The clebsch momenta approach to fluid lagrangians.

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Abstract

The clebsch potential approach to fluid lagrangians is developed in order to establish contact with other approaches to fluids. Three variants of the perfect fluid approach are looked at. The first is an explicit linear lagrangian constructed directly from the clebsch potentials, this has fixed equation of state and explicit expression for the pressure but is less general than a perfect fluid. The second is lagrangians more general than that of a perfect fluid which are constructed from higher powers of the comoving vector. The third is lagrangians depending on two vector fields which can represent both density flow and entropy flow.

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1 Introduction

1.1 Motivation

The motivation of this paper is to provided a unified approach to the various ways that fluids are described in physics. In particular the methods used by relativists, fluid mechanists, and nuclear physicists have grown distinct. In many areas of physics a unified approach is provided by the lagrange method, which for fluids is developed here.

1.2 Methodology

The methodology used is first to simplify a perfect fluid in order to investigate if methods of field theory can be applied to it; and then to generalize a perfect fluid to try and establish contact with more physical fluids.

1.3 Other approaches to fluids

A perfect fluid has a variational formulation [18, 36, 38, 39] which uses the first law of thermodynamics. In such a formulation clebsch potentials [10, 24, 4, 5, 17, 37, 13] for the comoving fluid vector field are used. Here this approach is both applied to less general fluids and to more general fluids. Other approaches to fluids include the following twelve. The first uses lagragians dependent on combinations of clebsh potentials which do not necessarily form a vector [34]. The second is that the comoving vector can be thought of as $U^a = \dot{x}^a$, so that a perfect fluid is a type of generalization of a point particle, then there turns out to be a fluid generalization of a membrane [35]. The third is that the charge substitution $\partial_a \rightarrow \partial_a + ieA_a$ can be applied to fluids as well as fields and this leads to a model of symmetry breaking [32]. The fourth is that the navier-stokes equation has a lagrangian formulation [8, 11, 12, 15, 40], but the lagrangian has different measure and also image fields. The fifth is that hydrodynamics can be expressed using a grad expansion [16, 22, 21, 14] which needs an entropy vector. The sixth is that contemporary bjorken models use the grad expansion [7, 27, 28, 29, 20]. The seventh is fluid plasmas [3, 1]. The eighth is elastic models [2]§3, where the density rather than the pressure is used as the lagrangian. The ninth is other quantization methods such as brst and path integral applied to fluids [6]. The tenth is superfluids [9]. The eleventh is spinning fluids [19, 23, 30]. The twelfth is cosmology [25], where clebsch potentials have been used.
1.4 Conventions

The word potential is disambiguated by referring to potentials for a vector field as clebsch potentials and potentials that occur in lagrange theory as coefficient functions. When a measure is suppressed it is $\int \sqrt{-g} dx^4$ not $\int d\tau$ unless otherwise stated. $\mu$ is density and $\mathcal{P}$ is the pressure. $q$ is a clebsch potential. $\sigma$ is used for a clebsch potential, a pauli matrix and the shear of a vector, to disambiguate the pauli matrix is always $\sigma_p$ and the shear labels with which vector it is with respect to $\sigma$. Capital $\Pi$ indicates a momentum with respect to the proper time $\tau$ not the coordinate time $t$. $a, b, c, \ldots$ are spacetime indices, $i, j, k, \ldots$ label sets of fields and momenta, and $\iota, \kappa, \ldots$ label constraints. The signature is $-+++$.

2 The perfect fluid

For a perfect fluid the lagrangian is taken to be the pressure $L = \mathcal{P}$, and the action is

$$I = \int dx^4 \mathcal{P}. \tag{1}$$

The clebsch potentials are given by

$$hV_a = W_a = \sigma_a + \theta s_a, \quad V_a V^a = -1, \tag{2}$$

where if more potentials are needed it is straightforward to instate them; there are several sign conventions for 2. The clebsch potentials are sometimes given names: $\sigma$ is called the higgs because it has a similar role to the higgs field in symmetry breaking using fluids [32, 34], $\theta$ is called the thermasy and $s$ the entropy [38]. Variation is achieved via the first law of thermodynamics

$$\delta \mathcal{P} = n \delta h - n T \delta s = -n V_a \delta W^a - n T \delta s, \quad nh = \mu + \mathcal{P}, \tag{3}$$

where $n$ is the particle number and $h$ is the enthalpy. Metrical variation yields the stress

