The $n$-tuple laws

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$^1$Based on ‘Exact and asymptotic $n$-tuple laws at first and last passage’ by K., Pardo and Rivero, to appear in Annals of Applied Probability
Lévy processes: "nice theory, no examples of the complicated stuff"?
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- If $X$ is a two-sided jumping strictly stable process with index $\alpha \in (0, 2)$ and positivity constant $\rho = \mathbb{P}(X_t \geq 0) \in (0, 1)$ then:
  For $x \in (0, b)$, $u \in [0, b - x]$, $v \in [u, b)$ and $y > 0$,

$$
\mathbb{P}_x(b - \overline{X}_{\tau^+_b} \in du, b - X_{\tau^+_b} \in dv, X_{\tau^+_b} - b \in dy, \tau^+_b < \tau^-_0) = \frac{\sin(\pi \alpha \rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1 - \rho))} \times \frac{x^{\alpha(1 - \rho)}(b - x - u)^{\alpha \rho - 1}(v - u)^{\alpha(1 - \rho) - 1}(b - v)^{\alpha \rho}}{(b - u)^\alpha (y + v)^{\alpha + 1}} du \, dv \, dy.
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- More interestingly, how would you prove such a result? And can you get more examples out of such a proof?
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- In this talk: address the previous bullet point by examining ‘$n$-tuple laws’ for Lévy processes and positive self-similar Markov processes.
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- Standard theory allows us to construct a local time at zero, say $L$, for the strong Markov Process $\overline{X} - X$. Then defining $H_t = X_{L_t^{-1}}$ (with the formality $H_\infty := \infty$) gives us the ascending ladder height processes $(L^{-1}, H)$. The pair $(L^{-1}, H)$ is a (killed) bivariate subordinator with potential measure denoted by

$$V(ds, dx) = \int_0^\infty dt \cdot P(L_t^{-1} \in ds, H_t \in dx)$$
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- The ladder processes has (amongst other things) hidden information about the distribution of $\overline{X}_t$, $\tau_x^+$ and

$$\overline{G}_t = \sup\{s < t : X_s = \overline{X}_s\}.$$
The quintuple law at first passage
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**Theorem (Doney and K. 2006)**

For each \( x > 0 \) we have on \( u > 0, v \geq y, y \in [0, x], s, t \geq 0, \)

\[
P(\tau^+_x - \overline{G}^+_x \in dt, \overline{G}^+_x \in ds, X^+_{\tau_x} - x \in du, x - X^+_{\tau_x} \in dv, x - \overline{X}^+_{\tau_x} \in dy) \\
= V(ds, x - dy)\overline{V}(dt, dv - y)\Pi(du + v)
\]

where the equality holds up to a multiplicative constant.
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- Assume that $X$ does not drift to $-\infty$ under $\mathbb{P}$. 
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- We can define a new law $P^\uparrow_x$ on the space of non-negative cadlag paths initialized at $x > 0$ via the semi-group

$$P^\uparrow_x(X_t \in dz) = \frac{\hat{V}(z)}{\hat{V}(x)} P_x(X_t \in z; \tau_0^- > t)$$

where $z > 0$ and $\hat{V}(z) = \hat{V}(\mathbb{R}_+, [0, z])$. 

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- Work of Bertoin, Chaumont, Doney and others help us justify the claim that $(X, \mathbb{P}^\uparrow_x)$ as a Doob $h$-transform is the result of ”conditioning” $X$ to stay non-negative.
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- Moreover, in the sense of weak convergence with respect to the Skorohod topology, they have also shown that $\mathbb{P}^\uparrow := \lim_{x \downarrow 0} \mathbb{P}_x$ is well defined.
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- The Tanaka-Doney pathwise construction of $(X, P^\uparrow)$ from $(X, P)$ replaces excursions of $X$ from $\overline{X}$ by their time-reversed dual.
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- Moreover, in the sense of weak convergence with respect to the Skorohod topology, they have also shown that $P^\uparrow := \lim_{x \downarrow 0} P_x$ is well defined.
- The Tanaka-Doney pathwise construction of $(X, P^\uparrow)$ from $(X, P)$ replaces excursions of $X$ from $-X$ by their time-reversed dual.
- Tanaka-Doney construction of $P^\uparrow$ together with the quintuple law at first passage gives us a quintuple law at last passage.
The quintuple law at last passage
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Let

