Cosmological Redshift in FRW Metrics with Constant Spacetime Curvature

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ABSTRACT

Cosmological redshift $z$ grows as the Universe expands and is conventionally viewed as a third form of redshift, beyond the more traditional Doppler and gravitational effects seen in other applications of general relativity. In this paper, we examine the origin of redshift in the Friedmann-Robertson-Walker metrics with constant spacetime curvature, and show that—at least for the static spacetimes—the interpretation of $z$ as due to the “stretching” of space is coordinate dependent. Namely, we prove that redshift may also be calculated solely from the effects of kinematics and gravitational acceleration. This suggests that its dependence on the expansion factor is simply a manifestation of the high degree of symmetry in FRW, and ought not be viewed as evidence in support of the idea that space itself is expanding.

Key words: cosmic microwave background, cosmological parameters, cosmology: observations, cosmology: redshift, cosmology: theory, distance scale

1 INTRODUCTION

Standard cosmology is based on the Friedmann-Robertson-Walker (FRW) metric for a spatially homogeneous and isotropic three-dimensional space. In terms of the proper time $t$ measured by a comoving observer, and the corresponding radial ($r$) and angular ($\theta$ and $\phi$) coordinates in the comoving frame, the interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

where $g_{\mu\nu} (\mu, \nu = 0, 1, 2, 3)$ are the metric coefficients, may be written as

$$ds^2 = c^2 dt^2 - a^2(t)[dr^2(1 - kr^2)^{-1} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)].$$

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The expansion factor \( a(t) \) is a function of cosmic time \( t \), whereas the spatial coordinates \( (r, \theta, \phi) \) in this frame remain “fixed” for all particles in the cosmos. The constant \( k \) is +1 for a closed universe, 0 for a flat, open universe, or −1 for an open universe.

This representation of proper distance as a product of a universal expansion factor (independent of position) and an unchanging set of comoving coordinates, is often interpreted as meaning that space itself is dynamic, expanding with time. This view, however, is not universally accepted because the difference between this situation—in which particles are fixed in an expanding space—and the alternative interpretation—in which the particles move through a fixed space—is more than merely semantic. Each has its own particular set of consequences, some of which have been explored elsewhere, e.g., by Chodorowski (2007).

Underlying much of the discussion concerning the expansion of space (see, e.g., Davis & Lineweaver 2004; Harrison 1995; Chodorowski 2007; Baryshev 2008; Bunn & Hogg 2009; Cook & Burns 2009) is the nature of cosmological redshift \( z \), defined as

\[
  z = \frac{\nu_e - \nu_o}{\nu_o},
\]

where \( \nu_o \) and \( \nu_e \) are the observed and emitted radiation frequencies, respectively. It is not difficult to show (Weinberg 1972) that

\[
  1 + z = \frac{a(t_o)}{a(t_e)},
\]

in terms of the expansion factor \( a(t) \), where \( t_o \) and \( t_e \) represent, respectively, the cosmic time at which the radiation is observed and that at which it was emitted. It is this formulation, in particular, that seems to suggest that \( z \) is due to the aforementioned stretching of space, because it doesn’t look like any of the other forms of redshift we have encountered before. But is cosmological redshift really due to “stretching,” and therefore a different type of wavelength extension beyond those expected from Doppler and gravitational effects? Or is this different formulation—and therefore its interpretation—merely due to our choice of coordinates? In other words, is it possible to use another set of coordinates to cast the cosmological redshift into a form more like the “traditional” lapse function used in other applications of general relativity? This is the principal question we wish to explore in this paper.

But finding a resolution to this important issue is quite difficult, as others have already discovered (see, e.g., Bunn & Hogg 2009; Cook & Burns 2009). In this paper, we will seek a partial answer to this question by considering a subset of FRW metrics—those that have a constant space-time curvature and can therefore be written in static form. The complete treatment, including also those FRW metrics whose curvature changes with time, will be discussed elsewhere. For these
static FRW metrics, we will prove that the cosmological redshift can be calculated—with equal validity—either from the “usual” expression (Equation 4) involving the expansion factor $a(t)$, or from the well-known effects of kinematic and gravitational time dilation, using a transformed set of coordinates $(cT, \eta, \theta, \phi)$, for which the metric coefficients $g_{\mu\nu}$ are independent of time $T$. We will therefore show for the static FRW metrics, that the interpretation of $z$ as a stretching of space is coordinate-dependent. A different picture emerges when we derive $z$ directly as a “lapse function” due to Doppler and gravitational effects.

