Positive periodic solutions for systems of impulsive delay differential equations

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Abstract
A class of periodic differential $n$-dimensional systems with patch structure with (possibly infinite) delay and nonlinear impulses is considered. These systems incorporate very general nonlinearities and impulses whose signs may vary. Criteria for the existence of at least one positive periodic solution are presented, extending and improving previous ones established for the scalar case. Applications to systems inspired in mathematical biology models, such as impulsive hematopoiesis and Nicholson-type systems, are also included.

Keywords: delay differential equations, impulses, positive periodic solutions, Krasnoselskii’s fixed point theorem, Nicholson systems.

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1 Introduction
Recently, differential equations with delays and impulses have been proposed as models in population dynamics, artificial neural networks, disease systems, chemical processes and in a number of other scientific settings. They often lead to very realistic models for evolutionary systems which go through sudden changes, caused by either natural phenomena, drug administration or other artificial inputs. Due to the real world interpretation of such equations, in many contexts only positive solutions are of interest. In the case of periodic models, without and with impulses, whether there exists any positive periodic solution is a prime question in applications.

This paper is concerned with a class of impulsive delay differential systems written in abstract form as

\begin{equation}
\begin{cases}
x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t)x_j(t) + g_i(t, x_{it}) \text{ for } t \neq t_k, \\
\Delta(x_i(t_k)) := x_i(t_k^+) - x_i(t_k) = I_{ik}(x_i(t_k)), \quad k \in \mathbb{Z}, \quad i = 1, \ldots, n,
\end{cases}
\end{equation}

where $(t_k)_{k \in \mathbb{Z}} \subset \mathbb{R}$ is a strictly increasing sequence, the functions $d_i, a_{ij}$ and $g_i$ are continuous, nonnegative and periodic in $t \in \mathbb{R}$ (with a common period $\omega > 0$), and, as usual, $x_t = (x_{1t}, \ldots, x_{nt})$.
denotes the past history segment of the solution given by \( x_t(s) = x(t + s) \) for \( s \in [-\tau, 0] \), where \( \tau \) is the maximum time-delay. The consideration of equations with infinite delay, in which case \( x_t(s) = x(t + s) \) for \( s \leq 0 \), is also possible. The impulses are supposed to occur with periodicity \( \omega \) and satisfy some additional conditions introduced in the next section. Clearly, appropriate phase spaces and conditions on the impulses have to be chosen for delay differential equations (DDEs) with impulses, so that the existence of solutions for the usual initial value problems is ensured [14,23,25].

Eq. (1.1) may refer to growth models of one or multiple populations, distributed over \( n \) classes or patches with migration of the populations among them. For each \( i \), \( x_i(t) \) is the density of the population in class \( i \), \( a_{ij}(t) \) \( (j \neq i) \) are the migration coefficients from class \( j \) to class \( i \), \( d_i(t) \) the coefficient of instantaneous loss for class \( i \) (which includes the death rate and the emigration rates for the population leaving class \( i \)), and \( g_i \) is the so called birth or production function. Since (1.1) is considered within the framework of mathematical biology or other natural sciences, only positive (or non-negative) solutions of (1.1) are meaningful. Note that (1.1) encompasses some relevant models, such as Nicholson or Mackey-Glass-type systems with patch structure and impulses.

Our main goal is to give sufficient conditions for the existence of at least one positive \( \omega \)-periodic solution to (1.1), extending to \( n \)-dimensional systems previous results established in [5,11] for very broad classes of scalar impulsive DDEs. The method used here relies on Krasnosel’skiǐ’s fixed point theorem in cones, which is applied to a convenient and original operator constructed here, whose fixed points are precisely the \( \omega \)-periodic solutions of (1.1).

Among other techniques, several fixed point theorems have been extensively used to derive the existence of solutions, as well as the existence of periodic or almost periodic solutions, both for scalar and multidimensional DDEs. For periodic scalar DDEs, this has been the subject of many researches, see e.g. [6,27,29] and [5,11,18,20,32], respectively for models without and with impulses, and references therein. We remark that, even in the scalar impulsive case, many authors restrict their analysis to DDEs with discrete delays and linear impulses. In the setting of nonimpulsive systems of DDEs, Li [19] employed the continuation theorem of coincidence degree to show that a positive periodic solution must exist for a family of periodic competitive \( n \)-dimensional Lotka-Volterra systems with distributed delay, and later Krasnosel’skiǐ’s fixed point theorem was applied in [26] and [3] to some classes of Lotka-Volterra systems with discrete delays. Recently, degree techniques were also used in [2] to investigate the existence of a nontrivial periodic solution to systems with a single discrete delay \( \tau > 0 \) in the general form \( x'(t) = f(t, x(t), x(t - \tau)) \), with \( f \) non-negative, continuous and periodic in \( t \). We also refer to [7,8,11,25,30] for the treatment of periodic or almost periodic multidimensional Nicholson systems.

The literature is however practically nonexistent in what concerns the use of fixed point methods to address the existence of a positive periodic solution for impulsive systems of DDEs, the paper of Zhang et al. [34] (on a planar impulsive Nicholson system) being an exception. As far as the authors know, the new methodology proposed here for the first time allows handling very broad classes of impulsive systems of DDEs, with very mild constraints on the impulses.

The organization of this paper is now described. Section 2 is a section of preliminaries, where the main hypotheses for (1.1) are introduced, a suitable operator \( \Phi \) on a cone \( K \) is defined and its major properties deduced. Section 3 contains the main results of the paper, which establish easily verifiable sufficient conditions for the existence of positive fixed points of \( \Phi \), i.e., \( \omega \)-periodic solutions to (1.1). A version of the Krasnosel’skiǐ theorem in [1] is used, both in its compressive and expansive forms. Moreover, as simple consequences of the main results, criteria based on either a
pointwise or an average comparison (for \( t \in [0, \omega] \)) of the coefficients in (1) are derived. In Section 4, we analyse some families of systems with bounded linearities. Section 5 presents some selected examples inspired in mathematical biology models, within the framework of impulsive hematopoiesis and Nicholson-type systems. A short section of conclusions ends the paper.

2 Preliminaries

We first set some notation. For a compact interval \( I = [\alpha, \beta] \) \((\alpha < \beta)\) and \( n \in \mathbb{N} \), consider the space of the piecewise continuous functions on \( I \) which are left-continuous in \((\alpha, \beta]\)

\[
PC(I, \mathbb{R}^n) := \{ \varphi : I \to \mathbb{R}^n | \varphi \text{ is continuous except for a finite number of points } \}
\]

for which there are \( \varphi(s^-) = \varphi(s), \varphi(s^+) \}

with the norm \(|\varphi|_\infty = \max_{t \in I} |\varphi(t)|\), for some fixed norm \(|\cdot|\) in \( \mathbb{R}^n \). For \( \omega > 0 \), denote \( PC_\omega(\mathbb{R}, \mathbb{R}^n) = \{ x : \mathbb{R} \to \mathbb{R}^n | x \text{ is } \omega\text{-periodic} \} \), with the supremum norm in \([0, \omega] \). We write \( \mathbb{R}^+ = [0, \infty), \mathbb{R}^- = (-\infty, 0] \). For \( v \in \mathbb{R}^n \), \( v \geq 0 \) stands for \( v \in (\mathbb{R}^+)^n \) and \( v > 0 \) for \( v \in (0, \infty)^n \); for a function \( x : I \to \mathbb{R}^n \), \( x \geq 0 \), \( x > 0 \) stand for \( x(t) \geq 0 \), \( x(t) > 0 \) for all \( t \in I \), respectively.

Consider the cone of non-negative elements \( PC_\omega^+(\mathbb{R}, \mathbb{R}^n) = \{ x \in PC_\omega(\mathbb{R}, \mathbb{R}^n) : x \geq 0 \} \). For spaces of continuous (rather than piecewise continuous) functions, the similar notations \( C_\omega(\mathbb{R}, \mathbb{R}^n) \) and \( C_\omega^+(\mathbb{R}, \mathbb{R}^n) \) will be used. When \( n = 1 \), we also write \( C_\omega(\mathbb{R}) = C_\omega(\mathbb{R}, \mathbb{R}), C_\omega^+(\mathbb{R}) = C_\omega^+(\mathbb{R}, \mathbb{R}) \) and \( PC_\omega(\mathbb{R}) = PC_\omega(\mathbb{R}, \mathbb{R}), PC_\omega^+(\mathbb{R}) = PC_\omega^+(\mathbb{R}, \mathbb{R}) \). Here, \( \mathbb{R}^n \) is also seen as the set of constant functions (defined on an interval \( I \) or \( \mathbb{R} \)).

For a fixed finite time-delay \( \tau > 0 \), set \( PC := PC([-\tau, 0], \mathbb{R}^n) \) as the phase space. Consider the \( n \)-dimensional DDE with impulses (1.1), for which initial conditions have the form \( x_\sigma = \varphi \) for \((\sigma, \varphi) \in \mathbb{R} \times PC \). The following hypotheses on (1.1) will be assumed:

(H1) The functions \( I_{ik} : \mathbb{R}^+ \to \mathbb{R} \) are continuous and there is a positive integer \( p \) such that \( 0 \leq t_1 < \cdots < t_p < \omega \) (for some \( \omega > 0 \)) and \( t_{k+p} = t_k + \omega, \ I_{i,k+p} = I_{ik}, \ k \in \mathbb{Z}, i = 1, \ldots, n; \)

(H2) There exist constants \( \alpha_{ik} > -1 \) and \( \eta_{ik} \) such that \( \alpha_{ik} u \leq I_{ik}(u) \leq \eta_{ik} u \) for \( u \geq 0 \) and there are the limits \( \lim_{u \to 0^+} \frac{u}{I_{ik}(u)}, \) for \( i = 1, \ldots, n, k = 1, \ldots, p; \)

(H3) \( \prod_{k=1}^p (1 + \eta_{ik}) < e^{\int_0^\omega d_i(t) \, dt}, i = 1, \ldots, n; \)

(H4) (i) For \( i, j = 1, \ldots, n, d_i, a_{ij} \in C_\omega^+(\mathbb{R}) \) with \( \int_0^\omega d_i(s) \, ds > 0 \), the functions \( g_i : \mathbb{R} \times PC([0, \omega], \mathbb{R}) \to \mathbb{R}^+ \) are continuous, \( \omega \)-periodic in \( t \in \mathbb{R} \) and

\[
g(t, x_t) := (g_1(t, x_{1t}), \ldots, g_n(t, x_{nt}))
\]

is bounded on bounded sets of \( \mathbb{R} \times PC; \)

(ii) moreover, if \( n > 1 \), either \( \int_0^\omega a_{ij}(s) \, ds > 0 \) for all \( i \neq j \) or \( \int_0^\omega g_i(s, 0) \, ds > 0 \), for each \( i = 1, \ldots, n. \)

For fixed \( \omega > 0 \), \( n \in \mathbb{N} \) and a sequence \((t_k)_{k \in \mathbb{Z}}\) as in (H1), define the space

\[
X := X(\mathbb{R}^n) = \{ x : \mathbb{R} \to \mathbb{R}^n | x \text{ is } \omega \text{-periodic, continuous for all } t \neq t_k, \text{ and } x(t_k^-) = x(t_k), \ x(t_k^+) \in \mathbb{R}, \text{ for } k \in \mathbb{Z} \}.
\]
and the cone
\[ X^+ := X(\mathbb{R}^n)^+ = \{ x \in X : x(t) \geq 0, \ t \in [0, \omega] \}. \] (2.2)

Hereafter, \( X \) is endowed with the norm \( \| \cdot \| \infty \) (where the maximum norm in taken in \( \mathbb{R}^n \)), simply denoted by \( \| \cdot \| \), and with the partial order induced by the cone \( X^+ \).

