We study the response of a two-dimensional hexagonal packing of rigid, frictionless spherical grains due to a vertically downward point force on a single grain at the top layer. We use a statistical approach, where each configuration of the contact forces is equally likely. We find that the response is double-peaked, independently of the details of boundary conditions. The two peaks lie precisely on the downward lattice directions emanating from the point of application of the force. We examine the influence of the confining pressure applied to the packing.

1 INTRODUCTION

Force transmissions in (static) granular packings have attracted a lot of attention in recent years (Jaeger & Nagel 1996; Bouchaud 2003; Mueth 1998; Blair 2001; Vanel et al. 1999; Coppersmith 1996; Nedderman 1982; Goldenberg & Goldhirsch 2002; Geng 2001; Geng 2003). Granular packings are assemblies of macroscopic particles that interact only via mechanical repulsion effected through physical contacts. Experimental and numerical studies of these systems have identified two main characteristics.

First, large fluctuations are found to occur in the magnitudes of inter-grain forces, implying that the probability distribution of the force magnitudes is rather broad (Mueth 1998; Blair 2001). Secondly, the average propagation of forces — studied via the response to a single external force — is strongly dependent on the underlying geometry (Vanel et al. 1999; Geng 2001; Geng 2003).

Most of the available theoretical models capture either one or the other of these two aspects. The scalar $q$-model (Coppersmith 1996) reproduces the observed force distribution reasonably well, but yields diffusive propagation of forces, in conflict with experiments (Geng 2001). Similarly, continuum elastic and elasto-plastic theories (Nedderman 1982) have been used by engineers for years, but they provide no information on the distribution of force magnitudes. More ad-hoc “stress-only” models (Bouchaud 2003) include structural randomness, but its consequences on the distribution of forces are unclear.

A simple conjecture, which provides a fundamental principle for the study of both fluctuations and propagation of forces, has been put forward by Edwards years ago (Edwards & Oakeshott 1989). The idea is to consider all “jammed” configurations equally likely. A priori, there is no justification for this ergodic hypothesis, but its application to models of jamming and compaction has been rather successful (Makse & Kurchan 2002). Its extension to forces in granular packings is in principle straightforward: sets of forces belonging to all mechanically stable configurations have equal probability. However, in an ensemble of stable granular packings, two levels of randomness are generally present (Bouchaud 2003). First, the force geometry clearly depends on the underlying geometrical contact network, which is different in different packings. Secondly, randomness in the values of the forces is present even in a fixed contact network, since the forces are not necessarily uniquely determined from the contact network. Instead of considering both levels of randomness simultaneously, a natural first step is thus to obtain the averages for a fixed contact geometry, and then possibly average over the contact geometries.

Such a method has been shown to produce single inter-grain force probability distributions in fixed geometry that compare well with experiments.
one first identifies an orthonormal basis that can be constructed via the three following steps: In that case, there exists a whole set of solutions column vector representing the external forces. If we elements. These elements satisfy geometrical configuration of dimensional disks of arbitrary radii — as in the hexagonal packing we study two cases: (i) mechanical equilibrium. Obviously, we study two cases: (i) mechanical equilibrium. (ii) (Ostoje & Panja 2005), in agreement with experiments (Geng 2001). Our definition of the response of the packing is \( G(i, j) = \left( W_{i,j} - \left( W_{k,l}^{(0)} \right) \right) / F \) where \( W_{i,j} \) and \( W_{i,j}^{(0)} \) are the vertical force transmitted by the \((i, j)\)th grain to the layer below it respectively with and without the external overload \( F \). The angular brackets denote averaging with equal probability over all configurations of repulsive contact forces in mechanical equilibrium.

Below we summarize the method and the results regarding the effects of confining pressure. More precisely, we study two cases: (i) \( p_2 = 0 \); (ii) \( p_1 = p_2 \equiv p \).

