Normal bases and irreducible polynomials

Hua Huang, Shanheng Han, Wei Cao*

Department of Mathematics, Ningbo University, Ningbo, Zhejiang 315211, P.R. China

Abstract

Let $\mathbb{F}_q$ denote the finite field of $q$ elements and $\mathbb{F}_{q^n}$ the degree $n$ extension of $\mathbb{F}_q$. A normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is a basis of the form $\{\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}}\}$. An irreducible polynomial in $\mathbb{F}_q[x]$ is called an $N$-polynomial if its roots are linearly independent over $\mathbb{F}_q$. Let $p$ be the characteristic of $\mathbb{F}_q$. Pelis et al. showed that every monic irreducible polynomial with degree $n$ and nonzero trace is an $N$-polynomial provided that $n$ is either a power of $p$ or a prime different from $p$ and $q$ is a primitive root modulo $n$. Chang et al. proved that the converse is also true. By comparing the number of $N$-polynomials with that of irreducible polynomials with nonzero traces, we present an alternative treatment to this problem and show that all the results mentioned above can be easily deduced from our main theorem.

Key words: Finite field, Normal basis, $N$-polynomial, $q$-polynomial

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1. Introduction

Let $p$ be a prime number, $n \geq 2$ be an integer. Let $\mathbb{F}_q$ denote the finite field of $q$ elements with characteristic $p$, and $\mathbb{F}_{q^n}$ be its extension of degree $n$. A normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is a basis of the form $\{\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}}\}$, i.e., a basis consisting of all the algebraic conjugates of a fixed element. We say that $\alpha$ generates a normal basis, or $\alpha$ is a normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. In either case we are referring to the fact that the elements $\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}}$ are linearly independent over $\mathbb{F}_q$. In 1850, Eisenstein [3] first conjectured the existence of normal bases for finite fields, and its proof was given by Schönemann [13] later in 1850 for the case $\mathbb{F}_p$ and then by Hensel [6] in 1888 for arbitrary finite fields. Normal bases

*Corresponding author.
Email address: caowei@nbu.edu.cn (Wei Cao)

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over finite fields have proved very useful for fast arithmetic computations with potential applications to coding theory and to cryptography, see, e.g., [4, 7, 8].

An irreducible polynomial in \( F_q[x] \) is called an \( N \)-polynomial if its roots are linearly independent over \( F_q \). The minimal polynomial of any element in a normal basis \( \alpha, \alpha^q, \ldots, \alpha^{q^n-1} \) is \( m(x) = \prod_{i=0}^{n-1} (x - \alpha^q) \in F_q[x] \), which is irreducible over \( F_q \). The elements in a normal basis are exactly the roots of an \( N \)-polynomial. Hence an \( N \)-polynomial is just another way of describing a normal basis. In general, it is not easy to check whether an irreducible polynomial is an \( N \)-polynomial. However in certain cases, the thing may be very simple according to Theorems 1.1 and 1.2 below.

**Theorem 1.1.** (Pelis [11]) Let \( n = p^e \) with \( e \geq 1 \). Then an irreducible polynomial \( f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in F_q[x] \) is an \( N \)-polynomial if and only if \( a_1 \neq 0 \).

**Theorem 1.2.** (Pei, Wang and Omura [10]) Let \( n \) be a prime different from \( p \) and \( q \) be a primitive root modulo \( n \). Then an irreducible polynomial \( f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in F_q[x] \) is an \( N \)-polynomial if and only if \( a_1 \neq 0 \).

In 2001, Chang, Truong and Reed [2] furthermore proved that the conditions in Theorems 1.1 and 1.2 are also necessary.

**Theorem 1.3.** If every irreducible polynomial \( f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in F_q[x] \) with \( a_1 \neq 0 \) is an \( N \)-polynomial, then \( n \) is either a power of \( p \) or a prime different from \( p \) and \( q \) is a primitive root modulo \( n \).

By comparing the number of \( N \)-polynomials with that of irreducible polynomials over \( F_q \), we will present an alternative treatment to this problem. Throughout the rest of the paper, write \( n = mp^e \) with \( p \nmid m \). It will be seen that Theorems 1.1, 1.2 and 1.3 are all the direct consequences of the following theorem.

