Quantization of Fields by Averaging Classical Evolution
Equations on a de Sitter Spacetime

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Abstract

This paper extends the formalism for quantizing field theories via a microcanonical quantum field theory and Hamilton’s principle to evolution equations on de Sitter spacetime. These are based on the well-known correspondence under a Wick rotation between quantum field theories and 4-D statistical mechanical theories. By placing quantum field theories on a 4+1-D or de Sitter manifold, under Wick rotation to 5-D, expectations of observables are calculated for a microcanonical field theory averaging Hamiltonian flow over a fifth spacelike dimension, a technique common in lattice gauge simulations but not in perturbation theory. In a novel demonstration, averaging pairs of external lines in the classical Feynman diagrams over the fifth dimension generates diagrams with loops and vacuum fluctuations identical to Standard Model diagrams. Virtual fermions are also demonstrated to have momentum in the 5th dimension while on-mass shell fermions do not, a novel observation. Because it is microcanonical, this approach, while equivalent for standard quantum fields theories in the Standard Model, is able to quantize theories that have no canonical quantization. It is also unique in representing expectations as averages over solutions to an ordinary, classical PDE rather than a path integral or operator based approaches. Hence, this approach draws a clear connection between quantum field theory and classical field theory in higher dimensions which has implications towards how quantum effects are interpreted.

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I. INTRODUCTION

Microcanonical quantum field theory first appears in the literature in the early 1980’s. Strominger developed a perturbation theory for the $\phi^4$ theory in the microcanonical ensemble hoping to extend it to quantum gravity [1]. Iwazaki later proved the existence of the perturbation series [2] and applied it to fermions [3]. Creutz developed his demon-algorithm based microcanonical quantum field theory around the same time [4]. Meanwhile, in a pair of seminal papers, Callaway developed microcanonical quantum field theory for lattice gauge computations [5][6]. In his method, Callaway proposes that discrete fields such as the electromagnetic vector potential, $\phi_{n,\mu}$, have conjugate momenta with respect to a second time dimension $\tau$, $p_{n,\mu} = \partial \phi_{n,\mu} / \partial \tau$. He points out that in the canonical quantization any quantity independent of the field $\phi$ can be added to the action. Thus, an observable expectation with respect to an action $S$,

$$
\langle O \rangle = Z^{-1} \int D\phi O \exp[-S]
$$

($\hbar = 1$), can also be given by $\langle O \rangle = Z'^{-1} \int D\phi Dp O \exp[-\beta H]$ where the energy functional, $H = T[p] + S[\phi]/\beta$, and partition function, $Z' = \int D\phi Dp \exp[-\beta H]$. He connects this equivalent theory to the microcanonical ensemble. These methods have since evolved into the hybrid Monte Carlo or molecular dynamics approach to lattice gauge simulation [7].

Approaching the problem as Hamiltonian flow in an additional dimension of $1...N$ lattice points of a 4-D lattice, the microcanonical quantum field theory is the integral over potential values on the lattice and their conjugate momenta:

$$
\Omega = \int d\phi_{1,\mu} \ldots d\phi_{N,\mu} dp_{1,\mu} \ldots dp_{N,\mu} \delta(H_0 - H),
$$

where the Hamiltonian is $H = T + V$ ($L = T - V$ is the Lagrangian by a Legendre transform), $T = \frac{1}{2} \sum_{n=1}^{N} |p_{n,\mu}|^2$ and $V = \sum \text{Re}(1 - U_{n,\mu}U_{n+\nu,\mu}^{-1}U_{n,\nu}^{-1})$ where $U_{n,\mu} = \exp i\phi_{n,\mu}$. The action is $S = \beta V$ where $\beta = 1/g_0^2$ with $g_0$ a coupling constant. This formulation is for $U(1)$ lattice gauge theories but extends to SU(2) and SU(3).
Observables on the lattice are given by,
\[ \langle O \rangle = \Omega^{-1} \int d\phi_{1,\mu} \cdots d\phi_{N,\mu} dp_{1,\mu} \cdots dp_{N,\mu} O \delta(E - H). \] (2)

The Hamiltonian is invariant under local gauge transformations \( U_{n,\mu} + W_n U_{n,\mu} W_n^\dagger \) if \( W_n \in U(1) \) and independent of \( \tau \).

By Hamilton’s equation for the flow through the second time dimensional \( \tau \),
\[ \frac{d^2 \phi_{n,\mu}}{d\tau^2} = \dot{p}_{n,\mu} = -\frac{\partial V\{\phi\}}{\partial \phi_{n,\mu}}. \]

The flow explores the \((2N - 1)\) dimensional hypersurface of constant energy \( H[\phi, p] = E \) and any expectation value is given by the average,
\[ \langle O \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau d\tau' O[\phi(\tau'), p(\tau')], \] (3)
and is a unique function of \( E \).

The equivalence of equations 2 and 3 follows from what Callaway refers to as the “principle of equal weight” which states that given \( E \), trajectories given by solutions to Hamilton’s equations cover the fixed energy hypersurface with equal density. That expectation values can be computed in either ensemble is discussed in [5] and references cited therein. This principle follows from the ergodic hypothesis. As long as the initial conditions are chosen appropriately and the system has sufficient mixing, all points on the energy hypersurface will be visited with equal probability over infinite \( \tau \). In this case, a time average is equivalent to an average over all phase space.

