ON A FAMILY OF CALDERO-CHAPOTON ALGEBRAS THAT HAVE THE LAURENT PHENOMENON

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Abstract. We realize a family of generalized cluster algebras as Caldero-Chapoton algebras of quivers with relations. Each member of this family arises from an unpunctured polygon with one orbifold point of order 3, and is realized as a Caldero-Chapoton algebra of a quiver with relations naturally associated to any triangulation of the alluded polygon. The realization is done by defining for every arc $j$ on the polygon with orbifold point a representation $M(j)$ of the referred quiver with relations, and by proving that for every triangulation $\tau$ and every arc $j \in \tau$, the product of the Caldero-Chapoton functions of $M(j)$ and $M(j')$, where $j'$ is the arc that replaces $j$ when we flip $j$ in $\tau$, equals the corresponding exchange polynomial of Chekhov-Shapiro in the generalized cluster algebra. Furthermore, we show that there is a bijection between the set of generalized cluster variables and the isomorphism classes of $E$-rigid indecomposable decorated representations of $\Lambda$.

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1. Introduction

The cluster algebras of Sergey Fomin and Andrei Zelevinsky have pervaded several areas of Mathematics and even Physics in the past 15 years. They are defined through a recursive process by iterating an algebraic-combinatorial operation called cluster mutation. Each cluster mutation produces a rational function by means of an exchange binomial dictated by a skew-symmetrizable matrix. Cluster algebras satisfy the remarkable Laurent phenomenon: all of the rational functions obtained during the process can be expressed as Laurent polynomials, cf [19, 21].

In [13], Leonid Chekhov and Michael Shapiro have defined the notion of generalized cluster algebra. The cluster mutation rule inside a generalized cluster algebra allows exchange polynomials to not be binomials, in contrast to cluster algebras, where all exchange polynomials are required to be binomials. Generalized cluster algebras, to which we shall often refer as Chekhov-Shapiro algebras, possess the remarkable Laurent phenomenon. Chekhov-Shapiro have shown that the “Lambda length coordinate ring” of a surface with marked points and arbitrary-order orbifold points carries a natural structure of Chekhov-Shapiro algebra, thus generalizing previous works of Penner, Fomin-Shapiro-Thurston and Felikson-Shapiro-Tumarkin. Thomas Lam and Pavlo Pylyavskyy have shown that the generalized cluster algebras of Chekhov-Shapiro fit into a more general framework of Laurent phenomenon algebras (LP-algebras for short).

Works of Caldero-Chapoton, Caldero-Chapoton-Schiffler, Derksen-Weyman-Zelevinsky, Palu, Plamondon and Schiffler, among others, have established a very fruitful representation-theoretic approach to cluster algebras through representations of quivers. By now, it is very well known that every cluster monomial in a skew-symmetric cluster algebra can be expressed as the Caldero-Chapoton function of a representation of a suitable quiver, and that, for a non-degenerate quiver with potential \((Q, S)\), the Caldero-Chapoton algebra of \(Q\) sits between the cluster algebra and the upper cluster algebra of \(Q\). In this paper we prove that the generalized cluster algebra associated by Chekhov-Shapiro to a polygon with one orbifold point of order 3 is equal to the Caldero-Chapoton algebra of a quiver with relations naturally arising from the referred polygon.

Let us to give a more specific context for the paper. Let \(Q\) be a quiver of type ADE, and let \(\Lambda = \mathbb{C}(Q)\) be the path algebra of \(Q\) over \(\mathbb{C}\). In [8], Caldero-Chapoton defined a Laurent polynomial \(C_\Lambda(M)\) (Caldero-Chapoton function for us) for every representation \(M\) of \(\Lambda\) and showed that there is a bijection, given by \(C_\Lambda(-)\), between the cluster variables of the cluster algebra \(A_Q\) and the indecomposable representations of \(\Lambda\). Recall that the cluster algebras are defined by an inductive procedure called cluster mutation, see [19, 21], but the Caldero-Chapoton functions are defined without an inductive procedure. The coefficients of \(C_\Lambda(M)\) are given by the complex Euler characteristics of Grassmannians of submodules of quiver representations.

The class of cluster algebras which cluster variables have been described with this idea has been extended, for example see [17, 24, 34, 36].

In [10] the authors define the Caldero-Chapoton algebra associated to a basic algebra \(\Lambda\) and take account the strongly reduced irreducible components \(\text{decIrrt}(\Lambda)\) of decorated representations of \(\Lambda\). With this data they define a generic cluster structure on \(\text{decIrrt}(\Lambda)\), namely they define the component clusters and CC-clusters of \(\Lambda\), as generalizations of clusters and cluster variables in cluster algebras, respectively. The strongly reduced components were introduced in [24]. In [37] the strongly reduced components are parametrized in terms of the \(g\)-vector of irreducible components for finite-dimensional algebras. In [10] the results of [37] are generalized for basic algebras.

These algebras with cluster structures have been related with surfaces in different contexts. Recently Charles Paquette and Ralf Schiffler introduced some notion of generalized cluster algebra that is different from the one mentioned above. In [18], cluster algebras are related with surfaces with marked points and orbifold points of order two. In [39], non-orientable surfaces are related with LP-algebras. In [13], some surfaces with arbitrary many orbifold points of arbitrary order are related with cluster generalized algebras. The interaction between the cluster algebraic structures and cluster combinatorial structures have been extensively studied. In this paper we show one more of those interactions.

In order to write our main result, see Theorem 1.1, let us introduce some notation. Let \(\Sigma_n\) be a \((n+1)\)-polygon with one orbifold point of order three. For any triangulation \(\tau\) of \(\Sigma_n\) we define an algebra \(\Lambda(\tau)\). This algebra is given by a quiver associated to \(\tau\) and relations given by the internal triangles of the triangulation. Our main result is:

**Theorem 1.1.** For any triangulation \(\tau\) of \(\Sigma_n\) the Caldero-Chapoton algebra of \(\Lambda(\tau)\) is a Chekhov-Shapiro algebra naturally associated to \(\Sigma_n\).
The paper is organized as follows. In the first five sections we recall some facts about Caldero-Chapoton algebras, Galois $G$-covering functors, string algebras, cluster generalized algebras and surfaces with marked points and orbifold points of order 3 that we need for state and prove our results. In Section 3 we recall the non-standard definition of basic algebra introduced in [10] and its representations. We also recall some facts about Galois $G$-coverings that will be crucial to deal with the $E$-invariant and $g$-vectors of arc representations with respect to an arbitrary triangulation of $\Sigma_n$. Section 4 is dedicated to recall the definition of Caldero-Chapoton algebra, see Example 4.5. In Section 5 we recall some results of [7] about string algebras, the concept of string is useful for us because the algebra $\Lambda(\tau)$ associated to a triangulation $\tau$ of $\Sigma_n$ is a string algebra. In Section 6 we recall the definition of Chekhov-Shapiro algebras introduced in [13], the reader can find interesting relations of these algebras with cluster algebras in [32, 33]. In Section 7 we recall some definitions and facts about surfaces with marked points and orbifold points. Also we define a natural Jacobian algebra for any triangulation denoted by $\Lambda(\tau)$.

The goal of Section 8 is to present the $\Lambda(\tau)$ as an orbit Jacobian algebra, see [35] and, give the definition of the arc representations of $\Lambda(\tau)$, which are (decorated) $\Lambda(\tau)$-modules that one can associate to arbitrary arcs on $\Sigma_n$. In Section 9 we study the arc representations of $\Lambda(\tau)$ for a specific choice of triangulation $\tau_0$; we show that on these representations the action of the Auslander-Reiten translation of $\Lambda(\tau_0)$ is given by rotation$^1$, compute their $g$-vectors, show that they are $E$-rigid, and prove that their orbits are dense in their respective irreducible components.

In Section 10 we show that the results of Section 9 hold for any triangulation $\tau$ of $\Sigma_n$ and not only for the specific triangulation $\tau_0$. Here, Galois coverings, both of quivers and surfaces, come into play.

In Section 11 we state and prove our main result for any triangulation $\tau$. We close the paper with Section 12, which contains an example with explicit computations illustrating our main result; the example can be thought of as a complement to [10, Example 9.4.2].

Sections 9 and 10 deserve a few words: all the results in Section 9, about the $E$-invariant, are particular cases of results from Section 10. We have decided to not omit 9, and rather present a particular instance followed by the general treatment, because the Galois covering techniques used in Section 10 are not needed when establishing the same results for the specific triangulation $\tau_0$ chosen in 9.

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3. Background

In this section we fix notation and we recall some basic definitions and facts about algebras and quiver representations that we will use throughout the work. The reader can find more details in [10].

A quiver $Q = (Q_0, Q_1, t, h)$ consists of a finite set of vertices $Q_0$, a finite set of arrows $Q_1$ and two maps $t, h : Q_1 \to Q_0$ (tail and head). For each $a \in Q_1$ we write $a : t(a) \to h(a)$. If $Q_0 = \{1, \ldots, n\}$, we define the skew-symmetric matrix $C_Q = (c_{i,j}) \in \text{Mat}_{n \times n}(\mathbb{Z})$ from $Q$ as follows

$$c_{i,j} = |\{a \in Q_1 : h(a) = i, t(a) = j\}| - |\{a \in Q_1 : h(a) = j, t(a) = i\}|.$$ 

We say that a sequence of arrows $\alpha = a_1a_{l-1} \cdots a_2a_1$, is a path of $Q$ if $t(a_{k+1}) = h(a_k)$, for $k = 1, \ldots, l - 1$, in this case, we define the length of $\alpha$ as $l$. We say that $\alpha$ is a cycle if $h(a_l) = t(a_1)$. In this work we deal with quivers with loops, that is, quivers where there is an arrow $a \in Q_1$ such that $h(a) = t(a)$.

For $m \in \mathbb{N}$ let $C_m$ be the set of paths of length $m$ and let $CC_m$ be the vector space with basis $C_m$. The path algebra of a quiver $Q$ is denoted by $\mathbb{C}(Q)$ and it is defined as $\mathbb{C}$-vector space as

$$\mathbb{C}(Q) = \bigoplus_{m \geq 0} CC_m,$$

where the product is given by the concatenation of paths. The completed path algebra of a quiver $Q$ is defined as vector space as

$$\mathbb{C}(\langle Q \rangle) = \prod_{m \geq 0} CC_m,$$

$^1$In sync with previous results of Brüstle-Qiu [6] and Caldero-Chapoton-Schiffler [9].
where the elements are written as infinite sums $\sum_{m \geq 0} x_m$ with $x_m \in \mathbb{C}C_m$ and the product in $\mathbb{C}\langle \langle Q \rangle \rangle$ is defined as

$$(\sum_{l \geq 0} b_l)(\sum_{m \geq 0} a_m) = \sum_{k \geq 0} \sum_{l+m=k} b_la_m.$$ 

Let $\mathfrak{M} = \prod_{a \geq 1} \mathbb{C}C_m$ be the two-sided ideal of $\mathbb{C}\langle \langle Q \rangle \rangle$ generated by arrows of $Q$. Then $\mathbb{C}\langle \langle Q \rangle \rangle$ can be viewed as a topological $\mathbb{C}$-algebra with the powers of $\mathfrak{M}$ as a basic system of open neighborhoods of 0. This topology is known as $\mathfrak{M}$-adic topology. Let $I$ be a subset of $\mathbb{C}\langle \langle Q \rangle \rangle$, we can calculate the closure of $I$ as $I = \bigcap_{l \geq 0} (I + \mathfrak{M}^l)$.

A two-sided ideal $I$ of $\mathbb{C}\langle \langle Q \rangle \rangle$ is semi-admissible if $I \subseteq \mathfrak{M}^2$ and it is admissible if some power of $\mathfrak{M}$ is a subset of $I$. Following [10] we call an algebra $\Lambda$ basic if $\Lambda = \mathbb{C}\langle \langle Q \rangle \rangle/I$ for some quiver $Q$ and some semi-admissible ideal $I$.

A finite-dimensional representation of $Q$ over $\mathbb{C}$ is a pair $((M_i)_{i \in Q_0}, (M_a)_{a \in Q_1})$ where $M_i$ is a finite-dimensional $\mathbb{C}$-vector space for each $i \in Q_0$ and $M_a : M_{t(a)} \to M_{h(a)}$ is a $\mathbb{C}$-linear map. Here the word representation means finite-dimensional representation.

The dimension vector of a representation $M$ of $Q$ is given by $\dim(M) = (\dim(M_1), \ldots, \dim(M_n))$. We define $\dim(M) = \sum_{i=1}^n \dim(M_i)$ as the dimension of $M$. We say $M$ is a nilpotent representation if there is an $n > 0$ such that for every path $a_n a_{n-1} \ldots a_1$ of length $n$ in $Q$ we have $M_{a_n} M_{a_{n-1}} \cdots M_{a_1} = 0$. A subrepresentation of $M$ is an $n$-tuple of $\mathbb{C}$-vector spaces $N = (N_i)_{i \in Q_0}$ such that $N_i \subseteq M_i$ for each $i \in Q_0$ and $M_a(N_{t(a)}) \subseteq N_{h(a)}$ for every $a \in Q_0$.

We denote by $\text{nil}_C(Q)$ the category of nilpotent representations of $Q$, and by $\mathbb{C}\langle \langle Q \rangle \rangle$-mod the category of finite-dimensional left $\mathbb{C}\langle \langle Q \rangle \rangle$-modules. It is known that the category of representations of $Q$ and the category of $\mathbb{C}\langle \langle Q \rangle \rangle$-modules are equivalent. In [17, Section 10] it was observed that $\text{nil}_C(Q)$ and $\mathbb{C}\langle \langle Q \rangle \rangle$-mod are equivalent.

Given a basic algebra $\Lambda = \mathbb{C}\langle \langle Q \rangle \rangle/I$ we define a representation of $\Lambda$ as a nilpotent representation of $Q$ which is annihilated by $I$. We consider the category $\text{mod}(\Lambda)$ of finite-dimensional left modules as the category $\text{rep}(\Lambda)$ of representations of $\Lambda$.

Let $\Lambda = \mathbb{C}\langle \langle Q \rangle \rangle/I$ be a basic algebra. We say $M = (M, V)$ is a decorated representation of $\Lambda$ if $M$ is a representation of $\Lambda$ and $V = (V_1, \ldots, V_n)$ is an $n$-tuple of finite-dimensional $\mathbb{C}$-vector spaces. We can think on $V$ as a representation of a quiver with no arrows and no vertices. Let $\text{dec}(\Lambda)$ be the category of decorated representations of $\Lambda$.

Let $M = (M, V)$ be a decorated representation of $\Lambda$. If $V = 0$, we write $M$ instead $\mathcal{M}$. For $i \in \{1, \ldots, n\}$ we define the negative simple representation of $\Lambda$ as $S_i^- = (0, S_i)$ where $(S_i)_j \in \mathbb{C}$ if $j = i$ and $(S_i)_j = 0$ in other wise.

For a representation $M = ((M_i)_{i \in Q_0}, (M_a)_{a \in Q_1})$ of $\Lambda$ and a vector $e \in \mathbb{N}^n$ let $\text{Gr}_e(M)$ be the quiver Grassmannian of subrepresentations $N$ of $M$ such that $\dim(N) = e$. We denote the Euler characteristic of $\text{Gr}_e(M)$ by $\chi(\text{Gr}_e(M))$. About Euler characteristic we are going to need the following result, see [5].

Lemma 3.1 (Białynicki-Birula). Let $T$ be an algebraic torus acting on an algebraic variety $X$. If we denote by $X^T$ the set of fixed points of the action, then $\chi(X^T) = \chi(X)$.

The following definition plays a crucial role in some computations that will be involved later. It was introduced in [10, Section 2.4].

Definition 3.2. Given a basic algebra $\Lambda = \mathbb{C}\langle \langle Q \rangle \rangle/I$ and $p \geq 2$ we say

$$\Lambda_p = \mathbb{C}\langle \langle Q \rangle \rangle/(I + \mathfrak{M}^p)$$

is the $p$-truncation of $\Lambda$.