2 THE COSMOLOGICAL LAPSE FUNCTION

Our procedure for finding the cosmological redshift as a lapse function involves three essential steps. First, we find a set of coordinates permitting us to write the metric in stationary form. It goes without saying that Equation (2) is not adequate for our purposes because the metric coefficients $g_{\mu\nu}$ generally depend on time $t$, through the expansion factor $a(t)$. Second, we use this transformed metric to calculate the time dilation at the emitter’s location relative to the proper time in a local free-falling frame. Finally, we obtain the apparent time dilation, which differs from its counterpart at the emitter’s location because the motion of the source alters the relative arrival times of the photon’s wave crests. Steps two and three are rather standard in relativity (see, e.g., Weinberg 1972). The most complicated portion of this procedure is the search for an appropriate coordinate transformation that renders the FRW metric static.

It is not difficult to show that there are exactly six FRW metrics with constant spacetime curvature; in each of these cases, a transformation of coordinates permits us to write these solutions in static form (Florides 1980). We will consider each of these special cases in turn, including the Minkowski spacetime, the Milne Universe, de Sitter space, the Lanczos Universe, and anti-de Sitter space. It is important to stress as we proceed through this exercise that although the spacetime curvature is constant in the cases we consider here, it is generally nonzero. This is a crucial point because the cosmological redshift is therefore not just a kinematic effect (as in the Milne Universe); it is generally a combination of Doppler and gravitational effects (as one finds in de Sitter and Lanczos). Static FRW metrics therefore do not simplify the redshift by eliminating one or more of the contributors. Gravitational effects are present even when the FRW metric is time-independent, as is well known from the Schwarzschild and Kerr spacetimes.
3 THE SIX STATIC FRW METRICS

3.1 Minkowski Spacetime

The Minkowski spacetime is spatially flat ($k = 0$) and is not expanding, $a(t) = 1$, so

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\Omega^2,$$

where, for simplicity, we have introduced the notation $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. This metric is already in static form, so there is no need to find a new set of coordinates. Quite trivially, then, $z = 0$ everywhere (from Equation 4). The Doppler and gravitational redshifts are also trivially zero in this case, since there is no expansion or spacetime curvature.

3.2 The Milne Universe

A universe with $\rho = 0$ and $k = -1$ corresponds to a simple solution of Einstein’s equations, in which

$$a(t) = ct,$$

i.e., the scale factor grows linearly in time. Since the “acceleration” $\ddot{a}(t)$ is therefore zero in this cosmology, first introduced by Milne (1933), one might expect such a universe to be flat and a mere re-parametrization of Minkowski space. Indeed, Milne intended this type of expansion to be informed only by special relativity, without any constraints imposed by the more general theory. This cosmology has been the subject of many past analyses, including two recent publications (Abramowicz et al. 2007; Cook & Burns 2009) that considered it in its manifestly flat form, obtained through a straightforward coordinate transformation that we describe as follows.

We first introduce the co-moving distance variable $\chi$, defined in terms of $r$ according to

$$r = \sinh\chi,$$

which allows us to write the FRW metric for Milne in the form

$$ds^2 = c^2 dt^2 - c^2 t^2 [d\chi^2 + \sinh^2\chi d\Omega^2].$$

The transformation that brings Equation (8) into a stationary (and manifestly flat) form is

$$T = t \cosh\chi$$
$$\eta = ct \sinh\chi,$$

for then

$$ds^2 = c^2 dT^2 - d\eta^2 - \eta^2 d\Omega^2.$$
the generally understood identity between the two or, as we alluded to above, the fact that one is a re-parametrization of the other. Neither Minkowski nor Milne have any spacetime curvature, and may therefore be transformed into each other with an appropriate set of coordinates (as we have just seen). However, we will affirm on several occasions in the following sections that the measurement of redshift depends critically on the observer and the coordinates he/she is using. Thus, even though Minkowski and Milne are equivalent, the source is moving (with the Hubble flow) relative to an observer in the latter, but not the former, and the two observers therefore do measure a different kinematic redshift.

