Polarization phase matrices for radiation scattering on atoms in external magnetic fields:
*The case of forbidden transitions in astrophysics*

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Abstract. Using a quantum electrodynamical approach, we derive the scattering phase matrices for polarized radiation in forbidden line transitions and in the presence of an external magnetic fields. The case of \((J = 0 \rightarrow 2 \rightarrow 0)\) scattering is considered as an example. The non-magnetic Rayleigh scattering phase matrix is also presented. The Stokes profiles in a single scattering event are computed for the strong field (Zeeman) and weak field (Hanle) limits, covering also the regime of intermediate field strengths (Hanle-Zeeman).

1. Introduction

In astronomical spectroscopy, the so-called forbidden transitions are as important as the allowed transitions [1]. They are seen [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] in the spectra of rarefied gases and plasmas under special conditions, such as are found in some nebulae, solar corona, parts of active galactic nuclei and the extreme upper atmosphere of the earth. Historically, the term “forbidden” was associated with all those atomic transitions which do not involve emission or absorption of the electric dipole radiation. The forbidden lines may some times account for 90% or more of the total visual brightness of an object like a planetary nebula, so much so that, when such lines were first seen in the 1860’s, they were thought to be due to a new element which was called as Nebulium. Forbidden lines are often unaffected by departures from local thermodynamic equilibrium (LTE) and are also insensitive to atmospheric temperature uncertainties as well as micro-turbulence. As such they have been quite useful to measure abundances, because at low densities of the order of a few atoms per cubic centimeter, the strength of a spectral line grows as \(N^2\) where \(N\) denotes the total number of atoms in the concerned atomic state, whereas at high densities it grows as \(N\). Scattering of radiation by atoms involves absorption followed by spontaneous emission and the scattering phase function has to be generalized [18] to a \(4 \times 4\) phase matrix to include polarization state of the incoming or outgoing radiation.
With technological advances in spectropolarimetry, accuracies of the order $10^{-5}$ have been achieved in measuring the polarization expressed through Stokes parameters. Using such instruments, it has been possible to discover a wealth of new and unexpected phenomena that are taking place in the outer layers of the solar atmosphere \cite{19}. These recent observations have opened a new window to the diagnosis of the polarized light scattering phenomena in the Sun and the stars \cite{20,21,22,23,24,25,26,27,28,29}. It is well-known that magnetic fields are present almost everywhere in the universe \cite{30}. Therefore we focus our attention on scattering in the presence of magnetic fields. In our previous papers we developed a quantum electrodynamical (QED) approach to line scattering in the presence of external electric and magnetic fields of arbitrary strengths \cite{31,32}. That approach can take account of all multipole type transitions, apart from the dominant one, namely, the dipole scattering. In this paper, we restrict our attention to the case of coherent scattering in the laboratory frame. The QED theory of dipole scattering, for the more realistic case of complete frequency redistribution (CRD) and partial frequency redistribution (PRD) are respectively developed by Landi Degl’Innocenti & Landolfi \cite{28}, and references cited therein) and by Bommier \cite{35,36,37,38}. These authors consider Hanle scattering in the presence of only magnetic fields of arbitrary strength. The classical non-perturbative theory of Hanle-Zeeman scattering with PRD in atom’s rest frame is presented in \cite{39}. Sampoorna et al. \cite{40,41} present the corresponding laboratory frame expressions in a form suitable for astrophysical applications.

A classical theory for the coronal forbidden emission lines that arise from magnetic dipole ($M1$) transition is derived in \cite{9} and \cite{10}. The purpose of the present paper is mainly to derive the phase matrices characterizing atomic scattering of polarized radiation for forbidden transitions in external magnetic fields. This approach can handle all multipoles in the transition. In particular, we present the explicit forms of the phase matrices for $E2$ and $M2$ type transitions, as they are not available in the literature. In sections 2 and 3 we present the theoretical framework briefly. In section 4 we describe 3 limiting cases of scattering in magnetic fields. The numerical examples illustrating the application of these formulae are given in section 5.

2. Theory of scattering for forbidden transitions

It is well-known that an energy level of an atom with total angular momentum $J$ splits into $(2J + 1)$ equally spaced components (Zeeman splitting) when the atom is exposed to an external magnetic field $B$. These levels are characterized by states $|Jm\rangle$, where $m$ denotes the magnetic quantum number which is projection of the total angular momentum operator $J$ along $B$, which is taken along the $Z-$axis. If we consider atomic scattering $J_i \rightarrow J \rightarrow J_f$ of radiation, the elements of the on-energy-shell transition matrix $T$ are given \cite{31,32} by

$$
\langle J_f m_f | T | J_i m_i \rangle = \sum_{m} \langle J_f m_f | \mathcal{E}(k, \mu) | J m \rangle \varphi_m \langle J m | \mathcal{A}(k', \mu') | J_i m_i \rangle , \quad (1)
$$
where $\varphi_m$ denotes the profile function

$$\varphi_m = (\omega_{mm} - \omega - i\Gamma)^{-1}; \quad \omega_{mm} = E_m - E_f. \quad (2)$$

