On the Complexity and Volume of Hyperbolic 3-Manifolds.

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Abstract

We compare the volume of a hyperbolic 3-manifold $M$ of finite volume and the complexity of its fundamental group.

1 Introduction.

Complexity of 3-manifolds and groups. One of the most striking corollaries of the recent solution of the geometrization conjecture for 3-manifolds is the fact that every aspherical 3-manifold is uniquely determined by its fundamental group. It seems to be natural to think that a topological/geometrical description of a 3-manifold $M$ produces the simplest way to describe its fundamental group $\pi_1(M)$; on the other hand, the simplest way to define the group $\pi_1(M)$ gives rise to the most efficient way to describe $M$. More precisely, we want to compare the complexity of 3-manifolds and their fundamental groups.

The study of the complexity of 3-manifolds goes back to the classical work of H. Kneser [K]. Recall that the Kneser complexity invariant $k(M)$ is defined to be the minimal number of simplices of a triangulation of the manifold $M$. The main result of Kneser is that this complexity serves as a bound of the number of embedded incompressible 2-spheres in $M$, and bounds the numbers of factors in a decomposition of $M$ as a connected sum. A version of this complexity was used by W. Haken to prove the existence of hierarchies for a large class of compact 3-manifolds (called since then Haken manifolds). Another measure of the complexity $c(M)$ for the 3-manifold $M$ is due to S. Matveev. It is the minimal number of vertices of a special spine of $M$ [Ma]. It is shown that in many important cases (e.g. if $M$ is a non-compact hyperbolic 3-manifold of finite volume) one has $k(M) = c(M)$ [Ma].

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The rank (minimal number of generators) is also a measure of complexity of a finitely generated group. According to the classical theorem of I. Grushko [Gr], the rank of a free product of groups is the sum of their ranks. This immediately implies that every finitely generated group is a free product of finitely many freely indecomposable factors, which is an algebraic analogue of Kneser theorem.

For a finitely presented group $G$ a measure of complexity of $G$ was defined in [De]. Here is its definition:

**Definition 1.1.** Let $G$ be a finitely presented group. We say that $T(G) \leq t$ if there exists a simply-connected 2-dimensional complex $P$ such that $G$ acts freely and simplicially on $P$ and the number of 2-faces of the quotient $\Pi = P/G$ is less than $t$.

If the group $G$ is defined by a presentation $\langle a_1, ... a_r; R_1, ..., R_n \rangle$ the sum $\Sigma(|R_i| - 2)$ serves as a natural bound for $T(G)$.

Note that an inequality between Kneser complexity and this invariant is obvious. Indeed, by contracting a maximal subtree of the 2-dimensional skeleton of a triangulation of $M$ one obtains a triangular presentation of the group $\pi_1(M)$. Since every 3-simplex has four 2-faces it follows

$$T(\pi_1(M)) \leq 4k(M).$$

In order to compare the complexity of a manifold and that of its fundamental group, it is enough to find a function $\theta$ such that $\theta(\pi_1(M)) \leq T(\pi_1(M))$. Note that the existence of such a function follows from G. Perelman’s solution of the geometrization conjecture [Pe 1-3]. Indeed there could exist at most finitely many different 3-manifolds having the fundamental groups isomorphic to the same group $G$ (for irreducible 3-manifolds with boundary this was shown much earlier in [Swa]). The question which still remains open is to describe the asymptotic behavior of the function $\theta$.

Note that for certain lens spaces the following inequality is proven in [PP]:

$$c(L_{n,1}) \leq \ln n \approx \text{const} \cdot T(\mathbb{Z}/n\mathbb{Z}).$$

However, the above problem remains widely open for irreducible 3-manifolds with infinite fundamental group. If $M$ is a compact hyperbolic 3-manifold, D. Cooper showed [C]:

$$\text{Vol} M \leq \pi \cdot T(\pi_1(M))$$

where $\text{Vol} M$ is the hyperbolic volume of $M$. Note that the converse inequality in dimension 3 is not true: there exists infinite sequences of different hyperbolic 3-manifolds $M_n$ obtained by Dehn filling on a fixed finite volume hyperbolic manifold $M$ with cusps such that $\text{Vol} M_n < \text{Vol} M$ [Th]. The ranks of the groups $\pi_1(M_n)$ are all bounded by $\text{rank}(\pi_1(M))$ and since $\pi_1(M_n)$ are not isomorphic, we must have $T(\pi_1(M_n)) \to \infty$. So the invariant $T(\pi_1(M))$ is not comparable.
with the volume of hyperbolic 3-manifolds. This difficulty can be overcome using the following relative version of the invariant $T$ introduced in [De]:

**Definition 1.2.** Let $G$ be a finitely presented group, and $\mathcal{E}$ be a family of subgroups. We say that $T(G, \mathcal{E}) \leq t$ if there exists a simply-connected 2-dimensional complex $P$ such that $G$ acts simplicially on $P$, the number of 2-faces of the quotient (an orbihedron) $\Pi = P/G$ is less than $t$, and the stabilizers of vertices of $P$ are elements of $\mathcal{E}$.

The main goal of the present paper is to obtain uniform constants comparing the volume of a hyperbolic 3-manifold $M$ of finite volume and the relative invariant $T(\pi_1(M), \mathcal{E})$ where $\mathcal{E}$ is the family of its elementary subgroups.

To finish our historical discussion let us point out that the relative invariant $T(G, \mathcal{E})$ allows one to prove the accessibility of a finitely presented group $G$ without 2-torsion over elementary subgroups [DePo1]. Using these methods it was shown recently that for hyperbolic groups without 2-torsion any canonical hierarchy over finite subgroups and one-ended subgroups is finite [Va]. The relative invariant $T$ and the hierarchical accessibility was used in [DePo2] to give a criterion of the co-Hopf property for geometrically finite discrete subgroups of Isom($\mathbb{H}^n$).

**Main Results.** Let $M$ be a hyperbolic 3-manifold of finite volume. We consider the family $E_\mu$ of all elementary subgroups of $\pi_1(M)$ having translation length less than the Margulis constant $\mu = \mu(3)$. The family $E_\mu$ includes all parabolic subgroups of $G$ as well as cyclic loxodromic ones representing geodesics in $M$ of length less than $\mu$ (see also the next Section).

The first result of the paper is the following:

**Theorem A.** There exists a constant $C$ such that for every hyperbolic 3-manifold $M$ of finite volume the following inequality holds:

$$C^{-1}T(G, E_\mu) \leq \text{Vol}(M) \leq CT(G, E_\mu) \quad (*)$$

The following are corollaries of Theorem A.

**Corollary 1.3.** Suppose $M_n \xrightarrow{f_n} M$ is a sequence of finite coverings over a finite volume 3-manifold $M$ such that $\text{deg} f_n \to +\infty$. Then $T(\pi_1(M_n), E_n) \to +\infty$, where $E_n$ is the above system of elementary subgroups of $\pi_1(M_n)$ whose translation length is less than $\mu$.

**Proof:** The statement follows immediately from the right-hand side of $(*)$ since $\text{Vol}(M_n) \to \infty$. QED.
Corollary 1.4. Let \( M_n \) be a sequence of different hyperbolic 3-manifolds obtained by Dehn surgery on a cusped hyperbolic 3-manifold of finite volume \( M \). Then
\[
T(\pi_1(M_n), E_n) \leq C \cdot \text{Vol}(M) < +\infty.
\]

Proof: The left-hand side of (*) gives
\[
T(\pi_1(M_n), E_n) \leq C \cdot \text{Vol}(M_n),
\]
and by [Th] one has \( \text{Vol}(M_n) < \text{Vol}(M) \). QED.

