Naked Singularities in Four-dimensional String Backgrounds

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Abstract

It is shown that gauged nonlinear sigma models can be always deformed by terms proportional to the field strength of the gauge fields (nonminimal gauging). These deformations can be interpreted as perturbations, by marginal operators, of conformal coset models. When applied to the $SL(2,R) \times SU(2)/(U(1) \times U(1))$ WZNW model, a large class of four-dimensional curved spacetime backgrounds are obtained. In particular, a naked singularity may form at a time when the volume of the universe is different from zero.

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1 Introduction

It is well-established by now that string propagation on curved spacetimes is described by a two-dimensional nonlinear sigma model. The conformal invariance conditions (the equations for the vanishing of the beta functions) for the sigma model are then interpreted as defining consistent backgrounds for string propagation [1,2]. These equations, however, are in general not easy to solve, especially when they include higher order terms in perturbation theory. One of the few techniques for finding these backgrounds is provided by non-compact gauged WZWN models (non-compact cosets) [3]. This is mainly because WZWN models, which are two-dimensional sigma models, are conformally invariant to all orders in perturbation theory [4,5].

The programme for constructing curved spacetimes for string theories received a tremendous boost when a two-dimensional black hole was discovered in the $SL(2, R)/U(1)$ coset model [6] and as solution to the string beta functions [7,8]. Since then many other papers have used gauged WZWN models to give one-loop solutions to the conformal invariance conditions [9-26]. Some of these solutions are related to each other by a kind of duality transformations [27-34]. An exact form for the backgrounds, consistent with the loop expansion [35,36], are also found using the so called “operator approach” [10,37,38].

This interest in coset models has subsequently led to constructing string theories on four-dimensional curved spacetime backgrounds and analysing their cosmological implications [39-43]. (Cosmological string backgrounds were also considered in another context in [44-48]). One of such models is the coset $SL(2, R) \times SU(2)/ (U(1) \times U(1))$, investigated by Nappi and Witten at the one-loop level [42] and its exact backgrounds were recently found by Bars and Sfetsos [43]. This model describes a closed, inhomogeneous expanding and recollapsing universe in four dimensions. The main feature of this model, however, is the formation of a naked singularity at the time when the universe collapses. This last result was also confirmed in [49].

The purpose of this paper is to investigate the effects of adding, to the gauged $SL(2, R) \times SU(2)/ (U(1) \times U(1))$ model, terms proportional to the field strength of the gauge fields. This alters drastically the singularity structure of the four-dimensional model. We find, in particular, that a naked singularity may form as well when the volume of the universe
is different from zero (the universe is not in a collapsed state). This might be a possible candidate for the violation of cosmic censorship.

The reason behind adding these extra terms to the $SL(2, R) \times SU(2)/ (U(1) \times U(1))$ coset model, and to any gauged WZWN model in general, has its roots in the mathematical formalism of gauging an isometry subgroup of a general nonlinear sigma model [50,51]. These extra terms are also a neat and mathematically appealing method for incorporating marginal perturbations of coset models [52,53].

In section two, we give the formalism for gauging two abelian isometries of a nonlinear sigma model with an antisymmetric field. We show that the usual minimal coupling of the gauge fields (replacing ordinary derivatives by gauge covariant derivatives) does not lead to a gauge invariant theory. The gauge invariant action is instead found by an explicit use of Noether’s method. At the end of this section we show how the extra terms, involving the field strength of the gauge fields, arise in this formalism. We apply this formalism, in section three, to the $SL(2, R) \times SU(2)/ (U(1) \times U(1))$ coset model. We obtain in this manner a four-dimensional curved spacetime whose singularity structure is discussed in the last section.

