Directional emission from weakly eccentric resonators

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It is shown that when a circular resonator is deformed in a nonintegrable way, a symmetry breaking of escaping rays occurs which can dramatically modulate the outgoing wave even for small perturbations. The underlying mechanism does not occur in integrable models for which the ray families can be computed exactly and is described in this Letter on the basis of canonical perturbation theory. Emission from deformed resonators is currently of immense practical interest in the context of whispering-gallery optical resonances of dielectric cavities and the approach outlined here promises simple analytical characterisations in the important case of small deformations.

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Short-wavelength approximations have proved invaluable in understanding and predicting the properties of optical resonators [1, 2]. They have in particular provided essential insight into the directional emission patterns that are observed when resonators are sufficiently asymmetric that ray dynamics display chaotic as well as regular behaviour [3], where features of wave chaos such as chaos-assisted tunnelling [4–6], scarring [7, 8] and dynamical localisation [6, 9] have been shown to play a role. Surprisingly, emission characteristics in the apparently benign limit of very weakly asymmetric resonators cannot readily be described using this approach [10]. The reason is that emission is essentially a tunnelling phenomenon whose short-wavelength approximation demands that we extend the underlying ray families into complex space. It has been established, however, that natural boundaries generically intervene [11], even in very slightly perturbed problems, which prevent analytic continuation as far as needed in the complex domain [12, 13]. In other words, the ray-dynamical data demanded by short-wavelength approximations of emission simply does not exist.

In view of the current technological interest in optical resonators [2], simple analytical approaches to this problem are clearly of great value. We outline one such solution in this Letter which works by substituting for the nonexistent classical data approximate solutions that can be extended as far as required in the complex domain while remaining sufficiently accurate for use in WKB approximations. A related approach has recently proved successful in predicting tunnel splittings in near-integrable potentials [14]. Although restricted to relatively small deformations, the solution is capable of predicting strongly directional emission patterns.

A feature of the solution is that there is a dramatic difference between the behaviours of integrable and non-integrable deformations. This is true even of problems whose Poincaré plots look quite similar. To illustrate this, emission patterns from two perturbations of a circular resonator, one integrable and the other nonintegrable, are compared in Fig. 1. The details of these systems are described later but here we simply point out that even though the Poincaré plot of the nonintegrable problem seems to deviate much less from the perfectly circular limit, it has a dramatically more directional emission pattern. The enormous qualitative difference between integrable and nonintegrable systems, irrespective of any gross similarity of the real ray dynamics, indicates that simple geometrical characteristics of the deformation, such as eccentricity, boundary curvature or the existence of particular island chains, do not transparently determine emission characteristics in this regime.

![FIG. 1: Emission patterns are shown for two perturbations of a resonance of $V_0(r) = -(r^2 - 1)^2$, with the generic perturbation $\varepsilon x$ in (a) and the integrable perturbation $\varepsilon x^2$ in (b). In each case the thick line shows the envelope predicted by first-order canonical perturbation theory and the thin line shows the angular dependence of $|\psi_n(m)|^2$, normalised to have unit average, with $n = 2$ and $m = 17$ for $\varepsilon = -1/40$ and $h = 1/40$. The respective Poincaré plots, shown in (c) and (d), are defined by setting $p_r = 0$. Even though the perturbations are of similar strength in each case, the primary islands near $L = 0$ appear only at second order in the nonintegrable case and are much smaller, growing as $\varepsilon$ rather than $\sqrt{\varepsilon}$.](image)
logical difference between the complexified dynamics of nonintegrable and integrable perturbations, which essentially amounts to a symmetry-breaking of the rays escaping to asymptopia. These escaping rays are real in the integrable case but are slightly complex for nonintegrable perturbations and the resulting complex eikonal phase strongly modulates the outgoing wave.

As a simple model of a deformed resonator we consider the Schrödinger equation for a perturbation $V(x) = V_0(r) + \varepsilon V_1(x)$ of a central potential well $V_0(r)$. This is analogous to a scalar optical problem $\nabla^2 \psi(x) + k^2 n^2(x)\psi(x) = 0$ with refractive index and wave number defined by $k^2 n^2(x) = 2(E - V)/\hbar^2$ (assuming unit mass). Of course, optical resonators have sharp material interfaces which should be modelled by step potentials in the Schrödinger equation. We present numerical evidence that a similar mechanism is at play for such systems in Fig. 2, where strong directionality is found in emission patterns of a circular resonator (Fig 2(a) and 2(b)) which are so weak that there is little sign of nonintegrability in Poincaré plots (Fig. (2(d)). We confine our detailed analytical work in this Letter however to smooth (or more accurately, analytic) potentials. The underlying idea can be set out more transparently this way and we will treat step potentials more properly in a future publication. For similar reasons, we will present details for the two-dimensional case only and assert that it will be obvious how to adapt the discussion to three dimensions.

