Solution of Wheeler–De Witt Equation, Potential Well and Tunnel Effect

HUANG Yong-Chang$^{1,2,*}$ and WENG Gang$^1$

$^1$Institute of Applied Mathematics and Physics, Beijing University of Technology, Beijing 100022, China
$^2$CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China

(Received February 24, 2005)

Abstract This paper uses the relation of the cosmic scale factor and scalar field to solve Wheeler–De Witt equation, gives the tunnel effect of the cosmic scale factor a and quantum potential well of scalar field, and makes it fit with the physics of cosmic quantum birth. By solving Wheeler–De Witt equation we achieve a general probability distribution of the cosmic birth, and give the analysis of cosmic quantum birth.

PACS numbers: 98.80.Bp
Key words: Wheeler–De Witt equation, cosmology, potential well, tunnel effect

1 Introduction

The standard cosmological theories successfully explain the evolvement of the universe from the big Bang up to now. The theories can explain the phenomena such as the origin of the elements (for example, helium), the redshift of galaxy spectrum, 2.7 K microwave background radiation, the galaxy count, the homogeneous property, isotropy in great scale and so on.

However, there exist unfathomed difficult problems. Especially there exists the problem about the cosmic evolvement details of the period from $t = 0$ to $10^{-10}$ second. The inflationary cosmological models can solve the problems such as the magnetic monopole problem, the horizon difficulty and the problem of flat property.

The quantum cosmology gives a hope to solve the problem of the birth of the universe. The cosmic wave function describes the quantum state of the universe. The cosmic wave function satisfies the cosmological Wheeler–De Witt equation. By confirming boundary conditions, one can quantitatively study the problem about the birth of the universe. Wheeler–De Witt equation is a Schrödinger equation of zero eigenvalue.$^{[1]}$

Confirming the boundary conditions is to confirm the integral path of the ground state wave function, the path integral of the quantum mechanics is to sum the history.

The present universe is isotropic, in great scale, homogenous and almost flat. The natures are that the universe had experienced the inflation of extreme high temperature and matter density.$^{[2]}$ The inflation makes temperature descend and the matter becomes sparse. After the big Bang the universe enters grand unified period, namely, unification period of the strong, weak and electromagnetic interactions. Because the CP breaking makes the universes have more baryons than anti-baryons and more positive matter than anti-matter, atomic nuclei are formed when temperature further declines, such as helium and deuteron nuclei. One of the missions of the quantum cosmology is to ensure the probability distribution of the early cosmic state. The early cosmic state can be related by scalar field $\Phi.$$^{[3–6]}$

In chaotic inflationary theories,$^{[7]}$ the probability of the cosmic quantum birth is marked by $\rho.$ When $\Phi$ is the greatest, $\rho$ may tend to infinitude, the universe inflates extremely, the scalar field $\Phi$ can no longer roll down.$^{[8]}$ Therefore, the great field needs to be cut off, which is called the great field difficulty.

Wheeler–De Witt equation$^{[9]}$ beneath is established to describe the quantum birth of the universe,

$$\frac{\partial^2 \Psi}{\partial a^2} - \frac{6}{k^2 a^2} \frac{\partial^2 \Psi}{\partial \Phi^2} - \frac{144\pi^4}{k^4} \left( k_c a^2 - \frac{k^2}{3} a^4 V(\Phi) \right) \Psi = 0, \quad (1)$$

where $a$ is the cosmic scale factor, $\Phi$ the scalar field, $\Psi$ the cosmic wave function, and $V(\Phi)$ the scalar potential, $k_c$ may equal one, $k$ equals the reciprocal of Planck mass $M_p.$ In fact, equation (1) satisfies quantitative causal relation, which means that changes (cause) of some quantities in Eq. (1) must cause relative changes (result) of the other quantities in Eq. (1) so that the equation’s right side keeps invariant. The cosmic naissance experienced an inflationary epoch and then entered the slow rolling dilation, which may be the gentle dilation or the frequent stir dilation.

