The Best Approximation of an Objective State With a Given Set of Quantum States

Li-qiang Zhang, Nan-nan Zhou, and Chang-shui Yu*

Approximating a quantum state by the convex mixing of some given states has strong experimental significance and provides alternative understandings of quantum resource theory. It is essentially a complex optimal problem which, up to now, has only partially solved for qubit states. Here, the most general case is focused on that the approximation of a d-dimensional objective quantum state by the given state set consisting of any number of (mixed-) states. The problem is thoroughly solved with a closed solution of the minimal distance in the sense of $l_2$ norm between the objective state and the set. In particular, the minimal number of states in the given set is presented to achieve the optimal distance. The validity of this closed solution is further verified numerically by several randomly generated quantum states.

1. Introduction

In the past few decades, quantum information technology has been developed rapidly. The essence of quantum information processing is the preparation and the manipulation of quantum states. However, due to inherent limitations, technical, or economic reasons in practical scenario, the required quantum state could not be prepared exactly as we expected. One alternative approach could be the approximate preparation of the state by convex mixing of some disposable quantum states.

In addition, the approximation of a state is also widely implied in the quantum resource theory. As we know, the quantification of quantum features is the core of the resource theory. Many important quantum features such as quantum entanglement,[1–6] quantum coherence,[7–12] quantum discord,[13–18] and so on,[19,20] have been quantitatively studied from the point of resource theory of view. One of the most common methods is to measure the nearest distance between the target state and the free state set.[21–23] For example, the entanglement measure can be quantified by the smallest distance between the target state and the separable state set.[24–27] Quantum discord of a quantum state can be measured by its closest distance from the set of classically correlated quantum states.[13,28–36] Quantum coherence can be described based on the minimal distance between the target quantum state and the convex combinations of the given orthogonal basis.[37–43] Quantum superposition measures the nearest distance between the given state and some linearly independent states.[44–46] Therefore, a most general question extracted is how to optimally approximate an objective quantum state by the convex mixing of some given quantum states.

The optimal approximation of a quantum state with limited states has been addressed in various cases.[47–52] In refs. [47, 48], optimally approximating an unavailable quantum state $\rho$ (quantum channel $\Phi$) by the convex mixing of states (channels) drawn from a set of available states $\{\varphi_i\}$ (channels $\{\Psi\}$) was considered. The choice of available quantum states is a key problem. The approximation of a state by the six eigenstates of three Pauli matrices was studied in ref. [48], which was further revised and supplemented in ref. [49]. Then the approximation by the eigenstates of any two Pauli matrices was investigated in ref. [50] and some interesting trade-off relations were also proposed. Later, the disposable quantum state set was extended from the eigenstates of the Pauli matrix to the eigenstates of any quantum logic gate,[51] and then to arbitrary quantum states without any restriction.[52] Up to now, all the relevant contributions have been only restricted to the qubit states. It remains open whether such an optimal problem could have a closed solution for a general high-dimensional state.

In this paper, we study the optimal approximation of a general d-dimensional quantum state by convex mixing the states in a given state set. We employ the $l_2$ norm to measure the distance between two quantum states. With any given state set, we give the closed solution to the question, that is, we find the minimal distance between the objective state and the optimal state mixed with the states in the set. In particular, we can give the minimal number of the states in the set to achieve the optimal distance. We also prove that the case with the state set including more than $d^2$ states can be transformed into the case with the set including no more than $d^2$ states. In order to validate our closed solution, we investigate several examples with different dimensions in a numerical way. All the examples demonstrate the perfect consistency with our closed solution. The remaining of this paper is organized as follows. In Section 2, we give a brief description of the question of the convex approximation question and the closed solution to the question. In Section 3, we consider several randomly

DOI: 10.1002/andp.202100407
generated examples to test our closed results. The discussion and conclusion are given in Section 4.

2. The Approximation of the Given Objective State

The problem.-To begin with, we would like to first introduce the optimal problem. Let $\rho$ denote an objective state and $S := \{\rho_i, i = 1, 2, \ldots, N\}$ denote a given set of quantum states. Our goal is to prepare a quantum state $\sigma = x_1, x_2, \ldots, x_K = \sum_{i=1}^{K} x_i \rho_i$ with the sub-
scripts of $x$ in increasing order by the convex mixing of $K \leq N$ quantum states in the set $S$ so that the distance between the objective quantum state $\rho$ and prepared quantum state $\sigma$ is the closest, and the vector $\vec{p} = \{p_1, p_2, \ldots, p_K\}$ is the probability vector to be optimized.

