Exponential decay of correlations for a real-valued dynamical system generated by a $k$ dimensional system

Jager Lisette, Maes Jules, Ninet Alain

To cite this version:

Jager Lisette, Maes Jules, Ninet Alain. Exponential decay of correlations for a real-valued dynamical system generated by a $k$ dimensional system. Acta Applicandae Mathematicae, 2019, 160, pp.21-34. 10.1007/s10440-018-0192-z. hal-01881754

HAL Id: hal-01881754
https://hal.science/hal-01881754
Submitted on 26 Sep 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Exponential decay of correlations for a real-valued dynamical system generated by a $k$ dimensional system

JAGER Lisette, MAES Jules, NINET Alain

8 novembre 2017

Abstract

As a first step towards modelling real time series, we study a class of real, bounded-variables processes $\{X_n, n \in \mathbb{N}\}$ defined by a $k$-term recurrence relation $X_{n+k} = \varphi(X_n, \ldots, X_{n+k-1})$. These processes are noise-free. We immerse such a dynamical system into $\mathbb{R}^k$ in a slightly distorted way, which allows us to apply the multidimensional techniques introduced by Saussol [SAU] for deterministic transformations.

The hypotheses we need are, most of them, purely analytic and consist in estimates satisfied by the function $\varphi$ and by products of its first-order partial derivatives. They ensure that the induced transformation $T$ is dilating.

Under these conditions, $T$ admits a greatest absolutely continuous invariant measure (ACIM). This implies the existence of an invariant density for $X_n$, satisfying integral compatibility conditions. Moreover, if $T$ is mixing, one obtains the exponential decay of correlations.

2010 Mathematical Subject Classification : 37C40.
Keywords and phrases : ACIM, dynamical systems, decay of correlations, dilating transforms.

Address : Laboratoire de Mathématiques, FR CNRS 3399, EA 4535, Université de Reims Champagne-Ardenne, Moulin de la Housse, B. P. 1039, F-51687, Reims, France

1 Introduction

In this work, we are concerned with a deterministic $k \geq 2$ terms induction rather than with the more classical study of a dynamical system $x_0, \varphi(x_0), \ldots, \varphi^n(x_0), \ldots$. We thus write the model in a probabilistic manner, $X_{n+k} = \varphi(X_n, \ldots, X_{n+k-1})$, where $X_0, \ldots, X_{k-1}$ are real random variables. We aim at proving, for this model, the exponential decay of correlations, principally under analytic assumptions about the partial derivatives of the function $\varphi$.

The decay of correlations has been treated by many authors for the classical model. Among the most recent works we may refer to the works of Alves, Freitas, Luzzatto, Vaienti [AFLV], Gouëzel [GO], Sarig [SAR], Young [YOU] and Saussol [SAU]. We shall use, most particularly, the results of the last author.

In a first paper [JMN], we have proved the result for a two-terms induction, by studying a system imbedded into $\mathbb{R}^2$. We adopt here the same method and imbed our system into $\mathbb{R}^k$, in...
a non-canonical way and we introduce a transformation $T : Z \mapsto T(Z)$, defined on a compact subset of $\mathbb{R}^k$. But the hypotheses which allowed us to conclude in the 2-terms case are not relevant for a general $k$. Indeed, the proof of the result requires to locate very precisely the eigenvectors of a real $k$ dimensional matrix, which is quite easy when $k = 2$ and much less so for higher orders. We only consider here orders $k$ greater than or equal to 3.

We first obtain (Theorem 2-4.) the existence of a greatest absolute continuous invariant measure (ACIM) $\mu$ for the transformation $T$. If $T$ is mixing, one obtains (Theorem 2-7.) that, for well-chosen applications $f$ and $g$, there exist constants $C = C(f, h) > 0$ and $\rho \in [0, 1]$ such that :

$$
\left| \int f \circ T^n h \, d\mu - \int f \, d\mu \, \int h \, d\mu \right| \leq C \rho^n.
$$

As a consequence, if a given $\mathbb{R}^k$-valued random variable $Z_0$ has distribution $\mu$ (the ACIM), if $F$ and $H$ are convenient real valued functions, one gets (Theorem 5 for $r = s = 1$) :

$$
|\text{Cov}(F(X_n), H(X_0))| \leq C \rho^n.
$$

The corresponding inequalities are more complicated when $T$ is not mixing (Theorem 2-6., Lemma 4).

Let us note, too, the existence of integral identities satisfied by the invariant measure (Theorem 3).

We apply our results to a non linear example (Section 3).

## 2 Hypotheses and results

Let $L \in \mathbb{R}^*_+$ and let us consider an application $\varphi : [-L, L]^k \to [-L, L]$, defined piecewise on $[-L, L]^k$.

Under conjugation by an affine function, similar results could be obtained for an application $\varphi$ defined on $[a, b]^k$, with values in $[a, b]$. Recall that $k \geq 3$.

Suppose that all following conditions are fulfilled :

1. There exists $d \in \mathbb{N}^*$ such that

$$
[-L, L]^k = \bigcup_{j=1}^{d} O_j \cup \mathcal{N},
$$

where the $O_j$ are nonempty open subsets, $\mathcal{N}$ is Lebesgue negligible and the union is disjoint. The boundary of each $O_j$ is contained in a compact, $C^1$, $(k - 1)$-dimensional submanifold of $\mathbb{R}^k$.

2. There exists $\varepsilon_1 > 0$ such that, for every $j \in \{1, \ldots, d\}$, there exists a map $\varphi_j$ defined on $B_{\varepsilon_1}(\overline{O_j}) = \{(x_1, \ldots, x_k) \in \mathbb{R}^k, \, d((x_1, \ldots, x_k), \overline{O_j}) \leq \varepsilon_1\}$ with values in $\mathbb{R}$ and satisfying $\varphi_j|_{O_j} = \varphi|_{O_j}$.

