Universality for eigenvalue correlations at the origin of the spectrum

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Abstract

We establish universality of local eigenvalue correlations in unitary random matrix ensembles
\[ \frac{1}{Z_n} \det M |^{2\alpha} e^{-n \text{tr} V(M)} dM \] near the origin of the spectrum. If \( V \) is even, and if the recurrence coefficients of the orthogonal polynomials associated with \( |x|^{2\alpha} e^{-n V(x)} \) have a regular limiting behavior, then it is known from work of Akemann et al., and Kanzieper and Freilikher that the local eigenvalue correlations have universal behavior described in terms of Bessel functions. We extend this to a much wider class of confining potentials \( V \). Our approach is based on the steepest descent method of Deift and Zhou for the asymptotic analysis of Riemann-Hilbert problems. This method was used by Deift et al. to establish universality in the bulk of the spectrum. A main part of the present work is devoted to the analysis of a local Riemann-Hilbert problem near the origin.

1 Introduction

In the present paper we consider the following unitary ensemble of random matrices, cf. \cite{2, 3}
\[ \frac{1}{Z_n} \det M |^{2\alpha} e^{-n \text{tr} V(M)} dM, \quad \alpha > -1/2. \tag{1.1} \]

The matrices \( M \) are \( n \times n \) Hermitian and \( dM \) is the associated flat Lebesgue measure on the space of \( n \times n \) Hermitian matrices, and \( Z_n \) is a normalizing constant (partition function). The confining potential \( V \) in (1.1) is a real valued function with enough increase at infinity, for example a polynomial of even degree with positive leading coefficient. Random matrix ensembles are important in many branches of mathematics and physics, see the recent survey paper \cite{15}. The specific ensemble (1.1) is relevant in three-dimensional quantum chromodynamics \cite{39}.

The ensemble (1.1) induces a probability density function on the \( n \) eigenvalues \( x_1, \ldots, x_n \) of \( M \), given by
\[ P^{(n)}(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{j=1}^{n} w_n(x_j) \prod_{i<j} |x_i - x_j|^2, \]

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where \( \hat{Z}_n \) is the normalizing constant, and where \( w_n \) is the following varying weight on the real line,

\[
w_n(x) = |x|^{2\alpha} e^{-nV(x)}, \quad \text{for } x \in \mathbb{R}.
\] (1.2)

Of particular interest are the local correlations between the eigenvalues of the ensemble \([1,1]\) of random matrices, when their size tends to infinity, at the origin of the spectrum, see \([2,22,32]\).

The correlations between eigenvalues can be expressed in terms of the orthonormal polynomials \( p_{k,n}(x) = \gamma_{k,n} x^k + \cdots \) with \( \gamma_{k,n} > 0 \) with respect to \( w_n \), that is

\[
\int p_{k,n}(x)p_{j,n}(x)|x|^{2\alpha} e^{-nV(x)}dx = \delta_{jk}.
\]

Namely, for \( 1 \leq m \leq n-1 \), the \( m \)-point correlation function

\[
\mathcal{R}_{n,m}(y_1, \ldots, y_m) = \frac{n!}{(n-m)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p^{(n)}(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n)dx_{m+1} \ldots dx_n,
\]

satisfies, by a well-known computation of Gaudin and Mehta \([29]\),

\[
\mathcal{R}_{n,m}(y_1, \ldots, y_m) = \det(K_n(y_i, y_j))_{1 \leq i, j \leq m},
\]

where

\[
K_n(x, y) = \sqrt{w_n(x)} \sqrt{w_n(y)} \sum_{j=0}^{n-1} p_{j,n}(x)p_{j,n}(y)
\]

\[
= \sqrt{w_n(x)} \sqrt{w_n(y)} \frac{\gamma_{n-1,n} p_{n,n}(x)p_{n-1,n}(y) - p_{n-1,n}(x)p_{n,n}(y)}{x-y},
\] (1.3)

which gives the connection with orthogonal polynomials. The second equality in (1.3) follows from the Christoffel-Darboux formula \([35]\).

Akmenn et al. \([2]\) showed that the local eigenvalue correlations at the origin of the spectrum have a universal behavior, described in terms of the following Bessel kernel

\[
\mathbb{J}_\alpha(u, v) = \pi \sqrt{u} \sqrt{v} \frac{J_{\alpha - \frac{1}{2}}(\pi u)J_{\alpha - \frac{1}{2}}(\pi v) - J_{\alpha + \frac{1}{2}}(\pi u)J_{\alpha + \frac{1}{2}}(\pi v)}{2(u-v)},
\] (1.4)

where \( J_{\alpha \pm \frac{1}{2}} \) denotes the usual Bessel function of order \( \alpha \pm \frac{1}{2} \). In \([2]\) it was assumed that the parameter \( \alpha \) is a non-negative integer, that the potential \( V \) is even, and that the coefficients \( c_{k,n} \) in the recurrence relation

\[
x p_{k,n}(x) = c_{k+1,n}p_{k+1,n} + c_{k,n}p_{k-1,n}
\]
satisfied by the orthonormal polynomials have a limiting behavior in the sense that the limit \( c_{k,n} \) exists whenever \( k, n \to \infty \) such that \( k/n \to t \) for some \( t > 0 \). The restriction that \( \alpha \) is a non-negative integer was removed by Kanzieper and Freilikher \([22]\), but they still required the assumption that \( V \) is even and that the recurrence coefficients have a limiting behavior. In fact, their method of proof (which they call Shohat’s method) relies heavily on these recurrence coefficients.

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It is the goal of this paper to establish the universality of the Bessel kernel \((1.4)\) at the origin of the spectrum without any assumption on the recurrence coefficients. We can also allow \(V\) to be quite arbitrarily. We assume the following

\[
V : \mathbb{R} \to \mathbb{R} \text{ is real analytic,} \tag{1.5}
\]

\[
\lim_{|x| \to \infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty, \tag{1.6}
\]

\[
\psi(0) > 0, \tag{1.7}
\]

where \(\psi\) is the density of the equilibrium measure in the presence of the external field \(V\), \([11, 34]\).

Let us explain the condition \((1.7)\). Denote the space of all probability measures on \(\mathbb{R}\) by \(M_1(\mathbb{R})\), and consider the following minimization problem

\[
\inf_{\mu \in M_1(\mathbb{R})} \left( \int \int \log \left| \frac{1}{s-t} \right| d\mu(s)d\mu(t) + \int V(t)d\mu(t) \right). \tag{1.8}
\]

Under the assumptions \((1.5)\) and \((1.6)\) it is known that the infimum is achieved \([9, 34]\) uniquely at the equilibrium measure \(\mu_V \in M_1(\mathbb{R})\) for \(V\). The measure \(\mu_V\) has compact support, and since \(V\) is real analytic, it is supported on a finite union of intervals. In addition it is absolutely continuous with respect to the Lebesgue measure, i.e.

\[
d\mu_V(x) = \psi(x)dx,
\]

and \(\psi\) is real analytic on the interior of the support of \(\mu_V\), see \([11, 12]\). The importance of the equilibrium measure lies in the fact that \(\psi\) is the limiting (as \(n \to \infty\)) mean eigenvalue density of the matrix ensemble \((1.1)\), cf. \([9, 12]\). The condition \((1.7)\) then says that the mean eigenvalue density \(\psi\) should be strictly positive there. If the origin belongs to the interior of the support of \(\mu_V\) but the mean eigenvalue density vanishes there, then the potential is called multicritical, see \([3, 4, 21]\). This case will not be treated in this paper.

The regular behavior of the recurrence coefficients assumed in \([2, 22]\) is probably satisfied if \(V\) is even and if the support of \(\mu_V\) consists of one single interval. Note that we make no assumptions on the nature of the support of \(\mu_V\). It can consist of any (finite) number of intervals.

Our main result is the following.

**Theorem 1.1** Assume that the conditions \((1.5) - (1.7)\) are satisfied. Let \(w_n\) be the varying weight \((1.2)\), let \(K_n\) be the kernel \((1.3)\) associated with \(w_n\), and let \(\psi\) be the density of the equilibrium measure for \(V\). Then, for \(u, v \in (0, \infty)\),

\[
\frac{1}{n\psi(0)}K_n \left( \frac{u}{n\psi(0)}, \frac{v}{n\psi(0)} \right) = J_\alpha^o(u,v) + O \left( \frac{u^\alpha v^\alpha}{n} \right), \quad \text{as } n \to \infty, \tag{1.9}
\]

where \(J_\alpha^o\) is the Bessel kernel given by \((1.4)\). The error term in \((1.9)\) is uniform for \(u,v\) in bounded subsets of \((0,\infty)\).

Other types of universal correlations have been established in the bulk \([8, 12, 22, 33]\), at the soft edge of the spectrum \([7,17,22,30,36]\), and at the hard edge \([17,28,31,37]\). The universality at the hard edge is also described in terms of a Bessel kernel, which we have denoted in \([28]\) by \(J_\alpha\), namely

\[
J_\alpha(u,v) = J_\alpha(\sqrt{u})\sqrt{v}J'_\alpha(\sqrt{v}) - J_\alpha(\sqrt{v})\sqrt{u}J'_\alpha(\sqrt{u}).
\]

To distinguish with this Bessel kernel, we use \(J_\alpha^o\) to denote the Bessel kernel \((1.4)\) relevant at the origin of the spectrum.
Remark 1.2 The universality (1.9) is restricted to $u,v > 0$. It can be extended to arbitrary real $u$ and $v$ in the following way. For $u,v \in \mathbb{R}$, we have that

$$|u|^{-\alpha} |v|^{-\alpha} \frac{1}{n\psi(0)} K_n \left( \frac{u}{n\psi(0)}, \frac{v}{n\psi(0)} \right) = u^{-\alpha} v^{-\alpha} J_\alpha^2(u,v) + O \left( \frac{1}{n} \right), \quad \text{as } n \to \infty, \quad (1.10)$$

and the error term holds uniformly for $u,v$ in compact subsets of $\mathbb{R}$. We will restrict ourselves to proving (1.9), but the same methods allow us to establish (1.10).

Our proof of Theorem 1.1 is based on the characterization of the orthogonal polynomials via a Riemann-Hilbert problem (RH problem) for $2 \times 2$ matrix valued functions, due to Fokas, Its and Kitaev [16], and on an application of the steepest descent method of Deift and Zhou [14]. See [9, 24] for an introduction. The Riemann-Hilbert approach gives asymptotics for the orthogonal polynomials in all regions of the complex plane, and it has been applied before on orthogonal polynomials by a number of authors, see for example [5, 12, 13, 23, 26, 27, 38]. Bleher and Its [6] and Deift et al [12] were the first to apply Riemann-Hilbert problems to universality results in random matrix theory. Later developments include [5, 14, 28].

