A first integration of some knot soliton models

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Abstract

Recently it has been shown that there exists a sector within the Faddeev-Niemi model for which the equations of motion may be reduced to first order equations. However, no solutions to that sector have been given. It is not even known whether this sector contains topologically nontrivial solutions, at all.

Here, we show that two models with analytically known Hopf solitons, namely the Nicole and the Aratyn-Ferreira-Zimerman models, possess sectors which can be integrated to first order partial differential equations. The main result is that these sectors are topologically nontrivial. In fact, all analytically known hopfions belong to them.

Key words: Hopf solitons
PACS: 05.45.Yv
1. Introduction

The well-known Faddeev-Niemi model \[1\] is a nonlinear field theory in 3+1 dimensions with the two-sphere \(S^2\) as its target space. The maps from one-point compactified three-dimensional Euclidean space (this compactification is needed for static configurations to have finite energy) to the target space \(S^2\) are classified by a topological index (the Hopf index), and, therefore, topological solitons with a knot structure are expected to exist.

The Lagrange density of the Faddeev-Niemi model is

\[
L_{FN} = L_2 - \lambda L_4, \tag{1}
\]

where

\[
L_2 = (\partial_\mu \vec{n})^2 = 4 \frac{\partial_\mu u \partial_\mu \bar{u}}{(1 + |u|^2)^2} \tag{2}
\]

and

\[
L_4 = |\vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n})|^2 = 4 \frac{(\partial_\mu u \partial_\mu \bar{u})^2 - (\partial_\mu u \partial_\mu u)(\partial_\mu \bar{u} \partial_\mu \bar{u})}{(1 + |u|^2)^4}. \tag{3}
\]

Moreover, \(\lambda\) is a real coupling constant. Here the topological three component unit vector field \(\vec{n}\) is expressed by a scalar complex \(u\) via stereographic projection

\[
\vec{n} = \frac{1}{1 + |u|^2} \left( u + \bar{u}, -i(u - \bar{u}), |u|^2 - 1 \right). \tag{4}
\]

Due to its nonlinearity and nontrivial topological content the derivation of knotted solutions in the Faddeev-Niemi model is a rather difficult problem. Simultaneously with numerical works, which indicated some knotted configurations as candidates for the pertinent ground states in sectors with fixed topological charge \[2\]-\[7\], some analytical investigations have been performed. One analytical approach to the FN model was mainly based on the application of the concept of integrability to higher dimensions \[8\]. In practice it means that some additional constraints for the complex field are introduced. As a consequence, one deals with subsectors of the full theory, which appear to be integrable, in the sense that infinitely many conserved currents exist \[9\]. Although the constraints, known as integrability conditions, provide very interesting knotted and linked structures with arbitrary Hopf index \[10\], \[11\], these submodels are still quite complicated and no exact solutions were found up to now.

On the other hand, recently Hirayama and Shi proposed an alternatively way of solving the FN model. It may be summarize as follows.

1) The static equation of motion can be written in the form

\[
\nabla \cdot \vec{\alpha} + \vec{\beta} \cdot \vec{\alpha} = 0, \tag{5}
\]

where the vectors \(\vec{\alpha}, \vec{\beta}\) depend on the fields \(u, \bar{u}\) and their derivatives. Notice, that \(\vec{\alpha}\) is not unique. Rather, it can be chosen in many different ways. But when its particular form is fixed the vector \(\vec{\beta}\) is defined uniquely (For some models there still remains some freedom for \(\vec{\beta}\). This happens, e.g., for the Aratyn-Ferreira-Zimerman model, see Section 3 for a discussion).

2) Now, the idea is to find vectors \(\vec{\alpha}_s\) which identically satisfy

\[
\nabla \cdot \vec{\alpha}_s + \vec{\beta} \cdot \vec{\alpha}_s = 0, \tag{6}
\]

with a fixed \(\vec{\beta}\). On this stage some arbitrary, 'external' complex fields are introduced on which the \(\vec{\alpha}_s\) depend.

Solutions are found using vector identities obeyed by gradients in three dimensional space. Therefore, one should remember that this procedure works only in three dimensions.

