Crossed Products Related to Cyclically Ordered Semi Groups

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Abstract. Suppose \((G, R)\) is a cyclically ordered abelian group. Thru unwinding method of Rieger, we construct a semi group crossed product related with \((G, R)\).

1. Introduction

The theory of crossed product by semigroup of endomorphisms is one of important and interesting area in modern theory of operator algebras. This theory has a plenty of applications such as in Cuntz algebras [1], Toeplitz algebras (Adji, et al.[2], Luca and Raeburn [3], Adj [4], Adj and Raeburn [5], Rosjanuardi [6], Adj and Rosjanuardi [7], Rosjanuardi and Albania [8] and Hecke algebras arising in number theory (Laca and Raeburn [9], Larsen and Raeburn [10]). This theory is a generalisation of the theory of crossed product by group of automorphism, which is a well developed area of the theory of operator algebras, and has rich of examples.

In some cases, the theory of crossed product by semigroup of endomorphisms heavily relies on theory of totally ordered group, for examples Adji et. al [2], Adj [4], Adj and Raeburn [5], Rosjanuardi [6], Adj and Rosjanuardi [7]. Rosjanuardi and Albania [8], Rosjanuardi [11] are among others. The concept of order is essentials as the semigroup crossed product here is a generalisation of Stacey’s [12] crossed products of single endomorphism which is related with the action of semigroup of natural numbers \(\mathbb{N}\).

A generalisation of the notion of totally ordered group is cyclically ordered group. Cyclically ordered group which was originally introduced by Rieger in 1947 has attracted great deal of attention, for example Swierczkowski [13], Fuchs [14], Harmic [15], Jakubik [16, 17], Cernak [18], Giraudet, Leloup, Lucas [19] and Leloup, Lucas [20] are among others. The concept of this order is different with regular linear order, as it is a ternary relation instead of binary relation.

Given a cyclically ordered group \((G, R)\), we will make use the theorem of Rieger to get its unwound totally ordered group \(uw(G)\) to obtain a dynamical system and its semigroup crossed product.

2. Preliminaries on Cyclically Ordered Group

Definition 1 A cyclically ordered group is a pair \((G, R)\) consists of a group \(G\) and a ternary relation \(R\) on \(G\) satisfying the following axioms:

- R1: \(\forall x, y, z, R(x, y, z) \Rightarrow x \neq y \neq z \neq x\) (\(R\) is strict),
- R2: \(\forall x, y, z, x \neq y \neq z \neq x \Rightarrow R(x, y, z)\) or \(R(x, z, y)\) (\(R\) is total),
- R3: \(\forall x, y, z, R(x, y, z) \Rightarrow R(y, z, x)\) (\(R\) is cyclic),
- R4: \(\forall x, y, z, u, R(x, y, z)\) and \(R(y, u, z) \Rightarrow R(x, u, z)\) (\(R\) is transitive),
- R5: \(\forall x, y, z, u, v, R(x, y, z) \Rightarrow R(uxv, uyz, uzv)\) (\(R\) is compatible with the operation in \(G\)
Riegerin Leloup and Lucas [20] states that every cyclically ordered group can be obtained from a totally ordered group. Let $(G, \leq)$ be a totally ordered group and $g_0$ be a positive, central and cofinal element of $G$. The quotient group $G/(g_0)$ can be cyclically ordered by setting 

$$R(a(g_0), b(g_0), c(g_0)) \iff \exists a' \in a(g_0), b' \in b(g_0), c' \in c(g_0) \text{such that } 1_G \leq a' < g_0, 1_G \leq b' < g_0, 1_G \leq c' < g_0$$

and either

$$a' < b' < c' \text{ or } b' < c' < a' \text{ or } c' < a' < b'.$$

The cyclically ordered group $G/(g_0)$ is called the wound round associated to $G$ and $g_0$.

Moreover, theorem of Rieger says that for every cyclically ordered group $(G, R)$, there is a totally ordered group $(G, \leq)$ and an element $z_G$ which is positive, central and cofinal in $uw(G)$ such that $(G, R)$ is isomorphic to the cyclically ordered $uw(G)/(z_G)$.