We are going to need some basic definitions about quivers with potential, for all details the reader can see [16]. Let $Q$ be a quiver. We say that $S \in \mathbb{C}\langle \langle Q \rangle \rangle$ is a potential for $Q$ if $S$ is a, possibly infinite, $\mathbb{C}$-linear combination of cycles in $Q$. Given two potentials $S$ and $W$ we say that they are cyclically equivalent and write $S \sim_{\text{cyc}} W$, if $S - W$ is in the closure of the sub-vector space of $\mathbb{C}\langle \langle Q \rangle \rangle$ generated by all elements of the form $a_1a_2 \cdots a_n-a_2 \cdots a_n a_1 a_1$, with $a_1a_2 \cdots a_n a_1$ a cycle on $Q$.

Definition 3.3. We say $(Q, S)$ is a quiver with potential (QP) if $S$ is a potential for $Q$ and if any two different cycles appearing with non-zero coefficient in $S$ are not cyclically equivalent.
Given an arrow $a \in Q_1$ and a cycle $a_1a_{n_1-1} \cdots a_1$ in $Q$, define the cyclic derivative of $a_1a_{n_1-1} \cdots a_1$ with respect to $a$ as follows:

$$\delta_a(a_1a_{n_1-1} \cdots a_1) = \sum_{k=1}^{n} \delta_{a_1a_{k-1}a_k \cdots a_{n_1}}.$$ 

We extend this definition by $\mathbb{C}$-linearity and continuity to all potentials for $Q$.

**Definition 3.4.** Let $(Q, S)$ be a quiver with potential. We define the Jacobian ideal $\mathcal{J}(Q, S)$ as the closure of the ideal on $\mathbb{C}\langle Q \rangle$ generated by all cyclic derivatives $\delta_a(S)$ with $a \in Q_1$. The quotient $\mathbb{C}\langle Q \rangle/\mathcal{J}(Q, S)$ is called the Jacobian algebra of $(Q, S)$ and is denoted as $\mathcal{P}(Q, S)$.

### 3.1. Varieties of representations.

Let $\Lambda = \mathbb{C}\langle Q \rangle/I$ and $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ be a basic algebra and a vector of no negative integers. The representations $M$ of $\Lambda$ with dimension vector $d$ can be seen as points of the affine space

$$\text{rep}_d(Q) = \prod_{a \in Q_1} \text{Hom}_\mathbb{C}(\mathbb{C}^{d_1(a)}, \mathbb{C}^{d_2(a)}).$$

Now, let $\text{rep}_d(\Lambda)$ be the Zariski closed subset of $\text{rep}_d(Q)$ given by the representations $N$ of $\Lambda$ with $\dim(N) = d$.

In $\text{rep}_d(\Lambda)$ we have the action of $G_d = \prod_{a \in Q_0} GL(\mathbb{C}^{d_1})$ by conjugation. If $g = (g_1, \ldots, g_n) \in G_d$ and $M = (\{M_i\}_{i \in Q_0}, \{M_a\}_{a \in Q_1}) \in \text{rep}_d(\Lambda)$, then

$$g \cdot M = ((M_i)_{i \in Q_0}, (g^{-1}_a M_a g(a))_{a \in Q_1}).$$

From definitions follows that the isomorphism classes of representations of $\Lambda$ with dimension vector $d$ are in bijection with the $G_d$-orbits in $\text{rep}_d(\Lambda)$. If $M \in \text{rep}_d(\Lambda)$, its $G_d$-orbit is denoted by $\mathcal{O}(M)$. For $(d, v) \in \mathbb{N}^n \times \mathbb{N}^n$ let $\text{decrep}_d, v(\Lambda)$ be the decorated representations variety of $\Lambda$. If $v = (v_1, \ldots, v_n)$, then

$$\text{decrep}_d, v(\Lambda) = \text{rep}_d(\Lambda) \times \{\mathbb{C}^{v_1}, \ldots, \mathbb{C}^{v_n}\}.$$

We have an action of $G_d$ on $\text{decrep}_d, v(\Lambda)$ given by $g \cdot M = (g \cdot M, V)$ where $M = (M, V) \in \text{decrep}_d, v(\Lambda)$ and $g \in G_d$.

### 3.2. Galois $G$-covering.

In this section we are going to make a reminder of Galois $G$-covering. For a nice review of this theory the reader can see the introductions of [1, 4]. For our convenience we are going to present some results from [4].

In this section $G$ will denote a finite group (in the general theory this assumption is not required). A category $\mathcal{A}$ is $\mathbb{C}$-linear or a $\mathbb{C}$-category if it is a category whose set of morphisms are $\mathbb{C}$-modules and the composition of morphisms is $\mathbb{C}$-linear. We assume that we have a morphism $\rho: G \to \text{Aut}(\mathcal{A})$ from $G$ to the group of automorphisms of $\mathcal{A}$, not the group of auto-equivalences. That means we have an action of $G$ on $\mathcal{A}$. We will abuse of notation and we will write $g$ instead of $\rho(g): \mathcal{A} \to \mathcal{A}$ for every $g \in G$. The action of $G$ on $\mathcal{A}$ is called free provided $g \cdot X$ is not isomorphic to $X$ for every non-trivial element $g \in G$ and for any indecomposable object $X$ of $\mathcal{A}$. The next definitions are due to Asashiba, see [1, Definition 1.1, Definition 1.7].

**Definition 3.5.** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{C}$-categories with $G$ acting on $\mathcal{A}$. A functor $F: \mathcal{A} \to \mathcal{B}$ is called $G$-stable if there exist a functorial isomorphism $\delta_g: Fg \to F$ such that $\delta_{gh} \delta_{g}X = \delta_{gh}X$ for any $g, h \in G$ and any object $X$ in $\mathcal{A}$. In this case $\delta = (\delta_g)_{g \in G}$ is called a $G$-stabilizer. If $\delta_g = \text{id}_F$ for every $g \in G$, we say that $F$ is $G$-invariant.

$$\xymatrix{ Fgh \ar[r]^= \ar[dr]_{\delta_{gh}} & Fh \\
Fh \ar[u]_{\delta_{gh}} & Fh \ar[u]_{\delta_{gh}} }$$

**Definition 3.6.** Let $\mathcal{A}, \mathcal{B}$ be $\mathbb{C}$-categories with a group $G$ acting on $\mathcal{A}$. Let $F: \mathcal{A} \to \mathcal{B}$ be a functor with stabilizer $\delta$. 

$$\xymatrix{ Fgh \ar[r]^= \ar[dr]_{\delta_{gh}} & Fh \\
Fh \ar[u]_{\delta_{gh}} & Fh \ar[u]_{\delta_{gh}} }$$
We say that \( \mathcal{F} \) is a G-precovering if the following maps are isomorphisms for any \( X, Y \) objects in \( \mathcal{A} \):

\[
\mathcal{F}_{X,Y} : \bigoplus_{g \in G} \mathcal{A}(X, g \cdot Y) \rightarrow \mathcal{B}(\mathcal{F}(X), \mathcal{F}(Y)); (u_g)_{g \in G} \mapsto \sum_{g \in G} \delta_{g,Y} \mathcal{F}(u_g),
\]

\[
\mathcal{F}^{X,Y} : \bigoplus_{g \in G} \mathcal{A}(g \cdot X, Y) \rightarrow \mathcal{B}(\mathcal{F}(X), \mathcal{F}(Y)); (v_g)_{g \in G} \mapsto \sum_{g \in G} \mathcal{F}(v_g)\delta_{g^{-1},X}.
\]

(2) A G-precovering \( \mathcal{F} \) is called a Galois G-covering if \( \mathcal{F} \) has the following three conditions:

(i) The functor \( \mathcal{F} \) is almost dense. It means that any indecomposable object of \( \mathcal{B} \) is isomorphic to someone in the image of \( \mathcal{F} \).

(ii) If \( X \) is indecomposable in \( \mathcal{A} \), then \( \mathcal{F}(X) \) is indecomposable in \( \mathcal{B} \).

(iii) For any indecomposable objects \( X, Y \) in \( \mathcal{A} \) such that \( \mathcal{F}(X) \cong \mathcal{F}(Y) \), there exist \( g \in G \) such that \( g \cdot X \cong Y \).

**Remark 3.7.**

(1) In [1, Proposition 1.6] is proved that \( \mathcal{F}^{X,Y} \) is an isomorphism if and only if \( \mathcal{F}_{X,Y} \) is an isomorphism. Note that a G-precovering is a faithful functor, see [4, Lemma 2.6].

(2) In Krull-Schmidt categories a functor is almost dense if and only if it is dense.

The following lemma allows us to find examples of a Galois G-covering from a G-precovering between module categories, see [4, Lemma 2.9]

**Lemma 3.8.** Let \( \mathcal{A}, \mathcal{B} \) be Krull-Schmidt \( \mathbb{C} \)-categories with a group \( G \) acting freely on \( \mathcal{A} \) and let \( \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B} \) be a G-precovering. Assume \( X \) is an object in \( \mathcal{A} \) such that \( \text{End}_{\mathcal{A}}(X) \) is local and it has nilpotent radical. Then \( \text{End}_{\mathcal{B}}(\mathcal{F}(X)) \) is local with nilpotent radical and if \( Y \) is an object in \( \mathcal{A} \) such that \( \mathcal{F}(X) \cong \mathcal{F}(Y) \), there exist \( g \in G \) such that \( g \cdot X \cong Y \).

The next theorem shows an interesting application of Galois G-covering in Auslander-Reiten theory, see [4, Theorem 3.7].

**Theorem 3.9 (Bautista-Liu, 2014).** Let \( \mathcal{A}, \mathcal{B} \) be Krull-Schmidt \( \mathbb{C} \)-categories with a group \( G \) acting freely on \( \mathcal{A} \) and let \( \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B} \) be a Galois G-covering. Then

(1) A short exact sequence \( \eta \) in \( \mathcal{A} \) is almost split if and only if \( \mathcal{F}(\eta) \) is almost split.

(2) An object \( X \) in \( \mathcal{A} \) is the starting or ending term of an almost split sequence if and only if \( \mathcal{F}(X) \) is the starting or ending term of an almost split sequence, respectively.

Given a \( \mathbb{C} \)-algebra \( \Lambda \) we consider it as a \( \mathbb{C} \)-category in the usual way, in other words, objects in \( \Lambda \) are a complete family of orthogonal and primitive idempotents and the set of morphisms is given by \( \Lambda(e_i, e_j) = e_j \Lambda e_i \). In this context, the category of left \( \Lambda \)-modules \( \Lambda \)-mod can be considered as the category of functors from \( \Lambda \) to \( \mathbb{C} \)-mod. An action of a group \( G \) on \( \Lambda \) induces an action of \( G \) on \( \Lambda \)-mod in the following way. Given an \( \Lambda \)-module \( \mathcal{M} : \Lambda \rightarrow \mathbb{C} \)-mod we define \( g \cdot \mathcal{M} := \mathcal{M}g^{-1} \), remember that \( g \) is thought as an automorphism of \( \Lambda \); for a morphism \( u : \mathcal{N} \rightarrow \mathcal{M} \) of \( \Lambda \)-mod, we define \( g \cdot u(x) = u(g^{-1}x) \) for \( x \) an object of \( \Lambda \). So if we have a G-precovering \( \pi : \Lambda \rightarrow \mathcal{A} \), Gabriel and Bongarzt defined the push-down functor \( \pi_* : \Lambda \text{-mod} \rightarrow \mathcal{A} \text{-mod} \).

We have a nice property for \( \pi_* \), see [4, Lemma 6.3].

**Lemma 3.10.** Let \( \Lambda, \mathcal{A} \) be finite dimensional \( \mathbb{C} \)-algebras with a group \( G \) acting on \( \Lambda \). Assume the action of \( G \) is free. If \( \pi : \Lambda \rightarrow \mathcal{A} \) is a Galois G-precovering, then the push-down functor \( \pi_* \) admits a G-stabilizer \( \delta \).

With the following lemma we can construct a G-precovering from Galois a G-covering, see [4, Theorem 6.5].

**Lemma 3.11.** Let \( \Lambda, \mathcal{A} \) be finite dimensional \( \mathbb{C} \)-algebras with a group \( G \) acting on \( \Lambda \). Assume the action of \( G \) is free. If \( \pi : \Lambda \rightarrow \mathcal{A} \) is a Galois G-precovering, then

\[ \pi_* : \Lambda \text{-mod} \rightarrow \mathcal{A} \text{-mod} \]

is a G-precovering.

**Remark 3.12.** For convenience we are going to define explicitly the push-down functor. Let \( \Lambda = \mathbb{C}(Q)/I \) be a finite dimensional algebra. Suppose a finite group \( G \) is acting freely on \( \Lambda \). Let \( \pi : \Lambda \rightarrow \Lambda_G \) the canonical projection of this action. Assume \( \pi \) is a Galois G-covering. We define the push-down functor \( \pi_* : \Lambda \text{-mod} \rightarrow \Lambda_G \text{-mod} \) as follows.

For objects: let \( M \in \Lambda \text{-mod} \) be a \( \Lambda \)-representation. For \( i \in Q_0 \) we define \( \pi_* (M)_{G,i} = \bigoplus_{g \in G} M_{g \cdot i} \). Let \( \alpha : j \rightarrow i \) be an arrow of \( Q \). We are going to define \( \pi_* (M)_{G \alpha} : \bigoplus_{g \in G} M_{g \cdot j} \rightarrow \bigoplus_{h \in G} M_{h \cdot i} \). Now, by definition,
for any \( h \in G \) we have an isomorphism \( \pi_{j,h,i} : \bigoplus_{g \in G} \Lambda(g \cdot j, h \cdot i) \to \Lambda_{G}(G \cdot i, G \cdot j) \). So \( G \cdot \alpha = \sum_{g \in G} \pi(\alpha_{h,g}) \) for any \( h \in G \) and we define \( \pi_{a}(\alpha)_{M,a} = (\alpha_{h,g})_{h \in G} \).

For morphisms: let \( f : M \to N \) be a morphism in \( \Lambda \)-mod. For any \( i \in Q_{0} \) we need to define \( \pi_{a}(f)_{G,i} : \bigoplus_{g \in G} M_{g,i} \to \bigoplus_{h \in G} N_{h,i} \). We define \( \pi_{a}(f)_{G,i} = \text{diag}(f_{g,i} : g \in G) \) as a diagonal map.

4. E-invariant and Caldero-Chapoton functions

In this section we recall some definitions that we work with. These definitions were introduced in [10, Section 3.4]. They were motivated by the theory of mutation of quivers with potential developed in [17] and the Caldero-Chapoton functions introduced in [8]. In this section let \( \Lambda = \mathbb{C} \langle \langle Q \rangle \rangle / I \) a basic algebra.

4.1. g-vectors. For a decorated representation \( \mathcal{M} = (M, V) \) of \( \Lambda \) the \( g \)-vector of \( \mathcal{M} \) is given by \( g_{\Lambda}(\mathcal{M}) = (g_{1}, \ldots, g_{n}) \) where

\[
g_{i} := g_{i}(\mathcal{M}) = - \dim \text{Hom}_{\Lambda}(S_{i}, M) + \dim \text{Ext}_{\Lambda}^{1}(S_{i}, M) + \dim(V_{i})\]

It is clear that \( g_{\Lambda}(\mathcal{M}) \in \mathbb{Z}^{n} \). We denote by \( I_{i} \) to the injective envelope of the simple representation \( S_{i} \) in \( \Lambda \)-mod. We recall an interesting result to compute the \( g \)-vector of \( \mathcal{M} \), see [10, Lemma 3.4] for a general version.

**Lemma 4.1.** Let \( \mathcal{M} = (M, V) \) be a decorated representation of a finite dimensional algebra \( \Lambda \) and let \( g_{\Lambda}(\mathcal{M}) = (g_{1}, \ldots, g_{n}) \) be its \( g \)-vector. Assume we have a minimal injective presentation of \( \mathcal{M} \)

\[
0 \to M \to I_{0}(M) \to I_{1}(M),
\]

where \( I_{0}(M) = \bigoplus_{i=1}^{n} I_{i}^{a_{i}} \) and \( I_{1}(M) = \bigoplus_{i=1}^{n} I_{i}^{b_{i}} \). Then

\[
g_{i} = -a_{i} + b_{i} + \dim(V_{i}).
\]

4.2. The \( E \)-invariant. For decorated representations \( \mathcal{M} = (M, V) \) and \( \mathcal{N} = (N, W) \) of \( \Lambda \) let

\[
E_{\Lambda}(\mathcal{M}, \mathcal{N}) = \dim \text{Hom}_{\Lambda}(M, N) + \sum_{i=1}^{n} \dim(M_{i}) \dim(W_{i}).
\]

The \( E \)-invariant of \( \mathcal{M} \) is defined as \( E_{\Lambda}(\mathcal{M}) = E_{\Lambda}(\mathcal{M}, \mathcal{M}) \).

