Two-Point Vorticity Statistics in the Inverse Turbulent Cascade

R. Friedrich,1 M. Voßkuhle,2 O. Kamps,3 and M. Wilczek1

1Institute for Theoretical Physics, University of Münster, Wilhelm-Klemm-Str. 9, D-48149 Münster, Germany
2Laboratoire de Physique, ENS de Lyon, 46 allée d’Italie F-69007 Lyon, France
3Center for Nonlinear Science, University of Münster, Corrensstr. 2, D-48149 Münster, Germany

(Dated: December 16, 2010)

We present a statistical analysis of the two-point vorticity probability density of the vorticity field generated in the inverse cascade of stationary two-dimensional turbulence.

PACS numbers:

I. INTRODUCTION

Fully developed turbulent flows are systems far from equilibrium giving rise to a transport of energy or enstrophy across scales. Up to now, no generally accepted theoretical description of this transport process has emerged although the phenomenological theory based on the works of Kolmogorov, Onsager, Heisenberg and their successors is quite successful in describing the gross features of turbulent fields (we refer to the monographs [1], [2]).

Two-dimensional turbulent flows play a prominent role due to the existence of a direct as well as an inverse cascade, as has been emphasized in the seminal work of Kraichnan [3]. The coexistence of both cascades has been convincingly demonstrated by Boffetta and Musacchio [4]. Recently, a detailed analysis of the contour lines of zero vorticity has revealed interesting scaling behaviour pointing to the existence of nontrivial multi-point statistics of vorticity in the inverse cascade [5]. The inverse cascade is a central issue in classical nonequilibrium physics [6].

The present contribution is concerned with the two-point statistics of vorticity in the inverse cascade. Starting from the hierarchy of evolution equations for multi-point probability distributions of vorticity, we shall perform a detailed examination of the partial differential equation defining the probability distribution of the vorticity increment relying on input from direct numerical simulation (DNS).

II. RESULTS FROM DNS

The spatio-temporal dynamics of the turbulent field is governed by the two-dimensional vorticity equation

\[ \frac{\partial}{\partial t} \omega(x,t) + u(x,t) \cdot \nabla_x \omega(x,t) = L(-\Delta)\omega(x,t) + F(x,t) \]

(1)

The velocity field \( u(x,t) \) is determined from vorticity \( \omega(x,t) \) via Biot-Savart’s law,

\[ u(x,t) = \int d^2x' e_z \times \frac{x - x'}{2\pi|x - x'|^2} \omega(x',t) \]

(2)

Viscosity is taken into account by the operator \( L(-\Delta) \), where we allow for the presence of hyperviscosity as well as friction acting on the large scales. In order to achieve a constant energy flux in the inertial range we have used a hyperviscosity of the form \( \nu_H(-\Delta)^8 \) and a large scale friction \( -\nu_V(-\Delta)^{-1} \) to extract energy transported to the large scales by the inverse cascade. Energy is injected into the system by the forcing term \( F(x,t) \). Here we employ a forcing that acts only on a narrow wavenumber band around \( k_0 \) in Fourier space. Within this band the amplitudes \( |\omega_k| \) of the Fourier modes

\[ \omega_k(t) = |\omega_k| e^{i\phi_k(t)} \]

(3)

are kept constant by resetting them every time step to the constant \( \omega_0 \) while the phases can evolve freely according to the vorticity equation [8]. In Fourier-space, the evolution equations can be written for amplitudes and phases.
as
\[
\frac{d}{dt} |\omega_k| + N_k = L(-k^2)|\omega_k| + \frac{1}{T}[F(k) - |\omega_k|] \Theta(k, k_0)
\]
\[
|\omega_k| \frac{d}{dt} \Phi_k + M_k = 0
\]
(4)

Here, \(N_k, M_k\) are the real and imaginary parts of the Fourier transform of the convective term \(u \cdot \nabla \omega\). \(F(k)\) is a real function centered around \(k_0\) and \(T \to 0\) is a time relaxation constant. The function \(\Theta(k, k_0)\) is different from zero only in the forcing interval. From this representation it becomes obvious that the forcing can be considered to be deterministic.

In most simulations (see e.g. \([7, 9]\)) the forcing is located at a wavenumber \(k_0\) that is close to the highest wavenumber resolved. In combination with hyperviscosity this allows for a wide inertial range and suppression of the direct enstrophy cascade. However, this leads to a poor numerical resolution of the forcing scale. To circumvent this flaw we inject the energy at an intermediate length scale and used a hyperviscosity that, for numerical reasons saturates at a maximal value. Comparison with other numerical simulations showed that this procedure does not affect the results while yielding a better resolution at the forcing scale.

The vorticity equation is numerically solved by means of a standard pseudospectral method with 2/3-dealiasing in a periodic box with side length \(2\pi\) and a resolution of 2048\(^2\) grid points. Time stepping is performed by a memory saving Runge-Kutta scheme of third order \([10]\). The hyperviscous term is treated by an integrating factor.

Data-analysis has been performed over more than 200 snapshots of the vorticity field taken from a simulation in a statistically stationary state. The snapshots are equally separated by approximately 1.4 large-eddy turnover times.

Figure 1 exhibits the energy spectrum in Fourier space in the inverse cascade, as obtained from our numerical solution. The spectral energy flux is constant and demonstrates the existence of the inverse cascade. The obtained Kolmogorov constant \(K_o\), defined by \(E(k) = K_o c^{5/3} k^{-5/3}\) is consistent with the results reported in the literature, see e.g. \([7, 9]\). The probability distributions of the vorticity increments, defined according to \(\Omega\), are exhibited in fig. 2 as a function of scale \(r = |\mathbf{r}|\). Apparentely, the vorticity increment is nearly Gaussian distributed. Furthermore, the second moment of the vorticity increment exhibits oscillations as a function of the distance \(r\), which are clearly a result of the small but visible oscillations in the shape of the pdfs. It may well be that the wings of the pdfs are super-Gaussian. In the following we shall be concerned with the behaviour represented by the center of the pdf, where a Gaussian representation of the statistics is possible. The behaviour of the tails of the pdf may be accessible by instanton calculations, as has been discussed in, e.g., \([11]\).

