Analysis and modeling of Fano resonances in coupled cantilevers

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Abstract. The coupled oscillators model has been used for decades to interpret the Fano interference in a variety of systems: optical, plasmonic, and microwave. Hence, Fano resonance can be modeled with a weak or tightly coupled mechanical oscillators system, which provide insight into the interaction dynamics of a radioactive continuum of propagation modes and a discrete state. Therefore, the coupled cantilevers model was implemented in a FEM routine, in order to study and discuss aspects of the Fano resonance. The study of the Fano resonance in coupled cantilevers shows that this model may be applied in the field of micromechanical sensors.

1. Introduction
The phenomenon of resonance is widely studied in quantum systems. In particular, the interference of Fano in semiconductor heterostructures of continuous energy levels with the discrete energy levels of quantum dots has already been studied. Analogous phenomena of interaction between oscillators are found in classical systems. Thus, it seems of interest to investigate analogs of these phenomena in a much simpler framework of MEMS theory for elastic cantilevers, such as an Euler-Bernoulli system describing two beams of finite length immersed in a fluid interacting with each other.

2. Euler-Bernoulli beam model
The transversal oscillations of a beam are described by the Euler-Bernoulli beam model given by the following equation

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0,$$

(1)

where $\rho$ is the density of the beam material, $A$ is the cross-section, $E$ is Young’s module of the beam material, and $I$ is the second moment of cross-section of the beam about the neutral axis. The solution of the equation (1) has the form

$$y(x,t) = y(t)(A_1 e^{\beta x} + A_2 e^{-\beta x} + A_3 e^{i\beta x} + A_4 e^{-i\beta x}),$$

(2)

where

$$\beta = \left( \frac{\omega^2 \rho A}{EI} \right)^{\frac{1}{4}},$$

(3)
2.1. Uniform cantilever beam

When the beam is fixed at one end and free at the other, the following boundary conditions apply

\[ y|_{x=0} = y'|_{x=0} = 0 \]  
\[ y''|_{x=l} = y'''|_{x=l} = 0 \]  

The cantilever has vibration modes with eigenfrequencies and wave eigenvectors, given by

\[ \omega_n \approx \left( n - \frac{1}{2} \right)^2 \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho A}} \]  
\[ \beta_n = \left( \omega_n \sqrt{\frac{\rho A}{EI}} \right)^{\frac{1}{2}} \approx \left( n - \frac{1}{2} \right) \frac{\pi}{l}, \quad n = 1, 2, 3... \]  

When the cantilever is immersed in a fluid, external friction forces appear on the beam. Consequently, a term that is proportional to velocity is introduced into the Euler-Bernoulli equation. Then, the equation of the cantilever motion is

\[ \rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} + \alpha_E \frac{\partial y}{\partial t} = 0, \]  

where \( \alpha_E \) is the friction coefficient due to fluid action on the cantilever. The equation of cantilever motion can be rewritten as follows

\[ \ddot{y} + \gamma \dot{y} + \omega_n^2 y = 0, \]  

where \( \gamma = \frac{\alpha_E}{\rho A} \) is the damping coefficient.

3. Model of two coupled cantilevers

There are two cantilevers in a fluid, both of the same material (\( \rho \) and \( E \)) and the same cross-section (\( A \) and \( I \)), but with lengths \( l_1 \) and \( l_2 \) slightly different (\( l_1 \neq l_2 \)). The first cantilever is forced with an external harmonic force with an angular frequency \( \omega \) at the free end, where \( x = l_1 \). The following is the set of differential equations representing the motion equations of two coupled cantilevers.
\begin{align*}
\ddot{y}_1 + \gamma_1 \dot{y}_1 + \omega_{11}^2 y_1 + k_{12} y_2 &= q(x,t) \\
\ddot{y}_2 + \gamma_2 \dot{y}_2 + \omega_{12}^2 y_2 + k_{12} y_1 &= 0,
\end{align*}
\tag{10}
\tag{11}

with
\[ q(x,t) = q(x) \cos(\omega t) \tag{12} \]

The magnitude \( k_{12} \) describes the coupling between beams, due to the interaction of the beams with the fluid and the distance \( d \) that separates them.

