Cosmological density versus velocity-divergence relation in the Zel’dovich approximation

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ABSTRACT

I derive a relation, both ‘forward’ and ‘inverse’, between the density and the divergence of the peculiar velocity which results from the Zel’dovich approximation. My calculations assume Gaussian initial conditions. The forward relation expresses the density (strictly speaking, the expectation value of the continuity density given the velocity divergence) in terms of the velocity divergence, while the inverse relation expresses the velocity divergence in terms of the density. The predicted scatter in the relations is small, hence the inverse relation is close to, though not identical with, a mathematical inversion of the forward one. The forward relation is equivalent to the well-known ‘standard’ density–velocity relation in the Zel’dovich approximation. The inverse relation, however, is successfully derived for the first time and constitutes a potentially interesting alternative to an inverse relation derived by Chodorowski et al., based on third-order perturbation theory. Specifically, it may better recover the peculiar velocity from the associated density field, when smoothed over scales as small as a few megaparsecs.

Key words: galaxies: clusters: general – galaxies: formation – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

In the gravitational instability scenario for the formation of structure in the Universe, the peculiar motions of galaxies are tightly related to the large-scale mass distribution. The comparison between the density and the velocity fields can serve as a test of the gravitational instability hypothesis and as a method for estimating the cosmological parameter \( \Omega \) (Dekel et al. 1993). In linear regime, the relation between the density and the velocity fields is

\[
\delta(x) = -f(\Omega) \cdot \nabla \cdot \mathbf{v}(x),
\]

where \( f(\Omega) \approx \Omega^{0.6} \) and I express distances in units of km s\(^{-1}\). This equation is applicable only when the density fluctuations are small compared to unity. However, sampling of galaxies in current redshift surveys and random errors in peculiar velocity catalogs enable reliable dynamical analysis with smoothing scale of several \( h^{-1} \) Mpc, where fluctuations slightly exceed the regime of applicability of linear theory.

Relation (1) has been recently extended for the mildly non-linear regime by Chodorowski & Lokas (1997; hereafter CL). Let us define a variable proportional to the velocity divergence,

\[
\vartheta \equiv -f(\Omega)^{-1} \nabla \cdot \mathbf{v}(x).
\]

CL rigorously computed the mean \( \delta(x) \) given \( \vartheta(x) \), i.e., \( \langle \delta | \vartheta \rangle \), up to third order in (Eulerian) perturbation theory (hereafter PT), assuming Gaussian initial conditions. The resulting formula is

\[
\langle \delta | \vartheta \rangle = a_1 \vartheta + a_2 (\vartheta^2 - \varepsilon_\vartheta^2) + a_3 \vartheta^3,
\]

where \( \varepsilon_\vartheta^2 \) is the variance of the field \( \vartheta \). The coefficients \( a_i \) entering the above expansion are given by some combinations of the joint moments of \( \delta \) and \( \vartheta \) and were explicitly calculated by CL. Gaussian initial conditions are also assumed in the present paper.

Mildly non-linear relation between the density and the velocity divergence is, in contrast to linear relation (1), non-local. The local estimator of density (3) has thus a non-zero variance. Therefore, to obtain an unbiased inverse estimator, i.e., of the velocity divergence from the density, we cannot simply invert expression (3). The inverse estimator was explicitly constructed up to third order in PT by Chodorowski et al. (1998a; hereafter CLPN), who also computed the expected scatter in the relation.

Having said that, it may seem difficult to understand why to investigate the density versus velocity-divergence relation (hereafter DVDR) in the Zel’dovich approximation (Zel’dovich 1970; hereafter ZA). This approximation is first order in Lagrangian PT and therefore provides only par-
tial answers for higher-order perturbative contributions to the density contrast and the velocity divergence. Having solved the problem rigorously, why to resort to approximate schemes again?

There are a few reasons for which the ZA is still worth studying. Firstly, due to its simplicity, it is very popular and in wide use. In particular, the density–velocity relation, resulting from an Eulerian version of this approximation (Nusser et al. 1991), is used in the potent reconstruction of the mass density from peculiar velocity data (Sigad et al. 1998). (Strictly speaking, Sigad et al. use a formula based on the ZA, with the coefficients slightly adjusted to best fit N-body results.) It is therefore interesting to see how the ZA-based estimator of density relates to perturbative formula (3).

Second, N-body simulations have shown that the ZA is apparently quite successful in recovering the density from the corresponding velocity field. An estimator of density resulting from the ZA (the continuity density) happens to be merely slightly biased, even for smoothing scales as small as a few $h^{-1}$ Mpc (Mancinelli et al. 1994, Ganon et al. 1998), where PT is expected to break down. For a Gaussian smoothing length of $5 \ h^{-1}$ Mpc, the rms fluctuation of a density field is already close to unity and in the perturbative expansion for $\delta$ and $\vartheta$, terms of all orders become comparable. Indeed, for smoothing scales smaller than $5 \ h^{-1}$ Mpc, the value of the coefficient $a_2$ estimated from N-body data starts to deviate significantly from the predicted value (Chodorowski & Stompor 1998). This is not a problem for formula (3), which is applicable to the mass density reconstruction from peculiar velocities, a part of so-called density–velocity comparisons. These comparisons (e.g., IRAS–potent) currently employ Gaussian smoothing length of 12 $h^{-1}$ Mpc. For such a smoothing length, one may hope formula (3) to be even better estimator of density than the formula used on the ZA. Velocity–velocity comparisons, however, employ smoothing lengths as small as 5, or even 3, $h^{-1}$ Mpc (e.g., Willick & Strauss 1998). For such small scales, an inverted version (i.e., an estimator of velocity from density) of the ZA may prove to do better than a formula based on rigorous third-order PT. It is therefore important to invert the ZA and to test its performance, relative to a third-order perturbative formula, against N-body simulations.

