Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
An epidemic prediction from analysis of a combined HIV-COVID-19 co-infection model via ABC-fractional operator

Idris Ahmed\textsuperscript{a,b,c,d}, Emile F. Doungmo Goufo\textsuperscript{e}, Abdullahi Yusuf\textsuperscript{f,g}, Poom Kumam\textsuperscript{b,c,h,\*}, Parin Chaipanya\textsuperscript{a,e,i}, Kamsing Nonlaopon\textsuperscript{j}

\textsuperscript{a} Department of Mathematics, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
\textsuperscript{b} Fixed Point Research Laboratory, Fixed Point Theory and Applications, Research Group, Center of Excellence in Theoretical and Computational Science, (TaCS-CoE), Faculty of Science, King Mongkut’s University of Technology Thonburi, (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
\textsuperscript{c} Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Science Laboratory Building, Faculty of Science, King Mongkut’s University of Technology Thonburi(KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
\textsuperscript{d} Department of Mathematics and Computer Science, Sule Lamido University, P. M. B 048 Kafin-Hausa, Jigawa State, Nigeria
\textsuperscript{e} Department of Mathematical Sciences, University of South Africa, Florida 0003, South Africa
\textsuperscript{f} Department of Computer Engineering, Biruni University, Istanbul 34010, Turkey
\textsuperscript{g} Department of Mathematics, Federal University Dutse, Jigawa 7156, Nigeria
\textsuperscript{h} Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
\textsuperscript{i} NCAO Research Center, Fixed Point Theory and Applications Research Group, Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
\textsuperscript{j} Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

Received 30 December 2020; revised 16 January 2021; accepted 21 January 2021
Available online 29 January 2021

KEYWORDS
Existence and uniqueness; ABC fractional derivative; Fixed point theorems; Coronavirus; HIV

Abstract The whole world is still shaken by the new corona virus and many countries are starting opting for the lockdown again after the first wave that already killed thousands of people. New observations also show that the virus spreads quickly during the cold period closer to winter season. On the other side, the number of new infections decreases considerably during hot period closer to summer time. The geographic structure of our planet is such that when some countries (in a hemisphere) are in their winter season, others in the other hemisphere are in their summer season.

https://doi.org/10.1016/j.aej.2021.01.041
1110-0168 © 2021 THE AUTHORS. Published by Elsevier BV on behalf of Faculty of Engineering, Alexandria University.
This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
However, we have observed in the world some countries undertaking national lockdown during their summer time, which result in their economy to be hugely hit. Other factors, beside the lockdown, have also impacted negatively the socio-economic situation in affected countries. These include, among others, the human immunodeficiency virus (HIV) susceptible to combine to the new corona virus. The new corona virus is indeed recent and many of its effect and impact on the society are still unknown and are still to be uncovered. Hence we use here the of Atangana-Baleanu fractional derivative to mathematically express and analyses a model of HIV disease combined with COVID-19 to assess the pandemic situation in many countries affected, such as South Africa, United Kingdom (UK), China, Spain, United States of America (USA), and Italy. A way to achieve that is to perform stability and bifurcation analysis. It is also possible to investigate in which conditions the combined model contains a forward and a backward bifurcation. Moreover, utilizing the techniques of Schaefer and Banach fixed point theorems, existence and uniqueness of solutions of the generalized fractional model were presented. Also, the Atangana-Baleanu fractional (generalized) HIV-COVID-19 co-infection model is solved numerically via well-known and effective numerical scheme and a predicted prevalence for the COVID-19 is provided. The global trend shown by the numerical simulation proves that the disease will stabilize at a later stage when adequate measures are taken.

© 2021 THE AUTHORS. Published by Elsevier BV on behalf of Faculty of Engineering, Alexandria University. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

All started in December 2019 where new pneumonia disease called the coronavirus disease 2019 started in Wuhan, a city in central China. Known as COVID-19, the new disease as so far infected about 50 million people worldwide as of 09 November 2020. Out of those 50 million infected people, about 34 million have recovered and 1.3 million have succumbed to the disease. COVID-19 is a highly contagious disease caused by the so-called SARS-CoV-2 meaning the severe acute respiratory syndrome coronavirus 2. Although the total virus dynamics and variations remains widely unknown, the research are continuing. But some research and also ground observations have shown that the SARS-CoV-2 can spread easily under the cold conditions, confirming what we are currently seeing in Europe with the increase of number of infection from mid-October Some works [40] recently proved that the SARS-CoV-2 easily spreads in environment with temperature relatively low, near 5 to 11 degrees Celsius, all associated with low specific, near 3 to 6 g/kg and absolute humidity near 4 to 7 g/m³. Hence, it is an unprecedented pandemic situation, since the 1918 Spanish flu pandemic [38], which may have been avoided if the whistle blow launched by Li Wenliang [23] was taken into consideration on time by Chinese authorities.

Genesis: First warning alert by Li Wenliang, the hero who tried to save the Humanity. It is the story of a young physician who has become an icon, Li Wenliang, 34-year-old, an ophthalmologist who was living a joyful and stable life in the city of Wuhan in the Hubei province in China. Unfortunately, his life took a tragic turn overnight because, at the end of December 2019, Li Wenliang was the first to launch the alert on a mysterious virus of unknown origin, strongly resembling SARS. He was later arrested, along with seven other medical colleagues, all of whom were accused of spreading rumors and false information. Chinese authorities did not respond until three weeks later when the epidemic had gotten out of hand. A fatal error that Li Wenliang paid with his life. To make sure that the story of this hero is never forgotten, we must mention here the tragic fate of this doctor who tried to save humanity at the expense of his life. On December 30, 2019, no one in China, or in the rest of the world, suspected that an unprecedented epidemic was going to shake the world and cause total chaos, taking modern companies and successful organizations by surprise. Nonetheless, somewhere in central China in Hubei province, more precisely in the 11 million cities of Wuhan, a young ophthalmic doctor is alarmed. Li Wenliang then working at the central hospital, was married, father, and his wife was expecting a second child for four months. He is worried because, for the past few days, he has noticed among the patients he meets daily, the appearance of a disease taking the form of viral pneumonia resistant to the usual treatments. Pneumonia that strangely reminded him of SARS, which appeared in China in late 2002. One of the emergency workers who work with him also shares the same fears. She also sent him, as well as other colleagues and doctors of the infectious disease control department, very disturbing photos of the analysis results of certain patients. Photos that only reinforced his suspicions. It was that night that Doctor Wenliang decided to face the situation and launch the alert with his colleagues via his personal messaging “WeChat”; a social network in China equivalent to “Whatsapp”.

The ophthalmologist was then trying to prevent the community and authorities that he realized that seven people from the same seafood market in the city, were infected with a new virus hitherto unknown, which had many similarities to the SARS which had caused in 2002 and 2003 nearly 800 victims in China. He also added in this message that these infected patients were in quarantine at the hospital where he worked in Wuhan city. Thus, he simply advised his doctor friends to warn their loved ones of the dangers of this new disease. A priori, these were simple words, started from a good intention which quickly transformed into a wave of photos and messages, sowing in the process a real mockery in social networks. It didn’t take long for this to backfire and two days later in the middle of the night, the eight doctors were arrested by the authorities, all accused of spreading rumors and seriously disturbing public order. After being questioned and threatened for a long time, Doctor Wenliang would get out of it by signing
a document, a sort of public confession in which he undertook not to commit any more acts contrary to the law. In the doc-
ument was written the following words: “You write false com-
ments on the Internet, will you be able to stop these illegal
actions? We strongly warn you: If you continue your illegal
activities, you will be prosecuted by the law”. Dr. Li Wenliang
accepted despite himself and signed. But unfortunately, his
fears became real since what was initially only a rumor accord-
ing to the authorities, became a few days later one of the great-
est pandemics that human history has known. Free and
allowed to return to work, Dr. Li Wenliang then resumed
his consultations at the hospital without knowing that a few
days later he would be among his patients. The day after their
arrest, the Chinese government informed the World Health
Organization that several cases of viral-like pneumonia have
been reported in Wuhan city. It was unfortunately not a coin-
cidence that, a few days later, on January 07, 2020, the Chinese
authorities officially declared that a new virus had appeared in
the country. On January 10, 2020, Doctor Li Wenliang took
charge of a patient who came to seek treatment for glaucoma
without knowing that she was also infected with the coron-
avirus. The one who has been silenced begins a few days later
to feel in a very feverish state. Once back home, the doctor also
unknowingly infected his parents, his five-year-old son, and his
pregnant wife. The young doctor’s entire family was now
infected with the new coronavirus. Doctor Li Wenliang’s situ-
ation worsened and he was finally hospitalized on January 12,
2020. That is how he then decided to make his experience pub-
lic. He criticized the Chinese government’s inaction, continued
to report on the virus, and was proclaimed hero overnight.
Conscious of being in a very precarious state, the young doctor
said however that he wanted to resume, once recovered from
the disease, his post at the hospital and confided that he would
be on the front line to confront the monster.

