RATIONALLY CONNECTED FOLIATIONS AFTER BOGOMOLOV AND
MCQUILLAN

STEFAN KEBEKUS, LUIS SOLÁ CONDE AND MATEI TOMA

ABSTRACT. This paper is concerned with sufficient criteria to guarantee that a given foliation on a normal variety has algebraic and rationally connected leaves. Following ideas from a preprint of Bogomolov-McQuillan and using the recent work of Graber-Harris-Starr, we give a clean, short and simple proof of previous results. Apart from a new vanishing theorem for vector bundles in positive characteristic, our proof employs only standard techniques of Mori theory and does not make any reference to the more involved properties of foliations in characteristic $p$.

We also give a new sufficient condition to ensure that all leaves are algebraic.

The results are then applied to show that $\mathbb{Q}$-Fano varieties with unstable tangent bundles always admit a sequence of partial rational quotients naturally associated to the Harder-Narasimhan filtration.

CONTENTS

1. Introduction 1
2. A vanishing criterion for vector bundles in positive characteristic 3
3. Relative deformations of curves 4
4. Proof of Theorem 1 5
5. Proof of Theorem 2 10
6. Proof of Corollary 5 11
7. A geometric consequence of unstability 13
References 14

1. INTRODUCTION

Since Ekedahl’s work on foliations in positive characteristic and Miyaoka’s landmark paper [Miy85], foliations of algebraic varieties have met with considerable interest from algebraic geometers and number theorists alike.

This paper is concerned with a sufficient criterion to guarantee that a given singular foliation on a normal variety has algebraic and rationally connected leaves. More precisely, using a vanishing theorem for vector bundles on curves in characteristic $p$, we give a simple proof of the following result.

**Theorem 1** (cf. [BM01 thm. 0.1]). Let $X$ be a normal complex projective variety, $C \subset X$ a complete curve which is entirely contained in the smooth locus $X_{\text{reg}}$, and $\mathcal{F} \subset T_X$ a (possibly singular) foliation which is regular along $C$. Assume that the restriction $\mathcal{F}|_C$ is an ample vector bundle on $C$. If $x \in C$ is any point, the leaf through $x$ is algebraic. If $x \in C$ is general, the closure of the leaf is rationally connected.

---

Date: September 9, 2018.

The authors thank their hosting institution, Universität zu Köln. The first two authors were supported in full or in part by the Forschungsschwerpunkt “Globale Methoden in der komplexen Analysis” of the Deutsche Forschungsgemeinschaft. A part of this paper was worked out while Stefan Kebekus visited the Korea Institute for Advanced Study. He would like to thank Jun-Muk Hwang for the invitation.
Bogomolov and McQuillan state a stronger result where the foliation is allowed to have singularities at points of $C$. Since this paper strives for simplicity, not completeness, we do not consider this extra complication here.

On the other hand, if $F$ is non-singular, we can use the Reeb stability theorem to prove the algebraicity of all leaves.

**Theorem 2.** In the setup of Theorem 1 if $F$ is regular, then all leaves are rationally connected submanifolds.

In fact, a stronger statement holds, see Theorem in section 5.

As an immediate corollary to Theorem 1 we prove a refinement of Miyaoka’s characterization of uniruled varieties. Before stating this corollary, we need to introduce the following notation.

**Notation 3.** Let $X$ be a normal projective variety, and let $q : X \rightarrow Q$ be the rationally connected quotient, defined through the maximally rationally connected fibration (“MRC-fibration”) of a desingularization of $X$, cf. [Kol96, IV . 5.3, 5.5]. Further, suppose that $C \subset X$ is a subvariety which is not contained in the singular locus of $X$, and not contained in the indeterminacy locus of $q$, and that $F \subset T_{X|C}$ is a subsheaf of the restriction of the tangent sheaf to $C$. We say that $F$ is *vertical with respect to the rationally connected quotient*, if $F$ is contained in $T_{X|Q}$ at the general point of $C$.

**Notation 4.** If $X$ is normal, we consider *general complete intersection curves* in the sense of Mehta-Ramanathan, $C \subset X$. These are reduced, irreducible curves of the form $C = H_1 \cap \cdots \cap H_{\dim X-1}$, where the $H_i \in |m_i \cdot L_i|$ are general, the $L_i \in \text{Pic}(X)$ are ample and the $m_i \in \mathbb{N}$ large enough, so that the Harder-Narasimhan filtration of $T_X$ commutes with restriction to $C$. If the $L_i$ are chosen, we also call $C$ a *general complete intersection curve with respect to $(L_1, \ldots, L_{\dim X-1})$.

We refer to [Fle84] and [Lan04] for a discussion and an explicit bound for the $m_i$.

**Corollary 5.** Let $X$ be a normal complex-projective variety and $C \subset X$ a general complete intersection curve. Assume that the restriction $T_{X|C}$ contains an ample locally free subsheaf $F_C$. Then $F_C$ is vertical with respect to the rationally connected quotient of $X$.

This statement appeared first implicitly in [Kol92, chap. 9], but see Remark 23. To our best knowledge, the argument presented here gives the first complete proof of this important result.

We would like to thank Thomas Peternell for discussions and the anonymous referee for careful reading and a number of helpful comments. In particular, we owe the referee an elementary proof of our vanishing result, Proposition 8, our previous approach relied on more involved results of Langer, [Lan04]. The statement of Theorem 2 was suggested to us by János Kollár. We would like to thank him very much for the discussion.

1.1. **Outline of the paper.** All our results here are principally based on a vanishing theorem for vector bundles in finite characteristic, Proposition 9 in Section 2 below. In Section 3 we have gathered a number of standard facts about the space of relative deformations of morphisms.

With these preparations at hand, the proof of Theorem 1 which we give in Section 4 becomes reasonably short and quite intuitive. The line of argumentation follows [BM01], but differs in one important aspect: we employ Mori’s standard method of “reduction modulo characteristic $p$”, but using our vanishing result, we do not have to use the partial Frobenius morphism for foliated varieties that was introduced by Miyaoka. We believe that this, and our use of the Graber-Harris-Starr result makes the proof much cleaner and easier to understand.

