On Some Summability Methods for a $q$-Analogue of an Integral Type Operator Based on Multivariate $q$-Lagrange Polynomials†

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ABSTRACT
The present paper considers a $q$-analogue of an operator defined by Erkus¸-Duman et al. (Calcolo, 45(1):53–67, 2008) involving $q$-Lagrange polynomials in several variables. The Korovkin type theorems in the settings of deferred weighted $A$-statistical convergence and the power series method are investigated.

1. Introduction
The past two decades have witnessed to the constructions of linear positive operators by means of multivariate-Lagrange polynomials (and their $q$-analogue) and to the investigation of their approximation behavior. Let $(\mathcal{C}(I), ||.||)$ be the Banach space of all continuous functions on $I = [0, 1]$ with the sup-norm $||.||$. After the introduction of celebrated multivariate Lagrange polynomials (widely known as Chan-Chyan-Srivastava polynomials) [1], for $f \in \mathcal{C}(I)$, Erkus et al. [2] defined the following sequence of linear positive operators as;
Altin et al. [3] proposed \( q \)-multivariable Lagrange polynomials in the following manner:

\[
\Omega_n^{\beta^{(1)}, \ldots, \beta^{(r)}}(f(s); x) = \left\{ \prod_{k=1}^{r} \left( 1 - x \beta_n^{(k)} \right)^n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+l_2+\ldots+l_r=p} f \left( \frac{l_r}{n + l_r - 1} \right) \prod_{s=1}^{r} \left( \beta_n^{(s)} \right)^{l_s} \frac{(n)_l}{l_s!} \right\} x^p, \tag{1.1}
\]

where \( x \in I, \beta^{(j)} = \langle \beta_n^{(j)} \rangle \) are sequences in \((0, 1)\) for each \( j = 1, 2, \ldots, r \), and \( (\rho)_s \) denotes the standard Pochhammer symbol. The authors studied the approximation properties of the above operator in the \( A \)-statistical settings. For every \( q \in \mathbb{R} \) such that \(|q| < 1\) and \( n \in \mathbb{N}^0 = \{0, 1, 2, \ldots\} \), the \( q \)-Pochhammer symbol \( (\rho; q)_n \) is given by

\[
(\rho; q)_n = \begin{cases} 
1, & \text{if } n = 0, \\
(1-\rho)(1-\rho q) \cdots (1-\rho q^{n-1}), & \text{if } n \in \mathbb{N},
\end{cases}
\]

and the \( q \)-analogue of a natural number \((q\text{-integer})\) is defined by

\[
[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.
\]

Altin et al. [3] proposed \( q \)-multivariable Lagrange polynomials \( h_{n,q}^{(\eta_1, \ldots, \eta_r)}(z_1, z_2, \ldots, z_r) \) as follows,

\[
h_{n,q}^{(\eta_1, \ldots, \eta_r)}(z_1, z_2, \ldots, z_r) = \sum_{l_1+l_2+\ldots+l_r=n} \left\{ \prod_{k=1}^{r} (q^{\eta_k}, q)_l \frac{(z_k)_l}{(q, q)_l} \right\}. \tag{1.2}
\]

The above \( q \)-multivariate polynomials have the generating function of the following form

\[
\prod_{k=1}^{r} \frac{1}{(t^k; q)_{\eta_k}} = \sum_{n=0}^{\infty} h_{n,q}^{(\eta_1, \ldots, \eta_r)}(z_1, z_2, \ldots, z_r) t^n, \tag{1.3}
\]

where \(|t| < \min\{|z_1|^{-1}, \ldots, |z_r|^{-1}\} \).

Erkus-Duman et al. [4] proposed an integral type generalization of the operator (1.1) in the following manner:

\[
\mathcal{C}_n^{\beta^{(1)}, \ldots, \beta^{(r)}} (f(s); x) = \left\{ \prod_{k=1}^{r} \left( 1 - x \beta_n^{(k)} \right)^n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+l_2+\ldots+l_r=p} (n + l_r - 1) \left( \prod_{s=1}^{r} \left( \beta_n^{(s)} \right)^{l_s} \frac{(n)_l}{l_s!} \right) \int_{ \frac{l_r+1}{n+l_r-1} }^{ \frac{l_r}{n+l_r-1} } f(s) ds \right\} x^p, \tag{1.4}
\]

and studied its statistical approximation properties by means of modulus of continuity and Peetre’s \( K \)-functional. Using the generating function given
by (1.3), Erkuş-Duman [5] studied the following $q$-analogue of the operator $\mathcal{Q}_n^{\beta^{(1)}, \ldots, \beta^{(r)}}$

$$\mathcal{Q}_n^{\beta^{(1)}, \ldots, \beta^{(r)}} (f(s); x) = \left\{ \prod_{k=1}^{r} \frac{(x \beta_n^{(k)}; q)_n}{(q; q)_n (q; q)_l \cdots (q; q)_l} \sum_{p=0}^{\infty} \sum_{l_1 + \ldots + l_r + \ldots = p} \left( q^n; q \right)_{l_1} (q^n; q)_{l_2} \cdots (q^n; q)_{l_r} \right\} x^p,$$

(1.5)

It is to be noted that the $q$-analogue of the operator (1.1) was also independently proposed by Mursaleen et al. [6] but unfortunately their definition was shown to be incorrect by Behar et al. [7]. The authors [7] extended the study of Erkuş-Duman’s operator (1.5) to the bi-variate and GBS (Generalized Boolean Sum) cases.

The prime objective of this paper is to define a $q$-analogue of the operators (1.4) by means of Riemann type $q$-integral, and study the convergence of such operators via summability methods. In Sec. 2, we construct the operator and establish some important lemmas essential to prove the main results. In Sec. 3, the Korovkin type theorems in the deferred weighted $A$-statistical approximation are studied for these operators. In the last section, we establish the basic convergence theorem and an estimate of error in the approximation by using power series summability method.

