A GLEASON–KAHANE–ŽELAZKO THEOREM FOR REPRODUCING KERNEL HILBERT SPACES

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Abstract. We establish the following Hilbert-space analogue of the Gleason–Kahane–Želazko theorem. If $H$ is a reproducing kernel Hilbert space with a normalized complete Pick kernel, and if $\Lambda$ is a linear functional on $H$ such that $\Lambda(1) = 1$ and $\Lambda(f) \neq 0$ for all cyclic functions $f \in H$, then $\Lambda$ is multiplicative, in the sense that $\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f, g \in H$ such that $fg \in H$. Moreover $\Lambda$ is automatically continuous. We give examples to show that the theorem fails if the hypothesis of a complete Pick kernel is omitted. We also discuss conditions under which $\Lambda$ has to be a point evaluation.

1. Introduction and statement of main result

The following result, known as the Gleason–Kahane–Želazko (GKZ) theorem, characterizes multiplicativity of a linear functional on a Banach algebra.

**Theorem 1.1** ([11] [15] [24]). Let $A$ be a complex unital Banach algebra, and let $\Lambda : A \to \mathbb{C}$ be a linear functional such that $\Lambda \neq 0$. The following statements are equivalent:

(i) $\Lambda(1) = 1$ and $\Lambda(a) \neq 0$ for all invertible elements $a \in A$.

(ii) $\Lambda(ab) = \Lambda(a)\Lambda(b)$ for all $a, b \in A$.

Continuity of the functional is not assumed, but it is a consequence of the theorem, since characters on a Banach algebra are always continuous.

Analogues of this result for certain holomorphic function spaces that are not algebras were obtained in [16] [17]. Our goal in this paper is to establish the following theorem, which is a version for abstract function spaces.

**Theorem 1.2.** Let $H$ be a reproducing kernel Hilbert space with a normalized complete Pick kernel, and let $\Lambda : H \to \mathbb{C}$ be a linear functional such that $\Lambda \neq 0$. The following statements are equivalent:

(i) $\Lambda(1) = 1$ and $\Lambda(f) \neq 0$ for all cyclic elements $f \in H$.

(ii) $\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f, g \in H$ such that $fg \in H$.

Moreover, a functional $\Lambda$ satisfying (i) or (ii) is automatically continuous.
The terminology is explained in §2, the automatic continuity of $\Lambda$ is established in §3, and the rest of the theorem proved in §4. In §5 we present examples showing that the theorem may fail if we drop the assumption that $\mathcal{H}$ has a complete Pick kernel. Finally, in §6 we discuss conditions under which we may conclude that $\Lambda$ is a point evaluation at a point of $X$.

2. Background on reproducing kernel Hilbert spaces

In this section we give a very brief introduction to reproducing kernel Hilbert spaces and complete Pick kernels. For more details we refer to the books of Paulsen–Raghupathi [19] and Agler–McCarthy [3].

2.1. Reproducing kernel Hilbert spaces. A reproducing kernel Hilbert space (RKHS for short) is a complex Hilbert space $H$ of functions on a set $X$ with the property that, for each $x \in X$, the evaluation functional $f \mapsto f(x)$ is continuous on $H$. By the Riesz representation theorem, there is a uniquely determined element $k_x \in H$ such that $f(x) = \langle f, k_x \rangle$ for all $f \in H$. The reproducing kernel associated to $H$ is the function $K : X \times X \to \mathbb{C}$ defined by $K(x, y) := k_y(x) = \langle k_y, k_x \rangle$.

A kernel $K$ is automatically a positive semi-definite function. This means that, for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of $X$, the $n \times n$ matrix $(K(x_i, x_j))$ is positive semi-definite.

We shall always assume that $K(x, x) > 0$ for all $x \in X$. This is equivalent to supposing that, for each $x \in X$, there exists $f \in H$ such that $f(x) \neq 0$. In particular, this is the case if $K$ is normalized (see §2.4 below).

2.2. Multipliers. Let $H$ be a RKHS on $X$. A multiplier of $H$ is a function $h : X \to \mathbb{C}$ such that $hf \in H$ whenever $f \in H$. We denote by $\mathcal{M}$ the set of multipliers. By the closed graph theorem, if $h \in \mathcal{M}$, then $M_h : f \mapsto hf$ is a bounded linear operator on $H$. We define $\|h\|_\mathcal{M}$ to be the operator norm of $M_h$. With this norm $\mathcal{M}$ becomes a Banach algebra.

If $h \in \mathcal{M}$ and $x \in X$, then a standard calculation shows that $M_h k_x = \overline{h(x)} k_x$, so $h(x)$ is an eigenvalue of $M_h$, and $|h(x)| \leq \|h\|_\mathcal{M}$. Consequently, every multiplier $h$ is a bounded function on $X$ with $\sup_X |h| \leq \|h\|_\mathcal{M}$.

