Poincaré gauge invariance and gravitation in Minkowski spacetime

C. Wiesendanger*
Dublin Institute for Advanced Studies
School of Theoretical Physics
10 Burlington Road
Dublin 4, Ireland

May 5, 1995

Abstract

A formulation of Poincaré symmetry as an inner symmetry of field theories defined on a fixed Minkowski spacetime is given. Local P gauge transformations and the corresponding covariant derivative with P gauge fields are introduced. The renormalization properties of scalar, spinor and vector fields in P gauge field backgrounds are determined. A minimal gauge field dynamics consistent with the renormalization constraints is given.

PACS numbers: 02.30.Mv,04.50.+h,11.10.Hi,11.30.Cp

DIAS-STP-95-18

*e-mail address: wie@stp.dias.ie
1 Introduction

In physics the description of a class of phenomena may often be based on different a priori conventions, hence yielding complementary pictures of these phenomena. In this context Poincaré pointed out the purely conventional role of spacetime geometry in the description of the behaviour of matter [1]. In fact two points of view are possible [2].

Either, one defines the line element $ds^2$ to be of Minkowskian form. Accordingly, in a gravitational field material rods will shrink and clocks slow down w.r.t. this metric. Hence, one defines the geometry of spacetime to be Minkowskian, whereas the behaviour of physical rods and clocks has to be determined by experiments.

Or, one defines rods or clocks to have one and the same length or period at any point of spacetime. Accordingly, a measurement of the line element $ds^2$ using these rods and clocks will yield that the geometry of spacetime is curved in general. This is the convention Einstein introduced to describe gravitation. Apart from global topological questions the two complementary points of view are equivalent.

The general theory of relativity and its extensions are based on the second point of view and yield a geometric description of the gravitational interaction [3]. This is incorporated in the theory by requiring that the behaviour of matter in gravitational backgrounds has to be described by equations which are form-invariant under the groups of general coordinate transformations and local $SO(1,3)$ frame rotations [3]. In this conception the gravitational field is closely connected to the metric tensor.

As long as one is interested in the macroscopic aspects of gravitation this point of view is very natural [4]. Its limitation shows up at the quantum level. It is very difficult to extend a picture so intimately related to classical concepts such as rods and clocks to a simple microscopic understanding of gravitation. In microphysics spacetime geometry enters only as a background concept necessary in defining a field theory. It cannot be subject to direct measurements in this context.

Hence, at the quantum level one is naturally led to the first point of view avoiding the interrelation of spacetime structure and gravitational phenomena. Here free matter is described by local, causal fields defined on Minkowski spacetime and its interactions are introduced using the gauge principle which allows a far-reaching generalization of the connection between conservation laws and global symmetry requirements [5].

To obtain a gauge theory of gravitation [6] one first ensures the conser-
vation of energy-momentum and angular momentum by the requirement of
global covariance of the free matter field theory under the Poincaré group P.
In a second step one gauges $P$ \cite{7}-\cite{21}, the translation subgroup $T$ \cite{22}-\cite{25}, the Lorentz subgroup $L$ \cite{26}, or even a larger group, e.g. \cite{27}-\cite{40}.

Most of the existing gauge theories of gravitation adopt the second point
of view yielding a geometric description of gravity \cite{7}-\cite{15}, \cite{22}. This is
related to the fact that $P$ is usually conceived partly as a spacetime partly
as an inner symmetry group. The local extension of its spacetime part
becomes then the diffeomorphism group, such that the gauged theory is
invariant under general coordinate transformations and local $SO(1,3)$ frame
rotations. This local symmetry group is then necessarily linked with the
geometry of spacetime.

Adopting the second point of view one can build up gauge theories
of gravitation erecting a principal bundle with $T$ \cite{24}-\cite{25}, $P$ \cite{16}-\cite{21} or some
other group as structural group \cite{35}-\cite{40}. One difficulty is now to link the
connection corresponding to the purely inner symmetry with the vierbein
and spin connection in gravity. Another one comes with choosing an action
for the gauge fields natural from the bundle point of view. It may come out
to be inconsistent with renormalization properties of matter fields in such
backgrounds.

Therefore we restrict ourselves to recast $P$ symmetry and its conse-
quences in the form of an inner symmetry (section 2) extending a previous
work on gauging the translations alone \cite{23}. This leads to a complementary
description of the global action of $P$ which is in complete analogy to the
description of the action of inner symmetry groups as groups of generalized
'rotations' in field space \cite{5}. In particular the coordinate system used to
specify the spacetime events is not affected anymore by $P$ transformations.

We next introduce local $P$ gauge transformations and demand the invari-
ance of physical processes under those (sections 3 and 4). This necessarily
leads to the existence of gauge fields with definite behaviour under local $P$
gauge transformations. Their coupling to any other field is essentially fixed
as in the case of other gauge field theories (section 5).

To obtain a gauge field dynamics consistent with renormalization prop-
erties of matter fields we next determine the changes of one-loop parti-
tion functions under rescaling (section 7). In a renormalizable theory the
anomalous contributions to these changes may be absorbed in the classical
actions for the different fields (e.g. \cite{41}). Using heat kernel methods and
the $\zeta$-function renormalization for one-loop determinants shortly presented
in appendices A and B we determine the contributions of the two gauge
fields explicitly. We finally give a minimal gauge field action consistent with theses contributions (section 8).

In order to get the interpretation of the resulting theory as a gauge theory of gravitation we show that it may be recast in usual geometrical terms replacing local $\mathbf{P}$ gauge invariance by invariance under general coordinate transformations and local $SO(1,3)$ frame rotations, the symmetry requirements in the general theory of relativity or its extensions (section 6). Hence, the $\mathbf{P}$ gauge theory of gravitation allows a complementary description of gravitational effects in which the mathematical structure of the underlying spacetime is not affected by physical events (in this context we refer to \[42\] - \[43\]).

We work on Minkowski spacetime $(\mathbb{R}^4, \eta)$ with Cartesian coordinates throughout, such that $\eta = \text{diag}(1, -1, -1, -1)$. Indices $\alpha, \beta, \gamma, \ldots$ from the first half of the Greek alphabet denote quantities defined on $(\mathbb{R}^4, \eta)$ which transform covariantly w.r.t. the Lorentz group. They are correspondingly raised and lowered with $\eta$.

# 2 Global Poincaré invariance as an inner symmetry

In this section we extend the two complementary conceptions of global translation invariance in field theory discussed in \[23\] to the full Poincaré group including fields with spin. The corresponding conserved currents, the canonical energy-momentum tensor $\Theta_{\gamma \alpha}$ and the angular momentum tensor $\mathcal{M}_{\gamma \alpha \beta}$ coincide in the two conceptions.

First let us state the Noether theorem in a quite general form. Consider a set of fields $\varphi_j(x)$ with $j = 1, \ldots, n$. Their dynamics shall be specified by the action $S_M = \int d^4x \ L_M(x, \varphi_j, \partial_\alpha \varphi_j)$. $\delta S_M = 0$ yields then the equations of motion. Consider further the infinitesimal transformations

$$ x^\alpha \rightarrow x'^\alpha = x^\alpha + \delta x^\alpha(x), \quad (1) $$
$$ \varphi_j(x) \rightarrow \varphi'_j(x') = \varphi_j(x) + \delta \varphi_j(x) $$

of the coordinates and the fields. If there are functions $\delta f^\gamma(x)$ for which

$$ d^4x' \ L_M(x', \varphi'_j(x'), \partial'_\alpha \varphi'_j(x')) = d^4x \ \{ L_M(x, \varphi_j(x), \partial_\alpha \varphi_j(x)) + \partial_\gamma \delta f^\gamma(x) \} \quad (2) $$
holds, then there is a conserved current $J^\gamma$ found to be

$$ J^\gamma = -\frac{\partial L_M}{\partial (\partial_\gamma \varphi_j)} \cdot \delta \varphi_j + \delta f^\gamma + \Theta^\gamma_{\alpha} \cdot \delta x^\alpha $$  \hspace{1cm} (3) $$

and, with appropriate boundary conditions, a conserved charge $Q$ given by

$$ Q = \int_{x^0 = \text{const}} d^3x \ J^0. $$  \hspace{1cm} (4) $$

The fields $\varphi_j$ above must obey the equations of motion. $\Theta^\gamma_{\alpha}$ is the canonical energy-momentum tensor

$$ \Theta^\gamma_{\alpha} = \frac{\partial L_M}{\partial (\partial_\gamma \varphi_j)} \cdot \partial_\alpha \varphi_j - \eta^\gamma_{\alpha} \cdot L_M. $$  \hspace{1cm} (5) $$

We apply this theorem now in two different ways to a globally Poincaré invariant theory of the fields $\varphi_j$ where $L_M$ thus depends on $x$ only through the fields.

The usual conception of Poincaré symmetry partly as a spacetime partly as an inner symmetry is expressed in the transformation formulae

$$ x^\alpha \rightarrow x'^\alpha = x^\alpha + \varepsilon^\alpha + \omega^\alpha_{\beta} x^\beta, \hspace{1cm} (6) $$

$$ \varphi_j(x) \rightarrow \varphi'_j(x') = \varphi_j(x) - \frac{i}{4} \omega^\alpha_{\beta} \Sigma_{\alpha\beta} \varphi_j(x). $$

$\delta x^\alpha = \varepsilon^\alpha + \omega^\alpha_{\beta} x^\beta$ is the change of $x$ under the combination of a global infinitesimal spacetime translation and a global infinitesimal Lorentz rotation, $\delta \varphi_j = -\frac{i}{4} \omega^\alpha_{\beta} \Sigma_{\alpha\beta} \varphi_j(x)$ the corresponding change of $\varphi_j$ in field space. $\Sigma_{\gamma\delta}$ are the representations of the generators of the Lie algebra so(1,3) in inner field space normalized to fulfil the commutation relations

$$ [\Sigma_{\gamma\delta}, \Sigma_{\varepsilon\zeta}] = 2i \{ \eta_{\delta\varepsilon} \Sigma_{\gamma\zeta} - \eta_{\delta\zeta} \Sigma_{\gamma\varepsilon} + \eta_{\gamma\varepsilon} \Sigma_{\zeta\delta} - \eta_{\gamma\zeta} \Sigma_{\varepsilon\delta} \}. $$  \hspace{1cm} (7) $$

One easily convinces oneself now that eqn.(2) holds for $\delta f^\gamma = 0$ and obtains with (3) the conserved current

$$ J^\gamma = \Theta^\gamma_{\alpha} \cdot \varepsilon^\alpha + \frac{1}{2} \mathcal{M}^\gamma_{\alpha\beta} \cdot \omega^\alpha_{\beta} $$  \hspace{1cm} (8) $$

where $\mathcal{M}^\gamma_{\alpha\beta}$ is the canonical angular momentum tensor

$$ \mathcal{M}^\gamma_{\alpha\beta} = \Theta^\gamma_{\alpha} x_\beta - \Theta^\gamma_{\beta} x_\alpha + \frac{i}{2} \frac{\partial L_M}{\partial (\partial_\gamma \varphi_j)} \Sigma_{\alpha\beta} \varphi_j. $$  \hspace{1cm} (9) $$
Next, we introduce global infinitesimal $\mathbf{P}$ gauge transformations

\[ x^\alpha \rightarrow x'^\alpha = x^\alpha, \]
\[ \varphi_j(x) \rightarrow \varphi'_j(x) = \varphi_j(x) - \{\varepsilon^\alpha + \omega^\alpha_{\beta x^\beta}\} \cdot \partial_\alpha \varphi_j(x) \] \[ - \frac{i}{4} \omega^{\alpha\beta} \Sigma_{\alpha\beta} \varphi_j(x). \]  

