Some special Euler sums and $\zeta^*(r + 2, \{2\}^n)$

Kwang-Wu Chen  
Department of Mathematics, University of Taipei  
No. 1, Ai-Guo West Road, Taipei 10048, Taiwan

Minking Eie  
Department of Mathematics, National Chung Cheng University  
168 University Rd., Minhsiung, Chiayi 62102, Taiwan

Abstract  
In this paper, we investigate the Euler sums

$$G_{n+2}(p, q) = \sum_{1 \leq k_1 < k_2 < \cdots < k_{p+1}} \frac{1}{k_1 k_2 \cdots k_p k_{p+1}} \sum_{1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_q \leq k_{p+1}} \frac{1}{\ell_1 \ell_2 \cdots \ell_q}.$$  

We give another two representations, a reflection formula, and some other properties. Then we use these results to calculate $\zeta^*(r + 2, \{2\}^n)$, for $r = 0, 1, 2$, as our applications.

Keywords: Euler sums, multiple zeta values, multiple zeta star values  
2010 MSC: 11M32, 11M41, 33B15

1. Introduction  
The multiple zeta values (MZVs) and the multiple zeta-star values (MZSVs) are defined by [9, 11, 12, 13]

$$\zeta(\alpha_1, \alpha_2, \ldots, \alpha_r) = \sum_{1 \leq k_1 < k_2 < \cdots < k_r} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r}$$

and

$$\zeta^*(\alpha_1, \alpha_2, \ldots, \alpha_r) = \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r}$$

with positive integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ and $\alpha_r \geq 2$ for the sake of convergence. The numbers $r$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ are the depth and weight of $\zeta(\alpha)$. For our convenience, we let $\{a\}^k$ be $k$ repetitions of $a$, for example, $\zeta(\{2\}^3) = \zeta(2, 2, 2)$.

MZVs of length one and two were already known to Euler. A systematic study of MZVs began in the early 1990s with the works of Hoffman [9] and Zagier [13]. Thereafter these
numbers have emerged in several mathematical areas including algebraic geometry, Lie group theory, advanced algebra, and combinatorics. See [8, 15] for introductory reviews.

The finite MZVs and MZSVs are defined as follows.

\[ \zeta_n(\alpha_1, \alpha_2, \ldots, \alpha_r) = \sum_{1 \leq k_1 < k_2 < \cdots < k_r \leq n} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r}, \]

\[ \zeta^*(\alpha_1, \alpha_2, \ldots, \alpha_r) = \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r \leq n} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r}. \]

The generalized harmonic numbers \( H_n^{(s)} \) of order \( s \) which are defined by \( H_n^{(s)} = \sum_{j=1}^{n} j^{-s} \). It is known that (ref. [2, Lemma 1])

\[ \zeta_n(\{1\}^m) = P_m(H_n^{(1)}, -H_n^{(2)}, \ldots, (-1)^{m+1}H_n^{(m)}), \]

\[ \zeta^*_n(\{1\}^m) = P_m(H_n^{(1)}, H_n^{(2)}, \ldots, H_n^{(m)}), \]

where the modified Bell polynomials \( P_m(x_1, x_2, \ldots, x_m) \) are defined by [2, 6, 7]

\[ \exp \left( \sum_{k=1}^{\infty} \frac{x_k}{k} z^k \right) = \sum_{m=0}^{\infty} P_m(x_1, x_2, \ldots, x_m) z^m. \]

Recently Choi [5] and Hoffman [10] investigated some special cases of the following Euler sums

\[ \sum_{m=1}^{\infty} \frac{\zeta_m(\{1\}^p) \zeta^*_m(\{1\}^q)}{m^{\alpha_1}(m+1)^{\alpha_2} \cdots (m+r-1)^{\alpha_r}}. \]

In this paper, we consider the little different Euler sums:

\[ G_{n+2}(p, q) = \sum_{1 \leq k_1 < k_2 < \cdots < k_{p+1}} \frac{1}{k_1 k_2 \cdots k_p k_{p+1}} \sum_{1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_q \leq k_{p+1}} \frac{1}{\ell_1 \ell_2 \cdots \ell_q}, \]

where \( n, p, q \) are nonnegative integers. It is noted that \( G_{n+2}(p, 0) = \zeta(\{1\}^p, n + 2) \) and \( G_{n+2}(0, q) = \zeta^*(\{1\}^q, n + 2) \).

This Euler sums have the other two representations:

**Theorem A.**

\[ G_{n+2}(p, q) = \sum_{r=p+1}^{p+q+1} \binom{r-1}{p} \sum_{|\alpha|=p+q+1} \zeta(\alpha_1, \ldots, \alpha_r + n + 1) \]

\[ = \frac{1}{p!q!n!} \int_{E_2} \left( \log \frac{1}{1-t_1} \right)^p \left( \log \frac{1}{1-t_2} \right)^q \left( \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1-t_1)t_2}. \]

The function \( G_{n+2}(p, q) \) has a reflection formula as follows.
Theorem B. For a pair of positive integers $p$, $q$, and an integer $k \geq 0$, we have

$$G_{k+3}(p-1, q) + (-1)^k G_{k+3}(q-1, p) = \sum_{a+b=k} (-1)^b \zeta(\{1\}^{p-1}, a+2) \zeta(\{1\}^{q-1}, b+2).$$

We evaluate $\zeta^*(r+2, \{2\}^m)$, for $r = 0, 1, \text{and} 2$, as applications of $G_{n+2}(p, q)$. We show that the generating function of $\zeta^*(r+2, \{2\}^m)$ is

$$\sum_{k=1}^{\infty} \frac{1}{k^r} \frac{\Gamma(k+x)\Gamma(k-x)}{\Gamma(k)^2}.$$

From this generating function we have

$$\zeta^*(r+2, \{2\}^m) = \sum_{p+q=2m, a+b=r} (-1)^{q+b} \binom{p+b}{p} G_{q+2}(p+b, a).$$