In an interview with the Caixing investigative magazine
[24], the doctor, who became patient overnight, declared that
he felt confused because at that time the government continued
to repeat that there is no human-to-human transmission and
that no member of the medical profession was contaminated.
Sick, helpless, and without support, this is how the doctor will
spend the terrible last days of his short life. A few days later,
the first official deaths were reported in China. Then, it was
the turn of other neighboring countries like Thailand to
announce their contamination with the coronavirus. Remem-
ber the rumor, it has, unfortunately, come true. On January
20, 2020, the Chinese government finally recognized that
human-to-human transmission is possible. Three days later,
to everyone’s surprise, the city of Wuhan was quarantined.
The Chinese government then took emergency measures to
stop the spread of the virus. But perfectly aware that this one
is virulent and more powerful than we thought, on Jan-
uary 24, 2020, the government decided to take the lead, to
be one step ahead of the enemy and launched a crazy project,
that of building a hospital that can accommodate up to 1,500
patients in just 10 days. On January 28, the first confessions
surfaced, the country’s supreme court said that the country
could have done better at the epidemic and prepared in
advance if the words of Dr. Li Wenliang had been taken seri-
ously. Forced to justify themselves to the general public, their
reaction is judged to be bad, belated, and lax, the leaders of the
province of Hubei tried to take responsibility by declaring that
the doctors had not been arrested, but simply reprimanded for
having used the term SARS and thus alarming citizens of
information that had not yet been officially confirmed. As
for Doctor Li Wenliang, his state of health worsened consid-
erably in a few days to the point that he had to be transferred to
intensive care in early February. On February 7, 2020, the
young doctor, unfortunately, succumbed to the virus whose
he had sought to reveal the existence to protect the whole of
Humanity.

In China, there is chaos, since the announcement of the
death of the young doctor of 34 years, tributes multiply in
front of the hospital and on social networks. Thousands of
sad messages can be found there, as well as hateful comments
from angry citizens who are very angry at the Chinese govern-
ment’s actions, which they considered to be disastrous in the
face of this situation. The rare thing, Chinese people usually
threatened by censorship, do not hesitate to express their
hatred and anger against the authorities who underestimated
the power of the virus and who especially tried to silence the
one who wanted to save China from the catastrophe. In the
messages, many were thankful and others said that he deserves
history to remember him. The WHO also responded to this
unfortunate event by expressing its deep sadness. As for the
city hall of Wuhan, it expressed its condolences to the family
of the doctor. The central government did not react to it, pre-
ferring to limit itself to platonic condolences. And other than
condolences, fellow citizens and institutions decided to com-
 pense the family of the deceased for their loss and had volun-
teed to provide them with financial assistance thanks, in
particular to a collection carried out among former students
of Wuhan University who have thus raised several thousand
dollars in just a few hours.

To calm the wave of indignation and somehow catch up
with this fatal error, which was not to disseminate information
promptly, and to cover up the error, Beijing was forced to
order an investigation within the discipline committee of the
Chinese Communist Party, to make an exhaustive inventory
and a report on the circumstances which led to the death of
doctor Li Wenliang and on the days which preceded his infe-
cction. The local authorities finally qualified as inappropriate
the treatments to which the doctor had been subjected, because
the hospitals of Wuhan were in a chaotic situation, also disavow-
ing the police who ended up apologizing to his family. The
mayor of Wuhan acknowledged his share of responsibility by
apologizing for delaying the release of information about the
new coronavirus in an interview on Chinese TV. However,
he recalled that he did not assume full responsibility for the
fact that he had not been authorized to disclose the informa-
tion. But the apologies were unfortunately too late and the
recovery project not effective enough and considered insuffi-
cient to calm the wave of indignation caused by the disap-
pearance of the young doctor, but also by the arrest of journalists
and other people who dared cry out against injustice and
openly criticized the party’s crisis management strategy. So
the revelations about Dr. Li Wenliang’s attempt to raise the
alarm, the pressures he had been subjected to, his contamina-
tion, and his death in just a few days, all these elements per-
fectedly added-up to transform the young doctor into a doubly
emblematic figure. At the same time martyr of the virus, whose
he had so well discovered the dangerousness and at the same
time defender of freedom of expression, that the authorities
cowardly forced themselves to muzzle. The world should never
experience this again.
Mathematical modeling of the spread of this virus would be one way to help mitigate and avoid the disease from plaguing the world by recognizing the danger posed by this virus, along with the fact that there are no medications or vaccines available for its treatment. Mathematical and computational modeling of infectious diseases provides deeper mechanistic insights into transmission pathways, preventive approaches, and prediction of dissemination. Many researchers have used the mathematical model to describe and predict the behavior of infectious diseases, (see [6,25,36,45] and references therein).

One of the fastest-growing fields of mathematics is fractional calculus, which generalizes classical calculus and handles the integrals and derivatives of non-integer order. The motives for such rapid growth are the ramifications obtained when some scientists used fractional operators to model such phenomena in the real world [2,8,10–12,14,18,29,39,42,43,48]. In dealing with practical problems, the fractional-order in the sense of Atangana-Baleanu derivative have the same initial and integer-order derivatives. More importantly, due to its nonlocal and nonsingular kernel, the accuracy superseded the other kinds of fractional operators like Caputo, Riemann–Liouville, Caputo-Fabrizio, and many others [37]. The aforementioned operator has been successfully applied to several complicated models in the field of science and engineering, we refer to [7,19,21,22,25,27,30–32] and the references therein.

2. Formulation of the model

It has been shown that fractional-order derivatives have a broad range of applications in the modeling of many chaotic dynamic systems, such as engineering, biology, medicine and many other fields see for example some of the recent results [13,15,21,26,28,33]. We recall some preliminaries concepts before introducing the proposed fractional epidemic model.

2.1. Preliminaries

Definition 2.1. [16] Suppose $g \in L^1(0, T)$ and $0 < q \leq 1$. Then the operator

\[ AB_{\alpha}^q g(t) = \frac{1 - q}{N(q)} g(t) + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} g(s)ds, \quad t > 0, \]

is referred to AB fractional integral of order $q$ with the lower limit $0^+$ for a function $g$ and $\Gamma(\cdot)$ denotes the gamma function, $N(q)$ is the defined as $N(q) = \frac{1}{\gamma(q)}$ and $E(\cdot)$ is the Mittag-Leffler function of the form

\[ E_q(g) = \sum_{m=0}^{\infty} \frac{g^m}{\Gamma(qm + 1)}, \quad Re(q) > 0. \quad (2.1) \]

Definition 2.2. [16] Suppose $g \in H^1(0, T)$ and $0 < q \leq 1$. Then the operator

\[ AB_{\alpha}^q g(t) = \frac{N(q)}{(1 - q)} \int_0^t E_q \left( \frac{-q}{1 - q} (t - s)^q \right) g(s)ds, \quad t > 0, \]

called the ABC fractional derivative of order $q$ with the lower limit $0^+$ for a function $g$.

