The Fidelity and Trace Norm Distances for Quantifying Coherence

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We investigate the coherence measures induced by fidelity and trace norm, based on the recent proposed coherence quantification in [Phys. Rev. Lett. 113, 140401, 2014]. We show that the fidelity of coherence does not in general satisfy the monotonicity requirement as a measure of coherence under the sub-selection of measurements condition. We find that the trace norm of coherence can act as a measure of coherence for qubit case and some special class of qudits.

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I. INTRODUCTION

Coherence arising from quantum superposition which plays a central role for quantum mechanics. Quantum coherence is an important subject in quantum theory and quantum information science which is a common necessary condition for both entanglement and other types of quantum correlations. It has been shown that a good definition of coherence does not only depend on the state of the system but also depends on the fixed basis for the quantum system. Up to now, several themes of coherence have been considered such as witnessing coherence, catalytic coherence, the thermodynamics of quantum coherence, and the role of coherence in biological system. There seems no well-accepted efficient method for quantifying coherence until recently. Girolami proposed a measure of quantum coherence based on the Wigner-Yanase-Dyson skew information. It is not only in theoretical but also an experimental scheme implementable with current technology. Baumgratz et al. introduced a rigorous framework for quantification of coherence and proposed several measures of coherence, which are based on the well-behaved metrics including the $l_p$-norm, relative entropy, trace norm and fidelity. The quantification of coherence promoted in a unified and rigorous framework thus stimulated a lot of further considerations about quantum coherence.

From the view point of the definition, one can straightforwardly quantify the coherence in a given basis by measuring the distance between the quantum state $\rho$ and its nearest incoherent state. This property is similar as that of the well studied measures of the quantum correlation, e.g., entanglement and quantum discord. We remark that the coherence measures are to be applied to one quantum system but quantum correlation measures naturally involve more than two parties. We know that several basic criteria are proposed which should be satisfied by any measure of the entanglement. In comparison, the coherence measures also need to satisfy the following four necessary criteria as presented in Ref. [1].

(C1) $C(\delta) = 0$ for all $\delta \in \mathcal{I}$.

(C2a) Monotonicity under all the incoherent completely positive and trace preserving (ICPTP) maps $\Phi$: $C(\rho) \geq C(\Phi(\rho))$.

(C2b) Monotonicity for average coherence under sub-selection based on measurements outcomes: $C(\rho) \geq \sum p_n C(\rho_n)$ for all $\{K_n\}$ with $\sum_n K_n^\dagger K_n = I$ and $K_n^\dagger K_n \subset \mathcal{I}$.

(C3) Non-increasing under mixing of quantum states: $\sum p_n C(\rho_n) \geq C(\sum p_n p_n)$ for any set of states $\rho_n$ and any $p_n \geq 0$ with $\sum_n p_n = 1$.

As shown in [1], the condition (C2b) is important as it allows for sub-selection based on measurement outcomes, a process available in well controlled quantum experiments. It has been shown that the quantum relative entropy and $l_1$-norm satisfy this condition. The squared Hilbert-Schmidt norm does not satisfy (C2b). However, it is still an open question whether some other coherence measures satisfy (C2b). In this paper, we will show that the measure of coherence induced by fidelity defined distance does not satisfy condition (C2b). Explicit example is presented. We will also show that trace norm of coherence for qubit satisfies condition (C2b), the case of qudit, which is in three-dimensional Hilbert space, is in general unknown, but for some special qudits, trace norm of coherence satisfies this condition.

This paper is organized as follows. In Sec. [II] we illustrate that the fidelity of coherence is not a good measure for quantum coherence by presenting an example that condition (C2b) is not satisfied. In Sec. [III] we show that condition (C2b) can be satisfied in qubit case and some special qudits for trace norm of coherence. We summarize our results in Sec. [IV].