Then we get a sum formula which was first appeared in [11, Theorem 2] and then Zagier regained it in [14].

$$\sum_{a+b=n} (2 + \delta_{0a}) \zeta^*(\{2\}^a, 3, \{2\}^b) = 2(2n+2) \left(1 - \frac{1}{2^{2n+2}}\right) \zeta(2n+3).$$

Our paper is organized as follows. In Section 2, we present some basic properties of $G_{n+2}(p, q)$. We investigate the generating function of $\zeta^*(r+2, \{2\}^m)$ in Section 3. When we calculate the values of $G_{n+2}(p, q)$ we need some sum formulas of MZVs and MZSVs of height one. We write these sum formulas in Section 4. In order to evaluate $\zeta^*(r+2, \{2\}^m)$ we calculate some sum formulas of $G_{n+2}(p, q)$ in Section 5. In the last section we evaluate the values of $\zeta^*(r+2, \{2\}^n)$, for $r = 0, 1, 2$, as our applications.

2. Properties of $G_{n+2}(p, q)$

In this section we will give some interesting properties of $G_{n+2}(p, q)$. We need a proposition in [8] to transform a sum of MVZs to a integral representation.

Proposition 2.1. [8, Proposition 6.5.1]

$$\sum_{|\alpha|=m} \zeta(\{1\}^p, \alpha_1, \alpha_2, \ldots, \alpha_q + n) = \frac{1}{p!(q-1)!(m-q)!(n-1)!} \times \int_{E_2} \left(\log \frac{1}{1-t_1}\right)^p \left(\log \frac{t_2}{t_1}\right)^{m-q} \left(\log \frac{1-t_1}{1-t_2}\right)^q \left(\log \frac{1}{t_2}\right)^{n-1} dt_1 dt_2.$$

The following we give two other representations of $G_{n+2}(p, q)$:
Theorem 2.2.

\[ G_{n+2}(p,q) = \sum_{r=p+1}^{p+q+1} \binom{r-1}{p} \sum_{|\alpha|=p+q+1} \zeta(\alpha_1, \ldots, \alpha_r + n + 1) \] (2)

\[ = \frac{1}{p!q!n!} \int_{E_2} \left( \log \frac{1}{1-t_1} \right)^p \left( \log \frac{1}{1-t_2} \right)^q \left( \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1-t_1)t_2}. \] (3)

Proof. Since \( G_{n+2}(p,q) \) is a product of a multiple zeta value and a multiple zeta-star value, we use the shuffle product relation (ref. [4]) and then we get

\[ G_{n+2}(p,q) = \sum_{r=p+1}^{p+q+1} \binom{r-1}{p} \sum_{|\alpha|=p+q+1} \zeta(\alpha_1, \ldots, \alpha_r + n + 1). \]

Using Proposition 2.1 we can transform this summation to an integral representation

\[ G_{n+2}(p,q) = \sum_{r=p+1}^{p+q+1} \binom{r-1}{p} \frac{1}{(r-1)!(p+q-r+1)!n!} \int_{E_2} \left( \log \frac{t_2}{t_1} \right)^{p+q-r+1} \left( \log \frac{1-t_1}{1-t_2} \right)^{r-1} \left( \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1-t_1)t_2}. \]

Using the binomial theorem we have

\[ G_{n+2}(p,q) = \frac{1}{p!q!n!} \int_{E_2} \left( \log \frac{t_2}{t_1} + \log \frac{1-t_1}{1-t_2} \right)^q \left( \log \frac{1-t_1}{1-t_2} \right)^p \left( \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1-t_1)t_2}. \]

We change the variables (ref. [3])

\[ \frac{1-t_1}{1-t_2} = \frac{1}{1-u_1} \quad \text{and} \quad \frac{1}{t_2} = \frac{u_2}{u_1}, \]

then we get the final desired form. \( \square \)

Therefore we can give a new integral representation of \( \zeta^*(\{1\}^q, n+2) \).

Corollary 2.3. Let \( q \) and \( n \) be nonnegative integers. Then

\[ \zeta^*(\{1\}^q, n+2) = \sum_{r=1}^{q+1} \sum_{|\alpha|=q+1} \zeta(\alpha_1, \ldots, \alpha_r + n + 1) \] (4)

\[ = \frac{1}{q!n!} \int_{E_2} \left( \log \frac{1}{1-t_1} \right)^q \left( \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1-t_1)t_2}. \] (5)

The function \( G_{n+2}(p,q) \) has a reflection formula as follows.

Proposition 2.4. For a pair of positive integers \( p, q \), and an integer \( k \geq 0 \), we have

\[ G_{k+3}(p-1,q) + (-1)^k G_{k+3}(q-1,p) = \sum_{a+b=k} (-1)^b \zeta(\{1\}^{p-1}, a+2) \zeta(\{1\}^{q-1}, b+2). \] (6)
Proof. Consider the double integral
\[
\frac{1}{k!p!q!} \int_0^1 \int_0^1 \left( \log \frac{u}{t} \right)^k \left( \log \frac{1}{1-t} \right)^p \left( \log \frac{1}{1-u} \right)^q \frac{dt}{t} \frac{du}{u}.
\]
Replace the factor
\[
\left( \log \frac{u}{t} \right)^k = \left( \log \frac{1}{t} - \log \frac{1}{u} \right)^k
\]
by its binomial expansion
\[
\sum_{a+b=k} \frac{k!}{a!b!} \left( \log \frac{1}{t} \right)^a \left( \log \frac{1}{u} \right)^b
\]
we see immediate that its value is given by
\[
\sum_{a+b=k} (-1)^b \zeta(\{1\}^{p-1}, a + 2) \zeta(\{1\}^{q-1}, b + 2).
\]
Now we decompose the square \([0, 1] \times [0, 1]\) into union of two simplices
\[
D_1 : 0 < t < u < 1 \quad \text{and} \quad D_2 : 0 < u < t < 1.
\]
On \(D_1 : 0 < t < u < 1\), the corresponding integral
\[
\frac{1}{p!q!k!} \int_{0 < t < u < 1} \left( \log \frac{1}{1-t} \right)^{p-1} \left( \log \frac{1}{1-u} \right)^q \left( \log \frac{u}{t} \right)^k \frac{dt}{t} \frac{du}{u}
\]
can be rewritten as
\[
\frac{1}{(p-1)!q!(k+1)!} \int_{0 < t < u < 1} \left( \log \frac{1}{1-t} \right)^{p-1} \left( \log \frac{1}{1-u} \right)^q \left( \log \frac{u}{t} \right)^{k+1} \frac{dt}{1-t} \frac{du}{u}
\]
which is equal to \(G_{k+3}(p-1, q)\). In the same manner, the corresponding integral on \(D_2\) is \((-1)^k G_{k+3}(q-1, p)\). \(\square\)

