Implied Volatility Surface Estimation via Quantile Regularization

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Abstract  The implied volatility function and the implied volatility surface are both key tools for analyzing financial and derivative markets and various approaches were proposed to estimate these quantities. On the other hand, theoretical, practical, and also computational pitfalls occur in most of them. An innovative estimation method based on an idea of a sparse estimation and an atomic pursuit approach is introduced to overcome some of these limits: the quantile LASSO estimation implies robustness with respect to common market anomalies; the panel data structure allows for a time dependent modeling; changepoints introduce some additional flexibility in order to capture some sudden changes in the market and linear constraints ensure the arbitrage-free validity; last but not least, the interpolated implied volatility concept overcomes the problem of consecutive maturities when observing the implied volatility over time. Some theoretical backgrounds for the quantile LASSO estimation method are presented, the idea of the interpolated volatilities is introduced, and the proposed estimation approach is applied to estimate the implied volatility of the Erste Group Bank AG call options quoted in EUREX Deutschland Market.

Keywords  Quantile regression · LASSO regularization · Panel data
          Implied volatility · Options
1 Introduction

The empirical econometrics and financial experts rely on many different analytical tools. For instance, considering option tradings and various derivative markets, the most fundamental tools are the option pricing strategies and the implied volatility estimation (see [3]). The most common approaches are usually based on the well-known Black-Scholes model introduced in [2] even despite the fact that this model is considered by practitioners to be unrealistic from the theoretical point of view. Many alternative approaches were, therefore, proposed in order to overcome some obvious drawbacks of the Black-Scholes model.

Semiparametric or nonparametric option pricing approaches are commonly considered instead (see, for instance, [1, 6], or [10]) while the arbitrage-free market validity is guaranteed by using some additional pre-defined shape constraints. On the other hand, the corresponding implied volatility function (or the implied volatility surface respectively) is usually obtained either similarly, in terms of some constrained optimization problem (for instance, [11, 19]), or alternatively, it can be interpolated directly from the estimated option pricing model (see [6] or [9]).

In this paper we focus on the implied volatility surface estimation (the implied volatility function which evolves in time over some fixed observational period) and we advocate and combine various ideas to construct the overall model: sparse estimation with LASSO regularization and changepoints, quantile regression with panel data structure, or interpolated implied volatility values with a constant maturity. From the theoretical point of view, the presented method is motivated by the concept of a regularized changepoint detection proposed in [7] and further elaborated for the conditional quantile estimation in [4, 5]. A similar idea of the sparse estimation was also recently presented in [18] to estimate the option price function using a standard squared loss objective function while the conditional quantile estimation approach was proposed in [12]. The quantile estimation is, in general, considered to be robust and it also offers a more complex insight into the underlying data as it can estimate any arbitrary conditional quantile rather than just the conditional mean. The robustness property is also useful as the final model is not too sensitive with respect to various market anomalies (such as bid-ask spreads, discrete ticks in price, non-synchronous trading, etc.). The panel data structure allows for a time dependent modeling and changepoints introduce some additional flexibility which is convenient for reflecting some occasional sudden changes in the market (caused by various financial, economical, political, or natural causes). Finally, the natural evolution of the implied volatility over time, from the issuing date of the option until its maturity, shows some increase in convexity of the implied volatility smile. In order to avoid this issue and to focus on the changes that are due to some exogenous effects the implied volatility of an artificial option with a constant maturity of 30 days is introduced. Such implied volatility is computed by interpolating the implied volatility of options at consecutive maturities. The artificial options with the constant maturity are later used to estimated the corresponding implied volatility surface.
The rest of the paper is organized as follows: the quantile LASSO model is described in Sect. 2 and two important model modifications for estimating a single implied volatility function or the overall time-dependent two-dimension surface are presented in Sect. 3. Both situations are considered: a model without arbitrage-free restrictions and also a model which complies with the financial theory on the arbitrage-free market scenarios. Finally, in Sect. 4, the model is applied to estimate the time dependent profile of the implied volatility function for the Erste Group Bank AG call options quoted in the EUREX Deutschland market and some inference tools are used to decide whether there is some significant sudden change over the given profile or not.

