GEOMETRIC MEAN OF STATES AND TRANSITION AMPLITUDES

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ABSTRACT. The transition amplitude between square roots of states, which is an analogue of Hellinger integral in classical measure theory, is investigated in connection with operator-algebraic representation theory. A variational expression based on geometric mean of positive forms is utilized to obtain an approximation formula for transition amplitudes.

INTRODUCTION

Geometric mean of positive operators is introduced by Pusz and Woronowicz in terms of associated positive sesquilinear forms, which is later generalized in various directions (see [3, 6, 7] for example). When this is applied to normalized positive functionals on a *-algebra (so-called states), it leads us to an approach to the theory of positive cones in the modular theory of W*-algebras, which turns out to be closely related to A. Uhlmann’s transition probability (or fidelity) between states (see [1, 15, 11]).

We here first clarify how inner products between square roots of states (which is referred to as “transition amplitude” in this paper just from its superficial appearance but without any physically serious justification) is relevant in representation theory of C*-algebras, which is then combined with Pusz-Woronowicz’ geometric mean to get a variational expression for our transition amplitudes. The result is especially useful in establishing an approximation formula, which says that, if states \( \varphi \) and \( \psi \) of a C*-algebra \( A \) are restricted to an increasing sequence of C*-subalgebras \( A_n \) with the induced states of \( A_n \) denoted by \( \varphi_n \) and \( \psi_n \) respectively, then we have

\[
\langle \varphi^{1/2} | \psi^{1/2} \rangle = \lim_{n \to \infty} \langle \varphi_n^{1/2} | \psi_n^{1/2} \rangle
\]

under a mild assumption on the density of \( \bigcup_n A_n \) in \( A \).

A decomposition theory of transition amplitudes is also described in the framework of W*-algebras for further applications.

1. GEOMETRIC MEAN OF POSITIVE FORMS

We begin with reviewing Pusz-Woronowicz’ geometric mean on positive forms in a slightly modified fashion from the original account.

Let \( \alpha, \beta \) be positive (sesquilinear) forms on a complex vector space \( H \). By a \textbf{representation} of an unordered pair \( \{ \alpha, \beta \} \), we shall mean a linear map \( j : H \to K \) of \( H \) into a Hilbert space \( K \) together with (possibly unbounded) positive self-adjoint
operators $A$, $B$ on $K$ such that $A$ commutes with $B$ in the strong sense, $j(H)$ is a core for the self-adjoint operator $A + B$ and

$$\alpha(x, y) = \langle j(x) | Aj(y) \rangle, \quad \beta(x, y) = \langle j(x) | Bj(y) \rangle$$

for $x, y \in H$. Note that, from the core condition, $j(H)$ is included in the domains of $A = \frac{A}{A + B + I}$, $B = \frac{B}{A + B + I}$ ($I$ being the identity operator) and therefore in the domains of $A^{1/2}$ and $B^{1/2}$. When $A$ and $B$ are bounded, we say that the representation is bounded. Note that, the core condition is reduced to the density of $j(H)$ in $K$ for a bounded representation.

A hermitian form $\gamma$ on $H$ is said to be dominated by $\{\alpha, \beta\}$ if $|\gamma(x, y)|^2 \leq \alpha(x, x) \beta(y, y)$ for $x, y \in H$. Note that the order of $\alpha$ and $\beta$ is irrelevant in the domination.

**Theorem 1.1** (Pusz-Woronowicz). Let $(j : H \to K, A, B)$ be a representation of positive forms $\alpha, \beta$ on $H$. Then, for $x \in H$, we have the following variational expression.

$$(A^{1/2}j(x)|B^{1/2}j(x)) = \sup \{\gamma(x, x); \gamma \text{ is positive and dominated by } \{\alpha, \beta\}\}$$

$$= \sup \{\gamma(x, x); \gamma \text{ is dominated by } \{\alpha, \beta\}\}.$$

The positive form defined by the right hand side of the theorem is called the geometric mean of $\{\alpha, \beta\}$ and denoted by $\sqrt{\alpha\beta} = \sqrt{\alpha \beta}$.

**2. $L^2$-Analysis on Quasi-equivalence of States**

Associated to a $W^*$-algebra $M$, we have the standard Hilbert space $L^2(M)$ so that its positive cone consists of symbols $\varphi^{1/2}$ where $\varphi$ varies in the set $M_+^+$ of normal positive linear functionals of $M$. On the Hilbert space $L^2(M)$, $M$ is represented by compatible left and right actions in such a way that

$$(\varphi^{1/2}|x\varphi^{1/2}) = \varphi(x) = (\varphi^{1/2}|\varphi^{1/2}x) \quad \text{for } x \in M$$

(inner products being linear in the second variable by our convention). This type of vectors are known to satisfy the following inequalities (Powers-Størmer-Araki)

$$\|\varphi^{1/2} - \psi^{1/2}\| \leq \|\varphi - \psi\| \leq \|\varphi^{1/2} - \psi^{1/2}\| \|\varphi^{1/2} + \psi^{1/2}\|.$$

Note that, given a central projection $q$ in $M$, we have the following natural identifications for the reduced $W^*$-algebra $qM = Mq$:

$$(qM)_* = qM_* = M_*q, \quad L^2(qM) = qL^2(M) = L^2(M)q.$$

Also note that there is a natural bilinear map $L^2(N) \times L^2(N) \to N_* = L^1(N)$ such that $\varphi^{1/2} \times \varphi^{1/2}$ is mapped to $\varphi$. The evaluation map $N_* \ni \varphi \mapsto \varphi(1) \in \mathbb{C}$ is also denoted by $\langle \varphi \rangle = \varphi(1)$ in this paper, which satisfies trace property $\langle \varphi^{1/2}\psi^{1/2} \rangle = \langle \psi^{1/2}\varphi^{1/2} \rangle$.

