PACKING AND COVERING ODD CYCLES IN CUBIC PLANE GRAPHS WITH SMALL FACES

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Abstract. We show that any 3-connected cubic plane graph on \( n \) vertices, with all faces of size at most 6, can be made bipartite by deleting no more than \( \sqrt{(p + 3t)n/5} \) edges, where \( p \) and \( t \) are the numbers of pentagonal and triangular faces, respectively. In particular, any such graph can be made bipartite by deleting at most \( \sqrt{12n/5} \) edges. This bound is tight, and we characterise the extremal graphs. We deduce tight lower bounds on the size of a maximum cut and a maximum independent set for this class of graphs. This extends and sharpens the results of Faria, Klein and Stehlík [SIAM J. Discrete Math. 26 (2012) 1458–1469].

1. Introduction

A set of edges intersecting every odd cycle in a graph is known as an odd cycle (edge) transversal, or odd cycle cover, and the minimum size of such a set is denoted by \( \tau_{\text{odd}} \). A set of edge-disjoint odd cycles in a graph is called a packing of odd cycles, and the maximum size of such a family is denoted by \( \nu_{\text{odd}} \). Clearly, \( \tau_{\text{odd}} \geq \nu_{\text{odd}} \). Dejter and Neumann-Lara [6] and independently Reed [17] showed that in general, \( \tau_{\text{odd}} \) cannot be bounded by any function of \( \nu_{\text{odd}} \), i.e., they do not satisfy the Erdős–Pósa property. However, for planar graphs, Kráľ and Voss [14] proved the (tight) bound \( \tau_{\text{odd}} \leq 2\nu_{\text{odd}} \).

In this paper we focus on packing and covering of odd cycles in 3-connected cubic plane graphs with all faces of size at most 6. Such graphs—and their dual triangulations—are a very natural class to consider, as they correspond to surfaces of genus 0 of non-negative curvature (see e.g. [21]).

A much-studied subclass of cubic plane graphs with all faces of size at most 6 is the class of fullerene graphs, which only have faces of size 5 and 6. Faria, Klein and Stehlík [9] showed that any fullerene graph on \( n \) vertices has an odd cycle transversal with no more than \( \sqrt{12n/5} \) edges, and characterised the extremal graphs. Our main result is the following extension and sharpening of their result to all 3-connected cubic plane graphs with all faces of size at most 6.

**Theorem 1.1.** Let \( G \) be a 3-connected cubic plane graph on \( n \) vertices with all faces of size at most 6, with \( p \) pentagonal and \( t \) triangular faces. Then

\[
\tau_{\text{odd}}(G) \leq \sqrt{(p + 3t)n/5}.
\]

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In particular, $\tau_{\text{odd}}(G) \leq \sqrt{\frac{12n}{5}}$ always holds, with equality if and only if all faces have size 5 and 6, $n = 60k^2$ for some $k \in \mathbb{N}$, and $\text{Aut}(G) \cong I_h$.

If $G$ is a fullerene graph, then $t = 0$ and Euler’s formula implies that $p = 12$, so Theorem 1.1 does indeed generalise the result of Faria, Klein and Stehlík [9]. We also remark that the smallest 3-connected cubic plane graph with all faces of size at most 6 achieving the bound $\tau_{\text{odd}}(G) = \sqrt{\frac{12n}{5}}$ in Theorem 1.1 is the ubiquitous buckminsterfullerene graph (on 60 vertices).

The rest of the paper is organised as follows. In Section 2, we introduce the basic notation and terminology, as well as the key concepts from combinatorial optimisation and topology. In Section 3, we introduce the notions of patches and moats, and prove bounds on the area of moats. Then, in Section 4, we use these bounds to prove an upper bound on the maximum size of a packing of $T$-cuts in triangulations of the sphere with maximum degree at most 6. Using a theorem of Seymour [19], we deduce, in Section 5, an upper bound on the minimum size of a $T$-join in triangulations of the sphere with maximum degree at most 6, and then dualise to complete the proof of Theorem 1.1. In Section 6, we deduce lower bounds on the size of a maximum cut and a maximum independent set in 3-connected cubic plane graphs with no faces of size more than 6. Finally, in Section 7, we show why the condition on the face size cannot be relaxed, and briefly discuss the special case when the graph contains no pentagonal faces.

2. Preliminaries

Most of our graph-theoretic terminology is standard and follows [1]. All graphs are finite and simple, i.e., have no loops and parallel edges. The degree of a vertex $u$ in a graph $G$ is denoted by $d_G(u)$. If all vertices in $G$ have degree 3, then $G$ is a cubic graph. The set of edges in $G$ with exactly one end vertex in $X$ is denoted by $\delta_G(X)$. A set $C$ of edges is a cut of $G$ if $C = \delta_G(X)$, for some $X \subseteq V(G)$. When there is no risk of ambiguity, we may omit the subscripts in the above notation.

The set of all automorphisms of a graph $G$ forms a group, known as the automorphism group $\text{Aut}(G)$. The full icosahedral group $I_h \cong A_5 \times C_2$ is the group of all symmetries (including reflections) of the regular icosahedron. The full tetrahedral group $T_d \cong S_4$ is the group of all symmetries (including reflections) of the regular tetrahedron.

A polygonal surface $K$ is a simply connected 2-manifold, possibly with a boundary, which is obtained from a finite collection of disjoint simple polygons in $\mathbb{R}^2$ by identifying them along edges of equal length. We denote by $|K|$ the union of all polygons in $K$, and remark that $|K|$ is a surface.

