REGULARITY OF TRANSITION SEMIGROUPS ASSOCIATED TO A 3D STOCHASTIC NAVIER-STOKES EQUATION

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ABSTRACT. A 3D stochastic Navier-Stokes equation with a suitable non degenerate additive noise is considered. The regularity in the initial conditions of every Markov transition kernel associated to the equation is studied by a simple direct approach. A by-product of the technique is the equivalence of all transition probabilities associated to every Markov transition kernel.

1. INTRODUCTION

An old dream in stochastic fluid dynamics is to prove the well posedness of a stochastic version of the 3D Navier-Stokes equations, taking advantage of the noise, as one can do for finite dimensional stochastic equations with non regular drift (see for instance Stroock & Varadhan [20]). The problem is still open, although some intriguing results have been recently proved, see for instance Da Prato & Debussche [3], Mikulevicius & Rozovski [15], Flandoli & Romito [11] (see also [10]). We recall here the framework constructed in [11] and prove some additional results.

We consider a viscous, incompressible, homogeneous, Newtonian fluid described by the stochastic Navier-Stokes equations on the torus $T = [0, L]^3$, $L > 0$,

\begin{equation}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = \nu \Delta u + \sum_{i=1}^{\infty} \sigma_i h_i(x) \beta_i(t)
\end{equation}

with $\text{div} u = 0$ and periodic boundary conditions, with suitable fields $h_i(x)$ and independent Brownian motions $\beta_i(t)$. The 3D random vector field $u = u(t, x)$ is the velocity of the fluid and the random scalar field $p = p(t, x)$ is the pressure. To simplify the exposition, we avoid generality and focus on one of the simplest set of assumptions:

\[
\sigma_i^2 = \lambda_i^{-3}
\]

where $\lambda_i$ are the eigenvalues of the Stokes operator (see the next section). This assumption also allows us to compare more closely the results in Da Prato & Debussche [3] and Flandoli [8]. However, following Flandoli & Romito [11], we could treat any power law for $\sigma_i$. Under this assumption,

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one can associate a transition probability kernel $P(t,x,\cdot)$ to equation (2.2), which is the abstract version of (1.1), in $D(A)$ (see the definitions in Section 2.1 below), satisfying the Chapman-Kolmogorov equation. In other words, there exists a Markov selection in $D(A)$ for equation (2.2). To avoid misunderstandings, this does not mean that equation (2.2) has been solved in $D(A)$ with continuous trajectories: this would imply well posedness. What has been proved is that the law of weak martingale solutions is supported on $D(A)$ for all times, with a number of related additional properties, but a priori the typical trajectory may sometimes blow-up in the topology of $D(A)$.

The transition probabilities $P(t,x,\cdot)$ are irreducible and strong Feller, hence equivalent, in $D(A)$. These results and the existence of $P(t,x,\cdot)$ have been proved first in Da Prato & Debussche [3] and Debussche & Odasso [6] by a careful selection from the Galerkin scheme. Then another proof by an abstract selection principle and the local-in-time regularity of equation (2.2) has been given in Flandoli & Romito [11]. More precisely, first one proves the existence of a Markov kernel $P(t,x,\cdot)$ by means of a general and abstract method, then one proves that any such kernel is irreducible and strong Feller, hence equivalent, in $D(A)$.

We complement here the approach of [11] with two results. First, the simple idea used in [11] to prove the strong Feller property is here developed further, to show a weak form of Lipschitz continuity of $P(t,x,\cdot)$ in $x \in D(A)$. More precisely, we prove the estimate

$$\tag{1.2} |P(t,x_0 + h,\Gamma) - P(t,x_0,\Gamma)| \leq \frac{C_T}{t \wedge 1} (1 + |Ax_0|^6)|Ah| \log(|Ah|^{-1})$$

for $t \in (0,T]$, $x_0, h \in D(A)$, with $|Ah| \leq 1$. This result has been proved in a stronger version in Da Prato & Debussche [3] for the transition kernel constructed from the Galerkin scheme, and also in Flandoli [8] for any Markov kernel associated to equation (2.2). In both cases the proof is based on the very powerful approach introduced in [3] which however requires a considerable amount of technical work. Here we give a rather elementary proof along the lines of Flandoli & Romito [11], based on the following simple idea: given $x_0, h \in D(A)$, for a short random time the solution is regular, unique and differentiable in the initial conditions; then the propagation of regularity in $x$ from small time to arbitrary time is due to the Markov property. Unfortunately we cannot prove in this way the stronger estimate obtained in [3] (where the right-hand-side of (1.2) has the form $t^{-1+\varepsilon}(1 + |Ax_0|^2)|Ah|$), so our first result here has mostly a pedagogical character, since the proof is conceptually very easy.

The second result, which follows from the same main estimates used to prove (1.2), is the equivalence

$$P^{(1)}(t,x,\cdot) \sim P^{(2)}(t',x',\cdot)$$

for any $t, t' > 0$ and $x, x' \in D(A)$, when $P^{(i)}(t,x,\cdot), i = 1,2$, are any two Markov transition kernels associated to equation (2.2) in $D(A)$. We have
not proved yet the existence of invariant measures associated to such kernels\(^1\), but if we assume to have such invariant measures, it also follows that they are equivalent. This result and the gradient estimates discussed above could be steps to understand better the open question of well posedness for equation (2.2). In particular, it seems to be not so easy to produce examples of stochastic differential equations without uniqueness but where all Markov solutions are equivalent.

Among the open problems related to this research we mention the relation between the regularity results for \(P(t,x,\cdot)\) in the initial condition discussed above and the properties of Malliavin derivatives, investigated for stochastic 3D Navier-Stokes equations by Mikulevicius and Rozovsky in [14] and [15].

2. Preliminaries

2.1. Notations. Denote by \(\mathcal{T} = [0,1)^3\) the three-dimensional torus, and let \(L^2(\mathcal{T})\) be the space of vector fields \(u: \mathcal{T} \to \mathbb{R}^3\) with \(L^2(\mathcal{T})\)-components. For every \(\alpha > 0\), let \(H^\alpha(\mathcal{T})\) be the space of fields \(u \in L^2(\mathcal{T})\) with components in the Sobolev space \(H^\alpha(\mathcal{T}) = W^{\alpha,2}(\mathcal{T})\).

