A LOCAL-GLOBAL PRINCIPLE FOR TORSORS UNDER GEOMETRIC PROSOLVABLE FUNDAMENTAL GROUPS II

Mohamed Saïdi

Abstract. We prove a local-global principle for torsors under the prosolvable geometric fundamental group of an affine curve over a number field.

Contents

§0. Introduction
§1. Prosolvable geometric tame fundamental groups
§2. Geometrically prosolvable arithmetic fundamental groups
§3. Proof of Theorem B

§0. Introduction. Let $k$ be a characteristic 0 field and $X \to \text{Spec } k$ a separated, smooth, and geometrically connected curve over $k$. Let $\eta$ be a geometric point of $X$ with values in its generic point. Thus, $\eta$ determines an algebraic closure $\bar{k}$ of $k$, and a geometric point $\bar{\eta}$ of $X_{\bar{k}} \defeq X \times_{\text{Spec } k} \text{Spec } \bar{k}$. There exists a canonical exact sequence of profinite groups (cf. [Grothendieck], Exposé IX, Théorème 6.1)

$$1 \to \pi_1(X_{\bar{k}}, \bar{\eta}) \to \pi_1(X, \eta) \to G_k \to 1.$$ 

Here, $\pi_1(X, \eta)$ denotes the arithmetic étale fundamental group of $X$ with base point $\eta$, $\pi_1(X_{\bar{k}}, \bar{\eta})$ denotes the étale fundamental group of $X_{\bar{k}}$ with base point $\bar{\eta}$, and $G_k \defeq \text{Gal}(\bar{k} / k)$ denotes the absolute Galois group of $k$. Write

$$\Delta \defeq \pi_1(X_{\bar{k}}, \bar{\eta}) \quad \text{and} \quad \Pi \defeq \pi_1(X, \eta).$$

We have the above natural exact sequence

$$(0.1) \quad 1 \to \Delta \to \Pi \to G_k \to 1.$$ 

Suppose that the sequence (0.1) splits [for example assume that $X(k) \neq \emptyset$]. Let $s : G_k \to \Pi$ be a section of the projection $\Pi \to G_k$. We view $\Delta$ as a $G_k$-group via the conjugation action of $s(G_k)$.

Assume that $k$ is a number field; i.e., $k$ is a finite extension of $\mathbb{Q}$. Let $v$ be a prime of $k$, $k_v$ the completion of $k$ at $v$, and $G_{k_v} \subset G_k$ a decomposition group associated to $v$. Thus, $G_{k_v}$ is only defined up to conjugation. We view $\Delta$ as a
$G_k$-group via the conjugation action of $s(G_{k_v})$. For each prime $v$ of $k$ we have a natural restriction map (of pointed non-abelian cohomology sets)

$$\text{Res}_v : H^1(G_k, \Delta) \to H^1(G_{k_v}, \Delta),$$

and a natural map

$$\prod_{\text{all v}} \text{Res}_v : H^1(G_k, \Delta) \to \prod_{\text{all v}} H^1(G_{k_v}, \Delta),$$

where the product is over all primes $v$ of $k$. The main problem we are concerned with in this paper is the following.

**Question A.** Is the map

$$\prod_{\text{all v}} \text{Res}_v : H^1(G_k, \Delta) \to \prod_{\text{all v}} H^1(G_{k_v}, \Delta)$$

injective?

As explained in [Saïdi] §0, the above question is related to the Grothendieck anabelian section conjecture. Let $\Delta^{\text{sol}}$ be the maximal prosolvable quotient of $\Delta$, which is a characteristic quotient. The above $G_k$ (resp. $G_{k_v}$)-group structure on $\Delta$ induces naturally a $G_k$ (resp. $G_{k_v}$)-group structure on $\Delta^{\text{sol}}$. Let $\mathfrak{Primes}_k$ be the set of primes of $k$ and $S \subseteq \mathfrak{Primes}_k$ a non-empty subset, we have as above a natural restriction map

$$\prod_{v \in S} \text{Res}_v^{\text{sol}} : H^1(G_k, \Delta^{\text{sol}}) \to \prod_{v \in S} H^1(G_{k_v}, \Delta^{\text{sol}}).$$

In [Saïdi] we proved the following result.

**Theorem A.** Assume $k$ is a number field, $X$ is proper, and $S \subseteq \mathfrak{Primes}_k$ is a set of primes of $k$ of density 1. Then the map

$$\prod_{v \in S} \text{Res}_v^{\text{sol}} : H^1(G_k, \Delta^{\text{sol}}) \to \prod_{v \in S} H^1(G_{k_v}, \Delta^{\text{sol}})$$

is injective.

In this note we generalise Theorem A by removing the assumption therein that $X$ is proper. Our main result is the following.

**Theorem B.** Assume $k$ is a number field, $X$ is affine, and $S \subseteq \mathfrak{Primes}_k$ is a set of primes of $k$ of density 1. Then the map

$$\prod_{v \in S} \text{Res}_v^{\text{sol}} : H^1(G_k, \Delta^{\text{sol}}) \to \prod_{v \in S} H^1(G_{k_v}, \Delta^{\text{sol}})$$

is injective.

Our proof of Theorem B relies on a devissage argument, and a careful analysis of the structure of the geometric prosolvable (resp. geometrically prosolvable arithmetic) (tame) fundamental group of an affine curve which is established in §1 (resp. §2). The results established in §1 and §2 may be of interest independently of the question discussed in this paper. The abelian analog of Theorem B is a consequence of results of Serre (cf. proof of Proposition 3.2). Theorem B is proved in §3.

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Notations. The following notations will be used throughout this paper (unless we specify otherwise). For a profinite group $H$:

- we denote by $[H, H]$ the closed subgroup of $H$ which is topologically generated by the commutator subgroup of $H$.
- we denote by $H^{ab} \overset{\text{def}}{=} H/[H, H]$ the maximal abelian quotient of $H$.
- we denote by $H^{\text{sol}}$ the maximal pro-solvable quotient of $H$.
- For an exact sequence $1 \to H' \to H \xrightarrow{\text{pr}} G \to 1$ of profinite groups we will refer to a continuous homomorphism $s : G \to H$ such that $\text{pr} \circ s = \text{id}_G$ as a group-theoretic section, or simply section, of the natural projection $\text{pr} : H \onto G$.