As it is pointed out in Corollary 1.3 above we must have \( T(\pi_1(M_n)) \to +\infty \) for the absolute invariant. Our next result is the following:

Theorem B. (Generalized Cooper inequality) Let \( E \) be the family of elementary subgroups of \( G \), then one has
\[
\text{Vol}(M) \leq \pi \cdot T(\pi_1(M), E) \tag{**}
\]

Note that Theorem B gives a generalization of the Cooper inequality (C) for the relative invariant \( T(G, E) \). Furthermore, if one puts \( E = E_\mu \), then Theorem B implies the right-hand side of (*) in Theorem A. Theorems A and B together have several immediate consequences:

Corollary 1.5. For the constant \( C \) from Theorem A the following statements hold:

i) Let \( M \) be a finite volume hyperbolic 3-manifold and \( E_\mu \) and \( E \) be the above families of elementary subgroups of \( \pi_1(M) \). Then
\[
T(\pi_1(M), E_\mu) \leq C \cdot \pi \cdot T(\pi_1(M), E).
\]

ii) Let \( M \) be a hyperbolic 3-manifold such that \( M = M_\text{thick} \), i.e. every loop in \( M \) of length less than \( \mu \) is homotopically trivial. Then
\[
T(\pi_1(M)) \leq C \cdot \pi \cdot T(\pi_1(M), E).
\]
Proof: i) By Theorems A and B we have

\[ T(\pi_1(M), E_\mu) \leq C \text{Vol}(M) \leq C \cdot \pi \cdot T(\pi_1(M), E). \quad QED. \]

ii) Since \( E_\mu = \emptyset \) the result follows from i). \( QED. \)

Let us now briefly describe the content of the paper. In Section 2 we provide some preliminary results needed in the future. The proof of Theorem B is given in Section 3, it provides a ”simplicial blow-up” procedure for an orbihedron. In Section 4 we prove the left-hand side of the inequality (*) using some standard techniques and the results of Section 2. In the last Section 5 we discuss some open questions related to the present paper.

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2 Preliminary results.

Let us recall few standard definitions which we will use in the future. We say that \( G \) splits as a graph of groups \( X = (X, (C_e)_{e \in X^1}, (G_v)_{v \in X^0}) \) (where \( C_e \) and \( G_v \) denote respectively edge and vertex groups of the graph \( X \)) if \( G \) is isomorphic to the fundamental group \( \pi_1(X) \) in the sense of Serre [Sc]. The Bass-Serre tree \( T \) is the universal cover of the graph \( X = T/G \). When \( X \) has only one edge, we will say that \( G \) splits as an amalgamated free product (resp. an HNN-extension) if \( X \) has two vertices (resp. one vertex).

Definition 2.1. Let \( G \) be a group acting on a tree \( T \). A subset \( H \) of \( G \) is elliptic (resp. hyperbolic) in \( T \) (and in the graph \( T/G \)) if \( H \) fixes a point in \( T \) (resp. does not fix a point in \( T \)). If \( T \) is the Bass-Serre tree of a splitting of \( G \) as a graph of groups, \( H \) is elliptic if and only if it is conjugate into a vertex group of this graph.

We say that \( G \) splits relatively to a family of subgroups \( (E_1, \ldots, E_n) \), or that the pair \( (G, (E_i)_{1 \leq i \leq n}) \) splits as a graph of groups, if \( G \) splits as a graph of groups such that all the groups \( E_i \) are elliptic in this splitting. A \( (G, (E_i)_{1 \leq i \leq n}) \)-tree is a \( G \)-tree in which \( E_i \) are elliptic for all \( i \).

Definition 2.2. Suppose \( G \) splits as a graph of groups

\[ G = \pi_1(X, C_e, G_v) \]  \hspace{1cm} (1)

relatively to a family of subgroups \( E_i \) \( \{i = 1, \ldots, n\} \).
The decomposition (1) such that all edge groups are non-trivial is called reduced if every vertex group $G_v$ cannot be decomposed relatively to the subgroups $E_i \in G_v$ as a graph of groups having one of the subgroups $C_e$ as a vertex group.

The decomposition (1) is called rigid if whenever one has a $(G, (E_i)_{i \in \{1, \ldots, n\}})$-tree $T^*$ such that the subgroup $C_e$ contains a non-trivial edge stabilizer then $C_e$ acts elliptically on $T^*$.

It was shown in [De] that the sum of relative $T$-invariants of the vertex groups of a reduced splitting is less than or equal to the absolute invariant of $G$.

Recall that the Margulis constant $\mu = \mu(n)$ is a number for which any $n$-dimensional hyperbolic manifold $M$ can be decomposed into thick and thin parts: $M = M_{\text{thick}} \cup M_{\text{thin}}$ such that the injectivity radius at each point of $M_{\text{thin}}$ is less than $\mu/2$, and $M_{\text{thick}} = M \setminus M_{\text{thin}}$. By the Margulis Lemma the components of $M_{\text{thin}}$ are either parabolic cusps or regular neighborhoods (tubes) of closed geodesics of $M$ of length less than $\mu$. We will denote by $E = E(\pi_1(M))$ (respectively $E_\mu = E_\mu(\pi_1(M))$) the system of elementary subgroups of $\pi_1(M)$ (respectively the systems of subgroups of $\pi_1 M_{\text{thick}}$). We will need the following:

**Lemma 2.3.** Let $H$ be a group admitting the following splitting as a graph of groups:

$$H = \pi_1(X, C_e, G_v),$$

where each vertex group $G_v$ is a lattice in $\text{Isom}(\mathbb{H}^n)$ ($n > 2$) and $C_e \in E(G_v)$ ($n > 2$).

Then (2) is a reduced and rigid splitting of the couple $(H, \mathcal{E})$ where $\mathcal{E} = \bigcup_v E(G_v)$.

**Remark 2.4.** The above Lemma will be further used in a very particular geometric situation when the group $H$ is the fundamental group of the double of the thick part $M_{\text{thick}}$ of $M$ along its boundary.

**Proof:** We first claim that it is enough to prove that every vertex group $G_v$ of the graph $X$ cannot split non-trivially over an elementary subgroup. Indeed, if it is the case then obviously (2) is reduced. If it is not rigid, then the couple $(H, \mathcal{E})$ acts on a simplicial tree $T^*$ such that one of the groups $C_e$ contains an edge stabiliser $C_e^*$ of $T^*$ and therefore acts hyperbolically on $T^*$. It follows that the vertex group $G_v$ containing $C_e$ also acts hyperbolically on $T^*$ and so is decomposable over elementary subgroups.

Let us now fix a vertex $v$ and set $G = G_v$. The Lemma now follows from the following statement:

**Sublemma 2.5.** [Be] Let $G$ be the fundamental group of a Riemannian manifold $M$ of finite volume of dimension $n > 2$ with pinched sectional curvature within $[a, b]$ for $a \leq b < 0$. Then $G$ does not split over a virtually nilpotent group.
Proof: We provide below a direct proof of this Sublemma in the case of the constant curvature. Suppose, on the contrary, that
\[ G = A \ast_C B \quad \text{or} \quad G = A \ast_C, \]
where \( C \) is an elementary subgroup. Let \( \tilde{C} \) be the maximal elementary subgroup containing \( C \). The group \( \tilde{C} \) is virtually abelian and contains a maximal abelian subgroup \( \tilde{C}_0 \) of finite index. We have the following

Claim 2.6. The group \( \tilde{C}_0 \) is separable in \( G \).

