REPRESENTATION OF LINEAR SYSTEMS BY MEANS OF DIFFERENTIAL EQUATIONS AND GENERATION OF RANDOM PROCESSES

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Linear systems are characterized by the impulse response $h(t, u)$ or simply $h(t)$ in the case of time-constant parameters. A distinctive feature of this description is that the input signal is considered known in the interval $-\infty < t < \infty$. The impulse response $h(t, u)$ is simply a solution to the differential equation when the input signal is a pulse at time $u$.

There are three ways to describe systems using differential equations.

The first way is related to initial conditions and state variables when considering dynamical systems [1]. The state of the system is defined as the minimum amount of information regarding the effects of the previous signals at the input of the system, necessary for a complete description of the output signal at $t \geq 0$. Variables containing this information are state variables. If the state of the system is specified at time $t_0$, and the input signal is in the interval from $t_0$ to $t_1$, then both the output signal and the state of the system at time $t_1$ can be found.

The second way is reduced to the implementation (or modelling) of the differential equation using an analog calculator. It can be represented as a system consisting of integrators, circuits with time-varying coefficients, adders, and nonlinear inertialess devices combined to reproduce the desired ratio between input and output signals. The initial condition $y(t_0)$ is here the bias at the output of the integrator. The biased output voltage of the integrator is a system state variable [1, 2].

The third way relates to the issue of generating a random process. If $u(t)$ is a random process or $y(t_0)$ is a random variable (or they are both random), then $y(t)$ is also a random process.

To determine the coefficients of the system of differential equations describing a dynamic system, it is necessary to obtain an equation that does not depend on the input signal.

Consider a system described by a differential equation of the form:

$$y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \ldots + p_1(t)y(t) + p_0(t) = b_0u(t),$$

where: $y^{(n)}(t)$ – $n$th derivative of the $y(t)$; $p_i(t)$ – differentiation operator; $u(t)$ – system input signal; $b_0$ – weight coefficient.

To determine the solution to an equation of order $n$, it is necessary to know the values of $y(t), \ldots, y^{(n-1)}(t)$ at time $t_0$. The first step in finding an implementation in the
form of an analog calculator is to model the terms on the left side of this equation. The next step is to interconnect these quantities in such a way that the specified equation is satisfied. The differential equation determines the input voltage at the adder. Introduce the initial conditions by setting certain displacements at the outputs of the integrator. The state variables are biased voltages at the output of the integrator.

It is easier to work with a first-order vector differential equation than a scalar differential equation $n$.

Let be:

$$
\begin{align*}
    x_1(t) &= y(t), \\
    x_2(t) &= y(t) = x_1(t), \\
    & \vdots \\
    x_n(t) &= y^{(n-1)}(t) = x_{n-1}(t),
\end{align*}
$$

$$
\dot{x}_n(t) = y^{(n)}(t) = -\sum_{k=1}^{n} p_{k-1} y^{(k-1)}(t) + b_0 = -\sum_{k=1}^{n} p_{k-1} x_k(t) + b_0 u(t),
$$

Denoting the system $x_i(t)$ using the column matrix, we note that the $n$-order scalar equation is equivalent to the next $n$-dimensional first order vector equation $[1, 3]$:

$$
\frac{dx(t)}{dt} = x(t) = Ax(t) + Bu(t),
$$

where:

- $A$ – system state matrix; $B$ – control matrix (input).

The vector $x(t)$ is the state vector for the given linear system, and (1) – is the equation of the system state. Any non-singular linear transformation of the vector $x(t)$ gives another state vector. The output voltage $y(t)$ is related to the state vector by the equation $[3 – 6]$:

$$
y(t) = Cx(t),
$$

where:

- $C$ – measurement matrix.

Equation (2) is the original equation of the system. Equations – (1) and (2) – completely define the system.

For systems with time-varying parameters, as the basic representation, consider the vector equations $[3, 7]$:

$$
\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t),
$$

$$
y(t) = C(t)x(t),
$$

where:

- $x(t)$ – state vector; $A(t)$ and $B(t)$ – variable matrices of the differential equation; $u(t)$ – signal at the input of the system, a process like white noise; $C(t)$ – measurement matrix.

Equation (3) is the equation of the system state, and (4) is the output equation of the system.

Using a white noise exposure as input

$$
E[u(t)u(\tau)] = q\delta(t - \tau),
$$
it is possible to simulate some non-stationary random processes. A nonstationary process can appear even when the matrices $A$ and $B$ are constant, and $x_0(t)$ is the deterministic quantity $[1, 4]$.

