ADJACENCY ENERGY OF HYPERGRAPHS
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Abstract. In this paper, we define and obtain several properties of the (adjacency) energy of a hypergraph. In particular, bounds for this energy are obtained as functions of structural and spectral parameters, such as Zagreb index and spectral radius. We also study how the energy of a hypergraph varies when a vertex/edge is removed or when an edge is divided. In addition, we solved the extremal problem energy for the class of hyperstars, and show that the energy of a hypergraph is never an odd number.

Keywords. Hypergraphs; Adjacency energy; Edge division; Power graph.

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1. Introduction

In Spectral graph theory, the structure of graphs is studied through the eigenvalues/eigenvectors of matrices associated with them. Many researchers around the world, motivated by this theory, have defined some matrices associated with hypergraph, aiming to develop a spectral hypergraph theory [8, 23, 24]. In 2012, Cooper and Dutle [6] proposed the study of hypergraphs by means of the adjacency tensor. It is known, however, that to obtain eigenvalues of tensors has a high computational and theoretical cost [12]. Perhaps for this reason, recently, some authors have renewed the interest to study matrix representations of hypergraphs, as for example in [1, 7, 16, 18, 19, 25, 26].

The study of molecular orbital energy levels of \( \pi \)-electrons in conjugated hydrocarbons may be seen as one of the oldest applications of spectral graph theory \([17, 14]\). In those studies, graphs were used to represent hydrocarbon molecules and it was shown that an approximation of the total \( \pi \)-electron energy may be computed from the eigenvalues of the graph. Based on this chemical concept, in 1977 Gutman \([10]\) defined graph energy. In 2007, Nikiforov \([20]\) extended the concept of graph energy to matrices. For a matrix \( \mathbf{M} \), its energy \( \mathcal{E}(\mathbf{M}) \), is defined as the sum of its singular values.

Let \( \mathcal{H} \) be a hypergraph and \( \mathbf{A} \) its adjacency matrix. We define here the energy of \( \mathcal{H} \) as \( \mathcal{E}(\mathbf{A}) \) and denote it by \( \mathcal{E}(\mathcal{H}) \).

The main purpose of this paper is to discuss this natural definition of (adjacency) energy of a hypergraph and to understand whether known properties of graph energy may be extended to the structure of hypergraphs.

We highlight some of the results we obtain. We start by studying a particular class of hypergraphs and determine which hyperstar has highest and lowest energy among this class. By understanding the operations sum and product of hypergraphs, we obtain restrictions on the values that \( \mathcal{E}(\mathcal{H}) \) may have. In particular, we show that \( \mathcal{E}(\mathcal{H}) \) is never an odd number. In addition, we determine bounds for the variation of the energy of a hypergraph \( \mathcal{H} \), when we perform operations on its edges or vertices. Our main contribution is to obtain several lower and upper bounds relating \( \mathcal{E}(\mathcal{H}) \) to important structural and spectral parameters.

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The remaining of the paper is organized as follows. In Section 2, we recall some basic definitions about hypergraphs and matrices that will be used. In Section 3, we study the energy of a hyperstar. In Section 4, we prove that the energy of a hypergraph can never be an odd number. In sections 5 and 6, we study what happens when we delete or divide an edge of a hypergraph. Finally, in Section 7, we will obtain bounds for the energy of a hypergraph. These bounds are related to important spectral and structural parameters, such as Zagreb index and spectral radius.

2. Preliminaries

In this section, we shall present some basic definitions about hypergraphs and matrices, as well as terminology, notation and concepts that will be useful in our proofs.

A hypergraph $\mathcal{H} = (V, E)$ is a pair composed by a set of vertices $V(\mathcal{H})$ and a set of (hyper)edges $E(\mathcal{H})$, where each edge is a subset of $V(\mathcal{H})$, with cardinality greater than or equal 2. The rank and the co-rank of a hypergraph are defined as the largest and smallest cardinality of its edges, respectively. $\mathcal{H}$ is said to be a $k$-uniform hypergraph (or a $k$-graph) for $k \geq 2$, if all edges have the same cardinality $k$. Let $\mathcal{H} = (V, E)$ and $\mathcal{H}' = (V', E')$ be hypergraphs, if $V' \subseteq V$ and $E' \subseteq E$, then $\mathcal{H}'$ is a subgraph of $\mathcal{H}$. The complete $k$-graph $K_n^k$ on $n$ vertices, is a hypergraph, such that any subset of $k$ vertices is an edge. A hypergraph $\mathcal{H}$ is linear if each pair of edges has at most one common vertex.

A multi-hypergraph $\mathcal{H} = (V, E)$ is a pair composed by a set of vertices $V(\mathcal{H})$ and a multi-set of (multi-)edges $E(\mathcal{H})$, where each edge is a subset of $V(\mathcal{H})$.

Notice that in a hypergraph the edges must have at least two vertices and distinct edges cannot have exactly the same vertices, while in a multi-hypergraph there may be edges containing exactly the same vertices. Additionally, there could be edges with one or zero vertices.

Let $\mathcal{H} = (V, E)$ be a (multi-)hypergraph. The edge neighborhood of a vertex $v \in V$, denoted by $E_{[v]}$, is the set of all edges that contains $v$. The degree of a vertex $v \in V$, denoted by $d(v)$, is the number of edges that contain $v$. More precisely, $d(v) = |E_{[v]}|$. A hypergraph is $r$-regular if $d(v) = r$ for all $v \in V$. We define the maximum, minimum and average degrees, respectively, as

$$
\Delta(\mathcal{H}) = \max_{v \in V}\{d(v)\}, \quad \delta(\mathcal{H}) = \min_{v \in V}\{d(v)\}, \quad d(\mathcal{H}) = \frac{1}{|V|} \sum_{v \in V} d(v).
$$

Let $\alpha = \{v_1, \ldots, v_l\} \subset V$ be a subset of vertices. We define the degree of a set $\alpha$, as the number of edges that contain simultaneously all vertices of $\alpha$ and denote it for $d(\alpha)$.

Let $\mathcal{H}$ be a hypergraph. A walk of length $l$ is a sequence of vertices and edges $v_0e_1v_1e_2\cdots e_lv_l$ where $v_{i-1}$ and $v_i$ are distinct vertices contained in $e_i$ for each $i = 1, \ldots, l$. A cycle is a walk where $v_0 = v_l$. The hypergraph is connected, if for each pair of vertices $u, w$ there is a walk $v_0e_1v_1e_2\cdots e_lv_l$ where $u = v_0$ and $w = v_l$. Otherwise, the hypergraph is disconnected.

Let $\mathcal{H} = (V, E)$ be a (multi-)hypergraph with $n$ vertices. The adjacency matrix $A(\mathcal{H}) = (a_{ij})$ is defined as a square matrix with $n$ rows. For each pair of vertices $i, j \in V$, if $i = j$, then $a_{ii} = 0$, and if $i \neq j$, then $a_{ij} = d(\{i, j\})$. We denote its characteristic polynomial by $P_A(\lambda) = \det(\lambda I_n - A)$. Its eigenvalues will be denoted by
\( \lambda_1(A) \geq \cdots \geq \lambda_n(A) \). If \( x \) is an eigenvector of \( \lambda \), we call the pair \( (\lambda, x) \), as an eigenpair of \( A \). The spectral radius \( \rho(A) \), is the largest absolute value of its eigenvalues. The (adjacency) energy of a hypergraph is the sum of the singular values of its adjacency matrix. We notice that \( A \) is a real and symmetric square matrix, then its energy is the sum of the absolute values of the eigenvalues, in other words

\[
E(H) = \sum_{i=1}^{n} |\lambda_i|.
\]

Let \( H = (V, E) \) be a hypergraph with \( n \) vertices. For a non-empty subset of vertices \( \alpha = \{v_1, \ldots, v_t\} \subset V \) and a vector \( x = (x_i) \) of dimension \( n \), we denote \( x(\alpha) = x_{v_1} + \cdots + x_{v_t} \). So we have,

\[
(A(H)x)_u = \sum_{e \in E[u]} x(e - \{u\}), \quad \forall u \in V(H).
\]

3. The energy of a hyperstar

In this section, we will obtain the energy of a hyperstar and determine which hyperstar with \( n \) vertices has the highest energy.

