Deep holes and MDS extensions of Reed-Solomon codes.

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Abstract

We study the problem of classifying deep holes of Reed-Solomon codes. We show that this problem is equivalent to the problem of classifying MDS extensions of Reed-Solomon codes by one digit. This equivalence allows us to improve recent results on the former problem. In particular, we classify deep holes of Reed-Solomon codes of dimension greater than half the alphabet size. We also give a complete classification of deep holes of Reed Solomon codes with redundancy three in all dimensions.

Index Terms

MDS codes, Reed Solomon codes, Covering Radius.

I. INTRODUCTION

A deep-hole for a code $C$ is a received vector whose distance from $C$ attains the maximum possible value viz. the covering radius of $C$. A $[n, k, D]_q$ RS (Reed-Solomon) code $C$ consists of codewords $(f(x_1), f(x_2), \ldots, f(x_n))$ as $f(X)$ runs over the set of univariate polynomials of degree at most $k-1$ with $GF(q)$-coefficients. The evaluation set $D = \{x_1, \ldots, x_n\}$ consists of $n$ distinct and ordered points of $GF(q)$. The covering radius of $C$ can be shown to be $n-k$. By a generating polynomial $u(x)$ for a received word $u \in GF(q)^n$ of $C$, we mean the Lagrange interpolation polynomial of degree at most $n-1$ for the data points $\{(x_1, u_1), \ldots, (x_n, u_n)\}$. It was shown in [1] that the problem of determining whether a received word is a deep hole of a given Reed Solomon code is NP-hard. Several authors have studied the problem of classifying deep holes of RS codes. Cheng and Murray conjectured that:

Conjecture 1 (Cheng and Murray [2]). All deep-holes of a $[k, q, D = GF(q)]_q$ RS code with $2 \leq k \leq q-2$ have generating polynomials of degree $k$, except when $q$ is even and $k = q-3$.

The exception was not part of [2], but is well known. In fact this conjecture is equivalent to a central conjecture in finite geometry (See conjecture 1). It may be somewhat surprising to note that the conjecture for $k = q-3$ and $q$ odd, is equivalent to Beniamino Segre’s foundational theorem of finite geometry that states – in coding theory terminology– that any 3-dimensional MDS code of length $q+1$ is Reed-Solomon. Recently, Zhuang, Cheng and Li obtained the following result:

Theorem (Zhuang, Cheng and Li [3]). Let $C$ be a $[n, k, D]_q$ RS code with $k \geq \lfloor (q-1)/2 \rfloor$. If $q > 2$ is a prime number and then the generating polynomials of deep holes of $C$ are

$$u(x) = au_1(X) + u_2(X), \ a \in GF(q)^{\times}, \ \deg(u_2) \leq k-1,$$

and $u_1(X)$ equals either $X^k$ or the generating polynomial for the data points $\{(x_i, \frac{1}{x_i-\delta}) : 1 \leq i \leq n\}$ for some $\delta \in GF(q) \setminus D$.

The techniques used in [3] necessitate the condition that the alphabet size is an odd prime, and they leave the problem open for $GF(q)$. Our method (which uses the work of Roth and Seroussi [4]) allows us to work with any finite field alphabet. The contributions of this work are as follows:

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Theorem 1.

1) The above theorem of Zhang et al. holds not only for GF(p) but for any GF(q) with q odd.
2) For q even, the result of the above theorem still holds except when n = k + 3. In this case $u_1(X)$ may also equal $X^{k+1} + \left(\sum_{i=1}^{n} x_i\right)X^k$ in addition to choices for $u_1(X)$ mentioned therein.

In Theorem 2, we give a geometric interpretation of Theorem 1. Our method uses the observation that for a $[n, k, D]$ RS code with $(n < q)$ and a parity check matrix $H$ of $C$, a received vector $u \in GF(q)^n$ is a deep hole of $C$ if and only if the syndrome $Hu$ has the property that the matrix $[H \mid Hu]$ is an MDS extension of the RS code $C^\perp$ by one digit. In the case $k \geq \lceil(q - 1)/2\rceil$, the results of Roth and Seroussi [4], allow us to obtain Theorem 1.

For $[n, k]_q$ RS codes $C$ of length $q + 1$ (here the evaluation set is $D = PG(1, q)$), it is not always true that the covering radius of $C$ is $n - k$. Sometimes the covering radius equals $n - k - 1$. In the latter case the equivalence between deep holes of $C$ and MDS extensions of $C^\perp$ breaks down. In Section 4 we present some results for RS codes of length $q + 1$.

In Section 5 we classify deep-holes of a $[n, k]_q$ RS codes with redundancy $n - k = 3$ (Theorem 5). We also obtain canonical forms for non GRS 3-dimensional MDS codes extending a GRS code by one digit. (Theorem 6).

II. Notation

Throughout this article, the dimension $k$ of a RS code is in the range $2 \leq k \leq q - 1$. As for the length $n$, we note that the cases where the length $n$ equals $k$ or $k + 1$ are uninteresting: In the former case there are no deep holes, and in the later case deep-holes are just those received words which are not codewords. Therefore, we impose the condition $k + 2 \leq n \leq q + 1$ on the length of $C$.

Generalized Reed Solomon codes (GRS codes) are obtained from RS codes by applying a diagonal Hamming isometry $u \mapsto \text{diag}(\nu_1, \ldots, \nu_n)u$ of $GF(q)^n$. Clearly the deep holes of the resulting GRS code are obtained from those of the original RS code by applying the same transformation. Thus, for the purposes of studying deep holes, it is sufficient to consider only RS codes. The dual of a RS code, is a GRS code. The definition of a $[n, k, D]_q$ RS code is easily extended to allow the evaluation set $D$ to be a subset of the projective line $PG(1, q) = GF(q) \cup \infty$. Here it is understood that the value of a message polynomial $f(X)$ at $\infty$ is the coefficient of $X^{k-1}$. The evaluation set $D$ of a GRS code is not unique: if $\varphi(x) = (c + dx)/(a + bx)$ is a fractional linear transformation of $PG(1, q)$ then $\varphi(D)$ is also an evaluation set (for example see Proposition 2.5 of [5]). Using this freedom we can (and will) always assume $D$ to be free of $\infty$ provided $n \neq q + 1$.

Given an evaluation set $D = \{x_1, \ldots, x_n\} \subset GF(q)$, we associate to $D$ some numbers $s_0, \ldots, s_n$ and $\nu_1, \ldots, \nu_n$ defined by:

$$\prod_{i=1}^{n} (X - x_i) = \sum_{j=0}^{n} s_j X^{n-j}, \quad \nu_j = \prod_{\{i : i \neq j\}} (x_j - x_i)^{-1}. \quad (1)$$

In order to give geometric interpretation to our results, it will be convenient to introduce some terminology. We recall that an $[n, k]_q$ MDS code is a $k$-dimensional code of length $n$ whose minimum distance is $n - k + 1$. Equivalently any generator matrix of a $[n, k]_q$ MDS code has the property that all its $k \times k$ minors are non-zero. An $n$-arc in $PG(k - 1, q)$ is an unordered collection of $n$ points of $PG(k - 1, q)$ such that any $k \times n$ matrix whose columns are lifts of the $n$ points of the arc to $GF(q)^n$, generates a $[n, k]_q$ MDS code. Thus there is a bijective correspondence between monomial equivalence classes of $[n, k]_q$ MDS
codes and projective equivalence classes of $n$-arcs in $PG(k - 1, q)$. For each element $x \in GF(q) \cup \infty$ we define vectors $c_m(x) \in GF(q)^m$ by:

$$c_m(x) = \begin{cases} (1, x, x^2, \ldots, x^{n-1})^T & \text{if } x \in GF(q) \\ (0, 0, \ldots, 0, 1)^T & \text{if } x = \infty. \end{cases}$$

