Characteristic classes of transitive Lie algebroids. Categorical point of view

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Outline

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Transitive Lie algebroids have specific properties that allow to look at the transitive Lie algebroid as an element of the object of a homotopy functor. Roughly speaking each transitive Lie algebroids can be described as a vector bundle over the tangent bundle of the manifold which is endowed with additional structures.

Therefore transitive Lie algebroids admits a construction of inverse image generated by a smooth mapping of smooth manifolds.
The construction can be managed as a homotopy functor $\mathcal{TLA}_g$ from category of smooth manifolds to the transitive Lie algebroids. The functor $\mathcal{TLA}_g$ associates with each smooth manifold $M$ the set $\mathcal{TLA}_g(M)$ of all transitive algebroids with fixed structural finite dimensional Lie algebra $\mathfrak{g}$. 
The intention of my talk is to use a homotopy classification of transitive Lie algebroids due to K. Mackenzie

Mackenzie, K.C.H., *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press, (2005).

and on this basis to construct a classifying space.
The realization of the intention allows to describe characteristic classes of transitive Lie algebroids form the point of view a natural transformation of functors similar to the classical abstract characteristic classes for vector bundles and to compare them with that derived from the Chern-Weil homomorphism by J.Kubarski.

Kubarski, J., *The Chern-Weil homomorphism of regular Lie algebroids*, Publications du Department de Mathematiques, Universite Claude Bernard - Lyon-1, (1991) pp.4–63.
Outline

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Given smooth manifold $M$ let

$$E \xrightarrow{\alpha} TM \xrightarrow{p_T} M$$

be a vector bundle over $TM$ with fiber $g$. The fiber $g$ has the structure of a finite dimensional Lie algebra and the structural group of the bundle $E$ is $Aut(g)$, the group of all automorphisms of the Lie algebra $g$. Let $p_E = p_T \cdot \alpha$. So we have a commutative diagram of two vector bundles

$$E \xrightarrow{\alpha} TM \quad TM \xrightarrow{p_T} M$$

$$\downarrow p_E \quad \downarrow p_T$$

$$M \quad \rightarrow M$$
The diagram is endowed with additional structure (commutator braces) and then is called (Mackenzie, definition 3.3.1, Kubarski, definition 1.1.1) transitive Lie algebroid

\[ \mathcal{A} = \left\{ \begin{array}{c}
E \xrightarrow{a} TM \\
p_E \\
M
\end{array} \right\} \]

\[ \begin{array}{c}
\downarrow p_E \\
M \rightarrow M
\end{array} \]

\{\bullet, \bullet\}
The braces \{\bullet, \bullet\} satisfy the natural properties, such that the space $\Gamma^\infty(E)$ with braces \{\bullet, \bullet\} forms an infinite dimensional Lie algebra with structure of the $C^\infty(M)$ – module, that is
Definitions

that is

1. Skew commutativity: for two smooth sections \( \sigma_1, \sigma_2 \in \Gamma^\infty(E) \) one has
   \[
   \{\sigma_1, \sigma_2\} = -\{\sigma_2, \sigma_1\} \in \Gamma^\infty(E),
   \] (1)

2. Jacobi identity: for three smooth sections \( \sigma_1, \sigma_2, \sigma_3 \in \Gamma^\infty(E) \) one has
   \[
   \{\sigma_1, \{\sigma_2, \sigma_3\}\} + \{\sigma_3, \{\sigma_1, \sigma_2\}\} + \{\sigma_2, \{\sigma_3, \sigma_1\}\} = 0,
   \] (2)

3. Differentiation: for two smooth sections \( \sigma_1, \sigma_2 \in \Gamma^\infty(E) \) and smooth function \( f \in C^\infty(M) \) one has
   \[
   \{\sigma_1, f \cdot \sigma_2\} = a(\sigma_1)(f) \cdot \sigma_2 + f \cdot \{\sigma_1, \sigma_2\} \in \Gamma^\infty(E).
   \] (3)
Let $f : M' \rightarrow M$ be a smooth map. Then one can define an inverse image (pullback) of the Lie algebroid (Mackenzie, page 156, Kubarski, definition 1.1.4), $f^!(A)$.

