Berezin-type quantization on even-dimensional compact manifolds

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Abstract. In this article we show that a Berezin-type quantization can be achieved on a compact even dimensional manifold $M^{2d}$ by removing a skeleton $M_0$ of lower dimension such that what remains is diffeomorphic to $\mathbb{R}^{2d}$ (cell decomposition) which we identify with $\mathbb{C}^d$ and embed in $CP^d$. A local Poisson structure and Berezin-type quantization are induced from $CP^d$. Thus we have a Hilbert space with a reproducing kernel. The symbols of bounded linear operators on the Hilbert space have a star product which satisfies the correspondence principle outside a set of measure zero. This construction depends on the diffeomorphism. One needs to keep track of the global holonomy and hence the cell decomposition of the manifold. As an example, we illustrate this type of quantization of the torus. We exhibit Berezin-Toeplitz quantization of a complex manifold in the same spirit as above.

1. Introduction

Berezin quantization \cite{1} is a method of defining a star product on the symbol of operators acting on a Hilbert space (with a reproducing kernel) on a Kähler manifold under certain conditions such that the star product satisfies the correspondence principle. The literature on subsequent work after \cite{1} on Berezin quantization is vast. We mention that in \cite{6} the conditions have been relaxed considerably. Another direction this field has expanded is Berezin-Toeplitz quantization, see for instance \cite{2}, \cite{9}.

Some quantum systems donot come from quantizing classical systems (which are expected to have a symplectic structure) but there is a semi-classical limit of the quantum system. For instance there is a semi-classical limit of the quantum system of spin, see for instance, \cite{12} ($S \rightarrow \infty$ in Radcliffe’s notation). We wish to include systems which donot have symplectic structure (or group action) and study if they are a semi-classical limit of some quantum system as $\hbar$ goes to zero. This is the motivation of considering manifolds which have no symplectic or Poisson structure. We induce local Poisson structure on the manifold by embedding parts of it (i.e. removing sets of measure zero) into $CP^n$ or $\mathbb{C}^n$ (depending on whether we expect a finite dimensional or an infinite dimensional Hilbert space) and induce the Berezin quantization from one of these two spaces.

The other motivation of the work is that sometimes the Hilbert space of the problem turns out to be different from what the actual manifold of parameter space should prescribe. The Hilbert space could be just obtained from geometric quantization of $\mathbb{C}^n$, or $CP^n$, whereas the parameter
space is not $C^n$ or $CP^n$. Roughly speaking in these two cases (namely $C^n$ or $CP^n$), the Hilbert space consists of polynomials. For some situations this could be at least a good approximation, for example the Quantum Hall Effect (where polynomials suffice for lowest Landau levels [17]). The global holonomy needs to be calculated, which we explain in the example of the torus.

In this article we show that a Berezin quantization can be achieved on a compact even dimensional manifold $M^{2d}$ by removing a skeleton of lower dimension such that what remains is diffeomorphic to $R^{2d}$ which we identify with $C^d$ and embed in $CP^d$. We get an induced Berezin quantization from $CP^d$. In other words, we obtain a Hilbert space with a reproducing kernel and a star product on the symbol of bounded linear operators on the Hilbert space which satisfy the correspondence principle. The Berezin quantization depends on the diffeomorphism of $M \setminus M_0$ to $R^{2d}$ but if we choose a different diffeomorphism of $M \setminus M_0$ to $R^{2d}$ then we obtain a quantization with another reproducing kernel with star product on symbols which satisfy the correspondence principle. These two quantizations need not be equivalent in the sense that there may not be a natural map between the Hilbert spaces which preserve the reproducing kernel.

The set of measure zero which we remove is the lower dimensional skeleton in cell decomposition so that what remains is a top dimensional cell which we identify with $U_0 \subset CP^n$, one of the homogeneous charts. We pull back the polynomials on $U_0$ to $X$ for the quantization. However we have to keep track of the cell decomposition because of global holonomy. The loops may pass through the sets of measure zero in $M_0$ which we have removed. But this can be handled if we remember the lower dimensional skeleton we had removed. Thus, even though we remove a set of measure zero, it plays an important role in determining the global holonomy. We illustrate with the torus.

In this context we recall that in [4] we had considered totally real submanifolds of $CP^n$ and defined pull back operators and their $CP^n$-symbols and showed that they satisfied the correspondence principle.

In this article we also exhibit Berezin-Toeplitz quantization on a compact complex manifold. This work is part of Kohinoor Ghosh’s thesis [8].

It has many interesting applications in harmonic analysis and non-commutative geometry. This is work in progress.

Role of $CP^n$ or $C^n$ can be taken by other appropriate manifolds too.

2. Review of Berezin quantization on $CP^n$

This section is a review based on ideas from [1]. In Berezin [1], the quantization on $CP^n$ is achieved thinking of it as a homogeneous space. In this section we give an explicit path to the quantization using a local description.

Let $\Phi_{FS}$ be a local Kähler potential for the Fubini-Study Kähler form $\Omega_{FS}$ on $CP^n$. Let us recall how this looks in local coordinates.

