Symplectic Polar Duality, Quantum Blobs, and Generalized Gaussians

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Contents

1 Introduction 2

2 A Geometric Quantum Phase Space 3
   2.1 Polar duality and quantum states ..................... 3
   2.2 Symplectic polar duality ............................. 6
   2.3 Polar duality and the symplectic camel .............. 10

3 Projections and Intersections of Quantum Blobs 12
   3.1 Block matrix notation ............................... 12
   3.2 Reconstruction of quantum blobs: discussion ......... 13
   3.3 Intersections with Lagrangian planes ................ 15

4 Gaussian Quantum Phase Space 17
   4.1 Generalized Gaussians and their Wigner transforms ... 17
   4.2 Gaussian density operators ........................... 19
   4.3 A characterization of Gaussian density operators ...... 21

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Abstract

We apply the notion of polar duality from convex geometry to the study of quantum covariance ellipsoids in symplectic phase space. We consider in particular the case of “quantum blobs” introduced in previous work; quantum blobs are the smallest symplectic invariant regions of the phase space compatible with the uncertainty principle in its strong Robertson–Schrödinger form. We show that these phase space units can be characterized by a simple condition of reflexivity using polar duality, thus improving previous results. We apply these geometric constructions to the characterization of pure Gaussian states in terms of partial information on the covariance ellipsoid, which allows us to formulate statements related to symplectic tomography.

Keywords: polar duality; Lagrangian plane; symplectic capacity; John ellipsoid; uncertainty principle

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1 Introduction

In a recent paper [16] we discussed the usefulness of the geometric notion of polar duality in expressing the uncertainty principle of quantum mechanics. We suggested that a quantum system localized in the position representation in a set $X$ cannot be localized in the momentum representation in a set smaller than its polar dual $X^\ast$, the latter being defined as the set of all $p$ in momentum space such that $px \leq \hbar$ for all $x \in X$. In the present work we go several steps further by studying the product sets $X \times X^\ast$. The first observation is that when $X$ is an ellipsoid, then the John ellipsoid of $X \times X^\ast$ is a “quantum blob”, to which one canonically associates a squeezed coherent state. This leads us to study more general phase space ellipsoids $\Omega$ viewed as covariance ellipsoids of a quantum state, and we find that the usual quantum condition for such ellipsoids can be restated in a simple way using polar duality between intersections with coordinate planes and orthogonal projection. Thus, we arrive at a purely geometric characterization of quantization.

The main results of this paper are:

- In Theorem 6 we use the notion of “symplectic polar duality” to characterize those phase space ellipsoids who arise as covariance ellipsoids
$\Omega$ of a quantum state. This result is very much related to what is called in quantum physics “symplectic tomography” \cite{23} since it gives global information by studying the local information obtained by considering the intersection of $\Omega$ with a Lagrangian plane;

- **Theorem 13** we prove that a centered phase space ellipsoid $\Omega$ is a quantum blob (i.e. a symplectic ball with radius $\sqrt{\hbar}$ \cite{11,13,18}) if and only if the polar dual of the projection of $\Omega$ on the position space is the intersection of $\Omega$ with the momentum space; this considerably strengthens a previous result obtained in \cite{16};

- **Theorem 16**; it is an analytical version of Theorem 13 which we use to give a simple characterization of pure Gaussian states in terms of partial information on the covariance ellipsoid of a Gaussian state. This result is related to the so-called “Pauli problem”.

**Notation 1** The configuration space of a system with $n$ degrees of freedom will in general be written $\mathbb{R}_x^n$, and its dual (the momentum space) $\mathbb{R}_p^n$. The position variables will be written $x = (x_1, \ldots, x_n)$ and the momentum variables $p = (p_1, \ldots, p_n)$. The duality form (identified with the usual inner product) is $p \cdot x = p_1x_1 + \cdots + p_nx_n$. The product $\mathbb{R}_x^n \times \mathbb{R}_p^n$ is identified with $\mathbb{R}^{2n}$ and is equipped with the standard symplectic form $\sigma$ defined by $\omega(z, z') = p \cdot x' - p' \cdot x$ if $z = (x, p)$, $z' = (x', p')$. The corresponding symplectic group is denoted $\text{Sp}(n)$. $S \in \text{Sp}(n)$ if and only $\omega(Sz, Sz') = \omega(z, z')$ for all $z, z'$. We denote by $\text{Sym}_{++}(n, \mathbb{R})$ the cone of real positive definite symmetric $n \times n$ matrices, and by $\text{GL}(n, \mathbb{R})$ the general (real) linear group (the invertible real $n \times n$ matrices).

## 2 A Geometric Quantum Phase Space

### 2.1 Polar duality and quantum states

Let $X \subset \mathbb{R}_x^n$ be a convex body: $X$ is compact and convex and has non-empty interior $\text{int}(X)$. If $0 \in \text{int}(X)$ we define the $\hbar$-polar dual $X^\hbar \subset \mathbb{R}_p^n$ of $X$ by

$$X^\hbar = \{ p \in \mathbb{R}_p^n : \sup_{x \in X} \langle p, x \rangle \leq \hbar \}$$

where $\hbar$ is a positive constant (we have $X^\hbar = \hbar X^o$ where $X^o$ is the traditional polar dual dual from convex geometry). The following properties of polar duality are obvious \cite{27}:
• \((X^h)^h = X\) (reflexivity) and \(X \subset Y \implies Y^h \subset X^h\) (anti-monotonicity),

• For all \(L \in GL(n, \mathbb{R})\):

\[
(LX)^h = (L^T)^{-1}X^h
\]

(scaling property). In particular \((\lambda X)^h = \lambda^{-1}X^h\) for all \(\lambda \in \mathbb{R}, \lambda \neq 0\).

We can view \(X\) and \(X^h\) as subsets of phase space by the identifications \(\mathbb{R}_x^n \equiv \mathbb{R}_x^n \times 0\) and \(\mathbb{R}_p^n \equiv 0 \times \mathbb{R}_p^n\). Writing \(\ell_X = \mathbb{R}_x^n \times 0\) and \(\ell_P = 0 \times \mathbb{R}_p^n\) the transformation \(X \rightarrow X^h\) is a mapping \(\ell_X \rightarrow \ell_P\). With this interpretation formula (2) can be rewritten in symplectic form as

\[
(M_{L^{-1}}X)^h = M_L^T X^h
\]

where \(M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}\) is in \(\text{Sp}(n)\). Notice that \(M_L : \ell_X \rightarrow \ell_X\) and \(M_L : \ell_P \rightarrow \ell_P\).

Suppose now that \(X\) is an ellipsoid centered at the origin:

\[
X = \{x \in \mathbb{R}_x^n : Ax \cdot x \leq \hbar\}
\]

where \(A \in \text{Sym}_+(n, \mathbb{R})\). The polar dual \(X^h\) is the ellipsoid

\[
X^h = \{p \in \mathbb{R}_p^n : A^{-1}p \cdot p \leq \hbar\}.
\]

In particular the polar dual of the ball \(B_x^n(\sqrt{\hbar}) = \{x : |x| \leq \sqrt{\hbar}\}\) is \((B_x^n(\sqrt{\hbar}))^h = B_p^n(\sqrt{\hbar})\).