Here $\omega = 2\pi\nu$ denotes the frequency of the scattered radiation, $E_i$, $E_m$ and $E_f$ denote respectively the energies associated with the initial $|J_i m_i\rangle$, intermediate $|J m\rangle$ and final $|J_f m_f\rangle$ states of the atom, and $\Gamma$ denotes the natural width associated with the intermediate state. The vectors $k'$ $(k', \theta', \phi')$ and $k$ $(k, \theta, \phi)$ are photon momenta of the incident and scattered radiation. The states of circular polarization associated with $k'$ and $k$ are denoted respectively by $\mu'$, $\mu = \pm 1$ following the convention employed by Rose [42]. If $\rho(k')$ denotes the $2 \times 2$ density matrix representing the state of polarization of the incident radiation, the density matrix of the scattered radiation is given by

$$\rho(k) = T \rho(k') T^\dagger, \quad (3)$$

where $T^\dagger$ denotes the hermitian conjugate of $T$ considered as a $2 \times 2$ matrix with respect to the basis states of circular polarization. Following [31, 32, 33, 34], the matrix elements for emission and absorption processes in Eq. (1) are given by

$$\langle J_f m_f | \mathcal{E}(k, \mu) | J_m \rangle = \sum_L C(J_f L J; m_f M m) D^L_{M \mu}(\phi, \theta, 0) (-i\mu)^{h(L)} J_L(\omega)^*, \quad (4)$$

$$\langle J_m | \mathcal{A}(k', \mu') | J_i m_i \rangle = \sum_{L'} C(J_i L' J; m_i M' m) D^{L'}_{M' \mu'}(\phi', \theta', 0) (-i\mu')^{h(L')} J_{L'}(\omega'), \quad (5)$$

where $\omega = k, \omega' = k'$ and $J_L(\omega)$ denotes the reduced matrix elements for the atomic transition from a lower level to an upper level given by Eq. (31) of [31]. Further

$$h(L) = \frac{1}{2}[1 + \pi_f \pi (-1)^L]; \quad h(L') = \frac{1}{2}[1 + \pi_i \pi (-1)^{L'}], \quad (6)$$

with $\pi_i, \pi$ and $\pi_f$ denoting respectively the parities of the initial, intermediate and final atomic states. All transitions other than $L = L' = \pi \pi_f = \pi \pi_i = 1$ are referred to as forbidden, following the traditional usage.

### 3. Hanle-Zeeman scattering phase matrix

Any density matrix $\rho$ for polarized radiation may be expressed in terms of the corresponding Stokes parameters $S_p$ with $p = 0, 1, 2, 3$, through

$$\rho = \frac{1}{2} \sum_{p=0}^{3} \sigma_p S_p, \quad (7)$$

where $\sigma_0$ denotes the $2 \times 2$ unit matrix and $\sigma_{1,2,3}$ denote the well-known Pauli matrices. The parameters $(S_0, S_1, S_2, S_3) \equiv (I, Q, U, V)$ denote the Stokes vector $S$. The $4 \times 4$ scattering matrix $R$ is now defined through the relation

$$S(k) = R S'(k'), \quad (8)$$

Noting that

$$S_p = tr(\sigma_p \rho), \quad (9)$$
where $tr$ denotes the trace, the elements of the scattering phase matrix are given by
\[
R_{pp'} = \frac{1}{2} tr(\sigma_p T \sigma_{p'} T^\dagger) = \frac{1}{2} \sum_{mm'} \varphi_m \varphi_{m'}^* \mathcal{P}_{pp'}^{mm'},
\] (10)
where the $4 \times 4$ matrix $\mathcal{P}^{mm'}$ is the phase matrix for line scattering, which is also called ‘scattering matrix’ in the literature. The product $\varphi_m \varphi_{m'}^*$ can be converted into a sum using the conversion formula of \cite{43, 39}, to which a Doppler convolution is applied. In scattering on a two-level atom ($J_i = J_f = 0$), the elements of $\mathcal{P}^{mm'}$ can be expressed in an elegant form
\[
\mathcal{P}_{pp'}^{mm'} = \mathcal{G}_p^{mm'}(k)^* \mathcal{G}_{p'}^{mm'}(k'),
\] (11)
where
\[
\mathcal{G}_p^{mm'}(k) = |J_L(\omega)|^2 \sum_{\mu\mu'} (\sigma_p)_{\mu\mu'} (-1)^{\mu''-m'} (\mu\mu'')^h(L)
\times \sum_{l} C(LL; m - m'm_l) C(LL; \mu - \mu''\mu_l) D_{m\mu l}(\phi, \theta, 0),
\] (12)
and $\mathcal{G}_{p'}^{mm'}(k')$ is given by a similar expression where $p$, $L$, $\omega$ are now replaced by $p'$, $L'$, $\omega'$ respectively. Equation (10) can be used to consider scattering in magnetic fields of arbitrary strength and orientation. Therefore we refer to $R$ as the Hanle-Zeeman phase matrix. It is interesting to consider the particular case of $0 \rightarrow 2 \rightarrow 0$ scattering for forbidden transitions with $L = L' = 2$ and study the phase matrix. We consider three domains of the field strength $B$, namely

(i) the case where $B$ is sufficiently strong so that the levels with different $m$ are distinct, leading to “Zeeman scattering” with $m = m'$ as the summation over $m$ in Eq. (10) drops out (no $m$-state interference);
(ii) the weak field limit, where the magnetic splitting of the upper level is of the same order as the natural width $\Gamma$, so that the quantum interferences between magnetic sublevels $m$ and $m'$ take place leading to the well-known “Hanle scattering”; and
(iii) “Rayleigh scattering” in the absence of magnetic field, when the levels with different $m$ are degenerate, so that there is only a single phase matrix $\mathcal{P}^{mm'} \equiv \mathcal{P}$.