2. Construction of the operators and important lemmas

Marinković et al. [8] introduced the following Riemann type $q$-integral

$$\int_{\alpha}^{\beta} f(s) dq_s^q = (1-q)(\alpha+\beta) \sum_{j=0}^{\infty} f(\alpha + (\beta - \alpha) q^j) q^j,$$  

(2.1)

where $\alpha, \beta, q \in \mathbb{R}$ such that $0 < \alpha < \beta$ and $q \in (0, 1)$. This definition of $q$-integral is appropriate to derive some $q$-analogs of well-known integral inequalities [9]. Using the $q$-Riemann type integral, for $f \in \mathcal{C}(I)$, we propose a $q$-analogue of the operators (1.4) as follows:

$$R_n^{\beta^{(1)}, \ldots, \beta^{(r)}} (f(s); x) = \left\{ \prod_{k=1}^{r} \frac{(x \beta_n^{(k)}; q)_n}{(q; q)_n (q; q)_l \cdots (q; q)_l} \sum_{p=0}^{\infty} \sum_{l_1 + \ldots + l_r + \ldots = p} \left( q^n; q \right)_{l_1} (q^n; q)_{l_2} \cdots (q^n; q)_{l_r} \right\} x^p.$$  

(2.2)
In order to derive the main results, we first prove the following auxiliary results.

**Lemma 1.** The operators $\mathcal{R}_{n,q}^{\beta^{(1)}, \ldots, \beta^{(r)}} (\cdot, x)$ verify the assertions:

(i) $\mathcal{R}_{n,q}^{\beta^{(1)}, \ldots, \beta^{(r)}} (1; x) = 1$;

(ii) $\mathcal{R}_{n,q}^{\beta^{(1)}, \ldots, \beta^{(r)}} (s; x) \leq x\beta_n^r + \frac{1}{[2]_q [n]_q}$. Moreover,

$$|\mathcal{R}_{n,q}^{\beta^{(1)}, \ldots, \beta^{(r)}} (s; x) - x| \leq x(1 - \beta_n^r) + \frac{1}{[2]_q [n]_q}.$$

(iii) $\mathcal{R}_{n,q}^{\beta^{(1)}, \ldots, \beta^{(r)}} (s^2; x) \leq \frac{1}{[3]_q [n]_q} + \frac{x\beta_n^r}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right) + q(x\beta_n^r)^2$. Also,

$$|\mathcal{R}_{n,q}^{\beta^{(1)}, \ldots, \beta^{(r)}} (s^2; x) - x^2| \leq \frac{1}{[3]_q [n]_q} + \frac{x\beta_n^r}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right) + 2x^2(1 - \beta_n^r).$$

**Proof.**

(i) Using the definition of $q$-Riemann integral defined by (2.1) and combining (1.2)–(1.3), we obtain the desired result.

(ii) From (1.5), we have

$$\mathcal{R}_{n,q}^{\beta^{(1)}, \ldots, \beta^{(r)}} (s; x) = \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \sum_{p=0}^{\infty} \left\{ \sum_{l_i + \ldots + l_p = p} (q^n; q)_{l_i} \cdots (q^n; q)_{l_p} |n + l_p - 1|_q q^{-l_p} \right\} \right\} x^p$$

$$\times \left( \frac{\beta^{(1)}_n}{(q; q)_{l_1} \cdots (q; q)_{l_p}} \frac{1}{[n]_q} \sum_{l_i + \ldots + l_p = p} \left\{ q^n [l_1]_q + \frac{q^{2n}}{[2]_q} \right\} \right) x^p$$

$$= \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \sum_{p=0}^{\infty} \left\{ \sum_{l_i + \ldots + l_p = p} (q^n; q)_{l_1} \cdots (q^n; q)_{l_p} \right\} \right\} x^p$$

$$\times \left( \frac{[l_1]_q}{[n + l_p - 1]_q} \frac{(\beta^{(1)}_n)^{l_1} \cdots (\beta^{(r)}_n)^{l_p}}{(q; q)_{l_1} \cdots (q; q)_{l_p}} \right) x^p$$

$$+ \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \sum_{p=0}^{\infty} \left\{ \sum_{l_i + \ldots + l_p = p} (q^n; q)_{l_i} \cdots (q^n; q)_{l_p} \right\} \right\} x^p \times \frac{q^{l_p}}{[2]_q [n + l_p - 1]_q} \frac{(\beta^{(1)}_n)^{l_1} \cdots (\beta^{(r)}_n)^{l_p}}{(q; q)_{l_1} \cdots (q; q)_{l_p}} x^p.$$

Since, $\frac{[l_1]_q}{[n]_q} = \frac{1}{(1-q)(q;q)_{l_1-1}}, \frac{(q^n; q)_{l_1}}{[n + l_p - 1]_q} = (1-q)(q^n; q)_{l_p-1}$ and $\frac{q^{l_p}}{[n + l_p - 1]_q} \leq$
\[
R_{n,q}^{\beta^{(1)},\ldots,\beta^{(r)}}(s;x) \leq x f_n^{(r)} \left\{ \prod_{k=1}^{r} (x f_n^{(k)} : q)_n \right\} \sum_{s=1}^{\infty} \left\{ \sum_{l_1+\ldots+l_r=p} \prod_{k=1}^{r} (q^n;q)_l \right\} x^{p-1} \\
+ \frac{1}{2[q]_q} \left\{ \prod_{k=1}^{r} (x f_n^{(k)} : q)_n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+\ldots+l_r=p} \prod_{k=1}^{r} (q^n;q)_l \right\} x^{p} \\
\leq x f_n^{(r)} \left\{ \prod_{k=1}^{r} (x f_n^{(k)} : q)_n \right\} \sum_{p=0}^{\infty} h_{p-1,q}^{(n,\ldots,n)} (\beta_n^{(1)},\beta_n^{(2)},\ldots,\beta_n^{(r)}) x^{p-1} \\
+ \frac{1}{2[q]_q} \left\{ \prod_{k=1}^{r} (x f_n^{(k)} : q)_n \right\} \sum_{p=0}^{\infty} h_{p,q}^{(n,\ldots,n)} (\beta_n^{(1)},\beta_n^{(2)},\ldots,\beta_n^{(r)}) x^{p}, \text{ from (1.2)} \\
\leq x f_n^{(r)} + \frac{1}{2[q]_q}, \text{ in view of (i)}. 
\]

As a consequence of the above inequality, we have
\[
R_{n,q}^{\beta^{(1)},\ldots,\beta^{(r)}}(s;x) - x \leq x (\beta_n^{(r)} - 1) + \frac{1}{2[q]_q}. \tag{2.3} 
\]

On the other hand, we have
\[
R_{n,q}^{\beta^{(1)},\ldots,\beta^{(r)}}(s;x) \geq \left\{ \prod_{k=1}^{r} (x f_n^{(k)} : q)_n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+\ldots+l_r=p} \prod_{k=1}^{r} (q^n;q)_l \right\} x^{p} \\
\times \left[ \frac{[l]_q}{[n+l-1]_q} \frac{(q^n;q)_l}{(q;q)_l} \right] x^{p} \\
\geq x f_n^{(r)} \tag{2.4} 
\]

Thus by combining Equations (2.3) and (2.4), we get
\[
| R_{n,q}^{\beta^{(1)},\ldots,\beta^{(r)}}(s;x) - x | \leq x (1 - \beta_n^{(r)}) + \frac{1}{2[q]_q}, 
\]

which proves the second assertion.