References [10] and [11] firstly fix $\Phi$, namely, took $\partial \Psi/\partial \Phi = 0$, then it follows from Eq. (1) that

$$\frac{\partial^2 \Psi}{\partial a^2} = \frac{144\pi^4}{k^4} \left( k_c a^2 - \frac{k^2}{3} a^4 V(\Phi) \right) \Psi. \quad (2)$$

$^*$Corresponding author, E-mail: ychuang@bjut.edu.cn
Therefore the tunneling probability can be obtained \((k_c = 1)\)\(^{[11]}\) as follows:

\[
\rho_a = c_1 \exp \left(\frac{-24\pi^2}{k^2 V(\Phi)}\right). \tag{3}
\]

If one takes

\[
V(\Phi) = \frac{m^2 \Phi^2}{2}, \tag{4}
\]

when \(\Phi_0 \to \infty\), at the beginning the universe inflates extremely. It has the great field difficulty.\(^{[10,11]}\)

Then the literatures\(^{[10,11]}\) fix \(a\), i.e. set \(\partial \Psi/\partial a = 0\), it follows that

\[
\partial^2 \Psi \over \partial \Phi^2 = 8\pi^4 a^6 \left[V(\Phi) - \frac{3}{a^2 k^2}\right] \Psi. \tag{5}
\]

In succession reference \([11]\) resolves the formula (5), and discusses the property of the resolution. It is obvious that such study is quite a special study.

This paper follows the authors of Refs. \([6]\), \([7]\), \([10]\), and \([11]\), generalizes their studies to more general situation, and investigates the cosmic birth. The concrete arrangement is as follows. Section 2 is solution of the Wheeler–De Witt equation for the cosmic evolution dynamics. Section 3 studies the tunneling and potential well effects of cosmic quantum birth and evolution, the solution of cosmic quantum birth and evolution is discussed in different situations in Sec. 4, and the last section is a summary and conclusion.

### 2 Solution of Wheeler–De Witt Equation for Cosmic Evolution Dynamics

According to the standpoint of the inflationary cosmology, the universe originates from the decay of scalar field \(\Phi\), thus the cosmic scalar factor depends on scalar field \(\Phi\). Many authors such as those of Refs. \([1]\), \([7]\), and \([10]\) adopt \(a = \alpha/V(\Phi)\) (\(\alpha\) is a parameter), while adopting the potential \(V = \lambda_0 \Phi^2\) (\(\lambda_0\) is a parameter).\(^{[7,11]}\) The potential is mainly adopted in many inflationary cosmological models, thus we can acquire the function representation

\[
a = \alpha_0 \Phi^{-2}. \tag{6}
\]

It follows that \(a = \alpha_0 \Phi^{-2} = \alpha_0 \lambda_0/V\), that is, \(\alpha_0 = \alpha/\lambda_0\) serves as the parameter. In this way, \(a\) is guaranteed simultaneously to be invariant when \(\Phi\) is transformed into \(-\Phi\). And \(a\) is largend when \(\Phi\) is dwin-dled. As a consequence it follows that

\[
\left(\frac{\partial a}{\partial \Phi}\right) = -2\alpha_0 \Phi^{-3}. \tag{7}
\]

The cosmic Weeler–De Witt equation can be changed by using

\[
\frac{1}{a^2} \frac{\partial^2 \Psi}{\partial \Phi^2} = \frac{1}{a^2} \left(\frac{\partial a}{\partial \Phi}\right)^2 \frac{\partial^2 
\Psi}{\partial a^2} = \frac{144\pi^4}{k^4} \left(k_c a^2 - \frac{k^2}{3} \alpha \lambda_0 \Phi\right). \tag{8}
\]

Using \(\Phi = (a/\alpha_0)^{-1/2}\) we can obtain

\[
\frac{6}{k^2 a^2} \frac{\partial^2 \Psi}{\partial a^2} = \frac{24a}{\alpha^2 k^2} \Psi. \tag{9}
\]

