An Interplay of Symmetry, Topology and Quantized Geometric Phase for the Kitaev Chain in Presence of Interaction

Rahul S,1, 2 Ranjith Kumar R,1, 2 Y R Kartik,1, 2 Amitava Banerjee,3 and Sujit Sarkar1

1Poornaprajna Institute of Scientific Research, 4, Sadashivanagar, Bangalore-560 080, India.
2Manipal Academy of Higher Education, Madhava Nagar, Manipal - 576104, India.
3Department of Physics, University of Maryland, College Park MD, 20742, USA.

(Dated: October 5, 2018)

Study of symmetry, topology and geometric phase can reveal many new and interesting results of the topological states of matter. Here we present a completely new and interesting result of symmetry, topology and quantization of geometric phase along with the physical explanation for the Kitaev chain in presence of interaction. We consider a very interesting nature of interaction which may provide a very good platform for quantum simulation physics. We show explicitly that the non-interacting symmetry presentation is not a proper criterion to characterize the topological properties of the interacting Kitaev chain. We present a detailed study of the auxiliary space in the presence of interaction. We show explicitly that the origin of the auxiliary space inside the curve is only a necessary condition but it is not a sufficient condition. We observe the emergence of extra symmetries in the auxiliary space. This work provides a new perspective on the study of topological state of interacting quantum matter.

Keywords: Geometric phase, Topological quantization, Quantum phase transition, Auxiliary space.
Introduction:
Symmetry plays an important role in the study of topological states of matter [1]. Topological insulators and superconductors are the important topological states of matter which can be classified based on the symmetry constraints [2, 3]. On the basis of presence or absence of non-spatial symmetries like time-reversal, particle-hole and chiral, one can classify a single-particle Hamiltonian into different symmetry classes [4]. In each symmetry class one can distinguish between topological distinct phases using the topological invariant. There are ten distinct symmetry classes of random matrices, which can be interpreted as first-quantized Hamiltonians of certain non-interacting fermionic systems [5]. Among the ten symmetry classes for 1D Hamiltonians, only the AIII, BDI, D, DIII and CI classes show topological states [5]. Thus there will be topological quantum phase transition between two distinct phases within a symmetry class by closing the gap. This can also be characterized by the quantization of geometric phase. This is the physics in the non-interacting picture. We will study how this physics changes drastically in presence of interaction in the following sections.

Geometric phase more commonly known as Berry phase [6], is a phase difference acquired by the state when subjected to cyclic adiabatic process [7–9]. The geometric phase in a 1D Bloch band system is called Zak phase [10]. For a given Bloch wave $\psi_k$ with a quasi momentum ‘$k$’, reciprocal lattice vector ‘$G$’, lattice spacing ‘$d$’, the Zak phase can be expressed as

$$\phi_{Zak} = i \int \frac{G}{2\pi} \langle u_k | \partial_k | u_k \rangle,$$

where $u_k(x) = e^{-ikx}\psi_k(x)$ and $G = \frac{2\pi}{d}$. The physics of geometric phase reveals many important aspects of topological state of matter [11].

There have been many breakthroughs in the field of topological quantum condensed matter starting with integer quantum Hall effect [12] and fractional quantum Hall effect [13], and later the idea of topological insulators [14–20]. One of the classic examples of this kind is 1D topological superconductor [3]. In this case topologically trivial and non-trivial phases are distinguished by the gap closing. This can be characterized by the Pfaffian of Majorana representation matrix. In general, the Pfaffian $Pf[A]$ of a $2n \times 2n$ skew-symmetric matrix $A$ is defined as,

$$Pf[A] = \frac{1}{2^n n!} \sum_{\sigma \in \Pi_{2n}} sign \Pi_{i=1}^n A_{\sigma(2i-1), \sigma(2i)}.$$  

(1)
If $A$ is a $2 \times 2$ matrix, $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then $Pf[A] = a$. In the case of topological states of matter Pfaffian of Majorana representation matrix is a topological invariant. On the other hand, one can also write the Hamiltonian in the form of Bogoliubov de Gennes (BdG) mean-field Hamiltonian using Nambu spinor. Then the anti-unitary particle-hole constraint of the BdG Hamiltonian gives rise to the quantization of geometric phase [21], which indicates the topological quantum phase transition.

The topological configuration space of a system gives rise to the particular value of topological invariant quantity like winding number. The closed curves in the configuration space are the auxiliary space curves which also specify the winding number of the system. Auxiliary space curves have a unique way of representing the topological quantum phase transition. When the system is in the topological state, the auxiliary space curve enclircles the origin; for the non-topological state, the origin lies outside the space curve; and at the point of phase transition, the origin lies on the space curve in which case the topological invariant number cannot be defined [22]. Here the auxiliary space curve is used to study the topological phase transition of the system both in presence and absence of interaction.

