DERIVATIONS ON TERNARY RINGS OF OPERATORS

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Abstract. To each projection $p$ in a $C^*$-algebra $A$ we associate a family of derivations on $A$, called $p$-derivations, and relate them to the space of triple derivations on $pA(1 - p)$. We then show that every derivation on a ternary ring of operators is spatial and we investigate whether every such derivation on a weakly closed ternary ring of operators is inner.

1. $S$-derivations on $C^*$-algebras

If $A$ is a $C^*$-algebra, we let $D(A)$ denote the Banach Lie algebra of derivations on $A$. To be more precise $D(A)$ consists of all operators $\delta \in B(A)$ that satisfy $\delta(xy) = \delta(x)y + x\delta(y)$ for every $x, y$ in $A$. $B(A)$ denotes the bounded linear operators on $A$.

A derivation $\delta \in D(A)$ is called self-adjoint if $\delta = \delta^*$, where $\delta^*$ is the derivation defined by $\delta^*(x) = \delta(x^*)^*$ for every $x$ in $A$. The space of all self-adjoint derivations on $A$ is a real Banach Lie subalgebra of $D(A)$ and is denoted $D^*(A)$.

Derivations on $C^*$-algebras have suitable counterparts in a more general setting of ternary rings of operators, or TROs for short, where they are sometimes termed triple derivations. However, in this paper we shall use the term triple derivation to denote a derivation of a Jordan triple system. For example, if $X$ is a Banach subspace of a $C^*$-algebra and $xy^*z + zy^*x \in X$ for every $x, y, z$ in $X$, then $X$ is called a JC$^*$-triple and a triple derivation on $X$ is an operator $\tau \in B(X)$ satisfying

$$\tau(\{xyz\}) = \{\tau(x)y^*z\} + \{x\tau(y)^*z\} + \{xy^*\tau(z)\}$$

for every $x, y, z$ in $X$, where $\{xyz\} = (xy^*z + zy^*x)/2$.

We shall use the term TRO-derivation, as follows: If $X$ is a Banach subspace of a $C^*$-algebra and $xy^*z \in X$ for every $x, y, z$ in $X$, then $X$ is called a TRO and a TRO-derivation on $X$ is an operator $\tau \in B(X)$ satisfying

$$\tau(xy^*z) = \tau(x)y^*z + x\tau(y)^*z + xy^*\tau(z)$$

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for every \( x, y, z \) in \( X \).

It is clear that a TRO (resp. JC*-triple) can also be defined as a Banach subspace of \( B(H, K) \), the bounded operators from Hilbert space \( H \) to Hilbert space \( K \), which is closed under the triple product \( xy^*z \) (resp. \( (xy^*z + zy^*x)/2 \)). If a TRO is weakly closed, it is called a W*-TRO.

In this section we will introduce the class of \( S \)-derivations on a \( C^* \)-algebra \( A \) associated with a subspace \( S \subseteq A \). Of particular interest will be the case \( S = pAp \) for a projection \( p \) in \( A \). We will seek to determine the relationship between the class of \( pAp \)-derivations (which we call \( p \)-derivations for short) on \( A \) and the class of TRO-derivations on \( pA(1-p) \).

**Definition 1.1.** Let \( A \) be a \( C^* \)-algebra and let \( S \) be a subspace of \( A \). We say that a derivation \( \delta \in D(A) \) is associated with \( S \), or simply that \( \delta \) is an \( S \)-derivation, if \( \delta \) leaves \( S \) invariant in the sense that \( \delta(S) \subseteq S \).

We use \( D_S(A) \) to denote the set of all \( S \)-derivations. In order to simplify the notation, we write \( D_e(A) \) for \( D_{eAe}(A) \) in case \( S = eAe \), for some idempotent \( e \in A \), and we abuse the terminology slightly by referring to the elements of \( D_e(A) \) simply as \( e \)-derivations.

To repeat, given an arbitrary idempotent \( e \) in a \( C^* \)-algebra \( A \), which in particular may be a projection, by an \( e \)-derivation on \( A \) we mean a derivation \( \delta \in D(A) \) satisfying \( \delta(eAe) \subseteq eAe \). This condition is easily seen to be equivalent to the requirement that \( \delta(e) = 0 \).

**Example 1.2.** Let \( A \) be a \( C^* \)-algebra and let \( e \in A \) be an idempotent. Fix \( a \in eAe \) and \( b \in (1-e)A(1-e) = \{ x - xe - ex + exe : x \in A \} \). Then \( \delta : A \to A \) defined by \( \delta(x) = (a+b)x - x(a+b) \) is an \( e \)-derivation.

**Lemma 1.3.** Let \( A \) be a \( C^* \)-algebra and let \( S \) be a subalgebra with an identity element \( 1_S \) (possibly different from the identity element of \( A \) if \( A \) is unital). Let \( \delta \in D(A) \) be a derivation. The following statements hold.

1. If \( \delta(S) \subseteq S \) then \( \delta(1_S) = 0 \).
2. If \( \delta(1_S) = 0 \) then \( \delta(S) \subseteq 1_SA1_S \).

**Proof.** A straightforward consequence of the derivation property. \( \square \)

**Lemma 1.4.** Let \( A \) be a \( C^* \)-algebra and let \( e \in A \) be an idempotent. Let \( \delta \in D(A) \) be a derivation. The following statements hold.

1. If \( \delta(e) = 0 \), then \( \delta \) leaves invariant the following subspaces
   \[ eAe, \quad eA(1-e), \quad (1-e)Ae, \quad (1-e)A(1-e). \]
(2) If \( \delta \) leaves invariant \( eAe \) or \( (1 - e)A(1 - e) \), then \( \delta(e) = 0 \).