Finally, any galaxy density field is derived from a redshift survey, i.e. given originally in the redshift space. To compare the galaxy density field with the real-space mass density inferred from peculiar velocity data, the galaxy field must be first reconstructed in the real space. The redshifts of galaxies differ from the true distances by the peculiar velocities, induced themselves by the fluctuations in the density field. Hence, the real-space galaxy density reconstruction requires a self-consistent solution for the real space density and velocity fields. The velocity field remains irrotational when smoothed over large enough scales, so given the field $\vartheta$, defined in equation (4), and appropriate boundary conditions, the velocity is

$$v(x) = \frac{f(\Omega)}{4\pi} \int d^3 x' \vartheta(x') \frac{x' - x}{|x' - x|^3}. \quad (4)$$

To proceed further, we need a local estimator of the velocity divergence from density. Thus, even in density-density comparisons, an inverse estimator is indispensable. In linear regime $\vartheta = \delta$, hence

$$v(x) = \frac{f(\Omega)}{4\pi} \int d^3 x' \delta(x') \frac{x' - x}{|x' - x|^3}. \quad (5)$$

Yahil et al. (1991) and Strauss et al. (1992) describe an iterative technique of simultaneously solving for the real space density and velocity fields, in which they use equation (3). However, in the present version of the IRAS–potent comparison, Sigad et al. (1998) include nonlinear corrections to this equation. The nonlinear formula for the velocity divergence in terms of the density they use is a purely phenomenological fit to CDM N-body simulations. If the ZA is applied to predict density from velocity (the potent reconstruction), why not to apply it to the inverse case as well, i.e. to predict velocity from density (the IRAS reconstruction)? The reason why Sigad et al. do not do this is simply that thus far, nobody has succeeded in inverting the ZA. For example, Nusser et al. (1991) tried to invert it, but failed.

In the present paper I express density in the ZA as a function of the velocity scalars: the expansion (divergence) and the shear. This enables me to derive easily the ‘forward’ DVDR in the ZA, i.e., an analog of formula (3). This also helps me to invert the ZA, i.e., to compute the mean velocity divergence given the density contrast in the ZA (‘inverse’ DVDR). Such an estimator of the velocity divergence from density has a scatter, but I show the scatter to be inevitable if the estimator is to be local. Moreover, I explicitly compute the scatter and find it to be small. In a follow-up paper, we test both perturbative and derived here, ZA-based DVDRs against N-body simulations (Chodorowski et al. 1998b).

The paper is organized as follows: in Section 3 I express the density in the ZA as a local function of the velocity scalars. In Section 4 I average this expression to obtain the mean density given the velocity divergence. In Section 5 I compute the expected scatter in this forward DVDR. In Section 6 I derive an inverse DVDR, i.e., the mean velocity divergence directly in terms of the density. Summary and conclusions are given in Section 7.

## 2 DENSITY IN TERMS OF THE VELOCITY SCALARS

In a Lagrangian approach to PT (Moutarde et al. 1991, Bouchet et al. 1992, Bouchet et al. 1995), instead of expanding the density contrast, one expands the trajectory of a particle,

\[ x = q + D\psi^{(1)}(q) + D^2\psi^{(2)}(q) + \ldots \quad (6) \]

Here, $q$ is particle’s unperturbed Lagrangian coordinate, $x$ is its final (Eulerian) position, $D(t)$ is the linear growth-factor of density fluctuations, and $\psi^{(1)}(q)$ and $\psi^{(2)}(q)$ are the corresponding values of the displacement fields $\psi^{(1)}$ and $\psi^{(2)}$. The point of the ZA is that only the field $\psi^{(1)}$ is retained. Then, the velocity of a particle at an Eulerian position $x$ is simply

\[ v = f(\Omega)H \psi^{(1)}(q), \quad (7) \]

where $H$ is the Hubble constant, and, expressing distances in units of km s$^{-1}$, in Eulerian space equation (3) takes the form
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\[ q(x) = x - f^{-1}v(x). \]  

Hence, the continuity equation reads (Nusser et al. 1991)

\[ \delta(x) = \frac{||\partial q/\partial x||}{1 = ||I - f^{-1}\partial v/\partial x||} - 1, \]  

where the double vertical bars denote the determinant and \( I \) is the unit matrix. Expanding the determinant in powers of products of velocity derivatives we have (following the notation of Sigad et al. 1998)

\[ \delta(x) = -f^{-1}\nabla \cdot v + f^{-2}\Delta_2 + f^{-3}\Delta_3, \]

where

\[ \Delta_2(x) = \sum_{i,j}^3 (v_{i,j}v_{j,i} - v_{i,i}^2), \]

\[ \Delta_3(x) = \sum_{i,j,k}^3 (v_i v_{j,k} v_{k,j} - v_i v_{j,i} v_{k,k}), \]

and \( v_{i,j} \equiv \partial v_i/\partial x_j. \)

