Abstract

Let \((M, \partial M)\) be a 3-manifold with incompressible boundary that admits a convex co-compact hyperbolic metric. We consider the hyperbolic metrics on \(M\) such that \(\partial M\) looks locally like a hyperideal polyhedron, and we characterize the possible dihedral angles.

We find as special cases the results of Bao and Bonahon \cite{BB02} on hyperideal polyhedra, and those of Rousset \cite{Rou02} on fuchsian hyperideal polyhedra. Our results can also be stated in terms of circle configurations on \(\partial M\), they provide an extension of the Koebe theorem on circle packings.

The proof uses some elementary properties of the hyperbolic volume, in particular the Schl"afli formula and the fact that the volume of (truncated) hyperideal simplices is a concave function of the dihedral angles.

Résumé

Soit \((M, \partial M)\) une variété de dimension 3 à bord incompressible, qui admet une métrique hyperbolique convexe co-compacte. On considère les métriques hyperboliques sur \(M\) pour lesquelles le bord ressemble localement à un polyèdre hyperbolique hyperidéal, et on caractérise les angles diédres possibles.

On retrouve comme cas particulier les résultats récents de Bao et Bonahon \cite{BB02} pour les polyèdres hyperidéaux, et de Rousset \cite{Rou02} pour les polyèdres hyperidéaux fuchsiens. Nos résultats peuvent aussi s’exprimer en terme de configurations de cercles sur le bord de \(M\), ils donnent une extension du théorème de Koebe sur les empilements de cercles.

La preuve repose sur les propriétés élémentaires du volume hyperbolique, en particulier sur la formule de Schl"afli et sur le fait que le volume des simplexes hyperidéaux (tronqués) est une fonction concave des angles diédres.

Contents

1 Hyperideal manifolds 6
2 Hyperideal triangles 9
3 Hyperideal simplices 12
4 Hyperideal polyhedra 18
5 Rigidity 24
6 Compactness 26
7 Spaces of polyhedra 33
8 Induced metrics 38
9 Circle configurations 42
10 Remarks 44

*Laboratoire Emile Picard, UMR CNRS 5580, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4, France. schlenker@picard.ups-tlse.fr; http://picard.ups-tlse.fr/~schlenker.
A The concavity of the volume at a regular simplex

Hyperbolic manifolds with boundary In all this paper, we will consider a 3-manifold with boundary \((M, \partial M)\). We suppose that it admits a complete, convex co-compact hyperbolic metric. This is a topological assumption, which could be stated in purely topological terms (see e.g. [Tim97]).

A natural question is to understand all the hyperbolic metrics on \(M\) in terms of quantities that can be read on the boundary. For complete, convex co-compact metrics, this is achieved by the (hyperbolic version of) the Ahlfors-Bers theorem \([Ahl66]\), which states that those metrics are uniquely determined by the conformal structure induced on \(\partial M\).

When we consider hyperbolic metrics such that \(\partial M\) is smooth and strictly convex (i.e. the boundary is at finite distance) there are some related results. First, the induced metrics on the boundary are exactly the metrics with curvature \(K > -1\), and each is obtained in exactly one way \([Lab92, Sch02]\). In addition, the third fundamental forms of the boundary (see section 1) are exactly the metrics with curvature \(K < 1\) and closed, contractible geodesics of length \(L > 2\pi\) \([Sch96, LS00, Sch02]\); this means that \(\partial M\) is globally CAT(1).

Hyperideal boundaries It looks like those statements should not be restricted to the case where the boundary is smooth; moreover, they should allow some situations where the boundary of \(M\) has some points at infinity. For instance, one can consider "ideal hyperbolic manifolds" in the following sense. First note that, given a hyperbolic metric \(g\) with convex boundary on \(M\), there is a unique complete, convex co-compact hyperbolic manifold \(E(M)\) in which \((M, g)\) can be isometrically embedded in such a way that the induced morphism \(\pi_1 M \to \pi_1 E(M)\) is an isomorphism. Then \((E(M), g) = H^3\), and \(\pi_1 M\) has a natural action on \(H^3\) by isometries.

Definition 0.1. Let \(g\) be a hyperbolic metric with convex boundary on \(M\). We say that \((M, g)\) is an ideal hyperbolic manifold if:

- for each convex ball \(\Omega \subset H^3\) and each isometric embedding \(\phi : \Omega \to E(M)\), the intersection of \(M\) with \(\phi(\Omega)\) the image by \(\phi\) of the intersection with \(\Omega\) of an ideal polyhedron \(P \subset H^3\).

- \(\partial M\) contains no closed curve which is a geodesic of \(M\).

This definition is a natural extension of the notion of ideal polyhedra in \(H^3\). As for ideal polyhedra, the third fundamental forms of those manifolds is a measure located on the edges; understanding it is equivalent to understanding the dihedral angles. Another rather simple example is given by what can be called the "fuchsian" case, when \(E(M)\) is the quotient of \(H^3\) by the \(\pi_1\) of a closed surface which acts on \(H^3\) fixing a totally geodesic plane \(P_0\), and the universal cover of \(M\), seen as a subset in \(H^3\), is invariant under the reflection in \(P_0\).

For hyperbolic polyhedra, the description of the third fundamental form reduces to the condition that the dual graph of the polyhedron is CAT(1), see \([Rav90]\). More precisely, that its closed paths have length \(L \geq 2\pi\), with equality exactly when they bound a face. In the more general case of an ideal manifold, the result is similar \([Sch01b]\).

We will consider analogs of the ideal manifolds, but replacing the notion of ideal polyhedron by the more general notion of hyperideal polyhedron. A hyperideal polyhedron can be defined in at least two equivalent ways.

- As the intersection of a finite set of half-spaces in \(H^3\), with the condition that, for each end \(E\), either all faces adjacent to \(E\) intersect in one ideal point, or there exists a plane which is orthogonal to all the faces adjacent to \(E\).

- Using the projective model of \(H^3\) as the open unit ball \(B^3\) in \(\mathbb{R}^3\), the hyperideal polyhedra are the intersections with \(H^3\) of the (convex) polyhedra in \(\mathbb{R}^3\) with all vertices outside \(B^3\), but with all edges intersecting \(B^3\).

Note that those two definitions allow for some ideal vertices, i.e. vertices on the boundary at infinity of \(H^3\). The other vertices are called "strictly hyperideal", and a hyperideal polyhedron with no ideal vertex is called "strictly hyperideal".

The same definition as for ideal manifolds can be used to define "hyperideal manifolds", i.e. hyperbolic manifold with a boundary that looks locally like a hyperideal hyperbolic polyhedron.

Definition 0.2. Let \(g\) be a hyperbolic metric with convex boundary on \(M\). We say that \((M, g)\) is a hyperideal hyperbolic manifold if:
• for each convex ball \( \Omega \subset H^3 \) and each isometric embedding \( \phi : \Omega \to E(M) \), the intersection of \( M \) with \( \phi(\Omega) \) is the image by \( \phi \) of the intersection with \( \Omega \) of a hyperideal polyhedron \( P \subset H^3 \).

• \( \partial M \) contains no closed curve which is a geodesic of \( M \).

This definition, and in particular the second point, is designed to exclude some "bad" situations where \( \partial M \) has non-empty intersection with the convex core of \( M \). More details on this can be found in section 7. Given a 3-manifold \( M \) with boundary, a "hyperideal metric" on \( M \) is a hyperbolic metric such that \((M, g)\) is a hyperideal hyperbolic manifold. We will sometimes call this a "hyperideal hyperbolic structure" on \( M \).

The main goal of this paper is to understand the possible dihedral angles of hyperideal manifolds, and to obtain a result similar to the result obtained for hyperideal polyhedra in [BB02].

**Dihedral angles** Before stating the main results, we have to define some sequences of edges which play a special role. We consider now a cellulation \( \sigma \) of \( \partial M \), i.e. a decomposition of \( \partial M \) in the union of a finite number of embedded images of the interior of polygons in \( \mathbb{R}^2 \), with disjoint interior, such that the intersection of two adjacent polygons is an edge of each. We will call \( \sigma_1 \) the 1-skeleton of \( \sigma \), which is a graph, and \( \sigma^*_1 \) the dual graph.

**Definition 0.3.** A circuit in \( \sigma \) is a sequence \( e_0, e_1, \cdots, e_n = e_0 \) which corresponds to the successive edges of a closed path in \( \sigma^*_1 \) which is homotopically trivial in \( M \). A circuit is elementary if the dual closed path in \( \sigma^*_1 \) bounds a face.

**Definition 0.4.** A simple path is a sequence of edges \( e_1, \cdots, e_n \) in \( \sigma_1 \) corresponding to the successive edges of a path in \( \sigma^*_1 \), which:

• begins and ends at boundary points of a face \( f \) of \( \sigma^*_1 \).
• is not included in the boundary of \( f \).
• is homotopic in \( M \) to a segment in \( f \).

We can now state our main result.

**Theorem 0.5.** Suppose that \( M \) has incompressible boundary. Let \( \sigma \) be a cellulation of \( \partial M \), and let \( w : \sigma_1 \to (0, \pi) \) be a map on the set of edges of \( \sigma \). There exists a hyperideal hyperbolic structure on \( M \), with boundary combinatorics given by \( \sigma \) and exterior dihedral angles given by \( w \), if and only if:

• the sum of the values of \( w \) on each circuit in \( \sigma_1 \) is greater than \( 2\pi \), and strictly greater if the circuit is non-elementary.

• The sum of the values of \( w \) on each simple path in \( \sigma_1 \) is strictly larger than \( \pi \).

This hyperideal structure is then unique.

This is an extension of the main result of [Sch01b], which concerns ideal manifolds only.

**Outline of the proof** The proof is related to the method used in [Sch01b]; the starting point is the Schläfli formula, which describes the first-order variations of the volume of a polyhedron in terms of the variation of its dihedral angles (see section 1). A simple consequence, obtained in section 3, is that the volume of hyperideal simplices (the definition is below) is a strictly concave function of the dihedral angles. This fact was well known to be true for ideal simplices, and this is the basis for several important constructions concerning ideal polyhedra (see e.g. [Thu97], chapter 7, [Riv94, Riv96]).

**Lemma 3.17.** For each \( i \in \{0, \cdots, 4\} \), the volume \( V \) is a strictly concave function on the space of hyperideal simplices having exactly \( i \) ideal vertices \( v_1, \cdots, v_i \), parametrized by the dihedral angles.

A consequence of the concavity of the volume is obtained using an interesting technique, based on deformations among singular hyperbolic structures to get hyperbolic metrics; those ideas can be traced back to the work of Thurston [Thu97] on the Andreev theorem [And71], and then of Colin de Verdière [CaV01], Brügger [Brü92], and Rivin [Riv94]. Applying those ideas to hyperideal polyhedra, one obtains the following result, which we will prove in section 4 since this proof is partly different from the one given by Bao and Bonahon. It is a special case of theorem 0.5 but also a tool in its proof.
Lemma 4.1 (Bao, Bonahon [BB02]). Let $\sigma$ be a cellulation of $S^2$, and let $w: \sigma_1 \to (0, \pi)$ be a map on the set of edges of $\sigma$. There exists a hyperideal polyhedron with combinatorics given by $\sigma$ and exterior dihedral angles given by $w$ if and only if:

- the sum of the values of $w$ on each circuit in $\sigma_1$ is greater than $2\pi$, and strictly greater if the circuit is non-elementary.
- The sum of the values of $w$ on each simple path in $\sigma_1$ is strictly larger than $\pi$.

This hyperideal polyhedron is then unique.

Since the sum of a finite number of concave functions is concave, lemma 3.17 can be used to prove that the volume of any hyperideal polyhedron is also a concave function of the dihedral angles, and this actually also applies to hyperideal manifolds.

Lemma 4.2 Let $\sigma$ be a cellulation of $S^2$. The volume is a strictly concave function of the dihedral angles, on the space of hyperideal polyhedra with combinatorics given by $\sigma$.

We will define in section 4 hyperideal cellulations as decompositions of a hyperideal manifolds in isometric images of hyperideal polyhedra, with some non-degeneracy conditions. Section 4 contains the proof of the:

Lemma 4.12 Any hyperideal manifold admits a hyperideal cellulation.

Using this, the concavity of the volume of polyhedra and the Schl"afli formula, we will obtain in section 5 an infinitesimal rigidity statement.

Lemma 5.1 Let $M$ be a hyperideal manifold. Any first-order deformation of its dihedral angles is obtained by a unique first-order deformation of $M$.

This can be considered as the key lemma of this paper. The infinitesimal rigidity of hyperbolic manifolds with convex boundary is a problem that can be tackled in different ways. For instance, for polyhedra, it is proved in [BB02] by the Cauchy-Legendre method; in [Sch02], the result is obtained using some analytic estimates and transformations defined by Pogorelov to translate rigidity problems from $H^3$ to $\mathbb{R}^3$. The method used here is completely different.

Another important element, as in many deformation proofs, is to obtain a compactness result. This is done in section 6, where the following result is proved.

Lemma 6.1 Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of hyperideal structures on $M$, with the same boundary combinatorics. For each $n$, let $\alpha_n$ be the function which associates to each boundary edge of $(M, g_n)$ its exterior dihedral angle, and suppose that $\alpha_n \to \alpha$, where $\alpha$ still satisfies the hypothesis of theorem 4.3. Then, after taking a subsequence, $g_n$ converges to a hyperideal structure $g$ on $M$.

Finally, a technically important point is to prove that the space of dihedral angle assignments appearing in the hypothesis of theorem 5.3 is connected. The key point is that, when $M$ has incompressible boundary, the conditions on the dihedral angles on the various boundary components of $\partial M$ behave independently, so that it is sufficient to prove the connectedness of the space of angles in the "fuchsian" case, i.e. when one considers a manifold which is topologically $S \times \mathbb{R}$, where $S$ is a compact surface of genus at least 2, with an isometric involution fixing a compact surface. But this case is well understood thanks to a result of Rousset, who proved in this case the analog of theorem 6.3 although by very different methods.

Theorem 7.15 (M. Rousset [Rou02]). Let $S$ be a compact surface of genus at least 2, let $\sigma$ be a cellulation of $S$, and let $w: \sigma_1 \to (0, \pi)$ be a map on the set of edges of $\sigma$. There exists a hyperideal fuchsian realization of $S$, with boundary combinatorics given by $\sigma$ and exterior dihedral angles given by $w$, if and only if:

- The sum of the values of $w$ on each circuit in $\sigma_1$ is greater than $2\pi$, and strictly greater if the circuit is non-elementary.
- The sum of the values of $w$ on each simple path in $\sigma_1$ is strictly larger than $\pi$.

This hyperideal realization is then unique.

The Koebe circle packing theorem The hyperideal polyhedra are related to the Koebe circle packing theorem. Recall that a circle packing in $S^2$ is a finite set of circles bounding disjoint open disks, and the incidence graph of a circle packing is the (combinatorially defined) graph with one vertex for each circle, and an edge between two circles if and only if the circles are tangent.
Theorem 0.6 (Koebe [Koe36]). Let \( \Gamma \) be the 1-skeleton of a triangulation of \( S^2 \). There is a unique circle packing in \( S^2 \) with incidence graph \( \Gamma \).

The uniqueness here is up to Möbius transformations. This theorem has an extension to graphs which are the 1-skeleton of a cellulation of \( S^2 \), but the uniqueness demands some additional hypothesis. Given a circle packing, an interstice is a connected component of the complement of the (closed) disks bounded by the circles.

Theorem 0.7 (Koebe [Koe36]). Let \( \Gamma \) be the 1-skeleton of a cellulation of \( S^2 \). There is a unique circle packing \( C \) in \( S^2 \) such that:

- the incidence graph of \( C \) is \( \Gamma \).
- for each interstice \( I \), there is a circle orthogonal to each circle of \( C \) adjacent to \( I \).

Thurston [Thu97] realized that the Koebe circle packing is related to ideal polyhedra and the Andreev theorem. There is however a simpler relationship between circle packings and hyperideal polyhedra. Let \( \sigma \) be a cellulation of \( S^2 \). Consider the function \( w : \sigma_1 \to (0, \pi) \) defined by \( w(e) = \pi - \epsilon \) for each edge \( e \) of \( \sigma \). By lemma \ref{lemma:epsilon} if \( \epsilon \) is small enough, there is a unique hyperideal polyhedron \( P_w \) with combinatorics given by \( \sigma \) and exterior dihedral angles by \( w \).

The intersections of the faces of \( P_w \) (or more precisely of the planes containing the faces) with \( \partial_\infty H^3 \) are circles; when two circles correspond to faces with a common edge \( e \), they intersect with angles \( w(e) \). Moreover, the truncated faces (see sections 1 and 3) correspond to another family of circles, which intersect the circles of the first family orthogonally.

Taking the limit as \( w \to \pi \) on each edge, we find two families \( F_1, F_2 \) of circles on \( S^2 \), such that:

- the circles of \( F_1 \) correspond to the faces of \( \sigma \). Those circles intersect if and only if the corresponding faces of \( \sigma \) share an edge, and they are then tangent.
- the circles of \( F_2 \) correspond to the vertices of \( \sigma \). They intersect if and only if the corresponding vertices are adjacent, and they are then tangent.
- a circle \( c_1 \) of \( F_1 \) intersects a circle \( c_2 \) of \( F_2 \) if and only if the face corresponding to \( c_1 \) contains the vertex corresponding to \( c_2 \). The intersection is then orthogonal.

This is another description of theorem \ref{theorem:interstice}. Note that, by the Schlafli formula (see equations \ref{equation:slf} in section 3), the limit taken here corresponds to letting the volume of the polyhedra go to its maximal value. The same line of reasoning leads from theorem \ref{theorem:interstice} to the following extension of the Koebe circle packing theorem. We first state the simpler form where we only consider triangulations.

Theorem 0.8. Suppose that \( M \) has incompressible boundary. Let \( \Gamma \) be the 1-skeleton of a triangulation of \( \partial M \). There exists a unique couple \((c, C)\), where \( c \) is a \( \mathbb{CP}^1 \)-structure on \( \partial M \) induced by a complete, convex co-compact hyperbolic metric on \( M \), and \( C \) is a circle packing of \((\partial M, c)\) with incidence graph \( \Gamma \).

Theorem 0.9. Suppose that \( M \) has incompressible boundary. Let \( \sigma \) be a cellulation of \( \partial M \). There exists a unique triple \((c, F_1, F_2)\), where:

- \( c \) is a \( \mathbb{CP}^1 \)-structure on \( \partial M \) induced by a complete, convex co-compact hyperbolic metric on \( M \).
- \( F_1 \) and \( F_2 \) are circle packings of \((\partial M, c)\).
- the circles of \( F_1 \) correspond to the faces of \( \sigma \). Those circles intersect if and only if the corresponding faces of \( \sigma \) share an edge, and they are then tangent.
- the circles of \( F_2 \) correspond to the vertices of \( \sigma \). They intersect if and only if the corresponding vertices are adjacent, and they are then tangent.
- a circle \( c_1 \) of \( F_1 \) intersects a circle \( c_2 \) of \( F_2 \) if and only if the face corresponding to \( c_1 \) contains the vertex corresponding to \( c_2 \). The intersection is then orthogonal.

This result contains as a special case, obtained by considering the "fuchsian" situation, some known results on circle packings on surfaces of genus at least 2 with a hyperbolic metric (see [Thi97, Cav91]). A slightly more general example is obtained by considering a closed surface \( S \) of genus \( g \geq 2 \), and the 3-manifold \( M := S \times \mathbb{R} \). Then \( M \) has two boundary components, which we call \( S_1 \) and \( S_2 \), each diffeomorphic to \( S \). Let \( \Gamma_1 \) and \( \Gamma_2 \)
be the 1-skeletons of finite triangulations, in $S_1$ and $S_2$ respectively. There is then a unique quasi-fuchsian hyperbolic metric on $M$, inducing $CP^1$-structures $c_1$, $c_2$ on $S_1$ and $S_2$, a unique circle packing $C_1$ in $S_1$ for $c_1$ and a unique circle packing $C_2$ in $S_2$ for $c_2$, such that the incidence graph of $C_1$ is $\Gamma_1$, the incidence graph of $C_2$ is $\Gamma_2$.

Theorem 9.9 is actually a limit case of a more general statement on configurations of circles, theorem 9.10, which is a direct translation of theorem 9.3. The configurations appearing there have two families of circles, and are more general than those usually associated to (generalizations of) the Andreev theorem on ideal polyhedra. To each such circle configuration one associates a volume, and theorem 9.9 is obtained as the limit case when the volume is maximal. The proof of theorem 9.9 along with some additional details, can be found in section 9.

**Induced metrics** Once we know that the hyperideal manifolds, with given boundary combinatorics, are parametrized by their dihedral angles, the fact that the volume has non-degenerate hessian can be translated, using the Schl"afli formula, into an **infinitesimal rigidity** statement: any non-zero first-order deformation induces a non-trivial deformation of the metric induced on the boundary.

**Lemma 8.7.** Let $(M, g)$ be a hyperideal manifold. It has no first-order deformation (among hyperideal manifolds) which does not change the induced metric on $\partial M$.

Using this lemma, we can recover rather simply a result describing the induced metrics on hyperideal polyhedra (see [Sch98a]).

**Theorem 8.9.** Let $h$ be a complete hyperbolic metric on $S^2$ minus a finite number of points. There is a unique hyperideal polyhedron on $H^3$ whose induced metric is $h$.

The same arguments also yield an analogous result for the fuchsian case.

**Theorem 8.10.** Let $S$ be a compact surface with non-empty boundary of genus at least 2, and let $h$ be a complete hyperbolic metric on $S$ minus a finite number of points. There is a unique fuchsian hyperideal manifold $(M, g)$ such that the induced metric on both connected components of $\partial M$ is $h$.

## 1 Hyperideal manifolds

We recall here some basic facts about hyperbolic geometry, in particular hyperideal polyhedra.

**Hyperbolic 3-space and the de Sitter space** Hyperbolic 3-space can be constructed as a quadric in the Minkowski 4-space $\mathbb{R}^4_1$, with the induced metric:

$$H^3 := \{ x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = -1 \land x_0 > 0 \}.$$  

But $\mathbb{R}^4_1$ also contains another quadric, the de Sitter space of dimension 3:

$$S^3_1 := \{ x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = 1 \}.$$

By construction, it is Lorentzian and has an action of $SO(3, 1)$ which is transitive on orthonormal frames, so it has constant curvature; one can easily check that its curvature is 1. $S^3_1$ contains many space-like totally geodesic 2-planes, each isometric to $S^2$ with its canonical round metric. Each separates $S^3_1$ into two “hemispheres”, each isometric to a model which we will denote by $S^3_{1,+}$. For instance, in the quadric above, the set of points $x \in S^3_{1,+}$ with positive first coordinate $x_0 > 0$ is a hemisphere.

**The third fundamental form and the dual metric** Consider a smooth surface $S \subset H^3$. The Riemannian metric on $H^3$ defines by restriction a Riemannian metric on $S$, which is called the induced metric, or first fundamental form, of $S$. We will denote it by $I$.

There is another metric, the third fundamental form, which is defined on a smooth, strictly convex surface $S$ in $H^3$. To define it, let $N$ be a unit normal vector field to $S$, and let $D$ be the Levi-Civit\'a connection of $H^3$; the second fundamental form of $S$ is defined by:

$$\forall s \in S, \forall x, y \in T_s S, \quad \langle x, y \rangle = -I(D_x N, y) = -I(x, D_y N).$$
and the third fundamental form by:

$$\forall s \in S, \forall x, y \in T_{x}S, \mathcal{III}(x, y) = I(D_{x}N, D_{y}N)$$.

The same definition applies in $S^3_1$.

There is a polyhedral analog of the third fundamental form. For a compact polyhedron $P \subset H^3$, it can be defined by gluing, for each vertex $v$ of $P$, the interior of a spherical polygon which is the dual of the link of $P$ at $v$. The duality which is used here is the projective duality in $S^2$, so that the polygon has an edge of length $\alpha$ for each edge adjacent to $v$ with exterior dihedral angle $\alpha$. This third fundamental form is also often called the "dual metric" of the polyhedron. The definition of the dual metric for ideal or hyperideal polyhedra is outlined below using the hyperbolic-de Sitter duality.

**The hyperbolic-de Sitter duality** The understanding of the third fundamental form of surfaces in $H^3$ relies heavily on an important duality between $H^3$ and the de Sitter space $S^3_1$. It associates to each point $x \in H^3$ a space-like, totally geodesic plane in $S^3_1$, and to each point $y \in S^3_1$ an oriented totally geodesic plane in $H^3$.

It can be defined using the quadric models of $H^3$ and $S^3_1$. Let $x \in H^3$; define $d_x$ as the line in $\mathbb{R}^4_x$ going through 0 and $x$. $d_x^*$ is a time-like line; call $d_x^*$ the orthogonal space in $\mathbb{R}^4_x$, which is a space-like 3-plane. So $d_x^*$ intersects $S^3_1$ in a space-like totally geodesic 2-plane, which we call $x^*$, and which is the dual of $x$. Conversely, given a space-like totally geodesic plane $p \in S^3_1$, it is the intersection with $S^3_1$ of a space-like 3-plane $P \ni 0$. Let $d$ be its orthogonal, which is a time-like line; the dual $p^*$ of $p$ is the intersection $d \cap H^3$.

