Nonlinear Acoustics: Blackstock–Crighton Equations with a Periodic Forcing Term

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Abstract. Blackstock–Crighton equations describe the motion of a viscous, heat-conducting, compressible fluid. They are used as models for acoustic wave propagation in a medium in which both nonlinear and dissipative effects are taken into account. In this article, a mathematical analysis of the Blackstock–Crighton equations with a time-periodic forcing term is carried out. For time-periodic data sufficiently restricted in size it is shown that a time-periodic solution of the same period always exists. This implies that the dissipative effects are sufficient to avoid resonance within the Blackstock–Crighton models. The equations are considered in a bounded domain with both non-homogeneous Dirichlet and Neumann boundary values. Existence of a solution is obtained via a fixed-point argument based on appropriate a priori estimates for the linearized equations.

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1. Introduction

The motion of a viscous, heat-conducting fluid is governed by mass conservation, momentum conservation, energy conservation, and a thermodynamic equation of state. The compressible Navier–Stokes equations describe the conservation of mass and momentum when viscous effects are taken into account. The Kirchhoff–Fourier equations describe the conservation of energy when heat-conducting effects are taken into account. Using the equation of state of an ideal fluid, and assuming the flow is irrotational, Blackstock [2] eliminated all but one dependent variable from the Navier–Stokes and Kirchhoff–Fourier equations to obtain the following model for a viscous, heat-conducting fluid:

\[ (a\Delta - \partial_t) \left( \partial_t^2 u - c^2\Delta u - b\partial_t \Delta u \right) - \partial_t^2 \left( \frac{1}{c^2} \frac{B}{2A} (\partial_t u)^2 + |\nabla u|^2 \right) = f. \]  \tag{BCK}

Here, \( u \) denotes the potential of the velocity field, and \( f \) a forcing term. The constant \( a \) is the heat conductivity of the fluid, and \( c \) the speed of sound. The diffusivity of sound \( b \) is a measure of energy dissipation due to viscosity and heat conduction in the fluid. Finally, the so-called (acoustic) parameter of nonlinearity \( B/A \), introduced by Beyер [1] (see also [8]), is the quotient of the second and first coefficient in the Taylor expansion of the pressure–density relationship.

Equation (BCK) is called the Blackstock–Crighton–Kuznetsov equation and is used as a model for acoustic wave propagation in a medium in which both nonlinear and dissipative effects are taken into account. Assuming that \( c^2|\nabla u|^2 \approx (\partial_t u)^2 \), one obtains the simplified Blackstock–Crighton–Westervelt equation:

\[ (a\Delta - \partial_t) \left( \partial_t^2 u - c^2\Delta u - b\partial_t \Delta u \right) - \partial_t^2 \left( \frac{1}{c^2} \frac{1}{2A} (\partial_t u)^2 \right) = f. \]  \tag{BCW}

If both nonlinear and dissipative terms are neglected in (BCK) (the latter is obtained by setting \( b = 0 \)), the model reduces to the classical wave equation.
In the present article we investigate if the dissipative effects present in (BCK) and (BCW) are sufficient to avoid resonance, that is, the occurrence of an unbounded solution when the system is excited by a periodic force. For undamped linear hyperbolic systems such as the wave equation, it is easy to show existence of unbounded solutions when a periodic force of a certain frequency (the eigenfrequency) is applied to it. In physical terms, energy conservation in such systems means that even a minor amount of work by the force over each periodic can accumulate in the system and lead to an unbounded response, i.e., resonance. The systems (BCK) and (BCW) are damped by the term $b\partial_t \Delta u$, which introduces a source of energy dissipation. The question thus arises whether this dissipation dampens the system sufficiently to avoid resonance. By showing existence of a periodic, and hence bounded, solution for periodic forces of arbitrary frequency, we establish that this is indeed the case; at least when the magnitude of the force is sufficiently restricted. We may therefore conclude that the dissipative effects of viscosity and heat conduction in the Blackstock–Crighton PDE model brings about a sufficient energy absorption mechanism to avoid resonance.

The initial-value problems corresponding to the Blackstock–Crighton equations (BCK) and (BCW) have been investigated recently by Brunnhuber and Kaltenbacher [4], Brunnhuber [3] [in particular we refer to [3, Section 2] for a derivation of (BCK)], and Brunnhuber and Meyer [5]. To our knowledge, the investigation of time-periodic solutions to the Blackstock–Crighton equations in a setting of time-periodic data is new. The Kuznetsov equation, which is a simpler model for nonlinear acoustic wave propagation distances is derived and analyzed.

We consider (BCK) and (BCW) with inhomogeneous Dirichlet and Neumann boundary conditions. We shall work in a setting of time-periodic functions and therefore take the whole of $\mathbb{R}$ as time-axis. We let $n \geq 2$ and denote by $\Omega \subset \mathbb{R}^n$ a spatial domain. In the following, $(t,x) \in \mathbb{R} \times \Omega$ will always denote a time-variable $t$ and spatial variable $x$, respectively. The Blackstock–Crighton equation (in generalized form) with Dirichlet boundary condition then reads

\[
\begin{aligned}
(a\Delta - \partial_t) \left( \partial_t^2 u - c^2 \Delta u - b\partial_t \Delta u \right) - \partial_t^2 \left( k(\partial_t u)^2 + s|\nabla u|^2 \right) &= f & \text{in } \mathbb{R} \times \Omega, \\
(u, \Delta u) &= (g,h) & \text{on } \mathbb{R} \times \partial \Omega,
\end{aligned}
\]

(BCD)

where we have used the same notation $k := \frac{1}{c^2} \left( (1-s) + \frac{B}{2s} \right)$ and $s \in \{0,1\}$ as in [3]. The corresponding Neumann problem reads

\[
\begin{aligned}
(a\Delta - \partial_t) \left( \partial_t^2 u - c^2 \Delta u - b\partial_t \Delta u \right) - \partial_t^2 \left( k(\partial_t u)^2 + s|\nabla u|^2 \right) &= f & \text{in } \mathbb{R} \times \Omega, \\
(\partial_n u, \partial_n \Delta u) &= (g,h) & \text{on } \mathbb{R} \times \partial \Omega.
\end{aligned}
\]

(BCN)

Here $f: \mathbb{R} \times \Omega \to \mathbb{R}$ and $g, h: \mathbb{R} \times \partial \Omega \to \mathbb{R}$ are given, and $u: \mathbb{R} \times \Omega \to \mathbb{R}$ is the unknown.

