Complexity of the FIFO Stack-Up Problem

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Abstract

We study the combinatorial FIFO stack-up problem. In delivery industry, bins have to be stacked-up from conveyor belts onto pallets. Given \( k \) sequences \( q_1, \ldots, q_k \) of labeled bins and a positive integer \( p \), the aim is to stack-up the bins by iteratively removing the first bin of one of the \( k \) sequences and put it onto an initially empty pallet located at one of \( p \) stack-up places. Bins with different pallet labels have to be placed on different pallets, bins with the same pallet label have to be placed on the same pallet. After all bins for a pallet have been removed from the given sequences, the corresponding stack-up place will be cleared and becomes available for a further pallet. The FIFO stack-up problem is to find a stack-up sequence such that all pallets can be build-up with the available \( p \) stack-up places. In this paper, we show that the FIFO stack-up problem is NP-complete in general, even if there are at most 6 bins for every pallet. The problem can be solved in polynomial time, if the number \( k \) of sequences or the number \( p \) of stack-up places is assumed to be fixed.

Keywords: computational complexity; combinatorial optimization; directed pathwidth; discrete algorithms

1 Introduction

We consider the combinatorial problem of stacking up bins from a set of conveyor belts onto pallets. This problem originally appears in stack-up systems that play an important role in delivery industry and warehouses. A detailed description of the practical background of this work is given in \[3, 11\].

The bins that have to be stacked up onto pallets arrive at the stack-up system on the main conveyor of an order-picking system. At the end of the main conveyor they enter a cyclic storage conveyor, see Figure 1. From the storage conveyor the bins are pushed out to buffer conveyors, where they are queued. The bins are picked-up by stacker cranes from the end of a buffer conveyor and moved onto pallets, which are located at some stack-up places. There is one buffer conveyor for each stack-up place. Automated guided vehicles (AGVs) take full pallets from stack-up places, put them onto trucks and bring new empty pallets to the stack-up places.

The stack-up system that has initiated our research is located at Bertelsmann Distribution GmbH in Gütersloh, Germany. On certain days, several thousands of bins are stacked-up using a storage conveyor with a capacity of approximately 60 bins and 24 stack-up places, while up to 32 bins are destined for a pallet.

The cyclic storage conveyor enables a smooth stack-up process irrespective of the real speed the cranes and conveyors are moving. Such details are unnecessary for computing an order in which...
the bins can be palletized. For the sake of simplicity, we disregard the cyclic storage conveyor, and for the sake of generality, we do not restrict the number of stack-up places to the number of sequences. The number of sequences can be larger than or less than the number of stack-up places.

Logistic experiences over 30 years lead to such high flexible conveyor systems in delivery industry. We do not intend to modify the architecture of existing systems, but try to develop efficient algorithms to control them. Figure 2 shows a sketch of a simplified stack-up system with 2 buffer conveyors and 3 stack-up places.

Figure 1: A general stack-up system. The colors represent the pallet labels. Bins with different colors have to be placed on different pallets, bins with the same color have to be placed on the same pallet.

Figure 2: The FIFO stack-up system analyzed in this paper. The system is blocked, because the pallet for the red bins cannot be opened and the pallets for the green, yellow, and blue bins cannot be closed.

Our aim in controlling stack-up systems is to avoid blocking situations. The system is blocked, if all stack-up places are occupied by pallets, and none of the first bins of any sequence is destined for a pallet already located on a stack-up place. To unblock the system, bins have to be picked-up manually and moved to pallets by human workers. Such a blocking situation is sketched in Figure 2.

From a theoretical point of view, we are given $k$ sequences $q_1, \ldots, q_k$ of bins and a positive integer $p$. Each bin $b \in q_i$, $1 \leq i \leq k$, is destined for exactly one pallet. The FIFO stack-up problem is to decide whether one can remove iteratively the bins of the $k$ sequences such that in each step only the first bin of one of the sequences will be removed and after each step at most $p$ pallets are open. A pallet $t$ is called open, if at least one bin for pallet $t$ has already been removed.
A sequence \( q \) is labeled with a pallet symbol \( \text{plt} \) representing the buffer queues (handled by the buffer conveyors) in real stack-up systems. Each bin \( b \) say bin \( b_i \) from one of the given sequences, and if at least one bin for pallet \( t \) is still contained in one of the remaining sequences. If a bin \( b \) is removed from a sequence then all bins located behind \( b \) are moved-up one position to the front (cf. Section 2 for the formal definition). Integer \( p \) represents the number of available stack-up places.

Our model is the second attempt to capture important parameters necessary for an efficient and provable good algorithmic controlling of stack-up systems. The only theoretical model for stack-up systems known to us uses a random access storage instead of buffer queues \([11][12][13]\). We think that buffer queues represent the real stack-up system more realistic than a random access storage. The FIFO stack-up problem seems to be not investigated by other authors up to know, although stack-up systems play an important role in delivery industry and warehouses. However, many facts are known about stack-up systems that use a random access storage instead of buffer queues. In \([11]\) it is shown that the stack-up problem with a random access storage is NP-complete, but can be solved efficiently if the storage capacity \( s \) or the number of stack-up places \( p \) is fixed. It remains NP-complete as shown in \([12]\), even if the sequence contains at most 9 bins per pallet. In \([12]\), a polynomial-time off-line approximation algorithm for minimizing the number of stack-up places is introduced. This algorithm yields a processing that is optimal up to a factor bounded by \( \log(p) \). In \([13]\) the performances of some simple on-line stack-up algorithms are compared with optimal off-line solutions by a competitive analysis \([2][4]\).

In this paper, we introduce two digraph models which are useful to give solutions for the FIFO stack-up problem, namely the processing graph and the sequence graph. The processing graph of a given list of sequences \( Q \) has a vertex for every possible configuration and represents all the possible assignments of pallets. If the number \( k \) of sequences is fixed, the processing graph allows to give a polynomial time algorithm for the FIFO stack-up problem. For this case, we also give a non-deterministic algorithm that uses only logarithmic memory. The sequence graph of a given list of sequences \( Q \) has a vertex for every pallet and displays the order in which the pallets of \( Q \) can be opened. We show that there is a processing of some list \( Q \) with at most \( p \) stack-up places if and only if the sequence graph of \( Q \) has directed pathwidth at most \( p - 1 \) (cf. Section 3.2 for the formal definition). This connection implies that the FIFO stack-up problem can also be solved in polynomial time, if the number \( p \) of given stack-up places is assumed to be fixed. Further this connection is used to show, that the FIFO stack-up problem is NP-complete in general, even if all sequences together contain at most 6 bins destined for the same pallet.

