Electromagnetic waves in Born Electrodynamics

Hedvika Kadlecová

1 Institute of Physics of the ASCR, ELI-Beamlines project, Na Slovance 2, 18221, Prague, Czech Republic

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We study two counter-propagating electromagnetic waves in the vacuum within the framework of the Born–Infeld theory in quantum electrodynamics. By choosing the crossed field case \( \mathbf{E} \cdot \mathbf{B} = 0 \), i.e. \( \mathcal{G}^2 = 0 \), the Born–Infeld Lagrangian reduces to the Born Lagrangian, therefore for this special case we present study which is identical for the Born–Infeld and the Born electrodynamics. In this paper, we show that the non-linear field equations decouple for ordinary wave case using self-similar solutions and we investigate the shock wave steepening. We show that the only solutions are exceptional traveling wave solutions which propagate with constant speed and which do not turn into shocks. We discuss the phase shift and the cross section of the process to be measured together with the our proposed direct detection of the photon–photon scattering \[^1][^2]\). This work serves as a preliminary study towards analyzing the waves in the more general case in the Born–Infeld theory.

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I. INTRODUCTION

The photon–photon scattering in a vacuum occurs via the generation of virtual electron–positron pair creation resulting in vacuum polarization \[^3\]. The process breaks the linearity of the Maxwell equations and is one of the oldest predictions of quantum electrodynamics (QED). It is convenient to use Heisenberg–Euler approach in QED \[^4\] \[^5\] to investigate such process.

The indirect measurement of the process was achieved just until very recently. In 2013 \[^6\] \[^7\], it was proposed to look for the light-by-light scattering in ultra-peripheral heavy-ion collisions at the LHC. Off–shell photon–photon scattering \[^8\] was indirectly observed in collisions of heavy ions accelerated by standard charged particles accelerators with 4.4 \(\sigma\) significance, see review article \[^9\] and results of experiments obtained with the ATLAS detector at the Large Hadron Collider \[^10\] \[^11\] in 2017, where the cross section was measured and was identified as compatible with standard model QED predictions \[^12\] \[^13\] \[^14\].

Next to the development of the Heisenberg–Euler general expression for the quantum nonlinearities in the Lagrangian of QED, \[^4,8,15\], there was an interest in a theory of QED with the upper limit on the strength of the electromagnetic field, today known as the Born–Infeld theory, which represents very unique nonlinear modification of the QED Lagrangian.

After being forgotten for several decades the Born–Infeld theory gained a new interest in 1985, when it was found as limiting case of string theory. It has been found in \[^16\] that the Born–Infeld Lagrangian is the exact solution of a constant Abelian external vector field problem in the open Bose string theory with the number of spacetime dimension \( D = 26 \). This happens in models inspired by M-theory where particles are localized on a brane in lower dimension separated by the distance \( \sim 1/\sqrt{\phi} = 1/M \) in an extra dimension where the constant \( b \) is an unknown parameter with the dimension of \( [\text{mass}]^2 \), \( b \equiv M^2 \) in the Born–Infeld Lagrangian.

Besides the theoretical application to the string theory, there is also an interest in experimental research in the Born–Infeld theory. Next to the search for the photon–photon scattering in vacuum there is also need to test QED and non–standard models like Born–Infeld theory and scenarios where mini charged particles are involved or axion–like bosons \[^17\]. Newly, the PVLAS experiment \[^18\] measured new limits also on the existence of hypothetical particles which couple to two photons, axion-like and milli-charged particles, besides casting upper limits on the magnetic birefringence predicted by QED.

In other words, the photon–photon scattering provides a tool for the search for a new physics, in which new particles can participate, see search for the process in X–ray region \[^19\]. Though Heisenberg–Euler theory is most probably preferred over the Born–Infeld theory in nature, the experimental observation and precision tests of the parameter for the Born–Infeld in the low energy effective Lagrangian are still waiting for reaching necessary sensitivity for its measurement in the process of photon–photon scattering.

In the limit of extremely intense electromagnetic fields the Maxwell equations are modified due to the nonlinear process of photon-photon scattering that makes the vacuum refraction index depend on the field amplitude. Due to the nonlinearity of the field equations, the electromagnetic field interacts with itself and generates deformations in the light cone \[^20\]. In the context of the nonlinear electrodynamics, introducing the background field affects the propagation velocity of the electromagnetic wave and creates the birefringence effect, i.e. the speed of wave propagation depends on the wave polarization, \[^21\] \[^22\] and in presence of electromagnetic waves with small but finite wavenumbers the vacuum behaves as a dispersive medium. This phenomenon exists in all physically acceptable nonlinear electrodynamics includ-
ing the Born one except for the Born–Infeld electrodynamics [23].

In the Heisenberg–Euler approximation of QED [4], where the birefringence effect exists, the electromagnetic fields propagate in the dispersionless media whose refractive index depends on the electromagnetic field, the electromagnetic wave evolves into a configuration with gradient singularities [23] and leads to formation of a shock wave. The formation of singularities in the Heisenberg–Euler theory was noticed in [22] where a particular solution of field equations from Heisenberg–Euler Lagrangian was obtained. The shock wave creation is accompanied by subsequent generation of high order harmonics. High frequency harmonics generation can be a powerful tool to explore the physics in vacuum. The highest harmonics can be used to probe the high energy region because they are naturally co–propagating and allow powerful measurements of QED effects in the coherent harmonic focus, [20, 27].

The analysis of shock waves behavior in the Born–Infeld non–linear electrodynamics was investigated in several papers. The early theoretical analysis was made by Boillat who showed that both polarization modes travel along the light cone of one optical metric in exceptional nonlinear electrodynamic like in the Born–Infeld [23, 28]. The shock waves were analyzed in the Born and Born–Infeld theories separately in [29] where they showed that the Born–Infeld exhibits no birefringence and the Born model itself surprisingly does. In general, the propagation of shock waves in nonlinear theories is given by optical metrics and polarization conditions describing the propagation of two differently polarized waves, with birefringence as an option to happen. The optical metrics reduce to one for the Born–Infeld electrodynamics meaning that this electrodynamics is a special case which does not show the effect of birefringence.

The exceptional means that no shocks are formed (in the sense of Lax representation) [23, 30] and the fields on the wave–front always satisfy the field equations. Born–Infeld electrodynamics is called completely exceptional and it is the only completely exceptional regular non–linear electrodynamics [31]. The electrodynamics shows special features as the absence of shock waves and birefringence. In [32], the study was extended to the motion of more general discontinuity fronts. Considering the convexity of energy density, they derived relations concerning exceptional waves (linearly degenerated) and shock fronts with the discontinuities of the field. They showed that the characteristic shocks, which are moving with the wave velocity, are unbounded, they allow arbitrary coefficients for the Born–Infeld electrodynamics. The discontinuities do not evolve into shocks, but when the shock exists at some initial time it propagates on characteristic surfaces, i.e. the Cauchy problem is well posed.

In general, the shock wave development problem is still not completed till today. Just recently, the general problem of shock formation in 3D dimensions was solved by D. Christodoulou in [33], it was proved that the shock waves are absent in 3D space for the Chaplygin gas known also as scalar Born–Infeld theory. In this work, it provided complete description of the maximal development of the initial data. This description sets up the problem of continuing the solution beyond the point where the solution ceases to be regular. It belongs to the category of free boundary problems but possesses the additional property of having singular initial data due to the behavior of the solution at the blow up surface. The same problem was discussed for the case with spherical symmetry for a barotropic fluid [34]. A complete description of the singularities associated to the development of shocks in terms of smooth functions is given. Its important to note that Boillat in his works assumed that Lax [22] one dimensional wave motion theorems would generalize to higher dimensional problems, therefore its not settled yet, and its not completely unreasonable to expect solutions with shocks from the nonlinear electrodynamic field equations of Boillat multi–parameter family.

More specifically in [35], the formation of singularities in the Born and Born–Infeld electrodynamics was studied for plane wave–pulse motions along one spatial direction. The results are based on earlier work on conservation laws in [36]. The set of Born–Infeld equations could be augmented as a $10 \times 10$ system of hyperbolic conservation laws, which share some similarities with magnetogyro–dynamics equations as the existence of convex entropy, galilean invariance and full linear degeneracy.

Meanwhile, the global existence of classical smooth finite–energy solutions to the 3D for small amplitude initial data in the Maxwell–Born–Infeld system has been proved in [37]. The Born–Infeld equations were solved for transverse plane waves in a rectangular waveguide. Waveguides can be employed to test non-linear effects in electrodynamics. It was shown that the energy velocity acquires a dependence on the amplitude, and harmonic components appear as a consequence of the non-linear behavior [38].