3) Finally, one has to compare a superposition of the obtained \(\vec{\alpha}_s\)'s with the original \(\vec{\alpha}\)

\[
\vec{\alpha} = \sum_{s=1..n} A_s \vec{\alpha}_s. \tag{7}
\]
This equation is a new equation for the complex field which by construction satisfies the original equation of motion. Indeed, this approach allows to integrate the equations of motion to first order partial differential equations. More precisely, this happens provided that the simplest non trivial $\vec{\beta}$ is chosen. By taking more complicated $\vec{\beta}$, instead, one arrives at different equations which, in principle, can be of any order. Let us also emphasize that, although all $\vec{\alpha}$ obeying (7) fulfill the original field equations, the implication in the opposite direction does not have to be true. Therefore, this procedure defines a submodel of the FN theory. However, no finite energy solutions to this submodel have been found. (Unfortunately, the final results in [12] are not correct due to an error in the derivation, see [13] for a detailed discussion.) It is not even known whether this submodel allows for any topologically nontrivial solutions at all.

In our work we apply this framework to two related models where exact Hopf solitons are known, namely, the Nicole [14], [15], and the Aratyn-Ferreira-Zimerman model [16], [17]. We show that they can be integrated to first order equations. Further, we show that all analytically known hopfions belong, in fact, to these submodels.

2. Nicole model

2.1. General solutions

The Nicole model is a scale invariant model built of the kinetic part $L_2$ taken to a fractional power

$$L_{N_i} = (L_2)^{\frac{2}{3}}. \quad (8)$$

The equations of motion are

$$\nabla \cdot \left[ \frac{(\nabla u \nabla \bar{u})^{\frac{2}{3}}}{(1 + |u|^2)^3} \nabla \bar{u} \right] + \frac{2\bar{u}}{(1 + |u|^2)^4} (\nabla u \nabla \bar{u})^{\frac{2}{3}} = 0 \quad (9)$$

and its complex conjugate. Now, let us introduce a function $g$ which is assumed to depend on $|u|^2$ only. In principle one may consider a more general case when $g$ is a function of $u$ and $\bar{u}$ independently, or allow for dependence on derivatives. This last possibility leads to higher than first order partial differential equation for the complex field. In this paper we restrict ourselves to the first case.

Now, the last expression can be rewritten as

$$\nabla \cdot \left[ g(\nabla u \nabla \bar{u})^{\frac{2}{3}} \nabla \bar{u} \right] + g(\nabla u \nabla \bar{u})^{\frac{2}{3}} \nabla \bar{u} \cdot \left[ \nabla u \left( -\frac{g'\bar{u}}{g} - \frac{\bar{u}}{1 + |u|^2} \right) + \nabla \bar{u} \left( -\frac{g'u}{g} - \frac{3u}{1 + |u|^2} \right) \right] = 0. \quad (10)$$

Here the prime denotes differentiation with respect to $|u|^2$. In other words, we expressed the equation of motion in the required form

$$\nabla \cdot \vec{\alpha} + \vec{\beta} \cdot \vec{\alpha} = 0$$

with the following vectors

$$\vec{\alpha} = g(\nabla u \nabla \bar{u})^{\frac{2}{3}} \nabla \bar{u}, \quad (11)$$

$$\vec{\beta} = -\left( \frac{g'\bar{u}}{g} + \frac{\bar{u}}{1 + |u|^2} \right) \nabla u - \left( \frac{g'u}{g} + \frac{3u}{1 + |u|^2} \right) \nabla \bar{u}. \quad (12)$$

After the polar decomposition

$$u = Re^{i\Phi} \quad (13)$$

we get

$$\vec{\beta} = -2 \left( \frac{g'}{g} + \frac{2}{1 + R^2} \right) R \nabla R + \frac{2iR^2}{1 + R^2} \nabla \Phi. \quad (14)$$

3
Here, the prime stands for differentiation with respect to $R^2$. Now, we solve equation (5) by introducing 'external' complex fields $\mu$ and $\rho$, analogous to those of Ref. [12]. There are two solutions

\[ \vec{\alpha}_1 = \nabla \Phi \times \nabla \mu + i \left( \frac{g'}{g} + \frac{2}{1 + R^2} \right) \frac{1 + R^2}{R} \nabla R \times \nabla \mu \] (15)

\[ \vec{\alpha}_2 = K(\Phi) g(1 + R^2)^2 \nabla \Phi \times \nabla \rho + \frac{\hat{K}(R)}{1 + R^2} e^{-2i\frac{R^2}{1 + R^2}} \nabla R \times \nabla \rho. \] (16)