Let $U_0 \subset CP^n$ given by $U_0 \equiv \{\mu_0 \neq 0\}$ where $[\mu_0, \ldots, \mu_n]$ are homogeneous coordinates on $CP^n$.

Let $\Phi_{FS}(\mu, \bar{\mu}) = \ln (1 + \sum_{i=1}^n |\mu|^2)$ be the Kähler potential and the Kähler metric $G$ is given by $g^{FS}_{\mu_j \bar{\mu}_j} = \frac{\partial^2 \Phi_{FS}}{\partial \mu_j \partial \bar{\mu}_j}$.

The Fubini-Study form is given by $\Omega_{FS} = \sum_{i,j=1}^n \Omega_{ij}^{FS} d\mu_i \wedge d\bar{\mu}_j$, where the Kähler metric $G$ and the Kähler form $\Omega_{FS}$ are related by $\Omega_{FS}(X, Y) = G (IX, Y)$.

The coefficients of the inverse matrix $\Omega^{-1}_{FS}$ appears in the definition of the Poisson bracket of two functions $t$ and $s$:

$$\{t, s\}_{FS} = \sum_{i,j=1}^n \Omega^{-1}_{ij}^{FS} \left( \frac{\partial t}{\partial \mu_i} \frac{\partial s}{\partial \bar{\mu}_j} - \frac{\partial t}{\partial \bar{\mu}_j} \frac{\partial s}{\partial \mu_i} \right).$$

Let $T = \{ (\mu, \nu) \in C^n \times C^n | \mu \cdot \bar{\nu} = -1\}$ and $S = (C^n \times C^n) \setminus T$. Note that the diagonal $\Delta \subset S$.

For $(\mu, \nu) \in S$, we can define (taking a branch of the logarithm) $\Phi_{FS}(\mu, \bar{\nu}) = \ln (1 + \mu \cdot \bar{\nu})$. 

2
Let $H^\otimes m$ be the $m$-th tensor product of the hyperplane bundle $H$ on $CP^n$. Then recall that $m\Omega_{FS}$ is its curvature form and $m\Phi_{FS}$ is a local Kähler potential. Let $\Gamma_{hol}$ be holomorphic sections on it. Let $\{\psi_i\}_{i=1}^N$ be an orthonormal basis for it. On $U_0$ the sections of $H^\otimes m$ are functions since the bundles are trivial when restricted to $U_0$.

Let $\hbar = \frac{1}{m}$ be a parameter. Thus each $\{\psi_i\}$ implicitly depend on $\hbar$. We define
\[
dV(\mu) = |\Omega_{FS}^\mu|_{U_0} = G(\mu)_{\mu=1}^{m} |d\mu| = G(\mu)|d\mu| = \frac{|d\mu|}{(\hbar + |d\mu|)^{m+1}}
\]
to be a volume form on $\mathcal{C}^m$, where $G = \det[g^{ij}(\mu)]$.

Let $V = \int_{C^m} dV = \int_{C^m} \frac{|d\mu|}{(\hbar + |d\mu|)^m} < \infty$.

Let $(\mu_1, \mu_2, \ldots, \mu_n)$ be coordinates on $U_0 \equiv \mathcal{C}^n$ such that $[1, \mu_1, \mu_2, \ldots, \mu_n] \in U_0$.

Let $(c(m))^{-1} = \int_{U_0} \frac{1}{(1 + |\nu|^2)^m} dV(\nu) = \int_{U_0} e^{-m\Phi_{FS}(\nu, \nu)} dV(\nu)$ where recall $e^{m\Phi_{FS}(\nu, \nu)} = (1 + |\nu|^2)^m$. Also, $D_{(q_1, q_2, \ldots, q_m)} = c(m) \int_{U_0} |\nu|^{q_1 + \cdots + q_m} dV(\nu)$, where $q_i$'s are all possible positive integers such that $q_1 + \cdots + q_n = q; q = 0, \ldots, m$.

Let $\Psi_{(q_1, q_2, \ldots, q_n, q)}(\mu) = \frac{1}{\sqrt{D_{(q_1, q_2, \ldots, q_n, q)}}} \mu^{q_1} \cdots \mu^{q_n}$ where $q_1 + \cdots + q_n = q; q = 0, \ldots, m$.

For shorthand we will use $I_q$ for the multi-index $I_q = (q_1, ..., q_n; q)$ which runs over the set $q_1 + \cdots + q_n = q; q = 0, \ldots, m$.

Then $D_I = c(m) \int_{U_0} \frac{|\nu|^{2I}}{(1 + |\nu|^2)^m} dV(\nu)$.

Let an innerproduct on the space of functions on $U_0$ be defined as
\[
\langle f, g \rangle = c(m) \int_{U_0} \frac{(\bar{f}(\nu)g(\nu))}{(1 + |\nu|^2)^m} dV(\nu) = \int_{U_0} \bar{f}(\nu)g(\nu)e^{-m\Phi_{FS}(\nu, \nu)} dV(\nu).
\]

It is easy to check that $\{\Psi_{(q_1, q_2, \ldots, q_n, q)}\}$ are orthonormal in $\mathcal{C}^m$ with respect to the inner product defined as above and are restriction of a basis for sections of $H^\otimes m$ to $U_0$. This forms a Hilbert space.

