Cosmological models and stability

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I would already have concluded my researches about world harmony, had not Tycho’s astronomy so shackled me that I nearly went out of my mind

Johannes Kepler
The perfect cosmological principle:

The universe is isotropic in space and time

The cosmological principle:

The universe is homogenous and isotropic

The standard model in cosmology:
- laws of general relativity
- cosmological principle
- observations

→ Universe is approximated by Friedmann model with positive cosmological constant $\Lambda$:

$$\Omega_m^0 \sim 0.3, \quad \Omega_\Lambda^0 \sim 0.7$$
Cosmological principles and the standard model

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Cosmological principles and the standard model
Cosmological models and inhomogeneity

- weaker cosmological principles:
  - statistical homogeneity
  - no special (matter bound) observer

Inhomogeneity in cosmological models
- fitting problem
- backreaction
- averaging
- “discrete” vs. “smooth” matter distribution
- effect of inhomogeneities on observations
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Mathematical problems
- (almost) EGS
- stability/instability
- asymptotics
  - at singularity (eg. BKL proposal – cosmic censorship)
  - in expanding direction:

What does an observer in the late universe see?
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Asymptotics of cosmological models

We appear not to be in an asymptotic regime...

however we may study the mathematical problem...
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Friedmann dust:

\[
\frac{H^2}{H_0^2} = \Omega_{0m} \left( \frac{a_0}{a} \right)^3 + \Omega_{0\Lambda} + \Omega_{0\kappa} \left( \frac{a_0}{a} \right)^2
\]

If \( \Lambda > 0 \), \( \Omega_{\Lambda} \) dominates as \( a \to \infty \)

Restrict to \( \Omega_{\Lambda} = 0 \):

- Einstein-de Sitter (matter dominated, \( \Omega_{\kappa} = 0 \)):
  - unstable within Friedmann models, slow volume growth
  - \( a \sim t^{2/3} \)

- Milne (empty universe, \( \Omega_{\kappa} = 1 \)):
  - stable within Friedmann models, rapid volume growth
  - \( a \sim t \)
Asymptotics of cosmological models

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If \( \Lambda > 0 \), \( \Omega_\Lambda \) dominates as \( a \to \infty \). Restrict to \( \Omega_\Lambda = 0 \):

- \( \Omega_m = 0 \)
- \( \Omega_m = 1 \)

\[ \kappa = -1 \quad \kappa = 0 \quad \kappa = 1 \]

- Einstein-de Sitter (matter dominated, \( \Omega_\kappa = 0 \)): unstable within Friedmann models, slow volume growth \( a \sim t^{2/3} \)
- Milne (empty universe, \( \Omega_\kappa = 1 \)): stable within Friedmann models, rapid volume growth \( a \sim t \)
Asymptotics of cosmological models

- Milne is the flat interior of the lightcone in Minkowski space

Cosmological time level

Line element $ds^2 = -dt^2 + t^2 g_{H^3}$

- Deformed Milne
Asymptotics of cosmological models

- Milne is the flat interior of the lightcone in Minkowski space
  
- Cosmological time level

- Line element $ds^2 = -dt^2 + t^2 g_{H^3}$

- Deformed Milne

Cosmological time level
Asymptotics of cosmological models

$\text{deformed region has slow volume growth}$

$\text{deformed Milne is flat and empty – but not homogenous and isotropic}$
Asymptotics of cosmological models

More general flat models, e.g. compact quotient of deformed Milne (Mess, 1990), (LA, 2002), (Barbot, 2005), (LA, Barbot, Beguin & Zeghib, 2012)

Neck region – slow volume growth

Hyperbolic region

- asymptotically, hyperbolic (thick) regions dominate
- “neck regions” (thin) become insignificant
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Einstein flow

The Lorentzian Einstein equations define a flow on the space of (scale free) geometries

- Consider vacuum spacetimes \((M, g_{ab})\)
  \[
  R_{ab} = 0
  \]
  with compact Cauchy surface \((M, g_{ij}, K_{ij})\)
- Use logarithmic constant mean curvature (Hubble) time
  \[
  T = -\ln(\tau/\tau_0)
  \]

- Consider the evolution of the scale free geometry \([g] = \tau^2 g\)
- The Lorentzian Einstein equations define a flow
  \[
  T \mapsto [g](T)
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Einstein flow

