,p) \) and

\[
\lambda := \begin{cases} 
\mu, & \text{if } \mu > 0, \\
\varepsilon, & \text{if } \mu = 0.
\end{cases} \tag{4.4}
\]

Moreover, we have

\[
|a_\varepsilon(r) - a(r)| \leq 2c(p)M\varepsilon \max \{1, e^{p-2}\} (\lambda^2 + r^2)^{\frac{p-2}{2}} \tag{4.5}
\]

for any \( r > 0 \) (see also Appendix A for the proof).

### 4.3 Weak solutions to the regularized problems

Here, we assume that

\[
u \equiv 0 \text{ on } (\partial \Omega)_{T} \cap Q_{2\varphi}(x_0),
\]

where \( Q_{2\varphi}(z_0) \) is a parabolic cylinder with \( z_0 \in \partial \Omega \) and \( (t_0 - 4\varphi^2, t_0) \subset (0,T) \). For a cutoff function \( \tilde{\eta} \in C^\infty(0,\infty;[0,1]) \) with \( \tilde{\eta} \equiv 1 \text{ on } [2,\infty) \) and \( \tilde{\eta} \equiv 0 \text{ on } (0,1) \), we consider the boundary values

\[
g_\varepsilon(x,t) := \eta_\varepsilon(x)u(x,t) \quad \text{with} \quad \eta_\varepsilon(x) := \tilde{\eta}\left(\frac{\text{dist}(x,\partial\Omega)}{\varepsilon}\right) \text{ for } x \in \Omega. \tag{4.6}
\]

We extend this function to \( \mathbb{R}^n \times (0,T) \) by letting \( g_\varepsilon \equiv 0 \text{ on } (\mathbb{R}^n \setminus \Omega)_{T} \). Note that the extension satisfies \( g_\varepsilon \in L^p(0,T;W_{0}^{1,p}(\Omega_\varepsilon,\mathbb{R}^N)) \). For the inhomogeneity, we consider the regularization \( b_\varepsilon := \phi_{\varepsilon/2} \ast b \), for a standard mollifier in space–time \( \phi \in C_0^\infty(Q_1,\mathbb{R}) \). Due to the construction of \( \Omega_\varepsilon \) (see (4.1)), we have

\[
spt b_\varepsilon \in \Omega_\varepsilon \times \mathbb{R}. \tag{4.7}
\]

We let \( a_\varepsilon \) be the coefficients constructed in Section 4.2. By

\[
u_\varepsilon \in g_\varepsilon + L^p \left( (t_0 - \varphi^2, t_0; W_{0}^{1,p}(\Omega_\varepsilon \cap B_\varphi(x_0), \mathbb{R}^N)) \right),
\]

we denote the weak solution to the Cauchy–Dirichlet problem

\[
\begin{cases}
\partial_t u_\varepsilon - \text{div} (a_\varepsilon(|Du_\varepsilon|)Du_\varepsilon) = b_\varepsilon & \text{in } (\Omega_\varepsilon)_{T} \cap Q_\varphi(z_0) \\
u_\varepsilon = g_\varepsilon & \text{on } \partial_p((\Omega_\varepsilon)_{T} \cap Q_\varphi(z_0)).
\end{cases} \tag{4.8}
\]
Note that \( u_\varepsilon = 0 \) on \((\partial \Omega_\varepsilon)_T \cap Q_\varepsilon(z_0)\) and \( u_\varepsilon = \eta_\varepsilon u \) on \((\Omega_\varepsilon)_T \cap \partial_p Q_\varepsilon(z_0)\). Using a reflection argument, interior regularity theory and up-to-the-boundary Schauder estimates we can show that \( u_\varepsilon \) is smooth up to the boundary component \((\partial \Omega_\varepsilon)_T \cap Q_\varepsilon(z_0)\) (see Appendix B).

5 | PROOF OF THEOREM 1.2

The proof of the gradient estimate will be achieved in Subsection 5.2. Prior to that, we shall prove an energy estimate for \( u_\varepsilon \).

5.1 | An energy estimate for the approximating solutions

Throughout this section, we omit the reference to the center \( z_0 \) in our notation. From [26, Corollary 3.11], we recall the following result; note that the constant \( \gamma \) in [26, Corollary 3.11] can be chosen as \( \gamma = \frac{1}{2} \) due to the convexity of \( \Omega \) (see [26, Example 3.6 (4)]).

**Lemma 5.1** (Hardy’s inequality). Let \( 1 < p < \infty \) and suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded open convex set. Then there is a constant \( c \) depending on \( n \) and \( p \) such that whenever \( u \in W^{1,p}_0(\Omega) \) there holds

\[
\int_\Omega \left( \frac{|u(x)|}{\text{dist}(x, \partial \Omega)} \right)^p \, dx \leq c \int_\Omega |Du(x)|^p \, dx.
\]

In the following, we let

\[
V_\varepsilon := L^\infty(t_0 - \varepsilon^2, t_0; L^2(\Omega_\varepsilon \cap B_\varepsilon, \mathbb{R}^N)) \cap L^p(t_0 - \varepsilon^2, t_0; W^{1,p}_0(\Omega_\varepsilon \cap B_\varepsilon, \mathbb{R}^N))
\]

with norm

\[
\| \varphi \|_{V_\varepsilon} := \| \varphi \|_{L^\infty-L^2} + \| \varphi \|_{L^p-W^{1,p}}.
\]

We start with an estimate for the spatial gradient of the boundary values.

**Lemma 5.2.** Let \( u \) be a weak solution to (1.1) in \( \Omega_T \) with \( u \equiv 0 \) on \((\partial \Omega)_T \cap Q_2\varepsilon(z_0)\) in the sense of traces, and \( g_\varepsilon = \eta_\varepsilon u \) be constructed as in (4.6). Then we have

\[
\iint_{(\Omega_T \cap Q_\varepsilon)} |Dg_\varepsilon|^p \, dx \, dt \leq c \iint_{(\Omega_T \cap Q_{2\varepsilon})} [ |Du|^p + \varphi^{-p} |u|^p ] \, dx \, dt,
\]

with a constant \( c = c(n, p) \).

**Proof.** We choose a standard cutoff function \( \xi \in C^\infty_0(B_2, [0,1]) \) with \( \xi \equiv 1 \) on \( B_\varepsilon \) and \( |D\xi| \leq \frac{2}{\varepsilon} \) on \( B_2 \). Then we apply Hardy’s inequality from Lemma 5.1 to the function \( \xi u \) on the time slices
$\Omega \times \{t\}$ for a.e. $t \in (t_o - \varphi^2, t_o)$, with the result
\[
\iint_{\Omega \cap Q_T} |Dg_\varepsilon|^p \, dx \, dt \leq c \iint_{\Omega \cap Q_T} |Du|^p \, dx \, dt + \frac{c}{\varepsilon^p} \iint_{\Omega \cap Q_T \cap \text{spt}(D\eta_\varepsilon)} |\xi u|^p \, dx \, dt
\]
\[
\leq c \iint_{\Omega \cap Q_T} |Du|^p \, dx \, dt + \iint_{\Omega \times (t_o - \varphi^2, t_o)} \left( \frac{|\xi u|}{\text{dist}(x, \partial \Omega)} \right)^p \, dx \, dt
\]
\[
\leq c \iint_{\Omega \cap Q_T} |Du|^p \, dx \, dt + \iint_{\Omega \cap Q_T} |D(\xi u)|^p \, dx \, dt
\]
\[
\leq c \iint_{\Omega \cap Q_T} [ |Du|^p + \varphi^{-p} |u|^p ] \, dx \, dt.
\]
This proves the claimed estimate. \(\square\)

In the next lemma, we provide an estimate for the distributional time derivative of the boundary values.

**Lemma 5.3.** Let $u$ be a weak solution to (1.1) in $\Omega_T$, and $g_\varepsilon = \eta_\varepsilon u$ be constructed as in (4.6). Then, for any $\varphi \in C^\infty_0((\Omega_\varepsilon)_T \cap Q_\varphi, \mathbb{R}^N)$ we have
\[
\left| \iint_{\Omega_T} g_\varepsilon \cdot \partial_t \varphi \, dx \, dt \right| \leq c \left[ \iint_{\Omega_T \cap \text{spt} \varphi} (\lambda^p + |Du|^p) \, dx \, dt \right]^{\frac{p-1}{p}} \|D\varphi\|_{L^p((\Omega_\varepsilon)_T \cap Q_\varphi)} \quad (5.1)
\]
\[
+ \|b\|_{L^{\frac{p(n+2)}{p(n+2) - n}}((\Omega_\varepsilon)_T \cap Q_\varphi)} \|\varphi\|_{L^{\frac{p(n+2)}{n}}((\Omega_\varepsilon)_T \cap Q_\varphi)}
\]
with a constant $c = c(n, p, M)$ and the parameter $\lambda$ from (4.4). In particular, $\partial_t g_\varepsilon \in V'_\varepsilon$.

Note that $b \in L^{\frac{p(n+2)}{p(n+2) - n}}((\Omega_\varepsilon)_T)$, since $\sigma > n + 2 > \frac{p(n+2)}{p(n+2) - n}$. Therefore, the right-hand side of (5.1) is finite.