2 Quantile LASSO Model for Implied Volatilities

Let us firstly briefly summarize the idea of using the quantile estimation and the LASSO type regularization for the regression estimation in general. A standard linear regression model where, in addition, the unknown vector parameter can change along the available observations \( i \in \{1, \ldots, n\} \), which are somehow naturally ordered, can be expressed as

\[
Y_i = x_i^\top \beta_i + \varepsilon_i, \quad i = 1, \ldots, n, \tag{1}
\]

where \( \beta_i \in \mathbb{R}^p \) is a \( p \)-dimensional parameter (the dimension does not depend on \( n \in \mathbb{N} \)) and \( x_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^\top \) is the subject’s specific vector of covariates. The random error terms \( \{\varepsilon_i\}_{i=1}^n \) are usually independent, centered, and identically distributed with some unknown distribution function \( F \). It is also assumed that there is some form of sparsity in the unknown parameters \( \beta_i \)'s, such that \( \beta_i = \beta_{i-1} \), for most of the indexes \( i \in \{2, \ldots, n\} \), but some few exceptions—changepoints. The model in (1) can be seen as a straightforward generalization of a simpler piece-wise constant model from [7] or, from the econometrics perspective, a more common trend model in [13]. The same model as in (1), however, for the dependent time series data, is also considered in [17].

The model in (1) is assumed to have \( K^* \in \mathbb{N} \) changepoints in total, located at some unknown indexes \( t_1^* < \cdots < t_K^* \in \{1, \ldots, n\} \), such that

\[
\beta_i = \beta_i^*, \quad \forall i = t_k^*, t_k^* + 1, \ldots, t_{k+1}^* - 1, \quad k = 0, 1, \ldots, K^*, \tag{2}
\]

with \( t_0^* = 1, t_{K^*+1}^* = n \), and \( \beta_n = \beta_{t_{K^*+1}}^* \). In general, the number of true changepoints \( K^* \in \mathbb{N} \) and their locations \( t_1^*, \ldots, t_{K^*+1}^* \) are all unknown. The true values of \( \beta_i \) are denoted by \( \beta_i^* \) and \( K^* = Card\{i \in \{2, \ldots, n\}; \beta_i^* \neq \beta_{i-1}^*\} \). The idea of the estimation method is to recover the unknown changepoint locations and to estimate the underlying model phases—the vector parameters which are associated with the conditional quantiles of interest. For this purpose, the following optimization problem is formulated
\[
\hat{\beta}^n = \text{Argmin}_{\beta_i \in \mathbb{R}^p \atop i = 1, \ldots, n} \sum_{i=1}^{n} \rho_\tau (Y_i - x_i^\top \beta_i) + n\lambda_n \sum_{i=2}^{n} \|\beta_i - \beta_{i-1}\|_2,
\]

where, for simplicity, \( \hat{\beta}^n = (\hat{\beta}^\top_1, \ldots, \hat{\beta}^\top_n)^\top \in \mathbb{R}^{np} \), \( \rho_\tau (u) = u(\tau - \mathbb{I}_{\{u < 0\}}) \), for \( \tau \in (0, 1) \), is the standard check function used for the quantile regression, \( \| \cdot \|_2 \) stands for the classical \( L_2 \) norm, and \( \lambda_n > 0 \) is the tuning parameter which controls for the overall number of changepoints (the sparsity level) occurring in the final model: for \( \lambda_n \to 0 \) there will be \( \hat{\beta}_i \neq \hat{\beta}_{i-1} \) for each \( i \in \{2, \ldots, n\} \), while for \( \lambda_n \to \infty \) no changepoints are expected to occur in the final model and, thus, \( \hat{\beta}_i = \hat{\beta}_{i-1} \) for all \( i \in \{2, \ldots, n\} \). The corresponding estimators for the changepoint locations are the observations \( i \in \{2, \ldots, n\} \), where \( \hat{\beta}_i \neq \hat{\beta}_{i-1} \). Let us, therefore, define the set

\[
\hat{A}_n = \{i \in \{2, \ldots, n\}; \hat{\beta}_i \neq \hat{\beta}_{i-1}\} = \{\hat{t}_1 < \cdots < \hat{t}_{|\hat{A}_n|}\},
\]