For the second step, let us now evaluate the time dilation in the coordinate frame \((cT, \eta, \theta, \phi)\), assuming only radial motion, i.e., \(d\theta = d\phi = 0\). In the co-moving frame, the cosmic time \(t\) is also the proper time (what we would conventionally call \(\tau\)). Thus, for an interval associated with proper time only, (i.e., \(ds^2 = c^2dt^2\)), we have from Equation (10)

\[
\frac{dt}{dT} = \left[1 - \frac{1}{c^2} \left(\frac{dn}{dT}\right)^2\right]^{1/2}.
\tag{11}
\]

This time dilation, however, evaluated at the emitter’s location (and at the time when the light was produced), is not necessarily equal to the \textit{apparent} time dilation. These two are equal only when the source is instantaneously at rest with respect to the observer. If the source is moving (as it is here), then the time between emission of successive wave fronts is indeed given by \(dT\) in Equation (11), but during this interval, the proper distance (as measured in the \(\eta - T\) frame) from the observer to the light source also increases by an amount \(v_\eta \sqrt{g_{\eta T}} \, dT\), where

\[
v_\eta \equiv \sqrt{\frac{g_{\eta \eta}}{g_{TT}}} \, dn \bigg|_{T_e} \tag{12}
\]

is the component of (proper) velocity (proper distance per unit proper time) measured in this frame along the line-of-sight to the source.

Thus, the ratio of the frequency of light actually measured by the observer to that emitted is

\[
\frac{\nu_o}{\nu_e} = \left(1 + \frac{v_\eta}{c}\right)^{-1} \frac{dt}{dT} \bigg|_{T_e},
\tag{13}
\]
a simple expression made possible by the static form of the metric in Equation (10). If the metric coefficients \(g_{\mu\nu}\) had been dependent on \(T\), other multiplicative factors would have needed to be introduced into Equation (13). (Note that the quantities on the right-hand side of this equation formally must all be evaluated at the time, \(T_e\) or, equivalently, \(t_e\), when the light was emitted. In the Milne Universe, the expansion velocity at a fixed \(\chi\) is trivially constant in time. This criterion is much more important for the curved spacetimes we will consider next.)
Assuming that the source is moving with the Hubble flow, i.e., that \( dr = 0 \), we now see that

\[
\frac{d\eta}{dT} = \frac{\partial \eta}{\partial t} \frac{dt}{dT} = c \tanh \chi
\]  
(14)

and Equations (11) and (14) are therefore trivially consistent with

\[
\frac{dt}{dT} = \frac{1}{\cosh \chi}.
\]  
(15)

Since in the Milne cosmology \( g_{\eta\eta} = g_{TT} = 1 \), the apparent frequency shift is therefore

\[
\frac{\nu_o}{\nu_e} = \frac{1}{1 + \tanh \chi} \cosh^{-1} \chi = e^{-\chi}.
\]  
(16)

So according to this procedure for finding \( z \) via the lapse function, the cosmological redshift is given by

\[
1 + z \equiv \frac{\nu_e}{\nu_o} = e^\chi.
\]  
(17)

How does this compare with the expression one would conventionally derive from Equation (4), based on the rate of universal expansion between the emission (\( t_e \)) and observation (\( t_o \)) times?

In starting its propagation from the source at time \( t_e \), the emitted light travels along a null geodesic \( (ds = 0) \) until it reaches the observer at time \( t_0 \). Therefore, from Equation (8) with \( d\Omega = 0 \), we see that

\[
\int_0^\chi d\chi' = \int_{t_e}^{t_0} \frac{dt'}{\tau'},
\]  
(18)

the cancelling minus sign arising because the light is approaching us. That is,

\[
\chi = \ln \left( \frac{t_o}{t_e} \right).
\]  
(19)

Thus, according to Equation (4), the cosmological redshift is

\[
1 + z = \frac{a(t_o)}{a(t_e)} = \frac{t_o}{t_e} = e^\chi,
\]  
(20)

fully consistent with the result we derived in Equation (17) through a consideration of the time dilation between moving frames (Equation 11) and its subsequent modification as a result of the shift in arrival times (Equation 13). So in the Milne cosmology, the redshift may be calculated either from knowledge of the expansion factor \( a(t) \), or by using a more direct approach already understood in the context of general relativity that does not involve the assumption of an expanding space.
3.3 de Sitter Space

The de Sitter spacetime is the first of the six static FRW solutions we will encounter that has a constant, but nonzero, curvature. Unlike the Milne Universe, which describes a flat universe with no gravitational acceleration, de Sitter has $\rho \neq 0$, and objects not only recede from one another, but also accelerate under the influence of gravity. Since the spacetime in de Sitter is curved, this FRW metric provides us with an important validation of our method, complementary to the Milne example.

The de Sitter cosmology (de Sitter 1917) corresponds to a universe devoid of matter and radiation, but filled with a cosmological constant whose principal property is the equation of state $p = -\rho$. The FRW metric in this case may be written

$$ds^2 = c^2 dt^2 - e^{2Ht}[dr^2 + r^2 d\Omega^2],$$

(21)

where $k = 0$ and the expansion factor has the specific form

$$a(t) = e^{Ht},$$

(22)

in terms of the Hubble constant $H$. This cosmology may represent the Universe’s terminal state, and may also have corresponded to its early inflationary phase, where it would have produced an exponentiation in size due to the expansion factor $\exp(HT)$.