Proof: Recall that the subgroup \( \tilde{C}_0 \) is said separable if \( \forall g \in G \setminus \tilde{C}_0 \) there exists a subgroup of finite index \( G_0 < G \) such that \( \tilde{C}_0 < G_0 \) and \( g \notin G_0 \). Since \( \tilde{C}_0 \) is a maximal abelian subgroup of \( G \), and \( g \notin \tilde{C}_0 \), it follows that there exists \( h \in \tilde{C}_0 \) such that \( \gamma = gh_0g^{-1}h_0^{-1} \neq 1 \). The group \( G \) is residually finite, so there exists an epimorphism \( \tau : G \to K \) to a finite group \( K \) such that \( \tau(\gamma) \neq 1 \). Since \( \tau(\tilde{C}_0) \) is abelian, \( \tau(\gamma) \notin \tau(\tilde{C}_0) \) and the subgroup \( G_0 = \tau^{-1}(\tau(\tilde{C}_0)) \) satisfies our Claim. QED.

Denote \( C_0 = C \cap \tilde{C}_0 \) (the maximal abelian subgroup of \( C \)). We have \( \tilde{C} = \bigcup_{i=1}^{m} C_0 C_0 \cup C_0 \). So by the Claim we can find a subgroup of finite index \( G_0 \) of \( G \) containing \( C_0 \) such that \( c_i \notin G_0 \) \((i = 1, \ldots, m)\). Then \( G_0 \cap \tilde{C} = C_0 \) is abelian group and by the Subgroup Theorem \( \text{[SW]} \) we have that \( G_0 \) splits as :
\[ G_0 = A_0 \ast_{C_0} B_0 \quad \text{or} \quad G_0 = A_0 \ast_{C_0}, \]
where \( C_0' < C_0 \) is also abelian. Suppose first that \( G_0 = A_0 \ast_{C_0} B_0 \), since \( G_0 \) is not elementary group, one of the vertex subgroups of this splitting, say \( A_0 \) is not elementary too. Then the map \( \varphi : G_0 \to (c A_0 c^{-1}) \ast_{C_0} B_0, \ c \in C_0 \), such that \( \varphi|_{A_0} = c A_0 c^{-1} \) and \( \varphi|_{B_0} = id \) is an exterior automorphism (as \( c \) commutes with every element of \( C_0 \)) of infinite order. So the group of the exterior automorphisms \( \text{Out}(G_0) \) is infinite. This contradicts to the Mostow rigidity as \( G_0 \) is still a lattice. In the case of HNN-extension \( G_0 = A_0 \ast_{C_0} = < A_0, t | t C_0 t^{-1} = \psi(C'_0) > \) suppose first that \( t \) does not belong to the centralizer \( Z(C_0) \) of \( C_0 \) in \( G_0 \). Then we put \( \varphi|_{A_0} = c A_0 c^{-1} \) for some \( c \in C_0 \) such that \( [c, t] \neq 1 \) and \( \varphi(t) = t \). Since \( t \notin Z(C'_0) \) we obtain again that \( \varphi \) is an infinite order exterior automorphism which is impossible. If, finally, \( t \in Z(C'_0) \) then put \( \varphi|_{A_0} = id \) and \( \varphi(t) = t^2 \) and it is easy to see that \( G'_0 = \varphi(G_0) \) is a subgroup of index 2 of \( G_0 \) isomorphic to \( G_0 \). Then \( \text{Vol}(\mathbb{H}^n / \varphi(G_0)) < +\infty \) and again by Mostow rigidity we must have \( \text{Vol}(\mathbb{H}^n / G_0) = \text{Vol}(\mathbb{H}^n / \varphi(G_0)) \), and so \( \varphi : G_0 \to G_0 \) should be surjective. A contradiction. The Sublemma 2.5 and Lemma 2.3 follow. QED.

\(^2\)The argument is due to M. Kapovich and one of the authors is thankful for sharing it with him (about 20 years ago).
3 Proof of the generalized Cooper inequality.

The aim of this Section is to prove Theorem B stated in the Introduction:

**Theorem B.** Let $E$ be an arbitrary family of elementary subgroups of $G$, then

$$\text{Vol}(M) \leq \pi \cdot T(\pi_1(M), E)$$  \hspace{1cm} (1)

**Proof:** If $E = \emptyset$, then $\text{Vol}(M) < \pi \cdot (L - 2n)$, where $L$ is the sum of the word-lengths of the relations of $\pi_1(M)$ and $n$ is the number of relations $[C]$. Let $D$ be a disk representing a relation in the presentation complex $R$ of $\pi_1(M)$. Then, triangulating $D$ by triangles having vertices on $\partial D$, we obtain $|D| - 2$ triangles. So $L - 2n$ represents the total number of triangles in $R$. Thus Cooper’s result implies $\text{Vol}(M) \leq \pi \cdot T(\pi_1(M))$.

Suppose now that $M = \mathbb{H}^3/G$ where $G < \text{Isom}(\mathbb{H}^3)$ is a lattice (uniform or not) and let $E$ be a family of elementary subgroups of $G$. Let $P$ be a simply-connected 2-dimensional polyhedron admitting a simplicial action of $G$ such that the vertex stabilizers are elements of the system $E$. Let us also assume that the quotient $\Pi = P/G$ is a finite orbihedron. We will need the following:

**Lemma 3.1.** There exists a $G$-equivariant simplicial continuous map $f : P \to \mathbb{H}^3 \cup \partial \mathbb{H}^3$ such that the images of the 2-simplices of $P$ are geodesic triangles or ideal triangles of $\mathbb{H}^3$.

**Proof:** Let us first construct a $G$-equivariant continuous map $f : P \to \mathbb{H}^3 = \mathbb{H}^3 \cup \partial \mathbb{H}^3$ such that the image of the fixed points for the action $G$ on $P$ belong to $\partial \mathbb{H}^3$. To do it we apply the construction from [DePo, Lemma 1.6] where instead of a tree as the goal space we will use the hyperbolic space $\mathbb{H}^3$. Let us first construct a map $\rho : E \to \mathbb{H}^3$ as follows. Since the group $G$ is torsion-free we can assume that all non-trivial groups in $E$ are infinite. Then for every elementary group $E_0 \in E$ we put $\rho(E_0) = x \in \partial \mathbb{H}^3$ to be one of the fixed points for the action of $E_0$ on $\partial \mathbb{H}^3$ (by fixing a point $O \in \partial \mathbb{H}^3$ for the image of the trivial group $\rho(id)$). The map $\rho$ has the following obvious properties:

a) $\forall E_1, E_2 \in E$ if $E_1 \cap E_2 \neq \emptyset$ then $\rho(E_1) = \rho(E_2)$;

b) if $\tilde{E}_0$ is a maximal elementary subgroup then $\rho(E_0) = \rho(\tilde{E}_0)$ and $\rho(g\tilde{E}_0g^{-1}) = g\rho(\tilde{E}_0)$ ($g \in G$).

We now choose the set of $G$-non-equivalent vertices $\{p_1, ..., p_l\} \subset P$ representing all vertices of $\Pi = P/G$. We first construct a map $f$ on zero-skeleton $P^{(0)}$ of the complex $P$ by putting $f(p_i) = \rho(E_i)$ and then extend it equivariantly $f(gp_i) = gf(p_i)$ ($g \in G$).
Suppose now $y = (q_1, q_2)$ ($q_1, q_2 \in P^{(0)}$) is an edge of $P$. To define $f$ on $y$ we distinguish two cases: 1) $H = \text{Stab}(y) \neq 1$ and 2) $H = 1$.