2.3. Complete Pick kernels. Let $K$ be the reproducing kernel of a RKHS $\mathcal{H}$ on $X$. It is a complete Pick kernel if $K(x, y) \neq 0$ for all $x, y \in X$ and if there exists $x_0 \in X$ such that

$$F(x, y) := 1 - \frac{K(x, x_0) K(x_0, y)}{K(x, y) K(x_0, x_0)}$$

is a positive semi-definite function on $X \times X$. If this condition holds for one $x_0 \in X$, then it automatically holds for all $x_0 \in X$.

Complete Pick kernels are so called because of their relation with the solution of Pick interpolation problems. This is described in detail in [3]. Examples of RKHS with complete Pick kernels include the Hardy space $H^2$, the
Drury–Arveson spaces $H^2_d$ [3], the classical Dirichlet space $D$ [1], all Dirichlet spaces with superharmonic weights $D_w$ [23], the Sobolev space $W^1_1(0,1)$ [20], as well as certain de Branges–Rovnyak spaces $H(b)$ [8].

2.4. Normalized kernels. A reproducing kernel $K$ for a RKHS $H$ on $X$ is said to be normalized if there exists $x_0 \in X$ such that $K(x, x_0) = 1$ for all $x \in X$. In this case $k_{x_0}(x) := K(x, x_0)$ for all $x \in X$. Since $k_{x_0} \in H$, it follows that $H$ contains the constant functions, and hence that $\mathcal{M} \subset H$.

We require the following well-known fact, which follows for instance from the discussion surrounding Equation (2.2) in [12]. For the convenience of the reader, we provide a short proof.

**Proposition 2.1.** Let $H$ be a RKHS on $X$ with a normalized complete Pick kernel $K$ and let $\mathcal{M}$ be its multiplier algebra. Then $k_y = K(\cdot, y) \in \mathcal{M}$ for all $y \in X$. In particular, $\mathcal{M}$ is dense in $H$.

**Proof.** Since $K$ is a normalized complete Pick kernel, a theorem of Agler and McCarthy [2, Theorem 4.2] shows that there exists an auxiliary Hilbert space $E$ and a function $b : X \to B(E, \mathbb{C})$ with $\|b(x)\| < 1$ for all $x \in X$ such that

$$K(x, y) = \frac{1}{1 - b(x)b(y)^*}$$

for all $x, y \in X$. Indeed, this follows by choosing $x_0$ in Equation (1) to be the normalization point, defining $E$ to be the RKHS with reproducing kernel $F$ and $b(x)$ to be the functional on $E$ of evaluation at $x$.

Since $$(x, y) \mapsto (1 - b(x)b(y)^*)K(x, y) = 1$$ is a positive semi-definite function on $X \times X$, the function $b$ is a multiplier from $H \otimes E$ into $H$ of norm at most 1 (see e.g. [19, Theorem 6.28]). From this, we deduce that for each $y \in X$, the scalar-valued function $x \mapsto b(x)b(y)^*$ belongs to $\mathcal{M}$ and

$$\|b(\cdot)b(y)^*\|_\mathcal{M} \leq \|b(y)^*\|_E < 1.$$ 

Therefore, the series

$$K(\cdot, y) = \sum_{n=0}^{\infty} (b(\cdot)b(y)^*)^n$$

converges absolutely in the Banach algebra $\mathcal{M}$. Hence $K(\cdot, y) \in \mathcal{M}$ for all $y \in X$. Since the linear span of the kernel functions is dense in any RKHS, it follows in particular that $\mathcal{M}$ is dense in $H$. □

2.5. Cyclic functions. Let $H$ be a RKHS on $X$ with multiplier algebra $\mathcal{M}$. Given $f \in H$, we write $[f] := \overline{\mathcal{M}f}$, the closed $\mathcal{M}$-invariant subspace generated by $f$. Note that, if $h \in \mathcal{M}$, then $h[f] \subset [hf]$. We say $f$ is cyclic if $[f] = H$. Clearly, if $f \in H$ is cyclic, then $f(x) \neq 0$ for all $x \in X$. In general, the reverse is not true.

It should be remarked that some authors (notably those in [4]) define a multiplier $h \in \mathcal{M}$ to be ‘cyclic’ if $\overline{h\mathcal{M}} = H$. If $\mathcal{M}$ is dense in $H$, then the
two notions of cyclic coincide for all $h \in \mathcal{M}$. In particular, this is the case whenever $\mathcal{H}$ has a normalized complete Pick kernel, by Proposition 2.1.

Cyclic functions play a role in RKHS analogous to that of invertible elements in algebras. In the case of the Hardy space $H^2$, a function is cyclic if and only if it is an outer function. This is a consequence of Beurling’s classification of the closed, shift-invariant subspaces of $H^2$. In many other spaces, it is difficult to characterize which functions are cyclic. In particular, in the case of the classical Dirichlet space, this problem is the subject of a well-known conjecture of Brown and Shields [7].