We shall consider data that are time-periodic with the period $T > 0$, that is, functions satisfying $v(t + T, \cdot) = v(t, \cdot)$ for all $t \in \mathbb{R}$. As the main result in this article, we show for given $T$-time-periodic data $f, g$ and $h$ in appropriate function spaces, and sufficiently restricted in size, the existence of a time-periodic solution $u$ to (BCD) and (BCN). In the case of Dirichlet boundary conditions, the result can be stated as follows (the subscript per indicates that a function space consists of $T$-time-periodic functions; see Sect. 2):

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a $C^4$-smooth boundary, and $\max\{2, \frac{n}{2}\} < p < \infty$. There is an $\varepsilon > 0$ such that for all $f \in L^p_{\text{per}}(\mathbb{R}; L^p(\Omega))$, $g \in T^p_{\text{per}, D_1}(\mathbb{R} \times \partial \Omega)$ and $h \in T^p_{\text{per}, D_2}(\mathbb{R} \times \partial \Omega)$ satisfying

\[
\|f\|_{L^p_{\text{per}}(\mathbb{R}; L^p(\Omega))} + \|g\|_{T^p_{\text{per}, D_1}(\mathbb{R} \times \partial \Omega)} + \|h\|_{T^p_{\text{per}, D_2}(\mathbb{R} \times \partial \Omega)} \leq \varepsilon
\]

there exists a time-periodic solution $u$ to the Blackstock–Crighton equations (BCK) and (BCW) for periodic forces $f, g, h$. Moreover, $u$ is bounded in $L^p$-norm and $C^1$-norm, and the energy $\frac{1}{2} \|\nabla u\|_{L^p}^2 + \frac{1}{2} \|u\|_{L^p}^2$ is uniformly bounded by $\varepsilon$.
there is a solution

\[ u \in W^{3,p}_{\text{per}}(\mathbb{R};L^p(\Omega)) \cap W^{1,p}_{\text{per}}(\mathbb{R};W^{4,p}(\Omega)) \]

to (BCD).

A solution to the Neumann problem (BCN) only exists when the data satisfies certain compatibility conditions. More precisely, we obtain:

**Theorem 1.2.** Let \(n, \Omega\) and \(p\) be as in Theorem 1.1. There is an \(\varepsilon > 0\) such that for all \(f \in L^p_{\text{per}}(\mathbb{R};L^p(\Omega))\), \(g \in T^p_{\text{per},N_1}(\mathbb{R} \times \partial \Omega)\) and \(h \in T^p_{\text{per},N_2}(\mathbb{R} \times \partial \Omega)\) satisfying

\[ \|f\|_{L^p_{\text{per}}(\mathbb{R};L^p(\Omega))} + \|g\|_{T^p_{\text{per},N_1}(\mathbb{R} \times \partial \Omega)} + \|h\|_{T^p_{\text{per},N_2}(\mathbb{R} \times \partial \Omega)} \leq \varepsilon \]

and

\[ \int_0^T \int_\Omega f \, dx \, dt + ac^2 \int_0^T \int_{\partial \Omega} h \, dS \, dt = 0 \]

there is a solution

\[ u \in W^{3,p}_{\text{per}}(\mathbb{R};L^p(\Omega)) \cap W^{1,p}_{\text{per}}(\mathbb{R};W^{4,p}(\Omega)) \]

to (BCN).

Our proofs are based on a priori \(L^p\)-estimates for the corresponding linearizations

\[ \begin{cases} \quad (a\Delta_t - \partial_t) \left( \partial_t^2 u - c^2 \Delta u - b\partial_t \Delta u \right) = f & \text{in} \ \mathbb{R} \times \Omega, \\
\quad (u, \Delta u) = (g, h) & \text{on} \ \mathbb{R} \times \partial \Omega, \end{cases} \tag{1.1} \]

and

\[ \begin{cases} \quad (a\Delta_t - \partial_t) \left( \partial_t^2 u - c^2 \Delta u - b\partial_t \Delta u \right) = f & \text{in} \ \mathbb{R} \times \Omega, \\
\quad (\partial_\nu u, \partial_\nu \Delta u) = (g, h) & \text{on} \ \mathbb{R} \times \partial \Omega, \end{cases} \tag{1.2} \]

of (BCD) and (BCN), respectively, and an application of the contraction mapping principle. Our analysis relies on a decomposition of (1.1) and (1.2) into a coupled system consisting of the time-periodic heat equation and the time-periodic Kuznetsov equation. The time-periodic heat equation was investigated in [11], and the time-periodic Kuznetsov equation in [6]. In the following, we shall employ the results obtained in these two articles.

**2. Function Spaces**

In the following, \(\Omega \subset \mathbb{R}^n, n \geq 2\), always denotes a domain with a \(C^4\)-smooth boundary. The outer normal on \(\partial \Omega\) is denoted by \(\nu\). Points in \(\mathbb{R} \times \Omega\) are generally denoted by \((t, x)\), with \(t\) being referred to as time, and \(x\) as the spatial variable. A time period \(T > 0\) remains fixed.

Classical Lebesgue and Sobolev spaces are denoted by \(L^p(\Omega)\) and \(W^{k,p}(\Omega)\), respectively. We write \(\|\cdot\|_p\) and \(\|\cdot\|_{k,p}\) instead of \(\|\cdot\|_{L^p(\Omega)}\) and \(\|\cdot\|_{W^{k,p}(\Omega)}\).

For a Lebesgue or Sobolev space \(E(\Omega)\), we define the space of smooth \(T\)-time-periodic vector-valued functions by

\[ C^\infty_{\text{per}}(\mathbb{R};E(\Omega)) := \{ f \in C^\infty(\mathbb{R};E(\Omega)) \mid f(t + T, x) = f(t, x) \}. \]
For $p \in (1, \infty)$, we let
\[
\|f\|_{L^p_{\text{per}}(\mathbb{R}; E(\Omega))} := \left(\frac{1}{T} \int_0^T \|f(t)\|_{E(\Omega)}^p \, dt\right)^{\frac{1}{p}},
\]
\[
\|f\|_{W^k_{\text{per},p}(\mathbb{R}; E(\Omega))} := \left(\sum_{\alpha=0}^k \|\partial^\alpha f\|_{L^p_{\text{per}}(\mathbb{R}; E(\Omega))}^p\right)^{\frac{1}{p}}.
\]
(2.1)
(2.2)

As one may verify, $\|\cdot\|_{L^p_{\text{per}}(\mathbb{R}; E(\Omega))}$ and $\|\cdot\|_{W^k_{\text{per},p}(\mathbb{R}; E(\Omega))}$ define norms on $C^\infty_{\text{per}}(\mathbb{R}; E(\Omega))$. We set
\[
L^p_{\text{per}}(\mathbb{R}; E(\Omega)) := C^\infty_{\text{per}}(\mathbb{R}; E(\Omega)) / \|\cdot\|_{L^p_{\text{per}}(\mathbb{R}; E(\Omega))}.
\]
(2.3)

For simplicity, we write $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p_{\text{per}}(\mathbb{R}; L^p(\Omega))}$. We also introduce Sobolev spaces of vector-valued time-periodic functions:
\[
W^{k,p}_{\text{per}}(\mathbb{R}; E(\Omega)) := C^\infty_{\text{per}}(\mathbb{R}; E(\Omega)) / \|\cdot\|_{W^{k,1,\alpha}_p(\mathbb{R}; E(\Omega))}.
\]
(2.4)

Corresponding Sobolev-Slobodeckii spaces are defined in the usual way by real interpolation $(k \in \mathbb{N}_0, \alpha \in (0, 1))$:
\[
W^{k+\alpha,p}_{\text{per}}(\mathbb{R}; E(\Omega)) := (W^{k+1,p}_{\text{per}}(\mathbb{R}; E(\Omega)), W^{k,p}_{\text{per}}(\mathbb{R}; E(\Omega)))_{1-\alpha, p}.
\]
Moreover, we let
\[
X^p_{\text{per}}(\mathbb{R} \times \Omega) := W^{3,p}_{\text{per}}(\mathbb{R}; L^p(\Omega)) \cap W^{1,p}_{\text{per}}(\mathbb{R}; W^{4,p}(\Omega)).
\]

Embedding properties are collected in the following lemma.

