STATIC PERFECT FLUIDS WITH SYMMETRIES

ADRIANO, L. 1, BARBOZA, M. 2, AND TOKURA, W. 3

Abstract. In this paper we utilize symmetries in order to exhibit exact solutions to Einstein’s equation of a perfect fluid on a static manifold whose spatial factor is conformal to a Riemannian space of constant sectional curvature. It’s virtually possible to obtain infinitely many solutions via this approach since the equation is reduced into an ordinary differential equation that, essentially, is of the Riccati type. Three examples are shown in detail.

1. Main Results

Einstein’s gravitational tensor of a Lorentzian manifold \((\bar{M}, \bar{g})\) is

\[(1) \quad G_{\bar{g}} = ric_{\bar{g}} - \frac{scal_{\bar{g}}}{2} \bar{g},\]

where \(ric_{\bar{g}}\) and \(scal_{\bar{g}}\) stand for, respectively, the Ricci tensor and the scalar curvature of the metric \(\bar{g}\). It should be noticed that the above tensor has zero divergence. General relativity flows from Einstein’s equation

\[(2) \quad G_{\bar{g}} = T,\]

of a manifold \((\bar{M}, \bar{g})\) filled up with matter represented by the stress-energy tensor \(T\). According to general relativity, the geometric properties of the universe are not independent, but rather determined by matter. Therefore, it’s only possible to infer something about the geometric nature of the universe when the state of matter is supposed to be known. We consider Einstein’s equation of a perfect fluid (see [3], [4], [5]) on \((\bar{M}, \bar{g})\), i.e.,

\[(3) \quad ric_{\bar{g}} - \frac{scal_{\bar{g}}}{2} \bar{g} = (\mu + \nu)\bar{g}(\cdot, X) \otimes \bar{g}(\cdot, X) + \nu \bar{g},\]

where functions \(\mu, \nu \in C^\infty(\bar{M})\) measure each a specific feature of the fluid, those being energy density in the case of \(\mu\) and pressure in that of \(\nu\), and the vector field \(X \in \mathfrak{X}^\infty(\bar{M})\), whose flow represents the dynamics of the fluid, accomplishes

\[\bar{g}(X, X) = -1,\]

all along \(\bar{M}\). The definition of a perfect fluid, however, does not tell how to build out a model of one. That is the reason why we have chosen to stick

\[Date: May 2, 2019.\]

\[3 \quad \text{Supported by CAPES/Brazil.}\]
with the so called static manifolds. A Lorentzian manifold \((\tilde{M}, \tilde{g})\) is called (globally) static (see [3], [4]) if

\[
\begin{cases}
\tilde{M} = M \times \mathbb{R}, \\
\tilde{g} = x^*g - f(x)^2dt^2,
\end{cases}
\]

where

\[
\begin{align*}
x & : \tilde{M} \to M \quad \text{and} \\
(x, t) & \mapsto x \\
(t, x) & \mapsto t
\end{align*}
\]

are the natural projections, \((M, g)\) is Riemannian and \(f \in \mathcal{C}^\infty(M)\) is positive.

It is known (see [4], [5]) that under (4), (3) is equivalent to

\[
\text{ric}_g - \frac{\text{scal}_g}{n}g = \frac{1}{f} \left( (\nabla^2 f)_g - \frac{(\Delta f)_g}{n}g \right) \quad \text{on} \quad M,
\]

with relations

\[
\mu = \frac{\text{scal}_g}{2} \quad \text{and} \quad \nu = \frac{n - 1}{n} \frac{(\Delta f)_g}{f} - \frac{n - 2}{n} \mu,
\]

where \((\nabla^2 f)_g\) is the Hessian and \((\Delta f)_g\) is the Laplacian of \(f\) with respect to the metric \(g\). In the scope of conformal geometry we suppose that

\[
(M, g) = (M^n, h^{-2}g_\kappa),
\]

where \((M^n_\kappa, g_\kappa)\) is a geodesically complete, simply connected Riemannian space of constant sectional curvature \(\kappa \in \{-1, 0, 1\}\) and \(h \in \mathcal{C}^\infty(M^n_\kappa)\) is a positive function yet to be found. Therefore, we have that:

