Noiseless scattering states in a chaotic cavity

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Shot noise in a chaotic cavity (Lyapunov exponent \( \lambda \), level spacing \( \delta \), linear dimension \( L \)), coupled by two \( N \)-mode point contacts to electron reservoirs, is studied as a measure of the crossover from stochastic quantum transport to deterministic classical transport. The transition proceeds through the formation of fully transmitted or reflected scattering states, which we construct explicitly. The fully transmitted states contribute to the mean current \( \bar{I} \), but not to the shot-noise power \( S \). We find that these noiseless transmission channels do not exist for \( N \lesssim \sqrt{k_F L} \), where we expect the random-matrix result \( S/2e\bar{I} = 1/4 \). For \( N \gtrsim \sqrt{k_F L} \) we predict a suppression of the noise \( \propto (k_F L/N^2)^{N\delta/\hbar\pi} \).

This nonlinear contact dependence of the noise could help to distinguish ballistic chaotic scattering from random impurity scattering in quantum transport.

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Shot noise can distinguish deterministic scattering, characteristic for particles, from stochastic scattering, characteristic for waves. Particle dynamics is deterministic: A given initial position and momentum fixes the entire trajectory. In particular, it fixes whether the particle will be transmitted or reflected, so the scattering is noiseless. Wave dynamics is stochastic: The quantum uncertainty in position and momentum introduces a probabilistic element into the dynamics, so it is noisy.

The suppression of shot noise in a conductor with deterministic scattering was predicted many years ago from this qualitative argument [4]. A better understanding, and a quantitative description, of how shot noise measures the transition from particle to wave dynamics in a chaotic quantum dot was put forward by Agam, Aleiner, and Larkin [2], and developed further in Ref. [3]. The key concept is the Ehrenfest time \( \tau_E \), which is the characteristic time scale of quantum chaos [4]. The noise power \( S \propto \exp(-\tau_E/\tau_D) \) was predicted to vanish exponentially with the ratio of \( \tau_E \) and the mean dwell time \( \tau_D = \pi\hbar/N\delta \) in the quantum dot (with \( \delta \) the level spacing and \( N \) the number of modes in each of the two point contacts through which the current is passed). A recent measurement of the \( N \)-dependence of \( S \) is consistent with this prediction for \( \tau_E < \tau_D \), although an alternative explanation in terms of short-range impurity scattering describes the data equally well [5].

The theory of Ref. [2] introduces the stochastic element by means of long-range impurity scattering and adjusts the scattering rate so as to mimic the effect of a finite Ehrenfest time. Here we take the alternative approach of explicitly constructing noiseless channels in a chaotic quantum dot. These are scattering states which are either fully transmitted or fully reflected in the semiclassical limit. They are not described by random matrix theory [3]. By determining what fraction of the available channels is noiseless, we can deduce a precise upper bound for the shot-noise power. A random matrix conjecture for the remaining noisy channels gives an explicit form of \( S(N) \). We find that the onset of the classical suppression of the noise is described not only by the Ehrenfest time, but by the difference of \( \tau_E \) and the ergodic time \( \tau_0 \), which we introduce and calculate in the paper. The resulting nonlinear dependence of \( \ln S \) on \( N \) may help to distinguish between the competing explanations of the experimental data [5].

We illustrate the construction of noiseless scattering states for the two-dimensional billiard with smooth confining potential \( U(x,y) \) shown in Fig. 1. The outer equipotential defines the area in the \( x-y \) plane which is classically accessible at the Fermi energy \( E_F = p_F^2/2m \) (with \( p_F = \hbar k_F \) the Fermi momentum). The motion in the closed billiard is chaotic with Lyapunov exponent \( \lambda \). We assume the billiard to be connected at \( x = 0 \) and \( x = L \) by two similar point contacts to leads of width \( W \).
The accessible values of the two-dimensional section of phase space shown in Fig. 2. The ratio \( \frac{p}{W/L} \) cross-section of the billiard. While a hard-wall potential typically has a typical transmission (or reflection) band \( \frac{p}{W/L} \) less than \( \frac{F}{L} \), with a smooth potential, the number of channels \( N \approx pF \) typically has \( \frac{p}{W/L} \) depends on details of the potential near the point contact. If \( \frac{p}{W/L} \ll 1 \) in general, the ratio \( \frac{p}{W/L} \) typically \( \frac{p}{W/L} \) collimation. In Ref. [7] the symmetric case \( \frac{p}{W/L} \) as \( \frac{1}{r} \). We will see below, the area of the band decreases with \( t_j \) as

\[
A_j \approx A_0 \exp(-\lambda t_j) \quad \text{if} \quad t_j \gg 1/\lambda, t_W. \tag{1}
\]