Strominger and Iwazaki present continuum microcanonical quantum field theories as limits of discretized lattice theories. Likewise, Callaway’s discrete Hamiltonian system can be extended to continuum field theory by taking the limit of the number of lattice sites to infinity, \( N \to \infty \), and the lattice spacings to zero. Hamilton’s principle can then be applied via the Euler-Lagrange equations in the standard field theoretic way,
\[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial p} \right) + \frac{d}{d x_\mu} \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) - \frac{\partial L}{\partial \phi} = 0, \] (4)
where \( L \) is the Lagrangian \( L = T - V \). In this case, we derive a partial differential equation indexed by coordinates \( x \) rather than a set of ordinary differential equations indexed by lattice sites \( n \). Provided that \( L \) is not directly dependent on \( \tau \), energy \( H = E \) is conserved by
these equations. This provides a starting point for a perturbation theory or a computational solution to the PDE.

In this paper, I extend the efforts of Strominger, Iwazaki, Creutz, and Callaway and others to quantize the Standard Model into a set of continuum flow equations, leading to a nonlinear classical equations in 4+1-D. Under Wick rotation to 5-D, the statistics of these flow equations is equivalent to quantum field theory in the microcanonical path integral formulation which in turn is equivalent to the canonical path integral formulation for the Standard Model [1]. Averages over perturbations about plane wave solutions show that it is equivalent to the standard perturbation series. Before averaging, perturbation series solutions to classical Klein-Gordon equations generate Feynman diagrams similar to those for quantum field theory, but they lack any loops or vacuum fluctuations. I show how averaging together (correlating) classical diagrams converts external lines into loops by connecting pairs of lines, turning external lines into internal ones. This method is extended to quantum electrodynamics (QED) including fermions and non-Abelian gauge theories. These equivalences show a novel connection between quantum field theory in Lorentz spacetime and classical theories on de Sitter spacetime.

II. METHOD

We begin by assuming that all field theories exist on a 4+1-D flat manifold, such that the usual spacelike coordinates, \( x, y, z \), are joined by a fourth spacelike dimension \( w \) and the metric has signature \((- + + + +)\). We could take \( w \) to be a timelike dimension as well or follow [2] and leave it undetermined. In the case of a scalar theory it makes little difference, but, for other theories of interest, particularly gravity, this may create problems such as negative cosmological constants. More immediately, it is easier to present fermions on a de Sitter rather than anti-de Sitter spacetime [9]. We use capital letters \( A, B, C \) as 5-vector and 5-tensor subscripts, numbered 0 through 4 with \( x_4 = w \). Small Greek letters, \( \mu, \nu, \lambda \), represent 4-vector and 4-tensors, numbered 0 through 3. All fields, \( \psi, \phi, A_B, g_{AB}, \ldots \), are parameterized by this space-like dimension, \( w \). At every slice \( w \) is a single instance or microstate of the classical 3+1-D universe. All vectors and tensors are assumed to have indexes from 0 through 4 (five indexes). We show below how assumptions on initial conditions, choice of gauge, and averaging can remove the additional index. Under Wick rotation
\( t \rightarrow i\sigma \), all fields are in equilibrium in the \( w \) dimension; hence, actions are integrals over the four usual dimensions, \( x_\mu = (t, x, y, z) \), which we write as \( S = \int d^4x L \).

We will work in double Wick rotated space to put the equations in more familiar statistical territory: \( t \rightarrow i\sigma \) and \( w \rightarrow i\tau \), making \( \tau \) timelike and \( \sigma \) spacelike. This has the benefit that reversing the Wick rotation of time gives us quantum field theoretic results while having the additional dimension timelike makes the calculations look more natural (wave rather than Poisson equations) while not affecting the results. This means the signature becomes \((+++-)\).

Given any standard action, \( S \), over a field \( \phi \) and coupling \( g_0 \) the corresponding Hamiltonian is the kinetic added to a potential energy, \( V \),

\[
H = T + V,
\]

where \( S = \beta V \), \( \beta = 1/g_0^2 \), and \( T = \frac{1}{2} \int d^4x |p|^2 \) where \( p = \partial \phi / \partial \tau \). By a Legendre transform, \( L = [\int d^4x |p\dot{x}|] - H \), we also obtain a Lagrangian,

\[
L = T - V.
\]

If the field is a vector \( A_\mu \), then \( T = \frac{1}{2} \int d^4xp_\mu p^\mu \) and likewise for higher order tensors.

The \( \phi^4 \) Lagrangian is well-known

\[
\mathcal{L}[\phi] = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 - J\phi,
\]

Let \( p = \partial \phi / \partial \tau \) in which case \( L = \int d^4x \frac{1}{2} p^2 + \mathcal{L}[\phi] \). By the Euler-Lagrange equations,

\[
\ddot{\phi} = \partial_\alpha (\partial \mathcal{L} / \partial (\partial_\alpha \phi)) - \partial \mathcal{L} / \partial \phi,
\]

where \( \ddot{\phi} = \partial^2 \phi / \partial \tau^2 \), the final evolution equation in terms of \( \phi \) alone over the 4+1-D space is then the Klein-Gordon equation:

\[
\ddot{\phi} = (\partial^\alpha \partial_\alpha - m^2)\phi - \frac{\lambda}{3!} \phi^3 + J. \tag{5}
\]

or

\[
0 = (\Box_{4+1} + m^2)\phi + \frac{\lambda}{3!} \phi^3 - J, \tag{6}
\]

where \( \Box_{4+1} = \partial_\tau^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 - \partial_\sigma^2 \).
III. PERTURBATION THEORY

In this section, the derivation the perturbation theory is given for the scalar field. Let $J = 0$,

$$0 = (\Box_{4+1} + m^2)\phi + \frac{\lambda}{3!} \phi^3. \quad (7)$$

This equation has a perturbation solution.

