Dirac operator spectrum on a nilmanifold

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ABSTRACT: We obtain the spectrum of the Dirac operator on the three-dimensional Heisenberg nilmanifold $\mathcal{M}_3$, and its complete dependence on the metric moduli. As an application, we construct the four-dimensional low-energy effective action obtained by compactification of a seven-dimensional gauge-fermion theory on $\mathcal{M}_3$.

KEYWORDS: nilmanifolds, Kaluza-Klein spectrum, Dirac operator
1 Introduction and summary

Compact spaces of negative scalar curvature have been considered in the context of extra-dimensional models in the past, because of their extremely interesting properties for realistic model building, nevertheless they remain much less studied than their positively-curved counterparts. The present paper focuses in particular on the three-dimensional Heisenberg nilmanifold $M_3$. Nilmanifolds are group manifolds based on nilpotent Lie algebras, see e.g. [1] for a review. From the physics point of view, see [2–6] for previous works, they present an ideal playground for compactification and Kaluza-Klein (KK) reduction, as they are arguably the simplest non-trivial examples of negatively-curved manifolds on which exact calculations are possible.

Obtaining the four-dimensional low-energy effective theory from KK reduction of a higher-dimensional one, requires knowledge of the spectrum of certain differential operators on the internal manifold, with the eigenmodes and eigenvalues corresponding to the fields and masses of the four-dimensional theory. In [7, 8] we studied the scalar and one-form spectrum on $M_3$. Knowledge of this part of the spectrum already allows for certain phenomenologically interesting applications to dark matter [7] and gauge-Higgs models [9, 10]. However these works did not include the study of the fermion spectrum, whose knowledge is indispensable for realistic model building. In the present paper we fill this gap by studying the spectrum of the Dirac operator on $M_3$.

Harmonic analysis on nilmanifolds, and $M_3$ in particular, has been considered before in the mathematical literature [11–16], however the results are not always presented in
a way accessible to physicists. Moreover the spectrum is typically computed using the canonical metric (the analogue of a square three-torus) and thus misses the dependence of the spectrum on the metric moduli. However, this dependence is an important piece of information for physical applications, as it affects the masses of the fields of the theory. Obtaining the complete metric moduli dependence of the spectrum is the main result of our paper.

Furthermore, as already discussed in [8], the spectrum admits a low-energy truncation to massless and light massive modes. As an application of our results, we use this truncation to construct the four-dimensional effective action obtained by compactification of a seven-dimensional gauge-fermion theory on $\mathcal{M}_3$.

The present work is part of the program initiated in [7–10] to explore the phenomenology of nilmanifold compactifications. As argued in those papers, compactification on nilmanifolds, or more general group manifolds, present several attractive features, in particular in the context of gauge-Higgs unification. Besides their purely mathematical interest, our results will allow to complete these models by taking into account the KK reduction of the fermionic sector of the seven-dimensional theory, thus taking a major step towards a more realistic phenomenology.

The outline of the paper is as follows. In §2.1 we review the scalar spectrum on the Heisenberg nilmanifold and its dependence on the metric moduli. In §2.2 we derive the spectrum of the Dirac operator in the simplified case, where only the dependence of the metric on the radii is taken into account. The general dependence of the spectrum on all metric moduli is obtained in §2.3. The four-dimensional effective action arising from reduction of a seven-dimensional gauge-fermion theory is contained in §3. Our spinor conventions and the spectrum of the light eigen one-forms on $\mathcal{M}_3$ are discussed in the appendices §A and §B respectively.

2 Dirac operator spectrum on the Heisenberg nilmanifold

Before addressing the spectrum of the Dirac operator on $\mathcal{M}_3$ in §2.2 and §2.3, it will be useful to give a brief review of the most general metric and the associated scalar Laplacian eigenbases in §2.1.

2.1 Review of the scalar spectrum on $\mathcal{M}_3$

The three-dimensional nilmanifold $\mathcal{M}_3$ is built from the nilpotent Heisenberg algebra

$$[V_1,V_2] = -f V_3 , \quad [V_1,V_3] = [V_2,V_3] = 0 ,$$

with structure constant $f = -f_{12}^3$. The Maurer–Cartan one-forms $e^a=1,2,3$, which are dual to the vectors $V_a$ above, satisfy

$$de^3 = f e^1 \wedge e^2 ; \quad de^1 = 0 ; \quad de^2 = 0 .$$

These vectors and one-forms furnish bases of the tangent and cotangent spaces of $\mathcal{M}_3$ respectively. Using coordinates $x^m=1,2,3 \in [0,1]$, and constant radii $r^m=1,2,3$, we parametrize,

$$e^1 = r^1 dx^1 ; \quad e^2 = r^2 dx^2 ; \quad e^3 = r^3 (dx^3 + N x^1 dx^2) ; \quad N = \frac{r^1 r^2}{r^3} f \in \mathbb{Z}^* .$$

