Langevin Simulations of Two Dimensional Vortex Fluctuations:
Anomalous Dynamics and a New IV-exponent

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Abstract

The dynamics of two dimensional (2D) vortex fluctuations are investigated through simulations of the 2D Coulomb gas model in which vortices are represented by soft disks with logarithmic interactions. The simulations strongly support a recent suggestion that 2D vortex fluctuations obey an intrinsic anomalous dynamics manifested in a long range $1/t$-tail in the vortex correlations. A new non-linear IV-exponent $a$, which is different from the commonly used AHNS exponent $a_{AHNS}$ and is given by $a = 2a_{AHNS} - 3$, is confirmed by the simulations. The results are discussed in the context of earlier simulations, experiments and a phenomenological description.

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I. INTRODUCTION

Superconducting films, 2D Josephson junctions, and $^4$He films undergo a Kosterlitz-Thouless (KT) type transition from the superfluid to the normal state. This transition is driven by thermally created vortex-antivortex pairs which start to unbind at the transition. The high-$T_c$ materials can to some extent be regarded as weakly coupled superconducting planes, which raises the question to what extent the 2D vortex fluctuations are important also for this new class of materials. In order to assess such questions it is important to understand the properties of the vortex fluctuations in the pure 2D case. One motivation for the present investigation is to gain such understanding.

The KT-transition is driven by the vortex fluctuations and this means that the large distance and long time behaviour in the transition region is dominated by the properties of the vortices. The static properties at the transition are described by the Kosterlitz renormalization group (RG) equations and are rather well understood. The properties related to the dynamics of the vortices constitute a much more open question. We have in the present paper performed extensive simulations on a simple dynamical model of vortex fluctuations in order to gain some further insight.

So far the most widespread view on the dynamics of vortex fluctuations derives from series of papers by Ambegaokar et al. We refer to this as the AHNS phenomenology. Somewhat later a variant was devised by one of us which we will refer to as the MP-description. These two phenomenologies in fact give very different predictions. Experimental data seem to favor the MP-description. Particularly clear experimental evidence for this was given by Théron et al in case of a 2D array of Josephson junctions. The MP-description has also been clearly borne out in computer simulations on 2D XY-type models. It has been argued that at the heart of the MP-description is a long range $1/t$-tail in the vortex correlations below the transition temperature. This $1/t$-tail reflects an anomalous diffusion of the vortex fluctuations and is then the key to the difference between the AHNS and the MP. In the present paper we demonstrate that this anomalous diffusion is already a
property of the “simplest” possible dynamical model of vortex fluctuations: A charge neutral system of positive and negative particles with logarithmic particle interaction and Langevin dynamics.

Another main prediction of the AHNS is the exponent $a_{AHNS}$ of the non-linear IV-characteristics i.e. $V \propto I^{a_{AHNS}}$. However, given the anomalous diffusion and the $1/t$-tail of the vortex correlations, it has been argued that this prediction is no longer correct. \[12\] A scaling argument suggests that the non-linear IV-exponent $a$ consistent with the $1/t$-tail is given by $a = 2a_{AHNS} - 3$. \[12\] This prediction is borne out to an excellent degree in the present simulations. \[13\]

The content of the present paper is as follows: In section 2 we describe the model and the simulation procedure. Sections 3-6 contain the results from the simulations. In section 3 we show that the model and the simulations correctly reproduce the static KT-transition \textit{per se}. In particular we verify the power law decay of the correlations for large distances in the low temperature phase. Good agreement with the predicted power law indices are found. Then in section 4 we present the results for the non-linear IV-exponent $a$ and verify the correctness of the new scaling exponent. In section 5 we verify the $1/t$ tail in the vortex correlations. In particular we show how the decay of the temporal correlations depend on the wavevector $k$ for small $k$. Section 6 gives the frequency dependence of the basic linear response function describing the coupling to an external electromagnetic field. The results are shown to be very well represented by the functional form of the MP description in the small $k$ limit. We also show how the response function crosses over to a a more Drude like behaviour as the wavevector $k$ is increased. Finally section 7 contains some concluding remarks.

\section*{II. MODEL AND SIMULATIONS}

In accordance with AHNS we assume that the dynamics of the vortices to good approximation is described by the Langevin equation
\[ \frac{d\mathbf{r}(t)}{dt} = \frac{D}{T} \mathbf{F}_{\text{tot}}(t) + \eta(t) \]  

(1)

where \( \mathbf{r} \) is the position of a vortex and \( \mathbf{F}_{\text{tot}} \) is the total force acting on it due to all the other particles as well as any externally imposed force, \( D \) is the diffusion constant, \( T \) is the temperature (unit system such that the Boltzmann constant \( k_B = 1 \)), and \( \eta \) is a random force obeying

\[ \langle \eta^\alpha(t)\eta^\beta(t') \rangle = 2DT \delta_{\alpha\beta} \delta(t - t') \]  

(2)

where \( \alpha \) and \( \beta \) denote the Cartesian components. This equation describes the strong friction limit in which the vortex motion is perpendicular to the applied external current and should be a good approximation of a 2D superconductor. [6,5,14,2]

According to the vortex-Coulomb gas analogy, the vortices can be described as a gas of 2D Coulomb charges with logarithmic interaction. [2] The two possible vorticities \( s \pm 1 \) of a 2D vortex corresponds to positive and negative Coulomb gas charge. The thermally created vortex configurations have zero total vorticity which corresponds to a neutral Coulomb gas. [2] In our model the Coulomb gas charges are taken to be disks of extension \( r_0 \). These disks correspond to the vortex cores and are such that the force acting between two particles \( i \) and \( j \) with charges \( s_i \) and \( s_j \) respectively (in units such that the charge is \( s = \pm 1 \)) and separated by the distance \( r \) is given by

\[ F_{ij} = s_i s_j \left( \frac{1}{r} - \frac{1}{r_0} K_1(r/r_0) \right) \]  

(3)

where \( K_1 \) is a modified Bessel function of order 1. This means that the charge distribution of a Coulomb gas particle is soft, which is in accordance with the precise vortex-Coulomb gas particle analogy. [2] Consequently the force between two particles vanishes for \( r = 0 \) and is proportional to \( 1/r \) for \( r >> r_0 \). Alternatively one may express the two particle interaction corresponding to Eq.(3) in terms of a potential \( U(r) \)

\[ U(r) = -\ln(r/r_0) + K_0(r/r_0) \]  

(4)

where \( K_0 \) is a modified Bessel function of order 0,
\[ F_{ij} = -s_i s_j \frac{r_{ij}}{r_{ij}} \frac{\partial}{\partial r_{ij}} U(r_{ij}) \]  

and \( r_{ij} \) is the position vector from particle \( i \) to particle \( j \). In the present paper length is in units of \( r_0 \) and time in units of \( t_0 \equiv r_0^2 / D \).

