Pesin-type relation for subexponential instability

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Abstract. We address here the problem of extending the Pesin relation among positive Lyapunov exponents and the Kolmogorov–Sinai entropy to the case of dynamical systems exhibiting subexponential instabilities. By using a recent rigorous result due to Zweim"uller, we show that the usual Pesin relation can be extended straightforwardly for weakly chaotic one-dimensional systems of the Pomeau–Manneville type, provided one introduces a convenient subexponential generalization of the Kolmogorov–Sinai entropy. We show, furthermore, that Zweim"uller’s result provides an efficient prescription for the evaluation of the algorithm complexity for such systems. Our results are confirmed by exhaustive numerical simulations. We also point out and correct a misleading extension of the Pesin relation based on the Krengel entropy that has appeared recently in the literature.

Keywords: dynamical processes (theory)
1. Introduction

One-dimensional chaotic dynamics are usually characterized by the existence of a positive Lyapunov exponent, which indicates exponential separation of initially nearby trajectories [1]. In recent years, we have witnessed a rapid development in the study and characterization of dynamical unpredictable systems in which the separation of trajectories is weaker than exponential [2]. For these systems, generically dubbed ‘weakly chaotic’ in the physical literature, the conventional Lyapunov exponent vanishes and new concepts and ideas for the characterization of dynamical instabilities are necessary for a deeper understanding of their global dynamics. Many results of infinite ergodic theory [3] come out as powerful tools in this context, shedding light on several apparently unrelated results in the physical literature. Among them, the Aaronson–Darling–Kac (ADK) theorem [3] certainly has a central role in these problems.

The main purpose of this work is to extend the well-known Pesin relation [4] for the case of weakly chaotic one-dimensional systems, a matter that has attracted considerable attention recently (see [5] for references) and even some controversy [6]. For the usual one-dimensional chaotic systems, the Pesin relation is given simply by \( h = \lambda \), with \( h \) and \( \lambda \) standing, respectively, for the Kolmogorov–Sinai (KS) entropy and the usual Lyapunov exponent. We will show that adequate subexponential generalizations of the KS entropy and of the Lyapunov exponent will obey exactly the same Pesin-type relation, for almost all trajectories. It is important to stress that the existence of such a generalization is far from intuitive since we are dealing with nonergodic systems for which the typical dynamical quantities do depend on the specific trajectory. We also show that the extension based on Krengel entropy proposed in [5] for weakly chaotic systems is incorrect. The source of the problem is identified and the correct expression is presented. We close by comparing our proposed Pesin-type relation based on the subexponential KS entropy and the proposal involving the Krengel entropy.

2. Pesin-type relation and statistics

We will consider here the general class of Pomeau–Manneville (PM) [7] maps \( x_{t+1} = f(x_t) \), with \( f: [0, 1] \rightarrow [0, 1] \) such that
\[
f(x) \sim x(1 + ax^{z-1}),
\] (1)
for $x \to 0$, with $a > 0$ and $z > 1$. From the physical point of view, the original PM system ($z = 2$) is paradigmatic since it corresponds to certain Poincaré sections related to the Lorenz attractor [7]. Systems of the type (1) exhibit exactly the kind of subexponential instability for nearby trajectories that we are concerned here: $\delta x_t \sim \delta x_0 \exp(\lambda t^\alpha)$, with $0 < \alpha < 1$. A distinctive characteristic of such a class of maps is the presence of an indifferent (neutral) fixed point at $x = 0$, i.e. $f(0) = 0$ and $f'(0) = 1$. The global form of $f$ is irrelevant for our purposes, provided it respects the axioms of an AFN system [8].

These systems are known to have invariant densities $\omega(x)$ such that

$$
\omega(x) \sim bx^{-1/\alpha}, \quad \alpha = (z - 1)^{-1},
$$

(2)

near the indifferent fixed point $x = 0$ [9]. Clearly, the corresponding invariant measures diverge for $z > 2$. In these cases, we have pictorially two qualitative distinct dynamical behavior coexisting, namely a laminar regime near $x = 0$ and a turbulent one elsewhere, resulting eventually in nonergodicity and subexponential separation of initially close trajectories. It is noteworthy here that it was recently shown that subexponential instability does imply an infinite invariant measure [10]. On the other hand, $1 < z < 2$ leads to a finite invariant measure, which is naturally related to ergodicity and positivity of the ordinary Lyapunov exponent.