$K$ and $\hat{K}$ are arbitrary complex functions of $\Phi$ and $R$, respectively. There is also a third solution perpendicular to the gradients of $\Phi$ and $R$ which does not depend on those external complex functions

\[ \vec{\alpha}_3 = G(R, \Phi) \nabla \Phi \times \nabla R. \] (17)

Interestingly, $\vec{\alpha}_1$ and $\vec{\alpha}_3$ even solve the stronger equation

\[ \nabla \cdot \vec{\alpha} = 0, \quad \vec{\beta} \cdot \vec{\alpha} = 0. \] (18)

Moreover, one can observe that the second solution can be trivially generalized to

\[ \vec{\alpha}_2 = K(\Phi, \rho) g(1 + R^2)^2 \nabla \Phi \times \nabla \rho + \frac{\hat{K}(R, \rho)}{1 + R^2} e^{-2i\frac{R^2}{1 + R^2}} \nabla R \times \nabla \rho. \] (19)

To summarize this subsection, we have derived the following first order partial differential equation whose solutions identically solve the original second order equations of motion

\[ g \sqrt{\nabla (\nabla R)^2 + R^2 (\nabla \Phi)^2} \nabla R - i R \nabla \Phi | e^{- i \Phi} = A_1 \left( \nabla \Phi \times \nabla \mu + i \left( \frac{g'}{g} + \frac{2}{1 + R^2} \right) \frac{1 + R^2}{R} \nabla R \times \nabla \mu \right) + \] (20)

\[ A_2 \left( K(\Phi, \rho) g(1 + R^2)^2 \nabla \Phi \times \nabla \rho + \frac{\hat{K}(R, \rho)}{1 + R^2} e^{-2i\frac{R^2}{1 + R^2}} \nabla R \times \nabla \rho \right) + \] (21)

\[ A_3 G(R, \Phi) \nabla \Phi \times \nabla R, \] (22)

where $A_1, A_2, A_3$ are complex constants.

The standard Hirayama and Shi type solution can be rederived if we assume $g = (1 + |u|^2)^{-2}$ and $K = \hat{K} = 1, G = 0$. Then

\[ \vec{\alpha}_1 = \nabla \Phi \times \nabla \mu \] (24)

\[ \vec{\alpha}_2 = \nabla \Phi \times \nabla \rho + e^{-2i\frac{R^2}{1 + R^2}} \nabla R \times \nabla \rho. \] (25)

Thus the pertinent vector is

\[ \vec{\alpha}_H = \vec{\alpha}_1(\mu) + \vec{\alpha}_2(\rho) - \vec{\alpha}_1(\rho) = \nabla \Phi \times \nabla \mu + e^{-2i\frac{R^2}{1 + R^2}} \nabla R \times \nabla \rho. \] (26)

2.2. $Q = 1$ hopfion

In this part of the paper we want to discuss the problem whether the well-known soliton solution with topological charge one does or does not belong to the solutions of the first order equation (21)-(23). As we will see it is sufficient to consider the simplest case (20).

For the discussion of the unit charge Hopf soliton it is useful to introduce toroidal coordinates,

\[ x = q^{-1} \sin \eta \cos \varphi, \quad y = q^{-1} \sin \eta \sin \varphi \]

\[ z = q^{-1} \sin \xi; \quad q = \cosh \eta - \cos \xi. \] (27)
Then the Hopf soliton takes the form
\[ u = \frac{1}{\sinh \eta} e^{i(\varphi + \xi)}. \]

Thus
\[ R = \frac{1}{\sinh \eta}, \quad \Phi = \xi + \varphi. \]

The condition \( \vec{\alpha} = \vec{\alpha}_H \) calculated for the hopfion is equivalent to the following three first order partial but linear differential equations
\[ -\sqrt{2} \frac{1}{\cosh^2 \eta} e^{-i(\varphi + \xi)} = \frac{1}{\sinh \eta} (\mu - \mu_\xi), \]
\[ -i \sqrt{2} \frac{\sinh \eta}{\cosh^3 \eta} e^{-i(\varphi + \xi)} = \frac{1}{\sinh \eta} \mu_\eta + \frac{\cosh \eta}{\sinh^3 \eta} e^{-\frac{2i}{\cos^2 \eta}(\xi + \varphi)} \rho_\varphi, \]
\[ -i \sqrt{2} \frac{1}{\cosh^3 \eta} e^{-i(\varphi + \xi)} = -\mu_\eta - \frac{\cosh \eta}{\sinh^2 \eta} e^{-\frac{2i}{\cos^2 \eta}(\xi + \varphi)} \rho_\xi. \]

