COEFFICIENT ESTIMATES FOR MEROMORPHIC BI-UNIVALENT FUNCTIONS

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Abstract. A univalent meromorphic function defined on \( \Delta := \{z \in \mathbb{C} : 1 < |z| < \infty \} \) with univalent inverse defined on \( \Delta \) is bi-univalent meromorphic in \( \Delta \). For certain subclasses of meromorphic bi-univalent functions, estimates on the initial coefficients are obtained.

1. Introduction

An analytic function defined on some open set \( D \) that maps different points of \( D \) to different points is called univalent in \( D \) and \( \mathcal{S} \) denote the class of univalent functions \( f \) defined on the open unit disk \( \mathbb{D} := \{z \in \mathbb{C} : |z| < 1 \} \) of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k.
\]

The well-known Koebe one-quarter theorem asserts that the function \( f \in \mathcal{S} \) has an inverse defined on disk \( \mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho \} \), \( \rho \geq \frac{1}{4} \). Thus, the inverse of \( f \in \mathcal{S} \) is a univalent analytic function on the disk \( \mathbb{D}_\rho \). The function \( f \in \mathcal{S} \) is called bi-univalent in \( \mathbb{D} \) if \( f^{-1} \) is also univalent in the whole disk \( \mathbb{D} \). The class \( \Sigma \) of bi-univalent analytic functions was introduced in 1967 by Lewin [10] and he showed that, for every functions \( f \in \sigma \) of the form (1), the second coefficient of \( f \) satisfy the inequality \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie [3] improved Lewin’s result by showing \( |a_2| \leq \sqrt{2} \). Later, Netanyahu [11] proved that \( \max_{f \in \sigma} |a_2| = 4/3 \). Since then, several authors such as Brannan and Taha [4], Taha [17] investigated the subclasses of bi-univalent analytic functions and found estimates on the initial coefficients for functions in these subclasses. Recently, Ali et al. [1], Frasin and Aouf [6], Srivastava et al. [16] also introduced new subclasses of bi-univalent functions and found estimates on the coefficients \( a_2 \) and \( a_3 \) for functions in these classes.

In this paper, the concept of bi-univalency is extended to the class of meromorphic functions defined on \( \Delta := \{z \in \mathbb{C} : 1 < |z| < \infty \} \). For this purpose, let \( \Sigma \) denote the class of all meromorphic univalent functions \( g \) of the form

\[
g(z) = z + \sum_{n=0}^{\infty} b_n z^n,
\]

defined on the domain \( \Delta \). Since \( g \in \Sigma \) is univalent, it has an inverse \( g^{-1} \) that satisfy

\[
g^{-1}(g(z)) = z \quad (z \in \Delta),
\]

and

\[
g(g^{-1}(w)) = w \quad (M < |w| < \infty, \ M > 0).
\]

Furthermore, the inverse function \( g^{-1} \) has a series expansion of the form

\[
g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{n!} w^n,
\]

where \( M < |w| < \infty \). Analogous to the bi-univalent analytic functions, a function \( g \in \Sigma \) is said to be meromorphic bi-univalent if \( g^{-1} \in \Sigma \). The class of all meromorphic bi-univalent functions is denote by \( \Sigma_{\mathcal{B}} \).

Estimates on the coefficient of meromorphic univalent functions were investigated in the literature; for example, Schiffer [12] obtained the estimate \( |b_2| \leq 2/3 \) for meromorphic univalent functions \( g \in \Sigma \) with \( b_0 = 0 \). In

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1971, Duren [5] gave an elementary proof of the inequality $|b_n| \leq 2/(n + 1)$ on the coefficient of meromorphic univalent functions $g \in \Sigma$ with $b_k = 0$ for $1 \leq k < n/2$. For the coefficient of the inverse of meromorphic univalent functions, Springer [15] proved that

$$|B_3| \leq 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n - 2)!}{n!(n - 1)!} \quad (n = 1, 2, \ldots).$$

In 1977, Kubota [9] has proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober [13] obtained a sharp bounds for the coefficients $B_{2n-1}$, $1 \leq n \leq 7$, of the inverse of meromorphic univalent functions in $\Delta$. Recently, Kapoor and Mishra [8] found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order $\alpha$ in $\mathbb{D}$.