**Remark 2.1.** In the case of infinite delay, as phase space we may take any admissible Banach space \((\mathcal{B}, \| \cdot \|_\mathcal{B})\) (in the sense of Hale and Kato definition [13]) of functions from \( \mathbb{R}^- \) to \( \mathbb{R}^n \), such that \( \mathcal{B} \) contains the space \( PC_\omega(\mathbb{R}^-, \mathbb{R}^n) \) of piecewise continuous, \( \omega \)-periodic functions \( x : \mathbb{R}^- \to \mathbb{R}^n \), and such that the norms \( \| \cdot \| \) and \( \| \cdot \|_\mathcal{B} \) are equivalent in \( PC_\omega(\mathbb{R}^-, \mathbb{R}^n) \). See [5, 14] for details. To simplify the exposition, below we only consider systems with finite delay, although straightforward adjustments can be effected to deal with the infinite delay case.

We remark that (H2) implies that \( I_{ik}(u) < -u \) for \( u > 0 \), hence a positive solution of (1.1) will remain positive after suffering an impulse at each instant \( t_k \). If \( n > 1 \), without loss of generality we take \( a_{ii} \equiv 0 \) for \( i = 1, \ldots, n \). For \( d_i \in C^+_\omega(\mathbb{R}) \), the requirement \( \int_0^\omega d_i(s) \, ds > 0 \) guarantees that \( d_i \) is not identically zero. Similarly, with \( a_{ij}, g_i(\cdot, 0) \in C^+_\omega(\mathbb{R}) \), \( a_{ij} \neq 0 \) if \( \int_0^\omega a_{ij}(s) \, ds > 0 \) (\( j \neq i \)) and \( g_i(\cdot, 0) \neq 0 \) if \( \int_0^\omega g_i(s, 0) \, ds > 0 \). The role of assumption (H4)(ii) is to preclude the existence of periodic solutions with one component positive but with others that may vanish. See additional comments on Remark 2.2.

In order to simplify the exposition, for \( i = 1, \ldots, n, k = 1, \ldots, p, t \in \mathbb{R} \), consider the following auxiliary functions:

\[
D_i(t) = \int_0^t d_i(s) \, ds, \quad J_{ik}(u) = \begin{cases} \frac{u}{u + I_{ik}(u)}, & u > 0, \\ \lim_{u \to 0^+} \frac{u}{u + I_{ik}(u)}, & u = 0 \end{cases} \] (2.3)

\[
B_i(t; x_i) = \prod_{k: t_k \in [0, t)} J_{ik}(x_i(t_k)) \quad \text{and} \quad B_i(t; x_i) = \frac{B_i(s; x_i)}{B_i(t; x_i)} = \prod_{k: t_k \in [t, s]} J_{ik}(x_i(t_k)) \quad \text{for } 0 \leq t \leq s \leq t + \omega, x \in X^+, \] (2.4)

and the autonomous quantities

\[
D_i(\omega) = \int_0^\omega d_i(s) \, ds, \quad B_i(\omega; x_i) = \prod_{k=1}^p J_{ik}(x_i(t_k)), \] (2.6)

\[
\Gamma_i(x_i) = \left( B_i(\omega; x_i)e^{D_i(\omega)} - 1 \right)^{-1} \quad \text{for } i = 1, \ldots, n, x \in X^+. \] (2.7)

We adopt the usual convention that a product is equal to one when the number of factors is zero.

For systems without impulses, clearly \( J_{ik}(u) \equiv 1, B_i(t; x_i) \equiv 1, \Gamma_i(x_i) \equiv (e^{D_i(\omega)} - 1)^{-1} \). On the other hand, when all impulses are linear, i.e., \( I_{ik}(u) = \eta_{ik} u \) for some constants \( \eta_{ik} > -1 \) with (H1),(H3) fulfilled, \( J_{ik} \) are also constants, \( J_{ik} \equiv (1 + \eta_{ik})^{-1} \), thus the functions \( B_i \) and \( \tilde{B}_i \) do not depend on \( x \).

We state some properties of these auxiliary functions, whose validity is easily verified by adapting the arguments for the scalar version of (1.1); the reader is referred to [5] to complete a proof of the properties below.
Lemma 2.1. Assume (H1)–(H4). For \( i = 1, \ldots, n, k \in \mathbb{Z}, x = (x_1, \ldots, x_n) \in X^+ \):
(i) \( J_{ik} : \mathbb{R}^+ \to (0, \infty), \Gamma_i : X^+(\mathbb{R}) \to (0, \infty) \) are continuous and satisfy
\[
(1 + \eta_{ik})^{-1} \leq J_{ik}(u) \leq (1 + \alpha_{ik})^{-1}, \quad u \geq 0,
\]
where \( \Gamma_i := \left( \prod_{k=1}^p (1 + \alpha_{ik})^{-1} e^{D_i(\omega)} - 1 \right)^{-1} \),
\( \overline{\Gamma}_i := \left( \prod_{k=1}^p (1 + \eta_{ik})^{-1} e^{D_i(\omega)} - 1 \right)^{-1} \);
(ii) \( B_i(t + \omega; x_i) = B_i(t; x_i)B(\omega; x_i) \) for \( t \in \mathbb{R} \);
(iii) \( B_i(t_k + \varepsilon; x_i) = B_i(t_k; x_i)J_{ik}(x_i(t_k))^{-1} \) for \( 0 < \varepsilon < \min_{1 \leq k \leq p} (t_{k+1} - t_k) \);
(iv) \( \tilde{B}_i(s; x_i) \) are bounded functions on \( D \times X^+(\mathbb{R}) \), where \( D = \{ (s, t) \in \mathbb{R}^2 : t \leq s \leq t + \omega \} \),
with
\[
B_i \leq \tilde{B}_i(s; t; x_i) \leq \overline{B}_i \quad \text{for} \quad (s, t, x_i) \in D \times X^+(\mathbb{R}),
\]
where \( B_i := \min \{ \prod_{k=j}^{j+l-1} (1 + \eta_{ik})^{-1} : j = 1, \ldots, p, l = 0, \ldots, p \} \),
\( \overline{B}_i := \max \{ \prod_{k=j}^{j+l-1} (1 + \alpha_{ik})^{-1} : j = 1, \ldots, p \} \);
(v) \( \tilde{B}_i(s + \omega, t + \omega; x_i) = \tilde{B}_i(s, t; x_i) \) for \( (s, t, x_i) \in D \times X^+(\mathbb{R}) \);
(vi) if \( x(t) = (x_1(t), \ldots, x_n(t)) \) is a solution of (1.1), the function \( y(t) = (y_1(t), \ldots, y_n(t)) \), where
\[
y_i(t) = B_i(t; x_i)x_i(t), \quad i = 1, \ldots, n,
\]
is continuous.

From Lemma 2.1 we obtain:

Lemma 2.2. Assume (H1)–(H4). The operator \( \Phi : X^+ \to X^+ \) given by
\[
\Phi = (\Phi_1, \ldots, \Phi_n),
\]
\[
(\Phi_i x)(t) = \Gamma_i(x_i) \int_t^{t+\omega} \tilde{B}_i(s, t; x_i) e^{\int_t^s d_*(r)} ds, \quad t \in \mathbb{R},
\]
is well defined. Moreover, \( x \) is a nonnegative \( \omega \)-periodic solution of (1.1) if and only if \( x \) is a fixed point of \( \Phi \).

Proof. We argue along the major lines in [3,11]. Let \( x = (x_1, \ldots, x_n) \in X^+ \). Clearly, \( \Phi x \geq 0 \), \( t \mapsto (\Phi x)(t) \) is continuous for \( t \neq t_k \) and left-continuous on \( t_k, \, k = 1, \ldots, p \). The properties in Lemma 2.1 show that \( (\Phi x)(t) \) is \( \omega \)-periodic and that
\[
(\Phi_i x)(t_k^+) = \lim_{\varepsilon \to 0^+} (\Phi_i x)(t_k + \varepsilon) = J_{ik}(x_i(t_k))^{-1}(\Phi_i x)(t_k)
\]
for all \( i \in \{1, \ldots, n\} \) and \( k \in \mathbb{Z} \), thus \( \Phi(X^+) \subset X^+ \).

Take \( x = (x_1, \ldots, x_n) \in X^+ \) and suppose that \( x \) is a solution of (1.1). For the continuous function \( y(t) \) with components as in (2.8) and \( t \neq t_k \) \( (k \in \mathbb{Z}) \), we have
\[
\left( y_i'(t) + d_i(t)y_i(t) \right) e^{D_i(t)} = B_i(t; x_i) e^{D_i(t)} \left( \sum_{j \neq i} a_{ij}(t)x_j(t) + g_i(t, x_{it}) \right).
\]
Since \(x_i(t)\) is \(\omega\)-periodic and \(g_i(t)\) is continuous, integration over intervals \([t, t + \omega]\), the properties in Lemma 2.1 and computations as in [5] lead to

\[
x_i(t)B_i(t; x_i)e^{D_i(t)}
\]

\[
\int_t^{t+\omega} B_i(s; x_i)e^{D_i(s)} \left( \sum_{j \neq i} a_{ij}(s)x_j(s) + g_i(s, x_{is}) \right) ds,
\]

thus \(x_i(t) = (\Phi_i x)(t)\) for all \(i\) and \(t\), and \(x\) is a fixed point of \(\Phi\).

Conversely if \(x \in X^+\) is a fixed point of \(\Phi\), for \(t \neq t_k\) differentiation of \(x_i(t) (1 \leq i \leq n)\) gives

\[
x_i'(t) = (\Phi_i x)'(t)
\]

\[
= -d_i(t)(\Phi_i x)(t) + \Gamma_i(x_i) \left( B_i(\omega; x_i)e^{D_i(\omega)} - 1 \right) \left( \sum_{j \neq i} a_{ij}(t)x_j(t) + g_i(t, x_{it}) \right)
\]

\[
= -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + g_i(t, x_{it}).
\]

On the other hand, for \(t = t_k\), from (2.10) we get

\[
\Delta x_i(t_k) = (\Phi_i x)(t_k^+) - x_i(t_k)
\]

\[
= J_{ik}(x_i(t_k))^{-1} (\Phi_i x)(t_k) - x_i(t_k)
\]

\[
= [J_{ik}(x_i(t_k))^{-1} - 1] x_i(t_k) = I_{ik}(x_i(t_k)).
\]

Therefore, \(x\) is a solution of (1.1).

For any \(\sigma = (\sigma_1, \ldots, \sigma_n) \in (0, 1)^n\), consider a new cone \(K(\sigma)\) in \(X\) given by

\[
K(\sigma) := \{ x \in X^+ : x_i(t) \geq \sigma_i \| x_i \|, t \in [0, \omega], i = 1, \ldots, n \}.
\]

(2.11)

If \(\sigma \in (0, 1)^n\) is fixed, we denote \(K(\sigma)\) simply by \(K\) and

\[
K_0 = K_0(\sigma) := \{ x \in K : x_i(t) > 0, t \in [0, \omega], i = 1, \ldots, n \}.
\]

The aim is to prove the existence of a fixed point of \(\Phi\) in \(K_0\), so that a positive \(\omega\)-periodic solution of (1.1) exists. For this, a Krasnoselskii fixed point theorem in the version in [1, Theorems 7.3 and 7.6], which includes both the compressive and expansive forms, will be used.

**Theorem 2.1.** [3] Let \(K\) be a closed cone in a Banach space, \(r, R \in \mathbb{R}^+\) with \(r \neq R\), \(r_0 = \text{min}\{r, R\}\), \(R_0 = \text{max}\{r, R\}\) and \(K_{R_0} := \{ x \in K : \|x\| \leq R_0 \}\). Assume that \(T : K_{R_0} \rightarrow K\) is a completely continuous operator such that

1. \(Tx \neq \lambda x\) for all \(x \in K\) with \(\|x\| = R\) and all \(\lambda > 1\);

2. There exists \(\psi \in K \setminus \{0\}\) such that \(x \neq Tx + \lambda \psi\) for all \(x \in K\) with \(\|x\| = r\) and all \(\lambda > 0\).