The simulations are performed for a fixed number of particles \( N \) and constant temperature \( T \). The particles are contained in a 2D quadratic box of side length \( L \) with periodic boundary conditions. The numerical solutions were obtained by discretizing time into small time steps \( \Delta t \) and introducing a random noise \( \eta(t) \) which acts independently on each particle at each time step. The Langevin equation (1) is then turned into a finite difference equation for the particle system

\[ r_i(t + \Delta t) = r_i(t) + \Delta t \sum_{j=1}^{N} F(r_{ij}(t)) + \Delta t F_{\text{ext}}(t) + \eta_i(t), \]  

where the indices \( i \) and \( j \) numerate the particles and the diffusion constant \( D \) has been absorbed into the time scale and the random force. \( F(r_{ij}(t)) = s_i s_j \frac{r_{ij}}{r_{ij}} F_{ij} \) is the force acting at time \( t \) on particle \( i \) due to particle \( j \) and \( F_{\text{ext}} \) is any external force. The random force in Eq.(4) can thus be treated as a random displacement vector \( \eta_i(t) \) which obeys (compare Eq.(2))

\[ \langle \eta_i^\alpha(t) \eta_j^\beta(t') \rangle = 2T \delta_{\alpha\beta} \delta_{ij} \delta(t - t'), \]  

and is sampled from a Gaussian distribution. This equation is then solved on the computer by using a standard Euler integration method. [16] For each temperature of interest the value of \( \Delta t \) was halved repeatedly until no dependence of the time step could be monitored (usually \( \Delta t \approx 0.01 \ t_0 \)). The number of time steps needed for convergence is usually \( 1 - 5 \times 10^6 \), but for the long time correlation functions as many as \( 15 \times 10^6 \) steps were needed in order to obtain decently converged time tails. In practice one has to strike a balance between choosing \( \Delta t \) small enough to ensure that the equation of motion is correctly solved yet as large as possible in order to achieve as large time sequences as possible. In practice we have been able to meet these conditions below \( T_c \) without too much problem. However,
just at and slightly above $T_c$ this turned out to be very computer time consuming. Another practical problem is to keep track on the influence of the boundary. Here particular care has to be taken because the two particle interaction is long range i.e. $U(r) \propto \ln r$ for large $r$. To this end we found it expedient to modify the interaction by a large distance exponential cut off $\lambda_c$ which could then be varied in order to check the dependence on the largest length scales. Thus Eq.(3) was modified into

$$F_{ij} = s_is_j\left(\frac{1}{\lambda_c}K_1(r/\lambda_c) - \frac{1}{r_0}K_1(r/r_0)\right)$$

(8)

corresponding to

$$U(r) = K_0(r/\lambda_c) - K_0(r_0/\lambda_c) - K_0(r/r_0)$$

(9)

Typical parameters in the simulations are $N = 512$ and $L/r_0 \approx 320$ which correspond to a particle density $n = 5 \times 10^{-3}r_0^{-2}$. The ratio $\lambda_c/L = 0.35$ turned out to be an efficient choice. The size dependence of the results was checked by varying $L$ for fixed $n$ and ratio $\lambda_c/L$. The size $L/r_0 = 320$ was in practice large enough to avoid finite size effects except very close to the phase transition. In fact we found that simulations on a $N = 512$ system were for practical purposes large enough for the parameter range we are investigating. However, to be on the safe side a fair amount of the numerical data was obtained for $N = 1024$ and occasional checks for $N = 2048$ were also performed.

The Coulomb gas is often discussed in terms of a fugacity variable $z$ where $z^2/\Delta^2$ is the probability of creating a dipole pair with particle separation $r_0$ and $\Delta$ is the phase space division for a particle. This means that in our model there exists a non trivial relation between $nr_0^2$ and $z(nr_0^2, T)$. However, in our present simulations $n$ and $T$ are the fundamental variables.

The basic correlation function which we obtain from the simulations is the Fourier transform of the charge density correlation function $g(r,t)$ defined as

$$\hat{g}(k,t) = \frac{1}{L^2} \sum_{i,j} s_is_j e^{-ik(r_i(t) - r_j(0))}.$$  

(10)
In principle this function has a slight directional dependence on \( \mathbf{k} \) due to our choice of periodic boundary on a quadratic box. However, in practice our simulation results are to good approximation spherical symmetric so that \( \hat{g}(\mathbf{k}, t) = \hat{g}(k, t) \).

The results are conveniently discussed in terms of the complex frequency dependent dielectric constant \( \frac{1}{\hat{\epsilon}(k, \omega)} \) of the Coulomb gas model which is the basic response function. This is related to the correlation function \( \hat{g}(k, t) \) by

\[
\frac{1}{\hat{\epsilon}(k, \omega = 0)} = \text{Re} \left[ \frac{1}{\hat{\epsilon}(k, \omega = 0)} \right] = 1 - \frac{\hat{U}(k)}{T} \hat{g}(k, t = 0) \quad \text{(11)}
\]

\[
\text{Re} \left[ \frac{1}{\hat{\epsilon}(k, \omega)} \right] = \text{Re} \left[ \frac{1}{\hat{\epsilon}(k, \omega = 0)} \right] + \frac{\omega \hat{U}(k)}{T} \int_0^\infty dt \sin \omega t \hat{g}(k, t) \quad \text{(12)}
\]

\[
\text{Im} \left[ \frac{1}{\hat{\epsilon}(k, \omega)} \right] = -\frac{\omega \hat{U}(k)}{T} \int_0^\infty dt \cos \omega t \hat{g}(k, t) \quad \text{(13)}
\]

The first equation (11) gives the static result which contains the information on the thermodynamic KT-transition. The following two, (12) and (13), contain the information specific to the dynamics of the model.

In addition to the linear response given by Eqs (11-13) we calculate the non-linear response for the case when the model is subject to an external force \( \mathbf{F}_{\text{ext}} = s_i \mathbf{E} \) where \( \mathbf{E} \) is constant in space and time. In this case we calculate the average particle charge current \( I_p \) per particle

\[
I_p = \frac{1}{N} \left( \sum_{i=1}^N s_i \frac{dr_i(t)}{dt} \right) = \frac{1}{N} \frac{D}{T} \left( \sum_{i=1}^N \mathbf{F}_{i,\text{tot}}(t) \right) \quad \text{(14)}
\]

where the first equality is the definition of \( I_p \) and the second follows directly from Eq.(1). \( \mathbf{F}_{i,\text{tot}} \) is the total force acting on the particle \( i \) and the brackets \( \langle \rangle \) denote a time average.

The results from these simulations are presented in the following three sections.

