dS/CFT correspondence from the Brick Wall method

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ABSTRACT

We research the entropy of a black hole in curved space-times by ’t Hooft’s approach, so-called the brick wall method. One of these space-time, a asymptotically dS space-time has two physical horizons; one is a black hole horizon and the other is a cosmological horizon. The others have only one horizon, a black hole horizon. Using this model, we calculate all thermodynamic quantities containing the background geometric effect and show that entropy is proportional to area of each boundary. Furthermore, we show that the Cardy-Verlinde formula can be rederived from the physical quantities of the brick wall method and this fact becomes a evidence of the dS/CFT correspondence.

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1 Introduction

Recently, much attention have been focused on studying the de Sitter (dS) space-time and asymptotically dS space-time. The motivation of this subject can be described by two aspects. First, some astronomical observations tell us that our universe with black hole might be asymptotically de Sitter space-time and approach to a pure de Sitter space-time as time goes \[1, 2, 3, 4, 5\]. Secondly as an analogy of AdS/CFT correspondence\[6\], dS/CFT correspondence\[7, 8\] between quantum gravity on de Sitter space-time and the Euclidean conformal field theory (CFT) on a boundary of the de Sitter space-time\[1, 10, 11, 12, 13, 14, 15\] was suggested.

Moreover, Verlinde\[16\] argued that the Cardy formula\[17\], describing the entropy of a 1+1-dimensional CFT, can be generalized to an arbitrary dimension and this generalization gives a so called Cardy-Verlinde formula. This argument was applied to the quantum gravity in a dS space-time, which is dual to a certain Euclidean CFT living on the space-like boundary of the dS space-time and some investigations on this subject have been carried out recently\[18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\].

It is well known that in the dS space-time, there is a cosmological horizon, which has the similar thermodynamic properties like the black hole horizon. So the thermodynamics of the cosmological horizon in asymptotically dS space can be identified with that of a certain Euclidean CFT living on a spacelike boundary by the dS/CFT correspondence. But in cases of the flat and the Anti de Sitter space-time, there is no such a cosmological horizon so that we can not find such a dS/CFT correspondence. When we define the Cardy-Verlinde formula, we have to face a difficulty to calculate some conserved charges including the mass (total energy) of the gravitational field of the asymptotically dS space-time. In the spirit of the dS/CFT correspondence, these conserved charges of the gravitational field can be identified with those of the corresponding Euclidean CFT. The difficulty arises due to the absence of the spatial infinity and the global timelike killing vector. In ref\[23\], authors used the AD mass prescription\[31\] on a black hole horizon and the BBM mass prescription on a cosmological horizon\[32\] so that the entropy for each horizon can be written as the Cardy-Verlinde formula. But in this paper, following the brick wall method\[33, 34, 35\], we will calculate the total energy of the gravitational field. Since the energy obtained here is not the mass of the black hole but the thermal energy of the gravitational field near the each horizon, so this is not exactly the same as the mass of the black hole. Moreover, due to the curved background geometry the extra term is added to the thermodynamic quantities, which can be removed in the large radius limit, in other word,
the flat space-time. Although the definition of the total energy is slightly modified, the similar Cardy-Verlinde formula can be obtained from this brick wall method. This fact that we can obtain the similar Cardy-Verlinde formula using various methods, becomes a good evidence for the dS/CFT correspondence.

The organization of this paper is as follow. In the next section, we review the brick wall method for Schwarzschild space-time. In section 3, applying the brick wall method, we calculate thermodynamic quantities of the curved background space-time and in the asymptotically dS space-time. In this space-time where instability occurs, we show that the asymptotically dS space-time with a black hole must naturally evolve to a pure dS space-time. Finally, using these thermodynamic quantities the Cardy-Verlinde formula is obtained. In section 4, we conclude with some discussion.

2 Revisited 4-D Schwarzschild space-time

In this section, we review the brick wall method in ref[33]. In ref[33] 't Hooft contrived the good exercise to ratiocinate the quantum statistical quantities of a Schwarzschild black hole. His basic idea is that he identify the entropy calculated by counting energy states of a particle near horizon with that of a black hole. However, since the number of energy level for a particle to occupy at the horizon diverges, for the energy level to be physically acceptable, one need the UV cutoff $h_+$ of which the role is to make the energy level finite. Moreover, one must impose an IR cutoff $L$ to make the infinite volume factor finite in a free energy. This gives us the consistent result for the Schwarzschild black hole. Somehow, to compare thermodynamics of 4-D Schwarzschild space-time with the other space-times with non-zero cosmological constant, we will review the brick wall method.

