EXISTENCE OF GROUNDSTATES FOR A CLASS OF NONLINEAR
CHOQUARD EQUATIONS IN THE PLANE

LUCA BATTAGLIA AND JEAN VAN SCHAFTINGEN

Abstract. We prove the existence of a nontrivial groundstate solution for the class of nonlinear Choquard equation

\[ -\Delta u + u = (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^2, \]

where \( I_\alpha \) is the Riesz potential of order \( \alpha \) on the plane \( \mathbb{R}^2 \) under general nontriviality, growth and subcriticality on the nonlinearity \( F \in C^1(\mathbb{R}, \mathbb{R}) \).

1. Introduction

We are interested in the existence of nontrivial solutions to the class of nonlinear Choquard equations of the form

\[ (P) \quad -\Delta u + u = (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^N, \]

where \( N \in \mathbb{N} = \{1, 2, \ldots \} \), \( \Delta \) is the standard Laplacian operator on the Euclidean space \( \mathbb{R}^N \), \( I_\alpha : \mathbb{R}^N \to \mathbb{R} \) is the Riesz potential of order \( \alpha \in (0, N) \) defined for each \( x \in \mathbb{R}^N \setminus \{0\} \) by

\[ I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{N}{2}\right)\pi^{\frac{N-\alpha}{2}}} |x|^{N-\alpha}, \]

and a nonlinearity is described by the function \( F \in C^1(\mathbb{R}, \mathbb{R}) \). Solutions of the equation \((P)\) are at least formally critical points of the energy functional defined for a function \( u : \mathbb{R}^N \to \mathbb{R} \) by

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u). \]

In the particular case where for each \( s \in \mathbb{R} \), \( F(s) = s^2/2 \), solutions to the Choquard equation \((P)\) are standing waves solutions of the Hartree equation. In particular when \( N = 3 \) and \( \alpha = 2 \), the problem \((P)\) has arisen in various fields of physics: quantum mechanics [20], one-component plasma [11] and self-gravitating matter [15]. In these cases, many existence results have been obtained in literature, with both variational [11, 13, 14] and ordinary differential equations techniques [6, 15, 21] (see also the review [18]). Such methods extend also to the case of homogeneous nonlinearities [16].
When the nonlinearity $F$ is not any more homogeneous, it has been shown that the Choquard equation (2) has a nontrivial solution if the nonlinearity $F$ satisfies the following hypotheses \[17\]:

- \(F_0\) there exists \(s_0 \in \mathbb{R}\) such that \(F(s_0) \neq 0\);
- \(F_1\) there exists \(C \in \mathbb{R}\) such that \(|F'(s)| \leq C(|s|^{N/2} + |s|^{N+\alpha})\) for every \(s > 0\);
- \(F_2\) \(\lim_{s \to 0} F(s)/|s|^{1+\frac{\alpha}{2}} = 0 = \lim_{s \to 0} F(s)/|s|^{\frac{N+\alpha}{N-2}}\).

The solution \(u\) is a **groundstate**, in the sense that \(u\) minimizes the value of the functional \(I\) among all nontrivial solutions. The assumptions \((F_0')\), \((F_1')\) and \((F_2')\) are rather mild and reasonable and are “almost necessary” in the sense of Berestycki and Lions [3]: the nontriviality of the nonlinearity condition \((F_0')\) is clearly necessary to have a nontrivial solution; the assumption \((F_1')\) secures a proper variational formulation of the problem (2) by ensuring that the energy functional \(I\) is well-defined on the natural Sobolev space \(H^1(\mathbb{R}^N)\) through the Hardy–Littlewood–Sobolev and Sobolev inequalities; the condition \((F_2')\) is a sort of **subcriticality** condition with respect to the limiting-case embeddings. The analysis by a Pohožaev identity shows that the assumptions \((F_1')\) and \((F_2')\) are necessary in the homogeneous case \(F(s) = s^p/p\) [16].

The results in [17] can thus be seen as a counterpart for Choquard-type equations of the result of Berestycki and Lions [3] which give similar “almost necessary” conditions for the existence of a groundstate to the equation

\[
- \Delta u + u = G'(u) \quad \text{in } \mathbb{R}^N.
\]

The latter equation can be at least formally be obtained by (2) by passing to the limit as \(\alpha \to 0\) and setting \(G = F^2/2\).

Whereas the above-mentioned almost necessary conditions for existence of the Choquard equation (2) and for the scalar field equation (2) have been obtained in higher dimensions \(N \geq 3\), the latter result has been extended to the two-dimensional case [4], under the following assumptions

- \((G_0)\) there exists \(s_0 \in \mathbb{R}\) such that \(G(s_0) > \frac{|s_0|^2}{2}\);
- \((G_1)\) for every \(\theta > 0\) there exists \(C = C_\theta > 0\) such that \(|G'(s)| \leq C_\theta \min\{1, \theta |s|^2\} e^{\theta |s|^2}\) for every \(s > 0\);
- \((G_2)\) \(\lim_{s \to 0} G(s)/|s|^2 < 1/2\).

This raises naturally the question whether there is a similar existence result for the Choquard equation (2) in the planar case.

In the present work, we provide a general existence result for groundstate solutions of problem (2) in the planar case \(N = 2\), which is a two-dimensional counterpart of [17] and a counterpart for the Choquard equation of [4]. The counterparts of \((F_0')\), \((F_1')\), \((F_2')\) we need are the following:

- \((F_0)\) there exists \(s_0 \in \mathbb{R}\) such that \(F(s_0) \neq 0\);
- \((F_1)\) for every \(\theta > 0\) there exists \(C = C_\theta > 0\) such that \(|F'(s)| \leq C_\theta \min\{1, |s|^2\} e^{\theta |s|^2}\) for every \(s > 0\);
- \((F_2)\) \(\lim_{s \to 0} F(s)/|s|^{1+\frac{\alpha}{2}} = 0\).

Our main result reads as follows:
Theorem 1.1. If \( N = 2 \) and \( F \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies the conditions \((F_0)\), \((F_1)\) and \((F_2)\), then the problem \((P)\) has a groundstate solution \( u \in H^1(\mathbb{R}^2) \setminus \{0\} \), namely the function \( u \) solves \((P)\) and

\[
I(u) = c := \inf \{ I(v) \mid v \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ is a solution of } (P) \}.
\]

Let us discuss the assumptions of Theorem 1.1. As above, the assumption \((F_0)\) is necessary for the existence of a nontrivial solution. As before, the condition \((F_1)\) ensures needed the well-defineteness of the energy functional on the whole space \( H^1(\mathbb{R}^2) \). It has a different shape, because in dimension \( N = 2 \), the critical nonlinearity for Sobolev embeddings is not anymore a power but rather an exponential-type nonlinearity. More precisely, the integral of \( \min\{1, u^2\}e^{\theta |u|^2} \) on \( \mathbb{R}^2 \) is uniformly controlled on \( H^1_0(B_1) \) if and only if \( \theta \int_{B_1} |
abla u|^2 \leq 4\pi \) (see [1, 19]); this is why the parameter \( \theta > 0 \) appears in condition \((F_1)\). It will appear that the condition \((F_1)\) is strong enough at infinity. Indeed, by integrating the function \( F' \), it is possible to observe that for every \( \theta > 0 \),

\[
\lim_{|s| \to \infty} \frac{|F(s)| + |F'(s)||s|}{e^{\theta |s|^2}} = 0.
\]

A subcriticality condition still needs to be imposed around 0; that is the goal of the subcriticality condition \((F_2)\).

