Sampling expansions associated with quaternion difference equations

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ABSTRACT
Starting with a quaternion difference equation with boundary conditions, a parameterized sequence that is complete in the finite-dimensional quaternion Hilbert space is derived. By employing the parameterized sequence as the kernel of the discrete transform, we construct a quaternion function space whose elements have sampling expansions. It is shown that the sample points of the sampling expansions are the eigenvalues of an associated quaternion difference operator. For the operator that has multiple standard eigenvalues, we provide a method to find the ‘missing sample points’. Through formulating the boundary-value problems, we make a connection between a class of tridiagonal quaternion matrices and the polynomials with quaternion coefficients. We prove that for a tridiagonal symmetric quaternion matrix, one can always associate a quaternion characteristic polynomial whose roots are the eigenvalues of the matrix. By representing colour image pixels as quaternions, an application of the proposed sampling theorem in colour image encryption is discussed. Throughout the paper, examples are given to illustrate the results.

1. Introduction
Sampling theorems for general integral transforms other than the Fourier one were discussed by Kramer [1]. Kramer’s result generalizes the Shannon sampling theorem and it can be flexibly used in many areas of physics and engineering. A necessary condition for Kramer’s sampling theorem is that the kernel of the integral transform has to be capable of generating an orthogonal basis in a certain Hilbert space. It is known that the kernels in Kramer’s theorem can be extracted from some boundary-value problems of differential equations (see e.g. [2]). Accordingly, sampling theorems associated with several types of boundary-value problems were extensively investigated [2,3]. Based on difference operators, the author in [4] provided a discrete version of Kramer’s theorem, and numerous
studies of sampling expansions associated with difference equations were introduced in a series of papers [5–7].

The purpose of this paper is to derive the sampling expansions associated with quaternion difference equations. The quaternion algebra is a hypercomplex number system, more precisely, a four-dimensional associative normed division algebra over the real numbers. It has wide applications in many fields like colour image processing [8] and computer graphics [9]. The quaternion-based model for colour images not only preserves the correlation among colour channels but also the orthogonal property for the coefficients of different channels, which achieves a structured representation [8]. The representations of rotations by quaternions [9] are more compact than the representations by Euler angles. Recently, the theory of quaternion dynamic equations (including quaternion difference equations) [10–13] has received a lot of attention because many physical problems can be described as quaternion dynamic models [14–17]. The current paper discusses the sampling theory of the quaternion-valued functions and provides a novel treatment for the functions from \( \mathbb{R}^4 \) to \( \mathbb{R}^4 \), where \( \mathbb{R} \) denotes the totality of real numbers. Under the framework of quaternions and difference equations, a multidimensional function \( (\mathbb{R}^4 \rightarrow \mathbb{R}^4) \) is treated as a quaternion-valued function of one variable, that is, a function from \( \mathbb{H} \) to \( \mathbb{H} \), where \( \mathbb{H} \) denotes the totality of quaternions. Since \( \mathbb{R} \) can be viewed as a subset of \( \mathbb{H} \), some results of this paper will reduce to the results of the classical one-dimensional case. It is worth noting that the quaternion-based treatment in this paper could deal with the problems that were unmanageable by the traditional approaches. For example, in the traditional case, only the self-adjoint second-order difference operators can produce sampling formulas, or more concretely, if the difference operator is normal, the sampling formula will encounter a problem of lacking samples. In this paper, however, if the difference operator is normal, the ‘missing sample points’ can be found by a quaternion-based method. Since this paper studies the sampling formulas for general discrete transforms other than the Fourier one. The sampling formulas are different from those concerning the bandlimited functions in the sense of Fourier transform [18,19] and linear canonical transform [20,21].

In the quaternion analysis setting, the study of difference equations [22], matrix theory [23–25] and polynomial root-finding [26,27] is essentially different from the traditional case due to the non-commutative property of quaternions. To study the sampling theory for the quaternion-valued functions defined on the quaternion skew field (or equivalently \( \mathbb{R}^4 \)), we cannot directly use the related results such as the spectral theorem of matrices and the structure of zeros of polynomials in the real or complex fields. Thus formulating the sampling expansions for the quaternion-valued functions has to be based on the distinctive structure of quaternions. This paper involves the eigenvalue problem of quaternion matrices, the spectral property of operators in quaternion Hilbert spaces and the zeros of quaternion polynomials. Starting with a quaternion difference equation with boundary conditions, a parameterized sequence that is complete in the finite-dimensional quaternion Hilbert space is obtained. By employing the parameterized sequence as the kernel of the discrete transform, we construct a quaternion function space whose elements have sampling expansions. The sample points are not only the eigenvalues of a certain quaternion matrix but also the zeros of a relevant quaternion polynomial. We show that for a tridiagonal symmetric quaternion matrix, one can always associate a quaternion polynomial whose roots are the eigenvalues of the matrix. Besides, an application of the proposed
sampling theorem in colour image encryption is discussed. The main contributions of this paper are highlighted as follows:

1. We present the sampling expansions associated with quaternion difference equations (Theorem 3.6). The quaternion-valued functions considered in the paper are defined on the quaternion skew field and the sample points involved in the interpolation formulas are non-uniformly spaced in \( \mathbb{R}^4 \).
2. We provide a method to compute the sampling expansions (see the computing process presented before Remark 3.6). This method is applicable even for the case that the associated quaternion difference operator has multiple standard eigenvalues.
3. We investigate the characteristic polynomials for the tridiagonal symmetric quaternion matrices (Theorem 4.2). It is shown that \( \gamma \) is an eigenvalue of such a matrix if and only if \( \gamma \) is similar to a zero of the characteristic polynomial.
4. By representing colour image pixels as quaternions, an application of the proposed sampling theorem in colour image encryption is discussed (see Section 5).

The rest of the paper is organized as follows. In Section 2, some useful results of the quaternion algebra are reviewed. Besides, a lemma of quaternion polynomials is presented. Section 3 is devoted to the sampling expansions associated with quaternion difference equations. In Section 4, we investigate the characteristic polynomials for the tridiagonal symmetric quaternion matrices. An application of the proposed sampling theorem in colour image encryption is discussed in Section 5. Finally, conclusion and discussion are drawn at the end of the paper.

2. Preliminaries

2.1. Quaternions and matrices of quaternions

The skew field of quaternions [28] denoted by \( \mathbb{H} \) is the four-dimensional algebra over \( \mathbb{R} \) with basis \( \{1, i, j, k\} \). The elements \( i, j \) and \( k \) obey Hamilton’s multiplication rules:

\[
i j = -ji = k, \quad jk = -kj = i, \quad ki = -ik, \quad i^2 = j^2 = ijk = -1.
\]

For each quaternion \( q = q_0 + q_1 i + q_2 j + q_3 k \), its scalar part and vector part are \( q_0 \) and \( q_1 i + q_2 j + q_3 k \) respectively. The conjugate of \( q \) is defined by \( \overline{q} = q_0 - q_1 i - q_2 j - q_3 k \) and its norm is given by \( |q| = \sqrt{\overline{q}q} \). Using the conjugate and norm of \( q \), one can define the inverse of \( q \in \mathbb{H} \setminus \{0\} \) by \( q^{-1} = \overline{q}/|q|^2 \). Observe that the set \( \mathbb{C} \) of complex numbers appears as a sub-algebra of \( \mathbb{H} \): \( \mathbb{C} = \text{Span}_{\mathbb{R}} \{1, i\} \), thus we will view \( \mathbb{C} \) as a subset of \( \mathbb{H} \).

Let \( M_{m \times n}(\mathbb{H}) \), simply \( M_n(\mathbb{H}) \) when \( m = n \), denote the set of all \( m \) by \( n \) matrices with entries from \( \mathbb{H} \). Just as with the complex case, a square matrix \( A \in M_n(\mathbb{H}) \) is said to be normal if \( A^*A = AA^* \), unitary if \( AA^* = I \) and invertible (non-singular) if \( AB = BA = I \) for some \( B \in M_n(\mathbb{H}) \). Here \( A^* \) is the conjugate transpose of \( A \). The concept of eigenvalues for quaternion matrices is somewhat different from the complex case. Owing to the non-commutativity of quaternions, there are two types (left and right) of eigenvalues for quaternion matrices. A vector \( \xi \in \mathbb{H}^n \setminus \{0\} \) is said to be a right (left) eigenvector of \( A \) corresponding to the right (left) eigenvalue \( \lambda \in \mathbb{H} \) provided that

\[
A\xi = \xi \lambda \quad (A^*\xi = \lambda \xi)
\]
holds. In the sequel we will only consider the right eigenvalues and the right eigenvectors, so we will use the terminology eigenvalues and eigenvectors for simplicity.