Now, we start with the 4-D Schwarzschild black hole metric;

$$ds^2 = -g(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega_2^2,$$

where

$$g(r) = 1 - \frac{2M}{r}.$$  \hspace{1cm} (2.1)

$M$ is the black hole mass. Since $M > 0$, the metric function $g(r)$ has one positive root $R_+$ which is the black hole horizon.

To calculate the number of the energy level in the vicinity of the black hole horizon, we consider the wave equation of massless scalar field $\Phi$, which describes the quantum
fluctuation of the space-time;

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \Phi) = 0. \quad (2.3)$$

To solve this wave equation, we impose the spherical symmetry and use the following ansatz

$$\Phi(x) = e^{-iEt} R(r) Y_{lm}(\theta, \phi). \quad (2.4)$$

As in ref[33], we restrict the wave function to the region $R_+ + h_+ < r < L$;

$$\Phi(x) = 0 \quad \text{if} \quad r \leq R_+ + h_+ \quad \text{and} \quad r \geq L \quad (2.5)$$

By inserting the equation(2.4) into the equation(2.3), we get the radial equation;

$$\frac{E^2}{g(r)} R(r) + \frac{1}{r^2} \partial_r \left[ r^2 g(r) \partial_r R(r) \right] - \left[ \frac{l(l + 1)}{r^2} \right] R(r) = 0. \quad (2.6)$$

To use WKB approximation, we put $R(r) = A(r) e^{iS(r)}$ where $A(r)$ is a slowly varying amplitude and $S(r)$ is a fast varying frequency. Then we look a little complicated equation;

$$\frac{E^2}{g(r)} A(r) + g(r) \left[ \partial_r^2 A(r) + 2i \left( \partial_r A(r) \right) \left( \partial_r S(r) \right) + i \left( \partial_r^2 S(r) \right) A(r) - \left( \partial_r S(r) \right)^2 A(r) \right] + \frac{2}{r} g(r) \left[ \partial_r A(r) + i \left( \partial_r S(r) \right) A(r) \right] + \left[ \partial_r g(r) \right] \cdot \left[ \partial_r A(r) + i \left( \partial_r S(r) \right) A(r) \right] - \left[ \frac{l(l + 1)}{r^2} \right] A(r) = 0.$$

Finally, if we assume that most contribution of above equation is the quadratic term of $\partial_r S(r)$, we can define the radial wave number $k^2(r)$ given by $[\partial_r S(r)]^2$;

$$k^2(r) = \frac{1}{g(r)} \left[ \frac{E^2}{g(r)} - \frac{l(l + 1)}{r^2} \right]. \quad (2.7)$$

The number of radial mode $n_r$ by counting the number of nodes in the radial wave function is given by

$$\pi n_r = \int_{R_+ + h_+}^{L} \sqrt{\frac{r}{(r - 2M)}} \left[ \frac{r E^2}{r - 2M} - \frac{l(l + 1)}{r^2} \right] dr, \quad (2.8)$$
where $L$ and $h_+$ are introduced to regularize the above integration. If there are no such regulators, some divergences occur at $r \to R_+ (= 2M)$ and $r \to \infty$ in the equation (2.8).

The density of states, $\omega(E_n)$ at each energy level $E_n$ are given by

$$\omega(E_n) = \int \pi n_s (2l + 1) dl$$

$$= \frac{2E_n^3}{3} \int_{R_+ + h_+}^{L} \frac{r^2}{g^2(r)} dr. \tag{2.9}$$

In the equation (2.9) the sum over the angular quantum number $l$ is approximated by integral and the factor $2l + 1$ implies the degeneracy of $L_z$. Hence the free energy $F$ reads

$$F = -\frac{2\pi^3}{45\beta^4} \int_{L}^{R_+ + h_+} \frac{r^2}{(1 - \frac{2M}{r})^2} dr. \tag{2.10}$$

For convenience, we set $g(r) = \frac{f(r)}{r}$ and $f(r) = r - 2M$. If we expand $f(r)$ near the horizon ($r = R_+ + x$, for small $x > 0$), then we get

$$f(r) = f(R_+) + xf'(R_+)$$

$$= \alpha_+ x \tag{2.11}$$

where $\alpha_+ \equiv f'(R_+) = 1$.

