Autonomous Dynamical System Approach for $f(R)$ Gravity

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In this work we shall investigate the cosmological dynamical system of $f(R)$ gravity, by constructing it in such a way so that it is rendered autonomous. We shall study the vacuum $f(R)$ gravity case, but also the case that matter and radiation perfect fluids are present along with the $f(R)$ gravity. The dynamical system is constructed in such a way so that the time-dependence of the system is contained in a single parameter which depends on the Hubble rate and it’s second derivative. The autonomous structure of the dynamical system is achieved when this parameter is constant, therefore we focus on these cases. For the vacuum $f(R)$ case, we investigate two cases with the first leading to a stable de Sitter attractor fixed point but also to an unstable de Sitter fixed point, and the second is related to a matter dominated era. The stable de Sitter attractor is also found for the $f(R)$ gravity in the presence of matter and radiation perfect fluids. In all the cases we performed a detailed numerical analysis of the dynamical system and we also investigate in detail the stability of the resulting fixed points. Also, we present an exceptional feature of the $R^2$ gravity model, in the absence of perfect fluids. Finally, we investigate what is the approximate form of the $f(R)$ gravities near the stable and the unstable de Sitter fixed points.

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I. INTRODUCTION

Since the observation of the late-time acceleration\cite{1} that our Universe experiences, the focus in modern theoretical cosmology is to find a theoretical framework that can consistently harbor this late-time acceleration era. Many theoretical proposals can potentially provide a consistent description of the so-called dark energy era, with the modified gravity being the main appealing approach, for reviews see\cite{2,3}. Among the various modified gravity descriptions, $f(R)$ gravity is the most important and most studied approach, in the context of which it is possible to provide the unification of the early with the late-time acceleration of the Universe\cite{6,7}.

Due to the importance of $f(R)$ gravity, it is compelling to know the general structure of the phase space of the cosmological dynamical system of the theory. Many studies already exist in the literature, studying various aspects of dynamical systems in cosmology of modified gravity\cite{8–26}, see also\cite{27}, which focus on the existence of fixed points and also their stability. In this paper we shall attempt an alternative approach for the study of the $f(R)$ gravity dynamical system case, in the presence or in the absence of perfect matter fluids. Particularly, we shall investigate the phase space in the limit that the corresponding dynamical system is strictly autonomous. The motivation for studying the autonomous limit of the cosmological dynamical system, comes mainly from the fact that if a dynamical system is non-autonomous, then the stability of the fixed points is not guaranteed by using the theorems which hold true in the autonomous case. Hence, the stability of a fixed point corresponding to a non-autonomous system is rather a problem without a solution, unless highly non-trivial techniques are employed. Let us explain our argument by exploiting a very simple example which can be found in\cite{28}, so consider the one dimensional dynamical system $\dot{x} = -x + t$. The solution can be easily found to be $x(t) = t - 1 + e^{-t}(x_0 + 1)$, from which it is obvious that all the solutions asymptotically approach $t - 1$ for $t \to \infty$. Also it is easy to see that the only fixed point is the time-dependent solution $x = t$, which however is not a solution to the dynamical system. In addition, a standard analysis by using the fixed point theorems, shows that the vector field actually move away from the attractor $x(t) = t - 1$, which is simply wrong. Therefore, it is conceivable that knowing the fixed points of a non-autonomous system, and then applying the Hartman-Grobman theorems, does not provide a consistent solution to the stability problem of the fixed points, and in effect, the structure of the cosmological phase space cannot be revealed in such a way.

The purpose of this paper is to construct a dynamical system for $f(R)$ gravity, and we aim to study the cases for which the dynamical system is rendered autonomous. Particularly, we shall quantify the whole time-dependence of
the dynamical system in terms of one single parameter \( m \), which turns out to be equal to \( m = -\frac{\pi}{R^2} \), where \( H \) is the Hubble rate of the Universe. Then we will focus on various values of this parameter with some physical significance, and specifically we focus on \( m \approx 0 \) and \( m = -\frac{\pi}{2} \). The first case turns out to describe an inflationary de Sitter final attractor which is stable, in both the vacuum and matter fluid \( f(R) \) gravity. We need to note that not only the de Sitter or a quasi de Sitter cosmology yield \( m \approx 0 \), so the general study we perform covers all the cosmologies which may yield \( m \approx 0 \). This means that the case \( m \approx 0 \) may describe an approximate solution of some specific \( f(R) \) gravity, for example during the slow-roll era, or similar. In all the cases, we shall find the fixed points of the dynamical system, and the fact that the dynamical system is autonomous will enable us to use the fixed point theorems and thus reveal the stability structure of the system. Also we shall use various sets of initial conditions, emphasizing for \( e \)-foldings numbers in the range \( N = [0, 60] \), and we shall present in detail the stability properties of the fixed points in terms of the resulting trajectories. As a side study, we shall also consider in brief the case \( m = -\frac{\pi}{2} \), which as we show, describes a matter dominated era, so we shall investigate the \( f(R) \) gravity phase space properties, and we also show that the stability of the matter domination final attractor can be achieved. The latter study clearly demonstrates that a vacuum \( f(R) \) gravity may successfully describe a matter dominated era, however caution is needed for the stability study of the fixed point. In principle there are other cases for which the \( f(R) \) gravity dynamical system can be rendered autonomous, but we do not cover all the different cases for brevity. We thoroughly discuss how an autonomous dynamical system can be obtained in the context of \( f(R) \) gravity in the presence of matter and radiation perfect fluids, and we mainly focus on de Sitter attractors. Also, along with the stable de Sitter attractor, in the vacuum \( f(R) \) gravity case, there exists also an unstable de Sitter fixed point. Intriguingly enough, as we explicitly demonstrate at a later section, when a vacuum \( R^2 \) gravity is considered, the dynamical system reaches the unstable de Sitter attractor. As we argue, this feature may be a sign of graceful exit from inflation. Also we shall investigate the approximate forms of the \( f(R) \) gravities near the stable and the unstable de Sitter fixed points, and also we discuss the role that play terms of the \( R^2 \) form, in the graceful exit issue.

This paper is organized as follows: In section II, we present the way to obtain an autonomous dynamical system from a general vacuum \( f(R) \) gravity. In section III, we study in detail the phase space of vacuum \( f(R) \) gravity, and we focus on the cases \( m \approx 0 \) and \( m = -\frac{\pi}{2} \). We find the fixed points in each case and we investigate the stability structure of the fixed points, by analyzing many possible initial conditions. Moreover, we investigate which are the approximate forms of the \( f(R) \) gravities, near the stable and the unstable de Sitter fixed points, and also we thoroughly discuss the graceful exit issue, in view of the presence of \( R^2 \) terms. Moreover, we present a specific example of an \( f(R) \) cosmology, which satisfies \( m \approx 0 \), and we examine the behavior of the dynamical variables which were introduced in the previous sections. In addition, other attractors apart from the de Sitter one, are also discussed. In section IV we thoroughly study the structure of the \( f(R) \) gravity autonomous dynamical system, in the presence of matter and radiation perfect fluids, emphasizing on the de Sitter attractors. Finally, the conclusions follow in the end of the paper.