In this paper we use many of the ideas of [12]. That paper deals with the varying weights $e^{-nV(x)}$ with $V$ satisfying (1.5) and (1.6). The steepest descent method for Riemann-Hilbert problems is used to establish universality of the sine kernel in the bulk of the spectrum for the associated unitary matrix ensembles. In our case the general scheme of the analysis is the same, and we refer to [12, 13] for some of the details and motivations. The extra factor $|x|^{2\alpha}$ in our weights $|x|^{2\alpha} e^{-nV(x)}$ gives rise to two important technical differences. The first difference lies in the construction of the so-called parametrix for the outside region. To compensate for the factor $|x|^{2\alpha}$ we need to construct a Szegő function on multiple intervals associated to $|x|^{2\alpha}$. The second and most important difference lies in the fact that we have to do a local analysis near the origin. This is where the Bessel functions $J_{\alpha \pm \frac{1}{2}}$ come in. The construction of the local parametrix near the origin is analogous to the construction of the parametrix near the algebraic singularities of the generalized Jacobi weight, recently done by one of us in [38]. The local parametrix determines the asymptotics of the orthonormal polynomials near the origin, and thus also governs the universality at the origin of the spectrum.

The rest of the paper is organized as follows. In Section 2.1 we characterize the orthogonal polynomials via a RH problem, due to Fokas, Its and Kitaev [16]. Via a series of transformations, we perform the asymptotic analysis of the RH problem as in [12, 13]. The first transformation will be done in Section 2.2, the second transformation in Section 3. Next, we construct the parametrices for the outside region and near the origin in Section 4 and Section 5 respectively. The final transformation will be done in Section 6. Then we have all the ingredients to prove Theorem 1.1 in Section 7. Here we use some techniques from [28].

2 Associated RH problem and first transformation $Y \mapsto T$

In this section we will characterize the orthonormal polynomials $p_{k,n}$ with respect to the weight (1.2) as a solution of a RH problem for a $2 \times 2$ matrix valued function $Y(z) = Y(z;n,w)$, due to Fokas, Its and Kitaev [16], and do the first transformation in the asymptotic analysis of this RH problem.

2.1 Associated RH problem

We seek a $2 \times 2$ matrix valued function $Y$ that satisfies the following RH problem.
RH problem for $Y$:

(a) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $Y$ possesses continuous boundary values for $x \in \mathbb{R} \setminus \{0\}$ denoted by $Y_+(x)$ and $Y_-(x)$, where $Y_+(x)$ and $Y_-(x)$ denote the limiting values of $Y(z')$ as $z'$ approaches $x$ from above and below, respectively, and

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & |x|^{2\alpha} e^{-nV(x)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in \mathbb{R} \setminus \{0\}. \quad (2.1)$$

(c) $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = \left( I + O\left( \frac{1}{z} \right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad \text{as } z \to \infty. \quad (2.2)$$

(d) $Y(z)$ has the following behavior near $z = 0$:

$$Y(z) = \begin{cases} O\left( \begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix} \right), & \text{if } \alpha < 0, \\ O\left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right), & \text{if } \alpha > 0, \end{cases} \quad (2.3)$$

as $z \to 0$, $z \in \mathbb{C} \setminus \mathbb{R}$.

Compared with the case of no singularity at the origin, see [12], we now have an extra condition (2.3) near the origin. This condition is used to control the behavior near the origin, see also [27, 38].

Remark 2.1 The $O$-terms in (2.3) are to be taken entrywise. So for example $Y(z) = O\left( \begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix} \right)$ means that $Y_{11}(z) = O(1)$, $Y_{12}(z) = O(|z|^{2\alpha})$, etc.

The unique solution of the RH problem is given by

$$Y(z) = \begin{pmatrix} \frac{1}{\gamma_{n,n}} p_{n,n}(z) & \frac{1}{2\pi i \gamma_{n,n}} \int_{-1}^{1} \frac{p_{n,n}(x)w_n(x)}{x-z} dx \\ -2\pi i \gamma_{n-1,n} p_{n-1,n}(z) & -\gamma_{n-1,n} \int_{-1}^{1} \frac{p_{n-1,n}(x)w_n(x)}{x-z} dx \end{pmatrix}, \quad (2.4)$$

where $p_{k,n}$ is the $k$th degree orthonormal polynomial with respect to the varying weight $w_n$, and where $\gamma_{k,n}$ is the leading coefficient of the orthonormal polynomial $p_{k,n}$. The solution (2.4) is due to Fokas, Its and Kitaev [16], see also [9, 12, 13]. See [24, 27] for the condition (2.3).

Note that (2.4) contains the orthonormal polynomials of degrees $n-1$ and $n$. By (1.3) it is then possible to write $K_n$ in terms of the first column of $Y$. So in order to prove Theorem 1.1 an asymptotic analysis of the RH problem for $Y$ is necessary. Via a series of transformations $Y \to T \to S \to R$ we want to obtain a RH problem for $R$ which is normalized at infinity (i.e., $R(z) \to I$ as $z \to \infty$), and with jumps uniformly close to the identity matrix, as $n \to \infty$. Then $R$ is also uniformly close to the identity matrix, as $n \to \infty$. Unfolding the series of transformations, we obtain the asymptotics of $Y$. In particular, we need the asymptotic behavior of $Y$ near the origin, which follows from the parametrix near the origin.
2.2 First transformation $Y \to T$

We first need some properties of the equilibrium measure $\mu_V$ for $V$. Its support is a finite union of disjoint intervals, say $\bigcup_{j=1}^{N+1} [b_{j-1}, a_j]$. So the support consists of $N + 1$ intervals and we refer to these as the bands. The complementary $N$ intervals $(a_j, b_j)$ are the gaps. Following [12], we define

$$J = \bigcup_{j=1}^{N+1} (b_{j-1}, a_j)$$

so that $J$ is the interior of the support. The density $\psi$ of $\mu_V$ has the form [12]

$$\psi(x) = \frac{1}{2\pi i} R_+^{1/2}(x) h(x), \quad \text{for } x \in J,$$

where

$$R(z) = \prod_{j=1}^{N+1} (z - b_{j-1})(z - a_j),$$

and where $h$ is real analytic on $\mathbb{R}$. In this paper we use $R^{1/2}$ to denote the branch of $\sqrt{R}$ which behaves like $z^{N+1}$ as $z \to \infty$ and which is defined and analytic on $\mathbb{C} \setminus \bar{J}$. In (2.5) we have that $R_+^{1/2}$ denotes the boundary value of $R^{1/2}$ on $J$ from above. There exists an explicit expression for $h$ in terms of $V$, see [11], but we will not need that here.

The equilibrium measure minimizes the weighted energy (1.8). The associated Euler-Lagrange variational conditions state that there exists a constant $\ell \in \mathbb{R}$ such that

$$2 \int \log |x - s| \psi(s) ds - V(x) = \ell, \quad \text{for } x \in \bar{J},$$

(2.7)

$$2 \int \log |x - s| \psi(s) ds - V(x) \leq \ell, \quad \text{for } x \in \mathbb{R} \setminus \bar{J}.$$  (2.8)

The external field $V$ is called regular if the inequality in (2.8) is strict for every $x \in \mathbb{R} \setminus \bar{J}$, and if $h(x) \neq 0$ for every $x \in \bar{J}$. Otherwise, $V$ is called singular. The regular case holds generically [25]. In the singular case there are a finite number of singular points. Singular points in $J$ are such that $h$ vanishes there. Singular points in $\mathbb{R} \setminus \bar{J}$ are such that equality holds in (2.8).

In order to do the first transformation, we introduce the so-called $g$-function [12, Section 3.2]

$$g(z) = \int \log(z - s) \psi(s) ds, \quad \text{for } z \in \mathbb{C} \setminus (-\infty, a_{N+1}],$$

(2.9)

where $\psi(s) ds$ is the equilibrium measure for $V$. In (2.9) we take the principal branch of the logarithm, so that $g$ is analytic on $\mathbb{C} \setminus (-\infty, a_{N+1}]$.

We now give properties of $g$ which are crucial in the following, [12, Section 3.2]. From the Euler-Lagrange conditions (2.7) and (2.8) it follows that

$$g_+(x) + g_-(x) - V(x) - \ell = 0, \quad \text{for } x \in \bar{J},$$

(2.10)

$$g_+(x) + g_-(x) - V(x) - \ell \leq 0, \quad \text{for } x \in \mathbb{R} \setminus \bar{J}.$$  (2.11)

A second crucial property is that

$$g_+(x) - g_-(x) = 2\pi i \int_x^{a_{N+1}} d\mu_V(s), \quad \text{for } x \in (-\infty, a_{N+1}),$$

(2.12)
so that \( g_+(x) - g_-(x) \) is purely imaginary for all \( x \in \mathbb{R} \) and constant in each of the gaps, namely

\[
g_+(x) - g_-(x) = \begin{cases} 2\pi i, & \text{for } x < b_0, \\ 2\pi i \int_{b_j}^{a_{N+1}} d\mu_V(s) := 2\pi i \Omega_j, & \text{for } x \in (a_j, b_j), j = 1 \ldots N, \\ 0, & \text{for } x > a_{N+1}. \end{cases} \tag{2.13}
\]

From (2.13) we see that \( \Omega_j \) is the total \( \mu_V \)-mass of the \( N + 1 - j \) largest bands. These constants all belong to \((0, 1)\). Note that \( \Omega_j \) was defined with an extra factor \( 2\pi \) in [12].

As in [12, Section 3.3], we define the matrix valued function \( T \) as

\[
T(z) = e^{-\frac{\alpha}{2}\sigma_3}Y(z)e^{\frac{\alpha}{2}\sigma_3}e^{-n(z)\sigma_3}, \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.14}
\]

where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is the Pauli matrix. Then \( T \) is the unique solution of the following equivalent RH problem.

**RH problem for \( T \):**

(a) \( T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) \( T \) satisfies the following jump relations on \( \mathbb{R} \):

\[
T_+(x) = T_-(x) \begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & 0 \\ |x|^{2\alpha} e^{n(g_+(x)-g_-(x))} & e^n(x) \end{pmatrix}, \quad \text{for } x \in \tilde{J} \setminus \{0\}, \tag{2.15}
\]

\[
T_+(x) = T_-(x) \begin{pmatrix} e^{-2\pi i n \Omega_j} & |x|^{2\alpha} e^{n(g_+(x)+g_-(x))-V(x)-\ell} \\ 0 & e^{2\pi i n \Omega_j} \end{pmatrix}, \quad \text{for } x \in (a_j, b_j), j = 1 \ldots N, \tag{2.16}
\]

\[
T_+(x) = T_-(x) \begin{pmatrix} 1 & |x|^{2\alpha} e^{n(g_+(x)+g_-(x))-V(x)-\ell} \\ 0 & 1 \end{pmatrix}, \quad \text{for } x < b_0 \text{ or } x > a_{N+1}. \tag{2.17}
\]

(c) \( T(z) = I + O(1/z) \), as \( z \to \infty \).

(d) \( T(z) \) has the same behavior as \( Y(z) \) as \( z \to 0 \), given by (2.3).