In this paper, by the ‘density in the ZA’ I always mean the continuity density. It is different from the dynamical density in the ZA (i.e., resulting from the equation of motion), since the ZA conserves mass and momentum simultaneously only to first order. The dynamical density, \( \delta_d \), is exactly as in linear theory, \( \delta_d = -f^{-1}\nabla \cdot v. \) Moscardini et al. (1996) used the dynamical density (specifically, the resulting solution for the velocity in terms of the ZA-predicted density) to model the velocity field of clusters of galaxies. A large radius, 20 \( h^{-1} \) Mpc, of a Gaussian window with which they smoothed the velocity field makes this approximation indeed applicable. Density–velocity comparisons, however, employ considerably smaller smoothing lengths, where the linear DVDR is no longer valid. As already stated, one approach to find a mildly nonlinear extension of the linear DVDR is to rigorously derive higher-order perturbative corrections. A complementary, less rigorous, but more intuitive and also promising approach is offered by the ZA, since the continuity density in the ZA successfully recovers the true density in N-body simulations by Mancinelli et al. (1994) and Ganon et al. (1998).

Before shell crossing, the velocity field remains irrotational. This implies that the velocity deformation tensor is symmetric and we can decompose it into expansion, \( \theta \), and shear (the traceless part), \( \sigma_{ij} \):

\[ v_{i,j} = \frac{1}{2}\theta\delta_{ij} + \sigma_{ij}, \]

where in general

\[ \sigma_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i}) - \frac{1}{2}\theta\delta_{ij} \]

and

\[ \theta \equiv \nabla \cdot v = v_{k,k}. \]

Here, the symbol \( \delta_{ij} \) denotes the Kronecker’s delta. Note that \( \theta = -f\theta \), \( \theta \) being defined by equation (2). I will now use decomposition (13) in expressions (10), (11), using the methods and the results of Chodorowski (1997; hereafter C97). For irrotational fields, the quantity \( \Delta_2 \) equals to the quantity \( m_{\delta} \) introduced by Gramann (1993). (Note that an expression for \( \Delta_2 \) in Sigad et al. 1998 has the wrong sign.) C97 showed that \( m_{\delta} = \theta^2/3 - \sigma^2/2 \) (eq.[37] of C97 with the vorticity term equal to zero), where \( \sigma^2 \) is the shear scalar, \( \sigma^2 = \sigma_{ij}\sigma_{ij} \)

and I use Einstein’s summation convention. Hence, we have

\[ \Delta_2 = \frac{1}{3}(\theta^2 - \frac{2}{3}\sigma^2). \]

Substitution of decomposition (13) in equation (12) yields

\[ \Delta_3 = -||\sigma_{ij}|| \]

\[ + \frac{1}{2}(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 - \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} - \sigma_{22}\sigma_{11}). \theta \]

where I have used the property \( \sigma_{ii} = 0 \). By definition,

\[ \sigma^2 = 2(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) + \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2. \]

Using the above equation and the identity

\[ (\sigma_{11} + \sigma_{22} + \sigma_{33})^2 = 0, \]

we can cast the second term in equation (18) to the form \( \sigma^2\theta/6 \). We thus obtain

\[ \delta(x) = -f^{-1}\theta + \frac{1}{3}f^{-2}(\theta^2 - \frac{2}{3}\sigma^2) \]

\[ + f^{-3}(\frac{1}{2}||\sigma_{ij}|| + \frac{1}{6}\sigma^2\theta - \frac{1}{27}\theta^3), \]

or, using the variables \( \vartheta \) and

\[ \Sigma_{ij} \equiv -f^{-1}\sigma_{ij}, \]

\[ \delta(x) = \vartheta + \frac{1}{3}(\vartheta^2 - \frac{2}{3}\Sigma^2) + ||\Sigma_{ij}|| - \frac{1}{6}\Sigma^2\vartheta + \frac{1}{27}\vartheta^3, \]

where

\[ \Sigma^2 = \Sigma_{ij}\Sigma_{ij}. \]

Thus, the density contrast is a function of three scalars, constructed from the velocity deformation tensor: the expansion scalar (the velocity divergence), the shear scalar, and the determinant of the shear matrix. The above equation is our starting point to derive the DVDR within the ZA.

3 DENSITY IN TERMS OF THE VELOCITY DIVERGENCE

The ZA yields expression (11) for the density in terms of the velocity derivatives. The resulting expression for the mean density in terms of the velocity divergence obtains by averaging both sides of equation (10) given the velocity divergence. Having transformed this equation to the form (23), the conditional averaging is straightforward. We have

\[ \langle \delta \rangle |_\vartheta = \vartheta + \frac{1}{3}(\vartheta^2 - \frac{2}{3}\Sigma^2) |_\vartheta + ||\Sigma_{ij}|| |_\vartheta - \frac{1}{6}(\Sigma^2) |_\vartheta \vartheta \]

\[ + \frac{1}{27}\vartheta^3. \]