The same construction works in the opposite direction. Given a point $y \in S^3_1$, we call $d_y$ the oriented line going through 0 and $y$, and $d_y^*$ its orthogonal, which is an oriented time-like 3-plane. Then $y^* := d_y^* \cap H^3$ is an oriented totally geodesic plane.

We can then define the duality on surfaces. Given a smooth, oriented surface $S \subset H^3$, its dual $S^*$ is the set of points in $S^3_1$ which are the dual of the oriented planes which are tangent to $S$. If $S$ is smooth and strictly convex, then $S^*$ is smooth, space-like, and strictly convex. Conversely, given a smooth, space-like surface $\Sigma \subset S^3_1$, its dual is the set $\Sigma^*$ of points in $H^3$ which are the duals of the planes tangent to $\Sigma$.

One of the main properties of this duality, on smooth surfaces, is that the induced metric on $\Sigma^*$ is the third fundamental form of $\Sigma$, while the third fundamental form of $\Sigma^*$ in $S^3_1$ is the induced metric on $\Sigma$.

**The dual metric of hyperbolic polyhedra** The hyperbolic-de Sitter duality works also for compact polyhedra in $H^3$ (for which it was introduced in [Riv86, RH93]). Given a compact polyhedron $P \subset H^3$, its dual is the convex, space-like polyhedron $P^* \subset S^3_1$ with:

- as vertices, the duals of the planes containing the faces of $P$;
- as edges, segments of the geodesics dual to the geodesics containing the edges of $P$;
- as faces, polygons in the planes dual to the vertices of $P$.

$P^*$ can be defined as the set of points dual to the support planes of $P$. The main point is that the induced metric on $P^*$ is the dual metric of $P$. It is a spherical cone-metric, with singular points corresponding to the vertices of $P^*$, where the singular curvature is negative.

When $P$ has ideal vertices, the set of points duals to the support planes of $P$ has a "hole" for each ideal vertex $v$ of $P$. This hole corresponds, in the projective model of $H^3$, to the face of the dual of $P$ (defined in projective terms) tangent to $\partial D^3$ at $v$. The length $l$ of the boundary of this face is equal to the sum of the exterior angles of the link of $P$ at $v$, which is Euclidean; thus $l = 2\pi$. For instance, if $P$ is an ideal polyhedron, the set of dual points of its support planes is a graph, which we will call its dual graph. It defines a cellulation of $S^2$, with each face of length $2\pi$. Figure 1 represents, in the projective model of $H^2$, the dual of an ideal polygon, which is a finite set of points; it should help understand what the dual of an ideal polyhedron in $H^3$ is.

When $P$ has some ideal points, we will define its dual metric $\mathcal{III}$ as the metric obtained by gluing in the "holes" corresponding to the ideal points a hemisphere (i.e. isometric to a hemisphere of $S^2$ with its canonical round metric). The result is a metric space which obviously has negative singular curvature at its singular points, because the singular points correspond to the vertices of the graph, and the total angle around those points is $\pi$ times the number of faces. $\mathcal{III}$ is the “natural” third fundamental form for instance in a limit sense, as follows:
Property 1.1. Let \((\Omega_n)_{n \in \mathbb{N}}\) be an increasing sequence of open subsets of the interior of \(P\) with smooth, convex boundary, such that \(\cup_n \Omega_n\) is the interior of \(P\). Then the third fundamental forms of \(\partial \Omega_n\) converge to the dual metric of \(P\).

We leave the proof to the reader.

For a hyperideal polyhedron \(P\), one can still consider the set of support planes, which is a graph in the de Sitter space. The total length of the edges of a face, however, is strictly greater than \(2\pi\) for faces corresponding to strictly hyperideal vertices. To define the dual metric, one has to glue in each of the faces a "singular hemisphere", obtained as the quotient by a rotation of angle \(\theta > 2\pi\) of the universal cover of the complement of the center in a hemisphere.

Truncated hyperideal polyhedra Given a hyperideal polyhedron \(P \subset H^3\), there is, for each strictly hyperideal \(v\) vertex of \(P\), a plane \(p\) which is orthogonal to all the faces of \(P\) adjacent to \(v\). \(p\) is the dual of \(v\) in the hyperbolic-de Sitter duality. Following [BB02], we call \(P_t\) the polyhedron obtained by cutting all the ends of \(P\) by this plane. \(P_t\) is a polyhedron of finite volume in \(H^3\), and is compact if and only if \(P\) is strictly hyperideal. The dual metric of \(P\) can also be defined as the dual metric of \(P_t\).

The hyperbolic polyhedra which can be obtained as truncated hyperideal polyhedra are quite special. The set of their faces can be split into two subset, the "real" faces and the "cuts". Each "cut" is adjacent to "real faces" only, while each non-ideal vertex is adjacent to exactly one "cut". Moreover, the dihedral angle between the "cuts" and the "real faces" is always \(\pi/2\).

The dual metric of a hyperideal polyhedron is therefore quite special too. It is constructed by gluing pieces which are "singular hemispheres", i.e. the quotient of the universal cover of the complement of the center in a spherical hemisphere by a rotation of angle strictly larger than \(2\pi\). Actually the dual metric of \(P_t\) is the same as the dual metric of \(P\), defined by gluing singular hemispheres in the faces of the dual graph. More details about this can be found in [Rou02].

The dual metrics of hyperideal manifolds The previous considerations on duality were local, so they extend from polyhedra to the boundaries of ideal or hyperideal manifolds. Here we consider a hyperideal manifold \(M\). We can define the dual graph of its universal cover as the set of points in \(S^3_1\) dual to the support planes of \(\tilde{M} \subset H^3\), and the dual graph of \(M\) as the quotient of the dual graph of \(\tilde{M}\) by the action of \(\pi_1 \tilde{M}\) on \(H^3\) and \(S^3_1\).

The dual metric — which we will still call the third fundamental form of the boundary — can still be defined by gluing in each face of the dual graph a singular hemisphere. It lifts to a CAT(1) metric on the boundary of the universal cover of \(M\). Indeed, this splits into a local curvature condition — which is satisfied by the local convexity, because the singular curvature is negative at each vertex — and a global condition on the length of the closed geodesics, which is also true here because of lemma 1.2 below.

This dual metric can also be defined as the dual metric of the hyperbolic manifold with convex, polyhedral boundary obtained by truncating the strictly hyperideal ends of a hyperideal manifold.

Necessary conditions on the angles The third fundamental form defined in this way has the important properties below.
Lemma 1.2. Let $M$ be a hyperideal manifold. Its dual metric $\mathcal{M}$ lifts to a CAT(1) metric on the boundary of the universal cover of $M$.

Consider the hyperbolic manifold with boundary $N$ obtained by truncating the strictly hyperideal ends of $M$. The universal cover $\tilde{N}$ of $N$ has a canonical embedding into $H^3$, because $N$ carries a hyperbolic manifold with convex boundary. Consider its boundary $\partial \tilde{N}$, it is a convex surface in $H^3$. Therefore, its dual metric is CAT(1). Indeed:

- it can be considered as the induced metric on the dual surface in $S^3_1$, and then the non-smooth version of the Gauss formula shows that it is locally CAT(1).
- its closed geodesics have length $L > 2\pi$, as shown by various geometric arguments, see e.g. [CD95, RH93, Sch90, Sch98a, Sch01a].

As a consequence of the description of the dual metrics of the hyperideal manifolds, it is clear that the simple paths and the circuits which appear in theorem 0.5 correspond to geodesics segments, resp. to closed geodesics, of the dual metric. Therefore, since the dual metrics are CAT(1), it is already clear that the conditions on the lengths of the circuits and of the simple paths in theorem 0.5 are necessary.

The dual metric and the dihedral angles A rather important point is that, while knowing the dual metric (and the combinatorics of the polyhedral surface) determines the dihedral angles, the converse is not true — the dihedral angles determine the length of the edges of the dual surface, but it does not determine the shape of the polyhedral surface with convex boundary. Consider its boundary $\partial \tilde{N}$, because

The second approach uses the cross-ratio. If $x, y, a$ and $b$ are four real numbers, their cross-ratio $[x, y; a, b]$ is defined as:

$$[x, y; a, b] := \frac{(x-a)(b-y)}{(y-a)(b-x)}.$$ Given four points on a line in $\mathbb{R}^2$, their cross-ratio is defined in the same way, by replacing $x - a$ by the oriented distance between $a$ and $x$, etc. An important property is that the cross-ratio is invariant under projective transformations.

If $x, y \in D^2$ are distinct, let $l$ be the line containing them, and let $a, b$ be the intersection points of $l$ with $S^1 = \partial D^2$. Suppose that $a, x, y, b$ are in this order on $l$, and define the Hilbert distance $d_H(x, y)$ as half the log of the cross-ratio of those four points on $l$:

$$d_H(x, y) = -\frac{1}{2}\log[x, y; a, b].$$

It is a result of Hilbert that $(D^2, d_H)$ is isometric to $H^2$, and that its geodesics are the segments of $D^2$.

To each geodesic $g$ in $H^2$, one can associate a unique point in $\mathbb{R}^2 \setminus \overline{D^2}$, which we will call the dual of $g$, by a simple and classical projective construction. If $g$ is, in the projective model, the intersection of $D^2$ with a line $l$, let $a, b$ be the intersections of $l$ with $S^1 = \partial D^2$; the dual of $g$ is the intersection point of the tangents to $S^1$
at \( a \) and \( b \). Note that this point can be at infinity if \( l \ni 0 \). Given a line \( l \subset \mathbb{R}^2 \) which intersects \( D^2 \), we will also call "dual of \( l \)" the point obtained in \( \mathbb{R}^2 \setminus \overline{D^2} \) by this construction. Given a point \( v \in \mathbb{R}^2 \setminus \overline{D^2} \), there is a unique line \( l \) such that the dual of \( l \) is \( v \); we will call \( l \) the dual of \( v \). The dual of a point \( v \) will be denoted by \( v^* \), and the dual of a line \( l \) by \( l^* \).

There is another, equivalent definition of the dual of a point \( v \in \mathbb{R}^2 \setminus \overline{D^2} \) in terms of the cross-ratio. For each point \( x \in D^2 \), let \( l \) be the line containing \( v \) and \( x \), and let \( a, b \) be the intersection points between \( l \) and \( S^1 \), chosen so that \( x, a, v, b \) appear in this order on \( l \). \( v^* \) is then the set of all points \( x \in D^2 \) such that \( [x, v; a, b] = -1 \). To check that it does not contradict the previous definition, check the case when \( v \) is a point at infinity and use the projective invariance of both definitions.

Of course, this duality is the same as the one defined in the previous section by considering \( H^3 \) and \( S^3 \) as quadrics.

**Proposition 2.1.** Let \( v \in \mathbb{R}^2 \setminus D^2 \), and let \( d \) be a line going through \( v \) and intersecting \( D^2 \). Then \( v^* \) intersects \( d \) orthogonally for the hyperbolic metric on \( D^2 \).

**Proof.** By projective invariance of the cross-ratio (and thus of the hyperbolic metric) we only need to prove the proposition when \( v \) is a point at infinity, for instance if it corresponds to the vertical direction in \( \mathbb{R}^2 \). In this case, for any \( x \in D^2 \), if \( a \) and \( b \) are the intersection of \( \partial D^2 \) with the vertical line containing \( x \), then:

\[
x \in v^* \iff \frac{ax}{xb} = 1.
\]

Thus \( v^* \) is the horizontal diameter \( d \) of \( D^2 \). An elementary symmetry argument shows that \( d \) is orthogonal, for the hyperbolic metric \( d_{\mathbb{H}} \), to the vertical lines.

**Proposition 2.2.** Let \( s \) be a segment in \( \mathbb{R}^2 \), with endpoints \( x, y \in \mathbb{R}^2 \setminus \overline{D^2} \), but with \( s \) intersecting \( D^2 \). Then the lines \( x^* \) and \( y^* \), dual to \( x \) and \( y \) respectively, do not intersect in \( D^2 \).

**Proof.** This is a consequence of the "geometric" definition of the dual of a point. Using the projective invariance, we can suppose for instance that both \( x \) and \( y \) lie on the \( x \)-axis, on opposite sides of \( D^2 \). Then \( x^* \) and \( y^* \) lie on opposite sides of the \( y \)-axis, and can therefore not intersect.

**Some definitions** The analog in dimension 2 of hyperideal polyhedra are the hyperideal polygons; in particular, the hyperideal triangles.

**Definition 2.3.** A **hyperideal triangle** is a triangle in \( \mathbb{R}^2 \) whose vertices are outside the open disk \( D^2 \) and whose edges all intersect \( D^2 \). It is **strictly hyperideal** if its vertices are outside the closure \( \overline{D^2} \).

The following definition is made possible by proposition 2.2.

**Definition 2.4.** Let \( T \) be a strictly hyperideal triangle. The **truncated hyperideal triangle** associated to \( T \) is the hexagon obtained by taking the intersection of \( T \) with the half-planes bounded by the lines dual to its vertices (and not containing the vertices).

![Figure 2: A hyperideal triangle and its truncated version](image)

**Figure 2:** A hyperideal triangle and its truncated version

Proposition 2.1 indicates that the truncated hyperideal triangles are actually right-angle hexagons. They have two kinds of kinds of edges: the remaining parts of the edges of the non-truncated triangle \( T \), which we
will call real edges, and the intersections of \( T \) with the lines dual to its vertices, which we will call cuts. The edges lengths of a truncated hyperideal triangle are the lengths of its real edges; the edge lengths of a hyperideal triangle are the edge lengths of the corresponding truncated triangle.

**Edge lengths** We will need to understand what are the possible edge lengths of hyperideal triangles.

**Proposition 2.5.** The edge lengths of any strictly hyperideal triangle are positive real numbers. For each \( l_1, l_2, l_3 \in \mathbb{R}^+ \setminus \{0\} \), there is at most one hyperideal triangle with edge lengths \( l_1, l_2, l_3 \). Moreover, given a strictly hyperideal triangle, it has no infinitesimal deformation which does not change its edge lengths.

**Proof.** The first statement is a consequence of proposition 2.2. For the second statement, let \( v \in \mathbb{R}^2 \setminus \overline{D^2} \) and choose a number \( l > 0 \). Now let \( x \in \mathbb{R}^2 \) be such that the line \( (vx) \) intersects \( S^1 \) at two points \( a, b \) such that \( v, a, b, x \) lie on \( (vx) \) in that order. Notice then that \([v, x; a, b] = e^{-2l} \) if and only if \( x \) lies on an ellipse which is tangent to \( S^1 \) at the points of \( S^1 \cap v^* \). Checking this can be done by choosing \( v \) as a point at infinity and then using the projective invariance again.

Now let \( l_1, l_2 \) and \( l_3 \) be positive real numbers, and choose a point \( x_1 \) in \( \mathbb{R}^2 \setminus \overline{D^2} \). The points \( x_2 \in \mathbb{R}^2 \setminus \overline{D^2} \) such that \( [x_1, x_2] \) intersects \( D^2 \) and that its length is \( l_3 \) form part of an ellipse (tangent to \( S^1 \) at \( S^1 \cap x_1^* \)); choose one of them.

The points \( x \in \mathbb{R}^2 \setminus \overline{D^2} \) such that \( [x_1, x] \) intersects \( D^2 \) and that its length is \( l_2 \) then form a subset \( E_1 \) of an ellipse tangent to \( S^1 \) at \( S^1 \cap x_1^* \), while the points \( y \in \mathbb{R}^2 \setminus \overline{D^2} \) such that \( [x_2, x] \) intersects \( D^2 \) and that its length is \( l_1 \) form a subset \( E_2 \) of an ellipse tangent to \( S^1 \) at \( S^1 \cap x_2^* \). \( E_1 \) and \( E_2 \) intersect at at most two points (see figure 3). Taking either of those intersections as \( x_3 \) yields a triangle (with vertices \( x_1, x_2 \) and \( x_3 \)) with edge lengths \( l_1, l_2 \) and \( l_3 \).

To prove the uniqueness of this triangle and the third statement, just notice that the choices of the points \( x_1 \) and \( x_2 \) are equivalent under the action of the group of orientation-preserving projective transformations of \( \mathbb{R}^2 \) fixing \( S^1 \), while the two possible choices of \( x_3 \) lead to triangles which are equivalent under an orientation reversing projective transformation fixing \( S^1 \).

---

**Figure 3:** How to position \( x_3 \) knowing \( x_1 \) and \( x_2 \)

---

**A projective model of the de Sitter space** The construction of a projective model for the hyperbolic plane, which was described above, extends with few modifications to the de Sitter plane. Recall that the de Sitter plane \( S^2_1 \) is a constant curvature 1 Lorentz surface; it can be obtained, much like the hyperbolic plane, as a quadric in \( \mathbb{R}^3_1 \), with the induced metric:

\[
S^2_1 \simeq \{ x \in \mathbb{R}^3_1 \mid \langle x, x \rangle = 1 \}.
\]

Its geodesics are the intersections with \( S^2_1 \) of the planes containing 0 in \( \mathbb{R}^3_1 \). They are of three kind: the space-like, light-like and time-like geodesics, corresponding respectively to the space-like, light-like and time-like planes in \( \mathbb{R}^3_1 \).
The projective model which we will consider is actually only for "half" the de Sitter plane. It can be obtained by projecting the part of \( S^2 \) which lies in the half-space \( \{ x_0 > 0 \} \) to the plane \( P_0 := \{ x_0 = 1 \} \) along the direction of 0, i.e. by sending a point \((x_0, x_1, x_2)\) with \( x_0 > 0 \) to the point \((1, x_1/x_0, x_2/x_0)\) \(\in P_0\). Note that another possible construction uses the Hilbert distance (see e.g. [Sch98a]).

**Triangles in the de Sitter space** Some questions concerning the dihedral angles of polyhedra, which we will encounter below, can be understood using elementary properties of the triangles in the de Sitter plane. We state some such results here for future reference.

**Definition 2.6.** A *space-like* triangle in the de Sitter plane is the image by the projective model of a triangle in \( \mathbb{R}^2 \) which bounds an open convex set which contains \( \overline{D^2} \).

Note that the edges of the "space-like" triangles are segments of space-like geodesics; but there are triangles in \( S^2 \) with space-like edges which are not "space-like triangles" as we have just defined them.

**Proposition 2.7.** The sum of the edge lengths of any space-like triangle is strictly greater than \(2\pi\). Conversely, given \( l_1, l_2, l_3 \in (0, \pi) \) with sum strictly greater than \(2\pi\), there is a unique space-like triangle in \( S^2 \) with those edge lengths.

**Proof.** The space-like triangles are exactly the triangles, in the de Sitter plane, which are duals of compact hyperbolic triangles. So, using this duality, proving the proposition is equivalent to showing that:

- the sum of the exterior angles of any hyperbolic triangle is strictly larger than \(2\pi\).
- for each triple \((\alpha_1, \alpha_2, \alpha_3) \in (0, \pi)^3\) such that \(\alpha_1 + \alpha_2 + \alpha_3 > 2\pi\), there is a unique hyperbolic triangle with \(\alpha_1, \alpha_2\) and \(\alpha_3\) as its exterior angles.

This is easily seen to be true by switching to the interior angles. The uniqueness is of course up to the global hyperbolic isometries. \(\square\)

**A topology on the space of hyperideal triangles** There are several natural ways to put a topology on the space of hyperideal triangles. One can use for instance:

- the lengths of the edges (which uniquely characterize the triangles up to orientation).
- the positions of the vertices in \( \mathbb{R}^2 \), modulo the action of the projective transformations leaving \( D^2 \) invariant.
- the angles at the vertices, which also characterize the triangles up to isometry.

We will use here the third solution. It has the advantage that the triangles with one or more ideal vertices appear simply as the triangles with one or more angles equal to 0. In terms of the lengths of the edges, the triangles with one or more ideal vertices have some infinite lengths and this makes things a little more troublesome.

## 3 Hyperideal simplices

**Definitions** We will be using in this section the projective model of \( H^3 \). Since it is quite analogous to the projective model of \( H^2 \) defined in the previous section, we will not give any further detail on its construction or on its basic properties. Here \( D^3 \) is the open ball of radius 1 in \( \mathbb{R}^3 \), which in the projective model is the image of \( H^3 \).

**Definition 3.1.** A *hyperideal simplex* is a simplex in \( \mathbb{R}^3 \) with all its vertices in \( \mathbb{R}^3 \setminus D^3 \), and such that all its edges intersect \( D^3 \). It is *strictly hyperideal* if none of its vertices lie on \( S^2 = \partial D^3 \). An hyperideal simplex is *degenerate* if it lies in a hyperbolic plane.

We will call \( S \) the set of hyperideal simplices, considered up to global hyperbolic isometries. \( S \) can be decomposed as \( S = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \), where \( S_i \) is the set of hyperideal simplices with exactly \( i \) ideal vertices; for instance, \( S_0 \) is the set of strictly hyperideal vertices, while \( S_4 \) is the set of ideal simplices.

Given a point \( v \in \mathbb{R}^3 \setminus \overline{D^3} \), its dual is a hyperbolic totally geodesic plane which can be defined, as in the previous section, in many ways including the following two:
• if \( C \) is the set of points in \( x \in S^2 \) such that the line \((xv)\) is tangent to \( S^2 \) at \( x \), then \( C \) is a circle on \( S^2 \) — i.e. the boundary of a geodesic ball. It is thus the boundary at infinity of a plane, which we define to be \( v^* \).

• \( v^* \) is the set of points \( x \in D^3 \) such that \([x,v;a,b] = -1\), where \( a \) and \( b \) are the intersections with \( S^2 \) of the line \((v,x)\).

The analog of proposition 2.2 can be proved in the same way:

**Proposition 3.2.** Let \( s \) be a segment in \( \mathbb{R}^3 \), with endpoints \( x, y \in \mathbb{R}^3 \setminus D^3 \), but with \( s \) intersecting \( D^3 \). Then the duals of \( x \) and \( y \) do not intersect.

**Truncated simplices** Given a hyperideal simplex \( S \), we define (following Bao and Bonahon \cite{BB02}) the associated truncated hyperideal simplex \( S' \) as the intersection of \( S \) with the half-spaces bounded by the duals of its vertices (and which do not contain those vertices).

By proposition 2.2, the truncated strictly hyperideal simplices are compact; they have two kinds of faces, hexagons, which we will call ”real faces”, and the triangles which are the intersections of \( S \) with the planes dual to its vertices. They also have two kinds of edges: the ”real edges” are the intersections of the edges of \( S \) with \( S' \), and the ”edge lengths” of \( S \) are the lengths of the ”real edges” of \( S' \).

On the other hand, truncation does not change the ideal ends of a simplex, and if \( S \) is a simplex with at least one ideal vertex, then its associated truncated simplex has finite volume but is not compact.

**Edge lengths** As a direct consequence of proposition 2.2, we find an analogous rigidity statement in dimension 3. Before proving it, we need a preliminary statement on the set of points at a given distance from a hyperideal or ideal point.

Note that, given two points \( x, y \in \mathbb{R}^3 \setminus D \), one can define the distance between them. This can be done in at least two ways, which can directly be checked to be equivalent.

• using the Hilbert distance, as in section 2.

• from the distance between their dual planes, when the segment \([x,y]\) intersects \( D \).

Similarly, there is a natural notion of distance between a point in \( D \) and a point in \( \mathbb{R}^3 \setminus D \), which can be defined from the Hilbert metric or from the distance from the hyperbolic point to the dual of the hyperideal point.

There is also a notion of distance between an ideal point \( x \) and a hyperbolic point \( y \), but it is not canonically defined and depends on the choice of a horosphere \( H \) centered at \( x \); once \( H \) is given, the distance between \( x \) and \( y \) is defined as the distance between \( y \) and \( H \). Replacing \( H \) by another horosphere \( H' \) changes the distances from all points in \( H^3 \) to \( x \) by the same constant (which is the distance between \( H \) and \( H' \)).

**Proposition 3.3.** Consider the projective model of \( H^3 \) and \( S^3_{1,+} \).

1. Let \( x \in S^3_{1,+} \) be a hyperideal point, and let \( d > 0 \). The set of points in \( S^3_{1,+} \) at constant distance \( d \) from \( x \) is, in the projective model, an ellipsoid of revolution, which is tangent to \( \partial_\infty H^3 \) along the intersection of \( \partial_\infty H^3 \) with the tangent cone with vertex \( x \).

2. Let \( x \in \partial_\infty H^3 \) be an ideal point, and let \( H \) be a horosphere at \( x \). The set of points \( y \in S^3_{1,+} \) such that the distance from \( y \) to \( x \) relative to \( H \) is fixed is an ellipsoid of revolution, which is outside \( B(0,1) \), but tangent to \( S^2 \) at \( x \).

3. Suppose that \( x_1, x_2, x_3 \) are either ideal or hyperideal points, and let \( d_1, d_2, d_3 \in \mathbb{R} \). For each ideal point in \( \{x_1,x_2,x_3\} \), choose a horosphere \( H_i \) centered at \( x_i \). Suppose that the intersection of the ellipsoids \( E_i \) of points in the projective model, at distance \( d_i \) from \( x_i \) (relative to \( H_i \)) contains more than two points. Then \( x_1, x_2 \) and \( x_3 \) lie on a line.

