Chiral expansion of the nucleon mass to order $O(q^6)$

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Abstract

We present the results of a complete two-loop calculation at order $O(q^6)$ of the nucleon mass in manifestly Lorentz-invariant chiral perturbation theory. The renormalization is performed using the reformulated infrared renormalization, which allows for the treatment of two-loop integrals while preserving all relevant symmetries, in particular chiral symmetry.

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I. INTRODUCTION

Chiral perturbation theory (ChPT) \cite{1, 2, 3} is the effective field theory for the strong interactions at low energies and has been highly successful in the application to purely mesonic processes (for a review see, e.g., Ref. \cite{4}). The extension to the one-nucleon sector was first addressed in Ref. \cite{5} and originally seemed to be problematic due to the presence of the nucleon mass as an additional mass scale that does not vanish in the chiral limit. When using dimensional regularization in combination with the modified minimal subtraction scheme there is no direct correspondence between the loop expansion and the chiral expansion. The first solution to this power counting problem was given by heavy-baryon ChPT (HBChPT) \cite{6, 7}, in which an expansion in inverse powers of the nucleon mass is performed in the Lagrangian. Later, several manifestly Lorentz-invariant renormalization schemes have been developed that also result in a proper power counting \cite{8, 9, 10, 11, 12, 13}, with the infrared (IR) regularization of Ref. \cite{9} the most commonly used scheme. In its original formulation IR regularization is applied to one-loop integrals, while the reformulated version of Ref. \cite{14} is also applicable to multi-loop diagrams \cite{15}. A different generalization of IR regularization to two-loop diagrams was suggested in Ref. \cite{16}.

Calculations at the two-loop level can be used to test the convergence behavior of the chiral expansion, which, compared to the purely mesonic part of the theory, seems to be slower in the baryonic sector. This is also of interest to lattice calculations, where chiral extrapolations are performed to obtain quantities at physical pion masses (see, e.g., Ref. \cite{17}). The nucleon mass at order $O(q^5)$ has been analyzed in the framework of HBChPT \cite{18}, including the evaluation of two-loop diagrams. In Ref. \cite{19} renormalization group techniques were used to determine the leading nonanalytic contributions to the nucleon axial-vector coupling constant $g_A$ at the two-loop level, which are independent of the applied renormalization scheme. To the best of our knowledge, however, no complete two-loop calculation in a manifestly Lorentz-invariant formulation of baryon ChPT (BChPT) has been performed so far. The determination of the nucleon mass up to a given order is one of the simplest calculations that can be performed in BChPT up to that order. This makes it the ideal physical quantity to perform a complete and consistent calculation at the two-loop level. In this Letter we present the results of such a calculation up to and including order $O(q^6)$ using the reformulated infrared renormalization.

II. LAGRANGIAN AND POWER COUNTING

The effective Lagrangian relevant for the calculation of the nucleon mass to order $O(q^6)$ is given by the sum of a purely mesonic and a one-nucleon part,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \mathcal{L}^{(3)}_{\pi N} + \mathcal{L}^{(4)}_{\pi N} + \mathcal{L}^{(5)}_{\pi N} + \mathcal{L}^{(6)}_{\pi N} + \cdots .$$  \hspace{1cm} (1)

The purely mesonic Lagrangian at order $O(q^2)$ is given in Ref. \cite{2}. Reference \cite{3} contains the mesonic Lagrangian at order $O(q^4)$ as well as the lowest-order and next-to-leading-order nucleonic Lagrangians. The Lagrangians of the nucleon sector at order $O(q^3)$ and $O(q^4)$ can be found in Refs. \cite{20, 21}. The complete Lagrangians at order $O(q^5)$ and $O(q^6)$ have not yet been constructed. However, up to the order we are considering, the nucleon mass does not receive any contributions from the Lagrangian at order $O(q^5)$, since only even powers of the pion mass can be generated from contact terms. The contributions from the Lagrangian at
order $O(q^6)$ will be of the form $\hat{g}_1 M^6$, where $\hat{g}_1$ denotes a linear combination of low-energy coupling constants (LECs) from $\mathcal{L}_{\pi\pi N}^{(6)}$.

We use the following standard power counting [22, 23]: Each loop integration in $n$ dimensions is counted as $q^n$, a pion propagator as $q^{-2}$, a nucleon propagator as $q^{-1}$ and vertices derived from $\mathcal{L}_i$ and $\mathcal{L}_{\pi N}^{(j)}$ as $q^i$ and $q^j$, respectively.