Based on this construction, $K$ may be viewed as a graph embedded in the surface $|K|$. Accordingly, we denote its set of vertices, edges, and faces by $V(K)$, $E(K)$, and $F(K)$, respectively. If every face of $K$ is incident to three edges, $K$ is a triangulated surface, or a triangulation of $|K|$. In this case, $K$ can be viewed as a simplicial complex. If $K$ is a simplicial complex and $X \subseteq V(K)$, then $K[X]$ is the subcomplex induced by $X$, and $K \setminus X$ is the subcomplex obtained by deleting $X$ and all incident simplices. If $L$ is a subcomplex of $K$, then we simply write $K \setminus L$ instead of $K \setminus V(L)$. 
If $K$ is a graph embedded in a surface $|K|$ without boundary, the dual graph $K^*$ is the graph with vertex set $F(K)$, such that $fg \in E(K^*)$ if and only if $f$ and $g$ share an edge in $K$. The size of a face $f \in F(K)$ is defined as the number of edges on its boundary walk, and is denoted by $d_K(f)$. Note that $d_K(f) = d_{K^*}(f^*)$.

Any polygonal surface homeomorphic to a sphere corresponds to a plane graph via the stereographic projection. Therefore, terms such as ‘plane triangulation’ and ‘triangulation of the sphere’ can be used interchangeably.

We shall make the convention to use the term ‘cubic plane graphs’ because it is so widespread, but refer to the dual graphs as ‘triangulations of the sphere’ because it reflects better our geometric viewpoint.

Given a polygonal surface $K$, the boundary $\partial K$ is the set of all edges in $K$ which are not incident to two triangles; the number of edges in the boundary is denoted by $|\partial K|$. With a slight abuse of notation, $\partial K$ will also denote the set of vertices incident to edges in $\partial K$. The set of interior vertices is defined as $\text{int}(K) = V(K) \setminus \partial K$.

Given a triangulated surface $K$, we define $\text{area}(K)$ to be the number of faces in $K$, and the combinatorial curvature of $K$ as $\sum_{u \in \text{int}(K)}(6 - d_K(u))$. Recall that the Euler characteristic $\chi(K)$ of a polygonal surface $K$ is equal to $|V(K)| - |E(K)| + |F(K)|$. It can be shown that $\chi$ is a topological invariant: it only depends on the surface $|K|$, not on the polygonal decomposition of $K$. If $X$ is any contractible space, then $\chi(X) = 1$, and if $S^2$ is the standard 2-dimensional sphere, then $\chi(S^2) = 2$. The following lemma is an easy consequence of Euler’s formula and double counting, and we leave its verification to the reader.

**Lemma 2.1.** Let $K$ be a triangulated surface with (a possibly empty) boundary $\partial K$. Then

$$\sum_{v \in \text{int}(K)}(6 - d(v)) + \sum_{v \in \partial K}(4 - d(v)) = 6\chi(K).$$

We remark that, if we multiply both sides of the equation by $\pi/3$, we obtain a discrete version of the Gauss–Bonnet theorem (see e.g. [15]), where the curvature is concentrated at the vertices.

In order to prove Theorem 1.1, it is more convenient to work with the dual graphs, which are characterised by the following simple lemma. The proof is an easy exercise, which we leave to the reader.

**Lemma 2.2.** If $G$ is a 3-connected simple cubic plane graph with all faces of size at most 6, then the dual graph $G^*$ is a simple triangulation of the sphere with all vertices of degree at least 3 and at most 6.

We will use the following important concept from combinatorial optimisation. Given a graph $G = (V, E)$ with a distinguished set $T$ of vertices of even cardinality, a $T$-join of $G$ is a subset $J \subseteq E$ such that $T$ is equal to the set of odd-degree vertices in $(V, J)$. The minimum size of a $T$-join of $G$ is denoted by $\tau(G, T)$. When $T$ is the set of odd-degree vertices of $G$, a $T$-join is known as a postman set. A $T$-cut is an edge cut $\delta(X)$ such that $|T \cap X|$ is odd. A packing of $T$-cuts is a disjoint collection $\delta(F) = \{\delta(X) \mid X \in F\}$ of $T$-cuts of $G$; the maximum size of a packing of $T$-cuts is denoted by $\nu(G, T)$.

A family of sets $F$ is said to be laminar if, for every pair $X, Y \in F$, either $X \subseteq Y$, $Y \subseteq X$, or $X \cap Y = \emptyset$. A $T$-cut $\delta(X)$ is inclusion-wise minimal if
no $T$-cut is properly contained in $\delta(X)$. For more information on $T$-joins and $T$-cuts, the reader is referred to [3, 16, 18].

3. Patches and moats

From now on assume that $K$ is a triangulation of the sphere with all vertices of degree at most 6. We define a subcomplex $L \subseteq K$ to be a patch if in the dual complex $K^*$, the faces corresponding to $V(L)$ form a subcomplex homeomorphic to a disc. (Equivalently, one could say that $L \subseteq K$ is a patch if $L$ is an induced, contractible subcomplex of $K$.) A patch $L \subseteq K$ such that $c = \sum_{u \in V(K)}(6 - d_K(u))$ is called a $c$-patch. We remark that a $c$-patch $L$ has combinatorial curvature $c$ if and only if all vertices in the boundary $\partial L$ have degree 6 in $K$. If $u \in V(K)$ has degree $6 - c$, and the set $X$ of vertices at distance at most $r$ from $u$ contains only vertices of degree 6, then the $c$-patch $K[\{u\} \cup X]$ is denoted by $D_r(c)$. The subcomplex of the dual complex $K^*$ formed by the faces in $V(D_r(c))$ is denoted by $D^*_r(c)$; see Figure 3.1.

The following isoperimetric inequality follows from the work of Justus [13, Theorem 3.2.3 and Table 3.1].

**Lemma 3.1** (Justus [13]). Let $K^*$ be a polygonal surface homeomorphic to a disc, with all internal vertices of degree 3 and with $n$ faces, all of size at most 6. Let $c = \sum_{f \in F(K^*)}(6 - d(f))$, and suppose that $c \leq 5$. Then

$$|\partial K^*| \geq \sqrt{8(6 - c)(n - 1) + (6 - c)^2}.$$  

Equality holds if $K^* \cong D^*_r(c)$, for some integer $r \geq 0$, and only if at most one face in $K^*$ has size less than 6.