2 CONCEPTUAL MODEL

To start with, we describe a method for assigning the uniform probability measure on the ensemble \( \mathcal{E} \) of stable repulsive contact forces pertaining to a fixed geometrical configuration of \( P \) rigid, frictionless twodimensional disks of arbitrary radii. The directions of the forces are fixed at each of the \( Q \) contact points, and one can represent any force configuration by a column vector \( \mathbf{F} \) consisting of \( Q \) non-negative force magnitudes \( \{ F_k \} \) (with \( k = 1, \ldots, Q \)) as its individual elements. These elements satisfy \( 2P \) Newton’s equations, which can be represented as \( \mathbf{A} \cdot \mathbf{F} = \mathbf{F}_{\text{ext}} \). Here, \( \mathbf{A} \) is a \( 2P \times Q \) matrix, and \( \mathbf{F}_{\text{ext}} \) is a \( 2P \)-dimensional column vector representing the external forces. If we assume \( 2P < Q \) — as in the hexagonal packing we consider — then there is no unique solution for \( \mathbf{F} \).

In that case, there exists a whole set of solutions that can be constructed via the three following steps: (1) one first identifies an orthonormal basis \( \{ \mathbf{F}^{(l)} \} \) that spans the space of Ker(\( \mathbf{A} \)); (2) one then determines a unique solution \( \mathbf{F}^{(0)} \) of \( \mathbf{A} \cdot \mathbf{F} = \mathbf{F}_{\text{ext}} \) by requiring \( \mathbf{F}^{(0)} \mathbf{F}^{(l)} = 0 \) for \( l = 1, \ldots, d_K \); and (3) one finally obtains all solutions of \( \mathbf{A} \cdot \mathbf{F} = \mathbf{F}_{\text{ext}} \) as \( \mathbf{F} = \mathbf{F}^{(0)} + \sum_{l=1}^{Q-2P} f_l \mathbf{F}^{(l)} \), where \( f_l \), for \( l = 1 \ldots Q - 2P \), are real numbers. This implies that \( \mathcal{E} \) is parametrized by the \( f_l \)'s belonging to a set \( \mathcal{S} \) obeying the non-negativity conditions for all forces, i.e. \( \mathbf{F}^{(0)} + \sum_{l=1}^{Q-2P} f_l \mathbf{F}^{(l)} \geq 0 \), \( \forall k = 1 \ldots Q \). The uniform measure on \( \mathcal{E} \) is thus equivalent to the uniform measure \( d\mu = \prod_k dF_k \delta(\mathbf{A} \cdot \mathbf{F} - \mathbf{F}_{\text{ext}}) \Theta(F_k) = \prod_l df_l \) on \( \mathcal{S} \).

3 ANALYSIS

Figure 2. Schematically shown forces on the \( j \)th grain in the \( n \)th layer: (a) \( i = 0 \), \( W_{\text{ext}}^{(j)} = p_2 + \delta_{j,0} F \) (b) \( i \leq N \), (c) \( i = N \); \( F^{(i,j)}_{m} \geq 0 \) \( \forall m \).

We will now apply the general method described above to a hexagonal packing of monodisperse, massless, rigid and frictionless disks subject to uniform confining pressures \( p_1 \) from the sides and \( p_2 \) from the top (see Fig. 1). Our aim is to calculate the mean of \( W_{i,j} = \frac{\sqrt{3}}{2} \left[ F_{1}^{(i,j)} + F_{2}^{(i,j)} \right] \) (see Fig. 2) when an overload \( F \) is applied on the top of the packing.

Force balance on individual grains (see Fig. 2) can be expressed as

\[
F_{5}^{(i,j)} = \frac{1}{\sqrt{3}} W_{\text{ext}}^{j} + [F_{4}^{(1,j)} - F_{3}^{(1,j)}] \\
F_{6}^{(i,j)} = \frac{1}{\sqrt{3}} W_{\text{ext}}^{j} - [F_{4}^{(1,j)} - F_{3}^{(1,j)}] \\
F_{5}^{(i,j)} = F_{2}^{(i,j)} + [F_{4}^{(i,j)} - F_{3}^{(i,j)}] \\
F_{6}^{(i,j)} = F_{1}^{(i,j)} - [F_{4}^{(i,j)} - F_{3}^{(i,j)}] \\
W_{N,j} = \sqrt{3} \left[ F_{1}^{(N,j)} + F_{2}^{(N,j)} \right] / 2 \\
F_{4}^{(N,j)} - F_{3}^{(N,j)} = \left[ F_{1}^{(N,j)} - F_{2}^{(N,j)} \right] / 2. \tag{1}
\]

