The index of leafwise $G$-transversally elliptic operators on foliations

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Abstract

We introduce and study the index morphism for $G$-invariant leafwise $G$-transversally elliptic operators on smooth closed foliated manifolds which are endowed with leafwise actions of the compact group $G$. We prove the usual axioms of excision, multiplicativity and induction for closed subgroups. In the case of free actions, we relate our index class with the Connes-Skandalis index class of the corresponding leafwise elliptic operator on the quotient foliation. Finally we prove the compatibility of our index morphism with the Gysin Thom isomorphism and reduce its computation to the case of tori actions. We also construct a topological candidate for an index theorem using the Kasparov Dirac element for euclidean $G$-representations.

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Introduction

Index theory for leafwise elliptic pseudodifferential operators on smooth foliations of closed manifolds was initiated by A. Connes in the late seventies, first in [20] where he proved a topological formula in the presence of a holonomy invariant transverse measure, and then in his fundamental paper [21] where he introduced all the needed tools and constructions to prove an index theorem in the K-theory of the (noncommutative) space of leaves. The complete proof of such index theorem was later carried out by A. Connes and G. Skandalis in [23], fully using Kasparov’s bivariant K-theory and especially the existence of the associative Kasparov product. Bibrations of smooth manifolds correspond to simple foliations which are the local models for general foliations, therefore the Connes-Skandalis index theorem represents a far reaching generalisation of the Atiyah-Singer families index theorem [5], taking into account the global high complexity of the transverse structure of the foliation. It is worthpointing out that the K-theory group of the space of leaves is in general hard to compute, and inorder to deduce the expected important consequences in topology, riemannian geometry and number theory for instance, one needs to pair the Connes-Skandalis index formula with appropriate (higher) traces inorder to provide scalar topological index formulae in terms of characteristic classes, see for instance [22, 27, 13, 24, 11].

We are mainly interested here in the generalization of the Connes-Skandalis index theory to the class of $G$-invariant leafwise $G$-transversally elliptic operators where $G$ is an additional compact group acting by leafwise diffeomorphisms. Notice first that the Connes-Skandalis index theorem is already valid for foliations which admit an additional foliated action of a compact group $G$ and for $G$-invariant leafwise elliptic
operators. The $G$-index theorem in this case has applications to fixed point theorems in the spirit of the Atiyah-Segal formulae as proved in [3]. On the other hand, the Atiyah-Segal formula as well as its generalization to foliations obtained in [11] only involves the group $G$ in an algebraic way and no harmonic analysis is involved. M. Atiyah and I. Singer then proposed to use $G$-invariant $G$-transversally elliptic operators to produce more complicated index invariants. This theory was initiated in Atiyah's seminal lecture series at the Institute for Advanced Studies in 1971, published in the LNM-401 [1] and as explained there, based on joint (unpublished) work with I. Singer. The lack of full ellipticity for $G$-transversally elliptic operators implies that the kernel $G$-representation is not finite dimensional in general. However, by combining the invariance of the operator along the orbits and its ellipticity in the directions transverse to these orbits, this kernel turns out to contain finite dimensional representations only with finite multiplicities. Moreover, Atiyah showed that the index can actually be well defined as a central distribution on $G$ which lives in a Sobolev space of $G$. When the operator is fully elliptic, one recovers the Atiyah-Singer $G$-index and the index distribution is given by the character of the finite dimensional kernel, a smooth central function on $G$. When $G$ is finite, this theory reduces to the equivariant index theory for elliptic operators alluded to above, and in general it strictly contains it since $G$-invariant elliptic operators are $G$-transversally elliptic. When the manifold is for instance a homogeneous $G$-manifold $G/H$, the whole theory reduces to the index problem for the zero operator and gives the reciprocity formula for induced representations from the closed subgroup $H$. When $G$ acts freely on a general closed manifold, the $G$-invariant $G$-transversally elliptic operators correspond to elliptic operators on the quotient manifold and the whole theory is rather standard and reduces to the elliptic index theory on the quotient. When the action is not free though, we get a suitable option for the index theory of the singular quotient manifold.

Atiyah proved in [11] a list of important properties for his index distribution and reduced the computation of the index distribution to the case of linear actions of tori on finite dimensional vector spaces. Later on, N. Berline and M. Vergne proved a delocalized cohomological formula for $G$-invariant $G$-transversally elliptic operators, see [17, 18, 19]. This formula computes the Atiyah index distribution around a given $s \in G$ as an integral of equivariant characteristic classes, by extending the similar formula for $G$-invariant elliptic operators that they obtained out of the Atiyah-Segal-Singer theorem by using the Kirillov localization principle. The similar formula was also obtained using the topological description of equivariant cohomology in [2]. It is worthpointing out the close relation of these delocalized index formulae with the Duistermaat-Heckman theorem for tori actions on compact symplectic manifolds [25]. These formulae give important insights into the Atiyah index distribution but don’t provide the explicit computation of the multiplicities building blocks of this distribution. To the authors knowledge, besides the case of circle actions which was already solved in [1], the general computation of the index distribution by an explicit formula including topology and harmonic analysis is still an open problem. More recently in [37], G. Kasparov applied the classical "Bott ↔ Dirac" approach to the Atiyah-Singer theorem to investigate the index problem now for $G$-invariant $G$-transversally elliptic operators. His approach is new and computes the K-homology index class introduced by P. Julg in [33], an equivalent index invariant which carries the allowed multiplicities of all the irreducible representations of $G$, in terms of the stable homotopy class of the principal symbol. Moreover, Kasparov actually considered the more general case of locally compact groups acting properly and cocompactly on smooth manifolds and succeeded in this wide generality to compute the Julg index class as a cup product of the symbol class by a fundamental Dirac element. This is again another important progress towards the full explicit computation of the multiplicities.

As explained above, the present paper is devoted to the generalization of the Connes leafwise index theory to the setting of $G$-invariant leafwise $G$-transversally elliptic operators. It is our aim that the transversally elliptic leafwise theory will provide interesting invariants for some singular foliations. We are more precisely interested in leafwise pseudodifferential operators which are $G$-invariant and leafwise $G$-transversally elliptic, for a given leafwise action of a compact group $G$. The starting point of this work was the observation that, in the presence of a holonomy invariant transverse measure, the Connes machinery [20] can be displayed to produce a measured distributional index for $G$-invariant leafwise $G$-transversally elliptic operators, mix-
ing the Atiyah approach with the Murray-von Neumann dimension theory. In the case of a compact Lie group, exactly as $G$-invariant $G$-transversally elliptic operators on closed manifolds yield type I spectral triples on $\mathcal{C}^\infty(G)$, in the foliated case the holonomy invariant measure allows to see any $G$-invariant leafwise $G$-transversally elliptic operator as a semi-finite spectral triple \[12\]. Now in the lack of such holonomy invariant measure, and exactly as in the Connes-Skandalis work, we were naturally led to the construction of an index theory taking place in appropriate bivariant K-theory groups. We prove in the present paper all the needed axioms to reduce the computation of our bivariant index class to the case of tori actions on $G$-representations where all the previously listed results can be applied. Moreover, we also construct a topological candidate for an index theorem in our setting. When the foliation is top dimensional, our results reduce to the now classical Atiyah results but replacing the cohomological viewpoint of distributions by the K-homology viewpoint introduced by P. Julg \[33\] and privileged in \[37\]. With no surprise and following the Connes-Skandalis method, we again fully exploit Kasparov’s bivariant theory and especially the deep associativity of the Kasparov product \[35\]. In order to keep this paper in a reasonable size, the (higher) distributional approach will be dealt with in a forthcoming paper where we also develop the cohomological viewpoint in the spirit of \[18\] \[19\]. We point out that this approach has been successfully carried out by the first author in the case of closed fibrations, including an extension of the Berline-Vergne formula, see \[7\] \[8\].

Let us explain in more details some results. Given a smooth foliation $\mathcal{F}$ of a smooth closed riemannian manifold $M$ together with a smooth isometric action of a compact group $G$ by leaf-preserving diffeomorphisms, we consider a classical leafwise pseudodifferential operator $P$ acting between the smooth sections of $G$-equivariant vector bundles over $M$, which is $G$-invariant and leafwise $G$-transversally elliptic. The leafwise $G$-transversality condition means that the restriction $P_L$ of $P$ to every leaf $L$ of $(M, \mathcal{F})$ is $G$-transversally elliptic in the usual sense. We denote by $F$ the total space of the leafwise tangent bundle that we identify as usual with its dual bundle $F^*$ by using the $G$-invariant metric. The subspace $F_G \subset F$ is composed of the leafwise tangent vectors which are orthogonal to the orbits of $G$, in the classical notations $F_G = F \cap T_G(M)$, see \[4\]. We shall always assume that the $G$-action is a holonomy action in the sense of \[15\], a technical condition which, when $G$ is connected, is automatically satisfied for all foliations. For general $G$, it is also satisfied for a large class of foliations, see again \[15\]. We then prove that $P$ admits a well defined index class which lives in the equivariant Kasparov bivariant group $\text{KK}_G(C^*(G), C^*(M, \mathcal{F}))$ where $G$ acts on the convolution group $C^*$-algebra $C^*G$ by conjugation. As usual, we verify that the index class only depends on the stable homotopy class of the principal symbol of $P$ when restricted to $F_G$ and composing with the forgetful map, we end up with our privileged index morphism

$$\text{Ind}^\mathcal{F} : \text{Ind}_{G}(F_G) \rightarrow \text{KK}(C^*(G), C^*(M, \mathcal{F})).$$

Therefore, the index of $P$ can be evaluated at any irreducible representation of $G$ and yields the K-theory multiplicity $m_P : \hat{G} \rightarrow \text{K}(C^*(M, \mathcal{F}))$ which extends the usual integer valued multiplicity map. If the foliation $(M, \mathcal{F})$ is gifted with a holonomy invariant Borel transverse measure $\Lambda$, then we recover the notion of $\Lambda$-multiplicity $m^\Lambda_P : \hat{G} \rightarrow \mathbb{R}$ which assigns to any $V \in \hat{G}$ the $\Lambda$-index in the Murray-von Neumann sense of the restriction of $P \otimes V$ to the invariant sections.

The main properties of our index morphism $\text{Ind}^\mathcal{F}$ are investigated and we prove the famous excision and multiplicativity axioms as well as the compatibility with the elliptic theory in the case of free actions. This latter compatibility theorem implies in particular that if $(M, \mathcal{F}) \rightarrow (B, \mathcal{F}_B)$ is a principal G-bundle so that $G$ preserves the leaves of $\mathcal{F}$ and induces the foliation $\mathcal{F}_B$ downstairs, and if $P$ is a $G$-invariant leafwise $G$-transversally elliptic operator on $(M, \mathcal{F})$ whose symbol corresponds to a leafwise elliptic operator $P_0$ on $(B, \mathcal{F}_B)$ then the Connes-Skandalis leafwise index of $P_0$ can be recovered from the index class of $P$ upstairs by evaluation at the trivial representation, modulo the obvious Morita morphism. The multiplicativity axiom can also be stated in a natural way as follows. Given two closed $G$-foliations $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ as before and assuming that an extra compact group $H$ acts on $(M, \mathcal{F})$ so that this action commutes with the $G$-action, any $G \times H$-invariant leafwise $H$-transversally (and not only $G \times H$-transversally) elliptic operator $P$ on $(M, \mathcal{F})$ turns out to have an index class in $\text{KK}_G(C^*H, C^*(M, \mathcal{F}))$ which yields an index class $\text{Ind}^\mathcal{F}_G(P)$
living in $\text{KK}(C^*(G \times H), C^*(M, \mathcal{F}) \otimes C^*G)$. On the other hand any $G$-invariant $G$-transversally elliptic operator $P'$ on $(M', \mathcal{F}')$ has the index class $\text{Ind}^{\mathcal{F}'}(P')$ in $\text{KK}(C^*G, C^*(M, \mathcal{F}))$. Therefore, we may consider the Kasparov product

$$\text{Ind}^{\mathcal{F}}(P) \otimes \text{Ind}(P') \in \text{KK}(C^*G, C^*(M, \mathcal{F}) \otimes C^*(M', \mathcal{F}'))) = \text{KK}(C^*G, C^*(M \times M', \mathcal{F} \times \mathcal{F}')).$$

The multiplicativity axiom then tells us that this latter class is the index class of a $G \times H$-invariant $G \times H$-transversally elliptic operator on $(M \times M', \mathcal{F} \times \mathcal{F}')$ which, as expected, is the operator whose principal symbol class is the cup product of the principal symbol classes of $P$ and $P'$ as defined in [3]. See Theorem 3.14 for the more precise statement. The previous axioms enabled us to prove the compatibility of our index morphism with the Gysin map associated with $G$-equivariant embeddings of foliations. More precisely, we prove the following

**Theorem 0.1.** Let $\iota : (M, \mathcal{F}) \hookrightarrow (M', \mathcal{F}')$ be a $G$-equivariant embedding of smooth foliations such that $\iota^*(TM'/T\mathcal{F}') \simeq TM/T\mathcal{F}$. Assume furthermore that $M$ is compact, then for any $j \in \mathbb{Z}_2$, the following diagram commutes:

$$
\begin{array}{ccc}
\text{KK}^j(C^*G, C^*(M, \mathcal{F})) & \xrightarrow{\iota^*} & \text{KK}^j(C^*G, C^*(M', \mathcal{F}')) \\
\text{Ind}^\mathcal{F} & \downarrow & \text{Ind}^\mathcal{F}' \\
\end{array}
$$

The class $\iota^*$ is a quasi-trivial Morita extension, see Section 3. Another important feature of the index morphism is the generalized reciprocity formula for closed subgroups as well as its good behaviour with respect to the restriction to a maximal torus. Assuming $G$ connected with a maximal torus $T$, we obtain for instance the following

**Theorem 0.2.** Denote by $r^G_\mathcal{F} : K^j_G(F_G) \rightarrow K^j_T(F_T)$ the map defined in Section 4 using the Dolbeault operator associated with the complex structure on $G/T$. Then for $j \in \mathbb{Z}_2$ the following diagram commutes:

$$
\begin{array}{ccc}
\text{KK}^j(C^*G, C^*(M, \mathcal{F})) & \xrightarrow{r^G_\mathcal{F}} & \text{KK}^j(C^*G, C^*(M, \mathcal{F})) \\
\text{Ind}^\mathcal{F} & \downarrow & \text{Ind}^\mathcal{F} \\
\end{array}
$$

where $[i] \in \text{KK}(C^*G, C^*(T))$ is the induction class.

This theorem allows to reduce the computation of the index morphism to the case of tori actions. We finally end our paper by the construction of a topological candidate for an index theorem. More precisely, using a $G$-embedding of $M$ in a euclidean $G$-representation $E$, we show that there exists a topological transversal $\mathcal{N}_G$ for the lamination $(M \times T_G(E), \mathcal{F} \times 0)$ together with a well defined Gysin morphism associated with a K-oriented $G$-embedding

$$\iota : K^j_G(F_G) \rightarrow K^j_G(\mathcal{N}_G).$$

Hence composition of $\iota$ with the quasi-trivial Morita map $K^j_G(\mathcal{N}_G) \rightarrow K^j_G(C^*(M \times T_G(E), \mathcal{F} \times 0))$ yields the $R(G)$-morphism

$$K^j_G(F_G) \rightarrow K^j_G(C_0(T_G(E)) \otimes C^*(M, \mathcal{F})).$$

Now, the topological index morphism is obtained by composition of this morphism with the Dirac morphism defined in [37] on $E$, through the Kasparov descent morphism.
Even in the case of closed fibrations as considered in [7], this topological construction is new and completes the bivariant approach to the families Atiyah problem that was investigated in [7].

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1 Some preliminaries about actions on foliations

We gather in this first section many classical results about group actions on foliations that will be used in the sequel. Most of them are well known to experts.
1.1 Holonomy actions and Hilbert $G$-modules

This first paragraph is devoted to a brief review of some standard results. For most of the classical properties of Hilbert $C^*$-modules and regular operators between them, we refer the reader to [35] and [15]. The constructions given below extend the standard ones, see for instance [4, 8, 26, 28, 34]. Our hermitian scalar products will always be linear in the second variable and anti-linear in the first. Let $G$ be a compact group with a fixed bi-invariant Haar measure $dg$. The convolution $*$-algebra $L^1(G)$ is defined as usual with respect to the rules

$$(\varphi \psi)(g) := \int_{g_1 \in G} \varphi(g_1) \psi(g_1^{-1} g) dg_1 \text{ and } \varphi^*(g) := \varphi(g^{-1}).$$

We denote by $C^*G$ the $C^*$-algebra associated with $G$, which is the operator-norm closure of the range of $L^1(G)$ in the bounded operators on $L^2(G)$. A classical construction shows that any finite-dimensional unitary representation of $G$ naturally identifies with a finitely generated projective module on $C^*G$ [32]. There is a well defined action of $G$ by automorphisms of the $C^*$-algebra $C^*G$ which is induced by the adjoint action on $C(G)$ given by

$$(Ad_g \varphi)(k) := \varphi(g^{-1} k g), \quad \text{for } \varphi \in C(G) \text{ and } g, k \in G.$$

Let now $M$ be a smooth compact manifold and let $\mathcal{F}$ be a given smooth foliation of $M$. We assume that $G$ acts smoothly on $M$ by leaf-preserving diffeomorphisms, so any element $g \in G$ preserves each leaf of $(M, \mathcal{F})$. We denote by $F$ the subbundle of $TM$ composed of all the vectors tangent to the leaves of $\mathcal{F}$, this is the tangent bundle of our foliation and its dual bundle is the cotangent bundle of the foliation and is denoted as usual by $T^*$. We fix a $G$-invariant riemannian metric on $M$ so that $G$ acts by isometries of $M$, and so that we can identify $F^*$ with a $G$-subbundle of $T^*M$ when needed. We denote by $\mathcal{G}$ the holonomy groupoid that will be confused with the manifold of its arrows. We assume for simplicity that $\mathcal{G}$ is Hausdorff so that $M = G^{(0)}$ can be identified with a closed subspace (and a submanifold) of $\mathcal{G}$. We denote as usual by $r$ and $s$ respectively the range and source maps of $\mathcal{G}$ and by $\mathcal{G}_x := s^{-1}(x)$ and $\mathcal{G}^x := r^{-1}(x)$. The compact group $G$ acts obviously on $\mathcal{G}$ by groupoid diffeomorphisms, hence $r$ and $s$ are $G$-equivariant submersions. The $G$-invariant riemannian metric induces a $G$-invariant riemannian metric on the leaves. We also choose and fix a leafwise Lebesgue class measure which is $G$-invariant and which allows to define a $G$-invariant Haar system $\nu$ on $\mathcal{G}$. More precisely, on each holonomy cover $s : \mathcal{G}^x := r^{-1}(x) \to L_x$ of the leaf $L_x$ through $x \in M$, we have the well defined "pull-back" measure $\nu^x$, see for instance [20]. The family $\nu^x := (\nu_x)_{x \in M}$ is then easily seen to be a (continuous and even smooth) Haar system for $\mathcal{G}$ in the sense of [12]. Similarly we may define the measure $\nu_x$ on the holonomy cover $r : \mathcal{G}_x \to L_x$ but this latter can also be seen as the image of $\nu^x$ under the diffeomorphism $\gamma : r \to \gamma^{-1}$. The $G$-invariance of the Haar system means that $g_* \nu^x = \nu^{g x}$ for any $(g, x) \in G \times M$ or said differently, that for any $f \in C_c(\mathcal{G})$ one has

$$\int_{\mathcal{G}^x} f(\gamma) d\nu^x(\gamma) = \int_{\mathcal{G}^x} f(g \gamma) d\nu^x(\gamma).$$

The space $C_c(\mathcal{G})$ of compactly supported continuous functions on $\mathcal{G}$ is endowed with the usual structure of an involutive convolution algebra for the rules

$$(f_1 f_2)(\gamma) := \int_{\gamma_1 \in \mathcal{G}^{r(\gamma)}} f_1(\gamma_1) f_2(\gamma_1^{-1} \gamma) d\nu^{r(\gamma)}(\gamma_1) \quad \text{and} \quad f^*(\gamma) := \overline{f(\gamma^{-1})}.$$ 

Moreover, for any given $x \in M$, we have a $*$-representation $\lambda_x : C_c(\mathcal{G}) \to \mathfrak{L}(L^2(\mathcal{G}_x))$ given by

$$\lambda_x(f)(\xi)(\gamma) := \int_{\gamma_2 \in \mathcal{G}_x} f(\gamma \gamma_2^{-1}) \xi(\gamma_2) d\nu_x(\gamma_2).$$

The completion of $C_c(\mathcal{G})$ in the direct sum representation $\oplus_{x \in M} \lambda_x$ is then a well defined $C^*$-algebra called the Connes algebra of the foliation $(M, \mathcal{F})$ and denoted $C^*(M, \mathcal{F})$, see [20] for more details.

Let $\pi : E = E^+ \oplus E^- \to M$ be a $\mathbb{Z}_2$-graded hermitian vector bundle on $M$ which is assumed to be $G$-equivariant with a $G$-invariant hermitian structure. Then, there is a classical $G$-equivariant Hilbert
Lemma 1.1. \[15\] The leaf-preserving diffeomorphism \( f \) is a holonomy diffeomorphism in the following cases:

1. When the holonomy is trivial, and the foliation is tame.
2. When the foliation is Riemannian.

3. When \( f \) belongs to a connected (Lie) group which acts on \( V \) by leaf-preserving diffeomorphisms. More generally, if \( f \) belongs to the path connected component of a holonomy diffeomorphism \( g \) in the group of leaf-preserving diffeomorphisms.

4. When restricted to the saturation \( \text{sat}(V^f) \) of the fixed point submanifold \( V^f \), that is the union of the leaves that intersect \( V^f \).

As an obvious corollary for instance, we see that when the compact Lie group \( G \) is connected, then its leaf-preserving action is automatically a holonomy action. As for the non-foliated case, we are mainly interested, especially for the cohomological index formula, in the case of the action of a compact connected Lie group \( G \). However, this assumption is not needed yet and only the holonomy assumption will be necessary.

From now on, we shall assume that the leafwise \( G \)-action is a holonomy action.

Using \( \theta \), we get for any \( x \in M \) an action of \( G \) on the manifold \( G_x \) by setting

\[
\Phi : G \times G_x \to G_x \text{ given by } \Phi(g, \gamma) := (g^\theta)(x).
\]

Indeed, one has

\[
\Phi(g, \Phi(k, \gamma)) = g(\Phi(k, \gamma)) = [g(k)\gamma] g^\theta(x) = [(gk)\gamma] g^\theta(x) = [(gk)\gamma] g^\theta(x) = \Phi(gk, \gamma).
\]

The holonomy covering map \( r : G_x \to L_x \) is then \( G \)-equivariant, so that \( \Phi \) can be understood as an \( r \)-lift of the original \( G \) action which fixes the source map \( s \). Using the \( G \)-invariance of the leafwise Lebesgue-class measure, it is then easy to check using the definition of the measure \( \nu_x \) that this latter is \( \Phi \)-invariant, i.e.

\[
\int_{G_x} f(\Phi(g, \gamma)) d\nu_x(\gamma) = \int_{G_x} f(\gamma) d\nu_x(\gamma).
\]

We can now define our unitary \( G \)-action \( U_x \) on the Hilbert space \( L^2(G_x, r^*E) \) by setting

\[
(U_x \eta)(\gamma) := g \eta(\Phi(g^{-1}, \gamma)), \quad \eta \in C_c(G_x, r^*E), \quad g \in G, \quad \text{and } \gamma \in G_x.
\]

The family \( U = (U_x)_{x \in M} \) actually represents the group \( G \) in the unitary adjointable operators on the Hilbert module \( \mathcal{E} \). More precisely:

**Lemma 1.2.** For the trivial action of \( G \) on \( C^*(M, \mathcal{F}) \), the Hilbert module \( \mathcal{E} \) is a \( G \)-Hilbert module. Indeed, for any \( \eta, \eta' \in C_c(G, r^*E) \) and \( g \in G \) we have:

\[
\langle U_g \eta, \eta' \rangle = \langle \eta, U_g^{-1} \eta' \rangle,
\]

so in particular, the operator \( U_g \) extends to an adjointable (unitary) operator on the Hilbert module \( \mathcal{E} \).

**Proof.** For \( \eta \in \mathcal{E} \), \( f \in C^*(M, \mathcal{F}) \) and \( g \in G \), the relation \( (U_g \eta) \cdot f = U_g(\eta \cdot f) \) can be easily verified by direct computation, however this will be automatically satisfied since the operator \( U_g \) is adjointable. More precisely, we have

\[
\langle U_g \eta, \eta' \rangle(\gamma) = \int_{G_x(\gamma)} \left\langle g \eta \left( (g^{-1}\gamma') \theta^\gamma r(\gamma) \right), \eta'(\gamma') \right\rangle d\nu_{r(\gamma)}(\gamma').
\]
We use the $G$-invariance of the metric on $E$ and make the change of variable $\gamma_1 := (g^{-1}\gamma)\theta^g(r(\gamma))$ so as to get:

$$\langle U_g \eta, \eta' \rangle(\gamma) = \int_{G(\gamma)} \langle \eta(\gamma), g^{-1}\eta' ((g\gamma_1)\theta^g(g^{-1}r(\gamma))\gamma) \rangle_E^{\gamma_1} d\nu_\gamma(\gamma_1)$$

$$= \int_{G(\gamma)} \langle \eta(\gamma), g^{-1}\eta' ((g(\gamma_1))\theta^g(s(\gamma))) \rangle_E^{\gamma_1} d\nu_\gamma(\gamma_1)$$

$$= \langle \eta, U_{g^{-1}}\eta' \rangle(\gamma).$$

Hence in particular $U_g$ is $C^*(M,F)$-linear.

**Corollary 1.3.** The $G$-action on the foliation $C^*$-algebra $C^*(M,F)$ is internal, i.e. it is implemented by unitary multipliers and we have $\lambda(g.f) = U_g \circ \lambda(f) \circ U_{g^{-1}}$ for any $f \in C_c(G)$.