This means that given the finite dimensional Lie algebra $\mathfrak{g}$ there is the functor $\mathcal{TLA}_g$ such that with any manifold $M$ it assigns the family $\mathcal{TLA}_g(M)$ of all transitive Lie algebroids with fixed Lie algebra $\mathfrak{g}$. 
The following statement can be proved, see for example

W. Walas

*Algebry Liego-Rineharta i pierwsze klasy charakterystyczne,*
PhD manuscript, Lodz, Poland, 2007.

**Theorem**

*Each transitive Lie algebroid is locally trivial.*
This means that for a small neighborhood there is a trivialization of the vector bundles $E$, $TM$, $\ker a = L \cong g \times M$ such that

$$E \cong TM \oplus L,$$

and the Lie braces are defined by the formula:

$$[(X, u), (Y, v)] = ([X, Y], [u, v] + X(v) - Y(u)).$$
Homotopy of pullback

Using the construction of pullback and the idea by Allen Hatcher

Allen Hatcher,

*Vector bundles and K-theory,*

Available at

http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html, 2003. [Proposition 1.7]

one can prove that the functor $\mathcal{T}\mathcal{L}\mathcal{A}_g$ is the homotopic functor. More exactly for two homotopic smooth maps $f_0, f_1 : M_1 \to M_2$ and for the transitive Lie algebroid

$$\mathcal{A} = (E \xrightarrow{a} TM_2 \to M_2; \{\bullet, \bullet\})$$

two inverse images $f_0''(\mathcal{A})$ and $f_1''(\mathcal{A})$ are isomorphic.
Let $\mathcal{TLA}_g$ be the category of all transitive Lie algebroids and morphisms between them. The objects $\mathcal{A} \in \text{Obj}(\mathcal{TLA}_g)$ are Lie algebroids $\mathcal{A} = \left\{ \begin{array}{c} E \xrightarrow{a} TM \\ p_E \downarrow \quad \downarrow p_T; \{\bullet, \bullet\} \end{array} \right\}$,\[ M = M(\mathcal{A}) . \]
The morphisms $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ between the Lie algebroids are commutative diagrams

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\tilde{f} = E(\varphi)} & E_2 \\
\downarrow p_{E_1} & & \downarrow p_{E_2} \\
M_1 & \xrightarrow{f = M(\varphi)} & M_2
\end{array}
\]
Then one can define the direct limits

\[ \mathcal{B}_g \overset{\text{def}}{=} \lim_{\rightarrow} (M(A); M(\varphi)) ; A \in \text{Obj}(\mathcal{T}LA_g), \varphi \in \text{Morph}(\mathcal{T}LA_g) \]
Hence the final classifying space $B_g$ has the property that the family of all transitive Lie algebroids with fixed Lie algebra $g$ over the manifold $M$ has one-to-one correspondence with the family of homotopy classes of continuous maps $[M, B_g]$: 

$$\mathcal{TLA}_g(M) \approx [M, B_g].$$
Using this observation one can describe the family of all characteristic classes of a transitive Lie algebroids in terms of cohomologies of the classifying space $B_g$. Really, from the point of view of category theory a characteristic class $\alpha$ is a natural transformation from the functor $T\mathcal{L}A_g$ to the cohomology functor $H^*$. 
This means that for the transitive Lie algebroid \( \mathcal{A} = (E \xrightarrow{\alpha} TM \longrightarrow M; \{\bullet, \bullet\}) \) the value of the characteristic class \( \alpha(\mathcal{A}) \) is a cohomology class

\[
\alpha(\mathcal{A}) \in H^*(M),
\]

such that for smooth map \( f : M_1 \longrightarrow M \) we have

\[
\alpha(f^0!!(\mathcal{A})) = f^*(\alpha(\mathcal{A})) \in H^*(M_1).
\]
Hence the family of all characteristic classes \( \{\alpha\} \) for transitive Lie algebroids with fixed Lie algebra \( \mathfrak{g} \) has a one-to-one correspondence with the cohomology group \( \mathcal{H}^*(B_\mathfrak{g}) \).

On the base of these abstract considerations a natural problem can be formulated.
Problem

Given finite dimensional Lie algebra $\mathfrak{g}$ describe the classifying space $\mathcal{B}_\mathfrak{g}$ for transitive Lie algebroids in more or less understandable terms.

