A UNIFORM GENERALIZATION OF SOME COMBINATORIAL HOPF ALGEBRAS

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Abstract. We generalize the Hopf algebras of free quasisymmetric functions, quasisymmetric functions, noncommutative symmetric functions, and symmetric functions to certain representations of the category of finite Coxeter systems and its dual category. We investigate their connections with the representation theory of 0-Hecke algebras of finite Coxeter systems. Restricted to type B and D we obtain dual graded modules and comodules over the corresponding Hopf algebras in type A. Representation of categories and free quasisymmetric function and quasisymmetric function and noncommutative symmetric function and symmetric function and Malvenuto–Reutenauer algebra and descent algebra and 0-Hecke algebra and induction and restriction and Coxeter group and type B and type D

1. Introduction

The self-dual graded Hopf algebra Sym of symmetric functions plays a significant role in algebraic combinatorics. Analogues of Sym include the graded Hopf algebra QSym of quasisymmetric functions and its dual graded Hopf algebra NSym of noncommutative symmetric functions, as well as the self-dual graded Hopf algebra FQSym of free quasisymmetric functions. The relation among these Hopf algebras is illustrated by the following commutative diagram.

\[
\begin{array}{ccc}
\text{FQSym} & \xrightarrow{\chi} & \text{QSym} \\
\text{NSym} & \xrightarrow{\text{dual}} & \text{Sym} \\
\end{array}
\]

See Grinberg and Reiner [11] for details.

On the other hand, there is a graded Hopf algebra structure on \(\mathbb{ZS}\), the direct sum of group algebras of the symmetric groups \(S_n\) for all \(n \geq 0\). This is the Malvenuto–Reutenauer Hopf algebra [10]. One has a Hopf algebra isomorphism between FQSym and \(\mathbb{ZS}\), sending NSym onto the descent algebra \(\Sigma(S)\) of type A. There is a surjection \(\chi'\) from \(\mathbb{ZS}\) to the dual \(\Sigma^*(S)\) of \(\Sigma(S)\), and we denote \(\Lambda(S) := \chi'(\Sigma(S))\). Then one has the following commutative diagram of graded Hopf algebras isomorphic to (1.1).

\[
\begin{array}{ccc}
\mathbb{ZS} & \xrightarrow{\chi'} & \Sigma^*(S) \\
\Sigma(S) & \xrightarrow{\text{dual}} & \Lambda(S) \\
\end{array}
\]

There are interesting connections between these Hopf algebras and representation theory. The classic Frobenius correspondence provides a Hopf algebra isomorphism from the Grothendieck group \(G_0(\mathbb{S}_\bullet)\) of the category of finitely generated (complex) representations of the symmetric groups \(\mathbb{S}_n\) to the Hopf algebra Sym. Analogously to this correspondence, Krob and Thibon [14] introduced two characteristic maps

\[
\text{Ch} : G_0(H_\bullet(0)) \xrightarrow{\sim} \text{QSym} \quad \text{and} \quad \text{ch} : K_0(H_\bullet(0)) \xrightarrow{\sim} \text{NSym}
\]

giving Hopf algebra isomorphisms from the Grothendieck groups \(G_0(H_\bullet(0))\) and \(K_0(H_\bullet(0))\) of the categories of finitely generated (projective) representations of the 0-Hecke algebras \(H_n(0)\) of type \(A\) to the dual Hopf
algebras $QSym$ and $NSym$. Here the 0-Hecke algebras are certain deformations of the group algebras of finite Coxeter groups, whose representation theory were studied by Norton [13].

In this paper we generalize the previously mentioned results from type A to finite Coxeter systems. An abstract finite Coxeter system is a pair $(W, S)$ where $W$ is an abstract group with a Coxeter presentation $W = \langle S \mid (st)^{m_{st}} = 1, \forall s, t \in S \rangle$. Each isomorphism class of finite Coxeter systems contains a unique abstract finite Coxeter system. We define a category $Cox$ whose objects are abstract finite Coxeter systems and whose morphisms are embeddings of abstract finite Coxeter systems. Also let $Cox^{op}$ be the opposite/dual category of $Cox$. We define a representation $\Omega (\Sigma \text{ resp.})$ of both $Cox$ and $Cox^{op}$ by sending abstract finite Coxeter systems to their group algebras (descent algebras resp.) and sending embeddings of abstract finite Coxeter systems to certain linear maps which generalize the product and coproduct of the Malvenuto–Reutenauer Hopf algebra (descent algebra of type A resp.). The inclusion of the descent algebras into the group algebras induces a natural transformation $\iota: \Sigma \to \Omega$. On the other hand, we have a representation $\Sigma^*$ dual to $\Sigma$ and a natural transformation $\chi': \Omega \to \Sigma^*$ dual to $\iota$. Furthermore, applying $\chi' \circ \iota$ to $\Sigma$ gives a representation $\Lambda$ of both $Cox$ and $Cox^{op}$ such that the diagram below is commutative.

(1.4)

Restricting this diagram to type A recovers the commutative diagram (1.2) of Hopf algebras.

Next, using the $P$-partition theory for finite Coxeter groups by Reiner [22], we generalize $FQSym$ to a representation $FQSym$ of both $Cox$ and $Cox^{op}$. There is a natural isomorphism between $\Omega$ and $FQSym$. Applying it to (1.4) gives the following commutative diagram, which is in natural isomorphism with (1.4).

(1.5)

Restricting this diagram to type A recovers the commutative diagram (1.1) of Hopf algebras.

We also obtain representations $G_0$ and $K_0$ of both $Cox$ and $Cox^{op}$ from the representation theory of 0-Hecke algebras of finite Coxeter groups, together with natural isomorphisms $Ch: G_0 \to QSym$ and $ch: K_0 \to N Sym$.

Restricting $Ch$ and $ch$ to type A recovers the characteristic maps in (1.3).

Our generalization not only recovers known results on Hopf algebras in type A, but also leads to new results in type B and type D on certain modules and comodules over the corresponding Hopf algebras in type A. We summarize our results in type B below; the results in type D are similar. In fact, restricting (1.4) and (1.5) to type B gives the following commutative diagrams.

Here each entry is a graded right module and comodule over the corresponding type A Hopf algebra. We also have characteristic maps analogous to (1.3), providing isomorphisms of graded modules and comodules:

$Ch^B: G_0(H_0^B(0)) \xrightarrow{\sim} QSym^B$ and $ch^B: K_0(H_0^B(0)) \xrightarrow{\sim} N Sym^B$.

In our earlier work [12] we obtained partial results on these characteristic maps using a tableau approach to the representation theory of 0-Hecke algebras; now we are able to get more complete results on them.
For each fixed finite Coxeter system \((W, S)\), Aguiar and Mahajan [1, Theorem 5.7.1] obtained a commutative diagram of vector spaces over a field of characteristic zero by a very interesting approach different from ours. Their result agrees with what we have when applying our diagrams (1.4) and (1.5) to the fixed finite Coxeter system \((W, S)\). This allows us to obtain free \(\mathbb{Z}\)-bases for \(\Lambda(W) \cong \text{Sym}(W)\). See Section 2.4.

We do not have a representation theoretic interpretation for \(\text{Sym}(W, S)\) except in type A. It is mentioned in [1, §5.1.2] that the construction of Hopf algebra structures is special to type A. In this paper we use embeddings and restrictions of Coxeter systems to realize (1.4) and (1.5) as commutative diagrams of representations of the category \(\text{Cox}\) and its opposite category \(\text{Cox}^{op}\). It is natural from our perspective that we do not get Hopf algebras in type B and D, as parabolic subsystems in type A are disjoint unions of Coxeter systems of the same type but this is not the case in type B and D.

2. Uniform Results

In this section we provide our main results for finite Coxeter systems.

2.1. Group algebras and descent algebras of finite Coxeter groups. A finite Coxeter group is a finite group \(W\) with a Coxeter presentation

\[
W := \langle S \mid (st)^{m_{st}} = 1, \forall s, t \in S \rangle
\]

where \(S\) is a finite generating set, \(m_{ss} = 1\) for all \(s \in S\), and \(m_{st} = m_{ts} \in \{2, 3, \ldots\}\) for all \(s, t \in S\). The relations for \(W\) are equivalent to the quadratic relations \(s^2 = 1\) for all \(s \in S\) and the braid relations \((sts \cdots)_{m_{st}} = (tst \cdots)_{m_{st}}\) for all \(s, t \in S\), where \((aba \cdots)_m\) denotes an alternating product of \(m\) terms.
The pair \((W, S)\) is called a finite Coxeter system, which is encoded by an edge-labeled graph called the Coxeter diagram of \((W, S)\). The vertex set of the Coxeter diagram of \((W, S)\) is \(S\) and there is an edge labeled with \(m_{st}\) connecting two vertices \(s\) and \(t\) if \(m_{st} \geq 3\). When \(m_{st} \in \{3, 4, 5\}\) the labeled edge between \(s\) and \(t\) is often drawn as \(m_{st} - 2\) multiple edges.

If the Coxeter diagram of \((W, S)\) is connected then \((W, S)\) is irreducible. There is a well-known classification of finite irreducible Coxeter systems. In general if \(S_1, \ldots, S_k\) are the vertex sets of the connected components of the Coxeter diagram of \((W, S)\) then \((W_{S_i}, S_i)\) is an irreducible Coxeter system for each \(i \in [k] := \{1, \ldots, k\}\) and \(W \cong W_{S_1} \times \cdots \times W_{S_k}\).

Every \(w \in W\) can be written as \(w = s_1 \cdots s_k\) where \(s_1, \ldots, s_k \in S\); if \(k\) is minimum then such an expression is called a reduced expression of \(w\) and \(\ell(w) := k\) is the length of \(w\). A generator \(s \in S\) is a descent of \(w\) if \(\ell(ws) < \ell(w)\), or equivalently, if some reduced expression of \(w\) ends with \(s\). The descent set of \(w\), denoted by \(D(w)\), consists of all descents of \(w\).

Let \(I \subseteq S\) and denote \(I^c := S \setminus I\). The parabolic subgroup \(W_I\) of \(W\) is generated by \(I\). It is also a finite Coxeter group whose elements have the same lengths and descents as in \(W\). A left or right \(W_I\)-coset in \(W\) has a unique representative of minimal length. The length-minimal representatives for the left and right \(W_I\)-cosets in \(W\) are given respectively by

\[
W^I := \{w \in W : D(w) \subseteq I^c\} \quad \text{and} \quad W_I^I := \{w \in W : D(w^{-1}) \subseteq I^c\}.
\]

**Proposition 2.1.1** ([5] Proposition 2.4.4, [15] Lemma 9.7). Every element \(w \in W\) can be written uniquely as \(w = w^I \cdot _I w\) such that \(w^I \in W^I\) and \(_I w \in W_I\), and this expression implies \(\ell(w) = \ell(w^I) + \ell(_I w)\). Similarly, every element \(w \in W\) can be written uniquely as \(w = w^I \cdot _I w\) such that \(w^I \in W_I\) and \(_I w \in W_I^I\), and this expression implies \(\ell(w) = \ell(w^I) + \ell(_I w)\).

The group algebra \(ZW\) is self-dual under the positive definite bilinear form defined by \(\langle u, v \rangle := \delta_{u,v}\) for all \(u, v \in W\). For convenience of notation we identify a subset of \(W\) with \(I\) if \(I\) is a subset of \(W\). Every element \(g \in G\) is self-dual under the positive definite bilinear form defined by \(\langle u, v \rangle := \delta_{u,v}\) for all \(u, v \in W\).

**Definition 2.1.2.** Given \(I \subseteq S\), we define the following four linear maps:

\[
\begin{align*}
\mu_I^S : ZW_I &\rightarrow ZW, & \rho_I^S : ZW &\rightarrow ZW_I, \\
\mu_I^S : ZW_I &\rightarrow ZW, & \rho_I^S : ZW &\rightarrow ZW_I,
\end{align*}
\]

**Proposition 2.1.3.** The following diagram is commutative and rotating it 180° gives a dual diagram, i.e., \(\mu_I^S\) is dual to \(\tilde{\mu}_I^S\), \(\rho_I^S\) is dual to \(\tilde{\rho}_I^S\), and \((\ )^{-1}\) is self-dual.

**Proof.** The result follows from Proposition 2.1.1.

**Lemma 2.1.4.** If \(I \subseteq S\) and \(w \in W\) then \(D(w) \cap I = D(\_I w)\).

**Proof.** Assume \(s \in I\). Then \(\ell(w) = \ell(w^I) + \ell(_I w)\) and \(\ell(ws) = \ell(w^I) + \ell(_I w)\) by Proposition 2.1.1. Hence \(\ell(ws) < \ell(w)\) if and only if \(\ell(_I ws) < \ell(_I w)\). The result follows.

**Proposition 2.1.5.** If \(I \subseteq J \subseteq S\) then \(\mu_J^S \circ \mu_I^J = \mu_I^S\), \(\mu_J^S \circ \mu_I^J = \mu_I^S\), \(\rho_J^S \circ \rho_I^J = \rho_I^S\), and \(\rho_J^S \circ \rho_I^J = \rho_I^S\).

**Proof.** Assume \(I \subseteq J \subseteq S\). Every element \(z \in W\) can be written uniquely as \(z = z' \cdot _J z\). Lemma 2.1.4 implies \(D(\_J z) = D(z) \cap J\). Hence \(z \in W^J\) if and only if \(_J z \in W^J\). Then for any \(u \in W_I\) one has

\[
\mu_J^S(\mu_I^J(u)) = W^J \cdot W^J \cdot u = W^J \cdot u = \mu_J^S(u).
\]

Hence \(\mu_J^S \circ \mu_I^J = \mu_I^S\). By Proposition 2.1.3 applying the maps \((\ )^{-1}\) to this gives \(\tilde{\mu}_J^S \circ \tilde{\mu}_I^J = \tilde{\mu}_I^S\), and then taking the duals gives \(\rho_J^S \circ \rho_I^J = \rho_I^S\) and \(\tilde{\rho}_J^S \circ \tilde{\rho}_I^J = \tilde{\rho}_I^S\).
Definition 2.1.6. An abstract finite Coxeter system is a pair \((W, S)\) where \(W\) is an abstract group with a Coxeter presentation \(W = \langle S \mid (st)^{n_{st}} = 1, \forall s, t \in S \rangle\). We identify a finite Coxeter system with the unique abstract finite Coxeter system isomorphic to it. We define a category \(\text{Cox}\) whose objects are abstract finite Coxeter systems and whose morphisms are embeddings of abstract finite Coxeter systems. Given such an embedding \((W', S') \hookrightarrow (W, S)\), there exists a unique \(I \subseteq S\) such that \((W', S') \cong (W_I, I)\), and thus we can identify this embedding with the inclusion \((W_I, I) \hookrightarrow (W, S)\). We also denote by \(\text{Cox}^{op}\) the opposite/dual category of \(\text{Cox}\) whose morphisms are restrictions \((W, S) \rightarrow (W_I, I)\) with \(I \subseteq S\).

A representation of a category \(C\) is a covariant functor \(R\) from \(C\) to the category of \(\mathbb{Z}\)-modules. The dual representation \(R^*\) of \(R\) is a representation of the dual category \(\text{C}^{op}\) which sends each object \(O\) to the dual \(\mathbb{Z}\)-module \(R(O)^*\) and sends each morphism \(f^* : O_2 \rightarrow O_1\) dual to \(f : O_1 \rightarrow O_2\) to the morphism \(R(f)^* : R(O_2)^* \rightarrow R(O_1)^*\) dual to \(R(f) : R(O_1) \rightarrow R(O_2)\). See commutative diagrams below.

![Commutative Diagram]

\[
\begin{array}{ccc}
O_1 & \xrightarrow{f} & O_2 \\
\downarrow{R} & & \downarrow{R} \\
R(O_1) & \xrightarrow{R(f)} & R(O_2)
\end{array}
\quad
\begin{array}{ccc}
O_1 & \xrightarrow{f^*} & O_2 \\
\downarrow{R^*} & & \downarrow{R^*} \\
R(O_1)^* & \xrightarrow{R(f)^*} & R(O_2)^*
\end{array}
\]

Definition 2.1.7. By Proposition 2.1.8 we have a representation \(\Omega\) of \(\text{Cox}\) sending an abstract finite Coxeter system \((W, S)\) to \(\mathbb{Z}W\) and sending an inclusion \((W_I, I) \hookrightarrow (W, S)\) to \(\mu_I^S\) whenever \(I \subseteq S\). By abuse of notation, also let \(\Omega\) be the representation of \(\text{Cox}^{op}\) sending an abstract finite Coxeter system \((W, S)\) to \(\mathbb{Z}W\) and sending a restriction \((W, S) \rightarrow (W_I, I)\) to \(\rho_I^S\) whenever \(I \subseteq S\). By Proposition 2.1.8 replacing \(\mu_I^S\) and \(\rho_I^S\) gives the dual representation \(\Omega^*\) of \(\Omega\), and \(w \mapsto w^{-1}\) induces a natural isomorphism between \(\Omega\) and \(\Omega^*\) denoted by \(\cdot^*\).

Given \(I \subseteq S\), the descent class of \(I\) in \(W\) is \(D_I(W) := \{w \in W \mid D(w) = I\}\).

Proposition 2.1.8. For every \(I \subseteq S\) the descent class \(D_I(W)\) is nonempty and becomes an interval \([w_0(I), w_1(I)]\) under the left weak order of \(W\), where \(w_0(I)\) and \(w_1(I)\) are the longest elements in \(W_I\) and \(W^I\), respectively.

Proof. By Lusztig [15] Lemma 9.8], the parabolic subgroup \(W_I\) has a unique longest element \(w_0(I)\) satisfying \(w_0(I) = w_0(I)^{-1}\) and \(D(w_0(I)) = I\). Hence \(D_I(W) \neq \emptyset\). The rest of the result follows from Björner and Wachs [14] Theorem 6.2. \(\square\)

We identify \(D_I(W)\) with the sum of its elements in the group algebra \(\mathbb{Z}W\). Let \(\Sigma(W)\) be the \(\mathbb{Z}\)-span \(\{D_I(W) \mid I \subseteq S\}\). Proposition 2.1.8 implies that this spanning set is indeed a free \(\mathbb{Z}\)-basis for \(\Sigma(W)\). One has an injection \(\iota : \Sigma(W) \hookrightarrow \mathbb{Z}W\) by inclusion. Solomon [23] showed that \(\Sigma(W)\) is a subalgebra of the group algebra \(\mathbb{Z}W\), called the descent algebra of \(W\). We will not use this algebra structure in this paper, but instead we will restrict the linear maps \(\mu_I^S\) and \(\rho_I^S\) to \(\Sigma(W_I)\) and \(\Sigma(W)\).

On the other hand, let \(\Sigma^*(W)\) be the dual \(\mathbb{Z}\)-module of \(\Sigma(W)\) with a free \(\mathbb{Z}\)-basis \(\{D_I^*(W) \mid I \subseteq S\}\) dual to \(\{D_I(W) \mid I \subseteq S\}\). If \(w \in W\) then write \(D_w^*(W) := D_I^*(W)\) where \(I = D(w)\). Dual to the injection \(\iota : \Sigma(W) \hookrightarrow \mathbb{Z}W\) is a surjection

\[
\chi : \mathbb{Z}W \twoheadrightarrow \Sigma^*(W), \quad w \mapsto D_w^*(W).
\]

Proposition 2.1.9. Let \(I \subseteq S\). Then \(\mu_I^S : \mathbb{Z}W_I \rightarrow \mathbb{Z}W\) restricts to \(\mu_I^S : \Sigma(W_I) \rightarrow \Sigma(W)\) and \(\rho_I^S : \mathbb{Z}W \rightarrow \mathbb{Z}W_I\) descends to

\[
\begin{align*}
\bar{\mu}_I^S & : \Sigma^*(W_I) \rightarrow \Sigma^*(W), \\
D_w^*(W_I) & \mapsto D_w^*(W_I).
\end{align*}
\]

If \(J \subseteq I\) and \(K \subseteq S\) then

\[
\bar{\mu}_I^S(D_J(W_I)) = \sum_{J \cap I = J} D_J(W) \quad \text{and} \quad \bar{\rho}_I^S(D_K^*(W)) = D_K^*(W_I).
\]

Proof. The result follows easily from Proposition 2.1.8 and Lemma 2.1.4. \(\square\)

To restrict \(\rho_I^S\) to descent algebras we need to first develop some lemmas.

Lemma 2.1.10. Suppose that \(I \subseteq S, z \in W^I, s \in D(z^{-1})^c, \) and \(sz \notin W^I\). Then \(D(sz) \cap I = \{z^{-1}sz\}\).
Proof. One has $D(sz) \cap I \neq \emptyset$ since $sz \notin W^I$. Let $r \in D(sz) \cap I$. Then $\ell(szr) < \ell(sz)$ and $\ell(zr) > \ell(z)$. One also has $\ell(sz) > \ell(z)$ since $s \in D(z^{-1})^c$. By Lusztig [15, Proposition 1.10] (applied to $w = z$ and $t = r$), one has $sz = zr$, i.e., $r = z^{-1}sz$. □

Lemma 2.1.11. Let $I, K \subseteq S$, $u \in W_I$, and $z \in \ell W$. Then $D(uz) = K$ if and only if (i)–(iii) all hold.

(i) If $s \in D(z)$ then $s \in K$, i.e. $D(z) \subseteq K$.

(ii) If $s \in D(z)^c$ and $D(sz^{-1}) \subseteq I^c$ then $s \in K^c$.

(iii) If $s \in D(z)^c$ and $D(sz^{-1}) \not\subseteq I^c$ then $s \in K \Leftrightarrow zsz^{-1} \in D(u)$.

Moreover, condition (iii) is equivalent to:

(iii') Let $L(z, I, K) \subseteq D(u) \subseteq L'(z, I, K)$, where $L(z, I, K)$ and $L'(z, I, K)$ are two subsets of $I$ defined as

$$L(z, I, K) := \bigcup_{s \in D(z)^c \cap K} (D(sz^{-1}) \cap I) \text{ and } L'(z, I, K) := \bigcap_{s \in D(z)^c \setminus K} (I \setminus D(sz^{-1})).$$

Consequently, $z \in \ell D_K := \{\ell w : w \in D_K(W)\}$ if and only if (i), (ii), and $L(z, I, K) \subseteq L'(z, I, K)$.

Proof. If $s \in D(z)$ then $\ell(uzs) \leq \ell(u) + \ell(sz) < \ell(u) + \ell(z) = \ell(uz)$. Hence $D(uz) = K$ implies (i).

If $s \in D(z)^c$ and $D(sz^{-1}) \subseteq I^c$ then $zs \in \ell W$ and thus $\ell(uzs) = \ell(u) + \ell(sz) > \ell(uz)$. Hence $D(uz) = K$ implies (ii).

If $s \in D(z)^c$ and $D(sz^{-1}) \not\subseteq I^c$, then Lemma 2.1.10 implies that $D(sz^{-1}) \cap I = \{r\}$ where $r = zsz^{-1}$. Since

$$\ell(uzs) = \ell(izr) = \ell(u) + \ell(z) \quad \text{ and } \quad \ell(uz) = \ell(u) + \ell(z)$$

one has $s \in D(uz)$ if and only if $r \in D(u)$. Hence $D(uz) = K$ implies (iii).

Conversely, one sees that (i)–(iii) imply $s \in D(uz) \Leftrightarrow s \in K$ in the above three cases. Thus $D(uz) = K$ is equivalent to (i)–(iii).

Next, assume (iii') holds. Let $s \in D(z)^c$ such that $D(sz^{-1}) \not\subseteq I^c$. Then $D(sz^{-1}) \cap I = \{r = zsz^{-1}\}$ by Lemma 2.1.10. If $s \in K$ then $r \in L(z, I, K) \subseteq D(u)$. If $s \in K^c$ then $r \in I \setminus L'(z, I, K) \subseteq I \setminus D(u)$. Thus (iii) holds.

Conversely, assume (iii) holds and let $s \in D(z)^c$. If there exists $r = zsz^{-1}$ then $r \in D(u)$ if only if $s \in K$. This implies $L(z, I, K) \subseteq D(u)$ and $I \setminus L'(z, I, K) \subseteq I \setminus D(u)$. Thus (iii') holds.

Finally, $z \in \ell D_K$ if and only if there exists an element $u \in W_I$ such that $D(uz) = K$. We know $D(uz) = K$ is equivalent to (i), (ii), and (iii'). By Proposition 2.1.12 any descent class is nonempty. Thus the existence of $u$ is equivalent to (i), (ii), and (iii'), and $L(z, I, K) \subseteq L'(z, I, K)$.

□

Proposition 2.1.12. Let $I \subseteq S$. Then $\rho^{S}_I : ZW \rightarrow ZW_I$ restricts to $\rho^{S}_{I} : \Sigma(W) \rightarrow \Sigma(W_I)$ such that

$$\rho^{S}_I(D_K(W)) = \sum_{z \in \ell D_K} \sum_{L(z, I, K) \subseteq K' \subseteq L'(z, I, K)} D_{K'}(W_I), \quad \forall K \subseteq S$$

and $\bar{\rho}^{S}_I : ZW_I \rightarrow ZW$ descends to the following map dual to $\rho^{S}_I : \Sigma(W) \rightarrow \Sigma(W_I)$:

$$\bar{\rho}^{S}_I : \Sigma^*(W_I) \rightarrow \Sigma^*(W), \quad D^{*}_{I}(W) \rightarrow \sum_{z \in \ell W} D^{*}_{I}(zaz(W)).$$

Proof. Let $w \in W$ with $w_I = u$ and $\ell w = z$. By Lemma 2.1.11 one has $D(w) = K$ if and only if $z \in \ell D_K$ and $L(z, I, K) \subseteq D(u) \subseteq L'(z, I, K)$. Thus $\rho^{S}_I : ZW \rightarrow ZW_I$ restricts to $\rho^{S}_{I} : \Sigma(W) \rightarrow \Sigma(W_I)$ and the desired formula for $\rho^{S}_I(D_K(W))$ holds.

Next, fix $z \in \ell W$. Lemma 2.1.11 implies that $D(uz)$ depends only on $z$ and $D(u)$ for any $u \in W_I$. Hence $\rho^{S}_I : ZW \rightarrow ZW$ descends to $\bar{\rho}^{S}_I : \Sigma^*(W_I) \rightarrow \Sigma^*(W)$.

Finally, if $K \subseteq S$ and $u \in W_I$ then

$$\langle D_K(W), \rho^{S}_{I}(D^{*}_{I}(W_I)) \rangle = \# \{ z \in \ell W : D(uz) = K \} \quad \text{ and } \quad \langle \rho^{S}_I(D_K(W)), D^{*}_{I}(W_I) \rangle = \# \{ z \in \ell D_K : L(z, I, K) \subseteq D(u) \subseteq L'(z, I, K) \} .$$

The two sets on the right hand side are the same by Lemma 2.1.11. The duality follows. □
Theorem 2.1.13. The diagram below is commutative and rotating it $180^\circ$ gives a dual diagram.

\[
\begin{array}{cccc}
\Sigma(W) &\rightarrow & \Sigma(W) &\rightarrow \Sigma^*(W) \\
\downarrow \mu^\delta_i & & \downarrow \mu^\delta_i & \downarrow \chi^* \\
\Sigma(W) &\rightarrow & \Sigma(W) &\rightarrow \Sigma^*(W) \\
\downarrow \rho^\delta_i & & \downarrow \rho^\delta_i & \downarrow \rho^\delta_i \\
\Sigma(W)^{-1} &\rightarrow & \Sigma(W)^{-1} &\rightarrow \Sigma^*(W) \\
\end{array}
\]

Proof. Combine the previous results obtained in this subsection. \qed

Definition 2.1.14. By Theorem 2.1.13 one has a representation $\Sigma$ of both the category $\text{Cox}$ and its dual category $\text{Cox}^{\text{op}}$ by sending each abstract finite Coxeter system $(W, S)$ to $\Sigma(W)$ and sending an inclusion $(W_I, I) \rightarrow (W, S)$ to $\mu^\delta_i : \Sigma(W_I) \rightarrow \Sigma(W)$ and a restriction $(W, S) \rightarrow (W_I, I)$ to $\rho^\delta_i : \Sigma(W) \rightarrow \Sigma(W_I)$ whenever $I \subseteq S$. By Theorem 2.1.13 replacing $(W, S)$, $\mu^\delta_i$, and $\rho^\delta_i$ with $\Sigma^*(W)$, $\bar{\mu}^\delta_i$, and $\bar{\rho}^\delta_i$ gives the dual representation $\Sigma^*$ of $\Sigma$. Theorem 2.1.13 also gives dual natural transformations $\iota : \Omega \rightarrow \Sigma^*$ and $\chi : \Omega^* \rightarrow \Sigma^*$, as well as a natural transformation $\chi' : \Omega \rightarrow \Sigma^*$ induced by $\chi' : \Sigma(W)^{-1} \rightarrow \Sigma(W)^{-1}$.