If $\varphi$ is faithful, we denote by $\Delta_\varphi$ and $J_\varphi$ the associated modular operator and modular conjugation respectively. The positive (self-adjoint) operator $\Delta_\varphi^{1/2}$ has a linear subspace $M\varphi^{1/2}$ as a core and we see

$$\Delta_\varphi^{1/2}(x\varphi^{1/2}) = \varphi^{1/2}x \quad \text{and} \quad J_\varphi(x\varphi^{1/2}) = \varphi^{1/2}x^*.$$

More generally, if $\psi$ is another positive normal functional of $M$, then the half-powered relative modular operator $\Delta_{\psi, \varphi}^{1/2}$ contains $M\varphi^{1/2}$ as a core and we have
\(\Delta_{\psi,\varphi}(x\varphi^{1/2}) = \psi^{1/2}x\) for \(x \in M\). Consult [13, 14] for systematic accounts on all these operations other than the standard texts on modular theory such as [11, 13, 14].

Lemma 2.1. For a positive normal functional \(\omega\) of a \(W^*\)-algebra, let \(e\) and \(z\) be its support and central support respectively. Then we have the equalities

\[ L^2(zM) = zL^2(M), \quad eL^2(M) = \omega^{1/2}z, \quad L^2(M)e = \omega^{1/2}eL^2(M) = eL^2(eMe). \]

Here bar denotes the closure in \(L^2(M)\).

Furthermore, \(\overline{M\omega^{1/2}} = \omega^{1/2}M\) if and only if \(\omega\) is faithful on \(zM = Mz\).

Proof. For \(x \in M\),

\[(\omega^{1/2}x|\omega^{1/2}e) = \omega(ex^*) = \omega(x^*) = (\omega^{1/2}x|\omega^{1/2})\]

shows that \(\omega^{1/2}(1 - e) = 0; \overline{M\omega^{1/2}} \subset L^2(M)e\).

Let \(\pi\) be a normal representation of \(M\) on \(\overline{M\omega^{1/2}}\) given by left multiplication. Since the projection to the subspace \(\overline{M\omega^{1/2}}\) commutes with the left action of \(M\), we can find a projection \(p\) in \(M\) such that \(\overline{M\omega^{1/2}} = L^2(M)p\). Particularly we have \(\omega(1 - p) = \omega^{1/2}\omega^{1/2}(1 - p) = 0\) and therefore \(e \leq p\).

Let \(Q\) be the projection to the subspace \(\overline{M\omega^{1/2}} \subset L^2(M)\). Then \(Q\) is realized by multiplication of a central projection \(q\) of \(M\). From \((1 - q)\omega = (1 - q)\omega^{1/2}\omega^{1/2} = 0\), we see that \(z \leq q; L^2(M)z \subset \overline{M\omega^{1/2}}\). On the other hand, \(x\omega^{1/2}yz = x\omega^{1/2}z\omega^{1/2} = x\omega^{1/2}y\) shows the reverse inclusion.

Assume that \(\overline{M\omega^{1/2}} = \omega^{1/2}M\). If \(x \in zM\) satisfies \(\omega(x^*x) = 0\), i.e., \(x\omega^{1/2} = 0\), then

\[zL^2(zM) = xM\omega^{1/2}M = x\omega^{1/2}MM = 0\]

and hence \(x = 0\). Conversely, if \(\omega\) is faithful on \(zM\), the associated vector \(\omega^{1/2}\) is cyclic and separating for \(z\); \(M\omega^{1/2} = L^2(zM) = \omega^{1/2}M\).

Let \(\omega_e\) be the restriction of \(\omega\) to \(eMe\), which is faithful. Since \(e\) commutes with \(\omega\), the relation \(a\omega^{1/2} = \omega^{1/2}a\) with \(a, b \in eMe\) implies \(a\omega^{1/2} = \omega^{1/2}b\) by the reduction relation for modular operators (a consequence of Connes’ 2 \(\times 2\) matrix analysis and more results can be found in [9]), which gives the unitarity of

\[L^2(eMe) \ni x\omega^{1/2}y \mapsto x\omega^{1/2}y \in eL^2(M)e.\]

Remark 1. The support projection \(e\) is characterized as the minimal one among projections \(p\) in \(M\) satisfying \(\overline{M\omega^{1/2}} = L^2(M)p\).

Corollary 2.2. Let \(\varphi\) and \(\psi\) be states of a \(C^*\)-algebra \(A\).

(i) \(\varphi\) and \(\psi\) are disjoint if and only if \(A\varphi^{1/2}A\) and \(A\psi^{1/2}A\) are orthogonal.

(ii) \(\varphi\) and \(\psi\) are quasi-equivalent if and only if \(\overline{A\varphi^{1/2}A} = \overline{A\psi^{1/2}A}\).

(iii) The state \(\varphi\) is pure if and only if \(\overline{A\varphi^{1/2}} \cap \overline{\psi^{1/2}A} = C\varphi^{1/2}\).

Proof. Given a state \(\varphi\) of a \(C^*\)-algebra \(A\), let \(z(\varphi)\) be the central support of \(\varphi\) in the universal envelope \(A^{**}\). Then it is well-known (see [5, Chapter 3] for example) that \(\varphi\) and \(\psi\) are disjoint (resp. quasi-equivalent) if and only if \(z(\varphi)z(\psi) = 0\).
(resp. \( z(\varphi) = z(\psi) \)). Since \( A\varphi^{1/2}A = A^{**}\varphi^{1/2}A^{**} \) in \( L^2(A^{**}) \), (i) and (ii) are consequences of the lemma.

Let \( e \) be the support of \( \varphi \) in \( A^{**} \). Then the identity
\[
A\varphi^{1/2} \cap \varphi^{1/2}A = L^2(A^{**})e \cap eL^2(A^{**}) = L^2(eA^{**}e)
\]
shows that the condition in (iii) is equivalent to \( eA^{**}e = \mathbb{C}e \), i.e., the purity of \( \varphi \).