Let \(\Omega\) be a convex body in \(\mathbb{R}^{2n}\). Recall [4] that the John ellipsoid \(\Omega_{\text{John}}\) is the unique ellipsoid in \(\mathbb{R}^{2n}\) with maximum volume contained in \(\Omega\). If \(M \in GL(2n, \mathbb{R})\) then

\[
(M(\Omega))_{\text{John}} = M(\Omega_{\text{John}}).
\]

In previous work [13, 18] we called the image of the phase space ball \(B_x^{2n}(\sqrt{\hbar})\) by some \(S \in \text{Sp}(n)\) a “quantum blob”. Quantum blobs are minimum quantum uncertainty phase space units, and can be used to restate the uncertainty principle of quantum mechanics in a symplectically invariant form [12]. The product \(X \times X^h\) contains a unique quantum blob:
Proposition 2 Let \( X = \{ x : Ax \cdot x \leq \hbar \} \). The John ellipsoid of the quantum state \( X \times X^h \) is a a quantum blob, namely

\[
(X \times X^h)_{\text{John}} = M_{A^{1/2}}(B^{2n}(\sqrt{\hbar}))
\]

where \( M_{A^{1/2}} = \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \in \text{Sp}(n). \)

**Proof.** That \( S_{A^{1/2}} \in \text{Sp}(n) \) is clear. Let \( B^n_X(\sqrt{\hbar}) \) and \( B^n_P(\sqrt{\hbar}) \) be the balls with radius \( \sqrt{\hbar} \) in \( \mathbb{R}_x^n \) and \( \mathbb{R}_p^n \), respectively. We have, by (4), (5), and (6),

\[
(X \times X^h)_{\text{John}} = (A^{-1/2}B^n_X(\sqrt{\hbar}) \times A^{1/2}B^n_P(\sqrt{\hbar}))_{\text{John}}
\]

Let us show that

\[
(B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar}))_{\text{John}} = B^{2n}(\sqrt{\hbar});
\]

this will prove our assertion. The inclusion \( B^{2n}(\sqrt{\hbar}) \subset B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar}) \) is obvious, and we cannot have \( B^{2n}(R) \subset B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar}) \) if \( R > 1 \). Assume now that the John ellipsoid \( \Omega_{\text{John}} \) of \( \Omega = B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar}) \) is defined by

\[
Ax \cdot x + Bx \cdot p + Cp \cdot p \leq \hbar
\]

where \( A, C \in \text{Sym}_{++}(n, \mathbb{R}) \) and \( B \) are real \( n \times n \) matrices. Since \( \Omega \) is invariant by the transformation \( (x, p) \mapsto (x, p) \) so is \( \Omega_{\text{John}} \) and we must thus have \( A = C \) and \( B = B^T \). Similarly, \( \Omega \) being invariant by the partial reflection \( (x, p) \mapsto (-x, p) \) we get \( B = 0 \) so \( \Omega_{\text{John}} \) is defined by \( Ax \cdot x + Ap \cdot p \leq \hbar \).

The last step is to observe that \( \Omega \), and hence \( \Omega_{\text{John}} \), are invariant under all symplectic rotations \( (x, p) \mapsto (Hx, HP) \) where \( H \in O(n, \mathbb{R}) \) so we must have \( AH = HA \) for all \( H \in O(n, \mathbb{R}) \), but this is only possible if \( A = \lambda I_{n \times n} \) for some \( \lambda \in \mathbb{R} \). The John ellipsoid of \( \Omega \) is thus of the type \( B^{2n}(\sqrt{\hbar}/\lambda) \) for some \( \lambda \geq 1 \) and this concludes the proof in view of the inclusion \( B^{2n}(\sqrt{\hbar}) \subset B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar}) \) since we cannot have \( \lambda > 1 \). \( \blacksquare \)

**Remark 3** The John ellipsoid \( (X \times X^h)_{\text{John}} \) is the set of all \( (x, p) \in \mathbb{R}_x^{2n} \) such that \( Ax \cdot x + A^{-1}p \cdot p \leq \hbar \). The orthogonal projections of \( (X \times X^h)_{\text{John}} \) on the coordinate planes \( \ell_X = \mathbb{R}_x^n \times 0 \) and \( \ell_P = 0 \times \mathbb{R}_p^n \) are therefore \( \Pi_X(X \times X^h)_{\text{John}} = X \) and \( \Pi_P(X \times X^h)_{\text{John}} = X^h \).
The construction above shows that we have a canonical identification between the ellipsoids \( X = \{ x : Ax \cdot x \leq \hbar \} \) and the squeezed coherent states 
\[
\phi_A(x) = (\pi \hbar)^{-n/4} (\det A)^{1/4} e^{-Ax \cdot x/2\hbar}.
\]

In fact, the covariance ellipsoid \([24, 11]\) of \( \phi_A \) is precisely the John ellipsoid of the product \( X \times X^h \) as can be seen calculating the Wigner transform of \( \phi_A \)
\[
W\phi_A(z) = (\pi \hbar)^{-n} (\det A)^{1/4} \exp \left[ -\frac{1}{\hbar} (Ax \cdot x + A^{-1}p \cdot p) \right]
\]

which corresponds to the canonical bijection
\[
X \mapsto (X \times X^h)_{\text{John}}
\]

between (centered) configuration space ellipsoids \( X \) and John ellipsoids of \( X \times X^h \) (we will have more to say about this correspondence in the forthcoming sections).

### 2.2 Symplectic polar duality

Let \( \Omega \) be a symmetric convex body in the phase space \((\mathbb{R}^{2n}, \omega)\). We define the symplectic polar dual \( \Omega^{h,\omega} \) of \( \Omega \) as the set
\[
\Omega^{h,\omega} = \{ z' \in \mathbb{R}^{2n} : \sup_{z \in \Omega} \omega(z, z') \leq \hbar \}.
\]

It is straightforward to verify that \( \Omega^{h,\omega} \) is related to the ordinary polar dual \( \Omega^{h} \) (calculated by identifying \( \mathbb{R}^{2n} \) with its own dual) by the formula
\[
\Omega^{h,\omega} = (J\Omega)^h = J(\Omega^h).
\]

The properties of symplectic polar duality are easily deduced from those of ordinary polar duality. This notion is particularly interesting because it enjoys a property of “symplectic covariance”:

**Proposition 4** Let \( S \in \text{Sp}(n) \) and \( \Omega \) a symmetric convex body. (i) We have
\[
(S\Omega)^{h,\omega} = S(\Omega^{h,\omega})
\]

(ii) The quantum blobs \( S(B^{2n}(\sqrt{\hbar})) \), \( S \in \text{Sp}(n) \), are the only fixed points of the transformation \( \Omega \mapsto \Omega^{h,\omega} \).
Proof. (i) The condition $S \in \text{Sp}(n)$ is equivalent to $S^T JS = J$ hence $JS = (S^T)^{-1} J$. Now, using the scaling property (2) and the equality (11) we get

$$(S\Omega)^{h,\omega} = J(S(\Omega)^h = J(S^T)^{-1}(\Omega^h)$$

$$= SJ(\Omega^h) = S(\Omega^{h,\omega})$$

which is (12). (ii) In particular, since $B^{2n}(\sqrt{\hbar})^{h} = B^{2n}(\sqrt{\hbar})$ we have

$$(S(B^{2n}(\sqrt{\hbar})))^{h,\omega} = S(B^{2n}(\sqrt{\hbar})).$$

Let us introduce some terminology. Let $M \in \text{Sym}_{++}(2n, \mathbb{R})$ and consider the centered phase space ellipsoid

$$\Omega = \{ z \in \mathbb{R}^{2n} : Mz \cdot z \leq \hbar \}.$$  \hspace{1cm} (14)

Setting $M = \frac{1}{2} \hbar \Sigma^{-1}$ we can visualize $\Omega$ as the covariance matrix of a (classical or quantum) state:

$$\Omega = \{ z \in \mathbb{R}^{2n} : \frac{1}{2} \Sigma^{-1} z \cdot z \leq 1 \}.$$  \hspace{1cm} (15)

We will say that $\Omega$ is quantized if it contains a quantum blob, i.e. if there exists $S \in \text{Sp}(n)$ such that $S(B^{2n}(\sqrt{\hbar})) \subset \Omega$. This condition is equivalent to the uncertainty principle in its strong Robertson–Schrödinger form when $\Sigma$ is viewed as the covariance matrix of a quantum state [12, 11, 18].