### III. RESULTS AND DISCUSSION

Figure 1 shows the one-loop diagrams which generate nonvanishing contributions to the nucleon mass up to order $O(q^6)$. Diagrams with mass insertions in the nucleon propagator are taken into account by shifting the mass in the undressed nucleon propagator [9]. Diagrams (a) and (d) are of order $O(q^3)$ and $O(q^4)$, respectively. Diagrams (b) and (c) are of order $O(q^5)$, while diagram (e) counts as $O(q^6)$. The relevant two-loop diagrams are shown in Fig. 2. According to the power counting there are additional diagrams at the given order, which however give vanishing contributions to the nucleon mass up to this order. An example would be diagram 2 (c) with one first-order vertex replaced by a second-order one.

The renormalization of one- and two-loop integrals is performed using the reformulated infrared renormalization of Ref. [14]. Details of the renormalization procedure will be given in a forthcoming publication [24].

Since the perturbative expansion is performed around a ground state realized in the Nambu-Goldstone mode, the quark mass expansion of physical quantities contains analytic as well as nonanalytic terms [25]. The chiral expansion of the nucleon mass up to order $O(q^6)$ reads

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2 \frac{M}{\mu} + k_8 M^6 \ln \frac{M}{\mu} + k_9 M^6,$$

where $M^2 = 2B\hat{m}$ is the leading-order expression for the pion mass squared in terms of the average light quark mass, $\hat{m} = (m_u + m_d)/2$ [2], and $m$ stands for the nucleon mass in the chiral limit. Combining the contributions from contact interactions with the one- and two-loop results the coefficients $k_i$ are given by

$$k_1 = -4c_1,$$

where $c_1$ is a coefficient from the contact interaction. The terms $k_2, k_3, \ldots, k_9$ are nonanalytic corrections arising from the higher-order terms in the chiral expansion.
\begin{align*}
k_2 &= -\frac{3 g_A^2}{32 \pi F^2}, \\
k_3 &= -\frac{3}{32 \pi^2 F^2 m} \left( g_A^2 - 8 c_1 m + c_2 m + 4 c_3 m \right), \\
k_4 &= -\hat{e}_1 - \frac{3}{128 \pi^2 F^2 m} \left( 2 g_A^2 - c_2 m \right), \\
k_5 &= \frac{3 g_A^2}{1024 \pi^4 F^4} \left( 16 g_A^2 - 3 \right), \\
k_6 &= \frac{3 g_A^2}{256 \pi^3 F^4} \left[ g_A^2 + \frac{\pi^2 F^2}{m^2} - 8 \pi^2 (3 l_3 - 2 l_4) - \frac{32 \pi^2 F^2}{g_A} (2 d_{16} - d_{18}) \right], \\
k_7 &= -\frac{3}{256 \pi^3 F^4 m} \left[ g_A^2 - 6 c_1 m + c_2 m + 4 c_3 m \right], \\
k_8 &= -\frac{g_A^4}{64 \pi^4 F^4 m} - \frac{g_A^2}{1024 \pi^4 F^4 m^2} \left[ 384 \pi^2 F^2 c_1 + 5 m + 192 \pi^2 m (2 l_3 - l_4) \right] \\
&\quad - \frac{3 g_A}{8 \pi^2 F^2 m} [2 d_{16} - d_{18}] + \frac{3}{256 \pi^4 F^4} [2 c_1 - c_3] \\
&\quad + \frac{1}{8 \pi^2 F^2 m} [6 c_1 c_2 - 12 \hat{e}_2 m - 6 \hat{e}_3 m - e_{16} m], \\
k_9 &= \hat{g}_1 - \frac{g_A^4}{24576 \pi^4 F^4 m} \left( 49 + 288 \pi^2 \right) - \frac{3 g_A}{16 \pi^2 F^2 m} (2 d_{16} - d_{18}) \\
&\quad - \frac{g_A^2}{1536 \pi^4 F^4 m^3} \left[ m^2 (1 + 18 \pi^2) - 12 \pi^2 F^2 + 144 \pi^2 m^2 (3 l_3 - l_4) \\
&\quad + 288 \pi^2 F^2 m c_1 - 24 \pi^2 m^3 (c_3 - 2 c_4) \right],
\end{align*}

FIG. 2: Two-loop diagrams contributing to the nucleon mass up to order $O(g^6)$. 

