Distance-2 Irregular chromatic numbers for some graphs

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Abstract. Let G be a graph and let c: V(G) → {1, 2, …, k} be a coloring of the vertices of G for some positive integer k (where adjacent vertices may be colored the same). The color code of a vertex v of G (with respect to c) is the ordered (k+1)-tuple code (v) = (a₀, a₁, a₂, …, aₖ) where a₀ is the color assigned to v and for 1 ≤ i ≤ k, aᵢ is the number of the vertices of G adjacent to that are colored i. The coloring c is called recognizable if distinct vertices have distinct color codes and the recognition number of G is the minimum positive integer k for which G has a recognizable k-coloring. In this paper we introduced a new variation of above parameter namely distance-2 irregular coloring. We initiate a study of this parameter and also find the distance 2-irregular chromatic number of some standard graphs.

Keywords: Irregular coloring, chromatic number, Path, Cycle.

1. Introduction.

Graph coloring is one of the most important areas of research in graph theory. The vertex coloring of a graph is coloring the vertices of the graph in such a way that adjacent vertices have different colors. The distance-coloring of graphs was introduced in 1969. A distance-coloring relative to distance p of a graph G(V,E) is a mapping of V in a set of colors in such a way that any two vertices of G of distance not greater than p have distinct colors. G.Chartrand and P.Zhang was introduced to recognize the vertices of a graph. Let G be a graph and let c: V(G) → {1, 2, …, k} be a coloring of the vertices of G for some positive integer k (where adjacent vertices may be colored the same). The color code of a vertex v of G (with respect to c) is the ordered (k+1)-tuple code (v) = (a₀, a₁, a₂, …, aₖ) where a₀ is the color assigned to v and for 1 ≤ i ≤ k, aᵢ is the number of the vertices of G adjacent to that are colored i. The coloring c is called recognizable if distinct vertices have distinct color codes and the recognition number of G is the minimum positive integer k for which G has a recognizable k-coloring. In this paper we introduced a new variation of above parameter namely distance-2 irregular coloring. We initiate a study of this parameter and also find the distance 2-irregular chromatic number of some standard graphs.
number of $G$ is the minimum positive integer $k$ for which $G$ has a recognizable $k$-coloring. In this paper, we present some results on distance-2 irregular Chromatic numbers of graphs.

2. Preliminaries.

Definition 2.1 A (proper) coloring of a graph $G$ is a function $c: V(G) \rightarrow \mathbb{N}$ having the property that $c(u) \neq c(v)$ for every pair $u, v$ of adjacent vertices of $G$. A $k$-coloring of $G$ uses $k$ colors. The chromatic number $\chi(G)$ of $G$ is the minimum positive integer $k$ for which $G$ admits a $k$-coloring.

Definition 2.2 A 2-distance coloring of a graph $G(V,E)$ is a proper coloring of the vertices such that any two vertices at a distance at most 2, receive distinct colors and the 2-distance chromatic number $\chi^2(G)$ is the minimum positive integer $k$ for which $G$ has 2-distance $k$-coloring.

Definition 2.3 Let $G$ be a connected graph and let $c: V(G) \rightarrow \{1, 2, 3, \ldots, k\}$ be a proper coloring of the vertices of $G$ for some positive integer $k$. The color code of a vertex $v$ of $G$ (with respect to $c$) is the ordered $(k+1)$-tuple code $c(v) = (a_0, a_1, a_2, \ldots, a_k)$, where $a_0$ is the color assigned to $v$ and for $1 \leq i \leq k$, $a_i$ is the number of the vertices of $G$ adjacent to that are colored $i$. The coloring $c$ is irregular if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has an irregular $k$-coloring.

Observation 2.1 Let $c$ be a coloring of the vertices of a graph $G$. If $u$ and $v$ are two vertices of $G$ with $c(u) \neq c(v)$ then code $(u) \neq$ code $(v)$.

Observation 2.2 Let $c$ be a coloring of the vertices of a graph $G$. If $u$ and $v$ are two vertices of $G$ with $\deg(u) \neq \deg(v)$ then code $(u) \neq$ code $(v)$.