### 3 Second transformation \( T \to S \)

In this section we transform the oscillatory diagonal entries of the jump matrix in (2.15) into exponentially decaying off-diagonal entries. This lies at the heart of the steepest descent method for RH problems of Deift and Zhou [14], and this step is often referred to as the opening of the lens.

For every \( z \in \mathbb{C} \setminus \mathbb{R} \) lying in the region of analyticity of \( h \), we define

\[
\phi(z) = \frac{1}{2} \int_{z}^{a_{N+1}} R(s)^{1/2} h(s) ds
\]
where the path of integration does not cross the real axis. Since \( \int_{a_k}^{b_k} R^{1/2}(s)h(s)ds = 0 \) for every \( k = 1, \ldots, N \), (this follows easily from the formulas in \( \frac{1}{2} \) Sections 3.1 and 3.2), and \( \int_{b_j}^{a_{N+1}} R^{1/2}(s)h(s)ds = 2\pi i \Omega_j \), we find that for every \( j \),

\[
2\phi(z) = \int_z^{a_j} R^{1/2}(s)h(s)ds + 2\pi i \Omega_j, \quad \text{if } \text{Im } z > 0, \quad \text{(3.1)}
\]

\[
2\phi(z) = \int_z^{a_j} R^{1/2}(s)h(s)ds - 2\pi i \Omega_j, \quad \text{if } \text{Im } z < 0. \quad \text{(3.2)}
\]

Note that in \( \frac{1}{2} \) a function \( G \) is defined which is analytic through the bands. We found it more convenient to have a function with branch cuts along the bands, see also \( \frac{1}{9} \). The functions \( G \) and \( \phi \) also differ by a factor \( \pm 2 \).

The point of the function \( \phi \) is that \( \phi_+ \) and \( \phi_- \) are purely imaginary on the bands, and that

\[
2\phi_+ = -2\phi_- = g_+ - g_. \quad \text{(3.3)}
\]

This means that \( 2\phi \) and \(-2\phi \) provide analytic extensions of \( g_+ - g_- \) into the upper half-plane and lower half-plane, respectively. We also have that for \( z \) in a neighborhood of a regular point \( x \in J \), (see \( \frac{1}{12} \) Section 3.3) for details) that

\[
\text{Re } \phi(z) > 0, \quad \text{if } \text{Im } z \neq 0, \quad \text{(3.4)}
\]

We will now discuss the opening of the lens in the regular case. In the singular case we need to modify the opening of the lens somewhat, since we have to take into account the singular points. We do not open the lens around singular points that belong to \( J \), see \( \frac{12}{12} \) Section 4) for details.

For \( V \) regular, there is a suitable neighborhood \( U \) of \( J \) such that the inequality in \( \frac{3.1}{3.1} \) holds for every \( z \in U \). The opening of the lens is based on the factorization of the jump matrix \( \frac{24}{24} \) into the following product of three matrices, see also \( \frac{38}{38} \),

\[
\begin{pmatrix}
  e^{-n(g_+(x) - g_-(x))} & |x|^{2\alpha} \\
  0 & e^{n(g_+(x) - g_-(x))}
\end{pmatrix}
\begin{pmatrix}
  e^{-2n\phi_+(x)} & |x|^{2\alpha} \\
  0 & e^{-2n\phi_-(x)}
\end{pmatrix}
\begin{pmatrix}
  e^{-2n\phi_+(x)} & |x|^{2\alpha} \\
  0 & e^{-2n\phi_-(x)}
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  |x|^{-2\alpha} e^{-2n\phi_-(x)} & 1
\end{pmatrix}
\begin{pmatrix}
  0 & 0 \\
  -|x|^{-2\alpha} & 1
\end{pmatrix}
\begin{pmatrix}
  0 & 0 \\
  1 & -|x|^{-2\alpha} e^{-2n\phi_+(x)}
\end{pmatrix}
\begin{pmatrix}
  0 & 0 \\
  1 & 1
\end{pmatrix}
\]

As in \( \frac{38}{38} \) we take an analytic continuation of the factor \( |x|^{2\alpha} \) by defining

\[
\omega(z) = \begin{cases} 
   (-z)^{2\alpha}, & \text{if } \text{Re } z < 0, \\
   z^{2\alpha}, & \text{if } \text{Re } z > 0,
\end{cases} \quad \text{(3.6)}
\]

with principal branches of powers. In contrast to the situation in \( \frac{12}{12} \), here we have to open the lens also going through the origin, cf. \( \frac{38}{38} \). This follows from the fact that \( |x|^{2\alpha} \) does not have an analytic continuation to a full neighborhood of the origin.

We thus transform the RH problem for \( T \) into a RH problem for \( S \) with jumps on the oriented contour \( \Sigma \), shown in Figure III. The precise form of the lens is not yet defined, but it will be contained in \( U \).

Define the piecewise analytic matrix valued function \( S \) as

\[
S(z) = \begin{cases}
   T(z), & \text{for } z \text{ outside the lens}, \\
   T(z) \left( -\omega(z)^{-1} e^{-2n\phi(z)} 1 \right), & \text{for } z \text{ in the upper parts of the lens}, \\
   T(z) \left( \omega(z)^{-1} e^{-2n\phi(z)} 1 \right), & \text{for } z \text{ in the lower parts of the lens}.
\end{cases} \quad \text{(3.7)}
\]

Then, \( S \) is the unique solution of the following equivalent RH problem. In \( \frac{38}{38} \), \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) are used to denote the upper half-plane \( \{ \text{Im } z > 0 \} \) and the lower half-plane \( \{ \text{Im } z < 0 \} \), respectively.
RH problem for $S$:

(a) $S : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $S$ satisfies the following jump relations on $\Sigma$:

\[
S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1} e^{-2n\phi(z)} & 1 \end{pmatrix}, \quad \text{for } z \in \Sigma \cap \mathbb{C}_\pm, \tag{3.8}
\]

\[
S_+(x) = S_-(x) \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix}, \quad \text{for } x \in J \setminus \{0\}, \tag{3.9}
\]

\[
S_+(x) = S_-(x) \begin{pmatrix} e^{-2\pi in\Omega_j} |x|^{2\alpha} e^{n(g_+(x)+g_-(x)-V(x)-\ell)} \\ 0 \end{pmatrix}, \quad \text{for } x \in (a_j, b_j), j = 1 \ldots N, \tag{3.10}
\]

\[
S_+(x) = S_-(x) \begin{pmatrix} 1 & |x|^{2\alpha} e^{n(g_+(x)+g_-(x)-V(x)-\ell)} \\ 0 \end{pmatrix}, \quad \text{for } x < b_0 \text{ or } x > a_{N+1}. \tag{3.11}
\]

(c) $S(z) = I + O(1/z)$, as $z \to \infty$.

(d) For $\alpha < 0$, the matrix function $S(z)$ has the following behavior as $z \to 0$:

\[
S(z) = O \begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix}, \quad \text{as } z \to 0, z \in \mathbb{C} \setminus \Sigma. \tag{3.12}
\]

For $\alpha > 0$, the matrix function $S(z)$ has the following behavior as $z \to 0$:

\[
S(z) = \begin{cases} 
O \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{as } z \to 0 \text{ from outside the lens}, \\
O \begin{pmatrix} |z|^{-2\alpha} \\ |z|^{-2\alpha} \end{pmatrix}, & \text{as } z \to 0 \text{ from inside the lens}.
\end{cases} \tag{3.13}
\]

(e) $S$ remains bounded near each of the endpoints $a_i$, $b_j$.

By (3.8) the factor $e^{-2n\phi(z)}$ is exponentially decaying for $z \in \Sigma \cap \mathbb{C}_\pm$ as $n \to \infty$. This implies that the jump matrix for $S$ converges exponentially fast to the identity matrix on the lips of the lens. Since $V$ is regular, we have the strict inequality

\[
g_+(x) + g_-(x) - V(x) - \ell < 0, \quad \text{for } x \in \mathbb{R} \setminus \bar{J}, \tag{3.14}
\]

so that the factor $e^{n(g_+(x)+g_-(x)-V(x)-\ell)}$ in (3.10) and (3.11) is also exponentially decaying as $n \to \infty$. 

Figure 1: Part of the contour $\Sigma$. 

9
4 Parametrix for the outside region

From the discussion at the end of the previous section we expect that the leading order asymptotics are determined by the solution of the following RH problem.

RH problem for $P^{(\infty)}$:

(a) $P^{(\infty)} : \mathbb{C} \setminus [b_0, a_{N+1}] \rightarrow \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}$ satisfies the following jump relations:

\[ P^{(\infty)}_+(x) = P^{(\infty)}_-(x) \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix}, \quad \text{for } x \in J \setminus \{0\}, \quad (4.1) \]

\[ P^{(\infty)}_+(x) = e^{2\pi i \Omega_j} P^{(\infty)}_-(x) \begin{pmatrix} 0 & e^{2\pi i \Omega_j} \\ -e^{-2\pi i \Omega_j} & 0 \end{pmatrix}, \quad \text{for } x \in (a_j, b_j), j = 1, \ldots, N, \quad (4.2) \]

(c) $P^{(\infty)}(z) = I + O\left(\frac{1}{z}\right)$, as $z \rightarrow \infty$.

The solution of this RH problem is referred to as the parametrix for the outside region, and will be constructed using the so-called Szegő function on the union of disjoint intervals $J$, associated to $|x|^{2\alpha}$. The importance of the Szegő function is that it transforms this RH problem into a RH problem with jump matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) on $J$.

4.1 The Szegő function

We seek a scalar function $D : \mathbb{C} \setminus [b_0, a_{N+1}] \rightarrow \mathbb{C}$ that solves the following RH problem.

RH problem for $D$

(a) $D$ is non-zero and analytic on $\mathbb{C} \setminus [b_0, a_{N+1}]$

(b) $D$ satisfies the following jump relations:

\[ D_+(x)D_-(x) = |x|^{2\alpha}, \quad \text{for } x \in J \setminus \{0\}, \quad (4.3) \]

\[ D_+(x) = e^{2\pi i \xi_j} D_-(x), \quad \text{for } x \in (a_j, b_j), j = 1, \ldots, N, \quad (4.4) \]

for certain unknown constants $\xi_1, \ldots, \xi_N \in \mathbb{R}$. The selection of $\xi_1, \ldots, \xi_N$ is part of the problem. We should choose them such that it is possible to construct $D$.

(c) $D$ and $D^{-1}$ remain bounded near the endpoints $a_i, b_j$ of $J$, and

\[ D_{\infty} := \lim_{z \rightarrow \infty} D(z) \quad (4.5) \]

exists and is non-zero.