3. Automatic continuity

As remarked in the introduction, it is well known (and easy to prove) that a multiplicative linear functional on a unital Banach algebra is automatically continuous. The corresponding result for a RKHS with a normalized, complete Pick kernel is also true, but not quite so obvious. In fact, it is even true under the weaker multiplicativity hypothesis that $\Lambda(hf) = \Lambda(h)\Lambda(f)$ for all $f \in \mathcal{H}$ and all $h \in \mathcal{M}$. As it will actually play a role in our proof of Theorem 1.2, we stop here to prove it first.

**Theorem 3.1.** Let $\mathcal{H}$ be a RKHS with a normalized complete Pick kernel, let $\mathcal{M}$ be its multiplier algebra, and let $\Lambda : \mathcal{H} \to \mathbb{C}$ be a linear functional such that $\Lambda(hf) = \Lambda(h)\Lambda(f)$ for all $f \in \mathcal{H}$ and all $h \in \mathcal{M}$. Then $\Lambda$ is continuous on $\mathcal{H}$.

The proof of Theorem 3.1 hinges on the following factorization theorem due to Jury and Martin [14, Theorem 1.1]. This result was proved by using non-commutative factorization theorems due to Arias and Popescu [8] and Davidson and Pitts [10].

**Theorem 3.2.** Let $\mathcal{H}$ be a RKHS with a normalized complete Pick kernel, and let $\mathcal{M}$ be its multiplier algebra. If $(f_n)$ is a sequence in $\mathcal{H}$ such that $\sum_n \|f_n\|^2_{\mathcal{H}} < \infty$, then there exists a sequence $(h_n)$ in $\mathcal{M}$ and a cyclic function $g \in \mathcal{H}$ such that:

- $f_n = h_n g$ for all $n$;
- $\sum_n \|f_n\|^2_{\mathcal{H}} = \|g\|^2_{\mathcal{H}}$;
- $\sum_n \|h_n f\|^2_{\mathcal{H}} \leq \|f\|^2_{\mathcal{H}}$ for all $f \in \mathcal{H}$, and in particular $\|h_n\|_{\mathcal{M}} \leq 1$ for all $n$.

**Proof of Theorem 3.1.** Suppose, if possible, that $\Lambda$ is not continuous on $\mathcal{H}$. Then there exists a sequence $(f_n)$ in $\mathcal{H}$ such that $\sum_n \|f_n\|^2_{\mathcal{H}} < \infty$ but $|\Lambda(f_n)| \to \infty$. By Theorem 3.2 we can factorize $f_n$ as $f_n = h_ng$, where $g \in \mathcal{H}$ and $(h_n)$ is a sequence in the multiplier algebra $\mathcal{M}$ of $\mathcal{H}$ such that $\|h_n\|_{\mathcal{M}} \leq 1$ for all $n$. By the multiplicativity hypothesis on $\Lambda$, we have $\Lambda(f_n) = \Lambda(h_n)\Lambda(g)$ for all $n$, which forces $|\Lambda(h_n)| \to \infty$. On the other hand, $\Lambda|_{\mathcal{M}}$ is a character on the Banach algebra $\mathcal{M}$, so it is automatically continuous on $\mathcal{M}$, and as the sequence $(h_n)$ is bounded in the norm of $\mathcal{M}$, it follows that $\Lambda(h_n)$ is bounded too. We have arrived at a contradiction. □
The proof of Theorem 1.2 is based on the following result, which is a version of the classical GKZ theorem for modules [17, Theorem 1.2].

**Theorem 4.1.** Let $A$ be a complex unital Banach algebra, let $M$ be a left $A$-module, and let $S$ be a non-empty subset of $M$ satisfying the following conditions:

(S1) $S$ generates $M$ as an $A$-module;

(S2) if $a \in A$ is invertible and $s \in S$, then $as \in S$;

(S3) for all $s_1, s_2 \in S$, there exist $a_1, a_2 \in A$ such that $a_j S \subset S$ ($j = 1, 2$) and $a_1s_1 = a_2s_2$.

Let $\Lambda : M \to \mathbb{C}$ be a linear functional such that $\Lambda(s) \neq 0$ for all $s \in S$. Then there exists a unique character $\chi$ on $A$ such that

$$\Lambda(am) = \chi(a)\Lambda(m) \quad (a \in A, \ m \in M).$$

We also need another factorization theorem for RKHS's, this one due to Aleman, McCarthy, Richter and the second author [4, Lemma 2.3 and Theorem 3.1]. See also §2.5 regarding the definition of cyclicity used in [4].