The fluctuations of \( t \) around the average are of the order of the time \( t_W \approx mL/pF \) to cross the point contact, which is typically \( \ll t_j \). As we will see below, the area of the band decreases with \( t_j \) as

\[
A_j \approx A_0 \exp(-\lambda t_j) \quad \text{if} \quad t_j \gg 1/\lambda, t_W. \tag{1}
\]

The requirement that \( \psi \) is single-valued as one winds around the contour imposes a quantization condition on the enclosed area,

\[
\oint_C p_y \, dy = (n + 1/2)\hbar. \tag{4}
\]

The increment \( 1/2 \) accounts for the phase shift acquired at the two turning points on the contour. The quantum
number \( n = 0, 1, 2, \ldots \) is the channel index. The largest value of \( n \) occurs for a contour enclosing an area \( A_j \). The number of transmission channels \( N_j \) within band \( j \) is therefore given by \( A_j / h \), with an accuracy of order unity. In view of Eq. (11) we have

\[
N_j \approx (A_0 / h) \exp(-At_j), \quad \text{for} \quad t_j < \tau_E; \quad (5a)
\]
\[
N_j = 0, \quad \text{for} \quad t_j > \tau_E. \quad (5b)
\]

The time
\[
\tau_E = \lambda^{-1} \ln(A_0 / h) = \lambda^{-1} \ln(Nr_{\min} / r_{\max}) \quad (6)
\]
above which there are no fully transmitted channels is the Ehrenfest time of this problem.

By decomposing one of these \( N_j \) scattering states into a given basis of transverse modes in the lead one constructs an eigenvector of the transmission matrix product \( tt^\dagger \). The corresponding eigenvalue \( \tau_{j,n} \) equals unity with exponential accuracy in the semiclassical limit \( n \gg 1 \). Because of the degeneracy of this eigenvalue any linear combination of eigenvectors is again an eigenvector. This manifests itself in our construction as an arbitrariness in the choice of \( C \).

We observe in Fig. 1 that the spatial density profile \( \rho(x, y) \) of a fully transmitted scattering state is highly non-uniform. The flux tube is broad (width of order \( W \)) at the two openings, but is squeezed down to very small width inside the billiard. A similar effect was noted \(^7\) in the excited states of an Andreev billiard (a cavity connected to a superconductor). Following the same argument we estimate the minimal width of the flux tube as \( W_{\min} \approx L \sqrt{N_j / k_F L} \).

The total number
\[
N_0 = \sum_j N_j = N \int_0^{\tau_E} P(t) \, dt \quad (7)
\]
of fully transmitted and reflected channels is determined by the dwell-time distribution \( P(t) \). Fig. 3 shows this distribution in our billiard. One sees three different time scales. The narrow peaks represent individual transmission (reflection) bands. They consist of an abrupt jump followed by an exponential decay with time constant \( t_W \). These exponential tails correspond to the borders of the bands, where the trajectory bounces many times between the sides of the point contact. If we smooth \( P(t) \) over such short time intervals, an exponential decay with time constant \( t_D = \pi h / N \delta \) is obtained (inset). The decay starts at the so called “ergodic time” \( \tau_0 \). There are no trajectories leaving the cavity for \( t < \tau_0 \). So the smoothed dwell-time distribution has the form

\[
P(t) = \tau_D^{-1} \exp((\tau_0 - t) / \tau_D) \theta(t - \tau_0), \quad (8)
\]
with \( \theta(t) \) the unit step function.

In order to find \( \tau_0 \) we consider Fig. 4, where the section of phase space along a cut through the middle of the billiard is shown (line \( b \) in Fig. 1). It is convenient to measure the momentum and coordinate along \( b \) in units of \( p_F \) and \( L \). The injected beam crosses the section for the first time over an area \( O_{\text{initial}} \) of size \( r_{\max} \times r_{\min} = hN / p_F L \). (Fig. 4 has \( r_{\min} \approx r_{\max} \), but the estimates hold for any \( r_{\min} < r_{\max} < 1 \).) Further crossings consist of increasingly more elongated areas. The fifth crossing is shown in Fig. 4. The flux tube intersects line \( b \) in a few disjoint areas \( O_j \), of width \( r_{\min} e^{-\lambda t} \) and total length \( r_{\max} e^{\lambda t} \). (Due to conservation of the integral \( \oint p \cdot d\mathbf{r} \) enclosing the flux tube, the total area \( \sum_j O_j \) decreases only when particles leave the billiard.) The typical separation of adjacent areas is \( (r_{\max} e^{\lambda t})^{-1} \). To leave the billiard (through the right contact) without further crossing of \( b \) a particle should pass through an area \( O_{\text{final}} \approx r_{\max} \times r_{\min} \). This is highly improbable \(^8\) until the separation of the areas \( O_j \) becomes of order \( r_{\max} \), leading to the ergodic time

\[
\tau_0 = \lambda^{-1} \ln r_{\max}^{-2}. \quad (9)
\]

The ergodic time varies from \( \tau_0 \approx \lambda^{-1} r_{\max}^{-1} \) to \( \tau_0 = \lambda^{-1} \ln(k_F L / N) \) for \( r_{\min} \approx r_{\max} \). The overlap of the areas \( O_j \) and \( O_{\text{final}} \) is the mapping of the transmission band onto the surface of section \( b \). It has an area \( p_F L r_{\min} e^{-\lambda t} = A(r_{\min} / r_{\max}) e^{-\lambda t} \), leading to Eq. (10).