Near the horizon, the free energy is rewritten as

$$F \approx -\frac{2\pi^3}{45\beta^4} \int_{h_+}^{L-R_+} \frac{(R_+ + x)^4}{\alpha_+^4 x^2} dx$$

$$= -\frac{2\pi^3}{45\beta^4 \alpha_+^4} \left[ \frac{R_+^{L-R_+}}{x} \right]_{h_+}^L - \frac{2\pi^3}{45\beta^4 \alpha_+^4} \left[ \frac{1}{3} x^3 \right]_{h_+}^{L-R_+} + \cdots$$

If we consider only leading terms, we see that the free energy is

$$F \approx -\frac{2\pi^3}{45h_+} \left( \frac{R_+^4}{\alpha_+^4 \beta^4} \right) - \frac{2\pi^3}{135} \frac{1}{\alpha_+^4 \beta^4} \left[ (L - R_+)^3 - h_+^3 \right]. \tag{2.12}$$

Since the first term grows up when $h_+$ decreases, it can be regarded as the contribution of the black hole to the free energy. The second term contains the finite volume effect. However, we are only interested in the black hole thermodynamics and so the second
term proportional to the volume is ignored. Then, the free energy of the black hole, $F_+$ is given by
\begin{equation}
F_+ = -\frac{2\pi^3}{45h_+} \left( \frac{R_+^4}{\alpha_+^2 \beta_+^4} \right)
\end{equation}
where the inverse temperature $\beta_+$ at horizon is
\begin{equation}
\frac{1}{\beta_+} = T_+ = \frac{1}{4\pi} \left[ \frac{d}{dr} g(r) \right]_{r=R_+} = \frac{\alpha_+}{4\pi R_+}.
\end{equation}
Using the usual thermodynamic formula, we find that the total energy $U_+$ and the entropy $S_+$ are
\begin{align}
U_+ &= \frac{\partial}{\partial \beta_+} (\beta_+ F_+) \\
&= \frac{2\pi^3}{15h_+} \left( \frac{R_+^4}{\alpha_+^2 \beta_+^4} \right) \\
S_+ &= \beta_+ (U_+ - F_+) = \beta_+^2 \frac{\partial F_+}{\partial \beta_+} \\
&= \frac{8\pi^3}{45h_+} \left( \frac{R_+^4}{\alpha_+^2 \beta_+^4} \right)
\end{align}
For the entropy satisfying the area law\cite{36, 37}, we take the UV cutoff $h_+$ to be $1/360\pi R_+$. Accordingly, the free energy $F_+$, the total energy $U_+$ and the entropy $S_+$ result into
\begin{align}
F_+ &= -\frac{1}{16} \alpha_+ R_+ \\
U_+ &= \frac{3}{16} (\alpha_+ R_+) \\
&= \frac{3}{8} M \\
S_+ &= \pi R_+^2 = \frac{A_+}{4}
\end{align}
where $A_+$ is the area of the black hole horizon. Notice that $U$ is the thermal energy of the quantum fluctuations and this is not equal to the black hole mass.
Next, we calculate the invariant distance of the brick wall which is independent of coordinates as ref\[33\].

\[
\int_{R_+}^{R_+ + h_+} ds = \int_{R_+}^{R_+ + h_+} \sqrt{\frac{r}{r - 2M}} dr \\
\approx \sqrt{\frac{1}{90\pi}}
\]

(2.20)

3 Thermodynamics of the black hole

3.1 Thermodynamic quantities

In this section, we will consider more general black hole metric in the curved background space-time. The metric is given by

\[
ds^2 = -g(r)dt^2 + \frac{1}{g(r)} dr^2 + r^2 d\Omega_2^2,
\]

(3.21)

where

\[
g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{k r^2}{l^2}.
\]

(3.22)

Here, \(M\) and \(Q\) are the mass and the charge of the black hole respectively and \(l\) is a space-time curvature radius. \(k\) implies the sign of the cosmological constant: \(k > 0\) and \(k < 0\) mean a de Sitter (dS) and a Anti de Sitter (AdS) space-time respectively and \(k = 0\) is a asymptotically flat space-time.