The assumptions \((F_0)\), \((F_1)\) and \((F_2)\) are still almost necessary: in the case \( F(s) = \frac{s^p}{p} \), they are satisfied if and only if \( p > 1 + \frac{\alpha}{2} \), and for \( p \leq 1 + \frac{\alpha}{2} \) the Choquard equation \((P)\) has no nontrivial solutions (see [16]).

In order to prove Theorem 1.1 the constraint minimization technique used in [3, 4] for the local problem \((2)\) does not seem to work, as it introduces a Lagrange multiplier that cannot be absorbed through a suitable dilation because of the presence of three different scalings in the equation and of the nonhomogeneity of the nonlinearity.

Following [17], we use a mountain-pass construction. We start by constructing a Palais–Smale sequence for the mountain-pass level

\[
b := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),
\]

where

\[
\Gamma := \{ \gamma \in C([0,1], H^1(\mathbb{R}^2)) \mid \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \}.
\]

To avoid relying on an Ambrosetti–Rabinowitz superlinearity condition, we use a scaling trick due to Jeanjean [2], which allows to construct Pohožaev–Palais–Smale sequence (Proposition 3.1), namely a Palais–Smale sequence which, in addition, satisfies asymptoticly the Pohožaev identity

\[
P(u) := \int_{\mathbb{R}^2} |u|^2 - \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u))F(u) = 0.
\]

Such a condition will imply quite directly the boundedness of the sequence in the space \( H^1(\mathbb{R}^2) \) and it will be crucial to get the convergence, hence the existence of a solution (Proposition 4.1).

We are left with showing that the solution \( u \) is actually a groundstate. To prove this, we first show that the solution \( u \) itself satisfies the Pohožaev identity (Proposition 5.2).
This will follow by simple calculations once a suitable regularity result is established (Proposition 5.1); this regularity turns out to be easier to prove from the assumption $(F_1)$ than in the higher-dimensional case [17] where a suitable nonlocal Brezis–Kato regularity had to be proved. The last ingredient that we need is an optimal path $\gamma_v \in \Gamma$ associated to any solution $v$ of $(P)$. The construction of such paths (Proposition 5.3) is inspired by [10, 17] but it is more delicate in our two-dimensional case than in the higher dimensions $N \geq 3$, because dilations $t \mapsto v(\cdot/t) \in H^1(R^N)$ are not anymore continuous at $t = 0$ when $N = 2$.

The content of the paper is the following: in Section 2 we provide some technical preliminaries; in Section 3 we construct the Pohožaev–Palais–Smale sequence; in Section 4 we show that the sequence converges to a solution of $(P)$; in Section 5 we prove that $u$ is actually a groundstate. In the last section we also state some qualitative result concerning the solutions, which can be proved directly following [17].

2. Preliminaries

In this section we present some preliminary results which we will need throughout the rest of this paper. We start by reformulating in a more convenient form the Moser–Trudinger inequality of Adachi and Tanaka [1]. This quantitative estimate will play a crucial role throughout the paper.

Proposition 2.1 (Moser–Trudinger inequality). For any $\beta \in (0, 4\pi)$ there exists $C = C_\beta > 0$ such that for every $u \in H^1(R^2)$ satisfying

$$\int_{R^2} |\nabla u|^2 \leq 1,$$

one has

$$\int_{R^2} \min\{1, |u|^2\} e^{\beta|u|^2} \leq C_\beta \int_{R^2} |u|^2.$$

Proof. The result follows the fact [1, Theorem 0.1] that under the conditions of the theorem,

$$\int_{R^2} (e^{\beta|u|^2} - 1) \leq C \int_{R^2} |u|^2.$$

together with the elementary inequalities valid for every $s \geq 0$,

$$\left(1 - \frac{1}{e}\right) \max\{1, s\} e^s \leq e^s - 1 \leq \max\{1, s\} e^s. \quad \square$$

We will also use the Hardy–Littlewood–Sobolev inequality to deal with the nonlocal term (see for example [12, Theorem 4.3]):

Proposition 2.2 (Hardy–Littlewood–Sobolev inequality). For any $p \in [1, \frac{2}{\alpha})$ and $f \in L^p(R^2)$ there exists a constant $C = C_{\alpha, p}$ such that

$$\|I_\alpha \ast f\|_{L^{2p/(2p-\alpha)}(R^2)} \leq C\|f\|_{L^p(R^2)}.$$

Combining the last two results with the assumption on $F$ and (3) we deduce that the energy functional is well-defined on $H^1(R^2)$:
Proposition 2.3. If $F$ satisfies $(F_1)$, then the energy functional $I$ defined by (1) is well-defined and continuously differentiable.

Proof. We first consider the superposition map $E$ defined for each $u \in H^1(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ by $E(u)(x) = F'(u(x))$. We claim that $E$ is well-defined and continuous as a map from $H^1(\mathbb{R}^2)$ to $L^{4/\alpha}(\mathbb{R}^2)$. Indeed by assumption $(F_1)$, for every $\theta > 0$, and $s \in \mathbb{R}$, we have

$$|F'(s)|^{\frac{1}{\theta}} \leq C_{\theta}^{\frac{1}{\theta}} \min\{1, s^2\} e^{\frac{\theta}{\alpha}|s|^2}.$$

If $u \in H^1(\mathbb{R}^2)$, we take $\theta > 0$ such that $\int_{\mathbb{R}^2} \nabla u|^2 < \frac{\theta}{\alpha}$. We observe that

$$|F'(u)|^{\frac{1}{\theta}} \leq C_{\theta}^{\frac{1}{\theta}} \min\{1, |u|^2\} e^{\frac{\theta}{\alpha}|u|^2}$$

on $\mathbb{R}^2$, where the right-hand side is integrable in view of the Moser–Trudinger inequality (Proposition 2.1); therefore the map $E : H^1(\mathbb{R}^2) \to L^{4/\alpha}(\mathbb{R}^2)$ is well-defined.