In the Hanle scattering case, one has to consider totally 25 pairs of $m$ and $m'$ for each element of the $4 \times 4$ phase matrix $\mathcal{P}^{mm'}$ given by Eqs. (11, 12) with $L = 2$ and $m = m' = 2, 1, 0, -1, -2$, while for the Zeeman scattering, one has 5 phase matrices $\mathcal{P}^{mm} \equiv \mathcal{P}^m$ with $m = \pm 2, \pm 1, 0$. We present these 3 limiting cases in the following sections.
4. Limiting Cases

4.1. Zeeman scattering phase matrix (Rayleigh scattering in strong fields)

The Zeeman scattering matrix for \( J = 0 \rightarrow 2 \rightarrow 0 \) transition in the strong field limit can be written as

\[
\mathbf{R}_{\text{Zeeman}} = \frac{5}{4} \sum_{m} \mathbf{P}^{m} H(v_{m}, a),
\]

where \( \mathbf{P}^{m} \) and \( H(v_{m}, a) \) represent the scattering phase matrices and the Voigt profile functions containing the energies of the upper states \( |Jm\rangle \) with \( m = -2, \ldots, 2 \) respectively. The normalization constant is 5/4. The frequency dependent shift \( v_{m} \) and the damping parameter \( a \) in Voigt profile function are defined following [43].

Zeeman scattering phase matrices \( \mathbf{P}^{m} \) for electric quadrupole (E2) and the magnetic quadrupole (M2) forbidden line transitions are given below. In the matrix elements, the upper sign represents \( M2 \) transition and the lower sign an \( E2 \) transition. \( M2 \) and \( E2 \) transition have the same expression for some matrix elements (eg. the \( (1, 1) \) element).

\[
\mathbf{P}^{2} = \frac{1}{2}(1 - \delta'^{2})(1 - \delta^{2}) \begin{pmatrix}
\frac{1}{2}(1 + \delta'^{2})(1 + \delta^{2}) & \frac{1}{2}(1 - \delta'^{2})(1 + \delta^{2}) & 0 & 0 \\
\frac{1}{2}(1 + \delta'^{2})(1 - \delta^{2}) & \frac{1}{2}(1 - \delta'^{2})(1 - \delta^{2}) & 0 & \pm \delta'(1 - \delta^{2}) \\
0 & 0 & -\delta(1 + \delta'^{2}) & 0 \\
0 & 0 & 0 & 2\delta' \\
\end{pmatrix},
\]

\[
\mathbf{P}^{-2} = \frac{1}{2}(1 - \delta'^{2})(1 - \delta^{2}) \begin{pmatrix}
\frac{1}{2}(1 + \delta'^{2})(1 + \delta^{2}) & \frac{1}{2}(1 - \delta'^{2})(1 - \delta^{2}) & 0 & 0 \\
0 & 0 & 0 & -\delta'(1 - \delta^{2}) \\
0 & 0 & 0 & 0 \\
-\frac{1}{2}\delta(1 - \delta'^{2}) & -\frac{1}{2}\delta'^{2}(1 - \delta^{2}) & -\frac{1}{2}\delta^{2}(1 - \delta'^{2}) & 0 \\
\end{pmatrix},
\]

\[
\mathbf{P}^{1} = \begin{pmatrix}
\frac{1}{4}[1 - 2\delta'^{2}][1 - 2\delta^{2}][1 - \delta^{2}] & \frac{1}{4}(1 - 2\delta'^{2})[1 - \delta^{2}] & 0 & 0 & 0 \\
\frac{1}{4}(1 - 2\delta'^{2})[1 - \delta^{2}] & \frac{1}{4}(1 - 2\delta'^{2})[1 - \delta^{2}] & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}\delta(1 - \delta'^{2}) & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}\delta'^{2}(1 - \delta^{2}) & 0 \\
0 & 0 & 0 & 0 & \delta^{2}(1 - \delta'^{2})(1 - \delta^{2}) \\
\end{pmatrix},
\]

\[
\mathbf{P}^{-1} = \begin{pmatrix}
\frac{1}{4}(1 - 2\delta'^{2})[1 - 2\delta^{2}][1 - \delta^{2}] & \frac{1}{4}(1 - 2\delta'^{2})[1 - \delta^{2}] & 0 & 0 & 0 \\
\frac{1}{4}(1 - 2\delta'^{2})[1 - \delta^{2}] & \frac{1}{4}(1 - 2\delta'^{2})[1 - \delta^{2}] & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta(1 - \delta'^{2}) \\
0 & 0 & 0 & 0 & \delta(1 - \delta'^{2}) \\
0 & 0 & 0 & 0 & \delta'^{2}(1 - \delta^{2}) \\
\end{pmatrix},
\]

\[
\mathbf{P}^{0} = 9\delta^{2}\delta'^{2}(1 - \delta^{2})(1 - \delta^{2}) \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \pm 1 & 0 & 0 & 0 \\
0 & 0 & \pm 1 & 0 & 0 \\
0 & 0 & 0 & \pm 1 & 0 \\
0 & 0 & 0 & 0 & \pm 1 \\
\end{pmatrix},
\]
where the symbols $\delta' = \cos \theta'$ and $\delta = \cos \theta$, with $\theta'$ and $\theta$ denoting the angle made by the incoming and outgoing ray with respect to the magnetic field chosen to be along the $Z$-axis of a coordinate system.

