Strong vortex-antivortex fluctuations in the type II superconducting film

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Abstract

The small size vortex-antivortex pairs proliferation in type II superconducting film is considered for the wide interval of temperatures below \( T_c \). The corresponding contribution to free energy is calculated. It is shown that these fluctuations give the main contribution to the heat capacity of the film both at low temperatures and in the vicinity of transition.

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The important role of the vortex-antivortex pairs in the 2D phase transitions is well known. For instance their presence restores the phenomenon of superconductivity in spite of the destruction of the long range order by phase fluctuations [1]. Let us consider the case of dirty superconducting film with the thickness \( d \ll \xi_{\text{BCS}}(0) \sim D/T_c \), where \( D = v_F\tau_{\text{tr}}/3 \) is the diffusion coefficient and \( \tau_{\text{tr}} \) is the transport collision time.

Slightly below the BCS critical temperature vortex-antivortex pairing becomes energetically profitable and the BKT transition to the state with the finite stiffness takes place. Let us recall that this transition is determined by the large scale vortex-antivortex pairs, which binding energy has the logarithmic form. The standard description of the BKT transition is based on the corresponding logarithmic expression for free energy [1]

\[
F_{\text{BKT}} = \left[ \frac{\pi n_{s2}(T)}{2m} - 2T \right] \ln \frac{R}{\xi(T)},
\]

where \( n_{s2}(T) \) is the superfluid density and \( \xi(T) \) is the coherence length, which in the vicinity of transition has the form:

\[
\xi^2(T) = \frac{\pi D}{16T_c \tau}
\]

with \( \tau = 1 - T/T_c \) as the reduced temperature. Let us underline that this definition differs twice from the standard GL expression. The Eq. (1) is valid for large size vortex-antivortex pairs, when \( R \gg \xi \). Namely these pairs determine the thermodynamical and transport properties of the 2D superconductor in the vicinity of the BKT transition.

In this letter we will demonstrate that below the immediate vicinity of the BKT transition, in contrast to the current opinion, the essential role in fluctuation properties of 2D superconductor play the small size vortex-antivortex pairs. The cornerstone of presented theory consists in the fact that the energy of the vortex-antivortex pairs turns zero when the distance between their centers becomes less than \( \xi \) [2, 3]. As the consequence such ”cheap” pairs become ”affordable” for thermal fluctuations and their proliferation takes place even at so low temperatures when other quasiparticles are frozen. In result namely this small size vortex-antivortex pairs give the dominant contribution with respect to other fluctuations in the Ginzburg-Landau region below \( T_{\text{BKT}} \) and even prevail over the BCS exponential tail in the heat capacity of the 2D superconducting system at low temperatures.

In order to take into account the specifics of the mentioned fluctuation processes let us
start from the general expression for the partition function

\[ Z = \int \mathcal{D} \Delta (\mathbf{r}) \int \mathcal{D} \Delta (\mathbf{r})^* \exp \left\{ -\frac{F(\Delta (\mathbf{r}), \Delta (\mathbf{r})^*)}{T} \right\}. \]

with \( F(\Delta (\mathbf{r}), \Delta (\mathbf{r})^*) \) as the Ginzburg-Landau functional, which is valid in the vicinity of \( T_c \), and then we will generalize the result obtained for the case of low temperatures. In contrast to the usual Ginzburg-Landau type long wave-length approximation the calculation of the functional integral here has to take into account also the vast variety of the order parameter functions \( \Delta_p (\mathbf{r}) \) corresponding to the specific realization of the vortex-antivortex pairs in the film. In other words side by side with the usual Ginzburg-Landau order parameter fluctuations we will take into account additionally some specific short wave-length fluctuations.

Let us first separate the partition function \( Z_0 \) of the superconducting film without fluctuations:

\[ Z = Z_0 \cdot Z_{(fl)}. \]

The partition function \( Z_{(fl)} \) we will calculate in the gas approximation. Namely we will assume that the main contribution to it appears from the small pairs neglecting their overlap. Hence

\[ Z_{(fl)} = Z_p^{S/[\pi \xi^2(T)]}, \]

where

\[ Z_p = \int d\Delta_\delta (\mathbf{r}) \int d\Delta_\delta (\mathbf{r})^* \exp \left\{ -\frac{F_p(\Delta_\delta (\mathbf{r}), \Delta_\delta (\mathbf{r})^*)}{T} \right\} \]

is the contribution to the partition function of the isolated single pairs of all possible sizes \( \delta \cdot \xi \), with \( 0 \leq \delta \ll 1 \). The power \( S/(\pi \xi^2) \) in Eq.(2) takes into account the combinatorial factor corresponding to the independent formation of such pairs. The order parameter \( \Delta_\delta (\mathbf{r}) \) must have at the distance \( 2\delta \xi \) two zeros of the opposite vorticity (i.e. the total vorticity calculated at large distances is equal to zero).