**III. KT-TRANSITION.**

We will first focus on the static dielectric function \( 1/\hat{\epsilon}(k) \equiv 1/\hat{\epsilon}(k, \omega = 0) \) given by Eq.(11). This function is related to the linearly screened two particle interaction by
\[
\frac{1}{\tilde{\varepsilon}(k)} = \frac{\hat{U}_{ef}(k)}{\hat{U}(k)}
\]  

(15)

The “bare” interaction \(\hat{U}(k)\) is in our case given by (compare Eq.(9))

\[
\hat{U}(k) = \frac{2\pi}{k^2 + \lambda_c^{-2}} - \frac{2\pi}{k^2 + r_0^{-2}},
\]

(16)

provided \(L = \infty\). In practice we use the numerical transform for finite \(L\) and periodic boundary conditions. The linearly screened interaction is for small \(k\) given by \[2\]

\[
\hat{U}_{ef}(k) = \frac{1}{\tilde{\varepsilon}} \frac{2\pi}{k^2 + \lambda^{-2} + O(k^4)}
\]

(17)

where \(\lambda \leq \lambda_c\) is the screening length. Consequently we expect that the static dielectric function for small \(k\) is of the form

\[
\frac{1}{\tilde{\varepsilon}(k)} = \frac{1}{\tilde{\varepsilon}} \frac{\hat{U}(k)}{k^2 + \lambda^{-2}}
\]

(18)

Figure 1 shows data for \(1/\tilde{\varepsilon}(k)\) obtained from our simulations for a sequence of temperatures at a fixed density \(n\). The filled circles represent the data and the full curves are fits to Eq.(18). From these fits we obtain \(1/\tilde{\varepsilon}\) and the screening length due to free charges \(\lambda_F\) defined as \(\lambda_F^{-2} = \lambda^{-2} - \lambda_c^{-2}\). These two quantities are the key quantities describing the KT charge unbinding transition; \(1/\tilde{\varepsilon}\) may be interpreted as describing the polarization due to bound dipole pairs whereas \(\lambda_F\) can be interpreted as the Debye screening length related to the density \(n_F\) of “free” charges \(\lambda_F^{-2} = 2\pi n_F/\tilde{\varepsilon}T\). \[2\] In the thermodynamic limit \(L \propto \lambda_c \rightarrow \infty\) all particles are bound into dipole pairs below the KT transition at \(T_c\) whereas above \(T_c\) some pairs are broken. \[1\] This means that \(\lambda_F = \infty\) for \(T < T_c\) and \(\lambda_F < \infty\) for \(T > T_c\). In accordance with this figure 2 shows how \(\lambda_F^{-2}\) obtained in our simulations rapidly decreases as the KT transition is approached from above. Precisely at the KT transition one has the condition \(\tilde{\varepsilon}T_c = 1/4\). \[1,2\] This is illustrated in figure 3 which shows \(\tilde{\varepsilon}\) as a function of \(T\) for a sequence of constant particle densities. One notes that \(\tilde{\varepsilon}\) increases monotonously with increasing \(T\) for low temperatures, goes through a maximum and then decreases towards \(\tilde{\varepsilon} = 1\) for higher \(T\). Roughly this means that first the polarization due to bound pairs
increases because the average separation between the particles in a bound pair increases and then the polarization decreases because the number of bound pairs decreases due to thermal pair breaking at higher temperatures. The full curve in figure 3 corresponds to the condition $\bar{\epsilon}T = 1/4$ and we use this as the determination of $T_c$. This determination gives the phase transition line in the $(n, T)$-plane, as shown in the insert of figure 3. In the thermodynamic limit $\bar{\epsilon}$ has the critical behaviour

$$\bar{\epsilon}(T) - \bar{\epsilon}(T_c) \propto \pm \sqrt{|T - T_c|}$$

(19)

where + and - refer to above and below $T_c$. As seen in figure 3, the weak singular behaviour implied by Eq.(19) cannot be resolved by our present simulations. One notes, however, that the determined $T_c$ is close to the inflection point of the numerically obtained $\bar{\epsilon}$-curve in accordance with Eq.(19). Associated with the weak singular behaviour of Eq.(19) is a corresponding singular behaviour of $\lambda_F$:

$$\ln \lambda_F^{-2} \propto -\frac{1}{\sqrt{T - T_c}}$$

(20)

as $T_c$ is approached from above. In figure 4 $|\ln \lambda_F^{-2}|$ is plotted against $1/\sqrt{T - T_c}$ with $T_c$ determined from $\bar{\epsilon}T_c = 1/4$. As seen the critical behaviour given by Eq.(20) is not discernible in the simulations. However, this result is expected because the true critical behaviour associated with Eq.(20) should in practice be extremely hard to resolve as a consequence of the extreme narrowness of the KT critical region. In figure 4 we have also analyzed the data with respect to Eq.(20) following a commonly used procedure in the context of superconducting films and simulations on the 2D XY model: $|\ln \lambda_F^{-2}|$ is plotted against $1/\sqrt{T - T_c}$ where $T_c$ is a free parameter. As seen in figure 4 it is by this procedure possible to get a very good fit to Eq.(20). Such fits are frequently claimed to be evidence for a KT critical behaviour. However, as discussed in ref. such fits do usually not reflect a critical KT property per se, but rather a property of the 2D Coulomb gas well outside the KT critical region. As is apparent from figure 4, our present simulations are consistent with this latter interpretation.
The low temperature phase displays a “quasi” 2D order in the sense that the correlations for large distances fall off like power laws. In case of the charge density correlations we have

\[ g(r, t = 0) \propto \frac{1}{r^{x(T)}} \]  

for \( r >> r_0 \) where

\[ x(T) = \frac{1}{\tilde{\epsilon} T} \]  

From a renormalization group (RG) point of view this means that each \( T \leq T_c \) corresponds to a fixed point in the RG-flow. The RG-flow is towards vanishing density \( n \) so that for \( T \leq T_c \) the line \( (n = 0, T) \) in the \((n, T)\)-plane is a line of fixed points. Each such fixed point corresponds to a particular value of the critical index \( x(T) \). In figure 5 we show \( g(r, 0) \) as a function of \( r \) for a \( T \) below \( T_c \). The function \( g(r, 0) \) was obtain by directly measuring the charge correlations as a function of distance for the configurations generated by the simulation. The data for \( g(r, 0) \) is plotted as \( \ln g(r, 0) \) against \( \ln r \) and according to Eq.(21) the data should then fall on a straight line with slope \( x(T) \) for large \( r \). As seen in figure 5 this prediction is borne out. The broken straight lines in figure 5 has the slopes given by \( 1/\tilde{\epsilon} T \) where \( \tilde{\epsilon} \) has been determined from \( 1/\tilde{\epsilon}(k, \omega = 0) \) as described in connection with figure 3. Thus the prediction \( x(T) = 1/\tilde{\epsilon} T \) is supported to high degree by our simulations. The fact that the power law decay of the correlations with distance and the power law index come out correctly gives us confidence in the present simulations.

**IV. IV-EXponent.**

Next we consider the non-linear response of the system when it is subject to an external force \( \mathbf{F}_{ext} = s_i \mathbf{E} \) where \( \mathbf{E} \) is constant in space and time. This force generates a particle charge current \( I_p \). The charge current is in our simulations obtained from Eq.(14). The
prediction is that below $T_c$ the generated charge particle current is a power law in the limit of small magnitudes of the force

$$I_p \propto F_{ext}^a$$

(23)

In the context of a 2D superconductor the voltage $V$ is proportional to the flux flow so that $V \propto I_p$ whereas the force $F_{ext}$ is proportional to the Lorentz force so that $F_{ext} \propto I$ where $I$ is the external current applied to the superconductor. Thus in the context of a 2D superconductor Eq.(23) corresponds to the non-linear $IV$-characteristics for small $I$ i.e. $V \propto I^a$.

