TROPICAL GAUSSIANS: A BRIEF SURVEY

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ABSTRACT. We survey ways to define the analogue of the Gaussian measure in the tropical semiring.

1. INTRODUCTION

Tropical mathematics have found many applications in both pure and applied areas. Many applications are considered ‘classic’ and have been well-documented by a number of monographs on tropical geometry and algebraic geometry [BP16, Gro11, Huh16, MS15], discrete event systems [BCOQ92, But10], large deviations and calculus of variations [KM97, Puh01], and combinatorial optimization [Jos14]. At the same time, new applications are emerging in areas including phylogenetics [LMY18, YZZ17], statistics [Hoo17], economics [BK13, CT16, EVDD04, GMS13, Jos17, Shi15, Tra13, TY15], game theory and complexity theory [ABGJ18, AGG12]. There is a growing need for a systematic study of probability distributions in tropical settings. Over the classical algebra, the Gaussian measure is arguably the most important distribution to both theoretical probability and applied statistics. In this work, we review the existing analogues of the Gaussian measure in the tropical semiring and outline various research directions.

1.1. Characterizations of the classical Gaussian. Fix a vector \( \mu \in \mathbb{R}^n \) and a positive definite matrix \( \Sigma \in \mathbb{R}^{n \times n} \). The Gaussian measure, also called the normal distribution, with mean \( \mu \) and covariance \( \Sigma \), denoted \( \mathcal{N}(\mu, \Sigma) \) is the probability distribution with density

\[
 f_{\Sigma, \mu}(x) \propto \exp\left( -\frac{1}{2} (x - \mu)^\top \Sigma^{-1} (x - \mu) \right), \quad x \in \mathbb{R}^n.
\]

Let \( I \) denote the identity matrix, and \( 0 \in \mathbb{R}^n \) the zero vector. Measures \( \mathcal{N}(0, \Sigma) \) are called centered Gaussians, while \( \mathcal{N}(0, I) \) is the standard Gaussian. Any Gaussian can be standardized by an affine linear transformation.

**Lemma 1.1.** Let \( \Sigma = U \Lambda U^\top \) be the eigendecomposition of \( \Sigma \). Then \( X \sim \mathcal{N}(\mu, \Sigma) \) if and only if \( (U \Lambda^{1/2})^{-1} (X - \mu) \sim \mathcal{N}(0, I) \).

The standard Gaussian has two important properties. First, if \( X \) is a standard Gaussian in \( \mathbb{R}^n \), then its coordinates \( X_1, \ldots, X_n \) are \( n \) independent and identically distributed (i.i.d) random variables. Second, for any orthonormal matrix \( A \), \( AX \overset{d}{=} X \). These two properties, being a product measure and spherically symmetric, completely characterize the standard Gaussian [Kal06, Proposition 11.2].

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**Theorem 1.2** (Maxwell). Let $X_1, \ldots, X_n$ be i.i.d random variables, where $n \geq 2$. Then the distribution of $X = (X_1, \ldots, X_n)$ is spherically symmetric iff the $X_i$’s are centered Gaussians on $\mathbb{R}$.

Maxwell discovered the above theorem while studying velocity distribution for gas molecules [Kal06, §11]. From a statistical perspective, Lemma 1.1 and Theorem 1.2 essentially reduce working with data from the Gaussian measure to doing linear algebra. In particular, if data points come from a Gaussian measure, then it is the affine linear transformation of a standard Gaussian, whose coordinates are always independent regardless of the orthonormal basis that it is represented in. These properties are fundamental to Principal Component Analysis, an important statistical technique whose tropical analogue is actively being studied [YZZ17].

There are numerous other characterizations of the Gaussians whose ingredients are only orthogonality and independence, see [Bog98, §1.9] and references therein. One example is Kac’s theorem [Kac39]. This is a special case of the Darmois-Skitovich theorem [LOR77], which characterize Gaussians (not necessarily centered) in terms of independence of linear combinations.