Before starting our study, let us note that we will use a flat Friedmann-Robertson-Walker (FRW) metric, as a geometric background, the line element of which is,

\[
\text{d}s^2 = -\text{d}t^2 + a(t)^2 \sum_{i=1,2,3} (\text{d}x^i)^2, \tag{1}
\]

where \( a(t) \) is the scale factor. Also for a FRW metric, the Ricci scalar is,

\[
R = 6 \left( \dot{H} + 2H^2 \right), \tag{2}
\]

where \( H = \frac{\dot{a}}{a} \), is the Hubble rate.

II. THE AUTONOMOUS DYNAMICAL SYSTEM OF VACUUM \( f(R) \) GRAVITY

In this section we shall present the general structure of the vacuum \( f(R) \) gravity, and we will discuss the cases that may render this an autonomous system. The vacuum \( f(R) \) gravity action is,

\[
S = \frac{1}{2\kappa^2} \int \text{d}^4x \sqrt{-g} f(R), \tag{3}
\]

where \( \kappa^2 = 8\pi G = \frac{1}{M_p^2} \) and also \( M_p \) is the Planck mass scale. In the context of the metric formalism, the equations of motion are obtained by varying the gravitational action \( S \) with respect to the metric tensor \( g_{\mu\nu} \), so by doing so we obtain the following equations of motion,

\[
F(R) R_{\mu\nu}(g) - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} f(R) + g_{\mu\nu} \Box F(R) = 0, \tag{4}
\]
which can be written as follows,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{\kappa^2}{F(R)} \left( T_{\mu\nu} + \frac{1}{\kappa^2} \left( \frac{f(R) - RF(R)}{2} g_{\mu\nu} + \nabla_\mu \nabla_\nu F(R) - g_{\mu\nu} \Box F(R) \right) \right),$$

(5)

with the prime indicating differentiation with respect to the Ricci scalar. For the FRW metric of Eq. (1), the cosmological equations of motion become,

$$0 = -\frac{f(R)}{2} + 3 \left( H^2 + \dot{H} \right) F(R) - 18 \left( 4H^2 \dot{H} + H \ddot{H} \right) F'(R),$$

(6)

$$0 = \frac{f(R)}{2} - \left( \dot{H} + 3H^2 \right) F(R) + 6 \left( 8H^2 \dot{H} + 4 \dot{H}^2 + 6H \ddot{H} + \dddot{H} \right) F'(R) + 36 \left( 4H \dot{H} + \dddot{H} \right)^2 F'(R),$$

(7)

where $F(R) = \frac{\partial f}{\partial R}$, $F'(R) = \frac{\partial F}{\partial R}$, and $F''(R) = \frac{\partial^2 F}{\partial R^2}$. In order to reveal the autonomous structure of the cosmological dynamical system described by the equations of motion (6), we shall introduce the following variables,

$$x_1 = -\frac{\dot{F}(R)}{F(R)H}, \quad x_2 = -\frac{f(R)}{6F(R)H^2}, \quad x_3 = \frac{R}{6H^2}.$$

(8)

In the following we shall use the e-foldings number $N$, instead of the cosmic time, so the derivative with respect to the e-foldings number can be expressed as follows,

$$\frac{d}{dN} = \frac{1}{H} \frac{dt}{dt},$$

(9)

which shall be useful. Hence, by using the variables (8) and also the equations of motion (6), after some algebra we obtain the following dynamical system,

$$\frac{dx_1}{dN} = -4 - 3x_1 + 2x_3 - x_1 x_3 + x_1^2,$$

(10)

$$\frac{dx_2}{dN} = 8 + m - 4x_3 + x_2 x_3 - 2x_2 x_3 + 4x_2,$$

$$\frac{dx_3}{dN} = -8 - m + 8x_3 - 2x_3^2,$$

where the parameter $m$ is equal to,

$$m = -\frac{\dddot{H}}{H^3}.$$

(11)

By looking the dynamical system (10), it is obvious that the only $N$-dependence (or time dependence) is contained in the parameter $m$. Also we did not expressed $m$ as a function of $N$, since we shall assume that this parameter will take constant values. Hence if the parameter $m$ is a constant, the dynamical system is rendered autonomous since no explicit time dependence exist in it.

The effective equation of state (EoS) for a general $f(R)$ gravity theory is,

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2},$$

(12)

and it can be written in terms of the variable $x_3$ as follows,

$$w_{\text{eff}} = -\frac{1}{3}(2x_3 - 1).$$

(13)

By using the dynamical system (10) and the EoS (13), given the value of the parameter $m$, in the following sections we shall investigate the structure of the phase space corresponding to the vacuum $f(R)$ gravity, and we shall discuss in detail the physical significance and implications of the results.
III. STUDY OF THE VACUUM $f(R)$ GRAVITY PHASE SPACE

The parameter $m$ appearing in the non-linear dynamical system (10) plays an important role, since it is the only source of time-dependence in the dynamical system. Let us note that for certain cosmological evolutions this parameter is constant. For example, a quasi de Sitter evolution, in which case the scale factor is,

$$a(t) = e^{H_0 t - H_i t^2},$$  \hspace{1cm} (14)

the parameter $m$ is equal to zero, and the same applies for a de Sitter evolution. However, in this section we shall not assume that the scale factor has a specific form, but we shall study in general the cases $m \simeq 0$ and $m = -\frac{9}{2}$. As we shall see, these two cases have special physical significance. With regard to the $m \simeq 0$ case, this is easy to check, since if we solve the differential equation $\frac{H}{H^2} = 0$, this yields the solution,

$$H(t) = H_0 - H_i t,$$  \hspace{1cm} (15)

This means that we focus on cosmologies for which the approximate solution for the evolution is a quasi de Sitter evolution. This does not mean that the exact Hubble rate is a quasi-de Sitter evolution, but the approximate $f(R)$ gravity which drives the evolution, leads to an approximate quasi-de Sitter evolution. Interestingly enough, for the quasi-de Sitter evolution (15), the following conditions hold true,

$$H \dot{H} \gg \dot{H}, \quad \dot{H} \ll H^2,$$  \hspace{1cm} (16)

which are the slow-roll conditions. Hence the $m \simeq 0$ case is related to the slow-roll condition on the inflationary era. In the following sections we shall perform an in depth analysis of the $m \simeq 0$, since this is the most interest scenario.

Before discussing the details of the two cases $m \simeq 0$ and $m = -\frac{9}{2}$, let us present in brief the classical approach to the classification of fixed points. According to the classical approach, the linearization procedure applies [28], which can also be applied in our case, if the fixed point is hyperbolic. Also we briefly discuss some essential features of the stability theory for dynamical systems.

With the stability theory of dynamical systems, basically one reveals the stability of the solutions and of the trajectories of dynamical systems. Basically, the stability of the solutions-trajectories, addresses the behavior of these, if small perturbations of the initial conditions are performed. Also it is important to have a solid grasp of the asymptotic behavior of the solutions and trajectories of a dynamical system, with asymptotic meaning after a long period of time. The duration and determination of this long period of time is usually determined by the physical scales in the theory. For the purposes of this paper, since we are interested in inflationary dynamics, the asymptotic behavior corresponds to approximately $N \simeq 60$ $e$-foldings. The simplest and most valuable behavior of the solutions-trajectories is exhibited by the fixed points of the dynamical system at hand, and also by periodic orbits.