(iii) Considering the definition (1.5), we have
\[
R_{n,q}^{\beta^{(1)},\ldots,\beta^{(r)}}(s^2;x) = \left\{ \prod_{k=1}^{r} (x f_n^{(k)} : q)_n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+\ldots+l_r=p} \prod_{k=1}^{r} (q^n;q)_l \right\} x^{p} \\
\times [n+l-1]_q q^{-l} \left[ \frac{(q^n;q)_l}{(q;q)_l} \right] x^{p} \\
\times \left[ \frac{[n+l]_q}{[n]_q} \right] x^{p}, 
\]
Consequently,

$$s_{n,q}^{(r)}(t^2; x) = \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+\cdots+l_r=p} (q^n; q)_{l_1} \cdots (q^n; q)_{l_r} [n + l_r - 1]_q q^{-l_r} \right\} x^p$$

$$= x\beta_n^{(r)} \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+\cdots+l_{r-1}=p-1} (q^n; q)_{l_1} \cdots (q^n; q)_{l_{r-1}} \right\} x^{p-1}$$

$$+ \frac{2}{3} \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+\cdots+l_r=p} (q^n; q)_{l_1} \cdots (q^n; q)_{l_r} \right\} x^p$$

$$+ \frac{1}{2} \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \right\} \sum_{p=0}^{\infty} \left\{ \sum_{l_1+\cdots+l_r=p} (q^n; q)_{l_1} \cdots (q^n; q)_{l_r} \right\} x^p$$

$$= \sum_1 + \sum_2 + \sum_3 + \sum_4 \quad \text{say.}$$

Now,

$$\sum_1 = x\beta_n^{(r)} \left\{ \prod_{k=1}^{r} (x\beta_n^{(k)}; q)_n \right\} \sum_{p=1}^{\infty} \left\{ \sum_{l_1+\cdots+l_{r-1}=p-1} (q^n; q)_{l_1} \cdots (q^n; q)_{l_{r-1}} \right\} x^{p-1}$$

$$= \frac{1}{[n]_q} [n + l_r - 1]_q (q; q)_{l_1} \cdots (q; q)_{l_{r-1}}$$

$$\leq \frac{1}{[n]_q} \leq \frac{1}{[n]_q}$$

and (i). 

(2.5)

Also,
\[
\sum_{2} = qx \beta_n^{(r)} \left\{ \prod_{k=1}^{r} (x \beta_n^{(k)}; q)_n \right\} \sum_{p=1}^{\infty} \left\{ \sum_{l_1+\cdots+l_{p-1}=p-1}^{L \geq 1} (q^n; q)_{l_1} \cdots (q^n; q)_{l_{p-1}} \right\}
\times \frac{[l_r-1]_q}{[n+l_r-1]_q} \frac{(\beta_n^{(1)})_{l_r} \cdots (\beta_n^{(r)})_{l_r-1}}{(q; q)_{l_r} \cdots (q; q)_{l_1-1}} x^{p-1}
\]

\[
= q(x \beta_n^{(r)})^2 \left\{ \prod_{k=1}^{r} (x \beta_n^{(k)}; q)_n \right\} \sum_{p=2}^{\infty} \left\{ \sum_{l_1+\cdots+l_{p-2}=p-2}^{L \geq 1} (q^n; q)_{l_1} \cdots (q^n; q)_{l_{p-2}} \right\}
\times \frac{1-q^{n+L-2}}{1-q^n} \frac{(1-q^{n+L-2}) (\beta_n^{(1)})_{l_2} \cdots (\beta_n^{(r)})_{l_2-1}}{(q; q)_{l_2} \cdots (q; q)_{l_1-1}} x^{p-2}
\]

\[
= q(x \beta_n^{(r)})^2 \left\{ \prod_{k=1}^{r} (x \beta_n^{(k)}; q)_n \right\} \sum_{p=2}^{\infty} \left\{ \sum_{l_1+\cdots+l_{p-2}=p-2}^{L \geq 1} (q^n; q)_{l_1} \cdots (q^n; q)_{l_{p-2}} \right\}
\times \frac{[n+l_r-2]_q}{[n+l_r-1]_q} \frac{(\beta_n^{(1)})_{l_r} \cdots (\beta_n^{(r)})_{l_r-2}}{(q; q)_{l_r} \cdots (q; q)_{l_1-2}} x^{p-2}.
\]

Since \( \frac{[n+l_r-2]_q}{[n+l_r-1]_q} < 1 \), using (i), we get

\[
\sum_{2} \leq q(x \beta_n^{(r)})^2.
\]

Similarly, using \( \frac{1}{[n+l_r-1]_q} \leq \frac{1}{|m|_q} \) and \( \frac{1}{[n+l_r-1]_q} \leq \frac{1}{|n|_q} \), we obtain

\[
\sum_{3} \leq x \beta_n^{(r)} \frac{2}{[3]_q [n]_q},
\]

and

\[
\sum_{4} \leq \frac{1}{[3]_q [n]_q^2},
\]

respectively.

Finally, combining the equations (2.5)–(2.8), we have

\[
K_{n, q}^{\beta_n^{(1)}, \cdots, \beta_n^{(r)}} (s^2; x) \leq \frac{1}{[3]_q [n]_q} + \frac{x \beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right) + q(x \beta_n^{(r)})^2.
\]

Now, from (2.9)

\[
K_{n, q}^{\beta_n^{(1)}, \cdots, \beta_n^{(r)}} (s^2; x) - x^2 \leq x^2(q(\beta_n^{(r)})^2 - 1) + \frac{1}{[3]_q [n]_q^2} + \frac{x \beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right)
\]

\[
= -x^2(1-q(\beta_n^{(r)})^2) + \frac{1}{[3]_q [n]_q^2} + \frac{x \beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right).
\]
Since \( q, \beta_n^{(r)} \in (0, 1) \), we get
\[
\mathcal{R}_{n,q}^{(1),\ldots,(r)}(s^2; x) - x^2 \leq \frac{1}{[3]_q[n]_q^2} + \frac{x\beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right).
\] (2.10)