Below we suggest a way of solution the problem and consider some trivial examples.
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Each transitive Lie algebroid

\[ A = \left\{ \begin{array}{c}
E \xrightarrow{a} TM \\
p^E \\
M \\
\end{array} \right\}, \{\bullet, \bullet\} \cdot \]

one can represent as an exact sequence of bundles

\[ 0 \rightarrow L \rightarrow E \xrightarrow{a} TM \rightarrow 0. \]
Coupling

The bundle $E$ can be represent as a direct sum of bundles

$$E = L \oplus TM,$$

Then each section $\sigma \in \Gamma(E)$ one can represent as the pair of sections

$$\sigma = (u, X), \quad u \in \Gamma(L), \quad X \in \Gamma(TM).$$
Then the commutator brace for the pair of the sections
\( \sigma_1 = (u_1, X_1), \quad \sigma_2 = (u_2, X_2) \) can be written by the formula

\[
\{\sigma_1, \sigma_2\} = \{(u_1, X_1), (u_2, X_2)\} = \\
= ([u_1, u_2] + \nabla_{X_1}(u_2) - \nabla_{X_2}(u_1) + \Omega(X_1, X_2), [X_1, X_2]).
\]
Here

\[ \nabla_X : \Gamma(L) \longrightarrow \Gamma(L) \]

is the covariant gradient of fiberwise differentiation of sections, where

\[ \Omega(X_1, X_2) \in \Gamma(L) \]

is classical two-dimensional differential form with values in the fibers of the bundle \( L \).

The covariant derivative for fiberwise differentiation of the sections or so called linear connection in the bundle \( L \)

\[ \nabla_X : \Gamma(L) \longrightarrow \Gamma(L) \]

is an operator that satisfies the following natural conditions:
1. Fiberwise differentiation with respect to multiplication in the Lie algebra structure of the fibre:

\[ \nabla_X([u_1, u_2]) = [\nabla_X(u_1), u_2] + [u_1, \nabla_X(u_2)], \]

\[ u_1, u_2 \in \Gamma(L). \]

2. Differentiation of sections in the space \( \Gamma(L) \) as module over the function algebra \( \mathcal{C}^\infty(M) \):

\[ \nabla_X(f \cdot u) = X(f) \cdot u + f \cdot \nabla_X(u), \]

\[ u \in \Gamma(L), \ f \in \mathcal{C}^\infty(M). \]

3. Linear dependence on vector fields:

\[ \nabla_{f \cdot X_1 + g \cdot X_2} = f \cdot \nabla_{X_1} + g \cdot \nabla_{X_2}, \]

that is

\[ \nabla_{f \cdot X_1 + g \cdot X_2}(u) = f \cdot \nabla_{X_1}(u) + g \cdot \nabla_{X_2}(u), \]

\[ X_1, X_2 \in \Gamma(TM), \ f, g \in \mathcal{C}^\infty(M), \ u \in \Gamma(L). \]
More exactly for each vector field $X \in \Gamma(TM)$ the covariant derivative associates a pair $\nabla_X = (\mathcal{D}, X)$,

$$
\mathcal{D} : \Gamma(L) \rightarrow \Gamma(L), \quad X : C^\infty(M) \rightarrow C^\infty(M),
$$

that satisfies the conditions

1. 

$$
\mathcal{D}([u_1, u_2]) = [\mathcal{D}(u_1), u_2] + [u_1, c\mathcal{D}(u_2)], \quad u_1, u_2 \in \Gamma(L).
$$

2. 

$$
\mathcal{D}(f \cdot u) = X(f) \cdot u + f \cdot \mathcal{D}(u), \quad u \in \Gamma(L), \quad f \in C^\infty(M).
$$
The association $\nabla_X = (\mathcal{D}, X) = (\mathcal{D}(X), X)$ satisfies the last condition

$$\nabla_f \cdot X_1 + g \cdot X_2 = f \cdot \nabla X_1 + g \cdot \nabla X_2,$$
The family of all covariant derivatives of fiberwise differentiation forms the infinite dimensional Lie algebra with respect to operations of summation and composition. Really, let $(\mathcal{D}_1, X_1)$ and $(\mathcal{D}_2, X_2)$ be two covariant derivatives. The sum of derivatives $(\mathcal{D}_3, X_3)$ is defined by the formula

1. $X_3 = X_1 + X_2$,
2. $\mathcal{D}_3 = \mathcal{D}_1 + \mathcal{D}_2$

The commutator brace $(\mathcal{D}_3, X_3) = \{(\mathcal{D}_1, X_1), (\mathcal{D}_2, X_3)\}$ is defined by the formula:

1. $X_3 = [X_1, X_2]$,
2. $\mathcal{D}_3 = [\mathcal{D}_1, \mathcal{D}_2]$
The family of all covariant derivatives of fiberwise differentiation is the space of sections of a transitive Lie algebroid, namely $\mathcal{D}_{\text{der}}(L) \to TM$, that is there is a bundle $\mathcal{D}_{\text{der}}(L) \to TM$, such that one has the exact sequence

$$0 \to \text{Aut}(L) \to \mathcal{D}_{\text{der}}(L) \to TM \to 0.$$ 

The bundle $\mathcal{D}_{\text{der}}(L)$ can be constructed as a union of fibres where each fiber $\mathcal{D}_{\text{der}}(L)_x$ in the point $x \in M$ consists of all covariant derivatives $(\mathcal{D}, X)$ in the point $x \in M$. 
That is the pair \((\mathcal{D}, X)\) is the pair of operators

\[
\mathcal{D} : \Gamma(L) \to L_x = \Gamma(x, L), \quad X : C^\infty \to C,
\]

which satisfy the conditions

\[
\mathcal{D}([u_1, u_2]) = [\mathcal{D}(u_1), u_2(x)] + [u_1(x), \mathcal{D}(u_2)], \quad u_1, u_2 \in \Gamma(L),
\]

\[
\mathcal{D}(f \cdot u) = X(f) \cdot u(x) + f(x) \cdot \mathcal{D}(u),
\]

\[
u \in \Gamma(L) \quad f \in C^\infty(M).
\]
The fiber $\mathcal{D}_{\text{der}}(L)_x$ has finite dimension since belongs to the exact sequence

$$0 \rightarrow \text{Aut}(L_x) \rightarrow \mathcal{D}_{\text{der}}(L)_x \rightarrow T_x M \rightarrow 0,$$

and the kernel $\text{Aut}(L_x)$ consist of the operators of the form $(\mathcal{D}, X)$, that satisfy the conditions

$$\mathcal{D}([u_1, u_2]) = [\mathcal{D}(u_1), u_2(x)] + [u_1(x), \mathcal{D}(u_2)], \quad u_1, u_2 \in \Gamma(L),$$

$$\mathcal{D}(f \cdot u) = f(x) \cdot \mathcal{D}(u),$$

$$u \in \Gamma(L) \quad f \in C^\infty(M),$$

hence consists of automorphisms of the finitely dimensional Lie algebra $L_x$.
The exact sequence of the bundles

\[ 0 \rightarrow Aut(L) \rightarrow \mathcal{D}_{der}(L) \rightarrow TM \rightarrow 0. \]

can be included in the exact diagram
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\[ ZL \overset{=} \longrightarrow ZL \]
\[ i \downarrow \quad i \downarrow \]
\[ L \overset{=} \longrightarrow L \]
\[ ad \downarrow \quad ad \downarrow \]
\[ 0 \longrightarrow \text{Aut}(L) \overset{j} \longrightarrow \mathcal{D}_{\text{Der}}(L) \overset{a} \longrightarrow T\mathcal{M} \longrightarrow 0 \]
\[ \overset{\hat{b}^0} \downarrow \quad \overset{\hat{b}} \downarrow \quad \overset{=} \downarrow \]
\[ 0 \longrightarrow \text{Out}(L) \overset{j} \longrightarrow \text{Out}\mathcal{D}_{\text{Der}}(L) \overset{\bar{a}} \longrightarrow T\mathcal{M} \longrightarrow 0 \]

\[ 0 \quad 0 \quad 0 \]

A.S. Mishchenko (Harbin Institute of Technology, Moscow State University) Talk at the International Conference: Characteristic classes of transitive Lie algebroids. Categories.
The splitting of the algebroid $E$ in a direct sum

$$E = L \oplus TM,$$

means that there is a splitting map in the exact sequence

$$0 \rightarrow L \rightarrow E \xrightarrow{a} TM \rightarrow 0,$$

$T : TM \rightarrow E$, $a \circ T = \text{Id}$, that is called transversal.
There is a morphism of the algebroids

\[ E \xrightarrow{ad} \mathcal{D}_{der}(L) \]

\[ TM \]