Lemma 2.3. [16] If $q \in (0, 1]$. Then

\[ AB_{\alpha}^q g(t) = g(t) - c_0. \quad (2.2) \]

Thus, the given problem

\[ \begin{cases} AB_{\alpha}^q g(t) = h(t), \\ g(0) = g_0, \end{cases} \quad (2.3) \]

does not give the solution in the form:

\[ g(t) = g_0 + \frac{1 - q}{N(q)} h(t) + \frac{q}{N(q)} \int_0^t (t - \xi)^{q-1} h(\xi)d\xi. \quad (2.4) \]

Theorem 2.4. [34] Consider the following fractional-order system:

\[ \begin{cases} AB_{\alpha}^q g(t) = h(g(t)), 0 < q \leq 1, \\ g(0) = g_0 \in \mathbb{R}^N, \end{cases} \quad (2.5) \]

where $g(t) = (g_1, g_2, \ldots, g_n)(t) \in \mathbb{R}^N$ and $z : [h_1, h_2, \ldots, h_n] \in \mathbb{R}^N \rightarrow \mathbb{R}^N$. The solutions to $h(\cdot)$ are said to be the equilibrium points of the system. An equilibrium point $F$ is locally asymptotically stable if and only if all eigenvalues $z_i$ of the Jacobian matrix $J := \frac{\partial g(t)}{\partial t}$ evaluated at $F$ satisfy the following equation $|arg(z_i)| > \frac{\pi}{2}$.

2.2. Fractional-order HIV and COVID-19 model.

Inspired by the above-mentioned useful application of some fractional operators in modeling the real word problems. In this paper, we are analyzing the behavior of the HIV and Shift- ing epicenters for the COVID-19 model proposed by Goufo et al. [20] in the form of the system of nonlinear differential equations via ABC fractional derivative of the following form:

\[ \begin{aligned}
AB_{\alpha}^q S_c(t) &= \Lambda_c - \gamma I_c - (\lambda_c + \beta_h + \mu_c) S_c, \\
AB_{\alpha}^q I_h(t) &= \lambda_c S_c - (\alpha + \beta_h + \kappa_c + \gamma) I_h, \\
AB_{\alpha}^q I_r(t) &= \beta_h S_h - (\theta + \alpha + \mu_h + \kappa_h) I_h, \\
AB_{\alpha}^q I_{\nu}(t) &= \alpha I_h - \beta_h I_{\nu} - (\mu_h + \kappa_h + \nu_h + \sigma) I_{\nu}, \\
AB_{\alpha}^q I_{\theta}(t) &= \kappa_h S_h - \gamma I_h, \\
AB_{\alpha}^q I_{\mu}(t) &= \mu_h I_h, \\
\end{aligned} \quad (2.6) \]

where

\[ \lambda_c = \frac{\beta_h \alpha}{N}, \quad \lambda_h = \frac{\beta_h (\alpha + \kappa_h)}{N}, \quad \lambda_{\nu} = \frac{\beta_h \kappa_h}{N}, \]

and $AB_{\alpha}^q (\cdot)$ denotes the fractional derivative of order $(0 < q \leq 1)$ in the sense of Atangana-Baleanu operator with nonnegative variables and initial condition. The meaning of each biological parameters as well as the state variables described in the Table 1.

3. Analysis of the proposed model

In this section, we study the dynamical analysis of the proposed model (2.6), that is Well-posedness, feasibility region,
and stability analysis. We start by analyzing the HIV and COVID-19 model only.

3.1. HIV model only

Setting the variables \( I_x = I_h = S_x = L = 0 \) in system (2.6) gives

\[
\begin{align*}
\frac{dS_p}{dt} &= \Lambda_p - (\dot{\lambda}_h + \lambda_h + \mu_b)S_p, \\
\frac{dI_h}{dt} &= \dot{\lambda}_hS_p - (\mu_b + \kappa_h)I_h,
\end{align*}
\]

where

\[ \dot{\lambda}_h = \frac{\beta_b I_h}{N_p}, \]

and \( N_p = S_p + I_h \).

3.1.1. Invariant region

Since the model (3.1) described the population of human, it is important to show the negativity of the state variables \( S_p, I_h \) for all \( t \geq 0 \). That is adding all the equations in model (3.1) yield

\[
\frac{dN_p}{dt} \leq \left( -\frac{\Lambda_p}{\mu_b} \right) + \frac{\Lambda_p}{\mu_b},
\]

which implies that

\[
N_p(t) \leq \left( N_p(0) - \frac{\Lambda_p}{\mu_b} \right) e^{-\frac{\mu_b t}{\mu_b}} + \frac{\Lambda_p}{\mu_b}.
\]

So one has the following feasible region given by

\[
\Gamma_h = \left\{ (S_p(t), I_h(t)) \in \mathbb{R}^2 : 0 \leq N_p = S_p + I_h \leq \frac{\Lambda_p}{\mu_b} \right\}.
\]

This shows that the model (3.1) is epidemiologically well posed and the solutions \( (S_p(t), I_h(t)) \in \mathbb{R}^2 \) remains in \( \Gamma_h \).

3.1.2. Stability analysis

Setting \( I_h = 0 \) the disease-free equilibrium (DFE) of system (3.1) archive as

\[
E_0 = (S_p(0), 0) = \left( \frac{\Lambda_p}{\mu_b}, 0 \right).
\]

Employing the strategy of the next generation matrix \( FV^{-1} \) as detailed in [46], we determine the value of basic reproduction number denoted by \( R_0^h \) as \( \rho(FV^{-1}) \) where \( \rho \) denotes the spectral radius of matrix \( FV^{-1} \). Setting \( ABC \frac{d}{dt} X(t) = F(X) - V(X) \), where

\[
F(X) = \begin{bmatrix} 0 \\ \beta_h \end{bmatrix},
\]

and

\[
V(X) = \begin{bmatrix} -\Lambda_p + \dot{\lambda}_h S_p + \mu_b S_p \\ \mu_b I_h + \kappa_h I_h \end{bmatrix}.
\]

This implies that the Jacobian matrix evaluated at the DFE \( E_0 \) is given by

\[
F = \begin{bmatrix} 0 & 0 \\ 0 & \beta_h \end{bmatrix}
\]

and

\[
V = \begin{bmatrix} \mu_b & \beta_h \\ 0 & \mu_b + \kappa_h \end{bmatrix}.
\]

So, the basic reproduction number \( R_0^h \) is computed as

\[
R_0^h = \rho \left[ \begin{array}{cc} 0 & 0 \\ 0 & \beta_h \end{array} \right] \begin{bmatrix} \frac{1}{\mu_b} & -\beta_h \\ 0 & \frac{1}{\mu_b + \kappa_h} \end{bmatrix} = \frac{\beta_h}{\mu_b + \kappa_h} > 0.
\]

**Theorem 3.1.** The disease-free equilibrium \( E_0 \) for the model (3.1) is locally asymptotically stable if \( R_0^h < 1 \) and unstable if \( R_0^h > 1 \).

**Proof.** Computing the Jacobian of the system (3.1) and evaluating at \( E_0 \) yields

\[
J_{E_0} = \begin{bmatrix} -\mu_b & -\beta_h \\ 0 & \beta_h - (\mu_b + \kappa_h) \end{bmatrix}.
\]

Thus, the eigenvalues for \( J_{E_0} \) are

\[
z_1 = -\mu_b < 0,
\]

\[
z_2 = \beta_h - (\mu_b + \kappa_h) = (R_0^h - 1)(\mu_b + \kappa_h) < 0 \iff R_0^h < 1.
\]

Thus, the inequality condition of Theorem 2.4 is true. Hence, system (3.1) is locally asymptotically stable at \( E_0 \). \( \square \)

**Theorem 3.2.** If \( R_0^h \leq 1 \) the disease-free equilibrium \( E_0 \) of the HIV model (3.1) is globally asymptotically stable.
Proof. Let $X = (S_p, I_o)^T$ and consider the Lyapunov function

$$L(X) = (\mu_k + k_o)I_o,$$

then, the derivative of $L(X)$ is

$$ABC D^\nu_t L(X) = (\mu_k + k_o)ABC D^\nu_t I_o$$

$$= (\mu_k + k_o)(\dot{S}_p - (\mu_k + k_o)I_o)$$

$$= \left( \frac{\beta I_o}{N_p} \right) - 1 \right) \frac{\beta I_o(S_p^2 + 2(S_p I_p) N_p)}{N_p}$$

$$\leq \left( R^q_0 - 1 \right) \frac{\beta I_o(S_p^2 + 2(S_p I_p) N_p)}{N_p} < 0,$$

if $R^q_0 < 1$ and the fact that $\frac{k_o}{N_p} \leq 1$. Moreover, $ABC D^\nu_t L(X) = 0$ if and only if $I_o = 0$. By Lasalle invariance principle [47], all trajectories of solutions to the system (3.1) that start in $R^2$ approach $E_0$ as $t \to +\infty$. □