Motivation and Goals:

One of the main emphases of our study is that the topologically non-trivial state transits to the topologically trivial state in presence of interaction. At the same time we search for symmetry changes in the presence of interaction. The primary motivation is to consider Kitaev’s 1D lattice chain as our model system because it helps in the realization of topological state of matter. Here we study how interactions modify and destroy the non-interacting feature of the topological state. We observe that the symmetry classification for the topological characterization for the non-interacting system remains same for the few cases of the Hamiltonian, but the system does not possess the topological properties. Thus the symmetry classification to characterise the topological state for the non-interacting system is not the proper characterization for the interacting system [23] [25]. At the same time, ask whether there is any smooth connection between the physics of non-interacting and interacting Hamiltonians.

In this study, we also want to realise the question at the fundamental level, where the topological properties of the system depend on the related symmetry properties of the model Hamiltonian or the periodicity of the Brillouin zone boundary. To the best of our knowledge,
this is the first time this question has been raised. We also want to study how the auxiliary space behaviour changes with interaction and the emergence of extra symmetries. The other goal of this study is to stimulate the science of quantum simulation physics from the results of this study. This work provides a new perspective on the study of topological state of interacting quantum matter.

**Model Hamiltonian and basic aspects of Geometric phase**

**Model Hamiltonian:**

We consider the Kitaev’s chain as our model Hamiltonian \[26\],

\[
H_0 = \sum_n -t(c_n^\dagger c_{n+1} + h.c) - \mu c_n^\dagger c_n + |\Delta|(c_n c_{n+1} + h.c),
\]

where \(t\) is the hopping matrix element, \(\mu\) is the chemical potential, and \(|\Delta|\) is the magnitude of the superconducting gap. We write the Hamiltonian in the momentum space as

\[
H_1 = \sum_{k>0} \left( \mu + 2t \cos k \right) (\psi_k^\dagger \psi_k + \psi_{-k}^\dagger \psi_{-k}) + 2i\Delta \sum_{k>0} \sin k (\psi_k^\dagger \psi_{-k}^\dagger + \psi_k \psi_{-k}),
\]

where \(\psi_k^\dagger \psi_k\) is the creation (annihilation) operator of the spinless fermion of momentum \(k\).

**Results and discussion for Kitaev’s chain:**

The symmetry properties of the Kitaev’s chain reveals its topological characters. Kitaev’s chain in matrix form can be written as

\[
\mathcal{H}(k) = \begin{pmatrix}
-2t \cos(k) - \mu & 2i\Delta \sin(k) \\
-2i\Delta \sin(k) & 2t \cos(k) + \mu
\end{pmatrix}.
\]

Hamiltonian \(\mathcal{H}(k)\) is invariant under time-reversal (\(T\)), particle-hole (\(C\)) and chiral (\(S\)) symmetry operations, i.e.

\[
T \mathcal{H}(k) T^{-1} = \mathcal{H}(k), \quad C \mathcal{H}(k) C^{-1} = -\mathcal{H}(k), \quad S \mathcal{H}(k) S^{-1} = -\mathcal{H}(k).
\]

Thus the Hamiltonian falls under the symmetry class BDI of the ten symmetry classes of topological insulators and superconductors \[5\] with topological invariant number \(Z\), which takes integer values. Each value of \(Z\) indicates a set of \(\mathcal{H}(k)\) which can be interpolated continuously without breaking the symmetries and without closing the energy gap. Topological
quantum phase transition can be observed by tuning the parameters of $\mathcal{H}(k)$ to get gapless state. This closing the gap involves changes in $Z$ by one unit. To get the condition for the parameters which distinguishes between topological trivial and non-trivial phases one can calculate the Majorana number ($\mathcal{M}$).

One can rewrite eq. 2 in the Majorana fermion operators by using the relation $\gamma_{2i-1} = c_i^+ + c_i$ and $\gamma_{2i} = i(c_i^+ - c_i)$, where $\gamma_j$ represent real fermions with properties $\gamma_j^\dagger = \gamma_j$ and $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. Thus eq. 2 can be written in the Majorana representation as

$$H(k) = \frac{i}{2} \sum_{\alpha,\beta} \gamma_\alpha A_{\gamma\beta} \gamma_\beta,$$

where $A$ is a real and anti-symmetric matrix.