**Theorem 4.2.** Let $H$ be a RKHS on $X$ with a normalized complete Pick kernel, and let $M$ be its multiplier algebra. Then, given $f \in H$, there exist $h_1, h_2 \in M$, with $h_2$ cyclic in $H$, such that $f = h_1/h_2$.

**Proof of Theorem 1.2.** We begin with the implication (i)⇒(ii). Assume that (i) holds, namely $\Lambda(1) = 1$ and $\Lambda(f) \neq 0$ for all cyclic functions $f \in H$.

The plan is to apply Theorem 4.1 with $M = H$, taking $A$ to be the multiplier algebra $M$ of $H$, and $S$ to be the set of cyclic functions in $H$. We verify the three conditions in that theorem:

(S1) Given $f \in H$, Theorem 3.2 implies that we can write $f = hg$, where $h \in M$ and $g \in S$.

(S2) If $h$ is invertible in $M$, then $M(hf) = Mf$. In particular, if $f$ is cyclic, so is $hf$.

(S3) Given $f_1, f_2 \in S$, Theorem 3.2 shows that we can write $f_1 = a_2 g$ and $f_2 = a_1 g$, where $a_1, a_2 \in M$ and $g \in S$. Clearly we have $a_1 f_1 = a_2 f_2$. It remains to check that $a_j S \subset S$. Let $f \in S$. Then $[a_1 f] \supset [a_1 f] = a_1 H \supset a_1 g = f_2$, and so $[a_1 f] \supset [f_2] = H$. Hence $a_1 f \in S$. Likewise for $a_2$.

By Theorem 4.1, there exists a character $\chi$ on $M$ such that (2) holds, i.e.,

$$\Lambda(hf) = \chi(h)\Lambda(f) \quad (h \in M, \ f \in H).$$

In particular, taking $f = 1$, we see that $\chi(h) = \Lambda(h)$ for all $h \in M$. It follows that

$$\Lambda(hf) = \Lambda(h)\Lambda(f) \quad (h \in M, \ f \in H).$$

This is nearly the desired conclusion (ii). To get the full conclusion, we employ a trick borrowed from [4]. Let $f, g \in H$ be such that $fg \in H$. By
Theorem 4.2, we can write \( f = h_1 / h_2 \), where \( h_1, h_2 \in \mathcal{M} \) and \( h_2 \) is cyclic in \( \mathcal{H} \). Then, using (3) several times, we have

\[
\Lambda(h_2)\Lambda(fg) = \Lambda(h_2fg) = \Lambda(h_1)\Lambda(g) = \Lambda(h_2)\Lambda(f)\Lambda(g).
\]

Since \( h_2 \) is cyclic, we have \( \Lambda(h_2) \neq 0 \), so we can cancel off the \( \Lambda(h_2) \) terms and conclude that \( \Lambda(fg) = \Lambda(f)\Lambda(g) \), as desired. This completes the proof that (i) \( \Rightarrow \) (ii).

We now turn to the reverse implication in Theorem 1.2, namely (ii) \( \Rightarrow \) (i). Assume that (ii) holds, i.e., \( \Lambda(fg) = \Lambda(f)\Lambda(g) \) for all \( f, g \in \mathcal{H} \) with \( fg \in \mathcal{H} \).

Since \( \Lambda(1) = \Lambda(1 \cdot 1) = \Lambda(1)^2 \), we have \( \Lambda(1) = 0 \) or \( 1 \). In the former case, we have \( \Lambda(f) = \Lambda(1 \cdot f) = \Lambda(1)\Lambda(f) = 0 \) for all \( f \in \mathcal{H} \), contrary to the hypothesis that \( \Lambda \neq 0 \). So \( \Lambda(1) = 1 \).

Let \( f \) be a cyclic function in \( \mathcal{H} \). From the definition of cyclicity, there exists a sequence of multipliers \( (h_n) \) such that \( h_n f \to 1 \) in \( \mathcal{H} \). By Theorem 3.1, the assumption that \( \Lambda \) is multiplicative implies that \( \Lambda \) is continuous on \( \mathcal{H} \), and so \( \Lambda(h_n)\Lambda(f) = \Lambda(h_n f) \to \Lambda(1) = 1 \). This implies that \( \Lambda(f) \neq 0 \), and completes the proof that (ii) \( \Rightarrow \) (i).

Finally, the continuity statement in Theorem 1.2 follows directly from Theorem 3.1.

5. Two examples on the complete Pick kernel condition

In this section we present examples showing that Theorems 1.2 and 3.1 are no longer true if we drop the assumption that \( \mathcal{H} \) has a complete Pick kernel. We begin with an example relating to Theorem 1.2.