If we average over all the incoming angles, we obtain the emission coefficients $\eta_{I,Q,V}$ used in the thermal radiation limit (LTE). $\eta_V = 0$ for the particular choice of the coordinate system used in this calculation. However, $\eta_U$ can be generated by a clockwise rotation of the coordinate system through an azimuthal angle $\chi$. We have verified that the new emission coefficients $\eta_{I,Q,U,V}$ obtained after rotation are identical with those derived by Beckers [44] for the quadrupole emission in Zeeman effect regime.

### 4.2. Rayleigh scattering phase matrix (the non-magnetic case)

The Rayleigh scattering matrix for $J = 0 \rightarrow 2 \rightarrow 0$ transition in the absence of magnetic fields is given by

\[
\mathcal{R}_{\text{Rayleigh}} = \frac{5}{4} \mathcal{P} H(v,a). \tag{19}
\]

In this case all the magnetic sublevels of the $J = 2$ are degenerate. Thus the profile functions $\varphi_m$ with $m = -2, -1, 0, 1, 2$ become identical and equal to $\varphi_0$. Therefore we obtain only one scattering phase matrix $\mathcal{P}$, and the profile function $H(v,a)$ corresponding to $\varphi_0$. The elements of the Rayleigh scattering phase matrix are given by

\[
\mathcal{P}_{00} = \frac{1}{2} (1 - 3\delta^2 + 4\delta^4)(1 - 5\delta^2 + 4\delta^4) + (1 - \delta^2)(1 - \delta^2) \left[ \frac{1}{2} (1 + \delta^2)(1 + \delta^2) + 9\delta^2 \delta^2 \right] \\
+ c_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} [4\delta^3 \delta^3 + 6\delta^3 (1 - 2\delta^2)(1 - 2\delta^2)] \\
+ c_2 \left[ 6(1 - \delta^2)(1 - \delta^2) \delta^2 \delta^2 + \frac{1}{2} (1 - 5\delta^2 + 4\delta^4)(1 - 5\delta^2 + 4\delta^4) \right] \\
+ 4c_3 (1 - \delta^2)^{3/2} (1 - \delta^2)^{3/2} \delta^2 \delta^2 + \frac{1}{2} c_4 (1 - \delta^2)^2 (1 - \delta^2)^2, \tag{20}
\]

\[
\mathcal{P}_{01} = \pm \frac{1}{2} (1 - 3\delta^2 + 4\delta^4)(1 - 5\delta^2 + 4\delta^4) + (1 - \delta^2)(1 - \delta^2) \left[ \frac{1}{2} (1 + \delta^2)(1 + \delta^2) \mp 9\delta^2 \delta^2 \right] \\
+ c_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} [\pm 4\delta^3 \delta^3 (1 - \delta^2) \mp 6\delta^3 (1 - 2\delta^2)(1 - 2\delta^2)] \\
\mp c_2 \left[ 6(1 - \delta^2)(1 - \delta^2) \delta^2 \delta^2 + \frac{1}{2} (1 - 5\delta^2 + 4\delta^4)(1 - 5\delta^2 + 4\delta^4) \right] \\
\pm 4c_3 (1 - \delta^2)^{3/2} (1 - \delta^2)^{1/2} \delta^2 \delta^2 \pm c_4 \frac{1}{2} (1 - \delta^2)^2 (1 - \delta^2)^2, \tag{21}
\]

\[
\mathcal{P}_{02} = \pm s_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} [2\delta^3 (1 - \delta^2) + 6\delta^2 (1 - 2\delta^2)] \\
+ s_2 [\pm \delta^3 (1 - 2\delta^2)(1 - 5\delta^2 + 4\delta^4) \mp 6(1 - \delta^2)(1 - \delta^2) \delta^2 \delta^2] \\
\mp 2s_3 (1 - \delta^2)^{3/2} (1 - \delta^2)^{1/2} \delta (1 - 3\delta^2) \pm s_4 (1 - \delta^2)^2 (1 - \delta^2) \delta \delta \tag{22},
\]

\[
\mathcal{P}_{10} = \pm \frac{1}{2} (1 - 3\delta^2 + 4\delta^4)(1 - 5\delta^2 + 4\delta^4) + (1 - \delta^2)(1 - \delta^2) \left[ \pm \frac{1}{2} (1 - \delta^2)(1 + \delta^2) \mp 9\delta^2 \delta^2 \right] \\
+ c_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} [\pm 4\delta^3 \delta^3 (1 - \delta^2) \mp 6\delta^3 (1 - 2\delta^2)(1 - 2\delta^2)] \\
\mp c_2 \left[ 6(1 - \delta^2)(1 - \delta^2) \delta^2 \delta^2 + \frac{1}{2} (1 - 5\delta^2 + 4\delta^4)(1 - 5\delta^2 + 4\delta^4) \right] \\
\pm 4c_3 (1 - \delta^2)^{3/2} (1 - \delta^2)^{3/2} \delta^2 \delta^2 \pm \frac{1}{2} c_4 (1 - \delta^2)(1 - \delta^2)^2 (1 - \delta^2), \tag{23}
\]
\[ P_{11} = \frac{1}{2} (1 - 5\delta^2 + 4\delta') (1 - 5\delta^2 + 4\delta'') + (1 - \delta^2) (1 - \delta^2) \left[ \frac{1}{2} (1 - \delta^2) (1 - \delta^2) + 9\delta^2 \delta^2 \right] \\
+ c_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \left[ 4\delta' (1 - \delta^2) (1 - \delta^2) + 6\delta' (1 - 2\delta^2) (1 - 2\delta^2) \right] \\
+ c_2 \left[ 6 (1 - \delta^2) (1 - \delta^2) \delta^2 \delta^2 + \frac{1}{2} (1 - 3\delta^2 + 4\delta') (1 - 3\delta^2 + 4\delta'') \right] \\
+ 4c_3 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \delta^3 \delta' + \frac{1}{2} c_4 (1 - \delta^2) (1 - \delta^2) (1 + \delta^2) (1 + \delta^2), \quad (24) \]