$\mathbf{P}$ acts now as a group of generalized 'phase rotations' in field space only and leaves the spacetime coordinates $x$ unchanged. As it is again a symmetry transformation of Poincaré invariant actions we are led to the complementary conception of Poincaré symmetry as a purely inner symmetry. We now have $\delta x^\alpha = 0, \delta \varphi_j = -\{\varepsilon^\alpha + \omega^\alpha_{\beta x^\beta}\} \cdot \partial_\alpha \varphi_j - \frac{i}{4} \omega^{\alpha\beta} \Sigma_{\alpha\beta} \varphi_j$ and

\[ d^4x L_M(\varphi'_j(x), \partial_\alpha \varphi'_j(x)) = \]
\[ d^4x \{L_M(\varphi_j(x), \partial_\alpha \varphi_j(x)) - \{\varepsilon^\gamma + \omega^\gamma_{\beta x^\beta}\} \cdot \partial_\gamma L_M(\varphi_j(x), \partial_\alpha \varphi_j(x))\} \]

so that eqn.(2) holds with $\delta f^\gamma = -\delta^\gamma_\alpha \cdot \{\varepsilon^\alpha + \omega^\alpha_{\beta x^\beta}\} L_M$. The conserved current is found to be the same $J^\gamma$ as in eqn.(8). This shows that the two complementary conceptions are equivalent w.r.t. their physical consequences. In both cases the conserved charges are found to be the energy-momentum

\[ P_\alpha = \int d^3x \Theta^0_\alpha \] \[ \text{(12)} \]

and the angular momentum

\[ M_{\alpha\beta} = \int d^3x M^0_{\alpha\beta}. \] \[ \text{(13)} \]

We have obtained a description of the global action of $\mathbf{P}$ which resembles very much the manner the action of well-known inner symmetries is usually described in field theory (see e.g. [5]). Let us now go one step further and gauge the Poincaré group $\mathbf{P}$ extending the discussion of gauging $\mathbf{T}$ in [23].

3 Local $\mathbf{P}$ gauge invariance. The covariant derivative $\tilde{\nabla}_\alpha$ and its decomposition w.r.t. $p_\gamma$ and $m_{\gamma\delta}$

In this section we introduce local $\mathbf{P}$ gauge transformations and the corresponding covariant derivative $\tilde{\nabla}_\alpha = \partial_\alpha + B_\alpha$ respecting the local $\mathbf{P}$ gauge symmetry. We give the decomposition of the compensating field $B_\alpha$ w.r.t.
the \( p \) generators \( p_\gamma \) and \( m_{\gamma\delta} \) and determine its behaviour under local \( P \)
gauge transformations.

In the previous section we recast \( P \) symmetry in the form of an inner
symmetry. Only in this conception it is possible to rewrite the formulae (10)
for global infinitesimal \( P \) gauge transformations as

\[
x^{\alpha} \rightarrow x'^{\alpha} = x^{\alpha}
\]

\[
\varphi_j(x) \rightarrow \varphi'_j(x) = \left((1 + \Theta)\varphi_j\right)(x).
\]

Hence, we can introduce in complete analogy to notions used in non-abelian
gauge field theory the unitary infinitesimal transformations in field space \((1 + \Theta)\)
forming a Lie group, where the infinitesimal hermitean gauge operators

\[
\Theta = -\{\varepsilon^\gamma + \omega^{\gamma\delta}x_\delta\} \cdot \partial_\gamma - \frac{i}{4}\omega^{\gamma\delta}\Sigma_{\gamma\delta}
\]

\[
= i\varepsilon^\gamma \cdot p_\gamma - i\frac{1}{2}\omega^{\gamma\delta} \cdot m_{\gamma\delta}
\]

are decomposed in two equivalent ways for later use. The corresponding
generators of the Lie algebra \( p \) in field space

\[
p_\gamma = i\partial_\gamma, \quad m_{\gamma\delta} = i(x_\gamma \partial_\delta - x_\delta \partial_\gamma) + \frac{1}{2}\Sigma_{\gamma\delta}
\]

are normalized to fulfil the usual commutation relations of the \( p \) generators

\[
[p_\gamma, p_\delta] = 0 \quad [p_\gamma, m_{\gamma\zeta}] = i\{\eta_{\gamma\zeta} p_\zeta - \eta_{\gamma\xi} p_\xi\}
\]

\[
[m_{\gamma\delta}, m_{\gamma\xi}] = i\{\eta_{\delta\xi} m_{\gamma\xi} - \eta_{\delta\xi} m_{\gamma\xi} + \eta_{\gamma\xi} m_{\gamma\gamma} - \eta_{\gamma\xi} m_{\xi\xi}\}.
\]

Above hermiticity and unitarity are understood w.r.t. the usual scalar product
in field space.

Let us extend now \( P \) to a Lie group of local infinitesimal gauge transformations allowing \( \varepsilon(x) \) and \( \omega(x) \) to vary with \( x \). We thus consider from now on

\[
\Theta(x) = -\{\varepsilon^\gamma(x) + \omega^{\gamma\delta}(x)x_\delta\} \cdot \partial_\gamma - \frac{i}{4}\omega^{\gamma\delta}(x)\Sigma_{\gamma\delta}
\]

\[
= i\varepsilon^\gamma(x) \cdot p_\gamma - i\frac{1}{2}\omega^{\gamma\delta}(x) \cdot m_{\gamma\delta}.
\]
Note that the algebra of the $\Theta(x)$ does close again. There is a new element of non-commutativity in the algebra of the $\Theta$ besides the one expressed in (17) as, contrary to the usual case, the local parameters $\varepsilon(x)$ and $\omega(x)$ don’t commute with the generators of the algebra given in (17). The emerging ordering problem is overcome by the convention that $\Theta(x)$ in its above form only acts to the right. This convention is motivated by demanding equivalence of the algebra of the $\Theta(x)$ to the diffeomorphism times $\mathfrak{so}(1,3)$ algebra.

In order to recast a given matter theory in a locally $\mathcal{P}$ gauge invariant form we must introduce a covariant derivative $\tilde{\nabla}_\alpha$. To be more precise we demand that the Lagrangian with covariant derivatives $\tilde{\nabla}_\alpha$ replacing the usual ones behave under local infinitesimal $\mathcal{P}$ transformations the same way it behaves with the usual derivatives $\partial_\alpha$ under global infinitesimal $\mathcal{P}$ transformations. Hence, we must ensure

\begin{equation}
\mathcal{L}_M(\varphi_j'(x), \tilde{\nabla}'_\alpha \varphi_j(x)) = \mathcal{L}_M(\varphi_j(x), \tilde{\nabla}_\alpha \varphi_j(x)) - \{\varepsilon^\gamma(x) + \omega^{\gamma\delta}(x) x_\delta\} \cdot \partial_\gamma \mathcal{L}_M(\varphi_j(x), \tilde{\nabla}_\alpha \varphi_j(x))
\end{equation}

where $\tilde{\nabla}_\alpha'$ denotes the gauge transformed covariant derivative. Note that (19) alone does not lead to the local $\mathcal{P}$ invariance of the original action $S_M = \int \mathcal{L}_M$ for the second term in eqn.(19) is no longer a pure divergence as it was in the case of global infinitesimal transformations.

As usual it is sufficient to construct a covariant derivative which fulfils

\begin{equation}
\tilde{\nabla}'_\alpha (1 + \Theta(x)) = (1 + \Theta(x)) \tilde{\nabla}_\alpha.
\end{equation}

Because $\tilde{\nabla}_\alpha$ transforms as a Lorentz vector we have to supplement the generators $\Sigma_{\gamma\delta}$ of $\mathfrak{so}(1,3)$ in matter field space occuring in the decomposition of $\Theta(x)$ with the corresponding generators $\Sigma_{\gamma\delta}$ acting on vectors to obtain the appropriate product representation as we will always do where necessary. For the infinitesimal transformations considered, eqn.(20) ensures indeed the proper transformation behaviour (19). As $\Theta(x)$ in eqn.(18) may be decomposed w.r.t. the Lie algebra generators as usual we are led to try the ansatz

\begin{equation}
\partial_\alpha \rightarrow \tilde{\nabla}_\alpha = \partial_\alpha + B_\alpha
\end{equation}

together with the decomposition of $B_\alpha$ w.r.t. $p_\gamma$ and $m_{\gamma\delta}$

\begin{equation}
B_\alpha \equiv -i B_\alpha ^\gamma \cdot p_\gamma + \frac{i}{2} B_\alpha ^{\gamma\delta} \cdot m_{\gamma\delta}
\end{equation}
introducing the 40 compensating fields \( B_\alpha^\gamma \) and \( B_\alpha^\gamma\delta \). We emphasize that this decomposition, relevant for any perturbative calculation, yields as fundamental compensating fields the 16 \( B_\alpha^\gamma \) for the local translations and the 24 \( B_\alpha^\gamma\delta \) for the local Lorentz rotations. It is only possible in the context of gauging the Poincaré group as an inner symmetry group. The spin generators \( \Sigma_{\gamma\delta} \) occurring in the decomposition of \( B_\alpha \) always have to be adjusted to the Lorentz group representation upon which they act, hence manifestly ensuring the covariant transformation behaviour of \( \tilde{\nabla}_\alpha \) throughout. Note that \( B_\alpha \) acts here not only as a matrix but also as a differential operator in field space.

We are ready now to discuss the behaviour of \( B_\alpha \) under local gauge transformations. Inserting the ansatz (21) for \( \tilde{\nabla}_\alpha \) in eqn. (20) we obtain the transformation law

\[
\delta B_\alpha = B'_\alpha - B_\alpha = [\Theta, \partial_\alpha + B_\alpha]
\]

(23)

where we remark that the second term in (23) just cancels the last term in

\[
\partial_\alpha \Theta = i \partial_\alpha \varepsilon^\gamma \cdot p_\gamma - \frac{i}{2} \partial_\alpha \omega^{\gamma\delta} \cdot m_{\gamma\delta} - \frac{i}{2} \omega^{\gamma\delta} \cdot \partial_\alpha m_{\gamma\delta}.
\]

Hence, there are only the derivatives of \( \varepsilon(x) \) and \( \omega(x) \) occuring above as expected. Eqn. (23) defines the representation of the local Poincaré group \( P \) in the gauge field space. Note the structural similarity of the results obtained up to now to similar ones in the discussion of non-abelian gauge symmetry [5]. Next we decompose \( \delta B_\alpha \) w.r.t. the generators \( p_\gamma \) and \( m_{\gamma\delta} \) of \( P \)

\[
\delta B_\alpha = -i \delta B_\alpha^\gamma \cdot p_\gamma + \frac{i}{2} \delta B_\alpha^{\gamma\delta} \cdot m_{\gamma\delta}.
\]

(24)

The quite lengthy evaluation of all the commutators shows that \( \delta B_\alpha \) has indeed the required decomposition and we obtain

\[
\delta B_\alpha^\gamma = \partial_\alpha^\gamma \varepsilon^\gamma + B_\alpha^\gamma \varepsilon^\gamma - B_\alpha^\gamma (x) \cdot \partial_\alpha (x) \cdot x_\gamma - \varepsilon^\gamma \cdot \partial_\alpha B_\alpha^\gamma
\]

(25)

and

\[
\delta B_\alpha^{\gamma\delta} = \partial_\alpha^{\gamma\delta} + B_\alpha^{\gamma\delta} - B_\alpha^{\gamma\delta} (x) \cdot \partial_\alpha (x) \cdot x_\gamma - \varepsilon^\gamma \cdot \partial_\alpha B_\alpha^{\gamma\delta}
\]

(26)

\[+ \omega^{\gamma\delta} \cdot \partial_\alpha B_\alpha^{\gamma\delta} + \omega^{\gamma\delta} B_\alpha^{\gamma\delta} \]

\[+ \omega^{\gamma\delta} B_\alpha^{\gamma\delta} + \omega^{\gamma\delta} B_\alpha^{\gamma\delta} \]

\[+ \omega^{\gamma\delta} B_\alpha^{\gamma\delta} + \omega^{\gamma\delta} B_\alpha^{\gamma\delta} \]
Compared with the corresponding transformation formula in the translation gauge invariant theory \[23\] eqns. (25) and (26) become quite involved and again strongly differ from the analogous ones in non-abelian gauge field theory.