**Lemma 2.1.** Let $p \in (1, \infty)$, $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^4$-smooth boundary. The embeddings
\[
X^p_{\text{per}}(\mathbb{R} \times \Omega) \hookrightarrow W^{2,p}_{\text{per}}(\mathbb{R}; W^{2,p}(\Omega)),
\]
(2.5)
\[
W^{2,p}_{\text{per}}(\mathbb{R}; L^p(\Omega)) \cap L^p_{\text{per}}(\mathbb{R}; W^{4,p}(\Omega)) \hookrightarrow W^{1,p}_{\text{per}}(\mathbb{R}; W^{2,p}(\Omega))
\]
(2.6)

and $(l \in \{1, 2, 3\})$
\[
L^p(\mathbb{R}^+; X^{p}_{\text{per}}(\mathbb{R} \times \mathbb{R}^{n-1})) \cap W^{4,p}(\mathbb{R}^+; W^{1,p}_{\text{per}}(\mathbb{R}; L^p(\mathbb{R}^{n-1})))
\]
\[
\hookrightarrow W^{4,l,p}(\mathbb{R}^+; W^{1,p}_{\text{per}}(\mathbb{R}; W^{4-l,p}(\mathbb{R}^{n-1})))
\]
(2.7)

are continuous.

**Proof.** The regularity of $\Omega$ suffices to ensure the existence of a continuous extension operator $E : X^p_{\text{per}}(\mathbb{R} \times \Omega) \rightarrow X^p_{\text{per}}(\mathbb{R} \times \mathbb{R}^n)$ as in the case of classical Sobolev spaces. Consequently, it suffices to prove Lemma 2.1 in the whole-space case $\Omega = \mathbb{R}^n$. For this purpose, it is convenient to replace the time-axis $\mathbb{R}$ with the torus $\mathbb{T} := \mathbb{R}/\mathbb{T} \mathbb{Z}$ in the function spaces of $\mathbb{T}$-time-periodic functions. The torus $\mathbb{T}$ canonically inherits a topology and differentiable structure from $\mathbb{R}$ via the quotient mapping $\pi : \mathbb{R} \rightarrow \mathbb{T}$ in such a way that
\[
C^\infty(\mathbb{T}; E(\mathbb{R}^n)) = \{ f : \mathbb{T} \rightarrow E(\mathbb{R}^n) \mid f \circ \pi \in C^\infty(\mathbb{R}; E(\mathbb{R}^n)) \}
\]

for any generic Sobolev space $E(\mathbb{R}^n)$. Moreover, if $\mathbb{T}$ is equipped with the normalized Haar measure, we may introduce the norms $\|\cdot\|_{L^p(\mathbb{T}; E(\mathbb{R}^n))}$ and $\|\cdot\|_{W^{k,p}(\mathbb{T}; E(\mathbb{R}^n))}$ on $C^\infty(\mathbb{T}; E(\mathbb{R}^n))$ by the same expressions as in (2.1)–(2.2). Lebesgue spaces $L^p(\mathbb{T}; E(\mathbb{R}^n))$ and Sobolev spaces $W^{k,p}(\mathbb{T}; E(\mathbb{R}^n))$ on the torus are then defined as in (2.3) and (2.4), respectively. The quotient map $\pi$ employed as a lifting operator acts as an isometric isomorphism between $C^\infty_{\text{per}}(\mathbb{R}; E(\mathbb{R}^n))$ and $C^\infty(\mathbb{T}; E(\mathbb{R}^n))$, and consequently also between the Sobolev spaces $W^{k,p}_{\text{per}}(\mathbb{R}; E(\mathbb{R}^n))$ and $W^{k,p}(\mathbb{T}; E(\mathbb{R}^n))$. We may therefore verify the embeddings in the
setting of function spaces where the time-axis has been replaced with the torus. In this setting, we can employ the Fourier transform \( \mathcal{F}_{\mathbb{T} \times \mathbb{R}^n} \) in time and space to characterize the Sobolev spaces
\[
W^{2,p}(\mathbb{T} \times \mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{T} \times \mathbb{R}^n) \mid \mathcal{F}_{\mathbb{T} \times \mathbb{R}^n}^{-1}[(1 + |k|^2|\xi|^2)f] \in L^p(\mathbb{T} \times \mathbb{R}^n) \right\}
\]
and
\[
X^p(\mathbb{T} \times \mathbb{R}^n) = W^{3,p}(\mathbb{T} ; L^p(\mathbb{R}^n)) \cap W^{1,p}(\mathbb{T} ; W^{4,p}(\mathbb{R}^n)),
\]
where \( \mathcal{S}' \) denotes the space of tempered distributions and \( \mathcal{F}_{\mathbb{T} \times \mathbb{R}^n}^{-1} \) is the inverse Fourier transform.

Here, \( \mathcal{S}'(\mathbb{T} \times \mathbb{R}^n) \) denotes the space of Schwartz-Bruhat distributions on the locally compact abelian group \( \mathbb{T} \times \mathbb{R}^n \), and \((k, \xi) \in \mathbb{Z} \times \mathbb{R}^n \) an element of the corresponding dual group. Now observe that
\[
\|f\|_{W^{2,p}(\mathbb{T} \times \mathbb{R}^n)} \leq C \left\| \mathcal{F}_{\mathbb{T} \times \mathbb{R}^n}^{-1}[(1 + |k|^2|\xi|^2)f] \right\|_{L^p(\mathbb{T} \times \mathbb{R}^n)},
\]
(2.8)
The multiplier
\[
m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \quad m(\eta, \xi) := \frac{1 + |\eta|^2|\xi|^2}{1 + |\eta|^2 + |\eta||\xi|^2}
\]
satisfies the condition of Marcinkiewicz’s Multiplier Theorem ([13, Chapter IV, §6]). Indeed, by Young’s inequality \( |\eta|^2|\xi|^2 \leq \frac{1}{3} |\eta|^3 + \frac{2}{3} |\eta||\xi|^4 \), whence \( \|m\|_{\infty} < \infty \). Similarly, one may verify that
\[
\max_{\varepsilon \in \{0,1\}^n + \{1\}} |\xi_1^{\varepsilon_1} \cdots \xi_n^{\varepsilon_n} \eta^{\varepsilon_n + 1} \partial_1^{\varepsilon_1} \cdots \partial_n^{\varepsilon_n} \partial_{\eta}^{1} m(\eta, \xi)\|_{\infty} < \infty.
\]
Consequently, \( m \) is an \( L^p(\mathbb{R} \times \mathbb{R}^n) \) multiplier. By de Leeuw’s Transference Principle for Fourier multipliers on locally compact abelian groups (see for example [7, Theorem B.2.1]), it follows that the restriction \( m|_{\mathbb{Z} \times \mathbb{R}^n} \) is an \( L^p(\mathbb{T} \times \mathbb{R}^n) \) multiplier. From (2.8) we thus deduce
\[
\|f\|_{W^{2,p}(\mathbb{T} \times \mathbb{R}^n)} \leq C \left\| \mathcal{F}_{\mathbb{T} \times \mathbb{R}^n}^{-1}[(1 + |k|^3 + |k||\xi|^4)f] \right\|_{L^p(\mathbb{T} \times \mathbb{R}^n)}
\]
and we conclude (2.5). The embeddings (2.6)–(2.7) can be established in a completely similar manner.

