Monogamy of the entanglement of formation

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We show that any measure of entanglement that on pure bipartite states is given by a strictly concave function of the reduced density matrix is monogamous on pure tripartite states. This includes the important class of bipartite measures of entanglement that reduce to the (von Neumann) entropy of entanglement. Moreover, we show that the convex roof extension of such measures (e.g., entanglement of formation) are monogamous also on mixed tripartite states. To prove our results, we use the definition of monogamy without inequalities, recently put forward [Gour and Guo, Quantum 2, 81 (2018)]. Our results promote the concept that monogamy of entanglement is a property of quantum entanglement and not an attribute of some particular measures of entanglement.

Quantum entanglement is one of the most counter-intuitive phenomena of quantum theory. In the early days of quantum mechanics it was recognized as "the characteristic trait of quantum mechanics the one that enforces its entire departure from classical lines of thought." \cite{1}. On the other hand, in recent years it was identified as the key resource of many quantum information processing tasks \cite{2,4}. One of its key features is that it cannot be freely shared among many parties, unlike classical correlations. This is the so-called monogamy law \cite{5} and is one of the fundamental traits of entanglement and of quantum mechanics itself \cite{6}. This shareability relation has been explored extensively \cite{7–31} ever since Coffman, Kundu, and Wootters proved the first quantitative monogamy relation \cite{7} for three-qubit states.

An important question in the study of monogamy of entanglement is to determine whether or not a given entanglement measure is monogamous. For multipartite systems, almost all the known entanglement measures are monogamous. These include the entanglement of formation \cite{32,34}, concurrence \cite{33,35}, tangle \cite{36}, negativity \cite{37,38}, convex-roof extended negativity \cite{39}, Tsallis-q entropy of entanglement \cite{40}, Rényi-\(\alpha\) entropy of entanglement \cite{41,42}, squashed entanglement \cite{43}, and one-way distillable entanglement \cite{44,45,46,47}. So far, in addition to the one-way distillable entanglement \cite{48} and squashed entanglement \cite{49}, we know only that the G-concurrence \cite{50} is monogamous in all finite dimensions \cite{51}. The latter are monogamous according to the definition given in Ref. \cite{49, Definition 1} (also see Eq. (3) below). That is, the monogamy of other measures of entanglement in higher-dimensional systems still remain unknown even for well-known operational measures, such as the entanglement of formation.

Let \(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \equiv \mathcal{H}^{ABC}\) be a tripartite Hilbert space with finite dimension, where \(A, B, C\) are three subsystems of a composite quantum system, and \(S(\mathcal{H}^{ABC}) \equiv S^{ABC}\) be the set of density matrices acting on \(\mathcal{H}^{ABC}\). Recall that the original monogamy relation of entanglement measure \(E\) is quantitatively displayed as an inequality of the following form:

\[
E(\rho^{A|BC}) \geq E(\rho^{AB}) + E(\rho^{AC}),
\]

where the vertical bar indicates the bipartite split across which the (bipartite) entanglement is measured. However, Eq. (1) is not valid for many entanglement measures \cite{8,9,10,11}. This may give the impression that monogamy of entanglement is not a property of entanglement itself but of the function that is used to quantify it.

Moreover, in Ref. \cite{12} the problem of faithfulness versus monogamy was raised by showing that many measures of entanglement, such as the entanglement of formation, cannot satisfy any relation of the form

\[
E(\rho^{A|BC}) \geq f[E(\rho^{AB}), E(\rho^{AC})],
\]

where \(f: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\) is some fixed function that is independent on the dimension of the underlying Hilbert space, and that it is continuous, and satisfies \(f(x, y) \geq \max\{x, y\}\) with strict inequality at some ranges of \(x\) and \(y\). Despite this remarkable result, we show here that the entanglement of formation, and many other measures of entanglement that are defined in terms of convex roof extensions, are monogamous according to the definition recently put forward in Ref. \cite{12}.

According to the definition in Ref. \cite{13} of monogamy (without inequalities), a measure of entanglement \(E\) is monogamous if for any \(\rho^{ABC} \in S^{ABC}\) that satisfies the disentangling condition, i.e.,

\[
E(\rho^{A|BC}) = E(\rho^{AB}),
\]

we have that \(E(\rho^{AC}) = 0\). With respect to this definition, if the entanglement between system \(A\) and the composite system \(BC\) is as much as the entanglement

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that system $A$ shares just with $B$, then there is no entanglement left for $A$ to share just with $C$.