Unlike the Minkowski and Milne models, the de Sitter cosmology contains mass-energy (in the form of a cosmological constant). However, an observer using only comoving coordinates is in free fall and is unaware of the gravitational acceleration. This was Einstein’s “happiest thought of his life” that lead to the Principle of Equivalence, which states that the spacetime in a free falling frame is locally Minkowskian, consistent with special relativity. But we realize that gravity plays an important role when we instead move to a different set of coordinates (Melia 2007; Melia & Abdelqader 2009), which may include the proper radius $\eta(t) = a(t)r$.

Let us first present the transformation that casts this metric into its static form, and then discuss the physical meaning of the new coordinates. With the transformation

$$\eta = a(t)r$$

$$T = t - \frac{1}{2H} \ln \Phi,$$

(23)

where

$$\Phi \equiv 1 - \left(\frac{\eta}{R_h}\right)^2,$$

(24)
and

\[ R_h \equiv \frac{c}{H} \quad (25) \]

is the gravitational (or Hubble) radius, the de Sitter metric becomes

\[
    ds^2 = \Phi \frac{c^2}{\Phi^{-1}} dT^2 - \Phi^{-1} d\eta^2 - \eta^2 d\Omega^2. \quad (26)
\]

Clearly, \( g_{TT} = \Phi \) and \( g_{\eta\eta} = \Phi^{-1} \), both independent of \( T \).

Written in this way, the metric explicitly reveals the spacetime curvature most elegantly inferred from the corollary to Birkhoff’s theorem (Birkhoff 1923). This theorem states that in a spherically symmetric spacetime, the only solution to the Einstein equations is the Schwarzschild exterior solution, which is static. What is relevant to our discussion here is not so much the Birkhoff theorem itself, but rather its very important corollary. The latter is a generalization of a well-known result of Newtonian theory, that the gravitational field of a spherical shell vanishes inside the shell. The corollary to Birkhoff’s theorem states that the metric inside an empty spherical cavity, at the center of a spherically symmetric system, must be equivalent to the flat-space Minkowski metric. Space must be flat in a spherical cavity even if the system is infinite. It matters not what the constituents of the medium outside the cavity are, as long as the medium is spherically symmetric.

If one then imagines placing a spherically symmetric mass at the center of this cavity, according to Birkhoff’s theorem and its corollary, the metric between this mass and the edge of the cavity is necessarily of the Schwarzschild type. Thus, the worldlines linked to an observer in this region are curved relative to the center of the cavity in a manner determined solely by the mass we have placed there. This situation may appear to contradict our assumption of isotropy, which one might naively take to mean that the spacetime curvature within the medium should cancel since the observer sees mass-energy equally distributed in all directions. In fact, the observer’s worldlines are curved in every direction because, according to the corollary to Birkhoff’s theorem, only the mass energy between any given pair of points in this medium affects the path linking those points.

The form of the metric in Equation (26) is how de Sitter himself first presented his now famous solution. One can almost see the inspiration for it by considering Schwarzschild’s solution describing the spacetime around an enclosed, spherically symmetric object of mass \( M \):

\[
    ds^2 = c^2 dT^2 \left[ 1 - 2GM/c^2\eta \right] - d\eta^2 \left[ 1 - 2GM/c^2\eta \right]^{-1} - \eta^2 d\Omega^2. \quad (27)
\]

De Sitter’s metric describes the spacetime around a radially dependent enclosed mass \( M(\eta) \). In a medium with uniform mass-energy density,

\[
    M(\eta) = M(R_h) (\eta/R_h)^3. \quad (28)
\]
for which the Schwarzschild factor \(1 - (2GM/c^2\eta)\) transitions into \(1 - (\eta/R_h)^2\), what we have here called \(\Phi\) (see Equation 24). It should be emphasized that this Equation implicitly contains the restriction that no mass energy beyond \(\eta\) should contribute to the gravitational acceleration inside of this radius, as required by the corollary to Birkhoff’s theorem. For a given interval \(ds\), the time \(T\) clearly diverges as \(\eta\) approaches \(R_h\), which therefore represents the limiting distance beyond which the spacetime curvature prevents any signal from ever reaching us. Though we know it as the Hubble radius, \(R_h\) is actually defined as a Schwarzschild radius, by the condition
\[
\frac{2GM(R_h)}{c^2} = R_h.
\]
That is, \(R_h\) is in fact the distance at which the enclosed mass-energy is sufficient to turn it into the Schwarzschild radius for an observer at the origin of the coordinates. And it is trivial to show that for \(k = 0\), \(R_h\) reduces to its more recognizable Hubble manifestation in Equation (25).