2 Gauging Two Abelian Isometries of a Sigma Model

In order to get a four-dimensional manifold from the $SL(2, R) \times SU(2)$ WZWN model, one has to gauge two abelian isometries of this group. Here we examine the general formalism of gauging any two abelian isometries of a general sigma model. The action for a general bosonic two-dimensional nonlinear sigma model is given by

$$S = \frac{1}{4\pi} \int d^2x \left( \sqrt{\gamma} \gamma^{\mu\nu} G_{ij} + \epsilon^{\mu\nu} B_{ij} \right) \partial_\mu \phi^i \partial_\nu \phi^j.$$  (2.1)

The metric $G_{ij}$ and the antisymmetric tensor $B_{ij}$ are the massless modes of the bosonic string theory. This action is invariant under the infinitesimal global isometries

$$\delta \phi^i = \varepsilon K^i(\phi) + \tilde{\varepsilon} \tilde{K}^i(\phi)$$  (2.2)

provided that $K^i$ and $\tilde{K}^i$ are Killing vectors of the metric $G_{ij}$, $\nabla_i K_j = \nabla_i \tilde{K}_j = 0$, and the antisymmetric tensor $B_{ij}$ satisfies

$$\partial_t B_{ij} K^l + B_{ij} \partial_t K^l + B_{il} \partial_j K^l = \nabla_i L_j - \nabla_j L_i$$
\[ \partial_i B_{ij} \tilde{K}^i + B_{ij} \partial_i \tilde{K}^i + B_{il} \partial_j \tilde{K}^i = \nabla_i \tilde{L}_j - \nabla_j \tilde{L}_i \] (2.3)

for some two vectors \( L_i \) and \( \tilde{L}_i \) [54]. These last two equations are consequences of the invariance of the action (but not the Lagrangian) under

\[ B_{ij} \rightarrow B_{ij} + \nabla_{[i} V_{j]} . \] (2.4)

We would like now to gauge the above infinitesimal transformations (the parameters \( \varepsilon \) and \( \tilde{\varepsilon} \) would now be local parameters \( \varepsilon(x) \) and \( \tilde{\varepsilon}(x) \)). For this gauging to take place, the generators of the two abelian isometries must commute

\[ [K^i \partial_i, \tilde{K}^j \partial_j] = 0 . \] (2.5)

We also introduce two abelian gauge fields \( A_\mu \) and \( \tilde{A}_\mu \) transforming as

\[ \delta A_\mu = -\partial_\mu \varepsilon , \quad \delta \tilde{A}_\mu = -\partial_\mu \tilde{\varepsilon} . \] (2.6)

Usually, the gauging is carried out by replacing ordinary derivatives by gauge covariant derivatives (minimal coupling)

\[ \partial_\mu \phi \rightarrow D_\mu \phi^i = \partial_\mu \phi + A_\mu K^i + \tilde{A}_\mu \tilde{K}^i . \] (2.7)

However, here and due to the presence of the antisymmetric tensor \( B_{ij} \), the minimal coupling covariantisation will not lead to a gauge invariant theory [50,51].

We therefore postulate the most general action involving the gauge fields \( A_\mu \) and \( \tilde{A}_\mu \) to have the form (Noether’s method)

\[ S_{gauged} = \frac{1}{4\pi} \int d^2 x \left\{ \sqrt{\gamma} \gamma^{\mu\nu} G_{ij} D_\mu \phi^i D_\nu \phi^j + \epsilon^{\mu\nu} B_{ij} \partial_\mu \phi^i \partial_\nu \phi^j - 2\epsilon^{\mu\nu} C_i A_\mu \partial_\nu \phi^i - 2\epsilon^{\mu\nu} \tilde{C}_i \tilde{A}_\mu \partial_\nu \phi^i - 2\epsilon^{\mu\nu} H(\phi) A_\mu \tilde{A}_\nu \right\} . \] (2.8)

The commutation relation (2.5) insures that the gauge covariant derivatives transforms in the correct way

\[ \delta (D_\mu \phi^i) = \left( \varepsilon \partial_j K^i + \tilde{\varepsilon} \partial_j \tilde{K}^i \right) D_\mu \phi^j \] (2.9)

and this guarantees the gauge invariance of the first term in the action (2.8). The quantities \( C_i(\phi) \), \( \tilde{C}_i(\phi) \) and \( H(\phi) \) are then determined by requiring gauge invariance of the rest of the action. First of all, \( C_i(\phi) \) and \( \tilde{C}_i(\phi) \) are defined by

\[ C_i = B_{ij} K^j + L_i \]
\[ \tilde{C}_i = B_{ij} \tilde{K}^j + \tilde{L}_i . \] (2.10)
Secondly, the Lie derivatives of $C_i(\phi)$ and $\tilde{C}_i(\phi)$ must vanish along the directions of $K^i$ and $\tilde{K}^i$

\[
\begin{align*}
\partial_j C_i K^j + C_j \partial_i K^j &= 0 \\
\partial_j \tilde{C}_i \tilde{K}^j + \tilde{C}_j \partial_i \tilde{K}^j &= 0 \\
\partial_j C_i \tilde{K}^j + C_j \partial_i K^j &= 0 \\
\partial_j \tilde{C}_i K^j + \tilde{C}_j \partial_i K^j &= 0 
\end{align*}
\] (2.11)