Additionally, we make use of the following embedding properties:

**Lemma 2.2.** Let \( p \in (1, \infty) \) and \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a bounded domain with a \( C^4 \)-smooth boundary. For \( k = 1, 2 \) let \( q_k, r_k \in [p, \infty] \) satisfy for some \( \alpha_k \in [0, k] \):
\[
\begin{cases}
  r_k \leq \frac{2p}{2 - \alpha_k p} & \text{if } \alpha_k p < 2, \\
  r_k < \infty & \text{if } \alpha_k p = 2, \\
  r_k \leq \infty & \text{if } \alpha_k p > 2,
\end{cases}
\]
\[
\begin{cases}
  q_k \leq \frac{np}{n - (k - \alpha_k)p} & \text{if } (k - \alpha_k)p < n, \\
  q_k < \infty & \text{if } (k - \alpha_k)p = n, \\
  q_k \leq \infty & \text{if } (k - \alpha_k)p > n.
\end{cases}
\]
For \( k = 3, 4 \) let \( q_k, r_k \in [p, \infty] \) satisfy for some \( \alpha_k \in [0, k] \):
\[
\begin{cases}
  r_k \leq \frac{4p}{4 - \alpha_k p} & \text{if } \alpha_k p < 4, \\
  r_k < \infty & \text{if } \alpha_k p = 4, \\
  r_k \leq \infty & \text{if } \alpha_k p > 4,
\end{cases}
\]
\[
\begin{cases}
  q_k \leq \frac{np}{n - (k - \alpha_k)p} & \text{if } (k - \alpha_k)p < n, \\
  q_k < \infty & \text{if } (k - \alpha_k)p = n, \\
  q_k \leq \infty & \text{if } (k - \alpha_k)p > n.
\end{cases}
\]
Then all \( u \in W_{\text{per}}^{3,p}(\mathbb{R}; L^p(\Omega)) \cap W_{\text{per}}^{1,4}(\mathbb{R}; W^{4,p}(\Omega)) \) satisfy
\[
\left\| \partial_t^2 \nabla u \right\|_{L^{r_4,p}(\mathbb{R}; L^{r_4,p}(\Omega))} + \left\| \partial_t^2 u \right\|_{L^{r_2,p}(\mathbb{R}; L^{r_2,p}(\Omega))} + \left\| \partial_t \nabla u \right\|_{L^{r_3,p}(\mathbb{R}; L^{r_3,p}(\Omega))} \leq C_1 \left\| u \right\|_{W_{\text{per}}^{3,p}(\mathbb{R}; L^p(\Omega)) \cap W_{\text{per}}^{1,4}(\mathbb{R}; W^{4,p}(\Omega))},
\]
with \( C_1 = C_1(T, \Omega, r_1, q_1, r_2, q_2, r_3, q_3, r_4, q_4) \).

**Proof.** The lemma extends similar embeddings established in [9, Theorem 4.1] and can be shown by a similar technique. We sketch here the proof of the embedding of \( \partial_t u \). The others follow analogously. As in the proof of Lemma 2.1, it suffices to prove the assertion in the whole-space case \( \Omega = \mathbb{R}^n \) and in the setting where the time-axis \( \mathbb{R} \) has been replaced by the torus \( T \). In this setting we can employ the Fourier transform to obtain the representation
\[
\partial_t u = \mathcal{F}_T^{-1}[(1 - \delta_Z(k))|k|^{-\frac{1}{2} \alpha}] \ast_T \mathcal{F}^{-1}_\mathbb{R}^n[|\xi|^{\alpha - 4}] \ast_{\mathbb{R}^n} F,
\]
where
\[
F := \mathcal{F}^{-1}_{\mathbb{T} \times \mathbb{R}^n} \left[ m(k, \xi), \mathcal{F}_{\mathbb{T} \times \mathbb{R}^n}[(\partial_t + \Delta^2)\partial_t u] \right],
\]
\[
m(k, \xi) := \frac{(1 - \delta_Z(k))|k|^{\frac{1}{2} \alpha} |\xi|^{4 - \alpha}}{i^{\frac{2p}{p}} k + |\xi|^4}.
\]
Here, \( \delta_Z \) denotes the delta distribution on \( \mathbb{Z} \), that is, \( \delta_Z(0) = 1 \) and \( \delta_Z(k) = 0 \) for \( k \neq 0 \). Observer that \( \partial_t u = \mathcal{F}_T^{-1}[(1 - \delta_Z(k)) \mathcal{F}_T[\partial_t u]] \). Let \( \chi \) be a “cut-off” function with
\[
\chi \in C_0^\infty(\mathbb{R}; \mathbb{R}), \quad \chi(\eta) = 1 \text{ for } |\eta| \leq \frac{\pi}{T}, \quad \chi(\eta) = 0 \text{ for } |\eta| \geq \frac{2\pi}{T}.
\]
The function
\[
M : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \quad M(\eta, \xi) := \frac{(1 - \chi(\eta))|\eta|^{\frac{1}{4} \alpha} |\xi|^{4 - \alpha}}{|\eta|^4 + i^{\frac{2p}{p}} \eta}
\]
satisfies the condition of Marcinkiewicz’s Multiplier Theorem ([13, Chapter IV, §6]). By de Leeuw’s Transference Principle [7, Theorem B.2.1], \( M \) is therefore an \( L^p(\mathbb{T} \times \mathbb{R}^n) \)-multiplier. Furthermore, as in [9, Proof of Theorem 4.1] one may verify that for \( r, q \in (1, \infty) \) satisfying
\[
\frac{1}{r} = \left(1 - \frac{1}{4 \alpha}\right) + \frac{1}{p} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{4 - \alpha}{n}
\]
the mappings \( v \mapsto \mathcal{F}_T^{-1}[(1 - \delta_Z(k))|k|^{-\frac{1}{2} \alpha}] \ast_T v \) and \( w \mapsto \mathcal{F}_\mathbb{R}^{-1}[|\xi|^{\alpha - 4}] \ast_{\mathbb{R}^n} w \) are bounded operators \( L^p(\mathbb{T}) \to L^r(\mathbb{T}) \) and \( L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \), respectively. Thus, we obtain
\[
\left\| \partial_t u \right\|_{L^{r_4,p}(\mathbb{R}; L^{r_4,p}(\Omega))} \leq c_0 \left\| u \right\|_{W_{\text{per}}^{3,p}(\mathbb{R}; L^p(\Omega)) \cap W_{\text{per}}^{1,4}(\mathbb{R}; W^{4,p}(\Omega))}
\]
when \( r_4, q_4 \) and \( \alpha_4 \) satisfy the assumptions of the lemma. \( \square \)