$$T_{ab} = (\mu + \mathcal{P}) V_a V_b + \mathcal{P} g_{ab}, \tag{4}$$

the nöther currents $j^a = \delta I / \delta q_a$ are

$$j^a_\sigma = -n V^a, \quad j^a_\theta = 0, \quad j^a_s = -n \theta V^a, \tag{5}$$

variation with respect to the clebsch potentials gives

$$(n V^a)_\alpha = \dot{n} + n \dot{\Theta} = 0, \quad \dot{s} = 0, \quad \dot{\theta} = T, \tag{6}$$
where \( \Theta \equiv V^a_{,a} \) is the vectors expansion: thus the conservation of the nother currents 5 gives the same equations 6 as varying the clebsch potentials; the normalization condition \( V^a V_a = -1 \) and 6 give
\[
\dot{\sigma} = -\hbar. \tag{7}
\]
The bianchi identity is
\[
T^{ab}_{\cdots;b} = n\dot{W}^a + \mathcal{P}^a, \tag{8}
\]
subsituting for \( W \) using 2 and for \( \mathcal{P} \) using 3 this vanishes identically. If one attempts to apply existing scalar field fourier oscillator quantization procedures to the above there is the equation
\[
W^a_{,a} = \Box \sigma + \Theta a s^a + \Theta a s = (hV^a)_a = \dot{h} + h\Theta = \dot{h} - \frac{n}{\hbar}h = h \left( \ln \left( \frac{\hbar}{n} \right) \right)^\circ, \tag{9}
\]
and if this vanishes the enthalpy \( n \) is proportional to the particle number \( n \), for an example of the see the next section §3. The pressure \( \mathcal{P} \) and density \( \mu \) are only implicity defined in terms of the clebsch potentials so it is not clear what operators should correspond to them. Another possibility is to note that 6 are first order differential equations and to try and replace them with spinorial equations; however this would require a spinorial absolute derivative in place of the vectorial absolute derivative, see [31]§4.4.

The canonical clebsch momenta are given by \( \Pi^i = \delta I/\delta \dot{q}^i \)
\[
\Pi^\sigma = -n, \quad \Pi^0 = 0, \quad \Pi^a = -n\theta, \tag{10}
\]
and these allow the nother currents 5 to be expressed as
\[
\dot{j}^a = \Pi^a V^a. \tag{11}
\]
The standard poisson bracket is defined by
\[
\{ A, B \} \equiv \frac{\delta A}{\delta q_i} \frac{\delta B}{\delta \Pi^i} - \frac{\delta A}{\delta \Pi^i} \frac{\delta B}{\delta q_i}, \tag{12}
\]
where \( i \) which labels each field is summed; the dirac matrix is defined by
\[
C_{\kappa\kappa} \equiv \{ \phi_\kappa, \phi_\kappa \}, \tag{13}
\]
and the dirac bracket is defined by
\[
\{ A, B \}^* \equiv \{ A, B \} - \{ A, \phi_\kappa \} \text{Inv} (C_{\kappa\kappa}) \{ \phi_\kappa, B \}, \tag{14}
\]
where $\text{Inv}(C_{\iota\kappa})$ denotes the inverse of $C_{\iota\kappa}$. For a perfect fluid the constraints are

$$\phi_1 = \Pi^s - \theta \Pi^\sigma, \quad \phi_2 = \Pi^\theta,$$

and the Dirac matrix 13 becomes

$$C_{\iota\kappa} = -i \sigma_{p2} \Pi^\sigma, \quad \text{Inv}C_{\iota\kappa} = +i \frac{\sigma_{p2}}{\Pi^\sigma},$$

(16)

where $\sigma_{p2}$ is the Pauli matrix

$$\sigma_{p1} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{p2} \equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_{p3} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(17)

The Dirac bracket 14 for 15 is

$$\{ A, B \}^* = \{ A, B \} + \frac{1}{\Pi^\sigma} \frac{\delta B}{\delta \theta} \left( \frac{\delta A}{\delta s} - \theta \frac{\delta A}{\delta \sigma} + \Pi^\sigma \frac{\delta A}{\delta \Pi^\theta} \right) - A \leftrightarrow B,$$

(18)

acting on the Clebsch momenta and Clebsch potentials

$$\{ \Pi^\sigma, \sigma \}^* = -1, \quad \{ \Pi^\theta, \theta \}^* = 0, \quad \{ \Pi^s, s \}^* = -1,$$

$$\Pi^\sigma \{ \sigma, \theta \}^* = -\theta, \quad \Pi^\sigma \{ \theta, s \}^* = -1, \quad \Pi^\sigma \{ \sigma, s \}^* = 0,$$