We seek $D$ in the form $D(z) = \exp \Phi(z)$. Then the problem is reduced to constructing a scalar function $\Phi$, analytic on $\mathbb{C} \setminus [b_0, a_{N+1}]$, remaining bounded near the endpoints $a_i, b_j$ of $J$ and at infinity, and having the following jumps

\[ \Phi_+(x) + \Phi_-(x) = 2\alpha \log |x|, \quad \text{for } x \in J \setminus \{0\}, \quad (4.6) \]

\[ \Phi_+(x) = \Phi_-(x) + 2\pi i \xi_j, \quad \text{for } x \in (a_j, b_j), j = 1, \ldots, N. \quad (4.7) \]
We can easily check, using Cauchy’s formula, the Sokhotskii-Plemelj formula \[20\], and the fact that \( R_{-1/2}(x) = -R_{+1/2}(x) \) for \( x \in J \), see (2.9), that \( \Phi \) defined by

\[
\Phi(z) = R^{1/2}(z) \left( \frac{1}{2\pi i} \int_J \frac{2\alpha \log |x|}{R_{+1/2}(x)} \frac{dx}{x-z} + \sum_{j=1}^N \xi_j \int_{a_j}^{b_j} \frac{1}{R^{1/2}(x)} \frac{dx}{x-z} \right),
\]

satisfies the jump conditions (4.6) and (4.7). We note that \( \Phi \) is analytic on \( \mathbb{C} \setminus [b_0,a_{N+1}] \) and remains bounded near the endpoints \( a_i, b_j \) of \( J \). We use the freedom we have in choosing the constants \( \xi_1, \ldots, \xi_N \) to ensure that \( \Phi \) remains bounded at infinity. Since \( R^{1/2}(z) \) behaves like \( z^{N+1} \) as \( z \to \infty \), and since

\[
\frac{1}{x-z} = -\sum_{k=0}^{N-1} \frac{x^k}{z^{k+1}} + O \left( \frac{1}{z^{N+1}} \right), \quad \text{as } z \to \infty,
\]

we have to choose \( \xi_1, \ldots, \xi_N \) such that the \( N \) conditions

\[
\frac{1}{2\pi i} \int_J \frac{2\alpha \log |x|}{R_{+1/2}(x)} x^k dx + \sum_{j=1}^N \xi_j \int_{a_j}^{b_j} \frac{x^k dx}{R^{1/2}(x)} = 0, \quad k = 0, \ldots, N-1,
\]

are satisfied. Note that (4.9) represents a system of \( N \) linear equations with coefficient matrix

\[
A = \begin{pmatrix}
\int_{a_1}^{b_1} \frac{dx}{R^{1/2}(x)} & \int_{a_2}^{b_2} \frac{dx}{R^{1/2}(x)} & \cdots & \int_{a_N}^{b_N} \frac{dx}{R^{1/2}(x)} \\
\int_{a_1}^{b_1} \frac{x dx}{R^{1/2}(x)} & \int_{a_2}^{b_2} \frac{x dx}{R^{1/2}(x)} & \cdots & \int_{a_N}^{b_N} \frac{x dx}{R^{1/2}(x)} \\
\vdots & \vdots & \ddots & \vdots \\
\int_{a_1}^{b_1} x^{N-1} dx & \int_{a_2}^{b_2} x^{N-1} dx & \cdots & \int_{a_N}^{b_N} x^{N-1} dx
\end{pmatrix}.
\]

By the multilinearity of the determinant, we have

\[
det A = \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{pmatrix} \frac{dx_1}{R^{1/2}(x_1)} \cdots \frac{dx_N}{R^{1/2}(x_N)}
\]

\[
= \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} \prod_{j<k} (x_k - x_j) \frac{dx_1}{R^{1/2}(x_1)} \cdots \frac{dx_N}{R^{1/2}(x_N)}.
\]

Since the gaps \((a_j, b_j)\) are disjoint, we have \( x_j < x_k \) for \( j < k \), so that \( \prod_{j<k} (x_k - x_j) > 0 \). Using the fact that \( R^{1/2} \) does not change sign on each of the gaps \((a_j, b_j)\), it then follows that the integrand in (4.11) has a constant sign in the region of integration, so that \( \det A \neq 0 \) and \( A \) is invertible.

We then define \( \xi_1, \ldots, \xi_N \) as follows

\[
\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix} = -A^{-1} \begin{pmatrix} \frac{1}{2\pi i} \int_J \frac{2\alpha \log |x|}{R_{+1/2}(x)} dx \\ \frac{1}{2\pi i} \int_J \frac{2\alpha \log |x|}{R_{+1/2}(x)} x dx \\ \vdots \\ \frac{1}{2\pi i} \int_J \frac{2\alpha \log |x|}{R_{+1/2}(x)} x^{N-1} dx \end{pmatrix}.
\]

(4.12)
Figure 2: The oriented contour $\gamma_\delta$.

Note that $R^{1/2}$ is real on each of the gaps, so that by (4.10) all entries of $A$ are real. This implies, from (4.12) and the fact that $R^{1/2}_+$ is purely imaginary on $J$, that the constants $\xi_1, \ldots, \xi_N$ are real.

We proved the following

**Theorem 4.1** The scalar function $D(z) = \exp \Phi(z)$, where $\Phi$ is given by (4.8), and the constants $\xi_1, \ldots, \xi_N$ by (4.12), solves the RH problem for $D$.

For later use we state the following lemma.

**Lemma 4.2** We have that $z^{-\alpha}D(z)$ and $z^\alpha D(z)^{-1}$ remain bounded near the origin.

**Proof.** For definiteness, suppose that the origin lies on the band $(b_j, a_{j+1})$ with $j \in \{0, \ldots, N\}$. Since $D(z) = \exp \Phi(z)$, with $\Phi$ given by (4.8), it is sufficient to prove that

$$\frac{1}{2\pi i} \int_{b_j}^{a_{j+1}} 2\alpha \log |x| \frac{dx}{R^{1/2}_+(x)} x - z = \alpha \log z \frac{R^{1/2}_+(z)}{R^{1/2}_-(z)} + F(z),$$

with $F$ analytic near the origin. Fix $z$ near the origin with Im $z \neq 0$, and let $\gamma_\delta$ be the oriented contour shown in Figure 2 with $\delta > 0$ small. Cauchy’s formula implies

$$\frac{1}{2\pi i} \int_{\gamma_\delta} \log \zeta \frac{d\zeta}{R^{1/2}(\zeta) \zeta - z} = \frac{\log z}{R^{1/2}(z)}.$$

Letting $\delta \to 0$, we then have, since $R^{-1/2}_+(x) = -R^{1/2}_-(x)$ for $x \in (b_j, a_{j+1})$,

$$\tilde{F}(z) + \frac{1}{2\pi i} \int_{b_j}^{a_{j+1}} \log |x| \frac{dx}{R^{1/2}_+(x)} x - z + \frac{1}{2\pi i} \int_0^{a_{j+1}} \log |x| \frac{dx}{R^{1/2}_+(x)} x - z$$

$$- \frac{1}{2\pi i} \int_{a_{j+1}}^0 \log |x| \frac{dx}{R^{1/2}_-(x)} x - z - \frac{1}{2\pi i} \int_0^{b_j} \log |x| - i\pi \frac{dx}{R^{1/2}_-(x)} x - z = \frac{\log z}{R^{1/2}(z)},$$

with $\tilde{F}$ analytic near the origin. Hence

$$\frac{1}{2\pi i} \int_{b_j}^{a_{j+1}} \log |x| \frac{dx}{R^{1/2}_+(x)} x - z = \frac{1}{2} \frac{\log z}{R^{1/2}(z)} - \frac{1}{2} \tilde{F}(z),$$

so that (4.13) holds with $F(z) = -\alpha \tilde{F}(z)$, which proves the lemma. \qed

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4.2 Construction of \( P^{(\infty)} \)

We now use the Szegő function \( D \) from the previous subsection, to transform the RH problem for \( P^{(\infty)} \) into a RH problem with jump matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) on \( J \). We seek \( P^{(\infty)} \) in the form, cf. [27, 38]

\[
P^{(\infty)}(z) = D^{\sigma_3} \tilde{P}^{(\infty)}(z) D^{-\sigma_3}, \quad \text{for } z \in \mathbb{C} \setminus [b_0, a_{N+1}].
\] (4.14)

Then, by (4.1)–(4.4) the problem is reduced to constructing a solution of the following RH problem.

**RH problem for \( \tilde{P}^{(\infty)} \)**

(a) \( \tilde{P}^{(\infty)} : \mathbb{C} \setminus [b_0, a_{N+1}] \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) \( \tilde{P}^{(\infty)} \) satisfies the following jump relations:

\[
\tilde{P}^{(\infty)}_+(x) = \tilde{P}^{(\infty)}_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } x \in J \setminus \{0\},
\] (4.15)

\[
\tilde{P}^{(\infty)}_+(x) = \tilde{P}^{(\infty)}_-(x) \begin{pmatrix} e^{-2\pi i \Omega_j} e^{2\pi i \xi_j} & 0 \\ 0 & e^{2\pi i \Omega_j} e^{-2\pi i \xi_j} \end{pmatrix},
\] (4.16)

\[
\text{for } x \in (a_j, b_j), \ j = 1, \ldots, N.
\]

(c) \( \tilde{P}^{(\infty)}(z) = I + O(1/z), \) as \( z \to \infty. \)

This corresponds to the RH problem [12, (4.24)–(4.26)], which has been solved there using Riemann theta functions. Note that, in contrast to the RH problem [12, (4.24)–(4.26)], the jump matrix in (4.16) contains extra factors \( \exp(\pm 2\pi i \xi_j) \) in the diagonal entries, which come from the Szegő function \( D \). However, this does not create any problems.

In order to formulate the solution of the RH problem for \( \tilde{P}^{(\infty)} \) we need to introduce some additional notations. Here we closely follow [12], see also [10].

Let \( J = \mathbb{R} \setminus \bar{J} \) be the complement of \( J \), and \( a_0 \equiv a_{N+1} \). Letting the point \( \infty \) lie on the interval \( (a_0, b_0) \), \( \bar{J} \) can be displayed as a union of intervals on the Riemann sphere. Let \( X \) be the two-sheeted Riemann surface of genus \( N \) associated to \( \sqrt{R(z)} \), obtained by gluing together two copies of the slit plane \( \mathbb{C} \setminus \bar{J} \) along \( \bar{J} \). We draw cycles \( A_j \) winding once, in the negative direction, around the slit \( (a_j, b_j) \) in the first sheet, and cycles \( B_j \) starting from a point on the slit \( (a_j, b_j) \) going on the first sheet through a point on the slit \( (a_0, b_0) \), and returning on the second sheet to the original point, as indicated in Figure 3. The cycles \( \{A_i, B_j\}_{1 \leq i,j \leq N} \) form a canonical homology basis for \( X \), see [13].