An important statistical relationship is the analog of the von Kármán-Howarth equation formulated for vorticity. It explicitly reads
\[
\begin{align*}
\nabla_x \cdot (u(x_1, t) \omega(x_1, t) \omega(x_2, t)) + \\
\nabla_x \cdot (u(x_2, t) \omega(x_1, t) \omega(x_2, t)) = \\
\langle \omega(x_1, t) L(-\Delta x_2) \omega(x_2, t) \rangle \\
+ \langle \omega(x_2, t) L(-\Delta x_1) \omega(x_1, t) \rangle \\
+ \langle \omega(x_1, t) F(x_2, t) \rangle + \langle \omega(x_2, t) F(x_1, t) \rangle
\end{align*}
\]
(5)

and can be derived in a straightforward manner from the evolution equation for vorticity. The transport terms of the left hand side of the equation are responsible for the transport process in the cascade. They essentially depend on the three-point vorticity correlations
\[
\langle \omega(x', t) \omega(x_1, t) \omega(x_2, t) \rangle
\]
(6)
since the velocity field is determined from Biot-Savart’s law.

### III. STATISTICAL TREATMENT

We shall base our statistical description on the N-point vorticity pdf \(f^N(\{\omega_i, x_i\}, t)\), which we define as an ensemble average \(\langle f^N(\{\omega_i, x_i\}, t) \rangle\) over the fine-grained probability distribution \(f\)
\[
\hat{f}^N(\{\omega_i, x_i\}, t) = \Pi_{i=1}^N \delta(\omega_i - \omega(x_i, t))
\]
(7)
Here, $\omega(x_i, t)$ denotes the solution of the vorticity equation at spatial point $x_i$ and time $t$.

It is straightforward to derive the following evolution equation for the fine-grained pdf (see e.g. [12]). A short discussion of the procedure can be found in the appendix.

$$
\frac{\partial}{\partial t} \mathcal{F}(\omega_i, x_i, t) = -\frac{\partial}{\partial \omega_i} \left[ \mathcal{F}(\omega_i, x_i, t) \right] + \left( \mathcal{F}(\omega_i, x_i, t) \right) + \mathcal{F}(\omega_i, x_i, t)
$$

In order to obtain the evolution equation for the coarse-grained pdf $\mathcal{F}^N(\{\omega_i, x_i\}, t)$, one has to perform an ensemble average of this equation. This averaging procedure immediately confronts us with the closure problem of turbulence, since the following unclosed expectations arise

$$
\langle u(x, t) \mathcal{F}(\omega_i, x_i, t) \rangle = \langle u, x \rangle \mathcal{F}(\omega_i, x_i, t)
$$

$$
\mathcal{F}(\omega_i, x_i, t) = \langle \mathcal{F}(\omega_i, x_i, t) \rangle
$$

It is convenient to express these unclosed expectations, as shown above, in terms of conditional averages of velocity, $\langle u, x \rangle \mathcal{F}(\omega_i, x_i, t)$, dissipation $\mathcal{F}(\omega_i, x_i, t)$ and forcing $\mathcal{F}(\omega_i, x_i, t)$. The introduction of these quantities is justified by the fact that conditional averages can be estimated from DNS (see e.g. [13, 16]), and for a recent application to turbulent Rayleigh-Bénard convection [17].

For the case of a white noise forcing, the term related to the conditional forcing may contain second-order derivatives with respect to $\omega_i$ and then takes the form of a diffusion operator, $\sum_{ij} \frac{\partial^2}{\partial x_i^2} \mathcal{F}(x_i - x_j) \frac{\partial}{\partial x_i}$. An alternative way is to write down the whole chain of evolution equations for higher N-point functions, a procedure which is well documented in literature [11, 18, 29]. However, up to now conclusive truncations of the hierarchy are lacking. Additionally, we want to mention that a similar approach for the study of the velocity increment statistics of the inverse cascade has been undertaken by Boffetta et al. [21] based on theoretical approaches by V. Yakhov [22, 23]. Here, numerical input for the conditional pressure term is used to close the equation for the probability distribution of the velocity increment.

Using the conditional expectations (9) we arrive at the following partial differential equation determining the pdf

$$
\frac{\partial}{\partial t} \mathcal{F}(\omega_i, x_i, t) = -\frac{\partial}{\partial \omega_i} \left[ \mathcal{F}(\omega_i, x_i, t) \right] + \mathcal{F}(\omega_i, x_i, t)
$$

These partial differential equations can be solved by the method of characteristics. The corresponding characteristic equations read

$$
\dot{\omega}_i = \langle L(-\Delta) \omega, x_i | \{\omega_i, x_i\}, t \rangle + \mathcal{F}(\omega_i, x_i, t)
$$

Along these characteristic curves the probability distribution changes according to

$$
\frac{d}{dt} \mathcal{F}(\omega_i, x_i, t) = -\frac{\partial}{\partial \omega_i} \left[ \mathcal{F}(\omega_i, x_i, t) \right] + \mathcal{F}(\omega_i, x_i, t)
$$

For the case of a stochastic forcing, the characteristic equations are stochastic differential equations. In the following we shall also include a discussion of this case, where the characteristic equations take the form

$$
\dot{x}_i = \langle u, x_i | \{\omega_i, x_i\}, t \rangle
$$

$$
\dot{\omega}_i = \langle L(-\Delta) \omega, x_i | \{\omega_i, x_i\}, t \rangle + F_i(t)
$$

Here, $F_i(t)$ are stochastic white noise forces with correlation function

$$
\langle F_i(t) F_j(t') \rangle = 2Q(x_i - x_j) \delta(t - t')
$$

In fact, the expression for the conditional force will allow such an interpretation. In the following we shall discuss an interpretation of these stochastic differential equations as a kind of stochastic point vortex model.