Theorem 2 is proven in section 5 using Reeb stability and standard facts on rationally connected submanifolds. After this paper was accepted for publication we learned that
2. A VANISHING CRITERION FOR VECTOR BUNDLES IN POSITIVE CHARACTERISTIC

Throughout the present section, let $C$ be a smooth curve of genus $g$ defined over an algebraically closed field $k$ of characteristic $p > 0$. Further, let $E$ be a vector bundle on $C$. Even if $E$ is ample, it is of course generally false that $H^1(C, E) = 0$ or that $E$ has any sections. However, in Proposition 9 we will give a criterion to guarantee that the pull-back of $E$ via the Frobenius morphism is globally generated and satisfies a strong vanishing statement, even if the pull-back is twisted with certain ideal sheaves.

We will use $\mathbb{Q}$-twists of vector bundles as presented in Lazkoz [1], II, 6.2, to which we refer for details. In our case, we identify rational numbers $\delta$ with numerical classes $\delta \cdot [P] \in N^1_{\mathbb{Q}}(C)$, where $P$ is a point in $C$. For every $\delta \in \mathbb{Q}$, the $\mathbb{Q}$-twist $E(\delta)$ is defined as the ordered pair of $E$ and $\delta$. $\mathbb{Q}$-twisted vector bundle is said to be ample if the class $c_1(C_{\mathbb{P}(E)}(1)) + \pi^*(\delta)$ is ample on the projectivized bundle $\mathbb{P}(E)$, where $\pi$ denotes the natural projection. One defines the degree $\deg(E(\delta)) := \deg(E) + \text{rank}(E)\delta$.

A quotient of $E(\delta)$ is a $\mathbb{Q}$-twisted vector bundle of the form $E'(\delta)$ where $E'$ is a quotient of $E$. Pull-backs of $\mathbb{Q}$-twisted vector bundles are defined in the obvious way.

In order to state the main result of this section, we introduce the following notation.

**Notation 6.** Let $F : C[1] \to C$ be the $k$-linear Frobenius morphism.

**Definition 7** (Vanishing threshold). Given a rational number $\delta$ we define

$$b_p(\delta) := p\delta - 2g + 1,$$

where $g$ denotes the genus of $C$.

The following trivial observation will later become important in the applications, when $C$ and $E$ are reductions modulo $p$ of objects that were initially defined in characteristic 0.

**Remark 8.** If we view the number $b_p(\delta)$ purely numerically as a function of integers $g$, $p$ and $\delta$, then

$$\lim_{p \to \infty} \frac{p}{b_p(\delta)} = \lim_{p \to \infty} \frac{p}{\lfloor b_p(\delta) \rfloor} = \frac{1}{\delta},$$

where $\lfloor . \rfloor$ means “round-down”.

**Proposition 9** (Vanishing after pull-back via Frobenius). Let $E$ be a vector bundle of rank $r$ over $C$, and $\delta$ a positive rational number. Assume that $E(\delta)$ is ample and that the vanishing threshold $b_p(\delta)$ is non-negative. Then for every subscheme $B \subseteq C[1]$ of length smaller than or equal to $b_p(\delta)$ we have

$$H^1(C[1], F^*(E) \otimes I_B) = \{0\}.$$

Further, $F^*(E) \otimes I_B$ is globally generated.
Lemma 10 (Vanishing after pull-back via Frobenius in families). Let \((E_t)_{t \in T}\) be an irreducible family of vector bundles on \(C\) is given by an algebraic variety \(T\) and a vector bundle \(E\) on the product \(C \times T\). Using that the vanishing statement involves all subschemes \(B \subset C\) of bounded length, this follows from the properness of the Hilbert-scheme by standard arguments.

3. Relative deformations of curves

The aim of this section is to fix notation and to gather some definitions and results about relative deformations of morphisms from curves. While all of this is fairly standard, the relative case is not covered in great detail in the standard references. We have therefore chosen to say a few words here.

The setup we are considering is that of a surjective morphism \(\sigma : C \to X\) of \(Y\)-schemes,

\[
\begin{array}{c}
C \xrightarrow{\sigma} X \\
\pi \downarrow \quad \quad \quad \downarrow \pi \\
Y 
\end{array}
\]

Throughout this section we work over an algebraically closed field \(k\) of arbitrary characteristic.

The main definitions concern the space of \(Y\)-morphisms.

Notation 11. Let \(\text{Hom}_Y(C, X) \subset \text{Hom}(C, X)\) be the scheme of \(Y\)-morphisms \(\sigma : C \to X\). We use \([\sigma]\) to denote the point associated with \(\sigma\).

Notation 12. Given any such morphism \(\sigma\) and a subscheme \(B \subset C\), we consider the subscheme \(\text{Hom}_Y(C, X, \sigma|_B) \subset \text{Hom}_Y(C, X)\) of \(Y\)-morphisms \(\sigma'\) whose restriction to \(B\) agrees with \(\sigma\), i.e. \(\sigma'|_B = \sigma|_B\). To shorten notation, let \(\mathcal{H}_{\sigma, B} \subset \text{Hom}_Y(C, X, \sigma|_B)\) be the connected component that contains \([\sigma]\).
When discussing separably uniruled smooth varieties, one of the most important notions is that of a “free morphism”. When the domain is a curve, the main feature of a free morphism is that it allows a large number of deformations. There is an obvious relative version of this notion with similar properties that is not discussed much in the literature.

**Definition 13.** A $Y$-morphism $\sigma : C \to X$ is called relatively free over $\sigma|_B$, if

1. $X$ and $\pi$ is smooth along the image of $\sigma$,
2. $H^1(C, \sigma^* (T_{X|Y}) \otimes I_B) = 0$, and
3. $\sigma^* (T_{X|Y}) \otimes I_B$ is globally generated.