**Proof.** When $E$ is the trivial line bundle, the previous lemma shows that $G$ acts by unitary multipliers $(U_g)_{g \in G}$ of the foliation $C^*$-algebra $C^*(M,F)$. We now compute

$$(U_g \circ \lambda(f) \circ U_{g^{-1}})(\xi)(\gamma) = g \left[ \lambda(f)(U_{g^{-1}}\xi) \right] \left( (g^{-1}\gamma)\theta^g(s(\gamma)) \right)$$

$$= \int_{G(\gamma)} f \left( (g^{-1}\gamma)\theta^g(s(\gamma))^1 \gamma_1 \right) \xi( (g\gamma_1)\theta^g(s(\gamma)) ) d\nu_\gamma(\gamma_1)$$

$$= \int_{G(\gamma)} f \left( g^{-1}(\gamma\gamma_2) \right) \xi(\gamma_2) d\nu_\gamma(\gamma_2).$$

Hence the result. Notice that the last equality is obtained by putting $\gamma_2 = \left( \theta_{g^{-1}}(gr(\gamma)) \right)^1 \gamma_1 = (g\gamma_1)\theta^g(s(\gamma)).$

**Proposition 1.4.** We set for $\eta \in C_c(G,r^*E)$ and $\varphi \in C(G)$:

$$\pi(\varphi)(\eta) := \int_G \varphi(g)(U_g \eta) dg. \tag{1}$$

Then $\pi$ extends to an involutive representation $\pi$ of $C^*-G$ in the Hilbert module $E$. More precisely, $\pi$ is a continuous $*$-homomorphism into the $C^*$-algebra of adjointable operators. Moreover, if we endow $C^*G$ with the conjugation action $Ad$ of $G$, then the representation $\pi$ is $G$-equivariant, i.e.

$$\pi(Ad_g \varphi)(\eta) = U_g \circ \pi(\varphi) \circ U_{g^{-1}}, \quad \text{for } \varphi \in C^*(G), \eta \in E \text{ and } g \in G.$$

**Proof.** Since $U_g$ is adjointable with $U_g^* = U_{g^{-1}}$, we obtain that $\pi(\varphi)$ is also adjointable with $\pi(\varphi^*) = \pi(\varphi)^*$. The relation $\pi(\varphi * \psi) = \pi(\varphi) \circ \pi(\psi)$ is also immediately verified. It follows that $\pi$ is an $*$-homomorphism which satisfies, by its very definition, the estimate $\|\pi(\varphi)\| \leq \|\varphi\|_{L^1(G)}$. Hence we get a well defined continuous $*$-representation of the $C^*$-algebra $C^*G$.

Finally, we have $\pi(g \cdot \varphi) \eta = \int_G \varphi(g^{-1}h) U_h \eta dh$, so setting $k = g^{-1}h$ we get

$$\pi(g \cdot \varphi) \eta = \int_G \varphi(k) U_{gkg^{-1}} \eta dk = U_g \left( \int_G \varphi(k) U_k dk \right) U_{g^{-1}} \eta.$$
**Definition 1.5.** [20] Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded vector bundle over $M$. A (classical) pseudodifferential $G$-operator $P$ of order $m$ acting from $E^+$ to $E^-$ is a family $(P_x)_{x \in M}$, where

$$P_x : C_c^\infty(G_x, r^*E^+) \rightarrow C_c^\infty(G_x, r^*E^-),$$

is a (uniformly supported and classical) pseudodifferential operator of order $m$, with the right $G$-invariance property:

$$P_{\gamma(x)} R_x = R_x P_{\gamma(x)}.$$

The uniform support is assumed here for simplicity and proper support would suffice in order to preserve the space of compactly supported sections, see [11]. We shall denote by $P^m(M, E^+ , E^-)$ the space of (classical) pseudodifferential $G$-operators on $M$ of order $m$. So such pseudodifferential $G$ operators correspond to longitudinal pseudodifferential operators on the graph manifold $\mathcal{G}$ with respect to the foliation $r^*F$, but which are $G$-invariant so that they induce operators downstairs acting over the leaves of $(M, F)$. We shall also sometimes call the elements of $P^m(M, E^+ , E^-)$ longitudinal or leafwise pseudodifferential operators on $(M, F)$ since no confusion can occur.

The principal symbol of such a longitudinal operator $P$ of order $m$ is defined as usual by the formula:

$$\sigma_m(P)(x, \xi) = \sigma_{m'}(P_x)(x, \xi), \quad \text{for } (x, \xi) \in T^*_x \mathcal{G},$$

where $\sigma_{m'}(P_x)$ is the principal symbol of the $m'$-th order classical pseudodifferential operator $P_x$ acting on the manifold $\mathcal{G}_x$.

When $E$ is a $G$-equivariant $\mathbb{Z}_2$-graded hermitian vector bundle, the induced action of $G$ on the smooth-sections yields a family of unitaries

$$V_{g, x} : C_c^\infty(G_x, r^*E) \rightarrow C_c^\infty(G_{gx}, r^*E) \text{ given by } (V_{g, x}\xi)(\gamma) := g\xi(g^{-1}\gamma).$$

It is then easy to check that for any operator $P \in P^m(M, E^+ , E^-)$, the operator $g \cdot P$ given by

$$(g \cdot P)_x := V_{g, g^{-1}x} \circ P_{g^{-1}x} \circ V_{g^{-1}, x},$$

is again an element of $P^m(M, E^+ , E^-)$ whose principal symbol coincides with $g \cdot \sigma_m(P)$ where the action here is as usual through the $G$ action on the leafwise cotangent bundle $F^*$. We shall say that $P$ is $G$-invariant if $g \cdot P = P$ for any $g \in G$.

**Remark 1.6.** It is easy to check the following relation (see [15]):

$$V_{g, g^{-1}x} \circ P_{g^{-1}x} \circ V_{g^{-1}, x} = U_{g, x} \circ P_x \circ U_{g^{-1}, x}.$$

A zero-th order longitudinal pseudodifferential operator $P_0 : C_c^\infty(\mathcal{G}, r^*E^+) \rightarrow C_c^\infty(\mathcal{G}, r^*E^-)$ extends into an adjointable operator, still denoted $P_0$, between the Hilbert modules $\mathcal{E}^+$ and $\mathcal{E}^-$ corresponding to the vector bundles $E^+$ and $E^-$ respectively [23][20]. The formal adjoint of $P_0$ defined over each $\mathcal{G}_x$, with respect to the hermitian structures and the Haar system, is then again a zero-th order longitudinal pseudodifferential operator acting from $E^-$ to $E^+$. Moreover, its extension to an adjointable operator from $\mathcal{E}^-$ to $\mathcal{E}^+$ is just the adjoint of $P_0$ with respect to the Hilbert module structures. So, if we denote by $P$ the operator $P := \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$, then $P$ is an adjointable operator on $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ which is by construction odd for the $\mathbb{Z}_2$-grading.

**Lemma 1.7.** With the previous notations, if we assume in addition that $P_0$ is $G$-invariant, then for any $\varphi \in C^*G$, we have $[\pi(\varphi), P] = 0$.

**Proof.** Since we have for any $x \in M$ and any $g \in G$:

$$V_{g, g^{-1}x} \circ P_{g^{-1}x} \circ V_{g^{-1}, x} = U_{g, x} \circ P_x \circ U_{g^{-1}, x},$$

the operator $P$ is $G$-invariant if and only if $P$ commutes with the unitary $U$ of $\mathcal{E}$ corresponding to the family of unitaries $(U_{g, x})_{(g, x) \in G \times M}$. Now, let $\varphi \in L^1(G)$, then by definition of $\pi(\varphi)$ we deduce that $\pi(\varphi) \circ P = P \circ \pi(\varphi)$. Therefore this commutation relation also holds for any $\varphi \in C^*G$ by continuity. \[\square\]
1.2 The moment map and some standard $G$-operators

Assume now that $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$, and that the action of $G$ on $M$ preserves the leaves and is through holonomy diffeomorphisms as explained in the previous section. This is for instance the case for any compact connected Lie group. We start by extending some results from [37, Section 6] to our foliation setting, and for the convenience of the reader we shall use Kasparov’s notations from there. For $x \in M$, we hence denote by $f_x : G \to M$ the map given by $f_x(g) = gx$ and by $f'_x : G \to T_xM$ its tangent map at the neutral element of $G$. The dual map of $f'_x$ is denoted by $f''_x : T_x^*M \to \mathfrak{g}^*$. So, any $X \in \mathfrak{g}$ defines as usual the vector field $X^*$ given by $X^*_x := f''_x(X)$ which, under our assumptions, is tangent to the leaves, i.e. $X^*_x \in F_x$ for any $x \in M$. Notice also that $g \cdot f''_x(X) = f''_g(Ad(g)X)$, for $g \in G$, $x \in M$ and $X \in \mathfrak{g}$ [37].

Let $\mathfrak{g}_M := \mathfrak{g} \times M$ be the $G$-equivariant trivial bundle of Lie algebras on $M$, associated to $\mathfrak{g}$ for the action $g \cdot (x, v) = (g x, Ad(g)v)$. The map $f' : \mathfrak{g}_M \to TM$ defined by $f'(x, v) = f'_x(v)$ is a $G$-equivariant vector bundle morphism. We endow $\mathfrak{g}_M$ with a $G$-invariant metric and we denote by $\| \cdot \|$ the associated family of Euclidean norms. Up to normalization, we can always assume that $\forall v \in \mathfrak{g}$, $\|f'_x(v)\| \leq \|v\|$. Here $\|f'_x(v)\|$ is the norm given by the riemannian metric at $x$. We thus assume from now on that $\|f'_x\| \leq 1$, $\forall x \in M$. These metrics on $\mathfrak{g}_M$ and $TM$ are also used to identify $\mathfrak{g}_M$ with $\mathfrak{g}_M^*$ and $TM$ with $T^*M$. Then we can define the map $\phi : T^*M \to T^*M$ by setting $\phi_x = f''_xf''_x^{-1}$. Again according to Kasparov’s notations [37], we introduce the quadratic form $q = (q_x)_{x \in M}$ on the fibers of $T^*M$ by setting:

$$q_x(\xi) = \|f''_x(\xi)\|^2_2, \forall (x, \xi) \in T^*M.$$  

If $\xi \in T^*_xM$, then it is easy to see that $\xi$ is orthogonal to the $G$-orbit of $m$ if and only if $q_x(\xi) = 0$. Notice also that we have $q_x(\xi) \leq \|\xi\|^2$.

As in the seminal book [1], we introduce a second order $G$-invariant longitudinal differential operator $\Delta_G$, whose symbol coincides with $q$. This is achieved for instance by using an orthonormal basis of $\mathfrak{g}$ for a bi-invariant metric on the compact Lie group $G$ and by considering the first order differential operators which are the Lie derivatives of the $G$-action, see again [1, page 12]. Recall that if $X \in \mathfrak{g}$ and $\eta \in C^\infty(\mathfrak{g}, r^*E)$, then the Lie derivative $\mathcal{L}(X)(\eta)$ is defined as

$$\mathcal{L}(X)(\eta)(\gamma) := \frac{d}{dt}|_{t=0}(e^{tX} \cdot \eta)(\gamma) = \frac{d}{dt}|_{t=0}e^{tX} \left( \eta(\Phi(e^{-tX}, \gamma)) \right).$$

So, $\mathcal{L}(X)$ preserves each space $C^\infty_c(\mathfrak{g}_x, r^*E)$ and the corresponding family of first order differential operators is clearly right $G$-invariant. Note indeed that $\Phi(\gamma, \gamma')(\eta) = \Phi(\gamma, \gamma')(\eta)$. Therefore,

$$R_{\gamma}(\mathcal{L}(X)_{\eta}(\gamma')) \equiv \mathcal{L}(X)_{\eta(\gamma')}(\gamma') = \frac{d}{dt}|_{t=0}e^{tX}(\eta(\Phi(e^{-tX}, \gamma'))) = \frac{d}{dt}|_{t=0}^{e^{tX}}(\Phi(e^{-tX}, \gamma'))$$

Then, for any orthonormal basis $\{V_k\}$ of $\mathfrak{g}$ with dual basis $\{v_k\}$, we define a longitudinal differential operator $d_G$ by considering the right $G$-invariant family

$$d_G = \{d_G, x : C^\infty_c(\mathfrak{g}_x, r^*E) \to C^\infty_c(\mathfrak{g}_x, r^*(E \otimes \mathfrak{g}_M^*))\}_{x \in M}$$

of differential operators between $E$ and $E \otimes \mathfrak{g}_M^*$ given by

$$d_G(x)(\eta) := \sum_k \mathcal{L}(V_k)\eta \otimes v_k, \forall \eta \in C^\infty_c(\mathfrak{g}_x, r^*E).$$

This definition is independent of the choice of the orthonormal basis of $\mathfrak{g}$.
Remark 1.8. We may take for $\Delta_G$ the operator $d_G^*d_G$. Indeed, it is easy to see that the symbol of $d_G$ at $(x,\xi) \in F$ is given by $\sqrt{-1} \operatorname{ext}(f_x^*(\xi))$. So the symbol of $d_G^*$ is given by $-\sqrt{-1} \operatorname{int}(f_x^*(\xi))$ and hence the principal symbol of $d_G^*d_G$ is given by $\langle f_x^*(\xi), f_x^*(\xi) \rangle = q_x(\xi)$.

In the same way and working on the manifold $G$ itself with its $G$-action by left translations, the orthonormal basis $\{V_k\}$ of $\mathfrak{g}$ (with dual basis $\{v_k\}$) allows to define the exterior differential of the manifold $G$ as follows. For any $\phi \in C^\infty(G)$, let $\frac{\partial \phi}{\partial V_k}$ be the derivative along the one-parameter subgroup of $G$ corresponding to the vector $V \in \mathfrak{g}$, then we get the first-order differential operator $d$ actions on smooth functions on $G$ and valued in $\mathfrak{g}^*$-valued smooth functions on $G$, by setting

$$d\phi = \sum_k \frac{\partial \phi}{\partial V_k} \otimes v_k.$$}

We may tensor the representation $\pi : C^\infty(G) \to \mathcal{L}_{C^\infty(M,F)}(\mathcal{E})$ with the identity of the vector space $\mathfrak{g}^*$ and get the extended map

$$\pi : C^\infty(G) \otimes \mathfrak{g}^* \to \mathcal{L}_{C^\infty(M,F)}(\mathcal{E} \otimes \mathfrak{g}^*) \simeq \mathcal{L}_{C^\infty(M,F)}(\mathcal{E}) \otimes \mathfrak{g}^*.$$}

Said differently, we simply set for $\psi \in L^1(G)$ and $v \in \mathfrak{g}^*$:

$$\pi(\psi \otimes v)\eta = \pi(\psi)\eta \otimes v = \int_G \psi(g)(U_g\eta \otimes v) \, dg, \quad \forall \eta \in \mathcal{E}.$$}

Here again the map $\pi$ corresponds to a family $(\pi_x)_{x \in M}$ of maps

$$\pi_x : C^\infty(G) \otimes \mathfrak{g}^* \to \mathcal{L}(L^2(G_x, r^*E)) \otimes \mathfrak{g}^*.$$}

Proposition 1.9. For $\phi \in C^\infty(G)$ and $V \in \mathfrak{g}$, we have

$$\mathcal{L}(V) \circ \pi(\phi) = \pi \left( \frac{\partial \phi}{\partial V} \right).$$}

In particular, $d_G(\pi(\phi)\eta) = \pi(d\phi)\eta$ for any $\eta \in C^\infty_c(G, r^*E)$, or equivalently $(d_G(x)[\pi_x(\phi)] = \pi_x(d\phi))_{x \in M}$.

Proof. We only need to check the first relation with the Lie derivatives. But we have for $V \in \mathfrak{g}$, $\phi \in C^\infty(G)$ and $\eta \in C^\infty_c(G, r^*E)$:

$$\mathcal{L}(V)\pi(\phi)\eta(\gamma) = \frac{d}{dt} \bigg|_{t=0} e^{tV} \left( \pi(\phi)\eta(\Phi(e^{-tV}, \gamma)) \right) \right.$$}

$$= \frac{d}{dt} \bigg|_{t=0} \int_G \phi(g)(e^{tV}g)(\eta(\Phi(g^{-1}, \Phi(e^{-tV}, \gamma)))) \, dg \right.$$}

$$= \frac{d}{dt} \bigg|_{t=0} \int_G \phi(g)(e^{tV}g)(\eta(\Phi(g^{-1}e^{-tV}, \gamma))) \, dg \right.$$}

$$= \int_G \frac{d}{dt} \bigg|_{t=0} \phi(e^{-tV}h)(\eta(\Phi(h^{-1}, \gamma))) \, dh.$$}

In the second to third line we have used the relation $\Phi(g^{-1}, \Phi(e^{-tV}, \gamma)) = \Phi(g^{-1}e^{-tV}, \gamma)$, and in the last equality, we have substituted $e^{tV}g = h$ and used the $G$-invariance of the Haar measure on $G$. Therefore, we get

$$\mathcal{L}(V)\pi(\phi)\eta(\gamma) = \int_G \frac{\partial \phi}{\partial V}(h)\dot{U}_h\eta(\gamma) \, dh = \left[ \pi \left( \frac{\partial \phi}{\partial V} \right) \eta \right](\gamma).$$}


Recall that we are given for any $g \in G$ and any $x \in M$ a holonomy class $\theta^g(x) \in G^x_\pi$ with the natural properties recalled in the previous section. So, for any $X \in \mathfrak{g}$, we have
\[ \theta^{e^{-ix}}(x) \in G^x_\pi \]
and $t \mapsto \theta^{e^{-ix}}(x)$ is a smooth path in $G_x$ which starts at $x$ viewed in $G_x$. Therefore, we defined a vector $\tilde{X}_x \in T_xG_x$ and hence in $F_x$ by setting
\[ \tilde{X}(x) := \frac{d}{dt}_{t=0} \theta^{e^{-ix}}(x). \]
An easy inspection in a local chart allows to see that the vector field $\tilde{X}$ coincides with the vector field $X_M$.

2 The index morphism

In this section we define the index class of a $G$-invariant leafwise $G$-transversally elliptic operator and we also introduce the K-multiplicity of any unitary irreductible representation in the index class.

2.1 Leafwise $G$-transversally elliptic operators

Definition 2.1. Denote as in [1] by $T_G M$ the closed subspace of $TM$ composed of vectors transverse to the $G$-orbits. More precisely, $T_G M := \{ (x, \xi) \in TM \text{ such that } q_x(\xi) = 0 \}$. We consider the subspace $F_G$ of leafwise vectors which are transverse to the $G$-orbits, i.e.
\[ F_G := T_G M \cap F. \]
Recall that any zero-th order longitudinal pseudodifferential operator $P_0$ gives rise to the self-adjoint operator that we have denoted by $P$ and which is defined following the usual convention, see [37]. More precisely, in the even case, say when $P_0$ acts from the sections of the hermitian vector bundle $E^+$ to the sections of the hermitian vector bundle $E^-$, we consider the $\mathbb{Z}_2$-graded Hilbert module $E = E^+ \oplus E^-$ associated with the $\mathbb{Z}_2$-grading $E = E^+ \oplus E^-$, and the operator $P$ is odd for the grading and given by $P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$.

In the ungraded case, $E^+ = E^- = E$ and $P = P_0$ is assumed to be a selfadjoint operator, say the bounded extension of a leafwise (formally) selfadjoint operator $P_0 : C_c^\infty(G, r^* E) \to C_c^\infty(G, r^* E)$. We shall refer to this convention as convention (K). The notion of $G$-invariant $G$-transversally elliptic operator was introduced and studied in [1]. In our case of foliations, we need to assume that the principal symbol of such $G$-invariant longitudinal operator be invertible away from the “zero section” of $F_G$. So for $(x, \xi) \in F_G$ and $\xi$ large enough one requires that the principal symbol of our operator is invertible. When working with classical polyhomogeneous symbols, this is equivalent to the invertibility off $M$ in $F_G$. So the first guess is to say that a given zero-th order $G$-invariant longitudinal pseudodifferential operator $P_0$ acting from the sections of $E^+$ to the sections $E^-$ is a longitudinal $G$-transversally elliptic operator or a leafwise $G$-transversally elliptic operator, if the symbol of the associated self-adjoint operator $P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ on $E = E^+ \oplus E^-$ satisfies the following condition
\[ \sup_{x \in M} \| \sigma_P(x, \xi)^2 - \text{id}_{E_x} \| \to 0 \text{ when } \xi \to \infty \text{ in } F_{G,x}. \]
(3)
The principal symbol of such leafwise $G$-transversally elliptic operator $P_0$ (as in Definition [3]) represents a class in the $G$-equivariant Kasparov bivariant group $KK_G(C_0(F_G))$ and is represented by the Kasparov even cycle $(C_0(\pi^* E), \sigma_A)$, where $C_0(\pi^* E)$ is the space of continuous sections of the continuous bundle $\pi^* E \to F_G$ which vanish at infinity. Hence, using the isomorphism $KK_G(C_0(F_G)) \simeq K_G(F_G)$, this naive
Then the following are equivalent:

\[ \sigma(P_0) \in K_G(F_G). \]

As already observed for a single operator, it is well known though that Definition 6.2 does not fully exploit the specific properties of pseudodifferential operators. See 18 where the notion of good symbol is needed to prove the localized index theorem, and see also the more recent technical definition introduced in 37 and which applies as well to non-compact proper actions. We shall follow here this latter technical definition and a leafwise G-transversally elliptic operator will be for us defined as follows. Recall the quadratic form \( q \) defined in (2). We shall say that a given leafwise tangent vector \( \xi \in F_x \) is orthogonal to the G-orbit through \( x \in M \) if \( q_x(\xi) = 0 \).

**Definition 2.2.** A G-invariant zero-th order selfadjoint longitudinal pseudodifferential operator \( P \), is a leafwise G-transversally elliptic operator if its principal symbol \( \sigma_P \) satisfies the following condition:

\[
\forall \varepsilon > 0, \exists c > 0 \text{ such that } \|\sigma_P^2(x, \xi) - \text{id}\|_{(x, \xi)} \leq c \frac{1 + q_x(\xi)}{1 + \|\xi\|^2} + \varepsilon, \text{ for any } (x, \xi) \in F.
\]

Here \( \| \cdot \|_{(x, \xi)} \) means the operator norm on \( E_x \).

In the ungraded case, this definition applies to the self-adjoint operator \( P = P_0 \) and we get an odd Kasparov cycle and hence a symbol class \( [\sigma(P)] \in \text{KK}_G^0(C, C_0(F_G)) \simeq K_G^0(F_G) \). In the graded case and again using Convention (K) above, we shall say that the G-invariant longitudinal zero-th order pseudodifferential operator \( P_0 : C_c^\infty(G, r^*E^+) \to C_c^\infty(G, r^*E^-) \) is leafwise G-transversally elliptic if the corresponding self-adjoint operator \( P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix} \) is G-transversely elliptic along the leaves. Hence in this case, we obtain a symbol class

\[
[\sigma(P)] \in \text{KK}_G^0(C, C_0(F_G)) = K_G^0(F_G) =: K_G(F_G).
\]

The following lemma was proved in 37 for a single operator.

**Lemma 2.3 (37, Definition-Lemma 6.2.).** Let \( F^W \) be a given smooth foliation of the not necessary compact manifold \( W \). Denote by \( \pi : F^W \to W \) the tangent bundle to \( F^W \). Let \( \sigma \in C^\infty_0(F^W, \pi^*E) \) be a given smooth bounded leafwise symbol, acting on a hermitian vector bundle \( E \) over \( W \), which is assumed to be compactly supported in the \( x \)-variable, and such that

(a) the exterior derivative \( d_x \sigma(x, \xi) \) is uniformly bounded in \( \xi \),

(b) the exterior derivative \( d_\xi \sigma(x, \xi) \) satisfies \( \|d_\xi \sigma(x, \xi)\| \leq C(1 + \|\xi\|)^{-1} \), with \( C \) only depending on \( \sigma \).

Then the following are equivalent:

1. The restriction \( \sigma|_{F^W_G} \) belongs to \( C_0(F^W_G, \pi^*E) \);

2. \( \forall \varepsilon > 0, \exists c > 0 \) such that for any \( (x, \xi) \in F^W \): \( \|\sigma(x, \xi)\| \leq c \frac{1 + q_x(\xi)}{1 + \|\xi\|^2} + \varepsilon \).

**Proof** [37]. 2. \( \Rightarrow \) 1. If \( \xi \in F^W_G \) that is if \( q_x(\xi) = 0 \) then \( \sigma(x, \xi) \to 0 \) when \( \xi \to \infty \).

1. \( \Rightarrow \) 2. Since \( \sigma(x, \xi) \) is compactly supported in the \( x \)-variable, we fix a compact subspace \( K \subset W \) such that \( \sigma \) is trivial off \( F^W|_K \). Assume that \( \sigma|_{F^W_G} \in C_0(F^W_G, \pi^*E) \) and suppose that condition 2. is not satisfied. Then there exists \( \varepsilon > 0 \) such that for any sequence \( c_n \) of positive numbers increasing to \( +\infty \), we can find a sequence \( (x_n, \xi_n) \in F^W|_K \) with

\[
\|\sigma(x_n, \xi_n)\| > c_n \frac{1 + q_{x_n}(\xi_n)}{1 + \|\xi_n\|^2} + \varepsilon.
\]
Since $\sigma$ is bounded, the sequence $(\|\xi_n\|)_n$ cannot be bounded and hence admits a subsequence which diverges to $+\infty$. Let us assume that $0 < \|\xi_n\| \to +\infty$ and put for any $n$, $\eta_n = \xi_n/\|\xi_n\|$ so that $(x_n, \eta_n)_n$ is a sequence living in a compact subspace in $F^W$, since $(x)_n$ lives in the compact subspace $K$. Up to a choice of a subsequence, we may again assume that $(x_n, \eta_n)_n$ is convergent to some point $(x, \eta) \in F^W|_K$. Notice then that $\|\eta\| = 1$. Now we have:

$$\lim_{n \to \infty} \frac{1 + q_{x_n}(\xi_n)}{1 + \|\xi_n\|} = q_x(\eta).$$

Since $e_n \to \infty$ and since our symbol $\sigma$ is bounded, we deduce that necessarily $q_x(\eta) = 0$, that is to say $(x, \eta) \in F^W_G$. Therefore we also have $(x, \|\xi_n\|\eta) \in F^W_G$ for any $n$. By condition 1., $\|\sigma(x, \|\xi_n\|\eta)\| \to 0$ when $n \to \infty$. But for $n$ large enough, we may assume that $x_n$ belongs to a small neighbourhood of $x$ where $F^W$ and $E$ are trivialized as products and we may hence see each $\xi_n$ as an element of the fixed euclidean vector space $F^W_x$ and we deduce the following estimate:

$$\|\sigma(x_n, \xi_n) - \sigma(x, \|\xi_n\|\eta)\| = \|\sigma(x_n, \xi_n) - \sigma(x, \xi_n) + \sigma(x, \xi_n) - \sigma(x, \|\xi_n\|\eta)\| \leq \|\sigma(x_n, \xi_n) - \sigma(x, \xi_n)\| + \|\sigma(x, \xi_n) - \sigma(x, \|\xi_n\|\eta)\|.$$  

Now, by the first assumption on $d_{x}\sigma$, there exists a constant $M \geq 0$ such that

$$\|\sigma(x_n, \xi_n) - \sigma(x, \xi_n)\| \leq M d(x_n, x_0),$$

and by the second assumption on $d_{\xi}\sigma$, we also deduce that

$$\|\sigma(x_n, \xi_n) - \sigma(x, \|\xi_n\|\eta)\| \leq C \frac{\|\xi_n\|\|\eta_n - \eta\|}{1 + \|\xi_n\|}.$$  

Therefore, we finally get that $\|\sigma(x_n, \xi_n)\| \to 0$. This is impossible since $\|\sigma(x_n, \xi_n)\| > \varepsilon$.  

\[ \square \]

### 2.2 The index class

We fix a $G$-invariant selfadjoint longitudinal pseudodifferential operator $Q$ acting on the sections of the vector bundle $E$ and with principal symbol given by $\sigma_Q(x, \xi) = \frac{1 + q(x, \xi)}{1 + \|\xi\|} \times \text{id}_E$. This can be achieved by using for instance the usual quantization map, see for instance [23].