Furthermore, the values of \(G_2(p, q)\) can be easily calculated.

**Proposition 2.5.**

\[
G_2(p, q) = \binom{p + q + 1}{q} \zeta(p + q + 2) = G_2(q - 1, p + 1).
\]  \hspace{1cm} (7)

**Proof.** Since \(G_2(p, q)\) have the following integral representation
\[
\frac{1}{p!q!} \int_{E_2} \left( \log \frac{1}{1-t_1} \right)^p \left( \log \frac{1}{1-t_2} \right)^q \frac{dt_1 dt_2}{(1-t_1)t_2}.
\]
With \( \log \frac{1}{1-t_2} = \log \frac{1-t_1}{1-t_2} + \log \frac{1}{1-t_1} \) we use the binomial theorem to decompose the second factor in the above representation, then we have
\[
\sum_{j=0}^{q} \binom{p+j}{p} \zeta(\{1\}^{p+q}, 2).
\]
Since
\[
\sum_{j=0}^{q} \binom{p+j}{p} = \binom{p+q+1}{q}
\]
and using the dual theorem
\[
\zeta(\{1\}^{p+q}, 2) = \zeta(p + q + 2)
\]
we conclude the first identity. Since \( \binom{p+q+1}{q} = \binom{p+q+1}{p+1} \) we get the second identity.

There is an easy result:
\[
\zeta^*(\{1\}^q, 2) = G_2(0, q) = (q + 1)\zeta(q + 2).
\]

3. The generating function of \( \zeta^*(r + 2, \{2\}^m) \)

Let \( r \) and \( m \) be nonnegative integers. We define the generating function of \( \zeta^*(r + 2, \{2\}^m) \) to be
\[
G_r^*(x) := \sum_{m=0}^{\infty} \zeta^*(r + 2, \{2\}^m) x^{2m}.
\]
This generating function can be expressed as the following.

**Proposition 3.1.**

\[
G_r^*(x) = \sum_{k=1}^{\infty} \frac{1}{k^{r+2}} \frac{\Gamma(k+x)\Gamma(k-x)}{\Gamma(k)^2}.
\]

**Proof.** From the well-known formula
\[
\frac{1}{\Gamma(s+1)} = e^{\gamma s} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{n} \right) e^{-\frac{s}{n}},
\]
we have
\[
\frac{\Gamma(k+x)\Gamma(k-x)}{\Gamma(k)^2} = \frac{k^2}{k^2 - x^2} \frac{\Gamma(k+x+1)\Gamma(k-x+1)}{\Gamma(k+1)^2} = \frac{k^2}{k^2 - x^2} \prod_{n=1}^{\infty} \frac{(1 + \frac{k+x}{n})(1 + \frac{k-x}{n})}{(1 + \frac{k}{n})^2} = \frac{1}{1 - \left( \frac{x}{n} \right)^2} \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{(n+k)^2} \right) = \prod_{n \geq k} \left( 1 - \frac{x^2}{n^2} \right)^{-1}.
\]
Since
\[
\sum_{m=0}^{\infty} \zeta^*(r + 2, \{2\}^m)x^{2m} = \sum_{k=1}^{\infty} \frac{1}{k^{r+2}} \prod_{n \geq k} \left(1 - \frac{x^2}{n^2}\right)^{-1} = \sum_{k=1}^{\infty} \frac{\Gamma(k + x)\Gamma(k - x)}{k^{r+2}\Gamma(k^2)}.
\]
Thus we have this conclusion.

This identity can give us the integral representation of \(\zeta^*(r + 2, \{2\}^m)\).

**Theorem 3.2.** For integers \(n, r \geq 0\), we have
\[
\frac{1}{r!n!} \int_{E_2} \left(\log \frac{1 - t_1}{1 - t_2}\right)^r \left(\log \frac{1 - t_1}{t_1} - \log \frac{t_2}{t_1}\right)^n \frac{dt_1dt_2}{(1 - t_1)t_2} = \begin{cases} 
\zeta^*(r + 2, \{2\}^m), & \text{if } n = 2m, \\
0, & \text{if } n = 2m + 1,
\end{cases}
\]

where \(E_2 = \{(t_1, t_2) \in \mathbb{R}^2 | 0 < t_1 < t_2 < 1\}\).