If now the sequence $(u_n)_{n \in \mathbb{N}}$ converges to $u$ in $H^1(\mathbb{R}^2)$, then we can assume without loss of generality that $\nu := \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} \nabla u|^2 < \frac{\theta}{\alpha}$ and that $(u_n)_{n \in \mathbb{N}}$ converges to $u$ almost everywhere. We have then for some constant $C \geq 0$,

$$C(\min\{1, |u|^2\} e^{\frac{\theta}{\alpha}|u|^2} + \min\{1, |u_n|^2\} e^{\frac{\theta}{\alpha}|u_n|^2}) - |F'(u) - F'(u_n)|^{\frac{1}{\theta}} \geq 0,$$

for each $n \in \mathbb{N}$ almost everywhere in $\mathbb{R}^2$. By Fatou’s lemma we get

$$\limsup_{n \to \infty} \int_{\mathbb{R}^2} C(\min\{1, |u|^2\} e^{\frac{\theta}{\alpha}|u|^2} + \min\{1, |u_n|^2\} e^{\frac{\theta}{\alpha}|u_n|^2}) - |F'(u) - F'(u_n)|^{\frac{1}{\theta}}$$

$$\geq 2C \int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\frac{\theta}{\alpha}|u|^2}$$

and therefore

$$\limsup_{n \to \infty} \int_{\mathbb{R}^2} |F'(u) - F'(u_n)|^{\frac{1}{\theta}} \leq C \limsup_{n \to \infty} \int_{\mathbb{R}^2} \min\{1, |u_n|^2\} e^{\frac{\theta}{\alpha}|u_n|^2} - \min\{1, |u|^2\} e^{\frac{\theta}{\alpha}|u|^2}. $$

If we consider the set $A^\lambda_n = \{x \in \mathbb{R}^2 \mid |u_n(x)| \geq \lambda\}$, we have by Lebesgue’s dominated convergence theorem, for every $\lambda > 0$,

$$\limsup_{n \to \infty} \int_{\mathbb{R}^2 \setminus A^\lambda_n} \min\{1, |u_n|^2\} e^{\frac{\theta}{\alpha}|u_n|^2}$$

$$\leq \limsup_{n \to \infty} \int_{\mathbb{R}^2 \setminus A^\lambda_n} (\min\{1, |u_n|^2\} e^{\frac{\theta}{\alpha}|u_n|^2} - \min\{1, |u|^2\} e^{\frac{\theta}{\alpha}\min(|u|^2, \lambda^2)})$$

$$+ \int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\frac{\theta}{\alpha}|u|^2}$$

$$\leq \int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\frac{\theta}{\alpha}|u|^2}.$$ 

On the other hand, we have by the Cauchy–Schwarz inequality, the Chebyshev inequality and the Moser–Trudinger inequality (Proposition 2.1)

$$\int_{A^\lambda_n} \min\{1, |u_n|^2\} e^{\frac{\theta}{\alpha}|u_n|^2} \leq |A^\lambda_n|^{\frac{1}{\theta}} \left( \int_{A^\lambda_n} \min\{1, |u_n|^2\} e^{\frac{\theta}{\alpha}|u_n|^2} \right)^{\frac{1}{\theta}} \leq \frac{C}{\lambda} \int_{\mathbb{R}^2} |u_n|^2.$$

This allows to conclude that the map $E : H^1(\mathbb{R}^2) \to L^{4/\alpha}(\mathbb{R}^2)$ is continuous.
We now consider the map $F : H^1(\mathbb{R}^2) \to L^{4/(2+\alpha)}(\mathbb{R}^2)$ defined for each $u \in H^1(\mathbb{R}^2)$ by $F(u) = F \circ u$. We observe that for every $s \in \mathbb{R}$,

$$F(s) = \int_0^1 F'(\tau s) s \, d\tau,$$

and thus for almost every $x \in \mathbb{R}^2$,

$$F(u(x)) = \int_0^1 F'(\tau u(x)) u(x) \, d\tau.$$

It follows thus from the first part of the proof that $F$ is well-defined from $H^1(\mathbb{R}^2)$ to $L^{4/(2+\alpha)}(\mathbb{R}^2)$.

For the differentiability we consider a sequence $(u_n)_{n \in \mathbb{N}}$ converging strongly to $u$ in $H^1(\mathbb{R}^2)$. We observe that for each $n \in \mathbb{N}$,

$$F(u_n) - F(u) - \mathcal{E}(u)(u_n - u) = \int_0^1 (\mathcal{E}((1 - \tau)u + \tau(u_n)) - \mathcal{E}(u))(u_n - u) \, d\tau,$$

and thus by Hölder’s inequality

$$\|F(u_n) - F(u) - \mathcal{E}(u)(u_n - u)\|_{L^{4/(2+\alpha)}} \leq \int_0^1 \|\mathcal{E}((1 - \tau)u + \tau(u_n)) - \mathcal{E}(u)\|_{L^{4/\alpha}} \|u_n - u\|_{L^2}.$$

By the convergence of the sequence $(u_n)_{n \in \mathbb{N}}$ and the continuity of the functional $\mathcal{E}$, it follows that, as $n \to \infty$,

$$\|F(u_n) - F(u) - \mathcal{E}(u_n)(u_n - u)\|_{L^{4/(2+\alpha)}} = o(\|u_n - u\|_{L^2}),$$

that is, $\mathcal{E}$ represents the Fréchet differential of the functional $F$. Since $\mathcal{E}$ is continuous, it follows that $F$ is of class $C^1$.

Finally, we consider the quadratic form $Q$ defined for $f \in L^{4/(2+\alpha)}$ by

$$Q(f) = \int_{\mathbb{R}^2} (I_\alpha * f)f.$$

By the Hardy–Littlewood–Sobolev inequality (Proposition 2.2), the quadratic form $Q$ is bounded on bounded sets of the space $L^{4/(2+\alpha)}(\mathbb{R}^2)$. This implies that $Q$ is continuously differentiable and thus the functional

$$u \in H^1(\mathbb{R}^2) \mapsto Q(F(u), F(u)) = \int_{\mathbb{R}^2} (I_\alpha * F(u)) F(u)$$

is continuously differentiable. By the smoothness of the norm on a Hilbert space, we conclude that the functional $I$ is continuously differentiable. \qed

Finally, we will use the following improvement of Proposition 2.2 when one has some more $L^p$ integrability:

**Proposition 2.4.** For any $p \in [1, \frac{2}{\alpha})$, $q \in (\frac{2}{\alpha}, +\infty)$ and $f \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ there exists $C = C_{\alpha,p,q}$ such that

$$\|I_\alpha * f\|_{L^\infty(\mathbb{R}^2)} \leq C(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^q(\mathbb{R}^2)}).$$
Proposition 3.1. If the function \(p, q\) by choosing \(\frac{2}{q} < (2 - \alpha) \frac{p}{p-1}\) and therefore, through splitting the integral and Hölder inequality we get for every \(x \in \mathbb{R}^2\)
\[
|I_\alpha * f(x)| \leq C \int_{\mathbb{R}^2} \frac{|f(x - y)|}{|y|^{2-\alpha}} \, dy \\
\leq C \left( \int_{B_1} \frac{dy}{|y|^{(2-\alpha) \frac{q}{q-1}}} \right)^{\frac{1}{1 - \frac{1}{q}}} \|f\|_{L^q(B_1(x))} \\
+ C \left( \int_{B_1} \frac{dy}{|y|^{(2-\alpha) \frac{p}{p-1}}} \right)^{\frac{1}{1 - \frac{1}{p}}} \|f\|_{L^p(\mathbb{R}^2 \setminus B_1(x))} \\
\leq C'(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^q(\mathbb{R}^2)}).
\]

\[\square\]

3. Construction of a Pohožaev–Palais–Smale sequence

In this section we show the existence of a Pohožaev–Palais–Smale sequence at the level \(b\) defined by \(I\). In other words, we construct a sequence of almost critical points which asymptotically satisfies the equation \(\mathcal{P}\) and the Pohožaev identity \(\mathcal{I}\).