First of all, we will present the definition of a power graph, and obtain an algebraic property of its eigenvalues.

**Definition 3.1.** Let \( G = (V, E) \) be a graph and let \( k \geq 2 \) be an integer. We define the power graph \( G^k \) as the \( k \)-graph with the following sets of vertices and edges

\[
V(G^k) = V(G) \cup \left( \bigcup_{e \in E(G)} \varsigma_e \right) \quad \text{and} \quad E(G^k) = \{e \cup \varsigma_e : e \in E(G)\},
\]

where \( \varsigma_e = \{v_1^e, \ldots, v_{k-2}^e\} \) for each edge \( e \in E(G) \).

We may say that \( G^k \) is obtained from a base graph \( G = (V, E) \), adding \( k - 2 \) new vertices to each edge \( e \in E(G) \). For each edge \( e \in E(G) \), we denote \( e^k = e \cup \varsigma_e \in E(G^k) \). The spectrum of this class has already been studied, see for example \([3, 5]\).

We define a hyperstar as a power graph of a star.

**Example 3.2.** The power graph \( (S_4)^3 \) of the star \( S_4 \) is illustrated in Figure 1.
Lemma 1. Let $\mathcal{H}$ be a $k$-graph having two vertices $u$ and $v$ which are contained exactly in the same edges. If $(\lambda, x)$ is an eigenpair of $A(\mathcal{H})$ with $\lambda \neq -d(u)$, then $x_u = x_v$.

Proof. We observe that,

$$(\lambda + d(u))x_u = \sum_{e \in E_u} x(e) = \sum_{e \in E_v} x(e) = (\lambda + d(v))x_v.$$ 

Since $d(u) = d(v)$ and $\lambda \neq -d(u)$, then the result follows. \hfill \Box

Theorem 2. Let $S_n$ be the star on $n$ vertices. If $k \geq 2$ is an integer, then spectrum of $(S_n)^k$ is:

$$\{(\lambda)_{n-1}^{(n-1)(k-2)}, (k-2)^{n-2}, r^+, r^-\},$$

where $r^+$ and $r^-$ are the roots of $x^2 - (k-2)x - (n-1)(k-1) = 0$.

Proof. Let $e \in E(S_n)$ be an edge, let $\{u_1, \ldots, u_{k-1}\}$ be the vertices of degree one in $e^k$ and $2 \leq i \leq k-1$. We can construct the following family of $k-2$ linearly independent eigenvectors.

$$x^i = \begin{cases} 
(x^i)_{u_1} = 1, \\
(x^i)_{u_i} = -1, \\
(x^i)_{u} = 0, \text{ for } u \in V((S_n)^k) - \{u_1, u_i\}.
\end{cases}$$

Repeating this construction for the other edges of $S_n$, we obtain $(n-1)(k-2)$ linearly independent eigenvectors, associated with $\lambda = -1$.

Let $E(S_n) = \{e_1, \ldots, e_{n-1}\}$ and $2 \leq j \leq n-1$. We can construct the following family of $n-2$ linearly independent eigenvectors.

$$z^j = \begin{cases} 
(z^j)_{u_1} = 1, \text{ if } u \text{ is a vertex of degree one in } (e_1)^k, \\
(z^j)_{u_j} = -1, \text{ if } v \text{ is a vertex of degree one in } (e_j)^k, \\
(z^j)_{u} = 0, \text{ if } w \text{ is not a vertex of degree one in } (e_1)^k \text{ or in } (e_j)^k.
\end{cases}$$

So, we have $n-2$ linearly independent eigenvectors, associated with $\lambda = k-2$.

Let $A((S_n)^k)$ be the adjacency matrix of the $k$th power of the star $S_n$, and let $x$ be the system of eigenvalues can be written as:

$$\begin{align*}
\lambda x_{u_1} &= x_v + (k-2)x_{u_1} \\
\lambda x_{u_2} &= x_v + (k-2)x_{u_2} \\
\vdots \\
\lambda x_{u_i} &= x_v + (k-2)x_{u_i} \\
\lambda x_{u} &= x_v + (k-1)x_{u_1} + \cdots + (k-1)x_{u_{n-1}}
\end{align*}$$

Notice that, if $\lambda \neq k-2$, then $x_{u_1} = \cdots = x_{u_{n-1}}$, so the system can be write as:

$$\begin{align*}
\lambda x_{u_1} &= x_v + (k-2)x_{u_1} \\
\lambda x_{v} &= (n-1)(k-1)x_{u_1}
\end{align*}$$

By the first equation we have that, $x_v = (\lambda - k + 2)x_{u_1}$, so by second equation we conclude that

$$\lambda (\lambda - k + 2)x_{u_1} = (n-1)(k-1)x_{u_1}.$$
so

$$\lambda^2 + \lambda(-k + 2) - (n - 1)(k - 1) = 0.$$ 

Finally, to conclude the result just notice that the sum of the multiplicities of the eigenvalues is $|V((S_n)^k)| = (n - 1)(k - 1) + 1$. 

**Corollary 3.** Let $S_n$ be the star on $n$ vertices. If $k \geq 2$ is an integer, then

$$\mathcal{E}((S_n)^k) = (k - 2)(2n - 3) + \sqrt{(k - 2)^2 + 4(n - 1)(k - 1)}.$$  

**Proof.** We just notice that

$$(n - 1)(k - 2) - 1 + (n - 2)|k - 2| = (k - 2)(2n - 3).$$

Now, observe that

$$r^+ = \frac{(k - 2) + \sqrt{(k - 2)^2 + 4(n - 1)(k - 1)}}{2} \geq 0$$

and

$$r^- = \frac{(k - 2) - \sqrt{(k - 2)^2 + 4(n - 1)(k - 1)}}{2} \leq 0,$$

so $|r^+| + |r^-| = r^+ - r^- = \sqrt{(k - 2)^2 + 4(n - 1)(k - 1)}$. 

**Corollary 4.** If $S$ is a hyperstar with $t$ vertices, then

$$2\sqrt{t - 1} = \mathcal{E}(S_t) \leq \mathcal{E}(S) \leq \mathcal{E}((S_2)^t) = 2(t - 1).$$

**Proof.** If $S$ is a hyperstar, then there is $2 \leq n \leq t$ and $2 \leq k \leq t$ such that $S = (S_n)^k$. In this way, we have that $t = (n - 1)(k - 1) + 1$, so $n = \frac{t - 1}{k - 1} + 1$, therefore

$$\mathcal{E}(S) = 2(t - 1) \left( \frac{k - 2}{k - 1} \right) - (k - 2) + \sqrt{(k - 2)^2 + 4(t - 1)}.$$ 

Consider the function $f : [2, t] \rightarrow \mathbb{R}$, defined by

$$f(x) = 2(t - 1) \left( \frac{x - 2}{x - 1} \right) - (x - 2) + \sqrt{(x - 2)^2 + 4(t - 1)}.$$ 

Computing its derivatives, we obtain

$$f'(x) = \frac{2(t - 1)}{(x - 1)^2} - 1 + \frac{x - 2}{\sqrt{(x - 2)^2 + 4(t - 1)}},$$

$$f''(x) = \frac{-4(t - 1)}{(x - 1)^3} + \frac{4(t - 1)}{(x - 2)^2 + 4(t - 1)} \frac{2}{x}.$$ 

We observe that $f''(x) < 0 \iff (x - 1)^2 < (x - 2)^2 + 4(t - 1) \iff 2x + 1 < 4t$, therefore $f'(x)$ is a decreasing function. Now, we notice that $f'(t) = \frac{2}{t - 1} - \frac{2}{t} > 0$, so $f'(x) > 0$ for all $x \in [2, t]$, so $f(x)$ is an increasing function and therefore the result follows. 

□
4. The energy of a hypergraph is never odd

In this section, we will prove that the energy of a hypergraph can never be an odd number. More precisely, we prove that the energy of a hypergraph can not be a $p$-th root of an odd number, and if it is a $p$-root of an even number then its even prime factors have power smaller than $p$. Similar results have already been obtained for graphs in [2, 22].

We start by defining some operations between hypergraphs, and computing the spectrum of the resulting hypergraph.