The point of all this is for us to recognize that de Sitter’s metric in Equation (26) is not only static (as we require for our procedure), but that it also contains the effects of gravitational curvature through the factor \(\Phi\). We will now follow steps 2 and 3 in our procedure, as we did with Milne, to derive the cosmological redshift in de Sitter based on the Doppler and gravitational effects.

The time dilation is here given as
\[
\frac{dt}{dT} = \left[ \Phi - \frac{1}{c^2} \Phi^{-1} \left( \frac{d\eta}{dT} \right)^2 \right]^{1/2},
\]
again assuming that the source moves only with the Hubble flow. Since \(r\) is therefore constant, we also have
\[
\frac{d\eta}{dT} = \frac{\partial \eta}{\partial t} \frac{dt}{dT} = \dot{\eta} \frac{dt}{dT} = H\eta \frac{dt}{dT}.
\]
Equations (23), (30), and (31) are therefore consistent with
\[
\frac{dt}{dT} = \Phi.
\]
This time dilation includes both the effects of gravity and the kinematics associated with the Hubble recession of the source. But as we learned previously, we cannot yet use this to infer the shift in frequency of the light without first finding the apparent time dilation, analogous to Equation (13).

In de Sitter, the proper velocity component of the source along our line-of-sight is
\[
v_\eta = \frac{\dot{\eta}}{\sqrt{g_{\eta\eta}} \sqrt{g_{TT}}} d\eta/dT,
\]
where now neither \(g_{\eta\eta}\) nor \(g_{TT}\) are equal to 1. Thus, all told
\[
\frac{v_o}{v_e} = \left(1 + \frac{v_\eta}{c}\right)^{-1} \frac{dt}{dT} \bigg|_{T_e}
= \left(1 + \frac{\eta(T_e)}{R_h}\right)^{-1} \Phi(T_e),
\]
where \(\Phi(T_e)\) is the Schwarzschild factor for the mean value of the radius at the expansion time of the source.

\(\Phi(T_e)\) is the Schwarzschild factor for the mean value of the radius at the expansion time of the source.
which means that in de Sitter

\[ 1 + z \equiv \frac{v_e}{v_o} = \left[ 1 - \frac{\eta(T_e)}{R_h} \right]^{-1}. \]  

(35)

According to Equation (4), this expression should be equivalent to \(a(t_0)/a(t_e)\), so let us see if this is indeed the case.

Along a null geodesic from \(t_e\) to \(t_o\), we have

\[ \int_0^r dr' = c \int_{t_e}^{t_o} dt' \exp(Ht') , \]  

(36)

so

\[ r = \frac{c}{H} \left( e^{-Ht_o} - e^{-Ht_e} \right). \]  

(37)

That is,

\[ \eta(t_e) = a(t_e)r = \frac{c}{H} \left( 1 - e^{-H(t_o-t_e)} \right). \]  

(38)

And therefore

\[ 1 + z = \frac{a(t_o)}{a(t_e)} = e^{H(t_o-t_e)} = \left[ 1 - \frac{\eta(T_e)}{R_h} \right]^{-1}, \]  

(39)

fully consistent with the result in Equation (35). As we found in the case of Milne, the cosmological redshift in de Sitter may be calculated either from the expansion factor \(a(t)\), or from the time dilation and frequency shift associated with motion of the source. For several reasons, the de Sitter case is even more important than Milne in this discussion because it clearly represents a situation in which the redshift is due to both gravitational and kinematic effects. We see in both cosmologies that the interpretation of redshift as an expansion of space is dependent upon the coordinates one chooses to calculate \(z\).

### 3.4 The Lanczos Universe

The Lanczos Universe (Lanczos 1924) is described by the metric

\[ ds^2 = c^2 dt^2 - (cb)^2 \cosh^2(t/b) \left[ \frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \right] , \]  

(40)

where \(b\) is a constant (though not the Hubble constant \(H \equiv \dot{a}/a\)) and \(k = +1\). The expansion factor is \(a(t) = (cb) \cosh(t/b)\), so \(H = (1/b) \tanh(t/b)\). This solution represents the gravitational field of a rigidly rotating dust cylinder coupled to a cosmological constant. We use the following transformation (see Florides 1980) to render this metric in static form:

\[ \eta = cbr \cosh(t/b) , \]  

(41)

and

\[ \tanh(T/b) = \left( 1 - r^2 \right)^{-1/2} \tanh(t/b) , \]  