Recently, we presented and analyzed an analytical solution of the non–linear field equations in the Heisenberg–Euler approximation in QED which describes the finite amplitude electromagnetic wave counter–propagating to the crossed electromagnetic field, [1], i.e. two counter–propagating waves in the QED quantum vacuum. The configuration corresponds to the collision of the short and long wavelength electromagnetic pulses. It may be considered as a model of interaction of the high intensity laser pulse with the X–ray pulse generated by XFEL.

We used the weak field approximation to the sixth order in the field amplitude to include four and six photon interactions to study the singularity formation.

We found the solution of the field equations in the form of a simple wave [24, 38, 41]. The resulting non–linear wave equation was solved by integration along the characteristics of the equation. It was demonstrated that the finite amplitude wave evolution of the non–linear wave solution describes high order harmonic generation, wave
steepening and formation of a shock wave in the vacuum.

Furthermore, we have found that the resulting electromagnetic wave breaking has a rarefaction shock wave character, the wave steepens and breaks in the backwards direction. At the shock wave front, where the approximation stops to be valid, the electron–positron pairs are being created during the Breit–Wheeler process, further they are accelerated by the electromagnetic wave and emit gamma–ray photons. Such emission leads to the electron–positron avalanche via the multi-photon Breit–Wheeler mechanism. In the proposed experiment we suggest to reach realistic energies of the electron–positron pair cross section instead of targeting directly the Schwinger limit \( E_S \) and detect the gamma–ray photons. The proposed experiment should serve for the direct detection of photon–photon scattering in the quantum vacuum which was not observed before.

In the subsequent paper [2], our analysis was widened by implementing an analytical method of solving the system of non–linear equations. We have shown that the non–linear field equations decouple for the ordinary wave case when we look for the solution in the form of a simple wave, i.e. Riemann wave.

The solution has a form of non–linear wave equation for the relatively short wavelength pulse in the linear approximation and generalizes our previous result in [1]. The solution was analyzed by method of characteristics and by perturbation method. We have demonstrated in more detail that the solution describes high order harmonic generation, wave steepening and formation of a shock wave.

The wave breaking was analyzed in the direction of the electromagnetic wave, it depends on the strength of the electromagnetic field \( E_0 \) (sign of \( f' \)) and has forward character for weak field and rarefaction shock wave character for stronger fields. The rarefaction character of the steepening in [1] is main and dominant character of the shock wave, the forward steepening happens on lower orders in the expansion of \( f' \).

In the same setup of two counter propagating electromagnetic waves we also investigated and found new relativistic electromagnetic soliton solutions and nonlinear waves dispersion in quantum vacuum, [11]. The balance between the vacuum polarization (dispersion and diffraction) showed and the nonlinear effects can produce the formation of one-dimensional Korteweg-de-Vries (KdV) type solitons, also multidimensional generalization of the KdV solutions, the Kadomtsev-Petviashvily solitons. This type of solitons can propagate over a large distance without changing their shape. The solutions have many implications from fluid mechanics to solid state physics, plasma physics and also in quantum field theory. There also exists electromagnetic photonic solitons in the Born–Infeld theory, [30, 42]. Photonic solitons are very important for nonlinear optics. They pass through one another without scattering. The existence of a limiting field strength now finds real justification in the increasing pairs production from vacuum [43].

In this paper, we investigate even further the problem of two counter–propagating waves in another equally important framework of Born–Infeld and implement some of the methods from [1, 2] to investigate the field equations for the ordinary wave case while looking for the solution in the self–similar form. By choosing the crossed field case \( E \cdot B = 0 \), i.e. \( \mathcal{G}^2 = 0 \), the Born–Infeld Lagrangian reduces to the Born Lagrangian and it is its special subcase. In fact, we present the solution of a problem which is identical for both the Born and the Born–Infeld electrodynamics for the crossed field condition \( \mathcal{G}^2 = 0 \). For simplicity, we will use just the name Born in relation to calculations knowing that it means the before mentioned identical solution. This is preliminary study leading us to investigate the problem in the more general case in the Born–Infeld theory.

The paper is organized as follows. In Section III we review the Born and the Born–Infeld theories which will be useful throughout this paper.

In Section IV we review the properties of the Born–Infeld theory such as canonical variables and Legendre transformations, duality rotations, a conservation law in Born–Infeld theory.

In Section V we derive the non–linear field equations in Born theory, we add small amplitude perturbations and linearize the coefficients.

Next, in Section VI we solve the Born field equations. We use the new analytical method of solving such system of equations introduced in [1, 2], we assume the solution in a form of a simple wave, we show that the system of equations decouple for the ordinary wave case and it can be solved exactly. The solution has a form of a non–linear wave without dispersion in the linear approximation.

In Section VII we concentrate on analyzing the solutions in the Born theory and compare it with the results in the Heisenberg–Euler theory. We analyze the solutions by the method of characteristics for two possible cases + and − corresponding to two orientations of the beams, co–propagating and counter–propagating beams. We discuss properties of the solutions, we demonstrate that the shock wave steepening does not take place and only the exceptional waves are created, the phase shift in the process and also contribution to the cross section of the photon–photon scattering. The Section VIII is devoted to the discussion of the experimental distinguishing of Born–Infeld and Heisenberg–Euler theories.

The main results of the paper are summarized in concluding Section VIII.

II. BORN AND BORN–INFELD LAGRANGIAN

The first model of a nonlinear electrodynamics was proposed by Born [44] in 1933 by the following choice of the Lagrangian,

\[
\mathcal{L}_B = -b^2 \left( \sqrt{1 - \frac{E^2 - B^2}{b^2}} - 1 \right),
\]
where \( \mathbf{E} \) and \( \mathbf{B} \) are electric and magnetic fields, \( b \) being a constant having the dimension of the electromagnetic field and the units \( c = \hbar = 1 \). In more detail, the Born theory was described in [43].

Born’s motivation was to find classical solutions representing electrically charged particles with finite self-energy. The mechanism restricting the particle’s velocity in relativistic mechanics to values smaller than \( c \) is going to restrict the electric field in the Born theory with \( \mathcal{L}_B \) [4] to values smaller than the critical field \( b \) (when \( \mathbf{B} = 0 \), [31].

A year later, the Born–Infeld electrodynamics was developed [31, 46] with the Lagrangian given by

\[
\mathcal{L}_{BI} = -b^2 \left( \sqrt{1 - \frac{E^2 - B^2}{b^2}} - \frac{E \cdot B}{b^4} - 1 \right),
\]

where a new pseudoscalar invariant was added while maintaining the Lagrangian as relativistic covariant.

The Born and the Born-Infeld theory reduce to the linear Maxwell theory for fields which are much weaker than the critical field \( b \), \( (b \rightarrow \infty, \text{i.e. classical linear electrodynamics}) \),

\[
\mathcal{L}_M = \frac{1}{2} (E^2 - B^2).
\]

The Born-Infeld theory is a unique nonlinear theory of electromagnetic field because it is the only theory which does not lead to birefringence effect, the propagation velocities in all directions do not depend on the wave polarization, i.e. the velocity of light in the Born-Infeld theory does not depend on its polarization. The Maxwell theory and the nonlinear electrodynamics of Born and Infeld are the only relativistic theories in which this takes place, [21].

III. PROPERTIES OF BORN–INFELD ELECTRODYNAMICS

Let’s list briefly the properties of the Born–Infeld electrodynamics, [31]:

A. General structure of Born–Infeld electrodynamics

The field equations for the Born–Infeld Lagrangian \( \mathcal{L}_{BI} \) [2] are given by

\[
\partial_{\mu}(\partial_{\nu}\mathcal{L}_{BI}/\partial(\partial_{\mu}\Phi)) - \partial\mathcal{L}_{BI}/\partial\Phi = 0,
\]

where

\[
\Phi = (-\phi, \mathbf{A}).
\]

Every theory of electrodynamic type is described by the source free Maxwell equations, the first pair of Maxwell field equations reads

\[
\nabla \cdot \mathbf{B} = 0,
\]

\[
\nabla \times \mathbf{E} = -\partial_t \mathbf{B}.
\]

The second pair can be found by varying the Lagrangian \( \mathcal{L}_{BI} \) [2] which gives the field equations. The second pair of equations can be written as

\[
\nabla \times \mathbf{H} = \partial_t \mathbf{D},
\]

\[
\nabla \cdot \mathbf{D} = 0,
\]

together with the nonlinear constitutive relations,

\[
\mathbf{E} = \mathbf{E}(\mathbf{D}, \mathbf{B}), \quad \mathbf{H} = \mathbf{H}(\mathbf{D}, \mathbf{B}).
\]