**Definition 4.1.** Let $H$ and $G$ be $k$-graphs. We define its sum $H \oplus G$ as the $k$-graph, with the sets of vertices $V(H \oplus G) = V(H) \times V(G)$ and edges

$$E(H \oplus G) = \{\{v\} \times e : v \in V(H), e \in E(G)\} \cup \{f \times \{u\} : f \in E(H), u \in V(G)\}.$$ 

**Example 4.2.** Let $H$ and $G$ be 3-graphs, such that $V(H) = \{1, 2, 3, 4\}$, $E(H) = \{123, 234\}$ and $V(G) = \{a, b, c\}$, $E(G) = \{abc\}$. The sum $H \oplus G$ has the following sets of vertices and edges

$$V(H \oplus G) = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\},$$

$$E(H \oplus G) = \begin{cases} 
\{(1, a), (1, b), (1, c)\}, & \{(1, a), (2, a), (3, a)\}, \\
\{(2, a), (2, b), (2, c)\}, & \{(1, b), (2, b), (3, b)\}, \\
\{(3, a), (3, b), (3, c)\}, & \{(1, c), (2, c), (3, c)\}, \\
\{(4, a), (4, b), (4, c)\}, & \{(2, a), (3, a), (4, a)\}, \\
\{(2, b), (3, b), (4, b)\}, & \{(2, c), (3, c), (4, c)\}, \\
\end{cases}$$

**Proposition 5.** If $G$ and $H$ are two $k$-graphs having eigenvalues $\mu$ and $\lambda$ with multiplicities $m_1$ and $m_2$, respectively, then $\mu + \lambda$ is an eigenvalue of $A(G \oplus H)$, with multiplicity $m_1 \cdot m_2$.

**Proof.** Suppose $x$ an eigenvector of $\lambda$ in $A(H)$ and $y$ an eigenvector of $\mu$ in $A(G)$. Consider $(v, u)$ a vertex of $G \oplus H$, define a vector $z$ by $z_{(v, u)} = y_v x_u$. Thus,

$$(Az)_{(v, u)} = \sum_{a \in E_{(v, u)}} z(\alpha - (v, u)) = \sum_{e \in E_{[v]}} y_e x(e - u) + \sum_{a \in E_{[v]}} y(a - v) x_u = (\mu + \lambda) y_v x_u.$$ 

Therefore, the result follows. \qed

**Definition 4.3.** Let $H$ and $G$ be $k$-graphs.

1. For each edge $e \in E(H)$, we say that a sequence $\alpha = (v_1, v_2, \ldots, v_k)$ is an ordered edge from $e$, if the set of its elements is equal to the edge $e$.

2. For each edge $e \in E(H)$, we denote by $S_H(e)$ the set of all ordered edges from $e$.

3. Let $e \in E(H)$ and $f \in E(G)$. For $\alpha = (v_1, \ldots, v_k) \in S_H(e)$ and $\beta = (u_1, \ldots, u_k) \in S_G(f)$, we define its product as the following set of ordered pairs

$$\alpha \otimes \beta = \{(v_1, u_1), \ldots, (v_k, u_k)\}.$$ 

4. We define the product $H \otimes G$ as the $k$-graph, with the sets of vertices $V(H \otimes G) = V(H) \times V(G)$ and edges

$$E(H \otimes G) = \{\alpha \otimes \beta : \alpha \in S_H(e), \text{ where } e \in E(H), \text{ and } \beta \in S_G(f), \text{ where } f \in E(G)\}.$$
**Example 4.4.** Let $\mathcal{H}$ and $\mathcal{G}$ be 3-graphs, such that $V(\mathcal{H}) = \{1, 2, 3, 4\}$, $E(\mathcal{H}) = \{123, 234\}$ and $V(\mathcal{G}) = \{a, b, c\}$, $E(\mathcal{G}) = \{abc\}$. The product $\mathcal{H} \otimes \mathcal{G}$ has the following sets of vertices and edges

$$V(\mathcal{H} \otimes \mathcal{G}) = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}.$$ 

$$E(\mathcal{H} \otimes \mathcal{G}) = \begin{cases} 
\{(1, a), (2, b), (3, c)\}, & \{(2, a), (3, b), (4, c)\}, \\
\{(1, a), (2, c), (3, b)\}, & \{(2, a), (3, c), (4, b)\}, \\
\{(1, b), (2, a), (3, c)\}, & \{(2, b), (3, a), (4, c)\}, \\
\{(1, b), (2, c), (3, a)\}, & \{(2, b), (3, c), (4, a)\}, \\
\{(1, c), (2, a), (3, b)\}, & \{(2, c), (3, a), (4, b)\}, \\
\{(1, c), (2, b), (3, a)\}, & \{(2, c), (3, b), (4, a)\} \end{cases}.$$

**Proposition 6.** If $\mathcal{G}$ and $\mathcal{H}$ are two $k$-graphs, with eigenvalues $\mu$ of multiplicity $m_1$ and $\lambda$ of multiplicity $m_2$ respectively, then $\mu \lambda$ is an eigenvalue of $A(\mathcal{G} \otimes \mathcal{H})$, with multiplicity $m_1 \cdot m_2$.

**Proof.** Suppose $x$ an eigenvector of $\lambda$ in $A(\mathcal{H})$ and $y$ an eigenvector of $\mu$ in $A(\mathcal{G})$. Consider $(v, u)$ a vertex of $\mathcal{G} \otimes \mathcal{H}$, define a vector $z$ by $z_{(v, u)} = y_v x_u$. Thus,

$$(Az)_{(v, u)} = \sum_{\alpha \in E_{\mathcal{H} \otimes \mathcal{G}}(v, u)} z(\alpha - (v, u)) = \sum_{\alpha \in E_{\mathcal{H}}(v)} \sum_{e \in E_{\mathcal{G}}(u)} y(\alpha - v)x(e - u) = \mu \lambda y_v x_u.$$ 

Therefore, the result is true. \hfill $\square$

**Lemma 7.** Let $\mathcal{H}$ be a $k$-graph. If $\lambda_1, \ldots, \lambda_t$ are all positive eigenvalues of $A(\mathcal{H})$, then $E(\mathcal{H}) = 2 \sum_{i=1}^t \lambda_i$.

**Proof.** This follows, because the trace of the adjacency matrix is zero and equals the sum of the eigenvalues. \hfill $\square$

**Remark 4.5.** We notice that the characteristic polynomial of $\mathcal{H}$ is a monic polynomial with integer coefficients. Thus, if an eigenvalue of $\mathcal{H}$ is a rational number, then it has to be an integer.

**Theorem 8.** Let $\mathcal{H}$ be a $k$-graph. If $E(\mathcal{H})$ is a rational number, then $E(\mathcal{H})$ is even.

**Proof.** Let $\lambda_1, \ldots, \lambda_t$ be all positive eigenvalues of $\mathcal{H}$, so $E(\mathcal{H}) = 2 \sum_{i=1}^t \lambda_i$, that is $\sum_{i=1}^t \lambda_i$ is a rational number. By Proposition 5, we have that $\sum_{i=1}^t \lambda_i$ is an eigenvalue of a hypergraph and by Remark 4.5 we conclude that it must be an integer, therefore $E(\mathcal{H}) = 2 \sum_{i=1}^t \lambda_i$ must be an even number. \hfill $\square$

**Theorem 9.** Let $p$ and $q$ be integers such that $p \geq 1$ and $0 \leq q \leq p - 1$ and $t$ be an odd integer. If $\mathcal{H}$ is a $k$-graph, then $E(\mathcal{H}) \neq \sqrt{2^t \lambda}$. 

**Proof.** Suppose by contradiction that $E(\mathcal{H}) = \sqrt{2^t \lambda}$. In this way, we have

$$(E(\mathcal{H}))^p = \left(2 \sum_{i=1}^t \lambda_i\right)^p = 2^t \lambda^p \Rightarrow \left(\sum_{i=1}^t \lambda_i\right)^p = \frac{t}{2^{p-q}}.$$ 

By Proposition 5 and 6, we have that $(\sum_{i=1}^t \lambda_i)^p$ is an eigenvalue of a hypergraph, however it is a non integral rational number, which is a contradiction. \hfill $\square$
In this section we will study the variation of the energy of a hypergraph when we delete a vertex or an edge of the hypergraph.