Using the positivity and linearity of the operators, and equation (2.4), we have
\[
0 \leq \mathcal{R}_{n,q}^{(1),\ldots,(r)}((s-x)^2; x) = \mathcal{R}_{n,q}^{(1),\beta(2)}(s^2; x) - 2x\mathcal{R}_{n,q}^{(1),\ldots,(r)}(s; x) + x^2
\]
or,
\[-2x^2(1-\beta_n^{(r)}) \leq \mathcal{R}_{n,q}^{(1),\ldots,(r)}(s^2; x) - x^2
\]
or,
\[-2x^2(1-\beta_n^{(r)}) - \frac{1}{[3]_q[n]_q^2} - \frac{x\beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right) \leq \mathcal{R}_{n,q}^{(1),\ldots,(r)}(s^2; x) - x^2.
\]

Hence, in view of (2.10)
\[
-2x^2(1-\beta_n^{(r)}) - \frac{1}{[3]_q[n]_q^2} - \frac{x\beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right)
\]
\[
\leq \mathcal{R}_{n,q}^{(1),\ldots,(r)}(s^2; x) - x^2
\]
\[
< \mathcal{R}_{n,q} \frac{1}{[3]_q[n]_q^2} + \frac{x\beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right) + 2x^2(1-\beta_n^{(r)}),
\]
thus
\[
|\mathcal{R}_{n,q}^{(1),\ldots,(r)}(s^2; x) - x^2| \leq \frac{1}{[3]_q[n]_q^2} + \frac{x\beta_n^{(r)}}{[n]_q} \left( 1 + \frac{2}{[2]_q} \right) + 2x^2(1-\beta_n^{(r)}).
\]

\[\square\]

**Lemma 2.** For the operators \( \mathcal{R}_{n,q}^{(1),\ldots,(r)}(\cdot; x) \), we have the inequality
\[
\mathcal{R}_{n,q}^{(1),\ldots,(r)}((s-x)^2; x) \leq 2x(1+x)(1-\beta_n^{(r)}) + \frac{x}{[n]_q} (\beta_n^{(r)} \left( 1 + \frac{2}{[2]_q} \right) + \frac{2}{[2]_q}) + \frac{1}{[3]_q[n]_q^2}
\]
\[
\leq 4(1-\beta_n^{(r)}) + \frac{1}{[n]_q} (\beta_n^{(r)} \left( 1 + \frac{2}{[2]_q} \right) + \frac{2}{[2]_q}) + \frac{1}{[3]_q[n]_q^2}
\]
\[
= \gamma_{n,q}(\beta_n^{(r)}), \; \text{say}.
\]

**Proof.** We may write
\[
\mathcal{R}_{n,q}^{(1),\ldots,(r)}((s-x)^2; x) \leq |\mathcal{R}_{n,q}^{(1),\ldots,(r)}(s^2-x^2; x)| + 2x|\mathcal{R}_{n,q}^{(1),\ldots,(r)}(s-x; x)|.
\]

Hence, using **Lemma 1**, we obtain the required inequality. \[\square\]
From now onwards, we assume that \( q = \langle q_n \rangle \in (0, 1) \) such that
\[
q_n \to 1 \quad \text{and} \quad q^n_n \to a \in [0, 1) \quad \text{as} \quad n \to \infty.
\]

### 3. Deferred weighted A-statistical approximation for \( \mathbb{R}^k_{n,q} \)

Let \( M \) be a subset of the set of natural numbers \( \mathbb{N} \) and for each \( n \in \mathbb{N} \), we define
\[
M_n = \{ m \in M : m \leq n \}.
\]

The density (or natural density) of the set \( M \), denoted by \( d(M) \), is defined by the limit (if it exists) of the sequence \( \langle |M_n|/n \rangle \), as \( n \to \infty \). More precisely,
\[
d(M) = \lim_{n \to \infty} \frac{|M_n|}{n}.
\]

There are many ways to define the density of the subsets of natural numbers and these definitions are playing a pivotal role in the areas of Number Theory and Graph Theory (see [10, 11]). A sequence of real or complex numbers \( \langle x_n \rangle \) is called statistically convergent to \( l \) if, for each \( \varepsilon > 0 \), we have the following
\[
\lim_{n \to \infty} \frac{|\{ k \in \mathbb{N} : k \leq n \text{ and } |x_k - l| \geq \varepsilon \}|}{n} = 0.
\]

In this case, we write \( \text{stat} - \lim_{n \to \infty} x_n = l \). The definition shows that every convergent sequence is always statistically convergent, while the converse need not be true. Karakaya et al. [12] gave the concept of weighted statistical convergence which was later modified by Mursaleen et al. in [13].

Let us assume that \( \langle s_k \rangle \) be a sequence such that \( s_k \geq 0, \forall k \in \mathbb{N} \), and\[
S_n = \sum_{k=1}^{n} s_k, \text{ with } s_1 > 0,
\]
be its partial sum. Now, set
\[
u_n = \frac{1}{S_n} \sum_{k=1}^{n} s_k x_k, \quad n \in \mathbb{N}.
\]

Then, the sequence \( \langle x_n \rangle \) is called weighted statistically convergent to a number \( l \) if, for any given \( \varepsilon > 0 \), the following holds
\[
\lim_{n \to \infty} \frac{|\{ k \in \mathbb{N} : k \leq S_n \text{ and } s_k|s_k - l| \geq \varepsilon \}|}{S_n} = 0,
\]
and we write \( \text{stat}_w - \lim x_n = l \). If \( X_1 \) and \( X_2 \) are sequence spaces such that for every infinite matrix \( A = (a_{n,k}) : X \to Y \), we have \( (Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k \). Then, the matrix \( A \) is called regular if \( \lim_{n \to \infty} (Ax)_n = l \).
whenever \( \lim_{n \to \infty} x_n = l \). For a non-negative regular matrix \( A = (a_{n,k}) \), Freedman et al. [14] defined the idea of \( A \)-statistical convergence. The sequence \( (x_n) \) is called \( A \)-statistically convergent to a number \( l \), denoted by \( \text{stat}_A \lim_{n \to \infty} x_n = l \), if for every \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \sum_{k:|x_n-l| \geq \epsilon} a_{n,k} = 0.
\]