Note that these equations can be solved \textit{gram by \textit{gram}} for increasing \( i \) and \( j \) starting from the top layer. For \( i = 1 \) and \( j = 1 \), the top and side forces are known from the boundary conditions: \( W_{\text{ext}}^{1} = p_2 \) and

(Snoeijer 2004). Following a similar path, we studied the response of a two-dimensional hexagonal packing of rigid, frictionless spherical grains (see Fig. 1) to a pointlike external force \( F \), and found that it is concentrated mainly on the two lattice directions emanating from the point of application of the force (Ostoje & Panja 2005), in agreement with experiments (Geng 2001). Our definition of the response of the packing is \( G(i, j) = \left( W_{i,j} - \left( W_{k,l}^{(0)} \right) \right) / F \). At the top, a uniform vertical pressure \( p_2 \) is added on one grain. On the sides, a uniform pressure \( p_1 \) is applied on the packing (little gray circles appear on interfaces where the contact forces are non-zero).
grains are chosen to be uncorrelated and identically

bounded and the uniform measure on $S$ is bounded and the uniform measure on $S$ is well-defined.

4 $q$-COORDINATES AND COMPUTATIONAL SCHEME

In order to evaluate $\langle W_{i,j} \rangle = \frac{1}{N} \int_S W_{i,j} \prod_{k,l} dF_{4}^{(k,l)}$ where $N = \int_S \prod_{i,j} dF_{4}^{(ij)}$ is the normalization constant, we define

$$q_{i,j} = \left[ \sqrt{3}(F_{4}^{(ij)} - F_{3}^{(ij)} + F_{2}^{(ij)}))/2 \right]/W_{i,j},$$

where $q_{i,j}$ is the fraction of $W_{i,j}$ that the $(i,j)$th grain transmits to the layer below it towards the left, i.e., $F_{4}^{(ij)} = 2q_{i,j}W_{i,j}/\sqrt{3}$ and $F_{6}^{(ij)} = 2(1 - q_{i,j})W_{i,j}/\sqrt{3}$. Clearly, $W_{0,j}$ are the external forces applied on the top layer. For $i > 0$, $W_{i,j}$ is a function of $q_{k,l}$ for $k < i$, since

$$W_{i,j} = (1 - q_{i-1,j-1}) W_{i-1,j-1} + q_{i-1,j} W_{i-1,j}.$$

The probability measure on the new coordinates $q_{i,j}$ is given by the Jacobian of the variable change

$$\prod_{i,j} dF_{4}^{(ij)} = 2^{NP} \prod_{i,j} dq_{i,j} W_{i,j}(q)/3^{NP/2}.$$
the peaks are more pronounced and propagate deeper with the inverse system size [Fig. 3(a); we how-

Figure 4. Simulation results for $G(i,j)$, for $p_1 = p_2 = 1$: (a) $G(i,j)$ for $F = 10$ as function of $j - j_0$, at four different values of $i$ for various system sizes; (b) $G(i,j)$ as function of $i$ for $j_0 - j = i$, for $F/p = 5, 10$ and $15$ and three different system sizes.

with the inverse system size [Fig. 3(a); we however show only two $z$ values], while the $G(x,z)$ values for $|x| = z/2$ lie on the same curve for all system sizes [Fig. 3(b)]. The data suggest that in the thermodynamic limit $N \to \infty$, the response $G(x,z)$ scales $\sim 1/N$ for $|x| < z/2$, but reaches a non-zero limiting value on $|x| = z/2$ for $\forall z < 1$. We thus expect $\lim_{N \to \infty} G(x,z)|_{x=z/2} > G(x,z)|_{|x|<z/2} \forall z < 1$; or equivalently, a double-peaked response at all depths $z < 1$ in the thermodynamic limit.

For $F/p_1 > \frac{\sqrt{3}}{2}$ and $p_2 = 0$, as the value of $p_1$ decreases, the $q_{i,j}$’s get restricted to narrower ranges within $(0, 1)$ with increasing values of $F/p_1$, and consequently, an increasing amount of vertically downward force carried by the grains is transferred to the boundary of the triangle. In the limit $F/p_1 \to \infty$, the non-negativity conditions of all the forces make all $F_{i,j}$ vanish, and the packing effectively becomes rectangular. In fact, the same behaviour also occurs for any value of $p_2 \neq 0$.