Then \(T\) has a fixed point in \(K_{r_0, R_0} := \{ x \in K : r_0 \leq \|x\| \leq R_0 \}\).

For \(g\) as in (H4), we also define

\[
G(t, x) = g(t, x_t) \quad \text{for} \quad t \in \mathbb{R}, x \in X^+.
\]

(2.12)

To derive the compactness of the operator \(\Phi\), an additional hypothesis on \(g\) is assumed:
(H5) The function \( t \mapsto G(t, x) \) is uniformly equicontinuous for \( t \in [0, \omega] \) on bounded sets of \( K \), in the sense that for any \( A \subset K \) bounded and \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( \max_{t \in [0, \omega]} |G(t, x) - G(t, y)| < \varepsilon \) for all \( x, y \in A \) with \( \|x - y\| < \delta \).

**Lemma 2.3.** Assume (H1)–(H4), consider \( \sigma = (\sigma_1, \ldots, \sigma_n) \) with \( 0 < \sigma_i \leq B_i \Gamma_i^{-1} e^{-D_i(\omega)} \) for \( i = 1, \ldots, n \), and \( K = K(\sigma) \). Then:

(i) \( \Phi(K) \subset K \).

(ii) If \( x \in K \setminus \{0\} \) is a fixed point of \( \Phi \), then \( x \) is a positive \( \omega \)-periodic solution of (1.1).

(iii) If in addition (H5) holds, \( \Phi \) is completely continuous.

**Proof.** (i) From Lemma 2.2 \( \Phi(X^+) \subset X^+ \). Now, take \( x = (x_1, \ldots, x_n) \in K \). For \( \sigma \) chosen as above,

\[
(\Phi_i x)(t) \leq \Gamma_i(x_i)e^{D_i(\omega)}B_i \int_0^\omega \left( \sum_{j \neq i} a_{ij}(s)x_j(s) + g_i(s, x_is) \right) ds
\]

and

\[
(\Phi_i x)(t) \geq \Gamma_i(x_i)B_i \int_0^\omega \left( \sum_{j \neq i} a_{ij}(s)x_j(s) + g_i(s, x_is) \right) ds, \tag{2.13}
\]

leading to \( (\Phi_i x)(t) \geq \sigma_i \|\Phi_i x\| \) for all \( i \) and all \( t \). Thus, \( \Phi(K) \subset K \).

(ii) If \( x \in K, x \neq 0 \) and \( x = \Phi x \), from Lemma 2.2 \( x(t) \) is a nontrivial \( \omega \)-periodic solution of (1.1). If \( n = 1 \), the definition of \( K \) implies that \( K \setminus \{0\} = K_0 \). If \( n > 1 \), there is \( i^* \in \{1, \ldots, n\} \) such that \( x_{i^*}(t) \geq \sigma_{i^*} \|x_{i^*}\| = \sigma_{i^*} \|x\| > 0 \), \( t \in [0, \omega] \). For \( i \neq i^* \), either \( x_i(t) \geq \|x_i\| > 0 \) for all \( t \), or \( x_i = (\Phi_i x) \equiv 0 \); in the latter case, from (H4)(ii) it then follows that

\[
0 = (\Phi_i x)(t) \geq \Gamma_i(x_i)B_i \int_0^\omega \left( a_{ii^*}(s)x_{i^*}(s) + g_i(s, 0) \right) ds > 0,
\]

for \( t \in [0, \omega] \), which is not possible. Hence \( x_i(t) \geq \sigma_i \|x_i\| > 0 \). Therefore, all the components of \( x \) are strictly positive on \([0, \omega] \), i.e., \( x \in K_0 \).

(iii) The proof follows by a straightforward adaptation of the arguments for the scalar case in [5], replacing a scalar function \( G(t, x) \) by the functions \( H_i(t, x) = \sum_{j \neq i} a_{ij}(t)x_j(t) + G_i(t, x_i) \) (\( 1 \leq i \leq n \)), for \( G \) as in (2.12). Clearly, the function \( H = (H_1, \ldots, H_n) \) also satisfies (H5), hence the proof in [5] applies to the present situation.

**Remark 2.2.** As previously mentioned, the role of (H4)(ii) is to preclude the existence of nontrivial fixed points of \( \Phi \) with one or more coordinates equal to zero. In this way, it can be replaced by any other assumption with the same outcome. We point out that some authors [3, 26] have imposed hypotheses and employed Krasnoselskii’s techniques to some classes of periodic systems of DDEs (without impulses), which however seem to only guarantee that a nontrivial, rather than positive, periodic solution must exist.

## 3 Main results

In this section, general criteria for the existence of positive periodic solutions of (1.1) are given. To use the compressive form of Krasnoselskii’s fixed point theorem, we impose the assumption:
(H6) There are constants $r_0, R_0$ with $0 < r_0 < R_0$ and functions $b_{1i}, b_{2i} \in C^0_+([0,\omega])$ with $\int_0^\omega b_{qi}(t)\,dt > 0$ ($q = 1, 2$), such that for $i = 1, \ldots, n$, $x \in K$ and $t \in [0,\omega]$ it holds:
\[
g_i(t,x_i) \geq b_{1i}(t)u \quad \text{if } 0 < u \leq x_i \leq r_0,
\]
\[
g_i(t,x_i) \leq b_{2i}(t)u \quad \text{if } R_0 \leq x_i \leq u.
\]

Bearing in mind the behaviour of the nonlinearities in (1.1) at infinity, when (H6) holds we say that (1.1) is sublinear – this is the situation of Mackey-Glass and Nicholson systems, as well as other important models from mathematical biology.

**Theorem 3.1.** Assume (H1)–(H6) and that, for $b_{1i}, b_{2i}$ as in (H6),
\[
\Gamma_iB_i \min_{t \in [0,\omega]} \int_t^{t+\omega} e^{\int_{\tau}^{t+\omega} d_i(\tau)\,d\tau} \left(\sum_{j \neq i} a_{ij}(s) + b_{1i}(s)\right) ds \geq 1,
\]
\[
\Gamma_iB_i \max_{t \in [0,\omega]} \int_t^{t+\omega} e^{\int_{\tau}^{t+\omega} d_i(\tau)\,d\tau} \left(\sum_{j \neq i} a_{ij}(s) + b_{2i}(s)\right) ds \leq 1, \quad i = 1, \ldots, n.
\]

Then there exists (at least) one positive $\omega$-periodic solution $x^*(t)$ of (1.1) satisfying
\[
\min_{t \in [0,\omega]} x_i^*(t) \geq \sigma_i \max_{t \in [0,\omega]} x_i^*(t), \quad i = 1, \ldots, n,
\]
for $0 < \sigma_i \leq B_i^{-1} e^{-D_i(\omega)} (1 \leq i \leq n)$ as in Lemma 2.3.

**Proof.** Fix $r_0, R_0$ as in (H6). Let $R \geq R_0 (\min_{1 \leq i \leq n} \sigma_i)^{-1}$ and $x \in K$ with $\|x\| = R$. Choose $i$ such that $\|x\| = \|x_i\| = R$. For such $i$, we have $x_i(t) \leq R$ and $x_i(t) \geq \sigma_i \|x_i\| = \sigma_i R \geq R_0$ for $t \in [0,\omega]$, therefore, from the second inequality in (3.1) we obtain
\[
g_i(t,x_i) \leq b_{2i}(t)R.
\]

Using the properties in Lemma 2.1 and (3.2), we have
\[
\|\Phi x\| \leq R \Gamma_iB_i \max_{t \in [0,\omega]} \int_t^{t+\omega} e^{\int_{\tau}^{t+\omega} d_i(\tau)\,d\tau} \left[\sum_{j \neq i} a_{ij}(s) + b_{2i}(s)\right] ds \leq R.
\]

In particular, we conclude that $\Phi x \neq \lambda x$ for all $\lambda > 1$ and $x \in K$ with $\|x\| = R$.

On the other hand, take $r \leq \min_{1 \leq i \leq n} \sigma_i r_0$, $\psi \equiv 1 := (1, \ldots, 1)$ and consider any $\lambda > 0$. For $x \in K$ with $\|x\| = r$, we claim that $x \neq \Phi x + \lambda \psi$.

Suppose otherwise that there are $\lambda > 0, x \in K$ with $\|x\| = r$ and $x = \Phi x + \lambda \psi$. Let $\mu := \min_{t \in [0,\omega]} \min_{1 \leq i \leq n} x_i(t)$. We first note that, for $t \in [0,\omega], i = 1, \ldots, n$, we have $0 < \lambda \leq \mu \leq x_i(t) \leq r \leq r_0$, thus the first inequality in (3.1) implies
\[
g_i(t,x_i) \geq b_{1i}(t)\mu,
\]
which, together with the first constraint in (3.2), yields for all $i = 1, \ldots, n$ and $t \in [0,\omega]$ that
\[
(\Phi x)(t) \geq \mu \Gamma_iB_i \min_{t \in [0,\omega]} \int_t^{t+\omega} e^{\int_{\tau}^{t+\omega} d_i(\tau)\,d\tau} \left[\sum_{j \neq i} a_{ij}(s) + b_{1i}(s)\right] ds \geq \mu.
\]
Next, choose $t^* \in [0, \omega]$ and $i^* \in \{1, \ldots, n\}$ such that $x_{i^*(t^*)} < \mu + \lambda$. We obtain
\[\mu > x_{i^*(t^*)} - \lambda = (\Phi_{i^*})^*(t^*) \geq \mu,\]
which is not possible. The claim is proven, thus Theorem 2.1 provides the existence of a fixed point $x^*$ for $\Phi$ in $K_{r,R} = \{x \in K : r \leq \|x\| \leq R\}$. From Lemma 2.3(ii), this fixed point is a positive $\omega$-periodic solution of (1.1).

A scaling of the variables allows us to obtain an algebraic variant of Theorem 3.1 which turns out to be very useful.

**Theorem 3.2.** Assume (H1)–(H6) and that there is $v = (v_1, \ldots, v_n) > 0$ such that, for $b_{i1}, b_{i2}$ as in (H6),
\[c^0_i(v) := \Gamma_i B_i \min_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^r d_i(s)} dr \left( \sum_{j \neq i} v_j^{-1} v_{ij}(s) + b_i(s) \right) ds \geq 1,\]
\[C^\infty_i(v) := \Gamma_i B_i \max_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^r d_i(s)} dr \left( \sum_{j \neq i} v_j^{-1} v_{ij}(s) + b_i(s) \right) ds \leq 1, \quad i = 1, \ldots, n. \tag{3.5}\]

Then there exists (at least) one positive $\omega$-periodic solution $x^*(t)$ of (1.1).

**Proof.** Effecting the change of variables $\bar{x}_i = v_i^{-1} x_i$ (1 $\leq i \leq n$) and dropping the bars for simplicity, system (1.1) becomes
\[
\begin{cases}
x'_i(t) = -d_i(t) x_i(t) + \sum_{j \neq i} v_j^{-1} v_{ij}(t) x_j(t) + \bar{g}_i(t, x_i) \quad &\text{for } t \neq t_k, \\
x_i(t_{k^+}) - x_i(t_k) = \bar{I}_{ik}(x_i(t_k)), \quad k \in \mathbb{Z},
\end{cases}
\tag{3.6}
\]
where $\bar{g}_i(t, u) = v_i^{-1} g_i(t, v_i u)$, $\bar{I}_{ik}(u) = v_i^{-1} I_{ik}(v_i u)$ for all $i, k$. On the one hand, the functions $\bar{I}_{ik}(u)$ satisfy hypotheses (H1)–(H3) with the same constants $\alpha_{ik}, \eta_{ik}$, and $\bar{J}_{ik}(u) := \frac{u}{u + J_{ik}(u)} = J_{ik}(v_i u)$ (for $J_{ik}$ as in (2.3)). On the other hand, if the functions $g_i(t, u)$ satisfy (3.1), then $\bar{g}_i(t, u)$ satisfy (3.1) as well. Consequently, Theorem 3.1 implies the result.