Before we proceed to prove the main results we recall the following symplectic diagonalization result (“Williamson diagonalization” [11]). For every $M \in \text{Sym}_{++}(2n, \mathbb{R})$ there exists $S_0 \in \text{Sp}(n)$ such that

$$M = S_0^T DS_0 \quad , \quad D = \begin{pmatrix} \Lambda^\omega & 0_{n \times n} \\ 0_{n \times n} & \Lambda^\omega \end{pmatrix}$$  \hspace{1cm} (16)

where $\Lambda^\omega = \text{diag}(\lambda_1^\omega, ..., \lambda_n^\omega)$; here $\lambda_1^\omega, ..., \lambda_n^\omega$ the symplectic eigenvalues of $M$ (i.e. the moduli of the usual eigenvalues of the matrix $JM$; they are the same as those of the antisymmetric matrix $M^{1/2}JM^{1/2}$ and hence of the type $\pm i \lambda$, $\lambda > 0$).

Proposition 5 Let $\Omega$ be a non-degenerate phase space ellipsoid. (i) $\Omega$ is quantized if and only if $\Omega^{h,\omega} \subset \Omega$ (i.e. if and only if $\Omega$ contains a quantum blob $S(B^{2n}(\sqrt{\hbar}))$). (ii) The equality $\Omega^{h,\omega} = \Omega$ holds if and only if there exists $S \in \text{Sp}(n)$ such that $\Omega = S(B^{2n}(\sqrt{\hbar}))(i.e. \text{if and only if } \Omega \text{ is a quantum blob}$. 

7
Proof. (i) Suppose that there exists $S \in \text{Sp}(n)$ such that $Q = S(B^{2n}(\sqrt{\hbar})) \subset \Omega$. By the anti-monotonicity of (symplectic) polar duality this implies that we have $\Omega^{h,\omega} \subset Q^{h,\omega} = Q \subset \Omega$, which proves the necessity of the condition. Suppose conversely that we have $\Omega^{h,\omega} \subset \Omega$. Then

$$\Omega^{h,\omega} = \{ z \in \mathbb{R}^{2n} : (-JM)z \cdot z \leq \hbar \}$$

(17)

hence the inclusion $\Omega^{h,\omega} \subset \Omega$ implies that $M \leq (-JM)$ ($\leq$ stands here for the L"owner ordering). Performing a symplectic diagonalization (16) of $M$ and using the relations $JS^{-1} = STJ$, $(ST)^{-1}J = JS$ this is equivalent to

$$M = S^TD^S \leq S^T(-JD^{-1}J)S$$

that is to $D \leq -JD^{-1}J$. In the notation in (16) this implies that we have $\Lambda^\omega \leq (\Lambda^\omega)^{-1}$ and hence $\lambda^\omega_j \leq 1$ for $1 \leq j \leq n$; thus $D \leq I$ and $M = S^TD^S \leq ST^S$. The inclusion $S(B^{2n}(\sqrt{\hbar})) \subset \Omega$ follows. (ii) The condition is sufficient since $S(B^{2n}(\sqrt{\hbar}))^{h,\omega} = S(B^{2n}(\sqrt{\hbar}))$. Assume conversely that $\Omega^{h,\omega} = \Omega$. Then there exists $S \in \text{Sp}(n)$ such that $Q = S(B^{2n}(\sqrt{\hbar})) \subset \Omega$. It follows that $\Omega^{h,\omega} \subset Q^{h,\omega} = Q$ hence $\Omega^{h,\omega} = \Omega \subset Q$ so we must have $\Omega = Q$.

We are going to prove a stronger statement, which can be seen as a “tomographic” result since it involves the intersection of the covariance ellipsoid with a subspace. Recall [11] that a subspace $\ell$ of the symplectic space $(\mathbb{R}^{2n},\omega)$ is a Lagrangian plane if $\text{dim} \ell = n$ and $\omega(z,z') = 0$ for all $(z,z') \in \ell \times \ell$. The coordinate spaces $\ell_X = \mathbb{R}^n_\times 0$ and $\ell_P = 0 \times \mathbb{R}^n_p$ are trivially Lagrangian planes. The set of all Lagrangian planes is denoted by $\text{Lag}(n)$ and is called the Lagrangian Grassmannian; it can be equipped with a topology making it diffeomorphic to the homogeneous space $U(n,\mathbb{C})/O(n,\mathbb{R})$. The symplectic group $\text{Sp}(n)$ acts transitively on $\text{Lag}(n)$; in particular for every $S \in \text{Sp}(n)$ the subspaces $\ell = S\ell_X$ and $\ell = S\ell_P$ are Lagrangian planes.

**Theorem 6** (i) The ellipsoid $\Omega$ contains a quantum blob $Q = S(B^{2n}(\sqrt{\hbar}))$ ($S \in \text{Sp}(n)$) if and only if there exists $\ell \in \text{Lag}(n)$ such that

$$\Omega^{h,\omega} \cap \ell \subset \Omega \cap \ell$$

(18)

in which case we have $\Omega^{h,\omega} \cap \ell \subset \Omega \cap \ell$ for all $\ell \in \text{Lag}(n)$. (ii) The equality $\Omega^{h,\omega} \cap \ell = \Omega \cap \ell$ holds if and only if $\Omega$ is a quantum blob.
Proof. (i) The necessity of the condition (18) is trivial (Proposition 5). Let us prove that the condition is sufficient. Setting $M = \frac{\hbar}{2} \Sigma^{-1}$ and

$$
\Omega_\Sigma = \{ z : Mz \cdot z \leq \hbar \} = \{ z : \frac{1}{2} \Sigma^{-1}z \cdot z \leq \hbar \}
$$

we have

$$
\Omega^{h,\omega}_\Sigma = \{ z \in \mathbb{R}^{2n} : (-JM^{-1}J)z \cdot z \leq \hbar \}. \quad (19)
$$

We now perform a symplectic diagonalization (16) of $M$, this leads to

$$
\Omega_\Sigma = S^{-1}\Omega_{hD-1/2}, \quad \Omega^{h,\omega}_\Sigma = S^{-1}(\Omega_{hD-1/2})^{h,\omega} \quad (21)
$$

where $\Omega_{hD-1/2}$ and its dual are explicitly given by

$$
\Omega_{hD-1/2} = \{ z \in \mathbb{R}^{2n} : Dz \cdot z \leq \hbar \}
$$

and

$$
(\Omega_{hD-1/2})^{h,\omega} = \{ z \in \mathbb{R}^{2n} : JD^{-1}JDz \cdot z \leq \hbar \}
$$

Let us first assume that $\ell = \ell_X = \mathbb{R}^n \times 0$. Then

$$
\Omega_{hD-1/2} \cap \ell_X = \{ x \in \mathbb{R}^n : \Lambda^\omega x \cdot x \leq \hbar \}
$$

and

$$
(\Omega_{hD-1/2})^{h,\omega} \cap \ell_X = \{ x \in \mathbb{R}^n : (\Lambda^\omega)^{-1}x \cdot x \leq \hbar \}.
$$

Now, the condition

$$(\Omega_{hD-1/2})^{h,\omega} \cap \ell_X \subset \Omega_{hD-1/2} \cap \ell_X$$

is equivalent to $(\Lambda^\omega)^{-1} \geq \Lambda^\omega$ that is to $D^{-1} \geq D$, which implies $(\Omega_{hD-1/2})^{h,\omega} \subset \Omega_{hD-1/2}$, and $\Omega_{hD-1/2}$ contains a quantum blob in view of Proposition 5. We have thus proven the theorem in the case where $\Sigma = hD^{-1/2}$ and $\ell = \ell_X$.