3. The application $\varphi_j$ is bounded and $C^{1, \alpha}$ on $B_{\varepsilon_1}(\overline{O_j})$ for an $\alpha \in [0, 1]^1$, which means that $\varphi_j$ is $C^1$ and that there exists $C_j > 0$ such that, for all $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ in $B_{\varepsilon_1}(\overline{O_j})$, all $i \in \{1, \ldots, k\}$ :

$$
\left| \frac{\partial \varphi_j}{\partial x_i}(u_1, \ldots, u_k) - \frac{\partial \varphi_j}{\partial x_i}(v_1, \ldots, v_k) \right| \leq C_j \|(u_1, \ldots, u_k) - (v_1, \ldots, v_k)\|^\alpha.
$$

4. The maximal number of $C^1$ arcs of $\mathcal{N}$ crossing is $Y \in \mathbb{N}^*$. Moreover, one sets

$$
\sigma > 1
$$

1. If $\varphi_j$ is $C^2$ on $B_{\varepsilon_1}(\overline{O_j})$, it is necessarily $C^{1, \alpha}$ on $B_{\varepsilon_1}(\overline{O_j})$ with $\alpha = 1$
and imposes that
\[ \eta := \left( \frac{1}{\sqrt{\sigma}} \right)^\alpha + \frac{4}{\sqrt{\sigma} - 1} \frac{Y \gamma_k^{-1}}{\gamma_k} < 1 \]

where \( \gamma_k = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2} + 1\right)} \) is the volume of the unit sphere of \( \mathbb{R}^k \).

5. Let \( A > 1 \) and \( \sigma > 1 \) satisfy \( A^{2/k} > \sigma \). Set
\[ \gamma = A^{-1/k} \text{ and } M_0(\sigma, A) = \frac{-(k - 1)\gamma^k + \sqrt{(k - 1)^2 \gamma^2 + 4(k - 2)\gamma^{2k} + (1/\sigma - \sigma)}}{2(k - 2)\gamma^{2k + 1}} > 0. \]

Let \( M \in [0, M_0(\sigma, A)] \).
Assume that, for all \( 2 \leq i \leq k \), all \( j \leq d \) and all \( (u_1, \ldots, u_k) \in B_{\varepsilon_1}(\overline{O}_j) \) :
\[ \left| \frac{\partial \varphi_j}{\partial x_1}(u_1, \ldots, u_k) \right| \geq A, \quad \left| \frac{\partial \varphi_j}{\partial x_1}(u_1, \ldots, u_k) \times \frac{\partial \varphi_j}{\partial x_i}(u_1, \ldots, u_k) \right| \leq M. \]

(These very tight conditions are due to the loss of precision in the localization of the eigenvalues of the matrix \( B \), see (8) - in the case when \( k > 2 \).)

6. The sets \( O_j \) satisfy the following geometrical condition: for all \( (u_1, u_2, \ldots, u_k) \) and \( (v_1, u_2, \ldots, u_k) \) in \( B_{\varepsilon_1}(\overline{O}_j) \), there exists a \( C^1 \) path \( \Gamma = (\Gamma_1, \ldots, \Gamma_k) : [0, 1] \rightarrow B_{\varepsilon_1}(\overline{O}_j) \) between \( (u_1, u_2, \ldots, u_k) \) and \( (v_1, u_2, \ldots, u_k) \), with nonzero gradient satisfying
\[ \forall t \in ]0, 1[, \quad \left| \Gamma_1'(t) \right| > \frac{M}{A^2} \sum_{i=2}^{k} \left| \Gamma_i'(t) \right|. \]

For every \( j \in \{1, \ldots, d\} \), one denotes by \( U_j \) (resp. \( W_j, N' \)) the image of \( O_j \) (resp. \( B_{\varepsilon_1}(\overline{O}_j), N \)) under the transformation which associates with \( (u_1, \ldots, u_k) \in \mathbb{R}^k \) the point \( (u_1, \gamma u_2, \ldots, \gamma^{k-1} u_k) \). The set \( \Omega = [-L, L] \times [-\gamma L, \gamma L] \times \cdots \times [-\gamma^{k-1} L, \gamma^{k-1} L] \), with which we shall work, is the image of \( [-L, L]^k \) under the same transformation.

For every non-negligible Borel set \( S \) of \( \mathbb{R}^k \), for every \( f \in L^1_m(\mathbb{R}^k, \mathbb{R}) \), one sets:
\[ \text{Osc}(f, S) = E_{S} \sup_{S} f - E_{S} \inf_{S} f \]

where \( E_{S} \sup \) et \( E_{S} \inf \) are the essential supremum and infimum on \( S \) with respect to the Lebesgue measure \( m \).

One then defines the norm \( \| \cdot \|_\alpha \) by
\[ |f|_\alpha = \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{\mathbb{R}^k} \text{Osc}(f, B_\varepsilon(x_1, \ldots, x_k)) \, dx_1 \ldots dx_k, \quad \| f \|_\alpha = \| f \|_{L^1_m} + |f|_\alpha \]

and the set \( V_\alpha = \{ f \in L^1_m(\mathbb{R}^k, \mathbb{R}); \| f \|_\alpha < +\infty \} \).

We introduce similar notions on \( \Omega \) : for every \( 0 < \varepsilon_0 < \gamma^{k-1} L \), for every \( g \in L^\infty_{m}(\Omega, \mathbb{R}) \), we define:
\[ N(g, \alpha, L) = \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{\Omega} \text{Osc}(g, B_\varepsilon(x_1, \ldots, x_k) \cap \Omega) \, dx_1 \ldots dx_k. \]

One then sets:
\[ \| g \|_{\alpha, L} = N(g, \alpha, L) + 2K(\Omega) \varepsilon_0^{1-\alpha} \| g \|_\infty + \| g \|_{L^1_m}. \]

---

2. In favorable cases, the geometrical hypothesis can be replaced by the following one, stronger but much simpler: for all points \( (u_1, u_2, \ldots, u_k) \) and \( (v_1, u_2, \ldots, u_k) \) in \( B_{\varepsilon_1}(\overline{O}_j) \), the segment \( [(u_1, u_2, \ldots, u_k), (v_1, u_2, \ldots, u_k)] \) is contained in \( B_{\varepsilon_1}(\overline{O}_j) \).
where \( K(\Omega) = 2^{k+2} \left( \sum_{i=1}^{k} 2 \gamma^{-1} L \right) k^{-1} = 2^{k+1} L^{k-1} \left( 1 - \frac{\gamma}{\gamma} \right)^{k-1} \).

The function \( g \) is said to belong to \( V_\alpha(\Omega) \) if this expression is finite. This set does not depend on the choice of \( \varepsilon_0 \), but \( N \) and \( \| \|_{\alpha,L} \) do.