§1. Prosolvable geometric tame fundamental groups. Let $\ell$ be an algebraically closed field of characteristic $p \geq 0$, $V \to \text{Spec} \, \ell$ a separated, smooth, and connected $\ell$-curve, $Z$ the smooth compactification of $V$, and $Z \setminus V$ the complement of $V$ in $Z$ which is either empty if $Z = V$ or otherwise $V$ is affine and $Z \setminus V$ consists of finitely many closed points of $Z$. Let $r \overset{\text{def}}{=} \text{Card}(Z \setminus V)$ be the cardinality of $Z \setminus V$, thus $r = 0$ if $Z = V$. Let $\xi$ be a geometric point of $V$ with values in its generic point, and $\Delta \overset{\text{def}}{=} \pi_1^{\text{tame}}(V, \xi)$ the tame fundamental group of $V$ with base point $\xi$. The geometric point $\xi$ determines a separable closure $K^{\text{sep}}$ of the function field $K$ of $Z$. Thus, $\Delta \overset{\text{def}}{=} \text{Gal}(K^{\text{tame}}/K)$ where $K^{\text{tame}}$ is the maximal sub-extension of $K^{\text{sep}}/K$ which is étale above $V$ and tamely ramified over the (discrete) valuations of $K$ corresponding to closed points of $Z \setminus V$.

Consider the derived series of $\Delta$

\[ \Delta(1) \subseteq \Delta(2) \subseteq \ldots \subseteq \Delta(i) \subseteq \ldots \subseteq \Delta(1) \subseteq \Delta(0) = \Delta \]

where

\[ \Delta(i + 1) = [\Delta(i), \Delta(i)], \]

for $i \geq 0$, is the $i + 1$-th derived subgroup which is a characteristic subgroup of $\Delta$.

Write

\[ \Delta_i \overset{\text{def}}{=} \Delta / \Delta(i). \]

Thus, $\Delta_i$ is the $i$-th step solvable quotient of $\Delta$, and $\Delta_1 \overset{\text{def}}{=} \Delta^{ab}$ is the maximal abelian quotient of $\Delta$. Note that there exists a natural exact sequence

\[ 1 \to \Delta^{i+1} \to \Delta_{i+1} \to \Delta_i \to 1 \]

where $\Delta^{i+1}$ is the subgroup $\Delta(i)/\Delta(i+1)$ of $\Delta_{i+1}$. In particular, $\Delta^{i+1}$ is abelian.

Lemma 1.1. We have a natural identification $\Delta^{\text{sol}} \overset{\sim}{\to} \lim_{i \geq 1} \Delta_i$. In particular, $\text{Ker}(\Delta \to \Delta^{\text{sol}}) = \bigcap_{i \geq 1} \Delta(i)$, and $\Delta^{\text{sol}}$ is a characteristic quotient of $\Delta$.

Proof. Follows from the various definitions. $\square$
Let $i \geq 1$ be an integer. The profinite group $\Delta_i$ is finitely generated as follows from the well-known finite generation of $\Delta$ which projects onto $\Delta_i$ (cf. [Grothendieck], Exposé XIII, Corollaire 2.12). Let $\{\Delta_i[n]\}_{n \geq 1}$ be a countable system of characteristic open subgroups of $\Delta_i$ such that
\[
\Delta_i[n + 1] \subseteq \Delta_i[n], \quad \Delta_i[1] \overset{\text{def}}{=} \Delta_i, \quad \text{and} \quad \bigcap_{n \geq 1} \Delta_i[n] = \{1\}.
\]

Write $\Delta_i[n]$ for the inverse image of $\widehat{\Delta}_i[n]$ in $\Delta$ via the map $\Delta \to \Delta_i$. Thus, $\Delta_i[n] \subseteq \Delta$ is an open subgroup corresponding to an étale Galois cover $V_{i,n} \to V$ which extends to a tamely ramified Galois cover $Z_{i,n} \to Z$ between proper, smooth, and connected $\ell$-curves (the cover $Z_{i,n} \to Z$ is étale if $r = 0$, i.e., if $V = Z$, or if $i = r = 1$). Note that since the étale cover $V_{i,n} \to V$ is defined via an open subgroup of $\pi^\text{tame}_1(V, \xi)$ it is a pointed étale cover, and $V_{i,n}$ (hence also $Z_{i,n}$) is naturally endowed with a geometric point $\xi_{i,n}$ above $\xi$.

If $i = r = 1$, let
\[
\Delta_i[n] \overset{\text{def}}{=} \Delta_i[n] = \pi^\text{tame}_1(V_{i,n}, \xi_{i,n})
\]
be the tame fundamental group of $V_{i,n}$ with base point $\xi_{i,n}$.

If $(i, r) \neq (1, 1)$, let
\[
\Delta_i[n] \overset{\text{def}}{=} \pi_1(Z_{i,n}, \xi_{i,n})
\]
be the étale fundamental group of $Z_{i,n}$ with geometric point $\xi_{i,n}$, which is a quotient of $\Delta_i[n] = \pi^\text{tame}_1(V_{i,n}, \xi_{i,n})$.

Let $i \geq 1$. We have, $\forall n \geq 1$, finite morphisms $V_{i,n+1} \to V_{i,n}$, and $Z_{i,n+1} \to Z_{i,n}$, which induce continuous homomorphisms $\Delta_i[n + 1] \to \Delta_i[n]$ and $\Delta_i[n + 1]^{\text{ab}} \to \Delta_i[n]^{\text{ab}}$. Thus, we have a projective system $\{\Delta_i[n]^{\text{ab}}\}_{n \geq 1}$.

Our main result in this section is the following description of the structure of $\Delta_i^{i+1}$; this description in the case $(i, r) \neq (1, 1)$ is in terms of the Tate modules of the jacobians of the $\{Z_{i,n}\}_{n \geq 1}$. Proposition 1.2 plays a key role in the proof of Theorem B and may be of interest independently of the question discussed in this paper.

**Proposition 1.2.** Assume $i \geq 1$. We have a natural identification
\[
\Delta_i^{i+1} \cong \varinjlim_{n \geq 1} \Delta_i[n]^{\text{ab}}.
\]

**Proof.** If $r = 0$, i.e., $V = Z$ is proper, or if $i = r = 1$, the assertion follows easily from the various Definitions. Observe that in the case $i = r = 1$ the natural projection $\Delta_1 = \pi^\text{tame}_1(V, \xi)^{\text{ab}} \to \pi_1(Z, \xi)^{\text{ab}}$ is an isomorphism.

Next, we assume $V = Z \setminus \{x_1, \ldots, x_r\}$ is affine and $V$ is the complement in $Z$ of a finite set $\{x_1, \ldots, x_r\}$ of closed points with $\max(i, r) > 1$. Let $G$ be a finite quotient of $\Delta_{i+1}$ which inserts in the following commutative diagram
\[
\begin{array}{cccccc}
1 & \longrightarrow & \Delta^{i+1} & \longrightarrow & \Delta_{i+1} & \longrightarrow & \Delta_{i} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 1
\end{array}
\]
where the vertical maps are surjective. The quotient $G$ corresponds to a finite étale Galois cover $V_1 \to V$ which extends to a tamely ramified finite Galois cover $Z_1 \to Z$ with Galois group $G$. The cover $Z_1 \to Z$ factorizes as $Z_1 \to Z'_1 \to Z$, where $Z'_1 \to Z$ is the sub-cover with Galois group $G'$, and $Z_1 \to Z'_1$ is a Galois abelian cover with group $A$. For $s \in \{1, \ldots, r\}$, let $I_{x_s} \subset G$ be an inertia subgroup associated to $x_s$. Thus, $I_{x_s}$ is only defined up to conjugation. Moreover, $I_{x_s}$ is cyclic of order $e_s \geq 1$ with $\gcd(e_s, p) = 1$ as the ramification is tame. The following claim follows immediately from the well-known structure of $\Delta$ (cf. [Grothendieck], Exposé XIII, Corollaire 2.12).