For convenience, we consider all the $d$-dimensional states in the representation (labeled by 'X') defined by some Hermitian matrix basis, for example, $\{\chi_i, i = 0, 1, 2, \ldots, d^2 - 1\}$ with $X_i = \frac{1}{\sqrt{d}} \chi_i$. Thus, a $d$-dimensional quantum state $\rho$ in the X representation can be expanded as $\rho = \sum_{i=0}^{d^2-1} r_i X_i$.

The Bloch representation is such a typical example in which case $r_i$ is the element of the Bloch vector. Similarly, we use $r_i$ to represent the vector of $\rho$ in the X representation with its elements $r_{ij} = \text{Tr}(\rho X_{ij})$ and $r_i$ ($r_i$ is its element) to denote the vector of the $i$th state in $S$. In this sense, the distance between our objective state $\rho$ and the state $\sigma$ to be prepared can be given based on the $L_2$ norm as

$$D(\rho, \sigma) = \frac{1}{2} \left\| r_{0} - \sum_{i=1}^{K} p_i r_i \right\|_2^2 \tag{1}$$

with $\|r\|_2 = \sqrt{\sum r_i}$. It is obvious that $D(\rho, \sigma) = 0$ for $\rho = \sigma$ and $D(\rho, \sigma) = 1$ for $\rho \perp \sigma$.

To construct the optimal quantum state, $\sigma$ which is the closest to the state $\rho$ is equivalent to achieve $\min_{\vec{p}} D(\rho, x_1, x_2, \ldots, x_K(\vec{p}))$. Within the linear constraint $\sum_{i=1}^{K} p_i - 1 = 0$, this minimization is a global convex optimization problem on $\vec{p}$, which can be verified by the Hessian matrix of this problem defined by

$$H = \frac{\partial^2 D}{\partial p_i^2} = R_i' R_i \tag{2}$$

with $R_i = \{r_1, r_2, \ldots, r_K\}$ being a $d \times K$ matrix ($K \leq N$), and the inequality convex constraint $-p_i \leq 0$. The global convex optimization provides a good property that the optimal solution can be obtained by the global optimal point if it satisfies the constraints, or at the constraint boundary if the global optimal point is not within the constraint range. A schematic illustration of the convex optimization process is given in Figure 1, where we consider a 2D convex function $D(p_1, p_2)$ as an example. It is shown that the global minimum on the intersection curve is not within the constraint $p_1, p_2 \geq 0$, so the optimal point of the system is just at the lower endpoint of the red curve, that is, $p_1 = 1, p_2 = 0$. In addition, we have to emphasize that $l_1$ norm due to its simple form allows the optimization problem to be analytically solved.

With the above knowledge, we can present our main results about the best approximation of the objective state as follows.

Figure 1. Schematic diagram of the convex optimization process. A convex function $D(p_1, p_2)$ is optimized over $p_1$ and $p_2$ subject to the constraint $p_1 + p_2 - 1 = 0$ shown by the plane C. The inequality constraint $p_1, p_2 \geq 0$ is illustrated by the cuboid. The minimum of $D$ is shown by the lower end-point of the red intersection curve instead of the minimum of the curve.

Theorem 1. Given an objective $d$-dimensional state $\rho$ and a state set $S$ composed of $N$ quantum states $\rho_i$ ($N \leq d^2$), in the X representation, one can define the vector $\vec{B}$ as $\vec{B}(i) = (r_1 - r_i)^2 r_i + \delta_{iK}$ and a matrix $A$ as $A(i, j) = (r_i - r_k)^2 r_j + \delta_{iK}$ with $K \leq N$ and $i, j = 1, 2, \ldots, K$.

The minimal distance between $\rho$ and $\sigma = \sum_{i=1}^{K} p_i \rho_i$ is given by

$$\min_{\vec{p} \in S} D(\rho, x_1, x_2, \ldots, x_K(\vec{p})) \tag{3}$$

where $\vec{p} = A^{-1} \vec{B}$ with rank($A$) = $K$ and $p_i > 0$ required, and $i_0$ denotes the $i$th element in the subset composed of $K$ states from the set $S$.