In the first case we have necessarily that $E_{q_1} \cap E_{q_2} = H_0$ is an infinite elementary group where $E_{q_i}$ is the stabilizer of $q_i$. Then there exist $g_i \in G$ such that $q_i = g_i(p_{k_i}) \ (i = 1, 2)$. So $E_{q_i} = g_iE_{p_{k_i}}g_i^{-1}$ and $g_1E_{p_1}g_1^{-1} \cap g_2E_{p_2}g_2^{-1} = H_0$. It follows that $E_{p_1} \cap g_1^{-1}g_2E_{p_1}g_2^{-1}g_1$ is an infinite group and, therefore $f(p_1) = g_1^{-1}g_2(f(p_2))$ implying that

$$f(q_1) = f(g_1p_1) = f(g_2p_2) = f(q_2).$$

In the case 2) the stabilizer of the infinite geodesic $l = \text{Stab}(y) \subseteq \mathcal{P}$ is trivial so we extend $f : y \to l$ by a piecewise-linear homeomorphism. Having defined the map $f$ as above on the maximal set of non-equivalent edges of $P^{(1)}$ under $G$, we extend it equivariantly to the 1-skeleton $P^{(1)}$ by putting $f(gy) = gf(y)$ ($g \in G$). Finally we extend $f$ piecewise linearly to the 2-skeleton $P^{(2)}$.

We obtain a $G$-equivariant continuous map $f : P \to \mathbb{H}^3$ such that the all 2-faces of the simplicial complex $f(P) \cap \mathbb{H}^3$ are ideal geodesic triangles. The Lemma is proved. \textit{QED.}

\textbf{Remarks 3.2.} 1. Note that the above Lemma is true in any dimension. We restricted our consideration to dimension 3 since the further argument will only concern this case.

2. If the system $E$ contains only parabolic subgroups one can claim that the action of $G$ on $f(P) \cap \mathbb{H}^3$ is in addition proper. Indeed, using the convex hull $\mathcal{P} \subseteq \mathbb{H}^3$ of the maximal family of non-equivalent parabolic points constructed in [EP] the above argument gives the map $f : P \to \overline{\mathcal{P}} \subseteq \mathbb{H}^3$. By [EP, Proposition 3.5] the set of faces of $\mathcal{P}$ is locally finite in $\mathbb{H}^3$. Since the boundary of each face of the 2-orbihedron $f(P)$ constructed above belongs to $\partial \mathcal{P}$, we obtain that the set of 2-faces of $f(P) \subseteq \mathbb{H}^3$ is locally finite in this case.

If now $W$ is the set of the fixed points for the action of $G$ on $P$, we put $P' = P \setminus W$ and $Q' = f(P') = f(P) \cap \mathbb{H}^3$. Let also $\nu : P' \to \Pi$ and $\pi : \mathbb{H}^3 \to M = \mathbb{H}^3/G$ denote the natural projections. Then by Lemma 3.1 the map $f$ projects to a simplicial map $F : (\Pi' = P'/G) \to Q'/G \subseteq M$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
P' & \xrightarrow{f|_{P'}} & Q' \subseteq \mathbb{H}^3 \\
\nu \downarrow & & \downarrow \pi \\
\Pi' & \xrightarrow{F} & Q'/G \subseteq M
\end{array}
$$

Note that, if $\Pi$ is a simplicial polyhedron, it is proved in [C] that the hyperbolic area of $F(\Pi)$ bounds the volume of the manifold $M$. This argument does not work if $\Pi$ is an orbihedron but
not a polyhedron. Indeed the complex \( Q' \) above is not necessarily simply connected. So the group \( G \) is not isomorphic to \( \pi_1(Q'/G) \) but is a non-trivial quotient of it. Our goal now is to construct a new simplicial polyhedron \( \Sigma \) with the fundamental group \( G \) whose image into \( M \) has area arbitrarily close to that of \( F(\Pi') \). So the main step in the proof of Theorem B is the following:

**Proposition 3.3.** (simplicial blow-up procedure). For every \( \varepsilon > 0 \) there exists a 2-dimensional complex \( \Sigma_\varepsilon \) and a simplicial map \( \varphi_\varepsilon : \Sigma_\varepsilon \to M \) such that

1) The induced map \( \varphi_\varepsilon : \pi_1 \Sigma_\varepsilon \to M \) is an isomorphism.

and

2) For the hyperbolic area one has:

\[ |\text{Area}(\varphi_\varepsilon(\Sigma_\varepsilon)) - \text{Area}(F(\Pi'))| < \varepsilon. \]

**Proof of the Proposition:** Let \( \Pi \) be a finite orbihedron with elementary vertex groups and such that \( \pi_1^{\text{orb}}(\Pi) \cong G \). Let us fix a vertex \( \sigma \) of \( \Pi \) and let \( \tilde{\sigma} \in \nu^{-1}(\sigma) \) be its lift in \( P \). We denote by \( G_\sigma \) the group of the vertex \( \sigma \) in \( G \). By Lemma 3.1 the point \( f(\tilde{\sigma}) \in \partial \mathbb{H}^3 \) is fixed by the elementary group \( G_\sigma \). We will distinguish between the two cases when the group \( G_\sigma \) is loxodromic cyclic or parabolic subgroup of rank 2.

**Case 1. The group \( G_\sigma \) is loxodromic.**

Let \( V \subset \Pi \) be a regular neighborhood of the vertex \( \sigma \). Then the punctured neighborhood \( V \setminus \sigma \) is homotopically equivalent to the one-skeleton \( L^{(1)} \) of the link \( L \) of \( \sigma \).

We will call realization of \( L \) a graph \( \Lambda \subset V \setminus \sigma \) such that the canonical map \( L \to \Lambda \) is a homeomorphism. Let us fix a maximal tree \( T \) in \( \Lambda \), and let \( y_i \) be the edges from \( \Lambda \setminus T \) which generate the group \( \pi_1(L) \) \((i = 1, \ldots, k)\).

By its very definition, the \( G \)-equivariant map \( f : P \to \mathbb{H}^3 \) sends the edges of \( P \) to geodesics of \( \mathbb{H}^3 \). So let \( G_\sigma = \langle g \rangle \) and let \( \gamma \subset M \) be the corresponding closed geodesic in \( M \). We denote by \( A_g \subset \mathbb{H}^3 \) the axis of the element \( g \) and by \( g^+, g^- \) its fixed points on \( \partial \mathbb{H}^3 \). Let us assume that \( f(\tilde{\sigma}) = g^+ \). For \( X \subset M \) we denote by \( \text{diam}(X) \) the diameter of \( X \) in the hyperbolic metric of \( M \).

Recall that the map \( f : P \to \mathbb{H}^3 \cup \partial \mathbb{H}^3 \) constructed in Lemma 3.1 induces the map \( F : \Pi' \to M \). We start with the following:

**Step 1.** For every \( \eta > 0 \) there exists a realization \( \Lambda \) of \( L \) in \( \Pi \) such that for the maximal tree \( T \) of \( \Lambda \) one has

\[ \text{diam}(F(T)) < \eta, \]
Furthermore, for every edge $y_i \in \Lambda \setminus T$ its image $F(y_i)$ is contained in a $\eta$-neighborhood $N_\eta(\gamma) \subset M$ of the geodesic $\gamma$ ($i=1,..,k$).