Note that if we choose a infinite radius (\(l \to \infty\)) in the case \(k \neq 0\), the dS and AdS go to the flat space-time (\(k = 0\)). So if we will have calculated physical quantities of the black hole in the non-zero cosmological constant background space-time, in the limit of \(l \to \infty\) these quantities must be consistent with those of the asymptotically flat space-time. Using this fact, we can indirectly prove that our calculations in curved background are correct. Though a black hole with a mass \(M\) and a charge \(Q\) has usually two horizons, the large one of these is called an event horizon of the black hole, \(R_+\) determined by

\[
g(R_+) = 1 - \frac{2M}{R_+} + \frac{Q^2}{R_+^2} - \frac{k R_+^2}{l^2} = 0,
\]

(3.23)

which is the radius of the black hole. Since in the inside of the event horizon there is no time-like curve, we consider the outside of the event horizon, \(r > R_+\). For the
dS space-time ($k > 0$), there exists another horizon called a cosmological horizon $R_c$ determined by

$$g(R_c) = 1 - \frac{2M}{R_c} + \frac{Q^2}{R_c^2} - \frac{R_c^2}{l^2} = 0 \quad (3.24)$$

where we set $k = 1$. Since the time-like curve can be defined in the inside of the cosmological horizon, for the dS space-time with a black hole, we consider the time-like region $R_+ < r < R_c$ only. When $R_+ = R_c = l/\sqrt{3}$ for $Q = 0$, this black hole becomes a Narai black hole. Since unlike a dS space-time a flat and an AdS space-time have no such a cosmological horizon, the time-like region is given by $r > R_+$.

To probe the energy level of a particle near one or two horizons, we consider the wave equation of massless scalar field $\Phi$;

$$\frac{1}{\sqrt{-g}}\partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = 0. \quad (3.25)$$

The solution ansatz of this wave equation under the spherical symmetry is given by

$$\Phi(x) = e^{-iEt} R(r) Y_{lm}(\theta, \phi). \quad (3.26)$$

Here we restrict the wave function to the region $R_+ + h_+ < r < R_c - h_c$;

$$\Phi(x) = 0 \quad \text{if} \quad r \leq R_+ + h_+ \quad \text{and} \quad r \geq R_c - h_c \quad (3.27)$$

By inserting the equation (3.26) into the equation (3.25), we get the radial equation

$$\frac{E^2}{g(r)} R(r) + \frac{1}{r^2} \partial_r \left[ r^2 g(r) \partial_r R(r) \right] - \left[ \frac{l(l + 1)}{r^2} \right] R(r) = 0. \quad (3.28)$$

Using the similar WKB approximation in section 2, the radial wave number is given by

$$k^2(r) = \frac{1}{g(r)} \left[ \frac{E^2}{g(r)} - \frac{l(l + 1)}{r^2} \right], \quad (3.29)$$

which depends on the metric. The equation (3.29) means that the frequency of a particle near the horizon is highly excited and diverge at the horizon. These infinities are the UV divergences similar to problem in renormalization of QFT contents[38, 39, 40]. Hence we need some UV cutoffs: for the dS space-time there are two cutoffs, $h_+$ and $h_c$ and for the flat and the AdS space-time there is only one cutoff. The introduction of the cutoff makes the number of energy state of $\Phi$ physically meaningful. Since in the asymptotically dS space-time the infinite volume factor does not occur,
we do not have to introduce the IR cutoff. But in the asymptotically flat and AdS space-time, there is no cosmological horizon and so we need an IR cutoff to regularize the infinite volume effect which is not important. Hence we can interpret $R_c$ as an IR cutoff and in this case, $h_c$ must be set to zero in the asymptotically AdS and flat space-time. Under the restriction on the wave function to $R_+ + h_+ < r < R_c - h_c$, the number of radial mode $n_r$ by counting the number of nodes in the radial wave function is given by

$$\pi n_r = \int_{R_++h_+}^{R_c-h_c} k(r,l,E)dr.$$  \hspace{1cm} (3.30)

And at each energy level $E_n$, the number of state $\omega(E_n)$ is given by

$$\omega(E_n) = \int \pi n_r (2l + 1)dl = \frac{2E_n^3}{3} \int_{R_++h_+}^{R_c-h_c} \frac{r^2}{g^2(r)}dr.$$  

