A HAMILTONIAN APPROACH FOR NONLINEAR ROTATIONAL CAPILLARY-GRAVITY WATER WAVES IN STRATIFIED FLOWS

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Abstract. Under consideration here are two-dimensional rotational stratified water flows driven by gravity and surface tension, bounded below by a rigid flat bed and above by a free surface. The distribution of vorticity and of density is piecewise constant— with a jump across the interface separating the fluid of bigger density from the lighter fluid adjacent to the free surface. The main result is that the governing equations for the two-layered rotational stratified flows, as described above, admit a Hamiltonian formulation.

1. Introduction. There is a diligent effort nowadays in pursuing the study of the tropical ocean dynamics. The investigations in this direction concern issues pertaining to modelling [2], existence of (exact) solutions [3, 5, 7, 17, 16], and stability/instability issues cf. [4, 18, 20].

Equatorial flows present several specific features. Apart from a strong stratification (with an interface, called thermocline, separating two layers of constant density), one is also confronted with the presence of strong depth-dependent underlying currents, cf. [23], with flow reversal at a depth of about 100 − 200 m; in the Pacific these currents are realistically modelled by two-dimensional flows with piecewise constant vorticity (see the discussion in [6]). The vorticity is a characteristic of wave-current interactions and captures the swirling motion beneath the surface wave, cf. [9, 33, 34].

Another key-feature of most water flows is their nonlinear character. While the linear wave theory predicts sinusoidal wave profiles, one needs a nonlinear setting to explain observed wave trains that are almost flat near the trough and exhibit a pronounced elevation near the crest.

The substantial difficulties caused by the presence of (piecewise constant) vorticity, by the fluid stratification and by the nonlinearities occurring in the governing equations can be mitigated if, for instance, a concise formulation that underlines the structural properties of the governing equations becomes available. A first step in this direction was made by Zakharov [38], who showed that the governing equations of two-dimensional irrotational gravity water waves over a flow of infinite

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depth possess a Hamiltonian structure with the canonical variables given by the surface elevation and the velocity potential evaluated at the free surface. A nearly-Hamiltonian formulation for two-dimensional periodic water waves with constant vorticity over flows with finite depth was achieved in [8]. The extent by which the nearly-Hamiltonian formulation in [8] fails to be Hamiltonian is given by the vorticity, the latter hindrance being dealt with in [35] where it was proved that the formulation in [8] is in fact Hamiltonian with respect to some non-canonical symplectic structure.

While a Hamiltonian formulation for water waves in stratified irrotational flows was achieved in [14], the corresponding formulation for stratified gravity flows with a piecewise constant vorticity distribution, thus allowing for linearly sheared currents, is very recent, cf. [12, 13].

These underlying currents are generated by the prevalent regular wind patterns (see the discussion in [6, 24, 25, 31]). Irregular wind bursts generate capillary waves that interact with these currents, and in their modelling it is reasonable to consider the inviscid setting since Reynolds numbers in geophysical fluid dynamics are very large, cf. [28].

While the Hamiltonian formulation for water waves over stratified flows driven only by capillarity and with piecewise constant vorticity was achieved in [26], we aim to show here that the nonlinear governing equations of water waves driven by gravity and surface tension over stratified flows with piecewise constant distribution of vorticity also have a Hamiltonian formulation. Note that wave-current interactions give rise to the possible appearance of critical layers, that is, surfaces where the wave speed equals the mean flow speed, see [10, 11, 21, 27, 29, 36] for a comprehensive account on critical layers in the context of a homogeneous fluid, and [6] for a discussion about their relevance for equatorial flows.

The Hamiltonian formulation is important in the study of the complicated equatorial ocean dynamics. We intend to pursue this direction in a future work, where we plan to include geophysical effects stemming from the rotation of the Earth, combined with the effects of gravity and surface tension, stratification and the presence of depth-dependent current fields.

2. The governing equations. Under consideration here is a two-dimensional periodic water flow, acted upon by gravity and surface tension. The water domain is bounded below by the rigid flat bed \( y = -h \), (with \( h > 0 \)), and above by the free surface \( y = \eta_1(x, t) + h_1 \), which is a perturbation of the flat free surface \( y = h_1 \).

The free surface wave propagates in the positive \( x \)-direction, while the \( y \) axis points vertically upwards. Here, \( h_1 > 0 \) is a constant, \( t \) stands for time and \( x \to \eta_1(x, t) \) is a periodic function in the spatial variable \( x \), of principal period \( L \), and which has mean zero, that is, \( \int_0^L \eta_1(x, t) \, dx = 0 \) for all \( t \geq 0 \) for all \( t \).

The stratification of the fluid is as follows: we assume that, neighboring the flat bed \( y = -h \), the water domain consists of a layer

\[ \Omega^* := \{ (x, y, t) : x \in \mathbb{R}, t \in \mathbb{R}, -h < y < \eta(x, t) \}, \]

of constant density \( \rho \), separated by the interface \( y = \eta(x, t) \) from the free-surface adjacent layer

\[ \Omega^*_1 := \{ (x, y, t) : x \in \mathbb{R}, t \in \mathbb{R}, \eta(x, t) < y < h_1 + \eta_1(x, t) \}, \]

of constant density \( \rho_1 < \rho \). The interface \( x \to \eta(x, t) \) is, at any fixed time \( t \), an \( L \) periodic function with zero mean.
Remark 1. The bold text notation used in due course of the paper refers to quantities that are defined in both layers $\Omega^*$ and $\Omega_1^*$.

Denoting with $(u(x, y, t), v(x, y, t))$ the velocity field and with $P = P(x, y, t)$ the pressure, the equations of motion are Euler’s equations
\[
\begin{align*}
\begin{cases}
u_t + \nu u_x + \nu u_y &= -\frac{1}{\rho} P_x, \\
v_t + \nu v_x + \nu v_y &= -\frac{1}{\rho} P_y - g,
\end{cases}
\end{align*}
\tag{1}
\]
supplemented by the equation of mass conservation
\[u_x + v_y = 0 \text{ in } \Omega^* \cup \Omega_1^*.
\tag{2}\]

In order to include underlying currents in our discussion and to treat wave-current interactions (see [9]) we need to allow the presence of vorticity in the flow. The vorticity is defined through
\[\gamma := u_y - v_x,\]
and measures local rotation. In our setting the vorticity is constant throughout each layer, but discontinuous across the interface. Therefore, it takes the form
\[\gamma = \begin{cases}
\gamma & \text{in } \Omega^*, \\
\gamma_1 & \text{in } \Omega_1^*,
\end{cases}\]
where $\gamma, \gamma_1 \in \mathbb{R}$ are constants with $\gamma \neq \gamma_1$.