Let $N$ is the dimension of the Hilbert space, i.e $N = \sum_{J} (1)$ where $J$ runs over the indices $J = (p_1, \ldots, p_n, p)$, $p_1 + \cdots + p_n = p = 0, \ldots, m$ and $\mathcal{V} = \int_{U_0} |\Omega|^m$.

**Definition:** The Rawlsley-type coherent states $[13], \ [15]$ are given on $U_0$ by $\psi_{\mu}$ reading as follows:
\[
\psi_{\mu}(\nu) := \sum_{q_1 + q_2 + \cdots + q_n = q; q = 0, 1, \ldots, m} \langle \nu | \Psi_{(q_1, q_2, \ldots, q_n, q)} \rangle (\mu) \Psi_{(q_1, q_2, \ldots, q_n, q)}(\nu).\]

In short hand notation $\psi_{\mu} := \sum_I \bar{\Psi}_I(\mu)\Psi_I$.

**Proposition 2.1** Reproducing kernel property. If $\Psi$ is any other section, then $\langle \psi_{\mu}, \Psi \rangle = \Psi(\mu)$. In particular, $\langle \psi_{\mu}, \psi_{\nu} \rangle = \delta(\mu, \nu)$.

**Proof** By linearity, it is enough to check this for $\Psi = \Psi_{I_0}$ a basis element.

$\langle \psi_{\mu}, \Psi_{I_0} \rangle = \langle \sum_I \bar{\Psi}_I(\mu)\Psi_I, \Psi_{I_0} \rangle = \sum_I \Psi_I(\mu) \langle \Psi_I, \Psi_{I_0} \rangle$. Now we observe that $\langle \Psi_I, \Psi_{I_0} \rangle = \delta_{II_0}$. Thus $\langle \psi_{\mu}, \Psi_{I_0} \rangle = \Psi_{I_0}(\mu)$.

**Proposition 2.2** Resolution of identity property:
\[
c(m) \int_{U_0} \langle \psi_{\mu}, \psi_{\nu} \rangle e^{-m\Phi_{FS}(\mu, \bar{\nu})} dV(\mu) = (\psi_{\mu}, \psi_{\nu}).
\]

In particular,
\[
c(m) \int_{U_0} \langle \psi_{\mu}, \psi_{\nu} \rangle e^{-m\Phi_{FS}(\mu, \bar{\nu})} dV(\mu) = (\psi_{\mu}, \psi_{\nu}).
\]

**Proof:** We know that $c(m) \int_{U_0} \langle \psi_{\mu}, \psi_{\nu} \rangle e^{-m\Phi_{FS}(\mu, \bar{\nu})} dV = c(m) \int_{U_0} \bar{\psi}_{\mu}(\mu)\psi_{\nu}(\mu)e^{-m\Phi_{FS}(\mu, \bar{\nu})} dV$ since by reproducing kernel property, $\langle \psi_{\mu}, \Psi \rangle = \Psi(\mu)$. The above integral is $\langle \Psi_{\mu}, \Psi \rangle$. The above integral is $\langle \Psi_{\mu}, \Psi \rangle$.

**Notation:** We denote by $L_{m}(\mu, \bar{\nu}) = \langle \psi_{\mu}, \psi_{\nu} \rangle = \psi_{\mu}(\mu)$. 

$\mathcal{L}_{m}(\mu, \bar{\nu}) = \langle \psi_{\mu}, \psi_{\nu} \rangle = \psi_{\mu}(\mu)$. 

\[
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\]
Let $\hat{A}$ be a bounded linear operator acting on $\mathcal{H}$. Then, as in [1], one can define a symbol of the operator as

$$A(\nu, \bar{\mu}) = \langle \psi_\nu, \hat{A}\psi_\mu \rangle.$$

One can show that one can recover the operator from the symbol by the formula [1]:

$$(\hat{A}f)(\mu) = c(m) \int_{U_0} A(\mu, \bar{\nu})f(\nu)L_m(\mu, \bar{\nu})e^{-m\Phi(\nu, \bar{\nu})}dV(\nu).$$

Let $\hat{A}_1, \hat{A}_2$ be two such operators and let $\hat{A}_1 \circ \hat{A}_2$ be their composition. Then the symbol of $\hat{A}_1 \circ \hat{A}_2$ will be given by the star product defined as in [1]:

$$(A_1 \ast A_2)(\mu, \bar{\mu}) = c(m) \int_{U_0} A_1(\mu, \bar{\nu})A_2(\nu, \bar{\bar{\nu}})L_m(\mu, \bar{\nu})L_m(\nu, \bar{\bar{\nu}})L_m(\nu, \bar{\bar{\nu}})e^{-m\Phi(\nu, \bar{\nu})}dV(\nu),$$

where recall $\frac{1}{c(m)} = \int_{U_0} e^{-m\Phi_{FS}(\nu, \bar{\nu})}dV(\nu)$.

This is the symbol of $\hat{A}_1 \circ \hat{A}_2$.