**Proof.** For (1), apply a projective transformation sending \( x \) to infinity, for instance to the point at infinity corresponding to the ”vertical” direction, without moving \( S^2 \). Then \( H^3 \) remains in a ball, while the cone tangent to \( \partial_\infty H^3 \) is sent to a vertical cylinder tangent to \( \partial_\infty H^3 \) along the intersection of \( \partial_\infty H^3 \) with the dual plane \( x^* \subset H^3 \).

We are interested in the set of points at constant distance from \( x \), i.e. in the set of points \( y \) such that, if \( D \) is the line through \( x \) and \( y \) and \( a, b \) are the intersections of \( D \) with \( \partial_\infty H^3 \) (such that \( x, a, b, y \) appear in this
order on $D$) the cross-ratio $[x, y; a, b]$ is equal to a constant $C$. After the projective transformation sending $x$ to infinity, this equation becomes: $ay = Cby$, which can be translated as: $(C−1)yb = Cab$. This is clearly the equation of an ellipsoid of revolution, tangent to $\partial_\infty H^3$ along $\partial_\infty H^3 \cap x^+$.

For (2), let $y \in S^3_{1, 4}$, and note that the distance from $x$ to $y$ relative to $H$ is $C$ if and only if the distance from $x$ to the plane $y^\perp$ relative to $H$ is $C$. Thus the set $S'$ of points at distance $C$ from $x$ relative to $H$ is the dual of the set $S$ of points $z \in H^3$ at distance $C$ from $x$ relative to $H$. But $S$ is clearly a horosphere centered at $x$. So, in the projective model, $S$ is an ellipsoid of revolution tangent to $\partial_\infty H^3$ at $x$. It is then a simple matter of projective geometry to check that $S'$ is also, in the projective model, an ellipsoid of revolution tangent to $\partial_\infty H^3$ at $x$.

For (3), note that we can suppose that, maybe after changing their labels, $x_1$ and $x_2$ are either both ideal points or both strictly hyperideal points. Moreover, after applying a projective transformation, we can suppose that $x_1$ and $x_2$ are either on the same vertical line going through 0, or symmetric with respect to the vertical line containing 0. In both cases, the sets of points at distance $d_i$ from $x_i$ (maybe relative to $H_i$), for $i \in \{1, 2\}$, is a circle $C$ in a plane in the projective model.

Since the $E_i$, $1 \leq i \leq 3$, are quadrics, their intersection contains either at most 2 points, or a whole curve. So, to prove the statement, we can suppose that $E_1 \cap E_2 \cap E_3 = C$. Now it is not difficult to check that the set of points at constant distance from a circle like $C$ is a line in the projective model; as a consequence, $x_1$, $x_2$ and $x_3$ are on a line.

We can describe more precisely the set of points at given distance from an ideal point $x$. First, a simple computation shows that, in the projective model, the horospheres centered at $x$ are simply the ellipsoids with radii $\lambda, \lambda$ and $\lambda^2$, for $\lambda \in (0, 1)$, which are tangent to $S^2$ at the end of the small axis. Then the projective duality shows that the sets of points in $S^3_1$ at constant distance from $x$ are the duals of those ellipsoids, which have the same description except that now $\lambda > 1$, so that the tangency to $S^2$ occurs at the end of the large axis.

We call $e_{12}, \cdots, e_{34}$ the edges of the simplex $S_0$ (considered as a combinatorial object).

**Proposition 3.4.** Let $l_{12}, \cdots, l_{34}$ be positive real numbers. There is at most one strictly hyperideal simplex $S$ such that the length of $e_{ij}$ is $l_{ij}$. There is no first-order deformation of $S$ which does not change its edge lengths.

**Proof.** Proposition 2.5 shows that there is at most one hyperideal triangle with edge lengths $l_{12}, l_{13}$ and $l_{23}$. Let $x_1, x_2, x_3$ be the three vertices obtained in this way. Finding the last vertex $x_4$ is equivalent to finding the intersection of the sets of points $E_1, E_2$ and $E_3$ at given distances $l_{14}, l_{24}$ and $l_{34}$, respectively, from $x_1, x_2$ and $x_3$.

According to point (3) of proposition 3.3 since $x_1, x_2$ and $x_3$ do not lie on a line, the intersection of $E_1, E_2$ and $E_3$ contains at most two points, which are exchanged by a hyperbolic transformation fixing $x_1, x_2$ and $x_3$. This proves the uniqueness of the simplex.

The same argument shows the infinitesimal rigidity statement, using the infinitesimal rigidity part of proposition 2.5.

**Corollary 3.5.** For each function $l : \{e_{12}, \cdots, e_{34}\} \to \mathbb{R}^+ \setminus \{0\}$, there is exactly one strictly hyperideal simplex with edge lengths given by $l$.

**Proof.** We consider the map $F$ sending a strictly hyperideal simplex (with vertices labeled from 1 to 4) to the set of its edge lengths. The statement follows from the following points, each of which can readily be checked.

1. the set of strictly hyperideal simplices, and the space of functions $l : \{e_{12}, \cdots, e_{34}\} \to \mathbb{R}^+ \setminus \{0\}$, both have dimension 6.

2. $F$ is locally injective (i.e., its differential is everywhere injective) according to the infinitesimal rigidity statement of proposition 3.4.

3. $F$ is proper, i.e. if a sequence of simplices $(S_i)_{i \in \mathbb{N}}$ is such that the corresponding length functions $(l_i)_{i \in \mathbb{N}}$ converges, then $(S_i)$ converges.

4. the space of strictly hyperideal simplices is connected, while the space of admissible distance functions is simply connected.

\[ \square \]
The hyperbolic-de Sitter duality

The (classical) construction which was given in section 2 of a projective model for the hyperbolic plane and the 2-dimensional de Sitter space extends, with minor modifications, to dimension 3. So does the duality between the hyperbolic and de Sitter space. The dual of a point in $H^3$ is now a 2-dimensional totally geodesic space-like plane in $S^3_1$, while the dual of a point in $S^3_1$ is an oriented totally geodesic plane in $H^3$.

One important property of this duality concerns the polyhedra; the next proposition was discovered by Rivin and Hodgson [Riv86, RH93]. It is an extension of the properties described in the previous section in dimension 2.

**Proposition 3.6.** Let $P$ be a compact polyhedron in $H^3$. Its dual is a compact, space-like polyhedron in $S^3_1$. To each edge $e$ of $P$ corresponds an edge $e^*$ of $P^*$, and the exterior dihedral angle at $e$ is the length of $e^*$.

The proof is can be found in a number of sources, e.g. [RH93, Sch98a], so we leave it to the reader.

This duality is valid not only for compact polyhedra, but also for ideal or hyperideal polyhedra, and also for smooth, strictly convex surfaces — the dual objects are then the smooth, strictly convex space-like surfaces in $S^3_1$, see e.g. [Sch96].

The group of orientation-preserving isometries of $H^3$, $SO(3,1)$, is also the group of orientation-preserving isometries of $S^3_1$ which do not exchange the two boundary components of $S^3_1$. Therefore, any group action on $H^3$ has an extension to $S^3_1$. Since the duality described above is defined "geometrically", it is not difficult to show that it "commutes" to the action of the isometries of $H^3$ resp. $S^3_1$. This means that, given a polyhedral surface in $H^3$ which is invariant under a subgroup $\Gamma$ of $SO(3,1)$, its dual is invariant under the action of $\Gamma$ on $S^3_1$.

**Definition 3.7.** A dual hyperideal simplex is a simplex in $\mathbb{R}^3$ with all vertices and edges in $\mathbb{R}^3 \setminus \overline{D^3}$ but with all faces intersecting $D^3$ (maybe at one point).

Note that we might use the same terminology to describe the intersection of a dual hyperideal simplex with $\mathbb{R}^3 \setminus \overline{D^3}$, and also the corresponding non-complete polyhedron in de Sitter space.

**Proposition 3.8.** The dual of the hyperideal simplices are the dual hyperideal simplices.

**Proof.** This follows from the elementary properties of the hyperbolic-de Sitter duality. If $S$ is a hyperideal simplex, each of its vertices is either strictly hyperideal, or ideal; so the dual planes intersect $H^3$, either on a disk (for strictly hyperideal vertices) or at a point (for ideal vertices). Moreover, each edge of $S$ intersects $H^3$, so that the dual edges are space-like geodesics in $S^3_1$, which remain outside $H^3$ in the projective model. The converse is proved in the same way.

The following proposition is a special case of results of [Sch98a], but we will outline its proof — which is elementary — for completeness.

**Proposition 3.9.** The possible edge lengths of a dual hyperideal simplex $S$ are the functions $l : \{e_{12}, \cdots, e_{34}\} \to (0, \pi)$ such that, for each face $f$ of $S$, the sum over the edges of $f$ of the values of $l$ is at least $2\pi$. For each such function, there is a unique simplex with those edge lengths.

The uniqueness in this statement is of course up to the global hyperbolic isometries.

**Proof.** Each face of a dual hyperideal simplex is isometric to a triangle:

- either in the projective model of $H^2$ and $S^2_{1,+}$, it is then the dual of a hyperbolic triangle (i.e. which contains $H^2$ in its interior).
- or in the degenerate space $H^2_{1,0}$, which is isometric to a light-like totally geodesic plane in $S^3_1$, and it then contains the limit point in its interior.

The edge lengths of those triangles are:

- in the first case, exactly the triples $(l_1, l_2, l_3)$ of numbers in $(0, \pi)$ such that $l_1 + l_2 + l_3 > 2\pi$.
- in the second case, the triples $(l_1, l_2, l_3)$ of numbers in $(0, \pi)$ such that $l_1 + l_2 + l_3 = 2\pi$.

Moreover, each triple determines a unique triangle. Thus, the lengths of the edges of a dual hyperideal simplex satisfy the hypothesis of the proposition, and moreover its faces are uniquely determined. It is then a simple matter to check that there is a unique way of gluing those faces to obtain a simplex.
Dihedral angles of hyperideal simplices  In a dual way, we need to understand the dihedral angles of the hyperideal simplices. The next lemma is a very special case of the main result of [BB02]. We will however give a direct proof for completeness. The proof is a direct consequence of the previous proposition and the hyperbolic-de Sitter duality.

Lemma 3.10. Let $S$ be a hyperideal simplex. Its exterior dihedral angles are such that, for each vertex $s$ of $S$, the sum of the angles of the edges containing $s$ is greater than $2\pi$, and equal to $2\pi$ if and only if $s$ is ideal. Moreover, given a map $\alpha : \{e_{12}, \cdots, e_{34}\} \to (0, \pi)$ such that, for each vertex $s$ of $S$, the sum of the values of $\alpha$ on the edges of $S_0$ incident to $s$ is at least $2\pi$, there exists a unique hyperideal simplex such that the exterior dihedral angle at each edge $e_{ij}$ is $\alpha(e_{ij})$.

Note that the path length condition which appears in lemma 1.1 or in theorem 0.5 is redundant here; a simple argument shows that it is always satisfied under the hypothesis of lemma 3.10.

Another consequence is that the strictly hyperideal or ideal nature of the vertices can only be read from the dihedral angles:

Remark 3.11. Let $S$ be a hyperideal simplex. A vertex $v$ of $S$ is ideal if and only if the sum of the exterior dihedral angles of the edges adjacent to $v$ is $2\pi$.

This clearly remains true for hyperideal polyhedra, or for hyperideal manifolds.

The Schlafli formula  It is a key element in this paper, so we recall it here. The reader might find a proof e.g. in [Mil94, Vin93].

Lemma 3.12. Let $P$ be compact hyperbolic polyhedron, with edge lengths $(L_i)$ and dihedral angles $(\theta_i)$. In a first-order deformation of $P$, the variation of its volume is given by:

$$ dV = -\frac{1}{2} \sum_i L_i d\theta_i . $$

(1)

To understand the Schlafli formula for hyperideal simplices with some ideal vertices, we introduce linear map as follows:

$$ \phi_4 : \mathbb{R}^4 \to \mathbb{R}^6 $$

$$(x, y, z, t) \mapsto (x + y, x + z, x + t, y + z, y + t, z + t) $$

We then call $\phi_1$ the restriction of $\phi_4$ to $\mathbb{R} \times \{(0, 0, 0)\}$ (identified with $\mathbb{R}$), $\phi_2$ the restriction of $\phi_4$ to $\mathbb{R}^2 \times \{(0, 0)\}$, and $\phi_3$ the restriction of $\phi_4$ to $\mathbb{R}^3 \times \{0\}$.

Now let $S$ be a hyperideal simplex with one ideal vertex $v$ exactly. Let $\bar{S}$ be the associated truncated hyperideal simplex, so that $\bar{S}$ is a finite volume polyhedron in $H^3$ with exactly one ideal vertex. Choose a horosphere $H$ centered at $v$ (which we will require to be small enough), and define the edge lengths of $S$ to be the lengths of the real edges of $\bar{S}$ of finite length, and the lengths of the segment of the other edges up to their intersection with $H$ (this intersection exists if $H$ is small enough). Of course the resulting edge lengths depend on the choice of $H$; but changing $H$ only adds a constant to the three edges incident to $v$, so that the lengths of the edges of $S$ are well defined as an element of $\mathbb{R}^6/\phi_1(\mathbb{R})$.

In the same way, if $S$ has two ideal simplices, its edge lengths are defined as an element of $\mathbb{R}^6/\phi_2(\mathbb{R}^2)$; for 3 ideal simplices they are in $\mathbb{R}^6/\phi_3(\mathbb{R}^3)$, and for 4 ideal vertices, in $\mathbb{R}^6/\phi_4(\mathbb{R}^4)$.

With those natural definitions, we can give an extension — also classically known — of lemma 3.12.

Lemma 3.13. Let $P$ be hyperideal polyhedron, with edge lengths $(L_i)$ and dihedral angles $(\theta_i)$. In a first-order deformation of $P$ which leaves its ideal vertices on the sphere at infinity, the variation of its volume is given by (1):

$$ dV = -\frac{1}{2} \sum_i L_i d\theta_i . $$

Note that this formula makes sense although the edge lengths are defined in general only up to the addition of a constant for each ideal vertex, because the sum of the dihedral angles of the edges containing an ideal vertex are constrained to be $2\pi$, so that the sum of their differential vanishes — the additive constant in the lengths therefore doesn’t make any difference.
**Infinitesimal rigidity for simplices** The definitions of edge lengths for hyperideal simplices also lead to an extension of proposition 3.4.

**Proposition 3.14.** Let \( i \in \{0, 1, 2, 3, 4\} \), and let \( S \in \mathcal{S}_i \) be a hyperideal simplex, with edge lengths \( l := (l_{ij}) \in \mathbb{R}^6/\phi_i(\mathbb{R}^4) \). There is no non-trivial first-order deformation of \( S \) in \( \mathcal{S}_i \) which does not change \( l \).

**Proof.** The proof depends on the number \( n \) of ideal vertices. In each case we consider an infinitesimal deformation \( \mathbf{S} \) of \( S \) which does not change the edge lengths, and then show that it is trivial.

- \( n = 0 \): this was already done in proposition 3.4.
- \( n = 1 \): let \( x_1 \) be the ideal vertex, so that \( x_2, x_3 \) and \( x_4 \) are strictly hyperideal. By proposition 3.5, we can apply a infinitesimal isometry to get in the situation where \( x_1, x_2 \) and \( x_3 \) are fixed by \( \mathbf{S} \). Moreover, we can fix the horosphere centered at \( x_1 \), so that the distance between \( x_1 \) and \( x_4 \) is well-defined. Then proposition 3.6 shows that \( x_4 \) has to be in the intersection of three ellipsoids (corresponding to the fixed distances to \( x_1, x_2 \) and \( x_3 \)) so that \( \mathbf{S} \) has to be trivial.
- \( n = 2 \): let \( x_1 \) and \( x_2 \) be ideal. By adding an infinitesimal isometry, we can suppose that \( \mathbf{S} \) fixes \( x_1, x_2 \) and \( x_3 \). We also fix horospheres centered at \( x_1 \) and \( x_2 \), and apply the same argument as for \( n = 1 \).
- \( n = 3 \): we use the same line of reasoning: note that, up to isometries, there is only one ideal triangle, so we can suppose that \( x_1, x_2 \) and \( x_3 \) are fixed by \( \mathbf{S} \). In addition there is a unique way to choose horospheres centered at \( x_1, x_2 \) and \( x_3 \) which are pairwise tangent. \( x_4 \) is then restrained to be in the intersection of three ellipsoids and so \( \mathbf{S} \) is trivial.
- \( n = 4 \): fix \( x_1, x_2 \) and \( x_3 \) (this is possible since they form an ideal triangle), and the corresponding horospheres so that they are pairwise disjoint. Then we can still suppose that the horosphere centered at \( x_4 \) is tangent e.g. to the horosphere centered at \( x_1 \). This leaves two conditions, corresponding to the distance between the horosphere centered at \( x_4 \) and those centered at \( x_2 \) and \( x_3 \). It is a rather simple matter of Euclidean geometry to check that those conditions are non-degenerate, i.e. that there is no possible displacement of \( x_4 \) which satisfies them at first order.

Note that the last case of the previous proposition, for ideal simplices, is well-known. One way to prove it (see [Sch01]) for a more general result on polyhedra or manifolds with ideal boundary) is in the direction opposite to what we do below: first show that the volume is a concave function of the dihedral angles, then use the Schlaffi formula to deduce the infinitesimal rigidity with respect to the edge lengths (or induced metric).

**The volume of simplices** Another key element, related to the Schlaffi formula and the rigidity phenomenon which we have just seen, is the fact that the volume of the hyperideal simplices (like the volume of the ideal simplices, see e.g. [Riv94]) is concave.

**Definition 3.15.** The volume of a hyperideal simplex is defined as the volume of the corresponding truncated hyperideal simplex.

Note that, for each \( i \in \{0, \cdots, 4\} \), \( \mathcal{S}_i \) is parametrized by the dihedral angles, subject to the condition that, for each ideal vertex, the sum of the angles of the incident edges is \( 2\pi \). By lemma 3.10 and remark 3.11, \( \mathcal{S}_i \) is affinely equivalent to a convex polytope in \( \mathbb{R}^{6-i} \). We call \( V \) the volume, seen as a function on each \( \mathcal{S}_i, 0 \leq i \leq 4 \).

**Proposition 3.16.** There exists a regular hyperideal simplex such that Hess(\( V \)) is negative definite.

**Proof.** Let \( S_0 \) be a regular hyperideal simplex, with edge lengths equal to \( l_0 \) and dihedral angles equal to \( \theta_0 \). By the Schlaffi formula, lemma 3.13 the matrix of Hess(\( V \)) is equal to:

\[
\text{Hess}(V) = -\frac{1}{2} \left[ \frac{\partial l_j}{\partial \theta_i} \right]_{i,j},
\]

where \( \theta_j \) is the exterior dihedral angle at the edge \( j \). So we only have to prove that, for some regular hyperideal simplex, the matrix \((\partial \theta_i/\partial l_j)_{1 \leq i,j \leq 5}\) is positive definite.

Using the symmetry of the regular simplices, this can be done as follows.

- consider a hyperideal simplex with five edge lengths equal to \( l_0 \), and one, say the length of \( e \), equal to \( l \).
- use the hyperideal version of some classical triangle formulas to compute the angles of the faces. There are three such angles, one, \( \alpha_0 \), which is the angle of the faces of the regular simplex with edge lengths \( l_0 \), and two others, \( \alpha \) and \( \beta \).
use $\alpha$ and $\beta$ to compute the dihedral angles of the simplex, using the same triangle formulas but for the links of the vertices (which are hyperbolic triangles). There are three angles to compute, the angle $\theta_e$ at $e$, the angle $\theta_{\bar{e}}$ at the edge opposite to $e$, and the angle $\theta$ at the four other edges.

- differentiate the three dihedral angles with respect to $l$, and then set $l = l_0$.
- find the eigenvalues of the matrix containing the derivatives of the dihedral angles with respect to the edge lengths, and check that they are all strictly positive.

Since this computation is a little tedious, we omit it here and give, in the appendix A, a little Maple program to do it.

Lemma 3.17. For each $i \in \{0, 1, 2, 3, 4\}$, the volume $V$ is a strictly concave function on $S_i$.

Proof. Let $S \in S_i$ be a hyperideal simplex. Suppose that there is a direction in $T_S S_i$ which is in the kernel of Hess($V$). Then by the Schl"afli formula (1), the corresponding first-order variation of the edge lengths vanishes, and this is impossible by proposition 3.14. Therefore, Hess($V$) has constant signature over each $S_i$, with maximal rank, so it only remains to check that Hess($V$) is negative definite at a point.

By proposition 3.16, Hess($V$) is negative definite at some regular hyperideal simplex. Therefore, Hess($V$) is negative definite on $S_0$, so $V$ is strictly concave on $S_0$.

But $S_1$ can be identified with one of the codimension 1 faces of $S_0$; so $V : S_1 \to \mathbb{R}_+$, as a limit of strictly concave functions, is concave. Since we have seen above that its hessian is non-degenerate, it is strictly concave. In the same way, $S_2$ is a codimension 1 face of $S_1$, so $V$ restricted to it is strictly concave, and the same can then be said of $S_3$, and then of $S_4$.

4 Hyperideal polyhedra

This section contains an extension of the result of the previous section from hyperideal simplices to hyperideal polyhedra. The main result of this section is the following lemma:

Lemma 4.1. Let $\sigma$ be a cellulation of $S^2$, and let $w : \sigma_1 \to (0, \pi)$ be a map on the set of edges of $\sigma$. There exists a hyperideal polyhedron with combinatorics given by $\sigma$ and exterior dihedral angles given by $w$ if and only if:

- the sum of the values of $w$ on each circuit in $\sigma_1$ is greater than $2\pi$, and strictly greater if the circuit is non-elementary.
- The sum of the values of $w$ on each simple path in $\sigma_1$ is strictly larger than $\pi$.

This hyperideal polyhedron is then unique.
Lemma 4.2. Let \( \sigma \) be a cellulation of \( S^2 \), and let \( V_i \) be a subset of the set of vertices of \( \sigma \). The volume is a strictly concave function of the dihedral angles, on the space of hyperideal polyhedra with combinatorics given by \( \sigma \), with ideal vertices at the points of \( V_i \), and strictly hyperideal vertices at the other vertices of \( \sigma \).

Note that, to prove lemmas 4.1 and 4.2 it is sufficient to prove them when \( \sigma \) is a triangulation of \( S^2 \), i.e. when all faces of \( \sigma \) are triangles. The general result then follows by adding the constraint that some exterior dihedral angles — on the edges which have been added to make \( \sigma \) a triangulation — are equal to 0. Given such a triangulation \( \sigma \), it is not difficult to find a triangulation \( \Sigma \) of the ball, i.e. a decomposition of the ball into simplices, such that \( \sigma \) is the ”trace” of \( \Sigma \) on the boundary. This can be done for instance by choosing a vertex \( x_0 \) of \( \sigma \), and adding one simplex with vertex \( x_0 \) for each face of \( \sigma \) not adjacent to \( x_0 \). In this section, \( \sigma \) and \( \Sigma \) will be fixed.

Definition 4.3. A sheared hyperbolic structure on the ball is a singular hyperbolic metric on \( B^3 \), defined by the choice, for each simplex of \( \Sigma \), of a diffeomorphism onto a hyperideal simplex in \( H^3 \), up to the isotopies fixing the vertices. The space of sheared hyperbolic structures is denoted by \( H_{sh} \).

Note that a sheared hyperbolic structure does not, in general, define a hyperbolic structure on \( B^3 \), or even on the complement of the edges of the triangulation. The obvious reason is that it is in general not possible to glue two hyperideal simplices along a face of each — this is a difference with the case of ideal triangulations. Actually it is not difficult to show that, given two hyperideal simplices \( S \) and \( S' \) and distinct vertices \( x_1, x_2, x_3 \) of \( S \) and \( x_1', x_2', x_3' \) of \( S' \), it is possible to glue \( S \) to \( S' \) in a way that identifies \( x_i \) to \( x_i' \) (for \( 1 \leq i \leq 3 \)) if and only if all the edges of \( S \) with endpoints in \( \{x_1, x_2, x_3\} \) either have infinite length, or have the same length as the corresponding edge of \( S' \), and conversely.

For each simplex of \( \Sigma \), the possible hyperbolic metrics on it are parametrized by the dihedral angles; this determines an affine structure on \( H_{sh} \).

Definition 4.4. Let \( \Theta : \Sigma_1 \to \mathbb{R}_+ \) be a map on the edges of \( \Sigma \), which takes values in \( (0, \pi) \) on the edges of \( \sigma \). We call \( H_{sh}(\Theta) \) the subspace of \( H_{sh} \) of sheared hyperbolic structures on \( \Sigma \) such that the exterior dihedral angle at each boundary edge \( e \) is \( \Theta(e) \), while the total angle around each interior edge \( e' \) is \( \Theta(e') \).