We note that a $[n, k, D]_q$ RS code with $D = \{x_1, \ldots, x_n\} \subset GF(q)$ has a generator and parity check matrix pair given by:

$$G_k(D) = [c_k(x_1) \mid c_k(x_2) \mid \ldots \mid c_k(x_n)]$$

$$G_k^T(D) = [\nu_1 c_{n-k}(x_1) \mid \nu_2 c_{n-k}(x_2) \mid \ldots \mid \nu_n c_{n-k}(x_n)]$$

where $\nu_i$ are as in (1). In the case $D = \{x_1, \ldots, x_q\} \cup \infty$, the $[q + 1, k, D]$ RS code $C$ has a generator and parity check matrix pair given by:

$$G_k(PG(1, q)) = [c_k(x_1) \mid c_k(x_2) \mid \ldots \mid c_k(x_q) \mid c_k(\infty)]$$

$$G_k^T(PG(1, q)) = [c_{q+1-k}(x_1) \mid c_{q+1-k}(x_2) \mid \ldots \mid c_{q+1-k}(x_q) \mid c_{q+1-k}(\infty)].$$

The standard RNC (rational normal curve) in $PG(m - 1, q)$ consists of $q + 1$ points of $PG(m - 1, q)$ given by $\{c_m(x) : x \in GF(q) \cup \infty\}$. (For a nonzero $v \in GF(q)^m$, we use the notation $[v] \in PG(m - 1, q)$ to denote the one-dimensional subspace of $GF(q)^m$ represented by $v$). By a RNC in $PG(m - 1, q)$, we mean the image of the standard RNC under a projective linear transformation of $PG(m - 1, q)$. Thus, in the correspondence between arcs and MDS codes, the $n$-arcs of $PG(k - 1, q)$ which correspond to $[n, k]_q$ RS codes, are those arcs which are contained in a RNC. A $n$-arc in $PG(k - 1, q)$ is said to be complete if it is not contained in a $n + 1$-arc. Equivalently, the corresponding $[n, k]_q$ MDS code cannot be extended to a $[n + 1, k]_q$ MDS code. Let $N_m \in GF(q)^m$ denote the vector $(0, \ldots, 0, 1)^T$. For $q$ even, the point $N_m \in PG(2, q)$ is known as the nucleus of the standard RNC in $PG(2, q)$.

**Definition 1.** For a $[n, k]$ code $C$, we say two received words $u_1, u_2 \in GF(q)^n$ are coset-equivalent if $u_2 - u_1 \in C$. We say $u_1, u_2$ are equivalent if $u_2 - au_1 \in C$ for some $a \in GF(q)^\times$.

(In this article we denote $GF(q) \setminus \{0\}$ by $GF(q)^\times$). In the case $D \subset GF(q)$, if $u(X)$ is the generating polynomial of a deep hole $u$, then the generating polynomials of words equivalent to $u$ are $\{au(X) + f(X) : a \neq 0, \deg(f) < k\}$. Thus there is a unique $v$ equivalent to $u$ whose generating polynomial is of the form $X^kP_u(X)$ with $P_u(X)$ monic of degree at most $n - k - 1$.

We use the notation $\rho(C)$ for the covering radius of a code $C$.

### III. Deep Holes and MDS Extensions

Let $C$ be a $[n, k, D]$ RS code. If $n \neq q + 1$, we recall the proof that the covering radius $\rho(C) = n - k$. Since $\rho(C) \leq n - k$ for any linear $[n, k]_q$ code, we just need to show there is a received word at a distance of $n - k$ from $C$. The word $(x_1, \ldots, x_n)$ is at a distance $n - k$ from $C$: the vector $(p(x_1) - x_1^k, \ldots, p(x_n) - x_n^k)$ for $\deg(p(X)) < k$ has at most $k$ zeros, and there is a $p(X)$ for which this vector has $k$ zeros.

In the case $n = q + 1$, let $D = \{x_1, \ldots, x_q, \infty\}$ with $\{x_1, \ldots, x_q\} = GF(q)$. Consider a word of the form $u = (x_1^k, \ldots, x_q^k, a)$ for any $a \in GF(q)$. We will show that the distance of $u$ from $C$ is $n - k - 1$. Let $k^\times GF(q)$ denote the subset of $GF(q)$ that consists of those elements which can be written as a sum of $k$ distinct elements of $GF(q)$. It is easy to see that $c \cdot (k^\times GF(q)) = k^\times GF(q)$ for all $c \in GF(q)^\times$ and hence $k^\times GF(q) = GF(q)$. Thus, there exist distinct elements $x_1, \ldots, x_k \in GF(q)$ such that $a = x_1 + \cdots + x_k$. Consider the polynomial $p(X) = X^k - \prod_{i=1}^k (X - x_i)$. This is a codeword (message word) of $C$ and the vector $(p(x_1) - x_1^k, \ldots, p(x_q) - x_q^k, p(\infty) - a)$ has exactly $k + 1$ zeros. This is because $p(\infty) = \sum_{i=1}^k x_i = a$. Thus, we conclude that $\rho(C)$ is either $n - k - 1$ or $n - k$. We will see in Section IV that both situations occur.
Lemma 1. Let $G_k(D)$ and $G_k^T(D)$ be a generator and parity check matrix for $C$ as given in (3) if $n \neq q + 1$ and (4) if $n = q + 1$. For a received word $u$ let $S_D(u) = G_k(D)u$ be the syndrome of $u$. When $u$ is a non-codeword, we use the term projective syndrome for $[S_D(u)] \in PG(n - k - 1, q)$.

**Proposition 1.** Let $C$ be a $[n, k, D]_q$ RS code. In case $n = q + 1$, suppose $\rho(C) = n - k$. For a received word $u \in GF(q)^n$, the augmented matrix:

$$G_k^T(D; u) := [G_k^T(D) | S_D(u)]$$

(5)

generates a $[n + 1, n - k]_q$ MDS code if and only if $u$ is deep-hole of $C$. Thus $S_D$ sets up a bijective correspondence between the set of coset-equivalence classes of deep-holes and the set of MDS extensions of the dual RS code $C^\perp$ by one digit.

**Proof:** We recall that the distance of $u$ from $C$ is the least number $m$ such that $S_D(u)$ can be written as a linear combination of some $m$ columns of $G_k(T^T(D))$. Suppose $\rho(C) = n - k$ (this is automatic if $n \neq q + 1$). It follows that $u$ is a deep hole of $C$ if and only if $[G_k^T(D) | S_D(u)]$ generates an $[n + 1, k]$ MDS code extending the GRS code $C^\perp$. The second assertion follows from the fact that two received words are coset equivalent if and only if their syndromes coincide.

We need the following result:

**Theorem** (Roth and Seroussi 1986 [4]). Let $C$ be a $[n, \ell, D]$ RS code. Suppose $2 \leq \ell \leq n - \lfloor (q - 1)/2 \rfloor$. Let $g \in GF(q)^\ell$ be a vector. The augmented matrix $[G_\ell(D) | g]$ generates an $[n + 1, \ell]_q$ MDS code if and only if:

1) (for $q$ odd) $[g] = [c_\ell(\delta)]$ for some $\delta \in PG(1, q) \setminus D$

2) (for $q$ even) $g$ is either as above, or additionally in case $\ell = 3$, $[g] = N_3$.