To specify the water wave problem we impose appropriate boundary conditions as follows: at the free surface, the dynamic boundary condition incorporates surface tension effects, reads as
\[P = P_{atm} - \sigma_1 \frac{\eta_{1,xx}}{(1 + \eta_{1,x})^{3/2}},\]
\tag{4}
(with $P_{atm}$ being the constant atmospheric pressure at the surface of the ocean and $\sigma_1$ the coefficient of surface tension at the free surface) and decouples the motion of the water from that of the air above. The kinematic boundary conditions, which refer to the flat bed $y = -h$, the interface $y = \eta(x, t)$ and the free surface $y = h_1 + \eta_1(x, t)$, ensure that a particle once on one of the three boundaries will remain confined to it; they read as
\[v_1 = \eta_{1,t} + u_1 \eta_{1,x} \text{ on } y = \eta_1(x, t) + h_1,\]
\[v_1 = \eta_t + u_1 \eta_x \text{ \ and } v = \eta_t + u \eta_x \text{ on } y = \eta(x, t),\]
\[v = 0 \text{ \ on } y = -h,\]
\tag{5, 6, 7}

with $(u, v)$ and $(u_1, v_1)$ being the velocity fields in $\Omega_1^*$ and $\Omega^*$, respectively. The balance of forces at the interface $y = \eta(x, t)$ is expressed by the continuity of the pressure along this internal boundary, that is,
\[P = P_1 \text{ \ on } y = \eta(x, t).\]
\tag{8}

2.1. Mathematical reformulation of the physical problem. We intend to write the velocity field with the help of the stream function and of the (generalized) velocity potential, to be defined in due course. The latter quantities will be then instrumental in bringing the physical problem into a mathematical form that can be handled easier.

The domains occupied by the two fluid layers at a fixed time $t$ are denoted by
\[\Omega = \Omega(t) := \{(x, y) : x \in (0, L), -h < y < \eta(x, t)\},\]
and \( \Omega_1 = \Omega_1(t) := \{(x, y) : x \in (0, L), \eta(x, t) < y < h_1 + \eta_1(x, t)\} \), respectively.

### 2.1.1. The generalized velocity potential

Due to (3) we are able to introduce in each layer a (generalized) velocity potential, denoted \( \phi \) in \( \Omega \) and \( \phi_1 \) in \( \Omega_1 \), that satisfies

\[
\begin{align*}
\left\{ \begin{array}{ll}
u = \phi_x + \gamma y & \text{in } \Omega, \\
u_1 = \phi_{1x} + \gamma_1 y & \text{in } \Omega_1,
\end{array} \right.
\end{align*}
\]

(9)

Up to functions that depend only on time, the generalized velocity potentials are given by

\[
\phi(x, y, t) = \int_0^x (u(l, -h, t) + \gamma h) dl + \int_{-h}^y v(x, l, t) dl, \quad \text{for } (x, y) \in \Omega,
\]

(10)

while, for \((x, y) \in \Omega_1\), we have

\[
\phi_1(x, y, t) = \int_0^x \left[ u_1(l, \eta(l, t), t) - \gamma_1 \eta(l, t) + v_1(l, \eta(l, t)) \eta_x(l, t) \right] dl
\]

\[
+ \int_{\eta(x, t)}^y v_1(x, l, t) dl.
\]

(11)

It follows from (10) that

\[
\phi(x + L, y, t) - \phi(x, y, t) = \kappa L,
\]

where

\[
\kappa := \frac{1}{L} \int_0^L \int_{-h}^y u(x, -h, t) + \gamma h dx,
\]

satisfies \( \kappa'(t) = 0 \) for all \( t \), as can be seen from the first equation in (1), (7) and the periodicity of the velocity field and of the pressure \( P \). As a consequence, the function \((x, y) \to \phi(x, y, t) - \kappa x\) is periodic in the \( x \)-variable, of period \( L \).

The quantity \( \kappa \) is also related to the current underlying the interface, denoted by \( U(y, t) \) and defined at the level \( y \) below the trough of the internal wave \( y = \eta(x, t) \) by

\[
U(y, t) := \frac{1}{L} \int_0^L u(x, y, t) dx.
\]

(12)

The above mentioned relation reads as

\[
U(y, t) = \gamma y + \kappa,
\]

(13)

and it is a consequence of the definition of \( \gamma \) and the periodicity of \( v \).

An analogous discussion can be performed for \( \phi_1 \), the potential associated to the upper layer \( \Omega_1 \), where the underlying current at level \( y \), above the crest of the internal wave \( y = \eta(x, t) \) and below the trough of the surface wave \( y = h_1 + \eta_1(x, t) \), is defined by

\[
U_1(y, t) := \frac{1}{L} \int_0^L u_1(x, y, t) dx.
\]

(14)

We set

\[
\kappa_1 := \frac{1}{L} \int_0^L \left[ u_1(x, \eta(x, t), t) + v_1(x, \eta(x, t), t) \eta_x(x, t) \right] dx.
\]

(15)

As with \( \kappa \), we see that \( \kappa_1 \) is also independent of \( t \), as it was showed in [13]. We also have that

\[
U_1(y, t) = \gamma_1 y + \kappa_1.
\]

(16)
\[ \varphi_1(x + L, y, t) - \varphi_1(x, y, t) = \kappa_1 L, \]

As a consequence, the function \( (x, y) \to \varphi_1(x, y, t) - \kappa_1 x \) is periodic in the \( x \)-variable, of period \( L \).

Motivated by the discussions in the previous remarks and in view of (9), we may write

\[
\left\{ \begin{array}{l}
\psi_1 = \tilde{\varphi}_1 + \gamma y + \kappa \\
u_1 = \tilde{\varphi}_1 + \gamma_1 y + \kappa_1
\end{array} \right., \ 	ext{in } \Omega,
\]

where the functions \( \tilde{\varphi}(x, y) := \varphi(x, y) - \kappa x \) and \( \tilde{\varphi}_1(x, y) := \varphi_1(x, y) - \kappa_1 x \) are periodic in the \( x \)-variable, of period \( L \). Thus, equation (18) represents a splitting of the velocity field into an underlying steady current component and a periodic harmonic wave velocity field.

The kinematic boundary conditions (5) and (6) can now be written as

\[ \eta_{1,t} = (\tilde{\varphi}_{1,y})_{s_1} - \eta_{1,x}[(\tilde{\varphi}_{1,x})_{s_1} + \gamma_1 (h_1 + \eta_1) + \kappa_1] \]  
and

\[ \eta_{t} = (\tilde{\varphi}_{1,y})_{s} - \eta_{x}[(\tilde{\varphi}_{1,x})_{s} + \gamma \eta + \kappa] = (\tilde{\varphi}_{y})_{s} - \eta_{x}[(\tilde{\varphi}_{x})_{s} + \gamma \eta + \kappa], \]

respectively; the subscript \( s_1 \) stands for traces on the free surface \( y = h_1 + \eta_1(x, t) \).