Lastly, let $\Lambda(W) := \chi'(\Sigma(W))$. Then $\Lambda(W)$ is spanned by $A_I(W) := \chi'(D_I(W)) = \sum_{w \in D_I(W)} D_{\omega^{-1}}^{\omega}(W), \quad \forall I \subseteq S.

Bases for $\Lambda(W)$ will be provided in Section 2.3. We have the following commutative diagram of $\mathbb{Z}$-modules.

(2.1)

We define a bilinear form on $\Lambda(W)$ by $\langle A_I(W), A_J(W) \rangle := c_{IJ}$ for all $I, J \subseteq S$, where $c_{IJ} := \# \{ w \in W : D(w^{-1}) = I, \ D(w) = J \}$.

The proof the following proposition is a straightforward exercise, left to the reader.

Proposition 2.1.15. The above bilinear form on $\Lambda(W)$ is well defined, symmetric, and nondegenerate. With this bilinear form the injection $\iota : \Lambda(W) \hookrightarrow \Sigma^*(W)$ and the surjection $\chi' : \Sigma(W) \twoheadrightarrow \Lambda(W)$ are dual to each other.

By Theorem 2.1.13 the linear maps $\bar{\mu}^\delta_i : \Sigma^*(W_I) \rightarrow \Sigma^*(W)$ and $\bar{\rho}^\delta_i : \Sigma^*(W) \rightarrow \Sigma^*(W_I)$ restrict to linear maps $\bar{\mu}^\delta_i : \Lambda(W_I) \rightarrow \Lambda(W)$ and $\bar{\rho}^\delta_i : \Lambda(W) \rightarrow \Lambda(W_I)$. More precisely, if $J \subseteq I \subseteq S$ and $K \subseteq S$ then

$\bar{\mu}^\delta_i(\Lambda_I(W)) := \bar{\mu}^\delta_i(\chi'(D_J(W_I))) = \chi'(\mu^\delta_i(D_J(W_I))),$

$\bar{\rho}^\delta_i(\Lambda_K(W)) := \bar{\rho}^\delta_i(\chi'(D_K(W))) = \chi'(\rho^\delta_i(D_K(W))).$

It follows from Proposition 2.1.9 and Proposition 2.1.12 that

$\bar{\mu}^\delta_i(\Lambda_J(W_I)) = \sum_{J' \cap I = J} A_{J'}(W) \quad \text{and} \quad \bar{\rho}^\delta_i(\Lambda_K(W)) = \sum_{z \in D_K} \sum_{L(z, I, K) \subseteq L(z, I, K)} \Lambda_K'(W).$

By diagram chasing in Theorem 2.1.13 one checks that $\mu^\delta_i : \Sigma(W_I) \rightarrow \Sigma(W)$ and $\rho^\delta_i : \Sigma(W) \rightarrow \Sigma(W_I)$ also descend to $\bar{\mu}^\delta_i : \Lambda(W_I) \rightarrow \Lambda(W)$ and $\bar{\rho}^\delta_i : \Lambda(W) \rightarrow \Lambda(W_I)$, and thus the following result holds.
Corollary 2.1.16. The diagram below is commutative and rotating it 180° gives a dual diagram.

\[
\begin{array}{c}
\Sigma(W_1) \xrightarrow{\chi'} \Lambda(W_1) \xrightarrow{\iota} \Sigma^*(W_1) \\
\downarrow \mu_1^W \quad \downarrow \rho_1^W \\
\Sigma(W) \xrightarrow{\chi'} \Lambda(W) \xrightarrow{\iota} \Sigma^*(W) \\
\downarrow \rho_i^W \\
\Sigma(W_I) \xrightarrow{\chi'} \Lambda(W_I) \xrightarrow{\iota} \Sigma^*(W_I)
\end{array}
\]

Definition 2.1.17. We define a representation \( \Lambda \) of both the category \( \text{Cox} \) and its dual category \( \text{Cox}^{op} \) by sending each abstract finite Coxeter system \((W, S)\) to \( \Lambda(W) \) and sending an inclusion \((W_1, I) \hookrightarrow (W, S)\) to \( \rho_i^W : \Lambda(W_1) \to \Lambda(W) \) and a restriction \((W, S) \to (W_I, I)\) to \( \mu_i^W : \Lambda(W) \to \Lambda(W_I) \) whenever \( I \subseteq S \). There are also dual natural transformations \( \iota : \Lambda \to \Sigma^* \) and \( \chi' : \Sigma \to \Lambda \).

In summary, we have a commutative diagram of representations of categories below.

\[
\begin{array}{c}
\Sigma \\
\downarrow \iota
\end{array}
\xrightarrow{\Omega}
\begin{array}{c}
\Lambda \\
\downarrow \chi' \\
\Sigma^*
\end{array}
\]

\(\downarrow \text{dual}\)

\[\begin{array}{c}
\Sigma \\
\downarrow \chi \\
\Lambda
\end{array}\]

2.2. \textbf{P-partitions and free quasisymmetric functions.} In this subsection we generalize free quasisymmetric functions from type \( \Lambda \) to finite Coxeter groups. We first review a generalization of the \( P \)-partition theory by Reiner \[22\], with some slight but not essential modifications. See Humphreys \[13\] for details on root systems.

Let \( E = \mathbb{R}^n \) be a finite dimensional (real) Euclidean space with standard inner product \((-,-)\). A \textit{root system} is a finite set \( \Phi \subset E \setminus \{0\} \) such that

- if \( \alpha \in \Phi \) then \( \mathbb{R} \alpha \cap \Phi = \{ \pm \alpha \} \), and
- if \( \alpha, \beta \in \Phi \) then \( s_\alpha \beta := \beta - \frac{2(\alpha, \beta)}{\langle \alpha, \alpha \rangle} \alpha \in \Phi \).

Here \( s_\alpha \) is the reflection across the hyperplane \( H_\alpha \) perpendicular to \( \alpha \). The elements of \( \Phi \) are called \textit{roots}.

One can choose a linearly independent subset \( \Delta \subset \Phi \), whose elements are called \textit{simple roots}, such that every root is either \textit{positive} or \textit{negative}, meaning that it is a linear combination of simple roots with coefficients either all nonnegative or all nonpositive. We do not require \( \Delta \) to be a basis for \( E \), as one can always restrict to the subspace of \( E \) spanned by \( \Delta \) if needed. We write \( \alpha > 0 \) (\( \alpha < 0 \) resp.) if \( \alpha \) is a positive (negative resp.) root. The root system \( \Phi \) is the disjoint union of the set \( \Phi^+ \) of positive roots and the set \( \Phi^- \) of negative roots. Let \( W \) be the group generated by the set \( S \) of \textit{simple reflections} \( s_\alpha \) for all \( \alpha \in \Delta \). Then \((W, S)\) is a finite Coxeter system. Note that \( \Phi = W \Delta \[13\] \S 1.5].

Conversely, every finite Coxeter system \((W, S)\) can be obtained in above way, i.e., one can realize \( W \) as a group generated by a set \( S \) of simple reflections of a Euclidean space \( E = \mathbb{R}^n \). This is called a \textit{geometric realization} of \((W, S)\). The simple root \( \alpha \in \Delta \) corresponding to a simple reflection \( s = s_\alpha \in S \) is denoted by \( \alpha_s \). The action of \( W \) on \( E \) is orthogonal, i.e., \( (wf, wg) = (f, g) \) for all \( w \in W \) and all \( f, g \in E \).

Proposition 2.2.1 \[13\] \S 1.6, \S 1.7]. Let \( s \in S \) and \( w \in W \). Then \( \ell(ws) > \ell(w) \Leftrightarrow w(\alpha_s) > 0 \).

In general there are different geometric realizations of the same finite Coxeter system. In this paper we fix one geometric realization for each abstract finite irreducible Coxeter system. We will explicitly give a “standard” geometric realization for type \( A, B, \) and \( D \) in \S 3.2 \S 4.2 and \S 5.2 but our main results are still valid if a different geometric realization is fixed. For other types the choice of a geometric realization is arbitrary for our purposes.

Now suppose that \((W, S)\) is an arbitrary abstract finite Coxeter system. Let \( S_1, \ldots, S_k \) be the vertex sets of the connected components of the Coxeter diagram of \((W, S)\). For each \( i \in [k] \) we have already fixed a geometric realization of the irreducible \( W_{S_i} \) as a reflection group of a Euclidean space \( \mathbb{R}^{n_i} \). This gives a realization of \( W \cong W_{S_1} \times \cdots \times W_{S_k} \) as a reflection group of the Euclidean space \( E = \mathbb{R}^n \) where
n = n_1 + \cdots + n_k$. Let $\Phi$ be the root system associated with this geometric realization of $(W, S)$ and fix a set $\Delta$ of simple roots.

Reiner [22] defined a parset (partial root system) of $\Phi$ to be a subset $P \subseteq \Phi$ such that

- if $\alpha \in P$ then $-\alpha \notin P$, and
- if $\sum_{i=1}^{k} c_i \alpha_i \in \Phi$ where $\alpha_i \in P$ and $c_i > 0$ for all $i = 1, \ldots, k$, then $\sum_{i=1}^{k} c_i \alpha_i \in P$.

Let $P \subseteq \Phi$ be a parset. The Jordan-Hölder set of $P$ is $\mathcal{L}(P) := \{w \in W : P \subseteq w \Phi^+\}$. Denote by $\mathcal{A}(P)$ the set of all $P$-partitions, where a $P$-partition is a function $f : [n] \to \mathbb{Z}$, identified with a vector $(f(1), \ldots, f(n)) \in \mathbb{Z}^n$, such that

- $(\alpha, f) \geq 0$ for all $\alpha \in P$, and
- $(\alpha, f) > 0$ for all $\alpha \in P \cap \Phi^-$.

The next result is well known in type $A$ and actually holds for all finite Coxeter systems. A proof can be found in [22, Proposition 3.1.1], which does not depend on the extra axiom for a root system adopted in [22]: $\Delta$ is a basis for $E$.

**Theorem 2.2.2** (Fundamental Theorem of $P$-partitions [22, Proposition 3.1.1]). For every parset $P$ of $\Phi$, the set $\mathcal{A}(P)$ of $P$-partitions is the disjoint union of $\mathcal{A}(w \Phi^+)$ for all $w \in \mathcal{L}(P)$.

Let $X = \{x_i : i \in \mathbb{Z}\}$ be a set of noncommutative variables. The free associative algebra $\mathbb{Z}[X]$ generated by $X$ has a free $\mathbb{Z}$-basis $\{x_a : a \in \mathbb{Z}^n, n \geq 0\}$, where $x_{a_1 \cdots a_n} := x_{a_1} \cdots x_{a_n}$. Denote by $\mathsf{FQSym}^S$ the $\mathbb{Z}$-span of $F_P^S$ for all parsets $P$ of $\Phi$, where

$$F_P^S := \sum_{\alpha \in \mathcal{A}(P)} x_{f(1)} \cdots x_{f(n)}$$

is the (noncommutative) generating function of $P$. Let $F_w^S := F_{w \Phi^+}^S$ and $s_w^S := F_{w^{-1}}^S$ for all $w \in W$.

**Lemma 2.2.3.** The set $\mathbb{Z}^n$ is a disjoint union of nonempty subsets: $\mathbb{Z}^n = \bigcup_{w \in W} \mathcal{A}(w \Phi^+)$.\hfill\Box

**Proof.** Applying Theorem 2.2.2 to $P = \emptyset$ one partitions $\mathbb{Z}^n$ into a disjoint union of $\mathcal{A}(w \Phi^+)$ for all $w \in W$. Since the intersection $\bigcap_{\alpha \in \Delta} \{f \in \mathbb{Z}^n : (f, \alpha) > 0\}$ is infinite, there exists $f \in \mathbb{Z}^n$ such that $f \in \mathcal{A}(\Phi^+)$. Then $\mathcal{A}(w \Phi^+)$ contains $wf$ for each $w \in W$. \hfill\Box

**Proposition 2.2.4.** One has two bases $\{F_w^S : w \in W\}$ and $\{s_w^S : w \in W\}$ for $\mathsf{FQSym}^S$.\hfill\Box

**Proof.** By Lemma 2.2.3 $\{F_w^S : w \in W\}$ is linearly independent. If $P$ is an arbitrary parset of $\Phi$ then Theorem 2.2.2 implies that $F_P^S$ is the sum of $F_w^S$ for all $w \in \mathcal{L}(P)$. The result follows. \hfill\Box

It follows that one has the following two isomorphisms of free $\mathbb{Z}$-modules:

$$F : \mathbf{Z}^W \xrightarrow{\sim} \mathsf{FQSym}^S \quad \text{and} \quad s : \mathbf{Z}^W \xrightarrow{\sim} \mathsf{FQSym}^S$$

$$w \mapsto F_w^S \quad \text{and} \quad w \mapsto s_w^S.$$

We will also need the following result when we study the $s$-basis in type $A, B,$ and $D$.

**Proposition 2.2.5.** Let $w \in W$ and $f \in \mathbb{Z}^n$. Then $f \in \mathcal{A}(w^{-1} \Phi^+)$ is equivalent to

$$\begin{cases} 
(f, \alpha) \geq 0, & \text{if } \alpha > 0, \omega \alpha > 0, \\
(f, \alpha) < 0, & \text{if } \alpha > 0, \omega \alpha < 0.
\end{cases}$$

**Proof.** This follows from the definition of $P$-partitions and the observation that $\Phi^+$ is the disjoint union of $\{\omega \alpha : \alpha > 0, \omega \alpha > 0\}$ and $\{-\omega \alpha : \alpha > 0, \omega \alpha < 0\}$. \hfill\Box

Next assume $I \subseteq S$. By Humphreys [13, §1.10], $W_I$ is isomorphic to the reflection group of $E$ with root system $\Phi_I = \Phi_I^+ \cup \Phi_I^-$, where $\Phi_I^+$ consists of all roots in $\Phi$ that are nonnegative linear combinations of $\{\alpha \in \Phi^+ : s_\alpha \in I\}$ and $\Phi_I^- = -\Phi_I^+$. For each $u \in W_I$ one can check that $u \Phi_I^+$ is a parset of $\Phi$.\hfill\Box

**Proposition 2.2.6.** Let $I \subseteq S$ and $u \in W_I$. Then $F_{u \Phi_I^+}^S = \sum_{z \in u W} F_{uz}^S$.

**Proof.** Each element $w \in W$ can be written as $w = uz$ with $z \in W$, and

$$w \Phi_I^+ \subseteq w \Phi^+ \iff z^{-1} \Phi_I^+ \subseteq \Phi^+ \iff z^{-1} \in W_I.$$
Define a symmetric bilinear form on $\text{FQSym}^S$ by $\langle F^S_w, s^S_v \rangle := \delta_{u,v}$ for all $u,v \in W$. This is not positive definite and thus not isomorphic to the bilinear form on $\mathbb{Z}W$ defined in [2.1]. With this bilinear form $\text{FQSym}^S$ becomes self-dual. Define linear maps

$$\mu^S_I : \text{FQSym}^I \to \text{FQSym}^S \quad \text{and} \quad \rho^S_I : \text{FQSym}^S \to \text{FQSym}^I$$



Proposition 2.2.7. The diagram below is commutative; rotating it $180^\circ$ gives a dual diagram.

$$\begin{array}{c}
\text{FQSym}^I \xrightarrow{\sim} \mathbb{Z}W \xrightarrow{(\cdot)^{-1}} \mathbb{Z}W \xrightarrow{\sim} \text{FQSym}^I \\
\mu^S_I \downarrow \quad \mu^S_I \downarrow \quad \mu^S_I \\
\text{FQSym}^S \xrightarrow{\sim} \mathbb{Z}W \xrightarrow{(\cdot)^{-1}} \mathbb{Z}W \xrightarrow{\sim} \text{FQSym}^S \\
\rho^S_I \downarrow \quad \rho^S_I \downarrow \quad \rho^S_I \\
\text{FQSym}^I \xrightarrow{\sim} \mathbb{Z}W \xrightarrow{(\cdot)^{-1}} \mathbb{Z}W \xrightarrow{\sim} \text{FQSym}^I \\
\end{array}$$

Proof. Apply Proposition 2.1.3 and Proposition 2.2.6.

Definition 2.2.8. We define a representation $\mathcal{FQSym}$ of both the category $\text{Cox}$ and its dual category $\text{Cox}^{op}$ by sending each abstract finite Coxeter system $(W,S)$ to $\text{FQSym}^S$ and sending an inclusion $(W,I) \hookrightarrow (W,S)$ to $\mu^S_I : \text{FQSym}^I \to \text{FQSym}^S$ and a restriction $(W,S) \to (W,I)$ to $\rho^S_I : \text{FQSym}^S \to \text{FQSym}^I$ whenever $I \subseteq S$.

Proposition 2.2.7 implies that $\mathcal{FQSym}$ is self-dual. It also shows that the isomorphisms $F : \mathbb{Z}W \to \text{FQSym}^S$ and $s : \mathbb{Z}W \to \text{FQSym}^S$ induce natural isomorphisms $F : \Omega^* \to \mathcal{FQSym}$ and $s : \Omega \to \mathcal{FQSym}$ of representations of categories.

Next, let $\text{NSym}^S$ be the $\mathbb{Z}$-submodule of $\text{FQSym}^S$ with two free $\mathbb{Z}$-bases consisting of $s^S_I := \sum_{w \in D_I(W)} s^S_w$ and $h^S_I := \sum_{J \subseteq I} s^S_J = F^S_{q^S_{I,J}}$, for all $I \subseteq S$.

where the last equality follows from Proposition 2.2.4. One has $\iota : \text{NSym}^S \hookrightarrow \text{FQSym}^S$ by inclusion. By Theorem 2.1.3 and Proposition 2.2.7 $\mu^S_I : \text{FQSym}^I \to \text{FQSym}^S$ and $\rho^S_I : \text{FQSym}^S \to \text{FQSym}^I$ restrict to $\mu^S_I : \text{NSym}^I \to \text{NSym}^S$ and $\rho^S_I : \text{NSym}^S \to \text{NSym}^I$.

Definition 2.2.9. We define a representation $\mathcal{NSym}$ of both the category $\text{Cox}$ and its dual category $\text{Cox}^{op}$ by sending each abstract finite Coxeter system $(W,S)$ to $\text{NSym}^S$ and sending an inclusion $(W,I) \hookrightarrow (W,S)$ to $\mu^S_I : \text{NSym}^I \to \text{NSym}^S$ and a restriction $(W,S) \to (W,I)$ to $\rho^S_I : \text{NSym}^S \to \text{NSym}^I$ whenever $I \subseteq S$.

The isomorphism $s : \mathbb{Z}W \to \text{FQSym}^S$ restricts to an isomorphism $s : \Sigma(W) \xrightarrow{\sim} \text{NSym}^S$. $D_I(W) \mapsto s^S_I$.

This induces a natural isomorphism of representations of categories $s : \Sigma \to \mathcal{NSym}$.

On the other hand, denote by $\text{QSym}^S$ the dual of $\text{NSym}^S$ with two bases $\{ F^S_I : I \subseteq S \}$ and $\{ M^S_I : I \subseteq S \}$ dual to $\{ s^S_I : I \subseteq S \}$ and $\{ h^S_I : I \subseteq S \}$, respectively. Dual to the injection $\iota : \text{NSym}^S \hookrightarrow \text{FQSym}^S$ is a surjection $\chi : \text{FQSym}^S \twoheadrightarrow \text{QSym}^S$.

By Theorem 2.1.3 and Proposition 2.2.7 $\overline{\mu}^S_I : \text{QSym}^I \to \text{QSym}^S$ and $\overline{\rho}^S_I : \text{QSym}^S \to \text{QSym}^I$ descend to $\overline{\mu}^S_I : \text{QSym}^I \to \text{QSym}^S$ and $\overline{\rho}^S_I : \text{QSym}^S \to \text{QSym}^I$.
The isomorphism $F : ZW \xrightarrow{\sim} FQSym^S$ descends to an isomorphism
\[
F : \Sigma^*(W) \xrightarrow{\sim} QSym^S \\
D^I(W) \mapsto F^S_i.
\]
This induces a natural isomorphism of representations of categories
\[
F : \Sigma^* \to QSym.
\]
Finally, let $\text{Sym}^S$ be the $Z$-span of
\[
s^S_I := \chi(s^S_I) = \sum_{w \in D^I(W)} F^S_{w^{-1}}, \quad \forall I \subseteq S.
\]
Another spanning set for $\text{Sym}^S$ consists of $h^S_I := \sum_{J \subseteq I} s^S_J$ for all $I \subseteq S$. One sees that the injection $\iota : \text{NSym}^S \hookrightarrow FQSym^S$ descends to $\iota : \text{Sym}^S \hookrightarrow QSym^S$ and the surjection $\chi : FQSym^S \twoheadrightarrow QSym^S$ restricts to $\chi : \text{NSym}^S \twoheadrightarrow \text{Sym}^S$. By Corollary 2.1.16, $\mu^S_I : QSym^I \to QSym^S$ and $\rho^S_I : QSym^S \to QSym^I$ restrict to $\mu^S_I : \text{Sym}^I \to \text{Sym}^S$ and $\rho^S_I : \text{Sym}^S \to \text{Sym}^I$, and $\mu^S_I : \text{NSym}^I \to \text{NSym}^S$ and $\rho^S_I : \text{NSym}^S \to \text{NSym}^I$ also descend to $\mu^S_I : \text{Sym}^I \to \text{Sym}^S$ and $\rho^S_I : \text{Sym}^S \to \text{Sym}^I$.

**Definition 2.2.11.** We define a representation $\text{Sym}$ of both the category $\text{Cox}$ and its dual category $\text{Cox}^{op}$ by sending each abstract finite Coxeter system $(W, S)$ to $\text{Sym}^S$ and sending an inclusion $(W_I, I) \hookrightarrow (W, S)$ to $\mu^S_I : \text{Sym}^I \to \text{Sym}^S$ and a restriction $(W, S) \to (W_I, I)$ to $\rho^S_I : \text{Sym}^S \to \text{Sym}^I$ whenever $I \subseteq S$.

One has an isomorphism
\[
s : \Lambda(W) \xrightarrow{\sim} \text{Sym}^S \\
A_I(W) \mapsto s^S_I
\]
which is compatible with both $s : \Sigma(W) \xrightarrow{\sim} \text{NSym}^S$ and $F : \Sigma^*(W) \xrightarrow{\sim} QSym^S$. This induces a natural isomorphism of representations of categories
\[
s : \Lambda \to \text{Sym}.
\]

In summary, one has the following commutative diagram of representations of categories, which are in natural isomorphism with $\text{QSym}$.

2.3. **Representation theory of 0-Hecke algebras.** Now we investigate connections of our previous results with the representation theory of 0-Hecke algebras.

We first review some general results on representation theory of associative algebras [2, §1]. Let $F$ be an arbitrary field and let $A$ be a (left) $F$-module. If $M$ is nonzero and has no submodules except 0 and itself, then $M$ is *simple*. If $M$ is a direct sum of simple $A$-modules then $M$ is *semisimple*. The algebra $A$ is *semisimple* if it is semisimple as an $A$-module. Every module over a semisimple algebra is also semisimple. If an $A$-module $M$ cannot be written as a direct sum of two nonzero $A$-submodules, then $M$ is *indecomposable*. If $M$ is a direct summand of a free $A$-module, then $M$ is *projective*.

One can write $A$ as a direct sum of indecomposable $A$-modules $P_1, \ldots, P_k$. The *top* of $P_i$, denoted by $C_i := \text{top}(P_i)$, is the quotient of $P_i$ by its unique maximal submodule, and hence simple [2 Proposition I.4.5 (c)]. Every projective indecomposable $A$-module is isomorphic to some $P_i$, and every simple $A$-module is isomorphic to some $C_i$.

The *Grothendieck group* $G_0(A)$ of the category of finitely generated $A$-modules is defined as the abelian group $F/R$, where $F$ is the free abelian group on the isomorphism classes $[M]$ of finitely generated $A$-modules $M$, and $R$ is the subgroup of $F$ generated by the elements $[M] - [L] - [N]$ corresponding to all exact sequences $0 \to L \to M \to N \to 0$ of finitely generated $A$-modules. The Grothendieck group $G_0(A)$ of the category of finitely generated projective $A$-modules is defined similarly. We often identify a finitely generated (projective)
A-module with the corresponding element in the Grothendieck group \( G_0(A) \) (\( K_0(A) \)). If \( L, M, \) and \( N \) are all projective \( A \)-modules, then \( 0 \to L \to M \to N \to 0 \) is equivalent to \( M \cong L \oplus N. \) If \( A \) is semisimple then \( G_0(A) = K_0(A). \)

Suppose that \( \{C_1, \ldots, C_\ell\} \) is a complete list of non-isomorphic simple \( A \)-modules, and \( \{P_1, \ldots, P_\ell\} \) is a complete list of pairwise non-isomorphic indecomposable \( A \)-modules, labeled in such a way that \( C_i = \text{top}(P_i) \) for each \( i. \) Then \( G_0(A) \) and \( K_0(A) \) are free abelian groups with bases \( \{C_1, \ldots, C_\ell\} \) and \( \{P_1, \ldots, P_\ell\}, \) respectively. One has

\[
dim \mathbb{F} \text{Hom}_A(P_i, C_j) = \dim \mathbb{F} \text{Hom}_A(C_i, C_j) = \delta_{i,j}.
\]

This defines a pairing between \( G_0(A) \) and \( K_0(A), \) denoted by \((-,-)\). One has \( (P,C) = \dim \mathbb{F} \text{Hom}_A(P,C) \) for any finite dimensional projective \( A \)-module \( P \) and any finite dimensional \( A \)-module \( C, \) since the hom functor \( \text{Hom}_A(\cdot,-) \) is exact.

Let \( B \) be a subalgebra of \( A. \) For any \( A \)-module \( M \) and \( B \)-module \( N, \) the induction \( N \uparrow_A^B \) of \( N \) from \( B \) to \( A \) is defined as the \( A \)-module \( A \otimes_B N, \) and the restriction \( M \downarrow_A^B \) of \( M \) from \( A \) to \( B \) is defined as \( M \) itself viewed as a \( B \)-module. The induction and restriction are both well defined for isomorphic classes of modules. The following result is well known.

**Frobenius Reciprocity.** \( \text{Hom}_A(N \uparrow_A^B, M) \cong \text{Hom}_B(N, M \downarrow_A^B). \)

Now recall that a finite Coxeter group \( W \) is generated by a finite set \( S \) with

- quadratic relations \( s^2 = 1 \) for all \( s \in S, \)
- braid relations \((st \cdots s)_m = (tst \cdots s)_m\) for all \( s, t \in S.\)

We focus on the \( 0 \)-**Hecke algebra** \( H_W(0) \) of the Coxeter system \((W,S), \) which is a deformation of the group algebra of \( W. \) It is the \( \mathbb{F} \)-algebra generated by \( \{\pi_s : s \in S\} \) with

- quadratic relations \( \pi_s^2 = -\pi_s \) for all \( s \in S, \)
- braid relations \( (\pi_s \pi_t \pi_s \cdots )_m = (\pi_t \pi_s \pi_t \cdots )_m \) for all \( s, t \in S.\)

If \( w \in W \) has a reduced expression \( w = s_1 \cdots s_\ell \), where \( s_1, \ldots, s_\ell \in S, \) then \( \pi_w := \pi_{s_1} \cdots \pi_{s_\ell} \) is well defined. In fact, by the Word Property of \( W \) \cite[Theorem 3.3.1]{3} or \cite[Theorem 1.9]{15}, \( \pi_w \) depends only on \( w, \) not on the choice of the above reduced expression of \( w. \) By this definition, for any \( s \in S \) and \( w \in S, \)

\[
\pi_s \pi_w = \begin{cases} 
\pi_{sw}, & \text{if } \ell(sw) > \ell(w), \\
-\pi_w, & \text{if } \ell(sw) < \ell(w).
\end{cases}
\]

One can show that \( \{\pi_w : w \in W\} \) is an \( \mathbb{F} \)-basis for \( H_W(0) \) using \cite[proof of Proposition 3.3]{15} with some straightforward modifications. See also Stembridge \cite[Proposition 2.1]{25}.