The superlinear case of (1.1) is dealt in a similar way, by using the expansive form of Krasnosel’skii’s theorem. The proof is omitted.

**Theorem 3.3.** Assume (H1)–(H5) and (H7) There are constants $r_0, R_0$ with $0 < r_0 < R_0$ and functions $b_{i1}, b_{i2} \in C^+_{\omega}(\mathbb{R})$ with $\int_0^{\omega} b_{qi}(t) dt > 0$ ($q = 1, 2$), such that for $i = 1, \ldots, n$, $x \in K$ and $t \in [0, \omega]$ it holds:
\[g_i(t, x_i) \leq b_{i1}(t) u \quad \text{if } 0 < x_i \leq u \leq r_0, \]
\[g_i(t, x_i) \geq b_{i2}(t) u \quad \text{if } x_i \geq u \geq R_0. \tag{3.7}\]

If there is a vector $v = (v_1, \ldots, v_n) > 0$ such that
\[c^0_i(v) := \Gamma_i B_i \min_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^r d_i(s)} dr \left( \sum_{j \neq i} v_j^{-1} v_{ij}(s) + b_{i1}(s) \right) ds \geq 1,\]
\[C^\infty_i(v) := \Gamma_i B_i \min_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^r d_i(s)} dr \left( \sum_{j \neq i} v_j^{-1} v_{ij}(s) + b_{i2}(s) \right) ds \geq 1, \quad i = 1, \ldots, n. \tag{3.8}\]
Though not as sharp as the ones above, the estimates in (3.5) are much easier to verify in practice.

Remark 3.2. Due to the version of Krasnoselskii’s theorem given in Theorem 2.1 we stress that the equalities to 1 are allowed in all conditions (3.5), (3.8). Hence, even for the scalar case, the above theorems lead to improvements of some criteria in [5], where the strict inequalities were required.

Remark 3.3. Under (H1)-(H6), it is clear that the sufficient conditions expressed by (3.2) or (3.5) are not optimal, since one can use sharper estimates for \( \Gamma_i \) and \( \tilde{\Gamma}_i \) if conditions (3.5) are replaced by (1.1), as in e.g. [18] and several other works. Theorem 3.4. Assume (H1)–(H6). For \( b_{11}(t), b_{21}(t) \) as in (H6), define

\[
B_1(t) = \text{diag} \left( b_{11}(t), \ldots, b_{1n}(t) \right), \quad B_2(t) = \text{diag} \left( b_{21}(t), \ldots, b_{2n}(t) \right),
\]

for \( t \in \mathbb{R} \). With \( D(t), A(t) \) as in (3.9) and some vector \( v > 0 \), assume one of conditions:

(a) either

\[
M_2 \left[ B_2(t) + A(t) \right] v \leq D(t)v \leq M_1 \left[ B_1(t) + A(t) \right] v,
\]

for

\[
M_1 = \text{diag}(m_{11}, \ldots, m_{1n}), \quad M_2 = \text{diag}(m_{21}, \ldots, m_{2n}),
\]

\[
m_{1i} := \Gamma_i B_i(e^{D_i(\omega)} - 1), \quad m_{2i} := \Gamma_i B_i(e^{D_i(\omega)} - 1), \quad i = 1, \ldots, n;
\]
(b) or
\[
\int_0^\omega N_2 \left[ B_2(t) + A(t) \right] v \, dt \leq v \leq \int_0^\omega N_1 \left[ B_1(t) + A(t) \right] v \, dt,
\]
for
\[
N_1 = \text{diag}(n_{11}, \ldots, n_{1n}), \quad N_2 = (n_{21}, \ldots, n_{2n}),
\]
\[
n_{1i} := \Gamma_i B_i; \quad n_{2i} := \Gamma_i B_i e^{D_i(\omega)}, \quad i = 1, \ldots, n.
\]

Then, (1.1) has (at least) one positive \(\omega\)-periodic solution.

**Proof.** Consider \(c_i^0(v), C_i^\infty(v)\) \((1 \leq i \leq n)\) defined in (3.5). From (a), we have
\[
\sum_{j \neq i} v_j a_{ij}(s) + v_i b_{1i}(s) \geq m_i^{-1} v_i d_i(s), \quad \sum_{j \neq i} v_j a_{ij}(s) + v_i b_{2i}(s) \leq m_i^{-1} v_i d_i(s),
\]
thus
\[
c_i^0(v) \geq m_i^{-1} \Gamma_i B_i \min_{t \in [0,\omega]} \int_t^{t+\omega} e^{\int_t^r d_i(r) \, dr} d_i(s) \, ds = m_i^{-1} \Gamma_i B_i (e^{D_i(\omega)} - 1) = 1,
\]
\[
C_i^\infty(v) \leq m_i^{-1} \Gamma_i B_i \max_{t \in [0,\omega]} \int_t^{t+\omega} e^{\int_t^r d_i(r) \, dr} d_i(s) \, ds = m_i^{-1} \Gamma_i B_i (e^{D_i(\omega)} - 1) = 1,
\]
for \(i = 1, \ldots, n\). If (b) is satisfied, for all \(i\) we obtain
\[
c_i^0(v) > v_i^{-1} \Gamma_i B_i \int_0^\omega \left( \sum_{j \neq i} v_j a_{ij}(s) + v_i b_{1i}(s) \right) \, ds \geq v_i^{-1} \Gamma_i B_i v_i n_i^{-1} = 1,
\]
\[
C_i^\infty(v) < v_i^{-1} \Gamma_i B_i e^{D_i(\omega)} \int_0^\omega \left( \sum_{j \neq i} v_j a_{ij}(s) + v_i b_{2i}(s) \right) \, ds \leq v_i^{-1} \Gamma_i B_i e^{D_i(\omega)} v_i n_i^{-1} = 1.
\]
The conclusion is drawn from Theorem 3.2. \(\square\)

For nonimpulsive systems
\[
x_i'(t) = -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + g_i(t, x_i(t), x_{it}), \quad i = 1, \ldots, n,
\]
conditions for the existence of a positive \(\omega\)-periodic solution are obtained by taking \(\Gamma_i = \Gamma_i \equiv (e^{D_i(\omega)} - 1)^{-1}\) and \(B_i = B_i = 1\) in the above theorem, leading to:

**Corollary 3.1.** Assume (H4)–(H6). For the matrices in (3.9), (3.10) suppose that for some \(v > 0\):

(a) either \(B_2(t) v \leq [D(t) - A(t)] v \leq B_1(t) v\);

(b) or
\[
\left\{ \begin{array}{l}
\int_0^\omega \left[ B_2(t) + A(t) \right] v \, dt \leq \text{diag} \left( 1 - e^{-D_1(\omega)}, \ldots, 1 - e^{-D_n(\omega)} \right) v
\\
\int_0^\omega \left[ B_1(t) + A(t) \right] v \, dt \geq \text{diag} \left( e^{D_1(\omega)} - 1, \ldots, e^{D_n(\omega)} - 1 \right) v.
\end{array} \right.
\]

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Then, there exists a positive \( \omega \)-periodic solution of \((3.15)\).

**Proof.** For the system with no impulses \((3.15)\), we have \(M_1 = M_2 = I\) and \(N_1^{-1} = \text{diag} \left( e^{D_1(\omega)} - 1, \ldots, e^{D_n(\omega)} - 1 \right), N_2^{-1} = \text{diag} \left( 1 - e^{-D_1(\omega)}, \ldots, 1 - e^{-D_n(\omega)} \right)\) for the matrices in Theorem 3.4.

**Remark 3.4.** Theorem 3.4 and Corollary 3.1 take into consideration the sublinear case; for the superlinear case, analised in Theorem 3.3, similar statements hold.

From Theorem 3.4 we retrieve some criteria which improve the ones obtained in [5] for the particular case of scalar equations.

**Corollary 3.2.** Consider the scalar impulsive DDE

\[
\begin{cases}
  x'(t) = -d(t)x(t) + g(t,x_t) \quad & \text{for } t \neq t_k, \\
  \Delta(x(t_k)) := x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad & k \in \mathbb{Z},
\end{cases}
\]

(3.16)

where \((t_k), (I_k)_{k \in \mathbb{Z}}, d \in C^+_{\omega}(\mathbb{R}), g : \mathbb{R} \times PC([-\tau,0],\mathbb{R}) \rightarrow \mathbb{R}^+\) satisfy the assumptions in \((H1)-(H5)\) (with \(i = 1\) and \(d(t) = d_1(t), g(t,\varphi) = g_1(t,\varphi), I_k = I_{1k}\) for \(k \in \mathbb{Z}\), and \((H6')\) There are constants \(r_0, R_0\) with \(0 < r_0 < R_0\) and functions \(b_1, b_2 \in C^+_{\omega}(\mathbb{R})\) with \(\int_0^\pi b_q(t) \, dt > 0\) \((q = 1, 2)\), such that for \(x \in K\) and \(t \in [0,\omega]\),

\[
g(t,x_t) \geq b_1(t)u \quad \text{if } 0 < u \leq x \leq r_0, \quad g(t,x_t) \leq b_2(t)u \quad \text{if } R_0 \leq x_t \leq u.
\]

(3.17)

With \(\Gamma = \Gamma_1, \mathbf{T} = \mathbf{T}_1\) and other obvious terminology as above, assume one of the following conditions:

(a) \(\Gamma B \left( e^{D(\omega)} - 1 \right) b_2(t) \leq d(t) \leq \Gamma B \left( e^{D(\omega)} - 1 \right) b_1(t)\);

(b) \(\Gamma B e^{D(\omega)} \int_0^\omega b_2(t) \, dt \leq 1 \leq \Gamma B \int_0^\omega b_1(t) \, dt\).

Then, \((3.16)\) has (at least) one positive \(\omega\)-periodic solution.

**Remark 3.5.** For a nonimpulsive scalar equation \(x'(t) = -d(t)x(t) + g(t,x_t)\), the conclusion is obtained by taking \(\Gamma = \mathbf{T} = \left( e^{D(\omega)} - 1 \right)^{-1}, B = \mathbf{T} = 1\) in Corollary 3.2 so that the conditions read as: (a) \(b_2(t) \leq d(t) \leq b_1(t)\) for \(t \in [0,\omega]\); (b) \(e^{D(\omega)} \int_0^\omega b_2(t) \, dt \leq e^{D(\omega)} - 1 \leq \int_0^\omega b_1(t) \, dt\).

**Example 3.1.** Consider a Nicholson’s blowflies equation

\[
x'(t) = -d(t)x(t) + p(t)x(t-\tau(t))e^{-x(t-\tau(t))}
\]

(3.18)

with \(d(t) = \sin^2 t, p(t) = 3 \cos^2 t\) and \(\tau \in C^+_{\omega}(\mathbb{R})\). For \(h(x) = xe^{-x}\), one has \(h'(0) = 1, h(\infty) = 0\). Fix any \(\varepsilon > 0\) small. Clearly, \((3.17)\) is satisfied with \(b_1(t) = (1-\varepsilon)p(t)\) and \(b_2(t) = \varepsilon\). On the other hand, \(\int_0^\pi p(t) \, dt = 3\pi/2, e^{\int_0^\tau d(t) \, dt} - 1 = e^\pi/2 - 1 \approx 3.81 < 3\pi/2\), and consequently \(\varepsilon\) can be chosen so that condition (b) in Remark 3.5 holds. Thus, \((3.18)\) has a positive \(\pi\)-periodic solution. Note however that condition \(p(t) > d(t)\) is not true for all \(t > 0\) – this, according to an assertion in [2], should imply that there is no positive \(\omega\)-periodic solution for \((3.18)\), which is contradicted by this example.
Here, we suppose that $\omega$ and $(\text{with and without impulses})$, with $M$ also possible to obtain generalisations to systems of the form

$$x'_i(t) = -d_i(t)x_i(t)h_i(t, x_i(t)) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t) + g_i(t, x_{il}), \ i = 1, \ldots, n$$

(with and without impulses), with $h_i(t, u)$ continuous, bounded above and below by positive constants and $\omega$-periodic in $t$, by a straightforward adjustment of the present technique. Of course, now the functions $b_{ij}(t)$ in (H6) should be multiplied by suitable constants. Details are left to the reader.