One can verify the existence of topological trivial and non-trivial phases by calculating the Pfaffian of Majorana representation matrix, with a real orthogonal transformation, $W$, and using the property of Pfaffian, i.e. $Pf[WA^TW^T] = Pf[A]\det[W]$. One can write the Majorana number $\mathcal{M} = \det(W(0))\det(W(\pi))$. The property of $W(k)$ i.e. $W(k)^* = W(-k)$, implies the quantized geometric phase indicating the topological quantum phase transition [27]. One can write the Hamiltonian in block-diagonal form by using the real orthogonal transformation,

$$A(k) = \begin{pmatrix}
0 & -2t \cos k - \mu + i2\Delta \sin k \\
2t \cos k + \mu + i2\Delta \sin k & 0
\end{pmatrix}.$$  

Majorana number can be expressed in terms of Pfaffian of the matrix $A(k)$ as

$$\mathcal{M} = \text{sign} \{ Pf[A(0)]Pf[A(\pi)]\}.$$  

Pfaffian of the matrix $A(k)$ at $k = 0$ and $k = \pi$ is calculated as,

$$Pf[A(0)] = -(2t + \mu), \quad Pf[A(\pi)] = 2t - \mu.$$

For the parametric condition $\mu > 2t$, the Majorana number $\mathcal{M} = +1$ indicates that the system is in the non-topological phase; also, for $\mu < 2t$, the Majorana number $\mathcal{M} = -1$ indicates that the system is in the topological phase. It is clear from this that the topological phase transition occurs at $\mu = \pm 2t$.

Existence of topological states can also be confirmed from the fig. 1. This shows the auxillary space of the Kitaev chain, which is a closed trajectory with origin inside the
curve. The integral along this trajectory takes quantized values indicating the topological quantum phase transition. The value of Zak phase is $\pi$ when the origin is inside the curve of the auxiliary space and 0 when it is outside. Since $H(k)$ has anti-unitary particle-hole symmetry it gives the reality condition for Majorana representation matrix $(A(k)^* = A(-k))$, i.e.

$$CA(k)C^{-1} = A(-k).$$

This results in the quantized Zak phase which is 0 for trivial and $\pi$ for non-trivial phases. We verify this by calculating the geometric phase for the system. First we write Kitaev’s chain in momentum space as

$$H_k = (\epsilon_k - \mu)\sigma_z - 2\Delta \sin k\sigma_y.$$

One can also write the same Hamiltonian in a rotated basis as,

$$H_k = (\epsilon_k - \mu)\sigma_x - 2\Delta \sin k\sigma_y,$$

where $\epsilon_k = -2t \cos k$. 

$$H_k = \begin{bmatrix} 0 & (\epsilon_k - \mu) \\ (\epsilon_k - \mu) & 0 \end{bmatrix} - 2\Delta \sin k \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$H_k = \begin{bmatrix} 0 & (\epsilon_k - \mu) + 2i\Delta \sin k \\ (\epsilon_k - \mu) - 2i\Delta \sin k & 0 \end{bmatrix}.$$
where $\epsilon_k - \mu = r \cos \theta$ and $2\Delta \sin k = r \sin \theta$. Now the Hamiltonian is reduced to

$$H = \begin{bmatrix} 0 & re^{i\theta} \\ re^{-i\theta} & 0 \end{bmatrix},$$

where $r = \sqrt{(2t \cos k + \mu)^2 + 4\Delta^2 \sin^2 k}$ and $\theta = -\tan^{-1}(\frac{2\Delta \sin k}{2t \cos k + \mu})$. Here we are calculating the geometric phase for the lowest energy eigen function. The basic aspects of the geometric phase are relegated to the appendix.

The Berry phase is given by

$$\gamma_n = i \oint \langle \Psi_n | \nabla_R | \Psi_n \rangle dR,$$

where $\Psi_n = \left(\frac{-1}{\sqrt{2}}\right) \left(-e^{i\tan^{-1}(\frac{2\Delta \sin k}{2t \cos k})}\right)$, and finally the Berry connection is given by,

$$A(k) = \left\langle \Psi_n \left| \frac{d\Psi_n}{dk} \right. \right\rangle = \frac{\Delta(2t + \mu \cos k)}{(\mu + 2t \cos k)^2 + 4\Delta^2 \sin^2 k}.$$

Finally,

$$\gamma_n = i \oint A \kappa = i \oint_C \frac{\Delta(2t + \mu \cos(k))}{(\mu + 2t \cos(k))^2 + 4\Delta^2 \sin^2(k)} \kappa.$$

FIG. 2. (color online) Variation of $\gamma$ with $\mu$. The red, blue, green, and black curves are for $t = 0.1, 0.2, 0.3$ and $0.4$ respectively.