Three types of trace operators are employed in the following:

\[
\begin{align*}
\text{Tr}_0 : C_0^\infty(\mathbb{R} \times \overline{\Omega}) & \to C_0^\infty(\mathbb{R} \times \partial \Omega), \quad \text{Tr}_0(u) := u|_{\mathbb{R} \times \partial \Omega}, \\
\text{Tr}_D : C_0^\infty(\mathbb{R} \times \overline{\Omega}) & \to C_0^\infty(\mathbb{R} \times \partial \Omega)^2, \quad \text{Tr}_D(u) := (u|_{\mathbb{R} \times \partial \Omega}, \Delta u|_{\mathbb{R} \times \partial \Omega}), \\
\text{Tr}_N : C_0^\infty(\mathbb{R} \times \overline{\Omega}) & \to C_0^\infty(\mathbb{R} \times \partial \Omega)^2, \quad \text{Tr}_N(u) := (\partial_n u|_{\mathbb{R} \times \partial \Omega}, \partial_n \Delta u|_{\mathbb{R} \times \partial \Omega}).
\end{align*}
\]

In order to characterize appropriate trace spaces, we introduce
\[
\begin{align*}
T_{\text{per}, D_1}(\mathbb{R} \times \partial \Omega) & := W_{\text{per}}^{3,\frac{p}{p}}(\mathbb{R}; L^p(\partial \Omega)) \cap W_{\text{per}}^{1,\frac{p}{p}}(\mathbb{R}; W^{4,\frac{p}{p}}(\partial \Omega)), \\
T_{\text{per}, D_2}(\mathbb{R} \times \partial \Omega) & := W_{\text{per}}^{2,\frac{p}{p}}(\mathbb{R}; L^p(\partial \Omega)) \cap W_{\text{per}}^{1,\frac{p}{p}}(\mathbb{R}; W^{2,\frac{p}{p}}(\partial \Omega)), \\
T_{\text{per}, N_1}(\mathbb{R} \times \partial \Omega) & := W_{\text{per}}^{\frac{5}{2},\frac{p}{p}}(\mathbb{R}; L^p(\partial \Omega)) \cap W_{\text{per}}^{1,\frac{p}{p}}(\mathbb{R}; W^{3,\frac{p}{p}}(\partial \Omega)), \\
T_{\text{per}, N_2}(\mathbb{R} \times \partial \Omega) & := W_{\text{per}}^{\frac{3}{2},\frac{p}{p}}(\mathbb{R}; L^p(\partial \Omega)) \cap W_{\text{per}}^{1,\frac{p}{p}}(\mathbb{R}; W^{1,\frac{p}{p}}(\partial \Omega)).
\end{align*}
\]
These spaces can be identified as trace spaces in the following sense:

**Lemma 2.3.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a $C^4$-smooth boundary. The trace operators extend to bounded operators:

\[
\begin{align*}
\text{Tr}_0 : X_{\text{per}}^p(\mathbb{R} \times \mathbb{R}_+^n) &\rightarrow T_{\text{per},D_1}(\mathbb{R} \times \partial \Omega), \\
\text{Tr}_0 : W_{\text{per}}^{2,p}(\mathbb{R}; L^p(\Omega)) \cap W_{\text{per}}^{1,p}(\mathbb{R}; W^{2,p}(\Omega)) &\rightarrow T_{\text{per},D_2}(\mathbb{R} \times \partial \Omega),
\end{align*}
\]

where the last equality is due to the embeddings (2.7). It follows from [15, Theorem 1.8.3] that $\text{Tr}_0$ extends to continuous operator

\[
\text{Tr}_0 : X_{\text{per}}^p(\mathbb{R} \times \mathbb{R}_+^n) \rightarrow \big( W_{\text{per}}^{1,p}(\mathbb{R}; L^p(\mathbb{R}^{n-1})), X_{\text{per}}^p(\mathbb{R} \times \mathbb{R}^{n-1}) \big)_{1-1/4p,p}.
\]

One may verify that $X_{\text{per}}^p(\mathbb{R} \times \mathbb{R}_+^n)$ and $W_{\text{per}}^{1,p}(\mathbb{R}; L^p(\mathbb{R}^{n-1}))$ form a quasilinearizable interpolation couple; see [15, Definition 1.8.4]. Indeed, an admissible operator in the sense of [15, Definition 1.8.4] is given by $V_1(\mu) := \mu^{-1}(\mu^{-1} - \partial^2_0 + \Delta^2)^{-1}$, where invertibility of $(\mu^{-1} - \partial_0^2 + \Delta^2) : X_{\text{per}}^p(\mathbb{R} \times \mathbb{R}^{n-1}) \rightarrow W_{\text{per}}^{1,p}(\mathbb{R}; L^p(\mathbb{R}^{n-1}))$ can be established by an analysis of the multiplier $(\eta, \xi) \rightarrow (\mu^{-1} + \eta^2 + |\xi|^4)^{-1}$ and an application of de Leeuw’s Transference Principle as in the proof of Lemma 2.1. Consequently, one obtains even better properties of the trace operator, namely that it possesses a continuous right-inverse; see [15, Theorem 1.8.5]. Moreover, we can utilize the property stated in [15, Theorem 1.12.1] concerning interpolation of intersections of spaces that form quasilinearizable interpolation couples to conclude

\[
\big( W_{\text{per}}^{1,p}(\mathbb{R}; L^p(\mathbb{R}^{n-1})), X_{\text{per}}^p(\mathbb{R} \times \mathbb{R}^{n-1}) \big)_{1-1/4p,p} = T_{\text{per},D_1}(\mathbb{R} \times \mathbb{R}^{n-1}),
\]

which verifies (2.9). The assertions (2.10)–(2.13) follow in a similar way.  

\[
\end{align*}
\]

**Lemma 2.4.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a $C^4$-smooth boundary. The embedding

\[
W_{\text{per}}^{2-\frac{1}{p},p}(\mathbb{R}; L^p(\partial \Omega)) \cap L_{\text{per}}^p(\mathbb{R}; W^{4-\frac{1}{p},p}(\partial \Omega)) \rightarrow T_{\text{per},D_2}(\mathbb{R} \times \partial \Omega)
\]

is continuous.