- 2+1 dimensional case: Einstein equations corresponds to time dependent Hamiltonian system on Teichmüller space (LA, Moncrief & Tromba, 1997)
Einstein flow

General scenario (Fischer & Moncrief, 2000), (M. Anderson, 2001)

- Non-collapsing case – negative Yamabe type
- For $T \to \infty$, $(M, [g])$ decomposes decomposition into hyperbolic pieces and Seyfert fibered pieces $\leftrightarrow$ (weak) geometrization
- Einstein flow in CMC time $\leadsto$ thick/thin decomposition of $M$
- Thick (hyperbolic) pieces have full volume growth
- $\Rightarrow$ in the far future, the hyperbolic pieces represent most of the volume of $M$
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Nonlinear stability problem for cosmological models:

*Prove that for Cauchy data close to Milne, the future Cauchy development is asymptotic to Milne*

Vacuum case: (LA & Moncrief, 2004), (LA & Moncrief, 2011)

More general question:

*For Cauchy data close to $\kappa \leq 0$ Friedmann, characterize the future Cauchy development*
Nonlinear stability problem for cosmological models:

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Nonlinear stability: Minkowski

- (Friedrich, 1986): nonlinear stability to the future of a hyperboloidal slice, regular $I^+$
- (Christodoulou & Klainerman, 1993; Klainerman & Nicolò, 2003): use null coordinates, Bel-Robinson energy. Get peeling if sufficiently regular at $i_0$
- (Lindblad & Rodnianski, 2005): use weak null condition, simple proof, matter can be added

conformal type: Minkowski diamond
Nonlinear stability: Dark Energy

future horizons, topology does not matter (but cf. (LA & Galloway, 2002))

Locality at $I^+$ ⇒ “small data” can be characterized locally in space

Results:

- (Friedrich, 1991), (M. Anderson & Chruściel, 2005) (Heinzle & Rendall, 2005): global stability
- (Starobinsky, 1983), (Rendall, 2006): expansions, Fuchsian
- (Ringström, 2007): “local” small data global existence (Einstein-$\Lambda$-scalar field)
Nonlinear stability: Dark Energy

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Locality at $I^+$ ⇒ “small data” can be characterized locally in space

Results:

- Einstein-$\Lambda$-irrotational fluid
  (Rodnianski & Speck, 2009)

- Einstein-$\Lambda$-Euler (Speck, 2011)

- Einstein-$\Lambda$-Vlasov
  (Ringström, 2012)

conformal type: finite cylinder
Nonlinear stability: Ordinary matter

- Example: Lorentz cone on compact hyperbolic: $ds^2 = -dt^2 + t^2 \gamma_{HH}$ ($\kappa = -1$ empty Friedmann)
- topology matters
- vacuum:
  - $U(1)$ (Choquet-Bruhat & Moncrief, 2001)
  - $G_0$ (LA & Moncrief, 2004), (LA & Moncrief, 2011)
- matter:
  - Einstein-Vlasov, Bianchi symmetry: (Rendall & Tod, 1999), (Heinzle & Uggla, 2006), (Nungesser, 2011)
  - 2+1 Einstein-Vlasov: (Fajman, 2012)
- test fluids on FLRW: (Speck, 2012)

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**Scale invariant variables**

- \((M, \gamma)\) \(n\)-dimensional negative Einstein
- \(\text{Ric}_\gamma = -\frac{n-1}{n^2}\gamma;\quad ds^2 = -dt^2 + \frac{t^2}{n^2}\gamma\) Lorentz cone over \(\gamma\)
- Line element \(ds^2 = -\tilde{N}^2 dt^2 + \tilde{g}_{ij}(dx^i + \tilde{X}^i dt)(dx^j + \tilde{X}^j dt)\)
  (physical) vacuum data: \((M, \tilde{g}, \tilde{K}, \tilde{N}, \tilde{X})\)
- \(\tau = \tilde{g}^{ij}\tilde{K}_{ij}\) mean curvature (assume CMC time gauge)