Clearly, this definition captures the spirit of monogamy of entanglement, and perhaps not surprisingly, can yield a family of monogamy relations similar to (1) by replacing $E$ with $E^\alpha$ for some $\alpha > 0$. More precisely, a continuous measure $E$ is monogamous according to this definition if and only if there exists $0 < \alpha < \infty$ such that

$$E^\alpha(\rho^{ABC}) \geq E^\alpha(\rho^{AB}) + E^\alpha(\rho^{AC}),$$

for all $\rho^{ABC}$ acting on the state space $\mathcal{H}^{ABC}$ with fixed dimension $\dim \mathcal{H}^{ABC} = d < \infty$ (see Theorem 1 in Ref. [46]). Note that (1) can be expressed in a similar form as in (2) with $f(x, y) = (x^\alpha + y^\alpha)^{1/\alpha}$. However, we stress here that (1) is not a special case of (2) since the exponent factor $\alpha$ depends on the underlying dimension of the Hilbert space. Note that there is no a priori physical reason to assume that the exponent factor is universal and independent on the dimension.

In this paper, we show that almost all entanglement monotones are monogamous on pure tripartite states, and furthermore, those that are based on convex roof extension [e.g., entanglement of formation, see also Eq. (5)], are also monogamous on mixed tripartite states according to our definition in terms of Eq. (3) [or equivalently Eq. (4)]. Our results indicate that monogamy is indeed a property of entanglement and not a consequence of a particular measure of entanglement.

A function $E : S^{AB} \to \mathbb{R}_+$ is called a measure of entanglement if (1) $E(\sigma^{AB}) = 0$ for any separable density matrix $\sigma^{AB} \in S^{AB}$, and (2) $E$ behaves monotonically under local operations and classical communications (LOCC). That is, for any given LOCC map $\Phi$ we have

$$E[\Phi(\rho^{AB})] \leq E(\rho^{AB}), \quad \forall \rho^{AB} \in S^{AB}.$$  

Moreover, convex measures of entanglement that do not increase on average under LOCC are called entanglement monotones.

Let $E$ be a measure of entanglement on bipartite states. The entanglement of formation $E_F$ associated with $E$ is defined by

$$E_F(\rho^{AB}) \equiv \min \sum_{j=1}^n p_j E(|\psi_j\rangle\langle\psi_j|^{AB}),$$

where the minimum is taken over all pure state decompositions of $\rho^{AB} = \sum_{j=1}^n p_j |\psi_j\rangle\langle\psi_j|^{AB}$. That is, $E_F$ is the convex roof extension of $E$. Vidal [48, Theorem 2] showed that $E_F$ above is an entanglement monotone on mixed bipartite states if the following concavity condition holds. For a pure state $|\psi^{AB}\rangle \in \mathcal{H}^{AB}$, $\rho^A = \text{Tr}_B|\psi\rangle\langle\psi|^{AB}$, define the function $h : S^A \to \mathbb{R}_+$ by

$$h(\rho^A) \equiv E(|\psi\rangle\langle\psi|^{AB}).$$

Note that since $E$ is invariant under local unitaries we must have

$$h(U\rho^A U^\dagger) = h(\rho^A)$$

for any unitary operator $U$ acting on $\mathcal{H}^A$. If $h$ is also concave, i.e.

$$h[\lambda \rho_1 + (1 - \lambda) \rho_2] \geq \lambda h(\rho_1) + (1 - \lambda) h(\rho_2)$$

for any states $\rho_1, \rho_2$, and any $0 \leq \lambda \leq 1$, then $E_F$ as defined in (3) is an entanglement monotone.

**Theorem.** Let $E$ be an entanglement monotone for which $h$, as defined in (4), is strictly concave; i.e., $h$ satisfies (8) with strict inequality whenever $\rho_1 \neq \rho_2$, and $0 < \lambda < 1$. Let also $E_F$ be as in (6). Then,

1. If $\rho^{ABC} = |\psi\rangle\langle\psi|^{ABC}$ is pure and (3) holds then

$$E_F(\rho^{ABC}) = E_F(\rho^{AB})$$

where $|\psi\rangle^{ABC} \in \mathcal{H}^{ABC}$ is a pure state on system $ABC$, and $E_F(\rho^{ABC})$ is monogamous for all $\rho^{ABC}$.