**Proof.** The minimum possible values of $|\partial K^*|$ are given in [13, Table 3.1], for all possible numbers of hexagonal, pentagonal, square, and triangular faces. In each case, our bound is satisfied. Moreover, it can be checked that equality holds only if at most one face in $K^*$ has size less than 6. Finally, if $K^* \cong D^*_r(c)$, then it can be shown that $|\partial K^*| = (6 - c)(2r + 1)$ and $f - 1 = (6 - r)(r + 1)/2$. Hence, $|\partial K^*| = \sqrt{8(6 - c)(f - 1) + (6 - c)^2}$. □

\footnote{Gunnar Brinkmann [2] has pointed out an error in the statement and proof of [8, Lemma 4.4] on which [13, Theorem 3.2.3] is based, but has sketched a different way to prove [13, Theorem 3.2.3].}
We can use Lemma 3.1 to deduce the following isoperimetric inequality for triangulations. Certain special cases of the inequality were already proved by Justus [13].

**Lemma 3.2.** Let $K$ be a triangulation of the sphere with all vertices of degree at most 6, and let $L \subseteq K$ be a patch of combinatorial curvature $c \leq 5$. Then

$$|\partial L| \geq \sqrt{(6 - c) \text{area}(L)}.$$  

Equality holds if $L \cong D_r(c)$, for some integer $r \geq 0$, and only if at most one vertex in $\text{int} \ L$ has degree less than 6.

**Proof.** Put $n = |V(K)|$, and let $L^*$ be the subcomplex of $K^*$ formed by the faces corresponding to $V(L)$. By Lemma 3.1

$$|\partial L^*| \geq \sqrt{8(6 - c)(n - 1) + (6 - c)^2}.$$  

Moreover, the following two equalities were shown by Justus [13, equations (3.8) and (3.11)]

$$2(n - 1) = \text{area}(L) - |\partial L|,$$  

$$|\partial L|^2 = \frac{1}{4}|\partial(L^*)|^2 + (6 - c)|\partial L| - \frac{1}{4}(6 - c)^2.$$  

So, combining (3.1), (3.2) and (3.3) gives

$$|\partial L|^2 \geq (6 - c) \text{area}(L).$$  

Equality holds in (3.4) if and only if equality holds in (3.1). The latter is true only if at most one face in $L^*$ has size less than 6, or equivalently, only if at most one vertex in $\text{int} \ L$ has degree less than 6. For the final part, it is enough to note that if $L \cong D_r(c)$, then $L^* \cong D_r^*(c)$, so equality holds in (3.1) and therefore in (3.4). □

Let $L \subseteq K$ be a patch. A *moat* of width 1 in $K$ surrounding $L$ is the set $\text{Mt}^1(L)$ of all the faces in $F(K) \setminus F(L)$ with at least one vertex in $V(K)$. More generally, we can define a moat of width $w$ in $K$ surrounding $L$ recursively as $\text{Mt}^w(L) = \text{Mt}^1(\text{Mt}^{w-1}(L) \cup L)$. With a slight abuse of notation, $\text{Mt}^w(L)$ will also denote the subcomplex of $K$ formed by the faces in $\text{Mt}^w(L)$. If $L$ is a $c$-patch, then $\text{Mt}^w(L)$ is a *c-moat* of width $w$ surrounding $L$. See Figure 3.2 for an example of a moat.

Under certain conditions, the area of a $c$-moat $\text{Mt}^w(L)$ can be bounded in terms of $c$, $w$, and $\text{area}(L)$.

**Lemma 3.3.** Let $K$ be a triangulation of the sphere with maximum degree at most 6, and suppose $L \subseteq K$ is a $c$-patch, for some $0 < c < 6$. If $L \cup \text{Mt}^i(L)$ is a $c$-patch, for every $0 \leq i \leq w - 1$, then

$$\text{area}(\text{Mt}^w(L)) \geq (6 - c)w^2 + 2w\sqrt{(6 - c) \text{area}(L)}.$$  

Equality holds if $L \cong D_r(c)$, for some integer $r \geq 0$, and only if at most one vertex in $\text{int} \ L$ has degree less than 6.
Proof. As $L$ is contractible, its Euler characteristic is $\chi(L) = 1$. We have

$$c = \sum_{u \in V(L)} (6 - d_K(u)) = \sum_{u \in \text{int } L} (6 - d_L(u)) + \sum_{u \in \partial L} (6 - d_L(u)) - \text{area}(\text{Mt}^1(L))$$

$$= \sum_{u \in \text{int } L} (6 - d_L(u)) + \sum_{u \in \partial L} (4 - d_L(u)) + 2|\partial L| - \text{area}(\text{Mt}^1(L)).$$

Hence, by Lemma 2.1,

$$2|\partial L| + 6 - c = \text{area}(\text{Mt}^1(L)). \quad (3.5)$$

The dual complex $K^*$ is homeomorphic to the sphere and the subcomplex $L^*$ formed by the faces corresponding to $V(L)$ is homeomorphic to a disc, so by the Jordan–Schoenflies theorem, the subcomplex formed by the faces corresponding to $V(K) \setminus V(L)$ is also homeomorphic to a disc. Hence, $K \setminus L$ is also a patch. Moreover, $K$ has Euler characteristic $\chi(K) = 2$, so by Lemma 2.1 $\sum_{u \in V(K)} (6 - d(u)) = 12$. Therefore, $\sum_{u \in V(K \setminus L)} (6 - d_K(u)) = 12 - c$, i.e., $K \setminus L$ is a $(12 - c)$-patch. Applying (3.5) to $L$ and to $K \setminus L$,