Let \( \omega = (\omega_1, \ldots, \omega_N) \) be the basis of holomorphic one-forms on \( X \) dual to the canonical homology basis, that is

\[
\int_{A_j} \omega_i = \delta_{ij}, \quad 1 \leq i, j \leq N.
\] (4.17)

The associated Riemann matrix of \( B \) periods, denoted by \( \tau \) and with entries

\[
\tau_{ij} = \int_{B_j} \omega_i, \quad 1 \leq i, j \leq N,
\] (4.18)
is symmetric with positive definite imaginary part, see [15]. The associated Riemann theta function is defined by

\[ \theta(z) = \sum_{m \in \mathbb{Z}^N} \exp(2\pi i \langle m, z \rangle + \frac{1}{2} \langle m, \tau m \rangle), \quad z \in \mathbb{C}^N, \tag{4.19} \]

where \( \langle \cdot, \cdot \rangle \) is the real scalar product, which defines an analytic function on \( \mathbb{C}^N \). The Riemann theta function has the periodicity properties [15] with respect to the lattice \( \mathbb{Z}^N + \tau \mathbb{Z}^N \)

\[ \theta(z + e_j) = \theta(z), \quad \theta(z \pm \tau_j) = e^{\pm 2\pi iz_j - \pi i \tau_j} \theta(z), \tag{4.20} \]

where \( z = (z_1, \ldots, z_N) \) and \( e_j \) is the \( j \)th unit vector in \( \mathbb{C}^N \) with 1 on the \( j \)th entry and zeros elsewhere, and where \( \tau_j \) is the \( j \)th column vector of \( \tau \).

Define the scalar function

\[ \gamma(z) = \left[ \prod_{i=1}^{N} \frac{(z - b_i)(z - a_{i+1})}{(z - a_i)(z - b_0)} \right]^{1/4} \tag{4.21} \]

which is analytic on \( \mathbb{C} \setminus \tilde{J} \), with \( \gamma(z) \sim 1 \) as \( z \to \infty, z \in \mathbb{C}_+ \). It is known [12, Lemma 4.1] that \( \gamma \) has the following properties:

- \( \gamma + \gamma^{-1} \) possesses \( N \) roots \( \{z_j^{(-)}\}_{j=1}^{N} \) with \( z_j^{(-)} \) on the \(-\) side of \((a_j, b_j)\),

- \( \gamma - \gamma^{-1} \) possesses \( N \) roots \( \{z_j^{(+)}\}_{j=1}^{N} \) with \( z_j^{(+)} \) on the \(+\) side of \((a_j, b_j)\).

Fix the base point for the Riemann surface \( X \) to be \( a_{N+1} = a_0 \), let \( K \) be the associated vector of Riemann constants [15], and define the multivalued function

\[ u(z) = \int_{a_{N+1}}^{z} \omega. \tag{4.22} \]

Here, we take the integral along any path from \( a_{N+1} \) to \( z \) on the first sheet. Since the integral is taken on the first sheet, \( u(z) \) is uniquely defined in \( \mathbb{C}^N/\mathbb{Z}^N \) because of [4.17]. Let \( d \) be defined as

\[ d = -K - \sum_{j=1}^{N} \int_{a_{N+1}}^{z_j^{(-)}} \omega, \tag{4.23} \]

where again the integrals are taken on the first sheet.
We now have introduced the necessary ingredients to formulate the solution of the RH problem for $\hat{P}^{(\infty)}$. Together with \[4.14\] this gives the parametrix $P^{(\infty)}$ for the outside region. The solution of the RH problem for $\hat{P}^{(\infty)}$ is given by, see \[12\] Lemma 4.3,

$$\hat{P}^{(\infty)}(z) = \text{diag} \left( \begin{array}{ccc} \frac{\theta(u_+^{(\infty)}+d)}{\theta(u_+^{(\infty)}-n\Omega+\xi+d)} & \frac{\theta(u_+^{(\infty)}+d)}{\theta(-u_+^{(\infty)}-n\Omega+\xi-d)} \\ \frac{\theta(u(z)-n\Omega+\xi+d)}{\theta(u(z)+d)} & \frac{\theta(u(z)-n\Omega+\xi-d)}{\theta(u(z)+d)} \end{array} \right),$$

(4.24)

for $z \in \mathbb{C}_+$, and

$$\hat{P}^{(\infty)}(z) = \text{diag} \left( \begin{array}{ccc} \frac{\theta(u_+^{(\infty)}+d)}{\theta(u_+^{(\infty)}-n\Omega+\xi+d)} & \frac{\theta(u_+^{(\infty)}+d)}{\theta(-u_+^{(\infty)}-n\Omega+\xi-d)} \\ \frac{\theta(-u(z)-n\Omega+\xi+d)}{\theta(-u(z)+d)} & \frac{\theta(-u(z)-n\Omega+\xi-d)}{\theta(-u(z)+d)} \end{array} \right),$$

(4.25)

for $z \in \mathbb{C}_-$. Here, $\Omega = (\Omega_1, \ldots, \Omega_N)$ and $\xi = (\xi_1, \ldots, \xi_N)$.

**Remark 4.3** In contrast to \[12\] we have an extra term $\xi$ in the Riemann theta functions. This comes from the slightly different jump matrix in \[12\] due to the Szegö function, as noted before. If $\xi \in \mathbb{Z}^N$ the factors $e^{\pm 2\pi i \xi_j}$ in \[1.16\] disappear and the RH problem for $\hat{P}^{(\infty)}$ is exactly the same as the RH problem \[12\] (4.24)–(4.26)]. Since the Riemann theta functions possess the periodicity properties (4.20), the term $\xi$ in (4.24) and (4.25) disappears in this case. This is in agreement with \[12\] Lemma 4.3.

For later use, we need $P^{(\infty)}$ to be invertible. In \[12\] Section 4.2 it has been shown that $\det \hat{P}^{(\infty)} \equiv 1$, so that by \[4.14\]

$$\det P^{(\infty)} \equiv 1.$$  

(4.26)

### 5 Parametrix near the origin

In this section we construct the parametrix near the origin. As noted in the introduction, it is similar to the construction of the parametrix near the algebraic singularities of the generalized Jacobi weight \[36], and we skip some details and motivations.

We surround the origin by a disk $U_\delta$ with radius $\delta > 0$. We assume that $\delta$ is small, so that in any case, we have that $[-\delta, \delta] \subset J$. We seek a matrix valued function $P$ that satisfies the following RH problem.

**RH problem for $P$:**

(a) $P(z)$ is defined and analytic for $z \in U_{\delta_0} \setminus \Sigma$ for some $\delta_0 > \delta$.

(b) On $\Sigma \cap U_{\delta}$, $P$ satisfies the same jump relations as $S$, that is,

$$P_+(z) = P_-(z) \begin{bmatrix} 1 & 0 \\ \omega(z)^{-1} e^{-2m\phi(z)} & 1 \end{bmatrix}, \quad \text{for } z \in \Sigma \cap (U_\delta \cap \mathbb{C}_\pm),$$

(5.1)

$$P_+(x) = P_-(x) \begin{bmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{bmatrix}, \quad \text{for } x \in (-\delta, \delta) \setminus \{0\}.$$  

(5.2)
(c) On $\partial U_\delta$ we have, as $n \to \infty$

$$P(z) \left( P^{(n)} \right)^{-1}(z) = I + O \left( \frac{1}{n} \right), \quad \text{uniformly for } z \in \partial U_\delta \setminus \Sigma. \quad (5.3)$$

(d) For $\alpha < 0$, the matrix function $P(z)$ has the following behavior as $z \to 0$:

$$P(z) = O \left( \frac{1}{|z|^{2\alpha}} \right), \quad \text{as } z \to 0. \quad (5.4)$$

For $\alpha > 0$, the matrix function $P(z)$ has the following behavior as $z \to 0$:

$$P(z) = \begin{cases} 
O \left( \frac{1}{|z|^{2\alpha}} \right), & \text{as } z \to 0 \text{ from outside the lens}, \\
O \left( \frac{|z|^{-2\alpha}}{1} \right), & \text{as } z \to 0 \text{ from inside the lens}. 
\end{cases} \quad (5.5)$$

We construct $P$ as follows. First, we focus on conditions (a), (b) and (d). We transform the RH problem for $P$ into a RH problem for $P^{(1)}$ with constant jump matrices, and solve the latter RH problem explicitly. Afterwards, we also consider the matching condition (c) of the RH problem.

We start with the following map $f$ defined on a neighborhood of the origin

$$f(z) = \begin{cases} 
i\phi(z) - i\phi_+(0), & \text{if } \text{Im } z > 0, \\
-i\phi(z) - i\phi_+(0), & \text{if } \text{Im } z < 0. 
\end{cases} \quad (5.6)$$

Since $\phi_+ = -\phi_-$, we have that $f$ is analytic for $z$ in a neighborhood of the origin. An easy calculation, based on the fact that $2\phi_+(x) = g_+(x) - g_-(x)$ and on (2.9) and (5.6), shows that

$$f(x) = \pi \int_0^x \psi(s)ds, \quad \text{for } x \in (-\delta, \delta), \quad (5.7)$$

which implies that $f'(0) = \pi \psi(0) > 0$. So, the behavior of $f$ near the origin is given by

$$f(z) = \pi \psi(0)z + O \left( z^2 \right), \quad \text{as } z \to 0. \quad (5.8)$$

So if we choose $\delta > 0$ sufficiently small, $\zeta = f(z)$ is a conformal mapping on $U_\delta$ onto a convex neighborhood of 0 in the complex $\zeta$-plane. We also note that $f(x)$ is real and positive (negative) for $x \in U_\delta$ positive (negative), which follows from (5.7).

Let $\Gamma_j, j = 1, \ldots, 8$ be the infinite ray

$$\Gamma_j = \{ \zeta \in \mathbb{C} \mid \text{arg } \zeta = (j - 1)\frac{\pi}{4} \}. \quad (5.9)$$

These rays divide the $\zeta$-plane into eight sectors I–VIII as shown in Figure 4. We define the contours $\Sigma_j, j = 1, 2, \ldots, 8$ as the preimages under the mapping $\zeta = f(z)$ of the part of the corresponding rays $\Gamma_j$ in $f(U_\delta)$, see Figure 5.

We have some freedom in the selection of the contour $\Sigma$. We now specify that we open the lens in such a way that

$$\Sigma \cap U_\delta = \bigcup_{j=1,2,4,5,6,8} \Sigma_j. \quad (5.10)$$

As a consequence we have that $f$ maps $\Sigma$ to part of the union of rays $\bigcup_j \Gamma_j$.\[16\]
Figure 4: The contour $\Gamma_\Psi$.

Figure 5: The conformal mapping $f$. Every $\Sigma_k$ is mapped onto the part of the corresponding ray $\Gamma_k$ in $f(U_\delta)$. 
In order to transform to constant jumps we use a piecewise analytic function \( W \) corresponding to the analytic continuation of \(|x|^{2\alpha}\). For \( z \in U_\delta \), we define
\[
W(z) = \begin{cases} 
  z^\alpha, & \text{if } \pi/2 < |\arg f(z)| < \pi, \\
  (-z)^\alpha, & \text{if } 0 < |\arg f(z)| < \pi/2,
\end{cases}
\]
(5.9)
with principal branches of powers. Then \( W \) is defined and analytic in \( U_\delta \setminus (\Sigma_1 \cup \Sigma_3 \cup \Sigma_5 \cup \Sigma_7) \).