Under the hypothesis of Proposition 1 and $k \geq \lfloor (q - 1)/2 \rfloor$, the matrix $G_k^T(D; u)$ in (5) generates a MDS code, if and only if $g = S_D(u)$ has the form given in the above theorem with $\ell = n - k$. Thus we have proved:

**Theorem 2** (Geometric form of Theorem 1). Under the hypothesis of Proposition 1 and and $k \geq \lfloor (q - 1)/2 \rfloor$, a received word $u$ is a deep hole of $C$ if and only if:

a) either $[S_D(u)] = [c_{n-k}(\delta)]$ lies on the standard RNC in $PG(n - k - 1, q)$ for some $\delta \in PG(1, q) \setminus D$.

b) or $q$ is even, $n = k + 3$, and $[S_D(u)]$ is as in part a) or equals the nucleus $N_3$ of the standard RNC in $PG(2, q)$.

Before, we prove Theorem 1 we need some lemmas. We now assume $C$ is a $[n, k, D]$ RS-code with $D \subset GF(q)$. Given a vector $u \in GF(q)^n$, let $u(X)$ be the generating polynomial of $u$. Clearly $u \mapsto u(X)$ is a linear isomorphism:

**Lemma 1.** A formula for $u(X)$ in terms of $u$, $s_0, \ldots, s_n$ and $v_1, \ldots, v_n$ is:

$$u(X) = [X^{n-1}, X^{n-2}, \ldots, X, 1]L_n [v_1c_n(x_1) | v_2c_n(x_2) | \ldots | v_nc_n(x_n)]u$$

(6)

where $L_n$ is the $n \times n$ lower triangular matrix given by $L_{ij} = s_{i-j}$. Moreover, writing $u(X) = u_1(X) + u_2(X)$ where $\deg(u_2) < k$ and $u_1$ only contains monomials $X^i$ for $i \geq k$, we see that

$$u_1(X) = [X^{n-1}, X^{n-2}, \ldots, X^k]L_{n-k}S_D(u),$$

(7)

where $L_{n-k}$ is the submatrix of $L_n$ on the first $n - k$ rows and columns.

**Proof:** The right hand side of (6) simplifies to the Lagrange interpolation polynomial

$$u(X) = \sum_{j=1}^{n} u_j \nu_j \prod_{\{i: i \neq j\}} (X - x_i)$$
Clearly, 

\[ u_1(X) = [X^{n-1}, X^{n-2}, \ldots, X^k] L_{n-k} G_k^1(D) u \]

which is the same as the formula in (7).

**Lemma 2.** Let \( u = (u_1, \ldots, u_n) \) be the vector with \( u_i = 1/(x_i - \delta) \) where \( \delta \in GF(q) \setminus D \). The generating polynomial of \( u \) is 

\[ u(X) = a [X^{n-1}, X^{n-2}, \ldots, X, 1] L_n c_n(\delta) \]  

(8)

where \( a = -1/\prod_{i=1}^{n}(\delta - x_i) \).

**Proof:** By Lagrange interpolation

\[ u(X) = \prod_{i=1}^{n} \nu_i(x_i - \delta)^{-1} \prod_{\{\mu \neq i\}} (X - x_{\mu}) \]

From the definition of the quantities \( \nu_i \), it follows that \( (X - \delta)u(X) \) is the Lagrange interpolation polynomial of degree at most \( n \) for the data \( \{(x, 1) : x \in D\} \cup \{(\delta, 0)\} \). In other words

\[ (X - \delta)u(X) = 1 + a \prod_{i=1}^{n} (X - x_i) = 1 + a \sum_{j=0}^{n} s_j X^{n-j} \]

Using this equation to determine the coefficients of \( u(X) \), we obtain

\[ u(X) = a \sum_{j=1}^{n} X^{n-j} \left( \sum_{i=0}^{n-j} \delta^{j-i} s_i \right) \]

which is the same as the formula (8)

**Proof: (of Theorem 1)** According to Theorem 2, for \( k \geq [(q - 1)/2] \), and \( n \neq k + 3 \) if \( q \) is even, \([S_D(u)] = [c_{n-k}(\delta)]\) lies on the standard RNC in \( PG(n - k - 1, q) \) for some \( \delta \in PG(1, q) \setminus D \). When \( \delta = \infty \), \( S_D(u) = ac_{n-k}(\infty) \) for some \( a \in GF(q)^\times \), and thus \( L_{n-k} S_D(u) \) is \( a \) times the last column of \( L_{n-k} \) which is again \( c_{n-k}(\infty) \). Thus the formula (7) implies that \( u_1(X) = aX^k \), as was to be shown. In case \( \delta \in GF(q) \setminus D \), \( S_D(u) = ac_{n-k}(\delta) \) for some \( a \in GF(q)^\times \). The generating polynomial \( u(X) \) is given by the formula (6). Comparing with equations (7) and (8) of Lemma 2, we see that \( u(X) \) is of the form \( u(X) = bf(X) + g(X) \), where \( b \in GF(q)^\times \), \( \deg(u_2) \leq k - 1 \), and \( f(X) \) is the generating polynomial for the data points \( \{(x_i, 1): x_i \in D \} \).

In the case \( q \) is even, \( n = k + 3 \) and \( k \geq [(q - 1)/2] \), and \([S_D(u)] \neq [c_{n-k}(\delta)] \) for \( \delta \in PG(1, q) \setminus D \), we must have \([S_D(u)] = (0:1:0)\) by Theorem 2. It follows from (7) that

\[ u(X) = a[X^{k+2}, X^{k+1}, X^k](\begin{pmatrix} s_1 & 1 & 0 & 0 \ 0 & s_2 & \delta & 1 \end{pmatrix}(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) + u_2(X) = a(X^{k+1} + s_1 X^k) + u_2(X) \]

for some polynomial \( u_2 \) of degree at most \( k - 1 \).

We now give a geometric restatement of the Conjecture [1] of Cheng and Murray that we mentioned in Section I. Since \( D = GF(q) \), we note that \( PG(1, q) \setminus D = \{\infty\} \). To say that the generating polynomial of a deep hole has degree exactly \( k \) is equivalent to the assertion that \([S_D(u)] = [c_{n-k}(\infty)] \) (by (7)). Thus Conjecture [1] is equivalent to the following conjecture (where \( m = q - k \))

**Conjecture.** 1' For \( 2 \leq m \leq q - 2 \), with the exception of \( m = 3 \) when \( q \) is even, any MDS extension (by one digit) of a \( m \) dimensional RS code with evaluation set \( GF(q) \) must itself be GRS. Equivalently, any \((q + 1)\)-arc in \( PG(m - 1, q) \) with \( q \) points on a RNC must have all its points on the RNC.

**Corollary 1.** (of Theorem 2) The conjecture of Cheng and Murray holds if \( k \geq [(q - 1)/2] \), except when \( q \) is even and \( k = q - 3 \).
Remark: We strongly believe that Conjecture 1 must hold for all \( q - 2 \geq k \geq 2 \), not just \( k \geq [(q-1)/2] \). We justify this belief with the next two Propositions.

Proposition 2. For \( q \) odd, Conjecture 1 holds for \( k = 2 \) (i.e. Conjecture 1' holds for \( m = q - 2 \), as it is equivalent to B. Segre's fundamental result [6] that any \( q + 1 \)-arc in \( PG(2, q) \) (for odd \( q \)) is a RNC. Equivalently, for odd \( q \) any \( [q + 1, 3]_q \) MDS code is GRS.