In the present investigation, certain subclasses of meromorphic bi-univalent functions are introduced and estimates for the coefficients $b_0$ and $b_1$ of functions in the newly introduced subclasses are obtained. These coefficients results are obtained by associating the given functions with the functions having positive real part. An analytic function $p$ of the form $p(z) = 1 + c_1z + c_2z^2 + \cdots$ is called a function with positive real part in $\mathbb{D}$ if $\text{Re} \, p(z) > 0$ for all $z \in \mathbb{D}$. The class of all functions with positive real part is denoted by $\mathcal{P}$. The following lemma for functions with positive real part will be useful in the sequel.

**Lemma 1.** [7] Theorem 3, p. 80] The coefficient $c_n$ of a function $p \in \mathcal{P}$ satisfy the sharp inequality

$$|c_n| \leq 2 \quad (n \geq 1).$$

2. **Coefficient estimates**

In this section, certain subclasses of the class $\Sigma_{\mathcal{P}}$ of meromorphic bi-univalent functions are introduced and estimates on the coefficient $b_0$ and $b_1$ for functions in these subclasses are obtained.

**Definition 1.** A function $g$ given by series expansion (2) is a meromorphic starlike bi-univalent functions of order $\alpha$, $0 \leq \alpha < 1$, if

$$\text{Re} \left( \frac{zg'(z)}{g(z)} \right) > \alpha \quad (z \in \Delta),$$

and

$$\text{Re} \left( \frac{wh'(w)}{h(w)} \right) > \alpha \quad (z \in \Delta),$$

where the function $h$ is the inverse of $g$ given by (3). The class of all meromorphic starlike bi-univalent functions of order $\alpha$ is denote by $\Sigma_{\mathcal{P}}(\alpha)$.

**Theorem 1.** If the function $g$ given by (2) is a meromorphic starlike bi-univalent function of order $\alpha$, $0 \leq \alpha < 1$, then the coefficients $b_0$ and $b_1$ satisfy the inequalities

$$|b_0| \leq 2(1 - \alpha), \quad \text{and} \quad |b_1| \leq (1 - \alpha)\sqrt{4\alpha^2 - 8\alpha + 5}.$$

**Proof.** Let $g$ be the meromorphic starlike bi-univalent function of order $\alpha$ given by (2). Then a calculation using Equation (2) shows that

$$\frac{zg'(z)}{g(z)} = 1 - \frac{b_0}{z} + \frac{b_0^2 - 2b_1}{z^2} - \frac{b_0^3 - 3b_1b_0 + 3b_2}{z^3} + \cdots \quad (z \in \Delta).$$

Since $h = g^{-1}$ is the inverse of $g$ whose series expansion is given by (3), a computation shows that

$$w = g(h(w)) = (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{B_3 - B_1B_0 + B_2}{w^2} + \frac{B_3 - B_1B_0 + B_2}{w^3} + \cdots.$$

Comparing the initial coefficients, the following relations are obtained:

$$b_0 + B_0 = 0,$$
(6) \[ b_1 + B_1 = 0, \]
(7) \[ B_2 - b_1 B_0 + b_2 = 0, \]
and
(8) \[ B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3 = 0. \]
Equations (5)–(8) yield
(9) \[ B_0 = -b_0, \]
(10) \[ B_1 = -b_1, \]
(11) \[ B_2 = -b_2 - b_0 b_1, \]
and
(12) \[ B_3 = -(b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2). \]
Use of Equations (9)–(12) shows that the series expansion for the function \( g^{-1} \) given by (3) becomes
(13) \[ h(w) = g^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \ldots. \]
A calculation using Equation (13) shows that
(14) \[ \frac{wh'(w)}{h(w)} = 1 + \frac{b_0}{w} + \frac{b_2 + 2b_1}{w^2} + \frac{b_3 + 6b_1 b_0 + 3b_2}{w^3} + \ldots \quad (z \in \Delta). \]
Since \( g \) is a bi-univalent meromorphic function of order \( \alpha \), there exist two functions \( p, q \) with positive real part in \( \Delta \) of the forms
(15) \[ p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta) \]
and
(16) \[ q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (z \in \Delta). \]
such that
(17) \[ \frac{zg'(z)}{g(z)} = \alpha + (1 - \alpha)p(z), \]
and
(18) \[ \frac{wh'(w)}{h(w)} = \alpha + (1 - \alpha)q(w). \]
Use of (15) in (17) shows that
(19) \[ \frac{zg'(z)}{g(z)} = 1 + \frac{(1 - \alpha)c_1}{z} + \frac{(1 - \alpha)c_2}{z^2} + \frac{(1 - \alpha)c_3}{z^3} + \cdots. \]
In view of the Equations (4) and (19), it is easy to see that
(20) \[ (1 - \alpha)c_1 = -b_0 \]
and
(21) \[ (1 - \alpha)c_2 = b_0^2 - 2b_1. \]
Similarly, use of (14), (16) in (18) immediately yields
(22) \[ (1 - \alpha)d_1 = b_0 \]
and
(23) \[ (1 - \alpha)d_2 = b_0^2 + 2b_1. \]
Equations (20) and (22) together yields
\[ c_1 = -d_1. \]
and
\[ b_0^2 = \frac{(1 - \alpha)^2}{2}(c_1^2 + d_1^2). \]
Since \( \text{Re} \ p(z) > 0 \) in \( \Delta \), the function \( p(1/z) \in \mathcal{P} \) and hence the coefficients \( c_n \) and similarly the coefficients \( d_n \) of the function \( q \) satisfy the inequality in Lemma 1 and this immediately yields the following estimate:
\[ |b_0^2| = \frac{(1 - \alpha)^2}{2}|c_1^2 + d_1^2| \leq 4(1 - \alpha)^2. \]
This readily yields the following estimate for \( b_0 \):
\[ |b_0| \leq 2(1 - \alpha). \]
The estimate \( |b_0| \leq 2(1 - \alpha) \) also follows directly from (20). Using Equations (21) and (23) yields
\[ 4b_0^2 = (1 - \alpha)^2c_2d_2, \]
or
\[ 4b_2^2 = -(1 - \alpha)^2c_2d_2 + b_0^4. \]
By Lemma 1, the estimates \( |c_2| = |d_2| \leq 2 \) holds. This estimate together with the estimate of \( b_0 \) imply that
\[ 4|b_2|^2 \leq 4(1 - \alpha)^2 + 16(1 - \alpha)^4. \]
Therefore
\[ |b_1| \leq (1 - \alpha)\sqrt{4\alpha^2 - 8\alpha + 5}. \]

**Definition 2.** The function \( g \) given by (2) is said to belong to class \( \tilde{\Sigma}_{\alpha}^\alpha(\alpha) \) of bi-univalent strongly starlike meromorphic functions of order \( \alpha, 0 < \alpha \leq 1 \), if
\[ \left| \arg \left( \frac{zg'(z)}{g(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \Delta), \]
and
\[ \left| \arg \left( \frac{wh'(w)}{h(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \Delta), \]
where the function \( h \) is the inverse of the function \( g \) given by (3).

The class considered in Definition 1 is related to starlikeness of order \( \alpha \) and the second subclass in Definition 2 is associated with strongly starlikeness of order \( \alpha \). It should be noted that meromorphic starlike bi-univalent functions of order 0 is essentially the same as meromorphic strongly starlike bi-univalent functions of order 1: \( \Sigma_{\alpha}^\alpha(0) = \tilde{\Sigma}_{\alpha}^\alpha(1) \). In view of this connection, it should be noticed that the class \( \Sigma_{\alpha}^\alpha(\alpha) \) provides a generalization of the class of meromorphic starlike bi-univalent functions in a different direction; the class \( \Sigma_{\alpha}^\alpha(\alpha) \) is associated with right half-planes while the class \( \tilde{\Sigma}_{\alpha}^\alpha(\alpha) \) associated with sectors. It is pertinent to see that the estimates of \( b_0 \) and \( b_1 \) in Theorem 1 when \( \alpha = 0 \) is the same as the corresponding estimates in Theorem 2 when \( \alpha = 1 \).