4 The covariant derivative $\tilde{\nabla}_\alpha$ and its decomposition w.r.t. $\partial_\gamma$ and $\Sigma_{\gamma\delta}$

In this section we decompose the covariant derivative $\tilde{\nabla}_\alpha$ w.r.t. $\partial_\gamma$ and $\Sigma_{\gamma\delta}$ and recast it in terms of the effective gauge fields $e_\alpha^\gamma$ and $B_\alpha^{\gamma\delta}$. We introduce the field strength operator and determine its behaviour under local P gauge transformations.

The decomposition of the covariant derivative $\tilde{\nabla}_\alpha$ given in the previous section emphasizes the relation of the fundamental compensating fields to the Poincaré algebra $\mathfrak{p}$ and is a crucial tool for all perturbative calculations in the present approach. To obtain the covariant objects of the theory in a compact form, however, it is suitable to recast $\tilde{\nabla}_\alpha$ from eqns. (21), (22)

\[
\tilde{\nabla}_\alpha = e_\alpha^\gamma \partial_\gamma + \frac{i}{4} B_\alpha^{\gamma\delta} \Sigma_{\gamma\delta}
\]  

(27)

introducing the effective matrix fields $e_\alpha^\gamma$

\[
e_\alpha^\gamma \equiv \delta_\alpha^\gamma + B_\alpha^\gamma + B_\alpha^{\gamma\delta} x_\delta.
\]  

(28)

Note that this decomposition of $\tilde{\nabla}_\alpha$ corresponds just to the first way of expressing the local gauge operator $\Theta$ in eqn. (18). Abbreviating

\[
d_\alpha \equiv e_\alpha^\gamma \partial_\gamma, \quad B_\alpha \equiv \frac{i}{4} B_\alpha^{\gamma\delta} \Sigma_{\gamma\delta},
\]  

(29)

where $\Sigma_{\gamma\delta}$ must be properly adjusted to the Lorentz group representation it acts upon, we write $\tilde{\nabla}_\alpha = d_\alpha + B_\alpha$ from now on. $d_\alpha$ is just the translation covariant derivative introduced in \[23\].

As in our conception coordinate and P gauge transformations are strictly separated we emphasize that the introduction of $B_\alpha^\gamma$, $B_\alpha^{\gamma\delta}$ and $e_\alpha^\gamma$ has neither implications on the structure of the underlying spacetime which we assumed to be $(\mathbb{R}^4, \eta)$ endowed with the Minkowski metric $\eta$. Nor has it implications on the maximal symmetry group of $(\mathbb{R}^4, \eta)$, which is the Poincaré group if we still restrict ourselves to the use of Cartesian coordinates only.
This fact will allow one to obtain the energy-momentum and angular momentum of the gauge fields by an application of the Noether theorem given in section 1.

Let us now recast the quite involved transformation behaviour of $B_{\alpha}^{\gamma}$ and $B_{\alpha}^{\gamma\delta}$ under local $P$ gauge transformations in terms of $e_{\alpha}^{\gamma}$ and $B_{\alpha}^{\gamma\delta}$. With the use of eqns. (25) and (26) the variation of $e_{\alpha}^{\gamma}$ becomes quite simple

$$\delta e_{\alpha}^{\gamma} = e_{\alpha}^{\zeta} \cdot \partial_{\zeta} \{e^{\gamma} + \omega^{\gamma\delta} x_{\delta}\}$$

and is expressed in terms of $e_{\alpha}^{\gamma}$ only. For the variation of $B_{\alpha}^{\gamma\delta}$ we obtain the result

$$\delta B_{\alpha}^{\gamma\delta} = e_{\alpha}^{\zeta} \cdot \partial_{\zeta} \omega^{\gamma\delta} - \{e^{\zeta} + \omega^{\zeta\eta} x_{\eta}\} \cdot \partial_{\zeta} B_{\alpha}^{\gamma\delta}$$

As the determinant $\det e^{-1}$ will enter the locally $P$ invariant actions we give its transformation behaviour already here

$$\delta \det e^{-1} = - \det e^{-1} \cdot \partial_{\zeta} \{e^{\zeta} + \omega^{\zeta\eta} x_{\eta}\}$$

Before turning to the field strength operator we introduce the non-covariant decomposition

$$[d_{\alpha}, d_{\beta}] \equiv H_{\alpha\beta}^{\gamma} d_{\gamma}$$

as in [23]. $H_{\alpha\beta}^{\gamma}$ is expressed in terms of $e_{\alpha}^{\gamma}$ as

$$H_{\alpha\beta}^{\gamma} = e^{-1\gamma} e (e_{\alpha}^{\zeta} \cdot \partial_{\zeta} e_{\beta}^{\varepsilon} - e_{\beta}^{\zeta} \cdot \partial_{\zeta} e_{\alpha}^{\varepsilon})$$

where $e^{-1\gamma} e$ is the matrix inverse to $e_{\alpha}^{\gamma}$, i.e. $e_{\alpha}^{\varepsilon} \cdot e^{-1\gamma} e = \delta_{\alpha}^{\gamma}$.

This allows us now to obtain the field strength operator and its decomposition. Taking into account the vector character of $\tilde{\nabla}_{\alpha}$ we obtain after a little algebra

$$S_{\alpha\beta} \equiv [\tilde{\nabla}_{\alpha}, \tilde{\nabla}_{\beta}]$$

$$= H_{\alpha\beta}^{\gamma} d_{\gamma} - (B_{\alpha\beta}^{\gamma} - B_{\beta\alpha}^{\gamma}) d_{\gamma}$$

$$= d_{\alpha} B_{\beta} - d_{\beta} B_{\alpha} + [B_{\alpha}, B_{\beta}].$$
Introducing the tensor coefficients of $d _ \gamma$

$$T _ {\alpha \beta} ^ {\gamma} \equiv B _ {\alpha \beta} ^ {\gamma} - B _ {\beta \alpha} ^ {\gamma} - H _ {\alpha \beta} ^ {\gamma} \tag{36}$$

we may rewrite $S _ {\alpha \beta}$ as

$$[ \tilde{\nabla} _ \alpha , \tilde{\nabla} _ \beta ] = - T _ {\alpha \beta} ^ {\gamma} \tilde{\nabla} _ \gamma + \frac{i}{4} \tilde{R} ^ {\gamma \delta} _ {\alpha \beta} \Sigma _ {\gamma \delta} , \tag{37}$$

where $\tilde{R} ^ {\gamma \delta} _ {\alpha \beta}$ is found to be

$$\tilde{R} ^ {\gamma \delta} _ {\alpha \beta} \equiv \frac{i}{4} \tilde{R} ^ {\gamma \delta} _ {\alpha \beta} \Sigma _ {\gamma \delta} . \tag{38}$$

For later use we finally introduce the shorthand notation

$$\tilde{R} _ {\alpha \beta} \equiv \frac{i}{4} \tilde{R} ^ {\gamma \delta} _ {\alpha \beta} \Sigma _ {\gamma \delta} . \tag{39}$$

As $S _ {\alpha \beta}$ has a decomposition w.r.t. $\tilde{\nabla} _ \gamma$ and $\Sigma _ {\gamma \delta}$ it acts in general not only as a matrix but also as a first order differential operator in field space. Only if $B _ {\alpha} ^ {\gamma \delta}$ is related to $H _ {\alpha \beta} ^ {\gamma}$ the coefficient $T _ {\alpha \beta} ^ {\gamma}$ of the operator part in eqn. (37) does vanish. Denoting this particular choice of $B _ {\alpha} ^ {\gamma \delta}$ being of much importance later with $C _ {\alpha} ^ {\gamma \delta}$ the required relation becomes

$$C _ {\alpha \beta} ^ {\gamma} - C _ {\beta \alpha} ^ {\gamma} = H _ {\alpha \beta} ^ {\gamma} . \tag{40}$$

We may now solve for $C _ {\alpha} ^ {\gamma \delta}$ with the result

$$C _ {\alpha} ^ {\gamma \delta} = \frac{1}{2} \left( H _ {\alpha} ^ {\gamma \delta} - H _ {\alpha} ^ {\delta \gamma} - H ^ {\gamma \delta} _ {\alpha} \right) . \tag{41}$$

For the special choice $B _ {\alpha} ^ {\gamma \delta} = C _ {\alpha} ^ {\gamma \delta}$ we omit the tilde, hence writing

$$\nabla _ \alpha \equiv d _ \alpha + C _ \alpha . \tag{42}$$

Obviously we obtain for $S _ {\alpha \beta}$ a matrix only

$$[ \nabla _ \alpha , \nabla _ \beta ] = \frac{i}{4} R ^ {\gamma \delta} _ {\alpha \beta} \Sigma _ {\gamma \delta} \equiv R _ {\alpha \beta} \tag{43}$$

where

$$R ^ {\gamma \delta} _ {\alpha \beta} = d _ \alpha C _ {\beta} ^ {\gamma \delta} - d _ \beta C _ {\alpha} ^ {\gamma \delta} + C _ {\alpha} ^ {\delta \epsilon} C ^ {\gamma \delta} _ {\beta \epsilon} - C _ {\alpha \beta} ^ {\epsilon \delta} (C _ {\alpha} ^ {\epsilon \delta} - C _ {\beta} ^ {\epsilon \delta}) C _ {\epsilon \delta} \tag{44}$$
is now expressed in terms of $C_{\alpha}^{\gamma \delta}$.

By construction $S_{\alpha \beta}$ transforms homogeneously under infinitesimal local $P$ gauge transformations

$$S'_{\alpha \beta} (1 + \Theta(x)) = (1 + \Theta(x)) S_{\alpha \beta} \tag{45}$$

and thus

$$\delta S_{\alpha \beta} = S'_{\alpha \beta} - S_{\alpha \beta} = [\Theta, S_{\alpha \beta}] \tag{46}$$

Decomposition of eqn.(46) w.r.t. $\tilde{\nabla}_\gamma$ and $\Sigma_{\alpha \delta}$ together with the use of the known transformation law eqn. (30) for $e_{\alpha}^{\gamma}$ leads to

$$\delta T_{\alpha \beta}^{\gamma} = -\{\varepsilon^{\zeta} + \omega^{\zeta \eta} x_\eta\} \cdot \partial_\zeta T_{\alpha \beta}^{\gamma} \tag{47}$$

$$+ \omega_\alpha^{\zeta} T_{\zeta \beta}^{\gamma} + \omega_\beta^{\zeta} T_{\alpha \zeta}^{\gamma} + \omega^{\gamma \zeta} T_{\alpha \beta}^{\zeta}$$

and to

$$\delta \tilde{R}_{\alpha \beta}^{\gamma \delta} = -\{\varepsilon^{\zeta} + \omega^{\zeta \eta} x_\eta\} \cdot \partial_\zeta \tilde{R}_{\alpha \beta}^{\gamma \delta} \tag{48}$$

$$+ \omega_\alpha^{\zeta} \tilde{R}_{\zeta \beta}^{\gamma \delta} + \omega_\beta^{\zeta} \tilde{R}_{\alpha \zeta}^{\gamma \delta} + \omega^{\gamma \zeta} \tilde{R}_{\beta \alpha}^{\zeta \delta}.$$  

$T_{\alpha \beta}^{\gamma}$ and $\tilde{R}_{\alpha \beta}^{\gamma \delta}$ transform homogeneously under infinitesimal local $P$ gauge transformations. We emphasize that the choice $T_{\alpha \beta}^{\gamma} = 0$ is indeed a gauge covariant statement as we implicitly assumed above introducing $C_{\alpha}^{\gamma \delta}$. As long as one works with regularizations respecting the gauge symmetry, as we will do later on, it is always possible to work consistently under the constraint $T = 0$.