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II. FIDELITY OF COHERENCE

As a measure of distance, the fidelity \( F(\rho, \delta) = |\langle \rho | \delta \rangle|^2 \) is non-decreasing under CPTP maps \( \varepsilon \), e.g., \( F(\varepsilon(\rho), \varepsilon(\delta)) \geq F(\rho, \delta) \). Then we know that the fidelity induced distance \( 1 - \sqrt{F(\rho, \delta)} \) is monotonic under ICPTP maps, and \( F(\rho, \delta) = 1 \) iff \( \rho = \delta \). Hence, the fidelity of coherence can be defined as:

\[
C_F(\rho) = \min_{\delta \in \mathcal{I}} D(\rho, \delta) = 1 - \sqrt{\max_{\delta \in \mathcal{I}} F(\rho, \delta)} \quad (1)
\]

It is easy to find that the fidelity of coherence fulfills (C1), (C2a) and (C3) \([1]\).

For the condition (C2b), without loss of generality, we consider the one-qubit system. It is known that for a qubit, the fidelity has a simple form. From the Bloch sphere representation of a quantum state, \( \rho \) and \( \delta \) can be expressed as \([12]\),

\[
\rho = \frac{I + \mathbf{r} \cdot \sigma}{2}, \quad \delta = \frac{I + \mathbf{s} \cdot \sigma}{2} \quad (2)
\]

where \( I \) is the identity operator, \( \mathbf{r} = (r_x, r_y, r_z) \) and \( \mathbf{s} = (s_x, s_y, s_z) \) are the Bloch vectors and \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is a vector of Pauli matrices. Then the fidelity for qubits has an elegant form,

\[
F(\rho, \delta) = \frac{1}{2} \left[ 1 + \mathbf{r} \cdot \mathbf{s} + \sqrt{1 - |\mathbf{r}|^2} \sqrt{1 - |\mathbf{s}|^2} \right] \quad (3)
\]

where \( \mathbf{r} \cdot \mathbf{s} \) is the inner product of \( \mathbf{r} \) and \( \mathbf{s} \). \( |\mathbf{r}| \) and \( |\mathbf{s}| \) is the magnitude of \( \mathbf{r} \) and \( \mathbf{s} \), respectively.

Because \( \delta \) is the incoherent state, then the Bloch vector \( \mathbf{s} \) can be expressed as \( \mathbf{s} = (0, 0, s_z) \), the Eq. (3) can be replaced as,

\[
F(\rho, \delta) = \frac{1}{2} \left[ 1 + r_z s_z + \sqrt{1 - r_z^2 - r_y^2} \right] \quad (4)
\]

In order to obtain \( \max_{\delta \in \mathcal{I}} F(\rho, \delta) \), we should take derivative with respect to \( s_z \), then we have,

\[
\frac{dF(\rho, \delta)}{ds_z} = \frac{1}{2} \left[ r_z - \sqrt{1 - r_z^2 - r_y^2} \right] \quad (5)
\]

After some simple algebraic operation, we can obtain,

\[
\max_{\delta \in \mathcal{I}} F(\rho, \delta) = \frac{1}{2} \left[ 1 + \sqrt{1 - r_z^2 - r_y^2} \right] \quad (6)
\]

Therefore, we obtain

\[
C_F(\rho) = 1 - \sqrt{\max_{\delta \in \mathcal{I}} F(\rho, \delta)} = 1 - \frac{\sqrt{2}}{2} \sqrt{1 + \sqrt{1 - r_z^2 - r_y^2}} \quad (7)
\]

This implies that the state \( \rho_{\text{diag}} \) is not necessarily optimized for the fidelity of coherence in the one-qubit system. Thus, in general, we have

\[
\min_{\delta \in \mathcal{I}} (1 - \sqrt{F(\rho, \delta)}) \neq 1 - \sqrt{F(\rho, \rho_{\text{diag}})} \quad (8)
\]

This makes that the sub-selection process becomes hard to verify. We should choose peculiar incoherent operations to simplify calculation.

