Axisymmetric models for galaxies by equipotential and equidensity methods

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Abstract

In this paper we outline equipotential and equidensity methods of constructing axisymmetric models for galaxies. The former method defines equipotentials, from which the corresponding densities of the galaxy models can be obtained using Poisson’s equation; the latter defines the equidensity surfaces of the galaxy models directly.

Key words: celestial mechanics, stellar dynamics – galaxies.

1 Introduction

Most of the earlier models for galaxies are spherical and purely empirical, in order to fit the surface brightness of galaxies observed. Although spherical models can be used to simulate the surface brightness of some observed galaxies, galaxies are mostly not spherical. Thus we construct axisymmetric models in order to advance our overall understanding of galaxy formation and evolution. There are several different ways to construct axisymmetric models for galaxies. One of the basic ideas is to extend existing spherical models to more general axisymmetric forms. The methods of this extension can usually be classified into two groups. One is the equipotential method\cite{16}, which was first introduced by Kuzmin\cite{11,12} and developed by Toomre\cite{22}, Miyamoto & Nagai\cite{17}, Satoh\cite{23}, Kutuzov & Ossipkov\cite{14,15}, Evans\cite{4}, Jiang\cite{7}, Ossipkov & Binney\cite{18}, Jiang & Moss\cite{10} and Jiang et al.\cite{9}. This device is to define equipotentials from which the corresponding densities of galaxy models can be derived using Poisson’s equation. The significance of this device is that it allows flattening to be achieved while maintaining a simple form of the potential. In particular, comparing the circular velocity profile generated from the potential of the model with the rotation curve of any very flattened galaxy, we can easily determine whether the model may be applied to describe the galaxy. An alternative is the equidensity method, which defines the equidensity surfaces of the galaxy models directly; this group includes the flattened $\gamma$-models\cite{3}. The advantage of this method is that the projected surface densities of the models can be easily derived from their equidensity surfaces, allowing determination of whether these surface densities can be used to fit the surface brightness of observed galaxies.

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2 The Equipotential Method

Although the construction of galaxy models by the equipotential method has been outlined by Kutuzov & Ossipkov\cite{Kutuzov1992}, we introduce below our rather different ideas about the method. Most of these ideas have been given by Jiang\cite{Jiang2003}. The equipotential method of constructing axisymmetric models for galaxies is currently being developed further. Up to now, the method can be roughly divided into three groups. The first is to use the axisymmetric radius of \( \sqrt{R^2 + (a + |z|)^2} \) in place of the spherical radius \( r = \sqrt{R^2 + z^2} \) in the potentials of the spherical models, where \( a > 0 \). This was originally introduced by Kuzmin\cite{Kuzmin1956, Kuzmin1957}. It is not directly relevant to elliptical galaxies, and it is used to model discs. The physical significance of this device is that at the point \((R, -|z|)\) below Kuzmin’s disc, the potential of Kuzmin’s models\cite{Kuzmin1957} is identical with that of a point mass located at distance \( a \) above the centre of the disc.

\[
\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}} \tag{2.1}
\]

is identical with that of a point mass located at distance \( a \) above the centre of the disc. This implies that \( \nabla^2 \Phi(R, z) = 0 \) for \( z \neq 0 \). As \( a \) tends to zero, Kuzmin’s models becomes point mass models. The second is to use the axisymmetric radius \( \sqrt{R^2 + (z/q)^2} \) in place of the spherical radius \( r \) of the potentials of spherical models, where \( q \) is the axial ratio, as in Binney’s logarithmic models\cite{Binney1987}. Evans’s power-law models\cite{Evans1915} are given by a simple power-law potential

\[
\Phi(m) = -\frac{\psi_c R_c^p}{(R_c^2 + m^2)^{p/2}}; \quad p > 0. \tag{2.2}
\]

where \( m \) is the axisymmetric radius given by \( m^2 = R^2 + (z/q)^2 \), \( \psi_c \) is the potential at the origin and \( R_c \) is the scale radius. In fact, Evans’s power-law models can be regarded as extensions of Plummer’s spherical models\cite{Plummer1911}, since (2.2) degenerates to the spherical Plummer models when \( q = p = 1 \). The third is to use the axisymmetric radius \( \sqrt{R^2 + (\sqrt{z^2 + c^2} + d)^2} \) in place of the spherical radius \( r \) of the potentials of the spherical models. This device was first introduced by Miyamoto and Nagai\cite{Miyamoto1975} for the Plummer potential, when they constructed the following potential-density pair

\[
\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (\sqrt{z^2 + c^2} + d)^2}}, \tag{2.3}
\]

and

\[
\rho(R, z) = \left(\frac{c^2 M}{4\pi}\right) \frac{dR^2 + (d + 3\sqrt{z^2 + c^2})(d + \sqrt{z^2 + c^2})^2}{[R^2 + (d + \sqrt{z^2 + c^2})^2]^{5/2} (z^2 + c^2)^{3/2}}. \tag{2.4}
\]

When \( c = 0 \), (2.3) reduces to Kuzumin’s potential mentioned above. When \( d = 0 \), (2.3) reduces to the spherical potential of Plummer.