Proof: We fix a sufficiently small neighborhood $V$ of a vertex $\sigma$ in $\Pi$ (the "smallness" will be specified later on). Let $\tilde{\sigma} \in \nu^{-1}(\sigma)$ be its lift to $P$ and let $\tilde{\Lambda}$ and $\tilde{T}$ be the lifts of $\Lambda$ and $T$ to a neighborhood $\tilde{V} \subset \nu^{-1}(V)$ of $\tilde{\sigma}.$ We are going first to show that, up to decreasing $V,$ the image $f(\tilde{T})$ belongs to a sufficiently small horosphere in $\mathbb{H}^3$ centered at the point $g^+.$

Let $\alpha$ be an edge of $\Pi$ having $\sigma$ as a vertex and $\tilde{\alpha}$ be its lift starting at a point $\tilde{\sigma}.$ Then $a = f(\tilde{\alpha}) \subset \mathbb{H}^3$ is the geodesic ray ending at the point $g^+,$ let $a(t)$ be its parametrization. For a given $t_0$ we fix a horosphere $S_{t_0}$ based at $g^+$ and passing through the point $a(t_0).$ Suppose there is a simplex in $P$ having two edges $\tilde{\alpha} = [\tilde{\sigma},s],[\tilde{\sigma},s_1]$ at the vertex $\tilde{\sigma}$ and an edge $[s,s_1]$ in $\Lambda.$ The horosphere $S_{t_0}$ is the level set of the Busemann function $\beta_{g^+}$ based at the point $g^+.$ So for the geodesic rays $a = f(\tilde{\alpha})$ and $a_1 = f(\tilde{\alpha}_1)$ issuing from the point $g^+$ we have that the points $f(s) = a(t_0)$ and $f(s_1) = a_1(t_0)$ belong to the horosphere $S_{t_0}.$ Proceeding in this way for all simplices whose edges share the vertex $\sigma,$ we obtain that $f(\tilde{T}(0)) \subset S_{t_0} \subset \mathbb{H}^3.$ Since $\Lambda$ is finite, so is the tree $\tilde{T}.$ By choosing $t_0$ sufficiently large ($t_0 > \Delta$) we may assume that $d(\alpha(i(t_0),\alpha(j(t_0))) < \eta$ and $d(\alpha(i(t_0),A) < \eta$ ($i,j = 1,\ldots,k$). We now connect all the vertices of $f(\tilde{T})$ by geodesic segments $b_i \subset \mathbb{H}^3.$ By convexity, and up to increasing the parameter $t_0,$ we also have $d(b_i,A) < \eta.$

By Lemma 3.1 the map $f$ sends the lifts $\tilde{y}_i \in \tilde{T}$ of the edges $y_i \in \Lambda \setminus T$ simplicially to $b_i$ ($i = 1,\ldots,k);$ and $f$ maps $G_\sigma$-equivariantly the preimage $\tilde{\Lambda} = \nu^{-1}(\Lambda)$ to $\mathbb{H}^3.$ Hence the map $f$ projects to the map $F: \Lambda \to M$ satisfying the claim of Step 1.

**Step 2. Definition of the polyhedron $\Pi^\ast$**

Using the initial orbihedron $\Pi$ we will construct a new polyhedron $\Pi^\ast$ having the following properties:

a) $\Pi^{(0)} = \Pi^{-0}$ and $\Pi = \Pi^\ast$ outside of $V$;

b) $\pi_1(L^*) = G_\sigma$, where $L^*$ is the link of $\sigma$ in $\Pi^\ast$;

c) $\pi_1(\Pi^\ast) \cong G$.

The graph $\Lambda$ realizes the link of the vertex $\sigma$ so there exists an epimorphism $\pi_1(\Lambda) \to \varphi G.$ Every edge $y_i \in \Lambda \setminus T$ which is a generator of the group $\pi_1 \Lambda$ is mapped onto $g^{n_{y_i}}$ in $G_\sigma$ ($i = 1,\ldots,k$). We now subdivide each edge $y_i$ by edges $y_{ij}$ ($i = 1,\ldots,k,j = 1,\ldots,n_{y_i}$), and denote by $\Lambda'$ the obtained graph. Let $S$ be a circle considered as a graph with one edge $e$ and one vertex $u.$ Then there exists a simplicial map from $\Lambda'$ to $S$ mapping simplicially each edge $y_{ij}$ onto $S$.

To construct polyhedron $\Pi^\ast,$ we replace the neighborhood $V$ by the cone of the above map. Namely, we first delete the vertex $\sigma$ from $\Pi$ as well as all edges connecting $\sigma$ with $L$. Then we connect the vertices of the edge $y_{ij}$ with the vertex $u \in S$ by edges which we call vertical.
(i = 1, ..., k, j = 1, ..., n_y). So \( \Pi' \) is the union of \( \Pi \setminus V \) and the rectangles \( R_{ij} \), which are bounded by \( y_{ij} \), two vertical edges and the loop \( S \). The set of rectangles \( \{ R_{ij} \mid i = 1, ... , k, j = 1, ... , n_y \} \) realizes the epimorphism \( \pi_1(L) \rightarrow G_\sigma \). By Van-Kampen theorem we have \( \pi_1(\Pi') \cong G_\sigma \), and the conditions a)-c) follow.

**Step 3.** There exists a constant \( c \) (depending only on the topology of \( \Pi \)) such that for all \( \eta > 0 \), there exists a map \( F^- : \Pi' \rightarrow M \) such that

1) \( F^- \) induces an isomorphism on the fundamental groups,

2) \( F^-|_{\Pi \setminus V} = F \),

3) \( \sum_{ij} \text{Area}(F^-(R_{ij})) < c \cdot \eta. \)  (2)

**Proof:** We choose a neighborhood \( V \) of the singular point \( \sigma \) and put \( F^- = F_{|_{\Pi \setminus V}} \). Using Step 2 we transform the orbihedron \( \Pi \) to \( \Pi' \) in the neighborhood \( V \) and let \( P^- \) be the universal covering of \( \Pi' \). Note that, by construction, \( P^- \) is obtained by adding the \( G \)-orbit of the rectangles \( R_{ij} \) to the preimage \( \tilde{\Pi}' = \nu^{-1}(\Pi') \) of the graph \( \Pi' \) (\( i = 1, ... , k, j = 1, ... , n_y \)).

We will now extend the map \( f \) defined on \( P \setminus V \) to the polyhedron \( P^- \setminus P \) as follows. We first subdivide every segment \( b_i \) in \( n_y \) by geodesic subsegments \( b_{ij} \subset b_i \) corresponding to the edges \( y_{ij} \). We now project orthogonally each \( b_{ij} \) to \( A_y \) and let \( \tilde{\gamma} \subset A_y \) denote its image. Let \( \tau_{ij} \subset \mathbb{H}^3 \) be the rectangle formed by \( b_{ij}, \tilde{\gamma} \) and these two orthogonal segments from \( b_{ij} \) to \( A_y \) whose lengths are by Step 1 less than \( \eta \). We extend the map \( f \) simplicially to a map \( F^- \) sending the rectangle \( \nu^{-1}(R_{ij}) \) to the rectangle \( \tau_{ij} \) (\( i = 1, ... , k, j = 1, ... , n_y \)). Note that by construction the lift \( \tilde{S} \) of the circle \( S \) is mapped on \( \tilde{\gamma} \). The map \( F^- \) descends to a map \( F^- : \Pi'_* \setminus \Pi \rightarrow N_\eta(\gamma) \). It induces the epimorphism \( \pi_1(\Pi') \rightarrow G_\sigma \).

Let us now make the area estimates for the added rectangles \( \tau_{ij} \). Each rectangle \( \tau = \tau_{ij} \) has four vertices \( A, B, C, D \) in \( \mathbb{H}^3 \) where \( B = gA, D = g(C) \) and the segment \( [A, B] \subset A_y \) is the orthogonal projection of \( [C, D] \) on \( A_y \). The rectangle \( \tau \) is bounded by these two and two perpendicular segments \( l_1 = [A, C] \) and \( l_2 = [B, D] \) to the geodesic \( A_y \) (\( l_2 = g(l_1) \)). We have \( \tau \subset ABC'D \) where \( \angle BDC' = \frac{\pi}{2} \) and \( \beta = \angle BC'D < \frac{\pi}{2} \). Then by [Be, Theorem 7.17.1] one has \( \cos(\beta) \leq \sinh(d(B, D)) \cdot \sinh(l(\gamma)). \) Therefore \( \text{Area}(\tau) < \frac{\pi}{2} - \beta, \) and \( \sin(\text{Area}(\tau)) \leq \sinh \eta \cdot \sinh l(\gamma). \) Summing up over all segments \( b_{ij} \) we arrive to the formula (2). This proves Case 1.