The assumption to choose canonical pair of variables as \( \mathbf{D} \) and \( \mathbf{B} \) leads to a consistent canonical formulation of the nonlinear theory. The consistency of the above equations and their relativistic covariance is guaranteed by the existence of invariant action principle, the existence of a scalar Lagrangian density \( \mathcal{L}_{BI} \), which may be any function of the scalar invariant \( \mathcal{S} \) and the pseudoscalar invariant of the electromagnetic field tensor \( \mathcal{G} \), the Poincaré invariants as \( \mathcal{L}_{BI}(\mathcal{S}, \mathcal{G}) \), [47].

\[
\mathcal{S} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2),
\]

\[
\mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B},
\]

\[
\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},
\]

where \( \varepsilon^{\mu\nu\rho\sigma} \) is the Levi-Civita symbol in four dimensions. The equations [8] follow from the assumption of existence of potentials and equations [11] follow from varying the Lagrange function \( \mathcal{L}_{BI}(\mathcal{S}, \mathcal{G}) \). The equations have a form in the relativistic tensor notation (Bianchi identities \( \partial_{\mu} F_{\nu\lambda} = 0 \)):

\[
\partial_{\mu} F_{\nu\lambda} + \partial_{\lambda} F_{\nu\mu} + \partial_{\nu} F_{\lambda\mu} = 0,
\]

\[
\partial_{\mu} h^{\mu\nu} = 0,
\]

where

\[
h^{\mu\nu} = \frac{\partial L}{\partial \tilde{F}^{\mu\nu}} = \frac{\partial L}{\partial \mathcal{S}} F^{\mu\nu} + \frac{\partial L}{\partial \mathcal{G}} \tilde{F}^{\mu\nu}.
\]

The Born [1] and the Born–Infeld [2] Lagrangians can be rewritten in terms of Poincaré invariants as

\[
\mathcal{L}_B = -b^2 \left( \sqrt{1 + \frac{2\mathcal{S}}{b^2}} - 1 \right),
\]

and

\[
\mathcal{L}_{BI} = -b^2 \left( \sqrt{1 + \frac{2\mathcal{S}}{b^2} - \frac{\mathcal{G}^2}{b^4}} - 1 \right).
\]
B. Legendre transformations and Duality rotations

We choose to have $L_{BI}(E, B)$ dependent on a pair of variables $(E, B)$. We can choose other three different pairs of variables $E, B, D$ and $H$, treating each set as a set of independent variables. Such transition between the choices can be described as an analogy with Legendre transformations.

The dependent variables $D, H$ are determined from the constitutive relations,

$$
H = -\frac{\partial L_{BI}}{\partial B}, \quad D = \frac{\partial L_{BI}}{\partial E}.
$$

By exchanging the dependence of $L_{BI}$ on the other three combinations of two independent variable pairs from $E, B, D$ and $H$, we can get the remaining Legendre transformations, in [31].

We can rewrite the Lagrangian as

$$
L_{BI} = b^2(-l + 1),
$$

$$
l = \sqrt{1 - \frac{E^2 - B^2}{b^2} - \frac{(E \cdot B)^2}{b^4}}.
$$

For our choice $L_{BI}(E, B)$, we obtain the constitutive relations, using (13),

$$
H = \frac{1}{l} \left( B - \frac{1}{b^2} (E \cdot B) E \right),
$$

$$
D = \frac{1}{l} \left( E + \frac{1}{b^2} (E \cdot B) B \right).
$$

Nonlinear electrodynamics has no internal symmetries but Born–Infeld electrodynamics has a new conservation law for some choices of the Lagrangian. The symmetry is the invariance of the field equations under the duality rotations of canonical fields $D$ and $B$ (Hodge duality rotation through an angle $\theta$), also called $\gamma$-invariance [48] and [31, 19],

$$
D + iB = e^{i\theta} (\bar{D} + i\bar{B}), \quad E + iH = e^{i\theta} (\bar{E} + i\bar{H}),
$$

which is a canonical transformation (and not an internal symmetry) and a symmetry transformation. The generator of the duality rotations represents an important constant of motion, the total charge, and it is also the generator of the phase transformations of the field.

The duality rotations [18] leads to identity in the Born–Infeld electrodynamics,

$$
E \cdot B = D \cdot H.
$$

IV. BORN FIELD EQUATIONS

In this Section, we will derive and analyze the Born field equations for the set up of two counter–propagating waves we have used in our previous papers, [1, 2, 41].

The motivation is to use our knowledge of solving the field equations for the Heisenberg–Euler Lagrangian of the two counter–propagating electromagnetic waves to solve the field equations in the Born electrodynamics.

We will work in the orthogonal coordinate system, $(x, y, z)$, where the two waves propagate along the $x-$axes. We are going to investigate the ordinary wave case in the birefringence problem, therefore we assume $E = (0, 0, E_z)$ and $B = (0, B_y, 0)$, which is the simple case of non–vanishing components $E_z$ and $B_y$, in order to investigate the crossed field case $E \cdot B = 0$ which is the setup we have made in our previous work. For this ansatz, the Born–Infeld Lagrangian [2] reduces to the Born Lagrangian [1], therefore the following investigation will be done in the Born theory.

The term $\delta^4$ can be neglected in situations where we are far away from singularities, [45]. Therefore we can say that we are far away from the creation of shock wave fronts (singularities) in our study by our choice of the crossed field ansatz $E \cdot B = 0$.

A. Derivation of Born field equations

The field equations were found by varying the Lagrangian [1] according to the potential $A$ (the first set comes from the set of equations [1]) and the second set from equations (21):

$$
\partial_t B_y - \partial_x E_z = 0,
$$

$$
- \left[ 1 + \frac{E_z^2}{b^2} \frac{1}{(1 - (E_z^2 - B_y^2)/b^2)} \right] \partial_t E_z
$$

$$
+ \left[ 1 - \frac{B_y^2}{b^2} \frac{1}{(1 - (E_z^2 - B_y^2)/b^2)} \right] \partial_x B_y
$$

$$
+ \frac{1}{b^2} \frac{E_z B_y}{(1 - (E_z^2 - B_y^2)/b^2)} (\partial_t B_y + \partial_x E_z) = 0,
$$

where we denote $E_z \equiv E$ and $B_y \equiv B$ and the condition $1 - 1/b^2(E^2 - B^2) > 0$ should be valid.

Subsequently, we add the small amplitude perturbation to the fields,

$$
E = E_0 + a_x(x, t), \quad B = B_0 + b_y(x, t),
$$

where the fields $E_0, B_0$ represent the constant electromagnetic background field and $a_x(x, t)$ and $b_y(x, t)$ are perturbations, functions of $x$ and $t$. The equations (20) can be rewritten in a form using the expressions (22) as

$$
\partial_t b_y(x, t) = \partial_x a_x(x, t),
$$

$$
\alpha \partial_t a_x(x, t) - \beta [\partial_x a_x(x, t) + \partial_t b_y(x, t)] - \gamma \partial_x b_y(x, t) = 0,
$$
where the coefficients $\alpha$, $\beta$ and $\gamma$ become

\[
\alpha = 1 + \frac{(E_0 + a_z)^2}{b^2} \frac{1}{\left(1 - \frac{1}{b^2}[(E_0 + a_z)^2 - (B_0 + b_y)^2]\right)},
\]

\[
\beta = \frac{1}{b^2} \frac{(E_0 + a_z)(B_0 + b_y)}{\left(1 - \frac{1}{b^2}[(E_0 + a_z)^2 - (B_0 + b_y)^2]\right)},
\]

\[
\gamma = 1 - \frac{(B_0 + b_y)^2}{b^2} \frac{1}{\left(1 - \frac{1}{b^2}[(E_0 + a_z)^2 - (B_0 + b_y)^2]\right)}.
\]

**B. Derivation of the phase velocity**

Now, we will derive the coefficients of the background field to calculate the phase velocities, we assume that $a_z(x, t) = b_y(x, t) = 0$ and obtain from Eqs. (28) that

\[
\alpha_0 = \frac{1 + \frac{E_0^2}{b^2}}{\left(1 - \frac{1}{b^2}(E_0^2 - B_0^2)\right)},
\]

\[
\beta_0 = \frac{E_0 B_0}{b^2} \frac{1}{\left(1 - \frac{1}{b^2}(E_0^2 - B_0^2)\right)},
\]

\[
\gamma_0 = \frac{1 - \frac{E_0^2}{b^2}}{\left(1 - \frac{1}{b^2}(E_0^2 - B_0^2)\right)}.
\]

Furthermore, for the crossed field case, we choose $B_0 = E_0$ for simplicity, we obtain

\[
\alpha_0 = 1 + \frac{E_0^2}{b^2},
\]

\[
\beta_0 = \frac{E_0^2}{b^2},
\]

\[
\gamma_0 = 1 - \frac{E_0^2}{b^2}.
\]

In order to find the wave phase velocity from the linearized equations (23) and (24), we look for the solutions in the form,

\[
a_z \propto \exp(-i\omega t + iqx), \quad b_y \propto \exp(-i\omega t + iqx),
\]

where $q$ is the wave number and $\omega$ is the frequency. Substituting (28) into the equations (23) and (24) where we assume the equations for the background field with $\alpha = \alpha_0, \beta = \beta_0$ and $\gamma = \gamma_0$ in a form (29) or for the crossed field $E \cdot B = 0$, where we use $B_0 = E_0$ for simplicity. We obtain algebraic set of equations for the wave velocity $v = \omega/q$. Since the medium is dispersionless in our study, we denote the phase $v_{ph} = \omega/q$ and the group $v_g = \partial \omega/\partial q$ velocity as one $v = v_{ph} = v_g$.