so called adjoint representation of the Lie algebroid \( E \).
This representation also can be included in an exact diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & TM & \longrightarrow & 0 \\
\downarrow{\text{ad}} & & \downarrow{\text{ad}} & & \downarrow{=} & & \\
0 & \longrightarrow & \text{Aut}(L) & \longrightarrow & \mathcal{D}_{\text{der}}(L) & \longrightarrow & TM & \longrightarrow & 0
\end{array}
\]
and can be extended to the following diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & E & \rightarrow & TM & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
ZL & \rightarrow & 0 & \rightarrow & Aut(L) & \rightarrow & D_{der}(L) & \rightarrow & TM & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Rightarrow & & \Rightarrow & & \Rightarrow & & \Rightarrow & & \Rightarrow & & \Rightarrow \\
Out(L) & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]
and can be closed to the following diagram

\[
\begin{array}{ccccccccc}
ZL & = & ZL \\
\downarrow & & \downarrow \\
0 & \rightarrow & L & \rightarrow & E & \rightarrow & TM & \rightarrow & 0 \\
\downarrow{ad} & & \downarrow{ad} & & \downarrow{=} \\
0 & \rightarrow & Aut(L) & \rightarrow & D_{der}(L) & \rightarrow & TM & \rightarrow & 0 \\
\downarrow{=} & & \downarrow{=} \\
Out(L) & = & Out(L) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
\]
Hence

\[
\begin{array}{ccccccc}
ZL & = & ZL & \Rightarrow & ZL & \Rightarrow & ZL \\
0 & \Rightarrow & L & \Rightarrow & E & \Rightarrow & TM & \Rightarrow & 0 \\
0 & \Rightarrow & Aut(L) & \Rightarrow & D_{der}(L) & \Rightarrow & OutD_{der}(L) & \Rightarrow & 0 \\
0 & \Rightarrow & Out(L) & \Rightarrow & Out(L) & \Rightarrow & Out(L) & \Rightarrow & 0
\end{array}
\]
All homomorphisms of the bundles are morphisms of the Lie algebroids. In particular the exact sequence

\[
0 \longrightarrow \text{Out}(L) \xrightarrow{j} \text{Out}\mathcal{D}_{\text{Der}}(L) \xrightarrow{\bar{a}} TM \longrightarrow 0 \quad (7)
\]

has a transversal $T$ which also is a morphism of the Lie algebroids. This means that $T$ is a coupling.

\[
0 \longrightarrow \text{Out}(L) \xrightarrow{j} \text{Out}\mathcal{D}_{\text{Der}}(L) \xrightarrow{T}{\bar{a}} TM \longrightarrow 0.
\]

Therefore the bundle $\text{Out}(L)$ is flat.
In (Mackenzie, Lemma 7.2.4) a cocycle $f_T \in \Omega^3(M, \rho^T, ZL)$ was defined that plays the role of the obstruction $f_T \in \text{Obs}(T) \in H^3(M, \rho^T, ZL)$ for existence of the structure of the Lie algebroids $A$. The key property of the cocycle $f_T$ is that it is functorial. Precisely, if $\varphi : M_1 \to M_2$, $L_1 = \varphi^*(L_2)$, $T_1$ is the inverse image of the coupling $T_2$ then

$$\varphi^*(\text{Obs}(T_2)) = \text{Obs}(T_1).$$
The bundle $L$ has the structural group $Out(L)^\delta$ with more fine topology such that the exact sequence

$$0 \rightarrow \mathbb{Z}g \rightarrow g \xrightarrow{ad} \text{Aut}(g) \xrightarrow{\delta^0} Out(g)_{\text{discrete}} \rightarrow 0$$

has the discrete topology in the group $Out(g)$. 
Hence the bundle $L$ can be described as inverse image from classifying space of the structural group $Out(\mathfrak{g})^\delta$:

$$\varphi : M \rightarrow BOut(\mathfrak{g})^\delta;$$

$L = \varphi^*(L^\infty)$
Theorem

There is a cohomology class $\text{Obs} \in \mathcal{H}^3(B\text{Out}(\mathfrak{g})^\delta, \rho, Z \mathfrak{g})$ such that

$$\text{Obs}(L) = \varphi^*(\text{Obs}) \in \Omega^3(M, \rho^T, ZL).$$
Due to

Yasumasa Hirashima *A note on cohomology with local coefficients*, Osaka J. Math., 16 (1979), 219–231.