The first equation (30) has the following solution
\[ \mu = \tilde{\mu}(\eta, \varphi + \xi) - \frac{\sqrt{2}}{\cosh^2 \eta} e^{-i(\varphi + \xi)} \varphi, \]
where \( \tilde{\mu} \) is an arbitrary function of \( \eta \) and \( \varphi + \xi \). Moreover, if we multiply (31) by \( \sinh \eta \) and add it to (32), then we obtain an equation for \( \rho \). Namely,
\[ -i \sqrt{2} \frac{\sinh^2 \eta}{\cosh^2 \eta} e^{-\frac{\sinh^2 \eta}{\cosh^2 \eta} \eta}(\xi + \varphi) = \rho_\varphi - \rho_\xi. \]

The corresponding solution reads
\[ \rho = \tilde{\rho}(\eta, \xi + \varphi) - i \sqrt{2} \frac{\sinh^2 \eta}{\cosh^2 \eta} e^{-\frac{\sinh^2 \eta}{\cosh^2 \eta} \eta}(\xi + \varphi) \varphi. \]

The last remaining step is to insert these solutions, e.g., into the third equation (32). One finds that
\[ -i \sqrt{2} \frac{1}{\cosh \eta} e^{-i(\xi + \varphi)} + \tilde{\mu}_\eta + \frac{\cosh \eta}{\sinh^2 \eta} e^{-\frac{\sinh^2 \eta}{\cosh^2 \eta} (\xi + \varphi)} \tilde{\rho}_\xi = \sqrt{2} e^{-i(\xi + \varphi)} \varphi \times \]
\[ \left[ \left( \frac{\sinh \eta}{\cosh^2 \eta} \right)' + \frac{\cosh \eta}{\sinh^2 \eta} \frac{\sinh^2 \eta}{\cosh^2 \eta} \sinh \eta \right]. \]

However, the right hand side of this equation is equal to zero, because the expression in brackets vanishes identically. In other words, the unit hopfion does belong to the Hirayama and Shi subsector, i.e., the hopfion is a solution of the first order PDE with \( \mu, \rho \) given by (33) and (35) respectively, where \( \tilde{\mu} \) and \( \tilde{\rho} \) obey
\[ -i \sqrt{2} \frac{1}{\cosh \eta} e^{-i(\xi + \varphi)} + \tilde{\mu}_\eta + \frac{\cosh \eta}{\sinh^2 \eta} e^{-\frac{\sinh^2 \eta}{\cosh^2 \eta} (\xi + \varphi)} \tilde{\rho}_\xi = 0. \]

3. AFZ model

Let us now consider another scale invariant model, namely the so-called Aratyn-Ferreira-Zimerman model presented in the introduction.
3.1. General solutions

With $L_4$ given by eq. (3), the Lagrangian reads

$$L_{AFZ} = -(L_4)^\frac{3}{4}.$$  (39)

The equations of motion are

$$\nabla \left[ \left( \hat{K} \nabla u \right)^{-\frac{1}{4}} \hat{K} \right] + \left( \hat{K} \nabla u \right)^{-\frac{3}{4}} \frac{2\bar{u}}{(1 + |u|^2)^{\frac{3}{4}}} = 0$$  (40)

and its complex conjugate. Here

$$\hat{K} = (\nabla \bar{u})^2 \nabla u - (\nabla u \nabla \bar{u}) \nabla \bar{u}.$$  (41)

Introducing an arbitrary function of the modulus, $g = g(|u|^2)$, the field equation may be rewritten as follows

$$\nabla [g(\hat{K} \nabla u)^{-\frac{1}{4}} \hat{K}] + g(\hat{K} \nabla u)^{-\frac{3}{4}} \bar{u} \left( -\frac{g'}{g} - \frac{1}{1 + |u|^2} \right) \hat{K} \cdot \nabla u = 0.$$  (42)

Thus, again we arrive at

$$\nabla \cdot \vec{\alpha} + \vec{\beta} \cdot \vec{\alpha} = 0,$$  (43)

where

$$\vec{\alpha} = g(\hat{K} \nabla u)^{-\frac{1}{4}} \hat{K}$$  (44)

and

$$\vec{\beta} = -\bar{u} \left( \frac{g'}{g} + \frac{1}{1 + |u|^2} \right) \nabla u.$$  (45)