Claim 1.2.2. There exists a finite quotient $\Delta_i \to P$ of $\Delta_i$ corresponding to a finite tamely ramified Galois cover $Z_2 \to Z$ which is étale above $V$ such that the followings hold. For $s \in \{1, \ldots, r\}$, if $I'_{x_s} \subset P$ is an inertia subgroup associated to $x_s$, then $I'_{x_s}$ is cyclic of order $f_s = e_s h_s$ a multiple of $e_s$ with $\gcd(f_s, p) = 1$.

Note that Claim 1.2.2, which holds under the assumption $\max(i, r) > 1$, is false if $i = r = 1$. Indeed, a finite abelian tamely ramified cover $C' \to C$ between proper and connected smooth $\ell$-curves which is étale above $C \setminus \{x\}$, where $x \in C$ is a closed point, is necessarily étale above $C$.

Next, let $K_1 \overset{\text{def}}{=} K_{Z_1}$ (resp. $K_2 \overset{\text{def}}{=} K_{Z_2}$) be the function field of $Z_1$ (resp. $Z_2$), $L = K_1.K_2$ the compositum of $K_1$ and $K_2$ in $K_{\text{sep}}$, and $\tilde{Z}$ the normalisation of $Z$ in $L$. Thus, $\tilde{Z} \to Z$ is a tamely ramified Galois cover with Galois group $H \subset G \times P$ which is étale above $V$.

Lemma 1.2.3. The quotient $\Delta \to H$ factorizes as $\Delta \to \Delta_{i+1} \to H$.

Proof of Lemma 1.2.3. This follows from the facts that $\tilde{Z} \to Z$ is étale above $V$, tamely ramified, $H \subset G \times P$, and $G \times P$ is $(i + 1)$-th step solvable. □

Let $H'$ be the image of $H$ in $G' \times P$. We have a commutative diagram of exact sequences where the vertical maps are natural inclusions.

\[
\begin{array}{cccccc}
1 & \longrightarrow & A_H & \overset{\text{def}}{=} & H \cap (A \times \{1\}) & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & A \times \{1\} & \longrightarrow & G \times P & \longrightarrow & G' \times P & \longrightarrow & 1
\end{array}
\]

Lemma 1.2.4. The group $H'$ is a quotient of $\Delta_i$. Moreover, the cover $\tilde{Z} \to Z$ factorizes as $\tilde{Z} \to \tilde{Z}' \to Z$ where $\tilde{Z}' \to Z$ is étale above $V$, tamely ramified, and Galois with Galois group $H'$, and $\tilde{Z} \to \tilde{Z}'$ is an abelian étale cover with Galois group $A_{H'}$.

Proof of Lemma 1.2.4. The first assertion, as well as the assertion concerning the cover $\tilde{Z}' \to Z$, follow from the various definitions. We prove the last assertion regarding the cover $\tilde{Z} \to \tilde{Z}'$. Recall the factorisation $Z_1 \to Z'_1 \to Z$ where $Z_1 \to Z'_1$ is Galois with abelian Galois group $A$ and $Z'_1 \to Z$ is Galois with group $G'$. Let $\tilde{Z}'$ be the normalisation of $Z$ in the compositum of the function fields of $Z'_1$ and $Z_2$ (in $K_{\text{sep}}$). Thus, $\tilde{Z}' \to Z$ is a Galois cover with Galois group $H'$ and we have the
following commutative diagram

\[
\begin{array}{ccc}
\tilde{Z} & \longrightarrow & Z_1 \\
\downarrow & & \downarrow \\
\tilde{Z}' & \longrightarrow & Z_1' \\
\downarrow & & \downarrow \\
Z_2 & \longrightarrow & Z \\
\end{array}
\]

of finite Galois covers. The ramification index in the Galois cover \(Z_2 \to Z\) above a branched (closed) point \(x_s \in Z\) is divisible by the ramification index above \(x_s\) in the Galois cover \(Z_1 \to Z\) (cf. the condition in Claim 1.2.2 that \(f_s\) is divisible by \(e_s\)). The fact that the cover \(\tilde{Z} \to Z_2\), and a fortiori \(\tilde{Z} \to \tilde{Z}'\) which is abelian with Galois group \(A_H\), is étale follows from Abhyankar’s Lemma (cf. [Grothendieck], Exposé X, Lemma 3.6). □

Going back to the proof of Proposition 1.2, the above discussion shows that the finite quotients \(\Delta_{i+1} \to H\) as in Lemma 1.2.3 form a cofinal system of finite quotients of \(\Delta_{i+1}\). Thus, \(\Delta_{i+1} \iso \varprojlim_H H\). Proposition 1.2 follows from the facts that the various \(H\) above fit in an exact sequence \(1 \to A_H \to H \to H' \to 1\), \(\Delta_i \iso \varprojlim_{H'} H'\), and the above Galois covers \(\tilde{Z} \to \tilde{Z}'\) with group \(A_H\) are étale abelian (cf. Lemma 1.2.4).

This finishes the proof of Proposition 1.2. □

\section{Geometrically prosolvable arithmetic fundamental groups.}

In this section we use the notations in §0: \(k\) is a field of characteristic 0, and \(X \to \text{Spec }k\) is a separated, smooth, and geometrically connected (not necessarily proper) curve. Consider the derived series of \(\Delta\) [recall \(\Delta = \pi_1(X_{\bar{k}}, \bar{\eta})\)]

\[(2.1) \qquad \ldots \subseteq \Delta(i + 1) \subseteq \Delta(i) \subseteq \ldots \subseteq \Delta(1) \subseteq \Delta(0) = \Delta\]

where

\[\Delta(i + 1) = [\Delta(i), \Delta(i)], \quad \forall i \geq 0.\]

For \(i \geq 0\) write

\[\Delta_i \overset{\text{def}}{=} \Delta/\Delta(i).\]

Thus, \(\Delta_i\) is the \(i\)-th step solvable quotient of \(\Delta\), and \(\Delta_1 \overset{\text{def}}{=} \Delta^{\text{ab}}\) is the maximal abelian quotient of \(\Delta\). There exists a natural exact sequence

\[(2.2) \qquad 1 \to \Delta^{i+1} \to \Delta_{i+1} \to \Delta_i \to 1\]

where \(\Delta^{i+1}\) is the subgroup \(\Delta(i)/\Delta(i + 1)\) of \(\Delta_{i+1}\) (cf. §1).