A matrix \( B_1 \) is said to be similar to a matrix \( B_2 \) if \( B_2 = S^{-1}B_1S \) for some non-singular matrix \( S \). In particular, we say that two quaternions \( p, q \) are similar if \( p = \alpha^{-1}q\alpha \) for some nonzero \( \alpha \in \mathbb{H} \). From [23,29,30], we know that any matrix \( A \in \mathbb{M}_n(\mathbb{H}) \) has exactly \( n \) eigenvalues (including multiplicity) which are complex numbers with nonnegative imaginary parts. These eigenvalues are called standard eigenvalues. Like the complex case, any two similar quaternion matrices have the same eigenvalues. We denote the totality of eigenvalues \( \lambda \). If any element of \( H \) is a right eigenvector of \( A \), it means that \( A\xi = (\xi\alpha)(\alpha^{-1}\lambda\alpha) \) for all nonzero \( \alpha \in \mathbb{H} \). This means that \( \xi\alpha \) is an eigenvector of \( A \) corresponding to eigenvalue \( \alpha^{-1}\lambda\alpha \) rather than \( \lambda \). For any \( q \in \mathbb{H} \), its similarity orbit [31] is defined by

\[
\theta(q) := \{ \alpha^{-1}q\alpha : \alpha \in \mathbb{H} \setminus \{0\} \} = \{ \alpha q\overline{\alpha} : \alpha \in \mathbb{H}, |\alpha| = 1 \}.
\]

It follows that if \( \theta(q) \cap \sigma(A) \neq \emptyset \), then \( \theta(q) \subset \sigma(A) \). The similarity orbit \( \theta(q) \) contains infinitely many elements for \( q \in \mathbb{H} \setminus \mathbb{R} \), but only two of them are complex numbers. Furthermore, two similarity orbits are disjoint, as described by the following lemmas.

**Lemma 2.1 ([31]):** For every \( q \in \mathbb{H} \setminus \mathbb{R} \), there is a non-real \( z \in \mathbb{C} \) such that \( \theta(q) \cap \mathbb{C} = \{z, \overline{z}\} \).

**Lemma 2.2 ([18]):** If \( \theta(p) \cap \theta(q) \neq \emptyset \), then \( \theta(p) = \theta(q) \).

### 2.2. Quaternion Hilbert spaces

An abelian group \( H \) is a right \( \mathbb{H} \)-module [31] if there is a right scalar multiplication map \((u, \alpha) \mapsto u\alpha \) from \( H \times \mathbb{H} \) into \( H \) such that for all \( u, v \in H \) and \( \alpha, \beta \in \mathbb{H} \)

\[
u(\alpha + \beta) = u\alpha + u\beta, \quad (u + v)\alpha = u\alpha + v\alpha, \quad u(\alpha\beta) = (u\alpha)\beta, \quad u1 = u.
\]

A subset \( V = \{u_1, u_2, \ldots, u_m\} \) of \( H \) is said to be (right) \( \mathbb{H} \)-independent if

\[
\sum_{l=1}^{m} u_l\alpha_l = 0, \quad \alpha_l \in \mathbb{H} \quad \text{implies that} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0.
\]

If any element of \( H \) can be expressed by a right \( \mathbb{H} \)-linear combination of \( V \subset H \), then \( V \) is called a basis of \( H \) and the dimension of \( H \) is \( m \).

A right \( \mathbb{H} \)-module \( H \) is called a quaternion pre-Hilbert space if there exists a quaternion-valued function (inner product) \( \langle \cdot, \cdot \rangle : H \times H \to \mathbb{H} \) such that for all \( u, v, w \in H \) and \( \alpha, \beta \in \mathbb{H} \):

1. \( \langle u, v \rangle = \overline{\langle v, u \rangle} \),
2. \( \langle u, \alpha v + w\beta \rangle = \langle u, v \rangle\alpha + \langle u, w \rangle\beta \),
3. \( \langle u, u \rangle \geq 0 \) and \( \langle u, u \rangle = 0 \) if and only if \( u = 0 \).

The function \( u \mapsto \|u\| = \sqrt{\langle u, u \rangle} \) is a norm on \( H \). Under this norm, the Cauchy–Schwarz inequality and the triangular inequality (see [32]) hold as \( |\langle u, v \rangle| \leq \langle u, u \rangle^{1/2} \langle v, v \rangle^{1/2} \).
and \( \|u + v\| \leq \|u\| + \|v\| \) respectively. Moreover, if \( H \) is complete under the norm \( \|\cdot\| \), then it is called a quaternion Hilbert space. For any \( n \)-dimensional Hilbert space \( H \), by the quaternion version of Gram–Schmidt theorem \cite{31}, there exists a basis \( V = \{u_1, u_2, \ldots, u_n\} \) of \( H \) such that \( \|u_k\| = 1 \) for every \( 1 \leq k \leq n \) and \( \langle u_s, u_t \rangle = 0 \) for \( s \neq t \). Such a basis is called an orthonormal basis. For every \( v \in H \), it can be expanded as \( v = u_1(v_1) + u_2(v_2) + \cdots + u_n(v_n) \).

A right \( \mathbb{H} \)-linear operator is a function \( T : H \to H \) such that \( T(uc + v\beta) = T(u)\alpha + T(v)\beta \) for all \( u, v \in H \) and \( \alpha, \beta \in \mathbb{H} \). Such an operator is also called an endomorphism, so we denote by \( E_n(H) \) the set of all endomorphisms on \( H \). For every \( T \in E_n(H) \), the Riesz representation theorem (see e.g. \cite{31,32}) guarantees that there exists a unique operator \( T^* \in E_n(H) \), such that for all \( u, v \in H \), \( \langle Tu, v \rangle = \langle u, T^*v \rangle \). As for quaternion matrices, an operator \( T \in E_n(H) \) is said to be self-adjoint if \( T = T^* \), normal if \( TT^* = T^*T \) and unitary if \( TT^* = T^*T = I \).

We recall the spectral theorem for normal operators \cite{31} as follows.

**Theorem 2.3:** Suppose that \( H \) is an \( n \)-dimensional quaternion Hilbert space. Then \( T \in E_n(H) \) is normal if and only if there is an orthonormal basis \( V = \{u_1, u_2, \ldots, u_n\} \subset H \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}^+ \) such that \( Tu_k = u_k\lambda_k \) for every \( 1 \leq k \leq n \).

Under the usual vector-scalar multiplication \( \langle \xi, \alpha \rangle \mapsto \xi\alpha, \mathbb{H}^n \) is a right \( \mathbb{H} \)-module. Furthermore, \( \mathbb{H}^n \) is a quaternion Hilbert space if it is embedded by inner product \( \langle \xi, \eta \rangle = \xi^*\eta \). Let \( A \in M_n(\mathbb{H}) \), \( \xi \in \mathbb{H}^n \) and define \( T_A = A\xi \). Then we have the spectral theorem for normal matrices as follows.

**Theorem 2.4:** The matrix \( A \in M_n(H) \) is normal if and only if there is a unitary matrix \( U \in M_n(\mathbb{H}) \) such that \( U^*AU = D \), where \( D \) is a diagonal matrix and its entries are complex numbers with nonnegative imaginary parts.

### 2.3. Quaternion polynomials

The polynomials with quaternion coefficients located on only the left side of powers are called simple quaternion polynomials \cite{27}. Let

\[
p_n(z) := \sum_{k=0}^{n} c_k z^k, \quad c_k \in \mathbb{H}, \quad 0 \leq k \leq n, \quad c_0 c_n \neq 0
\]

be a given simple quaternion polynomial of degree \( n \). It is known that a complex polynomial of degree \( n \) has exactly \( n \) complex zeros. For a quaternion polynomial, however, may have an infinite number of zeros. Consider the polynomial \( p(z) = z^2 + 1 \), it is easy to see that \( \pm i \) are zeros. In fact, all the quaternions similar with \( i \) are zeros of \( p(z) \) as well. This phenomenon leads to the classification of the zeros for simple quaternion polynomials.

It was shown in \cite{26} by Pogorui and Shapiro that the polynomials of type (2) may have two types of zeros: isolated and spherical zeros. Let \( z_0 \) be a zero of \( p_n \) defined by (2). If \( z_0 \notin \mathbb{R} \) and has the property that \( p_n(z) = 0 \) for all \( z \in \theta(z_0) \), then it is called a spherical zero. Otherwise, \( z_0 \) is called an isolated zero. If the zero set of \( p_n(z) \) intersects only with \( n_1 \) similarity orbits, we say that the number of zeros of \( p_n(z) \) is \( n_1 \). The authors in \cite{27} proved...
that \( n_1 \leq n \), and they also presented an effective algorithm for finding all zeros including their types without using iterations.

It should be noted that the general form of a quaternionic monomial would be \( a_0 \cdot z \cdot a_1 \cdot z \cdot a_2 \cdots a_{j-1} \cdot z \cdot a_j \). Thus the class of simple quaternion polynomials is only a very special type of quaternion polynomial. In this paper, we only consider simple quaternion polynomials, the readers could refer to [33] for more details of the general type of quaternion polynomials.

Let \( p_n(z, s) = \sum_{k=0}^{n} a_k z^k \). We introduce a lemma concerning the zeros of \( p_n(z, s_1) \) and \( p_n(z, s_2) \) for \( s_1 \neq s_2 \).

**Lemma 2.5:** Let \( p_n(z, s) \) be given above. Then \( z_0 \) is a zero of \( p_n(z, s_1) \) if and only if \( s^{-1}z_0s \) is a zero of \( p_n(z, s_1) \), where \( s = s_1^{-1}s_2 \). Besides, \( p_n(z, s_1) \) and \( p_n(z, s_2) \) have the same real and spherical zeros. Let \( Z_{iso}(s) \) be the totality of non-real isolated zeros of \( p_n(z, s) \). Then

\[
Z_{iso}(s_2) = \left\{ s^{-1}zs : s = s_1^{-1}s_2, z \in Z_{iso}(s_1) \right\}.
\]

In particular, \( Z_{iso}(s_2) = Z_{iso}(s_1) \) provided that \( s_1^{-1}s_2 \in \mathbb{R} \).