For later use we finally introduce the difference of the two gauge fields

$$K_{\alpha}^{\gamma \delta} \equiv B_{\alpha}^{\gamma \delta} - C_{\alpha}^{\gamma \delta} \tag{49}$$

which is related to $T_{\alpha \beta}^{\gamma}$ as

$$K_{\alpha \beta}^{\gamma} - K_{\beta \alpha}^{\gamma} = T_{\alpha \beta}^{\gamma} \tag{50}$$

with the obvious inversion

$$K_{\alpha \gamma \delta} = \frac{1}{2} (T_{\alpha \gamma \delta} - T_{\alpha \delta \gamma} - T_{\gamma \delta \alpha}). \tag{51}$$
5 P gauge invariant matter actions. Scalar, spinor and vector fields as examples

In this section we discuss the extension of globally P gauge invariant matter actions on \((\mathbb{R}^4, \eta)\) to locally P gauge invariant ones. We then apply the general framework to a scalar, spinor and vector field action in turn and determine their respective locally P gauge invariant forms.

Let us consider a globally P gauge invariant theory for \(n\) fields \(\varphi_j\) specified by the Lagrangian density \(L_M(\varphi_j, \partial_\alpha \varphi_j)\) assumed to be real \(L^*_M = L_M\). In section 3 we have constructed a covariant derivative \(\tilde{\nabla}_\alpha\) which respects the behaviour of \(L_M\) under global Poincaré gauge transformations extended to local ones as expressed in eqn.(19). But as we already mentioned (19) is not yet sufficient for the original action \(S_M = \int L_M\) to be locally P gauge invariant.

We have to complete the Lagrangian density with another term ensuring that the change of both parts together under a local P transformation will yield a pure divergence only. Using the transformation law (32) for \(\det e\) we get for the behaviour of the combination

\[
\det e^{-1} \cdot L_M(\varphi_j, \tilde{\nabla}_\alpha \varphi_j)
\]

under local P gauge transformations

\[
\det e'^{-1} \cdot L_M(\varphi'_j, \tilde{\nabla}'_\alpha \varphi'_j) = \det e^{-1} \cdot L_M(\varphi_j, \tilde{\nabla}_\alpha \varphi_j)
\]

\[-\partial_\gamma \left\{ e^\gamma(x) + \omega^\gamma_\delta(x) x_\delta \right\} \det e^{-1} \cdot L_M(\varphi_j, \tilde{\nabla}_\alpha \varphi_j)\],

i.e. the change of the combination (52) is indeed a pure divergence.

Therefore the minimally extended locally P gauge invariant matter action becomes

\[
S_M = \int d^4x \det e^{-1}(x) \cdot L_M(\varphi_j(x), \tilde{\nabla}_\alpha \varphi_j(x)).
\]

Of course, \(S_M\) remains invariant if we change from one to another inertial system by global coordinate translations or Lorentz rotations.

It is the conception of P symmetry as an inner symmetry together with the gauge principle which has led us to this minimal coupling prescription. In this conception the gauge fields and their transformation behaviour do not interfere with the spacetime structure \((\mathbb{R}^4, \eta)\) fixed by an a priori convention and the underlying geometry remains separated from the physics described...
by the $P$ gauge fields in the same manner it remains separated from the physics described by any usual matrix gauge field.

For later use we turn now to apply the general framework developed so far to a real massive scalar field, a massive Dirac spinor and a massive vector field. The globally $P$ invariant action for the scalar field $\varphi$ is given by

$$S_M = \int d^4x \left\{ \frac{1}{2} \partial_\alpha \varphi \cdot \partial^\alpha \varphi - \frac{1}{2} m^2 \varphi^2 \right\}. \quad (55)$$

The $\mathfrak{so}(1,3)$ generators are trivial and the $P$ covariant derivative is independent of $B_{\alpha \gamma \delta}$ as is the minimal extension of (55) to a locally $P$ gauge invariant action

$$S_M = \int d^4x \det e^{-1} \left\{ \frac{1}{2} d_\alpha \varphi \cdot d^\alpha \varphi - \frac{1}{2} m^2 \varphi^2 \right\}. \quad (56)$$

We obtain the same result as in the case of pure $T$ gauge invariance [23]. Note that in the presence of $e_{\alpha \gamma}$ the scalar product in real scalar field space becomes now $(\chi, \varphi)^e = \int d^4x \det e^{-1} \chi \cdot \varphi$.

The globally $P$ invariant action for a Dirac spinor with real Lagrangian density is given by

$$S_M = \int d^4x \left\{ \frac{i}{2} \bar{\psi} \gamma^\alpha (\partial_\alpha \psi) - \frac{i}{2} (\bar{\partial}_\alpha \psi) \gamma^\alpha \psi - m \bar{\psi} \psi \right\}. \quad (57)$$

The Dirac matrices fulfill the usual Clifford algebra $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}$ and the $\mathfrak{so}(1,3)$ generators become $\Sigma_{\alpha\beta} = \frac{i}{2} [\gamma_\alpha, \gamma_\beta]$. The minimal extension prescription yields the locally $P$ gauge invariant action

$$S_M = \int d^4x \det e^{-1} \left\{ \frac{i}{2} \bar{\psi} \gamma^\alpha (\bar{\nabla}_\alpha \psi) - \frac{i}{2} (\bar{\nabla}_\alpha \psi) \gamma^\alpha \psi - m \bar{\psi} \psi \right\}. \quad (58)$$

Due to spin $B_{\alpha \gamma \delta}$ enters now the action. We will further investigate this below in the context of quantum field theoretical considerations. Partially integrating $\bar{\nabla}_\alpha$ in the second term above leads to the usual form of the Dirac action

$$S_M = \int d^4x \det e^{-1} \bar{\psi} \left\{ i \gamma^\alpha (\bar{\nabla}_\alpha - \frac{1}{2} \gamma^\alpha K) - m \right\} \psi. \quad (59)$$

Note the occurrence of the tensor $K$ ensuring the hermiticity of the $P$ covariant Dirac operator w.r.t. $(\chi, \psi)^e = \int d^4x \det e^{-1} \chi \cdot \psi$.

We turn to the last example. The globally $P$ invariant action for a massive vector field is given by

$$S_M = \int d^4x \left\{ -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} m^2 A_\alpha A^\alpha \right\}. \quad (60)$$
where the field strength reads $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. The $\text{so}(1,3)$ generators in the vector representation are $(\Sigma_{\alpha\beta})^{\gamma\delta} = 2i(\eta_\alpha \eta_\gamma \eta_\beta \delta - \eta_\alpha \delta \eta_\beta \gamma)$ and the minimal extension of the action (39) to a locally \( P \) gauge invariant action yields

$$S_M = \int d^4x \det e^{-1} \left\{ -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} m^2 A_\alpha A^\alpha \right\}$$

(61)

together with the covariant field strength $F_{\alpha\beta} = \tilde{\nabla}_\alpha A_\beta - \tilde{\nabla}_\beta A_\alpha$. The covariant derivative in the vector representation is simply $\tilde{\nabla}_\alpha A_\beta = d_\alpha A_\beta - B_{\alpha\beta} \gamma A_\gamma$. We remark that under the \( P \) gauge covariant $U(1)$ gauge transformation

$$A_\alpha \rightarrow A_\alpha + \nabla_\alpha A$$

the field strength is in general not invariant

$$F_{\alpha\beta} \rightarrow F_{\alpha\beta} - T_{\alpha\beta} \gamma \nabla_\gamma A.$$

Only for $T = 0$ the Maxwell action with $m = 0$ is $U(1)$ gauge invariant. The scalar product in vector field space $(A, B)_\epsilon = -\int d^4x \det e^{-1} A_\alpha \eta^{\alpha\beta} B_\beta$ is chosen such that the physical polarizations have positive norm.

All the examples above and non-minimal extensions are discussed in the geometrical framework e.g. in [44].

6 Invariance under coordinate transformations and frame rotations as complementary conception

In this section the concept of local \( P \) gauge invariance is shown to be physically equivalent to the usual concepts of coordinate and local Lorentz invariance. This equivalence allows us to re-interpret the formalism in common geometrical terms.

Up to now we have relied on the conception of \( P \) symmetry as an inner symmetry expressed in the transformation behaviour eqn.(14) for matter fields and eqns. (30), (31) for the gauge fields. It allowed us to extend the framework of gauge theories of matrix groups to the operator gauge group \( P \). As the Poincaré group of global spacetime transformations relating different observers and the local gauge group \( P \) were strictly separated the a priori geometry of spacetime $(\mathbb{R}^4, \eta)$ chosen to be Minkowskian was not affected by the introduction of the \( P \) gauge fields $e_\alpha \gamma$ and $B_\alpha \gamma \delta$.

We turn now to the complementary conception of Poincaré symmetry partly as a spacetime partly as an inner symmetry $(\mathbb{I}, [I])$ and introduce
besides the orthonormal indices $\alpha, \beta, \gamma, \ldots$ used up to now the coordinate indices $\mu, \nu, \rho, \ldots$. The infinitesimal transformation formulae involve now both coordinates and fields as in eqn.(6)

$$x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu(x),$$

$$\varphi_j(x) \rightarrow \varphi'_j(x') = \varphi_j(x) - \frac{i}{4} \omega^{\alpha \beta}(x) \Sigma_{\alpha \beta} \varphi_j(x).$$

Note that $\varepsilon^\mu(x)$ parametrizes a general infinitesimal coordinate transformation and may contain effects of local translations as well as local Lorentz rotations of the coordinates which may actually no longer be distinguished (eqns. [7]-[15]). $\omega^{\alpha \beta}(x) = -\omega^{\beta \alpha}(x)$ parametrizes now a local orthonormal frame rotation and is in no respect related to $\varepsilon^\mu(x)$. This becomes manifest if we rewrite eqn.(30) in the equivalent form

$$e_\alpha^\mu(x) \rightarrow e'^\alpha_\mu(x') = (\delta_\alpha^\beta + \omega_\alpha^\beta)e_\beta^\nu(x)(\delta_\nu^\mu + \partial_\nu \varepsilon^\mu).$$

We find that coordinate indices $\mu, \nu, \rho, \ldots$ are transformed with the Jacobi matrix $\frac{\partial x'^\mu}{\partial x^\nu} = \delta_\nu^\mu + \partial_\nu \varepsilon^\mu$ resulting from the infinitesimal coordinate change and orthonormal indices with the infinitesimal frame rotation $\delta_\alpha^\beta + \omega_\alpha^\beta$.

Hence, the gauge group $P$ and the requirement of local $P$ gauge invariance are replaced by the groups of general (infinitesimal) coordinate transformations and local $SO(1, 3)$ frame rotations and the requirement of invariance under these groups $P$, $\tilde{P}$. Indeed, in many other gauge approaches to gravitation some combination of these two groups is used as the gauge group $P$-$\tilde{P}$.