We seek \( P \) in the form
\[
P(z) = E_n(z)P^{(1)}(z)W(z)^{-\sigma_3}e^{-n\phi(z)\sigma_3}.
\]
(5.10)
Here the matrix valued function \( E_n \) is analytic in a neighborhood of \( U_\delta \), and \( E_n \) will be determined below so that the matching condition (c) of the RH problem for \( P \) is satisfied. Similar considerations as in [38] show that \( P^{(1)} \) should satisfy the following RH problem, with jumps on the system of contours \( \bigcup_{i=1}^8 \Sigma_i \), oriented as in the left part of Figure 5. In (5.11)–(5.14), \( \Sigma_i^o \) is used to denote \( \Sigma_i \) without the origin.

RH problem for \( P^{(1)} \):

(a) \( P^{(1)}(z) \) is defined and analytic for \( z \in U_{\delta_0} \setminus (\Sigma \cup \Gamma) \) for some \( \delta_0 > \delta \).

(b) \( P^{(1)} \) satisfies the following jump relations on \( U_\delta \cap (\Sigma \cup \Gamma) \):

\[
P_+^{(1)}(x) = P_-^{(1)}(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } x \in \Sigma_1^o \cup \Sigma_5^o,
\]
(5.11)
\[
P_+^{(1)}(z) = P_-^{(1)}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\pi i \alpha} & 1 \end{pmatrix}, \quad \text{for } z \in \Sigma_2^o \cup \Sigma_6^o,
\]
(5.12)
\[
P_+^{(1)}(z) = P_-^{(1)}(z)e^{\pi i \sigma_3}, \quad \text{for } z \in \Sigma_3^o \cup \Sigma_7^o,
\]
(5.13)
\[
P_+^{(1)}(z) = P_-^{(1)}(z) \begin{pmatrix} 1 & 0 \\ e^{2\pi i \alpha} & 1 \end{pmatrix}, \quad \text{for } z \in \Sigma_4^o \cup \Sigma_8^o.
\]
(5.14)

(c) For \( \alpha < 0 \), \( P^{(1)}(z) \) has the following behavior as \( z \to 0 \):

\[
P^{(1)}(z) = O \left( \frac{|z|^\alpha}{|z|^\alpha} \right), \quad \text{as } z \to 0.
\]
(5.15)

For \( \alpha > 0 \), \( P^{(1)}(z) \) has the following behavior as \( z \to 0 \):

\[
P^{(1)}(z) = \begin{cases} 
  O \left( \frac{|z|^\alpha}{|z|^\alpha} \right), & \text{as } z \to 0 \text{ from outside the lens}, \\
  O \left( \frac{|z|^{-\alpha}}{|z|^{-\alpha}} \right), & \text{as } z \to 0 \text{ from inside the lens}.
\end{cases}
\]
(5.16)

Next we construct an explicit solution of the RH problem for \( P^{(1)} \). This is based on a model RH problem for \( \Psi_\alpha \) in the \( \zeta \)-plane, see [38]. We denote by \( \Gamma_\Psi \) the contour \( \bigcup_{j=1}^8 \Gamma_j \) oriented as shown in Figure 4.
RH problem for $\Psi_\alpha$:

(a) $\Psi_\alpha : \mathbb{C} \setminus \Gamma_\Psi \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $\Psi_\alpha$ satisfies the following jump relations on $\Gamma_\Psi$:

\[
\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_1 \cup \Gamma_5, \tag{5.17}
\]

\[
\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) \begin{pmatrix} 0 & e^{-2\pi i \alpha} \\ 1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_2 \cup \Gamma_6, \tag{5.18}
\]

\[
\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) e^{\pi i \alpha \sigma_3}, \quad \text{for } \zeta \in \Gamma_3 \cup \Gamma_7, \tag{5.19}
\]

\[
\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) \begin{pmatrix} 0 & e^{2\pi i \alpha} \\ 1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_4 \cup \Gamma_8. \tag{5.20}
\]

(c) For $\alpha < 0$ the matrix function $\Psi_\alpha(\zeta)$ has the following behavior as $\zeta \to 0$:

\[
\Psi_\alpha(\zeta) = O \left( \begin{pmatrix} |\zeta|^\alpha & |\zeta|^{-\alpha} \\ |\zeta|^\alpha & |\zeta|^{-\alpha} \end{pmatrix} \right), \quad \text{as } \zeta \to 0. \tag{5.21}
\]

For $\alpha > 0$ the matrix function $\Psi_\alpha(\zeta)$ has the following behavior as $\zeta \to 0$:

\[
\Psi_\alpha(\zeta) = \begin{cases} O \left( \begin{pmatrix} |\zeta|^\alpha & |\zeta|^{-\alpha} \\ |\zeta|^\alpha & |\zeta|^{-\alpha} \end{pmatrix} \right), & \text{as } \zeta \to 0 \text{ with } \zeta \in \Pi, \text{II, VI, VII,} \\
O \left( \begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \end{pmatrix} \right), & \text{as } \zeta \to 0 \text{ with } \zeta \in \text{I, IV, V, VIII}. \end{cases} \tag{5.22}
\]

This RH problem was solved in [38, formulas (4.26)–(4.33)]. It is built out of the modified Bessel functions $I_{\alpha\pm\frac{1}{2}}, K_{\alpha\pm\frac{1}{2}}$ and out of the Hankel functions $H^{(1)}_{\alpha\pm\frac{1}{2}}, H^{(2)}_{\alpha\pm\frac{1}{2}}$. For our purpose here, it suffices to know the explicit formula for $\Psi_\alpha$ in sector I. There we have

\[
\Psi_\alpha(\zeta) = \frac{1}{2} \sqrt{\pi} \zeta^{1/2} \begin{pmatrix} H^{(2)}_{\alpha+\frac{1}{2}}(\zeta) & -i H^{(1)}_{\alpha+\frac{1}{2}}(\zeta) \\ H^{(2)}_{\alpha-\frac{1}{2}}(\zeta) & -i H^{(1)}_{\alpha-\frac{1}{2}}(\zeta) \end{pmatrix} e^{-(\alpha+\frac{1}{4}) \pi i \sigma_3}, \quad \text{for } 0 < \arg \zeta < \frac{\pi}{4}. \tag{5.23}
\]

Starting from (5.23) we can find the solution in the other sectors by following the jumps (5.17)–(5.20). See [38] for explicit expressions.

Now we define

\[
P^{(1)}(z) = \Psi_\alpha(nf(z)), \tag{5.24}
\]

and $P^{(1)}$ will solve the RH problem for $P^{(1)}$. This ends the construction of $P^{(1)}$.

So far, we have proven that for every matrix valued function $E_n$ analytic in a neighborhood of $U_\delta$, the matrix valued function $P$ given by

\[
P(z) = E_n(z) \Psi_\alpha(nf(z)) W(z)^{-\sigma_3} e^{-n\phi(z)\sigma_3}, \tag{5.25}
\]

satisfies conditions (a), (b) and (d) of the RH problem for $P$. We now use the freedom we have in choosing $E_n$ to ensure that $P$, given by (5.25), also satisfies the matching condition (c) of
κ exists. For regular endpoints we can take

\[ E_n(z) = E(z) e^{\alpha \phi(x) (0) \sigma_3} e^{-\frac{\pi i}{4} \sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \]

(5.26)

where the matrix valued function \( E \) is given by

\[
\begin{align*}
E(z) &= P^{(\infty)}(z) W(z) \sigma_3 e^{\frac{1}{2} \pi i \sigma_3}, & \text{for } z \in f^{-1}(I \cup II), \\
E(z) &= P^{(\infty)}(z) W(z) \sigma_3 e^{-\frac{1}{2} \pi i \sigma_3}, & \text{for } z \in f^{-1}(III \cup IV), \\
E(z) &= P^{(\infty)}(z) W(z) \sigma_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-\frac{1}{2} \pi i \sigma_3}, & \text{for } z \in f^{-1}(V \cup VI), \\
E(z) &= P^{(\infty)}(z) W(z) \sigma_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\frac{1}{2} \pi i \sigma_3}, & \text{for } z \in f^{-1}(VII \cup VIII). 
\end{align*}
\]

(5.27) (5.28) (5.29) (5.30)

Following the proof of [38, Proposition 4.5], we obtain that \( E \) is analytic in a full neighborhood of \( U_\delta \). Here we need the fact that \( D(z)/W(z) \) and \( W(z)/D(z) \) remain bounded as \( z \to 0 \), which follows from [539] and Lemma 4.2. Then we see from [526] that \( E_n \) is also analytic in a neighborhood of \( U_\delta \). This completes the construction of the parametrix near the origin.

**Remark 5.1** Note that, in contrast to the case of the generalized Jacobi weight [38], here \( E \) depends on \( n \). This follows from the fact that the parametrix \( P^{(\infty)} \) for the outside region in our case depends on \( n \).

For later use we state, since \( E \) is analytic in \( U_\delta \) and from the explicit form of \( P^{(\infty)} \), cf. [12], that \( E(z) \) and \( \frac{d}{dz} E(z) \) are uniformly bounded for \( z \in U_\delta \), as \( n \to \infty \).

### 6 Third transformation \( S \to R \)

At each of the endpoints \( a_i, b_j \) of \( J \), we have to do a local analysis as well as at each of the singular points (if any). The endpoints and singular points are surrounded by small disks, say of radius \( \delta \), that do not overlap and that also do not overlap with the disk \( U_\delta \) around the region. Within each disk we construct a parametrix \( P \) which satisfies a local RH problem:

**RH problem for \( P \) near \( x_0 \) where \( x_0 \) is an endpoint or a singular point:**

(a) \( P(z) \) is defined and analytic for \( z \in \{|z - x_0| < \delta\} \setminus \Sigma \) for some \( \delta_0 > \delta \).

(b) \( P \) satisfies the same jump relations as \( S \) does on \( \Sigma \cap \{|z - x_0| < \delta\} \).

(c) There is \( \kappa > 0 \) such that we have as \( n \to \infty \):

\[
P(z) \left( P^{(\infty)} \right)^{-1}(z) = I + O \left( \frac{1}{n^\kappa} \right), \quad \text{uniformly for } |z - x_0| = \delta. \quad (6.1)
\]

The local RH problem near the regular endpoints \( a_i, b_j \) of \( J \) is similar to the situation in [12]. Here however, we have extra factors \( |x|^{\pm 2\alpha} \) and \( \omega(z)^{-1} \) in the jump matrices. These factors can easily be removed via an appropriate transformation, and the local RH problem is then solved as in [12] Section 4.3–Section 4.5 with the use of Airy functions. For our purpose, we do not need the explicit formulas for the parametrix near the endpoints. It suffices to know that \( P \) exists. For regular endpoints we can take \( \kappa = 1 \) in (6.1).
So, estimates then imply that \( v_z \) is normalized at infinity, we then find uniformly for \( \| P \|_\infty \equiv 1 \), see [12]. So the singular points lead to error terms with decay slower than for the regular points (\( \kappa = 1 \)). However, this has no influence on the universality result at the origin (not even in the error term), since that only depends on the leading order asymptotics.