**Proof.** Denote the embedding (2.6) by $i$. Recalling (2.10) and (2.11), we find that $\text{Tr}_0 \circ i = R_0$ yields the embedding (2.14).  

For functions $f$ defined on time-space domains, we let

\[
\mathcal{P} f(t,x) := \frac{1}{T} \int_0^T f(s,x) \, ds, \quad \mathcal{P}_\perp f(t,x) := f(t,x) - \mathcal{P} f(t,x)
\]
whenever the integral is well defined. Since $\mathcal{P}f$ is independent on time $t$, we shall implicitly treat $\mathcal{P}f$ as a function in the spatial variable $x$ only. Observe that $\mathcal{P}$ and $\mathcal{P}_\perp$ are complementary projections on the space $C_0^\infty(\mathbb{R};E(\Omega))$. We shall employ the projections to decompose the Lebesgue and Sobolev spaces introduced above. Since $\mathcal{P}f$ is time independent, we refer to $\mathcal{P}f$ as the *steady-state* part of $f$. We refer to $\mathcal{P}_\perp f$ as the *purely oscillatory* part of $f$. By continuity, $\mathcal{P}$ and $\mathcal{P}_\perp$ extend to bounded operators on $L^p_{\text{per}}(\mathbb{R};E(\Omega))$ and $W^{k,p}_{\text{per}}(\mathbb{R};E(\Omega))$.

### 3. Linear Problem

The linear Eqs. (1.1) and (1.2) can be decomposed into a time-periodic Kuznetsov equation coupled with a time-periodic heat equation. Based on this observation, we obtain the following linear theory:

**Theorem 3.1.** Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with a $C^4$-smooth boundary. Let $p \in (1, \infty)$. Then

$$A_D : \mathcal{P}_\perp X^p_{\text{per}}(\mathbb{R} \times \Omega) \rightarrow \mathcal{P}_\perp L^p_{\text{per}}(\mathbb{R};L^p(\Omega)) \times \mathcal{P}_\perp T^p_{\text{per},D_1}(\mathbb{R} \times \partial \Omega) \times \mathcal{P}_\perp T^p_{\text{per},D_2}(\mathbb{R} \times \partial \Omega),$$

$$A_D(u) := (\Delta u - a \partial_t u - c^2 \Delta u - b \partial_t \Delta u), \quad \text{Tr}_D u,$$

and

$$A_N : \mathcal{P}_\perp X^p_{\text{per}}(\mathbb{R} \times \Omega) \rightarrow \mathcal{P}_\perp L^p_{\text{per}}(\mathbb{R};L^p(\Omega)) \times \mathcal{P}_\perp T^p_{\text{per},N_1}(\mathbb{R} \times \partial \Omega) \times \mathcal{P}_\perp T^p_{\text{per},N_2}(\mathbb{R} \times \partial \Omega),$$

$$A_N(u) := (\Delta u - a \partial_t u - c^2 \Delta u - b \partial_t \Delta u), \quad \text{Tr}_N u,$$

are homeomorphisms.

**Proof.** On the strength of the embedding (2.5), we observe that $\Delta$ is a bounded operator

$$\Delta : \mathcal{P}_\perp X^p_{\text{per}}(\mathbb{R} \times \Omega) \rightarrow \mathcal{P}_\perp W^{2,p}_{\text{per}}(\mathbb{R};L^p(\Omega)) \cap \mathcal{P}_\perp W^{1,p}_{\text{per}}(\mathbb{R};W^{2,p}(\Omega)).$$

Together with the continuity of the trace operators established in Lemma 2.3, this implies that the operators $A_D$ and $A_N$ are well-defined as bounded operators in the given setting. We start by showing that the operators are surjective. We concentrate on $A_D$, as the operator $A_N$ can be treated in a completely similar manner. To this end, let

$$(f,g,h) \in \mathcal{P}_\perp L^p_{\text{per}}(\mathbb{R};L^p(\Omega)) \times \mathcal{P}_\perp T^p_{\text{per},D_1}(\mathbb{R} \times \partial \Omega) \times \mathcal{P}_\perp T^p_{\text{per},D_2}(\mathbb{R} \times \partial \Omega).$$

Consider the coupled equations

$$\begin{cases}
\partial_t^2 v - c^2 \Delta v - b \partial_t \Delta v = f & \text{in } \mathbb{R} \times \Omega, \\
v = a h - \Delta g & \text{on } \mathbb{R} \times \partial \Omega,
\end{cases} \quad (3.1)$$

and

$$\begin{cases}
a \Delta u - \partial_t u = v & \text{in } \mathbb{R} \times \Omega, \\
u = g & \text{on } \mathbb{R} \times \partial \Omega.
\end{cases} \quad (3.2)$$

We recognize (3.1) as the time-periodic strongly damped wave equation, which was studied in [6], and (3.2) as the time-periodic heat equation, which was studied in [11]. Recalling the embedding (2.14), we see that $\partial_t$ is bounded as an operator

$$\partial_t : \mathcal{P}_\perp T^p_{\text{per},D_1}(\mathbb{R} \times \partial \Omega) \rightarrow \mathcal{P}_\perp T^p_{\text{per},D_2}(\mathbb{R} \times \partial \Omega),$$

whence $ah - \partial_t g \in \mathcal{P}_\perp T^p_{\text{per},D_2}(\mathbb{R} \times \partial \Omega)$. Consequently, we obtain directly from [6, Theorem 3.1] existence of a unique solution

$$v \in \mathcal{P}_\perp W^{2,p}_{\text{per}}(\mathbb{R};L^p(\Omega)) \cap \mathcal{P}_\perp W^{1,p}_{\text{per}}(\mathbb{R};W^{2,p}(\Omega)).$$
Lemma 3.3. Let \( p \) be a homeomorphism for all such \( n \geq 2 \). We now turn to (3.2). From [11, Theorem 2.1] and a standard regularity and lifting argument, based on the mapping property of the trace operator \( \text{Tr} \), both the Fredholm index and the kernel are independent on \( p \). Since the operators \( \partial_t \) and \( \text{Tr}_0 \) commute on spaces of smooth functions, it follows from (3.3), (3.4) and the mapping property of the trace operator \( \text{Tr}_0 \) asserted in Lemma 2.3, that they commute in the setting

\[
\partial_t \circ \text{Tr}_0 = \text{Tr}_0 \circ \partial_t : \mathcal{P}_\perp \mathcal{X}^p_{\per}(\mathbb{R} \times \Omega) \rightarrow \mathcal{P}_\per W^2_p(\mathbb{R}; W^1_p(\Omega)) \cap \mathcal{P}_\per W^{1,p}(\mathbb{R}; W^2_p(\Omega)).
\]

Since the operators \( \partial_t \) and \( \text{Tr}_0 \) commute on spaces of smooth functions, it follows from (3.3), (3.4) and the mapping property of the trace operator \( \text{Tr}_0 \) asserted in Lemma 2.3, that they commute in the setting

\[
\partial_t \circ \text{Tr}_0 = \text{Tr}_0 \circ \partial_t : \mathcal{P}_\per \mathcal{X}^p_{\per}(\mathbb{R} \times \Omega) \rightarrow \mathcal{P}_\per T^p_{\per,\Omega}(\mathbb{R} \times \partial \Omega).
\]