One can show the following ([8])

**Proposition 2.3** $\psi_\mu(\nu) = (1 + \bar{\mu} \cdot \nu)^m.$

$$L_m(\mu, \bar{\nu}) = \langle \psi_\mu, \psi_\nu \rangle = \psi_\mu(\mu) = e^{m\Phi_{FS}(\mu, \bar{\nu})}$$ (1)

and for $(\mu, \nu) \in S$,

$$L_m(\mu, \bar{\nu}) = \langle \psi_\mu, \psi_\nu \rangle = \psi_\nu(\nu) = (1 + \mu \cdot \bar{\nu})^m = e^{m\Phi_{FS}(\mu, \bar{\nu})}.$$ (2)

Let $(\mu, \nu) \in S$.

Then we can define $\phi_{FS}(\mu, \bar{\mu}|\nu, \bar{\nu}) = \Phi_{FS}(\mu, \bar{\nu}) + \Phi_{FS}(\nu, \bar{\mu}) - \Phi_{FS}(\mu, \bar{\nu}) - \Phi_{FS}(\nu, \bar{\mu}).$

In fact, $\phi_{FS}(\mu, \bar{\mu}|\nu, \bar{\nu}) = \ln \left( \frac{(1+\nu \cdot \bar{\mu})}{(1+|\nu|^2)(1+|\bar{\nu}|^2)} \right).$

It is easy to show ( [8])

**Proposition 2.4** We have $\phi_{FS}$ is non-positive on $S$ and has a zero and a non-degenerate critical point ( as a function of $\nu$) at $\nu = \mu$.

**Proof:** We know $\phi_{FS}(\mu, \bar{\mu}|\nu, \bar{\nu}) = \ln \left( \frac{(1+\nu \cdot \bar{\mu})}{(1+|\nu|^2)(1+|\bar{\nu}|^2)} \right)$. By Cauchy-Schwartz inequality, we have $\phi_{FS}$ is non-positive definite. Fixing $\mu$, a straightforward calculation shows that $\frac{\partial^2 \phi_{FS}(\mu, \bar{\mu}|\nu, \bar{\nu})}{\partial \nu_i \partial \bar{\nu}_j}|_{\nu=\mu} = 0$ for each $i$ and $\frac{\partial^2 \phi_{FS}(\mu, \bar{\mu}|\nu, \bar{\nu})}{\partial \nu_i \partial \bar{\nu}_j}|_{\nu=\mu} \neq 0$.

Then we have $L_m(\mu, \bar{\nu})L_m(\nu, \bar{\nu})L_m(\nu, \bar{\nu}) = e^{m\phi_{FS}(\mu, \bar{\nu}|\nu, \bar{\nu})}$, where we have used equations (1, 2).

We also have, by the reproducing kernel property, that $\int_{U_0} \frac{L_m(\mu, \bar{\nu})L_m(\nu, \bar{\nu})L_m(\nu, \bar{\nu})}{L_m(\mu, \bar{\nu})L_m(\nu, \bar{\nu})}dV(\nu) = \frac{1}{c(m)}$.

**Theorem:** [Berezin]

Let $\mu \in C^n$. The star product satisfies the correspondence principle:

1. $\lim_{m \to \infty} (A_1 \ast A_2)(\mu, \bar{\mu}) = A_1(\mu, \bar{\mu})A_2(\mu, \bar{\mu}),$
2. $\lim_{m \to \infty} m(A_1 \ast A_2 - A_2 \ast A_1)(\mu, \bar{\mu}) = i\{A_1, A_2\}_{FS}(\mu, \bar{\mu}).$

See [1], [8] for proof.
3. Berezin-type quantization on compact even dimensional manifolds

Let \( M^{2d} \) be an even dimensional compact smooth manifold. We do not consider any symplectic structure or Poisson structure or group action on it. To obtain a Berezin-type quantization on it, first we embed the manifold (after perhaps removing a subset of measure zero) in \( CP^d \) and then induce a local Poisson structure on the embedded submanifold and induce the Berezin quantization from \( CP^d \). The Hilbert space of quantization is expected to be of finite dimension (since \( M \) is compact) and for that we choose \( CP^d \) and not \( C^d \).

Let \( M^{2d} \) be a compact topological manifold. Then by [5], there exists a skeleton \( M_0 \) of dimension at most \( 2d - 1 \) such that \( X = M \setminus M_0 \) is homeomorphic to \( R^{2d} \). We assume \( M^{2d} \) is equipped with a differentiable structure such that \( M \setminus M_0 \) is diffeomorphic to \( R^{2d} \) with standard smooth structure.

Let \( \tau \) be the diffeomorphism and \( Y = \tau(X) = R^{2d} \equiv C^d \). By abuse of notation, we name the coordinates on \( Y \) as \( (\tau_1, \tau_2, \ldots, \tau_d) \) where \( \tau_j = x_j + iy_j, \ j = 1, \ldots, d \), where \( (x_1, y_1, \ldots, x_d, y_d) \in R^{2d} \).