3.1.3. Stability of endemic equilibrium

To derive the endemic equilibrium $E^* = (S^*_p, I^*_o)$ of the system (3.1), the system (3.1) must satisfy the equations:

$$\begin{align*}
\dot{S}_p &= -((\lambda_c + \lambda_h + \mu_k)S_p) = 0, \\
\dot{I}_o &= -(\mu_k + k_o)I_o = 0.
\end{align*}$$

(3.4)

Upon simplification of system (3.4) gives

$$S^*_p = \frac{\Lambda_p}{\mu_k + k_o},$$

$$I^*_o = \frac{\Lambda_i}{(\mu_k + k_o) N_p},$$

(3.5)

where

$$\dot{S}^*_p = \frac{\beta I_o}{(I^*_o + S^*_p)},$$

$$\dot{I}^*_o = \frac{\beta I_o}{(I^*_o + S^*_p)},$$

(3.6)

In view of Eqs. (3.5) and (3.6), we get

$$\dot{S}^*_p = (1 - R^q_0)(\mu_k + k_o) = 0,$$

(3.7)

from which, we obtain

$$\dot{S}^*_p = (1 - R^q_0)/(\mu_k + k_o).$$

(3.8)

Substituting Eq. (3.8) in Eq. (3.5), yields

$$S^*_p = \frac{\Lambda_p}{\mu_k + k_o} R^q_0,$$

$$I^*_o = \frac{\Lambda_i}{(\mu_k + k_o) N_p},$$

(3.9)

which exist when $R^q_0 > 1$. Thus we have the following results.

Theorem 3.3. If $R^q_0 > 1$, the endemic equilibrium point $E^*$ is globally asymptotically stable.

3.2. COVID-19 model only

Substituting $I_o = I_k = 0$ in system (2.6) we get COVID-19 submodel only of the form:

$$\begin{align*}
ABC D^\nu_t S_p(t) &= \Lambda_p - \gamma I_o - (\lambda_c + \mu_k)S_p, \\
ABC D^\nu_t I_o(t) &= \lambda_c S_p - (\mu_k + k_o + \gamma)I_o, \\
ABC D^\nu_t S_o(t) &= \Lambda_s - (\mu_s + \lambda_s)S_o, \\
ABC D^\nu_t I_o(t) &= \lambda_s S_o - \mu_s I_o,
\end{align*}$$

(3.10)

where

$$\dot{S}^*_p = \frac{\beta I_o}{(I^*_o + S^*_p)},$$

$$\dot{I}^*_o = \frac{\beta I_o}{(I^*_o + S^*_p)},$$

(3.11)

Repeating the same analysis as in Theorem 3.2, we have the following result.

Theorem 3.4. The disease-free equilibrium DFE $E_0 = \left( \frac{\Lambda_p}{\mu_k}, 0, \frac{\Lambda_i}{N_p} \right)$ of the fractional model (3.10) is locally asymptotically stable if $R^q_0 < 1$ and unstable if $R^q_0 > 1$. 

3.2.2. Backward bifurcation for COVID-19 model

In this subsection, we investigate the possibility of backward bifurcation for the fractional model (3.10). Noting that this kind of bifurcation occurs only both stable DFE and stable
endemic equilibrium exist for some values of $\mathcal{R}_0 < 1$. In this regards, we first find the endemic equilibrium point $E' = (S_p', I_r', S_r', I_r')$ of (3.10) as

$$
\begin{align*}
L_x - \lambda_l - (\beta_l + \mu_l)S_b = 0, \\
\lambda_lS_b - (\lambda_b + \mu_b + \gamma_l)I_b = 0, \\
L_s - (\mu_s + \lambda_s)S_b = 0, \\
\lambda_sS_b - \mu_sI_b = 0.
\end{align*}
$$

Upon simplification gives

$$
\begin{align*}
S_p' = \frac{\lambda_l(\rho_x + \rho_b + \gamma l)}{\lambda_l + (\rho_x + \rho_b + \gamma l)}, \\
I_r' = \frac{\lambda_l(\rho_x + \rho_b + \gamma l)}{\lambda_l + (\rho_x + \rho_b + \gamma l)}, \\
S_r' = \frac{\lambda_l(\rho_x + \rho_b + \gamma l)}{\lambda_l + (\rho_x + \rho_b + \gamma l)}, \\
I_r' = \frac{\lambda_l(\rho_x + \rho_b + \gamma l)}{\lambda_l + (\rho_x + \rho_b + \gamma l)}.
\end{align*}
$$

where

$$
\begin{align*}
\lambda_l' = \frac{\beta_l\sigma I_b'}{S_p' + I_b'}, \\
\lambda_l' = \frac{\beta_l\sigma I_b'}{S_p' + I_b'}.
\end{align*}
$$

In view of Eqs. (3.13) and (3.14) yield

$$
\lambda_l^* = a_1(\lambda^*_s)^2 + a_2\lambda^*_s + a_3 = 0,
$$

where

$$
\begin{align*}
a_1 &= \frac{\lambda_l}{\rho_x + \rho_b + \gamma l} \\
a_2 &= (b - R_c) - \frac{\lambda_l^2 (\rho_x + \rho_b + \gamma l)}{\rho_x + \rho_b + \gamma l} \\
\lambda_l^* &= (1 - R_c) \lambda_l^2 (\rho_x + \rho_b + \gamma l)^2
\end{align*}
$$

and

$$
\begin{align*}
b &= \frac{\mu_b(2\mu_b + \beta_l)}{\mu_l(\rho_x + \rho_b + \gamma l)}.
\end{align*}
$$

Thus, we have

$$
\begin{align*}
a_1(\lambda_s^*)^2 + a_2\lambda_s^* + a_3 = 0, \quad \lambda_s^* = 0.
\end{align*}
$$

Let

$$
\sigma = \frac{\rho_x + \rho_b + \gamma l}{\beta_l \mu_b},
$$

this implies that clearly

$$
\begin{align*}
b > 1, \quad \text{if and only if} \quad \sigma < \sigma.
\end{align*}
$$

Then we state the following results.

**Lemma 3.5.** If $\sigma < \sigma$, then the fractional model (3.10) satisfies the necessary condition for the existence of sub-critical backward bifurcation. Moreover, if $0 < R_c < 1$, then:

- If $R_c = R_0$ then there exists one and only one endemic equilibrium point for system (3.10).
- If $R_c \in (R_0, 1)$ then system (3.10) has two endemic equilibrium points.
- If $R_c \geq R_0$ system (3.10) possesses at least one endemic equilibrium point.
- If $R_c < R_0$ then there exist no endemic equilibrium point of system (3.10).

**Proof.** In order to prove the above lemma, we use the relation (3.15), (3.16) and (3.17) where it is not difficult to see that $a_1 > 0$. In addition, if $b > R_0$ then $a_2 > 0$ and if $b < R_0$ then $a_2 < 0$. Furthermore, if $R_0 < 1$ then $a_3 > 0$ and if $R_0 > 1$ then $a_3 < 0$. Clearly, from relation (3.17) when $a < \sigma$ then $b < 1$. Thus, if $R_c > 1$ then we get $a_3 \leq 0$ this shows that $R_c \geq b$ and $a_1 \leq 0$ and Eq. (3.15) has one and only one positive root. Nevertheless, Eq. (3.15) has no positive root whenever $R_c \leq b < 1$ that is $a_3 \geq 0$ and $a_2 > 0$.

Consider the case $1 \geq R_c > b$, then we obtain $a_2 < 0$ and $a_3 \geq 0$. Let $D = F(R_0') = a_2^2 - 4a_1a_3$ where $D$ is the discriminant of (3.15) and a functional of $R_0'$ then

$$
F(R_0') = -2(b - R_0') \lambda_l^2 (\rho_x + \rho_b + \gamma l)^2 > 0, \quad R_0' \in (b, 1).
$$

Clearly, in $(b, 1)$ the functional $F > 0$ and $F(b) = -4a_1a_3 < 0$ and $F(1) = a_2^2 > 0$. Hence, the exists $R_0' \in (b, 1)$ such that $F(R_0') = 0$ if $0 \leq F \in (b, R_0')$ and $0 < F \in (R_0', 1)$. Therefore, Eq. (3.15) has two positive real roots, this shows that there exist two endemic equilibrium for system (3.12). Also, whenever $R_c = R_0$, Eq. (3.15) has one and only one positive real roots, otherwise there exist no positive real root whenever $R_c > R_0 > b$. Thus, the proof is complete.