Thirdly, we have the following constraints on $C_i(\phi)$ and $\tilde{C}_i(\phi)$

\[
\begin{align*}
C_j K^j &= 0 \\
\tilde{C}_j \tilde{K}^j &= 0
\end{align*}
\] (2.12)

Finally, the function $H(\phi)$ is determined from the equations

\[
\begin{align*}
C_i \tilde{K}^i - H &= 0 \\
\tilde{C}_i K^i + H &= 0
\end{align*}
\] (2.13)

The main feature of the above formalism of gauging a nonlinear sigma model with a WZ term is the following: By acting with $\partial_i$ on both sides of the two equations in (2.12) and then substituting for $C_j \partial_i K^j$ and $\tilde{C}_j \partial_i \tilde{K}^j$ in the first two equations in (2.11), we find

\[
\begin{align*}
(\partial_j C_i - \partial_i C_j) K^j &= 0 \\
(\partial_j \tilde{C}_i - \partial_i \tilde{C}_j) \tilde{K}^j &= 0
\end{align*}
\] (2.14)

Notice that these last two equations remain invariant under the replacements

\[
\begin{align*}
C_i &\rightarrow C_i + \partial_i X \\
\tilde{C}_i &\rightarrow \tilde{C}_i + \partial_i \tilde{X}
\end{align*}
\] (2.15)

This means that if $C_i$ and $\tilde{C}_i$ are solutions to the above gauge invariance conditions, then $C_i + \partial_i X$ and $\tilde{C}_i + \partial_i \tilde{X}$ are solutions, too. The only restriction on the two functions $X(\phi)$ and $\tilde{X}(\phi)$ come from the requirement that the rest of the equations involving $C_i$ and $\tilde{C}_i$ should remain invariant under these replacements. This is so, provided that

\[
\begin{align*}
\partial_i X K^i = \partial_i \tilde{X} \tilde{K}^i &= 0 \\
\partial_i \tilde{X} K^i = \partial_i X \tilde{K}^i &= 0
\end{align*}
\] (2.16)
The above freedom in determining $C_i$ and $\tilde{C}_i$ reflects simply the invariance of the equations (2.3) under the shifts $L_i \rightarrow L_i + \partial_i X$ and $\tilde{L}_i \rightarrow \tilde{L}_i + \partial_i \tilde{X}$.

At the level of the action, the above replacement amounts to adding the following extra terms

$$S_{\text{extra}} = \frac{1}{4\pi} \int d^2 x \epsilon^{\mu\nu} \left( F_{\mu\nu} X(\phi) + \tilde{F}_{\mu\nu} \tilde{X}(\phi) \right),$$

where $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are the abelian curvatures corresponding to the gauge fields $A_\mu$ and $\tilde{A}_\mu$, respectively. The appearance of this additional term is also motivated from a completely different point of view: In the quantum theory of the action (2.8), where the gauge fields are treated as fixed backgrounds, divergences proportional to $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are unavoidably generated [50]. Hence for a renormalisable theory we should add terms proportional to $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ to the classical action in order to absorb these divergences (see [50] for more details).

### 3 The Four-dimensional Curved Spacetime

The four-dimensional target space is obtained by taking a WZWN model defined on the group manifold $G = SL(2, R)_{k'} \times SU(2)_k/H$, where $H$ is a two-dimensional abelian subgroup of the group of isometries. The central charge for such a model is

$$c = \frac{3k'}{k'+2} + \frac{3k}{k+2} - 2$$

(3.1)

In the limit of very large $k$ and $k'$, the central charge is, as required, equal to four (see [42]).