(19)

replacing the Dirac brackets by quantum commutators there does not seem to be an easy representation of the resulting algebra as discussed in [33]. To quantize one replaces the Dirac bracket with a quantum commutator

$$\{ A, B \}^* \rightarrow \frac{1}{i \hbar} \left[ \hat{A} \hat{B} - \hat{B} \hat{A} \right],$$

(20)

in the present case the second equation of 19 suggests that $\Pi^\theta = 0$, which is assumed from now on. There is the problem of how to realize the coordinate commutation relations in the last three of 19 which does not seem possible using the Pauli matrices 17 and so this is left for now. Suppose that $\Pi^\sigma$ is replaced by a differential operator

$$\hat{\Pi}^\sigma = -i \hbar \partial_x,$$

(21)

possibilities for $x$ include spacetime coordinates $x_a$, the particle number $n$ or the proper time $\tau$ which restrict $\sigma$ as to a lesser extent does the Dirac operator $\gamma^a \partial_a$ so the simplest choice is taken $\partial_x$. To proceed it is necessary to have an explicit Hamiltonian which does not exist in the present case as the pressure $P$ and the density $\mu$ are not directly expressible in terms of the Clebsch potentials.
3 The explicit linear lagrangian

The lagrangian is taken to be linear in $W^2_a$

$$\mathcal{L} = -\frac{1}{2} \mathcal{F}(n, q) W^2_a - \bar{\mathcal{O}}(n, q),$$  \hspace{1cm} (22)

where $\mathcal{F}(n, q)$ and $\bar{\mathcal{O}}(n, q)$ are the first and zeroth order coefficient functions of the particle number and clebsch potentials respectively. Metric variation yields

$$T_{ab} = \mathcal{F} W_a W_b + \mathcal{L}_{g_{ab}} = \mathcal{F} h^2 V_a V_b + \mathcal{L}_{g_{ab}},$$ \hspace{1cm} (23)

requiring this stress to be that of a perfect fluid places the restriction

$$\mathcal{F} h^2 = n h = \mathcal{P} + \mu, \quad \text{or} \quad \mathcal{F} = \frac{n}{h},$$ \hspace{1cm} (24)

again requiring that the pressure is the lagrangian and using $W^2_a = -h^2$ gives

$$\mathcal{L} = \mathcal{P} = \frac{1}{2} n h - \bar{\mathcal{O}} = \frac{1}{2} (\mathcal{P} + \mu) - \bar{\mathcal{O}},$$ \hspace{1cm} (25)

which yields the linear equation of state

$$\mathcal{P} = \mu - 2 \bar{\mathcal{O}},$$ \hspace{1cm} (26)

this is a very restrictive equation of state as it does not have even simple cases such as $\mathcal{P} = (\gamma - 1) \mu$ as examples. Variation with respect to the clebsch potentials gives

$$\dot{n} + n \Theta - \bar{\mathcal{O}}_{,\sigma} = 0, \quad n \dot{s} + \bar{\mathcal{O}}_{,q} = 0, \quad n \dot{\theta} - \bar{\mathcal{O}}_{,s} + \theta \bar{\mathcal{O}}_{,\sigma} = 0,$$ \hspace{1cm} (27)

where

$$\bar{\mathcal{O}}_{,q} = \bar{\mathcal{O}}_{q,q} - \frac{1}{2} h^2 F_{,q},$$ \hspace{1cm} (28)

and 24 has been used. Using the equations of motion 27 the bianchi identities 8 become

$$T_{..ab}^{\text{th}} = \frac{1}{2} h^2 F_a - \left( \theta \bar{\mathcal{O}}_{\theta} + \frac{1}{2} h^2 F_s \right) s_a - n \theta \left( \frac{1}{n} \bar{\mathcal{O}}_{\theta} \right)_a - \sigma_a \bar{\mathcal{O}}_{,\sigma} - \theta_a \bar{\mathcal{O}}_{,\theta},$$ \hspace{1cm} (29)