Then we obtain the set of equations,

\[
v(b_y \beta_0 - a_z \alpha_0) - (a_z \beta_0 + b_y \gamma_0) = 0,
\]

which has two solutions,

\[
v_{1,2} = \frac{-\beta_0 \pm \sqrt{\beta_0^2 + \alpha_0 \gamma_0}}{\alpha_0}.
\]

The expression under the square root can be simplified using (26) as

\[
\beta_0^2 + \alpha_0 \gamma_0 = \frac{1}{\left(1 - \frac{1}{b^2}(E_0^2 - B_0^2)\right)^2},
\]

which results for crossed field $E_0 = B_0$, (27), in

\[
\beta_0^2 + \alpha_0 \gamma_0 = 1.
\]

The velocities (30) can be simplified by using the expressions (27) and (32),

\[
v_{1,2} = \frac{-\beta_0 \pm 1}{\alpha_0},
\]

then we find the velocities, the $-$ results as $v_1$ and the $+$ as $v_2$,

\[
v_1 = -1,
\]

\[
v_2 = \frac{\alpha_0}{\alpha_0} = \frac{1 - \frac{E_0^2}{b^2}}{\frac{E_0^2}{b^2}} \frac{1}{1 + \frac{E_0^2}{b^2}}.
\]

The phase velocities $v = v_{1,2}$ are the velocities for the wave propagation over the crossed background field in the Born theory. The solution $v_1$ corresponds to the co-propagating waves case and the solution $v_2$ corresponds to the case of counter propagating waves whose velocity is lower than speed of light $c$. The phase velocity and group velocity are equal in our calculation, $v = v_{ph} = v_g$ because of absence of dispersion. The phase velocities are similar to those for the Heisenberg–Euler approximation, (2), apart from the constants related to the theory.

The phase velocity also diminishes with the increasing field strength parameter $b$. In the limit $b \rightarrow \infty$, which leads to the linear Maxwell theory, the phase velocity $v_2 \rightarrow 1$. The obtained result is used further as a limit case for the background crossed field.

The Born–Infeld electrodynamics behaves as an isotropic medium with a polarization–independent refractive index. The individual plane wave propagating with speed of light in homogeneous isotropic scattering reduces its phase velocity uniformly [51]. For example, in the process of photon–photon scattering, a counter propagating circularly polarized monochromatic wave of the same helicity serves as an isotropic medium for the other counter propagating wave, studied on classical level in [51].
C. Linearization of the coefficients in the equations

Now, we perform the linearization of the coefficients $\alpha, \beta$ and $\gamma$ about the constant background field,

$$
\begin{align*}
\alpha &= \alpha_0 + \alpha_a a_z + \alpha_b b_y, \\
\beta &= \beta_0 + \beta_a a_z + \beta_b b_y, \\
\gamma &= \gamma_0 + \gamma_a a_z + \gamma_b b_y,
\end{align*}
$$

(35)

where we have denoted,

$$
\begin{align*}
\alpha_a &= (\partial_a \alpha)|_{a_z, b_y=0}, & \alpha_b &= (\partial_b \alpha)|_{a_z, b_y=0}, \\
\beta_a &= (\partial_a \beta)|_{a_z, b_y=0}, & \beta_b &= (\partial_b \beta)|_{a_z, b_y=0}, \\
\gamma_a &= (\partial_a \gamma)|_{a_z, b_y=0}, & \gamma_b &= (\partial_b \gamma)|_{a_z, b_y=0}.
\end{align*}
$$

(36)

In order to continue, we need to expand the following expression

$$
g(a_z, b_y) = \frac{1}{1 - \frac{1}{b} \{(E_0 + a_z)^2 - (B_0 + b_y)^2\}},
$$

(37)

into a Taylor series in two variables $a_z, b_y$ around the point $(a_z, b_y = 0)$, using this expansion, the parameters $\alpha, \beta$ and $\gamma$, then become

$$
\begin{align*}
\alpha &= 1 + \frac{(E_0 + a_z)^2}{b^2} g(a_z, b_y), \\
\beta &= \frac{1}{b^2} (E_0 + a_z)(B_0 + b_y) g(a_z, b_y), \\
\gamma &= 1 - \frac{(B_0 + b_y)^2}{b^2} g(a_z, b_y).
\end{align*}
$$

(38)

We perform the Taylor series for the special choice, $B_0 = E_0$, of the crossed field. The expansion then becomes

$$
g(a_z, b_y) \approx 1 + \frac{2E_0}{b^2} (a_z - b_y)
$$

$$
+ \frac{1}{b^4} \left[ (4E_0^2 - b^2)a_z^2 - 8E_0^2a_z b_y + (4E_0^2 + b^2)b_y^2 \right].
$$

(39)

In the following text, we will use just the first two linear terms of (39) for the linearization. We identify the coefficients $\alpha_{az}, \beta_{az}, \gamma_{az}$ and $\alpha_{by}, \beta_{by}, \gamma_{by}$, in the general formulae of $\alpha, \beta, \gamma$ (40), for the special choice, $B_0 = E_0$, of the crossed field.

The coefficients (40) have final form,

$$
\begin{align*}
\alpha_{az} &= \frac{2E_0}{b^2} \left( 1 + \frac{E_0^2}{b^2} \right), & \alpha_{by} &= -\frac{2E_0^3}{b^4}, \\
\beta_{az} &= \frac{E_0}{b^2} \left( 1 + \frac{2E_0^2}{b^2} \right), & \beta_{by} &= \frac{E_0}{b^2} \left( 1 - \frac{2E_0^2}{b^2} \right), \\
\gamma_{az} &= -\frac{2E_0^3}{b^4}, & \gamma_{by} &= \frac{2E_0}{b^2} \left( \frac{E_0^2}{b^2} - 1 \right).
\end{align*}
$$

(40)

We observe that the calculation is quite similar to the one performed in the Heisenberg–Euler approximation of QED theory in [2]. We have obtained the set of two differential equations (20) [21]. After adding the small amplitude perturbations (22), we obtained the same set of field equations (23, 24) with different coefficients $\alpha, \beta$ and $\gamma$ (25). The information from the Born Lagrangian (11) is hidden in these coefficients. Then when we have linearized the coefficient $\alpha, \beta$ and $\gamma$, the information from the Lagrangian propagates also into coefficients $\alpha_{az}, \alpha_{by}, \beta_{az}, \beta_{by}$ and $\gamma_{az}, \gamma_{by}$, (41), which we will use in the next Section.

V. BORN SELF–SIMILAR SOLUTIONS

In this Section, we will solve the field equations, we consider the field equations (23, 24) with parameter functions $\alpha(a_z, b_y), \beta(a_z, b_y)$ and $\gamma(a_z, b_y)$ in the linear approximation (33). As in [2], we approach solving the non–linear equations using Riemann wave (simple wave) well known in nonlinear wave theory [39, 40, 52]. The procedure is almost similar to that one developed in [2], it is thanks to the similar structure of field equations. The different form of the Born Lagrangian (11) comes in the form of the coefficients $\alpha_{az}, \alpha_{by}, \beta_{az}, \beta_{by}$ and $\gamma_{az}, \gamma_{by}$, (40).