Proof: Let \( u \) be a deep hole of a \( [q, 2]_q \) RS code \( C \) with \( D = GF(q) \). According to Proposition 1, the matrix \( G_2^\perp(D; u) \) generates a \( [q + 1, q - 2]_q \) MDS code \( C_1 \) extending the \( [q, q - 2]_q \) RS code \( C \). Consider the \( [q + 1, 3]_q \) MDS code \( C_1 \). Segre's theorem implies \( C_1 \) is GRS. Since the dual of a GRS code is GRS, it follows that \( C_1 \) is GRS. Thus the columns of the matrix \( G_2^\perp(D; u) \) represent the \( q + 1 \) points of some RNC in \( PG(q - 3, q) \). Now, a RNC in \( PG(m - 1, q) \) is uniquely determined by any \( m + 2 \) points on it. (see [7] Theorem 21.1.1 (v), or [5] Theorem 2.7) for a purely coding theoretic proof.) Since \( m + 2 = q \) here, and the first \( q \) columns of \( G_2^\perp(D; u) \) lie on the standard RNC in \( PG(q - 3, q) \), we conclude that the columns of \( G_2^\perp(D; u) \) represent the \( q + 1 \) points of the standard RNC.

Conversely, we show Segre’s theorem is equivalent to Conjecture 1 holding for \( k = 2 \). Given a \( q + 1 \)-arc in \( PG(2, q) \), or in other words a \( [q + 1, 3]_q \) MDS code \( C \), the dual code \( C^\perp \) is a \( [q + 1, q - 2] \) MDS code. Puncturing \( C^\perp \) on the last coordinate gives a \( [q, q - 2] \) MDS code \( C_1 \). The dual to this code is a \( [q, 2] \) MDS code and 2-dimensional MDS codes are always GRS. Thus \( C_1 \) is GRS. It follows that \( C^\perp \) is a \( [q + 1, q - 2] \) MDS code extending a \( [q, q - 2] \) GRS code. Assuming Conjecture 1 holds for \( k = 2 \), i.e Conjecture 1' holds for \( m = q - 2 \), we deduce that \( C^\perp \) is GRS. Thus \( C \) is GRS.

Before, we state our next proposition justifying the remark above, we present another conjecture. This conjecture is clearly implied by the MDS conjecture. (The MDS conjecture states that the maximum length of a \( k \)-dimensional MDS code with \( k < q \) is \( q + 1 \) except when \( q \) is even and \( k = 3, q - 1 \), in which cases it is \( q + 2 \).) Since the MDS conjecture is widely believed, the same can be said about Conjecture 2. We will show that Conjecture 1 implies Conjecture 2.

Conjecture 2 (implied by the MDS conjecture).

There is no \( [q + 1, m]_q \) MDS code extending a \( [q + 1, m]_q \) RS code, except when \( q \) is even and \( m = 3, q - 1 \). Equivalently, the RNC in \( PG(m - 1, q) \) is a complete arc unless \( q \) is even and \( m = 3, q - 1 \).

Proposition 3. Let \( 2 \leq m \leq q - 2 \). If Conjecture 1' holds for \( m \), then Conjecture 2 holds for \( m \).

Proof: Let \( C_1 \) be a \( [q + 1, m]_q \) RS code generated by a matrix \( G_m = G_m(PG(1, q)) \) as in (4). Let \( C \) be \( [q + 1, m]_q \) MDS code generated by the matrix \( [G_m \mid v] \) for some \( v \in GF(q)^m \). Let \( C_2 \) be the \( [q + 1, m]_q \) MDS code obtained by puncturing \( C \) on the \( q + 1 \)-th coordinate. We note that \( C_2 \) extends a \( [q, m, D]_q \) RS code with \( D = GF(q) \). Assuming Conjecture 1' holds for \( m \), it follows that \( v = ac_m(\infty) \) for some \( a \in GF(q)^k \), but then the last two columns of the matrix matrix \( [G_m \mid v] \) are linearly dependent, contradicting the MDS property of \( C \). Thus such a code \( [q + 2, m]_q \) MDS code \( C \) does not exist.

Proposition 4 in the next section presents some of the known answers to Conjecture 2.

Theorem 1 can be improved using results of Storme and Szönyi [8], [9]. These results state that for \( q \) odd sufficiently large and \( 4 \leq k \leq 0.09q + 3.09 \) any MDS extension of a \( [n, k] \) GRS code with \( n \geq (q + 3)/2 \) is GRS. Similarly for \( q \) even sufficiently large, any MDS extension of a \( [n, k] \) GRS code with \( n \geq q/2 + 2 \) if \( 5 \leq k \leq 0.09q + 3.59 \) or \( n \geq q/2 + 3 \) if \( k = 4 \), is GRS. Thus for such \( n, k, q \) the generating functions of deep holes of a \( [n, k]_q \) GRS code are as described in Theorem 1.

IV. REED SOLOMON CODES OF LENGTH \( q + 1 \)

In this section \( C \) is a \( [q + 1, k]_q \) RS code with evaluation set \( PG(1, q) \). In terms of arcs, \( C \) corresponds to a RNC in \( PG(k - 1, q) \). Let \( G_k = G_k(PG(1, q)) \) and \( G_k^\perp = G_k^\perp(PG(1, q)) \) denote the generator and parity check matrix of \( C \) as given in (4). As we showed in Section III the covering radius \( \rho(C) \) satisfies:

\[
q - k \leq \rho(C) \leq q + 1 - k.
\]
It follows that $\rho(C) = q + 1 - k \iff$ there exists a vector $u \in GF(q)^{q+1}$ at a distance of $q + 1 - k$ from $C \iff$ there exists a vector $v \in GF(q)^{q+1-k}$ such that the matrix $[G_k^\perp | v]$ generates a MDS code $\iff$ the RNC in $PG(q-k, q)$ is an incomplete arc. This establishes the following theorem due to A.Dür.

**Theorem** (Dür 1994 [10]). The covering radius of a $[q + 1, k]_q$ RS code $C$ is $q - k$ if and only if (any) RNC in $PG(q-k)$ is a complete arc. Equivalently there is no MDS extension of $C^\perp$ by one digit.

We can now restate Conjecture 2 as follows: (where $k = q + 1 - m$)

**Conjecture.** The covering radius of a $[q + 1, k]_q$ RS code is $q - k$ except when $q$ is even and $k = 2, q-2$ in which cases it is $q + 1 - k$.

We present some of the known answers to Conjecture 2.

**Proposition 4.** Conjecture 2 is true for

1) (Roth and Seroussi [4]) $m = 2$ and $3 \leq m \leq \lfloor q/2 \rfloor + 2$ except $m = 3$ when $q$ is even.
2) the exceptional cases $m = 3, q - 1$ with $q$ even.
3) (Segre [6]) $m = q - 1$ with $q$ odd.
4) (Segre [6]) $m = q - 2$ with $q$ odd.
5) (Segre [11]) $m = q - 3$ with $q$ odd.
6) (Storme and Thas [12], Storme [13]) $\lfloor q/2 \rfloor + 3 \leq m \leq q + 3 - 6\sqrt{q \ln q}$.
7) (S. Ball [14]) any $m < q$ if $q$ is a prime.
8) (S. Ball [15]) $m \leq 2p - 2$ where $q = p^h > p$ and $p$ is prime.
9) (Storme and Szőnyi) [16] $4 \leq m \leq 0.09q + 3.09$ with $q$ odd sufficiently large.
10) (Storme and Szőnyi [17]) $q$ even sufficiently large, either $m = 4$ or $5 \leq m \leq 0.09q + 3.59$.