For intermittent systems like (1), the statistics of a given observable $\vartheta$ for randomly distributed initial conditions has some peculiar properties. For ergodic systems, the time average $t^{-1} \sum_{k=0}^{t-1} \vartheta(f^k(x))$ converges to the spatial average $\int \vartheta \, d\mu$, with $d\mu = \omega(x) \, dx$. On the other hand, if the system has a diverging invariant measure, the time average will typically depend on the chosen trajectory. Nevertheless, the ADK theorem [3] ensures in this case that a suitable time-weighted average does converge in distribution terms towards a Mittag–Leffler distribution with unit first moment. More specifically, for a positive integrable function $\vartheta$ and a random variable $x$ with an absolutely continuous measure with respect to the Lebesgue measure on the interval of interest, there is a sequence $\{a_t\}$ for which

$$
\frac{1}{t^{\alpha}} \sum_{k=0}^{t-1} \vartheta(f^k(x)) \frac{d}{t^{\alpha}} \xi_{a_t},
$$

(3)

for $t \to \infty$, where $\xi_{a_t}$ is a non-negative Mittag–Leffler random variable with index $\alpha \in (0, 1]$ and unit expected value. Notice that, for $1 < z < 2$ (the ergodic regime), $a_t \sim t$ and the corresponding $\alpha = 1$ Mittag–Leffler distribution reduces to a Dirac $\delta$ function. For the subexponential regime ($z > 2$), we have $a_t \sim t^\alpha$ [8] and, by choosing $\vartheta = \ln|f'|$, the ADK theorem ensures the convergence in distribution terms toward a Mittag–Leffler distribution of the subexponential finite-time Lyapunov exponent:

$$
\lambda_t^{(\alpha)}(x) = \frac{1}{t^{\alpha}} \sum_{k=0}^{t-1} \ln|f'(f^k(x))|,
$$

(4)

for $t \to \infty$. The generalized Lyapunov exponent (4) plays for intermittent systems the same role played by the usual exponent (corresponding to $\alpha = 1$ in (4)) for one-dimensional chaotic systems, see [11] and references therein for a recent discussion.

In order to investigate the connection between subexponential instability and the corresponding degree of randomness of an intermittent dynamical system like (1), we will

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consider the Kolmogorov–Chaitin concept of complexity [1]. Let us assume that the phase space of the map (1) is partitioned and completely covered by a set of non-overlapping ordered cells. A given trajectory \{x_t\} generated by the map (1) can be represented by a sequence of symbols \{s_t\}, which we assume to be integers such that \(s_t\) corresponds to the cell where \(x_t\) belongs. The next step in the analysis consists in eliminating redundancies that may appear in \{s_t\} by performing a compression of information. This can be done, for instance, by introducing the so-called algorithmic complexity function \(C_t(\{s_t\})\), which is defined as the length of the shortest possible program able to reconstruct the sequence \{s_t\} on a universal Turing machine [1]. Systems that exhibit some degree of regularity are able to generate sequences of symbols at a rate higher than needed for recording their programs. For example, a periodic sequence can be recreated by replaying the periodic pattern over the total length. Typically, for these cases, one has \(C_t \sim \ln t\). On the other hand, if the trajectory is completely random, there is no way of reproducing it other than memorizing the whole trajectory, resulting in a sequence length that increases linearly in time, i.e. \(C_t \sim t\). The finite-time KS entropy is defined simply as \(h_t = C_t/t\). An important recent rigorous result due to Zweimüller [12] unveils the relation between KS entropy and the Lyapunov exponent for systems exhibiting subexponential instability. According to this result, we have, for almost all initial conditions,

\[
\frac{C_t}{\sum_{k=0}^{t-1} \partial (f^k(x))} \to \frac{h_\mu(f)}{\int \partial d\mu}, \tag{5}
\]

for \(t \to \infty\), for any observable function \(\vartheta\), where \(h_\mu(f)\) stands for the Krengel entropy [13], which can be expressed by the so-called Rohlin’s formula [14]:

\[
h_\mu(f) = \int \ln |f'| \, d\mu. \tag{6}
\]