**Definition 5.1.** Let $\mathcal{H} = (V, E)$ be a hypergraph, $v \in V$ be a vertex and $e_1, \ldots, e_d$ be all edges containing $v$, we define $\mathcal{H} - v$ by $V(\mathcal{H} - v) = V(\mathcal{H}) - \{v\}$ and $E(\mathcal{H} - v) = (E(\mathcal{H}) - \{e_1, \ldots, e_d\}) \cup \{e_1 - \{v\}, \ldots, e_d - \{v\}\}$.

**Theorem 10.** Let $\mathcal{H}$ be a hypergraph and $v \in V(\mathcal{H})$ be a vertex, then $E(\mathcal{H}) \geq E(\mathcal{H} - v)$.

**Proof.** First notice that, $A(\mathcal{H} - v)$ is a principal sub-matrix of $A(\mathcal{H})$. Let $\mu_1, \ldots, \mu_t$ be the positive eigenvalues of $A(\mathcal{H} - v)$ and let $\mu_{t+1}, \ldots, \mu_n - 1$ be the non positive eigenvalues of $A(\mathcal{H} - v)$. From the Interlace Theorem (see section 6.4 in [9]), we have that

$$\lambda_1(\mathcal{H}) \geq \mu_1, \quad \lambda_2(\mathcal{H}) \geq \mu_2, \ldots, \lambda_t(\mathcal{H}) \geq \mu_t.$$  

$$|\lambda_{n-1}(\mathcal{H})| \geq |\mu_{n-1}|, \quad |\lambda_{n-2}(\mathcal{H})| \geq |\mu_{n-2}|, \ldots, |\lambda_{t+2}(\mathcal{H})| \geq |\mu_{t+1}|.$$  

Therefore, $E(\mathcal{H}) \geq E(\mathcal{H} - v)$. \hfill $\square$

Now, we will bound the variation in energy when we delete an edge in the hypergraph.

The energy of a matrix $M$ is a generalization of the energy of graphs, it is computed as the sum of its singular values. In particular, if $M$ is a real and symmetric square matrix, its energy is also the sum of the absolute values of the eigenvalues.

**Lemma 11** (Lemma 2.21, [21]). If $M$ and $N$ are square matrices, then $E(M + N) \leq E(M) + E(N)$, $|E(M) - E(N)| \leq E(M - N)$.

**Definition 5.2.** Let $\mathcal{H} = (V, E)$ be a hypergraph and $e \in E$ be an edge, we define $\mathcal{H} - e = (V, E - \{e\})$.

**Theorem 12.** Let $\mathcal{H}$ be a hypergraph and $e \in E(\mathcal{H})$ be an edge, then $|E(\mathcal{H}) - E(\mathcal{H} - e)| \leq 2|e| - 2$.

**Proof.** First, we observe that $|E(\mathcal{H}) - E(\mathcal{H} - e)| \leq E(A(\mathcal{H}) - A(\mathcal{H} - e))$.

The inequality above follows from Lemma 11. Now, we observe that

$$M := A(\mathcal{H}) - A(\mathcal{H} - e) = \begin{bmatrix} 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$  

That is, the eigenvalues of $M$, are $|e| - 1$ with multiplicity 1 and $-1$ with multiplicity $|e| - 1$, thus the energy of this matrix is $E(A(\mathcal{H}) - A(\mathcal{H} - e)) = |e| - 1 + (|e| - 1) - 1 = 2|e| - 2$. Therefore, the result follows. \hfill $\square$
Example 5.3. Removing a vertex always implies that \( \mathcal{E}(\mathcal{H}) \geq \mathcal{E}(\mathcal{H} - v) \). For edge deletion, this is no longer true as we see bellow.

1. \( \mathcal{E}(\mathcal{H}) > \mathcal{E}(\mathcal{H} - e) \) removing any edge \( e \in E \):
   - Let \( \mathcal{K}_5^{[3]} \) be the complete 3-uniform hypergraph with 5 vertices and let \( e \in E. \)
   - Notice that \( \mathcal{E}(\mathcal{K}_5^{[3]}) = 24 \), and \( \mathcal{E}(\mathcal{K}_5^{[3]} - e) = 21.731. \)

2. \( \mathcal{E}(\mathcal{H}) < \mathcal{E}(\mathcal{H} - e) \) removing any edge \( e \in E \):
   - Let \( \mathcal{H} = (V, E) \) be the hypergraph given by \( V(\mathcal{H}) = \{1, 2, 3, 4, 5, 6\} \) and \( E = \{135, 136, 145, 146, 235, 236, 245, 246\}. \) We have that \( \mathcal{E}(\mathcal{H}) = 16. \) For \( e \in E, \) we have \( \mathcal{E}(\mathcal{H} - e) = 16, 4926. \)

3. \( \mathcal{E}(\mathcal{H}) = \mathcal{E}(\mathcal{H} - e) \) removing removing a particular edge \( e \):
   - Let \( \mathcal{H} = (V, E) \) be the hypergraph given by two copies of \( \mathcal{K}_5^{[3]} \) with 3 distinct edges of cardinality 2 with no common vertices connecting the two copies. Taking \( e \in E \) as one of those edges, we have that \( \mathcal{E}(\mathcal{H}) = \mathcal{E}(\mathcal{H} - e). \)

It is known that for graphs the same is true, that is, the energy of the graph when removing an edge, can increase, decrease or remain the same, [15].

6. Edge division

Some important concepts of graph theory can be generalized in more than one way. In particular, removing edges of a graph can be generalized, in the context of hypergraphs, as the deletion of edges (seen in the previous section) or as the division of edges, that we will define in this section.

Definition 6.1. Let \( \mathcal{H} = (V, E) \) be a hypergraph and \( F = \{e_1, \ldots, e_t\} \subseteq E \) a subset of edges. If we can divide each edge \( e_i = e'_i \cup e''_i \) so that \( |e'_i| > 0, |e''_i| > 0 \) and \( e'_i \cap e''_i = \emptyset \), we define the following multi-hypergraph
\[
\mathcal{H} \triangleleft (F, F', F'') \ := \ (V, (E \cup F' \cup F'') - F),
\]
where \( F' = \{e'_1, \ldots, e'_t\} \) and \( F'' = \{e''_1, \ldots, e''_t\}. \) Under these conditions, we say that the multi-hypergraph \( \mathcal{H} \triangleleft (F, F', F'') \) is obtained from \( \mathcal{H} \) by dividing the edges in \( F \). When is not necessary to explicitly say how the edges in \( F \) are divided, we will denote simply by \( \mathcal{H} \triangleleft F \). If \( F \) is an unitary set, \( F = \{e\} \), we denote by \( \mathcal{H} \triangleleft e \).

Example 6.2. Consider the hypergraph \( \mathcal{H} \) with the following vertices set \( V(\mathcal{H}) = \{0, \ldots, 9\} \) and edges \( E(\mathcal{H}) = \{012, 234567, 789\} \), in the Figure 2 we represent the hypergraph \( \mathcal{H} \triangleleft (\{234567\}, \{2345\}, \{67\}). \)

![Figure 2. Edge division](image)

Remark 6.3. Let \( J \) be the matrix with all entries equal to 1 and consider the \( n \times n \) block matrix given by \( M = \begin{bmatrix} 0_p & J_{p \times q} \\ J_{q \times p} & 0_q \end{bmatrix} \), where \( p + q = n. \) \( M \) has only two non zero eigenvalues which are given by \( \sqrt{pq} \) and \( -\sqrt{pq}. \)
Theorem 13. Let $\mathcal{H}$ be a hypergraph, $e \in E(\mathcal{H})$ an edge and $e = e' \cup e''$ a division of this edge. Then

$$\left| E(\mathcal{H}) - E(\mathcal{H} \setminus e) \right| \leq 2\sqrt{|e'||e''|}.$$ 

Proof. First, notice that

$$\left| E(\mathcal{H}) - E(\mathcal{H} \setminus e) \right| \leq E(A(\mathcal{H}) - A(\mathcal{H} \setminus e)).$$

We can assume that $e = \{1, 2, \ldots, r\}$, where $r = |e'| + |e''|$. Now, also notice that

$$A(\mathcal{H}) - A(\mathcal{H} \setminus e) = \begin{bmatrix} 0_{|e'|} & J_{|e'| \times |e''|} & 0 \\ J_{|e''| \times |e'|} & 0_{|e'|} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

From Remark 6.3, we have that $E(A(\mathcal{H}) - A(\mathcal{H} \setminus e)) = 2\sqrt{|e'||e''|}$. \hfill $\square$

Lemma 14. Let $\mathcal{H}$ be a hypergraph and $e \in E(\mathcal{H})$ an isolated edge (that is, the intersection of $e$ with any other edge is empty), then $|E(\mathcal{H}) - E(\mathcal{H} \setminus e)| = 2$.