Hence $F$ is

$$F = -\frac{2\pi^3}{45\beta^4} \int_{R_++h_+}^{R_c-h_c} \frac{r^2}{g^2(r)}dr.$$  \hspace{1cm} (3.31)

For the dS space-time $(k = 1)$, the integration part is rewritten as

$$\int_{R_++h_+}^{R_c-h_c} \frac{r^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{k r^2}{r^4}\right)^2}dr = \int_{R_++h_+}^{L} \frac{r^6}{f^2(r)}dr + \int_{L}^{R_c-h_c} \frac{r^6}{f^2(r)}dr$$  \hspace{1cm} (3.32)

where for convenience we set $g(r) = \frac{f(r)}{r^2}$ and $f(r) = \left(r^2 - 2Mr + Q^2 - \frac{r^4}{r^4}\right)$. Note that $L$ does not represent the IR cutoff. In the cases of the flat $(k = 0)$ and the AdS space-time $(k = -1)$, because there is no cosmological horizon the integration part is written as

$$\int_{R_++h_+}^{R_c} \frac{r^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{k r^2}{r^4}\right)^2}dr = \int_{R_++h_+}^{L} \frac{r^6}{f^2(r)}dr + \int_{L}^{R_c} \frac{r^6}{f^2(r)}dr.$$  \hspace{1cm} (3.33)

Here $R_c$ in these flat and AdS background space-times is not a physical radius but an IR cutoff and so the second term of the above equation gives term proportional to the volume at leading order. Because this volume effect must be removed from our calculations like the Schwarzschild black hole case, all thermodynamic quantities come from the first term of the above equation, in other words, the calculation near the event horizon of the black hole $R_+$. 

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Now in the dS space-time, we expand $f(r)$ near two horizons $R_+$ and $R_c$. Near the event horizon $r = R_+ + x$, the function $f(r)$ is given by
\[ f(r) \simeq \alpha_+ x, \quad (3.34) \]
and near the cosmological horizon $r = R_c - x$, $f(r)$ becomes
\[ f(r) \simeq -\alpha_c x, \quad (3.35) \]
where $\alpha_+ \equiv (2R_+ - 2M - \frac{4k}{l^2}R_+^3) > 0$ and $\alpha_c \equiv (2R_c - 2M - \frac{4}{l^2}R_c^3) < 0$. Notice that since only the dS space-time has the cosmological horizon, $\alpha_c$ appears in the case of $k > 0$ only and we can set $k = 1$ in the definition of $\alpha_c$. If $M$ is considered as a function of $R_+$ and $R_c$ these two factors can be rewritten as
\[ \alpha_+ = \left( R_+ - \frac{Q^2}{R_+} - \frac{3kR_+^3}{l^2} \right) > 0 \quad \text{and} \quad \alpha_c = \left( R_c - \frac{Q^2}{R_c} - \frac{3R_c^3}{l^2} \right) < 0. \]
Consequently, the equation (3.32) for the asymptotically dS space-time is given by
\[
\frac{1}{\alpha_+^2} \int_{h_+}^{L-R_+} \frac{x + R_+}{x^2} \, dx + \frac{1}{\alpha_c^2} \int_{R_c-L}^{h_c} \frac{(R_c - x)}{x^2} \, d(-x)
= \frac{R_+^6}{\alpha_+^2} \left[ - \frac{1}{x} \right]_{h_+}^{L-R_+} + \frac{5R_+^5}{\alpha_+^2} \left[ \frac{x^3}{3} \right]_{h_+}^{L-R_+} + \frac{6R_+^5}{\alpha_+^2} \left[ \log x \right]_{h_+}^{L-R_+}
+ \frac{R_c^6}{\alpha_c^2} \left[ - \frac{1}{x} \right]_{h_c}^{R_c-L} + \frac{5R_c^5}{\alpha_c^2} \left[ \frac{x^3}{3} \right]_{h_c}^{R_c-L} - \frac{6R_c^5}{\alpha_c^2} \left[ \log x \right]_{h_c}^{R_c-L} + \cdots
\]
where the second and the fourth terms are proportional to the volume. Therefore at the leading order the free energy becomes
\[
F = F_+ + F_c
\]
\[
F_+ = -\frac{2\pi^3}{45h_+} \left( \frac{R_+^6}{\alpha_+^2 \beta_+^4} \right)
\]
\[
F_c = -\frac{2\pi^3}{45h_c} \left( \frac{R_c^6}{\alpha_c^2 \beta_c^4} \right),
\quad (3.36)
\]
where $F_+$ and $F_c$ are originated from of the black hole and the dS space-time respectively. In the flat and AdS space-time, due to the absence of the cosmological horizon, $F_c$ does not appears. Hence the free energy of the black hole in the asymptotically flat and AdS space-time is given by $F = F_+$. Applying the thermodynamic formula to calculate the temperature $T$, the total energy $U$, the chemical potential $\varphi$ and the
entropy $S$, we find that near the event horizon of the black hole