and let \( |\hat{A}_n| \) be the cardinality of \( \hat{A}_n \). For each \( k = 0, \ldots, |\hat{A}_n| \) we can also define the \((k+1)\)-st model phase (observations indexed by the set \( \{\hat{t}_k, \ldots, \hat{t}_{k+1} - 1\} \), where \( \hat{t}_0 = 1 \) and \( \hat{t}_{|\hat{A}_n|+1} = n \) ), with the corresponding vector of estimated parameters \( \hat{\beta}_{\hat{t}_k} \). The minimization problem formulated in (1) is convex and it can be effectively solved by using some standard optimization toolboxes (see, for instance, [8]). The theoretical properties are studied in detail in [5]. Under some reasonable assumptions, the method is consistent in terms of the changepoint detection and, also, in terms of the parameter estimation. Nevertheless, the regularization parameter in the LASSO problems should be chosen, in general, differently when aiming at the changepoint recovery or the underlying model estimation: for the former one, larger values are preferred to avoid the overestimation issue and false changepoint detection while for the estimation purposes, slightly smaller values of \( \lambda_n > 0 \) are needed in order to limit the shrinkage effect and to improve the estimation bias performance. The value of \( \lambda_n > 0 \) which satisfies the set of assumptions used in [5] is, for instance, \( \lambda_n = (1/n) \cdot (\log n)^{5/2} \). The role of the regularization parameter is crucial and various approaches can be used to determine a proper value for a given data. However, its importance can be suppressed by using some alternative regularization source. This is, for instance, also the case for the option pricing problem where the final model must satisfy some pre-defined shape restrictions in order to comply with the financial theory on the arbitrage-free market scenarios. In the next sections we present two important modifications of the quantile LASSO model which can be directly used to estimate the implied volatility function and the implied volatility surface respectively. Both these quantities serve as key tools for analyzing financial markets and derivative tradings in general.
3 Quantile LASSO and Implied Volatility Estimation

Firstly, we consider a situation where the implied volatility values are only observed for some specific day from the observational period. The data can be represented as a sample \( \{(Y_i, x_i); \; i = 1, \ldots, n\} \), where \( Y_i \) stands for the observed implied volatility at the given strike \( x_i \). Thus, there are \( n \in \mathbb{N} \) observations in total for \( n \) unique strikes. The aim is to use the observed volatility values and to estimate the implied volatility function. The quantile fused LASSO presented in Sect. 2 is used, however, some modifications are needed in order to obtain the model which complies with the arbitrage free market conditions. Theses conditions imply that the estimated implied volatility function must be convex with respect to the strikes.

The quantile LASSO method provides a robust estimate which is not sensitive to various derivative market anomalies (such as bid-ask spreads, discrete ticks in price, or non-synchronous trading, or heavy tailed error distributions). The value of \( \tau = 0.5 \) is used to construct the conditional median in (3), which is, from the theoretical point of view, for a symmetric density of the error terms, the same quantity as the conditional mean. Nevertheless, the convexity of the final estimate is not automatically guaranteed in (3) and some additional linear constraints can be used to enforce the volatility smile in the final model.

3.1 Arbitrage-Free Market Restrictions

Let the available strikes \( \{x_i\}_{i=1}^n \) be all from some compact domain \( D \) with some functional basis \( \{\varphi_j(x); \; j = 1, \ldots, p\} \) defined on \( D \). Let \( x_i = (\varphi_1(x_i), \ldots, \varphi_p(x_i))^\top \) (for instance, let \( x_i = (1, x_i, x_i^2)^\top \), for \( i = 1, \ldots, n \), where \( p = 3 \), which gives a standard quadratic fit). For each strike the quantile fused LASSO in (3) assumes the corresponding vector of unknown parameters \( \beta_i = (\beta_{i1}, \beta_{i2}, \beta_{i3})^\top \in \mathbb{R}^3 \). This brings a huge amount of flexibility and the final model would be too much haphazard if no additional restrictions on the parameter vectors were imposed. Therefore, the regularization penalty in (3) is adopted. Another form of regularization can be applied if, for instance, some specific properties for the final fit are assumed (e.g., the convexity or the volatility smile respectively).

For \( \mathbf{0} \in \mathbb{R}^p \) being a zero vector of the length \( p \in \mathbb{N} \), we can easily define the model matrix

\[
X = \begin{bmatrix}
x_1^\top & 0^\top & \cdots & 0^\top \\
0^\top & x_2^\top & \cdots & 0^\top \\
\vdots & \vdots & \ddots & \vdots \\
0^\top & 0^\top & \cdots & x_n^\top
\end{bmatrix}
\]

and the model from (1) can be equivalently expressed as

\[
Y = X\beta^\top + \varepsilon. \tag{5}
\]
where \( Y = (Y_1, \ldots, Y_n)^\top, \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^\top \), and \( \beta^n = (\beta_1^n, \ldots, \beta_n^n)^\top \in \mathbb{R}^{np} \).