Let \( \pi_s := \pi_s + 1 \) for each \( s \in S. \) Then \( \pi_s \pi_s = \pi_s \pi_s = 0. \) For any \( u \) and \( w \) in \( W, \) write \( u \leq w \) if some reduced expression \( w \) contains a subword equal to \( u. \) This is the well-known **Bruhat order** of \( W. \) If \( w \in W \) has a reduced expression \( w = s_1 \cdots s_\ell \) then \( \pi_w := \pi_{s_1} \cdots \pi_{s_\ell} \) is well defined as Stembridge \cite[Lemma 3.2]{25} showed that

\[
\pi_w = \sum_{u \leq w} \pi_u
\]

which does not depend on the choice of the reduced expression \( w = s_1 \cdots s_\ell. \) This implies that \( \{\pi_s : s \in S\} \) is another generating set for \( H_W(0) \) satisfying the quadratic relations \( \pi_s^2 = \pi_s \) for all \( s \in S \) and the same braid relations as \( \{\pi_s : s \in S\} \) \cite[Lemma 3.3]{25}.

Norton \cite[4.12, 4.13]{18} obtained an \( H_W(0) \)-module decomposition

\[
H_W(0) = \bigoplus_{I \subseteq S} P_I^S,
\]

where \( P_I^S := H_W(0)\pi_{w_0(I)} \pi_{w_0(I^c)} \) is an indecomposable \( H_W(0) \)-module with an \( \mathbb{F} \)-basis

\[
\{\pi_w \pi_{w_0(I^c)} : w \in W, D(w) = I\}.
\]

The top of \( P_I^S, \) denoted by \( C_I^S, \) is a one-dimensional simple \( H_W(0) \)-module on which \( \pi_i \) acts by \(-1\) if \( i \in I \) or by \( 0 \) if \( i \not\in I. \) Thus \( \{P_I^S : I \subseteq S\} \) and \( \{C_I^S : I \subseteq S\} \) are complete lists of pairwise non-isomorphic projective indecomposable and simple \( H_W(0) \)-modules, respectively.
If \( w \in W \) then denote by \( C^S_w \) the simple \( H_W(0) \)-module indexed by \( D(w) \subseteq S \). Sending \( C^S_w \) to \( D^*_w(W) \) for all \( w \in W \) gives an isomorphism \( G_0(H_W(0)) \cong \Sigma^*(W) \) of free \( \mathbb{Z} \)-modules. Similarly, sending \( P^S_I \) to \( D_I(W) \) for all \( I \subseteq S \) gives an isomorphism \( K_0(H_W(0)) \cong \Sigma(W) \) of free \( \mathbb{Z} \)-modules.

Let \( I \subseteq S \). The parabolic subalgebra \( H_{W_I}(0) \) of the 0-Hecke algebra \( H_W(0) \) is generated by \( \{ \pi_s : s \in I \} \) and it is also isomorphic to the 0-Hecke algebra of the parabolic subgroup \( W_I \) of \( W \). By Proposition 2.1.1

\[
(2.3) \quad H_W(0) = \bigoplus_{w \in W^I} \pi_w H_{W_I}(0) = \bigoplus_{w \in W^I} H_{W_I}(0) \pi_w.
\]

We define two linear maps

\[
(2.4) \quad \bar{\mu}^S_I : G_0(H_{W_I}(0)) \to G_0(H_W(0)) \quad \text{and} \quad \bar{\rho}^S_I : G_0(H_W(0)) \to G_0(H_{W_I}(0))
\]

Proposition 2.3.1. The linear maps \( \bar{\mu}^S_I \) and \( \bar{\rho}^S_I \) are well-defined by (2.3). Moreover, they restrict to linear maps \( \mu^S_I : K_0(H_{W_I}(0)) \to K_0(H_W(0)) \) and \( \rho^S_I : K_0(H_W(0)) \to K_0(H_{W_I}(0)) \).

Proof. This can be proved similarly as Bergeron and Li [4, §3].

Proposition 2.3.2. If \( I \subseteq S \) and \( w \in W_I \) then \( \bar{\mu}^S_I(C^I_w) = \sum_{z \in I} w^I \bar{C}^S_{wz} \).

Proof. Fix \( w \in W_I \) and write \( C^I_w = F a \). The \( \pi_s(a) \) equals \(- a \) if \( s \in D(w) \) or 0 otherwise. Let \( M = \bar{\mu}(C^I_w) \).

Proposition 2.3.3. Let \( I, K \subseteq S \) and \( w \in W \). Then \( \bar{\rho}^S_I(C^S_K) = C^I_{K \cap R} \) and \( \bar{\rho}^S_I(C^S_w) = C^I_{w^I} \).

Proof. The first equality follows immediately from the definition of \( C^S_K \). The second equality is equivalent to the first one since \( D(I \cap R) = D(I) \cap R \) by Lemma 2.1.10.

Next, we consider the induction of projective modules. Let \( I, J \subseteq S \). We define a cyclic \( H_W(0) \)-module \( P^S_{I,J} := H_W(0) \pi_{w_{0(I)}} \pi_{w_{0(J)}} \). We may assume \( I \subseteq J \subseteq S \) without loss of generality, since \( P^S_{I,J} = P^S_{I,I \cup J} \). One sees that \( P^S_{I,S} = P_I \). In general, we showed the following result in [12, Theorem 3.2].

Proposition 2.3.4. [12 Theorem 3.2] If \( I, J \subseteq S \) then \( P^S_{I,J} \) has a basis

\[
\{ \pi_w \pi_{w_0(J \setminus I)} : w \in W, \ I \subseteq D(w) \subseteq (S \setminus J) \cup I \}.
\]

If \( I \subseteq J \subseteq S \)

\[
\mu^S_J(P^S_I) \cong P^S_{I,J} \cong \bigoplus_{K \subseteq S \setminus J} P_{I \cup K}.
\]

Finally, we investigate the restriction of a projective indecomposable module.
Proposition 2.3.5. Let $I, K \subseteq S$. Write $L(z) = L(z, I, K)$ and $L'(z) = L'(z, I, K)$. Then
\[
\rho^S_I(P^S_K) \cong \bigoplus_{z \in I^D_K} P^I_{L(z), I \setminus L'(z)} \cong \bigoplus_{z \in I^D_K} P^I_{L(z), I \setminus L'(z)}.
\]

Proof. Since $P^S_K$ has an $\mathbb{F}$-basis $\{\pi_w \pi_w(0) : w \in D_K(W)\}$, one has a decomposition
\[
\rho^S_I(P^S_K) = \bigoplus_{z \in I^D_K} P^I_{K, z}
\]
of vector spaces, where each $P^I_{K, z}$ has an $\mathbb{F}$-basis
\[
\{\pi_w \pi_w(0) : w \in D_K(W), \ i w = z\}.
\]
By Proposition 2.3.4 and Proposition 2.3.8 one has an isomorphism $\phi : P^I_{K, z} \rightarrow P^I_{L(z), I \setminus L'(z)}$ of vector spaces by
\[
(2.5) \quad \pi_w \pi_w(0) \mapsto \pi_w \pi_w(0), \quad \forall w \in D_K(W) \text{ with } i w = z.
\]
It remains to show that $\phi$ preserves $H_{W_I}(0)$-actions. Let $w \in D_K(W)$, $w_I = u$, $i w = z$, and $s \in I$.

If $s \in D(u^{-1})$ then $s \in D(w^{-1})$ and thus $\pi_s$ acts by $-1$ on both sides of $(2.5)$.

Next we assume $s \notin D(u^{-1})$ and $D(su) \subseteq L'(z)$. Then $\ell(sw) = \ell(su) + \ell(z) > \ell(w)$.

Thus $\pi_s \pi_u \pi_w(0) = \pi_u \pi_w(0)$ and $\pi_s \pi_u \pi_w(0) = \pi_u \pi_w(0)$. One also has $sw = su = s$, and $L(z) \subseteq D(u) \subseteq D(su) \subseteq L'(z)$. Hence $D(sw) = K$ by Lemma 2.1.1

Finally assume $s \notin D(u^{-1})$ and $D(su) \subseteq L'(z)$. Lemma 2.1.10 implies that $D(su) \setminus L'(z) = \{r\}$ where $r = u^{-1}su$, i.e. $ur = su$. The definition of $L(z)$ implies that $r \in D(tz^{-1}) \cap I$ for some $t \in D(z) \subseteq K$. Then applying Lemma 2.1.10 again gives $z^{-1} = z^{-1}r$. Hence $sw = su = uzr = uzt$. It follows that $\pi_s \pi_u \pi_w(0) = \pi_u \pi_w(0)$ and $\pi_s \pi_u \pi_w(0) = \pi_u \pi_w(0)$. Therefore $\phi$ is indeed an isomorphism of $H_{W_I}(0)$-modules. This shows the first desired isomorphism. The second one follows immediately from Proposition 2.3.4.

Theorem 2.3.6. The following two diagrams are commutative and dual to each other:

\[
\begin{array}{ccc}
G_0(H_{W_I}(0)) & \xrightarrow{\sim} & \Sigma^*(W_I) \\
\overrightarrow{\mu^S_I} & \downarrow & \overrightarrow{\mu^S_I} \\
G_0(H_W(0)) & \xrightarrow{\sim} & \Sigma^*(W) \\
\overrightarrow{\mu^S_I} & \downarrow & \overrightarrow{\mu^S_I} \\
G_0(H_W(0)) & \xrightarrow{\sim} & \Sigma^*(W_I) \\
\end{array}
\]

Proof. Compare Proposition 2.3.2, 2.3.3, 2.3.4 and 2.3.5 with Proposition 2.1.1 and 2.1.2.

Remark 2.3.7. By Theorem 2.3.6 if $J \subseteq I \subseteq S$ and $K \subseteq S$ then
\[
\langle \mu^S_I(P^J_I), C^S_K \rangle = \langle P^J_I, \mu^S_I(C^S_K) \rangle \quad \text{and} \quad \langle P^S_K, \mu^S_I(C^J_I) \rangle = \langle \rho^S_I(P^S_K), C^J_I \rangle.
\]
Note that the first equality is also a consequence of the Frobenius reciprocity, but the second one is not.

Definition 2.3.8. We define a representation $G_0(K_0)$ resp. of both the category $\text{Cox}$ and its dual category $\text{Cox}^{op}$ by sending each abstract finite Coxeter system $(W, S)$ to the Grothendieck group $G_0(H_W(0))$(resp. $K_0(H_W(0))$) resp. and sending an inclusion $(W_I, I) \hookrightarrow (W, S)$ to $\overrightarrow{\mu^S_I}$ (resp. $\mu^S_I$) resp. and sending a restriction $(W, S) \rightarrow (W_I, I)$ to $\overrightarrow{\rho^S_I}$ (resp. $\rho^S_I$) whenever $I \subseteq S$.

Theorem 2.3.9 gives a natural isomorphism between $G_0(K_0)$ resp. and $\Sigma^*(\Sigma)$ resp. Combining this with the natural isomorphism $F : \Sigma^* \rightarrow \text{QSym}$ and $s : \Sigma \rightarrow \text{NSym}$ one has two natural isomorphisms
\[
\text{Ch} : G_0 \rightarrow \text{QSym} \quad \text{and} \quad \text{ch} : K_0 \rightarrow \text{NSym}
\]
which are induced by the following characteristic maps
\[
\text{Ch} : \ G_0(H_W(0)) \to QSym^S \quad \text{and} \quad \text{ch} : \ K_0(H_W(0)) \to NSym^S
\]
\[
C_w^S \mapsto F_w^S.
\]
Lastly, we provide a result on \(\text{Ch}(\mathcal{P}_I^S)\) for later use.

**Proposition 2.3.9.** If \(I \subseteq S\) then \(\text{Ch}(\mathcal{P}_I^S) = \chi(\text{ch}(\mathcal{P}_I^S)) = s_I^S\).

**Proof.** Recall that \(\mathcal{P}_I^S\) has a basis \(\{\tau_w \pi_{w_0(I^c)} : w \in D_I(W)\}\). One has a filtration of \(\mathcal{P}_I^S\) by the length of \(w\) for all \(w \in D_I(W)\). If \(s \in S\) and \(w \in D_I(W)\) then
\[
\pi_s \pi_w \pi_{w_0(I^c)} = \begin{cases} 
-\pi_w \pi_{w_0(I^c)}, & \text{if } s \in D(w^{-1}), \\
0, & \text{if } s \notin D(w^{-1}), D(sw) \not\subseteq I,
\end{cases}
\]
It follows that \(\text{Ch}(\mathcal{P}_I^S) = \sum_{w \in D_I(W)} F_w^{S^{-1}}.\) Hence the result holds. \(\square\)

2.4. Connections to work of Aguiar and Mahajan. Let \((W, S)\) be a finite Coxeter system. In this subsection we compare the following diagrams.

(2.6)

\[
\begin{array}{ccc}
\text{FQSym}^S &=& \text{FQSym}^S \\
\text{NSym}^S & \leftarrow & \text{QSym}^S \\
\text{Supp}^S & \leftarrow & \text{Sym}^S
\end{array}
\]

\[
\begin{array}{ccc}
\text{KW} & \leftrightarrow & \text{KW}^* \\
\text{KQ} & \leftarrow & \text{KQ}^* \\
\text{Supp} & \leftarrow & \text{Supp}^*
\end{array}
\]

The first diagram is a commutative diagram of free \(\mathbb{Z}\)-modules obtained from our results in Section 2.2 and is equivalent to the diagram (2.1). The second diagram is a commutative diagram of vector spaces over a field \(\mathbb{K}\) of characteristic zero obtained by Aguiar and Mahajan [1, Theorem 5.7.1], which can be generalized to left regular bands. The remaining of this subsection is devoted to the proof of the following result.

**Proposition 2.4.1.** In (2.6), with an extension of scalars from \(\mathbb{Z}\) to \(\mathbb{K}\), the first diagram becomes isomorphic to the second one.

Recall that \(\text{FQSym}^S\) has dual bases \(\{F_w^S : w \in W\}\) and \(\{s_w^S = F_{w^{-1}}^S : w \in W\}\), and \(\text{NSym}^S\) and \(\text{QSym}^S\) have dual bases \(\{s_I^S : I \subseteq S\}\) and \(\{F_I^S : I \subseteq S\}\). One has \(\text{NSym}^S \hookrightarrow \text{FQSym}^S\) defined by \(s_I^S = \sum_{w \in D_I(W)} s_w^S\) and \(\chi : \text{FQSym}^S \to \text{QSym}^S\) defined by \(\chi(F_w^S) := F_{D(w)}^S\). We keep the superscript \(S\) even though the Coxeter system \((W, S)\) is fixed in this subsection.

Write \(U_{\mathbb{K}} := U \otimes_\mathbb{Z} \mathbb{K}\) for any \(\mathbb{Z}\)-module \(U\). By [1, §5.7.1], \(\text{KW}\) and \(\mathbb{K}\text{Q}\) have canonical bases \(\{K_w : w \in W\}\) and \(\{K_I : I \subseteq S\}\), and \(\text{KW}^*\) and \(\mathbb{K}\text{Q}^*\) have dual bases \(\{F_w : w \in W\}\) and \(\{F_I : I \subseteq S\}\). There are dual bases \(\{H_I : I \subseteq S\}\) and \(\{M_I : I \subseteq S\}\) for \(\mathbb{K}\text{Q}\) and \(\mathbb{K}\text{Q}^*\) given by
\[
H_I = \sum_{J \subseteq I} K_J \quad \text{and} \quad F_I = \sum_{I \subseteq J} M_J.
\]
There is no superscript \(S\) in these bases. One has isomorphisms of vector spaces:
\[
\text{NSym}_{\mathbb{K}}^S \cong \mathbb{K}\text{Q}, \quad \text{FQSym}_{\mathbb{K}}^S \cong \mathbb{K}W, \quad \text{FQSym}_{\mathbb{K}}^S \cong \mathbb{K}W^*, \quad \text{QSym}_{\mathbb{K}}^S \cong \mathbb{K}\text{Q}^*.
\]
By the definitions of \(\theta, s, \) and \(\text{des}\) [1, §5.7.4], one has the following commutative diagram.

(2.7)

\[
\begin{array}{ccc}
\text{NSym}_{\mathbb{K}}^S & \leftrightarrow & \text{FQSym}_{\mathbb{K}}^S \\
\text{KQ} & \theta & \leftrightarrow & \text{KW} \leftarrow s \to \text{KW}^* \leftarrow \text{des} \to \text{KQ}
\end{array}
\]

This shows that the top halves of the two diagrams in (2.6) agree with each other.

Next we study the bottom halves of the two diagrams in (2.6). We first define \(\text{Supp}\) and \(\text{Supp}^*\). Recall that \(W\) can be realized as a group generated by a set \(S\) of reflections of a Euclidean space \(E = \mathbb{R}^n\). The reflection arrangement associated with \((W, S)\) consists of all hyperplanes such that the reflections across
these hyperplanes belong to $W$. This hyperplane arrangement gives rise a simplicial complex called the Coxeter complex $\Sigma$ of $(W, S)$, which can be also constructed algebraically by partially ordering the parabolic cosets $wW_I$ for all $w \in W$ and all $I \subseteq S$ using reverse inclusion. A flat is an intersection of a subset of reflection hyperplanes of $(W, S)$. The intersection lattice $L$ consists of all flats ordered by inclusion. There is a surjection $\text{supp} : \Sigma \rightarrow L$ sending a face of $\Sigma$ to its linear span, i.e., the intersection of the reflection hyperplanes containing this face.

The $W$-action on the Euclidean space $E$ induces $W$-actions on $\Sigma$ and $L$. If $W$ acts on a vector space $V$ then $V^W := \{ v \in V : v(w) = v, \forall w \in W \}$ is the $W$-fixed subspace of $V$. By Barcelo and Ihrig [3], we can identify the set $\{ wW_Iw^{-1} : w \in W, I \subseteq S \}$ ordered by reverse inclusion with the intersection lattice $L$ via $wW_Iw^{-1} \mapsto E^{wW_Iw^{-1}}$, and the $W$-action on $L$ corresponds to the conjugate action of $W$ on parabolic subgroups. The following algebraic interpretation of $\text{supp} : \Sigma \rightarrow L$ has been generalized by Miller [17] to well-generated complex reflection groups:

$$(2.8) \quad \text{supp}(wW_I) = E^{wW_Iw^{-1}}, \quad \forall w \in W, \forall I \subseteq S.$$

Next, one sees that $(\mathbb{K}\Sigma)^W$ has a basis consisting of

$$(2.9) \quad \sigma_I := \sum_{w \in W^I} wW_I, \quad \forall I \subseteq S.$$

Sending $H_I$ to $\sigma_I$ for all $I \subseteq S$ gives an isomorphism $\mathbb{K}\Sigma \cong (\mathbb{K}\Sigma)^W$. On the other hand, let $\mathcal{L}$ be the set of the orbits of the $W$-action on $L$. Sending an orbit to the sum of its elements gives $\mathbb{K}\mathcal{L} \cong (\mathbb{K}L)^W$. Then $\text{supp} : \Sigma \rightarrow L$ induces $\text{supp} : \mathbb{K}\Sigma \rightarrow \mathbb{K}\mathcal{L}$, as illustrated below.

$$
\begin{array}{ccc}
\mathbb{K}\mathcal{L} & \cong & (\mathbb{K}L)^W \\
\downarrow & & \downarrow \\
\mathbb{K}\Sigma & \cong & (\mathbb{K}\Sigma)^W
\end{array}
$$

Write $h_I := \text{supp}(H_I)$ for all $I \subseteq S$. Then $\mathbb{K}\mathcal{L}$ is $\mathbb{K}$-spanned by $\{ h_I : I \subseteq S \}$. We write $I \sim J$ if two parabolic subgroups $W_I$ and $W_J$ are conjugate in $W$. This defines an equivalence relation on subsets of $S$ and we denote by $\Pi(W, S)$ the set of all equivalence classes. Then $|\mathcal{L}| = |\Pi(W, S)|$. By (2.8) and (2.9), $\text{supp} : \mathbb{K}\Sigma \rightarrow \mathbb{K}\mathcal{L}$ sends $H_I$ to the conjugacy class $\{ wW_Iw^{-1} : w \in W \}$ and thus $h_I = h_J$ if and only if $I \sim J$. Hence $h_\lambda := h_I$ is well defined if $\lambda \in \Pi(W, S)$ and $I \subseteq \lambda$, and $\mathbb{K}\mathcal{L}$ has a basis $\{ h_\lambda : \lambda \in \Pi(W, S) \}$.

By [11 §2.6], $\mathbb{K}\mathcal{L}$ has a bilinear form defined by

$$(h_I, h_J) := \# \{ w \in W : D(w) \subseteq I, \text{ des}(w^{-1}) \subseteq J \}, \quad \forall I, J \subseteq S.$$

The dual space $\mathbb{K}\mathcal{L}^*$ has a basis $\{ m_\lambda : \lambda \in \Pi(W, S) \}$ dual to $\{ h_\lambda : \lambda \in \Pi(W, S) \}$. The map $\phi : \mathbb{K}\mathcal{L} \rightarrow \mathbb{K}\mathcal{L}^*$ and the map $\text{supp} \ast : \mathbb{K}\Sigma \rightarrow \mathbb{K}\mathcal{L}$ are defined by

$$\phi(h_\lambda) := \sum_{\mu \in \Pi(W, S)} \langle h_\lambda, h_\mu \rangle m_\mu \quad \text{and} \quad \text{supp} \ast (m_\lambda) := \sum_{I \subseteq \lambda} M_I.$$

Next, we construct a free $\mathbb{Z}$-basis for $\text{Sym}^S$. Recall that $\text{NSym}^S$ and $\text{QSym}^S$ have dual bases $\{ h_I^S : I \subseteq S \}$ and $\{ M_I^S : I \subseteq S \}$ defined by $h_I^S = \sum_{J \subseteq I} s_J^I$ and $F_I^S = \sum_{J \subseteq I} M_J^S$, and $\text{Sym}^S$ is the $\mathbb{Z}$-span of $\{ h_I^S : I \subseteq S \}$ where $h_I^S := \chi(h_I^T)$.

**Proposition 2.4.2.** The element $h_\lambda^S := h_I^S$ is well defined whenever $\lambda \in \Pi(W, S)$ and $I \subseteq \lambda$, and the set $\{ h_\lambda^S : \lambda \in \Pi(W, S) \}$ is a free $\mathbb{Z}$-basis for $\text{Sym}^S$.

**Proof.** We first need an easy group theory result. If $L$ and $R$ are two subgroups of $W$ then we write $L \backslash W/R := \{ LwR : w \in W \}$ for the set of all double $(L, R)$-cosets in $W$. If $R' = uRu^{-1}$ for some $u \in W$ then $|L \backslash W/R| = |L \backslash W/R'|$ as there is a bijection between $L \backslash W/R$ and $L \backslash W/R'$ by $LwR \mapsto LwuR^{-1} = Luw^{-1}R'$ for all $w \in W$. Now if $I, J \subseteq S$ then

$$\langle h_I^S, h_J^S \rangle = \sum_{K \subseteq I} \sum_{w \in W} \sum_{D(w) \subseteq J} \langle s_K^I, P_{w^{-1}} \rangle = | \{ w \in W : D(w^{-1}) \subseteq I, D(w) \subseteq J \} | = |W_{I^c} \backslash W/W_{J^c}|.$$
If $J \sim J'$ then $\langle h_J^S, h_J' \rangle = \langle h_J^S, h_J' \rangle$, i.e., $h_J^S$ and $h_J'$ have the same expansion in the basis $\{ M_I : I \subseteq S \}$ for $\text{QSym}^S$, and thus $h_J^S = h_J'$. Then $h_J^S := h_J$ is well defined if $\lambda \in \Pi(W, S)$ and $I \in \lambda$. This gives a spanning set $\{ h_{\lambda}^S : \lambda \in \Pi(W, S) \}$ for $\text{Sym}^S$. Since

$$\text{Sym}^S_k \cong \chi \circ \iota(\text{NSym}^S_k) \cong \deg \circ \theta(\mathbb{K}Q) \cong \text{supp}(\mathbb{K}Q) \cong \mathbb{K}L$$

and $|\Pi(W, S)| = |L|$, this spanning set must be linearly independent. \hfill \Box

Recall that $\Lambda(W)$ has a symmetric bilinear form defined by $\langle \Lambda_I(W), \Lambda_J(W) \rangle := c_{I,J}$. Applying the isomorphism $s : \Lambda(W) \cong \text{Sym}^S$ defined by $\Lambda_I(W) \mapsto s_I$ one has

$$\langle h_I^S, h_J^S \rangle = \# \{ w \in W : D(w) \subseteq I, D(w^{-1}) \subseteq J \}, \quad \forall I, J \subseteq S.$$

This is compatible with the pairing between $\text{NSym}^S$ and $\text{QSym}^S$ by Proposition 2.1.15 and the isomorphisms $\Sigma(W) \cong \text{NSym}^S$ and $\Sigma^*(W) \cong \text{QSym}^S$.

There is a linearly independent set $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$ in $\text{QSym}^S$, where

$$m_{\lambda}^S := \sum_{i \in \lambda} M_i^S, \quad \forall \lambda \in \Pi(W, S).$$

**Proposition 2.4.3.** $\text{Sym}^S_k$ has dual $\mathbb{K}$-bases $\{ h_{\lambda}^S : \lambda \in \Pi(W, S) \}$ and $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$.

*Proof.* Let $I, I', J \subseteq S$ with $I \sim I'$. One shows $\langle h_I^S, h_{I'}^S \rangle = \langle h_I^S, h_J^S \rangle$ similarly as the proof of Proposition 2.4.2. Thus $h_J^S$ lies in the $\mathbb{Z}$-span of $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$. It follows that the free $\mathbb{Z}$-basis $\{ h_{\lambda}^S : \lambda \in \Pi(W, S) \}$ for $\text{Sym}^S$ is contained in the $\mathbb{Z}$-span of the linearly independent set $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$. This implies that $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$ is a $\mathbb{K}$-basis for $\text{Sym}^S_k$. If $\lambda, \mu \in \Pi(W, S)$ and $I \in \lambda$ then

$$\langle h_{\lambda}^S, m_{\mu}^S \rangle = \langle H_I, \sum_{J \subseteq \mu} M_J \rangle = \delta_{\lambda,\mu}.$$

Thus $\{ h_{\lambda}^S : \lambda \in \Pi(W, S) \}$ and $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$ are dual $\mathbb{K}$-bases for $\text{Sym}^S_k$. \hfill \Box

Now we have the following isomorphisms of vector spaces:

$$\text{NSym}^S_k \cong \mathbb{K}Q, \quad \text{Sym}^S_k \cong \mathbb{K}L, \quad \text{Sym}^S_k \cong \mathbb{K}L, \quad \text{QSym}^S_k \cong \mathbb{K}Q.$$

Then one can verify the following commutative diagram:

\[
\begin{array}{cccccc}
\text{NSym}^S_k & \xrightarrow{\chi} & \text{Sym}^S_k & \xrightarrow{\iota} & \text{Sym}^S_k & \xrightarrow{\phi} & \text{QSym}^S_k \\
\downarrow{\text{supp}} & & \downarrow{\text{supp}} & & \downarrow{\text{supp}} & & \downarrow{\text{supp}} \\
\mathbb{K}Q & \xrightarrow{\text{supp}} & \mathbb{K}L & \xrightarrow{\phi} & \mathbb{K}L & \xrightarrow{\phi} & \mathbb{K}Q
\end{array}
\]

Combining (2.7) and (2.10) one proves Proposition 2.4.1.

