Power Laws are Logarithmic Boltzmann Laws

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Multiplicative random processes in (not necessarily equilibrium or steady state) stochastic systems with many degrees of freedom lead to Boltzmann distributions when the dynamics is expressed in terms of the logarithm of the normalized elementary variables. In terms of the original variables this gives a power-law distribution. This mechanism implies certain relations between the constraints of the system, the power of the distribution and the dispersion law of the fluctuations. These predictions are validated by Monte Carlo simulations and experimental data. We speculate that stochastic multiplicative dynamics might be the natural origin for the emergence of criticality and scale hierarchies without fine-tuning.

In the last years researchers have found an exceedingly large number of power laws in very many natural and artificial (social, economic) systems.

The emergence of "scaling" properties was considered intriguing as in theoretically

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known models this is related usually with very special "critical" conditions. In the parameter space of typical equilibrium statistical models, critical systems correspond to subspaces of measure zero. Yet scaling systems seem to show up in nature much more often than this theoretically expected measure zero abundance. This lead researchers to coin the term self-organized criticality ([1], [2]).

In the present note we present a simple yet very general explanation of the emergence of power laws. According to our analysis, power-like systems are expected to arise as naturally as the Boltzmann distribution.

In fact we show that for a very large class of systems, their power law distribution is in a precise mathematical relation to a Boltzmann distribution when the measurables are represented on a logarithmic scale. This analysis implies additional relations which are confirmed experimentally.

Consider a system consisting of a large set of elements $i$ which are characterized each by a time-dependent variable $\omega_i(t)$ (for definiteness one can think of a set of investors $i = 1, ..., N$ each owning a wealth $\omega_i$ or $N$ towns containing each $\omega_i$ people).

Assume that the typical variations of $\omega$ are characterized effectively by a multiplicative stochastic law:

$$\omega_i(t+1) = \lambda \omega_i(t)$$

(1)

with $\nu$ being a stochastic variable with a finite support distribution of probability $\pi(\lambda)$.

The effective "transition probability" distribution $\pi(\lambda)$ is assumed not to depend on $i$ or on the actual value of $\omega_i$. However, we will see that our conclusions are not affected if the shape of $\pi(\lambda)$ varies in time during the process.

In order to isolate the shape of the distribution of $\omega$ even for situations in which there is an unbounded overall drift of the $\omega_i(t)$'s towards infinity, we will work in the sequel of the article with the distribution $P(w)$: which fulfills the master equation:

$$P(w, t+1) - P(w, t) = \int A \Pi(\lambda)P(w/\lambda, t)d\lambda - P(w, t)\int A \Pi(\lambda)d\lambda$$

(2)
where \( w \)'s are normalized \( \omega \)'s such as to fulfill at each time:

\[
\sum_i w_i(t) \equiv \int w P(w, t) dw = N \tag{3}
\]
i.e. in the wealth case one represents actually the \textit{relative} wealth of each investor. Correspondingly, the transition probability distribution \( \Pi(\lambda) \) for the new variables is related to \( \pi(\omega) \) by a shift in the argument.

Moreover, one limits from below the allowed values of \( w > w_0 \) (in the wealth case this consists in subsidizing individuals as not to fall below a certain poverty line \( w_0 \)). This implies appropriate changes in the transition probability for \( w_i \)'s in the immediate neighborhood of \( w_0 \).

In order to extract the implications of the dynamics (2) it is convenient to represent it on the logarithmic scale in terms of \( x = \ln w \) and \( \mu = \ln \lambda \). The corresponding probability distributions \( P \) and \( \Pi \) become in the new variables:

\[
P(x) = e^x P(e^x) \tag{4}
\]
and respectively \( \rho(\mu) = e^\mu \Pi(e^\mu) \). In terms of \( P, x, \rho, \mu \), the master equation (2) becomes:

\[
P(x, t + 1) - P(x, t) = \int_{\mu} \rho(\mu) P(x - \mu, t) d\mu - P(x, t) \int_{\mu} \rho(\mu) d\mu \tag{5}
\]

Not that this equation has the standard form of the master equation for an usual Monte Carlo process.

The iteration of the equation (5) for long time sequences projects upon the eigenmode with the largest eigenvalue of the time evolution operator:

\[
\Omega_{\rho} P(x) \equiv \int_{\mu} \rho(\mu) P(x - \mu) d\mu + P(x) \left( 1 - \int_{\mu} \rho(\mu) d\mu \right) \tag{6}
\]
This in turn leads to an asymptotic distribution of \( P \) which fulfills an equation of the form:
\[
\int \rho(\mu) P(x - \mu) d\mu = \Lambda P(x)
\] (7)

Ignoring for the moment the boundary and finite size effects, one can easily verify that the solution of this equation is:

\[ P(x) \sim e^{-x/T} \] (8)

with \( T \) determined by the condition

\[ \int e^{\mu/T} \rho(\mu) d\mu = \Lambda \] (9)

The uniqueness of the solution (8), (9) is insured by the normalization condition (3), the positivity of the density distribution \( P \) and by the fact that for positive \( \rho \) the left hand side in (9) is a convex function in \( \frac{1}{T} \). A rigorous proof that the equation (7) leads to (8) is given in [3] and is based on the extremal properties of the \( G - \) harmonic functions on non-compact groups (in our case the group of translations on \( \mathbb{R} \)).

