A DOOB-TYPE MAXIMAL INEQUALITY AND ITS APPLICATIONS TO VARIOUS STOCHASTIC PROCESSES

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ABSTRACT. We show how an improvement on Doob’s inequality leads to new inequalities for several stochastic processes: Lévy processes, processes with independent increments in general, branching processes, and time homogeneous Markov processes.

1. INTRODUCTION

The aim of this note is to show how an improvement on Doob’s inequality, already pointed out in [4], leads to new inequalities for various stochastic processes, such as Lévy processes, random walks, processes with independent increments in general, branching processes and continuous state branching processes, as well as some Markov processes, including geometric Brownian motion.

Despite the proofs being very simple, the inequalities obtained seem to be novel and quite useful.

2. IMPROVING DOOB’S INEQUALITY

Although the proofs of Theorems A and B below are almost identical to the ones of Theorems 5.2.1 and 7.1.9 in [4], we present their brief proofs for the sake of completeness. As usual, \( E[X; A] \) will denote \( E(X 1_A) \).

The following maximal inequality is precisely Doob’s inequality when \( a = 1 \); when \( a < 1 \), however, the submartingale property is no longer assumed and so Doob’s inequality is not available.

**Theorem A** (Improved Doob; discrete). Let \( N \geq 0 \). Let \( (X_n, \mathcal{F}_n, P)_{0 \leq n \leq N} \) be a discrete stochastic process with the last variable satisfying that \( 0 \leq X_N \in L^1(P) \), and assume that

\[
E(X_N \mid \mathcal{F}_n) \geq aX_n
\]

holds for all \( 0 \leq n < N \) with some \( 0 < a \). Then

\[
P \left( \max_{0 \leq n \leq N} X_n \geq \alpha \right) \leq \frac{1}{\alpha \tilde{a}} E \left[ X_N; \max_{0 \leq n \leq N} X_n \geq \alpha \right], \quad \alpha > 0,
\]

where \( \tilde{a} := \min\{a, 1\} \).

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Proof. It is enough to treat the case when $a < 1$, otherwise one is simply looking at Doob’s inequality. Define the mutually disjoint events

$$A_0 := \{X_0 \geq \alpha \} ;$$
$$A_n := \{X_n \geq \alpha \text{ but } \max_{0 \leq m < n} X_m < \alpha \} \in F_n, \ n = 1, 2,...$$

Since $a < 1$ and $X_N \geq 0$, the bound (1) holds even for $n = N$, and thus

$$P \left( \max_{0 \leq n \leq N} X_n \geq \alpha \right) = \sum_{n=0}^{N} P(A_n) \leq \sum_{n=0}^{N} \frac{E[X_n; A_n]}{\alpha} \leq \sum_{n=0}^{N} \frac{E[X_N; A_n]}{\alpha \tilde{a}}$$

$$\leq \frac{1}{\alpha \tilde{a}} E \left[ X_N; \max_{0 \leq n \leq N} X_n \geq \alpha \right],$$

as claimed. □

Next, we treat the continuous counterpart.

**Theorem B** (Improved Doob; continuous). Let $T > 0$. Let $(Z_t, F_t, P)_{t \in [0, T]}$ be a right-continuous stochastic process with the last variable satisfying that $0 \leq Z_T \in L^1(P)$, and assume that

$$E(Z_T | F_t) \geq a Z_t$$

holds for all $0 \leq t < T$ where $0 < a$. Then, for $\alpha > 0$,

$$P \left( \sup_{0 \leq s \leq T} Z_s \geq \alpha \right) \leq \frac{1}{\alpha \tilde{a}} E \left[ Z_T; \sup_{0 \leq s \leq T} Z_s \geq \alpha \right],$$

where $\tilde{a} := \min\{a, 1\}$.