Let \( \omega \) be a state of a C*-algebra \( A \) and \( \{ \tau_t \in \text{Aut}(A) \}_{t \in \mathbb{R}} \) be a one-parameter group of *-isomorphisms. Recall that \( \omega \) and \( \{ \tau_t \} \) satisfy the KMS-condition if the following requirements are satisfied: Given \( x, y \in A \), the function \( \mathbb{R} \ni t \mapsto \omega(x\tau_t(y)) \) is analytically extended to a continuous function on the strip \( \{ \zeta \in \mathbb{C}; -1 \leq \Im \zeta \leq 0 \} \) so that \( \omega(x\tau_t(y))|_{t=-i} = \omega(xy) \).

If one replaces \( y \) with \( \tau_s(y) \) and \( x \) with \( 1 \), then the condition takes the form \( \omega(\tau_s^{-1}(y)) = \omega(\tau_s(y)) \) for \( s \in \mathbb{R} \) and we see that the analytic function \( \omega(\tau_t(y)) \) is periodically extended to an entire analytic function. Thus \( \omega(\tau_t(y)) \) is a constant function of \( t \); the automorphisms \( \tau_t \) make \( \omega \) invariant.

**Lemma 2.3.** If \( \omega \) satisfies the KMS-condition, then \( A\varphi^{1/2} = \varphi^{1/2}A \).

**Proof.** We argue as in [5]: By the invariance of \( \omega \), a unitary operator \( u(t) \) in \( A\varphi^{1/2} \) is defined by \( u(t)(x\varphi^{1/2}) = \tau_t(x)\varphi^{1/2} \), which is continuous in \( t \) from the continuity assumption on the function \( \omega(x\tau_t(y)) \). Moreover, the function \( \mathbb{R} \ni t \mapsto u(t)x\varphi^{1/2} \) is analytically continued to the strip \( \{ -1 \leq \Im \zeta \leq 0 \} \). By Kaplansky’s density theorem and analyticity preservation for local uniform convergence, the same property holds for \( x \in A^{**} \) and the KMS-condition takes the form
\[
(x\varphi^{1/2}|u(t)y\varphi^{1/2})|_{t=-i} = (\varphi^{1/2}x|\varphi^{1/2}y) \quad \text{for } x, y \in A^{**}.
\]

Let \( z \) be the central support of \( \omega \) in \( A^{**} \) and assume that \( a \in zA^{**} \) satisfies \( a\varphi^{1/2} = 0 \). Then, \( xa\varphi^{1/2} = 0 \) for \( x \in A^{**} \) and therefore \( (\varphi^{1/2}(xa)|\varphi^{1/2}y) = 0 \) for any \( y \in A^{**} \) by analytic continuation, whence \( \varphi^{1/2}xa = 0 \) for \( x \in A^{**} \). Thus \( zL^2(A^{**})a = 0 \) and we have \( a = 0 \).

As a simple application of our analysis, we record here a formula which describes transition amplitude between purified states. First recall the notion of purification on states introduced by S.L. Woronowicz ([17]): Given a state \( \varphi \) of a C*-algebra \( A \), its purification \( \Phi \) is a state on \( A \otimes A^\circ \) defined by
\[
\Phi(a \otimes b^\circ) = (\varphi^{1/2}a\varphi^{1/2}b).
\]
Here \( A^\circ \) denotes the opposite algebra of \( A \) with \( a \mapsto a^\circ \) denoting the natural antimultiplicative isomorphism.

From the above definition, \( (a \otimes b^\circ)\Phi^{1/2} = a\varphi^{1/2}b \) gives rise to a unitary isomorphism \( (A \otimes A^\circ)\Phi^{1/2} \cong A\varphi^{1/2}A \) and the GNS-representation of \( A \otimes A^\circ \) with respect to \( \Phi \) generates the von Neumann algebra \( M \vee M' \) with \( M = A^{**}z(\varphi) \) represented on \( A\varphi^{1/2}A \) by left multiplication. Thus \( \varphi \) is a factor state if and only if \( \Phi \) is a pure state. Moreover, two factor states \( \varphi \) and \( \psi \) of \( A \) are quasi-equivalent if and only if their purifications are equivalent.

**Proposition 2.4.** Let \( \varphi \) and \( \psi \) be factor states of a C*-algebra \( A \) with their purifications denoted by \( \Phi \) and \( \Psi \) respectively. Then we have
\[
(\Phi^{1/2}|\Psi^{1/2}) = (\varphi^{1/2}|\psi^{1/2})^2.
\]
Proof. In view of the equalities
\[(\varphi^{1/2}|\psi^{1/2}) = 0 = (\Phi^{1/2}|\Psi^{1/2}).\]
for disjoint \(\varphi\) and \(\psi\), we need to consider the case that \(\varphi\) and \(\psi\) are quasi-equivalent, i.e., \(z(\varphi) = z(\psi)\). Since \(\varphi\) and \(\psi\) are assumed to be factor states, their purifications \(\Phi\) and \(\Psi\) are pure with the associated GNS-representations of \(A \otimes A^0\) generate the full operator algebra \(L(L^2(M))\). Thus, through the obvious identification \(L^2(L^2(M))) = L^2(M) \otimes L^2(M)\), \(\Phi^{1/2}\) and \(\Psi^{1/2}\) correspond to \(\varphi^{1/2} \otimes \psi^{1/2}\) and \(\psi^{1/2} \otimes \varphi^{1/2}\) respectively, whence we have
\[(\Phi^{1/2}|\Psi^{1/2}) = (\varphi^{1/2} \otimes \varphi^{1/2}|\psi^{1/2} \otimes \psi^{1/2}) = (\varphi^{1/2}|\psi^{1/2})^2.\]
\[\square\]

Remark 2. For purifications of states on a commutative C*-algebra, we have \((\Phi^{1/2}|\Psi^{1/2}) = (\varphi^{1/2}|\psi^{1/2})^2\). The general case is a mixture of these two formulas.