Recall \(\Pi \overset{\text{def}}{=} \pi_1(X, \eta)\) [cf. exact sequence (0.1)]. Denote

\[\Pi_i \overset{\text{def}}{=} \Pi/\Delta(i)\]
which inserts in the following exact sequence

$$1 \rightarrow \Delta_i \rightarrow \Pi_i \rightarrow G_k \rightarrow 1.$$  

We will refer to $\Pi_i$ as the **geometrically $i$-th step solvable** fundamental group of $X$. We have natural commutative diagrams of exact sequences

$$
\begin{array}{c}
1 & \rightarrow & \Delta & \rightarrow & \Pi & \rightarrow & G_k & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \Delta_i & \rightarrow & \Pi_i & \rightarrow & G_k & \rightarrow & 1
\end{array}
$$

(2.3)

and

$$
\begin{array}{c}
1 & \rightarrow & \Delta_i+1 & \rightarrow & \Pi_i+1 & \rightarrow & G_k & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \Delta_i & \rightarrow & \Pi_i & \rightarrow & G_k & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \Delta & \rightarrow & \Pi & \rightarrow & G_k & \rightarrow & 1
\end{array}
$$

(2.4)

where the left and middle vertical maps in diagram (2.3) are the natural surjections.

Consider the pushout diagram

$$
\begin{array}{c}
1 & \rightarrow & \Delta & \rightarrow & \Pi & \rightarrow & G_k & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \Delta^\text{sol} & \rightarrow & \Pi^\text{(sol)} & \rightarrow & G_k & \rightarrow & 1
\end{array}
$$

(2.5)

which defines the **geometrically prosolvable** fundamental group $\Pi^\text{(sol)}$ of $X$.

**Lemma 2.1.** We have a natural identification $\Pi^\text{(sol)} \xrightarrow{\sim} \varprojlim_{i \geq 1} \Pi_i$.

**Proof.** Follows from Lemma 1.1 and the various definitions. □

For the rest of this section we will assume that $X$ is **affine** and denote by $Y$ the **smooth compactification** of $X$. Thus, $Y \setminus X = \{P_1, \ldots, P_m\}$ consists of $m$-distinct closed points of $Y$, $m \geq 1$. The geometric point $\eta$ of $X$ induces geometric points $\eta$ and $\bar{\eta}$ of $Y$, and $Y_k \overset{\text{def}}{=} Y \times_{\text{Spec } k} \text{Spec } \bar{k}$; respectively. We have a natural exact sequence of fundamental groups

$$1 \rightarrow \Delta^\text{et} \overset{\text{def}}{=} \pi_1(Y_k, \bar{\eta}) \rightarrow \Pi^\text{et} \overset{\text{def}}{=} \pi_1(Y, \eta) \rightarrow G_k \rightarrow 1.$$
By pushing this sequence by the natural projection $\Delta^{\text{et}} \to \Delta_1^{\text{et}} \overset{\text{def}}{=} \pi_1(Y, \bar{\eta})^{\text{ab}}$ we obtain an exact sequence

$$1 \to \Delta_1^{\text{et}} \to \Pi_1^{\text{et}} \to G_k \to 1$$

where $\Pi_1^{\text{et}} \overset{\text{def}}{=} \pi_1(Y, \eta)^{(\text{ab})}$ is the geometrically abelian quotient of $\pi_1(Y, \eta)$. We have an exact sequence

$$1 \to I_X \to \Pi_1 \to \Pi_1^{\text{et}} \to 1$$

where $I_X \overset{\text{def}}{=} \text{Ker}(\Pi_1 \to \Pi_1^{\text{et}}) = \text{Ker}(\Delta_1 \to \Delta_1^{\text{et}})$ is the inertia subgroup of $\Pi_1$. Further, we have an exact sequence of $G_k$-modules

$$0 \to \hat{\mathbb{Z}}(1) \to \prod_{i=1}^m \text{Ind}_{k(P_i)}^k \hat{\mathbb{Z}}(1) \to I_X \to 0$$

as follows from the well-known structure of $\Delta_1 = \pi_1(X_{\overline{\mathbb{Q}}}, \bar{\eta})^{\text{ab}}$ (cf. the discussion in [Saïdi1], §0). Here $\hat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$ where the product is over all prime numbers and $(1)$ is a Tate twist.

Next, let $i \geq 1$ be an integer. The profinite group $\Delta_i$ is finitely generated as follows from the well-known finite generation of $\Delta$ which projects onto $\Delta_i$ (cf. the discussion after Lemma 1.1 and the references therein). Let $\{\hat{\Delta}_i[n]\}_{n \geq 1}$ be a countable system of characteristic open subgroups of $\Delta_i$ such that

$$\hat{\Delta}_i[n+1] \subseteq \hat{\Delta}_i[n], \quad \hat{\Delta}_i[1] \overset{\text{def}}{=} \Delta_i, \quad \text{and} \quad \bigcap_{n \geq 1} \hat{\Delta}_i[n] = \{1\}.$$

Write $\Delta_{i,n} \overset{\text{def}}{=} \Delta_i/\hat{\Delta}_i[n]$. Thus, $\Delta_{i,n}$ is a finite characteristic quotient of $\Delta_i$ which is an $i$-th step solvable group. We have a pushout diagram of exact sequences

$$\begin{array}{cccccc}
1 & \longrightarrow & \Delta_i & \longrightarrow & \Pi_i & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta_{i,n} & \longrightarrow & \Pi_{i,n} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

(2.8)

where the lower sequence defines a (geometrically finite) quotient $\Pi_{i,n}$ of $\Pi_i$.

In the following discussion we fix an integer $i \geq 1$. Suppose that the exact sequence $1 \to \Delta_i \to \Pi_i \to G_k \to 1$ splits. Let

$$s_i : G_k \to \Pi_i$$

be a section of the natural projection $\Pi_i \to G_k$ which induces a section

$$s_{i,n} : G_k \to \Pi_{i,n}$$

of the projection $\Pi_{i,n} \to G_k$ [cf. diagram (2.8)], for each $n \geq 1$. Write

$$\hat{\Pi}_i[n] \overset{\text{def}}{=} \hat{\Pi}_i[n][s_i] \overset{\text{def}}{=} \hat{\Delta}_i[n].s_i(G_k).$$
Thus, $\hat{\Pi}_i[n] \subseteq \Pi_i$ is an open subgroup which contains the image $s_i(G_k)$ of $s_i$. Write

$$\Pi_i[n] \overset{\text{def}}{=} \Pi_i[n][s_i]$$

for the inverse image of $\hat{\Pi}_i[n]$ in $\Pi$ [cf. diagram (2.3)]. Thus, $\Pi_i[n] \subseteq \Pi$ is an open subgroup corresponding to an étale cover $X_{i,n} \to X$, which extends to a (possibly ramified) cover $Y_{i,n} \to Y$

between proper and smooth curves with $Y_{i,n}$ geometrically irreducible (since $\Pi_i[n]$ maps onto $G_k$ via the natural projection $\Pi \to G_k$ by the very definition of $\Pi_i[n]$). Note that the finite cover $(Y_{i,n})_k \overset{\text{def}}{=} Y_{i,n} \times_{\text{Spec} k} \text{Spec } \overline{k} \to Y_k$ is Galois with Galois group $\Delta_{i,n}$, and we have a commutative diagram of covers

$$
\begin{array}{ccc}
(Y_{i,n})_k & \longrightarrow & Y_k \\
\downarrow & & \downarrow \\
Y_{i,n} & \longrightarrow & Y
\end{array}
$$

where $(Y_{i,n})_k \to Y$ is Galois with Galois group $\Pi_{i,n}$ and $(Y_{i,n})_k \to Y_{i,n}$ is Galois with Galois group $s_{i,n}(G_k)$.