For the model in (5) we can directly use the minimization formulation in (3) but the solution is, in general, not smooth and the volatility smile required for the arbitrage-free market scenario is also not automatically guaranteed. The implied volatility function is assumed to be smooth and convex which can be both enforced by minimizing (3) with respect to some specific linear constraints defined for the functional basis \( \{ \varphi_j(x); j = 1, \ldots, p \} \). The overall smoothness property can be achieved by the right choice of the functional basis (e.g., polynomials, or splines of some specific degree, which is large enough). Moreover, it is easy to see, that the solution in (3) will be also convex if, in addition, the estimated vector of parameters \( \hat{\beta}^n \in \mathbb{R}^{np} \) obeys
\[
C \hat{\beta}^n \geq 0, \tag{6}
\]
which holds element-wise for
\[
C = \begin{bmatrix}
\tilde{x}_1^\top & 0^\top & \ldots & 0^\top \\
0^\top & \tilde{x}_2^\top & \ldots & 0^\top \\
\vdots & \vdots & \ddots & \vdots \\
0^\top & 0^\top & \ldots & \tilde{x}_n^\top
\end{bmatrix},
\]
where \( \tilde{x}_i = (\varphi_1''(x_i), \ldots, \varphi_p''(x_i))^\top \) denotes the vector of the second derivatives of the functional basis functions \( \varphi_j(x) \) for \( j = 1, \ldots, p \) which are evaluated again at the given strike \( x_i \in \mathcal{D} \), for \( i = 1, \ldots, n \). The minimization (3) together with the linear constraints given in (6) is again convex and an effective solution can be obtained by adopting some standard optimization toolboxes.

For illustration, the quantile fused LASSO is applied for the Erste Group Bank AG call options quoted in EUREX Deutschland Market and the implied volatility function is estimated for two specific trading days—September 21st, 2018 and October 18th, 2018 (see Fig. 1 for illustration). It is clear from Fig. 1 that the arbitrage-free conditions are not automatically guaranteed by the data themselves and, indeed, the convexity property (volatility smile) must be enforced by the linear constraints in (6).

In practical applications, the estimated implied volatility function can change over time reflecting various trends or anomalies on the derivative market. Therefore, in the next section, we introduce another modification of the quantile LASSO model described above in order to estimate the implied volatility function for a set of consecutive days from some fixed period. We assume \( n \in \mathbb{N} \) independent panels (one panel for each strike) and the strike specific implied volatilities are observed over some trading interval \([0, T]\), for some fixed \( T > 0 \).
Fig. 1 The illustration of the Quantile LASSO performance when applied for the estimation of the implied volatility function. Two situations are considered: the first day of the observational period on the left panel, and $t = 20$ (volatility peak) on the right panel. The estimation is considered without any linear restrictions (dashed blue lines) and with the linear constraints—which enforce the volatility smile (solid red lines) and, thus, arbitrage-free validity.

3.2 Time Dependent Implied Volatility Surface

The implied volatility values are now represented as a sample $\{(Y_{ti}, x_{iti}); \ t = 1, \ldots, T; \ i = 1, \ldots, n\}$, where $Y_{ti}$ stands for the implied volatility at some specific time $t \in \{1, \ldots, T\}$ and the given strike $x_{iti} \in D$, for $i \in \{1, \ldots, n\}$. For simplicity, the quoted strikes are common over time, therefore, we have that $x_{iti} \equiv x_i$, for $i = 1, \ldots, n$. For each quoted strike $x_i \in D$ there is a strike specific panel of the implied volatilities observed over time $t \in \{1, \ldots, T\}$. The value of $T \in \mathbb{N}$ represents, for instance, the number of trading days available in the data. The underlying panel data model takes the form

$$Y_{ti} = x_i^\top \beta_t + \varepsilon_{ti}, \quad \text{for} \ t = 1, \ldots, T \text{ and } i = 1, \ldots, n,$$

where again $x_i = (\varphi_1(x_i), \ldots, \varphi_p(x_i))^\top$ is the given functional basis on $D$, and $\beta_t = (\beta_{t1}, \ldots, \beta_{tp})^\top \in \mathbb{R}^p$ is the vector of unknown parameters which can now also change over time $t \in \{1, \ldots, T\}$. The error vectors $\varepsilon_t = [\varepsilon_{t1}, \ldots, \varepsilon_{Ti}]$ are assumed to be independently distributed across panels $i \in \{1, \ldots, n\}$.