Proof. We will write $Z(s)$ instead of $Z_s$ for convenience. Let $n \in \mathbb{N}$ be given and apply Theorem A to the discrete parameter process $(W_m, G_m, P)_{0 \leq m \leq 2^n} := \left( Z \left( \frac{mT}{2^n} \right), F_{\frac{mT}{2^n}}, P \right)$ and $N := 2^n$, yielding

$$P \left( \max_{0 \leq m \leq 2^n} W_m \geq \alpha \right) \leq \frac{1}{\alpha \tilde{a}} E \left[ Z_T; \max_{0 \leq m \leq 2^n} W_m \geq \alpha \right].$$

Exploiting right-continuity, one has

$$\max_{0 \leq m \leq 2^n} W_m = \max_{0 \leq m \leq 2^n} Z \left( \frac{mT}{2^n} \right) \succ \sup_{0 \leq s \leq T} Z(s), \text{ as } n \to \infty,$$

and we are done by letting $n \to \infty$ and using dominated convergence theorem. □

3. Applications to various processes

We now present some useful inequalities which are straightforward applications of Theorems A and B.

3.1. Application to processes with independent increments. If the right-continuous process $(Z_t, F_t, P)$ on $[0, T]$ has independent increments, then

$$E(e^{Z_T} | F_s) = E \left( e^{Z_T - Z_s} | F_s \right) = E \left( e^{Z_T - Z_s} \right).$$

Let

$$a := \inf_{0 \leq s \leq T} E(e^{Z_T - Z_s}).$$
If $0 < a$, then the conditions of Thm B are satisfied for the process $\hat{Z} := e^Z$. Therefore, we have

**Theorem 1** (Independent increments). If the right-continuos process $(Z_t, F_t, P)$ on $[0, T]$ has independent increments, then for $\alpha \in \mathbb{R}$,

$$P \left( \sup_{0 \leq s \leq T} Z_s \geq \alpha \right) \leq \frac{e^{-\alpha}}{\alpha} E \left[ e^{\hat{Z}_T}; \sup_{0 \leq s \leq T} Z_s \geq \alpha \right].$$

**Remark.** If the righthand side is infinite, we still consider the bound valid in the broader sense, and therefore we do not assume any moment condition on $Z_T$.

As a particular case, we let $(S_n, F_n, P)_{0 \leq n \leq N}$ be a random walk on $\mathbb{Z}$ with $S_0 = 0$. Let the steps $Y_n := S_{n+1} - S_n$ be independent, and define $\phi_n := E e^{Y_n}$ and $\pi_n := \prod_{i=1}^{N-1} \phi_i$. Choosing

$$a := \min_{0 \leq n \leq N} E (e^{S_n} - S_n) = \min_{0 \leq n \leq N} \pi_n,$$

we obtain

**Corollary 1** (Random walks with time-inhomogeneous steps). For $\alpha \in \mathbb{R}$,

$$P \left( \max_{0 \leq n \leq N} S_n \geq \alpha \right) \leq e^{-\alpha} \max \left\{ 1, \max_{0 \leq n \leq N} (\pi_n)^{-1} \right\} E \left[ S_N; \max_{0 \leq n \leq N} S_n \geq \alpha \right].$$

### 3.2. Application to Lévy-processes

If we assume even more, namely that $Z$ is actually a Lévy-process, then, for $T > 1$, the Lévy-Khintchine Theorem implies that

$$a = \inf_{0 \leq s \leq T} E \left( e^{Z_T - Z_s} \right) = \inf_{0 \leq s \leq T} E \left( e^{Z_T - Z_s} \right) = \inf_{0 \leq s \leq T} \left( E(e^{Z_1}) \right)^{T-s}.$$

Observe that the infimum is either at 0 or at $T$. So, assuming as usual, that $Z_0 = 0$, in terms of the Lévy exponent $\psi$, we obtain that

- $a = 1$ when $\psi(1) \geq 0$;
- $a = E e^{Z_T}$ when $\psi(1) \leq 0$.