Let \( h \in H_{sh} \) be a sheared hyperbolic structure on \( \Sigma \). Suppose that it is possible to glue the simplices of \( \Sigma \), i.e. the condition stated above on the length of the edges is satisfied. Clearly, the singularities of \( h \) are concentrated on the interior edges of \( \Sigma \). Those singularities have two parts, which can be expressed in terms of the holonomy \( \rho(e) \) of the developing map at an edge \( e \):

- the total angle around \( e \) can be different from \( 2\pi \); this can be expressed in terms of the component of \( \rho(e) \) around \( e \).
- for each edge having as endpoints two ideal vertices, there might be a translation component of \( \rho(e) \) along \( e \); this means that, when one ”makes one turn around \( e \)”, one ends up some point away from the starting point. So one can associate to each interior edge of \( \Sigma \) a number, corresponding to this translation length, which we call the shear of \( h \) at \( e \). Note that it does not depend on the orientation chosen for \( e \).

Definition 4.5. An exact hyperbolic structure on \( \Sigma \) is a sheared hyperbolic structure such that:

- the hyperideal simplices can be glued along their common faces.
- the shear at all interior edges of \( \Sigma \) vanishes.

This result was proved by Bao and Bonahon [BB02] using a ”direct” deformation approach, with the key infinitesimal rigidity lemma proved using the Legendre-Cauchy method. Another proof was given recently by Rousset [Rou02], who reduced this result to the description of the dual metrics of compact polyhedra previously achieved by Rivin and Hodgson [RH99] (following related work of Andreev [And70]). Rousset [Rou02] also extended the description of the dihedral angles of hyperideal polyhedra to the fuchsian case.

We give here yet another proof of lemma 4.1 — with parts in common with the proof of [BB02], in particular in the use of the Andreev theorem at the end. The infinitesimal rigidity is based here on the remarkable properties of the volume of the hyperideal simplices and polyhedra. This is similar to what was done for ideal polyhedra previously (see [Sch01b] [Riv86, RH99]). The main point here is that this also works for hyperideal polyhedra, thanks to lemma 3.1. We mostly use here the same notations as in [Sch01b].

Another important result of this section, which is related to lemma 4.1, is that the volume is a concave function not only for simplices, but also for polyhedra.
The space of exact hyperbolic structures is denoted by \( \mathcal{H}_{\text{ex}} \). We call \( \mathcal{H}_{\text{ex}}(\Theta) := \mathcal{H}_{\text{ex}} \cap \mathcal{H}_{\text{sh}}(\Theta) \).

The "smooth" hyperbolic structures on \( \Sigma \) are, by definition, the exact hyperbolic structures such that the total angle around each interior vertex of \( \Sigma \) is \( 2\pi \). Thus it is an affine submanifold of the space of exact hyperbolic structures on \( \Sigma \).

A key point is that the exact hyperbolic structures are exactly the critical points of the volume, seen as a function on \( \mathcal{H}_{\text{sh}}(\Theta) \).

**Proposition 4.6.** Let \( h \in \mathcal{H}_{\text{sh}} \) be a sheared hyperbolic structure, with boundary dihedral angles given by \( \Theta \). Then \( h \) is in \( \mathcal{H}_{\text{ex}} \) if and only if \( h \) is a critical point of \( V \) restricted to \( \mathcal{H}_{\text{sh}}(\Theta) \).

We will only sketch the proof here, and refer the reader to \(^{1}\)[Sch01b] for the details. The proof was done there for ideal simplices, but applies just as well for hyperideal simplices. The only difference is that in the present case there are less constraints on the dihedral angles of the simplices, so that, for instance, if all simplices were supposed to be strictly hyperideal, the proof would be much simpler.

**Sketch of the proof.** Suppose that \( h \in \mathcal{H}_{\text{ex}} \). Let \( v_1, \cdots, v_n \) be the ideal vertices of \( \Sigma \). Since the shear of \( h \) at all interior edges of \( \Sigma \) vanishes, one can choose for each \( i \) and each simplex \( s \) having \( v_i \) as one of its vertex a (part of) horosphere \( H \) in \( s \) centered at \( v_i \), in a way such that, for the different choices of \( s \), the horospheres coincide on the codimension 1 faces of \( \Sigma \).

One can then apply the Schl" afl formula of lemma \(^{2}\)[BB01] to check that, in any deformation of \( h \) which does not change the total angles at the vertices of \( \Sigma \), the volume remains constant (at first order).

Conversely, suppose that \( h \) is a critical point of \( V \) restricted to \( \mathcal{H}_{\text{sh}}(\Theta) \). We have to prove that \( h \in \mathcal{H}_{\text{ex}} \), i.e. that corresponding faces of two simplices can be glued, and that the shear at all interior edges vanishes.

Let \( e \) be an edge of \( \Sigma \), with endpoints two strictly hyperideal vertices; suppose that \( e \) is in two simplices \( S_1, S_2 \) for which its length is different. Consider the deformation of the angles of \( S_1 \) and \( S_2 \) which increases the angle of \( S_1 \) at \( e \) at speed 1, decreases the angle of \( S_2 \) at \( e \) at speed one, and does not change any other angle. This deformation is clearly in the tangent space \( T\mathcal{H}_{\text{sh}}(\Theta) \), but by the Schl" afl formula it changes the volume, a contradiction. So the faces can be glued.

Now let \( e \) be an interior edge of \( \Sigma \) with endpoints two ideal vertices. We call \( e_- \) and \( e_+ \) the endpoints of \( e \), and use a special type of deformation of the simplices which are adjacent to \( e \). To describe those deformations, we call \( s \) one of those simplices adjacent to \( e \), and \( t_- \) and \( t_+ \) the 2-faces of \( s \) which are adjacent to \( e_- \) and \( e_+ \), respectively, but do not contain \( e \). We orient the vertices of those triangles in a way compatible with the orientation of \( s \). The deformation is as follows.

- the angle at \( e \) and at the opposite edge do not vary.
- the angle at the edge of \( t_+ \) "before" \( e_+ \), and at the edge of \( t_- \) "after" \( e_- \), varies at speed +1.
- the angle at the edge of \( t_+ \) "after" \( e_+ \), and at the edge of \( t_- \) "before" \( e_- \), varies at speed −1.

The same description applies to all the simplices containing \( e \). This deformation is compatible with the angle conditions on the simplices even when all vertices are ideal. Moreover, a direct computation using the Schl" afl formula and an adequate choice of horospheres in the simplices with ideal vertices (as in \(^{1}\)[Sch01b]) shows that the first-order variation of the volume is proportional to the shear of \( h \) at \( e \), and that it vanishes if and only if the shear of \( h \) at \( e \) vanishes. This proves the proposition.

This in turns implies a rigidity result for hyperideal polyhedra, with respect to their dihedral angles. It is one of the basic tools in the proof of lemma \(^{1}\)[BB02]. This rigidity result can also be obtained by other methods, for instance it was proved in \(^{2}\)[Can13] and Legendre \(^{2}\)[LegII] to prove the global rigidity of polyhedra in \( \mathbb{R}^3 \). But the proof given here has the advantage of extending from hyperideal polyhedra to manifolds with hyperideal boundary.

**Corollary 4.7.** Let \( P \) be a hyperideal hyperbolic polyhedron. There is no non-trivial infinitesimal deformation of \( P \) which changes neither its combinatorics nor its dihedral angles.

**Proof.** Let \( h \) be the exact hyperbolic structure on \( \Sigma \) corresponding to \( P \). An infinitesimal deformation of \( P \) which does not change its dihedral angles would be equivalent to a first-order deformation of \( h \), in \( \mathcal{H}_{\text{sh}}(\Theta) \) (where \( \Theta \) is given by the dihedral angles of \( P \) on the boundary edges of \( \Sigma \), and is equal to \( 2\pi \) on the interior edges) such that the volume remains critical at first order. This would contradict the strict concavity of the volume.

\(^{1}\)The fact that it was mostly due to Legendre was recently discovered by I. Sabitov.
The proof of lemma 4.1 also requires a compactness result, which we now state.

**Proposition 4.8.** Let \((P_n)_{n \in \mathbb{N}}\) be a sequence of hyperideal polyhedra with the same combinatorics. Suppose that the sequence of dihedral angles \((\theta_n)_{n \in \mathbb{N}}\) converges to a limit \(\theta\). Then:

- either some subsequence of \((P_n)\) converges.
- or there exists a non-elementary circuit in \(\sigma_1\) on which the sum of the values of \(\theta\) is \(2\pi\).
- or there exists a simple path in \(\sigma_1\) on which the sum of the values of \(\theta\) is \(\pi\).

We will skip the proof here, since this proposition is a consequence of a compactness lemma in [Sch98a] (which is stated in a more general setting). Note however that it is also a special case of a compactness result for manifolds with hyperideal boundary, lemma 6.1 below.

Corollary 4.7 and proposition 4.8 show that, for each combinatorics, the map sending a hyperideal polyhedron to its dihedral angles is a covering. To prove that it is one-to-one, we will show that there exists a specific set of dihedral angles which has a unique inverse image.

**Proposition 4.9.** Let \(\sigma\) be a cellulation of \(S^2\), along which a subset \(V_\sigma\) of the vertices of \(\sigma\). There exists at least one angle assignation satisfying the hypothesis of lemma 4.1 which is realized as the dihedral angles of a unique hyperideal polyhedron, the ideal vertices of which are the elements of \(V_\sigma\).

Of course the uniqueness here is again up to the global hyperbolic isometries.

**Proof.** We use the same proof as the one given by Bao and Bonahon [BB02], so we only sketch the proof and refer the reader to [BB02] for more details. We will use a result on compact polyhedra (as in [And70, RH93]) to obtain a compact polyhedron which is the truncated version of a hyperideal polyhedron with the right combinatorics.

We associate to \(\sigma\) and \(V_\sigma\) another cellulation, \(\overline{\sigma}\), defined as follows.

- \(\overline{\sigma}\) has one face for each face of \(\sigma\), and one for each vertex of \(\sigma\) which is not in \(V_\sigma\) (those vertices correspond to the strictly hyperideal vertices of \(\sigma\)).
- it has one vertex for each element of \(V_\sigma\), and one for each couple \((e, v)\), where \(v\) is a vertex of \(\sigma\) which is not in \(V_\sigma\) and \(e\) is an edge containing \(v\).
- it has one edge for each edge of \(\sigma\), and one for each couple \((f, v)\), where \(v\) is a vertex of \(\sigma\) not in \(V_\sigma\) and \(f\) is a face of \(\sigma\) containing \(v\).

Note that this transformation is the same as the transformation sending the combinatorics of a hyperideal polyhedron to the combinatorics of its truncated polyhedron. In figure 11 the hyperideal vertices are represented as “bigger” dots. Now we can put a weight \(2\pi/3\) on each edge of \(\overline{\sigma}\) which is also an edge of \(\sigma\), and a weight \(\pi/2\) on each edge of \(\overline{\sigma}\) which is not in \(\sigma\). Then apply the Andreev theorem [And70, RH93] to obtain that there is a unique compact polyhedron \(P_{\overline{\sigma}}\) with combinatorics given by \(\overline{\sigma}\) and with those dihedral angles. We leave it to the reader to check that the hypothesis of the Andreev theorem apply, as in [BB02].

**Proof of lemma 4.7.** First note that it is sufficient to prove the result when all faces of \(\sigma\) are triangles. Indeed, if \(\sigma\) has some non-triangular faces, one can add some edges so as to obtain triangles. By restricting the possible deformations to those which do not change the angles at those additional edges, the lemma as it is stated will follow from the result when the faces are triangles.

Consider a triangulation \(\sigma\) of \(S^2\), with a subset \(V_\sigma\) of the set of vertices of \(\sigma\). Let \(q := |V_\sigma|\), and let \(p\) be the number of vertices of \(\sigma\) which are not in \(V_\sigma\). We call:

- \(\mathcal{P}_{\sigma, V_\sigma}\) the space of hyperideal polyhedra with combinatorics given by \(\sigma\), with ideal vertices exactly the elements of \(V_\sigma\).
- \(\mathcal{A}_{\sigma, V_\sigma}\) the space of angle assignations on the edges of \(\sigma\), satisfying the hypothesis of lemma 4.1 such that, for each vertex \(v\) of \(\sigma\), the sum of the angles on the edges adjacent to \(v\) is \(2\pi\) if and only if \(v \in V_\sigma\).
- \(F : \mathcal{P}_{\sigma, V_\sigma} \rightarrow \mathcal{A}_{\sigma, V_\sigma}\) the map sending a hyperideal polyhedron to the set of its dihedral angles.
Both $P_{\sigma,V_i}$ and $A_{\sigma,V_i}$ are locally smooth manifolds of dimension $3p + 2q - 6$. For $P_{\sigma,V_i}$ it is clear, because $P_{\sigma,V_i}$ is parametrized by the position of the $p$ hyperideal vertices and the $q$ ideal vertices, modulo the action of the 6-dimensional group of isometries of $H^3$. $A_{\sigma,V_i}$ is parametrized by the dihedral angles on the edges of $\sigma$, subject to the condition that the sum of the angles at each ideal vertex is $2\pi$. But a simple computation using the Euler formula shows that the number of edges of $\sigma$ is $3p + 3q - 6$. So we only have to prove that the conditions at the ideal vertices are independent; or, in other terms, that if $a_1, \cdots, a_p \in \mathbb{R}$ are coefficients associated to the ideal vertices such that, for any edge $e$, the sum of the coefficients of the ideal vertices in $e$ is 0, then all $a_i$ are zero. Then, for each face $f$ of $\sigma$:

- either $f$ has at least one strictly hyperideal vertex, and then the coefficients of the two other vertices of $f$ are 0.
- or all the vertices are ideal, and we use the fact that all faces of $\sigma$ are triangles, so an elementary parity argument shows again that all coefficients are 0.

This ends the proof that $\dim(A_{\sigma,V_i}) = \dim(P_{\sigma,V_i})$.

Corollary 4.7 shows that $F$ is a local homeomorphism. Moreover, proposition 4.8 shows that $F$ is proper, so that it is a covering. But $A_{\sigma,V_i}$ is the interior of a convex polytope in some $\mathbb{R}^N$ for some $N$, so this covering is a homeomorphism between each connected component of $P_{\sigma,V_i}$ and $A_{\sigma,V_i}$. Finally proposition 4.9 shows that one element of $A_{\sigma,V_i}$ has a unique inverse image, and therefore $F$ is a homeomorphism. □

Finally, lemma 4.2 is a direct consequence of the construction which we have made, and of the following elementary remark, which we have taken from [Sch01b].

**Remark 4.10.** Let $\Omega \in \mathbb{R}^N$ be a convex subset, and let $f : \Omega \to \mathbb{R}$ be a smooth, strictly concave function. Let $\rho : \mathbb{R}^N \to \mathbb{R}^p$ be a linear map, with $p < N$, and let $\overline{\Omega} := \rho(\Omega)$. Define a function:

$$
\overline{f} : \overline{\Omega} \to \mathbb{R}
$$

$$
\overline{y} \mapsto \max_{x \in \rho^{-1}(\overline{y})} f(x)
$$

Then $\overline{\Omega}$ is convex, and $\overline{f}$ is a smooth, strictly concave function on $\overline{\Omega}$.

An important point is that this strict concavity extends to the case where some exterior dihedral angles are zero, i.e. when some edges are just segments drawn in a face.

**From polyhedra to hyperideal manifolds** We will show below that the proof just given for lemma 4.1 applies, with some modifications, to the proof of theorem 0.5. Just as we have used a decomposition of hyperideal polyhedra into hyperideal simplices, the "building blocks" for hyperideal manifolds will be hyperideal polyhedra. It would be more natural *a priori* to use hyperideal simplices here too, but it turns out to be a problem to prove
that any hyperideal manifold admits a non-degenerate triangulation by hyperideal simplices. Another solution is used in [Sch01b] for ideal manifolds — the point there is that any ideal manifold has a finite cover which admits an ideal triangulation.

**Definition 4.11.** Let $M$ be a hyperideal hyperbolic manifold. A **hyperideal cellulation** of $M$ is a decomposition of $M$ as the union of hyperideal polyhedra with non-empty interior, isometrically glued along their faces, such that the intersection of two polyhedra is always a face of each of them.

This definition implies a non-degeneracy condition: we do not allow degenerate hyperideal polyhedra, or two polyhedra having in common a triangle which is not a face of one of them.

**Lemma 4.12.** Each hyperideal manifold admits a hyperideal cellulation.

**Proof.** It is done along the ideas of Epstein and Penner [EP88]; the situation here is simpler since the action of $\pi_1 M$ on $S^2 \setminus \Lambda$ is discrete. On the other hand, one has to take into account the hyperideal vertices. Another, slightly more complicated but maybe more geometric version of the same idea, was used in [Sch01b].

Let $M$ be a hyperideal hyperbolic manifold. Then $M$ is isometric to the convex hull of a set $\{x_1, \ldots, x_p\}$ of ideal points and a set $\{y_1, \ldots, y_q\}$ of hyperideal points in $E(M)$, where $E(M)$ is the extension of $M$. By "hyperideal point" in $E(M)$, we mean here the orbit of a point in $S^3_1$ under $\pi_1 (E(M))$, seen as a group acting by isometries on $H^3$ and thus also on its "extension" $S^3_3$.

For each $i \in \{1, 2, \ldots, p\}$, choose a "small" horoball $b_i \subset E(M)$ centered at the ideal point $x_i$; we suppose that the $b_i$ are small enough to be embedded and disjoint. Let $B_i$ be the set of inverse images of $b_i$ under the quotient of $H^3$ by $\pi_1 E(M)$.

Let $b$ be one of the horoballs in the collection $B_i$. In the Minkowski model of $H^3$, $b$ is the intersection with $H^3$ of an affine light-like hyperplane $h$. There exists a unique vector $u(b) \in \mathbb{R}^4_1$ such that:

$$h := \{x \in \mathbb{R}^4_1 \mid \langle x, u(b) \rangle = 1\} .$$

Note that, in the projective model of $H^3$, $u(b)$ projects to the point at infinity of $b$. Let:

$$\overline{B}_i := \{u(b) \mid b \in B_i\} .$$

For each $j \in \{1, 2, \ldots, q\}$, the hyperideal vertex $y_j$ of $M$ lifts to an equivariant set $C_j$ of hyperideal points in $H^3$, or, in other terms, to an equivariant set of points in $S^3_3$. In the Minkowski model of $H^3$ we also consider the de Sitter space as the submanifold:

$$S^3_1 = \{x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = 1\} ,$$

with the induced metric. Again note that $C_j$ projects, in the natural extension of the projective model described above to $S^3_3$, to a lift of $y_j$.

Let $C := (\bigcup_{i=1}^p \overline{B}_i) \cup (\bigcup_{j=1}^q C_j)$. It is a discrete set of $\mathbb{R}^4_1$ which is invariant under the action of $\pi_1 M$, and the projections on $\{x_1 = 1\}$ (in the direction of 0) of its points are exactly the vertices of $\tilde{M}$.

Let $CH$ be the convex hull of $C$ in $\mathbb{R}^4_1$. The radial projection on $\{x_1 = 1\}$ of $CH$ is the convex hull of the radial projections of the elements of $C$, so its intersection with the radius 1 ball is exactly $M$. Moreover, $CH$ is polyhedral, and its boundary has a decomposition into 3-dimensional polyhedra. Since the radial projection sends the geodesics of $\mathbb{R}^4_1$ to lines in $\{x_1 = 1\}$, this decomposition projects to $\tilde{M}$ as a cellulation $\Sigma$ of $\tilde{M}$, i.e. a decomposition of $\tilde{M}$ into 3-dimensional polyhedra. By construction $\Sigma$ is invariant under the action of $\pi_1 M$, so $\Sigma$ determines a cellulation of $M$, which we also call $\Sigma$.

Also by construction, for each $h > 0$, there is a finite set of points of $C$ with first coordinate $x_1 \leq h$. So there is also a finite set of polyhedra in $\partial CH$ which contain a point with $x_1 \leq h$. If $K \subset \tilde{M}$ is a compact subset, its radial projection on $\partial CH$ has first coordinate bounded by some $h > 0$, so $\Sigma$ is locally finite — each compact subset of $\tilde{M}$ intersects only a finite number of polyhedra.

This implies that each polyhedron has only a finite number of vertices. Otherwise, one of the polyhedra, say $P$, having an infinite set of vertices, would intersect an infinite set of disjoint fundamental domains of $M$, and it would follow that each fundamental domain of $M$ intersects an infinite set of images of $P$ under elements of $\pi_1 M$.

Note that this construction is far from clear if there is a closed curve in $\partial M$ which is a geodesic of $M$, because in this case the endpoints of that geodesic are in the limit set $\Lambda$. 

23
5 Rigidity

An outline We can now proceed to lemma 5.1 which is the main step in the proof of theorem 0.5. We recall the statement for the reader’s convenience.

Lemma 5.1. Let \( M \) be a hyperideal manifold. Any first-order deformation of its dihedral angles is obtained by a unique first-order deformation of \( M \).

To prove it we will consider a hyperideal manifold \( M \). By lemma 4.12 it admits a decomposition into hyperideal polyhedra. Moreover, if those polyhedra have non-triangular faces, we can further subdivide those faces, so as to obtain only polyhedra with triangular faces — they will then have some (exterior) dihedral angles equal to 0, but it won’t make any difference in the proof.

By lemma 4.11 the dihedral angles of the hyperideal polyhedra provide a parametrization of their deformations. However, since some of the faces are hyperideal triangles, it is not always possible to glue them to obtain a hyperbolic structure — even singular — on \( M \); there are compatibility conditions related to the lengths of the edges. Of course, if such a gluing is possible, the hyperbolic structure obtained in this way will in general be singular at the interior edges of \( \Sigma \), where the total angle will in general be different from \( 2\pi \) and where a “shear” might occur along an edge having two ideal endpoints.

But, just as in section 4, the condition that the total angle is \( 2\pi \) is an affine condition on the space of dihedral angles of the polyhedra, while the lengths are compatible and the shear vanishes at all interior edges of \( \Sigma \) if and only if the volume, restricted to the deformation which do not change the total angles at the edges of \( \Sigma \), is critical. Since the volume of each polyhedron is strictly concave, the volume of \( M \) (as a function of the dihedral angles of the polyhedra) is also strictly concave, and this will allow the same argument as the one given in the proof of corollary 4.7. The existence of deformations inducing a given first-order variation of the dihedral angles will follow from this rigidity statement and a dimension-counting argument.

The space of angle assignations From now on, we consider a hyperideal manifold \( M \), along with a decomposition \( \Sigma \) of \( M \) into a finite set of polyhedra, for instance as provided by lemma 4.12 if \( M \) is supposed to be a hyperideal manifold. We also suppose given a subset \( V \) of the set of vertices of \( \Sigma \). \( V \) will appear later as the set of vertices of \( \Sigma \) which correspond to ideal points for the hyperbolic metrics that will appear on \( M \).

Definition 5.2. An angle assignation on \( \Sigma \) is the choice, for each polyhedron \( p \) of \( \Sigma \), of a set of (exterior) dihedral angles on the edges, subject to the condition that:

- it satisfies the hypothesis of lemma 4.1
- for each vertex \( v \) of \( p \) which is in \( V \), the sum of the angles assigned to the edges of \( p \) containing \( v \) is equal to \( 2\pi \).
- for each vertex \( v \) of \( p \) which is not in \( V \), the sum of the angles assigned to the edges of \( p \) containing \( v \) is strictly larger than \( 2\pi \).

The set of angle assignations on \( \Sigma \) will be denoted by \( \mathcal{A}_\Sigma \).

An angle assignation on \( \partial \Sigma \) is the choice, for each exterior edge of \( \Sigma \), of an angle in \( (0, \pi) \), in a way that satisfies the conditions in theorem 0.5. The set of angle assignations on \( \partial \Sigma \) will be denoted by \( \mathcal{A}_{\partial \Sigma} \).

By lemma 4.1 each angle assignation on \( \Sigma \) defines, for each polyhedron \( p \) of \( \Sigma \), a homeomorphism from \( p \) to a hyperideal polyhedron (up to isotopies fixing the vertices), which we can also consider as a "hyperideal metric" on \( p \). Each vertex of \( p \) is then either ideal — if the sum of the angles on the adjacent edges is \( 2\pi \), i.e. if \( v \in V \) — or strictly hyperideal.

For some angle assignations, the hyperideal polyhedra associated to the polyhedra of \( \Sigma \) can be glued along their faces. This happens exactly when, for each edge \( e \) of \( \Sigma \) with two endpoints which are not in \( V \), the lengths of \( e \) for the hyperideal metrics on the polyhedra of \( \Sigma \) containing \( e \) are the same. The next definition describes assignations such that the total angle around each interior edge of \( \Sigma \) is as needed.

Definition 5.3. Let \( \theta \in \mathcal{A}_\Sigma \) be an angle assignation on \( \Sigma \). \( \theta \) is:

- coherent, if the lengths assigned to the edges with no endpoint in \( V \) by all the polyhedra that contains it are the same. The set of coherent angle assignations will be denoted by \( \mathcal{A}_{\Sigma}^c \).
- regular if the total angle around each interior edge is \( 2\pi \). The set of regular angle assignations will be denoted by \( \mathcal{A}_{\Sigma}^{reg} \).
Note that the set of regular angle assignments is an affine subspace of $A_\Sigma$. When $\theta$ is coherent, it is possible to glue isometrically the faces of the polyhedra (according to the combinatorics given by $\Sigma$).