Proof. For $N \leq d^2$, Equation (1) can be rewritten as

$$D(\rho, \sigma) = \frac{1}{2} \sum_{i=1}^{N} (p_i r_i^T r_j - 2 p_i r_i^T r_j + r_j^T r_j) \tag{4}$$

Consider the Lagrangian function

$$L(p_i, \lambda, \lambda_i) = D(\rho, \sigma) - \sum_{i=1}^{N} \lambda p_i + \lambda \left( \sum_{i=1}^{N} p_i - 1 \right) \tag{5}$$

where $\lambda$ and $\lambda_i$ are the Lagrangian multipliers. The Karush–Kuhn–Tucker conditions are given by

$$\frac{\partial L}{\partial p_i} = \sum_{j} r_j^T r_j - r_j^T r_j - \lambda_i + \lambda = 0$$
\[
\lambda_i p_i = 0, \lambda_i \geq 0, p_i \geq 0, \sum_{j} p_j - 1 = 0, i = 1, 2, 3, \ldots, N
\] (6)

After eliminating \( \lambda \) by \( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} = 0 \), we have
\[
\sum_{j} (p_j r_j)^T (r_i - r_N) = r_j^T (r_i - r_N) + \lambda_i - \lambda_N, i = 1, 2, 3, \ldots, N - 1
\] (7)

For convenience, we first consider the case all \( p_i \neq 0 \) which mean \( \lambda_i = 0 \) for \( i = 1, 2, 3, \ldots, N \). We rewrite Equation (7) above in matrix form \( A \mathbf{p} = \mathbf{B} \) with \( P(\mathbf{i}) = p_i \) for \( i = 1, 2, 3, \ldots, N \). The determinant of matrix \( A \) is
\[
\det(A) = \det(R_i^T R_i)
\] (8)

with \( R_i = (r_1 - r_N, r_2 - r_N, \ldots, r_{N-1} - r_N) \) being a \( d^2 \times (N - 1) \) matrix.

Case 1: \( \det(A) = 0 \). This means that each column of the matrix \( R_i \) is linearly related. Any quantum state represented by \( \mathbf{r}_i = \sum_{j} p_j \mathbf{r}_j \), with \( p_j \in [0, 1] \) and \( \sum_j p_j = 1 \) must be represented by \( \mathbf{r}_i = \sum_{j} q_j \mathbf{r}_j \) with \( q_j \in [0, 1] \) and \( \sum q_j = 1 \), which is implied by the Caratheodory theorem or explicitly shown in our latter theorem 2. In other words, the optimal solution of \( N \) quantum states is equivalent to that of \( N - 1 \) quantum states. The optimal distance is given by
\[
\min_{i_1 < i_2 < \ldots < i_{N-1}} D(\rho, \chi_{i_1i_2\ldots i_{N-1}}(\tilde{\mathbf{p}}), i_a = 1, 2, 3, \ldots, N - 1
\] (9)

Case 2: \( \det(A) \neq 0 \). One can first calculate
\[
\mathbf{P} = A^{-1} \mathbf{B}
\] (10)

If all \( \tilde{p}_i \in [0, 1] \) in \( \mathbf{P} \), then the optimal weights \( \tilde{p}_i = \tilde{p}_i \). By substituting the optimal weights \( \tilde{p}_i \) into the prepared quantum state \( \chi_{i_1i_2\ldots i_{N-1}}(\tilde{\mathbf{p}}) = \sum_{j} \tilde{p}_j \mathbf{r}_j \), we can obtain the optimal distance \( D(\rho, \chi_{i_1i_2\ldots i_{N-1}}(\tilde{\mathbf{p}})) \). If not all \( \tilde{p}_i \in [0, 1] \), it means that the optimal weights should be at the boundary, that is, at least one of the probabilities \( p_i \) is 0. Thus, one need to consider the mixing of \( N - 1 \) states and the optimization problem is converted to
\[
\min_{i_1 < i_2 < \ldots < i_{N-1}} D(\rho, \chi_{i_1i_2\ldots i_{N-1}}(\tilde{\mathbf{p}})), i_a = 1, 2, 3, \ldots, N - 1
\] (11)