**Proof:** For any fixed \( z_0 \in \mathbb{H} \), we have that \( (s^{-1}z_0s)^k = s^{-1}z_0^k s \). It follows that

\[
p_n(s^{-1}z_0s, s_2) = \sum_{k=0}^{n} c_k s_2 (s^{-1}z_0s)^k = \sum_{k=0}^{n} c_k s_2 s^{-1}z_0^k s = \sum_{k=0}^{n} c_k s_2 s_2^{-1}z_0^k s = p_n(z_0, s_1)s.
\]

Therefore, \( z_0 \) is a zero of \( p_n(z, s_1) \) if and only if \( s^{-1}z_0s \) is a zero of \( p_n(z, s_2) \). If \( z_0 \) is real, then \( s^{-1}z_0s = z_0 \). Thus \( p_n(z, s_1) \) and \( p_n(z, s_2) \) have the same real zeros. Since \( z_0 \) and \( s^{-1}z_0s \) belong to the similarity orbit \( \theta(z_0) \), then \( p_n(z, s_1) \) and \( p_n(z, s_2) \) have the same spherical zeros. They have the same non-real isolated zeros if \( s_1^{-1}s_2 \in \mathbb{R} \).

**3. The sampling theorem**

We study the quaternion difference equation

\[
b(k)x(k+1) + a(k)x(k) + b(k-1)x(k-1) = x(k)\lambda, \quad k = 1, 2, \ldots, N,
\]

where \( a(k) \in \mathbb{H} \), \( b(k) \in \mathbb{H} \setminus \{0\} \) for \( 0 \leq k \leq N \) and \( \lambda \in \mathbb{H} \) is a parameter. The equations of type (3), which have two-sided coefficients, are difficult to solve [34]. To solve (3), we need to consider the boundary conditions of form

\[
x(0) + h_1x(1) = 0, \quad k = 1, 2, \ldots, N,
\]

where \( h_1, h_2 \) are quaternion numbers. Note that \( x(1) \) and \( x(N) \) cannot be zero, otherwise there is only a trivial solution \( x(k) = 0 \) for the boundary value problem (BVP) (3)–(5). Therefore if \( h_1 = 0 \) (which implies that \( x(0) = 0 \), we have to restrict \( x(1) \neq 0 \). Similar considerations are needed for the cases when \( h_2 = 0 \) or \( h_1 = h_2 = 0 \).
For fixed \( h_1, h_2 \in \mathbb{H} \), we define an operator \( L \) for any \( x = [x(1), x(2), \ldots, x(N)]^\top \in \mathbb{H}^n \) as

\[
(Lx)(k) := b(k)x(k + 1) + a(k)x(k) + b(k - 1)x(k - 1), \quad k = 1, 2, \ldots, N,
\]

where \( x(0), x(N + 1) \) involved in \( L \) are taken from (4) and (5). In the traditional case, the BVP (3)–(5) is called a regular Sturm–Liouville problem if \( a(k), b(k), h_1, h_2 \) are restricted to be real and the corresponding operator \( L \) is self-adjoint. In the present study, \( L \) is not necessarily to be self-adjoint. In fact, by Theorem 2.3, we know that if \( L \) is normal, then it can produce an orthonormal basis whose elements are solutions of (3).

Since \( L \) is a right \( \mathbb{H} \)-linear operator on \( \mathbb{H}^N \), it has a matrix form

\[
L = \begin{pmatrix}
  d_1 & b(1) & 0 & & & \\
  b(1) & a(2) & b(2) & & & \\
  b(2) & a(3) & b(3) & & & \\
  & \ddots & \ddots & \ddots & & \\
  0 & b(N - 2) & a(N - 1) & b(N - 1) & \\
  b(N - 1) & & & & d_2
\end{pmatrix},
\]

where \( d_1 = a(1) - b(0)h_1 \) and \( d_2 = a(N) - b(N)h_2 \). It follows that \( L \) is normal if and only if \( L \) is normal. Next, we introduce a necessary and sufficient condition for \( L \) to be normal.

**Theorem 3.1:** Let \( L = L_0 + iL_1 + jL_2 + kL_3 \in M_N(\mathbb{H}) \) be a symmetric quaternion matrix with \( L_0, L_1, L_2, L_3 \) being real matrices. Then \( L \) is normal if and only if \( L_0 \) is commutative with \( L_1, L_2, L_3 \), respectively.

**Proof:** By direct computations, we have that

\[
LL^* - L^*L = 0 \iff LL^* - L^*L = L^*L - LL^* \iff (L + L^*)(L - L^*) = (L - L^*)(L + L^*).
\]

Since \( L \) is symmetric, then \( L_0, L_1, L_2, L_3 \) are symmetric. It follows that

\[
L^* = L_0^\top - iL_1^\top - jL_2^\top - kL_3^\top = L_0 - iL_1 - jL_2 - kL_3.
\]

Therefore \( L + L^* = 2L_0 \) and \( L - L^* = 2iL_1 + 2jL_2 + 2kL_3 \). It follows that \( L + L^* \) is commutative with \( L - L^* \) if and only if \( L_0 \) is commutative with \( L_1, L_2, L_3 \). The proof is complete.

For any second-order quaternion difference operator defined by (6), we can use Theorem 3.1 to determine whether \( L \) is normal. Nevertheless, it’s more convenient to give some sufficient conditions on the coefficients of the quaternion difference Equation (3) such that \( L \) is normal.

**Proposition 3.2:** If \( b(1), b(2), \ldots, b(N - 1) \) are pure imaginary quaternions and \( a(2), a(3), \ldots, a(N - 1), a(1) - b(0)h_1, a(N) - b(N)h_2 \) have the same real part, then \( L \) is normal.

**Proof:** Under the given conditions, \( L_0 \) is a scalar matrix, so it is commutative with all real matrices. Thus \( L \) is normal by Theorem 3.1.
Remark 3.1: A majority of quaternion-based colour image processing methods encode the red, green and blue channel pixel values on the three imaginary parts of a quaternion (see e.g. [35]). That is, \( w = w_r i + w_g j + w_b k \), where \( w \) denotes a colour pixel, \( w_r, w_g \) and \( w_b \) are the red, green and blue channel pixel values, respectively. Therefore, the conditions mentioned in Proposition 3.2 meet considerable applications of quaternion-based colour image processing.

Without doubt, the conditions for \( a(n), b(n), h_1, h_2 \) such that \( L \) to be normal are much more than those mentioned in Proposition 3.2, more general conditions deserve further exploration.

Let \( \phi(k, \lambda, s) (0 \leq k \leq N + 1) \) be the unique solution of (3)–(4) satisfying \( \phi(1, \lambda, s) = s \). By direct computations, we have that \( \phi(k, \lambda, s) \) is a simple quaternion polynomial with the form of \( \sum_{j=0}^{k} c(j, k) s \lambda^k \) for \( k \geq 1 \), where \( c(0, k), c(1, k), \ldots, c(k, k) \) are undetermined coefficients. Therefore, for \( \phi(k, \lambda_0, s) \) to be the solution of (3)–(5), it is necessary that \( \lambda_0 \) is a zero of the polynomial \( \phi(N + 1, \lambda, s) + h_2 \phi(N, \lambda, s) \).

Let
\[
\rho_N(\lambda, s) = \phi(N + 1, \lambda, s) + h_2 \phi(N, \lambda, s)
\]
and
\[
\varphi(\lambda, s) = [\phi(1, \lambda, s), \phi(2, \lambda, s), \ldots, \phi(N, \lambda, s)]^\top.
\]
The relationship between \( \varphi(\lambda, s) \) and \( L \) is described as follows.

Theorem 3.3: Let \( \rho_N(\lambda, s), \varphi(\lambda, s), L \) be given above. Then for \( s_0 \in \mathbb{H} \setminus \{0\} \) the following assertions are equivalent.

1. \( \varphi(\lambda_0, s_0) \) is a solution of (3)–(5).
2. \( L \varphi(\lambda_0, s_0) = \varphi(\lambda_0, s_0) \lambda_0 \).
3. \( \rho_N(\lambda_0, s_0) = 0 \).

Proof: If we regard (3)–(5) as a system of equations, then it involves \( N + 2 \) equations and \( N + 3 \) variables \( x(0), x(1), \ldots, x(N + 1), \lambda \). Solving (4) and (5) with respect to \( x(0) \) and \( x(N + 1) \) and plugging the results into (3) yields the equation \( Lx = x \lambda \). Thus statement 1 implies statement 2.