**Definition 5.4.** An angle assignation $\theta$ on $\Sigma$ is **exact** if it is coherent and, in addition, the singular hyperbolic structure obtained by gluing the polyhedra has zero shear at all the interior edges of $\Sigma$. The space of exact angles assignations on $\Sigma$ will be denoted by $A^e_\Sigma$.

The condition that the shear is non-zero is non-trivial only for the edges with two endpoints which are in $V_i$.

The smooth hyperbolic structures on $M$ correspond to the angle assignations which are both exact and regular; in that case, at each interior edge of $\Sigma$, the shear vanishes because the structure is exact, while the total angle is $2\pi$ since the structure is regular.

**The volume** As a consequence of proposition 8.4, the volume is concave over the space of angle assignations on $M$.

**Proposition 5.5.** $V$ is a strictly concave function on $A_\Sigma$, and also on $A^e_\Sigma$.

**Proof.** The concavity on $A_\Sigma$ is clear since $V$ is the sum of the volumes of the simplices, which are concave functions. The concavity over $A^e_\Sigma$ is a consequence since $A^e_\Sigma$ is an affine subset of $A_\Sigma$.

**Definition 5.6.** Let $\alpha : \sigma_1 \to (0, \pi)$ be an angle assignation on $\partial \Sigma$. We call $A^e_\Sigma(\alpha)$ the set of angle assignations on $M$ such that the total (interior) angle at each boundary edge $e$ is $\pi - \alpha(e)$, and $A^e_\Sigma(\alpha)$ (resp. $A^e_\Sigma(\alpha)$) the space of those angle assignations which are regular (resp. coherent).

**Proposition 5.7.** Let $\theta \in A^e_\Sigma$, and let $\alpha : \sigma_1 \to (0, \pi)$ be the function sending a boundary edge to its exterior dihedral angle. Then $\theta$ is exact if and only if it is a critical point of $V$ restricted to $A^e_\Sigma(\alpha)$.

**Proof.** Suppose first that $\theta$ is exact. By construction, $\theta$ determines a hyperbolic metric on each polyhedron of $\Sigma$. $\theta$ is coherent, so that, by definition, the hyperideal polyhedra obtained can be glued along their faces. Since $\theta$ is exact and regular, the resulting singular hyperbolic structure on $M$ is actually smooth. Choose a horosphere centered at each of the ideal vertices of $\Sigma$, and let $\delta \in T_\theta A^e_\Sigma(\alpha)$ be an infinitesimal variation of $\theta$. The Schl"afli formula shows that the total first-order variation of the volume vanishes. This shows that $\theta$ is a critical point of $V$ restricted to $A^e_\Sigma(\alpha)$.

Suppose now that $\theta$ is not exact. Then either $\theta$ is not coherent, or it is coherent but not exact.

Suppose first that $\theta$ is not coherent. Then there is an edge $e$ of $\Sigma$ which is contained in two polyhedra $p_1$ and $p_2$ of $\Sigma$ which, for the angle assignations determined by $\theta$, give different lengths to $e$. This implies that both endpoints of $e$ are strictly hyperideal. Consider the first-order deformation $\delta \theta$ of $\theta$ which

- increases the angle of $p_1$ at $e$ at speed 1.
- decreases the angle of $p_2$ at $e$ at speed 1.
- does not change any other angle.

A short check shows that $\delta \theta \in T_\theta A^e_\Sigma(\alpha)$. Moreover, the Schl"afli formula shows that $dV(\delta \theta) \neq 0$, so $\theta$ is not a critical point of $V$ restricted to $A^e_\Sigma(\alpha)$.

Suppose now that $\theta$ is coherent but not exact. There is then an interior edge $e$ of $\Sigma$ at which the shear of the singular hyperbolic structure defined by $\theta$ is not zero. We call $e_-$ and $e_+$ the endpoints of $e$, which are both ideal. Let $p_1, \cdots, p_r$ be the polyhedra of $\Sigma$ containing $e$, in cyclic order; we set $p_0 := p_r, p_{r+1} = p_1$. For each $i \in \{1, \cdots, r\}$, let $e^+_i$ be the edge of $\Sigma$ which is common to $p_{i-1}$ and $p_i$ and contains $e_+$, and let $e^-_i$ be the edge of $\Sigma$ which is common to $p_{i+1}$ and $p_i$ and contains $e_-$. For each $i \in \{1, \cdots, r\}$, let $f_i$ be the 2-face of $\Sigma$ containing $e, e^+_i$ and $e^-_i$.

For each couple $(p, s)$, where $p$ is a polyhedron of $\Sigma$ and $s$ is an ideal vertex of $\Sigma$ contained in $p$, choose a horosphere in $p$ centered at $s$, in such a way that:

- for all $i \in \{2, \cdots, r\}$, the intersections with $f_i$ of the horospheres centered at $e_-$ and $e_+$ on both sides of $f_i$ coincide.
- for all $i \in \{1, \cdots, r\}$, the intersections with $f_i$ of the horospheres centered at the endpoints of $e^+_i$ and $e^-_i$ opposite to $e_+$ and $e_-$ coincide.

25
Since the shear of \( \theta \) at \( e \) does not vanish, the horospheres centered at \( e_+ \) (resp. \( e_- \)) on both sides of \( f_1 \) do not coincide: they are at a constant distance equal to the shear of \( \theta \) at \( e \). Consider the first-order variation \( \dot{\theta} \) of \( \theta \) which, for each \( i \in \{1, \ldots, r\} \):

- increases at speed one the angles of \( p_i \) at \( e_i^+ \) and at \( e_{i+1}^- \).
- decreases at speed one the angles of \( p_i \) at \( e_{i+1}^+ \) and at \( e_i^- \).
- does not change any other angle.

This deformation is in \( T_\theta \mathcal{A}_\Sigma \), i.e. it respects the condition that, for each polyhedron \( p \) of \( \Sigma \), the sum of the dihedral angles at each vertex of \( p \) which is in \( V_i \) remains \( 2\pi \). It is also in \( T_\theta \mathcal{A}_\Sigma(\alpha) \), because the dihedral angle on the boundary edges of \( \Sigma \) do not change. It is even in \( T_\theta \mathcal{A}_\Sigma^c(\alpha) \), because the total angle around the interior edges of \( \Sigma \) do not change — the variations corresponding to \( p_i \) and \( p_{i+1} \) always cancel.

Now apply the Schlafli formula (lemma 3.13) to \( \dot{\theta} \) with this choice of horosphere. There are two contributions for each edge \( e_i^+ \) and \( e_i^- \), one for each side of \( f_i \). But it is clear that those contributions cancel except for \( e_i^+ \) and \( e_i^- \), where the fact that the horospheres centered at \( e_- \) and \( e_+ \) on both side do not coincide on \( f_1 \) means that there is a discrepancy which is proportional to the shear at \( e \). This shows that, when \( \theta \) is not exact, it is not a critical point of \( V \) restricted to \( \mathcal{A}_\Sigma^c(\alpha) \).

\[ \square \]

**Proof of lemma 2.12** We start from a hyperideal manifold \( M \), with a decomposition into hyperideal polyhedra, as given by lemma 1.12.

The dihedral angles of the polyhedra of \( \Sigma \), for the hyperbolic metric on \( M \), define an element \( \theta \) of \( \mathcal{A}_\Sigma^c \cap \mathcal{A}_\Sigma^0 \), and, by proposition 5.7, it is a critical point of \( V \) restricted to \( \mathcal{A}_\Sigma^c(\alpha) \), where \( \alpha \) is the function sending a boundary edge of \( \Sigma \) to its exterior dihedral angle.

Let \( \theta \) be a first-order deformation of \( \alpha \), and let \( (\alpha_t)_{t \in [0,1]} \) be a 1-parameter deformation of \( \alpha \) with \( \dot{\alpha} = d\alpha/ dt \). For \( t \) small enough, \( \mathcal{A}_\Sigma^c(\alpha_t) \) is an affine subspace of \( \mathcal{A}_\Sigma^c(\alpha) \), and the strict concavity of \( V \) shows that there is a unique maximum \( \theta_t \) of \( V \) restricted to \( \mathcal{A}_\Sigma^c(\alpha_t) \). By proposition 5.7, \( \theta_t \) determines a hyperideal structure on \( M \).

Again by the strict concavity of \( V \), \( (\theta_t) \) is a smooth 1-parameter deformation of \( \theta \); if \( \dot{\theta} : = d\theta_t/ dt \), \( \dot{\theta} \) is a first-order deformation of \( \theta \) such that the induced variation of the boundary dihedral angles is \( \dot{\alpha} \).

**Remark 5.8.** \( V \) is a strictly concave function on \( \mathcal{A}_\Sigma^c \cap \mathcal{A}_\Sigma^0 \), parametrized by the boundary dihedral angles.

**Proof.** As in section 4, the proof is a direct consequence of remark 1.10

\[ \square \]

### 6 Compactness

This section contains the proof of the basic compactness result which we need for the proof of theorem 0.5; we recall it first.

**Lemma 6.1.** Let \((g_n)_{n \in \mathbb{N}}\) be a sequence of hyperideal structures on \( M \), with the same boundary combinatorics. For each \( n \), let \( \alpha_n \) be the function which associates to each boundary edge of \( (M, g_n) \) its exterior dihedral angle, and suppose that \( \alpha_n \to \alpha \), where \( \alpha \) still satisfies the hypothesis of theorem 0.5. Then, after taking a subsequence, \( g_n \) converges to a hyperideal structure \( g \) on \( M \).

This result uses the following natural notion of convergence of manifolds with hyperideal boundary. Let \((g_n)_{n \in \mathbb{N}}\) be a sequence of hyperbolic metrics on \( M \), such that the \((M, g_n)\) are manifolds with hyperideal boundary. Let \((M, g)\) be a hyperbolic manifold with hyperideal boundary. We say that the \((M, g_n)\) converges to \((M, g)\) if, for each compact subset \( K \subset E(M, g) \) in the extension of \( E(M, g) \) and each \( \epsilon > 0 \), for each \( n \) large enough, there exists a compact subset \( K_n \subset E(M, g_n) \) such that \( K_n \cap (M, g_n) \) is \( \epsilon \)-close to \( K \cap (M, g) \) in the Gromov-Hausdorff distance.

The proof uses another compactness lemma, concerning sequences of hyperbolic manifolds with a convex, polyhedral boundary (which is not hyperideal). It states that if the third fundamental forms of the boundary converge to a reasonable limit, then the sequence converges. Lemma 6.1 will follow by truncating the hyperideal ends, so as to obtain a sequence of manifolds with polyhedral, non hyperideal boundary. It will be necessary later for some applications to circle packings.

We will also give a slightly more general compactness result, in which some geodesics for the limit of the third fundamental forms have length \( 2\pi \).
Manifolds with polyhedral boundary  Lemma 6.1 will follow from the next lemma, which is also of independent interest and might be useful when dealing with manifolds with a boundary that is locally like a compact or ideal polyhedron.

Definition 6.2. Let \((M, g)\) be a hyperbolic 3-dimensional manifold with convex boundary, and let \(E(M)\) be its extension. We say that \((M, g)\) is a manifold with polyhedral boundary if:

- for each convex ball \(\Omega \subset H^3\) and each isometric embedding \(\phi : \Omega \to E(M)\), the intersection of \(M\) with \(\phi(\Omega)\) is the image by \(\phi\) of the intersection with \(\Omega\) of a semi-ideal polyhedron \(P \subset H^3\).

- \(\partial M\) contains no closed curve which is a geodesic of \(M\).

A semi-ideal polyhedron in \(H^3\) is a polyhedron which has vertices which can be either in hyperbolic space, or on its boundary (ideal points). For instance, compact polyhedra and ideal polyhedra are semi-ideal. In this definition, the second condition is necessary because, otherwise, the boundary of the convex hull of the vertices could intersect the boundary of the convex core of \(M\); this is a case we want to exclude because our rigidity proof then fails.

We use the same notion of convergence as for manifolds with hyperideal boundary, as described near the end of this section, i.e. Gromov-Hausdorff convergence on compact subsets.

Lemma 6.3. Let \((g_n)_{n \in \mathbb{N}}\) be a sequence of metrics on \(M\), such that \((M, g_n)\) are manifolds with polyhedral boundary. Suppose that:

- for all \(n\), \((M, g_n)\) has the same boundary combinatorics, and the ideal vertices of \(\partial M\) remain the same.

- the third fundamental forms \(\mathbb{I}_n\) of \(\partial M\) for the \(g_n\) converge to a limit \(\mathbb{I}_\infty\), which is a spherical metric with conical singularities.

- the closed geodesics of \(\mathbb{I}_\infty\) which are contractible in \(M\) have length \(L \geq 2\pi\), and \(L > 2\pi\) except when they bound a hemisphere.

Then, after taking a subsequence, \((M, g_n)_{n \in \mathbb{N}}\) converges to a manifold with polyhedral boundary \((M, g)\).

The proof will be done below, after we show how the proof of lemma 6.1 follows from this lemma.

Truncating hyperideal ends  Let \((M, g)\) be a manifold with hyperideal boundary. By definition it has a finite number of hyperideal ends, and, if \(E\) is one of them, there is a totally geodesic plane \(P\) in the extension \(E(M, g)\) which is orthogonal to all the faces and edges adjacent to \(E\). \(P\) is the plane dual to the hyperideal vertex at \(E\).

One can then truncate \((M, g)\) by each of the planes dual to its hyperideal ends. One obtains in this manner a manifold with polyhedral boundary, which we call the truncated manifold \((M, g_T)\) associated to \((M, g)\). It has two kinds of faces: the "black" faces, which are what remains of the faces of \((M, g)\) after truncation, and the "red" faces, which are where the truncation happened. Each "red" face is adjacent to "black" faces only. When a "black" face shares an edge with a "red" face, the dihedral angle between them is always \(\pi/2\).

We will also call "red" edges the edges between a black and a red face, and "black" edges the edges between two black faces. So the "black" edges are what remains of the edges of \((M, g)\) after truncation.

Consider the universal cover \(\tilde{M}\) of \((M, g_T)\) as a convex subset of \(H^3\), and then its boundary \(\partial \tilde{M}\). Let \(\partial^* \tilde{M}\) be the dual polyhedron in the de Sitter space, which is invariant under the natural action of \(\pi_1 \tilde{M}\) on \(S^3_1\). Taking the quotient, we find a compact polyhedron (in a quotient of \(S^3_1\)) for each boundary component of \(\tilde{M}\). This polyhedron has:

- a "red" vertex for each "red" face of \((M, g_T)\), i.e. for each strictly hyperideal vertex of \((M, g)\).

- a "black" vertex for each "black" face of \((M, g_T)\), i.e. for each face of \((M, g)\).

- a "red" edge of length \(\pi/2\) between any red vertex and any adjacent black vertex.

- a "black" edge between each two adjacent black vertices, of length equal to the exterior dihedral angle between the corresponding faces of \((M, g)\).
Of course, the faces have a metric of constant curvature 1. Thus we see that, as for hyperideal polyhedra (see [BB02, Rou02]), the dual metric of the boundary of \((M, g_T)\) has a very special form. Its "black" vertices, and the "black" edges between them, form a graph, which is combinatorially and metrically the dual graph of the boundary of \((M, g)\) (i.e. before truncation). Each face of this graph has a boundary of length at least \(2\pi\) (and strictly larger except for faces corresponding to an ideal vertex of \((M, g)\)).

Moreover, each of those faces with boundary length strictly larger than \(2\pi\) contains exactly one "red" vertex, which is connected to each of the "black" boundary vertices by a "red" edge of length exactly \(\pi/2\).

Thus we have a pretty simple picture of the third fundamental form of the boundary of the truncated manifold \((M, g_T)\). It has:

- one hemisphere for each ideal vertex of \((M, g)\).
- one "singular hemisphere", obtained as a quotient of the universal cover of a hemisphere minus its "center", for each strictly hyperideal vertex of \((M, g)\). The boundary length is then strictly larger than \(2\pi\). The center of the "singular hemispheres" are the "red" vertices. Those "singular hemispheres" have geodesic boundary.
- the "singular hemispheres" (and the hemisphere corresponding to ideal vertices of \((M, g)\)) are glued along the dual graph of the boundary of \((M, g)\).

The main points of the discussion above are in the first 3 columns of the table below; the fourth column is for later reference.

| Boundary of \((M, g)\) | Boundary of \((M, g_T)\) | III of \((M, g_T)\) | Circle packing limit |
|------------------------|------------------------|---------------------|---------------------|
| Strictly hyperideal vertices | faces | "Central" vertices | "Red" circles |
| Ideal vertices | Ideal vertices | Hemispheres | Tangency points |
| Faces | "Black" faces | "boundary" vertices | "Black" circles |
| Edges | "Black" edges | Length = exterior angle | Intersection of \(\geq 3\) circles |
| (vertex, face) | "Red" edges | Length \(L = \pi/2\) | Orthogonal intersection |
| (vertex, edge) | Vertices | Faces | |

A fundamental remark, made in [Rou02] for hyperideal polyhedra and hyperideal fuchsian polyhedra, is that the length condition which appears in lemma 6.3, when applied to \((M, g_T)\), is equivalent to a statement on the dihedral angles of the boundary of \((M, g)\). We only outline the proof, since the proof given in [Rou02] extends to the situation we consider with only minimal modifications.

**Lemma 6.4.** The following statements are equivalent.

1. Each closed geodesic of \((\partial M, \hat{M})\), which is contractible in \(M\), has length \(L > 2\pi\).

2. The dihedral angles of the boundary of \((M, g)\) satisfy the conditions of theorem 0.5.

**Outline of the proof.** A key point is that the intersection of a geodesic with the interior of a singular hemisphere (or a hemisphere) has length exactly \(\pi\). Therefore, a closed geodesic of length \(L \leq 2\pi\) can either remain on the graph made of the "black" edges, or it can enter only one singular hemisphere. If it enters one singular hemisphere, the remaining path outside it has length at most \(\pi\). It follows that \((2) \Rightarrow (1)\).

Conversely, any path made of "black" edges is a geodesic of \(\hat{M}\), since, at each vertex, each side is made of at least one singular hemisphere, so that the total angle on each side is at least \(\pi\). So any circuit in the dual graph of the boundary of \((M, g)\) is a geodesic of \(\hat{M}\). Moreover, two vertices in the boundary of a singular hemisphere \(f\) are either at distance less than \(\pi\) along the boundary of \(f\), or are joined by a geodesic of length exactly \(\pi\) going through the center of \(f\). Therefore the simple paths in the dual graph of \((M, g)\) also correspond to closed geodesics of \(\hat{M}\). This shows that condition (1) implies (2).

We can now prove lemma 6.1, admitting lemma 6.3, which we will prove below.

**Proof of lemma.** We consider a sequence \((M, g_n)\) of hyperideal manifolds. Let \((M, g_T, n)\) be the truncated manifold associated to \((M, g_n)\), and let \(\hat{M}^\infty_n\) be its dual metric.

Consider the graph \(G\) on \(\partial M\) dual to the combinatorics of \(\partial(M, g)\), with, for each edge, a length equal to the limit exterior angle \(\alpha\). Define a metric \(\hat{M}_\infty^\infty\) on \(\partial M\) by gluing in each face of \(G\), a singular hemisphere (there is a unique singular hemisphere with the right boundary length). Then \(\hat{M}_\infty^\infty = \lim_{n \to \infty} \hat{M}^\infty_n\).
The hypothesis of lemma 6.1 along with lemma 6.4 show that \( \mathcal{M}_\infty \) satisfies the hypothesis of lemma 6.3. Its closed geodesics which are contractible in \( M \) have length \( L \geq 2\pi \), and \( L > 2\pi \) unless they bound a hemisphere. Applying lemma 6.3 then shows that the sequence of manifolds with polyhedral boundary \((M, g_{T,n})\) converges to a limit \((M, g_T)\).

Since \( \mathcal{M}_\infty \) has a very special form — it is made by gluing singular hemispheres as described above — it is the dual metric of a manifold with polyhedral boundary which is obtained by truncating a hyperideal manifold \((M, g)\). So \((M, g_T)\) is the truncated manifold associated to \((M, g)\).

**Metrics on the boundary** We now turn to the proof of lemma 6.3. We consider a sequence \((g_n)\) of hyperbolic metrics with polyhedral boundary on \( M \), with the same boundary combinatorics \( \Sigma \).

Since, for each \( n \), \((M, g_n)\) is a hyperbolic manifold with polyhedral boundary, the faces are semi-ideal triangles — each is isometric to a semi-ideal hyperbolic triangle, i.e. a hyperbolic triangle having vertices which can be either hyperbolic or ideal points. There are 3 types of edges: those joining two "finite" vertices, which we call "finite edges", those which connect a finite vertex to an ideal vertex, which will be called "semi-ideal", and the "ideal" edges connecting two ideal vertices.

The metric induced by \( g_n \) on \( \partial M \) is however not completely determined in general by the metrics on those triangles; this is particularly clear when all the vertices are ideal. To reconstruct the metric, one needs some additional information, related to the following two definitions.

**Definition 6.5.** Let \( T \) be a semi-ideal triangle, let \( e \) be an edge of \( T \), let \( v \) be the vertex of \( T \) opposite to \( e \), and let \( \overline{e} \) be the complete hyperbolic geodesic containing \( e \). There is a unique point \( c \in \overline{e} \) such that the normal to \( \overline{e} \) at \( c \) contains the vertex \( v \). We will call \( c \) the *projection* of \( v \) on \( e \).

**Definition 6.6.** Let \( e \) be an edge of \( \Sigma \), and let \( \overline{e} \) be the complete geodesic containing \( e \). Orient it, and let \( T_+ \) and \( T_- \) be the triangles on the "positive" and "negative" sides of \( e \), respectively. The *shift* of the metric \( h \) at \( e \) is the oriented distance between the projections on \( \overline{e} \) of the vertices opposite to \( e \) in \( T_+ \) and \( T_- \).

Note that this definition does not depend on the orientation chosen on \( \overline{e} \). The shifts along the ideal edges, along with the metrics on the triangles, are the data necessary to understand the metric induced on \( \partial M \), since they describe how the ideal edges are glued. This implies in particular the next proposition, which implicitly uses the hypothesis of lemma 6.3.

**Proposition 6.7.** Suppose that:

- the lengths of all compact edges of the boundary converge.
- the shifts of all ideal edges of the boundary converge.
- each dihedral angle converges.

Then, after taking a subsequence, the sequence of metrics \((g_n)\) converges.

**Proof.** Let \( h_n \) be the sequence of metrics induced on \( \partial M \). When two triangles share an edge which is either finite or semi-ideal, there is a unique way of gluing them isometrically along their common boundary. When they share an ideal edge, the gluing is uniquely determined by the shift along this edge. Moreover, the lengths of the edges uniquely determine the metric on the triangles. So the \( h_n \) are determined uniquely by the edge lengths and the shifts along the ideal edges.

So, under the hypothesis of the proposition, \((h_n)\) converges to a metric \( h \). We have also supposed that the dihedral angles of all edges converge. So, for each connected component \( \partial_i M \) of \( \partial M \), after we compose with a sequence of hyperbolic isometries, the lift to \( H^3 \) of the universal cover of \( \partial_i M \) converges on compact subsets, as a sequence of convex, polyhedral surfaces in \( H^3 \).

We now consider the conformal structure at infinity on each connected component of \( \partial E(M) \). Those conformal structures can be reconstructed from the induced metric \( h_n \) and the dihedral angles, by a procedure known as "grafting": one should "open" each edge, and glue in a strip of width equal to the exterior dihedral angle. One should also glue in the hole corresponding to each vertex the interior of a spherical polygon with edge lengths given by the dihedral angles of the adjacent edges. The conformal structure of the resulting metric is the conformal structure at infinity of the corresponding boundary component of \( E(M) \).

By our hypothesis, this shows that the sequence of conformal structures \( c_n \) at infinity of \( E(M) \) converges. By the Ahlfors-Bers theorem [Ahlf63], \( E(M) \) converges. Thus, after extracting a subsequence, each boundary component of \( M \) converges, as a polyhedral surface in a converging sequence of hyperbolic ends.
This means that the universal cover of $M$ for the metrics $g_n$, seen as convex subsets of $H^3$, converge on compact subsets to a convex domain, on which $\pi_1 M$ acts by isometries. The quotient provides the limit hyperbolic metric $g$ on $M$.

Consequences of short closed curves As already mentioned above, the proof of lemma 6.8 is based on the idea that the existence of a short simple closed curve implies the appearance of a sequence of geodesics for $\mathbb{H}$ which are either contractible in $M$, with length converging to $2\pi$, or non-contractible in $M$, with length converging to 0. We state here two propositions clarifying both aspects of this phenomenon.

**Proposition 6.8.** Let $c$ be a simple closed curve in $\partial M$ which is contractible in $M$. Let $(c_n)_{n \in \mathbb{N}}$ be sequence of curves, each homotopic to $c$, such that the length of $c_n$ for $h_n$ converges to 0, and:

- either $c$ is non-contractible in $\partial M$.
- or $c_n$ bounds a disk in $\tilde{\partial} M$ which contains a point at distance at least 1 from $c_n$ (for the induced metrics $h_n$).