## 2 Preliminaries

Unless otherwise stated, \( k \) and \( p \) are some positive integers throughout the paper. We consider *sequences* \( q_1 = (b_{1,1}, \ldots, b_{1,n_1}), \ldots, q_k = (b_{k,n_{k-1}+1}, \ldots, b_{k,n_k}) \) of pairwise distinct *bins*. These sequences represent the buffer queues (handled by the buffer conveyors) in real stack-up systems. Each bin \( b \) is labeled with a *pallet symbol* \( \text{plt}(b) \) which is, without loss of generality, some positive integer. We say bin \( b \) is destined for pallet \( \text{plt}(b) \). The labels of the pallets can be chosen arbitrarily, because we only need to know whether two bins are destined for the same pallet or for different pallets. In order to avoid confusion between indices (or positions) and pallet symbols, we use in our examples characters for pallet symbols. The set of all pallets of the bins in some sequence \( q_i \) is denoted by

\[
\text{plts}(q_i) = \{ \text{plt}(b) \mid b \in q_i \}.
\]

For a list of sequences \( Q = (q_1, \ldots, q_k) \) we denote

\[
\text{plts}(Q) = \text{plts}(q_1) \cup \cdots \cup \text{plts}(q_k).
\]

For some sequence \( q = (b_{1,1}, \ldots, b_{n}) \), we say bin \( b_i \) is on the left of bin \( b_j \) in sequence \( q \) if \( i < j \). A sequence \( q' = (b_{j+1}, \ldots, b_n) \), \( j \geq 1 \), is called a subsequence of sequence \( q = (b_{1,1}, \ldots, b_{n}) \). We define \( q - q' = (b_{1}, \ldots, b_{j-1}) \).

Let \( Q = (q_1, \ldots, q_k) \) and \( Q' = (q_1', \ldots, q_k') \) be two lists of sequences of bins such that each sequence \( q_j \), \( 1 \leq j \leq k \), is a subsequence of sequence \( q_j \). Each such pair \((Q, Q')\) is called a
configuration. In every configuration \((Q, Q')\) the first entry \(Q\) is the initial list of sequences of bins and the second entry \(Q'\) is the list of sequences that remain to be processed. A pallet \(t\) is called open in configuration \((Q, Q')\), if a bin of pallet \(t\) is contained in some \(q'_i \in Q'\) and if another bin of pallet \(t\) is contained in some \(q_j - q'_j\) for \(q_j \in Q, q'_j \in Q'\). The set of open pallets in configuration \((Q, Q')\) is denoted by \(\text{open}(Q, Q')\). A pallet \(t \in \text{plts}(Q)\) is called closed in configuration \((Q, Q')\), if \(t \not\in \text{plts}(Q')\), i.e. no sequence of \(Q'\) contains a bin for pallet \(t\). Initially all pallets are unprocessed. After the first bin of a pallet \(t\) has been removed from one of the sequences, pallet \(t\) is either open or closed.

In view of the practical background, we only consider lists of sequences that together contain at least two bins for each pallet.

Throughout the paper \(Q\) always denotes a list of some sequences of bins.

The FIFO Stack-up Problem

Consider a configuration \((Q, Q')\). The removal of the first bin from one subsequence \(q' \in Q'\) is called transformation step. A sequence of transformation steps that transforms the list \(Q\) of \(k\) sequences into \(k\) empty subsequences is called a processing of \(Q\).

**Name:** FIFO stack-up

**Instance:** A list \(Q = (q_1, \ldots, q_k)\) of sequences and a positive integer \(p\).

**Question:** Is there a processing of \(Q\), such that in each configuration \((Q, Q')\) during the processing at most \(p\) pallets are open?

We use the following variables: \(k\) denotes the number of sequences, and \(p\) stands for the number of stack up places. Furthermore, \(m\) represents the number of pallets in \(\text{plts}(Q)\), and \(n\) denotes the number of bins in all sequences. Finally, \(N = \max\{|q_1|, \ldots, |q_k|\}\) is the maximum sequence length. In view of the practical background, it holds \(p < m, k < m, m < n, \text{ and } N < n\).

It is often convenient to use pallet identifications instead of bin identifications to represent a sequence \(q\). For \(n\) not necessarily distinct pallets \(t_1, \ldots, t_n\) let \([t_1, \ldots, t_n]\) denote some sequence of \(n\) pairwise distinct bins \((b_1, \ldots, b_n)\), such that \(\text{plt}(b_i) = t_i\) for \(i = 1, \ldots, n\). We use this notion for lists of sequences as well. For the sequences \(q_1 = [t_1, \ldots, t_{n_1}], \ldots, q_k = [t_{n_{k-1}+1}, \ldots, t_{n_k}]\) of pallets we define \(q_1 = (b_1, \ldots, b_{n_1}), \ldots, q_k = (b_{n_{k-1}+1}, \ldots, b_{n_k})\) to be sequences of bins such that \(\text{plt}(b_i) = t_i\) for \(i = 1, \ldots, n_k\), and all bins are pairwise distinct.

For some list of subsequences \(Q'\) we define \(\text{front}(Q')\) to be the pallets of the first bins of the queues of \(Q'\).

**Example 2.1** Consider list \(Q = (q_1, q_2)\) of sequences

\[ q_1 = (b_1, \ldots, b_4) = [a, a, b, b]\]

and

\[ q_2 = (b_5, \ldots, b_{12}) = [c, d, e, c, a, d, b, e].\]

Table I shows a processing of \(Q\) with 3 stack-up places. The underlined bin is always the bin that will be removed in the next transformation step. We denote \(Q_i = (q_i^1, q_i^2)\), thus each row represents a configuration \((Q, Q_i)\).

Consider a processing of a list \(Q\) of sequences. Let \(B = (b_{\pi(1)}, \ldots, b_{\pi(m)})\) be the order in which the bins are removed during the processing of \(Q\), and let \(T = (t_1, \ldots, t_m)\) be the order in which the pallets are opened during the processing of \(Q\). Then \(B\) is called a bin solution of \(Q\), and \(T\) is called a pallet solution of \(Q\). The transformation in Table I defines the bin solution

\[ B = (b_5, b_6, b_7, b_8, b_1, b_9, b_{10}, b_{11}, b_3, b_4, b_{12}), \]

and the pallet solution \(T = (c, d, e, a, b)\).
Table 1: A processing of $Q = (q_1, q_2)$ from Example 24 with 3 stack-up places. There is no processing of $Q$ that needs less than 3 stack-up places.

