COMPUTING COVARIANT LYAPUNOV VECTORS IN HILBERT SPACES

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(Communicated by Gary Froyland)

Abstract. Covariant Lyapunov Vectors (CLVs) are intrinsic modes that describe long-term linear perturbations of solutions of dynamical systems. With recent advances in the context of semi-invertible multiplicative ergodic theorems, existence of CLVs has been proved for various infinite-dimensional scenarios. Possible applications include the derivation of coherent structures via transfer operators or the stability analysis of linear perturbations in models of increasingly higher resolutions.

We generalize the concept of Ginelli’s algorithm to compute CLVs in Hilbert spaces. Our main result is a convergence theorem in the setting of [19]. The theorem relates the speed of convergence to the spectral gap between Lyapunov exponents. While the theorem is restricted to the above setting, our proof requires only basic properties that are given in many other versions of the multiplicative ergodic theorem.

1. Introduction. In this article we generalize the concept of Ginelli’s algorithm [16], the most commonly used CLV-algorithm, to Hilbert spaces and give a convergence proof thereof as a first step towards computing CLVs in infinite-dimensional settings. While the algorithm has already been studied in finite-dimensional settings from an applied [11] and a theoretical [25] point of view, CLV-algorithms in the context of infinite-dimensional dynamical systems are still widely unexplored.

Covariant Lyapunov Vectors (CLVs) characterize the asymptotically most expanding or contracting directions in tangent space along trajectories in dynamical systems. They naturally relate to Lyapunov Exponents (LEs) and generalize the classical stability theory of steady states and periodic orbits to more general background trajectories. In applications, CLVs have been described as the “physically relevant” modes for dissipative systems [33] and have been used to detect coherent structures, i.e., slowly mixing sets, via the Perron-Frobenius operator [12, 13, 17, 3], the dual of the Koopman operator. Recent research on coherent structures includes the analysis of large-scale features of the ocean and atmosphere [17, chapter 6]. Apart from techniques involving transfer operators, CLVs have been used directly to analyze instabilities in coupled models. Two examples are the assessment of

2020 Mathematics Subject Classification. Primary: 37H15; Secondary: 37M25.

Key words and phrases. Covariant Lyapunov Vectors (CLVs), multiplicative ergodic theorem, Ginelli’s algorithm, Hilbert spaces, semi-invertible cocycle.

The author is supported by DFG grant 274762653.

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long-term predictability in ocean-atmosphere models [32, 36, 6] and the decoupling of instabilities into modes associated to different timescales to analyze mixing in a two-scale Lorenz 96 model [5].

On the theoretical side, existence and uniqueness of CLVs are ensured by the *Multiplicative Ergodic Theorem* (MET). The theorem not only provides requirements for CLV-algorithms, but also plays an integral part in theoretical studies. While the original MET from 1968 is due to Oseledets [27], various versions emerged until today. They differ in their settings and proof techniques. Several versions follow Raghunathan’s approach [29], which uses a singular value decomposition of the linear propagator and relates finite-time optimal growth rates given by singular values to LEs and singular vectors to CLVs. Exploiting these relations, [25] derives a first, mathematically rigorous convergence proof of Ginelli’s algorithm for finite-dimensional dynamical systems. The proof relies on the two-sided, invertible version of the MET found in [1]. The invertible version requires an invertible base flow and an invertible linear propagator. On the other hand, there are non-invertible versions of the MET, which require neither of those. Though, this comes at a disadvantage. Instead of an Oseledets splitting, non-invertible versions only yield an *Oseledets filtration*. Consequently, there is no natural notion of CLVs.

Most METs can be categorized into invertible and non-invertible versions. However, in this article we target a new kind of MET. Recently so-called *semi-invertible* METs emerged to adjust to the settings of transfer operators. They require an invertible base flow while the linear propagator is allowed to be non-invertible. Semi-invertible METs still provide Oseledets splittings and, hence, also CLVs. The first semi-invertible MET was published by Froyland, Lloyd and Quas for finite-dimensional systems [12]. Semi-invertible versions for infinite-dimensional systems followed by Froyland, Lloyd, Quas and González-Tokman [13, 19, 18]. Since their METs are formulated on Banach spaces, they require new techniques that do not involve the use of singular value decompositions. Instead, some form of compactness is assumed as in the first MET on Hilbert spaces by Ruelle [31] and in the first MET on Banach spaces by Mañe [23].

Here, we focus on the semi-invertible MET by González-Tokman and Quas [19] for quasi-compact cocycles (linear propagators) on separable Banach spaces. The proof of their MET inspired some ideas for this article. Especially their technique of pushing forward so-called *good complements* of the Oseledets filtration from the past to the present in order to obtain the Oseledets splitting resembles basic ideas from Ginelli’s algorithm.

While the semi-invertible setting on Banach spaces is much more general than the invertible setting in finite dimensions from the previous convergence proof [25], it also brings new challenges. The fundamental differences when jumping from finite to infinite dimensions are the lack of a singular value decomposition of the linear propagator and the lack of a Lebesgue measure for general Banach or Hilbert spaces. Moreover, the linear propagator along trajectories does not need to be invertible due to the semi-invertible setting. In particular, we cannot simply obtain backward-time estimates via forward-time estimates of the time-reversed system as it was done in [25]. Moreover, the Lyapunov spectrum may consist of only a few,
at most countably many exceptional LEs (the highest LEs) until a possibly non-discrete part of the spectrum is reached. This restricts our analysis to CLVs of the exceptional LEs.

In spite of these challenges, we managed to derive a similar convergence theorem as in [25] but for a much wider class of systems, where the notion of *almost everywhere* now needs to be understood in the sense of *prevalence* [28].

**Main Theorem.** *In the setting of [19] for Hilbert spaces, Ginelli’s algorithm convergence for almost every initial configuration. The convergence is exponentially fast with a rate given by the spectral gap between corresponding Lyapunov exponents.*

Although our new convergence proof is formulated in the setting of [19], we remark that it only requires basic asymptotic characterizations of LEs and of Oseledets spaces that can be found in most versions of the MET. The only additional restriction we make is using Hilbert spaces instead of Banach spaces. On the one hand, this restriction is of a technical nature. On the other hand, Hilbert spaces are a natural assumption when looking at the implementation of Ginelli’s algorithm, since it uses an orthonormalization procedure. Nevertheless, many concepts of the algorithm and of our proof apply to systems on Banach spaces. Hence, we formulate many of our arguments at the level of Banach spaces.

We begin our article by building the framework. Section 2 introduces Grassmannians for Banach spaces. We concentrate on closed complemented subspaces whose dimension or codimension is finite. Instead of input vectors for Ginelli’s algorithm that stay close to certain singular vectors, which was central to the old convergence proof in [25], we seek input vectors that stay far from spaces of the Oseledets filtration. To this end, we introduce the notion of *well-separating common complements* [26]. Those are common complements for families of subspaces of finite codimension such that the degree of transversality, which describes the separation between complementary subspaces, decays at most subexponentially. In our proof the family of subspaces is given by spaces of the Oseledets filtration for different initial times with respect to Ginelli’s algorithm.

In section 3 we state the MET from [19] and extract basic asymptotic properties from the proof. More precisely, we need uniform bounds for growth rates of perturbations inside Oseledets spaces and inside spaces of the Oseledets filtration. Although such bounds are used in [19] and hints for their derivation is provided, the actual proof is not carried out. We make up for the missing details by executing the suggested ideas. Let us stress again that the properties required for our convergence proof are not unique to [19], but can also be found in other versions of the MET.

Section 4 covers the basic ideas and concepts behind Ginelli’s algorithm. We give a general intuition of how the algorithm operates at the level of Grassmannians before giving a more precise definition for Hilbert spaces.

Section 5 covers our new convergence theorem of Ginelli’s algorithm. After stating the theorem, we devote the remaining subsections to prove it. Subsection 5.1 treats forward propagation, whereas subsection 5.2 adds backward propagation along bundles of certain subspaces to the forward propagation. Both subsections are formulated in the context of maps on Banach spaces. In subsection 5.3 we combine the derived tools to come up with a convergence proof of Ginelli’s algorithm on Hilbert spaces.

Finally, the results of this article are summarized and discussed in section 6.
2. Grassmannians. This section introduces the topological framework in which we analyze convergence of Ginelli’s algorithm. The algorithm approximates Oseledets spaces, which are finite-dimensional subspaces complemented by closed subspaces of the Oseledets filtration. In particular, Oseledets spaces are elements of the Grassmannians.

Definition 2.1. Let \((X, \|\cdot\|)\) be a Banach space. The Grassmannian \(G(X)\) is the set of closed complemented subspaces of \(X\), i.e., closed subspaces \(V \subset X\) such that there is a closed subspace \(W \subset X\) with \(X = V \oplus W\). It contains \(G_k(X)\), the set of \(k\)-dimensional subspaces, and \(G^k(X)\), the set of closed subspaces of codimension \(k\).

The Grassmannian \(G(X)\) can be equipped with a metric \(d_G(V, W)\) via the Hausdorff distance between \(V \cap B\) and \(W \cap B\), where \(B\) denotes the closed unit ball in \(X\) [19, appendix B]:

\[
d_G(V, W) := d_H(V \cap B, W \cap B) = \max \left( \sup_{v \in V \cap B} d(v, W \cap B), \sup_{w \in W \cap B} d(w, V \cap B) \right)
= \max \left( \sup_{v \in V \cap B} \inf_{w \in W \cap B} \|v - w\|, \sup_{w \in W \cap B} \inf_{v \in V \cap B} \|w - v\| \right)
\]

for \(V, W \in G(X)\). Another metric \(d_\hat{G}\) is given by exchanging \(B\) with the unit sphere \(S\) in the above definition. In fact, Kato shows that \(G(X)\) equipped with \(d_\hat{G}\) is a complete metric space [20, chapter IV, §2.1]. Moreover, he relates \(d_G\) to the gap between subspaces, which is defined as

\[
\hat{\delta}(V, W) := \max \left( \sup_{v \in V \cap S} d(v, W), \sup_{w \in W \cap S} d(w, V) \right).
\]

In general the gap is not a metric. However, if \(X\) is a Hilbert space, it coincides with the metric given by taking differences between associated orthogonal projections: \(\hat{\delta}(V, W) = \|P_V - P_W\|\) [20, 7, 14]. The gap-metric was used in the previous convergence proof [25]. All three concepts of distances on \(G(X)\) are related via \(\delta \leq d_G \leq d_\hat{G} \leq 2\delta\). This follows from

\[
\sup_{v \in V \cap S} d(v, W) \leq \sup_{v \in V \cap B} d(v, W \cap B) \leq \sup_{v \in V \cap S} d(v, W \cap S) \leq 2 \sup_{v \in V \cap S} d(v, W), \quad (1)
\]

see [20]. In particular, \(d_G\) and \(d_\hat{G}\) induce the same topology. Hence, \(G(X)\) is complete with respect to the metric \(d_G\), which we will use in our convergence analysis.

The symmetry of \(d_G\) is an immediate consequence of its definition. We cannot reduce the definition to only one term since the terms \(\sup_{v \in V \cap B} d(v, W \cap B)\) and \(\sup_{w \in W \cap B} d(w, V \cap B)\) are different in general. However, if one term is small, then so is the other [19, lemma B.7]:

Lemma 2.2 ([19]). If \(V, W \in G_k(X)\) are subspaces of dimension \(k\), then

\[
\sup_{v \in V \cap B} d(v, W \cap B) =: r < 3^{-k}/4 \implies d_G(V, W) < 4 \cdot 3^k r.
\]

If \(V, W \in G^k(X)\) are closed subspaces of codimension \(k\), then

\[
\sup_{v \in V \cap B} d(v, W \cap B) =: r < 3^{-k}/8 \implies d_\hat{G}(V, W) < 8 \cdot 3^k r.
\]
Thus, when investigating convergence inside $G_k(X)$ or $G^k(X)$, it is enough to estimate only one of the two terms in the definition of $d_G$.

Ultimately, we want to approximate Oseledets spaces, which are complementary to spaces of the Oseledets filtration and are of finite dimension. Hence, we are working with tuples of the set

$$\text{Comp}_k(X) := \{ (C, V) \in G_k(X) \times G^k(X) \mid X = C \oplus V \}$$

for $k \in \mathbb{N}$. Given such a tuple, each $x \in X$ can be written uniquely as $x = c + v$ according to the associated splitting. In particular, we get two projections $\Pi_C|V : X \to C$ and $\Pi_V|C : X \to V$, which are bounded linear operators by the closed graph theorem. It can be shown that they are stable with respect to perturbations of the tuple $(C, V)$ [19, lemma B.18]:

**Lemma 2.3** ([19]). The mapping $\text{Comp}_k(X) \to L(X)$ given by $(C, V) \mapsto \Pi_C|V$ is continuous, where $\text{Comp}_k(X)$ has the product topology induced by $G(X)$ and where the space $L(X)$ of bounded linear operators on $X$ is equipped with the norm topology.