Observation 2.3 Let $c$ be an irregular coloring of a graph $G$. If $u$ and $v$ are distinct vertices of $G$ with $N(u) = N(v)$ then $c(u) \neq c(v)$.

3. Main Result

Definition 3.1 A distance-2 irregular coloring of a graph is an irregular coloring of the vertices such that any two vertices at a distance at most 2 receive distinct color codes and the distance-2 irregular chromatic number $\chi^2_{ir}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has the distance-2 irregular $k$-coloring.
Theorem 3.1 For any graph $G$, $\chi^2_{ir}(G) = n$ if and only if $\text{diam}(G) \leq 2$.

Proof. First we assume that $\text{diam}(G) \leq 2$, then by definition of $\chi^2_{ir}(G)$ clearly $\chi^2_{ir}(G) = n$. Conversely, assume that $\chi^2_{ir}(G) = n$. We need to prove that $\text{diam}(G) \leq 2$. Suppose not, let $u \in V(G)$, since $\text{diam}(G) > 2$ there exist at least one vertex $v \in V(G)$ such that $d(u, v) > 2$. If $\deg(u) \neq \deg(v)$ then we assign $c(u) = c(v)$ and hence $\chi^2_{ir}(G) \neq n$, which is a contradiction to $\chi^2_{ir}(G) = n$. If $\deg(u) = \deg(v)$, let $N(u) = \{u_1, u_2, \ldots, u_k\}$ and $N(v) = \{v_1, v_2, v_3, \ldots, v_k\}$. In this case assign $c(u) = c(v)$ and also assign a color to the vertex $u_i \in N(u)$ such that $c(u_i) \neq c(v_i)$ for $1 \leq i \leq k$, clearly code $(u) \neq \text{code}(v)$. Hence $\chi^2_{ir}(G) \neq n$ which is a contradiction. Hence $\chi^2_{ir}(G) = n$.

Theorem 3.2 For any graph $G$, $\chi(G) = \chi^2_{ir}(G)$ if and only if $G$ is a complete graph.

Proof. First we assume that $G$ is complete graph. We need to prove that $\chi(G) = \chi^2_{ir}(G)$. Clearly we know $\chi(G) = n$-------(1) In $G$, $\text{diam}(G) \leq 2$ then by theorem (3.1) $\chi^2_{ir}(G) = n$-------(2). From (1) and (2) $\chi(G) = \chi^2_{ir}(G)$.

Conversely, assume that $\chi(G) = \chi^2_{ir}(G)$. We claim that $G$ is complete graph. Suppose not, then there exist at least two non-adjacent vertices say $u, v$ of $G$.

Case(i) Suppose $d(u, v) = 2$. By definition of $\chi^2_{ir}(G)$, $c(u) \neq c(v)$. But same coloring is possible for $u$ and $v$ in $G$ this leads to $\chi(G) \leq n-1$ which is contradiction to $\chi(G) = \chi^2_{ir}(G)$.

Case(ii) Suppose $d(u, v) > 2$. That is, $N(u) \neq N(v)$. The path connecting the vertices $u$ and $v$ is of length greater than or equal to 3. Therefore $\chi^2_{ir}(G) \geq 3$. But $\chi(G) \geq 2$ which is contradiction to $\chi(G) = \chi^2_{ir}(G)$. Hence $u$ and $v$ are adjacent vertices. Thus $G$ is a complete graph.

Theorem 3.3 Let $n \geq 6$, If $k$ is an integer such that $n > \frac{k(k-1)(k-2)}{2} + 2$ then $\chi^2_{ir}(P_n) \geq k + 1$

Proof. Let $c$ be an distance-2 irregular coloring of the path $P_n$ of order $n$, then $P_n$ contains at most $\frac{k(k-1)(k-2)}{2}$ vertices of degree 2 and so $n \leq \frac{k(k-1)(k-2)}{2} + 2$.

Thus, if $n > \frac{k(k-1)(k-2)}{2} + 2$ then $\chi^2_{ir}(P_n) \geq k + 1$.