There exist relations between the sets \( V_\alpha(\Omega) \) and \( V_\alpha \). Indeed, one can prove the following result using Proposition 3.4 of [SAU]:

**Proposition 1**

1. If \( g \in V_\alpha(\Omega) \) and if one extends \( g \) to a function \( f \) defined on \( \mathbb{R}^k \), setting \( f(x) = 0 \) if \( x \notin \Omega \), then \( f \in V_\alpha \) and

\[
\| f \|_{\alpha} \leq \| g \|_{\alpha,L}.
\]

2. Conversely let \( f \in V_\alpha \) and set \( g = f 1_\Omega \). Then \( g \in V_\alpha(\Omega) \) and the following holds:

\[
\| g \|_{\alpha,L} \leq \left( 1 + 2K(\Omega) \frac{\max(1, \varepsilon_0^\alpha)}{\gamma k^{\alpha k-1+\alpha}} \right) \| f \|_{\alpha}.
\]

Under the hypotheses 1-5 listed above, we obtain a first result:

**Theorem 2** Let \( T \) be the transformation defined on \( \Omega \) by \( \forall u = (u_1, \ldots, u_k) \in U_j \):

\[
T(u) = T_j(u) = \left( \frac{u_2}{\gamma}, \ldots, \frac{u_k}{\gamma}, \gamma^{k-1} \varphi_j(u_1, \frac{u_2}{\gamma}, \ldots, \frac{u_k}{\gamma^{k-1}}) \right).
\]

The applications \( T_j \) can be defined naturally on \( W_j \) by the same formula. Then

1. The Frobenius-Perron operator \( P : L^1_{m}(\Omega) \rightarrow L^1_{m}(\Omega) \) associated with \( T \) has a finite number of eigenvalues of modulus 1, \( \lambda_1, \ldots, \lambda_r \).

2. For every \( i \in \{1, \ldots, r\} \), the eigenspace \( E_i = \{ f \in L^1_{m}(\Omega) : Pf = \lambda_i f \} \) associated with the eigenvalue \( \lambda_i \) is finite-dimensional and contained in \( V_\alpha(\Omega) \).

3. The operator \( P \) decomposes as

\[
P = \sum_{i=1}^{r} \lambda_i P_i + Q,
\]

where the \( P_i \) are projections on the spaces \( E_i \), \( \| P_i \|_1 \leq 1 \) and \( Q \) is a linear operator defined on \( L^1_{m}(\Omega) \), such that \( Q(V_\alpha(\Omega)) \subseteq V_\alpha(\Omega) \), \( \sup \| Q^n \|_1 < \infty \) and \( \| Q^n \|_{\alpha,L} = O(q^n) \) when \( n \rightarrow +\infty \), for a given \( q \in [0,1] \). Moreover, \( P_i P_j = 0 \) if \( i \neq j \), \( P_i Q = Q P_i = 0 \) for every \( i \).

4. The operator \( P \) has the eigenvalue 1. Set \( \lambda_1 = 1 \), let \( h_\ast = P_1 1_\Omega \) and \( dm = h_\ast dm \). Then \( \mu \) is the greatest absolutely continuous invariant measure (ACIM) of \( T \), which means that, if \( \nu << m \) and if \( \nu \) is \( T \)-invariant, then \( \nu << \mu \).

5. The support of \( \mu \) can be decomposed into a finite number of mutually disjoint measurable sets, on which a power of \( T \) is mixing. More precisely, for every \( j \in \{1, 2, \ldots, \dim(E_1)\} \), there exist a number \( L_j \in \mathbb{N}^* \) and \( L_j \) mutually disjoint sets \( W_{j,l} (0 \leq l \leq L_j - 1) \), satisfying \( T(W_{j,l}) = W_{j,l+1 \mod (L_j)} \), \( T^{L_j} \) being mixing on every \( W_{j,l} \). One denotes by \( \mu_{j,l} \) the normalized restriction of \( \mu \) on \( W_{j,l} \), defined by

\[
\mu_{j,l}(B) = \frac{\mu(B \cap W_{j,l})}{\mu(W_{j,l})}, \quad d\mu_{j,l} = \frac{h_\ast 1_{W_{j,l}}}{\mu(W_{j,l})} dm.
\]

Saying that \( T^{L_j} \) is mixing on every \( W_{j,l} \) means that, for every \( f \in L^1_{\mu_{j,l}}(W_{j,l}) \) and every \( h \in L^\infty_{\mu_{j,l}}(W_{j,l}) \),

\[
\lim_{n \rightarrow +\infty} < T^{nL_j} f, h >_{\mu_{j,l}} = < f, 1 >_{\mu_{j,l}} = < 1, h >_{\mu_{j,l}}
\]

with indifferently used notations : \( < f, g >_\mu = \mu'(fg) = \int fg \, dm' \).
6. Moreover, there exist real constants $C > 0$ and $0 < \rho < 1$ such that, for every $h$ in $V_\alpha(\Omega)$ and $f \in L^1_\mu(\Omega)$, the following holds:

$$\left| \int_{\Omega} f \circ T^n x \, d\mu - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) < f, 1 > \mu_{j,l} < 1, h > \mu_{j,l} \right| \leq C \| h \|_{\alpha, \Omega} \| f \|_{L^1_\mu(\Omega)} \rho^n.$$

7. If, moreover, $T$ is mixing, the preceding result can be stated as: there exist real constants $C > 0$ and $0 < \rho < 1$ such that, for every $h$ in $V_\alpha(\Omega)$ and $f \in L^1_\mu(\Omega)$, one has:

$$\left| \int_{\Omega} f \circ T^n h \, d\mu - \int_{\Omega} f \, d\mu \int_{\Omega} h \, d\mu \right| \leq C \| h \|_{\alpha, \Omega} \| f \|_{L^1_\mu(\Omega)} \rho^n.$$  

Let us now come back to the initial system and let us try to deduce the invariant law associated with $X_n$. If the sequence $(X_n)_n$ is defined by the initial terms $X_0, \ldots, X_{k-1}$, with values in $[-L, L]$, and the recurrence relation $X_{n+k} = \varphi(X_n, \ldots, X_{n+k-1})$, then $(Z_n)_n$ satisfies the recurrence relation $Z_{n+1} = T(Z_n)$, which yields the following result:

**Theorem 3** If the random variable $Z_0 = (\gamma^{j-1} X_{j-1})_{1 \leq j \leq k}$ has density $h_*$, then, for every $n \geq 0$, $Z_n$ has density $h_*$. Computing the marginal distributions, we get as a consequence that for every $n \in \mathbb{N}$, $X_n$ has a density $h_{\text{inv}}$ which has the following expressions: for every $j \in \{0, \ldots, k-1\}$

$$\forall u \in [-L, L], \quad h_{\text{inv}}(u) = \gamma^j \int_{\mathbb{R}^{k-1}} h_*(z_1, \ldots, \gamma^j u, \ldots, z_k) \, d\tilde{z}_{j+1}$$

where $d\tilde{z}_{j+1}$ means that one integrates with respect to all coordinates of $z$ but $z_{j+1}$.