As in the finite-dimensional convergence proof [25], we need to keep track of angles between subspaces.

**Definition 2.4.** Let $C, V \subset X$ be two subspaces. The sine of the minimal angle from $C$ to $V$ is defined as $\inf_{c \in C \cap S} d(c, V)$.

For $(C, V) \in \text{Comp}_k(X)$, we call the sine of the minimal angle degree of transversality. It is equal to $1/\|\Pi_C|V\|$ [4] and describes the quality of the splitting $X = C \oplus V$. The degree of complementing subspaces is always positive, since $\inf_{c \in C \cap S} d(c, V) = 0$ would imply that $C \cap V \neq \{0\}$. On the other hand, if $X$ is a Hilbert space, a degree of 1 implies $C = V^\perp$. Thus, we prefer complements with a high degree of transversality (close to 1) as they are better separated.

Essential to the old convergence proof [25] was to find initial vectors for Ginelli’s algorithm (represented by a subspace $C$) that would stay well-separated from the Oseledets filtration at different initial times. This was ensured by a rigorous, yet quite technical analysis involving singular vectors and the proof of the MET from [1]. Here, we take a similar, but more elegant approach that skips the technical details of [25]. Without involving singular vectors, we look for subspaces such that the degree of transversality to spaces of the Oseledets filtration at different initial times decays at most subexponentially. More abstractly, given a sequence of subspaces $(V_n)_{n \in \mathbb{N}} \subset G^k(X)$, we ask for common complements, i.e., subspaces $C \subset X$ with $(C, V_n) \in \text{Comp}_k(X)$ for all $n$, such that the degree of transversality of $(C, V_n)$ decays at most subexponentially with $n$.

**Definition 2.5.** Let $(V_n)_{n \in \mathbb{N}} \subset G^k(X)$ be given. A common complement $C \in G_k(X)$ for $(V_n)_{n \in \mathbb{N}}$ is called well-separating with respect to $(V_n)_{n \in \mathbb{N}}$ if

$$\lim_{n \to \infty} \frac{1}{n} \log \inf_{c \in C \cap S} d(c, V_n) = 0.$$

Well-separating common complements can be used without interfering on exponential scales that are important for our convergence proof. Natural questions are the existence and the genericity of well-separating common complements. While the

\[2\text{Note that the minimal angle from } C \text{ to } V \text{ is generally not the same as the minimal angle from } V \text{ to } C. \text{ However, in standard euclidean space } (\mathbb{R}^d, \| \cdot \|_2) \text{ it coincides with the notion of minimal angle used in [25]. In this case, symmetry of the minimal angle holds and we can speak of the minimal angle between } C \text{ and } V.\]
existence of common complements for families of subspaces has already been studied in various scenarios [9, 10, 21, 30, 35], results that combine qualitative features with genericity have been missing. A new work that was created along this article fills this gap. [26] proves that well-separated common complements are generic for Hilbert spaces (in the sense of prevalence [28]).

**Theorem 2.6.** Let $H$ be a Hilbert space and let $(V_n)_{n \in \mathbb{N}} \subset \mathcal{G}^k(H)$. Almost every tuple $(x_1, \ldots, x_k) \in H^k$ induces a well-separating common complement of $(V_n)_{n \in \mathbb{N}}$ via $\text{span}(x_1, \ldots, x_k)$.

This theorem plays a crucial role in our convergence proof. In fact, [26] shows that existence of well-separating common complements in Banach spaces would suffice to prove a version of theorem 2.6 for Banach spaces. Hence, we formulate a large part of our convergence proof in section 5 at the level of Banach spaces while leaving open the question of generality until subsection 5.3, where we restrict ourselves to Hilbert spaces.

3. **Multiplicative ergodic theorem.** The first MET on Hilbert spaces was published by Ruelle in 1982 [31]. One year later, Mañé generalized the MET to Banach spaces [23]. Extending Thieullen’s work [34], the first infinite-dimensional semi-invertible MET for quasi-compact operators on Banach spaces followed by Froyland, Lloyd and Quas in 2013 [13]. In this section, we present the semi-invertible MET by González-Tokman and Quas from 2014 [19]. We chose their MET, since its proof resembles basic ideas from Ginelli’s algorithm. After stating the theorem, we derive uniform bounds that are needed during our convergence proof later on.

3.1. **A semi-invertible MET.** The MET from [19] requires a strongly measurable random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$, which consists of a base (flow) and a cocycle (linear propagator) describing the tangent linear dynamics. The base $\sigma : \Omega \to \Omega$ is a probability-preserving transformation of a Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$. It is linked to the cocycle via the generator, which is a strongly measurable map $\mathcal{L} : \Omega \to L(X)$, i.e., $\mathcal{L}(\cdot)x : \Omega \to X$ is $(\mathcal{F}, \mathcal{B}_X)$-measurable for every $x \in X$. Iterative applications of $\mathcal{L}$ along trajectories yield the cocycle $\mathcal{L}^{(n)}_\omega := \mathcal{L}(\sigma^{n-1}\omega) \circ \cdots \circ \mathcal{L}(\omega)$, which describes the evolution of linear perturbations along the orbit of $\omega$ for $n$ timesteps. Moreover, we call the random dynamical system separable if the Banach space $X$ is separable.

Compared to the finite-dimensional case, systems on Banach spaces exhibit Lyapunov spectra with possibly non-discrete parts. In fact, an Oseledets splitting exists only for the first, at most countably many exceptional LEs that are isolated from the rest of the spectrum. To discern the exceptional LEs, we need the notion of quasi-compactness.

Let $(X, \|\cdot\|)$ be a Banach space. Write $B \subset X$ for the unit ball and $S \subset X$ for the unit sphere in $X$. Given a bounded linear operator $A \in L(X)$ on $X$, we define the index of compactness of $A$ as

$$\|A\|_{ic(X)} := \inf \{r > 0 \mid A(B) \text{ can be covered by finitely many balls of radius } r \}.$$ 

The index gives us a measure of how close $A$ is to being a compact operator. In fact, the index of compact operators, such as operators on $\mathbb{R}^d$ or operators with finite range, is always zero.

The following result extends the index of compactness to cocycles:
Proposition 1 ([19]). Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ be a separable strongly measurable random dynamical system such that $\log^+ \|\mathcal{L}\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

For $\mathbb{P}$-almost every $\omega \in \Omega$, the maximal Lyapunov exponent

$$\lambda(\omega) := \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}^{(n)}\|$$

and the index of compactness of the cocycle \[34\]

$$\kappa(\omega) := \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}^{(n)}\|_{ic(X)}$$

exist. Furthermore, $\lambda$ and $\kappa$ are measurable and $\sigma$-invariant.

If $\sigma$ is ergodic\(^3\), then $\lambda$ and $\kappa$ are constant $\mathbb{P}$-almost everywhere. Denote these constants by $\lambda^*$ and $\kappa^*$. It holds $\kappa^* \leq \lambda^* < \infty$.

We call a separable strongly measurable random dynamical system with ergodic base quasi compact if $\kappa^* < \lambda^*$. For such a system, Doan derives the existence of an Oseledets filtration [8] as a corollary of the two-sided MET by Lian and Lu [22]. With the additional assumption that the base is invertible, [19] proves a semi-invertible MET with a splitting that is similar to the Oseledets splitting obtained in fully invertible METs:

**Theorem 3.1 (Semi-invertible MET [19]).** Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ be a separable strongly measurable random dynamical system over an ergodic invertible base such that $\log^+ \|\mathcal{L}(\omega)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, assume that $\mathcal{R}$ is quasi-compact.

There exist $1 \leq p \leq \infty$ exceptional Lyapunov exponents $\lambda^* = \lambda_1 > \cdots > \lambda_p > \kappa^*$ (or if $p = \infty$: $\lambda_1 > \lambda_2 > \cdots > \kappa^*$ and $\lim_{n \to \infty} \lambda_i = \kappa^*$), multiplicities $d_1, \ldots, d_p \in \mathbb{N}$, and a unique measurable splitting of $X$ into closed subspaces

$$X = \bigoplus_{i=1}^p Y_i(\omega) \oplus V(\omega)$$

defined on a $\sigma$-invariant subset $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure such that the following hold for each $\omega \in \Omega'$:

1. The splitting is equivariant, i.e., $\mathcal{L}(\omega)V(\omega) \subset V(\sigma\omega)$ and $\mathcal{L}(\omega)Y_i(\omega) = Y_i(\sigma\omega)$,
2. $\dim Y_i(\omega) = d_i$,
3. $\lim_{n \to \infty} (1/n) \log \|\mathcal{L}^{(n)}y\| = \lambda_i$ for $y \in Y_i(\omega) \setminus \{0\}$,
4. $\limsup_{n \to \infty} (1/n) \log \|\mathcal{L}^{(n)}v\| \leq \kappa^*$ for $v \in V(\omega)$,
5. The norms of the projections associated to the splitting are tempered with respect to $\sigma$, where a function $f : \Omega \to \mathbb{R}$ is called tempered if

$$\lim_{n \to \pm \infty} (1/n) \log |f(\sigma^n\omega)| = 0$$

for $\mathbb{P}$-almost every $\omega$.

We call the above splitting Oseledets splitting and the spaces $Y_i(\omega)$ Oseledets spaces. The Oseledets filtration $X = V_1(\omega) \supset \cdots \supset V_p(\omega) \supset V_{p+1}(\omega)$ from Doan’s theorem can be reconstructed via $V_{p+1}(\omega) = V(\omega)$ and

$$V_i(\omega) = \bigoplus_{j=1}^p Y_j(\omega) \oplus V(\omega)$$

for $1 \leq i \leq p$. Inside of the Oseledets spaces $Y_i(\omega)$, we find the CLVs.

\(^3\)A metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ (see [1, Appendix A.1]) is called ergodic if all measurable, flow-invariant subsets of $\Omega$ have either probability 0 or 1.
**Definition 3.2.** Normalized basis vectors, which are covariant and chosen subject to the Oseledets splitting, are called *Covariant Lyapunov Vectors* (CLVs). We call a family of unit vectors \( u : \Omega \to X \) covariant if \( \mathcal{L}_\omega^{(n)} u(\omega) \) and \( u(\sigma_n \omega) \) coincide up to normalization for almost every \( \omega \) and for all \( n \in \mathbb{N} \).

Later on we generalize the concept of Ginelli’s algorithm to compute CLVs (or more generally Oseledets spaces) in Hilbert spaces for a fixed \( \omega \in \Omega' \). So far, the restriction to Hilbert spaces is of a technical nature and may be lifted in the future. Our convergence analysis requires cocycle data along the trajectory of \( \omega \) and basic asymptotic properties that appear, e.g., in the METs from [8] and [19]. That is, we need uniform upper bounds for asymptotics of the Oseledets filtration (eqs. 3, 4, 6 and 7) and uniform lower bounds for asymptotics of the Oseledets splitting (eqs. 5 and 8). In the following, we give a short overview over the needed bounds. A more detailed derivation can be found in the next two subsections, which are included for the sake of completeness but not essential for understanding Ginelli’s algorithm and our convergence analysis.

While bounds for the Oseledets filtration are recovered from Doan’s work [8]:

\[
\lim_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}_\omega^{(n)} |_{V_i(\omega)} \| = \lambda_i
\]

for \( 1 \leq i \leq p \) and

\[
\limsup_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}_\omega^{(n)} |_{V(\omega)} \| \leq \kappa^*,
\]

bounds for the Oseledets splitting are due to [19]. By choosing a suitable basis, González-Tokman and Quas reduce the cocycle along \( Y_i(\omega) \) to a cocycle of matrices (similar to [13, lemma 19]) for which uniform estimates are known. By applying the same arguments to the sum of Oseledets spaces \( Y_1(\omega) \oplus \cdots \oplus Y_i(\omega) \), we get uniform lower bounds of growth rates inside sums of Oseledets spaces:

\[
\liminf_{n \to \infty} \inf_{y \in Y_1(\omega) \oplus \cdots \oplus Y_i(\omega) \cap S} \frac{1}{n} \log \| \mathcal{L}_\omega^{(n)} y \| = \lambda_i.
\]

In addition to the bounds for \( \mathcal{L}_\omega^{(n)} \), we need similar bounds for \( \mathcal{L}_{\sigma_n \omega}^{(n)} \). These can be obtained via [12, lemma 8.2]. We get

\[
\lim_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}_{\sigma_n \omega}^{(n)} |_{V_i(\sigma_n \omega)} \| = \lambda_i
\]

for \( 1 \leq i \leq p \) and

\[
\limsup_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}_{\sigma_n \omega}^{(n)} |_{V(\sigma_n \omega)} \| \leq \kappa^*
\]

for \( \mathbb{P} \)-almost every \( \omega \). Uniform lower bounds for the Oseledets splitting are again obtained from reduced systems via matrix cocycles. We have

\[
\liminf_{n \to \infty} \inf_{y \in Y_1(\sigma_n \omega) \oplus \cdots \oplus Y_i(\sigma_n \omega) \cap S} \frac{1}{n} \log \| \mathcal{L}_{\sigma_n \omega}^{(n)} y \| = \lambda_i.
\]

Observe that \( \ker \mathcal{L}_\omega^{(n)} \subset V(\omega) \) and \( \ker \mathcal{L}_{\sigma_n \omega}^{(n)} \subset V(\sigma_n \omega) \) for every \( n \in \mathbb{N} \). Indeed, \( \mathcal{L}_\omega^{(n)} \subset V(\omega) \) follows from the different growth rates of vectors inside the Oseledets spaces. Since \( \ker \mathcal{L}_\omega^{(n)} \subset V(\omega) \) holds on a \( \sigma \)-invariant subset of \( \Omega \), we get \( \ker \mathcal{L}_{\sigma_n \omega}^{(n)} \subset V(\sigma_n \omega) \).