As the next step we disregard the axial asymmetry of the vortex-anivortex pair. We will see below that the characteristic size of the pairs which mainly contribute to the partition function \( r_{eff} \ll \xi \) what will justify our gas approximation \( \delta \ll 1 \). The corresponding to
Eq. (2) free energy functional is

\[
F_p = \nu d \int d^2r \left[ -\tau |\Delta_\delta (r)|^2 + \frac{\pi D}{8T_c} |\partial_+ |\Delta_\delta (r)|^2 + \frac{7\zeta (3)}{16\pi^2 T_c^2} |\Delta_\delta (r)|^4 \right].
\]  

(3)

Here \( \nu \) is the density of states and \( \partial_+ = \partial / \partial r - 2ieA \), \( \zeta (x) \) is the Riemann zeta-function.

One can demonstrate \[2\] that the energy of \( \Delta_\delta (r) \) profile with two zeros at the fixed distance \( 2\delta \xi \) is majorized by that one of the disc of radius \( \delta \xi \) and below we will estimate the latter.

The GL functional minimization procedure is equivalent to the solution of the 2D Ginzburg-Landau equation with the boundary condition \( \Delta_\delta (r = \delta \xi) = 0 \). Let us assume the order parameter in the form

\[
\Delta_\delta (\rho) = \Delta_0 (T) f (\rho)
\]

and write down the dimensionless GL equation:

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) f + \frac{1}{2} f - \frac{1}{2} f^3 = 0.
\]  

(4)

Here \( \rho = r/\xi \) is the dimensionless radius, while \( \Delta_0 (T) \) is the BCS gap.

Let us investigate the solutions of Eq. (4) in two limiting cases: small \( \rho \sim \delta \ll 1 \), and large \( \rho \rightarrow \infty \). In the region of small \( \rho \sim \delta \ll 1 \) the function \( f (\rho) \rightarrow 0 \) and it can be found as the solution of the linearized Eq. (4):

\[
f (\rho) = C_1 J_0 \left( \frac{\rho}{\sqrt{2}} \right) + C_2 N_0 \left( \frac{\rho}{\sqrt{2}} \right),
\]  

(5)

where \( J_0 \) and \( N_0 \) are the Bessel and Neumann functions correspondingly.

When \( \rho \gtrsim 1 \), the solution of Eq.(4) can be looked for in the form \( f (\rho) = 1 - f_1 (\rho) \). Substituting this expression to Eq.(4) one finds

\[
f (\rho) = 1 - f_1 (\rho) = 1 - C_3 K_0 (\rho),
\]  

(6)

where \( K_0 \) is the Bessel function. The constant term corresponds to the usual long wavelength order parameter contribution which will appear in the result of our consideration side by side with the contributions of the small vortex-antivortex pairs.

One can see that the asymptotic of Eq.(6), being calculated for \( \rho \gtrsim 1 \) nevertheless extends to the region of small values of \( \rho \). It is why we can match both solutions Eq.(5) and Eq.(6) at \( \rho \sim \delta \ll 1 \). In order to do this let us use the expressions for all Bessel functions \( J_0, N_0 \) and \( K_0 \) at small arguments \[4\]. The comparison of the coefficients at logarithmic term and of
constants gives two equations for coefficients $C_{1,2,3}$ the third equation is determined by the boundary condition $f(\delta) = 0$ which corresponds to our disc model for the vortex-antivortex pair. Finally

$$C_i = \delta_{i1} + \frac{1}{2 \ln \frac{2}{\gamma \delta}} \begin{cases} \ln 2 \\ \pi \\ 2 \end{cases}$$

Here $\gamma = \exp (C)$, where $C = 0.577215...$ is the Euler constant.

Now we can find the energy of disc (vortex-antivortex pair) as the function of its dimensionless radius $\delta$. In order to do this one can use the general expression (3) and the fact that $\Delta = \Delta_0 (T) f$ satisfies the nonlinear Eq. (4). Using this and performing the integration of the derivative by parts one can find the energy of vortex-antivortex pair:

$$F_p (\delta, T) = \frac{B (T)}{\ln \frac{2}{\gamma \delta}} \int_0^\infty xK_0 (x) \, dx = \frac{B (T)}{\ln \frac{2}{\gamma \delta}},$$

with

$$B (T) = \frac{\pi^2 \nu d \Delta_0^2 (T)}{4T_c}.$$

Let us stress that namely presence of the large logarithm in denominator of Eq. (7) does such localized vortex-antivortex type fluctuations of the order parameter more energetically favorable than the almost homogeneous (long wave-length) Ginzburg-Landau ones.