During a processing of a list $Q$ of sequences there are often configurations $(Q, Q')$ for which it is easy to find a bin $b$ that can be removed from $Q'$ such that a further processing with $p$ stack-up places is still possible. This is the case if bin $b$ is destined for an already open pallet, see configuration $(Q, Q_3), (Q, Q_5), (Q, Q_6), (Q, Q_7), (Q, Q_9), (Q, Q_{10})$, or $(Q, Q_{11})$ in Table 1. In the following we show:

- If one of the first bins of the sequences is destined for an already open pallet then this bin can be removed without increasing the number of stack-up places necessary to further process the sequences.
- If there are more than one bin at choice for already open pallets then the order, in which those bins are removed is arbitrary.

Both rules are useful, i.e. within the following two simple algorithms for the FIFO stack-up problem.

**Algorithm 1** Generate all possible bin orders $(b_{\pi(1)}, \ldots, b_{\pi(n)})$ and verify, whether this can be a processing. This leads to a simple but very inefficient algorithm with running time $O(n^2 \cdot m!)$, where $O(n^2)$ time is needed for each test.

**Algorithm 2** Generate all possible pallet orders $(t_{\pi(1)}, \ldots, t_{\pi(m)})$ and test, whether this can be a processing. This leads to a further simple but less inefficient algorithm with running time $O(n^2 \cdot m!)$.

The running time of Algorithm 2 is much better than those of Algorithm 1 and Algorithm 2 is only possible because of the second rule. If bins for already open pallets must be removed in a special order, Algorithm 2 would not be possible.

To show the rules, consider a processing of some list $Q$ of sequences with $p$ stack-up places. Let

$$(b_{\pi(1)}, \ldots, b_{\pi(i-1)}, b_{\pi(i)}, \ldots, b_{\pi(l-1)}, b_{\pi(l)}, b_{\pi(l+1)}, \ldots, b_{\pi(n)})$$

be the order in which the bins are removed from the sequences during the processing, and let $(Q, Q_j), 1 \leq j \leq n$ denote the configuration such that bin $b_{\pi(j)}$ is removed in the next transformation step. Suppose bin $b_{\pi(i)}$ will be removed in some transformation step although bin $b_{\pi(l)}$, $l > i$, for some already open pallet $\text{plt}(b_{\pi(i)}) \in \text{open}(Q, Q_i)$ could be removed next. We define a modified processing

$$(b_{\pi(1)}, \ldots, b_{\pi(i-1)}, b_{\pi(l)}, b_{\pi(i)}, \ldots, b_{\pi(l-1)}, b_{\pi(l+1)}, \ldots, b_{\pi(n)})$$

| $i$ | $q_1^i$ | $q_2^i$ | $\text{front}(Q_i)$ | remove | $\text{open}(Q, Q_i)$ |
|-----|---------|---------|----------------------|--------|----------------------|
| 0   | $b, b, a$ | $a, b, a$ | $\{a, c\}$ | $b_5$ | $\emptyset$ |
| 1   | $a, b, b$ | $d, e, a$ | $\{a, d\}$ | $b_6$ | $\{c\}$ |
| 2   | $a, b, b$ | $b, c, a$ | $\{a, e\}$ | $b_7$ | $\{c, d\}$ |
| 3   | $a, b, b$ | $b, c, a$ | $\{a, c\}$ | $b_8$ | $\{c, d, e\}$ |
| 4   | $a, b, b$ | $a, d, b$ | $\{a\}$ | $b_9$ | $\{d, e\}$ |
| 5   | $a, b, b$ | $a, d, b$ | $\{a\}$ | $b_9$ | $\{a, d, e\}$ |
| 6   | $b, b, b$ | $a, d, b$ | $\{a, b\}$ | $b_{10}$ | $\{d, e\}$ |
| 7   | $b, b, b$ | $a, d, b$ | $\{a, b\}$ | $b_{10}$ | $\{a, d, e\}$ |
| 8   | $b, b$ | $b, e$ | $\{b\}$ | $b_{11}$ | $\{e\}$ |
| 9   | $b, b$ | $b, e$ | $\{b\}$ | $b_{11}$ | $\{b, e\}$ |
| 10  | $b$ | $b, e$ | $\{b\}$ | $b_{12}$ | $\{e\}$ |
| 11  | $\|$ | $\|$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 12  | $\|$ | $\|$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
by first removing bin \( b_{\pi(i)} \), and afterwards the bins \( b_{\pi(i-1)}, \ldots, b_{\pi(1)} \) in the given order. Obviously, in each configuration during the modified processing there are at most \( p \) pallets open. To remove first some bin of an already open pallet is a kind of priority rule.

A configuration \((Q, Q')\) is called a decision configuration, if the first bin of each sequence \( q' \in Q' \) is destined for a non-open pallet, see configurations \((Q, Q_0)\), \((Q, Q_1)\), \((Q, Q_2)\), \((Q, Q_4)\), and \((Q, Q_8)\) in Table1 i.e. \( \text{front}(Q') \cap \text{open}(Q, Q') = \emptyset \). We can restrict FIFO stack-up algorithms to deal with such decision configurations, in all other configurations the algorithms automatically remove a bin for some already open pallet.

If we have a pallet solution computed by some FIFO stack-up algorithm, we need to convert the pallet solution into a sequence of transformation steps, i.e. a processing of \( Q \). This is done by algorithm Transform in Figure 3. Given a list of sequences \( Q = (q_1, \ldots, q_k) \) and a pallet solution \( T = (t_1, \ldots, t_m) \) algorithm Transform gives us a bin solution of \( Q \), i.e. a processing of \( Q \).

**Algorithm Transform**

\[
\begin{align*}
q'_1 & := q_1, \ldots, q'_k := q_k \\
n & := 1 \\
T' & := \{t_1\}
\end{align*}
\]

repeat the following steps until \( q'_1 = \emptyset, \ldots, q'_k = \emptyset \):

1. if there is a sequence \( q'_i \) such that the first bin \( b \in q'_j \) is destined for a pallet in \( T' \), i.e. \( plt(b) \in T' \), then remove bin \( b \) from sequence \( q'_i \); and output \( b \)
   
   Comment: Bins for already open pallets are removed automatically.

2. otherwise set \( n := n + 1 \) and \( T' := T' \cup \{t_n\} \)
   
   Comment: If the first bin of each subsequence \( q'_i \) is destined for a non-open pallet, the next pallet of the pallet solution has to be opened.

Figure 3: Algorithm for transforming a pallet solution into a bin solution.

Obviously, there is no other processing of \( Q \) that also defines pallet solution \( T \) but takes less stack-up places.

## 3 Digraph Models

In this section, we introduce two digraph models which are useful to give solutions for the FIFO stack-up problem.