Now we give an example to show that the condition (C2b) is violated. As we know that the depolarizing, the phase-damping, and the amplitude-damping channels are the qubit incoherent operations. We choose the amplitude-damping-like operation as incoherent operations, its operation elements are expressed as,

\[
K_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} . \quad (9)
\]

After applying it on the one-qubit, we obtained the output state

\[
\rho_1 = \frac{a^2 \rho + b^2 \rho - 2ab \rho^*}{|a|^2 + |b|^2} \quad (10)
\]

with the probability

\[
p_1 = \text{tr}(K_1 \rho K_1^+) = \frac{1}{2} \left( |a|^2 (1 + r_z) + |b|^2 (1 - r_z) \right) . \quad (11)
\]

In order to obtain the quantity, \( C_F(\rho_1) \), we should transform \( \rho_1 \) to the Bloch representation. Then the Bloch vector for \( \rho_1 \) can be given by,

\[
\begin{align*}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{a^2 (1 + r_z) + b^2 (1 - r_z)}{|a|^2 (1 + r_z) + |b|^2 (1 - r_z)} \quad (12)
\end{align*}
\]

The fidelity of coherence for \( \rho_1 \) is obtained by substituting \( x \) and \( y \) as,

\[
C_F(\rho_1) = 1 - \frac{\sqrt{2}}{2} \sqrt{1 + \sqrt{1 - x^2 - y^2}} \quad (13)
\]

Because that \( K_i \) should satisfy \( \sum K_i K_i^* = I \), then we have \( |a|^2 = 1, |b|^2 + |c|^2 = 1 \). Let \( |b|^2 = \frac{1}{4}, |c|^2 = \frac{3}{4} \), then we have,

\[
C_F(\rho_1) = 1 - \frac{3\sqrt{2}}{16} \sqrt{1 - \frac{1}{2} \left[ 1 + \sqrt{1 - \frac{32}{(10 - 3\sqrt{2})^2}} \right]} \approx 0.08273 \quad (14)
\]

and

\[
C_F(\rho) = 1 - \frac{\sqrt{2}}{2} \left[ 1 + \sqrt{\frac{7}{2}} \right] \approx 0.07612 . \quad (15)
\]
For the one-qubit quantum system, the eigenvalues of the trace norm of coherence can be simplified as

$$C_{tr}(\rho) = \min_{\delta \in \mathbb{I}} \sqrt{r_x^2 + r_y^2 + (r_z - s_z)^2}$$

$$= \|\rho - \rho_{diag}\|_{tr} = \sqrt{r_x^2 + r_y^2} \quad (19)$$

Note that $C_{tr}(\rho)$ has the same form of expression with the $l_1$ norm of coherence $C_1(\rho) = \sum |i,j,i \neq j| \rho_{ij}$ for the one-qubit case. So in this situation, $C_{tr}(\rho)$ satisfies the condition (C2b). Here we simply conclude that trace norm can act as a coherence measure for a qubit.

For the one-qutrit quantum system, the eigenvalues of the qutrit density matrices have complex expressions. It seems difficult to estimate the optimal incoherent state. Fortunately, we can find some special density matrices whose optimal incoherent states can be obtained.

**Theorem 1.** For the following three classes of qutrit states

$$\rho_X = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{2,2} & 0 \\ a_{13}^* & 0 & a_{33} \end{pmatrix}, \quad (20)$$

$$\rho_Y = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \quad (21)$$

$$\rho_Z = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{2,2} & a_{23} \\ 0 & a_{23} & a_{33} \end{pmatrix}, \quad (22)$$

the optimal incoherent state of the trace norm of coherence is of the form $\rho_{diag}$.