IV. LINEAR GAUSSIAN APPROXIMATION

Since the two-point statistics of vorticity is close to Gaussian, it is tempting to approximate the conditional averages [9] based on the assumption of Gaussian statistics. The conditional velocity field can be expressed via Biot-Savart’s law in terms of the conditional vorticity expectation $\langle \omega, x' | \{\omega_i, x_i\}, t \rangle$ according to

$$
\langle u, x | \{\omega_i, x_i\}, t \rangle = \int dx' \mathcal{F}(\omega, x') \frac{2\pi}{|x - x'|^2} \delta(t - t')
$$

In the case of Gaussian statistics all quantities depend on the vorticity-vorticity correlation function which we denote by $C(x - x')$. However, for Gaussian statistics
with circulation $\Gamma_i$ and distribution $\text{C}$.

The vorticity is color-coded, while the velocity is visualized as a field of a quasi-vortex in the sense of a Landau quasi-particle taking into account the surrounding turbulence by an effective velocity profile. The complete conditional velocity field is a linear superposition of velocity fields of such screened (or dressed) vortices. Alternatively, one could also say that Biot-Savart’s law is changed by the effective consideration of the turbulent background field. We mention that a change of Biot-Savart’s law has been used by Chevillard et al. [27] in the modeling of the statistics of velocity increments in three dimensional turbulence.

Let us explicitly determine the conditional velocity field conditioned on vorticity $\omega_1, \omega_2$ at positions $x_1, x_2$:

\[
\langle u, x | \{\omega_1, x_1\}, \{\omega_2, x_2\}\rangle = U(x - x_1) - \frac{C(0)\omega_1 - C(x_1 - x_2)\omega_2}{C(0)^2 - C(x_1 - x_2)^2} + U(x - x_2) \frac{C(0)\omega_2 - C(x_2 - x_1)\omega_1}{C(0)^2 - C(x_1 - x_2)^2}.
\]

Furthermore, we note that $U(0) = 0$, due to the fact that $C(0)$ is finite, c.f. eq. (19).

Furthermore, a dissipation field $\gamma(x - x_i)$ is defined according to

\[
\gamma(x - x_i) = -L(-\Delta)C(x - x_i).
\]

The characteristic equations then take the form

\[
x_i = \sum_{kl} U(x_i - x_k)C(x_k - x_l)^{-1}\omega_l
\]

\[
\dot{\omega}_i = -\sum_{kl} \gamma(x_i - x_k)C(x_k - x_l)^{-1}\omega_l + F(x_i, t)
\]

We have determined the conditional vorticity field from DNS (for the case of two points) and have compared it with the one obtained from the linear Gaussian approximation. Visually, there is no difference, as can be seen from Figs. (3), (4) and (5), (6), respectively. However,
as we shall see, the linear Gaussian approximation is not able to describe the turbulent cascade, at least if one restricts oneself to the treatment of two-point statistics. The reason is that the characteristic equation for the case \( N = 2 \) takes the form

\[
\begin{align*}
\dot{x}_1 &= U(x_1 - x_2)G(\omega_1, \omega_2, |x_1 - x_2|) \\
\dot{x}_2 &= U(x_2 - x_1)G(\omega_2, \omega_1, |x_1 - x_2|)
\end{align*}
\]

(23)

with

\[
G(\omega_1, \omega_2, |x_1 - x_2|) = \frac{C(0)\omega_2 - C(x_1 - x_2)\omega_1}{C(0)^2 - C(x_1 - x_2)^2}
\]

(24)

It describes two-vortex motions of vortices in a screened velocity field. This motion does not allow for a variation of the distance, i.e. there is no transport process in scale. This can be seen by invoking the analog of the van Kármán-Howard equation for vorticity. The transport terms read under consideration of the conditional velocity field

\[
\begin{align*}
\int d\omega_1 d\omega_2 \omega_1 \omega_2 [U(x_1 - x_2)G(\omega_2, \omega_1, |x_1 - x_2|) \cdot \nabla_{x_1} + U(x_2 - x_1)G(\omega_2, \omega_1, |x_1 - x_2|) \cdot \nabla_{x_2}]f(\omega_1, x_1; \omega_2, x_2)
\end{align*}
\]

(25)

However, since for homogeneous isotropic statistics the probability distribution depends on \(|x_1 - x_2|\), the transport term vanishes since

\[
U(x_1 - x_2) = e_z \times \frac{x_1 - x_2}{h(|x_1 - x_2|)}
\]

(26)

and the gradient of \( f(\omega_1, x_1; \omega_2, x_2) \) points in the direction of the vector \( x_1 - x_2 \) such that

\[
U(x_1 - x_2) \cdot \nabla_{x_i} f(\omega_1, x_1; \omega_2, x_2) = 0
\]

(27)

The characteristic equations (22) can be interpreted in a Lagrangian sense. Neglecting the temporal variation of the vorticities \( \omega_i \) the set of differential equations for the positions \( x_i \) can be viewed as a modified point vortex system, where the velocity field of an individual vortex is not the one of a point vortex, but is the velocity field of a dressed vortex. This perspective can be seen in close analogy to the notion of a Landau quasi-particle. This notion is widely used in theories of many particle systems, where a part of the many-particle interaction can be absorbed into effective properties of single particles. In the present case the reduction from a continuum description of vorticity to the description of the statistics of the field at finite spatial points quite naturally leads to an effective point vortex dynamics. However, since we are still in the realm of the linear Gaussian approximation, we are not able to obtain the inverse cascade on the basis of a two-point closure due to the vanishing transport terms discussed above. One might think of obtaining the inverse cascade starting from the three vortex problem in the linear Gaussian approximation. The three-vortex problem certainly brings in new scales, however, we conjecture that this is not the correct way to proceed.