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The vielbein $e^a_m$ and its inverse $e^m_a \equiv (e^{-1})^m_a$ are given by $e^a = e^a_m \, dx^m$, $V_a = e^m_a \partial_m$. We use letters from the beginning, the middle of the latin alphabet for flat, curved indices respectively. Explicitly the vectors read

$$V_1 = \frac{\partial}{\partial X^1}, \quad V_2 = \frac{\partial}{\partial X^2} - f X^1 \frac{\partial}{\partial X^1}; \quad V_3 = \frac{\partial}{\partial X^3},$$

where we have defined $X^{m=1,2,3} = r^m x^m$. Subject to the discrete identifications

$$x^1 \sim x^1 + n^1; \quad x^2 \sim x^2 + n^2; \quad x^3 \sim x^3 + n^3 - n^1 N x^2; \quad n^1, n^2, n^3 \in \{0, 1\},$$

the manifold $M_3$ is compact, and is topologically a twisted (for $f \neq 0$) circle fibration, with fiber parameterized by $x^3$, over a two torus base parameterized by $x^{1,2}$. The one-forms $e^a$ are invariant under (2.5), thus globally defined. The most general metric on $M_3$ is given by [7]

$$ds^2 = (e^1 + ae^3)^2 + (e^2 + be^3)^2 + (e^3)^2, \quad a, b \in \mathbb{R}.$$  

As explained in [7], the parameters $a$, $b$ are moduli related to complex deformations of $T^2 \subset M_3$. Let us also note that that $\sqrt{g} = r^1 r^2 r^3$, so that the volume is given by

$$V = \int d^3x \sqrt{g} = r^1 r^2 r^3.$$  

The eigen-modes of the scalar Laplacian operator on $M_3$ are given by two distinct sets of orthonormal eigenfunctions. The first set has a non-trivial dependence on $x^3$ and is given by [7]

$$U_{k,l,n}(x^1, x^2, x^3) = \sqrt{\frac{r^2}{|N|V}}\frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{2\pi K i (X^3 + \sqrt{f} X^1 X^2)} e^{2\pi i a X^1} \sum_{m \in \mathbb{Z}} e^{2\pi K M i X^1}$$

$$\times \exp \left[ -\frac{i \pi K b}{b^2 + 1} \left( X^2 + \frac{M}{f} + \frac{L}{K f} \right) \left( f a \left( X^2 + \frac{M}{f} + \frac{L}{K f} \right) - 2 \right) \right]$$

$$\times \Phi_n \left( X^2 + \frac{M}{f} + \frac{L}{K f} - \frac{a}{f(a^2 + b^2 + 1)} \right);$$

$$l = 0, \ldots, |k| - 1, \ k \in \mathbb{Z}, \ n \in \mathbb{N},$$

where

$$i = 1, 2, 3, \quad K = \frac{k}{r^3}, \quad L = \frac{l}{r^1}, \quad M = \frac{r^3}{r^1 m}; \quad m, k, l \in \mathbb{Z},$$

and

$$\sigma = \frac{2\pi K}{b^2 + 1}(a^2 + b^2 + 1)^{\frac{1}{2}}.$$  

The function $\Phi_n^\sigma$ is given by

$$\Phi_n^\sigma(z) = |\sigma|^{\frac{1}{2}} \Phi_n(|\sigma|^{-\frac{1}{2}} z), \quad \Phi_n(z) = e^{-\frac{z^2}{2}} H_n(z), \quad n \in \mathbb{N},$$

where $H_n$ are the Hermite polynomials: $H_n(y) = (-1)^n e^{y^2} \partial_y^n e^{-y^2}$. The Laplacian eigenvalues of the $U$-eigenfunctions are given by

$$(\nabla^2 + M^2_{k,l,n}) U_{k,l,n} = 0; \quad M^2_{k,l,n} = \frac{4\pi^2 k^2}{(r^3)^2(a^2 + b^2 + 1)} \left[ 1 + \frac{(2n + 1)^2 |f|}{2\pi |k|} (a^2 + b^2 + 1)^{-\frac{1}{2}} \right].$$  

(2.12)
The second set of eigenfunctions of the scalar Laplacian on $M_3$ have no dependence on the $x^3$ coordinate. They are given by

$$V_{p,q}(x^1, x^2) = \frac{1}{\sqrt{V}} e^{iP X^1} e^{iQ X^2}, \quad P = \frac{2\pi p}{r^1}, \quad Q = \frac{2\pi q}{r^2}, \quad p, q \in \mathbb{Z},$$  \hspace{1cm} (2.13)

with eigenvalues

$$(\nabla^2 + \mu_{p,q}^2)V_{p,q} = 0, \quad \mu_{p,q}^2 = \frac{p^2}{(r^1)^2} + \frac{q^2}{(r^2)^2} + \left(\frac{a p}{r^1} + \frac{b q}{r^2}\right)^2.$$  \hspace{1cm} (2.14)

**2.2 Simple case: $a = b = 0$**

As a warmup let us consider the Dirac operator eigenvalue problem on $M_3$ in the simplified case $a = b = 0$. The Dirac eigenvalue equation reads,

$$(D - \lambda)\psi = 0; \quad D \equiv \gamma^a e^m_a (\partial_m + \frac{1}{4} \omega_{mbc} \gamma^bc),$$  \hspace{1cm} (2.15)

where $\omega_{mbc}$ is the spin connection. The independent non-vanishing components of the flat spin connection $\omega_{abc} = e^m_a \omega_{mbc}$ on $M_3$ are given by

$$\omega_{123} = \omega_{231} = -\omega_{312} = \frac{f}{2}.$$  \hspace{1cm} (2.16)

The Dirac operator then takes the form

$$D = \gamma^a V_a + i \frac{f}{4},$$  \hspace{1cm} (2.17)

where the $V_a$'s were given in (2.4). The $\gamma^a$'s can be taken to be the Pauli matrices, so that (2.15) reduces to the following set of equations,

$$V_3 \psi_1 + (V_1 - i V_2) \psi_2 - (\lambda + \frac{f}{4}) \psi_1 = 0$$

$$-V_3 \psi_2 + (V_1 + i V_2) \psi_1 - (\lambda + \frac{f}{4}) \psi_2 = 0.$$  \hspace{1cm} (2.18)