**Theorem 1.4.** (Main Theorem) The following inequality holds

\[
q^{n-m} \prod_{d|m} (q^{\tau(d)} - 1)^{\phi(d)/\tau(d)} \leq \frac{q-1}{q} \sum_{d|m} \mu(d)q^{n/d},
\]

where \( \tau(d) \) is the order of \( q \) modulo \( d \), \( \phi(d) \) is the Euler totient function, and \( \mu(d) \) is the Möbius function. Furthermore, (1) becomes an equality if and only if \( n = p^e \), or \( n \) is a prime different from \( p \) and \( q \) is a primitive root modulo \( n \).
2. Two counting formulae and partial proof of Theorem 1.4

In this section, we will first explain two counting formulae appearing in Theorem 1.4 and then give a partial proof to it. Let $v(n, q)$ denote the number of normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Hensel [6] and Ore [9] obtained an expression of $v(n, q)$ by the factorization of $x^n - 1$. Akbik [1] and Gathen and Giesbrecht [5] gave the explicit formula for $v(n, q)$ as follows, which need not factorize $x^n - 1$:

**Theorem 2.1.** The number of normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is given by

$$v(n, q) = q^{n-m} \prod_{d|\tau(m)} (q^{\tau(d)} - 1)^{\phi(d)/\tau(d)}.$$

Since every element in a normal basis generates the same basis, we get

**Corollary 2.2.** The number of normal bases of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is given by

$$\frac{v(n, q)}{n} = q^{n-m} \prod_{d|\tau(m)} (q^{\tau(d)} - 1)^{\phi(d)/\tau(d)}.$$

The trace of a degree $n$ polynomial $f(x)$ over $\mathbb{F}_q$ is defined to be the coefficient of $x^{n-1}$. For a given $t \in \mathbb{F}_q^*$, let $I_q(n, t)$ denote the number of monic irreducible polynomials of degree $n$ over $\mathbb{F}_q$ with trace $t$. Relying on a generalized Möbius inversion formula, Ruskey, Miers and Sawada [12] showed that

**Theorem 2.3.** Let $t \in \mathbb{F}_q^*$, then

$$I_q(n, t) = \frac{1}{qn} \sum_{d|m} \mu(d) q^{n/d}. \quad (2)$$

Observing that $I_q(n, t)$ in (2) is independent of $t$, we have

**Corollary 2.4.** The number of monic irreducible polynomials of degree $n$ over $\mathbb{F}_q$ with nonzero traces is given by

$$\sum_{t \in \mathbb{F}_q^*} I_q(n, t) = q - 1 \frac{\sum_{d|m} \mu(d) q^{n/d}}{qn}. $$

**Partial Proof of Theorem 1.4** Inequality (1) comes from the fact that an $N$-polynomials must be an irreducible polynomial with nonzero trace and Corollaries 2.2 and 2.4. Let $\mathcal{L}$ and $\mathfrak{R}$ denote respectively the left- and right-hand sides of
For the “if” part, it is trivial to verify. So we focus on the “only if” part. Given an integer \( t \), denote by \( v_q(t) \) the \( q \)-adic valuation of \( t \). Assume \( n = mp^e \) with \( e \geq 1 \) and \( \mathfrak{L} = \mathfrak{R} \). We will briefly listed below; refer to [7, Chapters 2-3] for details. A polynomial of the form \( L(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{F}_q[x] \) is called a \( q \)-polynomial (or a linearized polynomial), and its conventional \( q \)-associate is defined to be \( l(x) = \sum_{i=0}^{n} c_i x^i \). Given two \( q \)-polynomials \( L_1(x) \) and \( L_2(x) \), we define symbolic multiplication by \( L_1(x) \otimes L_2(x) = L_1(L_2(x)) \). Similarly, we can define symbolic division, symbolic factorization, symbolic irreducibility, etc. for \( q \)-polynomials.

**Theorem 3.1.** ([7, Lemma 3.59]) Let \( L_1(x) \) and \( L_2(x) \) be \( q \)-polynomials over \( \mathbb{F}_q \) with conventional \( q \)-associates \( l_1(x) \) and \( l_2(x) \). Then \( l(x) = l_1(x)l_2(x) \) and \( L(x) = L_1(x) \otimes L_2(x) \) are \( q \)-associates of each other.

The following criterion is an immediate consequence of the theorem above.

**Corollary 3.2.** Let \( L_1(x) \) and \( L(x) \) be \( q \)-polynomials over \( \mathbb{F}_q \) with conventional \( q \)-associates \( l_1(x) \) and \( l(x) \). Then \( L_1(x) \) symbolically divides \( L(x) \) if and only if \( l_1(x) \) divides \( l(x) \). In particular, \( L(x) \) is symbolically irreducible over \( \mathbb{F}_q \) if and only if \( l_1(x) \) is irreducible over \( \mathbb{F}_q \).