\[ P_{12} = s_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \left[ 2(1 - \delta^2) (1 - \delta^2) - 6\delta^2 (1 - 2\delta^2) \right] \\
- s_2 \left[ (1 - 3\delta^2 + 4\delta') \delta' (1 - 2\delta^2) - 6(1 - \delta^2) (1 - \delta^2) \delta^2 \delta' \right] \\
- 2s_3 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \delta^3 (1 - 3\delta^2) + s_4 (1 - \delta^2) (1 - \delta^2) \delta' (1 + \delta^2) \quad , \quad (25) \]

\[ P_{20} = \pm s_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \left[ 2(1 - \delta^2) (1 - \delta^2) + 6\delta^2 \delta' (1 - 2\delta^2) \right] \\
+ s_2 \left[ 2(1 - \delta^2) (1 - \delta^2) \delta^2 \delta^2 + \delta (1 - 2\delta^2) (1 - 5\delta^2 + 4\delta') \right] \\
\pm 2s_3 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \delta^3 (1 - 3\delta^2) \mp s_4 (1 - \delta^2) (1 - \delta^2) \delta (1 + \delta^2) \quad , \quad (26) \]

\[ P_{21} = - s_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \left[ 2\delta' (1 - \delta^2) (1 - \delta^2) - 6\delta^2 \delta' (1 - 2\delta^2) \right] \\
+ s_2 \left[ \delta (1 - 2\delta^2) (1 - 3\delta^2 + 4\delta') - 6(1 - \delta^2) (1 - \delta^2) \delta^2 \delta' \right] \\
+ 2s_3 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \delta^3 (1 - 3\delta^2) - s_4 (1 - \delta^2) (1 - \delta^2) \delta (1 + \delta^2) \quad , \quad (27) \]

\[ P_{22} = c_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \left[ (1 - \delta^2) (1 - \delta^2) + 6\delta^2 \delta^2 \right] \\
+ c_2 \left[ 6(1 - \delta^2) (1 - \delta^2) \delta^2 \delta' + 2\delta' (1 - 2\delta^2) (1 - 2\delta^2) \right] \\
+ c_3 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} (1 - 3\delta^2) (1 - 3\delta^2) + 2c_4 (1 - \delta^2) (1 - \delta^2) \delta \delta', \quad (28) \]

\[ P_{33} = 2\delta' (1 - \delta^2) (1 - \delta^2) + 2\delta' (1 - 2\delta^2) (1 - 2\delta^2) \\
+ c_1 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} \left[ (1 - 3\delta^2) (1 - 3\delta^2) + 6\delta^2 \delta^2 \right] \\
+ 6c_2 (1 - \delta^2) (1 - \delta^2) \delta \delta' + c_3 (1 - \delta^2)^{1/2} (1 - \delta^2)^{1/2} (1 - \delta^2) (1 - \delta^2), \quad (29) \]

\[ P_{03} = P_{30} = P_{13} = P_{31} = P_{23} = P_{32} = 0 \quad . \quad (30) \]

In the above equations, the symbols \( c_n = \cos n(\phi - \phi') \) and \( s_n = \sin n(\phi - \phi') \). The incoming and outgoing rays make an angle \((\theta', \phi')\) and \((\theta, \phi)\) with respect to the Z-axis of a coordinate system.

4.3. Hanle scattering phase matrix (in weak field limit)

In the weak-field limit, the magnetic splitting is of the same order as the natural line width of an atomic state. Therefore there exists a frequency coupling between the magnetic sublevels of the upper energy level. If the product of profile functions \( \varphi_m \varphi_{m'}^* \) is simplified as mentioned before (see the paragraph following Eq. (10)), the weak field Hanle phase matrix is obtained.
5. Numerical results for a $0 \rightarrow 2 \rightarrow 0$ transition

We consider the example of single scattering involving a $0 \rightarrow 2 \rightarrow 0$ forbidden transition. An unpolarized beam of radiation is incident on the atom. Therefore the four Stokes parameters $(I, Q, U, V)$ of the scattered radiation are $(R_{00}, R_{10}, R_{20}, R_{30})$ respectively. In the expressions (31) - (37) for the Stokes parameters, the upper sign represents $M_2$ type transition and the lower sign represents $E_2$ type transition. The different curves in all the figures represent different values of magnetic field strength defined by a splitting parameter $v_B = g\nu_L/\Delta\nu_D$, where $\nu_L$ is the Larmor frequency, $g$ the Landé factor, and $\Delta\nu_D$ the Doppler width. The solid curve corresponds to $v_B = 0.004$, the dotted curve to $v_B = 0.02$, the dot-dashed curve to 0.1, the dashed curve to 0.5 and the long-dashed curve to 2.5. The damping parameter of the Voigt profile function $a = 0.004$. The direction of the magnetic field is chosen along the $Z$-axis of the co-ordinate system.