**Remark 2.4.4.** (i) By Geck and Pfeiffer [9], Chapter 2 and Appendix A], the conjugacy class of a parabolic subgroup $W_I$ of $W$ is determined by the type of the Coxeter system $(W_I, I)$ except that in type $D_{2m}$ there are two conjugacy classes of parabolic subgroups of type $A_{k_1} \times \cdots \times A_{k_r}$ with all $k_i$ odd such that $\sum_{i=1}^{r}(k_i+1) = 2m$ and in type $E_7$ there are two conjugacy classes of parabolic subgroups of type $A_1 \times A_1 \times A_1$, $A_1 \times A_3$, and $A_5$.

(ii) One can obtain dual orthogonal bases $\{ q_{\lambda}^S : \lambda \in \Pi(W, S) \}$ and $\{ p_{\lambda}^S : \lambda \in \Pi(W, S) \}$ for $\text{Sym}^S_k$ using work of Aguiar and Mahajan [11] §5.7]. In fact, if $\lambda \in \Pi(W, S)$ and $I \subseteq \lambda$ then

$$p_{\lambda}^S := \sum_{J \subseteq I} |\lambda(J)| m_{\lambda(J)}$$

where $\lambda(J)$ is the equivalent class in $\Pi(W, S)$ containing $J$ and $|\lambda(J)|$ is the quotient of $|W^J|$ by the size of the conjugacy class of the parabolic subgroup $W_J$ of $W$. In type $A$ this recovers the well-known power sum basis $\{ p_\lambda \}$ for $\text{Sym}_k$.

(iii) Though $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$ is a $\mathbb{K}$-basis for $\text{Sym}^S_k$, we will show in type $B$ that it is not a free $\mathbb{Z}$-basis for $\text{Sym}^S$ because the $\mathbb{Z}$-span of $\{ m_{\lambda}^S : \lambda \in \Pi(W, S) \}$ strictly contains $\text{Sym}^S$. 


3. Type A

In this section we recover some well-known results in type A from our results in Section 2.

3.1. Malvenuto–Reutenauer algebra and descent algebra. The symmetric group $S_n$ consists of all permutations of the set $[n] := \{1, 2, \ldots, n\}$ and is generated by $\{s_1, \ldots, s_{n-1}\}$ where $s_i$ is the adjacent transposition $(i, i+1)$. This gives the finite irreducible Coxeter system of type $A_{n-1}$, whose Coxeter diagram is drawn below.

\[
\begin{array}{ccccccc}
s_1 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} \\
\end{array}
\]

Let $w$ be a permutation in $S_n$. The one-line notation for $w$ is the word $w(1) \cdots w(n)$. The descent set of $w$ is $\{s_i : i \in [n-1], w(i) > w(i+1)\}$. The length of $w$ equals inv$(w)$, where

\[
\text{inv}(a_1 \cdots a_n) := \# \{(i, j) : 1 \leq i < j \leq n, a_i > a_j\}.
\]

It is often convenient to use compositions of $n$ to index subsets of the generating set $\{s_1, \ldots, s_{n-1}\}$. A composition is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers $\alpha_1, \ldots, \alpha_\ell$. The length of $\alpha$ is $\ell(\alpha) := \ell$ and the size of $\alpha$ is $|\alpha| := \alpha_1 + \cdots + \alpha_\ell$. If the size of $\alpha$ is $n$ then say $\alpha$ is a composition of $n$ and write $\alpha \vdash n$. The only composition of $n = 0$ is the empty composition $\alpha = \varnothing$. Assume $n \geq 1$. The descent set of $\alpha$ is $D(\alpha) := (\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_\ell - 1)$.

The map $\alpha \mapsto D(\alpha)$ is a bijection between compositions of $n$ and subsets of $[n-1]$. The complement of $\alpha$ is the composition $\alpha^c$ of $n$ with $D(\alpha^c) = [n-1] \setminus D(\alpha)$.

The parabolic subgroup $S_\alpha \cong S_{\alpha_1} \times \cdots \times S_{\alpha_\ell}$ is generated by $\{s_i : i \in D(\alpha^c)\}$. The set of minimal representatives of left $S_\alpha$-cosets in $S_n$ is $S_\alpha := \{w \in S_n : D(w) \subseteq D(\alpha)\}$.

A ribbon is a connected skew Young diagram without $2 \times 2$ boxes. A composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ can be identified with a ribbon whose rows have length $\alpha_1, \ldots, \alpha_\ell$ from bottom to top. See the example below.

\[
\begin{array}{cccc}
\alpha = (2, 3, 1, 1) & \alpha^c = (1, 2, 1, 3)
\end{array}
\]

If $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \ldots, \beta_k)$ are two nonempty compositions then define

\[
\alpha \cdot \beta := (\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_k) \text{ and } \\
\alpha \vdash \beta := (\alpha_1, \ldots, \alpha_\ell - 1, \alpha_\ell + \beta_1, \beta_2, \ldots, \beta_k).
\]

Let $\alpha \cdot \varnothing := \alpha$ and $\varnothing \cdot \beta := \beta$. Also set $\alpha \vdash \varnothing$ and $\varnothing \vdash \beta$ undefined. We write $\alpha \leq \beta$ if $\alpha$ and $\beta$ are compositions of the same size and $D(\alpha) \subseteq D(\beta)$, that is, $\alpha$ is refined by $\beta$.

The Malvenuto–Reutenauer algebra \[\mathcal{H}\] is a self-dual graded Hopf algebra whose underlying space is the free $\mathbb{Z}$-module $\mathcal{H} := \bigoplus_{n \geq 0} \mathbb{Z}S_n$ with a basis $\mathcal{S} := \bigsqcup_{n \geq 0} S_n$. We will first review its product and coproduct, and then recover them from the linear maps defined in Section 2.1.

Let $a = a_1 \cdots a_n \in \mathbb{Z}^n$ be a word of $n$ integers. Let $i$ and $j$ be two nonnegative integers. We write $a[i, j] := a_i \cdots a_j$ and denote by $a[i, j]$ the subword of $a$ obtained by keeping only the letters whose absolute values belong to the interval $[i, j]$. In case $i > j$ we define both $a[i, j]$ and $a[j, i]$ to be the empty word $\varnothing$.

The standardization $\text{st}(a)$ of the word $a$ is the unique permutation $w \in S_n$ such that

\[
w(i) < w(j) \iff a_i \leq a_j \quad \text{whenever} \quad 1 \leq i < j \leq n.
\]

One can obtain $\text{st}(a)$ by reading the letters of $a$ from smallest to largest, breaking up ties from left to right. For example, one has $\text{st}(3223625) = 4125736$.

We identify a set of permutations with the sum of its elements inside $\mathbb{Z}\mathcal{S}$. Define

\[
\begin{align*}
\text{st}(u \cup v) & := \{w \in S_{m+n} : w|[1, m] = u, \text{ st}(w|[m+1, m+n]) = v\} \quad \text{and} \\
\text{st}(\sqcap u) & := \sum_{0 \leq i \leq m} \text{st}(u[1, i]) \otimes \text{st}(u[i+1, m])
\end{align*}
\]
Thus Proposition 2.1.5 implies the associativity and coassociativity of \( u \). There is another graded Hopf algebra \((\mathbb{Z}\mathfrak{S}, \u, \m)\) defined by

\[
u := \{ w \in \mathfrak{S}_{m+n} : \text{st}(w[1,m]) = u, \text{st}(w[m+1,n]) = v \},
\]

\[
\u \cdot u := \sum_{0 \leq i \leq m} u[1,i] \otimes \text{st}(u[i+1,m]),
\]

for all \( u \in \mathfrak{S}_m \) and \( v \in \mathfrak{S}_n \). For example, one has

\[
\begin{align*}
21 \mathbin{\u} 12 &= 2134 + 2314 + 3214 + 3241 + 3421 + 34 \quad \text{and} \\
\m (231) &= \emptyset \otimes 2431 + 1 \otimes 321 + 12 \otimes 21 + 132 \otimes 1 + 2431 \otimes \emptyset.
\end{align*}
\]

There is another graded Hopf algebra \((\mathbb{Z}\mathfrak{S}, \u, \m)\) defined by

\[
u := \{ w \in \mathfrak{S}_{m+n} : \text{st}(w[1,m]) = u, \text{st}(w[m+1,n]) = v \},
\]

\[
\m \cdot u := \sum_{0 \leq i \leq m} u[1,i] \otimes \text{st}(u[i+1,m]),
\]

for all \( u \in \mathfrak{S}_m \) and \( v \in \mathfrak{S}_n \). For example, one has

\[
\begin{align*}
21 \mathbin{\u} 12 &= 2134 + 3124 + 3214 + 4123 + 4213 + 4312 \quad \text{and} \\
\m (231) &= \emptyset \otimes 2431 + 1 \otimes 321 + 12 \otimes 21 + 231 \otimes 1 + 2431 \otimes \emptyset.
\end{align*}
\]

Let \( u \in \mathfrak{S}_m \) and \( v \in \mathfrak{S}_n \). An element \((u, v) \in \mathfrak{S}_m \times \mathfrak{S}_n\) can be identified with a permutation \( u \cdot v \in \mathfrak{S}_{m+n}\) whose one-line notation is \( u(1) \cdots u(m)|v(m+1) \cdots (m+n) \). This gives an isomorphism between \( \mathfrak{S}_m \times \mathfrak{S}_n\) and the parabolic subgroup \( \mathfrak{S}_{m,n}\) of \( \mathfrak{S}_{m+n}\), and hence an embedding \( \mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}\). Let \( w \in \mathfrak{S}_{m+n}\).

Denote by \( m_m \) the \( m \)-th term in the coproduct \( \m \), and similarly for \( \u \). Using the linear maps defined in Definition 2.1.2, with \( S = \{ s_1, \ldots, s_{m+n-1} \} \) and \( I = S \setminus \{ s_m \} \), one can check that

\[
u = \mu \mathbin{\u} v = \mu \mathbin{\u} \nu, \quad m_m w = \mu \mathbin{\u} m_m w = \mu \mathbin{\u} \nu.
\]

Thus Proposition 2.1.5 implies the associativity and coassociativity of \( \mathbin{\u}, \m \). Moreover, Proposition 2.1.3 implies that the two graded Hopf algebras \((\mathbb{Z}\mathfrak{S}, \mathbin{\u}, \m)\) and \((\mathbb{Z}\mathfrak{S}, \u, \m)\) are isomorphic to each other via the map \( w \mapsto w^{-1} \) for all \( w \in \mathfrak{S}\), and dual to each other via the bilinear form \( \langle u, v \rangle = \delta_{u,v}, \forall u, v \in \mathfrak{S}\).
Finally, restricting the commutative diagram of representations of categories to type $A$ gives the following commutative diagram of graded Hopf algebras.

\[
\begin{array}{c}
\Sigma(\mathfrak{S}) \\ \downarrow \text{dual} \\
\Lambda(\mathfrak{S})
\end{array}
\]

(3.3)

Reflecting this diagram across the vertical line through $\mathbb{Z}\mathfrak{S}$ and $\Lambda(\mathfrak{S})$ gives a dual diagram.

3.2. Free quasisymmetric functions and related results. Let $(W,S)$ be the Coxeter system of type $A_{n-1}$, where $W = \mathfrak{S}_n$ and $S = \{s_1, \ldots, s_{n-1}\}$. Let $E = \mathbb{R}^n$ be a Euclidean space whose standard basis is $\{e_1, \ldots, e_n\}$. One can realize $\mathfrak{S}_n$ as a reflection group of $E$ whose root system $\Phi = \Phi^+ \cup \Phi^-$ is the disjoint union of $\Phi^+ = \{e_j - e_i : 1 \leq i < j \leq n\}$ and $\Phi^- = -\Phi^+$. The set $\Delta = \{e_i + e_j : 1 \leq i < j \leq n\}$ of simple roots corresponds to the generating set $S$ of simple reflections.

A parset $P$ in the root system $\Phi$ is equivalent to a partial order $\mathcal{P}(W)$. The Jordan-Hölder set $\mathcal{L}(P)$ consists of all linear extensions of the partial order $P$. A $P$-partition is a function $f : [n] \to \mathbb{Z}$ satisfying $i < j \Rightarrow f(i) \leq f(j)$ and $(i < p, j \neq i) \Rightarrow f(i) < f(j)$.

Let $X = \{x_i : i \in \mathbb{Z}\}$ be a set of noncommutative variables. For each $w \in \mathfrak{S}_n$ the generating function of $\mathcal{A}(P_w)$ is

\[
F_w := \sum_{f(w(1)) \leq \cdots \leq f(w(n))} x_{f(1)} \cdots x_{f(n)}.
\]

Let $\mathbf{FQSym} = \mathbf{FQSym}(X)$ be the free $\mathbb{Z}$-module with a basis $\{F_w : w \in \mathfrak{S}\}$. If $u \in \mathfrak{S}_m$ and $v \in \mathfrak{S}_{n-m}$ then $u \times v \in W \cong \mathfrak{S}_m \times \mathfrak{S}_{n-m}$ where $I = S \setminus \{s_m\}$, and thus Proposition 2.2.2 implies

\[
F_u \cdot F_v = F_{(u \times v)\Phi^+} = \sum_{w \in u \cup v} F_w.
\]

Thus $\mathbf{FQSym}$ is a graded algebra isomorphic to $(\mathbb{Z}\mathfrak{S}, \cup, \cap)$.

Then $\mathbf{FQSym}$ becomes a self-dual graded Hopf algebra isomorphic to the Malvenuto–Reutenauer algebra $(\mathbb{Z}\mathfrak{S}, \cup, \cap)$ via $F_w \mapsto w$ for all $w \in \mathfrak{S}$.

Remark 3.2.1. Replacing $X$ in the above definition of $\mathbf{FQSym}$ with any totally ordered countably infinite set $Z$ of noncommutative variables gives a Hopf algebra $\mathbf{FQSym}(Z)$ isomorphic to $\mathbf{FQSym}$. In fact, Duchamp, Hivert, and Thibon used $X_{\geq 0} := \{x_i : i \in \mathbb{Z}_{\geq 0}\}$ when they introduced $\mathbf{FQSym}$. We will also use $X_{\geq 0} := \{x_i : i \in \mathbb{Z}_{\geq 0}\}$ later.

Remark 3.2.2. One can define $[11, 8.1]$ the coproduct of $\mathbf{FQSym}$ by $\Delta F := F(X + Y)$ for all $F \in \mathbf{FQSym}$, where $X + Y$ is the union of two totally ordered countable sets of noncommutative variables $X = \{x_i : i \in \mathbb{Z}\}$ and $Y = \{y_i : i \in \mathbb{Z}\}$ with $x_i < y_j$ and $x_i y_j = y_j x_i$ for all $i, j \in \mathbb{Z}$. To avoid technicality we simply use $\mathbf{FQSym}$ as the definition of the coproduct of $\mathbf{FQSym}$, and will similarly deal with this issue in type B and D.

Applying Proposition 2.2.2 to $\alpha = e_i - e_j$ for all pairs $i, j \in [n]$ satisfying $1 \leq i < j \leq n$ shows that $f \in \mathcal{A}(w^{-1}\Phi^+)$ if and only if $\text{st}(f) = w$ for all $w \in \mathfrak{S}_n$. Thus one has

\[
\Delta F_w := \sum_{1 \leq i \leq n} F_{\text{st}(w[1,i])} \otimes F_{\text{st}(w[i+1,n])}, \quad \forall w \in \mathfrak{S}_n.
\]

This leads to a Hopf algebra isomorphism $\mathbf{FQSym} \cong (\mathbb{Z}\mathfrak{S}, \cup, \cap)$ by $s_w \mapsto w$ for all $w \in \mathfrak{S}$.

The graded Hopf algebra $\mathbf{NSym}$ of noncommutative symmetric functions is the free associative algebra $\mathbb{Z} \langle h_1, h_2, \ldots \rangle$ generated by $h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$ for all $k \geq 1$. It has a free $\mathbb{Z}$-basis consisting of the
complete homogeneous noncommutative symmetric functions $h_{\alpha} := h_{\alpha_1} \cdots h_{\alpha_\ell}$ and another basis consisting of the noncommutative ribbon Schur functions

$$s_\alpha := \sum_{\beta \leq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} h_\beta$$

where $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ runs through all compositions. Note that $s_{\emptyset} = h_{\emptyset} := 1$.

The product of $\text{NSym}$ is determined by the following two equivalent formulas

$$h_\alpha h_\beta = h_{\alpha \cdot \beta} \quad \text{and} \quad s_\alpha s_\beta = s_{\alpha \cdot \beta} + s_{\alpha \triangleright \beta}$$

for all compositions $\alpha$ and $\beta$, where the last term is treated as zero when $\alpha \triangleright \beta$ is undefined. The coproduct of $\text{NSym}$ is defined by $\Delta h_k = \sum_{i=0}^k h_i \otimes h_{k-i}$, where $h_0 := 1$. A more explicit coproduct formula was provided in our earlier work [12].

Given a composition $\alpha$, a tableau $\tau$ of shape $\alpha$ is a filling of the ribbon diagram of $\alpha$ with integers. Reading these integers from the bottom row to the top row and proceeding from left to right within each row gives the reading word $w(\tau)$ of $\tau$. We call $\tau$ a semistandard tableau of shape $\alpha$ if each row is weakly increasing from left to right and each column is strictly increasing from top to bottom. One sees that $s_\alpha$ is the sum of $x_{w(\tau)}$ for all semistandard tableaux $\tau$ of shape $\alpha$. Moreover, each $f \in \mathbb{Z}^n$ corresponds to a unique tableau $\tau$ of shape $\alpha$ such that $w(\tau) = f$; one has $D(\text{st}(f)) = D(\alpha)$ if and only if $\tau$ is semistandard. Thus $s_\alpha$ equals the sum of $s_{x_\alpha}$ for all $x_\alpha$ in the descent class of $\alpha$. It follows that $\text{NSym}$ is a Hopf subalgebra of $\text{FQSym}$ isomorphic to the descent algebra $\Sigma(\mathcal{S})$.

Let $X_{>0} = \{x_1, x_2, \ldots\}$ be a set of commutative variables. The graded Hopf algebra of quasisymmetric functions is defined as $\text{QSym} := \chi(\text{FQSym})$, where $\chi$ replaces $X$ with $X_{>0}$. If $\alpha = (\alpha_1, \ldots, \alpha_\ell) \triangleright n$ then one has the monomial quasisymmetric function

$$M_{\alpha} := \sum_{0 < i_1 < \cdots < i_\ell} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}$$

and the fundamental quasisymmetric function

$$F_{\alpha} := \sum_{\alpha \leq \beta} M_\beta = \sum_{0 < i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}.$$ 

One sees that $F_w := \chi(F_w)$ equals $F_{\alpha}$ if $w$ is in the descent class of $\alpha$. Hence $\text{QSym}$ has two bases $\{M_\alpha\}$ and $\{F_\alpha\}$ where $\alpha$ runs through all compositions. The product and coproduct of $F_\alpha$ can be easily obtained by applying $\chi$ to $\text{FQSym}$. Thus $\text{QSym}$ is isomorphic to the dual $\Sigma^*(\mathcal{S})$ of the descent algebra $\Sigma(\mathcal{S})$. The self-duality of $\text{FQSym}$ induces the duality between $\text{QSym}$ and $\text{NSym}$ via $\langle h_\alpha, M_\beta \rangle = \langle s_\alpha, F_\beta \rangle := \delta_{\alpha,\beta}$ for all compositions $\alpha$ and $\beta$.

A partition $\lambda$ of $n$, denoted by $\lambda \vdash n$, is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that its size $|\lambda| := \lambda_1 + \cdots + \lambda_\ell = n$. Denote by $\lambda(\alpha)$ the partition obtained from a composition $\alpha$ by rearranging its parts. One sees that two parabolic subgroups $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are conjugate if and only if $\lambda(\alpha) = \lambda(\beta)$.

The graded Hopf algebra $\text{Sym} := \chi(\text{NSym})$ of symmetric functions is the $\mathbb{Z}$-span of the ribbon Schur functions $s_\alpha := \chi(s_\alpha)$ for all compositions $\alpha$, where $\chi : \text{NSym} \to \text{Sym}$ is a surjection of Hopf algebras defined by replacing $X$ with $X_{>0}$. One sees that $\text{Sym}$ is isomorphic to $\Lambda(\mathcal{S})$ defined in Section [24]. If $\lambda$ is a partition then the complete homogeneous symmetric function $h_\lambda := \chi(h_\lambda)$ is well defined for any composition $\alpha$ with $\lambda(\alpha) = \lambda$. Hence $\text{Sym}$ has a free $\mathbb{Z}$-basis consisting of $h_\lambda$ for all partitions $\lambda$. Another free $\mathbb{Z}$-basis for $\text{Sym}$ consists of the monomial symmetric functions $m_\lambda$ for all partitions $\lambda$, where $m_\lambda$ is the sum of $M_\alpha$ for all compositions $\alpha$ with $\lambda(\alpha) = \lambda$. Hence $\text{Sym}$ is a Hopf subalgebra of $\text{QSym}$. The product and coproduct of $\text{Sym}$ can be obtained from $\text{QSym}$ and $\text{NSym}$ by Corollary [24.16]. The pairing between $\text{NSym}$ and $\text{QSym}$ induces a bilinear form on $\text{Sym}$ such that $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$ for all partitions $\lambda$ and $\mu$. With this bilinear form $\text{Sym}$ becomes self-dual. The inclusion $\iota : \text{Sym} \to \text{QSym}$ and the surjection $\chi : \text{NSym} \to \text{Sym}$ are also dual to each other.
In summary, one has the following commutative diagram of graded Hopf algebras isomorphic to \( \mathbb{Q} \).

\[
\begin{array}{ccc}
\mathbb{FQSym} & \xleftarrow{\Delta} & \mathbb{QSym} \\
\xrightarrow{\mu} & \xrightarrow{\epsilon} & \\
\mathbb{NSym} & \xrightarrow{\mu} & \mathbb{QSym}
\end{array}
\]

\[(3.6)\]

3.3. Representation theory. We first briefly review the (complex) representation theory of \( S_n \); see, e.g., Stanley [24 Chapter 7] for details. The group algebra \( \mathbb{C}[S_n] \) is semisimple. The simple \( \mathbb{C}[S_n] \)-modules \( S_\lambda \) are indexed by partitions \( \lambda \vdash n \). The Grothendieck group \( G_0(\mathbb{C}[S_\bullet]) \) of the tower \( \mathbb{C}[S_0] \hookrightarrow \mathbb{C}[S_1] \hookrightarrow \mathbb{C}[S_2] \hookrightarrow \cdots \) of algebras is a graded Hopf algebra, whose product and coproduct are given by induction and restriction along natural embeddings \( \mathbb{C}[S_m \times S_n] \hookrightarrow \mathbb{C}[S_m \times S_n] \). This Hopf algebra is self-dual under the bilinear form defined by \( \langle S_\lambda, S_\mu \rangle := \delta_{\lambda, \mu} \) for all partitions \( \lambda \) and \( \mu \). The Frobenius characteristic is a graded Hopf algebra isomorphism \( G_0(\mathbb{C}[S_\bullet]) \cong \text{Sym} \) defined by sending \( S_\lambda \) to the Schur function \( \text{ch} \).

Now let \( H_n(0) \) be the 0-Hecke algebra of the Coxeter system \((W, S)\) of type \( A_{n-1} \), where \( W = S_n \) and \( S = \{s_1, \ldots, s_{n-1}\} \). The generators \( \pi_1, \ldots, \pi_{n-1} \) for \( H_n(0) \) can be interpreted as the bubble-sorting operators: \( \pi_i \) swaps adjacent positions \( a_i \) and \( a_{i+1} \) in a word \( a_1 \cdots a_n \in \mathbb{Z}^n \) if \( a_i < a_{i+1} \), or fixes the word otherwise.

The projective indecomposable \( H_n(0) \)-modules and simple \( H_n(0) \)-modules are given by \( \mathcal{P}_\alpha := \mathcal{P}_\alpha^0 \) and \( \mathcal{S}_\alpha := \mathcal{C}_\alpha^0 \), respectively, where \( I = \{s_i : i \in D(\alpha)\} \), for all \( \alpha \vdash n \). One can realize \( \mathcal{P}_\alpha \) as the space of standard tableaux of ribbon shape \( \alpha \) with an appropriate \( H_n(0) \)-action \( [14] \).

The parabolic subalgebra \( H_\alpha(0) \) of \( H_n(0) \) is generated by \( \{\pi_i : i \in [n-1] \setminus D(\alpha)\} \). One has an embedding \( H_m(0) \otimes H_n(0) \cong H_{m+n}(0) \subseteq H_{m+n}(0) \) if \( m \) and \( n \) are nonnegative integers.

Associated with the tower of algebras \( H_\bullet(0) : H_0(0) \hookrightarrow H_1(0) \hookrightarrow H_2(0) \hookrightarrow \cdots \) are Grothendieck groups

\[
G_0(H_\bullet(0)) := \bigoplus_{n \geq 0} G_0(H_n(0)) \quad \text{and} \quad K_0(H_\bullet(0)) := \bigoplus_{n \geq 0} K_0(H_n(0)).
\]

The Grothendieck groups \( G_0(H_\bullet(0)) \) and \( K_0(H_\bullet(0)) \) are both graded Hopf algebras, whose product \( \otimes \) and coproduct \( \Delta \) are defined by

\[
M \otimes N := (M \otimes N) \uparrow H_{m+n}(0) \quad \text{and} \quad \Delta(M) := \sum_{0 \leq i \leq m} M \downarrow H_{m}(0) H_{m-i}(0)
\]

for all finitely generated (projective) modules \( M \) and \( N \) over \( H_m(0) \) and \( H_n(0) \), respectively. Using the linear maps \( \bar{\mu}_i^S : G_0(H_W(0)) \to G_0(H_W(0)) \) and \( \bar{\rho}_i^S : G_0(H_W(0)) \to G_0(H_W(0)) \) defined in [24], with \( S = \{s_1, \ldots, s_{m+n-1}\} \) and \( I = S \setminus \{s_m\} \), one has

\[
M \otimes N = \bar{\mu}_i^S(M \otimes N) \quad \text{and} \quad \Delta_m(Q) := Q \downarrow H_{m+n}(0) = \bar{\rho}_i^S(Q)
\]

where \( Q \) is a finitely generated \( H_{m+n}(0) \)-module. Also recall from Proposition 2.3.1 that \( \bar{\mu}_i^S \) and \( \bar{\rho}_i^S \) restrict to \( \mu_i^\mathbb{S} : K_0(H_W(0)) \to K_0(H_W(0)) \) and \( \rho_i^\mathbb{S} : K_0(H_W(0)) \to K_0(H_W(0)) \).

In this section we apply our results in Section 2 to type B and get some new results.

4. Type B
4.1. Malvenuto–Reutenauer algebra and descent algebra of type B. A signed permutation $w$ of $[n]$ is a bijection of the set $[\pm n] := \{\pm 1, \ldots, \pm n\}$ onto itself such that $w(-i) = -w(i)$ for all $i \in [\pm n]$. We set $w(0) = 0$. Since $w$ is determined by where it sends $1, \ldots, n$, one can identify $w$ with $w(1) \cdots w(n)$ or $[w(1), \ldots, w(n)]$, where a negative integer $-k < 0$ is often written as $k^\circ$. This is the window notation of $w$.