Indeed, $\gamma^j X_n$ is the $(j+1)$-th coordinate of $Z_{n-j}$ if $j = 0, \ldots, k-1$. Let us consider a Borel set $A$ of $\mathbb{R}$. Then, for $j \in \{0, \ldots, k-1\}$,

$$P(X_n \in A) = P(Z_{n-j} \in \mathbb{R}^j \times \gamma^j A \times \mathbb{R}^{k-j-1})$$

$$= \int_{\mathbb{R}^j \times \gamma^j A \times \mathbb{R}^{k-j-1}} h_*(z_1, \ldots, z_k) \, dz_1 \ldots dz_k$$

$$= \int_{\mathbb{R}^j \times A \times \mathbb{R}^{k-j-1}} h_*(z_1, \ldots, \gamma^j u, \ldots, z_k) \, d\tilde{z}_{j+1} \gamma^j du \quad \text{with} \quad z_{j+1} = \gamma^j u$$

$$= \int_A \left( \int_{\mathbb{R}^{k-1}} h_*(z_1, \ldots, \gamma^j u, \ldots, z_k) \, d\tilde{z}_{j+1} \right) \gamma^j du,$$

which gives the desired result.

If $F$ is defined on $[-L, L]$ and if $s \in \{1, \ldots, k\}$, let us denote by $T_s F$ the function defined on $\Omega$ by

$$T_s F(z) = T_s F(z_1, \ldots, z_k) = F(z_s \gamma^{1-s}). \quad (2)$$

The following Lemma is then a direct consequence of point 6 in Theorem 2, applied to $T_s F$ and $T_r H$ for $s, r \in \{1, \ldots, k\}$:

**Lemma 4** For every Borel set $B$ of $[-L, L]$ and every interval $I$ of $[-L, L]$, if $Z_0$ has the invariant distribution, one has:

$$P \left( X_{n \times \text{ppcm}(L_i) + s - 1} \in B, X_{r-1} \in I \right) = \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) < T_s 1_B, 1 > \mu_{j,l} < 1, T_r 1_I > \mu_{j,l} \right| \leq \left( 2L \right)^{k} \gamma^k \left( \frac{1}{2} \right)^{k-1} + 4(2L)^{k-1} \gamma^{1-r+k(k-1)/2} \varepsilon_0^{1-\alpha} + 2k L^{k-1} \left( \frac{1-\gamma^k}{1-k} \right)^{k-1} \varepsilon_0^{1-\alpha} C \rho^n.$$

3. Which is equivalent to : if 1 is the only modulus-1 eigenvalue of $P$ and if, additionnaly, it is simple.
More generally, let $F$, defined and measurable on $[-L, L]$, be such that $T_{s} F$ belongs to $L_{\mu}^{1}(\Omega)$. Let $H \in L_{\overline{\mu}}^{\infty}([-L, L])$ be such that $\sup_{0<\delta<\varepsilon_{0} \gamma^{1-r}} \int_{[-L,L]} \text{Osc}(H, x-\delta, x+\delta[\cap[-L,L]) dx < +\infty$. Then $T_{r} H \in V_{\alpha}(\Omega)$ and

$$E(F(X_{n} \times pcm(L)+s-1))H(X_{r}-1)) - \sum_{j=1}^{\dim(E_{1})} \sum_{l=0}^{L_{j}-1} \mu(W_{j,l}) \mu_{j,l}(T_{s}F) \mu_{j,l}(T_{r}H) \leq C(F, H, L) \rho^{n}$$

with

$$C(F, H, L) = C\|T_{s} F\|_{L_{\mu}^{1}} (2L)^{k-1} \gamma^{k-1/2} \|H\|_{L_{\overline{\mu}}^{\infty}([-L,L])}$$

$$+(2L)^{k-1} \gamma^{(k-1)/2} \alpha^{(r-1)} \sup_{0<\delta<\varepsilon_{0} \gamma^{1-r}} \int_{[-L,L]} \text{Osc}(H, x-\delta, x+\delta[\cap[-L,L]) dx$$

$$+2^{k} L^{k-1} \left( \frac{1-\gamma}{1-\gamma} \right)^{k-1} \varepsilon^{1-\alpha}_{0} \|H\|_{L_{\overline{\mu}}^{\infty}([-L,L])}.$$ 

This last result, which gives the exponential decay of correlations, is a straightforward consequence of Lemma 4 and of the remark in point 7, Theorem 2.

**Theorem 5** If, moreover, $T$ is mixing, then for all $r, s \in \{1, \ldots k\}$

$$|\text{Cov}(F(X_{n+s-1}), H(X_{r}-1))| \leq C(F, H, L) \rho^{n}.$$ 

### 3 A nonlinear example

We can state the result :

**Theorem 6** Let $\sigma > 1$ be such that

$$\eta := \frac{1}{\sqrt{\sigma}} + 4 \frac{(k+1)}{\sqrt{\sigma}-1} \frac{\gamma^{k-1}}{\gamma_{k}} < 1. \quad (3)$$

Let $A > \sigma^{k}$. Set $\gamma = A^{-\frac{1}{k}}$ and

$$M_{0}(\sigma, A) = \frac{-(k-1)\gamma^{k-1} + \sqrt{(k-1)2^{2k-2} + 4(k-2)\gamma^{2k+1}(\frac{1}{\gamma^{2}} - \sigma)}}{2(k-2)\gamma^{2k+1}}.$$ 

Suppose $M \in [0, M_{0}(\sigma, A)]$.