$$\frac{1}{\beta_+} \equiv T_+ = \frac{\alpha_+}{4\pi R^2_+} \quad (3.37)$$

$$U_+ = \frac{\partial}{\partial \beta_+}(\beta_+ F_+) = \frac{2\pi^3}{15 h_+} \left( \frac{R_+^6}{\alpha_+^2 \beta_+^4} \right) \quad (3.38)$$

$$\varphi_+ = \left( \frac{\partial}{\partial Q} F_+ \right)_{T_+} = -\frac{2Q}{R_+} \left( \frac{4\pi^3}{45 h_+} \cdot \frac{R_+^6}{\beta_+^4 \alpha_+^3} \right) \quad (3.39)$$

$$S_+ = \beta_+^2 \frac{\partial F_+}{\partial \beta_+} = \frac{8\pi^3}{45 h_+} \left( \frac{R_+^6}{\alpha_+^2 \beta_+^3} \right) \quad (3.40)$$

and near the cosmological horizon of the asymptotically dS space-time

$$\frac{1}{\beta_c} \equiv T_c = \frac{-\alpha_c}{4\pi R^2_c} \quad (3.41)$$

$$U_c = \frac{\partial}{\partial \beta_c}(\beta_c F_c) = \frac{2\pi^3}{15 h_c} \left( \frac{R_c^6}{\alpha_c^2 \beta_c^4} \right) \quad (3.42)$$

$$\varphi_c = \left( \frac{\partial}{\partial Q} F_c \right)_{T_c} = -\frac{2Q}{R_c} \left( \frac{4\pi^3}{45 h_c} \cdot \frac{R_c^6}{\beta_c^4 \alpha_c^3} \right) \quad (3.43)$$

$$S_c = \beta_c^2 \frac{\partial F_c}{\partial \beta_c} = \frac{8\pi^3}{45 h_c} \left( \frac{R_c^6}{\alpha_c^2 \beta_c^3} \right) \quad (3.44)$$
Notice that all thermodynamic quantities are locally defined at each horizon. So in the case of the asymptotically dS space-time containing the black hole there are two temperatures, which cause the instability of this system. In the next section, we will comment this instability.

In the same way in section 2, we choose the invariant distances of the brick walls

\[ h_+ = \frac{\alpha_+}{360\pi R^2} \quad \text{and} \quad h_c = \frac{-\alpha_c}{360\pi R^2_c}, \tag{3.45} \]

then we find that the free energy \( F_+ \), the total energy \( U_+ \), the chemical potential \( \varphi_+ \) and the entropy \( S_+ \) near the event horizon are

\[
egin{align*}
F_+ & = \frac{-1}{16} \left( R_+ - \frac{Q^2}{R_+} - \frac{3R^3_+}{l^2} \right) \tag{3.46} \\
U_+ & = \frac{3}{16} \left( R_+ - \frac{Q^2}{R_+} - \frac{3R^3_+}{l^2} \right) \tag{3.47} \\
\varphi_+ & = -\frac{Q}{4R_+} \tag{3.48} \\
S_+ & = \pi R^2
\end{align*}
\]

and those near the cosmological horizon are

\[
egin{align*}
F_c & = \frac{1}{16} \left( R_c - \frac{Q^2}{R_c} - \frac{3R^3_c}{l^2} \right) \tag{3.50} \\
U_c & = -\frac{3}{16} \left( R_c - \frac{Q^2}{R_c} - \frac{3R^3_c}{l^2} \right) > 0 \tag{3.51} \\
\varphi_c & = \frac{Q}{4R_c} \tag{3.52} \\
S_c & = \pi R^2_c \tag{3.53}
\end{align*}
\]

Notice that the entropy at each horizon satisfies the area law.