Let \(\mathcal{D}^\infty\) be the space of infinitely differentiable divergence free periodic fields \(u\) on \(\mathcal{T}\), with zero mean. Let \(H\) be the closure of \(\mathcal{D}^\infty\) in the topology of \(L^2(\mathcal{T})\): it is the space of all zero mean fields \(u \in L^2(\mathcal{T})\) such that \(\text{div}u = 0\) and \(u \cdot n\) on the boundary is periodic. We denote by \(\langle \cdot, \cdot \rangle_H\) and \(|\cdot|_H\) (or simply by \(\langle \cdot, \cdot \rangle\) and \(|\cdot|\)) the usual \(L^2\)-inner product and norm in \(H\). Let \(V\) (resp. \(D(A)\)) be the closure of \(\mathcal{D}^\infty\) in the topology of \(H^1(\mathcal{T})\) (in the topology of \(H^2(\mathcal{T})\), respectively): it is the space of divergence free, zero mean, periodic elements of \(H^1(\mathcal{T})\) (respectively of \(H^2(\mathcal{T})\)). The spaces \(V\) and \(D(A)\) are dense and compactly embedded in \(H\). From Poincaré inequality we may endow \(V\) with the norm \(||u||_V^2 := \int_{\mathcal{T}} |Du(x)|^2\ dx\).

Let \(A : D(A) \subset H \to H\) be the operator \(Au = -\triangle u\) (component wise). Since \(A\) is a selfadjoint positive operator in \(H\), there is a complete orthonormal system \((h_i)_{i \in \mathbb{N}}\) of eigenfunctions of \(A\), with eigenvalues \(0 < \lambda_1 \leq \lambda_2 \leq \ldots\) (that is, \(Ah_i = \lambda_i h_i\)). The fields \(h_i\) in equation (2.2) will be these eigenfunctions. We have

\[
\langle Au, u \rangle_H = ||u||_V^2
\]

for every \(u \in D(A)\).

Let \(V'\) be the dual of \(V\); with proper identifications we have \(V \subset H \subset V'\) with continuous injections, and the scalar product \(\langle \cdot, \cdot \rangle_H\) extends to the dual pairing \(\langle \cdot, \cdot \rangle_{V,V'}\) between \(V\) and \(V'\). We may enlarge this scheme to \(D(A) \subset V \subset H \subset V' \subset D(A)'\). Let \(B(\cdot, \cdot) : V \times V \to V'\) be the bilinear operator defined as

\[
\langle w, B(u,v) \rangle_{V,V'} = \sum_{i,j=1}^3 \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j \ dx
\]

\(^1\)This is apparently due to technical reasons and it is the subject of a work in progress.
for every \( u, v, w \in V \). We shall repeatedly use the following inequality:

\[
A_{1/2} B (u, v) \leq C_0 |Au| |Av|
\]

for \( u, v \in D(A) \). The proof is elementary (see Flandoli [9]).

2.2. Definitions, assumptions and known results. We (formally) rewrite equations (1.1) as an abstract stochastic evolution equation in \( H \),

\[
du(t) + [vAu(t) + B (u(t), u(t))] dt = \sum_{i=1}^{\infty} \sigma_i h_i d\beta_i (t).
\]

Let us set

\[
\Omega = C([0, \infty); D(A)')
\]

and denote by \( (\xi_t)_{t \geq 0} \) the canonical process on \( \Omega \), defined as \( \xi_t (\omega) = \omega(t) \), by \( \mathcal{F} \) the Borel \( \sigma \)-algebra in \( \Omega \) and by \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by the events \( \{ \xi_s \in A \} \) with \( s \in [0, t] \) and \( A \) a Borel set of \( D(A)' \). Finally, denote by \( \mathcal{B}(D(A)) \) the Borel \( \sigma \)-algebra of \( D(A) \) and by \( \mathcal{B}_b(D(A)) \) the set of all real valued bounded measurable functions on \( D(A) \).

**Definition 1.** Given a probability measure \( \mu_0 \) on \( H \), we say that a probability measure \( P \) on \( (\Omega, \mathcal{F}) \) is a solution to the martingale problem associated to equation (2.2) with initial law \( \mu_0 \) if

- [MP1] \( P[\xi \in L^\infty_{\text{loc}}([0, \infty); H) \cap L^2_{\text{loc}}([0, \infty); V)] = 1 \),
- [MP2] for each \( \varphi \in \mathcal{D}^\infty \) the process \( (M^\varphi_t, \mathcal{F}_t, P)_{t \geq 0} \), defined \( P \)-a. s. on \( (\Omega, \mathcal{F}) \) as

\[
M^\varphi_t := \langle \xi_t - \xi_0, \varphi \rangle_H + \int_0^t v \langle \xi_s, A\varphi \rangle_H ds - \int_0^t \langle B(\xi_s, \varphi), \xi_s \rangle_H ds
\]

is a continuous square integrable martingale with quadratic variation

\[
[M^\varphi]_t = t \sum_{i \in \mathbb{N}} \sigma_i^2 |\langle \varphi, h_i \rangle|^2,
\]

- [MP3] the marginal of \( P \) at time 0 is \( \mu_0 \).

**Remark 2.** Among all test functions in property [MP2], we can choose \( \varphi = h_i \). Set for all \( i \), \( \beta_i (t) = \frac{1}{\sigma_i} M^{h_i}_t \) (and 0 if \( \sigma_i = 0 \)). The \( (\beta_i)_{i \in \mathbb{N}} \) are a sequence of independent standard Brownian motions. Under the assumption \( \sum_i \sigma_i^2 < \infty \), the series \( \sum_{i=1}^{\infty} \sigma_i h_i \beta_i (t) \) defines an \( H \)-valued Brownian motion on \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \), that we shall denote by \( W(t) \). The canonical process \( (\xi_t) \) is a weak martingale solution of (2.2), in the sense that it satisfies (2.2) in the following weak form: there exists a Borel set \( \Omega_0 \subset \Omega \) with \( P(\Omega_0) = 1 \) such that on \( \Omega_0 \) for every \( \varphi \in \mathcal{D}^\infty \) and \( t \geq 0 \) we have

\[
\langle \xi_t - \xi_0, \varphi \rangle_H + \int_0^t v \langle \xi_s, A\varphi \rangle_H ds - \int_0^t \langle B(\xi_s, \varphi), \xi_s \rangle_H ds = \langle W(t), \varphi \rangle_H.
\]

The following theorem is well known, see for instance the survey paper of Flandoli [9] and the reference therein.
Theorem 3. Assume $\sum_i \sigma_i^2 < \infty$. Let $\mu$ be a probability measure on $H$ such that $\int_H |x|^2_H \mu(dx) < \infty$. Then there exists at least one solution to the martingale problem with initial condition $\mu$.

Definition 4. We say that $P (\cdot, \cdot, \cdot) : [0, \infty) \times D(A) \times \mathcal{B}(D(A)) \rightarrow [0,1]$ is a Markov kernel in $D(A)$ of transition probabilities associated to equation (1.1) if $P (\cdot, \cdot, \Gamma)$ is Borel measurable for every $\Gamma \in \mathcal{B}(D(A))$, $P(t,x,\cdot)$ is a probability measure on $\mathcal{B}(D(A))$ for every $(t,x) \in [0,\infty) \times D(A)$, the Chapman-Kolmogorov equation

$$P(t+s,x,\Gamma) = \int_{D(A)} P(t,x,dy) P(s,y,\Gamma)$$

holds for every $t,s \geq 0$, $x \in D(A)$, $\Gamma \in \mathcal{B}(D(A))$, and for every $x \in D(A)$ there is a solution $P_t$ on $(\Omega,F)$ of the martingale problem associated to equation (2.2) with initial condition $\xi$ such that

$$P(t,x,\Gamma) = P_t(\xi \in \Gamma)$$

for all $t \geq 0$.