Then, after extracting a subsequence, there exists a sequence of closed curves $(\gamma_n)_{n \in \mathbb{N}}$, each homotopic to $c$, converging to a geodesic of length $2\pi$ for $\mathbb{H}_\infty$.

Proof. Choose $n \in \mathbb{N}$. $c_n$ is contractible in $M$, so it lifts to a closed curve in $\tilde{\partial} M$, which we also call $c_n$, with $\lim_{n \to \infty} L(c_n) = 0$.

The universal cover $\tilde{M}$ of $M$ has convex boundary. If $c$ is non-contractible in $\tilde{\partial} M$, there exists a complete geodesic $\gamma$ in $\tilde{M}$, which does not intersect $\tilde{\partial} M$, but such that $c$ is not contractible in $\tilde{M} \setminus \gamma$. Otherwise, $c_n$ bounds a disk in $\tilde{\partial} M$ which contains a point $x_n$ at distance at least 1 from $c_n$; in this case let $\gamma$ be a geodesic ray in $\tilde{M}$ starting from $x_n$, such that $c_n$ is not contractible in $\tilde{M} \setminus \gamma$.

Let $p_n$ be a point in $\gamma$ such that the distance between $p_n$ and $c_n$ is minimal. Let $P_k$ be the hyperbolic plane orthogonal to $\gamma$ at $p_n$. Then $\tau_n \equiv \tilde{\partial} M \cap P_n$ is a curve in $\tilde{\partial} M$, homotopic to $c_n$, which for $n \to \infty$ is arbitrarily close to $c_n$. Moreover, $\lim_{n \to \infty} L(\tau_n) = 0$.

For each $n$, let $\tau_n^*$ be the curve in $S^1_1$ made of points dual to the support planes of $\tilde{\partial} M$ along $\tau_n$. As $n \to \infty$, $\tau_n^*$ converges to the dual $\gamma^*$ of $\gamma$ in $S^1_1$, which is geodesic, and therefore of length $2\pi$ like all geodesics of $S^1_1$. Therefore, $(\tau_n^*)$ converges to a geodesic of $(\partial M)^*$, so $(\tau_n)$ converges to a geodesic of length $2\pi$ of $\mathbb{H}_\infty$.

**Proposition 6.9.** Let $c$ be a closed curve in $\partial M$, which is not contractible in $M$. Suppose that there exists a family $(c_n)$ of curves homotopic to $c$, each $c_n$ being geodesic for $h_n$, such that the length of $c_n$ for $h_n$ converges to 0. Then the length of $c_n$ for $\mathbb{H}_n$ goes to 0.

Proof. Fix $n$, and consider the universal cover $\tilde{M}$ of $M$ as a subset of $H^3$. Each $c_n$ lifts to a geodesic in $(\partial \tilde{M}, h_n)$, with endpoints in $\partial_\infty H^3$. The result is obtained by applying to those lifted curves the following elementary statement of hyperbolic geometry (see Pau93, EM86): there exists a constant $C > 0$ such that, if $\Omega \subset H^3$ is a convex set and $g$ is a complete geodesic in $\partial \Omega$ with endpoints on $\partial H^3$, then, for each segment $s$ of $g$ of length $l$, the total bending of $s$ is at most $C(l+1)$.

Appearance of short closed curves In order to prove lemma 6.8 we now investigates two situations where the metric degenerates — when the length of an edge diverges, and when a shift goes to infinity — to show that, in each case, a short geodesic appears.

**Proposition 6.10.** Suppose that there is a finite edge $e$ in $\partial M$ whose length goes to infinity for the induced metrics $(h_n)$. Then, after taking a subsequence, there exists a closed curve $c$ in $\partial M$ whose length converges to 0, and which either is homotopically non-trivial, or bounds a disk $D$ which, for $n$ large enough, contains a non-ideal vertex at distance at least 1 from $\partial D$ (for $h_n$).

Proof. Since the length of $e$ goes to infinity, and since the number of vertices of the triangulation is bounded, we can find arbitrarily long segments of $e$ which are far enough from all the vertices. More precisely, for any $L > 0$, after taking a subsequence, we can find a segment $s$ of $e$ of length $L$ such that $s$ is at distance at least 1 from all the vertices.

The triangulation of $\partial M$ is by a finite (i.e. bounded) number of triangles; since the area of a hyperbolic triangle is bounded, the area of $\partial M$ for the metrics $h_n$ is uniformly bounded. Therefore, the infimum of the injectivity radius of the metrics $h_n$ at the points of $s$ goes to 0, and the existence of the requested curve follows.
Proposition 6.11. Suppose that the sequence of shifts of \((h_n)\) does not converge. Then, after taking a subsequence, there is a closed curve \(c_0\) in \(\partial M\) whose length goes to 0, and which either is homotopically non-trivial, or which bounds a disk \(D\) containing at least two vertices at distance at least 1 from \(\partial D\).

Proof. Same as for proposition 6.10 but with a segment \(s\) of \(e\) which lies between the projections on \(e\) of the vertices opposite to \(e\) in the triangles adjacent to \(e\).

Proof of lemma 6.3. Suppose that the sequence of metrics \((h_n)\) does not converge. Proposition 6.1 shows that either there is a finite edge whose length goes to infinity, or there exists an ideal edge whose shift goes to infinity. In the first case, proposition 6.10 provides a short curve which, thanks to propositions 6.8 and 6.9 is clearly seen to be forbidden by the statement of lemma 6.3. In the second case the same happens but with proposition 6.11.

A more general compactness statement. To prove theorem 0.9 we will need in section 9 a compactness statement more general than lemma 6.3. This is because theorem 0.9 is based on the limit of circle configurations when many angles converge to 0, which, in terms of truncated hyperideal manifolds, means that some edges converge to ideal vertices — a phenomenon which is absent from lemma 6.3. We give here a more general compactness statement, allowing the appearance of some geodesics of length \(2\pi\) for \(\Im\). This leads to the appearance of cusps in the induced metric on the boundary, with parts of the boundary disappearing by going to infinity.

Lemma 6.12. Let \((g_n)_{n \in \mathbb{N}}\) be a sequence of metrics on \(M\), such that \((M, g_n)\) are manifolds with polyhedral boundary. Suppose that:

- for all \(n\), \((M, g_n)\) has the same boundary combinatorics \(\Sigma\).
- the third fundamental forms \(\Im_n\) of \(\partial M\) for the \(g_n\) converge to a limit \(\Im_\infty\), which is a spherical metric with conical singularities.
- there exists a finite family \(D_1, \ldots, D_p\) of disjoint disks in \(\partial M\) such that, for each \(i\), \(\partial D_i\) is a geodesic of \(\Im_\infty\) of length \(2\pi\).
- \(\partial M \setminus \bigcup_{i=1}^p D_i\) contains no closed geodesic of \(\Im_\infty\) of length \(L \leq 2\pi\), contractible in \(M\), except its boundary components.

Then, after taking a subsequence, \((M, g_n)_{n \in \mathbb{N}}\) converges to manifold with polyhedral boundary \((M, g)\). The combinatorics of the boundary of \((M, g)\) is obtained from \(\Sigma\) by replacing each disk \(D_i\) by an ideal vertex. Its third fundamental form is obtained from \(\Im_\infty\) by replacing each \(D_i\) by a hemisphere.

As above, the convergence is as defined near the beginning of this section, i.e. Gromov-Hausdorff convergence of compact subsets. The proof will be given below; it follows the proof of lemma 6.3 the only additional point is that, if \(\Im_\infty\) has a closed geodesic of length \(2\pi\) which does not bound a hemisphere, then the vertices it contains “go to infinity”, so that they are replaced in the limit manifold by only one ideal point.

Cusps in the limit boundary. We need to prove that, whenever a sequence of surfaces has a sequence of closed geodesics of the third fundamental form with length going to \(2\pi\), there is a corresponding sequence of closed curves with length, for the induced metrics, going to 0. This is a converse to proposition 6.8.

Proposition 6.13. Let \((\phi_n)_{n \in \mathbb{N}}\) be a sequence of complete, convex embeddings in \(H^3\) of a surface \(S\). Let \(I_n\) and \(\Im_n\) be the induced metric and the third fundamental form of \(\phi_n\), and suppose that \(\Im_n\) converges to a smooth limit \(\Im_\infty\). Let \(c_n\) be a sequence of curves in \(S\), converging to a limit \(c\), which is a geodesic of \(\Im_\infty\) of length \(2\pi\). Then:

- the length of \(c_n\) for \(I_n\) converges to 0.
- for any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that, for \(n \geq N\), the injectivity radius of \((S, I_n)\) is at most \(\epsilon\) at all points within distance at most \(1/\epsilon\) from \(c_n\).
The proof of this proposition follows ideas which can be found in [KHH93, CD95, Sch96, Sch98a], so we only outline here. Consider the dual embeddings $\phi_n^*: n \in \mathbb{N}$ of $S$ in the de Sitter space $S^3_1$. The images of the $\phi_n^*$ are convex surfaces, and the induced metrics are the $\mathfrak{M}_n$. It is proved in the references cited above that the closed geodesics of convex surfaces in $S^3_1$ have length $L > 2\pi$, and that, if a sequence of simple geodesics $\gamma_n$ with $\gamma_n \subset \phi_n^*(S) \subset S^3_1$ has lengths going to 2$\pi$, then (after applying a sequence of isometries of $S^3_1$) its images converge to a geodesic $\gamma_0$ of $S^3_1$.

Then, by convexity, the surfaces $\phi_n(S)$ are contained in the interiors of the “cylinders” $C_n$ dual to the curves $\gamma_n$. But the $C_n$ converge, as $n \to \infty$, to the geodesic $\gamma_0$ dual to $\gamma_0$. The estimate on the injectivity radius follows.

**Proof of lemma. 6.12.** The proof is based on lemma 6.3 but uses also proposition 6.13.

By proposition 6.13, for each $i \in \{1, \ldots, p\}$, the boundary of $D_i$ is homological to a curve whose length goes to 0 for the induced metrics. Consider the boundary of the universal cover of $M$, which is a convex surface in $H^3$. Proposition 6.13 applied to the boundary of each of the disks $D_i$, shows that each $D_i$ “goes to infinity”, with a thin tube connecting it to the complement of the $D_i$ in $\partial M$.

For $n$ large enough, we can (equivariantly) ”cut” the thin tubes by a plane which is almost orthogonal to each of the edges which it intersects, and glue a small polygon (of diameter going to 0 as $n \to \infty$). The consequence on the third fundamental forms $\mathfrak{M}_n$ is to replace each disk $D_i$ by a spherical polygon which is almost a hemisphere.

Now we can apply lemma 6.3 to the manifolds with polyhedral boundary obtained after this surgery, and we obtain the result.

**Application to circle packings** Lemma 6.12 leads easily to the compactness statement which will be useful later on. We will consider a notion of configuration of circles, on which more details will be given in section 9.

Given a surface $S$ with a $\mathbb{C}P^1$-structure $c$, a **configuration of circles** on $(S, c)$ is a finite set of oriented circles $C_1, \ldots, C_n$ on $S$ for $c$, such that:

- each point of $S$ is in a most 2 closed disks bounded by the $C_i$,
- for each interstice (i.e. connected component of the complement of the disks bounded by the $C_i$) $I$, there exists a circle $C$ orthogonal to each of the circles adjacent to $I$.

The second condition is reminiscent of the statement of theorem 1.9. It is empty when all interstices are bounded by only 3 circles.

**Lemma 6.14.** Let $(c_n, C_n)$ be a sequence of couples, where, for each $n$, $c_n$ is a $\mathbb{C}P^1$-structure on $\partial M$ induced by a complete, convex co-compact hyperbolic metric, and $C_n$ is a circle configuration for $c_n$. Suppose that the incidence graph of the $C_n$ remains the same, and that all intersection angles go to 0. Then, after taking a subsequence, $(c_n)$ converges to a $\mathbb{C}P^1$-structure $c_\infty$ on $\partial M$, and $(C_n)$ converges to a circle packing $C_\infty$ for $c_\infty$.

**Proof.** By definition of a configuration of circles, for each $n$, we can consider another family $C'_n$ of circles, with one circle corresponding to each interstice of $C_n$, and with the circles of $C'_n$ orthogonal to the circles of $C_n$ when they intersect.

Let $g_n$ be a complete, convex co-compact hyperbolic metric on $M$ corresponding to $c_n$. For each circle $\sigma$ of $C_n$ or $C'_n$, consider the oriented totally geodesic plane in $(M, g)$ with boundary at infinity $\sigma$. Since the disks bounded by the circles of $C_n$ and $C'_n$ cover $\partial M$, the complement of the half-spaces bounded by those planes is a compact submanifold of $(M, g)$ with polyhedral boundary. Let $g_n'$ be the metric on this manifold.

The third fundamental form $\mathfrak{M}_n$ of $\partial M$ for the metric $g_n'$ can quite easily be described; $\partial M$ has two kind of faces (corresponding respectively to the circles of $C_n$ and of $C'_n$) and the faces of one kind intersect the faces of the other kind orthogonally. The dihedral angles between the faces corresponding to the circles of $C_n$ are the intersection angles between the corresponding circles. On the other hand, the circles of $C'_n$ are always disjoint, although they might become tangent in the limit when $n \to \infty$. This completely determines $\mathfrak{M}_n$ since the vertices of $\partial M$ are trivalent.

Let $\sigma, \bar{\sigma}$ be two intersecting circles of $C_n$. There are two circles $\sigma', \bar{\sigma}'$ of $C'_n$ which intersect both $\sigma$ and $\bar{\sigma}$ orthogonally. Consider a curve in $(\partial M, \mathfrak{M}_n)$ which follows the edges dual to the edge between the faces bounded by $\sigma, \sigma', \bar{\sigma}, \bar{\sigma}'$. Its total length is $4 \times \pi/2 = 2\pi$. As $n \to \infty$, the restriction of $\mathfrak{M}_n$ to the domain bounded by this curve converges to the metric on a hemisphere. Moreover, by construction there is no other geodesic path in $\mathfrak{M}_\infty$ of length $2\pi$. 

32
Now apply lemma 6.12 taking as $D_i$ all the disks bounded by a curve like the one we have just described; there is one such curve for each edge of the incidence graph of the $C_n$. Lemma 6.12 indicates that the hyperbolic metrics $g'_n$ converge to the hyperbolic metric $g'_\infty$ on a manifold with polyhedral boundary, with one ideal vertex for each disk $D_i$, i.e. for each edge of the incidence graph of the $C_n$.

So all vertices of $(M, g'_\infty)$ are ideal vertices, and each is adjacent to 4 faces. Two of those faces correspond to limits of circles of the $C_n$, and two to limits of circles of the $C'_n$. Taking the boundary of those faces, we find 4 circles, each tangent to exactly one other, each circle orthogonal to the circles of the other pair.

So the boundary of the faces of $(M, g'_n)$ corresponding to the circles of the $C_n$ constitute a circle packing on $\partial M$ for the $\mathbb{C}P^1$-structure on $\partial M$ defined by the complete hyperbolic metric which is the extension of $g'_n$.

7 Spaces of polyhedra

We are concerned in this section with the spaces of angle assignments and of hyperideal manifolds which appear in theorem 0.5. It is basically necessary to show that those spaces are connected. In practice, however, what we will prove is a little weaker.

For the spaces of hyperideal manifolds, we will actually prove that, given two hyperideal manifolds (with the same underlying topology) it is possible to connect them by a path in a space of hyperideal manifolds with a larger number of ideal or hyperideal vertices.

For the space of angle assignations, the approach used here, as in [Sch11], is to remark that, when $M$ has incompressible boundary, the conditions on the various connected components of the boundary are "independent", so that it is sufficient to prove this in the simpler, "fuchsian" situation. In this special case the analog of theorem 0.5 was proved by [Rou02] using other methods, and the connectedness follows.

Spaces of hyperideal manifolds We will later prove that the following spaces of polyhedra are "weakly connected". First we introduce a handy notation.

Definition 7.1. We call $\mathcal{P}$ the set of couples $(p, q)$, where $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n)$ are such that, for all $1 \leq i \leq n, \ p_i, q_i \in \mathbb{N}$, and $p_i + q_i \geq 1$.

Definition 7.2. Let $(p, q) \in \mathcal{P}$. We call $\mathcal{M}_{p,q}$ the space of hyperideal manifolds diffeomorphic to $M$, with, for each $1 \leq i \leq n$, have $p_i$ ideal vertices and $q_i$ hyperideal ends on $\partial_i M$.

Hyperideal manifolds and configurations of points and circles First we remark that hyperideal manifolds are associated to configurations of points and disks in $\partial M$ with a $\mathbb{C}P^1$-structure, and we will clarify to what extent the reverse correspondence is also valid.

Let $n$ be the number of connected components of $\partial M$, for each $i \in \{1, \ldots, n\}$, we call $\partial_i M$ the $i$-th connected component of $\partial M$.

Definition 7.3. Let $(p, q) \in \mathcal{P}$. We call $\mathcal{C}_{p,q}$ the space of triples $(c, P, Q)$, where $c$ is the $\mathbb{C}P^1$-structure on $M$ corresponding to a complete, convex co-compact hyperbolic metric on $M$, $P$ is a finite set of distinct points in $\partial M$, with $p_i$ points in $\partial_i M$, and $Q$ is a finite set of disjoint closed disks in $\partial M$ for $c$, with $q_i$ disks in $\partial_i M$, such that no point in $P$ lies in a disk in $Q$.

Let $g \in \mathcal{M}_{p,q}$. The extension $E(M)$ of $(M, g)$ is a complete, convex co-compact hyperbolic manifold homeomorphic to $M$. Under the isometric embedding of $(M, g)$ in $E(M)$, each ideal vertex of $M$ goes to an ideal point in $E(M)$. Moreover, for each hyperideal vertex of $M$, one can consider the dual plane; its boundary at infinity is a circle in $\partial_{\infty E(M)}$. It is clear that the ideal vertices are outside the disks in $\partial_{\infty E(M)}$ corresponding to the hyperideal vertices. So $g$ determines an element of $\mathcal{C}_{p,q}$, which we call $\Phi_{M, C}(g)$. This defines a map $\Phi_{M, C} : \mathcal{M}_{p,q} \rightarrow \mathcal{C}_{p,q}$.

Conversely, given $\Phi_{M, C}(g)$, one can reconstruct $g$ by taking the convex hull in $E(M)$ of the ideal points in $\Phi_{M, C}(g)$ and of the hyperideal points dual to the disks in $\Phi_{M, C}(g)$, so $\Phi_{M, C}$ is injective. On the other hand, $\Phi_{M, C}$ is in general not surjective, because, when one takes the convex hull $N$ of a set of ideal or hyperideal points in $E(M)$, $N$ might not be a hyperideal manifold. The reason is that $\partial N$ might have a non-empty intersection with the convex core of $E(M)$, and thus have parts looking like a typical convex core of hyperbolic 3-manifold, e.g. with a pleating lamination. Some arguments showing that this indeed happens can be found in [Sch11].

Note that taking the convex hull of the ideal and the hyperideal points in the definition of an element $\gamma = (c, P, Q) \in \mathcal{C}_{p,q}$ has an interpretation in terms of the $\mathbb{C}P^1$-structure $c$, the points in $P$ and the disks in $Q$. 33
Indeed, if \( g \) is the hyperbolic metric on \( M \) corresponding to \( c \), it is not difficult to check that the faces of the convex hull correspond to the maximal disks in \( \partial_{\infty}M \) such that:

- their interiors do not contain any point in \( P \).
- when they intersect a disk in \( Q \), the intersection has its two angles at most \( \pi/2 \).

Let \( \gamma = (c, P, Q) \in C_{p,q} \). Let \( g \) be the complete, convex co-compact hyperbolic metric on \( M \) corresponding to \( c \), and let \( \Lambda \) be the limit set, in \( \partial_{\infty}H^3 \), of the action of \( \pi_1 M \) on \( H^3 \) with quotient \( M \). \( \tilde{M} \), with the lifted metric, is isometric to \( H^3 \). Thus \( \partial \tilde{M} \) is projectively equivalent to \( S^2 \), and \( P \) and \( Q \) lift to sets \( \tilde{P} \) and \( \tilde{Q} \) of points and disks, respectively, in \( S^2 \setminus \Lambda \).

**Definition 7.4.** \( \gamma \) is free if there is no closed disk \( D \subset S^2 \) such that:

- \( D \cap \Lambda \neq \emptyset \).
- \( \text{int}(D) \cap \Lambda = \emptyset \).
- \( \text{int}(D) \) contains no point of \( \tilde{P} \).
- if \( \text{int}(D) \) has non-empty intersection with a disk \( D_1 \in \tilde{Q} \), then \( D \cap D_1 \) has both its angles acute (i.e. at most \( \pi/2 \)).

**Proposition 7.5.** Let \( \gamma = (c, P, Q) \in C_{p,q} \). Let \( g \) be the convex co-compact hyperbolic metric on \( M \) corresponding to \( c \), and let \( N \) be the convex hull in \((M,g)\) of the points in \( P \) and of the hyperideal points corresponding to the elements of \( Q \). The following statements are equivalent.

1. \( \partial N \cap C(M) = \emptyset \).
2. \( \partial \tilde{N} \cap C(\tilde{M}) \) contains no complete geodesic of \( H^3 \).
3. \( \gamma \) is free.
4. \( \gamma \) is in the image of \( \Phi_{\tilde{M},\tilde{C}} \).

**Proof.** (1) \( \iff \) (2): clearly \( C(\tilde{M}) \subset \tilde{N} \), \( C(\tilde{M}) \) is the convex hull of \( \Lambda \), while \( \tilde{N} \) is convex. So the intersection between \( \partial \tilde{N} \) and \( \partial C(\tilde{M}) \), if it is not empty, contains a line, i.e. a complete hyperbolic geodesic.

(1) \( \Rightarrow \) (3): suppose that \( \gamma \) is not free. Let \( D \subset \partial_{\infty}H^3 \) be a disk as in definition 7.4. Let \( P \) be the corresponding hyperbolic plane, i.e. the plane in \( H^3 \) with boundary at infinity the boundary of \( D \). Then \( P \cap N = \emptyset \). But \( \partial P \cap \Lambda \neq \emptyset \), so that \( d(P, \partial C(M)) = 0 \). Therefore, \( d(\partial \tilde{N}, C(M)) = 0 \), so that \( d(\partial N, C(M)) = 0 \). Since both \( \partial N \) and \( C(M) \) are compact, \( \partial N \cap C(M) \neq \emptyset \).

(3) \( \Rightarrow \) (2): if \( \partial \tilde{N} \cap C(\tilde{M}) \) contains a complete hyperbolic geodesic, then \( \partial \tilde{N} \) contains a complete geodesic \( \gamma_0 \) with endpoints in \( \Lambda \subset \partial_{\infty}H^3 \). Then by convexity there is a totally geodesic plane \( P \subset H^3 \) containing \( \gamma_0 \) but not intersecting the interior of \( \tilde{N} \). The corresponding disk in \( \partial_{\infty}H^3 \) contains in its boundary the endpoints of \( \gamma_0 \), and thus \( \gamma \) is not free.

(4) \( \Rightarrow \) (2): suppose that \( \partial \tilde{N} \cap C(\tilde{M}) \) contains a complete hyperbolic geodesic \( \gamma_0 \). Let \( \gamma_1 \) be the projection of \( \gamma_0 \) on the quotient \( C(M) \). If \( \gamma_1 \) is a closed geodesic, it contradicts point (2) of definition 7.2. Otherwise it contradicts point (1), since \( \gamma_1 \) has some accumulation points in \( \partial N \) where \( \partial N \) is not locally polyhedral.

(3) \( \Rightarrow \) (4): suppose that \( \gamma \) is free. Let \( x \in \partial_{\infty}H^3 \setminus \Lambda \) be such that its projection in \( (\partial_{\infty}H^3 \setminus \Lambda)/\pi_1 M \) is outside the points of \( P \) and the disks in \( Q \). Consider the disks in \( \partial_{\infty}H^3/\pi_1 M \) which are maximal among the disks which:

- have interiors which do not contain any point in \( P \).
- when they meet a disk in \( Q \), have an intersection which has angles at most \( \pi/2 \).

The boundary of each of those maximal disks either:

- contains at least 3 points of \( P \).
- contains at least one point of \( P \), and is orthogonal to at least one circle in \( Q \).
- is orthogonal to at least 2 circles in \( Q \).
Since \( \gamma \) is free, there is a finite set of orbits of such maximal disks under the action of \( \pi_1 M \). As a consequence, each point of \( \partial N \) has a neighborhood \( U \) such that \( U \cap \partial N \) is contained in a finite number of planes, so that \( \partial N \) is locally polyhedral. Moreover, we already know that \( \partial N \cap C(M) \) contains no complete geodesic, so that \( \partial N \) contains no closed geodesic, and thus point (2) of definition \( 1.2 \) is satisfied. So \( N \) is a hyperideal manifold, and this shows that \( \gamma \in \Phi_{M,C}(M_{p,q}) \).

A topology on the spaces of hyperideal manifolds For each choice of \( p, q \), there is a natural topology on \( C_{p,q} \), which comes from the topology on the space of hyperbolic, convex co-compact metrics on \( M \), and of points and disks configurations on \( \partial M \). In addition, given \( p, q \), there are certain values of \( p', q' \) such that \( C_{p',q'} \) can be naturally embedded in the boundary of \( C_{p,q} \). This happens if there exists \( i_0 \in \{1, \cdots, n\} \) such that:

1. For each choice of \( \gamma \) there is an element \( \gamma' \) such that \( \gamma' \in \Phi_{M,C}(M_{p',q'}) \).
2. \( \gamma' \) can be embedded in the boundary of \( C_{p,q} \).