We are interested in monomial potentials. Let $u(x, \tau)$ be a nonlinear potential such that our KG equation is,

$$0 = (\Box_{4+1} + m^2)\phi + u(x, \tau). \quad (8)$$

Applying Green’s function method to solve the nonhomogeneous equation,

$$\phi(x, \tau) = \int d^4y \int d\chi - u(y, \chi)D(x - y; \tau - \chi), \quad (9)$$

where the propagator (Green’s function) is,

$$D(x - y; \tau - \chi) = \int \frac{d\omega}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} e^{i(\omega(\tau - \chi) + (x - y)k)} - \omega^2 + k^2 + m^2 \quad (10)$$

and $k^2 = k_0^2 + k_1^2 + k_2^2 + k_3^2$, and $kx = k_\mu x^\mu$.

Now, we want to solve (7) when $0 < \lambda \ll m^2$. This method is straightforward perturbation theory \[10\]. Let the solution be the perturbation series,

$$\phi = \sum_{n=0}^{\infty} \phi_n \lambda^n. \quad (11)$$

When we plug this into the equation we get,

$$\sum_{n=0}^{\infty} (\Box_{4+1} + m^2)\phi_n \lambda^n = -\lambda \left(\sum_{n=0}^{\infty} \phi_n \lambda^n\right)^3$$

$$= -\sum_{n=0}^{\infty} \left(\sum_{k,l,m} \phi_k \phi_l \phi_m\right) \lambda^n. \quad (12)$$

Collecting coefficients we find,

$$(\Box_{4+1} + m^2)\phi_n = -\sum_{k,l,m \atop k+l+m+1=n} \phi_k \phi_l \phi_m. \quad (13)$$

Because $k, l, m < n$, this creates an iterative solution. Starting with the free solution,

$$(\Box_{4+1} + m^2)\phi_0 = 0, \quad (14)$$
given by a sum of plane waves:

\[ \hat{\phi}_0(k, \tau) = \frac{1}{2\omega(k)} \left\{ A(k)e^{i\omega(k)\tau} + A^*(k)e^{-i\omega(k)\tau} \right\}, \tag{11} \]

where \( A(k) \) is any function, \( \hat{\cdot} = \mathcal{F}[\cdot] \) is the Fourier transform \( x_\mu \to k_\mu \), and \( \omega(k) = \sqrt{k^2 + m^2} \).

The position space free solution is \( \phi_0(x, \tau) = \int d^4k/(2\pi)^4 e^{ikx} \hat{\phi}_0(k, \tau) \). We then use that solution to compute the next solution,

\[ (\Box_{4+1} + m^2)\phi_1 = -\phi_0^3, \]

which is

\[ \phi_1(x, \tau) = \int d^4y \int d\chi - \phi_0^3(y, \chi)D(x - y; \tau - \chi), \]

then that for the following solution,

\[ (\Box_{4+1} + m^2)\phi_2 = -3\phi_0^2\phi_1. \]

which allows our previous solution to substitute for \( \phi_1 \),

\[ \phi_2 = 3 \int d^4y \int d\chi \int d^4y' \int d\chi' \]

\[ D(x - y; \tau - \chi)\phi_0^2(y, \chi)D(y - y'; \chi - \chi')\phi_0^3(y', \chi'), \]

and so on so that all solutions can be found in terms of interactions of the free solution \( \phi_0 \). The integrals over \( \chi \) and \( \chi' \) indicate interactions between different slices of \( \tau \), the classical 3+1-D universes.

Let the perturbation solution be truncated to level \( N \), such that

\[ \phi(x, \tau) \approx \sum_{n=0}^{N} \phi_n \lambda^n, \]

and discard all solutions of order \( O(\lambda^{n+1}) \). (In general, the solution will diverge as \( N \to \infty \) as in standard perturbation theory.)

The 2\( M \)-correlation Green's functions against the vacuum \( \Omega \) in momentum space are,

\[ \langle \Omega| \hat{\phi}(k_1) \cdots \hat{\phi}(k_{2M})|\Omega \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau d\tau' \hat{\phi}(k_1, \tau) \cdots \hat{\phi}(k_{2M}, \tau). \]
Therefore, the expected value of the correlation of two plane wave solutions averaged over \( \tau \) is,

\[
\langle \hat{\phi}_0(k)\hat{\phi}_0(k') \rangle = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} d\tau' \hat{\phi}_0(k, \tau')\hat{\phi}_0(k', \tau')
\]

\[
= \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} d\tau' \frac{1}{4\sqrt{k^2 + m^2} \sqrt{k'^2 + m^2}}
\times \left\{ A(k)A(k')e^{i(\omega(k) + \omega(k'))\tau'} + A(k)A^*(k')e^{i(\omega(k) - \omega(k'))\tau'} + A^*(k)A(k')e^{i(-\omega(k) + \omega(k'))\tau'} + A^*(k)A^*(k')e^{i(-\omega(k) - \omega(k'))\tau'} \right\}
\]