for the thermodynamical case 6 and 27 give zeroth order coefficient function $\bar{\mathcal{O}} = n T s$ and then the bianchi identity reduces to $F_a = 0$ or $n = k h$, thus what in lagrangian 22 looked like an arbitrary function $\mathcal{F}$ has been reduced to a constant and this will be further considered in the next section. Variation with respect to everything else except the particle number $n$ gives
the same as the implicit approach of the previous section §2. For variation with respect to the particle number \( n \) there are three choices: the first is to simply do it in which case, for lagrangians in which \( n \) is separated and linear, the vectors \( W, V \) are forced to be null and the system is no longer that of a perfect fluid, the second is to ignore variation with respect to \( n \), for lagrangians in which \( n \) is separated and linear this amounts to assuming a first law of the form 3 and one is essentially working in the implicit formalism of the previous section §2, the third is to alter lagrangians with coefficient functions \( F, \mathcal{V} \) that depend on \( n \), then variation with respect to \( n \) can be chosen so that \( h^2 \mathcal{F} = -2 \mathcal{V} \). Using 10 for \( n \), 7 for \( h \) and 2 for \( V_a^2 \) the lagrangian is

\[
\mathcal{L} = \mathcal{P} = \frac{1}{2} \dot{\sigma} \Pi^\sigma - \mathcal{U},
\]

using 27 the hamiltonian is

\[
\mathcal{H} = \frac{1}{2} \dot{\sigma} \Pi^\sigma + \mathcal{U} - \frac{\theta}{n} \Pi^\sigma \mathcal{U},
\]

when \( \mathcal{U}, \theta = 0 \) the hamiltonian equals the density \( \mathcal{H} = \mu \). For this explicit form 31 it is possible to use the operator substitution 21 so that

\[
\mathcal{H} \Psi = -\frac{1}{2} i \hbar \dot{\sigma} \Psi + \mathcal{U} \Psi = 0,
\]

taking \( \dot{\sigma} \Psi = \Psi, 32 \) becomes

\[
i \hbar \Psi = 2 \mathcal{U} \Psi,
\]

integrating

\[
\Psi = A \exp \left( -\frac{2i}{\hbar} \int \mathcal{U} d\tau \right),
\]

this wavefunction turns out to be too restrictive to be of any use.

### 4 Simplest three clebsch potential fluid

The explicit linear thermodynamic lagrangian is

\[
\mathcal{L} = -\frac{1}{2} W_a^2 - nTs,
\]

see the remarks after 29. Varying with respect to the potentials gives

\[
W_a^\sigma = \Box \sigma + \theta_a s^\sigma + \theta \Box s = 0, \quad s^\sigma (\sigma_a + \theta s_a) = 0, \quad \theta^\sigma (\sigma_a + \theta s_a) = hT.
\]

7
If these equations are thought of as functions of one variable then the second equation gives either \( s_a = 0 \) or \( W_a = 0 \) both of which are of no practical use: therefore two variables are used specifically to seek plane wave solutions. A simple choice is

\[
\sigma = A \exp i (k_0 t + k_1 x), \quad \theta = B \exp ai (k_0 t + k_1 x), \quad s = C \exp (1-a)i (k_0 t + k_1 x),
\]

which gives

\[
W_a = i [k_0, k_1, 0, 0] K, \quad K \equiv \left( \frac{A}{BC} + 1 - a \right) \theta s,
\]

the first and second equations of motion 36 give

\[
(k_0^2 - k_1^2) K = 0,
\]

and the third gives

\[
(k_0^2 - k_1^2) K = \frac{T}{a \theta},
\]

thus the system is decoupled, in other words as 39 forces it to be null the terms in the first equation of 36 vanish separately, and by 40 the system is at zero temperature. Working through in the same manner with the clebsch potentials having both left and right movers, i.e. like 48, the same problems arise and there is no indication that mixed movers can generate non-zero temperature. Working through without the restriction that the clebsch potentials are co-directional produces equations too general to proceed with.

The hamiltonian is

\[
\mathcal{H} = -\frac{1}{2} \dot{\sigma} \Pi^\sigma - Ts \Pi^\sigma,
\]

in the present case \( \dot{\sigma} = h = n = -\Pi^\sigma \) so that

\[
\mathcal{H} = \frac{1}{2} \Pi^\sigma - Ts \Pi^\sigma
\]

using the substitution 21 with \( x = \sigma \) gives

\[
\mathcal{H} \Psi = -\frac{i h}{2} \Psi_{\sigma \sigma} + Ts \Psi_\sigma = 0,
\]

which has solution

\[
\Psi = A \sigma \exp \left( \frac{2i Ts}{h} \right) + B,
\]

which is again too restrictive.
5 One clebsch potential fluid with two coefficient functions

The lagrangian is taken to be

\[ \mathcal{L} = -\frac{1}{2} \mathcal{F}(\sigma)\sigma_a^2 - \mathcal{O}(\sigma), \]  