Additionally, let \( \delta = \delta^* \) and \( e = e^* \). Then the following statement holds.

(3) If \( \delta \) leaves invariant \( eA(1 - e) \) or \( (1 - e)Ae \), then \( \delta(e) = 0 \).

Proof. The assertions (1) and (2) are straightforward consequences of the derivation property. To prove (3), assume that \( eA(1 - e) \) is invariant for \( \delta = \delta^* \), and \( e = e^* \). Since \( \delta(e) = \delta(e)e + e\delta(e) \), we have \( e\delta(e)e = 0 \) and hence

\[
\delta(e) = e\delta(e)(1 - e) + (1 - e)\delta(e)e.
\]

This shows that both \( e\delta(e) \) and \( \delta(e)(1 - e) \) are equal to \( e\delta(e)(1 - e) \), and so both \( e\delta(e) \) and \( \delta(e)(1 - e) \) are elements of the subspace \( eA(1 - e) \) which is invariant under \( \delta \).

We will show that \( \delta(e) = 0 \) by showing that \( \delta(e)^2 = 0 \). For this, we identify \( A \) with \( \begin{pmatrix} eAe & eA(1 - e) \\ (1 - e)Ae & (1 - e)A(1 - e) \end{pmatrix} \) and write \( \delta(e) \) and \( \delta^2(e) \) as

\[
\delta(e) = \begin{pmatrix} 0 & \delta(e)(1 - e) \\ (1 - e)\delta(e)e & 0 \end{pmatrix}, \quad \delta^2(e) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Then \( \delta(e)^2 = \begin{pmatrix} e\delta(e)(1 - e)\delta(e)e & 0 \\ 0 & (1 - e)\delta(e)e\delta(e)(1 - e) \end{pmatrix} \) and since

\[
\delta(e) = \begin{pmatrix} e\delta(e)(1 - e)\delta(e)e & 0 \\ 0 & (1 - e)\delta(e)e\delta(e)(1 - e) \end{pmatrix},
\]

\[
\delta\delta(e)(1 - e) = \begin{pmatrix} -e\delta(e)(1 - e)\delta(e)e & 0 \\ 0 & d(1 - e) - (1 - e)\delta(e)e\delta(e)(1 - e) \end{pmatrix}
\]

it follows that \((1 - e)\delta(e)e\delta(e)(1 - e) = 0 = e\delta(e)(1 - e)\delta(e)e \). Thus \( \delta(e)^2 = 0 \), as desired. \( \square \)

If \( A \) is a \( C^* \)-algebra and \( p \in A \) is a projection, we let \( D_p^*(A) \) denote the (real) Banach Lie algebra of self-adjoint \( p \)-derivations on \( A \). To be more precise \( D_p^*(A) \) consists of all derivations \( \delta \in D(A) \) that satisfy \( \delta(p) = 0 \) and \( \delta = \delta^* \). If \( X \) is a TRO, we use \( D_{TRO}(X) \) to denote the (real) Banach Lie algebra of all TRO-derivations on \( X \).

Remark 1.5. Let \( A \) be a unital \( C^* \)-algebra and let \( p \in A \) be a projection. Then the map

\[
\Delta: D_p^*(A) \to D_{TRO}(pA(1 - p)), \quad \Delta(\delta) = \delta|_{pA(1 - p)}
\]

is a homomorphism of Banach Lie algebras.

Example 1.6. Let \( A = M_2(\mathbb{C}) \), \( p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). The set of all \( p \)-derivations on \( A \) is:

\[
D_p(A) = \{ \delta \in D(A) : \delta(p) = 0 \}
\]

\[
\simeq \{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \}
\]

\( \simeq \) a complex Banach Lie algebra.
The set of all self-adjoint $p$-derivations is:

$$D_p^*(A) = \{ \delta \in D_p(A) : \delta = \delta^* \}$$

$$\simeq \left\{ \left( \begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix} \right) : \alpha, \beta \in \mathbb{C} \text{ with } \mathfrak{R}(\alpha) = \mathfrak{R}(\beta) \right\}$$

$$= \text{ a real Banach Lie algebra.}$$

The mapping

$$\Delta : D_p^*(A) \to D_{TRO}(X), \quad \Delta(\delta) = \delta|_X$$

defines a linear surjection between the self-adjoint $p$-derivations on $A$ and the TRO-derivations on $X = pA(1-p) = \left( \begin{smallmatrix} 0 & \mathbb{C} \\ \mathbb{C} & 0 \end{smallmatrix} \right)$. The kernel of $\Delta$ is isomorphic to the center of $A$, i.e.,

$$\ker \Delta = Z(A) = \left\{ \left( \begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix} \right) : \alpha \in \mathbb{C} \right\}.$$}

In other words, the TRO-derivations on $X = pA(1-p) = \left( \begin{smallmatrix} 0 & \mathbb{C} \\ \mathbb{C} & 0 \end{smallmatrix} \right)$ are precisely the self-adjoint $p$-derivations on the linking algebra $(X^* X X^* X) = A = M_2(\mathbb{C})$.

**Example 1.7.** Let $A = M_5(\mathbb{C})$, and let $p \in A$ be the projection matrix with 1 in the $(1,1)$ and $(2,2)$ position and zero’s elsewhere. The set of all $p$-derivations on $A$ is:

$$D_p(A) = \{ \delta \in D(A) : \delta(p) = 0 \}$$

$$\simeq \left\{ \left( \begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix} \right) : A \in M_2(\mathbb{C}), B \in M_3(\mathbb{C}) \right\}$$

$$= \text{ a complex Banach Lie algebra.}$$

The set of all self-adjoint $p$-derivations is $D_p^*(A) = \{ \delta \in D_p(A) : \delta = \delta^* \}$ and it can be identified with the real Banach Lie algebra consisting of all matrices of the form $\left( \begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix} \right)$ where $A \in M_2(\mathbb{C})$, $B \in M_3(\mathbb{C})$, and $\left( \begin{smallmatrix} A^* + A^* & 0 \\ 0 & B + B^* \end{smallmatrix} \right)$ is in the center of $A$.

