Projective robustness for quantum channels and measurements and their operational significance

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Abstract. Recently, the projective robustness of quantum states has been introduced in [arXiv:2109.04481 (2021)]. It shows that the projective robustness is a useful resource monotone and can comprehensively characterize capabilities and limitations of probabilistic protocols manipulating quantum resources deterministically. In this paper, we will extend the projective robustness to any convex resource theories of quantum channels and measurements. First, we introduce the projective robustness of quantum channels and prove that it satisfies some good properties, especially sub- or supermultiplicativity under any free quantum process. Moreover, we use the projective robustness of channels to give lower bounds on the errors and overheads in any channel resource distillation. Meanwhile, we show that the projective robustness of channels quantifies the maximal advantage that a given channel outperforms all free channels in simultaneous discrimination and exclusion of a fixed state ensemble. Second, we define the projective robustness of quantum measurements and prove that it exactly quantifies the maximal advantage that a given measurement provides over all free measurements in simultaneous discrimination and exclusion of two fixed state ensembles. Finally, within a specific channel resource setting based on measurement incompatibility, we show that the projective robustness of quantum channels coincides with the projective robustness of measurement incompatibility.

Keywords: projective robustness, discrimination, exclusion, measurement incompatibility

I. INTRODUCTION

The resource-theoretic framework is a powerful tool to help us understand and manipulate resources. Various specific resource theories have been proposed and discussed under different physical constraints, such as well-known entanglement, coherence, magic, non-Gaussianity, quantum thermodynamics, steering, and imaginary. Moreover, due to the generality of this framework, it has been applied to study quantum measurements and quantum channels, and various resource theories about quantum measurements and channels have been proposed, such as measurement simulability and incompatibility, measurement informativeness, measurement coherence, the magic and coherence of quantum channels, and quantum memory of channels. These resource theories have significantly promoted the prosperous development of quantum informatics. To explore the common characteristics of resources, general resource theories have attracted much attention in recent years, including quantification and manipulation of general resource objects.

Resource quantification plays a vital role in quantum resource theories. A common and effective approach to accomplish this goal is to define resource measures (or monotones) to measure and compare the resourcefulness contained in a given physical object. At present, ones have proposed plenty of resource measures. Such a way can effectively quantify the amount of resource contained in a quantum object, but it often lacks a direct relation with the operational significance of resources. The generalized robustness measure and weight measure not only fully overcome this shortcoming but also apply to general state resource quantification, even resource quantifications of general measurements and channels. These two measures have wide applications in operational tasks of quantum resource theories, for the former such as catalytic resource erasure, min-accessible information, discrimination tasks, one shot distillation in general state resource theories, and one shot manipulation of channel resources, for the latter such as single-shot information theory, exclusion tasks, and channel resource purification.

In particular, in broad classes of resource theories, they can directly quantify the practical advantages that the resourceful object outperforms all resourceless objects in specific discrimination (or exclusion) tasks, and can provide constraints on error rates and overheads for various state and channel distillation protocols. Searching for such resource measures is always one of the cores of resource theories.

Hilbert projection metric introduced by Ref. has provided a sufficient and necessary condition for probabilistic transformations of pairs of quantum states, it has not been applied as a resource monotone until recently projective robustness of quantum states was introduced in. It is shown that the projective robustness of quantum states not only satisfies a number of useful properties but also has significant applications in any probabilistic manipulation protocol and simultaneous discrimination and exclusion of a fixed channel ensemble. On the one hand, the quantum channels lie at the heart of the manipulation of quantum states. On the other hand, any meaningful quantum information processing task includes a measurement at the end. So it is quite natural and necessary to study the quantification and manipulation of quantum channels and measurements resources.
ever, the projective robustness has not been studied in the con-
text of channels and measurements resource theories. In this
paper, we will introduce two types of projective robustness,
namely projective robustness for channels and measurements.
For the former, we will study its operational applications in
any deterministic channel distillation protocol and simultane-
ous discrimination and exclusion of a fixed state ensemble.
These two operational tasks form the foundation of the op-
erational aspects of quantum theory. For the latter, we will
study its operational applications in simultaneous discrimina-
and exclusion of two fixed state ensembles. It will pro-
mote our understanding of the common characteristics of re-
sources and lay a good foundation for in-depth research to
manipulate channel and measurement resources.

This paper is arranged as follows. Section II introduces
all of the relevant concepts, including resource theories, some
resource monotones, and state discrimination and exclusion
tasks. In Sect. III, we define the projective robustness of chan-
els and present that the measure obeys several useful prop-
erties. Moreover, we study its two operational applications.
In Sect. IV, we introduce the projective robustness of mea-
surements and its corresponding operational interpretation is
given. Particularly, within a specific channel resource theory
setting based on measurement incompatibility, we discuss the
relationship between projective robustness for channels and
projective robustness for measurements. Finally, Section V
offers the conclusions.