A. Self–similar solutions

In other words, we are assuming the dependence

$$
b_y = b_y(a_z), \quad \partial_t b_y = (\partial b_y/\partial a_z) \partial_t a_z, \quad \text{and} \quad \partial_x b_y = (\partial b_y/\partial a_z) \partial_x a_z.
$$

The field Eqs. (23, 24) then become

$$
\partial_t a_z = \frac{d a_z}{d b_y} \partial_x a_z,
$$

(41)

$$
\partial_t a_z = \frac{1}{\alpha} \left( 2\beta + \gamma \frac{d b_y}{d a_z} \right) \partial_x a_z.
$$

(42)

When we compare the two above equations, we get a quadratic equation for function $b_y(a_z)$ in a form,

$$
\gamma \left( \frac{d b_y}{d a_z} \right)^2 + 2\beta \frac{d b_y}{d a_z} - \alpha = 0,
$$

(43)

which has two unique solutions

$$
\left( \frac{d b_y}{d a_z} \right) = -\beta \pm \sqrt{\beta^2 + \alpha\gamma}.
$$

(44)

Furthermore, we use the weak and finite amplitude approximation, when we assume the solution in a form

$$
\left( \frac{d b_y}{d a_z} \right) = \nu,
$$

(45)

where we assume $\nu$ in the linearized form as the other previous parameters $\alpha, \beta$ and $\gamma$ Eq. (55),

$$
\nu = \nu_0 + \nu_a a_z + \nu_b b_y,
$$

(46)
with new parameters $\nu_0$, $\nu_a$, and $\nu_b$, which we will derive explicitly later. For the two solutions for $db_y/da_z$ \[44\], we will obtain two sets of parameters $\nu_0$, $\nu_a$, and $\nu_b$. We will discuss the two of them in the next Section where we investigate the wave steepening of the separate solutions.

In the following calculation, we use the definition of tangent to a surface at a point $(\alpha_0, \beta_0, \gamma_0)$ as

$$f(\alpha, \beta, \gamma) = f(\alpha_0, \beta_0, \gamma_0) + \partial f|_{\alpha_0, \beta_0, \gamma_0}(\alpha - \alpha_0) + \partial f|_{\alpha_0, \beta_0, \gamma_0} (\beta - \beta_0) + \partial f|_{\alpha_0, \beta_0, \gamma_0} (\gamma - \gamma_0),$$

(47)

where $db_y/da_z = f(\alpha, \beta, \gamma)$.

We obtain the resulting coefficients as

$$\nu_0 = f|_{\alpha_0, \beta_0, \gamma_0} = -\frac{\beta_0 + 1}{\gamma_0},$$

(48)

and

$$\partial f|_{\alpha_0, \beta_0, \gamma_0} = \pm \frac{1}{2},$$

(49)

$$\partial f|_{\alpha_0, \beta_0, \gamma_0} = \frac{1}{\gamma_0} (-1 \pm \beta_0),$$

(50)

$$\partial f|_{\alpha_0, \beta_0, \gamma_0} = \pm \frac{\alpha_0}{2\gamma_0} \left(-\frac{\beta_0 + 1}{\gamma_0}\right),$$

(51)

where we can rewrite the expressions \[35\] as

$$\alpha - \alpha_0 = \alpha_a \alpha + \alpha_b \beta,$$

$$\beta - \beta_0 = \beta_a \alpha + \beta_b \beta,$$

$$\gamma - \gamma_0 = \gamma_a \alpha + \gamma_b \beta,$$

and we have used the relation

$$\beta_0^2 + \alpha_0 \gamma_0 = 1.$$}

The linear coefficients $\nu_0$, $\nu_a$, and $\nu_b$ then have a final form,

$$\nu_0 = f|_{\alpha_0, \beta_0, \gamma_0},$$

$$\nu_a = \alpha \alpha_a \alpha + \alpha \beta_a \beta + \alpha \gamma_a \gamma,$$

$$\nu_b = \alpha \beta_a \beta + \beta \beta_a \beta + \beta \gamma_a \gamma,$$

where we denoted the derivatives $f_\alpha = \partial f|_{\alpha_0, \beta_0, \gamma_0}$, $f_\beta = \partial f|_{\alpha_0, \beta_0, \gamma_0}$, $f_\gamma = \partial f|_{\alpha_0, \beta_0, \gamma_0}$.

The explicit expressions for $f_\alpha$, $f_\beta$, $f_\gamma$, by using expressions \[48\], \[49\], \[50\] and \[51\], have a form,

$$f_\alpha = \pm \frac{1}{2},$$

$$f_\beta = \frac{1}{\gamma_0} (-1 \pm \beta_0),$$

$$f_\gamma = \pm \frac{\alpha_0}{2\gamma_0} \left(-\frac{\beta_0 + 1}{\gamma_0}\right).$$

The problem reduces to finding a solution to the differential equation \[44\]. The equation has a form of total differential, therefore it can be solved by the method of integration factor, which we choose as $m(a_z) = \exp(-\nu_b a_z)$.

The dependence $b_y = b_y(a_z)$, which solves the equation has a structure,

$$\frac{1}{\nu_b} \exp(-\nu_b a_z) \left(\nu_0 + \nu_a b_0 + \frac{\nu_a}{\nu_b} (\nu_b a_z + 1)\right) = \delta,$$

(56)

where $\delta$ is arbitrary constant. We can rewrite it and get the function $b_y = b_y(a_z)$ in an explicit form,

$$b_y = \delta \exp(\nu_b a_z) - \frac{\nu_a}{\nu_b} (\nu_b a_z + 1) - \frac{\nu_0}{\nu_b}.$$  

(57)

We determine the constant $\delta$ thanks to the initial condition $b_y|_{a_z = 0} = 0$,

$$\delta = \frac{\nu_a + \nu_0 \nu_b}{\nu_b^2}.$$  

(58)

In order to use and stay in the weak amplitude approximation, we will perform Taylor expansion of the first term in \[57\] to the first order,

$$\exp(\nu_b a_z) \approx 1 + \nu_b a_z + ...,$$

(59)

it produces simplified first term in \[57\], as

$$b_y = \delta (\nu_0 a_z + 1) - \frac{\nu_a}{\nu_b} (\nu_0 a_z + 1) - \frac{\nu_0}{\nu_b}.$$  

(60)

In order to simplify the expression for $b_y$ \[60\] even more, we substitute \[58\] into \[60\], and we obtain a simplified solution which shows a linear relation between $a_z$ and $b_y$,

$$b_y = \nu_0 a_z.$$  

(61)

Let’s get back to the field equations \[41\] and \[42\] which we aimed to solve. It is more convenient to use the first Eq. \[41\] in the set, we can rewrite it as

$$\partial a_z - \frac{1}{\nu} \partial x a_z = 0,$$

(62)

where it is denoted $\nu = \nu_0 + \nu_a a_z + \nu_b b_y$. \[46\]

In order to continue, we perform another linearization of a factor $1/\nu$ as

$$f(\nu) = f(\nu)|_{\nu_0} + \partial f|_{\nu_0}(\nu - \nu_0),$$

(63)

and subsequently we obtain

$$\frac{1}{\nu} = \frac{1}{\nu_0} \left(1 - a_z \frac{\nu_a}{\nu_0} + \nu b \nu_b \nu_b\right).$$  

(64)

B. The final form of the nonlinear wave

Then we can write Eq. \[62\] with the factor $1/\nu \ [64]$ in the final form,

$$\partial a_z + f(a_z) \partial x a_z = 0,$$

(65)
with the factor \( f(a_z) \),

\[
f(a_z) = -\frac{1}{\nu_0} \left[ 1 - a_z \left( \frac{\nu_{a_z} + \nu_0 b_y}{\nu_0} \right) \right].
\] (66)

This is the final form of the equation which we will work with and analyze it further. In the limit \( a_z = 0 \), the wave moves with the phase velocity of the unperturbed case \(-1/\nu_0\). It contains the two possible solutions for \( db_y/da_z \), the case \(-\) for counter–propagating waves and the case \(+\) for co–propagating waves. We have obtained two different sets of parameters \( \nu_0, \nu_{a_z} \) and \( b_y \) for them. In general, this form of equation contains the information whether the shock waves are being created, the wave steepening takes place and high–order harmonics are being generated. The two possible resulting wave equations have similar structure and the properties of the waves are hidden in the two sets of parameters \( \nu_0, \nu_{a_z} \) and \( b_y \) for the \(+\) and \(-\) solutions. We will discuss the two branches of solutions in the next Section VI where we investigate the wave steepening of the two possible solutions in more detail.

VI. PROPERTIES OF BORN SELF–SIMILAR SOLUTIONS

In this Section, we analyze the properties of the equation (65). The equation can be analyzed by the method of characteristics, therefore we will just shortly review the method here and also review the wave breaking. Furthermore, we will concentrate on analyzing the properties of the non-linear electromagnetic wave in more detail.