The time dependent implied volatility surface can be now estimated simultaneously, such that the final model will obey the shape restrictions required for the arbitrage-free market. The corresponding minimization problem takes the form

$$\min_{\beta_t \in \mathbb{R}^p} \sum_{t=1}^{T} \sum_{i=1}^{n} \rho_t \left( Y_{ti} - x_i^\top \beta_t \right) + n\lambda_n \sum_{t=2}^{T} \| \beta_t - \beta_{t-1} \|_2$$

with respect to
\[ C \beta_t \geq 0, \quad t = 1, \ldots, T; \quad \text{(convexity in the strike over time)} \]  

(9)

where \( C \) is defined analogously as in (6). The overall vector of the estimated parameters \( \hat{\beta}^n = (\hat{\beta}_1^T, \ldots, \hat{\beta}_T^T)^T \in \mathbb{R}^{T \times p} \) represents the set of all panels while \( \hat{\beta}_t \in \mathbb{R}^p \) is only associated with the estimated volatility function for some specific time \( t \in \{1, \ldots, T\} \). The implied volatility function is obviously allowed to evolve over time to reflect possible changes on the market with no specific restrictions what so ever. Moreover, there is again a specific sparsity structure assumed: for situations where \( \hat{\beta}_t \neq \hat{\beta}_{t-1} \) the estimated implied volatility function changes from time \( (t - 1) \) to time \( t \) to adapt for the situation at the market and, otherwise, the estimated implied volatility remains the same. The regularization parameter in (8) controls the amount of such changes and the shape constraints in (9) are responsible for the additional source of regularization by enforcing convexity of the estimated volatility function for each time point \( t \in \{1, \ldots, T\} \). The minimization problem in (8) together with the linear constraints in (9) is again convex and the optimal solution can be obtained by the standard optimization software. The Karush-Kuhn-Tucker (KKT) optimality conditions can be easily derived and they are formulated by the following lemma.

Lemma 1 (a) For any \( l \in \{1, \ldots, |\mathcal{A}_n|\} \), \( n \in \mathbb{N} \), and \( \lambda_n > 0 \) the following holds with probability one:

\[
\tau(T - \hat{t}_l + 1) \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \sum_{k=\hat{t}_l}^{T} x_i \mathbb{I}_{\{Y_{ik} \leq x_i^\top \hat{\beta}_k\}} = n\lambda_n \frac{\hat{\theta}_l}{\|\hat{\theta}_l\|_2},
\]

for a reparametrization \( \hat{\theta}_l = \sum_{i=1}^{l} \hat{\beta}_i \) for any \( t \in \{1, \ldots, T\} \);

(b) For any \( t \in \{1, \ldots, T\} \), \( n \in \mathbb{N} \), and \( \lambda_n > 0 \), the following holds with probability one:

\[
\left\| \tau(T - t + 1) \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \sum_{k=t}^{T} x_i \mathbb{I}_{\{Y_{ik} \leq x_i^\top \hat{\beta}_k\}} \right\|_2 \leq n\lambda_n.
\]

The proof of Lemma 1 is straightforward and it is omitted. More details can be found in [12]. Let us, however, briefly state some technical assumptions which are needed to prove the estimation consistency of the proposed method.

Assumptions:

(A1) The errors \( \epsilon_i = [\epsilon_{1i}, \ldots, \epsilon_{Ti}] \) are independent copies of some strictly stationary sequence \( \epsilon = [\epsilon_1, \ldots, \epsilon_T] \) with the continuous marginal distribution functions \( F_{\epsilon_t}(x) \) and \( F_{(\epsilon_t, \epsilon_{t+1})}(x, y) \), for \( x, y \in \mathbb{R}, t, k \in \{1, \ldots, T\} \), and \( k \geq 1 \). Moreover, \( F_{\epsilon_t}(0) = \mathbb{P}[\epsilon_t < 0] = \tau_t \), for \( \tau \in (0, 1) \). The corresponding density functions \( f(\cdot) \) and \( f(\cdot, \cdot) \) are bounded and strictly positive in the neighborhood of zero;

(A2) There exist two constants \( c, C \in \mathbb{R} \) such that

\[ 0 < c \leq \mu_{\min}(\mathbb{E}[X_n]) \leq \mu_{\max}(\mathbb{E}[X_n]) \leq C < \infty, \]
where $\mu_{\text{min}}$ and $\mu_{\text{max}}$ stand for the minimum and maximum eigenvalue of the matrix in the argument and $\mathbb{X}_n = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top$. Moreover, $\max_{1 \leq i \leq n} \|x_i\|_\infty < \infty$.

**(A3)** There are two deterministic positive sequences $(\lambda_n)$ and $(\delta_n)$, such that $\lambda_n \to 0$, $\delta_n \to 0$, $n^{1/2} \delta_n \to \infty$, and $\lambda_n / \delta_n \to 0$ as $n \to \infty$.