2. If $\rho^{ABC}$ is a mixed tripartite state and

$$E_F(\rho^{ABC}) = E_F(\rho^{AB}),$$

then

$$\rho^{ABC} = \sum_x p_x |\psi_x\rangle\langle\psi_x|^{ABC},$$

where $\{p_x\}$ is some probability distribution, and for each $x$ the Hilbert space $\mathcal{H}^x$ has a subspace isomorphic to $\mathcal{H}^{B_1(x)} \otimes \mathcal{H}^{B_2(x)}$, each pure state $|\psi_x\rangle^{ABC}$ is given by

$$|\psi_x\rangle^{ABC} = |\psi_x\rangle^{B_1(x)} |\psi_x\rangle^{B_2(x)} C.$$

In particular, the marginal state $\rho^{AC}$ is separable so that $E_F$ is monogamous (on mixed tripartite states).

**Remark 1.** The condition that $E$ in the theorem above is an entanglement monotone can be replaced with a weaker condition that the measure of entanglement $E$ satisfies $E \leq E_F$ on all bipartite density matrices. This is due to the fact that both Theorem 4 and Corollary 5 in Ref. [46] are still true if we assume only that $E$ satisfies $E \leq E_F$ and $E$ is not necessarily an entanglement monotone.

**Remark 2.** Part 1 of the Theorem indicates that, if $E$ is an entanglement monotone with $h$ is strictly concave, then for pure state $|\psi\rangle^{ABC}$, $E_F(\rho^{AB}) = E_F(\rho^{ABC})$ provided that $E(\langle\psi\rangle^{ABC}) = E_F(\rho^{ABC})$, that is, $E(\langle\psi\rangle^{ABC}) = E(\rho^{AB})$ is equivalent to $E(\langle\psi\rangle^{ABC}) = E_F(\rho^{ABC})$ in such a case. Corollary 5 in Ref. [46] proved only that if $E(\langle\psi\rangle^{ABC}) = E_F(\rho^{ABC})$, then $E(\rho^{ABC}) = E_F(\rho^{ABC})$, but it is unknown whether $E(\langle\psi\rangle^{ABC}) = E_F(\rho^{ABC})$ can imply $E(\rho^{ABC}) = E_F(\rho^{ABC})$ since in general we have only $E(\rho^{ABC}) \leq E_F(\rho^{ABC})$ (e.g., for the negativity $N$, we have $N \leq N_F$).
Proof. Part 1. In Ref. [46] it was shown that if (3) holds for a pure tripartite state \( \rho^{ABC} \equiv |\psi\rangle\langle\psi|^{ABC} \) then all pure state decompositions of \( \rho^{AB} \) must have the same average entanglement. Let \( \rho^{AB} = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j|^{AB} \) be an arbitrary pure state decomposition of \( \rho^{AB} \) with \( n = \text{Rank}(\rho^{AB}) \). Then,

\[
E(\rho^{AB}) \leq E_F(\rho^{AB}) = \sum_{j=1}^{n} p_j E(|\psi_j\rangle\langle\psi_j|^ {AB}),
\]

where the inequality follows from the convexity of \( E \), and the equality holds since all pure state decompositions of \( \rho^{AB} \) have the same average entanglement. Moreover, since \( E_F \) is an entanglement monotone, we must have

\[
E_F(\rho^{AB}) \leq E(|\psi\rangle\langle\psi|^{ABC}) = h(\rho^A).
\]

Therefore, denoting by \( \rho_j^A \equiv \text{Tr}_B |\psi_j\rangle\langle\psi_j|^ {AB} \), we conclude that if (3) holds then we must have

\[
\sum_{j=1}^{n} p_j h(\rho_j^A) = h(\rho^A).
\]

Given that \( \rho^A = \sum_{j=1}^{n} p_j \rho_j^A \) and \( h \) is strictly concave we must have

\[
\rho_j^A = \rho^A, \quad j = 1, \ldots, n.
\]  