**Proposition 2.4.** Let $A$ be a $G$-invariant selfadjoint longitudinal pseudodifferential operators of order 0 acting on the sections of the bundle $E$ over $M$. Suppose that the principal symbol $\sigma_A$ of $A$ satisfies

$$\forall \varepsilon > 0, \exists c > 0 \text{ such that } \forall (x, \xi) \in F : \|\sigma_A(x, \xi)\| \leq c \sigma_Q(x, \xi) + \varepsilon. \quad (4)$$

Then $\forall \varepsilon > 0$, there exist two $G$-invariant selfadjoint compact operators $R_1$ and $R_2$ on the Hilbert module $\mathcal{E}$ such that:

$$- (cQ + \varepsilon + R_1) \leq A \leq cQ + \varepsilon + R_2 \text{ as self-adjoint operators on } \mathcal{E}.$$

**Proof.** It is a classical result for a single operator even on non compact manifolds but with the proper support that such operators $R_1$ and $R_2$ exist as smoothing properly supported operators, see [31], [32], [37]. Since we shall only need the condition of compactness and since our ambient manifold is compact here, the proof is immediate. Indeed for any $\varepsilon > 0$, there exists $c > 0$ such that the principal symbol of the operators $cQ + \varepsilon \text{id}_E \pm A$ are non-negative as elements of the $C^*$-algebra $C(S^*F, \text{End}(E))$ of continuous sections of the algebra bundle $\text{END}(E) = \pi^*\text{End}(E)$ over the cosphere bundle $S^*F$ of the longitudinal bundle $F$. Now, a classical result of Connes [20], [29] gives us a $C^*$-algebra short exact sequence obtained out of the closure of the zero-th-order pseudodifferential operators along the leaves of $\mathcal{F}$:

$$0 \to K_{C^*(M, F)}(\mathcal{E}) \to \Psi^0(M, F; E) \xrightarrow{\sigma} C(S^*F, \text{END}(E)) \to 0,$$

where $\Psi^0(M, F; E)$ is the closure in $L_{C^*(M, F)}(\mathcal{E})$ of the $*$-algebra of zero-th order pseudodifferential operators along the leaves (acting on the sections of $E$) and $\sigma$ is the principal symbol map. Hence, we deduce that the operators $cQ + \varepsilon \text{id}_E \pm A$ are non-negative up to compact operators and hence the conclusion.  

\[ \square \]
Remark 2.5. We shall also need the previous result for a non compact foliated manifold in Section 3.2. We have given in Appendix B Proposition 13.1 the needed easy generalization of Proposition 2.4 by using results from [23].

We are now in position to state the following important result:

Theorem 2.6. The triple \( (\mathcal{E}, \pi, P) \) is a \( G \)-equivariant even Kasparov cycle for the \( C^* \)-algebras \( C^* G \) and \( C^* (M, F) \), where \( C^* G \) is endowed with the conjugation action as before and \( C^* (M, F) \) is endowed with the trivial \( G \)-action. It thus defines a class in

\[
[\mathcal{E}, \pi, P] \in KK_G(C^* G, C^* (M, F)).
\]

Notice that we are in the graded case so that \( [\mathcal{E}, \pi, P] = \left[ \mathcal{E}^+ \oplus \mathcal{E}^-, \pi, \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix} \right] \). In the ungraded case, we similarly have a class \( [\mathcal{E}, \pi, P] = [\mathcal{E}, \pi, P_0] \in KK^1_G(C^* G, C^* (M, F)) \).

Proof. We have already proven that \( \mathcal{E} \) is a Hilbert \( G \)-module if the \( C^* \)-algebra \( C^* (M, F) \) is endowed with the trivial \( G \)-action. By Lemma 1.4, we know that \( |\pi(\varphi)| = 0 \) for any \( \varphi \in C^* G \). Moreover, \( P \) is selfadjoint and odd for the \( \mathbb{Z}_2 \)-grading while \( \pi \) obviously respects the \( \mathbb{Z}_2 \)-grading. It thus remains to check that \( (\text{id} - P^2) \circ \pi(\varphi) \in KK_{C^* (M, F)}(\mathcal{E}) \), where \( KK_{C^* (M, F)}(\mathcal{E}) \) is the \( C^* \)-algebra composed of the compact operators in the Hilbert module \( \mathcal{E} \). As \( P \) is a leafwise \( G \)-transversally elliptic operator, the principal symbol of \( \text{id} - P^2 \) which coincides with \( \text{id} - \sigma^2 P \), satisfies the assumption of Proposition 2.4. Therefore \( \forall \varepsilon > 0 \), there exist \( c_1, c_2 > 0 \) and compact operators \( R_1 \) and \( R_2 \) on the Hilbert module (in fact leafwise smoothing operators) such that

\[
-(c_1 Q + \varepsilon + R_1) \leq 1 - P^2 \leq c_2 Q + \varepsilon + R_2.
\]

Let us take for \( \Delta_G \) the operator \( d_G^* d_G \), see Remark 1.8. Denote also by \( \Delta \) a \( G \)-invariant longitudinal second order differential operator with principal symbol \( \|\xi\|_2 \times \text{id}_{\mathcal{E}} \) for \( (x, \xi) \in F_x \). Modulo longitudinal pseudodifferential operators of negative order, the longitudinal pseudodifferential operator \( Q \) then coincides with the operator \( d_G^* (1 + \Delta)^{-1} d_G \).

By Proposition 1.9, we know that \( d_G \circ \pi(\varphi) = \pi(d \varphi) \) and hence this latter is a bounded operator on \( \mathcal{E} \). Moreover, \( d_G^* (1 + \Delta)^{-1} \) has negative order, so by Corollary 3 of [19], it is a compact operator of the Hilbert module \( \mathcal{E} \). It follow that \( d_G^* (1 + \Delta)^{-1} d_G \pi(\varphi) \) is compact as well. Again since longitudinal pseudodifferential operators of negative order extend to compact operators on the Hilbert module \( \mathcal{E} \), we deduce that \( Q \pi(\varphi) \) is compact. In order to show that the operator \( (\text{id} - P^2) \circ \pi(\varphi) \) is compact, we first notice that for \( \psi = \varphi^* \varphi \), we have:

\[
-\pi(\varphi)^* \circ (c_1 Q + \varepsilon + R_1) \circ \pi(\varphi) \leq \pi(\varphi)^* \circ (\text{id} - P^2) \circ \pi(\varphi) \leq \pi(\varphi)^* \circ (c_2 Q + \varepsilon + R_2) \circ \pi(\varphi),
\]

i.e.

\[
-(c_1 Q + \varepsilon + R_1) \circ \pi(\psi) \leq (\text{id} - P^2) \circ \pi(\psi) \leq (c_2 Q + \varepsilon + R_2) \circ \pi(\psi),
\]

since all the operators are \( G \)-invariant. Therefore, projecting in the Calkin algebra and letting \( \varepsilon \to 0 \), we deduce that \( (\text{id} - P^2) \circ \pi(\psi) \) is compact for any non-negative \( \psi \in C^* G \). Now, using the spectral theorem, we may write any \( \varphi \in C^* G \) as a linear combination of non-negative elements and conclude.

Definition 2.7. The index class \( \text{Ind}_{C^* G}^F (P_0) \) of the \( G \)-invariant leafwise \( G \)-transversally elliptic operator \( P_0 \) is defined as:

\[
\text{Ind}_{C^* G}^F (P_0) := [\mathcal{E}, \pi, P] \in KK^i_{\text{mathrm{G}}} (C^* G, C^* (M, F)),
\]

with \( i \in \mathbb{Z}_2 \) according to Convention (K). We also denote by \( \text{Ind}_{C^* G}^F (P_0) \) the image of \( \text{Ind}_{C^* G}^F (P_0) \) in \( KK^i (C^* G, C^* (M, F)) \) obtained by forgetting the adjoint \( G \) action on \( C^* G \).
2.3 The index map

**Proposition 2.8.** The index class \( \text{Ind}_G^F(P) \) only depends on the \( K \)-theory class \([\sigma(P)]\) of the principal symbol \( \sigma(P) \), and this induces for \( i = 0,1 \), a group homomorphism:

\[
\text{Ind}_G^i : K^i_G(F_G) \rightarrow KK^i_G(C^*G, C^*(M, F)).
\]

More precisely, the map \([\sigma] \mapsto \text{Ind}_G^F(P(\sigma))\) is well defined by using any quantization \( P_0 \) of \( \sigma \).

**Proof.** This is classical and we follow \[4\] and \[1\]. We only give the graded case, the ungraded being similar and easier. Let \( C(F_G) \) be the semigroup of 0-homogeneous homotopy classes of transversally elliptic symbols of order 0 and let \( C_0(F_G) \subset C(F_G) \) be the classes of such symbols whose restriction to the sphere bundle of \( F_G \), is induced by a bundle isomorphism over \( M \). By a standard argument, see Remark 2.9 below, we know that \( K_G(F_G) := C(F_G)/C_0(F_G) \). Let now \( \sigma \) be a homotopy of leafwise 0-th order \( G \)-transversally elliptic symbols, then the quantization of this homotopy gives an operator homotopy and hence by the very definition of the Kasparov group \( KK_G(C^*G, C^*(M, F)) \), the index classes of \( \sigma \) and \( \sigma_1 \) coincide in \( KK_G(C^*G, C^*(M, F)) \). On the other hand, given two 0-th order \( G \)-invariant leafwise \( G \)-transversally elliptic operators \( P : C^{\infty,0}(G, r^*E) \rightarrow C^{\infty,0}(G, r^*E) \) and \( P' : C^{\infty,0}(G, r^*E') \rightarrow C^{\infty,0}(G, r^*E') \), we obviously have

\[
\text{Ind}_G^F(P \oplus P') = [(\mathcal{E} \oplus \mathcal{E}', \piE \oplus \piE', P \oplus P')] = [(\mathcal{E}, \piE, P)] + [(\mathcal{E}', \piE', P')] = \text{Ind}_G^F(P) + \text{Ind}_G^F(P').
\]

Finally, it is clear that any zero-th order \( G \)-invariant longitudinal pseudodifferential operator whose symbol is induced by a bundle isomorphism over \( M \) has \( \text{Ind}_G^F(P) = 0 \), for more details see for instance \[5\]. \( \square \)

**Remark 2.9.** If we denote for \( k \in \mathbb{Z} \) by \( kC(F_G) \) the semigroup of \( k \)-homogeneous homotopy classes of \( k \)-homogeneous transversally elliptic symbols and by \( kC_0(F_G) \subset kC(F_G) \) those classes whose restriction to the sphere bundle of \( F_G \) are induced by bundle isomorphisms over \( M \), then the classical argument, see for instance \[3\], easily adapts to show that (\( M \) is compact here)

\[
K_G(F_G) \simeq kC(F_G)/kC_0(F_G).
\]

We may state the similar proposition when an extra compact group acts on the whole data. More precisely, we have:

**Proposition 2.10.** Assume that the compact group \( G_1 \) acts as before by holonomy diffeomorphisms, and that an extra compact group \( G_2 \) acts also on \( M \) by \( F \)-preserving isometries (not necessarily preserving the leaves) such that this \( G_2 \)-action commutes with the action of \( G_1 \), then the previous construction yields, for \( G_1 \times G_2 \)-invariant \( G_1 \)-transversally elliptic operators along the leaves of \((M, F)\), to a well defined \( G_2 \)-equivariant index map

\[
\text{Ind}^{F,G_2}_G : K^i_{G_1 \times G_2}(F_{G_1}) \rightarrow KK^i_{G_1 \times G_2}(C^*G_1, C^*(M, F)), \quad i \in \mathbb{Z}_2.
\]

The \( G_2 \)-equivariant index class of the \( G_1 \times G_2 \)-invariant leafwise pseudodifferential operator \( P_0 \) on \((M, F)\) which is \( G_1 \)-transversally elliptic is represented again by the cycle \((\mathcal{E}, \pi, P)\) which is now in addition \( G_2 \)-equivariant. Indeed, the Hilbert module \( \mathcal{E} \) is automatically endowed with the extra \( G_2 \)-action so that \( \mathcal{E} \) is a \( G_2 \)-equivariant Hilbert module over the \( G_2 \)-algebra \( C^*(M, F) \). Here of course the actions are the usual ones induced from the action on the holonomy groupoid \( G \) and on the bundle and no need to assume that the action of \( G_2 \) preserves the leaves. When the group \( G_1 \) is for instance the trivial group, then we recover the \( G_2 \)-equivariant index class for \( G_2 \)-invariant leafwise elliptic operators as considered for instance in \[10\].

We shall denote by \( \text{Ind}^{F,G_2}_G \) the composition of the above equivariant index map with the forgetful map for the conjugation action of \( G_1 \) on \( C^*G_1 \), i.e.

\[
\text{Ind}^{F,G_2}_G : K^i_{G_1 \times G_2}(F_{G_1}) \rightarrow KK^i_{G_2}(C^*G_1, C^*(M, F)).
\]

When \( G_2 \) is the trivial group, we obtain our previously defined index map \((G = G_1)\):

\[
\text{Ind}^F_G : K^i_{G_1}(F_{G_1}) \rightarrow KK^i(C^*G_1, C^*(M, F)).
\]
Lemma 2.15. The inner product satisfies the standard formula

$$\left< \sum v_i \otimes \eta_i, \sum w_j \otimes \xi_j \right> = \sum \left< \eta_i, \pi(C_{v_i,w_j}) \xi_j \right>, \quad \text{for} \quad \sum v_i \otimes \eta_i \in X \otimes \mathcal{E} \quad \text{and} \quad \sum w_j \otimes \xi_j \in \mathcal{E}_X^G.$$

Remark 2.11. It is an obvious observation that the index morphism $\text{Ind}^F$ factors through $\text{Ind}^{M,F} : K_{G^i}(F_{G^i}) \to \text{KK}(C(M) \rtimes G, C^*(M,F)).$

Remark 2.12. When the $G$-action is locally free, it generates a smooth regular subfoliation $\mathcal{F}'$ of the foliation $\mathcal{F}$ and the index morphism $\text{Ind}^{M,F}$ can be recast as valued in $\text{KK}(C^*(M,\mathcal{F}'), C^*(M,F))$ (cf. [30]).

Remark 2.13. If we assume in the previous proposition that the extra group $G_2$ also acts by holonomy diffeomorphisms on $(M, \mathcal{F})$, then exactly as for the $G_1$-action, we can arrange the $G_2$-action on $\mathcal{E}$ so that it becomes a $G_2$-equivariant Hilbert module over the $C^*$-algebra $C^*(M,\mathcal{F})$ but now endowed with the trivial $G_2$-action. Hence in this case, there are two ways to define the equivariant index class. Since the crossed product $C^*$-algebra $C^*(M,\mathcal{F}) \rtimes G_2$ for the usual $G_2$-action is here isomorphic to the $C^*$-algebra $C^*(M,\mathcal{F}) \otimes C^*G_2$ corresponding to the trivial $G_2$-action, it is easy to check that the two classes yield the same class once pushed by the Kasparov descent map, see Subsection 3.3 below.

Remark 2.14. We shall see in Subsection 3.2 that the index morphism is also well defined when $M$ is not necessarily compact as a morphism on the (compactly supported) equivariant $K$-theory of the space $F_G$.

2.4 The $K$-theory multiplicity of a representation

For any irreducible unitary representation of $G$, we now proceed to define a class in $K_i(C^*(M,\mathcal{F}))$ playing the role of its multiplicity in the index class $\text{Ind}^F(P)$, and which coincides with the usual multiplicity as defined by Atiyah in [1] in the case of a single operator, corresponding for us to the maximal foliation with a single leaf.

So let $\rho : G \to U(X)$ be an (irreducible) unitary representation of $G$ in the finite dimensional space $X$. For simplicity, we shall refer to such representation by $X$ when no confusion can occur. Recall that the space of isomorphism classes of irreducible unitary representations of $G$ is the discrete dual $\hat{G}$ of $G$, hence we have fixed $X \in \hat{G}$. The space $X \otimes \mathcal{E}$ is endowed with the obvious structure of a Hilbert module over $C^*(M,\mathcal{F})$ given by

$$(v \otimes \xi)a := v \otimes \xi a \quad \text{and} \quad (v \otimes \xi, v' \otimes \xi') := \langle v, v' \rangle \xi, \xi' \in \mathcal{E}, v, v' \in X \quad \text{and} \quad a \in C^*(M,\mathcal{F}).$$

Let $\mathcal{E}_X^G$ be the subspace of $X \otimes \mathcal{E}$ composed of the $G$-invariant elements for the action of $G$ given by $\rho \otimes U$ where $U$ has been introduced in Section 3.1, using that the action is by holonomy diffeomorphisms, i.e. with the previous notations,

$$\mathcal{E}_X^G = (X \otimes \mathcal{E})^G := \{ \xi \in X \otimes \mathcal{E} \text{ such that } (\rho(g) \otimes U_g)\xi = \xi, \forall g \in G \}.$$ 

The subspace $\mathcal{E}_X^G$ is then a Hilbert $C^*(M,\mathcal{F})$-submodule of $X \otimes \mathcal{E}$. We denote as usual by $C_{v,w} : g \mapsto \langle v, \rho(g)w \rangle$ the coefficient of the representation $X$ corresponding to the vectors $v, w \in X$. We quote the following lemma for later use.

Lemma 2.15. The inner product satisfies the standard formula

$$\left< \sum v_i \otimes \eta_i, \sum w_j \otimes \xi_j \right> = \sum \left< \eta_i, \pi(C_{v_i,w_j}) \xi_j \right>, \quad \text{for} \quad \sum v_i \otimes \eta_i \in X \otimes \mathcal{E} \quad \text{and} \quad \sum w_j \otimes \xi_j \in \mathcal{E}_X^G.$$
Proof. Indeed, we have for any $\gamma \in G$ and since $\sum_j \int_G \rho(g)(w_j) \otimes U_g(\xi_j)dg = \sum_j w_j \otimes \xi_j$:

$$\langle \sum v_i \otimes \eta_i, \sum w_j \otimes \xi_j \rangle (\gamma) = \int_{\gamma(\gamma)} \left\langle \sum_i (v_i \otimes \eta_i)(\gamma_1), \sum_j (w_j \otimes \xi_j)(\gamma_1) \right\rangle dv_{\gamma}(\gamma_1)$$

$$= \int_{\gamma(\gamma)} \left\langle \sum_i (v_i \otimes \eta_i)(\gamma_1), \int_G \sum_j (\rho(g)(w_j) \otimes U_g(\xi_j))(\gamma_1)dg \right\rangle dv_{\gamma}(\gamma_1)$$

$$= \int_{\gamma(\gamma)} \int_G \sum_{i,j} \langle (v_i \otimes \eta_i)(\gamma_1), (\rho(g)(w_j) \otimes U_g(\xi_j))(\gamma_1) \rangle dg dv_{\gamma}(\gamma_1)$$

$$= \int_{\gamma(\gamma)} \sum_{i,j} \langle \eta_i(\gamma_1), \int_G C_{v_i,w_j}(g)(U_g(\xi_j))(\gamma_1)dg \rangle dv_{\gamma}(\gamma_1)$$

$$= \int_{\gamma(\gamma)} \int_G \sum_{i,j} \eta_i(\gamma_1), \pi(C_{v_i,w_j})(\xi_j)(\gamma_1) dv_{\gamma}(\gamma_1)$$

$$= \sum \langle \eta_i, \pi(C_{v_i,w_j})\xi_j \rangle. \Box$$

By restricting the operator $\text{id}_X \otimes P \in \mathcal{L}_{C^*(M,F)}(X \otimes \mathcal{E})$ to the $G$-invariant elements, we get the adjointable operator $P^G_X \in \mathcal{L}_{C^*(M,F)}(\mathcal{E}^G_X)$, i.e.

$$P^G_X := (\text{id}_X \otimes P)|_{\mathcal{E}^G_X}.$$

Lemma 2.16. For any (irreducible) unitary representation $\rho : G \rightarrow U(X)$, $(\mathcal{E}^G_X, P^G_X)$ is a Kasparov cycle which defines a class in $\text{KK}^i(\mathbb{C}, C^*(M,F))$.

Proof. We only need to show that $(\text{id}_X \otimes (P^2 - 1))|_{\mathcal{E}^G_X} : \mathcal{E}^G_X \rightarrow \mathcal{E}^G_X$ is compact. We shall use the obvious observation that $\mathcal{E}^G_X$ is an orthocomplemented Hilbert submodule. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $X$. Let $\eta = \sum_i e_i \otimes \eta_i$ be an element of $\mathcal{E}^G_X$. Then we have $\int_X (\rho(g) \otimes U_g)\eta dg = \eta$ so that by an easy computation we get

$$\eta = \sum_{i,j} e_j \otimes \pi(C_{e_j,e_i})\eta_i.$$

We denote by $\lambda^*_j : X \otimes \mathcal{E} \rightarrow \mathcal{E}$ the map $\sum_i e_i \otimes \eta_i \mapsto \eta_j$ and by $\lambda_j : \mathcal{E} \rightarrow X \otimes \mathcal{E}$ the map $\xi \mapsto e_j \otimes \xi$. Then $\lambda_j$ and $\lambda^*_j$ are adjointable operators (which are adjoint of each other). Moreover, we can write

$$((P^G_X)^2 - \text{id}_{X \otimes \mathcal{E}})\eta = (\text{id}_X \otimes (P^2 - 1))\sum_{i,j} e_j \otimes \pi(C_{e_j,e_i})\eta_i$$

$$= \sum_{i,j} e_j \otimes (P^2 - 1) \circ \pi(C_{e_j,e_i})\eta_i.$$}

Hence

$$(P^G_X)^2 - \text{id}_{X \otimes \mathcal{E}} = \left[ \sum_{i,j} \lambda_j \circ (P^2 - 1) \circ \pi(C_{e_j,e_i}) \circ \lambda^*_i \right]|_{\mathcal{E}^G_X}.$$

Since for any $i, j$ the operator $(P^2 - 1) \circ \pi(C_{e_j,e_i})$ is a compact operator on the Hilbert module $\mathcal{E}$, the operator

$$\sum_{i,j} \lambda_j \circ (P^2 - 1) \circ \pi(C_{e_j,e_i}) \circ \lambda^*_i$$
is a compact operator on the Hilbert module $X \otimes \mathcal{E}$. But this operator restricts to $\mathcal{E}_X^G$ where it coincides with $(P_X^G)^2 - \text{id}_X \otimes \mathcal{E}$. Therefore, we conclude that the operator $(P_X^G)^2 - \text{id}_X \otimes \mathcal{E}$ is a compact operator on the (orthocomplemented) Hilbert module $\mathcal{E}_X^G$.

**Definition 2.17.** The K-multiplicity $m_{P_0}(X)$ of the irreducible unitary representation $\rho : G \to U(X)$ in the index class $\text{Ind}^F(P_0)$ is the image of the class $[(\mathcal{E}_X^G, P_X^G)] \in \text{KK}^i(\mathbb{C}, C^*(M, \mathcal{F}))$ under the isomorphism $\text{KK}^i(\mathbb{C}, C^*(M, \mathcal{F})) \overset{\cong}{\longrightarrow} K_i(C^*(M, \mathcal{F}))$. Hence we end up with the well defined K-multiplicity map:

$$m_{P_0} : \hat{G} \to K_i(C^*(M, \mathcal{F}))$$

which assigns to $X$ the multiplicity $m_{P_0}(X) \in K_i(C^*(M, \mathcal{F}))$.

**Remark 2.18.** The K-multiplicity map is another interpretation of the index class $\text{Ind}^F(P_0)$, more in the spirit of Atiyah’s distributional index.

Recall on the other hand that the representation $X$ defines a class in the $K$-theory of the $C^*$-algebra $C^*G$ (see [32]), or also an element, denoted by $[X]$, of the Kasparov group $\text{KK}(\mathbb{C}, C^*G)$ given by $[(X, 0)]$. Notice that the right module structure is given by

$$v \varphi := \int_G \varphi(g)\rho(g^{-1})(v)dg, \quad \text{for } v \in X \text{ and } \varphi \in L^1(G).$$

**Proposition 2.19.** The K-multiplicity of $X$ in the index of $P_0$ is nothing but the $K$-theory class in $K_i(C^*(M, \mathcal{F}))$ corresponding to the Kasparov product $[X] \otimes \text{Ind}^F(P_0)$ under the isomorphism

$$\text{KK}^i(\mathbb{C}, C^*(M, \mathcal{F})) \overset{\cong}{\longrightarrow} K_i(C^*(M, \mathcal{F})).$$

**Proof.** The class $[X]$ is given by the cycle $[(X, 0)] \in \text{KK}(\mathbb{C}, C^*G)$ and we have an isomorphism

$$Av : X \otimes_{C^*G} \mathcal{E} \longrightarrow \mathcal{E}_X^G = (X \otimes \mathcal{E})^G$$

given by $\int_G (\rho(g) \otimes U_g)(\bullet) \, dg$.