We recall the following result from Da Prato & Debussche [3], Debussche & Odasso [6] or Flandoli & Romito [11]:

Theorem 5. There exists at least one Markov kernel $P(t,x,\Gamma)$ in $D(A)$ of transition probabilities associated to equation (1.1).

We recall that a $P(t,x,\Gamma)$ is called irreducible in $D(A)$ if for every $t > 0$, $x_0, x_1 \in D(A)$, $\varepsilon > 0$, we have

$$P(t,x_0,B_A(x_1,\varepsilon)) > 0,$$

where $B_A(x_1,\varepsilon)$ is the ball in $D(A)$ of centre $x_1$ and radius $\varepsilon$.

We say that $P(t,x,\Gamma)$ is strong Feller in $D(A)$ if

$$x \mapsto \int_{D(A)} \phi(y) P(t,x,dy)$$

is continuous on $D(A)$ for every bounded measurable function $\phi : D(A) \rightarrow \mathbb{R}$ and for every $t > 0$. It is well known (see for example Da Prato & Zabczyk [5], Proposition 4.1.1)) that irreducibility and strong Feller in $D(A)$ imply that the laws $P(t,x,\cdot)$ are all mutually equivalent, as $(t,x)$ varies in $(0,\infty) \times D(A)$. Because of this equivalence property, we say that $P(t,x,\Gamma)$ is regular.

We recall also that $P(t,x,\Gamma)$ is called stochastically continuous in $D(A)$ if $\lim_{t \rightarrow 0} P(t,x,B_A(x,\varepsilon)) = 1$ for every $x \in D(A)$ and $\varepsilon > 0$.

In Da Prato & Debussche [3], the transition probability kernel constructed by Galerkin approximations is proved to be stochastically continuous, irreducible and strong Feller in $D(A)$, hence regular. More generally (see Flandoli & Romito [11]):

Theorem 6. Every Markov kernel $P(t,x,\Gamma)$ in $D(A)$ of transition probabilities associated to equation (1.1) is stochastically continuous, irreducible and strong Feller in $D(A)$, hence regular.
3. The Log-Lipschitz estimate

**Theorem 7.** Let $P(t,x,\Gamma)$ be a Markov kernel in $D(A)$ of transition probabilities associated to equation (1.1). Then, given $T > 0$, there is a constant $C_T$ such that the inequality

$$|P(t,x_0 + h, \Gamma) - P(t,x_0, \Gamma)| \leq \frac{C_T}{t \wedge 1} (1 + |Ax_0|^6)|Ah| \log(|Ah|^{-1})$$

holds for every $t \in (0,T)$, $x_0, h \in D(A)$, with $|Ah| \leq 1$, and $\Gamma \in \mathcal{B}(D(A))$.

We explain here only the logical skeleton of the proof, which is very simple. The two main technical ingredients will be treated in the next two separate subsections. The first idea is to decompose:

$$P(t,x_0 + h, \Gamma) - P(t,x_0, \Gamma) =$$

$$\int_{D(A)} [P(\varepsilon,x_0 + h, dy) - P(\varepsilon,x_0, dy)] P(t - \varepsilon, y, \Gamma).$$

To shorten some notation, let us write

$$(P_\varepsilon \varphi)(x) = \int_{D(A)} \varphi(y) P(t,x, dy)$$

so, with the function $\varphi(x) = 1_{\{x \in \Gamma\}}$ the previous identity reads

$$(3.1) \quad (P_\varepsilon \varphi)(x_0 + h) - (P_\varepsilon \varphi)(x_0) = (P_\varepsilon (P_{t-\varepsilon} \varphi))(x_0 + h) - (P_\varepsilon (P_{t-\varepsilon} \varphi))(x_0).$$

It is now sufficient to estimate

$$(P_\varepsilon \psi)(x_0 + h) - (P_\varepsilon \psi)(x_0)$$

uniformly in $\psi \in B_b(D(A))$. The value of $\varepsilon$ has to be chosen depending on the size of $x_0$ and $h$, as we shall see.

The second idea is to use an initial coupling: we introduce the equation with cut-off $\chi_R(|A u|^2)$, where $\chi_R(r) : [0, \infty) \to [0, 1]$ is a non-increasing smooth function equal to 1 over $[0,R]$, to 0 over $[R+2, \infty)$, and with derivative bounded by 1. The equation is

$$(3.2) \quad du + \left[Au + B(u,u)\chi_R \left(|A u|^2\right)\right] dt = \sum_{i=1}^\infty \sigma_i h_i d\beta_i(t),$$

$$u(0) = x.$$  

The definition of martingale problem for this equation is the same (with obvious adaptations) as the definition given above for equation (1.1). Let $\tau_R : \Omega \to [0, \infty]$ be defined as

$$\tau_R(\omega) = \inf\{t \geq 0 : |A \omega(t)| \geq R\}.$$  

We recall the following result from Flandoli & Romito [11], Lemma 5.11:

**Lemma 8.** For every $x \in D(A)$ there is a unique solution $P_x^{(R)}$ of the martingale problem associated to equation (3.2), with the additional property

$$P_x^{(R)}[\xi \in C([0, \infty); D(A))] = 1.$$
Let $P_x$ be any solution on $(\Omega,F)$ of the martingale problem associated to equation (2.2) with initial condition $x$. Then

$$\mathbb{E}^P_x[\Phi(\xi_t)\mathbb{1}_{\{\tau_R \geq t\}}] = \mathbb{E}^P_x[\Phi(\xi_t)\mathbb{1}_{\{\tau_R \geq t\}}]$$

for every $t \geq 0$ and $\phi \in B_b(D(A))$.