If $(g_1, g_2) \in SL(2, R)_{k'} \times SU(2)_k$, then the subgroup is chosen to be infinitesimally generated by [42]

$$\begin{align*}
\delta g_1 &= \varepsilon \sigma_3 g_1 + (\tilde{\varepsilon} \cos \alpha + \varepsilon \sin \alpha) g_1 \sigma_3 \\
\delta g_2 &= i\tilde{\varepsilon} \sigma_2 g_2 + i (-\tilde{\varepsilon} \sin \alpha + \varepsilon \cos \alpha) g_2 \sigma_2 .
\end{align*}$$

(3.2)

The local gauge parameters are $\varepsilon$ and $\tilde{\varepsilon}$, while $\alpha$ is an arbitrary constant and $\sigma_i$, $i = 1, \ldots, 3$, are the usual $2 \times 2$ Pauli matrices ($\sigma^*_2 = -\sigma_2$). These transformations generate an anomaly-free subgroup only when $k' = -k$ [42].
We would like now to apply the formalism of the previous section to the WZWN model defined on the group manifold $G$. For this purpose, we parametrize the $SL(2, R)$ and the $SU(2)$ groups by

$$g_1 = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \quad ab + uv = 1$$

$$g_2 = \exp \left( \frac{i}{2} (\rho + \lambda) \sigma_2 \right) \exp (i \sigma_3) \exp \left( \frac{i}{2} (\rho - \lambda) \sigma_2 \right). \quad (3.3)$$

It is then easy to read off the Killing vectors from the expressions of $\delta g_1$ and $\delta g_2$. These are given by (our six coordinates are $a, u, v, \rho, \lambda, s$)

$$K^a = (1 + \sin \alpha) a, \quad K^u = (1 - \sin \alpha) u, \quad K^v = -(1 - \sin \alpha) v, \quad K^\rho = \cos \alpha, \quad K^\lambda = -\cos \alpha, \quad K^s = 0$$

$$\tilde{K}^a = a \cos \alpha, \quad \tilde{K}^u = -u \cos \alpha, \quad \tilde{K}^v = v \cos \alpha, \quad \tilde{K}^\rho = (1 - \sin \alpha), \quad \tilde{K}^\lambda = (1 + \sin \alpha), \quad \tilde{K}^s = 0. \quad (3.4)$$

Our aim now is to see whether it is possible to find two functions $X$ and $\tilde{X}$ satisfying the gauge conditions in (2.16). Using the above Killing vectors and (2.16), we find

$$[u \cos \alpha \partial_u - v \cos \alpha \partial_v - (1 + \sin \alpha) \partial_\lambda] X = 0$$

$$[a \cos \alpha \partial_a + (1 - \sin \alpha) \partial_\rho] X = 0 \quad (3.5)$$

and two similar equations for $\tilde{X}$. The general solution to these differential equations is given by

$$X = X \left( ae^{-\frac{\cos \alpha}{1 + \sin \alpha} \rho}, u^\beta v^\gamma e^{\frac{\cos \alpha}{1 + \sin \alpha} (\beta - \gamma) \lambda}, u^{\beta'} v'^\gamma e^{\frac{\cos \alpha}{1 + \sin \alpha} (\beta' - \gamma') \lambda}, s \right), \quad (3.6)$$

where the parameters $(\beta, \beta', \gamma, \gamma')$ satisfy

$$\beta \gamma - \beta' \gamma' \neq 0. \quad (3.7)$$

A similar expression holds for $\tilde{X}$. As expected $X$ and $\tilde{X}$ depend on four variables only (instead of six). This is because the gauging reduces the number of degrees of freedom by two. Notice also that the arguments of $X$ and $\tilde{X}$ are invariant under the gauge transformations (3.2). Hence, $X$ and $\tilde{X}$ are functions of the invariants (under the gauge transformations (3.2)) of the group $SL(2, R) \times SU(2)$. 