We compute for $v \in X$ and $\eta \in \mathcal{E}$:

$$\int_G (\rho(g) \otimes U_g)(v \cdot \varphi \otimes \eta)dg = \int_G (\rho(g) \otimes U_g)(v \otimes \pi(\varphi) \eta) \, dg$$

and

$$\int_G (\rho(g) \otimes U_g)(v \otimes \eta)dg = \sum_k e_k \otimes \pi(C_{e_k,v}) \eta$$

where $(e_k)_k$ is again a given orthonormal basis of $X$. The first relation shows that the map $Av$ is well defined and we observe that the range of $Av$ is exactly equal to $\mathcal{E}_X^G$. Using the second relation, Lemma 2.13 and the standard fact that for any $(v, w) \in X^2$ we have $\sum_k C_{e_k,v}C_{e_k,w} = C_{v, w}$, in $C^*G$, we see that $Av$ is indeed an isometry between the Hilbert $C^*(M, \mathcal{F})$-modules $X \otimes \mathcal{E}$ and $\mathcal{E}_X^G$.

Moreover, the operator $\text{id}_X \otimes P$ is well defined because $P$ commute with the action of $C^*G$. Using the $C^*G$-module structure on $X$, it follows that $v = \sum_k e_k C_{e_k,v}$ for any $v \in X$. Therefore for $\eta = \sum_i v_i \otimes \eta_i$ in $X \otimes \mathcal{E}$, we have $\eta = \sum_i, k e_k \otimes \pi(C_{e_k,v_i}) \eta_i$, which allows to recover the properties of the operator $(\text{id}_X \otimes P)^2 - 1$, in particular, one immediately recovers the automatic property that $(\text{id}_X \otimes P)^2 - 1$ is compact by using that $X$ is a finitely generated projective $C^*G$-module.

**Remark 2.20.** If $(M, \mathcal{F})$ admits a holonomy invariant Borel transverse measure $\Lambda$, then applying the associated additive map $K_0(C^*(M, \mathcal{F})) \to \mathbb{R}$, we get a well defined $\Lambda$-multiplicity morphism, in the graded case, for the $G$-transversely elliptic operator $P_0$:

$$m_\Lambda^{P_0} : \hat{G} \to \mathbb{R},$$

in the spirit of the Murray-von Neumann dimension theory.
3 The Atiyah axioms for our index morphism

As before, we denote by $F_G$ the closed subspace of $F$ defined by $F \cap T_G M$.

3.1 The index for free actions

In this subsection, we let $G$ and $H$ be both compact Lie groups. Let $M$ be a smooth compact manifold and let $\mathcal{F}$ be a given smooth foliation of $M$. We suppose that the compact group $G \times H$ acts on $M$ by leaf-preserving diffeomorphisms that we may assume to be isometries of the ambient manifold $M$, by averaging the metric. We further assume that $H$ acts freely on $M$ so that the projection $q : M \to M/H$ corresponds to a $G$-equivariant principal $H$-fibration which sends leaves to leaves. So, we insist that we assume here that $H$ preserves the leaves upstairs and induces the corresponding leaves downstairs, this is automatic when $H$ is connected. Notice that the leaf of $(M, \mathcal{F})$ through a given point $m \in M$ coincides here with the inverse image of the leaf through $q(m)$ in the quotient manifold $M/H$. The induced foliation downstairs in $M/H$ will be denoted $\mathcal{F}/H$ in the sequel. We denote again by $\pi : F \to M$ the vector bundle projection and by $\tilde{q} : F/H \to M/H$ the induced vector bundle projection downstairs. The foliations $(M, \mathcal{F})$ and $(M/H, \mathcal{F}/H)$ then have the same codimension and under our assumptions do actually have the same space of leaves as we explain below. The action of $H$ on $F$ then preserves the subspace $F_G$ and we have an isomorphism

$$q^* : K_G^i((F/H)_G) \to K_{G \times H}^i(F_{G \times H}).$$

To be specific, this isomorphism identifies the classes of the $G$-invariant $G$-transversally elliptic $\mathcal{F}/H$-leafwise symbols over $M/H$, with those of the symbols of $G \times H$-invariant $G \times H$-transversally elliptic $\mathcal{F}$-leafwise symbols over $M$. At the level of cycles, $q^*$ associates with $(E, a)$ the cycle $(q^*E, q^*a)$ with $q^*a(m, \xi) = a(q(m), q_\ast \xi)$, identifying again covectors with vectors.

Let $\hat{H}$ be the set of isomorphism classes of irreducible unitary representations of the compact group $H$. We shall sometimes refer to an element $\alpha : H \to \text{End}(W_\alpha)$ of $\hat{H}$ simply as $\alpha$, and the corresponding character on $H$ will be denoted $\chi_\alpha$. Associated with such representation we have the homogeneous bundle $W_\alpha \to M/H$ associated with the principal $H$-bundle $q : M \to M/H$. We thus have the classical map:

$$R(H \times G) \quad \longmapsto \quad K_G(M/H)$$

$$V \quad \longmapsto \quad V^*$$

where $V^*$ is the dual representation.

Using a distinguished open cover for the foliated manifold $(M/H, \mathcal{F}/H)$ which trivializes the principal fibration $q : M \to M/H$ as well, it is easy to see that the foliations $(M, \mathcal{F})$ and $(M/H, \mathcal{F}/H)$ have Morita equivalent $C^*$-algebras. If we denote by $G(M/H, \mathcal{F}/H)$ the holonomy groupoid of the foliation $(M/H, \mathcal{F}/H)$, then this Morita equivalence is implemented by the Hilbert module associated with the graph space

$$G_q := \{(m, \alpha) \in M \times G(M/H, \mathcal{F}/H) \mid q(m) = r(\alpha)\}.$$

This is the graph of the morphism of groupoids induced by the projection $q : M \to M/H$. The action of $G$ on $G_q$ is given by $\gamma \cdot (m, \alpha) = (\gamma m, \varphi(\gamma)\alpha)$ and we leave it as an exercise for the interested reader to show that we get in this way a principal $G$-bundle in the sense of [42] and that this bundle indeed embodies the Morita equivalence. As a consequence, we can define the imprimitivity Hilbert bimodule which realizes the Morita equivalence between the corresponding $C^*$-algebras as the completion of the pre-Hilbert module $\mathcal{C}_c(G_q)$.

There is a left prehilbert $C_c(G)$-module structure on $C_c(G_q)$ given by

$$f \cdot \varphi(m, \alpha) = \int_{G_m} f(\gamma) \varphi(s(\gamma), q(\gamma)^{-1}\alpha) \, d\nu^m(\gamma)$$

and

$$g(\varphi, \psi)(\gamma) = \int_{G(M/H, \mathcal{F}/H) \cdot r(\gamma)} \varphi(s(\gamma), q(\gamma)^{-1}\beta) \psi(r(\gamma), \beta) \, d\nu^{r(\gamma)}(\beta).$$

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There is similarly a right prehilbert $C_c(G(M/H,F/H))$-module structure on $C_c(G_q)$ given by
\[
\varphi \cdot \xi(m,\alpha) = \int_{\beta \in G(M/H,F/H)^{\sim(\alpha)}} \varphi(m,\alpha\beta)\xi(\beta^{-1}) \, d\nu^m(\beta)
\]
and
\[
\langle \varphi, \psi \rangle_{G(M/H,F/H)}(\beta) = \int_{\gamma \in G^{m_0}} \overline{\varphi(s(\gamma),q(\gamma)^{-1})} \psi(s(\gamma),q(\gamma)^{-1}\beta) \, d\lambda^{m_0}(\gamma),
\]
for a chosen $m_0 \in q^{-1}\{r(\beta)\}$. Notice that the last integral does not depend on the choice of $m_0$ due to the $H$-invariance of our Haar system. We can now state our theorem.

**Theorem 3.1.** Denote by $\chi$ the class of the trivial representation in $KK(\mathbb{C},C^*H)$. Then for $i \in \mathbb{Z}_2$, the following diagram commutes:

\[
\begin{array}{ccc}
K^i_G((F/H)_G) & \xrightarrow{q^*} & K^i_G(F_G) \\
\text{Ind}^{F/H} & & \text{Ind}^{F/H}
\end{array}
\]

\[
KK^i(C^*G,C^*(M/H,F/H)) \xrightarrow{\otimes_{C^*(M,F)} \mathcal{E}_q} KK^i(C^*G,C^*(M,F))
\]

So if $a \in K^*_G((F/H)_G)$ then ignoring the quasi-trivial Morita isomorphism, we may write:

\[
\text{Ind}^{F/H}(a) \simeq \chi \otimes_{C^*H} \text{Ind}^{F}(q^*a).
\]

**Proof.** Recall that $H$ acts freely on $M$ and preserves the leaves of $F$. The holonomy groupoid upstairs is a principal $H$-fibration over $G_q$, this latter is an $H$-fibration over the holonomy groupoid downstairs. More precisely, $G$ can be identified with the smooth pull-back groupoid $\hat{G}$ given by

\[
\hat{G} := \{(m,\alpha,m') \in M \times G(M/H,F/H) \times M \mid q(m) = r(\alpha) \text{ and } q(m') = s(\alpha)\}.
\]

The Haar system on $G$ is supposed to be $H$-invariant and normalized. More precisely, we assume that the Haar system $(\nu^m)_{m \in M}$ on $\hat{G}$ combines the normalized Haar measure on $H$ with a Haar system downstairs $(\nu^m)_{\tilde{m} \in M/H}$ on $\tilde{G}(M/H,F/H)$. Let $A$ be a $G$-invariant leafwise $G$-transversally elliptic pseudodifferential operator representing $a$, of order 1 and supported as close as we please to the units $M/H$. So the operator $A$ can be seen as a $\hat{G}(M/H,F/H)$-invariant operator along the leaves of the groupoid $\tilde{G}(M/H,F/H)$, i.e.

\[
A = (A_x)_{x \in M/H}
\]

\[
A_x : C^\infty_c(\tilde{G}(M/H,F/H)_x,r^*E) \to C^\infty_c(\tilde{G}(M/H,F/H)_x,r^*E)
\]

and with the usual equivariance.

Using a partition of unity argument, we may lift $A$ to a $G \times H$-invariant leafwise $G \times H$-transversally elliptic pseudodifferential operator $\hat{A}$ on $(M,F)$, which represents $q^*a$. Roughly speaking, the operator $\hat{A}$ corresponds in the identification $G \simeq \hat{G}$ to tensoring locally by the identity on both sides and can be denoted abusively by $id \otimes A \otimes id$. The index class of $q^*a$ can then be represented in $KK'(C^*(G \times H),C^*(M,F))$ by the unbounded Kasparov cycle with the closure of $\hat{A}$ as an operator acting on the Hilbert module $\mathcal{E}$ corresponding to the pull-back bundle $q^*E$ over $M$, and with the usual representation $\pi_{G \times H}$ of $C^*(G \times H)$.

Let us first compute the Kasparov product $\text{Ind}^{F}(q^*a) \otimes_{C^*(M,F)} \mathcal{E}_q$. This is by definition the class of the triple

\[
\left(\mathcal{E} \otimes_{C^*(M,F)} C^*(G_q), \pi_{G \times H} \otimes_{C^*(M,F)} \text{id}, \hat{A} \otimes_{C^*(M,F)} \text{id}\right).
\]
If we denote as well by $r : G_q \to M/H$ the map $r(m, \alpha) := r(\alpha) = q(m)$ then it is easy to check that the Hilbert module $\mathcal{E}_{C(M,F)}$ is isomorphic to $C^*(G_q, r^*E)$, i.e. the completion of $C_c(G_q, r^*E)$ with respect to the prehilbertian structure given for $e_1, e_2 \in C_c(G_q, r^*E)$ by

$$\langle e_1, e_2 \rangle (\beta) = \int_{G_q} \langle e_1(s(\gamma), q(\gamma)^{-1}), e_2(s(\gamma), q(\gamma)^{-1}\beta) \rangle d\nu^m(\gamma),$$

(5)

To be specific, this identification can be described by a unitary $V$ which is given for $u \in C_c(G, r^*E)$ and $f \in C_c(G_q)$ by the formula

$$V(u \otimes f)(m, \beta) = \int_{G_q} u(\gamma) f(s(\gamma), q(\gamma)^{-1}\beta) d\nu^m(\gamma).$$

One then checks immediately that for any $\varphi \in C(G \times H)$, we have $V \circ (\pi_{G \times H}(\varphi) \otimes 1) = \tilde{\pi}_{G \times H}(\varphi) \circ V$, where

$$[\tilde{\pi}_{G \times H}(\varphi)(u)](m, \eta) = \int_{H \times G} \varphi(g, h) (g, h) \left( u((g, h)^{-1}(m, \eta)) \theta^{(g, h)^{-1}}_{G \times H}(s(\eta)) \right) dg \, dh.$$

Similarly, we have that $V \circ (\check{A} \otimes \mathcal{C}(M,F)) id = (id \otimes A) \circ V$, where $id \otimes A$ stands for a first order lift of the operator $A$ to $G_q$ using the $H$-fibration $G_q \to G(M/H, F/H)$. To sum up, we see that

$$\text{Ind}^F(q^*a) \otimes_{C^*(M,F)} \mathcal{E}_q = [C^*(G_q, r^*E), \tilde{\pi}_{G \times H}, id \otimes A].$$

It now remains to compute the Kasparov product of this latter class with the trivial representation of $H$. We shall use the identification

$$C^*(G_q, r^*E)^{\theta_H} \cong C^*(G(M/H, F/H), r^*E).$$

Notice indeed that for $\theta_H$-invariant sections $e_1$ and $e_2$, we have:

$$\langle e_1, e_2 \rangle (\beta) = \int_{G_q} \langle e_1(s(\gamma), q(\gamma)^{-1}), e_2(s(\gamma), q(\gamma)^{-1}\beta) \rangle d\nu^m(\gamma)$$

$$= \int_{G(M/H, F/H)^{\theta_H}} \langle e_1(\beta_1^{-1}), e_2(\beta_1^{-1}\beta) \rangle d\nu^m(\beta_1).$$

In the first expression $m_0 \in M$ is any chosen element of the fiber over $r(\beta)$, and the last equality is a consequence of the $\theta_H$-invariance together with our choice of Haar system upstairs which uses the normalized Haar measure on $H$. Now id $\otimes \tilde{\pi}_{G \times H}$ and id $\otimes (id \otimes A)$ both make sense and by using the previous isomorphism we can see that the first coincides with the representation $\pi_G$ of $C^*G$ on $C^*(G(M/H, F/H), r^*E)$ while the second is just the operator $A$. The verification is an exact rephrasing of the same proof for a single operator and is therefore omitted here. Whence we eventually get the allowed equality

$$\text{Ind}^{F/H}(a) = \chi_H \otimes_{C^*(H)} \left[ \text{Ind}^F(q^*a) \otimes_{C^*(M,F)} \mathcal{E}_q \right].$$

Associativity of the Kasparov product allows to conclude.

If we replace $a$ by the symbol corresponding to the twist of $a$ by a given unitary representation $(\alpha, W_\alpha)$, then the same proof yields to the following result:

**Theorem 3.2.** Let $(W_\alpha, \alpha)$ be a given finite dimensional unitary representation of $H$ and denote as before by $\chi_\alpha$ the corresponding class in $KK(\mathbb{C}, C^*H)$. Then the following diagram commutates:

$$\begin{array}{cccccc}
K^i_{B}(\mathbb{F}/H)_G & \overset{q^*}{\longrightarrow} & K^i_{B}(G \times H) & \overset{\text{Ind}^F}{\longrightarrow} & KK^i(C^*(G \times H), C^*(M, F)) \\
\overset{W_\alpha \otimes \bullet}{\downarrow} & & & & \overset{\chi_\alpha \otimes C^*H}{\downarrow} \\
K^i_{B}(\mathbb{F}/H)_G & \overset{\text{Ind}^{F/H}}{\longrightarrow} & KK^i(C^*G, C^*(M/H, F/H)) & \overset{\otimes_{C^*(M,F)} \mathcal{E}_q}{\longrightarrow} & KK^i(C^*G, C^*(M, F)).
\end{array}$$

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In other words, if \( a \in K^i_G((F/H)_G) \) and ignoring the Morita isomorphism \( \otimes \mathcal{E}_q \), we have
\[
\text{Ind}^{F/H}(W^*_\alpha \otimes a) \cong \chi_\alpha \otimes \text{Ind}^F(q^*a).
\]

**Remark 3.3.** In particular, as an element of \( \text{Hom}(R(H), \text{KK}^i(C^*G, C^*(M, F))) \) we have
\[
\text{Ind}^F(q^*a) = \sum_{\alpha \in H} \chi_\alpha \otimes \text{Ind}^{F/H}(W^*_\alpha \otimes a),
\]
where \( \chi_\alpha \) is the element of \( \text{Hom}(R(H), \mathbb{Z}) \) given by the usual multiplicity.

When the group \( G \) is the trivial group, we obtain the following expected relation between the Connes-Skandalis index of leafwise elliptic operators downstairs and the index of leafwise \( H \)-transversally elliptic operators upstairs.

**Corollary 3.4.** Let \( q : (M, F) \to (M/H, F/H) \) be as above a principal \( H \)-fibration of smooth foliations, recall that \( H \) preserves the leaves of \((M, F)\). Then for any leafwise elliptic pseudodifferential symbol \( \sigma \) on \((M/H, F/H)\) so that \( a = [\sigma] \in K^i(F/H) \), we have the following equality:
\[
\chi_\alpha \otimes \text{Ind}^{F/H}(q^*a) \otimes \mathcal{E}_q = \text{Ind}^F(W^*_\alpha \otimes a),
\]
where \( \text{Ind}^F(W^*_\alpha \otimes a) \in \text{KK}^i(C, C^*(M, F)) \cong K_i(C^*(M, F))) \) is the Connes-Skandalis index, as defined in [23], for the leafwise elliptic symbol \( W^*_\alpha \otimes \sigma \) on the compact foliated manifold \((M, F)\).

**Remark 3.5.** The previous corollary can as well be stated with the extra action of the compact Lie group \( G \) and gives a relation between the corresponding \( G \)-indices [27].

### 3.2 The excision property

In this subsection, we show an excision property for the index class of \( G \)-invariant leafwise \( G \)-transversally elliptic operators. More precisely, we shall first extend our definition of the index morphism to the case of any smooth foliated (open) manifold \((U, F^U)\) which is again endowed with a leaf-preserving action of \( G \) by holonomy diffeomorphisms, and obtain an index morphism
\[
\text{Ind}^{F^U} : K^i_G(F^U_G) \longrightarrow \text{KK}^i(C^*G, C^*(U, F^U)).
\]

Then we shall show the compatibility of this morphism with foliated open embeddings, in particular in closed foliated manifolds, this is the expected excision result. Again \( C^*(U, F^U) \) is the Connes algebra of the foliation \((U, F^U)\), i.e. the \( C^* \)-completion of the convolution algebra of compactly supported continuous functions on the holonomy groupoid \( G(U, F^U) \) of \((U, F^U)\). As usual, we have fixed a \( G \)-invariant metric on \( U \) and used it to identify for instance the colongitudinal bundle \( (F^U)^* \) with the longitudinal bundle \( F^U \).

We shall use the following classical lemma which is shown for instance in [1] lemma 3.6] in the non-foliated case, see also [34] for the original proof in the elliptic case and [23] for the leafwise elliptic case. The proof for the foliated \( G \)-transversely elliptic case is similar with the same standard techniques and hence is omitted.

Let \( \pi_U : F^U \to U \) be the projection of the tangent space to the foliation \( F^U \). We denote as before by \( F^U_G = F^U \cap T_G U \).

**Lemma 3.6.** Each element \( a \in K^i_G(F^U_G) \) can be represented by a \( G \)-equivariant zero-homogeneous morphism \( \pi_U^* E^+ \xrightarrow{\sigma} \pi_U^* E^- \) over the whole of \( F^U \), with \( E^\pm \) being \( G \)-equivariant vector bundles over \( U \), and such that:

- Outside some compact \( G \)-subspace \( L \) in \( U \), the bundles \( E^\pm \) are trivialized and the restriction of \( \sigma \) to \( \pi_U^{-1}(U \setminus L) \) is the identity morphism modulo the trivializations of \( E^\pm \).
The morphism \( \sigma(x, \xi) : E_x^+ \to E_x^- \) is an isomorphism for \((x, \xi) \in F_U^L \times U\).

So, the first item means that there exist bundle \(G\)-equivariant isomorphisms over \(U \setminus L\) (or rather over each of its connected components)

\[
\psi^\pm : E^\pm|_{U \setminus L} \to (U \setminus L) \times \mathbb{C}^{\dim(E^\pm)} \text{ such that } \forall (x, \xi) \in \pi^{-1}_U(U \setminus L) : \sigma(x, \xi) = (\psi_x^\pm)^{-1} \circ \psi_x^\pm : E_x^+ \to E_x^-.
\]

We endow the vector bundles \(E^\pm\) with \(G\)-invariant hermitian structures and consider the Hilbert modules \(E^\pm\) over \(\mathbb{C}^*(U, \mathcal{F}^L_U)\) which, as in the previous sections, are the completions of the prehilbertian \(C_c(\mathcal{G}(U, \mathcal{F}^L_U))\)-modules \(C_c(\mathcal{G}(U, \mathcal{F}^L_U), r^* E^\pm)\). Moreover, using the equivalence relation of stable homotopies with compact support as in [1], the bundle trivialization \(\psi^\pm\) can be assumed to be bounded and in fact even fiberwise unitaries for the hermitian structures. We thus assume as well that \(\sigma^*\sigma\) and \(\sigma^*\sigma^*\) are the identity bundle isomorphisms of \(E^+\) and \(E^-\) respectively, over \(U \setminus L\). By using the holonomy action as in Section 2 we can endow the Hilbert modules \(E^\pm\) with the structure of \(G\)-equivariant Hilbert modules when \(\mathbb{C}^*(U, \mathcal{F}^L_U)\) is trivially acted on by the compact Lie group \(G\). We can now quantize any such zero-degree homogeneous symbol \(\sigma\) and choose a uniformly supported zero-th order \(G\)-invariant pseudodifferential \(\mathcal{G}(U, \mathcal{F}^L_U)\)-operator \(P_0 : C_c^\infty(\mathcal{G}(U, \mathcal{F}^L_U), r^* E^+) \to C_c^\infty(\mathcal{G}(U, \mathcal{F}^L_U), r^* E^-)\) with the principal symbol equal to \(\sigma\). More precisely, uniform support is taken in the sense of [13], see also [23 Proposition 4.6]. Here, we can in fact insure that \(P_0\) is the identity operator outside some compact \(G\)-subspace \(L\) whose interior contains \(L\), i.e. that we have

\[
P_0(\eta)(\gamma) = (\psi_x^-)^{-1} \circ (\psi_x^+)(\eta(\gamma)), \text{ for any } \eta \in C_c^\infty(\mathcal{G}(U, \mathcal{F}^L_U)_{x, L'}^U, r^* E^+).
\]

Hence \(P_0^\dagger P_0\) and \(P_0 P_0^\dagger\) reduce to the identity operators on the sections which are supported above \(U \setminus L'\), say in \(r^{-1}(U \setminus L')\).

The operator \(P_0\) hence reduces to multiplication by the unitary bundle morphism \((\psi^-)^{-1} \circ (\psi^+)\) over \(U \setminus L'\), and it is easy to deduce that it extends to an adjointable \(G\)-equivariant operator from \(E^+\) to \(E^{-}\) [41] that we still denote by \(P_0\), see also [47] [13]. We denote as usual by \(P\) the self-adjoint \(G\)-invariant operator

\[
P := \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}
\]

acting on the Hilbert \(G\)-module \(E = E^+ \oplus E^-\). Since the action of \(G\) is assumed to be a holonomy action, we have the \(\ast\)-representation \(\pi\) of \(C^*G\) in the adjointable operators of the Hilbert module \(E\) defined as in Proposition [14] which respects the \(\mathbb{Z}_2\)-grading given by the decomposition \(E = E^+ \oplus E^-\). Recall that we are considering the trivial \(G\)-action on \(\mathbb{C}^*(U, \mathcal{F}^L_U)\) and that \(E\) is endowed with the structure of a \(\mathbb{Z}_2\)-graded \(G\)-module.