We are now ready to do the final transformation. As noted before, we surround the endpoints \( a_i, b_j \) of \( J \), the origin, and the singular points of the potential \( V \) by nonoverlapping small disks. Using the parametrix \( P^{(\infty)} \) for the outside region and the parametrix \( P \) defined inside each of the disks, we define the matrix valued function \( R \) as

\[
R(z) = \begin{cases} 
S(z) \left( P^{(\infty)} \right)^{-1}(z), & \text{for } z \text{ outside the disks,} \\
S(z)P^{-1}(z), & \text{for } z \text{ inside the disks.}
\end{cases}
\]  

(6.2)

Remark 6.1 It is known that the inverses of the parametrices \( P^{(\infty)} \) and \( P \) exist, since all matrices have determinant one. For \( P^{(\infty)} \), see [12]. For \( P \) within the disks around the endpoints \( a_i, b_j \) of \( J \), as well as within the disks around the singular points of \( V \) we refer to [12]. For \( P \) within the disk around the origin we refer to [38 Section 4].

Note that \( P^{(\infty)} \) and \( S \) have the same jumps on \( J \setminus \{0\} \), and that \( P \) and \( S \) have the same jumps on the lens \( \Sigma \) within the disks. This implies that \( R \) is analytic on the entire plane, except for jumps on the reduced system of contours \( \Sigma_R \), as shown in Figure 6; cf. [12], and except for a possible isolated singularity at the origin. Yet, as in [27, 38], it follows easily from the behavior of \( S \) and \( P \) near the origin, given by (3.12) and (3.13), and by (5.4) and (5.5), respectively, that the isolated singularity of \( R \) at the origin is removable. Therefore \( R \) is analytic on \( \mathbb{C} \setminus \Sigma_R \).

Recall that the matrix valued functions \( S \) and \( P^{(\infty)} \) are normalized at infinity. Since \( \det P^{(\infty)} \equiv 1 \), this implies, by (6.2), that also \( R \) is normalized at infinity.

Let \( v_R \) be the jump matrix for \( R \). It can be calculated explicitly for each component of \( \Sigma_R \). However, all that we require are the following estimates, cf. [12]

\[
\| v_R(z) \| = I + O(e^{-cn|z|}), \quad \text{as } n \to \infty, \ z \in \Sigma_R \setminus \text{circles},
\]

\[
\| v_R(z) \| = I + O(1/n^\kappa), \quad \text{as } n \to \infty, \ z \in \text{circles},
\]

for some \( c > 0 \) and \( 0 < \kappa \leq 1 \), and where \( \| \cdot \| \) is any matrix norm. We note that the extra factor \( |x|^{2\alpha} \), which we will meet in \( v_R \), does not cause any difficulties to obtain this behavior. These estimates then imply that \( v_R \) is uniformly close to the identity matrix as \( n \to \infty \), and, since \( R \) is normalized at infinity, we then find uniformly for \( z \in \mathbb{R} \setminus \Sigma_R \),

\[
R(z) = I + O(1/n^\kappa), \quad \text{as } n \to \infty.
\]  

(6.3)

So, \( R \) is uniformly bounded as \( n \to \infty \). We also have that \( \frac{d}{dz} R(z) \) is uniformly bounded as \( n \to \infty \). Another useful property is \( \det R \equiv 1 \), which follows from (6.2) and the fact that \( S, P^{(\infty)} \) and \( P \) all have determinant 1.

Figure 6: Part of the contour \( \Sigma_R \). The singular point \( z_1 \) corresponds to a point where \( h \) vanishes at the interior of \( J \), the singular point \( z_2 \) corresponds to a point where we obtain equality in (3.14).
7 Proof of Theorem 1.1

We now have all the ingredients necessary to prove Theorem 1.1. We point out that the general scheme of this proof is the same as the proof of [23, Theorem 1.1(c)]. We replace in the kernel $K_n$, given by (1.20), the orthonormal polynomials $p_{n-1,n}$ and $p_{n,n}$, together with their leading coefficients $\gamma_{n-1,n}$ and $\gamma_{n,n}$, by the appropriate entries of $Y$, given by (2.4), and find

$$K_n(x,y) = -\frac{1}{2\pi i} \sqrt{w_n(x)} \sqrt{w_n(y)} \frac{Y_{11}(y)Y_{21}(y) - Y_{21}(x)Y_{11}(y)}{x - y}. \quad (7.1)$$

This means that the kernel $K_n$ can be expressed in terms of the first column of $Y$. Hence, we want to know the asymptotic behavior of $Y$ near the origin. This will be determined in the following lemma.

Lemma 7.1 For $x \in (0, \delta)$,

$$
\begin{pmatrix}
Y_{11}(x) \\
Y_{21}(x)
\end{pmatrix} = e^{-\frac{\pi i}{4}} \sqrt{\frac{\pi}{w_n(x)}} e^{\frac{\nu}{2} \sigma_3} M_+(x) \left( (nf(x))^{1/2} J_{\alpha + \frac{1}{2}}(nf(x)) \right) \left( (nf(x))^{1/2} J_{\alpha - \frac{1}{2}}(nf(x)) \right),
\end{pmatrix} \quad (7.2)
$$

with $M(z)$ given by

$$M(z) = R(z) E(z) e^{\nu \phi(z) \sigma_3} e^{-\frac{\pi i}{4} \sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad (7.3)$$

where $R$ is the result of the transformations $Y \to T \to S \to R$ of the RH problem, and the matrix valued function $E$ is given by (5.27)–(5.30). The matrix valued function $M$ is analytic in $U_\delta$ with $M(z)$ and $\frac{d}{dz} M(z)$ uniformly bounded for $z \in U_\delta$ as $n \to \infty$. Furthermore,

$$\det M(z) \equiv 1. \quad (7.4)$$

Proof. We use the series of transformations $Y \to T \to S \to R$ and unfold them for $z$ inside the disk $U_\delta$ and in the right upper part of the lens, so that $z \in f^{-1}(I)$. Since $\omega(z) = z^{2\alpha}$ and $W(z) = z^{\alpha} e^{-\pi i \sigma}$ for our choice of $z$, see (3.6) and (5.3), we have by (2.14), (3.7), (5.26) and (6.2)

$$Y(z) = e^{\frac{\nu}{2} \sigma_3} R(z) E_n(z) \Psi_\alpha(n f(z)) e^{-\nu \phi(z) \sigma_3} z^{-\alpha \sigma_3} e^{\pi i \sigma_3} \times \begin{pmatrix}
1 \\
0
\end{pmatrix} e^{\frac{\nu}{2} \sigma_3} e^{ng(z) \sigma_3}. \quad (7.5)$$

We then get for the first column of $Y$,

$$\begin{pmatrix}
Y_{11}(z) \\
Y_{21}(z)
\end{pmatrix} = z^{-\alpha} e^{ng(z) - \phi(z) - \frac{\nu}{2} \sigma_3} e^{\frac{\nu}{2} \sigma_3} R(z) E_n(z) \Psi_\alpha(n f(z)) e^{\pi i \sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (7.6)$$

Since $z$ is in the right upper part of the lens and inside the disk $U_\delta$, we have $0 < \arg nf(z) < \pi/4$, cf. Figure 3 and we thus use (5.23) to evaluate $\Psi_\alpha(n f(z))$. Using the formulas 9.1.3 and 9.1.4 of [1] which connect the Hankel functions with the usual $J$-Bessel functions, we find

$$\Psi_\alpha(n f(z)) e^{\pi i \sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{-\frac{\pi i}{4}} \sqrt{\frac{\pi}{2}} \left( (nf(z))^{1/2} J_{\alpha + \frac{1}{2}}(nf(z)) \right) \left( (nf(z))^{1/2} J_{\alpha - \frac{1}{2}}(nf(z)) \right) \quad (7.7)$$

By (5.26) and (7.3) we have $R(z) E_n(z) = M(z)$. Inserting this and (7.7) into (7.6) we get

$$\begin{pmatrix}
Y_{11}(z) \\
Y_{21}(z)
\end{pmatrix} = e^{-\frac{\pi i}{4}} \sqrt{\frac{\pi}{2}} z^{-\alpha} e^{ng(z) - \phi(z) - \frac{\nu}{2} \sigma_3} e^{\frac{\nu}{2} \sigma_3} M(z) \left( (nf(z))^{1/2} J_{\alpha + \frac{1}{2}}(nf(z)) \right) \left( (nf(z))^{1/2} J_{\alpha - \frac{1}{2}}(nf(z)) \right) \quad (7.8)$$
Letting $z \to x \in (0, \delta)$, and noting that
\[ x - \alpha e^{n(g_+(x) - \phi_+(x) - \frac{\pi}{2})} = x - \alpha e^{\frac{\pi}{2} n(g_+(x) + g_-(x) - \ell)} = x - \alpha e^{\frac{\pi}{2} n V(x)} = w_n(x)^{-1/2}, \] (7.9)
which follows from the fact that $2\phi_+(x) = g_+(x) - g_-(x)$, see Section 3, and from (1.2) and (2.10), we obtain (7.9).

The matrix valued function $M$ is analytic in the disk $U_\delta$ since both $R$ and $E$ are analytic in this disk. So, we may write $M(x)$ instead of $M(x)^{\frac{\alpha}{2}}$ in (7.2).

We recall that $R(z), \frac{d}{dz} R(z), E(z)$ and $\frac{d}{dz} E(z)$ are uniformly bounded for $z \in U_\delta$ as $n \to \infty$, see Section 5 and Section 6. If we also use that $|e^{n\phi_+(0)}| = 1$, which follows from the fact that $\phi_+$ is purely imaginary on $J$, we have from (7.3) that $M(z)$ and $\frac{d}{dz} M(z)$ are uniformly bounded for $z \in U_\delta$ as $n \to \infty$.

Since $M$ is a product of five matrices all with determinant one, (7.4) is true. \[ \blacksquare \]

**Lemma 7.2** Let $u \in (0, \infty), u_n = \frac{u}{n\psi(0)}$ and $\tilde{u}_n = nf(u_n)$. Then
\[ \tilde{u}_n = \pi u + O\left(\frac{u^2}{n}\right), \quad \text{as } n \to \infty, \] (7.10)
\[ J_{\alpha + \frac{1}{2}}(\tilde{u}_n) = J_{\alpha + \frac{1}{2}}(\pi u) + O\left(\frac{u^{\alpha + \frac{2}{3}}}{n}\right), \quad \text{as } n \to \infty, \] (7.11)
\[ J_{\alpha - \frac{1}{2}}(\tilde{u}_n) = J_{\alpha - \frac{1}{2}}(\pi u) + O\left(\frac{u^{\alpha + \frac{2}{3}}}{n}\right), \quad \text{as } n \to \infty, \] (7.12)
where the error terms hold uniformly for $u$ in bounded subsets of $(0, \infty)$.

