In (2+1)-dimensional hydrodynamic systems with broken parity, the shear and bulk viscosity is joined by the Hall viscosity and curl viscosity. The dual holographic model has been constructed by coupling a pseudo scalar to the gravitational Chern-Simons term in (3+1)-dimensional bulk gravity. In this paper, we investigate the non-relativistic fluid with Hall viscosity and curl viscosity living on a finite radial cutoff surface in the bulk. Employing the non-relativistic hydrodynamic expansion method, we obtain the incompressible Navier-Stokes equations with Hall viscosity and curl viscosity. Unlike the shear viscosity, the ratio of the Hall viscosity over entropy density is found to be cutoff scale dependent, and it tends to zero when the cutoff surface approaches to the horizon of the background spacetime.

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Over the past years there have been a lot of studies on the gauge/gravity dualities [1, 2, 3]. In the long-wavelength limit, the fluid/gravity correspondence relates the boundary hydrodynamic equations to the perturbation equations in the bulk gravity [4]. Recently, the (2+1)-dimensional system with broken parity has attracted much attention, and it has been holographically realized through coupling a pseudo scalar to the gravitational Chern-Simons term in the (3+1)-dimensional bulk gravity [5, 6, 7, 8, 9, 10]. The dual gravity model is a generalization of the Chern-Simons modified gravity [11, 12], which has been widely studied in the context of cosmology, gravitational waves, and gravitational tests in the solar and earth systems, see e.g., [13, 14]. In the dual model, $\tilde{R}R$ is the parity-violating Pontryagin density [15, 16], which is a measure of the divergence of axial vector current of massless chiral fermions $\nabla_{\mu}j^{5\mu}$ in the gravitational anomaly [17].

The Hall viscosity $\eta_A$ is a non-dissipative viscosity coefficient analogous to Hall conductivity which exists in the (2+1)-dimension fluid with time reversal symmetry breaking. It does not contribute to the entropy production of the fluid. In the quantum Hall fluids, the ordinary dissipative viscosities $\eta$ and $\zeta$ are absent at zero temperature, while the non-dissipative Hall viscosity $\eta_A$ could exist [18]. It can be characterized as a rational number and provides a fundamental measure of incompressibility of the Hall fluid, and it is related to a Berry phase and proportional to the density of intrinsic angular momentum [19, 20, 21, 22, 23, 24]. It is also proposed that the Hall viscosity can be measured as the coefficient in front of $q^2$ term in the Hall conductivity in small wave number $q$ limit [25].

In the non-relativistic limit of the (2+1)-dimensional hydrodynamics, the incompressible Navier-Stokes equations will be corrected by the parity odd viscosities, and it would be interesting to deduce...
the equations of motion holographically. In this paper, we use the non-relativistic fluid expansion approach to study the (2+1)-dimensional non-relativistic hydrodynamics with Hall viscosity and curl viscosity, dual to gravity involving a pseudo scalar coupled to a topological gravitational Chern-Simons term. By using the non-relativistic hydrodynamic expansion method associated with a flat cutoff surface, we formally solve the gravity equations and the pseudo scalar equation up to second order in the non-relativistic hydrodynamic expansion parameter. The Brown-York stress tensor on the cutoff surface is identified with the stress energy tensor of the dual fluid, from which we can read out the general analytic expression for transport coefficient, such as shear viscosity $\eta$, Hall viscosity $\eta_A$ and curl viscosity $\zeta_A$. The incompressible Navier-Stokes equations with Hall viscosity and curl viscosity are obtained from the gravity side. In addition, it is shown that the Hall viscosity over entropy density is cutoff scale dependent, and tends to zero when the cutoff surface approaches the horizon of the background spacetime.

This paper is organized as follows. In section 2, the (2+1)-dimensional hydrodynamics with Hall viscosity and bulk viscosity in non-relativistic limit is investigated. Section 3 is the main part of this paper, where the incompressible Navier-Stokes equations with corrections are deduced from (3+1)-dimensional dual gravity. The case when the cutoff surface goes to the anti-de Sitter boundary is discussed in section 4. And some relevant further discussions are given in section 5. In appendix A, the holographic non-relativistic expansion procedure associated with a finite cutoff surface in (d+1)-dimensional gravity is briefly introduced. In this paper, we use the small letter $i, j, = 1, 2, ..., d - 1$ to denote the index of pure spatial coordinate $x^i$, the Greek symbols $\mu, \nu, = 0, 1, ..., d - 1$, to denote index of the ordinary space-time coordinate $x^\mu \sim (\tau, x^i)$, and the capital letter $M, N, = r, 0, 1, ..., d-1$ to denote the index of bulk space-time coordinates $x^M \sim (r, x^\mu)$.

## 2 Hydrodynamics with Hall Viscosity and Curl Viscosity

In a (2+1)-dimensional parity violating hydrodynamic system, the energy-momentum tensor of the fluid with the first order gradient expansion can be written as \[ T^{\mu\nu} = eu^\mu u^\nu + pP^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta A \Theta P^{\mu\nu} - 2\eta_A \sigma_A^{\mu\nu} - \zeta_A \Omega P^{\mu\nu}, \] where the thermodynamic quantities $e$ and $p$ are the energy density and pressure respectively, which depend on the local temperature of the fluid and relate to each other through the equation of state $p = p(e)$. The shear viscosity $\eta$ and bulk viscosity $\zeta$ are the canonical transport coefficients, while the Hall viscosity $\eta_A$ and curl viscosity $\zeta_A$ arise from the parity violating effect. In a (2+1)-dimensional flat space-time background, the velocity $u^\mu = (1, \beta^i)/\sqrt{1 - \beta^2}$ and the projection tensor $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ should be functions of the space-time coordinates $x^\mu \sim (\tau, x^i)$. The first-order gradient expansion tensors can be expressed as

\[ \sigma^{\mu\nu} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha - \Theta P_{\alpha\beta}), \quad \Theta = \eta^{\mu\nu} \partial_\mu u_\nu, \quad \Omega = -\epsilon^{\mu\nu\rho} u_\mu \partial_\nu u_\rho. \]

where we have used the convention $\epsilon^{xxy} = 1$, and chosen the Landau frame $T^{\mu\nu} u_\nu = -eu^\mu$.

In the thermal equilibrium state, we assume the fluid at rest has constant energy density $e_0$ and pressure $p_0$. Through introducing the heat function per unit volume $w_0 = e_0 + p_0$, we can define the
normalized fluctuations of the thermodynamic parameters as $\mathcal{E} = (e - e_0)/w_0$ and $\mathcal{P} = (p - p_0)/w_0$. In the non-relativistic hydrodynamic limit, the following scalings appear [26, 27],

$$\beta^i_\epsilon = \beta_i (e^x_i, \epsilon^2 \tau), \quad \mathcal{E}^\epsilon = \epsilon^2 \mathcal{E} (e^x_i, \epsilon^2 \tau), \quad \mathcal{P}^\epsilon = \epsilon^2 \mathcal{P} (e^x_i, \epsilon^2 \tau),$$

(4)

where $\epsilon \ll 1$ is a small parameter. The velocity $\beta_i$ could be regarded as a small vector fluctuation, while the thermodynamic quantities $\mathcal{E}$ and $\mathcal{P}$ basically come from the temperature fluctuation, and they would relate to each other through the equation of state $p = p(\epsilon)$ of the fluid. Up to the second order of the non-relativistic expansion parameter $\epsilon$, one has

$$u_\mu \sim (-1, 0) + (0, \beta_i) \epsilon + \frac{1}{2} (\beta^2, 0) \epsilon^2 + \mathcal{O}(\epsilon^3),$$

(5)

$$P_{\mu\nu} \sim \left( \begin{array}{cc} 0 & 0 \\ \delta_{ij} & \delta_{ij} \end{array} \right) - \left( \begin{array}{cc} 0 & \beta_j \\ \beta_i & 0 \end{array} \right) \epsilon + \left( \begin{array}{cc} \beta^2 & 0 \\ 0 & \beta_i \beta_j \end{array} \right) \epsilon^2 + \mathcal{O}(\epsilon^3).$$

(6)

The non-zero components of the normal traceless shear viscosity tensor and the traceless Hall viscosity tensor are

$$\sigma_{ij} \sim \frac{1}{2} \left( \begin{array}{cc} \partial_x \beta_x - \partial_y \beta_y & \partial_x \beta_y + \partial_y \beta_x \\ \partial_x \beta_y + \partial_y \beta_x & \partial_y \beta_y - \partial_x \beta_x \end{array} \right) \epsilon^2 + \mathcal{O}(\epsilon^3),$$

(7)

$$\sigma^A_{ij} \sim \frac{1}{2} \left( \begin{array}{cc} + \partial_x \beta_x + \partial_y \beta_y & - \partial_x \beta_x + \partial_y \beta_y \\ - \partial_x \beta_x + \partial_y \beta_y & - \partial_x \beta_y - \partial_y \beta_x \end{array} \right) \epsilon^2 + \mathcal{O}(\epsilon^3),$$

(8)

the divergence and the curl of the velocity become

$$\Theta = (\partial_x \beta_x + \partial_y \beta_y) \epsilon^2 + \mathcal{O}(\epsilon^3), \quad \Omega = (\partial_x \beta_y - \partial_y \beta_x) \epsilon^2 + \mathcal{O}(\epsilon^3).$$

(9)

The dynamical equations of the relativistic fluid are $\partial_\mu T^{\mu\nu} = F^\nu$, where $F^\nu$ is an external force density, and we can define the normalized force density as $f^\nu = F^\nu / w_0$. In the non-relativistic limit (4), $\partial_\mu T^{\mu\nu} = F^\nu$ will reduce to the incompressible Navier-Stokes equations with Hall viscosity and curl viscosity,

$$\partial_t \beta_i = 0, \quad \partial_t \beta_i + \beta_j \partial^j \beta_i + \partial_j \mathcal{P} - \nu \partial^2 \beta_i - \nu A \epsilon^{ij} \partial^2 \beta_j - \xi_A \epsilon^{ijk} \partial_i \partial_j \beta_k = f_i,$$

(10)

where $\nu \equiv \eta / w_0$, $\nu_A \equiv \eta_A / w_0$ and $\xi_A \equiv \zeta_A / w_0$ could be named as the kinematic shear viscosity, kinematic Hall viscosity and kinematic curl viscosity, respectively.

### 3 Parity Breaking Hydrodynamics from Gravity

In this section, we will deduce the (2+1)-dimensional non-relativistic parity breaking hydrodynamics described in the previous section from the (3+1)-dimensional gravity.