**Theorem 1.3** (Kac). Let $X, Y$ be i.i.d random variables in $\mathbb{R}^n$. Then $X$ is centered Gaussian if and only if for all $\phi \in \mathbb{R}$, $(X \sin \phi + Y \cos \phi, X \cos \phi - Y \sin \phi) \overset{d}{=} (X, Y)$.

**Theorem 1.4** (Darmois-Skitovich). Let $X_1, \ldots, X_n$ be independent random variables. Then the $X_i$’s are Gaussians if and only if there exists $\alpha, \beta \in \mathbb{R}^n$, $\alpha_i, \beta_i \neq 0$ for all $i = 1, \ldots, n$, such that $\sum_i \alpha_i X_i$ and $\sum_i \beta_i X_i$ are independent.

Another reason for the wide applicability of Gaussians in statistics is the Central Limit Theorem. An interesting historical account of its development can be found in [Kal06, §4]. From the Central Limit Theorem, one can derive yet other characterizations of the Gaussian, such as that which maximizes entropy subject constant variance [Bar86]. The appearance of the Gaussian in the Central Limit Theorem is fundamentally linked to its characterization as $\frac{1}{2}$-stable distributions. This is expressed in the following theorem by Pólya [P623]. There are a number of variants of this theorem, see [Bog98, Bry12] and discussions therein.

**Theorem 1.5** (Pólya). Suppose $X, Y \in \mathbb{R}^n$ are independent random variables. Then $X, Y$ and $(X + Y)/\sqrt{2}$ have the same distribution iff this distribution is the centered Gaussian.

From the perspective of stochastic analysis, the Gaussian measure can be characterized as the unique invariant measure for the Ornstein-Uhlenbeck semigroup [Bog98, §1]. Let us elaborate. Let $\gamma$ be a centered Gaussian measure on $\mathbb{R}^n$. The Ornstein-Uhlenbeck semigroup $(T_t, t \geq 0)$ is defined on $L^2(\gamma)$ by the Mehler formula

$$T_t f(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \gamma(dy), \quad t > 0$$

and $T_0$ is the identity operator. It characterizes the Gaussian measure in the following sense [Bog98, §1].

**Lemma 1.6.** $\gamma$ is the unique invariant probability measure for $(T_t, t \geq 0)$.

The Ornstein-Uhlenbeck semigroup is a powerful tool in proving hypercontractivity and log-Sobolev inequalities. In particular, the Gaussian density can be characterized as the function that satisfy such inequalities with the best constants [Bog98].
One can arrive at this semigroup without the Mehler’s formula as follows. Let \( \mathcal{D} = \{ f \in L^2(\gamma) : \lim_{t \to 0} \frac{T_t h - h}{t} \text{ exists in the norm of } L^2(\gamma) \} \). The linear operator \( L \) defined on \( \mathcal{D} \) by

\[
L h = \lim_{t \to 0} \frac{T_t h - h}{t}
\]
is called the generator of the semigroup \( (T_t, t \geq 0) \). The generator of the Ornstein-Uhlenbeck semigroup is given by

\[
L h(x) = \Delta h(x) - \langle x, \nabla h \rangle = \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2}(x) - \sum_{i=1}^{n} x_i \frac{\partial h}{\partial x_i}(x).
\]

This generator uniquely specifies the semigroup. Importantly, the two ingredients needed to define \( L \) are the Laplacian operator \( \Delta \), and the gradient operator \( \nabla \). Thus \( L \) can be defined on Riemannian manifolds, for instance. This opens up ways to define Gaussian on tropical curves, as discussed below.