(12)

Now, note that the integral is only non-zero if \( \omega(k) = \pm \omega(k') \). Since \( \omega(k) \geq 0 \), we have

\[
\langle \bar{\phi}_0(k)\bar{\phi}_0(k) \rangle = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} d\tau' \phi(x, \tau')\phi(y, \tau')
\]

\[= \frac{|A(k)|^2}{2|k^2 + m^2|}.
\]

(13)

Let \( A(k) = \sqrt{2} \), which gives the equivalent correlation to standard path integral quantization when \( \hbar = c = 1 \). This gives the initial condition for \( \phi \) as well (an amplitude scaling that turns out to be constant for all \( k \)). To regularize this for renormalization, which we will need to do, we will let \( A(k) \) fall to zero at some large \( |k| = \Lambda \), but for now we will let it remain constant.

The momentum space version of one of our solutions (keeping it in terms of \( \tau \)),

\[
\hat{\phi}_1(k, \tau) = -\int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \int d\chi
\times \hat{\phi}_0(k_1, \chi)\hat{\phi}_0(k_2, \chi)\hat{\phi}_0(k_3, \chi)\hat{D}(k; \tau - \chi)
\times \delta(k - k_1 - k_2 - k_3)
\]

where

\[
\hat{D}(k; \tau) = \int \frac{d\omega}{2\pi} \frac{e^{i\omega \tau}}{-\omega^2 + k^2 + m^2}
= -\frac{1}{i\sqrt{k^2 + m^2}} e^{i\sqrt{k^2 + m^2} |\tau|}.
\]

(14)

Taking the mean gives the propagator \( \hat{D}(k) = \langle \hat{D}(k; \tau) \rangle = \frac{1}{k^2 + m^2} \). Taking the negative gives the usual propagator after reverse Wick rotation. The delta function ensures conservation of momentum in the interaction. Higher order perturbations can be found similarly.
IV. FEYNMAN RULES

Suppose we want the Green’s function in momentum space up to order $\lambda$,

$$\langle \Omega | \hat{\phi}(k_1) \hat{\phi}(k_2) | \Omega \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau d\tau' \hat{\phi}(k_1, \tau) \hat{\phi}(k_2, \tau)$$

Carrying out the perturbative calculation, the correlation has three terms. The first, a zeroth order correlation, involves two free fields and no interaction. The second and third involve correlation between a free field and a single interaction.

These terms can be laboriously computed but there is a simpler approach with Feynman diagrams. Let the propagator be the field $\phi_g = -D(x - y; \tau - \chi)$. The solution for position space $\phi_1(x, \tau)$, for example, is

This represents bringing three plane wave solutions, $\phi_0$, to a single point $y$ and transporting that to $x$ using the Green’s function, $\phi_g$. This pattern can be iterated to higher orders adding more points that transport more particles together with symmetry factors representing the permutations of such graphs.

Helling [11] writes down the rules for these classical Feynman diagrams as follows:

1. Draw $n$ vertices for the expression for $\phi_n$ at order $\lambda^n$.

2. Each vertex gets one in-going line at the left and three outgoing lines to the right.

3. A line can either connect to the in-going port of another vertex or to the right-hand side of the diagram.

4. Write down an integral for the point of each vertex.
5. For a line connecting two vertices at points \( y_1, \tau_1 \) and \( y_2, \tau_2 \), write down a Greens function \( \phi_g(y_1 - y_2; \tau_1 - \tau_2) \).

6. For a line ending on the right, write down a factor of \( \phi_0 \) evaluated at the point of the vertex at the left of the line.

7. Multiply by the number of permutations of outgoing lines at the vertices which yield different diagrams (“symmetry factor”).

These rules are nearly identical to the Feynman rules for the equivalent scalar statistical theory with the only difference being that loops are prohibited because of rules #2 and #3 on incoming and outgoing lines. Relaxing these rules allows for loops.

When the expected value is taken by averaging over \( \tau \), we find that the correlation of pairs of plane wave solutions and the propagator, which are different in the 4+1-D evolution theory, become equal in the statistical theory. Reverse Wick rotating back to real time:

\[
\langle \phi_0(x)\phi_0(y) \rangle = \langle \phi_g(x-y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon},
\]

where \( i\epsilon \) is a small value to avoid integrating through poles and \( k^2 = -k_\mu k^\mu \). This can be shown by taking the \( \tau \)-average of 14 and comparing it to the correlation given by 13, then transforming to position space.

Expectations of correlations such as \( \langle \phi_1(x_1)\phi_0(x_2) \rangle \), for example, develop loops in the 3+1-D statistical theory from the pairs of instances of \( \phi_0 \) that were external lines in the 4+1-D theory. The preceding example generates two pairs of plane wave solutions and one propagator which, when averaged, become three propagators. Because of the linearity of the integrals in the average over \( \tau \), taking the expectation of correlations of perturbation solutions always reduces to products and sums of propagators provided the number of plane wave solutions is even.