Proposition 3.7. The triple \((E, \pi, P)\) is a \(\mathbb{Z}_2\)-graded Kasparov cycle over the pair of \(C^*\)-algebras \((C^*G, C^*(U, \mathcal{F}^L_U))\) and defines an index class in \(\text{KK}(C^*G, C^*(U, \mathcal{F}^L_U))\) which only depends on the \(G\)-equivariant stable homotopy class \(a = [\sigma] \in K_G(F_U^L)\) and is denoted \(\text{Ind}_{\mathcal{F}^L_U}(a)\). Moreover, we get in this way a well defined index morphism for the open foliation \((U, \mathcal{F}^L_U)\):

\[
\text{Ind}_{\mathcal{F}^L_U} : K_G(F_U^L) \to \text{KK}(C^*G, C^*(U, \mathcal{F}^L_U)).
\]

We have similarly a well defined (ungraded) index map

\[
\text{Ind}_{\mathcal{F}^L} : K_G(F_U^L) \to \text{KK}^1(C^*G, C^*(U, \mathcal{F}^L_U)).
\]

Proof. We freely use notations from Section 2 and only treat the even case. Since \(\sigma\) is bounded on \(L\) and assumed to be unitary outside \(L\), we get that \(\sigma\) is bounded. Now notice that \(\|\sigma(P)(x, \xi)^2 - \text{id}\| \neq 0\) only for \(x \in L\) which implies, using Lemma [23] that

\[
\forall \varepsilon > 0, \exists c > 0 \text{ such that } \|\sigma(P)(x, \xi)^2 - \text{id}\| \leq c\sigma(Q)(x, \xi) + \varepsilon,
\]
where \( \sigma(Q)(x, \xi) = \frac{1 + q_\varepsilon(\xi)}{1 + \|\xi\|} \). Using Proposition 13.1 we get

\[- (cQ + \varepsilon + R_1) \leq P^2 - \text{id} \leq cQ + \varepsilon + R_2 \text{ as self-adjoint operators on } \mathcal{E} \] (8)

where \( R_i \in \mathcal{K}_U(\mathcal{E}) \). Denote by \( \theta \in C^\infty_c(U, [0, 1]) \) a bump function which equals 1 on \( U' \). If \( \zeta \in C_c(U, [0, 1]) \) is any function which equals 1 on \( L \) and 0 outside \( L' \), we have \( \sigma(P)^2(r(\gamma), \xi) - \text{id} = \zeta(r(\gamma))(\sigma(P)^2(r(\gamma), \xi) - \text{id}) \) and \( \zeta = \zeta \). We use as usual an oscillatory integral to define the quantization map, see for instance [23]. More precisely, the \( G \)-invariant operator \( P^2 - \text{id} \) is given through its Schwartz kernel, a distribution \( k_{P^2 - \text{id}} \) on \( \mathcal{G} \) given by an expression of the following type

\[ k_{P^2 - \text{id}}(\gamma) = \int_{F^U(U, \gamma)} \chi(\gamma^{-1}) e^{-i(\Phi(\gamma^{-1}), \xi) \langle \sigma(P)^2(r(\gamma), \xi) - \text{id} \rangle} d\xi = k_{(P^2 - \text{id})r^*\zeta}(\gamma) = k_{r^*\zeta(P^2 - \text{id})}(\gamma), \]

where \( \Phi \) is a diffeomorphism from a uniform neighbourhood \( W \) of \( U \) in \( \mathcal{G}(U, F^U) \) to a neighbourhood of the zero section in \( F^U \) with \( d\Phi = \text{id} \) and \( \chi \) is a cut off function with support inside \( W \) which is equal to 1 in a smaller neighborhood of \( U \) whose closure is contained in \( W \), see [23] as well as [11] or [17]. With the appropriate choice of these small neighborhood of the unit manifold \( U \) in \( \mathcal{G}(U, F^U) \) in coherence with \( U' \), we obtain that

\[ r^*\theta(P^2 - \text{id}) = P^2 - \text{id} = (P^2 - \text{id})r^*\theta. \]

Multiplying on both sides the inequality \( 13 \) by \( r^*\theta \), we get

\[-(cr^*\theta Qr^*\theta + \varepsilon r^*\theta^2 + r^*\theta R_1 r^*\theta) \leq P^2 - \text{id} \leq cr^*\theta Qr^*\theta + \varepsilon r^*\theta^2 + r^*\theta R_2 r^*\theta. \]

Furthermore, modulo \( \mathcal{K}_U(\mathcal{E}) \), \( Q \) can be represented by \( d^*_G(1 + \Delta)^{-1}d_G \). Notice that

\[ \sigma(d^*_G(1 + \Delta)^{-1})(x, \xi) = \frac{-i \text{ int}(f^*_x(\xi))}{1 + \|\xi\|} \]

is bounded, therefore \( d^*_G(1 + \Delta)^{-1} \in \text{W}^0(U, F^U, \mathcal{E}) \) and its zero-th order symbol vanishes, so that \( d^*_G(1 + \Delta)^{-1} \in \mathcal{K}_U(\mathcal{E}) \). In particular, the operator \( r^*\theta d^*_G(1 + \Delta)^{-1} \) is compact. Now we can conclude exactly as in the proof of Theorem 26. Indeed, recall that we have

\[ d_G(r^*\theta \pi(\varphi)\eta)(\gamma) = \sum \frac{\partial \theta}{\partial z_k}(r(\gamma))\pi(\varphi)\eta \otimes v_k + \theta(r(\gamma))\pi(d_G\eta)(\gamma), \]

and since \( d_G r^*\theta \pi(\varphi) \) is bounded we deduce that \( r^*\theta d^*_G(1 + \Delta)^{-1}d_G r^*\theta \pi(\varphi) \) is compact. Moreover, the operators \( r^*\theta R_i r^*\theta \) are also compact since each \( R_i \in \mathcal{K}_U(\mathcal{E}) \).

Now for any non-negative \( \psi = \varphi^*\varphi \in C^G \),

\[ -\pi(\varphi)^*(cr^*Qr^*\varphi + \varepsilon r^*\theta^2 + r^*\theta R_1 r^*\theta)\pi(\varphi) \leq \pi(\varphi)^*(P^2 - \text{id})\pi(\varphi) \leq \pi(\varphi)^*(cr^*Qr^*\varphi + \varepsilon r^*\theta^2 + r^*\theta R_2 r^*\theta)\pi(\varphi). \]

Since \( P \) is \( G \)-invariant, this can be rewritten as

\[ -\pi(\varphi)^*(cr^*Qr^*\varphi + \varepsilon r^*\theta^2 + r^*\theta R_1 r^*\theta)\pi(\varphi) \leq (P^2 - \text{id})\pi(\psi) \leq \pi(\varphi)^*(c\theta Qr^*\varphi + \varepsilon r^*\theta^2 + r^*\theta R_2 r^*\theta)\pi(\varphi). \]

Therefore, projecting in the Calkin algebra and letting \( \varepsilon \to 0 \), we deduce that \((P^2 - \text{id}) \circ \pi(\psi) \) is compact. This implies, using the spectral theorem, that \((P^2 - \text{id}) \circ \pi(\varphi) \) is compact for any \( \varphi \in C^G \).

Assume now that there exists an open foliated \( G \)-embedding \( j : (U, F^U) \to (M, F) \) of smooth foliated manifolds. This means that \( j \) is a \( G \)-equivariant embedding of \( U \) as an open submanifold of the foliated \( G \)-manifold \( M \) which transports the foliation \( F^U \) into the restriction of the foliation \( F \) to the open submanifold \( j(U) \). We assume again that \( G \) acts on all foliations by holonomy diffeomorphisms. We shall mainly be interested in the present paper in the case \( M \) compact, but this is not needed so far. The embedding \( j \) then
induces an open embedding at the level of holonomy groupoids that we still denote by \( j \) for simplicity, i.e. \( j : G(U, F^U) \hookrightarrow G(M, F) = \mathcal{G} \). The \( C^* \)-algebra \( C^*(U, F^U) \) is hence isomorphic to the \( C^* \)-algebra of the foliation of \( j(U) \) induced by \( F \), but this latter can be seen as a \( C^* \)-subalgebra of \( C^*(M, F) \) in an obvious way. We therefore end-up with a well defined class \( j_1 \in KK(C^*(U, F^U), C^*(M, F)) \). In the notations of [23], the map \( j : U \to M \) induces in particular a submersion from \( U \) to \( M/F \), and we therefore have a well defined Morita extension Kasparov class which is exactly the class \( j_1 \), but the construction is simpler in our open embedding case. Finally, the differential \( d_j : F^U \to F^M = F \) of the \( G \)-embedding \( j \) restricts to an open \( G \)-embedding of the space \( F^U_G \) in the space \( F^M_G = F_G \), therefore and by functoriality, we get an \( R(G) \)-morphism

\[
 j_* : K^i_G(F^U_G) \longrightarrow K^i_G(F_G).
\]

We are now in position to state the main theorem of this subsection.

**Theorem 3.8.** Under the above assumptions and for any \( i \in \mathbb{Z}_2 \), the following diagram commutes:

\[
\begin{array}{ccc}
K^i_G(F^U_G) & \xrightarrow{j_*} & K^i_G(F_G) \\
\downarrow{\text{Ind}^U} & & \downarrow{\text{Ind}^F} \\
KK^i(C^*G, C^*(U, F^U)) & \xrightarrow{j_i} & KK^i(C^*G, C^*(M, F))
\end{array}
\]

**Remark 3.9.** If the action of \( G \) on \((U, F^U)\) is leafwise but not necessarily a holonomy action while it is a holonomy action on \((M, F)\), then the index morphism \( \text{Ind}^F \) is not well defined anymore, and one can use Theorem [23] precisely to define it for any given such embedding, as a class in \( KK^i(C^*G, C^*(M, F)) \) with the usual compatibility with embeddings.

Recall first the notion of support of a Kasparov \((A, B)\)-cycle \((\mathcal{E}, \pi, F)\), for given separable \( C^* \)-algebras \( A \) and \( B \), as introduced in [23, Appendix A]. This is the Hilbert submodule of \( \mathcal{E} \) which is generated by \( K_1 \mathcal{E} \), with \( K_1 \) the \( C^* \)-subalgebra of the \( C^* \)-algebra \( K(\mathcal{E}) \) of compact operators on the Hilbert \( B \)-module \( \mathcal{E} \), which is generated by the operators \([\pi(a), F] \); \( \pi(a)(F^2 - 1) \) and \( \pi(a)(F - F^*) \) and their multiples by \( A, F \) and \( F^* \).

Here \( a \) runs over \( A \) of course. Then obviously \( \mathcal{E}_1 \) is a Hilbert \((A, B)\)-bimodule and \( F \) restricts automatically to \( \mathcal{E}_1 \) to yield the operator \( F_1 \) so that \((\mathcal{E}_1, \pi, F_1)\) is again a Kasparov \((A, B)\)-cycle. We quote the following interesting observation from [23] which will be used in the sequel.

**Lemma 3.10.** [23] Let \((\mathcal{E}, \pi, F)\) be a Kasparov \((A, B)\)-cycle where \( A \) and \( B \) are given separable \( C^* \)-algebras. Let \((\mathcal{E}_1, \pi, F_1)\) be the Kasparov \((A, B)\)-cycle obtained by restricting to the support \( \mathcal{E}_1 \). Then \((\mathcal{E}_1, \pi, F_1)\) defines the same \( KK \)-class, i.e.

\[
[(\mathcal{E}_1, \pi, F_1)] = [(\mathcal{E}, \pi, F)] \in KK(A, B).
\]

**Proof of Theorem 3.8.** We concentrate again on the even case \( i = 0 \). Let \( a \in K^*_G(F^U_G) \) be fixed. We denote as before by \( \pi_U : F^U \to U \) and by \( \pi_M : F \to M \) the respective bundle projections. We start by representing \( a \) by a symbol of order 0 on \( F^U \) according to Lemma 3.6

\[
\pi_U^* E^+ \xrightarrow{\sigma} \pi_U^* E^-,
\]

which is thus trivial outside a compact set \( L \) of \( U \). By using the trivializations \( \psi^\pm \), a standard argument allows to extend the hermitian bundles \( E^\pm \) viewed over \( j(U) \) to hermitian \( G \)-equivariant vector bundles \( j_* E^\pm \) over \( M \) with the obvious extension \( j_* \sigma \) so that \((j_* E^+, j_* E^-, j_* \sigma)\) represents the push-forward class \( j_* a \), see for instance [41, 44]. The bundle trivializations \( \psi^\pm \) then give the extended bundle isomorphisms, still denoted \( \psi^\pm \), over \( M \setminus j(L) \). Associated with the hermitian \( G \)-bundles \( j_* E^\pm \), we then obtain the corresponding
Hilbert $G$-modules over the $C^*$-algebra $C^*(M, \mathcal{F})$ that we denote by $j_* \mathcal{E}^\pm$. Recall that $j$ induces as well a $*$-homomorphism

$$j_* : C^*(U, \mathcal{F}^U) \longrightarrow C^*(M, \mathcal{F}),$$

which allows to represent $C^*(U, \mathcal{F}^U)$ as adjointable operators on $C^*(M, \mathcal{F})$ when this latter is viewed as a Hilbert module over itself. We can therefore consider the $\mathbb{Z}_2$-graded Hilbert $G$-module over $C^*(M, \mathcal{F})$, obtained by composition, and denoted as usual $\mathcal{E} \otimes_{C^*(U, \mathcal{F}^U)} C^*(M, \mathcal{F})$. This latter Hilbert $G$-module can be identified with a Hilbert $G$-submodule of $j_* \mathcal{E}$, i.e. there is a Hilbert module isometry

$$V : \mathcal{E} \otimes_{C^*(U, \mathcal{F}^U)} C^*(M, \mathcal{F}) \longrightarrow j_* \mathcal{E},$$

which is given for $\eta \in C_c(\mathcal{G}(U, \mathcal{F}^U), r^* E)$ and $f \in C_c(\mathcal{G})$ by $V(\eta \otimes f) = \tilde{\eta} \cdot f$, that is, the convolution of $\tilde{\eta}$, the extension by 0 of $\eta$ outside $\mathcal{G}(U, \mathcal{F}^U)$, with $f$, i.e.

$$V(\eta \otimes f)(\gamma) := \int_{\mathcal{G}(\gamma)} \tilde{\eta}(\gamma_1) f(\gamma_1^{-1}\gamma) d\nu(\gamma)(\gamma_1).$$

We identify for simplicity $U$ with $j(U)$ for the rest of this proof. The Hilbert submodule $V \left( \mathcal{E} \otimes_{C^*(U, \mathcal{F}^U)} C^*(M, \mathcal{F}) \right)$ can be identified with the completion $C^*(\mathcal{G}^U, r^* E)$ of $C_c(\mathcal{G}(U, \mathcal{F}^U), r^* E)$ in $j_* \mathcal{E}$, where $\mathcal{G}^U$ is the space of elements of $\mathcal{G}$ which end inside $U$. See [23, Proposition 4.3]. To finish the proof, we only need to compare the supports of the two Kasparov cycles, and to apply Lemma 3.10.

We choose a uniformly supported $G$-invariant leafwise pseudodifferential operator $P_0$ on $(U, \mathcal{F}^U)$ with symbol $\sigma$ as in the above construction of the index class on $(U, \mathcal{F}^U)$. So, $P_0$ can be seen as a $\mathcal{G}(U, \mathcal{F}^U)$-operator in the sense of [20] that we denote again by

$$P_0 : C_c^\infty(\mathcal{G}(U, \mathcal{F}^U), r^* E) \longrightarrow C_c^\infty(\mathcal{G}(U, \mathcal{F}^U), r^* E),$$

which acts along the fibers of the groupoid and is an invariant family $(P_{0,x})_{x \in U}$. Here of course we assume, as we did in the construction of the index class, that $P_0$ is the identity outside some compact subspace $L'$ of $U$, modulo the trivializations $\psi^\pm$. For simplicity of notations, this operators is also the one over $j(U)$ with its foliation induced from $\mathcal{F}$. In order to quantize the pushforward class $j_* \alpha$, we can then consider the uniformly supported $G$-invariant leafwise operator on $M$ defined as follows.

Let $\theta \in C_c^\infty(M, [0, 1])$ be some $G$-invariant bump function which is equal to 1 on $L'$, and whose support is a compact subspace of $j(U)$ outside of which the operator $P_0$ is trivial. Denote by $\psi^\pm_t$ the isomorphisms $\psi^\pm$ viewed between the bundles $r^* E^\pm$ and which are only well defined over $r^{-1}(M \setminus j(L))$. Then $j_* P_0$ can be taken as the $G$-operator on $(M, \mathcal{F})$ defined by

$$j_* P_0 := \tilde{P}_0 r^* \theta + (\psi^-)^{-1} \circ \psi^+ (1 - r^* \theta).$$

We use here the same cut-off function used to extend $\sigma$ to $F$. Hence $j_* P_0$ is obviously a zero-th order leafwise $G$-operator which is $G$-invariant and has the principal symbol equal to $j_* \sigma = \sigma \theta + (\psi^-)^{-1} \circ \psi^+ (1 - \theta)$ and so represents $j_* \alpha$. Recall that the index class $\text{Ind}^{\tilde{P}_0}(a)$ is represented by the adjointable extension of the operator $P = \begin{pmatrix} 0 & P_0 \\ P_0 & 0 \end{pmatrix}$ acting on the Hilbert module $\mathcal{E}$, while the class $\text{Ind}(j_* \alpha)$ can obviously be represented by the adjointable extension of the operator $j_* P = \begin{pmatrix} 0 & j_* P_0 \\ j_* P_0 & 0 \end{pmatrix}$ acting on the Hilbert module $j_* \mathcal{E}$.
Notice that $\text{Ind}^{j_U}(a) \otimes j_U = \left[ [\mathcal{E}^{C^*_{\text{U}}}(M, F), \pi \otimes 1, P \otimes 1] \right]$ and using the isometry $V$ defined above we deduce that the Kasparov cycle $\left[ \mathcal{E}^{C^*_{\text{U}}}(M, F), \pi \otimes 1, P \otimes 1 \right]$ is unitarily equivalent to the cycle $\left[ C^*(G^U, r^*E), \pi, j_* P \right]_{C^*(G^U, r^*E)}$. Indeed, the representations of the $C^*$-algebra $C^*G$ are clearly compatible, and we have

$$V(P \eta \otimes f) = \tilde{P} \eta \cdot f = \tilde{P} \eta \otimes f = \tilde{P}(\eta \cdot f) = j_* P \left[ C^*(G^U, r^*E) \right] V(\eta \otimes f),$$

with $\tilde{P}$ being as before the $G$-operator obtain from $P$ by extending trivially its distributional kernel. To complete the proof, thanks to the Connes-Skandalis Lemma [31, 10] we only need to show that the supports of $\text{Ind}^{j_U}(j_* a)$ and $\left[ C^*(G^U, r^*E), \pi, j_* P \right]_{C^*(G^U, r^*E)}$ are the same. But using the cut off function $\theta$ which is supported in $U$, we can write

$$((j_* P)^2 - \text{id}) \tau^* \theta = (j_* P)^2 - \text{id}$$

and the same equality is true when replacing $j_* P$ by $j_* P \left[ C^*(G^U, r^*E) \right]$ and $\theta$ by $\theta_{|U}$. Therefore the supports do coincide as allowed.

\[ \square \]

### 3.3 Multiplicativity of the index morphism

Recall that $G$ is a compact Lie group. Let $M$ and $M'$ be two smooth closed manifolds endowed with smooth foliations that we denote respectively by $F$ and $F'$. We assume that $G$ acts by holonomy diffeomorphisms on $(M, F)$ and on $(M', F')$. We assume in addition that another compact Lie group $H$ acts on the first manifold $M$ also by holonomy diffeomorphisms, and that the actions of $G$ and $H$ commute. So said differently, the compact Lie groups $G \times H$ acts by holonomy diffeomorphisms which are isometries (for the ambient manifold metric) on $(M, F)$ and $(M', F')$ and we assume that the action of $H$ on the second manifold $M'$ is trivial. Recall that in this situation the compact Lie groups $G$ and $H$ act by inner automorphisms on the Connes' $C^*$-algebras of the foliations $(M, F)$ and $(M', F')$. We thus get for instance the following $C^*$-algebra isomorphism which will be used later on (see Corollary [13]):

$$\Psi : C^*(M, F) \times G \rightarrow C^*(M, F) \otimes C^*G,$$

and which is induced by the map $C^*(C^*(M, F)) \rightarrow C(G, C^*(M, F))$ for $f \in C(G, C^*(M, F))$ and $g \in G$ by $\Psi(f)(g) := f(g)U_g$. This isomorphism allows indeed to replace, the crossed product $C^*$-algebra $C^*(M, F) \rtimes G$ by the tensor product $C^*(M, F) \otimes C^*G$. We denote by $[\Psi] \in KK(C^*(M, F) \rtimes G, C^*(M, F) \otimes C^*G)$ the induced KK-equivalence.

Recall the Kasparov descent map [39] for given $G$-$C^*$-algebras $A$ and $B$. Let $\mathcal{E}$ be a Hilbert $G$-module on $B$. Define a right prehilbertian $C(G, B)$-module structure on the space $C(G, \mathcal{E})$ of continuous $\mathcal{E}$-valued functions on $G$, by setting

$$e \cdot d(s) = \int_G e(t) td(t^{-1}s) dt \quad \text{and} \quad \langle e_1, e_2 \rangle(s) = \int_G t^{-1} \langle e_1(t), e_2(t)s \rangle_{\mathcal{E}} dt,$$

for $e, e_1, e_2 \in C(G, \mathcal{E})$ and $d \in C(G, B)$. Then the completion of $C(G, \mathcal{E})$ with respect to this Hilbert structure defines by classical arguments, a $B \rtimes G$-Hilbert module that we denote by $\mathcal{E} \rtimes G$. If $\pi : A \rightarrow \mathcal{L}_B(\mathcal{E})$ is a $G$-equivariant star-morphism from $A$ to the $C^*$-algebra of adjointable operators on $\mathcal{E}$ then the map

$$\pi \rtimes G : A \rtimes G \rightarrow \mathcal{L}_B(\mathcal{E} \rtimes G)$$

is a star-morphism. Finally, if $T \in \mathcal{L}_B(\mathcal{E})$ then $T$ induces an operator $T \rtimes G \in \mathcal{L}_{B \rtimes G}(\mathcal{E} \rtimes G)$ defined by $(T \rtimes G)e(s) := T(e(s))$. It was then proved in [36] that if $(\mathcal{E}, \pi, T)$ is an $(A, B)$ Kasparov cycle, then $(\mathcal{E} \rtimes G, \pi \rtimes G, T \rtimes G)$ is an $(A \rtimes G, B \rtimes G)$ Kasparov cycle and that this induces a well defined group morphism at the level of KK-theory. More precisely,
Definition 3.11. For $i \in \mathbb{Z}_2$, the Kasparov descent map for the given $G$-algebras $A$ and $B$ is the well defined induced map

$$j^G : KK^i_G(A, B) \to KK^i(A \rtimes G, B \rtimes G)$$

is given by $[[\mathcal{E}, \pi, T]] \mapsto [[\mathcal{E} \rtimes G, \pi \rtimes G, T \rtimes G]]$.

Back to our foliations, recall from Proposition 2.10 the well defined $G$-equivariant index map for $G \times H$-invariant leafwise symbols on $(M, \mathcal{F})$ which are $H$-transversally elliptic along the leaves, i.e.

$$\text{Ind}^{F,G} : K_{G \times H}^i(F_H) \to KK^i(C^*(H), C^*(M, \mathcal{F})).$$

If we compose this index map with the Kasparov descent map for the trivial $G$-action and $C^*(M, \mathcal{F})$ for the standard action induced from the $G$-action along the leaves, and further use the isomorphism $\Psi$, then we end up with an index map

$$\tilde{\text{Ind}}^{F,G} : K_{G \times H}^i(F_H) \to KK^i(C^*(H \rtimes G), C^*(M, \mathcal{F}) \otimes C^*G).$$

Remark 3.12. Since $G$ acts by holonomy diffeomorphisms here, we can recast the representative of the index class given by Equation (5) so that it rather represents a $G$-equivariant class for the trivial $G$-action on $C^*(M, \mathcal{F})$. If we denote by $KK^i_{G,\text{trivial}}(C^*(H), C^*(M, \mathcal{F}))$ the equivariant Kasparov group for the trivial $G$-action, then this yields an index morphism

$$\text{Ind}^{F,G,\text{trivial}} : K_{G \times H}^i(F_H) \to KK^i_{G,\text{trivial}}(C^*(H), C^*(M, \mathcal{F})).$$

Lemma 3.13. Denote by $j^{G,\text{trivial}} : KK^i_{G,\text{trivial}}(C^*(H), C^*(M, \mathcal{F})) \to KK^i(C^*(H \rtimes G), C^*(M, \mathcal{F}) \otimes C^*G)$ the Kasparov descent morphism for the trivial $G$-action, then the following relation holds:

$$\tilde{\text{Ind}}^{F,G} = j^{G,\text{trivial}} \circ \text{Ind}^{F,G,\text{trivial}}.$$

Proof. Indeed, this is a consequence of the fact that $\mathcal{E} = eC^*(M, \mathcal{F})^N$ for a projector $e \in M_N(C^*(M, \mathcal{F}))$. Notice first that

$$\tilde{\text{Ind}}^{F,G}(a) = [\mathcal{E} \rtimes G \otimes (C^*(M, \mathcal{F}) \otimes C^*G), (\pi \rtimes G) \otimes \Psi, (A \rtimes G) \otimes \Psi]$$

while

$$j^{G,\text{trivial}}\left[\text{Ind}^{F,G,\text{trivial}}(a)\right] = [\mathcal{E} \rtimes G^{\text{trivial}}, \pi \rtimes G^{\text{trivial}}, A \rtimes G^{\text{trivial}}],$$

where $\mathcal{E} \rtimes G^{\text{trivial}}$ denotes the Hilbert module obtained using the modified action which allows to see $\mathcal{E}$ as a $G$-equivariant Hilbert module for the trivial $G$-action on $C^*(M, \mathcal{F})$. So the action of $G$ used in the definition of $\mathcal{E} \rtimes G^{\text{trivial}}$ is the one given by the unitary $U$ from Lemma 1.2. In the same way, the notations $\pi \rtimes G^{\text{trivial}}$ and $A \rtimes G^{\text{trivial}}$ should be clear.

We introduce the (unitary) isomorphism

$$\mathcal{R} : \mathcal{E} \rtimes G \otimes_{\Psi} (C^*(M, \mathcal{F}) \otimes C^*G) \to \mathcal{E} \rtimes G^{\text{trivial}},$$

which is well induced by the following formula on the elementary tensors:

$$\mathcal{R}(e \otimes f)(g) = \int_G e(r)U_r f(r^{-1}g)dr, \quad \text{for } e \in C(G, \mathcal{E}) \text{ and } f \in C(G, C^*(M, \mathcal{F})).$$

Here and as before, $U_r$ is the unitary multiplier implementing the internal action of $r \in G$ on $C^*(M, \mathcal{F})$. For the sake of completeness, we now check all the properties about $\mathcal{R}$ which show that it gives the allowed unitary equivalence between our two cycles over $(C^*(H \rtimes G), C^*(M, \mathcal{F}) \otimes C^*G)$. 