In all cases, how do orbits-trajectories or solutions behave near the fixed point, do these approach the fixed point, or alternatively said, are the trajectories attracted by the fixed point, or are these repelled from it? Also, if we perturb the orbit by using alternative initial conditions, how these perturbations affect the behavior of the trajectories near the fixed point? If the trajectory is always attracted by a fixed point, then this fixed point is called an attractor of the trajectories. Also if perturbed orbits tend asymptotically to a given orbit, then this orbit is called stable. The fixed points provide a characteristic picture of the structural stability of the dynamical system. The stability of a fixed point refers to the behavior of the trajectories near this point, if the trajectories are attracted to it, this is called stable (attractor), and if these are attracted to it asymptotically, this is called asymptotically stable (asymptotic attractor). In the case that trajectories are repelled from it, the fixed point is an unstable fixed point. One quantitative way to reveal the stability of a fixed point, is to apply the Hartman-Grobman theorem (see below), by linearizing the autonomous non-linear dynamical system. If the eigenvalues of the linearization matrix are negative real numbers, or even complex numbers with negative real parts, then the fixed point is stable (attractor). If none of the eigenvalues are purely imaginary, or equal to zero, then the fixed point can be an attractor or a repeller, and this is subject to the behavior of stable and unstable manifolds, see the book [28] for more details on this. The important thing to note is that the Hartman-Grobman theorem applies only on hyperbolic fixed points, which means that only in the case that the eigenvalues of the linearization matrix have non-zero real parts the Hartman-Grobman theorem may apply. It is the purpose of this paper to examine what happens in the cases that the Hartman-Grobman theorem does not apply.

As it is well known, the Hartman-Grobman linearization theorem determines the stability and the structure of the phase space, when hyperbolic fixed points are taken into account. Let $\Phi(t) \in \mathbb{R}^n$ be the solution to the following flow,

$$\frac{d\Phi}{dt} = g(\Phi(t)),$$  \hspace{1cm} (17)
with \( g(\Phi(t)) \) being a locally Lipschitz continuous map \( g: \mathbb{R}^n \to \mathbb{R}^n \). Let us denote with \( \phi_* \) all the fixed points of the dynamical system (17), and the corresponding Jacobian matrix, which we denote as \( J(g) \), is equal to,

\[
J = \sum_i \sum_j \left[ \frac{\partial f_i}{\partial x_j} \right].
\]

The Jacobian has to be calculated at the fixed points, and the eigenvalues \( \epsilon_i \) must satisfy \( \text{Re}(\epsilon_i) \neq 0 \). Let \( \sigma(A) \) denote the spectrum of the eigenvalues of \( A \), so a hyperbolic fixed point satisfies \( \text{Re}(\sigma(J)) \neq 0 \). The Hartman theorem ensures the existence of a homeomorphism \( \mathcal{F}: U \to \mathbb{R}^n \), where \( U \) is an open neighborhood of \( \phi_* \), such that \( \mathcal{F}(\phi_*) \). The homeomorphism generates a flow \( \frac{dh(u)}{dt} = Jh(u) \), which is,

\[
\frac{dh(u)}{dt} = Jh(u),
\]

and in addition it is a topologically conjugate flow to the one appearing in Eq. (17). Sometimes the Hartman theorem holds true for non-autonomous system, but the general rule is that it does not apply to non-autonomous systems, or should be applied with caution. A direct implication of the theorem is that the dynamical system of Eq. (17) may be written as follows,

\[
d\Phi = J(g)(\Phi) \bigg|_{\Phi=\phi_*} (\Phi - \phi_*) + S(\phi_*, t),
\]

where \( S(\phi, t) \) is a smooth map \([0, \infty) \times \mathbb{R}^n \). In effect, if the Jacobian \( J(g) \) satisfies \( \mathcal{R} (\sigma(J(g))) < 0 \) and also if

\[
\lim_{\Phi \to \phi_*} |S(\phi, t)| \to 0,
\]

the fixed point \( \phi_* \) corresponding to the dynamical flow \( \frac{d\Phi}{dt} = J(g)(\Phi) \bigg|_{\Phi=\phi_*} (\Phi - \phi_*) \), is a fixed point of the flow in Eq. (20), and it is stable asymptotically. It is conceivable that if any of the conditions we presented above is violated, further study of the phase space is needed in order to see whether stability is ensure or not.

### A. de Sitter Inflationary Attractors and their Stability

We start off with the case \( m \approx 0 \), which may possibly describe a quasi de Sitter evolution, however we shall analyze the dynamics of the system (10), for \( m \approx 0 \) without specifying the Hubble rate.

In the case of the dynamical system of Eq. (10), the matrix \( J = \sum_i \sum_j \left[ \frac{\partial f_i}{\partial x_j} \right] \) is equal to,

\[
J = \left( \begin{array}{ccc}
2x_1 - x_3 + 3 & 0 & 2 - x_1 \\
x_2 & x_1 - 2x_3 + 4 & -2x_2 - 4 \\
0 & 0 & 8 - 4x_3
\end{array} \right),
\]

where the functions \( f_i \) are,

\[
f_1 = -4 - 3x_1 + 2x_3 - x_1x_3 + x_1^2,
\]

\[
f_2 = 8 - 4x_3 + x_2x_1 - 2x_2x_3 + 4x_2,
\]

\[
f_3 = 8x_3 - 2x_3^2 - 8.
\]

The fixed points of the dynamical system (10), for general \( m \) are the following,

\[
\phi_{1*} = \left( \frac{1}{4} \left( -\sqrt{-2m} - \sqrt{-2m + 20\sqrt{-2m} - 4} + 2 \right) \right),
\]

\[
\phi_{2*} = \left( \frac{1}{4} \left( -\sqrt{-2m} + \sqrt{-2m + 20\sqrt{-2m} + 4} - 2 \right) \right),
\]

\[
\phi_{3*} = \left( \frac{1}{4} \left( \sqrt{-2m + \sqrt{-2m - 20\sqrt{-2m} + 4} + 2} \right) \right),
\]

\[
\phi_{4*} = \left( \frac{1}{4} \left( \sqrt{-2m + \sqrt{-2m - 20\sqrt{-2m} + 4} - 2} \right) \right).
\]
In the case $m \simeq 0$, the fixed points are,
\[
\phi_1^* = (-1, 0, 2), \quad \phi_2^* = (0, -1, 2).
\]
(25)
The eigenvalues for the fixed point $\phi_1^*$ are $(-1, -1, 0)$, while for the fixed point $\phi_2^*$ these are $(1, 0, 0)$. Hence both equilibria are non-hyperbolic, but as we show the fixed point $\phi_1^*$ is stable and $\phi_2^*$ is unstable.

Before we proceed let us discuss the physical significance of the two fixed points, and this can easily be revealed by observing that in both the equilibria $(20)$, we have $x_3 = 2$. By substituting $x_3 = 2$ in Eq. $(10)$, we get $w_{eff} = -1$, so effectively we have two de Sitter equilibria.