**Example 3.2.** Consider the family of differential systems with discrete delays and impulses given by:

$$\begin{align*}
  x'_i(t) &= -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + \sum_{l=1}^{m} f_{il}(t, x_i(t - \tau_{il}(t))), \ t \neq t_k, \\
  \Delta(x_i(t_k)) &= I_{ik}(x_i(t_k)), \ k \in \mathbb{Z},
\end{align*}
$$

(3.19)

Here, we suppose that $d_i, a_{ij}, \tau_{il} \in C^+_\omega(\mathbb{R})$, with $d_i(t) > 0$, $f_{il}(t, u)$ are nonnegative, continuous and $\omega$-periodic in $t$, and $t_k, I_{ik}(u)$ satisfy hypotheses (H1)-(H3), $i = 1, \ldots, n, l = 1, \ldots, m, k \in \mathbb{Z}$.

Define the values (in $[0, \infty]$) given by the limits

$$\begin{align*}
  f_i^0 &= \liminf_{u \to 0^+} \left( \min_{t \in [0, \omega]} \frac{F_i(t, u)}{d_i(t)} \right), \quad \tilde{F}_i^0 = \limsup_{u \to 0^+} \left( \max_{t \in [0, \omega]} \frac{F_i(t, u)}{d_i(t)} \right), \\
  f_i^\infty &= \liminf_{u \to \infty} \left( \min_{t \in [0, \omega]} \frac{F_i(t, u)}{d_i(t)} \right), \quad \tilde{F}_i^\infty = \limsup_{u \to \infty} \left( \max_{t \in [0, \omega]} \frac{F_i(t, u)}{d_i(t)} \right), \quad (3.20)
\end{align*}$$

where

$$F_i(t, u) = \sum_l f_{il}(t, u) \quad \text{for} \ i = 1, \ldots, n,$$

and $f_i^0, \tilde{F}_i^0, f_i^\infty, \tilde{F}_i^\infty$ the diagonal matrices with diagonal entries $f_i^0, \tilde{F}_i^0, f_i^\infty, \tilde{F}_i^\infty (1 \leq i \leq n)$, respectively.

**Theorem 3.5.** Consider (3.19) under the above assumptions, and assume also that:

(i) if $n > 1$, either $\int_0^\omega a_{ij}(s) \, ds > 0$ for all $i \neq j$ or $\int_0^\omega F_i(s, 0) \, ds > 0$, for each $i = 1, \ldots, n$;

(ii) there exists a vector $v > 0$ such that either

$$M_2[\tilde{F}^\infty D(t) + A(t)]v < D(t)v < M_1[f^0 D(t) + A(t)]v \quad (3.21)$$

or

$$M_1[f^\infty D(t) + A(t)]v > D(t)v > M_2[\tilde{F}^0 D(t) + A(t)]v, \quad (3.22)$$

where $M_1, M_2$ are as in (3.12). Then, system (3.19) has at least one positive $\omega$-periodic solution.

**Proof.** System (3.19) has the form (1.1) with $g_i(t, x_{il}) = \sum_{l=1}^{m} f_{il}(t, x_i(t - \tau_{il}(t)))$. From (i) and since $f_{il}(t, u)$ are uniformly continuous on bounded sets of $[0, \omega] \times \mathbb{R}$, clearly (H4),(H5) are satisfied.
Assume (3.21), for some \( v = (v_1, \ldots, v_n) > 0 \). In particular, this implies that \( f_i^0 > 0 \). For any fixed \( \varepsilon \in (0, f_i^0) \), let \( 0 < r_0 < R_0 \) be such that, for \( 1 \leq i \leq n \) and \( t \in [0, \omega) \), we have \( F_i(t, u) \leq (\tilde{f}_i^\infty + \varepsilon) d_i(t) u \) for \( u \geq R_0 \) and \( F_i(t, u) \geq (f_i^0 - \varepsilon) d_i(t) u \) for \( 0 < u \leq r_0 \). Then, (H6) is satisfied with \( b_2(t) = (\tilde{f}_i^\infty + \varepsilon) d_i(t), b_{1i}(t) = (f_i^0 - \varepsilon) d_i(t) \). Let \( \varepsilon \) be sufficiently small so that

\[
m_{2i}[v_i(\tilde{f}_i^\infty + \varepsilon)d_i(t) + \sum_j v_j a_{ij}(t)] < v_i d_i(t) < m_{1i}[v_i(f_i^0 - \varepsilon)d_i(t) + \sum_j v_j a_{ij}(t)]
\]

for all \( i \) and \( t \). The conclusion follows from Theorem 3.4(a). The superlinear case, where (3.22) holds, is handled in a similar way.

For (3.19) without impulses, as \( M_1 = M_2 = I \) in the above statement, we obtain:

**Corollary 3.3.** For the nonimpulsive version of (3.19),

\[
x_i'(t) = -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + \sum_{l=1}^{m} f_{il}(t, x_i(t - \tau_{il}(t))), i = 1, \ldots, n,
\]

(3.23)

with \( d_i, a_{ij}, \tau_{il}, f_{il}(t, u) \) as before, assume (i) in the above theorem. Then, there exists a positive \( \omega \)-periodic solution if there is \( v > 0 \) such that either

\[
\tilde{f}_i^\infty d_i(t) < d_i(t) - \sum_j v_i^{-1} v_j a_{ij}(t) < f_i^0 d_i(t) \quad (1 \leq i \leq n)
\]

or

\[
f_i^\infty d_i(t) > d_i(t) - \sum_j v_i^{-1} v_j a_{ij}(t) > \tilde{f}_i^0 d_i(t) \quad (1 \leq i \leq n).
\]

Theorem 3.5 recovers the criteria in [5] for the scalar version of (3.19) with a single delay,

\[
x'(t) = -d(t)x(t) + f(t, x(t - \tau(t))) \quad (t \neq t_k), \quad \Delta(x(t_k)) = I_k(x(t_k)) \quad (k \in \mathbb{Z}).
\]

(3.24)

See also [18, 32, 33] for the existence of positive periodic solutions for (3.24). Note however that Li et al. [18] consider the scalar model (3.24) only with nonnegative impulsive functions \( I_k(u) \geq 0 \); in this way, the criteria in [5, 18] are not always comparable, as explained in [5].

### 4 Systems with bounded nonlinearities

As an illustration with relevant applications, we study some classes of systems with bounded nonlinearities. Consider the following two families of impulsive systems:

\[
x_i'(t) = -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t)
\]

\[
+ \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) h_{il}(s, x_i(s)) d_s \nu_{il}(t, s), \quad t \neq t_k,
\]

(4.1)

\[
\Delta(x_i(t_k)) = I_{ik}(x_i(t_k)), \quad k \in \mathbb{Z}, \quad i = 1, \ldots, n,
\]
and
\[
x'_i(t) = -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + \sum_{l=1}^{m} \beta_{il}(t)h_{il}\left(t, \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s)x_i(s) \, ds \nu_{il}(t, s)\right),
\]
(4.2)
\[
\Delta(x_i(t_k)) = I_{ik}(x_i(t_k)), \quad k \in \mathbb{Z}, \quad i = 1, \ldots, n,
\]
where:

(h1) for \( i, j \in \{1, \ldots, n\}, l \in \{1, \ldots, m\}, d_i, a_{ij}, \tau_{il}, \beta_{il}, \gamma_{il} \in C^+_\omega(\mathbb{R}) \), with \( \tau_{il} \) bounded, \( d_i \neq 0 \) and \( a_{ij} \neq 0 \) for \( i \neq j \); \( \nu_{il}(t, s) \) are non-decreasing in \( s \), continuous and \( \omega \)-periodic in \( t \); \( h_{ik}(t, u) \) are continuous and \( \omega \)-periodic in \( t \);

(h2) for \( i \in \{1, \ldots, n\}, l \in \{1, \ldots, m\}, h_{il}(t, u) \) are bounded on \( \mathbb{R} \times \mathbb{R}^+ \) and
\[
b_i(t) := \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) \, ds \nu_{il}(t, s) > 0;
\]
(4.3)
(h3) for \( i \in \{1, \ldots, n\} \), there exist continuous functions \( h_i : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( h_i(0) = 0, h'_i(0) = 1, h_i(u) > 0 \) for \( u > 0 \), and such that
\[
h_{il}(t, u) \geq h_i(u), \quad t \in \mathbb{R}, u \geq 0, l = 1, \ldots, m;
\]
(h4) the sequences \( (t_k)_{k \in \mathbb{Z}}, (I_{ik})_{k \in \mathbb{Z}} \) satisfy (H1)–(H3), \( i = 1, \ldots, n \).

Here, the phase space is \( PC = PC([-\tau, 0], \mathbb{R}^n) \) with \( \tau = \max_{i, l} \max_{t \in [0, \omega]} \tau_{il}(t) \). However, the situation can be generalised in order to include DDEs with infinite delay, in which case \( t - \tau_{il}(t) \) are replaced by \( -\infty \) in the integrals in (4.1), (4.2) and (4.3).

For \( n \times n \) matrix-valued \( \omega \)-periodic functions \( M(t), N(t) \) and \( v \in \mathbb{R}^n \), we write
\[
M(t)v \leq_{\not=} N(t)v
\]
if \( M(t)v \leq N(t)v \) on \([0, \omega]\) and, for each \( i = 1, \ldots, n \), there is \( t_i \in [0, \omega] \) for which \( (M(t_i)v)_i < (N(t_i)v)_i \). The symbol \( \geq_{\not=} \) has an analogous meaning.

**Theorem 4.1.** Consider either (4.1) or (4.2), and assume (h1)–(h4). Suppose also that for some \( v = (v_1, \ldots, v_n) > 0 \) it holds
\[
\prod_{i=1}^{n} B_i \max_{t \in [0, \omega]} \int_{t}^{t+\omega} e^{\int_{t}^{s} d_i(r) \, dr} \left( \sum_{j \neq i} v_j^{-1}v_j a_{ij}(s) \right) \, ds < 1,
\]
(4.4)
\[
\prod_{i=1}^{n} B_i \min_{t \in [0, \omega]} \int_{t}^{t+\omega} e^{\int_{t}^{s} d_i(r) \, dr} \left( \sum_{j \neq i} v_j^{-1}v_j a_{ij}(s) + b_i(s) \right) \, ds > 1, \quad i = 1, \ldots, n.
\]
Then, the system has at least one positive \( \omega \)-periodic solution. In particular, for \( M_i, N_i \) \( (i = 1, 2) \) as in (3.12), (3.14) and \( B(t) = \text{diag}(b_1(t), \ldots, b_n(t)) \), this is the case if, for some \( v > 0 \), either
\[
M_2 A(t)v \leq_{\not=} D(t)v \leq_{\not=} M_1 \left[ B(t) + A(t) \right] v,
\]
(4.5)
or
\[
\int_{0}^{\omega} N_2 A(t)v \, dt \leq v \leq \int_{0}^{\omega} N_1 \left[ B(t) + A(t) \right] v \, dt.
\]
(4.6)
Proof. Systems (4.1), (4.2) are particular cases of (4.1) with, respectively

\[
g_i(t, x_0) = \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) h_{il}(s, x_i(s)) d\nu_{il}(t, s), \quad i = 1, \ldots, n, \tag{4.7}
\]

\[
g_i(t, x_0) = \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) x_i(s) d\nu_{il}(t, s), \quad i = 1, \ldots, n. \tag{4.8}
\]

From the above conditions (h1)-(h2), the nonlinearities (4.7), (4.8) are bounded and (H4)-(H5) are fulfilled. Moreover, the boundedness of all \(h_{il}\) also implies that, in both cases, for any \(\varepsilon > 0\) there exists \(R_0 > 0\) large, such that \(g_i(t, x_0) \leq \varepsilon u\) for \(R_0 \leq x_i \leq u\).