A. Method of characteristics and the wave breaking

We can solve the equation (65) by the method of characteristics. The characteristic equations for the Eq. (65) are

\[
\frac{dx}{dt} = f(a_z), \quad \frac{da_z}{dt} = 0.
\] (67)

Their solutions are \( a_z(x, t) = A_0(x_0) \) and \( x = f(A_0(x_0))t + x_0 \), where the function \( a_z(x, t) \) transfers along the characteristic \( x_0 \) without any distortion. Therefore for any differentiable function \( A = A(x) \), we can write solution \( a_z \) in a form

\[
a_z(x, t) = A_0(x_0) = A_0[x - f(a_z(x, t))t],
\] (68)

where \( A_0 \) is an arbitrary function determined by the initial condition, \( a_z(x)|_{t=0} = A_0(x) \).

The wave breaking is typical behavior of waves in non-linear dispersionless media. We can write the solution of equation (65) in an implicit form (65) with the Euler coordinate \( x \) dependent on the Lagrange coordinate \( x_0 \) and time \( t \). The location where the wave breaks is determined by the gradient of function \( a_z(x, t) \), the wave breaks when gradient becomes infinite, \( 52 \). We obtain such result by deriving (68), as

\[
\partial_t a_z = \frac{A_0'(x_0)}{1 + A_0'(x_0)f'(t)}
\] (69)

\[
t_{br} = -\frac{1}{A_0'(x_0)f'},
\] (70)

where it is denoted

\[
A'(x_0) = \frac{dA_0}{dx_0},
\] (71)

\[
f' = \partial_{a_z} f(a_z).
\] (72)

The gradient becomes infinite at time \( t_{br} \) when the denominator of (69) vanishes at some point \( x_{br} \). At the time \( t_{br} \) when the wave breaks the amplitude, \( a_z(x_{br}, t_{br}) = a_m \sin [k(x_{br} - f(a_z(x_{br}, t_{br})))] \), remains constant. Such singularity is called the wave breaking or the gradient catastrophe.

B. The character of the breaking wave for the counter–propagating waves: the \(-\) solutions in Born theory

In this Subsection, we will concentrate on the \(-\) solutions of Eq. (44). The \(-\) solutions correspond to the case of counter–propagating waves where the waves interact with each other and the photon–photon scattering process takes place. Thanks to the scattering the resulting phase velocity decreases and is less than speed of light \( c \). We identify the phase velocity as the phase velocity \( v_2 \) \cite{44}. We can also relate the parameter \( \nu_0^\prime \) with a phase velocity \( v_2 \) as

\[
\nu_0^\prime = -\frac{1}{v_2},
\] (73)

where \( v_2 > 0 \) it always positive.

We can rewrite \( f(a_z) \) in (65) by using the explicit expression for \( \nu_0^\prime \)\cite{43},

\[
f^\prime (a_z) = v_2 + a_z \left( \frac{\nu_{a_z}^\prime + \nu_0^\prime b_y^\prime}{\nu_0^\prime} \right).
\] (74)

The final equation (65) can be rewritten in the standard form corresponding to the equation of non-linear wave without dispersion, \( 39 \) \( 40 \),

\[
\partial_t a_z + (v_2 + a_z)\partial_x a_z = 0,
\] (75)

where we have denoted

\[
a_z = \frac{(\nu_{a_z}^\prime + \nu_0^\prime b_y^\prime)}{\nu_0^\prime} a_z.
\] (76)
Therefore the direction of the wave breaking is given by the sign in front of the function $f^-$ (72). In order to investigate the wave steepening, we analyze the expression (70) which we can rewrite using (72) as

$$a_z = f'(a), \quad f' = \frac{(v'^2_a + \nu_a \nu_b^e)}{v'^2_0}. \quad (77)$$

After performing the substitution $\alpha_0$, $\beta_0$ and $\gamma_0$ [22] into $f_\alpha$, $f_\beta$, $f_\gamma$, we observe that it is convenient to express the functions $f_\alpha$, $f_\beta$, $f_\gamma$ in terms of the phase velocity $v_2$. The explicit expressions then have a form,

$$f_\alpha = -\frac{1}{2}, \quad f_\beta = -\frac{1}{v_2}, \quad f_\gamma = -\frac{1}{2} \frac{1}{v_2^2}. \quad (78)$$

Then the coefficients $\nu_\alpha, \nu_\beta$ [53] become

$$\nu^-_\alpha = -\frac{E_0}{b^2 v^2_2} \left[ \left( 1 + \frac{E^2_2}{b^2} \right) v^2_2 + \left( 1 + \frac{2 E^2_2}{b^2} \right) v_2 + \frac{E^2_0}{b^2} \right],$$

$$\nu^-_\beta = \frac{E_0}{b^2 v^2_2} \left[ \frac{E^2_2}{b^2} v^2_2 - \left( 1 - \frac{2 E^2_2}{b^2} \right) v_2 + \left( \frac{E^2_0}{b^2} - 1 \right) \right]. \quad (79)$$

The function $f'^-$ (74) has a form,

$$f'^- = -\frac{E_0}{b^2} \left[ -\left( 1 + \frac{E^2_2}{b^2} \right) v^2_2 - \left( 1 + \frac{3 E^2_2}{b^2} \right) v_2 \right. \right.$$  

$$\left. \left. + \left( 1 - \frac{3 E^2_2}{b^2} \right) - \frac{E^2_0}{b^2} \right] \right] \quad (80)$$

We obtained the steepening factor $f'^-$ (80) in the general form, it is now expressed in terms of the phase velocity $v_2$ [34] and the $b$ is the Born–Infeld constant. In the case when the singularity is formed, the electromagnetic wave breaking forms a shock wave, which has a forward character for $f'^- > 0$ and rarefaction character for $f'^- < 0$. There is also a possibility that the factor $f' = 0$, then the shock waves are not created and just exceptional waves are the solutions of the equations, [23, 54].

In the limit $b \to \infty$, which leads to the linear Maxwell theory, the steepening factor $f'^- \to 0$ and the phase velocity $v \to 1$. This corresponds to the fact that the wave steepening does not happen in classical Maxwell theory. Subsequently, the resulting non-linear wave equation (76) with $f^-(a_z)$ (74) become, in the limit to the Maxwell theory, as

$$\partial_t a_z + v_2 \partial_x a_z = 0, \quad (81)$$

where

$$f^-(a_z)|_{b \to \infty} = v_2. \quad (82)$$

Now, we continue in the Born theory again. After we substitute the phase velocity $v_2$ into (79) and (80), we will obtain the coefficients $\nu_\alpha, \nu_\beta, \gamma_\alpha$ [29],

$$\nu^-_\alpha = -\frac{2 E_0}{b^2} \left[ \frac{E^2_0}{b^2} \right],$$

$$\nu^-_\beta = -\frac{2 E_0}{b^2} \left[ \frac{1}{b^2} \right].$$

and importantly the steepening factor becomes,

$$f'^- = 0. \quad (83)$$

This means that the only solutions for this case are exceptional waves, the exceptional traveling wave solutions which propagate with constant speed and which do not turn into shocks, [30]. The existence of just exceptional waves is in full correspondence with the known literature [23, 28, 54] and [52].

Lastly, lets have a look at the shock wave steepening analytically. It does not take place, we can show it explicitly. We obtain the gradient (69) and the time of wave breaking (70) as

$$\partial_x a_z = A'_0(x_0), \quad (84)$$

$$t_{br} = -\infty. \quad (85)$$

As mentioned earlier, the wave breaks when its gradient becomes infinite, the gradient (84) for the exceptional waves will never become infinite and the time of wave breaking is $-\infty$, therefore the wave will never break.

The final form of the non-linear wave equation for the counter–propagating case + has the following form,

$$\partial_t a_z + v_2 \partial_x a_z = 0, \quad (86)$$

and its solution (88),

$$a_z(x,t) = A_0(x_0) = A_0[x - v_2 t], \quad (87)$$

which propagates to the right hand side with the constant phase velocity $v_2$ along the increasing $x$-axes.

This exceptional wave is the real contribution to the outgoing radiation from the photon–photon scattering in the Born electrodynamics. Interestingly, the exceptional wave can form a singularity which does not develop into a shock wave front.