In Figs. 1 and 2, above that under some conditions the proposed model (3.10) has forward and backward bifurcation analysis. If $\lambda_l = 2500, \lambda_b = 1000, \beta_l = 0.03, \mu_b = 0.035, \mu_l = 0.0131$ and $\kappa_l = 0.000199$, the proposed model (3.10), undergoes forward bifurcation. Moreover, with $\lambda_l = 2500, \lambda_b = 1000, \beta_l = 0.05, \mu_b = 0.035, \mu_l = 0.0131$ and $\kappa_l = 0.000199$, the proposed model (3.10), undergoes backward bifurcation. In Fig. 3, we shown the stability analysis of the DFE $E_0 = (762224.14, 0, 707111.10)$ of proposed model (3.10) when $\beta_l = 0.05, \sigma = 0.217$ and $R_0 = 0.33379$.

Setting $\beta_l = 0.07, \sigma = 0.155$ and $R_0 = 0.50167$, we depicted in Fig. 4, that the proposed model (3.10) have stable endemic equilibrium $E_1 = (1938.62, 6571.11, 34941.36, 36511.93)$ and unstable endemic equilibrium $E_1 = (13687.91, 412.38, 8992.32)$. Moreover, from Fig. 5, when $\beta_l = 1.60, \sigma = 3.33 \times 10^{-13}$ and $R_0 = 1.07121$, the proposed model (3.10), has two endemic equilibrium point namely, globally stable DFE $E = (96211.13, 413.18, 68644.21, 6513.13)$ and unstable DFE $E_0$.

4. Existence and uniqueness results

In the fields of fractional calculus, existence and uniqueness are some of the most important qualitative studies of the fractional differential equation. Nevertheless, several researchers in the field of fractional calculus investigated the existence and uniqueness of solutions of fractional differential equation with different types of fractional operators, see among them the recent papers [1,3–5,9,35,41]. Hence, in this section, we demonstrate the existence and uniqueness results of the proposed model (2.6) by using the techniques of Schaefer’s and Banach fixed point theorems.

Before we proceed, the proposed generalized fractional model (2.6) can be reformulated as:
Fig. 1  Possibility of forward bifurcation analysis for system (3.10).

Fig. 2  Possibility of backward bifurcation analysis for system (3.10).

Fig. 3  Global stability analysis of DFE $E_0 = (76224.14, 0, 70711.11, 0)$ for system (3.10).
Fig. 4 Existence of stable endemic equilibrium $E_2 = (1938.62, 657.11, 34941.36, 36511.93)$ and unstable endemic equilibrium $E_1 = (13687.91, 412.38, 8992.32)$ for system (3.10).

Fig. 5 Existence of global endemic equilibrium $E^* = (96211.13, 413.18, 68644.21, 6513.13)$ and unstable $E_0$ for the system (3.10).

Thus, the nonlinear system to consider in view of (2.6) is of the form:

\[
\begin{align*}
\frac{dE}{dt} &= h_1(t, S_p, I, I_h, S, I), \\
\frac{dS_p}{dt} &= h_2(t, S_p, I, I_h, S, I), \\
\frac{dI}{dt} &= h_3(t, S_p, I, I_h, S, I), \\
\frac{dI_h}{dt} &= h_4(t, S_p, I, I_h, S, I), \\
\frac{dS}{dt} &= h_5(t, S_p, I, I_h, S, I), \\
\frac{dI_c}{dt} &= h_6(t, S_p, I, I_h, S, I),
\end{align*}
\]

(4.1)

where

\[
\begin{align*}
h_1(t, S_p, I, I_h, S, I) &= \lambda_p - \gamma I_c - (\lambda_b + \mu_b)S_p, \\
h_2(t, S_p, I, I_h, S, I) &= \lambda_S S_p - (\alpha + \mu_b + \kappa_c + \lambda_c)I, \\
h_3(t, S_p, I, I_h, S, I) &= \lambda_S S_p - \sigma I_h - (\alpha + \mu_b + \kappa_c + \lambda_c)I, \\
h_4(t, S_p, I, I_h, S, I) &= \xi \lambda I_t - \theta I_t - (\mu_b + \kappa_c + \lambda_c + \sigma)I_h, \\
h_5(t, S_p, I, I_h, S, I) &= \lambda_S - (\mu_b + \kappa_c)S, \\
h_6(t, S_p, I, I_h, S, I) &= \lambda_c S_c - \mu_c I_c.
\end{align*}
\]

(4.2)

Thus, the nonlinear system to consider in view of (2.6) is of the form:

\[
\begin{align*}
\frac{dE}{dt} &= F(t, g(t)), \\
g(0) &= g_0 \geq 0,
\end{align*}
\]

(4.3)

where

\[
\begin{align*}
g(t) &= (S_p, I, I_h, S, I)^T, \\
g_0 &= (S_{p0}, I_{0}, I_{h0}, S_{0}, I_{0})^T, \\
F(t, g(t)) &= (h_1(t, S_p, I, I_h, S, I))^T, j = 1, \ldots, 6.
\end{align*}
\]

(4.4)

Throughout the paper, we represents by $E = C([0, b], \mathbb{R}^6)$ the Banach space with the norm defined by

\[
\|g\| = \sup_{t \in [0, b]} |g(t)|,
\]

(4.5)

where $|g(t)| = |S_p| + |I| + |I_h| + |S_c| + |I|, S_p, I, I_h, S_c, I \in C[0, b]$ and $F : [0, b] \times \mathbb{R}^6 \to \mathbb{R}$ is a continuous function.

Thus, in view of Lemma 2.3, problem (4.3) is equivalent with the following integral equation given by
g(t) = g_0 + \frac{1 - q}{N(q)} \mathcal{F}(t, g(t)) + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} \mathcal{F}(\zeta, g(\zeta)) d\zeta.

Now, from Eq. (4.5), we define the operator Q : E → E by

\[
(Qg)(t) = g_0 + \frac{1 - q}{N(q)} \mathcal{F}(t, g(t)) + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} \mathcal{F}(\zeta, g(\zeta)) d\zeta.
\]

Remark 1. Note that, problem (4.3) has a solution if and only if the operator Q has a fixed point.

The existence result rely on the concepts of Schaefer’s fixed point theorem.

**Theorem 4.1.** Assume that:

(\mathcal{C}_1) There exists \( \rho_1, \rho_2 \in E \) such that \( \rho_1 = \sup_{t \in [0, b]} |\mathcal{F}(t, g(t))| \) and for every \( t \in [0, b], g \in E \),

\[ |\mathcal{F}(t, g(t))| \leq \rho_1(t) + \rho_2(t)|g(t)|. \]

Then, problem (4.3) (consequently, model (2.6)) has at least one solution.

Proof. The proof of the theorem is based on the steps given below:

Step 1. We show the continuity of the defined operator Q. Clearly, given any sequence \( g_n \in E \) such that \( g_n \to g \) and for each \( t \in [0, b] \). Then we get

\[
|(Qg_n)(t) - (Qg)(t)| \leq \frac{1 - q}{N(q)} |\mathcal{F}(t, g_n(t)) - \mathcal{F}(t, g(t))| + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} |\mathcal{F}(\zeta, g_n(\zeta)) - \mathcal{F}(\zeta, g(\zeta))| d\zeta.
\]

From the facts that \( \mathcal{F} \) is continuous, implies that \( \|Qg_n - Qg\| \to 0 \) as \( n \to +\infty \).

Step 2. Here, we show that the defined operator Q maps bounded set into bounded set.

Consider the set \( B_b \) as defined by \( B_b = \{ g \in E : \|g\| \leq k \} \), then it is enough to show that there exists \( \mathcal{P} > 0 \) such that \( \|Qg\| \leq \mathcal{P} \).