Clearly, $T > 1$ is not important in the argument above, and so we have obtained that $a = \min \{1, E e^{Z_T} \}$. This gives the following result.

**Theorem 2** (Lévy process). A Lévy process $(Z_t, F_t, P)$ on $[0, T]$ satisfies (assuming $Z_0 \equiv 0$) that for $\alpha \in \mathbb{R}$,

$$P \left( \sup_{0 \leq s \leq T} Z_s \geq \alpha \right) \leq e^{-\alpha} \frac{E \left[ e^{\hat{Z}_T}; \sup_{0 \leq s \leq T} Z_s \geq \alpha \right]}{\min \{1, E e^{Z_T} \}}.$$

In particular,

$$P \left( \sup_{0 \leq s \leq T} Z_s \geq \alpha \right) \leq e^{-\alpha} \max \{1, E e^{Z_T} \}.$$

**Remark.** (i) Again, the righthand sides of the bounds are allowed to be infinite, and so we make no moment assumptions on $Z_T$.

(ii) When $a = 1$, that is, $\psi(1) \geq 0$, the theorem is simply an exponential Doob’s inequality. For example, that is the case for standard Brownian motion. Nonetheless, when $E e^{Z_T} \leq 1$, i.e. $\psi(1) \leq 0$, we obtain a new inequality. Let $a \geq 0$, $b \in \mathbb{R}$, and $\Lambda$ be the characteristics of $Z_1$. That is, with $h(x) := x 1_{|x| \leq 1}$, let $Z_1$ have

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1For background on Lévy processes, see e.g. [3].
log-characteristic function $\Phi(t) = ibt - \frac{1}{2}at^2 - \int_{-\infty}^{\infty} e^{itx} - 1 - ith(x) \Lambda(dx)$. Then $\psi(1) \leq 0$ is equivalent to

$$b \leq -\frac{a}{2} - \int_{-\infty}^{\infty} (e^x - 1 - h(x)) \Lambda(dx).$$

(Of course, this condition is only meaningful when the Lévy measure has light enough tail, guaranteeing that $\int_{-\infty}^{\infty} (e^x - 1 - h(x)) \Lambda(dx) < \infty$.)

3.3. Application to subcritical branching processes. Let $(Z_t)_{t \geq 0}$ be a subcritical branching process, with mean offspring number $0 < \mu < 1$, and with exponential branching clock with rate $b > 0$. Let $m := \mu - 1 < 0$. Since, by the branching property, $E(Z_T | Z_s) = e^{bm(T-s)}Z_s$ for $T > s$, we pick $a = e^{bmT}$ and obtain that

**Theorem 3** (Subcritical branching processes). For $\alpha > 0$,

$$P\left( \sup_{0 \leq s \leq T} Z_s \geq \alpha \right) \leq \alpha^{-1} e^{-bmT} E\left[ Z_T; \sup_{0 \leq s \leq T} Z_s \geq \alpha \right].$$

Note: The righthand side is of course bounded by $\alpha^{-1}$ for any $T$ and $\mu < 1$, in accordance with the fact that for the $\mu = 1$ case, Doob’s inequality gives precisely the $\alpha^{-1}$-bound. But if $\alpha$ is large relative to $T$, our bound is much tighter, as the expectation term tends to zero as $\alpha \to \infty$.

**Remark** (CSBP’s). For a continuous state branching process (CSBP) $X$ (that can also be thought of as the total mass of a superprocess), with branching mechanism $\beta u - ku^2$ with $\beta < 0$, $k > 0$, we get, by a similar argument, that

$$P\left( \sup_{0 \leq s \leq T} X_s \geq \alpha \right) \leq \alpha^{-1} e^{-\beta T} E\left[ X_T; \sup_{0 \leq s \leq T} X_s \geq \alpha \right], \quad \alpha > 0.$$  

For background on CSBP’s, see [3]. For another, superprocess-related application, see [1].