Yet another way to characterize Gaussians is via the heat equation

\[
u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \quad u_t = \Delta u \iff \frac{\partial}{\partial t} u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u.
\]
The fundamental solution \( \Phi \) to the heat equation is given by [Eva98b] \( \Phi(x, t) = 0 \) for \( t < 0 \) and

\[
\Phi(x, t) = (4\pi t)^{n/2} \exp\left(-\frac{\|x\|^2}{4t}\right) \quad \text{for } x \in \mathbb{R}^n, \quad t > 0.
\]

In particular, for \( t > 0 \), \( \Phi(\cdot, t) \) is the Gaussian density with variance \( 2t \). While this only requires the Laplacian operator, the solution to the heat equation is not unique. Furthermore, the variance of the Gaussian measure obtained in the fundamental solution varies with time. Compared to the Ornstein-Uhlenbeck semigroup, the heat equation is the cheaper but imperfect way to get hold of Gaussians.

2. Tropical analogues of Gaussians

2.1. Tropicalizations of \( p \)-adic Gaussians. Evans [Eva01] used Theorem 1.3 as the definition of Gaussians to extend them to local fields. A local field is any locally compact, non-discrete field other than the field of real numbers or the field of complex numbers. All local fields are totally disconnected, and are finite algebraic extensions of either the field of \( p \)-adic numbers or the field of formal Laurent series with coefficients drawn from the finite field with \( p \) elements [Eva01]. In particular, local fields come with a tropical valuation, and thus one can define a tropical Gaussian to be the tropicalization of the Gaussian measure on a local field.

Let us consider an explicit example: tropicalization of the \( p \)-adic Gaussian. Fix a prime \( p \in \mathbb{N} \). A non-zero rational number \( r \in \mathbb{Q}\setminus\{0\} \) can be uniquely written as \( r = p^s(a/b) \) where \( a \) and \( b \) are not divisible by \( p \). The valuation of \( r \) is \( |r| := p^{-s} \). The completion of \( \mathbb{Q} \) under the metric \( (x, y) \mapsto |x - y| \) is the field of \( p \)-adic numbers, denoted \( \mathbb{Q}_p \). The tropical valuation of \( r \) is \( \text{val}(r) := s \). By [Eva01] Theorem 4.2, the family of \( \mathbb{Q}_p \)-valued Gaussians is indexed by \( \mathbb{Z} \). For each \( k \in \mathbb{Z} \), there is a unique \( \mathbb{Q}_p \)-valued Gaussian supported on the ball \( p^k \mathbb{Z}_p := \{ x \in \mathbb{Q}_p : |x| \leq p^{-k} \} \). Furthermore, the Gaussian is the normalized Haar measure on this support. As \( p^k \mathbb{Z}_p \) is made up of \( p \) translated copies of \( p^{k+1} \mathbb{Z}_p \), which in turn is made up of \( p \) translated copies of \( p^{k+2} \mathbb{Z}_p \), an immediate calculation gives the following.
Lemma 2.1 (Tropicalization of the $p$-adic Gaussian). Let $X$ be a $\mathbb{Q}_p$-valued Gaussian with index $k \in \mathbb{Z}$. Then $\text{val}(X)$ is a random variable supported on $\{k, k+1, k+2, \ldots\}$, and it is distributed as $k + \text{geometric}(1 - p^{-1})$. That is,

$$\text{Pr}(\text{val}(X) = k + s) = p^{-s}(1 - p^{-1}) \text{ for } s = 0, 1, 2, \ldots.$$  

Local fields were first studied in the context of number theory and the theory of group representations, see the historical notes and references in [Eva01]. There is a large and growing literature surrounding probability on local fields, or more generally, analysis on ultrametric spaces. They have found diverse applications, from spin glasses, protein dynamics, genetics, to cryptography and geology; see the recent comprehensive review [DKK+17] and references therein. The $p$-adic Gaussian was originally defined as steps towards building Brownian motions on $\mathbb{Q}_p$ [Eva01]. It would be interesting to use tools from tropical algebraic geometry to revisit and expand results involving random $p$-adic polynomials, such as the expected number of zeroes of a random $p$-adic polynomial system [Eva06], or determinant of matrices with i.i.d $p$-adic Gaussians [Eva02]. Previous work on random $p$-adic polynomials from a tropical perspective tend to consider systems with uniform valuations [AI11]. Lemma 2.1 hints that to connect the two literature, the geometric distribution may be more suitable.

