Hardness Results and Approximation Algorithms
for the Minimum Dominating Tree Problem

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Abstract. Given an undirected graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$, the Minimum Dominating Tree problem asks to find a minimum weight sub-tree of $G$, $T = (U, F)$, such that every $v \in V \setminus U$ is adjacent to at least one vertex in $U$. The special case when the weight function is uniform is known as the Minimum Connected Dominating Set problem.

Given an undirected graph $G = (V, E)$ with some subsets of vertices called groups, and a weight function $w : E \to \mathbb{R}$, the Group Steiner Tree problem is to find a minimum weight sub-tree of $G$ which contains at least one vertex from each group.

In this paper we show that the two problems are equivalents from approximability perspective. This improves upon both the best known approximation algorithm and the best inapproximability result for the Minimum Dominating Tree problem. We also consider two extrema variants of the Minimum Dominating Tree problem, namely, the Minimum Dominating Star and the Minimum Dominating Path problems which ask to find a minimum dominating star and path respectively.

1 Introduction

Given an undirected graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$, the Minimum Dominating Tree problem (MDT) asks to find a minimum weight sub-tree of $G$, $T = (U, F)$, such that every $v \in V \setminus U$ is adjacent to at least one vertex in $U$. The special case when the weight function is uniform is known as the Minimum Connected Dominating Set problem (CDS). Both CDS and MDT have many applications in routing for mobile ad-hoc networks, see for example [1, 2, 5, 6, 12]. Figure 1 shows an example instance and a possible solution to the problem.

Given an undirected graph $G = (V, E)$ with some subsets of vertices called groups, and a weight function $w : E \to \mathbb{R}$, the Group Steiner Tree problem (GST) is to find a minimum weight sub-tree of $G$ which contains at least one vertex from each group. GST is not approximable within $\Omega((\log 2^e n)^{\epsilon})$ unless NP admits quasi-polynomial-time Las-Vegas algorithm [10]. On the other hand, there is a $\log^3 n$-approximation algorithm for the problem [8].

In this paper we show that the two problems are equivalents from approximation algorithms perspective. This improves upon both the best known approximation algorithm and the best inapproximability result for MDT. We also
consider two extrema variants of MDT, namely, the Minimum Dominating Star (MDS) and the Minimum Dominating Path (MDP) problems which ask to find a minimum dominating star and path respectively.

**Previous Work:** CDS has a long history starting at the late 70s [11], and it is approximable within $\ln \Delta + 3$ [9] where $\Delta$ is the maximum degree in $G$, which is the best one can wish for if $P \neq NP$.

MDT, to the best of our knowledge, was introduced in [12]. In the same paper it was shown that the Minimum Weighted Dominating Set problem can be reduced to MDT in a way that preserve the approximation ratio. Thus, there is no $c \log n$-approximation algorithm, for some $c > 0$, for MDT unless $P = NP$. In the same paper it was shown that MDT can be reduced to the Minimum Directed Steiner Tree problem in a way that preserve the approximation ratio. Unfortunately, the current best approximation algorithm for the Minimum Directed Steiner Tree problem yields a $|S|^c$ approximation ratio [4], where $S$ is the set of terminals. To the best of our knowledge, this is the best approximation algorithm known for MDT. The existence of a dominating path in a graph was studied in several papers see [3,7] for example, but, to the best of our knowledge, it was never considered from algorithmic perspective.

**Our Result:** We show that the Minimum Dominating Tree problem is equivalent to the Group Steiner Tree problem from approximability perspective, by doing so we prove that MDT is inapproximable within $\log^{2-\epsilon} n$ unless $NP \subseteq ZTIME(n^{\polylog(n)})$. This also, directly leads to a $\log^2 n \log \Delta$-approximation algorithm, where $\Delta$ is the maximum degree in $G$. We also consider the Minimum Dominating Star problem, show that it is inapproximable within ratio of $c \log n$ for some $c > 0$ and show how to reduce it to the Minimum Set Cover problem to obtain a $O(\log n)$ approximation. Finally, we consider the Minimum Dominating Path problem and show that it is inapproximable at all.
2 Minimum Dominating Tree

Recall that instance of GST is a tuple \((G, w, \mathcal{G})\), where \(G = (V, E)\) is an undirected graph, \(w : E \to \mathbb{R}\) is a weight function, and \(\mathcal{G} \subseteq 2^V\) is a family of groups of vertices. A Steiner group tree of is a sub-tree of \(G\), \(T = (U, F)\) such that \(g \cap U \neq \emptyset\) for each \(g \in \mathcal{G}\). The cost of such tree is \(\sum_{e \in F} w(e)\). Given an instance of GST we are looking for a minimum cost Steiner group tree of \(G\).

In this section we show that MDT is equivalent to GST from approximability perspective, that is every approximation algorithm to one problem yields the same approximation ratio for the other problem. To show this we introduce approximation preserving reductions from MDT to GST and vice-versa.