**Theorem 2.** If the function \( g \) given by (2) is in the class \( \tilde{\Sigma}_{\alpha}^\alpha(\alpha) \), \( 0 < \alpha \leq 1 \), then the coefficients \( b_0 \) and \( b_1 \) satisfy the inequalities
\[ |b_0| \leq 2\alpha, \quad \text{and} \quad |b_1| \leq \sqrt{5} \alpha^2. \]

**Proof.** Consider the function \( g \in \tilde{\Sigma}_{\alpha}^\alpha(\alpha) \). Then, by definition of the class \( \tilde{\Sigma}_{\alpha}^\alpha(\alpha) \),
\[ \frac{zg'(z)}{g(z)} = (p(z))^\alpha \quad \text{and} \quad \frac{wh'(w)}{h(w)} = (q(w))^\alpha, \]
where \( p \) and \( q \) are functions with positive real part in \( \Delta \) and the series expansion of \( p \) and \( q \) are respectively given by
\[ p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta), \]
and
\[ q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (z \in \Delta). \]
A computation yields
\[(p(z))^α = 1 + \frac{αc_1}{z} + \frac{1}{2} \alpha(α-1)c_1^2 + \frac{αc_2}{z^2} + \frac{1}{3} \alpha(α-1)(α-2)c_1^3 + \alpha(α-1)c_1c_2 + αc_3 + \ldots \]
and, by definition of \(g\),
\[\frac{zg'(z)}{g(z)} = 1 - \frac{b_0}{z} + \frac{b_0^2 - 2b_1}{z^2} - \frac{b_0^3 - 3b_1b_0 + 3b_2}{z^3} + \cdots \quad (z \in Δ).\]
This equation with Equation (26) and first equation in (25) yield
\[(27) \quad αc_1 = -b_0,\]
and
\[(28) \quad \frac{1}{2} \alpha(α-1)c_1^2 + αc_2 = b_0^2 - 2b_1.\]
Similarly
\[(q(w))^α = 1 + \frac{αd_1}{w} + \frac{4α(α-1)d_1^2 + αd_2}{w^2} + \frac{1}{3} \alpha(α-1)(α-2)d_1^3 + \alpha(α-1)d_1d_2 + αd_3 + \ldots \]
and
\[\frac{wh'(w)}{h(w)} = 1 + \frac{b_0}{w} + \frac{b_0^2 + 2b_1}{w^2} + \frac{b_0^3 + 6b_1b_0 + 3b_2}{w^3} + \cdots \quad (z \in Δ).\]
The last equation and Equation (29) together with the second equation in (25) implies
\[(30) \quad αd_1 = b_0,\]
and
\[(31) \quad \frac{1}{2} \alpha(α-1)d_1^2 + αd_2 = b_0^2 + 2b_1.\]
Using the Equations (27) and (30), one gets
\[c_1 = -d_1\]
and
\[2b_0^2 = α^2(c_1^2 + d_1^2)\]
which implies
\[(32) \quad b_0^2 = \frac{α^2}{2}(c_1^2 + d_1^2).\]
By Lemma I \(|c_1| \leq 2\) and \(|d_1| \leq 2\) and using them in (32), it follows that
\[|b_0^2| = \frac{α^2}{2}|c_1^2 + d_1^2| \leq \frac{α^2}{2}(|c_1| + |d_1|) \leq 4α^2.\]
Hence
\[|b_0| \leq 2α.\]
Equations (28) and (30) together yield
\[(33) \quad 2b_0^2 + 8b_1^2 = \frac{α^2}{4}(α-1)^2(c_1^4 + d_1^4) + α^2(c_2^2 + d_2^2) + α^2(α-1)(c_1^2c_2 + d_1^2d_2).\]
In view of (32), the previous equation becomes
\[b_1^2 = \frac{α^2}{32}(α-1)^2(c_1^4 + d_1^4) + \frac{α^2}{8}(c_2^2 + d_2^2) + \frac{α^2}{8}(α-1)(c_1^2c_2 + d_1^2d_2) - \frac{α^4}{16}(c_1^4 + d_1^4) - \frac{α^4}{8}c_1^2d_1^2,\]
Lemma I again gives the estimates \(|c_i| = |d_i| \leq 2\) for \(i = 1, 2\), and using these in the above equation immediately yields
\[|b_1^2| \leq α^2(α-1)^2 + α^2 + 2α^2(α-1) + 2α^4 + 2α^4 = 5α^4.\]
This shows that
\[|b_1| \leq \sqrt{5} \alpha^2. \]
3. Meromorphic Bazilević bi-univalent functions