#### 3.1 The Dual Gravity Model

The dual gravity model is described by the Einstein gravity involving a pseudo scalar coupled to the topological gravitational Chern-Simons term. The action of the model can be expressed as\[1 \quad [5]

$$S_{\text{bulk}} = \frac{1}{2\kappa_4} \int d^4x \sqrt{-g} (R - 2\Lambda + L_{CS} + L_\theta),$$

(11)

\[1\] For the surface term, see, e.g., [12, 28, 29].
where \( g = \det g_{MN} \) is the determinant of the metric, \( R \) is the Ricci scalar, \( \Lambda \) is the cosmological constant\(^2\), \( \kappa_4 = 8\pi G_N \) with \( G_N \) the Newton’s gravitational constant. The Lagrangian density \( \mathcal{L}_{CS} \) and \( \mathcal{L}_\theta \) are
\[
\mathcal{L}_{CS} = \frac{\lambda}{4} \tilde{R}, \quad \mathcal{L}_\theta = -\frac{1}{2} g^{MN} \nabla_M \theta \nabla_N \theta - V(\theta),
\]
where \( \lambda \) is a coupling constant and \( \theta \) is the pseudo scalar field. The Pontryagin density \( \tilde{R} \) is defined as
\[
\tilde{R} = R^M_{NP} R^{NP}_M, \quad \tilde{R}^M_{NP} = \frac{1}{2} \epsilon^{PAB} R^{MN}_{AB},
\]
where \( \epsilon^{MNAB} \) is the four-dimensional Levi-Civita tensor in the bulk with the convention \( \epsilon^{rtxy} = 1/\sqrt{-g} \). Varying the action with respect to the metric and the pseudo scalar respectively leads to the equations of motion
\[
W_{MN} = 0, \quad W_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R + \Lambda g_{MN} + \lambda C_{MN} - \kappa_4 T^\theta_{MN},
\]
\[
W_\theta = 0, \quad W_\theta \equiv 2\kappa_4 \left( \nabla^2 \theta - \frac{dV(\theta)}{d\theta} \right) - \frac{\lambda}{4} \tilde{R},
\]
where the stress energy tensor of the pseudo scalar is
\[
T^\theta_{MN} \equiv -2 \frac{\delta \mathcal{L}_\theta}{\delta g^{MN}} + g_{MN} \mathcal{L}_\theta = \nabla_M \theta \nabla_N \theta - \left( \frac{1}{2} g^{AB} \nabla_A \theta \nabla_B \theta + V(\theta) \right) g_{MN},
\]
and the four-dimensional Cotton tensor \( C_{MN} \) is a symmetric traceless tensor of the second rank, which is defined through\(^3\)
\[
C^{MN} = -v_S \epsilon^{SAB(M} \nabla_A R^N_{B)} + v_{ST} \tilde{R}^{(MN)S},
\]
\[
v_S = \nabla_S \theta, \quad v_{ST} = \nabla_S \nabla_T \theta = \nabla_{(S} \nabla_{T)} \theta.
\]
The Cotton tensor’s divergence has a non-zero topological source as
\[
\nabla_M C^{MN} = \frac{v_N}{8} \tilde{R} = \frac{v_N}{4} \partial_M J^M,
\]
where \( J^M \) origins from the 4-divergence of the gravitational Chern-Simons topological current
\[
J^M = \epsilon^{MNPQ} (\Gamma^A_{NB} \partial_P \Gamma^B_{QA} + \frac{2}{3} \Gamma^A_{NB} \Gamma^B_{PS} \Gamma^S_{QA}).
\]

The gravity equations \( W_{MN} = 0 \) simultaneously imply \( \nabla^M W_{MN} = 0 \). Meanwhile, the Bianchi identity \( \nabla_{(L} R_{PQ)}_{MN} = 0 \) leads to \( \nabla^M (R_{MN} - \frac{1}{2} g_{MN} R + \Lambda g_{MN}) = 0 \). Thus, assuming that the parameters \( \lambda \) and \( \kappa_4 \) are constants, we have
\[
\kappa_4 \nabla^M T^\theta_{MN} = \lambda \nabla^M C_{MN}.
\]

\(^2\)If appropriate solutions exist in this system, we do not require the cosmological constant to be negative.

\(^3\)It was also named as C-Tensor in \([5]\).
On the other hand from Eq. (16) and Eq. (19) one has
\[
\nabla^M T_{MN}^\theta = v_N \left( \nabla^2 \theta - \frac{dV(\theta)}{d\theta} \right), \quad \nabla^M C_{MN} = \frac{v_N}{8} \tilde{R} R,
\]
from which together with Eq. (21), we arrive at either \( v_N = \nabla_N \theta = 0 \) or \( W_\theta = 0 \). As we are interested in the case where the pseudo scalar field \( \theta \) depends on the space-time coordinates, i.e. \( v_N \neq 0 \), thus, the Bianchi identity together with Einstein equations leads to the pseudo scalar equation \( W_\theta = 0 \).

### 3.2 Non-relativistic Hydrodynamics Expansion

From the above discussion we can see that the pseudo scalar equation is not independent. Thus, we will focus on the gravity equations henceforth. The Ricci scalar can be obtained from the trace of Eq. (14) as
\[
R - 4\Lambda + \lambda C_M^M - \kappa_4[(\partial \theta)^2 + 4V(\theta)] = 0,
\]
which leads to the trace-reversed form of the gravity equations: \( E_{MN} + \lambda c_{MN} = 0 \), where
\[
E_{MN} \equiv R_{MN} - \Lambda g_{MN} - \kappa_4 t_{MN}, \quad t_{MN} \equiv \partial_M \theta \partial_N \theta + g_{MN} V(\theta).
\]
The simplified Cotton tensor \( c_{MN} = C_{MN} \) is due to its traceless property \( C_M^M = 0 \). Then the equations of motion (EOMs) to be solved are \(^4\)
\[
\begin{align*}
\dot{W}_{MN} &= 0, \quad \dot{W}_{MN} \equiv E_{MN} + \lambda c_{MN}, \\
\dot{W}_\theta &= 0, \quad \dot{W}_\theta \equiv \nabla^2 \theta - \frac{dV(\theta)}{d\theta} - \frac{\lambda}{4} \tilde{R} R.
\end{align*}
\]
We assume that the following general (3+1)-dimensional black brane metric
\[
ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)\delta_{ij}dx^i dx^j, \quad i, j = 1, 2,
\]
as well as a radial coordinate dependent pseudo scalar field \( \theta(r) \) solves the EOMs. With Dirichlet boundary condition at the finite cutoff surface \( r = r_c \), keeping the induced metric flat, and employing the non-relativistic hydrodynamics expansion method (see appendix A for details), one can perturb the background metric up to the second order of expansion parameter \( \epsilon \ll 1 \),
\[
ds^2_{(b)} = g_{MN} dx^M dx^N = 2g_{r\tau}(r)dr d\tau - \frac{g_{tt}(r)}{g_{tt}(r_c)} d\tau^2 + \frac{g_{xx}(r)}{g_{xx}(r_c)} \delta_{ij} d\hat{x}^i d\hat{x}^j \\
- 2g_{r\tau}(r)\beta \delta_{ij} d\hat{x}^i dr + 2 \left( \frac{g_{tt}(r)}{g_{tt}(r_c)} - \frac{g_{xx}(r)}{g_{xx}(r_c)} \right) \beta \delta_{ij} d\hat{x}^i d\hat{\tau} \\
+ g_{r\tau}(r)\beta^2 d\tau d\hat{\tau} - \left( \frac{g_{tt}(r)}{g_{tt}(r_c)} - \frac{g_{xx}(r)}{g_{xx}(r_c)} \right) \left( \beta^2 d\tau^2 + \beta \delta_{ij} d\hat{x}^i d\hat{x}^j \right) \\
+ (\delta \hat{\tau}) g_{r\tau}(r) \frac{g_{tt}(r)}{g_{tt}(r_c)} dr d\tau + (\delta \hat{\tau}) \left( \frac{g_{tt}(r)}{g_{tt}(r_c)} - \frac{g_{tt}(r_c)}{g_{tt}(r)} \right) \left( g_{r\tau}(r) dr d\tau - \frac{g_{tt}(r)}{g_{tt}(r_c)} d\tau^2 \right) \\
+ (\delta \hat{\tau}) \left( \frac{g_{xx}(r)}{g_{xx}(r)} - \frac{g_{xx}(r_c)}{g_{xx}(r)} \right) \frac{g_{xx}(r)}{g_{xx}(r_c)} \delta_{ij} d\hat{x}^i d\hat{x}^j + O(\epsilon^3),
\]
\(^4\)In what follows, we will use the normalization \( 2\kappa_4 = 1 \), which means \( 16\pi G_N \equiv 1 \).
where we have defined $g_{r\hat{r}}(r) \equiv \sqrt{g_{rr}(r)g_{tt}(r)/g_{tt}(r_c)}$. The bulk coordinates are $x^M = (r, \hat{x}^\mu)$, the intrinsic coordinates $\hat{x}^\mu = (\hat{\tau}, \hat{x}^i) = (\hat{\tau}, \hat{x}, \hat{y})$ and we denote $\hat{\partial}_i \equiv \partial_{\hat{x}^i}$\textsuperscript{5}. The pseudo scalar field should also be expanded up to the second order

$$
\hat{\theta}(r) \equiv \theta(\hat{r}) = \theta(r) + \theta'(r)\delta \hat{r}.
$$

(29)

We have assumed that the perturbation parameters are intrinsic coordinates $\hat{x}^\mu$ dependent, where $\delta \hat{r} = \hat{r} - r = \delta \hat{r}(\hat{x}^\mu)$ is a scalar perturbation at order $\epsilon^2$ and $\beta^i = \beta^i(\hat{x}^\mu)$ is the Lorentz boost parameter at order $\epsilon^1$. Together with the non-relativistic low frequency, long wavelength limit, we use the following scalings

$$
\partial_r \sim \epsilon^0, \quad \partial_i \sim \beta_i(\hat{x}^\mu) \sim \epsilon^1, \quad \partial_{\hat{r}} \sim \delta \hat{r}(\hat{x}^\mu) \sim \epsilon^2.
$$

(30)

Then we can solve the EOMs order by order with the non-relativistic hydrodynamics expansion.