1. \((M^n_0, g_0)\) is the Euclidean \(n\)-space:
   \[
   M^n_0 = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R} \} = \mathbb{R}^n,
   \]
   with
   \[
   g_0 \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}
   \]
   in the parametrization
   \[
   x : \mathbb{R}^n \to \mathbb{R}^n, \quad p \mapsto p;
   \]
2. \((M^n_{-1}, g_{-1})\) is the hyperbolic \(n\)-space:
   \[
   M^n_{-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \} = \mathbb{H}^n,
   \]
   with
   \[
   g_{-1} \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = x_n^{-2} \delta_{ij},
   \]
   in the parametrization
   \[
   x : \mathbb{H}^n \to \mathbb{H}^n, \quad p \mapsto p,
   \]
   which is very similar in nature to that of the previous case;
3. \((M^n_1, g_1)\) is the euclidean \(n\)-sphere:

\[
M^n_1 = \left\{ (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\} = \mathbb{S}^n,
\]

with

\[
g_1 \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \left( \frac{1+r}{2} \right)^{-2} \delta_{ij},
\]

in the parametrization given as the inverse of the stereographic projection with respect to either of the poles \(\varepsilon e_{n+1} \in \mathbb{S}^n\), \(\varepsilon = \pm 1\):

\[
x : \mathbb{R}^n \longrightarrow \mathbb{S}^n \setminus \{-\varepsilon e_{n+1}\}, \quad p \longmapsto \frac{2}{1+r} p + \varepsilon \frac{1-r}{1+r} e_{n+1},
\]

where \(\{e_1, \ldots, e_n, e_{n+1}\}\) is the canonical linear basis of \(\mathbb{R}^{n+1}\),

\[
\mathbb{R}^n \equiv \left\{ (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0 \right\},
\]

and

\[
r : \mathbb{R}^n \longrightarrow [0, \infty), \quad (p_1, \ldots, p_n, 0) \longmapsto \sum_{i=1}^{n} p_i^2.
\]

In order to handle all these geometries at once we let \(x : U \subset \mathbb{R}^n \longrightarrow M^n_\kappa\) represent the selected parametrization according to the value of \(\kappa \in \{-1, 0, 1\}\) and, mostly important, we write

\[
g_\kappa \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \rho^{-2}_\kappa \delta_{ij},
\]

where

\[
\rho_\kappa(x_1, \ldots, x_n) = \begin{cases} 
1 & \text{on } \mathbb{R}^n \text{ if } \kappa = 0, \\
x_{n+1} & \text{on } \mathbb{H}^n \text{ if } \kappa = -1, \\
\frac{1+r}{2} & \text{on } \mathbb{R}^n \text{ if } \kappa = 1.
\end{cases}
\]

Here as in [1] we assume that there do exist functions

\[
(8) \quad \xi : M^n_\kappa \longrightarrow (a, b) \subset \mathbb{R} \quad \text{and} \quad f, h : (a, b) \subset \mathbb{R} \longrightarrow (0, \infty),
\]

making commutative diagrams out of those drawn below:

\[
\begin{array}{ccc}
M^n_\kappa & \xrightarrow{\xi} & (a, b) \\
\downarrow & & \downarrow \\
\begin{array}{c}
\xrightarrow{f} \\
\xleftarrow{f^{-1}} \end{array} & \xrightarrow{f \circ \xi} & \begin{array}{c}
\xleftarrow{f^{-1}} \\
\xrightarrow{f} \end{array}
\end{array}
\]

\[
\begin{array}{ccc}
M^n_\kappa & \xrightarrow{\xi} & (a, b) \\
\downarrow & & \downarrow \\
\begin{array}{c}
\xrightarrow{h} \\
\xleftarrow{h^{-1}} \end{array} & \xrightarrow{h \circ \xi} & \begin{array}{c}
\xleftarrow{h^{-1}} \\
\xrightarrow{h} \end{array}
\end{array}
\]

We then get, by the chain rule, expressions

\[
f_{,i} = \frac{\partial f}{\partial x_i} = \frac{df}{d\xi} \frac{\partial \xi}{\partial x_i} = f' \xi_{,i} \quad \text{and} \quad f_{,ij} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i} = f'' \xi_{,i} \xi_{,j} + f' \xi_{,ij},
\]

regarding the partial derivatives of the function \(f\) up to the second order (with analogous ones valid for \(h\)) in the parametrization \(x : U \subset \mathbb{R}^n \longrightarrow M^n_\kappa\). Now
that everything has been settled down, we believe it’s time to state the main result of this paper.