**Rescaled fields** \((g, \Sigma, N, X)\) :

- \(g_{ij} = \tau^2\tilde{g}_{ij}\) \\
- \(\Sigma_{ij} = \tau(\tilde{K}_{ij} - \frac{\tau}{n}\tilde{g}_{ij})\) \\
- \(X^i = \tau\tilde{X}^i\) \\
- \(N = \tau^2\tilde{N}\)

**Scale invariant time:** \(T = -\log(-\tau)\)

**At background:** \((g_{ij}, \Sigma_{ij}, X^i, N) = (\gamma_{ij}, 0, 0, n)\)
(\(M, \gamma\)) \(n\)-dimensional negative Einstein

\[
\text{Ric}_\gamma = -\frac{n-1}{n^2}\gamma; \quad ds^2 = -dt^2 + \frac{t^2}{n^2}\gamma
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Lorentz cone over \(\gamma\)

Line element \(ds^2 = -\tilde{\gamma}dt^2 + \tilde{\gamma}_{ij}(dx^i + \tilde{X}^i dt)(dx^j + \tilde{X}^j dt)\)

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N &= \tau^2\tilde{N}
\end{align*}
\]

Scale invariant time: \(T = -\log(-\tau)\)

At background: \((g_{ij}, \Sigma_{ij}, \chi^i, N) = (\gamma_{ij}, 0, 0, n)\)
Stable Riemannian Einstein spaces

- Require \textit{stability} of \((M, \gamma)\):

\[ \mathcal{L} \geq 0, \]

where \(\mathcal{L} h_{ac} = -\Delta h_{ac} - 2R_{abcd}h^{bd}\)

- Allow a nontrivial, integrable \textit{moduli space} of Riemannian Einstein structures

- Stability and integrability holds in all known cases for negative Einstein spaces

- \(n = 3\): Mostow rigidity \(\Rightarrow\) trivial moduli space

- \(n > 3\) The moduli space of Einstein structures corresponds to
  - center manifold for normalized Ricci flow
  - center manifold for Lorentzian Einstein flow
Stable Riemannian Einstein spaces

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  - center manifold for Lorentzian Einstein flow
Linearized stability analysis

- Linearizing the rescaled Einstein equations around background data \((g, \Sigma, N, X) = (\gamma, 0, n, 0)\) gives

\[
\ddot{X} + (n - 1)\dot{X} + n^2 \lambda X = 0
\]

damped oscillator equation with characteristic roots

\[
-(n - 1) \pm \sqrt{(n - 1)^2 - 4n^2 \lambda^2}
\]

- Energy

\[
E = \frac{1}{2} \dot{X}^2 + \frac{n^2 \lambda}{2} X^2 + c_E X \dot{X}
\]

- Energy decay: \(\frac{d}{dt} E \leq \alpha E\)
Linearized stability analysis

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  -(n - 1) \pm \sqrt{(n - 1)^2 - 4n^2\lambda} \over 2
  \]

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- Energy decay:
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Linearized stability analysis

Let $\lambda_0 = \text{smallest non-zero eigenvalue of } \mathcal{L}$. Define

$$c_E = \begin{cases} \frac{n-1}{2} \frac{\lambda_0}{n-1} \\ \frac{2n^2 \lambda_0}{n-1} \end{cases} \quad \alpha_+ = \begin{cases} -\frac{n-1}{2} \frac{1}{(n-1)+\sqrt{(n-1)^2-4n^2 \lambda_0}} \\ -\frac{1}{2} \frac{1}{(n-1)+\sqrt{(n-1)^2-4n^2 \lambda_0}} \end{cases}$$

universal anomalous
Energies

Use linearized analysis as a guide to defining energies:

- Write energies modelled on damped harmonic oscillator energy, in terms of variables $g - \gamma, \Sigma$
- Energy decay for small data
- Scale invariant geometry converges to Einstein geometry in the moduli space:

  $\Rightarrow$ Einstein spaces are attractors for the Einstein flow
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\[ \Rightarrow \text{Einstein spaces are attractors for the Einstein flow} \]
**Theorem (LA & Moncrief, 2011)**

Suppose \((M^n, \gamma^0)\) stable, integrable, \(\text{Ric}_{\gamma^0} = -\frac{n-1}{n^2} \gamma^0\) and let vacuum data \((\tilde{g}^0, \tilde{K}^0)\) be given.