The hyperoctahedral group $\mathfrak{S}_n^B$ consists of all signed permutations of $[n]$. It is generated by $s_0 = s_n^B := \bar{1}2 \cdots n$ and $s_i := [1, \ldots, i-1, i+1, i, i+2, \ldots, n]$ for all $i \in [n-1]$. The parabolic subgroup of $\mathfrak{S}_n^B$ generated by $s_1, \ldots, s_{n-1}$ is isomorphic to $\mathfrak{S}_n$. The pair $(\mathfrak{S}_n^B, [s_0, \ldots, s_{n-1}])$ is the finite irreducible Coxeter system of type $B_n$ whose Coxeter diagram is below.

$$s_0 = s_1 = s_2 = \cdots = s_{n-2} = s_{n-1}$$

Let $w \in \mathfrak{S}_n^B$. The descent set of $w$ is $\{s_i : i \in \{0, 1, \ldots, n-1\}, w(i) > w(i+1)\}$. Given a word $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we define $\text{neg}(a) := \# \text{Neg}(a)$ and $\text{nsp}(w) := \# \text{Nsp}(w)$ where

$$\text{Neg}(a) := \{i \in [n] : a_i < 0\} \quad \text{and} \quad \text{Nsp}(a) := \{(i,j) : 1 \leq i < j \leq n, a_i + a_j < 0\}.$$ 

Then the length of $w$ equals $\text{inv}(w) + \text{neg}(w) + \text{nsp}(w)$. Note that $\text{inv}(w)$ is defined by $[\text{3.1}].$

A pseudo-composition of $n$ is a sequence $\alpha = (a_1, \ldots, a_\ell)$ of integers such that $a_1 \geq 0, a_2, \ldots, a_\ell > 0$, and the size $|\alpha| := a_1 + \cdots + a_\ell$ equals $n$. This is denoted by $\alpha \vdash B n$. The length of $\alpha$ is $\ell(\alpha) := \ell$. The descent set of $\alpha$ is $D(\alpha) := \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_\ell\} \setminus \{0\}$. The map $\alpha \mapsto D(\alpha)$ is a bijection between pseudo-compositions of $n$ and the subsets of $\{0, 1, \ldots, n-1\}$. Analogously to type $A$, it is convenient to index subsets of the generating set $\{s_0, s_1, \ldots, s_{n-1}\}$ of $\mathfrak{S}_n^B$ by pseudo-compositions of $n$. If $\alpha = (a_1, \ldots, a_\ell) \vdash B n$ then the parabolic subgroup $\mathfrak{S}_n^{\alpha} \cong \mathfrak{S}_n^{a_1} \times \mathfrak{S}_n^{a_2} \times \cdots \times \mathfrak{S}_n^{a_\ell}$ of $\mathfrak{S}_n^B$ is generated by $s_i^\alpha := [s_i : 0 \leq i \leq n-1, i \not\in D(\alpha)]$. The minimal representatives for left $\mathfrak{S}_n^B$-cosets in $\mathfrak{S}_n^B$ form the set $(\mathfrak{S}_n^B)^\alpha := \{w \in \mathfrak{S}_n^B : D(w) \subseteq D(\alpha)\}$. A pseudo-composition $\alpha = (a_1, \ldots, a_\ell)$ can be identified with a pseudo-ribbon in the following way. If $a_1 > 0$ then $\alpha$ can be viewed as a composition which corresponds to a ribbon, and we draw an extra 0-box to the left of the bottom row of this ribbon. If $a_1 = 0$ then $(a_2, \ldots, a_\ell)$ corresponds to a ribbon, and we draw an extra 0-box below the leftmost column of this ribbon. Some examples are below.

$$\begin{array}{c}
(2,3,1,1) \leftrightarrow \\
\begin{array}{c}
\includegraphics{ribbon1.png}
\end{array}
\end{array}$$

$$(0,2,3,1,1) \leftrightarrow \\
\begin{array}{c}
\includegraphics{ribbon2.png}
\end{array}$$

If $\alpha = (a_1, \ldots, a_\ell) \vdash B m \geq 1$ and $\beta = (\beta_1, \ldots, \beta_k) \vdash n \geq 1$ then define

$$\begin{array}{c}
\alpha \cdot \beta := (a_1, \ldots, a_\ell, \beta_1, \ldots, \beta_k) \quad \text{and} \\
\alpha \triangleright \beta := (a_1, \ldots, a_\ell-1, a_\ell + \beta_1, \beta_2, \ldots, \beta_k).
\end{array}$$

Let $\alpha \cdot \emptyset := \alpha$ and $\emptyset \cdot \beta := \beta$. Also set $\alpha \triangleright \emptyset$ and $\emptyset \triangleright \beta$ undefined. If $\alpha$ and $\beta$ are pseudo-compositions of the same size and $D(\alpha) \subseteq D(\beta)$ then write $\alpha \preceq \beta$.

We define the signed standardization of a word $a \in \mathbb{Z}^n$, denoted by $\text{st}^B(a)$, to be the unique signed permutation $w \in \mathfrak{S}_n^B$ such that $\text{Neg}(a) = \text{Neg}(w)$ and the following holds whenever $1 \leq i < j \leq n$:

$$(4.1) \quad |w(i)| < |w(j)| \iff |a_i| < |a_j| \quad \text{or} \quad a_i = a_j \geq 0 \quad \text{or} \quad a_i = -a_j < 0.$$ 

Proposition [4.2.1] provides an interpretation of the signed standardization by P-partition theory.

One can obtain $\text{st}^B(a)$ by reading the letters in $a$ in the increasing order of their absolute values, breaking up ties first from right to left for the negative letters and then from left to right for the positive ones, and finally inserting negative signs at the same positions as in $a$. For example, one has $\text{st}^B(24320202) = 58741623$.

Let $a = a_1 \cdots a_n \in \mathbb{Z}^n$. Suppose that $\text{Neg}(a) = \{i_1, \ldots, i_k\}$, where $i_1 < \cdots < i_k$, and $[n] \setminus \text{Neg}(a) = \{j_1, \ldots, j_{n-k}\}$, where $j_1 < \cdots < j_{n-k}$. We define

$$\tilde{a} := \overline{a_{i_k} \cdots a_{i_1} a_{j_1} \cdots a_{j_{n-k}}}.$$ 

One sees that $\text{st}^B(a) = w$ if and only if $\text{Neg}(a) = \text{Neg}(w)$ and $\text{st}(\tilde{a}) = \tilde{w}$.

An element $(u, v) \in \mathfrak{S}_m^B \times \mathfrak{S}_n$ can be identified with a signed permutation $u \times v \in \mathfrak{S}_m^{B+n}$ defined by the window notation $[u(1), \ldots, u(m), m+v(1), \ldots, m+v(n)]$. This gives an isomorphism between $\mathfrak{S}_m^B \times \mathfrak{S}_n$ and
the parabolic subgroup $\mathfrak{S}_{m,n}^B$ of $\mathfrak{S}_{m+n}^B$ generated by the set \( \{ s_i : 0 \leq i \leq m + n - 1, i \neq m \} \). Thus one has an embedding $\mathfrak{S}_m^B \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}^B$. The set of minimal representatives for left $\mathfrak{S}_{m,n}^B$-cosets in $\mathfrak{S}_{m+n}^B$ is
\[
(\mathfrak{S}_{m,n}^B)^{m,n} := \{ z \in \mathfrak{S}_{m+n}^B : 0 < z(1) < \cdots < z(m), z(m+1) < \cdots < z(m+n) \}.
\]

**Proposition 4.1.1.** Every element $w \in \mathfrak{S}_{m+n}^B$ can be written uniquely as $w = (u \times v)$ where $u \in \mathfrak{S}_m^B$, $v \in \mathfrak{S}_n$, and $z^{-1} \in (\mathfrak{S}_{m,n}^B)^{m,n}$. Moreover, one has
\[
\begin{align*}
u & = w[1, m], \\
u^{-1} & = st^B(w^{-1}[1, m]), \\
v & = st(w[m+1, m+n]), \\
v^{-1} & = st(w^{-1}[m+1, m+n]).
\end{align*}
\]

**Proof.** Applying Proposition 2.1.1 to the parabolic subgroup $\mathfrak{S}_{m,n}^B$ shows that every element $w \in \mathfrak{S}_{m+n}^B$ can be written uniquely as $w = (u \times v)$ where $u \in \mathfrak{S}_m^B$, $v \in \mathfrak{S}_n$, and $z^{-1} \in (\mathfrak{S}_{m,n}^B)^{m,n}$.

Since $w(z^{-1}(i)) = u(i)$ for all $i \in [m]$ and $0 < z^{-1}(1) < \cdots < z^{-1}(m)$, one can obtain $u(1), \cdots, u(m)$ by reading from left to right those letters in $w(1), \ldots, w(m+n)$ with absolute values in $[m]$. This implies $u = w[1, m]$. Similarly, since $w(z^{-1}(m+j)) = m + v(j)$ for all $j \in [n]$ and $z^{-1}(m+1) < \cdots < z^{-1}(m+n)$, one can obtain $m + v(1), \ldots, m + v(n)$ by reading those letters in $w(1), \ldots, w(m+n)$ with absolute values in $[m+1, m+n]$, beginning with the negative ones from right to left, then followed by the positive ones from left to right. This implies $v = w[m+1, m+n]$.

On the other hand, one has $w^{-1} = z^{-1}(u^{-1} \times v^{-1})$. Since $0 < z^{-1}(1) < \cdots < z^{-1}(m)$, one sees that $w^{-1}(i) = z^{-1}(u^{-1}(i))$ and $u^{-1}(i)$ have the same sign for all $i \in [m]$, and if $1 \leq i < j \leq n$ then
\[
|w^{-1}(i)| < |w^{-1}(j)| \Leftrightarrow |u^{-1}(i)| < |u^{-1}(j)|.
\]

Thus $u^{-1} = st^B(w^{-1}[1, m])$. Similarly, one has $w^{-1}(m+i) = z^{-1}(m - v^{-1}(i))$ for all $i \in [n]$. It follows from $z^{-1}(m+1) < \cdots < z^{-1}(m+n)$ that
\[
 w^{-1}(m+i) < w^{-1}(m+j) \Leftrightarrow v^{-1}(i) < v^{-1}(j)
\]
whenever $1 \leq i < j \leq n$. Hence $v^{-1} = st(w^{-1}[m+1, m+n])$. 

Let $\mathcal{B} := \bigcup_{n \geq 0} \mathfrak{S}_n^B$. We use Proposition 4.1.1 to realize $\mathcal{Z} \mathcal{B} = \bigoplus_{n \geq 0} \mathcal{Z} \mathfrak{S}_n^B$ as a dual graded right module and comodule over the Malvenuto–Reutenauer algebra $\mathcal{Z} \mathcal{S}$. We define
\[
\begin{align*}
u \uparrow^B v & := \{ w \in \mathfrak{S}_{m+n}^B : w[1, m] = u, \ st(w[m+1, m+n]) = v \}, \\
u \downarrow^B v & := \{ w \in \mathfrak{S}_{m+n}^B : st^B(w[1, m]) = u, \ st(w[m+1, m+n]) = v \}, \\
\uparrow^B \uparrow^B u & := \sum_{0 \leq i \leq m} st^B(u[1, i]) \otimes st(u[i, 1, m]), \\
\uparrow^B \downarrow^B u & := \sum_{0 \leq i \leq m} u[1, i] \otimes st(u[i, 1, m])
\end{align*}
\]
for all $u \in \mathfrak{S}_m^B$ and $v \in \mathfrak{S}_n$. For example, one has
\[
\begin{align*}
\uparrow^B 21 &= 132 + 312 + 321 + 312 + 321 + 132 + 213 + 231 + 123 + 123 + 231 + 213 + 231, \\
\downarrow^B 21 &= 132 + 231 + 231 + 132 + 231 + 132 + 123 + 231 + 123 + 231 + 231 + 231 + 123. \\
\uparrow^B \uparrow^B 21 &= \emptyset \otimes 4123 + \emptyset \otimes 123 + \emptyset \otimes 41231 + \emptyset \otimes 1231 + \emptyset \otimes 41231, \\
\uparrow^B \downarrow^B 21 &= \emptyset \otimes 4321 + \emptyset \otimes 1231 + \emptyset \otimes 231 + \emptyset \otimes 231 + \emptyset \otimes 231.
\end{align*}
\]

Let $u \in \mathfrak{S}_m^B$, $v \in \mathfrak{S}_n$, and $w \in \mathfrak{S}_{m+n}^B$. Denote by $\uparrow^B_m w$ the $m$-th term in $\uparrow^B w$, and similarly for $\downarrow^B_m w$. Using Proposition 4.1.1 and the linear maps defined in Definition 2.1.2 with $S = \{ s_0, \ldots, s_{m+n-1} \}$ and $I = S \setminus \{ s_m \}$, one has
\[
\begin{align*}
u \uparrow^B v & = \mu^S_{r_u}(u \times v), \\
u \downarrow^B v & = \mu^S_{r_v}(u \times v), \\
u \uparrow^B w & = \mu^S_{r_u}(w), \\
u \downarrow^B w & = \mu^S_{r_v}(w).
\end{align*}
\]

**Proposition 4.1.2.** (i) $(\mathcal{Z} \mathcal{B}, \uparrow^B, \downarrow^B)$ is a graded right module and comodule over the graded Hopf algebra $(\mathcal{Z} \mathcal{S}, \uparrow, \downarrow)$.

(ii) $(\mathcal{Z} \mathcal{B}, \uparrow^B, \downarrow^B)$ is a graded right module and comodule over the graded Hopf algebra $(\mathcal{Z} \mathcal{S}, \uparrow^B, \downarrow^B)$ via the pairing $(u, v) := \delta_{u,v}$, $\forall u, v \in \mathfrak{S}_B$.

(iii) $(\mathcal{Z} \mathcal{B}, \uparrow^B, \downarrow^B)$ is dual to $(\mathcal{Z} \mathcal{B}, \uparrow^B, \downarrow^B)$ via the pairing $(u, v) := \delta_{u,v}$, $\forall u, v \in \mathfrak{S}_B$.

(iv) Sending $w$ to $w^{-1}$ gives an isomorphism between $(\mathcal{Z} \mathcal{B}, \uparrow^B, \downarrow^B)$ and $(\mathcal{Z} \mathcal{B}, \uparrow^B, \downarrow^B)$. 

Proof. It is clear that \( u \uparrow B \varnothing = u \) for any \( u \in B \), where \( \varnothing \in G_0 \) is the empty permutation. Let \( u \in B \), \( v \in \mathfrak{S}_n \), \( r \in \mathfrak{S}_k \). Proposition 2.1.5 implies \( (u \uparrow B v) \uparrow B r = u \uparrow B (v \uparrow B r) \). In fact, one can show that \( (u \uparrow B v) \uparrow B r \) and \( u \uparrow B (v \uparrow B r) \) both equal
\[
\{ w \in B : s^B (w[1]) = u, \ s^B (w[m + 1, m + n]) = v, \ s^B (w[m + n + 1, m + n + k]) = r \}.
\]
Hence \((B \uparrow B) \) is a graded right \((B, B)\)-module. The remaining results follow from Proposition 2.1.3.

The proof of \((\mathcal{G} \uparrow B) \neq (\mathcal{G} \uparrow B) \) implies \((\mathcal{G} \uparrow B, \uparrow B) \) is not a \((B, B)\)-Hopf module. Consequently, the dual \((\mathcal{G} \uparrow B, \uparrow B) \) is not a \((B, B)\)-Hopf module either.

Let \( \Sigma(B) := \bigoplus_{n \geq 0} \Sigma(B_n) \) where \( \Sigma(B_n) \) is the free \( \mathbb{Z} \)-module with a basis consisting of descent classes
\[
D_\alpha(B_n) := \{ w \in B_n : D(w) = D(\alpha) \}, \quad \forall \alpha \uparrow B n.
\]
One has an embedding \( : \Sigma(B) \rightarrow B \) by inclusion. If \( \alpha \uparrow B m \) and \( \beta \uparrow n \) then
\[
D_\alpha(B_m) \uparrow B D_\beta(B_n) = D_{\alpha \beta}(B_{m+n}) + D_{\alpha \beta}(B_{m+n})
\]
by Proposition 2.1.9 where the last term is treated as zero when \( \alpha \uparrow \beta \) is undefined.

Let \( \Sigma^*(B) := \bigoplus_{n \geq 0} \Sigma^*(B_n) \) where \( \Sigma^*(B_n) \) is the dual of \( \Sigma(B_n) \) with a dual basis \( \{ D_\alpha^*(B_n) : \alpha \uparrow B n \} \).

Dual to \( : \Sigma(B) \rightarrow B \) is a surjection \( \chi : B \rightarrow \Sigma^*(B) \) sending each \( w \in B_n \) to \( D_\alpha^*(B_n) := D_\alpha^*(B_n) \), where \( \alpha \uparrow B n \) satisfies \( D(w) = D(\alpha) \).

Recall from Section 2.1 that \( \Lambda(B) \) is the \( \mathbb{Z} \)-span of \( \Lambda_\alpha(B_n) := \chi'(D_\alpha(B_n)) \) for all \( \alpha \uparrow B n \), where \( \chi' := \chi \circ (\ )^{-1} \). Let \( \Lambda(B) := \bigoplus_{n \geq 0} \Lambda(B_n) \). For \( \alpha \uparrow B m \) and \( \beta \uparrow B n \) define
\[
\langle \Lambda_\alpha(B_m), \Lambda_\beta(B_n) \rangle := \begin{cases} 
\# \{ w \in B_n : D(w) = D(\alpha), \ D(w) = D(\beta) \}, & \text{if } m = n, \\
0, & \text{if } m \neq n.
\end{cases}
\]

By Proposition 2.1.11 this gives a well-defined symmetric nondegenerate bilinear form on \( \Lambda(B) \) such that \( : \Lambda(B) \rightarrow \Sigma^*(B) \) and \( \chi' : \Sigma(B) \rightarrow \Lambda(B) \) are dual to each other.

\textbf{Theorem 4.1.4.} The following diagram is commutative with each entry being a graded right module and comodule over the corresponding type A Hopf algebra in \( \mathbb{Z} \).

\[
\begin{array}{ccc}
\mathbb{Z} & \rightarrow & B \\
\downarrow \uparrow & & \downarrow \uparrow \\
\Sigma(B) & \leftarrow & \Sigma^*(B)
\end{array}
\]

Reflecting it across the vertical line through \( \mathbb{Z} \) and \( \Lambda(B) \) gives a dual diagram.

\textbf{Proof.} Apply Theorem 2.1.13 to \((B \uparrow B, \uparrow B, \uparrow B, \uparrow B, \uparrow B) \) and then use Corollary 2.1.10 \( \square \)

\subsection{Free quasisymmetric functions of type B and related results.}

In this subsection we obtain the following commutative diagram.

\[
\begin{array}{ccc}
\text{FQSym}_B & \leftarrow & \text{QSym}_B \\
\downarrow \uparrow & & \downarrow \uparrow \\
\text{NSym}_B & \leftarrow & \text{Sym}_B
\end{array}
\]

It is isomorphic to the diagram \((3.3)\) with each entry being a graded right module and comodule over the corresponding type A Hopf algebra in \( \mathbb{Z} \). Reflecting it across the vertical line through \( \text{FQSym}_B \) and \( \text{Sym}_B \) gives a dual diagram of graded modules and comodules.
4.2.1. Free quasisymmetric functions of type $B$. Let $(W, S)$ be the Coxeter system of type $B_n$, where $W = \mathfrak{S}_n^B$ and $S = \{s_0 = s_0^B, s_1, \ldots, s_{n-1}\}$. Let $E = \mathbb{R}^n$ be a Euclidean space with standard basis $\{e_1, \ldots, e_n\}$. The hyperoctahedral group $\mathfrak{S}_n^B$ can be realized as a reflection group of $E$ whose root system $\Phi$ is the disjoint union of $\Phi^+ = \{e_i, e_j \pm e_i : 1 \leq i < j \leq n\}$ and $\Phi^- = -\Phi^+$. The set of simple roots is $\Delta = \{e_1, e_2 - e_1, \ldots, e_n - e_{n-1}\}$, corresponding to the generating set $S$ of simple reflections.

Let $X = \{x_i : i \in \mathbb{Z}\}$ be a set of noncommutative variables. Define $\text{FQSym}^B$ to be the $\mathbb{Z}$-span of the generating functions $F^B_x$ for all sets $P$ of $\Phi$. Let $\text{FQSym}^B := \bigoplus_{n \geq 0} \text{FQSym}^B_n$. By Proposition 2.2.4 $\text{FQSym}^B$ has free $\mathbb{Z}$-basis $\{F^B_x : w \in \mathfrak{S}_n^B\}$ and $\{s^B_\mu : w \in \mathfrak{S}_n^B\}$, where $F^B_w$ is the generating function of the parset $w\Phi^+$ and $s^B_w := F^B_w$. Applying the definition of $f \in \mathcal{A}(w\Phi^+)$ to $u\alpha$ for all $\alpha \in \Delta$ gives

\begin{equation}
F^B_w = \sum_{f(w(0)) \leq f(w(1)) \leq \cdots \leq f(w(n)) \text{ for } i \in D(w) \Rightarrow f(u(i)) < f(w(i+1))} x_{f(1)} \cdots x_{f(n)}, \quad \forall w \in \mathfrak{S}_n^B.
\end{equation}

Here we set $w(0) = 0$, $f(0) = 0$, and $f(-i) = -f(i)$ for all $i \in [n]$ by convention.

**Proposition 4.2.1.** Let $w \in \mathfrak{S}_n^B$ and let $f \in \mathbb{Z}^n$. Then $f \in \mathcal{A}(w^{-1}\Phi^+)$ if and only if $\text{st}^B(f) = w$ and thus

$$s^B_w = \sum_{f \in \mathbb{Z}^n: \text{st}^B(f) = w} x_f.$$  

**Proof.** By Lemma 2.2.3 it suffices to show that $f \in \mathcal{A}(w^{-1}\Phi^+)$ implies $\text{st}^B(f) = w$. Applying Proposition 2.2.2 to $\alpha = e_i$ for all $i \in [n]$ gives $\text{Neg}(f) = \text{Neg}(w)$. Let $1 \leq i < j \leq n$ below. Apply Proposition 2.2.3 to $\alpha = e_j - e_i$ gives $w(i) < w(j) \Rightarrow f(i) \leq f(j)$, which implies \eqref{eq:st} when $w(i)$ and $w(j)$ have the same sign. Applying Proposition 2.2.3 to $\alpha = e_j + e_i$ gives $w(j) > w(i) \Rightarrow f(j) + f(i) \geq 0$, which implies \eqref{eq:st} when $w(i)$ and $w(j)$ have the opposite signs. Hence $\text{st}^B(f) = w$. \hfill $\Box$

**Corollary 4.2.2.** If $w \in \mathfrak{S}_n$ then $s^B_w$ equals the sum of $s^B_u$ for all $u \in \mathfrak{S}_n^B$ with $\text{st}(u) = w$. Consequently $\text{FQSym} \subseteq \text{FQSym}^B$.

**Proof.** The result follows from \eqref{eq:st} and Proposition 4.2.1 if we can prove $\text{st}(\text{st}^B(a)) = \text{st}(a)$ for all $a \in \mathbb{Z}^n$. Let $\text{st}^B(a) = u \in \mathfrak{S}_n^B$. Then $\text{Neg}(a) = \text{Neg}(u)$. Assume $1 \leq i < j \leq n$. If $a_i$ and $a_j$ are both positive then so are $u(i)$ and $u(j)$, and thus $u(i) < u(j) \Rightarrow a_i \leq a_j$ by \eqref{eq:st}. If $a_i$ and $a_j$ are both negative then so are $u(i)$ and $u(j)$, and thus $u(i) < u(j) \Rightarrow a_i \leq a_j$ by \eqref{eq:st}. If $a_i < 0 \leq a_j$ then $u(i) < 0 < u(j)$. Similarly, if $a_j < 0 \leq a_i$ then $u(j) < 0 < u(i)$. Therefore $u(i) < u(j)$ if and only if $a_i \leq a_j$. This implies $\text{st}(u) = \text{st}(a)$ and completes the proof. \hfill $\Box$

For example, one has $s_{12} = s^B_{12} + s^B_{13} + s^B_{23} + s^B_{13} + s^B_{12}$ and $s_{21} = s^B_{21} + s^B_{21} + s^B_{12} + s^B_{13}$.

Now we define an action and a coaction of $\text{FQSym}$ on $\text{FQSym}^B$. Proposition 2.2.6 implies

\begin{equation}
F^B_u \cdot F^B_v = \sum_{w \in \mathfrak{S}_m^B, v \in \mathfrak{S}_n^B} F^B_w, \quad \forall u \in \mathfrak{S}_m^B, \forall v \in \mathfrak{S}_n^B.
\end{equation}

This gives a right action of $\text{FQSym}$ on $\text{FQSym}^B$, which is denoted by $\circ \circ B$. Note that $F^B_v$ is $F^B_v(X)$ instead of $F^B_v(X_{>0})$ by our convention in this paper. If $u \in \mathfrak{S}_m^B$ then define

\begin{equation}
\Delta^B(F^B_u) = \sum_{0 \leq i \leq m} F^B_{\text{st}^B(u[i+1,m])} \circ F^B_{\text{st}(u[i+1,m])}.
\end{equation}

Also define a bilinear form on $\text{FQSym}^B$ by $\langle F^B_u, s^B_v \rangle := \delta_{u,v}$ for all $u, v \in \mathfrak{S}_n^B$.

**Proposition 4.2.3.** $(\text{FQSym}^B, \circ \circ B, \Delta^B)$ is a self-dual graded right module and comodule over $\text{FQSym}$ isomorphic to $(\mathcal{Z}\mathfrak{S}_n^B, \mathfrak{B}^B, \mathfrak{M}^B)$ via $F^B_u \mapsto u$ and to $(\mathcal{Z}\mathfrak{S}_n^B, \mathfrak{B}^B, \mathfrak{M}^B)$ via $s^B_u \mapsto u$.

**Proof.** This follows from \eqref{eq:st}, \eqref{eq:st}, and Proposition 4.2.2. (iv). \hfill $\Box$

It follows that if $u \in \mathfrak{S}_m^B$ and $v \in \mathfrak{S}_n^B$ then

\begin{equation}
s^B_u \cdot s^B_v = \sum_{w \in \mathfrak{S}_m^B, v \in \mathfrak{S}_n^B} s^B_w \quad \text{and} \quad \Delta^B(s^B_u) = \sum_{0 \leq i \leq m} s^B_{u[i+1,m]} \circ \mathfrak{S}_{\text{st}(u[i+1,m])}.
\end{equation}
4.2.2. Noncommutative symmetric functions of Type B. In our earlier work \[12\] we defined a type B analogue \( \text{NSym}^B \) of \( \text{NSym} \) with two free \( \mathbb{Z} \)-bases \( \{ h^B_\alpha \} \) and \( \{ s^B_\alpha \} \), where \( \alpha \) runs through all pseudo-compositions. If \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \) is a pseudo-composition then \( h^B_\alpha \) and \( s^B_\alpha \) are defined by
\[
h^B_\alpha = h^B_{\alpha_1} \cdot h^B_{\alpha_2} \cdots h^B_{\alpha_\ell} = \sum_{\beta \in \alpha} s^B_\beta
\]
where
\[
h^B_k = s^B_k := \sum_{0 \leq i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k} = \sum_{0 \leq i \leq k} x^i_k \cdot h^B_{k-i}, \quad \forall k \geq 0.
\]
It follows that \( \text{NSym}^B \) is a free right \( \text{NSym} \)-module with a basis \( \{ h^B_k : k \geq 0 \} \), and one has
\[
h^B_\alpha \cdot h^B_\beta = h^B_{\alpha \cdot \beta} \quad \text{and} \quad s^B_\alpha \cdot s^B_\beta = s^B_{\alpha \cdot \beta} + s^B_{\alpha \otimes \beta}
\]
for all \( \alpha \models^N \mathbb{Z} \) and \( \beta \models n \), where the last term is treated as zero when \( \alpha \otimes \beta \) is undefined. This right \( \text{NSym} \)-action on \( \text{NSym}^B \), denoted by \( \otimes^B \) for consistency of notation, was used by Chow \[7\] to define \( \text{NSym}^B \) abstractly (without a power series realization). We next provide an embedding of \( \text{NSym}^B \) into \( \text{FQSym}^B \), which will recover the above \( \text{NSym} \)-module structure and also induce a \( \text{NSym} \)-comodule structure on \( \text{NSym}^B \).