The question we are addressing with the present simulations is the value of the exponent $a$. There are two competing predictions: one is the AHNS-prediction

$$a_{AHNS} = \frac{1}{2\epsilon T} + 1$$

(24)

and the other is a scaling prediction

$$a = \frac{1}{\epsilon T} - 1$$

(25)

Figure 6 shows examples of the $I_pF_{ext}$-characteristics obtained from our simulations. The data is plotted as $\ln I_p$ against $\ln F_{ext}$ for a sequence of temperatures $T$. As is apparent from the figure the data fall to very good approximation on straight lines for small $F_{ext}$, as predicted by Eq.(23). The slopes of these lines give the values of the exponent $a$. The full curves in figure 6 are fits to the functional form

$$I_p = CF_{ext}e^{-(a-1)K_0(BF_{ext})}$$

(26)

where $a$ is the exponent and $B$ and $C$ are two constants. Fitting to this functional form turned out to be an expedient way of determining the exponent $a$: the $a$-values obtained by this fitting were the same as the ones obtained directly from the slope at small $F_{ext}$ but this latter procedure usually required much more computer time.
A heuristic motivation for Eq.(26) goes as follows: the particle current $I_p$ is proportional to the density of free particles $n_F$ and the force $F_{\text{ext}}$ i.e. $I_p \propto F_{\text{ext}} n_F$. The free particle density may be related to a self-energy $U_{\text{self}}$ for the creation of a free particle i.e. $\ln n_F \propto U_{\text{self}}$. The self-energy corresponding to the effective interaction in Eq.(17) is proportional to $K_0(r_0/\lambda)$ where the screening length $\lambda$ serves as an effective cut off of the particle interaction. $F_{\text{ext}}^{-1}$ has dimension of length and also serves as an effective cut off of the particle interaction. Consequently one may expect that whenever $F_{\text{ext}}^{-1} << \lambda$ the effective cut off in the self-energy is proportional to $F_{\text{ext}}^{-1}$. This argument suggests that $\ln n_F \propto K_0(B F_{\text{ext}})$ and Eq.(26) follows.

Figure 7 shows the obtained values for the exponent $a$ as a function of temperature $T$. These values are in the figure compared to the two competing predictions given Eqs (24) and (25), respectively. In this comparison we use the values of $\tilde{\epsilon}$ obtained as described in section 2. As is demonstrated by figure 7, the scaling prediction given by Eq.(25) (full curve in the figure) is borne out to high precision whereas the AHNS prediction of Eq.(24) (broken curve in the figure) clearly disagrees with the data. One notes that the two predictions agree precisely at the temperature corresponding to $1/\tilde{\epsilon} T = 4$ (crossing point between full and broken curve in figure 7). This corresponds to the critical condition for the KT-transition and to the universal jump value $a = 3$ at $T_c$. Above $T_c$ there are free charges even in the limit $F_{\text{ext}} = 0$. Consequently one has $I_p \propto n_F(F_{\text{ext}} = 0) \neq 0$ for very small $F_{\text{ext}}$ so that in principle $a = 1$ for $T > T_c$. Thus in principle the exponents $a$ jumps from 3 to 1 as $T_c$ is passed from below. However, as seen in figure 7, in practice the density of free charges $n_F$ is dominated by the pair breaking mechanism also above $T_c$ for small $F_{\text{ext}}$. This means that the exponent $a$ corresponding to pair breaking above $T_c$ can in practice be determined to very good precision, as is apparent from from figure 7. From figure 7 we infer that the pair breaking exponent $a$ is to very good approximation given by the scaling prediction Eq.(25) both below and above $T_c$.

The values of $a$ given in figure 7 are for a fixed density $n$. In general the exponent $a(T, n)$ is, of course, a function of both $T$ and $n$. Thus we can also test the prediction for $a$ as a
function of \( n \). In figure 8 the data is plotted as a function of \( 1/\varepsilon T \) for four different densities. The full straight line in figure 8 represents the scaling prediction of Eq.(25) and the broken straight line the AHNS prediction of Eq.(24). As seen in figure 8 the data falls clearly on the full straight line representing the scaling prediction for all the various densities. Thus we conclude that the present simulations strongly supports the scaling prediction.

An interesting observation in figure 8 is that the exponent \( a \) follows the scaling prediction (given by the full straight line) all the way down to \( a \approx 1 \) close to \( 1/\varepsilon T \approx 2 \) at which point there is an abrupt crossover to \( a = 1 \). This suggests an abrupt crossover behaviour at \( T = 1/2 \) for small particle densities. The 2D Coulomb gas model has an equation of state which to leading order in the particle density \( n \) is given by

\[
p = (T - \frac{1}{4})n \quad \text{for} \quad T > \frac{1}{2} \tag{27}
\]

and

\[
p = \frac{1}{2} nT \quad \text{for} \quad T < \frac{1}{2} \tag{28}
\]

where \( p \) is the pressure. For \( T < 1/2 \) this equation of state can be interpreted as the equation of state for an ideal gas of non-interacting dipole pairs. This suggests that the dominating part of the gas consists of dipole pairs in this small density limit. For such a gas of dipole pairs free charges can be generated by pair breaking caused by an external force. On the other hand for \( T > 1/2 \) the equation of state suggests a gas of free charges with no bound dipole pairs. This interpretation of the change of behaviour of the equation of state at \( T = 1/2 \) is in accordance with the sharp crossover at \( T = 1/2 \) of the exponent \( a \) which is seen in figure 8.

V. LARGE \( T \)-DEPENDENCE.

In this section we focus on the large \( t \)-dependence of the charge density correlations below \( T_c \). Figure 9 shows our numerical data for the Fourier transform \( \hat{g}(k,t) \). Our data suggest that the leading small \( k \) and large \( t \)-dependence is of the form...
\[ \hat{g}(k, t) \propto \frac{k^2 e^{-const k^2 t}}{t} \]  

(29)  

In order to establish this result we have in figure 9 plotted the logarithm of \( t\hat{g}(k, t)/k^2 \) against \( t \) for a sequence of fixed values of \( k \). The form given by Eq.(29) implies that the data, when plotted in this way, should for large \( t \) fall on straight lines. Furthermore the slope of these lines should vanish as \( k \) approaches zero. As seen in figure 9 these features are very consistent with the data and the data for the smallest \( k \)-values fall rather nearly on horizontal lines. The full straight lines in figure 9 are least square fits to the data in the region before too much noise sets in. For the four largest \( k \)-values in figure 9 such lines can be determined without much uncertainty. The corresponding slopes, together with estimated uncertainties, are in figure 10 plotted against \( k^2 \) (filled circles with error bars). The broken straight line in figure 10 is a line through the origin which is least square fitted to the determined slopes. The fact that the slopes rather closely follow this line suggests that the slopes are proportional to \( k^2 \) for small \( k \). The broken straight lines in figure 9 have the slopes given by the open circles in figure 10, i.e. they are the expected slopes for these smaller \( k \)-values based on the \( k^2 \)-extrapolation of the slopes for the larger \( k \)-values. As seen in figure 9 the broken lines also fit rather well to the data, which lends further support to the conclusion that the slopes are indeed proportional to \( k^2 \) all the way down to \( k = 0 \). Thus we conclude that the data in figures 9 and 10 support that the small \( k \) and large \( t \) behaviour of \( \hat{g}(k, t) \) to good approximation is given by Eq.(29). Simulations of the present type are of course always hampered by limited system sizes and time sequences. In particular we found that the smaller the \( k \)-value the harder it was to obtain a large \( t \)-value free of finite size effects. E.g. the large \( t \)-part of the two smallest \( k \)-values in figure 9 remain somewhat uncertain. Thus questions about logarithmic corrections to the leading \( t \)-dependence or non-leading terms appear to be outside the limitation of the present simulation precision. 