The group $uw(G)$ is called the unwound of $(G, R)$. More precisely, the unwound group $uw(G)$ of cyclically ordered group $(G, R)$ is given by

$$\mathbb{Z} \times G$$

together with the linear order

$$(n, c) < (n', c') \iff \text{either } n < n', \text{ or } n = n' \text{ and } R(1_G, c, c'), \text{ or } n = n' \text{ and } c = 1_G.$$  

3. Semigroup Crossed Products

**Definition 3** Let $\Gamma^+$ be the positive cone of a totally ordered abelian group. A dynamical system consists of a unital $C^*$-algebra $A$, and a semigroup homomorphism $\alpha : \Gamma^+ \to \text{Endo}(A)$. The homomorphism $\alpha$ is called an action of $\Gamma^+$ on $A$ by endomorphisms, or just action for short when the context is clear. We denote the dynamical system by $(A, \Gamma^+, \alpha)$.

**Definition 4** A covariant representation of $(A, \Gamma^+, \alpha)$ is a pair $(\pi, V)$ in which $\pi : A \to B(H)$ is a unital representation of $A$ and $V : \Gamma^+ \to \text{Isom}(B(H))$ is an isometric representation of $\Gamma^+$, that is, $V_\alpha V_t = V_{\alpha t}$, and the covariant condition $\pi(\alpha_1(a)) = V_t \pi(a) V_t^*$ for all $a \in A$, $t \in \Gamma^+$ is satisfied.

When we consider the semigroup $\mathbb{N}$, we have the dynamical system considered by Stacey [12]. Focusing on the dynamical systems considered by Stacey [12] allows us to find an example of a dynamical system that has no covariant representation. Stacey pointed out a dynamical system of no covariant representation, as we recited below.

**Definition 5** A covariant representation of $(A, \Gamma^+, \alpha)$ is a pair $(\pi, V)$ in which $\pi : A \to B(H)$ is a unital representation of $A$ and $V : \Gamma^+ \to \text{Isom}(B(H))$ is an isometric representation of $\Gamma^+$, that is, $V_\alpha V_t = V_{\alpha t}$, and the covariant condition $\pi(\alpha_1(a)) = V_t \pi(a) V_t^*$ for all $a \in A$, $t \in \Gamma^+$ is satisfied.

When we consider the semigroup $\mathbb{Z}$, we have the dynamical system considered by Stacey[12]. Focusing on the dynamical systems considered by Stacey [12] allows us to find an example of a dynamical system that has no covariant representation. Stacey pointed out a dynamical system of no covariant representation, as we recited below.

**Example 1** (Stacey [12]) Suppose $\alpha : c_0 \to c_0$ is the unilateral shift, defined by

$$\alpha((x_1, x_2, \ldots)) = (x_2, x_3, \ldots)$$  

(1)
Then for each \( x \in c_{00} \), the dense subalgebra of sequences of finite support, \( \alpha^n(x) = 0 \) for some \( n \).
Hence, if \( (\pi, \{T_i\}) \) is a covariant representation of \((c_{00}, \alpha)\) on \( H \) and \( \beta \) is the corresponding \( * \)-endomorphism of \( B(H) \) then \( \beta^n(\pi(x)) = 0 \) implies \( \pi(x) = 0 \). Hence, by the continuity, \( \pi \) is the trivial (zero) representation. Therefore \((c_{00}, \alpha, N)\) has no covariant representation.

The concept of semigroup crossed product of the dynamical system \((A, \Gamma^+, \alpha)\) was introduced by Adji et al [2]. It is a development of the crossed product by groups action of Raeburn [21]. This version of crossed product can also be viewed as the untwisted version of semigroup twisted crossed product of Rosjanuardi [6, 11], Adji and Rosjanuardi [7].

**Definition 4**
A crossed product of the dynamical system \((A, \Gamma^+, \alpha)\) is a triple \((B, i_A, i_V)\) of a unital \( C^* \)-algebra \( B \) together with a unital homomorphism \( i_A : A \to B \) and embedding of \( \Gamma^+ \) as isometries \( i_{\Gamma^+} : \Gamma^+ \to B \) such that:

1. \((i_A, i_{\Gamma^+})\) is a covariant pair for \((A, \Gamma^+, \alpha)\).
2. for any covariant representation \((\pi, V)\) of \((A, \Gamma^+, \alpha)\) there is a unital representation \( \pi \times V \) of \( B \) such that \((\pi \times V) \circ i_A = \pi \) and \((\pi \times V) \circ i_{\Gamma^+} = V\),
3. \( B \) is generated by \( i_A(A) \) and \( i_{\Gamma^+}(\Gamma^+)\).