Also it is worth to have a concrete idea on how the dynamical system behaves analytically. Actually, the third equation of the dynamical system $(10)$ is decoupled, and the solution of it reads,
\[
x_3(N) = \frac{4N - 2\omega + 1}{2N - \omega},
\]
(26)
where $\omega$ is an integration constant which can be fixed by the initial conditions. The asymptotic behavior of the solution $(20)$, that is for large $N$, is $x_3 \to 2$, which is exactly the behavior we indicated earlier. Also, by using the solution $(20)$ and substituting it in the first two coupled differential equations of the dynamical system $(10)$, we may obtain the analytic form of the solutions $x_1(N)$ and $x_2(N)$, which read,
\[
x_2(N) = c_1 - \frac{(-c_1 - 2) c_2 U \left(1 - \frac{-c_1 - 2}{c_1}, 2, N c_1 - \frac{c_1}{2}\right) - c_1 L_{c_1 - 2}^{1 - 1} \left(c_1 N - \frac{c_1 \omega}{2}\right)}{c_2 U \left(\frac{-c_1 - 2}{c_1}, 1, N c_1 - \frac{c_1 \omega}{2}\right) + L_{c_1 - 2}^{1 - 1} \left(c_1 N - \frac{c_1 \omega}{2}\right)},
\]
\[
x_1(N) = -\frac{(-c_1 - 2) c_2 U \left(1 - \frac{-c_1 - 2}{c_1}, 2, N c_1 - \frac{c_1 \omega}{2}\right) - c_1 L_{c_1 - 2}^{1 - 1} \left(c_1 N - \frac{c_1 \omega}{2}\right)}{c_2 U \left(\frac{-c_1 - 2}{c_1}, 1, N c_1 - \frac{c_1 \omega}{2}\right) + L_{c_1 - 2}^{1 - 1} \left(c_1 N - \frac{c_1 \omega}{2}\right)},
\]
(27)
where $c_1$ and $c_2$ are integration constants and also $U(a, b, z)$ and $L_n^\alpha(x)$ are the confluent hypergeometric function and the generalized Laguerre polynomial respectively. In order to have a direct command on the initial conditions, we shall proceed directly on the numerical analysis, but it can be shown that if the integration constants are appropriately chosen, the numerical and analytical results may coincide.

Now let us analyze the dynamics of the cosmological system, and for starters we numerically solve the dynamical system $(10)$ for various initial conditions and with the $e$-foldings number belonging to the interval $N = (0, 60)$. In Fig. 1 we present the numerical solutions for the dynamical system $(10)$, for the initial conditions $x_1(0) = -8$, $x_2(0) = 5$ and $x_3(0) = 2.6$. For the upper left plot of Fig. 1 the $e$-foldings number is chosen in the interval $N = (0, 10)$ and as it can be seen, the equilibrium $\phi_1^* = (x_1, x_2, x_3) = (-1, 0, 2)$ is not reached yet, and the same applies for the upper right plot in Fig. 1 in which case $N = (0, 40)$, although the situation is a bit better. In the third plot of Fig. 1 the $e$-foldings number is chosen in the interval $N = (0, 60)$, so by $N \sim 60$, the equilibrium $\phi_1^*$ is reached. The same can be seen in Fig. 2, where we plot $x_1 - x_3$, for various initial conditions. From Figs. 1 and 2 it already seems that the fixed point $\phi_1^*$ is a stable de Sitter attractor. As we show by using further numerical analysis, indeed this de Sitter equilibrium is the final attractor of the phase space. In Fig. 3 we plot the three dimensional flow, and as it can be seen, for various initial conditions, the global attractor behavior of $\phi_1^*$ can be seen. Also the stability of the attractor $\phi_1^*$ can be seen in the reduced dynamical system for $x_3 = 2$. Actually, the equation for $x_3$ can be solved directly since this is decoupled from the rest of the equations in $(10)$. The dynamics of the reduced system for $x_3 = 2$ are presented in Fig. 4 where we plot the vector field flow (left) and the trajectories in the $x_1 - x_2$ plane (right), for various initial conditions. It is obvious that the attractor behavior of the de Sitter fixed point $\phi_1^*$ is further supported by these flows in the reduced system. In conclusion, the case $m \simeq 0$ for the dynamical system $(10)$ results to a stable de Sitter attractor and also to an unstable de Sitter fixed point. This result is particularly interesting since it dictates that a general $f(R)$ gravity theory with Hubble rate that satisfies $m \simeq 0$ results to an asymptotic stable de Sitter attractor, which is reached at $N \sim 50 - 60$, as it can be seen in Fig. 1. This is quite appealing, if one considers an inflationary interpretation of the theory, for example if the scale factor is assumed to be a quasi-de Sitter one, namely $a(t) \sim e^{H_0 t - H_1 t^2}$. In the quasi-de Sitter case, the attractor property reveals that the phase space of a the $f(R)$ gravity has an asymptotic de Sitter attractor, which is stable, which clearly may be interpreted having an inflationary evolution in mind. However, not only a quasi-de Sitter cosmology results to $m \simeq 0$, consider for example the symmetric bounce cosmology $(29)$, in which case the scale factor is $a(t) = e^{\lambda t^2}$, with $\lambda > 0$. In the symmetric bounce, the parameter $m$ is also zero, but in this case the physical interpretation of the de Sitter attractor is different, since it can be a late-time de Sitter attractor. However, this requires different initial conditions and perhaps a re-parametrization of the dynamical system $(10)$, in terms of the cosmic time, in order to
Numerical solutions $x_1(N)$, $x_2(N)$ and $x_3(N)$ for the dynamical system, for the initial conditions $x_1(0) = -8$, $x_2(0) = 5$ and $x_3(0) = 2.6$, and for $m \simeq 0$. Avoid inconsistencies. We leave this task as a possible future work. Also, it is possible that for certain cosmological models, we may have asymptotically $m \simeq 0$, if the parameters of the theory are appropriately chosen. In such case, the interpretation of the fixed points needs special attention.

Let us note that there is also an unstable de Sitter fixed point, namely $\phi^2 = (0, -1, 2)$. So it may be possible that there are $f(R)$ gravities which may lead the dynamical system directly on the unstable de Sitter attractor. This can be seen in the 3-dimensional flow of Fig. 3, since there are trajectories that move away from the stable de Sitter attractor $\phi^1$. One example of this sort, we shall present at a later section.