In [10] it was shown that the \( E \)-invariant has a homological interpretation in terms of the Auslander-Reiten translation of truncations of \( \Lambda \), see Definition 3.2.

**Proposition 4.2.** [10, Proposition 3.5]. Let \( \mathcal{M} = (M, V) \) and \( \mathcal{N} = (N, W) \) be decorated representations of \( \Lambda \). If \( p > \dim(M), \dim(N) \), then

\[
E_{\Lambda}(\mathcal{M}, \mathcal{N}) = E_{\Lambda_{p}}(\mathcal{M}, \mathcal{N}) = \dim \text{Hom}_{\Lambda_{p}}(\tau_{\Lambda_{p}}^{-}(N), M) + \sum_{i=1}^{n} \dim(M_{i}) \dim(W_{i}).
\]

This proposition is quite useful for us because the basic algebras we consider satisfy \( \Lambda_{p} = \Lambda \) for a sufficiently large \( p \).

4.3. Caldero-Chapoton functions and algebras. Let \( \mathcal{M} = (M, V) \) be a decorated representation of \( \Lambda \). For \( f = (f_{1}, \ldots, f_{n}) \in \mathbb{Z}^{n} \) by \( \mathbf{x}^{f} \) we mean \( \prod_{i=1}^{n} x_{i}^{f_{i}} \). The Caldero-Chapoton function (CC function for short) associated to \( \mathcal{M} \) of \( \Lambda \) is the Laurent polynomial in \( n \)-variables \( x_{1}, \ldots, x_{n} \) defined by

\[
C_{\Lambda}(\mathcal{M}) = \mathbf{x}^{g_{\Lambda}(\mathcal{M})} \sum_{\alpha \in \Lambda} \chi(\text{Gr}_{\alpha}(M)) \mathbf{x}^{C_{\alpha} \epsilon},
\]

where \( C_{Q} \) is defined as in Section 4. From definitions we have \( C_{\Lambda}(\mathcal{M}) \in \mathbb{Z}[x_{1}^{\pm}, \ldots, x_{n}^{\pm}] \). Note \( C_{\Lambda}(S_{i}^{-}) = x_{i} \).

The set of Caldero-Chapoton functions associated to \( \Lambda \) is

\[
C_{\Lambda} = \{C_{\Lambda}(\mathcal{M}) : \mathcal{M} \in \text{decrep}(\Lambda)\}.
\]

The next lemma was shown in [10]. It is convenient for computations of \( g \)-vectors and Caldero-Chapoton functions.

**Lemma 4.3.** [10, Lemma 4.1]. If \( \mathcal{M} = (M, V) \) and \( \mathcal{N} = (N, W) \) are decorated representations of \( \Lambda \), then the following hold:

1. \( g_{\Lambda}(\mathcal{M} \oplus \mathcal{N}) = g_{\Lambda}(\mathcal{M}) + g_{\Lambda}(\mathcal{N}) \).
2. \( C_{\Lambda}(\mathcal{M}) = C_{\Lambda}(M, 0)C_{\Lambda}(0, V) \).
(3) $C_{A}(M \oplus N) = C_{A}(M)C_{A}(N)$.

**Definition 4.4.** The Caldero-Chapoton algebra $A_{A}$ associated to $A$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}[x_{1}^{\pm}, \ldots, x_{n}^{\pm}]$ generated by $C_{A}$.

From Lemma 4.3(3) follows that $C_{A}$ generates $A_{A}$ as $\mathbb{C}$-vector space, see [10, Lemma 4.2].

**Example 4.5.** Let $Q$ be the quiver

$$
1 \xrightarrow{a_{1}} 2 \xrightarrow{a_{2}} \cdots \xrightarrow{a_{n-1}} n-1 \xrightarrow{a_{n}} n
$$

and let $A = \mathbb{C}(Q)$. For each sub-interval $e = [i, j]$ of $[1, n]$ with $i \leq j$ we define an indecomposable representation $M_{e}$ of $A$ the following way. Let $(M_{e})_{k} = \mathbb{C}$ if $k \in e$ and $(M_{e})_{k} = 0$ if $k \notin e$, for $k \in [1, n]$. For an arrow $a_{l}$ with $l \in [1, n-1]$ define $(M_{e})_{a_{l}}$ as $\text{id}_{\mathbb{C}}$ if $t(a_{l})$, $h(a_{l}) \in e$ and zero and other wise. Note that the dimension vector of $M_{e}$ can be identified with the sub-interval $e$. If $1 < i$, then we have

$$g_{A}(M_{e})_{k} = \begin{cases} -1 & \text{if } k = j, \\ 1 & \text{if } k = i-1, \\ 0 & \text{in other wise}. \end{cases}$$

If $i = 1$, then $g_{A}(M_{e})_{j} = -1$ and $g_{A}(M_{e})_{k} = 0$ for $k \neq j$. We have $E_{A}(M_{e}) = 0$ for each sub-interval $e$ of $[1, n]$. From [8, Theorem 3.4] we have $A_{A}$ can be identified with the cluster algebra associated to $Q$.

5. **String algebras**

In this section we recall some definitions and results about string algebras exposed in [7]. Let $Q$ be a quiver.

Let $P$ be a subset of paths in $\mathbb{C}(Q)$ and denote by $\langle P \rangle$ the ideal generated by $P$. The algebra $A = \mathbb{C}(Q)/\langle P \rangle$ is called a string algebra if the following conditions hold:

(1) Any vertex $i \in Q_{0}$ is the tail or head point of at most two arrows of $Q$, that is, \(|\{a \in Q : t(a) = i\}| \leq 2\) and \(|\{a \in Q : h(a) = i\}| \leq 2\).

(2) For any arrow $a \in Q_{1}$ we have \(|\{b \in Q_{1} : t(a) = h(b) \text{ and } ab \notin P\}| \leq 1\) and \(|\{c \in Q_{1} : c \in Q\} : t(c) = h(a) \text{ and } ca \notin P\}| \leq 1\).

(3) The ideal $\langle P \rangle$ is admissible on $\mathbb{C}(Q)$.

To describe the finite-dimensional indecomposable $\Lambda$-modules we need the concept of string. We introduce an alphabet consisting of *direct letters* given by each arrow $a \in Q_{1}$ and *inverse letters* given by $a^{-1}$ for each arrow $a \in Q$. The head and tail functions extend to this alphabet in the obvious way, that is, $h(a^{-1}) = t(a)$ and $t(a^{-1}) = h(a)$ for every arrow $a \in Q_{1}$. For a letter $l$ in this alphabet we denote its inverse letter with $l^{-1}$ and we write $l$ instead of $(l^{-1})^{-1}$. A word in this alphabet of length $r \geq 1$ is a sequence of letters $l_{r} \cdots l_{1}$ such that $t(l_{i+1}) = h(l_{i})$ for $i = 1, \ldots, r-1$. For a word $W = l_{r} \cdots l_{1}$ we denote its inverse word by $W^{-1} = l_{1}^{-1} \cdots l_{r}^{-1}$. It is clear we can extend the head and tail functions to words. A string of length $r \geq 1$ is a word $W = l_{r} \cdots l_{1}$ such that $W$ and $W^{-1}$ do not contain sub-words of the form $ll^{-1}$ for a letter $l$ and no sub-words of $W$ belongs to $P$.

We introduce strings of length 0 in the following way. For each vertex $i \in Q_{0}$ we have two strings of length 0 denoted by $1_{(i,i)}$ with $t \in \{1, -1\}$. In this case $h(1_{(i,i)}) = i = t(1_{(i,i)})$. By definition $1_{(i,i)}^{-1} = 1_{(i,-i)}$.

We recall the definition of two functions to deal with strings. In [7] it is shown we can choose two functions \(\sigma, \epsilon : Q_{1} \to \{1, -1\}\) such that the following conditions are satisfied

(1) If $a_{1} \neq a_{2}$ are arrows with $t(a_{1}) = t(a_{2})$, then $\sigma(a_{1}) = -\sigma(a_{2})$.

(2) If $b_{1} \neq b_{2}$ are arrows with $h(b_{1}) = h(b_{2})$, then $\epsilon(b_{1}) = -\epsilon(b_{2})$.

(3) If $a, b \in Q$ are arrows with $t(b) = h(a)$ and $ba \notin P$, then $\sigma(b) = -\epsilon(a)$.

For an arrow $a \in Q_{1}$ we have $\sigma(a^{-1}) = \epsilon(a)$ and $\epsilon(a^{-1}) = \sigma(a)$. For a string $W = l_{r} \cdots l_{1}$ we define $\sigma(W) = \sigma(l_{r})$ and $\epsilon(W) = \epsilon(l_{r})$. Besides we have $\sigma(1_{(i,i)}) = -t$ and $\epsilon(1_{(i,i)}) = t$. Note that if $W_{1}$ and $W_{2}$ are strings such that $W_{2}W_{1}$ is a string, then $\sigma(W_{2}) = -\epsilon(W_{1})$. For $(i, t) \in Q_{0} \times \{1, -1\}$ let $W_{(i, t)}$ be the set of all strings $W$ with $h(W) = i$ and $\epsilon(W) = t$. Let $W$ be the set of all strings and define on $W$ an equivalence relation given by $W_{1} \sim W_{2}$ if and only if $W_{2} \in \{W_{1}, W_{1}^{-1}\}$. Let $\overline{W}$ be a complete set of representatives of the corresponding equivalence classes.
Remark 5.1. In this article we are not going to use this functions $\epsilon$ and $\sigma$, but it is useful to remember that for string algebras the strings can be thought of as sequence of signs.

In [7] also defined the set $B$ of bands. A string $W \in \mathcal{W}$ belongs to $B$ if length of $W$ is positive, $W^n \in \mathcal{W}$ for all $n \in \mathbb{N}$ and $W$ is not the power of some string of smaller length.

5.1. **Indecomposable string modules.** For a string $W$, in [7], it was defined a $\Lambda$-module $N(W)$, for convenience we repeat this definition. For the string $l_{(i,t)}$ we define $N(l_{(i,t)})$ as the simple representation $S_l$ at the vertex $i \in Q_0$. If $W = l_r \cdots l_1$, then $N(W)$ is a representation of $\mathbb{C}$-dimension $r + 1$. For describe the structure of $\Lambda$-module let $p_0 = t(1)$ and $p_k = h(k)$ for $k = 1, \ldots, r$ vertices of $Q$. By definition $\text{dim}(N(W)_i)$ is $\{k \in [1, r + 1] : p_k = i\}$. If $\{z_0, \cdots, z_n\}$ is a basis of $N(W)$ with $z_k \in N(W)_{p_k}$ for $k = 0, \ldots, r$, then the action of the arrows is given by the following way

$$ z_0 \xrightarrow{l_1} z_1 \xrightarrow{l_2} \cdots \xrightarrow{l_{n-1}} z_{n-1} \xrightarrow{l_n} z_n. $$

If $l_k$ is a direct letter, then $N(W)_{l_k}(z_{k-1}) = z_k$; if $l_k$ is a inverse letter, then $N(W)_{l_k}(z_{k-1}) = z_k$ with $k = 1, \ldots, n$; if $a \in Q_1$ and $N(W)_a(z_k)$ is not defined yet, then $N(W)_a(z_k) = 0$.

In [7] it was observed that $N(W)$ is isomorphic to $N(W^{-1})$. The modules $N(W)$ are called string modules.

**Theorem 5.2.** [7, Butler-Ringel] Let $\Lambda$ be a string algebra. If $B = \emptyset$, then the $\Lambda$-modules $N(W)$ with $W \in \mathcal{W}$ form a complete list of indecomposable $\Lambda$-modules pairwise non-isomorphic.

5.2. **Sub-strings.** For a string of positive length $W = l_r \cdots l_1$ we define its support as $\text{Supp}(W) = \{t(l_1)\} \cup \{h(l_k) : k = 1, \ldots, r\}$. If $W = l_{(i,t)}$, then $\text{Supp}(W) = \{i\}$. Given a string of positive length $W = l_r \cdots l_1$, we say that a string $V$ is a sub-string of $W$ if $V = l_{s_1} \cdots l_{s_t}$ is a subword of $W$ and there are no arrows $a, b \in Q_1$ such that $aV$ and $Vb^{-1}$ are subwords of $W$. For technical reasons we introduce the zero string $0$ which is a sub-string of any string. Now, given a string $W$ we denote by $\text{Cam}(W)$ the set of all sub-strings of $W$.

6. **Generalized cluster algebras**

For convenience we recall the definition of generalized cluster algebras introduced by Chekhov and Shapiro in [13].

We say a matrix $B \in \text{Mat}_{n \times n}(\mathbb{Z})$ is skew-symmetrizable if there exist positive integers $d_1, \ldots, d_n$ such that $DB$ is skew-symmetric with $D = \text{diag}(d_1, \ldots, d_n)$ a diagonal matrix. In this case we call $D$ a skew-symmetrizer of $B$.

Given a skew-simmetrizable matrix $B$ and an integer $k \in \{1, \ldots, n\}$, the mutation of $B$ with respect to $k$ is the matrix $\mu_k(B)$ with entries $b'_{ij}$ defined as follows

$$ b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = j \\
b_{ij} \left[ \frac{b_{ik}b_{kj} + h_kb_{ik}}{2} \right] & \text{if } i \neq k, j.
\end{cases} $$

For us it is enough work without coefficients. For a study of generalized cluster algebras with principal coefficients, in parallel with the one for cluster algebras, the reader can see [32].

Let $\mathcal{F}$ be the field of the rational functions in $n$ algebraic independent variables with coefficients in $\mathbb{Q}$.

**Definition 6.1.** A seed in $\mathcal{F}$ is a pair $(B, x)$ where $B$ is a skew-symmetrizable matrix and $x = (x_1, \ldots, x_n)$ is an $n$-tuple of algebraic independent elements of $\mathcal{F}$.

Now we assume that $B$ is skew-symmetric with skew-symmetrizer $D$ and $b_{ik}/d_k$ is an integer for all $i \in \{1, \ldots, n\}$.

For $k \in \{1, \ldots, n\}$ we define the polynomials

$$ v^+_k = \prod_{b_{ik} > 0} x_i^{b_{ik}/d_k}, \quad v^-_k = \prod_{b_{ik} < 0} x_i^{-b_{ik}/d_k} \quad \text{and} \quad \theta_k(u, v) = \sum_{l=0}^{d_k} u^l v^{d_k-l}. $$

**Definition 6.2** (Generalized cluster mutation). For a seed $(B, x)$ and $k \in \{1, \ldots, n\}$, the mutation of $(B, x)$ with respect to $k$ is the pair $\mu_k(B, x) = (\mu_k(B), \mu_k(x))$ where $\mu_k(x) = (x'_1, \ldots, x'_n)$ is the $n$-tuple of elements of $\mathcal{F}$ given by

$$ x'_i = \begin{cases} 
x_i & \text{if } k \neq i, \\
\theta_k(v^+_k, v^-_k) & \text{if } k = i.
\end{cases} $$
In this article the polynomials $\theta_k(v_k^+, v_k^-)$ are often called polynomials of Chekhov-Shapiro.

For a seed $(B, x)$, let

$$\chi = \{ x \in \mu_k \cdots \mu_k : k_r \in \{1, \ldots, n\} \text{ and } r \geq 0 \}.$$

**Definition 6.3.** The generalized cluster algebra associated to $(B, x)$, denoted by $A(B) = A(B, x)$, is the subring of $F$ generated by $\chi$.

Note that exchange polynomials do not have to be binomials. In [13, Theorem 2.5] it was shown a desired property

**Theorem 6.4 (The Laurent phenomenon).** Any generalized cluster variable can be expressed as a Laurent polynomial in the initial variables $x_i$.