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Since the two functions $X$ and $\tilde{X}$ have been found, we can therefore proceed to construct our gauged sigma model. Introducing two abelian gauge fields $A_\mu$ and $\bar{A}_\mu$ and taking $k' = -k$, the nonminimal gauged WZWN action takes the form

$$S_{\text{gauged}} + S_{\text{extra}} = -kS_{\text{WZWN}}(g_1) + kS_{\text{WZWN}}(g_2) + \frac{k}{2\pi} \int d^2z \left\{ A_\mu tr \left[ \partial g_1^{-1} \sigma_3 \right] + (\bar{A}_z \cos \alpha + A_z \sin \alpha) tr \left[ g_1^{-1} \partial g_1 \sigma_3 \right] \right\}$$

$$- \frac{ik}{2\pi} \int d^2z \left\{ \bar{A}_z tr \left[ \sigma_2 \partial g_2 \bar{g}_2^{-1} \right] + (A_z \cos \alpha - \bar{A}_z \sin \alpha) tr \left[ \sigma_2 g_2 \bar{g}_2^{-1} \partial g_2 \right] \right\} + \frac{k}{\pi} \int d^2z \left( A_z A_\mu + \bar{A}_z \bar{A}_\mu \right) + \frac{k}{\pi} \int d^2z \left( (\partial A_z - \partial A_\mu) X + (\partial \bar{A}_z - \partial \bar{A}_\mu) \bar{X} \right). \tag{3.8}$$

The standard WZWN action is

$$S_{\text{WZWN}}(g) = -\frac{1}{4\pi} \int \Sigma d^2z tr \left[ (g^{-1}\partial g)(g^{-1}\partial \bar{g}) \right] - \frac{i}{12\pi} \int_B tr \left[ (g^{-1}dg) \wedge (g^{-1}d\bar{g}) \wedge (g^{-1}d\bar{g}) \right]. \tag{3.9}$$

Using our parametrization for $g_1$ and $g_2$, the gauged WZWN action leads to a gauged nonlinear sigma model of the form (2.8) and it is only for $k' = -k$ that all the gauge invariance conditions of the previous section are fulfilled.

We would like now to fix a gauge and integrate out the gauge fields. A suitable gauge choice is found by taking $g_1$ to be given by

$$g_1 = \left( \begin{array}{cc} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{array} \right). \tag{3.10}$$

The integration over the gauge fields leads to a nonlinear sigma model having the following space-time metric

$$DG_{\psi \psi} = 2 \left( 1 - \cos 2\psi \cos 2s \right) + 2 \sin \alpha \left( \cos 2\psi - \cos 2s \right) + (\sin \alpha \cos 2s - 1) (\partial_\psi X)^2$$

$$- (\sin \alpha \cos 2s + 1) (\partial_\psi \bar{X})^2 + \cos \alpha \left( \cos 2s + \cos 2\psi \right) \partial_\psi X \partial_\psi \bar{X},$$

$$DG_{\psi s} = (\sin \alpha \cos 2s - 1) \partial_\psi X \partial_s X - (\sin \alpha \cos 2\psi + 1) \partial_\psi \bar{X} \partial_s \bar{X} + \frac{1}{2} \cos \alpha \left( \cos 2s + \cos 2\psi \right) \left( \partial_\psi X \partial_s \bar{X} + \partial_s X \partial_\psi \bar{X} \right),$$