2. **Derivations on TROs**

If $A$ is a unital C*-algebra and $e$ is a projection in $A$, then $X := eA(1-e)$ is a TRO. Conversely if $X \subset B(H,K)$ is a TRO, then with $X^* = \{ x^* : x \in X \} \subset B(H,K)$, $XX^* = \text{span} \{ xy^* : x, y \in X \} \subset B(H)$, $X^*X = \text{span} \{ z^*w : z, w \in X \} \subset B(K)$, $K_l(X) = XX^*$, $K_r(X) = X^*X$, we let

$$A_X = \left[ \begin{array}{cc} K_l(X) + \mathbb{C}1_H & X \\ X^* & K_r(X) + \mathbb{C}1_K \end{array} \right] \subset B(H \oplus K)$$

If $K_l(X)$ and $K_r(X)$ are unital subalgebras of $B(H)$ and $B(K)$ (resp.), and $X$ is nondegenerate, that is, $XX^*X$ is dense in $X$, then we take $A_X$ to be $\left[ \begin{array}{cc} K_l(X) & X \\ X^* & K_r(X) \end{array} \right]$. 

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1 If $K_l(X)$ and $K_r(X)$ are unital subalgebras of $B(H)$ and $B(K)$ (resp.), and $X$ is nondegenerate, that is, $XX^*X$ is dense in $X$, then we take $A_X$ to be $\left[ \begin{array}{cc} K_l(X) & X \\ X^* & K_r(X) \end{array} \right]$. 

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denote the (unital) linking C*-algebra of $X$. Then we have a TRO-isomorphism $X \simeq eA_X(1 - e)$, where $e = [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$.

**Lemma 2.1.** Let $X$ be a TRO and let $D : X \to X$ be a TRO-derivation of $X$. If $A_0 = (XX^* X^* X)$, then the map $\delta_0 : A_0 \to A_0$ given by

$$\left( \sum_i x_i y_i^* \sum_j z_j^* w_j \right) \mapsto \left( \sum_i (x_i(\overline{Dy_i}) + (\overline{Dx_i})y_i^*) \sum_j (z_j^*(Dw_j) + (Dz_j)^*w_j) \right)$$

is well defined and a bounded *-derivation of $A_0$, which extends $D$ (when $X$ is embedded in $A_X$ via $x \mapsto (0 \ 0)$), and which itself extends to a *-derivation $\delta$ of $A_X$. Thus, the Lie algebra homomorphism $\Delta : \delta \mapsto \delta|_X$ given in Remark 1.5 is onto.

**Proof.** If $\sum_i x_i y_i^* = 0$, then for every $z \in X$,

$$0 = D(\sum_i x_i y_i^* z) = \sum_i ((\overline{Dx_i})y_i^* z + x_i(\overline{Dy_i})*z + x_i y_i^*(Dz)) = (\sum_i ((\overline{Dx_i})y_i^* + x_i(\overline{Dy_i}))*z).$$

Since this is true for every $z$, we have $\sum_i ((\overline{Dx_i})y_i^* + x_i(\overline{Dy_i}))* = 0$ (see [4, Lemma 2.3(iv)]) and it follows that $\delta_0$ is well defined.

The map $\delta_0$ is self-adjoint since if $a = \left( \sum_i x_i y_i^* \sum_j z_j^* w_j \right)$, then

$$\delta_0(a^*) = \delta_0 \left( \sum_i y_i x_i^* \sum_j w_j^* z_j \right) = \left( \sum_i (y_i(\overline{Dx_i}) + (\overline{Dy_i})x_i^*) \sum_j (w_j^*(Dz_j) + (Dw_j)^*z_j) \right) = \delta_0(a)^*.$$  

It is easy to verify that $\delta_0(a^2) = \delta_0(a) + a\delta_0(a)$ so that $\delta_0$ is a Jordan *-derivation of $A_0$. (We omit that calculation.)

To see that $\delta_0$ is bounded, we first note that $D$ is bounded, since it is a Jordan triple derivation on the JB*-triple $X$ with the Jordan triple product $\{xyz\} = (xy^*z + zy^*x)/2$, and hence bounded by the theorem of Barton and Friedman [1]. Now denoting $\sum_i (x_i(\overline{Dy_i}) + (\overline{Dx_i})y_i^*)$ by
\[ \alpha, \text{ we have (by } [4, \text{ Lemma 2.3(iv)}] \text{ again) } \| \sum_i (x_i (D y_i)^* + (D x_i) y_i^*) \| = \| \alpha \| = \sup_{\| z \| \leq 1, z \in X} \| \alpha z \| = \sup_{\| z \| \leq 1, z \in X} \| \sum_i x_i y_i^* (D z) - \sum_i x_i y_i^* (D z) \| \]
\[ = \sup_{\| z \| \leq 1, z \in X} \| D \sum_i x_i y_i^* z - \| D \| \sum_i x_i y_i^* D z \| \leq 2 \| D \| \| \sum_i x_i y_i^* \|. \]
Thus \( \delta_0 \) is bounded and therefore extends to a bounded Jordan *-derivation \( \delta \) of \( A_0^n \) and hence to \( A_X \) by setting \( \delta(e) = 0 \), where \( e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). By the theorem of Sinclair ([15, Theorem 3.3]), \( \delta \) is a derivation of \( A_X \). \qed