For the general case we take $\ell = S^{-1}\ell_X$ where $S$ is a diagonalizing matrix; in view of (21) we have

$$
\Omega_\Sigma \cap \ell = S^{-1}\Omega_{hD-1/2} \cap S^{-1}\ell_X = S^{-1}(\Omega_{hD-1/2} \cap \ell_X)
$$

and hence $\Omega^{h,\omega}_\Sigma \cap \ell \subset \Omega_\Sigma \cap \ell$ if and only if $(\Omega_{hD-1/2})^{h,\omega} \subset \Omega_{hD-1/2}$. It now suffices to apply Proposition 5. To prove (ii) it is sufficient to note that the equality

$$(\Omega_{hD-1/2})^{h,\omega} \cap \ell_X = \Omega_{hD-1/2} \cap \ell_X$$

is equivalent to $(\Lambda^\omega)^{-1} = \Lambda^\omega$ that is to $\Lambda^\omega = I_{nxn}$ since we then have $M = S^TS_0$ in view of (16), the proof in the general case is then completed as above.

\[ 9 \]
2.3 Polar duality and the symplectic camel

Symplectic capacities (see for instance [8, 18]) are numerical invariants that serve as a fundamental tool in the study of various symplectic and Hamiltonian rigidity phenomena; they are closely related to Gromov’s symplectic non-squeezing theorem [19]; the latter is often referred to as the “principle of the symplectic camel” [12, 10, 18].

We denote $\text{Symp}(n)$ the group of all symplectomorphisms $(\mathbb{R}^{2n}, \omega) \rightarrow (\mathbb{R}^{2n}, \omega)$. That is, $f \in \text{Symp}(n)$ if and only if $f$ is a diffeomorphism of $\mathbb{R}^{2n}$ whose Jacobian matrix $Df(z)$ is in $\text{Sp}(n)$ for every $z \in \mathbb{R}^{2n}$.

A (normalized) symplectic capacity on $(\mathbb{R}^{2n}, \sigma)$ associates to every subset $\Omega \subset \mathbb{R}^{2n}$ a number $c(\Omega) \in [0, +\infty]$ such that the following properties hold:

- **SC1 Monotonicity:** If $\Omega \subset \Omega'$ then $c(\Omega) \leq c(\Omega')$;
- **SC2 Conformality:** For every $\lambda \in \mathbb{R}$ we have $c(\lambda \Omega) = \lambda^2 c(\Omega)$;
- **SC3 Symplectic invariance:** $c(f(\Omega)) = c(\Omega)$ for every $f \in \text{Symp}(n)$;
- **SC4 Normalization:** For $1 \leq j \leq n$ we have $c(B^{2n}(r)) = \pi r^2 = c(Z_j^{2n}(r))$ where $Z_j^{2n}(r)$ is the cylinder with radius $r$ based on the $x_j, p_j$ plane.

There exists a symplectic capacity, denoted by $c_{\text{max}}$, such that $c \leq c_{\text{max}}$ for every symplectic capacity. It is defined by

$$c_{\text{max}}(\Omega) = \inf_{f \in \text{Symp}(n)} \{ \pi r^2 : f(\Omega) \subset Z_j^{2n}(r) \} \quad (22)$$

where $Z_j^{2n}(r)$ is the phase space cylinder defined by $x_j^2 + p_j^2 \leq r^2$ and $\text{Symp}(n)$ the group of all symplectomorphisms of $\mathbb{R}^{2n}$ equipped with the standard symplectic structure. Similarly, there exists a smallest symplectic capacity $c_{\text{min}}$, it is defined by

$$c_{\text{min}}(\Omega) = \sup_{f \in \text{Symp}(n)} \{ \pi r^2 : f(B^{2n}(r)) \subset \Omega \}.$$

One shows [1, 2] that if $X \subset \mathbb{R}_x^n$ and $P \subset \mathbb{R}_p^n$ are centrally symmetric convex bodies then we have

$$c_{\text{max}}(X \times P) = 4h \sup \{ \lambda > 0 : \lambda X^h \subset P \}.$$  \( (23) \)

In particular,

$$c_{\text{max}}(X \times X^h) = 4h.$$  \( (24) \)
One also has the weaker notion of linear symplectic capacity, obtained by replacing condition (SC3) with

**SC3lin.** Linear symplectic invariance: \( c(S(\Omega)) = c(\Omega) \) for every \( S \in \text{Sp}(n) \) and \( c(\Omega + z) = c(\Omega) \) for every \( z \in \mathbb{R}^{2n} \).

One then defines the corresponding minimal and maximal linear symplectic capacities \( c_{\text{lin}}^{\text{min}} \) and \( c_{\text{lin}}^{\text{max}} \)

\[
c_{\text{lin}}^{\text{min}}(\Omega) = \sup_{S \in \text{Sp}(n)} \{ \pi R^2 : S(B^{2n}(z, R)) \subset \Omega, z \in \mathbb{R}^{2n} \} \quad (25)
\]

\[
c_{\text{lin}}^{\text{max}}(\Omega) = \inf_{f \in \text{Sp}(n)} \{ \pi r^2 : S(\Omega) \subset Z^{2n}_j(z, r), z \in \mathbb{R}^{2n} \}. \quad (26)
\]

It turns out that all symplectic capacities agree on ellipsoids. They are calculated as follows: assume that

\[
\Omega = \{ z \in \mathbb{R}^{2n} : Mz \cdot z \leq r^2 \}
\]

where \( M \in \text{Sym}^+(2n, \mathbb{R}) \), and let \( \lambda_1^\sigma, \lambda_2^\sigma, \ldots, \lambda_n^\sigma \) be the symplectic eigenvalue of \( M \), i.e. the numbers \( \lambda_j^\sigma > 0 \) (\( 1 \leq j \leq n \)) such that the \( \pm i\lambda_j^\sigma \) are the eigenvalues of the antisymmetric matrix \( M^{1/2}JM^{1/2} \). Then

\[
c(\Omega) = \frac{\pi r^2}{\lambda_{\text{max}}^\sigma} \quad (27)
\]

where \( \lambda_{\text{max}}^\sigma = \max\{\lambda_1^\sigma, \lambda_2^\sigma, \ldots, \lambda_n^\sigma\} \) (see [12, 18]). The following technical Lemma will allow us to prove a refinement of formula (24).

**Lemma 7.** Let \( \Omega \subset \mathbb{R}^{2n} \) be a centrally symmetric body. We have

\[
c_{\text{lin}}^{\text{min}}(\Omega) = \sup_{S \in \text{Sp}(n)} \{ \pi R^2 : S(B^{2n}(R)) \subset \Omega \}. \quad (28)
\]

**Proof.** Since \( \Omega \) is centrally symmetric we have \( S(B^{2n}(z_0, R)) \subset \Omega \) if and only if \( S(B^{2n}(-z_0, R)) \subset \Omega \). The ellipsoid \( S(B^{2n}(R)) \) is interpolated between \( S(B^{2n}(z_0, R)) \) and \( S(B^{2n}(-z_0, R)) \) using the mapping \( t \mapsto z(t) = z - 2tz_0 \) where \( z \in S(B^{2n}(z_0, R)) \), and is hence contained in \( \Omega \) by convexity.

**Proposition 8.** Let \( c_{\text{lin}}^{\text{min}} \) be the smallest linear symplectic capacity and \( X \subset \mathbb{R}^n_x \) a centered ellipsoid. We have

\[
c_{\text{lin}}^{\text{min}}(X \times X^h) = \pi h. \quad (29)
\]
Proof. In view of Lemma 7, $c_{\text{lin}}^\text{min}(X \times X^h)$ is the greatest number $\pi R^2$ such that $X \times X^h$ contains a symplectic ball $S(B^{2n}(R))$, $S \in \text{Sp}(n)$. In view of Proposition 2, $M_{A^{1/2}}(B^{2n}(\sqrt{\hbar}))$ is such a symplectic ball; since it is also the largest ellipsoid contained in $X \times X^h$ we must have
\[
c_{\text{lin}}^\text{min}(X \times X^h) = c_{\text{lin}}^\text{min}(M_{A^{1/2}}(B^{2n}(\sqrt{\hbar}))) = \pi \hbar.
\]

3 Projections and Intersections of Quantum Blobs

In this section we generalize the observation made in Remark 3.