Now the central question is, what is the physical interpretation of the above dynamics, and to which $f(R)$ gravities the case $m \simeq 0$ corresponds. It is conceivable that we can find approximate expressions of the $f(R)$ gravities near the fixed points, $\phi^1 = (-1, 0, 2)$ and $\phi^2 = (0, -1, 2)$. This is the subject of the next section.
In this section we shall analyze the physical implications of the dynamics we presented in the previous section. As we demonstrate, for the value of $m$ for which $m \simeq 0$, there are two de Sitter fixed points, one stable, which is $\phi_1^* = (-1, 0, 2)$ and also an unstable, which is $\phi_2^* = (0, -1, 2)$. Now we shall investigate the behavior of the $f(R)$ gravity near the fixed points and for $m \simeq 0$. Recall that as we discussed in Eq. (16), the case $m = 0$ is equivalent to
the slow-roll approximation, so effectively what we seek for is the behavior of the $f(R)$ gravities near the fixed points and with the slow-roll approximation holding true. Let us start with the first fixed point, namely $\phi_1^x = (-1, 0, 2)$, so the following differential equations must hold true simultaneously at the fixed point,

$$-\frac{d^2 f}{dR^2} \frac{\dot{R}}{H^2 \frac{df}{dR}} \simeq -1, \quad -\frac{df}{H^2 \frac{df}{dR}} \simeq 0,$$

which stem from the conditions $x_1 \simeq -1$ and $x_2 \simeq 0$. Since $m \simeq 0$ (or equivalently since the slow-roll approximation holds true), the left differential equation can be written as follows,

$$-24H^i \frac{d^2 f}{dR^2} - \frac{df}{dR} = 0,$$

which can easily be solved and it yields,

$$f(R) \simeq \Lambda_1 - 24 \Lambda_2 e^{-\frac{m}{\Lambda_2}}.$$

The $f(R)$ gravity solution (30) is nothing but the approximate form of the $f(R)$ gravity in the large curvature era, which generates the quasi-de Sitter evolution of Eq. (15) or equivalently, that yields $m \simeq 0$. So since we are interested in the large curvature era, the exponential term is subdominant, so the cosmological constant $\Lambda_1$ leads to the stable de Sitter attractor $\phi_1^x = (-1, 0, 2)$. By using the solution (30), it is easy to show that the second condition in Eq. (28) can be written as follows,

$$\frac{H_i \Lambda_1 e^{\frac{m^2}{\Lambda_2}}}{6H_0^3 \Lambda_2} \simeq 0,$$

so it follows that in order the above condition is true, we must have $H_0^3 \gg 1$ and also $H_0^3 \Lambda_2 \gg H_i \Lambda_1 e^{\frac{m^2}{\Lambda_2}}$. Notice that the final form of the $f(R)$ gravity is very similar to the exponential $f(R)$ gravity models of Ref. [30], in which case the full $f(R)$ gravity is,

$$f(R) = R - 2\Lambda(1 - e^{-2\frac{R}{m_0}}).$$

Now let us consider the case of the second de Sitter fixed point, namely $\phi_2^x = (0, -1, 2)$, and in this case the conditions $x_1 \simeq 0$ and $x_2 \simeq -1$ become,

$$-\frac{d^2 f}{dR^2} \frac{\dot{R}}{H^2 \frac{df}{dR}} \simeq 0, \quad -\frac{df}{H^2 \frac{df}{dR}} \simeq -1.$$

By using the fact that $R \simeq 12H^2$, when the quasi-de Sitter evolution is taken into account, the second differential equation can be written,

$$f \simeq \frac{df}{dR} \frac{R}{2},$$

which can be solved to yield,

$$f(R) \simeq \alpha R^2.$$

The solution (35) is not the exact form of the $f(R)$ gravity which leads the cosmological system to the fixed point, but it is the approximate form of the $f(R)$ gravity near the fixed point $\phi_2^x$ which corresponds to the case $m \simeq 0$. The approximate $f(R)$ gravity of Eq. (35) is very similar to the $R^2$ model. Finally, the first condition in Eq. (33), for the $f(R)$ gravity being of the form (35), becomes,

$$\frac{H_i}{H_0^2} \simeq 0,$$

which holds true if $H_0^2 \gg H_i$.

Thus by taking into account the resulting forms of the $f(R)$ gravities we discussed in this section, we may conclude that the $R^2$ terms related to quasi-de Sitter solutions always lead to an unstable de Sitter fixed attractor, while terms containing cosmological constants and exponentials, lead to a stable de Sitter attractor. This result is interesting, since it is well known [31] that $R^2$ corrections to viable $f(R)$ gravities, like the exponential, always trigger graceful exit from inflation, see the well-known viable Starobinsky inflation model [32]. Hence, the presence of an $R^2$ term leads to instability, which may be viewed as an indication of the graceful exit from the inflationary era. In the following section we shall analyze in detail the case of the $R^2$ gravity, and we will investigate whether the predicted behavior we found in the previous sections, holds true in this case.
2. Specific Example I: $R^2$ Gravity and Graceful Exit from Inflation

An intriguing example that leads to the picture we discussed in the previous section, occurs for the $R^2$ gravity, with regards to the final unstable de Sitter fixed point. As we will show, this $f(R)$ gravity makes the dynamical variables to end up to the unstable de Sitter fixed point. The functional form of the $R^2$ gravity is,

$$f(R) = R + \frac{R^2}{36H_i^2},$$ (37)

where the parameter $H_i$ has dimensions of mass$^2$. We shall focus on the vacuum $f(R)$ gravity case in the rest of this section. In order to investigate the behavior of the dynamical variables (8), we shall need the exact functional form of the Hubble rate, and also this technique will hold true only if the specific Hubble rate yields $m \simeq 0$. This issue needs to be further explained at this point. The behavior of the dynamical variables we described in the previous sections, holds true for all the cosmologies which yield $m \simeq 0$, without specifying the exact form of the $f(R)$ gravity, however the result shows us that there are $f(R)$ gravities, actually those of Eq. (30), which yield a Hubble rate for which $m \simeq 0$ and also the variables $x_1, x_2$ and $x_3$ approach the values $(x_1, x_2, x_3) = (-1, 0, 2)$, which correspond to the fixed point $\phi_i^1$. Also there are $f(R)$ gravities which lead to an unstable de Sitter fixed point, which can be approximated near the fixed point as in Eq. (35). However, finding the Hubble rate for a general $f(R)$ gravity functional form, is quite a formidable task. Nevertheless, for the $R^2$ gravity (37) this is easy when the slow-roll approximation is used.

The cosmological equations for the $R^2$ gravity in a FRW background metric, are,

$$\ddot{H} - \frac{H^2}{2} + 3H_i H = -3H \dot{H}, \quad \ddot{R} + 3HR + 6H_i R = 0,$$ (38)

so in the slow-roll case, since $\dot{H} \ll H \ddot{H}$ and $\ddot{H} \ll H^2$, the second equation of Eq. (38) can be solved to yield the following Hubble rate,

$$H(t) \simeq H_0 - H_i(t - t_k),$$ (39)

which is a quasi-de Sitter evolution. As it can be easily checked, and we also discussed this issue in the previous sections, the Hubble rate (39) yields $m = 0$. By using the Hubble rate (39), the functional form of the $R^2$ gravity (37) and also by expressing the cosmic time as a function of the e-foldings number $N$ (recall that $N = \ln a$), the dynamical variables (8) as a function of the e-foldings number $N$ become,

$$x_1(N) = \frac{2H_i}{H_0^2 - 2H_i N + H_i},$$

$$x_2(N) = \frac{(2H_0^2 + H_i(5 - 4N))(2H_0^2 - H_i(4N + 1))}{4(H_0^2 - 2H_i N)(H_0^2 - 2H_i N + H_i)},$$