Lemma 2.5. Let \(\pi : A \to M\) be a homomorphism from a C*-algebra \(A\) into a W*-algebra \(M\) and assume that \(\pi(A)\) is \(*\)-weakly dense in \(M\). Then we have an isometry \(T : L^2(M) \to L^2(A^{**})\) such that \(T(\pi(a)\varphi^{1/2}\pi(b)) = a(\varphi \circ \pi) \varphi^{1/2}b\) if \(\varphi, \pi(a)\varphi^{1/2}\pi(b)

Proof. Since the map \(M_* \ni \varphi \mapsto \varphi \circ \pi \in A^*\) is norm-continuous (in fact it is contractive), we have \(A^{**} \to M\) as its transposed map, which is \(\pi\) when restricted to \(A \subset A^{**}\). In other words, we see that \(\pi\) is extended to a normal homomorphism \(\tilde{\pi} : A^{**} \to M\) of W*-algebras in such a way that, if \(\varphi \circ \pi\) is regarded as a normal functional on \(A^{**}\), it is equal to \(\varphi \circ \tilde{\pi}\).

By our weak*-density assumption, \(\tilde{\pi}\) is surjective and and we can find a central projection \(z \in A^{**}\) so that \(\ker \tilde{\pi} = zA^{**}\) and \((1-z)A^{**} \cong M\) by \(\tilde{\pi}\). From the relation
\[z(\varphi \circ \pi) = z(\varphi \circ \tilde{\pi}) = \tilde{\pi}(z)(\varphi \circ \tilde{\pi}) = 0,\]
one sees that the isomorphism \(M_* \cong zA^{**}\) takes the form \(\varphi \mapsto (1-z)(\varphi \circ \pi) = \varphi \circ \pi\), which yields the formula in question by taking square roots. \[\square\]

Corollary 2.6. Let \(\pi : A \to B\) be a \(*\)-homomorphism between C*-algebras and \(\varphi, \psi\) be positive functionals of \(B\). Assume that, given \(a \in A\) and \(b \in B\), we can find a norm-bounded sequence \(\{a_n\}_{n \geq 1}\) in \(A\) such that
\[\lim_{n \to \infty} \pi(a_n)\pi(a)\varphi^{1/2} = b\pi(a)\varphi^{1/2}, \quad \lim_{n \to \infty} \pi(a_n)\pi(a)\psi^{1/2} = b\pi(a)\psi^{1/2}.\]
Then we have
\[(\varphi^{1/2}|\psi^{1/2}) = ((\varphi \circ \pi)^{1/2} | (\psi \circ \pi)^{1/2}).\]

Proof. Let \(z(\varphi)\) and \(z(\psi)\) be the central projections in \(B^{**}\) specified by
\[B\varphi^{1/2}B = z(\varphi)L^2(B^{**}), \quad B\psi^{1/2}B = z(\psi)L^2(B^{**}).\]
Let \(M = (z(\varphi) \vee z(\psi))B^{**}\) and \(\pi_M : A \to M\) be a homomorphism defined by \(\pi_M(a) = (z(\varphi) \vee z(\psi)) \pi(a)\).

Let \(\rho\) be the direct sum of GNS-representations associated to \(\varphi\) and \(\psi\). Then \(\rho\) is supported by \(z(\varphi) \vee z(\psi)\): \(\rho\) is extended to an isomorphism of \((z(\varphi) \vee z(\psi))B^{**}\) onto \(\rho(B''')\). On the other hand, by the approximation assumption, \(\rho(\pi(A))\) is dense in \(\rho(B)\) with respect to the strong operator topology. Thus \(\pi_M(A)\) is \(*\)-weakly dense in \(M\) and the lemma can be applied if one notices that \((z(\varphi) \vee z(\psi))\varphi = \varphi\) and \((z(\varphi) \vee z(\psi))\psi = \psi\) as identities in the predual of \(B^{**}\). \[\square\]
Example 2.7. Consider quasifree states $\varphi_0$ and $\varphi_T$ of a CCR C*-algebra $C^*(V, \sigma)$. Let $(\ | \ )$ be a positive inner product in $V$ majorizing both of $S + S^*$ and $T + T^*$. For example, one may take $(x|y) = (S + S^* + T + T^*)(x, y)$ as before. Then the presymplectic form $\sigma$ is continuous relative to $(\ | \ )$ and, if we let $V'$ be the associated Hilbert space (i.e., the completion of $V/\ker(\ | \ )$ with respect to $(\ | \ )$, $\sigma$ induces a presymplectic form $\sigma'$ on $V'$. Moreover, $S$ and $T$ also give rise to polarizations $S'$ and $T'$ on the presymplectic vector space $(V', \sigma')$ respectively.

Let $\pi : C^*(V, \sigma) \to C^*(V', \sigma')$ be the *-homomorphism induced from the canonical map $V \to V'$ ($\pi(e^{iv}) = e^{iv'}$ if $v'$ represents the quotient of $v$). Then $\pi$ satisfies the approximation condition with respect to quasifree states associated to $S'$ and $T'$ (see the proof of Proposition 4.3). Since $\varphi_0 = \varphi_{S'} \circ \pi$ and similarly for $T$, we obtain

$$
(\varphi_0^{1/2} | \varphi_T^{1/2}) = (\varphi_{S'}^{1/2} | \varphi_{T'}^{1/2}).
$$

Thus local positions of square roots of quasifree states are described under the assumption that $V$ is complete and $\sigma$ is continuous with respect to a non-degenerate inner product.

3. Transition Amplitude between States

Let $\omega$ be a positive functional of a C*-algebra $A$. According to [10], we introduce two positive sesquilinear forms $\omega_L$ and $\omega_R$ on $A$ defined by

$$
\omega_L(x, y) = \omega(x^*y), \quad \omega_R(x, y) = \omega(yx^*), \quad x, y \in A.
$$

**Lemma 3.1.** Let $M$ be a W*-algebra and let $\varphi, \psi$ be positive normal functionals of $M$. Then

$$
\varphi \psi R(x, y) = (\varphi^{1/2} x^* \psi^{1/2} y) \quad \text{for} \ x, y \in M.
$$

**Proof.** By the positivity $(\varphi^{1/2} x^* \psi^{1/2} x) = (x \varphi^{1/2} x^* \psi^{1/2} y) \geq 0$ and the Schwarz inequality $(1/2) |(\varphi^{1/2} x^* \psi^{1/2} y)|^2 \leq \varphi(x^*x) \psi(y^*y)$, the positive form $(x, y) \mapsto (\varphi^{1/2} x^* \psi^{1/2} y)$ is dominated by $(\varphi_L, \psi_R)$.