Let $r$ be the number of geometric points of $Y \setminus X$, i.e., $r$ is the cardinality of the finite set of closed points of $(Y \setminus X)_k \overset{\text{def}}{=} (Y \setminus X) \times_{\text{Spec} k} \text{Spec } \overline{k}$, where we view $Y \setminus X$ as a (reduced) closed sub-scheme of $Y$ (thus $r \geq m$).

If $(i,r) = (1,1)$, let

$$\tilde{\Delta}_i[n] \overset{\text{def}}{=} \pi_1((X_{i,n})_k, \eta_{i,n})$$

where $(X_{i,n})_k \overset{\text{def}}{=} X_{i,n} \times_{\text{Spec} k} \text{Spec } \overline{k}$, and

$$\tilde{\Pi}_i[n] \overset{\text{def}}{=} \pi_1(X_{i,n}, \eta_{i,n}).$$

If $(i,r) \neq (1,1)$, let

$$\tilde{\Delta}_i[n] \overset{\text{def}}{=} \pi_1((Y_{i,n})_k, \eta_{i,n})$$

and

$$\tilde{\Pi}_i[n] \overset{\text{def}}{=} \pi_1(Y_{i,n}, \eta_{i,n}).$$

We have, $\forall n \geq 1$, finite morphisms $X_{i,n+1} \to X_{i,n}$, and $Y_{i,n+1} \to Y_{i,n}$, which induce a commutative diagram of exact sequences

$$
\begin{array}{cccccc}
1 & \longrightarrow & \tilde{\Delta}_i[n+1] & \longrightarrow & \tilde{\Pi}_i[n+1] & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \tilde{\Delta}_i[n] & \longrightarrow & \tilde{\Pi}_i[n] & \longrightarrow & G_k & \longrightarrow & 1
\end{array}
$$

(2.9)

Note that since the étale cover $X_{i,n} \to X$ [resp. $(X_{i,n})_k \to X_k$] is defined via an open subgroup of $\pi_1(X, \eta)$ [resp. $\pi_1(X_k, \overline{\eta})$] it is a pointed étale cover and $X_{i,n}$ [resp. $(X_{i,n})_k$], hence also $Y_{i,n}$ [resp. $(Y_{i,n})_k$], is naturally endowed with a geometric point $\eta_{i,n}$ (resp. $\overline{\eta}_{i,n}$) above $\eta$ (resp. $\overline{\eta}$).
For each integer $n \geq 1$, consider the pushout diagram

$$
\begin{array}{c}
1 \longrightarrow \tilde{\Delta}_i[n] \longrightarrow \tilde{\Pi}_i[n] \longrightarrow G_k \longrightarrow 1
\end{array}
$$

(2.10)

$$
\begin{array}{c}
1 \longrightarrow \tilde{\Delta}_i[n]^{ab} \longrightarrow \tilde{\Pi}_i[n]^{(ab)} \longrightarrow G_k \longrightarrow 1
\end{array}
$$

where $\tilde{\Pi}_i[n]^{(ab)} = \pi_1(Y_{i,n}(\eta_{i,n})^{(ab)})$ (resp. $\tilde{\Pi}_i[n]^{(ab)} = \pi_1(X_{i,n}(\eta_{i,n})^{(ab)})$, if $i = r = 1$) is the geometrically abelian fundamental group of $Y_{i,n}$ (resp. $X_{i,n}$). Further, consider the following commutative diagram

$$
\begin{array}{c}
1 \longrightarrow \Delta_i^{i+1} \longrightarrow \mathcal{H}_i \overset{\text{def}}{=} \mathcal{H}_i[s_i] \longrightarrow G_k \longrightarrow 1
\end{array}
$$

(2.11)

$$
\begin{array}{c}
1 \longrightarrow \Delta_i^{i+1} \longrightarrow \Pi_{i+1} \longrightarrow \Pi_i \longrightarrow 1
\end{array}
$$

where the lower exact sequence is the sequence in diagram (2.3) and the right square is cartesian. Thus, (the group extension) $\mathcal{H}_i$ is the pullback of (the group extension) $\Pi_{i+1}$ via the section $s_i$.

**Proposition 2.2.** Assume $i \geq 1$. We have natural identifications

$$
\Delta_i^{i+1} \supseteq \varprojlim_{n \geq 1} \tilde{\Delta}_i[n]^{ab} \quad \text{and} \quad \mathcal{H}_i \supseteq \varprojlim_{n \geq 1} \tilde{\Pi}_i[n]^{(ab)}.
$$

*Proof.* The first assertion is Proposition 1.2 [note that $\Delta = \pi_1(V, \eta) = \pi_1^{\text{tame}}(V, \eta)$ since char($k$) = 0]. The second assertion follows from the first and the various Definitions. More precisely, assuming $(i, r) \neq (1, 1)$, let $\tilde{Y}_{i,n} \rightarrow Y_{i,n}$ be the pro-étale cover with Galois group $\tilde{\Pi}_i[n]^{(ab)} = \pi_1(Y_{i,n}(\eta_{i,n})^{(ab)})$. Then $\tilde{Y}_{i,n} \rightarrow Y$ is a Galois cover with Galois group $\tilde{\Pi}_{i,n}$. We have a commutative diagram of morphisms

$$
\begin{array}{c}
\tilde{Y}_{i,n} \longrightarrow (Y_{i,n})_k \longrightarrow Y_k \\
\downarrow \hspace{1cm} \downarrow \\
Y_{i,n} \longrightarrow Y
\end{array}
$$

which induces the following commutative diagram of exact sequences (recall the morphism $(Y_{i,n})_k \rightarrow Y$ is Galois with Galois group $\Pi_{i,n}$)

$$
\begin{array}{c}
1 \longrightarrow \tilde{\Delta}_i[n]^{ab} \longrightarrow \tilde{\Pi}_i[n]^{(ab)}
\end{array}
$$

$$
\begin{array}{c}
1 \longrightarrow \tilde{\Delta}_i[n]^{ab} \longrightarrow \tilde{\Pi}_i[n]^{(ab)} \longrightarrow G_k \longrightarrow 1
\end{array}
$$