**Remark 6.5.** The reader should be cautious by comparing the definitions of [13] with this ones because there they take skew-symmetrizer by the right and here we did by the left.

**Remark 6.6.** In [13, Lemma 3.1] the authors obtain exchange relations of the form $a^2 + 2 \cos(\frac{p}{a})ab + b^2$ in the case of orbifold points of order $p$, with $p$ grater than one. We are going to obtain exchange relations of the form $a^2 + ab + b^2$. Of course it is more natural to consider orbifold points of order three than order two as we made in the first version of this article.

### 7. Polygons with one orbifold point

We are working with polygons with one orbifold point of order three but for convenience we recall some definitions of surfaces with orbifold points. For more details about surfaces with orbifold points of order two or three and relations with generalized cluster algebras the reader can see [13] and references therein, for example [11, 12]. For an interesting and beautiful application of surfaces with orbifold points and group actions in some cluster structures the reader is kindly asked to look at [35].

#### 7.1. Basic definitions

Let $\Sigma$ be a compact connected oriented 2-dimensional real surface with possible empty boundary. The pair $(\Sigma, M)$ where $M$ is a finite subset of $\Sigma$ with at least one point from each connected component of the boundary of $\Sigma$ is called a bordered surface or just a surface. The points of $M$ are called marked points and the points of $M$ that lie in the interior of $M$ are called punctures. A triple $(\Sigma, M, O)$ where $(\Sigma, M)$ is a bordered surface and $O$ is a finite subset of $\Sigma \setminus (M \cup \partial \Sigma)$ is called a marked surface with orbifold points. The points of $O$ are called orbifold points and they will be denoted by a cross $\times$ in the surface. In this article we work with surfaces with boundary, without punctures and with just one orbifold point of order three.

**Definition 7.1.** A triangulation of $(\Sigma, M, O)$ is a maximal collection of pairwise compatible arcs.
Given a triangulation $\tau$ of the surface we define a triangle of $\tau$ as the closure of a connected component of $(\Sigma,\tau) \cup \{\text{the pending arcs of } \tau\}$. An orbifold triangle has one side that is a loop containing an orbifold point (a orbifold triangle is a degenerated triangle). A triangle without an orbifold point in its interior is called an ordinary triangle or just a triangle. If a triangle intersects to the boundary of the surface at most in a three points it is called an internal triangle.

Let $\tau$ be a triangulation of $(\Sigma,M,\varnothing)$. If $i$ is an arc of $\tau$, the flip of $i$ with respect to $\tau$ is the unique arc $i'$ such that $\sigma = (\tau \setminus \{i\}) \cup \{i'\}$ is a triangulation of $(\Sigma,M,\varnothing)$. In this case we denote $i' = \text{flip}_\tau(i)$ and we say that $\sigma$ is obtained from $\tau$ by a flip of $i \in \tau$. In our case, flips act transitively on triangulations of $(\Sigma,M,\varnothing)$, see [18, Theorem 4.2].

7.3. Our surfaces. Let $\Sigma_n$ be the surface, with $n \geq 2$, given by a disk with boundary, $n + 1$ marked points in its boundary, without punctures and one orbifold point of order three. In this work we often refer to $\Sigma_n$ as the $(n + 1)$-polygon with one orbifold point, the marked points are called vertices and they are denoted by $\{v_0,v_1,\ldots,v_n\}$. We orient the vertices in counterclockwise order. In pictures the orbifold point is drawn with the symbol $\times$.

Let $\tau$ be a triangulation of $\Sigma_n$. We have that $|\tau| = n$, see [35, Lemma 4.1]. In this case we have two types of admissible triangles for $\tau$, see Figure 1.

![Figure 1](image-url)

We associate a quiver $Q(\tau)$ to a triangulation $\tau$ of the orbifold $\Sigma_n$ in the following way: the set of vertices is given by the arcs of $\tau$ and the set of arrows is described as follows. For each triangle $\Delta$ of $\tau$ and arcs $i$ and $j$ in $\Delta$ we draw an arrow from $j$ to $i$ if $i$ succeeds $j$ in the clockwise orientation, with the understanding that no arrow incident to a boundary segment is drawn. Finally we draw a loop at the pending arc of $\tau$.

**Remark 7.2.** In the classical context of marked Riemann surfaces without orbifold points this loop is not drawn. For instance the quiver of a triangulation $T$ of a polygon $P$ without punctures and without orbifold points will be denoted by $Q(T)$ and it is constructed as above but, as we said, it does not have loops.

Denote by $H(\tau)$ the collection of all internal triangles $\Delta$ of a given triangulation $\tau$. Any element $\Delta$ of $H(\tau)$ define a 3-cycle $c_\Delta b_\Delta a_\Delta$ up to cyclical equivalence. If we denote by $\varepsilon$ the loop of $\tau$, then the potential associated to $\tau$ is $S(\tau) = \sum_{\Delta \in H(\tau)} c_\Delta b_\Delta a_\Delta + \varepsilon^3$.

**Definition 7.3.** For any triangulation $\tau$ of $\Sigma_n$ we define the basic algebra $\Lambda(\tau)$ associated to $\tau$ as the Jacobian algebra $\Lambda(\tau) = P(Q(\tau), S(\tau))$.

**Example 7.4.** Consider the triangulation $\tau$ of Figure 2. We see that the algebra $\Lambda(\tau)$ is is $\mathbb{C}(Q(\tau))/I$, where $I$ is the ideal generated by $ba, cb, ac$ and $\varepsilon^2$ (compare to [35, Example 2.3]).

**Definition 7.5.** A weighted quiver is a pair $(Q, d)$ where $Q$ is a quiver without loops and $d = (d_i)_{i \in Q_0}$ is an $n$-tuple of positive integers.

Let $Q(\tau)^*$ be the quiver obtained from $Q(\tau)$ by deleting the loop. Now we denote by $(Q(\tau)^*, d_\tau)$ the weighted quiver associated to $\tau$ where $(d_\tau)_r = 2$ if $r$ is the pending arc and $(d_\tau)_j = 1$ in other wise.

Fix an $n$-tuple $d = (d_1, \ldots, d_n)$, in [30, Lemma 2.3] it was proved that there is a bijection between the set of 2-acyclic weighted quivers $(Q, d)$ and the collection of skew-symmetrizable matrices $B$ with skew-symmetrizer given by $D = \text{diag}(d_1, \ldots, d_n)$. Indeed, given a quiver $Q$, if $c_{ij}$ is as in Section 4, then $b_{ij} = d_j c_{ij} / \text{gcd}(d_i, d_j)$ define a matrix $B_Q$ skew-symmetrized by $D$.

Following [30, Lemma 2.3], we denoted by $B(\tau)$ the skew-symmetrizable matrix associated to $(Q(\tau), d_\tau)$ and we call it the adjacency matrix associated to $\tau$. 

8. $\Lambda(\tau)$ as an Orbit Algebra

In this section we shall note that $\Lambda(\tau)$ can be seen as an *orbit Jacobian algebra*. With this observation we are going to obtain some results about Galois coverings. For details and missing definitions about orbit Jacobian algebras the reader can see [36]. At the end of the section we will define the arcs representations of $\Lambda(\tau)$.

Let $\tilde{\Sigma}_n$ be the regular $(3n + 3)$-gon with $u_1, u_2, \ldots, u_{3n+3}$ vertices in counterclockwise orientation and let $\theta$ be the rotation by $120^\circ$ on $\tilde{\Sigma}_n$ which sends a vertex $v_i$ to $v_{i+(n+1)}$ modulo $(3n + 3)$. In the terminology of [35], $\Sigma_n$ is the $\mathbb{Z}_3$-orbit space of $\tilde{\Sigma}_n$. We consider $\theta$ as a generator of $G = \mathbb{Z}_3$. We can see that $G$ acts freely on $\{u_1, u_2, \ldots, u_{3n+3}\}$, that is, if $g \in G \setminus \{e\}$, then $g \cdot u_i \neq u_i$ for $i \in [1, 3n + 3]$.

We say that an arc $j$ of $\tilde{\Sigma}_n$ is $G$-admissible or just admissible if $j$ belongs to some $G$-invariant triangulation $T$ of $\tilde{\Sigma}_n$.

Let $T$ be a triangulation of $\tilde{\Sigma}_n$ and suppose that $T$ is $G$-invariant. Consider $Q(T)$ the quiver associated to $T$, see Remark 7.2. We can define a potential for $Q(T)$ as $S(T) = \sum_{\Delta \in H(T)} \gamma_{\Delta} \beta_{\Delta} \alpha_{\Delta}$. Note that $G$ acts freely on $Q(T)$0 and for any $\alpha_{\Delta} \beta_{\Delta} \gamma_{\Delta}$ we have that $g \cdot (\alpha_{\Delta} \beta_{\Delta} \gamma_{\Delta})$ is again a summand of $S(T)$ for all $g \in G$.

We can define, [35, Section 2.1], the orbit quiver $Q(T)_G$ of $Q(T)$ in the obvious way. We define the potential $S(T)_G$ for $Q(T)_G$ as the image of $S(T)$ under the canonical morphism $\pi : \mathbb{C}\langle Q(T) \rangle \rightarrow \mathbb{C}\langle Q(T)_G \rangle$ induces by $\pi(i) = G \cdot i$ for $i \in Q(T)_0$ and $\pi(\alpha) = G \cdot a$ for $a \in Q(T)_1$, note that $\pi$ is a Galois $G$-covering. We define the orbit Jacobian algebra of the orbit quiver with potential as $\mathcal{P}(Q(T), S(T))_G = \mathcal{P}(Q(T)_G, S(T)_G)$. We make the following convention $\Lambda(T) = \mathcal{P}(Q(T), S(T))$ and $\Lambda(T)_G = \mathcal{P}(Q(T), S(T))_G$, see Example 8.3. The following result shows that we get a Galois covering, see [35, Proposition 3.1].

**Lemma 8.1** (Paquette-Schiffler). The Galois covering $\pi : \mathbb{C}\langle Q(T) \rangle \rightarrow \mathbb{C}\langle Q(T)_G \rangle$ induces a Galois G-covering $\pi : \Lambda(T) \rightarrow \Lambda(T)_G$.

**Remark 8.2.** Let $\tau$ be a triangulation of $\Sigma_n$ and let $T$ be the triangulation of $\tilde{\Sigma}_n$ such that $G \cdot T = \tau$. In $T$ there exist an unique triangle $\Delta_T$ such that it is $G$-invariant and the other triangles in $H(T)$ have a trivial stabilizer. The triangle $\Delta_T$ corresponds to the pending arc of $\tau$ and the $G$-orbit of any triangle $\Delta$ different to $\Delta_T$ corresponds with a triangle of $H(\tau)$. We conclude that $Q(\tau) = Q(T)_G$ and with the above observation we get that $\Lambda(\tau) = \Lambda(T)_G$.

**Example 8.3.** Let $\tau$ be the triangulation of $\Sigma_3$ depicted on the right of Figure 3. Let $T$ be the corresponding triangulation on $\tilde{\Sigma}_3$ depicted on the left of Figure 3. The quiver $Q(T)$ is drawn below
Consider \( S(T) = \varepsilon_1 \varepsilon_2 \varepsilon_3 \) the potential associated to \( T \). Let \( G = \langle \theta \rangle \) be the cyclic group of order 3 with generator \( \theta \). Then \( G \) acts freely on \( (Q,S) \) by increasing by one, module 3, the indices of the symbols. Passing to the orbit space of this action we get

\[
Q(T)_G : i^\beta \longrightarrow j^\alpha \longrightarrow k \quad \varepsilon \quad \text{and the potential } S(T)_G = \varepsilon^3,
\]

where \( i = G \cdot i_1, j = G \cdot j_1, k = G \cdot k_1, \alpha = G \cdot \alpha_1, \beta = G \cdot \beta_1 \) and \( \varepsilon = G \cdot \varepsilon_1 \). The orbit Jacobian algebra \( \mathcal{P}(Q(T)_G, S(T)_G) \) is not but \( \Lambda(\tau) = \mathbb{C}\langle \tau \rangle / \langle \varepsilon^2 \rangle \).

**Proposition 8.4.** Let \( \tau \) be a triangulation of \( \Sigma_n \). Then \( \Lambda(\tau) \) is finite dimensional.

**Proof.** From Lemma 8.1 we have that \( \pi : \Lambda(T) \rightarrow \Lambda(T)_G \) is a Galois \( G \)-covering. In particular we have isomorphism \( \pi_{i,j} : \bigoplus_{g \in G} \Lambda(T)(e_i, g \cdot e_j) \rightarrow \Lambda(T)(\pi(e_i), \pi(e_j)) \) for any idempotent \( e_i \) and \( e_j \) of \( \Lambda(T) \). We know \( \Lambda(T) \) is finite-dimensional (the reader can see the finite dimension of the Jacobian algebra associated to a triangulation in a more general context of surfaces with non-empty boundary in [28, Theorem 36]), so \( \Lambda(\tau) \) is finite dimensional. The proof of the lemma is completed. \( \Box \)

A string algebra \( B = \mathbb{C}\langle Q \rangle / \langle P \rangle \) is a gentle algebra if the following conditions are satisfied

(Gt1) \( P \) is generated by paths of length 2.

(Gt2) For any arrow \( a \in Q_1 \) we have \( \|\{b \in Q_1 : t(a) = h(b) \text{ and } ab \in P\}\| \leq 1 \) and \( \|\{c \in Q_1 : t(c) = h(a) \text{ and } ca \in P\}\| \leq 1 \).

**Proposition 8.5.** For any triangulation \( \tau \) of \( \Sigma_n \) we have that \( \Lambda(\tau) \) is a gentle algebra.

**Proof.** Let \( T \) be the triangulation of \( \tilde{\Sigma}_n \) such that \( G \cdot T = \tau \). The proof is an adaptation of proof [2, Lemma 2.5], from that lemma we have that \( \Lambda(T) \) is gentle. By definition \( \Lambda(\tau) = \mathbb{C}\langle Q_G \rangle / J(Q_G, S_G) \) and it is clear that \( J(Q_G, S_G) \) is generated by paths of length two. Since we have Proposition 8.4, only remains to prove (Gt2), (S1) and (S2).

(S1). First, let \( j \) be the pending arc of \( \tau \). We consider \( \tilde{j} \) an element in \( \tau^{-1}(j) \). We have that \( \tilde{j} \) is contained in two triangles of \( T \). One of those triangles has the other preimages of \( j \) as sides, say \( \Delta(j) \), in other words, \( \Delta(j) \) is invariant under \( G \). By the definition of \( S(T)_G \) we can conclude that there is a loop based at \( j \) and there is at most one arrow starting at \( j \) and one arrow ending at \( j \). Now, one component of \( \tilde{\Sigma}_n \setminus \tilde{j} \), precisely what no contain the other preimages of \( j \), it is a fundamental region for the action of \( G \) on \( T \). If \( k \) is not a pending arc of \( \tau \), we can consider a preimage of \( k \) in the fundamental region above, recall \( \Lambda(T) \) is gentle, this implies (S1) for \( k \). For this reason we just have to prove (S2) and (Gt2) in the orbifold triangle.

(S2) and (Gt2). This two properties follow from the fact that there is a loop based at the pending arc \( j \) and at most one arrow with starting at \( j \) and at most one arrow with ending at \( j \). This conclude the proof. \( \Box \)

By Lemma 3.11 and Lemma 8.1 we know the following.

**Lemma 8.6.** Let \( \tau \) be a triangulation of \( \Sigma_n \). Then the push-down functor \( \pi_\# : \Lambda(T)-\text{mod} \rightarrow \Lambda(\tau)-\text{mod} \) is a \( G \)-precovering.

We are going to define a string \( W_j(\tau) \) of \( \Lambda(\tau) \) for every arc \( j \notin \tau \) of \( \Sigma_n \). We denote by \( \pi : \tilde{\Sigma}_n \rightarrow \Sigma_n \) the canonical projection. We know that \( \pi^{-1}(j) = \{\tilde{j}, \tilde{j}_0, \tilde{j}_0 \} \). Recall that \( G = \langle \theta \rangle = \mathbb{Z}_3 \). Let \( T \) be the triangulation in \( \tilde{\Sigma}_n \) corresponding to \( \tau \), see Figure 4.