2.1.2. The stream function. From the equation of mass conservation (2) we conclude the existence of two stream functions denoted \( \psi \) in \( \Omega \), and \( \psi_1 \) in \( \Omega_1 \), satisfying

\[
\left\{ \begin{array}{l}
u = \psi_y, \\
u_1 = \psi_1_y, \\
u_1 = -\psi_{1,x}
\end{array} \right., \ 	ext{in } \Omega,
\]

In fact, from above and using also (7) we see that the expressions of the stream functions are

\[
\left\{ \begin{array}{l}
\psi(x, y, t) = \int_{-h}^{y} u(x, l, t) dl + \psi_{-}(t), \ (x, y) \in \Omega, \\
\psi_1(x, y, t) = \int_{z=0}^{x} [u_1(l, \eta(l, t), t) \eta(l, t) - v_1(l, \eta(l, t), t)] \\
+ \int_{y(x, t)}^{y} u_1(x, l, t) dl + \psi_{+}(t), \ (x, y) \in \Omega_1
\end{array} \right.
\]

for some smooth functions \( \psi_{-}, \psi_{+} \), that depend only on time.

Equalities (6) and (21) show that the stream functions \( \psi \) and \( \psi_1 \) differ only by a function of time at the interface \( y = \eta(x, t) \). Therefore, we can choose \( \psi_{-} \) and \( \psi_{+} \) above such that there exists \( \psi \in C(\overline{\Omega \cup \Omega_1}) \) with \( \psi = \psi \) in \( \overline{\Omega} \) and \( \psi = \psi_1 \) in \( \overline{\Omega_1} \). The latter remark justifies the notation

\[ \chi(x, t) := \psi(x, \eta(x, t), t) - \psi(0, \eta(0, t), t) = \psi_1(x, \eta(x, t), t) - \psi_1(0, \eta(0, t), t). \]

Setting also

\[ \chi_1(x, t) := \psi_1(x, h_1 + \eta_1(x, t), t) - \psi_1(0, h_1 + \eta_1(0, t), t), \]

we are now able to rewrite the kinematic boundary conditions (5) and (6) in a more concise form. Namely, they can be written as

\[ \chi(x, t) = -\int_{0}^{x} \eta_1(x', t) dx', \]

\[ \chi_1(x, t) = -\int_{0}^{x} \eta_{1,t}(x', t) dx', \]

formulas that will be useful later on.
Note also that, cf. [13], \( \psi \) is periodic of period \( L \) in the \( x \)-variable. The stream functions are related to the vorticity by the following relations

\[
\Delta \psi = \gamma \quad \text{in} \quad \Omega, \quad \Delta \psi_1 = \gamma_1 \quad \text{in} \quad \Omega_1.
\]

The generalized velocity potentials and the stream functions are beneficial in recasting Euler’s equations as

\[
\nabla \left[ \hat{\varphi}_t + \frac{1}{2} \nabla |\psi|^2 + \frac{P}{\rho} - \gamma \psi + gy \right] = 0 \quad \text{in} \quad \Omega \cup \Omega_1,
\]

and therefore

\[
\hat{\varphi}_t + \frac{1}{2} \nabla |\psi|^2 - \gamma \psi + \frac{P}{\rho} + gy = f(t) \quad \text{in} \quad \Omega,
\]

and

\[
\hat{\varphi}_{1,t} + \frac{1}{2} \nabla |\psi_1|^2 - \gamma_1 \psi_1 + \frac{P_1}{\rho_1} + gy = f_1(t) \quad \text{in} \quad \Omega_1,
\]

for some functions \( f \) and \( f_1 \). Using (4) and (29), we have that on \( y = h_1 + \eta_1(x,t) \)

\[
\hat{\varphi}_{1,t} + \frac{1}{2} \nabla |\psi_1|^2 - \gamma_1 \psi_1 - \frac{\sigma_1}{\rho_1} \frac{\eta_{1,xx}}{1 + \eta_{1,x}^2} + \eta \eta_1 = -\gamma_1 \psi_1(0, h_1 + \eta_1(0,t), t),
\]

provided

\[
f_1(t) = \frac{P_{atm}}{\rho_1} - \gamma_1 \psi_1(0, h_1 + \eta_1(0,t), t).
\]

With the help of the function \( \chi_1 \) introduced in (24) we can express (30) as

\[
\hat{\varphi}_{1,t} + \frac{1}{2} \nabla |\psi_1|^2 - \gamma_1 \chi_1 - \frac{\sigma_1}{\rho_1} \frac{\eta_{1,xx}}{1 + \eta_{1,x}^2} + \eta_1 = 0 \quad \text{on} \quad y = h_1 + \eta_1(x,t).
\]

Moreover, utilizing the function \( \chi \) from (23) we are able to reformulate (8) as

\[
\rho \left( \hat{\varphi}_t + \frac{1}{2} \nabla |\psi|^2 - \gamma \chi + gy \right) = \rho_1 \left( \hat{\varphi}_{1,t} + \frac{1}{2} \nabla |\psi_1|^2 - \gamma_1 \chi + gy \right),
\]

provided

\[
f(t) = \frac{\rho_1}{\rho} (f_1(t) + \gamma_1 \psi_1(0, \eta(0,t), t)) - \gamma \psi(0, \eta(0,t), t).
\]

**Remark 2.** The pressure in the fluid can be recovered by means of the stream functions \( \psi, \psi_1 \), of the perturbed velocity potentials \( \varphi, \varphi_1 \) and of \( \eta \) and \( \eta_1 \) by setting

\[
P_1 = P_{atm} - \rho_1 [\hat{\varphi}_{1,t} + \frac{1}{2} \nabla |\psi_1|^2 - \gamma_1 \psi_1 + gy + \gamma_1 \psi_1(0, h_1 + \eta_1(0,t), t)] \quad \text{in} \quad \Omega_1,
\]

\[
P = P_{atm} - \rho [\hat{\varphi}_t + \frac{1}{2} \nabla |\psi|^2 - \gamma \psi + gy] - \rho \gamma \psi(0, \eta(0,t), t)
\]

\[
- \rho_1 \gamma_1 \psi_1(0, h_1 + \eta_1(0,t), t) - \psi_1(0, \eta(0,t), t) \quad \text{in} \quad \Omega.
\]

3. **The Hamiltonian by means of the Dirichlet-Neumann operator.** The main object of study in this section is the Hamiltonian functional given by the total energy of the flow by means of the formula

\[
H = \int \int_{\Omega \cup \Omega_1} \rho \left( \frac{u^2 + v^2}{2} + gy \right) dydx + \sigma_1 \int_{h_1 + \eta_1} dl,
\]

where the first term in the double integral represents the kinetic energy (energy of motion), \( \rho gy \) is the gravitational potential energy (energy of position), while the second integral above stands for the free energy of the surface.
In due course we will be concerned with proving the existence of a density function \( \mathcal{h} \) which depends solely on the scalar variables \( \xi, \xi_1, \eta, \eta_1 \), on the trace of the (generalized) velocity potential \( \tilde{\varphi}_1 \) on the free surface, on the combined action of the two (generalized) potentials \( \tilde{\varphi} \) and \( \tilde{\varphi}_1 \) on the interface, as well as on the spatial derivatives of the mentioned quantities and which has the property that

\[
H = \int_0^L \mathcal{h} \, dx.
\]  

(35)

Formula (35) will be instrumental in proving the nearly-Hamiltonian formulation of the governing equations in Theorem 4.1.