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A first easy computation shows that for any \( e \in C(G,E) \) and \( f, \tilde{f} \in C(G,C^*(M,F)) \) one has
\[
\mathcal{R}(e \cdot \tilde{f} \otimes f) = \mathcal{R}(e \otimes \Psi(\tilde{f}) f).
\]
In a similar way, the straightforward computation shows that for \( e_1, e_2 \in C(G,E) \) and \( f_1, f_2 \in C(G,C^*(M,F)) \), one has
\[
\langle \mathcal{R}(e_1 \otimes f_1), \mathcal{R}(e_2 \otimes f_2) \rangle_{\mathcal{E} \times \mathcal{E}^{\text{trivial}}} = \int_{G^2} f_1(u^{-1}s)^*U_{u^{-1}} \langle e_1(u), e_2(uf) \rangle_{\mathcal{E}} U_uf f_2(v^{-1}ust) \, du \, dv \, ds.
\]
Hence setting \( r = u^{-1}s \), we get
\[
\langle \mathcal{R}(e_1 \otimes f_1), \mathcal{R}(e_2 \otimes f_2) \rangle_{\mathcal{E} \times \mathcal{E}^{\text{trivial}}} = \int_{G^2} f_1(r)^*U_{u^{-1}} \langle e_1(u), e_2(uf) \rangle_{\mathcal{E}} U_uf f_2(v^{-1}rt) \, du \, dv \, dr.
\]
But notice that \( U_{u^{-1}} \langle e_1(u), e_2(uf) \rangle_{\mathcal{E}} = u^{-1} \langle (e_1(u), e_2(uf))_{\mathcal{E}} \rangle \) and that on the other hand and by definition
\[
\langle e_1, e_2 \rangle_{\mathcal{E} \times \mathcal{E}}(v) = \int_G u^{-1} \langle (e_1(u), e_2(uf))_{\mathcal{E}} \rangle \, du
\]
so that we finally obtain
\[
\langle \mathcal{R}(e_1 \otimes f_1), \mathcal{R}(e_2 \otimes f_2) \rangle_{\mathcal{E} \times \mathcal{E}^{\text{trivial}}} = \int_{G^2} f_1(r)^* \langle e_1, e_2 \rangle_{\mathcal{E} \times \mathcal{E}}(v) U_r f_2(v^{-1}rt) \, dv \, dr.
\]
But this latter expression coincides with
\[
\langle f_1, \Psi(\langle e_1, e_2 \rangle_{\mathcal{E} \times \mathcal{E}} f_2) \rangle_{C^*(M,F) \otimes C^*G} = \langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{[\mathcal{E} \times \mathcal{E}]_{\mathcal{E} \times \mathcal{E}}} = \langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{[\mathcal{E} \times \mathcal{E} \otimes C^*(M,F) \otimes C^*G]}.
\]
This proves that \( \mathcal{R} \) extends to an isometric (adjointable) operator between the two Hilbert modules.

We now check that \( \mathcal{R} \) intertwines the representations and the operators. For \( \varphi \in C(G \times H) \), \( e \in C(G,E) \) and \( f \in C(G,C^*(M,F)) \), we can first compute:
\[
\mathcal{R}((\pi \times G)(\varphi)\cdot e \otimes f)(t) = \int_{G^2} \pi(\varphi(k))k(e(k^{-1}s))U_s f(s^{-1}t) \, ds \, dk = \int_{G^2} \pi(\varphi(k))k(e(k^{-1}s)U_{k^{-1}} f(s^{-1}t))U_k \, ds \, dk.
\]
Setting \( k^{-1}s = r \), we eventually obtain
\[
\mathcal{R}((\pi \times G)(\varphi)\cdot e \otimes f)(t) = \int_{G^2} \pi(\varphi(k))k(e(r)U_r f(r^{-1}k^{-1}t))U_k \, dr \, dk
\]
\[
= \int_G \pi(\varphi(k))k(\mathcal{R}(e \otimes f)(k^{-1}t))U_k \, dk = (\pi \times G^{\text{trivial}})(\varphi)\mathcal{R}(e \otimes f)(t).
\]
Similarly, we get using the \( C^*(M,F) \)-linearity of the operator \( A \),
\[
\mathcal{R}((A \times G) \otimes \text{id}(e \otimes f))(t) = \int_G A(e(s))U_s f(s^{-1}t) \, ds = (A \otimes \text{id})\mathcal{R}(e \otimes f)(t).
\]
It only remains to show that the isometry \( \mathcal{R} \) has dense range. For a given \( \xi \in C_c(G,r^*E) \) and for any \( g \in G \), we set
\[
\xi^g(\gamma) := \xi(\gamma \circ \theta^g \circ (gs(\gamma))) \in E_r(\gamma).
\]
Then \( \xi^g \in C_c(G,r^*E) \) and if we are now given \( \eta \in C(G,C_c(G,r^*E)) \), then we introduce \( \eta U^* \in C(G,C_c(G,r^*E)) \) by setting
\[
(\eta U^*)(g, \gamma) := \eta(g)^{g^{-1}}(\gamma).
\]

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We can then compute for any \( f \in C(G, C_c(G)) \) and any \( \gamma \in G \) with \( y = r(\gamma) \):

\[
V(\eta U^* \otimes f)(g, \gamma) = \int_{G \times G^y} \eta(k)^{-1}(\gamma_1) f \left( k^{-1} g, k^{-1} (\gamma_1^{-1} \gamma) \circ \theta(k^{-1} (s(\gamma))) \right) dk \, du^\eta(\gamma_1)
\]

\[
= \int_{G \times G^y} \eta(k, \gamma_2) f \left( k^{-1} g, \theta(k^{-1} (ks(\gamma_2))) \circ (\gamma_2 \theta(k^{-1} (ks(\gamma_2))))^{-1} \circ \gamma \right) dk \, du^\eta(\gamma_2)
\]

\[
= \int_{G \times G^y} \eta(k, \gamma_2) f(k^{-1} g, \gamma_2^{-1} \gamma) dk \, du^\eta(\gamma_2).
\]

Hence we get

\[
V(\eta U^* \otimes f) = \eta \cdot f,
\]

where \( \eta \cdot f \) is the right module product of \( \eta \in C(G^{\text{triv}}, \mathcal{E}) \) by \( f \in C^*G \otimes C^*(M, \mathcal{F}) \). Now, using an approximate unit for the \( C^*\)-algebra \( C^*G \otimes C^*(M, \mathcal{F}) \), we conclude that any \( \eta \in C(G, C_c(G, r^*E)) \) belongs to the closure of the range of \( V \). This allows to complete the proof by density of \( C(G, C_c(G, r^*E)) \) in \( \mathcal{E} \times G^{\text{triv}} \). \( \square \)

Using the action of \( G \) on the second foliation \((M', \mathcal{F}')\) we also have the index map for leafwise \( G \)-transversally elliptic symbols

\[
\text{Ind}_{\mathcal{F}'}^F: K_G(F_G') \to KK(C^*G, C^*(M', \mathcal{F}')).
\]

A classical construction then allows to build up from a \( G \times H \)-invariant leafwise \( H \)-transversally elliptic symbol \( a \) on \((M, \mathcal{F})\) and a \( G \)-invariant leafwise \( G \)-transversally elliptic symbol \( b \) on \((M', \mathcal{F}')\) a new symbol which is a leafwise symbol on the cartesian product \((M \times M', \mathcal{F} \times \mathcal{F}')\) of the two foliated manifolds, is \( G \times H \)-invariant and \( G \times H \)-transversally elliptic.

More precisely, there is a well defined product for all \( i, j \in \mathbb{Z}_2 \),

\[
K_{i,h}^G(F_H) \otimes K_j^G(F_H') \to K_{i+j}^{G \times H}(F \times F'),
\]

which assigns to \([\sigma] \otimes [\sigma']\) the class of the sharp product \( \sigma \sigma' \) that we proceed to recall now. The cartesian product \( F \times F' \) fibers over \( M \times M' \) and generates the foliation of \( M \times M' \) whose leaf through any given \((m, m')\) is just the cartesian product \( L_m \times L_{m'} \) of the leaf of \((M, \mathcal{F})\) through \( m \) by the leaf of \((M', \mathcal{F}')\) through \( m' \). The compact group \( G \times H \) acts obviously by leaf-preserving diffeomorphisms of this product foliation and the subspace \((F \times F')_{G \times H}\) of vectors transverse to this action, is well defined. Notice as well that this product action of \( G \times H \) is also a holonomy action. For the convenience of the reader, let us describe the above product in the case \( i = j = 0 \) for simplicity. We leave the other cases as an exercise. Recall that any class \( b \) in \( K_G(F_H') \) can be represented by a classical \( G \)-invariant pseudodifferential symbol \( \sigma' \) along the leaves of the foliation \( \mathcal{F}' \), that is defined over \( F' \), and whose restriction to \( F'_G \otimes M' \) is pointwise invertible. In the same way, any class \( a \) in \( K_{G \times H}(F_H) \) can be represented by a classical \( G \times H \)-invariant pseudodifferential symbol \( \sigma \) along the leaves of the foliation \( \mathcal{F} \), that is defined over \( F \), and whose restriction to \( F_H \otimes M \) is pointwise invertible, see [1]. The product \( a \# b \) is then the class in \( K_{G \times H}(F \times F')_{G \times H} \) which is represented by the leafwise \( G \times H \)-invariant symbol on the foliation \( \mathcal{F} \times \mathcal{F}' \) over \( M \times M' \) defined by:

\[
\sigma \# \sigma' := \begin{pmatrix} \sigma \otimes 1 & -1 \otimes \sigma'^* \\ 1 \otimes \sigma' & \sigma^* \otimes 1 \end{pmatrix}.
\]

This is the standard cup-product formula, used in [1] where it adapted the original Atiyah-Singer construction from the seminal paper [3] to the transversally elliptic context, and whose extension to the foliation setting is a routine exercise. In particular, the restriction of \( \sigma \sigma' \) to \((F \times F')_{G \times H} \setminus (M \times M')\) is pointwise invertible as allowed and hence represents our announced sharp product.

We are now in position to prove the multiplicativity axiom which computes the index of the sharp product \( a \# b \) in terms of the indices of \( a \) and \( b \). Notice that \( C^*(M \times M', \mathcal{F} \times \mathcal{F}') \cong C^*(M, \mathcal{F}) \otimes C^*(M', \mathcal{F}')\).
Theorem 3.14. For any $i, j \in \mathbb{Z}_2$, the following diagram commutes:

$$
\begin{array}{c}
\xymatrix{
K^i_{G \times H}(F_H) \otimes K^j_G(F'_G) \ar[r]^{\ast \ast} \ar[d]_{\text{Ind}^F \times \text{Ind}^{F'}} & K^{i+j}_{G \times H}((F \times F')_{G \times H}) \ar[d]_{\text{Ind}^{F \times F'}} \\
\text{KK}'(C^*(G \times H), C^*(M, \mathcal{F}) \otimes C^*G) \otimes \text{KK}'(C^*G, C^*(M', \mathcal{F})) \ar[r]_{\ast \otimes \ast} & \text{KK}^{i+j}(C^*(G \times H), C^*(M \times M', \mathcal{F} \times \mathcal{F}'))
}
\end{array}
$$

In other words, if $a \in K^i_{G \times H}(F_H)$ and $b \in K^j_G(F'_G)$ and if $a \ast b \in K^{i+j}_{G \times H}((F \times F')_{G \times H})$ is their sharp product, then we have

$$\text{Ind}^{F \times F'}(a \ast b) = \text{Ind}^{F, G}(a) \otimes \text{Ind}^{F'}(b).$$

Proof. We treat the case $i = 0 = j$, the other cases are similar. If $P_0$ is a longitudinal pseudodifferential operator of positive order then we denote again by $Q(P)$ the closure of the formally self-adjoint longitudinal operator $P_0$ in the corresponding Hilbert module. We also recall that the Woronowicz transform of $P$ is the adjointable operator $Q(P) = P(1 + P^2)^{-1/2}$.

Let $A_0 : C^\infty_c(G, r^*E^+) \to C^\infty_c(G, r^*E^-)$ be a $G \times H$-invariant, leafwise $H$-transversally elliptic operator of order 1 whose principal symbol represents the class $a$. Let similarly $B_0 : C^\infty_c(G', r^*E^+) \to C^\infty_c(G', r^*E^-)$ be a $G$-invariant, leafwise $G$-transversally elliptic operator of order 1 whose principal symbol lies in the class $b$. The index classes associated respectively are then by definition

$$\text{Ind}^{F \times F'}(A) = [(E, \pi_H, Q(A))] \quad \text{and} \quad \text{Ind}^{F'}(b) = [(E', \pi_G, Q(B))],$$

where the first class is $G$-equivariant for the $G^{\text{trivial}}$-action on $E$, i.e. viewed as a Hilbert $G$-module for the trivial $G$-action on $C^*(M, \mathcal{F})$ by using the holonomy hypothesis.

Hence, the image of $[(E, \pi_H, Q(A))]$ under the Kasparov descent is represented, with our previous notations and using Lemma 3.13 by the Kasparov $(C^*H \times G, C^*(M, \mathcal{F}) \otimes C^*G)$ cycle

$$(E \times G^{\text{trivial}}, \pi_H \times G^{\text{trivial}}, Q(A) \times G^{\text{trivial}}).$$

Recall that the action of $G$ on the $C^*$-algebra $C^*(M', \mathcal{F}')$ is also internal through unitary multipliers that we denote by $(U^n_g)_{g \in G}$. Let $U : C(G, E) \otimes \mathcal{E}' \to E \otimes \mathcal{E}'$ be the map defined by

$$U(\rho \otimes \eta') := \int_G \rho(g) \otimes U'_g \eta' dg, \quad \text{for} \ \rho \in C(G, E) \text{ and } \eta' \in \mathcal{E}'.
$$

Here the integral makes sense in the norm topology of the Hilbert module closure, denoted as usual $E \otimes \mathcal{E}'$, over the $C^*$-algebra $C^*(M, \mathcal{F}) \otimes C^*(M', \mathcal{F}')$. From the very definition of the representation $\pi_G$, we easily deduce that for $\varphi \in C(G)$, one has

$$U(\rho \cdot \varphi \otimes \eta') = U(\rho \otimes \pi_G(\varphi) \eta'),$$

so that $U$ is well defined. Moreover, we can check now that $U$ extends to a unitary isomorphism which identifies $(E \times G^{\text{trivial}}) \otimes \mathcal{E}'$ with the spatial tensor product Hilbert module $E \otimes \mathcal{E}'$. Let $\rho_1, \rho_2 \in E \times G^{\text{trivial}}$ and $\eta_1', \eta_2' \in \mathcal{E}'$. We have:

$$
\langle U(\rho_1 \otimes \eta_1'), U(\rho_2 \otimes \eta_2') \rangle_{E \otimes \mathcal{E}'}
= \int_{G^2} \langle \rho_1(h) \otimes U'_h \eta_1', \rho_2(k) \otimes U'_k \eta_2' \rangle_{E \otimes \mathcal{E}'} \, dh \, dk
= \int_{G^2} \langle \rho_1(h) \cdot \rho_2(k) \rangle_E \langle \eta_1', U_{h^{-1}} \eta_2' \rangle_{E'} \, dh \, dk
= \int_{G^2} \langle \rho_1(h) \cdot \rho_2(k) \rangle_E \langle \eta_1', U_{h^{-1}} \eta_2' \rangle_{E'} \, dh \, dk.
$$

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By substituting $g = h^{-1}k$, we get:

$$
\langle U(\rho_1 \otimes \eta^1_1), U(\rho_2 \otimes \eta^2_2) \rangle_{\mathcal{E} \otimes \mathcal{E}'} = \int_{G^2} \langle \rho_1(h), \rho_2(hg) \rangle_\mathcal{E} \langle \eta^1_1, U_g \eta^2_2 \rangle_{\mathcal{E}'} \, dh \, dg
$$

$$
= \int_{G^2} \langle \eta^1_1 \circ U, \eta^2_2 \rangle_{\mathcal{E} \otimes \mathcal{E}'} \, dh \, dg
$$

$$
= \langle \rho_1 \otimes \eta^1_1, \rho_2 \otimes \eta^2_2 \rangle_{(\mathcal{E} \otimes \mathcal{E}')^*_G}
$$

It is then easy to check that $\mathcal{U}(C(G, \mathcal{E}) \otimes \mathcal{E}')$ is dense in $\mathcal{E} \otimes \mathcal{E}'$. Indeed, given $\eta \in \mathcal{E}$ and $\eta' \in \mathcal{E}'$, we may use an approximate unit $(e_\alpha)_\alpha$ of the $C^*$-algebra $C^*_G$, composed of continuous functions on $G$ which are supported as close as we please to the neutral element of $G$, to see that $\pi_G(e_\alpha)(\eta')$ converges in $\mathcal{E}'$ to $\eta'$. Hence, the net $\mathcal{U}((\eta \otimes e_\alpha) \otimes \eta') = \eta \otimes \pi_G(e_\alpha)(\eta')$ converges to $\eta \otimes \eta'$ in the spatial tensor product $\mathcal{E} \otimes \mathcal{E}'$.

It thus remains to check that $\mathcal{U}$ intertwines operators as well. Let $\psi \in C(H)$ and $\varphi \in C(G)$, $\rho \in C(G, \mathcal{E})$ and $\eta' \in \mathcal{E}'$ and denote by $(U^H_h)_{h \in H}$ the unitary multipliers implementing the action of $H$ by holonomy diffeomorphisms. We have:

$$
\pi_{H \times G}(\psi \otimes \varphi)(\mathcal{U}(\rho \otimes \eta')) = \pi_{H \times G}(\psi \otimes \varphi) \left( \int_G \rho(t) \otimes U_t \eta' \, dt \right)
$$

$$
= \int_{H \times G} \int_G \psi(h) \varphi(g) U^H_g \rho(t) \otimes U_{gt} \eta' \, dt \, dg.
$$

Setting $gt = u$, we obtain:

$$
\pi_{H \times G}(\psi \otimes \varphi)(U(\rho \otimes \eta')) = \int_{H \times G} \int_G \psi(h) \varphi(g) U^H_g \rho(g^{-1} \cdot u) \otimes U'_u \eta' \, du \, dg
$$

$$
= \int_{G^2} \varphi(g) \pi_H(\psi) U_g \rho(g^{-1} \cdot u) \otimes U'_u \eta' \, du \, dg
$$

$$
= \int_G[(\pi_H \times G)(\psi \otimes \varphi)(\rho)](u) \otimes U'_u \eta' \, du
$$

$$
= \mathcal{U} \left( \left( \pi_H \times G)(\psi \otimes \varphi) \otimes \text{id} \right)(\rho \otimes \eta') \right),
$$

so that $\mathcal{U}$ intertwines indeed the representations. Let us now compute similarly $(A \otimes \text{id} \otimes \text{id} \otimes B) \circ \mathcal{U}$. We have with the previous notations and taking now $\rho \in C(G, C^\infty(G, r^* \mathcal{E}))$ and $\eta' \in C^\infty(G', r^* \mathcal{E}')$:

$$
[(A \otimes 1 + 1 \otimes B) \circ \mathcal{U}](\rho \otimes \eta') = (A \otimes 1 + 1 \otimes B) \left( \int_G \rho(t) \otimes U_t \eta' \, dt \right)
$$

$$
= \int_G [A(\rho(t)) \otimes U'_t \eta' + \rho(t) \otimes B(U'_t \eta')] \, dt
$$

$$
= \int_G [(A \times G^\text{trivial}) \rho(t) \otimes U'_t \eta' + \rho(t) \otimes U'_t B(\eta')] \, dt
$$

$$
= \left( \mathcal{U} \circ \left( (A \times G^\text{trivial}) \otimes \text{id} + \text{id} \otimes B \right)^*_G \right)(\rho \otimes \eta'),
$$

so that $\mathcal{U}$ intertwines operators as well.

The Kasparov product of $\text{Ind}^{\mathcal{E}, G}_H(A)$ and $\text{Ind}^{\mathcal{E}'}_{G'}(B)$ can be represented by the unbounded cycle

$$
\left( (\mathcal{E} \times G^\text{trivial})_{\otimes \mathcal{E}'} \right) \otimes (\pi_H \times G^\text{trivial}) \otimes \text{id}, (A \times G^\text{trivial})_{\otimes \text{id} + \text{id} \otimes B} \right)
$$

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where the operator $\id \otimes B$ is well defined here since $B$ commutes strictly with the representation $\pi_G$. Indeed, let us prove that the operator $Q((A \times G^{\text{trivial}})_{\pi_G} \otimes \id + \id \otimes B)$ is a $Q(B)$-connection on $(\mathcal{E} \times G^{\text{trivial}})_{\pi_G} \otimes \mathcal{E}'$. We may in fact write

$$Q((A \times G^{\text{trivial}})_{\pi_G} \otimes \id + \id \otimes B) = M^{1/2}Q(A \times G^{\text{trivial}})_{\pi_G} \otimes \id + N^{1/2} \id \otimes Q(B),$$

where

$$M = \left( \id + (A \times G^{\text{trivial}})_{\pi_G}^2 \otimes \id \right) \left( \id + (A \times G^{\text{trivial}})_{\pi_G}^2 \otimes \id + \id \otimes B^2 \right)^{-1}$$

and

$$N = \left( \id + \id \otimes B^2 \right) \left( \id + (A \times G^{\text{trivial}})_{\pi_G}^2 \otimes \id + \id \otimes B^2 \right)^{-1}.$$ 

Notice that the operators $M$ and $N$ are adjointable selfadjoint operators on $(\mathcal{E} \times G^{\text{trivial}})_{\pi_G} \otimes \mathcal{E}'$. Moreover $M$ and $N$ commute and we have

$$M + N = \id + (A \times G^{\text{trivial}})_{\pi_G}^2 \otimes \id + \id \otimes B^2 \right)^{-1}.$$ 

We thus need to show that $M$ is a $0$-connection and that $N$ is a $1$-connexion. That will imply then by a classical result on connections that $M^{1/2}$ is a $0$-connection and that $N^{1/2}$ is a $1$-connection, so that $Q((A \times G^{\text{trivial}}) \otimes \id + \id \otimes B)$ will be a $Q(B)$-connection as announced.

Let $\varphi \in C^\infty(G)$ and $\eta \in C_c^\infty(G, r^* E)$, then we set:

$$T_{\eta} : \mathcal{E}' \rightarrow \mathcal{E} \otimes \mathcal{E}'$$

$$\eta' \mapsto \eta \otimes \eta'$$

and viewing $\eta \otimes \varphi$ in $\mathcal{E} \times G^{\text{trivial}}$ we similarly set

$$T_{\eta \otimes \varphi} : \mathcal{E}' \rightarrow (\mathcal{E} \times G^{\text{trivial}})_{\pi_G} \otimes \mathcal{E}'$$

$$\eta' \mapsto (\eta \otimes \varphi) \otimes \eta'.$$

Notice on the other hand that

$$\mathcal{U} \circ M = (\id + A^2 \otimes \id) \left( \id + [A^2 \otimes \id + \id \otimes B^2] \right)^{-1} \circ \mathcal{U}.$$

Moreover, we also have the following obvious relation

$$\mathcal{U} \circ T_{\eta \otimes \varphi} = T_{\eta} \circ \pi_G(\varphi) : \mathcal{E}' \rightarrow \mathcal{E} \otimes \mathcal{E}' .$$

Hence we have:

$$\mathcal{U} \circ M \circ T_{\eta \otimes \varphi} = (\id + A^2 \otimes \id) \left( \id + [A^2 \otimes \id + \id \otimes B^2] \right)^{-1} \circ T_{\eta} \circ \pi_G(\varphi)$$

$$= (\id + \id \otimes B^2) \left( \id + [A^2 \otimes \id + \id \otimes B^2] \right)^{-1} \circ T_{\eta + A^2 \eta} \circ (\id + B^2)^{-1} \circ \pi_G(\varphi)$$

$$= (\mathcal{U} \circ N \circ \mathcal{U}^{-1}) \circ T_{\eta + A^2 \eta} \circ (\id + B^2)^{-1} \circ \pi_G(\varphi).$$

Since $(\id + B^2)^{-1} \circ \pi_G(\varphi)$ is a compact operator, we deduce that $M \circ T_{\eta \otimes \varphi}$ is a compact operator between the Hilbert modules $\mathcal{E}'$ and $(\mathcal{E} \times G^{\text{trivial}})_{\pi_G} \otimes \mathcal{E}'$. Moreover, the map which assigns to $\rho \in \mathcal{E} \times G^{\text{trivial}}$ the operator $M \circ T_{\rho}$ being continuous we deduce that $M$ is a $0$-connection.
To show that \( N \) is a 1-connection it is sufficient to check that the operator
\[
\left( \text{id} + \left( A \times G^{\text{trivial}} \right)^2 \otimes \text{id} + \text{id} \otimes B^2 \right)^{-1}
\]
is a 0-connection. This is obtained in a way similar to the previous computation, since
\[
\left( \text{id} + \left( A \times G^{\text{trivial}} \right)^2 \otimes \text{id} + \text{id} \otimes B^2 \right)^{-1} = N \circ (\text{id} + \text{id} \otimes B^2)^{-1}.
\]
The positivity condition is finally checked as follows. We have
\[
\left[ Q((A \times G^{\text{trivial}}) \otimes \text{id} + \text{id} \otimes B), Q(A \times G^{\text{trivial}}) \otimes \text{id} \right] = 2M^{1/2} \circ Q(A \times G^{\text{trivial}})^2 \otimes \text{id}.
\]
Hence for any \( \theta \in C(H \times G) \) we conclude that the operator
\[
\left( \pi_H \otimes G^{\text{trivial}}(\theta) \otimes \text{id} \right) \circ \left[ Q((A \times G^{\text{trivial}}) \otimes \text{id} + \text{id} \otimes B), Q(A \times G^{\text{trivial}}) \otimes \text{id} \right] \circ \left( \pi_H \otimes G^{\text{trivial}}(\theta) \otimes \text{id} \right)
\]
is positive modulo the compact operators of the Hilbert module \((\mathcal{E} \times G^{\text{trivial}}) \otimes \mathcal{E}'\).

\[\square\]

### 4 Reduction to tori actions

We now use the previous axioms to investigate the induction property of our index morphism with respect to closed subgroups, and then more specifically to a maximal torus.

We recall first some standard constructions from [11]. Let \( G \) be a compact connected Lie group and let \( H \) be a closed subgroup of \( G \). Denote by \( i : H \to G \) the inclusion. Then the functoriality class \([i] \in \text{KK}(C^*G, C^*H)\) is defined as follows, see [33]. We fix Haar measures on \( H, G \) and consider the right \( L^1(H)\)-module structure on the space \( C(G) \), which is induced by the right action of \( H \) on \( G \). More precisely, we set for \( f \in C(G) \) and \( \psi \in L^1(H) \):
\[
f \cdot \psi(g) = \int_H f(gh^{-1})\psi(h) \, dh,
\]
and define the \( L^1(H)\)-valued hermitian structure by setting for \( f_1, f_2 \in C(G) \):
\[
\langle f_1, f_2 \rangle(h) = \int_G \tilde{f}_1(g)f_2(gh) \, dg.
\]
The completion of this prehilbertian \( L^1(H)\)-module is then a Hilbert \( C^*H\)-module that we shall denote by \( J(G, H) \). The left action of \( G \) on itself by translation allows to define, after completing, the representation \( \pi_G : C^*G \to \mathcal{L}_{C^*H}(J(G, H)) \). The triple \((J(G, H), \pi_G, 0)\) is then a Kasparov cycle over the pair of \( C^*\)-algebras \((C^*H, C^*G)\), see again [33].