The resulting differential equations are:

\[
x(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t).
\]

The exciting function is vector.

To simulate the process, assume that the exciting function is white noise with a matrix covariance function $[5]$:

\[
E[u(t)u(\tau)] = Q \delta(t-\tau),
\]

where:
- $Q$ – negative definite matrix.

The scheme of the process modelling will look like this (Fig. 1):

**Fig. 1. Process generation scheme $y(t)$**

For random initial conditions, it is necessary to specify the covariance function and the mean value $E[x(t_0)]$ at the initial time moment at $t_0$ $[1, 3, 6]$:

\[
K_x(t_0, t_0) = E[x(t_0)x^T(t_0)].
\]

Related processes can be modelled by replacing the diagonal matrices in (5), (6) and (7) with matrices of the general form.

If equation (5) is a homogeneous equation with constant coefficients, then:

\[
x(t) = Ax(t),
\]

with the initial condition $x(t_0)$. If $x(t)$ and $A$ are scalars, then the solution has the form:

\[
x(t) = e^{A(t-t_0)}x(t_0).
\]

For the vector case, it can be shown that

\[
x(t) = e^{A(t-t_0)}x(t_0),
\]

where $e^{A\tau}$ is determined by the infinite series:

\[
e^{A\tau} = I + A\tau + \frac{A^2\tau^2}{2!} + \ldots,
\]

where:
- $I$ – identity matrix.

The function $e^{A(t-t_0)}$ is denoted by $\Phi(t-t_0) = \Phi(\tau)$. The function $\Phi(t-t_0)$ is the transition matrix of the system state, which is defined as a function of two variables $\Phi(t, t_0)$, which satisfies the differential equation
\[
\Phi(t,t_0) = A(t)\Phi(t_0) \tag{8}
\]

with the initial condition \( \Phi(t_0, t_0) = I \).

The solution at any time has the form:
\[
x(t,t_0) = \Phi(t,t_0)x(t_0) . \tag{9}
\]

For the inhomogeneous case, the general solution contains a homogeneous and particular solutions of the form:
\[
x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau . \tag{10}
\]

Linear systems with time-varying parameters are characterized by an impulse response \( h(t, \tau) \), provided that the input quantity is known over the interval from \(-\infty \) to \( t \) [7]. Thus,
\[
y(t) = \int_{-\infty}^{t} h(t, \tau)u(\tau)d\tau . \tag{11}
\]

In most cases, the influence of the initial condition \( x(-\infty) \) in (10) does not appear, therefore, accept it equal to zero. Then get
\[
y(t) = C(t) \int_{-\infty}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau . \tag{12}
\]

Comparing (11) and (12), have
\[
h(t, \tau) = \begin{cases} C(t)\Phi(t, \tau)B(\tau), & t \geq \tau, \\ 0, & \text{with others}. \tag{13} \end{cases}
\]

The matrices \( C(t), \Phi(t, \tau) \) and \( B(\tau) \) depend on the representation of the system, but the matrix impulse response is the only one.

Let us establish the statistical properties of vector processes \( x(t) \) and \( y(t) \), when \( u(t) \) is a sample function of a vector random process like white noise:
\[
E[u(t)u^T(\tau)] = Q\delta(t-\tau) .
\]

The cross-correlation between the state vector \( x(t) \) of the system excited by a white noise \( u(t) \) with zero mean, and the input variable \( u(\tau) \), equals
\[
K_{xx}(t, \tau) = E[x(t)u^T(\tau)]. \tag{14}
\]

This discontinuous function has the form
\[
K_{xx}(t, \tau) = \begin{cases} 0, & \tau > t, \\ \frac{1}{2}B(t)Q, & \tau = t, \\ \Phi(t, \tau)B(\tau)Q, & t_0 < \tau < t. \tag{15} \end{cases}
\]

Substituting (9) into definition (13), then obtain
\[
K_{xx}(t, \tau) = E\left\{ \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\alpha)B(\alpha)u(\alpha)d\alpha \right\}u^T(\tau),
\]

where:
\( \alpha \) - lag time.

Introduce the mathematical expectation under the integral sign and assume that the initial state \( x(t_0) \) does not depend on \( u(\tau) \) for \( \tau > t_0 \), then
\[
K_{xx}(t, \tau) = \int_{t_0}^{t} \Phi(t,\alpha)B(\alpha)E[u(\alpha)u^T(\tau)]d\alpha = \int_{t_0}^{t} \Phi(t,\alpha)B(\alpha)Q\delta(\alpha - \tau)d\alpha.
\]
For \( \tau > t \) this expression is equal to zero. If \( \tau = t \), and the delta function is symmetric, since it is the limit of the covariance function, then it is necessary to take only half of the area at the right limit point of the interval. Thus,
\[
K_{xx}(t,t) = \frac{1}{2} \Phi(t,t) B(t) Q .
\]

Using the result that follows from (8), obtain the expression located in the second line of formula (14).