**Proposition 14.** Let $\sigma$ be a $Y$-morphism which is relatively free over $\sigma|_B$. If $\dim Y = 1$ and $\pi \circ \sigma$ is surjective, then

1. the subscheme $\mathcal{H}_{\sigma,B}$ is smooth at $\sigma$,
2. the universal morphism $\mu : C \times \mathcal{H}_{\sigma,B} \to X$ dominates $X$, and
3. if $M \subset X$ is any subset of codim $M \geq 2$, and $[\sigma'] \in \mathcal{H}_{\sigma,B}$ a general point, then $\mu^{-1}(M) \subset B$.

**Proof.** This result is completely similar to [Kol96, II. props. 3.5 and 3.7] where the non-relative case is considered. The proof is analogous.

We close this section by recalling Mori’s famous Bend-and-Break argument which we use in our proof of Theorem 1. This result works for absolute and relative deformations alike.

**Theorem 15** (Bend-and-Break, [Kol91, prop. 3.3]). Assume that $C$ is smooth, not rational, that $\sigma$ is not constant and that there exists a number $b \in \mathbb{N}$ such that for every closed subscheme $B \subset C$ of length $b$, $\dim_{[\sigma]} \text{Hom}(C, X, \sigma|_B) > 0$. If $x \in C$ is a general point, and $H$ any ample divisor on $X$, then there exists a rational curve $R \subset X$ that contains $\sigma(x)$ and satisfies:

$$H \cdot R \leq 2 \frac{\deg f^* H}{b}.$$  

4. **Proof of Theorem 1**

We maintain the assumptions and notation from Theorem 1 throughout the present section. We will work over the complex number field, except in Sections 4.4–4.5, where we consider the reduction to fields of positive characteristic. Varieties that are defined over fields $k$ of positive characteristic will be marked with an index $k$.

In Section 4.1 we start by reducing ourselves to the situation where the curve $C$ is smooth and everywhere transversal to the given foliation $\mathcal{F}$. We will then, in Section 4.2, recall an old result of Hartshorne which immediately shows that all leaves through points of $C$ are algebraic. This argument also reduces us to the case where the foliation is the foliation of a morphism that is smooth along $C$.

The main point is then to show that the leaves are rationally connected. This is done by a reduction to the absurd: we assume in Section 4.3 that there is a non-trivial rationally connected quotient. Using Mori’s reduction modulo $p$ argument, and the vanishing result of Proposition 9, we can show in Sections 4.4–4.5 that the maximally rationally connected quotient is covered by rational curves. This contradicts a famous result of Graber-Harris-Starr.

As announced in the introduction, we end with a short section on the history of this problem, and with attributions.

4.1. **Reduction to the case of a normal foliation.** Let $\nu : \tilde{C} \to C \subset X$ be a non-constant morphism from a smooth curve of positive genus $g(\tilde{C}) > 0$ to $C$, and $Y$ denote the product
$\sigma := (\nu, \text{id})$

$p_2 =: \pi$

$\tilde{C} \subset C'$ leaves

**Figure 4.1.** Reduction to the case of a normal foliation

$X \times \tilde{C}$ with projections $p_1$ and $p_2$. We obtain a diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{p_1} & X \\
\sigma := (\nu, \text{id}) & \downarrow & \\
\tilde{C} & \xrightarrow{p_2} & \sigma := (\nu, \text{id}) \\
\end{array}
$$

Let $C'$ be the image of $\tilde{C}$ under $\sigma$. The sheaf $\mathcal{F}_Y := p_1^*(\mathcal{F})$ is naturally embedded into $T_Y$ via the relative tangent bundle $T_{Y|\tilde{C}}$. It is therefore identified with a foliation $\mathcal{F}_Y$ of $Y$ whose leaves lie in the fibers of $p_2$. This construction is depicted in Figure 4.1.

The leaves of $\mathcal{F}_Y$ are isomorphic to the leaves of $X$, so that it suffices to prove the theorem for $\mathcal{F}_Y \subset T_Y$ and the curve $C'$. Clearly the new foliation still fulfills the hypotheses of the theorem: it is regular along $C'$ and $\mathcal{F}_Y|_{C'} \cong \nu^*(\mathcal{F}|_{\tilde{C}})$ is the pull-back of an ample vector bundle by a finite map, and therefore ample.

The construction has the advantage that the curve $C'$ is smooth and transversal to $\mathcal{F}_Y$ since it is already transversal to the relative tangent bundle $T_{Y|\tilde{C}}$. Moreover, a leaf of $\mathcal{F}_Y$ passing by a point of $C'$ intersects it set-theoretically in exactly that point.

4.2. **The algebraicity of the leaves.** Bost and Bogomolov-McQuillan have independently observed that a result of Hartshorne on the function fields of formal schemes and the ampleness of $\mathcal{F}_Y|_{C'}$ imply that for any $x \in C'$, the leaf through $x$ is algebraic. This answers a question of Miyaoka, [Miy85, rem. 8.9.(1)]. The reader may want to look at [Bos01, thm. 3.5] or [BM01, sect. 2.1] for this. We restrict ourselves to a brief review of the argument.

Since $C'$ is everywhere transversal to $\mathcal{F}_Y$, the classical Frobenius theorem, see e.g. [War71] thm. 1.60, immediately yields the following.

**Fact 16.** There exists an irreducible analytic submanifold $W \subset Y$ containing $C'$ such that the restriction $\pi|_W$ is smooth, and such that its fibers are analytic open subsets of the leaves of the foliation passing through points of $C'$. Moreover, $\mathcal{F}_Y|_{C'} \cong \nu^*(\mathcal{F}|_{\tilde{C}}) \cong N_{C',W} \cong T_{W|\tilde{C}}|_{C'}$. □

Hartshorne’s result, applied to a formal neighborhood of $C'$ in $W$, then gives the exact dimension of the Zariski closure.
**Fact 17** ([Har68] thm. 6.7]). Let $\overline{W}$ be the Zariski closure of $W$. Since $N_{C', W}$ is ample we have $\dim(\overline{W}) = \dim(W)$. □

**Corollary 18.** If $x \in C'$ is any point, and $L \subset (\pi_{\overline{W}})^{-1}(\pi(x))$ the unique irreducible fiber component that contains $W_x := (\pi_{|W})^{-1}(\pi(x))$, then $L$ is exactly the leaf of $F_Y$ through $x$. In particular, all leaves of $F_Y$ through points of $C'$ are algebraic.