$$DG_{\psi \rho} = (\sin \alpha \cos 2s - 1) \partial_\psi X \partial_\rho X - (\sin \alpha \cos 2\psi + 1) \partial_\psi \bar{X} \partial_\rho \bar{X}.$$
\[\psi \lambda \] determines the singularities of the metric and is given by
\[DG_{\psi\lambda} = (\sin \alpha \cos 2s - 1) \partial_\psi \partial_\lambda X - (\sin \alpha \cos 2\psi + 1) \partial_\psi \tilde{X} \partial_\lambda \tilde{X}\]
\[+ \frac{1}{2} \cos \alpha (\cos 2s + \cos 2\psi) \left( \partial_\psi X \partial_\lambda \tilde{X} + \partial_\psi \tilde{X} \partial_\lambda X \right)\]
\[+ \cos^2 s (\cos 2\psi + 1) \left[ - \cos \alpha \partial_\psi X + (\sin \alpha + 1) \partial_\psi \tilde{X} \right]\]
\[DG_{ss} = -2 \left(1 - \cos 2\psi \cos 2s\right) - 2 \sin \alpha (\cos 2\psi - \cos 2s) + (\sin \alpha \cos 2s - 1) (\partial_s X)^2\]
\[- (\sin \alpha \cos 2\psi + 1) (\partial_s \tilde{X})^2 + \cos \alpha (\cos 2s + \cos 2\psi) \partial_s X \partial_s \tilde{X}\]
\[DG_{s\rho} = (\sin \alpha \cos 2s - 1) \partial_s X \partial_\rho X - (\sin \alpha \cos 2\psi + 1) \partial_s \tilde{X} \partial_\rho \tilde{X}\]
\[+ \frac{1}{2} \cos \alpha (\cos 2s + \cos 2\psi) \left( \partial_s X \partial_\rho \tilde{X} + \partial_\rho X \partial_s \tilde{X} \right)\]
\[+ \cos^2 s (\cos 2\psi + 1) \left[ - \cos \alpha \partial_s X + (\sin \alpha + 1) \partial_s \tilde{X} \right]\]
\[DG_{s\lambda} = (\sin \alpha \cos 2s - 1) \partial_s X \partial_\lambda X - (\sin \alpha \cos 2\psi + 1) \partial_s \tilde{X} \partial_\lambda \tilde{X}\]
\[+ \frac{1}{2} \cos \alpha (\cos 2s + \cos 2\psi) \left( \partial_s X \partial_\lambda \tilde{X} + \partial_\lambda X \partial_s \tilde{X} \right)\]
\[+ \sin^2 s (\cos 2\psi - 1) \left[ - \cos \alpha \partial_s X + (1 - \sin \alpha) \partial_s \tilde{X} \right]\]
\[DG_{\rho\rho} = -4 \cos^2 s \cos^2 \psi (1 + \sin \alpha) + (\sin \alpha \cos 2s - 1) (\partial_\rho X)^2 - (\sin \alpha \cos 2\psi + 1) (\partial_\rho \tilde{X})^2\]
\[+ \cos \alpha (\cos 2s + \cos 2\psi) \partial_\rho X \partial_\rho \tilde{X}\]
\[+ 2 \cos^2 s (\cos 2\psi + 1) \left[ - \cos \alpha \partial_\rho X + (\sin \alpha + 1) \partial_\rho \tilde{X} \right]\]
\[DG_{\rho\lambda} = (\sin \alpha \cos 2s - 1) \partial_\rho X \partial_\lambda X - (\sin \alpha \cos 2\psi + 1) \partial_\rho \tilde{X} \partial_\lambda \tilde{X}\]
\[+ \frac{1}{2} \cos \alpha (\cos 2s + \cos 2\psi) \left( \partial_\rho X \partial_\lambda \tilde{X} + \partial_\lambda X \partial_\rho \tilde{X} \right)\]
\[+ \cos^2 s (\cos 2\psi + 1) \left[ - \cos \alpha \partial_\rho X + (\sin \alpha + 1) \partial_\rho \tilde{X} \right]\]
\[+ \sin^2 s (\cos 2\psi - 1) \left[ - \cos \alpha \partial_\rho X + (\sin \alpha - 1) \partial_\rho \tilde{X} \right]\]
\[DG_{\lambda\lambda} = -4 \sin^2 s \sin^2 \psi (1 - \sin \alpha) + (\sin \alpha \cos 2s - 1) (\partial_\lambda X)^2 - (\sin \alpha \cos 2\psi + 1) (\partial_\lambda \tilde{X})^2\]
\[+ \cos \alpha (\cos 2s + \cos 2\psi) \partial_\lambda X \partial_\lambda \tilde{X}\]
\[+ 2 \sin^2 s (\cos 2\psi - 1) \left[ - \cos \alpha \partial_\lambda X + (\sin \alpha - 1) \partial_\lambda \tilde{X} \right]. \quad (3.11)\]

The factor \(D\) determines the singularities of the metric and is given by
\[D = -\frac{4\pi}{k} \left[ (1 - \cos 2\psi \cos 2s) + \sin \alpha (\cos 2\psi - \cos 2s) \right]. \quad (3.12)\]

Notice that if \(\psi = 0\) and \(s = 0\) or \(\psi = \frac{\pi}{2}\) and \(s = \frac{\pi}{2}\), then \(D\) vanishes and the metric is
singular. These singularities are interpreted, in the next section, as naked singularities.