$$x_3(N) = \frac{2H_0^2 - 4H_i N - H_i}{H_0^2 - 2H_i N}.$$

As it was shown in [33], for the choice $H_0 = 10^{13}\text{sec}^{-1}$ and $H_i = 10^{20}\text{sec}^{-2}$, a viable inflationary cosmology can be generated, so by using the aforementioned values, in Fig. [3] we plot the behavior of the variables (40) as a function of the e-foldings number. The red curve corresponds to $x_3(N)$, the black to $x_2(N)$ and finally the blue dashed curve corresponds to the variable $x_1(N)$. As it can be seen, the variables $x_1(N)$, $x_2(N)$ and $x_3(N)$ approach quite fast the fixed point values $(x_1, x_2, x_3) = (0, -1, 2)$, and therefore the $R^2$ gravity is an example of an $f(R)$ gravity which leads to an unstable de Sitter fixed point. Actually this is quite intriguing, since the unstable de Sitter fixed point is reached for the $R^2$ gravity, and not the stable de Sitter point. This feature could be seen as an indication that the graceful exit from inflation occurs for the $R^2$ gravity. In principle, the graceful exit issue cannot be addressed by studying solely the variables $x_i$, $i = 1, 2, 3$, since this may occur due to curvature perturbations, so it is an effect which can be seen in higher orders of curvature. For a detailed analysis on this, see [27]. However, the $R^2$ gravity shows an exceptional behavior, since for this case the unstable de Sitter fixed point is reached, and this is not accidental, since as we discussed earlier, $R^2$ terms are always related to the graceful exit from inflation, see [31, 32].

Let us briefly discuss our claim that connects the graceful exit and the $R^2$ gravity. In order to be more concise, there is indication for the graceful exit from inflation in the $R^2$ gravity case, thus our claim is more mild in rigidity. According to Refs. [27, 31], where a deeper analysis was performed, the graceful exit from inflation may be triggered by growing curvature perturbations around an unstable de Sitter solution of the theory. In this case, the de Sitter solution ceases to be the final attractor of the theory, and thus inflation ends as a consequence of that, since the de
FIG. 5: The behavior of $x_1(N)$, $x_2(N)$ and $x_3(N)$ for $H_0 = 10^{13}\text{sec}^{-1}$ and $H_i = 10^{20}\text{sec}^{-2}$. The red curve corresponds to $x_3(N)$, the black to $x_2(N)$ and finally the dashed blue curve corresponds to the variable $x_1(N)$.

The de Sitter solution is not the final attractor of the theory. What follows is part of the subsequent physical evolution, for example the reheating era, since the slow-roll era ends. In order to investigate the perturbations of the curvature, we perturb the de Sitter solution $H = H_0$ as follows,

$$H(t) = H_0 + \Delta H,$$

and we plug the perturbation in the first equation of motion of Eq. (38), and by keeping linear terms of the derivatives of $\Delta H$, we obtain the following differential equation,

$$3H_0H_i + H_0\Delta \dot{H}(t) + 3H_0\Delta \dot{H}(t) + 3H_i\Delta H(t) + \frac{\Delta \dot{H}(t)}{2} = 0 \quad (42)$$

which can be solved analytically and the solution is,

$$\Delta H(t) = c_1 \exp \left( \frac{t \left( -\sqrt{36H_0^2 - 48H_0H_i + 12H_0 + 1 - 6H_0 - 1} \right)}{4H_0} \right)$$

$$+ c_2 \exp \left( \frac{t \left( \sqrt{36H_0^2 - 48H_0H_i + 12H_0 + 1 - 6H_0 - 1} \right)}{4H_0} \right) - H_0,$$

where $c_1$ and $c_2$ are integration constants. As it can be seen from the solution of Eq. (43), the term proportional to $c_2$ is growing exponentially as a function of the cosmic time, therefore this supports our claim that the de Sitter point of the $R^2$ gravity is unstable, which we also demonstrated earlier by using the fixed point analysis. Basically, this unstable de Sitter point is the fixed point $(x_1, x_2, x_3) = (0, -1, 2)$. We need to stress once more that this is an indication that graceful exit is triggered by curvature perturbations, however this indication cannot be considered as a proof of the graceful exit from inflation.

**B. Other Possible Attractors and their Stability**

Let us now briefly discuss the case $m = -\frac{9}{2}$, which as we show corresponds to a matter dominated Universe. In the case at hand, the matrix $\mathcal{J} = \sum_i \sum_j \left[ \frac{\partial f}{\partial x_j} \right]$ is equal to,

$$\mathcal{J} = \begin{pmatrix} 2x_1 - x_3 + 3 & 0 & 2 - x_1 \\ x_2 & x_1 - 2x_3 + 4 & -2x_2 - 4 \\ 0 & 0 & 8 - 4x_3 \end{pmatrix}, \quad (44)$$
where the functions $f_i$ are in this case,
\begin{align}
    f_1 &= x_1^2 - x_1 x_3 + 3 x_1 + 2 x_3 - 4 , \\
    f_2 &= -\frac{9}{2} + x_1 x_2 - 2 x_2 x_3 + 4 x_2 - 4 x_3 + 8 , \\
    f_3 &= \frac{9}{2} - 2 x_3^2 + 8 x_3 - 8 .
\end{align}

The fixed points in this case are,
\begin{equation}
    \phi^1_* = \left(\frac{1}{4} \left(-5 - \sqrt{73}\right), \frac{1}{4} \left(7 + \sqrt{73}\right), \frac{1}{2}\right), \quad \phi^2_* = \left(\frac{1}{4} \left(\sqrt{73} - 5\right), \frac{1}{4} \left(7 - \sqrt{73}\right), \frac{1}{2}\right).
\end{equation}

For the first fixed point $\phi^1_*$ the eigenvalues are $(6, -4.272, -0.386001)$, while for the second fixed point $\phi^2_*$ these are $(6, 4.272, 3.886)$. Both equilibria are non-hyperbolic and the first seems to be stable. However as we show, if the initial conditions on the variable $x_3$ deviate from the equilibrium value $x_3 = \frac{1}{2}$, the dynamical system solutions simply blow-up.

The fixed points have some physical interpretation, since for $x_3 = \frac{1}{2}$, the EoS in Eq. (13) becomes $w_{eff} = 0$, which perfectly describes a matter dominated era. What now remains is to investigate the dynamics of the cosmological system.

As in the de Sitter case, it is possible to solve analytically the dynamical system, and in this case the solution for $x_3$ is,
\begin{equation}
    x_3(N) = \frac{7 e^{6 N} - \omega}{2 (e^{6 N} - \omega)} ,
\end{equation}
where $\omega$ is again an integration constant. The solutions for $x_2$ and $x_3$ can also be found analytically, but the general solutions are particularly complicated and lengthy to include them at this point. From the solution (47) it is obvious that asymptotically, $x_3 \approx 7/2$ which does not correspond to the matter dominated era value of $x_3$. Actually, this corresponds to an additional fixed point, which we shall consider below, however it has no physical significance, since it corresponds to a phantom evolution.