When one translates back the exponential "Boltzmann" law (8) in terms of the original variables \( w = e^x \) one gets according (4) a power-law distribution:

\[ P(w) \sim w^{(-1 - 1/T)} \] (10)

If one ignores the departures from (8) due to the (upper) boundary and finite size effects one can use the normalization conditions for the total "wealth", \( w \), eq. (3)

\[ C \int_{w_0}^{\infty} w^{-\frac{1}{2}} dw = N \] (11)

and for the total number of elements:

\[ C \int_{w_0}^{\infty} w^{-1-\frac{1}{2}} dw = N \] (12)

in order to express \( T \) in terms only of \( w_0 \):

\[ T = 1 - w_0 \] (13)
This power-law and the above relation are excellently confirmed by simulations in various systems for a wide range of $w_0$'s and is consistent with experimental data. It appears therefore that $T$ is largely independent on the shape of the transition probability distribution $\rho(\mu)$ (or $\Pi(\lambda)$). Physically, an intuitive understanding of this result can be achieved by thinking of eq. (5) in terms of a conservative system in which an energy $\mu$ can be absorbed or emitted by each degree of freedom $i$ according to the (“Monte Carlo”) emission-absorption probability distribution $\rho(\mu)$.

The emergence of a Boltzmann distribution is independent on the details of the energy exchange mechanism: it is more general than the details of the particular dynamical process leading to it. In fact, even if the process itself is not stationary and the ”transition probabilities” $\Pi(\lambda)$ and $\rho(\mu)$ depend on time, the distribution $P(w)$ can still converge: modifying during the process (or during a Monte Carlo simulation) the interactions from short range to infinite range from 2-body to many-body from direct interactions to interactions through the intermediary of a bath or of an ”energy reservoir” is known not affect the Boltzmann distribution.

One sees therefore that a power law is as natural and robust for a stochastic multiplicative process as the Boltzmann law is for an equilibrium statistical mechanics system. Far from being an exception and requiring fine tuning or sophisticated self-organizing mechanisms, this is the default.

For our general mechanism to apply to a scaling system, the system has to fulfill the effective stochastic multiplicative law (1). Yet, the mechanism by which each particular system is lead to fulfill (1) might differ. For instance in the towns example this might be related with interactions between town residents (residents moving upon marrying

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1 For very low values of $w_0$ the finite size effects and the upper bound cannot be ignored and equation (13) is modified. The modified relation is confirmed by Monte Carlo simulations too. In particular for $w_0 = 0$ one gets $T = \infty$. 

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somebody from another town, or upon entering a new employee-employer bound). In large scale universe structures, 2-body gravitational forces might lead to laws similar to \((1)\).

In a series of papers we have studied in detail theoretically and numerically these mechanisms in the context of the stock market (\([7]\), \([8]\), \([5]\)).

Our results turned out to explain in detail the Pareto power law distribution of the individual incomes (experimentally documented in \([9]\)) as well as the Lévy distribution in the market prices fluctuations (reported in \([10]\)).

The Pareto law arises as a consequence of eq. \((10)\) and of eq. \((1)\). These equations imply that the individual speculative incomes
\[
   r_i(t) = \int (\lambda - 1)\Pi(\lambda) w_i(t) d\lambda
\]
distributed by a power law \((10)\) too:
\[
   P(r) = r^{-1 - \frac{1}{1-w_0}}
\]
By a similar argument, one finds that the market price fluctuations induced by individuals are also distributed according to the \((14)\) law. According the generalized central limit theorem, a quantity which is a sum of random variables \(r\) distributed according to a probability distribution \(r^{-\alpha}\) converges to the Lévy distribution \(L_{1-\alpha}\) of characteristic exponent \(1 - \alpha\).

Our analysis implies therefore (and the experimental available data confirm) that if the individual wealth distribution is fitted by a power law of exponent \(-1 - \frac{1}{1-w_0}\) (Fig 1) then the speculative income distribution is governed by the same law \((14)\) (Fig 2) and the market fluctuations are given by a Lévy distribution of characteristic exponent \(-\frac{1}{1-w_0}\) Fig (3).

These relations are confirmed by the available experimental data (and by the Monte Carlo simulation of microscopic representations of the stock market \([3]\)) Figs. (1) (2) (3) with \(-\alpha = -\frac{1}{1-w_0} = -1.4 \text{ [11]}\).

We plan in a future publication to compare with experiment the relations which our mechanism predicts between the rate of inflation, the taxation policy and the lower
income bound (the poverty line $w_0$).

In the context of fundamental physics, one may hope that the extension [12] of the result (8)-(10) to $G$–harmonic distributions on general non-compact (Weyl gauge [13]) groups $G$ might lead microscopic models naturally, without fine tuning, to criticality and scale hierarchies.

Such discrete (lattice gauge) theories with non-compact group might provide a unified context for treating renormalization theory and time: the continuous re-scaling (3) of the ”running to infinity” degrees of freedom $w_i$ (1) suggests $\lambda^t$ as the microscopic stochastic origin of both time flow and renormalization flow (with the ”extremal” [12] distribution $P$ as the fixed point).

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Figure Captions

Figure 1:
The distribution of wealth (log-log scale). The solid line represents actual data from Great Britain (Source: Inland Revenue Statistics, 1970) Dots represent the wealth distribution in our simulations. The dashed line is a fit by a power law distribution with a slope of -2.40 ($\alpha = 1.4$).

Figure 2:
Empirical distribution of income (log-log scale). Data from Great Britain (Source: National Income and Expenditure 1970). Dashed line represents fit by a power law distribution with slope of -2.34.

Figure 3:
Distribution of returns on the stock (semi-logarithmic scale). The solid line represents the Lévy distribution with exponent $\alpha = 1.40$ (scale factor 0.00375). Dots represent distribution in simulation. Diamonds represent the empirical return distribution for the S&P 500 index during 1984 - 1989 as reported by Mantegna & Stanley [10]. The dashed line represents the Gaussian distribution with the empirical standard deviation ($\sigma = 0.05$).

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Figure 2

- Line with slope $-2.34$

Labels:
- Y-axis: thousands of pounds per 1 pound interval
- X-axis: income (pounds)
Figure 3