Let \( j \) be an arc of \( \Sigma_n \) such that \( j \notin \tau \). Choice \( \alpha \in \pi^{-1}(j) \), by definition \( \alpha \) is an arc of \( \tilde{\Sigma}_n \) and \( \alpha \) joints two vertices \( u_l \) and \( u_{l+r} \) of \( \tilde{\Sigma}_n \). Every time \( \alpha \) crosses two adjacent initial arcs \( \gamma : \tilde{i}_{s_1} \rightarrow \tilde{i}_{s_2} \) of \( \tau \), we write the letter \( G \cdot \gamma \) (a letter on \( Q(\tau) \)) if \( \alpha \) crosses \( \tilde{i}_{s_1} \) first from \( u_l \) to \( u_{l+r} \) or we write the letter \( G \cdot \gamma^{-1} \) in other wise, see Example 8.3. This construction does not depend of the choice of \( \alpha \) up to string equivalence, see Section 5. Denote by \( W_j(\tau) \) the string of \( \Lambda(\tau) \) obtained in this way. In Figure 4 we show an example of this construction.

**Definition 8.7.** Let \( \tau \) be a triangulation of \( \Sigma_n \). For any arc \( j \notin \tau \) we define the arc representation \( M(j, \tau) \) of \( j \) with respect to \( \tau \) as the string module associated to \( W_j(\tau) \), i.e \( M(j, \tau) = N(W_j(\tau)) \), see Section 5.1. Since a string and its inverse generate isomorphic string modules we have that \( M(j, \tau) \) is well defined up to isomorphism. Now, for any arc \( j \in \tau \) we define \( M(j, \tau) := S_j^{-} \) as the corresponding negative simple representation of \( \Lambda(\tau) \).

As long as there is no confusion we ease the notation and write \( W_j := W_j(\tau) \) and \( M(j) := M(j, \tau) \).
Figure 3. We obtain a triangulation of a square with one orbifold point of order 3 as the G-orbit space of a triangulation of a 12-gon.

Figure 4. Let j be the blue arc (right). We define W_j from the left. In this case α can be the blue, red or green arc. Note that W_j can be read directly from the right.

9. THE CALDERO-CHAPOTON ALGEBRA FOR A SPECIFIC TRIANGULATION

In this section we will study the Caldero-Chapoton algebra of a specific triangulation τ_0 of Σ_n. We will see that the arc representations of Λ(τ_0) τ_0 play a central role. Fix the vertices of Σ_n in counter clockwise order \{v_0, ..., v_n\}. Let i_n be the pending arc at v_0. We denote the pending at v_k as i_1^n, ..., i_n. With this notation we see that i_1^n = i_n. Let i_k be the arc from v_0 to v_k going in counterclockwise for k = 1, ..., n - 1. We define the special triangulation τ_0 of Σ_n as the collection of arcs \{i_1, ..., i_n\}, see on the right hand of Figure 3.

For τ_0 we have a nice description of the concepts introduced in Section 7, for instance, the weighted quiver associated to τ_0 looks like

\[
\begin{align*}
Q(\tau_0)^* : & \quad 1 \overset{a_1}{\longrightarrow} 2 \overset{a_2}{\longrightarrow} \cdots \overset{a_{n-2}}{\longrightarrow} n \overset{a_{n-1}}{\longrightarrow} n, \\
& \text{and } d_{\tau_0} = (1, 1, ..., 1, 2).
\end{align*}
\]

The matrix B(τ_0) is going to be our input to obtain the polynomials of Chekhov-Shapiro and we are going to describe a basic algebra associated to τ_0.

Let \( \Lambda := \Lambda(\tau_0) \) be the basic algebra associated to \( \tau_0 \), it is clear that \( \Lambda \) is given by \( \mathbb{C}[Q(\tau_0)]/I \) where \( Q(\tau_0) \) is the quiver

\[
\begin{align*}
1 \overset{a_1}{\longrightarrow} 2 \overset{a_2}{\longrightarrow} \cdots \overset{a_{n-2}}{\longrightarrow} n \overset{a_{n-1}}{\longrightarrow} n \overset{\epsilon}{\longrightarrow} n
\end{align*}
\]

and I is the ideal generated by \( \epsilon^2 \).
For every arc \( j \) of \( \Sigma_n \), we defined a decorated indecomposable representation \( M(j) \) of \( \Lambda \) with respect to \( \tau_0 \), see Definition 8.7.

**Remark 9.1.** For any arc \( j \notin \tau_0 \) Definition 8.7 can be rewritten up to isomorphism by counting intersection numbers directly in \( \Sigma_n \). We give this approach for convenience,

\[
\dim(M(j))_l = [i \cap j] \text{ in the interior of } \Sigma_n.
\]

Given an arrow \( a_l \) with \( l = 1, \ldots, n-1 \), we define \( M(j)_{a_l} \) as follows:

- if \( 0 < \dim(M(j)_{i(a_l)}) < \dim(M(j)_{h(a_l)}) \), then \( M(j)_{a_l} = (\varnothing) \);
- if \( 0 < \dim(M(j)_{i(a_l)}) = \dim(M(j)_{h(a_l)}) \), then \( M(j)_{a_l} \) acts as the corresponding identity;

\( M(j)_{a_l} = 0 \) in otherwise.

If \( \dim(M(j)_{a_l}) \neq 0 \), then \( M(j)_{a_l} = (\varnothing, 1) \).

**Remark 9.2.** From Definition 8.7 and Theorem 5.2 we know \( M(j) \) is indecomposable in decrep(\( \Lambda \)) for every arc \( j \). Remark 9.1 allow us to compute \( M(j) \) without \( \Sigma_n \).

For any arc \( j \notin \tau_0 \) we define the support of \( M(j) \) as \( \operatorname{Supp} M(j) = \{ l : M(j)_{l} \neq 0 \} \). The same argument of [9, Lemma 2.2] can be applied here to conclude that \( \operatorname{Supp} M(j) \) is connected as a subset of \( [1, n] \). So, we are going to think that \( \operatorname{Supp} M(j) \) is an interval.

**Example 9.3.** For \( n = 5 \), we compute \( M(j_2) \) with \( l = 1, 2 \) and 3, see Figure 5.

![Figure 5. Some arcs for n = 5.](image)

We have

\[
\begin{align*}
 M(j_1) & : 0 \longrightarrow C \xrightarrow{id} C \longrightarrow 0 \longrightarrow 0 \longrightarrow 0, \\
 M(j_2) & : 0 \longrightarrow C \xrightarrow{id} C \xrightarrow{\varnothing} C^2 \xrightarrow{id} C^2 \xrightarrow{\varnothing} (\varnothing, 1), \\
 M(j_3) & : 0 \longrightarrow C^2 \xrightarrow{id} C^2 \xrightarrow{id} C^2 \xrightarrow{id} C^2 \xrightarrow{\varnothing} (\varnothing, 1).
\end{align*}
\]

As illustration we have the Caldero-Chapoton function \( C_\Lambda(M(j_2)) \),

\[
\begin{aligned}
 x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_5 x_5 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + x_2 x_1 x_3 x_5 + x_2 x_1 x_3 x_5 + x_1 x_2 x_3 x_5 + x_1 x_2 x_3 x_5 + x_1 x_2 x_3 x_5.
\end{aligned}
\]

### 9.1. AR translations, E-invariant and g-vectors of arc representations.

Let \( j \) be an arc of \( \Sigma_n \). We introduce some notation. Given two vertices \( v_r \) and \( v_l \) of \( \Sigma_n \) with \( r + 1 < l \) and \( r \in \{1, \ldots, n-2\} \) we have two arcs from \( v_r \) to \( v_l \) denoted by \( [v_r, v_l]^* \) and \( [v_r, v_l]^\cdot \). Indeed, if \( 0 < r \), then \( [v_r, v_l]^\cdot \) does not intersect to \( i_n \) in the interior of \( \Sigma_n \) while \( [v_r, v_l]^* \) does. For example, in the Figure 5 we have \( j_1 = [v_2, v_5]^\cdot \) and \( j_2 = [v_2, v_4]^\cdot \). For \( r = 0 \) we say that \( i_k = [v_0, v_{k+1}]^\cdot \) and \( [v_0, v_{k+1}]^\cdot \) is the another arc from \( v_0 \) to \( v_{k+1} \) with \( k = 1, \ldots, n-1 \). In the case \( l = r + 1 \), we have \( [v_r, v_{r+1}]^\cdot \) is not a boundary segment.

**Remark 9.4.** If \( n \geq 4 \), with the above notation we can describe \( W_j \) explicitly for any \( j \notin \tau_0 \).

- \( W_{[v_r, v_l]^\cdot} = a_{n-3} \cdots a_1 \) and \( W_{[v_r, v_l]^*} = \varepsilon \cdots a_1 \)
- \( W_{[v_r, v_l]^\cdot} = a_{i-3} \cdots a_1 \) and \( W_{[v_r, v_l]^*} = a_l^{-1} \cdots a_{n-1} \varepsilon \cdots a_1 \) for \( 0 < i \) and \( i + 2 < l < n \);
- \( W_{[v_r, v_l]^\cdot} = 1_{(l, r)} \) and \( W_{[v_r, v_l]^*} = a_l^{-1} \cdots a_{n-1} \varepsilon \cdots a_r \) for \( l = i + 2 \) and \( i \leq n - 2 \);
- \( W_{[v_r, v_l]^\cdot} = a_l^{-1} \cdots a_{n-1} \varepsilon \cdots a_r \) for \( l = i + 1 \) and \( 0 < i < n - 2 \);
- \( W_{[v_r, v_l]^*} = \varepsilon a_{n-1} \)
- \( W_{[v_0, v_{k+1}]^\cdot} = a_{n-2} \cdots a_l \) for \( 1 \leq l < n - 1 \);
- \( W_{[v_0, v_{k+1}]^\cdot} = a_k^{-1} \cdots a_{n-1} \varepsilon \cdots a_k \) for \( 1 \leq k \leq n - 1 \);
- \( W_{[v_r, v_{r+1}]^\cdot} = \varepsilon \).
The reader can compare the following lemma with the $A_n$ case, [9, Theorem 2.13]. Given an arc $j$ we denote by $r^+(j)$ (in [9] it would be $r$) the arc of $\Sigma_n$ that we obtain by rotating $j$ in counterclockwise for an angle of $2\pi/(n+1)$. By $r^-(j)$ (in [9] it would be $r^+$) we denote the arc obtained from $j$ by rotating $j$ in clockwise for an angle of $2\pi/(n+1)$.

**Lemma 9.5.** Assume $j$ is not an initial arc of $\Sigma_n$.

(a) If $M(j)$ is not projective, then $\tau(M(j)) = M(r^+(j))$, where $\tau$ denotes the Auslander-Reiten translation.

(b) If $M(j)$ is not injective, then $\tau^-(M(j)) = M(r^-(j))$.

**Proof.** The lemma can be proved by cases using Remark 9.4 and the classification of the Auslander-Reiten sequences containing string modules from [7, p.p 170-172]. However, we proved this result in Corollary 10.3. □

**Lemma 9.6.** Assume that $j$ is an arc of $\Sigma_n$. Then $E_\Lambda(M(j)) = 0$.

**Proof.** We postpone this proof up to Corollary 10.5. It is worth mention that this lemma can be proved with the alluded results of [7]. □

**Lemma 9.7.** Assume $j$ is an arc of $\Sigma_n$ such that $j \notin \tau_0$. Then $\text{pd} M(j) \leq 1$ and $\text{id} M(j) \leq 1$. Here $\text{pd} M(j)$ (resp. $\text{id} M(j)$) denotes the projective dimension of $M(j)$ (resp. the injective dimension of $M(j)$).

**Proof.** The lemma follows from [25, Proposition 3.5] and the definition of $M(j)$. Indeed, in the language of [25], if we take

$$C = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -2 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 2 \end{pmatrix}$$

and $\Omega = \{(i+1, i) : 1 \leq i \leq n-1\}$, then we get $\Lambda = H(C, D, \Omega)$. By definition $M(j)$ is locally free $\Lambda$-module for every arc $j \notin \tau_0$ (see [25, Section 1.5]). □

**Remark 9.8.** Lemma 9.7 ensures that $\text{id} M(j) \leq 1$, now it can be seen that we have the following minimal injective presentation of $M(j)$ for each arc $j \notin \tau_0$, this is a consequence of [7].

(1) $j$ crosses to $i_n$: in this case $W_j = a_{n-1} \cdots a_{m_j} a_{n_j}$ with $m_j \leq n_j$. Then the following exact sequence is a minimal injective presentation of $M(j)$,

$$0 \to M(j) \to N(a_{n-1} \cdots a_{m_j}) \oplus N(a_{m_j-2} \cdots a_1).$$

Here we define $N(a_{m_j-2} \cdots a_1)$ as zero if $r = 1$ and it is the simple representation at 1 if $r = 2$.

(2) $j$ does not cross to $i_n$: in this case $W_j = a_{n_j} \cdots a_{m_j}$ with $n-2 \geq n_j \geq m_j$. Then the following exact sequence is a minimal injective presentation of $M(j)$,

$$0 \to M(j) \to N(a_{n_j} \cdots a_1) \to N(a_{m_j-2} \cdots a_1).$$

**Proposition 9.9.** If $j_1$ and $j_2$ are not arcs of $\tau_0$, then the following hold:

- There exists a $C$-linear isomorphism
  $$\text{Ext}_1^\Lambda(M(j_1), M(j_2)) \cong \text{Hom}_\Lambda(\tau^{-}(M(j_2)), M(j_1)).$$

- There exists a $C$-linear isomorphism
  $$\text{Hom}_\Lambda(M(j_1), \tau(M(j_2))) \cong \text{Hom}_\Lambda(\tau^{-}(M(j_1)), M(j_2)).$$

**Proof.** The proposition follows from Lemma 9.7, [3, Corollary (IV) 2.14] and [3, Corollary (IV) 2.15]. □

**Lemma 9.10.** Let $\tau$ be a triangulation of $\Sigma_n$. If $j_1$ and $j_2$ are arcs of $\tau$, then $E_\Lambda(M(j_1), M(j_2)) = 0$.

**Proof.** If $j_2 \in \tau_0$ or $M(j_2)$ is injective, then $E_\Lambda(M(j_1), M(j_2)) = 0$ for all arc $j_1 \in \tau$ by definitions and Proposition 4.2. So we can suppose $j_2$ is not in $\tau_0$ and $M(j_2)$ is not injective.

Case 1. $j_1 = i_k$ for some $1 \leq k \leq n$: in this case $M(j_1)$ is the negative simple representation of $\Lambda$ at $k$. It is clear that $E_\Lambda(M(j_2), M(i_k)) = \dim M(j_2)_k$ by Proposition 4.2, but $i_k, j_2 \in \tau$, then $\dim M(j_2)_k = 0.$
Case 2. \( j_1 \notin \tau_0 \) and \( M(j_1) \) is injective: then \( E_\Lambda(M(j_2), M(j_1)) = 0 \) for all \( j_2 \in \tau \). We have to prove \( E_\Lambda(M(j_1), M(j_2)) = 0 \). By Proposition 4.2 we get \( E_\Lambda(M(j_1), M(j_2)) = \dim \Hom_\Lambda(\tau^-\cdot M(j_2), M(j_1)) \). If \( M(j_1) \) is injective, then \( j_1 = [v_j, v_j]^+ \) with \( 2 < n \), \( j_1 = i_1 \) or \( j = [v_0, v_1]^+ \). Since \( j_1, j_2 \in \tau \) and \( M(j_2) \) is not injective, \( \Hom_\Lambda(\tau^-\cdot M(j_2), M(j_1)) = 0 \) and \( \dim \Hom_\Lambda(\tau^-\cdot M(j_2), M(j_1)) = 0 \).

Case 3. \( j_1 \notin \tau_0 \) and \( M(j_1) \) is not injective: we have to prove \( E_\Lambda(M(j_1), M(j_2)) = 0 \) and \( E_\Lambda(M(j_2), M(j_1)) = 0 \). For \( l = 1, 2 \), let \( m_l \) be the minimum positive integer such that \( \mathcal{M}(\tau^-\cdot M(j_1)) \) is injective.