Let $a_{1}, \ldots, a_{k}, b_{1}$ be nonnegative numbers such that

$$a_{1} \geq 2A^{2}, \quad (4)$$

$$b_{1} \geq 4LM\sqrt{k-1} + 2a_{1}L, \quad (5)$$

$$\sqrt{a_{1}a_{i}} \leq 2M \quad \forall i \in \{2, \ldots, k\}. \quad (6)$$

Set

$$\psi(x) = \left( \sum_{i=1}^{k} a_{i}x_{i}^{2} \right) + b_{1}x_{1} + \frac{b_{1}^{2}}{4a_{1}}.$$ 

Then $\psi$ is positive on $[-L, L]^{k}$. Set $\varphi_{0} = \sqrt{\psi}$. Let $\ell \in [-L, L]$. Define the transformation $\varphi$ on $[-L, L]^{k}$, piecewise, by

$$\varphi(x) = \ell + \varphi_{0}(x) - 2\ell \quad \text{if} \quad \ell + \varphi_{0}(x) \in [2\ell L - L, 2\ell L + L]. \quad (7)$$ 

The application $\varphi$ satisfies the hypotheses 1-6 of Theorem 2.
For example, for $k = 3$ and $L = 1$, one can take

$$\sigma = 150, \quad A = 1900, \quad M = 260$$

and

$$a_1 = 7250000, \quad a_2 = 0, \quad a_3 = 0.03, \quad b_1 = 15000000.$$

**Remark 7** Since this nonlinear transform admits, according to Theorem 2, a stationary density, it could be used as a pseudorandom number generator (cf [LM], [LY]).

The rest of this section will be dedicated to the proof. It appears in the proof that the parameters $\alpha$ and $Y$ of the hypotheses satisfy $\alpha = 1$ and $Y \leq k + 1$.

**Proof:** One sees that

$$\psi(x) = \frac{1}{4a_1} (2a_1 x_1 + b_1)^2 + \sum_{i=2}^{k} a_i x_i^2 \geq \frac{1}{4a_1} (-2a_1 L + b_1)^2$$

on $[-L, L]^k$, since, by (5), $2a_1 x_1 + b_1 \geq b_1 - 2a_1 L \geq 4LM \sqrt{k - 1} > 0$. Hence $\psi$ is positive on the compact set $[-L, L]^k$ and consequently on an open neighbourhood $U$ of $[-L, L]^k$. The function $\varphi_0 = \sqrt{\psi}$ is well defined and $C^\infty$ on $U$.

Hypothesis 2 is satisfied since, whatever the open subsets $O_j$ of $[-L, L]^k$ are, the expression (7) for $\varphi$ makes sense on the neighbourhood $U$ of $[-L, L]^k$ itself.

Since $\varphi_0$ is smooth, Hypothesis 3 is fully filled with $\alpha = 1$.

To prove that $\varphi$ satisfies Hypothesis 5, we only have to prove it for $\varphi_0$ on a neighbourhood of $[-L, L]^k$. One checks that

$$\frac{\partial \varphi_0}{\partial x_1} (x) = \frac{2a_1 x_1 + b_1}{2\sqrt{\psi(x)}} \geq \frac{2a_1 x_1 + b_1}{2\sqrt{\psi(x_1, L, \ldots, L)}}.$$

Denoting by $g = g(x_1)$ the function appearing in the right side above, one sees that $g'$ has the sign of $a_1 \sum_{i=2}^{k} a_i L^2$, which means that $g$ is an increasing function. To obtain the desired condition about $\frac{\partial \varphi_0}{\partial x_1}$, it suffices that $g(-L) > A$ on $[-L, L]^k$ (and hence on a neighbourhood of $[-L, L]^k$). Now, $g(-L) = \frac{-2a_1 L + b_1}{2\sqrt{\psi(-L, L, \ldots, L)}} > A$ if and only if

$$(-2a_1 L + b_1)^2 > 4A^2 \left( \frac{1}{4a_1} (-2a_1 L + b_1)^2 + \sum_{i=2}^{k} a_i L^2 \right)$$

$$\iff (-2a_1 L + b_1)^2 (a_1 - A^2) > 4A^2 a_1 \sum_{i=2}^{k} a_i L^2.$$

Using successively (4), (5) and (6), one gets

$$(-2a_1 L + b_1)^2 (a_1 - A^2) \geq (-2a_1 L + b_1)^2 (A^2) \geq 4A^2 (4M^2) (k - 1) L^2 > 4A^2 a_1 \sum_{i=2}^{k} a_i L^2,$$

which shows that $g(-L) > A$ and $\frac{\partial \varphi_0}{\partial x_1}(x) > A$ on $[-L, L]^k$ and on a neighbourhood of $[-L, L]^k$.

One has

$$\left| \frac{\partial \varphi_0}{\partial x_1}(x) \frac{\partial \varphi_0}{\partial x_i}(x) \right| = \frac{a_i |x_i|(2a_1 x_1 + b_1)}{2\psi(x)} \leq \frac{a_i |x_i|(2a_1 x_1 + b_1)}{2\psi(x_1, 0, \ldots, 0, x_i, 0, \ldots, 0)}.$$
This can be written as

\[
\left\lvert \frac{\partial \varphi_0}{\partial x_1}(x) \frac{\partial \varphi_0}{\partial x_i}(x) \right\rvert \leq \sqrt{a_i a_1} \frac{(\sqrt{a_1} x_1)(\sqrt{a_1} x_1 + b_1)}{2\sqrt{a_1}} + (\sqrt{a_1} x_1)^2,
\]

and it is easy to see that it is smaller than \(\frac{\sqrt{a_1} a_1}{2}\), hence strictly smaller than \(M\) according to (6) on \([-L, L]^k\) and on a neigbourhood. This achieves the proof that Hypothesis 5 is verified.

To verify Hypothesis 1, we must explicit the open sets. For \(p \in \mathbb{Z}\), define the open sets \(O_p\) by:

\[
O_p = \{x \in [-L, L]^k : \ell + \varphi_0(x) \in ]2pL - L, 2pL + L[\}.
\]

One sees that, for \(p \leq -1\), \(O_p\) is empty and that, otherwise,

\[
O_0 = \{x \in [-L, L]^k : \psi(x) < (L - \ell)^2\},
O_p = \{x \in [-L, L]^k : ((2p - 1)L - \ell)^2 < \psi(x) < ((2p + 1)L - \ell)^2\}, \quad p \geq 1.
\]

The sets \(O_p\) are open and may be empty.