![FIG. 2: Emission patterns are shown for two perturbations of a circular resonator with refractive index $n = 2$. An elliptical deformation is shown in (a) and a generic perturbation is $r = 1 + \varepsilon \cos 3\chi$ is shown in (b). In each case, $\varepsilon = 1/400$, where the elliptical perturbation is $r \approx 1 + \varepsilon \cos 2\chi$ at first order and the resonance has $m = 50$ and $\text{Re}(k) \approx 33.19$. A two-dimensional illustration and a Poincaré-Birkhoff plot are shown for the generic perturbation in (c) and (d) respectively. In (c), the intensity of the emitted wave has been exaggerated to make it visible.](image)

We consider a Gamow-Siegert state $\psi_{nm}(x)$, defined as a solution of the Schrödinger equation for a complex energy $E = E_0 - i\Gamma/2$ which satisfies radiating boundary conditions at infinity. Here $n$ and $m$ denote radial and azimuthal quantum numbers of the resonance in the limit of zero perturbation. In the short-wavelength limit we can associate this state with an invariant torus of classical rays moving in the plane configuration space in an annulus bounded by inner and outer caustics $\Gamma_1$ and $\Gamma_2$, as illustrated in Fig. 3.

![FIG. 3: Real and complex rays are described in the complex plane as shown. $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are caustics and $C_1$, $C_2$ and $C_3$ are contours in the complex $\theta$ plane which define loops on these tori as shown at left.](image)

Immediately outside $\Gamma_2$ is a band of complex rays where $\psi_{nm}(x)$ decays exponentially in the radial direction. We argue below that this band has an outer caustic $\Gamma_3$ which joins with a family of rays escaping to infinity (see Fig. 3). It is these escaping rays which determine the emission pattern. The wavefunction outside $\Gamma_3$ has an eikonal form

$$\psi_{nm}(x) \approx A(x)e^{iS(x)/\hbar}$$

where $S(x)$ is the action function of the escaping rays. We expand the action in the form

$$S(x) = S_0(x) + \varepsilon S_1(x) + \cdots,$$

where $S_1(x)$ is determined explicitly below using canonical perturbation theory. For integrable problems the escaping rays are necessarily real [17] and so therefore is $\nabla S$. This implies that any angular variation in the magnitude of the emitted wave derives from the amplitude $A(x)$ and is slight if $\varepsilon$ is small. If the escaping rays are complex on the other hand, as we will argue they are in the generic nonintegrable case, then the emission pattern is dominated by the imaginary part of the action

$$|\psi_{nm}(x)|^2 \propto e^{-2\varepsilon \text{Im} S_1(x)/\hbar}$$

at leading order. In this case a perturbation of the order of the small parameter $\hbar$ suffices to dramatically alter the envelope of the outgoing wave. We now set out the details of this calculation.

We let $(I, \theta)$ and $(L, \phi)$ respectively denote the action-angle pairs of variables for the radial and azimuthal degrees of freedom in the unperturbed limit and let $\omega$ and $\Omega$ denote the corresponding frequencies. Because the actions are fixed by the torus quantisation rules $I = (n + 1/2)\hbar$ and $L = m\hbar$ at leading order, we will suppress the actions notationally in much of what follows, writing $r(\theta)$ instead of $r(\theta, I)$ for example. We denote by $(r, \chi)$ the standard polar coordinates in the plane and point out that $\chi$ and $\phi$ are distinct but related by a shift

$$\chi = \phi + \gamma(\theta),$$
correction to the phase of

We first note that

where

\[ \gamma(\theta) = \frac{1}{\omega} \int_0^\theta \left( \frac{L}{r^2(\theta')} - \Omega \right) d\theta'. \]

We adapt the convention that \( \theta = 0 \) on the outer caustic \( \Gamma_2 \) of the real torus (Fig. 3). To characterise emission we must construct the complex rays in the classically forbidden region outside \( \Gamma_2 \). We first describe how this is achieved in the unperturbed limit.

Starting on the caustic \( \Gamma_2 \) where \( \theta = 0 \), and letting \( \theta \) move down the imaginary axis (the rays which define decaying WKB solutions are in the lower-half plane), we find that the radial coordinate \( r(\theta) \) evolves periodically. Denote the imaginary period by \( 2i\Theta \), so that

\[ r(\theta + 2i\Theta) = r(\theta). \]

As angles \( (\theta, \chi) \) respectively range over the imaginary and real axes, a two-dimensional complex extension of the real torus is swept out which itself has the topology of a torus and on which radial momentum \( p_r \) is imaginary and \( r \) is real. This second torus describes a band of evanescent decay immediately outside the resonator. It has an inner caustic \( \Gamma_3 \), where it touches the real torus, and an outer caustic \( \Gamma_3^* \). The caustic \( \Gamma_3 \) corresponds to \( \theta = -i\Theta \). If \( \theta \) evolves horizontally from \(-i\Theta\) (contour \( C_3 \) in Fig. 3) then we describe a family of real orbits escaping to infinity.