Introduce the notation

$$(P_t^{(R)}\phi)(x) = \mathbb{E}^P_{\xi_t}[\phi(\xi_t)].$$

The previous lemma implies that for every $\psi \in B_b(D(A))$ we have

$$(3.3) \quad \|(P_\varepsilon\psi)(x) - (P_\varepsilon^{(R)}\psi)(x)\| \leq 2P_x[\tau_R < \varepsilon]\|\psi\|_{\infty}.$$  

Summarising:

**Corollary 9.** For every $x_0, h \in D(A)$ and $\psi \in B_b(D(A))$ we have

$$\|(P_\varepsilon\psi)(x_0+h) - (P_\varepsilon\psi)(x_0)\| \leq 2(P_{x_0+h}[\tau_R < \varepsilon] + P_{x_0}[\tau_R < \varepsilon])\|\psi\|_{\infty}$$

$$+ \|(P_\varepsilon^{(R)}\psi)(x_0+h) - (P_\varepsilon^{(R)}\psi)(x_0)\|.$$

Let us give now the proof of Theorem 7. Assume $t \in (0,T], x_0, h \in D(A)$ be given, with $|Ah| \leq 1$. Let $K > 0$ be such that $|A(x_0)| + 1 \leq K$. We have $|A(x_0 + h)| \leq K$, so we may apply Proposition 11 below to both $x_0$ and $x_0 + h$. We thus get, for $\varepsilon \in (0,1/5C^*K^2)$, where $C^* > 0$ is the constant defined by (A.30), we have

$$P_{x_0 + h}[\tau_{2K} < \varepsilon] + P_{x_0}[\tau_{2K} < \varepsilon] \leq 2C\#e^{-\eta#K^2/4\varepsilon}.$$

Given $h$, $K$ and $t$ as above, let us look for a value $\varepsilon \in (0,1/5C^*K^2)$ such that $\varepsilon \leq t$ and the latter exponential quantity is smaller than $|Ah|$. We impose

$$\eta#K^2/4\varepsilon \geq \log(|Ah|^{-1})$$

hence it is sufficient to take

$$(3.4) \quad \varepsilon \leq \frac{\eta#K^2}{4\log(|Ah|^{-1})} \wedge \frac{t}{2} \wedge \frac{1}{5C^*K^2}.$$ 

We have proved so far the first claim of the following lemma. The second claim is a simple consequence of (3.1) and the previous corollary.

**Lemma 10.** Given $t > 0, x_0, h \in D(A)$, with $|Ah| \leq 1$, and $\Gamma \in \mathcal{B}(D(A))$, if $\varepsilon$ is chosen as in (3.4), then

$$P_{x_0 + h}[\tau_{2K} < \varepsilon] + P_{x_0}[\tau_{2K} < \varepsilon] \leq 2C\#|Ah|$$

and for $\phi(x) = \mathbb{1}_{\{x \in \Gamma\}}$ and $\psi = P_{t-\varepsilon}\phi$,

$$|P_t\phi(x_0+h) - P_t\phi(x_0)| \leq 4C\#|Ah|\|\phi\|_{\infty} + |P_\varepsilon^{(2K)}\psi(x_0+h) - P_\varepsilon^{(2K)}\psi(x_0)|.$$
Finally, from Proposition 12 below, renaming the constant $C$, with $\varphi(x) = 1_{\{x \in \Gamma\}}$ and $\psi = P_{t - \varepsilon} \varphi$,

$$\left| \left( P_{t}^{(2K)} \psi \right)(x_{0} + h) - \left( P_{t}^{(2K)} \psi \right)(x_{0}) \right| \leq \frac{C}{\varepsilon} |Ah| e^{CK^{6}\varepsilon}.$$

Thus, for $\varepsilon$ as in (3.4), we get

$$|P_{t} \varphi(x_{0} + h) - P_{t} \varphi(x_{0})| \leq 4C\#|Ah| + \frac{C}{\varepsilon} |Ah| e^{CK^{6}\varepsilon}.$$

Let us further restrict ourselves to

$$\varepsilon \leq \frac{\eta\#K^{2}}{4 \log(|Ah|^{-1})} \wedge \frac{t}{2} \wedge \frac{1}{5C^{*}K^{2}} \wedge \frac{1}{K^{6}},$$

so that we have

$$|P_{t} \varphi(x_{0} + h) - P_{t} \varphi(x_{0})| \leq 4C\#|Ah| + \frac{C}{\varepsilon} |Ah|.$$

The choice

$$\varepsilon = C \frac{t \wedge 1}{K^{6}\log(|Ah|^{-1})}$$

is admissible for a suitable constant $C > 0$, and we finally get (1.2). The proof of Theorem 7 is complete.

3.1. Probability of blow-up.

**Proposition 11.** Let $K \geq 1$ and assume that $x_{0} \in D(A)$ and $\varepsilon > 0$ are given such that $|Ax_{0}| \leq K$ and $\varepsilon \leq \frac{1}{5C^{*}K^{2}}$, where $C^{*}$ is the constant defined in (A.5). Then

$$P_{x_{0}} [\tau_{2K} < \varepsilon] \leq C\# e^{-\eta\#K^{2} \frac{\varepsilon}{4}},$$

for suitable universal constants $\eta > 0$ and $C\# > 0$.

**Proof.** From Corollary 17 we know that if $\varepsilon \leq \frac{1}{5C^{*}K^{2}}$ and $|Ax_{0}| \leq K$, then one has

$$\theta^{2}_{\varepsilon} \leq \frac{1}{4} K^{2} \quad \Rightarrow \quad |Au(s)| < 2K \quad \text{for} \quad s \in [0, \varepsilon] \quad \Rightarrow \quad \tau_{2K} \geq \varepsilon,$$

where $\theta_{\varepsilon}$ is defined in Section (A.1). Therefore, with the constraints $|Ax_{0}| \leq K$ and $\varepsilon \leq \frac{1}{5C^{*}K^{2}}$, by Proposition 15 one gets

$$P_{x_{0}} [\tau_{2K} < \varepsilon] \leq P_{x_{0}} \left[ \Theta^{2}_{\varepsilon} > \frac{1}{4} K^{2} \right] \leq C\# e^{-\eta\#K^{2} \frac{\varepsilon}{4}}.$$  

$\square$
3.2. Derivative of the regularised problem. Here we show the regularity of the transition semigroup associated to the regularised problem (3.2).

**Proposition 12.** For every \( R \geq 1 \) and \( x_0, h \in D(A) \),

\[
\| (P^{(R)}_t \psi)(x_0 + h) - (P^{(R)}_t \psi)(x_0) \| \leq \frac{C \| \psi \|_\infty}{\varepsilon} |Ah| e^{CR^6 \varepsilon},
\]

where \( C \) is a universal constant.

**Proof.** We write the following computations for the limit problem but the understanding is that we do it on the Galerkin approximations. For every \( \psi \in B_b(H), \varepsilon > 0 \), from the Bismut-Elworthy-Li formula (see Da Prato & Zabczyk [5]),

\[
\| (P^{(R)}_t \psi)(x_0 + h) - (P^{(R)}_t \psi)(x_0) \| \leq \frac{C \| \psi \|_\infty}{\varepsilon} \sup_{\eta \in [0,1]} \mathbb{E} \left[ \left( \int_0^\varepsilon \| A_\varepsilon^2 D_h u^{(R)}_{x_0 + \eta h}(s) \|^2 ds \right)^{\frac{1}{2}} \right],
\]

where, for each \( R \geq 1 \) and \( x \in D(A), u^{(R)}_x \) is the solution, starting at \( x \), of problem (3.2). From the regularised equation we have

\[
\frac{1}{2} \frac{d}{dt} |A D_h u^{(R)}_x(t)|^2 + |A_\varepsilon^2 D_h u^{(R)}_x(t)|^2 \leq \chi_R(|A u^{(R)}_x(t)|^2) |A D_h u^{(R)}_x(t), A B(D_h u^{(R)}_x, u^{(R)}_x) + A B(u^{(R)}_x, D_h u^{(R)}_x)|
\]