**Proof.** Fact 17 immediately implies that $\dim L = \operatorname{rank} F_Y$. Since $L$ contains the $F_Y$-invariant submanifold $W_x$, it is clear that $L$ itself is invariant under $F_Y$ wherever it is smooth. Hence the claim. □

By construction, Corollary 18 immediately implies that the leaves of $F$ through points of $C$ are algebraic. We will show in the rest of the present section that they are rationally connected, or equivalently, that the fibers of $\overline{W} \to \tilde{C}$ are rationally connected.

One technical problem with this approach is that $\overline{W}$ is not necessarily smooth along $C'$. In fact, it may happen that whole fibers of $\overline{W}$ over $\tilde{C}$ are contained in the non-normal locus. To overcome this difficulty, observe that images of rationally connected varieties are themselves rationally connected. It suffices therefore to prove rational connectedness for the fibers of the normalization of $\overline{W}$ over $\tilde{C}$.

**Remark 19.** The universal property of the normalization immediately yields an embedding $e : W \to \text{normalization of } \overline{W}$. In particular, the normalization of $\overline{W}$ is smooth along the image $e \circ \sigma(\tilde{C})$, and the normal bundle of $e \circ \sigma(\tilde{C})$ is isomorphic to $F_Y|_{C'}$.

### 4.3. Rational connectedness of the leaves: setup of notation

By the previous subsections, we may replace $X$ without loss of generality by a desingularization of the normalization of $\overline{W}$. Since we work in characteristic 0, Hartshorne’s characterization of ampleness [Laz04], II, 6.4.15] implies that if $E$ is any ample vector bundle on $C$, then $E(\frac{\operatorname{rank} E + 1}{2})$ is also ample. It therefore suffices to prove Theorem 1 under the following additional assumptions that we maintain through the end of the present section.

**Assumption 20.** In the setup of Theorem 1 assume additionally that $X$ is smooth, that $C$ is a smooth curve of genus $g(C) > 0$, and that the foliation $F$ is the foliation of a morphism $\pi : X \to C$ with connected fibers which is smooth about $C$ and with $\pi|_C = \text{id}_C$.

In particular, if $\sigma : C \to X$ is the inclusion, we assume that $\sigma^*(T_X|_C)$ and $\sigma^*(T_X|_C)(\frac{1}{\dim X})$ are ample vector bundles on $C$.

We will need to consider a maximally rationally connected fibration of $X$ ("MRC-fibration") $q : X \dashrightarrow Z$ which maps $X$ to a normal, projective variety $Z$; this is explained in detail in [Ko96a sect. IV.5] or [Deb01 sect. 5]. The functoriality of the MRC-fibration, [Ko96a thm. IV.5.5], and the assumption that $C$ is not rational, imply that $\pi$ factors via the morphism $q$. We obtain a diagram as follows,

$$
\begin{array}{ccc}
X & \xrightarrow{q} & Z \\
\sigma \downarrow \pi & & \\
C & \xrightarrow{\beta} & \\
\end{array}
$$

where all morphisms and maps but $\sigma$ are surjective, or dominant, respectively.

We observe that with these notations and assumptions, in order to prove Theorem 1 it is enough to show:

**Proposition 21.** The maximally rationally connected quotient $Z$ is a curve.
Recall [Deb01, thm. 5.13] that the MRC-fibration $q$ is an almost holomorphic map. Thus, if Proposition 21 holds true, then $q$ must be a morphism. Since both $q$ and $\pi$ have connected fibers, $\beta$ will be an isomorphism. This will complete the proof of Theorem 1.

The remainder of section 4 is concerned with the proof of Proposition 21. For that, we assume the contrary, i.e. that $\dim Z \geq 2$. We will then show that $Z$ is uniruled, which contradicts the well-known result of Graber, Harris and Starr, [GHS03]. The uniruledness of $Z$ will be established by a standard bend-and-break method in positive characteristic that is detailed below. The method requires us to bound the degrees of curves in $Z$. For this, we have to fix a polarization on $Z$.

Notation 22. Choose a point $z \in Z$. Further, choose a very ample line bundle $H_Z$ on $Z$. Let $H_X$ denote the pull-back of $H_Z$ by $q$; the rational map $q$ is then determined by a linear system $L \subset H^0(X, H_X)$, whose fixed locus is the indeterminacy locus of $q$.

Remark 23. The rational map $\beta$ induces a foliation on $Z$ whose leaves are fibers of $\beta$. We will implicitly show that there are lots of rational curves in $Z$ that are tangent to this foliation. This part of our proof is modelled on [Kol92, sect. 9], where a similar approach is employed in order to prove Corollary 5. While in our setup the existence of algebraic leaves on $X$ immediately yields the existence of Diagram 3 the construction of a foliation on the partial rationally connected quotient in [Kol92, p. 111] is problematic: the polarization of the $Z$-variety $Q$ discussed in [Kol92, p. 111] need not be stable under the action of the Galois group of $Q$ over $Z$. Accordingly, there is no obvious reason why the sheaf $L$ is. Also, it is not quite clear in [Kol92] why the images of the $C_t$ are again general complete intersection curves.

At this point of the argumentation, Bogomolov-McQuillan [BM01, p. 22] do not consider the MRC-fibration, but a rational map whose fibers are single rational curves. We had difficulties understanding that part of their paper.

4.4. Rational connectedness of the leaves: Reduction modulo $p$. In order to prove Proposition 21 we use Mori’s standard reduction mod $p$ argument, see [Mor79], [Kol96, II.5.10]. It is then enough to prove Proposition 21 assuming that all varieties and morphisms are defined over an algebraically closed field of large characteristic.