In real time, a factor of \( i \) is introduced in the example in the preceding paragraph:

\[
\langle \phi_1(x_1)\phi_0(x_2) \rangle = i\lambda \int d^4y \bar{\phi}_g(x_1 - y)\phi_g(y)\bar{\phi}_g(x_2 - y). \]

The diagrammatic equation correlates the first order diagram with a single, zeroth order plane wave solution (just a line),
\[
\left\langle \phi_g(x_1 - y) \phi_0(y) \times \phi_0(x_2) \right\rangle = x_1 y x_2
\]  

(15)

A pair correlation of plane waves at \( y \) in the classical solution (the two solutions above and below the left diagram) becomes a propagator from \( y \) back to itself in the quantum field solution on the right. Thus, it is as if two of the plane wave solutions in the diagram on the left connected to one another to form a propagator while the other plane wave solution at \( y \) on the right side of the left diagram connects to the solution at \( x_2 \).

The correlation also need not be between a diagram and an external line. As seen in the next section on QED, two complete diagrams can be correlated.

V. QUANTUM ELECTRODYNAMICS

Following [11] and [6], these scalar field results can be extended to other quantum theories, adding a kinetic energy term in \( \tau \) and solving the resulting PDE perturbatively, where the same transformation from pair correlation to propagator occurs in the \( \tau \) average. In quantum electrodynamics, there is a cubic term in the Lagrangian,

\[
\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - e \bar{\psi} \gamma_\mu A^\mu \psi - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. 
\]  

(16)

These produce the equations of motion,

\[
i \gamma^\mu \partial_\mu \psi - m \psi = e \gamma_\mu A^\mu \psi
\]

(17)

and

\[
\Box_{3+1} A^\mu = e \bar{\psi} \gamma^\mu \psi,
\]

(18)

assuming the Lorentz gauge condition, \( \partial_\mu A^\mu = 0 \).

Add another spatial dimension, \( x_4 = w \), a \( w \) dependency to the 5-vector field \( A^B \) and spinor field \( \psi \), and, in addition to the Lorentz condition, require the gauge condition \( A_4 = 0 \).
Now double Wick rotate, and the classical 4+1-D equation has the form,

$$\Box_{4+1} A^\mu = e \bar{\psi} \gamma^\mu \psi.$$  (19)

This equation can be extended with classical ghost fields \[12\] as well, but these do not become essential until the local gauge symmetry of non-Abelian theories demands them.

In molecular dynamical approaches (e.g., Hybrid Monte Carlo), the Dirac Lagrangian is not extended (\[7\] p. 62). Instead, bosonic term, $\frac{1}{2} p_\mu p^\mu$, where $p_\mu = \dot{A}_\mu$, is sufficient to drive the system to evolve in $\tau$ and maintain ergodicity.

The extended Dirac Lagrangian has an additional term $-\bar{\psi} \gamma^5 \dot{\psi}$ which couples the spinor to $\dot{\psi} = \partial_\tau \psi$ so that it cannot be, e.g., normalized out of the path integral.

It is possible, however, to use the de Sitter space Dirac equation and still achieve standard results. As far back as 1935 in his paper on the electron wave equation in de Sitter space \[9\], Dirac wrote down an alternative but equivalent form for the Dirac equation in 3+1-D,

$$\gamma^\mu (\partial_\mu - i e A_\mu) - i \gamma^5 \kappa \psi = 0,$$

where $\kappa$ is a mass term \[13\].

It turns out that we can derive this from averaging the 4+1-D equation where the momentum in the $\tau$ dimension is part of the mass term, $p_4 + m = \kappa$. The Dirac equation in 4+1-D, has gamma matrices obey the 4+1-D anti-commutation relation, $\{\Gamma^A, \Gamma^B\} = 2 \eta^{AB} I_4$ where $A$ and $B$ run over the indexes 0, 1, 2, 3, 4 and $x_4 = w$ is our extra dimension. From this, we will obtain the beautiful result that the Dirac matrices in 4+1-D are simply the Dirac matrices in 3+1-D plus $i$ times the chiral matrix $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$. Thus, let $\Gamma^\mu = \gamma^\mu$ and $\Gamma^4 = i \gamma^5$. The 4+1-D Dirac equation can be written,

$$i \Gamma^4 (\dot{\psi} - m) + i \Gamma^\mu \partial_\mu \psi = e \Gamma^\mu A^\mu \psi$$  (20)

with $\dot{\psi} = \partial_\tau \psi$. This equation also satisfies the Klein-Gordon equation in 4+1-D. The double Wick rotation multiplies $i$ by the relevant gamma matrices, $\Gamma^0 \rightarrow i \Gamma^0 = \Gamma^0$ and $\Gamma^4 \rightarrow i \Gamma^4 = \Gamma^4$ while $\Gamma^i = \Gamma^i$.