\[ X_t = \inf\{X_s : s \geq t\} \]

be the future infimum of \( X \),

\[ D_t = \inf\{s > t : X_s - X_t = 0\} \]

is the right end point of the excursion of \( X \) from its future infimum straddling time \( t \). Now define the last passage time

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Theorem

Suppose that \( X \) is a Lévy process which does not drift to \(-\infty\). For \( s, t \geq 0 \), \( 0 < y \leq x \), \( w \geq u > 0 \),

\[ \mathbb{P}^{\uparrow} ( \rightarrow D_{U_x} - U_x \in dt, U_x \in ds, \rightarrow X_{U_x} - x \in du, x - X_{U_x} - \in dy, X_{U_x} - x \in dw ) \]

\[ = V(ds, x - dy) \hat{V}(dt, w - du) \Pi(dw + y) \]

where the equality hold up to a multiplicative constant.
\[ \{ \frac{D}{U_x} - U_x \in dt, \ U_x \in ds, \ X_{U_x} - x \in du, \ x - X_{U_x} \in dy, \ X_{U_x} - x \in dw \} \]
The $n$-tuple laws

....and then one can start showing off and get septuple laws...
Corollary

Suppose that $X$ is a Lévy process which does not drift to $-\infty$. For $t > 0$, $x \geq z > 0$, $s > r > 0$, $0 \leq v \leq z \land x$, $0 < y \leq x - v$, $w \geq u > 0$,

\[
P^\uparrow_z (G_\infty \in dr, X_\infty \in dv, \rightarrow_{U_x} U_x \in dt, \\
U_x \in ds, X_{U_x} - x \in du, x - X_{U_x} - \in dy, X_{U_x} - x \in dw)
\]

\[= \hat{V}(z)^{-1} \hat{V}(dr, z - dv) V(ds - r, x - v - dy) \hat{V}(dt, w - du) \Pi(dw + y)
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■ Corollary

Suppose that $X$ is a Lévy process which does not drift to $-\infty$. For $t > 0, x \geq z > 0, s > r > 0, 0 \leq v \leq z \wedge x, 0 < y \leq x - v, w \geq u > 0,$

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■ Corollary

Suppose that $X$ is a Lévy process which drifts to $\infty$. For $t, x, v > 0, s > r > 0, 0 \leq y < x + v, w \geq u > 0,$

$$
\mathbb{P}(G_\infty \in dr, -X_\infty \in dv, D_{U_x} - U_x \in dt, U_x \in ds, X_{U_x} - x \in du, x - X_{U_x} \in dy, X_{U_x} - x \in dw)
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where the equality holds up to a multiplicative constant.
Playing with an idea of Lamperti, Caballero and Chaumont

Suppose that $X$ is a two-sided strictly stable process with index $\alpha \in (1, 2)$ and positivity parameter $\rho = \mathbb{P}(X_t \geq 0) \in (0, 1)$, then the following facts are known:
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- Its jump measure is given by

$$\Pi(dx) = 1_{(x>0)} \frac{c_+}{x^{1+\alpha}} dx + 1_{(x<0)} \frac{c_-}{|x|^{1+\alpha}} dx$$
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- Its renewal measures $V(dx) := V(\mathbb{R}_+, dx)$ and $\hat{V}(x) := \hat{V}(\mathbb{R}_+, dx)$ are known
  \[
  V(dx) = \frac{x^{\alpha \rho - 1}}{\Gamma(\alpha \rho)} dx \quad \text{and} \quad \hat{V}(dx) = \frac{x^{\alpha(1-\rho) - 1}}{\Gamma(\alpha(1 - \rho))} dx.
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and