**Case 2.** The group \( G_\sigma \) is parabolic.

The proof is similar and even simpler in this case. Let again \( T \) be the maximal tree of the graph \( \Lambda \) realizing the link \( L \) of the vertex \( \sigma \). We start by embedding a lift \( \tilde{T}^{(0)} \) of the zero-skeleton
of \( T^0 \) into a horosphere \( S_{t_0} \subset \mathbb{H}^3 \) based at the parabolic fixed point \( p \in \partial \mathbb{H}^3 \) of the group \( G_\sigma = \langle g_1, g_2 \rangle \cong \mathbb{Z} \times \mathbb{Z} \). Then, using Lemma 3.1, we construct an embedding \( f : \Lambda^{(0)} \to S_{t_0} \) of the zero-skeleton of the graph \( \Lambda = \nu^{-1}(\Lambda) \) into the same horosphere \( S_{t_0} \) invariant under \( G_\sigma \) (which was not so in the previous case). Since the number of vertices of \( \tilde{T} \) is finite, for any \( \eta > 0 \) we can choose a horosphere \( S_{t_0} \) \((t_0 > \Delta)\) such that \( \text{diam} \tilde{T} < \eta \). Fixing a point \( O \in S_{t_0} \), we can also assume that \( d(O, \tilde{T}^{(0)}) < \eta \).

Now, let us modify the orbihedron \( \Pi \) in the neighborhood \( V \) of \( \sigma \). First we delete the vertex \( \sigma \) from \( \Pi \) and all edges connecting \( \sigma \) with the graph \( \Lambda \). We then add to the obtained orbihedron a torus \( T \) with two intersecting loops \( C_1 \) and \( C_2 \) representing the generators of \( \pi_1(T, u) \) where \( u \in C_1 \cap C_2 \). To realize the epimorphism \( \pi_1 \Lambda \to G_\sigma \) in \( M \) we proceed as before. For any edge \( y \in \Lambda \setminus T \) corresponding to the element \( g = ng_1 + mg_2 \in G_\sigma \) we add a rectangle \( R \) bounded by \( y \), two edges connecting the end points of \( y \) with \( u \) and a loop \( C \subset T \) representing the element \( g \) in \( \pi_1(T, u) \). Let \( \Pi' \) denote the obtained orbihedron.

Coming back to \( \mathbb{H}^3 \), let us assume for simplicity that \( p = \infty \) and the horosphere \( S_{t_0} \) is a Euclidean plane. By Lemma 3.1 the map \( f \) sends the edges \( \tilde{y}_i \in \Lambda \setminus \tilde{T} \) to the geodesic edges \( b_i \) connecting the vertices of \( f(\tilde{T}) \).

We now construct the rectangles \( \tau_i \) by projecting the end points of the edges \( b_i \) to the corresponding vertices of the Euclidean lattice given by the orbit \( G_\sigma O \). Let us briefly describe this procedure in case of one rectangle \( \tau \). Suppose that the edge \( y \in \Lambda \setminus T \) represents the element \( g = ng_1 + mg_2 \in G_\sigma \). Let \( A \) and \( gA \) be vertices of \( f(\tilde{T}) \) belonging to \( S_{t_0} \) connected by a geodesic segment \( b \) corresponding to \( y \). Let \( \tau \subset \mathbb{H}^3 \) be the geodesic bounded by the edges \( b, l = [O, A], gl, gb \). We extend the map \( f^* : \tilde{R} \to \tau \) where \( \tilde{R} \) is a lift of the corresponding rectangle \( R \) added to \( \Pi' \). The map \( f^* \) descends now to a simplicial map \( F^* : \Pi' \to M \) sending the torus \( T \) into a cusp neighborhood of the manifold \( M \). Since the rectangle \( \tau \) belongs to \( \eta \)-neighborhood of the horosphere \( S_{t_0} \), its area, being close to the Euclidean one, is bounded by \( c \cdot \eta^2 \) for some constant \( c > 0 \). Summing up over all edges \( y_i \) we obtain that the area of added rectangles does not exceed \( k \cdot c \cdot \eta^2 \). This proves Case 2.

To finish the proof of Proposition 3.3, we note that the initial orbihedron \( \Pi \) is finite, so it has a finite number of vertices \( v_1, \ldots, v_l \) whose vertex groups are either loxodromic or parabolic. So for a fixed \( \varepsilon > 0 \), we apply the above simplicial ”blow-up” procedure in a neighborhood of each vertex \( v_i \) \((i = 1, \ldots, l)\). Finally, we obtain a 2-complex \( \Sigma_\varepsilon \); and the simplicial map \( \phi_\varepsilon : \Sigma_\varepsilon \to M \) which induces an isomorphism on the fundamental groups and such that \( |\text{Area}(\phi_\varepsilon(\Sigma_\varepsilon)) - \text{Area}(f(\Pi'))) < \psi(\eta) \), where \( \psi \) is a continuous function such that \( \lim_{\eta \to 0} \psi(\eta) = 0 \). So for \( \eta \) sufficiently small we have \( \psi(\eta) < \varepsilon \) which proves the Proposition. QED.

Proof of Theorem B. Let \( G \) be the fundamental group of a hyperbolic 3-manifold \( M \) of finite volume. Let \( \Pi = P/G \) be a finite orbihedron realizing the invariant \( T(G, E) \), i.e. \( \pi_1^{\text{orb}}(\Pi) \cong G \), all vertex groups of \( \Pi \) are elementary and \( |\Pi^{(2)}| = T(G, E) \). Hence \( \text{Area}(F(\Pi')) = \pi \cdot T(G, E) \).
Then by Proposition 3.3 for any \( \varepsilon > 0 \) there exists a 2-polyhedron \( \Sigma_\varepsilon \) and a map \( \psi_\varepsilon : \Sigma_\varepsilon \to M \) which induces an isomorphism on the fundamental groups and such that

\[
\text{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi T(G, E) + \varepsilon
\]

By [C] we have \( \text{Vol}M < \text{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi T(G, E) + \varepsilon \) (\( \forall \varepsilon > 0 \)). It follows \( \text{Vol}M \leq \pi T(G, E) \). Theorem B is proved. QED.

4 Proof of Theorem A.

In this Section we finish the proof of

**Theorem A.** There exists a constant \( C \) such that for every hyperbolic 3-manifold \( M \) of finite volume the following inequality holds:

\[
C^{-1}T(G, E_\mu) \leq \text{Vol}(M) \leq CT(G, E_\mu)
\]

(\( \ast \))

The right-hand side of the inequality (\( \ast \)) follows from Theorem B if one puts \( E = E_\mu \). So we only need to prove the left-hand side of (\( \ast \)). We start with the following Lemma dealing with \( n \)-dimensional hyperbolic manifolds:

**Lemma 4.1.** Let \( M \) be a \( n \)-dimensional hyperbolic manifold of finite volume. Then there exists a 2-dimensional triangular complex \( W \subset M_{\mu \text{thick}} \) such that \( \pi_1(W) \hookrightarrow \pi_1 M_{\mu \text{thick}} \) is an isomorphism and

\[
|W^2| \leq \sigma \cdot \text{Vol}(M),
\]

where \( |W^2| \) is the number of 2-simplices of \( W \) and \( \sigma = \sigma(\mu) \) is a constant depending only on \( \mu \).