As a consequence $e_\alpha^\mu$ has to be re-interpreted as vierbein and defines a metric tensor

$$g^{\mu \nu} = e_\alpha^\mu e^{\alpha \nu}. \quad (64)$$

The geometry of spacetime is now necessarily linked with the above discussed complementary symmetry requirements and Riemannian geometry becomes the natural framework to deal with this point of view. The geometric notations used here correspond to those in [15]. $\partial_\mu$ and $\hat{e}_\alpha \equiv d_\alpha$ fit in as coordinate and orthonormal non-coordinate basis vectors in the tangential spaces belonging to the Riemannian manifold $(\mathbb{R}^4, \hat{g})$. The manifold is endowed with the indefinite metric $g^{\mu \nu}$ defined in eqn.(64). $e_\alpha^\mu$ enters as the vierbein relating the two basis systems as $\hat{e}_\alpha = e_\alpha^\mu \partial_\mu$ and $c_{\alpha \beta \gamma} \equiv H_{\alpha \beta \gamma}$ as the anholonomy coefficients fulfilling $[\hat{e}_\alpha, \hat{e}_\beta] = c_{\alpha \beta \gamma} \hat{e}_\gamma$. The connection coefficients w.r.t. the frame $\hat{e}_\alpha$ are then to be identified as $\Gamma^{\gamma}_{\alpha \delta} \equiv -B_{\alpha \gamma \delta}$. Comparison with eqn.(31) shows that they transform indeed in the usual
way. Note that the antisymmetry of $B_{\alpha \gamma\delta}$ w.r.t. $\gamma$ and $\delta$ translates into the metric compatibility condition for the connection $\Gamma$. Hence, the $P$ gauge fields are always related to metric connections.

Introducing next the one-form basis $\hat{\theta}^\alpha$ dual to $\hat{e}_\alpha$ in cotangential space we can turn over to the Cartan formalism defining the connection one-form $\omega^\gamma_\delta \equiv \Gamma^\gamma_{\alpha \delta} \hat{\theta}^\alpha$ and may apply the subsequent calculus.

The components of the Riemann tensor are defined by the second of Cartan’s structure equations

$$d\omega^\gamma_\delta + \omega^\gamma_\varepsilon \wedge \omega^\varepsilon_\delta \equiv R^\gamma_\Gamma_\delta = \frac{1}{2} R^\gamma_\Gamma_{\delta \alpha \beta} \hat{\theta}^\alpha \wedge \hat{\theta}^\beta$$

and related as $R^\gamma_\Gamma_{\alpha \beta} = -\hat{R}^\gamma_{\alpha \beta}$ to the field strength components introduced in eqn.(65). The components of the torsion tensor given by the first Cartan structure equation

$$d\hat{\theta}^\gamma + \omega^\gamma_\varepsilon \wedge \hat{\theta}^\varepsilon \equiv T^\gamma_\Gamma = \frac{1}{2} T^\gamma_\Gamma_{\alpha \beta} \hat{\theta}^\alpha \wedge \hat{\theta}^\beta$$

are related to the components of $T$ introduced in eqn.(66) as $T^\gamma_\Gamma_{\alpha \beta} = -T_{\alpha \beta \gamma}$. We remark that the components of the tensor $K$ introduced in eqn.(49) translate into the contortion components $K^\alpha_\Gamma_{\gamma \delta} = -K_{\gamma \delta}^{\alpha \gamma \delta}$. The gauge potential $C_\alpha = \frac{i}{4} C_\alpha \gamma^\delta \Sigma_{\gamma \delta}$ thus corresponds just to the torsion free Levi-Civit\`a connection $\Gamma^\gamma_\Gamma_{\alpha \beta \delta} = -C_\alpha \gamma^\delta$. The $P$ covariant derivative finally becomes $\nabla_\alpha \equiv e_\alpha^\mu (\partial_\mu - \frac{i}{4} \Gamma^\gamma_\mu \gamma^\delta \Sigma_{\gamma \delta})$, i.e. the coordinate invariant and $SO(1,3)$ covariant derivative introduced e.g. in general relativity [15].

Hence, we have established the physical equivalence of our formulation to the geometrical introduction of gravitational interactions in the general theory of relativity relying on the principle of equivalence. This allows us to interpret the fundamental gauge fields $B_{\alpha \gamma}^\gamma$ and $B_{\alpha \gamma\delta}$ finally as gravitational potentials. The incorporation of the principle of equivalence in the present approach has been discussed in [23].

7 Matter partition functions in gauge field backgrounds and their scaling behaviour

In this section we express the scaling behaviour of the one-loop partition functions for the scalar, spinor and vector fields in the presence of $e_\alpha^\gamma$ and $B_{\alpha \gamma\delta}$ in terms of the $\zeta$-function belonging to the appropriate matter fluctuation operators.
The assumption that the interactions of the $P$ gauge fields with the different matter fields are renormalizable imposes strong conditions on the classical gauge field dynamics. For let us suppose that a given theory for a matter field and the gauge fields $B_\alpha^\gamma$ and $B_\alpha^{\gamma\delta}$ is perturbatively renormalizable. Then we know that the change of the partition function of the whole system under rescaling can be absorbed in its classical action yielding at most a nontrivial scale dependence of the different couplings, masses and wavefunction normalizations. Hence, the explicit computation of the change of the one-loop matter partition functions under rescaling will allow us to constrain the classical gauge field dynamics.

As technical subtleties arising in the necessary computations have already been discussed elsewhere \[41\] we may turn to the evaluation of the one-loop partition functions and their changes under rescaling for the locally $P$ invariant scalar, spinor and vector theories introduced in section 5.

The contribution of the scalar field to the partition function is given by

$$Z_{\varphi}[e] = \int \mathcal{D}\varphi e^{iS_M(\varphi; e)}. \tag{67}$$

Note that we omit possible normalizations in order to obtain the most general renormalization structure later. After a partial integration we may rewrite eqn.(54) for $S_M$ as

$$S_M(\varphi; e) = \frac{1}{2} (\varphi, M_\varphi(e)e) \tag{68}$$

introducing the hermitean hyperbolic fluctuation operator

$$M_\varphi(e) \equiv -\nabla_\alpha \nabla^\alpha - m^2. \tag{69}$$

Performing next the Gaussian integration formally yields

$$Z_{\varphi}[e] = e^{-\frac{1}{2} \log \det M_\varphi(e)}. \tag{70}$$

As we are only interested in the behaviour of $Z_{\varphi}[e]$ under rescaling the most suited renormalization of the ultraviolet divergent determinants above is based on the $\zeta-$function as it is a manifestly gauge invariant technique.

The scalar contribution to the partition function normalized at scale $\mu$ becomes with the use of eqn.(136) from appendix B

$$Z_{\varphi}[\mu; e] = e^{\frac{1}{2} \zeta'(0; \mu; M_\varphi(e))}. \tag{71}$$
Let us finally consider this contribution at the new scale $\tilde{\mu} = \lambda \mu$ and determine the corresponding change of $Z_{\psi}$. With the help of eqn. (139) this change becomes

$$Z_{\psi}[\tilde{\mu}; e] = Z_{\psi}[\mu; e] \cdot e^{\log \lambda \zeta(0; \mu; M_{\psi}(e))}.$$  \hfill (72)

We turn to the contribution of the spinor to the partition function. It is given by the Grassmann functional integral

$$Z_{\psi}[e, B] = \int D\overline{\psi} D\psi \cdot e^{iS_M(\overline{\psi}, \psi; e, B)}.$$  \hfill (73)

As $S_M$ is already of the usual quadratic form we may perform the Grassmann integral and formally obtain

$$Z_{\psi}[e, B] = e^{\frac{1}{2} \log \det M_{\psi}(e, B)}.$$  \hfill (74)

The hyperbolic fluctuation operator in the spinor case is obtained as usual by squaring the Dirac operator introduced in eqn. (59)

$$M_{\psi}(e, B) \equiv -\gamma^\alpha (\overline{\nabla}_\alpha - \frac{1}{2} T_{\gamma\alpha} \gamma) \cdot \gamma^\beta (\overline{\nabla}_\beta - \frac{1}{2} T_{\delta\beta} \delta) - m^2$$  \hfill (75)

and is hermitean w.r.t. $(,)_e$ due to the occurrence of $T$. We have to recast $M_{\psi}$ in the form of the general second order $P$ covariant operator considered in appendix A. Using $[\overline{\nabla}_\alpha, \gamma^\beta] = 0$ and $\gamma^\alpha \gamma^\beta = \eta^{\alpha\beta} - i \Sigma^{\alpha\beta}$ we obtain

$$M_{\psi}(e, B) = -\left( \overline{\nabla}_\alpha - \frac{1}{2} T_{\gamma\alpha} \gamma \right) \left( \overline{\nabla}^\alpha - \frac{1}{2} T^{\alpha\delta}_\delta \right)$$  \hfill (76)

$$+ \frac{i}{2} \Sigma^{\alpha\beta} \left( -T_{\alpha\beta} \delta \overline{\nabla}_\delta + \check{R}_{\alpha\beta} - \overline{\nabla}_\alpha K_{\delta\beta} \delta \right) - m^2,$$

where the matrix in spinor space $\check{R}_{\alpha\beta}$ has been defined in eqn. (39). Next we write $T_{\alpha} \equiv \frac{i}{4} T^{\gamma\delta} \gamma_{\alpha} \gamma_{\delta}$ and absorb the first order derivative term $-2T_{\alpha} \overline{\nabla}^\alpha$ in the second order one. Together with the use of the Jacobi identities for the covariant derivative $\overline{\nabla}_\alpha$ we then find the manifestly hermitean result

$$M_{\psi}(e, B) = -\left( \overline{\nabla}_\alpha + T_{\alpha} - \frac{1}{2} T_{\gamma\alpha} \gamma \right) \left( \overline{\nabla}^\alpha + T^\alpha - \frac{1}{2} T^{\alpha\delta}_\delta \right)$$  \hfill (77)

$$+ T_{\alpha} T^\alpha + \frac{i}{2} \Sigma^{\alpha\beta} \left( \check{R}_{\alpha\beta} + \check{R}^{\delta}_{\alpha\delta\beta} \right) - m^2.$$

To obtain the form discussed in appendix A we finally introduce

$$D_{\alpha} \equiv \nabla_{\alpha} + B_{\alpha} + T_{\alpha}$$  \hfill (78)
where \(B_\alpha\) shall only act on spinor indices and \(C_\alpha\) in \(\nabla_\alpha\) only on vector indices. With its use we obtain the desired form

\[
M_\psi(e, B) = -D_\alpha D_\alpha + \frac{1}{2} \nabla_\alpha T_\gamma \gamma - \frac{1}{4} T_\gamma_\alpha \gamma T_\delta \alpha_\delta \tag{79}
\]

Choosing \(B = C\) in eqn.(79) finally reduces \(M_\psi\) to the much simpler form

\[
M_\psi(e, C) = -D_\alpha D_\alpha + \frac{i}{2} \Sigma_\alpha \Sigma_\beta (\tilde{R}_\alpha_\beta + \tilde{R}_\beta_\alpha) - m^2. \tag{80}
\]

We now use eqn.(136) from appendix B to give the spinor contribution to the partition function normalized at scale \(\mu\)

\[
Z_\psi[\mu; e, B] = e^{-\frac{1}{2} \zeta(0; \mu; M_\psi(e, B))}. \tag{81}
\]

With the help of eqn.(139) we may finally express the change of \(Z_\psi\) corresponding to a change of scale \(\tilde{\mu} = \lambda \mu\) as

\[
Z_\psi[\tilde{\mu}; e, B] = Z_\psi[\mu; e, B] \cdot e^{-\log \lambda \zeta(0; \mu; M_\psi(e, B))}. \tag{82}
\]

The contribution of the vector field to the partition function is formally given by

\[
Z_A[e, B] = \int DA_\alpha e^{iS_M(A; e, B)} \tag{83}
\]

First we have to recast \(S_M\) and obtain after partially integrating \(\tilde{\nabla}_\alpha\) and re-arranging the different terms

\[
S_M(A; e, B) = \frac{1}{2} \int d^4x \det e^{-1} A_\xi \{ (\tilde{\nabla}_\alpha - T_\gamma_\alpha) \tilde{\nabla}_\alpha \eta_\xi - T_\gamma_\alpha \tilde{\nabla}_\alpha - \tilde{\nabla}_\gamma_\alpha \gamma + \tilde{R}_\xi + m^2 \eta_\xi \} A_\xi \tag{84}
\]

where \(\tilde{R}_\xi\) is a matrix in vector field space defined in eqn.(39). Using again \(T_\alpha = \frac{i}{4} T_\xi_\alpha \eta_\xi\) and absorbing the first order in the second order derivative term we can write the final result as

\[
S_M(A; e, B) = -\frac{1}{2} (A, M_{AA}(e, B) A) \tag{85}
\]

\[+ \frac{1}{2} \int d^4x \det e^{-1} \nabla_\alpha A_\alpha \cdot \nabla_\beta A_\beta.\]
In terms of the operator

\[(D_\alpha)^\varepsilon \eta \equiv \nabla_\alpha \eta^\varepsilon \eta + (B_\alpha + \frac{1}{2} T_\alpha)^\varepsilon \eta \quad (86)\]

the symmetric gauge field fluctuation operator

\[M_{AA}(e, B)^{\varepsilon\zeta} \equiv -(D_\alpha)^\varepsilon \eta (D_\alpha)^\eta \zeta + \frac{1}{4} (T_\alpha)^\varepsilon \eta (T^\alpha)^\eta \zeta \]

\[+ \frac{1}{2} \left( \nabla^\varepsilon T_\gamma^{\zeta\eta} - \tilde{R}^{\varepsilon \zeta} + \nabla^{\zeta} T_\gamma^{\varepsilon \eta} - \tilde{R}^{\zeta \varepsilon} \right) - m^2 \eta^{\varepsilon \zeta} \quad (87)\]

is of the general form considered in appendix A. Note that \(B, T, \tilde{R}\) only act on the vector indices \(\varepsilon, \zeta, \eta\) related to inner field space and \(C_\alpha\) in \(\nabla_\alpha\) only on the vector indices in derivatives.