For any C*-algebra \( A \subset B(H) \), the Lie algebra homomorphism \( \overline{A}^w \ni z \mapsto \text{ad } z \in D(\overline{A}^w) \) is onto (theorem of Kadison and Sakai ([14, 4.1.6])) and so we have the Lie algebra isomorphism
\[ \overline{A}^w / Z(\overline{A}^w) \simeq D(\overline{A}^w). \]
It follows (cf. [14, 4.1.7]) that
\[ \{ t \in \overline{A}^w : \text{ad } t(A) \subset A \} / Z(\overline{A}^w) \simeq D(A), \]
and
\[ \{ t \in \overline{A}^w : t^* = -t, \text{ad } t(A) \subset A \} / Z(\overline{A}^w) \simeq D^*(A). \]
Further, for a projection \( e \) in \( A \), we have
\[ \{ t \in \overline{A}^w : \text{et = te, t} = -t, \text{ad } t(A) \subset A \} / Z(\overline{A}^w) \simeq D^*_e(A). \]
Using these facts in the setting of Lemma [2.1] and noting that, by [9, page 268], \( \overline{A}_X^w = A_X^w = \left[ \begin{array}{c} K_i(X)^\prime \prime \end{array} \begin{array}{c} x^w \\
K_r(x)^\prime \prime \end{array} \right] \), we can now prove the following theorem.

**Theorem 2.2.** Every TRO-derivation of a TRO \( X \) is spatial in the sense that there exist \( \alpha \in K_i(X)^\prime \prime \) and \( \beta \in K_r(X)^\prime \prime \) such that \( \alpha^* = -\alpha, \beta^* = -\beta, \) and \( Dx = \alpha x + x \beta \) for every \( x \in X \).

\(^2\)It is also easy to verify directly, by (a more involved) calculation, that \( \delta_0 \) is a derivation, thereby avoiding the use of Sinclair’s theorem
Proof. If $D \in D_{TRO}(X)$, choose $\delta = \text{ad} \ t$ for some $t \in A_X^{\ast w}$ with $t^* = -t$, $te = et$ and
\[
\begin{bmatrix}
0 & Dx \\
0 & 0
\end{bmatrix} = \delta \begin{bmatrix}
0 & x \\
0 & 0
\end{bmatrix}.
\]
The conditions on $t$ imply that $t = \begin{bmatrix}
0 & 0 \\
0 & \beta
\end{bmatrix}$ with $\alpha^* = -\alpha$ and $\beta^* = -\beta$. Moreover
\[
\delta \begin{bmatrix}
0 & x \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix} \begin{bmatrix}
0 & x \\
0 & 0
\end{bmatrix} - \begin{bmatrix}
0 & x \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix} = \begin{bmatrix}
0 & ax + x(-\beta) \\
0 & 0
\end{bmatrix}.
\]
\[\square\]

A TRO derivation $D$ of a TRO $X$ is said to be an inner TRO derivation if there exist elements $\alpha, \beta \in X$ with $\alpha^* = -\alpha$ and $\beta^* = -\beta$ such that $Dx = \alpha x + x\beta$ for $x \in X$. Note that there exist $a_i, b_i, c_j, d_j \in X$, $1 \leq i \leq n, 1 \leq j \leq m$ such that $\alpha = \sum_{i=1}^{n}(a_i b_i^* - b_i a_i^*)$ and $\beta = \sum_{j=1}^{m}(c_j^* d_j - d_j^* c_j)$.

Corollary 2.3. Every TRO derivation of a $C^*$-algebra $A$ is of the form $A \ni x \mapsto \alpha x + x\beta$ with elements $\alpha, \beta \in A^{\ast w}$ with $\alpha^* = -\alpha, \beta^* = -\beta$. In particular, every TRO derivation of a von Neumann algebra is an inner TRO derivation.

Thus, every $W^*$-TRO which is TRO-isomorphic to a von Neumann algebra has only inner TRO derivations. For example, this is the case for the stable $W^*$-TROs of [12] (see subsection 3.2) and the weak*-closed right ideals in certain continuous von Neumann algebras acting on separable Hilbert spaces (see Theorem 3.3).

Theorem 2.2 is an improvement of [17], in which, although proved for the slightly more general case of derivation pairs, it is assumed that the TRO (called $B^*$-triple system in [17]) contains the finite rank operators. For the extension of Zalar's result to unbounded operators, see [16].

A triple derivation $\delta$ of a JC$^*$-triple $X$ is said to be an inner triple derivation if there exist finitely many elements $a_i, b_i \in X$, $1 \leq i \leq n$, such that $\delta x = \sum_{i=1}^{n} \{a_i b_i x\} - \{b_i a_i x\}$ for $x \in X$, where $\{xyz\} = (xy^* z + zy^* x)/2$. For convenience, we denote the inner triple derivation $x \mapsto \{abx\} - \{bax\}$ by $\delta(a, b)$. Thus
\[
\delta(a, b)(x) = (ab^* x + xb^* a - ba^* x - xa^* b)/2.
\]

Let $X$ be a TRO. As noted in the proof of Lemma 2.1, $X$ is a JC$^*$-triple in the triple product $(xy^* z + zy^* x)/2$, and every TRO-derivation of $X$ is obviously a triple derivation. On the other hand, every inner triple derivation is an inner TRO-derivation. Indeed, if
\[ \delta(x) = \{abx\} - \{bax\}, \text{ for some } a, b \in X, \text{ then } \delta(x) = Ax + xB, \text{ where } A = ab^* - ba^* \in XX^*, B = b^*a - a^*b \in X^*X \text{ with } A, B \text{ skew-hermitian.} \]

Moreover, since by [1, Theorem 4.6], every triple derivation \( \delta \) on \( X \) is the strong operator limit of a net \( \delta_{\alpha} \) of inner triple derivations, hence TRO-derivations, we have (i) and (ii) in the following proposition.