\[
+ 2 \chi'_R(|A u^{(R)}_x(t)|^2) |A D_h u^{(R)}_x, A B(u^{(R)}_x, u^{(R)}_x), A D_h u^{(R)}_x, A B(u^{(R)}_x, u^{(R)}_x)|
\]

\[
\leq C \chi_R(|A u^{(R)}_x(t)|^2) |A_\varepsilon^2 D_h u^{(R)}_x(t)| |A D_h u^{(R)}_x(t)| |A u^{(R)}_x(t)|
\]

\[
+ C \chi'_R(|A u^{(R)}_x(t)|^2) |A u^{(R)}_x(t)|^3 |A D_h u^{(R)}_x(t)| |A_\varepsilon^2 D_h u^{(R)}_x(t)|
\]

\[
\leq \frac{1}{2} |A_\varepsilon^2 D_h u^{(R)}_x(t)|^2 + C \chi_R(|A u^{(R)}_x(t)|^2) |A D_h u^{(R)}_x(t)|^2 |A u^{(R)}_x(t)|^2
\]

\[
+ C \chi'_R(|A u^{(R)}_x(t)|^2)^2 |A D_h u^{(R)}_x(t)|^2 |A u^{(R)}_x(t)|^6
\]

\[
\leq \frac{1}{2} |A_\varepsilon^2 D_h u^{(R)}_x(t)|^2 + CR^6 |A D_h u^{(R)}_x(t)|^2.
\]

Thus

\[
\frac{1}{2} \frac{d}{dt} |A D_h u^{(R)}_x(t)|^2 + \frac{1}{2} |A_\varepsilon^2 D_h u^{(R)}_x(t)|^2 \leq CR^6 |A D_h u^{(R)}_x(t)|^2.
\]

This implies

\[
|A D_h u^{(R)}_x(t)|^2 \leq e^{CR^6 t} |Ah|^2
\]

and

\[
\int_0^\varepsilon |A_\varepsilon^2 D_h u^{(R)}_{x_0 + \eta h}(s)|^2 ds \leq |Ah|^2 \left( 1 + \int_0^\varepsilon CR^6 e^{CR^6 s} ds \right) = |Ah|^2 e^{CR^6 \varepsilon}.
\]

Thus

\[
\| (P^{(R)}_t \psi)(x_0 + h) - (P^{(R)}_t \psi)(x_0) \| \leq \frac{C \| \psi \|_\infty}{\varepsilon} |Ah| e^{CR^6 \varepsilon}.
\]
Theorem 13. Let \( P^{(1)}(t,x,\Gamma) \) be two Markov kernels in \( D(A) \) of transition probabilities associated to equation \( \text{(1.1)} \). Assume they are stochastically continuous, irreducible and strong Feller in \( D(A) \). Then the probability measures \( P^{(1)}(t,x,\cdot) \) and \( P^{(2)}(t',x',\cdot) \) are equivalent, for any \( t, t' > 0 \) and \( x, x' \in D(A) \).

Proof. Step 1. Let \( \Gamma \) be a Borel set in \( D(A) \) such that \( P^{(2)}(t_0,x_0,\Gamma) = 0 \) for some \( t_0 > 0, x_0 \in D(A) \). It is sufficient to prove that \( P^{(1)}(t_0,x_0,\Gamma) = 0 \). We know that \( P^{(2)}(t,x,\Gamma) = 0 \) for every \( t > 0, x \in D(A) \).

Step 2. Since both \( P^{(1)}(\cdot,\cdot,\cdot) \) and \( P^{(2)}(\cdot,\cdot,\cdot) \) satisfy (3.3),

\[
P^{(1)}(t,x,\Gamma) = |P^{(1)}(t,x,\Gamma) - P^{(2)}(t,x,\Gamma)| \leq 2(P^{(1)}_x[\tau_R < t] + P^{(2)}_x[\tau_R < t]).
\]

Now, for every pair \( (\varepsilon,x) \), with \( \varepsilon > 0 \) and \( x \in D(A) \), such that \( 5C^*(1 + |Ax|)^2\varepsilon \leq 1 \) (the constant \( C^* \) is defined in \( (A.5) \), in the appendix), Proposition [11] implies that

\[
P^{(1)}(\varepsilon,x,\Gamma) \leq 2C_\#e^{-\frac{\varepsilon (1 + |Ax|)^2}{4e}} \leq 2C_\#e^{-\frac{1}{4}\eta_\#}.
\]

Step 3. For every \( \varepsilon < \frac{1}{5C^*} \), set \( A_\varepsilon = \{ x \in D(A) : 5C^*(1 + |Ax|)^2\varepsilon \leq 1 \} \), then by the Markov property and the previous step,

\[
P^{(1)}(t_0 + \varepsilon,x_0,\Gamma) = \int_{A_\varepsilon} P^{(1)}(\varepsilon,x,\Gamma) P^{(1)}(t_0,x_0,dx)
\]

\[
+ \int_{A_\varepsilon^c} P^{(1)}(\varepsilon,x,\Gamma) P^{(1)}(t_0,x_0,dx)
\]

\[
\leq 2C_\#e^{-\frac{1}{4}\eta_\#} + P^{(1)}(t_0,x_0,A_\varepsilon^c)
\]

Since \( P^{(1)}(x,x_0,D(A)) = 1 \), we have \( P^{(1)}(t_0,x_0,A_\varepsilon^c) \longrightarrow 0 \), as \( \varepsilon \to 0 \), and thus

\[
\lim_{\varepsilon \to 0} P^{(1)}(t_0 + \varepsilon,x_0,\Gamma) = 0.
\]

Step 4. By the Markov property, for every neighborhood \( G \) of \( x_0 \) in \( D(A) \),

\[
P^{(1)}(t_0 + \varepsilon,x_0,\Gamma) = \int P^{(1)}(t_0,y,\Gamma) P^{(1)}(\varepsilon,x_0,dy)
\]

\[
\geq P^{(1)}(\varepsilon,x_0,G) \inf_{y \in G} P^{(1)}(t_0,y,\Gamma).
\]
Since the kernel $P^{(1)}$ is stochastically continuous, $P^{(1)}(\varepsilon, x_0, G)$ converges to 1, as $\varepsilon \to 0$, and so, by the previous step, $\inf_{\gamma \in G} P^{(1)}(t_0, \gamma, \Gamma) \to 0$ as $\varepsilon \to 0$. By the strong Feller property, the map $y \mapsto P^{(1)}(t_0, y, \Gamma)$ is continuous, hence in conclusion $P^{(1)}(t_0, x_0, \Gamma) = 0$. The proof is complete. 