Put \(S_p = \{x \in \mathbb{R}^k : \psi(x) = ((2p - 1)L - \ell)^2\}\). If \(S_p \cap [-L, L]^k\) is not empty, \(\frac{\partial \psi}{\partial x_1}(x) > 0\) is valid for every point of \(S_p \cap [-L, L]^k\) (because of (5)), so \(x_1\) can be considered, locally, as a \(C^\infty\) function of the other \(x_i\) and \(S_p \cap [-L, L]^k\) is a finite union of \(C^\infty\) submanifolds. The edges of \([-L, L]^k\) are parts of hyperplanes, hence are \(C^\infty\) too. This gives Hypothesis 1.

A submanifold \(S_p\) crosses at most \(k\) hyperplanes, which implies that the maximal crossing number, \(Y\), is smaller than \(k + 1\). This, together with (3), gives Hypothesis 4.

Hypothesis 6 is satisfied under its simple form. Indeed, let \(U = (u_1, u_2, \ldots, u_k)\) and \(V = (v_1, v_2, \ldots, v_k)\) be two points of the same set \(O_p \subset [-L, L]^k\). On \([-L, L]^k\), \(\frac{\partial \psi}{\partial x_1}(x) > 0\). Hence, for \(t \in [0, 1]\), if one assumes that \(-L < u_1 < v_1 < L\),

\[
\psi(U) \leq \psi(tU + (1 - t)V) \leq \psi(V),
\]

since the only coordinate that changes is the first one. Therefore \(\psi(tU + (1 - t)V)\) is in the same interval as \(\psi(U)\) and \(\psi(V)\). Consequently, \(tU + (1 - t)V\) is in \(O_p\).

This achieves the proof of the theorem.

4 Proofs

Theorem 2 is a consequence of Theorems 5.1 and 6.1 of [SAU], which rely on [ITM], as well as [HK] in the case when \(d = 1\), where the use of bounded-variation functions is possible. The difficulty lies in verifying that \(T\) satisfies Hypotheses (PE1) to (PE5).

To prove that (PE2) is satisfied, we shall first establish that \(T_j\) is a \(C^1\) diffeomorphism on \(W_j\) onto \(T_j(W_j)\). Hypothesis 3 about \(\frac{\partial \varphi_j}{\partial x_1}\) assures that \(T_j\) is a local diffeomorphism. To check that it is injective, let us consider two different points \(u\) and \(v\) of \(W_j\), such that \(T_j(u) = T_j(v)\). Then \(u_i = v_i\) for every \(2 \leq i \leq k\) and

\[
\varphi_j \left( u_1, \frac{u_2}{\gamma}, \ldots, \frac{u_k}{\gamma^{k-1}} \right) = \varphi_j \left( v_1, \frac{v_2}{\gamma}, \ldots, \frac{u_k}{\gamma^{k-1}} \right).
\]

Using the geometrical hypothesis 6 and applying the fundamental theorem of calculus to \(t \mapsto \varphi_j(\Gamma(t))\) leads to a contradiction. The regularity hypotheses on the \(\varphi_j\) (and hence on the \(T_j\)) allow to prove that \(\det(D_T^{-1})\) is \(\alpha\)-Hölder, provided the domain is conveniently restricted. One can see that there exist, for each \(\beta_j > 0\), an open and relatively compact set \(V_j\) and a real constant \(c_j\) such that the following holds
One applies the fundamental theorem of calculus to the map defined on \([0,1]\) by
\[
\partial \phi \left( \begin{array}{c}
\beta_j(T_j(U_j)) \in T_j(V_j);
\end{array} \right)
\]

and of a negligible set.

For (PE4), we need two steps. We first prove that the map is locally expanding (when the
preimages in \(V_j\) and all \(x,y \in B_{\varepsilon}(z) \cap T_j(V_j)\),
\[
\left| \det(DT_j^{-1}(x)) - \det(DT_j^{-1}(y)) \right| \leq c_j \left| \det(DT_j^{-1}(z)) \right| \varepsilon^\alpha.
\]

Setting \(\beta = \min \beta_j > 0\) and \(c = \max c_j > 0\), one obtains constants which are convenient for every
\(j \in \{1, \ldots, d\}\). Hence (PE2) is satisfied.

This allows us to specify the open sets on which we shall work. There exists \(\varepsilon_2 > 0\) such that, for
every \(j \in \{1, \ldots, d\}, B_{2\varepsilon_2}(U_j) \subset V_j \subset W_j\). From now on, one sets \(V_j = B_{\varepsilon_2}(U_j)\). Then \(T_j(V_j)\)
is open and \(T_j(U_j)\) is compact and contained in \(T_j(V_j)\). One can find a positive \(\varepsilon_{0,1}\) such that
\(B_{\varepsilon_{0,1}}(T_j(U_j)) \subset T_j(V_j)\) for every \(j\), which proves that Hypothesis (PE1) is satisfied.

Hypothesis (PE3) is clearly fulfilled since \(\Omega = \bigcup_{j=1}^d U_j \cup \mathcal{N}'\) is a disjoint union of open sets
and of a negligible set.

For (PE4), we need two steps. We first prove that the map is locally expanding (when the
preimages in \(V_j\) are sufficiently near, Proposition 8). Then we prove the hypothesis itself (Propo-
sition 9), in the case when the images in \(T_j(V_j)\) are sufficiently near.

**Proposition 8** Let \(u\) and \(v \in V_j\) be such that the segment \([u,v]\) is contained in \(V_j\). Then
\[
||T_j(u) - T_j(v)||^2 \geq \sigma ||u - v||^2.
\]

**Proof** : One applies the fundamental theorem of calculus to the map defined on \([0,1]\) by \(t \mapsto \varphi_j(v_1 + t(u_1 - v_1), \frac{v_2 + tu_2 - v_2}{\gamma}, \ldots, \frac{v_k + tu_k - v_k}{\gamma^{k-1}})\), which yields a \(c \in [0,1]\) such that
\[
||T_j(u) - T_j(v)||^2 = (v_1 - u_1, \ldots, v_k - u_k)B \begin{pmatrix}
v_1 - u_1 \\
\vdots \\
v_k - u_k
\end{pmatrix}
\]

where \(B = (b_{i,l})_{1 \leq i,l,k}\) is the matrix with coefficients
\[
\begin{align*}
b_{i,l} &= \gamma^{2k-2} \left( \frac{\partial \varphi_j(M_c)}{\partial x_i} \right) \left( \frac{\partial \varphi_j(M_c)}{\partial x_l} \right) \quad \text{if } i \neq l, \\
b_{1,1} &= \gamma^{2k-2} \left( \frac{\partial \varphi_j(M_c)}{\partial x_1} \right)^2, \\
b_{i,i} &= \frac{1}{\gamma^2} + \gamma^{2(k-1)} \left( \frac{\partial \varphi_j(M_c)}{\partial x_i} \right)^2 \quad \text{if } i > 1,
\end{align*}
\]

with \(M_c = \left( v_1 + c(u_1 - v_1), \frac{v_2 + c(u_2 - v_2)}{\gamma}, \ldots, \frac{v_k + c(u_k - v_k)}{\gamma^{k-1}} \right)\).