Now,

\[
|(Qg)(t)| \leq \|g_0\| + \frac{1 - q}{N(q)} \|\mathcal{F}(t, g(t))\| + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} |\mathcal{F}(\zeta, g(\zeta))| d\zeta,
\]

make use of condition (\mathcal{C}_1), implies

\[ |\mathcal{F}(t, g(t))| \leq \rho_1(t) + \rho_2(t)|g(t)|. \]

Thus,

\[
|(Qg)(t)| \leq \|g_0\| + \frac{1 - q}{N(q)} \|\mathcal{F}(t, g(t))\| + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} |\mathcal{F}(\zeta, g(\zeta))| d\zeta.
\]

Step 3. In this part, we shows that Q maps bounded sets into equicontinuous set.

Now, given any \( t_1, t_2 \in [0, b] \) with \( t_1 \geq t_2 \) and \( g \in B_b \), we have

\[
|\mathcal{F}(t_1, g(t_1)) - \mathcal{F}(t_2, g(t_2))| \leq \frac{1 - q}{N(q)} \int_0^t (t - \zeta)^{q-1} |\mathcal{F}(\zeta, g(\zeta))| d\zeta
\]

Thus, as a results of Arzelá Ascoli Theorem together with Step 1–3, shows the completely continuous of the operator Q.

Step 4. Finally, we show that Q is a priori bounds.

Consider the set \( U = \{ g \in E : g = \pi(Qg), 0 < \pi < 1 \} \). So, it is enough to shows that U is bounded.

Clearly, for each \( t \in [0, b] \) and \( g \in U \) gives

\[
g(t) = \pi\left( g_0 + \frac{1 - q}{N(q)} \mathcal{F}(t, g(t)) + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} \mathcal{F}(\zeta, g(\zeta)) d\zeta \right), 0 < \pi < 1.
\]

In view of (\mathcal{C}_1), we get

\[
|\mathcal{F}(t, g(t))| \leq \|\mathcal{F}(t, g(t))\| \leq \|g_0\| + \frac{1 - q}{N(q)} \|\mathcal{F}(t, g(t))\| + \frac{q}{N(q)} \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} |\mathcal{F}(\zeta, g(\zeta))| d\zeta
\]

Hence, the desired result. Therefore, as a consequence of Schaefer’s fixed point theorem, problem (4.3) (consequently, model (2.6)) has at least one solution. □

Based on the techniques of Banach contraction principle, we state and prove the uniqueness results.

**Theorem 4.2.** Suppose that:

(\mathcal{C}_1) There exists \( \mathcal{L} > 0 \) (constant) such that

\[ \mathcal{F}(t, g_1(t)) - \mathcal{F}(t, g_2(t)) \leq \mathcal{L}|g_1(t) - g_2(t)|, \text{ for every } t \in [0, b] \text{ and } g_1, g_2 \in E. \]

Then problem (4.3) (consequently, problem (2.6)) has a unique solution provided that

\[
\left( \frac{1 - q}{N(q)} + \frac{b^q}{N(q)\Gamma(q)} \right) \mathcal{L} < 1.
\]
Proof. Now, by setting $\sup_{t\in[0,\theta]} |\mathcal{F}(t, 0)| = L < +\infty$ and $k \geq \frac{\alpha}{\tau + \frac{\alpha}{2}}$, we construct a neighborhood of radius $k$. Next, we show that $Q B_\alpha \subset B_k$, where $B_k = \{g \in E : \|g\| \leq k\}$. So, for any $g \in B_k$ and $t \in [0, b)$, gives
\[
\|(Qg)(t)\| \leq \sup_{t\in[0,b)]} |g(t)| + \frac{1}{\tau + \frac{\alpha}{2}} |\mathcal{F}(t, g(t))|
\]
\[
+ \frac{1}{\tau + \frac{\alpha}{2}} \int_0^t (t - \xi)^{\alpha - 1} |\mathcal{F}(\xi, g(\xi))|d\xi
\]
\[
\leq \|g\| + \frac{1}{\tau + \frac{\alpha}{2}} |\mathcal{F}(t, g(t))|b
\]
\[
+ \frac{1}{\tau + \frac{\alpha}{2}} \int_0^t (b - \xi)^{\alpha - 1} |\mathcal{F}(\xi, g(\xi))|d\xi
\]
\[
\leq \|g\| + \frac{1}{\tau + \frac{\alpha}{2}} |\mathcal{F}(t, g(t)) - \mathcal{F}(t, 0)| + |\mathcal{F}(t, 0)|
\]
\[
+ \frac{1}{\tau + \frac{\alpha}{2}} \int_0^t (b - \xi)^{\alpha - 1} |\mathcal{F}(\xi, g(\xi)) - \mathcal{F}(t, 0)| + |\mathcal{F}(t, 0)||d\xi
\]
\[
\leq \|g\| + (L\|g\| + L)\left(\frac{1}{\tau + \frac{\alpha}{2}} + \frac{\mu}{\tau + \frac{\alpha}{2}}\right)
\]
\[
\leq \|g\| + (L\|g\| + L)\Phi
\].
\[
(4.13)
\]
Thus, $Q B_\alpha \subset B_k$. In addition, let $g_1, g_2 \in E$. Then for every $t \in [0, b)$, yields
\[
\|(Qg_1)(t) - (Qg_2)(t)\| \leq \frac{1}{\tau + \frac{\alpha}{2}} |\mathcal{F}(t, g_1(t)) - \mathcal{F}(t, g_2(t))|
\]
\[
+ \frac{1}{\tau + \frac{\alpha}{2}} \int_0^t (t - \xi)^{\alpha - 1} |\mathcal{F}(\xi, g_1(\xi)) - \mathcal{F}(\xi, g_2(\xi))|d\xi
\]
\[
\leq L\left(\frac{1}{\tau + \frac{\alpha}{2}} + \frac{\mu}{\tau + \frac{\alpha}{2}}\right)\|g_1 - g_2\|
\].
\[
(4.14)
\]
Thus, from the hypothesis of the theorem above, implies that the operator $Q$ is a contraction. Therefore, as a results of Banach contraction principle, problem (4.3) (consequently, problem (2.6)) has a unique solution. □

5. Iterative scheme and discussion

To gain insights into the dynamic behavior of each state variables associated with the proposed model (2.6), we employed the recent and effective numerical scheme proposed in[44]. The numerical methods for the fractional-order model used in the present analysis are presented below
\[
S_p(t_{n+1}) = S_p(t_n) + \frac{1}{\tau + \frac{\alpha}{2}} \mathcal{W}_1(t_n, S_p(t_n))
\]
\[
+ \sum_{k=0}^{n} \frac{\alpha \mathcal{W}_1(t_k, S_p(t_k))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha}(n - k + 2 + q)
\right.
\]
\[
-
\left. (n - k)^{\alpha}(n - k + 2 + 2q)\right]
\]
\[
- \frac{\alpha \mathcal{W}_1(t_n, S_p(t_n))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha+1} - (n - k)^{\alpha}(n - k + 1 + q)\right],
\]
\[
(5.1)
\]
\[
I_v(t_{n+1}) = I_v(t_n) + \frac{1}{\tau + \frac{\alpha}{2}} \mathcal{W}_2(t_n, I_v(t_n))
\]
\[
+ \sum_{k=0}^{n} \frac{\alpha \mathcal{W}_2(t_k, I_v(t_k))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha}(n - k + 2 + q)
\right.
\]
\[
-
\left. (n - k)^{\alpha}(n - k + 2 + 2q)\right]
\]
\[
- \frac{\alpha \mathcal{W}_2(t_n, I_v(t_n))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha+1} - (n - k)^{\alpha}(n - k + 1 + q)\right],
\]
\[
(5.2)
\]
\[
I_h(t_{n+1}) = I_h(t_n) + \frac{1}{\tau + \frac{\alpha}{2}} \mathcal{W}_3(t_n, I_h(t_n))
\]
\[
+ \sum_{k=0}^{n} \frac{\alpha \mathcal{W}_3(t_k, I_h(t_k))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha}(n - k + 2 + q)
\right.
\]
\[
-
\left. (n - k)^{\alpha}(n - k + 2 + 2q)\right]
\]
\[
- \frac{\alpha \mathcal{W}_3(t_n, I_h(t_n))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha+1} - (n - k)^{\alpha}(n - k + 1 + q)\right],
\]
\[
(5.3)
\]
\[
I_h(t_{n+1}) = I_h(t_n) + \frac{1}{\tau + \frac{\alpha}{2}} \mathcal{W}_4(t_n, I_h(t_n))
\]
\[
+ \sum_{k=0}^{n} \frac{\alpha \mathcal{W}_4(t_k, I_h(t_k))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha}(n - k + 2 + q)
\right.
\]
\[
-
\left. (n - k)^{\alpha}(n - k + 2 + 2q)\right]
\]
\[
- \frac{\alpha \mathcal{W}_4(t_n, I_h(t_n))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha+1} - (n - k)^{\alpha}(n - k + 1 + q)\right],
\]
\[
(5.4)
\]
\[
S_i(t_{n+1}) = S_i(t_n) + \frac{1}{\tau + \frac{\alpha}{2}} \mathcal{W}_5(t_n, S_i(t_n))
\]
\[
+ \sum_{k=0}^{n} \frac{\alpha \mathcal{W}_5(t_k, S_i(t_k))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha}(n - k + 2 + q)
\right.
\]
\[
-
\left. (n - k)^{\alpha}(n - k + 2 + 2q)\right]
\]
\[
- \frac{\alpha \mathcal{W}_5(t_n, S_i(t_n))}{\tau + \frac{\alpha}{2}} \left[(n + 1 - k)^{\alpha+1} - (n - k)^{\alpha}(n - k + 1 + q)\right],
\]
\[
(5.5)
\]
Now, we implement the numerical scheme presented above using some recently published results in the literature as specified in Table 2. The initial conditions were assumed as follows:

| Parameters | Numerical Value |
|-----------|----------------|
| $\Lambda_p$ | 1000 |
| $\Lambda_v$ | 2500 |
| $\mu_h$ | 0.0131 |
| $\mu_c$ | 199 $\times 10^{-4}$ |
| $\kappa_i$ | 1.001 |
| $\kappa_h$ | 1.005 |
| $\gamma$ | 0.005 |
| $\sigma$ | 0.002 |
| $\theta$ | 1.002 |
| $\nu$ | 1.002 |
| $\tau$ | 1.001 |
| $\gamma$ | 1.001 |
| $\lambda_c$ | 0.05 |
| $\lambda_h$ | 1.6 |
| $\lambda_v$ | 1.004 |

Table 2 Parameters values.
fractional order $q = 1, 0.9, 0.8$ as shown in Figs. 6 and 7. Using the parameters values in Table 2 and the proposed generalized model (2.6), we shows the dynamics outlook of each compartments as shows in Figs. 6 and 7. Reference to the simulated results, one can observe that in some compartments increase of the fractional-order leads to a decrease in the compartment and vice versa. In Fig. 7, the susceptible people $S_s$ decreases this is due to the interaction with the infected animals, infected people with HIV, and natural death rate. The infected people due to COVID-19 compartment $I_c$ increase at some time and then decrease and become stable, as this compartment incorporated with the regulation parameter, natural death rate in human, death rate of the animal due to COVID-19 and recovered people due to COVID-19 respectively as seen in compartment $I_c$. Similar behavior observed in compartment $I_h$ and $I_{hc}$ respectively. The susceptible animal compartment increase due to the birth rate and contact with the infected animal with COVID-19 as seen in Fig. 7 compartment $S_r$ while this behavior it also appears in compartment $I_r$. This interesting hidden dynamic behavior appears due to the fractional-order incorporated with the generalized proposed model (2.6). As a consequence, fractional calculus could help us understand the mechanism of transmission of the novel COVID-19 disease. In contrast, the numerical scheme concerned could be seen as an appropriate mechanism to achieve a numerical simulation of this kind of complicated model.

6. Conclusions

We have exploited in this paper the triple combination of Atangana-Baleanu fractional derivative together with a model of HIV-COVID-19 co-infection to globally assess the pandemic situation in many countries affected by both diseases, such as South Africa, Brazil, and many other countries. A way to achieve that has been to start by performing stability and bifurcation analysis. Hence, the process we used allowed us to look at which conditions the combined model comprises a forward and a backward bifurcation. To push the epidemic

Fig. 6 Profiles for behavior of each state variable of the generalized fractional model for the ABC-version.

Fig. 7 The dynamics of the state variable of the combine generalized fractional model with different values of the fractional order $q$. 

fractional order $q = 1, 0.9, 0.8$ as shown in Figs. 6 and 7. Using the parameters values in Table 2 and the proposed generalized model (2.6), we shows the dynamics outlook of each compartments as shows in Figs. 6 and 7. Reference to the simulated results, one can observe that in some compartments increase of the fractional-order leads to a decrease in the compartment and vice versa. In Fig. 7, the susceptible people $S_s$ decreases this is due to the interaction with the infected animals, infected people with HIV, and natural death rate. The infected people due to COVID-19 compartment $I_c$ increase at some time and then decrease and become stable, as this compartment incorporated with the regulation parameter, natural death rate in human, death rate of the animal due to COVID-19 and recovered people due to COVID-19 respectively as seen in compartment $I_c$. Similar behavior observed in compartment $I_h$ and $I_{hc}$ respectively. The susceptible animal compartment increase due to the birth rate and contact with the infected animal with COVID-19 as seen in Fig. 7 compartment $S_r$ while this behavior it also appears in compartment $I_r$. This interesting hidden dynamic behavior appears due to the fractional-order incorporated with the generalized proposed model (2.6). As a consequence, fractional calculus could help us understand the mechanism of transmission of the novel COVID-19 disease. In contrast, the numerical scheme concerned could be seen as an appropriate mechanism to achieve a numerical simulation of this kind of complicated model.

6. Conclusions

We have exploited in this paper the triple combination of Atangana-Baleanu fractional derivative together with a model of HIV-COVID-19 co-infection to globally assess the pandemic situation in many countries affected by both diseases, such as South Africa, Brazil, and many other countries. A way to achieve that has been to start by performing stability and bifurcation analysis. Hence, the process we used allowed us to look at which conditions the combined model comprises a forward and a backward bifurcation. To push the epidemic
investigation one step further, we have made use of techniques of Schaefer and Banach contraction principle. Moreover, results for the existence and uniqueness of solutions of the generalized fractional model have been presented. Lastly, we have solved numerically the generalized Atangana-Baleanu fractional HIV-COVID-19 model and have been able to provide a predicted prevalence for the COVID-19 in those countries severely affected by both diseases. Fortunately, the numerical simulations performed here to prove that the disease will stabilize at a later stage when adequate protective measures are taken. This paper improves the preceding works with the introduction of such a triple combination of Atangana-Baleanu operator with HIV and COVID-19 diseases which are still negatively impacting the lives of missions of people around the world.

Author contributions

The authors contributed equally in writing this article. All authors read and approved the final manuscripts.

Funding

Petchra Pra Jom Klao Doctoral Scholarship for Ph.D. program of King Mongkut’s University of Technology Thonburi (KMUTT), Thailand Science Research and Innovation (TSRI) Basic Research Fund: The fiscal year 2021 under project number 64A306000005.

The Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: The fiscal year 2021 under project number 64A306000005. These works were done while the first author visits Cankaya University, Ankara, Turkey. The first author was supported by the “Petchra Pra Jom Klao Ph.D. Research Scholarship from King Mongkut’s University of Technology Thonburi”. (For Ph.D.Petchra Pra Jom Klao Doctoral Scholarship) (Grant No. 13/2561).