Due to the temporal variation of the vorticities \( \omega_i \) the characteristic equations form an extension of the problem of point vortex motion. We refer the reader to the review [24]. On the other side, if the vorticities are assumed to undergo white noise processes

\[
\langle \omega_i(t)\omega_j(t') \rangle = 2Q_{ij} \delta(t - t')
\]

(28)

one obtains an extension of the Kraichnan model [24], which has been investigated with respect to passive scalar
advection (for a review we refer the reader to [22]). As is transparent from [22], in the adiabatic limit the vorticities are directly related to the white noise forces
\[ \omega_i = \Gamma_{ii} F(x_i, t) \] (29)
where \( \Gamma_{ii} \) is the inverse matrix to \( \sum_{k l} \gamma(x_i - x_k) C(x_k - x_l) \). The result is the Fokker-Planck equation for the positions \( x_i \) of the point vortices
\[ \frac{\partial}{\partial t} p(x_i, t) = \sum_{ij} \nabla_x D_{ij} [\{x_i\}] \nabla_x p(x_i, t) \] (30)
In general, the diffusion matrix \( D_{ij} [\{x_i\}] \) depends on all positions \( \{x_i\} \) of the vortices. In the Kraichnan model, the diffusion matrix \( D_{ij} \) is only a function of the difference vector \( x_i - x_j \). The explicit expression of \( D_{ij} [\{x_i\}] \) is
\[ D_{ij} [\{x_i\}] = U(x_i - x_j) C^{-1} \Gamma_{mn} U(x_j - x_k) C^{-1} \Gamma_{pq} Q_{pq} \] (31)
We sum over dummy indices.

V. VORTICITY INCREMENTS

For the following we shall restrict our considerations to the case of two spatial points, since then the estimation of the conditional averages from DNS becomes feasible. Due to homogeneity and isotropy, the two-point pdf only depends on the distance \( r, r = x_1 - x_2 \) and the evolution equation for the two-point functions takes the form
\[ \nabla_r \cdot v(r, \omega_1, \omega_2) f(\omega_1, \omega_2, r) = - \left[ \frac{\partial}{\partial \omega_1} \mu(r, \omega_1, \omega_2) + \frac{\partial}{\partial \omega_2} \mu(r, \omega_2, \omega_1) \right] f(\omega_1, \omega_2, r) \] (32)
Thereby, we have defined the conditional velocity increment
\[ v(r, \omega_1, \omega_2) = \langle u, x_1 | \{\omega_1, x_i\} \rangle - \langle u, x_2 | \{\omega_1, x_i\} \rangle \] (33)
and have taken into account that the quantity \( \mu = L + F \) only depends on the distance \( r \) between the two points. Furthermore, due to radial symmetry of the pdf \( f(\omega_1, \omega_2, r) \), only the radial component of the conditional velocity \( v_r(\omega_1, \omega_2, r) e_r \) contributes (\( e_r \) denotes the unit vector in radial direction). It is convenient to introduce the pdf \( h(\omega_1, \omega_2, r) = r f(\omega_1, \omega_2, r) \).
It is further possible to reduce the complexity of the problem by considering the statistics of the vorticity increment \( \Omega = \omega_1 - \omega_2 \). The corresponding equation defining the statistics of \( \Omega \), \( H(\Omega, r) \) can be obtained via the definition
\[ H(\Omega, r) = \int d\omega_1 d\omega_2 \delta(\Omega - (\omega_1 - \omega_2)) h(\omega_1, \omega_2, r) \] (34)
These equations are linear in the vorticity and therefore can be solved directly.

V.1. Conditional pdf

The conditional pdf is important for the analysis of the conditional statistics of the vorticity increment. The pdf is given by
\[ p(\omega_1, \omega_2, r) = \frac{1}{Z(r)} \exp \left( - \frac{1}{2} \left( \frac{\omega_1 - \mu(r, \omega_1)}{\sigma(\omega_1, r)} \right)^2 - \frac{1}{2} \left( \frac{\omega_2 - \mu(r, \omega_2)}{\sigma(\omega_2, r)} \right)^2 \right) \] (35)
where \( Z(r) \) is the normalization factor.

V.2. Conditional statistics

The conditional statistics of the vorticity increment \( \Omega = \omega_1 - \omega_2 \) can be obtained via the conditional pdf
\[ \langle \Omega \rangle(r) = \int d\omega_1 d\omega_2 \omega_1 p(\omega_1, \omega_2, r) - \int d\omega_1 d\omega_2 \omega_2 p(\omega_1, \omega_2, r) \] (36)
\[ \langle \Omega^2 \rangle(r) = \int d\omega_1 d\omega_2 \omega_1^2 p(\omega_1, \omega_2, r) - 2 \int d\omega_1 d\omega_2 \omega_1 \omega_2 p(\omega_1, \omega_2, r) + \int d\omega_1 d\omega_2 \omega_2^2 p(\omega_1, \omega_2, r) \] (37)
These equations are used to analyze the conditional statistics of the vorticity increment.