2.1 MDT \(\leq_p\) GST

We start by showing an approximation-preserving reduction from MDT to GST. Given an instance of MDT, \((G, w)\), where \(G = (V, E)\), we define an instance of GST, \((G, w, \mathcal{G})\). We now have to define \(\mathcal{G}\): for each vertex \(v \in V\) we define \(g_v \in \mathcal{G}\) to be \(\{v\} \cup N(v)\). Figure 2 depicts this transformation. Note that the number of groups is \(n\) and that the size of the largest group is \(\Delta\). Clearly this transformation can be done in polynomial time. The following two claims show that this is an approximation preserving reduction.

\[\text{Fig. 2. From left to right: a) A MDT instance (weights are omitted). b) A corresponding GST instance on the same weighted graph. For each vertex } v \text{ we define a group that contains its neighborhood to ensure that in any group Steiner tree there is at least one vertex that dominate } v. g_2 \text{ is marked in the figure with a dashed blue line.}\]

Claim. Any dominating tree, \(T\), in \((G, w)\) is a feasible group Steiner tree in \((G, w, \mathcal{G})\).

Proof. Assume for contradiction that \(T\) is not feasible group Steiner tree, that is, there is a group \(g_v\) such that none of the vertices in \(g_v\) is spanned by \(T\), that
is $v$ is not in $T$ nor any of its neighbors and thus $T$ is not a dominating tree - contradiction.

Claim. Any feasible group Steiner tree in $(G, w, G)$, $T$, is a dominating tree in $(G, w)$.

Proof. Assume for contradiction that $T$ is not a dominating tree, that is, there is a vertex $v$ not in $T$ such that none of its neighbors belong to $T$, thus, $T$ does not cover $g_v$ - contradiction.

2.2 MDT $\geq_p$ GST

We now show an approximation-preserving reduction from GST to MDT. Given an instance of GST, $(G, w, G)$, where $G = (V, E)$, we define an instance of MDT, $(G', w')$ where $G' = (V', E')$, and:

- $V' = V \cup \{g_i : g_i \in G\}$
- $E' = E \cup \{vg_i : v \in g_i\} \cup (V \times V) \setminus E$
- $w' - w'(e) = \begin{cases} w(e) & e \in E \\ \infty & \text{otherwise} \end{cases}$

Figure 3 depicts this transformation. Clearly the above transformation can be done in polynomial time. The following two claims show that this is an approximation preserving reduction:

Claim. Any dominating tree with finite weight, $T$, in $(G', w')$ is a feasible group Steiner tree in $(G, w, G)$.

\[
\begin{align*}
\text{Fig. 3.} & \quad \text{From left to right: a) An instance of GST (weights are omitted), groups are marked by a dashed blue, dotted green, and dashed-dotted lines respectively. b) A corresponding MDT instance: we add a vertex for each group and connect it to all terminals in the group. Weights for the original edges remain intact, dashed edges have infinity weight.}
\end{align*}
\]
Proof. Observe first that \( T \) contains only original edges or otherwise its weight is infinite. Assume for contradiction that \( T \) is not feasible group Steiner tree, that is, there is a group \( g_i \) such that none of the vertices in \( g_i \) is spanned by \( T \), that is \( g_i \) is not in dominated by any vertex in \( T \) - contradiction.

Claim. Any feasible group Steiner tree in \( (G, w, \mathcal{G}) \), \( T \), is a dominating tree in \( (G', w') \).

Proof. First, observe that any original vertex dominate all other original vertices. Assume for contradiction that \( T \) is not a dominating tree, that is, there is a new vertex \( g_i \) that is not dominated by \( T \) which implies that \( T \) does not cover \( g_i \) - contradiction.

3 Minimum Dominating Star

In this section we consider the Minimum Dominating Star problem, that is the Minimum Dominating Tree problem when restricted to stars, i.e. given an undirected weighted graph \( (G, w) \) find a minimum weight dominating sub-star (a tree with diameter at most 2). We start by showing that this problem cannot be approximated within \( c \log n \) for some \( c > 0 \) unless \( P = NP \). Then we show how to reduce the problem to a set cover instance to achieve a \( O(\log n) \)-approximation algorithm.

3.1 Hardness

We show an approximation preserving reduction from the Minimum Dominating Set problem (DOM) to Minimum Dominating Star. Given an (un-weighted) instance of DOM \( G = (V, E) \) we create an instance of MDS \( (G', w) \) where \( G' = (L \cup R \cup c, E' \cup \{c\} \times L) \).

\[
\begin{align*}
L & = \{ v_l : v \in V \} \\
R & = \{ v_r : v \in V \} \\
E' & = \{ u_l v_r : uv \in E \}
\end{align*}
\]

We also set \( w(e) = \infty \) for every \( e \in E' \) and \( w(cv_l) = 1 \) for every \( v_l \in L \). Figure 3 depicts this transformation. Clearly the above transformation can be done in polynomial time. The following two claims show that this is an approximation preserving reduction:

Claim. If \( D \) is a dominating set in \( G \) then \( S = (\{c\} \cup \{v_l : v \in D\}, \{cv_l : v \in D\}) \) is a dominating star in \( G' \) that weigh \( |D| \).