the obstruction class $\text{Obs} \in \mathcal{H}^3(\text{BOut}(\mathfrak{g})^\delta, \rho, Z\mathfrak{g})$ can be identified with the homotopy class of continuous maps

$$\text{BOut}(\mathfrak{g})^\delta \xrightarrow{\varphi_{\text{Obs}}} \text{K}_{\text{Out}(\mathfrak{g})}(Z\mathfrak{g}, 3)$$

where $K_G(Z, 3)$ denotes the equivariant version of the Eilenberg-MacLane complex for $G$-module $Z$. 
The Eilenberg-MacLane complex $K_G(Z, 3)$ has the subcomplex $K(G, 1) \subset K_G(Z, 3)$ which pays the role of the point for classical case. Then we have the diagram of fibrations

$$
\begin{array}{ccc}
B\text{Out}(g)^{\delta} & \xrightarrow{\varphi_{\text{Obs}}} & K_{\text{Out}(g)}(Zg, 3) \\
\uparrow & & \uparrow \\
F(g) & \longrightarrow & K(\text{Out}(g), 1)
\end{array}
$$
For a transitive Lie algebroid

\[ \mathcal{A} = \left\{ \begin{array}{ccc} E & \xrightarrow{a} & TM \\ p_E & & \downarrow p^T \\ M & \xrightarrow{\bullet, \bullet} & M \end{array} \right\} \]

with fixed structural Lie algebra \( g \) one has a continuous map

\[ f : M \rightarrow F(g) \]
that is generated by a continuous map

\[ \mathcal{B}_g \longrightarrow F(g) \]

That is one has the commutative diagram

\[ \mathcal{T \mathcal{L} A}_g(M) \approx [M, \mathcal{B}_g] \longrightarrow [M, F(g)] \]
Due to Mackenzie theorem [7.3.18]

**Theorem**

Let $\mathcal{A}$ be a Lie algebroid on $M$, let $L$ be a Lie algebra bundle on $M$ and let $T$ be a coupling of $\mathcal{A}$ with $L$ such that

$$\text{Obs}(T) = 0 \in \mathcal{H}^3(\mathcal{A}, \rho^T, ZL).$$

Then the additive group $\mathcal{H}^2(\mathcal{A}, \rho^T, ZL)$ acts freely and transitively on the fiber of the map $\text{Coupling}(T)$. 
Therefore

\[ \mathcal{B}_g \approx F(g) \times K_{Out(g)}(Zg, 2) \]

**Proposition**

The family of all characteristic classes for transitive Lie algebroids \( \mathcal{T}\mathcal{L}A_g(M) \) can be identified with

\[ \mathcal{H}^*(F(g) \times K_{Out(g)}(Zg, 2)) \]
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The Chern-Weil homomorphism is a homomorphism from the algebra of invariants $\bigoplus (\Gamma \vee L^*)_I$ to the cohomology of the Lie algebroid $\mathcal{A}$, $\mathcal{H}^*(M, \mathcal{A})$:

$$h_{\mathcal{A}} : \bigoplus (\Gamma \vee L^*)_I \longrightarrow \mathcal{H}^*(M, \mathcal{A})$$

that gives a nonclassical example of the characteristic classes since the Chern-Weil homomorphism commutes with pullback of the Lie algebroids (Kubarski, theorem 4.2.2, theorem 4.3.1)
The difference consists on that the cohomology for Kubarski case depends on the choice of the Lie algebroid on $M$. So we should introduce here the cohomology of the classifying space $B_g$ with coefficients in nonexisting Lie algebroid on the $B_g$. 

Chern-Weil homomorphism
Chern-Weil homomorphism

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Mackenzie, K.C.H., *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press, (2005).

Kubarski, J., *The Chern-Weil homomorphism of regular Lie algebroids*, Publications du Department de Mathematiques, Universite Claude Bernard - Lyon-1, (1991) pp.4–63.

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W.Walas *Algebry Liego-Rineharta i pierwsze klasy charakterystyczne*, PhD manuscript, Lodz, Poland, 2007.
References

Allen Hatcher, *Vector bundles and K-theory*, Available at http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html, page 7, 2003.

Yasumasa Hirashima *A note on cohomology with local coefficients*, Osaka J. Math., 16 (1979), 219–231.