In the next two subsections we give a more detailed derivation of the above uniform bounds. Since [19] provides hints for their derivation and uses them in the proof of theorem 3.1, we think this is a good opportunity to provide the necessary
details. While uniform bounds for systems on \( \mathbb{R}^d \) follow from properties of the singular value decomposition, systems on Banach spaces require different approaches. We first derive bounds for the Oseledets filtration by invoking [8] and [22]. Then, we extract bounds for the Oseledets splitting via [12] and [19].

3.2. **Bounds for the Oseledets filtration.** In his dissertation Doan proves the existence of an Oseledets filtration for one-sided random dynamical systems on separable Banach spaces by embedding systems into larger systems that have injective cocycles [8]. For those enlarged systems, the MET by Lian and Lu provides existence of an Oseledets filtration with uniform bounds [22]. We state the necessary arguments to derive uniform bounds for the Oseledets filtration of Doan’s MET from the MET by Lian and Lu and refer to the selected references for full statements of the results and techniques.

**Theorem 3.3 (MET by Lian and Lu [22]).** Let \( \mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L}) \) be a separable strongly measurable random dynamical system over an ergodic invertible base such that \( \log^+ \| \mathcal{L} \| \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \). Furthermore, assume that \( \mathcal{L} : \Omega \to L(X) \) is injective \( \mathbb{P} \)-almost everywhere and \( \mathcal{R} \) is quasi-compact.

There exist \( 1 \leq p \leq \infty \) exceptional LEs \( \lambda^* = \lambda_1 > \cdots > \lambda_p > \kappa^* \) (or if \( p = \infty \): \( \lambda_1 > \lambda_2 > \cdots > \kappa^* \) and \( \lim_{n \to \infty} \lambda_i = \kappa^* \)), multiplicities \( d_1, \ldots, d_p \in \mathbb{N} \), and a unique measurable filtration of \( X \) into closed subspaces

\[
X = V_1(\omega) \supset \cdots \supset V_p(\omega) \supset V(\omega) \supset \{0\}
\]

defined on a \( \sigma \)-invariant subset \( \Omega' \subset \Omega \) of full \( \mathbb{P} \)-measure such that the following hold for \( \omega \in \Omega' \):

1. The splitting is equivariant, i.e., \( \mathcal{L}(\omega)V_i(\omega) \subset V_i(\sigma\omega) \) and \( \mathcal{L}(\omega)V(\omega) \subset V(\sigma\omega) \),
2. \( \text{codim}V_{i+1}(\omega) = d_1 + \cdots + d_i \),
3. \( \lim_{n \to \infty}(1/n) \log \| \mathcal{L}^n(\omega)v \| = \lambda_i \) for \( v \in V_i(\omega) \setminus V_{i+1}(\omega) \),
4. \( \lim_{n \to \infty}(1/n) \log \| \mathcal{L}^n(\omega) \| = \lambda_i \), and
5. \( \limsup_{n \to \infty}(1/n) \log \| \mathcal{L}^n(\omega) \| \leq \kappa^* \),

where we set \( V_{p+1}(\omega) := V(\omega) \).

The full MET by Lian and Lu even provides an Oseledets splitting that is related to the Oseledets filtration via eq. 2. For our purposes the filtration is enough.

Now, Doan’s theorem states that the above is still true if we drop the assumption of \( \mathcal{L} \) being injective \( \mathbb{P} \)-almost everywhere. To be precise, in Doan’s formulation the fourth and fifth properties about uniform bounds are left out. Since we need these properties, we derive them as products of Doan’s proof.

Given \( \gamma > 0 \), Doan enlarges the separable Banach space \( X \) to the space of sequences of elements of \( X \):

\[
X_\gamma := \left\{ x := (x_n)_{n \in \mathbb{N}_0} \mid \lim_{n \to \infty} e^{-\gamma n} x_n \text{ exists} \right\}.
\]

He shows that \( X_\gamma \) equipped with the norm

\[
\| x \|_\gamma := \sup_{n \in \mathbb{N}_0} e^{-\gamma n} \| x_n \|
\]

is a separable Banach space. On this Banach space an extended cocycle can be defined using the generator

\[
\hat{\mathcal{L}}_\omega x := (\mathcal{L}_\omega x_0, \alpha_0 x_0, \alpha_1 x_1, \ldots),
\]
where \((\alpha_n)_{n \in \mathbb{N}_0}\) is a descending sequence of positive scalars satisfying certain growth conditions. As Doan suggests, we set \(\alpha_n := e^{-\left(2n+1\right)}\). The generated cocycle has the form

\[
\hat{\mathcal{L}}^{(n)}_\omega x = \left(\mathcal{L}_{\omega}^{(n)}x_0, \alpha_0 \mathcal{L}_{\omega}^{(n-1)}x_0, \ldots, \alpha_{n-1} \ldots \alpha_0 x_0, \alpha_n \ldots \alpha_1 x_1, \ldots \right).
\]

If the original system \(\mathcal{R}\) is strongly measurable, has an ergodic invertible base, satisfies \(\log^+ \|\mathcal{L}\| \in L^1\), and is quasi compact, then the enlarged random dynamical system \(\hat{\mathcal{R}}\) fulfills the assumptions of theorem 3.3. Thus, we have LEs \(\lambda_1 > \cdots > \lambda_p > \kappa^*\) and an Oseledets filtration \(X_\gamma = \hat{V}_1(\omega) \supset \cdots \supset \hat{V}_p(\omega) \supset \hat{V}(\omega) \supset \{0\}\) of \(\hat{\mathcal{R}}\).

Doan proves that \(\pi x = x_0\) projects the Oseledets filtration of \(\hat{\mathcal{R}}\) onto a filtration of \(\mathcal{R}\) via \(V_\gamma(\omega) = \pi \hat{V}_\gamma(\omega)\) with almost the same properties (all except 4. and 5. of theorem 3.3). Hence, we call the projected filtration Oseledets filtration. The exceptional LEs of the original and of the enlarged system coincide. We now argue why properties 4. and 5. still hold for the projected filtration:

**Lemma 3.4.** In Doan’s MET [8] for one-sided random dynamical systems on Banach spaces, eqs. 3 and 4 hold for \(\mathbb{P}\)-almost every \(\omega\), i.e., we have

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}^{(n)}_\omega|_{V_\gamma(\omega)}\| = \lambda_i
\]

for \(1 \leq i \leq p\) and

\[
\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}^{(n)}_\omega|_{V_\gamma(\omega)}\| \leq \kappa^*.
\]

In particular, these bounds hold in the setting of theorem 3.1.

**Proof.** Since the third property of theorem 3.3 also holds in Doan’s MET, we get \(\lambda_i\) as a lower bound of the limit in eq. 3. It remains to prove that \(\lambda_i\) is an upper bound.

The main idea is to use the uniform bound coming from the enlarged system. For \(x \in \hat{V}_\gamma(\omega)\) and \(x_0 = \pi x\), we have

\[
\|\mathcal{L}^{(n)}_\omega x_0\| = \|\mathcal{L}^{(n)}_\omega \pi x\| = \|\pi \hat{\mathcal{L}}^{(n)}_\omega x\| \leq \|\pi\| \|\hat{\mathcal{L}}^{(n)}_\omega|_{\hat{V}_\gamma(\omega)}\| \|x\|_\gamma.
\]

Since \(\|\pi\|\) is a constant factor, it vanishes on exponential scales. The second factor on the right can be bounded according to the MET by Lian and Lu. To get rid of the last factor, we show that \(\pi \hat{V}_\gamma(\omega) \subset \hat{V}_\gamma(\omega)\), where \(\pi x_0 = (x_0, 0, 0, \ldots)\). If this is true, then \(\hat{V}_\gamma(\omega) = \pi \hat{V}_\gamma(\omega)\) and

\[
\|\mathcal{L}^{(n)}_\omega|_{\hat{V}_\gamma(\omega)}\| \leq \sup_{x_0 \in \hat{V}_\gamma(\omega) \cap B} \|\mathcal{L}^{(n)}_\omega x_0\| \leq \|\pi\| \|\hat{\mathcal{L}}^{(n)}_\omega|_{\hat{V}_\gamma(\omega)}\|
\]

since \(\|\pi x_0\|_\gamma = \|x_0\|\). In particular, this would prove the claim.\(^4\)

Let \(x \in \hat{V}_\gamma(\omega)\). To show that \(\pi x_0 \in \hat{V}_\gamma(\omega)\), it suffices to investigate its growth rate, since \(\hat{V}_\gamma(\omega)\) is the set of all elements whose exponential growth rate is at most \(\lambda_i\). Thus, if \(\pi x_0\) does not have a faster growth rate than \(x\), it is an element of \(\hat{V}_\gamma(\omega)\).

We have

\[
\|\hat{\mathcal{L}}^{(n)}_\omega x\|_\gamma = \max \left(\max_{0 \leq k \leq n} e^{-\gamma k} \|\mathcal{L}^{(n-k)}_\omega x_0\| \prod_{j=0}^{k-1} \alpha_j, \sup_{k \in \mathbb{N}} e^{-\gamma (n+k)} \|x_k\| \prod_{j=k}^{k+n-1} \alpha_j \right).
\]

\(^4\)The bound for \(V(\omega)\) follows analogously.
The second part decays superexponentially fast:
\[- \log \prod_{j=k}^{k+n-1} \alpha_j = (2k + 1) + (2k + 1 + 2) + \cdots + (2k + 1 + 2(n - 1)) = 2kn + n^2\]
and
\[
\frac{1}{n} \log \left( \sup_{k \in \mathbb{N}} e^{-\gamma(n+k)} \left\| x_k \right\| \prod_{j=k}^{k+n-1} \alpha_j \right) \leq \frac{1}{n} \log \left( \sup_{k \in \mathbb{N}_0} e^{-\gamma k} \left\| e^{-\gamma n} e^{-2kn-n^2} \right\| \right) \\
\leq \frac{\log \|x\|}{n} - \gamma - n \to -\infty.
\]
Hence, the exponential growth rate only depends on \(x_0\). We get \(iV_i(\omega) \subset \tilde{V}_i(\omega)\).

The uniform estimates for backward-time, i.e., eqs. 6 and 7, follow from a result by Froyland and others:

**Lemma 3.5** ([12]). Let \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) be an ergodic metric dynamical system over \(\mathbb{Z}\). If \((f_n)_{n \in \mathbb{N}}\) is a subadditive sequence of functions \(\Omega \to \mathbb{R} \cup \{\pm \infty\}\), i.e., \(f_{m+n}(\omega) \leq f_m(\sigma_n\omega) + f_n(\omega)\) for every \(n, m \in \mathbb{N}\) and \(\omega \in \Omega\), and \(f_1^+ \in L^1\), then there is a constant \(c \in \mathbb{R} \cup \{-\infty\}\) such that \(f_n(\omega)/n \to c\) and \(f_n(\sigma_n\omega)/n \to c\).