3.1 Block matrix notation

For $M \in \text{Sym}_{++}(2n, \mathbb{R})$ we consider again the phase space ellipsoid
\[
\Omega = \{z \in \mathbb{R}^{2n} : Mz \cdot z \leq \hbar\}.
\]

Let us write $M = \hbar \Sigma^{-1}$ and $\Sigma$ in block-matrix form
\[
M = \begin{pmatrix}
M_{XX} & M_{XP} \\
M_{PX} & M_{PP}
\end{pmatrix}, \quad \Sigma = \begin{pmatrix}
\Sigma_{XX} & \Sigma_{XP} \\
\Sigma_{PX} & \Sigma_{PP}
\end{pmatrix}
\]

where the blocks are $n \times n$ matrices. The condition $M \in \text{Sym}_{++}(2n, \mathbb{R})$ ensures us that $M_{XX} > 0$, $M_{PP} > 0$, and $M_{PX} = M_{XP}^T$ (resp. $\Sigma_{XX} > 0$, $\Sigma_{PP} > 0$, and $\Sigma_{PX} = \Sigma_{XP}^T$; see [28]). Using classical formulas for the inversion of block matrices [26] we have
\[
M^{-1} = \begin{pmatrix}
(M/M_{PP})^{-1} & -(M/M_{PP})^{-1}M_{XP}M_{PP}^{-1} \\
-M_{PP}^{-1}M_{PX}(M/M_{PP})^{-1} & (M/M_{XX})^{-1}
\end{pmatrix}
\]

where $M/M_{PP}$ and $M/M_{XX}$ are the Schur complements:
\[
M/M_{PP} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX},
\]
\[
M/M_{XX} = M_{PP} - M_{PX}M_{XX}^{-1}M_{XP}.
\]
Similarly,
\[
\Sigma^{-1} = \begin{pmatrix}
\frac{1}{\Sigma_P} & \frac{-\Sigma_X^T}{\Sigma_P} \\
\frac{-\Sigma^T}{\Sigma_P} & \frac{1}{\Sigma_X}
\end{pmatrix}
\]  
(35)

Notice that these formulas imply
\[
\Sigma_{XX} = \frac{\hbar}{2} (\Sigma/M_{PP})^{-1}, \quad \Sigma_{PP} = \frac{\hbar}{2} (\Sigma/M_{XX})^{-1}
\]  
(36)
\[
\Sigma_{XP} = -\frac{\hbar}{2} (\Sigma/M_{PP})^{-1} M_X P P^{-1}.
\]  
(37)

Let \( M \) be the symmetric positive definite matrix (31). The following results is well-known (see for instance [16]):

**Lemma 9** The orthogonal projections \( \Pi_{\ell_X} \Omega \) and \( \Pi_{\ell_P} \Omega \) on the coordinate subspaces \( \ell_X = \mathbb{R}^n_x \times 0 \) and \( \ell_P = 0 \times \mathbb{R}^n_p \) of \( \Omega \) are the ellipsoids
\[
\Pi_{\ell_X} \Omega = \{ x \in \mathbb{R}^n_x : (M/M_{PP}) x \cdot x \leq \hbar \}
\]  
(38)
\[
\Pi_{\ell_P} \Omega = \{ p \in \mathbb{R}^n_p : (M/M_{XX}) p \cdot p \leq \hbar \}.
\]  
(39)

In terms of the covariance matrix \( \Sigma \) and the formulas (36) this is
\[
\Pi_{\ell_X} \Omega = \{ x \in \mathbb{R}^n_x : \frac{1}{2} \Sigma^{-1}_{XX} x \cdot x \leq 1 \}
\]  
(40)
\[
\Pi_{\ell_P} \Omega = \{ p \in \mathbb{R}^n_p : \frac{1}{2} \Sigma^{-1}_{PP} p \cdot p \leq 1 \}.
\]  
(41)

### 3.2 Reconstruction of quantum blobs: discussion

We have seen in Proposition [2] that if \( X \subset \mathbb{R}^n \) is a centered ellipsoid then the John ellipsoid of \( X \times X^\hbar \) is a quantum blob. By construction, the orthogonal projections of this quantum blob on the position and momentum spaces are precisely \( X \) and \( X^\hbar \), respectively. In this section we address the following question: for a given ellipsoid \( X \) are there other quantum blobs projecting this way? The key to the answer lies in the following simple observation:

**Lemma 10** The ellipsoid \( \Omega \) is a quantum blob \( S(B^{2n}(\sqrt{\hbar})) \), \( S \in \text{Sp}(n) \) if and only if the block entries of \( M = (SS^T)^{-1} \) satisfy
\[
M_{XX} M_{PP} - M_{XP}^2 = I_{n \times n}, \quad M_{PX} M_{PP} = M_{PP} M_{XP}.
\]  
(42)

These relations are in turn equivalent to
\[
\Sigma_{XX} \Sigma_{PP} - \Sigma_{XP}^2 = \frac{1}{4} \hbar^2 I_{n \times n} \quad \text{and} \quad \Sigma_{PX} \Sigma_{PP} = \Sigma_{PP} \Sigma_{XP}
\]  
(43)

where \( M = (\hbar/2) \Sigma^{-1} \).
Proof. The ellipsoid $\Omega$ is the set of all $z \in \mathbb{R}^{2n}$ such that $(SS^T)^{-1}z \cdot z \leq \hbar$. The positive definite matrix $M = (S^T S)^{-1}$ is thus symplectic. This condition is equivalent to the matrix relation $MJM = J$, which is itself equivalent to the conditions (13). In this case the matrix $(2/\hbar)\Sigma$ is also symplectic, whence the conditions (43).

Remark 11 The conditions (43) constitute the matrix form of the saturated Robertson–Schrödinger uncertainty principle [12, 18].

Explicitly the ellipsoid $\Omega$ is the set of all $z = (x, p)$ such that
$$M_{XX}x \cdot x + (M_{XP} + M_{XP}^T)x \cdot p + M_{PP}p \cdot p \leq \hbar;$$
(44)
the necessary and sufficient conditions for $\Omega$ to be a quantum blob are given by the conditions in (12) in the lemma above. Let us now determine the orthogonal projection $X = \Pi_{\ell_X}\Omega$ of the quantum blob $\Omega$ on the position space $\ell_X = \mathbb{R}^n_x \times 0$. By formula (38) $X$ is the set of all $x$ such that $(M/M_{PP})x^2 \leq \hbar$. Using the relations (12) we have
$$M/M_{PP} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX}$$
$$= (M_{XX}M_{PP} - M_{XP}^2)M_{PP}^{-1}$$
$$= M_{PP}^{-1}$$
and hence
$$X = \{x \in \mathbb{R}^n_x : M_{PP}^{-1}x \cdot x \leq \hbar\}$$
$$X^h = \{p \in \mathbb{R}^n_p : M_{PP}p \cdot p \leq \hbar\}$$

Similarly, the projection $\Pi_P\Omega$ on $\ell_P = 0 \times \mathbb{R}^n_p$ is the momentum space ellipsoid
$$P = \{p \in \mathbb{R}^n_p : M_{XX}^{-1}p \cdot p \leq \hbar\}.$$ We thus have $X^h = P$ if and only if $M_{PP}M_{XX} = I_{n \times n}$ which is possible if and only if $M_{XP} = 0$, that is, $\Omega$ must be the John ellipsoid of $X \times X^h$. The latter is thus the only quantum blob projecting orthogonally on $X$ and $X^h$. This will be discussed in a more general setting in Theorem 13 below.