5. Conclusion and remarks

We have proved that the transition probabilities associated to any Markov selection are all equivalent to each other. However, the problem of uniqueness of Markov selections remains open. We stress that it would imply uniqueness of solutions to the martingale problem, by the argument that one can find in Stroock & Varadhan [20] Theorem 12.2.4.

The estimates proved in this work allows us at least to state a sufficient condition for uniqueness of Markov selections. The proof is inspired to a well known proof in semigroup theory as well as to the proof of uniqueness of Markov selections. The proof is inspired to a well known proof in semigroup theory as well as to the proof of uniqueness given by Bressan and co-authors (see for instance [1]).

**Proposition 14.** Assume that a Markov selection $(P_x)_{x \in A}$ has the following property: for every $t > 0$ and $x \in D(A)$,

$$
\lim_{n \to \infty} \sum_{k=1}^{n} P \left( t - \frac{k}{n}, x, B_A \left( 0, \sqrt{\frac{n}{t}} \right) \right) = 0
$$

where $B_A(0,n)$ is the ball in $D(A)$ of radius $n$. Then $(P_x)_{x \in A}$ coincides with any other Markov selection.

**Proof.** Let $(Q_x)_{x \in A}$ be another Markov selection. Let us rewrite, for $\phi \in C_c(D(A))$:

$$
P_t \phi - Q_t \phi = P_{t-\frac{k}{n}} \phi - P_{t-\frac{k}{n}} Q_{t-\frac{k}{n}} \phi + P_{t-\frac{k}{n}} Q_{t-\frac{k}{n}} \phi - Q_{t-\frac{k}{n}} Q_{t-\frac{k}{n}} \phi
$$

and so on iteratively until we have

$$
P_t \phi - Q_t \phi = \sum_{k=1}^{n} P_{t-\frac{k}{n}} \left( P_{\frac{k}{n}} \psi_{(k-1)n} - Q_{\frac{k}{n}} \psi_{(k-1)n} \right) \psi_{(k-1)n}
$$

where $\psi_s = Q_t \phi$. We have, by using (5.3) and Proposition [11]

$$
\left| P_{t-\frac{k}{n}} \left( P_{\frac{k}{n}} \psi_{(k-1)n} - Q_{\frac{k}{n}} \psi_{(k-1)n} \right) \psi_{(k-1)n} \right|
$$

$$
= \left| \mathbb{E} P_{\frac{k}{n}} \left( P_{\frac{k}{n}} \psi_{(k-1)n} - Q_{\frac{k}{n}} \psi_{(k-1)n} \right) \psi_{(k-1)n} \right|
$$

$$
\leq \mathbb{E} P_{\frac{k}{n}} \left[ \left| P_{\frac{k}{n}} \psi_{(k-1)n} - Q_{\frac{k}{n}} \psi_{(k-1)n} \right| \psi_{(k-1)n} \right] \mathbb{1} \{ \xi_{t-\frac{k}{n}} \in A \}
$$

$$
+ \mathbb{E} P_{\frac{k}{n}} \left[ \left| Q_{\frac{k}{n}} \psi_{(k-1)n} - Q_{\frac{k}{n}} \psi_{(k-1)n} \right| \psi_{(k-1)n} \right] \mathbb{1} \{ \xi_{t-\frac{k}{n}} \in \bar{A} \}
$$

$$
\leq 4C \mathbb{E} e^{-\frac{\eta}{2} t} + 2P \left( \xi_{t-\frac{k}{n}} \in \bar{A} \right)
$$

$$
\leq 4C \mathbb{E} e^{-\frac{\eta}{2} t} + 2P \left( \xi_{t-\frac{k}{n}}, B_A(0, \sqrt{\frac{n}{t}}) \right),
$$

with any other Markov selection.
where \( A_t = \{ 5C^n t(1 + |Ax|)^2 \leq 1 \} \) and, roughly, \( A_t \approx B_A(0, \sqrt{n}) \). Hence

\[
|P\phi(x) - Q\phi(x)| \leq 4nC\#e^{-\eta_#}n + 2 \sum_{k=1}^{n} P(t-k/n, x, B_A(0, \sqrt{n}t))
\]

which completes the proof of the proposition. \( \square \)

The criterion of this proposition is apparently not really useful at the present stage of our understanding. Indeed, if we apply Chebichev inequality we get the sufficient condition

\[
\lim_{n \to \infty} \sum_{k=1}^{n} (\frac{t}{n})^{1+\varepsilon} \mathbb{E}^P_x \left[ \left| \tilde{A}_t - \tilde{B}_t \right|^{2(1+\varepsilon)} \right] = 0
\]

with is implied by the condition

\[
\mathbb{E}^P_x \left[ \int_0^t |\tilde{A}_s|^{2(1+\varepsilon)} \, ds \right] < \infty
\]

which however would easily imply the well posedness of the 3D Navier-Stokes equation by direct estimates of the difference of two solutions.

**APPENDIX**

**A.1. A exponential tail estimate for the Stokes problem.** Consider the following Stokes problem

\[
dZ + AZ \, dt = A^{-\frac{3}{2}} dW, \quad Z(0) = 0,
\]

and set \( \Theta_t = \sup_{s \in [0,t]} |AZ(s)| \). The next result is well known, but we give a proof to keep track of the dependence on the constants of interest in this paper.

**Proposition 15.** There exist \( \eta_# > 0 \) and \( C_# > 0 \) such that for every \( K \geq \frac{1}{2} \) and \( \varepsilon > 0 \),

\[
\mathbb{P}[\Theta_t \geq K] \leq C_# e^{-\eta_# K^2 / t}.
\]

**Proof.** Step 1. Set \( y(t) = e^{-\frac{t}{2}} Z(\varepsilon t) \), then it is easy to see that \( y \) solves the equation \( dy + \varepsilon Ay \, dt = Q^{\frac{1}{2}} dW \). Next, fix a value \( \alpha \in (\frac{1}{6}, \frac{1}{4}) \), then by the factorisation method (see Da Prato & Zabczyk, Chapter 5),

\[
y(t) = \int_0^t e^{-(t-s)A} \, dW_s = C_\alpha \int_0^t e^{-(t-s)A} (t-s)^{\alpha-1} Y(s) \, ds,
\]

where \( Y(s) = \int_s^t e^{-(s-r)A} (s-r)^{-\alpha} \, dW_r \) and \( C_\alpha \) denotes a generic constant depending only on \( \alpha \) (it will keep changing value along the proof). For every \( t \in (0, 1] \), since \( \alpha > \frac{1}{6} \), it follows from Hölder’s inequality that

\[
|Ay(t)|_H \leq C_\alpha \int_0^t (t-s)^{\alpha-1} |AY(s)|_H \, ds \leq C_\alpha \left( \int_0^1 |AY(s)|^6_H \, ds \right)^{\frac{1}{6}}.
\]
In conclusion, since $\varepsilon^{-1} \Theta^2 = \sup_{t \in [0,1]} |A_y(t)|^2_{H}$, it follows by the above inequality and standard arguments that

\[(A.1) \quad \mathbb{P}[\Theta \geq K] \leq e^{-\frac{\alpha K^2}{2}} E\left[\exp(\bar{a}\left(\int_0^1 |A_y(s)|^6 ds\right)^{\frac{1}{2}})\right],\]

with a constant $\bar{a}$ that will be specified later (and $\bar{a} = a C_\alpha$).