Owing to the stratification of the fluid, the functional \( H \) equals

\[
H = \frac{1}{2} \int_0^L \int_{-h}^\eta \rho (u^2 + v^2) \, dy \, dx' + \frac{1}{2} \int_0^L \int_{\eta(x,t)}^{h_1 + \eta_1(x,t)} \rho_1 (u_1^2 + v_1^2) \, dy \, dx' \\
+ \int_0^L \int_{-h}^\eta g \rho y \, dy \, dx' + \int_0^L \int_{\eta(x,t)}^{h_1 + \eta_1(x,t)} g \rho_1 y_1 \, dy \, dx' \\
+ \frac{1}{2} \int_0^L \sqrt{1 + \eta_{1,x}^2} \, dx,
\]

(36)

which, by (18), can be rewritten as

\[
H = \frac{\rho_1}{2} \int_0^L \int_{-h}^\eta (\tilde{\varphi}_1)^2 \, dy \, dx + \rho \gamma \int_0^L \int_0^\eta y \tilde{\varphi}_x \, dy \, dx + \frac{\rho \gamma^2}{6} \int_0^L (\eta^3 + h^3) \, dx \\
+ \rho k \int_0^L \int_{-h}^\eta \tilde{\varphi}_x \, dy \, dx + \frac{\rho \gamma k}{2} \int_0^L (\eta^2 - h^2) \, dx + \frac{\rho k^2}{2} \int_0^L (\eta + h) \, dx \\
+ \frac{\rho_1}{2} \int_0^L \int_\eta^{h_1 + \eta_1} |\tilde{\varphi}_1|^2 \, dy \, dx + \rho_1 \gamma_1 \int_0^L \int_\eta^{h_1 + \eta_1} y_1 \tilde{\varphi}_1,x \, dy \, dx \\
+ \frac{\rho_1 \gamma_1^2}{2} \int_0^L (h_1 + \eta_1)^3 - \eta^3 \, dx/3 \\
+ \rho_1 \kappa_1 \int_0^L \int_\eta^{h_1 + \eta_1} \tilde{\varphi}_1,x \, dy \, dx + \frac{\rho_1 \gamma_1 \kappa_1}{2} \int_0^L ((h_1 + \eta_1)^2 - \eta^2) \, dx \\
+ \frac{\rho_1 k_1^2}{2} \int_0^L (h_1 + \eta_1 - \eta) \, dx \\
+ \frac{\rho g}{2} \int_0^L (\eta^2 - h^2) \, dx + \frac{\rho_1 g}{2} \int_0^L ((h_1 + \eta_1)^2 - \eta^2) \, dx \\
+ \sigma_1 \int_0^L \sqrt{1 + \eta_{1,x}^2} \, dx.
\]

(37)

We proceed now in proving the claim made in (35). In doing so, we will make use of the Dirichlet-Neumann operators associated to the lower layer and to the upper layer, respectively. For computational aspects pertaining to the Dirichlet-Neumann operators related to water waves we refer the reader to the recent survey paper [37] and to [30].

Given smooth, \( L \)-periodic scalar functions \( \Phi \) and \( \eta \) such that \( \eta(x) > -h \) for all \( x \in [0, L] \), we denote with \( \tilde{\varphi} \) the unique \( L \)-periodic in the \( x \) variable smooth solution
of the boundary value problem
\[
\begin{cases}
\Delta \tilde{\varphi} = 0 & \text{in } \Omega^*(\eta), \\
\tilde{\varphi} = \Phi & \text{on } y = \eta(x), \\
\frac{\partial \tilde{\varphi}}{\partial n} = 0 & \text{on } y = -h,
\end{cases}
\]
where \(\Omega^*(\eta) = \{(x,y) : x \in \mathbb{R}, -h < y < \eta(x)\}\). Denoting with \(n\) the outward pointing normal vector along the upper boundary \(y = \eta(x)\) of the domain \(\Omega\), the Dirichlet-Neumann operator \(G = G(\eta)\) associated to \(\Omega^* = \Omega^*(\eta)\) is defined by mapping the Dirichlet data \(\Phi\) to the normal derivative of the solution on the upper boundary,
\[
G\Phi := \sqrt{1 + \eta_x^2} \frac{\partial \tilde{\varphi}}{\partial n}_{y = \eta(x)}.
\]
To define the Dirichlet-Neumann operator associated to the upper layer, we consider \(L\)-periodic functions \(\eta_1, \Phi_1, \Phi_2\), such that \(\eta(x) < h_1 + \eta_1(x)\) for all \(x \in [0,L]\). We then set \(\tilde{\varphi}_1\) to be the unique \(L\)-periodic in the \(x\) variable solution of the Dirichlet boundary value problem
\[
\begin{cases}
\Delta \tilde{\varphi}_1 = 0 & \text{in } \Omega_1^*(\eta, \eta_1), \\
\tilde{\varphi}_1 = \Phi_1 & \text{on } y = \eta(x), \\
\tilde{\varphi}_1 = \Phi_2 & \text{on } y = h_1 + \eta_1(x),
\end{cases}
\]
where \(\Omega_1^*(\eta, \eta_1) = \{(x,y) : x \in \mathbb{R}, \eta(x) < y < h_1 + \eta_1(x)\}\). The Dirichlet-Neumann operator \(G_1 = G_1(\eta, \eta_1)\) associated to \(\Omega_1^* = \Omega_1^*(\eta, \eta_1)\) is defined by the normal derivatives of the solution on the lower and upper boundaries of the domain \(\Omega_1^*\) through the formula
\[
G_1(\Phi_1, \Phi_2) := \begin{pmatrix}
-\sqrt{1 + \eta_x^2} \frac{\partial \tilde{\varphi}_1}{\partial n}_{y = \eta(x)} \\
\sqrt{1 + \eta_1_x^2} \frac{\partial \tilde{\varphi}_1}{\partial n}_{y = \eta_1(x) + h_1}
\end{pmatrix},
\]
where \(n_1\) denotes the outward unit normal vector along the upper boundary \(y = h_1 + \eta_1(x)\) and \(n\) has the same meaning as in the definition of \(G\). The entries of the matrix operator \(G_1(\eta, \eta_1)\) are denoted by
\[
G_1(\eta, \eta_1) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.
\]
For an account on the properties of the Dirichlet-Neumann operators we refer the reader to [10, 14, 22].