For $p_2 \neq 0$ however, experimentally the most relevant case is $p_1 = p_2 = p$, since in practice a confining pressure has to be applied on the packing. In this case, the response clearly depends on the dimensionless parameter $F/p$. The simulation results for $N = 20, 30$ and $40$ are presented in Fig. 4.

Fig. 4(a) shows the response for $F/p = 10$ as function of $j - j_0$ for various values of $i$. For small $i$, the response displays two single-grain-diameter-wide symmetric peaks along the lattice directions emanating from the point of application of the overload. As $i$ increases, the magnitude of the peaks decreases, and the response for large $i$ becomes essentially flat. The dependance of the response on the system size in this case however is very different from the case $p_2 = 0$. Rather then being self-similar, the response is independent of $N$: $G(i,j)$ at a given depth $i$ is the same for all systems with $N > i$ layers.

The values of $G(i,j)$ on the lattice direction $i = j_0 - j$ as function of $i$ are plotted in Fig. 4(b), for $N = 20, 30$ and $40$, and $F/p = 5, 10$ and $15$. For increasing $F/p$, the values $G(i,j_0 - i)$ increase for fixed $i$ and their decay with increasing $i$ is slower, implying that the peaks are more pronounced and propagate deeper in the packing.

6 CONCLUSIONS

In summary, we find that assigning equal probability to all mechanically stable force configurations for rigid, frictionless spherical grains in a two-dimensional hexagonally close-packed geometry yields a double-peaked response independently of the details of the boundary condition. The peaks are single grain diameter wide, and they lie on the two downward lattice directions emanating from the point of application of $F$. In the case of zero top pressure, the response exhibits self-similarity, but when the packing is confined by uniform pressure, the peaks penetrate the packing deeper with larger $F$. Such a simple model is in good qualitative agreement with experiments in a reasonably robust manner. Whether grains with friction (Breton 2002) produce broadening of the peaks or not remains to be investigated.

REFERENCES

Blair, D. L. et al., 2001. Force distributions in 3d granular assemblies: Effects of packing order and inter-particle friction. Phys. Rev. E 63: 041304.

Bouchaud, J.-P. 2003. Granular media: some ideas from statistical physics. In J. Barrat (ed.), Les Houches, Session LXXVII. EDP Sciences.

Breton, L. et al., 2002. Stress response function of a two-dimensional ordered packing of frictional beads. Europhys. Lett. 60: 813.

Coppersmith, S. et al., 1996. Model for force fluctuations in bead packs. Phys. Rev. E 53: 4673.

Edwards, S. F. & Oakeshott, R. 1989. Theory of powders. Physica A 157: 1080.

Geng, J. et al., 2001. Footprints in sand: The response of a granular material to local perturbations. Phys. Rev. Lett. 87: 035506.

Geng. J. et al., 2003. Green’s function measurements of force transmission in 2d granular materials. Physica D 182: 274.

Goldenberg, C. & Goldhirsch, I. 2002. Force chains, microelasticity and macroelasticity. Phys. Rev. Lett. 89: 084302.

Jaeger, H. M. & Nagel, S. R. 1996. Granular solids, liquids, and gases. Rev. Mod. Phys. 68: 1259.

Makse, H. A. & Kurchan, J. 2002. Testing the thermodynamic approach to granular matter with a numerical model of a decisive experiment. Nature 415: 614.

Mueth, D. et al. 1998. Force distribution in a granular medium. Phys. Rev. E 57: 3164.

Nedderman, R. M. 1982. Statics and Kinematics of Granular Materials. Cambridge University Press.

Newman, M. E. J. & Barkema, G. 1999. Monte Carlo Methods in Statistical Physics. Oxford University Press.

Ostojic, S. & Panja, D. 2005. Response of a hexagonal granular packing under a localized external force: Exact results. J. Stat. Mech.: P01011.

Snoeijer, J. H. et al., 2004. Force network ensemble: a new approach to static granular matter. Phys. Rev. Lett. 92: 054302.

Vanel, L., Howell, D., Clark, D., Behringer, R. P., & Clement, E. 1999. Memories in sand: Experimental tests of construction history on stress distributions under sandpiles. Phys. Rev. E 60: R5040.