Let us next assume that we know: (i) the orthogonal projection $X = \Pi_{\ell_X}\Omega$ of the quantum blob $\Omega$ and (ii) the intersections $\Omega \cap \ell_X$ and $\Omega \cap \ell_P$ of the quantum blob with the position and momentum spaces:
$$\Omega \cap \ell_X = \{x \in \mathbb{R}^n_x : M_{XX}x \cdot x \leq \hbar\}$$
(45)
$$\Omega \cap \ell_P = \{p \in \mathbb{R}^n_p : M_{PP}p \cdot p \leq \hbar\}.$$
We observe that the knowledge of these intersections is not sufficient to determine $\Omega$. We have to complement these with the first relation (42) to get the lacking term $M_{XP}$. The solution is however not unique; for instance in the case $n = 1$ we have two solutions $M_{XP} = \pm (M_{XX} M_{PP} - 1)^{1/2}$ and the number of solutions increases with $n$. Observe that the case $M_{XP} = 0$ precisely corresponds to the John ellipsoid. This is closely related to the Pauli problem [25] for generalized Gaussians.

3.3 Intersections with Lagrangian planes

Orthogonal projections and intersections are exchanged by polar duality:

**Proposition 12** (i) For every linear subspace $F$ of $\mathbb{R}^n$ we have

$$(X \cap \ell)^h = \Pi_\ell(X^h) \text{ and } (\Pi_\ell X)^h = X^h \cap \ell$$

where $\Pi_\ell$ is the orthogonal projection $\mathbb{R}^n \to \ell$. (In both equalities, the operation of taking the polar set in the left hand side is made inside $\ell$). (ii) Let $\ell$ be a linear subspace of $\mathbb{R}^n$ and $\Omega$ a symmetric convex body in $\mathbb{R}^n$. We have

$$(\Omega \cap \ell)^{h,\omega} = \Pi_{J\ell}(\Omega^{h,\omega}) \text{ and } (\Pi_{J\ell} \Omega)^{h,\omega} = \Omega^{h,\omega} \cap \ell$$

where $JF$ is the orthogonal subspace to $F$.

**Proof.** (i) (See Vershynin [27]). Let us first show that $\Pi_\ell(X^h) \subset (X \cap \ell)^h$. Let $p \in X^h$. We have, for every $x \in X \cap \ell$,

$$x \cdot \Pi_\ell p = \Pi_\ell x \cdot p = x \cdot p \leq h$$

hence $\Pi_\ell p \in (X \cap \ell)^h$. To prove the inverse inclusion we note that it is sufficient, by the anti-monotonicity property of polar duality, to prove that $(\Pi_\ell(X^h))^h \subset X \cap \ell$. Let $x \in (\Pi_\ell(X^h))^h$; we have $x \cdot \Pi_\ell p \leq h$ for every $p \in X^h$. Since $x \in \ell$ (because the dual of a subset of $\ell$ is in $\ell$) we also have

$$h \geq x \cdot \Pi_\ell p = \Pi_\ell x \cdot p = x \cdot p$$

from which follows that $x \in (X^h)^h = X$, which shows that $x \in X \cap \ell$. This completes the proof of the first formula in (47). The second formula in (47) follows by duality, noting that in view of the reflexivity of polar duality we have

$$(X^h \cap \ell)^h = \Pi_\ell(X^h)^h = \Pi_\ell X$$
and hence $X^h \cap \ell = (\Pi_\ell X)^h$. (ii) We have $(\Omega \cap \ell)^h = \Pi_\ell(\Omega^h)$ and hence 

$$(\Omega \cap \ell)^{h,\omega} = J(\Omega \cap \ell)^h = J\Pi_\ell(\Omega^h)$$

hence the first formula (48) noting that 

$$\Pi_\ell(\Omega^h) = J\Pi_\ell J^{-1}(J\Omega^h) = \Pi_\ell \Omega^{h,\omega}.$$ 

The second formula (48) follows by duality. □

The following result considerably improves the statements we gave in [16]:

**Theorem 13** A centered phase space ellipsoid $\Omega = \{ z : Mz \cdot z \leq \hbar \}$ $(M \in \text{Sym}_{++}(2n, \mathbb{R}))$ is a quantum blob if and only if the equivalent conditions 

$$(\Pi_{\ell_x} \Omega)^h = \Omega \cap \ell_P , \quad (\Pi_{\ell_x} \Omega)^{h,\omega} = (J\Omega) \cap \ell_X .$$  \hspace{1cm} (49) 

are satisfied. In terms of the matrix $M$ these conditions are equivalent to the identity 

$$M_{PP}(M/M_{PP}) = I_{n \times n} .$$  \hspace{1cm} (50) 

**Proof.** That both conditions (49) are equivalent is from definition (11) of symplectic polar duality. Writing $M$ in block matrix form, the condition $z = (x,p) \in \Omega$ means that 

$$M_{XX}x^2 + (M_{XP} + M_{PX})xp + M_{PP}p^2 \leq \hbar$$  

(we are using the abbreviations $M_{XX}x \cdot x = M_{XX}x^2$, etc.) and the intersection $\Omega \cap \ell_P$ is therefore the set 

$$\Omega \cap \ell_P = \{ p : M_{PP}p^2 \leq \hbar \} .$$

On the other hand, in view of Lemma 9 

$$\Pi_{\ell_x} \Omega = \{ x : (M/M_{PP})x^2 \leq \hbar \}$$

and the polar dual $(\Pi_{\ell_x} \Omega)^h$ is 

$$(\Pi_{\ell_x} \Omega)^h = \{ p : (M/M_{PP})^{-1}p^2 \leq \hbar \}$$

so we have to prove that $\Omega$ is a quantum blob if and only if (50) holds. Using the explicit expression (33) of the Schur complement this is equivalent to the condition 

$$(M_{XX} - M_{XP}M_{PP}^{-1}M_{PX})M_{PP} = I_{n \times n} .$$  \hspace{1cm} (51)
Assume now that Ω is a quantum blob; then Ω = \( S(B^{2n}(\sqrt{\hbar})) \) for some \( S \in \text{Sp}(n) \); then \( z \in \Omega \) if and only if \( Mz \cdot z \leq \hbar \) where \( M = (S^T)^{-1}S^{-1} \). Since \( M \in \text{Sp}(n) \cap \text{Sym}_+^{++}(2n, \mathbb{R}) \) we have \( M_{PP}M_{XP} = M_{PX}M_{PP} \) (second formula \((12)\) in Lemma 10) and hence

\[
(M_{XX} - M_{XP}M_{PP}^{-1}M_{PX})M_{PP} = M_{XX}M_{PP} - M_{XX}M_{PP}^{-1}(M_{PX}M_{PP}) = M_{XX}M_{PP} - (M_{XP})^2.
\]

Using the first formula \((12)\) in Lemma 10 we thus have

\[
(M_{XX} - M_{XP}M_{PP}^{-1}M_{PX})M_{PP} = I_{n \times n}
\]

which implies that \((\Pi_{\ell x} \Omega)^h = \Omega \cap \ell_P\), so we have proven the necessity of the condition \((49)\). Let us prove that this condition is sufficient as well. In view of Williamson’s diagonalization result \((16)\) we have \( M = S_0^T \begin{pmatrix} \Lambda^\omega & 0 \\ 0 & \Lambda^\omega \end{pmatrix} S_0 \) for some \( S_0 \in \text{Sp}(n) \) where \( \Lambda^\omega \) is the diagonal matrix whose non-zero entries are the symplectic eigenvalues of \( M \). Since a symplectic automorphism transforms a quantum blob into another quantum blob, we can reduce the proof of the sufficiency of \((49)\) to the case where \( \Omega \) is the ellipsoid

\[
\Omega_0 = \{ z \in \mathbb{R}^{2n} : \Lambda^\omega x^2 + \Lambda^\omega p^2 \leq \hbar \}.
\]

We have here \( \Pi_{\ell x} \Omega_0 = \{ x : \Lambda^\omega x^2 \leq \hbar \} \) hence

\[
(\Pi_{\ell x} \Omega_0)^h = \{ x : (\Lambda^\omega)^{-1}x^2 \leq \hbar \}
\]

and \( \Omega \cap \ell_P = \{ x : \Lambda^\omega x^2 \leq \hbar \} \). The equality \((\Pi_{\ell x} \Omega_0)^h = \Omega \cap \ell_P \) thus implies that \( \Lambda^\omega = I_{n \times n} \) hence \( M = S_0^T S_0 \in \text{Sp}(n) \) and \( \Omega \) is thus the quantum blob \( S_0^{-1}(B^{2n}(\sqrt{\hbar})) \). ■

4 Gaussian Quantum Phase Space

In this section we apply some of our previous geometric results to the theory of Gaussian states.