4 Discussions

Let us now discuss the cosmological model in which the time parameter is $\psi$, taking values in the range $0 \leq \psi \leq \frac{\pi}{2}$. We distinguish therefore two different cases:

The first is when $X$ and $\tilde{X}$ are set to constants. This case was elaborated in details in ref.[42]. Here we give a brief summary. The target space line element in this case is given by

$$dl^2 = -d\psi^2 + ds^2 - \frac{4}{D} \left( (1 + \sin \alpha) \cos^2 s \cos^2 \psi d\rho^2 + (1 - \sin \alpha) \sin^2 s \sin^2 \psi d\lambda^2 \right). \quad (4.1)$$

The universe starts from a collapsed state (a big bang) at $\psi = 0$, since the determinant of the metric $G_{ij}$ vanishes (hence the volume of the universe). At $\psi = \frac{\pi}{2}$, the universe collapses again (a big crunch). More appealing is the presence of a possible candidate for a naked singularity and hence a possible violation of the cosmic censorship. Indeed, at $\psi = s = 0$ or $\psi = s = \frac{\pi}{2}$ the metric is not defined ($D = 0$) and since these singularities appear at definite values of the time $\psi$, they might stand as candidates for naked singularities. However, this is not the case since it is precisely at $\psi = 0$ or $\psi = \frac{\pi}{2}$ that the universe is in a collapsed state. Therefore the naked singularities cannot be seen as the whole universe collapses when they would have appeared.

The second case is when $X$ and $\tilde{X}$ are not constants (or at least one of the two is not). We would like to examine the fate of the above singularities when the action contains the extra nonminimal term. There is no reason, a priori, for the volume of the universe to vanish at $\psi = 0$ since the functions $X$ and $\tilde{X}$ can be chosen at will. We will provide, in what follows, an example in which the universe is not in a collapsed state at $\psi = 0$.

Notice that all the extra terms appearing in the metric (3.11) involve only the derivatives of $X$ and $\tilde{X}$. Let us therefore write down these derivatives in the gauge specified by (3.10).

We have for the derivatives of $X$ (and similarly for $\tilde{X}$)

$$\partial_\psi X = -e^{-\frac{\cos \alpha}{1 - \sin \alpha} \beta} (\sin \psi) \partial_\xi_1 X + (\beta + \gamma) e^{\frac{\cos \alpha}{1 - \sin \alpha} (\beta - \gamma) \lambda} (\cos \psi) (\sin \psi)^{\beta + \gamma - 1} \partial_\xi_2 X$$

$$+ (\beta' + \gamma') e^{\frac{\cos \alpha}{1 - \sin \alpha} (\beta' - \gamma') \lambda} (\cos \psi) (\sin \psi)^{\beta' + \gamma' - 1} \partial_\xi_3 X$$

$$\partial_\nu X = -\frac{\cos \alpha}{1 - \sin \alpha} e^{-\frac{\cos \alpha}{1 - \sin \alpha} \beta} (\cos \psi) \partial_\xi_1 X$$
\[
\partial_\lambda X = \frac{\cos \alpha}{1 + \sin \alpha} \left[ (\beta - \gamma) e^{\frac{\cos \alpha}{1 + \sin \alpha} (\beta - \gamma) \lambda} (\sin \psi)^{\beta + \gamma} \partial_{\xi_2} X \\
+ (\beta' - \gamma') e^{\frac{\cos \alpha}{1 + \sin \alpha} (\beta' - \gamma') \lambda} (\sin \psi)^{\beta' + \gamma'} \partial_{\xi_3} X \right]
\]
\[
\partial_{\xi_4} X = \partial_{\xi_4} X.
\] (4.2)

Here, the coordinates \( \xi_1, \xi_2, \xi_3 \) and \( \xi_4 \) are defined by
\[
\xi_1 = e^{-\frac{\sin \alpha}{1 + \sin \alpha} \beta} \cos \psi, \quad \xi_2 = e^{\frac{\cos \alpha}{1 + \sin \alpha} (\beta' - \gamma) \lambda} (\sin \psi)^{\beta + \gamma}, \\
\xi_3 = e^{\frac{\cos \alpha}{1 + \sin \alpha} (\beta - \gamma) \lambda} (\sin \psi)^{\beta' + \gamma'}, \quad \xi_4 = s.
\] (4.3)