Proof. We can assume $\mathcal{H} = \mathcal{H}' \cup e$, where $e = \{1, 2, \ldots, r\}$ and $r = |e'| + |e''|$. We have that

$$A(\mathcal{H}) = \begin{bmatrix} (J - I)_{|e| \times |e|} & 0 \\ 0 & A(\mathcal{H}') \end{bmatrix}$$ and $$A(\mathcal{H} \setminus e) = \begin{bmatrix} (J - I)_{|e'| \times |e'|} & 0 \\ 0 & (J - I)_{|e''| \times |e''|} & 0 \\ 0 & 0 & A(\mathcal{H}') \end{bmatrix}.$$ 

Therefore

$$E(\mathcal{H}) = E(\mathcal{H}') + 2|e| - 2 \quad \text{and} \quad E(\mathcal{H} \setminus e) = E(\mathcal{H}') + 2|e'| - 2 + 2|e''| - 2.$$ 

So

$$E(\mathcal{H}) - E(\mathcal{H} \setminus e) = 2|e| - 2 - 2(|e'| + |e''|) + 4 = 2.$$ \hfill $\square$

Remark 6.4. Let $\mathcal{H}$ be a hypergraph and $e \in E(\mathcal{H})$ an isolated edge, then the equality $|E(\mathcal{H}) - E(\mathcal{H} \setminus e)| = 2\sqrt{|e'||e''|}$ holds if, and only if, $|e| = 2$.

Definition 6.5. Let $\mathcal{H} = (V, E)$ be a connected hypergraph and $F \subset E$ a subset of edges. The ordered triple $(F, F', F'')$ is said to be a weak cut, if $\mathcal{H} \setminus (F, F', F'')$ is disconnected, and for no proper subset $P \subset F$, the multi-hypergraph $\mathcal{H} \setminus P$ is disconnected, considering any way to divide the edges of $P$.

For our next result we need the following theorem from [15].

Theorem 15. [15] For a real and symmetric partitioned matrix $C = \begin{bmatrix} M & X \\ Y & B \end{bmatrix}$ where both $M$ and $B$ are square matrices, we have

$$\sum_j |\lambda_j(M)| + \sum_j |\lambda_j(B)| \leq \sum_j |\lambda_j(C)|.$$ 

Moreover, equality holds if and only if there exist unitary matrices $U$ and $V$ such that

$$\begin{bmatrix} UM & UX \\ VY & VB \end{bmatrix}$$ is positive semi-definite.
Theorem 16. Let $\mathcal{H} = (V, E)$ be a connected hypergraph. If $F \subset E$ is a weak cut, then $\mathcal{E}(\mathcal{H} \triangle F) \leq \mathcal{E}(\mathcal{H})$.

Proof. Notice that $\mathcal{H} \triangle F$ is the disjoint union of two multi-hypergraphs, let say $\mathcal{H} \triangle F = \mathcal{H}_1 \cup \mathcal{H}_2$. So we have that

$$A(\mathcal{H} \triangle F) = \begin{bmatrix} A(\mathcal{H}_1) & 0 \\ 0 & A(\mathcal{H}_2) \end{bmatrix}$$

and

$$A(\mathcal{H}) = \begin{bmatrix} A(\mathcal{H}_1) & X \\ XT & A(\mathcal{H}_2) \end{bmatrix},$$

where $X = (x_{ij})$ is a matrix of order $|V(\mathcal{H}_1)| \times |V(\mathcal{H}_2)|$ and $x_{ij}$ is the number of edges in $F$ that contain the vertices $i \in V(\mathcal{H}_1)$ and $j \in V(\mathcal{H}_2)$.

Therefore from Theorem 15, we have that $\mathcal{E}(\mathcal{H}) \geq \mathcal{E}(\mathcal{H}_1) + \mathcal{E}(\mathcal{H}_2) = \mathcal{E}(\mathcal{H} \triangle F)$. \qed

Lemma 17 (Exercise 2 of Section 7.1, [13]). A positive semidefinite matrix has a zero entry on its main diagonal if and only if the entire row and column to which that entry belongs is zero.

Theorem 18. Let $\mathcal{H} = (V, E)$ be a connected hypergraph, $e \in E$ an edge and $v \in e$ a vertex. If $(e, e - \{v\}, \{v\})$ is a weak cut, then $\mathcal{E}(\mathcal{H} \triangle (e, e - \{v\}, \{v\})) < \mathcal{E}(\mathcal{H})$.

Proof. Notice that $\mathcal{H} \triangle (e, e - \{v\}, \{v\}) = \mathcal{H}_1 \cup \mathcal{H}_2$. So we have that

$$A(\mathcal{H} \triangle (e, e - \{v\}, \{v\})) = \begin{bmatrix} A(\mathcal{H}_1) & 0 \\ 0 & A(\mathcal{H}_2) \end{bmatrix}$$

and

$$A(\mathcal{H}) = \begin{bmatrix} A(\mathcal{H}_1) & X \\ XT & A(\mathcal{H}_2) \end{bmatrix},$$

with $x_{ij} = 1$ if $i \in e - \{v\}$ and $j = v$, and with $x_{ij} = 0$ otherwise. We can label the vertices in such a way that the first row of $X$ is the only non-null row.

Supposing that $\mathcal{E}(\mathcal{H} \triangle (e, e - \{v\}, \{v\})) = \mathcal{E}(\mathcal{H})$, from Theorem 15 we know that exist orthogonal matrices $U$ and $W$, such that

$$\begin{bmatrix} UA(\mathcal{H}_1) \quad UX \\ WX^T \quad WA(\mathcal{H}_1) \end{bmatrix}$$

is a positive semidefinite matrix. So we have that $(UX)^T = WX^T$, because of the structure of $X$ we have that $W = \begin{bmatrix} \alpha & 0 \\ 0 & W_1 \end{bmatrix}$, with $|\alpha| = 1$ and $W_1$ is an orthogonal matrix. So, we can write $A(\mathcal{H}_1) = \begin{bmatrix} 0 & y^T \\ y & A_1 \end{bmatrix}$, therefore $WA(\mathcal{H}_1) = \begin{bmatrix} 0 & \alpha y^T \\ W_1 y & W_1 A_1 \end{bmatrix}$.

From Lemma 17, we have that $y = 0$, but this implies that $v$ has no neighbor in $\mathcal{H}_2$, what contradicts the fact that $\mathcal{H}$ is connected. \qed

Definition 6.6. A hypergraph $\mathcal{T}$ is a hypertree if $\mathcal{T}$ is connected and has no cycles.

For a hypertree $\mathcal{T}$ we notice that:

1. $\mathcal{T}$ is a linear hypergraph, since if there exist two edges $e_1, e_2$ that have two (or more) common vertices $u \in e_1$, then we can construct the cycle $ue_1ve_2u$.

2. For any edge $e$ of a hypertree, if $v \in e$, then $(e, e - \{v\}, \{v\})$ is a weak cut.

Corollary 19. Let $\mathcal{T}$ be a hypertree. If $e \in E$ is an edge and $v \in e$ is a vertex, then $\mathcal{E}(\mathcal{T} \triangle (e, e - \{v\}, \{v\})) < \mathcal{E}(\mathcal{T})$.

Remark 6.7. Let $\mathcal{H} = (V, E)$ be a connected hypergraph and $e \in E$ an edge. If $v \in e$ has degree 1, then $(e, e - \{v\}, v)$ is a weak cut.