### 3.1 The processing graph

Our aim in controlling FIFO stack-up systems is to compute a processing of the given sequences of bins with a minimum number of stack-up places. Such an optimal processing can always be found by computing the processing graph and doing some calculation on it. Before we define the processing graph let us consider some general graph problem, that is strongly related to the FIFO stack-up problem.

Let \( G = (V, E, f) \) be a directed acyclic vertex-labeled graph. Function \( f : V \to \mathbb{Z} \) assigns to every vertex \( v \in V \) a value \( f(v) \). Let \( s \in V \) and \( t \in V \) be two vertices, where \( s \) stands for source, and \( t \) stands for target. For some vertex \( v \in V \) and some path \( P = (v_1, \ldots, v_l) \) with \( v_1 = s, v_l = v \) and \( (v_i, v_{i+1}) \in E \) we define

\[
val_P(v) := \max_{u \in P} f(u)
\]

to be the maximum value of \( f \) on that path. Let \( \mathcal{P}_s(v) \) denote the set of all paths from vertex \( s \) to vertex \( v \). Then we define

\[
val(v) := \min_{P \in \mathcal{P}_s(v)} (val_P(v)).
\]

If \( v \) is not reachable from \( s \), we define \( val(v) = \infty \). The problem is to compute the value \( val(t) \). A solution of this problem can be found by dynamic programming and solves also the FIFO stack-up
problem. The reason for this is given shortly. For some vertex \( v \in V \) let
\[ N^-(v) := \{ u \in V \mid (u, v) \in E \} \]
be the set of predecessors of \( v \) in digraph \( G \). The acronym \( N^- \) stands for in-neighborhood. Then the following holds:
\[ \text{val}(v) = \max \{ f(v), \min_{u \in N^-(v)} \text{val}(u) \} \]
That means, the value of \( \text{val}(v) \) can be processed recursively. But, if we would do that, subproblems often would be solved several times. So, we use dynamic programming to solve each subproblem only once, and put these solutions together to a solution of the original problem. This is possible, since the graph is directed and acyclic.

Let \( \text{topol} : V \rightarrow \mathbb{N} \) be an ordering of the vertices of digraph \( G \) such that \( \text{topol}(u) < \text{topol}(v) \) holds for each \( (u, v) \in E \). That means, \( \text{topol} \) is a topological ordering of the vertices.

**Algorithm Opt**

\[
\begin{align*}
\text{val}[s] & := f(s) \\
\text{for every vertex } v \neq s \text{ in order of } \text{topol} & \text{ do}
\begin{align*}
\text{val}[v] & := \infty \\
\quad \text{for every } u \in N^-(v) & \text{ do}
\begin{align*}
\quad \text{if } (\text{val}[u] < \text{val}[v]) & \text{ then}
\begin{align*}
\quad \quad \text{val}[v] & := \text{val}[u] \\
\quad \quad \text{pred}[v] & := u
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\]

\[ \text{if } (\text{val}[v] < f(v)) \text{ then } \text{val}[v] := f(v) \]

Figure 4: Finding an optimal processing by dynamic programming.

The value \( \text{val}(t) \) can be computed in polynomial time by Algorithm Opt given in Figure 4. A topological ordering of the vertices of digraph \( G \) can be found by a depth first search algorithm in time \( O(|V| + |E|) \). The remaining work of algorithm Opt can also be done in time \( O(|V| + |E|) \subseteq O(|V|^2) \), i.e. linear in the input size.

We need some additional terms, before we show how algorithm Opt can solve the FIFO stack-up problem. For a sequence \( q = (b_1, \ldots, b_n) \) let
\[ \text{left}(q, i) := (b_1, \ldots, b_i) \]
de note the sequence of the first \( i \) bins of sequence \( q \), and let
\[ \text{right}(q, i) := (b_{i+1}, \ldots, b_n) \]
de note the remaining bins of sequence \( q \) after removing the first \( i \) bins. Consider again Example 2.1 and Table 1. It can be seen that a configuration is well-defined by the number of bins that are removed from each sequence. Configuration \((Q, Q_6)\) can be described equivalently by the tuple \((2, 4)\), since in configuration \((Q, Q_6)\) 2 bins of sequence \( q_1 \) have been removed, and 4 bins of sequence \( q_2 \) have been removed.

The position of the first bin in some sequence \( q_i \) destined for some pallet \( t \) is denoted by \( \text{first}(q_i, t) \), similarly the position of the last bin for pallet \( t \) in sequence \( q_i \) is denoted by \( \text{last}(q_i, t) \).

**Example 3.1** Consider list \( Q = (q_1, q_2) \) of the sequences
\[ q_1 = [a, b, c, a, b, c] \]
and

$$q_2 = [d, e, f, d, e, f, a, b, c].$$

Then we get $left(q_1, 2) = [a, b]$, $right(q_1, 2) = [c, a, b, c]$, $left(q_2, 3) = [d, e, f]$, and $right(q_2, 3) = [d, e, f, a, b, c]$.

If we denote $q'_1 := right(q_1, 2)$, and $q'_2 := right(q_2, 3)$, then there are 5 pallets open in configuration $(Q, Q')$ with $Q' = (q'_1, q'_2)$, namely the pallets $a, b, d, e,$ and $f$.

We generalize this for a list $Q = (q_1, \ldots, q_k)$ of sequences and we define the cut

$$cut_Q(i_1, \ldots, i_k) := \{ t \in \text{plts}(Q) \mid \exists j, j', b \in left(q_j, i_j), b' \in right(q_{j'}, i_{j'}) : \text{plt}(b) = \text{plt}(b') = t\}$$

at some configuration $(i_1, \ldots, i_k)$ to be the set of pallets $t$ such that one bin for pallet $t$ has already been removed and another bin for pallet $t$ is still contained in some sequence. Let $\#cut_Q(i_1, \ldots, i_k)$ be the number of elements in $cut_Q(i_1, \ldots, i_k)$.