**Proof.** Since, see (5.8)
\[ f(x) = \pi\psi(0)x + O(x^2), \quad \text{as } x \to 0, \]
we have, uniformly for $u$ in bounded subsets of $(0, \infty)$,
\[ f\left(\frac{u}{n\psi(0)}\right) = \pi u + O\left(\frac{u^2}{n^2}\right), \quad \text{as } n \to \infty, \]
which proves (7.10).

We note [11, formula 9.1.10] that $J_{\alpha + \frac{1}{2}}(z) = z^{\alpha + \frac{1}{2}}H(z)$, with $H$ an entire function. It then follows from (7.10) that, as $n \to \infty$, uniformly for $u$ in bounded subsets of $(0, \infty)$,
\[ J_{\alpha + \frac{1}{2}}(\tilde{u}_n) = \left[(\pi u)^{\alpha + \frac{1}{2}} + O\left(\frac{u^{\alpha + \frac{2}{3}}}{n}\right)\right]\left[H(\pi u) + O\left(\frac{u^2}{n}\right)\right] \]
\[ = J_{\alpha + \frac{1}{2}}(\pi u) + O\left(\frac{u^{\alpha + \frac{2}{3}}}{n}\right), \]
so that equation (7.11) is proved. Similarly, we can prove (7.12). \[ \blacksquare \]

We are now able to prove Theorem 1.1.
**Proof of Theorem 1.1** Let $u, v \in (0, \infty)$ and define

$$u_n = \frac{u}{n \psi(0)}, \quad v_n = \frac{v}{n \psi(0)}, \quad \tilde{u}_n = n f(u_n), \quad \tilde{v}_n = n f(v_n).$$

We put

$$\hat{K}_n(u, v) = \frac{1}{n \psi(0)} K_n(u, v).$$

From (7.1) and (7.2) we then have

$$\hat{K}_n(u, v) = -\frac{1}{2\pi i (u-v)} \det \left( e^{-\frac{nt}{2}} \sqrt{w_n(u_n)} Y_{11}(u_n) \quad e^{-\frac{nt}{2}} \sqrt{w_n(v_n)} Y_{11}(v_n) \quad e^{-\frac{nt}{2}} \sqrt{w_n(v_n)} Y_{21}(v_n) \right).$$

The matrix in the determinant can be written as

$$M(v_n) \begin{bmatrix} \tilde{v}_{n \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\tilde{u}_n) & \tilde{v}_{n \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\tilde{v}_n) \\ \tilde{u}_{n \frac{1}{2}} J_{\alpha - \frac{1}{2}}(\tilde{u}_n) & \tilde{u}_{n \frac{1}{2}} J_{\alpha - \frac{1}{2}}(\tilde{v}_n) \end{bmatrix} + M(v_n)^{-1} (M(u_n) - M(v_n)) \begin{bmatrix} u_{n \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\tilde{u}_n) & 0 \\ u_{n \frac{1}{2}} J_{\alpha - \frac{1}{2}}(\tilde{u}_n) & 0 \end{bmatrix}. \quad (7.13)$$

We will now determine the asymptotics of the second term in (7.13). Since $\det M(v_n) = 1$ and since $M(z)$ is uniformly bounded for $z \in U_\delta$, see Lemma 7.1, the entries of $M(v_n)^{-1}$ are uniformly bounded. By Lemma 7.1 we also have that $\frac{d}{dz} M(z)$ is uniformly bounded for $z \in U_\delta$, so that from the mean value theorem $M(u_n) - M(v_n) = O \left( \frac{u-v}{n} \right)$. From Lemma 7.2 it follows that $\tilde{u}_{n \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\tilde{u}_n) = O(u^{\alpha+1})$ and $\tilde{u}_{n \frac{1}{2}} J_{\alpha - \frac{1}{2}}(\tilde{u}_n) = O(u^\alpha)$ uniformly for $u$ in bounded subsets of $(0, \infty)$ as $n \to \infty$. Hence we have, uniformly for $u, v$ in bounded subsets of $(0, \infty)$,

$$M(v_n)^{-1} (M(u_n) - M(v_n)) \begin{bmatrix} u_{n \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\tilde{u}_n) & 0 \\ u_{n \frac{1}{2}} J_{\alpha - \frac{1}{2}}(\tilde{u}_n) & 0 \end{bmatrix} = O \left( \frac{u-v u^\alpha}{n} \right). \quad (7.14)$$

Inserting this into (7.13), using the fact that $\det M(v_n) = 1$, and that $\tilde{v}_{n \frac{1}{2}} J_{\alpha \pm \frac{1}{2}}(\tilde{v}_n) = O(v^\alpha)$ as $n \to \infty$, we then find uniformly for $u, v$ in bounded subsets of $(0, \infty)$,

$$\hat{K}_n(u, v) = \frac{1}{2(u-v)} \det \begin{bmatrix} \tilde{u}_{n \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\tilde{u}_n) + O \left( \frac{u-v u^\alpha}{n} \right) & \tilde{v}_{n \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\tilde{v}_n) \\ \tilde{u}_{n \frac{1}{2}} J_{\alpha - \frac{1}{2}}(\tilde{u}_n) + O \left( \frac{u-v u^\alpha}{n} \right) & \tilde{v}_{n \frac{1}{2}} J_{\alpha - \frac{1}{2}}(\tilde{v}_n) \end{bmatrix}.$$

Using Lemma 7.2, we can replace in the determinant, $\tilde{u}_n$ by $\pi u$ and $\tilde{v}_n$ by $\pi v$. We then make an error which can be estimated by Lemma 7.2. However, since this estimate is not uniform for $u - v$ close to zero, we have to be more careful. We insert a factor $u^{-\alpha}$ in the
first column of the determinant, and a factor $v^{-\alpha}$ in the second. Then we subtract the second column from the first to obtain

$$
\hat{K}_n(u, v) = \frac{u^{\alpha}v^{\alpha}}{2(u-v)} \det \left( \begin{array}{ccc}
\alpha^{-\alpha/2} J_{\alpha+1/2}(u) - v^{-\alpha} J_{\alpha+1/2}(v) \\
v^{-\alpha} J_{\alpha-1/2}(u) - v^{-\alpha} J_{\alpha-1/2}(v)
\end{array} \right) + O\left( \frac{u^{\alpha}v^{\alpha}}{n} \right).
$$

(7.15)

One can check, using (7.14), (7.15) and the facts that $J_{\alpha+1/2}'(x_n) = J_{\alpha+1/2}(x_n) + O(x_n^{\alpha+1})$ and $\frac{d}{dx} x_n = \pi + O(\frac{1}{n})$, where we have put $x_n = n f(\frac{x}{n})$, that

$$
\frac{d}{dx} \left[ x^{-\alpha} J_{\alpha+1/2}(x_n) - x^{-\alpha} J_{\alpha+1/2}(\pi x) \right] = O\left( \frac{1}{n} \right), \quad \text{as } n \to \infty,
$$

uniformly for $x$ in bounded subsets of $(0, \infty)$. It then follows that the $(1,1)$–entry in the determinant of (7.15) is equal to

$$
u^{-\alpha}(\pi u)^{1/2} J_{\alpha+1/2}(\pi u) - v^{-\alpha}(\pi v)^{1/2} J_{\alpha+1/2}(\pi v) + O\left( \frac{u-v}{n} \right).
$$

Similarly, we have from

$$
\frac{d}{dx} \left[ x^{-\alpha} J_{\alpha+1/2}(x_n) - x^{-\alpha} J_{\alpha+1/2}(\pi x) \right] = O\left( \frac{1}{n} \right), \quad \text{as } n \to \infty,
$$

that the $(2,1)$–entry in the determinant of (7.15) is equal to

$$
u^{-\alpha}(\pi u)^{1/2} J_{\alpha-1/2}(\pi u) - v^{-\alpha}(\pi v)^{1/2} J_{\alpha-1/2}(\pi v) + O\left( \frac{u-v}{n} \right).
$$

From Lemma 7.2 it also follows that $\tilde{v}_n J_{\alpha+1/2}(\tilde{v}_n) = (\pi v) J_{\alpha+1/2}(\pi v) + O(1/n)$. Therefore, uniformly for $u, v$ in bounded subsets of $(0, \infty),

$$
\hat{K}_n(u, v) = \frac{u^{\alpha}v^{\alpha}}{2(u-v)}
$$

\times \det \left( \begin{array}{ccc}
\alpha^{-\alpha/2} J_{\alpha+1/2}(u) - v^{-\alpha} J_{\alpha+1/2}(v) + O\left( \frac{u-v}{n} \right) & \alpha^{-\alpha/2} J_{\alpha+1/2}(\pi u) - v^{-\alpha} J_{\alpha+1/2}(\pi v) + O\left( \frac{1}{n} \right) \\
v^{-\alpha} J_{\alpha-1/2}(u) - v^{-\alpha} J_{\alpha-1/2}(v) + O\left( \frac{u-v}{n} \right) & v^{-\alpha} J_{\alpha-1/2}(\pi u) - v^{-\alpha} J_{\alpha-1/2}(\pi v) + O\left( \frac{1}{n} \right)
\end{array} \right)
$$

+ O\left( \frac{u^{\alpha}v^{\alpha}}{n} \right)

= J_\alpha(u, v) + \frac{u^{\alpha}v^{\alpha}}{2(u-v)} \det \left( \begin{array}{ccc}
\alpha^{-\alpha/2} J_{\alpha+1/2}(u) - v^{-\alpha} J_{\alpha+1/2}(v) & \alpha^{-\alpha/2} J_{\alpha+1/2}(\pi u) - v^{-\alpha} J_{\alpha+1/2}(\pi v) + O\left( \frac{1}{n} \right) \\
v^{-\alpha} J_{\alpha-1/2}(u) - v^{-\alpha} J_{\alpha-1/2}(v) & v^{-\alpha} J_{\alpha-1/2}(\pi u) - v^{-\alpha} J_{\alpha-1/2}(\pi v) + O\left( \frac{1}{n} \right)
\end{array} \right)
$$

+ O\left( \frac{u^{\alpha}v^{\alpha}}{n} \right)

(7.16)

Since $z^{-\alpha+1/2} J_{\alpha+1/2}(z)$ is an entire function we have by the mean value theorem that

$$
\frac{u^{-\alpha}(\pi u)^{1/2} J_{\alpha+1/2}(\pi u) - v^{-\alpha}(\pi v)^{1/2} J_{\alpha+1/2}(\pi v)}{u-v}
$$

25
is bounded for $u,v$ in bounded subsets of $(0, \infty)$. From (7.16) we then have
\[
\hat{K}_n(u,v) = J_\alpha(u,v) + O\left(\frac{u^\alpha v^\alpha}{n}\right),
\]
uniformly for $u,v$ in bounded subsets of $(0, \infty)$, which completes the proof of Theorem 1.1. □

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