$$2|\partial L| + 6 - c = \text{area}(\text{Mt}^1(L))$$

$$= \text{area}(\text{Mt}^1(K \setminus L))$$

$$= 2|\partial(K \setminus L)| + 6 - (12 - c).$$

Hence, $|\partial(L \cup \text{Mt}^1(L))| = |\partial(K \setminus L)| = |\partial L| + 6 - c$, so by induction, and the fact that $L \cup \text{Mt}^i(L)$ is a patch for all $0 \leq i \leq w - 1$,

$$|\partial(L \cup \text{Mt}^i(L))| = |\partial L| + (6 - c)i. \quad (3.6)$$

By (3.5) and (3.6),

$$\text{area}(\text{Mt}^1(L \cup \text{Mt}^i(L))) = 2|\partial(L \cup \text{Mt}^i(L))| + 6 - c$$

$$= 2(|\partial L| + (6 - c)i) + 6 - c$$

$$= 2|\partial L| + (6 - c)(2i + 1),$$
so the area of $Mt^{w}(L)$ is
\[
\text{area}(Mt^{w}(L)) = \sum_{i=0}^{w-1} \text{area}(Mt^{1}(L \cup Mt^{i}(L)))
\]
\[
= \sum_{i=0}^{w-1} (2|\partial L| + (6 - c)(2i + 1))
\]
\[
= 2w|\partial L| + (6 - c)w^2.
\]
The combinatorial curvature of $L$ is at most $c$, so by Lemma 3.2
\[
|\partial L| \geq \sqrt{(6 - c)\text{area}(L)},
\]
with equality if $L \cong D_r(c)$, for some integer $r \geq 0$, and only if at most one vertex in $\text{int} L$ has degree less than 6. \hfill \Box

4. Packing odd cuts in triangulations of the sphere with maximum degree at most 6

We now relate certain special types of packings of $T$-cuts to packings of 1-, 3- and 5-moats.

**Lemma 4.1.** Let $K$ be a triangulation of the sphere with all vertices of degree at most 6, and let $T$ be the set of odd-degree vertices in $K$. There exists a family $\mathcal{F}$ on $V(K)$ and a vector $w \in \mathbb{N}^{[\mathcal{F}]}$ with the following properties.

(M1) $\mathcal{M} = \bigcup_{X \in \mathcal{F}} Mt^{w_{X}}(X)$ is a packing of moats in $K$;

(M2) The total width of $\mathcal{M}$ is $\sum_{X \in \mathcal{F}} w_{X} = \nu(K,T)$;

(M3) For every $X \in \mathcal{F}$, the subcomplex $K[X]$ is a patch;

(M4) Every $Mt^{w_{X}}(X) \in \mathcal{M}$ is a 1-, 3-, or 5-moat in $K$;

(M5) If $X$ is an inclusion-wise minimal element in $\mathcal{F}$, then $|X| = 1$;

(M6) $\mathcal{F}$ is laminar.

**Proof.** Consider a packing $\delta(\mathcal{F'})$ of inclusion-wise minimal $T$-cuts in $K$ of size of $\nu(K,T)$. Note that $\sum_{u \in X}(6 - d_{K}(u))$ is odd, for every $X \in \mathcal{F}$. Since $\sum_{u \in X}(6 - d_{K}(u)) = 12$ and $\delta(X) = \delta(V(K) \setminus X)$, we can assume that $\sum_{u \in X}(6 - d_{K}(u)) \leq 5$; otherwise we could replace $X$ by $V(K) \setminus X$ in $\delta(\mathcal{F'})$. Finally, we can also assume that, subject to the above conditions, $\mathcal{F'}$ minimises $\sum_{X \in \mathcal{F'}} |X|$.

We remark that $\mathcal{F'}$ is a laminar family. Indeed, suppose that $X, Y \in \mathcal{F'}$, $X \cap Y \neq \emptyset$, $X \subseteq Y$ and $Y \subseteq X$. Then $Mt^{1}(X) \cap Mt^{1}(Y) \neq \emptyset$, so there is a face $\{u,v,w\}$ of $K$ in $Mt^{1}(X) \cap Mt^{1}(Y)$. Since
\[
|\delta(X) \cap \{uv,uw,vw\}| = |\delta(Y) \cap \{uv,uw,vw\}| = 2,
\]
it follows that $\delta(X) \cap \delta(Y) \neq \emptyset$, contradicting the fact that $\mathcal{F'}$ is a packing of $T$-cuts. Hence, $\mathcal{F'}$ is laminar.

We summarise the properties of the family $\mathcal{F'}$ below.

(P1) $\delta(\mathcal{F'})$ is a packing of $T$-cuts;

(P2) $|\mathcal{F'}| = \nu(K,T)$;

(P3) $\delta(X)$ is an inclusion-wise minimal cut, for every $X \in \mathcal{F'}$;

(P4) $\sum_{u \in X}(6 - d_{K}(u)) \in \{1,3,5\}$ for all $X \in \mathcal{F'}$;

(P5) Subject to (P1) (P4) $\mathcal{F'}$ minimises $\sum_{X \in \mathcal{F'}} |X|$;

(P6) $\mathcal{F'}$ is laminar.
We let \( \mathcal{F} \) be the subfamily of \( \mathcal{F}' \) consisting of the elements \( X \in \mathcal{F}' \) such that
\[
\sum_{u \in Y} (6 - d_K(u)) < \sum_{u \in X} (6 - d_K(u)),
\]
for every \( Y \in \mathcal{F} \) such that \( Y \subseteq X \). For each \( X \in \mathcal{F} \), let
\[
\mathcal{F}'_X = \{ Y \in \mathcal{F}' : X \subseteq Y, \sum_{u \in Y} (6 - d_K(u)) = \sum_{u \in X} (6 - d_K(u)) \},
\]
and let \( w_X = |F'_X| \).