Let us recall that in similar models (see, for instance, [5], [4], or [18]) there is an additional assumption which requires that the span between two consecutive changepoints increases. Analogously, the overall number of changes in the model is usually considered to be fixed. However, as far as $T \in \mathbb{N}$ is assumed to be fixed these two assumptions are irrelevant for our specific situation. Given the assumptions above, the consistency can be formulated by the next theorem.

**Theorem 1** Let the assumptions in (A1)–(A3) be all satisfied. Then, for any $t = 1, \ldots, T$, it holds that

$$\|\hat{\beta}_t - \beta^*_t\|_1 = O_P \left( \sqrt{\frac{\log n}{n}} \right),$$

where $\hat{\beta}_t \in \mathbb{R}^p$ denotes the vector of the estimated parameters obtained by minimizing (8) and $\beta^*_t$ is the corresponding vector of the true values.

The theorem above specifies a proper converge rate for the estimates obtained by minimizing (8). An example of sequences $\{\lambda_n\}$ and $\{\delta_n\}$, which satisfy Assumption (A3) are $\lambda_n = n^{-1} \cdot (\log n)^{1/2}$ and $\delta_n = (n^{-1} \log n)^{1/2}$. For the proof of the theorem we only refer to [12].

In the next section we discuss an application of the proposed modified quantile fused LASSO method to simultaneously estimate a set of implied volatility panels where each panel represents implied volatilities observed over time for a given quoted strike. The observed implied volatilities are, however, firstly interpolated over consecutive maturities in order to obtain artificial call options with a fixed expiry date (30 days). Such smoothing suppresses the natural dynamics of the market (such as increasing convexity of the volatility smile when progressing towards expiry dates) and it gives an opportunity to focus on exogenous effects (changepoints) only.

### 4 Application: Implied Volatility with Constant Maturity

The proposed quantile fused LASSO approach is applied to estimate the implied volatility surface and to detect possible changes over time for the call options written on Erste Group Bank AG and quoted in the EUREX Deutschland market. The implied volatilities $z_{i,t,k}$, where $i$ represents the strike of the option, $k$ its maturity, and $t$ is the observing day, are downloaded from Thomson Reuters Datastream. The available call option strikes range from 30 Euro to 43.50 Euro with an equidistant step of 0.50 Euro, which gives 28 strikes all together ($n = 28$). There are three considered
maturities, for October 19th, 2018, November 16th, 2018, and December 21st, 2018. There are 37 trading days within the analyzed period from September 21st, 2018 till November 12th, 2018, thus $T = 37$. Such period is long enough to capture the dynamics of the volatility smile and to investigate possible changes in its shape. However, similar analysis could be also conducted on some other time periods using the corresponding data.

The first aim is to construct panels that report, for each strike $i \in \{1, \ldots, n\}$ and each observing day $t \in \{1, \ldots, T\}$, the implied volatility $Y_{i,t}$ of an artificial option having always a constant maturity of $K$ days. Therefore, for each day from the considered period, the observed implied volatilities $z_{i,t,k}$ of the two options that have their maturities immediately before and immediately after are interpolated using the linear combination defined as

$$Y_{i,t} = \frac{1}{(t + K) - k_b} \cdot z_{i,t,k_b} + \frac{1}{k_a - (t + K)} \cdot z_{i,t,k_a} + \frac{1}{(t + K) - k_b} + \frac{1}{k_a - (t + K)},$$

(10)

where $k_b$ is the maturity of the first option expiring before the time $t + K$, respectively, $k_a$ is the maturity of the first option expiring after the time point $t + K$. For the Erste Group Bank AG call options quoted in EUREX Deutschland the fixed maturity of $K = 30$ days is considered.

For example, for the first observing day ($t = 1$), which is September 21st, 2018, the artificial option expires in $t + 30$ days, i.e. October 21st, 2018. The two options used for the artificial volatility interpolation are those with the expiry dates October 19th, 2018 (which is denoted as $k_b$) and November 16th, 2018 (denoted as $k_a$). In this case, the distance between the artificial maturity (October 21st, 2018) and the maturity of the first option is $(t + 30) - k_b = 2$ trading days and the distance between the artificial maturity and the maturity of the second option is $k_a - (t + 30) = 19$ trading days.

