On fibre space structures of a projective irreducible symplectic manifold

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Abstract

In this note, we investigate fibre space structures of a projective irreducible symplectic manifold. We prove that an 2n-dimensional projective irreducible symplectic manifold admits only an n-dimensional fibration over a Fano variety which has only Q-factorial log-terminal singularities and whose Picard number is one. Moreover we prove that a general fibre is an abelian variety up to finite unramified cover, especially, a general fibre is an abelian surface for 4-fold.

1 Introduction

We first define an irreducible symplectic manifold.

Definition 1 A complex manifold X is called irreducible symplectic if X satisfies the following three conditions:

1. X is compact and Kähler.
2. X is simply connected.
3. $H^0(X, \Omega^2_X)$ is spanned by an everywhere non-degenerate two-from $\omega$.

Such a manifold can be considered as an unit of compact Kähler manifold X with $c_1(X) = 0$ due to the following Bogomolov decomposition theorem.

Theorem 1 (Bogomolov decomposition theorem [2]) A compact Kähler manifold X with $c_1(X) = 0$ admits a finite unramified covering of $\tilde{X}$ which is isomorphic to a product $T \times X_1 \times \cdots \times X_r \times A$ where T is a complex torus, $X_i$ are irreducible symplectic manifolds and A is a projective manifold with $h^0(A, \Omega^p) = 0$, $0 < p < \dim A$.

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In dimension 2, $K3$ surfaces are the only irreducible symplectic manifolds, and irreducible symplectic manifolds are considered as higher-dimensional analogies of $K3$ surfaces. In this note, we investigate fibre space structures of a projective irreducible symplectic manifolds.

**Definition 2** For an algebraic variety $X$, a fibre space structure of $X$ is a proper surjective morphism $f : X \to S$ which satisfies the following two conditions:

1. $X$ and $S$ are normal varieties such that $0 < \dim S < \dim X$
2. A general fibre of $f$ is connected.

Some of $K3$ surface $S$ has a fibre space structure $f : S \to \mathbb{P}^1$ whose general fibre is an elliptic curve. In higher dimensional analogy, we obtain the following results.

**Theorem 2** Let $f : X \to B$ be a fibre space structure of a projective irreducible symplectic $2n$-fold $X$ with projective base $B$. Then a general fibre $F$ of $f$ and $B$ satisfy the following three conditions:

1. $F$ is an abelian variety up to finite unramified cover and $K_F \sim O_F$.
2. $B$ is $n$-dimensional and has only $\mathbb{Q}$-factorial log-terminal singularities
3. $-K_B$ is ample and Picard number $\rho(B)$ is one.

Especially, if $X$ is 4-dimensional, a general fibre of $f$ is an abelian surface.

**Example.** Let $S$ be a $K3$ surface with an elliptic fibration $g : S \to \mathbb{P}^1$ and $S^{[n]}$ a $n$-pointed Hilbert scheme of $S$. It is known that $S^{[n]}$ is an irreducible symplectic $2n$-fold and there exists a birational morphism $\pi : S^{[n]} \to S^{(n)}$ where $S^{(n)}$ is the symmetric $n$-product of $S$ (cf. [1]). We can consider $n$-dimensional abelian fibration $g^{(n)} : S^{(n)} \to \mathbb{P}^n$ for the symmetric $n$-product of $S^{(n)}$. Then the composition morphism $g^{(n)} \circ \pi : S^{[n]} \to \mathbb{P}^n$ gives an example of a fibre space structure of an irreducible symplectic manifold.

**Remark.** Markushevich obtained some result of theorem 2 in [3, Theorem 1, Proposition 1] under the assumption $\dim X = 4$ and $f : X \to B$ is the moment map. In general, a fibre space structure of an irreducible symplectic manifold is not a moment map. Markushevich constructs in [4, Remark 4.2] counterexample.

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2 Proof of Theorems

First we introduce the following theorem due to Fujiki [3] and Beauville [4].

**Theorem 3** (3 Theorem 4.7, Lemma 4.11, Remark 4.12 4 Théorème 5) Let $X$ be an irreducible symplectic $2n$-fold. Then there exists a nondegenerate quadratic form $q_X$ of signature $(3, b_2(X) − 3)$ on $H^2(X, \mathbb{Z})$ which satisfies

$$
\alpha^{2n} = a_0 q_X(\alpha, \alpha)^n \\
c_{2i}(X)\alpha^{2n-2i} = a_i q_X(\alpha, \alpha)^{n-i} \quad (i \geq 1),
$$

where $\alpha \in H^2(X, \mathbb{Z})$ and $a_i$’s are constants depending on $X$.