**Theorem 1.** On each point of the parameter domain $U \subset \mathbb{R}^n$, if it’s not only true that

$$
\frac{(\Delta \xi)_{g_0}}{n} \delta_{ij} + \xi_i \frac{\rho_{\kappa,j}}{\rho_{\kappa}} + \frac{\rho_{\kappa,i}}{\rho_{\kappa}} \xi_j - \frac{2}{n} g_0 \left( \nabla \xi \right)_{g_0} \frac{(\nabla \rho_{\kappa})_{g_0}}{\rho_{\kappa}} \delta_{ij} = 0,
$$

for all $i, j \in \{1, \ldots, n\}$, but also that

$$
\xi_{,k} \xi_{,l} - \frac{\| (\nabla \xi)_{g_0} \|_2^2}{n} \delta_{kl} \neq 0,
$$

for some $k, l \in \{1, \ldots, n\}$, then $f$ is a solution to (5) on $(M, g) = (M_\kappa, h^{-2} g_\kappa)$ if, and only if,

$$
(n - 2) \frac{h''}{h} - 2 \frac{h'}{h} \frac{f'}{f} - \frac{f''}{f} = 0,
$$

on $\xi(x(U)) = \{ \xi(q) \mid q \in x(U) \} \subset (a, b)$. Energy density and pressure of the fluid are then given by:

$$
\mu = \frac{n - 1}{2} \left( \kappa nh^2 + 2h(\Delta h)_{g_\kappa} - n \| (\nabla h)_{g_\kappa} \|^2 \right),
$$

and

$$
\nu = \frac{n - 1}{n} \frac{(\Delta f)_{g_\kappa}}{f} - \frac{n - 2}{n} \mu,
$$

respectively.

**Remark 1.** Upon declaring

$$
\begin{align*}
  x &= h' & \text{and} & \quad y &= f',
\end{align*}
$$

we might observe that

(9) \hspace{1cm}
(10)

becomes

$$
y' = (n - 2)(x^2 + x') - 2xy - y'^2,
$$

which is Riccati in $y$ for a known $x$. Therefore, it’s general solution is of the form $y = y_0 + u$, where $y_0$ is a particular solution of (10) and $u$ must solve the linear equation

$$
\frac{d}{d\xi} \left( \frac{1}{u} \right) - 2(y_0 + x) \frac{1}{u} = 1.
$$

**Proposition 1.** If the local expression of $\xi : M_\kappa^n \rightarrow \mathbb{R}$ with respect to the parametrization $x : U \subset \mathbb{R}^n \rightarrow M_\kappa^n$ satisfies

$$
\xi_{ij} - \frac{(\Delta \xi)_{g_0}}{n} \delta_{ij} + \xi_i \frac{\rho_{\kappa,j}}{\rho_{\kappa}} + \frac{\rho_{\kappa,i}}{\rho_{\kappa}} \xi_j - \frac{2}{n} g_0 \left( \nabla \xi \right)_{g_0} \frac{(\nabla \rho_{\kappa})_{g_0}}{\rho_{\kappa}} \delta_{ij} = 0,
$$

on each point of the parameter domain $U \subset \mathbb{R}^n$ for all $i, j \in \{1, \ldots, n\}$, then:
1. κ = 0:

\[ \xi : \mathbb{R}^n \rightarrow \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \left( \frac{a_i}{2} x_i^2 + b_i x_i + c_i \right), \]

where \( a, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{R} \) are constants;

2. κ = −1:

\[ \xi : \mathbb{H}^n \rightarrow \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto \frac{1}{x_n} \sum_{i=1}^{n} \left( \frac{a_i}{2} x_i^2 + b_i x_i + c_i \right), \]

where \( a, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{R} \) are constants;

3. κ = 1:

\[ \xi : \mathbb{S}^n \rightarrow \mathbb{R}, \quad (p_1, \ldots, p_n, p_{n+1}) \mapsto p_{n+1}, \]

if, in addition, it’s assumed that there do exist \( a < 0 \) and \( \xi : (a, \infty) \subset \mathbb{R} \rightarrow \mathbb{R}, \)

such that

\[ \xi(x) = \xi(r(x)), \]

for all \( x \in \mathbb{R}^n, \)

where

\[ r : \mathbb{R}^n \rightarrow [0, \infty), \quad (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i^2. \]

2. Examples

Example 1. As for the case \( \kappa = 0, \) we might choose

\[ \xi : \mathbb{R}^n \rightarrow \mathbb{R}, \quad (p_1, \ldots, p_n) \mapsto \frac{1}{\sqrt{n}} p_1 + \frac{1}{\sqrt{n}} p_2 + \cdots + \frac{1}{\sqrt{n}} p_n, \]

and then

\[ h : \mathbb{R}^n \rightarrow \mathbb{R}, \quad p \mapsto \cos(\xi(p)), \]

from which we see that \( \mathbb{R}^n \) must be shrunk into one of its open subsets, let’s say,

\[ M^n = \left\{ p \in \mathbb{R}^n : -\frac{\pi}{2} < \xi(p) < \frac{\pi}{2} \right\}, \]

so that \( h \) gets to be strictly positive. This leaves us with the equation

\[ f'' - 2 \tan(\xi) f' + (n - 2) f = 0, \]