By choosing again \(\vartheta = \ln |f'|\), we get from (5) the surprisingly simple relation

\[
h_t^{(\alpha)} \to \lambda_t^{(\alpha)}, \tag{7}
\]

for \(t \to \infty\) and for almost all initial conditions, where

\[
h_t^{(\alpha)} = \frac{C_t}{t^\alpha} \tag{8}
\]

is the subexponential generalization of the finite-time KS entropy. The relation (7) is the most natural generalization of the Pesin relation for systems of the type (1). From the ADK theorem and (7), we have that both \(h_t^{(\alpha)}\) and \(\lambda_t^{(\alpha)}\) converge in distribution terms towards the same Mittag–Leffler distribution. Nevertheless, Zweimüller’s result is indeed stronger, ensuring that, for almost all trajectories, \(h_t^{(\alpha)}\) coincides with \(\lambda_t^{(\alpha)}\) in the limit \(t \to \infty\). In this way, the relation (5) does provide an efficient prescription for evaluating the algorithmic complexity of a given trajectory for one-dimensional maps, namely

\[
C_t \to \sum_{k=0}^{t-1} \ln |f'(f^k(x))|, \tag{9}
\]

for large \(t\). The power of the prescription (9) resides in the fact that it does provide, for the systems in question, a computable way for the calculation of the algorithmic complexity.
function $C_t$, a well-known non-computable function in general [1]. It is important also to recall that, for dynamical systems with an infinite invariant measure, the invariant density, and consequently the invariant measure, is defined up to an arbitrary multiplicative positive constant. In other words, the transformation $ω → ξω$ (implying, in this way, that $b → ξb$ in (2)), with $ξ > 0$, does not have any dynamical implication. Zweimüller’s construction, based in (5), is clearly invariant under such a transformation. Of course, such ‘symmetry’ is broken in the usual ergodic case due to the normalization of the invariant measure.

We notice that the relation (7) is compatible with the pioneering work of Gaspard and Wang [2], where the standard PM map $f(x) = x + axz$ (mod 1) was considered. They argue, in particular, that the algorithmic complexity $C_t$ for the PM map is proportional to $N_t$, the number of entrances into a given phase space cell during $t$ iterations of the PM map. By invoking some results from renewal theory [15], one has

$$\text{Prob} \left( N_t \geq \frac{t^α}{u^α} \right) \to G_α(u), \quad (10)$$

for $0 < α < 1$ and $t → ∞$, where $c$ is a positive constant and $G_α$ stands for the one-sided Lévy cumulative distribution function with index $α$. With the change of variable $u = rξ^{-1/α}$, where $r^α = αΓ(α)$, we have that the normalized random variable $ξ = N_t/(N_t)$ tends towards a Mittag–Leffler random variable with index $α$ and unit first moment for $t → ∞$ (see [16] for more details on the relations between one-sided Lévy and Mittag–Leffler distributions), in perfect agreement with the predictions of the ADK theorem. The possibility of estimating the algorithmic complexity function $C_t$ from $N_t$ also for generic systems of the type (1) is indeed confirmed by our exhaustive numerical explorations. (See also [8] and references therein.)

3. Numerical simulations

The ADK convergence of the generalized Lyapunov exponent (4) was exhaustively checked and confirmed by numerical simulations for different maps in [11]. The Zweimüller prescription for calculating the algorithmic complexity (9) ensures also an ADK-like convergence for the generalized KS entropy (8). A possible way of testing the consistence of the overall picture is to compare the Zweimüller prescription (9) with other independent prescription for calculating the algorithmic complexity $C_t$. For this purpose, we consider some simple realizations of the general maps of the type (1), namely the standard PM case discussed in [2], the Thaler map [9]

$$f(x) = x + axz \mod 1,$$

and the so-called modified Bernoulli map (see [17] for references)

$$f(x) = \begin{cases} 
  x + 2z^{-1}x^z, & 0 \leq x \leq \frac{1}{2}, \\
  x - 2z^{-1}(1 - x)^z, & \frac{1}{2} < x \leq 1.
\end{cases} \quad (12)$$

The modified Bernoulli map (12) has indeed two neutral fixed points at $x = 0$ and 1, but this does not alter our analysis since we still have $a_t \sim t^α$ for $z > 2$ in this case [8].