\[k_2 = -\frac{3 g_A^2}{32 \pi F^2},\]
\[k_3 = -\frac{3}{32 \pi^2 F^2 m} \left( g_A^2 - 8 c_1 m + c_2 m + 4 c_3 m \right),\]
\[k_4 = -\hat{e}_1 - \frac{3}{128 \pi^2 F^2 m} \left( 2 g_A^2 - c_2 m \right),\]
\[k_5 = \frac{3 g_A^2}{1024 \pi^4 F^4} \left( 16 g_A^2 - 3 \right),\]
\[k_6 = \frac{3 g_A^2}{256 \pi^3 F^4} \left[ g_A^2 + \frac{\pi^2 F^2}{m^2} - 8 \pi^2 (3 l_3 - 2 l_4) - \frac{32 \pi^2 F^2}{g_A} (2 d_{16} - d_{18}) \right],\]
\[k_7 = -\frac{3}{256 \pi^3 F^4 m} \left[ g_A^2 - 6 c_1 m + c_2 m + 4 c_3 m \right],\]
\[k_8 = -\frac{g_A^4}{64 \pi^4 F^4 m} - \frac{g_A^2}{1024 \pi^4 F^4 m^2} \left[ 384 \pi^2 F^2 c_1 + 5 m + 192 \pi^2 m (2 l_3 - l_4) \right] \\
&\quad - \frac{3 g_A}{8 \pi^2 F^2 m} [2 d_{16} - d_{18}] + \frac{3}{256 \pi^4 F^4} [2 c_1 - c_3] \\
&\quad + \frac{1}{8 \pi^2 F^2 m} [6 c_1 c_2 - 12 \hat{e}_2 m - 6 \hat{e}_3 m - e_{16} m],\]
\[k_9 = \hat{g}_1 - \frac{g_A^4}{24576 \pi^4 F^4 m} \left( 49 + 288 \pi^2 \right) - \frac{3 g_A}{16 \pi^2 F^2 m} (2 d_{16} - d_{18}) \\
&\quad - \frac{g_A^2}{1536 \pi^4 F^4 m^3} \left[ m^2 (1 + 18 \pi^2) - 12 \pi^2 F^2 + 144 \pi^2 m^2 (3 l_3 - l_4) \\
&\quad + 288 \pi^2 F^2 m c_1 - 24 \pi^2 m^3 (c_3 - 2 c_4) \right],\]
\[
+ \frac{1}{6144\pi^4 F^4 m} \left[ 3 - 1152\pi^2 F^2 c_1 c_2 + 1152\pi^2 F^2 m \hat{e}_3 + 320\pi^2 F^2 m e_{16} \right].
\] (3)

To simplify the notation we use

\[
\begin{align*}
\hat{e}_1 & = 16e_{38} + 2e_{115} + 2e_{116}, \\
\hat{e}_2 & = 2e_{14} + 2e_{19} - e_{36} - 4e_{38}, \\
\hat{e}_3 & = e_{15} + e_{20} + e_{35}
\end{align*}
\]

for combinations of fourth-order baryonic LECs, while \( \hat{g}_1 \) denotes a combination of LECs from the Lagrangian at order \( O(q^6) \).

In general, the expressions of the coefficients in the chiral expansion of a physical quantity differ in various renormalization schemes, since analytic contributions can be absorbed by redefining LECs. However, this is not possible for the leading nonanalytic terms, which therefore have to agree in all renormalization schemes. Comparing our result with the HBChPT calculation of \([18]\), we see that the expressions for the coefficients \( k_2, k_3, \) and \( k_5 \) agree as expected. At order \( O(q^6) \) also the coefficient \( k_7 \) has to be the same in all renormalization schemes. Note that, while \( k_6 M^5 \) and \( k_8 M^6 \ln \frac{M}{\mu} \) are nonanalytic in the quark masses, the algebraic form of the coefficients \( k_6 \) and \( k_8 \) are renormalization scheme dependent. This is due to the different treatment of one-loop diagrams. The counterterms for one-loop subdiagrams depend on the renormalization scheme and produce nonanalytic terms proportional to \( M^5 \) and \( M^6 \ln \frac{M}{\mu} \) when used as vertices in counterterm diagrams.

The numerical contributions from higher-order terms cannot be determined so far, since most expressions in Eq. (3) contain unknown LECs from the Lagrangians of order \( O(q^4) \) and higher. Only the coefficient \( k_5 \) is free of these higher-order LECs and is given in terms of the axial-vector coupling constant \( g_A \) and \( F \). While the values for both \( g_A \) and \( F \) should be taken in the chiral limit, we evaluate \( k_5 \) using the physical values \( g_A = 1.2695(29) \) \([26]\) and \( F_\pi = 92.42(26) \) MeV in order to get an estimate of higher-order corrections. Setting \( \mu = m_N, \quad m_N = (m_p + m_n)/2 = 938.92 \) MeV, and \( M = M_{\pi^+} = 139.57 \) MeV we obtain \( k_5 M^5 \ln(M/m_N) = -3.8 \) MeV. This amounts to approximately 25\% of the leading nonanalytic contribution at one-loop order, \( k_2 M^3 \). For a discussion of the importance of the terms at order \( O(q^5) \) at unphysical quark masses as used in lattice extrapolations see Ref. \([27]\).

IV. SUMMARY

Using the reformulated infrared regularization \([14]\) we have calculated the nucleon mass up to and including order \( O(q^6) \). This is the first complete two-loop calculation in manifestly Lorentz-invariant baryon chiral perturbation theory. The applied renormalization scheme preserves the standard power counting and respects all symmetries. Our results for the renormalization scheme independent terms agree with the HBChPT results of Ref. \([18]\).
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