C. The character of the breaking wave for the co–propagating waves: the + solutions in Born theory

The + solutions correspond to the co–propagating waves case where the waves do not interacting with each other and the photon–photon scattering is not happening. The phase velocity of the resulting wave is therefore
unphysical and has a value $v_1 = -1$. The parameter $\nu_0^+$ has the same value as the phase velocity $v_1$,

$$\nu_0^+ = v_1 = -1,$$  \hfill (88)

We can rewrite the function $f(a_z)$ \(\text{(60)}\) as

$$f^+(a_z) = f_0^+ + a_z(x,t)f'^+,$$  \hfill (89)

where

$$f_0^+ = -\frac{1}{\nu_0^+} = 1, \quad f'^+ = \frac{\nu_0^+ + \nu_0^- \nu_{b_0}^+}{\nu_0^2}.$$  \hfill (90)

Again we substitute $\alpha_0$, $\beta_0$ and $\gamma_0$ \(\text{[27]}\) into $f_\alpha$, $f_\beta$, $f_\gamma$ \(\text{[53]}\) for the + case using again $v_2$, we obtain

$$f_\alpha = \frac{1}{2}, \quad f_\beta = -1, \quad f_\gamma = \frac{1}{2v_2} - \frac{1}{1 - \frac{E_0^2}{b^2}}.$$  \hfill (91)

The coefficients $\nu_\alpha, \nu_\beta$ \(\text{[54]}\) become

$$\nu_\alpha^+ = \frac{E_0^3}{b^3 v_2} \left[ -1 - v + \frac{2v_2}{1 - \frac{E_0^2}{b^2}} \right],$$

$$\nu_\beta^+ = \frac{E_0}{b^2 v_2} \left[ \left( \frac{E_0^2}{b^2} + 1 \right) v_2 + \frac{E_0^2}{b^2} - 1 \right].$$  \hfill (92)

The function $f'^+$ \(\text{[53]}\), expressed in the phase velocity, has a form,

$$f'^+ = \frac{E_0}{b^2} \left[ 1 - \frac{E_0^2}{b^2} + v_2 \left( -1 - \frac{2E_0^2}{b^2} + \frac{E_0^2}{b^2} \right) \right].$$  \hfill (93)

The general form of the steepening factor $f'^+$ \(\text{[53]}\) is now expressed in the phase velocity $v_1$ \(\text{[34]}\) and the $b$ is the Born–Infeld constant.

In the limit $b \to \infty$, which leads to the linear Maxwell theory, the steepening factor $f'^+ \to 0$ and the phase velocity $v_1 \to 1$. The resulting equation becomes

$$\partial_t a_z - \frac{1}{v_1} \partial_x a_z = 0,$$  \hfill (94)

where

$$f^+(a_z)|_{b \to \infty} = -\frac{1}{v_1}.\hfill (95)\]$$

Now, we continue in the Born theory. After using the phase velocity $v_1$, the coefficients \(\text{[92]}\) and \(\text{[93]}\) reduce to

$$\nu_\alpha^+ = 0, \quad \nu_\beta^+ = 0,$$  \hfill (96)

and again $f'^+ \text{[90]}$ importantly,

$$f'^+ = 0.$$  \hfill (97)

This means again, that there is no shock wave steepening and just exceptional waves are created. The shock wave steepening should not take place in this case because the co–propagating waves do no interact and the photon–photon scattering is not happening.

The final form of the non-linear wave equation for the co–propagating case + has the following form,

$$\partial_t a_z + \partial_x a_z = 0,$$  \hfill (98)

and its solution,

$$a_z(x,t) = A_0(x_0) = A_0[x - t],$$  \hfill (99)

which propagates to the left hand side with the constant phase velocity $v_1 = -1$ along the x-axes. The analytical expressions for the shock wave steepening are the same as for the previous case --, \(\text{[33]}\) and \(\text{[53]}\), the steepening is not happening for the exceptional waves.

This exceptional wave is not real physical contribution to the outgoing radiation from the photon–photon scattering in the Born electrodynamics. The photon–photon scattering is not occurring in the case of two co–propagating waves.

D. The phase shift

By linearizing the Eqs. \(\text{[23–24]}\), it is easy to find the expressions describing the small amplitude wave shift $b_y - a_z$ for which we have obtained,

$$u(x,t) = u_0 \left[ x \left( 1 + \frac{E_0^2}{b^2} \right) - t \left( 1 - \frac{E_0^2}{b^2} \right) \right],$$  \hfill (100)

where we have introduced the light cone coordinates $x_+, x_-$ used in \(\text{[1]}\), the fields can be expressed in variables $u, v$ as

$$x_+ = \frac{1}{\sqrt{2}} (x + t), \quad x_- = \frac{1}{\sqrt{2}} (x - t),$$  \hfill (101)

$$u = \frac{1}{\sqrt{2}} (b_y - a_z), \quad w = \frac{1}{\sqrt{2}} (a_z + b_y).$$  \hfill (102)

Then using the field Eq. \(\text{[23]}\) rewritten in light cone coordinates as $\partial_{x_-} w = \partial_{x_+} u$, to derive the difference $b_y + a_z$,

$$w(x,t) = -\frac{E_0}{b^2} u_0 \left[ x \left( 1 + \frac{E_0^2}{b^2} \right) - t \left( 1 - \frac{E_0^2}{b^2} \right) \right] + w_0(x+t).$$  \hfill (103)

In Eqs. \(\text{[100–103]}\) the functions $u_0$ and $w_0$ are determined by the initial conditions. The function $u(x,t)$ depends on the combination of coordinates $(x, t)$, and then the phase shift $\psi(x,t)$ is then given by,

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left[ x \left( 1 + \frac{E_0^2}{b^2} \right) - t \left( 1 - \frac{E_0^2}{b^2} \right) \right],$$  \hfill (104)
which we can rewrite as

$$\psi(x,t) = \frac{1 + \frac{E_0^2}{b^2}}{\sqrt{2}} \left[ x - t \frac{1 - \frac{E_0^2}{b^2}}{1 + \frac{E_0^2}{b^2}} \right],$$  \hspace{1cm} (105)

where in the brackets we can find the phase \(\phi\),

$$\phi(x,t) = x - t \frac{1 - \frac{E_0^2}{b^2}}{1 + \frac{E_0^2}{b^2}}.$$  \hspace{1cm} (106)

The phase should not change in time,

$$\frac{d\phi}{dt} = 0,$$  \hspace{1cm} (107)
and we obtain

$$\frac{d\phi}{dt} = \frac{1 - \frac{E_0^2}{b^2}}{1 + \frac{E_0^2}{b^2}}.$$  \hspace{1cm} (108)

The constant phase condition shows that the wave propagates from the left to the right with the speed

$$v_W = \frac{1 - \frac{E_0^2}{b^2}}{1 + \frac{E_0^2}{b^2}} \approx 1 - 2 \frac{E_0^2}{b^2} + 2 \frac{E_0^4}{b^4}.$$  \hspace{1cm} (109)

It is less than unity, i.e., the phase (group) velocity is less than the speed of light in a vacuum. Measuring the phase difference between the phase of the electromagnetic pulse colliding with the counter-propagating wave and the phase of the pulse which does not interact with high intensity wave, it is equal to

$$\Delta \psi = 4\pi \frac{d}{\lambda} \frac{1}{b^2} \frac{E_0^2}{b^2},$$  \hspace{1cm} (110)

where \(\lambda\) is the wavelength of high frequency pulse and \(d\) is the interaction length, plays a central role in discussion of experimental verification of the QED vacuum birefringence \cite{53,54}. For 10 PW laser the radiation intensity can reach \(10^{24}\) W/cm\(^2\), for which \(E_0^2 \approx 10^{-5}\). Taking the ratio equal to \(d/\lambda \approx 10^4\), i.e. equal to the ratio between the optical and x-ray radiation wavelength, and using for \(1/b^2\) estimate that \(1/b^2 = 8\kappa = 2\kappa_2 \approx 10^{-4}\), (see \cite{1}, \cite{57} and \cite{58}), and the expression (110), we find that \(\psi \approx 10^{-4}\).

### E. The cross section

The motivation of this paper was the deeper understanding of the photon–photon scattering in the Born–Infeld electrodynamics. It is important because the photon–photon scattering also contributes to its effective cross section and we have to include it into discussion when proposing an experiment with PW lasers. The phase shift and the cross section of the process could be measured together in our proposed experiment for the direct detection of the photon–photon scattering \cite{1,2} at the same time.