**Proof.** The double integrals come from the differentiation of the following function \(H_r(x)\) with parameter \(x > -1\), defined by
\[
H_r(x) = \frac{1}{r!} \int_{E_2} \left(\log \frac{1 - t_1}{1 - t_2}\right)^r \left(\frac{t_1}{t_2}\right)^x (1 - t_1)^{-x} \frac{dt_1dt_2}{(1 - t_1)t_2}.
\]
Indeed, we have
\[
\frac{1}{m!} \left(\frac{d}{dx}\right)^m H_r(x)\bigg|_{x=0} = \frac{1}{r!m!} \int_{E_2} \left(\log \frac{1 - t_1}{1 - t_2}\right)^r \left(\log \frac{1 - t_1}{t_1} - \log \frac{t_2}{t_1}\right)^m \frac{dt_1dt_2}{(1 - t_1)t_2},
\]
since
\[
\frac{d}{dx} \left(\frac{t_1}{t_2}\right)^x (1 - t_1)^{-x} = \left(\log \frac{1 - t_1}{t_1} - \log \frac{t_2}{t_1}\right) \left(\frac{t_1}{t_2}\right)^x (1 - t_1)^{-x}.
\]
Under the change of variables \(u_1 = 1 - t_2\) and \(u_2 = 1 - t_1\), \(H_r(x)\) is transformed into
\[
\frac{1}{r!} \int_{E_2} \left(\log \frac{u_2}{u_1}\right)^r \left(\frac{1 - u_2}{1 - u_1}\right)^x u_2^{-x} \frac{du_1du_2}{(1 - u_1)u_2}
\]
and it can be evaluated as
\[
\sum_{k=1}^{\infty} \frac{1}{k^{r+2}} \frac{\Gamma(k + x)\Gamma(k - x)}{\Gamma(k^2)}.
\]
Along with our previous results, we conclude that
\[
G^*_r(x) = H_r(x) = \frac{1}{r!} \int_{E_2} \left(\log \frac{1 - t_1}{1 - t_2}\right)^r \left(\frac{t_1}{t_2}\right)^x (1 - t_1)^{-x} \frac{dt_1dt_2}{(1 - t_1)t_2}.
\]
Take the coefficients of \(x^{2m}\) of both sides, we obtain our double integral representation of \(\zeta^*(r + 2, \{2\}^m)\). Also note that \(H(x)\) is an even function of \(x\), so that its coefficients of odd powers vanish.
We can express $\zeta^*(r + 2, \{2\}^m)$ as a sum of $G_{n+2}(p, q)$.

**Proposition 3.3.**

$$\zeta^*(r + 2, \{2\}^m) = \sum_{p+q=2m \atop a+b=2r} (-1)^{q+b} \binom{p+b}{p} G_{q+2}(p+b, a).$$

**Proof.** The integral representation of $\zeta^*(r + 2, \{2\}^m)$ in Theorem 3.2 is

$$\zeta^*(r + 2, \{2\}^m) = \frac{1}{r!(2m)!} \int_{E_2} \left( \log \frac{1-t_1}{1-t_2} \right)^r \left( \log \frac{1}{1-t_1} - \log \frac{t_2}{t_1} \right)^{2m} \frac{dt_1 dt_2}{(1-t_1)t_2}.$$  

We use the binomial theorem to decompose the first factor in the above integral with

$$\log \frac{1-t_1}{1-t_2} = \log \frac{1}{1-t_2} - \log \frac{1}{1-t_1}.$$  

Then from the integral representation of $G_{n+2}(p, q)$ in Theorem , we get the conclusion. $\square$

**4. Sums of multiple zeta values of height one**

**Proposition 4.1.** For integers $n \geq 0$ and $s, r \geq 2$, we have

$$\sum_{a+b=n} \zeta^*(\{s\}^a) \zeta(sb + r) = \sum_{a+b=n} \zeta^*(\{s\}^a, r, \{s\}^b).$$

**Proof.** Let $F(x)$ be the generating function of $\zeta^*(\{s\}^a, r, \{s\}^b)$, where $s, r \geq 2$, that is

$$F(x) = \sum_{a,b \geq 0} \zeta^*(\{s\}^a, r, \{s\}^b) x^{(a+b)s}.$$  

From this definition we have

$$F(x) = \sum_{n=1}^{\infty} \prod_{0<k\leq n} \left(1 - \frac{x^s}{k^s}\right)^{-1} \cdot \frac{1}{n^r} \prod_{\ell \geq n} \left(1 - \frac{x^s}{\ell^s}\right)^{-1}$$  

$$= \prod_{k=1}^{\infty} \left(1 - \frac{x^s}{k^s}\right)^{-1} \cdot \sum_{n=1}^{\infty} \frac{1}{n^r} \left(1 - \frac{x^s}{n^s}\right)^{-1}$$  

$$= \sum_{n=0}^{\infty} \zeta^*(\{s\}^n) x^{ns} \cdot \sum_{k=0}^{\infty} \zeta(sk + r) x^{ks}$$  

$$= \sum_{a,b \geq 0} \zeta^*(\{s\}^a) \zeta(sb + r) x^{(a+b)s}.$$  

$\square$
We write $\zeta(\{1\}^r, n + 2 - r)$ in its double integral form, then

$$
\sum_{r=0}^{n} (-1)^{r+n} \zeta(\{1\}^r, n + 2 - r) = \frac{1}{n!} \int_{E_2} \left( \log \frac{1}{1 - t_1} - \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1 - t_1)t_2}.
$$

Using Theorem 3.2 we have the following proposition.