$$
\begin{array}{c}
1 \longrightarrow \Delta_i[n]^{ab} \longrightarrow \Pi_i[n]^{(ab)} \longrightarrow G_k \longrightarrow 1
\end{array}
$$
where \( \tilde{\Delta}_{i,n} \overset{\text{def}}{=} \text{Ker}(\tilde{\Pi}_{i,n} \to G_k) \). Recall the section \( s_{i,n} : G_k \to \Pi_{i,n} \) of the projection \( \Pi_{i,n} \to G_k \) induced by the section \( s_i \) [cf. discussion after diagram (2.8)] and consider the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Delta}_{i,n} & \longrightarrow & H_{i,n} \overset{\text{def}}{=} H_{i,n}[s_{i,n}] & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \tilde{\Delta}_{i,n} & \longrightarrow & \tilde{\Pi}_{i,n} & \longrightarrow & \Pi_{i,n} & \longrightarrow & 1 \\
\end{array}
\]

where the right square is cartesian. Thus, (the group extension) \( H_{i,n} \) is the pullback of (the group extension) \( \tilde{\Pi}_{i,n} \) via the section \( s_{i,n} \). It follows from the various Definitions that there exists a natural identification \( H_{i,n} = \tilde{\Pi}_{i,n}[s_{i,n}] \) and a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \Delta_{i+1}^+ & \longrightarrow & \Pi_{i+1} & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \tilde{\Delta}_{i,n} & \longrightarrow & \tilde{\Pi}_{i,n} & \longrightarrow & \Pi_{i,n} & \longrightarrow & 1 \\
\end{array}
\]

where the vertical maps are surjective. Taking projective limits, and using the fact \( \Delta_{i+1}^+ \cong \lim_{n \geq 1} \Delta_i[n]^\text{ab} \), it follows \( \mathcal{H}_i \cong \lim_{n \geq 1} \tilde{\Pi}_i[n]^{(ab)} \). The case \((i,r) = (1,1)\) is treated in an entirely similar way as above. □

Next, assume that the section \( s_i : G_k \to \Pi_i \) lifts to a section \( s_{i+1} : G_k \to \Pi_{i+1} \) of the natural projection \( \Pi_{i+1} \to G_k \), i.e., there exists a section \( s_{i+1} \) and a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \Delta_{i+1}^+ & \longrightarrow & \mathcal{H}_i & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \tilde{\Delta}_i[n]^\text{ab} & \longrightarrow & \tilde{\Pi}_i[n]^{(ab)} & \longrightarrow & G_k & \longrightarrow & 1 \\
\end{array}
\]

where the left and middle vertical map are surjective. We view \( \Delta_{i+1}^+ \) (hence also \( \Delta_i \) and \( \Delta_{i+1}^+ \)) as a \( G_k \)-group via the action of \( s_{i+1}(G_k) \) by conjugation. Thus, this \( G_k \)-group structure on \( \Delta_i \) is the one induced by the section \( s_i \). The above sequence is an exact sequence of \( G_k \)-groups.

**Lemma 2.3.** Assume that \( k \) is a **number field** or a \( p \)-adic local field (i.e., a finite extension of \( \mathbb{Q}_p \)). Then \( H^0(G_k, \Delta_{i+1}^+) = \{0\} \), \( \forall i \geq 0 \).
Proof. First, consider the case $i = 0$ ($\Delta_0 = \{1\}, \Delta^1 = \Delta_1$). We have exact sequences of $G_k$-modules $1 \to I_X \to \Delta_1 \to \Delta^{et}_1 \to 1$, and $1 \to \hat{\Z}(1) \to \prod_{i=1}^m \text{Ind}^k_{k(P_i)} \hat{\Z}(1) \to I_X \to 1$ (cf. discussion after Lemma 2.1). The group $H^0(G_k, \Delta^{et}_1) = \{0\}$ vanishes (see proof of Lemma 1.3 in [Sâidî]). Further, we have an exact sequence $0 \to H^0(G_k, \hat{\Z}(1)) \to H^0(G_k, \prod_{i=1}^m \text{Ind}^k_{k(P_i)} \hat{\Z}(1)) \to H^0(G_k, I_X) \to 0$ as follows from the fact that the map $H^1(G_k, \hat{\Z}(1)) \to H^1(G_k, \prod_{i=1}^m \text{Ind}^k_{k(P_i)} \hat{\Z}(1))$ is injective. Indeed, this latter map is identified (via Kummer theory) with the natural map $(k^\times)^\wedge \to \prod_{i=1}^m (k(P_i)^\times)^\wedge$; where for a field $\ell$ we write $(\ell^\times)^\wedge = \varprojlim_0 \ell^\times/(\ell^\times)^j$, and we use Shapiro’s Lemma to identify $H^1(G_k, \text{Ind}^k_{k(P_i)} \hat{\Z}(1))$ with $H^1(G_k(P_i), \hat{\Z}(1))$. Now the map $(k^\times)^\wedge \to \prod_{i=1}^m (k(P_i)^\times)^\wedge$ is injective if $k$ is a number field or a $p$-adic local field. This follows from the well-known structure of $k^\times$ if $k$ is a $p$-adic local field. If $k$ is a number field this follows from the $p$-adic local field case and the fact that the map $(k^\times)^\wedge \to \prod_{v \in \text{Primes}_k} (k_v^\times)^\wedge$ is injective (this follows from [Neukirch-Schmidt-Wingberg], (9.1.3)Theorem). Also $H^0(G_k, \hat{\Z}(1)) = H^0(G_k, \prod_{i=1}^m \text{Ind}^k_{k(P_i)} \hat{\Z}(1)) = \{0\}$, which implies $H^0(G_k, I_X) = \{0\}$, and hence $H^0(G_k, \Delta_1) = \{0\}$.

Next, assume $i \geq 1$. If $(i, r) \neq (1, 1)$, the group $H^0(G_k, \Delta_i^{i+1})$ is naturally identified with $\varprojlim_{n \geq 1} H^0(G_k, \Delta_i[n]^{ab})$ (cf. Proposition 2.2) where the $G_k$-module $\Delta_i[n]^{ab}$ is isomorphic to the Tate module of the jacobian of the curve $Y_{i,n}$. Further, $H^0(G_k, \Delta_i[n]^{ab}) = \{0\}$ (see proof of Lemma 1.3 in [Sâidî]). Thus, $H^0(G_k, \Delta_i^{i+1}) = \{0\}$ in this case. If $(i, r) = (1, 1)$, the group $H^0(G_k, \Delta_i^{i+1})$ in this case is naturally identified with $\varprojlim_{n \geq 1} H^0(G_k, \Delta_i[n]^{ab})$ (cf. Proposition 2.2) where the $G_k$-module $\Delta_i[n]^{ab}$ is by definition $\pi_1((X_{i,n}, k), \eta_{i,n})^{ab}$. The vanishing of $H^0(G_k, \Delta_i^{i+1})$ follows in this case from the case $i = 0$ discussed above. □

Lemma 2.4. Assume that $k$ is a number field or a $p$-adic local field (i.e., a finite extension of $\Q_p$). Then $H^0(G_k, \Delta_i) = \{1\}, \forall i \geq 0$.