4.1 Generalized Gaussians and their Wigner transforms

Recall that the Wigner transform (or function) of a square integrable function \( \psi : \mathbb{R}^n \rightarrow \mathbb{C} \) is the function \( W\psi \in C^0(\mathbb{R}^{2n}, \mathbb{R}) \) defined by the absolutely
convergent integral
\[ W\psi(x,p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}p y} \psi(x + \frac{1}{2} y) \psi^*(x - \frac{1}{2} y) d^n y. \] (53)

The Wigner transform satisfies the Moyal identity
\[ (W\psi|W\phi)_{L^2(\mathbb{R}^n)} = (2\pi\hbar)^{-n}|(\psi|\phi)|^2_{L^2(\mathbb{R}^n)} \] (54)
which implies, in particular, that
\[ ||W\psi||_{L^2(\mathbb{R}^n)} = (2\pi\hbar)^{-n/2} ||\psi||^2_{L^2(\mathbb{R}^n)}. \] (55)

An important property satisfied by the Wigner transform is its symplectic covariance: for every \(S \in \text{Sp}(n)\) and \(\psi \in L^2(\mathbb{R}^n)\) we have
\[ W\psi(S^{-1} z) = W(\hat{S}\psi)(z) \] (56)
where \(\hat{S} \in \text{Mp}(n)\) is one of the two metaplectic operators projecting onto \(S\) (recall \[11\] that \(\text{Mp}(n)\), the metaplectic group, is a unitary representation in \(L^2(\mathbb{R}^n)\) of the double cover of \(\text{Sp}(n)\)). The covering projection \(\pi^\text{Mp}: \text{Mp}(n) \rightarrow \text{Sp}(n)\) is uniquely determined by its action of the generators of \(\text{Mp}(n)\).

Here is a basic example. Let \(X \in \text{Sym}_+(n, \mathbb{R})\) and \(Y \in \text{Sym}(n, \mathbb{R})\). The associated generalized Gaussian \(\psi_{X,Y}\) is defined by
\[ \psi_{X,Y}(x) = (\pi\hbar)^{-n/4}(\det X)^{1/4} e^{-\frac{1}{\hbar\pi}(X+iY)x^2}. \] (57)
Its Wigner transform is given by \[5, 11, 14\]
\[ W\psi_{X,Y}(z) = (\pi\hbar)^{-n} e^{-\frac{1}{\hbar}Gz} \] (58)
where
\[ G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix}. \] (59)
It is essential to observe that \(G = G^T \in \text{Sp}(n)\); this is most easily seen using the factorization
\[ G = S^T S, \quad S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \in \text{Sp}(n). \] (60)
It can be shown by a direct calculation that the generalized Gaussians satisfy the second order partial differential equation \( \hat{H}_{X,Y} \psi_{X,Y} = 0 \) where \( \hat{H}_{X,Y} \) is the operator with Weyl symbol
\[
\hat{H}_{X,Y}(x,p) = (p + Yx) \cdot (p + Yx) + X^2 \cdot x - \hbar \text{Tr} X. 
\]
(\( \hat{H}_{X,Y} \) is the “Fermi function” \[6\] of \( \psi_{X,Y} \)).

Let us introduce the following notation:

- \( \text{Gauss}_0(n) \) is the set of all centered Gaussian functions \[57\]: \( \psi \in \text{Gauss}_0(n) \) if and only if there exist \( X, Y \) such that \( \psi = \psi_{X,Y} \);
- \( \text{Quant}_0(n) \) is the set of all centered quantum blobs: \( Q \in \text{Quant}(n) \) if and only if there exists \( S \in \text{Sp}(n) \) such that \( Q = S(B^{2n}(\sqrt{\hbar})) \).

In \[13\] we proved that:

**Proposition 14** There exists a bijection
\[
F : \text{Gauss}_0(n) \rightarrow \text{Quant}_0(n).
\]
That bijection is defined as follows: if \( \psi_{X,Y} \) satisfies
\[
W \psi_{X,Y}(z) = (\pi \hbar)^{-n} e^{-\frac{1}{\hbar} Gz \cdot z}
\]
then \( F[\psi_{X,Y}] = \{z : Gz \cdot z \leq \hbar\} \).

That \( F[\psi_{X,Y}] \in \text{Quant}_0(n) \) immediately follows from \[60\].

4.2 Gaussian density operators

Let \( \hat{\rho} \in L^1(L^2(\mathbb{R}^n)) \) be a trace class operator on \( L^2(\mathbb{R}^n) \). If \( \text{Tr}(\hat{\rho}) = 1 \) and \( \hat{\rho} \) is positive semidefinite (\( \hat{\rho} \geq 0 \)) one says that \( \hat{\rho} \) is a density operator (it represents the mixed states in quantum mechanics, for an up-to-date discussion of trace class operators and their applications to quantum mechanics see \[15\]).

One shows, using the spectral theorem for compact operators, that the Weyl symbol of \( \hat{\rho} \) can be written as \( (2\pi \hbar)^n \rho \) where \( \rho \) (the “Wigner distribution of \( \hat{\rho} \)”) is a convex sum
\[
\rho = \sum_j \lambda_j W \psi_j, \quad \lambda_j \geq 0, \quad \sum_j \lambda_j = 1
\]
where \((\psi_j)_j\) is an orthonormal set of vectors in \(L^2(\mathbb{R}^n)\) (the series is absolutely convergent in \(L^2(\mathbb{R}^n)\)). Of particular interest are Gaussian density operators, by definition these are the density operators whose Wigner distribution can be written

\[
\rho(z) = \frac{1}{(2\pi)^n\sqrt{\det \Sigma}} e^{-\frac{1}{2} \Sigma^{-1}(z-z_0)(z-z_0)}
\]

(62)

where \(z_0 \in \mathbb{R}_2^n\) and the covariance matrix \(\Sigma \in \text{Sym}_{++}(n, \mathbb{R})\) (we will from now on choose \(z_0 = 0\), but all the statements on the covariance matrix and ellipsoid that follow are not influenced by this assumption). While the operator \(\hat{\rho}\) with Weyl symbol \((2\pi \hbar)^n \rho\) automatically has trace one, the condition \(\hat{\rho} \geq 0\) is equivalent to \(\Sigma + \frac{i\hbar^2}{2} J \geq 0\).

\[
(63)
\]

(by that is, the eigenvalues of the Hermitian matrix \(\Sigma + \frac{i\hbar^2}{2} J\) are \(\geq 0\)).

By definition the purity of a density operator \(\rho\) is the number \(\mu(\hat{\rho}) = \text{Tr}(\hat{\rho}^2)\). We have \(0 < \mu(\hat{\rho}) \leq 1\) and \(\mu(\hat{\rho}) = 1\) if and only the Wigner distribution \(\hat{\rho}\) consists of a single term: \(\rho = W\psi\) for some \(\psi \in L^2(\mathbb{R}^n)\).