Proposition 3.1. If the function \(F \in C^1(\mathbb{R}, \mathbb{R})\) satisfies the assumptions \((F_0)\) and \((F_1)\), then there exists a sequence \((u_n)_{n \in \mathbb{N}}\) in \(H^1(\mathbb{R}^2)\) such that:

(a) \(\mathcal{I}(u_n) \to b\);
(b) \(\mathcal{I}'(u_n) \to 0\) strongly in \(H^1(\mathbb{R}^2)'\);
(c) \(\mathcal{P}(u_n) \to 0\).

To prove Proposition 3.1 we first need to show that the energy functional \(\mathcal{I}\) has the mountain pass geometry, namely that the mountain pass level \(b\) is well-defined and nontrivial:

Lemma 3.2. The critical level \(b\) defined by \(I\) satisfies \(b \in (0, +\infty)\).

Proof. We start by showing the finiteness of \(b\), which will be done as in \(\mathcal{I}\) Proposition 2.1. By the definition of the set \(b\), it is sufficient to show that \(\Gamma \neq \emptyset\), which in turn is equivalent to find \(u_0 \in H^1(\mathbb{R}^2)\) such that \(\mathcal{I}(u_0) < 0\). By the assumption \((F_0)\), we can take \(s_0\) such that \(F(s_0) \neq 0\) and we find
\[
\int_{\mathbb{R}^2} (I_\alpha * F(s_0 1_{B_1})) F(s_0 1_{B_1}) = F(s_0)^2 \int_{B_1} \int_{B_1} I_\alpha(x - y) \, dx \, dy > 0;
\]
therefore by density of smooth functions in \(L^q(\mathbb{R}^2)\) there will be \(v_0 \in H^1(\mathbb{R}^2)\) with \(\int_{\mathbb{R}^2} (I_\alpha * F(v_0)) F(v_0) > 0\). We consider now, for \(t > 0\), the function \(v_t: \mathbb{R}^2 \to \mathbb{R}\) defined for \(x \in \mathbb{R}^2\) by \(v_t(x) := v_0(\frac{x}{t})\). This function verifies
\[
\mathcal{I}(v_t) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_0|^2 + \frac{t^2}{2} \int_{\mathbb{R}^2} |v_0|^2 - \frac{t^{2+\alpha}}{2} \int_{\mathbb{R}^2} (I_\alpha * F(v_0)) F(v_0),
\]
therefore, for some \(t_0 \gg 0\), the function \(u_0 := v_{t_0}\) satisfies \(\mathcal{I}(u_0) < 0\).

Let us now show that \(b > 0\). By the definition of \(b\), it is equivalent to show that there exists \(\varepsilon > 0\) such that for every path \(\gamma \in \Gamma\) there exists \(t_\gamma \in [0, 1]\) with \(\mathcal{I}(\gamma(t_\gamma)) \geq \varepsilon > 0\).
We first assume that \( u \in H^1(\mathbb{R}^2) \) and \( \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) \leq \delta \ll 1 \). In particular, since \( \int_{\mathbb{R}^2} |\nabla u|^2 \leq 1 \), Proposition 2.1 applies to \( u \) with \( \beta = 2\pi \). Therefore, by Propositions 2.2 and 2.4 and by [3], we have

\[
\int_{\mathbb{R}^2} (I_\alpha * F(u)) F(u) \leq C \left( \int_{\mathbb{R}^2} |F(u)|^{\frac{2+\alpha}{\alpha+2}} \right)^{\frac{\alpha+2}{2+\alpha}} \leq C \left( \int_{\mathbb{R}^2} \min\{1, |u|^2 \} e^{2\pi |u|^2} \right)^{\frac{\alpha+2}{2+\alpha}},
\]

which is smaller than \( \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) \) if \( \delta \) is small enough. It follows then that if \( \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) \leq \delta \), we have

\[
\mathcal{I}(u) \geq \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2).
\]

We now take an arbitrary path \( \gamma \in \Gamma \). Since \( \mathcal{I}(\gamma(1)) < 0 < \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla \gamma(t_\gamma)|^2 + |\gamma(t_\gamma)|^2) \), we have

\[
\int_{\mathbb{R}^2} (|\nabla \gamma(1)|^2 + |\gamma(1)|^2) > \delta > 0 = \int_{\mathbb{R}^2} (|\nabla \gamma(0)|^2 + |\gamma(0)|^2);
\]

therefore, there exists \( t_\gamma \in (0, 1) \) such that \( \int_{\mathbb{R}^2} (|\nabla \gamma(t_\gamma)|^2 + |\gamma(t_\gamma)|^2) = \delta \), and hence \( \mathcal{I}(\gamma(t_\gamma)) \geq \frac{\delta}{4} \). The lemma follows by taking \( \varepsilon := \frac{\delta}{4} \). \( \square \)

Proof of Proposition 3.1. We follow [8] Chapter 4; [9] Chapter 2; [17] Proposition 2.1. We consider the map \( \Phi \) given by

\[
\Phi : \mathbb{R} \times H^1(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2)
\]

\[
(\sigma, v) \mapsto \Phi(\sigma, v)(x) := v(e^{-\sigma} x)
\]

and the functional \( \tilde{\mathcal{I}} = \mathcal{I} \circ \Phi \):

\[
\tilde{\mathcal{I}}(\sigma, v) = \mathcal{I}(\Phi(\sigma, u)) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{e^{2\sigma}}{2} \int_{\mathbb{R}^2} |v|^2 - \frac{e^{(2+\alpha)\sigma}}{2} \int_{\mathbb{R}^2} (I_\alpha * F(v)) F(v),
\]

which is well-defined and Fréchet-differentiable on the Hilbert space \( \mathbb{R} \times H^1(\mathbb{R}^2) \).

We define now the class of paths

\[
\tilde{\Gamma} := \{ \tilde{\gamma} \in C([0, 1], \mathbb{R} \times H^1(\mathbb{R}^2)) \mid \tilde{\gamma}(0) = (0, 0) \text{ and } \tilde{\mathcal{I}}(\tilde{\gamma}(1)) < 0 \};
\]

since we have \( \Gamma = \{ \Phi \circ \tilde{\gamma} \mid \tilde{\gamma} \in \tilde{\Gamma} \} \), the mountain pass levels of \( \mathcal{I} \) and \( \tilde{\mathcal{I}} \) coincide, namely

\[
b = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{t \in [0, 1]} \tilde{\mathcal{I}}(\tilde{\gamma}(t)).
\]

Since, by Lemma 3.2, the mountain pass level \( b \) is not trivial, we can thus apply the minimax principle ([24], Theorem 2.9) and we find a sequence \( (\sigma_n, v_n) \in \mathbb{R} \times H^1(\mathbb{R}^2) \) such that:

\[
\tilde{\mathcal{I}}(\sigma_n, v_n) \to b \quad \text{and} \quad \tilde{\mathcal{I}}(\sigma_n, v_n) \to 0 \text{ strongly in } (\mathbb{R} \times H^1(\mathbb{R}^2))' .
\]

By writing explicitly the derivative of \( \tilde{\mathcal{I}} \):

\[
\tilde{\mathcal{I}}'(\sigma_n, v_n)[h, w] = \mathcal{I}'(\Phi(\sigma_n, v_n))[\Phi(\sigma_n, v_n)h] + \mathcal{P}(\Phi(\sigma_n, v_n))h;
\]

we see that the conclusion follows by taking \( u_n = \Phi(\sigma_n, v_n) \). \( \square \)
4. Convergence of the Pohožaev–Palais–Smale sequence

In this Section we will construct a nontrivial solution of \((\text{P})\) from the sequence given by Proposition 3.1.