2.2. Gaussians via tropical linear algebra. Consider arithmetic done in the tropical algebra $(\mathbb{R}, \oplus, \odot)$, where $\mathbb{R}$ is $\mathbb{R}$ union with the additive identity. In the max-plus algebra $(\mathbb{R}, \oplus, \odot)$ where $a \oplus b = \max(a, b)$, for instance, $\mathbb{R} = \mathbb{R} \cup \{-\infty\}$. In the min-plus algebra $(\mathbb{R}, \oplus, \odot)$ where $a \oplus b = \min(a, b)$, we have $\mathbb{R} = \mathbb{R} \cup \{+\infty\}$. To avoid unnecessary technical details, in this section we choose focus on vectors taking values in $\mathbb{R}$ instead of $\mathbb{R}$.

Tropical linear algebra was developed by several communities with different motivations. It evolved as a linearization tool for certain problems in discrete event systems, queueing theory and combinatorial optimization, see the monographs [BCOQ92, But10], as well as the recent survey [KLBvdB18] and references therein. A large body of work focuses on using the tropical settings to find generalized versions of classical results in linear algebra and convex geometry. Many fundamental concepts have rich tropical analogues, including the spectral theory of matrices [ABG06, BCOQ92, But10], linear independence and projectors [AGK11, AGNS11, BS+07, Ser09], separation and duality theorems in convex analysis [BH08, CGQ04, GK11, NS07], matrix identities [Gau96, HK12a, MT16, Sim94], matrix rank [CJR11, DSS05, IR09, Shi11], and tensors [BF18, Tsu15]. Another research direction focuses on the combinatorics of objects arise in tropical convex geometry, such as polyhedra and hyperplane arrangements [AGG12, DS04, JL16, JSY07, ST12, Tra17]. These work have close connections to matroid theory and are at the interface of tropical linear algebra and tropical algebraic geometry [AD09, FR15, GG17, Ham15, LS18].

Overall, tropical linear algebra is a rich theory. However, in this section, we shall show that there is currently no satisfactory way to define the tropical Gaussian as a classical probability measure based on the characterizations of Gaussians via orthogonality and independence.

A natural definition for tropical linear combinations of $v_1, \ldots, v_m \in \mathbb{R}^n$ would be the set of vectors of the form

$$[v_1, \ldots, v_m] := \{a_1 \odot v_1 \oplus \cdots \oplus a_m \odot v_m \text{ for } a_1, \ldots, a_m \in \mathbb{R}\}.$$  

However, for finite $m$, $[v_1, \ldots, v_m]$ is always a compact set in $\text{TP}^{n-1} := \mathbb{R}^n / \mathbb{R} \cdot 1$ [DS04]. Thus, one cannot hope for finitely many vectors to ‘tropically span’ $\mathbb{R}^m$. In particular, this suggests that $(1)$ is a better analogue for the tropical convex hull of $v_1, \ldots, v_m$, and that the
natural ambient space for doing tropical convex geometry is not $\mathbb{R}^m$, but $\mathbb{TP}^{n-1}$, where a vector $x \in \mathbb{R}^m$ is identified with all of its scalar multiples $a \odot x$. This is indeed the direction taken by the literature.

Probability theory on classical projective spaces relies on group representation [BQ14]. Unfortunately, there is no satisfactory tropical analogue of the general linear group. Every invertible $n \times n$ matrices with entries in $\mathbb{R}$ is the composition of a diagonal matrix and a permutation of the standard basis of $\mathbb{R}^n$ [KM97]. We note that several authors have studied tropicalization of special linear group over a field with valuation [JSY07, Wer11]. It would be interesting to see whether this can be utilized to define probability measures on $\mathbb{TP}^{n-1}$.