Very recently, Srivastava et al. [15] defined a generalization of the concept of \( A \)-statistical convergence, called as deferred weighted \( A \)-statistical convergence. Suppose \( (b_n) \) and \( (c_n) \) are the sequences of non-negative integers satisfying the regularity conditions \( b_n < c_n, \forall n \in \mathbb{N} \) and \( \lim_{n \to \infty} c_n = \infty \). Now, we set

\[ S_n = \sum_{m=b_n+1}^{c_n} s_m, \]

for any given sequence \( (s_n) \) of non-negative real numbers then the deferred weighted mean of the sequence \( (x_n) \) is given by \( \rho_n = \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} s_m x_m \). The sequence \( (x_n) \) is called deferred weighted summable to \( l \) (denoted by \( \text{cDWS} - \lim_{n \to \infty} x_n = l \)) if \( \lim \rho_n = l \). Further, we call \( (x_n) \) to be deferred weighted \( A \)-summable to a number \( l \) (denoted by \( \text{cDWS}_A - \lim_{n \to \infty} x_n = l \)) if

\[
\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k=1}^{\infty} s_m a_{m,k} x_k = l.
\]

Let \( \text{cDWS} \) be the space of all deferred weighted summable sequences and \( (b_n), (c_n) \) be the sequences of non-negative integers. Then the infinite matrix \( A = (a_{n,k}) \), is called a deferred weighted regular matrix if

\[
(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k \in \text{cDWS} \quad \text{for every convergent sequence} \ x = (x_n),
\]

with

\[
\text{cDWS} - \lim_{n \to \infty} (Ax)_n = \text{stat}_A \lim_{n \to \infty} x_n.
\]

Let \( \epsilon > 0 \) be arbitrary. For a non-negative deferred weighted regular matrix \( A = (a_{n,k}) \) and \( K_\epsilon \subset \mathbb{N} = \{ k \in \mathbb{N} : |x_k-l| \geq \epsilon \} \), the sequence \( (x_n) \) is said to be deferred weighted \( A \)-statistically convergent to \( l \) (denoted by \( \text{stat}_A^{\text{DWS}} - \lim_{n \to \infty} x_n = l \)) if for each \( \epsilon > 0 \), the deferred weighted \( A \)-density of \( K_\epsilon \), denoted by \( d_{\text{DWS}}^A(K_\epsilon) \), is zero. That is,
In our further consideration, we assume $A = (a_{n,k})$ to be a non-negative deferred weighted regular matrix.

The following theorem shows the deferred weighted $A$-statistical convergence of the operators $R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (.; x)$ defined by (2.2).

**Theorem 1.** For $f \in \mathcal{C}(I)$, we have

$$\text{stat}_A^{D_W} - \lim_{n \to \infty} \|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f) - f\| = 0,$$

if and only if $\text{stat}_A^{D_W} - \lim_{n \to \infty} \|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f_i) - f_i\| = 0$, for $i = 1, 2$, where $f_i(s) = s^i$.

**Proof.** The necessary part is trivial. For the converse, let us assume that $\text{stat}_A^{D_W} - \lim_{n \to \infty} \|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f_i) - f_i\| = 0$, holds true. For $f \in \mathcal{C}(I)$, there exists a positive constant $M_f$ such that

$$|f(s) - f(x)| \leq 2M_f, \quad \text{for all } s, x \in I.$$

In view of the uniform continuity of $f$ on $I$, for any $\epsilon > 0$, $\exists \delta > 0$, such that $|f(s) - f(x)| < \epsilon$, whenever $|s - x| < \delta$. Hence, for all $s, x \in I$, we can write

$$|f(s) - f(x)| \leq \epsilon + \frac{2M_f}{\delta^2} (s-x)^2. \quad (3.1)$$

Thus, applying the operators $R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (.; x)$ on the above equation and using Lemma 1, we obtain

$$|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f; x) - f(x)| \leq \epsilon + \frac{2M_f}{\delta^2} (|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (s^2; x) - x^2| + 2|x||R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (s; x) - x|).$$

Hence considering the sup-norm on $I$, we have the following inequality

$$\|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f) - f\| \leq \epsilon + \frac{2M_f}{\delta^2} (\|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f_2) - f_2\| + 2\|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f_1) - f_1\|).$$

Now, for any $\epsilon' > 0$, we consider the following sets;

$$K_{\epsilon'} := \left\{k \in \mathbb{N} : \|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f) - f\| \geq \epsilon' \right\};$$

$$K_{\epsilon'}' := \left\{k \in \mathbb{N} : \epsilon + \frac{2M_f}{\delta^2} (\|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f_2) - f_2\| + 2\|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f_1) - f_1\|) \geq \epsilon' \right\};$$

$$K_{\epsilon'}'' := \left\{k \in \mathbb{N} : \frac{4M_f}{\delta^2} (\|R_{n,q_n}^{\beta^{(1)},\ldots,\beta^{(r)}} (f_1) - f_1\| \geq \epsilon' \right\},$$

thus $K_{\epsilon'} \subset K_{\epsilon'}' \cup K_{\epsilon'}''$ and therefore
\[
\frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K} s_m a_{m,k} \leq \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K'} s_m a_{m,k} + \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K''} s_m a_{m,k}. \tag{3.2}
\]

Now, using the hypothesis and Lemma 1, it is obvious that
\[
d_A^{D_W}(K') = \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K'} s_m a_{m,k} = 0;
\]
and
\[
d_A^{D_W}(K'') = \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K''} s_m a_{m,k} = 0.
\]

Hence, from Equation (3.2), we have
\[
d_A^{D_W}(K_r) = \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K_r} s_m a_{m,k} = \text{stat}_{D_W} A - \lim_{n \to \infty} \|R_{\beta^{(1)}, \ldots, \beta^{(r)}}(f) - f\|
\]

\[= 0.
\]

Let us recall the definition of usual modulus of continuity. For any continuous function \(f : I \to \mathbb{R}\) and a given \(\delta > 0\), the modulus of continuity \(\omega_f(\delta)\) is defined as
\[
w_f(\delta) := \sup_{|s-x| \leq \delta} \{|f(s) - f(x)| : s, x \in I\}.
\]

From the above definition, we have
\[
|f(s) - f(x)| \leq \left(1 + \frac{(s-x)^2}{\delta^2}\right) \omega_f(\delta). \tag{3.3}
\]

Let \(\langle x_n \rangle\) be a positive non-increasing sequence of real numbers and \(\langle \delta_n \rangle\) be any sequence of positive real numbers. Then, we say the sequence \(\langle \omega_f(\delta_n) \rangle\) is deferred weighted \(A\)-statistically convergent with \(o(x_n)\) if
\[
\text{stat}_{D_W}^{A} - \lim_{n \to \infty} \frac{\omega_f(\delta_n)}{x_n} = 0.
\]

**Theorem 2.** For the operator \(R_{\gamma_{n,q_n}(\beta^{(r)})}\), if
\[
\omega_f(\sqrt{\gamma_{n,q_n}(\beta^{(r)})}) = \text{stat}_{A}^{D} - o(x_n),
\]
then
\[
\|R_{\beta^{(1)}, \ldots, \beta^{(r)}}(f) - f\| = \text{stat}_{A}^{D} - o(x_n),
\]
where \(\gamma_{n,q_n}(\beta^{(r)})\) is as defined in Lemma 2.
Proof. For any \( d > 0 \), from the inequality (3.3) and Lemma 1, we obtain
\[
|\mathcal{R}_{n,q_n}^{\beta(1), \ldots, \beta(r)}(f; x) - f(x)| \leq \left( 1 + \frac{1}{d^2} \mathcal{R}_{n,q_n}^{\beta(1), \ldots, \beta(r)}((s-x)^2; x) \right) \omega_f(\delta),
\]
for all \( x \in I \).