In ref. [12] it was found that the function \( \lim_{k \to 0} \hat{g}(k, t)/k^2 \propto 1/t \) for large \( t \) in case of the 2D XY-model on a square lattice with TDGL (time-dependent Ginzburg-Landau type) dynamics. This result was in ref. [12] associated with the vortex fluctuations. In the present
paper we confirm this conclusion by establishing the result directly in the Coulomb gas model with Langevin dynamics. In addition we have obtained the leading small \( k \)-dependence for large \( t \).

The form given by Eq.\((29)\) implies that for large \( t \) the dominant contribution to \( g(r, t) \) comes from the small \( k \). Thus we expect that the leading large \( t \) contribution to \( g(r, t) \) is given by

\[
g(r, t) \propto \frac{1}{t} \int_0^{k_{\text{max}}} dk^2 k^2 e^{-tk^2 \text{const}} e^{ik \cdot r} \propto \frac{1}{t^3} \tag{30}
\]

for large \( t \). The charge density is obviously a conserved quantity in the Coulomb gas model. Thus a pile up of charge in one place can only decay by diffusing away. Ordinary diffusion in 2D leads to \( g(r, t) \propto 1/t \). However, from our simulations of the 2D Coulomb gas we conclude that the long range interaction between the particles changes this result into a more rapid decay \( g(r, t) \propto 1/t^3 \) in the low temperature phase \( T < T_c \). In the high temperature phase \( T > T_c \) the screening length \( \lambda \) is always finite due to the presence of free charges. Thus in this case the decay of the charge density correlations are expected to decay exponentially, where the decay is dominated by a factor \( \exp(-t\lambda^{-2}\text{const}) \). We have not been able to explicitly verify this result in the present simulations, since the simulations are harder to converge in the high temperature phase. We note that, since in the small density limit the dipole pairs dominate the response for \( T < 1/2 \) (see the end of the preceding section), one might likewise expect that in practice the behaviour \( g(r, t) \propto 1/t^3 \), which we associate with the dipole pairs, also dominates the response in a region somewhat above \( T_c \) (\( T_c \) is always smaller than 1/4) for not too large time scales. Thus one might expect that the frequency response for small but not too small frequencies are dominated by the dipole pair response also in a region somewhat above \( T_c \).

The scaling prediction for the exponent \( a \) given by Eq.\((25)\) was in ref. \([12]\) based on the assumption that the charge density correlations \( g(r, t) \) can be associated with a scaling function \( \lambda^{-z}\Phi(r\lambda^{-1}, t\lambda^{-z}) \) where \( z \) is the dynamical exponent and \( \lambda \) is the screening length which diverges for any \( T \) below \( T_c \) in the limit \( \lambda_c \rightarrow \infty \). Furthermore it was assumed that
the scaling function $\Phi(x, y)$ had the limits $\Phi(x, 0) \propto x^{2-1/T\tilde{\epsilon}}$ for large $x$ and $\Phi(0, y) \propto y^{-1}$ for large $y$. Consequently we can now infer that the relation between $g(r, t)$ and the invoked scaling form has to be

$$\lambda^{-z}\Phi(r\lambda^{-1}, t\lambda^{-1}) \propto r^2 t^2 g(r, t) \quad (31)$$

since $g(r_0, t) \propto t^{-3}$ for large $t$ and $g(r, t_0) \propto r^{1/T\tilde{\epsilon}}$ (compare discussion in connection with figure 5, $t_0 \equiv r_0^2 / D$ is the microscopic time scale and $r_0$ is the microscopic size of a particle).

From Eqs (12) and (13) one obtains the leading small $\omega$ dependence corresponding to Eq.(29). Since according to Eq.(29) $F(t) \equiv \lim_{k \to 0} \hat{U}(k)\hat{g}(k, t) \propto 1/t$ for large $t$, the leading small $\omega$-dependences are proportional to

$$\omega \int_0^\infty dt \sin \omega t \ F(t) \propto \omega \int_0^\infty dx \frac{\sin x}{x} = \omega \frac{\pi}{2} \quad (32)$$

respectively,

$$- \omega \int_0^\infty dt \cos \omega t \ F(t) \propto -\omega \int_{const}^{1/\omega} dt \frac{1}{t} \propto \omega \ln \omega \quad (33)$$

and consequently

$$Re \left[ \frac{1}{\hat{\epsilon}(0, \omega)} \right] - \frac{1}{\tilde{\epsilon}} = const \pi \frac{\omega}{2} \quad (34)$$

$$Im \left[ \frac{1}{\hat{\epsilon}(0, \omega)} \right] = const \omega \ln \omega \quad (35)$$

for small $\omega$. The constant const depends on $T$ and has the same value in the last two equations.

It is instructive to translate this result into the linear response function corresponding to the conductivity $\sigma(\omega)$ for a 2D superconductor. The connection is $\sigma(\omega) \propto [-i \omega \hat{\epsilon}(0, \omega)]^{-1}$. Consequently we have the leading small $\omega$-dependence given by

$$Re\sigma(\omega) \propto -\ln|\omega| \quad (36)$$

and
\[
I m \sigma(\omega) \propto \frac{\text{constant}}{\omega} + \frac{\pi}{2} \text{sign}(\omega) \tag{37}
\]

or equivalently

\[
\sigma(\omega) \propto \frac{\text{constant}}{-i\omega} - \ln(-i\omega) \tag{38}
\]

For a 2D superconductor the dissipation for small \(\omega\) are due to thermally created vortices. Hence Eq. (36) suggests that for a 2D superconductor the real part of the conductivity diverges logarithmically below \(T_c\).

VI. PHENOMENOLOGICAL DESCRIPTION.

As described above, we compute the Fourier transform of the charge density correlation function \(\hat{g}(k,t)\) in our simulations. By aid of Eqs. (12) and (13) we obtain the frequency dependence of the dielectric function \(1/\hat{\epsilon}(k,\omega)\). In figure 11 we present the result for \(Re\left[1/\hat{\epsilon}(k,\omega)\right]\) and \(Im\left[1/\hat{\epsilon}(k,\omega)\right]\) for the smallest \(k\) we managed to converge \((k = 0.039 \ (r_0^{-1}))\).