To be more precise, let $S$ be a ring of definition of all varieties and morphisms that appear in Diagram 3, and of $z$, $H_X$ and $H_Z$. Let $X_S$ be the associated scheme over $\text{Spec} S$. Given an algebraically closed field $k$ and a morphism $S \rightarrow k$, set $X_k := X_S \times_{\text{Spec} S} \text{Spec} k$. We use the same notation for $Z_k, H_{Z,k}$, etc.

The following Proposition will be shown in Section 4.5 below.

Proposition 24. The integer

$$d := 2 \deg \sigma^*(H_X) \cdot \dim X$$

is positive. There exists a number $p_0$ with the following property: If $k$ is the algebraic closure of a residue field of $S$ such that:

- all assumptions made in subsection 4.3 still hold for the geometric fibers over $k$,
- the characteristic of $k$ is larger than $p_0$, i.e. $\text{char} k > p_0$,

then the geometric fiber $Z_k$ is uniruled with curves of $H_{Z,k}$-degrees at most $d$.

If Proposition 24 holds true, Mori’s reduction argument implies that $Z = Z_C$ is uniruled with curves of $H_Z$-degree at most $d$. This will complete the proof of Proposition 21 and hence of Theorem 1.

4.5. Rational connectedness of the leaves: Proof in characteristic $p$. We will now prove Proposition 24. Recall from Remark 8 that there exists a number $p_0$ such that for all algebraically closed fields $k$ with $p := \text{char}(k) > p_0$, the vanishing threshold $b_p$, defined
on page 3 satisfies
\[ b_p(1/ \dim X) > 1 \quad \text{and} \quad \left| \frac{2p \cdot \deg \sigma^*(H_X)}{b_p(1/ \dim X)} - d \right| \leq \frac{1}{2}. \]

Now let \( k \) be the algebraic closure of a residue field of \( S \) that satisfies the assumptions of Proposition 24. As before, let \( F : C_k[1] \to C_k \) be the \( k \)-linear Frobenius morphism.

**Lemma 25.** There is an open neighborhood \( \Omega \subset \text{Hom}_{C_k}(C_k[1], X_k) \) of \( \sigma_k \circ F \) such that

1. If \( \sigma' \in \Omega \) is any morphism and \( B \subset C_k[1] \) any subscheme of length \( \#(B) \leq b_p(1/ \dim X) \), then \( \sigma' \) is relatively free over \( \sigma'_B \).
2. If \( T \subset X \) is the indeterminacy locus of the rational map \( q \) then the subset
   \[ \Omega^0 = \{ \sigma' \in \Omega \mid (\sigma')^{-1}(T_k) = \emptyset \} \]
   of morphisms whose images avoid \( T_k \) is again open in \( \text{Hom}_{C_k}(C_k[1], X_k) \).

**Proof.** The vanishing results of Proposition 24 and Lemma 10 applied to \( E = \sigma_k^*(T_{X_k}|_{C_k}) \) and \( \delta = \frac{1}{\dim X} \), yield the existence of an open set \( \Omega \) such that all \( \sigma' \in \Omega \) are relatively free over \( \sigma'_B \) if \( \#(B) \leq b_p(1/ \dim X) \). This shows (1). Assertion (2) follows from Proposition 14 (3). \( \square \)

**Notation 26.** Since morphisms \( \sigma' \in \Omega^0 \) avoid \( T \), we have a natural morphism:
\[
\eta : \Omega^0 \to \text{Hom}(C_k[1], Z_k)
\]
\[
[\sigma'] \mapsto [q_k \circ \sigma']
\]
We will also need to consider the associated evaluation morphism.
\[
\mu : C_k[1] \times \Omega^0 \to Z_k
\]
\[
(y, [\sigma']) \mapsto q_k \circ \sigma'(y)
\]

The most important properties of \( \eta \) and \( \mu \) are summarized in the following corollary to Lemma 25.

**Corollary 27.** With the notation introduced above, we have the following:

1. The evaluation morphism \( \mu : C_k[1] \times \Omega^0 \to Z_k \) dominates \( Z_k \).
2. For all \( \sigma' \in \Omega^0 \) and subschemes \( B \subset C_k[1] \) of length \( \#B \leq b_p(1/ \dim X) \), we have
   \[ \dim_{\eta([\sigma'])} \text{Hom}(C_k[1], Z_k, (q_k \circ \sigma')|_B) \geq 1. \]
3. For all morphisms \( \tau \) contained in the image \( \eta(\Omega^0) \), we have
   \[ \deg \tau^*(H_{Z,k}) = p \cdot \deg \sigma^*(H_X). \]

**Proof.** Assertions (1) and (2) are immediate consequences of Proposition 14 (2).

For Assertion (3), let \( [\tau] \in \eta(\Omega^0) \) be any element. The morphism \( \tau \) can then be written as \( \tau = q_k \circ \sigma' \), where \( \sigma' : C_k[1] \to X_k \) is a deformation of \( \sigma \circ F \). This immediately implies
\[
\deg \tau^*(H_{Z,k}) = \deg(q_k \circ \sigma')^*(H_{Z,k}) = \\
= \deg(\sigma')^*(H_{X,k}) = p \cdot \deg \sigma^*(H_X).
\]
\( \square \)

Let \( Z^0 \) be the maximal open set contained in the image of the evaluation morphism \( Z^0 \subset \mu(C_k[1] \times \Omega^0) \). By Corollary 27 (1) this is not empty. Now, if \( z \in Z^0 \) is any point, then there exists a morphism \( \tau : C_k[1] \to Z_k, [\tau] \in \eta(\Omega^0) \) whose image contains \( z \).
Bend-and-Break, Theorem 15, then implies that $z$ is contained in a rational curve $R \subset Z_k$ of degree

$$\begin{align*}
0 < H_{Z,k} \cdot R &\leq \frac{2 \cdot \deg \tau^*(H_{Z,k})}{[b_p(1/\dim X)]} \quad \text{Theorem 15 Corollary 27(2)} \\
&= \frac{2p \cdot \deg \sigma^*(H_X)}{[b_p(1/\dim X)]} \quad \text{Corollary 27(3)} \\
\Rightarrow 0 < H_{Z,k} \cdot R &\leq d \quad \text{I.h.s. is integral, Inequality (4)}.
\end{align*}$$

In particular, the integer $d$ is positive. This shows the first statement of Proposition 24.