Let \( Y \) be given by the coordinates \( (\tau_1, \ldots, \tau_d) \). Let \( U_0 \) be the open subset of \( CP^d \) given by \( \{w_0 \neq 0\} \) where \( \{w_0, \ldots, w_d\} \) is a local coordinate on \( CP^d \). Let \( U_0 = \{[1, \tau_1, \ldots, \tau_d]\} \equiv C^d \) where \( \tau_i = \frac{w_i}{w_0}, \ i = 1, \ldots, d \).

Let us give a metric on \( X = M \setminus M_0 \) by identifying it with its image \( Y = \tau(X) \equiv U_0 \). The volume form is \( dV = \frac{|d\tau|}{(1+|\tau|^2)^{d+1}} \) and \( V = \int_Y dV < \infty \).

**Algebra of operators on \( M \setminus M_0 \):**

On \( M \setminus M_0 \), we define the Hilbert space of quantization to be \( \mathcal{H}_\tau = \tau^*(\mathcal{H}_Y) \) (i.e. pulled back by the diffeomorphism \( \tau \)), where the volume form on \( M \setminus M_0 \) is induced from \( U_0 \subset CP^d \). Let \( \zeta \in X \). Let \( \tau = \tau(\zeta), s \in \mathcal{H}_Y \). Let \( dS(\zeta) \) be the volume form of \( M \) such that \( M_0 \) is of measure zero.

Let \( h(\zeta) > 0 \) be such that \( h(\zeta)dS(\zeta) = dV_\tau(\tau) = \frac{|d\tau|}{(1+|\tau|^2)^{d+1}} \). In other words, \( \int_X |\tau^*(s)|^2(h(\zeta)dS(\zeta)) = \int_\tau |s|^2 \frac{|d\tau|}{(1+|\tau|^2)^{d+1}} \).

Let \( \bar{s} \in \mathcal{H}_\tau \) such that \( \bar{s} = \tau^*(s) \). Then we define bounded linear operators \( \hat{A} \) on \( \mathcal{H}_\tau \) to be:

\[
\hat{A}(\bar{s})(p) = \hat{A}(s)(z),
\]

where \( z = \tau(p) \in U_0 \) and \( \hat{A} \) is a bounded operator on \( \mathcal{H}_Y \). It can be shown that given a bounded linear operator \( \hat{A} \) on \( \mathcal{H}_\tau \), there is a unique bounded linear operator \( \hat{A} \) on \( \mathcal{H}_Y \) such that \( \hat{A}(\bar{s})(p) = \hat{A}(s)(z) \).

Then symbols and star product can be defined for \( \hat{A} \) via \( \hat{A} \) and correspondence principle follows. Now we elaborate this.

The symbol of \( \hat{A} \) is defined to be \( \hat{A}(p, q) \equiv A(z, \bar{w}) \) where \( z = \tau(p), w = \tau(q) \).

Suppose we have two operators \( \hat{A}_1 \) and \( \hat{A}_2 \).

Then \( \hat{A}_1 \ast \hat{A}_2 \) is defined on \( (M \setminus M_0) \times (M \setminus M_0) \) to be \( \hat{A}_1 \ast \hat{A}_2)(p, p) \equiv (A_1 \ast A_2)(z, \bar{z}) \) in \( CP^d \).

In general the algebra of operators will depend on the diffeomorphism.

Then we can see that the star product satisfy the correspondence principle. The proof is exactly same as the previous section with \( n = d \).

**Proposition 3.1** Let \( \tau \in C^d \).

\( \hat{A}_1 \ast \hat{A}_2 = A_1 \ast A_2 \) satisfy the correspondence principle.

1. \( \lim_{m \to \infty} (A_1 \ast A_2)(\tau, \bar{\tau}) = A_1(\tau, \bar{\tau})A_2(\tau, \bar{\tau}) \).
2. \( \lim_{m \to \infty} m(A_1 \ast A_2 - A_2 \ast A_1)(\tau, \bar{\tau}) = \{A_1, A_2\}_FS(\tau, \bar{\tau}) \).

**Proof:** Set \( n = d \) in the previous section. The proof follows essentially from Lemma (2.1) in [1] as elaborated in [8].
3.1. Equivalence of two Berezin quantizations:

On a smooth (complex) manifold $M^{2d} \backslash M_0$, let there be a local Poisson structure and a Berezin-type quantization defined as above induced from $CP^d$. Suppose there are two diffeomorphisms (biholomorphisms if $M \backslash M_0$ is complex) which induce two such quantizations. Then there are two Hilbert spaces with reproducing kernels and star products on symbols of bounded linear operators which satisfy the correspondence principle. Suppose there exists a smooth (or biholomorphic) bijective map $\psi$ from $M \backslash M_0$ to $M \backslash M_0$ which preserve the local Poisson structures. If $\psi$ induces an isomorphism (i.e. a bijective linear map that preserves innerproduct) between the two Hilbert spaces such that the reproducing kernel maps to the corresponding reproducing kernel then we shall say the two Berezin quantizations are equivalent.