Assuming $e$ is a small parameter, a perturbation solution exists. Now, instead of a Green’s function for one field, we have two, one in each type of field $A_\mu$ and $\psi$ and each given by its own type of line. The 4+1-D Green’s functions (in the more common momentum space representation) are,

$$\dot{\psi}_g(\omega, p) = \frac{\Gamma^4 (\omega + m) + \Gamma^\mu p_\mu}{-\omega^2 + p^2 + m^2 + i \epsilon},$$
and

\[ \hat{A}_{g,\mu\nu}(\omega, p) = \frac{-i\eta_{\mu\nu}}{-\omega^2 + p^2 + i\epsilon}. \]

Taking the Fourier transform \( \omega \rightarrow \tau \) and mean over \( \tau \) of each gives the propagators,

\[ \hat{\psi}_g(p) = \frac{\Gamma^4 \kappa + \Gamma^\mu p_\mu}{p^2 + m^2 + i\epsilon}, \]

and

\[ \hat{A}_{g,\mu\nu}(p) = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}, \]

where \( \kappa = \omega(p_\mu) + m \) and \( \omega(p_\mu) = \sqrt{p^2 + m^2} \).

The perturbation theory now represents the interaction of the two fields. Let \( A^\mu = \sum_{n=0}^{\infty} A^{\mu,n} e^n \) and \( \psi = \sum_{n=0}^{\infty} \psi_n e^n \). The calculations are similar to the scalar theory and have the same result of reproducing quantum field theories in 3+1-D. Because the wave equation [19] has the right hand side, \( e\bar{\psi}\gamma^\mu \psi \), which is quadratic in the fermion, we have diagrams with two fermions on the right connecting to one photon propagator on the left where one fermion is always going in and one always coming out. The Dirac equation [20] is bilinear in the photon and fermion. Thus, it connects one photon to one fermion on the right and one fermion propagator on the left. These form tree structures as pictured in [11]. As shown below, using the power of averaging, diagrams can be correlated together to form the more complex diagrams of quantum field theory.

In the following we show that common scattering results such as fermion scattering \( \langle \bar{\psi}(x_1)\gamma^\mu \psi(x_1)\bar{\psi}(x_2)\gamma^\nu \psi(x_2)A_{g,\mu\nu}(x_1 - x_2) \rangle \) and Compton scattering \( \langle \bar{\psi}(x_1)\gamma^\mu \psi_g(x_1 - x_2)\gamma^\nu \psi(x_2)A_\mu(x_1)A_\nu(x_2) \rangle \) agree with those of standard quantizations. The correspondence of pair correlations of free, plane wave solutions and propagators in the average over \( \tau \),

\[ \langle \psi(x, \tau)\bar{\psi}(y, \tau) \rangle = \langle \bar{\psi}_g(x - y, \tau) \rangle = \psi_g(x - y) \] and \( \langle A_\mu(x, \tau)A_\nu(y, \tau) \rangle = \langle A_{g,\mu\nu}(x - y, \tau) \rangle = A_{g,\mu\nu}(x - y) \) also create loops in diagrams where there were none in the classical solution when \( x = y \) exactly as in the scalar theory of the previous section.

The plane wave solution for the vector field, \( A_\mu \), is just a vector of solutions in the form of [11] with bare mass zero, \( m = 0 \).

The relation \( \psi(x, \tau)\bar{\psi}(y, \tau) \) is an outer product of two four component spinors which, when composed of a pair of plane wave solutions, produces a matrix, which, in the average over \( \tau \), becomes the Dirac propagator in 3+1-D. Although the outer product to propagator equivalence can be found in many textbooks (e.g., [14]) for 3+1-D, adding the additional dimension requires a slightly different derivation shown in the following:
The Dirac equation has four plane wave solutions, the four combinations of spin-up, spin-down, positive energy (electrons), and “negative” energy (positrons). This is true in both 3+1-D and 4+1-D. For the following, we will work in real time and space assuming we can do the same in double Wick rotated spacetime. Consider the 5-momentum $p_A$ where $p_i = \pm E$ (with $c = 1$), $(p_1, p_2, p_3) = \vec{p} = (p_x, p_y, p_z)$ and the final momentum is $p_4 = p_\mu$. For this we will describe solutions in terms of the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

In the Dirac basis,

$$\Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix}$$

where $\sigma^i = (iI_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (-iI_2, -\vec{\sigma})$ and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ and

$$\Gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

By the anzatz, $\psi = u(p_\mu)e^{-ip_Ax^A}$, $u(p_\mu)$ is four component spinor independent of position, $x^A$. Let $m = 0$ for now. We will reintroduce it later. The Dirac equation in 4+1-D is now,

$$\Gamma^A p_A u = \begin{pmatrix} E & p_i \sigma^i \\ p_i \bar{\sigma}^i & -E \end{pmatrix} u = 0, \quad (21)$$

where $p_0 = E$.

Let $u^T = (u_1, u_2)$ with $u_i$ being 2-component spinors. The Dirac equation now reads

$$(p_i \sigma^i) u_2 = -E u_1, \quad (p_i \bar{\sigma}^i) u_1 = E u_2. \quad (22)$$

Each of these equations implies the other. In the following, we can use the identity $(p_i \sigma^i)(p_i \bar{\sigma}^i) = -p_i p^i = -E^2$ which is equivalent to $p_A p^A = 0$.