By [Gro95, 4(c)], the scheme $\text{Hom}_d(P^1_k, Z_k)$ of non-constant morphisms $f : P^1_k \to Z_k$ with $\deg f^*(H_{Z,k}) \leq d$ is quasi-projective. In particular, it contains only finitely many irreducible components. The existence of rational curves of degree $\leq d$ through every point of $Z^0$ therefore implies that there exists one component $H \subset \text{Hom}_d(P^1_k, Z_k)$ such that the evaluation morphism

$$\mu : H \times P^1_k \to Z_k$$

is dominant. [Kol96, IV. prop. 1.4] then asserts that $Z_k$ is uniruled with curves of $H_{Z,k}$-degree at most $d$. This shows Proposition 24 and therefore ends the proof of Theorem 1. \hfill \Box

4.6. Attributions. The reduction to the case of a normal foliation and the proof of the algebraicity of the leaves follow [BM01] closely. The setup that we use to prove the rational connectedness, however, differs somewhat from that of [BM01]; see Remark 23.

The reduction modulo $p$ that Bogomolov-McQuillan and that we employ to produce rational curves on the quotient is of course due to Mori [Mor79]. The argumentation in characteristic $p$, is different from theirs: using the vanishing result of Proposition 9 rather than [BM01, lem. 3.2.1], we can give an explicit bound $d$ for the degree of the curves constructed. It is obvious in our construction that the number $d$ does not depend on the characteristic; this is perhaps not so clear in [BM01]. Another advantage of our approach is that we construct the curves directly using the standard Bend-and-Break argument, and do not have to deal with the partial Frobenius morphism associated with a foliation.

5. Proof of Theorem 2

Theorem 2 is an immediate consequence of the following, stronger statement.

**Theorem 28.** Let $X$ be a complex projective manifold, and $\mathcal{F} \subset T_X$ a foliation. Assume that there exists a compact, rationally connected leaf $L$ which does not intersect the singular locus of $\mathcal{F}$. Then all leaves are algebraic. The set

$$V := X \setminus \bigcup_{L': \text{non-compact leaf}} L'$$

is a Zariski-open neighborhood of $L$, and the restriction $\mathcal{F}|_V$ is the foliation given by a proper submersion $\pi : V \to B$. All fibers of $\pi$ are rationally connected.

**Proof.** Recall the standard fact that rationally connected projective manifolds are simply connected, [Deb01 cor. 4.18]. The holonomy of the foliation along that leaf $L$ is therefore trivial, and Reeb’s Stability Theorem [CLN85 thm. IV.3] asserts that there exists a fundamental system of analytic neighborhoods of $L$ in $X$ that are saturated with respect to $\mathcal{F}$. The triviality of the holonomy implies that one of these neighborhoods, say $V^o$, admits an analytic slice $B^o$ and a proper submersion $\pi^o : V^o \to B^o$ that induces the foliation $\mathcal{F}|_{V^o}$. By [KMM92 cor. 2.4], all fibers of $\pi^o$ are rationally connected.

Since the normal bundle of $L$ in $X$ is trivial and $L$ is rationally connected, we have $h^1(L, N_{L/X}) = 0$. The Douady space $D(X) \cong \text{Hilb}(X)$ parametrizing compact analytic subspaces of $X$ is thus smooth at the point representing $L$, and it follows immediately that
\( B^\circ \) embeds as an analytic open subset in a component \( D \) of \( D(X) \). If \( U \subset D \times X \) is the universal family and \( \pi_1, \pi_2 \) the canonical projections, we obtain a diagram

\[
\begin{array}{ccc}
V^\circ & \xrightarrow{\text{anal. open inclusion}} & U \\
\pi^\circ & & \pi_2 \\
\text{ratl. comm. fibers} & & \pi_1 \\
B^\circ & \xrightarrow{\text{anal. open inclusion}} & D.
\end{array}
\]

If \( d \in D \) is any point, then the reduced subvariety \( X_d := (\pi_2(\pi_1^{-1}(d)))_{\text{red}} \) is of pure dimension \( \dim X_d = \text{rank} \mathcal{F} \) and is \( \mathcal{F} \)-integral at all points \( x \in X_d \) wherever both \( X_d \) and \( \mathcal{F} \) are regular. Consequence: if \( x \in X_d \) is a regular point of \( \mathcal{F} \), then \( X_d \) is smooth at \( x \) and contains the closure of the leaf through \( x \) as an irreducible component. It follows that all leaves of \( \mathcal{F} \) are algebraic.

On the other hand, if \( x \in V \) is any point, then the associated leaf \( L' \) is compact. If \( y \in \pi_2^{-1}(x) \) is any point of the fiber, then \( L' = X_{\pi_1(y)} \). By [Kol96 IV 3.5.2], \( L' \) is rationally chain connected. Since \( L' \) is smooth, it will then be rationally connected. The holonomy argument from above then shows that \( D \) is smooth at \( \pi_1(y) \), and that \( \pi_1 \) is submersive in a neighborhood of \( L' \). In particular, the projection \( \pi_2 \) is birational and isomorphic at \( y \).

The exceptional locus \( E \) of \( \pi_2 \) does not intersect \( V \) and therefore does not dominate \( D \). Let \( B \subset D \setminus \pi_1(E) \) be the subset of regular points of \( D \setminus \pi_1(E) \) over which \( \pi_1 \) is smooth. We have seen that \( \pi_1^{-1}(B) \) is isomorphic to \( V \) and \( \pi := \pi_1|_{\pi_1^{-1}(B)} \) verifies the required properties.

\[ \square \]

6. PROOF OF COROLLARY 5

The proof of Corollary 5 relies on a number of facts about the Harder-Narasimhan filtration of vector bundles on curves, which are possibly known to the experts. For lack of an adequate reference we have included full proofs in Section 6.1 below. We refer to [Ses82] for a detailed account of semistability and of the Harder-Narasimhan filtration of vector bundles on curves.