**Lemma 3.6.** In the setting of theorem 3.1, eqs. 6 and 7 hold for \(\mathbb{P}\)-almost every \(\omega\), i.e., we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \left\| \mathcal{L}_{\sigma_{-n}\omega}^{(n)} | V(\sigma_{-n}\omega) \right\| = \lambda_i
\]
for \(1 \leq i \leq p\) and
\[
\limsup_{n \to \infty} \frac{1}{n} \log \left\| \mathcal{L}_{\sigma_{-n}\omega}^{(n)} | V(\sigma_{-n}\omega) \right\| \leq \kappa^*.
\]

**Proof.** We apply lemma 3.5 to the sequences \(f_n := \log \left\| \mathcal{L}_{\omega}^{(n)} | V(\omega) \right\|\) and \(g_n := \log \left\| \mathcal{L}_{\omega}^{(n)} | V(\omega) \right\|\). Indeed, subadditivity follows from the cocycle property \(\mathcal{L}_{\sigma_n\omega}^{(n)} = \mathcal{L}_{\sigma_n\omega}^{(n)} \circ \mathcal{L}_\omega^{(1)}\) and from equivariance of the Oseledets filtration. \(f_1^+, g_1^+\) can both be bounded by the integrable function \(\log^+ \| L \|\). Now, according to lemma 3.5 it holds \(f_n(\omega)/n \to c_f, f_n(\sigma_{-n}\omega)/n \to c_f, g_n(\omega)/n \to c_g, \) and \(g_n(\sigma_{-n}\omega)/n \to c_g\) for \(\mathbb{P}\)-almost every \(\omega\). By lemma 3.4 we must have \(c_f = \lambda_i\) and \(c_g \leq \kappa^*\). This proves the claim.

### 3.3. Bounds for the Oseledets splitting

In their proof of the MET in [19] González-Tokman and Quas require uniform lower bounds of growth rates inside Oseledets spaces. Even though they argue why those bounds hold for forward-time [19, lemma 2.14], the details are left for the reader to complete. Here, we provide the missing details. Moreover, we show that uniform lower bounds hold for the sum of the first Oseledets spaces instead of single Oseledets spaces. The main idea is to find a basis of the first Oseledets spaces in which we reduce the cocycle to a cocycle on \(\mathbb{R}^d\). Then, the finite-dimensional theory applies and gives us uniform bounds via the singular value decomposition:

**Lemma 3.7.** Let \(X\) be a separable Banach space and \(Y(\omega) \subset X\) a measurable subspaces of dimension \(k\). There is a measurable map \(A: \Omega \times \mathbb{R}^k \to X\) such that \(A(\omega): \mathbb{R}^k \to Y(\omega)\) is a linear isomorphism with
\[
\frac{1}{2^{k+1} - 2} \|a\|_2 \leq \|A(\omega)a\| \leq \sqrt{k} \|a\|_2
\]
for every $a \in \mathbb{R}^k$.

We prove the lemma by finding an $\epsilon$-nice basis of $Y(\omega)$. Using this basis, we may identify $Y(\omega)$ with $\mathbb{R}^k$.

**Definition 3.8** ([19]). Let $Y$ be a Banach space of dimension $k$. A basis $(y_1, \ldots, y_k)$ is called $\epsilon$-nice if $1 - \epsilon < \|y_i\| < 1 + \epsilon$ and $d(y_i, \text{span}(y_1, \ldots, y_{i-1})) > 1 - \epsilon$ for each $i > 1$.

We need [19, lemma B.4] for the proof of lemma 3.7:

**Lemma 3.9** ([19]). If $(y_1, \ldots, y_k)$ is an $\epsilon$-nice basis with $\epsilon < 2^{-k-2}$, then

$$\left\| \sum_{i=1}^k a_i y_i \right\| \leq 1 \implies |a_i| \leq 2^{k+1-i} \text{ for each } i.$$

**Proof of lemma 3.7.** We inductively prove existence of a measurable $\epsilon$-nice basis $(y_1(\omega), \ldots, y_k(\omega))$ of $Y(\omega)$ with $\epsilon < 2^{-k-2}$. Additionally, we assume that $\|y_i\| = 1$. Fix a countable dense subset $(x_j)_{j \in \mathbb{N}}$ of the unit sphere $S \subset X$. Assume we already have the first $i-1$ basis vectors for some $i = 1, \ldots, k$. We show existence of the $i$th vector. Define

$$r_1(\omega) := \min \left\{ j \in \mathbb{N} \mid d(x_j, \text{span}(y_1(\omega), \ldots, y_{i-1}(\omega))) > 1 - \frac{\epsilon}{2} \text{ and } d(x_j, Y(\omega)) < \frac{\epsilon}{2} \right\}$$

and inductively set

$$r_s(\omega) := \min \left\{ j \in \mathbb{N} \mid d(x_j, x_{r_{s-1}(\omega)}) < \frac{\epsilon}{2^s} \text{ and } d(x_j, Y(\omega)) < \frac{\epsilon}{2^s} \right\}.$$

The sequence of measurable functions $(x_{r_s}(\omega))_{s \in \mathbb{N}}$ converges pointwise to a measurable function $y_i(\omega)$, which satisfies the required properties.

Now, let $A(\omega)f := \sum a_i y_i(\omega)$. We have

$$\|A(\omega)f\| \leq \sum_{i=1}^k |a_i| \|y_i(\omega)\| = \|a\|_1 \leq \sqrt{k} \|a\|_2$$

and by lemma 3.9

$$\frac{\|a\|_2}{2^{k+1} - 2} \leq \frac{\|a\|_1}{2^{k+1} - 2} \leq \sum_{i=1}^k 2^{k+1-i} = 1 = \|A(\omega)\|.$$ 

where $c := \|A(\omega)f\|$. Since $c$ is a scalar, the above chain of (in-)equalities can be scaled to eliminate $c$. The claim follows. $\square$

Let $R$ be a random dynamical system as in theorem 3.1. Using lemma 3.7, we get an identification of the sum of the first Oseledets spaces $Y_1(\omega) \oplus \cdots \oplus Y_i(\omega)$ with $\mathbb{R}^k$, where $k = d_1 + \cdots + d_i$. Moreover, the identification provides a new one-sided cocycle on $\mathbb{R}^k$ via $\hat{\mathcal{L}}^{(n)} := A(\omega)^{-1} \mathcal{L}^{(n)}(\omega) A(\omega)$. Here, $A(\omega)$ should be understood as an isomorphism $\mathbb{R}^k \to Y_1(\omega) \oplus \cdots \oplus Y_i(\omega)$.

By [19, lemma A.5] the composition of strongly measurable maps is again strongly measurable. However, the section on strong measurability in [19] is only formulated for operators acting on a single separable Banach space $X$. To show that $\hat{\mathcal{L}}$ is strongly measurable one needs to generalize the whole section to operators between potentially different separable Banach spaces. We leave this to the reader. Special care should be given to $A(\omega)^{-1}$. In fact, it suffices to show strong measurability of $A(\omega)^{-1} Y_{1}(\omega) \oplus \cdots \oplus Y_{i}(\omega)|_{V_{i+1}(\omega)}$, which is a well-defined map $\Omega \times X \to \mathbb{R}^k$. This
can be done in a similar fashion to the proof of lemma 3.7. Fix a countable dense subset \((b_j)_{j \in \mathbb{N}} \subset \mathbb{R}^k\). For \(x \in X\) and \(s \in \mathbb{N}\), define

\[
    r_s(x) := \min \left\{ j \in \mathbb{N} : \| A(x)b_j - \Pi Y_{1(\omega)} \cdots \Pi Y_{s(\omega)} Y_{s+1(\omega)}x \| < \frac{1}{s} \right\}.
\]

Writing \(y = \Pi Y_{1(\omega)} \cdots \Pi Y_{s(\omega)} Y_{s+1(\omega)}x\), lemma 3.7 implies that

\[
    \frac{1}{s} > \| A(x)b_{r_s(x)} - y \| = \| A(x)(b_{r_s(x)} - A(\omega)^{-1}y) \| \geq \frac{\| b_{r_s(x)} - A(\omega)^{-1}y \|}{2^{k+1} - 2}.
\]

In particular, \(b_{r_s(x)} \to A(\omega)^{-1}y\) pointwise in \(\omega\) for \(s \to \infty\). Since the projection onto the sum of the first Oseledets spaces is measurable, we have strong measurability of \(A(\omega)^{-1} \Pi Y_{1(\omega)} \cdots \Pi Y_{s(\omega)} Y_{s+1(\omega)}\). Thus, \(\mathcal{L}\) is strongly measurable. Finally, the word “strongly” can be omitted, because the strong operator topology and the norm topology on \(\mathbb{R}^{k \times k}\) coincide.

Besides the measurability of \(\mathcal{L}\), the other cocycle properties are inherited from the original cocycle. Moreover, the norm estimates in lemma 3.7 imply that \(\| \mathcal{L}(n)^{(n)} \|\) and \(\| \mathcal{L}(n)^{(n)} | Y_{1(\omega)} \cdots Y_{s(\omega)} \|\) differ by at most a positive constant depending only on \(k\). In particular, \(\log^+ \| \mathcal{L}^n \| \in L^1\) is integrable and we may apply case (A) of the one-sided MET from \([1]\). The LEs of the reduced cocycle coincide with the first \(i\) exceptional LEs of the original cocycle. Hence, the lowest \(^5\) singular value \(\delta_{1,ini}(\mathcal{L}(n))\) of the reduced cocycle grows exponentially with a rate given by \(\lambda_i\). Since \(\| \mathcal{L}(n)^{(n)}a \|_2 \geq \delta_{1,ini}(\mathcal{L}(n))\) for all \(\|a\|_2 = 1\), we get a uniform lower bound which can be transferred back to the original cocycle.

**Lemma 3.10.** In the setting of theorem 3.1, eq. 5 holds for \(\mathbb{P}\)-almost every \(\omega\), i.e., we have

\[
    \liminf_{n \to \infty} \inf_{y \in Y_{1(\omega)} \cdots Y_{s(\omega)}} \frac{1}{n} \log \| \mathcal{L}_n^{(n)} y \| = \lambda_i
\]

for \(1 \leq i \leq p\).

The last uniform bound (eq. 8) follows immediately from \([12, \text{lemma 8.3}]\) which relates singular values of \(\mathcal{L}_{\sigma-n}^{(n)}\) to those of \(\mathcal{L}_n^{(n)}\):

**Lemma 3.11 (\([12]\)).** Let \(\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{L})\) be an ergodic semi-invertible random dynamical system on \(\mathbb{R}^k\) with \(\log^+ \| \mathcal{L}^n \| \in L^1\). By the one-sided MET from \([1]\) the system admits a Lyapunov spectrum with exponents \(\lambda_i\). The singular values of \(\mathcal{L}_{\sigma-n}^{(n)}\) grow exponentially according to the LEs:

\[
    \forall i,j : \lim_{n \to \infty} \frac{1}{n} \log \delta_{ij} \left( \mathcal{L}_{\sigma-n}^{(n)} \right) = \lambda_i.
\]

**Lemma 3.12.** In the setting of theorem 3.1, eq. 8 holds for \(\mathbb{P}\)-almost every \(\omega\), i.e., we have

\[
    \liminf_{n \to \infty} \inf_{y \in Y_{1(\sigma-n)} \cdots Y_{s(\sigma-n)}} \frac{1}{n} \log \| \mathcal{L}_{\sigma-n}^{(n)} y \| = \lambda_i
\]

for \(1 \leq i \leq p\).

\(^5\)We sort singular values in decreasing order and group indices according to the multiplicities of LEs:

\[
    \delta_1 \geq \delta_2 \geq \cdots \geq \delta_{d_1} \geq \delta_1 \geq \cdots \geq \delta_{d_2} \geq \delta_1 \geq \cdots \geq \delta_{d_3} \geq \cdots \geq \delta_{d_i}.
\]
4. Ginelli’s algorithm. In this section we present a method to compute CLVs, or more generally Oseledets spaces, that arise in the MET. Due to newly developed algorithms, CLVs were made available and have gained increased interest in applications during the last years, see section 1. Among the most famous algorithms is the Ginelli algorithm [16], which uses a dynamical approach to approximate CLVs. Other algorithms, like the one by Wolfe and Samelson [37], combine the use of dynamical techniques with a singular value decomposition of the cocycle, its adjoint, or its inverse. Here, we focus on a purely dynamical approach which can easily be translated to infinite-dimensional scenarios.

We start by describing the concept of Ginelli’s algorithm on an analytical level. A precise mathematical formulation is derived for Hilbert spaces.

Ginelli’s algorithm requires cocycle data along a given trajectory for which Oseledets spaces exist. The cocycle may be part of a random dynamical system or simply a sequence of matrices in the finite-dimensional case. Independent of the setting the fundamental idea behind Ginelli’s algorithm is that almost every vector has a non-vanishing projection (subject to the Oseledets splitting) onto the first Oseledets space. Since vectors inside the first Oseledets space have the highest exponential growth rate, almost every vector will align with the first Oseledets space asymptotically in forward-time. Similarly, we expect the linear span of \( k = d_1 + \cdots + d_i \) randomly chosen vectors to align with the fastest expanding \( k \)-dimensional subspace, the sum of the first \( i \) Oseledets spaces, in forward-time. Reversing time, the fastest growing direction inside \( Y_1 \oplus \cdots \oplus Y_i \) is the slowest growing direction in forward-time, that is, \( Y_i \). Thus, we have a means to compute Oseledets spaces.