V.3. Conditional velocity field

The conditional velocity field \( v(r, \Omega) \) is important for the analysis of the conditional statistics of the velocity field. The pdf is given by
\[ p_v(\omega_1, \omega_2, r) = \frac{1}{Z_v(r)} \exp \left( - \frac{1}{2} \left( \frac{\omega_1 - \mu_v(r, \omega_1)}{\sigma_v(\omega_1, r)} \right)^2 - \frac{1}{2} \left( \frac{\omega_2 - \mu_v(r, \omega_2)}{\sigma_v(\omega_2, r)} \right)^2 \right) \] (38)
where \( Z_v(r) \) is the normalization factor.

V.4. Conditional statistics

The conditional statistics of the velocity field \( \omega = \omega_1 - \omega_2 \) can be obtained via the conditional pdf
\[ \langle \omega \rangle(r) = \int d\omega_1 d\omega_2 \omega_1 p_v(\omega_1, \omega_2, r) - \int d\omega_1 d\omega_2 \omega_2 p_v(\omega_1, \omega_2, r) \] (39)
\[ \langle \omega^2 \rangle(r) = \int d\omega_1 d\omega_2 \omega_1^2 p_v(\omega_1, \omega_2, r) - 2 \int d\omega_1 d\omega_2 \omega_1 \omega_2 p_v(\omega_1, \omega_2, r) + \int d\omega_1 d\omega_2 \omega_2^2 p_v(\omega_1, \omega_2, r) \] (40)
These equations are used to analyze the conditional statistics of the velocity field.

V.5. Conditional statistics

The conditional statistics of the vorticity increment \( \Omega = \omega_1 - \omega_2 \) and the velocity field \( \omega = \omega_1 - \omega_2 \) are important for the analysis of the conditional statistics of the flow. The conditional pdf is given by
\[ p_{\Omega \omega}(\omega_1, \omega_2, r) = \frac{1}{Z_{\Omega \omega}(r)} \exp \left( - \frac{1}{2} \left( \frac{\omega_1 - \mu_{\Omega \omega}(r, \omega_1)}{\sigma_{\Omega \omega}(\omega_1, r)} \right)^2 - \frac{1}{2} \left( \frac{\omega_2 - \mu_{\Omega \omega}(r, \omega_2)}{\sigma_{\Omega \omega}(\omega_2, r)} \right)^2 \right) \] (41)
where \( Z_{\Omega \omega}(r) \) is the normalization factor.

We have determined the conditional velocity field \( V_r(\Omega, r) \) as well as the conditional field \( \mu(\Omega, r) \) from DNS. For not too small values of \( r \) it is possible to perform the following quadratic fit in \( \Omega^2 \), see fig. (7):
\[ V_r(\Omega, r) = g(\Omega) [\Omega^2 - \langle \Omega(r, t)^2 \rangle] \] (42)
\[ \mu(\Omega, r) = \mu(r) \Omega \] (43)
This fit is consistent with the conditions (36) and can be viewed as a Taylor expansion in \( \Omega \). The quantity \( \mu(\Omega, r) \) can be represented by a linear fit in \( \Omega \).
It turns out that the function \( g(\Omega) \) is an oscillating function of \( r \) (see fig. 7) which can be well approximated in the inertial range by
\[ g(\Omega) = cr^{-\delta} \sin(kr - \Phi) \] (44)
The constant \( k = 2.1 \pi / \lambda \) is related to the width of the forcing function of the turbulence. The exponent \( \delta \) is of the order of \( \delta \approx 0.8 \). Amplitude and phase turn out to be \( c = 1.8 \times 10^{7} \sigma_{\infty} / \sigma_{\infty}^2, \Phi = 1.3 \pi \).

The function \( \mu(\Omega, r) \) consists of three contributions, namely forcing, dissipation, and friction. All contribute with a linear dependency in \( \Omega \) to \( \mu(\Omega, r) \). The various contributions are exhibited in fig. (8).
We also have performed a fit of the form

$$C(r) = ar^\alpha [c + \sin(\omega r + \Phi)]$$  \hspace{1cm} (39)$$

with the parameters $a = 0.0016$, $\omega = 104$, $c = 0.66$, $\Phi = 0.32$ and $\alpha = -4/3$ to the correlation function explicitly incorporating the expectation of scaling behaviour according to K41.

It is interesting to notice that we can find a solution to the pdf equation \[35\] with the ansatz \[37\] for values of the scale, for which the function $g(r)$ vanishes. In that case the characteristic equation degenerates to the differential equation

$$g'(r)[\Omega^2 - \langle\Omega^2(r,t)\rangle]H(\Omega, r) = -\frac{\partial}{\partial \Omega} \mu(r)\Omega H(\Omega, r)$$  \hspace{1cm} (40)$$

with the explicit Gaussian solution

$$H(\Omega, r) = \frac{e^{-\frac{r^2}{2\langle\Omega(r,t)^2\rangle}}}{\sqrt{2\pi\langle\Omega(r,t)^2\rangle}}$$  \hspace{1cm} (41)$$

The following relationship between conditional velocity field, conditional dissipation and forcing, and the correlation function $\langle\Omega(r,t)^2\rangle$ then has to hold:

$$\mu(r) = g'(r)\langle\Omega(r,t)^2\rangle$$  \hspace{1cm} (42)$$

A check of this equality is presented in fig. \[9\]. We want to point out that, strictly, the result is only valid for values of $r$, for which $g(r) = 0$, i.e. at the minima and maxima of the moment. In between these points, the pdf can be determined perturbatively.

FIG. 8: The three contributions to the function $\mu(r, \Omega)$, fitted by $\mu(r, \Omega) = \mu(r)\Omega$, due to forcing, viscosity, and friction.