Therefore, the equation from (10) takes the form

$$Y_{i,t} = \frac{1}{2} \cdot x_{i,t,k_b} + \frac{1}{19} \cdot x_{i,t,k_a},$$

(11)

The whole procedure is repeated for all strike panels $i \in \{1, \ldots, n\}$ and all trading days from the observational period $t \in \{1, \ldots, T\}$. The resulting panels of the artificial implied volatilities $\{Y_{i,t}^{n,T}\}_{i,t=1}$ are presented in Fig. 2. All together, there are 28 strike panels which are observed for $T = 37$ consecutive trading days.

In the second step, the proposed quantile fused LASSO estimation approach is used to estimate the overall time dependent implied volatility surface while the linear constraints from (9) are again employed to obtain the arbitrage-free valid model at the end. The linear constraints enforce the convexity of the estimated implied
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Fig. 2  The time development of the artificial implied volatility for the Erste Group Bank AG call options with a constant 30 days maturity. The analyzed period is from September 21st, 2018 till November 12th, 2018. All together, there are 28 strike panels ranging from 30.0 Euro up to 43.5 Euro observed for 37 consecutive trading days.

volatility surface—the convexity (i.e., the volatility smile) with respect to the strikes simultaneously for every day from the analyzed period. The volatility smiles for consecutive days are assumed to be mostly the same with only a few exceptions where the implied volatility function from the time $t$ changes at the time $t + 1$ to adapt for some existing underlying changes on the market. The estimated implied volatility panels are presented in Fig. 3. It is clear that the estimated implied volatility functions are, indeed, all convex for any time point $t \in \{1, \ldots, 37\}$. The overall surface is quite stable but there are also some obvious changes in the estimated volatility surface: some of them occur over time and others are present within the convexity of the volatility smile for some fixed trading days.

From the practical point of view, there are two different explanations for these changes: the changes occurring over time are most likely caused by some exogenous effects (such as the recent COVID-19 outbreaks or the President Trump tweets on additional 10% tariff to be placed on Chinese imports) while the changes in the volatility smile (increasing convexity when approaching the expiry dates) are still due to some natural dynamics of the market (i.e., high spikes for high strikes in Fig. 3). From the theoretical point of view, these two cases cannot be distinguished automatically therefore, we used the artificial options with the constant maturity of 30 days in order to suppress the changes caused by the natural dynamics of the market and, on the other hand, to highlight and detect the changes caused by the external causes. The natural market dynamics is still present in the estimated surface in term of a few high spikes, however, the rest of the surface can be effectively used to analyze the market with respect to external causes effecting the market.

Peripherally, one could be interested whether or not a change in the artificial implied volatilities occurred for some common trading day (cf. Fig. 3), assuming that the volatilities are approximately constant before and after the possible change for every strike. The ratio-type changepoint test statistics proposed in [14] as well as the bootstrap self-normalized changepoint test statistics form [15] both suggest
Fig. 3 The estimated panels of the implied volatilities: for each time point $t \in \{1, \ldots, 37\}$ the estimated implied volatility function is clearly convex in strikes and the overall surface is stable over time with just some few spikes for rather high strikes—detected changes in the volatility to reject the null hypothesis of no change in the panel means. Furthermore, the changepoint estimator developed in [16] reveals a change on the 20th trading day (see Fig. 4).

Alternatively, one could be also interested in some individual tests whether there is a change in some specific strike panel when the panels are considered separately. From Fig. 3 it is obvious that a sudden change (a spike or a wave respectively) occurring on the 20th trading day is only observed for high strikes (roughly the strike values above 38 Euro) while no such behavior is observed for lower strikes. Such panel specific tests are, however, all significant as the overall variability of the observed implied volatility values is relatively high and, more importantly, the raw volatility values do not reflect the arbitrage free market scenario which is implicitly accounted for in our model. In addition, the multiple testing problem should be considered taken care of properly. Therefore, more precise and more appropriate conclusions can be indeed drawn from the model presented in Fig. 3 rather than performing individual tests and considering individual strike panels separately.
5 Conclusion

The implied volatility function and the implied volatility surface are both fundamental tools for the empirical econometrics, the financial derivative markets in particular. A new method, based on the panel data structure, conditional quantile estimation, LASSO regularization, and artificial volatility interpolation is proposed to automatically estimate the time development of the implied volatility function over some specific (fixed) trading period.

This presented approach avoids some popular multistage techniques and nonparametric kernels which usually perform slowly. The sparsity principle and the LASSO fused-type penalty are used to firstly inflate the overall flexibility of the model but, later, the estimate is regularized in order to obtain the final model which fully complies with the financial theory developed for the arbitrage-free market scenarios. The model also implicitly incorporates a prior knowledge that the implied volatility function should not change too roughly and it should be, more or less, stable over time.