Proposition 4.1. If the function \(F \in C^1(\mathbb{R}, \mathbb{R})\) satisfies \((F_1)\) and \((F_2)\) and the sequence \((u_n)_{n \in \mathbb{N}}\) in \(H^1(\mathbb{R}^2)\) satisfies

(a) \(I(u_n)\) is uniformly bounded,

(b) \(I'(u_n) \rightharpoonup 0\) strongly in \((H^1(\mathbb{R}^2))'\),

(c) \(\mathcal{P}(u_n) \rightharpoonup 0\);

then, up to subsequences, one of the following occurs:

- either \(u_n \overset{n \to \infty}{\to} 0\) strongly in \(H^1(\mathbb{R}^2)\);
- or there exists \(u \in H^1(\mathbb{R}^2) \setminus \{0\}\) solving \((\text{P})\) and a sequence \((x_n)_{n \in \mathbb{N}}\) in \(\mathbb{R}^2\) such that \(u_n(\cdot - x_n) \overset{n \to \infty}{\to} u\) weakly in \(H^1(\mathbb{R}^2)\).

We follow the strategy of [17, Proposition 2.2]. Since the gradient does not appear in the Pohožaev identity \((\text{P})\), it will be more delicate to show that the nonlocal term does not vanish.

Proof of Proposition 4.1. We assume that the first alternative does not hold, namely

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2) > 0.
\]

By writing for each \(n \in \mathbb{N}\)

\[
\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{\alpha}{2(2 + \alpha)} \int_{\mathbb{R}^2} |u_n|^2 = I(u_n) - \frac{\mathcal{P}(u_n)}{2 + \alpha}
\]

we deduce that the sequence \((u_n)_{n \in \mathbb{N}}\) is bounded in the space \(H^1(\mathbb{R}^2)\). Since \(I'(u_n) \to 0\) in \(H^1(\mathbb{R}^2)'\) as \(n \to \infty\), we have \(I'(u_n)[u_n] \to 0\) as \(n \to \infty\), therefore

\[
\int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F'(u_n) u_n = \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2) - I'(u_n)[u_n] \geq \frac{1}{C}.
\]

Taking \(C_0 \geq \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2)\), we can apply Proposition 2.1 to \(\frac{1}{\sqrt{c_0}} u_n\) with \(\beta = 2\pi\) and we obtain for each \(n \in \mathbb{N}\)

\[
\int_{\mathbb{R}^2} \min\{1, u_n^2\} e^{\frac{\pi}{c_0} |u_n|^2} \leq C_2 \pi \int_{\mathbb{R}^2} |u_n|^2 \leq C_2 \pi,
\]

moreover, we also have, as \(n \to \infty\),

\[
\int_{\mathbb{R}^2} |u_n|^2 = \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) + \mathcal{P}(u_n)
\]

\[
= \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) + o(1).
\]
Therefore, from Proposition 2.2 and by (3) we get
\[
\frac{1}{C} \leq \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F'(u_n) u_n \leq C \left( \int_{\mathbb{R}^2} |F(u_n)| \right)^{\frac{\alpha}{\alpha+1}} \left( \int_{\mathbb{R}^2} (|F'(u_n)| u_n) \frac{2 \alpha}{\alpha+1} \right)^{\frac{\alpha+1}{\alpha}}
\]
(7)
\[
\leq C' \left( \int_{\mathbb{R}^2} \min \{ 1, |u_n|^2 \} e^{\frac{2 \alpha}{\alpha+1} |u_n|^2} \right)^{1+\frac{\alpha}{\alpha+1}} \leq C'' \left( \int_{\mathbb{R}^2} |u_n|^2 \right)^{1+\frac{\alpha}{\alpha+1}}
\]
\[
= C'' \left(1 + \alpha \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) + o(1) \right)^{1+\frac{\alpha}{\alpha+1}},
\]
namely \( \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) \) is bounded from above from zero when \( n \to \infty \).

We now want to prove that \( u_n \) does not vanish. We will use the following inequality [13 Lemma I.1] (see also [16 lemma 2.3; 22 (2.4); 24 lemma 1.21]):
\[
\int_{\mathbb{R}^2} |u_n|^p \leq C \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2) \left( \sup_{x \in \mathbb{R}^2} \int_{B_1(x)} |u_n|^2 \right)^{1-\frac{2}{p}}
\]
and we will show that the right-hand side term is bounded from below by a positive constant, for every \( p > 2 \). By the assumption \((F_2)\) and (3), for every \( \varepsilon > 0 \) there exists \( C_{\varepsilon, \theta} > 0 \) such that
\[
|F(s)|^{\frac{\alpha}{\alpha+1}} \leq \varepsilon \min \{ 1, |s|^2 \} e^{\theta |s|^2} + C_{\varepsilon, \theta} |s|^p;
\]
therefore
\[
\left( \sup_{x \in \mathbb{R}^2} \int_{B_1(x)} |u_n|^2 \right)^{1-\frac{2}{p}} \geq \frac{1}{C} \int_{\mathbb{R}^2} |u_n|^p \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2)
\]
\[
\geq \frac{1}{CC_0 C_{\varepsilon}} \left( \int_{\mathbb{R}^2} |F(u_n)|^{\frac{\alpha}{\alpha+1}} - \varepsilon \int_{\mathbb{R}^2} \min \{ 1, |u_n|^2 \} e^{\frac{2 \alpha}{\alpha+1} |u_n|^2} \right)
\]
\[
\geq \frac{1}{C_{\varepsilon}} \left( \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) \right)^{\frac{\alpha}{\alpha+1}} - \varepsilon C \int_{\mathbb{R}^2} |u_n|^2
\]
\[
\geq \frac{1}{C_{\varepsilon}} \left( \frac{1}{C} - \varepsilon CC_0 \right).
\]
The quantity \( \varepsilon \) being arbitrary, we get \( \int_{B_1(x_n)} |u_n|^p \geq \frac{1}{C_{\varepsilon}} \) for some \( x_n \in \mathbb{R}^2 \), for \( n \) large enough.

We can now consider the translated sequence \((u_n(\cdot - x_n))_{n \in \mathbb{N}}\). Since the problem \((P)\) is invariant by translation, this sequence will satisfy the hypotheses of the present proposition, hence we will still denote it as \((u_n)_{n \in \mathbb{N}}\) and we will assume that \( x_n = 0 \) for all \( n \in \mathbb{N} \). Since \( \lim \inf_{n \to \infty} \int_{B_1} |u_n|^p > 0 \), we can assume that this sequence \((u_n)_{n \in \mathbb{N}}\) converges weakly to \( u \in H^1(\mathbb{R}^2) \setminus \{0\} \). We just have to show that \( u \) solves \((P)\).