Now let us perform the functional integration in Eq.(2). Vortex-antivortex pairs proliferate below $T_c$ in fluctuation way and in our model this is equivalent to the account of the discs of the different radii:

$$Z_p = 2 \int \delta d\delta \exp \left\{ -\frac{F_p (\delta, T)}{T} \right\}.$$

The integral in Eq. (9) may be carried out by the method of the steepest descent what gives

$$Z_p = \frac{4 \cdot 2^{1/4} \pi^{1/2}}{\gamma^2} \left( \frac{B (T)}{T} \right)^{1/4} \exp \left\{ -2 \sqrt{\frac{2B (T)}{T}} \right\}$$

and finally, the corresponding contribution to the free energy is

$$\tilde{F}_p (T) = -T \ln Z_p^{S/\xi^2} = -\frac{TS}{\pi \xi^2 (T)} \left[ -2 \sqrt{\frac{2B (T)}{T}} + \frac{1}{4} \ln \frac{B (T)}{T} \right].$$
This expression is valid in the assumption $B(T) \gg T$ which is necessary for the applicability of the steepest descent method. Looking on Eq. (8) and recalling the definition of the characteristic for superconducting film $2D$ Ginzburg-Levanyuk number $Gi(2d)$

$$Gi(2d) = \frac{21\zeta(3)}{\pi^2} \frac{1}{p_F^2 d l_{tr}} \ll 1$$

one can see that this requirement is equivalent to $\tau \gg Gi(2d)$, i.e. our consideration is valid for temperatures below the critical region of transition.

At temperatures close to transition, in the GL region $1 \gg \tau \gg Gi(2d)$ the main temperature dependence of Eq. (10) originates from functions $\xi(T)$ and $\Delta_0(T)$. Corresponding vortex-antivortex pairs contribution to heat capacity is

$$C_p(T \rightarrow T_c) = -\frac{48ST_c}{\nu F l_{tr}} \frac{\partial^2}{\partial \tau^2} \left[ 4 \sqrt{\nu d D} - \frac{\tau}{4\pi^2} \ln \frac{2\pi^4 \nu d D \tau}{7\zeta(3)} \right]. \quad (12)$$

One can see that the differentiation of the second term in the square brackets of the Eq. (12) gives nothing else as the well known contribution to heat capacity occurs due to the GL long wave length order parameter fluctuations. Nevertheless the main fluctuation contribution to the heat capacity of a $2D$ superconducting film ($d \ll \xi(\tau)$) results from the first term of the Eq. (12), corresponding to the protagonists of this letter, i.e. short wave length vortex-antivortex type order parameter fluctuations:

$$\delta C_{(f_{fl})}^{(2)}(1 \gg \tau \gg Gi(2d)) = \frac{8\pi^2 S d}{7\zeta(3)} \nu T_c \left( \frac{Gi(2d)}{\tau} \right). \quad (13)$$

One can see that the contribution (13) dominates over (12) in the entire Ginzburg-Landau region $\tau \gg Gi(2d)$. Let us note that the negative sign of the vortex-antivortex fluctuation corrections means that the mean field heat capacity jump

In the critical region $\tau \lesssim Gi(2d)$ the main contribution to heat capacity belongs to the classical BKT large vortex-antivortex pairs:

$$C_{(BKT)}^{(v-a)}(\tau \ll Gi(2d)) \sim -\nu T_c S d \left( \frac{Gi(2d)}{\tau} \right)^{3/2}$$

which matches with Eq. (14) at $\tau \sim Gi(2d)$. Let us note that the negative sign of the vortex-antivortex fluctuation correction means that the mean field heat capacity jump
overestimates its true value and the vortex-antivortex pairs fluctuation contribution smears it out.

Now let us turn to discussion of the region of low temperatures. Unfortunately we cannot write down the exact functional for the superconductor’s free energy here, but it is almost evident that the main changes in it with respect to the Ginzburg-Landau one at $T \sim T_c$ occur through the different temperature dependence of $\xi(T)$ and $\Delta_0(T)$ at low temperatures. Hopefully the region of validity of the Eq. (10) in this way can be considerably extended and, at least as the estimate, we will still use it just substituting the GL parameters $\xi(T)$ and $\Delta_0(T)$ by their exact BCS values.