Let \( \tau \) be a tableau of pseudo-ribbon shape \( \alpha \), i.e., a filling of the pseudo-ribbon \( \alpha \) with integers. Reading these integers from the bottom row to the top row and proceeding from left to right within each row, \textit{excluding the extra } \theta \text{ in the pseudo-ribbon } \alpha \text{, gives the reading word } w(\tau) \text{ of } \tau \text{. We call } \tau \text{ (type B) semistandard} \text{ if each row is weakly increasing from left to right and each column is strictly increasing from top to bottom, including the extra } \theta \text{. We showed in } \[12\] \text{ that } s^B_\alpha \text{ is the sum of } x_{w(\tau)} \text{ for all semistandard tableaux } \tau \text{ of pseudo-ribbon shape } \alpha. \]

**Proposition 4.2.4.** If \( \alpha \models^N m \) then \( s^B_\alpha \) equals the sum of \( s^B_w \) for all \( w \in \mathcal{S}^B_m \) with \( D(w) = D(\alpha) \).

*Proof.* Each \( f \in \mathbb{Z}^n \) corresponds to a unique tableau \( \tau \) of shape \( \alpha \) such that \( w(\tau) = f \). One sees that \( D(s^B_f) = D(\alpha) \) if and only if \( \tau \) is semistandard. Thus the result follows from Proposition 4.2.1. \( \square \)

It follows that there is an injection \( \iota : \text{NSym}^B \hookrightarrow \text{FQSym}^B \) by inclusion.

**Proposition 4.2.5.** The graded right module and comodule \( \text{FQSym}^B \) over \( \text{FQSym} \) restricts to a graded right module and comodule \( (\text{NSym}^B, \otimes^B, \Delta^B) \) over \( \text{NSym} \), which is isomorphic to the graded right module and comodule \( (\Sigma(\mathcal{B}^B), \otimes^B, \Delta^B) \) over \( \Sigma(\mathcal{B}) \), \( \otimes, \Delta \) via the map \( s^B_\alpha \mapsto D_\alpha(\mathcal{S}^B_m), \forall \alpha \models^N m, \forall m \geq 0. \)

*Proof.* By Proposition 4.2.3 there is an isomorphism \( (\text{FQSym}^B, \otimes^B, \Delta^B) \cong (\mathbb{Z} \mathcal{B}^B, \otimes^B, \Delta^B) \) via \( s^B_w \mapsto w \), \( \forall w \in \mathcal{S}^B \). Restricting this isomorphism to \( \text{NSym}^B \) gives the result. \( \square \)

**Remark 4.2.6.** If \( k \) is a nonnegative integer then it follows from \[4.2.3\] and Proposition 4.2.4 that
\[
\Delta^B(h^B_k) = \Delta^B(s^B_k) = \sum_{0 \leq i \leq k} s^B_{i+1} \otimes s^B_{k-i+1} = \sum_{0 \leq i \leq k} h^B_i \otimes h^B_{k-i}.
\]
We do not have any explicit formula for \( \Delta^B(h^B_\alpha) \) or \( \Delta(s^B_\alpha) \) for an arbitrary \( \alpha \models^N n \).

4.2.3. Quasisymmetric functions of type B. Let \( X_{\alpha \models^N \mathbb{Z}} = \{ x_0, x_1, x_2, \ldots \} \) be a totally ordered set of commutative variables. Chow \[7\] introduced a type B analogue \( \text{QSym}^B \) of \( \text{QSym} \), which admits two free \( \mathbb{Z} \)-bases \( \{ F^B_\alpha \} \) and \( \{ M^B_\alpha \} \), where \( \alpha \) runs through all pseudo-compositions. If \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \) is a pseudo-composition of \( n \) then one has the \textit{type B monomial quasisymmetric function}
\[
M^B_\alpha := \sum_{0 \leq i_1 < \cdots < i_\ell} x^{\alpha_1}_{i_1} x^{\alpha_2}_{i_2} \cdots x^{\alpha_\ell}_{i_\ell} = x^{\alpha_1}_0 \cdot M^B_{(\alpha_2, \ldots, \alpha_\ell)}
\]
and the \textit{type B fundamental quasisymmetric function} (with \( i_0 := 0 \))
\[
F^B_\alpha := \sum_{\alpha \models \beta} M^B_\alpha = \sum_{0 \leq i_1 < \cdots < i_n \in \mathcal{D}(\alpha)} x_{i_1} \cdots x_{i_n}.
\]
There exist unique $\alpha \leq i \models B \cdot i$ and $\alpha > i \models n-i$ such that $\alpha \in \{\alpha \leq i \cdot \alpha > i, \alpha \leq \alpha > i\}$ for each $i \in \{0, 1, \ldots, n\}$, and one can show that $F^B = \sum_{0 \leq i \leq n} x^B_i \cdot F_{\alpha > i}$. By (4.10), one also has

$$Q\text{Sym}^B = \mathbb{Z}[x_0] \cdot Q\text{Sym} \cong \mathbb{Z}[x_0] \otimes \mathbb{Z} Q\text{Sym}.$$ 

We define an algebra map $\chi^B : \mathbb{Z}(X) \to \mathbb{Z}[X_{\geq 0}]$ by $x_i \mapsto x_{|i|}$ for all $i \in \mathbb{Z}$. If $w \in \mathcal{S}_n$ then

$$F^B_w := \chi^B(F^B_w) = \sum_{0 \leq f(w(1)) \leq \cdots \leq f(w(n)) \in D(w)^n = f(w(1)) \cdot f(w(n))} x_{f(w(1))} \cdots x_{f(w(n))}$$

by (4.13). One sees that $F^B_w = F^B_{\alpha}$ if $w \in \mathcal{S}_n$, $\alpha \models B \cdot n$, and $D(w) = D(\alpha)$. This gives a surjection $\chi^B : FQ\text{Sym}^B \to Q\text{Sym}^B$.

If $u \in \mathcal{S}_m$ and $v \in \mathcal{S}_n$ then applying the algebra homomorphism $\chi^B$ to (4.6) gives

$$F^B_u \circ^B F^B_v := \chi^B(F^B_v) = \sum_{w \in u \cup^B v} F^B_w.$$

This defines a right $Q\text{Sym}$-action on $Q\text{Sym}^B$, which is believed to be new.

Chow [2] introduced a right coaction of $Q\text{Sym}$ on $Q\text{Sym}$ by $\Delta^B F^B := F^B(X_{\geq 0} + Y_{> 0}), \forall F^B \in Q\text{Sym}^B$, where $X_{\geq 0} + Y_{> 0} = \{x_0, x_1, x_2, \ldots, y_1, y_2, \ldots\}$ is a totally ordered set of commutative variables. One can check that if $\alpha = (\alpha_1, \ldots, \alpha_\ell) \models B \cdot n$ then

$$\Delta^B M^B_\alpha = \sum_{1 \leq j \leq \ell} M^B_{(\alpha_1, \ldots, \alpha_j)} \otimes M^B_{(\alpha_{j+1}, \ldots, \alpha_\ell)} \quad \text{and} \quad \Delta^B F^B_\alpha = \sum_{0 \leq i \leq n} F^B_{\alpha \leq i} \otimes F^B_{\alpha > i}.$$ 

The second equality is equivalent to

$$\Delta^B F^B_w = \sum_{0 \leq i \leq n} F^B_{\text{st}_i(w[1, i])} \otimes F^B_{\text{st}_i(w[i+1, n])}, \quad \forall w \in \mathcal{S}_n.$$ 

Therefore this coaction is preserved by the surjections $\chi^B : FQ\text{Sym}^B \to Q\text{Sym}^B$ and $\chi : \text{FQSym} \to \text{QSym}$. We observe that this coaction can also be obtained by applying the coproduct of $Q\text{Sym}$ to the second tensor component of $Q\text{Sym}^B \cong \mathbb{Z}[x_0] \otimes \mathbb{Z} Q\text{Sym}$.

**Proposition 4.2.7.** $(Q\text{Sym}^B, \circ^B, \Delta^B)$ is a graded right module and comodule over $Q\text{Sym}$ isomorphic to the graded right module and comodule $(\Sigma^*(\mathcal{S}), \cup^B, \cap^B)$ over $(\Sigma^*(\mathcal{S}), \cup, \cap)$. The map $\chi^B$ induces a surjection from the graded right module and comodule $FQ\text{Sym}^B$ over $Q\text{Sym}$ onto $Q\text{Sym}^B$.

**Proof.** The result follows from (4.12) and (4.13). \(\square\)

Define a pairing between $\text{NSym}^B$ and $Q\text{Sym}^B$ by $\langle s^B_\alpha, F^B_\beta \rangle = \langle h^B_\alpha, M^B_\beta \rangle := \delta_{\alpha, \beta}$ for all pseudo-compositions $\alpha$ and $\beta$.

**Corollary 4.2.8.** The embedding $\iota : \text{NSym}^B \hookrightarrow FQ\text{Sym}^B$ and the surjection $\chi^B : FQ\text{Sym}^B \to Q\text{Sym}^B$ are dual morphisms of graded modules and comodules.

**Proof.** This follows from Theorem 4.1.1, Proposition 4.2.5, and Proposition 4.2.7. \(\square\)

4.2.4. Symmetric functions of type $B$. We next investigate $\text{Sym}^B := \chi^B(\text{NSym}^B) \subseteq Q\text{Sym}^B$, which is the $\mathbb{Z}$-span of $s^B_\alpha := \chi^B(s^B_\alpha)$ for all pseudo-compositions $\alpha$. A representation theoretic interpretation for $s^B_\alpha$ will be provided in Proposition 4.3.2. Another spanning set for $\text{Sym}^B$ consists of

$$h^B_\alpha := \chi^B(h^B_\alpha) = \sum_{\beta \leq \alpha} \chi^B(s^B_\beta) = \sum_{\beta \leq \alpha} s^B_\beta$$

for all pseudo-compositions $\alpha$. We have an isomorphism $\Lambda(\mathcal{S}) \cong \text{Sym}^B$ via $\Lambda_\alpha(\mathcal{S}_n) \hookrightarrow s^B_\alpha, \forall \alpha \models B \cdot n$, $\forall n \geq 0$. This and (4.2) give a nondegenerate symmetric bilinear form on $\text{Sym}^B$.

A pseudo-partition $\lambda$ of $n$, denoted by $\lambda \models B \cdot n$, is a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq 0, \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1$, and its size $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$ equals $n$. Given a pseudo-composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, denote by $\lambda^B(\alpha)$ the pseudo-partition obtained from $\alpha$ by rearranging $\alpha_2, \ldots, \alpha_\ell$.

**Proposition 4.2.9.** If $\lambda \models B \cdot n$ then $h^B_\lambda := h^B_\alpha$ is well defined for any $\alpha \models B \cdot n$ with $\lambda^B(\alpha) = \lambda$. Moreover, there is a free $\mathbb{Z}$-basis $\{h^B_\lambda : \lambda \models B \cdot n, \ n \geq 0\}$ for $\text{Sym}^B$. 

Proof. According to Remark 4.2.4 (i), two parabolic subgroups \( \mathcal{G}_\alpha^B \) and \( \mathcal{G}_\beta^B \) are conjugate in \( \mathcal{G}_n^B \) if and only if \( \lambda^B(\alpha) = \lambda^B(\beta) \). Thus the result follows from Proposition 4.2.2.

Proposition 4.2.10. We have \( \text{Sym}^B \subseteq \mathbb{Z}[x_0] \cdot \text{Sym} \cong \mathbb{Z}[x_0] \otimes_{\mathbb{Z}} \text{Sym} \) and if \( \mathbb{K} \) be a field of characteristic zero then \( \text{Sym}^B_{\mathbb{K}} = \mathbb{K}[x_0] \cdot \text{Sym}_\mathbb{K} \cong \mathbb{K}[x_0] \otimes_{\mathbb{K}} \text{Sym}_\mathbb{K} \).

Proof. For any nonnegative integer \( k \) it follows from (4.9) and the definition of \( h_k \) that
\[
(4.14) \quad h_k^B = \chi^B(h_k^B) = \sum_{0 \leq i \leq k} x_i^0 \cdot h_{k-i}
\]
and
\[
(4.15) \quad \chi^B(h_k) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k} = \sum_{a,b,c \geq 0, a+b+c=k} h_a \cdot x_0^b \cdot h_c.
\]
Hence \( h_\alpha = \chi^B(h_\alpha^B) \cdot \chi^B(h_{\alpha_2}) \cdots \chi^B(h_{\alpha_t}) \in \mathbb{Z}[x_0] \cdot \text{Sym} \) for all \( \alpha = (\alpha_1, \ldots, \alpha_t) \mid \mathbb{B} n \). This implies \( \text{Sym}^B \subseteq \mathbb{Z}[x_0] \cdot \text{Sym} \). By Proposition 4.2.9 the homogeneous component \( \text{Sym}^B_n \) of degree \( n \) of \( \text{Sym}^B \) is a free \( \mathbb{Z} \)-module of rank \( n \), equal to the sum of the numbers of partitions of \( k \) for all \( k = 0, 1, 2, \ldots, n \). Thus the equality between \( \text{Sym}^B_{\mathbb{K}} \) and \( \mathbb{K}[x_0] \cdot \text{Sym}_\mathbb{K} \) follows from a comparison of the dimensions of their homogeneous components.

If \( \lambda = (\lambda_1, \ldots, \lambda_t) \) is a pseudo-partition then define \( m_\lambda^B := \sum_{\lambda(\alpha)=\lambda} M_\alpha^B \). We have
\[
(4.16) \quad m_\lambda^B = x_0^{\lambda_1} \cdot \sum_{\lambda(\alpha_{1, \ldots, \alpha_t})=\lambda(\lambda_{1, \ldots, \lambda_t})} M_{(\alpha_{1, \ldots, \alpha_t})} = x_0^{\lambda_1} \cdot m_{(\lambda_{1, \ldots, \lambda_t})}.
\]
Hence \( \{m_\lambda^B\} \) is a free \( \mathbb{Z} \)-basis for \( \mathbb{Z}[x_0] \cdot \text{Sym} \), where \( \lambda \) runs through all pseudo-partitions.

Proposition 4.2.11. One has dual \( \mathbb{K} \)-bases \( \{h_\lambda^B\} \) and \( \{m_\lambda^B\} \) for \( \text{Sym}^B_{\mathbb{K}} \), where \( \lambda \) runs through all pseudo-partitions.

Proof. We already know that \( \{h_\lambda^B\} \) is a free \( \mathbb{Z} \)-basis for \( \text{Sym}^B \), and hence a \( \mathbb{K} \)-basis for \( \text{Sym}^B_{\mathbb{K}} \). By Proposition 4.2.10 and (4.16), \( \{m_\lambda^B\} \) is a \( \mathbb{K} \)-basis for \( \text{Sym}^B_{\mathbb{K}} = \mathbb{K}[x_0] \cdot \text{Sym}_\mathbb{K} \). Let \( \lambda \) and \( \mu \) be two pseudo-partitions, and let \( \alpha \) be a pseudo-composition with \( \lambda(\alpha) = \lambda \). Then
\[
\langle h_\lambda^B, m_\mu^B \rangle = \langle h_\alpha^B, \sum_{\lambda(\beta)=\mu} M_{(\beta)}^B \rangle = \delta_{\lambda, \mu}.
\]
Thus \( \{h_\lambda^B\} \) and \( \{m_\lambda^B\} \) are dual \( \mathbb{K} \)-bases for \( \text{Sym}^B_{\mathbb{K}} \).

Remark 4.2.12. (i) Proposition 4.2.3 implies that \( s_{1^k}^B = s_1^{1^k} \).
(ii) We know that \( \{h_\lambda^B\} \) is a free \( \mathbb{Z} \)-basis for \( \text{Sym}^B \) and \( \{m_\lambda^B\} \) is a free \( \mathbb{Z} \)-basis for \( \mathbb{Z}[x_0] \cdot \text{Sym} \). However, \( \{m_\lambda^B\} \) is not a free \( \mathbb{Z} \)-basis for \( \text{Sym}^B \), since \( \text{Sym}^B \subseteq \mathbb{Z}[x_0] \cdot \text{Sym} \) by the following calculation using (4.11) and (4.15).
\[
\begin{align*}
h_2^B &= x_0^2 + x_0 h_1 + h_2 & x_0^2 &= \frac{8}{3} h_2^B - \frac{4}{3} h_{11} - \frac{4}{3} h_{02} + h_{011} \\
h_1^B &= x_0^2 + 3x_0 h_1 + 2 h_{11} \quad \Rightarrow \quad h_0 h_1 &= -\frac{4}{3} h_2^B - \frac{4}{3} h_{11} + \frac{2}{3} h_{02} - h_{011} \\
h_0^B &= x_0^2 + 2x_0 h_1 + 2 h_2 + h_{11} & h_2 &= -\frac{4}{3} h_2^B - \frac{4}{3} h_{11} + \frac{2}{3} h_{02} \\
h_{01}^B &= x_0^2 + 2x_0 h_1 + 4 h_{11} & h_{11} &= -\frac{4}{3} h_2^B - \frac{4}{3} h_{11} + \frac{2}{3} h_{02} + h_{011}
\end{align*}
\]
For any pseudo-composition \( \alpha \) and composition \( \beta \), applying \( \chi^B \) to \( h_\alpha^B \cdot h_\beta = h_{\alpha \cdot \beta}^B \) gives
\[
h_\alpha^B \circ \chi^B \cdot h_\beta = h_{\alpha \cdot \beta}^B.
\]
Also define
\[
\Delta^B(s_\alpha^B) = \Delta^B(\chi^B(s_\alpha^B)) := (\chi^B \otimes \chi)(\Delta^B(s_\alpha^B)).
\]
Proposition 4.2.13. \( (\text{Sym}^B, \circ^B, \Delta^B) \) is a graded right module and comodule over \( \text{Sym} \) isomorphic to the graded right module and comodule \( (\mathcal{G}(\mathcal{E}^B)) \) over \( \Lambda(\mathcal{E}) \). The injection \( \iota : \text{Sym}^B \to Q\text{Sym}^B \) and the surjection \( \chi^B : \text{NSym}^B \to \text{Sym}^B \) are dual morphisms of graded right modules and comodules. There is a free basis \( \{h_\alpha^B : k \geq 0\} \) for \( \text{Sym}^B \) as a right \( \mathbb{K} \)-module.

Proof. Apply Theorem 4.1.4 and Propositions 4.2.5, 4.2.7 and 4.2.9.
4.3. **Representation theory.** Now we study the connections of $\text{NSym}^B$ and $\text{QSym}^B$ with the representation theory of 0-Hecke algebras of type B. Let $(W, S)$ be the finite Coxeter system of type $B_n$, where $W = S_n^B$ and $S = \{s_0 = s_0^B, s_1^B, \ldots, s_{n-1}^B\}$. Let $\mathbb{F}$ be a field. The 0-Hecke algebra $H_n(B)$ of $(W, S)$ is an $\mathbb{F}$-algebra with two generating sets $\{\pi_i : 0 \leq i \leq n - 1\}$ and $\{\pi_i : 0 \leq i \leq n - 1\}$. One can realize $\pi_0, \pi_1, \ldots, \pi_{n-1}$ as signed bubble-sorting operators on $\mathbb{Z}^n$: if $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ then

$$
\pi_i(a_1, \ldots, a_n) := \begin{cases} 
(-a_1, a_2, \ldots, a_n), & \text{if } i = 0, a_1 > 0, \\
(a_1, \ldots, a_{i+1}, a_i, a_{i+2}, \ldots, a_n), & \text{if } 1 \leq i \leq n-1, a_i < a_{i+1}, \\
(a_1, \ldots, a_n), & \text{otherwise}.
\end{cases}
$$

The projective indecomposable $H_n(B)$-modules and simple $H_n(B)$-modules are given by $P^B_\alpha := P^S_\alpha$ and $\mathcal{C}^B_\alpha := \mathcal{C}^S_\alpha$, respectively, where $I = \{s_i : i \in D(\alpha)\}$, for all $\alpha \vdash n$. One can realize $P^B_\alpha$ as the $\mathbb{F}$-space of standard tableaux of pseudo-ribbon shape $\alpha$ with an appropriate $H_n(B)$-action.

The parabolic subalgebra $H_n^B(0)$ of $H_n(B)$ is generated by $\{\pi_i : i \in \{0, 1, \ldots, n-1\}\} \setminus D(\alpha)$. If $m$ and $n$ are nonnegative integers one has $H_n(B) \otimes H_m(0) \cong H_n(B) \otimes H_{m+n}(0)$, giving an embedding $H_n(B) \otimes H_m(0) \hookrightarrow H_{m+n}(0)$.

Associated with the tower of algebras $H_n^B(0): H_0^B(0) \hookrightarrow H_1^B(0) \hookrightarrow H_2^B(0) \hookrightarrow \cdots$ are Grothendieck groups

$$
G_0(H_n^B(0)) := \bigoplus_{n \geq 0} G_0(H_n^B(0)) \quad \text{and} \quad K_0(H_n^B(0)) := \bigoplus_{n \geq 0} K_0(H_n^B(0)).
$$

Let $M$ and $N$ be finitely generated modules over $H_m(B)$ and $H_n(0)$, respectively. Define

$$
M \odot^B N := (M \otimes N) \uparrow_{H_{m+n}(0)}^{H_m(B) \otimes H_n(0)} \quad \text{and} \quad \Delta^B(M) := \sum_{0 \leq i \leq m} M \downarrow_{H_{m+n}(0)}^{H_m(B) \otimes H_n(0)}.
$$

Using the linear maps $\mu^S_i : G_0(H_n(0)) \to G_0(H_n(B))$ and $\rho^S_i : G_0(H_n(B)) \to G_0(H_n(0))$ defined in [24], with $S = \{s_0, \ldots, s_{m+n-1}\}$ and $I = S \setminus \{s_m\}$, one has

$$
M \odot^B N = \mu^S_i(M \otimes N) \quad \text{and} \quad \Delta^B(M) := \mu^S_i(M \otimes N) = \rho^S_i(Q)
$$

where $Q$ is a finitely generated $H_{m+n}(0)$-module. Also recall from Proposition 2.5.1 that $\mu^S_i$ and $\rho^S_i$ restrict to $\mu^B_i : K_0(H_n(0)) \to K_0(H_n(B))$ and $\rho^B_i : K_0(H_n(B)) \to K_0(H_n(0))$. If $M, N,$ and $Q$ are all projective then

$$
M \odot^B N = \mu^B_i(M \otimes N) \quad \text{and} \quad \Delta^B(M) = \rho^B_i(Q).
$$

Define the following characteristic maps, where $\alpha$ runs through all pseudo-compositions:

$$
\text{Ch} : G_0(H_n^B(0)) \to \text{QSym}^B \quad \text{and} \quad \text{ch} : K_0(H_n^B(0)) \to \text{NSym}^B.
$$

**Theorem 4.3.1.** (i) $(G_0(H_n^B(0)), \odot^B, \Delta^B)$ is a graded right module and comodule over the Hopf algebra $G_0(H_n(0))$.

(ii) $(K_0(H_n^B(0)), \odot^B, \Delta^B)$ is a graded right module and comodule over the Hopf algebra $K_0(H_n(0))$.

(iii) $(G_0(H_n^B(0)), \odot^B, \Delta^B)$ is dual to $(K_0(H_n^B(0)), \odot^B, \Delta^B)$ via the pairing $\langle P^B_\alpha, \mathcal{C}^B_\beta \rangle := \delta_{\alpha, \beta}$.

(iv) Both $\text{Ch}$ and $\text{ch}$ are isomorphisms of graded modules and comodules.

**Proof.** Apply Theorem 2.3.9 Theorem 4.1.3 Proposition 4.2.5 and Proposition 4.2.7. 

**Proposition 4.3.2.** If $\alpha$ is a pseudo-composition of $n$ then $\text{Ch}(P^B_\alpha) = s^B_\alpha$.

**Proof.** This follows from Proposition 2.3.9. 

4.4. **Other results.** Our results in type B are based on the embedding $S_m^B \times S_n \hookrightarrow S_{m+n}$ which identifies $S_m^B \times S_n$ with the parabolic subgroup $S_{m,n}^B$ of $S_{m+n}$. There is another embedding $S_m^B \times S_n^B \hookrightarrow S_{m+n}$ whose image is a non-parabolic subgroup of $S_{m+n}$. \[ s_m' := s_m \cdots s_1 s_0 s_1 \cdots s_m = 1 \cdots (m+1) \cdots (m+n). \]

It does not induce an embedding $H_n^B(0) \otimes H_m^B(0) \hookrightarrow H_{m+n}(0)$, as $\pi_m' := \pi_m \cdots \pi_1 \pi_0 \pi_1 \cdots \pi_m$ does not satisfy the relation $(\pi_m')^2 = \pi_m'$. For example, in $H_{1+1}^B(0)$ one has $\pi_1' = \pi_1 \pi_0 \pi_1$ and $(\pi_1')^2 = \pi_0 \pi_1 \pi_0 \pi_1 \neq \pi_1'$. 


Although the embedding $\mathcal{G}_n^B \times \mathcal{G}_n^B \rightarrow \mathcal{G}_n^{B+m+n}$ does not fit into our general theory in Section 2, it leads to a self-dual graded Hopf algebra structure on $\mathbb{Z} \mathcal{G}_m^B$. In fact, an element $(u, v) \in \mathcal{G}_n^B \times \mathcal{G}_n^B$ can be identified with $u \times v \in \mathcal{G}_m^{B+m}$, whose window notation is $[u(1), \ldots, u(m), v(1)+m, \ldots, v(n)+m]$ where $a^{m+n} = a + \text{sgn}(a) \cdot m$. This gives an embedding $\mathcal{G}_m^B \times \mathcal{G}_n^B \rightarrow \mathcal{G}_m^{B+m+n}$. One shows that every $w \in \mathcal{G}_m^{B+n}$ can be written uniquely as $w = z(u \times v)$, where $u \in \mathcal{G}_m^B$, $v \in \mathcal{G}_n^B$, and $z \in \mathcal{G}_m^{B+n}$ satisfying

$$0 < z(1) < \cdots < z(m) \quad \text{and} \quad 0 < z(m+1) < \cdots < z(m+n).$$

Given such an expression, one can check that

$$u = \text{st}^B([w, 1, m]), \quad v = \text{st}^B([w, m+1, n]), \quad u^{-1} = w^{-1}[1, m], \quad v^{-1} = \text{st}^B(w^{-1}[m+1, n]).$$

If $u \in \mathcal{G}_m^B$ and $v \in \mathcal{G}_n^B$ then we define

$$u \uplus v := \{ w \in \mathcal{G}_m^{B+n+m} : w[1, m] = u, \text{ st}^B([w, m+1, n]) = v \},$$

$$u \uplus v := \{ w \in \mathcal{G}_m^{B+n+m} : \text{ st}^B([w, 1, m]) = u, \text{ st}^B([w, m+1, n]) = v \},$$

$$\uplus (u) := \sum_{0 \leq i \leq m} \text{ st}^B([u[1, i]) \otimes \text{ st}^B([u[i+1, m]),$$

$$\text{ and } \quad \uplus (u) := \sum_{0 \leq i \leq m} u[1, i] \otimes \text{ st}^B([u[i+1, m]).$$

This is very similar to the Malvenuto-Rentkenauer algebra defined in Section 3 and hence we use the same notation. For example, one has

$$21 \uplus 12 = 2134 + 2314 + 3214 + 2341 + 3241 + 3421,$$

$$21 \uplus 12 = 2134 + 3124 + 3214 + 4123 + 4312,$$

$$\text{ and } \quad \uplus (2431) = \emptyset \otimes 2431 + \emptyset \otimes 321 + 321 \otimes 21 + 132 \otimes 1 + 2431 \otimes \emptyset,$$

$$\uplus (2431) = \emptyset \otimes 2431 + 1 \otimes 132 + 21 \otimes 21 + 231 \otimes 1 + 2431 \otimes \emptyset.$$

**Proposition 4.41.** $(\mathcal{Z} \mathcal{G}^B, \uplus, \uplus)$ and $(\mathcal{Z} \mathcal{G}^B, \uplus, \uplus)$ are graded Hopf algebras isomorphic to each other via $w \mapsto w^{-1}$, $\forall w \in \mathcal{G}^B$, and dual to each other via $(u, v) := \delta_{u,v}, \forall u, v \in \mathcal{G}^B$.