We will now specify the functions \(\Phi, \Phi_1, \Phi_2\) to represent the restriction of the (generalized) velocity potentials \(\tilde{\varphi}\) and \(\tilde{\varphi}_1\) to the free surface and to the interface, respectively. More precisely we set
\[
\begin{cases}
\Phi(x,t) = \tilde{\varphi}(x, \eta(x,t), t), \\
\Phi_1(x,t) = \tilde{\varphi}_1(x, \eta(x,t), t), \\
\Phi_2(x,t) = \tilde{\varphi}_1(x, h_1 + \eta_1(x,t), t).
\end{cases}
\]
To capture the effect of the two (generalized) velocity potentials \(\tilde{\varphi}\) and \(\tilde{\varphi}_1\) along the interface \(y = \eta(x,t)\) we set
\[
\xi := \rho \Phi - \rho_1 \Phi_1.
\]
We will also need to introduce the variable
\[
\xi_1 := \rho_1 \Phi_2.
\]
From the definition of the Dirichlet-Neumann operators and (20) we have that
\[ G_{11} \Phi_1 + G_{12} \Phi_2 = (\tilde{\varphi}_{1x})_s \eta_x - (\tilde{\varphi}_{1y})_s = -(\eta_t + \gamma_1 \eta_{tx}) - \kappa_1 \eta_x, \quad (46) \]
and
\[ G \Phi = -\tilde{\varphi}_x \eta_x + \tilde{\varphi}_y = \eta_t + \gamma \eta_{tx} + \kappa \eta_x. \quad (47) \]
Moreover, from (19) we have
\[ G_{21} \Phi_1 + G_{22} \Phi_2 = -(\tilde{\varphi}_{1x})_s \eta_{1x} + (\tilde{\varphi}_{1y})_{s_1} = \eta_{1t} + [\gamma_1 (h_1 + \eta_1) + \kappa_1] \eta_{1x}. \quad (48) \]
Adding up the relations (46) and (47), we obtain
\[ G_{11} \Phi_1 + G_{12} \Phi_2 + G \Phi = (\gamma - \gamma_1) \eta_x + (\kappa - \kappa_1) \eta_x \quad (49) \]
Denoting
\[ B = B(\eta, \eta_1) := \rho_1 G + \rho G_{11} \]
and making use of (44), (45) and (19) we can express \( \Phi, \Phi_1, \Phi_2 \) in terms of \( \xi \) and \( \xi_1 \) as follows
\[ \Phi = B^{-1} (G_{11} \xi - G_{12} \xi_1 + \rho_1 (\gamma - \gamma_1) \eta_{tx} + \rho_1 (\kappa - \kappa_1) \eta_x) \quad (50) \]
\[ \Phi_1 = B^{-1} \left( -G \xi - \frac{\rho}{\rho_1} G_{12} \xi_1 + \rho (\gamma - \gamma_1) \eta_{tx} + \rho (\kappa - \kappa_1) \eta_x \right) \quad (51) \]
\[ \Phi_2 = \frac{1}{\rho_1} \xi_1. \quad (52) \]
With the help of the Dirichlet-Neumann operators, using Green’s second identity and (46) - (48) we obtain,
\[
K_1 := \frac{\rho_1}{2} \int_0^L \int_{-h}^h |\nabla \tilde{\varphi}|^2 dy dx + \frac{\rho_1}{2} \int_0^L \int_{\eta}^{h_1 + \eta_1} |\nabla \tilde{\varphi}_1|^2 dy dx
\]
\[
= \frac{\rho}{2} \int_0^L \left( \frac{\partial \tilde{\varphi}}{\partial n} \right)_s \sqrt{1 + \eta_x^2} dx
\]
\[
+ \frac{\rho_1}{2} \int_0^L \left[ \left( \tilde{\varphi}_{1x} \right)_s \sqrt{1 + \eta_{tx}^2} - \left( \tilde{\varphi}_{1t} \right)_s \sqrt{1 + \eta_x^2} \right] dx
\]
\[
= \frac{\rho}{2} \int_0^L G \Phi \Phi dx + \frac{\rho_1}{2} \int_0^L \left( \Phi_1 \Phi_2 \right)^T \left( \begin{array}{cc}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{array} \right) \left( \begin{array}{c}
\Phi_1 \\
\Phi_2
\end{array} \right) dx
\]
\[
= \frac{\rho}{2} \int_0^L \Phi (\eta_t + \gamma \eta_{tx} + \kappa \eta_x) dx
\]
\[
+ \frac{\rho_1}{2} \int_0^L (\Phi_1 \Phi_2) \left( \begin{array}{c}
-\eta_t - \gamma_1 \eta_{tx} - \kappa_1 \eta_x \\
\eta_{1t} + \gamma_1 (h_1 + \eta_1) \eta_{1x} + \kappa_1 \eta_{11x}
\end{array} \right) dx
\]
\[
= \frac{1}{2} \int_0^L \left[ \xi \eta_t + \xi_1 (\eta_{1t} + \gamma_1 (h_1 + \eta_1) + \kappa_1) \eta_{1x}
\right.
\]
\[
\left. + (\rho \gamma \Phi - \rho_1 \gamma_1 \Phi_1) \eta_x + (\rho \kappa \Phi - \rho_1 \kappa_1 \Phi_1) \eta_x \right] dx
\]
\[
= \frac{1}{2} \int_0^L \left( \xi \eta \right) \left( \eta_{1x} + (h_1 + \eta_1 + \gamma_1) \eta_{11x} \right) dx
\]
\[
+ \frac{1}{2} \int_0^L (\rho \gamma \Phi - \rho_1 \gamma_1 \Phi_1) \eta_x dx + \frac{1}{2} \int_0^L (\rho \kappa \Phi - \rho_1 \kappa_1 \Phi_1) \eta_x dx
\]
Using now the periodicity of $\tilde{\varphi}, \tilde{\varphi}_1, \eta, \eta_1$ we obtain

$$K_2 := \rho \gamma \int_0^L \int_0^\eta y \tilde{\varphi}_x dy dx + \rho_1 \gamma_1 \int_0^L \int_0^{h_1 + \eta_1} y \tilde{\varphi}_{1,x} dy dx$$

$$= -\rho \gamma \int_0^L \Phi \eta_1 dx - \rho_1 \gamma_1 \int_0^L \Phi_2 (\eta_1 + h_1) \eta_{1,x} dx + \rho_1 \gamma_1 \int_0^L \Phi_1 \eta_{1,x}$$

where, to pass to the second equality, we have used the rule

$$\frac{\partial}{\partial x} \left[ \int_{f_1(x)}^{f_2(x)} F(x, y) dy \right] = \int_{f_1(x)}^{f_2(x)} F_x(x, y) dy$$

for differentiable functions $f_1, f_2$ and $F$.