Proposition 5.1. There exist a RKHS \( \mathcal{H} \) on a set \( X \) and a linear functional \( \Lambda : \mathcal{H} \to \mathbb{C} \) with the following properties:

(i) The reproducing kernel \( K : X \times X \to \mathbb{C} \) associated to \( \mathcal{H} \) is normalized, and satisfies \( K(x, y) \neq 0 \) for all \( x, y \in X \).

(ii) The linear functional \( \Lambda \) satisfies \( \Lambda(1) = 1 \) and \( \Lambda(f) \neq 0 \) for all cyclic elements \( f \in \mathcal{H} \). Furthermore \( \Lambda \) is continuous on \( \mathcal{H} \).

(iii) There exists a multiplier \( h \) of \( \mathcal{H} \) such that \( \Lambda(h^2) \neq \Lambda(h)^2 \).

Proof. (i) Let

\[
\mathcal{H} := \left\{ f(z_1, z_2) := \sum_{j,k \geq 0} a_{jk} z_1^j \overline{z_2}^k : \|f\|_{\mathcal{H}}^2 := \sum_{j,k \geq 0} j!|a_{jk}|^2 < \infty \right\}.
\]

This is a RKHS of holomorphic functions on \( \mathbb{C} \times \mathbb{D} \). It has the orthonormal basis \( \{z_1^j z_2^k / (j!^{1/2}) : j, k \geq 0\} \), from which we deduce that its reproducing kernel is given by

\[
K((z_1, z_2), (w_1, w_2)) = \sum_{j,k \geq 0} z_1^j \overline{z_2}^k \frac{w_1^j \overline{w_2}^k}{(j!)^{1/2} (j!)^{1/2}} = e^{z_1 w_1 - \overline{z_2} \overline{w_2}}.
\]
Clearly $K$ is normalized at $(0, 0)$, and it is non-zero everywhere on $\mathbb{C} \times \mathbb{D}$. (In fact $\mathcal{H}$ is the tensor product of the Segal–Bargmann space and the Hardy space, and so $K$ is the product of the kernels of these two spaces.)

(ii) Define $\Lambda : \mathcal{H} \to \mathbb{C}$ by

$$\Lambda \left( \sum_{j,k \geq 0} a_{jk} z_1^j z_2^k \right) := a_{00} + a_{01}.$$ 

This is a continuous linear functional on $\mathcal{H}$. Clearly $\Lambda(1) = 1$. Also, the condition that $\Lambda(f) \neq 0$ for all cyclic $f \in \mathcal{H}$ is satisfied vacuously, because there are no cyclic elements of $\mathcal{H}$. Indeed, every multiplier of $\mathcal{H}$ is a bounded holomorphic function on $\mathbb{C} \times \mathbb{D}$, so by Liouville’s theorem it must be independent of the first variable. This evidently precludes the existence of cyclic functions.

(iii) Let $h(z_1, z_2) := z_2$. Multiplication by $h$ is an isometry on $\mathcal{H}$, so $h$ is certainly a multiplier. Also, we have $\Lambda(h) = 1$ and $\Lambda(h^2) = 0$, so $\Lambda(h^2) \neq \Lambda(h)^2$. □

Remark. In the example constructed in the proof of Proposition 5.1, the multiplier algebra $\mathcal{M}$ is not dense in $\mathcal{H}$. This is no accident. Indeed, if $\mathcal{M}$ is dense in $\mathcal{H}$, then every invertible multiplier is a cyclic element of $\mathcal{H}$. Thus the classical GKZ theorem (Theorem 1.1), applied to the restriction of $\Lambda$ to $\mathcal{M}$, shows that

$$\Lambda(h_1 h_2) = \Lambda(h_1) \Lambda(h_2) \quad (h_1, h_2 \in \mathcal{M}),$$

and if, further, $\Lambda$ is continuous on $\mathcal{H}$, then it follows that

$$\Lambda(h f) = \Lambda(h) \Lambda(f) \quad (h \in \mathcal{M}, f \in \mathcal{H}).$$

Concerning the continuity of $\Lambda$, one might also ask if the automatic continuity result, Theorem 3.1, holds if we drop the assumption that $\mathcal{H}$ has a complete Pick kernel. The following example shows that this is not the case, even if the multiplier algebra $\mathcal{M}$ is dense in $\mathcal{H}$.

**Proposition 5.2.** There exist a RKHS $\mathcal{H}$ of holomorphic functions on $\mathbb{D}$ and a linear functional $\Lambda : \mathcal{H} \to \mathbb{C}$ with the following properties:

(i) The reproducing kernel $K$ associated to $\mathcal{H}$ is normalized at 0.

(ii) The multiplier algebra $\mathcal{M}$ of $\mathcal{H}$ equals $H^\infty$ and is dense in $\mathcal{H}$.

(iii) The linear functional $\Lambda$ satisfies $\Lambda(1) = 1$ and $\Lambda(h f) = \Lambda(h) \Lambda(f)$ for all $h \in \mathcal{M}$ and all $f \in \mathcal{H}$.