of which

\[ f : M^n \rightarrow (0, \infty), \quad p \mapsto \frac{e^{-\xi(p) \sqrt{n-1}}}{\cos(\xi(p))}, \]

is a positive solution. Therefore, the manifold

\[ \bar{M} = M \times \mathbb{R}, \]

munished with the metric tensor

\[ \bar{g} = \sec^2(\xi(x)) \left[ x^* g_0 - \left( e^{-\xi(x) \sqrt{n-1}} \right)^2 dt^2 \right], \]
solves Einstein’s equation for the perfect fluid
\[ T = (\mu + \nu)\bar{g}(\cdot, X) \otimes \bar{g}(\cdot, X) + \nu \bar{g} \]
\[ = \sec^2(\xi(x)) \left[ \nu(\xi(x))x^*g_0 + \mu(\xi(x)) \left( e^{-\xi(x)\sqrt{n-1}} \right)^2 dt^2 \right], \]
characterized by its energy density
\[ \mu(\xi) = \frac{n-1}{2} ((n-2)\cos^2(\xi) - n), \]
and the vector field \( X = \frac{1}{f'} \frac{\partial}{\partial t} \) where, in all of the above, \( x : \bar{M} \rightarrow M \) and \( t : \bar{M} \rightarrow \mathbb{R} \) denote the natural projections.

Example 2. By choosing
\[ \xi : \mathbb{H}^n \rightarrow \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto \frac{1}{x_n} \sum_{i=1}^{n-1} b_i x_i, \]
where \( b_1, \ldots, b_{n-1} \in \mathbb{R} \) are constants subject to
\[ \sum_{i=1}^{n-1} b_i^2 = 1, \]
and also
\[ h : \mathbb{H}^n \rightarrow (0, \infty), \quad x \mapsto \cosh(\xi(x)), \]
as representatives of the case \( \kappa = -1 \), we get the equation
\[ f'' + 2 \tanh(\xi(x))f' - (n-2)f = 0, \]
of which
\[ f : \mathbb{H}^n \rightarrow (0, \infty), \quad x \mapsto e^{-\xi(x)\sqrt{n-1}} \cosh(\xi(x)), \]
is a positive solution. Thus, the manifold
\[ M = \mathbb{H}^n \times \mathbb{R}, \]
equipped with the metric tensor
\[ \bar{g} = \operatorname{sech}(\xi(x)) \left[ x^* g_{-1} - \left( e^{-\xi(x)\sqrt{n-1}} \right)^2 dt^2 \right], \]
solves Einstein’s equation for the perfect fluid
\[ T = (\mu + \nu)\bar{g}(\cdot, X) \otimes \bar{g}(\cdot, X) + \nu \bar{g} \]
\[ = \operatorname{sech}^2(\xi(x)) \left[ \nu(\xi(x))x^* g_{-1} + \mu(\xi(x)) \left( e^{-\xi(x)\sqrt{n-1}} \right)^2 dt^2 \right], \]
characterized by its energy density
\[
\mu(\xi) = \frac{n - 1}{2} \left\{ -n \cosh^2(\xi) + 2 \cosh(\xi)((1 + \xi^2) \cosh(\xi) + n\xi \sinh(\xi)) + 
- n(1 + \xi^2) \sinh^2(\xi) \right\},
\]
pressure
\[
\nu(\xi) = \frac{(n - 1) \cosh^2(\xi)}{n} \left\{ (1 + \xi^2)[(\tanh(\xi) + \sqrt{n - 1})^2 - \text{sech}^2(\xi)] + 
+ n\xi(\tanh(\xi) + \sqrt{n - 1}) + (n - 2)(1 + \xi^2) \tanh(\xi)(\tanh(\xi) + \sqrt{n - 1}) \right\} + 
- \frac{n - 2}{n} \mu(\xi)
\]
and the vector field \(X = \frac{1}{f} \frac{\partial}{\partial t}\) where, as usual, \(x : \bar{M} \rightarrow \mathbb{H}^n\) and \(t : \bar{M} \rightarrow \mathbb{R}\) indicate the natural projections.