FIG. 9: The function $\mu(r)$ for the contributions due to forcing, viscosity, and friction compared with the relation $\mu(r) = \langle\Omega(r)^2\rangle g'(r)$. Scale is measured in terms of the Taylor scale.

FIG. 10: Characteristic curves of the characteristic equation \[22\]. The vertical lines mark the scale with $g(r) = 0$. The curve is the correlation $\langle\Omega(r,t)^2\rangle$.

VI. LAGRANGIAN MODEL

It is plausible to look for a relationship of the characteristic equation \[11\] with the Lagrangian description of the relative motion of two-point particles with relative vorticity $\Omega$. The conditional velocity field thereby is a type of averaged relative velocity field, whose single realization has a deterministic as well as a stochastic part. In order to formulate a model for these single stochastic realizations, we separate the conditional velocity field into two parts, a deterministic and a fluctuating part:

$$V_r(r, \Omega) = G(r)[\Omega^2 - \langle\Omega(r)^2\rangle] - D(r) \frac{1}{H(r, \Omega)} \frac{\partial}{\partial r} H(r, \Omega)$$  \hspace{1cm} (43)$$
If we take the Gaussian solution valid for the scale separation defined by $g(r) = 0$, we are led to the relation
\[
g(r) = G(r) - D(r) \frac{1}{2(\Omega(r,t)^2)^2} \frac{\partial}{\partial r} \langle \Omega(r,t)^2 \rangle
\approx G(r) + \frac{D(r)}{2} \frac{1}{\gamma} g''(r) \tag{44}
\]
At the present stage, we are not able to fix the function $D(r)$. Therefore, we take $D(r)$ as a small positive quantity in the following. A plausible choice would be to take $D(r)$ according to Kraichnan’s model of passive scalar advection [29].

A similar ansatz is made for the quantity $\mu(r,\Omega)$, i.e. we assume a linear damping and white noise forcing and model the Lagrangian statistics by the Langevin equations for relative position $r$ and relative vorticity $\Omega$:
\[
\dot{r} = G(r)[\Omega^2 - \langle \Omega(r,t)^2 \rangle] + \eta
\]
\[
\dot{\Omega} = -\gamma(r)\Omega + F \tag{45}
\]
We intend to derive the decomposition [44] and the statistics of $F$ using well-established methods for the determination of drift- and diffusion terms for diffusion processes [25] directly from Lagrangian data.

Thereby, the two stochastic forces $\eta, F$ are white noise forces and are characterized by their correlations $\langle F(r,t)F(r,t') \rangle = Q(r)\delta(t-t'), \langle \eta(r,t)\eta(r,t') \rangle = C(r)\delta(t-t')$. A comparison yields the relationship
\[
\mu(r) = -\gamma(r) + \frac{Q(r)}{\langle \Omega(r,t)^2 \rangle} \tag{46}
\]
We have used the Ito-interpretation. The corresponding Fokker-Planck equation for the Lagrangian joint position-vorticity increment $H(r,\Omega,t)$ reads
\[
\frac{\partial}{\partial t} H(r,\Omega,t) + \frac{\partial}{\partial r} G(r)[\Omega^2 - \langle \Omega(r,t)^2 \rangle] H(r,\Omega,t)
= \left[ \frac{\partial}{\partial \Omega} \gamma(r)\Omega + Q(r) \frac{\partial^2}{\partial \Omega^2} + \frac{\partial}{\partial r} C(r) \frac{\partial}{\partial r} \right] H(r,\Omega,t) \tag{47}
\]
It is interesting to note that the Langevin equations are prototype equations for so-called ratchets. The ratchet process is a genuinely nonlinear transport process underlying the motion of Brownian motors and rotors (we refer the reader to the reviews [29], [30]). According to [31], the present system is a pulsating ratchet.

It is interesting to compare the Lagrangian model [47] with the models presented by Castiglione and Pumir [34] and Pumir et al. [35], who are dealing with multi-point statistics. Our present analysis refines these models by an explicit consideration of the functional form of the conditional relative velocity field between two Lagrangian points.

On a more general level it would be interesting to understand the statistics of Lagrangian observables like velocity increments and acceleration, which have been investigated recently in two-dimensional flows [32] [33] [40] [41] in terms of the multi-point vorticity statistics.

VII. SUMMARY

A generalization of the Langevin Model to the two-point vorticity equation is straightforward and has the following structure
\[
\begin{align*}
\dot{x}_1 &= e_x U(r,\omega_1,\omega_2) + e_v V(r,\omega_1,\omega_2) + \eta_1 \\
\dot{x}_2 &= -e_x U(r,\omega_2,\omega_1) - e_v V(r,\omega_1,\omega_2) + \eta_2 \tag{48}
\end{align*}
\]
The vorticity variables undergo Ornstein-Uhlenbeck processes. $U(r,\omega_1,\omega_2)$ is an odd function with respect to $\omega_1$, whereas $V(r,\omega_1,\omega_2)$ is an even function. In the present paper we have identified and used the lowest order approximations
\[
\begin{align*}
e_x U(r,\omega_1,\omega_2) &= U(x_1 - x_2) \sum_i C_{2i}^{-1} \omega_i \\
V(r,\omega_1,\omega_2) &= G(r) \sum_{ij} a_{ij}(r) \langle \omega_i \omega_j - \langle \omega_i \rangle \langle \omega_j \rangle \rangle \tag{49}
\end{align*}
\]
Our analysis has led us to an effective point vortex description, which is in the spirit of the Landau effective particle approach for many-body systems. The effective azimuthal dynamics is a two vortex motion, however with a screened velocity field. The effective relative motion, which is absent for bare point vortices as well as for the vortex motion in the linear Gaussian approximation is the fingerprint of the inverse cascade. It determines the energy flux. The relative motion can be attracting or repelling, depending on the distance and the instantaneous vorticities. A directed flux results due to a ratchet effect, a genuinely nonlinear transport process. It remains a challenge to derive this relative motion between two effective point vortices directly from the stochastic Navier-Stokes dynamics.