To prove \( (P1) \), we use an argument very similar to the one we used to prove \( (P6) \). Clearly, for every \( X \in \mathcal{F} \), \( \text{Mt}^{\omega_X}(X) = \bigcup_{Y \in \mathcal{F}_X} \text{Mt}^{1}(Y) \) is a moat around \( X \) of width \( w_X \). Let \( X, Y \in \mathcal{F} \), and suppose that \( \text{Mt}^{\omega_X}(X) \cap \text{Mt}^{\omega_Y}(Y) \neq \emptyset \). Then there exists a face \( \{u, v, w\} \in F(K) \) and sets \( X', Y' \in \mathcal{F}' \) such that \( X \subseteq X', Y \subseteq Y' \), and
\[
|\delta(X') \cap \{uv, uw, vw\}| = |\delta(Y') \cap \{uw, uv, vw\}| = 2.
\]
But then \( \delta(X') \cap \delta(Y') \neq \emptyset \), so by \( (P1) \), \( X' = Y' \). Hence, by the construction of \( \mathcal{F} \), \( X = Y \). This proves \( (P1) \).

To prove \( (P2) \), it suffices to note that \( \sum_{X \in \mathcal{F}} w_X = |\mathcal{F}'| = \nu(K, T) \) by \( (P2) \).

The property \( (P3) \) follows immediately from \( (P3) \); indeed, since \( \delta(X) \) is an inclusion-wise minimal cut, the dual edges form a cycle, so by the Jordan–Schoenflies theorem, the subcomplex of \( K^* \) formed by the faces in \( X \) is homeomorphic to a disc, so \( K[X] \) is a patch.

Since \( \mathcal{F} \subseteq \mathcal{F}' \), \( (P4) \) follows immediately from \( (P4) \) and \( (P6) \) follows immediately from \( (P6) \).

To prove \( (P5) \), let \( X \) be an inclusion-wise minimal element of \( \mathcal{F} \). By the definition of \( \mathcal{F} \), \( X \) is also an inclusion-wise minimal element of \( \mathcal{F}' \). Since \( \sum_{u \in X} d(u) \) is odd, at least one vertex in \( X \) has odd degree. If \( |X| > 1 \), let \( u \) be a vertex of odd degree in \( X \), and let \( \mathcal{F}'' = (\mathcal{F}' \setminus X) \cup \{u\} \). Then \( \mathcal{F}'' \) satisfies \( (P1), (P4) \) but \( \sum_{X \in \mathcal{F}''} |X| < \sum_{X \in \mathcal{F}'} |X| \), contradicting \( (P5) \). \( \square \)

Lemmas 3.3 and 4.1 can be used to prove the following upper bound on the maximum size of a packing of odd cuts in spherical triangulations with all vertices of degree at most 6, which may be of independent interest. By taking the planar dual, we also get an upper bound on \( \nu_{\text{odd}} \) for the class of 3-connected cubic plane graphs with all faces of size at most 6.

**Theorem 4.2.** Let \( K \) be a triangulation of the sphere with maximum degree at most 6. If \( T \) is the set of odd-degree vertices of \( K \), then
\[
\nu(K, T) \leq \sqrt{\frac{1}{5} \sum_{u \in T} (6 - d(u)) \text{area}(K)}.
\]

In particular, \( \nu(K, T) \leq \sqrt{\frac{12 \text{area}(K)}{5}} \) always holds, with equality if and only if all vertices have degree 5 and 6, \( \text{area}(K) = 60k^2 \) for some \( k \in \mathbb{N} \), and \( \text{Aut}(K) \cong I_h \).

**Proof.** Let \( \mathcal{M} = \bigcup_{X \in \mathcal{F}} \text{Mt}^{\omega_X}(X) \) be a packing of 1-, 3- and 5-moats in \( K \) of total width \( \sum_{X \in \mathcal{F}} w_X = \nu(K, T) \), as guaranteed by Lemma 4.1. Let \( m_c \) be the total area of \( c \)-moats of \( \bigcup_{X \in \mathcal{F}} \text{Mt}^{\omega_X}(X) \), where \( c \in \{1, 3, 5\} \). Define
the incidence vectors \( r, s, t \in \mathbb{R}^{|T|} \) as follows: for every \( u \in T \), let \( r_u, s_u, t_u \) be the width of the 1-moat, 3-moat and 5-moat surrounding \( u \), respectively.

Define the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^{|T|} \) by \( \langle x, y \rangle = \sum_{u \in T} (6 - d(u)) x_u y_u \) and the norm \( \| \cdot \| \) by \( \|x\| = \langle x, x \rangle \). With this inner product, the total width of 1-, 3- and 5-moats in \( \bigcup_{X \in \mathcal{F}} M^t_K(X) \) can be expressed as \( \langle r, 1 \rangle, \frac{1}{3} \langle s, 1 \rangle \), and \( \frac{1}{5} \langle t, 1 \rangle \), respectively. Therefore,

\[
\nu(K, T) = \sum_{X \in \mathcal{F}} w_X = \langle r + \frac{1}{3}s + \frac{1}{5}t, 1 \rangle.
\]

To prove the inequality in Theorem 4.2, we compute lower bounds on \( m_1 \), \( m_3 \) and \( m_5 \) in terms of the vectors \( r, s \) and \( t \), and then use the fact that the moats are disjoint, so the sum \( m_1 + m_3 + m_5 \) cannot exceed \( f \), the number of faces of \( K \). Simplifying the inequality gives the desired bound.