(12)

Set \( r = \text{Rank}(\rho^A) \leq \text{dim} \mathcal{H}_B \), and let \( \mathcal{H}_B \) be an \( r \)-dimensional subspace of \( \mathcal{H}_B \) such that there exists a pure state \( |\phi\rangle^{AB_1} \in \mathcal{H}_B \) with marginal on part \( A \) being \( \rho^A \). Since all the reduced density matrices of \( \{ |\psi_j\rangle^{AB} \} \) have the same marginal on system \( A \) they must be related via local isometry on Bob’s side, to a purification \( |\phi\rangle^{AB_1} \) of \( \rho^A \). Therefore, there exists isometries \( \{ V_j^{B_1} \} \) such that

\[
|\psi_j\rangle^{AB} = (I^A \otimes V_j^{B_1}) |\phi\rangle^{AB_1}, \quad j = 1, \ldots, n.
\]  

(13)

Now, let \( \rho^{AB} = \sum_{k=1}^{n} q_k |\phi_k\rangle\langle\phi_k|^{AB} \) be another pure state decomposition of \( \rho \) with the same number of elements \( n \). For the same reasons leading to (13), there exists isometries \( W_k^{B_1} \) such that

\[
|\phi_k\rangle^{AB} = (I^A \otimes W_k^{B_1}) |\phi\rangle^{AB_1}, \quad k = 1, \ldots, n.
\]

On the other hand, since both decompositions \( \{ p_j, |\psi_j\rangle^{AB} \} \) and \( \{ q_k, |\phi_k\rangle^{AB} \} \) correspond to the same density matrix \( \rho^{AB} \), they must be related by a unitary matrix \( U = (u_{kj}) \) in the following way:

\[
\sqrt{q_k} |\phi_k\rangle^{AB} = \sum_{j=1}^{n} u_{kj} \sqrt{p_j} |\psi_j\rangle^{AB} = (I^A \otimes \sum_{j=1}^{n} u_{kj} \sqrt{p_j} V_j^{B_1}) |\phi\rangle^{AB_1}.
\]

Denoting by

\[
X_{k}^{B_1 \rightarrow B} \equiv \frac{1}{\sqrt{|q_k|}} \sum_{j=1}^{n} u_{kj} \sqrt{p_j} V_j^{B_1 \rightarrow B}, \quad k = 1, \ldots, n,
\]

we have

\[
(I^A \otimes X_{k}^{B_1 \rightarrow B}) |\phi\rangle^{AB_1} = (I^A \otimes W_k^{B_1 \rightarrow B}) |\phi\rangle^{AB_1}.
\]

Now, multiplying both sides of the equation above by \( (\rho^A)^{-1/2} \) (the inverse is understood to be on the support of \( \rho^A \)), we get

\[
(I^A \otimes X_{k}^{B_1 \rightarrow B}) |\phi_+\rangle^{AB_1} = (I^A \otimes W_k^{B_1 \rightarrow B}) |\phi_+\rangle^{AB_1},
\]

and we therefore conclude that \( X_{k}^{B_1 \rightarrow B} = W_k^{B_1 \rightarrow B} \). This means that \( X_{k}^{B_1 \rightarrow B} \) is an isometry for any choice of unitary matrix \( U = (u_{kj}) \). But since \( U = (u_{jk}) \) is an arbitrary unitary matrix, we can take its first row, \( \{ u_{1j} \} \), to be an arbitrary normalized vector. Hence, we conclude that any linear combination of the isometric matrices \( \{ V_j^{B_1 \rightarrow B} \} \) is proportional to an isometric matrix. We now discuss the consequence of this property on the form of \( \{ V_j^{B_1 \rightarrow B} \} \).