The solution of forced cantilever motion is as follows
\[ y_1 = c_1 e^{i\omega t} \tag{13} \]

The solution of coupled cantilever motion is
\[ y_2 = c_2 e^{i\omega t} \tag{14} \]

By entering the equations (12), (13) and (14) in the equations (10) and (11), the following values of the complex amplitudes are obtained
\[ c_1 = \frac{\omega_{12}^2 - \omega^2 + i\gamma_2 \omega}{(\omega_{11}^2 - \omega^2 + i\gamma_1 \omega)(\omega_{12}^2 - \omega^2 + i\gamma_2 \omega) - k_{12}^2} q(x) \tag{15} \]
\[ c_2 = \frac{k_{12}}{(\omega_{11}^2 - \omega^2 + i\gamma_1 \omega)(\omega_{12}^2 - \omega^2 + i\gamma_2 \omega) - k_{12}^2} q(x) \tag{16} \]

In this case, \( \omega_{11} \) and \( \omega_{12} \) represent the eigenfrequencies of forced and coupled cantilever first vibration mode, respectively. Likewise, the magnitudes \( \gamma_1 \) and \( \gamma_2 \) are the damping coefficients of each one.

4. Implementation and results

To know the dynamics of the cantilevers immersed in a fluid the model was implemented in a FEM program. One of the cantilevers is subjected to the action of an external force, meanwhile the second cantilever interacts with the first through the fluid depending on the distance \( d \) that separates the two cantilevers. In this case, the two coupled cantilevers were submerged in water and the following are the values of the parameters used in this problem.

| Material      | E [GPa] | \( \rho \) [\( K_{m} \)] | A [\( \mu m^{2} \)] | Length [\( \mu m \)] | Frequency [Hz] |
|---------------|---------|-----------------|-----------------|----------------|----------------|
| Silicon - Si  | 131     | 2329            | 900             | \( l_1 = 600 \) | \( f_{11} = 43380 \) |
|               |         |                 |                 | \( l_2 = 590 \) | \( f_{12} = 44760 \) |

\( l_1 \) - forced cantilever
\( l_2 \) - coupled cantilever

Figures (2) and (3) represent the absolute value of the amplitudes and phase of the two cantilevers when the distance separating them is 182 \( \mu m \). When the numerator of the equation (15), \( \omega_{12}^2 - \omega^2 + i\gamma_2 \omega \) tends to zero and \( \gamma_2 \ll \omega_{12} \), the amplitude of forced beam approaches to zero \( |c_1| \approx 0 \), as shown in the figure (4). This happens because the movement of forced oscillator
is drastically damped by second oscillator. Figures (5) and (6) represent the absolute value of the amplitudes and phase of the two cantilevers when the distance between cantilevers is 82 \( \mu m \).

(a) Forced cantilever normalized amplitude \( |c_1| \).  
(b) Coupled cantilever normalized amplitude \( |c_2| \).

Figure 2: Cantilevers normalized amplitudes

(a) Forced cantilever phase \( \phi_1 \).  
(b) Coupled cantilever phase \( \phi_2 \).

Figure 3: Cantilevers phases

According to the equations (15) and (16), \( c_1 = |c_1| e^{i\phi_1} \) and \( c_2 = |c_1| e^{i\phi_2} \). Thus

\[
\frac{c_1}{c_2} = \frac{\omega_1^2 - \omega_2^2 + i\gamma_2 \omega}{k_{12}} = |a| e^{i\theta},
\]

(17)

where

\[
\theta = \arctan \left( \frac{\gamma_2 \omega}{\omega_1^2 - \omega_2^2} \right)
\]

(18)

The values of \( c_1 \) y \( c_2 \) satisfy the following conditions

\[
\frac{|c_1|}{|c_2|} = |a|\]

(19)

\[
|\phi_2 - \phi_1| = \pi - \theta
\]

(20)
As can be seen in the figures (3) and (6), the phases of the cantilevers satisfy the condition given by the equation (20).

![Figure 4: Forced cantilever amplitude when $f \approx f_{12}$.](image)

![Figure 5: Cantilevers normalized amplitudes](image)

As the distance between the cantilevers decreases, the coupling coefficient increases and the $|c_{1,2}|_{max}$ move away from resonant frequencies $\omega_{11}$ and $\omega_{12}$. Likewise, if $\gamma_2 \approx 0$ and $k_{12} \approx 0$, the minimum amplitude $|c_1|_{min}$, is obtained when $\omega \approx \omega_{12}$. With the increase of $k_{12}$ the position of the minimum amplitude $|c_1|_{min}$ shifts from $\omega_{12}$. 
5. Conclusions
A model of two cantilevers coupled through a fluid was obtained. The interaction between the cantilevers was observed analogously to the Fano resonances. As in other studies, a phenomenon similar to the Fano resonances of quantum systems occurs in this classical problem.

6. Acknowledgments
This project is partially founded by Engineering Faculty of the Universidad Distrital Francisco José de Caldas de Bogotá, inside the Proyecto Curricular de Ingeniería Electrónica.

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