The microscopic BCS equation for $\Delta_0(T)$ is well known

$$\ln \frac{T_c}{T} = 2\pi T \sum_{\omega_n \geq 0} \left( \frac{1}{\omega_n} - \frac{1}{\sqrt{\omega_n^2 + \Delta_0^2(T)}} \right).$$

with $\omega_n = 2\pi T(n + 1/2)$. In the limiting cases

$$\Delta_0^2(T) = \begin{cases} \frac{8\pi^2 T^2}{7\zeta(3)} \tau, & \tau \ll 1, \\
(\pi T_c/\gamma)^2, & T \ll T_c. \end{cases}$$

In the region of low temperatures $T \ll T_c$ the temperature corrections to the order parameter are exponentially small by the parameter $\exp(-\Delta_0(0)/T)$.

The coherence length $\xi(T)$ in the most general case can be defined as the pole of the linear response operator kernel for the corresponding correction to modulus of $\Delta$. In the simplest case of dirty superconductor the corresponding equation takes form

$$\sum_{\omega_n > 0} \left[ \frac{\omega_n^2}{[\omega_n^2 + \Delta_0^2(T)](\sqrt{\omega_n^2 + \Delta_0^2(T)} - D/|2\xi^2(T)|)} - \frac{1}{\sqrt{\omega_n^2 + \Delta_0^2(T)}} \right] = 0.$$ 

In the limiting cases of low and high temperatures

$$\xi^2(T) = \begin{cases} \frac{\pi D}{16 T_c^2}, & T \to T_c, \\
\frac{D}{2\theta \Delta_0(0)}, & T \ll T_c, \end{cases}$$

where $\theta = 0.76595$ is the solution of the equation

$$\frac{\pi}{4} = \sqrt{1 - \theta^2} \left[ \frac{\pi}{2} - \arctan \left( \sqrt{\frac{1 - \theta}{1 + \theta}} \right) \right].$$
Now we are ready to calculate the contribution of the vortex-antivortex fluctuations to the heat capacity $C^{(v-a)}(T \ll T_c)$ of the 2D superconducting film ($d \ll \xi(0)$) at low temperatures. The values $\xi(T)$ and $\Delta_0(T)$ here can be assumed to be temperature independent: $\xi(T) \approx \xi(0) = \frac{D}{2d\Delta(0)}$ and $\Delta_0(T) \approx \Delta_0(0) = \pi T_c/\gamma$ and differentiation of the Eq. (10) results in

$$\begin{align*}
\frac{C^{(v-a)}(T \ll T_c)}{\Delta C(T_c)} = \frac{7\pi \zeta(3)\theta}{8(l_{tr}p_F)^2} \sqrt{\frac{T_c}{T}} \frac{3l_{tr}}{d} = 0.87 \left(\frac{G_i(2d)}{t}\right)^{1/2},
\end{align*}$$

(15)

where $t = T/T_c \ll 1$ and

$$\Delta C = \frac{8\pi^2 S d}{7\zeta(3)} T_c$$

is the BCS jump of heat capacity at $T_c$.

Contribution (15) requires a special discussion. First of all let us stress its power dependence on temperature and unusual growth with the decrease of temperature. Comparison of Eq. (15) with the BCS expression for heat capacity of superconductor shows that already at $T \lesssim T_c/\ln \left[G_i^{-1}(2d)\right]$ the former exceeds the latter, i.e. nonmonotoneous temperature dependence of a heat capacity can be expected. Naturally the formal divergence of Eq. (15) at very low temperatures has not been taken seriously: at low temperatures the quasiclassical approximation used for evaluation of the vortex-antivortex energy [2, 3] breaks down and these excitations probably acquire some small gap ($\sim \Delta^2/E_F$) in the spectrum (in complete analogy with the excitations in the vortex core in 2D superconductors [8, 9]).

In conclusion let us summarize the results obtained. We have demonstrated that the possibility of small vortex-antivortex pairs proliferation in 2D superconducting film results in appearance of the specific contribution to its heat capacity. Being gapless such excitations result in the appearance of the non-exponential heat capacity of superconducting film at low temperatures which turns to dominate the BCS term. At high temperatures the small vortex-antivortex pairs contribution to heat capacity dominates over all other fluctuation corrections in the entire GL region below $T_c$. Finally, only within the critical region the large BKT type vortex-antivortex pairs become of the first importance and determine the thermodynamics of the 2D superconducting film.

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