The Fourier transform of a shear component is \( \langle \Sigma_{ij} \rangle_k = (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}) \hat{\theta}_k \), where \( \hat{k}_i \equiv k_i/k \) and \( \hat{\theta}_k \) is the Fourier transform of the velocity divergence field. Hence,

\[ \langle \vartheta \Sigma_{ij} \rangle = \int \frac{d^3k}{(2\pi)^3} \left( \hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij} \right) P_\vartheta(k) \]

\[ = 0 \]

\[ (P_\vartheta(k) \) is the power spectrum of the velocity divergence field). This means that the shear components are uncorrelated with the velocity divergence. In the case of Gaussian random variables, and only in this case, it is a sufficient condition to be statistically independent. Since the initial
conditions are assumed to be Gaussian, in linear regime \( \vartheta \) and \( \Sigma_{ij} \) are independent. When the fields become non-linear, they become non-Gaussian as well (e.g., Bernardeau et al. 1995, Lokas et al. 1995). In the ZA, however, since the velocity field is proportional to the initial displacement field, equation (31), it remains linear even when the density field becomes non-linear. Derivatives of a Gaussian field, being its linear combinations, are also Gaussian, so \( \vartheta \) and \( \Sigma_{ij} \) remain Gaussian, thus independent. In effect, we can simply replace the conditional averages in equation (23) by the ordinary averages. We have \( \langle \Sigma^2 \rangle = (2/3)\epsilon_0^2 \). The mean value of the determinant of the shear matrix is hence,

\[
\langle \delta \rangle |_\vartheta = a_1^{(ZA)} \vartheta + a_2^{(ZA)} (y^2 - \epsilon_0^2) + a_3^{(ZA)} \vartheta^3,
\]

(27)

where

\[
a_1^{(ZA)} = 1 - \frac{1}{6} \epsilon_0^2, \tag{28}
\]

\[
a_2^{(ZA)} = \frac{1}{3}, \tag{29}
\]

and

\[
a_3^{(ZA)} = \frac{1}{27}. \tag{30}
\]

The mean density given the velocity divergence in the ZA is thus a third order polynomial in the velocity divergence, similarly to the third-order PT result (2). Also the coefficients of the polynomial are in many aspects similar to the corresponding coefficients resulting from perturbative calculations: they form a hierarchy \( a_2^{(ZA)} \ll a_1^{(ZA)} \ll a_1^{(ZA)} \), they are independent of \( \Omega \), and \( a_1^{(ZA)} \) has a corrective term \( \epsilon_0^2 \) which scales linearly with the variance of the velocity divergence field. (The corrective term is due to the term \( -\frac{1}{2} \Sigma^{ij} \partial_i \partial_j \) in equation (37), a third-order mixed term in the shear and the velocity divergence.) Quantitatively, however, the coefficients are different. As stated earlier, in a separate paper we use N-body simulations to test relative accuracy of both approximations (Chodorowski et al. 1998b).

4 SCATTER IN THE RELATION

Expression (27) for density in terms of the velocity divergence, since obtained by conditional averaging of equation (23), has clearly a scatter. The rms value of the scatter at a given value of the velocity divergence, \( s|_\vartheta \), is given by the square root of the conditional variance, \( s|_\vartheta = \left\langle (\delta - \langle \delta \rangle|_\vartheta)^2 \right\rangle ^{1/2} \). We have

\[
\left\langle (\delta - \langle \delta \rangle|_\vartheta)^2 \right\rangle |_\vartheta = \left\langle \left( \frac{1}{2} y + \frac{1}{6} \vartheta - ||\Sigma_0|| \right)^2 \right\rangle |_\vartheta, \tag{31}
\]

where

\[
y \equiv \Sigma^2 - \langle \Sigma^2 \rangle. \tag{32}
\]

For ‘typical’ fluctuations, the first term in parentheses in equation (31) is of the order of \( \epsilon_0^2 \), while the second and the third are \( \mathcal{O}(\epsilon_0^3) \). In large N-body simulations, however, one can trace statistical events of the velocity field which are many standard deviations away from the mean. In particular, Chodorowski et al. (1998b) reliably estimate the scatter as a function of \( \vartheta \) even for \( \vartheta \) well above unity. Therefore, in equation (31) I do not assume \( \vartheta \) to be small. Since \( \vartheta \) and \( \Sigma \) are statistically independent, we obtain

\[
\left\langle (\delta - \langle \delta \rangle|_\vartheta)^2 \right\rangle |_\vartheta = \frac{1}{2} \left( y^2 + \frac{1}{6} \vartheta \right)^2 + \mathcal{O}(\epsilon_0^3) . \tag{33}
\]

I recall that given Gaussian initial conditions, the velocity field in the ZA remains Gaussian. For such a field,

\[
\langle y^2 \rangle = \langle (\Sigma^2 - \langle \Sigma^2 \rangle)^2 \rangle = \frac{3}{45} \epsilon_0^4 \tag{34}
\]

(see C97 for details), hence

\[
s|_\vartheta = \frac{2}{45} \epsilon_0^4 \left( 1 + \frac{1}{6} \vartheta \right)^2 + \mathcal{O}(\epsilon_0^5). \tag{35}
\]