In fig. 2, we present the variation of geometric phase ($\gamma$) with $\mu$. It is clear from this study that there is a topological quantum phase transition from $\gamma = \pi$ to $\gamma = 0$ [28]. We have also observed that the transition occurs at $\mu = 2t$ [29]. The physical explanation of this transition can be understood in the following way: Fig. 3 describes the energy dispersion spectrum of this model Hamiltonian, where we observe the gapless state at $k = \pm \pi$. A gapless state is present if the value of the parameters obey the transition relation $\mu = \pm 2t$. This is when transition takes place from topological to non-topological state. If the values of the parameters do not obey the transition relation, then we observe a gapped state (non
FIG. 3. Dispersion curve $E_k$ vs $k$ for the Hamiltonian $H(k)$.

topological state), as shown in the lower panel of fig. 3. Here we clearly observe that the gapless states, in other words degenerate states, appear for discrete values of $\mu$, i.e. $\mu = \pm 2t$ for $k = 0$ and $k = \pm \pi$, not for the different values of $\mu$. Thus we justify the topological characterization of the Kitaev chain from the perspective of symmetry Pfaffian number calculation, winding number calculation and also the gapless state at the topological quantum phase transition point.

**The effect of interaction in the topological state of matter:**

The model Hamiltonian in the presence of interaction term can be written as [30]

$$H = H_0 + H_I,$$

where $H_0 = (\epsilon_k - \mu)\sigma_x - 2\Delta \sin k\sigma_y$, and $H_I$ is different for three different Hamiltonians. $H_I$ is proportional to the momentum $k$ and is given by,

$$H_I = \alpha k,$$

where $\alpha$ is interaction strength [31, 32].

The present problem of this interaction is only for theoretical study of spinless system. The basic Kitaev Hamiltonian is also a spinless fermion.

The topological state of quantum matter is a very rapidly growing field. One may quantum simulate this type of interaction in different physical systems [33–35]. The addition of the interaction term to the different components of the Hamiltonian reveals many interesting results which also helps one to quantum simulate different kind of quantum many-body
systems with interactions. Therefore the interaction term we present here, may get an importance in the study of quantum simulation physics.

Simulating quantum systems like topological state of matter are practically not possible using classical systems. One can also use quantum devices that mimic the evolution of other quantum systems as quantum simulators\cite{36}. Therefore the experimental realization of these quantum devices helps us in the understanding of many interesting quantum systems. This interaction looks apparently innocent but we will notice in the course of this study that it has significant effect on the topological properties and also on the emergent symmetries in the auxiliary space.

**First case:** Here we consider the interaction in the $\sigma_x$ component. The 1D version of the interaction Hamiltonian is given by

$$H_I = \alpha k \sigma_x.$$  

Kitaev’s Hamiltonian in presence of interaction becomes,

$$H^{(1)}(k) = \begin{pmatrix}
0 & (\epsilon_k - \mu + \alpha k) + 2i\Delta \sin k \\
(\epsilon_k - \mu + \alpha k) - 2i\Delta \sin k & 0
\end{pmatrix}. \tag{15}$$

Here the interaction is added to the kinetic component of the Hamiltonian.

**Second case:** Here we consider the interaction in the $\sigma_y$ component. This is also a plausible system to quantum simulate, since the interaction Hamiltonian is linear in momentum $k$.

With this motivation we consider the 1D version of interaction Hamiltonian, given by

$$H_R = \alpha k \sigma_y.$$  

Kitaev’s Hamiltonian with this interaction becomes

$$H^{(2)}(k) = \begin{pmatrix}
0 & (\epsilon_k - \mu) + 2i\Delta \sin k + i\alpha k \\
(\epsilon_k - \mu) - 2i\Delta \sin k - i\alpha k & 0
\end{pmatrix}. \tag{16}$$

Here interaction term is added to the potential component of the model Hamiltonian.

**Third case:** Here we consider the interaction in both $\sigma_x$ and $\sigma_y$ components, which gives

$$H^{(3)}(k) = \begin{pmatrix}
0 & (\xi_k + \alpha_1 k) + 2i\Delta \sin k + i\alpha_2 k \\
(\xi_k + \alpha_1 k) - 2i\Delta \sin k - i\alpha_2 k & 0
\end{pmatrix}, \tag{17}$$

where $\xi_k = \epsilon_k - \mu$. Here interaction term is added to both potential and kinetic parts of the Hamiltonian.