Repeating the process from Equation (7) to Equation (11), until the probabilities \( \tilde{p}_i \in [0, 1] \). Suppose the first valid probability vector \( \mathbf{P} \) is found when considering the mixing of \( M \) states, the optimal distance is taken as the minimal distance over all \( C_M^N \) combinations of the \( M \) states with \( M \leq N \). The proof is completed. \( \square \)

**Theorem 2.** If there are \( N > d^2 \) states in the set \( S \), the optimization approximation is determined by
\[
\min_{i_a < i_b < \ldots < i_{N-1}} D(\rho, \chi_{i_1i_2\ldots i_{N-1}}(\tilde{\mathbf{p}}))
\] (12)

where \( i_a \) denotes the \( i_a \)th state in the set \( S \).

**Proof.** In the \( X \) representation, the prepared states \( \sigma \) can be expressed as \( \mathbf{r}_i = \sum_{j} p_j \mathbf{r}_j \). Caratheodory theorem\(^{18,59}\) shows that \( \mathbf{r}_i \) can be represented by the convex combination of no more than \( d^2 + 1 \) vectors in the set \( \hat{S} := \{ \mathbf{r}_i | i = 1, 2, 3, \ldots, N \} \) such as
\[
\mathbf{r}_i = \sum_{j} q_j \mathbf{r}_j, \text{ with } q_j \geq 0 \text{ and } \sum_{j} q_j = 1
\]

Considering that \( d^2 + 1 \) vectors in \( \hat{S} \) must be linearly independent, there exist \( \ell_i \), \( i = 1, 2, \ldots, d^2 + 1 \) such that \( \sum_{j} \ell_j r_j = 0 \), which implies \( \sum_{i} \ell_i r_i = 0 \) due to \( X_0 = l_i / \sqrt{d} \). Thus one can obtain
\[
\mathbf{r}_i = \sum_{j} q_j \mathbf{r}_j - \alpha \sum_{j} |r_i - r_j| q_j(1 - \alpha \frac{l_i}{q_j}) r_j
\] (13)

Let \( \alpha = \frac{q_i}{\sum_j q_j} = \min_{1 \leq i \leq d^2+1} \left( \frac{q_i}{\sum_j q_j} > 0 \right) \), we will find that
\[
1 - \alpha \frac{l_i}{q_j} = 0, i = \ell' \text{ and } \sum_j q_j(1 - \alpha \frac{l_i}{q_j}) = 1 \text{ which mean that at most } d^2 \text{ vectors in the set } \hat{S} \text{ are enough to convexly construct } \mathbf{r}_i.
\]

It implies that for \( N > d^2 \), the convex mixing of only \( d^2 \) states in \( S \) is enough to achieve the optimal distance. Therefore, we can directly consider all potential combinations of only \( d^2 \) quantum states among the set \( S \). The minimal distance will give our expected optimal result. The proof is finished. \( \square \)

**3. Examples**

To verify the reliability of our theorems, we provide several randomly generated density matrices and compare our closed analytic results with the numerical results. In the following, the objective state \( \mathbf{r}_i \) in the \( X \) representation is given as
\[
\mathbf{r}_i = kr_{i1} + (1 - k) r_{i2}, k \in [0, 1]
\] (14)

where \( r_{i2} \) is given in several special cases and \( r_{i1} \) will be randomly generated by Matlab. The explicit expression of \( N \) quantum states in set \( S = \{ \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N \} \) in all the below examples are given in Appendix A.

(i) \( d = 2 \) and \( N = 3 \). According to theorem 1, we first consider two states, \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). The pseudo probability reads
\[
\tilde{p}_1 = \frac{(r_2 - r_1)^T (r_1 - r_3)}{\|r_1 - r_3\|^2_2}, \tilde{p}_2 = 1 - \tilde{p}_1
\] (15)

With all the three states in the set taken into account, the pseudo probability reads
\[
\tilde{p}_1 = \frac{1}{d} (r_1 - r_3)^T \times \left| (r_3 - r_2)(r_2 - r_3)^T - (r_2 - r_3)(r_3 - r_2)^T \right| (r_2 - r_3).
\]
\[
\tilde{p}_2 = \frac{1}{d} (r_2 - r_3)^T \times \left| (r_3 - r_1)(r_3 - r_1)^T - (r_2 - r_3)(r_1 - r_3)^T \right| (r_1 - r_3),
\]
\[
\tilde{p}_3 = 1 - \tilde{p}_1 - \tilde{p}_2.
\] (16)
where
\[ d = \| r_1 - r_2 \|_2^2 - \| r_1 - r_3 \|_2^2 - \| (r_1 - r_2) \|_2^2 \]