**Proof.** Let $\varphi \in C^\infty_0((\Omega_\varepsilon)_T \cap Q_\varphi, \mathbb{R}^N)$, and consider the cutoff function $\eta_\varepsilon(x)$ from (4.6). Testing the weak form of (1.1) with $\eta_\varepsilon \varphi$ and recalling (2.2), we estimate
\[
\left| \iint_{\Omega_T} g_\varepsilon \cdot \partial_t \varphi \, dx \, dt \right| \leq cM \iint_{\Omega_T} \left( \mu^2 + |Du|^2 \right)^{\frac{p-2}{2}} |Du| \, dx \, dt + \iint_{\Omega_T} |b||\varphi| \, dx \, dt
\]
\[ \leq cM \left[ \int_{\Omega_T \cap \text{spt} \varphi} (\lambda^p + |Du|^p) \, dx \, dt \right]^{p-1 \over p} \| D(\eta \varphi) \|_{L^p(\Omega_T)} + \| b \|_{L^{p(n+2)-n}((\Omega_T \cap \text{spt} \varphi))} \| \varphi \|_{L^{p(n+2)}((\Omega_T \cap \text{spt} \varphi))} \cdot \]

For the norm in the second-to-last term, we have

\[ \| D(\eta \varphi) \|_{L^p_\Omega} \leq c \int_{\Omega_T \cap Q_\varphi} |D\varphi|^p \, dx \, dt + c \varepsilon^p \int_{\Omega_T \cap Q_\varphi \cap \text{spt} (D\eta \varepsilon)} |\varphi|^p \, dx \, dt. \quad (5.3) \]

To bound the last integral, we observe that for points \( x \) in the domain of integration, we have

\[ \text{dist}(x, \partial \Omega \varepsilon) \leq 2\varepsilon + \text{dist}(x, \partial \Omega) \leq 4\varepsilon \]

by the construction of \( \Omega \varepsilon \) and since \( \text{spt}(D\eta \varepsilon) \) is contained in the \( 2\varepsilon \)-neighborhood of \( \partial \Omega \). Therefore, we can apply Hardy’s inequality from Lemma 5.1 on the time slices \( \Omega \varepsilon \times \{ t \} \) for a.e. \( t \in (t_0 - \varepsilon^2, t_0) \), after extending \( \varphi \) by zero on \( (\Omega \varepsilon \times \{ t \}) \setminus B_\varepsilon \). Note that the constant in Hardy’s inequality only depends on \( n \) and \( p \), but is independent of \( \varepsilon \). As a result, we obtain

\[ \frac{1}{\varepsilon^p} \int_{\Omega_T \cap Q_\varphi \cap \text{spt} (D\eta \varepsilon)} |\varphi|^p \, dx \, dt \leq c \int_{\Omega \varepsilon \times (t_0 - \varepsilon^2, t_0)} \left( \frac{|\varphi|}{\text{dist}(x, \partial \Omega \varepsilon)} \right)^p \, dx \, dt \]

\[ \leq c \int_{\Omega \varepsilon \times (t_0 - \varepsilon^2, t_0)} |D\varphi|^p \, dx \, dt. \]

Joining this bound with (5.3), we arrive at \( \| D(\eta \varphi) \|_{L^p_\Omega} \leq c \| D\varphi \|_{L^p_\Omega} \), for a constant \( c = c(n, p) \).

Using this in (5.2), we deduce the asserted estimate (5.1). Finally, we note that Gagliardo–Nirenberg’s inequality implies

\[ \| \varphi \|_{L^{p(n+2)}(\Omega_T \cap Q_\varphi)} \leq \left[ \int_{\Omega_T \cap Q_\varphi} |D\varphi|^p \, dx \, dt \left( \sup_{t \in (t_0 - \varepsilon^2, t_0)} \int_{\Omega \varepsilon \cap B_\varepsilon} |\varphi(\cdot, t)|^2 \, dx \right) \right]^{p \over n} \leq c \| \varphi \|_{W^1_{\varepsilon}}. \]

Therefore, the estimate (5.1) can be rewritten in the form

\[ \left| \int_{\Omega_T} g \varepsilon \cdot \partial_t \varphi \, dx \, dt \right| \]

\[ \leq c \left[ \left( \int_{\Omega_T \cap \text{spt} \varphi} (\lambda^p + |Du|^p) \, dx \, dt \right)^{p-1 \over p} + \| b \|_{L^{p(n+2)}((\Omega_T \cap \text{spt} \varphi))} \right] \| \varphi \|_{W^1_{\varepsilon}} \]

for any \( \varphi \in C_0^\infty((\Omega_T \varepsilon) \cap Q_\varphi, \mathbb{R}^N) \). This proves the assertion \( \partial_t g \varepsilon \in V_{\varepsilon}' \). \( \square \)

We use the preceding estimate of the distributional time derivative of \( g \varepsilon \) for the proof of the desired energy estimate. The difficulty comes from the fact that \( u \) and \( u \varepsilon \) are solutions on different
domains $\Omega_T$ and $(\Omega_\varepsilon)_T$. For ease of notation, we define

$$V_\lambda(A) := \left(\lambda^2 + |A|^2\right)^{\frac{p-2}{4}} A, \quad \text{for } A \in \mathbb{R}^{Nn}.$$

**Lemma 5.4** (Energy estimate). For any $\varepsilon > 0$ and any weak solution $u_\varepsilon$ to the Cauchy–Dirichlet problem (4.8), we have

$$\sup_{\tau \in (t_0 - \varphi^2, t_0)} \int_{(\Omega_\varepsilon \cap B_\varphi) \times \{\tau\}} |u_\varepsilon - g_\varepsilon|^2 \, dx + \iint_{(\Omega_\varepsilon)T \cap Q_\varphi} |V_\lambda(Du_\varepsilon)|^2 \, dx \, dt$$

$$\leq c \iint_{\Omega_\varepsilon \cap Q_{2\varphi}} \left[\lambda^p + |Du|^p + \varphi^{-p}|u|^p\right] \, dx \, dt + \|b\| + \|b_\varepsilon\| \frac{\|\partial_t u\|_{L^{\frac{p(n+2)}{p(n+2)-n-p}}(\Omega_\varepsilon)_T \cap Q_\varphi)}{L^{\frac{p(n+2)}{p(n+2)-n}}((\Omega_\varepsilon)_T \cap Q_\varphi)},$$

with a constant $c = c(n, p, m, M)$.

**Proof.** For fixed $\tau \in (t_0 - \varphi^2, t_0)$ and $\delta \in (0, t_0 - \tau)$, we let

$$\zeta_\delta(t) := \begin{cases} 1, & \text{for } t \in [t_0 - \varphi^2, \tau], \\ \frac{\tau + \delta - t}{\delta}, & \text{for } t \in (\tau, \tau + \delta), \\ 0, & \text{for } t \in [\tau + \delta, t_0]. \end{cases}$$

As in Lemma 5.3, one easily checks that solutions $u_\varepsilon$ to the parabolic systems (4.8) own a distributional time derivative $\partial_t u_\varepsilon \in V'_\varepsilon$. Therefore, the testing function $\zeta_\delta^2(u_\varepsilon - g_\varepsilon) \in V_\varepsilon$ is admissible in the weak form of (4.8), which implies

$$\langle \partial_t u_\varepsilon, \zeta_\delta^2(u_\varepsilon - g_\varepsilon) \rangle + \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} \zeta_\delta^2 a_\varepsilon(Du_\varepsilon)Du_\varepsilon \cdot (Du_\varepsilon - Dg_\varepsilon) \, dx \, dt$$

$$= \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} \zeta_\delta^2 b_\varepsilon \cdot (u_\varepsilon - g_\varepsilon) \, dx \, dt. \quad (5.4)$$

Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $V'_\varepsilon \times V_\varepsilon$. We rewrite the first term on the left-hand side in the form

$$\langle \partial_t u_\varepsilon, \zeta_\delta^2(u_\varepsilon - g_\varepsilon) \rangle = \langle \partial_t (u_\varepsilon - g_\varepsilon), \zeta_\delta^2(u_\varepsilon - g_\varepsilon) \rangle + \langle \partial_t g_\varepsilon, \zeta_\delta^2(u_\varepsilon - g_\varepsilon) \rangle$$

$$= \langle \partial_t \zeta_\delta(u_\varepsilon - g_\varepsilon), \zeta_\delta(u_\varepsilon - g_\varepsilon) \rangle - \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} \zeta_\delta' \zeta_\delta |u_\varepsilon - g_\varepsilon|^2 \, dx \, dt$$

$$+ \langle \partial_t g_\varepsilon, \zeta_\delta^2(u_\varepsilon - g_\varepsilon) \rangle$$

$$=: \text{II} + \text{II}(\delta) + \text{II}(\delta),$$

with the obvious meaning of $\text{I}(\delta)$ to $\text{III}(\delta)$. For the first term, we find

$$\text{I}(\delta) = \frac{1}{2} \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} \partial_t \zeta_\delta |u_\varepsilon - g_\varepsilon|^2 \, dx \, dt = 0,$$
since \( \zeta_\delta(t_o) = 0 \) and \( u_\varepsilon = g_\varepsilon \) on the initial time slice \( (\Omega_\varepsilon \cap B_\varphi) \times \{t_o - \varphi^2\} \). By the mean value theorem, we obtain

\[
\lim_{\delta \downarrow 0} \mathbf{II}(\delta) = \frac{1}{2} \int_{(\Omega_\varepsilon \cap B_\varphi) \times \{\tau\}} |u_\varepsilon - g_\varepsilon|^2 \, dx.
\]

Finally, we estimate the third term by means of Lemma 5.3, with the result

\[
|\mathbf{III}(\delta)| \leq c \left[ \int_{\Omega_T \cap Q_\varphi} (\lambda^p + |Du|^p) \, dx \, dt \right]^{\frac{p-1}{p}} \|Du_\varepsilon - Dg_\varepsilon\|_{L^p(\Omega_\varepsilon \cap Q_\varphi)}
\]

\[+ \|b\| \left. \right|_{L^{\frac{p(n+2)}{n}}((\Omega_\varepsilon)_T \cap Q_\varphi)} \|u_\varepsilon - g_\varepsilon\|_{L^{\frac{p(n+2)}{n}}((\Omega_\varepsilon)_T \cap Q_\varphi)}.
\]

For the last term in (5.4), a straightforward application of Hölder’s inequality yields

\[
\oint_{(\Omega_\varepsilon)_T \cap Q_\varphi} \zeta_\delta^2 b_\varepsilon \cdot (u_\varepsilon - g_\varepsilon) \, dx \, dt
\]

\[\leq \|b\| \left. \right|_{L^{\frac{p(n+2)}{n}}((\Omega_\varepsilon)_T \cap Q_\varphi)} \|u_\varepsilon - g_\varepsilon\|_{L^{\frac{p(n+2)}{n}}((\Omega_\varepsilon)_T \cap Q_\varphi)}.
\]