The isometries \( V_j^{B_1 \rightarrow B} \) can be expressed as

\[
V_j^{B_1 \rightarrow B} = \sum_k |v_{jk}\rangle \langle k|,
\]  

(14)

where \( \{ |k\rangle \} \) is an orthonormal basis of \( \mathcal{H}_B \), and for each \( j \), \( \{ |v_{jk}\rangle \} \) are some orthonormal vectors in \( \mathcal{H}_B \). Consider the arbitrary linear combination \( \sum_j c_j V_j \). It can be expressed as

\[
\sum_{j,k} c_j |v_{jk}\rangle \langle k| \equiv \sum_k |u_k\rangle \langle k|, \quad |u_k| = \sum_j c_j |v_{jk}|.
\]

Therefore, \( \sum_j c_j V_j \) is proportional to an isometry if and only if for all \( k \neq k' \), \( \langle u_{k'} | u_k \rangle = 0 \) and \( \| u_k \| = \| u_{k'} \| \). Observe that

\[
\langle u_{k'} | u_k \rangle = \sum_{j,j'} c_j c_{j'}^* \langle v_{k'j'} | v_{kj}\rangle.
\]

Now, for a fixed \( k \) and \( k' \), the above equation can be viewed as an inner product between a vector \( v_{k'k} \), whose components are \( v_{k'j'} | v_{kj}\rangle \), and a vector \( \tilde{e} \otimes c \), whose components are \( c_j c_{j'}^* \). Since \( v_{kk'} \) is orthogonal to any vector of the form \( \tilde{e} \otimes c \) whenever \( k \neq k' \), it must be equal to the zero vector. We therefore conclude that

\[
\langle v_{kk'} | v_{kj}\rangle = 0, \quad k \neq k'
\]

and

\[
\langle v_{kk'} | v_{kj}\rangle = d_{jj'} \neq 1
\]

for some \( d_{jj'} \) which are independent of \( k \). Note that we can assume with no loss of generality that \( \rho^{AB} = \rho^A \otimes \rho^B \).
Therefore, where \( K \equiv \{ |k\rangle \} \) holds for any given \( K \sim H \) conclude that \( \rho \) pure state decomposition of such that \( \rho \rangle \). Hence, if \( E_F(\rho^{A|BC}) = E_F(\rho^{AB}) \) then

\[
\sum_x p_x E(|\psi_x^x\rangle^{A|BC}) = E_F(\rho^{AB}) \leq \sum_x p_x E(\rho_x^x),
\]

where we used the convexity of \( E_F \). On the other hand, since \( E_F \) is a measure of entanglement for each \( x \) we have \( E(|\psi_x^x\rangle^{A|BC}) \geq E(\rho_x^x) \). Combining this with the equation above we conclude that

\[
E(|\psi_x^x\rangle^{A|BC}) = E_F(\rho_x^x), \quad \forall x.
\]

Therefore, the rest of the proof of part 2 follows from part 1 of the theorem.

**Corollary.** Using the same notations as in the theorem above, if \( E_F(\rho^{A|BC}) = E_F(\rho^{AB}) \) and \( \dim H^B \leq 3 \) then \( \rho^{ABC} \) is bi-separable, and in particular it admits the form

\[
\rho^{ABC} = t \sigma^{A|BC} + (1 - t) \gamma^{AB|C},
\]

where \( \sigma^{A|BC} \) is a pure state, then \( |\psi\rangle^{ABC} \) has the form \( |\psi\rangle^{AB|C} \) or \( |\phi\rangle^{A|B|C} \).

At last we discuss the strict concavity of the entanglement measures so far. Many operational measures of entanglement such as the relative entropy of entanglement, entanglement cost, and distillable entanglement, all reduce on a bipartite pure state to the entropy of entanglement given in terms of the von Neumann entropy of the reduced state, \( H(\rho) \equiv -\text{Tr}(\rho \log \rho) \). The von Neumann entropy is known to be strictly concave \([49]\) and therefore they are all monogamous on pure tripartite states. The first part of the theorem above generalizes a similar result that was proved in Ref. \([43]\) for the special case in which \( E \) is taken to be the negativity. It demonstrates that many measures of entanglement are monogamous on pure tripartite states, while their convex roof extensions as defined in \([5]\) are monogamous even on mixed tripartite states.