**Proof:** The Lemma is a quite standard fact, proved for \( n = 3 \) in [Th] and more generally in [C], [BGLM], [Ge]. We provide a short proof of it for the sake of completeness. Consider a maximal set of points \( \mathcal{A} = \{a_i \mid a_i \in M_{\mu \text{thick}}, \ d(a_i, a_j) > \mu/4 \} \) where \( d(\cdot, \cdot) \) is the hyperbolic distance of \( M \) restricted to \( M_{\mu \text{thick}} \). By the triangle inequality we obtain

\[
B(a_i, \mu/8) \cap B(a_j, \mu/8) = \emptyset \quad \text{if} \quad i \neq j,
\]

where \( B(a_i, \mu) \) is an embedded ball in \( M \) (isometric to a ball in \( \mathbb{H}^n \)) centered at \( a_i \) of radius \( \mu \). By the maximality of \( \mathcal{A} \) we have \( M_{\mu \text{thick}} \subset \mathcal{U} = \bigcup_i B(a_i, \mu/4) \). Recall that the nerve \( \mathcal{N} \mathcal{U} \) of the covering \( \mathcal{U} \) is constructed as follows. Let \( \mathcal{N} \mathcal{U}^0 = \mathcal{A} \) be the vertex set. The vertices \( a_{i_1}, \ldots, a_{i_{k+1}} \)
span a \( k \)-simplex if for the corresponding balls we have \( \bigcap_{j=1}^{k+1} B(a_{i_j}, \mu/4) \neq \emptyset \). Since the covering \( \mathcal{U} \) is given by balls embedded into \( M \), the nerve \( N \mathcal{U} \) is homotopy equivalent to \( \mathcal{U} \) [Hat, Corollary 4G.3].

Note that \( M^{\mu \text{thick}} \hookrightarrow \mathcal{U} \hookrightarrow M^{\frac{\mu}{2} \text{thick}} \). Indeed if \( x \in \partial B(a_i, \mu/4) \) then by the triangle inequality we have \( B(x, \mu/4) \subset B(a_i, \mu/2) \), and so both are embedded in \( M \). Then \( x \in M^{\frac{\mu}{2} \text{thick}} \). By the Margulis lemma, as the corresponding components of their th in parts are homeomorphic, the embedding \( M^{\mu \text{thick}} \hookrightarrow M^{\frac{\mu}{2} \text{thick}} \) is a homotopy equivalence. It implies that the complex \( N \mathcal{U} \) is homotopy equivalent to \( \hat{M}^{\mu \text{thick}} \). Let \( W \) denote the 2-skeleton of \( N \mathcal{U} \). Then it is a standard topology fact that \( W \) carries the fundamental group of \( N \mathcal{U} \) [Hat]. Therefore, \( \pi_1 W \cong \pi_1 M^{\mu \text{thick}} \).

It remains to count the number of 2-faces of \( W \). We have for the cardinality \( |\mathcal{A}| \) of the set \( \mathcal{A} \):

\[
|\mathcal{A}| \leq \frac{\text{Vol}(M^{\mu \text{thick}})}{\text{Vol}(B(\mu/8))} \leq \frac{\text{Vol}(M)}{\text{Vol}(B(\mu/8))},
\]

where \( B(\mu) \) denotes a ball of radius \( \mu \) in the hyperbolic space \( \mathbb{H}^n \). The number of faces of \( W \) containing a point of \( \mathcal{A} \) as a vertex is at most \( m = \frac{\text{Vol}(B(\mu/2))}{\text{Vol}(B(\mu/8))} \). Then

\[
|W^{(2)}| \leq C_m^2 \frac{\text{Vol}(M)}{\text{Vol}(B(\mu/8))} = \sigma \cdot \text{Vol}(M),
\]

where \( \sigma = \sigma(\mu) = \frac{C_m^2}{\text{Vol}(B(\mu/8))} \). This completes the proof of the Lemma.

Suppose now that \( M \) is a hyperbolic 3-manifold of finite volume and let \( \mu = \mu(3) \) be the 3-dimensional Margulis constant. We are going to use a result of [De] which we need to adapt to our Definition 1.2 of the invariant \( T \). So we start with the following:

**Remark 4.2.** In the definition of the invariant \( T \) in [De] there is one more additional condition compared to our Definition 1.2. Namely, it requires that every element of a system \( E \) fixes a vertex of \( P \). To be able to use the results of [De] we will denote by \( T_0(G, E) \) the invariant defined in [De] and keep the notation \( T(G, E) \) for that of our Definition 1.2. Notice that nothing changes for the absolute invariant \( T(G) \).

Let \( l_1, \ldots, l_k \) be the set of closed geodesics in \( M \) of length less than \( \mu \). Then by [Ko] the manifold \( M' = M \setminus \bigcup_i l_i \) is a complete hyperbolic manifold of finite volume and \( \pi_1 M^{\mu \text{thick}} \cong \pi_1(M') \).

Let \( \mathcal{E}_\mu \) denote the system \( \pi_1(\partial M^{\mu \text{thick}}) \) of fundamental groups of the boundary components of the thick part \( M^{\mu \text{thick}} \). We have the following:
Lemma 4.3.

\[ T_0(\pi_1(M), E_\mu) \leq T_0(\pi_1(M'), \pi_1(\partial M')) \leq T_0(\pi_1(M), E_\mu) + 2k. \]  

(5)

Proof: 1) Consider first the left-hand side. Let \( G = \pi_1(M) \) and \( G' = \pi_1(M') \). Let \( E'_\mu = \{E_{k+1}, ..., E_n\} \) be the set of fundamental groups of cusps of \( M_{\text{thin}} \). Let us fix a two-dimensional \((G', E'_\mu)\)-orbihedron \( P' \) containing \( T_0(G', E'_\mu) \) triangular 2-faces. The pair \((G', E'_\mu)\) acts on its orbihedral universal cover \( P' \). Let \( N(l_i) \) be a regular neighborhood of the geodesic \( l_i \in M \) \((i = 1, ..., k)\) and \( H_i = \langle \alpha_i, \beta_i \rangle \) be the fundamental group of the torus \( T_i = \partial N(l_i) \) where \( \alpha_i \) is freely homotopic to \( l_i \) in \( N(l_i) \). The group \( H_i \) fixes a point \( x_i \in P' \). We will now construct a 2-orbihedron \( P \) for the couple \((G, E_\mu)\) as follows. The group \( G \) is the quotient of \( G' \) by adding the relation \( \beta_i = 1 \) \((i = 1, ..., k)\). We identify the vertices of \( P' \) equivalent under the groups generated by \( \beta_i \) \((i = 1, ..., k)\). The natural projection map \( P' \rightarrow P \) consists of contracting each edge of \( P' \) of the type \((y, \beta(y)) \((y \in P^{(0)})\) to a point. The projection has connected fibres so the 2-orbihedron \( P \) is simply connected and the pair \((G, E_\mu)\) acts on it. The procedure did not increase the number of 2-faces, and we have: \(|\Pi^{(2)} = P/G| \leq |\Pi^{(2)} = P'/G'|. Thus \( T_0(\pi_1(M), E_\mu) \leq T_0(\pi_1(M'), \pi_1(\partial M')) = E'_\mu)). \)