If \( \tau < t \), get
\[
K_{xx}(t,\tau) = \frac{1}{2} \Phi(t,\tau) B(\tau) Q , \quad \tau < t ,
\]
which corresponds to the third line of the formula (14). The special case (15) holds, assuming \( \tau \to t \)
\[
\lim_{\tau \to t} K_{xx}(t,\tau) = B(t) Q .
\]

Hence the cross-correlation function of the output vector \( y(t) \) and \( u(t) \):
\[
K_{yu}(t,\tau) = C(t) K_{xx}(t,\tau) .
\]

Denote
\[
\Lambda_x(t) = K_x(t,t) .
\]

Hence,
\[
\Lambda_x(t) = E[x(t)x^T(t)] .
\]

Differentiating both sides of equation (16), obtain:
\[
\frac{d\Lambda_x(t)}{dt} = E\left[ \frac{dx(t)}{dt} x^T(t) \right] + E\left[ x(t) \frac{dx^T(t)}{dt} \right] .
\]

Substituting (5) into the first term (17), obtain:
\[
E\left[ \frac{dx(t)}{dt} x^T(t) \right] = E\{ [A(t)x(t) + B(t)u(t)] x^T(t) \} .
\]

Using property (15) as applied to the second term in (18), obtain
\[
E\left[ \frac{dx(t)}{dt} x^T(t) \right] = A(t)\Lambda_x(t) + \frac{1}{2} B(t) Q B^T(t) .
\]

Then the dispersion matrix of the state vector \( x(t) \) of system (5) satisfies the differential equation
\[
\dot{\Lambda}_x(t) = A(t)\Lambda_x(t) + \Lambda_x(t)A^T(t) + B(t)QB^T(t)
\]
with the initial condition
\[
\Lambda_x(t_0) = E[x(t_0)x^T(t_0)].
\]

**Conclusions.** Dispersion equation (19) does not contain the received signal, so it can be solved before receiving any information and used to find the transmission coefficients. The dispersion equation is the matrix Riccati equation, which, using the substitution method in [8], is reduced to a linear differential equation, the solution of which is transformed in the opposite direction into the Riccati equation solution, which is an improvement in the method for determining the basic matrices of a dynamical system using the Riccati equation.

**References:**

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[3] Димова, Г.О. (2020). Методи і моделі упорядкування експериментальної інформації для ідентифікації
The article studies the effect of cyclic loads on the strength characteristics and filtration in a porous medium, fatigue processes in the rock skeleton, and the prospects for developing technologies for active stimulation of formations in order to clean the bottom-hole zone and intensify oil and gas production. The issues of formation and growth of fatigue cracks in the rock under the influence of the pulse generator GKP-1 are also considered.

We studied the filtration processes in a porous medium during the action of cyclic loads with different frequencies and amplitudes. To obtain reliable results in the absence of oscillation interference (which are present when using installations for studying the permeability of core samples of the UIPK type), in IFNTUOG, together with the scientific and production company INTEX, a facility was developed for studying the permeability of a porous medium in the process of hydraulic impulse loads on the core UDC-2 [1].

The volume of the filtrate, which is filtered through the core over time, directly indicates the state of spatial permeability, which varies depending on the conditions under which the filtration occurs. Since, during filtration and simultaneous cyclic influence, the fluid moves in the pores and microcracks of the core, changes in its rheological properties, the movement of uncemented particles (pollutant or parent rock) [2], electrokinetic processes, opening, closing, development of new microcracks, the amount of filtrate per unit time can vary within certain limits. In this case, it is necessary to investigate what kind of the filtrate volume will be before, during, and after the treatment with pressure hydroimpulses, and evaluate the changes in comparison with the initial results. It is also necessary to determine the characteristics of materials removed during core filtration by applying the methods of lithological-petrographic analysis.

As it is seen from Figure 1, the processing of natural and artificial core mainly caused an increase in the filtrate volume during filtration for the same period of time. The highest growth was observed for artificial cores 2,4 during processing and is equal to 36-38%. After processing, this figure drops to 30-32%. However, for artificial cores 7,11,15, despite the increase in the filtrate volume during processing by 24%,