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Figure 1. The cells $A_0$ and $A_1$ for the Bernoulli map (12) with $z = 5/2$. Notice that the dynamics are regular inside each of the cells, with the trajectories departing monotonically from the respective fixed points. Nevertheless, the transition for one cell to the other is chaotic. The situation is similar for the PM map with $a = 1$ and for the Thaler map (11), even though for these cases the partitions are not symmetric as the present case. (See also [2].)

Motivated by the construction introduced in [2], let us consider the standard partition of the interval $[0,1]$ into two cells, $A_0 = [0,x_\star]$ and $A_1 = (x_\star, 1]$, where $x_\star$ is the point of discontinuity of the maps, i.e. $\lim_{x \to x_\star^-} f(x) = 1$, with $0 < x_\star < 1$. For the modified Bernoulli map (12), one has simply $x_\star = 1/2$, while for the other cases the value of $x_\star$ does depend on the map details, in particular on the value of $z$. The trajectories inside both cells $A_0$ and $A_1$ are typically regular, the turbulent behavior is associated with the transition from one cell to the other, see [2] and figure 1. Let $N_t$ be the number of entrances of a given trajectory into the cell $A_1$ during $t$ iterations of the map. Since the contributions for $C_t$ arising from the laminar parts of the trajectories contained inside the cells are subdominant for large $t$, it is natural to expect that, for weakly chaotic regimes, $C_t = \gamma N_t$ for large $t$, where $\gamma$ is some proportionality constant, independent of the specific trajectory considered, implying, in particular, that $C_t/\langle C_t \rangle = N_t/\langle N_t \rangle$ for large $t$. We could check by numerical simulations that the subexponential KS entropy (8) calculated directly from $N_t$, namely

$$\frac{h_t^{(a)}}{\langle h_t^{(a)} \rangle} = \frac{N_t}{\langle N_t \rangle},$$

(13)
does converge toward a Mittag–Leffler distribution with unit expected value. Figure 2 depicts a typical example. This convergence is robust and typically fast, see [11] for a recent discussion. Interestingly, equation (13) and its convergence issues do not depend on the specific partition introduced above, even though the specific value of $\gamma$ does. Figure 3 depicts the relation between the algorithm complexity calculated by using the

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6
Figure 2. Distribution of finite-time Kolmogorov–Sinai entropy $h_t^{(\alpha)}$ calculated from (13) for the Bernoulli map (12), with $z = 28/13$ ($\alpha = 13/15$), for $t = 6 \times 10^4$ and $2.5 \times 10^5$ uniformly distributed initial conditions. The histogram was built directly from the numerical data, while the dotted line is the corresponding Mittag–Leffler probability density computed with the algorithm of [16]. For convergence details, see [11]. Similar results hold also for the other considered maps.

Figure 3. Graphics of the algorithm complexity $C_t$, calculated by the Zweimüller prescription (9), as a function of $N_t$, the number of entrances of a given trajectory into the cell $A_1 = (x_*, 1]$, during $t = 10^6$ iterations of the Bernoulli map (12) with $z = 28/13$. For the sake of clarity, only 2500 points are shown for each case (a)–(d), which correspond, respectively, to $x_* = 1/2, 5/8, 3/4$ and $7/8$. The linear relation is evident. The situation for the other considered maps is similar.
Pesin-type relation for subexponential instability

Figure 4. The proportionality constant $\gamma$ between $C_t$ and $N_t$, calculated with respect to the standard partition, as a function of $\alpha$ for the PM (a), Thaler (b) and Bernoulli (c) maps. For the three cases, the typical uncertainty in $\gamma$ is about 1% for samples of $10^4$ trajectories (computed up to $t = 10^6$). The curves are calculated with increments of $10^{-2}$ in the values of $\alpha$.

Zweimüller’s prescription (9) and the values of $N_t$ for different partitions. As one can see, both quantities are indeed proportional, with very good accuracy, irrespective of the partition employed. We also notice that the value of $\gamma$ depends on the details of the maps, specifically on the value of $z$ and, consequently, of $\alpha$, see figure 4.