Proof. By induction on $i$, using Lemma 2.3, and the fact that we have an exact sequence of groups $1 \to H^0(G_k, \Delta_i^{i+1}) \to H^0(G_k, \Delta_i^{i+1}) \to H^0(G_k, \Delta_i)$. □

Lemma 2.5. Assume that $k$ is a number field or a $p$-adic local field (i.e., a finite extension of $\Q_p$). Then the natural map $H^1(G_k, \Delta_i^{i+1}) \to H^1(G_k, \Delta_i^{i+1})$ of pointed sets is injective, $\forall i \geq 0$.

Proof. There exists an exact sequence of pointed sets

$$H^0(G_k, \Delta_i) \to H^1(G_k, \Delta_i^{i+1}) \to H^1(G_k, \Delta_i^{i+1})$$

(cf. [Serre], I, §5, 5.5, Proposition 38). The proof follows from [Serre], I, §5, 5.5, Proposition 39 (ii), and the fact that $H^0(G_k, \Delta_i)$ is trivial (cf. Lemma 2.4). □

§3. Proof of Theorem B. This section is devoted to the proof of Theorem B (cf. §0). We use the same notations as in Theorem B, §0, and §2. We assume further that the set $S \subset \text{Primes}_k$ contains no real place.
Recall the commutative diagram of exact sequences of profinite groups

\[
\begin{array}{cccccc}
1 & \longrightarrow & \Delta \overset{\text{def}}{=} \pi_1(X_k, \eta) & \longrightarrow & \Pi \overset{\text{def}}{=} \pi_1(X, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^{\text{sol}} & \longrightarrow & \Pi^{(\text{sol})} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}
\]

[cf. diagram (2.5)] and the natural map (cf. §0)

\[
\prod_{v \in S} \text{Res}^v_{\text{sol}} : H^1(G_k, \Delta^{\text{sol}}) \to \prod_{v \in S} H^1(G_{k_v}, \Delta_v^{\text{sol}}).
\]

We will show this map is injective. Recall the definition of the \(i\)-th step solvable characteristic quotient \(\Delta_i\) of \(\Delta\) [cf. the discussion before the exact sequence (2.2)].

**Proposition 3.1.** The natural map

\[
\prod_{v \in S} \text{Res}^v_i : H^1(G_k, \Delta_i) \to \prod_{v \in S} H^1(G_{k_v}, \Delta_i)
\]

is injective for \(i \geq 1\).

We will prove Proposition 3.1 by an induction argument on \(i \geq 1\). The case \(i = 1\) is treated first in the following Proposition.

**Proposition 3.2.** The natural map

\[
\prod_{v \in S} \text{Res}^v_1 : H^1(G_k, \Delta_1) \to \prod_{v \in S} H^1(G_{k_v}, \Delta_1)
\]

is injective.

**Proof of Proposition 3.2.** Recall the exact sequence \(0 \to I_X \to \Delta_1 \to \Delta_1^{\text{et}} \to 0\) (cf. the discussion after Lemma 2.1). We have a commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(G_k, I_X) & \longrightarrow & H^1(G_k, \Delta_1) & \longrightarrow & H^1(G_k, \Delta_1^{\text{et}}) \\
\Pi_{v \in S} \text{Res}^v_1 & \downarrow & \Pi_{v \in S} \text{Res}^v_1 & \downarrow & \Pi_{v \in S} \text{Res}^v_1^{\text{et}} & \\
0 & \longrightarrow & \prod_{v \in S} H^1(G_{k_v}, I_X) & \longrightarrow & \prod_{v \in S} H^1(G_{k_v}, \Delta_1) & \longrightarrow & \prod_{v \in S} H^1(G_{k_v}, \Delta_1^{\text{et}})
\end{array}
\]

where the horizontal sequences arise from the cohomology exact sequences associated to the exact sequence \(0 \to I_X \to \Delta_1 \to \Delta_1^{\text{et}} \to 0\) of \(G_k\) (and \(G_{k_v}\))-modules, the vertical maps are restriction maps, and the injectivity on the left in the horizontal sequences follows from Lemma 1.4 in [Saïdi]. The injectivity of the left and right vertical maps in the above diagram would imply the injectivity of the middle vertical map. The map \(\prod_{v \in S} \text{Res}^v_1^{\text{et}} : H^1(G_k, \Delta_1^{\text{et}}) \to \prod_{v \in S} H^1(G_{k_v}, \Delta_1^{\text{et}})\) is injective by Proposition 2.2 in [Saïdi]. We shall prove the injectivity of the left vertical map \(\prod_{v \in S} \text{Res}^v_1 : H^1(G_k, I_X) \to \prod_{v \in S} H^1(G_{k_v}, I_X)\). To this end it suffices to prove, for a prime number \(p\), the injectivity of the map \(\prod_{v \in S} \text{Res}^v_1(p) : H^1(G_k, I_X(p)) \to \prod_{v \in S} H^1(G_{k_v}, I_X(p))\) where \(I_X(p)\) is the \(p\)-primary part of \(I_X\). This follows from
results of Serre in [Serre1] and [Serre2]. For a profinite group $G$ and a continuous $G$-module $N$ write $H^1_c(G, N) \overset{\text{def}}{=} \ker[H^1(G, N) \rightarrow \prod_v H^1(G, N)]$ where the product is over all pro-cyclic subgroups of $G$. Recall the exact sequence $0 \rightarrow \mathbb{Z}_p(1) \rightarrow \prod_i \text{Ind}_{k_i(P_i)}^k \rightarrow I_X(p) \rightarrow 0$ of $G_k$-modules (cf. discussion after Lemma 2.1). The $\mathbb{Z}_p$-module $I_X(p)$ is free of finite rank and is a $G_k$-module.