We turn to evaluate the functional integral for \(Z_A[e, B]\). As we are dealing with a massive vector field coupled to a general gauge field \(B_\alpha^\gamma \delta\) the action (61) is no longer gauge invariant under the transformations defined at the end of section 5 and in principle we would not have to fix a gauge. As we are also interested in the two limiting cases where the mass vanishes and where \(B = C\) we nevertheless apply the Faddeev-Popov procedure in order to be safe in taking the aforementioned limits restoring \(U(1)\) gauge invariance. Hence we choose a gauge condition \(F[A_{\alpha}^A] = G(x)\) and insert the identity \(1 = \int D\Lambda \delta(F[A_{\alpha}^A] - G(x)) \det M_F(A)\) into eqn.(83) which becomes

\[Z_A[e, B] = \int D\Lambda D\alpha \delta(F[A_{\alpha}^A] - G(x)) \det M_F(A) \cdot e^{iS_M(A; e, B)}. \quad (88)\]

As the gauge field measure and the Faddeev-Popov determinant are gauge invariant we may change therein the coordinates from \(A_\alpha\) to \(A_{\alpha}^A = A_\alpha + \nabla_\alpha A\) without affecting the result. If we express the action in the new fields we obtain

\[S_M(A^A, A; e, B) = \int d^4x \det e^{-1} \left\{ -\frac{1}{4} F_{\alpha \beta}^A F^{\alpha \beta} + \frac{1}{2} m^2 A_{\alpha}^A A^{\alpha} \right. \]

\[- m^2 A_{\alpha}^A \cdot \nabla^\alpha A + \frac{1}{2} F_{\alpha \beta}^A \cdot (T^\varepsilon)^{\alpha \beta} \nabla_\varepsilon A \]

\[+ \frac{1}{2} m^2 \nabla_\alpha A \cdot \nabla^\alpha A - \frac{1}{4} (T^\varepsilon)_{\alpha \beta} \nabla_\varepsilon A \cdot (T^\eta)^{\alpha \beta} \nabla_\eta A \right\} \quad (89)\]

displaying the \(T\)- and the \(A\)-dependence explicitly. We may now change the variable \(A_{\alpha}^A \rightarrow A_\alpha\) everywhere in the functional integral (88). To rewrite it in
Gaussian form we recast the result as a quadratic form in \( \Phi \equiv (A_\alpha, A) \)-space obtaining

\[
S_M(\Phi; e, B) = \frac{1}{2}(\Phi, M_\Phi(e, B)\Phi)_e + \frac{1}{2} \int d^4x \det e^{-1}\nabla_\alpha A^\alpha \cdot \nabla_\beta A^\beta.
\]

The fluctuation operator in \( \Phi \)-space has four entries

\[
M_\Phi(e, B) = \begin{pmatrix} M^{\varepsilon\zeta}_{AA} & M^{\varepsilon A}_{AA} \\ M^{A\zeta}_{AA} & M^{AA} \end{pmatrix}
\]

and is hermitean w.r.t. \((\cdot, \cdot)_e\). \( M^{AA}(e, B)^{\varepsilon\zeta} \) is given in eqn.(87) and the three other elements are found to be

\[
M^{AA}(e, B)^{\varepsilon}\equiv (\nabla_\eta - T^{\eta\zeta}_\zeta)(T^{\varphi\eta}_\varphi\varphi)\nabla_\varphi + m^2\nabla^\varepsilon
\]

and

\[
M^{AA}(e, B)^{\zeta}\equiv \nabla_\eta(T^{\eta\zeta}_\zeta)\nabla_\zeta - m^2\nabla^\zeta
\]

and

\[
M^{AA}(e, B)\equiv -\frac{1}{2}\nabla_\varepsilon(T^\zeta)^{\eta\varphi}(T^\varphi)^{\eta\varphi}\nabla_\zeta + m^2\nabla_\varepsilon\nabla^\zeta.
\]

Note that these three operators are of a more general form than those considered in appendix A such that their corresponding heat kernel coefficients would have to be computed in a different way.

To obtain the final form of the gauge fixed functional we multiply with the Gaussian weight \( e^{-\frac{1}{2} \int d^4x \det e^{-1}G^2} \) and integrate out the auxiliary field \( G \) with the result

\[
Z_{Agf}[e, B] = \int D\Phi \det M_F(e, B) \cdot e^{\frac{i}{2}(\Phi, M_\Phi(e, B)\Phi)_e}.
\]

Here we made use of the background gauge condition

\[
F[A_\alpha] \equiv -\nabla_\alpha A^\alpha
\]

to get rid of the last term in eqn.(85) for \( S_M \). The corresponding Faddeev-Popov operator is then independent of \( m, B, K \) and \( A \) itself as in free QED

\[
M_F(e, B) = \frac{\delta F[A_\alpha]}{\delta A} = -\nabla_\alpha \nabla^\alpha
\]
and its determinant may be taken out of the functional integral (94) being now of Gaussian form. We can perform it and formally obtain

\[ Z_{Agf}[e, B] = \det M_F(e, B) \cdot e^{-\frac{1}{2} \log \det M_\Phi(e, B)}. \] (97)

Although the \( \zeta \)-function technique discussed in appendix B may be used to renormalize the \( \det M_\Phi(e, B) \) we can not discuss the complete scaling behaviour of \( Z_{Agf}[e, B] \) as the fluctuation operator \( M_\Phi(e, B) \) is no longer of the form considered in appendix A and we do not know its heat kernel coefficient functions.

But recasting \( M_\Phi(e, B) \) as a product

\[ M_\Phi(e, B) = \left( \begin{array}{cc} M_{AA} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & (M_{AA}^{-1}) M_{AA} \\ M_{AA} & M_{AA} \end{array} \right) \] (98)

we may split off the contribution coming from \( \det M_{AA}(e, B) \) and obtain for the result regularized at scale \( \mu \)

\[ Z_{Agf}[\mu; e, B] = e^{-\zeta'(0; \mu; M_F(e, B)) + \zeta'(0; \mu; M_{AA}(e, B)) + o.t.} \] (99)

where o.t. denotes the other terms present due to nonvanishing \( T \) and \( m \).

At the new scale \( \bar{\mu} = \lambda \mu \) we then find

\[ Z_{Agf}[\bar{\mu}; e, B] = Z_{Agf}[\mu; e, B] \cdot e^{-2 \log \lambda \cdot \zeta(0; \mu; M_F(e, B))} \cdot e^{\log \lambda \cdot \zeta(0; \mu; M_{AA}(e, B)) + o.t.}. \] (100)

In particular we are now safe taking the limits \( m = 0 \) and \( B = C \) where all the extra terms simply drop out. The Faddeev-Popov operator (96) does not change whereas the gauge field fluctuation operator displayed in eqn.(87) reduces to the simple form

\[ M_{AA}(e, C)^{\xi \zeta} = -(D_\alpha)^{\xi} \eta(D^\alpha)^{\eta \kappa} - R^{\xi \zeta}. \] (101)

8 Renormalizability and the dynamics of the gauge fields. The minimal gravitational action

In this section we evaluate the \( \zeta \)-functions yielding the rescaling changes in terms of the P gauge fields. We then determine a minimal gauge field action compatible with renormalizability requirements.
In the previous section we expressed the changes under rescaling of the one-loop partition functions for scalar, spinor and vector fields in terms of different $\zeta$-functions. Renormalizability of any theory including dynamical gauge fields requires now at least that these anomalous contributions, which are local polynomials in $e_\alpha \gamma$ and $B_\alpha \gamma \delta$ and their derivatives, may be absorbed in the classical action for the gauge fields $e_\alpha \gamma$ and $B_\alpha \gamma \delta$. Hence, to determine explicitly a minimal gauge field dynamics consistent with renormalizability we finally have to evaluate the different $\zeta$-functions.

Let us begin with the scalar field. The corresponding fluctuation operator is given in eqn.(69) and is of the form of the general operator (118) in appendix A if we choose $A_\alpha = 0, E = -m^2$ there. The coefficient function $U_m(x) \equiv \text{tr}SC_2(x)$ obtained from eqn.(135) reduces to a quite simple form

$$U_m = -\frac{1}{30} \nabla_\gamma \nabla_\gamma R^\alpha_\beta \alpha_\beta + \frac{1}{72} R^\alpha_\beta \alpha_\beta \cdot R_\gamma \delta \gamma \delta + \frac{1}{180} R^\alpha_\beta \gamma \delta \cdot R^\gamma_\delta \alpha_\beta - \frac{1}{180} R^\alpha_\gamma \alpha_\delta \cdot R^\beta_\gamma \delta \delta - \frac{1}{6} m^2 \cdot R^\alpha_\beta \alpha_\beta + \frac{1}{2} m^4.$$ (102)

With the use of eqn.(141) from appendix B we next obtain the value of $\zeta(0; \mu; M_\phi(e))$ as the integral over $\text{tr}SC_2(x)$. Its insertion into eqn.(72) finally yields the anomalous term in the scalar case.