The probability distribution function for the velocity divergence has an abrupt cutoff at \( \vartheta = -1.5 \), as PT predicts (Bernardeau 1994) and N-body simulations confirm (Bernardeau & van de Weygaert 1996, Chodorowski & Stompor 1998). Therefore, \( 1 + \frac{1}{6} \vartheta \) is always positive and we obtain finally

\[
s|_\vartheta = \frac{1}{45} \epsilon_0^4 \left[ a_0^{(ZA)} \vartheta + a_1^{(ZA)} \vartheta + a_2^{(ZA)} \vartheta^2 \right], \tag{36}
\]

where

\[
a_0^{(ZA)} = \frac{1}{45} \sqrt{\frac{\pi}{2}} \simeq 0.21 \tag{37}
\]

and

\[
a_1^{(ZA)} = \frac{1}{45} \sqrt{\frac{\pi}{2}} \simeq 0.07 . \tag{38}
\]

Carrying calculations up to second order in PT, C97 derived a formula for a scatter in the DVDR similar to the first term in equation (23). Extending the calculations up to third order, CLPN derived a formula already containing the second term, but were unable to predict the value of the coefficient \( b_1 \). The ZA predicts, in a simple way, not only \( b_0 \), but the values of both coefficients \( b_0 \) and \( b_1 \) as well.

The rms value of the scatter relative to the rms value of the divergence, \( \epsilon_\vartheta \), vanishes in the limit \( \epsilon_\vartheta \to 0 \), as expected. More importantly, however, this ratio is substantially smaller than unity even for \( \epsilon_\vartheta \) close to unity. Thus, even when almost fully nonlinear, the density and the velocity divergence at a given point remain strongly correlated. In a follow-up paper we test the prediction of the ZA for a scatter in the DVDR against N-body simulations.

5 VELOCITY FROM DENSITY

Equation (25) can be perturbatively inverted to express the velocity divergence as a local function of the density and the shear. The resulting expansion for \( \vartheta \) has an infinite number of terms. Up to cubic terms, it is

\[
\vartheta(\varphi) = \delta - \frac{1}{2} \epsilon_0^2 \left( \delta^2 - \frac{1}{2} \Sigma^2 \right) - \frac{1}{2} \Sigma \delta + \frac{1}{4} \vartheta^2 \delta + \mathcal{O}(\epsilon_0^3) , \tag{39}
\]

where \( \epsilon_0^2 \equiv \langle \delta^2 \rangle \). Obviously, the velocity divergence is not a function of the density alone. Thus, like the forward relation studied in Section (3), a local estimator of the velocity divergence from the density will inevitably have a scatter. To obtain an expression for the divergence exclusively in terms

\[
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\]
of the density, I will average the above equation given the density. We have
\((\partial)_{\delta} = \delta - \frac{1}{3}\delta^3 + \frac{1}{3}(\Sigma^2)_{\delta} = -\frac{1}{3}(\Sigma^2)_{\delta} + \frac{1}{3}\delta^3 + O(\epsilon_3^4) \). (40)

Unlike \(\tilde{\nu}\) and \(\delta\), these are not independent, since the evolved density is a (mildly) non-Gaussian variable. The calculation of \((\Sigma^2)_{\delta}\) is a non-trivial problem; I present it in Appendix A. The result is
\((\Sigma^2)_{\delta} = \frac{2}{5}\delta^2 - \frac{3}{5}\epsilon_3^2\delta + O(\epsilon_3^3) \). (41)

Thus, the mean value of the shear scalar given the density weakly depends on the density. The term generating the dependence is of the order of \(\epsilon_3^2\), higher than the constant term. This is because at the linear order \(\delta\) and \(\Sigma\) are independent. For the same reason, \(\langle||\Sigma||\rangle\delta = \langle||\Sigma||\rangle + O(\epsilon_3) = O(\epsilon_3)\). Using this fact and substituting equation (21) in equation (40) yields
\((\partial)_{\delta} = r_1^{(ZA)}\delta + r_2^{(ZA)}(\delta^2 - \epsilon_3^2) + r_3^{(ZA)}\delta^3 + O(\epsilon_3^4) \), (42)

where
\[ r_1^{(ZA)} = 1 - \frac{7}{3\delta^2}, \] (43)
\[ r_2^{(ZA)} = -\frac{2}{3}, \] (44)
and
\[ r_3^{(ZA)} = \frac{6}{3\delta^2}. \] (45)

And if we instead invert formula (27), which expresses the mean value of density directly in terms of the velocity divergence? Straightforward inversion of (27) yields expansion (42), with the coefficients \(n_j^{(ZA)}\) which I will call the naive ones,
\[ n_1^{(ZA)} = -2 - \frac{2a_1}{a_2} = 1 - \frac{1}{3}\epsilon_3^2, \] (46)
\[ n_2^{(ZA)} = -a_2 - \frac{1}{3}, \] (47)
and
\[ n_3^{(ZA)} = -a_3 + 2a_2 = \frac{2a_2}{3\delta^2}. \] (48)

The coefficients \(n_2^{(ZA)}\) and \(n_3^{(ZA)}\) are equal to \(r_1^{(ZA)}\) and \(r_3^{(ZA)}\), respectively, but \(n_1^{(ZA)}\) is different from \(r_1^{(ZA)}\). This is a consequence of a scatter in the DVDR. If a relation between two random variables has a scatter, in general the inverse relation is not given by a mathematical inversion of the forward relation (e.g., forward and inverse Tully-Fisher relations). CLPN showed the true and the naive coefficients to be related in the following way:
\[ r_1 = n_1 + (b_2^2 - b_0^2)\epsilon_3^2, \] (49)
\[ r_2 = n_2 \] (50)
and
\[ r_3 = n_3 - b_2^2. \] (51)