**Proposition 4.2.** Let $n$ be a nonnegative integer. Then

$$
\sum_{r=0}^{n} (-1)^{r+n} \zeta(\{1\}^r, n + 2 - r) = \begin{cases} 
\zeta^*(\{2\}^{m+1}), & \text{if } n = 2m, \\
0, & \text{if } n = 2m + 1.
\end{cases} \quad (15)
$$

**Proposition 4.3.** Let $n$ be a nonnegative integer. Then

$$
\sum_{r=0}^{n} (-1)^{r+n} (r + 1) \zeta(\{1\}^r, n + 2 - r) = \begin{cases} 
(m + 1) \zeta^*(\{2\}^{m+2}), & \text{if } n = 2m + 1, \\
\sum_{a+b=m} \zeta^*(\{2\}^a, 3, \{2\}^b), & \text{if } n = 2m.
\end{cases} \quad (16)
$$

**Proof.** The alternating sum has the integral representation

$$
\frac{1}{n!} \int_{E_2} \left( \log \frac{1}{1 - t_1} \right) \left( \log \frac{1}{1 - t_1} - \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1 - t_1)t_2}.
$$

Consider the following function

$$
M(x, y) = \int_{E_2} \left( \frac{t_1}{t_2} \right)^x (1 - t_1)^{-x-y} \frac{dt_1 dt_2}{(1 - t_1)t_2},
$$

It can be seen that

$$
\frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n \left( \frac{\partial}{\partial y} \right) M(x, y) \bigg|_{x=0, y=0} = \frac{1}{n!} \int_{E_2} \left( \log \frac{1}{1 - t_1} \right) \left( \log \frac{1}{1 - t_1} - \log \frac{t_2}{t_1} \right)^n \frac{dt_1 dt_2}{(1 - t_1)t_2}.
$$

With the change of variables $t_1 = 1 - u_2$ and $t_2 = 1 - u_1$, we have the dual integral representation

$$
\int_{E_2} \left( \frac{1 - u_2}{1 - u_1} \right)^x u_2^{-x-y} \frac{du_1 du_2}{(1 - u_1)u_2}.
$$

It can be evaluated as

$$
M(x, y) = \sum_{k=1}^{\infty} \frac{\Gamma(k + x)\Gamma(k - x - y)}{\Gamma(k + 1)\Gamma(k + 1 - y)}.
$$

So that

$$
\frac{\partial}{\partial y} M(x, y) \bigg|_{y=0} = \sum_{k=1}^{\infty} \frac{\Gamma(k + x)\Gamma(k - x)}{\Gamma(k + 1)^2} \cdot [\psi(k + 1) - \psi(k - x)],
$$

where $\psi(x)$ is the digamma function

$$
\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.
$$
By Eq. (10) and for a positive integer \( p \)
\[
\left( \frac{d}{dx} \right)^p \psi(x) = (-1)^{p-1}p! \zeta(p + 1; x),
\]
we have
\[
\frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n \left( \frac{\partial}{\partial y} \right) M(x, y) \bigg|_{x=0,y=0} = \sum_{k=1}^{m+1} \zeta^*(\{2\}^r) \zeta(2m + 4 - 2r), \text{ if } n = 2m + 1,
\]
\[
\sum_{r=0}^{m} \zeta^*(\{2\}^r) \zeta(2m + 3 - 2r), \text{ if } n = 2m.
\]
Applying Proposition 4.1 we conclude the results.

We use Eq. (11) with \( n = 2m + 1 \) and decompose \( \log \left( \frac{1-t_1}{1-t_2} \right) = \log \frac{1}{1-t_2} - \log \frac{1}{1-t_1} \), we have
\[
\frac{1}{(2n+1)!} \int_{E_2} \left( \log \frac{1}{1-t_1} \right) \left( \log \frac{1}{1-t_1} - \log \frac{t_2}{t_1} \right)^{2n+1} \frac{dt_1 dt_2}{(1-t_1)t_2} = \frac{1}{(2n+1)!} \int_{E_2} \left( \log \frac{1}{1-t_2} \right) \left( \log \frac{1}{1-t_1} - \log \frac{t_2}{t_1} \right)^{2n+1} \frac{dt_1 dt_2}{(1-t_1)t_2}.
\]

In terms of multiple zeta values it is
\[
\sum_{p+q=2n+1} (-1)^q(p+1) \zeta(\{1\}^{p+1}, q + 2) = \sum_{p+q=2n+1} (-1)^q G_{q+2}(p, 1).
\]
The alternating sum in the left is equal to \( (n+1) \zeta^*(\{2\}^{n+2}) \) while the sum in the right is equal to
\[
\sum_{p+q=2n+1} \zeta^*(\{1\}^{p+1}, q + 2) - \sum_{0 \leq a+b \leq 2n} (-1)^b \zeta(a+2) \zeta(\{1\}^{2n-a-b}, b + 2)
\]
\[
= \sum_{p+q=2n+2} \zeta^*(\{1\}^{p}, q + 2) - \sum_{c+d=n+1} \zeta^*(\{2\}^c) \zeta(2d + 2)
\]
\[
= \sum_{p+q=2n+2} \zeta^*(\{1\}^{p}, q + 2) - (n+2) \zeta^*(\{2\}^{n+2}).
\]
This leads to
\[
\sum_{p+q=2n+2} \zeta^*(\{1\}^{p}, q + 2) = (2n+3) \zeta^*(\{2\}^{n+2}). \tag{17}
\]
Since
\[ \zeta^*(\{2\}^{n+2}) = 2 \left(1 - \frac{1}{2^{2n+3}}\right) \zeta(2n + 4), \]
we have
\[ \sum_{p+q=2n+2} \zeta^*(\{1\}^p, q + 2) = 2(2n + 3) \left(1 - \frac{1}{2^{2n+3}}\right) \zeta(2n + 4). \]
Hence we can conclude the above result as a proposition.

**Proposition 4.4.** For a nonnegative integer \( n \),
\[ \sum_{p+q=2n+2} \zeta^*(\{1\}^p, q + 2) = 2(2n + 3) \left(1 - \frac{1}{2^{2n+3}}\right) \zeta(2n + 4). \quad (18) \]

As a matter of fact, this result is just a special case of [1, Theorem 1] which was proved by Aoki and Oho.
\[ \sum_{k \in I_0(k,s)} \zeta^*(k) = 2 \left(\frac{k-1}{2s-1}\right)(1 - 2^{1-k})\zeta(k), \quad (19) \]
where \( I_0(k,s) \) is the set of admissible multi-indices \( k = (k_1, k_2, \ldots, k_n) \) with weight \( k = k_1 + k_2 + \cdots + k_n \) and height \( s = \#\{i \mid k_i > 1\} \). We include this result as an application of the function \( G_{n+2}(p,q) \).