The matrix \(B\) is real and symmetrical. Its eigenvalues are contained in the Gerschgorin disks and
hence in the following domain
\[
\bigcup_{i=1}^k \left[ b_{i,i} - \sum_{l \neq i} |b_{i,l}|, b_{i,i} + \sum_{l \neq i} |b_{i,l}| \right].
\]
We shall establish that all these intervals are contained in $[\sigma, +\infty[$. To that aim, it is sufficient to prove that $b_{i,l} - \sum_{l \neq i} |b_{i,l}| \geq \sigma$ for every $i$.

According to Hypothesis 3 one has, for every $l > 1$, $|\frac{\partial \phi}{\partial x_l}(M_c)| \leq \frac{M}{A^3}$, which implies that

$$\begin{cases}
  b_{1,l} - \sum_{l \neq 1} |b_{1,l}| &\geq \gamma^{2k-2} A^2 - M \sum_{l \neq 1} \gamma^{2k-1-l} \\
  b_{i,l} - \sum_{l \neq i} |b_{i,l}| &\geq \frac{1}{\gamma^2} - M \gamma^{2k-1} - \frac{M^2}{A^2} \sum_{l \neq i, l > 1} \gamma^{2k-i-l} 
\end{cases} \quad \text{for } i > 1.$$

Since $\gamma < 1$ and $2k - 1 - l \geq k - 1$ one eventually gets:

$$\begin{cases}
  b_{1,l} - \sum_{l \neq 1} |b_{1,l}| &\geq \gamma^{2k-2} A^2 - M(k-1) \gamma^{k-1} \\
  b_{i,l} - \sum_{l \neq i} |b_{i,l}| &\geq \frac{1}{\gamma^2} - M(k-1) \gamma^{k-1} - \frac{M^2}{A^2} (k-2) \gamma^{2k+1} \quad \text{for } i > 1.
\end{cases}$$

Since $\gamma = A^{-1/k}$, one derives the inequalities:

$$\begin{cases}
  b_{1,l} - \sum_{l \neq 1} |b_{1,l}| &\geq \frac{1}{\gamma^2} - M(k-1) \gamma^{k-1} \\
  b_{i,l} - \sum_{l \neq i} |b_{i,l}| &\geq \frac{1}{\gamma^2} - M(k-1) \gamma^{k-1} - M^2(k-2) \gamma^{2k+1} \quad \text{for } i > 1.
\end{cases}$$

Therefore, the eigenvalues of $B$ are all greater than or equal to $\frac{1}{\gamma^2} - M(k-1) \gamma^{k-1} - M^2(k-2) \gamma^{2k+1}$. Since $M \in [0, M_0(\sigma, A)]$, one has:

$$\frac{1}{\gamma^2} - M(k-1) \gamma^{k-1} - M^2(k-2) \gamma^{2k+1} \geq \sigma.$$

Consequently, the eigenvalues of $B$ are all greater than or equal to $\sigma$, which gives the desired result. Notice that this last inequality compels us to choose $\frac{1}{\gamma^2} > \sigma$. \hfill \Box

Compacity arguments give the existence of $\varepsilon_{0,2} > 0$ such that, for every $u \in V_j$,

$$B_{\varepsilon_{0,2}}(T_j(u)) \subset T_j(B_{\varepsilon_2}(u)).$$

**Proposition 9** Let $\varepsilon_0 = \min(\varepsilon_{0,1}, \varepsilon_{0,2}) > 0$. Recall that $\overline{U}_j \subset V_j \subset V_j \subset V_j \subset W_j$. As a consequence,

- for all $x, y \in T_j(V_j)$ satisfying $\|x - y\| < \varepsilon_0$, the following inequality is valid:

$$\frac{1}{\sigma} \|x - y\| > \|T_j^{-1}(x), T_j^{-1}(y)\|.$$

- $B_{\varepsilon_0}(T_j(\overline{U}_j)) \subset T_j(V_j)$.

**Proof:** The second statement comes from the fact that $\varepsilon_0 \leq \varepsilon_{0,1}$ and from the results we obtained in relation with (PE1).

Let us prove the first statement, which implies Condition (PE4) of Saussol. Let $x, y \in T_j(V_j)$ satisfy $\|x - y\| < \varepsilon_0$. Set $u = T_j^{-1}(x) \in V_j$. According to the preceding remark, as $\varepsilon_0$ is smaller than $\varepsilon_{0,2}$,

$$y \in B_{\varepsilon_0}(T_j(u)) \subset T_j(B_{\varepsilon_2}(u)).$$

Hence $v = T_j^{-1}(y) \in B_{\varepsilon_2}(u) \subset V_j$. According to Proposition 8,

$$\|x - y\|^2 = \|T_j(u) - T_j(v)\|^2 > \sigma \|u - v\|^2,$$
which proves the result.

To conclude, Hypothesis (PE5) is a consequence of Lemma 2.1 of Saussol and of Hypothesis 4.

Since the hypotheses (PE1) to (PE5) are satisfied, Theorem 5.1 of [SAU] implies the properties 1 to 5 of Theorem 2 about \( V_\alpha \) and \( L^1_{m} \). Now, if \( f \in E_i \), \( f \) is equal to zero on \( \Omega \), which implies that \( f \in L^1_{m}(\Omega) \) and \( V_\alpha(\Omega) \).