Write $G \overset{\text{def}}{=} \text{Image}[G_k \rightarrow \text{Aut}(I_X(p))]$. Thus, $G$ is a $p$-adic Lie group. Write $G$ for the $p$-adic Lie algebra of $G$. The inflation map $H^1_c(G, I_X(p)) \rightarrow H^1_c(G_k, I_X(p))$ is an isomorphism (cf. [Serre1], Proposition 6). If $v \in \mathfrak{p} \text{rimes}_k$ is a prime of $k$ of residue characteristic $\neq p$, and $\text{Fr}_v \in G$ is a Frobenius at $v$, then its eigenvalues on $I_X(p)$ have complex absolute value $Nv$; where $Nv$ is the cardinality of the residue field $\kappa(v)$. Write $X(G, I)$, $S$ and $H$ using Proposition 3.2 and Proposition 2.2. Fix an integer $i \geq 1$. Consider the following commutative diagram of maps of pointed cohomology sets

$$
\begin{array}{cccc}
1 & \longrightarrow & H^1(G_k, \Delta^{i+1}) & \longrightarrow & H^1(G_k, \Delta_i) \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \prod_v H^1(G_{k_v}, \Delta^{i+1}) & \longrightarrow & \prod_v H^1(G_{k_v}, \Delta_i) \\
& & \text{Res}_v^{i+1} & & \text{Res}_v^i \\
\end{array}
$$

where the horizontal sequences are exact (cf. Lemma 2.5) and the vertical maps are the natural restriction maps. We assume by induction hypothesis that the right vertical map $\prod_v H^1(G_{k_v}, \Delta_i) \rightarrow \prod_v H^1(G_{k_v}, \Delta_i)$ is injective. We will show that the middle vertical map $\prod_v H^1(G_{k_v}, \Delta_{i+1}) \rightarrow \prod_v H^1(G_{k_v}, \Delta_{i+1})$ is injective. Let $[\rho], [\tau] \in H^1(G_k, \Delta_{i+1})$ be two cohomology classes such that

$$\prod_v \text{Res}_v^{i+1}([\tau]) = \prod_v \text{Res}_v^{i+1}([\rho]).$$

Write $s_j : G_k \rightarrow \Pi_j$ for the section of the projection $\Pi_j \rightarrow G_k$ induced by the section $s : G_k \rightarrow \Pi$, for $j \geq 1$. We will show $[\tau] = [\rho]$. We can (without loss of generality), and will, assume that $[\tau] = [s_{i+1}] = 1$ is the distinguished element of $H^1(G_k, \Delta_{i+1})$. The classes $[\rho]$ and $[\tau]$ map to the classes $[\rho_1]$ and $[\tau_1] = [s_i] = 1$ in $H^1(G_k, \Delta_i)$, respectively. In particular, we have the equality

$$\prod_v \text{Res}_v^i([\tau_1]) = \prod_v \text{Res}_v^i([\rho_1]).$$
hence $[\tau] = [\rho] = 1$ since $\prod_{v \in S} \text{Res}^i_v$ is injective by the induction hypothesis. Thus, there exist classes $[\tilde{\tau}] = 1$ and $[\tilde{\rho}]$ in $H^1(G_k, \Delta^{i+1})$ which map to $[\tau]$ and $[\rho]$, respectively in $H^1(G_k, \Delta_{i+1})$ (cf. exactness of horizontal sequences in the above diagram).

Next, and in order to show that $[\rho] = [\tau]$ in $H^1(G_k, \Delta_{i+1})$ it suffices to show $[\tilde{\rho}] = [\tilde{\tau}]$ in $H^1(G_k, \Delta^{i+1})$. Note that the assumption $\text{Res}^i_{\nu+1}([\tau]) = \text{Res}^i_{\nu+1}([\rho])$ implies that $\text{Res}^i_{\nu+1}([\tilde{\tau}]) = \text{Res}^i_{\nu+1}([\tilde{\rho}])$ in $H^1(G_{k_v}, \Delta^{i+1})$, for each place $v \in S$, as follows from the injectivity of the maps $H^1(G_{k_v}, \Delta^{i+1}) \to H^1(G_{k_v}, \Delta_{i+1})$ (cf. Lemma 2.5). We have a commutative diagram of group homomorphisms

\[
\begin{array}{ccc}
H^1(G_k, \Delta^{i+1}) & \longrightarrow & \lim_{n \geq 1} H^1(G_k, \tilde{\Delta}_i[n]^a) \\
\prod_{v \in S} \text{Res}^i_{\nu+1} & \downarrow & \\
\prod_{v \in S} H^1(G_{k_v}, \Delta^{i+1}) & \longrightarrow & \prod_{v \in S} (\lim_{n \geq 1} H^1(G_{k_v}, \tilde{\Delta}_i[n]^a))
\end{array}
\]

where the horizontal maps are induced by the identification $\mathcal{H}_i \sim \lim_{n \geq 1} \Pi_i[n]^a$ (cf. Proposition 2.2) and the vertical maps are restriction maps. The identification $\mathcal{H}_i \sim \lim_{n \geq 1} \Pi_i[n]^a$ induces isomorphisms $H^1(G_k, \Delta^{i+1}) \sim \lim_{n \geq 1} H^1(G_k, \tilde{\Delta}_i[n]^a)$, and $H^1(G_{k_v}, \Delta^{i+1}) \sim \lim_{n \geq 1} H^1(G_{k_v}, \tilde{\Delta}_i[n]^a)$ for each $v \in S$, where the transition homomorphisms in the projective limit are induced by the natural $G_k$ (resp. $G_{k_v}$)-homomorphisms $\tilde{\Delta}_i[n+1]^a \to \tilde{\Delta}_i[n]^a$ (cf. [Neukirch-Schmidt-Wingberg], Chapter II, (2.3.5)Corollary). The right vertical map in the above diagram is injective in the case $(i, r) \neq (1, 1)$ by Proposition 2.2 in [Saïdi] [recall in this case that the $G_k$-module $\tilde{\Delta}_i[n]^a$ is isomorphic to the Tate module of the jacobian of the curve $Y_{i,n}$ defined in §2]. In case $(i, r) = (1, 1)$ the injectivity of the right vertical map in the above diagram follows from Proposition 3.2 [recall; with the notations in §2, that in this case the $G_k$-module $\tilde{\Delta}_i[n]^a$ is $\tau_1(\{Y_{i,n}\}_k, \bar{\mathfrak{n}}_{i,n})^a$].

Thus, the left vertical map $H^1(G_k, \Delta^{i+1}) \to \prod_{v \in S} H^1(G_{k_v}, \Delta^{i+1})$ is injective. In particular, $[\tilde{\rho}] = [\tilde{\tau}]$ and $[\tilde{\rho}] = [\tilde{\tau}]$. This finishes the proof of Proposition 3.1. $\square$

Finally, going back to the proof of Theorem B, let

$[\alpha], [\beta] \in H^1(G_k, \Delta^\text{sol})$

be two cohomology classes such that

$\prod_{v \in S} \text{Res}^\text{sol}([\alpha]) = \prod_{v \in S} \text{Res}^\text{sol}([\beta])$.

One shows $[\alpha] = [\beta]$ by similar arguments to the ones used to conclude the proof of Theorem B in [Saïdi] (cf. loc. cit. discussion after the proof of Lemma 2.1), using Proposition 2.1 and Lemma 2.3 in loc. cit..

This finishes the Proof of Theorem B. $\square$

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Mohamed Saïdi
College of Engineering, Mathematics, and Physical Sciences
University of Exeter
Harrison Building
North Park Road
EXETER EX4 4QF
United Kingdom
M.Saidi@exeter.ac.uk