(iv) $\Lambda$ is discontinuous on $\mathcal{H}$.

**Proof.** Let $(a_n)_{n=0}^\infty$ be a sequence of strictly positive real numbers with the following properties:

(a) $a_0 = 1$,

(b) $a_n/a_{n+1} \leq 1$ for all $n \geq 0$,

(c) $\inf_{n \geq 0} a_n/a_{n+1} = 0$, and

(d) $\lim_{n \to \infty} a_n^{1/n} = 1$. 

For instance, we can define \( a_n := k! \) whenever \( k^2 \leq n < (k + 1)^2 \). Then properties (a) and (b) are obviously satisfied. Moreover, (c) holds since

\[
\frac{a_{(k+1)^2-1}}{a_{(k+1)^2}} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \xrightarrow{k \to \infty} 0.
\]

Finally, to see (d), notice that, if \( k^2 \leq n < (k + 1)^2 \), then

\[
1 \leq a_n^{1/n} \leq (k!)^{1/k^2} \leq k^{1/k} \xrightarrow{k \to \infty} 1.
\]

Next, we define a positive semi-definite kernel \( K \) on \( \mathbb{D} \) by

\[
K(z, w) := \sum_{n=0}^{\infty} a_n (z\bar{w})^n \quad (z, w \in \mathbb{D}).
\]

Condition (d) ensures that the power series defining \( K \) has radius of convergence 1. Let \( \mathcal{H} \) be the RKHS on \( \mathbb{D} \) with reproducing kernel \( K \). It is well known and not hard to see that \( \mathcal{H} \) consists of holomorphic functions on \( \mathbb{D} \), and that the monomials \( (z^n)_{n=0}^{\infty} \) form an orthogonal basis of \( \mathcal{H} \) with

\[
\|z^n\|^2 = \frac{1}{a_n} \quad (n \geq 0),
\]

see e.g. [19, §4.4.2]. Since \( a_0 = 1 \), the reproducing kernel \( K \) is normalized at 0.

We next show that the multiplier algebra \( \mathcal{M} \) of \( \mathcal{H} \) equals \( H^\infty \). Certainly \( \mathcal{M} \) is contractively contained in \( H^\infty \), as was already remarked in §2.2. On the other hand, condition (b) above can be interpreted as saying that \( a_n/a_{n+1} \leq b_n/b_{n+1} \) for all \( n \geq 0 \), where \( b_n := 1 \) for all \( n \geq 0 \), namely the power series coefficients of the reproducing kernel of \( H^2 \). In this setting, [5] Proposition 3.3 implies that \( H^\infty \), the multiplier algebra of \( H^2 \), is contractively contained in \( \mathcal{M} \). Hence equality holds. In particular, we see that \( \mathcal{M} \) is dense in \( \mathcal{H} \).

To construct the functional \( \Lambda \), we first show that the (non-closed) subspace \( z\mathcal{H} \) has infinite codimension in \( \mathcal{H} \). To this end, notice that \( M_z \) is an injective operator on \( \mathcal{H} \), and condition (c) and equation (4) together imply that it is not bounded below. Therefore the range of \( M_z \) is not closed and thus it has infinite codimension in \( \mathcal{H} \). Hence there exists a discontinuous linear functional \( \Lambda : \mathcal{H} \to \mathbb{C} \) such that

\[
\Lambda(\alpha + zf) = \alpha \quad (\alpha \in \mathbb{C}, \ f \in \mathcal{H}).
\]

We finish the proof by showing the multiplicativity statement in (iii). To this end, let \( h \in \mathcal{M} = H^\infty \) and let \( f \in \mathcal{H} \). Then there exists \( g \in H^\infty \) such that \( h - h(0) = zg \). In particular, \( \Lambda(h) = h(0) \). Moreover, \( hf = h(0)f + zgf \) and so

\[
\Lambda(hf) = \Lambda(h(0)f) + \Lambda(zgf) = h(0)\Lambda(f) = \Lambda(h)\Lambda(f),
\]

as claimed. \( \square \)
6. Point evaluations

Let $\mathcal{H}$ be a RKHS on $X$. If $\Lambda : \mathcal{H} \to \mathbb{C}$ is a point evaluation at a point $x_0$ of $X$, namely $\Lambda(f) := f(x_0)$ for all $f \in \mathcal{H}$, then $\Lambda$ satisfies the hypothesis (i) of Theorem 1.2. Indeed, as remarked earlier, if $f \in \mathcal{H}$ is cyclic, then $f(x) \neq 0$ for all $x \in X$, and so $\Lambda(f) \neq 0$. Obviously $\Lambda(1) = 1$ as well.