To bound \( m_1 \), recall that by property (M5) of Lemma 4.1, every 1-moat in \( M \) is of the form \( M^r_t u K(u) \), where \( u \) is a 5-vertex in \( K \). By Lemma 3.3,

\[
\text{area}(M^r_t u K(u)) = (6 - (6 - d(u))) r_u^2 = 5r_u^2,
\]

and summing over all 1-moats gives the equality

\[
m_1 = 5 \sum_{u \in T} (6 - d(u)) r_u^2 = 5\|r\|^2.
\]

To bound \( m_3 \), let \( M^s_t u K(u) \) be a non-empty 3-moat in \( M \), for some \( u \in T \cap X \). By the laminarity of \( M \), the 3-patch \( K[X] \) contains the (possibly empty) 1-moats \( M^r_t u K(u) \), for all 5-vertices \( u \in T \cap X \). All the moats are pairwise disjoint, so by (4.2) and the Cauchy–Schwarz inequality,

\[
\text{area}(K[X]) \geq \sum_{u \in T \cap X} \text{area}(M^r_t u K(u))
\geq 5 \sum_{u \in T \cap X} (6 - d(u)) r_u^2
\geq \frac{5}{3} \left( \sum_{u \in T \cap X} (6 - d(u)) r_u \right)^2.
\]

Hence, by Lemma 3.3

\[
\text{area}(M^s_t u K(u)) \geq 3s_u^2 + 2s_u \sqrt{3} \text{area}(K[X])
\geq \sum_{u \in T \cap X} (6 - d(u)) s_u^2 + 2\sqrt{5} \sum_{u \in T \cap X} (6 - d(u)) r_u s_u.
\]

Summing over all 3-moats gives the inequality

\[
m_3 \geq \|s\|^2 + 2\sqrt{5} \langle r, s \rangle.
\]

To bound \( m_5 \), let \( M^s_t u K(u) \) be a non-empty 5-moat in \( M \), for some \( u \in T \cap Y \). By the laminarity of \( M \), the 5-patch \( K[Y] \) contains at most one
non-empty 3-moat $Mt^n_K(X)$ of $M$. All the moats are pairwise disjoint, so by \(4.2\), \(4.4\) and the Cauchy–Schwarz inequality,

\[
\text{area}(K[Y]) \geq \sum_{u \in T \cap Y} \text{area}(Mt^n_K(u)) + \sum_{u \in T \cap Y} \text{area}(Mt^n_K(X)) \\
\geq 5 \sum_{u \in T \cap Y} (6 - d(u)) r_u^2 + \sum_{u \in T \cap Y} (6 - d(u)) \left(2\sqrt{5}r_u s_u + s_u^2\right) \\
= 5 \sum_{u \in T \cap Y} (6 - d(u)) \left(r_u + \frac{1}{\sqrt{5}}s_u\right)^2 \\
\geq \frac{5}{\sqrt{5}} \sum_{u \in T \cap Y} (6 - d(u)) \left(r_u + \frac{1}{\sqrt{5}}s_u\right)^2 \\
= \left(\sum_{u \in T \cap Y} (6 - d(u)) \left(r_u + \frac{1}{\sqrt{5}}s_u\right)^2\right).
\]

Using Lemma \(3.3\),

\[
\text{area}(Mt^n_K(Y)) \geq t_u^2 + 2t_u \sqrt{\text{area}(K[Y])} \\
\geq \frac{1}{5} \sum_{u \in T \cap Y} (6 - d(u)) r_u^2 + 2t_u \sum_{u \in T \cap Y} (6 - d(u)) \left(r_u + \frac{1}{\sqrt{5}}s_u\right) \\
= \sum_{u \in T \cap Y} (6 - d(u)) \left(\frac{1}{5}t_u^2 + 2t_u r_u + \frac{2}{\sqrt{5}}s_u t_u\right),
\]

with equality only if $t_u = 0$, because $K$ is a simple triangulation of the sphere, and as such has no vertex of degree 1. Summing over all 5-moats gives the inequality

\[
(4.5)
\]

\[
m_5 \geq \frac{1}{5} ||t||^2 + 2\langle r, t \rangle + \frac{2}{\sqrt{5}}\langle s, t \rangle,
\]

with equality only if $t = 0$.

The moats are disjoint, so by inequalities \(4.3\), \(4.5\) and \(4.6\),

\[
\text{area}(K) \geq m_1 + m_3 + m_5 \\
\geq 5||r||^2 + ||s||^2 + \frac{1}{5}||t||^2 + 2\sqrt{5}\langle r, s \rangle + 2\langle r, t \rangle + \frac{2}{\sqrt{5}}\langle s, t \rangle \\
= \left\|\sqrt{5}r + s + \frac{1}{\sqrt{5}}t\right\|^2.
\]

Hence, by the Cauchy–Schwarz inequality and \(4.4\),

\[
\sqrt{\frac{1}{5} \sum_{u \in T} (6 - d(u)) \text{area}(K)} \geq \sqrt{\sum_{u \in T} (6 - d(u)) \left\|r + \frac{1}{\sqrt{5}}s + \frac{1}{5}t\right\|} \\
\geq \left\langle r + \frac{1}{\sqrt{5}}s + \frac{1}{5}t, 1\right\rangle \\
\geq \left\langle r + \frac{1}{\sqrt{5}}s + \frac{1}{5}t, 1\right\rangle \\
= \nu(K, T).
\]

This completes the proof of the first part of Theorem \(4.2\).

To prove the inequality $\nu(K, T) \leq \sqrt{12 \text{area}(K)/5}$, it suffices to observe that $\sum_{u \in T} (6 - d(u)) \leq 12$ by Lemma \(2.4\). Now suppose that $\nu(K, T) = \sqrt{12 \text{area}(K)/5}$. By Lemma \(4.3\) there exists a packing $M = \bigcup_{X \in \mathcal{F}} Mt^n_K(X)$
of 1-, 3- and 5-moats in $K$ of total width $\sqrt{12 \text{area}(K)/5}$. Then $\sum_{u \in T} (6 - d(u)) = 12$, i.e., all vertices of degree less than 6 have odd degree, namely, 3 or 5. Equality holds in (4.6) and in (4.8), so $t = s = 0$. Furthermore, equality holds in (4.7), so there is a natural number $k \geq 1$ such that $r_u = k$ for every $u \in T$. Therefore, every $u \in T$ has degree 5, so $|T| = 12$. By Lemma 3.3 each moat $Mt^k(u) \in \mathcal{M}$ has area $5k^2$, so $\text{area}(K) = 12 \cdot 5k^2 = 60k^2$. Hence, $K$ is the union of twelve face-disjoint 1-moats $Mt^k(u)$, for $u \in T$ (see Figure 4.1). Each $Mt^k(u)$ can be identified with a face of a regular dodecahedron, which shows that Aut($K$) contains a subgroup isomorphic to $Ih$. On the other hand, the dual graph of $K$ is a fullerene graph, and it can be shown (see e.g. [7]) that the largest possible automorphism group of a fullerene graph is isomorphic to $Ih$. Hence, Aut($K$) $\cong Ih$.