**Results and discussion for the Hamiltonian $H^{(1)}(k)$:**

The matrix form of the Hamiltonian $H^{(1)}(k)$ can be written as

$$H^{(1)}(k) = \begin{pmatrix}
-2t \cos(k) - \mu - \alpha k & i2\Delta \sin(k) \\
-i2\Delta \sin(k) & 2t \cos(k) + \mu + \alpha k
\end{pmatrix}. \tag{18}$$
We observe that the $H^{(1)}(k)$ does not satisfy the condition for time-reversal and particle-hole symmetry operations but satisfies only the chiral symmetry condition:

$$\mathcal{T}H^{(1)}(k)\mathcal{T}^{-1} \neq H^{(1)}(k) \quad \mathcal{C}H^{(1)}(k)\mathcal{C}^{-1} \neq -H^{(1)}(k),$$

$$SH^{(1)}(k)S^{-1} = -H^{(1)}(k).$$

(19)

Thus from the symmetry analysis, $H^{(1)}(k)$ falls under AIII class in the ten symmetry classes with topological invariant number $\mathbb{Z}$. This indicates that there is a possibility for the topological quantum phase transition by tuning the parameters to obtain the gapless state with the change in the topological invariant number $\mathbb{Z}$. But we observe a strange behaviour of the system in that it does not allow one to calculate Majorana number $\mathcal{M}$ (eq. 9) to get the condition for the parameters. The Majorana representation matrix $A_{\alpha\beta}$ for $H^{(1)}(k)$ is given by

$$A(k) = \begin{pmatrix}
0 & \xi_k + \alpha k + i2\Delta \sin k \\
-\xi_k + \alpha k + i2\Delta \sin k & 0
\end{pmatrix},$$

(20)

where $\xi_k = -2t \cos k - \mu$. Here $A(k)$ loses the anti-symmetric property for $k = \pi$, which does not allow one to calculate the Pfaffian of the matrix. This indicates that there is no Majorana number which implies that the system is in non-tivial topological phase. This can also be verified by the trajectory of the system in the auxiliary space, given in fig. 4.

![FIG. 4. Parametric plots for $H^{(1)}(k)$ for different values of $\mu$, $\alpha$ and $t$.](image-url)
A very peculiar observation can be made from the fig. 4, although the auxiliary space curve in the upper panel encircles the origin, the curve is not closed. For the system to be in the topological state, the auxiliary space curve encircling the origin is a necessary condition, but the closeness of the curve is the sufficient condition, as we observe in the Kitaev chain (fig. 1).

The left and the right column represent the auxiliary space curves for positive and negative values of $\alpha$. Both the columns show mirror symmetric behaviour for positive and negative values of $\alpha$. Although the auxiliary space curve is symmetric with respect to positive and negative values of $\alpha$, they are not symmetric with respect to $k$. Thus we conclude that the topological properties of the system do not depend on the symmetry of the auxiliary space.

We observe that there is no closed trajectory in the auxiliary space. We also observe that the addition of interaction $\alpha k$ to Hamiltonian $H^{(1)}(k)$ results in the breaking of the periodicity of the Brillouin zone. One can observe this from energy dispersion curve (fig. 5) that $E(\pi) \neq E(-\pi)$. This lack of periodicity does not allow one to calculate the geometric/Zak phase [10, 37]. In other words the integral over the non-periodic Brillouin zone will not be a Cauchy integral and does not take the quantized value [35], which again indicates that there is no topological quantum phase transition. It is very clear from the study of $H^{(1)}(k)$, that the non-interacting symmetry parameters are not proper criterion to analyse the topological

FIG. 5. Dispersion curve of $E_k$ vs $k$ for the Hamiltonian $H^{(1)}(k)$. 

![Dispersion curve of $E_k$ vs $k$ for the Hamiltonian $H^{(1)}(k)$](image)
properties for the interacting topological states.

**Results and discussion for the Hamiltonian** $H^{(2)}(k)$:

The Hamiltonian $H^{(2)}(k)$ in the matrix form can be written as

$$H^{(2)}(k) = \begin{pmatrix} -2t \cos(k) - \mu & 2i \Delta \sin(k) + i \alpha k \\ -2i \Delta \sin(k) - i \alpha k & 2t \cos(k) + \mu \end{pmatrix}.$$  

(21)

This satisfies the conditions for time-reversal, particle-hole and chiral symmetry operations:

$$T H^{(2)}(k) T^{-1} = H^{(2)}(k), \quad C H^{(2)}(k) C^{-1} = -H^{(2)}(k),$$

$$S H^{(2)}(k) S^{-1} = -H^{(2)}(k).$$  

(22)

As in the case of Kitaev chain ($H(k)$), $H^{(2)}(k)$ also belongs to the symmetry class BDI [5] with the topological invariant $\mathbb{Z}$. One can also expect the topological quantum phase transition in $H^{(2)}(k)$. This system has anti-unitary particle-hole symmetry, thus one may consider that system is in the topological state and Zak phase must be quantized [10, 37]. But we observe that even though the symmetry class of the Hamiltonian $H(k)$ and $H^{(2)}(k)$ are the same, there is no topological non-trivial phase for this system. Addition of interaction breaks the anti-symmetric property of the Majorana representation matrix for $k = \pi$

$$A(\pi) = \begin{pmatrix} 0 & 2t - \mu + i \alpha \pi \\ -2t + \mu + i \alpha \pi & 0 \end{pmatrix}.$$  