Let $u_1 = \sqrt{-i(p_i \sigma^i)} \xi$ and $u_2 = \sqrt{i(p_i \bar{\sigma}^i)} \xi$ with $\xi$ a constant 2-component spinor. These solve the equations [22]

Now, introduce the basis, $\xi^s$ such that $s = 1, 2$ such that $\xi^s \xi^s = \delta^{rs}$ and $\sum_s \xi^s \xi^{s\dagger} = I_2$, e.g., $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we have solutions $u^s(p_A)$. Let $p_4 \rightarrow p_4 + m$. The
outer product of these solutions is,
\[
\sum_{s=1}^{2} u^s \bar{u}^s = \Gamma^4(p_4 + m) + \Gamma_\mu p^\mu,
\] (23)

which follows from direct calculation [14]. Negative energy solutions, \(v^1\) and \(v^2\) can be found likewise and these satisfy the completeness relation:
\[
\sum_{s=1}^{2} v^s \bar{v}^s = \Gamma^4(p_4 - m) + \Gamma_\mu p^\mu,
\] (24)

Now, let \(\psi(p_\mu, \tau) = A \int d\omega \delta(\omega^2 + p^2 - m^2) [(u^1 + u^2)e^{i\omega\tau}]\) where \(A\) is constant and \(\omega = p_4 = \sqrt{p^2 - m^2}\). Using the same methods and the result [23] one can show that
\[
\langle \psi(p_\mu, \tau) \bar{\psi}(p_\mu, \tau) \rangle = \frac{\Gamma^4(\omega(p_\mu) + m) + \Gamma_\mu p^\mu}{p^2 - m^2 + i\epsilon} = \frac{i\gamma^5(\omega(p_\mu) + m) + \gamma_\mu p^\mu}{p^2 - m^2 + i\epsilon},
\]
where we have added the small factor of \(\epsilon\) to avoid poles [14]. (We can also show this for the negative energy solutions \(v^s\). See pg. 108-9 of [14] for more details.) The additional term in the numerator \(i\gamma^5\sqrt{p^2 - m^2}\) modifies the mass. On mass shell \(p^2 = m^2\), however, it vanishes.

The above presents an interesting result. When particles are on mass shell, it means their momentum in the 5th dimension is zero, i.e., they are parallel to that dimension, since if \(p^2 = m^2\) then \(p_4 = 0\). Therefore, a necessary condition for particles to be “real” is that they not be propagating in that dimension, i.e., they are confined to a single microstate or universe.

The diagrams in the classical theory have no loops again, but they develop loops when averaged over \(\tau\). For example, using the Feynman rules described in the previous section on scalar theory, modified for QED to include fermion/photon vertexes, the following diagrammatic expression [25] shows the correlation of two second-order classical momentum space diagrams. These have been rotated 45 degrees so the starting fermion propagator is at the bottom. The diagram on the right has been reversed from the convention we have been using so that it starts on the right and moves to the left. This reversal makes it easier to see the transformation of the correlation of the two diagrams into the loop diagram of the quantum field theory.

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When integrated, the plane wave solutions form outer products which become propagators, and the correlation becomes a quantum diagram showing fermion scattering. An electron emits a photon which decays into an electron-positron pair and recombines into a photon before being absorbed into another electron:

This section has shown that the theory established for scalar fields that classical perturbation theory in 4+1-D reduces to quantum theory in 3+1-D on average over an additional dimension applies to quantum electrodynamics as well, including fermions. The rules for the Feynman diagrams are the same as in the standard theory with correlations of plane wave solutions to the 4+1-D equations becoming the loops in the statistical theory.

VI. NON-ABELIAN THEORY

The procedure for developing an equivalence between classical 4+1-D non-Abelian gauge theory such as SU(2)xU(1) electroweak theory or SU(3) Quantum Chromodynamics (QCD) is straightforward. The free equations of motion in 3+1-D are,

\[ \partial^\mu F^a_{\mu\nu} + g f^{abc} A^\mu_{\nu} F^c_{\mu\nu} = 0, \]  

(26)

with

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \]
where \( f^{abc} \) are the structure constants of the Lie algebra,

\[
\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c.
\]

For the 4+1-D case \cite{15}, it is convenient to choose the gauge \( A^a_4 = 0 \) and \( \partial^\mu A^a_\mu = 0 \) (or an \( R_\xi \) gauge), in which case the free equations are

\[
\Box_{4+1} A^a_\mu + g f^{abc} A^b_\nu \partial^\nu A^c_\mu + g f^{abc} A^{b,\nu} F^c_{\nu\mu} = 0 \tag{27}
\]

\[
f^{abc} A^b_\mu \dot{A}^c_\mu = 0 \tag{28}
\]

The first equation in \eqref{27} is the usual Yang-Mills equation with an “acceleration” term added to it. Thus, it fits within the overall approach from molecular dynamics.

The overcounting problem of the path integral method of quantization is addressed in molecular dynamics by a change of variables from dependent to independent \cite{6}. For perturbation theory, however, the problem requires a continuum approach that addresses all orders of loop renormalization. The problem itself is slightly less intuitive in the evolution equation approach than in the path integral approach where the functional integral is clearly overcounting. Here the overcounting occurs because the PDE is under-determined (\cite{16} pg. 40-1), a concept Dirac first proposed \cite{17}, in its evolution in \( \tau \).