**Proof:**

1) follows from the theorem of Roth and Seroussi above, together with the fact that for $m = 2$ every MDS code is GRS.
2) follows from the fact that for $q$ even, the matrices:

$$H_3 = [c_3(x_1) | \ldots | c_3(x_q) | c_3(\infty) | N_3] \text{ and } H_{q-1} = [c_{q-1}(x_1) | \ldots | c_{q-1}(x_q) | N_{q-1} | c_{q-1}(\infty)] \quad (9)$$

are parity check matrices of each other, and respectively generate a non-GRS $[q+2, 3]$ MDS extension of a $[q + 1, 3]_q$ GRS code, and a non-GRS $[q + 2, q - 1]$ MDS extension of a $[q + 1, q - 1]_q$ GRS code.
3) Suppose there is a $[q + 2, q - 1]$ MDS code. Its dual $C$ is a $[q + 2, 3]$ MDS code. By the theorem of Segre [6], it follows that the corresponding arc extends the RNC in $PG(2, q)$ contradicting the fact that the RNC in $PG(2, q)$ is a complete arc (Part 1).
4) follows from Proposition 3 and Proposition 2
5) The result we need from [11] is that any $[q, 3]$ MDS code is GRS for $q$ odd (For a proof see [16, Theorem 8.6.10]). It follows that any $[q, q - 3]$ MDS code is GRS for $q$ odd. Suppose the matrix $[G_{q-3}(PG(1, q)) | v]$ generates a $[q + 2, q - 3]$ MDS code. Puncturing on the first two coordinates gives a $[q, q - 3]$ MDS code, which is GRS. The corresponding arc is thus contained in a RNC of $PG(q - 4, q)$. This arc has $q - 1$ points on the standard RNC. Since $q - 1 = q - 3 + 2$, once again appealing to the fact that a RNC in $PG(k - 1, q)$ is uniquely determined by any $k + 2$ points on it, it follows that $v$ also lies on the standard RNC. Thus all $q + 2$ columns of $[G_{q-3}(PG(1, q)) | v]$ lie on the standard RNC in $PG(q - 4, q)$ thus contradicting the fact that this matrix generates a MDS code.
6) , 9)-10) the RNC is complete in the range indicated as proved in the cited articles.
7) -8) follow from the fact that the MDS conjecture holds in the parameter range indicated, as proved in the cited articles.
We now focus on the problem of classifying the deep holes of \( \mathcal{C} \) in the exceptional cases of Conjecture 2’. For \( q \) even, \( k = 2, q - 2 \), and \( \mathcal{C} \) a \([q + 1, k]_q\) RS code, the matrices \( H_{q-1}, H_3 \) of (9) generate MDS codes of length \( q + 2 \) extending \( \mathcal{C}^\perp \). Thus the theorem of Dür above implies that \( \rho(\mathcal{C}) \) is indeed \( q + 1 - k \). We recall that the coset equivalence class of a deep hole \( u \) is completely determined by its syndrome \( S_D(u) \), and the equivalence class of \( u \) is completely determined by the projective syndrome \( [S_D(u)] \).

**Theorem 3.** Let \( \mathcal{C} \) be a \([q + 1, k]_q\) RS code, where \( q \) is even and \( k = 2, q - 2 \). Let \( u \) be a deep hole of \( \mathcal{C} \).

1. If \( k = q - 2 \), the projective syndrome \( [S_D(u)] = (0 : 1 : 0) \). Thus there is only one equivalence class of deep holes of \( \mathcal{C} \).

2. If \( k = 2 \), there is a bijective correspondence between the set of equivalence classes of deep holes of \( \mathcal{C} \) and the set of projective equivalence classes of ordered hyperovals of \( PG(2, q) \).

**Proof:** Part 1): Here \( k = q - 2 \). Part 2) of the theorem of Roth and Seroussi from the previous section states that the code generated by \( H_3 \) is the only possible MDS extension of a \( \mathcal{C}^\perp \). Thus there is only one equivalence class of deep holes, represented by \([S_D(u)] = [0 : 1 : 0]\).

Part 2): Here \( k = 2 \). In this case the generator matrix is \( G_2 = (\begin{array}{ccc} 1 & \ldots & 1 \\ x_1 & \ldots & x_q \\ u_1 & \ldots & u_q \end{array}) \), with \( \{x_1, \ldots, x_q\} = GF(q) \). Let \( x_{q+1} = \infty \). An ordered hyperoval of \( PG(2, q) \) is the ordered set of \( q + 2 \) points of \( PG(2, q) \) represented by the columns of a generator matrix for a \([q+2,3]\) MDS code. By a result of B. Segre (see [16, Theorem 8.4.2]), up to projective equivalence any such hyperoval is represented by a matrix

\[ G = \begin{pmatrix} 1 & \ldots & 1 & 0 & 0 \\ x_1 & \ldots & x_q & 1 & 0 \\ u_1 & \ldots & u_q & 0 & 1 \end{pmatrix} \]

with the property that \( u_i = 0 \) if \( x_i = 0, \infty \) and \( u_i = 1 \) when \( x_i = 1 \), and that \( G \) generates a MDS code. The condition that \( G \) generates a MDS code can be stated as: there are at most two zero entries of \((bx_1 + a - u_1, \ldots, bx_q + a - u_q, b)\) for any \((a, b) \in GF(q)^2\). This is equivalent to \( u \) being a deep hole of \( \mathcal{C} \). Since, the equivalence class of a received word \( v \in GF(q)^{q+1} \) of \( \mathcal{C} \) (i.e. \( \{au + c : a \in GF(q)^\times, c \in \mathcal{C}\} \)) has a unique representative \( u \) such that \( u_i = 0 \) if \( x_i = 0, \infty \) and \( u_i = 1 \) when \( x_i = 1 \), it follows that equivalence classes of deep holes of \( \mathcal{C} \) are in bijective correspondence with projective equivalence classes of ordered hyperovals of \( PG(2, q) \).

The problem of classifying deep holes of a \([q + 1, 2]\) RS code for \( q \) even, is thus equivalent to the difficult problem of classifying hyperovals of \( PG(2, q) \). (See Section 2 of [17] for a survey of this problem). The equivalence classes of deep holes \( u \) are completely determined by their syndrome \([S_D(u)] \in PG(q-2, q) \). Thus, the hyperovals can be studied in terms of possible syndromes \( S_D(u) \). This is done in the work of Storme and Thas [18]. It is interesting to note that such a syndrome \([S_D(u)] = (a_0 : \cdots : a_{q-2}) \) necessarily satisfies \( a_0 = a_2 = \cdots = a_{q-2} = 0 \). (see Theorem 3.10 of [18]).

The problem of classifying deep holes of \( \mathcal{C} \) when \( \rho(\mathcal{C}) = q - k \) (for example Parts 1), 3)-5) of Proposition 4) is an open problem (since at least 1991, see Remark 5 of [19]). By turning to the syndromes of the deep holes, and setting \( m = q - k \), this problem is equivalent to finding all points of \( PG(m, q) \) which are not in the linear span of \( m - 1 \) points of the standard RNC in \( PG(m, q) \). We just consider the easiest case of this problem.

**Theorem 4.** For \( k = q - 2 \) and \( q \) odd, \( u = (u_1, \ldots, u_{q+1}) \) is a deep hole of \( \mathcal{C} \) if and only if its projective syndrome \([S_D(u)] \) does not lie on the standard RNC in \( PG(2, q) \). Thus there are exactly \( q^2 \) equivalence classes of deep holes of \( \mathcal{C} \).

**Proof:** Let \( m = q - k = 2 \). A point of \( PG(2, q) \) which is not in the linear span of \( m - 1 = 1 \) points of the standard RNC, is just a point which does not lie on the RNC.
V. CLASSIFICATION OF DEEP HOLES OF RS CODES OF REDUNDANCY 3

In this section we will classify deep holes of $[n, k; D]_q$ RS codes $C$ of redundancy $n - k$ at most 3. As remarked earlier, the cases $n - k$ being 0 and 1 are uninteresting; in the former case there are no deep holes, and in the latter case the deep holes are all received words which are not codewords. A generator and parity check matrix for $C$ is as given in (3), (4). Since the projective syndrome $[S_D(u)]$ completely determines the equivalence class of a deep hole $u$, we will focus on determining the possible values for $[S_D(u)]$.