The crucial feature in this description (of the unperturbed case) is that the orbits escaping to infinity are real. This is related in the following way to the fact that the radial coordinate and other phase space functions are biparabolic functions of \( \theta \) — in addition to the usual real period \( r(\theta + 2\pi) = r(\theta) \) there is an imaginary period expressed in (2). We have stated that the escaping rays are launched from the contour \( C_3 \) in the complex plane (see Fig. 3). They are real if they coincide with their complex conjugates, which are found along the conjugate contour \( C_3^* \). Since \( C_3^* = C_3 + 2i\Theta \), the rays are therefore real if there is an imaginary period, as claimed. We will now show that if the rays are approximated using canonical perturbation theory, this second period is destroyed at first order and the rays therefore become complex.

We use a type-two generating function \( F_2(\theta, \phi, \bar{I}, \bar{L}) = \theta \bar{I} + \phi \bar{L} + \varepsilon G(\theta, \phi, \bar{I}, \bar{L}) \) to generate the transformation to action-angle variables \( (\theta, \phi, \bar{I}, \bar{L}) \) for the perturbed system. At first order, \( G(\theta, \phi, \bar{I}, \bar{L}) \) provides the leading correction to the phase of \( \psi_{nm}(x) \) and is a solution of

\[ \omega \frac{\partial G}{\partial \theta} + \Omega \frac{\partial G}{\partial \phi} = -U(\theta, \phi), \tag{3} \]

where \( U(\theta, \phi) \) is the oscillating part of \( V_1(x) \), expressed in action-angle variables [14]. As in the unperturbed problem, \( (\bar{I}, \bar{L}) \) are fixed by torus quantisation conditions and are suppressed notationally from now on.

The perturbing potential \( U(\theta, \phi) \) inherits a biparabolic structure from the radial degree of freedom as follows. We first note that

\[ \gamma(\theta + i\Theta) = \gamma(\theta) - i\Phi, \]

where

\[ \Phi = \int_0^{\Theta/\omega} \left( \Omega - \frac{L}{r^2(\theta')} \right) d\tau \]

is real. The potential is periodic with respect to \( (\theta, \chi) \rightarrow (\theta + 2i\Theta, \chi) \), which, in terms of dynamical angles, gives

\[ U(\theta + 2i\Theta, \phi + 2i\Phi) = U(\theta, \phi). \]

Let the perturbing potential have the form

\[ U(r, \chi) = \sum_k U_k(r)e^{ik\chi} = \sum_k u_k(\theta)e^{ik\phi} \]

where \( u_k(\theta) = U_k(r(\theta))e^{ik\gamma(\theta)} \). We substitute

\[ G(\theta, \phi) = \sum_k g_k(\theta)e^{ik\phi} \]

in (3) and solve the resulting equation for \( g_k(\theta) \) to get

\[ g_k(\theta) = e^{-ik\Omega\theta/\omega}g_0(\theta) - \frac{e^{-ik\Omega\theta/\omega}}{\omega} \int_0^\theta e^{ik\Omega\theta'/\omega}u_k(\theta')d\theta'. \]

The integration constant \( g_0(\theta) \) is fixed by imposing the period \( g_k(\theta + 2\pi) = g_k(\theta) \) on the real axis. The resulting generating function is single-valued on the real axis, where it gives the real torus between caustics \( \Gamma_1 \) and \( \Gamma_2 \).

Emission patterns are dominated by the imaginary part of \( G(\theta, \phi) \) along \( C_3 \). In light of \( C_3^* = C_3 + 2i\Theta \), this can be written \( \text{Im} G(\theta, \phi) = \Delta G(\theta, \phi)/(2i) \), where

\[ \Delta G(\theta, \phi) = G(\theta, \phi) - G(\theta + 2i\Theta, \phi + 2i\Phi). \]

It can be shown that

\[ \Delta G(\theta, \phi) = i\kappa_0 + iK(\theta, \phi), \]

where

\[ \kappa_0 = \frac{1}{\omega} \int_0^{2\Theta} u_0(\theta) d\theta = \int_0^{2\Theta/\omega} u_0(i\omega\tau) d\tau \]

is a real constant and

\[ K(\theta, \phi) = \sum_{k\neq 0} \frac{e^{ik(\phi - \Omega\theta'/\omega)}}{1 - e^{2\pi ik\Omega/\omega}} \frac{1}{i\omega} \int_C e^{ik\Omega\theta'/\omega}u_k(\theta')d\theta'. \]

The contour \( C \) is a clockwise circuit of the rectangle at right in Fig. 3 (a derivation of an analogous result in the context of double-well splittings can be found in [14]).