(42)
which together allow us to write the interval in the form
\[ ds^2 = \left[ 1 - \left( \frac{\eta}{c b} \right)^2 \right] c^2 dT^2 - \left[ 1 - \left( \frac{\eta}{c b} \right)^2 \right]^{-1} d\eta^2 - \eta^2 d\Omega^2. \] (43)

We now follow the steps used for the Minkowski, Milne, and de Sitter metrics, and first calculate the time dilation
\[ \frac{dt}{dT} = \left[ \left( 1 - \left( \frac{\eta}{c b} \right)^2 \right) - \frac{1}{c^2} \left( 1 - \left( \frac{\eta}{c b} \right)^2 \right)^{-1} \left( \frac{d\eta}{dT} \right) \right]^{1/2}. \] (44)

But since \( dr = 0 \) (and therefore \( dr/dT = 0 \)) in the Hubble flow, we have
\[ \frac{d\eta}{dT} = cr \sinh(t/b) \frac{dt}{dT}. \] (45)

From Equations (44) and (45) we therefore see that
\[ \frac{dt}{dT} = \frac{1 - r^2 \cosh^2(t/b)}{\sqrt{1 - r^2}} \] (46)

(which may also be confirmed directly from Equation 42). Thus, following the same argument as before, the ratio of the frequency of light actually measured by the observer to that emitted is given by Equation (13), where now
\[ v_\eta = \sqrt{\frac{g_{\eta\eta}}{g_{TT}}} \frac{d\eta}{dT} = \frac{cr \sinh(t/b)}{\sqrt{1 - r^2}}. \] (47)

Therefore
\[ \frac{v_0}{v_e} = \left. \frac{1 - r^2 \cosh^2(t/b)}{\sqrt{1 - r^2} + r \sinh(t/b)} \right|_{r_e}, \] (48)

and the redshift in this cosmology is given by
\[ 1 + z = \frac{v_e}{v_0} = \left. \frac{\sqrt{1 - r^2} + r \sinh(t/b)}{1 - r^2 \cosh^2(t/b)} \right|_{r_e}. \] (49)

To compare this expression with the result we would have obtained from Equation (4), consider the propagation of a light signal from its emission at comoving distance \( r_e \) at time \( t_e \), on its way to the observer at \( r = 0 \) and time \( t_0 \). The geodesic equation describing this trajectory (derived from Equation 40) is
\[ \int_0^{r_e} \frac{dr}{\sqrt{1 - r^2}} = \int_{t_0/b}^{t_e/b} \frac{du}{\cosh(u)}, \] (50)

whose solution may be written
\[ \sin^{-1} (r_e) = 2 \tan^{-1} \left( e_{b/0}^{b/0} \right) - 2 \tan^{-1} \left( e_{e/b}^{e/b} \right). \] (51)

Therefore,
\[ r_e = 2 \sin \left[ \tan^{-1} \left( e_{b/0}^{b/0} \right) - \tan^{-1} \left( e_{e/b}^{e/b} \right) \right] \cos \left[ \tan^{-1} \left( e_{b/0}^{b/0} \right) - \tan^{-1} \left( e_{e/b}^{e/b} \right) \right], \] (52)

and after some algebra, using the identities \( \sin (\tan^{-1} [x]) = x (1 + x^2)^{-1/2} \) and \( \cos (\tan^{-1} [x]) = (1 + x^2)^{-1/2} \), one finds that
\[ r_e = 2 \left( e_{b/0}^{b/0} - e_{e/b}^{e/b} \right) \left( 1 + e^{(b/0 + t_e)/b} \right) \left( 1 + e^{2b/0} \right)^{-1} \left( 1 + e^{2t_e/b} \right)^{-1}. \] (53)
With further lengthy algebraic manipulations, substituting this expression into Equation (49) produces the final result,

$$1 + z = \frac{\cosh(t_0/b)}{\cosh(t_e/b)} ,$$

(54)

which is the correct form of Equation (4) for the Lanczos expansion factor $a(t) = (cb) \cosh(t/b)$.