5.1. Forward scattering

Figure 1 shows the scattered Stokes profiles in a forward scattering event ($\theta' = 0^\circ$, $\phi' = 0^\circ$; $\theta = 0^\circ$, $\phi = 0^\circ$). For this choice of scattering geometry, the Stokes parameters, given by the following expressions, are the same for both $M_2$ and $E_2$ type transitions. For the case of non-magnetic Rayleigh scattering:

$$I = \frac{5}{2} [\varphi_0 \varphi_0^*]; \quad Q = 0; \quad U = 0; \quad V = 0.$$  (31)

Likewise, the scattered Stokes profiles for the Hanle-Zeeman scattering are:

$$I = \frac{5}{4} [\varphi_1 \varphi_1^* + \varphi_{-1} \varphi_{-1}^*]; \quad Q = 0; \quad U = 0; \quad V = \frac{5}{4} [\varphi_1 \varphi_1^* - \varphi_{-1} \varphi_{-1}^*].$$  (32)

In the particular geometry of forward scattering, the same expressions hold good for the Zeeman scattering case also. Therefore we show only $I$ and $V/I$ profiles corresponding to the Hanle-Zeeman case. The Stokes $I$ for the Rayleigh scattering case is nearly same as the Stokes $I$ of the Hanle-Zeeman case (the solid curve). The Stokes $Q/I = 0$ in both the cases. It can be understood from the classical theory that the degree of linear polarization is zero for the forward Rayleigh scattering. As the line of sight is taken parallel to the magnetic field, the standard doublet pattern of longitudinal Zeeman effect is seen in $I$. The $V/I$ profiles show anti-symmetric pattern (oppositely circularly polarized). These components are produced due to the scattering involving $|0, 0\rangle$ and $|2, \pm 1\rangle$ states. The resolution of $I$ into 2 components takes place only for very strong fields like $v_B = 2.5$. For weaker fields, it appears only as a slight broadening of the non-magnetic $I$ profiles. All the weak field cases for $v_B \leq 0.1$ merge, and are not distinguishable graphically.

5.2. Scattering at 90°

This scattering geometry corresponds to the maximum degree of linear polarization. The angles chosen for this case are: $\theta' = 0^\circ$, $\phi' = 0^\circ$ and $\theta = 90^\circ$, $\phi = 45^\circ$. In this case,
In Figs. 2 and 3 we show the results for

\[ I = \frac{5}{4} \varepsilon_0 \varepsilon_0^*; \quad Q = \mp \frac{5}{4} \varepsilon_0 \varepsilon_0^*; \quad U = 0; \quad V = 0, \quad (33) \]

for Rayleigh scattering case, and

\[ I = \frac{5}{8} [\varepsilon_1 \varepsilon_1^* + \varepsilon_{-1} \varepsilon_{-1}^*]; \quad Q = \mp \frac{5}{8} [\varepsilon_1 \varepsilon_1^* + \varepsilon_{-1} \varepsilon_{-1}^*]; \quad U = 0; \quad V = 0, \quad (34) \]

for the Hanle-Zeeman as well as the Zeeman scattering cases.

It is clearly seen that the maximum degree of linear polarization \( Q/I = -1 \) in all the three cases. The \( Q/I \) ratio is independent of frequency as well as the magnetic field strength. From Eqs. (33) and (34), we observe that the linear polarization \( (Q) \) profiles for \( M2 \) and \( E2 \) type transitions differ only in sign. As the \( I \) profiles look very similar to the \( I \) profiles of Fig. 1, except for the fact that \( U/I = V/I = 0 \) and \( Q/I = -1 \), we do not show these profiles again.

5.3. Scattering at an arbitrary angle

In Figs. 2 and 3 we show the results for \( M2 \) and \( E2 \) type transitions respectively. They correspond to the choice of scattering geometry: \( \theta' = 45^\circ, \phi' = 0^\circ \) and \( \theta = 90^\circ, \phi = 45^\circ \). For this specific choice of angles, the scattered Stokes parameters are given by the following simple expressions:

\[ I = \frac{5}{8} \varepsilon_0 \varepsilon_0^*; \quad Q = 0; \quad U = 0; \quad V = 0, \quad (35) \]

for non-magnetic Rayleigh scattering case. For the Zeeman scattering case we have

\[ I = \frac{5}{4} [0.125 (\varepsilon_1 \varepsilon_1^* + \varepsilon_{-1} \varepsilon_{-1}^*) + 0.1875 (\varepsilon_2 \varepsilon_2^* + \varepsilon_{-2} \varepsilon_{-2}^*)], \]

\[ Q = \frac{5}{4} \mp 0.125 (\varepsilon_1 \varepsilon_1^* + \varepsilon_{-1} \varepsilon_{-1}^*) \pm 0.1875 (\varepsilon_2 \varepsilon_2^* + \varepsilon_{-2} \varepsilon_{-2}^*), \quad (36) \]