Next we turn to the spinor sector. The operator (118) of appendix A coincides with the spinor fluctuation operator given in eqn.(79) provided that we set

$$A_\alpha \equiv B_\alpha + T_\alpha = \frac{i}{4} \Sigma_{\gamma \delta} (B_\alpha \gamma \delta + T^\gamma_\delta \alpha),$$ (103)

where $\Sigma_{\gamma \delta}$ acts on the spinor indices only, and

$$E \equiv V_1 + V_2 + V_3 - m^2.$$ (104)

Here we introduced

$$V_1 = -\frac{1}{8} \Sigma^\alpha \beta \Sigma^\gamma \delta V_1_{\gamma \delta \alpha \beta}, \quad V_1_{\gamma \delta \alpha \beta} = \tilde{R}_{\gamma \delta \alpha \beta} + \frac{1}{2} T_{\gamma \delta \eta \tau} T_\alpha_\beta \eta \tau,$$

$$V_2 = \frac{i}{4} \Sigma^\alpha \beta V_2_{\alpha \beta}, \quad V_2_{\alpha \beta} = \frac{1}{2} (V^\eta_1 \alpha \eta \beta - V^\eta_1 \beta \eta \alpha),$$ (105)

$$V_3 = \frac{1}{2} \tilde{\nabla}_\alpha T_\gamma \alpha \gamma - \frac{1}{4} T_{\gamma \alpha \gamma} T^{\alpha \delta} \delta.$$
We also need the field strength \( F_{\alpha\beta} \) corresponding to \( A_{\alpha} \) defined in eqn.(134)

\[
F_{\alpha\beta} = i \sum_{\gamma\delta} F_{\gamma\delta\alpha\beta} \\
F_{\gamma\delta\alpha\beta} = \hat{R}_{\gamma\delta\alpha\beta} + \hat{\nabla}_{\alpha} T_{\gamma\delta\beta} - \hat{\nabla}_{\beta} T_{\gamma\delta\alpha} + T_{\alpha\beta} \eta^\gamma T_{\gamma^\delta\eta} + T_{\eta^\gamma\alpha} T_{\gamma^\delta\beta} - T_{\eta^\gamma\beta} T_{\gamma^\delta\alpha}.
\]

Now we insert the above expressions into eqn.(135) for \( c^2(x) \) and take the Dirac trace

\[
\text{tr}_{D} c^2 = 4U_m + \left( \frac{1}{6} R^{\alpha\beta}_{\alpha\beta} - m^2 \right) \cdot (4V_3 - V_1^\gamma \gamma^\delta) - \frac{1}{6} \nabla_\alpha (4V_3 - V_1^\gamma \gamma^\delta) - \frac{1}{24} F_{\alpha\beta\gamma\delta} \cdot F_{\gamma\delta\alpha\beta} - V_3 \cdot V_1^\gamma \gamma^\delta - \frac{1}{8} V_2 \cdot V_2^\alpha \beta + \frac{1}{8} V_1^\gamma \alpha \beta \cdot V_1^\gamma \gamma^\delta + 2V_3^2 + \frac{1}{8} V_1^\alpha \beta \gamma^\delta \cdot (V_1^\alpha \beta \gamma^\delta + V_1^\alpha \delta \beta + V_1^\gamma \beta \delta) + \frac{1}{8} m^2 \cdot R^{\alpha\beta}_{\alpha\beta} + 2m^4.
\]

With the use of eqn.(141) from appendix B we next obtain the value of \( \zeta(0; \mu; M_\psi(e, B)) \) which finally yields the anomalous term in eqn.(82) in the spinor case. We remark that for \( T \neq 0 \) this result contains a huge number of different terms if we recast it in the natural variables \( \hat{R} \) and \( T \). Only for \( T = 0 \) it reduces to a simple form with

\[
\text{tr}_{D} c^2 = \frac{1}{30} \nabla_\gamma \nabla^\gamma R^{\alpha\beta}_{\alpha\beta} + \frac{1}{72} R^{\alpha\beta}_{\alpha\beta} \cdot R_{\gamma\delta}^\gamma \gamma^\delta + \frac{7}{360} R_{\alpha\beta\gamma\delta} \cdot R^{\alpha\beta\gamma\delta} - \frac{1}{45} R_{\alpha\gamma}^\alpha \delta \cdot R_{\beta}^\gamma \delta^\beta + \frac{1}{3} m^2 \cdot R^{\alpha\beta}_{\alpha\beta} + 2m^4.
\]

In the vector case we have to evaluate both the \( \zeta \)-functions belonging to the ghost operator \( M_F \) and the vector operator \( M_{AA} \). The former has been obtained in eqn.(94) and coincides with the operator (118) of appendix A if we choose \( A_{\alpha} = E = 0 \) whereas the latter, given in eqn.(87), coincides with the operator (118) provided that we set

\[
A_\alpha \equiv B_\alpha + \frac{1}{2} T_\alpha = i \sum_{\gamma\delta} (B_\alpha^\gamma \gamma^\delta + \frac{1}{2} T^\gamma \gamma^\delta_\alpha),
\]

where \( \sum_{\gamma\delta} \) acts on inner vector indices only, and

\[
E^{\alpha] \zeta} \equiv V^{\gamma \zeta} - m^2 \eta^{\gamma \zeta}.
\]
Here we set
\[ V^{\zeta} = \frac{1}{2} \left( \nabla^e T_\gamma \zeta^\gamma - \tilde{R}_\gamma \zeta^e + \tilde{\nabla}^e T_\gamma \zeta^\gamma - \tilde{\nabla}_\gamma \zeta^e \right) - \frac{1}{4} T_\gamma^e \zeta^\gamma T^{\zeta} \gamma^\delta. \] (111)

The field strength \( F_{\alpha\beta} \) corresponding to \( A_\alpha \) is found to be
\[ F_{\alpha\beta} = i \sum_{\gamma} T^\gamma_{\gamma \alpha \beta} + \tilde{R}_{\gamma \alpha \beta} \]
\[ F_{\gamma \delta \alpha \beta} = \tilde{R}_{\gamma \delta \alpha \beta} + \frac{1}{2} \nabla_\gamma T_{\delta \beta} - \frac{1}{2} \nabla_\beta T_{\gamma \alpha} \]
\[ + \frac{1}{2} T_{\alpha \beta} \eta \gamma \delta \eta + \frac{1}{4} T_{\gamma \delta} \eta \alpha \beta - \frac{1}{4} T_{\gamma \beta} \eta \delta \alpha. \] (112)

Inserting the above expressions into eqn.(135) for \( c_2(x) \) and taking the respective traces we get in the ghost case the same result as in the scalar one for \( m = 0 \)
\[ \text{tr}_{Gc_2} = U_0, \] (113)
whereas the result in the vector case is
\[ \text{tr}_{M c_2} = 4U_m + \left( \frac{1}{6} R_{\alpha \beta}^{\alpha \beta} - m^2 \right) V_\gamma^\gamma \]
\[ - \frac{1}{6} \nabla_\alpha \nabla^\alpha V_\gamma^\gamma - \frac{1}{12} F_{\alpha \beta \gamma \delta} \cdot F^{\alpha \beta \gamma \delta} \]
\[ + \frac{1}{2} V_{\alpha \beta} \cdot V_{\alpha \beta}. \] (114)

With the use of eqn.(141) from appendix B we next obtain the values of \( \zeta(0; \mu; M_\lambda(e)) \) and \( \zeta(0; \mu; M_{AA}(e, B)) \) which finally yield the anomalous terms in eqn.(100) for the vector case. Again, for \( T \neq 0 \) the result (114) contains a huge number of different terms if we recast it in the natural variables \( \tilde{R} \) and \( T \). Only in the \( U(1) \) gauge invariant case, for \( T = m = 0 \), it reduces to the simple form
\[ \text{tr}_{M c_2} = \frac{1}{30} \nabla_\gamma V_\gamma^\gamma R^{\alpha \beta}_{\alpha \beta} - \frac{1}{9} R_{\alpha \beta}^{\alpha \beta} \cdot R_{\gamma \delta}^{\gamma \delta} \]
\[ - \frac{11}{180} R_{\alpha \beta \gamma \delta} \cdot R^{\alpha \beta \gamma \delta} - \frac{43}{90} R_{\alpha \gamma}^{\alpha \delta} \cdot R_{\beta \gamma \delta}. \] (115)

The results eqns.(102), (108) and (115) for \( T = 0 \) are contained in [46] as special cases.

In eqns.(102), (107), (113) and (114) we have explicitly obtained the different anomalous contributions to the rescaled partition functions as local
gauge invariant polynomials in the fields \( e_\alpha^\gamma \) and \( B_\alpha^\gamma \). As discussed above, they also must be present in any classical gauge field dynamics consistent with renormalizability of the matter sectors. Hence, we are finally led to construct a minimal action for the gauge fields just in terms of these \( P \) gauge invariant polynomials. Note that this reasoning yields in the case of non-abelian matrix groups indeed the usual Yang-Mills action.

For \( T \neq 0 \) we restrict ourselves to the contributions of \( O(\partial^0, \partial^2) \) in the derivatives and obtain as minimal classical action to this order

\[
S_G(e, B) = \int \det e^{-1} \{ \Lambda - \frac{1}{\kappa^2} \cdot \tilde{R}^{\alpha\beta^\delta} + \beta_1 \cdot T^{\gamma\delta} T_\delta^\alpha + \beta_2 \cdot T_\alpha^\beta T_\beta^\gamma T^\gamma_\alpha + O(\partial^4) \},
\]

skipping possible total divergence terms. Here we have to introduce different couplings \( \kappa, \beta_1, \beta_2, \beta_3 \) and the constant \( \Lambda \) which are independently renormalized by the one-loop contributions we determined above. Note that our reasoning automatically enforces a cosmological constant as to be expected from general renormalization considerations. The action eqn. (116) describes the classical gauge field dynamics correctly at sufficiently low momentum scales and small values of the couplings. Nevertheless, only a dynamics containing the huge number of different \( O(\partial^4) \) terms as well, coming along with the same number of independent couplings, will be consistent with renormalizability [44].

If we set \( T = 0 \) the minimal classical action must contain the terms

\[
S_G(e) = \int \det e^{-1} \{ \Lambda - \frac{1}{\kappa^2} \cdot R^{\alpha\beta^\delta} + \alpha_1 \cdot R^{\alpha\beta^\gamma} R_\gamma^\delta + \alpha_2 \cdot R^{\alpha\gamma^\delta} R^{\gamma\beta^\delta} + \alpha_3 \cdot R^{\alpha\beta^\gamma} R^{\alpha\beta^\delta} \},
\]

if discarding total divergencies. The couplings \( \kappa, \alpha_1, \alpha_2, \alpha_3 \) and the constant \( \Lambda \) obtain again contributions from the one-loop scale anomalies which have been determined above. We emphasize that \( S_G \) is an action for gauge fields defined on the Minkowski spacetime \((\mathbb{R}^4, \eta)\) and is invariant on one hand under local \( P \) gauge transformations, on the other hand under global Poincaré transformations reflecting the symmetries of the underlying spacetime.

Important aspects of the quantized theory (117) such as one-loop divergencies and \( \beta \)-functions and its unitarity problems are discussed in [44] and references given there.
9 Conclusions

Based on the complementary conception of Poincaré symmetry as a purely inner symmetry we have developed a $P$ gauge theory of gravitation. The gravitational interaction is mediated by gauge fields defined on a fixed Minkowski spacetime. Their dynamics has been determined imposing consistency requirements with renormalization properties of matter fields in gravitational backgrounds. In an appropriate low energy limit it reduces to a form yielding the same observational predictions as made in general relativity.

In our conception there is no direct interrelation between gravity and the structure of spacetime. E.g., only if asking about the behaviour of rods and clocks at the classical level one is led to introduce an effective metric containing the desired information [23]. On the other hand, at the quantum level it may conceptually be easier to deal with a field theoretical description of gravitation free of any geometrical aspects.

This may shed some new light on questions related to the causality structure of spacetime at the quantum level, or the question of energy-momentum carried by the gravitational fields. Namely, the separation of the local gauge group $P$ from the global Poincaré symmetry group of the underlying Minkowski spacetime will allow us to obtain the energy-momentum carried by the gauge potentials in the usual Noether way.

In the determination of the scaling behaviour of the one-loop vector field partition function we obtained fluctuation operators of a more general form than usually investigated. Working out the coefficient functions occurring in the asymptotic expansion of the corresponding heat kernels poses an interesting technical problem in its own and is a necessary ingredient of a determination of the full scaling behaviour for $T \neq 0, m \neq 0$.

The most serious drawback of the present approach is of course the necessity of including the terms quadratic in the field strength in the classical gauge field action. Although the corresponding quantum theory is known to be renormalizable, the occurrence of negative energy or negative norm ghost states has destroyed up to now any attempt of establishing unitarity and hence a physical interpretation of the theory [44].
Acknowledgments

This work has partially been supported by Schweizerischer Nationalfonds. I am indebted to S. Dürr and R. Bärtschi for their encouragement, to F. Krahe for intensive discussions and a careful reading of the manuscript. L. O’Raifeartaigh helped me clarifying some group theoretical aspects. Helpful remarks of J. Kambor, J. Stern and A. Wipf are acknowledged.