At an abstract level on Grassmannians, Ginelli’s algorithm pushes forward a randomly chosen subspace \( W \subset X \) to get an approximation of \( Y_1 \oplus \cdots \oplus Y_i \) and then pushes backward a second randomly chosen subspace \( \tilde{W} \) inside the forward propagated subspace of \( W \) to extract an approximation of \( Y_i \) from the approximation of \( Y_1 \oplus \cdots \oplus Y_i \) (see figure 1).

In practice we express \( W \) in terms of a basis \( (x_1, \ldots, x_k) \). By propagating these vectors, we can track the evolution of \( W \). Similarly, we express \( \tilde{W} \) in terms of a basis. The corresponding vectors can be described as coefficients of the propagated vectors of \( W \). Hence, the backward propagation can be done completely inside a finite-dimensional coefficient space.

Let \( X = H \) be a Hilbert space. To avoid that all vectors \( x_1, \ldots, x_k \) collapse onto the first Oseledets space, which renders them numerically indistinguishable, Ginelli and others suggest to orthonormalize them between smaller propagation steps. While this procedure does not change the outcome of Ginelli’s algorithm analytically, as the involved spaces remain the same, it helps with numerical stability. In particular, they use a QR-decomposition to store orthonormalized vectors in a matrix \( Q \) and the cocycle on coefficient space in a matrix \( R \) for each propagation step. The upper triangular \( R \)-matrices can easily be inverted to perform the backward propagation in coefficient space. Using the identification between vectors and coefficients, we substitute initial vectors for the backward propagation by an upper triangular matrix representing their coefficients. For more details on the implementation see [16, 15].

**Definition 4.1.** Taking the above into account, we define (the analytical kernel of) Ginelli’s algorithm on Hilbert spaces as

\[
G_{\omega,k}^{n_1,n_2} : H_k \times \mathbb{R}_{ru}^{k \times k} \to H_k,
\]
Figure 1. Ginelli’s algorithm at the level of Grassmannians. The algorithm approximates CLVs (or Oseledets spaces) by pushing forward and backward linear perturbations along the \( \sigma \)-trajectory of a state \( \omega \) via the linear propagator \( L \). First, a randomly chosen subspace \( W \) of dimension \( d_1 + \cdots + d_i \) is propagated from the past to the future. Then, a new randomly chosen subspace \( \tilde{W} \) of dimension \( d_i \) inside the forward propagated subspace of \( W \) is propagated backward from the future to the present to provide an approximation of the \( i \)th Oseledets space at \( \omega \). Here, \( d_1, \ldots, d_i \) are the multiplicities of the first \( i \) Lyapunov exponents.

where \( \omega \in \Omega \) defines the trajectory, \( k \in \mathbb{N} \) is the number of CLVs we wish to compute, \( n_1 \in \mathbb{N} \) is the amount of past data, \( n_2 \in \mathbb{N} \) is the amount of future data, and \( \mathbb{R}^{k \times k} \) denotes the set of upper triangular \( k \times k \)-matrices. \( G_{\omega,k}^{n_1,n_2} \) operates on \((x_1, \ldots, x_k), (r_{ij})_{i,j=1}^k\) via the following steps:

1. Forward propagation from \( \sigma_{-n_1} \omega \) to \( \omega \):
\[
(x_1, \ldots, x_k) := \left( L_{\sigma_{-n_1} \omega}^{(n_1)} x_1, \ldots, L_{\sigma_{-n_1} \omega}^{(n_1)} x_k \right).
\]

2. Forward propagation from \( \omega \) to \( \sigma_{n_2} \omega \):
\[
(x_1, \ldots, x_k) := \left( L_{\omega}^{(n_2)} x_1, \ldots, L_{\omega}^{(n_2)} x_k \right).
\]

3. Orthonormalization\(^6\):
\[
(x_1, \ldots, x_k) := \text{orth} \left( x_1, \ldots, x_k \right).
\]

4. Initialization of vectors for backward propagation:
\[
(y_1, y_2, \ldots, y_k) := \left( r_{11} x_1^3, r_{12} x_1^3 + r_{22} x_2^3, \ldots, \sum_{j=1}^k r_{jk} x_j^3 \right).
\]

\(^6\)Any orthonormalization procedure respecting the order of the tuple is feasible. For example, this includes the QR-decomposition and the Gram-Schmidt procedure.
5. Backward propagation from \( \sigma_{n_2} \omega \) to \( \omega \):

\[
(y^2_1, \ldots, y^2_k) := \left( \left( \mathcal{L}^{(n_2)}_{\omega} | W^1 \right)^{-1} y^1_1, \ldots, \left( \mathcal{L}^{(n_2)}_{\omega} | W^1 \right)^{-1} y^1_k \right),
\]

where \( W^1 := \text{span} (x^1_1, \ldots, x^1_k) \).

6. Normalization:

\[
(y^3_1, \ldots, y^3_k) := \left( \frac{y^2_1}{\|y^2_1\|}, \ldots, \frac{y^2_k}{\|y^2_k\|} \right).
\]

We set \( G_{\omega,k}^{n_1,n_2}((x_1, \ldots, x_k), (r_{ij})_{i,j=1}^k) := (y^3_1, \ldots, y^3_k) \) as our approximation of the first \( k \) CLVs at \( \omega \).

Ginelli’s algorithm requires two types of inputs: a tuple of vectors \((x_1, \ldots, x_k)\) as initial vectors for forward propagation and a coefficient matrix \((r_{ij})_{i,j=1}^k\) to initialize vectors for backward propagation. In the above definition we denoted new vectors obtained during the different propagation steps by \( x \) and \( y \) with corresponding indices. \( x \)-vectors were used during forward propagation and \( y \)-vectors were used during backward propagation.

Let us remark that whenever \( G_{\omega,k+1}^{n_1,n_2}((x_1, \ldots, x_{k+1}), (r_{ij})_{i,j=1}^{k+1}) \) is well-defined, its first \( k \) components coincide with \( G_{\omega,k}^{n_1,n_2}((x_1, \ldots, x_k), (r_{ij})_{i,j=1}^k) \). Thus, it suffices to assume \( k = d_1 + \cdots + d_l \) for our convergence analysis.

In the next section we provide a convergence proof of the algorithm as \( \min(n_1, n_2) \) goes to infinity.

5. **Convergence theorem.** We now derive a convergence theorem for Ginelli’s algorithm in the setting of theorem 3.1. The main statement is similar to the one for finite dimensions [25]: exponentially fast convergence of the algorithm for almost every input. This time, however, convergence is analyzed using well-separating common complements instead of admissibility. Since we know about existence and prevalence of well-separating common complements in Hilbert spaces and since we used an orthonormalization procedure in the definition of Ginelli’s algorithm, the upcoming convergence theorem is only for Hilbert spaces.

To shorten our notation, we group indices according to the multiplicities of LEs:

\[
(x_{11}, x_{12}, \ldots, x_{1d_1}, x_{21}, \ldots, x_{2d_2}, x_{31}, \ldots, x_{p_{d_p}}).
\]

Let us state the main result of this article:

**Theorem 5.1** (Convergence a.e. of Ginelli’s algorithm on Hilbert spaces). Let \( R = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, H, \mathcal{L}) \) satisfy the assumptions of theorem 3.1 and let \( k = d_1 + \cdots + d_l \) for some finite \( l \leq p \). Moreover, set \( \lambda_0 := \infty \) and \( \lambda_{p+1} := \kappa^* \).

On a subset \( \Omega' \subset \Omega \) of full \( \mathbb{P} \)-measure, Ginelli’s algorithm converges for almost every input. That is, fixing \( \omega \in \Omega' \), for almost every tuple \((x_1, \ldots, x_k) \in H^k \), for almost every \( R \in \mathbb{P}_{r_{x}^k} \), and for all \( i \leq l \), it holds

\[
\lim_{N \to \infty} \sup_{n_1, n_2 \geq N} \frac{1}{\min(n_1, n_2)} \log d_{\mathcal{F}} \left( \text{span} \left\{ \left( G_{\omega,k}^{n_1,n_2} \right)_{i,j} \right\}, Y_i(\omega) \right) \leq - \min \{ |\lambda_i - \lambda_{i-1}|, |\lambda_i - \lambda_{i+1}| \}
\]

at \((x_1, \ldots, x_k), R\).
There are three concepts of “almost every” in the statement of the theorem. Firstly, the algorithm fixes \( \omega \) from a set of full \( \mathcal{F} \)-measure to determine the trajectory along which Ginelli’s algorithm is applied. Secondly and thirdly, the algorithm requires a tuple \( (x_1, \ldots, x_k) \in H^k \) and an upper triangular matrix \( R \in \mathbb{R}^{k \times k} \) as inputs. “Almost every” with respect to the tuple is understood in terms of prevalence [28], whereas “almost every” with respect to the matrix is meant in the usual Lebesgue sense. If \( H \) is finite-dimensional, the two previous notions coincide and we recover the convergence theorem from [25] with the addition that our new theorem also includes the semi-invertible case.

The theorem tells us that, generically, output vectors of Ginelli’s algorithm span subspaces that are exponentially close to the Oseledets spaces. Hence, the algorithm approximates CLVs. To get a good approximation, it is necessary to increase both \( n_1 \) and \( n_2 \). In other words, the algorithm needs sufficient data along the past and the future of the trajectory. Moreover, theorem 5.1 reveals that the speed of convergence to the \( i \)th Oseledets space \( Y_i(\omega) \) is at least exponentially fast in proportion to the spectral gap between \( \lambda_i \) and neighboring LEs.

5.1. Forward-time estimates. The forward- and backward-time estimates are proved for general Banach spaces \( (X, \| \cdot \|) \). Our first result investigates how certain subspaces evolve in the presence of an equivariant splitting under a given map. The estimates consist of terms that are well-understood when the splitting is the Oseledets splitting. As before, we write \( B \subset X \) for the unit ball and \( S \subset X \) for the unit sphere in \( X \).

**Lemma 5.2.** Let \( (Y, V), (Y', V') \in \text{Comp}_k(X) \) be two pairs of closed complemented subspaces. Assume we have a bounded linear map \( \mathcal{L} \in \mathbb{L}(X) \) respecting the splittings, i.e., \( \mathcal{L}Y \subset Y' \) and \( \mathcal{L}V \subset V' \), such that \( \ker \mathcal{L} \subset V \).

If \( W \in \mathcal{G}_k(X) \) is a complement to \( V \) such that the degree of transversality satisfies

\[
\inf_{w \in W \cap S} d(w, V) \geq 2 \| \Pi_V^{|Y} \| \frac{\| \mathcal{L} | V \|}{\inf_{y \in Y \cap S} \| \mathcal{L} y \|},
\]

then

\[
\sup_{w' \in \mathcal{L}W \cap B} d(w', Y' \cap B) \leq 4 \| \Pi_V^{|Y} \| \frac{\| \mathcal{L} | V \|}{\inf_{y \in Y \cap S} \| \mathcal{L} y \|}.
\]

**Proof.** If \( \mathcal{L}|_V = 0 \), then \( \ker \mathcal{L} = V \). Thus, \( \mathcal{L} \) restricts to an isomorphism between any complement \( W \) to \( V \) and \( Y' \). In this case the claim is trivially satisfied.