The preceding considerations allow us to pass to the limit \( \delta \downarrow 0 \) in (5.4). In the term not yet considered, that is, the one containing the coefficients \( a_\varepsilon \), the passage to the limit under the integral can be justified by dominated convergence. Overall, we get

\[
\frac{1}{2} \int_{(\Omega_\varepsilon \cap B_\varphi) \times \{\tau\}} |u_\varepsilon - g_\varepsilon|^2 \, dx + \oint_{(\Omega_\varepsilon)_T \cap Q_\varphi} a_\varepsilon(Du_\varepsilon)Du_\varepsilon \cdot (Du_\varepsilon - Dg_\varepsilon) \, dx \, dt
\]

\[\leq c \left[ \int_{\Omega_T \cap Q_\varphi} (\lambda^p + |Du|^p) \, dx \, dt \right]^{\frac{p-1}{p}} \|Du_\varepsilon - Dg_\varepsilon\|_{L^p(\Omega_\varepsilon)_T \cap Q_\varphi}
\]

\[+ \frac{2\|b\|^2 \left. \right|_{L^{\frac{p(n+2)}{n}}((\Omega_\varepsilon)_T \cap Q_\varphi)} \|u_\varepsilon - g_\varepsilon\|_{L^{\frac{p(n+2)}{n}}((\Omega_\varepsilon)_T \cap Q_\varphi)}.
\]

for any \( \tau \in (t_o - \varphi^2, t_o) \). By the growth properties (4.3) of \( a_\varepsilon \) and Young’s inequality for the \( V_\lambda \)-function [2, Lemma 2.3], we obtain for the diffusion term

\[
\oint_{(\Omega_\varepsilon)_T \cap Q_\varphi} a_\varepsilon(Du_\varepsilon)Du_\varepsilon \cdot (Du_\varepsilon - Dg_\varepsilon) \, dx \, dt
\]

\[\geq \frac{m}{\varepsilon} \oint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |V_\lambda(Du_\varepsilon)|^2 \, dx \, dt - c \oint_{(\Omega_\varepsilon)_T \cap Q_\varphi} \left( \lambda^2 + |Du_\varepsilon|^2 \right)^{\frac{p-2}{2}} |Du_\varepsilon| |Dg_\varepsilon| \, dx \, dt
\]

\[\geq \frac{m}{2\varepsilon} \oint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |V_\lambda(Du_\varepsilon)|^2 \, dx \, dt - c \oint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |V_\lambda(Dg_\varepsilon)|^2 \, dx \, dt.
\]
We join the two preceding estimates, take the supremum over $\tau \in (t_o - \varphi^2, t_o)$ in the first term on the left-hand side and let $\tau \uparrow t_o$ in the second one. This gives

$$S + \iint_{(\Omega_\tau) \cap Q_\varphi} |V_A(Du_\varphi)|^2 \, dx \, dt \tag{5.5}$$

$$\leq c \left[ \iint_{\Omega_\tau \cap Q_\varphi} (\lambda^p + |Du|^p) \, dx \, dt \right]^{\frac{1}{p-1}} \|Du_\varphi - Dg_\varphi\|_{L^p((\Omega_\tau) \cap Q_\varphi)}$$

$$+ c \|b\| + |b_\varphi| \left[ \iint_{\Omega_\tau \cap Q_\varphi} |u_\varphi - g_\varphi|^{\frac{p(n+2)}{p(n+2)-n}} \, dx \, dt \right]^{\frac{p(n+2)-n}{p(n+2)}}$$

$$+ c \iint_{\Omega_\tau \cap Q_\varphi} |V_A(Dg_\varphi)|^2 \, dx \, dt$$

with the abbreviation

$$S := \sup_{\tau \in (t_o - \varphi^2, t_o)} \int_{(\Omega_\tau \cap B_\varphi) \times \{\tau\}} |u_\varphi - g_\varphi|^2 \, dx.$$

The inequalities of Gagliardo–Nirenberg and Young provide us with the estimate

$$\|u_\varphi - g_\varphi\|_{L^p((\Omega_\tau) \cap Q_\varphi)} \leq c \left[ S \iint_{(\Omega_\tau) \cap Q_\varphi} |Du_\varphi - Dg_\varphi|^p \, dx \, dt \right]^{\frac{n}{p(n+2)}}$$

$$\leq c \left[ S + \iint_{(\Omega_\tau) \cap Q_\varphi} |Du_\varphi - Dg_\varphi|^p \, dx \, dt \right]^{\frac{n}{p(n+2)}} \iint_{(\Omega_\tau) \cap Q_\varphi} |V_A(Dg_\varphi)|^2 \, dx \, dt + c \iint_{\Omega_\tau \cap Q_\varphi} |V_A(Dg_\varphi)|^2 \, dx \, dt$$

We use this bound in (5.5) and apply Young’s inequality, with the result

$$S + \iint_{(\Omega_\tau) \cap Q_\varphi} |V_A(Du_\varphi)|^2 \, dx \, dt$$

$$\leq \frac{1}{2} S + \frac{1}{2} \iint_{(\Omega_\tau) \cap Q_\varphi} |V_A(Du_\varphi)|^2 \, dx \, dt + c \iint_{\Omega_\tau \cap Q_\varphi} (\lambda^p + |Du|^p) \, dx \, dt$$

$$+ c \iint_{\Omega_\tau \cap Q_\varphi} |V_A(Dg_\varphi)|^2 \, dx \, dt + c \|b\| + |b_\varphi| \left[ \iint_{\Omega_\tau \cap Q_\varphi} |u_\varphi - g_\varphi|^{\frac{p(n+2)}{p(n+2)-n}} \, dx \, dt \right]^{\frac{p(n+2)-n}{p(n+2)}}.$$
Remark 5.5. The same arguments yield the following local (in time) energy estimate
\[
\sup_{\tau \in (t_0 - \varphi^2, t_0]} \int_{(\Omega_\varepsilon \cap B_\varphi) \times \{\tau\}} |u_\varepsilon - g_\varepsilon|^2 \, dx + \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |V_\lambda(Du_\varepsilon)|^2 \, dx \, dt \\
\leq c \iint_{\Omega_\varepsilon \cap Q_{2\varphi}} [\lambda^p + |Du|^p + \varphi^{-p}|u|^p] \, dx \, dt + c \|b\| + \|b_\varepsilon\| \frac{p(n+2)}{p(n+2)-n-p} \left[ L_{p(n+2)-n}((\Omega_\varepsilon)_T \cap Q_\varphi) \right]^d \varphi.
\]
for any \( s \in (t_0 - \varphi^2, t_0] \).

5.2 Proof of the gradient estimate

We recall (4.2), (4.3), and (4.7) and the fact \( u_\varepsilon \in C^3((\overline{\Omega_\varepsilon})_T \cap Q_\varphi(z_\circ)) \) (see Appendix B). Therefore, Proposition 3.1 is applicable with \( u, a, b, \Omega, \Theta \) replaced by \( u_\varepsilon, a_\varepsilon, b_\varepsilon, \Omega_\varepsilon, \Theta_{\varphi/2}(x_\circ) \). We thus obtain the gradient estimate
\[
\sup_{(\Omega_\varepsilon)_T \cap Q_{\varphi/2}} \left( 1 + |Du_\varepsilon|^2 \right)^{\frac{1}{2}} \leq C \left[ \left( 1 + \varphi^{n+2} \|b_\varepsilon\|_{L^2((\Omega_\varepsilon)_T \cap Q_\varphi)} \right) \iint_{(\Omega_\varepsilon)_T \cap Q_{3\varphi/4}} (1 + |Du_\varepsilon|^p) \, dx \, dt \right]^{\frac{d}{p}}. \tag{5.6}
\]

In view of (4.2) and (4.3), the constant \( C \) in the preceding inequality depends only on \( n, N, p, m, M \), and \( \Theta_{\varphi/2}(x_\circ) \), but is independent of \( \varepsilon \). The energy estimate from Lemma 5.4 implies
\[
\sup_{\tau \in (t_0 - \varphi^2, t_0]} \int_{(\Omega_\varepsilon \cap B_\varphi) \times \{\tau\}} |u_\varepsilon - g_\varepsilon|^2 \, dx + \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |Du_\varepsilon|^p \, dx \, dt \leq C, \tag{5.7}
\]
with a constant \( C \) independent of \( \varepsilon \in (0, 1) \); note in particular that \( \|b_\varepsilon\|_{L^2} \) is bounded independently of \( \varepsilon \). We combine this with the gradient sup-estimate from (5.6) replacing \( (\varphi^2, \varphi) \) by \( (\varphi^{3/4}, \varphi) \). This yields the uniform bound
\[
\sup_{(\Omega_\varepsilon)_T \cap Q_{3\varphi/4}} |Du_\varepsilon| \leq C, \tag{5.8}
\]
with a constant \( C \) independent of \( \varepsilon \). From the construction of the boundary values \( g_\varepsilon \), it is clear that \( g_\varepsilon \to u \) in \( L^p(\Omega_T \cap Q_\varphi) \) as \( \varepsilon \downarrow 0 \). Moreover, Lemma 5.2 ensures that
\[
\iint_{\Omega_T \cap Q_\varphi} [\lambda |Dg_\varepsilon|^p + |g_\varepsilon|^p] \, dx \, dt \leq c \iint_{\Omega_T \cap Q_{2\varphi}} [\lambda |Du|^p + \varphi^{-p}|u|^p] \, dx \, dt, \tag{5.9}
\]
for every \( \varepsilon \in (0, 1) \). We therefore deduce
\[
g_\varepsilon \to u \text{ weakly in } L^p(t_0 - \varphi^2, t_0; W^{1,p}(\Omega \cap B_\varphi, \mathbb{R}^N)) \text{ as } \varepsilon \downarrow 0. \tag{5.10}
\]
Moreover, Poincaré’s inequality and (5.9) yield the bound

\[
\iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |u_\varepsilon|^p \, dx \, dt \leq c \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |u_\varepsilon - g_\varepsilon|^p \, dx \, dt + c \iint_{(\Omega_\varepsilon)_T \cap Q_\varphi} |g_\varepsilon|^p \, dx \, dt
\]

(5.11)