The main advantage of the proposed method is that it does not apriori assume the arbitrage-free input data. The estimated implied volatility function, which satisfies the arbitrage-free conditions (so called volatility smile) is obtained automatically in a straightforward and data-driven manner by minimizing the objective function together with some appropriate linear constraints which enforce the convex property. This is crucial for the implied volatility estimation because the volatilities violating the natural market conditions would have serious consequences.
The proposed quantile LASSO method for the panel data structures serves as an innovative and pioneering approach for the option pricing problem and the implied volatility estimation in particular. In addition, the interpolated implied volatilities with the fixed maturity over time offer a much more stable insight into the true market conditions. The proposed estimation approach can easily serve for both, the estimation under the arbitrage-free restrictions or the situation without such restrictions and the presented application shows a straightforward all-in-once implementation for real data cases.

Acknowledgements The authors would like to express thanks to the reviewers for their interesting comments and suggestions. The research of Matúš Maciak and Michal Pešta has been partially supported by the Czech Science Foundation project GAČR No. 18-00522Y. The work of Sebastiano Vitali was supported by MIUR-ex60% 2019 and 2020 sci. resp. Sebastiano Vitali.

References
1. Benko, M., Fengler, M., Härdle, W., Kopa, M.: On extracting information implied in options. Comput. Statistics 4(22), 543–553 (2007)
2. Black, F., Scholes, M.: The pricing of options and corporate liabilities. J. Polit. Econ. 81, 637–654 (1973)
3. Britten-Jones, M., Neuberger, A.: Option prices, implied price process and stochastic volatility. J. Finance 55(2), 839–866 (2000)
4. Ciuperca, G., Maciak, M.: Change-point Detection by the Quantile LASSO Method. J. Statistical Theory Practice 14(11) (2020). https://doi.org/10.1007/s42519-019-0078-z
5. Ciuperca, G., Maciak, M.: Change-point detection in a linear model by adaptive fused quantile method. Scand. J. Statistics (2019). https://doi.org/10.1111/sjos.12412
6. Fengler, M.R.: Semiparametric Modeling of Implied Volatility. Springer, Berlin, 1st edn. ISBN: 978-3-540-26234-3 (2005)
7. Harchaoui, Z., Lévy-Leduc, C.: Multiple change-point estimation with a total variation penalty. J. Am. Statistical Assoc. 105(492), 1480–1493 (2010)
8. Huang, J., Ma, S., Xie, H., Zhang, C.: A group bridge approach for variable selection. Biometrika 96, 339–355 (2009)
9. Hull, C.J., White, A.: The pricing of options on assets with stochastic volatilities. J. Finance 42(1), 281–300 (1987)
10. Kahale, N.: An arbitrage-free interpolation of volatilities. Risk 5(17), 102–106 (2004)
11. Kopa, M., Vitali, S., Tichý, T., Hendrych, R.: Implied volatility and state price density estimation: arbitrage analysis. Comput. Manage. Sci. 14(4), 559–583 (2017)
12. Maciak, M.: Quantile LASSO with changepoints in panel data models applied to option pricing. Econometr. Statistics (2019). https://doi.org/10.1016/j.econsta.2019.12.005
13. Maciak, M., Mizera, I.: Regularization techniques in joinpoint regression. Statistical Papers 57(4), 939–955 (2016)
14. Maciak, M., Pešťová, B., Pešta, M.: Structural breaks in dependent, heteroscedastic, and extremal panel data. Kybernetika 54(6), 1106–1121 (2018)
15. Maciak, M., Pešta, M., Pešťová, B.: Changepoint in dependent and non-stationary panels. Statistical Papers (2020) https://doi.org/10.1007/s00362-020-01180-6
16. Pešta, M., Pešťová, B., Maciak, M.: Changepoint estimation for dependent and non-stationary panels. Appl. Math. (2020) https://doi.org/10.21136/AM.2020.0296-19
17. Qian, J., Su, L.: Structural change estimation in time series regression with endogenous variables. Econ. Lett. 125, 415–421 (2014)
18. Qian, J., Su, L.: Shrinkage estimation of common breaks in panel data models via adaptive group fused Lasso. J. Econometr. 191, 86–955 (2016)
19. Vitali, S., Kopa, M., Tichý, T.: State price density estimation for options with dividend yields. Central Europ. Rev. Econ. Issues 20(3), 81–90 (2017)