Similarly, as for the calculation of $K_2$, we obtain

$$K_3 := \rho \kappa \int_0^L \int_0^\eta \tilde{\varphi}_x dy dx + \rho_1 \kappa_1 \int_0^L \int_0^{h_1 + \eta_1} \tilde{\varphi}_{1,x} dy dx$$

$$= -\rho \kappa \int_0^L \Phi \eta_{1,x} dx - \rho_1 \kappa_1 \int_0^L \Phi_2 \eta_{1,x} dx + \rho_1 \kappa_1 \int_0^L \Phi_1 \eta_{1,x}$$

Summing up the relations for $K_1, K_2$ and $K_3$, we get

$$K_1 + K_2 + K_3 = \frac{1}{2} \int_0^L \left( \xi \right)^T \left( \begin{array}{c} -G_{11} & -G_{12} \\ G_{21} & G_{22} \end{array} \right) \left( \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right) dx$$

$$\quad - \frac{1}{2} \int_0^L \left( (\rho \gamma \Phi - \rho \gamma_1 \Phi_1) \eta + \rho \kappa \Phi - \rho_1 \kappa_1 \Phi_1 \right) \eta_x dx$$

With the notation

$$\mu = \mu(\eta) := (\gamma - \gamma_1) \eta + \kappa - \kappa_1) \eta_x,$$

taking into account (50)-(52), and in view of the equalities

$$\rho \gamma \Phi - \rho \gamma_1 \Phi_1 = \rho (\gamma - \gamma_1) \Phi + \xi \gamma_1, \quad \rho \kappa \Phi - \rho_1 \kappa_1 \Phi_1 = \rho (\kappa - \kappa_1) \Phi + \xi \kappa_1,$$
we obtain that
\[ K_1 + K_2 + K_3 = \frac{1}{2} \int_0^L \left( \frac{G_{11}B^{-1}G^*}{\rho_1} \rho_1 G_{11}B^{-1}G_{12} \right) (\xi) \, dx 
+ \frac{1}{2} \int_0^L \left( \frac{G_{11}B^{-1}G^*}{\rho_1} \rho_1 G_{11}B^{-1}G_{12} \right) (\xi) \, dx 
- \frac{1}{2} \int_0^L \left( \frac{G_{11}B^{-1}G^*}{\rho_1} \rho_1 G_{11}B^{-1}G_{12} \right) (\xi) \, dx \]

where, in the second equality, we have used the definition of \( B \) to derive that
\[ \frac{\rho}{\rho_1} G_{11}B^{-1}G_{12} - \frac{1}{\rho_1} G_{12} = \frac{1}{\rho_1} (B - \rho_1 G)B^{-1}G_{12} - \frac{1}{\rho_1} G_{12} = -GB^{-1}G_{12}, \]
and the last equality is obtained by employing again the definition of the operator \( B \), and using that the operators \( B^{-1}, G_{11} \) are self-adjoint, while \( G_{12}^* = G_{21} \), cf. (14).

The proof of the claim about the functional dependence of the total energy \( H \) made in (35) emerges now by summarizing the above computations in the formula
\[ H = K_1 + K_2 + K_3 + \frac{\rho^2}{6} \int_0^L (\eta^3 + h^3) \, dx + \frac{\rho_1 \gamma_1^2}{6} \int_0^L ((h_1 + \eta_1)^3 - \eta^3) \, dx \]
\[ + \frac{\rho (\gamma + \kappa)}{2} \int_0^L (\eta^2 - h^2) \, dx + \frac{\rho_1 (\gamma_1 \kappa + g)}{2} \int_0^L ((h_1 + \eta_1) - \eta^2) \, dx, \]
\[ + \frac{\rho_1 \gamma_1^2 h_0}{2} + \frac{\rho_1 \kappa_1^2 h_1 L}{2} + s_1 \int_0^L \sqrt{1 + \eta_1^2} \, dx. \]

4. Re-formulation of the governing equations and of their boundary conditions. The purpose of this section is to show that the governing equations for
water waves, given in the Section 2 may be reduced to a Hamiltonian system involving functions of one variable. The latter are the free surface $\eta_1$, the interface $\eta$ and the evaluations of the perturbed velocity potentials $\tilde{\varphi}$ and $\tilde{\varphi}_1$.

4.1. The nearly-Hamiltonian formulation. We first compute the variations of the Hamiltonian functional

$$H = \int_\Omega \rho \left( \frac{u^2 + v^2}{2} + gy \right) dydx + \sigma_1 \int_{h_1 + \eta_1} dl,$$

with respect to $\eta, \eta_1, \xi$, and $\xi_1$, respectively. To this end, we recall that the functions $\Phi, \Phi_1$ and $\Phi_2$, which will appear below, have the same meaning as in [23]. Moreover, $\xi$ and $\xi_1$ are defined in [44] and [45].

In order to compute the variations of $H$ we need a couple of formulas concerning harmonic functions and variational calculus, which we state in the next remark.

Remark 3. (i) If $F(x) = \int_{g_1(x)}^{g_2(x)} f(y) dy$, then (cf. [15]) it holds that

$$\delta F(x) = \int_{g_1(x)}^{g_2(x)} \delta f(y) dy + f(g_2(x))\delta g_2(x) - f(g_1(x))\delta g_1(x). \quad (62)$$

(ii) For harmonic functions $\theta_1$ and $\theta_2$, the identity $\nabla \cdot (\theta_1 \nabla \theta_2) = (\nabla \theta_1) \cdot (\nabla \theta_2)$ holds. From the latter formula we obtain that for harmonic variations $\delta \tilde{\varphi}$ of $\tilde{\varphi}$ we have

$$\delta((\nabla \tilde{\varphi}) \cdot (\nabla \varphi)) = 2\nabla \cdot (\delta \tilde{\varphi} \nabla \varphi). \quad (63)$$

Throughout this section we rely on formula (37). For computing the variation of the first term in (37), we use rule (63), apply the divergence theorem in the domain

$$D := \{(x, y) : 0 < x < L, 0 < y < \eta(x)\} \quad (with \ n \ being \ the \ outward \ unit \ normal \ to \ its \ boundary)$$

and owing to the periodicity of $\tilde{\varphi}$ obtain

$$\int_0^L \int_{-h}^\eta \delta |\nabla \tilde{\varphi}|^2 dydx = 2 \int_{\partial D} \delta \tilde{\varphi} \nabla \varphi \cdot n$$

$$= 2 \int_0^L [\tilde{\varphi}_y - \eta_x \tilde{\varphi}_x]_s (\delta \tilde{\varphi})_s dx \quad (64)$$

$$= 2 \int_0^L [\tilde{\varphi}_y - \eta_x \tilde{\varphi}_x]_s [\delta \Phi - (\tilde{\varphi}_y)_s \delta \eta] dx,$$

where, for the last equality above, we used

$$\frac{(\delta \Phi)(x, t)}{\epsilon} = \lim_{\epsilon \to 0} \frac{(\tilde{\varphi} + \epsilon \delta \tilde{\varphi})(x, \eta(x, t) + \epsilon \delta \eta, t) - (\tilde{\varphi}(x, \eta(x, t), t)}{\epsilon}$$

$$= (\tilde{\varphi}_y)_s \delta \eta + (\tilde{\varphi})_s. \quad (65)$$

Thus, from (62) and since the flat bed $y = -h$ is a fixed boundary, we have

$$\delta \left( \int_0^L \int_{-h}^\eta |\nabla \tilde{\varphi}|^2 dydx \right) = 2 \int_0^L [\tilde{\varphi}_y - \eta_x \tilde{\varphi}_x]_s [\delta \Phi - (\tilde{\varphi}_y)_s \delta \eta] dx$$

$$+ \int_0^L |\nabla \tilde{\varphi}|^2 \delta \eta dx. \quad (66)$$

Analogously, we have

$$\delta \left( \int_0^L \int_{\eta}^{h_1 + \eta_1} |\nabla \tilde{\varphi}_1|^2 dydx \right)$$
\[
\begin{align*}
\frac{\delta}{\delta \eta_1}\left( \int_0^L \sqrt{1 + \eta_1^2} \, dx \right) &= \int_0^L \frac{\partial \sqrt{1 + \eta_1^2}'}{\partial \eta_1} \cdot \frac{\delta \eta_1'(x')}{\delta \eta_1(x)} \, dx'
\end{align*}
\]
The governing equations admit the following nearly-Hamiltonian formulation

\[
\begin{align*}
\xi_t &= -\frac{\delta H}{\delta \eta} + (\rho \gamma - \rho_1 \gamma_1) \chi, \\
\eta_t &= \frac{\delta H}{\delta \xi}, \\
\xi_{1,t} &= -\frac{\delta H}{\delta \eta_1} + \rho_1 \gamma_1 \chi_1, \\
\eta_{1,t} &= \frac{\delta H}{\delta \xi_1}.
\end{align*}
\]