Corollary 20. Let $\mathcal{H} = (V, E)$ be a connected hypergraph and $e \in E$ an edge. If $v \in e$ has degree 1, then $\mathcal{E}(\mathcal{H} \triangle (e, e - \{v\}, \{v\})) < \mathcal{E}(\mathcal{H})$. 

Example 6.8. As it happens in $\mathcal{H} - e$, when we divide edges we can also have $\mathcal{E}(\mathcal{H}) > \mathcal{E}(\mathcal{H} \triangle e)$ and $\mathcal{E}(\mathcal{H}) < \mathcal{E}(\mathcal{H} \triangle e)$.

1) $\mathcal{E}(\mathcal{H}) > \mathcal{E}(\mathcal{H} \triangle e)$, by doing any division at any edge:
   Let $K_5^{[4]}$ be the complete 4-uniform hypergraph with 5 vertices and observe that $\mathcal{E}(K_5^{[4]}) = 24$. Let $e \in E$. Without loss of generality, we can assume that $e = \{1, 2, 3, 4\}$. We can divide $e$ in two different ways.

Case 1: If $e' = \{1, 2\} e'' = \{3, 4\}$, then $\mathcal{E}(K_5^{[4]} \triangle e) = 20, 8924$.

Case 2: If $e' = \{1\} e'' = \{2, 3, 4\}$, then $\mathcal{E}(K_5^{[4]} \triangle e) = 21, 7438$.

2) $\mathcal{E}(\mathcal{H}) < \mathcal{E}(\mathcal{H} \triangle e)$, by doing any division at any edge:
   
   Let $\mathcal{H} = (V, E)$ be the hypergraph give by $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and
   
   $E = \{1357, 1358, 1367, 1368, 1457, 1458, 1467, 1468, 2357, 2358, 2367, 2368, 2457, 2458, 2467, 2468\}$.

   We have that $\mathcal{E}(\mathcal{H}) = 48$. Now, let $e \in E$ be any edge. Without loss of generality, we can assume that $e = \{1, 3, 5, 7\}$. We can divide $e$ in two different ways.

Case 1: If $e' = \{1, 3\} e'' = \{5, 7\}$, then $\mathcal{E}(\mathcal{H}) = 48.3294$.

Case 2: If $e' = \{1\} e'' = \{3, 5, 7\}$, then $\mathcal{E}(\mathcal{H}) = 48.4723$.

3) $\mathcal{E}(\mathcal{H}) = \mathcal{E}(\mathcal{H} \triangle e)$ by doing a particular division at a particular edge:
   
   Example 5.3 (3) also works on this case. Notice that the edge $e$ has cardinality 2 and in this case we have that $A(\mathcal{H} \triangle e) = A(\mathcal{H} - e)$.

7. Sharp bounds for energy

In this section we will obtain bounds for the adjacency energy of a hypergraph. These bounds will be computed as functions of important and well known spectral and structural parameters.

Definition 7.1. Let $\mathcal{H}$ be a hypergraph. Its Zagreb index is defined as the sum of the squares of the degrees of its vertices. More precisely

$$Z(\mathcal{H}) = \sum_{v \in V(\mathcal{H})} d(v)^2.$$ 

This is an important parameter in graph theory, having chemistry applications, see the survey [11].

We will start by presenting some upper bounds for the energy of a hypergraph

Theorem 21. Let $\mathcal{H} = (V, E)$ be a hypergraph with rank $r$. If $|E| = m$ and $|V| = n$, then

$$\mathcal{E}(\mathcal{H}) \leq \sqrt{n(r-1)Z(\mathcal{H})} \leq \sqrt{nm(r^2 - r)\Delta}.$$ 

Equality holds if, and only if, $\mathcal{H}$ is a 2-graph with isolated edges and no isolated vertices or if $\mathcal{H}$ is an edgeless hypergraph.

Proof. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $A(\mathcal{H})$. Then,

$$\sum_{i=1}^{n} \lambda_i^2 = \text{Tr}(A^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} d(i, j)^2 \leq \sum_{i=1}^{n} \left( d(i) \sum_{j=1}^{n} d(i, j) \right)^{(*)} \leq (r - 1) \sum_{i=1}^{n} d(i)^2.$$
Notice that the equality (*) holds only on hypergraphs with the following property: If two vertices \( u \) and \( v \) are neighbors then they belong to the same edges. But this only happens on hypergraphs made of isolated edges (and possibly some isolated vertices). Also notice that the equality (**) holds only on uniform hypergraphs. Then,

\[
\sum_{i=1}^{n} \lambda_i^2 \leq (r-1)Z(H).
\]

Therefore,

\[
E(H)^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \leq n \left( \sum_{i=1}^{n} \lambda_i^2 \right) \leq n(r-1)Z(H).
\]

By Cauchy Schwarz inequality, we know that the equality (cs) only is true when all the eigenvalues have the same absolute values. For uniform hypergraphs made of isolated edges, this only happens when there is no isolated vertex and the edges have size 2.

Since,

\[
Z(H) = \sum_{i=1}^{n} d(i)^2 \leq \sum_{i=1}^{n} \Delta d(i) \leq \Delta rm
\]

And this equality occurs only on regulars hypergraphs, we conclude that

\[
E(H) \leq \sqrt{n(r-1)Z(H)} \leq \sqrt{nm(r^2 - r)}\Delta
\]

With all the equalities happening only if \( H \) is a 2-graph with isolated edges and no isolated vertices or if \( H \) has no edges.

\[ \square \]

**Theorem 22.** Let \( H = (V, E) \) be a hypergraph with rank \( r \) and \( |V| = n \), then

\[
E(H) \leq \lambda_1 + \sqrt{(n-1)[(r-1)Z(H) - \lambda_1^2]}.
\]

Equality holds if, and only if, \( H \) is a 2-graph with isolated edges and no isolated vertices or if \( H \) is an edgeless hypergraph.

**Proof.** From Theorem 21 we have that

\[
\lambda_1^2 + \sum_{i=2}^{n} \lambda_i^2 = \sum_{i=1}^{n} \lambda_i^2 \leq (r-1)Z(H).
\]

With equality holding only on uniform hypergraphs made of isolated edges and possibly some isolated vertices.

Therefore,

\[
(\mathcal{E}(H) - \lambda_1)^2 = \left( \sum_{i=2}^{n} |\lambda_i| \right)^2 \leq (n-1) \sum_{i=2}^{n} \lambda_i^2 \leq (n-1)[(r-1)Z(H) - \lambda_1^2].
\]

Notice that the equality (cs) only is true when \( |\lambda_2| = |\lambda_3| = \ldots = |\lambda_n| \). For uniform hypergraphs made of isolated edges, this can only occur when all the edges have size 2 and there is no isolated vertices.

\[ \square \]

**Lemma 23.** Let \( H \) be a hypergraph with rank \( r \) and co-rank \( s \), then

\[
(s-1)\delta \leq \lambda_1 \leq (r-1)\Delta.
\]
Proof. Let $x = (x_i)$ be the non negative eigenvector associated to the eigenvalue $\lambda_1$. Taking vertices $v, u \in V$ such that $x_v$ has maximum value and $x_u$ has minimum value, we have that

$$\lambda_1 x_v = (Ax)_v = \sum_{e \in E[v]} x(e - \{v\})$$

therefore

$$\lambda_1 = \sum_{e \in E[v]} \frac{x(e - \{v\})}{x_v} \leq \Delta \frac{(r-1)x_v}{x_v} = (r-1)\Delta.$$

The other inequality follows similarly, using $x_u$. \qed

Corollary 24. Let $\mathcal{H} = (V, E)$ be a $r$-uniform and $d$-regular hypergraph, then

$$\mathcal{E}(\mathcal{H}) \leq (r-1)d + \sqrt{(n-1)[(r-1)nd^2 - (r-1)^2d^2]}.$$

Now we will obtain some lower bounds for the adjacency energy.

Lemma 25. Let $\mathcal{H} = (V, E)$ be a hypergraph with $|V| = n$, then

$$\mathcal{E}(\mathcal{H})^2 \geq 2 \sum_{i=1}^{n} \lambda_i^2.$$

Equality holds only if $\mathcal{H}$ has at most one $2$-uniform complete bipartite connected component and possibly isolated vertices, or if $\mathcal{H}$ has no edges.