Now we proceed to the dynamical system analysis, and as we now show, if the initial conditions on $x_3$ deviate from the equilibrium value, the solutions tend to infinity. In Fig. 6 we plotted the behavior of the solutions for the initial conditions $x_1(0) = -3$, $x_2(0) = 1$ and $x_3(0) = 0.5$ (left plot) and for $x_1(0) = -3$, $x_2(0) = 1$ and $x_3(0) = 0.501$ (right plot). As it can be seen in Fig. 6 if the variable $x_3$ deviates from it’s equilibrium value, the solutions blow up and the fixed point is not a stable attractor. Hence, the stability of the system is ensured only if $x_3 = \frac{1}{2}$, and only for this case $w = 0$. Now we need to interpret this physically, and the interpretation is simple, since the behavior of the dynamical system indicates that when $x_3$ deviates from it’s equilibrium value $x_3 = \frac{1}{2}$, which is equivalent to say when $w_{eff} \neq 0$, then the vacuum $f(R)$ gravity does not approach the first fixed point $\phi^1_*$. Actually, this is reasonable, since it seems that the vacuum $f(R)$ gravity fails to describe a matter dominated era unless $w_{eff} = 0$. To our opinion,
the matter dominated attractor is reached only if \( x_3 = \frac{1}{2} \), and if this condition is not met, instability occurs and the system ceases to describe a matter dominated era.

Let us now demonstrate that the reduced dynamical system by fixing \( x_3 = \frac{1}{2} \), is indeed stable. In Fig. 7 we plot the vector field flow in the \( x_1 - x_2 \) plane by fixing \( x_3 = \frac{1}{2} \) (left plot), and various trajectories in the \( x_1 - x_2 \) plane. As it can be seen in both plots, the fixed point \( \phi_1^* \) is reached asymptotically. Finally, the strong \( x_3 \) dependence of the

![Image](image1.png)

**FIG. 7:** The vector field flow in the \( x_1 - x_2 \) plane with \( x_3 = \frac{1}{2} \) (left plot), and various trajectories in the \( x_1 - x_2 \) plane, with \( m = -9/2 \).

three dimensional system can be seen in the left and right plots of Fig. 8. As it can be seen, depending on the initial conditions, the stable fixed point \( \phi_1^* \) may or may not be reached. Having discussed the vacuum \( f(R) \) case, what now remains for completing the study is to add matter and radiation perfect fluids, and this is the subject of the next

![Image](image2.png)

**FIG. 8:** The three dimensional flow with \( m = -9/2 \).
section.

Before closing, we need to discuss another interesting point, having to do with the existence of two more fixed points of the dynamical system (11) for \( m = -9/2 \), which are,

\[
(x_1, x_2, x_3) = \left( \frac{1}{4} \left(1 - i \sqrt{47} \right), \frac{1}{4} \left(1 + i \sqrt{47} \right), \frac{7}{2} \right), \quad (x_1, x_2, x_3) = \left( \frac{1}{4} \left(1 + i \sqrt{47} \right), \frac{1}{4} \left(-11 - i \sqrt{47} \right), \frac{7}{2} \right).
\]  

(48)

The form of the new fixed points renders them physically unappealing, since the variables \( x_1 \) and \( x_2 \) take complex values. Actually, this means that the \( f(R) \) gravity is complex, and this is connected with phantom behavior, as was shown in Ref. [34]. Actually, the complex \( f(R) \) gravity is usually connected with a phantom scalar theory which develops a Big Rip singularity, and particularly, the complex \( f(R) \) gravity corresponds to the region, where the phantom scalar theory develops the crushing type Big Rip singularity. This cannot be accidental, since, as we already mentioned below Eq. (47) for \( x_3 = \frac{7}{2} \), the EoS becomes \( w_{eff} = -2 \), which clearly describes a phantom evolution. We shall not extend the analysis further, since this theory can be useful for a dark energy description, however from an inflationary point of view, it is not so appealing having a phantom theory at hand.

IV. THE NON-VACUUM \( f(R) \) GRAVITY AUTONOMOUS DYNAMICAL SYSTEM: THE DE SITTER ATTRACTOR CASE

In the previous sections we discussed the vacuum \( f(R) \) gravity case, and in this section we shall incorporate the radiation and matter domination perfect fluids for completeness. We shall focus on the case \( m \approx 0 \), since the other cases can be easily studied. So for the case \( m \approx 0 \) let us analyze the behavior of the phase space.

Let us first construct the dynamical system in the case that matter and radiation perfect fluids are considered. The equations of motion in this case become,

\[
0 = -\frac{f(R)}{2} + 3 \left ( H^2 + \dot{H} \right ) F(R) - 18 \left ( 4H^2 \dot{H} + H \ddot{H} \right ) F'(R) + \kappa^2 \rho_{matter}, \quad (49)
\]

\[
0 = \frac{f(R)}{2} - \left ( \dot{H} + 3H^2 \right ) F(R) + 6 \left ( 8H^2 \ddot{H} + 4H \dot{H}^2 + 6H \dot{H} + \ddot{H} \right ) F'(R) + 36 \left ( 4H \dddot{H} + \ddot{H} \right )^2 F'(R) + \kappa^2 \rho_{matter}, \quad (50)
\]

where \( \rho_{matter} \) and \( p_{matter} \) are the total effective energy density and the effective pressure of the radiation and matter perfect fluids. In order to reveal the autonomous structure of the \( f(R) \) gravity cosmological system, we introduce the following variables,

\[
x_1 = -\frac{\dot{F}(R)}{F(R)\dot{H}}, \quad x_2 = -\frac{f(R)}{6F(R)H^2}, \quad x_3 = \frac{R}{6H^2}, \quad x_4 = \frac{\kappa^2 \rho_r}{3FH^2}, \quad x_5 = \frac{\kappa^2 \rho_m}{3FH^2}, \quad (51)
\]

where \( \rho_r \) and \( \rho_m \) are the radiation and matter energy densities. In terms of the e-foldings number \( N \), the dynamical system in the case at hand reads,

\[
\frac{dx_1}{dN} = -4 - 3x_1 + 2x_3 - x_1x_3 + x_1^2 + 3x_5 + 4x_4, \quad (52)
\]

\[
\frac{dx_2}{dN} = 8 + m - 4x_3 + 2x_2, \quad \frac{dx_3}{dN} = -8 - m + 8x_3 - 2x_3^2, \quad \frac{dx_4}{dN} = x_4x_1 - 2x_4x_3, \quad \frac{dx_5}{dN} = x_5 + x_5x_1 - 2x_5x_3,
\]

where the parameter \( m \) is again as it appears in Eq. (11). The matrix \( \mathcal{J} = \sum_i \sum_j \left( \frac{\partial x_j}{\partial x_i} \right) \) in the case at hand is equal to,

\[
\mathcal{J} = \begin{pmatrix}
2x_1 - x_3 + 3 & 0 & 2 - x_1 & 4 & 3 \\
x_2 & x_1 - 2x_3 + 4 & -2x_2 - 4 & 0 & 0 \\
0 & 0 & 8 - 4x_3 & 0 & 0 \\
x_4 & 0 & -2x_4 & x_1 - 2x_3 & 0 \\
x_5 & 0 & -2x_5 & 0 & x_1 - 2x_3 + 1
\end{pmatrix}, \quad (53)
\]
and in this case the functions \( f_i \) for \( m \simeq 0 \) are,

\[
\begin{align*}
    f_1 &= -4 - 3x_1 + 2x_3 - x_1x_3 + x_1^2 + 3x_5 + 4x_4, \\
    f_2 &= 8 - 4x_3 + x_2x_1 - 2x_2x_3 + 4x_2, \\
    f_3 &= 8x_3 - 2x_3^2 - 8, \\
    f_4 &= x_4x_1 - 2x_4x_3, \\
    f_5 &= x_5 + x_5x_1 - 2x_5x_3.
\end{align*}
\]