As is apparent from figure 9 we expect that for such a small value of \(k\) the real part of \(1/\hat{\epsilon}\) should vanish linearly with \(\omega\), as discussed in connection with Eq. (34). For large \(\omega\), on the other hand, the dipole pairs have no time to respond so that in this limit we expect that \(Re\left[1/\hat{\epsilon}(k,\omega)\right] = 1\). Thus we expect the real part for small enough \(k\) to be of the form \(Re\left[1/\hat{\epsilon}(k,\omega)\right] \approx Re\left[1/\hat{\epsilon}(0,\omega)\right]\) where

\[
Re \left[ \frac{1}{\hat{\epsilon}(0,\omega)} \right] = \frac{1}{\hat{\epsilon}(0,0)} + \left[ 1 - \frac{1}{\hat{\epsilon}(0,0)} \right] \frac{\omega}{\omega + G(\omega)} \tag{39}
\]

provided \(0 < G(\omega) < \infty\) and \(G(\omega = 0) = \text{constant} > 0\). If \(G(\omega)\) only depends weakly on \(\omega\), we can approximate \(G(\omega)\) by a positive constant \(G(\omega) \approx \omega_0\). Using this approximation we can now obtain the corresponding approximation for the imaginary part by using the Kramers-Kronig relation. This leads to

\[
Re \left[ \frac{1}{\hat{\epsilon}(0,\omega)} \right] = \frac{1}{\hat{\epsilon}(0,0)} + \left[ 1 - \frac{1}{\hat{\epsilon}(0,0)} \right] \frac{\omega}{\omega + \omega_0} \tag{40}
\]

and
\[
\text{Im}\left[\frac{1}{\hat{\epsilon}(0,\omega)}\right] = -\left[1 - \frac{1}{\hat{\epsilon}(0,0)}\right] \frac{2}{\pi} \frac{\omega_0 \ln \omega}{\omega^2 - \omega_0^2}
\]

(41)

One notes that Eqs (40) and (41) correctly reduces to Eqs (34) and (35) in the small \(\omega\)-limit. The form of the \(\omega\)-dependence given by Eqs (40) and (41) is identical to the form given by the MP-description. The MP-description was originally motivated from a heuristical argument for the dipole pair response. As described here, we can also view the MP-form as a simple interpolation between two known limits. In figure 11 we have fitted Eqs (40) and (41) to the data from the simulations by using the two constants \(\omega_0\) and \(1 - \frac{1}{\hat{\epsilon}(0,0)}\) as fitting parameters. As seen from the figure, the data is very well described by the MP-form. From the fitting we obtain \(1/\hat{\epsilon}(0,0) = 0.91\). This is fairly close to the value obtained directly from the simulations \(1/\hat{\epsilon}(k,0) = 1/\hat{\epsilon} \approx 0.92\) (compare figure 9, \(k^{-2}\hat{g}(k, t = 0) \propto 1 - 1/\hat{\epsilon}(k, \omega = 0)\)). The value obtained for \(\omega_0\) from the fitting was \(\omega_0 \approx 0.36 \pm 0.02\). This can be compared to \(G'(0) \approx 0.24\) which is obtained directly from the simulations: From the data in figure 9 we get \(\hat{g}(k,t)/k^2 = A/t\) for small \(k\) and large \(t\). By aid of Eq. (32) we then obtain the value of the constant in front of the linear small \(\omega\) dependence. Finally, by using \(1/\hat{\epsilon}(k,0)\) from figure 1, we obtain \(G'(0)\) as \(G'(0) \approx (1 - 1/\hat{\epsilon}(k,0))/\text{constant}\). One notes that \(G'(0)\) is somewhat smaller but of the same magnitude as the \(\omega_0\) determined from the fitting to the data, which supports the assumption that \(G'(\omega)\) only has a quite weak \(\omega\)-dependence. Thus we conclude that Eqs (40) and (41), obtained from Eq. (39) by using the approximation \(G'(\omega) \approx \omega_0\), gives a very good and quite consistent description of the data from our present simulations.

So far we have focused on the \(\omega\)-dependence in the small \(k\)-limit. Next we consider how this \(\omega\) dependence is changed as \(k\) is increased. Figure 12 shows the real and imaginary part of \(1/\hat{\epsilon}(k,\omega)\) as a function of \(\omega\) for a sequence of increasing \(k\)-values. As seen in figure 12 the imaginary part appears to be remarkably independent of \(k\) over the range of \(k\)-values in the figure \((0.039 \leq k \leq 0.31)\), whereas the real part decreases significantly with increasing \(k\). However, since the real and imaginary part are related by a Kramer-Kronig relation, the change in the real part part must have a corresponding change in the imaginary part. The point is that the corresponding change in the imaginary part is concentrated to small
frequencies. Since the imaginary part is almost independent of \( k \) whereas the real part changes significantly, one concludes that Eqs \((40)\) and \((41)\) does not describe the data as \( k \) is increased, as expected from the motivation of these equations. In order to quantify this feature in a practical way we focus on the change with \( k \) precisely at the maximum of the imaginary part. As seen in figure 12 this frequency is close to \( \omega \approx 0.36 \) and is approximately independent of \( k \) over the \( k \) range in the figure. Now the MP-form given by Eqs \((40)\) and \((41)\) has the property that, at the maximum frequency for the absolute value of the imaginary part the, ratio between the absolute value of the imaginary and real part is precisely \( 2/\pi \). In figure 13 we have plotted this ratio as a function of \( k \). For small \( k \) this ratio approaches \( 2/\pi \) as expected from Eqs \((40)\) and \((41)\). However as \( k \) increases this ratio also increases. Ordinary diffusion corresponds to

\[
\frac{1}{\hat{\epsilon}(0, \omega)} = \frac{1}{\hat{\epsilon}(0, 0)} + \left[ 1 - \frac{1}{\hat{\epsilon}(0, 0)} \right] \frac{\omega}{\omega + iDk^2} \]

The real and imaginary part is for this case of the usual Drude form. The ratio between the absolute value of the imaginary and real part precisely at the maximum of the absolute value of the imaginary part is in the Drude case unity. It appears from figure 13 that the ratio approaches the ordinary Drude value unity as \( k \) increases. This change of behaviour from MP to Drude is somewhat reminiscent of Eq.\((29)\) where the diffusion like factor \( \exp(-const \ k^2 t) \) becomes more important with increasing \( k \).