Collecting the cases with all \( p_i = \tilde{p}_i \geq 0 \), one can find the minimal distance in terms of
\[ D(\rho, \sigma) = \frac{1}{2} \sum_{i=1}^{n} p_i r_i \]

For example, we can make
\[ r_{01} = (\frac{1}{\sqrt{2}}, 0, 0) \]

which is the maximally mixed state. Then three different kinds of \( r_{01} \) are randomly generated as
\[ r_{01}^1 = (1/\sqrt{2}, -0.0989, 0.1337, -0.1564)^\dagger \]
\[ r_{01}^2 = (1/\sqrt{2}, -0.1810, 0.0522, 0.2173)^\dagger \]
\[ r_{01}^3 = (1/\sqrt{2}, 0.2285, -0.0403, 0.2218)^\dagger \]

The optimal distance denoted by \( D(\rho) \) versus \( k \in [0, 1] \) is plotted in Figure 3a, where the dotted blue, solid red and dashed green lines correspond to \( r_{01}^1, r_{01}^2 \), and \( r_{01}^3 \), respectively. It is shown that our closed analytic solution is completely consistent with the numerical solution. In Figure 3b, we can find that for the optimal approximation of the objective quantum state, the minimal number of quantum states in set \( S \) is up to 4.

(iii) \( d = 3 \) and \( N = 15 \). In this case, \( r_{02} \) is randomly generated as
\[ r_{02} = (1/\sqrt{3}, 0.0568, 0.1463, 0.1405, -0.0456, -0.0531, -0.1342, 0.1669, -0.0918)^\dagger \]

and the three special quantum states are considered for \( r_{01} \) as
\[ r_{01}^1 = (1/\sqrt{3}, 0, 0, 0, 0, 0, 0)^\dagger \],
\[ r_{01}^2 = (1, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})/\sqrt{3} \]

The optimal distance denoted by \( D(\rho) \) versus \( k \in [0, 1] \) is plotted in Figure 4a, which validates our theorem based on the perfect consistency. The Figure 4b shows that the minimal number of quantum states in set \( S \) used to optimally approximate the objective quantum state is no more than 9.

(iv) \( d = 4 \) and \( N = 20 \). Due to the large dimension of the considered states, the concrete expressions of \( r_{01} \) and \( r_{02} \) are given in Appendix A. Similarly, we consider three special quantum states \( \{ r_{01}^1, r_{01}^2, r_{01}^3 \} \), and quantum state \( r_{01}^3 \) is the maximally mixed state. The optimal distance denoted by \( D(\rho) \) versus \( k \in [0, 1] \) is plotted in Figure 5a, which shows the perfect consistency between the numerical and the closed results, and further supports our theorem. As can be seen from
Figure 3. The optimal distance $D(\rho)$ versus various parameters $k$ in (a) for $d = 2$ and $N = 6$. The solid line corresponds to the strictly closed expressions given in our theorems, while the numerical solutions are marked with "+". The minimum number $n$ of quantum states in set $S$ needed for the optimal approximation of each objective state is shown in (b).

Figure 4. The optimal distance $D(\rho)$ versus various parameters $k$ in (a) for $d = 3$ and $N = 15$. The solid line corresponds to the strictly closed expressions, while the numerical solutions are marked with "+". The minimal number $n$ of quantum states needed for the optimal approximation of each objective state is shown in (b).

Figure 5b, the minimal number of quantum states in set $S$ used to approximate the target quantum state is less than or equal to 14. By comparing the figures, we can find that with the increase of the maximally mixed state ratio, the optimal approximate distance tends to 0.