### 3.5 A Lanczos Universe with $k = -1$

The application of our procedure to the next case is very similar to Lanczos, so there is no need to dwell on the various steps. The fifth static FRW metric is simply the Lanczos Universe with $k = -1$, for which

$$ds^2 = c^2 dt^2 - (cb)^2 \sinh^2(t/b) \left[ \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \right] ,$$

(55)

where $a(t) = (cb) \sinh(t/b)$. The metric may be written in static form with the transformation

$$\eta = cbr \sinh(t/b) ,$$

(56)

and

$$\tanh(T/b) = \left(1 + r^2\right)^{1/2} \tanh(t/b) ,$$

(57)

which together allow us to write the interval in the form

$$ds^2 = \left[1 - \left(\frac{\eta}{cb}\right)^2\right] c^2 dT^2 - \left[1 - \left(\eta/cb\right)^2\right]^{-1} d\eta^2 - \eta^2 d\Omega^2 ,$$

(58)

identical (in terms of $\eta$ and $T$) to the Lanczos metric in Equation (43). We see immediately that the redshift in this case is also given by Equation (44) though, of course, $\eta$ and $T$ are here given by Equations (56) and (57), respectively, instead of (41) and (42). Therefore, in this case we have

$$\frac{d\eta}{dT} = cr \cosh(t/b) \frac{dt}{dT} ,$$

(59)

and

$$\frac{dt}{dT} = \frac{1 - r^2 \sinh^2(t/b)}{\sqrt{1 + r^2}} ,$$

(60)

(which may also be confirmed directly from Equation 57).

The proper velocity is thus

$$v_\eta \equiv \sqrt{g_{\eta\eta}} \frac{d\eta}{dT} = \frac{cr \cosh(t/b)}{\sqrt{1 + r^2}} ,$$

(61)

and the redshift analogous to Equation (49) is

$$1 + z = \frac{v_e}{v_0} = \left. \frac{\sqrt{1 + r^2} + r \cosh(t/b)}{1 - r^2 \sinh^2(t/b)} \right|_{t_e} .$$

(62)

The exercise is completed by calculating $r_e \equiv r(t_e)$ from the geodesic equation

$$\int_{r_0}^{r_e} \frac{dr}{\sqrt{1 + r^2}} = \int_{t_0/b}^{t_e/b} \frac{du}{\sinh(u)} ,$$

(63)
whose solution is
\[
\sinh^{-1}(r_e) = \ln \left( \tanh \left[ t_0/2b \right] \right) - \ln \left( \tanh \left[ t_e/2b \right] \right).
\]
That is,
\[
r_e = \frac{1}{2} \left( \frac{\tanh(t_0/2b)}{\tanh(t_e/2b)} + \frac{\tanh(t_e/2b)}{\tanh(t_0/2b)} \right).
\]
Another lengthy algebraic manipulation following the substitution of this expression into Equation (62) produces the result
\[
1 + z = \frac{\sinh \left( t_0/b \right)}{\sinh \left( r_e/b \right)},
\]
which matches the correct form of Equation (4) for the expansion factor appropriate for this metric.

### 3.6 Anti-de Sitter Space (A Universe with Negative Mass Density)

The sixth, and final, static FRW metric is that for a Universe with negative mass density and spatial curvature $k = -1$. Known as anti-de Sitter space, due to its negative spacetime curvature, this metric is given by
\[
ds^2 = c^2 dt^2 - (cb)^2 \sin^2 \left( t/b \right) \left[ \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \right],
\]
where clearly the expansion factor is now $a(t) = cb \sin \left( t/b \right)$. The coordinate transformation
\[
\eta = cbr \sin \left( t/b \right),
\]
and
\[
\tan(T/b) = \left( 1 + r^2 \right)^{1/2} \tan(t/b),
\]
produces the static form of the metric,
\[
ds^2 = \left[ 1 + \left( \frac{\eta}{cb} \right)^2 \right] c^2 dT^2 - \left[ 1 + \left( \frac{\eta}{cb} \right)^2 \right]^{-1} \eta^2 d\Omega^2.
\]
The proper velocity is now
\[
v_\eta = \frac{cr \cos(t/b)}{\sqrt{1 + r^2}},
\]
where
\[
\frac{dt}{dT} = \frac{1 + r^2 \sin^2 \left( t/b \right)}{\sqrt{1 + r^2}}.
\]
For this metric, the redshift is therefore
\[
1 + z = \frac{\nu_e}{\nu_0} = \left. \frac{\sqrt{1 + r^2 + r \cos(t/b)}}{1 + r^2 \sin^2(t/b)} \right|_{T_e}.
\]
Now, along a geodesic,
\[
\int_0^{r_e} \frac{dr}{\sqrt{1 + r^2}} = \int_{t_e/b}^{t_0/b} \frac{du}{\sin(u)},
\]

\[
\]
which has the solution
\[
    r_e = \frac{1}{2} \left( \frac{\tan(t_0/2b)}{\tan(t_e/2b)} - \frac{\tan(t_e/2b)}{\tan(t_0/2b)} \right).
\]  
(75)

And substituting this expression for \(r_e\) into Equation (73) then gives
\[
    1 + z = \frac{\sin(t_0/b)}{\sin(t_e/b)},
\]  
(76)

which is again the correct form of the redshift in terms of the expansion factor for this metric.