First we consider redundancy 2 case, i.e. $[k + 2, k; D]_q$ RS code $C$ with $2 \leq k \leq q - 1$. If $k = q - 1$, then the length is $q + 1$, and the theorem of Dür stated in Section IV together with the fact that the RNC in $PG(1, q)$ is complete (i.e. there are no $[q + 2, 2]$ MDS codes) implies that $\rho(C) = 1$. Thus deep-holes of $C$ are those received words which are not codewords. For $k < q - 1$, the length $k + 2 \leq q$, and hence Proposition [I] implies that equivalence classes of deep holes of $C$ are in bijective correspondence with $[S_D(u)] \in PG(1, q)$ such that $[G_k \rho(D) | S_D(u)]$ generates a $[k + 3, 2]$ MDS code. Since every 2-dimensional MDS code is GRS, it follows that $[S_D(u)] \in PG(1, q) \setminus D$

Now we turn to RS codes of redundancy 3. Let $C$ be a $[k+3, k; D]$ RS code. Here $2 \leq k \leq q - 2$. We need some preliminary results and some notation. Let $\epsilon$ denote a fixed non-square element of $GF(q)^\times$ when $q$ is odd. The group $GL(2, q) = \{\bigl(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\bigl) : ad - bc \neq 0\}$ acts on $GF(q)^2$ in the standard manner $v \mapsto gv$. This induces an action of the group $PGL(2, q) = GL(2, q)/\{\pm(1 \ \ 0)\}$ on $PG(1, q)$ by $g \cdot x = (c + dx)/(a + bx)$. Here $x$ denotes $[c_2(x)]$. Consider the action of $GL(2, q)$ on $GF(q)^3$ given by:

$$g \cdot \xi = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + cb & bd \\ 2cd & d^2 \end{pmatrix} \xi, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \xi \in GF(q)^3$$

This induces an action of $PGL(2, q)$ on $PG(2, q)$ (see [5] Proposition 2.5-2.6 for details). Under this action, it is easy to see that

$$g \cdot [c_3(x)] = [c_3(g \cdot x)]$$

(which is $[c_3(\frac{c + dx}{a + bx})]$).

Since $PGL(2, q)$ acts transitively on $PG(1, q)$ it follows that $PGL(2, q)$ acts transitively on the standard RNC in $PG(2, q)$. Thus the standard RNC forms one orbit of the $PGL(2, q)$ action on $PG(2, q)$. We also note that for $q$ even each element of $PGL(2, q)$ fixes the nucleus $(0 : 1 : 0)$, and thus this gives an orbit of size 1.

**Lemma 3.** There are 3 orbits for the action of $PGL(2, q)$ on $PG(2, q)$ given by:

1) For $q$ even: i) the standard RNC, ii) the nucleus $(0 : 1 : 0)$, and iii) the orbit of $(1 : 0 : 1)$. The stabilizer of $(0 : 1 : 0)$ is $PGL(2, q)$, and the stabilizer of $(1 : 0 : 1)$ is

$$G_1 = \{a \in GF(q) : a \neq 1\}$$

isomorphic to the additive group of $GF(q)$

We will denote the union of the two orbits i) and ii) by $O_1$. The orbit iii) will be denote $O_4$.

2) For $q$ odd: i) the standard RNC, ii) the orbit of $(0 : 1 : 0)$, and iii) the orbit of $(1 : 0 : -\epsilon)$. The stabilizer of $(0 : 1 : 0)$ is

$$G_0 = \{x \mapsto ax^{\pm 1} : a \in GF(q)^\times\}$$

isomorphic to the dihedral group of order $2(q - 1)$

The stabilizer of $(1 : 0 : -\epsilon)$ is

$$G_\epsilon = \{(a \ b) : |a + c\sqrt{\epsilon}| = 1, (b, d) = \pm(\epsilon c, a)\},$$

isomorphic to the dihedral group of order $2(q + 1)$. Here $|| \cdot || : GF(q^2)^\times \simeq GF(q)[\sqrt{\epsilon}]^\times \to GF(q)^\times$ is the norm.
We will denote orbits i), ii) and iii) by $O_1, O_2$ and $O_3$ respectively.

Proof: We need to show that the orbits other than the RNC for $q$ odd, and the RNC and its nucleus for $q$ even are as described. Let $W$ denote the 3-dimensional space of symmetric bilinear forms on $GF(q)^2$. The group $GL(2, q)$ acts on $W$ by $(g \cdot B)(v, w) = B(g^{-1}v, g^{-1}w).$ We consider a linear isomorphism:

$$\Phi : GF(q)^3 \to W,$$

given by $\Phi(M, N, P)(v, w) = v^T(P \cdot N^{-M})w$.

The corresponding projective isomorphism will be also denoted $\Phi : PG(2, q) \to P_3W$. For later use, we record the formula

$$\Phi(M, N, P)((1, X)^T, (1, Y)^T) = \frac{\det(1 X, Y, N \cdot P)}{X \cdot Y} = MXY - N(X + Y) + P. \quad (11)$$

It is easy to check that $\Phi(g \cdot \xi) = \det(g)^2g \cdot \Phi(\xi)$, and thus at the projective level $\Phi(g \cdot [\xi]) = g \cdot [\Phi(\xi)]$. The bilinear form $\Phi(M, N, P)$ is degenerate if and only if $\det(P \cdot N^{-M}) = MP - N^2 = 0$. Thus $\Phi$ carries the orbit formed by the standard RNC to the projective space of degenerate symmetric bilinear forms on $GF(q)^2$. For the remaining orbits, it suffices to consider nondegenerate bilinear forms $B$. If $q$ is odd, it is well known that there exists $g \in GL(2, q)$ such that $(g \cdot B)(v, w) = v^T(1 1, 1, 1)w$ or $v^T(0 0, 0, 0)w$ depending on whether or not there is a nonzero vector $v$ with $B(v, v) = 0$ (for example see [20, Theorem 7.2.12]). Thus the orbits of $(0 : 1 : 0)$ and $(1 : 0 : -\epsilon)$ are the other orbits. The stabilizer of $(0 : 1 : 0)$ and $(1 : 0 : -\epsilon)$ are easy to compute and can also be found in [21, pp. 45-46].

For $q$ even, suppose $B(v, v) = 0$ for all $v \in GF(q)^2$. In this case $B(v, w) = \alpha v^T(1 0, 1)w$ for some $\alpha \in GF(q)^\times$ and thus $B = \Phi(0 : 1 : 0)$. If there exists a vector $v$ such that $B(v, v) \neq 0$, we may replace $v$ by $v/\sqrt{B(v, v)}$ to achieve $B(v, v) = 1$. The nondegeneracy implies there is a vector $w$ with $B(v, w) = 0$ and $B(w, w) = 0$. As above, we can assume $B(w, w) = 1$. Thus $B = \Phi(1 : 0 : 1)$. The stabilizer of $(1 : 0 : 1)$ is clearly all matrices satisfying $g^Tg = (1 0, 0, 0)$, which is as described in the statement. The map $(1+a \ a \ 1+a) \mapsto a$ is an isomorphism of $G_1$ with the additive group of $GF(q)$.

We define some subsets of $PG(2, q)$ associated with an evaluation set $D \subset PG(1, q)$ of size $k + 3$.

- $O_1(D) = \begin{cases} \{[c_3(\delta)] : \delta \in PG(1, q) \setminus D\} & \text{if } q \text{ is odd} \\ \{[c_3(\delta)] : \delta \in PG(1, q) \setminus D\} \cup \{0 : 1 : 0\} & \text{if } q \text{ is even} \end{cases}$

- $O_2(D) = \{g \cdot (0 : 1 : 0) : g \in PGL(2, q)/G_0, x \neq y \in g^{-1} \cdot D \Rightarrow x \neq -y\}$ for $q$ odd.