$$\hat{V}(dx) = \frac{x^{\alpha(1-\rho) - 1}}{\Gamma(\alpha(1-\rho))} dx.$$ 

- The process $(X, \mathbb{P}_x^\uparrow)$ is a positive self-similar Markov process with index $\alpha$ meaning for $k > 0$, the law of $(kX_{k^{-\alpha}t}, t \geq 0)$ under $\mathbb{P}_x^\uparrow$ is $\mathbb{P}_{kx}^\uparrow$ and that it respects the Lamperti representation

$$X_t = x \exp\{\xi_{\theta(tx^{-\alpha})}\}$$

where $\theta(t) = \inf\{s \geq 0 : \int_0^s \exp\{\alpha \xi_u\} du > t\}$. and $\xi$ is a Lévy process.
The Lamperti-stable process $\xi$

Many characteristics can be computed explicitly (see eg Caballero and Chaumont 2006).
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$$\nu(dx) = 1_{(x>0)} \frac{c_+ e^{(\alpha(1-\rho)+1)x}}{(e^x - 1)^{\alpha+1}} \, dx + 1_{(x<0)} \frac{c_- e^{-\alpha \rho x}}{(e^{-x} - 1)^{\alpha+1}} \, dx.$$
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- Using the law of the global infimum of a conditioned Lévy process applied to $(X, \mathbb{P}_x^\uparrow)$ one computes the law of the global infimum of $\xi$ by the Lamperti-transformation and thereby obtains

$$\hat{V}([0, x]) = (1 - e^{-x})^{\alpha(1-\rho)}.$$
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- One then uses Vigon’s equations amicales to give us an expression for the jump measure of $H$: $\Pi_H(x, \infty) = \int_0^\infty \widehat{V}(du) \nu(u + x, \infty)$ from which it turns out to be easy to compute the potential

$$V(dx) = \frac{\sin(\pi \alpha \rho)}{\pi \Gamma(\alpha + 1)} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1 - \rho) + 1)} (1 - e^{-x})^{\alpha \rho - 1} dx.$$
Generating new identities playing $X$ off against $\xi$

Note that none of the aforementioned $n$-tuple laws mentioned before accommodate for first passage of $(X, \mathbb{P}_x)$, nor the two sided exit problem.
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Note that none of the aforementioned $n$-tuple laws mentioned before accommodate for first passage of $(X, \mathbb{P}_x^\uparrow)$, nor the two sided exit problem.

- The quintuple law at first passage for $\xi$ marginalized to a triple law give us: For $y \in [0, x]$, $v \geq y$ and $u > 0$,

$$
\mathbb{P}(\xi_{\tau_x}^+ - x \in du, x - \xi_{\tau_x}^- \in dv, x - \xi_{\tau_x}^{\uparrow} \in dy)
\begin{align*}
&= \frac{\sin(\pi \alpha \rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1 - \rho))}
\left(1 - e^{-x+y}\alpha \rho - 1\right)
\left(1 - e^{-v+y}\alpha(1 - \rho) - 1\right)
\cdot e^{-v+y} e^{(\alpha(1 - \rho) + 1)(u+v)} (e^{u+v} - 1)^{-\alpha - 1} dy dv du.
\end{align*}
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\mathbb{P}(\xi_{\tau_x^+} - x \in du, x - \xi_{\tau_x^+} \in dv, x - \xi_{\tau_x^+} \in dy) = \frac{\sin(\pi \alpha \rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1 - \rho))} (1 - e^{-x+y})^{\alpha \rho - 1} (1 - e^{-v+y})^{\alpha(1-\rho) - 1} \\
\cdot e^{-v+y} e^{(\alpha(1-\rho)+1)(u+v)} (e^{u+v} - 1)^{-\alpha - 1} dy \ dv \ du.
$$