We shall now take each of the first three characterizations of Gaussians in Section 1 as definitions, and utilize them to define tropical Gaussians. For Maxwell and Kac characterizations, we need a concept of orthogonality. One could attempt to mimic orthogonality via the orthogonal decomposition theorem, as done in [Eva01] for the cas e of local fields discussed in Section 2.1. Namely, over a normed space $(Y, \|\cdot\|)$ over some field $K$, say that $y_1, \ldots, y_m \in Y$ are orthogonal if and only if for all $\alpha_i \in K$(2) $\| \sum_i \alpha_i y_i \| = \sum_i |\alpha_i| \| y_i \|$. The natural metric in tropical geometry is the projective Hilbert metric $d_H$ [CGQ00, CGQ04]. In the max-plus algebra, it is defined by

$$d_H(x, y) = \max_{i,j \in [n]} (x_i - y_i + y_j - x_j).$$

It is the logarithmic version of Hilbert’s metric in projective geometry. One main justification is that $d_H$ is compatible with the tropical notion of projection. Let us elaborate. Associated to a tropical polytope $V = [v_1, \ldots, v_m]$ defined in (1) is the canonical projector $P_V : \mathbb{R}^n \to V$, given by [CGQ04] $P_V(x) := \max(v \in V : v_i \leq x_i \text{ for all } i \in [n])$. For vectors $x, y$, the projector $P_{[y]}$ becomes

$$P_{[y]}(x) = x/y = \max(c \odot y : c \odot y \leq x).$$

This can be interpreted as the result of ‘subtracting $y$ from $x$’. For all $y \in \mathbb{TP}^{n-1}$ and $v \in V$, one can show that [CGQ04] $d_H(x, v) \leq d_H(x, P_V(x))$. In particular, if we define $d_H(x, V) = \inf_{v \in V} d_H(x, v)$, then [AGNST11] $d_H(x, V) = d_H(x, P_V(x))$. That is, $P_V(x)$ is a best-approximation of $x$ by points in $V$.

In classical linear algebra, best-approximations in the Euclidean distance can be written as a matrix-vector multiplication. In the tropical case, the max-plus projector $P_V$ also has such an expression when $V = \text{span}(A)$ where $A \in \mathbb{R}^{n \times n}$ is a Kleene star [Ser09]. In this case, $P_V$ is given by matrix-vector multiplication in the min-plus algebra

$$P_V(x) = (-A^\top) \odot x.$$ 

Kleene stars are the nicest possible $n \times n$ matrices from the viewpoint of tropical spectral theory: the image set of a Kleene star is a tropically and classically convex set called polytropes [JK10], whose tropical extreme points are its $n$ eigenvectors, see [BCOQ92, But10]. Polytropes are fundamental building blocks for tropical polytopes [DS04]. They are tightly
connected to the all-pairs shortest path problem [BCOQ92, But10, Tra17], which form vital
topics between tropical methods and various applications [ABGJ18, CT16].

Define the Hilbert projective norm $\| \cdot \|_H : \mathbb{R}^m \to \mathbb{R}$ via $\| x \|_H = d_H(x, 0)$. Since $d_H(x, y) = \max_i (x_i - y_i) - \min_j (x_j - y_j)$, we find that

$$\| x \|_H = \| x - \min_i x_i \|_\infty.$$  

This formulation shows that the projective Hilbert norm plays the role of the $\ell_\infty$-norm on $\mathbb{TP}^{n-1}$. The appearance of $\ell_\infty$, instead of $\ell_2$, agrees with the conventional ‘wisdom’ that generally in the tropical algebra, $\ell_2$ is replaced by $\ell_\infty$ [Eva01].