It is convenient to express the result using the angles

\[ \alpha(\theta, \phi) = \phi + i\Phi - \frac{\Omega}{\omega}(\theta + i\Theta), \]

\[ \beta(\theta) = \frac{\Omega}{\omega}(\theta + i\Theta) + \gamma(\theta + i\Theta) = \frac{1}{\omega} \int_0^\theta \frac{L}{r^2(\theta')} d\theta', \]

Note that \( \alpha \) is real and constant along the escaping rays of the unperturbed problem and is the value of the polar
angle \( \chi \) at which a given ray touches the caustic \( \Gamma_3 \). We also have \( \chi = \alpha + \beta \), so \( \beta \) measures the angular deflection of the escaping ray as it moves away from \( \Gamma_3 \). Then

\[
K(\theta, \phi) = \sum_{k \neq 0} \kappa_k e^{i k \alpha}
\]

where

\[
\kappa_k \neq 0 = \frac{1}{1 - e^{2 \pi i k \alpha / \omega}} \int_C e^{i k \beta'(\theta')} U_k(r(\theta')) d\theta'.
\]

The main conclusion of this Letter is that the asymptotic intensity of a WKB mode (1) is modulated by a function

\[
\mathcal{M}(\alpha) = e^{-\varepsilon K(\alpha)/h}
\]

which can vary strongly with \( \chi \) even for \( \varepsilon = O(h) \).

We illustrate this result numerically for the potential \( V_0(r) = -(r^2 - 1)^2 \). Although this potential does not vanish at infinity, it defines Gamow-Siegert states entirely analogous to those of asymptotically free problems and is used because solutions are easily obtained for it using complex rotation in a harmonic-oscillator basis. None of the calculations above are substantially affected by this choice. We consider two perturbations, \( V_1(x) = x \), which behaves generically, and \( V_2(x) = x^2 \), for which the total potential can be shown to be separable in elliptic coordinates and which therefore provides us with an example of an integrable perturbation. The respective angular distributions of the outgoing intensity, normalised to having unit average, are shown in Figs. 1(a) and 1(b). The heavy curves show the modulation function (4). Note that for the integrable perturbation we necessarily find that \( K(\alpha) = 0 \) and the small modulation in the numerical solution is a result of corrections to the amplitude \( A(x) \), which have not been taken into account. In the nonintegrable case, the modulation argument can be written \( K(\alpha) = 2 \kappa_1 \cos \alpha \) and provides a good description of the numerical emission pattern (Fig. 1(a)).

A similar modulation is seen in the numerical results for a deformed cavity shown in Fig. 2. The dielectric ellipse is a special case — it is not separable but the real and complex rays can nonetheless be calculated analytically. The envelope shown is \( \mathcal{M}(\alpha) = \exp[2 \pi k (n^2 - 1)/n^2 p^2 - 1 \cos 2\alpha] \), where \( p = m/k \), whose derivation will be described in a future publication. The envelope for the generically nonintegrable perturbation in (b) has not been calculated analytically but has been found numerically to be well described by a function of the form \( \mathcal{M}(\alpha) = \exp[\varepsilon K(\alpha) \cos 2\alpha] \), with a typical case shown in the figure. This is consistent with the mechanism set out in this Letter.

In conclusion, canonical perturbation theory married with complex WKB approximation can successfully describe strongly directional emission from weakly deformed resonators. The underlying mechanism is not transparently related to the structure of real phase space and there is a stark qualitative difference between integrable and nonintegrable systems, even when the signatures of nonintegrability are slight in real phase space [18]. Despite this sensitivity to dynamical detail, simple analytical formulas can be given for the observed output. The theory has been presented for smooth potentials but a similar mechanism is expected to govern sharp cavity problems and should provide a useful analytical tool to understand the evanescent field outside optical resonators, which is important to applications such as lasers and sensors [2].

\[\text{References}\]

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[17] Integrable systems are characterised by the existence of a global invariant which is real-valued on real phase space. The escaping family of rays is determined by fixing this invariant, along with energy, and is seen therefore to be real. (The very small imaginary part of the energy \( E = E_0 - i \Gamma/2 \) leads to some modulation along individual rays but does not affect angular dependence and is ignored.)
[18] The total area of chaotic layers in Fig. 1(a) is many orders of magnitude smaller than \( h \), for example.