The \( N \)th element of \( L \varphi(\lambda_0, s_0) \) is \( b(N - 1) \phi(N - 1, \lambda_0, s_0) + (a(N) - b(N) h_2) \phi(N, \lambda_0, s_0) \). Therefore \( L \varphi(\lambda_0, s_0) = \varphi(\lambda_0, s_0) \lambda_0 \) implies that
\[
b(N - 1) \phi(N - 1, \lambda_0, s_0) + (a(N) - b(N) h_2) \phi(N, \lambda_0, s_0) = \phi(N, \lambda_0, s_0) \lambda_0.
\]
Since \( \varphi(\lambda_0, s_0) \) satisfies (3), then
\[
b(N) \phi(N + 1, \lambda_0, s_0) + a(N) \phi(N, \lambda_0, s_0) + b(N - 1) \phi(N - 1, \lambda_0, s_0) = \phi(N, \lambda_0, s_0) \lambda_0.
\]
It follows that
\[
b(N) \phi(N + 1, \lambda_0, s_0) = -b(N) h_2 \phi(N, \lambda_0, s_0).
\]
Note that \( b(N) \neq 0 \), thus
\[
\phi(N + 1, \lambda_0, s_0) = -h_2 \phi(N, \lambda_0, s_0),
\]
which means that \( \rho_N(\lambda_0, s_0) = 0 \). Therefore statement 2 implies statement 3.
If $p_N(\lambda_0, s_0) = 0$, then $\varphi(\lambda_0, s_0)$ satisfies (5). Notice that $\varphi(\lambda_0, s_0)$ is defined to be a solution of (3)–(4), thus it is a solution of (3)–(5). It follows that statement 3 implies statement 1.

Thus the three assertions are equivalent. ■

As a consequence of Theorem 3.3, we obtain a corollary as follows.

**Corollary 3.4:** Let $p_N(\lambda, s), \varphi(\lambda, s), L$ be given above. Then the following assertions hold.

1. If $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$ is an eigenvector of $L$ corresponding to the eigenvalue $\lambda_0 \in \mathbb{H}$, then $\xi_1 \neq 0$, $\varphi(\lambda_0, \xi_1) = \xi$. Moreover, $\varphi(\lambda_0, \xi_1)$ is a solution of (3)–(5) and therefore $p_N(\lambda_0, \xi_1) = 0$.
2. If $p_N(\lambda_0, s_0) = 0$ for $s_0 \neq 0$, then $\varphi(\lambda_0, s_0)s_1 = \varphi(s_1^{-1}\lambda_0 s_1, s_0 s_1)$ for every $s_1 \in \mathbb{H} \setminus \{0\}$.

**Proof:** (1) We mentioned previously that $x(1)$ in (3)–(4) cannot be zero, otherwise there is only a trivial solution for the equations. We can also see this fact from the matrix $L$ directly. Since $L\xi = \lambda\xi$, then $d_1\xi_1 + b(1)\xi_2 = \xi_1\lambda$. It follows that $\xi_1 = 0$ will result in $\xi_2 = 0$, as $b(k) \in \mathbb{H} \setminus \{0\}$ for $0 \leq k \leq N$. By similar arguments, we have $\xi_2 = \xi_4 = \cdots = \xi_N = 0$ which leads to a contradiction with the assumption that $\xi$ is an eigenvector of $L$. Define $\xi_0 = -h_1\xi_1$ and $\xi_{N+1} = -h_2\xi_N$, then $x(k) = \xi_k (0 \leq k \leq N + 1)$ is a solution of (3)–(5) satisfying $x(1) = \xi_1$. By the definition of $\varphi$, we conclude that $\varphi(\lambda_0, \xi_1) = \xi$ for the uniqueness of the solution.

(2) If $p_N(\lambda_0, s_0) = 0$ for $s_0 \neq 0$, then $L\varphi(\lambda_0, s_0) = \varphi(\lambda_0, s_0)\lambda_0$ by Theorem 3.3. It follows that

$$L\varphi(\lambda_0, s_0)s_1 = \varphi(\lambda_0, s_0)\lambda_0 s_1 = \varphi(s_1^{-1}\lambda_0 s_1, s_0 s_1),$$

for every $s_1 \in \mathbb{H} \setminus \{0\}$. Denote $\varphi(\lambda_0, s_0)s_1$ and $s_1^{-1}\lambda_0 s_1$ by $\eta = (\eta_1, \eta_2, \ldots, \eta_N)^T$ and $\lambda_1$ respectively, then $L\eta = \eta \lambda_1$. This means that $\eta$ is an eigenvector of $L$ corresponding to the eigenvalue $\lambda_1$. By statement 1, we conclude that

$$\varphi(\lambda_1, \eta_1) = \eta.$$

Since $\eta = \varphi(\lambda_0, s_0)s_1$ and the first element of $\varphi(\lambda_0, s_0)$ is $s_0$, then the first element of $\eta$ is $s_0 s_1$, namely $\eta_1 = s_0 s_1$. Note that $\lambda_1 = s_1^{-1}\lambda_0 s_1$, it follows that

$$\varphi(s_1^{-1}\lambda_0 s_1, s_0 s_1) = \varphi(\lambda_1, \eta_1) = \eta = \varphi(\lambda_0, s_0)s_1.$$

The proof is complete. ■

**Remark 3.2:** From the statement 1 of Corollary 3.4, we see that if $\lambda_0$ is an eigenvalue of $L$, there always exists $s_0 \neq 0$ such that $\varphi(\lambda_0, s_0)$ is a solution of (3)–(5), where $s_0$ can be the first element of any eigenvector of $L$ corresponding to $\lambda_0$. Additionally, we can further find $t_1, t_2, \ldots$ such that $\varphi(\lambda_0, t_1), \varphi(\lambda_0, t_2), \ldots$ are solutions of (3)–(5) by the statement 2 of Corollary 3.4. This can be seen by setting $\xi_k = s_0 t_k$, where $s_k, k = 1, 2, \ldots$ are unequal nonzero real numbers. In particular, if $\lambda_0$ is real-valued, then $\varphi(\lambda_0, s)$ is a solution of (3)–(5) for every $s \in \mathbb{H} \setminus \{0\}$. 
The above result states that for any fixed eigenvalue $\lambda_0$, there exist infinitely many $t_1, t_2, \ldots$ such that $\varphi(\lambda_0, t_1), \varphi(\lambda_0, t_2), \ldots$ are solutions of (3)–(5). We will show that for any fixed $s \in \mathbb{H} \setminus \{0\}$, there exist $N$ distinct quaternion numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $\varphi(\lambda_1, s), \varphi(\lambda_2, s), \ldots, \varphi(\lambda_N, s)$ are solutions of (3)–(5). Furthermore, these solutions constitute an orthogonal basis of $\mathbb{H}^N$.

**Theorem 3.5:** Suppose that $L$ (or equivalently $L$) is normal. Then for any fixed $s \in \mathbb{H} \setminus \{0\}$, there exist $N$ distinct quaternion numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that

1. $V = \{\varphi(\lambda_1, s), \varphi(\lambda_2, s), \ldots, \varphi(\lambda_N, s)\}$ is an orthogonal basis of $\mathbb{H}^N$;
2. every element of $V$ is a solution of (3)–(5), therefore $\lambda_k$ is an eigenvalue of $L$ for $k = 1, 2, \ldots, N$.

**Proof:** Since $L$ is normal, by Theorem 2.4, there exists $U = [u_1, u_2, \ldots, u_N] \in M_n(\mathbb{H})$ such that $U^*LU = D$ where $D$ is a diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}$. Note that $U$ is unitary, we have $LU = UD$. It follows that $u_k = [u_{1k}, u_{2k}, \ldots, u_{Nk}]^\top$ is an eigenvector of $L$ corresponding to the eigenvalue $\alpha_k$ for $k = 1, 2, \ldots, N$. We may encounter $\alpha_i = \alpha_j$ for some $i \neq j$, as it is possible that $L$ has multiple standard eigenvalues. Thus we need to construct $\lambda_k$ from $\alpha_k$ so that $\lambda_1, \lambda_2, \ldots, \lambda_N$ can be distinct.

By Corollary 3.4, we know that $u_{11}, u_{12}, \ldots, u_{1N}$ are nonzero and $\varphi(\alpha_k, u_{1k}) = u_k$ is a solution of (3)–(5). Let $t_1, t_2, \ldots, t_N$ be the quaternion numbers such that

$$u_{11}t_1 = u_{12}t_2 = \cdots = u_{1N}t_N = s$$

and let $\lambda_k = t_k^{-1}\alpha_k t_k$ for $k = 1, 2, \ldots, N$. It follows from Corollary 3.4 that

$$\varphi(\lambda_k, s) = \varphi(t_k^{-1}\alpha_k t_k, u_{1k}t_k) = \varphi(\alpha_k, u_{1k})t_k = u_k t_k$$

is a solution of (3)–(5) for $k = 1, 2, \ldots, N$. Note that the columns of $U$ form an orthonormal basis of $\mathbb{H}^N$; we have that $V$ is an orthogonal basis and therefore $\lambda_1, \lambda_2, \ldots, \lambda_N$ have to be unequal. Otherwise, suppose that $\lambda_i = \lambda_j$ for some $i \neq j$, then $\varphi(\lambda_i, s) = \varphi(\lambda_j, s)$, which means they cannot be orthogonal. 

Having introduced several useful results about the solutions of (3)–(5), we are in a position to state the main result of this section: the sampling expansions associated with the quaternion difference equations.

**Theorem 3.6:** Let $\varphi(\lambda, s) = [\varphi(1, \lambda, s), \varphi(2, \lambda, s), \ldots, \varphi(N, \lambda, s)]^\top$ be given above and suppose that $L$ is normal. For any fixed $s \in \mathbb{H} \setminus \{0\}$, let $f_s(\lambda)$ be a function defined by the discrete transform

$$f_s(\lambda) := \sum_{k=1}^N \overline{F(k)}\varphi(k, \lambda, s), \quad F(k) \in \mathbb{H}, \quad k = 1, 2, \ldots, N. \quad (7)$$

Then there exist $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{H}$ which depend on $s$ such that

$$f_s(\lambda) = \sum_{k=1}^N f_s(\lambda_k)\psi_k(\lambda, s), \quad (8)$$
where \( \psi_k(\lambda, s) = \frac{\langle \varphi(\lambda_k, s), \varphi(\lambda_k, s) \rangle}{\| \varphi(\lambda_k, s) \|^2} \).