Since \( \Phi_{M,C} \) is an injective map from each \( M_{p,q} \) to \( C_{p,q} \), the topology on \( C \) determines a topology on \( M \), the union of the \( M_{p,q} \) for the possible choices of \( p, q \).

A weak connectivity property The main result of this section concerning the spaces of hyperideal manifolds is the next lemma. For \( p = (p_1, \cdots, p_n) \) and \( q = (q_1, \cdots, q_n) \) given, with \( p_i + q_i \geq 1 \) for all \( i \in \{1, \cdots, n\} \), we call:

\[
M_{p,q} := \{ \gamma' \in \Phi_{M,C}(M_{p',q'}) \mid p' \leq p \text{ and } q' \leq q \},
\]

with the topology induced by the topology on \( C \) described above. Here \( p' \leq p \) means that, for all \( i \in \{1, \cdots, n\} \), \( p_i' \leq p_i \).

**Lemma 7.6.** Let \( (p, q) \in \mathcal{P} \). Let \( m_0, m_1 \in M_{p,q} \). There exists \( p' \geq q, q' \geq q \) such that \( m_0 \) and \( m_1 \) can be connected by a continuous path in \( M_{p',q'} \).

The proof uses the following proposition.

**Proposition 7.7.** Choose \( p, q \) and \( \gamma = (c, P, Q) \in C_{p,q} \). Then:

1. suppose that \( \gamma \in \Phi_{M,C}(M_{p,q}) \). Let \( p' \geq p, q' \geq q \), and let \( \gamma' = (c, P', Q') \), with \( P \subset P', Q \subset Q' \). Then \( \gamma' \in \Phi_{M,C}(M_{p',q'}) \).
2. there exists \( p' \geq p \) and \( \gamma' = (c, P', Q') \in C_{p,q} \) such that \( P \subset P' \) and \( \gamma' \in \Phi_{M,C}(M_{p',q'}) \).

**Proof.** The first point is a direct consequence of proposition 7.6 and in particular of its point (3), since adding ideal or hyperideal points to an element \( \gamma \in C_{p,q} \) which is free obviously results in an element which is free.

The second point is also a consequence of point (3) of proposition 7.6 since adding enough ideal points to any element \( \gamma \in C_{p,q} \) eventually leads to an element which is free, and thus in the image of \( \Phi_{M,C} \).

**Proof of lemma 7.6** Let \( \gamma_0 := \Phi_{M,C}(m_0), \gamma_1 := \Phi_{M,C}(m_1) \). Since \( C_{p,q} \) is connected, there exists a path \( (\gamma_t)_{t \in [0,1]} \) in \( C_{p,q} \) connecting \( \gamma_0 \) to \( \gamma_1 \). Let \( \gamma_t = (c_t, P_t, Q_t) \).

Choose \( t_0 \in (0, 1) \). Point (2) of proposition 7.7 shows that there exists a finite subset \( P''_{t_0} \) of \( \partial M \) such that \( \gamma_{t_0}' := (c_{t_0}, P_{t_0} \cup P''_{t_0}, Q_{t_0}) \) is in \( \Phi_{M,C}(C_{p,q}) \), where \( P := p + \#(P''_{t_0}) \). Adding some more ideal vertices if necessary, this remains true for \( t \) close enough to \( t_0 \), so there exists an open interval \( I = (a, b) \ni t_0 \) and a family \((P_t')_{t \in (a,b)} \) such that, for all \( t \in (a, b) \), \( \gamma_t' := (c_t, P_t \cup P'_t, Q_t) \in \Phi_{M,C}(C_{p,q}) \).

Doing this for all values of \( t_0 \) and using the compactness of \([0,1] \), we find a finite sequence of intervals \( I_k = (a_k, b_k), 0 \leq k \leq N, \) covering \([0,1] \), with both \((a_k)\) and \((b_k)\) increasing, and a sequence of families \((P''_{k,t})_{t \in [0,1]} \) such that \((c_t, P_t \cup P''_{k,t}, Q_t) \in \Phi_{M,C}(C_{p(t),q(t)}) \) since \( \gamma_0, \gamma_1 \in \Phi_{M,C}(C_{p,q}) \), we can suppose moreover that \( P''_{1,t} = P_{N,t} = \emptyset \).

Let \( p_T := \sum_{k=1}^{N-1} p_k \). Define a family \((\mathcal{T}_t)_{t \in [0,1]} = (\mathcal{T}_{1,t}, \cdots, \mathcal{T}_{p_T,t})_{t \in [0,1]} \)
of \( p_T \)-uples of points of \( \partial M \) such that, for all \( k \in \{1, \cdots, N - 1\} \), all \( t \in I_k \), and all \( j \in \{1, \cdots, p_k\} \):

\[
\{ \overline{p}_{(j+\sum_{i=1}^{t-1} p_i),t} \mid 1 \leq j \leq p_k \} = P''_{k,t}.
\]

For all \( t \in [0,1] \), let:

\[
\Gamma_t := (c_t, P_t \cup \{ \overline{p}_{j,t} \mid 1 \leq j \leq p_T \}, Q_t).
\]

By the first point of proposition \ref{lem}, \( \Gamma_t \in \Phi_{M,\mathcal{C}}(C_{p+pr,q}) \) for all \( t \in [0,1] \). Taking the inverse image in \( M_{p+pr,q} \) yields the results.

**Spaces of fuchsian hyperideal manifolds** The relationship between the hyperideal metrics on \( M \) and the elements of \( C_{p,q} \) is simpler when one considers fuchsian manifolds, i.e. manifolds topologically of the form \( S \times \mathbb{R} \), where \( S \) is a surface of genus \( g \geq 2 \), with a hyperbolic metric which admits an isometric involution fixing a totally geodesic compact surface. Restricting one's attention to such manifolds means, in terms of \( \mathcal{C} \), that \( \partial M = S_+ \cup S_- \), with both \( S_- \) and \( S_+ \) homeomorphic to \( S \), and with \( c, P \) and \( Q \) invariant under a map sending \( S_- \) to \( S_+ \) (and conversely). We call \( M_{p,q} \) the space of fuchsian hyperideal manifolds with \( p \) ideal and \( q \) hyperideal vertices (clearly \( M_{p,q} = \emptyset \) unless \( p \) and \( q \) are even).

**Definition 7.8.** Let \( p, q \) be even numbers, \( p = 2p', q = 2q' \). Let \( M \) be diffeomorphic to \( S \times \mathbb{R} \), where \( S \) is a closed surface of genus \( g \geq 2 \). \( C^F_{p,q} \) is the subset of elements \( (c, P, Q) \in C_{p,q} \) such that there exists an involution \( \iota : \partial M \to \partial M \) which exchanges the two connected components while leaving invariant \( c, P \) and \( Q \).

The following remark is easy and left to the reader.

**Remark 7.9.** Let \( g \in M_{p,q} \), then \( \Phi_{M,\mathcal{C}}(g) \in C^F_{p,q} \) if and only if \( \gamma \in M^F_{p,q} \).

A more interesting fact is that, whenever one considers an element of \( C_{p,q} \) which is fuchsian, then the corresponding convex hull is always a hyperideal hyperbolic manifold (which of course is fuchsian by the previous remark).

**Proposition 7.10.** Let \( \gamma = (c, P, Q) \in C^F_{p,q} \), and let \( N \) be the complete, hyperbolic, convex co-compact metric on \( M \) determined by \( c \). Let \( M \) be the convex hull in \( N \) of the ideal points in \( P \) and the hyperideal points in \( Q \). Then \( M \) is a hyperideal hyperbolic manifold, i.e. \( \partial M \cap C(N) = \emptyset \), where \( C(N) \) is the convex hull of \( N \).

**Proof.** By construction, \( N = H^3/\Gamma \), where \( \Gamma \subset \text{Isom}(H^3) \) is the image of \( \pi_1 S \) by a morphism, and \( \Gamma \) leaves invariant a totally geodesic plane \( P_0 \subset H^3 \). The limit set of \( \Gamma \) in \( \partial_\infty H^3 \) is the boundary at infinity of \( P_0 \), i.e. a circle in \( S^2 \). Let \( h_- \) and \( h_+ \) be the hyperbolic metrics on the two connected components \( D_- \) and \( D_+ \) of \( S^2 \setminus \partial_\infty P_0 \) for which \( \Gamma \) acts isometrically.

Let \( D_1 \) be a disk in \( S^2 \) which is tangent to \( \partial_\infty P_0 \), suppose for instance that \( D_1 \subset D_- \). \( D_1 \) corresponds to a horodisk in the hyperbolic metric \( h_+ \), so that, if \( P \neq \emptyset \), \( D_1 \) contains a point of the lift \( \tilde{P} \) of \( P \) to \( D_+ \). If \( P = \emptyset \) then \( Q \neq \emptyset \), and for the same reason \( D_1 \) contains a disk in the lift \( \tilde{Q} \) of \( Q \) to \( D_+ \). So the proposition follows from proposition \ref{prop}.

We can sum up the previous two statements in the following lemma.

**Lemma 7.11.** Let \( p, q \) be even integers. Then \( C^F_{p,q} = \Phi_{M,\mathcal{C}}(M^F_{p,q}) \).

**Results on fuchsian hyperideal manifolds** First we define the natural notion of "fuchsian polyhedron".

**Definition 7.12.** A fuchsian subgroup of \( \text{Isom}(H^3) \) is a subgroup \( \Gamma \) of \( \text{Isom}(H^3) \) such that:

- there exists a totally geodesic 2-plane \( P_0 \) which is globally invariant under all elements of \( \Gamma \).
- \( P_0/\Gamma \) is a compact surface of genus \( g \geq 2 \).

**Definition 7.13.** A fuchsian polyhedron in \( H^3 \) is a convex, complete polyhedral surface \( P \subset H^3 \) (with an infinite number of faces) such that there exists a fuchsian subgroup \( \Gamma \) of \( \text{Isom}(H^3) \), leaving \( P \) globally invariant, and such that \( P/\Gamma \) has a finite number of faces. \( P \) is hyperideal if, for each end \( E \) of \( P \) of infinite area, there exists a totally geodesic plane which is orthogonal to all the faces of \( P \) at \( E \).
Note that hyperideal fuchsian polyhedra are obviously related to hyperideal hyperbolic manifolds. More precisely, if $P$ is a hyperideal fuchsian polyhedron, and $\sigma$ is the symmetry in the plane $P_0$ which is left invariant by the fuchsian subgroup of $\text{Isom}(H^3)$ associated to $P$, then $P$ and $\sigma(P)$ bound a convex domain $\Omega \subset H^3$ whose quotient by $\Gamma$ is a fuchsian hyperideal hyperbolic manifold.

Given a surface $S$, a **polyhedral embedding** of $S$ in $H^3$ is a topological embedding whose image is locally like a polyhedron.

**Definition 7.14.** Let $S$ be a compact surface of genus $g \geq 2$. A **hyperideal fuchsian embedding** of $S$ is a polyhedral embedding of $S \setminus C$ into $H^3$, where $C$ is the union of a finite number of orbits of the action of $\pi_1 S$ on $S$, whose image is a hyperideal fuchsian polyhedron.

Note that this definition allows the existence of some ideal vertices in the image of the embedding.

**Theorem 7.15 (M. Rousset [Rou02]).** Let $S$ be a compact surface of genus at least 2, let $\sigma$ be a cellulation of $S$, and let $w : \sigma_1 \to (0, \pi)$ be a map on the set of edges of $\sigma$. There exists a hyperideal fuchsian realization of $S$, with boundary combinatorics given by $\sigma$, and exterior dihedral angles given by $w$, if and only if:

- The sum of the values of $w$ on each circuit in $\sigma_1$ is at least $2\pi$.
- The sum of the values of $w$ on each simple path in $\sigma_1$ is strictly larger than $\pi$.

This hyperideal realization is then unique. A vertex is ideal in this realization if and only if the sum of the values of $w$ on the adjacent edges is equal to $2\pi$.

The proof given in [Rou02] uses a reduction to the case of manifolds with a boundary which has only “compact” points, for which a result was obtained in [Sch04]. There are however some subtle technical questions, in particular the infinitesimal rigidity of those polyhedra and the extension to the limit case where some points are ideal instead of “compact”, which are proved in [Rou02] using methods essentially coming from the work of Pogorelov [Pog73].

**Spaces of angle assignments** We now have all the tools necessary to define the spaces of angle assignations which we will need, and to state the relevant connectedness properties.

**Definition 7.16.** Let $\sigma$ be a cellulation of $\partial M$, and let $P = (P_i)_{1 \leq i \leq n}$, where, for each $i \in \{1, \ldots, n\}$, $P_i$ is a subset of the set of vertices of $\sigma$ in $\partial_i M$. We call $A_{\sigma, P}$ the set of functions $w$ from $\sigma_1$ to $(0, \pi)$ which satisfy the hypothesis of theorem 7.14, and such that a vertex is in $P_i$ if and only if the sum of the values of $w$ on the adjacent edges is $2\pi$. For $(p, q) \in P$, we also call $A_{p, q}$ the union of the $A_{\sigma, P}$ over all cellulations $\sigma$ with $p_i + q_i$ vertices in $\partial_i M$, and with card$(P_i) = p_i$, and we call $A := \cup_{\sigma} A_{p, q}$.

Here $P$ corresponds to ideal vertices in the hyperideal manifolds which will later turn out to be associated to the elements of $A_{\sigma, P}$. There is a natural topology on $A$, corresponding to some natural gluings of the “cells” $A_{\sigma, P}$:

- $A_{\sigma', P'}$ has a natural embedding in $\partial A_{\sigma, P}$ if $\sigma'$ is obtained from $\sigma$ by collapsing an edge $e$ of $\sigma$ with both endpoints in some $P_i$, and $P'$ is the same as $P$ except that the two vertices just mentioned are replaced by one. The angle on $e$ is considered to be $0$ in the limit where elements of $A_{\sigma, P}$ converge to $A_{\sigma', P'}$.
- $A_{\sigma', P'}$ has a natural embedding in $\partial A_{\sigma, P}$ if $\sigma'$ is obtained from $\sigma$ by removing an edge $e$. The angle on $e$ is considered to be $0$ for elements of $A_{\sigma', P'}$ considered as elements of $\partial A_{\sigma, P}$.

It should be noted that, in each case, the result of the transformation is indeed in $\partial A_{\sigma, P}$. We can now state and prove a proposition which will be necessary in the proof of the main result.

**Proposition 7.17.** For each $\sigma$ and $P$, $A_{\sigma, P}$ is affinely equivalent to the interior of a polytope in $\mathbb{R}^N$, for some $N$. If $M$ has incompressible boundary, then, for all $p, q \in P$, $A_{p, q}$ is connected.

**Proof.** The affine structure on each $A_{\sigma, P}$ comes from the parametrization by the values of the function $w$; the fact that the $A_{\sigma, P}$ are affinely equivalent to Euclidean polytopes is a direct consequence of the conditions of theorem 7.15.

Note that the conditions on $w$ on each connected components of $\partial M$ are independent. Since we have supposed that $M$ has incompressible boundary, the conditions on each connected component of $\partial M$ are the
same as in the fuchsian case. Therefore, to prove that $A_{p,q}$ is connected, it is sufficient to prove that it is so in the fuchsian case.

Now note that, in the fuchsian case, the space of hyperideal polyhedra with a fixed number of ideal and hyperideal vertices and a fixed genus is connected. This follows from lemma 6.11 and the connectedness of $C^p_{p,q}$, which one can readily check. Therefore theorem 7.13 implies that the set of dihedral angles assignments is also connected.

Proof of the main theorem First we consider a fixed cellulation $\sigma$ of $\partial M$, along with a subset $P$ of its vertices. Let $M_{\sigma, P}$ be the space of hyperideal hyperbolic manifolds with boundary combinatorics given by $\sigma$, and let $\Phi_{\sigma, P} : M_{\sigma, P} \to A_{\sigma, P}$ be the map sending a hyperideal hyperbolic manifold to the set of dihedral angles of the edges of $\sigma$. If $M_{\sigma, P}$ is non-empty, then $M_{\sigma, P}$ and $A_{\sigma, P}$ are manifolds with boundary of the same dimension, and lemma 5.1 shows that $\Phi_{\sigma, P}$ is a local homeomorphism between them.

Moreover, lemma 6.1 shows that $\Phi_{\sigma, P}$ is proper, so that $\Phi_{\sigma, P}$ is a covering of $A_{\sigma, P}$ by $M_{\sigma, P}$; all the elements of $A_{\sigma, P}$ have the same number $N_{\sigma, P}$ of inverse images, which can be 0 (if $M_{\sigma, P} = \emptyset$), 1, or larger.

By proposition 7.14, each $A_{p,q}$ is connected. Since $M$ has incompressible boundary, lemma 5.4 shows that the number $N_{\sigma, P}$ remains the same when one moves from one cell of $A_{p,q}$ — corresponding to a space $A_{\sigma, P}$ — to a neighboring cell — corresponding to a $A_{\sigma', P'}$. Therefore, the $N_{\sigma, P}$ are equal to a fixed number $N_{p,q}$ depending only on the number of ideal and hyperideal vertices in the triangulation of each connected component of $\partial M$.

When one goes from $A_{p,q}$ to $A_{p', q'}$, with $p_i' = p_i + 1$, $q_i' = q_i - 1$, and $p_i' = p_i$, $q_i' = q_i$ for $i \neq i_0$, the number $N_{p,q}$ can only increase — this follows from lemma 6.12 because a sequence of hyperideal manifolds with a hyperideal vertex which "becomes ideal" has a limit which is a hyperideal manifold.

Thus to prove the main theorem it is sufficient to remark that, by a result of [Sch01b], $N_{p,q} = 1$ when $q = 0$, i.e. when all $q_i = 0$, since this correspond to ideal hyperbolic manifolds.

8 Induced metrics

This section contains the proof of lemma 8.7 and then of theorems 8.9 and 8.10. They will follow from the tools introduced in the previous sections, once we have given some simple definitions and properties of complete hyperbolic metrics on triangulated surfaces. Those elements are generalizations of those in section 9 of [Sch01b], where only ideal triangles were considered.

In all this section we consider a compact surface $S$, with a triangulation $\sigma$ by a finite number of triangles, and a subset $V_1$ of the set of vertices of $\sigma$. $S$ is not necessarily connected, below it will be $\partial M$, $\sigma$ will be the triangulation induced on $\partial M$ by a cellulation of $M$, and $V_1$ will be the set of ideal vertices of a hyperideal metric on $M$.

Recall from section 2 that we can define a hyperideal triangle as the interior of a triangle in $\mathbb{R}^2$, with its vertices outside the open unit disk $D^2$, and all its edges intersecting $D^2$. One can then consider the metric coming from the projective model of $H^2$ (maybe extended by the de Sitter metric outside the disk). The triangle might have one, two or three ideal vertices, which then sit on the boundary $S^1$ of $D^2$.

If $T$ is a hyperideal triangle, and if $e$ is an edge of $T$ such that none of the endpoints of $e$ is an ideal vertex, we can define the length of $e$ as the distance between the hyperbolic geodesics which are dual to the vertices of $e$. It is a positive number, see section 2 and in particular proposition 2.8.

Definition 8.1. Let $T_0$ be the space of hyperideal triangles. We call $N$ be the space of maps from the set of triangles of $\sigma$ to $T_0$ such that, for each edge $e$ of $\sigma$ with no endpoint in $V_1$, the lengths of $e$ for the hyperideal triangles corresponding to both sides of $e$ are equal.

When $V_1 = \emptyset$, each element of $N$ determines a complete hyperbolic metric on the complement of the vertices of $\sigma$ in $S$, such that the area of each end is infinite. This metric is obtained by gluing adjacent hyperideal triangles in a way such that the lines dual to their vertices have the same endpoints — this gluing condition will be used in all this section. But when there are some ideal vertices, however, some additional care is needed because there might be more than one way to glue the triangles, and the hyperbolic metrics obtained might not be complete.

Definition 8.2. Let $g \in N$ and let $v$ be a vertex of $\sigma$. The shift of $g$ at $v$ is the sum of the shifts of $g$ at the edges containing $v$. We will say that $g$ is complete if its shift is zero at all the vertices in $V_1$. The set of complete structures will be denoted by $N_c$. 38
The definition of the shift of \( g \) at an edge can be found in definition 6.4. Of course the notion of completeness defined here is the same as the usual, topological notion. Indeed if the shift of \( g \) at a vertex \( v \) is non-zero, it is possible to use this — and the fact that ideal triangles are exponentially thin near their vertices — to attain \( v \) in a finite time, by “circling” around it to take opportunity of the shift. The converse is not difficult to prove either.

**The lengths of the edges of \( \sigma \) ** Let \( g \in \mathcal{N}_c \). Let \( v \) be a vertex of \( \sigma \) which is in \( V_i \), and let \( t_1, \ldots, t_n \) be the triangles of \( \sigma \) adjacent to \( v \), in cyclic order. Then, by definition of \( \mathcal{N}_c \), \( v \) is an ideal vertex of each of the \( t_i \). Consider \( t_1 \) as a triangle in \( H^2 \), and choose a horocycle \( H_1 \) centered at \( v \). Do the same thing in \( t_2 \), with a horocycle \( H_2 \) such that \( H_1 \) and \( H_2 \) meet at a point of the edge which is common to \( t_1 \) and \( t_2 \). Repeating this operation, one finds a sequence \( H_1, H_2, \ldots, H_n \) of horocycles in the \( t_i \). The fact that \( g \) is complete implies that \( H_n \) and \( H_1 \) meet at the edge which is common to \( t_n \) and \( t_1 \).

One can do the same for all the vertices of \( \sigma \) which are in \( V_i \), with the additional condition that the horocycles corresponding to different ideal vertices do not intersect. There is then a well-defined notion of length for all the edges of \( \sigma \) for \( g \):

- the length of an edge with no endpoint in \( V_i \) was defined above, as the distance between the hyperbolic lines dual to the two vertices in one of the triangles.
- the distance between a vertex \( v_i \) in \( V_i \) and a vertex \( v_h \) which is not in \( V_i \) is the distance, in either of the triangles bounded by the edge, between the hyperbolic line dual to \( v_h \) and the horocycle attached to \( v_i \).
- the distance between two vertices of \( V_i \) is the distance between the horocycles attached to each.

Of course the set of lengths of the edges depends on the choices of the horocycles at the vertices in \( V_i \). It does so, however, in a very simple way, since changing the horocycles at a vertex \( v \in V_i \) only adds the same constant to the lengths of all the edges adjacent to \( v \).

**Definition 8.3.** Let \( L_0 \) be the space of maps from the set of edges of \( \sigma \) to \( \mathbb{R} \). Let \( v_i := \# V_i \), and let \( L := L_0/\mathbb{R}^{v_i} \), where an element \((r_1, \ldots, r_{v_i}) \in \mathbb{R}^{v_i} \) acts on \( L_0 \) by adding \( r_i \) to the numbers attached to all the edges adjacent to the \( i^{th} \) element of \( V_i \).

The considerations above show that there is natural map \( l \) from \( \mathcal{N}_c \) to \( L \), defined by sending a complete hyperbolic metric to the lengths of its edges (defined up to one additive constant for each ideal vertex).

**Proposition 8.4.** \( l \) is a homeomorphism between \( \mathcal{N}_c \) and its image in \( L \).

The proof uses the following elementary property of hyperideal triangles in \( H^2 \), which is taken with a small generalization from [Sch01b].

**Sub-lemma 8.5.** Consider two hyperideal triangles \( x_1, x_2, x_3 \) and \( x_1, x_3, x_4 \) with disjoint interior, sharing an edge \((x_1, x_3)\) which has two ideal vertices at its endpoints. For each \( i \in \{1, 2, 3, 4\} \), let \( h_i \) be a horocycle centered at \( x_i \), if \( x_i \) ideal, and let \( h_i \) be the hyperbolic line dual to \( x_i \) otherwise. For \( i \neq j \), let \( l_{ij} \) be the distance between \( h_i \) and \( h_j \) along the geodesic going from \( v_i \) to \( v_j \) — which is negative if \( h_i \) and \( h_j \) overlap. Let \( \pi_2 \) and \( \pi_4 \) be the orthogonal projections on \((x_3, x_1)\) of \( x_2 \) and \( x_4 \) respectively, and let \( \delta \) be the oriented distance between \( \pi_2 \) and \( \pi_4 \) on \((x_3, x_1)\). Then:

\[
2\delta = l_{12} - l_{23} + l_{34} - l_{41} .
\]

**Proof.** It follows from figure 6 where numbers from 1 to 6 are attached to lengths of segments. Elementary properties of the ideal triangle show that:

\[
l_{12} - l_{23} + l_{34} - l_{41} = (1 + 2) - (2 + 3) + (4 + 5) - (5 + 6)
= 1 - 3 + 4 - 6
= (1 - 6) + (4 - 3)
= 2\delta .
\]

**Proof of proposition 8.4.** The sub-lemma shows that the lengths of the edges — in the sense defined above, with horocycles chosen around the ideal vertices — uniquely determine the shifts at the edges of \( \sigma \) which have an ideal vertex at each endpoint. The proposition clearly follows.
Figure 6: Hyperideal triangles (in the projective model of $H^2$)

Proposition 8.4 shows that the metrics on $\partial M$ are determined by the lengths of the edges, as they appear in the Schlafli formula. This means that the strict concavity of the volume translates as an infinitesimal rigidity statement relative to the metric induced on $\partial M$. To understand the situations where $\partial M$ is not triangulated, we will need the next lemma (it is formulated in a more general context, since it can be useful in different situations).