**Example 3.** Given that we have
\[
\xi : \mathbb{S}^n \rightarrow \mathbb{R}, \quad (p_1, \ldots, p_n, p_{n+1}) \mapsto p_{n+1},
\]
in case \(\kappa = 1\), we see that upon choosing
\[
h : \mathbb{S}^n \rightarrow (0, \infty), \quad p \mapsto \cos(\xi(p)),
\]
we are lead to the equation
\[
f'' - 2 \tan(\xi) f' + (n - 2) f = 0,
\]
of which
\[
f : \mathbb{S}^n \rightarrow (0, \infty), \quad p \mapsto \frac{e^{-\xi(p)\sqrt{n-1}}}{\cos(\xi(p))},
\]
is a positive solution. Henceforth, the manifold
\[
\bar{M} = \mathbb{S}^n \times \mathbb{R},
\]
furnished with metric tensor
\[
\bar{g} = \sec^2(\xi(x)) \left[ x^* g_1 - \left( e^{-\xi(x)\sqrt{n-1}} \right)^2 dt^2 \right],
\]
solves Einstein’s equation for the perfect fluid
\[
T = (\mu + \nu) \bar{g}(\cdot, X) \otimes \bar{g}(\cdot, X) + \nu \bar{g}
\]
\[
= \sec^2(\xi(x)) \left[ \nu(\xi(x)) x^* g_1 + \mu(\xi(x)) \left( e^{-\xi(x)\sqrt{n-1}} \right)^2 dt^2 \right],
\]
characterized by its energy density
\[
\mu(\xi) = \frac{(n - 1)}{n} \left\{ n \cos^2(\xi) + 2 \cos(\xi)[- (1 - \xi^2) \cos(\xi) + n\xi \sin(\xi)] + 
- n(1 - \xi^2) \sin^2(\xi) \right\}
\]
pressure

\[ \nu(\xi) = \frac{(n - 1) \cos^2(\xi)}{n} \left\{ (1 - \xi^2)[\tan(\xi) - \sqrt{n - 1}^2 + \sec^2(\xi)] + 
- n\xi(\tan(\xi) - \sqrt{n - 1}) + (n - 2)(1 - \xi^2)\tan(\xi)(\tan(\xi) - \sqrt{n - 1}) \right\} + 
- \frac{n - 2}{n} \mu(\xi), \]

and vector field \( X = \frac{1}{f} \frac{\partial}{\partial t} \) where, once more, \( x : \bar{M} \to \mathbb{S}^n \) and \( t : \bar{M} \to \mathbb{R} \) stand for the natural projections. Next, we plot the graphs of both \( \mu \) and \( \nu \) as functions of \( \xi \) in the dimensions \( n = 2, 3, 4, 5 \) and \( 6 \).

3. Proofs

Please, take a look at section 1 to familiarize yourself with both the notation we adopt and the various conventions we do make.

**Proof of Theorem**

Exactly how the change of metrics \( g_\kappa \rightleftharpoons g = h^{-2}g_\kappa \) seems to alter the tensors \( \text{ric}_{g_\kappa} = \kappa(n - 1)g_\kappa \) and \( (\nabla^2 f)_{g_\kappa} \) is something better comprehended with the help of the next 2 formulas (see [2]):

\[ \text{ric}_g = \kappa(n - 1)g_\kappa + 
\]

\[ + h^{-2}\left\{ (n - 2)h(\nabla^2 h)_{g_\kappa} + \left[h(\Delta h)_{g_\kappa} - (n - 1)\|(\nabla h)_{g_\kappa}\|^2 \right]g_\kappa \right\}, \]

and

\[ (\nabla^2 f)_g = (\nabla^2 f)_{g_\kappa} + 
\]

\[ + df \otimes d(\log h) + d(\log h) \otimes df - g_\kappa((\nabla f)_{g_\kappa}, (\nabla \log h)_{g_\kappa}). \]

Therefore, it’s readily seen that

\[ \text{scal}_g = \kappa n(n - 1)h^2 + (n - 1)\left[ 2h(\Delta h)_{g_\kappa} - n\|(\nabla h)_{g_\kappa}\|^2 \right], \]

and

\[ (\Delta f)_g = h^2\left[ (\Delta f)_{g_\kappa} - (n - 2)g_\kappa((\nabla f)_{g_\kappa}, (\nabla \log h)_{g_\kappa}) \right]. \]
From expressions (11) and (13) comes

\[
\text{ric}_g - \frac{\text{scal}_g}{n} g = \frac{n - 2}{h} \left[ (\nabla^2 h)_{g_n} - \frac{(\Delta h)_{g_n}}{n} g_n \right],
\]

whilst

\[
(\nabla^2 f)_g - \frac{(\Delta f)_g}{n} g = (\nabla^2 f)_{g_n} - \frac{(\Delta f)_{g_n}}{n} g_n + \\
+ df \otimes d(\log h) + d(\log h) \otimes df - \frac{2}{n} g_n ((\nabla f)_{g_n}, (\nabla \log h)_{g_n}) g_n.
\]

follows from (12) and (14). By (16), since

\[
g_n = \rho^{-2} g_0,
\]

on the open set \(x(U) \subset M^\kappa_n\), which is the image set of the parametrization \(x : U \subset \mathbb{R}^n \to M^\kappa_n\), (15) becomes