4. Discussion and Conclusion
Before the end, we would like to mention that we have studied the best approximation of an objective state by a limited state set based on $l_2$ norm, which provided an analytically solvable
global convex optimization. In contrast, the Fidelity is only convex for the probability vector $\tilde{p}$ which is not allowable by our method. In addition, both the Fidelity and Trace norm, despite the better properties like contractility, lead to quite complex optimized objective function which is impossible for an analytical solution to a high-dimensional system. However, $l_2$ norm perfectly avoids those shortcomings. Although $l_2$ norm is not contractive, it is undeniably a valid distance of two vectors and there is no problem for the static comparison of two states. In the literature, most of us take it for granted that the distance between two states had better be contractive, just like Fidelity and Trace norm. In fact, in many cases, the contractility is originated from a dynamic process, especially considering the quantum features which cannot be produced by classical operations. For example, in quantum resource theory, there is an explicit requirement akin to/directly related to contractility. Apart from such a dynamic consideration, there is no sufficient reason to require a contractive distance. In other words, contractility is also an additional requirement for the distance. If our approach is used to solve some problem in the resource theory, strictly speaking, our result at most provides the criterion of existence instead of a measure. If one is interested in the optimal distance of two states after some potential operations, one can safely perform the corresponding operations on both the objective state and the set of states, and then employ our approach which is also valid in this case. Even if someone employed our approach in resource theory, a closed expression should be more useful than that without an explicit expression, which is popular in the relevant researches on entanglement such as negativity in high dimensional systems. Here, we only consider the static comparison without any additional constraints such as a dynamic process allowed.

If the given set $S$ is defined by $d$ mutually orthogonal quantum states, the optimal distance provides an alternative measure of quantum coherence of the objective quantum states based on $l_2$ norm, which is equal to the trace norm in the 2D case.\(^7\) If $d$ linearly independent quantum states are given for the set $S$, our results provide a measure of the superposition of the objective states.\(^{44}\) In addition, in the paper, we only consider the approximation of a single-party system, it is worth considering the local or nonlocal approximation of states in a composite system.

In summary, we have given a closed solution to the approximation of a $d$-dimensional objective state by using a given state set. We have not only presented the optimal distance between the objective state and the prepared state, but also given the minimal number of states in the given set $S$ to achieve the best approximation. Numerical tests in several examples validate our closed solutions via perfect consistency. In addition, it is found that for the $d$-dimensional objective quantum state, the optimal distance can be achieved by the convex combination of no more than $d^2$ quantum states. Finally, we emphasize that our closed solution indicates the least number of quantum states to construct the target quantum state approximately, which is beneficial to the practical operation in the experiment.

**Appendix A: The Explicit Forms of the Set $S$ for Examples**

In example (i), the quantum state set $S$ includes 3 quantum states, which can be expressed as

$$r_1 = (1/\sqrt{2}, -0.0453, -0.0429, -0.0774)$$
$$r_2 = (1/\sqrt{2}, -0.3348, -0.2708, -0.2571)$$
$$r_3 = (1/\sqrt{2}, 0.0287, 0.2456, -0.0534)$$

(A1)

In example (ii), the quantum state set $S$ includes six quantum states, which are composed of eigenstates of three Pauli matrix $\{\sigma_x, \sigma_y, \sigma_z\}$, expressed as

$$r_1 = (1, 1, 0, 0)^\top/\sqrt{2}$$
$$r_2 = (1, -1, 0, 0)^\top/\sqrt{2}$$
$$r_3 = (1, 0, 1, 0)^\top/\sqrt{2}$$
$$r_4 = (1, 0, -1, 0)^\top/\sqrt{2}$$
$$r_5 = (1, 0, 0, 1)^\top/\sqrt{2}$$
$$r_6 = (1, 0, 0, -1)^\top/\sqrt{2}$$

(A2)

In example (iii), the quantum state set $S$ is given in operator Hilbert space as

$$r_1 = (0.5774, -0.0089, 0.0192, 0.0446, -0.0585, -0.0403, -0.0061, 0.0094, 0.0210)^\top$$
$$r_2 = (0.5774, -0.0679, -0.0568, 0.0278, -0.0335, 0.0708, -0.1094, 0.1036, -0.1595)^\top$$
$$r_3 = (0.5774, 0.3314, -0.2469, -0.0453, -0.0119, -0.2463, 0.1383, 0.0430, -0.1808)^\top$$
$$r_4 = (0.5774, 0.0138, 0.2443, -0.2903, 0.2369, -0.1108, -0.0694, 0.2347, -0.1709)^\top$$
$$r_5 = (0.5774, -0.1427, -0.0667, -0.4253, -0.3187, -0.1495, -0.3435, 0.3087, 0.2761)^\top$$