(73)

Proof. Throughout the proof we will use the variational formulas

\[
\delta \left( \int_0^L \eta^m \, dx \right) = m \int_0^L \eta^{m-1} \delta \eta \, dx,
\]

(74)

and

\[
\delta \left( \int_0^L (h_1 + \eta_1)^m \, dx \right) = m \int_0^L (h_1 + \eta_1)^{m-1} \delta \eta_1 \, dx,
\]

(75)

as well as the property established in (35).

We start by computing the variations of the functional \(H\) with respect to the variable \(\eta\).

\[
\frac{\delta H}{\delta \eta} = \rho (\tilde{\varphi}_y - \varphi_x \eta_x - \gamma \eta \eta_x)_s \cdot (-\tilde{\varphi}_y)_s + \rho \left( \frac{1}{2} |\nabla \tilde{\varphi}_1|^2 + \gamma \eta (\tilde{\varphi}_x)_s + \frac{\gamma^2}{2} \eta^2 + g \eta \right) \\
+ \rho_1 (\eta_x \tilde{\varphi}_1,x - \varphi_1,x + \gamma_1 \eta \eta_x)_s \cdot (-\tilde{\varphi}_1,y)_s \\
- \rho_1 \left( \frac{1}{2} |\nabla \tilde{\varphi}_1|^2 + \gamma_1 \eta (\tilde{\varphi}_1,x)_s + \frac{\gamma_1^2}{2} \eta^2 + g \eta \right) \\
+ \rho \kappa [(\tilde{\varphi}_x)_s + \eta_x (\tilde{\varphi}_y)_s] - \rho_1 \kappa_1 [(\tilde{\varphi}_1,x)_s + \eta_x (\tilde{\varphi}_1,y)_s] \\
+ \eta (\rho \kappa \gamma - \rho_1 \kappa_1 \gamma_1) + \frac{\rho \kappa^2}{2} - \frac{\rho_1 \kappa_1^2}{2}.
\]

Making use of formula (20) and of the identities

\[
\frac{1}{2} |\nabla \tilde{\varphi}_1|^2 + \frac{\gamma_1^2 \eta^2}{2} + \gamma_1 \eta (\tilde{\varphi}_1,x)_s = \frac{1}{2} |\nabla \psi|^2 - \frac{k^2}{2} - k (\tilde{\varphi}_x)_s - k \gamma \eta,
\]

(77)

\[
\frac{1}{2} |\nabla \tilde{\varphi}_1|^2 + \frac{\gamma^2 \eta^2}{2} + \gamma \eta (\tilde{\varphi}_x)_s = \frac{1}{2} |\nabla \psi|^2 - \frac{k^2}{2} - k (\tilde{\varphi}_x)_s - k \gamma \eta,
\]

(78)

we see that

\[
\frac{\delta H}{\delta \eta} = -\rho (\eta_t + \kappa \eta_x) (\tilde{\varphi}_y)_s + \rho_1 (\eta_t + \kappa_1 \eta_x) (\tilde{\varphi}_1,y)_s \\
+ \rho \left( \frac{1}{2} |\nabla \psi|^2 - \frac{k^2}{2} - k (\tilde{\varphi}_x)_s - k \gamma \eta + g \eta \right).
\]
Making use now of (30) and (72) yields
\begin{align*}
- \rho_1 \left( \frac{1}{2} |\nabla \psi_1|_{t_s}^2 - \frac{k_1^2}{2} - k_1(\tilde{\varphi}_{1,x})_s - k_1 \gamma_1 + g \eta \right) \\
+ \rho k \left[ (\tilde{\varphi}_x)_s + \eta_x(\tilde{\varphi}_y)_s \right] - \rho_1 \kappa_1 \left[ (\tilde{\varphi}_{1,x})_s + \eta_x(\tilde{\varphi}_{1,y})_s \right] \\
+ \eta (\rho k \gamma - \rho_1 \kappa_1 \gamma_1) + \frac{\rho k^2}{2} - \frac{\rho_1 \kappa_1^2}{2}.
\end{align*}
\tag{79}

Cancellations in the formula above and (33) lead further to
\begin{align*}
\frac{\delta H}{\delta \eta} &= - \rho \eta (\tilde{\varphi}_y)_s + \rho_1 \eta (\tilde{\varphi}_{1,y})_s \\
+ \rho \left( \frac{|\nabla \psi_1|_{t_s}^2}{2} + g \eta \right) - \rho_1 \left( \frac{|\nabla \psi_1|_{t_s}^2}{2} + g \eta \right) \\
&= - \rho [\eta (\tilde{\varphi}_y)_s + (\tilde{\varphi}_t)_s] + \rho_1 [\eta (\tilde{\varphi}_{1,y})_s + (\tilde{\varphi}_{1,t})_s] + \rho \gamma - \rho_1 \gamma_1 \chi,
\end{align*}
\tag{80}

which, by (43), (44) and (45) equals
\[ (\rho_1 \Phi - \rho \Phi)_t + (\rho \gamma - \rho_1 \gamma_1) \chi = - \xi_t + (\rho \gamma - \rho_1 \gamma_1) \chi. \]