\[
\text{ric}_g - \frac{\text{scal}_g}{n} g = \frac{n - 2}{h} \left[ (\nabla^2 h)_{g_0} - \frac{(\Delta h)_{g_0}}{n} g_0 + \\
+ dh \otimes d(\log \rho_\kappa) + d(\log \rho_\kappa) \otimes dh - \frac{2}{n} g_0 ((\nabla h)_{g_0}, (\nabla \log \rho_\kappa)_{g_0}) g_0 \right],
\]

and (16) itself gives

\[
(\nabla^2 f)_g - \frac{(\Delta f)_g}{n} g = (\nabla^2 f)_{g_0} - \frac{(\Delta f)_{g_0}}{n} g_0 + \\
+ df \otimes d(\log \rho_\kappa) + d(\log \rho_\kappa) \otimes df - \frac{2}{n} g_0 ((\nabla f)_{g_0}, (\nabla \log \rho_\kappa)_{g_0}) g_0 + \\
+ df \otimes d(\log h) + d(\log h) \otimes df - \frac{2}{n} g_0 ((\nabla f)_{g_0}, (\nabla \log h)_{g_0}) g_0,
\]

where, in the last expression, we have used the fact that

\[
(\nabla f)_{g_n} = \rho^2_{\kappa}(\nabla f)_{g_0} \quad \text{and} \quad (\nabla \log h)_{g_n} = \rho^2_{\kappa}(\nabla \log h)_{g_0}.
\]
As a result, (5) asserts that
\[
\begin{align*}
\frac{n - 2}{h} \left[ (\nabla^2 h)_{g_0} - \frac{(\Delta h)_{g_0}}{n} g_0 + \
+ dh \otimes d(\log \rho_\kappa) + d(\log \rho_\kappa) \otimes dh - \frac{2}{n} g_0 ((\nabla h)_{g_0}, (\nabla \log \rho_\kappa)_{g_0}) g_0 \right] = \\
= \frac{1}{f} \left[ (\nabla^2 f)_{g_0} - \frac{(\Delta f)_{g_0}}{n} g_0 + \
+ df \otimes d(\log \rho_\kappa) + d(\log \rho_\kappa) \otimes df - \frac{2}{n} g_0 ((\nabla f)_{g_0}, (\nabla \log \rho_\kappa)_{g_0}) g_0 + \
+ df \otimes d(\log h) + d(\log h) \otimes df - \frac{2}{n} g_0 ((\nabla f)_{g_0}, (\nabla \log h)_{g_0}) g_0 \right],
\end{align*}
\]
and because we have
\[
(\nabla f)_{g_0} = f'(\nabla \xi)_{g_0} \quad \text{and} \quad (\Delta f)_{g_0} = f''(\nabla \xi)_{g_0}^2 + f'(\Delta \xi)_{g_0},
\]
even that
\[
\left( \xi, \xi, j - \frac{\| (\nabla \xi)_{g_0} \|^2}{n} \delta_{ij} \right) \left[ (n - 2) k'' h - 2 \frac{k' f'}{f} - f'' \right] = \\
= \left( \frac{f'}{f} - (n - 2) \frac{k'}{h} \right) \left( \xi, i + \frac{\rho \kappa, i}{\rho_\kappa} + \frac{\rho_\kappa, i}{\rho_\kappa} \xi, j - \frac{2}{n} \rho_0 \left( (\nabla \xi)_{g_0}, (\nabla \rho_\kappa)_{g_0} \right) \delta_{ij} \right),
\]
for all \( i, j \in \{1, \ldots, n\} \). \( \square \)

**Proof of Proposition 1**

**Case \( \kappa = 0 \)**: From the fact that
\[
0 = \left( \frac{\Delta \xi}{n} \right)_{g_0} \delta_{ij} + \left( \frac{\rho_0, i}{\rho_0} + \frac{\rho_0, j}{\rho_0} \right) \xi, j - \frac{2}{n} g_0 \left( (\nabla \xi)_{g_0}, (\nabla \rho_0)_{g_0} \right) \delta_{ij} = \\
= \xi, i \delta_{ij},
\]
for every \( i, j \in \{1, \ldots, n\} \), we get that \( \xi, ij = 0 \) whenever \( i \neq j \). Thus,
\[
\xi(x_1, \ldots, x_n) = \sum_{i=1}^{n} F_i(x_i),
\]
and, as such,
\[
F''_i(x_i) = \frac{1}{n} \sum_{j=1}^{n} F''_j(x_j),
\]
for all \( i \in \{1, \ldots, n\} \). If we let
\[
F''_1 = \cdots = F''_n = a \in \mathbb{R},
\]
then
\[
x(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left( \frac{a}{2} x_i^2 + b_i x_i + c_i \right),
\]
where \(b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{R}\) are constants.