Hence in view of Lemma 2, we can write
\[
\|\mathcal{R}_{n,q_n}^{\beta(1), \ldots, \beta(r)}(f) - f\| \leq \omega_f(\delta)(1 + \frac{\gamma_{n,q_n}(\beta_n^{(r)})}{\delta^2}).
\]

Now, we choose \( \delta = \sqrt{\gamma_{n,q_n}(\beta_n^{(r)})} \) and consider the hypothesis \( \omega_f \left( \sqrt{\gamma_{n,q_n}(\beta_n^{(r)})} \right) = \text{stat}_A^{DW} - o(z_n) \), to reach the assertion. \( \square \)

4. Power series summability method for \( \mathcal{R}_{n,q_n}^{\beta(1), \ldots, \beta(r)} \)

Let \( \langle p_j \rangle \) be a sequence of real numbers such that \( p_1 > 0 \) and \( p_j \geq 0 \), \( \forall \ j = 2, 3, \ldots \). Also, suppose that the power series
\[
p(u) = \sum_{j=1}^{\infty} p_j u^{j-1}, \quad (4.1)
\]
has a radius of convergence \( R \in (0, \infty] \). Now, a sequence \( \langle \eta_j \rangle \) is said to be convergent to \( l \) in the sense of power series method (please see [16–18]) if
\[
\lim_{u \to R} \frac{1}{p(u)} \sum_{j=1}^{\infty} \eta_j p_j u^{j-1} = l, \quad \forall \ u \in (0, R).
\]

Further, the power series method is called regular [19] if and only if
\[
\lim_{u \to R} \frac{p_j u^{j-1}}{p(u)} = 0, \quad \forall \ j \in \mathbb{N}. \quad (4.2)
\]

Recently, the power series summability method of convergence has attracted the attention of many researchers due to its nature of generality over the classical convergence [20]. For some important contributions in this direction, the interested reader may refer to [18, 20–23]. The following result shows the convergence of our operators \( \mathcal{R}_{n,q_n}^{\beta(1), \ldots, \beta(r)}(.; x) \) by means of power series method.

Theorem 3. For \( f \in \mathcal{C}(I) \), the operators \( \mathcal{R}_{n,q_n}^{\beta(1), \ldots, \beta(r)}(.; x) \) satisfy
\[
\lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \|\mathcal{R}_{n,q_n}^{\beta(1), \ldots, \beta(r)}(f) - f\| p_n u^{n-1} = 0, \quad (4.3)
\]
if and only if
\[ \lim_{u \to R^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| R^{(1), \ldots, (r)}_{n, q_n} (f_i) - f_i \| p_n u^{n-1} = 0, \quad (4.4) \]

for \( i = 1, 2 \) where \( f_i(s) = s^i \).

**Proof.** First assume that (4.3) is true. Then (4.4) is obvious. Conversely, let the condition (4.4) be true. Now since \( f \in C(I) \), assuming \( \epsilon > 0 \), let us recall the inequality (3.1), where \( \delta > 0 \). Then, we can write

\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \| R^{(1), \ldots, (r)}_{n, q_n} (f(s); x) - f(x) \| p_n u^{n-1}
\leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left\{ \epsilon + \frac{2M_f}{\delta^2} \| R^{(1), \ldots, (r)}_{n, q_n} ((s-x)^2; x) \| p_n u^{n-1} \right\}
\leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left\{ \epsilon + \frac{2M_f}{\delta^2} \left( \| R^{(1), \ldots, (r)}_{n, q_n} (s^2; x) - x^2 \| + 2 \| R^{(1), \ldots, (r)}_{n, q_n} (s; x) - x \| \right) \right\} p_n u^{n-1},
\]

for all \( x \in I \). Considering sup-norm and taking the limit as \( u \to R^- \), in view of (4.1), we obtain

\[
\lim_{u \to R^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| R^{(1), \ldots, (r)}_{n, q_n} (f) - f \| p_n u^{n-1}
\leq \epsilon + \frac{2M_f}{\delta^2} \left\{ \lim_{u \to R^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| R^{(1), \ldots, (r)}_{n, q_n} (f_2) - f_2 \| p_n u^{n-1} \right\}
+ \lim_{u \to R^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| R^{(1), \ldots, (r)}_{n, q_n} (f_1) - f_1 \| p_n u^{n-1} \right\}.
\]

Now, the assertion easily follows on using the hypothesis (4.4) and the arbitrariness of \( \epsilon \).

\[ \square \]

**Remark 1.** Let us assume that \( \lim_{n \to \infty} \beta_n^{(r)} = 1 \). In order to show the convergence of \( \langle R^{(1), \ldots, (r)}_{n, q_n} (f) \rangle \) to \( f \), as \( u \to R^- \), on \( I \) by power series method, it is sufficient to establish the following;

\[
\lim_{u \to R^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| R^{(1), \ldots, (r)}_{n, q_n} (f_i) - f_i \| p_n u^{n-1} = 0, \quad \text{for } i = 1, 2.
\]

Using **Lemma 1,**

\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \| R^{(1), \ldots, (r)}_{n, q_n} (f_1) - f_1 \| p_n u^{n-1} \leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left( (1 - \beta_n^{(r)}) + \frac{1}{2[\beta_n^{(r)}]} \right) p_n u^{n-1} \quad (4.5)
\]

\[ = J_1 + J_2, \quad \text{say.} \]

Now, let us estimate
\[ J_1 = \frac{1}{p(u)} \sum_{n=1}^{\infty} (1 - \beta_n^{(r)}) p_n u^{n-1}. \]