**Step 2.** In order to estimate the expectation in (A.1), notice that

\[
\exp(\bar{a}\left[\int_0^1 |A_y(s)|^6 ds\right]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} \frac{\bar{a}^n}{n!}\left[\int_0^1 |A_y(s)|^6 ds\right]^{\frac{n}{2}} \leq \\
\sum_{n=0}^{\infty} \frac{\bar{a}^n}{n!}\left[\int_0^1 |A_y(s)|^6 ds\right]^{\frac{n}{2}} + \sum_{n=0}^{\infty} \frac{\bar{a}^n}{n!}\left[\int_0^1 |A_y(s)|^8 ds\right]^{\frac{n}{2}} + \int_0^1 e^{\bar{a}|A_y(s)|^2_{H}} ds
\]

**Step 3.** Now, $A_y(s)$ is a centered Gaussian process with covariance (cfr. proof of Theorem 5.9 in Da Prato & Zabczyk [4])

$$\hat{Q}_s = \int_0^r (s-r)^{-2\alpha} A^{-1} e^{-2\varepsilon(s-r)A} dr,$$

so that, by Proposition 2.16 of [4],

$$\mathbb{E}[e^{\bar{a}|A_y(s)|^2_{H}}] = e^{-\frac{1}{2}\text{Tr}[\log(1-2\bar{a}\hat{Q}_s)]},$$

provided that $\bar{a} \leq \inf_{\lambda \in \sigma(\hat{Q}_s)} \frac{1}{2\lambda}$, where $\sigma(\hat{Q}_s)$ is the spectrum of $\hat{Q}_s$. Similarly, $\mathbb{E}|A_y(s)|^{2p} = C_p(\text{Tr}(\hat{Q}_s))^p$, for all integers $p$.

In order to choose a suitable value of $a$, let $\mu \in \sigma(\hat{Q}_s)$, then there is an eigenvalue $\lambda$ of $A$ such that $\mu = \lambda(\lambda)$ is given by

$$\mu = \lambda^{2-2\alpha} r^{-2\alpha} e^{-2\varepsilon} dr = \lambda^{2-2\alpha} \int_0^{2\lambda \varepsilon} r^{-2\alpha} e^{-r} dr \leq C_\alpha \lambda^{-1},$$

where $\lambda_0$ is the smallest eigenvalue of $A$. Hence $a$ can be chosen as $C_\alpha \lambda_0$, for a suitable $C_\alpha$.

**Step 4.** We conclude the proof: we have that $-\text{Tr}[\log(1-2\bar{a}\hat{Q}_s)] \leq C_\alpha \text{Tr}[\hat{Q}_s]$ since $a$ is small enough, and, as in step 3,

$$\text{Tr}[Q_s] = \lambda \int_0^{2\lambda \varepsilon} r^{-2\alpha} e^{-r} dr \leq C_\alpha e^{(2-2\alpha)},$$

where the sum in $\lambda$ converges since $\alpha < \frac{1}{2}$ and $\sigma_\alpha \approx n^{2\alpha}$. Hence, by (A.1) and (A.2),

$$\mathbb{P}[\Theta \geq K] \leq e^{-\frac{\alpha K^2}{2}} E\left[\bar{a}\left(\int_0^1 |A_y(s)|^{10} ds\right)^{\frac{1}{2}} + \frac{\bar{a}^2}{2}\left[\int_0^1 |A_y(s)|^8 ds\right]^{\frac{1}{2}} + \int_0^1 e^{\bar{a}|A_y(s)|^2_{H}} ds\right] \leq C_\alpha e^{-\frac{\alpha K^2}{2}} (e^{C_\alpha e^{(1-2\alpha)}} + e^{(1-2\alpha)} + e^{-2(1-2\alpha)}) \leq C_\alpha e^{-\frac{\alpha K^2}{2}},$$
where \( \eta \# \) and \( C \# \) can be easily found, since \( K \geq \frac{1}{T} \).

A.2. The deterministic equation. The basic ingredient of our approach is the bunch of regular paths that every weak solution has for a positive local (random) time, when the initial condition is regular. It was called regular jet in Flandoli [8]. It is based on the solutions of the following deterministic equation

\[
(A.3) \quad u(t) + \int_0^t (Au(s) + B(u,u)) \, ds = x + w(t). 
\]

We say that \( u \in C([0, \infty); H_\sigma) \cap L^2_{loc}([0, \infty); V) \) is a weak solution of (A.3) if

\[
\langle u(t), \varphi \rangle + \int_0^t \left( \langle u(s), A\varphi \rangle - \langle B(u(s), \varphi), u(s) \rangle \right) \, ds = \langle x, \varphi \rangle + \langle w(t), \varphi \rangle 
\]

for every \( \varphi \in D^\infty \). Notice that all terms in the above definition are meaningful, included the quadratic one in \( u \) due to the estimate

\[
|\langle B(u,v), z \rangle| \leq C |Dv|_{L^\infty} |u|_{L^2} |z|_{L^2}.
\]

We take \( w \in \Omega^* \) where

\[
\Omega^* = \bigcap_{\beta \in (0, \frac{1}{2})} C^0([0, \infty); D(A^\beta)).
\]

Consider also the auxiliary Stokes equations

\[
z(t) + \int_0^t A z(s) \, ds = w(t)
\]

having the unique mild solution

\[
z(t) = e^{-tA} w(t) - \int_0^t A e^{-(t-s)A} (w(s) - w(t)) \, ds.
\]

From elementary arguments based on the analytic estimates \(|A^\alpha e^{-tA}| \leq \frac{Ca^T}{T^\alpha} \) for \( t \in (0, T) \), we have (see for instance Flandoli [7] for details)

\[
z \in C([0, \infty); D(A)).
\]

Let us set

\[
(A.4) \quad \Theta_T = \sup_{t \in [0,T]} |Az(t)|.
\]

Let \( C_0 > 0 \) be the constant of inequality (2.1) and let

\[
(A.5) \quad C^* := 4C_0^2.
\]
Lemma 16. Given $x \in D(A)$ and $w \in \Omega^*$, let $K \geq |Ax|$ and $\varepsilon > 0$ be such that

$$(K^2 + \theta_\varepsilon^2)(\frac{1}{2K^2} + C^*\varepsilon) < 1$$

Then there exists a solution $u \in C([0, \varepsilon]; D(A))$, which is unique in the class of weak solutions, and $|Au(s)| < 2K$ for $s \in [0, \varepsilon]$.

Proof. We show only the quantitative estimate, the other statements being standard in the theory of Navier-Stokes equations. For simplicity, all computations will be made on the limit problem, although they should be made on its Galerkin approximations. The uniqueness of local solution ensures that the procedure is nevertheless correct.