(23)

For this case one cannot calculate the Majorana number for this system since the Pfaffian does not exist due to the lack of anti-symmetric property. This also indicates that there is no topological non-trivial phase for this system. This result is also evident from the nature of auxiliary space of $H^{(2)}(k)$. Fig. 6 shows that the trajectory in the auxiliary space is not closed and the integral along the trajectory will not take quantized values. We observe an interesting feature from this behaviour in the auxiliary space: although the origin of auxiliary space is encircled by the curve, the system is in the non-topological state. At the same time there is no mirror symmetry with $\alpha$.

This can also be verified from the energy dispersion curve (fig. 7). It shows there is no gapless state for topological quantum phase transition to occur.

Unlike in the case of $H^{(1)}(k)$, the Hamiltonian $H^{(2)}(k)$ does not have the mirror symmetric auxiliary space curves for positive and negative values of $\alpha$. 


Results and discussion for the Hamiltonian $H^{(3)}(k)$:

The matrix form of the Hamiltonian $H^{(3)}(k)$ is given by

$$
\mathcal{H}^{(3)}(k) = \begin{pmatrix}
-2t \cos(k) - \mu - \alpha_1 k & 2i\Delta \sin(k) + i\alpha_2 k \\
-2i\Delta \sin(k) - i\alpha_2 k & 2t \cos(k) + \mu + \alpha_1 k
\end{pmatrix}.
$$

We observe that $H^{(3)}(k)$ belongs to the symmetry class AIII, i.e. it only obeys the chiral
symmetry condition:
\[ TH^{(3)}(k)T^{-1} \neq H^{(3)}(k), \quad CH^{(3)}(k)C^{-1} \neq -H^{(3)}(k), \]
\[ S\mathcal{H}^{(3)}(k)S^{-1} = -\mathcal{H}^{(3)}(k). \]  (25)

As in the case of \( H^{(1)}(k) \), here also one can expect the topological quantum phase transition with change in the value of topological invariant number \( Z \). But similar to \( H^{(1)}(k) \) and \( H^{(2)}(k) \), we observe the Majorana representation matrix breaks its anti-symmetric property as we add the interaction terms,
\[
A(\pi) = \begin{pmatrix}
0 & 2t - \mu + \alpha_1 \pi + i\alpha_2 \pi \\
-2t + \mu + \alpha_1 \pi + i\alpha_2 \pi & 0
\end{pmatrix}.
\]  (26)

Thus Pfaffian does not exist for this system, which shows there is no topological non-trivial phase.

FIG. 8. Parametric plots for \( H^{(3)}(k) \) for different values of \( \mu, \alpha \) and \( t \).
FIG. 9. Dispersion curve of $E_k$ vs $k$ for the $H^{(3)}(k)$ Hamiltonian.

Here the trajectory in the auxiliary space is not closed and integral along the trajectory is not quantized. Absence of the origin inside the perimeter of the trajectory in fig. 8 shows there is no topological states for the system. The curves in the auxiliary space are neither symmetric with $\alpha$ nor with $k$. A curve in the auxiliary space encircling the origin is a necessary condition but the closeness of the curve is the sufficient condition for the system to be in the topological state. This can also be verified from fig. 9 which shows the energy dispersion for $H^{(3)}(k)$. We observe there are no gapless states responsible for the topological quantum phase transition.

A Few Relevant Calculations and Discussion for the Topological Characterization of this Interacting System.

(A). Sufficient condition for the topological characterization from the behaviour of curves in auxiliary space.

Here we mathematically prove the sufficient condition for the topological characterization of the system form the behaviour of auxiliary space.
We have
\[ H^{(3)}(k) = (-2t \cos k - \mu + \alpha_1 k)\sigma_x + (\alpha_2 k - 2\Delta \sin k)\sigma_y. \]  
(27)