5. Sum formulas of \( G_{n+2}(p,q) \)

**Proposition 5.1.** Let \( n \) be a nonnegative integer. Then
\[ \sum_{p+q=2n} (-1)^q G_{q+2}(p,2) = \sum_{p+q=2n+1} G_{q+2}(1,p) - \sum_{a+b=n} \zeta(1,2a+3)\zeta^*(\{2\}^b). \]

**Proof.** By Proposition 2.4 and 2.5, the left hand side of the above identity is
\[
\sum_{p+q=2n} (-1)^q G_{q+2}(p,2) \\
= \sum_{q=0}^{2n} G_{2+q}(1,2n+1-q) - \sum_{q=1}^{2n} \sum_{a=0}^{q-1} (-1)^{q-2a} \zeta(1,a+2)\zeta(\{1\}^{2n-q}, q+1-a).
\]
After a suitable change the variable of indices and their orders, we can reach the following
\[
\sum_{q=0}^{2n} G_{2+q}(1,2n+1-q) - \sum_{\ell=0}^{2n-1} \sum_{r=0}^{\ell} (-1)^{r+\ell} \zeta(1,2n+1-\ell)\zeta(\{1\}^{r}, \ell + 2 - r).
\]
By Proposition 4.2 we have
\[
\sum_{p+q=2n} (-1)^q G_{q+2}(p,2) = \sum_{q=0}^{2n} G_{2+q}(1,2n+1-q) - \sum_{a=1}^{n} \zeta^*(\{2\}^a)\zeta(1,2n+3-2a).
\]
Since \( G_{2n+3}(1,0) = \zeta(1,2n+3) \), hence we complete the proof. \qed
Proposition 5.2.

\[
\sum_{p+q=2n} (-1)^q (p+1) G_{q+2}(p+1, 1) = \sum_{p+q=2n} (p+1) \zeta^*({\{1\}^{p+2}, q+2}) - \sum_{a+b=n} b \cdot \zeta(2a+2) \zeta^*({\{2\}^{b+1}}) - \sum_{a+b+c=n-1} \zeta^*({\{2\}^a, 3, \{2\}^b}) \zeta(2c+3).
\]

Proof. We first use Proposition 2.4 and 2.5 such that the sum of \(G_{q+2}(p+1, 1)\) in the left hand side of the above identity becomes

\[
\sum_{p+q=2n} (p+1) G_{q+2}(0, p+2) - \sum_{q=1}^{2n} \sum_{a=0}^{q-1} (-1)^{q-1-a} (2n+1-q) \zeta(a+2) \zeta({\{1\}^{2n+1-q}}, q+1-a).
\]

Since \(G_{q+2}(0, p+2) = \zeta^*({\{1\}^{p+2}}, q+2)\), we get the desired first factor in the right hand side of the above identity.

After a suitable change the variable of indices and their orders in the second factor, this factor can reach the following

\[
\sum_{\ell=0}^{2n-1} \zeta(2n+1-\ell) \sum_{r=0}^{\ell} (-1)^{\ell+r} (r+1) \zeta({\{1\}^{r+1}}, \ell+2-r).
\]

Using Proposition 4.3 we get the desired remaining terms.

Theorem 5.3. For a nonnegative integer \(n\), we have

\[
\sum_{p+q=2n+1} (p+1) \zeta^*({\{1\}^{p+1}}, q+2) - \sum_{p+q=2n+1} G_{q+2}(1, p)
\]

\[
= \binom{2n+4}{3} \zeta(2n+4) + \sum_{j=1}^{n} \sum_{|c_j|=2n+1-2j} (-1)^j \zeta(c_j+3, c_j+1, \ldots, c_j+2) W(c_j)
\]

with

\[
W(c_j) = W(c_{j0}, c_{j1}, \ldots, c_{jj}) = \binom{c_{j0}+3}{3} (c_{j1}+1) \cdots (c_{jj}+1).
\]

Proof. The difference in the left hand side has the integral representation

\[
\frac{1}{(2n+1)!} \int_{E_2} \left( \log \frac{1-t_1}{1-t_2} \right) \left( \log \frac{1-t_2}{1-t_1} + \log \frac{t_2}{t_1} \right) \frac{t_1^{2n+1}}{(1-t_1)t_2} dt_1 dt_2.
\]

So we begin with its generating function

\[
G(x, y) = \int_{E_2} \left( \frac{t_1}{t_2} \right)^x (1-t_1)^{-y} (1-t_2)^{x+y} \frac{t_1 dt_2}{(1-t_1)t_2}.
\]