4. Our method of quantization for the torus

Let $L$ be a line bundle on $CP^1$. $CP^1$ is homeomorphic to the sphere of radius 1 and let $N, S$ be the north and the south poles and $E$ the equatorial circle. Let $U_N = S^2 \backslash N$ and $U_S = S^2 \backslash S$ be two charts on the sphere such that $U_N$ is homeomorphic to the equatorial plane using the stereographic projection from $N$ and $U_S$ being the same using stereographic projection from $S$. The transition function $t_{NS}$ of the line bundle $L$ when restricted to the equator, winds the equatorial circle $E$ to $r$ times $U(1) \equiv S^1$, $r \in \mathbb{Z}$. This winding number characterises smooth line bundles on the sphere. For the transition function of $H$, the hyperplane line bundle, let the winding number be $r_0$. Then for $L = H^{\otimes m}$ the winding number is $q = r_0m$. (As an aside, the set of holomorphic sections of $H^{\otimes m}$ are in one to one correspondence with polynomials of degree $\leq m$ in one complex variable--for more details, see [11], p 500).

Let $i\theta$ be the imaginary valued connection 1-form for $H$ (curvature proportional to the Fubini-Study form $\omega_{FS}$). Let $i\theta = mi\theta_1$ the connection 1-form on $H^{\otimes m}$. Let $m = 2s$ be an even integer. Let $\psi$ be a section of $H^{\otimes m}$ on sphere which satisfies $(d + mi\theta_1)\psi = 0$. On integration on any closed loop $C_1$ on the sphere, $\psi = \exp(-im\int_{C_1} \theta_1)\psi_0$ where $\psi_0 = \psi(t_0)$. The phase factor is called holonomy and is well defined along this path because the curvature of the line bundle $\omega_{FS} = d\theta_1$ belongs to the integral cohomolgy $H^2(S^2, \mathbb{Z})$, in [18], p 158.

Let $\overline{U}_u$ be the upper hemisphere of the sphere with boundary $E$. $U_u$ is the interior of $\overline{U}_u$ which is diffeomorphic to a disc. As before let $\psi$ be a section of $H^{\otimes m}$ and $E$ be parametrized by $t$ such that $E_1$ and $E_2$ are parametrised by $0 \leq t \leq \frac{1}{2}$ and $\frac{1}{2} \leq t \leq 1$ respectively such that $E = E_1 + E_2$. Let $E_2 = -E_3$, i.e. $E_2$ with the reverse direction.

We note that $\exp(-i\int_{E_1} \theta) = \exp(-i\int_{E_2} 2s\theta_1) = \exp(-i\int_{U_u} 2s\omega_{FS}) = \exp(-is\frac{A_{FS}}{4}) = 1$ since $\int_{U_u} \omega_{FS} = \frac{A_{FS}}{4} = \pi$. Also $\exp(-i\int_{E_1} \theta)\exp(i\int_{E_2} \theta) = 1$ by the same reason. Thus $\exp(-i\int_{E_{1}} \theta) = \exp(-i\int_{E_{2}} \theta)$. One sees that $\psi(\frac{1}{2}) = \exp(-i\int_{E_1} \theta)\psi_0 = \exp(-i2s\int_{E_1} \theta_1)\psi_0$ and $\psi'(1) = \exp(-i\int_{E_2} \theta)\psi_0 = \exp(-i\int_{E_2} 2s\theta_1)\psi_0 = \psi_0$.

Let $E = E_1^1 + E_2^2$ where $E_1^1$ is half way of $E_1$ and $E_2^2$ is the other half of $E_1$ and $E_2 = E_2^1 + E_2^2$, where $E_2^1$ is half way of $E_2$ and $E_2^2$ is the other half of $E_2$.

Let $A$ and $B$ be the two representatives of the homology of the torus, $M_0 = A \cup B$. Let $X = T^2 \backslash M_0 = U_u$ (by a diffeomorphism). $\overline{U}_u \backslash U_u$ is $E$ and $T^2 \backslash X = A \cup B$. Let us identify points on $E$ with $A \cup B$ such that one-fourths of the equator $E_1^1$ is identified with $A$ and the other half $E_2^1$ is identified with $B$ (using a quotient map). Similarly, $E_2^1 = E_2^1$ is identified with $\bar{A} = -B$. This is in keeping with the cell decomposition of the torus. $E_1$ and $E_2$ are identified with loops $A + B$ and $-A - B$.

After identification, $\exp(-i\int_{A+B} \theta) = \exp(-i\int_{E_1} \theta)$ and $\exp(-i\int_{-A-B} \theta) = \exp(-i\int_{E_2} \theta)$. In other words, $\exp(-i\int_{E_1} \theta)\exp(i\int_{E_2} \theta) = \exp(-i\int_{E_2} \theta)$, as was shown before.

Take a loop $C$ on the torus such that $C = k_1A + k_2B$. This is identified with $\tilde{C} = k_1E_1^1 + k_2E_2^2$. $C = k_1A + k_2B$ be a closed loop of the torus, as before, parametrised by $0 \leq t \leq 1$ and $\psi$
be a section of $H^{\otimes m}$, $m = 2s$, an even integer. Then $\psi(1) = \exp(-i \int_{k_1} A \theta - i \int_{k_2} B \theta) \psi_0$ where $\psi_0 = \psi(0)$. Then the phase factor due to holonomy is $\exp(-i \int_{k_1} E_1^1 \theta - i \int_{k_2} E_2^2 \theta)$ which is well defined (because we can translate the question to that on the sphere).