- $O_3(D) = \{g \cdot (1 : 0 : -\epsilon) : g \in PGL(2, q)/G_0, x \neq y \in g^{-1} \cdot D \Rightarrow x \neq \epsilon/y\}$ for $q$ odd.

- $O_4(D) = \{g \cdot (1 : 0 : 1) : g \in PGL(2, q)/G_1, x \neq y \in g^{-1} \cdot D \Rightarrow x \neq 1/y\}$ for $q$ even.

We note that $O_i(D)$ is a subset of $O_i$. The notation of $O_i(D), i = 2, 3, 4$ needs some explanation. The notation $PGL(2, q)/G_0$ stands for the left cosets of the stabilizer $G_0$ of $(0 : 1 : 0)$ in $PGL(2, q)$. Similarly for $G_1$. It is easy to show that if a subset $A$ of $PG(1, q)$ has the property that for any pair of distinct elements $a, b \in A$, i) $a \neq -b$ (q odd), or ii) $a \neq \epsilon/b$ (q odd), or iii) $a \neq 1/b$ (q even), then for $g$ in i) $G_0$, ii) $G_1$, iii) $G_1$, the sets $gA$ also have the same property. Thus $O_i(D), i = 2, 3, 4$ are well-defined. Moreover, such a set $A$ has size at most i) $(q+3)/2$, ii) $(q+1)/2$, iii) $(q+2)/2$ respectively, by a simple application of pigeon hole principle. We recall the notation $D = \{x_1, \ldots, x_{k+3}\}$. The size of these subsets can be expressed as:

$$|O_1(D)| = q - k - 2 \text{ if } q \text{ is odd, and } q - k - 1 \text{ if } q \text{ is even.}$$

$$|O_2(D)| = |\{g \in G_0 \setminus PGL(2, q) : gx_i \neq -gx_j \forall i \neq j\}|.$$  \[(12)\]

$$|O_3(D)| = |\{g \in G_1 \setminus PGL(2, q) : gx_i \neq \epsilon/gx_j \forall i \neq j\}|.$$  

$$|O_4(D)| = |\{g \in G_1 \setminus PGL(2, q) : gx_i \neq 1/gx_j \forall i \neq j\}|.$$  

where $G_0 \setminus PGL(2, q)$ denotes the set of right cosets of $G_0$ in $PGL(2, q)$. The exact values of the sizes of $O_i(D), i = 2, 3, 4$ depends on the configuration of $D$ in $PG(1, q)$, and appears to be a hard problem.
Theorem 5. The set of possible values of $[S_D(u)]$ is:

1) \[
\begin{cases}
(0 : 1 : 0) & \text{if } q \text{ is even, } k = q - 2 \\
O_2 \cup O_3 & \text{if } q \text{ is odd, } k = q - 2
\end{cases}
\]

2) $O_1(D)$, if $q - 3 \leq k \leq \lfloor (q - 1)/2 \rfloor$.

3) $O_1(D) \cup O_2(D)$, if $k = (q - 3)/2$ with $q$ odd.

4) $O_1(D) \cup O_2(D) \cup O_3(D)$, if $2 \leq k \leq (q - 5)/2$ with $q$ odd.

5) $O_1(D) \cup O_4(D)$, if $2 \leq k \leq (q - 4)/2$ with $q$ even.

Proof: First we consider the length $q + 1$ case (i.e. $k = q - 2$). If $q$ is even, then the only possibility for $[S_D(u)]$ is the nucleus $(0 : 1 : 0)$ by Part 1 of Theorem 3. If $q$ is odd, then by Theorem 4 the possibilities for $[S_D(u)]$ is the complement of the standard RNC in $PG(2, q)$. This proves part 1.

Now we assume $k \leq q - 3$. The length $k + 3$ is then at most $q$ and Proposition 1 implies that $[G_k(D) \mid S_D(u)]$ generates a $[k + 4, 3]$ MDS code. This always holds if $[S_D(u)] \in O_1(D)$. It remains to consider other possibilities for $[S_D(u)]$. For $q - 3 \leq k \leq \lfloor (q - 1)/2 \rfloor$, Theorem 2 implies that there are no other possibilities. This proves part 2.

We now assume $2 \leq k \leq \lfloor (q - 3)/2 \rfloor$. Let $B(v, w)$ be the bilinear form on $GF(q)^2$ given by $\Phi(S_D(u))$. We are given that

$$B((1, x_i), (1, x_j)) \neq 0, \forall x_i \neq x_j \in D.$$ (13)

In case $[S_D(u)]$ lies on the standard RNC or RNC $\cup$ its nucleus if $q$ is even, it follows that $[S_D(u)] \in O_1(D)$, which we have already considered. Thus we assume $[S_D(u)] \notin O_1$. If $q$ is even, that leaves us with $[S_D(u)] \in O_4$. Writing $[S_D(u)] = g^{-1}(1 : 0 : 1)$ for some $g \in PGL(2, q)$, and let $g \cdot D = \{y_1, \ldots, y_{k+3}\}$. It follows that:

$$[G_k(D) \mid S_D(u)] = g^{-1}[\mu_1c_3(y_1) \mid \cdots \mid \mu_{k+3}c_3(y_{k+3}) \mid (1, 0, 1)^T],$$ (14)

for some $\mu_1, \ldots, \mu_{k+3} \in GF(q)^\times$. In this case the condition (13) is equivalent to $y_i \neq 1/y_j$ for $i \neq j$. (It follows from (11) that for $(M, N, P) = (1, 0, 1)$ the form $Mxy - N(x + y) + P = xy + 1 = xy - 1$.) Hence $[S_D(u)] \in O_4(D)$. This proves part 5.

Now we turn to the case $q$ odd, and $S_D(u) = (M, N, P)^T \notin O_1$. In case $(M, N, P) \in O_2$, let $(M, N, P)^T = g^{-1}(0, 1, 0)^T$ for some $g \in PGL(2, q)$, and let $g \cdot D = \{y_1, \ldots, y_{k+3}\}$. It follows that:

$$[G_k(D) \mid S_D(u)] = g^{-1}[\mu_1c_3(y_1) \mid \cdots \mid \mu_{k+3}c_3(y_{k+3}) \mid (0, 1, 0)^T],$$ (15)

for some $\mu_1, \ldots, \mu_{k+3} \in GF(q)^\times$. In this case the condition (13) is equivalent to $y_i \neq -y_j$ for $i \neq j$, because $Mxy - N(x + y) + P = -(x + y)$. Hence $[S_D(u)] \in O_2(D)$. Similarly, if $(M, N, P) \in O_3$, we get $(M, N, P) \in O_3(D)$. As mentioned above the set $O_3(D)$ is empty unless $k + 3 \leq (q + 1)/2$, thus for $k = (q - 3)/2$, the possibility $S_D(u) \in O_3$ does not occur. This proves parts 3)-4).

We record the following theorem about canonical forms of non GRS $[n + 1, 3]$ MDS codes extending a GRS $[n, 3]$ code. It will be useful to regard two codes $C, C'$ as diagonally equivalent if there is a diagonal Hamming isometry (a diagonal matrix in $GL(n, q)$) which carries $C$ to $C'$. Note that diagonally equivalent codes are monomially equivalent but the converse is not true in general. At the level of arcs, diagonal equivalence yields the notion of ordered arcs, whereas monomial equivalence yields the the notion of (unordered) arcs.