**Proposition 2.4.** Let \( X \) be a TRO.

(i): Every TRO-derivation is the strong operator limit of inner TRO-derivations.

(ii): The triple derivations on \( X \) coincide with the TRO-derivations.

(iii): The inner triple derivations on \( X \) coincide with the inner TRO-derivations

(iv): All TRO derivations of \( X \) are inner, if and only if, all triple derivations of \( X \) are inner.

**Proof.** Since (iv) is immediate from (ii) and (iii), we only need to show part of (iii), that is, that every inner TRO-derivation is an inner triple derivation. If \( D \) is an inner TRO-derivation, then \( Dx = \alpha x + x\beta \), with \( \alpha^* = -\alpha \in XX^* \) and \( \beta^* = -\beta \in X^*X \). We must show that there exist elements \( a_k, b_k \) such that \( Dx = \sum_{k=1}^{p} \delta(a_k, b_k)x \) where \( \delta(a_k, b_k) \) is the inner triple derivation \( x \mapsto \{a_kb_kx\} - \{b_ka_kx\} \). If \( \alpha = \sum_{i=1}^{n} x_iy_i^* \) and \( \beta = \sum_{j=1}^{m} z_j^*w_j \), then it suffices to take \( p = m + n \) and choose \( a_i = x_i/2, b_i = y_i \) for \( 1 \leq i \leq n \) and \( a_{n+i} = w_i, b_{n+i} = z_i/2 \) for \( 1 \leq i \leq m \). \( \square \)

3. Derivations on W*-TROs

A von Neumann algebra \( M \) is an example of a unital reversible \( JW^* \)-algebra, and as such, by [6, Theorem 2 and the first sentence in its proof], every triple derivation on \( M \) is an inner triple derivation. Hence we see that the last statement in Corollary 2.3 follows also from this and Proposition 2.4(iv). For completeness, we include a proof of the former result which avoids much of the Jordan theory, starting with the following lemma, the first part of which is straightforward.

**Lemma 3.1.** Let \( A \) be a unital Banach *-algebra equipped with the ternary product given by \( \{a, b, c\} = \frac{1}{2} (ab^*c + cb^*a) \) and the Jordan product \( a \circ b = (ab + ba)/2 \).

- Let \( D \) be an inner derivation, that is, \( D = ad_a : x \mapsto ax - xa \), for some \( a \) in \( A \). Then \( D = ad_a \) is a *-derivation whenever \( a^* = -a \). Conversely, if \( D \) is a *-derivation, then \( a^* = -a + z \) for some \( z \) in the center of \( A \).
- Every triple derivation is the sum of a Jordan *-derivation and an inner triple derivation.
Proof. To prove the second statement, we modify the proof in [7, Section 3] which is in a different context. We note first that for a triple derivation \( \delta \), \( \delta(1)^* = -\delta(1) \). Next, for a triple derivation \( \delta \), the mapping \( \delta_1(x) = \delta(1) \circ x \) is equal to the inner triple derivation \( -\frac{1}{2} \delta(\delta(1), 1) \) so that \( \delta_0 := \delta - \delta_1 \) is a triple derivation with \( \delta_0(1) = 0 \). Finally, any triple derivation which vanishes at 1 is a Jordan *-derivation. \( \square \)

Theorem 3.2. Every triple derivation on a von Neumann algebra is an inner triple derivation.

Proof. It suffices, by the second statement in Lemma 3.1, to show that every self-adjoint Jordan derivation is an inner triple derivation. If \( \delta \) is a self-adjoint Jordan derivation of \( M \), then \( \delta \) is an associative derivation (by the theorem of Sinclair, [15, Theorem 3.3]) and hence by the theorem of Kadison and Sakai ([14, 4.1.6]) and the first statement in Lemma 3.1, \( \delta(x) = ax - xa \) where \( a + a^* = z \) is a self-adjoint element of the center of \( M \). Since for every von Neumann algebra, we have \( M = Z(M) + [M, M] \), where \( Z(M) \) denotes the center of \( M \) (see [11, Section 3] for a discussion of this fact), we can therefore write

\[
a = z' + \sum_j [b_j + ic_j, b'_j + ic'_j]\]

\[
= z' + \sum_j ([b_j, b'_j] - [c_j, c'_j]) + i \sum_j ([c_j, b'_j] + [b_j, c'_j]),
\]

where \( b_j, b'_j, c_j, c'_j \) are self-adjoint elements of \( M \) and \( z' \in Z(M) \).

It follows that

\[
0 = a^* + a - z = (z')^* + z' - z + 2i \sum_j ([c_j, b'_j] + [b_j, c'_j])
\]

so that \( \sum_j ([c_j, b'_j] + [b_j, c'_j]) \) belongs to the center of \( M \). We now have

\[
\delta = \text{ad } a = \text{ad } \sum_j ([b_j, b'_j] - [c_j, c'_j]).
\]

A direct calculation shows that \( \delta \) is equal to the inner triple derivation

\[
\sum_j \left( \delta(b_j, 2b'_j) - \delta(c_j, 2c'_j) \right),
\]

completing the proof. \( \square \)

3.1. Weakly closed right ideals in von Neumann algebras. In this subsection, we shall consider the TRO \( pM \) where \( M \) is a von Neumann algebra and \( p \) is a projection in \( M \).

A TRO of the form \( pM \), with \( M \) a continuous von Neumann algebra, is classified into four types in [8] as follows.