Let \( L(x) \in \mathbb{F}_q[x] \) be a nonzero \( q \)-polynomial with \( q^n \) simple roots and let

\[
\mathfrak{S} : \quad L(x) = L_1(x) \otimes \cdots \otimes L_1(x) \otimes \cdots \otimes L_r(x) \otimes \cdots \otimes L_r(x)
\]

(3)
be the symbolic factorization of $L(x)$ with pairwise relatively prime (not necessarily symbolically irreducible) $q$-polynomials $L_i(x)$ over $\mathbb{F}_q$ and $\deg(L_i(x)) = q^{n_i}$. For $1 \leq i \leq r$, denote by $K_i(x)$ the polynomial obtained from the symbolic factorization $\mathfrak{F}$ in (3) of $L(x)$ by omitting the symbolic factor $L_i(x)$. Clearly, $\deg(K_i(x)) = q^{n_i}$. Let $S_{\mathfrak{F}}(L)$ be the set of the roots of $L(x)$ that are not the roots of some $K_i(x)$. By the inclusion-exclusion principle, the formula for $|S_{\mathfrak{F}}(L)|$ is given by

$$
|S_{\mathfrak{F}}(L)| = q^n - \sum_{i=1}^{r} q^{n_i} + \sum_{1 \leq i < j \leq r} q^{n_i-n_j} - \cdots + (-1)^r q^{n_{i_1} - \cdots - n_r} = q^n (1 - q^{-n_1}) \cdots (1 - q^{-n_r}).
$$

The expression can also be interpreted in a different way. Let $l(x)$ be the conventional $q$-associate of $L(x)$. Then

$$
f : \quad l(x) = l_1(x)^{e_1} l_2(x)^{e_2} \cdots l_r(x)^{e_r}
$$

is the canonical factorization of $l(x)$ in $\mathbb{F}_q[x]$, where $l_i(x)$, the conventional $q$-associate of $L_i(x)$, are pairwise relatively prime (not necessarily irreducible) and $\deg(l_i(x)) = n_i$. Let $S_f(l)$ denote the set of polynomials in $\mathbb{F}_q[x]$ that are of smaller degree than $l(x)$ as well as not divisible by any $l_i(x)$ for all $1 \leq i \leq r$ in the factorization $f$ in (5). The generalized Euler’s $\phi$-function for $l(x)$ under such factorization is defined to be $\Phi_f(l) := |S_f(l)|$.

**Remark 3.3.** The sets $S_{\mathfrak{F}}(L)$ and $S_f(l)$ depend on the concrete factorizations $\mathfrak{F}$ and $f$, respectively. If all the polynomials $L_i(x)$ in the symbolic factorization $\mathfrak{F}$ are symbolically irreducible, then all the polynomials $l_i(x)$ in the factorization $f$ are irreducible in the ordinary sense by Corollary 3.2 and we simply write $\Phi(l) := \Phi_f(l)$ for this case which coincides with the original definition in [7, p. 122].

**Theorem 3.4.**

(i) $|S_{\mathfrak{F}}(L)| = \Phi_f(l)$.

(ii) Suppose that $l_1(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are relatively prime with $\deg(g) \geq 1$ and $\deg(h) \geq 1$. Then for the new factorization

$$
f' : \quad l(x) = g(x)^{e_1} h(x)^{e_2} \cdots l_r(x)^{e_r}
$$

we have $\Phi_{f'}(l) < \Phi_f(l)$. In particular, $\Phi(l) \leq \Phi_f(l)$.

**Proof.** (i) The proof is similar to the proof of [7, Lemma 3.69 (iii)].

(ii) Clearly, $S_{\mathfrak{F}}(l) \subset S_f(l)$ and $g(x) \in S_f(l)$ but $g(x) \not\in S_{\mathfrak{F}}(l)$. So $S_{\mathfrak{F}}(l) \subsetneq S_f(l)$ and hence $\Phi_{f'}(l) < \Phi_f(l)$.
Recall the assumption that \( n = mp^e \) with \( p \nmid m \) and it remains to consider the case that \( e = 0 \) and \( n \) is square free.

**Lemma 3.5.** ([7, Theorems 2.45 and 2.47]) For \( e = 0 \), i.e., \( p \nmid n \), we have

\[
x^n - 1 = \prod_{d|n} \Phi_d(x) = \prod_{d|n} \prod_{j=1}^{\phi(d)/\tau(d)} h_{d,j}(x),
\]

where \( \Phi_d(x) \) denotes the \( d \)th cyclotomic polynomial of degree \( \phi(d) \), and has \( \phi(d)/\tau(d) \) distinct monic irreducible factors \( h_{d,j}(x) \), each of degree \( \tau(d) \).