6.1. Vector bundles over complex curves. To start, we show that any vector bundle on a smooth curve contains a maximally ample subbundle.

**Proposition 29.** Let \( C \) be a smooth complex-projective curve and \( E \) a vector bundle on \( C \), with Harder-Narasimhan filtration

\[
0 = E_0 \subset E_1 \subset \ldots \subset E_r = E
\]

and \( \mu_i := \mu(E_i/E_{i-1}) \) be the slopes of the Harder-Narasimhan quotients. Suppose that \( \mu_1 > 0 \) and let \( k := \max \{ i \mid \mu_i > 0 \} \). Then \( E_i \) is ample for all \( 1 \leq i \leq k \) and every ample subsheaf of \( E \) is contained in \( E_k \).

**Proof.** Hartshorne’s characterization of ampleness [Har71 thm. 2.4] says that \( E_i \) is ample iff all its quotients have positive degree. But the minimal slope of such a quotient is \( \mu_k \), which is positive for all \( 1 \leq i \leq k \).

Let now \( F \subset E \) be any ample subsheaf of \( E \) and \( j := \min \{ i \mid F \subset E_i, 1 \leq i \leq r \} \). We need to check that \( j \leq k \). By the definition of \( j \) and the ampleness of \( F \), the image of \( F \) in \( E_j/E_{j-1} \) has positive slope. The semi-stability of \( E_j/E_{j-1} \) therefore implies \( \mu_j > 0 \) and \( j \leq k \). \[ \square \]

Proposition 29 says that the first few terms in the Harder-Narasimhan filtration are ample. The following, related statement will be used later to construct foliations on certain manifolds.
Proposition 30. In the setup of Proposition 29 the vector bundles $E_j \otimes (E/E_i)^\vee$ are ample for all $0 < j \leq i < r$. In particular, if $E_i$ is any ample term in the Harder-Narasimhan Filtration of $E$, then $\text{Hom}(E_i, E/E_i)$ and $\text{Hom}(E_i \otimes E_i, E/E_i)$ are both zero.

Remark 31. If $X$ is a polarized manifold whose tangent bundle contains a subsheaf of positive slope, Proposition 30 shows that the first terms in the Harder-Narasimhan filtration of $T_X$ are special foliations in the sense of Miyaoka. [Miy85 sect. 8]. By [Miy85 thm. 8.5], this already implies that $X$ is dominated by rational curves that are tangent to these foliations.

Proof of Proposition 30. As a first step, we show that the vector bundle

$$F_{i,j} := (E_j / E_{j-1}) \otimes (E/E_i)^\vee$$

is ample. Assume not. Then, by Hartshorne’s ampleness criterion [Har71 prop. 2.1(ii)], there exists a quotient $A$ of $F_{i,j}$ of degree $\text{deg}_C A \leq 0$. Equivalently, there exists a non-trivial subbundle

$$\alpha : B \to F_{i,j}^\vee = (E_j / E_{j-1})^\vee \otimes (E/E_i)$$

with $\text{deg}_C B \geq 0$. Replacing $B$ by its maximally destabilizing subbundle, if necessary, we can assume without loss of generality that $B$ is semistable. In particular, $B$ has non-negative slope $\mu(B) \geq 0$. On the other hand, we have that $(E_j / E_{j-1})$ is semistable. The slope of the image of the induced morphism

$$B \otimes (E_j / E_{j-1}) \to (E/E_i)$$

will thus be larger than $\mu_{\text{max}}(E/E_i) = \mu(E_{i+1} / E_i)$. This shows that $\alpha$ must be zero, a contradiction which proves the ampleness of $F_{i,j}$.

With this preparation we will now prove Proposition 30 inductively.

Start of induction: $j = 1$. In this case, the above claim and the statement of Proposition 30 agree.

Inductive Step. Assume that $1 < j \leq i < r$ and that the statement was already shown for $j - 1$. Then consider the sequence

$$0 \to E_{j-1} \otimes (E/E_i)^\vee \to E_j \otimes (E/E_i)^\vee \to (E_j / E_{j-1}) \otimes (E/E_i)^\vee \to 0$$

But then also the middle term is ample, which shows Proposition 30. □

6.2. Proof of Corollary 5. We will show that the sheaf $F_C$, which is defined only on the curve $C$ is contained in a foliation $\mathcal{F}$ which is regular along $C$ and whose restriction to $C$ is likewise ample. Corollary 5 then follows immediately from Theorem 1.

An application of Proposition 29 to $E := T_X |_C$ yields the existence of a locally free term $E_i \subset T_X |_C$ in the Harder-Narasimhan filtration of $T_X |_C$ which contains $F_C$ and is ample. The choice of $C$ then guarantees that $E_i$ extends to a saturated subsheaf $\mathcal{F} \subset T_X$. The proof is thus finished if we show that $\mathcal{F}$ is a foliation, i.e. closed under the Lie-bracket. Equivalently, we need to show that the associated O’Neill-tensor

$$N : \mathcal{F} \otimes \mathcal{F} \to T_X / \mathcal{F}$$

vanishes. By Proposition 30, the restriction of the bundle

$$\text{Hom}(\mathcal{F} \otimes \mathcal{F}, T_X / \mathcal{F}) \cong (\mathcal{F} \otimes \mathcal{F})^\vee \otimes T_X / \mathcal{F}$$

to $C$ is anti-ample. Ampleness is an open property, [Gro66 cor. 9.6.4], so that the restriction of $\mathcal{F}$ to deformations $(C_t)_{t \in T}$ of $C$ stays ample for most $t \in T$. Since the $C_t$ dominate $X$, the claim follows. This ends the proof of Corollary 5. □
6.3. Attributions. The arguments used to derive Corollary from Theorem were certainly known to experts, and are implicitly contained in the literature, in particular [Miy85] and [Ko92]. We would like to thank Thomas Peternell for explaining the existence of a maximally ample subbundle to us.