- \( II_1^a \) if \( M \) is of type \( II_1 \) and \( p \) is (necessarily) finite.
- \( II_{\infty,1}^a \) if \( M \) is of type \( II_{\infty} \) and \( p \) is a finite projection.
• $II^a_\infty$ if $M$ is of type $II_\infty$ and $p$ is a properly infinite projection.
• $III^a$ if $M$ is of type $III$ and $p$ is a (necessarily) properly infinite projection.

Similarly, we also define types for $pM$ for $M$ of type I:
• $I_i^a$ if $M$ is finite of type $I$ and $p$ is (necessarily) finite.
• $I_{\infty,1}^a$ if $M$ is of type $I_{\infty}$ and $p$ is a finite projection.
• $I_{\infty}^a$ if $M$ is of type $I_{\infty}$ and $p$ is a properly infinite projection.

The following theorem involves the cases $II^a_\infty, III^a$ and when $M$ is a factor, the cases $I_1^a, I_{1,\infty}^a,$ and $I_{\infty}^a$.

**Theorem 3.3.** Let $X = pM$ be a TRO, where $M$ is a von Neumann algebra and $p$ is a projection in $M$.

(i): If $X$ is of type $II^a_\infty$ or $III^a$, and has a separable predual, then every TRO-derivation of $X$ is an inner TRO-derivation.

(ii): If $M$ is of type III and countably decomposable, then every TRO-derivation of $X = pM$ is an inner TRO-derivation.

(iii): If $M = B(H)$ is a factor of type I, then
1. If $\dim H < \infty$, then every TRO-derivation of $X = pM$ is an inner TRO-derivation.
2. If $\dim pH = \dim H$, then every TRO-derivation of $X = pM$ is an inner TRO-derivation.
3. If $\dim pH < \dim H = \infty$, then $X = pM$ admits outer TRO-derivations.

**Proof.** If $M$ is a continuous von Neumann algebra with a separable predual and $p$ is a properly infinite projection in $M$, then it is shown in [8, Theorem 5.16] that $pM$ is triple isomorphic to a von Neumann algebra, and hence by Theorem 3.2 every triple derivation is an inner triple derivation in this case. Consequently, by Proposition 2.4(iv), every TRO-derivation is an inner TRO-derivation. (Another way to see this latter fact is to note that by [8, Lemma 5.15], $pM$ is actually TRO-isomorphic to a von Neumann algebra, and to apply Corollary 2.3.) This proves (i).

To prove (ii), we note first that if $A$ is a von Neumann algebra with a projection $p \sim 1$, then $pA$ is TRO-isomorphic to $A$. Indeed, If $u$ is a partial isometry in $A$ with $uu^* = p$ and $u^*u = 1$, then $x \mapsto u^*x$ is a TRO-isomorphism from $pA$ onto $A$. Now if $A$ is of type III, then $\tilde{A} := c(p)A$ is of type III, $c(p)$ is the identity of $\tilde{A}$ and $pA = p\tilde{A}$. Further, if $A$ is countably decomposable, then by [14, 2.2.14], since in $\tilde{A}$, $c(p) = 1\tilde{A} = c(1\tilde{A})$, we have $p \sim 1\tilde{A}$, so $\tilde{A}$ is TRO-isomorphic to $p\tilde{A} = pA$. 

Finally, let $M = B(H)$. The first statement in (iii) follows from the fact that every finite dimensional semisimple Jordan triple system has only inner derivations. This result first appeared in [10, Chapter 11] (see also [13, Theorem 2.8, p. 136]). If $\dim pH = \dim H$, then $pM \simeq B(H)$ has only inner triple derivations by Theorem 3.2. On the other hand, if $\dim pH < \dim H = \infty$, then $pM \simeq B(H, pH)$ has outer triple derivations, as shown in [6, Corollary 3]. By Proposition 2.4(iv), this proves (iii).

Remark 3.4. Although it follows from Theorem 3.3, it is worth pointing out that the TROs $B(\mathbb{C}, H)$ and $B(H, \mathbb{C})$ support outer TRO derivations if and only if $\dim H = \infty$. According to [8, Lemma 5.15], if $B$ is a von Neumann algebra of type $II_\infty$ or III, and $H$ is a separable Hilbert space, then $B$ and $B \hat{\otimes} B(\mathbb{C}, H)$ are TRO-isomorphic. Corollary 2.3 shows that $B \hat{\otimes} B(\mathbb{C}, H)$ has only inner TRO-derivations and only inner triple derivations, although, as just noted, $B(\mathbb{C}, H)$ can have an outer TRO derivation and an outer triple derivation. This contrasts the situation of derivations on tensor products of $C^*$-algebras, as in [2, Proposition 3.2]).

3.2. W*-TROs of types I,II,III. We begin by recalling some concepts from [12]. If $R$ is a von Neumann algebra and $e$ is a projection in $R$, then $V := eR(1 - e)$ is a W*-TRO. Conversely if $V \subset B(K, H)$ is a W*-TRO, then with $V^* = \{x^*: x \in V\} \subset B(H, K)$, $M(V) = \overline{XX^{\text{sot}}} \subset B(H)$, $N(V) = \overline{X^*X^{\text{sot}}} \subset B(K)$, we let

$$R_V = \begin{bmatrix} M(V) & V \\ V^* & N(V) \end{bmatrix} \subset B(H \oplus K)$$

denote the linking von Neumann algebra of $V$. Then we have a SOT-continuous TRO-isomorphism $V \simeq eRe^\perp$, where $e = \begin{bmatrix} 1_H & 0 \\ 0 & 0 \end{bmatrix}$ and $e^\perp = \begin{bmatrix} 0 & 0 \\ 0 & 1_K \end{bmatrix}$.