This is better understood in the discrete case: for \( N \) independent degrees of freedom there are \( M > N \) pairs of Hamiltonian ODEs and \( M \) functions \( \phi_1(\tau), \ldots, \phi_M(\tau) \) (where \( \phi_i(\tau) \) is a discrete element of any field evolving in \( \tau \)) \cite{6}. It may appear by analogy with the path integral that the overcounting problem is in the time-average integral, i.e., that information is being counted at \( \tau_1 \) and later at some \( \tau_2 > \tau_1 \) redundantly counted again so that \( \phi_i(\tau_1) \) is gauge equivalent to \( \phi_i(\tau_2) \). That does not occur. Information (in terms of contributions to expectations of observables) is, instead, being repeated in the degrees of freedom that are evolving in \( \tau \) all along because of interdependencies between the \( M \) solutions. If we were naively to sum up all the contributions from all these equations to the time-average of an observable, we would get the wrong answer because there are more ODEs than actual degrees of freedom.

Likewise, in the continuous case, we need a way to constrain the contributions of redundant degrees of freedom in the PDE built into their functional space. Going from a set of ODEs to a single PDE, the extra functions \( \phi_{N+1}, \ldots, \phi_M \) in the discrete case become extra regions of the solution function’s \textit{image} (while the \( M \) indexes become the domain \( t, x, y, z \) or
that are gauge equivalent to other regions. Once we start using these solutions to do statistics, the redundancies in the image become a problem.

We can resolve the problem using a Faddeev-Popov method of spin-0 anti-commuting ghost fields. These fields remove the redundant dimensions from the PDE solution space. In the Abelian case, equations for the ghost fields are decoupled from the equations for the vector potential. In the non-Abelian case, however, there is no global gauge freedom and so the ghost fields couple to the potential. (A simpler dynamic ghost field can be added by a Bell-Treiman-Veltman transformation instead, but this is only beneficial in one-loop renormalization \[18\] and some computational schemes where it can condition matrices \[12\].)

The Lagrangian for the ghost fields is \[19\],

\[
\mathcal{L}_{\text{ghost}} = -\bar{c}^{a}(\Box_{4+1}c^{a} + gf^{abc}\partial^{A}(A^{b}_{A}c^{c}))
\]

where \(\bar{c}\) and \(c\) are scalar fermionic fields, Grassmann numbers. Taking variation with respect to \(\bar{c}^{a}\), these give the equations of motion,

\[
\Box_{4+1}c^{a} + gf^{abc}\partial^{A}(A^{b}_{A}c^{c}) = 0.
\]

Let the initial condition be \(\dot{A}_{4}^{a} = \ddot{A}_{4}^{a} = 0\) at \(\tau = 0\) and, thus, \(\dot{A}_{4}^{a} = 0\) for all \(\tau\). Adding this condition to the choice of gauge, these reduce to,

\[
\Box_{4+1}c^{a} + gf^{abc}\partial^{A}_{\mu}(A^{b}_{A}c^{c}) = 0.
\]

Now both \[27\] and \[29\] form the Yang-Mills equations with Faddeev-Popov ghosts with an acceleration term added which agrees with the molecular dynamics approach.

Thus, we have shown that we can reduce the equations to the desired results (no dependencies on the \(\tau\) dimension other than those internal to the fields which will be averaged and the acceleration term that will drop out in the average). This is necessary in order to achieve equivalence with standard perturbative QFT within the framework of a classical de Sitter spacetime.

Expanding \[27\] and \[29\] in terms of powers of \(g\), we can obtain a perturbation series for the classical 4+1-D equations. The approach is almost identical to the scalar case but with more kinds of vertices possible and now coordinate and gauge indexes on the lines. The equation \[27\] has quadratic terms in the gluons, \(A_{\mu}^{a}\), as well as a cubic term. Therefore, following the convention of the previous sections, diagrams begin on the left with a gluon
propagator which then leads to vertices that can have two or three gluon lines. These can be plane wave solutions forming external lines or propagators leading to other vertices. The contributions for these vertices turn out to be the same as in the quantum theory. Likewise, the ghost equations have the usual vertices with a vertex that is bilinear with the ghost field and gluon and contributes $-g f^{abc} p^\mu$. Averaging over $\tau$ has the same result as in the scalar theory of producing the usual propagator as well as allowing different diagrams to be correlated in order to form loops. Fermions can be added in the same way as in QED of the previous section.

Renormalization can be done after averaging (with regularization done before) in which case the approach is the same as in the Standard Model.

VII. CONCLUSION

Discrete Hamiltonian flow method of molecular dynamics has been extended to a continuum PDE method showing that classical Feynman diagrams reduce to quantum field theoretic ones when correlated. By doing so, a novel way of reducing quantum physics on a 3+1-D spacetime to classical physics on a 4+1-D de Sitter spacetime has been presented. Is this a mathematical trick or physical reality? One way to distinguish the fundamentally (and mysteriously) statistical nature of quantum physics in the standard formulation and the evolution approach presented in this paper is to look at areas where the equivalence breaks down. In particular, the equilibrium assumption allows for averaging over all $\tau$. If this assumption were to break down and the theory to become a non-equilibrium one, the additional dimension might be observable. Another potential mode of detecting is to measure the effect of virtual particles being particles that have momentum in the $\tau$ dimension and shown in the section on QED. This suggests that the de Sitter rather than Lorentz invariance may be measured when observing virtual particles that become real and vice versa. There may be a measurable effect because the velocity of a massless virtual particle in the familiar three dimensions would be less than the speed of light since some of that velocity is in the 4th spatial dimension.

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