The Cotton tensor firstly presents at order $\epsilon^2$, namely $C^{(0)}_{MN} \equiv 0$ and $C^{(1)}_{MN} \equiv 0$. Thus we obtain background equations

$$
E^{(0)}_{rr} = 0, \quad E^{(0)}_{\hat{r}\hat{r}} = 0, \quad E^{(0)}_{ii} = 0, \quad \hat{W}^{(0)}_{\theta} = 0,
$$

(31)

where three of them are independent, and

$$
E^{(0)}_{rr} = \frac{1}{2} \left(\frac{g'_{rr}(r)}{g_{rr}(r)} + \frac{g'_{tt}(r)}{g_{tt}(r)} + \frac{g'_{xx}(r)}{g_{xx}(r)}\right) \frac{g''_{xx}(r)}{g_{xx}(r)} - \frac{1}{2} \frac{g''_{tt}(r)}{g_{tt}(r)},
$$

(32)

$$
E^{(0)}_{\hat{r}\hat{r}} = \frac{g_{tt}(r)}{g_{tt}(r_c)} \left(\Lambda + \frac{1}{2} V(\theta(r)) + \frac{1}{2g_{rr}(r)} \left[\frac{g''_{rr}(r)}{g_{rr}(r)} + \frac{g''_{xx}(r)}{g_{xx}(r)} - \frac{1}{2} \left(\frac{g''_{rr}(r)}{g_{rr}(r)} + \frac{g''_{tt}(r)}{g_{tt}(r)}\right)\right]\right),
$$

(33)

$$
E^{(0)}_{ii} = -\frac{g_{xx}(r)}{g_{xx}(r_c)} \left(\Lambda + \frac{1}{2} V(\theta(r)) + \frac{1}{2g_{rr}(r)} \left[\frac{g''_{xx}(r)}{g_{xx}(r)} - \frac{1}{2} \left(\frac{g''_{xx}(r)}{g_{xx}(r)} + \frac{g''_{tt}(r)}{g_{tt}(r)} - \frac{1}{2} \left(\frac{g''_{rr}(r)}{g_{rr}(r)} + \frac{g''_{tt}(r)}{g_{tt}(r)}\right)\right)\right]\right),
$$

(34)

$$
\hat{W}^{(0)}_{\theta} = \frac{\theta''(r)}{g_{rr}(r)} + \left(\frac{g'_{xx}(r)}{g_{xx}(r)} + \frac{g'_{tt}(r)}{2g_{tt}(r)} - \frac{g'_{tt}(r)}{2g_{rr}(r)}\right) \frac{\theta'(r)}{d\theta(r)} - \frac{dV(\theta(r))}{d\theta(r)}.
$$

(35)

We assume that background equations are always satisfied and they can be used to simplify the perturbation equations at higher order. It turns out that the intrinsic coordinate dependent metric and pseudo scalar can only solve the EOMs (25) and (26) up to order $\epsilon^1$. Employing the spatial rotational $SO(2)$ symmetry between $\hat{x}^i$ directions, the nonzero traceless tensor sector and scalar sector appear in $\hat{W}_{MN}$ firstly at order $\epsilon^2$. Correction terms to the metric and pseudo scalar are required to cancel the corresponding structure. For the traceless shear tensor sector, the following tensor correction terms are needed in the metric at order $\epsilon^2$

$$
\sqrt{g_{rr}(r_c)} \frac{g_{xx}(r)}{g_{xx}(r_c)} \frac{2F(r)}{\sigma_{ij} + 2F_A(r) \sigma_{ij}^A} d\hat{x}^i d\hat{x}^j,
$$

(36)

$$
\sigma_{ij} = \partial_{(i}\beta_{j)} - \frac{1}{2} \delta_{ij} \partial_k \beta^k, \quad \sigma_{ij}^A = \frac{1}{2} (\epsilon_{ik} \sigma_{jk}^k + \epsilon_{kj} \sigma_{ik}^k),
$$

(37)

where $F(r)$ and $F_A(r)$ are chosen to cancel the ordinary shear tensor $\sigma_{ij}$ and Hall tensor $\sigma_{ij}^A$ of source terms at order $\epsilon^2$, respectively.

\textsuperscript{5}Here and henceforth, the subscript $i$ denotes $\hat{x}$ or $\hat{y}$. 

6
For the scalar sector, there are four perturbation equations. The Bianchi identity leads to a constraint among them, thus, only three are independent. Although one can add three corrections terms in the metric and one in the pseudo scalar, a gauge choice for these perturbations is needed. We choose a gauge so that no corrections appear in the pseudo scalar, and introduce the following three correction terms in the metric to cancel the residual curl scalar $\Omega$ appearing in the Cotton tensor at order $\epsilon^2$

$$ds^2_{(s,\epsilon^2)} = \left(2g_{r\hat{r}}(r)H_{r\hat{r}}(r)drd\hat{r} - \frac{g_{tt}(r)}{g_{tt}(r_c)}H_{tt}(r)d\tau^2 + \frac{g_{xx}(r)}{g_{xx}(r_c)}H_{\hat{x}\hat{x}}(r)d\bar{x}d\bar{x}\right)\Omega,$$  

$$\Omega \equiv \epsilon^{ij}\partial_i\beta_j = \partial_x\beta_\rho - \partial_\rho\beta_x.$$  

In summary, to solve the trace-reversed form of the gravity equations $\hat{W}_{MN} = 0$ and the pseudo scalar equation $\hat{W}_\theta = 0$, we will work with the perturbed metric and pseudo scalar field with corrections up to $\epsilon^2$ in the non-relativistic hydrodynamic limit,

$$\begin{align*}
    ds^2 &= ds^2_{(b)} + ds^2_{(t,\epsilon^2)} + ds^2_{(s,\epsilon^2)}, \\
    \hat{\theta}(r) &= \theta(r) + \theta'(r)\hat{\sigma},
\end{align*}$$

where the subscripts $(b), (t), (s)$ in the metric represent “background”, “tensor”, and “scalar”, respectively. $\hat{\theta}(r)$ is substituted for $\theta$ in EOMs. In addition, let us mention here that correction terms to the vector sector appear at order $\epsilon^3$, thus will not be considered in this paper.

### 3.2.1 Traceless Tensor Sector of the Perturbation Equations

By substituting the metric (40) and pseudo scalar field (41) into the EOMs $\hat{W}_{MN} = 0$, the traceless tensor sector gives the following second order ordinary differential equation

$$0 = \frac{d}{dr} \left[ g_{xx}(r) \left( \frac{\sqrt{g_{tt}(r)}}{\sqrt{g_{rr}(r)}} F'(r) + 1 \right) \right] \sigma_{ij}$$

$$+ \frac{d}{dr} \left[ g_{xx}(r) \left( \frac{\sqrt{g_{tt}(r)}}{\sqrt{g_{rr}(r)}} F_A'(r) + \frac{\lambda}{2g_{rr}(r)} \left( \frac{g_{tt}'(r)}{g_{tt}(r)} - \frac{g_{xx}'(r)}{g_{xx}(r)} \right) \theta'(r) \right) \right] \sigma_{ij}^A.$$  

As $\sigma_{ij}$ and $\sigma_{ij}^A$ have different tensor structure, we can solve the two second order differential equations, respectively. Here we are interested in the physics between the black brane horizon $\Sigma_h$ and the cutoff surface $\Sigma_c$. At the cutoff surface $\Sigma_c$, we impose the Dirichlet boundary condition to keep the induced metric $\hat{g}_{\mu\nu}(r_c)$ flat, i.e., $F(r_c) = 0$ and $F_A(r_c) = 0$. At the horizon $r = r_h$, we demand $F(r_h)$ and $F_A(r_h)$ to be finite. Then the function $F(r)$ and $F_A(r)$ can be determined as

$$F(r) = \int_r^{r_c} dy \sqrt{\frac{g_{rr}(y)}{g_{tt}(y)}} \left[ 1 - \frac{c_F}{g_{xx}(y)} \right],$$

$$F_A(r) = \int_r^{r_c} dy \sqrt{\frac{g_{rr}(y)}{g_{tt}(y)}} \left[ \frac{\lambda}{2g_{rr}(y)} \left( \frac{g_{tt}'(y)}{g_{tt}(y)} - \frac{g_{xx}'(y)}{g_{xx}(y)} \right) \theta'(y) - \frac{c_{FA}}{g_{xx}(y)} \right].$$
where the two integration constants are
\[
\begin{align*}
    c_F &= g_{xx}(r_h), \\
    c_{F_A} &= \frac{\lambda g_{xx}(r_h)}{2 g_{rr}(r_h)} \left( \frac{g'_{tt}(r_h)}{g_{tt}(r_h)} - \frac{g'_{xx}(r_h)}{g_{xx}(r_h)} \right) \theta'(r_h) = \frac{\lambda g_{xx}(r_h) g'_{tt}(r_h)}{2 g_{rr}(r_h) g_{tt}(r_h)} \theta'(r_h).
\end{align*}
\]

Here we have used the assumption that at the horizon of the black brane solution (27), \( g_{tt}(r_h) \) has the first order zero \( g_{tt}(r_h) = 0 \), and \( g_{rr}(r) \) has the first order pole \( g_{rr}^{-1}(r_h) = 0 \), while their product \( g_{tt}(r_h)g_{rr}(r_h) \) is finite.

### 3.2.2 Scalar Sector of the Perturbation Equations

After solving the traceless tensor perturbation equations, only scalar sectors are left at order \( \epsilon^2 \). At first, it is worthy to emphasize that the constraint equations on the scalar sector of gravity equations at order \( \epsilon^2 \) give
\[
\hat{N}^M \hat{W}_M = \frac{1}{2} w_0(r) \partial_i \beta^i = 0, \quad w_0(r) = \frac{1}{\sqrt{g_{rr}(r)}} \left( \frac{g'_{tt}(r)}{g_{tt}(r)} - \frac{g'_{xx}(r)}{g_{xx}(r)} \right),
\]

where \( \hat{N} \) is the unit normal vector of the constant \( r \) surface. As \( w_0(r) \) is not identically equal to zero, the constraint equation (47) leads to the incompressibility condition \( \Theta = \partial_i \beta^i = 0 \), which shows the velocity field is divergence-free while the vorticity \( \Omega = \epsilon^0 \partial_i \beta^j \) is still allowed.