To prove point 6, we shall apply Theorem 6.1 of [SAU] on every subset \( W_{j,l} \), on which a suitable power of \( T \) is mixing. Adopting the notations of Point 5 of Theorem 5.2 of [SAU], there exist real constants \( C > 0 \) and \( \rho \in ]0,1[ \) such that, for every \((j,l)\) satisfying \( 1 \leq j \leq \dim(E_1) \), \( 0 \leq l \leq L_j - 1 \), every function \( f \in L^1_{\mu_j,l}(\Omega) \) and every function \( h \in V_\alpha(\Omega) \),

\[
\left| \int_{\Omega} (f - \mu_{j,l}(f)) \circ T^{nL_j} h \, d\mu_{j,l} \right| \leq C \| f - \mu_{j,l}(f) \|_{L^1_{\mu_j,l}} \| h \|_{\alpha,L} \rho^{n}.
\]

Let us choose, then, \( h \in V_\alpha(\Omega) \) and \( f \in L^1_{\mu_j,l}(\Omega) \) (with the result that \( f \in L^1_{\mu_j,l}(\Omega) \) for every \( j,l \)). Taking the smallest common multiple \( L' \) of the \( L_j \) and summing the above inequalities, with \( n \) replaced with \( nL_j/L' \), we obtain that

\[
\left| \int_{\Omega} \left( f \circ T^{nL} h \right) \, d\mu - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(f) \mu_{j,l}(h) \right| \leq C \| f \|_{\alpha,L} \| h \|_{\alpha,L} \rho^{n},
\]

remarking that \( \| f - \mu_{j,l}(f) \|_{L^1_{\mu_j,l}} \leq 2 \| f \|_{L^1_{\mu_j,l}} \).

Point 7 is a straightforward consequence of Point 6, since \( \dim(E_1) = 1 \) and \( L_1 = 1 \). This concludes the proof of Theorem 2.

Let us now turn to Lemma 4. If \( Z_0 = (X_0, \ldots, \gamma^{k-1}X_{k-1}) \) has distribution \( \mu \), then this is the case for \( Z_n = (X_n, \ldots, \gamma^{k-1}X_{n+k-1}) \) as well. If \( f \in L^1_{\mu}(\Omega) \) and if \( h \in V_\alpha(\Omega) \), one has :

\[
\left| E(f(Z_nL'))h(Z_0) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(f) \mu_{j,l}(h) \right| \leq C \| f \|_{L^1_{\mu}} \| h \|_{\alpha,L} \rho^{n}.
\]

Let \( r, s \) be in \( \{1, \ldots, k\} \) and let \( F, H \) be measurable functions defined on \([-L, L] \). The application \( T_rH \) (defined in (2)) belongs to \( V_\alpha(\Omega) \) if and only if \( H \) is in \( L^{\infty}([-L, L], m) \) and satisfies

\[
\sup_{0 < \delta < \in \gamma^{1-r}} \delta^{-\alpha} \int_{[-L,L]} \text{Osc}(H) \left| x - \delta, x + \delta [\cap [-L, L]) \right| \, dx < +\infty.
\]

One then has

\[
\| T_r H \|_{\alpha,L} = (2L)^{k-1} \gamma^{(k-1)/2} \| H \|_{L^1_{\mu}}([-L,L]) + (2L)^{k-1} \gamma^{(k-1)/2 - \alpha(1-r)} \sup_{0 < \delta < \in \gamma^{1-r}} \delta^{-\alpha} \int_{[-L,L]} \text{Osc}(H) \left| x - \delta, x + \delta [\cap [-L, L]) \right| \, dx + 2^{2k} L^{k-1} \left( \frac{1 - \gamma^k}{1 - \gamma} \right)^{k-1} \varepsilon_0^{-\alpha} \| H \|_{L^\infty_{\mu}}([-L,L]).
\]

Consequently, if \( H \) satisfies these conditions and if \( F \) is such that \( T_sF \) belongs to \( L^1_{\mu}(\Omega) \), for example if \( F \) is measurable and bounded on \([-L, L] \), one gets the second statement of Lemma 4.
\[ E(F(X_n \times L') + s_1)H(X_{r-1}) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l})\mu_j(l(T_sF)\mu_j(l(T_sH)) \leq C||T_sF||_{L^1} \]

\[ ((2L)^{k-1} \gamma^{(k-1)/2} ||H||_{L^1([-L,L])}) \]

\[ + (2L)^{k-1} \gamma^{(k-1)/2} - \alpha(r-1) \sup_{0 < \delta < \epsilon_0 \rho_1 - \gamma} \int_{[-L,L]} \text{Osc}(H,x - \delta, x + \delta[\cap[-L,L]]) \, dx \]

\[ + 2^k \left( \frac{1 - \gamma^k}{1 - \alpha} \right)^{k-1} \epsilon_0^{-\alpha} ||H||_{L^\infty([-L,L])} \rho^n. \]

In particular, if \( H \) is the indicator function of an interval and \( F \), that of a Borel set, we obtain the first assertion of Lemma 4.

**Références**

[AFLV] ALVES José F., FREITAS Jorge M., LUZZATTO Stefano, VAIENTI Sandro, From rates of mixing to recurrence times via large deviations, Advances in Mathematics, 228 (2011), n 2 1203-1236.

[GO] GOUÉZEL Sébastien, Sharp polynomial estimates for the decay of correlations, Israel Journal of Mathematics Vol. 139 (2004), 29-65.

[HK] HOFBAUER Franz, KELLER Gerhard, Ergodic properties of invariant measures for piecewise monotonic transformations, Mathematische Zeitschrift 180 (1982), 119-140.

[ITM] IONESCU TULCEA C.T., MARINESCU G., Théorie ergodique pour des classes d’opérations non complètement continues, Annals of Mathematics Vol. 52, n2 (1950), 140-147.

[JMN] JAGER Lisette, MAES Jules, NINET Alain, Exponential decay of correlations for a real-valued dynamical system embedded in \( \mathbb{R}^2 \). Comptes rendus - Mathématique Vol. 353, n11 (2015), 1041-1045.

[LM] LASOTA Andrzej, MACKEY Michael C., Chaos, fractals and noise : stochastic aspects of dynamics, Springer Verlag, New York (1998)

[LY] LI Tien-Yien, YORKE James A., Ergodic maps on \([0,1]\) and nonlinear pseudo-random number generators, Nonlinear Analysis, Theory, Methods and Applications, 2 (1978) No. 4, pp 473-481

[SAR] SARIG Omri, Subexponential decay of correlations, Inventiones Mathematicae 150 (2002), 629-653.

[SAU] SAUSSOL Benoît, Absolutely continuous invariant measures for multidimensional expanding maps, Israel Journal of Mathematics 116 (2000), 223-248.

[YOU] YOUNG Lai-Sang, Recurrence times and rates of mixing, Israel Journal of Mathematics 110 (1999), 153-188.