Assume for the moment that $\varphi$ and $\psi$ are faithful and consider the embedding $j : M \ni x \mapsto x \varphi^{1/2} \in L^2(M)$. Then $\varphi_L$ is represented by the unitary operator, whereas $\psi(x^*x) = \|\psi^{1/2} x\|^2$ shows that $\psi_R$ is represented by the relative modular operator $\Delta$ with $\Delta^{1/2} (x \varphi^{1/2}) = \psi^{1/2} x$. Recall that $M \varphi^{1/2}$ is a core for $\Delta^{1/2}$. Thus Theorem 1.1 gives

$$
\varphi \psi R(x, y) = (x \varphi^{1/2} | \Delta^{1/2} (y \varphi^{1/2} )) = (x \varphi^{1/2} | \psi^{1/2} y) = (\varphi^{1/2} x^* \psi^{1/2} y).
$$

Now we relax $\varphi$ and $\psi$ to be not necessarily faithful. Let $e$ be the support projection of $\varphi + \psi$. Then it is the support for $\varphi_n = \varphi + 1/n \psi$ and $\psi_n = 1/n \varphi + \psi$ as well. In particular, $\varphi_n$ and $\psi_n$ are faithful on the reduced algebra $e Me$.

Let $\gamma$ be a positive form on $M$ dominated by $(\varphi_n)_L, (\psi_n)_R$. Then $\varphi_n(1 - e) = 0 = \psi_n(1 - e)$ shows that

$$(\gamma(x(1 - e), (1 - e)y)) \leq \varphi_n((1 - e)x^*x(1 - e)) \psi_n((1 - e)y^*y(1 - e)) = 0,$$

i.e., $\gamma(x, y) = \gamma(xe, ey)$ for $x, y \in M$, whence we have

$$
\gamma(x, y) = \gamma(xe, ey) = \gamma(ey, xe) = \gamma(ey, e) = \gamma(ex, e).
$$

Since the restriction $\gamma|_{eMe}$ is dominated by $(\varphi_n|_{eMe})_L$ and $(\psi_n|_{eMe})_R$ with $\varphi_n$ and $\psi_n$ faithful on $eMe$, we have

$$
\gamma(x, y) = \gamma(ex, ex) \leq (e \varphi_n^{1/2} ex^* e \psi_n^{1/2} ex) = (\varphi_n^{1/2} x^* \psi_n^{1/2} x).
$$
Proof. Combining two lemmas just proved, we have
\[ \langle \varphi^{1/2} x^* \psi^{1/2} x \rangle \leq \langle \varphi^{1/2} x^* \psi^{1/2} x \rangle \] in view of the Powers-Størmer inequality.

Remark 3.

(i) The case \( \varphi = \psi \) was dealt with in the proof of [10] Theorem 3.1] under the separability assumption on \( M \).
(ii) In the notation of [16], we have \( QF_t(\varphi_L, \psi_R)(x, y) = \langle \varphi^{1-t} x^* \psi^t y \rangle \) for \( 0 \leq t \leq 1 \) and \( x, y \in M \).

Given a positive functional \( \varphi \) of a C*-algebra \( A \), let \( \overline{\varphi} \) be the associated normal functional on the W*-envelope \( A^{**} \) through the canonical duality pairing.

Lemma 3.2. Let \( \varphi \) and \( \psi \) be positive functionals on a C*-algebra \( A \) with \( \overline{\varphi} \) and \( \psi \) the corresponding normal functionals on \( A^{**} \). Then
\[ \sqrt{\varphi Lv_R(x, y)} = \langle \varphi^{1/2} x^* \psi^{1/2} y \rangle \quad \text{for} \quad x, y \in A \subseteq A^{**}. \]

Proof. The positive form \( A \times A \ni (x, y) \mapsto \langle \varphi^{1/2} x^* \psi^{1/2} y \rangle \) (recall that \( x^* \psi^{1/2} x \) is in the positive cone to see the positivity) is dominated by \( \varphi_L \) and \( \psi_R \) because of

\[ |\langle \varphi^{1/2} x^* \psi^{1/2} y \rangle|^2 \leq \overline{\varphi}(x^* x) \overline{\psi}(yy^*) = \varphi(x^* x) \psi(yy^*). \]

Consequently,
\[ \langle \varphi^{1/2} x^* \psi^{1/2} y \rangle \leq \sqrt{\varphi L \psi R(x, x)} \quad \text{for} \quad x \in A. \]

To get the reverse inequality, let \( \gamma \) be a positive form on \( A \times A \) dominated by \( \varphi_L \) and \( \psi_R \). Then we have the domination inequality
\[ |\gamma(x, y)|^2 \leq \varphi(x^* x) \psi(yy^*) = \| x \overline{\varphi}^{1/2} \| \| \overline{\psi}^{1/2} y \|^2. \]

Since \( A \) is dense in \( A^{**} \) relative to the \( \sigma^* \)-topology, we see that \( \gamma \) is extended to a positive form \( \overline{\gamma} \) on \( A^{**} \times A^{**} \) so that
\[ |\overline{\gamma}(x, y)|^2 \leq \| x \overline{\varphi}^{1/2} \| \| \overline{\psi}^{1/2} y \|^2 \quad \text{for} \quad x, y \in A^{**}, \]
whence
\[ \gamma(x, x) = \overline{\gamma}(x, x) \leq \sqrt{\varphi_L \psi_R(x, x)} = \langle \varphi^{1/2} x^* \psi^{1/2} x \rangle \quad \text{for} \quad x \in A. \]

Maximization on \( \gamma \) then yields the inequality
\[ \sqrt{\varphi_L \psi_R(x, x)} \leq \langle \varphi^{1/2} x^* \psi^{1/2} x \rangle \quad \text{for} \quad x \in A \]
and we are done. \( \square \)

Corollary 3.3. Given a normal state \( \varphi \) of a W*-algebra \( M \), let \( \overline{\varphi} \) be the associated normal state of the second dual W*-algebra \( M^{**} \). Then
\[ L^2(M) \ni \varphi^{1/2} \mapsto \overline{\varphi}^{1/2} \in L^2(M^{**}) \]
defines an isometry of \( M \)-\( M \) bimodules.