Next we expand the two-component Dirac spinor on the complete basis of Laplacian eigenfunctions on $M_3$,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \sum_{k,l,n} \begin{pmatrix} C_{k,l,n}^1 \\ C_{k,l,n}^2 \end{pmatrix} u_{k,l,n} + \sum_{p,q} \begin{pmatrix} D_{p,q}^1 \\ D_{p,q}^2 \end{pmatrix} v_{p,q},$$  \hspace{1cm} (2.19)

where, following [8], we use the notation $u_{k,l,n}$ and $v_{p,q}$, for the $a, b \to 0$ limit of the functions given in (2.8) and (2.13) respectively. Since the two sets of functions are orthogonal to each other, we can treat the two cases independently. The action of the $V_a$ on the $v$-basis reads,

$$V_1 v_{p,q} = i P v_{p,q}; \quad V_2 v_{p,q} = i Q v_{p,q}; \quad V_3 v_{p,q} = 0,$$  \hspace{1cm} (2.20)

\footnote{This expansion implicitly assumes that the Dirac spinor on $M_3$ returns to its original value after a going around each of the three circles parameterized by $x^a$. This condition can be relaxed, leading to what is known as nontrivial spin structures [16].}
so that (2.18) reduces to

\[ i(P - iQ)D_{p,q}^2 - (\lambda_{p,q} - \frac{f}{4})D_{p,q}^1 = 0 \]

\[ i(P + iQ)D_{p,q}^1 - (\lambda_{p,q} - \frac{f}{4})D_{p,q}^2 = 0. \]

From this system we deduce the eigenvalues

\[ -i\lambda_{p,q} = \frac{f}{4} \pm \sqrt{P^2 + Q^2}, \]

and the corresponding eigen-spinors

\[ \psi_{p,q} = C \left( \begin{array}{c} v_{p,q} \\ \alpha v_{p,q} \end{array} \right) \]

\[ \alpha = \pm \frac{\sqrt{P^2 + Q^2}}{(P - iQ)}, \]

up to a normalization constant \( C \in \mathbb{C} \).

Let us now turn to the \( u_{k,l,n} \) series. Using the action of the \( V' \)'s on the basis,

\[ iV_3 u_{k,l,n} = -|\sigma| \text{sign}(\sigma) u_{k,l,n} \]

\[ V_2 u_{k,l,n} = \frac{|\sigma|}{2} \left( -\sqrt{2(n+1)} u_{k,l,n+1} + \sqrt{2n} u_{k,l,n-1} \right) \]

\[ iV_1 u_{k,l,n} = -\frac{|\sigma|}{2} \text{sign}(\sigma) \left( -\sqrt{2(n+1)} u_{k,l,n+1} + \sqrt{2n} u_{k,l,n-1} \right), \]

with \( \sigma \) defined in (2.10), we obtain

\[ [i\lambda_{k,l,n} + \left( \frac{f}{4} - \frac{\sigma}{4} \right)] C_{k,l,n}^1 = -|\sigma| \frac{1}{2} \left[ p^+(\sigma) \sqrt{2n} C_{k,l,n-1}^2 + p^-(\sigma) \sqrt{2(n+1)} C_{k,l,n+1}^2 \right] \]

\[ [i\lambda_{k,l,n} + \left( \frac{f}{4} + \frac{\sigma}{4} \right)] C_{k,l,n}^2 = -|\sigma| \frac{1}{2} \left[ p^-(\sigma) \sqrt{2n} C_{k,l,n-1}^1 + p^+(\sigma) \sqrt{2(n+1)} C_{k,l,n+1}^1 \right], \]

where we have defined \( p^\pm(\sigma) \equiv \frac{\text{sign}(\sigma)\pm1}{2} \). We thus obtain

\[ -i\lambda_{k,l,n} = \frac{f}{4} \pm \sqrt{\left( \frac{\sigma}{f} \right)^2 + |\sigma| \left( p^+(\sigma) 2n + p^-(\sigma) 2(n+1) \right)}. \]

Note that the set of eigenvalues is the same for either sign of \( \sigma \): they are just offset by one increment of \( n \). Moreover there is a degeneracy, since in there is no dependence of the eigenvalues on \( l \). The associated eigen-spinors read

\[ \psi_{k,l,n} = C \left( \begin{array}{c} u_{k,l,n} \\ p^+(\sigma) \alpha u_{k,l,n-1} + p^-(\sigma) \beta u_{k,l,n+1} \end{array} \right), \]

where \( C \in \mathbb{C} \) is a normalization constant and,

\[ \alpha = \frac{\sigma}{f} + \sqrt{\left( \frac{\sigma}{f} \right)^2 + |\sigma| 2n} \]

\[ \beta = \frac{\sigma}{f} + \sqrt{\left( \frac{\sigma}{f} \right)^2 + |\sigma| 2(n+1)} \].
2.3 Non-trivial metric

For the most general metric given in (2.6), the Dirac operator takes the form

\[ D = \gamma^a E_m^a (\partial_m + \frac{1}{4} \omega_{mbc} \gamma^b \gamma^c), \]  

(2.30)

where \( E_m^a \equiv (E^{-1})^m_a \) is the inverse vielbein associated with (2.6), so that [7],

\[ E^{-T} \partial = \begin{pmatrix} V_1 & V_2 \\ V_3 - aV_1 - bV_2 \end{pmatrix}, \]  