Set $v = u - z$, then

$$\frac{dv}{dt} + Av + B(u, u) = 0$$

and, by using (2.1),

$$\frac{d}{dt} |Av|^2 + 2 \|Av\|^2 \leq 2 |\langle Av, AB(u, u) \rangle| \leq 2 \|Av\|_V \left| A^{1/2} B(u, u) \right|$$

$$\leq 2C_0 \|Av\|_V \|Au\|^2 \leq \|Av\|^2 + C_0^2 \|Au\|^4$$

$$\leq \|Av\|^2 + C^* (|Av|^2 + |Az|^2)^2.$$ 

Hence on $[0, \varepsilon]$ we have that

$$\frac{d}{dt} |Av|^2 \leq C^* (|Av|^2 + \theta_\varepsilon^2)^2,$$

and so, if we set $y(t) = |Av(t)|^2 + \theta_\varepsilon^2$, it follows that

$$\frac{dy}{dt} \leq C^* y^2, \quad \text{on } [0, \varepsilon].$$

Consequently, since $y > 0$ (except for the irrelevant case $w \equiv 0$), we have

$$y(s) \leq \frac{y(0)}{1 - C^* s y(0)},$$

namely,

$$|A(u(s) - z(s))|^2 + \theta_\varepsilon^2 \leq \frac{|Ax|^2 + \theta_\varepsilon^2}{1 - C^* s \left( |Ax|^2 + \theta_\varepsilon^2 \right)}$$

for $s \in [0, \varepsilon]$. Therefore

$$|Au(s)|^2 \leq \frac{2(|Ax|^2 + \theta_\varepsilon^2)}{1 - C^* s \left( |Ax|^2 + \theta_\varepsilon^2 \right)} \leq \frac{2(K^2 + \theta_\varepsilon^2)}{1 - C^* s (K^2 + \theta_\varepsilon^2)}.$$

This result is true until $1 - C^* s (K^2 + \theta_\varepsilon^2) > 0$, namely for $s \in \left[0, \frac{1}{C^* (K^2 + \theta_\varepsilon^2)}\right]$. The assumption of the lemma ensures that $[0, \varepsilon]$ is included in this interval.
Thus the last inequality is true at least on $[0, \varepsilon]$. Moreover, again by the assumption of the lemma,

$$\frac{2(K^2 + \theta_\varepsilon^2)}{4K^2} < 1 - C^*s(K^2 + \theta_\varepsilon^2)$$

that implies

$$\frac{2(K^2 + \theta_\varepsilon^2)}{1 - C^*s(K^2 + \theta_\varepsilon^2)} < 4K^2,$$

and thus $|Au(s)|^2 < 4K^2$, for $s \in [0, \varepsilon]$.

**Corollary 17.** Assume there are $K > 0$ and $\varepsilon > 0$ such that

$$\varepsilon \leq \frac{1}{5C^*K^2} \quad \text{and} \quad \theta_\varepsilon^2 \leq \frac{1}{4}K^2,$$

then, for every $x \in D(A)$ such that $|Ax| \leq K$, we have $|Au(s)| < 2K$ for $s \in [0, \varepsilon]$.

**References**

1. A. Bressan, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, Oxford Lecture Series in Mathematics and its Applications, 20. Oxford University Press, Oxford, 2000.
2. G. Da Prato, *Kolmogorov equations for stochastic PDEs*, Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2004.
3. G. Da Prato, A. Debussche, *Ergodicity for the 3D stochastic Navier-Stokes equations*, J. Math. Pures Appl. (9) 82 (2003), no. 8, 877–947.
4. G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.
5. G. Da Prato, J. Zabczyk, *Ergodicity for infinite-dimensional systems*. London Mathematical Society Lecture Note Series, 229. Cambridge University Press, Cambridge, 1996.
6. A. Debussche, C. Odasso, *Markov solutions for the 3d stochastic Navier-Stokes equations with state dependent noise*, available on the arXiv preprint archive at the web address [http://www.arxiv.org/abs/math.AP/0512361](http://www.arxiv.org/abs/math.AP/0512361).
7. F. Flandoli, *Stochastic differential equations in fluid dynamics*, Rend. Sem. Mat. Fis. Milano 66 (1996), 121–148.
8. F. Flandoli, *On the method of Da Prato and Debussche for the 3D stochastic Navier Stokes equations*, 2006, to appear on J. Evol. Equ.
9. F. Flandoli, *An introduction to 3D stochastic fluid dynamics*, to appear on the proceedings of the CIME course on SPDE in hydrodynamics: recent progress and prospects. Lecture Notes in Mathematics, Springer. Available on the web page of CIME at the address [http://www.cime.unifi.it](http://www.cime.unifi.it).
10. F. Flandoli, M. Romito, *Markov selections and their regularity for the three-dimensional stochastic Navier-Stokes equations*, C. R. Math. Acad. Sci. Paris, Ser. I 343 (2006), 47–50.
11. F. Flandoli, M. Romito, *Markov selections for the three dimensional stochastic Navier Stokes equations*, available on the arXiv preprint archive at the web address [http://www.arxiv.org/abs/math.PR/0602612](http://www.arxiv.org/abs/math.PR/0602612).
12. J. L. Menaldi, S. S. Srinharan, *Stochastic 2-D Navier-Stokes equation*, Appl. Math. Optimiz. 46 (2002), 31-53.
13. M. Metivier, *Stochastic partial differential equations in infinite dimensional spaces*, Quaderni della Scuola Normale Superiore, Pisa, 1988.
14. R. Mikulevicius, B. L. Rozovsky, *Stochastic Navier-Stokes equations for turbulent flows*, SIAM J. Math. Anal. 35 (2004), no. 5, 1250–1310.
15. R. Mikulevicius, B. L. Rozovsky, *Global $L^2$-solutions of stochastic Navier-Stokes equations*, Annals of Probab. 33 (2005), no. 1, 137–176.
16. M. Röckner, Z. Sobol, *Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations*, to appear on The Annals of Probab.
17. M. Röckner, Z. Sobol, *A new approach to Kolmogorov equations in infinite dimensions and applications to stochastic generalized Burgers equations*, C. R. Math. Acad. Sci. Paris 338 (2004), no. 12, 945–949.
18. B. Schmalfuss, *Qualitative properties of the stochastic Navier-Stokes equation*, Nonlinear Analysis, TMA 28 (1997), 1545-1563.
19. S. S. Sritharan, *Deterministic and stochastic control of Navier-Stokes equations with linear, monotone and hyper viscosity*, Appl. Math. Optimiz. 41 (2000), 255-308.
20. D. W. Stroock, S. R. S. Varadhan, *Multidimensional diffusion processes*, Springer-Verlag, Berlin 1979.
21. R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, Second edition, CBMS-NSF Regional Conference Series in Applied Mathematics, 66. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.