The sequence \((u_n)_{n \in \mathbb{N}}\) being bounded in \( H^1(\mathbb{R}^2) \), the sequence \((F(u_n))_{n \in \mathbb{N}}\) is bounded in \( L^p(\mathbb{R}^2) \) for every \( p \geq \frac{4}{2+\alpha} \). Moreover, up to subsequences, \( u_n \to u \) almost everywhere as \( n \to \infty \), so by the continuity of the function \( F \) we also have \( F(u_n) \to F(u) \) almost everywhere as \( n \to \infty \); this implies that \( F(u_n) \to F(u) \) weakly in \( L^p(\mathbb{R}^2) \) for every such \( p \) as \( n \to \infty \). Since \( \frac{2}{\alpha} > \frac{4}{2+\alpha} \), by Propositions 2.2 and 2.4 we get \( I_\alpha * F(u_n) \to I_\alpha * F(u) \)
weakly in $L^{4/(2-\alpha)}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ as $n \to \infty$. By the condition (F1) and Proposition 2.1, the sequence $(F'(u_n))_{n \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^2)$ for every $p \in [\frac{4}{\alpha}, \infty)$, and by continuity $F'(u_n) \to F'(u)$ almost everywhere as $n \to \infty$; therefore, $F'(u_n) \to F'(u)$ strongly in $L^q_{\text{loc}}(\mathbb{R}^2)$ for every $q \in [1, +\infty)$ as $n \to \infty$, hence

$$(I_\alpha * F(u_n))F'(u_n) \underset{n \to \infty}{\to} (I_\alpha * F(u))F'(u) \quad \text{in } L^r_{\text{loc}}(\mathbb{R}^2) \quad \forall r \in [1, +\infty].$$

Therefore, for every $\varphi \in C^1_0(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} (\nabla u \cdot \nabla \varphi + u \varphi) = \lim_{n \to \infty} \int_{\mathbb{R}^2} (\nabla u_n \cdot \nabla \varphi + u_n \varphi)
$$

$$= \lim_{n \to \infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F'(u_n) \varphi = \int_{\mathbb{R}^2} (I_\alpha * F(u))F'(u) \varphi,$$

namely $u$ solves the Choquard equation (P).

Corollary 4.2. If $F$ satisfies the conditions (F0), (F1) and (F2), then problem (P) has a nontrivial solution $u \in H^1(\mathbb{R}^2)$.

Proof. By Proposition 5.1, $\mathcal{I}$ admits a Pohožaev–Palais–Smale sequence $(u_n)_{n \in \mathbb{N}}$ at the level $b$. We apply Proposition 4.1 to $(u_n)_{n \in \mathbb{N}}$. If the first alternative occurred, then we would have $\mathcal{I}(u_n) \to \mathcal{I}(0) = 0$ as $n \to \infty$, in contradiction with Lemma 2.2. Therefore, the second alternative must occur, and in particular we get a solution $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ of (P).

5. From solutions to groundstates

We start by providing a local regularity result for solution of (P). This result can be obtained quite directly because our growth assumption (F1) gives a good control on $I_\alpha * F(u)$ which, in turn, permits to apply a standard bootstrap method. The equivalent result in higher dimension $N \geq 3$ is more delicate to prove (see [17] Theorem 2) because of the relative weakness of assumption (F1).

Proposition 5.1. If $F \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the condition (F1) and if the function $u \in H^1(\mathbb{R}^2)$ solves the problem (P), then $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for every $p \geq 1$.

Proof. By (3) and Lemma 2.1 we deduce that if $v \in H^1(\mathbb{R}^2)$ then $F(v) \in L^p(\mathbb{R}^2)$ for every $p \geq \frac{4}{2-\alpha}$. Since $\frac{2}{\alpha} > \frac{4}{2-\alpha}$, by Proposition 2.4 inequality we get $I_\alpha * F(v) \in L^\infty(\mathbb{R}^2)$. Therefore, any solution $u$ of (P) verifies

$$|\Delta u + u| \leq C|F'(u)|,$$

with $F'(u) \in L^p_{\text{loc}}(\mathbb{R}^2)$ for every $p \geq 1$ because of (F1). By standard (interior) regularity theory on bounded domains (see for example [17] Chapter 9) we deduce that $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$.

The extra regularity just proved allows to prove that solutions of (P) satisfy the Pohožaev identity (6). The proof of the Pohožaev identity is classical and it is based on testing (P) against a suitable cut-off of $x \cdot \nabla u(x)$, therefore it will be skipped. Details can be found in [17] Theorem 3.
Proposition 5.2 (Pohožaev identity). If \( F \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies \((F_1)\) and \( u \in H^1(\mathbb{R}^2) \cap W^{2,2}_{\text{loc}}(\mathbb{R}^2) \) solves \((\mathcal{P})\), then
\[
\mathcal{P}(u) = \int_{\mathbb{R}^2} |u|^2 - \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_a * F(u)) F(u) = 0.
\]

The Pohožaev identity allows us to show that the mountain pass solution is actually a groundstate. We will argue like [10, Lemma 2.1; 17, Proposition 2.1], associating to any solution \( v \) a path \( \gamma_v \in \Gamma \) passing through \( v \). The main difficulty here is that the integral of \(|\nabla u|^2\) is invariant by dilation, therefore we are not allowed to join \( v \) with \( v(\tau) \). To overcome this difficulty, we will combine properly dilatations and multiplication by constants \([10]\).

Proposition 5.3. If \( F \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies \((F_1)\) and \( v \in H^1(\mathbb{R}^2) \setminus \{0\} \) solves \((\mathcal{P})\), then there exists a path \( \gamma_v \in C \left([0,1], H^1(\mathbb{R}^2)\right) \) such that:
(a) \( \gamma_v(0) = 0 \);
(b) \( \gamma_v(1/2) = v \);
(c) \( \mathcal{I}(\gamma_v(t)) < \mathcal{I}(v) \) for every \( t \in [0,1] \setminus \{1/2\} \);
(d) \( \mathcal{I}(\gamma_v(1)) < 0 \).

Proof. We consider the path \( \tilde{\gamma} : [0, +\infty) \to H^1(\mathbb{R}^2) \) given for each \( \tau \in [0, \infty) \) by
\[
(\tilde{\gamma}(\tau))(x) := \begin{cases} 
\frac{\tau}{\tau_0} v\left(\frac{x}{\tau}\right) & \text{if } \tau \leq \tau_0, \\
v\left(\frac{x}{\tau}\right) & \text{if } \tau \geq \tau_0.
\end{cases}
\]
with \( \tau_0 \ll 1 \) to be chosen later. The function \( \tilde{\gamma} \) is clearly continuous on the interval \([0, +\infty)\) and in particular at its boundary 0. For \( \tau \geq \tau_0 \), Proposition 5.2 gives
\[
\mathcal{I}(\tilde{\gamma}(\tau)) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau_0^2}{2} \int_{\mathbb{R}^2} |v|^2 - \frac{\tau_0^{2+\alpha}}{2} \int_{\mathbb{R}^2} (I_a * F(v)) F(v)
= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \left(\frac{\tau_0^2}{2} - \frac{\tau_0^{2+\alpha}}{2+\alpha}\right) \int_{\mathbb{R}^2} |v|^2,
\]
which attains its strict maximum in \( \tau = 1 \) and is negative for \( \tau \geq \tau_1 \), for some \( \tau_1 \gg 1 \).