**Proof:** For any fixed \( s \in \mathbb{H} \setminus \{0\} \), by Theorem 3.5, there exist \( \lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{H} \) such that \( V = \{ \varphi(\lambda_1, s), \varphi(\lambda_2, s), \ldots, \varphi(\lambda_N, s) \} \) is an orthogonal basis of \( \mathbb{H}^N \). It follows that

\[
F = [F(1), F(2), \ldots, F(N)]^T
\]

and \( \varphi(\lambda, s) \) has an expansion in terms of \( V \):

\[
\varphi(\lambda, s) = \sum_{k=1}^{N} \frac{\varphi(\lambda_k, s)}{\| \varphi(\lambda_k, s) \|^2} \langle \varphi(\lambda_k, s), \varphi(\lambda, s) \rangle = \sum_{k=1}^{N} \varphi(\lambda_k, s) \psi_k(\lambda, s).
\]

It follows that

\[
f_s(\lambda) = (F, \varphi(\lambda, s)) = \sum_{k=1}^{N} f_s(\lambda_k) \left( \frac{\varphi(\lambda_k, s)}{\| \varphi(\lambda_k, s) \|^2}, \sum_{j=1}^{N} \varphi(\lambda_j, s) \psi_j(\lambda, s) \right)
\]

\[
= \sum_{k=1}^{N} f_s(\lambda_k) \left( \frac{\varphi(\lambda_k, s)}{\| \varphi(\lambda_k, s) \|^2}, \varphi(\lambda_k, s) \psi_k(\lambda, s) \right)
\]

\[
= \sum_{k=1}^{N} f_s(\lambda_k) \psi_k(\lambda, s),
\]

which completes the proof.

**Remark 3.3:** We denote by \( H_s \) the totality of the quaternion functions of form (7). Then \( H_s \) is a left \( \mathbb{H} \)-module and every function in \( H_s \) has at least one sampling expansion with the form (8). It is known that the multiplication of two quaternions is non-commutative, but the scalar parts of \( pq \) and \( qp \) are equal for every \( p, q \in \mathbb{H} \). It follows that \( \sum_{k=1}^{N} f_s(\lambda_k) \psi_k(\lambda, s) = \sum_{k=1}^{N} \psi_k(\lambda, s) f_s(\lambda_k) \) provided that \( f_s(\lambda) \) is real-valued for all \( \lambda \). Otherwise \( f_s(\lambda) \neq \sum_{k=1}^{N} \psi_k(\lambda, s) f_s(\lambda_k) \).

The proof of Theorem 3.6 does not describe how to compute the sample points \( \lambda_k \) and the interpolation functions \( \psi_k(\lambda, s) \) for \( k = 1, 2, \ldots, N \). In the next part, some examples are presented to illustrate the sampling theorem. We discuss how to formulate the sampling expansions in practice and show the process of computation in detail.
Example 3.7: Consider the quaternion difference equation

\[-i\xi(k+1) + j\xi(k) - i\xi(k-1) = \xi(k)\lambda, \quad k = 1, 2, \ldots, N \tag{9}\]

with the boundary condition

\[\xi(0) = \xi(N+1) = 0. \tag{10}\]

It is easy to see that \(L\) defined by (6) is normal for Equation (9). For simplicity, we only show the result for \(N = 3\). By direct computations, we have that \(\phi(1, \lambda, s) = s, \phi(2, \lambda, s) = is\lambda - ks\) and

\[\phi(3, \lambda, s) = -s\lambda^2 - 2s, \quad \phi(4, \lambda, s) = -is\lambda^3 + k\lambda s^2 - 3is\lambda + 3ks.\]

Thus \(p_3(\lambda, s) = \phi(4, \lambda, s) + 0\phi(3, \lambda, s) = -is\lambda^3 + k\lambda s^2 - 3is\lambda + 3ks.\)

Let \(s = s_1 = 1 + k\). We consider the sampling expansion for

\[f_{s_1}(\lambda) := \sum_{k=1}^{N} F(k)\phi(k, \lambda, s_1).\]

We need to solve the equation

\[p_3(\lambda, s_1) = (j - i)\lambda^3 + (k - 1)\lambda^2 + 3(j - i)\lambda + 3(k - 1) = 0.\]

By applying the algorithm in [27] for computing the zeros, we get the zero set of \(p_3(\lambda, s_1)\): \(Z(s_1) = [i] \cup \theta(i\sqrt{3})\). It was shown in [18] that two eigenvectors of a normal operator are orthogonal if they correspond to two non-similar eigenvalues. So we may set \(\lambda_1 = i\). It is not easy to determine two elements \(\lambda_2, \lambda_3\) of \(\theta(i\sqrt{3}) \subset Z(s_1)\) such that \(\langle \phi(\lambda_2, s_1), \phi(\lambda_3, s_1) \rangle = 0\). We provide two methods to determine the sample points \(\lambda_2, \lambda_3\).

Method 3.1: Computing eigenvectors (eigenvalues) and applying quaternion Gram-Schmidt process. We need to compute the eigenvectors of

\[L = \begin{pmatrix} j & -i & 0 \\ -i & j & -i \\ 0 & -i & j \end{pmatrix}.\]

For \(\lambda = i\sqrt{3}\), we can get two linearly \(\mathbb{H}\)-independent eigenvectors

\[\xi_1 = \begin{pmatrix} i \\ -i\sqrt{3} - j \\ i \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -i\sqrt{3} - j \\ 2i \\ -i\sqrt{3} - j \end{pmatrix}.\]

It is fortunate that the inner product \(\langle \xi_1, \xi_2 \rangle = -4\sqrt{3}\) is real. So we can construct an eigenvector \(\xi_3\) corresponding to \(i\sqrt{3}\) from \(\{\xi_1, \xi_2\}\) such that \(\langle \xi_1, \xi_3 \rangle = 0\) by the quaternion Gram–Schmidt process [31]. Let \(\xi_3 = \xi_2 - \frac{\xi_1}{\|\xi_1\|^2} \langle \xi_1, \xi_2 \rangle = \frac{-i}{\sqrt{3}} - j, \frac{-2j}{\sqrt{3}}, \frac{-i}{\sqrt{3}} - j \rangle^\top\), then \(\langle \xi_1, \xi_3 \rangle = 0\) and

\[L\xi_3 = L(\xi_2 - \frac{\xi_1}{\|\xi_1\|^2} \langle \xi_1, \xi_2 \rangle) \tag{11}\]
\[
\begin{align*}
\xi_2i\sqrt{3} - \frac{\xi_1i\sqrt{3}}{\|\xi_1\|^2} \langle \xi_1, \xi_2 \rangle \\
= (\xi_2 - \frac{\xi_1}{\|\xi_1\|^2} \langle \xi_1, \xi_2 \rangle)i\sqrt{3} = \xi_3i\sqrt{3}.
\end{align*}
\]

In Equation (11), \(i\sqrt{3} \langle \xi_1, \xi_2 \rangle = \langle \xi_1, \xi_2 \rangle \) because \(\langle \xi_1, \xi_2 \rangle\) is real. Otherwise, to find suitable \(\xi\) we need repeating the eigenvector computing and the quaternion Gram–Schmidt process as the procedure of the diagonalization for quaternion normal matrices (see Theorem 3.3 in [31]).

By Corollary 3.4, we know that
\[
\phi(i\sqrt{3}, i) = \xi_1, \quad \phi(i\sqrt{3}, -\frac{i}{\sqrt{3}} - j) = \xi_3
\]
and they are solutions of the BVP (9)-(10). Let
\[
t_1 = i^{-1}s_1 = j - i, \quad t_2 = \left(-\frac{i}{\sqrt{3}} - j\right)^{-1}s_1 = \frac{1}{4} \left(\sqrt{3} + 3\right)i + \frac{1}{4} \left(3 - \sqrt{3}\right)j.
\]
Then we can select \(\lambda_2\) and \(\lambda_3\) to be
\[
\lambda_2 = t_1^{-1}i\sqrt{3}t_1 = -j\sqrt{3}, \quad \lambda_3 = t_2^{-1}i\sqrt{3}t_2 = \frac{3i + j\sqrt{3}}{2}.
\]
It follows that
\[
\phi(\lambda_2, s_1) = \begin{pmatrix} 1 + k \\ 1 - \sqrt{3} - (1 + \sqrt{3})k \\ 1 + k \end{pmatrix}, \quad \phi(\lambda_3, s_1) = \begin{pmatrix} 1 + k \\ \frac{\sqrt{3} - 1}{2} + \frac{1 + \sqrt{3}}{2}k \\ 1 + k \end{pmatrix}.
\]

Note that \(\phi(\lambda_1, s_1) = (1 + k, 0, -1 - k)^\top\) and by some direct computations, we obtain the interpolation functions:
\[
\psi_1(\lambda, s_1) = \frac{\lambda^2 + 3}{2},
\]
\[
\psi_2(\lambda, s_1) = \frac{-\lambda^2 + (i + j\sqrt{3})\lambda + k\sqrt{3}}{6},
\]
\[
\psi_3(\lambda, s_1) = \frac{-2\lambda^2 - (i + j\sqrt{3})\lambda - k\sqrt{3} - 3}{6}.
\]