Any function that can be expressed as

\[
H_{g}(\rho) = \text{Tr}[g(\rho)] = \sum_j g(p_j),
\]

where \( p_j \) are the eigenvalues of \( \rho \) is strictly concave if \( g''(\rho) < 0 \) for all \( 0 \leq p < 1 \). This includes the quantum Tsallis \( q \)-entropy \([50, 51]\) with \( q > 0 \). In particular, the linear entropy (or the Tsallis 2-entropy) is strictly concave, and therefore the tangle is a monogamous measure of entanglement since it is defined in terms of the convex roof extension. Another important example is the Rényi \( \alpha \)-entropy \([52, 54]\). For the Rényi parameter \( \alpha \in [0, 1] \) the Rényi entropies are strictly concave (see, e.g., Ref. \([48]\),
but in general, for $\alpha > 1$ the Rényi entropies are not even concave (although they are Schur-concave). To the authors’ knowledge, except for this case of Rényi $\alpha$-entropy of entanglement with $\alpha > 1$, all other measures of entanglement that have been studied intensively in literature, correspond on pure bipartite state to strict concave functions of the reduced density matrix. These include the negativity, tangle, concurrence (see the Appendix), $G$-concurrence, and the Tsallis entropy of entanglement.

In conclusion, we showed that many measures of entanglement, such as the entanglement of formation, that were believed not to be monogamous (irrespective of the specific monogamy relation \([17]\)), are in fact monogamous according to a new definition of monogamy without inequalities that we put forward in Ref. [46]. This new definition is equivalent to the quantitative inequality \([10]\), but with a key difference that the exponent factor $\alpha$ can depend on the underlying dimension. Therefore, the results presented here support this non-universal (i.e., dimension dependent) definition of monogamy. The fact that so many important measures of entanglement are not universally monogamous \([17]\) may give the impression that monogamy of entanglement cannot be attributed to entanglement itself but rather is a property of the particular measure that is used to quantify entanglement. Furthermore, as was shown in Ref. [17], measures of entanglement cannot be simultaneously faithful (as defined in Ref. [17]) and universally monogamous. Here we avoided all these issues by adopting a new definition of monogamy that allows for non-universal monogamy relations, while at the same time maintaining a quantitative way [as in [11]] to express monogamy relations.

While we were not able to show that all measures of entanglement are monogamous (according to our definition), we are also not aware of any continuous measures of entanglement that are not monogamous. It may be the case that all continuous measures of entanglement are monogamous, which will support our assertion that monogamy is a property of entanglement and not of some particular functions quantifying entanglement. Moreover, many important measures of entanglement, are not defined in terms of convex roof extensions. For such measures, our theorem does not provide any information regarding their monogamy on mixed tripartite states. One example of that is the negativity. Our theorem implies that the convex roof extended negativity is monogamous but we do not know if the negativity itself is monogamous.

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Appendix: Constructing monogamous measures of entanglement from other monogamous measures

For any entanglement monotone $E$, and any monotonically increasing function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the property that $g(x) = 0$ iff $x = 0$, denote

$$E_g(\rho^{AB}) \equiv \min \sum_j p_j g(E(|\psi_j\rangle\langle\psi_j|^{AB})), \quad (A.1)$$

where the minimum is taken over all pure state decompositions of $\rho^{AB} = \sum_{j=1}^n p_j |\psi_j\rangle\langle\psi_j|^{AB}$. Observe that if in addition $g$ is convex then we must have

$$E_g(\rho^{AB}) \geq g(E(\rho^{AB})). \quad (A.2)$$

Therefore, if $E_g$ is an entanglement measure and is monogamous on pure tripartite states then $E$ is also monogamous on pure tripartite states. To see why, note that if $E(|\psi^{A|BC}\rangle) = E(\rho^{AB})$ we also have

$$E_g(|\psi^{A|BC}\rangle) = g(E(|\psi^{A|BC}\rangle)) = g(E(\rho^{AB})) \leq E_g(\rho^{AB}). \quad (A.3)$$

But since $E_g$ is a measure of entanglement we must have $E_g(|\psi^{A|BC}\rangle) \geq E_g(\rho^{AB})$ so that we get $E_g(|\psi^{A|BC}\rangle) = E_g(\rho^{AB})$. Since we assume here that $E_g$ is monogamous, thus $E_g(\rho^{AC}) = 0$, which implies that $E(\rho^{AC}) = 0$. As a simple example of this, consider the function $g(x) = x^2$ and take $E = C^2$ be the concurrence as defined in Ref. [33]. Then, $E_g = C^2$ is the tangle which is monogamous (it is given in terms of the linear entropy, which is strictly concave). Hence, the above analysis implies that the concurrence $C$ is also monogamous.

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