This raises the question of whether every linear functional $\Lambda$ satisfying the hypotheses of Theorem 1.2 is a point evaluation at some point of $X$. The answer is affirmative in certain cases, notably the Hardy space $H^2$ on the unit disk [17, Theorem 2.1] and, more generally, the Drury–Arveson space $H^2_d$ on the unit ball in $\mathbb{C}^d$ for $1 \leq d \leq \infty$ (combine Theorem 1.2 above with [13, Lemma 5.3]). Under the added assumption that $\Lambda$ is continuous, further results of this kind have been established for various Banach function spaces on the unit disk [17, Theorem 3.1] and on various domains in $\mathbb{C}^n$ [22, §5].

However, it is unrealistic to hope for an affirmative answer in the general situation considered in Theorem 1.2. For example, if we think of $H^2$ as a RKHS on the punctured disk $\mathbb{D} \setminus \{x_0\}$ for some fixed point $x_0 \in \mathbb{D} \setminus \{0\}$, then it still has a normalized complete Pick kernel, and the linear functional $\Lambda(f) := f(x_0)$ is non-zero on cyclic functions, but is not given by point evaluation at any point of $\mathbb{D} \setminus \{x_0\}$.

A more ‘natural’ example is the case when $\mathcal{H} = D_\zeta$, the local Dirichlet space at a point $\zeta \in \mathbb{T}$ [21]. This is a RKHS on the unit disk $\mathbb{D}$, and by a result of Shimorin [23] it has a complete Pick kernel. In this case, the functional $\Lambda(f) := \lim_{z \to \zeta} f(z)$ is well-defined and continuous on $D_\zeta$, and is non-zero on cyclic functions, but it is not given by evaluation at any point of $D$.

These examples suggest that, if we want to conclude that $\Lambda$ is a point evaluation, then we need an assumption that ties $\Lambda$ more closely to the set $X$. One possibility is to suppose that $\Lambda(f) \neq 0$ for every $f \in \mathcal{H}$ that is nowhere zero on $X$. This eliminates the two counterexamples just mentioned. For $H^2$ on $\mathbb{D} \setminus \{x_0\}$ this is easy to see directly, and for $D_\zeta$ it is a special case of [16, Theorem 1.3]. However, as our final result shows, in the abstract setting of this paper, even this stronger hypothesis is not enough to yield the desired conclusion.

**Proposition 6.1.** There exist a RKHS $\mathcal{H}$ on a set $X$ and a linear functional $\Lambda : \mathcal{H} \to \mathbb{C}$ with the following properties:

(i) The reproducing kernel $K : X \times X \to \mathbb{C}$ associated to $\mathcal{H}$ is a normalized complete Pick kernel.

(ii) The linear functional $\Lambda$ satisfies $\Lambda(1) = 1$ and $\Lambda(f) \neq 0$ for each $f \in \mathcal{H}$ that is nowhere zero on $X$. Furthermore $\Lambda$ is continuous on $\mathcal{H}$.

(iii) $\Lambda$ is not given by point evaluation at any point of $X$.

**Proof.** Fix $d \geq 2$, and let $H^2_d$ denote the Drury–Arveson space on $\mathbb{B}_d$, the open unit ball in $\mathbb{C}^d$. Fix a point $x_0 \in \mathbb{B}_d \setminus \{0\}$, and set $X := \mathbb{B}_d \setminus \{x_0\}$ and $\mathcal{H} := H^2_d|_X$. 

(i) The space $\mathcal{H}$ is a RKHS on $X$ whose kernel is normalized at 0 and has the complete Pick property.

(ii) Each function $f \in \mathcal{H}$ has a unique extension to a holomorphic function on $\mathbb{B}_d$. Evaluating this extension at $x_0$ defines a continuous linear functional $\Lambda$ on $\mathcal{H}$ such that $\Lambda(1) = 1$. Moreover, since holomorphic functions in two or more variables have no isolated zeros, it follows that $\Lambda(f) \neq 0$ for every function $f \in \mathcal{H}$ that does not vanish on $X$.

(iii) By considering its action on the coordinate functions, we see that $\Lambda$ is not given by evaluation at any point of $X$. \hfill \Box

The counterexamples exhibited above are closely related to the notion of algebraic consistency or maximal domains. To elaborate somewhat, a non-zero continuous linear functional $\Lambda$ on a RKHS $\mathcal{H}$ on $X$ is said to be partially multiplicative if $\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f, g \in \mathcal{H}$ satisfying $fg \in \mathcal{H}$. Clearly, point evaluation at any point in $X$ defines a partially multiplicative functional. The set $X$ is said to be a maximal domain for $\mathcal{H}$ if every partially multiplicative functional is of this type.