(2.31)

with the \( V_a \)’s given in (2.4). Moreover the independent non-vanishing components of the flat spin connection \( \omega_{abc} = E_m^a \omega_{mbc} \) read

\[
\begin{align*}
\omega_{112} &= -f_a; & \omega_{113} &= f_{ab}; & \omega_{123} &= \frac{1}{2} f(-a^2 + b^2 + 1) \\
\omega_{212} &= -f_b; & \omega_{213} &= -\frac{1}{2} f(a^2 - b^2 + 1); & \omega_{223} &= -f_{ab} \\
\omega_{312} &= \frac{1}{2} f(a^2 + b^2 - 1); & \omega_{313} &= f_b; & \omega_{323} &= -f_a.
\end{align*}
\]  

(2.32)

Taking (2.31), (2.32) into account, (2.30) reduces to

\[ D = \gamma^1 V_1 + \gamma^2 V_2 + \gamma^3 (V_3 - aV_1 - bV_2) + i \frac{f}{4} (1 + a^2 + b^2). \]  

(2.33)

As in §2.2, the spectrum falls into two distinct series, depending on whether or not there is non-trivial dependence on the \( x^3 \) coordinate. Let us first examine the \( x^3 \)-independent case. We expand the Dirac spinor as follows

\[ \psi_{p,q} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} V_{p,q}, \]  

(2.34)

for some constants \( \alpha, \beta \in \mathbb{C} \). The action of the \( V_a \) on the \( V_{p,q} \)-basis (2.13) reads

\[ V_1 V_{p,q} = iP V_{p,q}; \quad V_2 V_{p,q} = iQ V_{p,q}, \quad V_3 V_{p,q} = 0, \]  

(2.35)

so that the eigenvalue equation

\[ D \psi = \lambda_{p,q} \psi, \]  

(2.36)

reduces to

\[
\begin{align*}
-aP \alpha + (P - iQ) \beta + [i\lambda_{p,q} + \frac{1}{4} f(a^2 + b^2 + 1)] \alpha &= 0 \\
bQ \beta + (P + iQ) \alpha + [i\lambda_{p,q} + \frac{1}{4} f(a^2 + b^2 + 1)] \beta &= 0,
\end{align*}
\]  

(2.37)

where we have taken (2.33) into account. From this system we deduce the eigenvalues

\[ -i\lambda_{p,q} = \frac{1}{4} f(a^2 + b^2 + 1) \pm \frac{1}{2} (bQ - aP) \pm \sqrt{P^2 + Q^2 + \frac{1}{4} (aP + bQ)^2}. \]  

(2.38)
The Dirac eigen-spinor (2.34) is determined, up to an overall normalization constant, by the equation

\[
\beta = \frac{1}{2} (aP + bQ) \pm \sqrt{\frac{P^2 + Q^2 + \frac{1}{4}(aP + bQ)^2}{P - iQ}} \alpha. \tag{2.39}
\]

In the case of nontrivial \(x^3\)-dependence the relevant basis is given by the polynomials (2.8).

In order not to clutter the notation, we will present the calculation of the spectrum for the case \(\sigma > 0\), cf. (2.10). The \(U\)-polynomials obey

\[
\begin{align*}
(V_3 - aV_1 - bV_2)U_{k,l,n} &= \kappa U_{k,l,n} + \sqrt{2} \left( w^* \sqrt{\frac{n+1}{n}} U_{k,l,n+1} - w \sqrt{n} U_{k,l,n-1} \right) \\
(V_1 + iV_2)U_{k,l,n} &= \kappa z U_{k,l,n} + \sqrt{2} \left( \sqrt{(n+1)}(B_- + bw^*)U_{k,l,n+1} + \sqrt{n}(B_+ - bw)U_{k,l,n-1} \right) \\
(V_1 - iV_2)U_{k,l,n} &= \kappa z^* U_{k,l,n} + \sqrt{2} \left( \sqrt{(n+1)}(B_+ - bw^*)U_{k,l,n+1} + \sqrt{n}(B_- + bw)U_{k,l,n-1} \right),
\end{align*}
\tag{2.40}
\]

where we have defined

\[
\begin{align*}
z &:= a + ib; \quad \kappa := \frac{2\pi iK}{1 + |z|^2}; \quad w := b + \frac{ia}{\sqrt{1 + |z|^2}}; \quad B_{\pm} := \pm (1 + b^2) + \frac{1 + b^2}{\sqrt{1 + |z|^2}}. \tag{2.41}
\end{align*}
\]

Let us now come to the Dirac eigenvalue problem. We start with the following ansatz for the Dirac spinor:

\[
\psi = \begin{pmatrix} \alpha U_{k,l,n} + \beta U_{k,l,n-1} \\ \gamma U_{k,l,n-1} + \delta U_{k,l,n} \end{pmatrix}. \tag{2.42}
\]