Conversely, suppose $K$ is a triangulation of the sphere with $\text{area}(K) = 60k^2$, all vertices of degree 5 and 6, and Aut($K$) $\cong Ih$. Then it can be shown (see [4, 10]) that $K$ can be constructed by pasting triangular regions of the (infinite) 6-regular triangulation of the plane into the faces of a regular icosahedron (this is sometimes known in the literature as the Goldberg–Coxeter construction). The construction is uniquely determined by a 2-dimensional vector $(i, j) \in \mathbb{Z}^2$, known as the Goldberg–Coxeter vector (see Figure 4.2). Since Aut($K$) $\cong Ih$, we must have $j = 0$ or $j = i$. The area of $K$ is given by the formula $\text{area}(K) = 20(i^2 + ij + j^2)$. The condition $\text{area}(K) = 60k^2$ implies that the Goldberg–Coxeter vector of $K$ is $(k, k)$, which means that the distance between any pair of 5-vertices in $K$ is at least $2k$. Therefore, $\bigcup_{u \in T} Mt^k(u)$ is a packing of 1-moats of total width $12k = 12\sqrt{\text{area}(K)/60} = \sqrt{12 \text{area}(K)/5}$, so $\nu(K, T) \geq \sqrt{12 \text{area}(K)/5}$. □

5. PROOF OF THEOREM 1.1

Given a triangulation $K$ of the sphere, we construct the refinement $\hat{K}$ as follows. First, we subdivide each edge of $K$, that is, we replace it by an internally disjoint path of length 2, and then we add three new edges inside every face, incident to the three vertices of degree 2. (For an illustration, see Figure 5.1.) Therefore, every face of $K$ is divided into four faces of $\hat{K}$. Observe that all the vertices in $V(\hat{K}) \setminus V(K)$ have degree 6 in $\hat{K}$, so if $T$ is the set of odd-degree vertices of $\hat{K}$, then $T$ is also the set of odd-degree vertices of $\hat{K}$.
Figure 4.2. The Goldberg–Coxeter construction with Goldberg–Coxeter vector \((3,1)\) (to go from the bottom left vertex to the vertex on the right, take three steps to the right, then make a 60 degree left turn and take one more step).

Figure 5.1. A face of a triangulation and its refinement.

The following lemma was proved in [9] using a theorem of Seymour [19].

**Lemma 5.1.** If \(K\) is a triangulation of the sphere and \(T \subseteq V(K)\) is a subset of even cardinality, then \(\tau(K,T) = \frac{1}{2} \nu(\hat{K}, T)\).

Theorem 4.2 and Lemma 5.1 immediately give the following tight upper bound on the minimum size of a postman set in a plane triangulation with maximum degree 6.

**Theorem 5.2.** Let \(K\) be a triangulation of the sphere with \(f\) faces and maximum degree at most 6. If \(T\) is the set of odd-degree vertices of \(G\), then

\[
\tau(K,T) \leq \sqrt{\frac{1}{5} \sum_{u \in T} (6 - d(u)) \text{area}(K)}.
\]

In particular, \(\tau(K,T) \leq \sqrt{12 \text{area}(K)}/5\) always holds, with equality if and only if all vertices have degree 5 and 6, \(\text{area}(K) = 60k^2\) for some \(k \in \mathbb{N}\), and \(\text{Aut}(G) \cong I_h\).

**Proof.** Let \(K\) be a triangulation of the sphere with maximum degree at most 6, and let \(\hat{K}\) be its refinement; observe that \(\text{area}(\hat{K}) = 4 \text{area}(K)\). By Lemma 5.1 and Theorem 4.2

\[
\tau(K,T) = \frac{1}{2} \nu(\hat{K}, T) \leq \sqrt{\frac{1}{5} \sum_{u \in T} (6 - d(u)) \text{area}(K)} \leq \sqrt{12 \text{area}(K)/5},
\]

as required.

If \(\tau(K,T) = \sqrt{12 \text{area}(K)}/5\), then \(\nu(\hat{K}, T) = \sqrt{12 \cdot 4 \text{area}(K)/5}\), so by the second part of Theorem 4.2 all vertices in \(\hat{K}\) have degree 5 and 6, and this must clearly hold in \(K\). Furthermore, \(4 \text{area}(K) = 60k^2\), for some
Corollary 6.1. Let $G$ be a 3-connected cubic plane graph on $n$ vertices with all faces of size at most 6, with $p$ pentagonal and $t$ triangular faces. By Lemma 2.2, the dual graph $G^*$ is a plane triangulation with $\text{area}(G^*) = n$ and all vertices of degree at most 6, having exactly $p$ vertices of degree 5 and $t$ vertices of degree 3. Let $T$ be the set of vertices of odd degree, $J^*$ a minimum $T$-join of $G^*$, and $J$ the set of edges of $G$ which correspond to $J^*$. Since $G^* \setminus J^*$ has no odd-degree vertices, $G \setminus J = (G^* \setminus J^*)^*$ has no odd faces, so is bipartite. By Theorem 5.2

$$|J| = |J^*| \leq \sqrt{\frac{1}{2} \sum_{u \in T} (6 - d(u))n} = \sqrt{\frac{1}{2} (p + 3t)n}.$$ 

In particular, $|J| \leq \sqrt{12n/5}$, with equality if and only if all faces have size 5 and 6, $n = 60k^2$ for some $k \in \mathbb{N}$, and $\text{Aut}(G) \cong I_h$. \hfill \square

6. Consequences for max-cut and independence number

A classic problem in combinatorial optimisation, known as max-cut, asks for the maximum size of an edge-cut in a given graph. This problem is known to be NP-hard, even when restricted to triangle-free cubic graphs [22]. However, for the class of planar graphs, the problem can be solved in polynomial time using standard tools from combinatorial optimisation (namely $T$-joins), as observed by Hadlock [11]. Cui and Wang [5] proved that every planar, cubic graph on $n$ vertices has a cut of size at least $39n/32 - 9/16$, improving an earlier bound of Thomassen [20]. However, when the face size is bounded by 6, we get the following improved bound.