**Case \(\kappa = -1\):** Notice that
\[
0 = \xi_{ij} - \frac{(\Delta \xi)_{g_0}}{n} \delta_{ij} + \xi_{i \rho} \rho^{-1} j \xi_{j \rho} - \frac{2}{n} g_0 \left( \nabla \xi \rho, \nabla \rho_{g_0} \right) \delta_{ij},
\]
turns out to be
\[(19)\]
\[
0 = \xi_{ij} - \frac{(\Delta \xi)_{g_0}}{n} \delta_{ij} + \xi_i j \delta_{jn} \xi_{j n} \frac{2}{n} x_n \delta_{ij},
\]
for all \(i, j \in \{1, \ldots, n\}\), now that \(\kappa = -1\). Therefore, \(1 \leq i \leq n - 1\) and \(j = n\) together give
\[
(\xi_i \cdot x_n)_{n} = \left( \xi_{in} + \frac{\xi_i}{x_n} \right) x_n = 0,
\]
from which we get that
\[
\xi_i(x_1, \ldots, x_n) = \frac{F_i(x_1, \ldots, x_{n-1})}{x_n},
\]
whenever \(i \in \{1, \ldots, n - 1\}\). But then,
\[
\frac{F_{i j}}{x_n} = \xi_{i j} = 0,
\]
for all \(1 \leq i \neq j \leq n - 1\), culminating in \(F_i = F_i(x_i)\) for every \(i \in \{1, \ldots, n - 1\}\). Upon taking \(1 \leq i = j \leq n - 1\) we see that
\[
\xi_{ii} - \frac{(\Delta \xi)_{g_0}}{n} \frac{2}{n} x_n = 0,
\]
implies that
\[
\frac{F_i'}{x_n} = \xi_{ii} = \frac{(\Delta \xi)_{g_0}}{n} + \frac{2}{n} \xi_{i n} x_n = \xi_{jj} = \frac{F_j'}{x_n},
\]
for every \(i, j \in \{1, \ldots, n - 1\}\). Henceforth, there must be an \(a \in \mathbb{R}\) such that
\[
F_1' = \cdots = F_{n-1}' = a,
\]
meaning that there does exist some \(b_i \in \mathbb{R}\) validating the identity
\[
F_i(x_i) = a x_i + b_i,
\]
for each \(i \in \{1, \ldots, n - 1\}\). So far, we may solely guarantee that
\[
\xi(x_1, \ldots, x_n) = \frac{1}{x_n} \sum_{i=1}^{n-1} \left( \frac{a}{2} x_i^2 + b_i x_i + c_i \right) + G(x_n),
\]
where \(c_1, \ldots, c_{n-1} \in \mathbb{R}\) are constants but, as \(i = j = n\) lead to
\[
\xi_{nn} - \frac{(\Delta \xi)_{g_0}}{n} + \xi_{n n} + \frac{\xi_n}{x_n} - \frac{2}{n} \xi_{n n} x_n = 0,
\]
once more by (19), we have that
\[ \xi_{,nn} + 2 \frac{\xi_{,n}}{x_n} = \frac{(\Delta \xi)_{g_0}}{n} + \frac{2 \xi_{,n}}{n x_n} = \xi_{,ii} = \frac{a}{x_n}, \]
so that
\[ (\xi_{,n} \cdot x_n^2)_{,n} = \left( \xi_{,nn} + 2 \frac{\xi_{,n}}{x_n} \right) x_n^2 = ax_n. \]
It’s now readily seen that
\[ [G' \cdot x_n^2]_{,n} = \left[ \left( - \frac{1}{x_n^2} \sum_{i=1}^{n-1} \left( \frac{a}{2} x_i^2 + b_i x_i \right) \right) + G' \right] x_n^2 = ax_n, \]
of which
\[ G(x_n) = \frac{a}{2} x_n + c_n \frac{x_n}{x_n} + b_n, \]
is the general solution, where \( b_n, c_n \in \mathbb{R} \) are constants. In conclusion, we have
the function
\[ \xi(x_1, \ldots, x_n) = \frac{1}{x_n} \sum_{i=1}^{n} \left( \frac{a}{2} x_i^2 + b_i x_i + c_i \right). \]