We thus deduce from (3.2) and the boundary condition in (3.1) that

\[
\text{Tr}_D u = (g, \text{Tr}_0(\Delta u)) = \left( g, \text{Tr}_0 \left( \frac{1}{a} \left[ \partial_t u + v \right] \right) \right) = \left( g, \frac{1}{a} \partial_t g + \frac{1}{a} v \right) = (g, h).
\]

It follows that \( A_D(u) = (f, g, h) \), and we conclude that \( A_D \) is surjective. To show that \( A_D \) is injective, consider \( u \in \mathcal{P}_\per \mathcal{X}^p_{\per}(\mathbb{R} \times \Omega) \) with \( A_D(u) = (0, 0, 0) \). Unique solvability of the time-periodic strongly damped wave equation [6, Theorem 3.1] implies \( a \Delta u - \partial_t u = 0 \). In turn, unique solvability of the time-periodic heat equation [11, Theorem 2.1] implies \( u = 0 \). Consequently, \( A_D \) is injective. By the open mapping theorem, \( A_D \) is a homeomorphism.

Next, we consider the bi-Laplacian Dirichlet problem

\[
\begin{cases}
- \alpha^2 \Delta^2 u = f & \text{in } \Omega, \\
(u, \Delta u) = (g, h) & \text{on } \partial \Omega,
\end{cases}
\]

and the corresponding Neumann problem

\[
\begin{cases}
- \alpha^2 \Delta^2 u = f & \text{in } \Omega, \\
(\partial_\nu u, \partial_\nu \Delta u) = (g, h) & \text{on } \partial \Omega.
\end{cases}
\]

Both of these boundary value problems are elliptic in the sense of Agmon, Douglis and Nirenberg, and the corresponding linear operators therefore Fredholm. Consequently, the following two lemmas can be established with standard methods:

**Lemma 3.2.** Let \( \Omega \) and \( p \) be as in Theorem 3.1. For any \( f \in L^p(\Omega) \), \( g \in W^{4-\frac{1}{p}}(\partial \Omega) \) and \( h \in W^{2-\frac{1}{p}}(\partial \Omega) \) there exists a unique solution \( u \in W^{4,p}(\Omega) \) to (3.5) satisfying

\[
\|u\|_{4,p} \leq C_2 \left( \|f\|_p + \|g\|_{4-\frac{1}{p},p} + \|h\|_{2-\frac{1}{p},p} \right),
\]

where \( C_2 = C_2(p, \Omega) > 0 \).

**Proof.** In the case \( p = 2 \), it can be verified directly that the operator

\[
A : W^{4,p}(\Omega) \rightarrow L^p(\Omega) \times W^{4-\frac{1}{p},p}(\partial \Omega) \times W^{2-\frac{1}{p},p}(\partial \Omega), \quad A u := \left( - \alpha^2 \Delta^2 u, u, \Delta u \right)
\]

is Fredholm of index 0; see [16, Example 16.6]. Moreover, it is easy to show that the kernel is trivial. The operator is therefore a homeomorphism in this case. By the celebrated result of Geymonat [10] (see also [12]), both the Fredholm index and the kernel are independent on \( p \in (1, \infty) \), whence the operator is in fact a homeomorphism for all such \( p \).

**Lemma 3.3.** Let \( \Omega \) and \( p \) be as in Theorem 3.1. For any \( f \in L^p(\Omega) \), \( g \in W^{3-\frac{1}{p},p}(\partial \Omega) \) and \( h \in W^{1-\frac{1}{p},p}(\partial \Omega) \) satisfying

\[
\int_\Omega f \, dx + \alpha^2 \int_{\partial \Omega} h \, dS = 0
\]

(3.8)
there exists a solution \( u \in W^{4,p}(\Omega) \) to (3.6) satisfying
\[
\|u\|_{4,p} \leq C_3 \left( \|f\|_p + \|g\|_{3-\frac{1}{p},p} + \|h\|_{1-\frac{1}{p},p} \right),
\]
where \( C_3 = C_3(p,\Omega) > 0 \).

Proof. In the case \( p = 2 \), it can be verified directly that also the operator
\[
A : W^{4,p}(\Omega) \to L^p(\Omega) \times W^{3-\frac{1}{p},p}(\partial \Omega) \times W^{1-\frac{1}{p},p}(\partial \Omega), \quad Au := \left( -ac^2 \Delta u, \partial_p u, \partial_p \Delta u \right)
\]
is Fredholm of index 0; see again [16, Example 16.6]. It is easy to see that its kernel consists of the constants, and is therefore one-dimensional. It follows that also the kernel of the adjoint operator is one-dimensional and is given by the span of the constant \((1,0,ac^2)\). Since \( \mathcal{A}(A) = \mathcal{N}(A^*) \), data \((f,g,h)\) satisfying (3.8) lie in the range of \( A \). Since the range of a Fredholm operator is closed, existence of a solution \( u \in W^{4,p}(\Omega) \) to (3.6) satisfying (3.9) follows.

4. Proof of Main Theorems

We shall now prove Theorems 1.1 and 1.2. For this purpose, we employ a fixed-point argument based on the estimates established for the linearized systems (1.1) and (1.2) in the previous section. The nonlinear terms in (BCD) and (BCN) can be estimated as follows:

Lemma 4.1. For \( n \geq 2 \) let \( \Omega \) and \( p \) be as in Theorem 1.1. Then
\[
\|\partial_t \nabla v \cdot \partial_t^2 u\|_p + \|\partial_t^2 v \partial_t^2 u\|_p + \|\partial_t \nabla v \cdot \partial_t \nabla u\|_p + \|\nabla v \cdot \partial_t^2 u\|_p \leq C_4 \|v\|_{X^p_{\text{per}}} \|u\|_{X^p_{\text{per}}}
\]
holds for any \( u, v \in X^p_{\text{per}}(\mathbb{R} \times \Omega) \).