(45)

the metric stress is

\[ T_{ab} = \mathcal{F}\sigma_a\sigma_b + \mathcal{L}g_{ab}, \]  

(46)

\( \mathcal{P}, \mu, n, h \) are recovered in the same way as in the last section. Varying with respect to \( \sigma \)

\[ \mathcal{F}\Box\sigma + \mathcal{F}^a\sigma_a + \mathcal{O}_\sigma = 0, \]  

(47)

which for simple \( \mathcal{F} \) has simple spherically symmetric solutions; for wave solutions use zeroth order coefficient function \( \mathcal{O} = m^2\sigma^2/2 \) and

\[ \sigma_\pm = A_\pm \exp(ik \cdot x) \pm A_- \exp(-ik \cdot x), \]  

(48)

with \( \sigma = \sigma_+ \), 47 becomes

\[ (\mathcal{F}k_a^2 + m^2)\sigma + \mathcal{F}'k_a\sigma_\sigma^2 = 0, \]  

(49)

the last term forces either \( A_+ \) or \( A_- \) to vanish.

6 One fluid described using one vector field

In the one fluid one vector approach one hopes to find a lagrangian which recovers as much of the stress [26]eq.22.16d as possible

\[ T_{ab} = (\mu + \mathcal{P} - \xi\Theta)U_aU_b - 2\eta U\sigma_{ab} + 2q_{(a}U_{b)} + (\mathcal{P} - \xi\Theta)g_{ab}, \]  

\[ S_a = nsU_a + \frac{q_a}{T}, \]  

(50)

where \( \mu \) is the density, \( \mathcal{P} \) is the pressure, \( \xi \geq 0 \) is the coefficient of bulk viscosity, \( \eta \geq 0 \) is the coefficient of dynamic viscosity, \( \sigma \) is the shear, \( q \) is the heat flux, \( S_a \) and \( s \) are the entropy vector and scalar and \( T \) is the temperature. Consider a fluid lagrangian dependent on one velocity which can be expanded

\[ \mathcal{L} = \mathcal{L}(V) = \mathcal{L}(V^{0+}, V^1, V^{1+}, V^2, \ldots) \]  

(51)
where
\begin{align*}
\mathcal{L}(V^{0+}) &= k^{0+} \mathcal{P}, \\
\mathcal{L}(V^{1}) &= k^{1} \Theta = k^1 V^a_{;a}, \\
\mathcal{L}(V^{1+}) &= k^{1+} \mathcal{P} \Theta, \\
\mathcal{L}(V^{2}) &= k^{2} \dot{\Theta} + k_2 R_{ab} V^a V^b + k_3 \omega^2 + k_4 \sigma^2 + k_5 \Theta^2 + k_6 \dot{V}^a_{;a}
\end{align*}

so that the integer superscript on the velocity \( V \) indicates the power in which it occurs in the lagrangian, the meaning of the superscript + will become apparent later. Here just the first three terms are considered, terms of second or Raychadhuri order and higher are ignored, as are any auxiliary, image, entropy or electromagnetic fields. From a technical point of view just the first term in this expansion has been considered in the previous section §2 and in this section the next two terms are considered. Metrical variation gives the stress
\begin{equation}
T_{ab} = (\mu + \mathcal{P}) \left( k^{0+} + k^{1+} \Theta \right) V^a V^b - 2 k^1 V_{(a;b)} + \mathcal{L}_{dab},
\end{equation}

53 does not bare much resemblance to 50 except that there is a term similar to bulk viscosity appearing. Variation with respect to the clebsch potentials, particularly of \( \mathcal{L}(V^{1}) \), gives long expressions which are not helpful. Variation with respect to the clebsch velocities gives the momenta
\begin{align*}
\Pi^\phi &= -n \left( k^{0+} + k^{1+} \Theta \right), \\
\Pi^s &= \theta \Pi^\phi, \\
\Pi^\theta &= 0,
\end{align*}

and these give two constraints
\begin{equation}
\phi_1 = \Pi^s - \theta \Pi^\phi, \quad \phi_2 = \Pi^\theta.
\end{equation}

In the present case, as the constraints 55 are the same as for the perfect fluid, the dirac bracket between the coordinates and momenta and thus the quantum relations are the same as for the perfect fluid as given above and in [33].