Thus there is

\[
\left(1 - \frac{24a}{\alpha_0 k^2}\right) \frac{d^2 \Psi}{da^2} = \frac{144\pi^4}{k^4} \left(k_c a^2 - \frac{k^2}{3} \alpha \lambda_0 \Phi\right). \tag{10}
\]

Because there is a relation

\[
\frac{1}{\Psi} \frac{d^2 \Psi}{da^2} = \frac{\left(d \ln \Psi\right)}{da^2} \left(\frac{d \ln \Psi}{da}\right)^2, \tag{11}
\]

then we can obtain

\[
\ln \Psi + ca + c_0 + \ln \Psi + c_1 = G, \tag{12}
\]

where \(c_0, c_1,\) and \(c\) are the constants of integration, and one has

\[
G = \int \frac{144\pi^4 k^{-4}}{1 - 24a/\alpha_0 k^2} \left(k_c a^2 - \frac{k^2}{3} \alpha \lambda_0 \Phi\right) da. \tag{13}
\]

By carefully calculating Eq. (13), we can obtain

\[
G(a) = \frac{\left(1 - \frac{24a}{\alpha_0 k^2}\right)}{k^2} \frac{k_c}{6} a^2 + \frac{k_c}{2} a^2 + \frac{k^2 \alpha_0}{24} \left(a - \frac{k^2 \alpha_0}{24}\right) \ln a - \frac{k^2 \alpha_0}{24} \tag{14}
\]

where \(s_0\) and \(s_1\) are the integration constants, the absolute value in Eq. (14) can guarantee that \(a\) can take less value than \(k^2 \alpha_0/24\).

Defining \(\ln \Psi = f\), formula (12) can be rewritten as

\[
f^2 + (2c_1 + 1)f + ca + c_0 - G = 0, \tag{15}
\]

\[
f = \frac{- (2c_1 + 1) \pm \sqrt{(2c_1 + 1)^2 - 4(\alpha a + c_0 - G)}}{2}. \tag{16}
\]
So we can obtain
\[ \Psi(a) = c_2 \exp\left\{ \pm \left[ G(a) - ca + \left( c_1 + \frac{1}{2} \right) - c_0 \right]^{1/2} \right\}, \]
where \( c_2 = \exp[-(c_1 + 1/2)] \). Because there is a relation \( a = \alpha_0 \Phi^{-2} \), we have
\[ \Psi(\Phi) = c_2 \exp\left\{ \pm \left[ G(\Phi) - \alpha_0 \Phi^{-2} + \left( c_1 + \frac{1}{2} \right) - c_0 \right]^{1/2} \right\}. \]

Substituting \( a = \alpha_0 \Phi^{-2} \) into Eq. (14) we obtain
\[
G(\Phi) = \frac{-6\pi^4 \alpha_0}{k^2} \left\{ k_c \left[ \alpha_0^2 \Phi^{-6} + \frac{k^2 \alpha_0}{24} \Omega^{-4} + \left( \frac{k^2 \alpha_0}{24} \right)^2 \ln |\alpha_0 \Phi^{-2} - \frac{k^2 \alpha_0}{24}| \right]
- \left( \frac{k^2 \alpha_0}{24} \right)^2 \frac{1}{6} \ln |\alpha_0 \Phi^{-2} - \frac{k^2 \alpha_0}{24}| \right) \right. \\
+ \left( \frac{k^2 \alpha_0}{24} \right)^3 \frac{1}{2} \ln |\alpha_0 \Phi^{-2} - \frac{k^2 \alpha_0}{24}| \right) + s_0 \alpha_0 \Phi^{-2} + s_1 \right\}. \]