Proof. Combining two lemmas just proved, we have
\[ \langle \varphi^{1/2} x^* \psi^{1/2} y \rangle = \sqrt{\varphi L \psi R(x, y)} = \langle \overline{\varphi}^{1/2} x^* \psi^{1/2} y \rangle \]
for \( x, y \in M \). \( \square \)
In what follows, $\varphi^{1/2}$ is identified with $\tilde{\varphi}^{1/2}$ via the isometry just established: Given a positive normal functional $\varphi$ of a W*-algebra $M$, $\varphi^{1/2}$ is used to stand for a vector commonly contained in the increasing sequence of Hilbert spaces

$$L^2(M) \subset L^2(M^{**}) \subset L^2(M^{****}) \subset \ldots$$

In accordance with this convention, the formula in the previous lemma is simply expressed by

$$(x\varphi^{1/2}|\psi^{1/2}y) = \sqrt{\varphi_L \psi_R (x, y)} \quad \text{for } x, y \in A.$$ 

Here the left hand side is the inner product in $L^2(A^{**})$, whereas the right hand side is the geometric mean of positive forms on the C*-algebra $A$. Note that, the formula is compatible with the invariance of geometric means: $\sqrt{\varphi_L \psi_R (x, y)} = \sqrt{\psi_L \varphi_R (y^*, x^*)}$.

**Remark 4.**

(i) When $\varphi$ and $\psi$ are vector states of a full operator algebra $L(\mathcal{H})$ associated to normalized vectors $\xi, \eta$ in $\mathcal{H}$, the inner product $(\varphi^{1/2}|\psi^{1/2})$ is reduced to the transition probability $|\langle \xi|\eta \rangle|^2$. Moreover, in view of the inequality $t\varphi^{1/2} + (1-t)\psi^{1/2} \leq (t\varphi + (1-t)\psi)^{1/2}$ for $0 \leq t \leq 1$ (which follows from $(\varphi^{1/2} - \psi^{1/2})^2 \geq 0$), our transition amplitude meets the requirements for transition probability listed in [12].

(ii) Let $P(\varphi, \psi)$ be the transition probability between states in the sense of A. Uhlmann ([15]). Then we have $P(\varphi, \psi) = |\langle \varphi^{1/2}|\psi^{1/2} \rangle|^2$ (cf. [11]) and

$$(\varphi^{1/2}|\psi^{1/2})^2 \leq P(\varphi, \psi) \leq (\varphi^{1/2}|\psi^{1/2}).$$

**4. APPROXIMATION ON TRANSITION AMPLITUDES**

In this section, we shall see how transition amplitudes are approximated by states obtained by restriction to subalgebras.

**Lemma 4.1** (cf. [16] Proposition 17). Let $\Phi : A \to B$ be a unital Schwarz map between unital C*-algebras. Then, for positive linear functionals $\varphi, \psi$ of $B$,

$$(\varphi^{1/2}|\psi^{1/2}) \leq ((\varphi \circ \Phi)^{1/2}|(\psi \circ \Phi)^{1/2}).$$

**Proof.** Let $\gamma : B \times B \to \mathbb{C}$ be a positive form dominated by $\{\varphi_L, \psi_R\}$. Then

$$|\gamma(\Phi(x), \Phi(y))|^2 \leq \varphi(\Phi(x)^* \Phi(x))\psi(\Phi(y)\Phi(y)^*) \leq \varphi(\Phi(x^* x))\psi(\Phi(y y^*))$$

shows that the positive form $A \times A \ni (x, y) \mapsto \gamma(\Phi(x), \Phi(y))$ is dominated by $\{(\varphi \circ \Phi)_L, (\psi \circ \Phi)_R\}$. Thus

$$\gamma(1, 1) = \gamma(\Phi(1), \Phi(1)) \leq \sqrt{(\varphi \circ \Phi)_L(\psi \circ \Phi)_R(1, 1)} = ((\varphi \circ \Phi)^{1/2}|(\psi \circ \Phi)^{1/2}).$$

Maximizing $\gamma(1, 1)$ with respect to $\gamma$, we obtain the inequality. \qed

**Theorem 4.2.** Let $\varphi$ and $\psi$ be positive functionals on a C*-algebra $A$ with unit $1_A$. Let $\{A_n\}_{n \geq 1}$ be an increasing sequence of C*-subalgebras of $A$ containing $1_A$ in common and assume that, given any $a \in A$, we can find a sequence $\{a_n \in A_n\}_{n \geq 1}$ satisfying

$$\lim_{n \to \infty} a_n \varphi^{1/2} = a \varphi^{1/2}, \quad \lim_{n \to \infty} \psi^{1/2} a_n = \psi^{1/2} a$$

in norm topology. Set $\varphi_n = \varphi|_{A_n}, \psi_n = \psi|_{A_n} \in A_n^*$. Then the sequence $\{(\varphi_n^{1/2}|\psi_n^{1/2})\}_{n \geq 1}$ is decreasing and converges to $(\varphi^{1/2}|\psi^{1/2})$. 


Proof. The sequence \( \{(\varphi_n^{1/2}|\psi_n^{1/2})\} \) is decreasing with \((\varphi^{1/2}|\psi^{1/2}) \) a lower bound by the previous lemma.