(A3)
and the random generated quantum state $r_0$ is

$\text{r}_0 = (0.5000, 0.3480, -0.0903, -0.2264, -0.0123, -0.3373, -0.1569, 0.2523, 0.0478, 0.3409, -0.1137, -0.0728, 0.0766, -0.3191, 0.0155, -0.2063)^T$ \hspace{1cm} (A7)

In addition, the quantum state set $S$ includes 20 randomly generated quantum states which are

$\text{r}_1 = (0.5000, 0.3401, 0.2281, 0.1506, -0.1592, 0.0518, 0.1435, -0.0227, 0.0123, -0.0654, -0.1799, -0.2337, -0.3263, -0.3152, -0.3466, -0.3052)^T$, $\text{r}_2 = (0.5000, 0.2630, 0.2397, -0.0116, 0.3371, -0.0435, -0.2855, 0.2873, 0.3155, 0.0002, -0.0044, 0.1938, -0.3148, 0.2518, -0.0674, -0.1989)^T$, $\text{r}_3 = (0.5000, -0.2184, -0.0078, 0.1946, 0.1876, 0.3716, 0.2003, -0.1040, -0.1933, -0.0483, 0.0962, 0.2114, -0.2066, 0.4027, -0.3182, -0.2010)^T$, $\text{r}_4 = (0.5000, -0.3085, 0.1438, -0.3556, -0.0276, -0.3483, -0.3359, 0.1912, 0.0149, -0.0655, 0.1375, 0.2116, 0.0990, -0.3137, -0.0414, 0.2411)^T$, $\text{r}_5 = (0.5000, -0.2741, -0.1356, 0.0618, 0.2636, -0.1240, -0.2177, 0.1307, 0.0534, 0.3311, 0.3185, 0.3498, 0.3522, 0.1482, -0.0426, -0.1383)^T$ \hspace{1cm} (A8)
\[ r_6 = (0.5000, 0.1112, 0.3263, 0.3190, -0.1469, -0.3317, 0.2964, -0.2435, -0.0953, -0.0306, 0.1241, 0.3175, -0.2460, 0.1503, 0.1725, 0.1075)^T \]

\[ r_7 = (0.5000, -0.3397, 0.1331, -0.3639, -0.0465, -0.3013, -0.0081, -0.1089, 0.0448, 0.2988, 0.2088, 0.1497, 0.3078, 0.3506, 0.0583, 0.0359)^T \]

\[ r_8 = (0.5000, -0.0234, -0.1351, -0.0322, 0.2409, -0.3197, -0.1314, -0.2997, -0.3042, -0.3151, -0.2814, -0.0517, -0.0072, 0.2808, -0.2075, 0.3309)^T \]

\[ r_9 = (0.5000, 0.0011, 0.0720, 0.0828, 0.3043, 0.3365, -0.2482, -0.2899, -0.1409, 0.1969, 0.2455, -0.1839, 0.2696, -0.2972, -0.2457, 0.1112)^T \]

\[ r_{10} = (0.5000, -0.0969, -0.2549, -0.1631, 0.3248, -0.1620, -0.2249, 0.0524, -0.3021, -0.2719, 0.2176, -0.3062, -0.1522, -0.0556, -0.1959, -0.3050)^T \]

(A9)

\[ r_{11} = (0.5000, -0.1085, -0.1859, -0.2159, -0.3368, -0.1676, -0.2280, 0.2087, -0.0767, 0.2550, -0.2157, -0.2451, -0.0837, -0.3492, -0.0487, -0.3334)^T \]

\[ r_{12} = (0.5000, -0.1116, 0.1404, -0.3683, -0.2396, 0.2121, 0.0992, -0.3009, 0.0709, -0.3062, -0.2525, 0.2682, 0.0603, -0.3460, -0.0274, -0.1445)^T \]

\[ r_{13} = (0.5000, 0.2993, -0.2373, -0.1311, -0.0875, -0.1534, 0.1559, 0.1657, -0.2268, 0.3667, -0.1520, 0.0617, -0.1243, 0.3017, -0.2709, -0.3336)^T \]

\[ r_{14} = (0.5000, 0.3539, 0.0031, -0.3141, -0.1757, -0.3098, 0.3047, -0.0659, 0.1496, -0.0462, -0.0051, -0.2103, 0.2230, -0.1621, 0.3111, 0.2460)^T \]

\[ r_{15} = (0.5000, 0.1284, 0.3134, 0.2846, 0.1660, -0.0162, -0.0902, 0.2459, -0.3016, 0.0968, -0.2279, 0.2806, -0.1189, 0.2264, -0.3173, -0.2467)^T \]