Case \( \kappa = 1 \): As for the last case, there remains to solve the equation
\[ 0 = \xi_{,ij} - \frac{(\Delta \xi)_{g_0}}{n} \delta_{ij} + \xi_{,i} \frac{\rho_{1,j}}{\rho_1} + \frac{\rho_{1,i} \xi_{,j}}{\rho_1} - 2 \frac{\rho_{j}}{n} \left( (\nabla \xi)_{g_0}, \frac{(\nabla \rho_1)_{g_0}}{\rho_1} \right) \delta_{ij} \]
\[ = \xi_{,ij} - \frac{(\Delta \xi)_{g_0}}{n} \delta_{ij} + \xi_{,i} \frac{2 x_j}{1 + r} + \frac{2 x_i}{1 + r} \xi_{,j} - 2 \frac{\rho_{j}}{n} \left( (\nabla \xi)_{g_0}, \frac{2 x_i}{1 + r} \right) \delta_{ij}, \]
and since it is a difficult one to get over with in all of its generality, we may tacitly assume that there does exist some function \( \xi : (a, \infty) \subset \mathbb{R} \rightarrow (0, \infty) \), where \( a < 0 \), such that the local expression of \( \xi : \mathbb{S}^n \rightarrow \mathbb{R} \) with respect to the parametrization \( x : \mathbb{R}^n \rightarrow \mathbb{S}^n \setminus \{-e_{n+1}\} \) satisfies
\[ \xi(u) = \xi(r(u)), \]
for all \( u \in \mathbb{R}^n \), where
\[ r : \mathbb{R}^n \rightarrow \mathbb{R}, \quad (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} u_i^2. \]
Because
\[ \xi_{,i} = \xi' 2 x_i \quad \text{and} \quad \xi_{,ij} = \xi'' 4 x_i x_j + \xi' 2 \delta_{ij}, \]
for every \( i, j \in \{1, \ldots, n\} \), we know that
\[ (\nabla \xi)_{g_0} = 2 \xi' x \quad \text{and} \quad (\Delta \xi)_{g_0} = 4 r \xi'' + 2 n \xi'. \]
Henceforth,
\[ 0 = \xi'' 4 x_i x_j + \xi' 2 \delta_{ij} - \frac{\xi'' 4 r + \xi' 2 n}{n} \delta_{ij} + \xi' 4 x_i x_j \frac{1}{1 + r} + \xi' 4 x_i x_j \frac{1}{1 + r} - 2 \frac{\xi'}{n} \frac{4 r}{1 + r} \delta_{ij} \]
\[ = 4 \left( \xi'' + \frac{2}{1 + r} \xi' \right) \left( x_i x_j - \frac{r}{n} \delta_{ij} \right). \]
Notice that
\[ \xi : S^n \rightarrow \mathbb{R},\quad (p_1, \ldots, p_n, p_{n+1}) \mapsto p_{n+1}, \]
owns the expression
\[ \xi(r) = \frac{2\varepsilon}{1 + r} - \varepsilon, \]
with respect to
\[ x : \mathbb{R}^n \rightarrow S^n \setminus \{-\varepsilon e_{n+1}\},\quad u \mapsto \frac{2}{1 + r} u + \frac{1 - r}{1 + r} \varepsilon e_{n+1}, \]
with which it’s seen that
\[ \xi' = \frac{d\xi}{dr} = -\frac{2\varepsilon}{(1 + r)^2} \quad \text{and} \quad \xi'' = \frac{d^2\xi}{dr^2} = \frac{4\varepsilon}{(1 + r)^3}, \]
therefore resulting in
\[ \xi'' + 2\frac{1}{1 + r} \xi' = 0, \]
on all of \( \mathbb{R}^n \) or, perhaps, we should say for all \( r \in \{r(u) \mid u \in \mathbb{R}^n\} = [0, \infty). \)

References

[1] BARBOZA, Marcelo; LEANDRO, Benedito; PINA, Romildo. Invariant solutions for the Einstein field equation. Journal of Mathematical Physics, v. 59, n. 6, p. 062501, 2018.
[2] BESSE, Arthur L. Einstein manifolds. Springer Science and Business Media, 2007.
[3] CHOQUET-BRUHAT, Yvonne. General relativity and the Einstein equations. Oxford University Press, 2009.
[4] KOBAYASHI, Osamu; OBATA, Morio. Conformally-flatness and static space-time. In: Manifolds and Lie groups. Birkhäuser, Boston, MA, 1981. p. 197-206.
[5] O’NEILL, Barrett. Semi-Riemannian geometry with applications to relativity. Academic press, 1983.

1 Universidade Federal de Goiás, IME, 131, 74001-970, Goiânia, GO, Brazil. E-mail address: levi@ufg.br
2 Instituto Federal Goiano, Rodovia Geraldo Silva Nascimento Km 2,5, 75790-000, Urutai, GO, Brazil. E-mail address: marcelo.barboza@ifgoiano.edu.br
3 Universidade Federal de Goiás, IME, 131, 74001-970, Goiânia, GO, Brazil. E-mail address: williamisaotokura@hotmail.com