Imposing

\[
\left[ D - \frac{i}{4}(1 + |z|^2) \right] \psi = \begin{pmatrix} \alpha' U_{k,l,n} + \beta' U_{k,l,n-1} \\ \gamma' U_{k,l,n-1} + \delta' U_{k,l,n} \end{pmatrix}, \tag{2.43}
\]

for arbitrary coefficients \(\alpha', \beta', \gamma', \delta',\) taking (2.40) into account, leads to a system of four homogeneous equations for the four coefficients \(\alpha, \ldots, \gamma\). Noting the identity

\[
(B_- + bw)(B_+ - bw) = w^2, \tag{2.44}
\]

this system turns out to be equivalent to the following two conditions

\[
\begin{align*}
\beta &= \frac{i(B_- + bw)}{w} \gamma; \quad \delta = \frac{i(B_- + bw^*)}{w^*} \alpha, \tag{2.45}
\end{align*}
\]

where it is assumed that \(a, b\) are not both zero. Imposing in addition the eigenvalue equation

\[
D\psi = \lambda_{k,l,n}\psi, \tag{2.46}
\]

taking (2.45) into account, leads to a system of four homogeneous equations for \(\alpha, \gamma\). Clearly this is highly overdetermined. Remarkably, however, the system admits a nontrivial solution, provided

\[
-i\lambda_{k,l,n} = \frac{1}{4} f(1 + |z|^2) \pm \sqrt{\frac{4\pi K^2}{1 + |z|^2} + 4\pi n K f \sqrt{1 + |z|^2}}, \tag{2.47}
\]
where we took (2.10) into account. As was observed in the case of the scalar spectrum [7], the eigenvalues depend on the $a, b$ parameters only through the norm of $z$. As in §2.2 we see that there is a degeneracy, since the eigenvalues are independent of $l$.

## 3 Reduction to four dimensions

We would now like to examine the 4D effective theory arising as the low-energy limit of a 7D gauge-fermion theory compactified on the Heisenberg nilmanifold $\mathcal{M}_3$. The 7D Lagrangian consists of a Yang-Mills term $L_{7D}^{YM}$ and a fermion term $L_{7D}^f$. The effective theory in four dimensions, $L_{4D}^{\text{eff}}$, will be given by

$$L_{4D}^{\text{eff}} = \int d^3 y \left( L_{7D}^{YM} + L_{7D}^f \right).$$

(3.1)

The right-hand side above indicates the KK reduction of the seven-dimensional theory, and involves integrating over the three-dimensional internal space parameterized by the $y$-coordinates.

Moreover, we will place ourselves in the small fiber/large base limit [8],

$$|\mathfrak{f}| \ll \frac{1}{r^i}, \ i = 1, 2, 3 \quad \Rightarrow \quad \frac{1}{r_{1.2}} \ll \frac{1}{r^3}.$$  

(3.2)

In this limit all fields whose masses carry an $r^i$ dependence (i.e. all the KK modes) decouple, leaving in the theory only those fields with masses of either the order of $|\mathfrak{f}|$, or zero.

Explicitly, the reduction ansatz for the gauge fields is given by [9]:

$$A^a = \frac{1}{\sqrt{V}} \left( A^a + \sum_{I=1}^{3} \phi^{aI} \tilde{E}^I \right),$$

(3.3)

where $A^a, a = 1, \ldots, \dim(G)$, is a 4D one-form and $\phi^{aI}, I = 1, 2, 3$, are three scalars in the adjoint of the Lie algebra of $G$. The $\tilde{E}^I$'s span the space of low-lying one-forms on $\mathcal{M}_3$, cf. B. They can be chosen so that $\tilde{E}^{1,2}$ are harmonic and $\tilde{E}^3$ has Laplacian eigenvalue $2 f^2 (a^2 + b^2 + 1)^2$. Integrating the Yang-Mills term over the internal space $\mathcal{M}_3$ we obtain [9]:

$$\int dy^3 L_{7D}^{YM} = \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} + \sum_{I=1}^{3} D_\mu \phi^{aI} D^{\mu} \phi^{aI} + M^2 (\phi^{a3})^2 + \mathcal{U},$$

(3.4)

where,

$$\mathcal{U} = \text{Tr} \left( -2 i g M [\phi^1, \phi^2] \phi^3 + \frac{1}{2} g^2 \sum_{I,J=1}^{3} [\phi^I, \phi^J] [\phi^I, \phi^J] \right),$$

(3.5)

with $F_{\mu\nu}^a = 2 \partial_{[\mu} A_{\nu]}^a + i g f_{abc} A_{\mu}^b A_{\nu}^c, D_\mu \phi^{aI} = \partial_\mu \phi^{aI} + i g f_{abc} A_{\mu}^b \phi^{aI}$, and $M = |\mathfrak{f}| (a^2 + b^2 + 1)$; $f^{abc}$ are the structure constants of the algebra of the gauge group. The 4D gauge coupling constant $g$ is related to the 7D coupling and the volume of $\mathcal{M}_3$ via

$$g = \frac{g_{7D}}{\sqrt{V}}$$

(3.6)

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$^2$In the present paper we reinstate dependence on the the $a, b$ parameters which were set to zero in [9].
The fermionic Lagrangian will be taken to be of the form

$$L^f_{\text{TD}} = \overline{\psi}_i \Gamma^M (\delta^{ij} \nabla_M + i A^a_{\mu} \rho^a_{ij}) \psi_j + \frac{1}{6} F_{MNP} \overline{\psi}_i \Gamma^{MNP} \psi_i + M_0 \overline{\psi}_i \psi_i,$$  \hspace{1cm} (3.7)$$

where the hermitian matrices $\rho^a_{ij}$ provide a representation $R$ of the Lie algebra of the gauge group, so that $i, j = 1, \ldots, \dim(R)$, and $\psi_i$ transforms in the $R$ representation. We have also allowed for a constant background zero-form flux (a mass term) $M_0$, and a three-form flux $F_{MNP}$, which will be assumed to be along the internal manifold $M_3$ in order not to break the 4D Lorentz invariance. \(^3\) This implies that the only non-vanishing component is given by

$$F_{mnp} = M_1 \varepsilon_{mnp},$$  \hspace{1cm} (3.8)$$

for some real constant $M_1$, and the Levi-Civita symbol is a tensor in our conventions.