**Continue Proof of Theorem 1.4** Let \( n = p_1 \ldots p_k \) where \( p_i \) are distinct primes different from \( p \), \( k \geq 2 \) or \( k = 1 \) but \( q \) is not a primitive root modulo \( n \). We need to show \( \mathcal{L} < \mathcal{R} \) where \( \mathcal{L} \) and \( \mathcal{R} \) denote the left- and right-hand sides of (1), respectively. By Theorem 3.4 and Lemma 3.5, we have

\[
\Phi(x^n - 1) = \prod_{d|n} (q^{\tau(d)} - 1) \phi(d)/\tau(d) \leq \prod_{d|n} (q^{\phi(d)} - 1). \tag{6}
\]

Furthermore, the second inequality in (6) becomes a strict inequality if \( \phi(d) > \tau(d) \) for some divisor \( d \), especially, \( k = 1 \) but \( q \) is not a primitive root modulo \( n \). Therefore it suffices to prove the strict inequality

\[
\prod_{d|n} (q^{\phi(d)} - 1) < \frac{q-1}{q} \sum_{d|n} \mu(d)q^{n/d} \tag{7}
\]

holds for \( k \geq 2 \). Let \( \Psi_d(x) \) denote the linearized \( q \)-associate of \( \Phi_d(x) \). By Theorem 3.1, the corresponding symbolic factorization of \( x^n - 1 = \prod_{d|n} \Phi_d(x) \) is

\[
\mathfrak{F} : \quad x^{q^n} - x = \bigotimes_{d|n} \Psi_d(x). \tag{8}
\]

Let \( \alpha \in S_{\mathfrak{F}}(x^{q^n} - x) \). We claim three assertions as below:

(i) The trace of \( \alpha \) is nonzero, i.e., \( \alpha q^{n-1} + \cdots + \alpha + \alpha \neq 0 \).
(ii) The degree of \( \alpha \) is \( n \), i.e., \( n \) is the least positive integer \( t \) such that \( \alpha^q = \alpha \).
(iii) There exists at least one element \( \beta \notin S_{\mathfrak{F}}(x^{q^n} - x) \) that satisfies both the above conditions (i) and (ii).
Assume that $\alpha^{q^{n-1}} + \cdots + \alpha^q + \alpha = 0$. It follows that $\alpha$ is a root of $\bigotimes_{d|n} \Psi_d(x) = x^{nq-1} + \cdots + x^q + x$ and this contradicts to the definition of $S_{\overline{G}}(x^{q^n} - x)$. So assertion (i) is proved. Now assume that $l < n$ is the least positive integer $t$ such that $\alpha^{q^t} = \alpha$. Then we have $l|\frac{n}{p^j}$ for some $1 \leq i \leq k$ and hence $\alpha$ is a root of $\bigotimes_{d|n} \Psi_d(x) = x^{\frac{n}{p^j}}^{q^j} - x$, again contradicting to the definition of $S_{\overline{G}}(x^{q^n} - x)$. So assertion (ii) is true. Assertions (i) and (ii) means that $\alpha$ and its conjugates over $F_{q^n}$ form the roots of an irreducible polynomial with nonzero trace over $F_q$. Thus by Corollary 2.4 and Theorem 3.4, we have

$$\#S_{\overline{G}}(x^{q^n} - x) = \prod_{d|n}(q^{\varphi(d)} - 1) \leq \frac{q-1}{q} \sum_{d|n} \mu(d)q^{n/d}. \quad (9)$$

To prove assertion (iii), we observe that the element $\alpha \in S_{\overline{G}}(x^{q^n} - x)$ also satisfies other conditions besides (i) and (ii), e.g., $\Psi_1(\alpha) \bigotimes \Psi_n(\alpha) \neq 0$ for $k \geq 2$. Set $A = \{ \beta \in F_{q^n} : \Psi_1(\beta) \bigotimes \Psi_n(\beta) = 0 \}, A_0 = \{ \beta \in A : \bigotimes_{1 < d|n} \Psi_d(\beta) = 0 \}$, and $A_i = \{ \beta \in A : \bigotimes_{d|n} \Psi_d(\beta) = 0 \}$ for $i = 1, \ldots, k$. Then by the inclusion-exclusion principle, we obtain

$$|A \setminus \bigcup_{i=0}^k A_i| = q^{1+\varphi(n)} - q^{\varphi(n)} + \sum_{i=1}^k \binom{k}{i} (-1)^i q^{1+\varphi(n)} - q^{\varphi(n)} - q \geq 1.$$  

This proves assertion (iii) which says that (9) can not be an equality. Thus the strict inequality (7) holds. The proof is finished.

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