The $\ell_\infty$-norm, unfortunately, does not work well with the usual notion of independence in probability. This lack of compatibility is to be expected, because the projective Hilbert norm was designed to be compatible with tropical arithmetic. In the Hilbert projective norm, (2) can be interpreted either as

$$\| \max_i (\alpha_i + y_i) \|_H = \max_i \| \alpha_i + y_i \|_H = \max_i \| y_i \|_H \quad (3)$$

or

$$\| \max_i (\alpha_i + y_i) \|_H = \max_i (\alpha_i + \| y_i \|_H). \quad (4)$$

Unfortunately, neither formulation give a satisfactory notion of orthogonality. In (3), as the norm is projective, the coefficients $\alpha_i$’s have disappeared from the RHS. This betrays the notion that over an orthogonal set, computing is norm of linear combinations is like computing norm of the vector of coefficients. In (4), for sufficiently large $\alpha_1$, the RHS increases without bound whereas the LHS is bounded, and thus equality cannot hold for all $\alpha_i \in \mathbb{R}$ over any generating set of $y_i$’s.

The Darmois-Skitovich characterization for Gaussians also does not generalize well. Note that the additive identity in $(\mathbb{R}, \oplus, \odot)$ is either $-\infty$ or $+\infty$, so the condition that $\alpha_i, \beta_i \neq 0$ becomes redundant. The following lemma states that the any compact distribution will satisfy the Darmois-Skitovich condition.

**Lemma 2.2.** Let $X_1, \ldots, X_n$ be independent random variables on $\mathbb{R}^n$. Then there exists $\alpha, \beta \in \mathbb{R}^n$ such that $\bigoplus_{i=1}^n \alpha_i \odot X_i$ and $\bigoplus_{i=1}^n \beta_i \odot X_i$ are independent if and only if $X_1, \ldots, X_n$ have compact support.

**Proof.** Let us sketch the proof for $n = 2$ under the min-plus algebra. Let $X = (X_1, X_2) \in \mathbb{R}^2$ and $Y = (Y_1, Y_2) \in \mathbb{R}^2$ be two independent variables. Define $\overline{F}_X, \overline{F}_Y : \mathbb{R}^2 \to [0, 1]$ via $\overline{F}_X(t) = \mathbb{P}(X \geq t)$ and $\overline{F}_Y(t) = \mathbb{P}(Y \geq t)$. Fix $\alpha, \beta \in \mathbb{R}^2$. Define vector-scalar addition element-wise. For $t \in \mathbb{R}^2$,

$$\mathbb{P}(\alpha_1 \odot X \oplus \alpha_2 \odot Y \geq t) = \mathbb{P}(\min(\alpha_1 + X, \alpha_2 + Y) \geq t) = \mathbb{P}(X \geq t - \alpha_1)\mathbb{P}(Y \geq t - \alpha_2) = \overline{F}_X(t - \alpha_1)\overline{F}_Y(t - \alpha_2).$$
Meanwhile, 
\[
\mathbb{P}(\alpha_1 \odot X \oplus \alpha_2 \odot Y \geq t, \beta_1 \odot X \oplus \beta_2 \odot Y \geq t) \\
= \mathbb{P}(\min(\alpha_1 + X, \alpha_2 + Y) \geq t, \min(\beta_1 + X, \beta_2 + Y) \geq t) \\
= \mathbb{P}(X \geq t - \alpha_1, X \geq t - \beta_1) \mathbb{P}(Y \geq t - \alpha_2, t - \beta_2) \\
= \min(F_X(t - \alpha_1), F_X(t - \beta_1)) \min(F_Y(t - \alpha_2), F_Y(t - \beta_2)).
\]