**Proof.** We only prove the case of state $\rho_X$, the states $\rho_Y$ and $\rho_Z$ are completely analogous. Since all qutrit incoherent states have the form as

$$\delta = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}, \quad (23)$$

then we can easily obtain the eigenvalues for $\rho_X - \delta$,

$$\lambda_1 = a_{22} - y,$$

$$\lambda_2 = \frac{a_{22} - y}{2} - \sqrt{\left(\frac{2x + y - 2a_{11} - a_{22}}{4|a_{11}|^2} \right)^2 + \left(\frac{4|a_{11}|a_{13}}{4|a_{11}|^2} \right)^2},$$

$$\lambda_3 = \frac{a_{22} - y}{2} + \sqrt{\left(\frac{2x + y - 2a_{11} - a_{22}}{4|a_{11}|^2} \right)^2 + \left(\frac{4|a_{11}|a_{13}}{4|a_{11}|^2} \right)^2}. \quad (24)$$

We know that $\rho_X - \delta$ is a normal matrix, its singular values are the modulus of the eigenvalues for $\rho_X - \delta$, then we have,

$$||\rho_X - \delta||_{tr} = |\lambda_1| + |\lambda_2| + |\lambda_3| \quad (25)$$

In order to minimize $||\rho_X - \delta||_{tr}$ over all the incoherent states, we should consider four cases as following.
Case 1. When \( y - a_{22} \geq \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \) and \( a_{22} \leq y \), we can simplify Eq. (25) as
\[
\|\rho_X - \delta\|_{tr} = 2y - 2a_{22} \geq 2\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2} \\
\geq 2|a_{13}|^2 = \|\rho_X - \rho_{\text{diag}}\|_{tr}. \tag{26}
\]

Case 2. When \( y - a_{22} \leq \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \) and \( a_{22} \leq y \), similar to case 1, we have
\[
\|\rho_X - \delta\|_{tr} = y - a_{22} + \sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2} \\
\geq 2\sqrt{|a_{13}|^2} = \|\rho_X - \rho_{\text{diag}}\|_{tr}. \tag{27}
\]

Case 3. When \( y - a_{22} \leq \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \), \( y \leq a_{22} \) and \( y - a_{22} + \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \geq 0 \), we have
\[
\|\rho_X - \delta\|_{tr} = a_{22} - y + \sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2} \\
\geq 2\sqrt{|a_{13}|^2} = \|\rho_X - \rho_{\text{diag}}\|_{tr}. \tag{28}
\]

Case 4. When \( y - a_{22} \leq \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \), \( y \leq a_{22} \) and \( y - a_{22} + \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \leq 0 \), we have
\[
\|\rho_X - \delta\|_{tr} \geq 2\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2} \\
\geq 2\sqrt{|a_{13}|^2} = \|\rho_X - \rho_{\text{diag}}\|_{tr}. \tag{29}
\]

Through the above analysis, we can obtain that the trace norm of coherence for \( \rho_X \) has the optimal incoherent state \( \rho_{\text{diag}} \). \( \square \)

According to the above theorem, we can also obtain an analytical expression of the trace norm of coherence for \( \rho_X \) as,
\[
C_{tr}(\rho_X) = D_{tr}(\rho_X, \rho_{\text{diag}}) = 2|a_{13}|. \tag{30}
\]

Note that \( C_{tr}(\rho_X) \) also has the same form of expression with the \( l_1 \) norm of coherence \( C_{l_1}(\rho) = \sum_{i,j,i\neq j} |\rho_{i,j}| \) for the \( \rho_X \). Based on this fact and as shown in Ref. [1], we know that \( C_{tr}(\rho_X) \) satisfies the condition (C2b). Similarly, we can verify the trace norm of coherence for \( \rho_Y \) and \( \rho_Z \) satisfies the condition (C2b).

IV. CONCLUSION

In this paper, we show that the fidelity of coherence does not satisfy condition (C2b) by presenting an example. We then conclude that the measure of coherence induced by fidelity is not a good measure for quantifying coherence. For the trace norm of coherence, we have shown that the qubit states and some special qutrit states can satisfy condition (C2b). Our results show that the trace norm of coherence is equivalent to \( l_1 \) norm of coherence for qubits and special qudits. It is unknown whether the coherence measure induced by trace norm can be applied for general quantum states. Our findings complement the results of coherence quantification in Ref. [1].

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