The intention of a processing graph $G = (V, E, f)$ is the following. Suppose each vertex $v \in V$ of digraph $G$ represents a configuration $(i_1, \ldots, i_k)$ during the processing of some set of sequences $Q$. Suppose further, an arc $(u, v) \in E$ represents a transformation step during this processing, such that a bin $b$ is removed from some sequence in configuration $u$ resulting in configuration $v$. Suppose also that each vertex $v$ is assigned the number of open pallets in configuration $v$, i.e. number $\#cut_Q(i_1, \ldots, i_k)$. If vertex $s$ represents the initial configuration $(0, 0, \ldots, 0)$, while vertex $t$ represents the final configuration $(|q_1|, |q_2|, \ldots, |q_k|)$, then we are searching a path $P$ from $s$ to $t$ such that the maximal number on path $P$ is minimal. Thus, an optimal processing of $Q$ can be found by Algorithm $\text{OPT}$ given in Figure 3

The processing graph has a vertex for each possible configuration. Each vertex $v$ of the processing graph is labeled by the vector $h(v) = (v_1, \ldots, v_k)$, where $v_i$ denotes the position of the bin that has been removed last from sequence $q_i$. Further each vertex $v$ of the processing graph is labeled by the value $f(v) = \#cut_Q(v_1, \ldots, v_k) = \#cut_Q(h(v))$. There is an arc from vertex $u$ labeled by $(u_1, \ldots, u_k)$ to vertex $v$ labeled by $(v_1, \ldots, v_k)$ if and only if $u_i = v_i - 1$ for exactly one element of the vector and for all other elements of the vector $u_j = v_j$. The arc is labeled with the pallet symbol of that bin, that will be removed in the corresponding transformation step. That means, the processing graph represents all the possible branchings through the decision process. For the sequences of Example 3.1 we get the processing graph of Figure 3. The processing graph is directed and acyclic, and we use this digraph to compute the values of $\#cut_Q(i_1, \ldots, i_k)$ iteratively in the following way.

![Processing Graph](attachment:image.png)

Figure 5: The processing graph of Example 3.1. For every vertex $v$ the vector $h(v)$ in the vertex represents the configuration of $v$ and every arc is labeled with the pallet symbol of that bin, that is removed in the corresponding transformation step.
First, since none of the bins has been removed from some sequence, we have \( \#\text{cut}_Q(0,\ldots,0) = 0 \). Since the processing graph is directed and acyclic, there exists a topological ordering \( \text{topol} \) of the vertices. The vertices are processed according to the order \( \text{topol} \). In each transformation step we remove exactly one bin for some pallet \( t \) from some sequence \( q_j \), thus

\[
\#\text{cut}_Q(i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_k) = \#\text{cut}_Q(i_1,\ldots,i_{j-1},i_j-1,i_{j+1},\ldots,i_k) + c_j
\]

where

\[
c_j = \begin{cases} 
1, & \text{if } \text{first}(q_j,t) = i_j \text{ and } (t \notin \text{plts}(q_i) \text{ or } \text{first}(q_i,t) > i_i) \forall i \neq j \\
-1, & \text{if } \text{last}(q_j,t) = i_j \text{ and } (t \notin \text{plts}(q_i) \text{ or } \text{last}(q_i,t) \leq i_i) \forall i \neq j \\
0, & \text{otherwise}. 
\end{cases}
\]

That means, \( c_j = 1 \) if pallet \( t \) has been opened in the last transformation step, and \( c_j = -1 \) if pallet \( t \) has been closed in the last transformation step. Otherwise, \( c_j \) is zero. That means, the calculation of value \( \#\text{cut}_Q(i_1,\ldots,i_k) \) for the vertex labeled \((i_1,\ldots,i_k)\) depends only on already calculated values. Figure 6 shows such a processing for the sequences of Example 3.1.

![Figure 6: The processing graph of Example 3.1](image)

For every vertex \( v \) the set in the vertex \( v \) shows the open pallets, the vector \( h(v) \) in the vertex represents the configuration of \( v \), and the number \( f(v) \) on the left of the vertex represents the number of open pallets in the corresponding configuration \( h(v) \).

To efficiently perform the processing of \( c_j \), we have to store for each pallet the first and last bin in each sequence. We use hash-tables to efficiently store such values without the need of initializing the values of absent pallets. Table 2 shows such hash-tables for the sequences of Example 3.1. The calculation of those tables can be done in linear time \( O(|q_1| + \ldots + |q_k|) = O(k \cdot N) \), where \( N = \max\{|q_1|,\ldots,|q_k|\} \) denotes the maximum sequence length. Afterwards, the computation of each value \( c_j \) can be done in time \( O(k) \).

| pallet \( t \) | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) |
|---------------|-------|-------|-------|-------|-------|-------|
| \( \text{first}(q_1,t) \) | 1     | 2     | 3     | .     | .     | .     |
| \( \text{first}(q_2,t) \) | 7     | 8     | 9     | 1     | 2     | 3     |
| \( \text{last}(q_1,t) \)  | 4     | 5     | 6     | .     | .     | .     |
| \( \text{last}(q_2,t) \)  | 7     | 8     | 9     | 4     | 5     | 6     |

Table 2: Additional hash-tables to perform the computation of \( c_j \) efficiently.
The computation of all at most \((N + 1)^k\) values \(#\text{cut}_Q(i_1, \ldots, i_k)\) can be performed in time \(O(k \cdot (N + 1)^k)\). After that we can use Algorithm Opt to compute the minimal number of stack-up places necessary to process the given FIFO stack-up problem in time \(O((N + 1)^{2k})\). If the size of the processing graph is polynomially bounded in the size of the input, the FIFO stack-up problem can be solved in polynomial time. In general the processing graph can be too large to efficiently solve the problem.

3.2 The Sequence Graph

Next we show a correlation between the used number of stack-up places for a processing of an instance \(Q\) and the directed pathwidth of a digraph \(G_Q\) defined by \(Q\). The notion of directed pathwidth (directed treewidth) was introduced by Johnson, Robertson, Seymour, and Thomas in [7].

This correlation implies that 1.) the decision version of the FIFO stack-up problem is NP-complete and 2.) a pallet solution can be computed in polynomial time if there are only a fixed number of stack-up places.

A directed path-decomposition of a digraph \(G = (V, E)\) is a sequence 
\[\langle X_1, \ldots, X_r \rangle\]
of subsets of \(V\), called bags, that satisfy the following three properties.

\begin{enumerate}[\textbf{(dpw-1)}]
\item \(X_1 \cup \ldots \cup X_r = V\)
\item for each arc \((u, v) \in E\) there are indices \(i, j\) with \(i \leq j\) such that \(u \in X_i\) and \(v \in X_j\)
\item if \(u \in X_i\) and \(u \in X_j\) for some node \(u\) and two indices \(i, j\) with \(i \leq j\), then \(u \in X_l\) for all indices \(l\) with \(i \leq l \leq j\)
\end{enumerate}

The width of a directed path-decomposition \(\mathcal{X} = \langle X_1, \ldots, X_r \rangle\) is
\[\max_{1 \leq i \leq r} |X_i| - 1.\]

The directed pathwidth of \(G\), d-pw\((G)\) for short, is the smallest integer \(k\) such that there is a directed path-decomposition for \(G\) of width \(k\). For symmetric digraphs, the directed pathwidth is equivalent to the undirected pathwidth of the corresponding undirected graph [8]. For each fixed integer \(k\), it is decidable in polynomial time whether a given digraph has directed pathwidth at most \(k\), see Tamaki [14].