Observe that $\hat{K}$ satisfies the identity

$$\hat{K} \cdot \nabla \bar{u} \equiv 0.$$  (46)

Thus, one can always include in $\vec{\beta}$ a part which is proportional to the gradient of $u$. That is, the most general $\vec{\beta}$ reads

$$\vec{\beta} = - \left( \frac{g'}{g} + \frac{1}{1 + |u|^2} \right) [\bar{u} \nabla u + h \nabla \bar{u}],$$  (47)

where $h$ is any function depending on arbitrary variables. However, for simplicity, in the subsequent analysis we put $h = -1$. Therefore, after the polar decomposition

$$\vec{\beta} = -2iR^2 \left( \frac{g'}{g} + \frac{1}{1 + R^2} \right) \nabla \Phi.$$  (48)

Similarly as for the Nicole model, we solve equation (43) for fixed $\vec{\beta}$. There are three solutions

$$\vec{\alpha}_1 = \nabla \Phi \times \nabla \mu,$$  (49)

$$\vec{\alpha}_2 = e^{2iR^2 \left( \frac{\Phi}{\Phi} + \frac{1}{1 + R^2} \right)} \Phi \nabla R \times \nabla \rho,$$  (50)

and

$$\vec{\alpha}_3 = G(\Phi, R) \nabla \Phi \times \nabla R.$$  (51)
Choosing a particular form for the $g$ function

$$g = \frac{1}{(1 + |u|^2)^3}$$

we find the Hirayama and Shi type solution

$$\vec{\alpha}_H = \nabla \Phi \times \nabla \mu + e^{\frac{-4i \mu^2}{\mu + \rho}} \nabla R \times \nabla \rho.$$  

(53)

In the next subsection we prove that the Hopf solitons can be indeed expressed in this form.

### 3.2. Hopfions

It is widely known that the AFZ model possesses infinitely many analytically known finite energy toroidal Hopf solitons which, in toroidal coordinates, read

$$u = f e^{i(m\xi + n\varphi)}.$$  

(54)

Here

$$f^2 = \frac{\cosh \eta - \sqrt{n^2/m^2 + \sinh^2 \eta}}{\sqrt{1 + m^2/n^2 \sinh^2 \eta - \cosh \eta}},$$

(55)

whereas $m, n$ are integer constants. The corresponding topological charge is $Q = -nm$. Hence, in our parametrization

$$R = f, \quad \Phi = m \xi + n \varphi.$$  

(56)

Now, we establish that

$$\vec{\alpha} = \vec{\alpha}_H$$

or in other words, that these hopfions belong to Hirayama and Shi submodel.

The last formula leads to three first order equations. Namely,

$$-i\sqrt{2}(f' f)^{\frac{1}{2}} \frac{f}{(1 + f^2)^{\frac{3}{2}}} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{\frac{1}{2}} e^{-i(m\xi + n\varphi)} = \frac{1}{\sinh \eta} (m \mu_\varphi - n \mu_\xi)$$

(58)

$$i\sqrt{2}m(f' f)^{\frac{1}{2}} \frac{f'}{(1 + f^2)^{\frac{3}{2}}} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{-\frac{1}{2}} e^{-i(m\xi + n\varphi)} = \frac{n}{\sinh \eta} \mu_\eta - \frac{f'}{\sinh \eta} \rho_\varphi e^{-\frac{4i f^2}{1 + f^2}(m\xi + n\varphi)},$$

(59)

$$i\sqrt{2}n(f' f)^{\frac{1}{2}} \frac{f'}{(1 + f^2)^{\frac{3}{2}} \sinh \eta} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{-\frac{1}{2}} e^{-i(m\xi + n\varphi)} = -m \mu_\eta + f' \rho_\xi e^{-\frac{4i f^2}{1 + f^2}(m\xi + n\varphi)}.$$  

(60)

The first equation may be integrated and gives

$$\mu = \tilde{\mu}(\eta, n\varphi + m\xi) - \sqrt{2}(f' f)^{\frac{1}{2}} \frac{f}{(1 + f^2)^{\frac{3}{2}}} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{\frac{1}{2}} e^{-i(m\xi + n\varphi)} \frac{\varphi}{m}.$$  

(61)