Theorem 3.4 For each integer $n \geq 6$, $\chi^2_{ir}(P_n) \leq \chi^2_{ir}(C_{n-2})$. 
Proof. Let \( v_1, v_2, \ldots, v_{n-2} \) be the vertices of the cycle of order \( n-2 \) and \( u_0, u_1, u_2, \ldots, u_{n-1} \) be the vertices of the path of order \( n \). Let \( c \) be the minimum distance-2 irregular coloring of \( C_{n-2} \). Now we define a coloring \( c' \) of \( P_n \) from \( c \) by

\[
c'(u_i) = \begin{cases} 
    c(v_{n-2}) & \text{if } i = 0 \\
    c(v_1) & \text{if } i = n-1 \\
    c(v_i) & \text{if } 1 \leq i \leq n-2
  \end{cases}
\]

Observe that the color codes of the vertices \( P_n \) are those of \( C_{n-2} \) except \( u_0 \) and \( u_{n-1} \). Since \( c'(u_0) \neq c'(u_{n-1}) \), it follows by “If \( c(u) \neq c(v) \) then code \( (u) \neq code (v) \)” that code \( c'(u_0) \neq code c'(u_{n-1}) \). Furthermore code \( c'(u_0) \neq code c'(u_i) \) and code \( c'(u_{n-1}) \neq code c'(u_i) \) for \( 1 \leq i \leq n-2 \) by “If \( deg_G u \neq deg_G v \) then code \( (u) \neq code (v) \)”. Since \( c \) is an distance-2 irregular coloring of \( C_{n-2} \) it follows that code \( c'(u_i) \neq code c'(u_j) \) for \( 1 \leq i \neq j \leq n-2 \). Thus \( c' \) is an distance-2 irregular coloring using \( \chi^2_{ir}(C_{n-2}) \) colors and so \( \chi^2_{ir}(P_n) \leq \chi^2_{ir}(C_{n-2}) \).

Theorem-3.5 Let \( k \geq 4 \) and \( n = \frac{k(k-1)(k-2)}{2} \). If \( \chi^2_{ir}(C_n) = k \) then \( \chi^2_{ir}(C_{n-1}) \leq k+1 \).

Proof. Let \( C_n: v_1,v_2,\ldots,v_n,v_1 \) be the cycle of order \( n \). If \( c \) be the distance-2 irregular coloring of the cycle \( C_n \). Since \( \frac{k(k-1)(k-2)}{2} \) is the largest possible value of \( n \) for which \( \chi^2_{ir}(C_n) = k \) it follows that \( c \) results in exactly \( \frac{k(k-1)(k-2)}{2} \) distinct color codes for the \( n \) vertices of \( C_n \). Let \( C_{n-1}: u_1,u_2,\ldots,u_{n-1},u_1 \) be the cycle of order \( n-1 \). We define a coloring \( c' \) of \( C_{n-1} \) from \( C_n \) by \( c'(u_i) = c(v_i), i=1,2,\ldots,n-2 \) and \( c'(u_{n-1}) = c(v_{n-1}) \) or \( c(v_n) \) if the color code is distinct. Otherwise \( c'(u_{n-1}) = k+1 \), then \( \chi^2_{ir}(C_{n-1}) \leq k+1 \).

Theorem-3.6 Let \( n \geq 6 \). If \( k \) is the unique integer such that

\[
\frac{(k-1)(k-2)(k-3)}{2} + 3 \leq n \leq \frac{k(k-1)(k-2)}{2} + 2 \text{ then } \chi^2_{ir}(P_n) = k.
\]