The cross section for low energy photon–photon scattering in BI and QED (unpolarized initial states with summation over final polarizations), \cite{50,59}, becomes

$$\sigma_{\gamma\gamma} = \left( \frac{1}{64b^4} + \frac{11\alpha^2}{720b^2m^4} + \frac{139\alpha^4}{32400m^8} \right) \frac{\omega^6}{\pi^2} \left( 3 + \cos^2 \theta \right)^2,$$  \hspace{1cm} (111)

where the expression depends on the scattering angle \(\theta\) and the photon frequency \(\omega\), the \(m\) and \(\alpha\) and the total cross section

$$\sigma_{\gamma\gamma}^\text{tot} = \left( \frac{7}{20b^4} + \frac{77\alpha^2}{225b^2m^4} + \frac{973\alpha^4}{10125m^8} \right) \frac{\omega^6}{\pi}.$$  \hspace{1cm} (112)

The additional terms with the free Born–Infeld parameter \(b\) in the formulae signify the additive character of the photon–photon process in the Born–Infeld electrodynamics and can be seen as a contribution from the beyond standard model (BSM) particles.

We have discussed the subsequent production of electron–positron pairs in the photon–photon scattering process in the Heisenberg–Euler electrodynamics in the low energy photon approximation \(\omega \ll m\) in \cite{1,2} at the shock wave fronts where the approximation is no longer valid. Therefore our results are limited to this low energy regime and will lose validity if we approach the Swinger limit \(E_S\).

When the low energy photon approximation breaks in QED, the photon–photon interaction can result in creation of real electron–positron pairs via Breit–Wheeler process \cite{60} thanks to the saturation of wave steepening and the electromagnetic shock wave formation. Reaching the energies for the electron–positron generation requires much less intense laser intensities than reaching Schwinger field \(E_S\) and can be achieved in the near future at ELI.

Similarly to the cross section formulae (111), we can expect that the cross section \(\sigma_{\gamma\gamma}^{\text{influence}}\) will include additive terms with the parameter \(b\). Therefore we might expect a contribution to the electron–positron pair production from the BSM physics, in our case from the Born–Infeld part, and contribution to the subsequent emission of the gamma-ray photons leading to the electron–positron avalanche thanks to the multiphoton Breit–Wheeler mechanism \cite{61}.

In our previous papers \cite{1,2}, we were proposing experiment to directly detect the photon–photon scattering by detecting the gamma-rays coming from the electron–positron pair formation. In this experiment, we might be able to study also the Born–Infeld (BSM) contribution to the process.
VII. ON EXPERIMENTAL DISTINGUISHING OF BORN–INFELD AND HEISENBERG–EULER THEORIES

There is an interest in experimental research to test QED and non–standard models like Born–Infeld theory and scenarios where mini charged particles are involved or axion–like bosons [17]. For example, the PVLAS experiment [18] has obtained limits on the parameter $b$ in the Born–Infeld and to the existence axion-like and millicharged particles next to setting upper limits on the magnetic birefringence predicted by QED.

It’s better to discuss the results with respect to the effective Lagrangian formed by the two theories Born–Infeld and Heisenberg–Euler Lagrangian because the quantized version of the Born–Infeld theory is missing and its hard to predict any connection to the real world besides the connection to the string theory which does not help in this context.

Part of the testing of QED is also the possible distinguishing of the two theories Born–Infeld (and other non–standard models together with scenarios involving minicharged particles or axion–like bosons) and the Heisenberg–Euler by precision test experiments. The effective Lagrangian is defined as

$$\mathcal{L}_{\text{eff}} \simeq -\mathcal{F} + \zeta_L \mathcal{F}^2 + \zeta_T \mathcal{G}^2,$$  \hspace{1cm} (113)

where the constants are parameters \(\zeta_L\) corresponding to the QED theory and \(\zeta_T\) to the Born–Infeld theory.

In order to distinguish the two theories we have to measure the two parameters as independent parameters. The previous experiments, [18], were sensitive only to a difference \(4\zeta_T - 7\zeta_L\) and therefore unable to set constraint on a pure Born–Infeld theory. In the Born theory we have \(\mathcal{G}^2 = 0\), therefore the parameter for the Born–Infeld \(\zeta_T\) can not be assumed and measured. To consider the full effective Lagrangian, we have to generalize our results to include \(\mathcal{G}^2 \neq 0\).

The search for photon–photon scattering in vacuum can be done by measuring phase shifts and ellipticities and also can be used to determine both the coefficients \(\zeta_L\) and \(\zeta_T\), in two counter–propagating waves which one of them represented an ultra high power beam. It will be possible to determine the precision estimates for \(\zeta_T\) and \(\zeta_L\), and estimate the QED parameter \(\kappa\) and the Born–Infeld free parameter \(b\) - upper and lower bounds.

To summarize the Section, the complete test of all the parameters appearing in the low energy effective Lagrangian could be done newly including the parameter for the Born–Infeld term thanks to the availability of the PW class lasers. The experiments can be performed at HERCULES [64, 65], laser ZEUS [66], at laser LUXE [67] at DESY, at ELI facility [68] or on 100 PW laser at SIOM [27] in the future, thus providing a new class of precision tests of the Standard Model and beyond.

VIII. CONCLUSION

In conclusion, we have investigated the two counter–propagating waves in another important framework of Born–Infeld electrodynamics. The study was motivated by our previous work on photon–photon scattering in vacuum, [11, 2, 41], in the Heisenberg–Euler theory. Previously, we have assumed the cross field case of the ordinary wave in the birefringence problem in the quantum vacuum, i. e. problem describing the finite amplitude electromagnetic wave counter-propagating to the crossed electromagnetic field.

For the crossed field ansatz \(E \cdot B = 0\), i.e. \(\mathcal{G}^2 = 0\), the Born–Infeld Lagrangian reduces to the Born Lagrangian and it is its special subcase. We have investigated the field equations of Born Lagrangian, which are identical to the equations for the Born–Infeld Lagrangian for the crossed field condition \(E \cdot B = 0\). For simplicity we used the name Born in relation to calculations. In general, the term \(\mathcal{G}^2\) can be neglected in situations where we are far away from singularities, [43], in our case, far away from the creation of shock wave fronts.

Let us mention that there is a similarity with exact solutions of Einstein equations called gyratons [69, 71] and [72, 73] which describe a gravitational field of spinning beam (light) whose metric terms \(g_{ui}\) can be set to zero locally using gauge transformation but they cannot be globally removed because the gauge invariant contour \(\oint g_{ui}(u, x_i) du^i\) around the position of the gyron (singularity) is proportional to the nonzero angular momentum density \(j_i\), which is nonvanishing.

We have solved the Born field equations analytically assuming the solution in a form of a simple wave, we added the small amplitude perturbations and linearized the coefficients to study the singularity formation. We have showed that the system of equations decoupled for the ordinary wave case. The solutions have a form of a non–linear wave without dispersion in the linear approximation.

We concentrated on analyzing the solutions in the Born theory for the + and – solutions corresponding to the counter–propagating waves in the case – and co–propagating in the case +. We have analyzed the wave breaking in detail giving the analytical formule for both cases + and – in question. In the case –, for the counter–propagating waves, the waves interact with each other and thus the photon–photon scattering process takes place. The wave steepening factor reduces to zero, \(f'' = 0\), therefore only exceptional waves are the solutions for – case. In the case +, the co–propagating waves do not interact with each other and no photon–photon scattering occurs, then we obtained trivial solution \(f'' = 0\) with coefficients \(\nu_{+} = \nu_{-} = 0\). We have showed explicitly that the only solutions for both cases are exceptional waves, the exceptional traveling wave solutions which propagate with constant speed and which do not turn into shocks, [25, 28, 51] and [52]. The existence of just exceptional waves is in full correspondence...
with the known literature. We have discussed the analytical details for the wave steepening for exceptional waves which never happens. In comparison to the Heisenberg–Euler approximation \cite{2}, where the shock wave development takes place and has rarefaction character, the shock wave development does not occur in the Born electrodynamics for our problem of two counter–propagating laser beams.

We have also presented analytical estimate for the measurement of the phase shift for the resulting electromagnetic wave in the Born theory for PW laser facilities such as ELI Beamlines.

This work serves as a preparatory study to the problem in the more general case in the Born–Infeld electrodynamics. This is important step forward because the photon–photon scattering in the Born–Infeld electrodynamics contributes also to the effective cross section of the photon–photon scattering.

The cross section for the low energy photon–photon scattering in Born–Infeld adds two terms with the parameter \(b\) to the standard cross section of the process in the Heisenberg-Euler approximation, therefore we might measure a contribution of the BSM physics in an experiment. The contribution will be present also in the subsequent production of the electron–positron pairs and gamma ray photons. By our recent proposal of the direct detection of the photon–photon scattering we might be able to study the contribution from the BSM physics represented here by the Born–Infeld electrodynamics, distinguish the Born–Infeld and Heisenberg–Euler theories, and also get new experimental estimates for the parameters in QED and other non-standard models and their precision tests.

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