with \( U = V = 0 \), and finally for the Hanle-Zeeman case we get

\[ I = \frac{5}{4} [0.125 (\varepsilon_1 \varepsilon_1^* + \varepsilon_{-1} \varepsilon_{-1}^*) + 0.1875 (\varepsilon_2 \varepsilon_2^* + \varepsilon_{-2} \varepsilon_{-2}^*)] + i \times 0.125 (\varepsilon_{-1} \varepsilon_{1}^* - \varepsilon_1 \varepsilon_{-1}^*) - 0.0625 (\varepsilon_2 \varepsilon_{-2}^* + \varepsilon_{-2} \varepsilon_2^*), \]

\[ Q = \frac{5}{4} \mp i \times 0.125 (\varepsilon_{-1} \varepsilon_{1}^* - \varepsilon_1 \varepsilon_{-1}^*) \mp 0.0625 (\varepsilon_2 \varepsilon_{-2}^* + \varepsilon_{-2} \varepsilon_2^*), \]

\[ U = \frac{5}{4} \times 0.0883883 [(1 - i) (\pm \varepsilon_{-2} \varepsilon_1^* \mp \varepsilon_{-2} \varepsilon_1^* \pm \varepsilon_{-2} \varepsilon_1^* \mp \varepsilon_{-1} \varepsilon_{-1}^*)] + (1 + i) (\pm \varepsilon_{-2} \varepsilon_1^* \mp \varepsilon_{-2} \varepsilon_1^* \pm \varepsilon_{-2} \varepsilon_1^* \mp \varepsilon_{-1} \varepsilon_{-1}^*), \]

\[ V = \frac{5}{4} \times 0.0883883 [(1 - i) (\varepsilon_{-2} \varepsilon_1^* - \varepsilon_{-2} \varepsilon_1^* + \varepsilon_{-2} \varepsilon_1^* - \varepsilon_{-2} \varepsilon_1^*)] + (1 + i) (\varepsilon_{-2} \varepsilon_1^* - \varepsilon_{-2} \varepsilon_1^* - \varepsilon_{-2} \varepsilon_1^* - \varepsilon_{-2} \varepsilon_1^*). \quad (37) \]

Clearly the \( Q/I \) and \( U/I \) in Figs. 2 and 3 for both Zeeman as well as Hanle-Zeeman cases differ only in sign (see Eqs. (36) and (37)). However the \( V/I \) remains the same.
for both $M2$ and $E2$ type transitions. In the following we discuss only Fig. 2 for $M2$ type transition.

In the cases of Zeeman and Hanle-Zeeman scattering, the Stokes $I$ profiles clearly show splitting in the strong fields. When the fields are weak, these components are not resolved. In this choice of geometry, the Stokes $Q$ gets contributions not only from the energy states $|2, \pm 1\rangle$ but also from $|2, \pm 2\rangle$ in both the cases (see Eqs. (36) and (37)). Rayleigh scattered Stokes $Q$ is zero in the $M2$ as well as $E2$ transition, even for a non-zero scattering angle (see Eq. (35)). The $I$ profile for Rayleigh scattering is similar to the solid curve of the general Hanle-Zeeman case, and hence is not shown again. Equations (36) and (37), differ only in the $m$-state interference terms. The effect of these extra terms in Eq. (37) becomes significant only for weak fields (compare solid, dotted and dot-dashed $Q/I$ curves in the 2nd row of Fig. 2). When the fields are stronger (dashed and long-dashed curves), the Zeeman and Hanle-Zeeman scattering give nearly same $Q/I$ profiles, showing that the interference terms gradually become negligible.

The Stokes $U$ and $V$ are zero in Rayleigh and Zeeman scattering cases due to our choice of the scattering geometry, but they appear in Hanle-Zeeman scattering cases in the weak field limit. The generation of these $U$ and $V$ in the Hanle-Zeeman scattering case is entirely due to the interference between different $m$-states. As the magnetic field strength increases, $U/I$ decreases and approaches small values. The $V/I$ is extremely small in the weak field limit and gradually develops the typical anti-symmetric profiles of the usual Zeeman effect as the field strength increases.

6. Conclusions

The polarization phase matrices for forbidden line transitions are derived. The 3 important regimes of Rayleigh, Zeeman, and Hanle-Zeeman scattering are studied. The regime of Hanle-Zeeman scattering provides a complete description of scattering theory in magnetic fields of arbitrary strengths, with the other two regimes serving as extreme limiting cases. We show that the linear polarization profiles ($Q$ and $U$) for $M2$ and $E2$ type transitions differ only in sign. The $V$ profiles are however, insensitive to the type of transition.

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References

[1] Duric N 2004 Advanced Astrophysics (Cambridge: University Press)
[2] Bransden B H and Jochain C J 1983 Physics of Atoms and Molecules (New York: Longman)
[3] Osterbrock D E 1989 *Astrophysics of gaseous nebulae and active galactic nuclei* (Mill Valley CA: University science Books)