Proof: This conjecture is true for \( 6 \leq n \leq 107 \) as we show next by proposition (3.2),
\(\chi^{2}_{ir}(P_{14}) \leq \chi^{2}_{ir}(C_{12}) = 4, \ \chi^{2}_{ir}(P_{32}) \leq \chi^{2}_{ir}(C_{30}) = 5, \ \chi^{2}_{ir}(P_{62}) \leq \chi^{2}_{ir}(C_{60}) = 6, \ \chi^{2}_{ir}(P_{105}) \leq \chi^{2}_{ir}(C_{107})\). On the other hand by proposition (3.1) if \(n = 6\) then \(\chi^{2}_{ir}(P_{n}) \geq 4\); if \(n \geq 15\), then \(\chi^{2}_{ir}(P_{n}) \geq 5\); if \(n \geq 33\), then \(\chi^{2}_{ir}(P_{n}) \geq 6\); if \(n \geq 63\), \(\chi^{2}_{ir}(P_{n}) \geq 7\). Thus \(\chi^{2}_{ir}(P_{14})=4, \ \chi^{2}_{ir}(P_{32})=5, \ \chi^{2}_{ir}(P_{62})=6, \ \chi^{2}_{ir}(P_{n})=7\) for \(63 \leq n \leq 105\). With the aid of these observations and the proof of proposition (3.2) we are able to determine the distance-2 irregular chromatic number of \(P_{n}\) for \(3 \leq n \leq 107\) as follows.

\[
\chi^{2}_{ir}(P_{n}) = \begin{cases} 
3 & \text{if } n=3to5 \\
4 & \text{if } n=6to14 \\
5 & \text{if } n=15to32 \\
6 & \text{if } n=33to62 \\
7 & \text{if } n=63to107 
\end{cases}
\]

Theorem 3.7 Let \(k \geq 4\) and \(n = \frac{k(k-1)(k-2)}{2}\). If \(\chi^{2}_{ir}(C_{n}) = k\) then \(\chi^{2}_{ir}(C_{n+2}) \leq k+1\).

Proof. Let \(c\) be the distance-2 irregular \(k\)-coloring of \(C_{n}\). Since \(\frac{k(k-1)(k-2)}{2}\) is the largest possible value of \(n\) for which \(\chi^{2}_{ir}(C_{n}) = k\) it follows that \(c\) results in exactly \(\frac{k(k-1)(k-2)}{2}\) distinct color codes for the \(n\) vertices of \(C_{n}\). Thus there exist five consecutive vertices \(s, t, u, v, w\) on \(C_{n}\) such that \(c(s) = c(v) = 1, c(t) = 2, c(u) = 3\) and \(c(w) = k \geq 4\). The cycle \(C_{n+2}\) can be constructed from \(C_{n}\) by identifying the vertices \(t, u\) and \(v\) resulting in a vertex \(x\), then the \((k+1)\)-coloring \(c'\) of \(C_{n+2}\) by defined by \(c'(y) = k+1\) if \(y = x\) and \(c'(y) = c(y)\) if \(y \neq x\) is distance-2 irregular and so \(\chi^{2}_{ir}(C_{n+2}) \leq k+1\).

**Conclusion.** In this paper, we have proved the distance-2 irregular chromatic numbers for paths and cycles and established a relation between proper coloring and distance-2 irregular coloring.

**References.**

[1] M. Aigner and E. Triesch, Irregular assignments and two problems a la Ringel. Topics in Combinatorics and Graph Theory. (R. Bodendiek and R. Henn, eds.). Physica, Heidelberg (1990) 29-36.
[2] M. Aigner, E. Triesch and Z Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs. Combinatorics '90: Recent Trends and Applications (A. Barlotti, A. Bichara, P.V. Ceccherini, and G. Tallini, eds.). Elsevier Science Pub., New York (1992) 1-9.

[3] M. Albertson and K. Collins, Symmetric breaking in graphs. Electron. J. Combin. 3 (1996) R18.

[4] A.C. Burris, On graphs with irregular coloring number 2. Congr Numer. 100 (1994) 129 – 140.

[5] A.C. Burris, The irregular coloring number of a tree. Discrete Math. 141 (1995) 279 – 283.

[6] G. Chartrand and P. Zhang, Introduction to Graph Theory. McGraw –Hill, Boston (2005).

[7] R.C. Entringer and L.D. Gassman, Line0critical point determining and point distinguishing graphs. Discrete Math. 10 (1974) 43-55.

[8] F. Harary and M. Plantholt, The point – distinguishing chromatic index. Graphs and applications. Wiley, New York (1985)147-162.