We extend \( u_\varepsilon \) by zero on \( Q_\varphi \setminus (\Omega_\varepsilon)_T \). Since \( u_\varepsilon = 0 \) on \( (\partial \Omega_\varepsilon)_T \cap Q_\varphi \) in the sense of traces, the extended maps satisfy \( u_\varepsilon \in L^p(t_0 - \varphi^2, t_0; W^{1,p}(B_\varphi, \mathbb{R}^N)) \) for every \( \varepsilon \in (0, 1) \). Moreover, estimates (5.7) and (5.11) imply that the family \( (u_\varepsilon)_{\varepsilon \in (0, 1)} \) is bounded in the latter space. Therefore, we find \( \varepsilon_i \downarrow 0 \) and a limit map \( \tilde{u} \in L^p(t_0 - \varphi^2, t_0; W^{1,p}(B_\varphi, \mathbb{R}^N)) \) such that

\[
u_{\varepsilon_i} \rightharpoonup \tilde{u} \text{ weakly in } L^p\left(t_0 - \varphi^2, t_0; W^{1,p}(B_\varphi, \mathbb{R}^N)\right) \text{ as } i \to \infty. \tag{5.12} \]

In view of the uniform \( L^\infty - L^2 \) bound (5.7), we can pass to a non-relabeled subsequence to deduce that \( \tilde{u} \in L^\infty(t_o - \varphi^2, t_o; L^2(B_\varphi, \mathbb{R}^N)) \) and

\[
u_{\varepsilon_i} - g_{\varepsilon_i} \rightharpoonup^* \tilde{u} - u \text{ weakly* in } L^\infty\left(t_o - \varphi^2, t_o; L^2(B_\varphi, \mathbb{R}^N)\right) \text{ as } i \to \infty. \]

By construction, the maps \( u_\varepsilon \) agree with \( g_\varepsilon \) on the lateral boundary in the sense that

\[
u - g_\varepsilon \in L^p\left(t_0 - \varphi^2, t_0; W^{1,p}(B_\varphi, \mathbb{R}^N)\right).
\]

Because of the weak convergences (5.10) and (5.12), this boundary condition is preserved in the limit \( \varepsilon_i \downarrow 0 \), from which we deduce

\[
u - u \in L^p\left(t_0 - \varphi^2, t_0; W^{1,p}_0(B_\varphi, \mathbb{R}^N)\right). \tag{5.13} \]

Now, let \( \varepsilon_o > 0 \) and consider the outer parallel set \( O_{2\varepsilon_o} := \{x \in B_\varphi: \text{dist}(x, \Omega) < 2\varepsilon_o\} \). Since \( \Omega_\varepsilon \subset O_{2\varepsilon_o} \) for every \( \varepsilon \in (0, \varepsilon_o) \), we have

\[
u_{\varepsilon} - g_\varepsilon = 0 \text{ a.e. on } Q_\varphi \setminus (O_{2\varepsilon_o})_T, \text{ for every } \varepsilon \in (0, \varepsilon_o).
\]

Also, this property is preserved in the limit \( \varepsilon_i \downarrow 0 \), which implies that \( \tilde{u} = u \) a.e. on \( Q_\varphi \setminus (O_{2\varepsilon_o})_T \) for every \( \varepsilon_o > 0 \). In turn, we conclude

\[
u = u \text{ a.e. on } Q_\varphi \setminus \Omega_T. \tag{5.14} \]

Combining the properties (5.13) and (5.14), we infer the desired boundary condition

\[
u \in u + L^p\left(t_0 - \varphi^2, t_0; W^{1,p}_0(\Omega \cap B_\varphi, \mathbb{R}^N)\right).
\]
for the limit map $\tilde{u}$. Our next goal is to show that the limit map $\tilde{u}$ attains the expected initial values at the initial time $t_0 - \varphi^2$. To this end, we exploit the lower semicontinuity of the $L^2$-norm with respect to weak convergence and the local (in time) energy estimate from Remark 5.5 to estimate

$$
\frac{1}{h} \int_{t_0 - \varphi^2}^{t_0 - \varphi^2 + h} \|u(t) - u(t)\|_{L^2(\Omega \cap B_\varphi)}^2 \, dt
\leq \liminf_{\varepsilon \downarrow 0} \frac{1}{h} \int_{t_0 - \varphi^2}^{t_0 - \varphi^2 + h} \|u_\varepsilon(t) - g_\varepsilon(t)\|_{L^2(\Omega \cap B_\varphi)}^2 \, dt
\leq c \int_{(\Omega \cap B_{3\varphi}) \times (t_0 - \varphi^2, t_0 - \varphi^2 + h)} \left[ \lambda^p + |Du|^p + \varphi^{-p} |u|^p \right] \, dx \, dt
+ c \|b\|_{L^p((\Omega \cap B_{3\varphi}) \times (t_0 - \varphi^2, t_0 - \varphi^2 + h))}^p
$$

for every $h \in (0, \varphi^2)$. Since the right-hand side of the last inequality converges to 0 as $h \downarrow 0$, we infer

$$
\lim_{h \downarrow 0} \frac{1}{h} \int_{t_0 - \varphi^2}^{t_0 - \varphi^2 + h} \|\tilde{u}(t) - u(t)\|_{L^2(\Omega \cap B_\varphi)}^2 \, dt = 0.
$$

Since $u \in C^0([0, T]; L^2(\Omega, \mathbb{R}^N))$ by assumption, this implies that $\tilde{u} = u$ on $(\Omega \cap B_\varphi) \times \{t_0 - \varphi^2\}$ in the usual $L^2$-sense. At this stage, it remains to verify the differential equation for the limit map $\tilde{u}$. For a fixed compact set $K \Subset \Omega_T \cap Q_\varphi$, the interior $C^{1,\alpha}$-estimates from [12, Chapter IX, Theorem 1.1 and Chapter VIII, Theorems 5.1 and 5.2'] and the uniform energy bound (5.7) imply

$$
\|Du_\varepsilon\|_{C^{0,\alpha}(K)} \leq C
$$

(5.15)

for every $\varepsilon \in (0, 1)$, for some Hölder exponent $\alpha \in (0, 1)$ and some constant $C > 0$, both independent of $\varepsilon$. This allows us to apply Ascoli-Arzéla’s theorem to conclude that $Du_\varepsilon$ converges uniformly to $D\tilde{u}$ on compact subsets of $\Omega_T \cap Q_\varphi$. In particular, we have $Du_\varepsilon \to D\tilde{u}$ pointwise in $\Omega_T \cap Q_\varphi$. In view of the uniform gradient bound on compact subsets contained in (5.15) and the property (4.5) of the regularized coefficients, we can use dominated convergence to pass to the limit in the weak formulation of the system (4.8). We conclude that the limit map $\tilde{u}$ is a weak solution to the system (1.1) on $\Omega_T \cap Q_\varphi$. Moreover, we know that $\tilde{u} = u$ on $\partial_{p}(\Omega_T \cap Q_\varphi)$. By uniqueness of solutions, this shows that $\tilde{u} \equiv u$ in $\Omega_T \cap Q_\varphi$.

Moreover, due to the sup-bound for the spatial gradient (5.8) we may apply the dominated convergence theorem to get

$$
Du_\varepsilon \to D\tilde{u} = Du \text{ strongly in } L^p(Q_{3\varphi/4}, \mathbb{R}^N) \text{ in the limit } \varepsilon_i \downarrow 0,
$$

where we extended $u_\varepsilon$ by zero on $Q_{3\varphi/4} \setminus (\Omega_\varepsilon)_T$. This strong convergence enables us to pass to the limit $\varepsilon_i \downarrow 0$ on the right-hand side of (5.6). Note that the construction of $b_\varepsilon$ ensures the convergence $\|b_\varepsilon\|_{L^p((\Omega_\varepsilon)_T \cap Q_\varphi)} \to \|b\|_{L^p(\Omega_T \cap Q_\varphi)}$. On the left-hand side of (5.6), we may pass to the limit due to the
pointwise convergence. In this way, we obtain
\[
\sup_{\Omega_T \cap Q_{3/4}} (1 + |Du|^2)^{1/2} \leq C \left( 1 + \varphi^{n+2} \| b \|_{L^\infty(\Omega_T \cap Q_{3/4})} \right) \int_{\Omega_T \cap Q_{3/4}} (1 + |Du|^p) \, dx \, dt \right)^{\frac{d}{p}}.
\]
This yields the asserted sup-estimate for the gradient of \( u \), and completes the proof of Theorem 1.2.

APPENDIX A: PROPERTIES OF THE REGULARIZED COEFFICIENTS

Here, we provide proofs for the properties of the regularized coefficients \( a_\varepsilon \) stated in Subsection 4.2. The first line in (4.3) follows directly from the definition of \( c_\varepsilon \) and the growth condition (2.2) for \( a \). The constant \( c \) can be chosen in the form \( c(p) \max\{1, e^{p-2}\} \) with the constant \( c(p) \) from (2.2). Concerning the ellipticity condition, we observe that
\[
a'_\varepsilon(r)r + a_\varepsilon(r) = c'_\varepsilon(\log r) + c_\varepsilon(\log r) = (\phi_\varepsilon * (c' + c))(\log r) \quad \text{for any } r > 0. \quad (A.1)
\]
For the function \( c' + c \) appearing on the right-hand side, we have in the case \( \mu = 0 \) that
\[
c'(s) + c(s) = \frac{1}{\varepsilon^2 + e^{2s}} \left[ e^{2s} \left( e^{2s} a'(e^s) + a(e^s) \right) + e^2 a \left( e^{2s} + e^{2s} \right) \right]
\]
for any \( s \in \mathbb{R} \). Using the lower bounds from (1.4) and (2.2), we deduce
\[
c'(s) + c(s) \geq c(p)^{-1} m(e^2 + e^{2s})^{\frac{p-2}{2}}. \quad (A.2)
\]
On the other hand, in the case \( \mu > 0 \) we have
\[
c'(s) + c(s) = a'(e^s)e^s + a(e^s) \geq m(\mu^2 + e^{2s})^{\frac{p-2}{2}} \quad (A.3)
\]
for any \( s \in \mathbb{R} \). Similarly as above, we infer from (A.1), (A.2), (A.3) and the definition of \( \lambda \) that
\[
a'_\varepsilon(r)r + a_\varepsilon(r) \geq c(p)^{-1} m \min \{1, e^{-(p-2)}(\lambda^2 + r^2)^{\frac{p-2}{2}} \quad \text{for any } r > 0.
\]
This yields the lower bound in (4.3). Similarly, by applying the upper bound from (1.4) (taking also into account the fact that \( a \) is non-negative, cf. (2.2)), we obtain
\[
(\phi_\varepsilon * c')(s) \leq c(p)M \max\{1, e^{p-2}\}(\lambda^2 + e^{2s})^{\frac{p-2}{2}} \quad \text{for any } s \in \mathbb{R}.
\]
From this, we deduce
\[
a'_\varepsilon(r)r = (\phi_\varepsilon * c')(\log r) \leq c(p)M \max\{1, e^{p-2}\}(\lambda^2 + r^2)^{\frac{p-2}{2}} \quad \text{for any } r > 0.
\]
which implies the asserted upper bound in (4.3). At this stage, it remains to derive the estimate for the second derivative $a''_\varepsilon$. To this end, we compute

$$a''_\varepsilon(r)^2 = c''_\varepsilon(\log r) - c'_\varepsilon(\log r) = ((\phi'_\varepsilon - \phi_\varepsilon) \ast c')(\log r).$$