Substituting Eq. (8) into Eq. (7) we arrive at the number \( N_0 \) of fully transmitted and reflected channels,

\[
N_0 = N \theta(\tau_E - \tau_0) \left[ 1 - e^{(\tau_0 - \tau_E) / \tau_0} \right], \quad (10)
\]
\[
\tau_E - \tau_0 = \lambda^{-1} \ln(N^2 / k_F L). \quad (11)
\]

There are no fully transmitted or reflected channels if
\( \tau_E < \tau_0 \), hence if \( N < \sqrt{k_F L} \). Notice that the dependence of \( \tau_E \) and \( \tau_0 \) separately on the degree of collimation drops out of the difference \( \tau_E - \tau_0 \). The number of noiseless channels is therefore insensitive to details of the confining potential. An Ehrenfest time \( \propto \ln(N^2/k_F L) \) has appeared before in connection with the Andreev bilayer [10], but the role of collimation (and the associated finite ergodic time) was not considered there.

Eqs. [1] and [11] imply that the majority of noiseless channels group in bands having \( N_1 \gg 1 \), which justifies the semiclassical approximation. The total number of these noiseless bands is \( (N-N_0)/\lambda\tau_D \), which is much less than both \( N-N_0 \) and \( N_0 \). Because of this inequality the relatively short trajectories contributing to the noiseless channels are well separated in phase space from other, longer trajectories (cf. Fig. 2).

The shot-noise power \( S \) is related to the transmission eigenvalues by

\[
S = 2eIg^{-1} \sum_{k=1}^{N} T_k(1-T_k),
\]

with \( I \) the time-averaged current and \( g = \sum_k T_k \) the dimensionless conductance. The \( N_0 \) fully transmitted or reflected channels have \( T_k = 1 \) or 0, hence they do not contribute to the noise. The remaining \( N-N_0 \) channels contribute at most \( 1/4 \) per channel to \( Sg/2eI \). Using that \( g = N/2 \) for large \( N \), we arrive at an upper bound for the noise power \( S < eI (1-N_0/N) \).

For a more quantitative description of the noise power we need to know the distribution \( P(T) \) of the transmission eigenvalues for the \( N-N_0 \) noisy channels, which can not be described semiclassically. We expect the distribution to have the same bimodal form \( P(T) = \pi^{-1} T^{-1/2}(1-T)^{-1/2} \) as in the case \( N_0 = 0 \) [12]. This expectation is motivated by the earlier observation that the \( N_0 \) noiseless channels are well separated in phase space from the \( N-N_0 \) noisy ones. Using this form of \( P(T) \) we find that the contribution to \( Sg/2eI \) per noisy channel equals \( \int_0^1 T(1-T)P(T)dT = 1/8 \), half the maximum value. The Fano factor \( F = S/2eI \) is thus estimated as

\[
F = \frac{1}{4}, \quad \text{for } N \lesssim \sqrt{k_F L}, \quad (13a)
\]

\[
F = \frac{1}{4}(k_F L/N^2)^{\delta/\pi h\lambda}, \quad \text{for } N \gtrsim \sqrt{k_F L}. \quad (13b)
\]

This result should be compared with that of Ref. [2]: \( F' = (1-q/T_D) = 1/4(1-\text{constant} \times N) \), where \( q \) is some \( N \)-independent time. Eq. [13] predicts a more complex \( N \)-dependence, a plateau followed by a decrease as \( LN \propto N \ln(N^2/k_F L) \), which could be observable if the experiment extends over a larger range of \( N \).

We mention two other experimentally observable features of the theory presented here. The reduction of the Fano factor described by Eq. [13] is the cumulative effect of many noiseless bands. The appearance of new bands with increasing \( N \) introduces a fine structure in \( F(N) \), consisting of a series of cusps with a square-root singularity near the cusp. The second feature is the highly nonuniform spatial extension of open channels, evident in Fig. 1, which could be observed with the STM technique of Ref. [12]. From a more general perspective the noiseless channels constructed in this paper show that the random matrix approach may be used in ballistic systems only for sufficiently small openings: \( N \lesssim \sqrt{k_F L} \) is required. For larger \( N \) the scattering becomes deterministic, rather than stochastic, and random matrix theory starts to break down.

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