**Proof.** The definition of $\cup$ implies $w \uplus \emptyset = \emptyset \uplus w = w$ for all $w \in \mathcal{G}^B$. If $u \in \mathcal{G}_m^B$, $v \in \mathcal{G}_n^B$, and $r \in \mathcal{G}_k^B$ then one can check that $(u \uplus v) \cup w$ and $u \uplus (v \cup w)$ both equal the sum of all $w \in \mathcal{G}_m^{B+n+k}$ such that

$$w[1, m] = u, \quad \text{ st}^B([w, m+1, n+m]) = v, \quad \text{ and } \quad \text{ st}^B([w, m+n+1, m+n+k]) = r.$$

Thus $(\mathcal{Z} \mathcal{G}^B, \cup)$ is a graded algebra, whose dual is the graded coalgebra $(\mathcal{Z} \mathcal{G}^B, \uplus)$. Applying $w \mapsto w^{-1}$ to $(\mathcal{Z} \mathcal{G}^B, \cup)$ and $(\mathcal{Z} \mathcal{G}^B, \uplus)$ gives the isomorphic graded algebra $(\mathcal{Z} \mathcal{G}^B, \cup)$ and its dual graded coalgebra $(\mathcal{Z} \mathcal{G}^B, \uplus)$. One checks that $\uplus (u \uplus v)$ and $(\uplus u) \uplus (\uplus v)$ both equal

$$\sum_{0 \leq i \leq m} \sum_{0 \leq j \leq n} \left[ \text{ st}^B([u[1, i]) \otimes \text{ st}^B([v[j, n]]) \right] \otimes \left[ \text{ st}^B([u[i+1, m]) \otimes \text{ st}^B([v[j+1, n]]) \right]$$

where the shuffle product $(\uplus u) \uplus (\uplus v)$ is defined tensor-component-wise. It follows that $(\mathcal{Z} \mathcal{G}^B, \cup, \uplus)$ is a Hopf algebra and so is the isomorphic $(\mathcal{Z} \mathcal{G}^B, \cup, \uplus)$. \hfill $\Box$

Next, let $ab$ be the concatenation of two words $a$ and $b$ in $\mathbb{Z}^n$. Assume $\text{ st}^B(ab) = w$ and $u = \text{ st}^B([w, 1, m])$. Since $[w(1)], \ldots, [w(m)]$ are distinct, one sees that $|u(i)| < |u(j)|$ if and only if $|w(i)| < |w(j)|$ whenever $i, j \in [m]$. Thus $\text{ st}^B(ab) = w = \text{ st}^B([w, 1, m])$ by restricting the definition of $\text{ st}^B(ab) = w$ to $i, j \in [m]$. Similarly one has $\text{ st}^B(b) = \text{ st}^B([w, m+1, n+m])$. Hence for all $u, v \in \mathcal{G}^B$, Proposition 4.1 implies

$$s^B_u \cdot s^B_v = \sum_{\text{ st}^B(f) = u \cdot \text{ st}^B(g) = v} \left[ x_f(1) \cdots x_f(n) y_g(1) \cdots y_g(n) \right] = \sum_{w \in \mathcal{G}^B} s^B_w.$$

If $w \in \mathcal{G}_n^B$ then define

$$\Delta s^B_w := \sum_{0 \leq i \leq n} s^B_w[i,i] \otimes s^B_w[i+1,n].$$
Proposition 4.4.2. There are Hopf algebra isomorphisms $(\text{FQSym}^B, \cdot, \Delta) \cong (\mathbb{Z}S^B, \mu, \eta)$ via $F^B_w \mapsto w$, $\forall w \in \mathbb{S}^B$, and $(\text{FQSym}^B, \cdot, \Delta) \cong (\mathbb{Z}S^B, \mu, \eta)$ via $s^B_w \mapsto w$, $\forall w \in \mathbb{S}^B$.

Proof. Comparing (4.17) and (4.18) with the definition of $(\mathbb{Z}S^B, \mu, \eta)$ gives the second isomorphism. Applying $w \mapsto w^{-1}$ gives the first one. \hfill \square

Remark 4.4.3. Novelli and Thibon [19] generalized FQSym to a family of graded Hopf algebras $\text{FQSym}^{(\ell)}$ using an $\ell$-colored standardization. The Hopf algebra $(\text{FQSym}^{(1)}, \cdot, \Delta)$ is isomorphic to $\text{FQSym}^{(2)}$, but the dual bases $(F^B_w)$ and $(s^B_w)$ for $\text{FQSym}^{(2)}$ are different from the dual bases for $\text{FQSym}^{(2)}$ provided in [19]. In fact, $F^B_w$ is the generating function of $A(w^{\Phi^+})$, and this forces $s^B_{w} := F^B_{w^{-1}}$ to involve the signed standardization, which is not a 2-colored standardization defined in [19].

By the duality between $(\mathbb{Z}S^B, \mu, \eta)$ and $(\mathbb{Z}S^B, \mu, \eta)$, if $u \in \mathbb{S}_m^B$ and $v \in \mathbb{S}_n^B$ then

$$ F^B_u \cdot F^B_v = \sum_{w \in u \cup v} F^B_w \quad \text{and} \quad \Delta F^B_u = \sum_{0 \leq i \leq m} F^B_{st^B_i(u | 1, i)} \otimes F^B_{st^B_i(u | i+1, n)}.$$

One can also directly verify the first equation. Applying $\chi^B: \text{FQSym}^B \rightarrow \text{QSym}^B$ gives

$$ F^B_u \cdot F^B_v = \sum_{w \in u \cup v} F^B_w, \quad \forall u, v \in \mathbb{S}^B. $$

This recovers a graded algebra structure for $\text{QSym}^B$ studied by Chow [7].

Next, we apply the embeddings $\mathbb{S}^B_m \otimes \mathbb{S}^B_n \hookrightarrow \mathbb{S}^B_{m+n}$ and $\mathbb{S}^B_m \otimes \mathbb{S}^B_n \hookrightarrow \mathbb{S}^B_{m+n}$ to the (complex) representation theory of hyperoctahedral groups. We first review the representation theory of the hyperoctahedral group $\mathbb{S}^B$ from Geissinger and Kincu [10]. Let $E(n)$ be the subgroup of $\mathbb{S}^B_n$ generated by the signed permutations $e(n, i) := 1 \cdots i \cdots n$ for all $i \in [n]$. Then $\mathbb{S}^B_n$ is a semidirect product of $\mathbb{S}^B_n$ and $E(n)$. For every $k \in [n]$ there is a one-dimensional $E(n)$-module $\chi(n, k)$ on which $e(n, i)$ acts by $-1$ if $1 \leq i \leq k$ or by $1$ if $k < i \leq n$. Let $\mathbb{S}^B_{n-k}$ be the subgroup of $\mathbb{S}^B_n$ consisting of those signed permutations $w \in \mathbb{S}^B_n$ satisfying $|w(i)| \leq k$ for all $i \in [k]$ and $|w(i)| > k$ for all $i \in [k+1, n]$. Then $\mathbb{S}^B_{k,n-k} \cong \mathbb{S}^B_k \times \mathbb{S}^B_{n-k}$. One sees that $\mathbb{S}^B_{k,n-k}$ is the semidirect product of $\mathbb{S}^B_k \times \mathbb{S}^B_{n-k}$. Hence one can define

$$ S^B_{\mu, \nu} := (\chi(n, k) \otimes S^B_k) \uparrow_{\mathbb{S}^B_{k,n-k}} \quad \in S^B_{k,n-k} $$

where $(\mu, \nu)$ is a double partition of $n$, i.e. an ordered pair of partitions $(\mu, \nu)$ such that $|\mu| + |\nu| = n$. A complete list of pairwise non-isomorphic simple $\mathbb{C}S^B$-modules are given by $S^B_{\mu, \nu}$ for all double partitions of $(\mu, \nu)$ of $n$.

The Grothendieck group $G_0(\mathbb{C}S^B)$ of the tower $\mathbb{C}S^B \hookrightarrow \mathbb{C}S^B_{1} \hookrightarrow \mathbb{C}S^B_{2} \hookrightarrow \cdots$ of algebras has a product and a coproduct given by the induction and restriction along the embeddings $\mathbb{S}^B_k \times \mathbb{S}^B_{n-k} \hookrightarrow \mathbb{S}^B_n$. This gives a Hopf algebra structure on $G_0(\mathbb{C}S^B)$, which turns out to be isomorphic to $G_0(\mathbb{C}S^B) \otimes G_0(\mathbb{C}S^B)$ via $\phi : S^B_{\mu, \nu} \otimes S^B_{\zeta, \eta} \mapsto S^B_{\mu, \nu} \otimes S^B_{\zeta, \eta}$ for all double partitions $(\mu, \nu)$ and $(\zeta, \eta)$. We denote by $\cdot$ and $\Delta$ the product and coproduct of both $G_0(\mathbb{C}S^B)$ and $G_0(\mathbb{C}S^B)$.

Now we define a right action and coaction of $G_0(\mathbb{C}S^B)$ on $G_0(\mathbb{C}S^B)$ by

$$ S^B_{\mu, \nu} \cdot S^B_{\lambda} := (S^B_{\mu, \nu} \otimes S^B_{\lambda}) \uparrow_{\mathbb{S}^B_{m+n}} \quad \text{and} \quad \Delta(S^B_{\mu, \nu}) := \bigoplus_{0 \leq i \leq m} S^B_{\mu, \nu} \downarrow_{\mathbb{S}^B_{i+m+n}} $$

for all double partitions $(\mu, \nu)$ of $m$ and all partitions $\lambda$ of $n$.

Proposition 4.4.4. $(G_0(\mathbb{C}S^B), \cdot, \Delta)$ is a dual graded right module and comodule over $G_0(\mathbb{C}S^B)$ such that if $(\mu, \nu)$ is a double partition and $\lambda$ is a partition then

$$ S^B_{\mu, \nu} \cdot S^B_{\lambda} = S^B_{\mu, \nu} \cdot \phi(\Delta(S^B_{\lambda})). $$

Proof. Let $(\zeta, \eta)$ be a double partition of $n$ with $|\zeta| = k$. By Mackey’s formula [11 §4.1.5],

$$ (\chi(n, k) \otimes S^B_{\zeta} \otimes S^B_{\eta}) \uparrow_{\mathbb{S}^B_{k,n-k}} \downarrow_{\mathbb{S}^B_{k,n-k}} \cong (\chi(n, k) \otimes S^B_{\zeta} \otimes S^B_{\eta}) \downarrow_{\mathbb{S}^B_{k,n-k}} \uparrow_{\mathbb{S}^B_{k,n-k}} \cong S^B_{\zeta} \cdot S^B_{\eta}, $$

since there is only one double $(\mathbb{S}^B_n, \mathbb{S}^B_{k,n-k})$-coset in $\mathbb{S}^B_n$. Hence

$$ S^B_{\zeta, \eta} \downarrow_{\mathbb{S}^B_n} \cong (S^B_{\zeta} \otimes S^B_{\eta}) \uparrow_{\mathbb{S}^B_{k,n-k}} \downarrow_{\mathbb{S}^B_{k,n-k}} \cong S^B_{\zeta} \cdot S^B_{\eta}. $$
Using Frobenius reciprocity, the self-duality of $G_0(\mathbb{C}S_\bullet)$, and the Hopf algebra isomorphism $\phi$, one obtains

$$
\langle S_{\lambda} \uparrow_{\mathbb{C}S_n}^{\mathbb{C}S_{n+\epsilon}}, S_{\eta} \rangle = \langle S_{\lambda}, S_{\xi} \ast S_{\eta} \rangle = \langle \Delta(S_{\lambda}), S_{\xi} \otimes S_{\eta} \rangle = \langle \phi(\Delta(S_{\lambda})), S_{\xi} \rangle
$$

for every partition $\lambda$ of $n$. Therefore for any double partition $(\mu, \nu)$ of $m$ one has

$$
S_{\mu, \nu} \circ^{B} S_{\lambda} := (S_{\mu, \nu} \otimes S_{\lambda}) \uparrow_{\mathbb{C}S_{m+n}}^{\mathbb{C}S_{m+n}} \otimes \mathbb{C}S_{n} = S_{\mu, \nu} \cdot (S_{\lambda} \uparrow_{\mathbb{C}S_{m+n}}^{\mathbb{C}S_{m+n}}) = S_{\mu, \nu} \cdot \phi(\Delta(S_{\lambda})).
$$

Using this formula we show that $(G_0(\mathbb{C}S_B), \circ^{B})$ is a graded right $G_0(\mathbb{C}S_\bullet)$-module. It is clear that $S_{\mu, \nu} \circ^{B} S_{\varnothing} = S_{\mu, \nu}$ where $\varnothing$ is the empty partition. If $\xi$ is another partition then

$$
S_{\mu, \nu} \circ^{B} (S_{\lambda} \cdot S_{\xi}) = S_{\mu, \nu} \cdot \phi(\Delta(S_{\lambda} \cdot S_{\xi})) = S_{\mu, \nu} \cdot \phi(\Delta(S_{\lambda})) \cdot \phi(\Delta(S_{\xi})) = (S_{\mu, \nu} \circ^{B} S_{\lambda}) \circ^{B} S_{\xi}.
$$

Hence $(G_0(\mathbb{C}S_B), \circ^{B})$ is a graded right module over $G_0(\mathbb{C}S_\bullet)$. Finally, by the Frobenius reciprocity, $(G_0(\mathbb{C}S_B), \Delta^{B})$ is dual to $(G_0(\mathbb{C}S_\bullet), \circ^{B})$, and hence a graded right comodule over $G_0(\mathbb{C}S_\bullet)$. \hfill $\Box$

Recall from Proposition 2.1.3. that Sym$^{B}$ is a graded module and comodule over Sym. Even though there is a Hopf algebra isomorphism Sym $\cong G_0(\mathbb{C}S_\bullet)$, there is no $\mathbb{Z}$-module isomorphism between $G_0(\mathbb{C}S^\bullet_\bullet)$ and Sym$^{B}$. One can also check that $(G_0(\mathbb{C}S^\bullet_\bullet), \circ^{B}, \Delta^{B})$ is not a Hopf module over the Hopf algebra $G_0(\mathbb{C}S_\bullet)$.

5. Type D

In this section we apply our results in Section 2 to type D and obtain some new results.

5.1. Malvenuto–Reutenauer algebra and descent algebra of type D. The hyperoctahedral group $\mathbb{S}^D_n$ admits a subgroup $\mathbb{S}^D_n$ consisting of all signed permutations $w \in \mathbb{S}^D_n$ with neg$(w)$ even. When $n \geq 2$ this subgroup $\mathbb{S}^D_n$ has a generating set $S$ consisting of $s_0^D := 213 \cdots n$ and $s_i := [1, \ldots, i-1, i+1, i, i+2, \ldots, n]$ for all $i \in [n-1]$. We write $s_0 = s_0^D$ and $s_1^D = 12 \cdots n$ in this section. The pair $(\mathbb{S}^D_n, S)$ is the finite irreducible Coxeter system of type $D_n$ whose Coxeter diagram is illustrated below.

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccccccc}
s_0 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} \\
s_1
\end{array}
\end{array}
\end{array}
$$

Let $w \in \mathbb{S}^D_n$ and set $w(0) := -w(2)$. The descent set of $w$ consists of all $s_i$ such that $i \in \{0, 1, \ldots, n-1\}$ and $w(i) > w(i+1)$. The length of $w$ equals inv$(w) + \text{nsp}(w)$. Subsets of $S = \{s_0, s_1, \ldots, s_{n-1}\}$ are indexed by pseudo-compositions of $n$. For consistency of notation we write $\alpha \vdash_D n$ for a pseudo-composition $\alpha$ of $n$ in this section. If $\alpha \vdash_D n$ then the parabolic subgroup $\mathbb{S}_\alpha^D$ of $\mathbb{S}^D_n$ is generated by $\{s_i : 0 \leq i \leq n-1, i \notin D(\alpha)\}$ and the set of minimal representatives for left $\mathbb{S}_\alpha^D$-cosets in $\mathbb{S}^D_n$ is

$$(\mathbb{S}^D_n)^\alpha := \{w \in D_n : D(w) \subseteq D(\alpha)\}.$$

We define two type $D$ standardizations $\text{D}st(a)$ and $\text{D}st^{B}(a)$ of a word $a \in \mathbb{Z}^n$ as follows. If $\text{D}st^{B}(a) \in \mathbb{S}^D_n$ then let $\text{D}st(a) = \text{D}st^{B}(a) := \text{D}st^{D}(a)$; otherwise let $\text{D}st(a) := s_0^D \text{D}st^{B}(a)$ and $\text{D}st^{D} := \text{D}st^{D}(a) s_0^B$. Then $\text{D}st(a)$ and $\text{D}st^{D}(a)$ are both elements of $\mathbb{S}_\alpha^D$. For example, one has

$$
\text{D}st^{B}(211321) = \text{D}st(211321) = \text{D}st^{D}(211321) = 423651 \quad \text{and} \quad \text{D}st^{B}(211321) = 432651, \quad \text{D}st(211321) = 432651, \quad \text{D}st^{D}(211321) = 432651.
$$

Assume $m \geq 2$ and $n \geq 0$. One sees that the embedding $\mathbb{S}^D_m \times \mathbb{S}_n \hookrightarrow \mathbb{S}^D_{m+n}$ restrict to an embedding $\mathbb{S}^D_m \times \mathbb{S}_n \hookrightarrow \mathbb{S}^D_{m+n}$, which identifies $(u, v) \in \mathbb{S}^D_m \times \mathbb{S}_n$ with an element $u \ast v$ of $\mathbb{S}^D_{m+n}$ whose window notation is $u \ast v := [u(1), \ldots, u(m), m + v(1), \ldots, m + v(n)]$. The image of this embedding is the parabolic subgroup $\mathbb{S}_\alpha^D_{m+n}$ of $\mathbb{S}^D_{m+n}$ generated by $\{s_i : 0 \leq i \leq n-1, i \neq m\}$. The set of minimal representatives for left $\mathbb{S}^D_{m+n}$-cosets in $\mathbb{S}^D_{m+n}$ is

$$
(\mathbb{S}^D_{m+n})^{\mu, \nu} := \{z \in \mathbb{S}^D_{m+n} : -z(2) < z(1) < \cdots < z(m), \ z(m+1) < \cdots < z(m+n)\}.
$$
Proposition 5.1.1. Assume $m \geq 2$ and $n \geq 0$. Then every element $w \in \mathcal{S}_{m+n}^D$ can be written uniquely as $w = (u \times v)z$ where $u \in \mathcal{S}_{m}^D$, $v \in \mathcal{S}_n$, and $z^{-1} \in (\mathcal{S}_m^D)^{m,n}$. Moreover, one has
\[
\begin{align*}
u &= \text{st}^D(w[1,m]), & v &= \text{st}(\hat{w}[m+1,m+n]), \\
u^{-1} &= D\text{st}(w^{-1}[1,m]), & v^{-1} &= \text{st}(w^{-1}[m+1,m+n]).
\end{align*}
\]

Proof. Applying Proposition 2.1.1 to the parabolic subgroup $\mathcal{S}_{m+n}$ shows that every element $w \in \mathcal{S}_{m+n}^D$ can be written uniquely as $w = (u \times v)z$ where $u \in \mathcal{S}_m^D, v \in \mathcal{S}_n$, and $z^{-1} \in (\mathcal{S}_m^D)^{m,n}$. By Proposition 4.1.1, one also has $w = (u' \times v')z'$ where
\[
\begin{align*}
u' &= w[1,m] \in \mathcal{S}_m^B, & v' &= \text{st}(\hat{w}[m+1,m+n]) \in \mathcal{S}_n, & (z')^{-1} \in (\mathcal{S}_m^B)^{m,n}.
\end{align*}
\]

First assume $u' \in \mathcal{S}_m^B$. Since $\text{neg}(z') \equiv \text{neg}(u') \equiv 0$ (mod 2) and $-(z')^{-1}(2) < 0 < (z')^{-1}(1)$, one has $z' \in (\mathcal{S}_m^B)^{m,n}$. It follows that
\[
\begin{align*}
u &= u' = \text{st}^D(w[1,m]), & v &= v', & z &= z'.
\end{align*}
\]
Since $\text{neg}((u')^{-1}) = \text{neg}(u')$, Proposition 4.1.1 implies
\[
\begin{align*}
u^{-1} &= \text{st}^D(w^{-1}[1,m]) = D\text{st}(w^{-1}[1,m]) & v^{-1} &= \text{st}(w^{-1}[m+1,m+n]).
\end{align*}
\]

Next assume $u' \notin \mathcal{S}_m^B$. Then $w = (w's_0^B \times v)s_0^B z'$. Since $\text{neg}(u') \equiv \text{neg}(z') \equiv 1$ (mod 2), one has $u's_0^B \in \mathcal{S}_m^B$ and $s_0^B z' \in \mathcal{S}_{m+n}^B$. Since $\text{neg}(s_0^B z')^{-1}(1) = -(z')^{-1}(1)$ and $\text{neg}(s_0^B z')^{-1}(i) = (z')^{-1}(i)$ if $2 \leq i \leq m+n$, one sees that $(z')^{-1} \in (\mathcal{S}_m^B)^{m,n}$ implies $(s_0^B z')^{-1} \in (\mathcal{S}_m^B)^{m,n}$. Hence
\[
\begin{align*}
u &= u's_0^B = \text{st}^D(w[1,m]), & v &= v', & z &= s_0^B z'.
\end{align*}
\]
Since $\text{neg}((u')^{-1}) = \text{neg}(u') \equiv 1$ (mod 2), Proposition 4.1.1 implies
\[
\begin{align*}
u^{-1} &= s_0^B (u')^{-1} = D\text{st}(w^{-1}[1,m]) & v^{-1} &= \text{st}(w^{-1}[m+1,m+n]).
\end{align*}
\]
This completes the proof. \qed

For example, $w = 25134 = (213 \times 21) \cdot 14325$ and $w^{-1} = 31452 = 13425 \cdot (213 \times 21)$.

Let $\mathcal{S}_m^D := \bigcup_{n \geq 2} \mathcal{S}_{m+n}^D$. We give $\mathcal{Z}\mathcal{S}_m^D = \bigoplus_{n \geq 2} \mathcal{Z}\mathcal{S}_{m+n}^D$ a dual graded right module and comodule structure over the Malvenuto–Reutenauer algebra $\mathcal{Z}\mathcal{S}_m^D$. Assume $m \geq 2$ and $n \geq 0$. Let $u \in \mathcal{S}_m^D$ and $v \in \mathcal{S}_n$. We define
\[
\begin{align*}
u \sqcup^D v &= \{ w \in \mathcal{S}_{m+n}^D : \text{st}^D(w[1,m]) = u, \text{st}(\hat{w}[m+1,m+n]) = v \}, \\
u \sqcup^D v &= \{ w \in \mathcal{S}_{m+n}^D : D\text{st}(w[1,m]) = u, \text{st}(w[m+1,m+n]) = v \}, \\
\sqcap^D u &= \sum_{2 \leq i \leq m} D\text{st}(u[1,i]) \otimes \text{st}(\hat{u}[i+1,m]), \\
\sqcap^D u &= \sum_{2 \leq i \leq m} \text{st}^D(u[1,i]) \otimes \text{st}(\hat{u}[i+1,m]).
\end{align*}
\]

For example, one has
\[
\begin{align*}231 \sqcup^D 1 = 2314 + 2341 + 2431 + 4231 + 2314 + 2341 + 2431 + 4231, \\
231 \sqcup^D 1 = 2314 + 2413 + 3412 + 3421 + 2314 + 2413 + 3412 + 3421, \\
\sqcap^D 231 &= 12 \otimes 12 + 132 \otimes 1 + 2431 \otimes \emptyset, \\
\sqcap^D 231 &= 21 \otimes 12 + 231 \otimes 1 + 2431 \otimes \emptyset.
\end{align*}
\]

Using Proposition 5.1.1 and the linear maps defined in Definition 2.1.2 with $S = \{s_0, \ldots, s_{m+n-1}\}$ and $I = S \setminus \{s_m\}$, one has
\[
\begin{align*}
u \sqcup^D v &= \rho^S_1 (u \times v), & \sqcap^D w &= \rho^S_1 (w), \\
\nu \sqcup^D v &= \rho^S_1 (u \times v), & \sqcap^D w &= \rho^S_1 (w).
\end{align*}
\]
where $w \in \mathcal{S}_{m+n}^D$, $\sqcap^D w$ is the $m$-th term in $\sqcup^D w$, and similarly for $\sqcap^D w$.

Proposition 5.1.2. (i) $(\mathcal{Z}\mathcal{S}_m^D, \sqcup^D, \sqcap^D)$ is a graded right module and comodule over the graded Hopf algebra $(\mathcal{Z}\mathcal{S}, \sqcup, \sqcap)$.

(ii) $(\mathcal{Z}\mathcal{S}_m^D, \psi^D, \sqcap^D)$ is a graded right module and comodule over the graded Hopf algebra $(\mathcal{Z}\mathcal{S}, \psi, \sqcap)$.

(iii) $(\mathcal{Z}\mathcal{S}_m^D, \sqcup^D, \sqcap^D)$ is dual to $(\mathcal{Z}\mathcal{S}, \psi^D, \sqcap^D)$ via the pairing $(u, v) := \delta_{u,v}, \forall u, v \in \mathcal{S}_m^D$.

(iv) $(\mathcal{Z}\mathcal{S}_m^D, \sqcup^D, \sqcap^D)$ is isomorphic to $(\mathcal{Z}\mathcal{S}_m^D, \psi^D, \sqcap^D)$ via $w \mapsto w^{-1}, \forall w \in \mathcal{S}_m^D \cup \mathcal{E}$. 


Proof. This is similar to the proof of Proposition 4.1.2.

Let $\Sigma(\mathcal{G}^D) := \bigoplus_{n \geq 2} \Sigma(\mathcal{G}_n^D)$ where $\Sigma(\mathcal{G}_n^D)$ is a free $\mathbb{Z}$-module with a basis consisting of descent classes

$$D_\alpha(\mathcal{G}_n^D) := \{ w \in \mathcal{G}_n^D : D(w) = D(\alpha) \}, \quad \forall \alpha \models n \geq 2.$$  

One has an embedding $\iota : \Sigma(\mathcal{G}^D) \hookrightarrow \mathbb{Z}\mathcal{G}^D$ by inclusion. If $\alpha \models m \geq 2$ and $\beta \models n \geq 0$ then

$$D_\alpha(\mathcal{G}_m^D) \cup D_\beta(\mathcal{G}_n^D) = D_{\alpha \triangleright \beta}(\mathcal{G}_{m+n}^D) + D_{\alpha \triangleright \beta}(\mathcal{G}_{m+n}^D)$$

by Proposition 2.1.9 where the last term is treated as zero when $\alpha \triangleright \beta$ is undefined.

Let $\Sigma^*(\mathcal{G}^D) := \bigoplus_{n \geq 2} \Sigma^*(\mathcal{G}_n^D)$ where $\Sigma^*(\mathcal{G}_n^D)$ is the dual of $\Sigma(\mathcal{G}_n^D)$ with a dual basis $\{ D^*_\alpha(\mathcal{G}_n^D) : \alpha \models D n \}$. Dual to $\iota : \Sigma(\mathcal{G}^D) \hookrightarrow \mathbb{Z}\mathcal{G}^D$ is a surjection $\chi : \mathbb{Z}\mathcal{G}^D \rightarrow \Sigma^*(\mathcal{G}^D)$ defined by sending each $w \in \mathcal{G}_n^D$ to $D^*_\alpha(\mathcal{G}_n^D)$, where $\alpha \models D n$ satisfies $D(w) = D(\alpha)$.