3.4. Application to time-homogeneous Markov processes. If $X$ is a time-homogenous Markov process, then our condition becomes

$$E_{X_s}(X_T) \geq aX_s, \quad s \in [0, T]$$

where $a = a(T) > 0$.

This, besides branching processes, is also satisfied for example by a geometric Brownian motion $S$ solving the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with $S_0 = z > 0$. Here $\mu \in \mathbb{R}$, $\sigma > 0$, while $W$ is a standard Brownian motion. Indeed,

$$E_{\mu}(S_T) = re^{\mu(T-s)} \geq ra, \quad 0 \leq s \leq T,$$

where $a := 1$ for $\mu \geq 0$ and $a := e^{\mu T}$ for $\mu < 0$.

In the latter case for instance, we obtain that,

**Theorem 4** (GBM; $\mu < 0$). Assume that the geometric Brownian motion $S$ has drift $\mu < 0$ and $S_0 = z$. Then, for $\alpha > z$,

$$P_z \left( \exists t > 0 : S_t \geq \alpha \right) \leq \frac{z}{\alpha}.$$
Proof. Using continuity,

\[ P_z(\exists t > 0: S_t \geq \alpha) = \lim_{T \to \infty} P_z(\max_{0 \leq t \leq T} S_t \geq \alpha). \]

Now, Theorem B along with the previous comments yields for \( \alpha > z \) and \( T > 0 \), that

\[ P_z(\max_{0 \leq t \leq T} S_t \geq \alpha) \leq \frac{E_z(S_T)}{\alpha} = \frac{ze^{\mu T}}{\mu T} = \frac{z}{\alpha}, \]

and we are done. \( \square \)

For some related results on geometric Brownian motion, see [2].

3.5. Application to proving limits. We conclude with a simple application to proving limit theorems. The point is that even though the process is defined for continuous times, there is no need to go through the ubiquitous (and rather unpleasant) ‘discrete time skeleton first’ procedure. This has been utilized in [1] for superprocesses.

Theorem 5 (Almost sure convergence). Let \((X_t, F_s, P)_{t \geq 0}\) be a nonnegative real valued, filtered stochastic process, such that \(m_t := EX_t < \infty\). Assume that

1. there is an \( a \in (0, 1] \) such that
   \[ E(X_t \mid F_s) \geq aX_s, \quad \forall 0 \leq s < t; \]
2. \( \exists (t_i)_{i \in \mathbb{N}} \) sequence such that \( t_i \uparrow \infty \) as \( i \to \infty \) and \( \sum_i m_{t_i} < \infty \).

Then \( \lim_{t \to \infty} X_t = 0 \) \( P \)-a.s.

Proof. By Borel-Cantelli, it is enough to show that for any given \( \epsilon > 0 \),

\[ \sum_{i \geq i_0} P \left( \sup_{s \in [t_{i-1}, t_i]} X_s > \epsilon \right) < \infty. \]

By our first assumption along with Theorem B, the lefthand side is bounded by

\( (ae)^{-1} \sum_{i \geq i_0} EX_{t_i} \), and we are done, given our second assumption. \( \square \)

References

[1] Englānder, J.; Ren, Y-X.; Song, R. Weak extinction versus global exponential growth of total mass for superdiffusions. Ann. Inst. Henri Poincaré Probab. Stat. 52 (2016), no. 1, 448–482.
[2] Graversen, S. E.; Peskir, G. Optimal stopping and maximal inequalities for geometric Brownian motion. J. Appl. Probab. 35 (1998), no. 4, 856–872.
[3] Kyprianou, A. E. Fluctuations of Lévy processes with applications. Introductory lectures. Second edition. Universitext. Springer, Heidelberg, 2014.
[4] Stroock, D. W. Probability theory. An analytic view. Second edition. Cambridge University Press, Cambridge, 2011.

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