A W*-TRO $V$ is stable if it is TRO-isomorphic to $B(\ell_2)\overline{\otimes} V$. A W*-TRO is of type I, II, or III, by definition, if its linking von Neumann algebra is of that type as a von Neumann algebra. There is a further classification of the types I and II depending on the types of $M(V)$ and $N(V)$ leading to the types $I_{m,n}, II_{\alpha,\beta}$ where $m, n$ are cardinal numbers and $\alpha, \beta \in \{1, \infty\}$. See [12, Section 4] for detail.

In what follows, for ultraweakly closed subspaces $A \subset M$ and $B \subset N$, where $M$ and $N$ are von Neumann algebras, $A \overline{\otimes} B$ denotes the ultraweak closure of the algebraic tensor product $A \otimes B$.

We shall use the following results from [12], which we summarize as a theorem.
Theorem 3.5 (Ruan [12]). Let $V$ be a $W^*$-TRO acting on separable Hilbert spaces.

(i) [12] Theorem 3.2] If $V$ is a stable $W^*$-TRO, then $V$ is TRO-isomorphic to $M(V)$ and to $N(V)$.

(ii) [12] Corollary 4.3] If $V$ is a $W^*$-TRO of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or $III$, then $V$ is a stable $W^*$-TRO, and hence TRO-isomorphic to a von Neumann algebra.

(iii) [12] Theorem 4.4] If $V$ is a $W^*$-TRO of type $II_1,\infty$ (respectively $II_{\infty,1}$), then $V$ is TRO-isomorphic to $B(H,\mathbb{C})\otimes M$ (respectively $B(\mathbb{C},H)\otimes N$), where $M$ (respectively $N$) is a von Neumann algebra of type $II_1$.

Because taking a transpose is a triple isomorphism, we have the following consequence of Theorem 3.5(iii).

Lemma 3.6. A $W^*$-TRO of type $II_{1,\infty}$ is triple isomorphic to a $W^*$-TRO of type $II_{\infty,1}$. More precisely, $B(H,\mathbb{C})\otimes M$ is triple isomorphic to $B(\mathbb{C},H)\otimes M^t$, where $x^t = Jx^*J$, for $x \in M \subset B(H)$ and $J$ is a conjugation on $H$.

Proposition 3.7. Let $V$ be a $W^*$-TRO.

(i): If $V$ acts on a separable Hilbert space and is of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or $III$, then every triple derivation of $V$ is an inner triple derivation and every TRO derivation of $V$ is an inner TRO-derivation.

(ii): If every TRO-derivation of any $W^*$-TRO of type $II_{1,\infty}$ has only inner TRO-derivations, then every TRO-derivation of any $W^*$-TRO of type $II_{\infty,1}$ has only inner TRO-derivations. The converse also holds.

Proof. (i) is an immediate consequence of Theorem 3.5(ii), Theorem 3.2 and Proposition 2.4(iv). (ii) is an immediate consequence of Lemma 3.6 and Proposition 2.4(iv).

It follows from Remark 3.4 that if $M$ is a von Neumann algebra of type $II_\infty$ or $III$ and $H$ is a separable Hilbert space, then $B(\mathbb{C},H)\otimes M$ is triple isomorphic to a von Neumann algebra and hence has only inner TRO-derivations, giving alternate proofs of parts of Proposition 3.7.

By [12, Theorem 4.1], if $V$ is a $W^*$-TRO of type $I$, then $V$ is TRO-isomorphic to $\oplus_\alpha L^\infty(\Omega_\alpha)\otimes B(K_\alpha,H_\alpha)$. In the next two results, we consider the related TRO $C(\Omega,B(H,K))$, where $\Omega$ is a compact Hausdorff space.

Lemma 3.8. Let $E$ be a TRO and $\Omega$ a compact Hausdorff space.
(i): If every TRO derivation of $V := C(\Omega, E)$ is an inner TRO derivation, then the same holds for $E$.

(ii): If every triple derivation of $V := C(\Omega, E)$ is an inner triple derivation, then the same holds for $E$.

Proof. By Proposition 2.4(iv), it is sufficient to prove (i).

If $D$ is a TRO derivation of $E$, then $\delta f(\omega) := D(f(\omega))$ is a TRO derivation of $V$, as is easily checked. Suppose every TRO derivation of $V$ is an inner TRO derivation. Then $\delta f = \alpha f + f \beta$, where $\alpha = -\alpha^* = \sum_i x_i y_i^*$ for some $x_i, y_i \in V$, and $\beta = -\beta^* = \sum_i z_j^* w_j$ for some $z_j, w_j \in V$.

For $a \in E$, let $1 \otimes a \in V$ be the constant function equal to $a$. Then $D(a) = D((1 \otimes a)(\omega)) = \delta((1 \otimes a)(\omega)) = \alpha(\omega)a + a\beta(\omega)$ for all $\omega \in \Omega$. Since $\alpha(\omega)^* = -\alpha(\omega) \in EE^*$ and $\beta(\omega)^* = -\beta(\omega) \in E^*E$, $D$ is an inner TRO derivation of $E$.

Recall from Theorem 3.3(iii) that the TRO $B(H, K)$ supports outer TRO derivations if and only if it is infinite dimensional and $\dim H \neq \dim K$.

Proposition 3.9. If $V = \bigoplus \alpha C(\Omega_\alpha, E_\alpha)$, where $E_\alpha = B(K_\alpha, H_\alpha)$ and if every triple derivation of $V$ is an inner triple derivation, then for every $\alpha$, either $\dim E_\alpha < \infty$ or $\dim K_\alpha = \dim H_\alpha$.