Our reduction ansatz for the 7D spinors is as in (A.20),

$$\psi_i = (\chi_{i+} + \theta_{i-}) \otimes \xi,$$  \hspace{1cm} (3.9)$$

where $\chi_i$, $\theta_i$ are Weyl 4D spinors and $\xi$ is a spinor on $M_3$. We can already see that the resulting 4D model will necessarily be non-chiral since, the 4D positive and negative chiralities of the spinors both transform in the same representation $R$.

We expand $\xi$ on a basis of normalized eigen-spinors of the Dirac operator in 3D

$$\sigma^m \nabla_m \xi = \lambda \xi,$$  \hspace{1cm} (3.10)$$

where the eigenvalues of the Dirac operator on $M_3$ were given in §2.3. In the limit (3.2) of decoupling of the KK modes, only the lowest eigenspinor is kept, corresponding to eigenvalue $\lambda = i \frac{1}{4} (a^2 + b^2 + 1)$, cf. (2.38). Moreover we assume that $\xi$ is normalized: $\int d^3 y \xi^\dagger \xi = 1$. Putting everything together we obtain

$$\int d^3 y \ L^f_{\text{TD}} = L_{\text{kinetic}} + L_{\text{Yukawa}} + L_{\text{mass}},$$  \hspace{1cm} (3.11)$$

where

$$L_{\text{kinetic}} = \overline{\chi}_{i+} \gamma^\mu (\delta^{ij} \partial_\mu + i g A^a_\mu \rho^a_{ij}) \chi_{j+} + \overline{\theta}_{i-} \gamma^\mu (\delta^{ij} \partial_\mu + i g A^a_\mu \rho^a_{ij}) \theta_{j-}$$

$$L_{\text{Yukawa}} = i g (\overline{\theta}_{i-} \chi_{j+} - \overline{\chi}_{i+} \theta_{j-}) \rho^a_{ij} \Phi^a$$

$$L_{\text{mass}} = M_c \overline{\chi}_{i+} \chi_{i+} + M_c^* \overline{\chi}_{i+} \theta_{i-},$$  \hspace{1cm} (3.12)$$

and we have defined a complex “mass” $M_c$

$$M_c = M_0 + i \left[ M_1 + \frac{1}{4} (a^2 + b^2 + 1) \right].$$  \hspace{1cm} (3.13)$$

The adjoint scalar $\Phi^a$ is defined by

$$\Phi^a := \sum_{I=1}^{3} \phi^a_I c^I,$$  \hspace{1cm} (3.14)$$

\(^3\)We may also allow for other types of fluxes, however this will result in a similar mass terms as in (3.16) below, so we do not introduce them independently.
where the three constants $c^I$ are given by $c^I := \xi^\dagger \sigma^I \xi$. To make this more explicit, let us give a parameterization for $\xi$. Up to an unimportant overall phase we may set

$$\xi = \left( e^{-i\beta} \cos \frac{\alpha}{2} \right), \quad (3.15)$$

for some angles $\alpha, \beta$. We then find $c^1 = \sin \alpha \cos \beta, c^2 = \sin \alpha \sin \beta, c^3 = \cos \alpha$. I.e. $\vec{c}$ can be thought of as a unit vector of $\mathbb{R}^3$.

The final expression for the fermionic 4D theory can also be expressed in terms of 4D Dirac spinors $\Psi_i := i\chi_i + \theta_i$.

$$\int d^3y \mathcal{L}^\xi_{7D} = \overline{\Psi}_i \Gamma^\mu (\delta^{ij} \partial_\mu + igA^a_{\mu} \rho^a_{ij}) \Psi_j + g\rho^a_{ij} \Phi^a \overline{\Psi}_i \Psi_j + [M_1 + \frac{\alpha}{2}(a^2 + b^2 + 1)] \overline{\Psi}_i \Psi_i - iM_0 \overline{\Psi}_i \gamma_5 \Psi_i. \quad (3.16)$$

The 7D parameters $M_{0,1}$ are free (up to flux quantization), and can be thought of as arising from the inclusion of constant background flux on the nilmanifold.

### A Fermion conventions

In a space of arbitrary dimension and Lorentzian signature, the gamma matrices are taken to satisfy

$$(\Gamma^M)^\dagger = \Gamma^0 \Gamma^M \Gamma^0. \quad (A.1)$$

We define the antisymmetric product of $n$ gamma matrices by

$$\Gamma_{M_1 \ldots M_n} := \Gamma_{[M_1 \ldots \Gamma_{M_n}]} \quad (A.2)$$

Given a spinor $\psi$ we define

$$\overline{\psi} := \psi + \Gamma^0. \quad (A.3)$$

Given a spinor $\psi$ in a space of arbitrary dimension and arbitrary signature, we define

$$\tilde{\psi} := \psi^T C, \quad (A.4)$$

where $C$ is the charge conjugation matrix. This has the property that for any spinors $\psi, \chi$, the bilinear $\tilde{\psi} \Gamma_{M_1 \ldots M_n} \chi$ is an antisymmetric tensor of order $n$.