The volume of the universe is different from zero at \( \psi = 0 \) only if all the derivatives of \( X \) and \( \widetilde{X} \) evaluated at \( \psi = 0 \) are well-defined and different from zero. This situation can always be realized by suitably choosing \( X \) and \( \widetilde{X} \). For example, by taking
\[
\beta + \gamma = 1, \quad \beta' + \gamma' = 0,
\] (4.4)
we see that all the derivatives of \( X \) and \( \widetilde{X} \) are different from zero at \( \psi = 0 \) provided that \( \partial_{\xi_i} X \) and \( \partial_{\xi_i} \widetilde{X} \), \( i = 1, \ldots, 4 \), are well-defined and different from zero at \( \psi = 0 \).

Therefore, we can always choose \( \partial_{\xi_i} X \) and \( \partial_{\xi_i} \widetilde{X} \) in such a way that the volume of the universe is different from zero at \( \psi = 0 \). This statement holds for generic values of \( s \). It is also easy to explicitly verify that if none of the derivatives of \( X \) and \( \widetilde{X} \) vanishes at \( \psi = 0 \) and \( s = 0 \), then all the components of the metric in (3.11) are nonvanishing at \( \psi = 0 \) and \( s = 0 \). Hence, we have provided an example in which a kind of naked singularity might form when the universe is not in a collapsed state. This is in contrast to the results reached in [42,43,49], where the universe was found to collapse right at the time when a naked singularity was about to form. We were not able to prove that the universe does not collapse at \( \psi = \frac{\pi}{2} \) and \( s = \frac{\pi}{2} \).

We would like now to address some issues related to our model. The first issue regards the gauge fixing. The gauge choice (3.10) covers only a patch of space-time and it is possible to continue past \( \psi = 0 \) and \( \psi = \frac{\pi}{2} \). The continuation past \( \psi = 0 \) is made by taking the gauge condition [42]
\[
g_1 = \frac{1}{1 + x} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\] (4.5)
The resulting target space-time metric is yet another complicated expression of the form (3.11). What is of interest to us here is the equivalent of the denominator $D$. This is given by

$$D = -\frac{8\pi}{k} \frac{1}{1+x} [(1-x) \cos 2s - (1+x) + \sin \alpha ((1+x) \cos 2s - (1-x))] . \quad (4.6)$$

Hence the metric is not defined for $x=0$ and $s=0$. Here also it is possible to find $X$ and $\tilde{X}$ such that the volume of the universe is different from zero at $x=0$ and $s=0$.

We mention here that none of the two chosen gauges covers the whole four-dimensional space-time (see ref.[43]). Therefore, the issue of gauge fixing in this model, and in gauged WZWN models in general, needs further examinations.

The second problem concerns the dilaton field in this model. So far, we have given only the metric field (the antisymmetric tensor can be extracted in the same way as the metric). Solving the one-loop beta functions using the metric (3.11) is not an easy task. However, this task could be made somehow simpler by putting some restrictions on the functions $X$ and $\tilde{X}$. Since $X$ and $\tilde{X}$ are scalars, we could require them to satisfy the equations of a massless scalar on the target space,

$$\nabla^2 X = \nabla^2 \tilde{X} = 0 . \quad (4.7)$$

The Laplacian is evaluated using the metric (4.1). This requirement is equivalent to demanding the vanishing of the one-loop counterterms proportional to $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ which would arise from the action (2.8) when the gauge fields are treated as backgrounds [50,51].

In the language of conformal field theory, and using the techniques of ref.[10], this is the same as requiring $X$ and $\tilde{X}$ to be conformal operators of dimension $(0,0)$ so that the extra terms in the action (3.8) are of dimension $(1,1)$. This provides a way of analysing perturbations, by marginal operators, in coset theories. This is an interesting issue to which we hope to return.

Finally, it is worth mentioning that the singularities of the $SL(2, R) \times SU(2)/(U(1) \times U(1))$ model (without the extra terms) were boosted away [55] through the introduction of a fifth coordinate and by using an $O(3,3)$ transformation. It is therefore of great interest to explore the effects of duality transformations on our metric.
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