Here, \(b_0\) and \(b_2\) are the coefficients entering the leading-order perturbative formula for the rms value of the scatter in the DVDR,
\[ s|\delta = b_0\delta + \left[1 + b_2^2\delta^2/(b_0^2\delta^2)\right]^{1/2} + O(\epsilon_3^3) \] (52)

(CLNP; note a slightly different notation used here). This expression was derived under an assumption that \(\nu\) is of the order of \(\epsilon_3\), so the second term under square root is of the order of unity. The formula does not account for the second term in expression (27) because it is already of third order in \(\epsilon_3\). Comparing expressions (52) and (43) we find that in the ZA the coefficient
\[ b_0^{(ZA)} = 0 \] (53)
and \(b_1^{(ZA)}\) is given by equation (27). From equations (51), (44) and (45) we have that indeed \(r_2^{(ZA)} = n_2^{(ZA)}\) and \(r_3^{(ZA)} = n_3^{(ZA)}\). Using equations (43), (44), (45) and (52) we obtain \(r_1^{(ZA)} = 1 - \frac{7}{3\delta^2}\epsilon_3^2\), in agreement with equation (53).

Thus, I rederived the values of the coefficients \(r_j^{(ZA)}\) in a different way.

Both ‘forward’ and ‘inverse’ relations, expressions (27) and (42), describe mean statistical properties of the matter field. They were derived by constrained averaging over all possible realizations of the density and velocity fields. It is instructive to compare them to the results obtained assuming spherical symmetry of perturbations. In this case, all the shear terms in equation (27) vanish, and it simplifies to the form
\[ \delta = (1 + \delta/3)^3 - 1. \] (54)

This equation is easily invertible,
\[ \nu = 3 \left[(1 + \delta)^{1/3} - 1\right]. \] (55)

The above expression is in agreement with a result of Bouchet et al. (1995) for a spherical top-hat (eq. [A31] of Bouchet et al. 1995). When expanded, it yields the values of the coefficients \(r_2\) and \(r_3\) equal to these given by equations (44) and (45). It does not, however, predict the correction to the leading-order value of the linear coefficient \(r_1\). More importantly, it does not involve a constant term. Such a term, \(r_2^{(ZA)}\epsilon_3^2\), is present in equation (43) and naturally assures the (ordinary) mean of the velocity divergence to vanish, the property which expression (27) lacks. The shear terms are thus generally important.

6 SUMMARY

I have derived the mildly nonlinear DVDR as predicted by the ZA, as well as a scatter in it. The ‘forward’ relation states that the mean density contrast, given the velocity divergence, is a third-order polynomial in the velocity divergence. This is a ‘law of Nature’ in the ZA, or, rather, a ‘law of the ZA’. In contrast, the ‘inverse’ relation, expressing the mean velocity divergence in terms of the density contrast, has an infinite number of terms; I have explicitly computed the coefficients of the first three. A relation between two mildly nonlinear variables should be described by a polynomial of third degree quite well. In any case, I will not attempt to calculate higher-order coefficients before testing the already computed ones against N-body simulations: the ZA is only an approximation, and modelling its prediction even more accurately is no guarantee of bringing us any closer to the truth.

The \(\Omega\)-dependence of the DVDR in the ZA enters only
via a factor $f(\Omega)$, used in the definition of the scaled velocity divergence. Similarly, in PT, the relation between the density and the scaled velocity divergence is practically $\Omega$-independent (Bernardeau 1992, Gramann 1993, CL, CLPN; cf. also Nusser & Colberg 1998).

I have explicitly computed a scatter in the forward relation; a scatter in the inverse relation can be computed analogously. I have computed only the ‘forward’ scatter because here we are mostly interested in the mean relations and the scatter is only an auxiliary quantity informing us about the limitations of our local estimators. The predicted scatter is relatively small, even for the fields which are almost fully nonlinear. Therefore, the inverse relation, when obtained by a straightforward inversion of the forward, will be only slightly biased. Indeed, a proper calculation of the inverse relation yields a minor correction to the value of the linear coefficient and no corrections to the quadratic and cubic coefficients. This offers an efficient way of deriving approximate values of the coefficients of the higher-order terms, if there is in future any need to include them.

I have not included the effects of smoothing the evolved density and velocity fields, while the fields inferred from observations are smoothed. In rigorous PT, smoothing slightly changes the values of the ‘forward’ and ‘inverse’ coefficients, making them weakly dependent on the underlying power spectrum of mass fluctuations and the window function used (CL, CLPN, Chodorowski et al. 1998b). The inclusion of smoothing in the ZA can be done in an analogous way to that in PT. Formula (40), which is clearly for unsmoothed fields, is however found a successful estimator of smoothed density from smoothed velocity in N-body simulations by Mancinelli et al. (1994) and Ganon et al. (1998). Thus, the apparent success of the ZA is somewhat accidental: neither higher orders in Lagrangian PT nor the smoothing effects are included, but it still works. If it really works, an inverse estimator based on the ZA may, on scales smaller than about 5 $h^{-1}$ Mpc, do better than the corresponding estimator based on third-order (Eulerian) PT. If so, it would be very useful in large-scale velocity–velocity and density–density comparisons. In a follow-up paper, we will test the performance of both approximations in N-body simulations (Chodorowski et al. 1998b).