This section is related to a general class called the class of meromorphic Bazilević bi-univalent functions. Let \( p \in \mathcal{P} \), \( h \in \mathcal{H}^* \), \( \alpha \) any real number and \( \beta > 0 \), Bazilević [2] introduced a subclass of \( \mathcal{A} \) consisting of the principal branch of the functions
\[
f(z) = \left( \frac{\beta}{1 + \alpha z} \right) \int_0^z \left( p(\xi) - \alpha i \xi \right)^{-1} h(\xi)^{-\beta} d\xi
\]
and he showed that each principal branch is univalent in \( \mathbb{D} \). In the case when \( \alpha = 0 \), a computation shows that
\[
z f'(z) = f(z)^{1-\beta} h(z)^\beta p(z)
\]
or
\[
(34) \quad \text{Re} \left( \frac{z f'(z)}{f(z)^{1-\beta} h(z)^\beta} \right) > 0.
\]
Thomas [18] called a function satisfying the condition (34) as a Bazilević function of type \( \beta \). Furthermore, if \( h(z) = z \) in (34), then the condition (34) becomes
\[
(35) \quad \text{Re} \left( \frac{z f'(z)}{f(z)^{1-\beta} z^\beta} \right) > 0.
\]
The class of all functions \( f \in \mathcal{A} \) satisfies (35) is introduced by Singh [14] and the class of all such functions is denoted by \( \mathcal{B}(\beta) \). In this section, the estimates for the initial coefficients of the meromorphic functions analogous to the functions belonging to the class \( \mathcal{B}(\beta) \) are obtained.

**Definition 3.** Let \( \beta > 0 \) and \( 0 < \alpha \leq 1 \). A meromorphic bi-univalent function \( g \) given by (2) is said to be in the class \( \Sigma_{\mathcal{B}}^\beta (\beta, \alpha) \) of meromorphic strongly Bazilević bi-univalent functions of type \( \beta \) and order \( \alpha \), if
\[
\left| \arg \left( \frac{z}{g(z)} \right)^{1-\beta} g'(z) \right| < \frac{\alpha \pi}{2} \quad (z \in \Delta)
\]
and
\[
\left| \arg \left( \frac{w}{h(w)} \right)^{1-\beta} h'(w) \right| < \frac{\alpha \pi}{2} \quad (z \in \Delta)
\]
where the function \( h \) is the inverse of \( g \) and given by (3).

**Theorem 3.** Let \( \beta > 0 \) and \( 0 < \alpha \leq 1 \). If \( g \in \Sigma_{\mathcal{B}}^\beta (\beta, \alpha) \), then the coefficients \( b_0 \) and \( b_1 \) satisfy the inequalities
\[
|b_0| \leq \frac{2\alpha}{1-\beta}, \quad \text{and} \quad |b_1| \leq \frac{2\alpha^2}{(1-\beta)(2-\beta)2\sqrt{2(1-\beta)(2-\beta)+1}}.
\]

**Proof.** Suppose \( g \in \Sigma_{\mathcal{B}}^\beta (\beta, \alpha) \) has a representation given by (2), then a computation shows that
\[
g'(z) = 1 - \frac{b_1}{z} + \frac{2b_2}{z^2} - \frac{3b_3}{z^3} + \cdots
\]
and
\[
\frac{z}{g(z)} = 1 - \frac{b_0}{z} - \frac{b_1^2 - b_1}{z^2} - \frac{b_2 - 2b_1 b_0 + b_2}{z^3} + \cdots.
\]
Furthermore
\[
\left( \frac{z}{g(z)} \right)^{1-\beta} = 1 - \frac{(1-\beta)b_0}{z} + \frac{(1-\beta)(2-\beta)b_0^2 - 2b_1}{2z^2} + \frac{(1-\beta)((1-\beta)^2 + 3(1-\beta) + 2)b_0^3 - 6(2-\beta)b_1 b_0 + 6b_2}{6z^3} + \cdots.
\]
Further calculations show that
\[
\left( \frac{z}{g(z)} \right)^{1-\beta} g'(z) = 1 - \frac{(1-\beta)b_0}{z} + \frac{(2-\beta)((1-\beta)b_0^2 - 2b_1)}{2z^2} - \frac{(3-\beta)((1-\beta)(2-\beta)b_0^3 + 6(1-\beta)b_1b_0 + 6b_2)}{6z^3} + \frac{1 - 12(1-\beta)((2-\beta)b_1b_0 - 2b_2^2 - 2b_0b_2)}{24z^4} + \ldots .
\]
(36)