[4] Shull J M and von Steenberg M 1982 *ApJ Suppl.* 48 95

[5] Suess Chandra 1983 *Astrophysics and Space Science* 92 223

[6] Bhatia A K and Kastner S O 1995 *ApJ Suppl.* 96 325

[7] Kastner S O and Bhatia A K 1997 *ApJ Suppl.* 109 241

[8] Judge P G 1998 *ApJ* 500 1009

[9] Lin H and Casini R 2000 *ApJ* 542 528

[10] Casini R and Lin H 2002 *ApJ* 571 540

[11] Charro E and Martín I 2002 *J. Phys. B: At. Mol. Opt. Phys.* 35 3227

[12] Ryde N 2006 *A & A* 455L 13R

[13] Ryde N 2006 arXiv: astro-ph/0607017 v1

[14] Ryde N 2006 arXiv: astro-ph/0611703 v1

[15] Aldenius M and Johnsson S 2007 *A & A* 467 753

[16] Caffau E and Ludwig H G 2007 *A & A* 467 L11

[17] Caffau E et al. 2007 arXiv: astro-ph/07042335 v1

[18] Chandrasekhar S 1950 *Radiative Transfer* (Oxford: Clarendon Press)

[19] Stenflo J O 2004 *Nature* 430 304

[20] Stenflo J O 1994 *Solar Magnetic Fields. Polarized Radiation Diagnostics* (Dordrecht: Kluwer)

[21] Stenflo J O and Nagendra K N (Eds.) 1996 *Solar Polarization. Proc. of International Workshop on Solar Polarization, St. Petersburg* (Dordrecht: Kluwer), also *Solar Physics*, 164

[22] Nagendra K N and Stenflo J O (Eds.) 1999 *Proc. of 2nd International Workshop on Solar Polarization* (Dordrecht: Kluwer), ASSl Series 243

[23] Trujillo-Bueno J and Sanchez Almeida J 2003 *Proc. of 3rd International Workshop on Solar Polarization* (ASP Conference Series), ASP Conf. Ser. 307

[24] Casini R and Lites B W (Eds.) 2006 *Proc. of 4th International Workshop on Solar Polarization* (ASP Conference Series), ASP Conf. Ser. 358

[25] Berdyugina S V, Nagendra K N and Ramelli R (Eds.) 2008 *Proc. of 5th International workshop on Solar Polarization* (ASP Conference Series), (in press)

[26] Kazantsev S A and Heoux J C 1995 *Polarization Spectroscopy of Ionized Gases* (Dordrecht: Kluwer)

[27] Trujillo Bueno J, Moreno Insertis F and Sánchez F (Eds.) 2002 *Astrophysical Spectropolarimetry* (Cambridge University Press)

[28] Landi Degl’Innocenti E and Landolfi M 2004 *Polarization in Spectral Lines* (Dordrecht: Kluwer)

[29] Gondorfer A 2000, 2002, 2005 *The Second Solar Spectrum 1, 2, 3* (Zurich: Zurich VdF)

[30] Richard Wielebinski and Rainer Beck 2005 *Cosmic Magnetic Fields* (Berlin: Springer)

[31] Yee Yee Oo, Sampoorna M, Nagendra K N, Ananthamurthy S and Ramachandran G 2007 *J. Quant. Spectrosc. Radiat. Transfer* 108 167

[32] Yee Yee Oo 2004 *Studies in Astrophysical Line Formation Theory* Ph. D. Thesis (Bangalore University: Unpublished)

[33] Yee Yee Oo, Nagendra K N, Ananthamurthy S, Vijayashankar R and Ramachandran G 2004 *J. Quant. Spectrosc. Radiat. Transfer* 84 35

[34] Yee Yee Oo, Nagendra K N, Ananthamurthy S, Swarnamala Sirsi, Vijayashankar R and Ramachandran G 2005 *J. Quant. Spectrosc. Radiat. Transfer* 90 343

[35] Bommier V 1997a *A & A* 328 706

[36] Bommier V 1997b *A & A* 328 726

[37] Bommier V 1999 Solar polarization *Proc. 2nd Int. Workshop on Solar Polarization, Bangalore (India)* ed K N Nagendra and J O Stenflo (Boston: Kluwer), ASSl 243, 43

[38] Bommier V 2003 Solar polarization *Proc. 3rd Int. Workshop on Solar Polarization, Tenerife (Spain)* ed J Trujillo Bueno and J Sánchez Almeida (San Francisco: ASP), ASP Conf. Ser. 307, 213
[39] Bommier V and Stenflo J O 1999 A & A 350 327
[40] Sampoorna M, Nagendra K N and Stenflo J O 2007 ApJ 663 625
[41] Sampoorna M, Nagendra K N and Stenflo J O 2007 ApJ 670 1485
[42] Rose M E 1957 Elementary theory of angular momentum (New York: John Wiley)
[43] Stenflo J O 1998 A & A 338 301
[44] Beckers J M 1969 Solar Phys. 9 372
Figure 1. The Stokes profiles in the regime of Hanle-Zeeman scattering. The case of forward scattering is considered. Different curves represent various values of the field strength $v_B$. See sections 5 and 5.1 for model parameters.
Figure 2. Zeeman (1st column) and Hanle-Zeeman scattering (2nd column) in a M2 type scattering event. The scattering geometry is defined through the choice of angles: $\theta' = 45^\circ, \phi' = 0^\circ$ and $\theta = 90^\circ, \phi = 45^\circ$. Different curves correspond to different values of the field strengths expressed through the parameter $v_B$. The solid curves: $v_B = 0.004$, dotted curves: 0.02, dot-dashed curves: 0.1, dashed curves: 0.5, and long dashed curves: 2.5. See section 5.3 for discussions.
Figure 3. Zeeman (1st column) and Hanle-Zeeman scattering (2nd column) in a $E_2$ type single scattering event. The scattering geometry and different curve types are the same as in Fig. [2]