(A10)

and

\[ r_{16} = (0.5000, -0.0621, 0.1726, -0.0117, 0.1752, 0.4750, 0.1905, 0.3538, -0.0945, 0.0778, -0.0987, -0.3450, -0.0469, -0.0181, 0.2563, 0.2941)^T \]

\[ r_{17} = (0.5000, -0.3708, -0.1331, -0.2681, 0.2228, -0.0237, 0.0812, -0.3113, -0.2672, -0.2705, 0.1829, 0.0050, -0.1646, -0.1872, 0.3579, 0.0310)^T \]

\[ r_{18} = (0.5000, 0.1800, 0.1847, -0.2499, -0.1384, 0.3771, 0.3414, -0.0918, -0.2475, 0.0291, -0.2798, -0.1287, -0.2310, -0.1423, -0.2583, -0.1931)^T \]

\[ r_{19} = (0.5000, -0.2897, 0.0626, -0.2482, 0.0671, 0.2700, 0.2767, -0.0325, 0.2770, -0.2419, 0.0868, -0.2809, 0.1914, 0.2186, -0.2852, -0.2412)^T \]

\[ r_{20} = (0.5000, 0.2981, -0.3500, 0.0817, -0.1222, -0.4227, -0.1678, 0.1315, 0.3039, 0.0275, 0.2742, -0.0520, 0.1699, 0.2849, 0.0298, 0.1050)^T \]

(A11)
Acknowledgements

This work was supported by the National Natural Science Foundation of China under Grant No. 11775040 and No. 1201130014, the Fundamental Research Fund for the Central Universities under Grant No. DUT20LAB203, and the Key Research and Development Project of Liaoning Province under Grant No. 2020J21000003.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Keywords

best convex approximation, quantum state preparation, quantum coherence

Received: September 1, 2021
Revised: November 5, 2021
Published online: December 30, 2021

[1] C. H. Bennett, H. J. Bernstein, S. Popescu, B. Schumacher, Phys. Rev. A 1996, 53, 2046.
[2] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, W. K. Wootters, Phys. Rev. A 1996, 54, 3824.
[3] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, W. K. Wootters, Phys. Rev. Lett. 1996, 76, 722.
[4] P. Horodecki, R. Horodecki, Quantum Info. Comput. 2001, 1, 45.
[5] W. K. Wootters, Quantum Info. Comput. 2001, 1, 27.
[6] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Rev. Mod. Phys. 2009, 81, 865.
[7] T. Baumgratz, M. Cramer, M. B. Plenio, Phys. Rev. Lett. 2014, 113, 140401.
[8] A. Winter, D. Yang, Phys. Rev. Lett. 2016, 116, 120404.
[9] E. Chitambar, M.-H. Hsieh, Phys. Rev. Lett. 2016, 117, 020402.
[10] A. Streltsov, S. Rana, P. Boes, J. Eisert, Phys. Rev. Lett. 2017, 119, 140402.
[11] K. B. Dana, M. G. Díaz, M. Mejatty, A. Winter, Phys. Rev. A 2017, 95, 062327.
[12] A. Streltsov, S. Rana, M. N. Bera, M. Lewenstein, Phys. Rev. X 2017, 7, 011024.
[13] L. Henderson, V. Vedral, J. Phys. A: Math. General 2001, 34, 356899.
[14] H. Ollivier, W. H. Zurek, Phys. Rev. Lett. 2001, 88, 017901.
[15] A. Datta, A. Shaji, C. M. Caves, Phys. Rev. Lett. 2008, 100, 050502.
[16] T. Tufarelli, D. Girolami, R. Vasile, S. Bose, G. Adesso, Phys. Rev. A 2012, 86, 052326.