For (4.1), (h3) implies that, for any fixed \(\varepsilon > 0\), there exists \(r_0 > 0\) such that for \(0 < u \leq x_i \leq r_0\) we have \(h_{il}(s, x_i(s)) \geq h_i(u) \geq (1 - \varepsilon)u\). Since \(\nu_{il}(t, s)\) are nondecreasing in \(s\), we have \(g_i(t, x_0) \geq (1 - \varepsilon)b_i(t)u\) for \(u \leq x_i \leq r_0\). Hence, (H6) holds with \(b_i(t) = (1 - \varepsilon)b_i(t)\) and \(b_{21}(t) = \varepsilon\), where \(b_i(t)\) is defined in (4.3). From (4.4), for \(\varepsilon > 0\) sufficiently small we have

\[
\begin{align*}
\epsilon_{ij}^0(v) &= \Gamma \left( \frac{-\epsilon B_i \min_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + (1 - \varepsilon) b_i(s) \right) ds \right) > 1, \\
C_{ij}^{\infty}(v) &= \Gamma \left( \frac{\max_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + \varepsilon \right) ds}{1} \right) < 1, \quad i = 1, \ldots, n, \tag{4.9}
\end{align*}
\]

hence (3.3) holds. Thus, Theorem 3.2 provides the existence of at least one positive periodic solution.

Moreover, in (3.10) we obtain \(B_1(t) = (1 - \varepsilon)B(t), B_2(t) = \varepsilon I\). Under conditions (4.5), note that

\[
\begin{align*}
\min_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + b_i(s) \right) ds &= m_i^{-1} \min_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + \epsilon \right) ds = \left( \frac{\Gamma_i}{\beta_i} \right)^{-1}, \\
\max_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + b_i(s) \right) ds &= m_i \max_{t \in [0, \omega]} \int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + \epsilon \right) ds = \left( \frac{\Gamma_i}{\beta_i} \right)^{-1},
\end{align*}
\]

thus one can find \(\varepsilon > 0\) small enough so that conditions (4.9) hold. In an analogous way, since

\[
\begin{align*}
\int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + b_i(s) \right) ds &= \int_t^{t+\omega} e^{\int_t^s d(r)} ds \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij}(s) + b_i(s) \right) ds,
\end{align*}
\]

for all \(i\) and \(t \in [0, \omega]\), conditions (4.9) follow under (4.6).

Similarly, for (4.2), again using (h3) and the fact that \(\nu_{il}(t, s)\) are nondecreasing in \(s\), for \(r_0 > 0\) sufficiently small we derive

\[
g_i(t, x_0) \geq \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) x_i(s) d\nu_{il}(t, s) \geq (1 - \varepsilon)u \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) ds \nu_{il}(t, s) = (1 - \varepsilon)b_i(t)u, \quad \text{for } u \leq x_i \leq r_0.
\]

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This means that (H6) still holds with the same $b_{1i}(t) = (1 - \varepsilon)b_i(t)$ and $b_{21}(t) = \varepsilon$. The statements follow again by the previous results.

From Corollary 3.1 we also conclude:

**Corollary 4.1.** Assume (h1)–(h3) and define $B(t) = \text{diag}(b_1(t), \ldots, b_n(t))$. If there is a vector $v = (v_1, \ldots, v_n) > 0$ such that:

(a) either $A(t)v < D(t)v$ and

$$
\gamma_i(t,v) := \frac{b_i(t)v_i}{d_i(t)v_i - \sum_j v_j a_{ij}(t)} \geq 1, \quad t \in [0, \omega], i = 1, \ldots, n, \tag{4.10}
$$

(b) or

$$
e\int_0^\omega \sum_j v_j a_{ij}(t)dt \leq v_i(e\int_0^\omega d_i(s)ds - 1) \leq \int_0^\omega [b_i(t)v_i + \sum_j v_j a_{ij}(t)] dt, \quad i = 1, \ldots, n, \tag{4.11}
$$

then the nonimpulsive system (3.15) where $g_i(t, x_{it})$ has one of the forms (4.7), (4.8) possesses at least one positive $\omega$-periodic solution.

Note that (a) is equivalent to $A(t)v < D(t)v \equiv [B(t) + A(t)]v$. For the case of bounded delay, we emphasize that the existence of such a periodic solution for the nonimpulsive version of system (4.1) was established in [8, Theorem 3.1] assuming (h1)–(h3) and that there are constants $\alpha, \gamma$ such that

$$1 < \alpha \leq \gamma_i(t,v) \leq \gamma, \quad t \in [0, \omega], i = 1, \ldots, n. \tag{4.12}
$$

Under the above general assumptions and the stronger requirement (4.12), the nonimpulsive version of system (4.1) was proven to be permanent [10]. Moreover, in this setting, a fixed point argument allows to conclude that the sufficient conditions for permanence also imply the existence of a positive periodic solution [8,35]. However, at least for non-periodic systems, condition (4.10) is not enough to guarantee the persistence of the system. In fact, in [10, Example 4] a counter-example was given: a nonimpulsive system of the form (4.1) with $n = 2$ and a single discrete delay, which is not persistent although it satisfies conditions (h1)-(h3) and $\gamma_i(t, 1) > 1$ for all $t \in \mathbb{R}^+$ and $i = 1, 2$.

5 Applications to natural sciences models

In this section, our results are illustrated with applications to some selected hematopoiesis models and Nicholson-type systems. Several other models appearing in natural sciences could have been analysed.

**Example 5.1.** A hematopoiesis-type model.

In the celebrated paper of Mackey and Glass [22], two basic scalar models of the form $x'(t) = -dx(t) + f(x(t - \tau))$ were proposed to describe the hematopoiesis process (production and specialisation of blood cells) taking place in the bone marrow, with either a monotone decreasing production function $f(x) = b_a x^\alpha$, or a unimodal function $f(x) = b a^{x^\alpha} + \beta x^{1+\alpha}$, for $d, a, b, \alpha, \tau > 0$ (after normalization of the coefficients, one can take $a = 1$). Numerous generalizations of the original Mackey-Glass
Proof. This equation falls into the framework of problem (1.1), with $i$, $l$, $\beta_{il}(t) \in C^+_\omega(\mathbb{R})$ and $c_{il}(t) = c_{il}(t) \alpha_{il} > 0$ for all $i = 1, \ldots, n$, $l = 1, \ldots, m$. If there is $v = (v_1, \ldots, v_n) > 0$ such that

$$
\text{max}_{t \in [0, \omega]} \left| \int_{t}^{t+\omega} e^{\int_{s}^{t} d_i(t) \, ds} \left( \sum_{j \neq i} v_i^{-1} v_j a_{ij} \right) \, ds \right| < 1,
$$

then there is a positive $\omega$-periodic solution of (5.1). In particular, this is the case if, for some vector $v > 0$, either

$$
\text{diag} \left( \prod_{i=1}^{m} B_i (e^{D_1(\omega)} - 1), \ldots, \prod_{n} B_n (e^{D_n(\omega)} - 1) \right) A(t) v \leq D(t) v,
$$

or

$$
\text{diag} \left( \prod_{i=1}^{m} B_i e^{D_1(\omega)}, \ldots, \prod_{n} B_n e^{D_n(\omega)} \right) \int_{0}^{\omega} A(t) v \, dt \leq v.
$$

Proof. This equation falls into the framework of problem (1.1), with $g_i(t, \varphi_i) = \sum_{l=1}^{m} h_{il}(t, \varphi_i)$ and

$$
h_{il}(t, \varphi_i) = \frac{\beta_{il}(t)}{1 + c_{il}(t) \left( \int_{-\tau_{il}(t)}^{0} \varphi_i(s) \, ds \right)^{\alpha_{il}}}, \quad i = 1, \ldots, n, \quad l = 1, \ldots, m,
$$

so $g_i$ are bounded functions. Since $\int_{0}^{\omega} g_i(t, 0) \, dt = \sum_{l=1}^{m} \int_{0}^{\omega} \beta_{il}(t) \, dt > 0$, clearly (H4)(ii) is satisfied. On the other hand, for $i, l$ fixed and all $t \in [0, \omega], x, y \in K$,

$$
|h_{il}(t, x_i(t)) - h_{il}(t, y_i(t))| = \beta_{il}(t) \left| \frac{1}{1 + c_{il}(t) X^{\alpha_{il}}} - \frac{1}{1 + c_{il}(t) Y^{\alpha_{il}}} \right|,
$$

where $X = \int_{-\tau_{il}(t)}^{0} x_i(t+s) \, ds$, $Y = \int_{-\tau_{il}(t)}^{0} y_i(t+s) \, ds$, and $|X - Y| \leq \int_{-\tau_{il}(t)}^{0} |x_i(t+s) - y_i(t+s)| \, ds \leq \tau \|x - y\|$. Furthermore, for each function of the form $f(t, x) := \frac{1}{1 + c(t) x^\alpha}, t \in [0, \omega], x \in \mathbb{R}^+$, with $c(t) \in C^+_\omega(\mathbb{R})$ and $\alpha > 0$, from its uniform continuity on compact sets we derive that for any $M > 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that, for $t \in [0, \omega], x, y \in [0, M]$ with $|x - y| < \delta$, we have $|f(t, x) - f(t, y)| \leq \varepsilon$.

If (5.2) is satisfied for some $v > 0$, it is possible to choose $0 < \varepsilon < M$ such that, with $b_{i1}(t) := M, b_{i2}(t) \equiv \varepsilon$ in the definition of the constants $C^\infty_i(v), c^0_i(v)$ as in (3.5), we have

$$
C^\infty_i(v) < 1 < c^0_i(v), \quad i = 1, \ldots, n.
$$

(5.3)
Since $0 < g_i(t, x_{il}) \leq g_i(t, 0)$, for any $0 < \varepsilon < M$ there are $0 < r_0 < R_0$ such that $g_i(t, x_{il}) \geq Mu$ if $0 \leq u \leq x_i \leq r_0$ and $g_i(t, x_{il}) \leq \varepsilon u$ if $R_0 \leq x_i \leq u$, for $t \in \mathbb{R}, x \in X^+, i = 1, \ldots, n$. Thus, (H6) is satisfied with the above choices $b_1(t) \equiv M$ and $b_2(t) \equiv \varepsilon$. The results are a consequence of (5.3) and Theorems 3.2 and 3.4.

Clearly, an analogous statement applies to a periodic hemotopoiesis system where the nonlinearities $g_i(t, \varphi_i)$ contain only discrete delays, so that $g_i(t, x_{it}) = \sum_{l=1}^{m} \frac{\beta_{il}(t)}{1 + c_{il}(t)x_{i}^{\alpha_i}(t-\tau_{il}(t))}$, for coefficients and delay functions as in the above theorem.