To convince that our result is reasonable, take the limit of the zero cosmological constant limit, \( \Lambda \to 0 \) (the curvature radius \( l \to \infty \)). In the SdS (Schwarzschild dS) black hole \( (Q = 0) \) case, using the equation (3.23), the total energy \( U_+ \) can be rewritten as

\[ U_+ = \frac{3}{8} \left( M - \frac{2R^3}{l^2} \right). \tag{3.54} \]

In the limit \( l \to \infty \), which in the equation (3.54) means that the dS space-time goes to the flat space-time, \( U_+ \) is exactly equal to that of Schwarzschild black hole in section 2 and \( R_+ \to 2M \). As mentioned before, in flat space-time the cosmological horizon
$R_c$ must be interpreted as the IR cutoff and so we have to ignore all quantities $F_c$, $U_c$ and $S_c$ near the cosmological horizon. As a result, in this limit we can check that the total energy $U_+$ and entropy $S_+$ are the same as those of the Schwarzschild black hole [33];

$$U_+ = \frac{3}{8} M \quad \text{and} \quad S_+ = \pi (R_+)^2. \quad (3.55)$$

In ref[24, 25], the black hole mass was identified with the energy of the black hole and the dS space-time, for example $U_+ = M$ and $U_c = -M$, which are different with our results. For the Schwarzschild black hole case, the used metric is valid to observer who live in the asymptotically flat space-time and so the black hole mass $M$ is a special quantity observed by such an observer. In generally curved space-time, since there is no global timelike killing vector, it is not clear how to define the black hole mass. But if the black hole mass is defined, it has to be that of the Schwarzschild black hole in the large $l$ limit and contain the effect of the background geometry. However, in equation (3.54), the first term, which survives in the limit of $l \to \infty$, is the effect of the black hole in the asymptotically flat space-time and is the exactly same as the result of [24, 25] up to $\frac{3}{8}$ factor. The second term is the background geometric effect which vanishes in the flat space-time. Therefore, if we ignore the factor $\frac{3}{8}$ - whose origin is not clear but we guess that it comes from the thermalization of the quantum fluctuations, all thermodynamic quantities obtained here are the most reliable.

### 3.2 Instability of the black hole in the dS space-time

As mentioned in the previous section, in the asymptotically dS space-time containing a black hole there are two temperatures defined locally at each horizon. These two temperatures imply that this system is not an equilibrium state and so the energy transfer can occur. In our calculation, it is shown that at the leading order the total entropy of this system is given by the sum of those at each horizon. For convenience considering only the Schwarzschild dS space-time ($Q = 0$), the total entropy at leading order is given by

$$S = S_+ + S_c = \pi R_+^2 + \pi R_c^2. \quad (3.56)$$

From the equation(3.23) and the equation(3.24), we get

$$R_+^2 + R_c R_+ + R_c^2 - l^2 = 0 \quad (3.57)$$
Inserting the equation (3.57) into (3.56) and after some calculation, $S$ is described as the function of $R_c$:

$$S = \pi l^2 - \pi R_c R_+$$

$$= \pi l^2 + \frac{\pi}{2} \left( R_c^2 - R_c \sqrt{4l^2 - 3R_c^2} \right)$$

(3.58)

where $\frac{l}{\sqrt{3}} < R_c < l$ and $S$ have no extremum value in this region. Consequently, when $R_c \to l$, $S$ approaches to maximum. So the maximal entropy $S_{\text{max}}$ reads

$$S \to S_{\text{max}} = \pi l^2.$$  

(3.59)

Note that this maximal entropy is the same as that of the pure dS space-time. According to the thermodynamic law, since all physical system evolves in the direction of increasing the entropy, the Schwarzschild dS space-time evolves to the pure dS space-time. This means that the black hole disappears due to its evaporation which decreases the mass and makes the cosmological horizon $l$. This result is equal to that of ref[27]. Therefore the Schwarzschild dS space-time is unstable, while the pure dS space-time is stable.