Proof. First, notice that

$$0 = \left( \sum_{i=1}^{n} \lambda_i \right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j.$$

Therefore,

$$\sum_{i=1}^{n} \lambda_i^2 = -2 \sum_{i<j} \lambda_i \lambda_j, \quad \Rightarrow \quad \sum_{i<j} |\lambda_i \lambda_j| \stackrel{(1)}{=} \left| \sum_{i<j} \lambda_i \lambda_j \right| = \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2.$$

Then,

$$\mathcal{E}(\mathcal{H})^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 = \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{i<j} |\lambda_i \lambda_j| \stackrel{(2)}{=} 2 \sum_{i=1}^{n} \lambda_i^2.$$

Notice that equality (1) and consequently (2) occur only if every non zero eigenvalue have the same signal. If $\mathcal{H}$ has at least three non zero eigenvalues (not necessarily distincts) that is impossible to happen. Therefore $\mathcal{H}$ must have at most one $2$-uniform complete bipartite connected component and possibly isolated vertices. \qed

In what follows, we present three lower bounds for the energy of a hypergraph, based on different parameters.

Theorem 26. Let $\mathcal{H} = (V, E)$ be a hypergraph with co-rank $s$, $|V| = n$ and average degree $d(\mathcal{H})$ then

$$\mathcal{E}(\mathcal{H}) \geq \sqrt{2n(s-1)d(\mathcal{H})}.$$

Equality holds if, and only if, $\mathcal{H}$ is the $2$-uniform hypergraph with at most one complete bipartite connected component.
Proof. It suffices to note that
\[
\mathcal{E}(\mathcal{H})^2 \geq 2 \sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} d(i, j)^2 \geq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} d(i, j) \geq 2n(s - 1)d(\mathcal{H}).
\]

\[\square\]

Theorem 27. Let \(\mathcal{H} = (V, E)\) be a hypergraph with co-rank \(s\) and \(|V| = n\), then
\[
\mathcal{E}(\mathcal{H}) \geq \sqrt{\frac{2(s - 1)^2}{n} Z(\mathcal{H})}.
\]
Equality holds if, and only if, \(\mathcal{H}\) is the 2-uniform hypergraph with only one edge and no isolated vertices or if \(\mathcal{H}\) has no edges.

Proof. It suffices to note that
\[
\mathcal{E}(\mathcal{H})^2 \geq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} d(i, j)^2 \geq 2 \sum_{i=1}^{n} \left( \frac{1}{n} \left( \sum_{j=1}^{n} d(i, j) \right) \right)^2 \geq \frac{2(s - 1)^2}{n} Z(\mathcal{H}).
\]

Note that equality (4) holds only if \(d(i, j)\) is constant for each \(i\). The only complete bipartite graph with this property is the edge. \(\square\)

Remark 7.2. Let \(\mathcal{H}\) be a hypergraph with \(n\) vertices and co-rank \(s\). We can compare the bounds provided by these two theorems. We define \(b(\mathcal{H}) = \sqrt{2n(s - 1)d(\mathcal{H})}\) and \(B(\mathcal{H}) = \sqrt{\frac{2(s - 1)^2}{n} Z(\mathcal{H})}\). It is easy to see that:

1. If \(\frac{\Delta^2}{\delta^2} \geq \frac{n}{s - 1}\), then \(b(\mathcal{H}) \leq \sqrt{2n(s - 1)\Delta} \leq \sqrt{2(s - 1)^2\delta^2} \leq B(\mathcal{H})\).
2. If \(\frac{\Delta^2}{\delta^2} \leq \frac{n}{s - 1}\), then \(b(\mathcal{H}) \leq \sqrt{2(s - 1)^2\Delta^2} \leq \sqrt{2n(s - 1)\delta} \leq B(\mathcal{H})\).
3. In particular, if \(\mathcal{H}\) is \(d\)-regular, then:
   (a) If \(d = \frac{n}{s - 1}\), then \(b(\mathcal{H}) = B(\mathcal{H})\).
   (b) If \(d > \frac{n}{s - 1}\), then \(b(\mathcal{H}) < B(\mathcal{H})\).
   (c) If \(d < \frac{n}{s - 1}\), then \(b(\mathcal{H}) > B(\mathcal{H})\).

Theorem 28. Let \(\mathcal{H}\) be a hypergraph, then
\[
\mathcal{E}(\mathcal{H}) \geq \sqrt{\frac{2n}{n - 1} \lambda_1^2}.
\]
Equality holds if, and only if, \(\mathcal{H}\) is the 2-uniform hypergraph with one edge and two vertices or if \(\mathcal{H}\) has no edges.

Proof. Notice that
\[
\sum_{i=1}^{n} \lambda_i^2 = \lambda_1^2 + \sum_{i=2}^{n} \lambda_i^2 \geq \lambda_1^2 + \frac{1}{n - 1} \left( \sum_{i=2}^{n} \lambda_i \right)^2 = \lambda_1^2 + \frac{(-\lambda_1)^2}{n - 1} = \frac{n}{n - 1} \lambda_1^2.
\]
Therefore,
\[
\mathcal{E}(\mathcal{H})^2 \geq 2 \sum_{i=1}^{n} \lambda_i^2 \geq \frac{2n}{n - 1} \lambda_1^2.
\]
Equality \((cs)\) only is true when \(|\lambda_2| = |\lambda_3| = \ldots = |\lambda_n|\). For complete bipartite graphs, this only holds when \(\mathcal{H} = K_2\). \(\square\)

Note that the proofs of the previous theorems leads immediately to distinct low bounds of \(\sum_{i=1}^{n} \lambda_i^2\). More specifically, we have the following lemma, which will be useful later.
Lemma 29. Let $\mathcal{H}$ be a hypergraph with co-rank $s$ and $|V| = n$.

1. $\sum_{i=1}^{n} \lambda_i^2 \geq n(s - 1)d(\mathcal{H})$.
2. $\sum_{i=1}^{n} \lambda_i^2 \geq \frac{(s-1)^2}{n} Z(\mathcal{H})$.
3. $\sum_{i=1}^{n} \lambda_i^2 \geq \frac{n}{n-1} \lambda_1^2$.

Lemma 30. Let $\mathcal{H} = (V, E)$ be a hypergraph with $|V| = n$, then

$$E(\mathcal{H})^2 \geq \sum_{i=1}^{n} \lambda_i^2 + n(n-1)\det(A)^{\frac{2}{n}}.$$ 

Equality holds if $|\lambda_i| = |\lambda_j|$ for every $i \neq j$, that is, if $\mathcal{H}$ is the graph with one edge and two vertices.

Proof. First, notice that

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i \lambda_j| \geq \prod_{i \neq j} |\lambda_i \lambda_j|^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} |\lambda_i|^\frac{2}{n} = |\det(A)|^{\frac{2}{n}}$$

Therefore,

$$E(\mathcal{H})^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i \neq j} |\lambda_i \lambda_j| \geq \sum_{i=1}^{n} \lambda_i^2 + n(n-1)\det(A)^{\frac{2}{n}}.$$ 

Where (1) holds because the arithmetic mean of non negative numbers is greater or equal than the geometric mean. That is:

$$\frac{x_1 + \ldots + x_n}{n} \geq (x_1 \ldots x_n)^\frac{1}{n}$$

and equality holds if, and only if, $x_1 = \ldots = x_n$. \qed

From these lemmas, we establish other lower bounds for the energy of a hypergraph, depending on the determinant of its adjacency matrix.