If $p \in X = T^2 \setminus (A \cup B)$, $\beta$ is a loop on the torus $T^2$ which is contained entirely in $T^2 \setminus (A \cup B)$ one can easily show there is well defined global holonomy, after the identification of $X$ with $U_\nu$. If $\beta$ is a loop starting and ending at $p \in T^2$ that intersects $A$ or $B$, by our identification of $A$ and $B$ with $E_1^1$ and $E_2^2$, the holonomy on $\beta$ is also well defined (as we can translate the question to that on the sphere).

5. Toeplitz quantization on compact complex manifolds

Let $M$ be a compact complex manifold of dimension $d$. Let $M_0$ be a set of measure zero such that $X = M \setminus M_0 \equiv R^{2d} \equiv C^d$ (diffeomorphism). Let $U_0 = \{[1, z_1, z_2, \ldots, z_d]\}$ be one of the homogeneous charts of $CP^d$. Thus we have an embedding, $\epsilon$, which maps $M \setminus M_0$ onto $U_0 \subset CP^d$.

Let $\mathcal{C}^m$ be endowed with Fubini-Study metric.

Note that $d\mathcal{H}^m_\epsilon \equiv \epsilon^*(\mathcal{H}^m)$. $\mathcal{H}^m_\epsilon \equiv \epsilon^*(\mathcal{H}^m)$.

$M \setminus M_0$ has an induced volume form as follows. Let $\Sigma = \epsilon(M \setminus M_0)$ and $h(\zeta) dS(\zeta) = dV_{CP}(\epsilon(\zeta))$, where $h > 0$ is a smooth function. Note that all pullback sections in $\mathcal{H}^m_\epsilon$ are square integrable w.r.t. the measure $d\mathcal{S}$ on $X = M \setminus M_0$.

Let $f, g$ be a smooth function on $CP^d$ restricted to $U_0$ and let $\tilde{f}, \tilde{g}$ be the smooth functions on $M \setminus M_0$, which are pulled back by $\epsilon$, i.e., for $\mu \in M \setminus M_0$, $\tilde{f}(\mu) \equiv f(\epsilon(\mu))$, similarly $\tilde{g}(\mu) \equiv g(\epsilon(\mu))$.

We claim $f$ is a unique function on $CP^d$ given $\tilde{f} = \epsilon^*(f)$. Suppose, $\tilde{f} = \epsilon^*(f_1) = \epsilon^*(f_2)$. Then $f_1 - f_2 = 0$ on $\Sigma = \epsilon(M \setminus M_0)$. But $\Sigma = U_0 \subset CP^d$ is an open set in $CP^m$ such that its complement is of measure zero. Since $f_1 - f_2$ is smooth, it extends to all of $CP^d$ and is identically 0.

Recall for $CP^d$ (restricted to $U_0$), $m$-th level Toeplitz operator of $f$, denoted by $T^m_\pi$, defined on $\mathcal{H}^m$, defined as $T^m_\pi(s) = \Pi^m(f,s)$, where $\Pi^m$ is the projector map from square integrable smooth sections onto $\mathcal{H}^m$ and $s \in \mathcal{H}^m$. Let $\tilde{s} = \epsilon^*s$. One can show that given $\tilde{s}$, $s$ is unique. This is because if $\tilde{s} = \epsilon^*s_1 = \epsilon^*s_2$. Then $s_1 - s_2 = 0$ on $\Sigma$ and thus on $CP^d$. But $s_1, s_2$ are global holomorphic sections of $H^m$ and can be extended to all of $CP^d$. Thus $s_1 - s_2 \equiv 0$.

For $X = M \setminus M_0$, we denote

$$||\tilde{s}||^2 = ||\tilde{s}||^2_X = \int_X |\tilde{s}|^2 h(\zeta) dS(\zeta) = \int_\Sigma |s_{|z_1}|^2 dV_{CP}(\epsilon(\zeta))$$

where recall $\Sigma = \epsilon(X)$.

But $\int_\Sigma |s_{|z_1}|^2 dV_{CP}(\epsilon(\zeta)) = \int_{CP^d} |s|^2 dV_{CP^d}$ since $\Sigma = U_0 \subset CP^d$ and $CP^d \setminus U_0$ is of measure zero.

Thus we have

$$||\tilde{s}||^2 = ||s||^2$$

(3)

where the first norm is in $X = M \setminus M_0$ and second norm is in $CP^d$.

For a functions $\tilde{f} \in C^\infty(X)$, we define a set of operators for $M \setminus M_0$, defined on $\mathcal{H}^m_\epsilon$, denoted by $\tilde{T}^m_\tilde{f}$.

Definition $\tilde{T}^m_\tilde{f}(\tilde{s}) = \tilde{\Pi}^m(\tilde{f} \tilde{s})$ where $\tilde{\Pi}^m \epsilon^* \equiv \epsilon^* \Pi^m$.