Let $P$ be hyperbolic polygon, with vertices $x_1, x_2, \ldots, x_n$ which can be in $H^2$, ideal or hyperideal, and let $h$ be the induced metric on $P$. Let $H_0$ be a totally geodesic plane in $H^3$; consider $P$ as a polygon in $H_0$. Let $\tau_1$ and $\tau_2$ be two triangulations of $P$, i.e. decompositions into triangles with disjoint interiors and vertices the $x_i$.

Lemma 8.6. Let $P$ be a first-order deformation of the $x_i$ as hyperbolic (resp. ideal, hyperideal) points in $H^3$. Let $h_1$ and $h_2$ be the first-order deformations of $h$ obtained through the deformations of the triangulated surfaces in $H^3$ defined by $\tau_1$ and $\tau_2$. Then $h_1 = h_2$.

Proof. It is quite easy to see that the first-order displacements of the $x_i$ orthogonally to $H_0$ induce no variation of the lengths of the edges of the $\tau_i$, and of the shifts at the edges of the $\tau_i$ between two ideal vertices. The arguments given above thus show that those orthogonal displacements do not contribute to $h_1$ and $h_2$.

But the displacements of the $x_i$ tangent to $H_0$ have the same influence on the deformations $h_1$ and $h_2$, and the lemma follows.

Lemma 8.7. Let $(M, g)$ be a hyperideal manifold. It has no first-order deformation (among hyperideal manifolds with the same ideal vertices) which does not change the induced metric on $\partial M$.

Proof. Theorem 0.5 shows that, if $\partial M$ is triangulated, the deformations of $M$ are parametrized by the deformations of the dihedral angles at the edges. If $\partial M$ is not triangulated, i.e. if some of its faces have at least 4 edges, then choose any triangulation of $\partial M$, obtained by adding some edges to subdivide the non-triangular faces.

Then consider the first-order deformations of the hyperideal metrics on $M$ which respect this triangulation, but for which $\partial M$ does not necessarily remain convex. The first-order deformations of the hyperideal metrics are parametrized by the first-order deformations of the angles at the edges of this triangulation, and lemma 8.6 shows that the first-order variation of the metric on $\partial M$ does not depend on the triangulation chosen.

By lemma 3.17, the volume is a strictly concave function on the space of hyperideal manifolds, parametrized by the dihedral angles at the edges of $\partial M$. By the Schlafli formula (lemma 3.13), this shows that, for each first-order deformation of the hyperideal metric on $M$, the lengths of the edges have a non-zero first-order variation. Proposition 8.4 then implies the lemma.

An elementary dimension-counting argument then shows that each first-order deformation of the hyperbolic induced on $\partial M$ can be obtained from a first-order deformation of the hyperideal metric on $M$.

Corollary 8.8. Let $(M, g)$ be a hyperideal manifold, and let $h$ be the induced metric on $\partial M$. Let $h$ be a first-order deformation of $h$ among the complete hyperbolic metrics on $\partial M$, which have finite area at each vertex
of $M$ which is ideal. There is a unique first-order deformation of $(M, g)$, among hyperideal manifolds with the same number of ideal and strictly hyperideal vertices, such that the variation of the induced metric on $\partial M$ is $h$.

Proof. By theorem 8.5, the deformations of the hyperideal metrics on $M$, with the same ideal vertices, are parametrized by the dihedral angles at the edges, under the condition that the sum of the exterior angles of the edges adjacent to each ideal vertex remains equal to $2\pi$. The same argument as in the proof of lemma 6.1 shows that those conditions are independent. The dimension of the space of hyperideal metrics on $M$, with the same ideal vertices, is thus $e - v_i$, where $v_i$ is the number of ideal vertices.

By proposition 8.4, the complete hyperbolic metrics on $\partial M$ (again with the same ideal vertices) are parametrized by the lengths of the edges of a triangulation, which are defined up one additive constant for each ideal vertex. The space of those metrics is therefore also equal to $e - v_i$. Lemma 8.7 therefore shows that any first-order deformation of the metric induced on $\partial M$ is obtained (uniquely) from a first-order of the hyperideal metric on $M$.

As a consequence, we can find a result describing the induced metrics on hyperideal polyhedra. It is a special case of results of Schulte, but the proof that we obtain here is different.

**Theorem 8.9.** Let $h$ be a complete hyperbolic metric on $S^2$ minus a finite number of points (at least 3). There is a unique hyperideal polyhedron on $H^3$ whose induced metric is $h$.

The same result can be obtained in the context of fushcan hyperideal manifolds. It was obtained in Scholz since they deal with the induced metrics instead of the dihedral angles. They are both easy to prove. We first deal with theorem 8.9.

For each $p, q \in \mathbb{N}$ with $p + q \geq 3$, let:
- $P_{p,q}$ be the space of hyperideal polyhedra with $p$ strictly hyperideal and $q$ ideal vertices, up to the isometries of $H^3$.
- $N_{p,q}$ be the space of complete hyperbolic metrics on $S^2$ minus $p + q$ points, with $q$ cusps and $p$ ends of infinite area.

**Lemma 8.11.** Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of elements of $P_{p,q}$. Let $(h_n)_{n \in \mathbb{N}}$ be the induced metrics, which are elements of $N_{p,q}$. Suppose that $(h_n)$ converges, as $n \to \infty$, to a metric $h \in N_{p,q}$. Then, after taking a subsequence, $(P_n)$ converges to a hyperbolic polyhedron with $p$ strictly hyperideal and $q$ ideal vertices.

Proof. Let $S$ be $S^2$ minus $p + q$ points, let $x \in S$, and let $x_n$ be the points in the $P_n$ corresponding to $x$. Choose a sequence $(\phi_n)_{n \in \mathbb{N}}$ of hyperbolic isometries such that $\phi_n(x_n)$ remains equal to a fixed point $y_0 \in H^3$.

Consider the projective model of $H^3$, with $y_0$ at the center. Suppose that the $(P_n)$ do not converge to a hyperideal polyhedron $P$, then, after taking a subsequence, at least two of the vertices, say $v_1$ and $v_2$, converge to the same point in $\mathbb{R}^3$.

Let $c$ be a closed curve in $S$ going through $x$ and such that $v_1$ and $v_2$ are in different connected components of the complement of $c$. Then as $n \to \infty$, the curves in $P_n$ homeomorphic to $c$ and containing $x$ have to go arbitrarily close to $\partial_{\infty}H^3$, so that their lengths have to go to infinity. Therefore $(h_n)$ can not converge.

From here on, we choose a fixed number $q \geq 2$, and call $S_g$ the closed (compact without boundary) surface of genus $g$. For each $p, q \in \mathbb{N}$ with $p + q \geq 1$, let:
- $P_{p,q}^F$ be space of fushcan hyperideal polyhedra of genus $g$ with $p$ strictly hyperideal and $q$ ideal vertices, up to the isometries of $H^3$.
- $N_{p,q}^F$ be the space of complete hyperbolic metrics on $S_g$ minus $p + q$ points, with $q$ cusps and $p$ ends of infinite area.

**Lemma 8.12.** Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of elements of $P_{p,q}^F$. Let $(h_n)_{n \in \mathbb{N}}$ be the induced metrics, which are elements of $N_{p,q}^F$. Suppose that $(h_n)$ converges, as $n \to \infty$, to a metric $h \in N_{p,q}^F$. Then, after taking a subsequence $(P_n)$ converges to a fushcan hyperbolic polyhedron with $p$ strictly hyperideal and $q$ ideal vertices.
Proof. The $P_n$ are fuchsian polyhedra with a representation that fixes a given totally geodesic plane in $H^3$, say $H_0$. So by definition the representation $\rho_n$ of each $P_n$ acts co-compactly on $H_0$.

Suppose that $(\rho_n)$ has no converging subsequence. Then there would exist a closed geodesic in $S_\rho$ whose lengths goes to infinity for the hyperbolic metrics corresponding to the quotient of $H_0$ by $\rho_n$. Since the projection from $P_n$ to $H_0$ is contracting, the length of the same curve in $P_n$ would also go to infinity, which would contradict the hypothesis. So, after taking a subsequence, $(\rho_n)$ converges.

Suppose that the distance between $P_n$ and $H_0$ does not remain bounded. Then the orthogonal projection from $P_n$ to $H_0$ would be contracting by a factor $c_n \to \infty$. But, after extracting a subsequence, the length of some closed curve in $H_0/\rho_n$ remains bounded from below by some positive constant. Therefore the length of the same curve in $P_n/\rho_n$ goes to infinity, a contradiction. So the distance between $P_n$ and $H_0$ remains bounded.

The lemma then follows from the same argument as in the proof of lemma 8.11.

Proof of theorem 8.9. Consider the map $F_{p,q} : P_{p,q} \to N_{p,q}$ sending a hyperideal polyhedron to its induced metric. By corollary 8.8, $P_{p,q}$ and $N_{p,q}$ are manifolds of the same dimension, and $F_{p,q}$ is a local homeomorphism. Moreover, by lemma 8.11, $F_{p,q}$ is proper, thus it is a covering.

Both $P_{p,q}$ and $N_{p,q}$ have retractions to the space of configurations of $p+q$ points on $S^2$; for $N_{p,q}$ it follows from considerations on the hyperbolic metrics in conformal classes, see [Tro91]. Both spaces have non-zero Euler characteristic since $p+q \geq 3$ (this can checked directly by a recursion argument). Therefore $F_{p,q}$ have degree $\pm 1$, and each element of $P_{p,q}$ has a unique inverse image.

Proof of theorem 8.10. The proof is the same as for theorem 8.9 except that the spaces $P_{p,q}^F$ and $N_{p,q}^F$ now have retractions on the space of configurations of $p+q$ points in a surface of genus at least 2; the argument is otherwise the same.

9 Circle configurations

Configurations of circles It is well known that the Andreev theorem can be formulated in terms of configurations of circles on $S^2$. Namely, one considers the decompositions of $S^2$ into a finite number of closed disks, such that each point of $S^2$ is contained in the interior of at most 2 of the disks.

![Figure 7: A "classical" configuration of circles](image)

The Andreev theorem [And71], as extended by Rivin [Riv96], provides an explicit description of the possible crossing angles between the circles. The points is that one can consider the ideal polyhedron in $H^3$ with faces the planes bounded by the circles, and the dihedral angle between two faces is the equal to the angle between the corresponding circles in $S^2$.

This can also be done in hyperbolic surfaces, and corresponds to fuchsian ideal polyhedra, see [Thu97] and [CdV91]. It is also possible to give a more general statement on circle configurations in the boundary of a 3-manifold which admits a complete, convex co-compact hyperbolic metric [Sch01b].

42
Going from ideal to hyperideal polyhedra leads to another, more general kind of circle configurations. One now considers two families of circles on $S^2$:

- “red” circles, which never intersect one another.
- “black” circles, which intersect the red circles orthogonally and such that the each point of $S^2$ is contained in the interiors of at most 2 of the black circles.

We also demand that the sphere is covered by the closed disks bounded by those circles. We will call this setup a “configuration of red and black circles”. The red circles correspond to the hyperideal ends of the polyhedron — more precisely they are the boundary of the dual hyperbolic planes — while the black circles are the boundary of the planes containing the faces of the polyhedron.

To each configuration of red and black circles, one can associate a graph with red vertices and black vertices, corresponding to the red and to the black circles respectively, with an edge between two vertices if and only if the corresponding circles intersect. No edge can have red vertices at both endpoints, since we have specified that red circles do not intersect.

To each edge we can also associate an angle, which we choose to be $\pi$ minus the interior angle of the intersection between the corresponding two circles. For edges which have a red vertex as one of their endpoints and a black vertex at the other, the angle is $\pi/2$ since we have specified that the black circles intersect the red circles orthogonally.

In the picture below, the “red” circle are drawn in thicker black ink, and the “red” vertices are drawn with a bigger dot.

![Figure 8: A configuration of red and black circles, and the corresponding graph](image)

This is not limited to $S^2$, and can also be done in a hyperbolic surface, or more generally in the boundary $\partial M$ of a 3-manifolds $M$. To be able to speak about circles, we need to equip $\partial M$ with a $\mathbb{C}P^1$-structure. We can then reformulate theorem 0.5 as in theorem 9.1 below.

To state it, one must define analogs of the notions of “circuits” and ”simple paths” defined for hyperideal manifolds in the introduction. The translation is obvious once one remarks that the graph corresponding to a red and black circle configuration is almost the graph dual to the cellulation defined from an hyperideal manifold: the black vertices are indeed associated to the faces of the polyhedron, but the red vertices correspond to the hyperideal vertices. Therefore, one defines:

- a **circuit** as a closed curve made of segments of the graph having black vertices at both endpoints, and which is contractible in $M$.

- an **elementary circuit** as a circuit which is made of the segments bounding a face.

- a **simple path** is a closed curve made of segments of the graph, with exactly one red vertex, which is contractible in $M$.
Theorem 9.1. Let $M$ be a 3-manifold with incompressible boundary, which admits a complete, convex co-compact hyperbolic metric. Let $\sigma$ be a cellulation of $\partial M$, with a subset $R$ of the set of its vertices, and let $w : \sigma_1 \to (0, \pi)$ be a map on the set of edges of $\sigma$ with no endpoint in $R$. There exists a complete, convex co-compact hyperbolic metric $g$ on $M$, inducing a $CP^1$-structure $c$ on $\partial M$, and a configuration of red and black circles $C$ for $c$, with combinatorics given by $\sigma$, red vertices at the points of $R$, and intersection angles by $w$, if and only if:

- no edge of $\sigma$ has both vertices in $R$.
- the sum of the values of $w$ on each circuit in $\sigma_1$ is greater than $2\pi$, and strictly greater if the circuit is non-elementary.
- The sum of the values of $w$ on each simple path in $\sigma_1$ is strictly larger than $\pi$.

There is then a unique possible choice of $g$ and of the circle packing $C$.

Given a configuration of red and black circles, one can associate to it a volume, which is of course defined as the hyperbolic volume of the corresponding hyperbolic (truncated) hyperideal manifold. From the Schl"afli formula (lemma 5.13), the volume increases when the exterior dihedral angles of the edges increase.

Consider a cellulation $\sigma$ of $\partial M$. Add one red vertex for each face of $\sigma$, with an edge going to each of the vertices of the face. Let $\overline{\sigma}$ be the cellulation — with red and black vertices — obtained, and let $\overline{\sigma}_1$ be its 1-skeleton. Attach to each edge an angle which is close enough to $\pi$. Theorem 9.1 shows that there is a unique configuration of red and black circles in $\partial M$ (for a $CP^1$-structure coming from a complete convex co-compact hyperbolic metric on $M$) with the incidence graph $\overline{\sigma}_1$ and the prescribed angles.

Moreover, if one increases the angles up to $\pi$, the conditions of theorem 9.1 remain satisfied, so that the circle configurations still exist. In the limit case where all angles are equal to $\pi$, the black circles are tangent (when they intersect) and the volume is maximal. It is easy to check that this configuration is exactly the one given by the extended Koebe theorem 0.9 — in the simple case where $M$ is a ball it is the Koebe theorem, as classically extended to non-triangular interstices.

The nice point about this proof is that it gives directly the two families of circles in the Koebe theorem; one, the black circles, are the circles in the circle packing, while the other, the red circles, are the circles orthogonal to the circles in the packing.

10 Remarks

The strictly hyperideal case An interesting remark that came up during a conversation with Francis Bonahon is that, in the strictly hyperideal case, theorem 9.1 can be proved very simply using the results of [BO01] on the convex cores of hyperbolic manifolds, which are complete when the bending lamination is along closed curves. Indeed, if $M$ is a strictly hyperideal manifold, one can consider the associated truncated manifold $M_0$, and glue two copies of $M_0$ along the cuts; one obtains the convex core of a hyperbolic manifolds, and the convex cores that can be obtained in this way are characterized by an elementary symmetry properties. The results of [BO01] thus lead to theorem 9.1.

It is interesting to compare those two ways of looking at hyperideal manifolds with strictly hyperideal vertices, in particular concerning the infinitesimal rigidity. The fact that both a volume argument and an argument based on [HK98] work might indicate the possibility of proving infinitesimal rigidity statements other geometric objects, like hyperbolic cone-manifolds, by methods based on the volume (or some version of the Hilbert-Einstein functional, which is the same for hyperbolic metrics).

The ideal case As already mentioned, the proof given here is slightly different from the approach used in [Sch01b]. The main difference is in the decomposition of the ideal manifolds, which were cut in simplices in [Sch01b] and in polyhedra here. It was necessary in [Sch01b] to consider finite covers of those manifolds to ensure that the decomposition was possible, while it is not necessary here.

Smooth surfaces There is an analog of lemma 4.1 for smooth surfaces, see [Sch98b]; it is a result describing the third fundamental form of some complete surfaces in $H^3$ whose boundary at infinity is a disjoint union of circles — this seems to be a natural analog of the condition that, for each end, all edges going to infinity meet at a hyperideal vertex. There is also a result on the induced metric.
It appears that this result should also hold in the setting of hyperbolic manifolds with boundary. Then one considers complete, constant curvature $K$ metrics on the boundary minus a finite number of points. Each such metric should be realized uniquely as the induced metric (if $K > -1$) or the third fundamental form (for $K \leq 1$) on the boundary, for a hyperbolic metric on $M$ such that each connected component of $\partial_\infty M$ is a circle for the Möbius structure of $\partial_\infty H^3$, see [MS02].

A The concavity of the volume at a regular simplex

We give here the maple code to check proposition 3.16, i.e. to make sure that, for at least one regular hyperideal simplex, the matrix $((\partial \theta_i / \partial l_j)_{i,j}$ is positive definite.

with(linalg);

al:=arccosh((cosh(l0)^2+cosh(l))/sinh(l0)^2);
bl:=arccosh(cosh(l0)*(cosh(l)+1)/(sinh(l)*sinh(l0)));
a0:=arccosh(cosh(l0)*(cosh(l0)+1)/sinh(l0)^2);
t1:=arccos((-cosh(a0)+cosh(bl)^2)/sinh(bl)^2);
t2:=arccos(cosh(bl)*(cosh(a0)-1)/(sinh(bl)*sinh(a0)));
t3:=arccos((cosh(a0)^2-cosh(al))/sinh(a0)^2);

a:=simplify(subs(al=a0,bl=b0,l=l0,diff(t1, l)));
b:=simplify(subs(al=a0,bl=b0,l=l0,diff(t2, l)));
c:=simplify(subs(al=a0,bl=b0,l=l0,diff(t3, l)));

ae:=evalf(subs(l0=1.2, a));
be:=evalf(subs(l0=1.2, b));
ce:=evalf(subs(l0=1.2, c));

M:=matrix([[ae,be,be,be,be,be],[be,ae,be,be,ce,be],[be,be,ae,be,be,ce],[ce,be,be,ae,be,be],[be,ce,be,be,ae,be],[be,be,ce,be,be,ae]]);

V:=eigenvals(M);

Feeding this into maple returns a set of clearly positive eigenvalues.

Acknowledgments

I would like to thank Francis Bonahon and Igor Rivin for some very interesting conversations related to this work.

References

[Ahl66] L. V. Ahlfors. Lectures on quasiconformal mappings. D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10.

[And70] E.M. Andreev. Convex polyhedra in Lobacevskii space. Mat. Sb.(N.S.), 81 (123):445–478, 1970.

[And71] E.M. Andreev. On convex polyhedra of finite volume in Lobacevskii space. Math. USSR Sbornik, 12 (3):225–259, 1971.

[BB02] X. Bao and F. Bonahon. Hyperideal polyhedra in hyperbolic 3-space. Preprint available at http://math.usc.edu/~fbonahon Bull. Soc. Math. France, to appear, 2002.

[BO01] F. Bonahon and J.-P. Otal. Laminations mesurées de plissage des variétés hyperboliques de dimension 3. http://math.usc.edu/~fbonahon 2001.

[Brä92] Walter Brägger. Kreispackungen und Triangulierungen. Enseign. Math. (2), 38(3-4):201–217, 1992.

[Cau13] A. L. Cauchy. Sur les polygones et polyèdres, second mémoire. Journal de l’Ecole Polytechnique, 19:87–98, 1813.
[CD95] R. Charney and M. Davis. The polar dual of a convex polyhedral set in hyperbolic space. *Michigan Math. J.*, 42:479–510, 1995.

[CdV91] Yves Colin de Verdière. Un principe variationnel pour les empilements de cercles. *Invent. Math.*, 104(3):655–669, 1991.

[EM86] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces. In D. B. A. Epstein, editor, *Analytical and geometric aspects of hyperbolic space*, volume 111 of *L.M.S. Lecture Note Series*. Cambridge University Press, 1986.

[EP88] D. B. A. Epstein and R. C. Penner. Euclidean decompositions of noncompact hyperbolic manifolds. *J. Differential Geom.*, 27(1):67–80, 1988.

[HK98] C. D. Hodgson and S. P. Kerckhoff. Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery. *J. Differential Geom.*, 48:1–60, 1998.

[Koe36] P. Koebe. Kontaktprobleme der konformen Abbildung. *Abh. Sachs. Akad. Wiss. Leipzig Math.-Natur. Kl.*, 88:141–164, 1936.

[Lab92] F. Labourie. Métriques prescrites sur le bord des variétés hyperboliques de dimension 3. *J. Differential Geom.*, 35:609–626, 1992.

[LegII] A.-M. Legendre. *Eléments de géométrie*. Paris, 1793 (an II). Première édition, note XII, pp.321–334.

[LS00] F. Labourie and Jean-Marc Schlenker. Surfaces convexes fuchsiennes dans les espaces lorentziens à courbure constante. *Math. Annalen*, 316:465–483, 2000.

[Mil94] J. Milnor. The Schlafli differential equality. In *Collected papers, vol. 1*. Publish or Perish, 1994.

[MS02] Greg McShane and Jean-Marc Schlenker. Hyperbolic manifolds with constant curvature boundaries. In preparation, 2002.

[Pog73] A. V. Pogorelov. *Extrinsic Geometry of Convex Surfaces*. American Mathematical Society, 1973. Translations of Mathematical Monographs. Vol. 35.

[RH93] I. Rivin and C. D. Hodgson. A characterization of compact convex polyhedra in hyperbolic 3-space. *Invent. Math.*, 111:77–111, 1993.

[Riv86] I. Rivin. *Thesis*. PhD thesis, Princeton University, 1986.

[Riv94] I. Rivin. Euclidean structures on simplicial surfaces and hyperbolic volume. *Annals of Math.*, 139:553–580, 1994.

[Riv96] I. Rivin. A characterization of ideal polyhedra in hyperbolic 3-space. *Annals of Math.*, 143:51–70, 1996.

[Rou02] Mathias Rousset. Sur la rigidité de polyèdres hyperboliques en dimension 3 : cas de volume fini, cas hyperidéal, cas fuchsien. math.GT/0211280 2002.

[Sch96] Jean-Marc Schlenker. Surfaces convexes dans des espaces lorentziens à courbure constante. *Commun. Anal. and Geom.*, 4:285–331, 1996.

[Sch98a] Jean-Marc Schlenker. Métriques sur les polyèdres hyperboliques convexes. *J. Differential Geom.*, 48(2):323–405, 1998.

[Sch98b] Jean-Marc Schlenker. Représentations de surfaces hyperboliques complètes dans $H^3$. *Annales de l'Institut Fourier*, 48(3):837–860, 1998.

[Sch00] Jean-Marc Schlenker. Dihedral angles of convex polyhedra. *Discrete Comput. Geom.*, 23(3):409–417, 2000.

[Sch01a] Jean-Marc Schlenker. Convex polyhedra in Lorentzian space-forms. *Asian J. of Math.*, 5:327–364, 2001.
Jean-Marc Schlenker. Hyperbolic manifolds with polyhedral boundary. math.GT/0111136 available at http://picard.ups-tlse.fr/~schlenker, 2001.

Jean-Marc Schlenker. Hyperbolic manifolds with convex boundary. preprint, math.DG/0205305 available at http://picard.ups-tlse.fr/~schlenker, 2002.

J. J. Stoker. Geometrical problems concerning polyhedra in the large. Comm. Pure Appl. Math., 21:119–168, 1968.

William P. Thurston. Three-dimensional geometry and topology. Recent version of the 1980 notes. http://www.msri.org/publications/books/gt3m/ 1997.

Marc Troyanov. Prescribing curvature on compact surfaces with conical singularities. Trans. Amer. Math. Soc., 324(2):793–821, 1991.

E.B. Vinberg, editor. Geometry II, Geometry of Spaces of Constant Curvature, volume 29 of Encyclopaedia of Mathematical Sciences. Springer, 1993.