**Proposition 15** Let \(\hat{\rho}\) be a Gaussian density operator with covariance matrix \(\Sigma\). (i) The condition \(\Sigma + \frac{i\hbar^2}{2} J \geq 0\) holds if and only if the covariance ellipsoid \(\Omega\) associated with \(\Sigma\) contains a quantum blob. (ii) We have \(\mu(\hat{\rho}) = 1\) if and only \(\Omega\) is a quantum blob and we have in this case \(\rho = W\psi_{X,Y}\) for some pair of matrices \((X,Y)\).

**Proof.** We have proven part (i) in [11, 12] (also see [18]). To prove (ii) we note that the purity of a Gaussian state \(\hat{\rho}\) is \(\mu(\hat{\rho}) = \left(\frac{\hbar}{2}\right)^n (\det \Sigma)^{-1/2}\)

hence \(\mu(\hat{\rho}) = 1\) if and only if \(\det \Sigma = (\hbar/2)^{2n}\). Let \(\lambda_1^\omega, \ldots, \lambda_n^\omega\) be the symplectic eigenvalues of \(\Sigma\) as in the proof of Theorem [13] in view of Williamson’s symplectic diagonalization theorem there exists \(S \in \text{Sp}(n)\) such that \(\Sigma = S^{-1}D(S^T)^{-1}\) where \(D = \begin{pmatrix} \Lambda^\omega & 0 \\ 0 & \Lambda^\omega \end{pmatrix}\) with \(\Lambda^\omega = \text{diag}(\lambda_1^\omega, \ldots, \lambda_n^\omega)\). The quantum condition (63) is equivalent to \(\lambda_j^\omega \geq \hbar/2\) for all \(j\) hence

\[
\det \Sigma = (\lambda_1^\omega)^2 \cdots (\lambda_n^\omega)^2 = 1
\]

if and only if \(\lambda_j^\omega = \hbar/2\) for all \(j\), hence \(\Sigma = \frac{\hbar}{2} S^{-1}(S^T)^{-1}\) and \(\Omega = S(B^{2n}(\sqrt{\hbar}))\) is a quantum blob. \(\blacksquare\)
4.3 A characterization of Gaussian density operators

We are going to apply Theorem 13 to characterize pure Gaussian density operators without prior knowledge of the full covariance matrix. This is related to the so-called “Pauli reconstruction problem” [25] we have discussed in [17]. The latter can be reformulated in terms of the Wigner transform as follows: given a function $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ whose Fourier transform is also in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ the question is whether we reconstruct $\psi$ from the knowledge of the marginal distributions

$$\int W\psi(x,p)d^n p = |\psi(x)|^2, \quad \int W\psi(x,p)d^n x = |\hat{\psi}(p)|^2$$

where the Fourier transform $\hat{\psi}$ of $\psi$ is given by

$$\hat{\psi}(p) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int e^{-\frac{i}{\hbar}px} \psi(x)d^n x.$$  

The answer to Pauli’s question is negative; the study of this problem has led to many developments, one of them being the theory of symplectic quantum tomography (see e.g. [23]). The following result is essentially an analytic restatement of Theorem 13:

**Theorem 16** Let $\hat{\rho} \in L^1(L^2(\mathbb{R}^n))$ be a density operator with Gaussian Wigner distribution

$$\rho(z) = \frac{1}{(2\pi)^n\sqrt{\det \Sigma}} e^{-\frac{1}{2}z^t \Sigma^{-1} z}.$$  

Then $\hat{\rho}$ is a pure density operator if and only if

$$\Phi(x) = 2^n \int \rho(x,p)d^n p$$

where $\Phi$ is the Fourier transform of the function $p \mapsto \rho(0,p/2)$.

**Proof.** We begin by noting that by the well-known formula about marginals in probability theory we have

$$\int \rho(x,p)d^n p = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma_{XX}}} e^{-\frac{1}{2}x^t \Sigma^{-1}_{XX} x}.$$
Returning to the notation $M = \frac{1}{2} \Sigma^{-1}$ we have

$$
\rho(z) = (\pi \hbar)^{-n} (\det M)^{1/2} e^{-\frac{1}{\hbar} M z \cdot z}
$$

and the margin formula (67) reads

$$
\int \rho(x, p) d^n p = (\pi \hbar)^{-n/2} (\det M/M_{PP})^{1/2} e^{-\frac{1}{\hbar} (M/M_{PP}) x \cdot x}. \tag{68}
$$

Assume now that $\hat{\rho}$ is a pure density operator and let us show that (66) holds (also see Remark 17 below). In view of Proposition 15 we then have

$$
\rho(0, p/2) = (\pi \hbar)^{-n} e^{-\frac{1}{\hbar} X x \cdot x} \frac{1}{12}
$$

and its Fourier transform is

$$
\Phi(p) = \left( \frac{2}{\pi \hbar} \right)^n (\det M)^{1/2} e^{-\frac{1}{\hbar} M_{PP}^1 x \cdot x}. \tag{66}
$$

The equality (66) requires that

$$
\det M = (\det M_{PP})^{1/2} e^{-\frac{1}{\hbar} M_{PP}^1 x \cdot x} \det M/M_{PP}^{1/2} = (\det M_{PP})^{1/2} e^{-\frac{1}{\hbar} (M/M_{PP}) x \cdot x}
$$

that is, equivalently,

$$
M_{PP}^{-1} = (M/M_{PP}) \quad (\det M)^{1/2} (\det M_{PP})^{-1/2} = (\det M/M_{PP})^{1/2}.
$$

The first of these two conditions implies that the covariance ellipsoid $\Omega$ is a quantum blob (formula (60)) in Theorem 13); the second condition is then automatically satisfied since $\det M = 1$ in this case.
Remark 17 Condition (66) is actually satisfied by all even Wigner transformations (and hence by all pure density operators corresponding to an even function $\psi$). Suppose indeed that $\rho = W\psi$ for some suitable even function $\psi \in L^2(\mathbb{R}^n)$. Then

$$W\psi(0, p/2) = (\pi \hbar)^{-n} \int e^{-\frac{\pi p \cdot y}{\hbar}} |\psi(y)|^2 d^n y;$$

Taking the Fourier transform of both sides and using the first marginal property (64) yields the identity (66).

5 Perspectives and Comments

Among all states (classical, or quantum) the Gaussians are those which are entirely characterized by their covariance matrices. The notion of polar duality thus appears informally as being a generalization of the uncertainty principle of quantum mechanics as expressed in terms of variances and covariances. Polar duality actually is a more general concept than the usual uncertainty principle, expressed in terms of covariances and variances of position and momentum variables (and the derived notion of quantum blob). As was already in the work of Uffink and Hilgevoord [20, 21], variances and covariances are satisfactory measures of uncertainties only for Gaussian (or almost Gaussian) distributions. For more general distributions having nonvanishing “tails” they can lead to gross errors and misinterpretation. Another advantage of the notion of polar duality is that it might precisely be extended to study uncertainties when non-Gaussianity appears (for an interesting characterization of non-Gaussianity see [22]). Instead of considering ellipsoids $X$ in configuration space $\mathbb{R}_x^n$ one might want to consider sets $X$ which are only convex. In this case the polar dual $X^h$ is still well-defined and one might envisage, using the machinery of the Minkowski functional to generalize the results presented here to general non-centrally symmetric convex bodies in $\mathbb{R}_x^n$. The difficulty comes from the fact that we then need to choose the correct center with respect to which the polar duality is defined since there is no privileged “center” [3]; different choices may lead to polar duals with very different sizes and volumes. These are difficult questions, but they may lead to a better understanding of very general uncertainty principles for the density operators of quantum mechanics.
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