Then we need to solve scalar sector of the perturbation equations in (25)
\[
\hat{W}^{(2)}_{rr} = 0, \quad \hat{W}^{(2)}_{\tau\tau} = 0, \quad \hat{W}^{(2)}_{xx} + \hat{W}^{(2)}_{yy} = 0, \quad \hat{W}^{(2)}_{\theta \theta} = 0.
\]

Among them, only three are independent. Using the solutions (40) and (41), we can obtain the scalar sector of gravity equations defined in (24) at order \( \epsilon^2 \),
\[
\begin{align*}
    E^{(2)}_{rr} &= \left[ \frac{g'_{xx}(r)}{g_{xx}(r)} H'_{rr}(r) + \frac{g'_{rr}(r)}{2 g_{rr}(r)} + \frac{g'_{tt}(r)}{2 g_{tt}(r)} - \frac{g'_{xx}(r)}{g_{xx}(r)} \right] H_{xx}(r) - H_{xx}(r) \right] \Omega, \\
    E^{(2)}_{\tau \tau} &= \left[ \frac{g'_{tt}(r)}{g_{tt}(r)} H'_{\tau \tau}(r) - H'_{\tau \tau}(r) + \frac{g_{tt}(r)}{2 g_{tt}(r)} \left( \frac{g'_{xx}(r)}{g_{xx}(r)} - \frac{1}{2} \frac{g'_{rr}(r)}{g_{rr}(r)} \right) \right] \frac{H_{\tau \tau}(r) - 2 H_{\tau \tau}(r)}{2 g_{rr}(r)} \right] \Omega, \\
    E^{(2)}_{xx} &= \left[ \frac{g'_{xx}(r)}{g_{xx}(r)} H'_{xx}(r) + \frac{1}{4} \frac{g_{xx}(r)}{g_{rr}(r)} \left( \frac{g'_{tt}(r)}{g_{tt}(r)} - \frac{g'_{rr}(r)}{g_{rr}(r)} \right) - \frac{g'_{xx}(r)}{g_{xx}(r)} \right] \frac{2 H_{xx}(r)}{g_{rr}(r)} \right] \Omega, \\
    E^{(2)}_{yy} &= \left[ \frac{g'_{xx}(r)}{g_{xx}(r)} H'_{yy}(r) + \frac{1}{4} \frac{g_{xx}(r)}{g_{rr}(r)} \left( \frac{g'_{tt}(r)}{g_{tt}(r)} - \frac{g'_{rr}(r)}{g_{rr}(r)} \right) - \frac{g'_{xx}(r)}{g_{xx}(r)} \right] \frac{2 H_{yy}(r)}{g_{rr}(r)} \right] \Omega, \\
    E^{(2)}_{\theta \theta} &= \left[ \frac{g'_{xx}(r)}{g_{xx}(r)} H'_{\theta \theta}(r) + \frac{1}{4} \frac{g_{xx}(r)}{g_{rr}(r)} \left( \frac{g'_{tt}(r)}{g_{tt}(r)} - \frac{g'_{rr}(r)}{g_{rr}(r)} \right) - \frac{g'_{xx}(r)}{g_{xx}(r)} \right] \frac{2 H_{\theta \theta}(r)}{g_{rr}(r)} \right] \Omega,
\end{align*}
\]

and we have imposed the incompressibility condition here and henceforth. The corresponding scalar
sector of Cotton tensor defined in Eq. (17), appears firstly at order $\epsilon^2$, can be expressed as

$$C^{(2)}_{rr} = \frac{-1}{2\sqrt{g_{rr}(r)}} \sqrt{g_{tt}(r)} g_{xx}(r_c) g_{rt}(r) \frac{d}{dr} \left[ \frac{g_{xx}(r)}{g_{tt}(r)} \left( \frac{g_{tt}'(r)}{g_{tt}(r)} - \frac{g_{xx}'(r)}{g_{xx}(r)} \right) \theta'(r) \right] \Omega,$$

$$C^{(2)}_{r\tau} = \frac{-1}{2\sqrt{g_{rr}(r)}} g_{tt}(r_c) g_{xx}(r) \frac{d}{dr} \left[ \sqrt{g_{tt}(r)} \sqrt{g_{xx}(r)} \left( \frac{g_{tt}'(r)}{g_{tt}(r)} - \frac{g_{xx}'(r)}{g_{xx}(r)} \right) \theta'(r) \right] \Omega,$$

$$C^{(2)}_{xx} + C^{(2)}_{yy} = \frac{-1}{2\sqrt{g_{rr}(r)}} \frac{1}{g_{tt}(r)} \frac{1}{g_{tt}(r)} \frac{d}{dr} \left[ \frac{g_{tt}(r)}{g_{tt}(r)} \left( \frac{g_{tt}'(r)}{g_{tt}(r)} - \frac{g_{xx}'(r)}{g_{xx}(r)} \right) \theta'(r) \right] \Omega. \quad (50)$$

In addition, we have the pseudo scalar equation defined in Eq. (26) at order $\epsilon^2$,

$$\tilde{W}^{(2)}_\theta = \left[ \frac{\theta'(r)}{g_{rr}(r)} (H'_{r\tau} + H'_{r\bar{c}}) + \frac{dV(\theta(r))}{d\theta(r)} (H'_{r\tau} - 2H_{r\tau}(r)) \right] + \frac{\lambda}{4} \frac{g_{xx}(r_c) g_{xx}'(r)}{g_{tt}(r_c) g_{xx}'(r)} \sqrt{g_{tt}(r)} \sqrt{g_{tt}(r)} \left( \frac{g_{tt}'(r)}{g_{tt}(r)} - \frac{g_{xx}'(r)}{g_{xx}(r)} \right)^2 \Omega. \quad (51)$$

Now we have three functions to be determined through four equations, for the latter, only three of which are independent due to the Bianchi identity. With the Dirichlet boundary condition at the cutoff surface, namely $H_{r\tau}(r_c) = 0$, $H_{r\bar{c}}(r_c) = 0$ and the boundary condition at the horizon, in principle we can determine the three functions $H_{r\tau}(r)$, $H_{r\bar{c}}(r)$ and $H_{r\bar{c}}(r)$ in terms of the background metric functions $g_{tt}(r)$, $g_{rr}(r)$ and $g_{xx}(r)$. But due to the complexity of equations, here we do not intend to present these expressions explicitly. Instead we just assume that the three functions $H_{r\tau}(r)$, $H_{r\bar{c}}(r)$ and $H_{r\bar{c}}(r)$ solve the three independent scalar perturbation equations in (48), as the assumption that the black brane metric (27) solves the background equations of motion [30].

### 3.3 Incompressible Navier-Stokes at the Cutoff Surface

We now study the (2+1)-dimensional hydrodynamics in the non-relativistic limit dual to the background gravity solution. At the cutoff surface $\Sigma_c$, the Brown-York tensor [31]

$$T_{\mu\nu} = 2 \left( \tilde{K}_{\mu\nu}(r_c) - \tilde{K}_{\mu\nu} + C \tilde{g}_{\mu\nu}(r_c) \right) \quad (52)$$

is identified with the stress energy of the dual fluid in Ref. [32], where $\tilde{K}_{\mu\nu}$ is the extrinsic curvature tensor, $\tilde{g}_{\mu\nu}(r_c) = \eta_{\mu\nu}$ is the induced metric on the cutoff surface, and $C$ is a constant. In the non-relativistic limit, the non-zero components at order $\epsilon^0$ turn out to be

$$T^{(0)}_{\tau\tau} = e_0(r_c), \quad e_0(r_c) = -\frac{2}{\sqrt{g_{rr}(r_c) g_{xx}(r_c)}} \frac{g_{xx}'(r_c)}{g_{xx}(r_c)} \sqrt{g_{tt}(r_c)} \left( \frac{g_{tt}'(r_c)}{g_{tt}(r_c)} + \frac{g_{xx}'(r_c)}{g_{xx}(r_c)} \right) + 2C, \quad (53)$$

$$T^{(0)}_{ij} = p_0(r_c) \delta_{ij}, \quad p_0(r_c) = \frac{1}{\sqrt{g_{rr}(r_c)}} \left( \frac{g_{tt}'(r_c)}{g_{tt}(r_c)} + \frac{g_{xx}'(r_c)}{g_{xx}(r_c)} \right) + 2C. \quad (54)$$

$e_0(r_c)$ and $p_0(r_c)$ could be regarded as the energy density and pressure of the dual fluid at rest on the cutoff surface in the thermal equilibrium state. They are relate to each other through the equation
of state of the fluid, which comes from the Hamiltonian constraint (see Eq. (106) in appendix A). At the zeroth order it is given by
\[
\frac{1}{16} \left( e_0(r_c) + 2C \right) (e_0(r_c) + 4p_0(r_c) - 6C') = \Lambda + \frac{V(\theta)}{2} + \frac{1}{2} \left( 1 - \frac{1}{g_{\tau\tau}(r_c)} \right) \theta^2(r_c) \tag{55}
\]

The entropy density and temperature of the dual fluid is defined as the local physical quantities at the cutoff surface \( \Sigma \) of the black brane \([32, 33]\)

\[
s_0(r_c) = \frac{1}{4G_N} g_{xx}(r_h), \quad T(r_c) = \frac{T_H}{\sqrt{g_{tt}(r_c)}}, \quad T_H = \lim_{r \to r_h} \frac{g''_t(r)}{4\pi \sqrt{g_{tt}(r)g_{\tau\tau}(r)}}, \tag{56}
\]

where a factor in the entropy density \( s_0(r_c) \) has been chosen to meet with the Bekenstein-Hawking entropy formula, while the local temperature at the cutoff surface \( T(r_c) \) relates with the Hawking temperature \( T_H \) through the Tolman redshift relation.

At order \( \epsilon^1 \), only vector components appear in the dual stress energy tensor

\[
\mathcal{T}^{(1)}_{\dot{r}i} = \mathcal{T}^{(1)}_{\dot{r}r} = -w_0(r_c) \beta_i, \quad w_0(r_c) = e_0(r_c) + p_0(r_c) = \frac{1}{\sqrt{g_{\tau\tau}(r_c)}} \left( \frac{g'_{tt}(r_c)}{g_{tt}(r_c)} - \frac{g'_{xx}(r_c)}{g_{xx}(r_c)} \right), \tag{57}
\]

where \( w_0(r_c) \) is the heat function per unit volume. While at order \( \epsilon^2 \), the scalar and tensor modes are

\[
\begin{align*}
\mathcal{T}^{(2)}_{\dot{x}x} &= w_0(r_c) (\mathcal{E} + \beta^2) + \zeta_A(r_c) \Omega, \\
\mathcal{T}^{(2)}_{ij} &= w_0(r_c) (\mathcal{P} + \beta_i \beta_j) - 2\eta(r_c) \sigma_{ij} - 2\eta_A(r_c) \sigma^A_{ij} - \zeta_A(r_c) \Omega \delta_{ij}, \tag{58, 59}
\end{align*}
\]

The normalized energy density and pressure perturbations are defined as

\[
\mathcal{E} = \frac{de_0(r_c)}{dr_c} \frac{\delta \hat{\rho}(\hat{x}^\mu)}{w_0(r_c)}, \quad \mathcal{P} = \frac{dp_0(r_c)}{dr_c} \frac{\delta \hat{p}(\hat{x}^\mu)}{w_0(r_c)}, \tag{60}
\]

where \( \delta \hat{\rho}(\hat{x}^\mu) \) could be regarded as the thermodynamic parameter due to the temperature perturbation.