Theorem 31. Let $\mathcal{H} = (V, E)$ be a hypergraph with co-rank $s$ and $|V| = n$, then

$$E(\mathcal{H}) \geq \sqrt{n(s - 1)d(\mathcal{H}) + n(n-1)\det(A)^{\frac{2}{n}}}.$$ 

Proof. Use Lemma 30 and Remark 29 (1). \qed

Theorem 32. Let $\mathcal{H} = (V, E)$ be a hypergraph with co-rank $s$ and $|V| = n$, then

$$E(\mathcal{H}) \geq \sqrt{\frac{(s-1)^2}{n} Z(\mathcal{H}) + n(n-1)\det(A)^{\frac{2}{n}}}.$$ 

Proof. Use Lemma 30 and Remark 29 (2). \qed

Theorem 33. Let $\mathcal{H} = (V, E)$ be a hypergraph, then

$$E(\mathcal{H}) \geq \sqrt{\frac{n}{n-1} \lambda_1^2 + n(n-1)\det(A)^{\frac{2}{n}}}.$$ 

Proof. Use Lemma 30 and Remark 29 (3). \qed

Note that the bounds obtained in Theorems 31 and 32 can also be compared, following the same cases listed in Remark 7.2.

Finally, the next remark allow to compare the bounds of the last three theorems with the bounds of the three preceding theorems.
Remark 7.3. Regarding the bounds of Lemmas 25 and 30, we have four cases:

1. If $\mathcal{H}$ is a hypergraph that has zero as an eigenvalue, we have that the bound of Lemma 25 is sharper than the bound of Lemma 30 (consequently the bounds from Theorems 26, 27 and 28 are better than the respective bounds from Theorems 31, 32 and 33).

2. If $\mathcal{G}$ is a graph with $|\det(A(\mathcal{G}))| \geq 1$ then the bound from Lemma 30 is better than the bound from Lemma 25.

   Indeed, we can assume that $\mathcal{G}$ has no isolated vertices. Saying that 30 is better than Lemma 25 means that
   \[ E(\mathcal{H})^2 \geq \sum_{i=1}^{n} \lambda_i^2 + n(n-1)|\det(A)|^{\frac{2}{n}} \geq 2 \sum_{i=1}^{n} \lambda_i^2. \]

   And this is true if, and only if,
   \[ n(n-1)|\det(A)|^{\frac{2}{n}} \geq \sum_{i=1}^{n} \lambda_i^2. \]

   and in fact
   \[ n(n-1)|\det(A)|^{\frac{2}{n}} \geq n(n-1) = 2 \frac{n(n-1)}{2} \geq 2m = \sum_{i=1}^{n} \lambda_i^2. \]

3. We have that (2) does not hold for hypergraphs in general. For example: Let $\mathcal{H} = (V, E)$ be a hypergraph with vertices $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and edges $E = \{1234, 1237, 1238, 1256, 126, 1278, 3456, 3478, 5678\}$,

   we have $|\det(A(\mathcal{H}))| = 252$ and Lemma 25 is sharper than Lemma 30, because
   \[ \sum_{i=1}^{n} \lambda_i^2 > 253 > 224 > n(n-1)|\det(A(\mathcal{H}))|^{\frac{2}{n}} \]

4. Let $\mathcal{H}$ be a hypergraph with rank $r$ and $|\det(A(\mathcal{H}))| \geq 1$. If $n \geq (r-1)\Delta^2 + 1$, the the bound of Lemma 30 is better than Lemma 25.

   Indeed, notice that
   \[ \sum_{i=1}^{n} \lambda_i^2 \leq (r-1)n\Delta^2 \]

   Therefore
   \[ n(n-1)|\det(A)|^{\frac{2}{n}} \geq n(n-1) \geq n(r-1)\Delta^2 \geq \sum_{i=1}^{n} \lambda_i^2. \]

8. Conclusion

In this paper, we made contributions to spectral hypergraph theory. More precisely, we define the energy of a hypergraph as the energy of its adjacency matrix, and study its properties. We obtain which hypergraphs have the highest and lowest energy within the class of hyperstars. By obtaining spectral properties of operations sum and product of hypergraphs, we prove that the energy of a hypergraph is never an odd number. We study how the hypergraph energy varies when we remove a vertex or an edge from it, and we further define an edge division operation and analyze how an edge division impacts the energy of a hypergraph. Our main results are the determination of bounds
for the energy. These bounds are functions of well known parameters, such as maximum and minimum degree, Zagreb index and spectral radius.

We end this paper presenting some open problems about energies of hypergraphs.

(1) In [5], the authors define the signless Laplacian matrix for general hypergraphs, so we believe that many results proven in [4] can be generalized to non-uniform hypergraphs using some of the techniques that we used here.

(2) We have determined which hyperstar with \( t \) vertices have highest and lowest energy. An interesting problem is to determine in other classes, which is the hypergraph with the highest and lowest energy.

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REFERENCES

[1] Banerjee, A. On the spectrum of hypergraphs. *Linear Algebra and its Applications* (2020), doi.org/10.1016/j.laa.2020.01.012.
[2] Bapat, R., and Pati, S. Energy of a graph is never an odd integer. *Bull. Kerala Math. Assoc.* 1 (2004), 129–132.
[3] Cardoso, K., Hoppen, C., and Trevisan, V. The spectrum of a class of uniform hypergraphs. *Linear Algebra and its Applications* 590 (2020), 243–257.
[4] Cardoso, K., and Trevisan, V. Energies of hypergraphs. *arXiv:1912.03224* (2019).
[5] Cardoso, K., and Trevisan, V. The signless laplacian matrix of hypergraphs. *arXiv:1909.00246* (2019).
[6] Cooper, J., and Dutle, A. Spectra of uniform hypergraphs. *Linear Algebra Appl.* 436 (2012), 3268–3292.
[7] Duttweiler, L., and Reff, N. Spectra of cycle and path families of oriented hypergraphs. *Linear Algebra and its Applications* 578 (2019), 251–271.
[8] Feng, K., Ching, W., and Li, W. Spectra of hypergraphs and applications. *Journal of number theory* 60 (1996), 1–22.
[9] Franklin, J. *Matrix theory*. Dover Publications, 2000.
[10] Gutman, I. The energy of a graph. *Ber. Math.-Statist. Sekt. Forschungsz. Graz* 103 (1978), 1–22.
[11] Gutman, I., and Das, C. The first zagreb index 30 years after. *MATCH Communications in Mathematical and in Computer Chemistry* 50 (2004), 83–92.
[12] Hillar, C., and Lim, L. Most tensor problems are np-hard. *Journal of the ACM* 60 (2013), 1–39.
[13] Horn, R., and Johnson, C. *Matrix Analysis*. Cambridge, 2013.
[14] Hückel, E. Quantentheoretische beitrage zum benzolproblem. *Z. Phys. 70* (1931), 204–286.
[15] Jane Day, W. S. Graph energy change due to edge deletion. *Linear Algebra and its Applications* 428 (2008), 2070–2078.
[16] Kumar, K., and Varghese, R. Spectrum of (k,r)-regular hypergraphs. *International Journal of Mathematical Combinatorics* 2 (2017), 52–59.
[17] Li, X., Shi, Y., and Gutman, I. *Graph Energy*. Springer, 2012.
[18] Lin, H., and Zhou, B. Spectral radius of uniform hypergraphs. *Linear Algebra and its Applications* 527 (2017), 32–52.
[19] Lu, H., Xue, N., and Zhu, Z. On the signless laplacian estrada index of uniform hypergraphs. *International Journal of Quantum Chemistry n/a, n/a, e26579*.

[20] Nikiforov, V. The energy of graphs and matrices. *Journal of Mathematical Analysis and Applications* 326 (2007), 1472–1475.

[21] Pinheiro, L. *Energia laplaciana sem sinal de grafos*. Doctoral thesis - Universidade Federal do Rio Grande do Sul (UFRGS), 2018.

[22] Pirzada, S., and Gutman, I. Energy of a graph is never the square root of an odd integer. *Applicable Analysis and Discrete Mathematics* 2 (2008), 118–121.

[23] Reff, N. Spectral properties of oriented hypergraphs. *Electronic Journal of Linear Algebra* 27 (2014), 373–391.

[24] Rodriguez, J. On the laplacian eigenvalues and metric parameters of hypergraphs. *Linear and Multilinear Algebra* 50 (2002), 1–14.

[25] Wang, Y., and Zhou, B. On distance spectral radius of hypergraphs. *Linear and Multilinear Algebra* 66 (2018), 2232–2246.

[26] Zhu, Q. Extremal k-uniform hypertrees on incidence energy. *International Journal of Quantum Chemistry n/a, n/a, e26592*.

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