A Heat kernel coefficients of P gauge covariant differential operators

In this appendix we determine the heat kernel coefficients \( c_1 \) and \( c_2 \) belonging to a general hermitean \( P \) covariant second order differential operator \( M \) defined on the \( d \)-dimensional Minkowski spacetime \((\mathbb{R}^d, \eta)\). We adapt here well-known techniques developed in a geometrical context to our case \([47]-[50]\).

Let us consider the \( P \) covariant hermitean operator

\[
M = -D_\alpha D^\alpha + E, \quad D_\alpha = \nabla_\alpha + A_\alpha. \tag{118}
\]

\( \nabla_\alpha = d_\alpha + C_\alpha \) is the \( P \) covariant derivative defined in eqn.(42). We emphasize that \( C_\alpha = \frac{i}{4} C_\alpha^{\gamma\delta} \Sigma_{\gamma\delta} \) is throughout understood to be adjusted to the Lorentz group representation it acts upon to ensure the covariant transformation properties of \( D_\alpha \). The anti-hermitean matrix-valued four-vector \( A_\alpha \), on the other hand, is kept fixed. Finally, \( E \) is a general hermitean matrix field.

The heat kernel \( K(is; x, y), s > 0 \), belonging to \( M_x \) fulfils

\[
\left( \frac{\partial}{\partial(is)} + M_x \right) K(is; x, y) = 0 \tag{119}
\]

together with the initial condition \( \lim_{s \to 0} K(is; x, y) = \frac{1}{\det e^{-\frac{1}{2}}} \delta(x - y) \). We are interested in the small \( s \)-expansion of \( K(is; x, y) \) in the coincidence limit \( y \to x \). Asymptotically this expansion is of the form

\[
K(is; x, y) \overset{s \to 0}{\sim} \frac{i}{(4\pi is)^{\frac{d}{2}}} e^{\frac{r^2(x, y)}{4is}} \sum_{k=0}^{\infty} (is)^k c_k(x, y). \tag{120}
\]

Hence, the task is to evaluate \( r^2(x, y) \) and the coefficient functions \( c_k(x, y) \). Inserting the expansion (120) in eqn.(119) and equating equal powers of \( s \)
we obtain the three \( P \) covariant relations

\[
2\nabla_\alpha r^2(x, y) = \frac{1}{4} \nabla_\alpha r^2(x, y) \cdot \nabla_\alpha r^2(x, y),
\]

\[
\frac{1}{2} \nabla_\alpha r^2(x, y) \cdot D^\alpha c_0(x, y) = \left\{ \frac{d}{2} - \frac{1}{4} \nabla_\alpha \nabla_\alpha r^2(x, y) \right\} \cdot c_0(x, y),
\]

\[
\frac{1}{2} \nabla_\alpha r^2(x, y) \cdot D^\alpha c_{k+1}(x, y) = \left\{ \frac{d}{2} - k - 1 - \frac{1}{4} \nabla_\alpha \nabla_\alpha r^2(x, y) \right\} \cdot c_{k+1}(x, y) - M x c_k(x, y).
\]

The first equation allows to evaluate \( r^2 \) and all its covariant derivatives at \( y = x \) whereas the two other relations (122) and (123) allow a recursive determination of \( c_k \) and all its covariant derivatives again at \( y = x \). \( c_0 = 1 \) at \( y = x \) ensures the correct initial condition. Note the introduction of the shorthand notation \( f(x) \equiv f(x, x) \) for functions taken in the coincidence limit \( y = x \).

We turn to the calculation of \( r^2 \) and its covariant derivatives. Differentiation of eqn.(121) leads to the relations

\[
2\nabla_\alpha r^2 = \nabla_\alpha \nabla^\gamma r^2 \cdot \nabla_\gamma r^2,
\]

\[
2\nabla_\beta r^2 = \nabla_\beta \nabla^\alpha r^2 \cdot \nabla_\alpha r^2 + \nabla_\beta r^2 \cdot \nabla_\gamma r^2,
\]

\[
2\nabla_\gamma r^2 = \nabla_\gamma \nabla^\alpha r^2 \cdot \nabla_\alpha r^2 + \nabla_\gamma r^2 \cdot \nabla_\beta r^2 + \nabla_\gamma r^2 \cdot \nabla_\beta r^2 + \nabla_\gamma r^2 \cdot \nabla_\beta r^2.
\]

Here we introduced the shorthand notations \( \nabla_\beta = \nabla_\beta \nabla_\alpha \) etc.. As the initial value is \( r^2(x) = 0 \), the relation (124) leads to

\[
\nabla_\alpha r^2(x) = 0
\]

which is consistent with (121). The use of eqn.(127) in the second relation (123) yields now \( 2\nabla_\beta r^2(x) = \nabla_\alpha \nabla^\gamma r^2 \cdot \nabla_\gamma r^2(x) \) and is solved by

\[
\nabla_\beta r^2(x) = 2\eta_\beta \alpha.
\]

As \( \eta_\beta \alpha \) is the only second rank tensor with the desired covariance properties this solution is in fact unique. Using the above results the third relation (126) becomes \( \nabla_\gamma r^2(x) = \nabla_\beta \alpha r^2(x) + \nabla_\gamma r^2(x) + \nabla_\gamma r^2(x) \). To solve it we commute the covariant derivatives according to eqn.(43). This yields e.g. \( \nabla_\alpha r^2(x) = \nabla_\gamma r^2(x) + \nabla_\gamma (R_\alpha r^2(x)) = \nabla_\gamma r^2(x) \), for \( r^2(x) \) is a scalar and thus \( R_\alpha r^2(x) = 0 \). Hence, we find

\[
\nabla_\gamma r^2(x) = 0
\]
expressing simply the fact that no homogeneously transforming third rank tensor built from \( e_\alpha^\gamma \) exists. In the same way we obtain
\[
\nabla_{\beta\gamma\alpha} r^2(x) = \frac{2}{3} (R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\beta\delta})
\]
(130)
and higher derivatives.

We turn to the computation of \( c_1(x) \). Appropriate differentiation of the relation (123) for \( k = 0 \) and the use of the results eqns. (127) - (129) for the covariant derivatives of \( r^2 \) lead in the limit \( y \to x \) to
\[
c_1(x) = D_\alpha^\alpha c_0(x) - E \cdot c_0(x).
\]
(131)
Note the introduction of the shorthand notations \( D_\alpha^\beta = D_\beta D_\alpha \) etc. There remain the different higher derivatives of \( c_0 \) to be determined. We now differentiate the relation (122) for \( c_0 \) and obtain together with the results eqns. (127) - (129) in the coincidence limit
\[
D_\alpha c_0(x) = 0,
\]
\[
D_\alpha^\alpha c_0(x) = -\frac{1}{8} \nabla_\alpha \nabla_\beta r^2(x) \cdot c_0(x).
\]
Inserting the initial condition \( c_0(x) = 1 \) finally yields
\[
c_1(x) = -\frac{1}{6} R^{\alpha\beta} - E.
\]
(133)
The calculation of \( c_2(x) \) is algebraically more involved. We thus restrict ourselves to note that it requires the use of the commutation relation for the covariant derivative \( D_\alpha \)
\[
[D_\alpha, D_\beta] = [\nabla_\alpha, \nabla_\beta] + \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta] = R_{\alpha\beta} + F_{\alpha\beta}
\]
(134)
which defines the field strength \( F_{\alpha\beta} \) belonging to the gauge field \( A_\alpha \). The Jacobi identities for the covariant derivative \( \nabla_\alpha \) lead to \( R_{\eta\alpha\beta\gamma} + R_{\gamma\alpha\beta\eta} + R_{\eta\beta\gamma\alpha} = 0 \) and \( \nabla_\gamma R^{\alpha\beta}_{\alpha\beta} = 2 \nabla_\alpha R^{\alpha\beta}_{\gamma\beta} \) allowing then to bring the result for \( c_2(x) \) into the simple form
\[
c_2(x) = -\frac{1}{30} \nabla_\gamma \nabla_\alpha^\gamma R^{\alpha\beta}_{\alpha\beta} + \frac{1}{72} R_{\alpha\beta} \cdot R^{\alpha\beta}_{\gamma\delta} - \frac{1}{180} R_{\alpha\gamma\delta} \cdot R^{\alpha\beta}_{\gamma\delta} - \frac{1}{180} R_{\alpha\gamma} \cdot R^{\alpha\beta}_{\gamma\delta} \cdot R^{\alpha\beta}_{\gamma\delta}
\]
\[
+ \frac{1}{12} R_{\alpha\beta} \cdot F^{\alpha\beta} + \frac{1}{6} R^{\alpha\beta}_{\alpha\beta} \cdot E + \frac{1}{6} [D_\alpha, \{D^\alpha, E\}] + \frac{1}{2} E^2.
\]
(135)
We finally remark that unfortunately only $c_1$ has been computed directly for an operator $M$ with $D_\alpha = \tilde{\nabla}_\alpha + A_\alpha$, where $\tilde{\nabla}_\alpha = d_\alpha + B_\alpha$ includes torsion [51].

**B \ \ \zeta- function regularization of functional determinants and $\zeta(0)$.**

In this appendix we define the functional determinant belonging to $M$ in terms of the $\zeta-$function regularization technique. We then determine its change under a rescaling using the heat kernel expansion obtained in appendix A.

One can define the functional determinant of the operator $M = -D_\alpha D^\alpha + E$ introduced in appendix A to be [52], [53]

$$\log \det M \equiv -\lim_{u \to 0} \frac{d}{du} \zeta(u; \mu; M) \quad (136)$$

where the generalized $\zeta-$function belonging to $M$ is given by

$$\zeta(u; \mu; M) \equiv \mu^{2u} \text{Tr} M^{-u}. \quad (137)$$

The scale $\mu$ at which parameters such as couplings, masses and wavefunction normalizations have to be adjusted is introduced in order to keep the determinant dimensionless.

The above definition of $\zeta$ does not allow to take the $u$-derivative at $u = 0$ since the trace is defined only for $\text{Re } u > \frac{d}{2}$. The necessary analytic continuation is achieved by recasting $\zeta$ as the Mellin transformation of the heat kernel

$$\zeta(u; \mu; M) = \frac{i\mu^{2u}}{\Gamma(u)} \int_0^\infty ds (is)^{u-1} \text{Tr} e^{-isM} \quad (138)$$

and yields indeed the desired ultraviolet regularization.

Let us next consider the behaviour of the functional determinant under a change of scale $\tilde{\mu} = \lambda \mu$. One obtains

$$\zeta'(0; \tilde{\mu}; M) = \zeta'(0; \mu; M) + 2 \log \lambda \cdot \zeta(0; \mu; M). \quad (139)$$

The change of the functional determinant under a rescaling is thus fully determined by $\zeta(0; \mu; M)$.

To evaluate $\zeta(0; \mu; M)$ we use the representation eqn.(138). It is the singular part of the $s$-integration in eqn.(138) which yields a nonvanishing
value for $\zeta(0; \mu; M)$. As this singular part comes from the small $s$-region we may use the expansion for the trace of the heat kernel following from eqn. (120) in appendix A

$$\text{Tr} e^{-isM} \overset{s \to 0}{\sim} \frac{i}{(4\pi is)^{\frac{d}{2}}} \sum_{k=0}^{\infty} (is)^k \int d^d x \det e^{-1} \text{tr} c_k(x). \quad (140)$$

Performing the $s$-integration in (138) singles out the contribution for $k = \frac{d}{2}$ from the infinite sum and one obtains

$$\zeta(0; \mu; M) = \frac{i}{(4\pi)^{\frac{d}{2}}} \int d^d x \det e^{-1} \text{tr} c_{\frac{d}{2}}(x). \quad (141)$$

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