The perturbative derivation of the mildly nonlinear DVDR by CL and CLPN is very formal. In contrast, a local relation between the density and the velocity scalars in the ZA (though strictly valid for unsmoothed fields) enables one to derive the DVDR in an easy and intuitive way. In this picture, the source of the scatter in the DVDR is the shear and the velocity divergence is a source of the correction to the leading-order value, unity, of the linear coefficient, $a_1$, in the forward relation. The conditional average of this term, given the velocity divergence, yields an additional term linear in the velocity divergence, with the coefficient proportional to the variance of the velocity divergence field. The linear scaling of the correction to $a_1$ with the variance is indeed formally predicted by PT. Thus, even if the ZA fails to model quantitatively the mildly nonlinear DVDR, it will remain a useful tool to understand it.

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APPENDIX A: CONDITIONAL A VERAGE OF THE SHEAR SCALAR

I outline here a derivation of the expectation value of the velocity shear scalar given the velocity divergence or the density contrast. The only assumption made is that the density and the velocity fields remain in the mildly non-linear regime, so the calculation can be performed perturbatively. Besides that the derived formula is entirely general, i.e. it can be applied to any approximation of mildly nonlinear dynamics. Here, I apply it to the ZA.

The derivation of \((\Sigma^2)\) is greatly simplified when we introduce an auxiliary variable

\[ \beta \equiv \theta + \Sigma^2 - \langle \Sigma^2 \rangle, \tag{A1} \]

where \(\langle \Sigma^2 \rangle\) is an ordinary average of the shear scalar. Expanding shear components in a perturbative series, \(\Sigma_{ij} = \Sigma_{ij}^{(1)} + \Sigma_{ij}^{(2)} + \ldots\), yields \(\Sigma^2 - \langle \Sigma^2 \rangle = \Sigma^{(1)2} - \Sigma^{(1)2} + 2\Sigma^{(1)1}\Sigma^{(2)} + O(\varepsilon^3)\), where \(\Sigma^{(1)2} \equiv \Sigma^{(1)1}\Sigma^{(1)}\). Similarly, \(\theta = \theta^{(1)} + \theta^{(2)} + \theta^{(3)} + O(\varepsilon^4)\), or, for short, \(\theta = \theta_1 + \theta_2 + \theta_3 + O(\varepsilon^4)\). Hence,

\[ \beta = \beta_1 + \beta_2 + \beta_3 + O(\varepsilon^4), \tag{A2} \]

where

\[ \beta_1 = \theta_1, \tag{A3} \]

\[ \beta_2 = \theta_2 + \Sigma_2, \quad \Sigma_2 \equiv \langle \Sigma^{(1)2} \rangle - \langle \Sigma^{(2)} \rangle \tag{A4} \]

and

\[ \beta_3 = \theta_3 + \Sigma_3, \quad \Sigma_3 \equiv 2\Sigma_{ij}^{(1)}\Sigma_{ij}^{(2)}. \tag{A5} \]

By definition (A1),

\[ \langle \Sigma^2 \rangle = \langle \Sigma^2 \rangle + \langle \beta \rangle \theta - \theta, \tag{A6} \]

so our problem reduces to the calculation of mean \(\beta\) given \(\theta\), where both variables, of vanishing mean, are mildly non-linear and equal to each other at linear order. This problem was solved by CL. According to CL,

\[ \langle \beta \rangle \theta = c_1 \theta + c_2 (\theta^2 - \varepsilon^2) + c_3 \theta^3 + O(\varepsilon^4), \tag{A7} \]

where

\[ c_1 = \frac{1}{6} \left( 4Z_2 + \frac{\langle S_{33} - S_{36} \rangle \langle S_{36} \rangle}{3} - \frac{Z_4}{2} \right) \varepsilon^2, \tag{A8} \]

\[ c_2 = \frac{S_{36} - S_{36}}{6}, \tag{A9} \]

and

\[ c_3 = \frac{Z_4 - \langle S_{33} - S_{36} \rangle \langle S_{36} \rangle}{6}. \tag{A10} \]

(I use here slightly different notation.) The quantity \(S_{3\delta}\) is defined by

\[ \varepsilon^4 S_{3\delta} = 3(\theta_1^2 \theta_2), \tag{A11} \]

and \(S_{3\delta}\) is defined in an analogous way. The quantities \(Z_2\) and \(Z_4\) are given by

\[ \varepsilon^4 Z_2 = \langle \theta_2 \theta_2 \rangle - \langle \theta_2^2 \rangle + \langle \theta_1 \theta_3 \rangle - \langle \theta_1^2 \theta_3 \rangle \tag{A12} \]

and

\[ \varepsilon^6 Z_4 = 3(\theta_1^2 \theta_2 \theta_2) - 3(\theta_1^2 \theta_2 \theta_2) + \langle \theta_1 \theta_3 \rangle - \langle \theta_1^2 \theta_3 \rangle. \tag{A13} \]