2) Let \( \Pi \) be the 2-orbihedron which realizes \( T_0(\pi_1(M), E_\mu) \), and let \( P \) be its universal cover. To obtain a \((\pi_1(M'), E'_\mu)\)-orbihedron we modify \( P \) as follows. Let \( H_i = \langle h_i \rangle \) be the loxodromic subgroup corresponding to the geodesic \( l_i \subset M \) of length less than \( \mu \) \((i = 1, ..., k)\). Let \( x_i \in P \) be a vertex fixed by the subgroup \( H_i \). Notice that the group \( G' \) is generated by \( G \) and elements \( \beta_i \) such that \( [h_i, \beta_i] = 1 \) \((i = 1, ..., k)\). So we add to \( \Pi \) a new loop \( \beta_i \) (by identifying it with the corresponding element in \( G \)) and glue a disk whose boundary is the loop corresponding to \( [h_i, \beta_i] \). By triangulating each such a disk we add \( 2k \) new triangles to \( \Pi^{(2)} \). Thus the universal cover \( P' \) is obtained by adding to \( P \) a vertex \( y_i \) and its orbit \( \{gy_i\} \), so that the points \( \beta_i h_i gy_i \) are identified with \( h_i \beta_i gy_i \). We further add the rectangle \( gD_i \) \((g \in G)\) whose vertices are \( h_i gy_i, \beta_i h_i gy_i, \beta_i gy_i, gy_i \) and subdivide it by one of the diagonal edges, say \( (h_i gy_i, \beta_i gy_i) \) \((i = 1, ..., k)\). The construction gives a new 2-complex \( P' \) on which the pair \((G', E'_\mu)\) acts simplicially. We claim that \( P' \) is simply connected. Indeed if \( \alpha \) is a loop on it, since \( P \) is simply connected, \( \alpha \) is homotopic to a product of loops belonging to the disks \( gD_i \) so \( \alpha \) is a trivial loop. Since the 2-orbihedron \( \Pi' = P'/G' \) contains \(|\Pi^{(2)}| + 2k \) faces, we obtain \( T_0(\pi_1(M'), \pi_1(\partial M')) \leq T_0(\pi_1(M), E_\mu) + 2k \) which was promised. \( \text{QED.} \)

Remark 4.4. **It is worth pointing out that in the context of volumes of hyperbolic 3-manifolds the following inequality (similar to (5)) is known:**

\[ \text{Vol}(M) < \text{Vol}(M') < k \cdot (C_1(R) \cdot \text{Vol}(M) + C_2(R)), \]  

(†)

where \( R \) is the maximum of radii of the embedded tubes around the short geodesics \( l_i \) \((i = 1, ..., k)\) and \( C_i(R) \) are functions of \( R \) \((i = 1, 2)\). The left-hand side of (†) is classical and due to W. Thurston [Th], the right-hand side is proved recently by I. Agol, P. A. Storm, and W. Thurston [AST] \( \blacksquare \)

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Proof of the left-hand side of the inequality (*): By Lemma 4.1 the thick part $M_{\mu \text{thick}}$ of $M$ contains a 2-dimensional complex $W$ such that $\pi_1 W \hookrightarrow \pi_1 M_{\mu \text{thick}}$ is an isomorphism and $|W^{(2)}| < \sigma \cdot \text{Vol}(M)$ for some uniform constant $\sigma$. Consider now the double $N = DM_{\mu \text{thick}}$ of the manifold $M_{\mu \text{thick}}$ along the boundary $\partial M_{\mu \text{thick}}$. By repeating the argument of Lemma 4.1 to each half of $N$ we obtain two complexes $W$ and $\tau(W)$ embedded in $N$ where $\tau : N \rightarrow N$ is the involution such that $M_{\mu \text{thick}} = N/\tau$. By Van-Kampen theorem the fundamental group of the complex $V = W \cup \tau(W)$ is generated by $\pi_1 W$ and $\pi_1(\tau(W))$ and is isomorphic to $\pi_1(N)$. Furthermore, for the number of two-dimensional faces in $V$ we have $|N^{(2)}| = 2|W^{(2)}|$. So by Lemma 4.1 $T(\pi_1 N) \leq 2|V^{(2)}| < 2\sigma \cdot \text{Vol}(M)$. The group $\pi_1 N$ splits as the graph of groups whose two vertex groups are $\pi_1 M_{\mu \text{thick}}$. The edge groups of the graph of groups are given by the system $E_\mu$. As $\pi_1 M_{\mu \text{thick}} \cong \pi_1(M)'$ and $M'$ is a complete hyperbolic 3-manifold of finite volume it follows from Lemma 2.3 that the above splitting is reduced and rigid. So by [De] we have:

$$T(\pi_1 N) \geq 2T_0(\pi_1 M_{\mu \text{thick}}, E_\mu).$$

Then by Lemma 4.3 $T_0(\pi_1 M_{\mu \text{thick}}, E_\mu) \geq T_0(\pi_1(M), E_\mu)$, and therefore

$$\sigma^{-1} \cdot T_0(\pi_1(M), E_\mu) < \text{Vol}(M).$$

Recall that the initial system $E_\mu$ of elementary subgroups includes all elementary subgroups of $\pi_1(M)$ whose translation length is less than $\mu$. So $E_\mu \subset E_\mu$ implying that $T(\pi_1(M), E_\mu) \leq T_0(\pi_1(M), E_\mu)$. We finally obtain

$$C^{-1} \cdot T(\pi_1(M), E_\mu) < \text{Vol}(M),$$

where $C = \sigma$. The left-hand side of (*) is now proved. Theorem A follows. ■

5 Concluding remarks and questions.

The finiteness theorem of Wang affirms that there are only finitely many hyperbolic manifolds of dimension greater than 3 having the volume bounded by a fixed constant [W]. So it is natural to compare the volume of a hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ with the absolute invariant $T(\Gamma)$. In the case $n > 3$ the inequality

$$\text{const} \cdot T(\Gamma) \leq \text{Vol}(M)$$

follows from [Ge, Thm 1.7] (see also Section 2 above, where instead of $T(\pi_1(M), E)$ one needs to consider $T(\pi_1(M))$ and use the fact that $\pi_1 M_{\mu \text{thick}} \cong \pi_1(M)$). However, the result [C] is not known in higher dimensions. Thus we have the following:

**Question 5.1.** Is there a constant $C_n$ such that for every lattice $\Gamma$ in $\text{Isom}(\mathbb{H}^n)$ one has

$$\text{Vol}(\Gamma) \leq C_n \cdot T(\Gamma)?$$
Remark 5.2. (M. Gromov) The answer is positive if $M$ is a compact hyperbolic manifold of dimension 4. Indeed in this case by the Gauss-Bonnet formula one has $\text{Vol}(M) = \Omega_4 \cdot \chi(M)$, where $\Omega_4$ is the volume of the standard unit 4-sphere. Hence $\text{Vol}(M) < \frac{\Omega_4}{2} \cdot (2 - 2b_1 + b_2)$ where $b_i = \text{rank}(H_i(M, \mathbb{Z}))$ is the $i$-th Betti number of $M$ ($i = 1, 2$). Since $b_2 < T(\pi_1(M))$, one has $\text{Vol}(M) < \frac{\Omega_4}{2} \cdot (2 + b_2) < \Omega_4 \cdot T(\pi_1(M))$ (as $T(\pi_1(M)) > 1$).

Recently it was shown by D. Gabai, R. Meyerhoff, and P. Milley that the Matveev-Weeks 3-manifold $M_0$ is the unique closed 3-manifold of the smallest volume $[GMM]$. Furthermore, C. Cao and R. Meyerhoff found cusped 3-manifolds $m_{003}$ and $m_{004}$ of the smallest volume $[CM], [GMM]$. In this context we have the following:

**Question 5.3.** Is the invariant $T(\pi_1(M), E_\mu)$ on the set of compact hyperbolic 3-manifolds attained on the manifold $M_0$? Is the minimal relative invariant $T(\pi_1(M), E_\mu)$ on the set of cusped finite volume 3-manifolds attained on the manifolds $m_{003}$ and $m_{004}$?

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