The sequence graph \(G_Q = (V, E)\) for an instance \(Q = (q_1, \ldots, q_k)\) of the FIFO stack-up problem is defined by vertex set \(V = \text{plts}(Q)\) and the following set of arcs. There is an arc \((u, v) \in E\) if and only if there is a sequence \(q_i = (b_{n_{i-1}+1}, \ldots, b_{n_i})\) with two bins \(b_{j_1}, b_{j_2}\) such that
\begin{enumerate}
\item \(j_1 < j_2\),
\item \(b_{j_1}\) is destined for pallet \(u\),
\item \(b_{j_2}\) is destined for pallet \(v\), and
\item \(u \neq v\).
\end{enumerate}

If \(G_Q = (V, E)\) has an arc \((u, v) \in E\) then \(u \neq v\) and for every processing of \(Q\), pallet \(u\) is opened before pallet \(v\) is closed. Digraph \(G_Q = (V, E)\) can obviously be computed in optimal time, i.e., in time \(O(n + |E|) \subseteq O(n + m^2)\).

\textbf{Example 3.2} Figure [7] shows the sequence graph \(G_Q\) for \(Q = (q_1, q_2, q_3)\) with sequences \(q_1 = [a, a, d, e, d]\), \(q_2 = [b, b, d]\), and \(q_3 = [c, c, d, e, d]\).
**Theorem 3.3** A processing \((Q, Q_0), (Q, Q_1), \ldots, (Q, Q_l)\) of \(Q\) with \(Q_0 = Q, Q_l = (\emptyset, \ldots, \emptyset)\), and \(p\) stack-up places defines a directed path-decomposition

\[\mathcal{X} = (\text{open}(Q, Q_0), \ldots, \text{open}(Q, Q_l))\]

for \(G_Q\) of width \(p - 1\).

**Proof** We show that \(\mathcal{X} = (\text{open}(Q, Q_0), \ldots, \text{open}(Q, Q_l))\) satisfies all properties of a directed path-decomposition.

**(dpw-1)** \(\text{open}(Q, Q_0) \cup \cdots \cup \text{open}(Q, Q_l) = \text{plts}(Q)\), because every pallet is opened at least once.

**(dpw-2)** If \((u, v) \in E\) then there are indices \(i, j\) with \(i \leq j\) such that \(u \in \text{open}(Q, Q_i)\) and \(v \in \text{open}(Q, Q_j)\), because \(v\) can not be closed before \(u\) is opened.

**(dpw-3)** If \(u \in \text{open}(Q, Q_i)\) and \(u \in \text{open}(Q, Q_j)\) for some pallet \(u\) and two indices \(i, j\) with \(i \leq j\), then \(u \in \text{open}(Q, Q_l)\) for all indices \(l\) with \(i \leq l \leq j\), because every pallet is opened at most once.

Since \(Q\) is processed with \(p\) stack-up places, we have \(|\text{open}(Q, Q_i)| \leq p\) for \(0 \leq i \leq l\), and therefore \(\mathcal{X}\) has width at most \(p - 1\).

**Theorem 3.4** If there is a path-decomposition \(\mathcal{X} = (X_1, \ldots, X_r)\) for \(G_Q\) of width \(p - 1\) then there is a processing of \(Q\) with \(p\) stack-up places.

**Proof** For a pallet \(t\) let \(\alpha(\mathcal{X}, t)\) be the smallest \(i\) such that \(t \in X_i\) and \(\beta(\mathcal{X}, t)\) be the largest \(i\) such that \(t \in X_i\), see Example 3.5. Then \(t \in X_i\) if and only if \(\alpha(\mathcal{X}, t) \leq i \leq \beta(\mathcal{X}, t)\). If \((t_1, t_2)\) is an arc of \(G_Q\), then \(\alpha(\mathcal{X}, t_1) \leq \beta(\mathcal{X}, t_2)\). This follows by (dpw-2) of the definition of a directed path-decomposition.

Instance \(Q\) can be processed as follows. If it is necessary to open in a configuration \((Q, Q')\) a new pallet, then we open a pallet \(t\) of \(\text{front}(Q')\) for which \(\alpha(\mathcal{X}, t)\) is minimal.

We next show that for every configuration \((Q, Q')\) of the processing above there is a bag \(X_i\) in the directed path-decomposition \(\mathcal{X} = (X_1, \ldots, X_r)\), such that \(\text{open}(Q, Q') \subseteq X_i\). This implies that the processing uses at most \(p\) stack-up places.

First, let \((Q, Q')\) be a decision configuration, let \(t \in \text{front}(Q')\) such that \(\alpha(\mathcal{X}, t)\) is minimal, and let \(t' \in \text{open}(Q, Q')\) be an already open pallet. We show that the intervals \([\alpha(\mathcal{X}, t), \beta(\mathcal{X}, t)]\) and \([\alpha(\mathcal{X}, t'), \beta(\mathcal{X}, t')\) overlap, because \(\alpha(\mathcal{X}, t) \leq \beta(\mathcal{X}, t')\) and \(\alpha(\mathcal{X}, t') \leq \beta(\mathcal{X}, t)\).

1. \(\alpha(\mathcal{X}, t) \leq \beta(\mathcal{X}, t')\):
   Since \(t' \notin \text{front}(Q')\), there has to be a pallet \(t'' \in \text{front}(Q')\) such that \((t'', t')\) is an arc of \(G_Q\). This implies that \(\alpha(\mathcal{X}, t'') \leq \beta(\mathcal{X}, t')\). Since \(t\) is a pallet of \(\text{front}(Q')\) for which \(\alpha(\mathcal{X}, t)\) is minimal, we have \(\alpha(\mathcal{X}, t) \leq \alpha(\mathcal{X}, t'')\) and thus \(\alpha(\mathcal{X}, t) \leq \beta(\mathcal{X}, t')\).
(2) $\alpha(\mathcal{X}, t') \leq \beta(\mathcal{X}, t)$:

Let $(Q, Q')$ be the configuration in that $t'$ has been opened.

(a) $t \in \text{front}(Q')$. Since $t'$ is a pallet of $\text{front}(Q')$ for which $\alpha(\mathcal{X}, t')$ is minimal, we have $\alpha(\mathcal{X}, t') \leq \alpha(\mathcal{X}, t)$ and thus $\alpha(\mathcal{X}, t') \leq \beta(\mathcal{X}, t)$.