Now, assume \( \mathcal{L}|_V \neq 0 \). Let \( W \) be a complement as in the claim. For \( w \in W \cap S \), it holds

\[
\| \mathcal{L}\Pi_V^{|Y} w \| \leq \| \mathcal{L}|_V \| \| \Pi_V^{|Y} \| \| w \|
\]

and

\[
\| \mathcal{L}\Pi_Y^{|V} w \| \geq \inf_{y \in Y \cap S} \| \mathcal{L} y \| \| \Pi_Y^{|V} \| \| w \|
= \inf_{y \in Y \cap S} \| \mathcal{L} y \| \| w - \Pi_Y^{|V} w \|
\geq \inf_{y \in Y \cap S} \| \mathcal{L} y \| d(w, V)
\geq 2 \| \Pi_Y^{|V} \| \| \mathcal{L}|_V \| > 0.
\]

Combining both estimates, we get

\[
\frac{\| \mathcal{L}\Pi_Y^{|V} w \|}{\| \mathcal{L}\Pi_V^{|Y} w \|} \leq \frac{1}{2}.
\]
To derive eq. 11 it is enough to estimate \( d(Lw/\|Lw\|, Y' \cap B) \) for \( w \in W \cap S \). Write \( w = y + v \) according to the decomposition \( X = Y \oplus V \). We have
\[
\begin{align*}
d \left( \frac{Lw}{\|Lw\|}, Y' \cap B \right) &\leq \frac{\|Lw\|}{\|Lw\|} - \frac{Ly}{\|Ly\|} \\
&= \left| \frac{\|Lv\|}{\|Lw\|} - \left( \frac{1}{\|Ly\|} - \frac{1}{\|Lw\|} \right) \right| \frac{Ly}{\|Ly\|} \\
&\leq \frac{\|Lv\|}{\|Lw\|} + \left| 1 - \|Ly\| \right| \frac{\|Ly\|}{\|Lw\|}.
\end{align*}
\]
Since \( y \neq 0 \) and by eq. 12, we estimate the first term as follows:
\[
\frac{\|Lv\|}{\|Lw\|} \leq \frac{\|Lv\|}{\|Ly\| - \|Lw\|} = \frac{\|Lv\|}{\|Ly\|} \left( 1 - \frac{\|Lv\|}{\|Ly\|} \right)^{-1} \leq 2 \frac{\|Lv\|}{\|Ly\|}.
\]
For the other term, we distinguish between two cases. If \( \|Ly\|/\|Lw\| \leq 1 \), then
\[
1 - \frac{\|Ly\|}{\|Lw\|} \leq 1 - \frac{\|Ly\|}{\|Ly\| + \|Lw\|} = \frac{\|Lw\|}{\|Ly\| + \|Lw\|} \leq \frac{\|Lv\|}{\|Ly\|}.
\]
If \( \|Ly\|/\|Lw\| \geq 1 \), then
\[
\frac{\|Ly\|}{\|Lw\|} - 1 = \frac{\|Lw\|}{\|Ly\|} \leq \frac{\|Lv\|}{\|Lw\|} \leq 2 \frac{\|Lv\|}{\|Ly\|}.
\]
In total we get
\[
d \left( \frac{Lw}{\|Lw\|}, Y' \cap B \right) \leq 4 \frac{\|Lv\|}{\|Ly\|}.
\]
Since \( v = \Pi_{V' \cap Y} w \) and \( y = \Pi_{V' \cap V} w \), the claim follows from the estimates in the beginning. \( \square \)

**Corollary 1.** In the setting of lemma 5.2 it holds
\[
\|\Pi_{V' \cap Y} Lw\| \leq 2 \frac{\|\Pi_{V' \cap Y}\|}{\inf_{w \in W \cap S} d(w, V)} \frac{\|L|_V\|}{\inf_{y \in Y \cap S} \|L_y\|}.
\]

**Proof.** The corollary follows from
\[
\|\Pi_{V' \cap Y} Lw\| = \sup_{w \in W \cap S} \left\| \Pi_{V' \cap Y'} \frac{Lw}{\|Lw\|} \right\| = \sup_{w \in W \cap S} \left\| \frac{L \Pi_{V' \cap Y} w}{\|Lw\|} \right\|
\]
and from the estimate of \( \|Lv\|/\|Lw\| \) in the proof of lemma 5.2. \( \square \)

Next, we derive two lemmata that handle sequences of maps acting on equivariant splittings with different asymptotic growth rates. The first lemma is concerned with propagation from present to future, whereas the second lemma treats propagation from past to present.

**Lemma 5.3.** Let \((Y, V) \in \text{Comp}_k(X)\) and \((Y(n), V(n)) \in \text{Comp}_k(X)\) for \(n \in \mathbb{N}\). Assume we have bounded linear maps \( L(n) \in L(X) \) respecting the splittings, i.e., \( L(n)Y' \subset Y(n) \) and \( L(n)V \subset V(n) \), such that \( \text{ker} L(n) \subset V \). Furthermore, assume there are numbers \( \infty > \lambda_Y > \lambda_V \geq -\infty \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \|L(n)|_V\| \leq \lambda_V
\]
and
\[
\liminf_{n \to \infty} \inf_{y \in Y \cap S} \frac{1}{n} \log \|L(n)y\| \geq \lambda_Y.
\]
Then, we have
\[ \limsup_{n \to \infty} \frac{1}{n} \log d_{\varphi}(\mathcal{L}(n)W, Y(n)) \leq -|\lambda_Y - \lambda_V| \]
for any complement \( W \) to \( V \).

**Proof.** According to the assumptions we have
\[ \limsup_{n \to \infty} \frac{1}{n} \log \frac{\|\mathcal{L}(n)\|}{\inf_{y \in Y \cap S} \|\mathcal{L}(n)y\|} \leq -|\lambda_Y - \lambda_V| < 0, \]
i.e., the quotient \( \|\mathcal{L}(n)\|/\inf_{y \in Y \cap S} \|\mathcal{L}(n)y\| \) decays exponentially fast with \( n \).
Thus, for any complement \( W \) to \( V \), there is \( N > 0 \) such that eq. 10 from lemma 5.2 is satisfied for all \( n \geq N \). Applying the lemma, we get
\[ \limsup_{n \to \infty} \frac{1}{n} \log \sup_{w \in \mathcal{L}(n)W \cap B} d(w', Y(n) \cap B) \leq -|\lambda_Y - \lambda_V|. \]
The claim follows from lemma 2.2. \( \square \)

Lemma 5.3 implies that complements to spaces of the Oseledets filtration will align with Oseledets spaces asymptotically (at an exponential speed). Moreover, the lemma tells us that any two complements to \( V \) will align asymptotically if they have a uniformly higher growth rate than \( V \). Interestingly, we do not need the existence of an Oseledets splitting. In fact, the lemma may be applied to systems with a possibly non-invertible base (e.g., see [2, theorem 2] or [4]).

**Lemma 5.4.** Let \( (Y, V) \in \text{Comp}_k(\mathcal{X}) \) and \( (Y(-n), V(-n)) \in \text{Comp}_k(\mathcal{X}) \) for \( n \in \mathbb{N} \). Assume we have bounded linear maps \( \mathcal{L}(-n) \in L(\mathcal{X}) \) respecting the splittings, i.e., \( \mathcal{L}(-n)Y(-n) \subset Y \) and \( \mathcal{L}(-n)V(-n) \subset V \), such that \( \ker \mathcal{L}(-n) \subset V(-n) \). Furthermore, assume that
\[ \lim_{n \to \infty} \frac{1}{n} \log \|\Pi_{Y(-n)}\| = 0 \]
and that there are numbers \( \infty > \lambda_Y > \lambda_V \geq -\infty \) such that
\[ \limsup_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}(-n)\| \leq \lambda_V \]
and
\[ \liminf_{n \to \infty} \inf_{y \in Y(-n) \cap S} \frac{1}{n} \log \|\mathcal{L}(-n)y\| \geq \lambda_Y. \]

Then, we have
\[ \limsup_{n \to \infty} \frac{1}{n} \log d_{\varphi}(\mathcal{L}(-n)W, Y) \leq -|\lambda_Y - \lambda_V| \]
for any well-separating common complement \( W \) for \((V(-n))_{n \in \mathbb{N}}\).

**Proof.** As in lemma 5.3, we see that
\[ \limsup_{n \to \infty} \frac{1}{n} \log \frac{\|\mathcal{L}(-n)\|}{\inf_{y \in Y(-n) \cap S} \|\mathcal{L}(-n)y\|} \leq -|\lambda_Y - \lambda_V|. \]
By our assumption on the associated projections we get
\[ \limsup_{n \to \infty} \frac{1}{n} \log \left(2\|\Pi_{Y(-n)}\| \frac{\|\mathcal{L}(-n)\|}{\inf_{y \in Y(-n) \cap S} \|\mathcal{L}(-n)y\|} \right) \leq -|\lambda_Y - \lambda_V| < 0. \]
In particular, by definition 2.5 any well-separating common complement for the sequence \((V(-n))_{n \in \mathbb{N}}\) fulfills eq. 10 for \( n \) large enough as the degree of transversality
decays only subexponentially. The claim may be derived as in the proof of lemma 5.3. □

Corollary 2. In the setting of lemma 5.4, we have
\[
\limsup_{n \to \infty} \frac{1}{n} \log \| \Pi V \|_{\mathcal{L}(-n) W} \leq -|\lambda Y - \lambda V|
\]
for any well-separating common complement W for (V(-n))_{n \in \mathbb{N}}.

Proof. Since lemma 5.2 and corollary 1 give the same estimate up to a factor of 2, the proof of corollary 2 is the same as for lemma 5.4. □

The following theorem gives us convergence of certain subspaces of Banach spaces to the sum of the first Oseledets spaces in forward-time:

Theorem 5.5. Let R be as in theorem 3.1 and \( \omega \in \Omega \) such that the Oseledets splitting exists. Write \( \lambda_{p+1} := \kappa^* \) and fix some finite \( i \leq p \).
If eqs. 3-5 hold\(^7\), then
\[
\limsup_{n \to \infty} \frac{1}{n} \log d_\rho \left( \mathcal{L}^{(n)} \omega, Y_1(\sigma_n \omega) \oplus \cdots \oplus Y_i(\sigma_n \omega) \right) \leq -|\lambda_i - \lambda_{i+1}|
\]
for any complement W to \( V_{i+1}(\omega) \).
If eqs. 6-8 hold, then
\[
\limsup_{n \to \infty} \frac{1}{n} \log d_\rho \left( \mathcal{L}^{(n)} \omega, Y_1(\omega) \oplus \cdots \oplus Y_i(\omega) \right) \leq -|\lambda_i - \lambda_{i+1}|
\]
for any well-separating common complement W for (V_{i+1}(\sigma_n \omega))_{n \in \mathbb{N}}.

Proof. The proof is a direct application of lemma 5.3 and lemma 5.4 to the splittings \( (Y, V) = (Y_1(\omega) \oplus \cdots \oplus Y_i(\omega), V_{i+1}(\omega)), (Y(n), V(n)) := (Y_1(\sigma_n \omega) \oplus \cdots \oplus Y_i(\sigma_n \omega), V_{i+1}(\sigma_n \omega)) \) for \( n \in \mathbb{Z} \), and to the maps \( \mathcal{L}(n) := \mathcal{L}^{(n)} \omega \) and \( \mathcal{L}(-n) := \mathcal{L}^{(n)}_{\sigma - n \omega} \) for \( n \in \mathbb{N} \).

In view of theorem 2.6, theorem 5.5 for Hilbert spaces implies that we can compute the sum of the first Oseledets spaces \( Y_1 \oplus \cdots \oplus Y_i \) at \( \omega \) or asymptotically by pushing forwards a set of \( d_1 + \cdots + d_i \) randomly chosen vectors. The convergence is exponentially fast with a rate given by the spectral gap between the consecutive LEs \( \lambda_i \) and \( \lambda_{i+1} \).

5.2. Backward-time estimates. In this subsection we investigate backward propagation along bundles of certain subspaces. Since we did not assume a cocycle with invertible action, we cannot simply apply our results about forward propagation to a time-reversed system as it was done in the finite-dimensional case [25]. Instead, we derive new estimates for forward propagation to deduce properties for backward propagation.

Lemma 5.6. Let \((Y_1, V_1) \in \text{Comp}_{\kappa_k}(X)\) and \((Y_2, V_2) \in \text{Comp}_{\kappa_k}(V_1)\), so that \( X = Y_1 \oplus V_1 \) and \( V_1 = Y_2 \oplus V_2 \). Moreover, let \( W_i \) be a complement to \( V_i \) in \( X \) for \( i = 1, 2 \) such that \( W_1 \subset W_2 \). Assume we have a map \( \mathcal{L} \in L(X) \) with \( \ker \mathcal{L} \subset V_2 \).

If \( \bar{w} \in G_k(W_2) \) is a complement to \( W_1 \) in \( W_2 \) and if \( \bar{w} \in W \cap S \) satisfies
\[
d(\bar{w}, Y_2) \geq (2\|\Pi V_1\|_{W_1} + \|\Pi V_1\|_{Y_1} \|\Pi W_1\|_{V_1}) \frac{\|\mathcal{L}|_{V_1}\|}{\inf_{y \in Y_1 \cap S} \|\mathcal{L} y\|} + \|\Pi V_2\|_{Y_1} \|\Pi V_2\|_{V_2} \|w_2\|,
\]
\[
(13)
\]
\(^7\)We remark that eqs. 3-5 and eqs. 6-8 hold for \( \mathbb{P} \)-almost every \( \omega \in \Omega \).
then
\[ d \left( \frac{\mathcal{L} \tilde{w}}{\|\mathcal{L} \tilde{w}\|}, \mathcal{L} W_1 \right) \leq \frac{2 \|\mathcal{L}|_{V_1} \| \|\Pi_{V_1}|_{W_1}\|}{(\inf_{y \in Y_1 \cap S} \|\mathcal{L} y\|) (d(\tilde{w}, Y_2) - \|\Pi_{V_2}|_{Y_1 \cup Y_2}|_{W_2}\|) - \|\mathcal{L}|_{V_1} \| \|\Pi_{V_1}|_{Y_1}\| \|\Pi_{W_1}|_{V_1}\|} \quad (14) \]

Proof. Since \( V_1 = Y_2 \oplus V_2 \) is a splitting with \( Y_2 \neq \{0\} \) and \( \ker \mathcal{L} \subset V_2 \), it holds \( \mathcal{L}|_{V_1} \neq 0 \).