Existence of semigroup crossed product of a dynamical system heavily rely on the existence of non zero covariant representation. Adji et al. [2] gave an example of dynamical system in which the crossed product exists, and is universal for isometric representation of the semi group. The algebra in the dynamical system is the closed subspace of \( \ell^0(\Gamma) \) spanned by \( \{1_x : x \in \Gamma^+\} \), where

\[
1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise.} \end{cases}
\]

This algebra is denoted by \( B_{\Gamma^+} \). For each \( x \in \Gamma^+ \), the automorphism \( \tau_x \) of \( \ell^0(\Gamma) \) which is defined by

\[
\tau_x(f)(y) = f(y - x)
\]

satisfies \( \tau_x(1_y) = 1_{x+y} \).

(2)

which implies that \( \tau \) restricts to an action \( \alpha \) of \( \Gamma^+ \) by endomorphism of \( B_{\Gamma^+} \).

Suppose \( T: \Gamma^+ \to B(\ell^2(\Gamma^+)) \) defined by \( T_x(\delta_y) = \delta_{y+x} \), then \( T \) is an isometric representation of \( \Gamma^+ \). Proposition 2.2 of Adji, Laca, Nilsen, Raeburn [2] implies there is a representation \( \pi_V \) of \( B_{\Gamma^+} \) such that \((\pi_V, V)\) is a covariant representation of the dynamical system \((B_{\Gamma^+}, \Gamma^+, \alpha)\). Therefore the crossed product \( B_{\Gamma^+} \times_{\alpha'} \Gamma^+ \) exists.

4. Results and Discussion
Given a \( C^* \)-algebra \( A \), a cyclically ordered abelian group \((G, R)\) and an action \( \alpha \) of the positive cone \( P \) of \( G \). Since the class of the order is not a linear order, we can not directly make use the Definition 4 to obtain the semigroup crossed product of the dynamical system \((A, P, \alpha)\). But we can make use the method of Rieger to unwind the cyclic order to get the linear order related with \( R \).

If \( \text{uw}(G) \) is the unwound of \((G, R)\), then \( \text{uw}(G) \) is linearly ordered. We now consider the positive cone \((\text{uw}(G))^+ \). If \( \alpha' \) is an action of \((\text{uw}(G))^+ \) by endomorphisms of \( A \), and there is a nontrivial covariant representation \((\pi, V)\) of the dynamical system \((A, (\text{uw}(G))^+, \alpha')\), then the semigroup crossed \( A \times_{\alpha'} (\text{uw}(G))^+ \) exists. We call \( A \times_{\alpha'} (\text{uw}(G))^+ \) the semigroup crossed product related with the cyclically ordered abelian group \((G, R)\).
Example:
Given a cyclically ordered abelian group \((G, R)\). Let \(uw(G)\) be the unwound group of \((G, R)\). We will make use the method of Adji, Laca, Nilsen and Raeburn [2] in constructing the semi group crossed product which is universal for the isometric representation of the semi group \((uw(G))^+\), i.e. the positive cone of \(uw(G)\).

The algebra in the dynamical system is the closed subspace of \(\ell^\infty(uw(G))\) spanned by 
\[\{1_x : x \in (uw(G))^+\},\]
where 
\[1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise}. \end{cases}\]
This algebra is denoted by \(B_{(uw(G))^+}\). For each \(x \in (uw(G))^+\), the automorphism \(\tau_x\) of \(\ell^\infty(uw(G))\) which is defined by 
\[\tau_x(f)(y) = f(y - x),\]
which implies that \(\tau\) restricts to an action \(\alpha\) of \((uw(G))^+\) by endomorphism of \(B_{(uw(G))^+}\).

Suppose \(T: \Gamma^+ \to B(\ell^2(uw(G))^+)\) defined by \(T_x(\delta_y) = \delta_{y-x}\), then \(T\) is an isometric representation of \((uw(G))^+\). Proposition 2.2 of Adji, Laca, Raeburn (1994) implies there is a representation \(\pi_V\) of \(B_{uw(G)^+}\) such that \((\pi_V, V)\) is a covariant representation of the dynamical system \((B_{uw(G)^+}, (uw(G))^+ , \alpha)\). Therefore the crossed product \(B_{(uw(G))^+} \times_\alpha (uw(G))^+\) exists.

In particular, since the unwound group of \((G, R)\) is the linearly ordered group \(Z \times G\), then the semi group crossed product \(B_{(Z\oplus Z)^+} \times_\tau (Z \oplus Z)^+\) considered in Rosjanuardi and Albania [8] and Rosjanuardi [11] can be viewed as a semi group crossed product related with the cyclically ordered group \(Z\).

5. Conclusion
Given a cyclically ordered abelian group \((G, R)\). Thru unwinding method of Rieger, we can aplly the method Adji, et al [2] to obtain a semi group crossed product related with \((G, R)\).

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