7 One fluid described using two vector fields

If one tries to incorporate entropy by choosing it to move in the same direction as fluid flow then the entropy is proportional to the enthalpy and
\( s = kh \) and the first law becomes

\[
\delta p = n \delta h - n T \delta s = -n V_a \delta W^a - n T k \delta h
\]

\[
= -n V_a \delta W^a + n T k V_a \delta W^a = -n^* V_a \delta W^a,
\]

where \( n^* = (1 - k T)n \), \hspace{1cm} (56)

so the problem becomes the same as before with the particle number \( n \) replaced by \( n^* \). In the one fluid two vectors approach one proceeds as before except the clebsch decomposition 2 is replaced by

\[
h V_a = W_a = \sigma_a + \eta \chi_a, \quad V_a V^a = -1,
\]

\[
\ell U_a = X_a = \alpha_a + \beta \gamma_a, \quad U_a U^a = -1,
\]

and the first law 3 is replaced by

\[
\delta p = n \delta h - n T \delta S = -n V^a \delta W_a + n T U^a \delta X_a.
\]

(58)

Metrical variation gives

\[
T_{ab} = n h V_a V_b - n T \ell U_a U_b + \mathcal{P}_{ab},
\]

(59)

when either \( T \) or \( \ell \to 0 \) the second term vanishes. 59 can be re-written as

\[
T_{ab} = (\mathcal{P} + \mu) \left( V_a V_b - \frac{T}{h \ell} X_a X_b \right) + \mathcal{P}_{ab},
\]

(60)

Two choices firstly the free case where the clebsch potential are all independent, secondly the thermodynamic case where the clebsh potential are chosen so that the thermasy and entropy change are non-vanishing. The free case essentially duplicates the one vector case of the perfect fluid §2.

For the thermodynamic case choose

\[
\theta = \eta = \beta, \quad s = \chi = \gamma,
\]

(61)

in 57. Defining two absolute derivatives

\[
\dot{T}_{abc...} = V^e T_{abc...;e}, \quad T'_{abc...} = U^e T_{abc...;e},
\]

(62)

variation with respect to \( \sigma, \alpha, \theta \) and \( s \) gives

\[
\dot{n} + n \dot{\Theta} = 0, \quad (n T)' + n T \dot{\Theta} = 0, \quad \dot{s} = T s', \quad \dot{\theta} = T \theta',
\]

(63)

respectively. The relationship between dot and dash derivatives is needed, note

\[
X_a = (\alpha - \sigma)_a + W_a,
\]

(64)
thus for an arbitrary tensor $A_{abc...}$ there is the relationship between the derivatives

$$\ell A_{abc...}' = (\alpha - \sigma)^c A_{abc...} + h \dot{A}_{abc...}, \quad (65)$$

using 65, 63 becomes

$$h V \Theta - \ell U = (\ln(nT))^a (\alpha - \sigma)_a + h (\ln(T))^\circ,$$

$$s = \frac{\ell(\alpha - \sigma)^a s_a}{\ell - Th}, \quad \dot{\theta} = \frac{\ell(\alpha - \sigma)^a \theta_a}{\ell - Th}. \quad (66)$$

In the limit that the two vectors coincide $\alpha \to \sigma$ and 64 and 66 give $\dot{T} = \dot{s} = \dot{\theta} = 0$ so that the thermasy relation is not recovered unless $T = 0$. The dot momenta are the same as that given by 10, the dashed momenta are

$$\Pi^\alpha(\text{dashed}) = nT, \quad \Pi^s(\text{dashed}) = n\theta T, \quad (67)$$

it is necessary to convert these to dot momenta using

$$\Pi^i \equiv \frac{\delta I}{\delta q^i} = \frac{\delta q'^i}{\delta q^i} \frac{\delta I}{\delta q'^i} \Pi^i(\text{dashed}) \equiv f(q) \Pi^i(\text{dashed}), \quad (68)$$

where $i$ is not summed and $f$ is an undetermined function of the Clebsch potentials; collecting together 10,67,68 gives the total momenta

$$\Pi^\sigma = -n, \quad \Pi^\alpha = nTf, \quad \Pi^\theta = 0, \quad \Pi^s = -n\theta + n\theta Tf, \quad (69)$$

and the three constraints

$$\phi_1 = \Pi^s - \theta \Pi^\sigma - \theta \Pi^\alpha, \quad \phi_2 = \Pi^\theta, \quad \phi_3 = \Pi^\alpha + Tf \Pi^\sigma, \quad (70)$$

the Dirac matrix is

$$C_{12} = -\Pi^\sigma - \Pi^\alpha, C_{13} = \left(-\frac{\delta f}{\delta s} + \theta \frac{\delta f}{\delta \sigma} + \theta \frac{\delta f}{\delta \alpha}\right) T \Pi^\sigma, C_{23} = -\frac{\delta f}{\delta \theta} T \Pi^\sigma, \quad (71)$$

the inverse Inv$C$ can be found, but there is no simplification in further expressions.