Let \( \epsilon > 0 \) be arbitrary. Then from the hypothesis \( \exists n_0(\epsilon) \) such that \( \frac{|1 - \beta_n^{(r)}|}{4} \leq \epsilon \), for all \( n > n_0(\epsilon) \). Hence,

\[ J_1 \leq \frac{1}{p(u)} \sum_{n=1}^{n_0} (1 - \beta_n^{(r)}) p_n u^{n-1} + \frac{\epsilon}{4p(u)} \sum_{n=n_0+1}^{\infty} p_n u^{n-1} \]

\[ < \frac{1}{p(u)} \sum_{n=1}^{n_0} (1 - \beta_n^{(r)}) p_n u^{n-1} + \frac{\epsilon}{4p(u)} \sum_{n=1}^{\infty} p_n u^{n-1}. \]

Let \( M_1 = \max_{1 \leq n \leq n_0} (1 - \beta_n^{(r)}) \) then using (4.1),

\[ J_1 < \frac{M_1}{p(u)} \sum_{n=1}^{n_0} p_n u^{n-1} + \frac{\epsilon}{4}. \]

In view of regularity condition given by (4.2), there exists \( \delta_j(\epsilon) > 0 \), such that \( \frac{p_{n-1}}{p_n} < \epsilon \) for all \( R - \delta_j(\epsilon) < u < R \) and \( j = 1, 2, \ldots n_0(\epsilon) \). Let us consider \( \delta(\epsilon) = \min(\delta_1(\epsilon), \delta_2(\epsilon), \ldots, \delta_{n_0}(\epsilon)) \) then for every \( R - \delta(\epsilon) < u < R \) and for all \( n = 1, 2, \ldots n_0 \), we have

\[ J_1 < \frac{\epsilon}{4M_1 n_0} M_1 n_0 + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \]

Now, we estimate

\[ J_2 = \frac{1}{p(u)} \sum_{n=1}^{\infty} \frac{1}{[2]_{q_n} [n]_{q_n}} p_n u^{n-1}. \]

Since \( \frac{1}{[2]_{q_n} [n]_{q_n}} \to 0 \), as \( n \to \infty \) there exists \( n_1(\epsilon) \in \mathbb{N} \) such that \( \frac{1}{[2]_{q_n} [n]_{q_n}} < \frac{\epsilon}{4M_1 n_0} \), for all \( n > n_1(\epsilon) \). Let us set \( M_2 = \max_{1 \leq n \leq n_1(\epsilon)} \frac{1}{[2]_{q_n} [n]_{q_n}} \). Then

\[ J_2 < \frac{1}{p(u)} \sum_{n=1}^{n_1} \frac{1}{[2]_{q_n} [n]_{q_n}} p_n u^{n-1} + \frac{\epsilon}{4} \]

\[ < \frac{\epsilon}{4M_2 n_1} M_1 n_1 + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \]

for some \( \delta'(\epsilon) > 0 \) such that \( u \in (R - \delta', R) \), in view of the regularity condition (4.2). Now, let us choose \( \delta_0 = \min(\delta, \delta') \). Then, using the estimates of \( J_1 \) and \( J_2 \) in (4.5), we obtain

\[ \frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathcal{R}_{\beta_n^{(r)}}(f_1) - f_1 \| p_n u^{n-1} < \epsilon, \quad \forall \ u \in (R - \delta_0, R). \]
Next, using Lemma 1, we consider
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \|S_{n,q_n}^{(1),\ldots,(r)}(f_2;x) - f_2\|p u^{n-1} \leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left( \frac{1}{[3]_{q_n}[n]_{q_n}^2} + \frac{\beta_n^{(r)}}{n_{q_n}} \left( 1 + \frac{2}{2} \right) \right) + 2(1-\beta_n^{(r)}) p u^{n-1} \]
\[
= K_1 + K_2 + K_3, \quad \text{say.} \tag{4.6}
\]
Let \(\epsilon>0\) be given. First we estimate \(K_1\). Proceeding in a manner similar to the estimate of \(J_2\), we have
\[
K_1 = \frac{1}{p(u)} \sum_{n=1}^{\infty} \frac{1}{[3]_{q_n}[n]_{q_n}^2} p u^{n-1}
\leq \frac{1}{p(u)} \sum_{n=1}^{n_2} \frac{1}{[3]_{q_n}[n]_{q_n}^2} p u^{n-1} + \frac{\epsilon}{6}, \quad \forall \, n>n_2(\epsilon)
\leq \left( \max_{1 \leq n \leq n_2} \frac{1}{[3]_{q_n}[n]_{q_n}^2} \right) \frac{1}{p(u)} \sum_{n=1}^{n_2} p u^{n-1} + \frac{\epsilon}{6}
\leq \frac{M_3}{p(u)} \sum_{n=1}^{n_2} p u^{n-1} + \frac{\epsilon}{6}
\leq \frac{M_3 n_2 - \epsilon}{6M_3 n_2} + \frac{\epsilon}{6} = \frac{\epsilon}{3},
\]
for all \(u \in (R-\theta,R)\) and some \(\theta(\epsilon)>0\). Similarly, we can show that there exist some \(\theta'>0\) and \(\theta''>0\) such that
\[
K_2 = \frac{1}{p(u)} \sum_{n=1}^{\infty} \frac{\beta_n^{(r)}}{[n]_{q_n}} \left( 1 + \frac{2}{2} \right) p u^{n-1} < \frac{\epsilon}{3}, \quad \forall \, u \in (R-\theta',R),
\]
and
\[
K_3 = \frac{2}{p(u)} \sum_{n=1}^{\infty} (1 - \beta_n^{(r)}) p u^{n-1} < \frac{\epsilon}{3}, \quad \forall \, u \in (R-\theta'',R).
\]
Considering \(\theta_0(\epsilon) = \min\{\theta,\theta',\theta''\}\), and using the estimates \(K_1-K_3\) in (4.6), we reach
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \|S_{n,q_n}^{(1),\ldots,(r)}(f_2) - f_2\|p u^{n-1} < \epsilon, \quad \forall \, u \in (R-\theta_0,R).
\]
Next, we furnish an example to demonstrate that the Theorem 3 is a non-trivial generalization of the classical Korovkin theorem.