The fixed points of the dynamical system (52), for \( m \simeq 0 \) are the following,

\[
\phi_1^i = (-1, 0, 2, 0, 0), \\
\phi_2^i = (3, 0, 2, 0, -4), \\
\phi_3^i = (4, 0, 2, -5, 0).
\]

The corresponding eigenvalues for the fixed point \( \phi_1^i \) are \((-5, -4, -1, -1, 0)\), while for the fixed point \( \phi_2^i \) these are \((4, 3, 3, -1, 0)\) and finally for the fixed point \( \phi_3^i \) the eigenvalues are \((5, 4, 4, 1, 0)\). All the equilibria are non-hyperbolic and as we now show, only \( \phi_1^i \) is stable, which means that regardless the initial conditions used, the trajectories are attracted to the fixed point \( \phi_1^i \). The fact that \( x_3 \) is equal to \( x_3 = 2 \) for all the equilibria, shows that these are de Sitter equilibria, since \( w_{eff} = -1 \), as in the cases we studied in the previous section. Let us now focus on the dynamics of the configuration space spanned by the variables \( x_i, i = 1, \ldots, 5 \).

The analytical study in this case turns out to be quite complicated, and only the solution for \( x_3(N) \) can be found analytically, and this is given in Eq. (20).

Thus we proceed numerically, and after numerically solving the dynamical system (52) for various sets initial conditions in Fig. 11 we present the plots of the variables \( x_i(N) \) \( i = 1, \ldots, 5 \), for \( N = (0, 0, 0) \), by using the initial conditions \( x_1(0) = -10, x_2(0) = 5, x_3(0) = 2.6, x_4(0) = 0.1 \) and \( x_5(0) = 1 \). In Fig. 9 in both plots the red curve is the plot \( x_1 - N \), the black curve is \( x_2 - N \), the blue is \( x_3 - N \), the green is \( x_4 - N \) and finally the purple is \( x_5 - N \). In the left plot, \( N \) belongs in the interval \( N = (0, 7) \), so the equilibrium \( \phi_1^i \) has not been reached, however for large \( N \), that is for \( N \simeq 50 - 60 \), the equilibrium fixed point \( \phi_1^i \) is finally reached. Hence, this partially proves the stability, since the same behavior occurs regardless the initial conditions we use. In order to further support the stability of the fixed point \( \phi_1^i \), in Fig. 10, we plot the trajectories of the dynamical system in the \( x_1 - x_3 \) plane (left plot), in the \( x_3 - x_2 \) plane (right plot), and in the \( x_2 - x_5 \) plane (bottom plot), for various initial conditions. As it can be seen in all the plots of Fig. 10, the dynamical system approaches asymptotically the fixed point \( \phi_1^i \). Hence it seems that the matter-radiation \( f(R) \) gravity has a global de Sitter attractor. In order to further reveal the global stability of the attractor \( \phi_1^i \), in Fig. 11 we present two 3-dimensional plots of the trajectories in the space spanned by the variables \( x_1, x_4, x_5 \), by fixing \( (x_2, x_3) \) to their values at the fixed point \( \phi_1^i \), that is, for \( (x_2, x_3) = (0, -2) \). As it can be seen in the left plot but also clearer in the right plot, viewed by a different angle, the final attractor in the \( (x_1, x_4, x_5) \) configuration space, is the point \( (x_1, x_4, x_5) = (-1, 0, 0) \).

In conclusion, even when matter and radiation perfect fluids are added in the \( f(R) \) gravity, there is always a global stable de Sitter attractor of all the cosmologies that satisfy \( m \simeq 0 \), exactly as in the vacuum case. This is clearly an
inflationary attractor which is reached asymptotically for $N \sim 50 - 60$. Also we need to note that the fixed point is in the point $(x_4, x_5) = (0, 0)$, which is physically correct, since at a de Sitter point, neither the mass nor the radiation perfect fluids dominate the evolution. In principle the same study could be repeated for other values of $m$, like in the previous section, but we refer from going into details for brevity.

V. CONCLUSIONS

In this work we performed a detailed analysis of the $f(R)$ gravity phase space, by using an autonomous dynamical system approach. As we demonstrated, it is possible to contain all the time-dependence of the dynamical system in a single variable, which depends only on the Hubble rate. Hence, for specific values of this parameter, we investigated the existence and stability of the fixed points. We mainly focused on the case $m \simeq 0$, and we briefly discussed also the case $m = -\frac{9}{2}$. For the analysis, we used two forms of $f(R)$ gravities, firstly the vacuum $f(R)$ gravity and secondly the $f(R)$ gravity with matter and radiation perfect fluids present. In the vacuum $f(R)$ gravity case, the case $m \simeq 0$ results to a stable de Sitter final attractor. We scrutinized the problem of stability, by using various sets of initial conditions, and the numerical analysis we performed, supports our claim about the existence of an asymptotic de Sitter attractor. In fact the attractor is inflationary as we showed, and it is reached by the dynamical system at $N \sim 50 - 60$. The same behavior occurs for the $f(R)$ gravity in the presence of matter and radiation perfect fluids. As we demonstrated the dynamical system is attracted to an asymptotic stable de Sitter attractor, at which $(x_4, x_5) = (0, 0)$. Hence the de Sitter attractor is due to the $f(R)$ gravity solely.

An interesting outcome of the vacuum $f(R)$ gravity case is the existence of a stable and of an unstable de Sitter attractors. Actually, as we demonstrated, the approximated form of the $f(R)$ gravity that leads to the unstable de Sitter fixed points is $\sim R^2$, while the one that leads to the stable is an exponential plus a cosmological constant. As we discussed, the $R^2$ term is always related to growing curvature perturbations, which eventually may cause the graceful exit from inflation. At the dynamical system level in principle it is not possible to even see the possibility of graceful exit, since the $f(R)$ gravity is free from ghost instabilities and also it does not contain the Ostrogradsky instability. The same happens in a scalar inflationary theory, the de Sitter final attractor is stable at the level of dynamical system analysis, however, the graceful exit occurs when the slow-roll parameters become of the order one. In effect, the graceful exit is triggered by the breakdown of the slow-roll expansion, which is a perturbative expansion. In some sense, the dynamics of the graceful exit from inflation cannot be seen at the configuration space, so extra higher order
parameters are introduced in order to investigate the dynamical evolution at a perturbative level. Nevertheless, the existence of an $f(R)$ gravity which may lead to an unstable de Sitter fixed point, may provide hints that the graceful exit can occur even in pure $f(R)$ gravity. This result however, is, by all means, non-conclusive since it provides only hints but no proof. In modified gravities that contain ghosts, the instability can be seen in the dynamical system level, even in the configuration space. We have strong hints to believe this, and in fact in a mimetic $f(R)$ gravity context, there exist unstable manifolds folding around a stable de Sitter point. The presence of the unstable manifolds utterly changes the dynamics of a stable final attractor. We will report on this issue soon, including the $R^2$ case we discussed in a previous section. The fact that for the $R^2$ gravity, the dynamical variables tend to the unstable de Sitter fixed point, indicates the presence of a globally unstable manifold in the phase space.

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