For \( \tau \leq \tau_0 \) we use \([3]\) with \( \theta = \left(1 + \frac{\tau_0}{2}\right)\pi \) and then apply Proposition 2.1 to the function \( \tilde{\gamma}(\tau)/\left(\int_{\mathbb{R}^2} |\nabla \tilde{\gamma}(\tau)|^2\right)^{1/2} \):
\[
\int_{\mathbb{R}^2} |F(\tilde{\gamma}(\tau))|^{\frac{2+\alpha}{2+\alpha}} \leq C \int_{\mathbb{R}^2} \min\{1, |\tilde{\gamma}(\tau)|^2\} e^{2\pi |\tilde{\gamma}(\tau)|^2} \leq C \frac{\int_{\mathbb{R}^2} |\tilde{\gamma}(\tau)|^2}{\int_{\mathbb{R}^2} |\nabla \tilde{\gamma}(\tau)|^2} = C \tau_0 \int_{\mathbb{R}^2} |v|^2,
\]
therefore, because of the Pohožaev identity (Proposition 5.2) and the Hardy–Littlewood–Sobolev inequality (Proposition 2.2), we have
\[
\mathcal{I}(\tilde{\gamma}(\tau)) = \frac{\tau_0^2}{2\tau_0} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau_0^2}{2} \int_{\mathbb{R}^2} |v|^2 - \int_{\mathbb{R}^2} (I_a * F(\tilde{\gamma}(\tau))) F(\tilde{\gamma}(\tau))
\leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau_0^2}{2} \int_{\mathbb{R}^2} |v|^2 + C \left(\int_{\mathbb{R}^2} |F(\tilde{\gamma}(\tau))|^{\frac{2+\alpha}{2+\alpha}}\right)^{1+\frac{\alpha}{2+\alpha}}.
\]
Therefore, in view of (11) and the Pohožaev identity again, we deduce that
\[
\mathcal{I}(\tilde{\gamma}(\tau)) \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau_0^2}{2} \int_{\mathbb{R}^2} |v|^2 + C \tau_0^{2+\alpha} \left( \int_{\mathbb{R}^2} |v|^2 \right)^{\frac{1+\alpha}{2}},
\]
which is strictly less than \(\mathcal{I}(v)\) if \(\tau_0 = \tau_0(v)\) is chosen small enough. Therefore, the function \(\tilde{\gamma}\) verifies the following properties:

(a') \(\tilde{\gamma}(0) = 0\);
(b') \(\tilde{\gamma}(1) = v\);
(c') \(\mathcal{I}(\tilde{\gamma}(\tau)) < \mathcal{I}(v)\) for every \(\tau \in [0, \tau_1]\); 
(d') \(\mathcal{I}(\tilde{\gamma}(\tau_1)) < 0\).

To get the required \(\gamma_v\) it suffices to take a suitable change of variable \(\gamma_v(t) := \tilde{\gamma}(T(\tau))\) for some function \(T \in C([0, 1], \mathbb{R})\) satisfying \(T(0) = 0\), \(T(1) = 1/2\) and \(T(\tau_1) = 1\). \(\square\)

We are now in position to prove the main theorem of this work.

**Proof of Theorem 1.1.** Let \((u_n)_{n \in \mathbb{N}}\) be the Pohožaev–Palais–Smale sequence given by Proposition 5.1. Then, by Proposition 4.1, it converges weakly to a solution \(u \in H^1(\mathbb{R}^2)\) \(\setminus \{0\}\) of (P). By definition of groundstate, \(\mathcal{I}(u) \geq c\) and, by Proposition 5.2, we have \(\mathcal{P}(u) = 0\) (Proposition 5.2 is applicable in view of Proposition 5.1). Arguing as in [17, Theorem 1], we get successively

\[
\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\alpha}{2(2 + \alpha)} \int_{\mathbb{R}^2} |u|^2 \\
\leq \liminf_{n \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{\alpha}{2(2 + \alpha)} \int_{\mathbb{R}^2} |u_n|^2 \right) = \liminf_{n \to \infty} \left( \mathcal{I}(u_n) - \frac{\mathcal{P}(u_n)}{2 + \alpha} \right) = b.
\]

If \(v \in H^1(\mathbb{R}^2) \setminus \{0\}\) is another solution of the Choquard equation (P), we apply Proposition 5.3 to \(v\):

\[
\mathcal{I}(v) = \sup_{t \in [0, 1]} \mathcal{I}(\gamma_v(t)) \geq \inf_{\gamma \in \Gamma, \tau \in [0, 1]} \sup_{t \in [0, 1]} \mathcal{I}(\gamma(t)) = b.
\]

The solution \(v\) being arbitrary, by definition of groundstate one has \(b \leq c\). Putting everything together, we get

\[
c \leq \mathcal{I}(u) \leq b \leq c,
\]
hence \(\mathcal{I}(u) = b = c\). The proof is complete. \(\square\)

We point out as a corollary of the proof of Theorem 1.1 that the convergence in Proposition 4.1 turns out to be actually a strong convergence in \(H^1(\mathbb{R}^2)\) and that this gives as a byproduct a compactness property of the set of groundstates of (P).

**Corollary 5.4.** Let \((u_n)_{n \in \mathbb{N}}\) be a Pohožaev–Palais–Smale sequence satisfying the assumptions of Proposition 4.1 and in addition

\[
\lim_{n \to \infty} \mathcal{I}(u_n) = c.
\]
Then, there exists $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ solving \(\mathcal{P}\) and a sequence \((x_n)_{n \in \mathbb{N}}\) in $\mathbb{R}^2$ such that, up to subsequences, $u_n(\cdot - x_n) \rightharpoonup_{n \to \infty} u$ strongly in $H^1(\mathbb{R}^2)$.

Moreover, the set of groundstates

$$S_c := \{ u \in H^1(\mathbb{R}^2); u \text{ solves } \mathcal{P} \text{ and } \mathcal{I}(u) = c \}$$

is compact, up to translations, in $H^1(\mathbb{R}^2)$.

**Proof.** We apply Proposition 4.1, the first alternative is excluded by our assumption and the continuity of the functional $\mathcal{I}$ at $0$. Therefore we get, up to translations, $u_n \rightharpoonup u$ as $n \to \infty$ in $H^1(\mathbb{R}^2)$ and the function $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ solves \(\mathcal{P}\). As in the proof of Theorem 1.1 we get

$$\lim_{n \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{\alpha}{2(2 + \alpha)} \int_{\mathbb{R}^2} |u_n|^2 \right) \leq c = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\alpha}{2(2 + \alpha)} \int_{\mathbb{R}^2} |u|^2,$$

from which it follows that $u_n \rightharpoonup u$ strongly in $H^1(\mathbb{R}^2)$ as $n \to \infty$.