**Definition 4.1.** [33] The functoriality class \([i] \) is the class of the Kasparov cycle \((J(G, H), \pi_G, 0)\), i.e.
\[
[i] := [(J(G, H), \pi_G, 0)] \in \text{KK}(C^*G, C^*H).
\]

**Remark 4.2.** Since the crossed product \( C^*\)-algebra \( C(G/H) \rtimes G \) for the induced left action of \( G \) on the homogeneous manifold \( G/H \) is Morita equivalent to \( C^*H \), it is easy to reinterpret the class \([i]\) as the class induced through the descent map for the \( G \)-action, by the trivial representation of \( H \) viewed as a trivial \( G \)-equivariant vector bundle over \( G/H \).
Since the underlying closed manifold $G$ is endowed with the $G \times H$-action given by $(g, h) \cdot g' = gg' h^{-1}$ for $g, g' \in G$ and $h \in H$, we may use the product defined in Equation (11) for any given smooth foliation $\mathcal{F}$ on a closed manifold $M$ as soon as this latter is endowed with a smooth leaf-preserving $H$-action, which is a holonomy action. Indeed, we are considering here the trivial top-dimensional foliation on $G$ and we thus get the following product for $j \in \mathbb{Z}_2$

$$K_{G \times H}(T_G G) \otimes K_H^j(F_H) \to K_{G \times H}^j((T_G \times F)_{G \times H}).$$

(13)

Notice that the space $T_G G$ is just $G \times \{0\} \simeq G$, and hence since $H$ acts freely on $G$:

$$K_{G \times H}(T_G G) \simeq K_G(G/H) \simeq R(H).$$

Moreover, $H$ also acts freely on the cartesian product $G \times M$ preserving the product foliation $T_G \times F$ and the quotient manifold $Y := G \times_H M$ inherits a foliation that we denote by $\mathcal{F}_Y$ and which is automatically endowed with the action of $G$ by holonomy diffeomorphisms, as can be checked easily. The receptacle group $K_{G \times H}^j((T_G \times F)_{G \times H})$ in (13) is then given by

$$K_{G \times H}^j((T_G \times F)_{G \times H}) \simeq K_G^j(F_G^Y).$$

(14)

Notice that the space $F_G^Y = G \times_H F_H$ is $G$-equivariantly Morita equivalent as a groupoid to $(G \times F_H) \times H$ and we deduce the following list of Morita equivalences

$$F_G^Y \times G \simeq [(G \times F_H) \times H] \times G \simeq [(G \times F_H) \times G] \times H \simeq [(G \times F_H)/G] \times H \simeq F_H \times H.$$  

In particular, the group $K_G^j(F_G^Y)$ is isomorphic to the group $K_H^j(F_H)$, the isomorphism $i_* : K_H^j(F_H) \to K_G^j(F_G^Y)$ is given explicitly as follows. There is a privileged element in the group $K_{G \times H}(T_G G)$ which corresponds to the class, in $R(H)$, of the trivial representation of $H$. This class is in fact the class of the $G \times H$-equivariant $G$-transversally elliptic symbol on $G$, associated with the zero operator $0 : C^\infty(G) \to 0$. The product in (13) by this trivial class yields the allowed isomorphism

$$i_* : K_H^j(F_H) \to K_G^j(F_G^Y).$$

(15)

As we prove below, this isomorphism allows to reduce the index problem for leafwise $H$-transversally elliptic operators on foliated $H$-manifolds to the index problem for leafwise $G$-transversally elliptic operators on foliated $G$-manifolds. Notice that $Y$ is the base of the principal $H$-fibration $G \times M \to Y$ and we are exactly in position to apply the properties of the index morphism with respect to free actions, see Subsection 8.1. Furthermore, since the compact Lie group $G$ is assumed to be connected here, the $C^*$-algebra upstairs, that is $C^*(G \times M, G \times \mathcal{F})$ is Morita equivalent, and in fact isomorphic when $\mathcal{F}$ is not the zero foliation [11, 29], to $C^*(M, \mathcal{F})$. Hence we end up with a KK-equivalence that we denote by $\epsilon \in KK(C^*(M, \mathcal{F}), C^*(Y, \mathcal{F}_Y)).$

**Theorem 4.3.** [7] For $j \in \mathbb{Z}_2$, the following diagram commutes

$$\begin{array}{ccc}
K_H^j(F_H) & \xrightarrow{i_*} & K_G^j(F_G^Y) \\
\downarrow \text{Ind}^F & & \downarrow \text{Ind}^{F_Y} \\
KK^j(C^*(M, \mathcal{F})) \otimes_{C^*H} \otimes_{C^*(M, \mathcal{F})} & \xrightarrow{\epsilon} & KK^j(C^*(G, C^*(Y, \mathcal{F}_Y)))
\end{array}$$

**Proof.** Recall the Kasparov class

$$\mathcal{E}_q \in KK \left( (C^*(G \times M, G \times \mathcal{F}), C^*(Y, \mathcal{F}_Y)) \right) \simeq KK \left( K(L^2(G)) \otimes C^*(M, \mathcal{F}), C^*(Y, \mathcal{F}_Y) \right)$$

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introduced in Section 3.1 and associated here to the principal $H$-fibration $q : G \times M \to Y = G \times M$. If we denote by $\mu(G) \in KK(\mathbb{C}, K(L^2(G)))$ the standard KK-equivalence then we have by definition

$$\epsilon = \mu(G) \otimes_{K(L^2(G))} \mathcal{E}_q.$$ 

Let now $a \in K^0_H(F_H)$ be fixed. By Theorem 3.1 we know that

$$\text{Ind}^{F_Y}(i_*a) = \chi^H_1 \otimes_{C^*H} \text{Ind}^{G \times F}(q^*(i_*a))_{C^*(G \times M, G \times F)} \mathcal{E}_q,$$

where $\chi^H_1 \in KK(\mathbb{C}, C^*H)$ is the class of the trivial representation of $H$ and where in the present case $q^*(i_*a)$ is just the isomorphic class to $i_*a$ through the identification $\text{Ind}(\sigma) \in KK^G(C^*H \otimes C^*G, K(L^2(G))) \otimes C^*(M, F))$.

We can now apply the multiplicative property of the index from Theorem 3.14 to compute

$$\text{Ind}^{G \times F}([\sigma(0)] \cdot a) = \text{Ind}^{G \times F}((\sigma(0)) \otimes \text{Ind}^{F}(a).$$

For simplicity the KK-equivalence class $\mu(G)$ is often removed from the formulae, it is only used to naturally identify, in $K$-theory, $K(L^2(G))$ with $\mathbb{C}$. The index class $\text{Ind}^{G \times F}((\sigma(0)) \in KK(C^*G \otimes C^*H, C^*H)$ reduces here to the image under the Kasparov descent map, for the trivial $H$-action, of the $H$-equivariant index class in $KK_H(C^*G, \mathbb{C})$, of the $G$-transversally elliptic operator $0 : C^\infty(G) \to 0$. By gathering the previous equalities, we finally get

$$\text{Ind}^{F_Y}(i_*a) = \chi^H_1 \otimes_{C^*H} \text{Ind}^{G \times F}(q^*(i_*a))_{C^*(G \times M, G \times F)} \mathcal{E}_q$$

$$= \chi^H_1 \otimes_{C^*H} \left[ \text{Ind}^{G \times H}((\sigma(0)) \otimes \text{Ind}^{F}(a)_{C^*(G \times M, G \times F)} \mathcal{E}_q \right]$$

$$= \chi^H_1 \otimes_{C^*H} \left[ \text{Ind}^{G \times H}((\sigma(0)) \otimes \text{Ind}^{F}(a) \otimes \mu(G)_{C^*(G \times M, G \times F)} \mathcal{E}_q \right]$$

$$= \left( \chi^H_1 \otimes_{C^*H} \text{Ind}^{G \times H}((\sigma(0)) \otimes \text{Ind}^{F}(a)_{C^*(Y, F^Y)} (\mu(G) \otimes_{K(L^2(G))} \mathcal{E}_q \right)$$

where we have used associativity of the Kasparov product. The proof is now complete since we have

$$\chi^H_1 \otimes_{C^*H} \text{Ind}^{G \times H}((\sigma(0)) = [i] \text{ and } \mu(G) \otimes_{K(L^2(G))} \mathcal{E}_q = \epsilon.$$ 

**Remark 4.4.** Theorem 4.3 allows to extract information on the index morphism for the action of the compact Lie group $H$ using all such compact connected Lie groups $G$ and their induced actions on the Morita equivalent foliation $(Y, F^Y)$. Such $G$ always exists as any compact Lie group is isomorphic to a closed subgroup of a unitary group.

We fix for the rest of this section a compact connected Lie group $G$ and a smooth closed foliated manifold which is endowed with an action of $G$ by leaf-preserving diffeomorphisms. For simplicity, we shall denote this new $G$-foliation again by $(M, F)$ since we shall again need to build up the new foliation $(Y, F^Y)$ by using a particular closed subgroup of $G$, so no confusion should occur. Since $G$ is connected this action is
a holonomy action and we may apply all the results of the previous sections. In order to compute the index morphism for leafwise $G$-transversally elliptic operators, we shall use a maximal torus $T$ in $G$ and we use the induced action of $T$ to define the Morita equivalent $G$-foliation $(Y, F^Y)$ as explained above. However, since the action of $T$ on $(M, \mathcal{F})$ is now the restriction of an action of the whole group $G$, this foliation is easier to describe. More precisely, the map $(g, m) \mapsto (gH, g \cdot m)$ is a $G$-equivariant diffeomorphism which allows to identify the foliation $(Y, F^Y)$ with the foliation $(G/T \times M, G/T \times \mathcal{F})$. We quote for later use that $C^*(Y, F^Y)$ coincides here with $C^*(M, \mathcal{F}) \otimes K(L^2(G/T))$ which in turn, when $\mathcal{F}$ is not the zero foliation, is even isomorphic to $C^*(M, \mathcal{F})$. Notice also that there is hence a well defined product

$$K^j_G(F_G) \otimes K_G(T(G/T)) \rightarrow K^j_G(F^Y_G). \quad (16)$$

Recall that $G/T$ carries a complex structure and we may use the Dolbeault operator $\overline{\partial}$. This is an elliptic $G$-invariant operator on the rational variety $G/T$ whose $G$-index equals $1 \in R(G)$ since only the zero-degree Dolbeault cohomology space is non trivial, see [1], i.e.

$$\text{Ind}(\overline{\partial}) = 1 \in R(G).$$

The product by the symbol class $[\sigma(\overline{\partial})] \in K_G(T(G/H))$ in (16) allows to define the morphism

$$\beta : K_G(F_G) \rightarrow K_G(F^Y_G).$$

Recall the isomorphism $i_\ast$ defined in Equation (14) as well as the KK-class $[i]$ introduced in Definition 3.3. We use these notations for the torus closed subgroup $H = T$ to state the following

**Theorem 4.5.** Let $T$ be a maximal torus of the compact connected Lie group $G$. Denote by $r^G_T : K^j_G(F_G) \rightarrow K^j_T(F_T)$ the composite map $r^G_T := (i_\ast)^{-1} \circ \beta$. Then for $j \in \mathbb{Z}_2$ the following diagram commutes:

$$
\begin{array}{ccc}
K^j_G(F_G) & \xrightarrow{r^G_T} & K^j_T(F_T) \\
\text{Ind}_G & & \text{Ind}_G \\
\downarrow & & \downarrow \\
\text{KK}^j(C^*G, C^*(M, \mathcal{F})) & \overset{[i] \otimes 1}{\longrightarrow} & \text{KK}^j(C^*T, C^*(M, \mathcal{F}))
\end{array}
$$

**Proof.** We apply the multiplicative property of our index morphism from Theorem 3.14. In the notations of Theorem 3.14 we take for $H$ the trivial group, for $(M, \mathcal{F})$ the $G$-manifold $G/T$ with one leaf, and for $(M', \mathcal{F}')$ our $G$-foliation here, that is the foliation $(M, \mathcal{F})$ used in the statement of Theorem 4.5. Then we obtain the commutativity of the following diagram (recall that $C^*(Y, F^Y) = K(L^2(G/T)) \otimes C^*(M, \mathcal{F})$ and hence can be replaced by $C^*(M, \mathcal{F})$):

$$
\begin{array}{ccc}
K_G(T(G/T)) \otimes K^j_G(F_G) & \xrightarrow{\ast \ast} & K^j_G(F^Y_G) \\
\text{Ind}_G \otimes \text{Ind}_G & & \text{Ind}_G \\
\downarrow & & \downarrow \\
\text{KK}(C^*G, C^*G) \otimes \text{KK}^j(C^*G, C^*(M, \mathcal{F})) & \overset{C^*_G \otimes 1}{\longrightarrow} & \text{KK}^j(C^*G, C^*(M, \mathcal{F}))
\end{array}
$$

We recall that $\text{Ind}_G^C = j^G \circ \text{Ind}_G$ where $\text{Ind}_G^C : K_G(T(G/T)) \rightarrow R(G) \simeq \text{KK}_C(\mathbb{C}, \mathbb{C})$ is the usual Atiyah-Singer $G$-index that we view as valued in the Kasparov group $\text{KK}_C(\mathbb{C}, \mathbb{C})$ and $j^G : \text{KK}_C(\mathbb{C}, \mathbb{C}) \rightarrow \text{KK}(C^*G, C^*G)$ is the Kasparov descent map for the trivial $G$ action on $\mathbb{C}$. In particular, $\text{Ind}_G^C(\overline{\partial})$ coincides with the unit of the ring $\text{KK}(C^*G, C^*G)$. If we thus apply this multiplicativity result to a given $a \in K^j_G(F_G)$ and to the Dolbeault symbol, then we get

$$\text{Ind}_G^C \left( \beta(a) \right) = \text{Ind}_G^C(\overline{\partial}) \otimes \text{Ind}_G^C(a) = \text{Ind}_G^C(a)$$

and so $\text{Ind}_G^C \circ \beta = \text{Ind}_G^C$. The proof is now complete since we already proved in Theorem 4.3 the compatibility of the index morphism with the map $i_\ast$. 

$\square$
5 Naturality of the index morphism

We now apply the previous results to give the allowed topological construction of an index map which will be compared with our analytical index map from Proposition 2.8

5.1 Compatibility with Gysin maps

Let \( \iota : (M, \mathcal{F}) \hookrightarrow (M', \mathcal{F}') \) be a foliated embedding of \( G \)-foliations. So we assume that the compact Lie group acts on \( M \) and on \( M' \) by leaf-preserving holonomy diffeomorphisms and that \( \iota : M \hookrightarrow M' \) is a \( G \)-equivariant embedding which sends leaves inside leaves. We assume for simplicity that \( M \) is compact, since this is the only needed situation for the proof of our index theorem. We denote by \( N := \iota^* TM'/TM \) the normal bundle to \( \iota \). In view of the construction of the topological index in Subsection 5.1 we shall only need the case where the transverse bundles \( \tau := TM/F \) and \( \tau' = TM'/F' \) do fit under \( \iota \), i.e. that \( \iota^* \tau' \simeq \tau \). As a consequence, the \( G \)-equivariant embedding \( d\iota : F \to F' \), obtained by differentiating \( \iota \) and restricting to \( F \), is \( K \)-oriented by a \( G \)-equivariant complex structure. Indeed, under this assumption, the normal bundle \( N \) is identified with the normal bundle to the leaves of \( \mathcal{F} \) inside the leaves of \( \mathcal{F}' \), and it is easy then to see that the normal bundle \( N' \) to \( d\iota \) is isomorphic to the bundle \( \pi^*_F(N \otimes \mathbb{C}) \) with \( \pi_F : F \to M \) being the bundle projection. Following [1], we deduce for any \( j \in \mathbb{Z}_2 \), a well defined Thom \( R(G) \)-morphism

\[
\iota^j : K^j_G(F_G) \longrightarrow K^j_G(F'_G).
\]

More precisely, denote by \( \pi : N' \to F \) the bundle projection of the normal bundle \( N' \) to \( F \) in \( F' \), and let \( (\pi_F \circ \pi)^*(A^*(N \otimes \mathbb{C})) \) be the associated exterior algebra over \( N' \). Together with exterior multiplication by the underlying vector, this defines a complex over \( N' \) which is exact off the zero section \( F \subset N' \) and which is denoted \( \lambda(N \otimes \mathbb{C}) \) for simplicity. The usual Thom isomorphism \( K_G(F) \to K_G(N') \) is defined by assigning to a given compactly supported \( G \)-complex \((E, \sigma)\) over \( F \) the compactly supported \( G \)-complex over \( N' \) given by \( \pi^*((E, \sigma) \cdot \lambda(N \otimes \mathbb{C})) \). See [4] for more details. On the other hand, the total space of the fibration \( \pi : N' \to F \) is \( G \)-equivariantly diffeomorphic to a \( G \)-stable open tubular neighborhood \( p : U' \to F \) of \( d\iota(F) \) in \( F' \) and this allows to define classically the Gysin map \( \iota^j : K_G(F_G) \to K_G(F'_G) \). As explained in [1], if we only assume that \((E, \sigma)\) represents a class in \( K_G(F_G) \), then the complex \( \pi^*((E, \sigma) \cdot \lambda(N \otimes \mathbb{C}) \otimes \pi_F^*(\mathbb{C}(M')) \) extends to an element of \( K_G(F'_G) \). More precisely, if we assume that \((E, \sigma)\) is only compactly supported when restricted to \( F_G \), that is \( \text{Supp}(E, \sigma) \cap F_G \) is compact, then the \( G \)-complex \( \pi^*((E, \sigma) \cdot \lambda(N \otimes \mathbb{C}) \otimes \pi_F^*(\mathbb{C}(M')) \) is a \( G \)-complex over an open subspace \( U'_G \) of \( F'_G \) defined as follows. If we identify similarly the total space \( N \) with a \( G \)-stable open tubular neighborhood \( U \) of \( \iota(M) \) in \( M' \), then the foliation \( \mathcal{F} \) induces by restriction to the open submanifold \( U \) a foliation \( \mathcal{F}U \). Then \( U' \) can be naturally identified with the total space \( F'_U \) of the leafwise tangent bundle of the foliation \( \mathcal{F}U \). The subspace \( U'_G \) is then simply \( F'_G = F'_U \cap T_GU \). To sum up, we deduce in this way a well defined Thom homomorphism of \( R(G) \)-modules \( K_G(F_G) \to K_G(U'_G) \) (see again [1]). Since \( U'_G \) is an open subspace of the locally compact space \( F'_G \), the \( C^* \)-algebra \( C_0(U'_G) \) is a \( G \)-stable ideal in the \( G \)-algebra \( C_0(F_G) \) and we have the extension \( R(G) \)-morphism \( K_G(U'_G) \to K_G(F'_G) \). Composing the Thom homomorphism with this extension map, we end up with our Gysin \( R(G) \)-morphism

\[
\iota^j : K_G(F_G) \longrightarrow K_G(F'_G).
\]

Starting with a class in \( K^j_G(F_G) \) we get in the same way a class in \( K^j_G(F'_G) \) and we finally get the morphism

\[
\iota^j : K^j_G(F_G) \longrightarrow K^j_G(F'_G) \quad \text{for } j \in \mathbb{Z}_2.
\]

The \( G \)-embedding \( \iota \) gives a submersion \( M \to M'/F' \) in the sense of [23], we hence deduce from [23, Section 4] the well defined Connes-Skandalis Morita extension element \( \iota_* \in \text{KK}(C^*(M, \mathcal{F}), C^*(M', \mathcal{F}')) \). Indeed, the submanifold \( \iota(M) \) is automatically a transverse \( G \)-submanifold in \( (M', \mathcal{F}') \) which inherits a foliation \( \mathcal{F}^{(M)} \) which is diffeomorphic to \( (M, \mathcal{F}) \), hence identifying \( C^*(M, \mathcal{F}) \) with \( C^*(\iota(M), \mathcal{F}^{(M)}) \) and
using the $G$-equivariant Morita equivalence of $(\iota(M), \mathcal{F}^{(M)})$ with a foliation $(U, \mathcal{F}^U)$ obtained as an open tubular neighborhood of $\iota(M)$ in $M'$, we get the easy definition of the Connes-Skandalis map in our case.

**Theorem 5.1.** Let $\iota : (M, \mathcal{F}) \hookrightarrow (M', \mathcal{F}')$ be a $G$-equivariant embedding of smooth foliations as above, so we assume in particular that $\iota^* \mathcal{F}' \simeq \mathcal{F}$ and that $G$ acts by leaf-preserving holonomy diffeomorphisms on both foliations. Assume furthermore that $M$ is compact. Then for any $j \in \mathbb{Z}_2$, the following diagram commutes:

$$
\begin{align*}
\text{Ind}^{\mathcal{F}} & : K_j^G(F_G) \rightarrow K_j^G(F_G^U) \\
\text{Ind}^{\mathcal{F}'} & : K_j^G(C^*(\mathcal{F}', M', \mathcal{F}')) \rightarrow K_j^G(C^*(\mathcal{F}, M, \mathcal{F})).
\end{align*}
$$

Here the index morphism $\text{Ind}^{\mathcal{F}'}$ is defined according to Proposition [1].

**Proof.** For simplicity, we shall identify $(\iota(M), \mathcal{F}^{(M)})$ with $(M, \mathcal{F})$ and assume that $M$ is a smooth transverse submanifold to the foliation $(M', \mathcal{F}')$ whose foliation $\mathcal{F}$ coincides with the restricted foliation generated by $T\mathcal{F}|_{M'} \cap TM$. By definition of $\iota$, it is the composite map of a Thom morphism from $K_G(F_G) \rightarrow K_G(F_G^U)$ with the open extension corresponding to the inclusion of the open submanifold $U$. Notice that since the $G$-action on $(M, \mathcal{F})$ is a holonomy action, it is also a holonomy action on $(U, \mathcal{F}^U)$. By the excision theorem, the index morphism does automatically respect the latter open extension map and the following diagram commutes:

$$
\begin{align*}
\text{Ind}^{\mathcal{F}^U} & : K_j^G(F_G^U) \rightarrow K_j^G(F_G^U) \\
\text{Ind}^{\mathcal{F}'} & : K_j^G(C^*(G,C) \rightarrow K_j^G(C^*(G,C, \mathcal{F}))).
\end{align*}
$$

Now the open submanifold $U$ can be identified with a vector bundle $N \rightarrow M$ which is the normal bundle to $M$ in $M'$. Moreover, the foliation $\mathcal{F}^U$ is then identified with the foliation of $N$ whose leaves are given by the total spaces of the restrictions of the bundle $\pi_N : N \rightarrow M$ to the leaves of $(M, \mathcal{F})$. It is thus sufficient to show the theorem in the case of a real $G$-vector bundle $N$ over $M$ foliated by $F_N := \ker(TN \rightarrow TM/F) \simeq \pi_N(F \oplus N)$ and with $\iota : K_G(F_G) \rightarrow K_G(F_G^N)$ being the Thom homomorphism associated to the $0$-section $\iota : M \rightarrow N$. Following [1], we can write $N = P \times O(n) \mathbb{R}^n$, where $\pi_1 : P \rightarrow M$ is a $G$-equivariant $O(n)$-principal bundle over $M$, foliated by $FP := \ker(TP \rightarrow TM/F)$. Denote by $\pi_2 : P \times \mathbb{R}^n \rightarrow N$ the $G$-equivariant projection corresponding to moding out by the action of $O(n)$. Let $FP \times \mathbb{R}^n$ be the foliation given by $FP \times \mathbb{R}^n$ on $P \times \mathbb{R}^n$ and by $FP \times \mathbb{R}^n$ the tangent bundle to this foliation.