8 Conclusion

In §2 the lagrangian approach to perfect fluids was presented, the Clebsch potentials contain more information than is needed to describe the system so that it is constrained, Dirac constraint analysis removes the superfluous
degrees of freedom resulting in the algebra 19, see also [33], which when quantized via 20 does not seem to lead to a recognizable algebra; also as the pressure and density are implicit rather than explicit functions of the Clebsch potentials it is not clear what quantum operators should represent them. In §3 to overcome lack of explicit expressions for the pressure and density an explicit Lagrangian 22 was studied, this has a very restrictive equation of state 26, and for thermodynamic choices of the functions reduces to the example of the next section. In §4 the simplest three potential Lagrangian 35 was studied, it seems to lead to an unrealistic quantum theory 43. In §5 a one potential Lagrangian 43 was studied, it is just a simple generalization of the Klein Gordon Lagrangian, but the generalization is obstructive enough to prevent successful investigation of it by the Fourier oscillator method. In §6 the perfect fluid is generalized so that it depends on higher powers of the comoving fluid, a term similar to the bulk viscosity appears and the momentum constraints are the same as those for a perfect fluid. In §7 the perfect fluid is generalized to include two vector fields in the hope that these can represent both density and entropy flow, equating some of the Clebsch potentials in each vector leads to plausible thermodynamic equations 66.

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References

[1] A. Achterberg, Variational principle for relativistic magnetohydrodynamics, Phys. Rev. A 28 (1983) 2449-2458.

[2] Lars Andersson, Robert Bieg and Bernd G. Schmidt, Static Self-Gravitating Elastic Bodies in Einstein Gravity (2008), Math.Rev.2009f:83034, Comm. Pure Appl. Math. 61 (2008) 988-1023.

[3] Angelo Marcello Anile, Relativistic Fluids and magneto-fluids, Cambridge University Press (1989).

[4] P.R. Baldwin, (1993) Constructing Clebsch Potentials for Vector Fields. Print Physics/Akron.

[5] Eugene Balkovsky, Some notes on the Clebsch representation for incompressible fluids. Math.Rev.94k:76026 Phys. Lett. A 186 (1994) 135-136.
[6] Arjun Berera & David Hochberg, Gauge fixing, BRS invariance and Ward identities for randomly stirred flows, *Nucl. Phys. B* **814** (2009) 522-548.

[7] James D. Bjorken, Highly relativistic nucleus-nucleus collisions: The central rapidity region, *Phys.Rev.D* **27** (1983) 140.

[8] Carlos Cartes, Miguel D. Bustamante & Marc Étienne Brachet, Generalized Euler-Lagrangian description of Navier-Stokes dynamics, *Physics of Fluids* **19** (2007) 007101.

[9] Brandon Carter & I.M. Khalatnikov Momentum, vorticity, and helicity in covariant superfluid dynamics. Math.Rev.93m:83046 Ann. Physics **219**(1992) 243-265.

[10] Alfred Clebsch, Über eine allgemeine Transformation der Hydrodynamischen Gleichungen, Crelle’s Journal, liv (1859). Reine Angnew. Math. ("Crelle") **56**(1859) 1.

[11] Peter Constantin, An Eulerian-Lagrangian Approach to the Navier-Stokes Equations, Math.Rev.2002m:76023 Comm.Math.Phys. **216**(2001) 663-686.

[12] Hugh L. Dryden, Francis P. Murnaghan and H. Bateman, (1956) Hydrodynamics, Dover, Math.Rev.0077307 (17,1019f).

[13] Carl Eckart, Variational Principles of Hydrodynamics, Math.Rev.0114428 (22#5249), *Physics of Fluids* **3**(1960) 421 esp.Appendix.

[14] A. El, Z. Xu & C. Greiner, Third-order relativistic dissipative hydrodynamics, 0907.4500

[15] Bruce A. Finlayson, Existence of variational principles for the Navier-Stokes equation, Math.Rev.0326189 (48#4534), *Phys.Fluids* **15**(1977) 963-967.