The transport coefficient appears at the first order gradient expansion. In the traceless tensor modes, the normal shear viscosity reads

\[
\eta(r_c) \equiv 1 + \sqrt{\frac{g_{tt}(r_c)}{g_{\tau\tau}(r_c)}} F'(r_c) = g_{xx}(r_h) \frac{g_{xx}(r_h)}{g_{xx}(r_c)}, \quad \frac{\eta(r_c)}{s_0(r_c)} = 4G_N = \frac{1}{4\pi}, \tag{61}
\]

which shows that \( \eta/s_0 \) is cutoff scale independent. While the Hall viscosity turns out to be

\[
\begin{align*}
\eta_A(r_c) &= \sqrt{\frac{g_{tt}(r_c)}{g_{\tau\tau}(r_c)}} F'_A(r_c) = \frac{\lambda}{2} \frac{g_{xx}(r_h)}{g_{\tau\tau}(r_h)} \frac{g_{tt}(r_h)}{g_{rr}(r_h)} = \frac{\lambda}{2} \frac{w_0(r_c) \theta'(r_c)}{\sqrt{g_{\tau\tau}(r_c)}}, \\
\frac{\eta_A(r_c)}{s_0(r_c)} &= \frac{\lambda}{8\pi} \frac{g_{tt}(r_h)}{g_{\tau\tau}(r_h) g_{tt}(r_h)} - \frac{\lambda}{8\pi} \frac{g_{xx}(r_h) g_{\tau\tau}(r_h)}{g_{xx}(r_h) g_{rr}(r_h)} \left( \frac{g'_{tt}(r_c)}{g_{tt}(r_c)} - \frac{g'_{xx}(r_c)}{g_{xx}(r_c)} \right). \tag{62-63}
\end{align*}
\]

Thus, we see that \( \eta_A/s_0 \) is cutoff scale dependent due to the second term. When the cutoff surface approaches to the horizon, namely, \( r_c \to r_h \), one has \( \eta_A(r_c)/s_0(r_c) \to 0 \). Even without solving
the scalar perturbation equations in (48), we can still give the formula of curl viscosity with the formal functions in Eq. (38). In the dual fluid stress energy tensor (58) and (59), the scalar sector coefficients at order $\epsilon^2$ are

$$\hat{\zeta}_A(r_c) = \frac{2}{\sqrt{g_{rr}(r_c)}} \left( \frac{g'_{xx}(r_c)}{g_{xx}(r_c)} H_{r^r}(r_c) - H'_{xx}(r_c) \right),$$  
(64)

$$\zeta_A(r_c) = \frac{1}{\sqrt{g_{rr}(r_c)}} \left( \frac{g_{tt}(r_c)}{g_{tt}(r_c)} H_{r^r}(r_c) - H'_{tt}(r_c) \right) + \frac{\hat{\zeta}_A(r_c)}{2}. \quad (65)$$

Here we have used the Dirichlet boundary condition $H_{r^r}(r_c) = 0$ and $H_{xx}(r_c) = 0$. If the Landau frame is chosen that $\hat{\zeta}_A(r_c) = 0$, we can identify $\zeta_A(r_c)$ as the curl viscosity of the dual fluid.

According to the Gauss-Codazi equation in pure geometry sector (see Eq. (107) in appendix A) and the gravity equations in Eq. (14), we have

$$\partial^\mu T_{\mu\nu} \equiv -2\hat{N}^A R_{AB} h^B_\nu = \hat{N}^A \left( 2\lambda C_{AB} - T^\theta_{AB} \right) h^B_\nu,$$  
(66)

where $h_{AB} = g_{AB} - \hat{N}_A \hat{N}_B$ is the induced metric at the cutoff surface. With the gauge choice for the scalar perturbations in Eqs. (40)-(41), one can show that up to order $\epsilon^3$ of the source terms on the right-handed side, the Cotton tensor $C_{AB}$ contributes nothing, while the stress energy tensor $T^\theta_{AB}$ contributes an external force density as

$$\mathcal{F}_i(r_c) = -\hat{N}^A T^\theta_{AB} h^B_i = -\frac{\theta'(r_c)^2}{\sqrt{g_{rr}(r_c)}} \partial_i (\delta \hat{r}(\hat{x}^\mu)), \quad (67)$$

which comes from the scaling transformation parameter $\delta \hat{r}(\hat{x}^\mu)$, and the normalized force density is $f_i(r_c) = \mathcal{F}_i(r_c)/w_0(r_c)$. Thus finally we get the incompressible Navier-Stokes equations with Hall and curl viscosities in Eq. (10) from Einstein gravity with Chern-Simons term,

$$\partial_t \beta^i = 0, \quad \partial^i \beta_i + \beta_j \partial^j \beta_i + \partial_i \mathcal{P} - \nu(r_c) \partial^2 \beta_i - \nu_A(r_c) \epsilon^{ij} \partial^2 \beta_j - \xi_A(r_c) \epsilon^{ijk} \partial_i \partial_j \beta_k = f_i(r_c).$$  
(68)

where the kinematic viscosities are defined as

$$\nu(r_c) = \eta(r_c)/w_0(r_c), \quad \nu_A(r_c) = \eta_A(r_c)/w_0(r_c), \quad \xi_A(r_c) = \xi_A(r_c)/w_0(r_c), \quad (69)$$

respectively. The Reynolds number of the dual fluid associated with the shear viscosity and Hall viscosity can also be defined as

$$\mathcal{R} \equiv \frac{uL}{\nu} \propto \frac{1}{\nu(r_c)}, \quad \mathcal{R}_A \equiv \frac{uL}{\nu_A} \propto \frac{1}{\nu_A(r_c)}, \quad (70)$$

where $u$ is the characteristic velocity, $L$ is the characteristic scale. If we further set $\delta \hat{r}(\hat{x}^\mu) \equiv 0$, then the normalized pressure perturbation $\mathcal{P}$ and force density $f_i(r_c)$ vanish, the holographic incompressible Navier-Stokes equations will reduce into the incompressible Burger’s equations [34] with Hall viscosity and curl viscosity,

$$\partial_t \beta^i = 0, \quad \partial^i \beta_i + \beta_j \partial^j \beta_i - \nu(r_c) \partial^2 \beta_i - \nu_A(r_c) \epsilon^{ij} \partial^2 \beta_j - \xi_A(r_c) \epsilon^{ijk} \partial_i \partial_j \beta_k = 0. \quad (71)$$
4 Dual Fluid at the Infinite Boundary

To compare with results in Ref. [5], we take the limit with the cutoff surface $r_c \to \infty$. The ansatz of the bulk metric is

$$ds^2 = -r^2 f(r) dr^2 + 2H(r) dr d\tau + r^2 dx^i dx_i, \quad i, j = 1, 2, \quad (72)$$

$\theta = \theta(r)$ is a pseudo scalar coupled to the gravitational Chern-Simons term, and $\Lambda = -3/\ell^2$ is the negative cosmological constant. Under this ansatz, the Pontryagin density $\tilde{R}\tilde{R}$ is identically zero [12]. This metric is related to our generic metric in Eq. (87) through

$$g_{tt}(r) = r^2 f(r), \quad g_{xx}(r) = r^2, \quad g_{rr}(r) = \frac{H(r)^2}{r^2 f(r)}. \quad (73)$$

The pseudo scalar equation $\tilde{W}_\theta^{(0)} = 0$, with Eq. (35), becomes

$$\theta''(r) + \left(\frac{4}{r} + \frac{f'(r)}{f(r)} - \frac{H'(r)}{H(r)}\right) \theta'(r) - \frac{H(r)^2}{r^2 f(r)} \frac{dV(\theta(r))}{d\theta(r)} = 0. \quad (74)$$

The shear viscosity and Hall viscosity at the cutoff surface $\Sigma_c$ defined in Eqs. (61) and (62) turn out to be

$$\eta(r_c) = \frac{r_h}{r_c}, \quad \eta_A(r_c) = \frac{\lambda}{r_c^2} \left(\frac{r_h^4 f'(r_h) \theta'(r_h)}{2H(r_h)^2} - \frac{r_h^4 f'(r_c) \theta'(r_c)}{2H(r_c)^2}\right). \quad (75)$$

According to Eq. (56), the entropy density and temperature of the dual fluid at cutoff surface $\Sigma_c$ are

$$s_0(r_c) = \frac{1}{4G_N} \frac{r_h}{r_c}, \quad T(r_c) = \frac{T_H}{r_c \sqrt{f(r_c)}}, \quad T_H = \frac{r_h^2 f'(r_h)}{4\pi H(r_h)}. \quad (76)$$

Thus the shear and Hall viscosities over entropy density are given respectively by

$$\frac{\eta(r_c)}{s_0(r_c)} = \frac{1}{4\pi}, \quad \frac{\eta_A(r_c)}{s_0(r_c)} = \frac{\lambda}{4\pi r_h^2} \left(\frac{r_h^4 f'(r_h) \theta'(r_h)}{2H(r_h)^2} - \frac{r_h^4 f'(r_c) \theta'(r_c)}{2H(r_c)^2}\right). \quad (77)$$

where we have used normalization $16\pi G_N = 1$. The shear viscosity over entropy density of the dual fluid does not run with the cutoff surface; this is not the case for the Hall viscosity. In the infinity boundary limit $r_c \to \infty$, in order to compare with previous work, we make the assumption that

$$f(r_c) \to 1 - O(r_h^3/r_c^3), \quad H(r_c) \to 1 - O(r_h^3/r_c^3), \quad \theta(r_c) \to O(r_h^a/r_c^a), \quad (78)$$

when $a > 0$, we can drop the second term in $\eta_A/s$, thus recover the result in Ref. [5]. In that case, the Hall viscosity over entropy density is entirely determined by the near horizon region of the black brane geometry.

6There is a notation difference from Ref. [5]: $\lambda_{\text{here}} = -\lambda_{\text{there}}$. 