Analogously, we have
\begin{align*}
\frac{\delta H}{\delta \eta_1} &= \rho_1 (-\tilde{\varphi}_{1,y})_s \left[ (\tilde{\varphi}_{1,y})_s - \eta_{1,x}(\tilde{\varphi}_{1,x})_s - \gamma_1 (h_1 + \eta_1) \eta_{1,x} \right] \\
+ \rho_1 \gamma_1 (h_1 + \eta_1)(\tilde{\varphi}_{1,x})_s \\
+ \rho_1 \gamma_1 \left[ (\tilde{\varphi}_{1,x})_s + \eta_{1,x}(\tilde{\varphi}_{1,y})_s \right] + \rho_1 \kappa_1 \gamma_1 (h_1 + \eta_1) + \frac{\rho_1 \kappa_1^2}{2} \\
- \sigma_1 \frac{\eta_{1,xx}}{(1 + \eta_{1,x}^2)^{3/2}}
\end{align*}
\tag{81}

which by (19) and utilizing the formula
\[ \frac{1}{2} |\nabla \tilde{\varphi}|_{t_s}^2 + \frac{\eta_1 (h_1 + \eta_1)}{2} + \gamma_1 (h_1 + \eta_1)(\tilde{\varphi}_{1,x})_s, \]
\tag{82}

obtained by an algebraic computation from (18), we see that
\begin{align*}
\frac{\delta H}{\delta \eta_1} &= - \rho_1 (\tilde{\varphi}_{1,y})_s (\eta_{1,t} + \kappa_1 \eta_{1,x}) \\
+ \rho_1 \left[ \frac{1}{2} |\nabla \psi_{1,s}|_{t_s}^2 - \frac{k_1^2}{2} - k_1(\tilde{\varphi}_{1,x})_s + (g - \kappa_1 \gamma_1) (h_1 + \eta_1) \right] \\
+ \rho_1 \kappa_1 \left[ (\tilde{\varphi}_{1,x})_s + \eta_{1,x}(\tilde{\varphi}_{1,y})_s \right] + \rho_1 \kappa_1 \gamma_1 (h_1 + \eta_1) \\
+ \frac{\rho_1 \kappa_1^2}{2} - \sigma_1 \frac{\eta_{1,xx}}{(1 + \eta_{1,x}^2)^{3/2}}
\end{align*}
\tag{83}

Making use now of (30) and (72) yields
\begin{align*}
\frac{\delta H}{\delta \eta_1} &= - \rho_1 [(\tilde{\varphi}_{1,y})_s \eta_{1,t} + (\tilde{\varphi}_{1,t})_s] + \rho_1 \gamma_1 \chi = - \xi_{1,t} + \rho_1 \gamma_1 \chi,
\end{align*}
\tag{84}

where, the last equality is obtained by employing (43) and (45).
Collecting now all the factors of $\delta \Phi$ and $\delta \Phi_1$ from (66)-(71) and making use of (20) and of notation (44) we have

$$\frac{\delta H}{\delta \xi} = (\tilde{\varphi}_y)_{s} - \eta \varphi_x_{s} - \gamma \eta \varphi_x - \kappa \eta \varphi_x = (\tilde{\varphi}_y)_{s} - \eta \varphi_x_{s} - \gamma \eta \varphi_x - \kappa \eta \varphi_x = \eta_t. \quad (85)\tag{85}$$

Similarly, from (67), (69), (71) and using (45) we obtain that

$$\frac{\delta H}{\delta \xi_1} = (\tilde{\varphi}_1, y)_{s} - \eta_1 \varphi_x_{s} - \gamma \eta \varphi_x - \kappa \eta \varphi_x = (\tilde{\varphi}_1, y)_{s} - \eta_1 \varphi_x_{s} - \gamma \eta \varphi_x - \kappa \eta \varphi_x = \eta_1, \quad (86)\tag{86}$$

the last equality being in fact the formula (19). We summarize the previous computations as

$$\delta H = \int_0^L \left\{ \eta \delta \xi + \eta_1 \delta \xi_1 + \left[ -\xi_t + (\rho \gamma - \rho_1 \gamma_1) \chi \right] \delta \eta + \left[ -\xi_1, t + \rho_1 \gamma_1 \chi \right] \delta \eta_1 \right\}, \quad (87)\tag{87}$$

relation, which together with the definition of the variational derivative with respect to the inner product in the space $L^2[0, L]$ of square integrable functions, enforces (73).

4.2. The Hamiltonian formulation. We dedicate this section to showing that the water wave equations (1)-(2) together with their boundary conditions (4)-(8) have a Hamiltonian formulation.

**Theorem 4.2.** The change of variables

$$z = \xi + \frac{\rho_1 \gamma_1}{2} \int_0^x \eta(x', t) \, dx', \quad (88)$$

$$z_1 = \xi_1 + \frac{\rho_1 \gamma_1}{2} \int_0^x \eta_1(x', t) \, dx', \quad (89)$$

transforms the nearly-Hamiltonian system (73) in the Hamiltonian system

$$\begin{align*}
z_t &= -\frac{\delta H}{\delta \eta}, \\
\eta_t &= \frac{\delta H}{\delta z}, \\
z_{1,t} &= -\frac{\delta H}{\delta \eta_1}, \\
\eta_{1,t} &= \frac{\delta H}{\delta z_1},
\end{align*}\quad (90)\tag{90}$$

which is a re-formulation of the governing equations.

**Proof.** Setting $\alpha := \rho \gamma - \rho_1 \gamma_1$ and $\beta := \rho_1 \gamma_1$ in (87), we obtain by means of (88)

$$\delta H = \int_0^L \left[ \left( -\xi_t + \frac{\alpha}{2} \int_0^x \eta(x', t) \, dx' + \alpha \chi \right) \delta \eta \right] dx$$

$$+ \int_0^L \eta_t \left[ -z_t + \frac{\alpha}{2} \int_0^x \delta \eta(x', t) \, dx' \right] dx$$

$$+ \int_0^L \left[ -z_{1,t} + \frac{\beta}{2} \int_0^x \eta_1(x', t) \, dx' + \beta \chi_1 \right] \delta \eta_1 dx$$

$$+ \int_0^L \eta_{1,t} \left[ \delta z_1 - \frac{\beta}{2} \int_0^x \delta \eta_1(x', t) \, dx' \right] dx. \quad (90)\tag{90}$$

Note now that, due to (25) and since $\int_0^L \eta(x, t) \, dx = 0$, we have

$$\int_0^L \eta_t \left( \int_0^x \delta \eta(x', t) \, dx' \right) \, dx$$

$$= \int_0^L \frac{d}{dx} \left( \int_0^x \eta_t(x', t) \, dx' \right) \left( \int_0^x \delta \eta(x', t) \, dx' \right) \, dx$$
\[
\begin{align*}
&= - \int_0^L \left( \int_0^x \eta_t(x'',t) \, dx'' \right) \left( \delta \eta \right)(x,t) \, dx \\
&= \int_0^L \chi \delta \eta \, dx, \tag{91}
\end{align*}
\]
and, similarly
\[
\begin{align*}
\int_0^L \eta_1 \left( \int_0^x \delta \eta_1(x',t) \, dx' \right) \, dx \\
&= \int_0^L \frac{d}{dx} \left( \int_0^x \eta_1(x'',t) \, dx'' \right) \left( \int_0^x \delta \eta_1(x',t) \, dx' \right) \, dx \\
&= - \int_0^L \left( \int_0^x \eta_1(x'',t) \, dx'' \right) \left( \delta \eta_1 \right)(x,t) \, dx \\
&= \int_0^L \chi_1 \delta \eta_1 \, dx. \tag{92}
\end{align*}
\]
With the help of the previous two relations we can rewrite (90) as
\[
\delta H = \int_0^L (-z_t) \delta \eta \, dx + \int_0^L \eta_t \delta z \, dx + \int_0^L (-z_{1,t}) \delta \eta_1 \, dx + \int_0^L \eta_{1,t} \delta z_1 \, dx, \tag{93}
\]
from which our claim emerges.

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