[16] Harold Grad, On the kinetic theory of rarefied gases, Math.Rev.0033674 (11,4739), *Commun.Pure Appl.Math.* **2**(1949) 331.

[17] C. Robin Graham and Frank S. Henyey, Clebsch representation near points where the vorticity vanishes, Math.Rev.2000k:76033 *Physics of Fluids* **12**(2000) 744-746
[18] R. Hargreaves, A Pressure-integral as Kinetic Potential, Phil. Mag. 16 (1908) 436-444. 2

[19] Friedrich W. Hehl, Paul von der Heyde, G. David Kerlich and James M. Nester, Math.Rev.0439001 (55#11902), Rev.Mod.Phys.48 (1976) 393. 2

[20] Michal P. Heller, Romuald A. Janik and R. Peschanski, Hydrodynamic Flow of the Quark-Gluon Plasma and Gauge/Gravity Correspondence, 0811.3113 v2 2

[21] William A. Hiscock and Lee Lindblom, Math.Rev.88e:83056, Phys.Rev.D35 (1987) 3723-3737. 2

[22] W. Israel and J.M. Stewart, Transient relativistic thermodynamics and kinetic theory, Math.Rev.80d:80011, Ann.Phys.N.Y.118 (1979) 341. 2

[23] Roman Jackiw, Description of Vorticity by Grassmann Variables and an Extension to Supersymmetry, Fluctuating paths and fields, 553-563, World Sci.Pub., River Edge NJ (2001), Math.Rev.1868089 physics/0010079 v2 2

[24] Horace Lamb, Hydrodynamics, p.248, CUP, Cambridge 1932, Math.Rev.96f:76001. 2

[25] V.G. Lapchiniskii and V.A. Rubakov, Theor.Math.Phys.33 (1977) 1076. 2

[26] Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, Gravitation, W.H.Freeman & Company, New York (1970), Math.Rev.0418833 (54#6869). 9

[27] Ian G. Moss, Superfluidity in Super-Yang-Mills Theory, 0712.2568 2

[28] Azwinindini Muronga, Causal theories of dissipative relativistic fluid dynamics for nuclear collisions, Phys.Rev.C 69 (2004) 034903. 2

[29] Shin Nakamura and Sang-Jin Sin, A holographic dual of hydrodynamics, Math.Rev.2007e:81120, JHEP 09 (2006) 020. 2

[30] Tino S. Nyawelo, Jan-Willem van Holten and Stefan Groot Nibbelink, Relativistic fluid mechanics, Kähler manifolds and supersymmetry, hep-th/0307283, Math.Rev.2005b:83053 2

[31] Roger Penrose and Wolfgang Rindler, Spinor calculus and relativistic fields, CUP (1987), ISBN 0521-33707-0 4

15
[32] Mark D. Roberts, Fluid Symmetry Breaking II: Velocity Potential Method, *Hadronic J.* **20**(1997)73-84, hep-th/9904079 2, 3

[33] Mark D. Roberts, The Quantum Commutator Algebra of a Perfect Fluid. Math.Rev.2000j:81120 *Mathematical Physics, Analysis and Geometry* **1**(1999)367-373. gr-qc/9810089 5, 10, 13

[34] Mark D. Roberts, A Generalized Higgs Model. *Physics Essays*(2006)September Vol.19, No.3, p.1-5. hep-th/9904080 2, 3

[35] Mark D. Roberts, A Fluid Generalization of Membranes, hep-th/0406164 2

[36] Hanno Rund, D.R. Wells and L.C. Hawkins, *J Plasma Phys.* **20**(1978)329-344. 2

[37] Hanno Rund, Clebsch Representations and Relativistic Dynamical Systems, Math.Rev.80d:83054, *Arch. Rat. Mech. Anal.* **71**(1979)199-220. 2

[38] R.L. Salinger and Gerald B. Whitham, Variational Principles in Continuum Mechanics, *Proc. Roy. Soc. London* A**305**(1960)1-25. 2, 3

[39] Abraham Haskel Taub, General Relativistic Variational Principle for Perfect Fluids, Math.Rev. 0063824 (16, 185b) *Phys. Rev.* **6**(1954)1468-1470. 2

[40] J.R. Usher & Alex D.D. Craik, Nonlinear wave interactions in shear flows. Part 1. A Variational formulation, *J. Fluid Mech.* **66**(1974)209-221. 2

[41] Victor Yakhot and Vladimir Zakharov, Hidden conservation laws in hydrodynamics, Math.Rev.94a:76041, *Physica D* **64**(1993)379-394.