**Corollary 6.1.** If $G$ is a 3-connected cubic plane graph on $n$ vertices with all faces of size at most 6, with $p$ pentagonal and $t$ triangular faces, then $G$ has a cut of size at least

$$3n/2 - \sqrt{(p + 3t)n/5}.$$ 

In particular, $G$ has a cut of size at least $3n/2 - \sqrt{12n/5}$, with equality if and only if all faces have size 5 and 6, $n = 60k^2$ for some $k \in \mathbb{N}$, and $\text{Aut}(G) \cong I_h$.

**Proof of Corollary 6.1.** Let $G$ be a 3-connected cubic plane graph on $n$ vertices with all faces of size at most 6. Let $J \subseteq E(G)$ be an odd cycle transversal, and let $X$ be a colour class of $G \setminus J$. Then $|\delta_G(X)| = 3n/2 - |J|$. By Theorem 1.1 we can always find $J$ such that $|J| \leq \sqrt{12n/5}$, with equality if and only if all faces have size 5 and 6, $n = 60k^2$ for some $k \in \mathbb{N}$, and $\text{Aut}(G) \cong I_h$. \hfill \square

A set of vertices in a graph is independent if there is no edge between any of its vertices, and the maximum size of an independent set in $G$ is the
independence number $\alpha(G)$. Heckman and Thomas [12] showed that every triangle-free, cubic, planar graph has an independent set of size at least $3n/8$, and this bound is tight. Again, forbidding faces of size greater than 6 gives a much better bound.

**Corollary 6.2.** If $G$ is a 3-connected cubic plane graph on $n$ vertices with all faces of size at most 6, with $p$ pentagonal and $t$ triangular faces, then

$$\alpha(G) \geq n/2 - \sqrt{(p+3t)n}/20.$$  

In particular, $\alpha(G) \geq n/2 - \sqrt{3n}/5$, with equality if and only if all faces have size 5 and 6, $n = 60k^2$ for some $k \in \mathbb{N}$, and $\text{Aut}(G) \cong I_h$.

**Proof of Corollary 6.2.** Every graph $G$ contains an odd cycle vertex transversal $U$ such that $|U| \leq \tau_{\text{odd}}(G)$, so $\alpha(G) \geq \alpha(G \setminus U) \geq n/2 - \tau_{\text{odd}}(G)/2$. Therefore, by Theorem 1.1, $\alpha(G) \geq n/2 - \sqrt{3n}/5$, for every 3-connected cubic graph $G$ with all faces of size at most 6. When $J^*$ is a minimum $T$-join of $G^*$, every face of $G^*$ is incident to at most one edge of $J^*$. This means that the set $J \subset E(G)$ corresponding to $J^*$ is a matching of $G$. Therefore, by Theorem 1.1 equality holds if and only if all faces have size 5 and 6, $n = 60k^2$ for some $k \in \mathbb{N}$, and $\text{Aut}(G) \cong I_h$. □

7. Concluding remarks

Clearly, a necessary condition for $\tau_{\text{odd}} = O(\sqrt{n})$ is that $\nu_{\text{odd}} = O(\sqrt{n})$. In the case of planar graphs, the theorem of Král and Voss [14] mentioned in the introduction guarantees that it is also a sufficient condition. It can be shown that $\nu_{\text{odd}} = O(\sqrt{n})$ is also a necessary and sufficient condition for having a max-cut of size at least $3n/2 - O(\sqrt{n})$, and for having an independent set of size at least $n/2 - O(\sqrt{n})$.

It is not hard to construct an infinite family of 3-connected cubic plane graphs with all faces of size at most 7 such that $\tau_{\text{odd}} \geq \varepsilon n$, for a constant $\varepsilon > 0$. This shows that the condition on the size of faces in Theorem 1.1 and Corollaries 6.1 and 6.2 cannot be relaxed.

To construct such a family, consider the graphs $C$ and $R$ in Figure 7.1. Note that $C$ is embedded in a disc, and $R$ is embedded in a cylinder. There are ten vertices on the boundary of $C$ and also on each boundary of $R$, with the degree alternating between 2 and 3. We can paste $k$ copies of $R$ along their boundaries, and then paste a copy of $C$ on each boundary of the resulting cylinder. Assuming $k > 0$, this gives a 3-connected cubic plane graph $G$ on $n = 15 + 40k$ vertices with all faces of size 5 and 7, such that $\nu_{\text{odd}}(G) \geq 4 + 5k > \varepsilon n$.

Finally, we remark that bounding $\tau_{\text{odd}}$ is much simpler if the graph contains no pentagonal faces. In this case, the bound in Theorem 1.1 can be improved to $\tau_{\text{odd}}(G) \leq \sqrt{tn}/3$, where $t$ is the number of triangular faces. In particular, $\tau_{\text{odd}}(G) \leq \sqrt{4n}/3$, with equality if and only if all faces have size 3 and 6, $n = 12k^2$ for some $k \in \mathbb{N}$, and $\text{Aut}(G) \cong T_d$. Corollaries 6.1 and 6.2 can be strengthened in the same way.

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Figure 7.1. The graphs $C$ and $R$ used to construct an infinite family of 3-connected cubic plane graphs with all faces of size 5 and 7 such that $\nu_{\text{odd}} > \frac{1}{8}n$. 

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