The assumption \( g \in \Sigma_{\beta}^\mathbb{B}(\beta, \alpha) \) shows that there is a function \( p \) with \( \text{Re}(p(z)) > 0 \) such that
\[
\left( \frac{z}{g(z)} \right)^{1-\beta} g'(z) = (p(z))^\alpha ,
\]
(37)

where the function \( p \) has the representation given by
\[
p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \ldots .
\]

The Equations (36), (37) and (26) together yield the following:
\[
- (1-\beta)b_0 = \alpha c_1
\]
and
\[
\frac{1}{2}(2-\beta)((1-\beta)b_0^2 - 2b_1) = \frac{1}{2}\alpha(\alpha - 1)c_1^2 + \alpha c_2 .
\]

Similarly
\[
h'(w) = 1 + \frac{b_1}{w^2} + \frac{2(b_2 + b_0b_1)}{w^3} + \frac{3(b_2^2b_1 + 2b_0b_2 + b_1^2 + b_3)}{w^4} + \ldots .
\]
and
\[
\frac{w}{h(w)} = 1 + \frac{b_0}{w} + \frac{b_0^2 + b_1}{w^2} + \frac{b_0^3 + 3b_0b_1 + b_2}{w^3} + \ldots .
\]

Hence
\[
\left( \frac{w}{h(w)} \right)^{1-\beta} = 1 + \frac{(1-\beta)b_0}{w} + \frac{(1-\beta)((2-\beta)b_0^2 + 2b_1)}{2w^2} + \frac{(1-\beta)((2-\beta)(3-\beta)b_0^3 + 6(2-\beta)b_1b_0 + 6b_2)}{6w^3} + \ldots .
\]
and
\[
\left( \frac{w}{h(w)} \right)^{1-\beta} h'(w) = 1 + \frac{(1-\beta)b_0}{w} + \frac{(2-\beta)((1-\beta)b_0^2 + 2b_1)}{2w^2} + \frac{(3-\beta)((1-\beta)(2-\beta)b_0^3 + 6(2-\beta)b_1b_0 + 6b_2)}{6w^3} + \frac{1 - 12(1-\beta)((2-\beta)b_1b_0 - 2b_2^2 - 2b_0b_2)}{24w^4} + \ldots .
\]

The hypothesis \( g \in \Sigma_{\beta}^\mathbb{B}(\beta, \alpha) \) again implies that there exist a function \( q \) with \( \text{Re}(q(w)) > 0 \) satisfying
\[
\left( \frac{w}{h(w)} \right)^{1-\beta} h'(w) = (q(w))^\alpha ,
\]
where \( q \) has a series representation given by
\[
q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \ldots .
\]

Equations (11), (42) and (29) yield
\[
- (1-\beta)b_0 = \alpha d_1
\]
(43)
and
\begin{equation}
\frac{1}{2} (2 - \beta)((1 - \beta)b_0^2 + 2b_1) = \frac{1}{2} \alpha(\alpha - 1)d_1^2 + \alpha d_2.
\end{equation}

Equations (38) and (43) shows that
\[ c_1 = -d_1 \]
and
\[ 2(1 - \beta)^2 b_0^2 = \alpha^2(d_1^2 + c_1^2), \]
or
\begin{equation}
(b_0)^2 = \frac{\alpha^2}{2(1 - \beta)^2}(d_1^2 + c_1^2).
\end{equation}