References

[1] M.S. Abdo, K. Shah, H.A. Wahash, S.K. Panchal, On a comprehensive model of the novel coronavirus (covid-19) under mittag-leffler derivative, Chaos, Solitons Fractals (2020) 109867.
[2] A. Ahmed, B. Salam, M. Mohammad, A. Akgül, S.H. Khoshnaw, Analysis coronavirus disease (covid-19) model using numerical approaches and logistic model, AIMS Bioeng. 7 (3) (2020) 130–146.
[3] I. Ahmed, I.A. Baba, A. Yusuf, P. Kumam, W. Kumam, Analysis of caputo fractional-order model for covid-19 with lockdown, Adv. Diff. Eqns. 2020 (1) (2020) 1–14.
[4] I. Ahmed, P. Kumam, F. Jarad, P. Borisut, K. Sitthithakerngkiet, A. Ibrahim, Stability analysis for boundary value problems with generalized nonlocal condition via hilfer-katugampola fractional derivative, Adv. Diff. Eqns. 2020 (1) (2020) 1–18.
[5] I. Ahmed, P. Kumam, K. Shah, P. Borisut, K. Sitthithakerngkiet, M.A. Demba, Stability results for implicit fractional pantograph differential equations via φ-hilfer fractional derivative with a nonlocal riemann-liouville fractional integral condition, Mathematics 8 (1) (2020) 94.
[6] I. Ahmed, G.U. Modu, A. Yusuf, P. Kumam, I. Yusuf, A mathematical model of coronavirus disease (covid-19) containing asymptomatic and symptomatic classes, Res. Phys. 21 (2021) 103776.
[7] A. Akgül, A novel method for a fractional derivative with non-local and non-singular kernel, Chaos, Solitons Fractals 114 (2018) 478–482.
[8] E.K. Akgül, Solutions of the linear and nonlinear differential equations within the generalized fractional derivatives, Chaos: Interdiscip. J. Nonlinear Sci. 29 (2) (2019) 023108.
[9] M.A. Alqudah, T. Abdeljawad, K. Shah, F. Jarad, Q. Al-Mdallal, et al, Existence theory and approximate solution to prey–predator coupled system involving nonsingular kernel type derivative, Adv. Diff. Eqns. 2020 (1) (2020) 1–10.
[10] A. Atangana, Derivative with a New Parameter: Theory, Methods and Applications, Academic Press, 2015.
[11] A. Atangana, Fractional Operators with Constant and Variable Order with Application to Geo-hydrology, Academic Press, 2017.
[12] A. Atangana, Modelling the spread of covid-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination?, Chaos, Solitons Fractals 136 (2020) 109860.
[13] A. Atangana, A. Akgül, On solutions of fractal fractional differential equations, Discr. Contin. Dyn. Syst.-S (2018).
[14] A. Atangana, A. Akgül, Can transfer function and bode diagram be obtained from sumudu transform, Alexandria Eng. J. (2020).
[15] A. Atangana, A. Akgül, K.M. Owolabi, Analysis of fractal fractional differential equations, Alexandria Eng. J. (2020).
[16] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, Thermal Sci. 20 (2) (2016) 763–769.
[17] I.A. Baba, B.A. Nasidi, Fractional order epidemic model for the dynamics of novel covid-19, Alexandria Eng. J. (2020).
[18] K. Diethelm, A fractional calculus based model for the simulation of an outbreak of dengue fever, Nonlinear Dyn. 71 (4) (2013) 613–619.
[19] B. Ghanbari, S. Kumar, R. Kumar, A study of behaviour for immune and tumor cells in immunogenetic tumour model non-singular fractional derivative, Chaos, Solitons Fractals 133 (2020) 109619.
[20] E.F.D. Goufo, Y. Khan, Q.A. Chaudhry, Hiv and shifting epicenters for covid-19, an alert for some countries, Chaos, Solitons Fractals 110030, 2020.
[21] E.F.D. Goufo, S. Kumar, S. Mugisha, Similarities in a fifth-order evolution equation with and with no singular kernel, Chaos, Solitons Fractals 130 (2020) 109467.
[22] E.F.D. Goufo, H. Tenkam, M. Khumalo, A behavioral analysis of ddb equation under the law of mittag-leffler function, Chaos, Solitons Fractals 125 (2019) 139–145.
[23] A. Green, Li wenliang, Lancet (London, England) 395 (10225) (2020) 682.
[24] Q. Jianhang, T. Shen, Whistleblower Li Wenliang: There should be more than one voice in a healthy society. https://www.caixinglobal.com/2020-02-06/after-being-punished-by-local-police-coronavirus-whistleblower-vindicated-by-top-court-101509986.html consulted 2 June 2020, 2020.

[25] M.A. Khan, A. Atangana, Modeling the dynamics of novel coronavirus (2019-ncov) with fractional derivative, Alexandria Eng. J. (2020).

[26] A. Kilbas, H. Srivastava, J. Trujillo, Theory and applications of fractional derivational equations, North-Holland Math. Stud. 204 (2006).

[27] I. Koca, Modelling the spread of ebola virus with atangana-baleanu fractional operators, Eur. Phys. J. Plus 133 (3) (2018) 1–11.

[28] S. Kumar, A new analytical modelling for fractional telegraph equation via laplace transform, Appl. Math. Model. 38 (13) (2014) 3154–3163.

[29] S. Kumar, R. Chauhan, S. Momani, S. Hadid, Numerical investigations on covid-19 model through singular and non-singular fractional operators, Num. Methods Partial Diff. Eqs. (2020).

[30] S. Kumar, R. Kumar, R.P. Agarwal, B. Samet, A study of fractional lotka-volterra population model using haar wavelet and adams-bashforth-moulton methods, Math. Methods Appl. Sci. 43 (8) (2020) 5564–5578.

[31] S. Kumar, M.M. Rashidi, New analytical method for gas dynamics equation arising in shock fronts, Comput. Phys. Commun. 185 (7) (2014) 1947–1954.

[32] K. Muhammad Altaf, A. Atangana, Dynamics of ebola disease in the framework of different fractional derivatives, Entropy 21 (3) (2019) 303.

[33] K.M. Owolabi, A. Atangana, A. Akgul, Modelling and analysis of fractal-fractional partial differential equations: Application to reaction-diffusion model, Alexandria Eng. J. (2020).

[34] I. Petrakis, Fractional-order Nonlinear Systems: Modeling, Analysis and Simulation, Springer Science & Business Media, 2011.

[35] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications, vol. 198, Elsevier, 1998.

[36] L. Qianying et al, A conceptual model for the coronavirus disease 2019 (covid-19) outbreak in wuhan. China with individual reaction and governmental action, Int. J. Infect. Dis. 93 (2020) 211–216.

[37] S. Qureshi, A. Yusuf, A.A. Shaikh, M. Inc, Transmission dynamics of varicella zoster virus modeled by classical and novel fractional operators using real statistical data, Physica A 534 (2019) 122149.

[38] M. Radosin, The spanish flu, Part II: The second and third wave, Vojnosanitetski Pregled 69 (10) (2012) 917–927.

[39] K.M. Safare, V.S. Betageri, D.G. Prakash, P. Veeresha, S. Kumar, A mathematical analysis of ongoing outbreak covid-19 in india through nonsingular derivative, Num. Methods Partial Diff. Eqs. (2020).

[40] M.M. Sajadi, P. Habibzadeh, A. Vintzileos, S. Shokouhi, F. Miralles-Wilhelm, A. Amoroso, Temperature and latitude analysis to predict potential spread and seasonality for covid-19. Available at SSRN 3550308 (2020).

[41] K. Shah, T. Abdeljawad, I. Mahariq, F. Jarad, Qualitative analysis of a mathematical model in the time of covid-19, BioMed Res. Int. 2020 (2020).

[42] K. Shah, M.A. Alqudah, F. Jarad, T. Abdeljawad, Semi-analytical study of pine wilt disease model with convex rate under caputo-febrizio fractional order derivative, Chaos, Solitons Fractals 135 (2020) 109754.

[43] K. Shah, F. Jarad, T. Abdeljawad, On a nonlinear fractional order model of dengue fever disease under caputo-fabrizio derivative, Alexandria Eng. J. (2020).

[44] M. Toufilk, A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: application to chaotic models, Eur. Phys. J. Plus 132 (10) (2017) 444.

[45] R. ud Din, K. Shah, M.A. Alqudah, F. Jarad, Mathematical study of sir epidemic model under convex incidence rate,AIMS Mathe. 5 (6) (2020) 7548–7561.

[46] P. Van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Mathe. Biosci. 180 (1–2) (2002) 29–48.

[47] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, vol. 2, Springer Science & Business Media, 2003.

[48] C. Xu, Y. Yu, Q. Yang, Z. Lu, Forecast analysis of the epidemics trend of covid-19 in the united states by a generalized fractional-order sir model. arXiv preprint arXiv:2004.12541 (2020).
An epidemic prediction from analysis of a combined HIV-COVID-19
An epidemic prediction from analysis of a combined HIV-COVID-19