Let \( \tilde{w} \in \tilde{W} \cap S \) be as in the claim, so that eq. 13 is satisfied. We estimate
\[ \|\mathcal{L}\Pi_{V_1}|_{W_1} \tilde{w}\| \leq \|\mathcal{L}|_{V_1} \| \|\Pi_{V_1}|_{W_1}\| \]

and
\[ \|\mathcal{L}\Pi_{W_1}|_{V_1} \tilde{w}\| = \|\mathcal{L}(\Pi_{V_1}|_{V_1} + \Pi_{V_1}|_{Y_1})\Pi_{W_1}|_{V_1} \tilde{w}\| \geq \|\Pi_{V_1}|_{V_1}\Pi_{W_1}|_{V_1} \tilde{w}\| - \|\mathcal{L}\Pi_{V_1}|_{V_1}\Pi_{W_1}|_{V_1} \tilde{w}\| \geq \left( \inf_{y \in Y_1 \cap S} \|\mathcal{L} y\| \right) \|\Pi_{V_1}|_{V_1}\Pi_{W_1}|_{V_1} \tilde{w}\| - \|\mathcal{L}|_{V_1} \| \|\Pi_{V_1}|_{V_1}\| \|\Pi_{W_1}|_{V_1}\|. \]

The term with two consecutive projections applied to \( \tilde{w} \) can be estimated further via
\[ \|\Pi_{V_1}|_{V_1}\Pi_{W_1}|_{V_1} \tilde{w}\| = \|\Pi_{W_1}|_{V_1} \tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{W_1}|_{V_1} \tilde{w}\| = \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{W_1}|_{V_1} \tilde{w}\| = \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_1} \tilde{w}\| = \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| = \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| = \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| = \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| \geq \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| - \|\Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| \geq \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| \geq \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| \geq \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| \geq \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| \geq \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\| \geq \|\tilde{w} - \Pi_{V_1}|_{V_1}\Pi_{V_2}|_{V_2} \tilde{w}\|. \]

Note that \( \Pi_{V_1}|_{V_2} \) and \( \Pi_{V_2}|_{V_2} \) are projections defined on \( V_1 \). By eq. 13 we have
\[ \|\Pi_{V_1}|_{V_1}\Pi_{W_1}|_{V_1} \tilde{w}\| \geq (2\|\Pi_{V_1}|_{W_1}\| + \|\Pi_{V_1}|_{V_1}\| \|\Pi_{W_1}|_{V_1}\|) \inf_{y \in Y_1 \cap S} \|\mathcal{L} y\| \|\Pi_{V_1}|_{W_1}\|. \]

Hence, we get
\[ \|\mathcal{L}\Pi_{W_1}|_{V_1} \tilde{w}\| \geq 2\|\Pi_{V_1}|_{W_1}\| \|\mathcal{L}|_{V_1}\| > 0 \]

and
\[ \|\mathcal{L}\Pi_{V_1}|_{W_1} \tilde{w}\| \|\Pi_{V_1}|_{W_1}\| = \frac{1}{2}. \]
Finally, it holds
\[ d \left( \frac{\mathcal{L}\tilde{w}}{\|\mathcal{L}\tilde{w}\|}, \mathcal{L}W_1 \right) \leq \left\| \frac{\mathcal{L}\tilde{w}}{\|\mathcal{L}\tilde{w}\|} - \frac{\mathcal{L}\Pi_{W_1||V_1}\tilde{w}}{\|\mathcal{L}\tilde{w}\|} \right\| \]
\[ = \frac{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|}{\|\mathcal{L}\tilde{w}\|} \]
\[ \leq \frac{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\| - \|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|}{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|} \]
\[ = \frac{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|}{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|} \left( 1 - \frac{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|}{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|} \right)^{-1} \]
\[ \leq 2 \frac{\|\mathcal{L}\Pi_{V_1||W_1}\tilde{w}\|}{\|\mathcal{L}\Pi_{W_1||V_1}\tilde{w}\|}. \]

Estimating the numerator and the denominator as in the beginning of the proof, we arrive at eq. 14. □

**Corollary 3.** Let \( Y_i, V_i, W_i \) for \( i = 1, 2 \) and \( \mathcal{L} \) be as in lemma 5.6.
If \( W \subset W_2 \) is a complement to \( W_1 \) in \( W_2 \) satisfying
\[ \inf_{\tilde{w}^i \in \mathcal{L}W \cap S} d(\tilde{w}^i, \mathcal{L}W_1) \geq \delta \]
for some \( 0 < \delta \leq 1 \), then
\[ \sup_{\tilde{w} \in \widetilde{W} \cap B} \left\| \frac{\mathcal{L}\tilde{w}}{\|\mathcal{L}\tilde{w}\|} \right\| \]
\[ \leq \left( \frac{2}{\delta} \|\mathcal{L}|_{V_1||W_1}^i \| + \|\mathcal{L}|_{V_1||V_1}^i \| \|\Pi_{W_1||V_1}\| \right) \frac{\|\mathcal{L}|_{V_1}\|}{\inf_{\tilde{w} \in \mathcal{L}W \cap S} \|\mathcal{L}|_{V_1}\|} + 2\|\Pi_{V_2||Y_1 \cap Y_2}|_{W_1}\|, \]
then by lemma 5.6
\[ \delta \leq d \left( \frac{\mathcal{L}\tilde{w}}{\|\mathcal{L}\tilde{w}\|}, \mathcal{L}W_1 \right) \]
\[ \leq \frac{2\|\mathcal{L}|_{V_1||W_1}^i \| \|\Pi_{V_1||W_1}^i \|}{\inf_{\tilde{w} \in \mathcal{L}W \cap S} \|\mathcal{L}|_{V_1}\|} \left( d(\tilde{w}, Y_2) - \|\Pi_{V_2||Y_1 \cap Y_2}|_{W_2}\| - \|\mathcal{L}|_{V_1}\| \|\Pi_{V_1||Y_1}\| \|\Pi_{W_1||V_1}\| \right). \]

However, the former would be strictly smaller than \( \delta \) by our assumption on \( d(\tilde{w}, Y_2) \). Thus, we must have
\[ \sup_{\tilde{w} \in \widetilde{W} \cap B} \left\| \frac{\mathcal{L}\tilde{w}}{\|\mathcal{L}\tilde{w}\|} \right\| \]
\[ \leq \left( \frac{2}{\delta} \|\mathcal{L}|_{V_1||W_1}^i \| + \|\mathcal{L}|_{V_1||Y_1}^i \| \|\Pi_{W_1||V_1}\| \right) \frac{\|\mathcal{L}|_{V_1}\|}{\inf_{\tilde{w} \in \mathcal{L}W \cap S} \|\mathcal{L}|_{V_1}\|} + 2\|\Pi_{V_2||Y_1 \cap Y_2}|_{W_2}\|. \]

The claim follows from eq. 1. □

From corollary 3 we can derive an upper bound of the distance between \( \widetilde{W} \) and \( Y_2 \) from a lower bound of the degree of transversality of \( (\mathcal{L}W, \mathcal{L}W_1) \) in \( \mathcal{L}W_2 \). Hence, the corollary describes backward propagation.

Next, we use the spaces \( W_1, W_2 \) to connect estimates from subsection 5.1 to backward propagation, ultimately giving us an understanding of Ginelli’s algorithm at the level of maps:
Lemma 5.7. Let \((Y_1, V_1) \in \text{Comp}_{k_1}(X), (Y_2, V_2) \in \text{Comp}_{k_2}(V_1)\), and \(\infty > \lambda_{Y_1} > \lambda_{Y_2} > \lambda_{V_1} \geq -\infty\).

For the past data, let \((Y_1(-n), V_1(-n)) \in \text{Comp}_{k_1}(X)\) and \((Y_2(-n), V_2(-n)) \in \text{Comp}_{k_2}(V_1(-n))\) for \(n \in \mathbb{N}\). Assume we have bounded linear maps \(L(n) \in L(X)\) respecting the splittings, i.e., \(L(-n)Y_i(-n) \subset Y_i\) for \(i = 1, 2\) and \(L(-n)V_2(-n) \subset V_2\), such that \(\ker L(-n) \subset V_2(-n)\) for \(n \in \mathbb{N}\). Moreover, assume that

1. \(\lim_{n \to \infty} (1/n) \log \|\Pi_{Y_1(-n)}Y_1(-n)\| = 0\),
2. \(\lim_{n \to \infty} (1/n) \log \|\Pi_{V_2(-n)}Y_1(-n) \oplus Y_2(-n)\| = 0\),
3. \(\limsup_{n \to \infty} (1/n) \log \|\mathcal{L}(n)Y_i(-n)\| \leq \lambda_{Y_i}\) for \(i = 1, 2\),
4. \(\liminf_{n \to \infty} \inf_{y \in Y_i(-n) \cap S} (1/n) \log \|\mathcal{L}(n)y\| \geq \lambda_{Y_i}\), and
5. \(\liminf_{n \to \infty} \inf_{y \in Y_i(-n) \cap S} (1/n) \log \|\mathcal{L}(n)y\| \geq \lambda_{Y_2}\).

For the future data, let \((Y_1(n), V_1(n)) \in \text{Comp}_{k_1}(X)\) and let \((Y_2(n), V_2(n)) \in \text{Comp}_{k_2}(V_1(n))\) for \(n \in \mathbb{N}\). Assume we have bounded linear maps \(\mathcal{L}(n) \in L(X)\) respecting the splittings, i.e., \(\mathcal{L}(n)Y_i \subset Y_i\) for \(i = 1, 2\) and \(\mathcal{L}(n)V_2 \subset V_2(n)\), such that \(\ker \mathcal{L}(n) \subset V_2(n)\) for \(n \in \mathbb{N}\). Moreover, assume that

6. \(\limsup_{n \to \infty} (1/n) \log \|\mathcal{L}(n)Y_i\| \leq \lambda_{Y_i}\) and
7. \(\liminf_{n \to \infty} \inf_{y \in Y_i \cap S} (1/n) \log \|\mathcal{L}(n)y\| \geq \lambda_{Y_1}\).

Let \(W_i\) be a well-separating common complement for \((V_i(-n))_{n \in \mathbb{N}}\) for \(i = 1, 2\) such that \(W_1 \subset W_2\). If \((\mathcal{W}(n_1, n_2))_{n_1, n_2 \in \mathbb{N}}\) is a family of subspaces such that \(\mathcal{L}(n_2)\mathcal{L}(-n_1)W_1 \oplus \mathcal{W}(n_1, n_2) = \mathcal{L}(n_2)\mathcal{L}(-n_1)W_2\), and if

\[
\inf_{\hat{w} \in \hat{\mathcal{W}}(n_1, n_2) \cap S} d(\hat{w}, \mathcal{L}(n_2)\mathcal{L}(-n_1)W_1) \geq \delta
\]

for some constant \(0 < \delta \leq 1\), then

\[
\limsup_{N \to \infty} \sup_{n_1, n_2 \geq N} \frac{1}{\min(n_1, n_2)} \log d_{\mathcal{G}} \left( \left( \mathcal{L}(n_2)\mathcal{L}(-n_1)W_2 \right)^{-1} \mathcal{W}(n_1, n_2), Y_2 \right) 
\leq -\min(|\lambda_{Y_2} - \lambda_{Y_1}|, |\lambda_{Y_2} - \lambda_{V_2}|). \tag{16}
\]

Proof. Let \(W_1\) and \(W_2\) be as in the claim. We apply lemma 5.4 to \((Y, V) = (Y_1, V_1)\) for \(W = W_1\) and to \((Y, V) = (Y_1 \oplus Y_2, V_2)\) for \(W = W_2\) with their respective spaces and mappings at \(-n\). It follows that

\[
\limsup_{n \to \infty} \frac{1}{n} \log d_{\mathcal{G}}(\mathcal{L}(-n)W_1, Y_1) \leq -|\lambda_{Y_1} - \lambda_{V_1}|
\]

and

\[
\limsup_{n \to \infty} \frac{1}{n} \log d_{\mathcal{G}}(\mathcal{L}(-n)W_2, Y_1 \oplus Y_2) \leq -|\lambda_{Y_2} - \lambda_{V_2}|.
\]

Thus, we have good approximations of \(Y_1\) and \(Y_1 \oplus Y_2\) from the past data. Moreover, by corollary 2 we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|\Pi_{V_2}Y_1 \oplus Y_2 \mathcal{L}(-n)W_2\| \leq -|\lambda_{Y_2} - \lambda_{V_2}|.
\]

Since \(\mathcal{L}(-n)W_1\) converges to \(Y_1\), the projections \(\Pi_{\mathcal{L}(-n)W_1}Y_1\) converge to \(\Pi_{Y_1}Y_1\) by lemma 2.3. In particular, \(\|\Pi_{\mathcal{L}(-n)W_1}Y_1\|\) and \(\|\Pi_{V_2}Y_1 \oplus Y_2 \mathcal{L}(-n)W_2\|\) are bounded from above by a constant independent of \(n\).