Therefore, for \( \alpha_1 \odot X \oplus \alpha_2 \odot Y \) and \( \beta_1 \odot X \oplus \beta_2 \odot Y \) to be independent, for all \( t \in \mathbb{R}^2 \), we need 
\[
F_X(t - \alpha_1) F_X(t - \beta_1) F_Y(t - \alpha_2) F_Y(t - \beta_2) \\
= \min(F_X(t - \alpha_1), F_X(t - \beta_1)) \min(F_Y(t - \alpha_2), F_Y(t - \beta_2)) \cdot \max(F_X(t - \alpha_1), F_X(t - \beta_1)) \max(F_Y(t - \alpha_2), F_Y(t - \beta_2)) \\
= \min(F_X(t - \alpha_1), F_X(t - \beta_1)) \min(F_Y(t - \alpha_2), F_Y(t - \beta_2)).
\]

But \( F_X \) and \( F_Y \) are non-increasing functions taking values between 0 and 1. So the LHS is strictly smaller than the RHS, unless if
\[
\max(F_X(t - \alpha_1), F_X(t - \beta_1)) = \max(F_Y(t - \alpha_2), F_Y(t - \beta_2)) = 1,
\]
else we must have
\[
\min(F_X(t - \alpha_1), F_X(t - \beta_1)) = 0, \text{ or } \min(F_Y(t - \alpha_2), F_Y(t - \beta_2)) = 0.
\]

As either of these scenarios must hold for each \( t \in \mathbb{R}^2 \), we conclude that \( X \) and \( Y \) must have compact supports. In that case, one can choose \( \alpha_1 = \beta_2 = 0 \) and \( \alpha_2 = \beta_1 \) be a sufficiently large number, so that
\[
\alpha_1 \odot X \oplus \alpha_2 \odot Y = X, \text{ and } \beta_1 \odot X \oplus \beta_2 \odot Y = Y.
\]

In this case, the Darmois-Skitovich condition holds trivially, as desired. \( \square \)

Now consider Polya’s condition. Here the Gaussian is characterized via stability under addition. When addition is replaced by minimum, it is well-known that this leads to the classical exponential distribution. One such characterization, which generalizes to distributions on arbitrary lattices, is the following [Bry12, Theorem 3.4.1].

**Theorem 2.3.** Suppose \( X, Y \) are independent and identically distributed nonnegative random variables. Then this distribution is the exponential if and only if for all \( a, b > 0 \) such that \( a + b = 1, \min(X/a, Y/b) \) has the same distribution as \( X \).

By consider \( \log(X) \) and \( \log(Y) \), one could restate this theorem in terms of the min-plus algebra, though the condition \( a + b = 1 \) does not have an obvious tropical interpretation. It indicates that in the tropical algebra, searching for the analogue of the exponential distribution maybe a fruitful direction.

### 2.3. Gaussians in idempotent probability

Idempotent probability is a branch of idempotent analysis, which is functional analysis over idempotent semirings [KM97]. It was developed by Litvinov, Maslov and Shipz [LMS98] in relation to problems of calculus of variations. Closely related are the work on large deviations [Puh01], which have found applications in queueing theory, as well as fuzzy measure theory and logic [DP12, WK13]. The work we discussed in this section is based on that of Akian, Quadrat and Viot and co-authors.
The theory of idempotent probability exists in complete parallel to classical probability. All fundamental concepts of probability have an idempotent analogue, see [AQV94] and references therein. For a flavor of this theory, we compare the concept of a measure. In classical settings, a probability measure \( \mu \) is a map from the \( \sigma \)-algebra on a space \( \Omega \) to \( \mathbb{R}_{\geq 0} \) that satisfy three properties: (i) \( \mu(\emptyset) = 0 \), (ii) \( \mu(\Omega) = 1 \), and (iii) for a countable sequence \( (E_i) \) of pairwise disjoint sets,

\[
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).
\]

The analogous object in the min-plus probability is the cost measure \( \mathbb{K} \) defined by three axioms: (i) \( \mathbb{K}(\emptyset) = +\infty \), (ii) \( \mathbb{K}(\Omega) = 0 \), and

\[
\mathbb{K}\left(\bigcup_{i} E_i\right) = \bigoplus_{i} \mathbb{K}(E_i) = \inf_{i} \mathbb{K}(E_i).
\]

