NON-POLYNOMIAL ENTIRE SOLUTIONS TO $\sigma_k$ EQUATIONS

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ABSTRACT. For $2k = n + 1$, we exhibit non-polynomial solutions to the Hessian equation

$$\sigma_k(D^2u) = 1$$

on all of $\mathbb{R}^n$.

1. INTRODUCTION

In this note, we demonstrate the following.

Theorem 1. For

$$n \geq 2k - 1,$$

there exist non-polynomial elliptic entire solutions to the equation

$$\sigma_k(D^2u) = 1$$

on $\mathbb{R}^n$.

Corollary 2. For all $n \geq 3$, there exist on $\mathbb{R}^n$ non-polynomial entire solutions to

$$\sigma_2(D^2u) = 1.$$ 

For $k = 1$, the entire harmonic functions in the plane arising as real parts of analytic functions are classically known. For $k = n$, the famous Bernstein result of Jörgens [5], Calabi [1], and Pogorelov [6] states that all entire solutions to the Monge-Ampère equation are quadratic. Chang and Yuan [2] have shown that any entire convex solution to (2) in any dimension must be quadratic. To the best of our knowledge, for $1 < k < n$, the examples presented here are the first known non-trivial entire solutions to $\sigma_k$ equations.

The special Lagrangian equation is the following

$$\sum_{i=1}^{n} \arctan \lambda_i = \theta$$

(here $\lambda_i$ are eigenvalues of $D^2u$) for

$$\theta \in \left( -\frac{n}{2} \pi, \frac{n}{2} \pi \right)$$

a constant. Fu [3] showed that when $n = 2$ and $\theta \neq 0$ all solutions are quadratic. When $n = 2$ and $\theta = 0$ the equation (3) becomes simply the Laplace equation, which admits well-known non-polynomial solutions. Yuan [8] showed that all convex solutions to special Lagrangian equations are quadratic.

The critical phase for special Lagrangian equations is

$$\theta = \frac{n - 2}{2} \pi.$$ 

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Yuan [9] has shown that for values above the critical phase, all entire solutions are quadratic. On the other hand, by adding a quadratic to a harmonic function, one can construct nontrivial entire solutions for phases 

\[ |\theta| < \frac{n-2}{2} \pi. \]

By [4] when \( n = 3 \), the critical equation

\[ \sum_{i=1}^{3} \arctan \lambda_i = \frac{\pi}{2} \]

is equivalent to the equation (2). Thus Corollary 2 answers the critical phase Bernstein question when \( n = 3 \). In the process, we also show the following.

**Theorem 3.** There exists a special Lagrangian graph in \( \mathbb{C}^3 \) over \( \mathbb{R}^3 \) that does not graphically split.

Harvey and Lawson [4], show that a graph

\[(x, \nabla u(x)) \subset \mathbb{C}^n\]

is special Lagrangian and a minimizing surface if and only if \( u \) satisfies (3). We say a graph splits graphically when the function \( u \) can be written the sum of two functions in independent variables.

There are still many holes in the Bernstein picture for \( \sigma_k \) equations. To begin with, when \( n = 4 \) the existence of interesting solutions to \( \sigma_3 = 1 \). For special Lagrangian equations the existence of critical phase solutions when \( n \geq 4 \) is open.

2. **Proof**

We will assume that \( n \) is odd and

\[ 2k = n + 1. \]

We construct a solution \( u \) on \( \mathbb{R}^n \). The general result will follow by noting that if we define

\[ \tilde{u} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \]

via

\[ \tilde{u}(z, w) = u(z) \]

then

\[ \sigma_k (D^2 \tilde{u}) = \sigma_k (D^2 u) = 1. \]

Consider functions on \( \mathbb{R}^{n-1} \times \mathbb{R} \) of the form

\[ u(x, t) = r^2 e^t + h(t) \]

where

\[ r = (x_1^2 + x_2^2 + \ldots + x_{n-1}^2)^{1/2}. \]
Compute the Hessian, rotating $\mathbb{R}^{n-1}$ so that $x_1 = r$:

$$
D^2u = \begin{pmatrix}
2e^t & 0 & \ldots & 0 & 2re^t \\
0 & 2e^t & 0 & \ldots & 0 \\
\ldots & 0 & \ldots & 0 & \ldots \\
0 & \ldots & 0 & 2e^t & 0 \\
2re^t & 0 & \ldots & 0 & r^2 e^t + h''(t)
\end{pmatrix}
$$

(4)

$$
= e^t \begin{pmatrix}
2 & 0 & \ldots & 0 & 2r \\
0 & 2 & 0 & \ldots & 0 \\
\ldots & 0 & \ldots & 0 & \ldots \\
0 & \ldots & 0 & 2 & 0 \\
2r & 0 & \ldots & 0 & r^2 + e^{-t}h''(t)
\end{pmatrix} ,
$$

We then compute. The $k$-th symmetric polynomial is given by the sum of $k$-minors. Let

(5)

$$
S = \{ \alpha \subset \{1, \ldots, n\} : |\alpha| = k \} ,
$$

and let

$$
A = \{ \alpha \in S : 1 \in \alpha \} \\
B = \{ \alpha \in S : n \in \alpha \} .
$$