We shall prove theorem 2 in five steps.

1. $\dim B = n$ and $B$ has only log-terminal singularities;
2. A general fibre $F$ of $f$ is an abelian variety up to unramified finite cover and $K_F \sim O_F$;
3. $\rho(B) = 1$;
4. $B$ is $\mathbb{Q}$-factorial;
5. $-K_B$ is ample.

**Step 1.** $\dim B = n$ and $B$ has only log-terminal singularities.

**Lemma 1** Let $X$ be an irreducible symplectic projective $2n$-fold and $E$ be a divisor on $X$ such that $E^{2n} = 0$. Then,

1. If $E.A^{2n-1} = 0$ for some ample divisor $A$, $E \equiv 0$.
2. If $E.A^{2n-1} > 0$ for an ample divisor $A$ on $X$, then

\[
\begin{aligned}
E^m A^{2n-m} &= 0 \quad (m > n) \\
&> 0 \quad (m \leq n)
\end{aligned}
\]

**Proof of Lemma.** Let $V := \{E \in H^2(X, \mathbb{Z})|E.A^{2n-1} = 0\}$. By [3, Lemma 4.13], $q_X$ is negative definite on $W$ where $V = H^{2,0} \oplus H^{0,2} \oplus W$. Thus, if $E.A^{2n-1} = 0$ and $E^{2n} = 0$, $E \equiv 0$. Next we prove (2). From Theorem 3, for every integer $t$,

\[
(tE + A)^{2n} = a_0(q_X(tE + A, tE + A))^n.
\]
Because $E^{2n} = a_0(q_X(E, E))^n = 0$,

$$q_X(tE + A, tE + A) = 2tq_X(E, A) + q_X(A, A).$$

Thus the right hand side of the equation \([\text{II}]\) has order at most \(n\). Comparing the both hand side of the equation \([\text{II}]\), we can obtain \(E^m.A^{2n-m} = 0\) for \(m > n\). If \(E.A^{2n-1} > 0\), comparing the first order term of \(t\) of both hand of the equation \([\text{II}]\) we can obtain \(q_X(E, A) > 0\). Because coefficients of other terms of left hand side of \([\text{II}]\) can be written \(q_X(E, A)\) and \(q_X(A, A)\), we can obtain \(E^m.A^{2n-m} > 0\) for \(0 < m \leq n\). \(\square\)

Let \(H\) be a very ample divisor on \(B\). Then \(f^*H\) is a nef divisor such that \((f^*H)^{2n} = 0\), \((f^*H).A^{2n-1} > 0\) for an ample divisor \(A\) on \(X\). Thus \(\dim B = n\). From \([8, \text{Theorem 2}]\), \(B\) has only log-terminal singularities.

Step 2. A general fibre \(F\) of \(f\) is an abelian variety up to unramified finite cover and \(K_F \sim O_F\).

By adjunction, \(K_F \sim 0\). Moreover

$$c_2(F) = c_2(X)(f^*H)^{2n-2} = a_1(q_X(f^*H, f^*H))^{n-1} = 0,$$

by Theorem \([3]\). Thus \(F\) has an étale cover \(\tilde{F} \to F\) such that \(\tilde{F}\) is an Abelian variety by \([3]\).

Step 3. \(\rho(B) = 1\).

Lemma 2 Let \(E\) be a divisor of \(X\) such that \(E^{2n} = 0\) and \(E^n.(f^*H)^n = 0\). Then \(E \sim_\mathbb{Q} \lambda f^*H\) for some rational number \(\lambda\).

Proof of Lemma. Considering the following equation

$$(E - \lambda f^*H)^{2n} = a_0q_X(E - \lambda f^*H, E - \lambda f^*H)^n$$

$$= a_0(2\lambda q_X(E, f^*H))^n,$$

we can obtain \(q_X(E, f^*H) = cE^n.(f^*H)^{n} = 0\) where \(c\) is a constant. Thus \((E - \lambda f^*H)^{2n} = 0\). Because \(f^*H.A^{2n-1} > 0\) for every ample divisor \(A\) on \(X\), we can take a rational number \(\lambda\) such that \((E - \lambda f^*H).A^{2n-1} = 0\) Then \(E - \lambda f^*H \equiv 0\) by lemma \([4]\).

Let \(D\) be a Cartier divisor on \(B\). Then \((f^*D)^{2n} = 0\) and \((f^*D)^n.(f^*H)^n = 0\), thus \(E \sim_\mathbb{Q} \lambda H\) and \(\rho(B) = 1\).