By using the product defined in [1] with $G$ replaced by $G \times O(n)$ and with trivial $H$, we obtain the well-defined product:

$$
K^j_{G \times O(n)}(F_{G \times O(n)}^P \otimes K_{G \times O(n)}(TP^\mathbb{R}^n) \rightarrow K^j_{G \times O(n)}(F_{G \times O(n)}^P \otimes K_{G \times O(n)}(TP^\mathbb{R}^n)).
$$

(17)

But since $O(n)$ acts freely on $P$, we also have the following identifications:

$$
q_1^* : K^j_{G}(F_G) \xrightarrow{\simeq} K^j_{G \times O(n)}(F_{G \times O(n)}^P \otimes K_{G \times O(n)}(TP^\mathbb{R}^n))
$$

and

$$
q_2^* : K^j_{G}(F_G) \xrightarrow{\simeq} K^j_{G \times O(n)}(F_{G \times O(n)}^N \otimes K_{G \times O(n)}(TP^\mathbb{R}^n)).
$$

Therefore, we end up with the product:

$$
K^j_{G}(F_G) \otimes K_{G \times O(n)}(TP^\mathbb{R}^n) \rightarrow K^j_{G}(F_G^N).
$$

(18)

Since $G$ acts trivially on $\mathbb{R}^n$, the inclusion $i : \{0\} \rightarrow \mathbb{R}^n$ induces the Bott morphism $i^* : R(G \times O(n)) \rightarrow K_{G \times O(n)}(TP^\mathbb{R}^n)$ and we have $\text{Ind}(i^*(1)) = 1 \in K_{G \times O(n)}(C, C)$, see [1]. Now, multiplication by $i^*(1)$ in (18) is exactly the Thom morphism that we denote $\zeta$ since $\zeta$ is the zero section here, i.e.

$$
\zeta : K^j_{G}(F_G) \rightarrow K^j_{G}(F_G^N).
$$

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Moreover, we may as well consider the multiplication by \( i_1(1) \) in the product \( \mathbb{R}^n \), and we then obviously have the following commutative diagram:

\[
\begin{array}{c}
K^j_G(F_G) \xrightarrow{\zeta i} K^j_G(F^N_G) \\
q_1 \downarrow \quad \downarrow q_2 \\
K^j_{G \times O(n)}(F^P_{G \times O(n)}) \xrightarrow{\mu(1)} K^j_{G \times O(n)}(F^P_{G \times O(n)})
\end{array}
\]

We deduce that for any \( a \in K^j_G(F_G) \):

\[
\text{Ind}^{P \times \mathbb{R}^n}(q_2^*(\zeta(a))) = \text{Ind}^{P \times \mathbb{R}^n}(q_1^*(a) \cdot i_1(1)) = j^{G \times O(n)}(1) \otimes \text{Ind}^P(q_1^*(a)) \otimes \mu(\mathbb{R}^n),
\]

where the last equality is a consequence of the multiplicativity axiom satisfied by our index morphism, as stated in Theorem 3.14 and where \( \mu(\mathbb{R}^n) \in \text{KK}(\mathbb{C}, C^*(\mathbb{R}^n \times \mathbb{R}^n)) \) is the Morita equivalence. On the other hand, by the axiom for free actions stated in Theorem 3.11 and denoting by \( \chi^O(O(n)) \) the trivial representation of \( O(n) \), we have:

\[
\text{Ind}^F(a) = \chi^O(O(n)) \otimes \text{Ind}^F(q_1^*(a)) \otimes \hat{\mathcal{E}}_{q_1}, \quad \text{while}
\]

\[
\text{Ind}^{F^N}(\zeta(a)) = \chi^O(O(n)) \otimes \text{Ind}^{F^N}(q_2^*(\zeta(a))) \otimes \hat{\mathcal{E}}_{q_2}
\]

We finally conclude by gathering the previous relations as follows:

\[
\text{Ind}^{F^N}(\zeta(a)) = \chi^O(O(n)) \otimes \left( \text{Ind}^F(q_1^*(a)) \otimes \mu(\mathbb{R}^n) \right) \otimes \hat{\mathcal{E}}_{q_2},
\]

\[
= \chi^O(O(n)) \otimes \text{Ind}^F(q_1^*(a)) \otimes \mu(\mathbb{R}^n) \otimes \hat{\mathcal{E}}_{q_1}
\]

\[
= \chi^O(O(n)) \otimes \text{Ind}^F(q_1^*(a)) \otimes \mu(\mathbb{R}^n) \otimes \hat{\mathcal{E}}_{q_1} \otimes \hat{\mathcal{E}}_{\zeta}
\]

\[
= \chi^O(O(n)) \otimes \text{Ind}^F(q_1^*(a)) \otimes \mu(\mathbb{R}^n) \otimes \hat{\mathcal{E}}_{q_1} \otimes \hat{\mathcal{E}}_{\zeta}
\]

\[
= \text{Ind}^F(a) \otimes \hat{\mathcal{E}}_{\zeta}
\]

We have used \( \hat{\mathcal{E}}_{q_1} \otimes \hat{\mathcal{E}}_{\zeta} = \mu(\mathbb{R}^n) \otimes \hat{\mathcal{E}}_{q_2} \) which is a consequence of the equality \( \zeta \circ q_1 = q_2 \circ s_0 \) where \( s_0 : P \to P \times \mathbb{R}^n \) is the zero section of this trivial bundle.

\[\Box\]

### 5.2 A topological index morphism

We prove the following important proposition.

**Proposition 5.2.** Let \((M, \mathcal{F})\) be a smooth foliated riemannian \(G\)-manifold such that \(G\) acts by leaf-preserving holonomy diffeomorphisms. Assume that we are given an isometric \(G\)-embedding \(M \to E\) of \(M\) in a finite dimensional euclidean \(G\)-representation \(E\). Let \(B \to F\) be the normal bundle to the \(G\)-embedding \(F \to T(E)\), obtained by restricting to \(F\), the induced \(G\)-embedding \(TM \to T(E)\). Denote by \(A\) the total space of the vector \(G\)-bundle over \(T(M)\) which is the pull-back, under the \(G\)-projection \(T(M) \to F\) given by the \(G\)-invariant metric, of the vector \(G\)-bundle \(B \to F\). Then the composite map of the zero section \(T(M) \to A\) with the obvious inclusion \(F \to T(M)\), yields the \(G\)-embedding \(i : F \to A\) which satisfies the following properties:
1. The map $i$ is $K$-oriented by a complex $G$-structure;

2. $A$ is diffeomorphic to a smooth $G$-submanifold $\mathcal{A}$ of the cartesian product $M \times T(E)$, which is an open transversal to the smooth foliation $\mathcal{F} \times 0$;

3. Setting $\mathcal{A}_G := \mathcal{A} \cap (M \times T_G(E))$, the usual Thom construction yields, for $j \in \mathbb{Z}_2$, a well defined $R(G)$-homomorphism $\iota : K^j_G(F_G) \to K^j_G(\mathcal{A}_G)$;

4. The space $\mathcal{A}_G$ is a topological ($G$-stable) transversal to the lamination $(M \times T_G(E), \mathcal{F} \times 0)$.

Before giving the proof of Theorem 5.2, we point out that, exactly as in the case of smooth foliations, the topological transversal $\mathcal{A}_G$ to the lamination $(M \times T_G(E), \mathcal{F} \times 0)$, obtained in the fourth item, gives rise to a well defined quasi-trivial Morita extension class $\epsilon \in \text{KK}(C_0(\mathcal{A}_G), C^* (M \times T_G(E), \mathcal{F} \times 0))$ and hence the $R(G)$-morphism

$$\epsilon : K^j_G(\mathcal{A}_G) \to K^j_G(\mathcal{A}_G)$$

The class $\epsilon$ can be described as follows. By using that the normal bundle to $\mathcal{A}$ in $M \times T(E)$ is isomorphic to the vector bundle $F \times 0$ that we restrict to $\mathcal{A}$, we may consider an open tubular neighborhood $\mathcal{N}$ of $\mathcal{A}$ in $M \times T(E)$ which is a disc-bundle over $\mathcal{A}$ whose fibers are small disc-placques which correspond to the restricted foliation $\mathcal{F} \times 0$. It is then easy to construct by clear transversality that $\mathcal{N}_G := \mathcal{N} \cap (M \times T_G(E))$ is also a disc-fibration by the same plaques but now over the space $\mathcal{A}_G$, so the base is no more a smooth manifold. The $C^*$-algebra $C^*(\mathcal{N}_G, \mathcal{F}^{\text{NG}})$ of the open subspace $\mathcal{N}_G$ which is the restriction of the foliation $\mathcal{F} \times 0$ of $M \times T_G(E)$, is then Morita equivalent to $C_0(\mathcal{A}_G)$. Hence using the trivial extension map

$$K^j_G(C^*(\mathcal{N}_G, \mathcal{F}^{\text{NG}})) \to K^j_G(C^*(M \times T_G(E), \mathcal{F} \times 0)) \simeq K^j_G(C_0(T_G(E), C^*(M, \mathcal{F})))$$

corresponding to the open subspace $\mathcal{N}_G$ in the space $M \times T_G(E)$, we finally obtain the allowed quasi-trivial $G$-equivariant extension map $\epsilon$.

Proof. Denote by $i$ the embedding $M \hookrightarrow E$ and by $i_*$ its tangent map and let $\nu_F = TM/F$ be the normal bundle to the foliation $(M, \mathcal{F})$ that we identify with the orthogonal bundle of $F$ in $TM$. Then we identify the bundle $B \to F$ with $\pi_F^*(N(i) \oplus N)$ where $N(i)$ is the normal bundle to the embedding $i$ and $N$ is the normal bundle to the leaves of $(M, \mathcal{F})$ inside $E$, so $N = N(i) \oplus \nu_F$ and all direct sums are orthogonal here. As we shall see below we only need to work with small enough sections $W$ of $\pi_F^*(N(i) \oplus N)$. A straightforward computation shows that the normal $G$-bundle to $F$ in the composite embedding $F \hookrightarrow T(M) \hookrightarrow A$ is $G$-equivariantly isomorphic to the $G$-bundle

$$\pi_F^*(N \oplus \nu_F) \simeq \pi_F^*(N \oplus \mathbb{C}).$$

This proves the first item.

Let now $h : A \hookrightarrow M \times TE$ be the smooth map defined by

$$h((x, \xi, \eta), W) := (x, (i(x) + i_*(\eta) + W_1, i_*(\xi) + W_2)),$$

for $(x, \xi, \eta) \in TM = F \oplus \nu_F$ and for $W = (W_1, W_2) \in B_{(x, \xi)} \simeq N(i)_x \oplus N_x$. Since the sums $i_*(\eta) + W_1$ and $i_*(\xi) + W_2$ are orthogonal, the map $h$ is clearly injective. Moreover, by a similar but slightly more involved verification, one also shows that $h$ is immersive and that it is a $G$-embedding. The restriction $h_0$ of $h$ to $F$, viewed as submanifold of the zero section, is given by

$$h_0 : (x, \xi) \mapsto (x, (i(x), i_*(\xi))).$$

This is clearly transverse to the foliation $\mathcal{F} \times 0$ of $M \times T(E)$. Hence the same is true for the range of the zero section $T(M)$, and therefore a small enough open neighborhood of $T(M)$ will still be transverse to $\mathcal{F} \times 0$, with dimension exactly equal to the codimension of this latter foliation of $M \times T(E)$. Hence the second item is now proved. The third and fourth items are eventually easily deduced by standard arguments that we already explained in the previous section, see [1] and [24].
Following Kasparov, we define a Dirac element $[D_E] \in \text{KK}(C_0(T_G(E)) \rtimes G, \mathbb{C})$ which, according to the main result of [37], computes the index of $G$-invariant $G$-transversally elliptic operators on the orthogonal $G$-representation $E$, through the descent morphism $j^G$. There are though some technical details which are passed over here and which would need to be expanded elsewhere. One especially needs to replace $C_0(T_G(E)$ by a better (although non-separable) symbol $C^*$-algebra denoted by $\mathcal{S}_G(E)$ in [37], and therefore one needs as well to use the extended version of Kasparov’s KK-theory, adapted to non-separable algebras. All these details with their generalizations to foliations will be dealt with in a forthcoming paper.

We only mention here that since $C^*(M,F)$ is endowed with the trivial $G$-action, we have a well defined morphism

$$K^G_j(C_0(T_G(E)) \otimes C^*(M,F)) \xrightarrow{j^G} \text{KK}^j(C^*G, [C_0(T_G(E)) \rtimes G] \otimes C^*(M,F)) \xrightarrow{[D_E]} \text{KK}^j(C^*G, C^*(M,F)),$$

that we denote by $\partial_E \otimes C^*(M,F)$. Roughly speaking and using the main result of [37], the map $\partial_E \otimes C^*(M,F)$ is the expected index map for $G$-invariant $G$-transversally elliptic operators on $E$ with coefficients in the $G$-trivial $C^*$-algebra $C^*(M,F)$.

**Remark 5.3.** The composite morphism $\text{Ind}^{F,\text{top}}$:

$$\text{Ind}^{F,\text{top}} : K^j_G(F_G) \xrightarrow{i_G} K^j_G(A_G) \xrightarrow{\varphi} K^j_G(C_0(T_G(E)) \otimes C^*(M,F)) \xrightarrow{\partial_E \otimes C^*(M,F)} \text{KK}^j(C^*G, C^*(M,F))$$

is independent of the choice of euclidean $G$-representation $E$ with the isometric $G$-embedding $i$.

**Definition 5.1.** The $R(G)$-morphism

$$\text{Ind}^{F,\text{top}} : K^j_G(F_G) \longrightarrow \text{KK}^j(C^*G, C^*(M,F)),$$

will be called the topological index morphism for $G$-invariant leafwise $G$-transversally elliptic operators.

If $G$ is the trivial group then the topological index morphism reduces to the topological index morphism for leafwise elliptic operators as defined in [23], and it then coincides with the analytic index morphism $\text{Ind}^F$, this is precisely the Connes-Skandalis index theorem. For general $G$ and when the foliation is top dimensional, the naturality of the index distribution proved in [11] together with the Kasparov index theorem proved in [37] implies again the equality of the topological index morphism with the analytical one.

**Remark 5.4.** When $G$ and $M$ are no more compact, but the $G$-action is supposed to be proper and cocompact as in [37], then the proofs given here allow to still define the index morphism

$$\text{Ind}^{M,F} : K^j_G(F_G) \longrightarrow \text{KK}^j(C_0(M) \rtimes G, C^*(M,F)).$$

We finally point out that most of the constructions given in the present paper apply, with minor changes, to the more general category of foliated spaces (or laminations) as studied in [10] using sections and operators which are leafwise smooth and transversally continuous. However, the construction of the topological index for instance is not clear in general since the $G$-embedding in $E$ is not insured a priori.

## A Unbounded version of the index class

We define in this appendix the index class for operators of order 1, using the unbounded version of Kasparov’s theory [6]. The unbounded version simplifies the computation of some Kasparov products and was used in the present paper.

**Definition A.1** (Unbounded Kasparov module [6]). Let $A$ and $B$ be $C^*$-algebras. An $(A,B)$-unbounded Kasparov cycle $(E, \phi, D)$ is a triple where $E$ is a Hilbert $B$-module, $\phi : A \rightarrow \mathcal{L}(E)$ is a graded $*$-homomorphism and $(D, \text{dom}(D))$ is an unbounded regular seladjoint operator such that:
1. \((1 + D^2)^{-1} \phi(a) \in k(E), \forall a \in A,\)

2. The subspace of \(A\) composed of the elements \(a \in A\) such that \(\phi(a)(\text{dom}(D)) \subset \text{dom}(D)\) and \([D, \phi(a)] = D\phi(a) - \phi(a)D\) is densely defined and extends to an adjointable operator on \(E\), is dense in \(A\).

When \(E\) is \(\mathbb{Z}_2\)-graded with \(D\) odd and \(\pi(a)\) even for any \(a\), we say that the Kasparov cycle is even. Otherwise, it is odd.

In \([6]\), appropriate equivalence relations are introduced on such (even/odd) unbounded Kasparov cycles, which allow to recover the groups \(KK^* (A, B)\). When the compact group \(G\) acts on all the above data, one recovers similarly \(KK^*_G (A, B)\) by using the equivariant version of the Baaj-Julg unbounded cycles of the previous definition.

To defined our index class for positive order operators, we first need to show that \(G\)-invariant leafwise \(G\)-transversally elliptic operators do define regular operators. We will use here notations and discussions from \([46, 47]\). We work with \(E = M \times \mathbb{C}\) for simplicity. Let \(P\) be a leafwise pseudodifferential operator on \(M\), we denote by \(P^\sharp\) the formal adjoint of \(P\). Recall that we have

\[
\langle P\eta, \eta' \rangle = \langle \eta, P^\sharp \eta' \rangle, \quad \eta, \eta' \in \mathcal{C}_c^\infty (G).
\]

The operator \(P\) is densely defined with domain \(\mathcal{C}_c^\infty (G)\) and has a well defined closure \(\bar{P}\) with graph

\[
G (\bar{P}) = \overline{G(P)} = \{ (u, v) \in C^*(M, F)^2, \exists (u_n) \in \mathcal{C}_c^\infty (G), \| u_n - u\| \to 0 \text{ and } \| Pu_n - v\| \to 0 \}.
\]

The same observation holds for the leafwise pseudodifferential operator \(P^\sharp\). So we obtain by continuity that

\[
\langle u, y \rangle = \langle v, x \rangle, \quad \forall (u, v) \in G(\bar{P}) \text{ and } (x, y) \in G(\overline{P^\sharp})).
\]

So \(\overline{P^\sharp} \subset P^*\). The operators \(\bar{P}\) and \(\overline{P^\sharp}\) being densely defined., we recall Lemma 20 of \([47]\) in the case of integer order.

**Lemma A.2.** \([47]\) Let \(A, B\) be compactly supported pseudodifferential operator on \(G\), such that \(\text{ord} A + \text{ord} B \leq 0\) and \(\text{ord} B \leq 0\). Then we have \(\overline{AB} = \overline{BA}\), an equality of adjointable operators.

The following theorem and lemma are generalizations of Proposition 3.4.9 and Lemma 3.4.10 of \([46]\), see also \([47]\).

**Theorem A.3.** Let \(P\) be a formally selfadjoint, \(G\)-invariant leafwise \(G\)-transversally elliptic operator. Then the closure \(\bar{P}\) of \(P\) is a regular operator.

**Lemma A.4.** Let \(P\) be a formally selfadjoint, \(G\)-invariant leafwise \(G\)-transversally elliptic operator of order \(m > 0\). Let \(\Delta_G\) be the Laplace operator along the orbits introduced in Subsection 1.2 and set \(D_G = (1 + \Delta)^{-m/2} \Delta_G (1 + \Delta)^{-m/2}\). Then the operator \(A := (P, D_G) : \mathcal{C}_c^\infty (G) \to \mathcal{C}_c^\infty (G) \oplus \mathcal{C}_c^\infty (G)\) has injective symbol and \(A^2 A = P^2 + D_G^2\) is leafwise elliptic. Moreover, denote by \(Q\) a parametrix for \(A^2A\) and set \(R = \text{id} - (P^2 + D_G^2)Q\) and \(S = \text{id} - Q(P^2 + D_G^2)\), then we have:

1. \(\overline{P^2} Q = \overline{P^2} Q = \overline{P^2} Q\)
2. \(\text{dom} (\overline{P^2}) = \text{im} (\overline{Q}) + \text{im} (\overline{S})\)
3. \(\overline{P^2} = \overline{P^2}\)
4. \(\overline{P^2} = \overline{P^2} = (P^*)^2 = (P^2)^*\).
Theorem A.3. Let us prove that $\bar{P}$ is regular, this will imply that $\bar{P}$ is regular by the spectral theorem, see [15] Theorem 14.29. Indeed, notice that 1 + $|P|^2 = 1 + P^2$. We have $G(\bar{P}) = \{(Qx + Sy, P^2Qx + P^2Sy) : (x, y) \in \mathcal{E} \times \mathcal{E}\}$. Denote by $U = \left(\frac{Q}{P^2Q}, \frac{S}{P^2S}\right)$. Then $U \in \mathcal{L}(\mathcal{E} \oplus \mathcal{E})$, since $Q$, $S$, $P^2Q$, $P^2S$ are leafwise pseudodifferential operators of negative order and extend to adjointable operators of $\mathcal{E}$. Moreover, the range of $U$ is closed. In fact, $U(x_n, y_n)$ converges if and only if $(Qx_n + Sy_n, P^2Qx_n + P^2Sy_n)$ converges and $G(\bar{P})$ is closed. This implies that $G(\bar{P}) = \text{im}(U)$ is orthocomplemented in $\mathcal{E} \oplus \mathcal{E}$. Therefore using Lemma 14.12 of [15] we conclude.
In the sequel we will denote simply by $P$ the regular operator obtained from a formally selfadjoint $G$-invariant leafwise $G$-transversally elliptic operator. We are now in position to state the main result of this appendix.

**Theorem A.6.** Let $P_0 : C^\infty_c(G, r^*E^+) \to C^\infty_c(G, r^*E^-)$ be a $G$-invariant leafwise $G$-transversally elliptic pseudodifferential operators of order 1, and let $P$ be the associated regular self-adjoint operator defined by

$$
\begin{pmatrix}
0 & P_0^* \\
P_0 & 0
\end{pmatrix}.
$$

Then the triple $(\mathcal{E}, \pi, P)$ is an even $(C^*(G), C^*(M, \mathcal{F}))$-bounded Kasparov cycle, which defines a class in $\text{KK}_G(C^*(G), C^*(M, \mathcal{F}))$. The similar statement holds in the ungraded case giving a class in $\text{KK}^1_G(C^*(G), C^*(M, \mathcal{F}))$.

**Proof.** By Proposition 1.4, the representation $\pi$ is $G$-equivariant. For any $\varphi \in C(G)$, it is easy to see that $\pi(\varphi)$ preserves the domain of $P$ and by Remark 1.5, we have $[\pi(\varphi), P] = 0$. It remains to check that $(1 + P^2)^{-1} \circ \pi(\varphi) \in K(\mathcal{E})$. We may take for our operator $\Delta_G$ the Casimir operator, which is a leafwise differential operator of order 2. We already noticed that the operator $P^2 + \Delta_G$ is elliptic since $A = (P, d_G)$ has injective symbol and $A^*A = P^2 + d_G d_G = P^2 + \Delta_G$ is leafwise elliptic. We hence deduce that the resolvent $(1 + P^2 + \Delta_G)^{-1}$ is a compact operator in $\mathcal{E}$.

We now show that for any $\varphi \in C^\infty(G)$, the operator

$$(1 + P^2 + \Delta_G)^{-1} \circ \pi(\varphi) - (1 + P^2)^{-1} \circ \pi(\varphi),$$

is compact, which will insure that $(1 + P^2) \circ \pi(\varphi)$ is also compact. Denote by $\tilde{\Delta}_G$ the Laplacian on $G$ viewed as a riemannian $G$-manifold. Using that $\Delta_G \pi(\varphi) = \pi(\tilde{\Delta}_G \varphi)$ and that $[\pi(\varphi), P] = 0$, we have:

$$(1 + P^2 + \Delta_G)^{-1} \circ \pi(\varphi) - (1 + P^2)^{-1} \circ \pi(\varphi) = - (1 + P^2 + \Delta_G)^{-1} \Delta_G (1 + P^2)^{-1} \circ \pi(\varphi) = -(1 + P^2 + \Delta_G)^{-1} \pi(\tilde{\Delta}_G \varphi)(1 + P^2)^{-1}.$$

Now since $\pi(\tilde{\Delta}_G \varphi)$ and $(1 + P^2)^{-1}$ are adjointable operators and since $(1 + P^2 + \Delta_G)^{-1}$ is compact, we deduce that $(1 + P^2 + \Delta_G)^{-1} \circ \pi(\varphi) - (1 + P^2)^{-1} \circ \pi(\varphi)$ is a compact operator on $\mathcal{E}$.

**Remark A.7.** The previous proof can be easily adapted to any positive order. We though have restricted ourselves to first order operators since this is the only case used in the present paper.

**Definition A.8.** The index class $\text{Ind}_G^G(P_0)$ of a $G$-invariant leafwise $G$-transversally elliptic pseudodifferential operator of positive order, is the class in $\text{KK}^*_G(C^*(G), C^*(M, \mathcal{F}))$ of the $(C^*(G), C^*(M, \mathcal{F}))$-bounded Kasparov cycle $(\mathcal{E}, \pi, P)$.

The relation with the bounded version is obtained by using the Woronowicz transform, see [6]. More precisely, if $P_0$ is a $G$-invariant leafwise $G$-transversally elliptic pseudodifferential operator of order 1 for instance, then the triple $(\mathcal{E}, \pi, P(1 + P^2)^{-1/2})$ is a (bounded) Kasparov cycle.

**B The technical proposition in the non-compact case**

Let us fix a non compact foliated manifold $(U, \mathcal{F}^U)$ and denote by $\mathcal{P}^0(U, \mathcal{F}^U, \mathcal{E})$ the $C^*$-subalgebra of the adjointable operators $\mathcal{L}_{C^*_{c}(U, \mathcal{F}^U)}(\mathcal{E})$, which is generated by the closures of the zero-th order pseudodifferential operator with symbols in $C^\infty_c(S^*F^U, \text{End}(\mathcal{E}))$, see [23] [41] [47]. Here $\mathcal{E}$ is the Hilbert $C^*(U, \mathcal{F}^U)$-module defined before for the foliation $(U, \mathcal{F}^U)$. Then setting

$$K_U(\mathcal{E}) := \{T \in \mathcal{L}_{C^*_{c}(U, \mathcal{F}^U)}(\mathcal{E}) \mid Tf \& ft \in K_{C^*(U, \mathcal{F}^U)}(\mathcal{E}), \forall f \in C_0(U)\}$$

and

$$\mathcal{P}^0(U, \mathcal{F}^U, \mathcal{E}_U) := \{T \in \mathcal{L}_{C^*_{c}(U, \mathcal{F}^U)}(\mathcal{E}) \mid Tf \& ft \in \mathcal{P}^0(U, \mathcal{F}^U, \mathcal{E}), \forall f \in C_0(U)\},$$

47
the following exact sequence holds (see [23 Proposition 4.6]):

\[ 0 \to \mathcal{K}_U(E) \xrightarrow{\Psi^0(U,F^U,E)_U} \mathcal{F}_b(S^*F^U,\text{End}(E)) \to 0. \tag{19} \]

**Proposition B.1.** Let \((U,F^U)\) be a (non compact) foliated manifold. Let \(A \in \Psi^0(U,F^U,E)_U\) be selfadjoint. Suppose that the principal symbol \(\sigma_A\) of \(A\) satisfies

\[ \forall \varepsilon > 0, \exists c > 0 \text{ such that } \forall (x, \xi) \in F^U : \|\sigma_A(x, \xi)\| \leq c \|\sigma_A(x, \xi) + \varepsilon. \tag{20} \]

Then \(\forall \varepsilon > 0\), there exist two selfadjoint operators \(R_1\) and \(R_2 \in \mathcal{K}_U(E)\) such that:

\[ -(cQ + \varepsilon + R_1) \leq A \leq cQ + \varepsilon + R_2 \text{ as self-adjoint operators on } E. \]

**Proof.** Notice that \(\sigma(Q)(x, \xi) \leq 1\). Therefore the inequality (20) is true in \(\mathcal{F}_b(S^*F^U,\text{End}(E))\). This implies that \(0 \leq \sigma(A)(x, \xi) + c\sigma(Q)(x, \xi) + \varepsilon\) again in \(\mathcal{F}_b(S^*F^U,\text{End}(E))\). Now using the exact sequence (19), we get that there is \(R_1 \in \mathcal{K}(E)_U\) such that \(0 \leq A + cQ + \varepsilon + R_1\), in other words we get

\[ -(R_1 + cQ + \varepsilon) \leq A. \]

Now replacing \(\sigma(A)\) by \(-\sigma(A)\) we also get the existence of an \(R_2 \in \mathcal{K}_U(E)\) such that \(0 \leq -A + cQ + \varepsilon + R_2\), in other words we get

\[ A \leq (R_2 + cQ + \varepsilon). \]

Gathering this inequalities as in Proposition [23] we hence obtain

\[ -(cQ + \varepsilon + R_1) \leq A \leq cQ + \varepsilon + R_2 \text{ as self-adjoint operators on } E \tag{21} \]

with each \(R_i \in \mathcal{K}_U(E)\) as allowed.

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