If \( m_1 < m_2 \), then by [3, Corollary (IV) 2.15 (c)] and Proposition 4.2 we have

\[
E_\Lambda(M(j_1), M(j_2)) = \dim \Hom_\Lambda(\tau^-\cdot M(j_2), M(j_1)).
\]

[3, Corollary (IV) 2.14 (b)] implies \( \dim \Ext^1(\tau^-\cdot M(j_1), \tau^-\cdot M(j_2)) = E_\Lambda(M(j_1), M(j_2)) \).

Since \( \tau^-\cdot M(j_1) \) is injective, we get \( E_\Lambda(M(j_1), M(j_2)) = 0 \). Now,

\[
E_\Lambda(M(j_2), M(j_1)) = \dim \Hom_\Lambda(\tau^-\cdot M(j_1), M(j_2)).
\]

Since \( m_1 < m_2 \), we apply [3, Corollary (IV) 2.15 (c)] to obtain \( E_\Lambda(M(j_2), M(j_1)) = 0 \). The case \( m_2 < m_1 \) is similar.

This proves the lemma.

\begin{lemma}
Given an arc \( j \notin \tau_0 \) we have \( \Ext^1(M(j), M(j)) = 0 \).
\end{lemma}

\begin{proof}
By Proposition 4.2 we have \( E_\Lambda(M(j)) = \dim \Hom_\Lambda(\tau^-\cdot M(j), M(j)) \). Proposition 9.9 implies \( E_\Lambda(M(j)) = \dim \Ext^1(M(j), M(j)) \), the lemma follows from Lemma 9.6.
\end{proof}

\begin{lemma}
Assume \( j \notin \tau_0 \). Then the \( G_d \)-orbit \( \mathcal{O}(M(j)) \) is open in \( \text{rep}_d(\Lambda) \).
\end{lemma}

\begin{proof}
By Lemma 9.11 we have \( \Ext^1(M(j), M(j)) = 0 \), this implies that \( \mathcal{O}(M) \) is open, for example see [15, 1.7 Corollary 3].
\end{proof}

By Lemma 9.6 we have examples of \( E \)-rigid indecomposable \( \Lambda \)-modules i.e.\( E_\Lambda(M(j)) = 0 \). The next result shows that we already know all \( E \)-rigid \( \Lambda \)-modules, this result can be seen a consequence of Lemma Lem13.1, but we have been mentioned that for \( \tau_0 \) the proof of some result may be made with explicit computations, here we give an example.

\begin{proposition}
If \( N \) is a indecomposable \( \Lambda \)-module and \( N \) is not of the form \( M(j) \) for some arc \( j \) of \( \Sigma_n \), then \( E_\Lambda(N) > 0 \).
\end{proposition}

\begin{proof}
Let \( W \) be the string associated to \( N \) and assume that \( N \) is not \( E \)-rigid. Given a non-initial arc \( j \) of \( \Sigma_n \) we have the string \( W_j \) is one of the following

\[
1_{(i, +)}, \quad \varepsilon \cdots a_{m_j}, \quad \varepsilon \cdots a_{m_j} \quad \text{with} \quad m_j \in [1, n-1],
\]

\[
a_{n_j} \cdots \varepsilon \cdots a_{m_j}, \quad \text{with} \quad m_j \leq n_j \quad \text{and} \quad m_j \in [1, n-1],
\]

\[
a_{n_j} \cdots \varepsilon \cdots a_{m_j}, \quad \text{with} \quad n - 2 \leq n_j \quad \geq m_j.
\]

Therefore \( W \) is different to \( W_j \) for any arc \( j \) of \( \Sigma_n \), here we use Remark 9.4. If \( W = a_{n-1} \cdots a_l \) with \( l > 1 \) we have \( N(W) \) looks like

\[
0 \rightarrow \cdots 0 \rightarrow \mathbb{C} \rightarrow \cdots \rightarrow \mathbb{C}.
\]

By [7] we get \( \tau^{-1}(N(W)) = N(e^{-1}a_{n-1} \cdots a_{l-1}) \). Since

\[
N(e^{-1}a_{n-1} \cdots a_{l-1}) : 0 \rightarrow \cdots 0 \rightarrow \mathbb{C} \xrightarrow{id} \cdots \xrightarrow{id} \mathbb{C} \xrightarrow{\left( \begin{array}{c} 1 \, 1 \\ 0 \, 1 \end{array} \right)} \mathbb{C}^2 \xrightarrow{\left( \begin{array}{c} 0 \, 1 \\ 0 \, 0 \end{array} \right)} 0
\]

we have \( \Hom_\Lambda(\tau^{-1}(N(W)), N(W)) \neq 0 \). Proposition 4.2 implies \( E_\Lambda(N(W)) > 0 \). The case when \( l = 1 \) is similar and follows from [7].

If \( W = a_{n_W}^{-1} \cdots e^{-1} \cdots a_{m_W} \) with \( 1 < m_W < n_W \), then

\[
N(W) : 0 \rightarrow \cdots 0 \rightarrow \mathbb{C} \xrightarrow{id} \cdots \xrightarrow{id} \mathbb{C} \xrightarrow{\left( \begin{array}{c} 1 \, 1 \\ 0 \, 1 \end{array} \right)} \mathbb{C}^2 \xrightarrow{\left( \begin{array}{c} 0 \, 1 \\ 0 \, 0 \end{array} \right)} 0
\]

and by [7] we get \( \tau^{-1}(N(W)) = N(a_{n_W-1}^{-1} Wa_{m_W-1}) \). From definitions we get \( \Hom_\Lambda(\tau^{-1}(N(W)), N(W)) \neq 0 \), so \( E_\Lambda(N(W)) > 0 \). The case when \( m_W = 1 \) is similar and follows from [7]. Since the indecomposable \( \Lambda \)-modules are parametrized by strings, the proposition follows from Theorem 5.2. \qed
Now we interpret the $g$-vector of a representation $M(j)$ in terms of intersection numbers. The three lemmas below follow from Remark 9.8 and Lemma 4.1. For instance

**Lemma 9.14.** For a pending arc $i_k'$ with $k \in \{1, 2, \ldots, n\}$ we have

$$g_\Lambda(M(i_k'))_l = \begin{cases} 2 & \text{if } l = k - 1, \\ -1 & \text{if } l = n, \\ 0 & \text{in otherwise.} \end{cases}$$

**Proof.** We start with the pending arc $i_k'$; from Remark 9.8 we obtain $I_n = N(a_1^{-1} \cdots \varepsilon \cdots a_1)$ and $N(a_{n-2} \cdots a_1) = N(a_{n-2} \cdots a_1)$ because $n_j = k = m_j$, remember that the string associated to $i_k'$ is $a_k^{-1} \cdots \varepsilon \cdots a_k$. By the other hand $I_{k-1} = N((a_{k-2} \cdots a_1))$. The result follow from Lemma 4.1. □

**Lemma 9.15.** Let $j$ be an arc. If $j$ is not a initial arc, it is not a pending arc and it intersects to $i_n$ in the interior of $\Sigma_n$, then we have

$$g_\Lambda(M(j))_l = \begin{cases} 1 & \text{if } l + 1 \text{ is the minimum of } k \text{ such that } \dim(M(j))_k = 1, \\ 1 & \text{if } l + 1 \text{ is the minimum of } k \text{ such that } \dim(M(j))_k = 2, \\ -1 & \text{if } l = n, \\ 0 & \text{in otherwise.} \end{cases}$$

**Lemma 9.16.** Let $j$ be an arc. If $j$ is not an initial arc, it is not a pending arc and it does not intersect to $i_n$, then we have

$$g_\Lambda(M(j))_l = \begin{cases} 1 & \text{if } l + 1 \text{ is the minimum of } k \text{ such that } \dim(M(j))_k = 1, \\ -1 & \text{if } l \text{ is the maximum of } k \text{ such that } \dim(M(j))_k \neq 0, \\ 0 & \text{in otherwise.} \end{cases}$$

**Proposition 9.17.** The set

$$\{C_\Lambda(M(j)) : j \text{ is an arc of } \Sigma_n\}$$

is linearly independent over $\mathbb{C}$.

**Proof.** From the three lemmas above we have that the $g$-vectors $g_\Lambda(M(j))$ are pairwise different. For $n$ even the result follows directly from [10, Proposition 4.3] since $\ker(C_\Lambda) = 0$. For $n$ arbitrary we can adapt the argument in proof of [10, Proposition 4.3] as follows. Define

$$Q_0^n = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0 \text{ for all } i\},$$

$$Q_{\text{succ}}^n = \{(x_1, x_2, \ldots, x_n) : x_i \neq 0 \text{ implies } x_{i+1} \neq 0\},$$

$$Q_0^n = \{(x_1, x_2, \ldots, x_n) : x_i = 0\}.$$

We can define two partial orders in $\mathbb{Z}^n$. Let $a, b \in \mathbb{Z}^n$ be vectors. We say $a \approx b$ if there exist some $e \in Q_{\text{succ}}^n$ such that $a = b + C_Q e$ and $a \leq b$ if there exist some $f \in Q_{\text{succ}}^n$ such that $a = b + C_Q f$. These two orders induce two partial orders on the set of Laurent monomials in $n$ variables $x_1, x_2, \ldots, x_n$. We say $x^a \leq x^b$ if $a \approx b$ and $x^a \leq x^b$ if $a \leq b$. We define the degree of $x^a$ as $\deg(x^a) = a$.

If $\text{soc}(M(j)) = S_n$ (the socle of $M(j)$), then $C_\Lambda(M(j))$ has an unique monomial of maximal degree with respect to $\approx$, namely $g_\Lambda(M(j))$. If $\text{soc}(M(j)) = S_i$ with $i \neq n$, then $C_\Lambda(M(j))$ has an unique monomial of maximal degree with respect to $\leq$ given by $g_\Lambda(M(j))$. Since the $g$-vectors are pairwise different we have that the Caldero-Chapoton functions are pairwise different. Now assume $\lambda_i C_\Lambda(M(j)) + \cdots + \lambda_i C_\Lambda(M(j)) = 0$ for some $\lambda_i \in \mathbb{C}$. We can assume that $\lambda_i \neq 0$ for all $l$.

It can be seen that if $\text{soc}(M(j)) = S_n$, then $x^{g_\Lambda(M(j))}$ does not occur as a summand of any $C_\Lambda(M(k))$ with $\text{soc}(M(k)) = S_i$ and $i < n$. If there exist an index $s$ such that $M(j_n)$ has socle $S_n$, then there exist an index $s$ such that $x^{g_\Lambda(M(j))}$ is $\leq$-maximal in the set of $\{x^{g_\Lambda(M(j))} : \text{soc}(M(j)) = S_n\}$. Since the $g$-vectors are pairwise different we can conclude that $\lambda_s = 0$. Indeed, $x^{g_\Lambda(M(j))}$ does not occur as a summand of any $C_\Lambda(M(j))$ with $l \neq s$, which is a contradiction.

If $\text{soc}(M(j)) \neq S_n$ for all $l$, then there exist an index $r$ such that $x^{g_\Lambda(M(j))}$ is $\leq$-maximal in the set of $\{x^{g_\Lambda(M(j))} : 1 \leq l \leq t\}$. Since the $g$-vectors are pairwise different we have that $x^{g_\Lambda(M(j))}$ does not occur as a summand of any of the $C_\Lambda(M(j))$ with $l \neq r$. Thus $\lambda_r = 0$, a contradiction. Therefore $C_\Lambda(M(j_1)), \ldots, C_\Lambda(M(j_t))$ are linearly independent. □
9.2. Generic version. The results of this section are based and motivated on [10]. For this reason we summarize some results of our particular case in the language of [10, Section 5]. We suggesting to the interested reader see this work for details and missing definitions. In this section we obtain generic version of the results of the last section. Given a triangulation $\tau \in \Sigma_n$ we construct an irreducible component strongly reduced of $\text{decrep}(\Lambda)$, see [10, Section 5.1].

Denote by $Z_j$ the irreducible component of $\text{decrep}(\Lambda)$ that contain to $\mathcal{O}(M(j))$. We know $\mathcal{O}(M(j))$ is open, so it is dense in $Z_j$. Then $E_\Lambda(Z_j) = E_\Lambda(M(j)) = 0$. In the notation of [10, Section 5] that means, in particular, $Z_j$ is an irreducible component strongly reduced of $\text{decrep}(\Lambda)$. We can think that some generic homological data of $Z_j$ is encoded in the homological data of $M(j)$.

**Proposition 9.18.** Given a triangulation $\tau \in \Sigma_n$ and two arcs $j_1, j_2 \in \tau$ we have $E_\Lambda(Z_{j_1}, Z_{j_2}) = 0$.

**Proof.** By Lemma 9.10 we know $E_\Lambda(M(j_1), M(j_2)) = 0$. It can be seen that $\mathcal{O}(M(j_1)) \times \mathcal{O}(M(j_2))$ is open in $Z_{j_1} \times Z_{j_2}$. If $(M, N) \in \mathcal{O}(M(j_1)) \times \mathcal{O}(M(j_2))$, then $E_\Lambda(M, N) = 0$. Since $Z_{j_1}$ and $Z_{j_2}$ are irreducible, we have $\mathcal{O}(M(j_1)) \times \mathcal{O}(M(j_2))$ is dense in $Z_{j_1} \times Z_{j_2}$. Then $E_\Lambda(Z_{j_1}, Z_{j_2}) = 0$. \(\square\)

The next result is the strongly reduced version (see [10, Theorem 5.11]) in our case of [14, Theorem 1.2] for irreducible components. It follows from [10, Theorem 5.11] and Proposition 9.18.

**Proposition 9.19.** Given a triangulation $\tau = \{j_1, \ldots, j_n\}$ of $\Sigma_n$ we have

$$Z_\tau = Z_{j_1} \oplus \cdots \oplus Z_{j_n}$$

is a strongly reduced irreducible component of $\text{decrep}(\Lambda)$.

The next proposition generalizes [10, Proposition 9.4].

**Proposition 9.20.** The set

$$\{C_\Lambda(Z) : Z \in \text{decl}\mathcal{r}_{\tau}(\Lambda), E_\Lambda(Z) = 0\}$$

generates the Caldero-Chapoton algebra $\mathcal{A}_\Lambda$ as $\mathbb{C}$-algebra. Where $\text{decl}\mathcal{r}_{\tau}(\Lambda)$ denotes the irreducible components strongly reduced of $\text{decrep}(\Lambda)$.

**Proof.** Only remains to prove that the Caldero-Chapoton functions of the non-$E$-rigid representations can be expressed in terms of the Caldero-Chapoton functions of the $E$-rigid representations. Let $L_1 = a_n^1 \cdots \varepsilon a_{n_1}$ with $m_1 < n_1$ be a string and let $m_2 \leq n$ be an integer. A direct calculation yields the following equations

$$C_\Lambda(N(L_1)) = C_\Lambda(N(L_1)) + C_\Lambda(N(W_{[m_1, n_1 - 2]})) \quad C_\Lambda(N(W_{[m_2, m_2 - 1]})) = C_\Lambda(N(W_{[m_2, m_2 - 1]})) + C_\Lambda(S_{m_2 - 1})$$

Here we set $W_{\emptyset} = 0$ and $S_0 := 0$. The proposition follows from Proposition 9.13. \(\square\)

10. The Caldero-Chapoton algebra for an arbitrary triangulation

In this section we will extend the main results of the previous section. Let $\tau$ be any triangulation of $\Sigma_n$ and let $\Lambda(\tau)$ be the algebra associated to $\tau$. We can find a triangulation $T$ of $\Sigma_n$ such that $G \cdot T = \tau$. By Lemma 8.6 we have that the push-down functor $\pi_\tau : \Lambda(T) \text{-mod} \to \Lambda(\tau) \text{-mod}$ is a $G$-covering. Recall that $G = Z_3$ acts on $\Sigma_n$ by an appropriate rotation. In this section we are going to prove that $\pi_\tau$ is a $G$-covering. We are going to use this to characterize the $E_\Lambda(\tau)$-rigid representations.