Since $\tilde{f} \tilde{s} = \epsilon^*(f s)$ for a unique $f s \in \mathcal{H}^m$, we have that $\tilde{T}^m_\tilde{f}(\tilde{s})$ is well defined.
Let $\omega$ be a compact integral Kähler manifold with Kähler form $\omega$. Let $L$ be a geometric quantum bundle whose curvature is proportional to $\omega$. Let $\mathbb{H}^m$ be a geometric quantum bundle whose curvature is proportional to $m\omega$.

There is a corollary to a theorem by several mathematicians [16, 14, 19, 10, 3] as mentioned in [7], page 131. It goes as follows.

**Theorem** [Tian, Ruan, Zelditch, Lu, Catlin] For large $m$, an orthonormal basis of $\mathbb{H}^0(M,L^m)$ gives a map $\mathbb{H}^m : M \rightarrow \mathbb{C}P^d$, where $N_m + 1 = \dim \mathbb{H}^0(M,L^m)$ and

$$\frac{1}{m} \epsilon_m(\Omega_{FS}) - 2\pi \omega = O(m^{-2}) \text{ in } C^\infty.$$

Let $\Omega_m = \epsilon_m^*(\Omega_{FS})$.

For each $m$ and $t, s$ two smooth functions on $M$, let the two Poisson brackets on $M$ be

$$\{ t, s \}_{PB} = \sum_{ij} \Omega_{m_{ij}} \left( \frac{\partial t}{\partial s_i} \frac{\partial s}{\partial y_j} - \frac{\partial s}{\partial y_i} \frac{\partial t}{\partial y_j} \right) .$$
\{t, s\}_PB2 = \sum_{ij} m \omega_{ij} \left( \frac{\partial t}{\partial \tau_i} \frac{\partial s}{\partial \tau_j} - \frac{\partial t}{\partial \bar{\tau}_j} \frac{\partial s}{\partial \tau_i} \right)

Then motivated by the theorem above we have the following conjecture.

**Conjecture:** Suppose \( M \) has a Berezin quantization as defined in \([1]\) induced by the Kähler form \( \omega \). Also in \([4]\) we defined a pullback Berezin-type quantization on \( M \), in this case pull back from \( CP^N \). (One can show that one does not need the totally real condition in this case). We conjecture that these two quantizations are equivalent, in the sense that there is an isomorphism of the Hilbert spaces with reproducing kernels and the Poisson brackets (which appear in the correspondence principle) are the same in the limit \( m \to \infty \).

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**References**

[1] Berezin F A 1974 Quantization *Math USSR Izv.* 8 1109
[2] Bordemann M, Meinrenkene E and Schlichenmaier M 1994 Toeplitz Quantization of Kähler manifolds and \( gl(N) \). \( N \to \infty \) limits *Comm. Math. Phys.* 165 281
[3] Catlin D 1997 The Bergman kernel and a theorem of Tian *Analysis and geometry in several complex variables* (Trends Math. ) (Boston: Birkhauser Boston Inc.)101
[4] Dey R and Ghosh K 2022 Pull back coherent states and squeezed states and quantization *Symm. Integr. Geom.: Meth. and Appl.* 8 028 01
[5] Doyle P H and Hocking J G 1962 A Decomposition Theorem for \( n \)-dimensional manifolds *Proc. Amer. Math. Soc.* 13 469
[6] Englis M 1996 Berezin Quantization and Reproducing Kernel on Complex Domains *Trans. Amer. Math. Soc.* 348 411
[7] Gabor S 2014 *An introduction to extremal Kähler metrics* (Graduate Studies in Mathematics) vol 152 ( Providence: American Mathematical Society)
[8] Ghosh K 2023 Berezin-type quantization of even-dimensional manifolds and pullback coherent states, *Thesis ICTS-TIFR* https://thesis.icts.res.in/
[9] Kordyukov Y A 2022 Berezin-Toeplitz quantization associated with higher Landau levels of the Bochner Laplacian *J. Spectr. Theory* 12 143
[10] Lu Z 1998 On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch *Amer. J. Math.* 122 235
[11] Nair V P 2004 *Quantum Field Theory: A Modern Persective*(New York: Springer)
[12] Radcliffe J M 1971 Some Properties of Coherent Spin States *J. Phys. A: Gen. Phys.* 4 313
[13] Rawnsley J H 1977 Coherent States and Kähler Manifolds *Quart. J. Math.* 28 403
[14] Ruan W D 1998 Canonical coordinates and Bergman metrics *Comm. Anal. Geom.* 6 589
[15] Spera M 2000 On Kählerian Coherent States *Proc. Int. Conf. Geom.Integr. Quant* (Sofia: Bulg. Acad. Scien.)
[16] Tian G 1990 On a set of polarized Kähler metrics on algebraic manifolds *J. Diff. Geom.* 32 99
[17] Tong D 2016 The Quantum Hall Effect *Tata Infogqs Lectures* http://www.damtp.cam.ac.uk/user/tong/qhe.html
[18] Woodhouse N M J 2007 *Geometric Quantization* (Ox. Math. Mono.) (Oxford : Claredon Press) .
[19] Zelditch S 1998 Szego kernel and a theorem of Tian *Int. Math. Res. Notices* 6 317