Proof. \( S \) weights \( D \) by the definition of \( G' \) and \( S' \). Assume for contradiction that it is not a dominating star and let \( v \) be a non dominated vertex then \( v \) is also not dominated in \( G \) under \( D \) - contradiction.

Claim. If \( S = (U, F) \) is a dominating star in \( G' \) of weight \( k < \infty \) then \( \{ v : v_l \in U \setminus \{c\} \} \) is a dominating set in \( G \) of size \( k \).
Fig. 4. From left to right: a) An instance of DOM (unweighted graph) b) The corresponding instance of MDS, the weight on the original edges (solid black) is infinity and the weight on the new edges is 1.

Proof. Observe that any star that contains more than 3 vertices must be centered at $c$ or otherwise its weight is infinity. Thus the leaf of the star dominate all vertices in $R$, by construction, this mean that the corresponding vertices in the original graph dominate all other vertices.

3.2 $\log n$-Approximation

We now show how to reduce MDS to an instance of the MINIMUM SET COVER problem (SC) in order to obtain a $O(\log n)$-approximation algorithm. This is the best one can hope for if $P \neq NP$.

Without loss of generality, we assume that the center of the dominating star is known, or otherwise we can solve the problem for every vertex in the graph assuming it is the center. Given an undirected weighted graph $(G, w)$ and a the center of the dominating star $c \in V$ we create the following an instance of SC $(U, S, w')$ as follow:

- $U = V \setminus N(c)$
- $S = \{S_v : cv \in E\}$
- $S_v = N(v) \cap U$
- $w' = w'(S_v) = w(cv)$

Clearly the above transformation can be done in polynomial time. The following two claims show that this is an approximation preserving reduction:

Claim. If $S = (c, L)$ is a dominating star in $G$ then $C = \{S_v : v \in L\}$ is a set cover in $(U, S, w')$, moreover, $w(S) = w'(C)$.

Proof. $w(S) = w'(C)$ by construction. Now, assume for contradiction that $C$ is not a set cover and let $v$ be an uncover element then $v$ is also not dominated by $S$ - contradiction.
Claim. If $C$ is a set cover in $(U, S, w')$ then $S = (e, \{ v : S_v \in C \})$ is a dominating star in $G$, moreover, $w(S) = w'(C)$.

Proof. $w(S) = w'(C)$ by construction. Now, assume for contradiction that $S$ is not a dominating star and let $v$ be an undominated vertex then $v$ is also uncovered by $C$ - contradiction.

4 Minimum Dominating Path

We show that the Minimum Dominating Path problem (MDP) cannot be approximated at all unless $P = NP$. We show a reduction from the Hamiltonian Path problem (HP). In HP we are given an undirected graph $G = (V, E)$ and we are asked to decide if there is an Hamiltonian path (a simple path traversing all the vertices in $V$) in $G$ or not. HP is one of the classical NP-hard problems.

Given an instance of HP, $G = (V, E)$, we define an instance of MDP, $(G', w)$, where $G' = (V \cup \{ v' : v \in V \}, E \cup \{ vv' : v \in V \})$. We set $w(e) = 0$ for every edge $e \in E$ and set $w(e) = \infty$ otherwise. Figure 5 depicts this transformation. We now claim that any (multiplicative) approximation algorithm for MDP can solve HP. Let $G$ be an instance of (decision problem) HP, and denote by $A(G', w)$ the value of (approximation) algorithm, $A$, on the corresponded MDP instance, then:

Claim. $A(G', w) = 0 \iff G \in HP$.

Proof. Let $P$ be a dominating path in $(G', w)$ with value 0, then it uses only edges of $E$, moreover $P$ is an hamiltonian path in $G$ or otherwise there is a vertex $v'$ that is not dominated by $P$. Now, let $P$ be an hamiltonian path in $G$ then $P$ is a dominating path in $G'$ with value 0, thus, any (multiplicative) approximation algorithm must also find a path value 0.

5 Conclusion

Our main result shows that on general graphs any approximation algorithm for the Minimum Dominating Tree problem yields the same approximation result for the Group Steiner Tree problem and vice versa. This result might give another perspective and maybe shed some light on the Group Steiner Tree problem.

We remark, however, that the two problems are not equivalent. A good example is when the input graph is a tree, GST is known to be as hard to approximate as SC even in this case while MDT is trivially solvable on trees. Thus, there is also a place to studying each of the problems on its own for particular families of graphs.
Fig. 5. From left to right a) An instance of the Hamiltonian Path problem. b) The corresponded Minimum Dominating Path instance, original edges have zero weight, dashed edges have infinite weight.

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