We are following the notation of [35, Section 5]. For $\tau = \{t_1, t_2, \ldots, t_n\}$ we write, unless we say something else, the triangulation $T$ according to its orbits, namely $T = \{t_{1,1}, t_{1,2}, t_{1,3}, \ldots, t_{n,1}, t_{n,2}, t_{n,3}\}$. If we denote by $x_{i,j}$ the initial cluster variable associated with the arc $t_{i,j}$, then the initial cluster will be $x_0 = (x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3})$. If we associated the variable $z_i$ to the arc $t_i$, then we obtain a morphism of algebras $\pi : \mathbb{C}[x_{1,1}^\pm] \to \mathbb{C}[z_{1,1}^\pm]$, given by $\pi(x_{i,j}) = z_i$ for $i = 1, \ldots, n$ and $j = 1, 2, 3$. The action of $Z_3$ on $T$ allow us to define the following function

$$\pi : \mathbb{N}^3 \to \mathbb{N}^n, \quad (a_{1,1}, a_{1,2}, a_{1,3}, \ldots, a_{n,1}, a_{n,2}, a_{n,3})' = a_{i,1} + a_{i,2} + a_{i,3}.$$
every block is a multiple of the identity of size 3, except for the block $n, n$ that corresponds to the adjacency of the 3-cycle of $Q(T)$ with vertices $t_{n,1}, t_{n,2}, t_{n,3}$.

**Lemma 10.1.** The push down functor $\pi_* : \Lambda(T)\text{-mod} \to \Lambda(\tau)\text{-mod}$ is a Galois-covering.

*Proof.* By Lemma 3.8 we only need to prove that $\pi_*$ is dense. Well, by Proposition 8.4 and Proposition 8.5 we know that $\Lambda(\tau)$ is a finite-dimensional gentle algebra. From Theorem 5.2 we have that the strings parametrize the indecomposable modules of $\Lambda(\tau)$. Since we work in Krull-Schmidt categories we need to prove that the Galois G-covering $\pi : \Lambda(T) \to \Lambda(\tau)$ induces a surjective function between the set of all string of those algebras and that $\pi_*(N(W)) \cong N(W)$ where $W$ is a string of $\Lambda(T)$ such that $G \cdot W = W$, recall that $N(W)$ denotes the string module associated to $W$. The last fact follows from definitions.

Suppose the pending arc of $\tau$ is based at $v_i$. Assume $W = W_2 \varepsilon^{k(W)} W_1$ is a string for $\Lambda(\tau)$ with $W_1$ a string without the letter $\varepsilon$ for $i = 1, 2$ and $k(W) \in \{-1, 0, +1\}$. Recall that $\varepsilon$ is the loop based at the pending arc of $\tau$. It is clear that if $W_1$ does not contain the letter $\varepsilon$, then $W_1$ can be lifted to a string with letters contained in one of the three fundamental region divided by the dashed blue lines, see Figure 6, say that is contained in the region that contains to $[u_i,u_{i+n+1}]$. Note that the final letter of $W_1$ must be $a_1$ or $b_i^{-1}$, see Figure 6. Now, if $k(W) = 0$, then $W$ itself can be lifted to a word in that region. If $k(W) = 1$, we choose $\varepsilon_{3,1}$ and $W_2$ can be lifted to a string in the third fundamental region containing $[u_i,u_{i+2(n+1)}]$. If $k(W) = -1$, we put the letter $\varepsilon_{2,1}$ and it is clear that $W_2$ can be lifted to a string of $\Lambda(T)$ with letter of the second fundamental region containing $[u_{i+n+1}, u_{i+2(n+1)}]$. Therefore the string $W$ can be lifted to one string $\hat{W}$ of $\Lambda(T)$. Note that $\hat{W}$ depends on where we lifted the tail point of $W_1$. The proof of the lemma is completed. □

**Lemma 10.2.** The push down functor $\pi_* : \Lambda(T)\text{-mod} \to \Lambda(\tau)\text{-mod}$ induces a Galois G-covering $\pi_* : \text{dec} \Lambda(T) \to \text{dec} \Lambda(\tau)$.

*Proof.* Let $R = \mathbb{C}^Q(T)_v$ be the vertex span of $\Lambda(T)$. We can see that $R_{Q(\tau)} = \mathbb{C}^Q(\tau)_v$ is the vertex span of $\Lambda(\tau)$. With this notation it is clear that a decorated representation $(M, V)$ is a pair where $M \in \Lambda(T)\text{-mod}$ and $V \in R\text{-mod}$. For $V \in R\text{-mod}$ we can define $\pi_*(V) \in R_{\text{G-covering}}\text{-mod}$ as in Remark 3.12. We put $\pi_*(M, V) = (\pi_*(M), \pi_*(V))$ and the lemma follows from the fact that $\text{Hom}_{\text{dec} \Lambda(T)}((M, V), (N, W)) = \text{Hom}_{\Lambda(T)}(M, N) \oplus \text{Hom}_{R}(V, W)$. □

![Figure 6](image_url)

**Figure 6.** Fundamental regions in $\widetilde{\Sigma}_m$. The sub-index in the red arrows indicates which fundamental regions they connect.

Now, we can extend Lemma 9.5 to other triangulations.

**Corollary 10.3.** Let $\sigma$ be a triangulation of $\Sigma_m$ and let $j$ be an arc of $\Sigma_m$ not in $\sigma$.

(a) Assume $M(j, \sigma)$ is not projective, then $\tau(M(j, \sigma)) = M(r^+(j), \sigma)$.

(b) Assume $M(j, \sigma)$ is not injective, then $\tau^-(M(j, \sigma)) = M(r^-(j), \sigma)$. 
This is a consequence of the $A_n$ case from [9, Theorem 2.13] and Theorem 3.9. Let $\tilde{j}$ be a lifting of $j$ in $\tilde{\Sigma}$ and let $\tilde{\sigma}$ be the lifting of $\sigma$.

(a) From [9, Theorem 2.13] we see that $\tau(M(\tilde{j}, \tilde{\sigma})) = M(r^+(\tilde{j}), \tilde{\sigma})$ and by applying $\pi_*$, from [4, Theorem 4.7 (1)], we get what we want, $\tau(M(j, \sigma)) = M(r^+(j), \sigma)$. The proof of (b) is similar.

The reader can compare the next proposition and Proposition [35, Proposition 7.15].

**Proposition 10.4.** Let $\tau$ be a triangulation of $\Sigma_n$ and let $\Lambda(\tau)$ be the algebra associated to $\tau$. Suppose $T$ is the triangulation of $\tilde{\Sigma}$ such that $G \cdot T = \tau$. If $M$ is an indecomposable representation of $\Lambda$-mod, then $\pi_*(M)$ is $E_{\Lambda(\tau)}$-rigid if and only if $E_{\Lambda(T)}(g \cdot M) = 0$ for any $g \in G$.

**Proof.** The proposition follows from Proposition 4.2 and the following equalities

$$\dim \text{Hom}_{\Lambda(\tau)}(\tau^-(\pi_*(M)), \pi_*(M)) = \dim \text{Hom}_{\Lambda(T)}(g \cdot \tau^- M), \pi_*(M))$$

$$= \dim \bigoplus_{g \in G} \text{Hom}_{\Lambda(T)}(g \cdot \tau^- M), M)$$

$$= \sum_{g \in G} \dim \text{Hom}_{\Lambda(T)}(\tau^- (g \cdot M), M).$$

Since $\pi_*$ is a Galois $G$-covering, Lemma 10.1, the first equality follows from Theorem 3.9, see [4, Theorem 4.7 (1)]. The second line follows from definition of $G$-precovering and the third line is a consequence that $g \cdot -$ is an isomorphism of categories. This conclude the proof.

**Lemma 10.5.** Let $N$ be an indecomposable representation of $\Lambda(\tau)$. Then $N$ is $E_{\Lambda}$-rigid if and only if $N = M(j, \tau')$ for some arc $j$ of $\Sigma_n$.

**Proof.** Let $M$ be a representation of $\Lambda(\tau)$ such that $\pi_*(M) = N$, recall that $\pi_*$ is dense. From [9, Corollary 2.12] we know that there is a bijection between the indecomposable representations of $\Lambda(T)$ and the diagonals of $\tilde{\Sigma}$ not in $T$. Then $M = M(j, T)$ for some arc $j$ of $\tilde{\Sigma}$. By Proposition 10.4 we know that $N$ is $E_{\Lambda(T)}$-rigid if and only if $E_{\Lambda(T)}(M, g \cdot M) = 0$. We need to analyze $\dim \text{Hom}_{\Lambda(T)}(\tau^{-1}(g \cdot M), M)$. Suppose $j = [u_l, u_{l+1}]$ for some $l \in [0, \ldots, n^\prime + 1]$, then $g \cdot M = M([u_l-n\tau^-, u_{l+n\tau^-}])$ and $\tau^{-1}(g \cdot M) = M([u_{l-n\tau^+}, u_{l+n\tau^+}])$. This means, in particular, that $g \cdot M$ is an arc representation. By [9, Lemma 2.5], the Auslander-Reiten formulas and [9, Remark 2.15] if $E_{\Lambda(T)}(M, g \cdot M) = 0$ for any $g \in I_3$, then we can conclude that $\tilde{j}$ has to be an admissible arc of $\tilde{\Sigma}$, therefore $G \cdot \tilde{j} = j$ is an arc of $\Sigma_n$ and $N = \pi_*(M(j, T)) = M(j, \tau)$. The proof of the lemma is completed.

**Remark 10.6.** Let $\mathcal{F} : A \to B$ be a Galois $G$-precovering. Then $\mathcal{F}$ is faithful, see [4, Lemma 2.6 (2)].

**Lemma 10.7.** Let $I_l$ be the indecomposable injective at $l \in Q(T)$. Then $\pi_*(I_l) \cong I_{G \cdot l}$, where $I_{G \cdot l}$ is the indecomposable injective at $G \cdot l$.

**Proof.** Let $D = \text{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ be the standard $\mathcal{C}$-dual functor. Remember that $I_l(k) = D \text{Hom}(\mathcal{C}^\vee, \mathcal{C}^\vee)$ and $\pi_*(I_l)(G \cdot k) = \bigoplus_{g \in \mathcal{Z}_3} D \text{Hom}(g \cdot \mathcal{C}^\vee, \mathcal{C}^\vee)$. By definition there exist an isomorphism $\pi_*^{k,l} : \bigoplus_{g \in \mathcal{Z}_3} \text{Hom}(g \cdot \mathcal{C}^\vee, \mathcal{C}^\vee) \to \text{Hom}(G \cdot \mathcal{C}^\vee, G \cdot \mathcal{C}^\vee)$. In other words, for any $k \in Q(T)$ we get an isomorphism $\pi_*^{k,l} : I_{G \cdot l}(G \cdot k) \to \pi_*(I_l)(G \cdot k)$. Let $\alpha : k_1 \to k_2 \in Q(T)$ be an arrow. It can be shown that $\pi_*^{k_2,l} \circ (I_{G \cdot l})_{G \cdot \alpha} = \pi_*(I_l)_{G \cdot \alpha} \circ \pi_*^{k_1,l}$. 

**Lemma 10.8.** If $f : M \to N$ is injective in $\Lambda(T)$-mod, then $\pi_*(f) : \pi_*(M) \to \pi_*(N)$ is injective in $\Lambda(\tau)$-mod.

**Proof.** Remember that $f = (f_i)_{i \in Q(T)}$ is a $3n$-tuple of linear transformations and by hypothesis we have that $\dim \text{rank } f_i = \dim M_i$. The lemma follows from Remark 3.12 and that $\dim \text{rank } \pi_*(f)_{G \cdot i} = \sum_{g \in \mathcal{Z}_3} \dim \text{rank } f_{g \cdot i}$. 

**Lemma 10.9.** Let $\tilde{j}$ be an arc of $\tilde{\Sigma}$. For $M := M(\tilde{j}) \in \Lambda(T)$-mod, let

$$0 \to M \to I_0 \to I_1$$

be a minimal injective presentation of $M$. Then

$$0 \to \pi_*(M) \to \pi_*(I_0) \to \pi_*(I_1)$$

is a minimal injective presentation of $\pi_*(M(\alpha))$. 

\end{document}
Proof. The lemma follows from the fact that \( \pi \) is dense, Lemma 10.8 and Lemma 10.7.

We shall discuss about the Caldero-Chapoton algebra associated to different triangulation. Let \( T_1 \) and \( T_2 \) be triangulations of \( \Sigma_n \). Denote by \( A_i := A(T_i) \) the Caldero-Chapoton algebra corresponding for \( i = 1, 2 \). Let \( D_i \) the \( \mathbb{C} \)-subalgebra of \( A_i \) generated by \( C_{A(T_j)}(M(\tilde{j}, T_i)) \) for any admissible arc \( \tilde{j} \) of \( \Sigma_n \) for \( i = 1, 2 \).

**Lemma 10.10.** With the above notation \( D_1 \) and \( D_2 \) are isomorphic as \( \mathbb{C} \)-algebras.

Proof. Let \( \varphi : A_1 \to A_2 \) be the corresponding isomorphism of cluster algebras. This isomorphism sends a cluster variable to the corresponding Laurent Polynomial in the initial seed associated to \( T_2 \), i.e \( x_i \mapsto C_{A(T_2)}(M(\tilde{i}, T_2)) \) for any \( \tilde{i} \) in \( T_1 \). Suppose that we can get \( T_2 \) from \( T_1 \) by the flip sequence \( (s_1, s_{l-1}, \ldots, s_1) \).

By [17] we conclude that \( C_{A(T_1)}(M(\tilde{j}, T_1)) = C_{A(T_2)}(\mu_s, \mu_s, \ldots, \mu_s(M(\tilde{j}, T_2))) \), then

\[ \varphi(C_{A(T_1)}(M(\tilde{j}, T_1))) = C_{A(T_2)}(M(\tilde{j}, T_2)). \]

That means that \( \varphi \) can be restricted to \( D_1 \) and we obtain the isomorphism desired. The lemma is completed.

\[ \square \]

**Remark 10.11.** Let \( W \) be a string of \( \Sigma_n \). Consider a lifting \( \tilde{W} \) of \( W \) on \( \tilde{\Sigma}_n \). If \( f \) is a dimension vector of some sub-representation of \( N(W) \), then \( \pi(y^{C_{Q(T)}f}) = z^{C_{Q(T)}(\pi f)} \).

Indeed, we need to prove that

\[ \pi(y_{i,1}^{C_{Q(T)}f}, y_{i,2}^{C_{Q(T)}f}, y_{i,3}^{C_{Q(T)}f}) = z_{i}^{C_{Q(T)}(\pi f)} \]

for any \( i \in [1, n] \), here \( C_{Q(T)}f \) denote the \((i, j)\)-th row of the matrix \( C_{Q(T)} \). For \( i < n \) the calculation is straightforward and we will concentrate in the case when \( i = n \). Assume \( i = n \), we orient the arcs \( t_{n,1}, t_{n,2}, t_{n,3} \) in counter clockwise on \( \tilde{\Sigma}_n \). This orientation determines the \( n, n \) block of \( C_{Q(T)} \). In order to obtain (10.1) we need that

\[ (f_{n,2} - f_{n,1}) + (f_{n,3} - f_{n,1}) = 0. \]

The observation is that (10.2) is true in case \( f \) is the dimension vector of an indecomposable representation of \( \Lambda(T) \).

**Lemma 10.12.** Let \( \tau \) be a triangulation of \( \Sigma_n \) and let \( \Lambda(\tau) \) be the algebra associated to \( \tau \). Let \( W \) be a string on \( Q(\tau) \) and let \( \tilde{W} \) be a lifting of \( W \) in \( Q(T) \), where \( T \) is the triangulation of \( \tilde{\Sigma}_n \) such that \( G \cdot T = \tau \). Then for any dimension vector \( e \) of \( N(W) \) we get

\[ \sum_{f : \pi(f) = e} \chi(\text{Gr}_f(N(\tilde{W}))) = \chi(\text{Gr}_e(N(W))). \]

Proof. First, we are going to introduce some notation. We write \( j := j(W) \) for the arc determined by \( W \), note that this arc can have self intersections. We will denote \( M(j) := N(W) \). Suppose \( j \) connects \( v_k \) and \( v_l \) with \( k \leq l \). So, we orient \( j \) from \( v_k \) to \( v_l \). Let \( x_{p_1} \) be the first intersection point between \( j \) and the pending arc \( p(\tau) \) of \( \tau \). Let \( x_{p_2} \) be the second intersection point between \( j \) and \( p(\tau) \). We divide the arc \( j \) in three parts:

- The top part \( j_{1,0} = [v_k, x_{p_1}] \).
- The center part \( j_{1,1} = [x_{p_1}, x_{p_2}] \).
- The bottom part \( j_{0,1} = [x_{p_2}, v_l] \).