12
5 Discussions

In this paper, we have investigated the non-relativistic hydrodynamics with Hall viscosity and curl viscosity living on a finite cutoff surface, dual to (3 + 1)-dimensional Einstein gravity with a pseudo scalar coupled to a topological gravitational Chern-Simons term. The topological Pontryagin density $\tilde{R}R$ is totally depicted by the Weyl tensor, which describes the traceless part of the Riemann tensor. The Ricci tensor is the Riemann tensor’s trace, and together with the Weyl tensor, provides a complete description of the curvature of the space-time. In (3+1)-dimensional space-time, the Weyl tensor and its dual can be written as

$$C_{ABCD} = R_{ABCD} - (g_{A[C}R_{D]B} - g_{A[D}R_{C]B}) + \frac{1}{3}g_{A[C}g_{D]B}R,$$

(79)

$$\tilde{C}_{ABCD} = \frac{1}{2}\epsilon^{CDEF}C^{AB}_{EF}, \quad \tilde{C}C \equiv \tilde{C}_{ABCD}C_{BACD},$$

(80)

and it can be shown that the Pontryagin density $\tilde{R}R = \tilde{C}C$ (for example see [14]). One can further define the gravito-electric and gravito-magnetic field as

$$E_{AC} + iB_{AC} \equiv (C_{ABCD} + i\tilde{C}_{ABCD})u^Bu^D = (C_{ABCD} + i\tilde{C}_{ABCD})P^{BD},$$

(81)

where $u^B$ is a normalized time-like 4-velocity, and its projection $P^{BD} = g^{BD} + u^Bu^D$. The last identity is true due to the traceless properties of the Weyl tensor. Thus, the Lagrangian density in Eq. (12) in the bulk can be expressed as

$$\mathcal{L}_{CS} \equiv \frac{\lambda}{4} \theta \tilde{R}R = \frac{\lambda}{4} \theta \tilde{C}C = -4\lambda \theta E_{AC}B^{AC},$$

(82)

which is analogous to that in $U(1)$ gauge theory, $\mathcal{L}_{CS}^{(\kappa)} = \kappa \theta \tilde{F}^{\mu\nu}F_{\mu\nu} = -4\kappa \theta (\tilde{E} \cdot \tilde{B})$. With suitable coefficient and a constant $\theta$-vacuum, $\mathcal{L}_{CS}^{(\kappa)}$ can be used to describe the electromagnetic response of (3+1)-dimensional topological insulators (TIs), while $\mathcal{L}_{CS}$ can be used to describe the gravitational response of (3+1)-dimensional topological superconductors (TSCs), a half quantized thermal Hall effect may appear on the surface [35, 36]. Alternatively, if we consider the stress response of (3+1)-dimensional TIs to an external torsion field, it was shown in Ref. [37] that the time-reversal invariant TIs will exhibit a quantum Hall viscosity on their surfaces. Thus it would be interesting to see whether the bulk gravity in this paper could describe a deformed TIs or TSCs, with the surface quantum Hall effect as the dual description. In our general perturbative metric (28), the Lagrangian of the anomaly term firstly appears at order $\epsilon^2$ as

$$\mathcal{L}_{CS} = \frac{\lambda}{2} \theta(r) \left(\frac{g_{xx}(r_c)}{g_{tt}(r_c)} g_{x_x}'(r) \sqrt{\frac{g_{tt}(r)}{g_{tt}(r_c)}} \left(\frac{g_{tt}(r)}{g_{tt}(r_c)} - \frac{g_{xx}'(r)}{g_{xx}(r)}\right)\right)^2 \Omega,$$

(83)

which has been simplified through the background equations in (31). To get a non-vanishing Hall viscosity $\eta_A$ in Eq. (62), the pseudo scalar field $\theta(r)$ is required to be coordinate $r$-dependent. Otherwise, when $\theta(r)$ vanishes, parity is not broken in the bulk and in the dual boundary theory, and when $\theta(r)$ is a non-zero constant, the gravitational Chern-Simons term is just a surface term in the action which violates the parity in the bulk but contributes nothing to the equations of motion in the bulk, hence leads to a vanishing Hall viscosity. In Ref. [7], by including a gauge field in the
bulk gravity such that the solution is free from the violation of the positive energy theorem [38], the pseudo scalar hair would break parity spontaneously.

Inheriting the spirit of holographic Wilson’s renormalization group (RG) approach in Ref. [32], the re-scaled cutoff size could be regarded as the RG running scale. The non-vanishing trace of the symmetric stress energy tensor leads to the trace anomaly in quantum field theory, where the corresponding $\beta$-function can be obtained from the coefficients in front of the trace. In our dual gravity model, Setting $\lambda = 0$ to turn off the gravitational Chern-Simons term in the bulk, we can go back to the d-dimensional fluid corresponding to (d+1)-dimensional bulk gravity with a scalar field $\theta(r)$, the relation between the trace $T^\mu_\mu$ in Eq. (105) of dual Brown-York stress energy tensor at zeroth order and $\beta(e)$ is,

$$ T^\mu_\mu = -\beta(e) \frac{d}{r_c \left[ g_x^{d/2}(r_c) \right]} + \frac{2g_{xx}(r_c) \theta'(r_c)^2}{g_{xx}'(r_c) \sqrt{g_{rr}(r_c)}}, \quad \beta(e) = r_c \frac{\partial}{\partial r_c} \left[ e(r_c)g_x^{d/2}(r_c) \right], \quad (84) $$

where $e(r_c)$ is given in Eq. (103), $r_c$ is analogous to the cutoff energy scale. Considering there exists a conformal factor difference in the definition of the dual stress tensor in Ref. [39] and transforming to that frame, we can conclude that the trace of the fluid stress tensor is generated by the energy density as well as an extra term contributed by the scalar field. This relation could also be generalized at higher order of hydrodynamic expansion, and in order to consider the effects of parity violating interactions, one can turn on the gravitational Chern-Simons term in the bulk. As the ratio of the Hall viscosity over entropy density in Eq. (63) is found to be cutoff scale dependent, we can also define the following $\beta$-function

$$ \beta(\eta_A/s) = r_c \frac{\partial}{\partial r_c} \left[ \frac{\eta_A(r_c)}{s_0(r_c)} \right] = -\frac{\lambda r_c}{8\pi} \frac{\partial}{\partial r_c} \left[ \frac{g_{xx}(r_c) w_0(r_c) \theta'(r_c)}{g_{xx}(r_h) \sqrt{g_{rr}(r_c)}} \right] , \quad (85) $$

to represent its non-trivial evolution, where $w_0(r_c)$ is the heat function per unit volume in Eq. (57). When the cutoff surface approaches to the horizon of the background spacetime, $\eta_A(r_h)/s_0(r_h) = 0$, but $\beta(\eta_A/s)$ could be non-zero.

Note that in our model, the ratio of Hall viscosity to entropy density depends on the cutoff scale, while the ratio of shear viscosity to entropy density does not. This means that the shear viscosity has a same dependence of the cutoff as the entropy density, but it is not for the Hall viscosity. Thus it is of some interest to ask whether one can construct a quantity involving the Hall viscosity, which is cutoff independent. While having considered the form (62), at the moment we have no idea on the Hall viscosity. Here we just mention that a similar phenomena appears in charged fluid with anomalous current [40]. It was recently shown that the anomaly vortical coefficient $\xi$ depends on the cutoff scale, while the ratio of $\xi$ to a function of thermodynamic quantities is cutoff independent [41]. A similar denominator for the curl viscosity is expected to be found, but the approach does not applies to the Hall viscosity since it is non-dissipative. Certainly it is required to further understand the dependence of the cutoff for the ratio of Hall viscosity over entropy density in the parity broken fluid [8] or weak-isospin incompressible quantum liquid [42].
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A Non-relativistic Hydrodynamic from Gravity

This appendix briefly introduces the non-relativistic expansion procedure associated with a cutoff surface \[33\]. For further details, see Refs. \[26, 27, 32, 39, 43, 44, 45, 46, 47, 48\].

A.1 Bulk Geometry and a Finite Cutoff Surface

In order to study the fluid in \(d\)-dimensional flat space-time, let us consider a \((d+1)\)-dimensional background geometry with a generic metric
\[
ds_{d+1}^2 = -g_{tt}(r)dt^2 + g_{xx}(r)\delta_{ij}dx^idx^j + g_{rr}(r)dr^2, \quad i, j = 1, \ldots d - 1,
\]
where the metric components depend only on radial coordinate \(r\). In addition, we assume the fully spatial rotational \(SO(d-1)\) symmetry in \(x^i\) directions, and require that the geometry of the space-time manifold \(\mathcal{M}\) has a well-defined future horizon located at \(r = r_h\), where \(g_{tt}(r)\) has the first-order zero \(g_{tt}(r_h) = 0\), and \(g_{rr}(r)\) has the first-order pole \(g_{rr}^{-1}(r_h) = 0\) \[43\]. By using the Eddington-Finkelstein coordinate \(\tau\) defined through
\[
d\tau = dt + \sqrt{g_{rr}(r)} \, dr,
\]
we can rewrite the bulk metric as
\[
ds_{d+1}^2 = -g_{tt}(r)d\tau^2 + g_{xx}(r)dx_idx^i + 2\sqrt{g_{tt}(r)g_{rr}(r)}d\tau dr.
\]
(87)

Let us define an arbitrary hyper-surface at a constant radial coordinate \(r\) as \(\Sigma_r\), and introduce a specific finite cutoff hyper-surface \(\Sigma_c\) at \(r = r_c\) outside the horizon \((r_c > r_h)\), where the associated intrinsic coordinates \(\hat{x}^\mu \sim (\hat{\tau}, \hat{x}^i)\) on \(\Sigma_c\) are
\[
\hat{x}^0 = \hat{\tau} = \tau \sqrt{g_{tt}(r_c)}, \quad \hat{x}^i = x^i \sqrt{g_{xx}(r_c)}, \quad i = 1, 2, \ldots, d - 1.
\]
(88)

The bulk metric in Eq. (87) in the intrinsic coordinates becomes
\[
ds_{d+1}^2 = -\frac{g_{tt}(r)}{g_{tt}(r_c)} d\hat{\tau}^2 + \frac{g_{xx}(r)}{g_{xx}(r_c)} d\hat{x}_id\hat{x}^i + 2\sqrt{\frac{g_{tt}(r)}{g_{tt}(r_c)}} \sqrt{g_{rr}(r)}d\hat{\tau} d\hat{r}.
\]
(89)
We adopt the Arnowitt-Deser-Misner (ADM) decomposition along \( r \) direction \(^{49}\)

\[
d s_{d+1}^2 = \alpha^2(r) dr^2 + \gamma_{\mu\nu}(r)(d\hat{x}^\mu - \beta^\mu(r)dr)(d\hat{x}^\nu - \beta^\nu(r)dr),
\]

(90)

with the “lapse function” \( \alpha(r) = \sqrt{g_{tt}(r)} \) and the “shift vector” \( \beta^\mu(r) = \sqrt{g_{tt}(r_c)} \sqrt{g_{rr}(r)} \delta^\mu_r \) respectively. The generic induced metric \( \gamma_{\mu\nu}(r) \) is an analytically extension from \( \Sigma_c \) to \( \Sigma_r \), where

\[
\gamma_{\mu\nu}(r)d\hat{x}^\mu d\hat{x}^\nu \equiv -\frac{g_{tt}(r)}{g_{tt}(r_c)}d\hat{\tau}^2 + \frac{g_{xx}(r)}{g_{xx}(r_c)}d\hat{x}^i d\hat{x}^i.
\]

(91)

It is worthy to notice that \( \gamma_{\mu\nu}(r_c) = \eta_{\mu\nu} \equiv (-1, \delta_{ij}) \), which implies that the induced metric \( \gamma_{\mu\nu}(r) \) on \( \Sigma_r \) reduces to a flat metric on \( \Sigma_c \).