#### A.1 Spinors in 4D Minkowski space

The charge conjugation matrix in 1 + 3 dimensions satisfies

$$C^\dagger = -C; \quad (C \gamma^\mu)^\dagger = -C \gamma^\mu. \quad (A.5)$$

The chirality matrix is defined by

$$\gamma^5 := i\gamma^0 \ldots \gamma^3; \quad (\gamma^5)^2 = 1. \quad (A.6)$$

The fundamental, positive-chirality (Weyl), two-component, spinor representation $\psi_+$ is complex, meaning that its complex conjugate $\psi_-$ has negative chirality. The complex conjugate $\psi_-$ of $\psi_+$ is defined by

$$\tilde{\psi}_- := \overline{\psi}_+. \quad (A.7)$$
which also implies

$$\tilde{\psi}_+ = -\bar{\psi}_-. \quad (A.8)$$

We stress that these are not reality conditions: they simply define $\tilde{\psi}_-$ in terms of $\tilde{\psi}_+$ or vice-versa. Indeed a reality condition would equate (up to a constant) $\bar{\psi}_+$ and $\bar{\psi}_+$, which is impossible in four dimensions.

Let $\chi_\pm, \psi_\pm$ be arbitrary anticommuting Weyl spinors of positive or negative chirality. We have the following useful relations

$$\tilde{\psi}_\pm \chi_\mp = 0; \quad \tilde{\psi}_\pm \gamma_\mu \chi_\pm = 0. \quad (A.9)$$

The following symmetry relations are valid for Weyl spinors of any chirality

$$\bar{\psi} \chi = \bar{\chi} \psi; \quad \bar{\psi} \gamma_\mu \chi = \bar{\chi} \gamma_\mu \psi. \quad (A.10)$$

It is also useful to note the following complex conjugation relations

$$(\psi_\pm \gamma_\mu \chi_\pm)^* = -\bar{\chi}_\pm \gamma_\mu \psi_\pm; \quad (\psi_\pm \chi_\mp)^* = \bar{\psi}_\mp \psi_\pm. \quad (A.11)$$

Let us now consider an arbitrary anticommuting Dirac spinor $\psi_D$. It can be written in terms of two arbitrary Weyl spinors $\chi_+$ and $\theta_+$

$$\psi_D = \chi_+ + \theta_-, \quad (A.12)$$

where $\theta_-$ is the complex conjugate of $\theta_+$, given by (A.7). The “Dirac mass” is given by

$$\bar{\psi}_D \psi_D = \bar{\chi}_+ \theta_- + \bar{\theta}_- \chi_+ = \bar{\chi}_- \theta_- + \bar{\theta}_+ \chi_+, \quad (A.13)$$

which is real, as can be verified using (A.11).

A Weyl spinor can be considered as a special case of a Dirac spinor whose component of negative or positive chirality vanishes. Therefore it is sometimes said that the Dirac mass of a Weyl spinor vanishes: indeed setting $\chi$ or $\theta$ to zero would make the right hand side of (A.13) vanish. Nevertheless a mass term can be defined for a single Weyl spinor: it suffices to set $\theta_\pm = \chi_\pm$ in (A.13). This is sometimes described as defining a Majorana spinor

$$\psi_M = \chi_+ + \chi_-, \quad (A.14)$$

which is nothing other than a Dirac spinor whose negative-chirality component is the complex conjugate of its positive-chirality component.\footnote{In our conventions the Majorana spinor satisfies the reality condition: $\gamma_5 \psi_M = \tilde{\psi}_M$.} A real mass term for a Weyl spinor $\chi$ can then be written in terms of $\psi_M$,

$$\bar{\psi}_M \psi_M = \bar{\chi}_+ \chi_- + \bar{\chi}_- \chi_+ = \bar{\chi}_- \chi_- - \bar{\chi}_+ \chi_+. \quad (A.15)$$

This is sometimes called the Majorana mass.
A.2 Spinors in 3D Riemannian space

In a 3D space of Euclidean signature the gamma matrices can be taken to be the Pauli matrices, while the charge conjugation matrix can be taken as \( C = i \sigma_2 \). We have

\[
C^\text{Tr} = -C; \quad (C \gamma^m)^\text{Tr} = C \gamma^m. \tag{A.16}
\]

The irreducible spinor representation of \( \text{Spin}(3) \) has two complex components. We thus have the following useful symmetry properties

\[
\tilde{\psi} \gamma^{m_1 \ldots m_p} \chi = (-1) \frac{1}{2} (p-1)(p-2) \chi \gamma^{m_1 \ldots m_p} \psi, \tag{A.17}
\]

where \( \chi, \psi \) are arbitrary commuting spinors. We define the complex conjugate \( \psi_c \) of \( \psi \) via

\[
\tilde{\psi}_c = \psi^\dagger, \tag{A.18}
\]

so that \( \psi_c \) transforms as a spinor. We then have the complex conjugation properties,

\[
(\overline{\tilde{\psi}} \gamma^{m_1 \ldots m_p} \chi)^* = -(−1)^p \overline{\tilde{\psi}} \gamma^{m_1 \ldots m_p} \chi_c. \tag{A.19}
\]

A.3 Spinors in 7D Lorentzian space

The irreducible spinor representation of \( \text{Spin}(1, 6) \) has eight complex components. In terms of an \( \text{Spin}(1, 6) \to \text{Spin}(1, 3) \times \text{Spin}(3) \) decomposition, the 7D spinor \( \psi \) decomposes as

\[
\psi = (\chi_+ + \theta_-) \otimes \xi, \tag{A.20}
\]

where \( \chi, \theta \) are irreducible Weyl spinors of \( \text{Spin}(1, 3) \) and \( \xi \) is an irreducible spinor of \( \text{Spin}(3) \). The seven-dimensional gamma matrices \( \Gamma^M \) decompose as

\[
\Gamma^\mu = \gamma^\mu \otimes 1_2; \quad \Gamma^{m+3} = \gamma^5 \otimes \gamma^m, \tag{A.21}
\]

where \( \mu = 0, \ldots, 3 \) and \( m = 1, 2, 3 \).