When the mass of the black hole $M$ increases, the event horizon expands and the the cosmological horizon shrinks. Finally when the event horizon meets the cosmological horizon $R_c = R_+ = \frac{l}{\sqrt{3}}$, the black hole becomes the Narai black hole which have the maximal size in the dS space-time. Although the Narai black hole by itself has the maximal entropy, the total entropy of this space-time becomes a minimal value;

$$S_{\text{min}} = \frac{2\pi}{3} l^2.$$  

(3.60)

3.3 The Cardy-Verlinde formula

In this subsection, the Cardy-Verlinde formula in ref[16] is rederived from the brick wall method. The total energy $U(S, V)$ may be written as a sum of two terms

$$U(S, V) = E_{\text{ext}}(S, V) + E_{\text{sub}}(S, V)$$

(3.61)

where $E_{\text{sub}}(S, V)$ is the sub-extensive energy which can contains not only the casimir energy $E_c(S, V)$ but also other sub-extensive energy. In the general charged dS black hole, the total energy near the black hole is given by

$$U = E_{\text{ext}} + E_Q + \frac{1}{3} E_C.$$  

(3.62)
where
\[
\begin{align*}
E_{\text{ext}} & = -\frac{9}{16l^2}R_+^3 \\
E_Q & = -\frac{3Q^2}{16R_+} \\
\frac{E_C}{3} & = \frac{3}{16}R_+.
\end{align*}
\]

(3.63)

Notice that \(E_Q\) is another sub-extensive energy and the factor \(\frac{1}{3}\) in front of the casimir energy is included because we consider four-dimensional space-time with three spatial dimensions. From these energies, we find the Cardy-verlinde entropy \((S_+)_{C-V}\)
\[
(S_+)_{C-V} = \frac{2\pi l}{3} \sqrt{\frac{E_C}{3} \left\{ 3(U_+ - E_Q) - E_C \right\}}.
\]

(3.64)

After more some calculation, we find that the Cardy-Verlinde entropy is equal to the entropy obtained in the brick wall method, \((S_+)_{b-w}\), up to some factor;
\[
(S_+)_{C-V} = \frac{3}{8}(S_+)_{b-w}.
\]

(3.65)

Similarly we reproduce above the procedure near the cosmological horizon. From the total energy(3.50) near the cosmological horizon,
\[
U = E_{\text{ext}} + E_Q + \frac{1}{3}E_C
\]

(3.66)

where
\[
\begin{align*}
E_{\text{ext}} & = \frac{9}{16l^2}R_c^3 \\
E_Q & = \frac{3Q^2}{16R_c} \\
\frac{E_C}{3} & = -\frac{3}{16}R_c,
\end{align*}
\]

(3.67)

the relation between the Cardy-Verlinde entropy \((S_c)_{C-V}\) and the brick wall entropy \((S_c)_{b-w}\) is given by
\[
(S_c)_{C-V} = \frac{2\pi l}{3} \sqrt{\frac{E_C}{3} \left\{ 3(U_c - E_Q) - E_C \right\}} = \frac{3}{8}(S_c)_{b-w}.
\]

(3.68)

Here using the independent way, that is, the brick wall method, we have obtained the thermodynamical quantities. Since these quantities, especially the entropy, satisfy the Cardy-Verlinde formula, the result in this paper can be a good evidence for the dS/CFT correspondence.
4 Discussion

Using the ’t Hooft’s brick wall approach, we have calculated all thermodynamic quantities of asymptotically flat and non-flat space-time except rotating space-time. Asymptotically de Sitter space-time has two physical horizons; one is a event horizon of the black hole and the other is a cosmological horizon. And the others have the only event horizon of the black hole. The thermodynamic quantities at each horizon is similar to those of other papers. But since instead of the black hole mass the thermal energy of the gravitational field near the horizons is defined as a total energy, the energy is slightly modified containing the factor $3/8$ which, we guess, is the thermalization factor and the background geometrical effect. These geometrical effect disappears in the large radius limit (the flat space-time) and in this limit, if we ignored the factor $3/8$ which always appears in the brick wall method, the energy obtained here is exactly the same as the black hole mass. Generally the mass of the Schwarzschild black hole is defined by the observer living in the flat space-time which is the asymptotic space-time of the Schwarzschild black hole. So in the large radius limit, the results of the curved space-time is exactly the same as those of the flat space-time. In this paper, we have shown that our result is consistent with these fact. Therefore our result is reasonable and is the generalized one containing the background geometric effect.

Moreover, in spite of some different results, it is very interesting that the Cardy-Verlide formula can be rederived from these results up to some factor. This fact is a good evidence for the dS/CFT correspondence.

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