Then we use (1.4) and (2.2) to derive in the case $\mu = 0$ the bound

$$|c'(s)| = \left| a'(\sqrt{\varepsilon^2 + e^{2s}}) \frac{e^{2s}}{\sqrt{\varepsilon^2 + e^{2s}}} \right|$$

$$\leq \left| \sqrt{\varepsilon^2 + e^{2s}} a'(\sqrt{\varepsilon^2 + e^{2s}}) + a(\sqrt{\varepsilon^2 + e^{2s}}) \right|$$

$$\leq (1 + c(p))M(\varepsilon^2 + e^{2s})^{\frac{p-2}{2}} \leq 2c(p)M(\varepsilon^2 + e^{2s})^{\frac{p-2}{2}},$$

while in the case $\mu > 0$, we obtain

$$|c'(s)| = |a'(e^s)e^s| \leq |e^s a'(e^s) + a(e^s)| = |a(e^s)|$$

$$\leq (1 + c(p))M(\mu^2 + e^{2s})^{\frac{p-2}{2}} \leq 2c(p)M(\mu^2 + e^{2s})^{\frac{p-2}{2}}.$$

Hence, in both cases we have

$$|c'(s)| \leq 2c(p)M(\lambda^2 + e^{2s})^{\frac{p-2}{2}}.$$  \hspace{1cm} (A.4)

From this we deduce, similarly as above, that

$$|a''_\varepsilon(r)|^2 = |((\phi'_\varepsilon - \phi_\varepsilon) \ast c')(s)|$$

$$\leq 2c(p)M\max\{1, e^{p-2}\}(\lambda^2 + e^{2s})^{\frac{p-2}{2}} \int_\mathbb{R} |\phi'_\varepsilon - \phi_\varepsilon| \, ds$$

$$\leq 2c(p)M\left(\frac{2}{\varepsilon} ||\phi'||_{L^\infty} + 1\right)\max\{1, e^{p-2}\}(\lambda^2 + e^{2s})^{\frac{p-2}{2}}.$$

The proof of the claim (4.3) is thus complete. Finally, we analyze the convergence of $a_\varepsilon(r)$ in the limit $\varepsilon \downarrow 0$ and thereby prove (4.5). For any $s \in \mathbb{R}$, we estimate

$$|c_\varepsilon(s) - c(s)| \leq \sup_{s-\varepsilon < r < s + \varepsilon} |c(r) - c(s)| = \sup_{s-\varepsilon < r < s + \varepsilon} \left| \int_s^r c'(\tau) \, d\tau \right|$$

$$\leq 2c(p)M\sup_{s-\varepsilon < r < s + \varepsilon} \left| \int_s^r (e^{2\tau} + e^{s\tau})^{\frac{p-2}{2}} \, d\tau \right|$$

$$\leq 2c(p)M\varepsilon \max\{1, e^{p-2}\}(\varepsilon^2 + e^{2s})^{\frac{p-2}{2}},$$

where the second-to-last step follows from (A.4). This gives the desired estimate (4.5).
APPENDIX B: REGULARITY UP TO THE BOUNDARY

Here, we show that solutions to the regularized problem (4.8) are smooth up to the boundary as claimed at the end of Section 4. To this end, we follow the strategy of Banerjee and Lewis [3, Appendix, Proof of (2.7)] to flatten the boundary and then to reduce the problem of boundary regularity to the interior case by a reflection argument.

B.1 Schauder estimates for linear parabolic systems

In this section, we explain Schauder estimates for linear parabolic systems of the type

$$\partial_t u^i - \sum_{\alpha,\beta=1}^{n} \sum_{j=1}^{N} \left[ A^{ij}_{\alpha\beta} u^j x_\alpha \right] x_\beta = \sum_{\alpha=1}^{n} \sum_{j=1}^{N} b^{ij}_\alpha u^j x_\alpha + \sum_{\alpha=1}^{n} (f^i_\alpha) x_\alpha + c^i \quad \text{in } \Omega_T,$$

for $i = 1, 2, \ldots, N$, where the coefficients $A^{ij}_{\alpha\beta} : \Omega_T \to \mathbb{R}$ satisfy for some $0 < \nu \leq L$ the ellipticity and boundedness condition

$$\nu |\xi|^2 \leq \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} A^{ij}_{\alpha\beta} \xi_i \xi_j \leq L |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{Nn}. \quad (B.2)$$

We will assume that the functions $c^i : \Omega_T \to \mathbb{R}$ belong to a parabolic Campanato–Morrey space, which is defined as follows.

**Definition B.1.** With $q \geq 1$, $\theta \geq 0$, a measurable map $w : \Omega_T \to \mathbb{R}^k$, $k \geq 1$, belongs to the (parabolic) Morrey space $L^{q,\theta}(\Omega_T, \mathbb{R}^k)$ if and only if

$$\|w\|_{L^{q,\theta}(\Omega_T, \mathbb{R}^k)} := \sup_{z_0 \in \Omega_T, 0 < \varphi < \text{diam}(\Omega_T)} \varphi^{-\theta} \int_{\Omega_T \cap Q_\varphi(z_0)} |w|^q \, dx \, dt < \infty.$$

By $C^{0,\mu}$ we mean Hölder continuity with respect to the parabolic metric, that is, with Hölder exponent $\mu$ in space and $\frac{\mu}{2}$ in time. With these notions at hand, we state the following parabolic Schauder estimates, which can be proved by standard comparison and freezing techniques (cf. [7, 29, 31]).

**Theorem B.2.** Suppose $A^{ij}_{\alpha\beta}$ and $f^i_\alpha$ are in $C^{0,\mu}(\Omega_T)$ for some $\mu \in (0, 1)$, whereas $b^{ij}_\alpha \in L^\infty(\Omega_T)$ and $c^i \in L^{2,\theta}(\Omega_T)$ for $\theta := n + 2\mu$. Let $u$ be a weak solution to (B.1) under the assumption (B.2). Then $Du \in C^{0,\mu}_\text{loc}(\Omega_T)$ and moreover for any compact set $K \subset \Omega_T$, we have

$$[Du]_{\mu,K} \leq C \left[ \|Dv\|_{L^2(\Omega_T)} + M \right],$$

where $C$ depends on $n$, $\nu$, $L$ and $\text{dist}(K, \Omega_T)$, and $M$ depends on the norms of $A^{ij}_{\alpha\beta}$, $b^{ij}_\alpha$, $f^i_\alpha$, and $c^i$ in their corresponding spaces.

B.2 Flattening of the boundary

Before we start with the actual construction of local boundary coordinates, we introduce a few abbreviations. By $B^{(n-1)}_\delta(0) \equiv B^{(n-1)}_\delta$, we denote the ball of radius $\delta > 0$ centered at the origin.
in \( \mathbb{R}^{n-1} \). Then, for \( \eta > 0 \) we define \( C_{\delta,\eta}^{+} := B_{\delta}^{(n-1)} \times (-\eta, \eta) \), and similarly \( C_{\delta,\eta}^{-} := B_{\delta}^{(n-1)} \times (-\eta, 0) \). Cylinders in \( \mathbb{R}^{n+1} \) of height \( T > 0 \) with base \( C_{\delta,\eta} \), \( C_{\delta,\eta}^{\pm} \) are denoted by \( Q_{\delta,\eta}^{\pm} \), \( Q_{\delta,\eta}^{\pm} \), so that \( Q_{\delta,\eta} := C_{\delta,\eta} \times (0, T) \).