In [11], the following scalar hemotopoiesis model with linear impulses and discrete delays was considered:

\[
\begin{align*}
    x'(t) &= -d(t)x(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + \sum_{l=1}^{m} \frac{\beta_{il}(t)}{1 + c_{il}(t)x_{i}^{\alpha_i}(t-\tau_{il}(t))}, \quad t \neq t_k, \\
    x(t_k^+) - x(t_k) &= b_kx(t_k), \quad k \in \mathbb{Z},
\end{align*}
\]

where $(b_k), (t_k)$ are $\omega$-periodic sequences, with $0 \leq t_1 < \cdots < t_p < \omega$ for some $p$, $\alpha_i$ are positive constants, $d, \beta_i, \tau_i, c_i \in C^+_{\omega}(\mathbb{R})$ and $d \neq 0, \sum \beta_i \neq 0, c_i(t) > 0$, for all $t \in [0, \omega], l = 1, \ldots, m$. From the version of Theorem 5.1 for discrete delays, we conclude the existence of a positive periodic solution for (5.4) if $b_k > -1$ and $\prod_{k=1}^{p}(1 + b_k) < e^{J_{\omega} \int dt}$, recovering the result in [11] Theorem 3.1.

**Example 5.2. Nicholson blowflies systems.**

Recently, there has been an increasing interest in periodic (or almost periodic) Nicholson-type systems with patch structure, and several authors have addressed the topics of existence, uniqueness and/or exponential stability of (almost) positive periodic solutions see e.g. [7,9,15,28,30,31].

Here, we consider a generalised Nicholson system with distributed delays given by

\[
\begin{align*}
    x_i'(t) &= -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s)x_i(s)e^{-c_{il}(s)x_i(s)}ds, \quad t \neq t_k, \\
    x_i(t_k^+) - x_i(t_k) &= I_{ik}(x_i(t_k)), \quad k \in \mathbb{Z}, \quad i = 1, \ldots, n.
\end{align*}
\]

**Theorem 5.2.** Assume that $(t_k), I_{ik}(u)$ satisfy (H1)–(H3), $d_i, a_{ij}, \beta_{il}, \gamma_{il}, \tau_i, c_i \in C^+_{\omega}(\mathbb{R})$ with $d_i \neq 0, a_{ij} \neq 0 (j \neq i), \sum \beta_{il} \neq 0, c_i(t) > 0, 0 \leq \tau_{il}(t) \leq \tau$ on $[0, \omega], \tau > 0$. Assume also that either (4.3), (4.5) or (4.6) is satisfied with

\[
b_i(t) = \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s)ds, \quad t \geq 0, \quad i = 1, \ldots, n.
\]

Then (5.5) has a positive $\omega$-periodic solution.

**Proof.** Note that (5.5) has the form (4.1) with $h_{il}(s, u) = ue^{-c_{il}(s)u}, \nu_{il}(t, s) = s$. Let $c_{il}^+ = \max_{l \in [0, \omega]} c_{il}^+(t)$ and $c_{il}^- = \max_{l \in [0, \omega]} c_{il}^-(t)$, $i = 1, \ldots, n, l = 1, \ldots, m$. Then (h3) is satisfied with $h_i(u) = ue^{-c_{il}^+ u}$. The result is a consequence of the criteria in Theorem 4.1.

A similar result holds for e.g. Mackey-Glass-type systems with patch structure (1.1) with

\[
g_i(t, x_{it}) = \sum_{l=1}^{m} \frac{\beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} x_i(s)ds}{1 + c_{il}(t) \left( \int_{t-\tau_{il}(t)}^{t} x_i(s)ds \right)^{\alpha_i}},
\]
where $\alpha_d$ are positive constants, $d_i, a_{ij}, \beta_d, c_d, \tau_d \in C_{d}^{+}(\mathbb{R})$, which are included in the family (4.2). Theorem 4.1 is applicable with $b_i(t)$ in (4.3) given by $b_i(t) = \sum_{l=1}^{m} \beta_{il}(t)\tau_{il}(t)$, $i = 1, \ldots, n$.

Example 5.3. Nicholson systems with mixed monotonicity.

For the last years, there has been an increasing interest in DDEs with the nonlinearities given by functions $f(t, x, y)$ with mixed monotonicity in the spatial variables, i.e., with $f$ increasing in the variable $x$ and decreasing in $y$. See e.g. [4] for some relevant features and applications of scalar DDEs with mixed monotonicity.

In the innovative work of Chen [6], a criterion for the existence of a positive periodic solution for the periodic Nicholson equation

$$x'(t) = -d(t)x(t) + b(t)x(t - \tau(t))e^{-c(t)x(t - \theta(t))}$$

was established. Such criterion was generalised and improved in [11] for impulsive Nicholson equations with multiple pairs of discrete delays,

$$x'(t) = -d(t)x(t) + \sum_{l=1}^{m} \beta_{il}(t)x(t - \tau_l(t))e^{-c_l(t)x(t - \theta_l(t))},$$

and linear impulses, and in [5] for the case of distributed delays and more general impulses as in [11]. The aim here is to state a result for systems, as an illustration of Theorem 3.2 with the coefficients $\sigma_i$ in the definition of the cone $K$ having an active role.

We start with no impulses, and consider periodic Nicholson’s blowflies systems with mixed monotonicity as follows:

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + \sum_{l=1}^{m} \beta_{il}(t)x_i(t - \tau_{il}(t))e^{-c_{il}(t)x_i(t - \theta_{il}(t))}, \quad i = 1, \ldots, n,$$

where $d_i, a_{ij}, \beta_{il}, c_{il}, \tau_{il}, \theta_{il} \in C_{\omega}^{+}(\mathbb{R})$ with $d_i \neq 0, a_{ij} \neq 0, \sum_{l} \beta_{il} \neq 0, c_{il}(t) > 0$ on $[0, \omega]$. Define

$$b_i(t) = \sum_{l=1}^{m} \beta_{il}(t), \quad i = 1, \ldots, n.$$

Consider the cone $K = K(\sigma)$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\sigma_i = e^{-\int_{0}^{s} d_i(s)ds} = e^{-D_i(\omega)}$. Below, we consider the functions $g_i(t, \varphi_i) = \sum_{l=1}^{m} \beta_{il}(t)\varphi_i(-\tau_{il}(t))e^{-c_{il}(t)\varphi_i(-\theta_{il}(t))}$ and the positive constants $c_{il}^+ = \max_{t \in [0, \omega]} c_{il}^+ (t)$, $c_{il}^- = \min_{t \in [0, \omega]} c_{il}^- (t)$ and $c_{il}^+ = \max_{l \leq il \leq m} c_{il}^+$, $c_{il}^- = \min_{l \leq il \leq m} c_{il}^-$ ($1 \leq l \leq m, 1 \leq i \leq n$). Now, let $x = (x_1, \ldots, x_n) \in K$, and recall that $x_i \geq \sigma_i \|x_i\|$ for all $i$. Then, for any $\varepsilon > 0$, there are $0 < r_0 < R_0$ such that

$$g_i(t, x_i) \leq \sum_{l=1}^{m} \beta_{il}(t)\|x_i\|e^{-c_i^-\sigma_i\|x_i\|} < \varepsilon u \quad \text{if} \quad R_0 \leq x_i \leq u,$$

$$g_i(t, x_i) \geq \sum_{l=1}^{m} \beta_{il}(t)\sigma_i\|x_i\|e^{-c_i^+\|x_i\|} > (1 - \varepsilon)\sigma_i b_i(t)u \quad \text{if} \quad 0 < u \leq x_i \leq r_0.$$

Hence, (H6) is satisfied with $b_{1i}(t) = (1 - \varepsilon)\sigma_i b_i(t)$ and $b_{2i}(t) = \varepsilon$. Reasoning as in the proof of Theorem 4.1, we obtain the following result:
Theorem 5.3. Consider (5.8), with all the coefficients and delays satisfying the above general conditions. With $D_i(\omega) = \int_0^\omega d_i(t) dt, b_i(t) = \sum_{l=1}^m \beta_{il}(t) (1 \leq i \leq n)$, assume that there is a vector $v = (v_1, \ldots, v_n) > 0$ such that one of the following conditions is satisfied:

(a) \[ \left\{ \begin{array}{l} \int_t^{t+\omega} e^{\int_r^s d_i(r) dr} \left( \sum_{j \neq i} v_j a_{ij}(s) \right) ds < \epsilon D_i(\omega) - 1, \\
\int_t^{t+\omega} e^{\int_r^s d_i(r) dr} \left( \sum_{j \neq i} v_j a_{ij}(s) + e^{-D_i(\omega)} b_i(s) \right) ds > \epsilon D_i(\omega) - 1, \quad t \in [0, \omega], i = 1, \ldots, n; \\
\end{array} \right. \]

(b) \[ 0 \leq v_i d_i(t) - \sum_{j \neq i} v_j a_{ij}(t) \leq \sqrt[n]{v_i e^{-D_i(\omega)} b_i(t), \quad t \in [0, \omega], i = 1, \ldots, n; \]

(c) \[ \left\{ \begin{array}{l} \int_0^\omega \sum_{j \neq i} v_j a_{ij}(t) dt \leq v_i (1 - e^{-D_i(\omega)}) \\
\int_0^\omega b_i(t) dt + \sum_{j \neq i} v_j \int_0^\omega a_{ij}(t) dt \geq v_i (e^{D_i(\omega)} - 1), \quad i = 1, \ldots, n. \end{array} \right. \]

Then (5.8) has at least one positive $\omega$-periodic solution.

Corollary 5.1. Consider the periodic Nicholson equation with multiple pairs of delays (5.7), where $d, \beta_1, c_1, \tau_1, d_1 \in C^+_\omega(\mathbb{R})$ with $d \neq 0, \sum_{l=1}^m \beta_l \neq 0, c_i(t) > 0$ on $[0, \omega]$. Assume that one of the following conditions is satisfied:

(a) \[ \sum_{l=1}^m \int_t^{t+\omega} \beta_l(s) e^{-\int_r^s d_i(r) dr} ds \geq \epsilon \int_0^\omega d(s) ds - 1, \quad t \in [0, \omega); \]

(b) \[ \sum_{l=1}^m \beta_l(t) \geq \epsilon d(t) \int_0^\omega d(s) ds, \quad t \in [0, \omega]; \]

(c) \[ \sum_{l=1}^m \int_0^\omega \beta_l(t) dt \geq \epsilon \int_0^\omega d(s) ds \left( \epsilon \int_0^\omega d(s) ds - 1 \right). \]

Then, (5.7) has at least one positive $\omega$-periodic solution.

Even for the scalar case, Corollary 5.1 improves results in [6, 11]. In fact, Chen [6] showed that a positive $\omega$-periodic solution for (5.6) exists if

\[ \int_0^\omega b(t) dt > e^2 \int_0^\omega d(s) ds \int_0^\omega d(s) ds \]

and in [11] the existence of a positive $\omega$-periodic solution for (5.7) was established under the strict inequalities “$>$” in (a),(b) or (c) above. Clearly, for impulsive systems

\[
\begin{cases}
  x_i'(t) = -d_i(t) x_i(t) + \sum_{j \neq i} a_{ij}(t) x_j(t) + \sum_{l=1}^m \beta_{il}(t) x_i(t - \tau_{il}(t)) e^{-c_{il}(t)(t - \theta_{il}(t))}, & t \neq t_k \\
  x_i(t_k^+) - x_i(t_k) = I_{ik}(x_i(t_k)), & k \in \mathbb{Z}, \quad i = 1, \ldots, n,
\end{cases}
\]

similar criteria can be stated as in Theorem 4.1, with each $\sigma_i$ above replaced by $\sigma_i = B_i / \prod_{l=1}^m e^{-D_l(\omega)}$.

Example 5.4. A planar Nicholson system with discrete delays.