So formula (17) can be concretely rewritten as
\[
\Psi(a) = c_2 \exp\left\{ \pm \left[ -\frac{6\pi^4 \alpha_0}{k^2} \left\{ k_c \left[ \frac{a^2}{6} + \frac{k^2 \alpha_0}{24} \right] \Omega^{-2} + \left( \frac{k^2 \alpha_0}{24} \right)^2 \ln |a - \frac{k^2 \alpha_0}{24}| \right]
- \left( \frac{k^2 \alpha_0}{24} \right)^2 \frac{1}{6} \ln |a - \frac{k^2 \alpha_0}{24}| \right) \right. \\
- \left( \frac{k^2 \alpha_0}{24} \right)^3 \frac{1}{2} \ln |a - \frac{k^2 \alpha_0}{24}| \right) + s_0 a + s_1 \right\}. \]

We further discuss the tunneling and potential well effects of the cosmic generation as follows.

### 3 Tunneling and Potential Well of Cosmic Quantum Generation and Evolution

From Eq. (10) we can obtain
\[
\frac{d^2 \Psi}{da^2} = \frac{144 \pi^4}{k^4} \left( k_c a^2 - \frac{k^2}{3} a^3 \alpha_0 \lambda_0 \right)
\times \left( 1 - \frac{24 \alpha}{\alpha_0 k^2} \right)^{-1} \Psi. \tag{21}
\]

By using Schrödinger equation
\[
H \Psi = E \Psi, \quad H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x), \tag{22}
\]
it follows that\[11]
\[
\frac{d^2 \Psi}{dx^2} = \frac{2m}{\hbar^2} (U(x) - E) \Psi. \tag{23}
\]

Then we can obtain effective potential
\[
U_1(a) = \frac{144 \pi^4}{k^4} \left( k_c a^2 - \frac{k^2}{3} a^3 \alpha_0 \lambda_0 \right)
\times \left( 1 - \frac{24 \alpha}{\alpha_0 k^2} \right)^{-1}, \quad E_a = 0. \tag{24}
\]

When \( a_m = \alpha_0 k^2 / 24 \), \( U_1(a) \) has the maximum value thus we find that \( U_1(a) \) is an infinitely deep potential well. This is the conclusion that people did not obtain in the past, for instance, literatures[1], [7], [10], and[11] did not have this conclusion. When \( a \) changes from \( a = 0 \) to \( a_0 \) and \( a_0 > a_m \), \( U(a) \) alters into \( U(a_0) = 0 \) and there is tunneling through barrier.

By using \( \alpha = \alpha_0 \Phi^{-2} \) and \( da = -2 \alpha_0 \Phi^{-3} d \Phi \) we obtain
\[
\frac{d^2 \Psi}{d \Phi^2} = \frac{576 \pi^4}{k^4} \left( k_c \alpha_0^2 \Phi^{-4} - \frac{k^2}{3} \alpha_0^4 \lambda_0 \Phi^{-6} \right)
\times \left( 1 - \frac{24 \alpha}{\alpha_0 k^2} \right)^{-1} \Phi^2 \Phi^{-6} \Psi. \tag{25}
\]

Similarly we can obtain the effective potential about the scalar field \( \Phi \) as follows:
\[
U_2(\Phi) = \frac{576 \pi^4}{k^4} \left( k_c \alpha_0^2 - \frac{k^2}{3} \alpha_0^4 \lambda_0 \Phi^{-2} \right)
\times \left( 1 - \frac{24 \alpha}{\alpha_0 k^2} \right)^{-1} \Phi^{-10}. \tag{26}
\]

It is obvious that \( U_2(\Phi) \) tends to infinitude when \( \Phi = 0 \) and \( \Phi^2 = k^2 / 24 \). Therefore there is an infinitely deep potential well on the situation of \( \Phi \in (0, \sqrt{2}\kappa/k) \). When \( a \) changes from \( a = 0 \) to \( a_0 \), \( U(a) \) alters into \( U(a_0) = 0 \) and there is tunneling through barrier. Then \( \Phi \) is changed into \( \Phi_0 \) and \( \Phi(\Phi_0) \) is changed into \( U(\Phi_0) = 0 \). Namely, there is quantum well effect. By means of \( \alpha = \alpha_0 \Phi^{-2} \), we obtain the conclusion that \( \Phi \) decreases when \( a \) agrandizes.