**VII. CONCLUDING REMARKS.**

The model investigated is a model of 2D vortex fluctuations. Consequently, the properties of this model should be directly reflected in measurements on 2D superconductors like superconducting films and 2D Josephson junction arrays. In our simulations we verified that the KT critical region is very narrow. For the resistance \( R \) of a superconductor the predicted KT-critical behaviour is

\[
\ln R \propto \frac{1}{\sqrt{T_{CG} - T_C}} \]

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where $T^{CG}$ is the effective temperature variable for the vortices. The narrowness of the critical region means that in practice the KT-critical behaviour probably cannot be resolved in resistance measurements, as has been pointed out earlier. In our simulations we obtained $T_c$ to good precision. However, if we instead used $T_c$ as a free parameter, then we showed that the data could indeed be fitted to the KT-critical behaviour over several decades. This procedure of using $T_c$ as a free parameter and fitting to Eq. (43) is the commonly used way of establishing the KT-critical behaviour for the resistance $R$ of 2D superconductors. It is frequently found that by treating $T_c$ as a free parameter the data can be nicely fitted to Eq. (43), as was also the case for our simulations. However, in our simulations we also found that the $T_c$ from the fit was significantly different from (i.e. $13\%$ lower than) the true $T_c$. Thus we conclude that these type of fits have in fact no direct bearing on the “real” critical behaviour. It should also be noted that although the fitted $T_c$ for a superconductor can appear to be quite close to the true $T_c$ in real temperatures, the corresponding difference in the effective vortex temperature variable $T^{CG}$ is usually much larger. The crucial point here is that our simulations suggest that, if the true $T^{CG}$ is used, then the data does not follow the functional form given by Eq. (43).

We also obtained the non-linear $IV$-exponent $a$ from the simulations and verified that $a = 1/\tilde{e}T - 1$, as proposed in ref. [12] and which is quite different from the earlier AHNS-prediction $a_{AHNS}$. However, precisely at $T_c$ both prediction give $a = 3$ since $a = 2a_{AHNS} - 3$. Below $T_c$ the new value is larger than the AHNS-value. In principle $a = 1$ above $T_c$ due to usual flux flow resistance of free vortices. Nonetheless, we found that in practice a non-linear $IV$-exponent describing the pair breaking could also be determined to very good precision above $T_c$ up to roughly $2T_c$ at which point there was a rapid cross over to $a = 1$. All the way up to roughly $2T_c$ we found $a = 1/\tilde{e}T - 1$ to very good approximation.

For a 2D superconductor the exponent $a$ is directly related to $1/\tilde{e}T^{CG} \propto \rho(T)/T$ where the proportionality constant is a combination of fundamental physical constants and $\rho(T)$ is the 2D superfluid density. Thus in order to test the prediction for $a$ one needs to know the temperature dependence of $\rho(T)$. One way is to measure the complex impedance $Z(T, \omega)$
since \[2\]  
\[
Z(T, \omega) = -i\omega L_k(T)\hat{\epsilon}(k = 0, \omega, T) \propto \frac{T^{CG}}{T}\hat{\epsilon}(k = 0, \omega, T)
\]  
(44)

where \(L_k(T)\) is the sheet kinetic inductance and the proportionality factor is again just a combination of fundamental physical constants. Consequently, if \(Z(T, \omega)\) is measured for very small frequencies for a sample and the non-linear IV-characteristics for the same sample is measured to high precision, then the prediction for \(a\) for \(T \leq T_c\) can be directly put to experimental test. Alternatively, if \(Z(T, \omega)\) is measured for a small but finite frequencies, then the \(\omega = 0\)-limit can be extracted by using Eqs (40) and (41). For \(T \geq T_c\) one needs both \(L_k(T)\) and \(\bar{\epsilon}(T)\). The sheet kinetic inductance \(L_k(T)\) can often be quite well determined from the complex impedance. The qualitative behaviour of \(\bar{\epsilon}(T)\) is clear from figure 3. It is also possible to make a somewhat more quantitative determination of \(\bar{\epsilon}\) above \(T_c\) by starting from \(\hat{\epsilon}(k = 0, \omega, T)\) at a finite small frequency and fitting to the MP-phenomenology. \[22, 11\]

We concluded from our model that 2D vortex fluctuations has a long range \(1/t\)-tail in the vortex correlations below \(T_c\). This means that the conductivity \(\sigma(\omega)\) is of the form given by Eq. (38). The crucial feature is that the next leading term for small \(\omega\) is a logarithm \(\ln(-i\omega)\). Such a logarithmic term is strongly supported by experiments on a 2D array of Josephson junctions. \[9\] These experiments are in fact very well described by the MP-phenomenology. \[9\] In connection with these experiments there has been other proposal for the origin of this logarithm, e.g like vortex-spin wave coupling \[23\] or as a specific single vortex property of a proximity coupled array \[24\]. The point we are making here is that this logarithm is an intrinsic collective property of 2D vortex fluctuations which is strongly linked to the long range logarithmic vortex interaction.

The general form of the frequency dependence is given by Eq. (39) which reduces to the MP-form of Eqs (40) and (41) for \(G(\omega) = \text{const.}\). The point here is that, since there is no a priori characteristic frequency scale other than the microscopic \(t_0^{-1}\) one expects that \(G(\omega) \approx G(0)\) as long as \(\omega << t_0^{-1}\). We believe that the fact that the MP-form describes a variety of experimental data as well as simulations very well \[4, 7, 11\], reflects this aspect: the
MP-form describes the dynamics for frequencies much smaller than the basic microscopic frequency scale and the functional form is independent of the microscopic details of the dynamics. In this sense it describes a universal behaviour of 2D vortex fluctuations.

One may also note that $\omega_0(T)$ in Eqs (40) and (41) depend on $T$. Thus the MP-form can be tested either for fixed $T$ by varying $\omega$ or for fixed $\omega$ by varying $T$. This latter way is more common in experiments. A particular feature of the MP-form is the fact that the ratio between the imaginary and real part of the response is $2/\pi$ precisely at the dissipation peak. The position of this peak is often quite well defined in the experiments and peak ratios close to $2/\pi$ have been found both in experiments and simulations. In the present paper we directly verify that the peak ratio $2\pi$ is an intrinsic dynamical small frequency property of 2D vortex fluctuations.

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FIGURES

FIG. 1. The static dielectric response function $1/\tilde{\epsilon}(k)$ plotted as a function of the wave vector $k$ (in units of $2\pi/L$ where $L$ is the system size) at a small constant particle density, $n = 0.005 \left(r_0^{-2}\right)$. The filled circles represent the data from simulations at $T = 0.12$, 0.14, 0.16, 0.18, 0.20, 0.23, 0.26, 0.29, 0.32, 0.35, 0.38, 0.41, 0.50, from top to bottom, respectively. The full curves are fits of equation 18.

FIG. 2. The inverse square of the screening length $\lambda_F^{-2}$ extracted from the data in figure 1 by using Eq.(18). The open circles represent $\lambda_F^{-2}$ (in units of $(2\pi/L)^2$ and $n = 0.005 \left(r_0^{-2}\right)$) plotted as a function of temperature $T$. The critical temperature $T_c$ is also shown ( $T_c$ is determined as described in connection with figure 3). The figure illustrates the charge unbinding transition: Below $T_c$ the charges are bound together in dipole pairs. These pairs start to unbind at $T_c$, resulting in a rapid increase in the density of free charges $n_F \propto \lambda_F^{-2}$ as $T$ passes $T_c$ from below.