Remark 2. Suppose that \(\lim_{n \to \infty} \beta_n^{(r)} = 1\). For \(f \in \mathcal{C}(I)\), consider the following sequence of auxiliary operators defined by
\[ \mathcal{Q}_{n, q_n}^{\beta(1), \ldots, \beta(r)}(f; x) = (1 + x_n) R_{n, q_n}^{\beta(1), \ldots, \beta(r)}(f; x), \quad (4.7) \]

where \( \langle x_n \rangle = \begin{cases} 1, & \text{if } n = m^2, m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \) Now if we take \( p_n = 1 \) for all \( n \in \mathbb{N}, \) then we obtain

\[ p(u) = \sum_{n=1}^{\infty} p_n u^{n-1} = \frac{1}{1 - u}, |u| < 1 \]

which implies that \( R = 1. \) Further, we note that

\[ \frac{1}{p(u)} \sum_{n=1}^{\infty} p_n u^{n-1} x_n = \frac{(1-u)}{u} \sum_{m=1}^{\infty} u^{m_2}. \]

Since by Cauchy’s root test, the series \( \sum_{m=1}^{\infty} u^{m_2} \) is absolutely convergent in the interval \( |u| < 1, \) it follows that

\[ \lim_{u \to 1^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} p_n u^{n-1} x_n = \lim_{u \to 1^-} \frac{(1-u)}{u} \sum_{m=1}^{\infty} u^{m_2} = 0. \quad (4.8) \]

Hence, the sequence \( \langle x_n \rangle \) converges to zero in the sense of power series method. Using the definition of auxiliary operators and (4.8), we conclude that

\[ \lim_{u \to 1^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} p_n u^{n-1} \| \mathcal{Q}_{n, q_n}^{\beta(1), \ldots, \beta(r)}(f_0) - f_0 \| = 0. \]

Moreover, from Equation (4.7) and Lemma 1, we have

\[ \| \mathcal{Q}_{n, q_n}^{\beta(1), \ldots, \beta(r)}(f_1) - f_1 \| \leq (1 - \beta_n^{(r)}) \frac{1}{2 |q_n|} + \frac{1}{2 |q_n|^2} + x_n \beta_n^{(r)} + \frac{1}{2 |q_n|}. \]

Since \( \langle (1 - \beta_n^{(r)}) + \frac{1}{2 |q_n|} \rangle \) converges to 0, as \( n \to \infty, \) it will also converge to 0 in the sense of power series method. Further \( \beta_n^{(r)} + \frac{1}{2 |q_n|} \leq 2, \) for each \( n \in \mathbb{N}, \) hence in view of (4.8), we have

\[ \lim_{u \to 1^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} p_n u^{n-1} \| \mathcal{Q}_{n, q_n}^{\beta(1), \ldots, \beta(r)}(f_1) - f_1 \| = 0. \]

Again, using the definition (4.7) and Lemma 1, we obtain

\[ \| \mathcal{Q}_{n, q_n}^{\beta(1), \ldots, \beta(r)}(f_2) - f_2 \| \leq \left( \frac{1}{3 |q_n|} + \frac{\beta_n^{(r)}}{|q_n|^2} \right) \left( 1 + \frac{2}{|q_n|} \right) + 2(1 - \beta_n^{(r)}) \]

\[ + x_n \left( 1 + \frac{1}{|q_n|} \right) \left( 1 + \frac{2}{|q_n|} \right) + \frac{1}{3 |q_n|^2}. \]

We see that the sequence \( \frac{\beta_n^{(r)} + \beta_n^{(r)}}{|q_n|} (1 + \frac{2}{|q_n|}) + 2(1 - \beta_n^{(r)}) \) converges to 0, in the sense of power series method and \( \left( 1 + \frac{1}{|q_n|} \left( 1 + \frac{2}{|q_n|} \right) + \frac{1}{3 |q_n|^2} \right) \leq 5 \) for each \( n \in \mathbb{N}, \) therefore using (4.8), we have
\[
\lim_{u \to 1} \frac{1}{p(u)} \sum_{n=1}^{\infty} p_n u^{n-1} \| \Psi_{\beta_{n,q_n}}^{(1), \ldots, (r)}(f_2) - f_2 \| = 0.
\]

This confirms that the auxiliary operator defined by (4.7) satisfies all the sufficient conditions given by (4.4) of Theorem 3, therefore
\[
\lim_{u \to 1} \frac{1}{p(u)} \sum_{n=1}^{\infty} p_n u^{n-1} \| \Psi_{\beta_{n,q_n}}^{(1), \ldots, (r)}(f) - f \| = 0.
\]

However, \( \langle x_n \rangle \) is not convergent to 0, as \( n \to \infty \) (in the usual sense). Thus, the Korovkin theorem for linear positive operators does not work for the auxiliary operator defined in (4.7). Hence, our Theorem 3 is a non-trivial generalization of the classical Korovkin theorem.

**Theorem 4.** For \( f \in \mathcal{C}(I) \), if
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \omega_f(\sqrt{\gamma_{n,q_n}(\beta_{n,r})}) p_n u^{n-1} = O(\Omega(u)), \quad \text{as } u \to R-
\]
then,
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \| \Psi_{\beta_{n,q_n}}^{(1), \ldots, (r)}(f) - f \| p_n u^{n-1} = O(\Omega(u)),
\]
as \( u \to R- \), where \( \Omega(u) \) is some positive function on \((0, R)\) and \( \gamma_{n,q_n}(\beta_{n,r}) \) is as defined in Lemma 2.

**Proof.** For \( f \in \mathcal{C}(I) \) and any \( \delta > 0 \), using Lemma 2
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \| \Psi_{\beta_{n,q_n}}^{(1), \ldots, (r)}(f) - f \| p_n u^{n-1} \leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left\{ 1 + \frac{1}{\delta^2} \| \Psi_{\beta_{n,q_n}}^{(1), \ldots, (r)}(f_i - f)^2 \| \right\} \omega_f(\delta) p_n u^{n-1} 
\leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left\{ 1 + \frac{1}{\delta^2} \gamma_{n,q_n}(\beta_{n,r}) \right\} \omega_f(\delta) p_n u^{n-1}
\]
for every \( u \in (0, R) \). Taking \( \delta = \sqrt{\gamma_{n,q_n}(\beta_{n,r})} \), we obtain
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \| \Psi_{\beta_{n,q_n}}^{(1), \ldots, (r)}(f) - f \| p_n u^{n-1} \leq \frac{2}{p(u)} \sum_{n=1}^{\infty} \omega_f(\sqrt{\gamma_{n,q_n}(\beta_{n,r})}) p_n u^{n-1}.
\]
Hence in view of our hypothesis, the required result follows. \( \square \)

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**Authors’ contributions**

All the authors have equally contributed to the conceptualization, framing and writing of the manuscript.

**Data availability**

We assert that no data sets were generated or analyzed during the preparation of the manuscript.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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