**Method 3.2:** The method of undetermined coefficients and solving simple quaternion polynomials. It is also natural to construct an orthogonal basis \(V = \{\phi(\lambda_k, s_1) : k = 1, 2, 3\}\) from \(\phi(\lambda, s_1)\) directly without solving \(L\xi = \xi\lambda\). Let \(\lambda_1 = \lambda_3 = i, \lambda_2 = i\sqrt{3}\). To find a
\( \tilde{\lambda}_3 \in \theta(i\sqrt{3}) \) such that \( \varphi(\tilde{\lambda}_2, s_1) \) and \( \varphi(\tilde{\lambda}_3, s_1) \) are orthogonal, we simply need to solve

\[
(\varphi(\tilde{\lambda}_2, s_1), \varphi(\tilde{\lambda}_3, s_1)) = -2\tilde{\lambda}_3^2 + (2 - 2\sqrt{3})i\tilde{\lambda}_3 - 2\sqrt{3} = 0.
\]

The solution set is \( \{i, -i\sqrt{3}\} \). Note that \( \tilde{\lambda}_3 \) has to be in \( \theta(i\sqrt{3}) \). It follows that \( \tilde{\lambda}_3 = -i\sqrt{3} \).

By some direct computations, we obtain the corresponding interpolation functions:

\[
\begin{align*}
\tilde{\psi}_1(\lambda, s_1) &= \frac{\lambda^2 + 3}{2}, \\
\tilde{\psi}_2(\lambda, s_1) &= \frac{-3 + \sqrt{3})\lambda^2 - 2\sqrt{3}i\lambda - 3(1 + \sqrt{3})}{12}, \\
\tilde{\psi}_3(\lambda, s_1) &= \frac{(\sqrt{3} - 3)\lambda^2 + 2\sqrt{3}i\lambda + 3(\sqrt{3} - 1)}{12}.
\end{align*}
\]

Having introduced the sampling expansions for \( f_{s_1}(\lambda) \), we now discuss how to derive the sampling expansions for \( f_{s_2}(\lambda) (s_2 \neq s_1) \) by making use of the relationship between \( f_{s_1}(\lambda) \) and \( f_{s_2}(\lambda) \). Without loss of generality, we let \( s_2 = 2j \) and \( t = s_1^{-1}s_2 = i + j \), by Lemma 2.5, the zero set of \( p_3(\lambda, s_2) \) is

\[
Z(s_2) = \{z = t^{-1}z_1 : z_1 \in Z(s_1)\} = \{j\} \cup \theta(i\sqrt{3}).
\]

It follows from Corollary 3.4 that \( W_1 = \{\varphi(\beta_k, s_2) : k = 1, 2, 3\} \) is an orthogonal basis where

\[
\beta_1 = t^{-1}\lambda_1 t = j, \quad \beta_2 = t^{-1}\lambda_2 t = -i\sqrt{3}, \quad \beta_3 = t^{-1}\lambda_3 t = \frac{\sqrt{3}i + 3j}{2}.
\]

Similarly, another orthogonal basis is \( W_2 = \{\varphi(\tilde{\beta}_k, s_2) : k = 1, 2, 3\} \), where

\[
\tilde{\beta}_1 = t^{-1}\tilde{\lambda}_1 t = j, \quad \tilde{\beta}_2 = t^{-1}\tilde{\lambda}_2 t = j\sqrt{3}, \quad \tilde{\beta}_3 = t^{-1}\tilde{\lambda}_3 t = -j\sqrt{3}.
\]

The corresponding interpolation functions can be computed by the definition, we omit the details.

**Remark 3.4:** Two sampling expansions are given for \( f_{s_1}(\lambda) \) by different methods. In fact, regardless of which method we apply, we may obtain various sampling expansions by using different initial candidates of the sample points in \( \theta(i\sqrt{3}) \). The non-uniqueness of sampling expansions will be further discussed in Proposition 3.9. Another observation of this example is that the sampling expansions for \( f_{s_2}(\lambda) (s_2 \neq s_1) \) can be easily obtained by making use of the relationship between \( f_{s_1}(\lambda) \) and \( f_{s_2}(\lambda) \).

**Example 3.8:** Consider the BVP (3)–(5) with the coefficients as follows.

| \( a(1) \) | \( a(2) \) | \( a(3) \) | \( b(0) \) | \( b(1) \) | \( b(2) \) | \( b(3) \) | \( h_1 \) | \( h_2 \) |
|---|---|---|---|---|---|---|---|---|
| \( j \) | \( i \) | \( -k \) | \( i + j \) | \( \sqrt{3}j \) | \( j - k \) | \( 1 + j \) | \( -k \) | \( -i \) |
Table 1. Computing the sample points by Method 3.2.

| Zeros of $p_N(\lambda, s)$ | Sample point | Candidates of sample point | Zero set of $h$ | Sample point |
|--------------------------|--------------|----------------------------|----------------|--------------|
| $\lambda_1 = j$ (isolated) | $\lambda_{1,1} = j$ | $S_{1,1} = \bigcup_{k \in \mathbb{Z}} \theta(\lambda_k)$ | $S_{1,1} \cap \theta(\lambda_1) = \emptyset$ | N/A |
| $\lambda_2 = 1.17557i$ (spherical) | $\lambda_{2,1} = 1.17557i$ | $S_{2,1} = \{j\} \cup \{-0.188511i + 1.16036j\} \cup \theta(\lambda_3) \cup \theta(\lambda_4) \cup \theta(\lambda_5)$ | $S_{2,1} \cap \theta(\lambda_2) = \{-0.188511i + 1.16036j\}$ | $\lambda_{2,2} = -0.188511i + 1.16036j$ |
| $\lambda_3 = 1.54336i$ (spherical) | $\lambda_{3,1} = 1.54336i$ | $S_{3,1} = \{j\} \cup \{-0.630661i + 1.40863j\} \cup \theta(\lambda_4) \cup \theta(\lambda_5)$ | $S_{3,1} \cap \theta(\lambda_3) = \{-0.630661i + 1.40863j\}$ | $\lambda_{3,2} = -0.630661i + 1.40863j$ |
| $\lambda_4 = 1.90211i$ (spherical) | $\lambda_{4,1} = 1.90211i$ | $S_{4,1} = \{j\} \cup \{-1.07834i + 1.56692j\} \cup \theta(\lambda_5)$ | $S_{4,1} \cap \theta(\lambda_4) = \{-1.07834i + 1.56692j\}$ | $\lambda_{4,2} = -1.07834i + 1.56692j$ |
| $\lambda_5 = 2.14896i$ (spherical) | $\lambda_{5,1} = 2.14896i$ | $S_{5,1} = \{j\} \cup \{-1.38394i + 1.644j\} \cup \theta(\lambda_2) \cup \theta(\lambda_3)$ | $S_{5,1} \cap \theta(\lambda_5) = \{-1.38394i + 1.644j\}$ | $\lambda_{5,2} = -1.38394i + 1.644j$ |

Then we have $\phi(1, \lambda, s) = s$, $\phi(2, \lambda, s) = -\frac{\sqrt{3}}{3} j s \lambda - \frac{\sqrt{3}}{3} k s$, $\phi(3, \lambda, s) = \frac{\sqrt{3}}{6} (i - 1) s \lambda^2 + \frac{2\sqrt{3}}{3} (i - 1) s$ and

$$\phi(4, \lambda, s) = \frac{\sqrt{3}}{12} (k + j + i - 1) s \lambda^3 + \frac{\sqrt{3}}{12} (1 + i + j - k) s \lambda^2 + \frac{\sqrt{3}}{2} (k + j + i - 1) s \lambda + \frac{\sqrt{3}}{6} (3 + 3i + j - k) s.$$  

It follows that

$$p_3(\lambda, s) = \phi(4, \lambda, s) + h_2 \phi(3, \lambda, s)$$

$$= \frac{\sqrt{3}}{12} (k + j + i - 1) s \lambda^3 + \frac{\sqrt{3}}{12} (3 + 3i + j - k) s \lambda^2 + \frac{\sqrt{3}}{2} (k + j + i - 1) s \lambda + \frac{\sqrt{3}}{6} (7 + 7i + j - k) s.$$  

Let $s_1 = -k$, the zero set of $p_3(\lambda, s_1)$ is $Z(s_1) = \{-i - j, -i + 2j, -i - 3j\}$. Let $\lambda_1 = -i - j, \lambda_2 = -i + 2j, \lambda_3 = -i - 3j$. Then

$$\varphi(\lambda_1, s_1) = \begin{pmatrix} -k \\ -\frac{\sqrt{3}}{3} k \\ \frac{\sqrt{3}}{3} (j + k) \end{pmatrix}, \quad \varphi(\lambda_2, s_1) = \begin{pmatrix} -k \\ \frac{2\sqrt{3}}{3} k \\ -\frac{\sqrt{3}}{3} (j + k) \end{pmatrix},$$

$$\varphi(\lambda_3, s_1) = \begin{pmatrix} -k \\ -\frac{\sqrt{3}}{3} k \\ -\frac{\sqrt{3}}{3} (j + k) \end{pmatrix},$$

are orthogonal as $\lambda_1, \lambda_2, \lambda_3$ belong to different similarity orbits. By some direct computations, we obtain the interpolation functions:

$$\psi_1(\lambda, s_1) = \frac{\lambda^2 + 7j\lambda + 7 - k}{6},$$

$$\psi_2(\lambda, s_1) = \frac{-\lambda^2 - 4j\lambda + 4k + 2}{15},$$
\[ \psi_3(\lambda, s_1) = \frac{-\lambda^2 + j\lambda - k - 3}{10}. \]

**Remark 3.5:** If the number of zeros of \( p_N(\lambda, s) \) is equal to \( N \) (which means that \( p_N \) has \( N \) distinct quaternion numbers belonging to \( N \) different similarity orbits), it is easy to find \( N \) distinct quaternion numbers \( \lambda_1, \lambda_2, \ldots, \lambda_N \) satisfying the conditions of the proposed sampling theorem. In particular, if \( p_N(\lambda, s) \) has \( N \) isolated zeros, then the sampling expansion (8) for \( f_s(\lambda) \) is unique.