### 4 Discussion of Solution of Cosmic Quantum Generation

Because equation (20) is a general wave function about the cosmic quantum generation, we need to discuss the problem about sign \pm in the exponential wave function. In the location where there is \( a \rightarrow \varepsilon \) from zero, known as the infinitesimal value of \( a \) when the universe just forms the micromolecule, equation (20) can keep only the terms that have the main contribution for the wave function, that is, we can remove the high-order term of \( a \). Then we can obtain

\[
\frac{d^2 \Psi}{d \Phi^2} = \frac{576 \pi^4}{k^4} \left( k_c \alpha_0^2 \Phi^{-4} - \frac{k^2}{3} \alpha_0^4 \lambda_0 \Phi^{-6} \right)
\times \left( 1 - \frac{24 \alpha}{\alpha_0 k^2} \right)^{-1} \Phi^2 \Phi^{-6} \Psi. \tag{25}
\]
\[ \Psi(a) = c_2 \exp \left\{ \pm \frac{-6\pi^4\alpha_0}{k^2} \left[ \frac{k^2\alpha_0}{24} \right]^2 \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^2 a ight\} 
- \frac{k^2}{3} \alpha_0 \alpha_0 \left[ \left( \frac{k^2\alpha_0}{24} \right)^3 \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^3 a \right] + s_0 a + s_1 \right\}. 
\] (27)

When there is
\[ -\frac{6\pi^4\alpha_0}{k^2} \left[ \frac{k^2\alpha_0}{24} \right]^2 \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^2 a \times \left( a - \frac{k^2\alpha_0}{24} \right) \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^3 a \right] + s_0 a + s_1 \right\} - ca + \left( c_1 + \frac{1}{2} \right)^2 - c_0 = -L < 0, \]

one has
\[ \Psi(a) = c_2 \exp \left\{ \pm \frac{\left( \frac{\pi^4\alpha_0^2}{4\alpha_0^4} \right)^{1/2}}{6} \right\}. \] (29)

When the exponent of \( \Psi(a) \) takes the positive sign, \( a \) increases, \( \Psi(a) \) exponentially increases, then the probability exponentially increases so that the universe aggrandizes to inflate.\(^{[9,10]}\)

\[ \rho(a) = c_2^2 \exp \left\{ \frac{2}{k^2} \left[ -6\pi^4\alpha_0 \left\{ \frac{k^2\alpha_0}{24} \right]^2 \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^2 a \right\} 
- \left( \frac{k^2\alpha_0}{24} \right)^2 a \times \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^3 a \right] + s_1 \right\} - ca + \left( c_1 + \frac{1}{2} \right)^2 - c_0 \right\} \right\}. \] (31)

\[ \Psi(\Phi) = c_2 \exp \left\{ \frac{-6\pi^4\alpha_0}{k^2} \left[ \frac{k^2\alpha_0}{24} \right]^2 \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^2 a \times \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^3 a \right\} \right\}. \] (32)

Due to \( a = \alpha_0 \Phi^{-2} \), its corresponding wave function about \( \Phi \) is as follows. At this point, there is \( \Phi \rightarrow \varepsilon_1 \) and \( \varepsilon_1 \) is the minimum thus \( \Phi \) can be adopted as
\[ \Psi(\Phi) = c_2 \exp \left\{ \frac{\left( \frac{\pi^4\alpha_0^2}{4\alpha_0^4} \right)^{1/2}}{6\Phi^2} \right\}. \] (30)