It was shown in [18, §2] that $X$ can always be enlarged to a maximal domain by regarding $\mathcal{H}$ as a space of functions on its set of partially multiplicative functionals. For concrete spaces of holomorphic functions on domains in $\mathbb{C}^d$, one can sometimes determine the maximal domain by considering all points $\lambda \in \mathbb{C}^d$ such that the functional $p \mapsto p(\lambda)$, defined for all polynomials $p$, is continuous with respect to the norm of $\mathcal{H}$; see [22, §3]. Moreover, in the case when $\mathcal{H}$ has a normalized complete Pick kernel, there are alternative constructions of the maximal domain, using weak-$\ast$ continuous characters on the multiplier algebra $\mathcal{M}$, Gleason parts of the character space of $\mathcal{M}$, or a Zariski type closure; see Proposition 4.2 and Remark 4.3 in [4]. For more information on maximal domains, we refer the reader to [4], [9, Def.1.5], [13, §5], [18, §2], and [22, §3].

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**References**

[1] J Agler. Some interpolation theorems of Nevanlinna–Pick type. Unpublished manuscript, 1988.

[2] J. Agler and J. E. McCarthy. Complete Nevanlinna-Pick kernels. J. Funct. Anal., 175(1):111–124, 2000.

[3] J. Agler and J. E. McCarthy. Pick interpolation and Hilbert function spaces, volume 44 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002.
[4] A. Aleman, M. Hartz, J. E. McCarthy, and S. Richter. The Smirnov class for spaces with the complete Pick property. *J. Lond. Math. Soc.* (2), 96(1):228–242, 2017.

[5] A. Aleman, M. Hartz, J. E. McCarthy, and S. Richter. Radially weighted Besov spaces and the Pick property. In *Analysis of Operators on Function Spaces*, pages 29–61, Springer International Publishing, 2019.

[6] A. Arias and G. Popescu. Factorization and reflexivity on Fock spaces. *Integral Equations Operator Theory*, 23(3):268–286, 1995.

[7] L. Brown and A. L. Shields. Cyclic vectors in the Dirichlet space. *Trans. Amer. Math. Soc.*, 285(1):269–303, 1984.

[8] C. Chu. Which de Branges-Rovnyak spaces have complete Nevanlinna-Pick property? *J. Funct. Anal.*, 279(6):108608, 15, 2020.

[9] C. C. Cowen and B. D. MacCluer. *Composition operators on spaces of analytic functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[10] K. R. Davidson and D. R. Pitts. Invariant subspaces and hyper-reflexivity for free semigroup algebras. *Proc. London Math. Soc.* (3), 78(2):401–430, 1999.

[11] A. M. Gleason. A characterization of maximal ideals. *J. Analyse Math.*, 19:171–172, 1967.

[12] D. C. V. Greene, S. Richter, and C. Sundberg. The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels. *J. Funct. Anal.*, 194(2):311–331, 2002.

[13] M. Hartz. On the isomorphism problem for multiplier algebras of Nevanlinna-Pick spaces. *Canad. J. Math.*, 69(1):54–106, 2017.

[14] M. T. Jury and R. T. W. Martin. Factorization in weak products of complete Pick spaces. *Bull. Lond. Math. Soc.*, 51(2):223–229, 2019.

[15] J.-P. Kahane and W. Želazko. A characterization of maximal ideals in commutative Banach algebras. *Studia Math.*, 29:339–343, 1968.

[16] J. Mashreghi, J. Ransford, and T. Ransford. A Gleason-Kahane-Żelazko theorem for the Dirichlet space. *J. Funct. Anal.*, 274(11):3254–3262, 2018.

[17] J. Mashreghi and T. Ransford. A Gleason-Kahane-Żelazko theorem for modules and applications to holomorphic function spaces. *Bull. Lond. Math. Soc.*, 47(6):1014–1020, 2015.

[18] J. E. McCarthy and O. M. Shalit. Spaces of Dirichlet series with the complete Pick property. *Israel J. Math.*, 220(2):509–530, 2017.

[19] V. I. Paulsen and M. Raghupathi. *An introduction to the theory of reproducing kernel Hilbert spaces*, volume 152 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[20] P. Quiggin. For which reproducing kernel Hilbert spaces is Pick’s theorem true? *Integral Equations Operator Theory*, 16(2):244–266, 1993.

[21] S. Richter and C. Sundberg. A formula for the local Dirichlet integral. *Michigan Math. J.*, 38(3):355–379, 1991.

[22] J. Sampat. Cyclicity preserving operators on spaces of analytic functions in $C^n$. *Integral Equations Operator Theory*, 93(2):Paper No. 14, 20, 2021.

[23] S. Shimorin. Complete Nevanlinna-Pick property of Dirichlet-type spaces. *J. Funct. Anal.*, 191(2):276–296, 2002.

[24] W. Želazko. A characterization of multiplicative linear functionals in complex Banach algebras. *Studia Math.*, 30:83–85, 1968.
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