Let \( e_n \) and \( f_n \) be projections on \( L^2(A^{**}) \) defined by
\[
e_nL^2(A^{**}) = A_n\varphi^{1/2}, \quad f_nL^2(A^{**}) = \psi^{1/2}A_n.
\]
Choose positive forms \( \gamma_n : A_n \times A_n \to \mathbb{C} \) for \( n \geq 1 \) so that \( \gamma_n \) is dominated by \( \{(\varphi_n)_L, (\psi_n)_R\} \) and satisfies
\[
\gamma_n(1,1) \geq (\varphi_n^{1/2}|\psi_n^{1/2}) - 1/n.
\]
From the domination estimate on \( \gamma_n \), we can find a linear map \( C'_n : \psi^{1/2}A_n \to A_n\varphi^{1/2} \) such that
\[
\gamma_n(x,y) = (x\varphi^{1/2}|C'_n(\psi^{1/2}y)) \quad \text{for } x, y \in A_n,
\]
which satisfies \( \|C'_n\| \leq 1 \). Let \( C_n = e_nC'_nf_n : \psi^{1/2}A \to A\varphi^{1/2} \). Since \( \|C_n\| \leq 1 \), we may assume that \( C_n \to C \) in weak operator topology by passing to a subsequence if necessary. Now set
\[
\gamma(x,y) = (x\varphi^{1/2}|C(\psi^{1/2}y)),
\]
which is a sesquilinear form on \( A \) satisfying \( |\gamma(x,y)| \leq \|x\varphi^{1/2}\| \|\psi^{1/2}y\| \). Moreover, if \( x \in A_m \) for some \( m \geq 1 \),
\[
\gamma(x,x) = \lim_{n \to \infty} (x\varphi^{1/2}|C_n(\psi^{1/2}x)) = \lim_{n \to \infty} \gamma_n(x,x) \geq 0
\]
shows that \( \gamma \) is positive on \( \bigcup_{m \geq 1} A_m \) and hence on \( A \) by the approximation assumption. Thus, \( \gamma \) is a positive form dominated by \( \{(\varphi_L, \psi_R)\} \) and we have
\[
(\varphi^{1/2}|\psi^{1/2}) \geq \gamma(1,1) = \lim_{n \to \infty} (\varphi^{1/2}|C_n\psi^{1/2}) = \lim_{n \to \infty} \gamma_n(1,1)
\]
\[
\geq \lim_{n \to \infty} \left((\varphi_n^{1/2}|\psi_n^{1/2}) - \frac{1}{n}\right) = \lim_{n \to \infty} (\varphi_n^{1/2}|\psi_n^{1/2}).
\]

As a concrete example, we have the following situation in mind: Let \((V, \sigma)\) be a real presymplectic vector space and \( C^*(V, \sigma) \) be the associated \( C^* \)-algebra. Let \( \varphi \) and \( \psi \) be quasifree states of \( C^*(V, \sigma) \) associated to covariance forms \( S \) and \( T \) respectively:
\[
\varphi(e^{ix}) = e^{-S(x,x)/2}, \quad \psi(e^{ix}) = e^{-T(x,x)/2} \quad \text{for } x \in V.
\]
Note that \( S \) is a positive form on the complexification \( V \mathbb{C} \) satisfying \( S(x, y) - \overline{S(x, y)} = i\sigma(x, y) \) for \( x, y \in V \) and similarly for \( T \).

Let \( \{V_n\}_{n \geq 1} \) be an increasing sequence of subspaces of \( V \) and assume that \( \bigcup_{n \geq 1} V_n \) is dense in \( V \) with respect to the inner product
\[
V \times V \ni (x, y) \mapsto (x|y) = S(x, y) + \overline{S(x, y)} + T(x, y) + \overline{T(x, y)} \in \mathbb{R}.
\]
Note that \((x|x)\) may vanish on non-zero \( x \in V \). Let \( A_n \) be the \( C^* \)-algebra of \( A = C^*(V, \sigma) \) generated by \( \{e^{ix}; x \in V_n\} \).

**Proposition 4.3.** In the setting described above, the increasing sequence \( \{A_n\}_{n \geq 1} \) satisfies the approximation property with respect to \( \varphi, \psi \).
Proof. For \( x \in V \), choose a sequence \( x_n \in V_n \) so that \( (x_n - x | x_n - x) \to 0 \). Then, for any \( y \in V \),

\[
\| (e^{ix_n} - e^{ix})e^{iy} \psi^{1/2} \|^2 = 2 - 2e^{-S(x_n-x)/2}Re^{i\sigma(x_n-x,y+x/2)} \to 0
\]

because of the continuity of \( \sigma \) with respect to \( (| \cdot |) \). Similarly, we see \( \| \psi^{1/2}e^{iy}(e^{ix_n} - e^{ix})i^{2} \| \to 0 \). Since any \( a \in C^*(V, \sigma) \) is approximated in norm by a finite linear combination of \( e^{ix} \)'s, we are done. \( \square \)

5. Central Decomposition

In this final section, we describe a decomposition theory for transition amplitudes between normal states, which will be effectively used in [20].

Let \( M \) be a W*-algebra with a separable predual and \( Z \) be a central W*-subalgebra of \( M \). Since \( Z \) is a commutative W*-algebra, we have an expression \( Z = L^\infty(\Omega) \) with \( \Omega \) a measurable space furnished with a measure class \( \omega \). If we choose a measure \( \mu \) representing the measure class \( \omega \), then it further induces a decomposition of the form

\[
\int \oplus_{L^\infty(\Omega)} M_\omega d\omega \text{ on } L^2(M) = \int \oplus_{L^2(\Omega)} L^2(M_\omega) \mu(d\omega).
\]

Here \( \{ M_\omega \} \) is a measurable family of W*-algebras and each normal functional \( \varphi \) of \( M \) is expressed by a measurable family \( \{ \varphi_\omega \}_{\omega \in \Omega} \) of normal functional in such a way that

\[
\varphi(x) = \int _\Omega \varphi_\omega(x_\omega) \mu(d\omega) \text{ if } x = \int \oplus_{L^\infty(\Omega)} x_\omega d\omega
\]

and the \( L^2 \)-identification is given by

\[
(\varphi^{1/2} | \psi^{1/2}) = \int _\Omega (\varphi^{1/2}_\omega | \psi^{1/2}_\omega) \mu(d\omega).
\]

This can be seen as follows:

Lemma 5.1. Let \( \{ M_\omega, H_\omega \} \) be a measurable family of von Neumann algebras and set

\[
M = \int \oplus_{L^\infty} M_\omega d\omega, \quad H = \int \oplus_{L^2} H_\omega \mu(d\omega).
\]

Let \( \xi = \int \oplus \xi_\omega \mu(d\omega) \) be a vector in \( H \). Then \( \xi \) is cyclic for \( M \) if and only if \( \xi_\omega \) is a cyclic vector of \( M_\omega \) for a.e. \( \omega \).