Proof. For \( \max\{2, \frac{n}{2}\} < p < \infty \) Lemma 2.2 yields
\[
\alpha_1 = 1 : \quad \|\partial_t^2 \nabla u\|_{L^\infty_{\text{per}}(\mathbb{R};L^p(\Omega))} \leq c_0 \|v\|_{X^p_{\text{per}}(\mathbb{R} \times \Omega)},
\]
\[
\alpha_2 = 0 : \quad \|\partial_t^2 u\|_{L^p_{\text{per}}(\mathbb{R};L^\infty(\Omega))} \leq c_1 \|u\|_{X^p_{\text{per}}(\mathbb{R} \times \Omega)},
\]
\[
\alpha_3 = 2 : \quad \|\partial_t^2 v\|_{L^p_{\text{per}}(\mathbb{R};L^p(\Omega))} \leq \|v\|_{X^p_{\text{per}}(\mathbb{R} \times \Omega)},
\]
\[
\alpha_4 = 0 : \quad \|\partial_t \nabla u\|_{L^p_{\text{per}}(\mathbb{R};L^p(\Omega))} \leq \|u\|_{X^p_{\text{per}}(\mathbb{R} \times \Omega)},
\]
\[
\alpha_5 = 3 : \quad \|\partial_t \nabla u\|_{L^p_{\text{per}}(\mathbb{R};L^\infty(\Omega))} \leq \|u\|_{X^p_{\text{per}}(\mathbb{R} \times \Omega)}.
\]

The restriction \( p > 2 \) is required to obtain the first and last estimates above, and \( p > \frac{n}{2} \) is needed for the remaining ones. Utilizing H"older’s inequality, we can therefore deduce
\[
\|\nabla v \cdot \partial_t^2 u\|_p \leq \|\nabla v\|_{L^p_{\text{per}}(\mathbb{R};L^\infty(\Omega))} \|\partial_t^2 u\|_{L^p_{\text{per}}(\mathbb{R};L^p(\Omega))} + \|\partial_t \nabla v\|_p \|\partial_t^2 u\|_p
\]
\[
+ \|\partial_t^2 v\|_{L^p_{\text{per}}(\mathbb{R};L^p(\Omega))} \|\partial_t^2 u\|_{L^p_{\text{per}}(\mathbb{R};L^\infty(\Omega))} + \|\partial_t \nabla v\|_{L^p_{\text{per}}(\mathbb{R};L^\infty(\Omega))} \|\partial_t u\|_{L^\infty_{\text{per}}(\mathbb{R};L^p(\Omega))},
\]
\[
\leq C_4 \|v\|_{X^p_{\text{per}}} \|u\|_{X^p_{\text{per}}},
\]
with \( C_4 = c_0 c_1 + c_1 c_2 + c_3 c_4 + c_5 \).

Proof of Theorem 1.1. Consider functions \( f \in L^p_{\text{per}}(\mathbb{R};L^p(\Omega)) \), \( g \in T^p_{\text{per},D_1}(\mathbb{R} \times \partial \Omega) \) and \( h \in T^p_{\text{per},D_2}(\mathbb{R} \times \partial \Omega) \) with \( \|f\|_p + \|g\|_{T^p_{\text{per},D_1}} + \|h\|_{T^p_{\text{per},D_2}} \leq \varepsilon \), where \( \varepsilon \) is to be specified later. Put \( f_s := \mathcal{P} f \), \( g_s := \mathcal{P} g \), \( h_s := \mathcal{P} h \) and \( f_p := \mathcal{P}_\perp f \), \( g_p := \mathcal{P}_\perp g \), \( h_p := \mathcal{P}_\perp h \). We shall establish existence of a solution \( u \) to (BCD)
on the form $u = u_s + u_p$, where $u_s \in W^{1,p}(\Omega)$ is a solution to the elliptic problem (3.5) with $f_s$ and $(g_s, h_s)$ as right-hand side, and $u_p \in \mathcal{P}_\perp \mathcal{X}_p^p(\mathbb{R} \times \Omega)$ is a solution to the purely oscillatory problem
\begin{align}
(a\Delta - \partial_t)(\partial_t^2 u_p - c^2\Delta u_p - b\partial_t \Delta u_p) \\
-2s\nabla u_s \cdot \nabla \partial_t^2 u_p - \partial_t^2 (k(\partial_t u_p)^2 + s|\nabla u_p|^2) = f_p \\
(u_p, \Delta u_p) = (g_p, h_p)
\end{align}

in $\mathbb{R} \times \Omega$, \quad (4.1)

Lemma 3.2 yields a solution $u_s \in W^{1,p}(\Omega)$ to (3.5) satisfying (3.7). Sobolev’s embedding theorem implies
\[ \|\nabla u_s\|_\infty \leq c_0 \|\nabla u_s\|_{3,p} \leq c_0 \|u_s\|_{4,p} \leq c_1 \varepsilon. \]

The solution to (4.1) shall be obtained as a fixed point of the mapping
\[ N: \mathcal{P}_\perp \mathcal{X}_p^p(\mathbb{R} \times \Omega) \to \mathcal{P}_\perp \mathcal{X}_p^p(\mathbb{R} \times \Omega), \]
\[ N(u_p) := A_\perp^{-1}\left(\mathcal{N}(u_p)\right), \]
with $A_\perp$ as in Theorem 3.1. For this purpose, let $p > 0$ and consider some $u_p \in \mathcal{P}_\perp \mathcal{X}_p^p(\mathbb{R} \times \Omega) \cap B_p$.

Since $A_\perp$ is a homeomorphism, we conclude from Lemmas 4.1 and 4.2 the estimate
\[ \|N(u_p)\|_{\mathcal{X}_p^p} \leq \|A_\perp^{-1}\| \left( \|\partial_t u_p \partial_t^2 u_p\|_p + \|\partial_t^2 u_p\|_p + \|\partial_t \nabla u_p\|_p + \|f_p\|_p \right) \]
\[ \leq c_2 \left( \|u_p\|_{\mathcal{X}_p^p}^2 + \|\nabla u_s\|_{L^\infty(\Omega)} \|\nabla \partial_t^2 u_p\|_{L^p(\mathbb{R}; L^p(\Omega))} + \varepsilon \right) \]
\[ \leq c_3 \left( \rho^2 + \varepsilon \rho + \varepsilon \right). \]

Choosing $\rho := \sqrt{\varepsilon}$ and $\varepsilon$ sufficiently small, we have $c_3 (\rho^2 + \varepsilon \rho + \varepsilon) \leq \rho$, i.e., $N$ becomes a self-mapping on $B_p$. Furthermore,
\[ \|N(u_p) - N(v_p)\|_{\mathcal{X}_p^p} \leq c_4 \|A_\perp^{-1}\| \left( \|\partial_t u_p \partial_t^2 u_p - \partial_t v_p \partial_t^2 v_p\|_p \right) \]
\[ \leq c_5 \left( \|\partial_t u_p \partial_t^2 (u_p - v_p)\|_p + \|\partial_t^2 v_p \partial_t (u_p - v_p)\|_p + \|\partial_t u_p \cdot \nabla \partial_t (u_p - v_p)\|_p \right) \]
\[ \leq c_6 \left( \|u_p - v_p\|_{\mathcal{X}_p^p} + \varepsilon \|u_p - v_p\|_{\mathcal{X}_p^p} \right). \]

Therefore, if $\varepsilon$ is sufficiently small $N$ also becomes a contracting self-mapping. By the contraction mapping principle, existence of a fixed point for $N$ follows. This concludes the proof.

**Proof of Theorem 1.2.** Similar to the proof of Theorem 1.1. Employing the Neumann operator $A_N$ instead of $A_\perp$, and utilizing Lemma 3.3 instead of Lemma 3.2, we obtain Theorem 1.2 by the same argument as above.

**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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