When the exponent of \( \Psi(\Phi) \) still aggrandizes, the fact that the matter field \( \Phi \) decays causes the universe to inflate. Therefore, the two kinds of discussions above are consistent, and our researches do not have the great field difficulty in the literatures \([10,11]\). The general probability of the cosmic quantum generation is

\[ \rho(\Phi) = c_2^2 \exp \left\{ \frac{2}{k^2} \left[ -6\pi^4\alpha_0 \left[ \frac{k^2\alpha_0}{24} \right]^2 \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^2 a \times \ln \left| a - \frac{k^2\alpha_0}{24} \right| - \left( \frac{k^2\alpha_0}{24} \right)^3 a \right] + s_0 a + s_1 \right\} - ca + \left( c_1 + \frac{1}{2} \right)^2 - c_0 \right\} \right\}. \] (33)
It is obvious that our method is more general and differs from the method of “twice loose shoe” in the literatures [10] and [11]. Because the universe originates from the decay of scalar field $\Phi$, the cosmic radius $a$ depends upon the scalar field $\Phi$.

In other words, the universe goes through extreme inflation, or chaotic inflation, then enters the slow rolling dilatation, finally there is only minor quantum stir left, the universe further enters the standard stage described by the cosmological model of the big Bang.

5 Summary and Conclusion

This paper solves Wheeler–De Witt equation of describing the cosmic quantum birth and the inflation that the universe has experienced, and overcomes the scarcity of the method of “twice loose shoe”. The method of “twice loose shoe”[10,11] seeks to use, in turn, the cosmic scale factor $a$ (in the same time, demands $\partial \Psi / \partial \Phi = 0$) and the scalar field $\Phi$ (in the same time, demands $\partial \Psi / \partial a = 0$) to solve the Wheeler–De Witt equation, and expects to conquer the great field difficulty that is caused by only considering the tunneling effect of the cosmic quantum generation, namely, using the potential well effect of $\Phi$ to offer the truncation of the great field.[11]

This paper uses the relation of the cosmic scale factor and scalar field to solve Wheeler–De Witt equation, gives the tunnel effect of the cosmic scale factor $a$ and quantum potential well of scalar field, and makes it match the physics of cosmic quantum birth. By solving Wheeler–De Witt equation we achieve the probability distribution of the cosmic generation, and give the analysis of cosmic quantum birth related to Wheeler–De Witt equation.

Because of $a = \alpha_0 \Phi^{-2}$, when $a$ aggrandizes, $\Phi$ dwindles. After the infinitesimal universe born, the universe experiences inflation, or critical chaotic inflation, and then enters the slow rolling stage of the dilatation. It ends in the small quantum stir, then enters the standard stage described by the big Bang cosmology.

Acknowledgments

One of the authors (Y.C. Huang) is grateful to Profs. R.G. Cai and D.H. Zhang for useful discussions.

References

[1] D.L. Wiltshire, “An introduction to quantum cosmology”, gr-qc/0101003.
[2] P.J.E. Peebles, Physical Cosmology, Princeton University Press, Princeton (1993).
[3] A. Vilenkin, Phys. Rev. D 37 (1988) 888.
[4] Y.G. Shen and H.G. Ding, Science in China (Series A) 23 (1993) 299.
[5] S.W. Hawking, Nucl. Phys. B 239 (1984) 257.
[6] S.W. Hawking and Z.C. Wu, Phys. Lett. B 151 (1985) 15.
[7] A. Linde, Particle Physics and Inflationary Cosmology, Harwood Academic Publishers, Berkshire (1990).
[8] A. Linde, gr-qc/982038.
[9] B.S. De Witt, Phys. Rev. 160 (1967) 1113; J.A. Wheeler, Relativity, Groups and Topology, eds. C.M. De Witt and J.A. Wheeler, Benjamin, New York (1968).
[10] E.W. Kolb and M.S. Turner, The Early Universe, Addison Wesley, California (1990).
[11] D.H. Zhang, Commun. Theor. Phys. (Beijing, China) 35 (2001) 635.