Step 4. \(B\) is \(\mathbb{Q}\)-factorial.
Let $D$ be an irreducible and reduced Weil divisor on $B$ and $D_i$, $(1 \leq i \leq k)$ divisors on $X$ whose supports are contained in $f^{-1}(D)$. We construct a divisor $\tilde{D} := \sum \lambda_i D_i$, $(\tilde{D} \neq 0)$ such that $\tilde{D}^{2n} = 0$. Let $A$ be a very ample divisor on $X$, $S := A^{-1}.(f^*H)^{n-1}$ and $C := H^{n-1}$. Then there exists a surjective morphism $f' : S \to C$. If we choose $H$ and $A$ general, we may assume that $S$ and $C$ are smooth and $C \cap D$ and are contained smooth loci of $B$. Because $D$ is a Cartier divisor in a neighborhood of $C \cap D$, we can define $f^*D$ in a neighborhood $U$ of $S$. We can express $f^*D = \sum \lambda_i D_i$ in $U$ and let $\tilde{D} := \sum \lambda_i D_i$. Note that if $\lambda_i > 0$, $f(D_i) = D$ because we choose $C$ generally. Comparing the $n$th order term of $t$ of the both hand side of the following equation

$$(\tilde{D} + tf^*H)^{2n} = a_0q_X(\tilde{D} + tf^*H, \tilde{D} + tf^*H)$$

$$= a_0(q_X(\tilde{D}, \tilde{D}) + 2tq_X(\tilde{D}, f^*H))^n,$$

we can see that $\tilde{D}^n.(f^*H)^n = cq_X(\tilde{D}, f^*H)$. Since $f(\tilde{D}) = D$, $\tilde{D}^n.(f^*H)^n = 0$ and $q_X(\tilde{D}, f^*H) = 0$. Considering the following equation

$$(s\tilde{D} + tA + f^*H)^{2n} = a_0q_X(s\tilde{D} + tA + f^*H, s\tilde{D} + tA + f^*H)^n$$

$$= a_0(s^2q_X(\tilde{D}, \tilde{D}) + t^2q_X(A, A) + 2stq_X(\tilde{D}, A) + 2tq_X(A, f^*H))^n,$$

we can obtain $q_X(\tilde{D}, \tilde{D})q_X(A, f^*H) = c\tilde{D}^2.A^{n-1}.(f^*H)^{n-1}$ where $c$ is a constant. Since $\tilde{D}.A^{n-1}.(f^*H)^{n-1}$ is a fibre of $f'$, $\tilde{D}^2.A^{n-1}.(f^*H)^{n-1} = 0$. Thus $a_0q_X(\tilde{D}, \tilde{D}) = \tilde{D}^{2n} = 0$. Considering $\tilde{D}^n(f^*H)^n = 0$, we can obtain $\tilde{D} \sim_\mathbb{Q} \lambda f^*H$ by Lemma [5] and $D \sim_\mathbb{Q} \lambda H$ because $f(\tilde{D}) = D$. Therefore $B$ is $\mathbb{Q}$-factorial.

**Step 5.** $-K_B$ is ample.

From Step 3,4, we can write $-K_B \sim_\mathbb{Q} tH$. It is enough to prove $t > 0$. Because $K_X \sim \mathcal{O}_X$ and a general fibre of $f : X \to B$ is a minimal model, $\kappa(B) \leq 0$ by [4, Theorem 1.1] and $t \geq 0$. Assume that $t = 0$. If $K_B \not\sim \mathcal{O}_B$, we can consider the following diagram:

$$
\begin{array}{ccc}
X & \to & B \\
\alpha & \uparrow & \beta \\
\tilde{X} & \to & \tilde{B}
\end{array}
$$

where $\beta$ is an unramified finite cover and $K_{\tilde{B}} \sim \mathcal{O}_{\tilde{B}}$. Because $\pi_1(X) = \{1\}$, $\tilde{X}$ is the direct sum of $X$. Thus there exists a morphism from $X$ to $\tilde{B}$ and we may assume that $K_B \sim \mathcal{O}_B$. Then there exists a holomorphic $n$-form $\omega'$ on $X$ coming from $B$. However, if $n$ is odd, it is a contradiction because there exist no holomorphic $(2k-1)$-form on $X$. If $n$ is even, it is also a contradiction because $\omega'$ dose not generated by $\omega \in H^0(X, \Omega^2)$. Thus $t > 0$ and we completed the proof of Theorem 2.
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