Indeed, our integral is equal to

\[
\frac{1}{(2n+1)!} \left( \frac{\partial}{\partial x} \right)^{2n+1} G(x, y) \bigg|_{x=y=0}.
\]
We change the variables $t_1 = 1 - u_2$, $t_2 = 1 - u_1$, the dual form of $G(x, y)$ is given by
\[
\int_{E_2} \left(\frac{1 - u_2}{1 - u_1}\right)^x u_1^{x+y} u_2^{1-y} \frac{du_1 du_2}{(1 - u_1) u_2}
\]
and it can be evaluated as
\[
\sum_{k=1}^{\infty} \frac{1}{(k + x + y) \Gamma(k) \Gamma(k + 2x + 1)} \frac{\Gamma(k + x)^2}{\Gamma(k + 2x + 1)}
\]
As
\[
\left(-\frac{\partial}{\partial y}\right) G(x, y) \bigg|_{y=0} = \sum_{k=1}^{\infty} \frac{1}{(k + x)^2} \frac{\Gamma(k + x)^2}{\Gamma(k) \Gamma(k + 2x + 1)}
\]
so our evaluation is equivalent to find the coefficient of $x^{2n+1}$ of the above function up to the sign $(-1)^{2n+1}$. To do so we have to express the quotient of gamma functions as an infinite product through the infinite product formula of the gamma function. The procedure is
\[
\sum_{k=1}^{\infty} \frac{1}{(k + x)^2(2k + 2x)} \frac{\Gamma(k + x)^2}{\Gamma(k) \Gamma(k + 2x + 1)} = \sum_{k=1}^{\infty} \frac{k}{(k + x)^4} \prod_{v=1}^{\infty} \frac{(1 + k/v)(1 + (k + 2x)/v)}{(1 + (k + x)/v)^2} \frac{\Gamma(k + x + 1)^2}{\Gamma(k) \Gamma(k + 2x + 1)}
\]
\[
= \sum_{k=1}^{\infty} \frac{k}{(k + x)^4} \prod_{v=1}^{\infty} \frac{(1 + 2x/(v + k))}{(1 + x/(v + k))^2} \frac{\Gamma(k + x + 1)^2}{\Gamma(k) \Gamma(k + 2x + 1)}
\]
\[
= \sum_{k=1}^{\infty} \frac{k}{(k + x)^4} \prod_{v=1}^{\infty} \left\{1 - \frac{x^2}{(v + k)^2} \left(1 + \frac{x}{v + k}\right)^{-2}\right\}.
\]
From the final infinite product its coefficient of $(-x)^{2n+1}$ is given by
\[
\left(\binom{2n + 4}{3}\right) \zeta(2n + 4) + \sum_{j=1}^{n} (-1)^j \sum_{|c_j|=2n+1-2j} \zeta(c_{j0} + 3, c_{j1} + 2, \ldots, c_{jj} + 2) W(c_j).
\]

6. Evaluations of $\zeta^*(r + 2, \{2\}^n)$

In this section we give the evaluations of $\zeta^*(r + 2, \{2\}^n)$, for $r = 0, 1$, and 2 using the formula in Proposition 3.3.

Firstly, let $r = 0$. Then we have
\[
\zeta^*(2, \{2\}^n) = \sum_{p+q=2n} (-1)^q G_{q+2}(p, 0)
\]
\[
= \sum_{p+q=2n} (-1)^q \zeta(\{1\}^p, q + 2).
\]
This is exactly the even case in Proposition 4.2.

Secondly, let \( r = 1 \). Then we have

\[
\zeta^*(3, \{2\}^n) = H_0 + H_1,
\]

where

\[
H_0 = \sum_{p+q=2n} (-1)^{q+1}(p + 1)G_{q+2}(p + 1, 0), \quad \text{and}
\]

\[
H_1 = \sum_{p+q=2n} (-1)^q G_{q+2}(p, 1).
\]

Since \( G_{q+2}(p, 0) = \zeta(\{1\}^p, q + 2) \), \( H_0 \) is the negative of the even case in Proposition 4.3.

\[
H_0 = - \sum_{a+b=n} \zeta^*(\{2\}^a, 3, \{2\}^b).
\]

On the other hand, we use a similar method as the proof of Proposition 5.1 to treat \( H_1 \).

\[
H_1 = \sum_{q=0}^{2n} G_{q+2}(0, 2n + 1 - q) - \sum_{q=1}^{2n} \sum_{a=0}^{q-1} (-1)^{q-a} \zeta(a + 2) \zeta(\{1\}^{2n-q}, q + 1 - a).
\]

After a suitable change the variable of indices and their orders, we can reach the following

\[
\sum_{p+q=2n+1} \zeta^*(\{1\}^p, q + 2) - \sum_{\ell=0}^{2n-1} \sum_{r=0}^{\ell} (-1)^{\ell+r} \zeta(2n + 1 - \ell) \zeta(\{1\}^r, \ell + 2 - r).
\]

By Proposition 4.2 we have

\[
H_1 = \sum_{p+q=2n+1} \zeta^*(\{1\}^p, q + 2) - \sum_{a=0}^{n} \zeta^*(\{2\}^a) \zeta(2n + 3 - 2a).
\]

We use Eq. (19) to write the first term as a Riemann zeta value times a constant and apply Proposition 4.1 to the second term, then we have

\[
H_1 = 2(2n + 2) \left( 1 - \frac{1}{2^{2n+2}} \right) \zeta(2n + 3) - \sum_{a+b=n} \zeta^*(\{2\}^a, 3, \{2\}^b).
\]

Hence we conclude the value of \( \zeta^*(3, \{2\}^n) \) as

\[
\zeta^*(3, \{2\}^n) = 2(2n + 2) \left( 1 - \frac{1}{2^{2n+2}} \right) \zeta(2n + 3) - 2 \sum_{a+b=n} \zeta^*(\{2\}^a, 3, \{2\}^b).
\]

Note that the form \( \zeta^*(3, \{2\}^n) \) is the special case in the last sum with \( a = 0, b = n \). Thus we can collect them together and give a beautiful sum formula which was first appeared in [11, Theorem 2] and then Zagier regained it in [14].

\[
\sum_{a+b=n} (2 + \delta_{0a}) \zeta^*(\{2\}^a, 3, \{2\}^b) = 2(2n + 2) \left( 1 - \frac{1}{2^{2n+2}} \right) \zeta(2n + 3). \quad (20)
\]
Thirdly, let \( r = 2 \). Then we have

\[
\zeta^*(4, \{2\}^n) = A_0 - A_1 + A_2,
\]

where

\[
A_0 = \sum_{p+q=2n} (-1)^q \binom{p+2}{2} \zeta(\{1\}^{p+2}, q+2),
\]

\[
A_1 = \sum_{p+q=2n} (-1)^q(p+1)G_{q+2}(p+1, 1), \quad \text{and}
\]

\[
A_2 = \sum_{p+q=2n} (-1)^qG_{q+2}(p, 2).
\]