(b) $t \notin \text{front}(Q')$. Then there is a pallet $t'' \in \text{front}(Q')$ such that $(t'', t)$ is an arc of $G_Q$. This implies that $\alpha(\mathcal{X}, t'') \leq \beta(\mathcal{X}, t)$. Since $t'$ is a pallet of $\text{front}(Q')$ for which $\alpha(\mathcal{X}, t')$ is minimal, we have $\alpha(\mathcal{X}, t') \leq \alpha(\mathcal{X}, t'') \leq \beta(\mathcal{X}, t)$.

Finally, let $(Q, \hat{Q})$ be an arbitrary configuration of the processing of $Q$. By the discussion above, we can conclude that for every pair $t_i, t_j \in \text{open}(Q, \hat{Q})$ the intervals $[\alpha(\mathcal{X}, t_i), \beta(\mathcal{X}, t_i)]$ and $[\alpha(\mathcal{X}, t_j), \beta(\mathcal{X}, t_j)]$ overlap, because $t_i$ is opened before $t_j$ or vice versa. Since the cut of all intervals $[\alpha(\mathcal{X}, t_i), \beta(\mathcal{X}, t_i)]$, $t_i \in \text{open}(Q, \hat{Q})$ mutually overlap, there is a bag $X_j$ in the directed path-decomposition $\mathcal{X}$ such that $\text{open}(Q, \hat{Q}) \subseteq X_j$. \hfill $\Box$

**Example 3.5** We consider digraph $G_Q$ for $Q = (q_1, q_2, q_3)$ with sequences $q_1 = [a, a, d, e, d]$, $q_2 = [b, b, d]$, and $q_3 = [c, c, d, e, d]$ from Example 3.2. Sequence $\{(a, e), \{b, e\}, \{c, e\}, \{d, e\}\}$ is a directed path-decomposition of width 1, which implies the following values for $\alpha$ and $\beta$ used in the proof of Theorem 3.4.

| pallet t | \(\alpha(\mathcal{X}, t)\) | \(\beta(\mathcal{X}, t)\) |
|----------|----------------|----------------|
|          | 1   | 2   | 3   | 4   | 1   |

Table 3 shows a processing of $Q$ with 2 stack-up places and pallet solution $S = (a, b, c, d, e)$. The underlined bin is always the bin that will be removed in the next transformation step. We denote $Q_i = (q_1^i, q_2^i, q_3^i)$, thus each row represents a configuration $(Q, Q_i)$.

| $i$ | $q_1^i$ | $q_2^i$ | $q_3^i$ | \(\text{front}(Q')\) | \(\text{open}(Q, Q_i)\) |
|-----|---------|---------|---------|----------------|----------------|
| 0   | $[a, a, d, e, d]$ | $[b, b, d]$ | $[c, c, d, e, d]$ | \{a, b, c\} | $\emptyset$ |
| 1   | $[a, a, d, e, d]$ | $[b, b, d]$ | $[c, c, d, e, d]$ | $\{a\}$ | $\emptyset$ |
| 2   | $[d, e, d]$ | $[b, b, d]$ | $[c, c, d, e, d]$ | $\{b, c, d\}$ | $\emptyset$ |
| 3   | $[d, e, d]$ | $[b, b, d]$ | $[c, c, d, e, d]$ | $\{b, c, d\}$ | $\{b\}$ |
| 4   | $[d, e, d]$ | $[c, c, d, e, d]$ | $[c, c, d, e, d]$ | $\{c\}$ | $\emptyset$ |
| 5   | $[d, e, d]$ | $[c, c, d, e, d]$ | $[c, c, d, e, d]$ | $\{c\}$ | $\emptyset$ |
| 6   | $[d, e, d]$ | $[d, e, d]$ | $[d, e, d]$ | $\{d\}$ | $\emptyset$ |
| 7   | $[e, d]$ | $[e, d]$ | $[e, d]$ | $\{e\}$ | $\{d\}$ |
| 8   | $[e, d]$ | $[e, d]$ | $[e, d]$ | $\{e\}$ | $\{d\}$ |
| 9   | $[e, d]$ | $[e, d]$ | $[e, d]$ | $\{e\}$ | $\{d\}$ |
| 10  | $[e, d]$ | $[d, e]$ | $[d, e]$ | $\{d, e\}$ | $\emptyset$ |
| 11  | $[e, d]$ | $[d, e]$ | $[d, e]$ | $\{d, e\}$ | $\emptyset$ |
| 12  | $[d]$ | $[d]$ | $[d]$ | $\{d\}$ | $\emptyset$ |
| 13  | $[d]$ | $[d]$ | $[d]$ | $\{d\}$ | $\emptyset$ |

Table 3: A processing of $Q$ with respect to a given directed path-decomposition for $G_Q$ of Example 3.5

**4 Hardness Result**

Let $G = (V, E)$ be a digraph. We will assume that $G = (V, E)$ does not contain any vertex with only outgoing arcs and not contain any vertex with only incoming arcs. This is only for technical
reasons and the removal of such vertices will not change the directed pathwidth of $G$, because a vertex $u$ with only outgoing arcs can be placed in a singleton $X_i = \{u\}$ at the beginning of the directed path-decomposition and a vertex $u$ with only incoming arcs can be placed in a singleton $X_i = \{u\}$ at the end of the directed path-decomposition, without to change its width.

The queue system $Q_G = (q_1, \ldots, q_l)$ for some given digraph $G = (V, E)$ is defined as follows.
1. $\text{plts}(Q) = V$.
2. There is a queue $q_i = (b_{2i-1}, b_{2i})$, $1 \leq i \leq l$, in $Q$ if and only if there is a directed edge $(\text{plt}(b_{2i-1}), \text{plt}(b_{2i})) \in E$.

The definition of queue system $Q_G$ and sequence graph $G_Q$, defined in Section 3.2, now imply the following proposition.

**Proposition 4.1** For every digraph $G$, $G = G_{Q_G}$.

**Example 4.2** Consider the digraph $G$ of Figure 8. The corresponding queue system is $Q_G = (q_1, q_2, q_3, q_4, q_5, q_6, q_7)$, where
$q_1 = [a, b]$, $q_2 = [b, c]$, $q_3 = [c, d]$, $q_4 = [d, e]$, $q_5 = [e, a]$, $q_6 = [e, f]$, $q_7 = [f, a]$.

The sequence graph of $Q_G$ is $G$.

![Figure 8: Digraph G of Example 4.2](image)

**Lemma 4.3** There is a directed path-decomposition for $G$ of width $p - 1$ if and only if there is a processing of $Q_G$ with at most $p$ stack-up places.