Since \( \partial \Omega_{\varepsilon} \) is a smooth closed \((n - 1)\)-dimensional submanifold of \( \mathbb{R}^{n} \), it can locally be written as graph of a smooth function \( \phi \in C^\infty(B_{\delta}^{n-1}) \) after a suitable rigid motion. More precisely, for any point \( y_o \in \partial \Omega_{\varepsilon} \cap B_{\delta}(x_o) \), there is a neighborhood \( N_o \) of \( y_o \), so that \( \Omega_{\varepsilon} \cap N_o = \Phi(C_{\delta,\eta}^{\pm}) \) with the parameterization \( \Phi : C_{\delta,\eta} \to N_o \subset \mathbb{R}^n \) defined by

\[
\Phi(y', y_n) := (y', \phi(y')) + \nu(y', \phi(y')) y_n, \quad \text{for } y' \in B_{\delta}^{n-1} \text{ and } y_n \in (-\eta, \eta),
\]

where \( \nu : \partial \Omega_{\varepsilon} \to \mathbb{R}^n \) denotes the outward unit normal on \( \partial \Omega_{\varepsilon} \). By another rigid motion we can achieve that \( y_o = 0 \) and \( \nu(0) = e_n \). The inverse mapping \( \Psi := \Phi^{-1} : N_o \to C_{\delta,\eta} \) is given by

\[
\Psi(x) = \left( x_1 - d_{x_1}(x)d(x), ..., x_{n-1} - d_{x_{n-1}}(x)d(x), d(x) \right) \quad \text{for } x \in N_o,
\]

where \( d \) denotes the signed distance to \( \partial \Omega_{\varepsilon} \). A straightforward computation yields

\[
D\Psi(0) = \text{id}_{\mathbb{R}^n},
\]

and

\[
Q(x) := D\Psi(x)' \cdot D\Psi(x) = \begin{bmatrix} (D\Psi_\alpha(x) \cdot D\Psi_\beta(x))_{1 \leq \alpha, \beta \leq n-1} & 0 \\ 0 & 1 \end{bmatrix}. \tag{B.4}
\]

For a more detailed derivation of these properties, we refer to [5, Section 5.1]. In what follows, we use the short-hand notations

\[
Q_x(\xi, \zeta) := \sum_{i=1}^{N} \sum_{\alpha, \beta=1}^{n} Q_{\alpha}(x)\xi_\alpha \zeta_\beta \quad \text{and} \quad |\xi|_{Q_x} := \sqrt{Q(x)(\xi, \xi)} \tag{B.5}
\]

for matrices \( \xi, \zeta \in \mathbb{R}^{Nn} \). Now we define

\[
\hat{u}(y, t) := u_\varepsilon(\Phi(y), t) \quad \iff \quad u_\varepsilon(x) = \hat{u}(\Psi(x), t)
\]

for \( y \in C_{\delta,\eta}^{\pm} \) and \( t \in [0, T] \), and analogously

\[
\hat{\varphi}(y, t) := \varphi(\Phi(y), t) \quad \iff \quad \varphi(x, t) = \hat{\varphi}(\Psi(x), t)
\]

for any \( \varphi \in L^p(0, T; W_{0}^{1,p}(\Omega_{\varepsilon}, \mathbb{R}^N)) \). Then, \( \hat{u} \in L^p(0, T; W^{1,p}(C_{\delta,\eta}^{\pm})) \) and \( \hat{u} = 0 \) in the sense of traces on \( B_{\delta}^{(n-1)} \times \{0\} \times (0, T) \). For the derivatives in spatial directions, we have

\[
Du_\varepsilon(x, t) \cdot D\varphi(x, t) = Q_x(D\hat{u}(\Psi(x), t), D\hat{\varphi}(\Psi(x), t)).
\]
Moreover, for a.e. \( x \in N_\varepsilon \) and \( t \in (0, T) \) we have

\[
u_\varepsilon(x, t) \cdot \partial_t \varphi(x, t) = \hat{u}(\Psi(x), t) \cdot \partial_t \hat{\varphi}(\Psi(x), t).
\]

Using the two preceding formulae and applying the transformation \( x = \Phi(y) \) on a fixed time slice, we infer

\[
\int_{\Phi(C^-_{\delta,\eta}) \times \{t\}} [u_\varepsilon \cdot \partial_t \varphi - a_\varepsilon(|D u_\varepsilon|) D u_\varepsilon \cdot D \varphi] \, dx
\]

\[
= \int_{C^-_{\delta,\eta} \times \{t\}} [\hat{u} \cdot \partial_t \hat{\varphi} - a_\varepsilon(|D \hat{u}|_{Q_\varphi}) Q_\varphi(D \hat{u}, D \hat{\varphi})] J_n \Phi \, dy
\]

for a.e. \( t \in (0, T) \), where \( J_n \Phi := |\det D \Phi| \) denotes the Jacobian of \( \Phi \). Integrating this identity with respect to \( t \in (0, T) \), we obtain the left-hand side of (4.8). Diminishing \( \eta > 0 \) if necessary, we can achieve that the right-hand side \( b_\varepsilon \) in (4.8) vanishes in a tubular neighborhood of \( \delta \Omega_\varepsilon \times (0, T) \) by construction (cf. (4.7)). Consequently, (4.8) turns into

\[
\iint_{Q^-_{\delta,\eta}} \left[ \hat{u} \cdot \partial_t \hat{\varphi} - a_\varepsilon(|D \hat{u}|_{Q_\varphi}) Q_\varphi(D \hat{u}, D \hat{\varphi}) \right] J_n \Phi \, dy \, dt = 0. \tag{B.6}
\]

In this equation, the testing function \( \hat{\varphi} \) can be chosen as an arbitrary smooth function with compact support in \( Q^-_{\delta,\eta} \). By an approximation argument, we can also verify it for every \( \hat{\varphi} \in L^p(0, T; W^{1,p}_0(C^-_{\delta,\eta})) \) with \( \hat{\varphi}(0) = 0 = \hat{\varphi}(T) \). Next, for an arbitrary testing function \( \psi \in L^p(0, T; W^{1,p}_0(C^-_{\delta,\eta})) \) with \( \psi_t \in L^2(Q^-_{\delta,\eta}) \) and \( \psi(0) = 0 = \psi(T) \), we test (B.6) with \( \hat{\varphi} := (J_n \Phi)^{-1} \psi \), which is admissible since \( J_n \Phi \) is a positive Lipschitz function. This leads to

\[
\iint_{Q^-_{\delta,\eta}} \left[ \hat{\varphi} \cdot \partial_t \varphi - a_\varepsilon(|D \hat{u}|_{Q_\varphi}) Q_\varphi(D \hat{u}, D \varphi) \right] \, dy \, dt
\]

\[
= - \iint_{Q^-_{\delta,\eta}} a_\varepsilon(|D \hat{u}|_{Q_\varphi}) Q_\varphi(D \hat{u}, \psi \otimes D[J_n \Phi]) \, (J_n \Phi)^{-1} \, dy \, dt, \tag{B.7}
\]

for every \( \psi \in L^p(0, T; W^{1,p}_0(C^-_{\delta,\eta})) \) with \( \psi_t \in L^2(Q^-_{\delta,\eta}) \) and \( \psi(0) = 0 = \psi(T) \).

### B.3 Reflection and reduction to the interior

Next, we extend \( Q_\varphi \) and \( J_n \Phi \) to \( C^+_{\delta,\eta} \) by an even reflection across \( \Gamma_\delta := B^{(n-1)}_\delta \times \{0\} \). To this aim, we define

\[
Q_\varphi(y', y_n) := Q_\varphi(y', -y_n) \quad \text{and} \quad J_n \Phi(y', y_n) := J_n \Phi(y', -y_n) \quad \text{for any } (y', y_n) \in C^+_{\delta,\eta}.
\]

Note that the functions \( Q_\varphi \) and \( J_n \Phi \) are smooth on \( B^{(n-1)}_\delta \times (-\eta, 0] \), and therefore their extensions are also smooth on \( \Gamma_\delta \). However, the extensions are in general only Lipschitz continuous on \( C_{\delta,\eta}^+ \). Only the horizontal derivatives of the extended Jacobian are continuous across \( \Gamma_\delta \), since they are even functions as the Jacobian itself. Next, we extend the solution \( \hat{u} \) by an odd reflection across
the boundary $\Gamma_\delta$ on each time-slice. More precisely, we let

$$\hat{u}(y', y_n, t) := -\hat{u}(y', -y_n, t) \text{ for } (y', y_n) \in C^+_{\delta, \eta}.$$  

Now we consider testing functions $\psi \in L^p(0, T; W^{1,p}(C^+_{\delta, \eta}))$ with $\partial_t \psi \in L^2(Q^+_{\delta, \eta})$ and $\psi(0) = 0 = \psi(T)$. We decompose $\psi = \psi_e + \psi_o$ into its even part $\psi_e$ and odd part $\psi_o$ with respect to reflection across $\Gamma_\delta$. According to $Q^+_{\delta, \eta} = Q^+_{\delta, \eta} \cup Q^-_{\delta, \eta}$, we write

$$I := \iint_{Q^+_{\delta, \eta}} \left[ \hat{u} \cdot \partial_i \psi - a_e \left( |D\hat{u}|_{Q^+_{\delta, \eta}} \right) Q_\Phi(D\hat{u}, D\psi) \right] dy dt = I_e^+ + I_e^- + I_o^+ + I_o^-.$$  

The right-hand side integrals are defined as follows: For any sign $\{+,-\}$ and any symmetry type $\{e,o\}$, one has to replace $Q_{\delta, \eta}, \psi$ in $I$ by the corresponding half cylinder $\{Q^+_{\delta, \eta}, Q^-_{\delta, \eta}\}$ and the corresponding even or odd part $\{\psi_e, \psi_o\}$ of $\psi$. In the last two terms, we observe that $Q_\Phi(D\hat{u}, D\psi_o)$ is an even function with respect to $y_n$ because the derivatives of $\hat{u}$ and $\psi_o$ in direction of $y_i$ with $i \in \{1, \ldots, n-1\}$ are odd and the derivatives in the direction of $y_n$ are even. Furthermore, the structure of $Q$ from (B.4) does not lead to mixed terms with both types of derivatives. For the same reason, $|D\hat{u}|_{Q^o_{\delta, \eta}}$ is an even function, and by definition we have that $\hat{u} \cdot \partial_i \psi_o$ is even as well. Consequently, the integrands of the last two integrals are even, which implies $I_o^- = -I_o^+$. Similarly, using the facts that $\hat{u}$ is odd and $\psi_e$ is even, we deduce that $I_e^- = I_e^+$. Therefore, we obtain

$$\iint_{Q^+_{\delta, \eta}} \left[ \hat{u} \cdot \partial_i \psi - a_e \left( |D\hat{u}|_{Q^o_{\delta, \eta}} \right) Q_\Phi(D\hat{u}, D\psi) \right] dy dt = 2 \iint_{Q^-_{\delta, \eta}} \left[ \hat{u} \cdot \partial_i \psi_o - a_e \left( |D\hat{u}|_{Q^o_{\delta, \eta}} \right) Q_\Phi(D\hat{u}, D\psi_o) \right] dy dt.$$  

Note that the right-hand side coincides with the left-hand side of (B.8) with $\psi_o$ in place of $\psi$. Analogously to the decomposition of $I$, we write

$$\iint_{Q^+_{\delta, \eta}} a_e \left( |D\hat{u}|_{Q^o_{\delta, \eta}} \right) Q_\Phi(D\hat{u}, \psi \otimes D[J_n \Phi])(J_n \Phi)^{-1} dy dt = \Pi_e^+ + \Pi_e^- + \Pi_o^+ + \Pi_o^-.$$  