Proposition 5.2 gives an evaluation of \( A_1 \) and makes it to be

\[
A_1 = \sum_{p+q=2n+1} (p+1)\zeta^*(\{1\}^{p+2}, q+2)
\]

\[-\sum_{a+b=n} b \cdot \zeta(2a+2)\zeta^*({2}^{b+1}) - \sum_{a+b+c=n-1} \zeta^*({2}^a, 3, {2}^b)\zeta(2c+3).
\]

We decompose the first term as

\[
\sum_{p+q=2n} (p+1)\zeta^*(\{1\}^{p+2}, q+2)
\]

\[
= \sum_{p+q=2n+1} (p+1)\zeta^*(\{1\}^{p+1}, q+2) - \sum_{p+q=2n+2} \zeta^*({1}^p, q+2) + \zeta(2n+4).
\]

Therefore \( A_1 \) becomes

\[
A_1 = \sum_{p+q=2n+1} (p+1)\zeta^*(\{1\}^{p+1}, q+2) - \sum_{p+q=2n+2} \zeta^*({1}^p, q+2) + \zeta(2n+4)
\]

\[-\sum_{a+b=n} b \cdot \zeta(2a+2)\zeta^*({2}^{b+1}) - \sum_{a+b+c=n-1} \zeta^*({2}^a, 3, {2}^b)\zeta(2c+3).
\]

The value of \( A_2 \) is calculated in Proposition 5.1:

\[
A_2 = \sum_{p+q=2n+1} G_{q+2}(1, p) - \sum_{a+b=n} \zeta(1, 2a+3)\zeta^*({2}^b).
\]

We put the first terms of \( A_1 \) and \( A_2 \) together, we have

\[
\sum_{p+q=2n+1} G_{q+2}(1, p) - \sum_{p+q=2n+1} (p+1)\zeta^*({1}^{p+1}, q+2).
\]
This value is just the left hand side of the identity in Theorem 5.3. Applying the result of Theorem 5.3 and Proposition 4.4, we get the final form of \( \zeta^*(4, \{2\}^n) \) as

\[
\zeta^*(4, \{2\}^n) = -\binom{2n+4}{3} \zeta(2n+4) + 2(2n+3) \left( 1 - \frac{1}{2^{2n+3}} \right) \zeta(2n+4) - \zeta(2n+4) \\
+ \sum_{j=1}^{n} (-1)^{j+1} \sum_{|c_j|=2n+1-2j} \zeta(c_{j0}+3, c_{j1}+2, \ldots, c_{jj}+2) W(c_j) \\
+ \sum_{p+q=2n} (-1)^q \binom{p+2}{2} \zeta(\{1\}^{p+2}, q+2) - \sum_{a+b=n} \zeta(1, 2a+3) \zeta^*(\{2\}^b) \\
+ \sum_{a+b=n} b \zeta(2a+2) \zeta^*(\{2\}^{b+1}) + \sum_{a+b+c=n-1} \zeta^*(\{2\}^a, 3, \{2\}^b) \zeta(2c+3).
\]

Acknowledgements

Chen was funded by the Ministry of Science and Technology, Taiwan, Republic of China, through grant MOST 106-2115-M-845-001.

References

[1] T. Aoki, Y. Ohno, Sum relations for multiple zeta values and connection formulas for the Gauss hypergeometric functions, *Publ. RIMS, Kyoto Univ.*, 41 (2005), 329–337.

[2] K.-W. Chen, Generalized harmonic numbers and Euler sums, *Int. J. Number Theory*, 13 (2) (2017), 513–528. DOI:10.1142/S1793042116500883.

[3] K.-W. Chen, C.-L. Chung, M. Eie, Sum formulas and duality theorems of multiple zeta values, *J. Number Theory*, 158 (2016), 33–53.

[4] K.-W. Chen, C.-L. Chung, M. Eie, Combinatorial implications of decomposition theorems of multiple zeta values, *J. Comb. Number Theory*, 7 (2) (2016), 111–129.

[5] J. Choi, Summation formulas involving binomial coefficients, harmonic numbers, and generalized harmonic numbers, *Abst. Appl. Anal.*, 2014 (2014), Article ID 501906, 10 pages.

[6] M.-A. Coppo, B. Candelpergher, The Arakawa-Kaneko zeta function, *Ramanujan J.*, 22 (2) (2010), 153–162. DOI: 10.1007/S11139-009-9205-X.

[7] M.-A. Coppo, B. Candelpergher, Inverse binomial series and values of Arakawa-Kaneko zeta functions, *J. Number Theory*, 150 (2015), 98–119.
[8] M. Eie, *Topics in Number Theory*, Monographs in Number Theory, vol. 2, World Scientific, Singapore, 2009.

[9] M. E. Hoffman, Multiple harmonic series, *Pac. J. Math.*, 152 (1992), 275–290.

[10] M. E. Hoffman, Harmonic-number summation identities, symmetric functions, and multiple zeta values, *Ramanujan J.*, 42 (2017), 501–526.

[11] K. Ihara, J. Kajikawa, Y. Ohno, J.-I. Okuda, Multiple zeta values vs. multiple zeta-star values, *J. Algebra*, 332 (2011), 187–208.

[12] S. Muneta, Algebraic setup of non-strict multiple zeta values, *Acta Arith.*, 136 (2009), 7–18.

[13] D. Zagier, Values of zeta functions and their applications, In: *First European Congress of Mathematics, vol. II, Paris, 1992*, Progr. Math. 120. Birkhuser, Basel 1994, (1992), 497–512.

[14] D. Zagier, Evaluation of the multiple zeta value \( \zeta(2, \ldots, 2, 3, 2, \ldots, 2) \), *Annals of Mathematics*, 175 (2012), 977-1000.

[15] J. Zhao, *Multiple Zeta Functions, Multiple Polylogarithms and Their Special Values*, Series on Number Theory and Its Applications, World Scientific Publishing Co. Pte. Ltd, 2016.