Idempotent probability is rich and has interesting connections with dynamic programming and optimization. For instance, tropical matrix-vector multiplication can be interpreted as an update step in a Markov chain, so the Bellman equation plays the analogue of the Kolmogorov-Chapman equation [AQV94]. In particular, the analogue of the Gaussian density in idempotent probability is the classical quadratic form \((x - y)^2 / 2\sigma^2\). In parallel to the classical theory, this density defines a stable distribution, and it is the unique density that is invariant under the Legendre-Fenchel transform, which is the tropical analogue of the Fourier transform.

2.4. **Tropical curves, metric graphs and Brownian motions.** In tropical algebraic geometry, an abstract tropical curve is a metric graph [MZ08]. There are some minor variants: with vertex weights [BMV11, Cha12], or just the compact part [BF06]. An embedded tropical curve is a balanced weighted one-dimensional complex in \( \mathbb{R}^n \).

There are several constructions of tropical curves. In particular, they arise as limits of amoebas through a process called Maslov dequantization in idempotent analysis [LMS98]. Tropical algebraic geometry took off with the landmark paper of Mikhalkin [Mik05], who used tropical curves to compute Gromov-Witten invariants of the plane \( \mathbb{P}^2 \). Since then, tropical curves and more generally, tropical varieties, have been studied in connections to mirror and symplectic geometry [Gro11]. Another heavily explored aspect of tropical curves is their divisors and Riemann-Roch theory [BN07, BP16, GK08, MZ08]. This theory is connected to chip-firing and sandpile, which were initially conceived as deterministic models of random walks on graphs [CS06].

Metric graphs are Riemannian manifolds with singularity [BF06]. Brownian motions defined on metric graph, heat semigroup on graph and graph Laplacian are an active research area [KPS12, Pos08]. As of current, however, the author is unaware of an analogue of the Ornstein-Uhlenbeck semigroup and its invariant measure on graphs. It would also be interesting to study what Brownian motion on graphs reveals about tropical curves and their Jacobian.

2.5. **Further open directions.** One difficult with the idempotent algebra is the lack of the additive inverse. Some authors have put back the additive inverse and developed a theory
of linear algebra in this new algebra, called the supertropical algebra [IR08]. It would be interesting to study matrix groups and their actions under this algebra, and in particular, pursue the definition of Gaussians as invariant measures under actions of the orthogonal group.

In a more applied direction, $\mathbb{T}_p^{n-1}$ is a natural ambient space to study problems in economics, network flow and phylogenetics. Thus one may want an axiomatic approach to finding distributions on $\mathbb{T}_p^{n-1}$ tailored for specific applications. For instance, in shape-constrained density estimation, log-concave multivariate totally positive of order two (MTP2) distributions are those whose density $f : \mathbb{R}^d \to \mathbb{R}$ is log-concave and satisfy the inequality

$$f(x)f(y) \leq f(x \lor y)f(x \land y)$$

for all $x, y \in \mathbb{R}^d$.

A variety of distributions, including the Gaussians, belong to this family. Requiring that such inequalities hold for all $x, y \in \mathbb{T}_p^{n-1}$ leads to the stronger condition of $L^3$-concavity

$$f(x)f(y) \leq f((x + \alpha \mathbf{1}) \lor y)f(x \land (y - \alpha \mathbf{1}))$$

for all $x, y \in \mathbb{R}^d, \alpha \geq 0$.

All Gaussians are log-concave MTP2, however, only diagonally dominant Gaussians are $L^3$-concave [Mur03, §2]. This subclass of densities have nice properties that make them algorithmically tractable in Gaussian graphical models [MJW06, WF01]. In particular, density estimation for $L^3$-concave distributions are significantly easier than that for log-concave MTP2 [RSTU18]. It would be interesting to pursue this direction to define distributions on the space of phylogenetic trees, for instance.

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