For these integrals, we can use the similar symmetry considerations as above. Since $\psi_o \otimes D[J_n \Phi]$ enjoys the same symmetry properties as $D\psi_o$, we infer $\Pi_o^+ = \Pi_o^-$. Similarly, we deduce $\Pi_e^+ = -\Pi_e^-$. This implies

$$\iint_{Q^+_{\delta, \eta}} a_e \left( |D\hat{u}|_{Q^o_{\delta, \eta}} \right) Q_\Phi(D\hat{u}, \psi \otimes D[J_n \Phi])(J_n \Phi)^{-1} dy dt = 2 \iint_{Q^-_{\delta, \eta}} a_e \left( |D\hat{u}|_{Q^o_{\delta, \eta}} \right) Q_\Phi(D\hat{u}, \psi_o \otimes D[J_n \Phi])(J_n \Phi)^{-1} dy dt.$$  

Note that $\psi_o = 0$ on $\Gamma_\delta$, which makes $\psi_o$ admissible in the transformed parabolic system (B.7). This means that the right-hand sides of (B.8) and (B.9) coincide. Thus, we conclude that the
extended map $\hat{u}$ satisfies

$$
\int_{Q_{\delta,\eta}} \left[ \hat{u} \cdot \partial_t \psi - a_\varepsilon \left( |D\hat{u}|_{Q_0} \right) Q_\Phi(D\hat{u}, D\psi) \right] dy dt
$$

$$
= \int_{Q_{\delta,\eta}} a_\varepsilon \left( |D\hat{u}|_{Q_0} \right) Q_\Phi(D\hat{u}, \psi \otimes D[J_n \Phi])(J_n \Phi)^{-1} dy dt \quad (B.10)
$$

for every $\psi \in L^p(0,T; W^{1,p}_0(C_{\delta,\eta}))$ with $\partial_t \psi \in L^2(Q_{\delta,\eta})$ and $\psi(0) = 0 = \psi(T)$. Dropping the $\Phi$ on $Q$ for ease of notation, (B.10) is the weak form of the parabolic system

$$
\partial_t \hat{u}^i - \sum_{\alpha,\beta=1}^n \left[ a_\varepsilon \left( |\hat{u}|_Q \right) Q_{x\alpha\beta} \hat{u}^i_{y\alpha} \right]_{y\beta}
$$

$$
= \sum_{\alpha,\beta=1}^n a_\varepsilon \left( |\hat{u}|_Q \right) Q_{x\alpha\beta} \frac{[J_n \Phi]_{y\beta}}{(J_n \Phi)} \quad \text{in } Q_{\delta,\eta}. \quad (B.11)
$$

B.4 Smoothness of $u_\varepsilon$ up to the lateral boundary

We first observe that $Q_{\Phi(0)}(\xi,\xi) = |\xi|^2$, since $D\Psi(0) = \text{id}_{\mathbb{R}^n}$ by (B.3). By shrinking $\delta$ and $\eta$ if necessary, we can achieve

$$
\frac{1}{2} |\xi| \leq |\xi|_{Q_{\Phi(y)}} \leq 2 |\xi| \quad \text{for any } \xi \in \mathbb{R}^{Nn} \text{ and } y \in C_{\delta,\eta}, \text{ and } \text{Lip} \left( Q_\Phi \right) \leq \Lambda \quad (B.12)
$$

for some universal constant $\Lambda < \infty$. This implies that assumptions (1.7)–(1.9) from [32] are fulfilled if we replace the functions $b$, $q(\xi)$ used by Tolksdorf by the functions $Q$, $|\xi|^2_Q$ defined in (B.5). Similarly, we have

$$
\frac{1}{2} \leq J_n \Phi(y) \leq 2 \quad \text{for any } y \in C_{\delta,\eta}, \quad \text{and } \text{Lip} \left( J_n \Phi \right) \leq \Lambda \quad (B.13)
$$

for some universal constant $\Lambda < \infty$. Furthermore, the estimates (4.3) for the coefficients $a_\varepsilon(t)$ imply that assumptions (1.10)–(1.12) from Tolksdorf [32] hold true. For the inhomogeneous term, we observe that

$$
a^i(x,\xi) = \sum_{\alpha,\beta=1}^n a_\varepsilon \left( |\xi|_Q \right) Q_{x\alpha\beta} \xi_{\alpha} \frac{[J_n \Phi]_{y\beta}}{J_n \Phi}.
$$

Again, by (B.12) and (B.13), we will find the desired positive constant in order to verify (1.13) from [32]. Having arrived at this stage, we can apply the $C^{1,\alpha}$-regularity results from [12]. Indeed, as pointed out by DiBenedetto in the monograph [12, Chapter VIII.7], the statement of [12, Chapter IX, Theorem 1.1] continues to hold under these assumptions. The application of the theorem yields $D_j \hat{u} \in C_{1,\alpha}^{0}((Q_{\delta,\eta}))$. Hence $u_\varepsilon$ enjoys the same degree of regularity in the vicinity of $(\partial \Omega_\varepsilon \cap B_{\frac{1}{2}}(x_0)) \times (0,T)$. A further application of the interior regularity from [12, Chapter IX] directly to $u_\varepsilon$ gives $Du_\varepsilon \in C^{0,\alpha}_{1}(\Omega_\varepsilon \cap B_{\frac{1}{2}}(x_0) \times (\tau,T))$ for some $\tau > 0$.

Up to now, all the above regularity results also hold for the degenerate or singular case, and solutions cannot be expected to be more regular in this case. However, since the regularized
problem is non-degenerate, we can show higher regularity of solutions. We begin by noting that a standard application of the difference quotient technique yields the weak differentiability of $V_\lambda(D\hat{u}) = (\lambda^2 + |D\hat{u}|^2) \frac{p-2}{2} D\hat{u}$ with $D[V_\lambda(D\hat{u})] \in L^2_{\text{loc}}(Q_{\delta,\eta}, \mathbb{R}^{Nn})$; see, for instance, [18, Lemma 5.1] and [30, Theorem 1.1] for the cases $p \geq 2$ and $\frac{2n}{n+2} < p < 2$, respectively. By using the fact $\lambda > 0$ in the case $p > 2$ and the local boundedness of $|D\hat{u}|$ if $p < 2$, we deduce that the second spatial derivatives of the solution satisfies $D^2\hat{u} \in L^2_{\text{loc}}(Q_{\delta,\eta}, \mathbb{R}^{Nn})$.

Having second spatial derivates in $L^2_{\text{loc}}$ and first spatial derivatives locally bounded, we are allowed to perform an integration by parts in (B.10) in the diffusion term. After that we shift all terms except the one containing the time derivative to the right-hand side. In this way, we obtain an estimate of the form

$$\left| \int_{Q_{\delta,\eta}} \hat{u} \cdot \partial_t \psi \, dy \, dt \right| \leq C \|\psi\|_{L^2(Q_{\delta,\eta})}$$

for any $\psi \in C^\infty_0(Q_{\delta,\eta}, \mathbb{R}^N)$. This implies that $\partial_t \hat{u} \in L^2_{\text{loc}}(Q_{\delta,\eta})$.

The main ingredient for the higher regularity is the Schauder estimates for linear parabolic systems stated in Theorem B.2. We begin by differentiating (B.10) in tangential directions, that is, with respect to $y_\ell$ for $\ell = 1, 2, \ldots, n - 1$. As before we omit the $\Phi$ on $\mathcal{Q}$. Since $D^2\hat{u} \in L^2_{\text{loc}}(Q_{\delta,\eta})$, we infer that $v := \hat{u}_{y_\ell}$ is a weak solution to the following parabolic system:

$$\partial_t v^i - \sum_{\alpha, \beta = 1}^n \sum_{j=1}^N \left[ A_{\alpha \beta}^{ij} v_j^i \right] = \sum_{\alpha=1}^n \sum_{j=1}^N b_{\alpha}^{ij} v_j^i + \sum_{\alpha=1}^n (f^i)_\alpha + c^i \quad \text{(B.15)}$$

in $Q_{\delta,\eta}$ and for $i = 1, \ldots, N$, where the coefficients are given by

$$A_{\alpha \beta}^{ij} := a_\epsilon(|D\hat{u}|_Q)Q_{\alpha \beta} \delta_{ij} + \frac{a_\epsilon'(|D\hat{u}|_Q)}{|D\hat{u}|_Q} \sum_{\gamma, \delta = 1}^n Q_{\alpha \gamma} Q_{\beta \delta} \hat{u}_i^\gamma \hat{u}_j^\delta,$$

and

$$b_{\alpha}^{ij} := \sum_{\beta=1}^n \frac{[J_n \Phi]_{y_\beta}^\beta}{J_n \Phi} \left[ \left. a_\epsilon(|D\hat{u}|_Q)Q_{\alpha \beta} \delta_{ij} + \sum_{\gamma, \delta = 1}^n \right] \frac{a_\epsilon'(|D\hat{u}|_Q)}{|D\hat{u}|_Q} Q_{\alpha \gamma} Q_{\beta \delta} \hat{u}_i^\gamma \hat{u}_j^\delta \right]. \quad \text{(B.17)}$$