Recall from Section 2.1 that $\Lambda(\mathcal{G}_n^D)$ be the $\mathbb{Z}$-span of $\Lambda_\alpha(\mathcal{G}_n^D) := \chi'(D_\alpha(\mathcal{G}_n^D))$ for all $\alpha \models D n$, where $\chi' \coloneqq \chi \circ (\cdot)^{-1}$. For $\alpha \models D m$ and $\beta \models D n$ we define

$$\langle \Lambda_\alpha(\mathcal{G}_m^D), \Lambda_\beta(\mathcal{G}_n^D) \rangle := \begin{cases} \# \{ w \in \mathcal{G}_n^D : D(w^{-1}) = D(\alpha) \}, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

By Proposition 2.1.14 this gives a well-defined nondegenerate symmetric bilinear form on the free $\mathbb{Z}$-module $\Lambda(\mathcal{G}^D) := \bigoplus_{n \geq 2} \Lambda(\mathcal{G}_n^D)$ such that $\iota : \Lambda(\mathcal{G}^D) \rightarrow \Sigma^*(\mathcal{G}^D)$ and $\chi' : \Sigma(\mathcal{G}^D) \rightarrow \Lambda(\mathcal{G}^D)$ are dual to each other.

**Theorem 5.1.3.** The following diagram is commutative with each entry being a graded right module and comodule over the corresponding type $A$ Hopf algebra in (3.3).

\[
\begin{array}{ccc}
\mathbb{Z}\mathcal{G}^D & \xrightarrow{\chi} & \Sigma(\mathcal{G}^D) \xleftarrow{\text{dual}} & \Sigma^*(\mathcal{G}^D) \\
\downarrow{\iota} & & \uparrow{\iota} & \\
\Lambda(\mathcal{G}^D) & \xleftarrow{\chi'} & \Sigma(\mathcal{G}^D) & \xrightarrow{\text{dual}} & \Lambda(\mathcal{G}^D)
\end{array}
\]

Reflecting it across the vertical line through $\mathbb{Z}\mathcal{G}^D$ and $\Lambda(\mathcal{G}^D)$ gives a dual diagram.

**Proof.** Apply Theorem 2.1.13 to $(\mathbb{Z}\mathcal{G}^D, \mathfrak{m}^D, \mathfrak{n}^D, \mathfrak{p}^D, \mathfrak{q}^D)$ and then use Corollary 2.1.16.

5.2. **Free quasisymmetric functions of type D and related results.** In this subsection we obtain the following diagram.

\[
\begin{array}{ccc}
\mathbb{F}\text{QSym}^D & \xrightarrow{\chi} & \text{NSym}^D \xleftarrow{\text{dual}} & \text{Sym}^D \\
\downarrow{\iota} & & \uparrow{\iota} & \\
\mathbb{Q}\text{Sym}^D & \xleftarrow{\chi'} & \text{NSym}^D & \xrightarrow{\text{dual}} & \text{Sym}^D
\end{array}
\]

It is isomorphic to the diagram (5.1), with each entry being a graded right module and comodule over the corresponding type $A$ Hopf algebra in (3.6). Reflecting it across the vertical line through $\mathbb{F}\text{QSym}^D$ and $\text{Sym}^D$ gives a dual diagram of graded modules and comodules.

5.2.1. **Free quasisymmetric functions of type D.** Let $(W, S)$ be the Coxeter system of type $D_n$, where $W = \mathcal{G}_n^D$ and $S = \{ s_0 = s_0^D, s_1, \ldots, s_{n-1} \}$, with $n \geq 2$. Let $E = \mathbb{R}^n$ be a Euclidean space with a standard basis $\{ e_1, \ldots, e_n \}$. The group $\mathcal{G}_n^D$ can be realized as a reflection group of $E$ whose root system $\Phi$ is the disjoint union of $\Phi^+ = \{ e_j \pm e_i : 1 \leq i < j \leq n \}$ and $\Phi^- = -\Phi^+$. The set $\Delta = \{ e_1 + e_2, e_2 - e_1, \ldots, e_2 - e_1 \}$ of simple roots corresponds to the generating set $S$ of simple reflections.

Let $X = \{ x_i : i \in \mathbb{Z} \}$ be a set of noncommutative variables. We define $\mathbb{F}\text{QSym}_n^D$ to be the $\mathbb{Z}$-span of $\mathbb{F}\text{QSym}_n^D$ for all parsets $P$ of $\Phi$ and define $\mathbb{F}\text{QSym}_n^D := \bigoplus_{n \geq 2} \mathbb{F}\text{QSym}_n^D$. By Proposition 2.2.4, $\mathbb{F}\text{QSym}_n^D$ has free
\[ F'_{w} = \sum_{f(0), f(1), \ldots, f(n)} x_{f(1)} \cdots x_{f(n)}, \quad \forall w \in D_n. \]

Here we set \( w(0) = -w(2) \) and \( f(-i) = (i) \) for all \( i \in [n] \) by convention.

**Proposition 5.2.1.** \( \text{If } w \in D_n \ (n \geq 2) \text{ and } f \in A(w^{-1} \Phi^+) \text{ then } f \in A(w \Phi^+) \iff \text{st}(f) = w \text{ and thus} \]

\[ s^D_w = \sum_{f \in \mathbb{Z}^n : \text{st}(f) = w} x_f. \]

**Proof.** Let \( f \in \mathbb{Z}^n \). By Lemma 5.2.1 there exists a unique \( w \in D_n \) such that \( f \in A(w^{-1} \Phi^+) \), i.e., the following holds for all \( \alpha = e_j \pm e_i, 1 \leq i < j \leq n \):

\[ \begin{cases} (f, w^{-1} \alpha) \geq 0, & \text{if } w^{-1} \alpha > 0, \\ (f, w^{-1} \alpha) > 0, & \text{if } w^{-1} \alpha < 0. \end{cases} \]

Let \( u = \text{st}^B(f) \). Since a positive root of \( D_n \) is also a positive root of \( B_n \), Proposition 5.2.1 implies (5.4) if \( w = u \). Thus if \( u \in D_n \) then (5.4) holds for \( w = u = \text{st}^B(f) \). On the other hand, if \( u \notin D_n \) then (5.4) still holds for \( w = s^B_0 u = \text{st}^B(f) \), since if \( \alpha = e_j \pm e_i \), where \( 1 < j \leq n \), then \( w^{-1}(\alpha) = u^{-1}(e_j \mp e_i) \), and if \( \alpha = e_j \pm e_i \), where \( 1 < i < j \leq n \), then \( w^{-1}(\alpha) = u^{-1}(\alpha) \). Hence the result holds.

**Corollary 5.2.2.** \( \text{If } w \in D_n \ (n \geq 2) \text{ then } s^D_w = s^B_w + s^D_{s^B(w)} \text{ and } F^D_w = F^B_w + F^D_{s^B(w)}. \text{ Consequently, one has } \text{FQSym}^D \subseteq \text{FQSym}^B. \)

**Proof.** Let \( w \in D_n \ (n \geq 2) \). It follows from Proposition 5.2.1 that

\[ s^D_w = \sum_{\text{st}(f) = w} x_f + \sum_{s^B(f) = s^D_w} x_f = s^B_w + s^D_{s^B(w)} \]

and applying \( w \mapsto w^{-1} \) gives \( F^D_w = F^B_w + F^D_{s^B(w)} \). Hence \( \text{FQSym}^D \subseteq \text{FQSym}^B. \)

Let \( m \geq 2 \) and \( n \geq 0 \). By Proposition 5.2.6 if \( u \in D_m \) and \( v \in D_n \) then

\[ F^D_u \cdot F^D_v = \sum_{w \in u \cdot v} F^D_w. \]

This gives a right action of \( \text{FQSym} \) on \( \text{FQSym}^D \). For consistency of notation we denote this action by \( \circ \). Note that \( F_v \) is \( F_v(X) \) instead of \( F_v(X_{>0}) \) by our convention in this paper. If \( u \in D_m \) then define

\[ \Delta^D(F^D_u) := \sum_{2 \leq i \leq m} F^D_{\text{st}(u[1,i])} \otimes F^D_{\text{st}(u[i+1,m])}. \]

This gives a right coaction of \( \text{FQSym} \) on \( \text{FQSym}^D \).

**Proposition 5.2.3.** \( (\text{FQSym}^D, \circ, \Delta^D) \) is a self-dual graded right module and comodule over \( \text{FQSym} \)

isomorphic to \( (\mathbb{Z} \times G, \otimes, \bullet) \) via \( F^D_w \mapsto w \) and to \( (\mathbb{Z} \times G, \circ, \otimes) \) via \( s^D_w \mapsto w \).

**Proof.** This follows from (5.5), (5.6) and Proposition 6.1.2 (iv).

It follows that if \( u \in D_m \ (m \geq 2) \) and \( v \in D_n \) then

\[ s^D_u \cdot s^D_v = \sum_{u \cdot v} s^D_w \text{ and } \Delta^D(s^D_u) = \sum_{2 \leq i \leq m} s^D_{\text{st}(u[1,i])} \otimes s^D_{\text{st}(u[i+1,m])}. \]

**Remark 5.2.4.** Given \( u \in D_m \ (m \geq 2) \), it follows from (5.7), (5.8), and Corollary 5.2.2 that

\[ \Delta^D(F^D_u) = \sum_{2 \leq i \leq m} F^D_{\text{st}(u[1,i])} \otimes F^D_{\text{st}(u[i+1,m])} \text{ and } \Delta^D(s^D_u) = \sum_{2 \leq i \leq m} s^D_{\text{st}(u[1,i])} \otimes s^D_{\text{st}(u[i+1,m])}. \]

This does not give the desired coaction of \( \text{FQSym} \) on \( \text{FQSym}^D \).
5.2.2. Noncommutative symmetric functions of type D. Let \( \alpha \vdash^D n \geq 2 \). A tableau \( \tau \) of pseudo-ribbon shape \( \alpha \) is type D semistandard if each row is weakly increasing from left to right and each column is strictly increasing from top to bottom, with the extra 0-entry interpreted as \(-w(\tau)(2)\). Let \( s^D_\alpha \) be the sum of \( x_w(\tau) \) for all type D semistandard tableaux \( \tau \) of pseudo-ribbon shape \( \alpha \) and define \( h^D_\alpha := \sum_{\beta \vdash^D n} s^D_\beta \).

In our earlier work [12] we defined a type D analogue \( \text{NSym}^D \) of \( \text{NSym} \), which has two free \( \mathbb{Z} \)-bases consisting of \( s^D_\alpha \) and \( h^D_\alpha \), respectively, for all \( \alpha \vdash^D n \) and all \( n \geq 2 \). We showed \( h^D_\alpha \cdot h_\beta = h^D_{\alpha \beta} \) and \( s^D_\alpha \cdot s_\beta = s^D_{\alpha \beta} + s^D_{\alpha \beta} \) for \( \alpha \vdash^D n \geq 2 \) and \( \beta \vdash \geq n \geq 0 \), where the last term is treated as zero when \( \alpha \triangleright \beta \) is undefined. This gives a right \( \text{NSym} \)-action on \( \text{NSym}^D \), which is denoted by \( \circ^D \) for consistency of notation. Now we study the relation between \( \text{NSym}^D \) and \( \text{FQSym}^D \), and use it to obtain a \( \text{NSym} \)-coaction on \( \text{NSym}^D \).

**Proposition 5.2.5.** Let \( \alpha \vdash^D n \geq 2 \). Then \( s^D_\alpha \) equals the sum of \( s^D_w \) for all \( w \in D_\alpha (\mathbb{S}_n^D) \).

*Proof.* Each \( f \in \mathbb{Z}^n \) corresponds to a unique tableau \( \tau \) of shape \( \alpha \) such that \( w(\tau) = f \). Let \( u = \text{st}_B(f) \) and \( w = \text{st}_D(f) \). By Proposition 5.2.11 it suffices to show that

\[
D(w) = D(\alpha) \quad \text{if and only if} \quad \tau \text{ is type D semistandard.}
\]

One sees that (5.8) holds when \( w = u \in \mathbb{S}_n^D \). If \( u \notin D_n \) then \( w = s^D_u \) and one still has (5.8) as negating \( \pm 1 \) does not change any descent. \( \square \)

It follows that there is an injection \( : \text{NSym}^D \hookrightarrow \text{FQSym}^D \) by inclusion.

**Proposition 5.2.6.** The graded right module and comodule \( \text{FQSym}^D \) over \( \text{FQSym} \) restricts to a graded right module and comodule \( \text{NSym}^D \) over \( \text{NSym} \), which is isomorphic to the graded right module and comodule \( (\Sigma(\mathbb{S}_n), \psi^D, \circ^D) \) over \( \text{NSym} \) via the map \( s^D_\alpha \mapsto D_\alpha (\mathbb{S}_n^{D}) \), \( \forall \alpha \vdash^D m, \forall m \geq 2 \).

*Proof.* By Proposition 5.2.3 there is an isomorphism \( (\text{FQSym}^D, \circ^D, \Delta^D) \cong (\mathbb{Z}\mathbb{S}_n, \psi^D, \circ^D) \) via \( s^D_w \mapsto w \), \( \forall w \in \mathbb{S}_n^D \). Restricting this isomorphism to \( \text{NSym}^D \) gives the result. \( \square \)

**Remark 5.2.7.** If \( k \) is a nonnegative integer then it follows from (5.8) and Proposition 5.2.6 that

\[
\Delta^D(h^D_\alpha) = \Delta^D(s^D_{\tau_{12} \cdots k}) = \sum_{0 \leq i \leq k} s^D_{12 \cdots i} \circ \text{st}_D(i+1, \ldots, k) = \sum_{0 \leq i \leq k} h^D_i \circ h_{k-i}.
\]

We do not have any explicit formula for \( \Delta^D(h^D_\alpha) \) or \( \Delta^D(s^D_\alpha) \) for an arbitrary \( \alpha \vdash^D n \geq 2 \).

5.2.3. Quasisymmetric functions of type D. Let \( X = \{ x_i : i \in \mathbb{Z} \} \) be a totally ordered set of commutative variables. If \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is a pseudo-composition of \( n \geq 2 \) and \( i_0 := -i_2 \) then define

\[
M^D_\alpha := \sum_{-i_2 \leq i_1 \leq \cdots \leq i_n \atop j \in D(\alpha)} x_{i_1} x_{i_2} \cdots x_{i_n} \quad \text{and} \quad F^D_\alpha := \sum_{-i_2 \leq i_1 \leq \cdots \leq i_n \atop j \in D(\alpha)} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\alpha \leq \beta} M^D_{\beta}.
\]

One can check that \( M^D_\alpha = M^B_\alpha \) if \( \alpha \notin D(\alpha) \). In fact, one has

\[
M^D_\alpha = \begin{cases} x_{i_0} \cdot M_{(\alpha_2, \ldots, \alpha_k)} = M^B_\alpha, & \text{if } \alpha_1 \geq 2, \\ \sum_{0 < j_2 < \cdots < j_k} x_{j_2} x_{j_3} \cdots x_{j_k} & \text{if } \alpha_1 = 1, \\ M_{(\alpha_2, \ldots, \alpha_k)} = M^B_\alpha, & \text{if } \alpha_1 = 0 \text{ and } \alpha_2 \geq 2, \\ \sum_{j_2 < j_3 < \cdots < j_k} x_{j_2} x_{j_3} \cdots x_{j_k} & \text{if } \alpha_1 = 0 \text{ and } \alpha_2 = 1. \end{cases}
\]

In our earlier work [12] we defined a type D analogue \( \text{QSym}^D \) of \( \text{QSym} \), which admits two free \( \mathbb{Z} \)-bases consisting of \( M^D_\alpha \) and \( F^D_\alpha \), respectively, for all \( \alpha \vdash^D n \) and all \( n \geq 2 \).

Recall that \( \chi^B : \text{FQSym}^B \rightarrow \text{NSym}^B \) is defined by \( x_i \mapsto x_{\lfloor i \rfloor} \) for all \( i \in \mathbb{Z} \). If \( f \in \mathbb{Z}^n \) one can write \( \chi^B(x_f) = x_{i_1} \cdots x_{i_n} \) such that \( 0 \leq i_1 \leq \cdots \leq i_n \), and we define \( \chi^D(x_f) := x_{i_1} x_{i_2} \cdots x_{i_n} \), where the sign of \( i_1 \) is the same as \( (-1)^{\text{neg}(f)} \). This gives a linear maps \( \chi^D : \mathbb{Z}(X) \rightarrow \mathbb{Z}[X] \), which is not an algebra homomorphism, as \( \chi^D(x_2 x_1) = x_1 x_2 \neq x_2 x_1 = \chi^D(x_2) \chi^D(x_1) \).
Proposition 5.2.8. Let \( w \in \mathfrak{S}_n^D \) and \( \alpha \models^D n \geq 2 \) with \( D(w) = D(\alpha) \). Then \( F_w^D := \chi^D(F_w^D) = F_\alpha^D \).

Proof. Suppose that \( f \in \mathcal{A}(w \Phi^+) \). Then \( -f(w(2)) \leq f(w(1)) \leq f(w(2)) \leq \cdots \leq f(w(n)) \). This implies \( 0 \leq |f(w(1))| \leq f(w(2)) \leq \cdots \leq f(w(n)) \). Since \( \text{neg}(w) \) is even, it follows that \( f(w(1)) \) has the same sign as \((-1)^{\text{neg}(f)}\). Therefore \( \chi^D(x_f) = x_{f(w(1))} \cdots x_{f(w(n))} \). Then comparing (5.3) with (5.4) gives the result. \( \square \)

Let \( X + Y_{>0} = \{ \ldots , x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots , y_1, y_2, \ldots \} \) be a totally ordered set of commutative variables. We defined in [12] a right coaction \( \Delta^D \) of \( \mathfrak{QSym}^D \).

The second map above is induced by the canonical projection

\[
\mathbb{Z}[X + Y_{>0}] \cong \mathbb{Z}[X] \otimes \mathbb{Z}[Y_{>0}] \rightarrow \mathbb{Z}[X]_{\geq 2} \otimes \mathbb{Z}[Y_{>0}]
\]

where \( \mathbb{Z}[X]_{\geq 2} \) is the \( \mathbb{Z} \)-span of those polynomials in \( X \) with degree at least 2. If \( \alpha = (\alpha_1, \ldots , \alpha_\ell) \models^D n \geq 2 \) then

\[
\Delta^D M_\alpha^D = \sum_{k \leq j \leq \ell} M_{(\alpha_1, \ldots , \alpha_j)}^D \otimes M_{(\alpha_{j+1}, \ldots , \alpha_\ell)}^D
\]

and

\[
\Delta^D F_\alpha^D = \sum_{2 \leq i \leq n} F_{\alpha_{\leq i}}^D \otimes F_{\alpha_{> i}}^D.
\]

The second equality is equivalent to

\[
\Delta^D F_w^D = \sum_{2 \leq i \leq n} F_{\text{st}(w[i:i],i)}^D \otimes F_{\text{st}(w[i+1:n])}^D.
\]

Now we define a \( \mathfrak{QSym} \)-action on \( \mathfrak{QSym}^D \). Let \( u \in \mathfrak{S}_m^D \) and \( v \in \mathfrak{S}_n^D \). Applying \( \chi^D \) to (5.5) gives

\[
F_u^D \circ^D F_v := \chi^D(F_u^D \cdot F_v^D) = \sum_{w \in u \cdot v} F_w^D.
\]

Proposition 5.2.9. \( (\mathfrak{QSym}^D, \circ^D, \Delta^D) \) is a graded right module and comodule over the Hopf algebra \( \mathfrak{QSym} \) isomorphic to the graded right module and comodule \((\Sigma^*(\mathfrak{S}_m^D), \otimes^D, \tau^D)\) over \((\Sigma^*(\mathfrak{S}), \otimes, \tau)\). The map \( \chi^D \) induces a surjection from the graded right module and comodule \( \mathfrak{FQSym}^D \) onto \( \mathfrak{QSym}^D \).

Proof. Compare (5.2) and (5.3) with the definition of \((\Sigma^*(\mathfrak{S}_m^D), \otimes^D, \tau^D)\) and with (5.5) and (5.6). \( \square \)

Corollary 5.2.10. The embedding \( \iota : \mathfrak{NSym}^D \rightarrow \mathfrak{FQSym}^D \) and the surjection \( \chi^D : \mathfrak{FQSym}^D \rightarrow \mathfrak{QSym}^D \) are dual morphisms of graded modules and comodules.

Proof. This follows from Theorem 5.1.3, Proposition 5.2.6 and Proposition 5.2.9. \( \square \)

5.2.4. Symmetric functions of type \( D \). We define \( \text{Sym}^D := \chi^D(\mathfrak{NSym}^D) \subseteq \mathfrak{QSym}^D \). It has two spanning sets \( \{ s_\alpha^D : \alpha \models^D n, n \geq 2 \} \) and \( \{ h_\alpha^D : \alpha \models^D n, n \geq 2 \} \), where

\[
s_\alpha^D := \chi^D(s_\alpha^D) \quad \text{and} \quad h_\alpha^D := \chi^D(h_\alpha^D) = \sum_{\beta \leq \alpha} \chi^D(s_\beta^D) = \sum_{\beta \leq \alpha} s_\beta^D.
\]

We will give a representation theoretic interpretation for \( s_\alpha^D \) later.

Suppose that \( \alpha \models^D m \geq 2 \) and \( \beta \models n \geq 0 \). We define

\[
h_\alpha^D \circ^D h_\beta^D := \chi^D(h_\alpha^D \cdot h_\beta^D) = h_{\alpha \beta}^D \quad \text{and} \quad \Delta^D(s_\alpha^D) = \Delta^D(\chi^D(s_\alpha^D)) := (\chi^D \otimes \chi)(\Delta^D(s_\alpha^D)).
\]

Proposition 5.2.11. \( (\text{Sym}^D, \circ^D, \Delta^D) \) is a graded right module and comodule over the Hopf algebra \( \text{Sym} \) isomorphic to the graded right module and comodule \( \Lambda(\mathfrak{S}_m^D) \) over \( \Lambda(\mathfrak{S}) \). The injection \( \iota : \text{Sym}^D \rightarrow \mathfrak{QSym}^D \) and the surjection \( \chi^D : \mathfrak{NSym}^D \rightarrow \text{Sym}^D \) are dual morphisms of graded right modules and comodules.

Proof. The result follows from Theorem 5.1.3, Proposition 5.2.6 and Proposition 5.2.9. \( \square \)
5.3. **Representation theory of 0-Hecke algebras of type D.** Assume $n \geq 2$ and let $(W, S)$ be the finite Coxeter system of type $D_n$, where $W = \mathcal{S}_n^D$ and $S = \{s_0 = s_0^D, s_1, \ldots, s_{n-1}\}$. The 0-Hecke algebra $H_n^D(0)$ of $(W, S)$ admits two generating sets $\{\pi_0 = \pi_0^D, \pi_1, \ldots, \pi_{n-1}\}$ and $\{\bar{\pi}_0 = \bar{\pi}_0^D, \bar{\pi}_1, \ldots, \bar{\pi}_{n-1}\}$. One can realize $\pi_1, \ldots, \pi_{n-1}$ as signed bubble-sorting operators on $\mathbb{Z}^n$ in the same way as in type B and $\pi_0$ acts on $\mathbb{Z}^n$ by

$$\pi_0(a_1, \ldots, a_n) := \begin{cases} (-a_2, -a_1, a_3, \ldots, a_n), & \text{if } a_1 + a_2 > 0, \\ (a_1, \ldots, a_n), & \text{if } a_1 + a_2 \leq 0. \end{cases}$$

The projective indecomposable $H_n^D(0)$-modules and simple $H_n^D(0)$-modules are given by $\mathcal{P}_\alpha^D := \mathcal{P}_i^D$ and $\mathcal{C}_\alpha^D := C_i^D$, respectively, where $I = \{s_i : i \in D(\alpha)\}$, for all $\alpha \vdash_D n \geq 2$. One can realize $\mathcal{P}_\alpha^D$ as the space of type D standard tableaux of pseudo-ribbon shape $\alpha$ with an appropriate $H_n^D(0)$-action [22].

The parabolic subalgebra $H_0^D(0)$ of $H_n^D(0)$ is generated by $\{\pi_i : i \in \{0, 1, \ldots, n-1\} \setminus D(\alpha)\}$. If $m \geq 2$ and $n \geq 0$ then one has $H_m^D(0) \otimes H_n^D(0) \cong H_{m+n}^D(0)$ and hence an embedding $H_m^D(0) \otimes H_n^D(0) \hookrightarrow H_{m+n}^D(0)$.

Associated with the tower $H_0^D(0) : H_2^D(0) \to H_4^D(0) \to \cdots$ are Grothendieck groups

$$G_0(H_n^D(0)) := \bigoplus_{n \geq 2} G_0(H_n^D(0)) \quad \text{and} \quad K_0(H_n^D(0)) := \bigoplus_{n \geq 2} K_0(H_n^D(0)).$$

Assume $m \geq 2$ and $n \geq 0$. Let $M$ and $N$ be finitely generated modules over $H_m^D(0)$ and $H_n^D(0)$, respectively. Define

$$M \otimes^D N := (M \otimes N) \uparrow_{H_{m+n}(0)}^{H_m^D(0)} \quad \text{and} \quad \Delta^D(M) := \sum_{2 \leq i \leq m} M \downarrow_{H_{i-1}^D(0)}^{H_m^D(0)}.$$

Using the linear maps $\tilde{\mu}_i^S : G_0(H_{W_i}(0)) \to G_0(H_{W}(0))$ and $\tilde{\rho}_i^S : G_0(H_{W}(0)) \to G_0(H_{W_i}(0))$ defined in [24], with $S = \{s_0, \ldots, s_{m+n-1}\}$ and $I = S \setminus \{s_m\}$, one has

$$M \otimes^D N = \tilde{\mu}_i^S(M \otimes N) \quad \text{and} \quad \Delta^D_m(Q) := Q \downarrow_{H_{m+n}(0)}^{H_{m}^D(0)} = \tilde{\rho}_i^S(Q).$$

where $Q$ is a finitely generated $H_{m+n}^D(0)$-module. Also recall from Proposition 2.3.1 that $\tilde{\mu}_i^S$ and $\tilde{\rho}_i^S$ restrict to $\mu_i^S : K_0(H_{W_i}(0)) \to K_0(H_{W}(0))$ and $\rho_i^S : K_0(H_{W}(0)) \to K_0(H_{W_i}(0))$. If $M$, $N$, and $Q$ are all projective then

$$M \otimes^D N = \mu_i^S(M \otimes N) \quad \text{and} \quad \Delta_m^D(Q) = \rho_i^S(Q).$$

Define two characteristic maps below, where $\alpha \vdash_D n \geq 2$:

$${\text{Ch}} : \quad G_0(H_n^D(0)) \to Q\mathrm{Sym}^D \quad \text{and} \quad \mathbf{ch} : \quad K_0(H_n^D(0)) \to \mathbf{N}\mathrm{Sym}_D^D.$$

**Theorem 5.3.1.** (i) $(G_0(H_n^D(0)), \otimes^D, \Delta^D)$ is a graded right module and comodule over the graded Hopf algebra $G_0(H_n^D(0))$.  
(ii) $(K_0(H_n^D(0)), \otimes^D, \Delta^D)$ is a graded right module and comodule over the graded Hopf algebra $K_0(H_n^D(0))$.  
(iii) $(G_0(H_n^D(0)), \otimes^D, \Delta^D)$ is dual to $(K_0(H_n^D(0)), \otimes^D, \Delta^D)$ via $(\mathcal{P}_\alpha^D, C^D_\alpha) := \delta_{\alpha, \beta}$.

(iv) Both $\text{Ch}$ and $\mathbf{ch}$ are isomorphisms of graded modules and comodules.

**Proof.** Apply Theorem 2.3.6, Theorem 1.1.1, Proposition 5.2.6, and Proposition 5.2.9 \hfill \Box

**Proposition 5.3.2.** If $\alpha$ is a pseudo-composition of $n \geq 2$ then $\text{Ch}(\mathcal{P}_\alpha^D) = s_n^D$.

**Proof.** This follows from Proposition 2.3.9 \hfill \Box

**Remark 5.3.3.** We obtained in Section 1.4 a self-dual graded Hopf algebra structure on $\mathcal{S}_n^D$ using the embedding $\mathcal{S}_m^D \times \mathcal{S}_n^D \hookrightarrow \mathcal{S}_{m+n}^D$. This embedding restricts to $\mathcal{S}_m^D \times \mathcal{S}_n^D \hookrightarrow \mathcal{S}_{m+n}^D$, which may be used to define dual graded algebra and coalgebra structures on $\mathcal{S}_n^D$ similarly as in type B. However, the result is not a Hopf algebra.
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