Assume \(g^0 = \tau^2 \tilde{g}^0, \Sigma^0 = \tau (\tilde{K} - \frac{\tau}{n})\) are close to \((\gamma^0, 0)\),

Then, the maximal vacuum Cauchy development \((M, g)\) of \((M, \tilde{g}^0, \tilde{K}^0)\)

- has a global CMC foliation to the future,
- is future causally geodesically complete,
- \(g(T) \to \gamma_\infty \in \mathcal{N}_{\gamma^0}\), as \(T \to \infty\).
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Generalized Kasner spacetimes

(LA & Heinzle, 2006)

Generalized Kasner spaces: $\mathbb{M} \cong \mathbb{R} \times M \times N$, with

- $(M^m, g)$, $(N^n, h)$, $D = d + 1 = m + n + 1$
- $\text{Ric}_g = -(m + n - 1)g$, $\text{Ric}_h = -(m + n - 1)h$
- Line element $ds^2 = -dt^2 + a^2(t)g + b^2(t)h$

- Introduce scale invariant variables

$p = -\dot{a}/a$, $q = -\dot{b}/b$, $P = p/H$, $Q = q/H$, $A = \frac{1}{aH}$, $B = \frac{1}{bH}$

- Einstein equations $\Rightarrow$ autonomous system for $(P, Q, A, B)$ with 2 constraints.
Generalized Kasner spacetimes

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- introduce scale invariant variables
  \[ p = -\dot{a}/a, q = -\dot{b}/b, \]
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- Einstein equations \(\Rightarrow\) autonomous system for $(P, Q, A, B)$ with 2 constraints.
Generalized Kasner spacetimes

Dynamical systems analysis shows the generic orbit is
- generalized Kasner ($a \sim t^p$, $b \sim t^q$) at singularity,
- Friedmann ($\cong$ cone) in expanding direction
- Friedmann is stable node only if spacetime dimension $D \geq 11$

\[ (F_1) \quad (F_2) \quad (F_A) \quad (F_B) \]

\[ \begin{align*}
D < 10 & \quad D > 10
\end{align*} \]
Generalized Kasner spacetimes

Quiescent singularities:

(Demaret, Henneaux, & Spindel, 1985): condition for quiescent behavior at singularity in $D = d + 1$ dimensions:

$$1 + p_1 - p_d - p_{d-1} > 0 \quad (1)$$

where $p_a =$ generalized Kasner exponents at singularity.

Heuristic: Eq. (1) holds in vacuum only if $D \geq 11 \Rightarrow$

- generic vacuum, $D < 11$ spacetime has oscillatory singularity,
- generic vacuum, $D \geq 11$ spacetime has quiescent singularity

- generic $D = 4$ spacetime with scalar field has quiescent singularity (LA & Rendall, 2001), Fuchsian analysis

- generic $D \geq 11$ vacuum spacetime has quiescent singularity (Damour, Henneaux, Rendall, & Weaver, 2002), Fuchsian analysis
Generalized Kasner spacetimes

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- generic $D \geq 11$ vacuum spacetime has quiescent singularity (Damour et al., 2002), Fuchsian analysis
Global nonlinear stability in the real analytic category:

**Theorem (LA, 2009)**

Suppose $M^m, N^n$ stable, integrable with $D = m + n + 1 \geq 11$. Then there is a full-parameter family of $C^\omega$ Cauchy data on $M \times N$, such that the maximal Cauchy development $(M, g)$

- has global CMC time function,
- has quiescent, crushing singularity,
- is future causally complete,
- is asymptotically Friedmann to the future,
- $g(T) \to \gamma^M_\infty + \gamma^N_\infty$, as $T \to \infty$.

This applies to a large variety of factors $M, N$, and can easily be generalized to multiple factors to give rich future asymptotics.
From $\alpha$ to $\omega$

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Concluding remarks/Open problems

- Future asymptotics of cosmological models well understood in highly symmetric cases: Friedmann, Bianchi, Gowdy, $T^2$, $U(1)$ – full 3+1 case mostly open

- Prove nonlinear stability of Milne for Einstein-Vlasov

- Characterize future evolution of inhomogenous Einstein-matter spacetimes close to Friedmann. Which cases are nonlinearly stable?

- Numerical studies of cosmological models in GR beyond LTB/spherical symmetry
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Thank You
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