In the expressions above, \(\varepsilon^2\) is the linear variance of the velocity divergence field, \(\varepsilon^2 = \langle \theta_1^2 \rangle\), and the symbol \(\langle \cdot \rangle_c\) stands for the connected (reduced) part of the moments. From equations (A6) and (A7) we have

\[ \langle \Sigma^2 \rangle = \langle \Sigma^2 \rangle + (c_1 - 1) \theta + c_2 (\theta^2 - \varepsilon^2) + c_3 \theta^3. \tag{A14} \]

Using expansion (A2) of \(\beta\) in the expression for the coefficient \(c_2\) yields

\[ 2\varepsilon^2 c_2 = \langle \beta_1^2 \beta_2 \rangle - \langle \theta_1^2 \theta_2 \rangle \]

\[ = \langle \theta_1^2 (\theta_2 - \theta_2) \rangle \]

\[ = \langle \theta_1^2 \rangle \langle \Sigma^{(1)2} - \langle \Sigma^{(1)2} \rangle \rangle \]

\[ = \langle \theta_1^2 \rangle \langle \Sigma^{(1)2} - \langle \Sigma^{(1)2} \rangle \rangle \]

\[ = 0. \tag{A15} \]

Thus, the average value of the shear scalar given the velocity divergence is equal to its unconstrained average plus the corrective terms, dependent on the divergence. These terms are of the order of \(\varepsilon^3\), higher than the first term, which is of the order of \(\varepsilon^2\). This is again due to the fact that at linear order, \(\Sigma\) and \(\theta\) are statistically independent.

Though the variable \(\theta\) denotes the velocity divergence, the only specific property of \(\theta\) I have used thus far was its independency of \(\Sigma\) at linear order. This property is also shared by the variable \(\delta\), since \(\delta_1 = \theta_1\). Therefore, an expression for the mean value of the shear scalar given the density contrast can immediately be written by replacing the symbol \(\theta\) with \(\delta\) in expression (A1). Specifically,

\[ \langle \Sigma^2 \rangle = \langle \Sigma^2 \rangle + s_1 \varepsilon^2 \theta + s_3 \theta^3 + O(\varepsilon^4), \tag{A21} \]

where

\[ s_1 = Z_2 - \frac{1}{2} \varepsilon^4, \tag{A22} \]

\[ s_3 = \frac{1}{6} Z_4, \tag{A23} \]

with

\[ \varepsilon^4 Z_2 = \langle \theta_2 \Sigma_2 \rangle + \langle \theta_1 \Sigma_3 \rangle \tag{A24} \]

and

\[ \varepsilon^6 Z_4 = 3(\theta_1^2 \theta_2 \Sigma_2) + \langle \theta_1^3 \Sigma_3 \rangle. \tag{A25} \]
The formulas (A16) and (A21) are general in a sense that they are applicable to any approximation of mildly nonlinear dynamics (including rigorous PT). Here, I will apply them to the ZA.

In the ZA, the velocity field remains linear all the time, so \( \dot{\vartheta}^{(ZA)} = \Sigma^{(ZA)} = 0 \). (The quantity \( \dot{\vartheta}^{(ZA)} \) is non-zero because it is constructed from first-order quantities). Hence, \( Z_{\dot{\vartheta}}^{(ZA)} = Z_{\dot{\vartheta}}^{(ZA)} = 0 \) and

\[
(\Sigma^{(ZA)})^2 = \langle \Sigma^2 \rangle = 0.
\]

This result is otherwise obvious, since in the ZA, \( \Sigma \) and \( \vartheta \) remain independent in the nonlinear regime (see Section 3).

The case of \( \langle \Sigma^2 \rangle \), however, is not so trivial, because unlike the velocity field, the density field in the ZA is non-linear. From equation (23),

\[
\delta_{\dot{\vartheta}}^{(ZA)} = \frac{4}{3} \left( \vartheta^{(1)2} - \frac{4}{45} \Sigma^{(1)2} \right).
\]

This yields

\[
Z_{\dot{\vartheta}}^{(ZA)} = - \frac{2}{45}. \quad \text{(A29)}
\]

Analogous calculation shows that

\[
Z_{\dot{\vartheta}}^{(ZA)} = 0 \quad \text{(A30)}
\]

(I recall that \( Z_{\dot{\vartheta}} \) is constructed from the connected part of moments). This yields \( s_1^{(ZA)} = -4/45 \) and \( s_3^{(ZA)} = 0 \), hence

\[
(\Sigma^{(ZA)})^2 = \langle \Sigma^2 \rangle = \frac{4}{45} \varepsilon_\delta^2 \delta + \mathcal{O}(\varepsilon_\delta^3). \quad \text{(A31)}
\]

The ordinary average of the shear scalar is equal to \((2/3)\varepsilon_\delta^2\) and the variance of the velocity divergence field is not equal to the variance of the nonlinear density field, \( \varepsilon_\delta^4 \). The difference, however, is \( \mathcal{O}(\varepsilon_\delta^3) \). Therefore finally

\[
(\Sigma^{(ZA)})_5^2 = \frac{2}{3} \varepsilon_\delta^2 - \frac{4}{45} \varepsilon_\delta^2 \delta + \mathcal{O}(\varepsilon_\delta^3). \quad \text{(A32)}
\]