7. A GEOMETRIC CONSEQUENCE OF UNSTABILITY

Recall that a complex variety $X$ is called $\mathbb{Q}$-Fano if a sufficiently high multiple of the anticanonical divisor $-K_X$ is Cartier and ample. The methods introduced above immediately yield that $\mathbb{Q}$-Fano varieties whose tangent bundles are unstable allow sequences of rational maps with rationally connected fibers.

**Corollary 32.** Let $X$ be a normal complex $\mathbb{Q}$-Fano variety and $L_1, \ldots, L_{\dim X-1} \in \text{Pic}(X)$ be ample line bundles. Let
\[
\{0\} = E_{-1} = E_0 \subset E_1 \subset \cdots \subset E_m = T_X
\]
be the Harder-Narasimhan filtration of the tangent sheaf with respect to $L_1, \ldots, L_{\dim X-1}$ and set
\[
k := \max\{0 \leq i \leq m \mid \mu(E_i/E_{i-1}) > 0\}.
\]
Then $k > 0$, and there exists a commutative diagram of dominant rational maps
\[
\begin{array}{c}
X \\
| \\
| \\
\downarrow q_1 \\
| \\
\downarrow q_2 \\
| \\
\downarrow q_k \\
Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_k,
\end{array}
\]
with the following property: if $x \in X$ is a general point, and $F_i$ the closure of the $q_i$-fiber through $x$, then $F_i$ is rationally connected, and its tangent space at $x$ is exactly $E_i$, $T_{F_i}|_x = E_i|_x$.

**Proof.** Let $C \subset X$ be a general complete intersection curve with respect to $L_1, \ldots, L_{\dim X-1}$. Since $c_1(T_X) \cdot C > 0$, Proposition implies $k > 0$ and that the restrictions $E_1|_C, \ldots, E_k|_C$ are ample vector bundles. We have further seen in Section 6.2 that the $(E_i)_{1 \leq i \leq k}$ give a sequence of foliations with algebraic and rationally connected leaves.

To end the construction of Diagram, let $q_i : X \rightarrow \text{Chow}(X)$ be the map that sends a point $x$ to the $E_i$-leaf through $x$, and let $Q_i := \text{Image}(q_i)$. □

**Remark 33.** Corollary also holds in the more general setup where $X$ is a normal variety whose anti-canonical class is represented by a Weil divisor with positive rational coefficients.

It is of course conjectured that the tangent bundle of a Fano manifold $X$ with $b_2(X) = 1$ is stable. We are therefore interested in a converse to Corollary and ask the following.

**Question 34.** Given a $\mathbb{Q}$-Fano variety and a sequence of rational maps with rationally connected fibers as in Diagram, when does the diagram come from the unstability of $T_X$ with respect to a certain polarization? Is Diagram characterized by universal properties?

**Question 35.** To what extent does Diagram depend on the polarization chosen?

**Question 36.** If $X$ is a uniruled manifold or variety, is there a polarization such that the MRC-fibration comes from the Harder-Narasimhan filtration of $T_X$?
REFERENCES

[BM01] Feodor A. Bogomolov and Michael L. McQuillan. Rational curves on foliated varieties. IHES Preprint, February 2001.

[Bos01] Jean-Benoît Bost. Algebraic leaves of algebraic foliations over number fields. *Publ. Math. Inst. Hautes Études Sci.*, 93:161–221, 2001.

[CLN85] César Camacho and Alcides Lins Neto. *Geometric theory of foliations*. Birkhäuser Boston Inc., Boston, MA, 1985. Translated from the Portuguese by Sue E. Goodman.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

[Fle84] Hubert Flenner. Restrictions of semistable bundles on projective varieties. *Comment. Math. Helv.*, 59(4):635–650, 1984.

[GHS03] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67 (electronic), 2003.

[Gro66] Alexandre Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, 28:255, 1966.

[Gro95] Alexandre Grothendieck. Techniques de construction et théorèmes d’existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[Har68] Robin Hartshorne. Cohomological dimension of algebraic varieties. *Ann. of Math.*, 88(2):403–450, 1968.

[Har71] Robin Hartshorne. Ample vector bundles on curves. *Nagoya Math. J.*, 43:73–89, 1971.

[Hör05] Andreas Höring. Uniruled varieties with splitting tangent bundle. preprint math.AG/0505327, 2005.

[Kam91] János Kollár. Extremal rays on smooth threefolds. *Ann. Sci. École Norm. Sup. (4)*, 24(3):339–361, 1991.

[Kam92] János Kollár, editor. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Kam96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1996.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.

[Lan04] Adrian Langer. Semistable sheaves in positive characteristic. *Ann. of Math. (2)*, 159(1):251–276, 2004.

[Laz95] Robert Lazarsfeld. *Positivity in algebraic geometry*. Springer-Verlag, Berlin, 2004.

[Miy85] Yoichi Miyaoka. Deformation of a morphism along a foliation. In S. Bloch, editor, *Algebraic Geometry*, volume 46 of *Proceedings of Symposia in Pure Mathematics*, pages 245–269, Providence, Rhode Island, 1985. American Mathematical Society.

[Mor79] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)*, 110(3):593–606, 1979.

[Ses82] C. S. Seshadri. *Fibrés vectoriels sur les courbes algébriques*, volume 96 of *Astérisque*. Société Mathématique de France, Paris, 1982. Notes written by J.-M. Drezet from a course at the École Normale Supérieure, June 1980.

[War71] F. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Scott, Foresman and Company, Glenview, Illinois and London, 1971.

S. KEBEKUS, L. SOLÁ CONDE, AND M. TOMA

KÖLN, GERMANY

E-mail address: stefan.kebekus@math.uni-koeln.de
URL: http://www.mi.uni-koeln.de/~kebekus

L. SOLÁ CONDE, MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEWERTAL 86-90, 50931 KÖLN, GERMANY

E-mail address: lsola@math.uni-koeln.de

M. TOMA, MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEWERTAL 86-90, 50931 KÖLN, GERMANY AND MATHEMATICAL INSTITUTE OF THE ROMANIAN ACADEMY, BUCHAREST

E-mail address: matei@math.uni-koeln.de