In Example 3.7, the number of zeros for \( p_N(\lambda, s) \) is less than \( N \). We use different methods to obtain two sampling expansions for \( f_s(\lambda) \). Roughly speaking, the non-uniqueness is caused by the non-real similarity orbits in the zero set of \( p_N \).

**Proposition 3.9:** If \( p_N(\lambda, s) \) has at least one spherical zero, then there are infinitely many sampling expansions for \( f_s(\lambda) \).

**Proof:**

**Case 1:** If the number of zeros for \( p_N(\lambda, s) \) is equal to \( N \). Suppose that \( f_s(\lambda) \) has a sampling expansion at \( \lambda_1, \lambda_2, \ldots, \lambda_N \). Without loss of generality, assume that \( \lambda_1 \) is a spherical zero of \( p_N(\lambda, s) \). Then for any non-real \( \beta \), \( f_s(\lambda) \) has a sampling expansion at \( \beta^{-1}\lambda_1\beta, \lambda_2, \ldots, \lambda_N \).

**Case 2:** If the number of zeros for \( p_N(\lambda, s) \) is less than \( N \). Then there exists at least one non-real similarity orbit \( \theta(\lambda_0) \) containing at least two sample points, where \( \lambda_0 \) is a spherical zero of \( p_N(\lambda, s) \). Suppose that \( f_s(\lambda) \) has a sampling expansion at \( \lambda_1, \lambda_2, \ldots, \lambda_N \). Without loss of generality, assume that \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \theta(\lambda_0) \) with \( n_1 \geq 2 \). Pick \( \alpha_1 \in \theta(\lambda_0) \) and \( \alpha_1 \neq \lambda_i \) for all \( 1 \leq i \leq n_1 \). We denote the solution set of

\[ \langle \varphi(\alpha_1, s), \varphi(\alpha, s) \rangle = 0 \]

by \( S_1 \). Pick \( \alpha_2 \in S_1 \cap \theta(\lambda_0) \) and denote by \( S_2 \) the solution set of

\[ \langle \varphi(\alpha_2, s), \varphi(\alpha, s) \rangle = 0. \]

Then we can find \( \alpha_3 \in S_2 \cap S_1 \cap \theta(\lambda_0) \) such that \( \varphi(\alpha_1, s), \varphi(\alpha_2, s), \varphi(\alpha_3, s) \) are orthogonal. Repeating this process, we have \( \alpha_1, \alpha_2, \ldots, \alpha_{n_1} \in \theta(\lambda_0) \) such that \( f_s(\lambda) \) has a sampling expansion at \( \alpha_1, \alpha_2, \ldots, \alpha_{n_1}, \lambda_{n_1+1}, \lambda_{n_1+2}, \ldots, \lambda_N \). Since \( \theta(\lambda_0) \) contains infinitely many elements, we can always find new sample points to construct sampling expansions for \( f_s(\lambda) \). The proof is complete. \( \square \)

The key step to derive the sampling expansions is to find all the sample points. From the above examples and discussions, we see that if \( p_N(\lambda, s) \) has \( N \) zeros belonging to distinct similarities, the sample points can be easily computed. If the number of zeros for \( p_N(\lambda, s) \) is less than \( N \) or \( L \) has multiple standard eigenvalues, we have to pay more efforts to compute the sample points. To show all the parts of the sampling theorem in detail, we have given the above two examples with small \( N \). In fact, the proposed two methods in the examples are available to the general case with large \( N \). Nevertheless, since it is more convenient to use Method 3.2 to compute the sample points in practice, we will only present the whole process of Method 3.2.
Step 1. Compute the zeros for \( p_N(\lambda, s) \) and get the isolated zeros \( \lambda_1, \lambda_2, \ldots, \lambda_K \), the spherical zeros \( \lambda_{K+1}, \lambda_{K+2}, \ldots, \lambda_{K+M} \). Clearly, \( K + M \leq N \).

Step 2. For every \( K + 1 \leq n \leq K + M \), let \( \lambda_{n,1} = \lambda_n \). Construct a quaternion polynomial by \( h_{n,1}(\lambda, s) = \langle \varphi(\lambda_{n,1}, s), \varphi(\lambda, s) \rangle \) and denote its zero set by \( S_{n,1} \). Pick \( \lambda_{n,2} \in S_{n,1} \cap \theta(\lambda_{n,1}) \) and construct \( h_{n,2}(\lambda, s) = \langle \varphi(\lambda_{n,2}, s), \varphi(\lambda, s) \rangle \) and denote its zero set by \( S_{n,2} \). Pick \( \lambda_{n,3} \in S_{n,2} \cap S_{n,1} \cap \theta(\lambda_{n,2}) \). Repeat the above process and denote by \( l_n \) the number such that

\[
S_{n,l_n} \cap \ldots \cap S_{n,2} \cap S_{n,1} \cap \theta(\lambda_{n,1}) = \emptyset,
\]

\[
S_{n,l_n-1} \cap \ldots \cap S_{n,2} \cap S_{n,1} \cap \theta(\lambda_{n,1}) \neq \emptyset.
\]

Then we find \( l_n \) sample points associated with \( \lambda_n \). Thus

\[
\{ \lambda_n : 1 \leq n \leq K \} \bigcup \left( \bigcup_{n=K+1}^{K+M} \{ \lambda_{n,m} : 1 \leq m \leq l_n \} \right)
\]

contains the required sample points for the sampling expansion.

Remark 3.6: Method 3.1 is based on the diagonalization of quaternion normal matrices by the quaternion Gram–Schmidt process [31]. Method 3.2 is based on the approaches for solving simple quaternion polynomials [27]. Method 3.2 is more suitable to deal with the current problem. For the readers interested in the quaternion Gram–Schmidt process, please refer to [31].

We list the values of \( \lambda_n, \lambda_{n,m}, S_{n,m} \) for Example 3.7 with \( N = 9 \). We also provide the Matlab code\(^1\) program (based on the quaternion software package [36]) to evaluate the values of \( \lambda_n, \lambda_{n,m}, S_{n,m} \) for arbitrary \( N \).

For any isolated zero of \( p_N(\lambda, s) \), we can also view it as a spherical zero and use step 2 of Method 3.2 to produce the sample points (see Table 1). This ensures that Method 3.2 is still available under the situation that the types of zeros of \( p_N(\lambda, s) \) are unknown.

4. Characteristic polynomials for tridiagonal quaternion matrices

For any complex or real matrix \( B \), we can compute its eigenvalues by finding all the zeros of the polynomial \( \det(B - \lambda I) \). When considering a quaternion matrix \( A \in M_n(\mathbb{H}) \), however, we don’t know whether there exists a quaternion polynomial \( p_A \) of degree \( n \) having the property that \( p_A(\lambda) = 0 \) if and only if \( \lambda \) is a right eigenvalue of \( A \). It is difficult to find such a \( p_A \) by the traditional methods used in the complex case because the singularity of \( A - \lambda I \) does not mean \( A \xi = \xi \lambda \) for some \( \xi \neq 0 \).

Example 4.1: Let

\[
A = \begin{pmatrix} 0 & i \\ j & 1 \end{pmatrix},
\]

and \( \lambda = \frac{1}{2}(1 + i + j - k) \). It is easy to verify that

\[
A - \lambda I = \begin{pmatrix} -\frac{1}{2}(1 + i + j - k) & i \\ j & \frac{1}{2}(1 - i - j + k) \end{pmatrix}
\]
is singular. Suppose that there exists \( \xi = \begin{pmatrix} u \\ v \end{pmatrix} \) such that \( A\xi = \xi \lambda \). We have

\[
\begin{pmatrix} 0 & i \\ j & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \cdot \frac{1}{2} (1 + i + j - k).
\]

That is,

\[
v = u \cdot \frac{1}{2} (1 + i + j - k), \tag{12}
\]

\[
ju + v = v \cdot \frac{1}{2} (1 + i + j - k). \tag{13}
\]

Plugging Equation (12) into Equation (13), we obtain

\[(j + 1)u = 0.\]

It follows that

\[u = 0,\]

\[v = 0.\]

Thus, there is no nonzero vector \( \xi \) such that \( A\xi = \xi \lambda \).