The first and third devices described above have attracted the attention of many outstanding investigators. Thus several ideas of constructing axisymmetric potentials are
given and they are the development of these devices. A new potential-density pair can be obtained if (2.1) is differentiated with respect to $a^2$. Toomre’s model \( n \) was derived by Toomre\(^{22} \) by taking the \((n-1)\)st derivative of \((2.1)/a \) with respect to $a^2$. Similarly, if (2.3) is differentiated \( n \) times with respect to $c^2$, Satoh’s models\(^{21} \) can be obtained.

Following Kuzmin’s models mentioned above, Kuzmin & Kutuzov\(^{13} \) constructed a more general axisymmetric potential as follows

\[
\Phi(R, z) = -\frac{GM}{(R^2 + z^2 + a^2 + c^2 + 2\sqrt{a^2c^2 + c^2R^2 + a^2z^2})^{1/2}}.
\]

This is a Stäckel model. Similarly, Evans et al.\(^{5} \) obtained flattened isochrone models with the potential

\[
\Phi(R, z) = -\frac{X + aY + c^2}{Y(X + aY + a^2)}.
\]

where $X = \sqrt{a^2c^2 + c^2R^2 + a^2z^2}$ and $Y = \sqrt{a^2 + c^2 + R^2 + z^2 + 2X}$. In (2.5) and (2.6) \( a \) and \( c \) are non-negative length scales.

Recently, oblate Jaffe models for galaxies were given by Jiang\(^{7} \), using Miyamoto & Nagai’s device. Then Jiang & Moss\(^{10} \) constructed a class of prolate Jaffe models, obtained by replacing the spherical radius \( r \) of the potentials of Jaffe’s spherical models with a more general axisymmetric radius $\sqrt{(\sqrt{R^2 + a^2 + b})^2 + z^2}$, where \( a \) and \( b \) are two positive constants. This method is similar to that of Miyamoto & Nagai. Furthermore, Jiang et al.\(^{9} \) gave more general flattened Jaffe models with potentials

\[
\Phi(R^2, z) = \frac{GM}{r_J} \ln \left( \frac{\sqrt{(\sqrt{R^2 + a^2 + b})^2 + (\sqrt{z^2 + c^2 + d})^2}}{\sqrt{(\sqrt{R^2 + a^2 + b})^2 + (\sqrt{z^2 + c^2 + d})^2 + r_J}} \right),
\]

using a device similar to that of Miyamoto & Nagai, that is replacing the spherical radius \( r \) of the potentials of Jaffe’s models\(^{6} \) with a more general axisymmetric radius $m$, where $m = \sqrt{(\sqrt{R^2 + a^2 + b})^2 + (\sqrt{z^2 + c^2 + d})^2}$, and \( a \), \( b \), \( c \) and \( d \) are positive constants.

Since the spherical $\gamma$-model\(^{20,21} \) include Jaffe’s spherical model, we can also define a class of axisymmetric $\gamma$-models, replacing the spherical radius \( r \) of the potential of the spherical $\gamma$-model with the more general axisymmetric radius $m$ given above.

### 3 The Equidensity Method

The equidensity method directly defines the ellipsoidal equidensity surfaces of the galaxy models. For example, the axisymmetric radius $\sqrt{R^2 + (z/q)^2}$ can be used in place of the spherical radius \( r \) in the densities of the spherical models, where \( q \) is the axial ratio, as in Binney’s logarithmic models\(^{1} \). The family of densities of the flattened $\gamma$-model\(^{1} \) is

\[
\rho(m) = \frac{(3 - \gamma)M}{4\pi q} \frac{am^{-\gamma}}{(m + a)^{4-\gamma}},
\]

(3.1)
from the density of the spherical $\gamma$-model mentioned in Section 2, where $M$ and $a$ are total mass and scale radius, $m$ is the same as in (2.2), $q$ is the axial ratio and $0 \leq \gamma < 3$.

The advantage of this method is that the projected surface densities of the models are easily derived. Thus we can easily determine whether the surface densities can be used to fit the surface brightness of the galaxies observed. However, this method can lead to mathematical difficulties in finding analytical expressions for the potentials.

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