To show the compactness of the set of groundstates $S_c$, we consider an arbitrary sequence \((u_n)_{n \in \mathbb{N}}\) in $S_c$. Because of Proposition 5.2, it verifies $\mathcal{P}(u_n) = 0$ for every $n \in \mathbb{N}$, so it satisfies the hypotheses of Proposition 4.1 and of the first part of the present corollary; therefore, up to subsequences and translations it will converge to some $u$ which solves \(\mathcal{P}\) and, by the continuity of the functional $\mathcal{I}$ in $H^1(\mathbb{R}^2)$, we get $u \in S_c$. \(\square\)

We conclude this paper by the following result on additional qualitative properties of the solution $u$.

**Proposition 5.5.** If $F$ is even and nondecreasing on $(0, \infty)$ and $u$ is a groundstate solution of \(\mathcal{P}\), then $u$ has constant sign and is radially symmetric with respect to some point $a \in \mathbb{R}^N$.

**Proof.** The proof is the same as [17] Propositions 5.2 and 5.3. We briefly sketch the argument for the convenience of the reader.

To prove the constant-sign property, consider the path $\gamma_u$ defined in Proposition 5.3. Since $F$ is an even function, $\mathcal{I}(|v|) = \mathcal{I}(v)$ for every $v \in H^1(\mathbb{R}^2)$, hence $\mathcal{I}(|\gamma_u(t)|) < \mathcal{I}(|\gamma_u(1/2)|) = b$ for every $t \in [0,1] \setminus \{1/2\}$. From this, one easily deduces that the function $|u|$ is a groundstate solution of \(\mathcal{P}\); since $F' \geq 0$, we can apply the strong maximum principle and get $|u| > 0$, namely $u$ has constant sign. Without loss of generality we assume now that $u \geq 0$.

For the symmetry, we follow the strategy of Bartsch, Weth and Willem [2] and its adaptation to the Choquard equation [16,17]. For any closed half space $H \subset \mathbb{R}^2$ we consider the reflection $\sigma_H$ with respect to $H$ and define, for every $u \in H^1(\mathbb{R}^2)$, the polarization (see for example [5])

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{if } x \in H, \\ \min\{u(x), u(\sigma_H(x))\} & \text{if } x \notin H. \end{cases}$$

We first observe that [5] Lemma 5.3

$$\int_{\mathbb{R}^2} |\nabla u^H|^2 + |u^H|^2 = \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2$$
Moreover, since $F$ is nondecreasing on $(0, +\infty)$, we have $(F \circ u)^H = F \circ (u^H)$ and thus in view of the rearrangement inequality for the Riesz potential
\[
\int_{\mathbb{R}^2} (I_\alpha * F(u^H))F(u^H) = \int_{\mathbb{R}^2} (I_\alpha * F(u)^H)F(u)^H \leq \int_{\mathbb{R}^2} (I_\alpha * F(u))F(u)
\]
with equality if and only if either $(F \circ u)^H = F \circ u$ or $(F \circ u)^H = F \circ u \circ \sigma_H$ [16, lemma 5.3]. It follows thus that, $I(u^H) \leq I(u)$, with equality holding if and only if either $F(u^H) = F(u)$ or $F(u^H) = F(u \circ \sigma_H)$ on $\mathbb{R}^2$. From this and the definition of the level $b$, it follows that $u^H$ is a ground state solutions of (2), hence either $F(u^H) = F(u)$ or $F(u^H) = F(u \circ \sigma_H)$ on $\mathbb{R}^2$. In the former case we easily get $f(u^H) = f(u)$, hence $u^H = u$; in the later, we similarly get $u^H = u \circ \sigma_H$. The hyperplane $H$ being arbitrary, in either case we conclude that the function $u$ is radially symmetric with respect to some point $a \in \mathbb{R}^2$ [16, lemma 5.4; [23, proposition 3.15].

\[\square\]

References

[1] S. Adachi and K. Tanaka, *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents*, Proc. Amer. Math. Soc. 128 (2000), no. 7, 2051–2057.

[2] T. Bartsch, T. Weth, and M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math. 96 (2005), 1–18.

[3] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal. 82 (1983), no. 4, 347–375.

[4] H. Berestycki, T. Gallouët, and O. Kavian, *Équations de champs scalaires euclidiens non linéaires dans le plan*, C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), no. 5, 307–310.

[5] F. Brock and A. Yu. Solynin, *An approach to symmetrization via polarization*, Trans. Amer. Math. Soc. 352 (2000), no. 4, 1759–1796.

[6] P. Choquard, J. Stubbe, and M. Vuffray, *Stationary solutions of the Schrödinger-Newton model—an ODE approach*, Differential Integral Equations 21 (2008), no. 7–8, 665–679.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Second, Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer, Berlin, 1983.

[8] J. Hirata, N. Ikoma, and K. Tanaka, *Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches*, Topol. Methods Nonlinear Anal. 35 (2010), no. 2, 253–276.

[9] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. 28 (1997), no. 10, 1633–1659.

[10] L. Jeanjean and K. Tanaka, *A remark on least energy solutions in $\mathbb{R}^N$*, Proc. Amer. Math. Soc. 131 (2003), no. 8, 2399–2408.

[11] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation*, Studies in Appl. Math. 57 (1976/77), no. 2, 93–105.

[12] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, R.I., 2001.

[13] P.-L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. 4 (1980), no. 6, 1063–1072.

[14] G. P. Menzala, *On the nonexistence of solutions for an elliptic problem in unbounded domains*, Funkcial. Ekvac. 26 (1983), no. 3, 231–235.

[15] I. M. Moroz, R. Penrose, and P. Tod, *Spherically-symmetric solutions of the Schrödinger-Newton equations*, Classical Quantum Gravity 15 (1998), no. 9, 2733–2742. Topology of the Universe Conference (Cleveland, OH, 1997).

[16] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. 265 (2013), no. 2, 153–184.

[17] , *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. 367 (2015), no. 9, 6557–6579.
A guide to the Choquard equation, J. Fixed Point Theory Appl., posted on 2016, DOI 10.1007/s11784-016-0373-1, (to appear in print).

J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1970/71), 1077–1092.

S. I. Pekar, Untersuchungen über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.

P. Tod and I. M. Moroz, An analytical approach to the Schrödinger–Newton equations, Nonlinearity 12 (1999), no. 2, 201–216.

J. Van Schaftingen, Interpolation inequalities between Sobolev and Morrey-Campanato spaces: a common gateway to concentration-compactness and Gagliardo-Nirenberg interpolation inequalities, Port. Math. 71 (2014), no. 3-4, 159–175.

J. Van Schaftingen and M. Willem, Symmetry of solutions of semilinear elliptic problems, J. Eur. Math. Soc. (JEMS) 10 (2008), no. 2, 439–456.

M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser, Boston, Mass., 1996.

Sapienza Università di Roma, Dipartimento di Matematica, Piazzale Aldo Moro 5, 00185 Rome, Italy
E-mail address: battaglia@mat.uniroma1.it

Université catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du Cyclotron 2 bte L70.01.01, 1348 Louvain-la-Neuve, Belgium
E-mail address: Jean.VanSchaftingen@uclouvain.be