By Lemma 1, \(|c_1| = |d_1| \leq 2\) and use of this inequality in the Equation (45) immediately leads to the following estimate for \(b_0\):
\[ |b_0^2| = \frac{\alpha^2}{2(1 - \beta)^2}|d_1^2 + c_1^2| \]
\[ \leq \frac{\alpha^2}{2(1 - \beta)^2}(|d_1^2| + |c_1^2|) \]
\[ = \frac{4\alpha^2}{(1 - \beta)^2}. \]

This completes the proof of the inequality \(|b_0| \leq 2\alpha/(1 - \beta)\). Yet another calculation using (39) and (44) shows that
\[ \frac{1}{4}(2 - \beta)^2 ((1 - \beta)^2 b_0^4 - 4b_1^2) = \frac{\alpha^2(\alpha - 1)}{4}(d_1^4 c_1^2) + \frac{\alpha^2(\alpha - 1)}{2}(c_1^2 d_2 + d_1^2 c_2) + \alpha^2 c_2 d_2. \]

Use of (45) in the above equation leads to the following expression for \(b_1\):
\[ -(2 - \beta)^2 b_1^2 = \frac{\alpha^2(\alpha - 1)}{4}(d_1^4 c_1^2) + \frac{\alpha^2(\alpha - 1)}{2}(c_1^2 d_2 + d_1^2 c_2) \]
\[ + \alpha^2 c_2 d_2 - \frac{4(2 - \beta)^2 \alpha^4}{(1 - \beta)^2}. \]

Once again, an application of Lemma 1 immediately yields
\[ |b_1| \leq \frac{4\alpha^2(\alpha - 1)^2}{(2 - \beta)^2} + \frac{8\alpha^2(\alpha - 1)}{(2 - \beta)^3} + \frac{4\alpha^2}{(2 - \beta)^2} + \frac{4\alpha^4}{(1 - \beta)^2} \]
\[ = \frac{4\alpha^4(2(1 - \beta)(2 - \beta) + 1)}{(1 - \beta)^2(2 - \beta)^2}. \]

and therefore
\[ |b_1| \leq \frac{2\alpha^2}{(1 - \beta)(2 - \beta)} \sqrt{2(1 - \beta)(2 - \beta) + 1}. \]

\begin{remark}
If \(b_0 = 0\) for the function \(g \in \Sigma\), the series expansion (13) becomes
\[ g^{-1}(w) = w - \frac{b_1}{w} - \frac{b_2}{w^2} - \frac{b_3}{w^3} + \cdots. \]
This series expansion was obtained by Schober [13].
\end{remark}

\begin{example}
The function \(g(z) = z + 1/z\) is clearly a univalent meromorphic function. A direct calculation that
\[ g^{-1}(w) = \frac{w + \sqrt{w^2 - 4}}{2}. \]
This function shows \(g^{-1}\) has the series expansion given by
\[ g^{-1}(w) = w - \frac{1}{w} - \frac{1}{w^3} - \frac{2}{w^5} - \frac{5}{w^7} - \frac{14}{w^9} - \cdots. \]
\end{example}
Theorem 4. If \( g \) given by (2) is in the class \( \Sigma^\gamma_{\beta}(\alpha) \), \( 0 < \alpha \leq 1 \), and \( b_0 = 0 \), then
\[
|b_1| \leq \alpha.
\]

Proof. Assume that the function \( g = z + \sum_{n=1}^{\infty} b_n z^{-n} \in \Sigma^\gamma_{\beta}(\alpha) \) where \( 0 < \alpha \leq 1 \). Since \( b_0 = 0 \), \( c_1 = d_1 = 0 \) and the result can be verified by a direct calculation of (19).

Theorem 5. Let \( g \in \Sigma^\gamma_{\beta}(\alpha, \beta) \), where \( \alpha > 0 \) and \( 0 < \beta \leq 1 \). Then
\[
|b_1| \leq \frac{2\beta^2}{2 - \alpha}.
\]

Proof. Since the function \( g = z + \sum_{n=1}^{\infty} b_n z^{-n} \in \Sigma^\gamma_{\beta}(\alpha, \beta) \) where \( 0 < \alpha \leq 1 \) and \( b_0 = 0 \), it follows that \( c_1 = d_1 = 0 \). By replacing these values in Equation (33) and continuing as in the proof of Theorem 2 the result is obtained.

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