Appendix A: Kinetic Equations

For the sake of completeness, we present the derivation of the hierarchy of evolution equations for the vorticity probability distributions and explicitly discuss the introduction of the conditional averages. To this end we define the fine-grained N-point vorticity distributions
\[
\begin{align*}
\tilde{f}(\omega_1, x_1; t) &= \delta(\omega_1 - \omega(x_1,t)) \\
\tilde{f}(\{\omega_i, x_i\}; t) &= \Pi_i \delta(\omega_i - \omega(x_i,t)) \tag{A1}
\end{align*}
\]
The evolution equations for these quantities are obtained by differentiation with respect to time:
\[
\frac{\partial}{\partial t} \tilde{f}(\omega_1, x_1; t) = -\frac{\partial}{\partial t} \omega(x_1,t) \frac{\partial}{\partial \omega_1} \delta(\omega_1 - \omega(x_1,t)) \tag{A2}
\]
Taking into account the vorticity equation we obtain
\[
\begin{align*}
\left[ \frac{\partial}{\partial t} + \eta(x_1, t) \cdot \nabla x_1 \right] \tilde{f}(\omega_1, x_1; t) &= -\frac{\partial}{\partial \omega_1} [L(-\Delta)\omega(x_1,t) + F(x_1,t)] \tilde{f}(\omega_1, x_1; t) \tag{A3}
\end{align*}
\]
Thereby, we have used the fact that the velocity field is incompressible
\[ \frac{\partial}{\partial \omega_1} u(x_1, t) \cdot \nabla x_1 \omega(x_1, t) \delta(\omega - \omega(x_1, t)) = -u(x_1, t) \cdot \nabla x_1 \tilde{f}(\omega_1, x_1; t) \]  
(A4)

We now proceed to derive the evolution equations for the coarse-grained probability distribution. Averaging yields
\[ \frac{\partial}{\partial t} \langle u(x_1, t) \rangle + \nabla x_1 \cdot \langle u(x_1, t) \rangle \tilde{f}(\omega_1, x_1; t) = -\frac{\partial}{\partial \omega_1} \{ F(x_1, t) \tilde{f}(\omega_1, x_1; t) \} \]
(\text{A5})

The unclosed expectation value \( \langle u(x_1, t) \tilde{f}(\omega_1, x_1; t) \rangle \), can be expressed in terms of the conditional expectation
\[ \langle u(x_1, t) \tilde{f}(\omega_1, x_1; t) \rangle = U(x_1 | \omega_1 x_1; t) f(\omega_1, x_1; t) \]  
(A6)
or, alternatively, using the representation
\[ u(x_1, t) = \int dx' U(x_1 - x') \omega(x', t) \]  
(A7)

with
\[ U(x) = e_x \times \frac{x}{2\pi |x|^2} \]  
(A8)
in terms of the two-point probability distribution
\[ \langle u(x_1, t) \tilde{f}(\omega_1, x_1; t) \rangle = \int dx' \int d\omega_1 \omega' U(x_1 - x') f(\omega', x'_1; \omega_1, x_1; t) = \int dx' \int d\omega_1 \omega U(x_1 - x') \langle \omega, x' | \omega_1, x_1, t \rangle f(\omega_1, x_1; t) \]  
(A9)

In a similar way, the dissipative term can be expressed as
\[ \langle L(-\Delta x_1) \omega(x_1, t) \tilde{f}(\omega_1, x_1; t) \rangle = \int dx' \int d\omega_1 \delta(x_1 - x') \omega L(-\Delta x') f(\omega', x'_1; \omega_1, x_1; t) \]  
(A10)

Finally, we treat the forcing term. To this end we assume the force to be \( \delta \)-correlated in time. The evolution equation takes the form
\[ \frac{\partial}{\partial t} \tilde{f} = L \tilde{f} - \frac{\partial}{\partial \omega_1} F(x_1, t) \tilde{f} \]  
(A11)

This leads to
\[ \frac{\partial}{\partial t} f = L f + \frac{\partial^2}{\partial \omega_1^2} Q(0) f \]  
(A12)

where we have defined
\[ Q(x_1 - x_j) \delta(t - t') = \frac{1}{2} [F(x_1, t) F(x_j, t)] \]  
(A13)

This completes the derivation of the kinetic equation for the one-point probability distribution.

The derivation of the equation for the multi-point distribution is performed analogously. We obtain
\[ \left[ \frac{\partial}{\partial t} + \nabla x_i \cdot U(x_1 | \omega_1 x_1) \right] f(\omega_1, x_1; t) = -\sum_i \frac{\partial}{\partial \omega_i} \left[ \langle L(-\Delta) \omega \rangle x_1 | \omega_1, x_1 \right] f(\omega_1, x_1; t) + \langle F, x_1 | \omega_1, x_1, t \rangle f(\omega_1, x_1; t) \]  
(A14)

where the operator \( L_i \) is defined according to
\[ L_i = \frac{\partial}{\partial \omega_i} \omega_i + \sum_j Q(x_i - x_j) \frac{\partial^2}{\partial \omega_i \omega_j} \]  
(A15)

We remind the reader that, in general, the conditional velocity field \( U(x_1 | \omega_1 x_1) \) does not need to be solenoidal.