FIG. 3. The dielectric constant $\tilde{\epsilon}$ extracted from the data in figure 1 by using Eq.(18). The various symbols represent, from bottom to top, the determined value of $1/\tilde{\epsilon}(k)$ for the particle densities, $n = 0.001$, 0.005, 0.01, 0.025, 0.05, 0.075, 0.025, 0.05, 0.1 $\left(r_0^{-2}\right)$, respectively. The broken curves are guides to the eye. The full curve represent the KT critical condition $\tilde{\epsilon} = 1/4T_c$. The full curve crosses the broken curves close to the inflection points of the broken curves as expected for the KT-transition. The crossing points between the full curve and the broken curves gives the phase transition line in the $(n, T)$-plane as shown in the insert. The estimated error bars in the insert are due to numerical uncertainties as well as finite size effects. It should be noted that the uncertainties in the values of $T_c$ are very small.
FIG. 4. Test of KT-criticality: The quantity $|\ln \lambda_F^2|$ is plotted against $1/\sqrt{T - T_c}$. Plotted in this way the data should, according to Eq.(20), fall on a straight line, provided the data is in the critical region. The data for $\lambda_F$ is the same as in figure 2. The filled circles represent the case when the true $T_c$ is used ($T_c = 0.215(7)$, determined as described in connection with figure 3). The broken curve is a guide to the eye. As expected Eq.(20) does not describe the data because of the unusual narrowness of the KT critical region. The critical behaviour would only show up closer to $T_c$ for much larger values of $\lambda_F$ which cannot be converged in the present simulations. If instead $T_c$ is treated as a free parameter then data can indeed be manipulated to fall on a straight line. The open circles show the case for $T_c = 0.187$. In this case the data fall on a straight line over several decades as indicated by the dotted straight line in the figure.

FIG. 5. The charge density correlation function $g(r, t = 0)$ as a function of distance $r$ for $T = 0.18$, $n = 0.005$ (circles), and $T = 0.16$, $n = 0.025$ (diamonds), respectively. The data is plotted as $|\log g(r, 0)|$ against $\log r$ in order to test the prediction $g(r, 0) \propto r^{-1/\tilde{\epsilon}T}$ for large $r$. Plotted in this way the data should fall on straight lines for large $r$ and this prediction is borne out. The broken lines have the slopes given by $1/\tilde{\epsilon}T$ where $\tilde{\epsilon}$ has been determined as discussed in connection with figure 3 and fit the data very well. Consequently the prediction $g(r, 0) \propto r^{-1/\tilde{\epsilon}T}$ is verified to high degree by the present simulations.

FIG. 6. The charge current $I_p$ as a function of an external force $F_{ext} = s_i E$ for the temperatures $T = 0.12, 0.14, 0.16, 0.18, 0.23, 0.26, 0.29, 0.50$ at constant density $n = 0.005 \ (r_0^{-2})$. The data from the simulation are given by the filled circles with error bars and is plotted as $\log I_p$ against $\log F_{ext}$. The full curves are fits to Eq.(26). From these fits the exponents $a$ defined by $I_p \propto F_{ext}^a$ are determined. Alternatively the exponent $a$ can be determined directly from the slopes at small $F_{ext}$ (see text).
FIG. 7. The non-linear IV-exponent $a$ obtained from the data in figure 6. The exponent $a$ is plotted as a function of temperature (filled circles with error bars). The full curve is the scaling prediction Eq.(25) and the broken curve is the AHNS prediction Eq.(24). The values of $\bar{\epsilon}$ needed to make the comparison were determined as described in connection with figure 3. The data strongly favors the scaling prediction.

FIG. 8. The IV-exponent obtained from the simulations at four different particle densities ($n = 0.001, 0.005, 0.01, \text{and} 0.025 (r_0^{-2})$). The data is plotted as $a$ against $1/\bar{\epsilon}T$. Plotted in this way the scaling prediction corresponds to the full straight line and the AHNS prediction to the broken straight line. The vertical error bars estimate the uncertainty in the value $a$ for a given $T$ and the horizontal the uncertainty in $\bar{\epsilon}$. The data clearly verifies the scaling prediction. Precisely at $T_c$ one has $1/\bar{\epsilon}T = 4$ and both predictions give $a = 3$. Note that $a$ in practice can be determined also above $T_c$ (see text) and clearly follows the scaling prediction all the way upto $1/\bar{\epsilon}T \approx 2$ at which point there an abrupt cross over to $a = 1$.

FIG. 9. The time dependence of the charge density correlation function $g$ below $T_c$ ($T = 0.18$ and $n = 0.005 (r_0^{-2})$). In order to verify the $t$-dependence $\hat{g}(k, t) \propto k^2 e^{-\text{const}k^2t}/t$ given by Eq.(29) the data is plotted as $\log(\hat{g}(k, t)/k^2)$ against $t$. The data for large $t$ should then fall on straight lines where the slopes go towards zero as $k$ is decreased. The data is given by the open circles and the eight data sets correspond from top to bottom to the $k$-values $k = (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)$ and $(8, 8)$ (in units of $2\pi/L$). The $t$-dependence of Eq.(29) is apparently borne out to fairly good approximation. The full straight lines are fits to the data for the larger $k$-values before too much noise sets in. The broken straight lines have slopes extrapolated from the full straight lines using Eq.(24) as described in connection with figure 10. Also the broken straight lines fit the data fairly well.
FIG. 10. The $k$-dependence of the slopes in figure 9. The slopes of the full straight lines in figure 9 are plotted against $k^2$ (filled circles with error bars). According to Eq. (29) these slopes should extrapolate linearly to zero. The broken straight line is a line through zero which is fitted to the filled circles and shows that the slopes apparently to good approximation are proportional to $k^2$. The open circles give the expected slopes for some smaller $k$-values and correspond to the broken straight lines in figure 9.

FIG. 11. The real (circles) and imaginary (diamonds) part of the frequency dependent dielectric constant $1/\hat{\epsilon}(k, \omega)$ for a small $k$ (the $k$-vector is $(1, 1)$ (in units of $2\pi/L$) and the data is the same as the top data set in figure 9). The full drawn curves are fits to the MP-form given by Eqs (40) and (41) using the two constants $\omega_0$ and $1 - 1/\hat{\epsilon}(0, 0)$ as free parameters. The MP-form describes the data very well.

FIG. 12. Real and imaginary parts of $1/\hat{\epsilon}(k, \omega)$ for the same parameters as in figure 9. The real part of $1/\epsilon(k, \omega)$ is represented by the upper set of curves. The amplitude decreases with increasing $k$ (the $k$-values are the same as in figure 9). The uppermost curve is the same data as was shown to be well described by the MP-form in figure 11. The $k$-dependence of the imaginary part (lower set of curves) is by contrast very small. Note that the absolute value of the imaginary part has a maximum at $\omega \approx 0.36$ for all $k$-values in the figure. The figure illustrates that the MP-form as expected only describes the data in the limit of small $k$.

FIG. 13. The peak ratio of the imaginary and real parts of $1/\hat{\epsilon}(k, \omega)$ as a function of $k$. The peak ratios are obtained from the data shown in figure 12 and are denoted by open circles. The broken curve is a guide to the eye. For small $k$ the ratio approaches the predicted MP value of $2/\pi$ whereas the Drude ratio of unity is approached for large $k$. 