Consider

\[
\begin{cases}
  x_1'(t) = -d_1(t) x_1(t) + a_1(t) x_2(t) + \sum_{l=1}^m \beta_{1l}(t) x_1(t - \tau_{1l}(t)) e^{-c_{1l}(t) x_1(t - \theta_{1l}(t))}, & t \neq t_k, \\
  x_2'(t) = -d_2(t) x_2(t) + a_2(t) x_1(t) + \sum_{l=1}^m \beta_{2l}(t) x_2(t - \tau_{2l}(t)) e^{-c_{2l}(t) x_2(t - \theta_{2l}(t))}, & t \neq t_k, \\
  x_i(t_k^+) - x_i(t_k) = I_{ik}(x_i(t_k)), & k \in \mathbb{Z}, \quad i = 1, 2,
\end{cases}
\]
where all the coefficients and delays are in $C_+^1(\mathbb{R})$, $c_{il}(t) > 0$, $d_i(t) > 0$, $\int_0^\omega a_i(t) dt > 0$ and, as before, define $b_i(t) = \sum_{l=1}^m \beta_{il}(t)$, $i = 1, 2$. Theorem 3.5 leads to the criterion below.

**Theorem 5.4.** Under the above conditions, suppose that $t_k$, $I_{1k}$, $I_{2k}$ satisfy (H1)-(H3) and that, for $m_{qi}$, $q, i = 1, 2$ defined as in (3.12), there is $v = (v_1, v_2) > 0$ such that:

\[
\begin{align*}
 m_{21}\max_{t \in [0, \omega]} \frac{v_1^{-1}v_2a_1(t)}{d_1(t)} &< 1 < m_{11}\min_{t \in [0, \omega]} \frac{b_1(t) + v_1^{-1}v_2a_1(t)}{d_1(t)}, \\
 m_{22}\max_{t \in [0, \omega]} \frac{v_2^{-1}v_1a_2(t)}{d_2(t)} &< 1 < m_{12}\min_{t \in [0, \omega]} \frac{b_2(t) + v_2^{-1}v_1a_2(t)}{d_2(t)}.
\end{align*}
\]

(5.12)

Then there exists a positive $\omega$-periodic solution of (5.11).

**Proof.** With the notation in (3.20), for $v = (v_1, v_2) > 0$ we have $I^0_i = \min_{t \in [0, \omega]} \frac{b_i(t)}{a_i(t)}$, $\mathcal{B}_i^\infty = 0$ ($i = 1, 2$). Conditions (5.12) imply that the requirements in (3.21) are satisfied.

For the planar system with no impulses, with the particular choice of $v = (1, 1)$, conditions (5.12) reduce to $\min_{t \in [0, \omega]} \frac{a_i(t)}{d_i(t)} < 1 < \min_{t \in [0, \omega]} \frac{b_i(t) + a_i(t)}{d_i(t)}$ for $i = 1, 2$.

**Remark 5.1.** We observe that a very particular case of (5.11) was considered by Zhang et al. [34], under the following requirements:

(i) the impulses are linear, $I_{ik}(u) = \eta_{ik}u$ ($i = 1, 2, k \in \mathbb{Z}$) with $\eta_{ik} > -1$, and (H1) holds;

(ii) the functions $t \mapsto \prod_{k: t_k \in [0, t]} (1 + \eta_{ik})$ ($i = 1, 2$) are $\omega$-periodic;

(iii) $d_i(t), a_i(t), \beta_{il}(t), c_{il}(t), \tau_{il}(t)$ are strictly positive functions in $C_\omega(\mathbb{R})$, for $i = 1, 2, l = 1, \ldots, m$;

(iv) $\frac{a_i^+ a_2^+}{d_1 d_2} < 1$

(5.13)

where $a_i^+ = \max_t a_i(t), d_i^- = \min_t d_i(t)$.

Recall that for linear impulses as in (i) above, we have $J_{ik}(u) \equiv (1 + \eta_{ik})^{-1}$, $B_i(t) := B_i(t; u) = \prod_{k: t_k \in [0, t]} (1 + \eta_{ik})^{-1}$ and $\Gamma_i(u) \equiv (B_i(\omega)e^{D_i(\omega)} - 1)^{-1}$, in particular these functions do not depend on $u \in \mathbb{R}^+$, $i = 1, 2$. Moreover, from (ii) it follows from Liu and Takeuchi [20] that $\prod_{k=1}^p (1 + \eta_{ik})^{-1} = 1$, thus $B_i(\omega) = 1$ and $m_{ii} = B_i, m_{21} = B_{21}$.

We also stress that in [34] the authors reduce the system to a system without impulses and nonlinearities with jumps, by the change of variables (2.8). However, in [34] initial conditions $x_t = \phi$ are taken with $\phi = (\phi_1, \phi_2)$ strictly positive and continuous, instead of piecewise continuous functions with jumps discontinuities at the instants $t_k$ – which seems not to be consistent with the problem. In this scenario, by using a Krasnoselski’s fixed point argument, Zhang et al. [34] claimed the existence of a positive $\omega$-periodic solution, without imposing any other restrictions on the impulses.

Note that (5.13) implies that it is possible to choose $v = (v_1, v_2) > 0$ such that:

\[
v_2 \frac{a_1^+}{d_1} < v_1 < v_2 \frac{d_2^-}{a_2^+},
\]

and therefore

\[
\begin{align*}
\max_{t \in [0, \omega]} \frac{v_1^{-1}v_2a_1(t)}{d_1(t)} &\leq v_1^{-1}v_2 \frac{a_1^+}{d_1} < 1, \\
\max_{t \in [0, \omega]} \frac{v_2^{-1}v_1a_2(t)}{d_2(t)} &\leq v_2^{-1}v_1 \frac{a_2^+}{d_2} < 1.
\end{align*}
\]
In particular, for the nonimpulsive situation, the first inequalities in both conditions (5.12) are satisfied. On the other hand, contrary to what is asserted in [34], the above impositions (i)-(iv) are not enough to guarantee the existence of a positive periodic solution, as the following simple counter-example shows. More elaborated examples for nonautonomous systems and with nonlinear impulses could also be given.

**Example 5.5.** An autonomous planar Nicholson system with and without impulses.

Consider the autonomous nonimpulsive planar system

\[
\begin{align*}
    x'_1(t) &= -d_1 x_1 + a_1 x_2 + \beta_1 x_1(t - \tau_1) e^{-c_1 x_1(t - \tau_1)} \\
    x'_2(t) &= -d_2 x_2 + a_2 x_1 + \beta_2 x_2(t - \tau_2) e^{-c_2 x_2(t - \tau_2)}
\end{align*}
\]  

(5.14)

with \(d_i, a_i, \beta_i, c_i, \tau_i > 0\), \(i = 1, 2\). For this system, condition (5.13) in [34] reduces to \(d_1 d_2 > a_1 a_2\). Define the so-called community matrix as \(M = \begin{bmatrix} \beta_1 - d_1 & a_1 \\ a_2 & \beta_2 - d_2 \end{bmatrix}\). From [12] it follows that \(s(M) \leq 0\) is a necessary and sufficient condition for 0 to be a globally asymptotically stable equilibrium of (5.14) (in the set of all nonnegative solutions), where \(s(M) = \max \{Re \lambda : \lambda \in \sigma(M)\}\). Choose e.g. \(d_i = 2, a_i = \beta_i = 1, i = 1, 2\). Clearly (5.13) is satisfied. However, since \(\sigma(M) = \{0, -2\}\), 0 is a global attractor for (5.14) – in particular, the claim in [34] is not valid for the nonimpulsive case.

On the other hand, fix any \(\omega > 0\) and add to system (5.14) e.g. a single linear, constant, positive impulse on each component and on each interval of length \(\omega\):

\[
\Delta x_i(t_k) = \eta_i x_i(t_k), \quad i = 1, 2, k \in \mathbb{Z}
\]  

(5.15)

with \(0 < t_1 < \omega, t_{k+1} = t_k + \omega, k \in \mathbb{Z}\), and \(0 < \eta_i < e^{2\omega} - 1\). In this situation, (H1)-(H3) hold. With the previous notation we have \(\Gamma_i = \Gamma_i = \left( (1 + \eta_i)^{-1} e^{2\omega} - 1 \right)^{-1}, B_i = (1 + \eta_i)^{-1}, B_i = 1\),

\[
m_{1i} = m_{1i}(\eta_i) = \frac{e^{2\omega} - 1}{e^{2\omega} - (1 + \eta_i)}, \quad m_{2i} = m_{2i}(\eta_i) = \frac{e^{2\omega} - 1}{(1 + \eta_i)^{-1} e^{2\omega} - 1}, \quad i = 1, 2.
\]  

(5.16)

Note that (5.12) is satisfied with \(v = (1, 1)\) if \(\frac{1}{2} m_{2i}(\eta_i) < 1 < m_{1i}(\eta_i)\), which holds if

\[0 < \eta_i < \frac{e^{2\omega} - 1}{e^{2\omega} + 1}, \quad i = 1, 2.\]  

From Theorem 5.3, this leads to the existence of a positive \(\omega\)-periodic solution for the impulsive Nicholson system. This shows that implementing a small constant, \(\omega\)-periodic positive impulse to system (5.14), for any periodicity \(\omega > 0\), can create a positive periodic solution, whereas populations are otherwise driven to extinction.

## 6 Conclusions

In the present paper, we consider \(\omega\)-periodic delayed systems (1.1), with either discrete or distributed delays and subject to \(\omega\)-periodic impulses. Under very general conditions on the nonlinearities and impulses, we prove that (1.1) possesses at least one positive \(\omega\)-periodic solution, by using
Krasnoselskii’s fixed point theorem. As far as the authors know, this is one of the first papers proving the existence of positive periodic solutions for systems of differential equations with delays and impulses. Moreover, the original method proposed here applies to very broad classes of impulsive systems of DDEs under very mild assumptions on the impulses, which are in general nonlinear and whose signs may vary. In fact, recently Zhang et al. [34] studied the particular planar Nicholson system (5.11) with linear impulses $I_{ik}(u) = \eta_{ik} u$, where the constants $\eta_{ik} > -1$ are subject to the additional restriction that the functions $t \mapsto \prod_{k: t_k \in [0,t]} (1 + \eta_{ik}) (i = 1, 2)$ are $\omega$-periodic. Contrary to the authors’ claim however, condition (5.13) is not sufficient to guarantee the existence of a positive periodic solution, as shown in Example 5.5.

The major novelty of our approach is based on the particular operator whose fixed points are the periodic solutions we are looking for. The construction of such an operator follows along the main lines in [5,11], however most of the arguments have to be modified, due to the multidimensional character of (1.1). This operator is far different from other ones constructed in the literature, see e.g. [3,18,26,32,34], since it departs from inserting the impulses in a multiplicative way (rather than additive), through the products of the auxiliary functions $J_{ik}(u)$ in (2.3).

Our results are illustrated and analysed within the context of some related works, showing the advantage and novelty of our approach. We have restricted ourselves to the presentation of a few selected examples to reduce the size of this manuscript; many other examples could have been given, e.g. multidimensional versions of the models treated in [11,18,32,34]. Most of the applications refer to systems with bounded delays, but our results apply with straightforward changes to impulsive systems (1.1) with infinite delay (see Remark 2.1). Once the existence of a positive periodic solution is established, a future line of investigation is to study sufficient conditions for its global attractivity: of course, this depends strongly on the particular nonlinearities $g_i$ in (1.1), as shown in [7,9,15,30] for nonimpulsive Nicholson systems.

Although fixed point theorems in cones have been employed in some works, mostly dealing with periodic competitive Lotka-Volterra systems of DDEs as in [3,26], the literature for impulsive versions of periodic multidimensional DDEs is almost nonexistent, so we believe that the new results presented here have significant outcomes, namely in addressing Nicholson-type systems. The present technique has the potential to treat other families of impulsive systems with delay, such as Lotka-Volterra models, Nicholson systems with patch structure and nonlinear mortality terms as mentioned in Remark 3.6 or hematopoiesis systems with harvesting terms (see [17] for a nonimpulsive very general model).

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