Let \( \text{up}_j = \{ x_i : x_i = j_{1,0} \cap i \text{ with } i \in \tau \} \) be the upper points of \( j \). Let \( \text{bp}_j = \{ y_i : y_i = j_{0,1} \cap i \text{ with } i \in \tau \} \) be the below points of \( j \). For convention if \( j \) does not cross \( p(\tau) \), then \( x_{p_1} = v_l, x_{p_2} = v_k \) and \( \text{bp}_j = \text{up}_j \), see Figure 7.

Let \( L \in \text{Gr}_e(M(j)) \) be a sub-representation of \( M(j) \) with dimension vector \( e \). We are going to define an action of \( \mathbb{C}^* \) on \( \text{Gr}_e(M(j)) \). For \( t \in \mathbb{C}^* \) we define \( t \cdot L \) as follows:

\[ (t \cdot L)_k = \begin{cases} (t^k) \cdot \mathbb{C} & \text{if } L_k = (0) \cdot \mathbb{C} \text{ and } \dim M(j)_k = 2, \\ L_k & \text{in other wise.} \end{cases} \]

Indeed, this define an action of \( \mathbb{C}^* \) on \( \text{Gr}_e(M(j)) \). By Lemma 3.1 we know that \( \chi(\text{Gr}_e(M(j))^{\mathbb{C}^*}) = \chi(\text{Gr}_e(M(j))) \). In this case \( \text{Gr}_e(M(j))^{\mathbb{C}^*} \) is a finite set, then the Euler characteristic is its cardinality. Denote by \( Q(j) \) the full sub-quiver of \( Q(\tau) \) defined by \( j \). We consider the lifting \( \tilde{j} \) of \( j \) on \( \tilde{\Sigma}_n \). On \( \tilde{\Sigma}_n \) we can also define the corresponding top, center and bottom part of \( \tilde{j} \).
Note that if the arc $j$ does not cross $p(\tau)$, then $\tilde{j}$ is completely contained in one fundamental region of the action and $\pi$ acts as a bijection between dimension vectors of sub-representations of $M(\tilde{j})$ and dimension vector of sub-representations of $M(j)$. In other words, there exist an unique $f$ such that $\pi(f) = e$. Therefore $\chi(\text{Gr}_e(N(\tilde{W}))) = \chi(\text{Gr}_e(N(W))) = 1$.

Let $L \in \text{Gr}_e(M(j))^e$ be a sub-representation of $M(j)$. It is clear that $L$ define a walk in $Q(j)$ and also an unique subset $F(L)$ of $bp_j \cup up_j$. Indeed, the action we have defined allows to indentify every subspace $L_i$ with points of $F(L) \subset bp_j \cup up_j$ in the following way. If $L_i$ is generated by $(1,0)^t$, then we take the corresponding upper point of $j$. If $L_i$ is generated by $(0,1)^t$, then we take the corresponding below point of $j$. In case $L_i$ is 2 dimensional, then we take both, the upper and below point of $j$. It is clear that $F(L)$ determines an unique vector $f_L$ of $M(\tilde{j})$ such that $\pi(f_L) = e$. This implies that

$$\sum_{f: \pi(f) = e} \chi(\text{Gr}_e(N(\tilde{W}))) \geq \chi(\text{Gr}_e(N(W))).$$

For any vector $f$ of some sub-representation of $N(\tilde{W})$ with $\pi(f) = e$ we can find a subset $D_f$ of $bp_j \cup up_j$ corresponding to a sub-representation of $N(W)$, namely we obtain the image under $\pi_a$ of the representation given by $f$. We make this by cutting the arc $j$ on $\Sigma_n$ along the boundary of the fundamental regions that it crosses and gluing that parts on $\Sigma_n$ according to the orientation we fixed on $j$. This subset corresponds to a sub-representation $L_f$ of $N(W)$. By the definition of the action we can conclude that $L_f \in \text{Gr}_e(M(j))^e$. This shows the another inequality. Hence the lemma is completed.

![Figure 7](image_url)

**Figure 7.** An arc $j$ on $\Sigma_n$ with respect to a triangulation $\tau$.

The next proposition follows from Lemma 10.9, Remark 10.11 and Lemma 10.12. The reader can compare this result with the discussion of [35, Remark 7.9].

**Proposition 10.13.** Let $\tau$ be a triangulation of $\Sigma_n$ and let $\Lambda(\tau)$ be the algebra associated to $\tau$. Assume $T$ is the triangulation of $\Sigma_n$ such that $G \cdot T = \tau$. Then for any string $W$ of $\Lambda(\tau)$ the following equation is true

$$\pi(\mathcal{C}_{\Lambda(T)}(N(\tilde{W}))) = \mathcal{C}_{\Lambda(\tau)}(N(W)).$$

**Proof.** By expanding $\mathcal{C}_{\Lambda(T)}(N(\tilde{W}))$ and reordering its monomials as in Lemma 10.12 we are able to apply Lemma 10.9 and Remark 10.11 to any monomial. Note that from Lemma 10.9 we have that $\pi(\mathcal{G}_{\Lambda(T)}(N(\tilde{W}))) = g_{\Lambda(\tau)}(N(W))$. The proposition is completed.

11. **The Caldero-Chapoton algebra is a Chekhov-Shapiro algebra**

In this section we will state and prove our main result. Before that, we need some previous propositions.

**Proposition 11.1.** Let $\tau$ be a triangulation of $\Sigma_n$ and let $\Lambda(\tau)$ be the algebra associated to $\tau$. Assume $T$ is the triangulation of $\Sigma_n$ such that $G \cdot T = \tau$ and $j \notin \tau$. Then the $G_d$-orbit $\mathcal{O}(M(j))$ is open in $\text{rep}_d(\Lambda)$.

**Proof.** By [15, 1.7 Corollary 3] we need to prove that for any arc $j$ of $\Sigma_n$ we have that $\text{Ext}_{\Lambda(\tau)}(M(j), M(j)) = 0$. This is clear by the Auslander-Reiten formulas since $E(M(j)) = 0 = \dim \text{Hom}(\tau^-(M(j)), M(j))$. \(\square\)
If we denote by $Z(j)$ the irreducible component containing $M(j)$, then $\mathcal{O}(M(j))$ is dense in $Z(j)$. Therefore $M(j)$ is generic and all its homological data is generic in $Z(j)$. We can take generic versions of the results of the above section as in Section 9.

Repeating the arguments of Proposition 10.4 and applying what we know for the $A_n$ case, for instance see [9, Remark 2.15], we have

**Proposition 11.2.** Given a triangulation $\sigma$ of $\Sigma_n$ and two arcs $j_1, j_2 \in \sigma$ we have $E_{\Lambda(\tau)}(Z_{j_1}, Z_{j_2}) = 0$.

The next proposition shows that the $E$-rigid representations generate the corresponding Caldero-Chapoton algebra.

**Proposition 11.3.** The set

$$\{ C_{\Lambda(\tau)}(Z) : Z \in \text{decIrr}^+ (\Lambda), E_{\Lambda(\tau)}(Z) = 0 \}$$

generates the Caldero-Chapoton algebra $\mathcal{A}_{\Lambda(\tau)}$ as $\mathbb{C}$-algebra.

**Proof.** As in Proposition 9.20 we are going to prove that the Caldero-Chapoton functions of $E$-rigid representations generate the remaining Caldero-Chapoton functions. Let $\tilde{j}$ be an arc of $\tilde{\Sigma}_n$ such that it does not belong to any triangulation invariant under the action of $Z_3$. By Proposition 10.4 from these arcs come all the non-$E$-rigid representations of $\Lambda(\tau)$, so what we need to do is to prove the result in this case. In other words, we are going to prove that the Caldero-Chapoton function of $\pi_* (M(\tilde{j}))$ can be expressed in terms of the Caldero-Chapoton functions of $E$-rigid representations. For $\tilde{j}$ we construct a quadrilateral in the following way: first, we choice an ending point of $\tilde{j}$, say $u_i$. Then we draw the triangle invariant under the $Z_3$-action incident to $u_i$ with sides given by $\tilde{j}_1$, $\tilde{j}'$ and $\tilde{j}_4$. Finally, we complete the quadrilateral with the other ending point of $\tilde{j}$ such that $\tilde{j}$ and $\tilde{j}'$ are the respective diagonals. We label the remaining sides with $\tilde{j}_2$ and $\tilde{j}_3$. This construction is depicted in Figure 8. From [9, 8] we have that

$$C_{\Lambda(\tau)}(M(\tilde{j})) C_{\Lambda(\tau)}(M(\tilde{j}')) = C_{\Lambda(\tau)}(M(\tilde{j}_1)) C_{\Lambda(\tau)}(M(\tilde{j}_3)) + C_{\Lambda(\tau)}(M(\tilde{j}_2)) C_{\Lambda(\tau)}(M(\tilde{j}_4)).$$

Note that $\tilde{j}_1$, $\tilde{j}'$ and $\tilde{j}_4$ are in the same orbit. By applying the algebras homomorphism $\pi$ to the above equation, from Proposition 10.13, we obtain

$$C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}))) C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}'))) = C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}_1))) C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}_3))) + C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}_2))) C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}_4))).$$

Since we are in an integral domain, we have the desired relation

$$C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}))) = C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}_1))) + C_{\Lambda(\tau)}(\pi_* (M(\tilde{j}_2))).$$

The proposition is completed. □

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{The quadrilateral with $\tilde{j}$ as diagonal and with two adjacent sides of one invariant triangle of $\tilde{\Sigma}_n$.}
\end{figure}

From the above proposition we obtain our main result.
Theorem 11.4. For any triangulation $\tau$ of $\Sigma_n$ we have that the Caldero-Chapoton algebra $A_{\Lambda(\tau)}$ is isomorphic to the Chekhov-Shapiro algebra $A_{\Lambda(\tau_0)}$.

Proof. Let $T_1$ and $T_2$ be triangulations of $\tilde{\Sigma}_n$. Denote by $A_i := A_{\Lambda(T_i)}$ the Caldero-Chapoton algebra corresponding for $i = 1, 2$. Let $D_i$ be the $C$-subalgebra of $A_i$ generated by $C_{\Lambda(T_j)}(\hat{M}(j, T_i))$ for any admissible arc $j$ of $\tilde{\Sigma}_n$ for $i = 1, 2$. By Lemma 10.10 we have that $D_1$ and $D_2$ are isomorphic. The previous Proposition and Proposition 10.13 show that the Caldero-Chapoton algebra associated to $\tau_i = G \cdot T_i$ is $\pi(D_i)$ for $i = 1, 2$. We conclude that the Caldero-Chapoton algebras $A_1$ and $A_2$ are isomorphic. Now, from [35] we know that the image of $\Lambda(\tau_0)$ under $\pi$ is a Chekhov-Shapiro generalized cluster algebra with initial seed $(B(\tau_0), d_{\tau_0})$. Actually, from [35, Lemma 5.6] and [35, Lemma 5.7] we know that the exchange polynomial for $\Lambda(\tau_0)$ under $\pi$ is isomorphic to $\Lambda(\tau_0)$ and $\Lambda(\tau_0)$ is isomorphic. The theorem is completed.

12. Example

The example in this section is meant to illustrate our main result. It can also be considered a complement to [10, Example 9.4.2]. Let $\tau_0$ be the special triangulation of $\Sigma_3$, see Figure 3, and let $\tau$ be the triangulation of Example 7.4.

It is clear that $\tau$ and $\tau_0$ are related by a flip at one arc. To ease the notation we set $\Lambda = \Lambda(\tau)$, see Example 7.4. From Theorem 5.2 we know that the indecomposable $\Lambda$-modules are parametrized by the strings of $\Lambda$. We say that a string $W$ is $E$-rigid if its string module $N(W)$ is $E$-rigid. There are 12 indecomposable $E$-rigid decorated representations of $\Lambda$ of which 9 are given by the $E$-rigid strings $1_1, 1_2, \varepsilon, a, eb, \varepsilon eb, b^{-1}eb$ and $\varepsilon ec^{-1}$; and the remaining three are the negative simple representations of $\Lambda$. The non-$E$-rigid strings are $1_3, b, c, b^{-1}\varepsilon, \varepsilon ec^{-1}$ and $b^{-1}\varepsilon c^{-1}$.

By definition $C_{\Lambda}(\Sigma^i) = y_i$ for $i = 1, 2, 3$. In Figure 9 we write the string module corresponding to every arc of $\Sigma_3$. The Caldero-Chapoton functions associated to the 9 $E$-rigid strings of $\Lambda$ are

$$C_{\Lambda}(S_1) = \frac{y_2 + y_3}{y_1},$$
$$C_{\Lambda}(S_2) = \frac{y_1 + y_3}{y_2},$$
$$C_{\Lambda}(N(a)) = \frac{y_2 + y_3 + y_1}{y_1 y_2},$$
$$C_{\Lambda}(N(\varepsilon)) = y_3^2 + y_1 y_2 + y_2 y_3,$$
$$C_{\Lambda}(N(\varepsilon eb)) = y_1 y_3 + y_1^2 + y_1 y_2 + y_2^2,$$
$$C_{\Lambda}(N(\varepsilon b)) = \frac{y_1 y_3 + y_1^2 + y_1 y_2 + y_2^2}{y_1 y_2 y_3},$$
$$C_{\Lambda}(N(\varepsilon ec^{-1})) = \frac{y_3^2 + 2 y_1 y_3 + y_1^2 + y_2 y_3 + y_1 y_2 + y_2^2}{y_1 y_2 y_3}.$$

Figure 9. The nine $E$-rigid representations of $\Lambda$ with respect to the triangulation $\tau$ on $\Sigma_3$. 
The Caldero-Chapoton functions associated to the non-$E$-rigid strings of $\Lambda$ are

\[
C_\Lambda(S_3) = y_1 + y_2, \\
C_\Lambda(N(b)) = \frac{y_1 + y_2 + y_3}{y_1}, \\
C_\Lambda(N(c)) = \frac{y_3 + y_1 + y_2}{y_2}, \\
C_\Lambda(N(b^{-1}c^{-1})) = \frac{y_1 y_3 + y_1^2 + y_1 y_2 + y_2 y_3 + y_3^2 + y_1 y_3 + y_2 y_3}{y_1 y_2 y_3}.
\]

Remark 12.1. According to Proposition 11.3 we have that the Caldero-Chapoton functions of indecomposable $E$-rigid representations generate the remaining Caldero-Chapoton functions. In this case we have the following relations

\[
C_\Lambda(S_3) = C_\Lambda(S_1^-) + C_\Lambda(S_2^-), \\
C_\Lambda(N(b)) = C_\Lambda(S_1) + 1, \\
C_\Lambda(N(c)) = C_\Lambda(S_2) + 1, \\
C_\Lambda(N(b^{-1}c)) = C_\Lambda(N(ec)) + 1, \\
C_\Lambda(N(ce)) = C_\Lambda(N(ce)) + 1, \\
C_\Lambda(N(b^{-1}c^{-1})) = C_\Lambda(N(ceb)) + C_\Lambda(N(a)).
\]

These relations correspond to the algorithm of the proof of Proposition 11.3. With the notation of [10, Example 9.4.2] we can define the following isomorphism of the corresponding Caldero-Chapoton algebras $\varphi : A_\Lambda(\tau) \to A_\Lambda(\tau_0)$. We set $y_1 \mapsto x_1$, $y_2 \mapsto C_\Lambda(\tau_0)(2) = \frac{x_1 + x_2}{x_2}$ and $y_3 \mapsto x_3$. This morphism sends the Caldero-Chapoton function of the arc representation associated to one arc of $\tau$ to the Caldero-Chapoton function with respect to $\Lambda(\tau_0)$ of the same arc.

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