Moreover, we multiply (59) by $m \cosh \eta$ and add it to (61). Then we get

$$-i\sqrt{2}(f' f)^{\frac{1}{2}} \frac{f'}{(1 + f^2)^{\frac{3}{2}}} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{\frac{1}{2}} \sinh \eta e^{-i(m\xi + n\varphi)} = (m \rho_\varphi - n \rho_\xi) e^{-\frac{4i f^2}{1 + f^2}(m\xi + n\varphi)},$$

(62)
which has the following solution

\[
\rho = \tilde{\rho}(\eta, m\xi + n\varphi) - i\sqrt{2}(f'f)^{1/4} \frac{f'}{1 + f^2} \frac{n^2}{\sinh^2 \eta} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{\frac{3}{4}} e^{-i(1 + \frac{m^2}{1 + f^2})(m\xi + n\varphi) \varphi}. \tag{63}
\]

Finally, inserting the obtained expressions into the third equation we arrive at

\[
\frac{i\sqrt{2}n(f'f)^{1/4} f'}{(1 + f^2)^3 \sinh \eta} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{-\frac{4}{3}} e^{-i(m\xi + n\varphi)} + m\tilde{\mu} - f' \tilde{\rho} e^{-4i\frac{m^2}{1 + f^2}(m\xi + n\varphi)} = \sqrt{2} e^{-i(m\xi + n\varphi) \varphi} \times \tag{64}
\]

\[
\left[ \left( \frac{f'f}{(1 + f^2)^3} \frac{n^2}{\sinh^2 \eta} \right)^{\frac{4}{3}} - \frac{(f'f)^{1/4} f'}{(1 + f^2)^3} \frac{n^2}{\sinh^2 \eta} \left( m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{\frac{4}{3}} \left( 1 - \frac{4f^2}{1 + f^2} \right) \right]. \tag{65}
\]

However, the right hand side of this last formula is zero if the expression for \( f \) is taken into account. To conclude, we have demonstrated that all hopfion solutions are solutions of the first order PDE defined by \( \tilde{\sigma}_H \).

4. Conclusions

The main achievement of this paper is the demonstration that all known exact Hopf solitons in the Nicole as well as the AFZ models may be expressed by solutions of the first order equations of Hirayama and Shi with \( \mu \) and \( \rho \) defined in the previous sections. As a consequence, this method of the reduction of the equations of motion appears to be applicable to models with knotted solitons, because topologically nontrivial solutions of the theory are taken into account by the Ansatz. Therefore, this framework probably will incorporate knotted configurations for the Faddeev-Niemi model, as well, and may be useful in the search for solutions or in further analytical investigations of that model.

Let us further remark that the method can be easily generalized to models with a potential. For example, if one adds a potential term \( 24V(uu^*) \) to the Nicole model, then this just modifies \( \tilde{\beta} \) to

\[
\tilde{\beta} = \tilde{\beta}_{old} - \left( 1 + |u|^2 \right)^3 V' \frac{u^* \nabla u}{(\nabla u^* \nabla u^*)^{3/2}}.
\]

This modification is quite simple, but since for realistic applications of the FN model a potential term is needed, this makes the procedure all the more interesting. Now, \( \tilde{\beta} \) depends on derivatives, but in the vectorial sense it is still proportional to \( \nabla u \) and \( \nabla u^* \).

Further generalizations are provided by allowing for more complicated \( g \). In principle, one may consider a case where \( g \) is a function not only of the modulus but depends on \( u, \bar{u} \) in an arbitrary way. This possibility might lead to configurations without toroidal symmetry which are quite typical for the FN model like, for instance, trefoil knots. Moreover, one can also allow for a dependence on (higher) derivatives. Then, after constructing solutions of Eq. \( \tilde{\beta} \) and comparing them with the primary \( \tilde{\alpha} \), one gets a more complicated set of partial differential equations. If \( g \) depends on first derivatives, the resulting PDE is of the second order. In general, we are able to find PDE of any order whose solutions identically solve the original theory. This might allow for the construction of a hierarchy of submodels.

Acknowledgements

C.A. and J.S.-G. thank MCyT (Spain) and FEDER (FPA2005-01963), and support from Xunta de Galicia (grant PGIDIT06PXIB296182PR and Conselleria de Educacion). A.W. is partially supported from Jagiellonian University (grant WRBW 41/07). Further, C.A. acknowledges support from the Austrian START award project FWF-Y-137-TEC and from the FWF project P161 05 NO 5 of N.J. Mauser.

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