4 CONCLUSIONS

The cosmological redshift has the same form in terms of the expansion factor regardless of whether the spacetime curvature is constant or not. In this paper, we have focused on the six static FRW metrics, and showed for them that the interpretation of \(z\) as due to the “stretching of light” in an expanding space is coordinate dependent. When calculated using an alternative set of coordinates, the redshift has precisely the same contributions—Doppler and gravitational shifts—that one would expect from the calculation of the lapse function in other applications of general relativity. This association may break down for the non-static FRW metrics, but it would be difficult to see why, given that the formulation of \(z\) in terms of the expansion factor \(a(t)\) is identical in all cases. Still, the proof we have presented here is only partial. It remains to be seen whether the cosmological redshift is a lapse function even when the spacetime curvature changes with time.

There are many reasons why the distinction between an expanding space and a fixed space through which particles move is dynamically important. For example, one sometimes hears statements to the effect that light in cosmology can be transported over vast distances faster than one would infer on the basis of \(c\) alone. The justification for this is that the speed of light is limited to \(c\) only in an inertial frame, but if space is expanding, then light can be carried along with the expansion at even higher speeds. However, it is not difficult to understand why such notions arise from the improper use of the coordinates. In general relativity, the velocity measured by an observer is the proper velocity (e.g., Equation 12), calculated in terms of the proper distance and proper time. For light, \(ds\) satisfies the null condition (i.e., \(ds = 0\)) and therefore \(v_\eta\) is always equal to \(c\), regardless of which coordinate system is being used, or even if the frame of reference is inertial or not. What is true is that the speed \(d\eta/dt\) is not restricted to \(c\). But this is not the proper speed measured by a single observer using solely his rulers and clocks, because the quantity \(\eta = a(t)r\) is a community distance, compiled from the infinitesimal contributions of myriads of observers lined up between the endpoints (see, e.g., Weinberg 1972). A demonstration that \(z\) is not due to the
stretches of space affirms these conclusions by removing the possibility that light may be “carried along” superluminally with the expansion.

Though we have only partially addressed the issue concerning the origin of cosmological redshift, we can now nonetheless turn these results around and ask the opposite question. If there really exists a third mechanism producing a redshift, beyond Doppler and gravity, why don’t we see it manifested in the static FRW metrics? After all, FRW spacetimes with constant curvature also satisfy Equation (4), just like the rest do. And if Equation (4) is evidence that \( z \) arises from the stretching of light in an expanding space, this process should happen regardless of whether the metric is static or not.

In closely related work, Chodorowski (2011) uses a very different technique to arrive at results similar to those reported in this paper. The fact that these two approaches lead to the same conclusions adds significantly to the validity of (his and) our thesis that cosmological redshift is not due to a new form of wavelength extension, beyond those from kinematic and gravitational effects. Chodorowski’s approach is beautifully complementary to that described here because very different coordinates systems are utilized in the derivations. We have sought metrics that can be written in static form, though the transformed coordinates do not necessarily describe a local inertial frame. Yet the velocity of the source may be expressible in terms of these coordinates, as long as we correctly use the proper distance and proper time to evaluate this (proper) velocity. (By the way, this is what we typically do with the Schwarzschild and Kerr metrics.) The decomposition of the cosmological redshift is then based on this proper velocity. If the velocity is zero in this coordinate system, then the time dilation is entirely due to the curvature (or gravity), but the cosmological redshift generally includes a second factor that enters because sources are moving with the Hubble flow. Chodorowski instead chooses to parallel-transport the source’s velocity into the local inertial frame of the central observer, thereby providing a means of calculating the “Dopplerian” redshift (as he calls it) in this frame, with “the rest” arising from the effects of curvature.

What’s interesting, of course, is that because the two sets of coordinates are different (one inertial, the other not), the two decompositions are generally not equal, but the final results are the same, as they should be because the underlying physics is identical.

Demonstrating that \( z \) is a lapse function even for the time-dependent FRW metrics is quite challenging. But given the importance of understanding the origin of cosmological redshift, it is a task worth undertaking. We mention in this regard that Mizony & Lachièze-Rey (2005) found a way of transforming an FRW metric into a local static form, which interestingly is equivalent to de Sitter in this limited domain. Following their approach may be a very useful intermediate step.
in the process of finding the lapse function globally in cases where the spacetime curvature is not constant. We will examine this question next and hope to report the results of these efforts in the near future.

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