The inhomogeneities are defined by

$$f^i_\alpha := \sum_{\beta, \gamma, \delta = 1}^{N} \frac{a_\epsilon'(|D\hat{u}|_Q)}{2|D\hat{u}|_Q} [Q_{\gamma \delta}]_{y_\gamma} Q_{\alpha \beta} \hat{u}_i^\beta \hat{u}_j^k \hat{u}_j^k + \sum_{\beta=1}^n a_\epsilon(|D\hat{u}|_Q)[Q_{\alpha \beta}]_{y_\beta} \hat{u}_i^\beta,$$

and

$$c^i := \sum_{\alpha, \beta = 1}^n a_\epsilon(|D\hat{u}|_Q) \left[ \left. \frac{[J_n \Phi]_{y_\beta}^\beta}{J_n \Phi} \right] \right] \hat{u}_i^\alpha + \frac{[J_n \Phi]_{y_\beta}^\beta a_\epsilon'(|D\hat{u}|_Q)}{2|D\hat{u}|_Q} \sum_{\alpha, \beta, \gamma, \delta = 1}^n [Q_{\gamma \delta}]_{y_\gamma} Q_{\alpha \beta} \hat{u}_i^\beta \hat{u}_j^j \hat{u}_j^i.$$
Note that the derivatives \((J_n \Phi)_{y \ell}\) and \(Q_{y \ell}\) are Lipschitz continuous on the whole domain \(B_\delta \times (-\eta, \eta)\) for any \(\ell = 1, 2, \ldots, n-1\). According to the \(C^{1,\alpha}\)-regularity of \(\tilde{u}\), the coefficients \(A^{ij}_{\alpha \beta}\) and the term \(f^2_i\) appearing in (B.15) are Hölder continuous, while the coefficients \(b^{ij}_{\alpha}\) and the inhomogeneities \(c^i\) are bounded. Moreover, for any \(\xi \in \mathbb{R}^{Nn}\), by (4.3)\(_{1,2}\) we have that

\[
\sum_{\alpha, \beta = 1}^{n} \sum_{i, j = 1}^{N} A^{ij}_{\alpha \beta} \xi^i \xi^j \geq m \varepsilon \left( \varepsilon + |D\tilde{u}|_Q^2 \right)^{\frac{p-2}{2}} |\xi|^2.
\]

Consequently, the interior Schauder estimates from Theorem B.2 yield the Hölder continuity of the spatial gradient \(Dv\) for some proper Hölder exponent. In particular, \(\tilde{u}_{y_\alpha y_\beta}\) is locally Hölder continuous on \(Q_{\delta, \eta}\), provided \(\alpha + \beta < 2n\).

Likewise, we may differentiate (B.11) with respect to \(t\). This procedure becomes legitimate if we can show \(D_y \partial_t \tilde{u} \in L^2_{loc}(Q_{\delta, \mu})\). Thanks to (B.14), this can be done by working with the difference quotient of \(w\) in the time variable. Indeed, let \(h > 0\) and define the finite difference in time by

\[
\tau_h \tilde{u}(t) := \tilde{u}(t + h) - \tilde{u}(t).
\]

Here and in the sequel, we keep silent of the dependence of \(\tilde{u}\) on \(x\). Taking finite differences in the time variable of (B.11), we obtain that the parabolic system

\[
\begin{align*}
\tau_h \partial_t \tilde{u}^i - \sum_{\alpha, \beta = 1}^{n} \tau_h \left[ a^i_{\alpha \beta} (|D\tilde{u}|_Q) Q_{\alpha \beta} \tilde{u}^i_{y_\alpha \beta} \right] \geq \sum_{\alpha, \beta = 1}^{n} \tau_h \left[ a^i_{\alpha \beta} (|D\tilde{u}|_Q) Q_{\alpha \beta} \tilde{u}^i_{y_\alpha \beta} \right] (J_n \Phi)_{y_\beta}.
\end{align*}
\]

is satisfied weakly in \(Q_{\delta, \eta}\). Next, for fixed \(t \) and \(h\), we introduce the quantity

\[
\Delta(s) := s D\tilde{u}(t + h) + (1 - s) D\tilde{u}(t) \in \mathbb{R}^{Nn}
\]

whose entries are \(\Delta^i_{\alpha}(s) := s \tilde{u}^i_{y_\alpha}(t + h) + (1 - s) \tilde{u}^i_{y_\alpha}(t)\), and calculate

\[
\tau_h \left[ a^i_{\alpha \beta} (|D\tilde{u}|_Q) Q_{\alpha \beta} \tilde{u}^i_{y_\alpha} \right] = \tau_h \tilde{u}^i_{y_\alpha} \int_0^1 \left[ a^i_{\alpha \beta} (|\Delta(s)|_Q) Q_{\alpha \beta} \delta^i_{\alpha \beta} + \frac{a^i_{\alpha \gamma} (|\Delta(s)|_Q) Q_{\alpha \beta} \delta^i_{\alpha \gamma}}{|\Delta(s)|_Q} Q_{\beta \gamma} \Delta^j(s) Q_{\beta \delta} \Delta^j_{\delta}(s) \right] ds
\]

\[
= \tau_h \tilde{u}^i_{y_\alpha} A^i_{\alpha \beta},
\]

for \(\beta = 1, \ldots, n\) and \(i = 1, \ldots, N\). It is not hard to verify that the matrix \(A^i_{\alpha \beta}\) satisfies

\[
\sum_{\alpha, \beta = 1}^{n} \sum_{i, j = 1}^{N} A^{ij}_{\alpha \beta} \xi^i \xi^j \geq m |\xi|^2 \int_0^1 \left( \varepsilon^2 + |\Delta(s)|_Q^2 \right)^{\frac{p-2}{2}} ds \geq C_o |\xi|^2
\]
and

$$|A_{\alpha\beta}^{ij}| \leq cM \int_0^1 \left( \varepsilon^2 + |\Delta(s)|_Q^2 \right)^{\frac{p-2}{2}} ds \leq C_1$$

for some positive constants $C_0$ and $C_1$ depending on $p$, $m$, $M$, $c$, $\varepsilon$, and $\|D\hat{u}\|_{L^\infty}$.

We may test (B.18) by $\tau_h \hat{u}^2 \zeta^2$ with $\zeta \in C_0^1(Q_{\delta,\eta})$. Employing the above growth conditions on $A_{\alpha\beta}^{ij}$ and the fact that $\hat{\sigma}_i \hat{u} \in L^2_{\text{loc}}(Q_{\delta,\eta})$, a standard calculation gives

$$\int_{Q_{\delta,\eta}} \zeta^2 |\tau_h \hat{u}|^2 dydt \leq C h^2$$

for some constant $C$ with dependence only on $C_0$, $C_1$, $\Lambda$, $\|\zeta\|_{L^\infty}$, $\|D\hat{\zeta}\|_{L^\infty}$, $\|\hat{\sigma}_i \zeta\|_{L^\infty}$ and $\|\hat{\sigma}_i \hat{u}\|_{L^2(\text{spt } \zeta)}$ but independent of $h$. Passing to the limit in the above estimate as $h \downarrow 0$, we conclude that $D\hat{\sigma}_i \hat{u} \in L^2_{\text{loc}}(Q_{\delta,\eta})$ as promised. Therefore, we may differentiate (B.10) with respect to $t$ and obtain, denoting $\tilde{v} := \hat{\sigma}_i \hat{u}$, that

$$\partial_t \tilde{v}^i - \sum_{\alpha,\beta=1}^N \sum_{j=1}^N \left[ A_{\alpha\beta}^{ij} \tilde{v}_y^j \right]_{y^\alpha} = \sum_{\alpha=1}^N \sum_{j=1}^N b_{\alpha j}^i \tilde{v}_{y^\alpha}^j$$

for $i = 1, 2, \ldots, N$ in $Q_{\delta,\mu}$, where $A_{\alpha\beta}^{ij}$ and $b_{\alpha j}^i$ are defined in (B.16) and (B.17), respectively. Then the interior Schauder estimates from Theorem B.2 yield the local Hölder continuity of $\hat{\sigma}_i D\hat{u}$ on $Q_{\delta,\mu}$.

To obtain Hölder regularity for $\hat{u}_{y^i y^j}$, we turn back to (B.7) in $Q^-_{\delta,\mu}$. Let us write it in non-divergence form and keep the terms with $\hat{u}_{y^i y^j}$ on the left-hand side, while we put all other terms on the right-hand side. As usual, we will omit $\Phi$ on $Q$. In this way, we may obtain an algebraic, linear system

$$\sum_{j=1}^N B_{ij}^j \hat{u}_{y^i y^j} = g^i \quad \text{in } Q^-_{\delta,\mu} \text{ for } i = 1, 2, \ldots, N,$$

where

$$B_{ij}^j = a_\varepsilon(|D\hat{u}|_Q)_{Q_{n\eta}} \delta_{ij} + a_\varepsilon'(|D\hat{u}|_Q)_{Q_{n\eta}} \hat{u}_{y^i}^j Q_{\alpha n} \hat{u}_{y^j}^i$$

and the right-hand side $g^i$ is a combination of first derivatives, second derivatives of $\hat{u}$ excluding $\hat{u}_{y^i y^j}$, together with $Q, J_{n\|D\hat{\zeta} \Phi}$ and their first derivatives. As a result, $g^i$ is Hölder continuous for all $i = 1, 2, \ldots, N$. On the other hand, we observe that the matrix $(B_{ij}^j)$ is positive definite and Hölder continuous in the closure of $Q^-_{\delta,\mu}$, provided we choose $\delta$ and $\mu$ sufficiently small. As a result, $\hat{u}_{y^i y^j}$ can be solved from the algebraic, linear system (B.19), and is also Hölder continuous in the closure of $Q^-_{\delta,\mu}$. Hence, we have shown that $\hat{u}_{y^i y^j}$ is Hölder continuous in the closure of $Q^-_{\delta,\mu}$ for all $i, j = 1, 2, \ldots, n$. Consequently, the same fact holds for $\hat{\sigma}_i \hat{u}$ due to the system (B.7). Transforming back to $u_\varepsilon$ we obtain that $\hat{\sigma}_i u_\varepsilon$ and $D^2 u_\varepsilon$ are Hölder continuous up to the lateral boundary $\{\partial \Omega_\varepsilon \cap N_0\} \times [\tau, T]$ for some $\tau > 0$. 

The sketched procedure can be iterated to give even higher regularity. To this end, we successively differentiate the linear system (B.15) in tangential directions and with respect to time and apply the Schauder estimate from Theorem B.2. This yields the H"{o}lder continuity for all derivatives except from the ones in normal directions. The H"{o}lder regularity of the remaining derivatives can then be deduced from the system (B.7) on $Q_{\delta,\eta}$. In this way, we inductively deduce $\hat{u} \in C^{k,\alpha}_{\text{loc}}(Q_{\delta,\eta})$ for any $k \in \mathbb{N}$, which yields the desired smoothness of the approximating solutions $u_\varepsilon$.

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