4. Final remarks

We close with some final remarks on the work [5] and what has led to its misleading conclusion that

$$h_\mu(f) = \alpha \langle \lambda^{(\alpha)} \rangle$$

(14)

for systems of the type (1). The first observation is that (14) is incompatible with the re-scaling transformation $\omega \to \xi \omega$, which must be a symmetry for the dynamics in the infinite measure case. The dynamical quantity on the right-handed side, whatever way the average is taken, must be invariant under such transformation, while the left-handed side is certainly not, see (6). Such a problem can be elucidated recalling that the ADK theorem gives (see, for instance, [11])

$$\langle \lambda^{(\alpha)} \rangle_{\text{ADK}} = \frac{1}{ba} \left( \frac{a}{a} \right)^{\alpha} \frac{\sin(\pi \alpha)}{\pi \alpha} \int \ln |f'(x)| \omega(x) dx,$$

(15)

from which Rohlin’s formula (6) for the Krengel entropy implies immediately that

$$\frac{1}{b} h_\mu(f) = a \left( \frac{a}{a} \right)^{\alpha} \frac{\pi \alpha}{\sin(\pi \alpha)} \langle \lambda^{(\alpha)} \rangle_{\text{ADK}},$$

(16)

doi:10.1088/1742-5468/2012/03/P03010

8
Pesin-type relation for subexponential instability

which is the correct relation between Krengel entropy and a dynamically meaningful average of subexponential Lyapunov exponents for maps of the type (1). The ADK average is not only a dynamically meaningful average, it is essentially the dynamically meaningful average for these systems. For instance, the average of the subexponential Lyapunov exponents (4) calculated for randomly chosen (with any absolutely continuous measure with respect to the usual Lebesgue measure on the interval [0, 1]) initial conditions x will converge to the ADK average for large t, see [11] for some recent applications of this important fact. Notice also that both sides of (16) are invariant under the symmetry $\omega \rightarrow \xi \omega$.

A closer inspection of [5] (see, in particular, their equation (10)) shows that they, when dealing with the continuous time stochastic linear model proposed in [18], tacitly choose a value for $\xi$ such that

$$b = \left(\frac{a}{\alpha}\right)^{\alpha-1} \frac{\sin(\pi \alpha)}{\pi \alpha}, \quad (17)$$

breaking the measure re-scaling symmetry of (16) and rendering it in its $\xi$-dependent form (14). However, one could have chosen any other value for $\xi$, leading to a distinct value of $b$ and to a completely different ‘relation’ between the Krengel entropy and the ADK average. Since these relations do depend on some specific multiplicative constant of the infinite invariant measure, they have no dynamical meaning. It is interesting to notice that $N_t$ is also considered as a Mittag–Leffler random variable in [5] by using renewal theory in a different manner, but its relation to $C_t$ is not stated. Instead, it is used as the hypothesis that $\sum_{k=0}^{t-1} \ln |f'(f^k(x))| \propto N_t$ in order to conclude that $\lambda_t^{(\alpha)}$ is Mittag–Leffler-distributed. Such an assumption presumes the convergence

$$\frac{\lambda_t^{(\alpha)}}{\langle \lambda_t^{(\alpha)} \rangle} \rightarrow \frac{N_t}{\langle N_t \rangle} \quad (18)$$

for almost all trajectories, which is indeed correct, but it is a very strong assumption without prior knowledge of Zweimüller’s relation (5).

We close by noticing that, comparing (7) and (16), it is clear that the subexponential KS entropy is the appropriate entropy for extending the Pesin relation for weakly chaotic systems. Relation (7) is simpler than (16) and, mainly, it is more powerful since it holds for almost all single trajectories, in contrast to (16), where a statistical description involving many trajectories is necessary (and, moreover, an invariant measure, which usually is not explicitly known, is required for the calculation of Krengel entropy). Furthermore, the ergodic transition $\alpha \rightarrow 1$ in (16) is rather awkward in comparison with the same transition for the relation (7), which is straightforward and natural since the Mittag–Leffler distribution tends to a Dirac $\delta$ function for $\alpha \rightarrow 1$.

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Pesin-type relation for subexponential instability

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