Appendix B: Conditional Expectations: Linear Gaussian Approximation

It is straightforward to evaluate the expectation necessary to calculate the conditional velocity field based on the assumption of Gaussian statistics using the Fourier representation of the distribution \[ \langle \omega', x' | \omega_1, x_1 \rangle \rangle = \int d\alpha' \Pi d\alpha' \langle \omega', x', \alpha' \rangle \rho(\alpha, \omega, x, (\alpha, x)) \]  
(B1)

The quantity \( W(\alpha', x', \alpha_1, x_1) \) is the cumulant generating function of the multi-point vorticity statistics. The conditional expectation of vorticity is then obtained by
\[ \int d\omega' \omega' \langle \omega', x', \omega_1, x_1; \ldots ; \omega_N, x_N | t \rangle = -\int \Pi d\alpha' \left[ \left[ \frac{1}{i} \frac{\partial}{\partial \alpha'} W(\alpha, x', \alpha_1, x_1) \right]_{\alpha' = 0} - \left[ \frac{1}{i} \frac{\partial}{\partial \alpha'} W(\alpha', x', \alpha_1, x_1) \right]_{\alpha' = 0} \right] f(\omega_1, x_1; t) \]  
(B2)

This is an exact relationship involving the generating function \( W(\alpha', x', \alpha_1, x_1) \) of the \( (N + 1) \)-point cumulants of vorticity.

For a Gaussian distribution we have
\[ W(\alpha', x', \alpha_1, x_1) = \frac{\alpha'^2}{2} C(0) \]
\[ + \sum_{i,j} \frac{\alpha_1 \alpha_j}{2} C(x_i - x_j) + \sum_i \alpha_i \alpha_i' C(x' - x_i) \]  
(B3)
and we obtain for the expectation

\[
\int d\omega' \omega' f(\omega', x'; \{\omega_i, x_i\}) = \\
= \int \Pi_i d\omega_i \ e^{\sum \omega_i \alpha_i} \times \\
\times \frac{1}{i} \sum_i C(x' - x_i) \alpha_i e^{-\sum_{i,j} \frac{\alpha_i \alpha_j}{2} C(x_i - x_j)} \\
= \sum_{im} C(x' - x_i) C^{-1}_{im}(\{x_i\}) \omega_m f(\{\omega_i, x_i\}) 
\]

(B4)

Here, \( C_{ij} = C(x_i - x_j) \) denotes the vorticity-vorticity correlation function at points \( x_i, x_j \), which completely determines a multivariate Gaussian vorticity correlation. \( C^{-1}_{im}(\{x_i\}) \) denotes the inverse matrix,

\[
\sum_k C(x_i - x_k) C^{-1}_{km}(\{x_i\}) = \delta_{im} 
\]

(B5)

We have used the identity

\[
\alpha_i e^{-\sum_{i,j} \frac{\alpha_i \alpha_j}{2} C(x_i - x_j)} = -\sum_{im} C^{-1}_{im} \partial \omega_m e^{-\sum_{i,j} \frac{\alpha_i \alpha_j}{2} C(x_i - x_j)} 
\]

(B6)

This expression for the conditional vorticity can be used to calculate the conditional velocity \((\{17\})\) as well as the conditional dissipation \((\{21\})\) in the Gaussian approximation. Since the conditional velocity field turns out to be linear in the vorticities, we shall denote the present approximation as linear Gaussian approximation (LGA).

[1] A.S. Monin, A. M. Yaglom, Statistical Fluid Mechanics, (Dover Publications, Mineola, 2007)
[2] A. Tsinober, An Informal Conceptual Introduction to Turbulence, (Springer Verlag Heidelberg, 2009)
[3] R. Kraichnan, Phys. Fluids 10, 1417 (1967)
[4] G. Boffetta, S. Mussachio, Phys. Rev. E 82, 016307 (2010)
[5] D. Bernard, G. Boffetta, A. Celani, G. Falkovich, Nature Physics 2, 124 (2006)
[6] J. Cardy, G. Falkovich, K. Gawedzki, Non-equilibrium Statistical Mechanics and Turbulence (Cambridge University Press 2008)
[7] G. Boffetta, A. Celani, M. Vergassola, Phys. Rev. E 61, R29-R32, (2000)
[8] S. Goto, J. C. Vassilicos, New Journal of Physics, 6, 65 (2004)
[9] J. Paret, P. Tabeling, Phys. Rev. Lett. 79, 4162 (1997)
[10] C-W. Shu and S. Osher, J. Compt. Phys. 77, 439 (1988)
[11] G. Falkovich, V. Lebedev, M. Stepanov, arXiv: 1011.3647
[12] R. Friedrich, Chem. Phys. 375, 587 (2010)
[13] A. Novikov and D.G. Dommermuth, Mod. Phys. Lett. B 23, 1395 (1994)
[14] Mui, R. C. Y. and Dommermuth, D. G. and Novikov, E. A., Phys. Rev. E 53, 2355-2359 (1996)
[15] M. Wilczek, R. Friedrich, Phys. Rev. E 80, 016316 (2009)
[16] Wilczek, M. and Daitche, A. and Friedrich, R., submitted to J. Fluid Mech., (2010)
[17] J. Lüllf, M. Wilczek, R. Friedrich, New J. Phys. (2011)
[18] P.R. Ulinich, B. Ya. Lyubimov, Soviet Physics JETP, 28, 494 (1969)
[19] T.S. Lundgren, Physics of Fluids 10, 969 (1967)
[20] E. A. Novikov, Sov. Phys. JETP 20, 1290 (1964)
[21] G. Boffetta, M. Cencini, J. Davoudi, Phys. Rev. E 66, 017301 (2002)
[22] V. Yakhot, Phys. Rev. E 63, 026307 (2001)
[23] V. Yakhot, Phys. Rev. E 60, 5544 (1999)