We plot the parametric plot \((x(k), y(k))\),
\[ x(k) = -2t \cos k - \mu + \alpha_1 k = r(k) \cos \theta(k), \]  
(28)
\[ y(k) = \alpha_2 k - 2\Delta \sin k = r(k) \sin \theta(k), \]  
(29)
so that, in the auxiliary plane,
\[ r^2(k) = (-2t \cos k - \mu + \alpha_1 k)^2 + (\alpha_2 k - 2\Delta \sin k)^2, \]  
(30)
\[ \theta(k) = \tan^{-1} \left( \frac{\alpha_2 k - 2\Delta \sin k}{-2t \cos k - \mu + \alpha_1 k} \right). \]  
(31)
To have a closed curve for \(k\) running between \([-\pi, \pi]\) the curve must come back to its starting point, i.e.
\[ r^2(k = \pi) = r^2(k = -\pi), \quad \theta(k = \pi) = \theta(k = -\pi) \mod(2\pi). \]  
(32)
Putting these two conditions in the expression of \(r^2(k)\) and \(\theta(k)\),
\[ (-2t \cos(-\pi) - \mu + \alpha_1(-\pi))^2 + (\alpha_2(-\pi) - 2\Delta \sin(-\pi))^2 \]  
\[ = (-2t \cos(\pi) - \mu + \alpha_1(\pi))^2 + (\alpha_2(\pi) - 2\Delta \sin(\pi))^2, \]  
(33)
and
\[ \tan^{-1} \left( \frac{\alpha_2(-\pi) - 2\Delta \sin(-\pi)}{-2t \cos(-\pi) - \mu + \alpha_1(-\pi)} \right) = \tan^{-1} \left( \frac{\alpha_2(\pi) - 2\Delta \sin(\pi)}{-2t \cos(\pi) - \mu + \alpha_1(\pi)} \right). \]  
(34)
The conditions for the curve to be closed, i.e. eq. 33 and eq. 34 can be simultaneously satisfied if \(\alpha_1 = \alpha_2 = 0\).

Thus it is clear from this study that this equation \((\alpha_1 = 0 = \alpha_2)\) is only satisfied by the Hamiltonian \(H\), i.e. Kitaev chain. To the best of our knowledge this study is the first in the literature to study the necessary and sufficient conditions for the topological characterization from the behaviour of auxiliary space.
(B). A general physical explanation for the existence of topological state: from the perspective of Berry connection and geometric phase. 

BdG Hamiltonian obeys an anti-unitary particle-hole symmetry, $C = \sigma_x K$ ($\sigma_x$ and $K$ are Pauli spin matrix and complex conjugate operator respectively),

$$\{H_{BdG}, C\},$$

with $C^2 = 1$. This symmetry implies that the bands below and above the energy gap are conjugates of each other, i.e.

$$C |\Psi^o(-k)\rangle = e^{-i\phi(k)} |\Psi^e(-k)\rangle.$$ \hspace{1cm} (36)

Here $|\Psi^o\rangle$ and $|\Psi^e\rangle$ are the Bloch states of the occupied and empty bands respectively. Using this abelian Berry connection \cite{39, 40} of the occupied bands can be written as

$$\mathcal{A}^o(k) = -i \sum_\alpha \langle \psi^o_\alpha(k) | \partial_k | \psi^o_\alpha(k) \rangle,$$ \hspace{1cm} (37)

where $\alpha = 1, ..., n$ represent the independent Bloch bands. One can observe the equivalence of the Berry connection of the occupied bands at $k$ and the empty bands at $-k$ up to a gauge transformation,

$$\mathcal{A}^o(-k) = \mathcal{A}^e(k) - \sum_\alpha \partial_k \phi_\alpha(k).$$ \hspace{1cm} (38)

This constraint implies that the Zak phase over half of the Brillouin zone can be written as

$$\gamma = \int_{-\pi}^{\pi} \mathcal{A}^o(k) dk = \int_{0}^{\pi} [\mathcal{A}(k) - \partial_k \phi(k)] dk,$$ \hspace{1cm} (39)

with $\mathcal{A}(k) = \mathcal{A}^o(k) + \mathcal{A}^e(k)$. In the Majorana representation one can write the $H_{BdG}$ in diagonal form as

$$W(k)H_{BdG}W^\dagger(k) = \sigma_z diag(\epsilon_\lambda),$$ \hspace{1cm} (40)

where from the particle-hole symmetry $W$ satisfies the condition

$$CW(k)C^{-1} = W(-k).$$ \hspace{1cm} (41)

This implies that the phase factor $\phi(k)$ in eq. \cite{39} vanishes. Thus in the presence of anti-unitary particle-hole symmetry the Zak phase is quantized to integer multiples of $\pi$ indicating the gapless state with topological quantum phase transition.
But for the present case, in presence of interaction, one cannot express the Berry connection $A^0(-k)$ to $A^e(k)$ by the eq.38 and also geometric phase, $\gamma$, as in eq.39.