### A.2 Diffeomorphisms Associated with the Cutoff Surface

With a flat induced metric given at the cutoff surface \( \Sigma_c \), we can take into consider two finite diffeomorphisms, which preserve the induced metric invariant on the cutoff surface. The first one is a linear scale transformation along the radial coordinate \( r \) and the rescaling of the intrinsic coordinates \( \hat{x}^\mu = (\hat{\tau}, \hat{x}^i) \),

\[
r \to \hat{r} \equiv k(r), \quad \hat{\tau} \to \hat{\tau} \sqrt{\frac{g_{tt}(r_c)}{g_{tt}(\hat{r}_c)}}, \quad \hat{x}^i \to \hat{x}^i \sqrt{\frac{g_{xx}(r_c)}{g_{xx}(\hat{r}_c)}},
\]

(92)

where \( k(r) \) is a linear function of \( r \), and a concrete form of \( k(r) \) will be chosen according to the specific global geometry of the bulk metric in Eq. (89). In addition, the notation \( \hat{r}_c \equiv k(r_c) \) is introduced. Under this diffeomorphism, the bulk metric in Eq. (89) is rescaled as

\[
d s_{d+1}^2 = -\frac{g_{tt}(\hat{r})}{g_{tt}(\hat{r}_c)}d\hat{\tau}^2 + \frac{g_{xx}(\hat{r})}{g_{xx}(\hat{r}_c)}d\hat{x}^i d\hat{x}^i + 2\sqrt{\frac{g_{tt}(\hat{r})}{g_{tt}(\hat{r}_c)}} g_{rr}(\hat{r})d\hat{\tau} d\hat{r}.
\]

(93)

The second diffeomorphism is a generic Lorentz transformation

\[
\hat{\tau} \to \gamma(\hat{\tau} - \beta_i \hat{x}^i), \quad \hat{x}^i \to \hat{x}^i - \gamma \beta^i \hat{\tau} + (\gamma - 1) \frac{\beta^i \beta_j}{\beta^2} \hat{x}^j,
\]

(94)

with the boost parameter \( \beta_j = \delta_{ij} \beta^i \) and Lorentz factor \( \gamma = 1/\sqrt{1 - \beta^2} \). We can define the \( d \)-dimensional velocity and \( d \)-dimensional polarization vector as

\[
u_\mu = \gamma(-1, \beta_i), \quad n^i_\mu = (-\gamma \beta^i, \delta^i_\mu + (\gamma - 1) \frac{\beta^i \beta_j}{\beta^2}),
\]

(95)

with \( u^\mu u_\mu = -1, \ n^\mu_\mu n^i_\mu = d - 1 \) and \( u^\mu n^i_\mu = 0 \). Then the Lorentz transformation on the cutoff hypersurface becomes \( \hat{\tau} \to -u_\mu \hat{x}^\mu \), and \( \hat{x}^i \to n^i_\mu \hat{x}^\mu \). The re-scaled bulk metric in Eq. (93) becomes

\[
d s_{d+1}^2 = -\frac{g_{tt}(\hat{r})}{g_{tt}(\hat{r}_c)}u_\mu u_\nu d\hat{x}^\mu d\hat{x}^\nu + \frac{g_{xx}(\hat{r})}{g_{xx}(\hat{r}_c)} P_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu - 2\sqrt{\frac{g_{tt}(\hat{r})}{g_{tt}(\hat{r}_c)}} g_{rr}(\hat{r})u_\mu d\hat{\tau} d\hat{r},
\]

(96)
where the projection tensor
\[ P_{\mu \nu} \equiv \eta_{\mu \nu} + u_\mu u_\nu = \delta_{ij} n^i_\mu n^j_\nu = \left( \begin{array}{cc} \gamma^2 \beta^2 & -\gamma^2 \beta^j \\ -\gamma^2 \beta^i & \delta^i_j + \gamma^2 \beta^i \beta^j \end{array} \right), \]  
(97)
satisfying \( P^\mu_\mu = d - 1 \). Under the two diffeomorphism transformations, the metric in the ADM-like decomposition with the “lapse” function \( \dot{\alpha}(r) \) and the “shift” vector \( \beta(r) \) becomes
\[ ds^2_{\hat{a}+1} = \dot{\alpha}^2(r) dx^2 + \dot{\gamma}_{\mu \nu}(r) \left( \dot{d}x^\mu - \hat{\beta}^\mu(r)d\hat{r} \right) \left( \dot{d}x^\nu - \hat{\beta}^\nu(r)d\hat{r} \right), \]
\[ \dot{\alpha}(r) = \sqrt{g_{tt}(\hat{r})}, \quad \dot{\beta}^\mu(r) = \sqrt{g_{tt}(\hat{r})} g^{tt}(\hat{r}) u^\mu, \]  
(98)
and the induced metric in Eq. (91) is transformed into \( \dot{\gamma}_{\mu \nu}(r) \) as
\[ \dot{\gamma}_{\mu \nu}(r) \equiv -\frac{g_{tt}(\hat{r})}{g^{tt}(\hat{r})} u_\mu u_\nu + \frac{g_{xx}(\hat{r})}{g^{xx}(\hat{r})} P_{\mu \nu}. \]  
(100)

At the cutoff \( r = r_c \), the induced metric \( \dot{\gamma}_{\mu \nu}(r_c) \) keeps flat.

### A.3 Dual Hydrodynamic in Non-Relativistic Limit

We perturb the diffeomorphism metric (96) in the non-relativistic limit. As \( r \) is the radial coordinate along the extra dimension orthogonal to the cutoff hyper-surface, we can define a scaling transformation parameter \( \delta \hat{r} \equiv \hat{r} - r \). Assuming the scalar parameter \( \delta \hat{r} \) and the velocity parameters \( \beta^i \) in the Lorentz boost are all functions of the intrinsic coordinates \( \hat{x}^\mu = (\hat{r}, \hat{x}^i) \), in the non-relativistic hydrodynamic limit we have the following scalings (here the subscript \( i \) present \( \hat{x}^i \))
\[ \partial_r \sim \epsilon^0, \quad \partial_i \sim \beta_i(\hat{x}^\mu) \sim \epsilon^1, \quad \partial_{\hat{r}} \sim \delta \hat{r}(\hat{x}^\mu) \sim \epsilon^2, \]  
(101)
where \( \epsilon \ll 1 \) is a small parameter. Therefore, the metric function can be expanded, i.e., \( g_{tt}(\hat{r}) = g_{tt}(r) + g'_{tt}(r) \delta \hat{r}(r) + \ldots \), then the diffeomorphism metric can be expanded order by order in the expansion parameter, up to order \( \epsilon^2 \), which gives the result in Eq. (28). In Einstein gravity, we assume that the generic metric in (86) solves Einstein’s equations. Then the diffeomorphism metric is also the solution. After demanding the transformation parameters \( \beta_i \) and \( \delta \hat{r} \) be coordinates \( \hat{x}^\mu \) dependent, the metric in (96) is no longer the solution of Einstein’s equations. We can solve the Einstein’s equations order by order in the non-relativistic hydrodynamic expansion, via adding new correction terms to the metric. Similar procedure is also applicable in the case when matter fields are present.

The intrinsic curvature of the bulk geometry is measured by Riemann tensor, while the extrinsic curvature depends on how the hyper-surface is embedded into the bulk geometry. Thus, when we perturb the background metric, we can choose the gauge to keep the induced metric \( \dot{\gamma}_{\mu \nu}(r_c) \) flat. The extrinsic curvature at the cutoff surface is \( \dot{K}_{\mu \nu} = \frac{1}{2} \mathcal{L}_N \dot{\gamma}_{\mu \nu}(r)|_{r=r_c} \), where \( \mathcal{L}_N \) is the Lie derivative along the outpointing normal vector \( \hat{N} \) of the cutoff surface, \( \dot{\gamma}_{\mu \nu}(r) \) is an analytic extension of the induced metric \( \dot{\gamma}_{\mu \nu}(r_c) \) on \( \Sigma_c \). And they are all associated with the perturbed metric with correction
terms. In \((d + 1)\)-dimensional Einstein gravity, if we use the unit \(16\pi G_N^{(d+1)} = 1\), the Brown-York tensor on the cutoff surface \(\Sigma_c\) is given by \[\tag{102}
\mathcal{T}^{BY}_{\mu\nu} = 2(\hat{K}_{\mu\nu}(r_c) - \hat{K}_{\mu\nu} + C\hat{\gamma}_{\mu\nu}(r_c)).
\]

It can be identified with the stress tensor of the dual fluid Ref. \[\text{[32]}\]. In the non-relativistic hydrodynamic expansion, we can also fix the physical stress energy tensor order by order. The zeroth order stress tensor can be written as \[\mathcal{T}^{BY}_{\mu\nu} = e(r_c) u^\mu u^\nu + p(r_c) g_{\mu\nu}, \]
where \(e(r_c) = -\frac{d-1}{\sqrt{g_{rr}(r_c)}} \frac{g'_{xx}(r_c)}{g_{xx}(r_c)} - 2C, \)
and \(p(r_c) = \frac{1}{\sqrt{g_{rr}(r_c)}} \left( \frac{g'_{tt}(r_c)}{g_{tt}(r_c)} + (d-2) \frac{g'_{xx}(r_c)}{g_{xx}(r_c)} \right) + 2C, \)
are the energy density and pressure of the dual fluid, respectively. And the trace of the dual stress energy tensor at the zeroth order is \[\mathcal{T}^\mu_\mu = -e(r_c) + (d-1)p(r_c). \]

In addition, the dual fluid is described by the constraint equations \[\text{[47, 50]}\]
\[\frac{1}{2} D^A (\hat{K} h_{AB} - \hat{K}_{AB}) = -2 G_{MN} \hat{N}^M \hat{N}^N \big|_{\Sigma_c}, \]
\[2 D^A (\hat{K} h_{AB} - \hat{K}_{AB}) = -2 G_{MN} \hat{N}^M h^N_B \big|_{\Sigma_c}. \]

where \(G_{MN}\) is the Einstein tensor, \(h_{AB} = g_{AB} - \hat{N}_A \hat{N}_B\) and \(D\) denotes the derivative operator associated with the induced metric \(h_{AB}\). The first one is the Hamiltonian constraint which gives the equation of state of fluid relating the pressure and energy density. The second one is the momentum constraint which gives the evolution equations of the fluid, and it reduces to the incompressible Navier-Stokes equations in the non-relativistic limit. Correction terms will appear if matters or higher order curvature terms are added to Einstein gravity. For example, the matters will provide external sources to the equations of motion, and the Gauss-Bonnet term will lead to an additional term for the shear viscosity \[\text{[33]}\].