B Laplacian eigen one-forms

In this section we work out the low-lying Laplacian eigen one-forms (i.e. those which do not descend form KK states) in the case of non-trivial metric. These are linear combinations of the coframe associated with the metric (2.6)

\[
A = \sum_{a=1}^3 c_a E^a, \tag{B.1}
\]

where the \( c_a \)'s are real constants to be determined in the following and,

\[
E^1 = e^1 + ae^3; \quad E^2 = e^2 + be^3; \quad E^3 = e^3. \tag{B.2}
\]

Recall that in the case of an undeformed metric \((a, b = 0)\) the low-lying Laplacian eigen one-forms are \( e^{1, 2} \), which are harmonic, and \( e^3 \), which has eigenvalue \( f^2 \) [8]. To see how
this spectrum is modified for a general metric \((a,b \neq 0)\), we need to calculate the action of the Laplacian \(\Delta\) on \(A\)

\[
\Delta A = \sum_{a=1}^{3} c_a (d^d + d^d) E^a ,
\]

(B.3)

where \(d^d \equiv \star d \star\). The Hodge star is calculated with respect to the deformed metric (2.6), and operates canonically on \(E^a\)

\[
\star E^a = \frac{1}{2} \sum_{b,c=1}^{3} \varepsilon^{abc} E^b \wedge E^c ,
\]

(B.4)

while the action of the exterior differential on \(E^a\) is calculated form (2.2), (B.2). It is then easily verified that the coframe is co-closed, \(d^d E^a = 0\), and,

\[
\Delta A = f^2 (a^2 + b^2 + 1) (ac_1 + bc_2 + c_3) (aE^1 + bE^2 + E^3).
\]

(B.5)

Setting \(c_1 = a, c_2 = b, c_3 = 1\), it follows that \(aE^1 + bE^2 + E^3\) is a Laplacian eigen one-form with eigenvalue \(f^2 (a^2 + b^2 + 1)^2\). Moreover, we obtain a two-dimensional space of harmonic one-forms, parameterized by the solutions of \(ac_1 + bc_2 + c_3 = 0\).

References

[1] Christoph Bock. On Low-Dimensional Solvmanifolds. *Asian J. Math.*, 20(2):199–262, 2016.
[2] Shamit Kachru, Michael B. Schulz, Prasanta K. Tripathy, and Sandip P. Trivedi. New supersymmetric string compactifications. *JHEP*, 03:061, 2003.
[3] Mariana Grana, Ruben Minasian, Michela Petrini, and Alessandro Tomasiello. A Scan for new \(N=1\) vacua on twisted tori. *JHEP*, 05:031, 2007.
[4] Claudio Caviezel, Paul Koerber, Simon Kors, Dieter Lüst, Dimitrios Tsimpis, and Marco Zagermann. The Effective theory of type IIA AdS(4) compactifications on nilmanifolds and cosets. *Class. Quant. Grav.*, 26:025014, 2009.
[5] Pablo G. Camara and Fernando Marchesano. Open string wavefunctions in flux compactifications. *JHEP*, 10:017, 2009.
[6] David Andriot. New supersymmetric vacua on solvmanifolds. *JHEP*, 02:112, 2016.
[7] David Andriot, Giacomo Cacciapaglia, Aldo Deandrea, Nicolas Deutschmann, and Dimitrios Tsimpis. Towards Kaluza-Klein Dark Matter on Nilmanifolds. *JHEP*, 06:169, 2016.
[8] David Andriot and Dimitrios Tsimpis. Laplacian spectrum on a nilmanifold, truncations and effective theories. *JHEP*, 09:096, 2018.
[9] David Andriot, Alan Cornell, Aldo Deandrea, Fabio Dogliotti, and Dimitrios Tsimpis. A new mechanism for symmetry breaking from nilmanifolds. *JHEP*, 05:122, 2020.
[10] Aldo Deandrea, Fabio Dogliotti, and Dimitrios Tsimpis. Twisting by the Higgs. *arXiv:2201.01151 [hep-ph]*, 1 2022.
[11] S. Thangavelu. Harmonic analysis on heisenberg nilmanifolds. *Revista de la Unión Matemática Argentina*, 50(2), 2009.
[12] C. S. Gordon and E. N. Wilson. The spectrum of the laplacian on riemannian heisenberg manifolds. *Michigan Math. J.*, 33, 1986.

[13] R. Gornet. A new construction of isospectral riemannian nilmanifolds with examples. *Michigan Math. J.*, 43, 1996.

[14] R. Gornet. The marked length spectrum vs. the laplace spectrum on forms on riemannian nilmanifolds. *arXiv:dg-ga/9510001*, 1995.

[15] M. M. Peloso D. Müller and F. Ricci. Eigenvalues of the hodge laplacian on the heisenberg group. *Collectanea Mathematica*, 57(327), 2006.

[16] Christian Baer Bernd Ammann. The dirac operator on nilmanifolds and collapsing circle bundles. *Ann. Global Anal. Geom.*, 16(3):221–253, 1998.