**Proof** By Proposition 4.1 we know that $G = G_{Q_G}$. If there is a directed path-decomposition for $G = G_{Q_G}$ of width $p - 1$ then by Theorem 3.3 there is a processing of $Q_G$ with at most $p$ stack-up places. If there is a processing of $Q_G$ with at most $p$ stack-up places then by Theorem 3.3 there is directed path-decomposition for $G$ of width $p - 1$. □

**Theorem 4.4** Given a list $Q = (q_1, \ldots, q_k)$ of $k$ sequences of bins and some positive integer $p$. The problem to decide whether there is a processing of $Q$ with at most $p$ stack-up places is NP-complete.

**Proof** The problem is obviously in NP. The NP-hardness follows from Lemma 4.3 because $Q_G$ can be constructed from $G$ in linear time. □

According to [1] it is shown in [9] that determining whether the pathwidth of some given (undirected) graph $G$ is at most some given value $w$ remains NP-complete even for planar graphs with maximum vertex degree 3. Thus the problem to decide, whether the directed pathwidth of some given symmetric digraph $G$ is at most some given value $w$ remains NP-complete even for planar digraphs with maximum vertex in-degree 3 and maximum vertex out-degree 3. Thus, by our transformation of graph $G$ into $Q_G$ we get sequences that contain together at most 6 bins per pallet. Hence, the FIFO stack-up problem is NP-complete even if the number of bins per pallet is bounded. Therefore we have proved the following statement.
Corollary 4.5 Given a list \( Q = (q_1, \ldots, q_k) \) of \( k \) sequences of bins and some positive integer \( p \). The problem to decide whether there is a processing of \( Q \) with at most \( p \) stack-up places is \( NP \)-complete, even if the sequences of \( Q \) contain together at most 6 bins per pallet.

5 Bounded FIFO stack-up systems

In this section we show that the FIFO stack-up problem can be solved in polynomial time, if the number \( k \) of sequences or the number \( p \) of stack-up places is assumed to be fixed.

5.1 Fixed number of sequences

In this section we assume that the number \( k \) of sequences is fixed.

In Section 3.1 we have shown that the FIFO stack-up problem can be solved by dynamic programming using the processing graph in time \( O((N+1)^{2k}) \). Thus, we have already shown the following result.

Theorem 5.1 Given a list \( Q = (q_1, \ldots, q_k) \) of sequences of bins for some fixed \( k \) and a number of stack-up places \( p \). The question whether the sequences can be processed with at most \( p \) stack-up places can be solved in polynomial time.

Next we improve this result.

Theorem 5.2 Given a list \( Q = (q_1, \ldots, q_k) \) of sequences of bins for some fixed \( k \) and a number of stack-up places \( p \). The question whether the sequences can be processed with at most \( p \) stack-up places is non-deterministically decidable using logarithmic work-space.

Proof We need \( k+1 \) variables, namely \( pos_1, \ldots, pos_k \) and \( open \). Each variable \( pos_i \) is used to store the position of the bin which has been removed last from sequence \( q_i \). Variable \( open \) is used to store the number of open pallets. These variables take \((k+1) \cdot \lceil \log(n) \rceil \) bits. The simulation starts with \( pos_1 := 0, \ldots, pos_k := 0 \) and \( open := 0 \).

(i) Choose any index \( i \) and increment variable \( pos_i \). Let \( b \) be the bin on position \( pos_i \) in sequence \( q_i \), and let \( t := \text{plt}(b) \) be the pallet symbol of bin \( b \).
   Comment: The next bin \( b \) from some sequence \( q_i \) will be removed.

(ii) If \( \text{first}(q_j, t) > pos_j \) or \( t \not\in \text{plts}(q_j) \) for each \( j \neq i, 1 \leq j \leq k \), and \( \text{first}(q_i, t) = pos_i \), then increment variable \( open \).
   Comment: If the removed bin \( b \) was the first bin of pallet \( t \) that ever has been removed from any sequence, then pallet \( t \) has just been opened.

(iii) If \( \text{last}(q_j, t) \leq pos_j \) or \( t \not\in \text{plts}(q_j) \) for each \( j, 1 \leq j \leq k \), then decrement variable \( open \).
   Comment: If bin \( b \) was the last one of pallet \( t \), then pallet \( t \) has just been closed.

If \( open \) is set to a value greater than \( p \) in Step (ii) of the algorithm then the execution is stopped in a non-accepting state. To execute Steps (ii) and (iii) we need a fixed number of additional variables. Thus, all steps can be executed non-deterministically using logarithmic work-space. \( \square \)

Theorem 5.2 implies that the FIFO stack-up problem with a fixed number of given sequences can be solved in polynomial time since NL is a subset of \( P \). The class NL is the set of problems decidable non-deterministically on logarithmic work-space. For example, reachability of nodes in a given graph is NL-complete, see [10]. Even more, it can be solved in parallel in polylogarithmic time with polynomial amount of total work, since NL is a subset of NC\(_2\). The class NC\(_2\) is the set of problems decidable in time \( O(\log^2(n)) \) on a parallel computer with a polynomial number of processors, see [10].
5.2 Fixed number of stack-up places

In [14] it is shown that the problem of determining the bounded directed pathwidth of a digraph is solvable in polynomial time. By Lemma 4.3 the FIFO stack-up problem with fixed number \( p \) of stack-up places is also solvable in polynomial time.

**Theorem 5.3** Given a list \( Q = (q_1, \ldots, q_k) \) of \( k \) sequences of bins and a fixed number of stack-up places \( p \). The question whether the sequences can be processed with at most \( p \) stack-up places can be solved in polynomial time.

6 Conclusion

In this paper, we have shown that the FIFO stack-up problem is NP-complete in general, even if in all sequences together there are at most 6 bins destined for the same pallet. The problem can be solved in polynomial time, if the number \( k \) of sequences or if the number \( p \) of stack-up places is assumed to be fixed.

In our future work, we want to find an answer to the very natural question, whether the FIFO stack-up problem is fixed-parameter tractable (fpt) when parameterized by the number \( k \) of sequences or when parameterized by the number \( p \) of stack-up places (cf. [5] for the definition of fixed-parameter tractability). We also want to analyze the existence of approximation algorithms for minimizing the number of stack-up places.

We are also interested in online algorithms for instances where we only know the first \( c \) bins of every sequence instead of the complete sequences. Especially, we are interested in the answer to the following question: Is there a \( d \)-competitive online algorithm? Such an algorithm must compute a processing of some \( Q \) with at most \( p \cdot d \) stack-up places, if \( Q \) can be processed with at most \( p \) stack-up places.

In real life the bins arrive at the stack-up system on the main conveyor of a pick-to-belt orderpicking system. That means, the distribution of bins to the sequences has to be computed. Up to now we consider the distribution as given. We intend to consider how to compute an optimal distribution of the bins from the main conveyor onto the sequences such that a minimum number of stack-up places is necessary to stack-up all bins from the sequences.

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