The growth rate assumptions on future data imply

\[
\limsup_{n \to \infty} \frac{1}{n} \log \frac{\|\mathcal{L}(n)Y_1\|}{\inf_{y \in Y_1 \cap S} \|\mathcal{L}(n)y\|} \leq -|\lambda_{Y_1} - \lambda_{V_1}|.
\]
Now, apply corollary 3 to \((Y_1, V_1), (Y_2, V_2)\), the two complements \(\mathcal{L}(−n_1)W_1\) to \(V_1\) and \(\mathcal{L}(−n_1)W_2\) to \(V_2\), \(\mathcal{L} = \mathcal{L}(n_2)\), and \(\tilde{W} = (\mathcal{L}(n_2)|\mathcal{L}(−n_1)W_2)^{-1} \tilde{W}(n_1, n_2)\). We get
\[
\sup_{\tilde{w} \in (\mathcal{L}(n_2)|\mathcal{L}(−n_1)W_2)^{-1}\tilde{W}(n_1, n_2) \cap B} d(\tilde{w}, Y_2 \cap B) \\
\leq 2 \left( \frac{2}{\delta} \|\Pi V_1\|_{\mathcal{L}(−n_1)W_1} + \|\Pi V_1\|_Y \|\Pi_{\mathcal{L}(−n_1)W_1}\|_V \right) \frac{\|\mathcal{L}(n_2)|V_1\|}{\inf_{y \in Y_2} \|\mathcal{L}(n_2)y\|} \\
+ 2\|\Pi V_2\|_{Y_1 \oplus Y_2} \mathcal{L}(−n_1)W_2 \|.
\] (17)

In view of lemma 2.2, all that remains to prove eq. 16 is to insert respective asymptotics into the terms of eq. 17. Indeed, the terms inside the large bracket are bounded from above by a constant, and the other terms can be estimated as above.

Lemma 5.7 provides an appropriate tool to study convergence of Ginelli’s algorithm in infinite dimensions. Since the algorithm initiates vectors for backward propagation inside spaces from the forward propagation, which vary with the chosen runtime, the domain for initial vectors is not constant. Hence, \(\tilde{W}\) varies with \(n_1\) and \(n_2\). This poses a problem when talking about convergence with respect to initial conditions. One way to solve this problem is to express initial vectors of the backward propagation in terms of runtime-independent coefficients. If \(X = H\) is a Hilbert space, then we may identify an orthonormal basis of \(\mathcal{L}(n_1)\mathcal{L}(−n_2)W_2\) with the standard basis of \((\mathbb{R}^{k_1+k_2}, \|\cdot\|_2)\) as it is done in our definition of Ginelli’s algorithm on Hilbert spaces (see definition 4.1). The identification defines an isometry leaving distances and angles invariant. In particular, we may represent \(\tilde{W}\) in terms of runtime-independent coefficients and check eq. 15 on the coefficient space.

5.3. Proof of Theorem. We now combine our tools to prove theorem 5.1:

Proof of theorem 5.1. Fix an element \(\omega\) of the subset \(\Omega' \subset \Omega\) of full \(\mathbb{P}\)-measure on which the Oseledets splitting is defined and on which eqs. 3-8 hold. We show convergence of Ginelli’s algorithm at \(\omega\) for almost every input.

Let \(\mathcal{F}_l \subset H^{d_1+\cdots+d_l}\) be the subset of all tuples inducing well-separating common complements for \((V_{i+1}(\sigma−n\omega))_{n \in \mathbb{N}}\) for \(i = 1, \ldots, l\). Then, the set
\[
\mathcal{F} := (\mathcal{F}_1 \times H^{d_2+\cdots+d_l}) \cap (\mathcal{F}_2 \times H^{d_3+\cdots+d_l}) \cap \cdots \cap \mathcal{F}_l \subset H^k
\]
consists of tuples \((x_1, \ldots, x_{d_l})\) such that \(\text{span}(x_1, \ldots, x_{d_l})\) is a well-separating common complement for \((V_{i+1}(\sigma−n\omega))_{n \in \mathbb{N}}\) for each \(i = 1, \ldots, l\). In particular, since products and intersections of prevalent sets are prevalent, theorem 2.6 implies that \(\mathcal{F}\) is prevalent. We use elements of \(\mathcal{F}\) as initial vectors for the forward phase of Ginelli’s algorithm.

Let \(\mathcal{B} \subset \mathbb{R}^{k \times k}_{\text{up}}\) be the subset of upper triangular matrices with non-zero diagonal elements, i.e., the subset of invertible upper triangular matrices. \(\mathcal{B}\) has full Lebesgue measure and is used in our proof for initial vectors for the backward phase of Ginelli’s algorithm.

Now, let \(((x_1, \ldots, x_k), R) \in \mathcal{F} \times \mathcal{B}\) be an input for Ginelli’s algorithm. According to theorem 5.5 the first set of vectors \((x_1, \ldots, x_{d_1})\) gives us an approximation of \(Y_1(\omega)\) via the first step of Ginelli’s algorithm. The remaining steps of Ginelli’s algorithm do not change this approximation. In fact, the first set of output vectors \((G_{\omega, j}^{n_1, n_2})_1\) for \(j = 1, \ldots, d_1\) at \(((x_1, \ldots, x_k), R)\) spans the same space as
Thus, we have

\[
\limsup_{N \to \infty} \sup_{n_1, n_2 \geq N} \frac{1}{\min(n_1, n_2)} \log d_G \left( \text{span} \left\{ \left( G^{n_1, n_2}_{\omega, j} \right)_{j=1}^{d_1} \right\}, Y_1(\omega) \right)
\]

\[
\leq -|\lambda_1 - \lambda_2|
\]

\[
= - \min(|\lambda_1 - \lambda_0|, |\lambda_1 - \lambda_2|)
\]

at \((x_1, \ldots, x_k, R)\).

Convergence of the remaining spaces is due to lemma 5.7. Fix some \(1 < i \leq l\). We set \(Y_1 = Y_1(\omega) \oplus \cdots \oplus Y_{i-1}(\omega), V_1 = V_i(\omega), Y_2 = Y_i(\omega), V_2 = V_{i+1}(\omega), L(-n) = L^{(n)}_{\sigma - \omega}, L(n) = L^{(n)}_{\omega}, \) and spaces \(Y_j(\pm n)\) and \(V_j(\pm n)\) for \(j = 1, 2\) accordingly. The growth rates in lemma 5.7 are given by uniform bounds obtained from theorem 3.1 and its proof. Furthermore, let \(W_1 = \text{span}(x_{i_1}, \ldots, x_{(i-1)d_{i-1}})\) and \(W_2 = \text{span}(x_{i_1}, \ldots, x_{i_d})\) be the well-separating common complements, which approximate \(Y_1\) and \(Y_2\) in the first step of Ginelli’s algorithm. The family of spaces \((W(n_1, n_2))_{n_1, n_2 \in \mathbb{N}}\) is given by \(\text{span}(y_{i_1}^1, \ldots, y_{i_d}^1)\) via vectors of the fourth step of the algorithm. Indeed, the \(i_1\)th to \(i_d\)th column

\[
[r_{i_1} \cdots r_{i_d}] := \begin{bmatrix}
* & \cdots & * \\
\vdots & & \vdots \\
* & \cdots & * \\
r_{i_1, i_1} & \cdots & r_{i_1, i_d} \\
\ddots & \cdots & \ddots \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix}
\]

of \(R\) give us coefficients with which we may express \(y_{i_1}^1, \ldots, y_{i_d}^1\) in terms of the orthonormalized vectors

\[
\text{orth}(L(n_2) L(-n_1) x_{i_1}, \ldots, L(n_2) L(-n_1) x_{i_d})
\]

\[
= \text{orth} \left( L^{(n_1+n_2)}_{\sigma - \omega} x_{i_1}, \ldots, L^{(n_1+n_2)}_{\sigma - \omega} x_{i_d} \right),
\]

that emerge in the third step of Ginelli’s algorithm. By means of this orthogonal transformation between coefficients and initial vectors, eq. 15 may be checked on coefficient space. Since \(L(n_2) L(-n_1) W_1\) is mapped to \(\mathbb{R}^{d_1+i_d-i_1} \times \{0\} \subset \mathbb{R}^k\) and \(L(n_2) L(-n_1) W_2\) to \(\mathbb{R}^{d_1+i_d-i_2} \times \{0\} \subset \mathbb{R}^k\), we need to check that

\[
\inf_{r \in \text{span}(r_{i_1} \cdots r_{i_d}) \cap S} \|pr_r\|_2 > 0,
\]

where \(pr_r : \mathbb{R}^k \to \{0\} \times \mathbb{R}^{d_1} \times \{0\}\) is the projection onto the \(i_1\)th to \(i_d\)th coordinates. This is easily verified, since \(R\) is an upper triangular matrix with non-zero elements on the diagonal. Thus, we may apply lemma 5.7 to see that the linear span of the
i_{1}^{th} to i_{d}^{th} vectors from the fifth step of Ginelli’s algorithm approximates $Y_{i}(\omega)$ at the desired speed. This concludes the proof.\footnote{The last step of Ginelli’s algorithm only normalizes computed vectors. It does not change their linear span and, thus, does not play a role in eq. 9. However, the step is a necessary part of the algorithm, since CLVs are defined as normalized basis vectors of $Y_{i}(\omega)$.}

6. Conclusions. With the emergence of semi-invertible METs, the concept of CLVs has been opened up to new settings. In particular, various infinite-dimensional versions of the METs have been proved. In this article we analyzed convergence of Ginelli’s algorithm to compute CLVs in the setting of the semi-invertible MET from [19]. Our main result is a convergence proof of the algorithm in the context of Hilbert spaces. The proof not only generalizes the previous convergence proof [25] to an infinite-dimensional setting, but also treats the case of non-invertible linear propagators. We formulated most arguments for maps on Banach spaces before connecting them to basic asymptotic properties of the Oseledets splitting. Since those properties appear in most versions of the MET, our convergence proof can be translated to other settings as well.

An important tool in our convergence proof were so-called well-separating common complements [26]. Those are subspaces that stay well-separated from a given sequence of subspaces. In particular, we applied this concept to the case where the sequence of subspaces is given by the Oseledets filtration at different initial times. Then, we used vectors subject to the obtained well-separating common complements as input vectors for Ginelli’s algorithm. Since almost every tuple of input vectors spans a well-separating common complement, describing convergence with respect to such complements is sufficient in Hilbert spaces.

The actual convergence proof was split into estimates for forward and for backward propagation. During forward propagation, almost every complement to spaces of the Oseledets filtration asymptotically aligns with the sum of the first Oseledets spaces. The fact that complements generically align in forward-time even holds if we only have an Oseledets filtration. For backward propagation, we had to restrict the propagator to bundles of certain subspaces, since it may not be globally invertible in a semi-invertible setting. Last but not least, we combined our estimates to form the convergence proof.

Throughout the proof, we connected estimates to the LEs. Thus, we were able to relate LEs to the speed of convergence. As in the finite-dimensional case, Ginelli’s algorithm converges exponentially fast with a rate given by the spectral gap between associated LEs.

While we successfully generalized and proved Ginelli’s algorithm for infinite dimensions, it is foremost an analytical tool. The numerical computation of CLVs brings its own set of challenges. Indeed, our results may be seen as a help to understand limit cases of applications of Ginelli’s algorithm for systems of increasingly higher resolutions. The transition between finite and infinite dimensions is still an open question and leads to the concept of stability of CLVs. Additionally, numerical inaccuracies in computing the linear propagator can result in a different output of Ginelli’s algorithm. We remind that CLVs may depend only measurably on the trajectory.

Despite the remaining challenges, we made a big step towards understanding the computation of CLVs in infinite dimensions. Through the connection to semi-invertible METs, our research applies to recent developments in the context of CLVs.
and paves the way for new advancements of both analytical and numerical aspects of CLV-algorithms.

Acknowledgments. This paper is a part of my dissertation [24] and a contribution to the project M1 (Instabilities across scales and statistical mechanics of multi-scale GFD systems) of the Collaborative Research Centre TRR 181 “Energy Transfer in Atmosphere and Ocean” funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 274762653.

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Received for publication August 2020; early access July 2021.

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