In general, to obtain the right eigenvalues of \( A \in M_n(\mathbb{H}) \), one needs to compute the eigenvalues of the complex adjoint matrix \( \chi_A \) of \( A \), where \( \chi_A \) is a \( 2n \times 2n \) complex matrices with the form

\[
\chi_A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}.
\]

Here \( A_1, A_2 \) stem from the unique representation \( A = A_1 + A_2j \), \( A_1, A_2 \in M_n(\mathbb{C}) \). The characteristic polynomial of the complex matrix \( \chi_A \) was defined to be a characteristic polynomial for \( A \) (see [23]). This means that for a \( n \times n \) quaternion matrix \( A \), the degree of its characteristic polynomial is \( 2n \). It is natural to ask: is there a polynomial with degree \( n \) which has the right eigenvalues of \( A \) as roots?

Theorem 3.3 gives us a hint to compute the right eigenvalues of \( n \times n \) tridiagonal symmetric quaternion matrices by finding the zeros of a simple quaternion polynomial with degree \( n \). The following result is a direct consequence of Theorem 3.3.

**Theorem 4.2:** Let \( A \in M_n(\mathbb{H}) \) be a tridiagonal symmetric quaternion matrix, there exists a simple quaternion polynomial \( p_n(A, z) \) such that

\[
\sigma(A) = \{ \alpha^{-1} \lambda \alpha : \alpha \neq 0, \lambda \in Z_p \},
\]

where \( \sigma(A) \) and \( Z_p \) are the right spectrum of \( A \) and the zero set of \( p_n(A, z) \), respectively.
Table 2. The zeros of $p_4$ and the standard eigenvalues of $A$ computed via $\chi_A$.

| Zeros of $p_4$                                      | Standard eigenvalues of $A$ |
|-----------------------------------------------------|----------------------------|
| $z_1$ $1.21826 - 0.378569i + 0.22245j - 0.321633k$ | $1.21826 + 0.544285i$      |
| $z_2$ $-0.208978 - 0.433043i + 0.412505j + 0.129384k$ | $-0.208978 + 0.611905i$    |
| $z_3$ $1.03613 + 1.08041i + 0.090633j + 0.326603k$ | $1.03613 + 1.13233i$      |
| $z_4$ $1.3011 + 1.8469i + 0.0828995j + 0.843984k$ | $1.3011 + 2.0323i$       |

Example 4.3: Consider the eigenvalue problem for

$$A = \begin{pmatrix} 1 & 1 + i & 0 & 0 \\ 1 + i & i & 1 + j & 0 \\ 0 & 1 + j & 1 & 1 + k \\ 0 & 0 & 1 + k & k \end{pmatrix}.$$ 

On the one hand, by solving the quaternion difference equation associated with $A$, we can construct a quaternion polynomial

$$p_4(A, z) = (i + k)z^4 + (3 - i - j - k)z^3 + (3 - i - j - k)z^2 + (j - 3 - 3i - 3k)z + 1 - 4i + j.$$ 

On the other hand, by computing the eigenvalues of

$$\chi_A = \begin{pmatrix} 1 & 1 + i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 + i & i & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & i \\ 0 & 0 & 1 & 0 & 0 & 0 & i & i \\ 0 & 0 & 0 & 0 & 1 & 1 - i & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 - i & -i & 1 & 0 \\ 0 & -1 & -1 & i & 0 & 1 & 0 & 1 \\ 0 & 0 & i & i & 0 & 0 & 1 & 0 \end{pmatrix},$$

we can find the standard eigenvalues of $A$.

Table 2 displays the zeros of $p_4$ and the standard eigenvalues of $A$. We see that $z_k$ is similar with $\lambda_k$ for $k = 1, 2, 3, 4$, since they have the same real part and the same norm [23]. It is known that $\gamma$ is an eigenvalue if and only if $\gamma$ is similar to a standard eigenvalue. By the transitive property of the quaternion similarity, we have that $\gamma$ is an eigenvalue of $A$ if and only if $\gamma$ is similar with a zero of $p_4(A, z)$. This validates the statement of Theorem 4.2.

5. Colour image encryption based on the quaternion discrete transform

A pixel value in a colour image can be represented as a quaternion $w = w_r, i + w_g, j + w_b, k$, where $w_r, w_g, w_b$ denote the red, green and blue components, respectively. Quaternion model treats a colour image in a holistic manner and preserves the structural correlation information among the different colour channels [8,35]. The grayscale image encryption algorithms related to difference operators are given in [7,37]. By employing the quaternion discrete transform derived from the difference Equation (3), we propose a colour image encryption method which treats the different colour channels holistically.
We regard the colour pixels (pure quaternions) of a colour image as the coefficients of the discrete transform, namely $F(k)$, then a quaternion function associated with the colour image is obtained. Note that this quaternion function can be recovered from its finite samples. It follows that these samples can be used as an encrypted version of the original colour image and the encryption key is the associated quaternion difference equation. For example, if we use the discrete transform in Example 3.7 to encrypt a colour image, the encryption key is $\{-i, j\}$. The procedure of the colour image encryption and decryption is stated as follows.

1. Let $I$ be a colour image (a quaternion matrix with pure imaginary quaternions), rearrange $I$ as a vector $v = (v_1, v_2, \ldots, v_N)$ and let $F(k) = v_k, \forall 1 \leq k \leq N$.

2. Select a second-order quaternion difference equation with constant coefficients:
   
   $$bx(k + 1) + ax(k) + bx(k - 1) = x(k)\lambda, \quad k = 1, 2, \ldots, N.$$ 

   Compute $\{\phi(k, \lambda, s) : 1 \leq k \leq N\}$ from the given quaternion difference equation. Here, $s$ is an arbitrary nonzero quaternion. For simplicity, let $s = 1$.

3. Construct $f_s(\lambda)$ by the discrete transform
   
   $$f_s(\lambda) := \sum_{k=1}^{N} F(k)\phi(k, \lambda, s), \quad 1 \leq k \leq N.$$ 

4. Compute $\{f_s(\lambda_k) : 1 \leq k \leq N\}$, where $\lambda_k$ is obtained from the associated quaternion difference equation. The calculation method for $\lambda_k$ has been comprehensively discussed in Section 3. Here, $\{f_s(\lambda_k) : 1 \leq k \leq N\}$ is the encrypted image for the original image $\{F(k) : 1 \leq k \leq N\}$.

5. The colour image can be decrypted by
   
   $$[F(1), F(2), \ldots, F(N)]^\top = \sum_{k=1}^{N} \frac{\varphi(\lambda_k, s)}{||\varphi(\lambda_k, s)||^2} f_s(\lambda_k),$$

   where $\varphi(\lambda, s) = [\phi(1, \lambda, s), \phi(2, \lambda, s), \ldots, \phi(N, \lambda, s)]^\top$.

**Remark 5.1:** For any $q \in \mathbb{H}$, $w = w_0 + w_1i + w_2j + w_3k, v = v_0 + v_1i + v_2j + v_3k \in \theta(q)$, we say $w > v$ if there exists $0 \leq k \leq 3$ such that $w_k > v_k$ and $w_l = v_l, \forall 0 \leq l < k$.

It is shown that there are numerous options for the sample points if $p_N(\lambda, s)$ has at least one spherical zero. Nevertheless, if we select the sample points in order of $>$, by applying Method 3.2, a fixed selection can be made. The problem caused by non-uniqueness could be settled.

We use the quaternion discrete transform associated with $-ix(k + 1) + jx(k) - ix(k - 1) = x(k)\lambda$ to encrypt colour images. The experimental results are presented in Figure 1. Although the original images are the matrices with pure imaginary quaternions, the transformed functions contain the quaternions with nonzero scalar parts. For a colour image, its encrypted version includes a grayscale image (scalar part) and a colour image (vector part) as shown in rows 2 and 3 of Figure 1. If the correct inverse discrete transform is
Figure 1. Colour image encryption based on quaternion discrete transform. The first row: the original test images. The second row and third row: the encrypted images (row 2: scalar part; row 3: vector part). The fourth row: the decrypted images by key \{-i, 0.9j\}. The fifth row: the decrypted images by key \{i, j\}. 
applied to the encrypted image, the original image could be recovered successfully (see row 5 of Figure 1). To show that only the correct key can decrypt the encrypted images, we also provide the decrypted images produced by the incorrect key \((-i, 0.9j)\) (see row 4 of Figure 1). Namely, the decrypted images are produced by applying the inverse quaternion discrete transform associated with \(-ix(k + 1) + 0.9jx(k) - ix(k - 1) = x(k)\lambda\). We see that the decrypted image produced by the incorrect key has very little in common with its corresponding original image, except for the base colour. It’s worth noting that each pixel of the encrypted image produced by the proposed quaternion-based method contains the cross-information among the different colour channels of the original image. This helps to improve the security of the colour image encryption.

We provide a preliminary attempt to use the results of Section 3 to encrypt colour images. It is noted that one can use the proposed method in conjunction with the other useful encryption techniques, such as pixel scrambling [38], double-encrypted algorithm [39], chaotic system [37], to improve the quaternion-based approach. This topic will be discussed in our further study.

6. Conclusion and discussion

In this paper, we investigate the relationship between sampling theory and quaternion difference equations. The sampling expansions associated with quaternion difference equations are derived. Through examples, we show the computational techniques for the proposed formula in detail. Additionally, we find the characteristic polynomials for tridiagonal symmetric quaternion matrices. An application of the proposed sampling theorem in colour image processing is also discussed.

Note

1. https://github.com/Codecd2022/samplingwithQDEs

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