**C. Topological characterization from the perspective of winding number**

In general BdG Hamiltonian in the symmetry class BDI can be written in the special form as

$$H(k) = \begin{pmatrix} h_0(k) & i\Delta(k) \\ -i\Delta(k) & -h_0(k) \end{pmatrix}.$$  

This can be written in the diagonal form as

$$H(k) = \begin{pmatrix} 0 & A(k) \\ A^T(-k) & 0 \end{pmatrix},$$

where $A(k) = h_0(k) + i\Delta(k)$, satisfying the condition $A^*(k) = A(-k)$. Winding number in this case can be defined from the phase of $A(k)$,

$$A(k) = |A(k)|e^{i\theta(k)}.$$  

Considering $z(k) = e^{i\theta(k)}$ one can define the winding number as

$$w = -\frac{i}{\pi} \int_0^\pi \frac{dz(k)}{z(k)} = \frac{1}{\pi}(\theta(\pi) - \theta(0)).$$

In the case of Kitaev chain, i.e. $H(k)$, we have

$$|A(k)| = \sqrt{(2t \cos k + \mu)^2 + (2\Delta \sin k)^2},$$

$$\theta(k) = \tan^{-1} \left[ \frac{2\Delta \sin k}{-2t \cos k - \mu} \right].$$

Thus the winding number for Kitaev chain is given by

$$w = \frac{1}{\pi}(\pi - 0) = 1.$$  

Since $\tan^{-1}(0)$ can be either 0 or $\pi$, winding number can take quantized values, $w = \pm 1, 0$.

In the case of $H^{(1)}(k)$, $H^{(2)}(k)$ and $H^{(3)}(k)$ one cannot define the integral in the eq. 45. Even if we try to calculate the winding by brute force for $H^{(3)}(k)$, then we have

$$\theta(k) = \tan^{-1} \left[ \frac{2\Delta \sin k + \alpha_2 k}{-2t \cos k - \mu + \alpha_1 k} \right].$$
The winding number is given by

\[
    w = \frac{1}{\pi} \left( \tan^{-1} \left( \frac{\alpha_2 \pi}{2t - \mu + \alpha_1 k} \right) - 0 \right)
    = \frac{1}{\pi} \tan^{-1} \left( \frac{\alpha_2 \pi}{2t - \mu + \alpha_1 k} \right).
\]  

(50)

We observe that the winding number is not quantized to integer values rather takes continuous values, thus the system will be in non-topological state for all non-zero values of \( \alpha_1 \) and \( \alpha_2 \).

**Conclusions:**

We have presented the results of symmetry, topology and quantization of geometric phase along with the physical explanation for interacting Kitaev’s chain. The nature of the interaction and the results motivated quantum simulation physics. We have shown explicitly that the symmetry criteria for the non-interacting physics are not sufficient to characterize the topological state of the interacting system. We have also presented the results based on auxiliary space to derive the necessary and sufficient conditions for topological characterization.

**Acknowledgments**

The authors would like to acknowledge DST (EMR/2017/000898) for the funding and RRI library for the books and journals. The authors would like to acknowledge Mr.N.Prakash who has read this manuscript critically. Finally authors would like to acknowledge ICTS Lectures/seminars/workshops/conferences/discussion meetings of different aspects of physics.

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Appendix

Berry phase in Bloch band

The basic Hamiltonian for the Bloch band can be written as $H = \frac{k^2}{2m} + V(r)$, with the value $V(r+a) = V(r)$, where $a$ is the distance between two lattice points. The Bloch state satisfies the following conditions for the edge state: $\psi_{nK}(r + a) = e^{iqa}\psi_{nk}(r)$, i.e. the wavefunction at the point $r$ and $r+a$ are related to the phase. One can also write the model Hamiltonian as $H(q) = \frac{(k+hq)^2}{2m} + V(r)$, the wavefunction $\psi_{nq}(r) = u_{nq}(r)e^{iqr}$.

$u_{nq}(r)$ is the free particle wave function in the presence of periodic potential, which also satisfies the following condition:

$u_{nq}(r + a) = u_{nq}(r)$.

The most interesting point to be noted is the Brillouin zone in the parameter space for the transformed Hamiltonian with the eigen basis $|u_{nq}\rangle$. The states $|\psi_n(q)\rangle$ and $|\psi_n(q + h)\rangle$ satisfy the same boundary condition as that of the torus. The crystal momentum $q$ is found to vary and the Bloch state picks up a Berry phase. This is nothing but the Zak phase

$$\gamma_n = \oint dq.\langle u_n(q)|i\Delta_q|u_n(q)\rangle.$$
For the Kitaev chain (Hamiltonian $H(k)$) the physics is allright, but in the presence of interaction the wavefunction

$$|\psi_n(q)\rangle \neq |\psi_n(q + h)\rangle,$$

and the parameter of the Zak phase is not valid.