Proof. Let $\delta$ be a triple derivation of $V$, and let $\delta_\alpha = \delta|_{C(\Omega_\alpha, E_\alpha)}$, which is a triple derivation of the weak*-closed ideal $C(\Omega_\alpha, E_\alpha)$. Then $\delta(\{f_\alpha\}) = \{\delta_\alpha f_\alpha\}$. Moreover if $\delta$ is an inner triple derivation, say $\delta = \sum_i \delta(a_i, b_i)$ for $a_i = \{a_{i\alpha}\}, b_i = \{b_{i\alpha}\} \in V$, then $\delta_\alpha = \sum_i \delta(a_{i\alpha}, b_{i\alpha})$ is an inner triple derivation of $C(\Omega_\alpha, E_\alpha)$.

Now suppose every that every triple derivation of $V$ is an inner triple derivation, and that for some $\alpha_0$, $E_{\alpha_0}$ is infinite dimensional and $\dim K_{\alpha_0} = \dim H_{\alpha_0}$. Then, as noted above, there is an outer triple derivation $D$ of $C(\Omega_{\alpha_0}, E_{\alpha_0})$. Then the triple derivation on $V$ which is zero on $C(\Omega_\alpha, E_\alpha)$ for $\alpha \neq \alpha_0$ and equal to $D$ on $C(\Omega_{\alpha_0}, E_{\alpha_0})$, cannot be inner by the preceding paragraph, which is a contradiction.

4. SOME QUESTIONS LEFT OPEN

Questions 1. It remains to complete the results of Theorem 3.3 to include the cases where $p$ is a finite projection in a continuous von Neumann algebra, or when $p$ is arbitrary and $M$ is a general von Neumann algebra of type I. As a possible tool for the first question, we note that there is an alternate proof of Proposition 2.4(ii), in the case $X = pM$, $p$ finite, using the technique in [5, Section II.B].
Questions 2. Besides the problem of extending the known cases to non separable Hilbert spaces, the cases left open in Proposition \ref{prop:open-cases} for arbitrary W*-TROs are those of types $II_{1,1}$ and $II_{1,\infty}$ (the latter being equivalent to $II_{\infty,1}$).

Questions 3. Let $E$ be a W*-TRO, and let $V = \oplus_\alpha L^\infty(\Omega) \overline{\otimes} B(K_\alpha, H_\alpha)$ be a W*-TRO of type I.

- If every derivation of the W*-TRO $L^\infty(\Omega) \overline{\otimes} E$ is inner, does it follow that every derivation of $E$ is inner?
- If every derivation of $V$ is inner, does it follow that $\dim B(K_\alpha, H_\alpha) < \infty$, for all $\alpha$; or $\sup_\alpha \dim B(K_\alpha, H_\alpha) < \infty$?
- If $\sup_\alpha \dim B(K_\alpha, H_\alpha) < \infty$, does it follow that every derivation of $V$ is inner?

Remark 4.1. With respect to Questions 3,

(i) In the first bullet, if $E$ had a separable predual, then a variant of \cite[1.22.13]{[14]} would state that $L^\infty(\Omega) \overline{\otimes} E = L^\infty(\Omega, E)$ and the technique in Lemma \ref{lem:tech-II} could be used.

(ii) In the first bullet, suppose that $E = pM$, with $M$ a von Neumann algebra in $B(H)$ and $p$ a projection in $M$, and let $D$ is a derivation of $E$. Then $\delta := id \otimes D$ is a derivation of $V = L^\infty(\Omega) \overline{\otimes} E$. Assuming that $\delta$ is inner, there exist $\alpha = -\alpha^* \in VV^* = L^\infty(\Omega) \overline{\otimes} (EE^*)$ ($EE^*$ denoting the weak closure) and $\beta \in V^*V = L^\infty(\Omega) \overline{\otimes} (E^*E)$, such that
\[
1 \otimes Dx = \alpha(1 \otimes x) + (1 \otimes x)\beta, \quad (x \in E).
\]

We have $EE^* = pMp \subset B(pH)$, $E^*E \subset B(H)$, and $L^\infty(\Omega) \subset B(L^2(\Omega))$. For each $\varphi \in B(L^2(\Omega))^*$, let $R_\varphi : B(L^2(\Omega) \otimes pH) \to B(pH)$ be the slice map of Tomiyama defined by $R_\varphi(f \otimes x) = \varphi(f)x$ (\cite[Lemma 7.2.2]{[3]}).

Since $VV^*$ is the ultraweak closure of $L^\infty(\Omega) \otimes EE^*$, by the weak*-continuity of $R_\varphi$, we have
\[
\varphi(1)Dx = R_\varphi(\alpha)x + xR_\varphi(\beta)
\]
with $R_\varphi(\alpha) \in EE^*$ and $R_\varphi(\beta) \in E^*E$. Thus, if $\dim H = \dim pH$, or if $E$ is finite dimensional, then $R_\varphi(\beta) \in E^*E$, so that $D$ is an inner TRO-derivation, (take $\varphi$ to be a normal state so that $R_\varphi$ is self-adjoint and $R_\varphi(\alpha)^* = -R_\varphi(\alpha)$ and $R_\varphi(\beta)^* = -R_\varphi(\beta)$.) In general, $D$ could be called a “quasi-inner” TRO-derivation.

(iii) In the second bullet, if each $B(K_\alpha, H_\alpha)$ had a separable predual, then a variant of \cite[1.22.13]{[14]} would state that $L^\infty(\Omega_\alpha) \overline{\otimes} B(K_\alpha, H_\alpha) = L^\infty(\Omega_\alpha, B(K_\alpha, H_\alpha))$ and the technique in Proposition \ref{prop:tech} could be used.
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