- Using the Lamperti representation this translates into a first passage problem for $(X, \mathbb{P}_x^\uparrow)$. Let $b > x > 0$. For $u \in [0, b - x]$, $v \in [u, b)$ and $y > 0$,

$$
\mathbb{P}_x^\uparrow(b - X_{\tau_b^+} \in du, b - X_{\tau_b^+} \in dv, X_{\tau_b^+} - b \in dy) = \frac{\sin(\pi \alpha \rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1 - \rho))} \\
\times \frac{(b - x - u)^{\alpha \rho - 1} (v - u)^{\alpha(1-\rho) - 1} (b - v)^{\alpha \rho} (y + b)^{\alpha(1-\rho)}}{(b - u)^{\alpha} (y + v)^{\alpha+1}} du \ dv \ dy,
$$
Generating new identities playing $X$ off against $\xi$
The \( \pi \)-tuple laws

Generating new identities playing \( X \) off against \( \xi \)

- Recalling \( \mathbb{P}_x^\uparrow (X_t \in dz) = (z/x)^{\alpha(1-\rho)} \mathbb{P}_x (X_t \in dz, t < \tau_0^-) \) we can use the last identity to deduce

\[
\mathbb{P}_x (b - X_{\tau_b^+}^+ \in du, b - X_{\tau_b^-}^- \in dv, X_{\tau_b^+} - b \in dy, \tau_b^+ < \tau_0^-) = \frac{\sin(\pi \alpha \rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1-\rho))} \\
\times \frac{x^{\alpha(1-\rho)} (b - x - u)^{\alpha \rho - 1} (v - u)^{\alpha(1-\rho) - 1} (b - v)^{\alpha \rho}}{(b - u)^{\alpha} (y + v)^{\alpha+1}} \, du \, dv \, dy.
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- Recalling $\mathbb{P}_x^\uparrow(X_t \in dz) = (z/x)^{\alpha(1-\rho)} \mathbb{P}_x(X_t \in dz, t < \tau_0^-)$ we can use the last identity to deduce

\[
\mathbb{P}_x(b - \bar{X}_{\tau_b^+} \in du, b - X_{\tau_b^+} \in dv, X_{\tau_b^+} - b \in dy, \tau_b^+ < \tau_0^-) = \frac{\sin(\pi \alpha \rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1 - \rho))} \\
\times \frac{x^{\alpha(1-\rho)}(b - x - u)^{\alpha \rho - 1}(v - u)^{\alpha(1-\rho)-1}(b - v)^{\alpha \rho}}{(b - u)^{\alpha}(y + v)^{\alpha+1}} du \, dv \, dy.
\]

- We can bring this identity back through the Doob h-transforms relating $\mathbb{P}_x^\uparrow$ and $\mathbb{P}_x$ and through the Lamperti transformation to give (with obvious notation): For $\theta \in [0, b], \theta \leq \phi < b - u$ and $\eta > 0$

\[
\mathbb{P}\left(b - \bar{\xi}_{T_b^+} \in d\theta, b - \xi_{T_b^+} \in d\phi, \xi_{T_b^+} - b \in d\eta, T_b^+ < T_u^-\right) \\
= \frac{\sin(\pi \alpha \rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha(1 - \rho))} e^b(1 - e^u)^{\alpha(1-\rho)} e^{-\theta - \phi} e^{(\alpha(1-\rho)+1)\eta} (e^{b-\theta} - 1)^{\alpha \rho - 1} (e^{-\theta} - e^{-\phi})^{\alpha \rho} \\
\times (e^{b-\phi} - e^u)^{\alpha \rho} (e^{b-\theta} - e^u)^{-\alpha} (e^{\eta} - e^{-\phi})^{-\alpha - 1} d\theta \, d\phi \, d\eta.
\]