Theorem 6. Let $C$ be a non GRS $[n + 1, 3]$ MDS code extending a $[n, 3]$ GRS code $C_1$ where $n \geq 5$. Up to diagonal equivalence, $C$ is the code generated by one of the families of matrices $M_1, M_2, M_3$ below.

Equivalently let $A$ be an ordered $n + 1$-arc in $PG(2, q)$ with the first $n$ points (but not the last) on a RNC (where $n \geq 5$), then $A$ is projectively equivalent to the ordered arc defined by the columns of one of the families of matrices $M_1, M_2, M_3$ below.
In the following, \( D = \{x_1, \ldots, x_n\} \subset PG(1, q) \) denotes a subset of \( n \geq 5 \) distinct points satisfying certain conditions.

1) \( D \) satisfies \( x_i \neq -x_j \) if \( i \neq j \). In this case \( n \leq (q + 3)/2 \) if \( q \) is odd and \( n \leq q + 1 \) if \( q \) is even.

\[
M_1 = \begin{pmatrix}
1 & \ldots & 1 & 0 \\
x_1 & \ldots & x_n & 1 \\
\alpha x_1^2 & \ldots & \alpha x_n^2 & 0
\end{pmatrix}
\]

(16)

2) \( q \) is odd, \( n \leq (q + 1)/2 \), and \( D \) satisfies \( x_i \neq \epsilon/x_j \) if \( i \neq j \).

\[
M_2 = \begin{pmatrix}
1 & \ldots & 1 & 1 \\
x_1 & \ldots & x_n & 0 \\
\alpha x_1^2 & \ldots & \alpha x_n^2 & -\epsilon
\end{pmatrix}
\]

(17)

3) \( q \) is even, \( n \leq (q + 2)/2 \), and \( D \) satisfies \( x_i \neq 1/x_j \) if \( i \neq j \).

\[
M_3 = \begin{pmatrix}
1 & \ldots & 1 & 1 \\
x_1 & \ldots & x_n & 0 \\
\alpha x_1^2 & \ldots & \alpha x_n^2 & 1
\end{pmatrix}
\]

(18)

Proof: From the fact that a RNC in \( PG(2, q) \) is uniquely determined by any 5 points on it, it follows that the matrices \( M_i \) above do not generate a GRS code for \( n \geq 5 \) (the corresponding arcs do not lie on a RNC). To prove that the code \( C \) in question is diagonally equivalent to the code generated by one of the matrices of the type \( M_i \), let \( C_1 \) be diagonally equivalent to the code generated by a matrix \( G = [c_3(t_1) | \ldots | c_3(t_n)] \). Thus there is a vector \( v \in GF(q)^3 \) such that \([G | v]\) generates the non-GRS code \( C \). The analysis of such matrices \([G | v]\) was carried out in the proof of Theorem 5 (see (14), (15)). It was shown that there are matrices \( P \in GL(3, q) \) and a diagonal matrix \( Q \in GL(n + 1, q) \) such that \( PG(v)Q \) is of the type \( M_1, M_2 \) or \( M_3 \). In other words \( C \) is diagonally equivalent to the code generated by one of the types of matrices \( M_i \).

We note that two distinct matrices of the type, say \( M_2 \) may represent the same MDS extension \( C \) of \( C_1 \). In order to count the diagonal equivalence classes of codes \((C_1, C)\) where \( C \) is a \([n + 1, 3]_q \) MDS and non-GRS code extending a \([n, 3]_q \) RS code \( C_1 \), we have to factor out the left action of \( G_0, G_\epsilon, G_1 \) on generator matrices of the type \( M_1, M_2, M_3 \). It is convenient to use the language of arcs. We will now count the number of projective equivalence classes of ordered arcs \((A_1, A)\) where \( A \) is an ordered \( n + 1 \)-arc not contained in a RNC, but its first \( n \) points form the arc \( A_1 \) which is contained in a RNC. Let \( M_i \) be the set of ordered arcs (without using projective equivalence) arising from matrices of the type \( M_i \). Let \( G_i \subset PGL(2, q) \) be the stabilizer of the point represented by the last column. It is easy to see that \( G_i \) acts freely (i.e. without fixed point) on \( M_i \). This is because the only element of \( PGL(2, q) \) which fixes 3 points is the identity transformation. The quotient \( G_i \backslash M_i \) gives the projective equivalence classes of ordered arc pairs \((A_1, A)\) that we are trying to count and which are of type \( M_i \). It is straightforward to count the relevant quantities: \(|M_1| = (q + 1)!/(q + 1 - n)!\) if \( q \) is even, and

\[
|M_1| = \frac{q^{-1/2}2^n}{(q^{-1/2} - n)!} + \frac{q^{-1}2n}{(q^{-1} - n)!} + \frac{q^{-1/2}2n-2}{(q^{-1/2} - n)!} n(n-1)\quad\text{if } q \text{ is odd.}
\]

Here we use the convention \((-m)! = \infty\) for natural numbers \( m \). We illustrate the method we use to obtain \(|M_1|\) for \( q \) odd. The other cases are similar. We may write \( PG(1, q) \) as the disjoint union of \((q + 3)/2\) sets of the form \( \{\infty\}, \{0\}, \{\pm\alpha_1\}, \ldots, \{\pm\alpha(q-1)/2\} \). We note that \( M_1 \) consists of \( n \)-tuples \((z_1, \ldots, z_n)\) such that we pick at most one element from each of the \((q + 3)/2\) sets above. By similar methods, we obtain

\[
|M_2| = \frac{(q+1)!2^n}{(q+1 - n)!} + \frac{(q+1)!2n}{(q+1 - n)!} + \frac{(q+1)!2n-2}{(q+1 - n)!} n(n-1)\quad\text{if } q \text{ is even,}
\]

\[
|M_3| = \frac{(q+1)!2^n}{(q+1 - n)!} + \frac{(q+1)!2n-1}{(q+1 - n)!} n(n-1)\quad\text{if } q \text{ is odd.}
\]
The groups $G_i$ have been computed previously: $G_1$ is $\text{PGL}(2, q)$ if $q$ is even and isomorphic to a dihedral group of order $2(q - 1)$ for odd $q$. The group $G_2$ isomorphic to a dihedral group of order $2(q + 1)$, and the group $G_3$ isomorphic to the additive group $(GF(q), +)$. Thus we obtain that the number of ordered arc pairs $(A_1, A)$ of the type $M_i$ equals:

1. $\frac{q-3}{2}! \cdot 2^{n-4} \cdot \frac{(q+1)(q+3-2n) + n(n-1)}{q^i-2n!}$ if $i = 1$ and $q$ is odd. Here $n \leq (q-1)/2$.

2. $\frac{q-1}{2}! \cdot 2^{n-2} \cdot \frac{(q^i+2n-1)!}{2n!}$ if $i = 2$. Here $n \leq (q+1)/2$.

3. $\frac{q-3}{2}! \cdot 2^{n-2} \cdot \frac{(q^i+2n-2)!}{2n!}$ if $i = 3$. Here $n \leq (q+2)/2$.

VI. CONCLUSION

We solve the problem of classifying deep holes of $[n, k]_q$ RS codes for $k \geq (q - 1)/2$ for non prime $q$, which was posed as an open problem in the concluding remarks of [3]. The problem for $k < (q - 1)/2$ is open. We solve the problem for $n = k + 3$ and all $k$. We also solve the problem for $k = 2, n = q$ with $q$ odd, by reducing it to Segre’s ‘oval equals conic’ theorem. For $k = 2, n = q + 1$ with $q$ even, we show that the problem is equivalent to the difficult problem of classifying hyperovals in projective planes. Finally, we obtain canonical forms for $[n + 1, 3]_q$ MDS but non-GRS codes extending a $[n, 3]_q$ GRS code.

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