Proof. The ‘only if’ part follows from the fact that

\[
\int \oplus_{L^2} (M_\omega \xi_\omega)^2 \mu(d\omega)
\]

is a subspace orthogonal to \( M\xi \).

For the ‘if’ part, we use the commutant formula

\[
M' = \int \oplus_{L^\infty} M'_\omega d\omega
\]

and check that, if \( \xi_\omega \) is a separating vector of \( M'_\omega \) for a.e. \( \omega \), then

\[
x' \xi = \int \oplus_{L^2} x'_\omega \xi_\omega \mu(d\omega) = 0
\]
for $x' = \int_{\omega}^{\oplus} x'_\omega \, d\omega \in M'$ implies $x'_\omega \xi_\omega = 0$ for a.e. $\omega$ and hence $x'_\omega = 0$ for a.e. $\omega$, i.e., $x' = 0$.

By replacing $(M_\omega, \mathcal{H}_\omega)$ with $(M_\omega \otimes 1_\mathcal{K}, \mathcal{H}_\omega \otimes \mathcal{K})$ ($\mathcal{K}$ being a Hilbert space) and then restricting to cyclic subspaces, we may assume that we can find a cyclic and separating $\xi$ for the von Neumann algebra $M$. Then $\xi_\omega \in \mathcal{H}_\omega$ is a cyclic and separating vector of $M_\omega$ for a.e. $\omega$. Let $J_\omega$ be the modular conjugation associated to $\xi_\omega$ (these are defined up to null sets). Then, from the relevant definitions on modular stuff, we see that $\{J_\omega\}$ is a measurable family of operators and

$$J = \int_{L^2(\Omega)} J_\omega \, d\omega$$

gives the modular conjugation associated to $\xi$.

Let

$$\varphi = \int_{L^1(\Omega)} \varphi_\omega \mu(d\omega), \quad \psi = \int_{L^1(\Omega)} \psi_\omega \mu(d\omega)$$

and choose $\xi$ so that both of $\varphi$ and $\psi$ is majorized by the functional $(\xi : \xi)$. By a Radon-Nikodym type theorem (cf. [2]), we can find $a, b \in M$ such that $\varphi, \psi$ are associated to the vectors $a\xi a^* = aJaJ\xi, b\xi b^* = bJbJ\xi$ respectively. (In the notation of [18], we can choose $a = \varphi^{1/4} \xi^{1/2}, b = \psi^{1/4} \xi^{-1/2}$.) Then $\varphi_\omega$ and $\psi_\omega$ are represented by vectors $a_\omega J_\omega a_\omega J_\omega \xi_\omega$ and $b_\omega J_\omega b_\omega J_\omega \xi_\omega$ for a.e. $\omega$. Thus, for $x, y \in M$,

$$\langle x_\omega \varphi_\omega^{1/2} y_\omega \psi_\omega^{1/2} \rangle = \langle a_\omega \xi_\omega a_\omega^* x_\omega | y_\omega b_\omega \xi_\omega b_\omega^* \rangle$$

is a measurable function of $\omega$ and we have

$$\langle x \varphi^{1/2} y \psi^{1/2} \rangle = \int_\Omega \langle x_\omega \varphi_\omega^{1/2} y_\omega \psi_\omega^{1/2} \rangle \mu(d\omega).$$

Consequently, $\{L^2(M_\omega)\}$ is a measurable family of Hilbert spaces in such a way that for any $\varphi \in M_\omega^*$, the decomposed components $\{\varphi_\omega^{1/2}\}$ is measurable. Moreover, we have a decomposable unitary

$$L^2(M) \ni x \varphi^{1/2} \mapsto \int_{L^2(\Omega)} x_\omega \varphi_\omega^{1/2} \, d\omega.$$
defines a state of $A$.

A measurable family of positive functionals $\{\varphi_\omega\}$ is said to be **disjoint** with respect to a probability measure $\mu$ of $\Omega$ if $\int_{\Omega'} \varphi_\omega \mu(d\omega)$ and $\int_{\Omega''} \varphi_\omega \mu(d\omega)$ are disjoint for $\Omega' \cap \Omega'' = \emptyset$.

**Proposition 5.2.**

(i) Given a probability measure $\mu$ and a $\mu$-disjoint family of separable states $\{\varphi_\omega\}$, the integrated state $\varphi = \int \varphi_\omega \mu(d\omega)$ is separable and we have a unitary map $A^{\varphi^{1/2}}A \ni a \mapsto \int_{L^2(\Omega)} a(\omega) \varphi_\omega^{1/2} b(\omega) \mu(d\omega) \in \int_{L^2(\Omega)} A(\omega) \varphi_\omega^{1/2} \mu(d\omega)$.

(ii) Let $\{\psi_\omega\}$ be another $\mu$-disjoint family of separable states with $\psi = \int \psi_\omega \mu(d\omega)$ the integrated state of $A$. Then, for $a, b \in A$, $\langle a(\omega) \varphi_\omega^{1/2} b(\omega) \psi_\omega^{1/2}\rangle$ is measurable and

$$\langle a(\omega) \varphi_\omega^{1/2} b(\omega) \psi_\omega^{1/2}\rangle = \int_{\Omega} \langle a(\omega) \varphi_\omega^{1/2} b(\omega) \psi_\omega^{1/2}\rangle \mu(d\omega).$$

**Proof.** Let $M$ be the von Neumann algebras generated by the left multiplication of $A$ on $A^{\varphi^{1/2}}A$. Then, by the disjointness of $\{\varphi_\omega\}$, $L^\infty(\Omega, \mu) \subset M$ and we can apply the results on W*-algebras. □

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