**References**

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. \textbf{2}, 231 (1998) [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B \textbf{428}, 105 (1998) [hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. \textbf{2}, 253 (1998) [hep-th/9802150].

[4] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” JHEP \textbf{0802}, 045 (2008) [arXiv:0712.2456 [hep-th]].
[5] O. Saremi and D. T. Son, “Hall viscosity from gauge/gravity duality,” JHEP 1204, 091 (2012) [arXiv:1103.4851 [hep-th]].

[6] T. Delsate, V. Cardoso and P. Pani, “Anti de Sitter black holes and branes in dynamical Chern-Simons gravity: perturbations, stability and the hydrodynamic modes,” JHEP 1106, 055 (2011) [arXiv:1103.5756 [hep-th]].

[7] J. -W. Chen, N. -E. Lee, D. Maity and W. -Y. Wen, “A Holographic Model For Hall Viscosity,” Phys. Lett. B 713, 47 (2012) [arXiv:1110.0793 [hep-th]].

[8] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, “Parity-Violating Hydrodynamics in 2+1 Dimensions,” JHEP 1205, 102 (2012) [arXiv:1112.4498 [hep-th]].

[9] R. G. Leigh, A. C. Petkou and P. M. Petropoulos, “Holographic Fluids with Vorticity and Analogue Gravity,” JHEP 1211, 121 (2012) [arXiv:1205.6140 [hep-th]].

[10] J. -W. Chen, S. -H. Dai, N. -E. Lee and D. Maity, “Novel Parity Violating Transport Coefficients in 2+1 Dimensions from Holography,” JHEP 1209, 096 (2012) [arXiv:1206.0850 [hep-th]].

[11] R. Jackiw and S. Y. Pi, “Chern-Simons modification of general relativity,” Phys. Rev. D 68, 104012 (2003) [arXiv:gr-qc/0308071].

[12] D. Grumiller, R. B. Mann and R. McNees, “Dirichlet boundary value problem for Chern-Simons modified gravity,” Phys. Rev. D 78, 081502 (2008) [arXiv:0803.1485 [gr-qc]].

[13] T. L. Smith, A. L. Erickcek, R. R. Caldwell and M. Kamionkowski, “The Effects of Chern-Simons gravity on bodies orbiting the Earth,” Phys. Rev. D 77, 024015 (2008) [arXiv:0708.0001 [astro-ph]].

[14] D. Grumiller and N. Yunes, “How do Black Holes Spin in Chern-Simons Modified Gravity?,” Phys. Rev. D 77, 044015 (2008) [arXiv:0711.1868 [gr-qc]].

[15] S. Deser, R. Jackiw and S. Templeton, “Three-Dimensional Massive Gauge Theories,” Phys. Rev. Lett. 48, 975 (1982).

[16] S. Deser, R. Jackiw and S. Templeton, “Topologically Massive Gauge Theories,” Annals Phys. 140, 372 (1982) [Erratum-ibid. 185, 406 (1988)] [Annals Phys. 185, 406 (1988)] [Annals Phys. 281, 409 (2000)].

[17] L. Alvarez-Gaume and E. Witten, “Gravitational Anomalies,” Nucl. Phys. B 234, 269 (1984).

[18] J. E. Avron, R. Seiler and P. G. Zograf, “Viscosity of quantum Hall fluids,” Phys. Rev. Lett. 75, 697 (1995).

[19] X. -G. Wen and A. Zee, “Shift and Spin Vector: New Topological Quantum Numbers for the Hall Fluids,” Phys. Rev. Lett. 69, 953 (1992).

[20] J. E. Avron, “Odd Viscosity,” J. Stat. Phys. 92, 543 (1998) [arXiv:physics/9712050].

[21] N. Read, “Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and p(x) + ip(y) paired superfluids,” Phys. Rev. B 79, 045308 (2009) [arXiv:0805.2507 [cond-mat.mes-hall]].

[22] F. D. M. Haldane, “Hall viscosity’ and intrinsic metric of incompressible fractional Hall fluids,” arXiv:0906.1854 [cond-mat.str-el].
[23] N. Read and E. H. Rezayi, “Hall viscosity, orbital spin, and geometry: paired superfluids and quantum Hall systems,” Phys. Rev. B 84, 085316 (2011) [arXiv:1008.0210 [cond-mat].]

[24] A. Nicolis and D. T. Son, “Hall viscosity from effective field theory,” arXiv:1103.2137 [hep-th].

[25] C. Hoyos and D. T. Son, “Hall Viscosity and Electromagnetic Response,” Phys. Rev. Lett. 108, 066805 (2012) [arXiv:1109.2651 [cond-mat.mes-hall]].

[26] S. Bhattacharyya, S. Minwalla and S. R. Wadia, “The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity,” JHEP 0908, 059 (2009) [arXiv:0810.1545 [hep-th]].

[27] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, “From Navier-Stokes To Einstein,” JHEP 1207, 146 (2012) [arXiv:1101.2451 [hep-th]].

[28] I. Bakas, “Energy-momentum/Cotton tensor duality for AdS(4) black holes,” JHEP 0901, 003 (2009) [arXiv:0809.4852 [hep-th]].

[29] O. Miskovic and R. Olea, “Topological regularization and self-duality in four-dimensional anti-de Sitter gravity,” Phys. Rev. D 79, 124020 (2009) [arXiv:0902.2082 [hep-th]].

[30] S. S. Gubser, S. S. Pufu and F. D. Rocha, “Bulk viscosity of strongly coupled plasmas with holographic duals,” JHEP 0808, 085 (2008) [arXiv:0806.0407 [hep-th]].

[31] J. D. Brown and J. W. York, Jr., “Quasilocal energy and conserved charges derived from the gravitational action,” Phys. Rev. D 47, 1407 (1993) [gr-qc/9209012].

[32] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, “Wilsonian Approach to Fluid/Gravity Duality,” JHEP 1103, 141 (2011) [arXiv:1006.1902 [hep-th]].

[33] R. -G. Cai, L. Li, Z. -Y. Nie and Y. -L. Zhang, “Holographic Forced Fluid Dynamics in Non-relativistic Limit,” Nucl. Phys. B 864, 260 (2012) [arXiv:1202.4091 [hep-th]].

[34] L. D. Landau, E. M. Lifshitz. “Fluid Mechanics,” Oxford: Pergamon Press (1987)

[35] S. Ryu, J. E. Moore, A. W. W. Ludwig, “Electromagnetic and gravitational responses and anomalies in topological insulators and superconductors” Phys. Rev. B 85, 045104 (2012) [arXiv:1010.0936 [cond-mat.str-el]].

[36] Z. Wang, X. -L. Qi and S. -C. Zhang, “Topological field theory and thermal responses of interacting topological superconductors,” Phys. Rev. B 84, 014527 (2011) [arXiv:1011.0586 [cond-mat.str-el]].

[37] T. L. Hughes, R. G. Leigh and E. Fradkin, “Torsional Response and Dissipationless Viscosity in Topological Insulators,” Phys. Rev. Lett. 107, 075502 (2011) [arXiv:1101.3541 [cond-mat.mes-hall]].

[38] T. Hertog, “Towards a Novel no-hair Theorem for Black Holes,” Phys. Rev. D 74, 084008 (2006) [gr-qc/0608075].

[39] D. Brattan, J. Camps, R. Loganayagam and M. Rangamani, “CFT dual of the AdS Dirichlet problem: Fluid/Gravity on cut-off surfaces,” JHEP 1112, 090 (2011) [arXiv:1106.2577 [hep-th]].

[40] D. T. Son and P. Surowka, “Hydrodynamics with Triangle Anomalies,” Phys. Rev. Lett. 103, 191601 (2009) [arXiv:0906.5044 [hep-th]].
[41] X. Bai, Y. -P. Hu, B. -H. Lee and Y. -L. Zhang, “Holographic Charged Fluid with Anomalous Current at Finite Cutoff Surface in Einstein-Maxwell Gravity,” JHEP 1211, 054 (2012) [arXiv:1207.5309 [hep-th]].

[42] M. Perelstein and Y. -H. Qi, “Topological Interactions in the Higgsless Model at the LHC,” Phys. Rev. D 82, 015004 (2010) [arXiv:1003.5725 [hep-ph]].

[43] N. Iqbal and H. Liu, “Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm,” Phys. Rev. D 79, 025023 (2009) [arXiv:0809.3808 [hep-th]].

[44] D. Nickel and D. T. Son, “Deconstructing holographic liquids,” New J. Phys. 13, 075010 (2011) [arXiv:1009.3094 [hep-th]].

[45] G. Compere, P. McFadden, K. Skenderis and M. Taylor, “The holographic fluid dual to vacuum Einstein gravity,” JHEP 1107, 050 (2011) [arXiv:1103.3022 [hep-th]].

[46] R. -G. Cai, L. Li and Y. -L. Zhang, “Non-Relativistic Fluid Dual to Asymptotically AdS Gravity at Finite Cutoff Surface,” JHEP 1107, 027 (2011) [arXiv:1104.3281 [hep-th]].

[47] V. Lysov and A. Strominger, “From Petrov-Einstein to Navier-Stokes,” [arXiv:1104.5502 [hep-th]].

[48] C. Niu, Y. Tian, X. -N. Wu and Y. Ling, “Incompressible Navier-Stokes Equation from Einstein-Maxwell and Gauss-Bonnet-Maxwell Theories,” Phys. Lett. B 711, 411 (2012) [arXiv:1107.1430 [hep-th]].

[49] R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: An Introduction to Current Research, edited by L. Witten (Wiley, New York, 1962)

[50] R. M. Wald, “General Relativity,” University of Chicago Press (1984).