We express $S$ as a disjoint union

$$
S = (A \cap B) \cup (B \setminus A) \cup (S \setminus B) .
$$

Define

$$
\sigma_k^{(\alpha)} = \det \left( \begin{array}{c}
\text{k \times k matrix with} \\
\text{row and columns} \\
\text{chosen from } \alpha
\end{array} \right) .
$$

For $\alpha \in (A \cap B)$ we have

$$
\sigma_k^{(\alpha)} = \det \left( e^t \begin{pmatrix}
2 & 0 & \ldots & 0 & 2r \\
0 & 2 & 0 & \ldots & 0 \\
\ldots & 0 & \ldots & 0 & \ldots \\
0 & \ldots & 0 & 2 & 0 \\
2r & 0 & \ldots & 0 & r^2 + e^{-t}h''(t)
\end{pmatrix} ,
$$

that is

$$
\sigma_k^{(\alpha)} = e^{kt} 2^{k-2} (2r^2 + 2e^{-t}h'' - 4r^2) .
$$

Next, for $\alpha \in B \setminus A$,

$$
\sigma_k^{(\alpha)} = \det \left( e^t \begin{pmatrix}
2 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 \\
\ldots & 0 & 2 & \ldots \\
0 & \ldots & 0 & r^2 + e^{-t}h''
\end{pmatrix} ,
$$

that is

$$
\sigma_k^{(\alpha)} = e^{kt} 2^{k-1} (r^2 + e^{-t}h'') .
$$

Finally, for $\alpha \in (S \setminus B)$ we have

$$
\sigma_k^{(\alpha)} = \det \left( e^t \begin{pmatrix}
2 & 0 & \ldots \\
0 & \ldots & 0 \\
\ldots & 0 & 2
\end{pmatrix} ,
$$
that is
\[ \sigma_k^{(\alpha)}(x) = e^{kt}x^k. \]

We sum these up:
\[ \sigma_k(D^2u) = \sum_{\alpha \in (A \cap B)} \sigma_k^{(\alpha)} + \sum_{\alpha \in (B \setminus A)} \sigma_k^{(\alpha)} + \sum_{\alpha \in (S \setminus B)} \sigma_k^{(\alpha)}. \]

Counting, we get
\[ \sigma_k(D^2u) = \sum_{\alpha \in (A \cap B)} \sigma_k^{(\alpha)} + \sum_{\alpha \in (B \setminus A)} \sigma_k^{(\alpha)} + \sum_{\alpha \in (S \setminus B)} \sigma_k^{(\alpha)}. \]

Grouping the terms, we see
\[ \sigma_k(D^2u) = e^{kt}2^{k-1} \left[ -\frac{(n-2)}{k-2} + \frac{(n-2)}{k-1} \right] r^2 + \left[ -\frac{(n-2)}{k-2} + \frac{(n-2)}{k-1} \right] e^{-t}h'' \]
\[ + e^{kt}2^{k-1} \left( \alpha \right). \]

Now
\[ -\frac{(n-2)}{k-2} + \frac{(n-2)}{k-1} = \frac{(n-2)(k-1)-(n-2)(k-2)}{(k-2)(k-1)} = \frac{(n-k)}{(k-1)!} = \frac{(n-k)}{(k-1)!}. \]

This vanishes if and only if
\[ 1 = \frac{(n-k)(k-2)}{(n-2)(k-2)} = \frac{(n-k)}{(k-1)!}\]
or precisely when
\[ n-k = k-1 \]
\[ 2k = n+1. \]

Thus for this choice of \( k \), (6) becomes
\[ \sigma_k(D^2u) = A_{n,k}e^{(k-1)t}h'' + B_{n,k}e^{kt} \]
for some constants \( A_{n,k}, B_{n,k} \). Setting to this expression to 1, we solve for \( h''(t) \)
\[ h''(t) = \frac{1 - B_{n,k}e^{kt}}{A_{n,k}e^{(k-1)t}}, \]
noting the right-hand side is a smooth function in \( t \). Integrating twice in \( t \) yields solutions to (7) and hence to (1).

To see that the equation is elliptic, we first note that inspecting (4) the \( n-2 \) eigenvalues in the middle must be positive. Of the remaining two, at least one must be positive as the diagonal (of the \( 2 \times 2 \) matrix) contains at least one positive entry. We then note the following.
Lemma 4. Suppose that
\[ \sigma_k(D^2 u) > 0 \]
and \( D^2 u \) has at most 1 negative eigenvalue. Then \( D^2 u \in \Gamma^+_k \).

Proof. Diagonalize \( D^2 u \) so that \( D^2 u = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) with \( 0 \leq \lambda_2 \leq \lambda_3 \ldots \leq \lambda_n \). Clearly
\[ \frac{d}{ds}\sigma_k(\text{diag}\{\lambda_1 + s, \lambda_2, \ldots, \lambda_n\}) \geq 0 \]
so we may deform \( D^2 u \) to a positive definite matrix \( D^2 u + M \), with \( \sigma_k(D^2 u + sM) > 0 \) for \( s \geq 0 \). Thus \( D^2 u \) is in the component of \( \sigma_k > 0 \) containing the positive cone, that is, \( D^2 u \in \Gamma^+_k \). □

Example 5. When \( n = 3 \) the function
\[ u(x, y, t) = (x^2 + y^2)e^t + \frac{1}{4}e^{-t} - e^t \]
solves
\[ \sigma_2(D^2 u) = 1. \]

Remark 6. This method allows one to construct solutions to complex Monge-Ampère equations as well. See [7].

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