II. SETTING

Given the Hilbert space of a finite-dimensional system \( A \) with
dimension \( d_A \), let \( \mathcal{L}(A) \) and \( \mathcal{D}(A) \) be the set of the linear
operators and the set of the density operators acting on this
space, respectively. We use CPTP(\( A \to B \)) to represent the
set of quantum channels, i.e., completely positive and trace-

preserving (CPTP) maps from \( \mathcal{L}(A) \) to \( \mathcal{L}(B) \).
For each channel \( \mathcal{E} : A \to B \), its Choi matrix is \( \mathcal{E}_C := \text{id} \otimes \mathcal{E}(\Phi^+) \in \mathcal{L}(RB) \),
where \( \Phi^+ = \sum_{ij} |ij\rangle\langle ij| \) is the unnormalized maxi-
tely entangled state in \( \mathcal{L}(RA) \) and \( R \cong A \). It is well known that
\( \mathcal{E}_C \) is completely positive if and only if \( \mathcal{E}_C \geq 0 \), trace preserv-
ing if and only if \( \text{Tr} \left( \mathcal{E}_C \right) = \mathbb{I} \). The normalized Choi state
type is then \( \mathcal{E}_C := \mathcal{E}_C/d_A \). Let \( \langle A, B \rangle := \text{Tr} \left( A^\dagger B \right) \) represent the
Hilbert-Schmidt inner product between operators. If a map always maps a quantum channel to a valid quantum channel,
then it is called a quantum superchannel \([55]\), that is, maps
CPTP(\( A \to B \)) \to CPTP(\( C \to D \)).

Throughout the paper, we will be interested in POVMs on
finite dimensional Hilbert space \( \mathbb{H} \cong \mathbb{C}^d \). A POVM \( \mathbb{M} \) is
a collection of POVM elements \( \mathbb{M} = \{ M_i \}_{i=1}^n \) with \( M_i \geq 0 \) \forall i
and \( \sum_i M_i = \mathbb{I} \). The operators \( M_i \) are called the elements of POVM
\( \mathbb{M} \). We denote the set of POVMs on \( \mathbb{C}^d \) with \( n \) outcomes
by \( \mathcal{M}(d,n) \). This set has a natural convex structure: given two
POVMs \( \mathbb{M}, \mathbb{N} \in \mathcal{M}(d,n) \), their convex combination \( p \mathbb{M} + (1 - p) \mathbb{N} \) is the POVM with \( i \)-th effect given by \( p M_i + (1 - p) N_i \).
Note that \( \mathcal{M}(d,n) \) is a convex subset of \( (\text{Herm} (\mathbb{H}))^{\otimes n} \) where
(\( \text{Herm}(\mathbb{H}) \)) is the set of Hermitian operator from \( \mathbb{H} \) to itself.
The space \( (\text{Herm}(\mathbb{H}))^{\otimes n} \) can be regarded as \( d^n \) dimensional

A. Resource theories

A general state resource theory consists of two parts under
some physical restrictions [23]. One is the free states which
are available freely and we denoted the set of free states as \( \mathcal{F} \).
The other is the free operations which are allowed within the
given physical constraints. To remain as general as possible,
we will only introduce two common and intuitive assumptions
about \( \mathcal{F} \): it is convex and closed. The former means that sim-
ply taking a sequence of free states generates no resource, and
the latter means that no resource can be generated by simply
probabilistically mixing free states. As for free operations, we
only assume the weakest possible constraint: a free operation
does not generate any resource by itself.

When we aim to study channel resources [13, 57, 58], in
the given physical setting, a specific subset of quantum chan-
nels \( \mathcal{O} \) was chose as the free channels. We will assume that
the set \( \mathcal{O} \) is compact and convex. The compactness of \( \mathcal{O} \) is a
technical assumption that allows us to formally state our re-

sults. Moreover, the free operations between the channels are
free superchannels. A free superchannel \( \Theta \) does not generate
any resource by itself; that is, for any free channel \( \mathcal{M} \in \mathcal{O} \), it
holds that \( \Theta (\mathcal{M}) \in \mathcal{O} \). Let \( \mathcal{S} \) represent the set of all such free
superchannels. The above assumptions will make our results
applicable to broad channel resource settings.

As for measurement resource [27, 31], we use a similar
approach to define the set of free measurements as \( \mathcal{M}_F \subseteq
\mathcal{M}(d,n) \), and assume that \( \mathcal{M}_F \) is convex and compact. A valid
transformation \( \mathcal{B} \) between measurements should transform a
POVM into a POVM, i.e., \( \Pi (\mathbb{M}) = \{ \Pi (M_i) \} \) is a valid POVM
for any \( \mathbb{M} = \{ M_i \} \in \mathcal{M}(d,n) \). It means that a valid trans-
formation between measurements should be a positive and
trace-preserving map. Moreover, we assume that the class of
free operations \( \mathcal{O}_F \) consists of mappings \( \Gamma : \mathcal{M}(d,n) \to
\mathcal{M}(d,n) \) that (i) preserve the set of free measurements i.e.,
\( \Gamma (\mathbb{M}) \in \mathcal{M}_F \) for all \( \mathbb{M} \in \mathcal{M}_F \), (ii) are convex-linear i.e.,
\( \Gamma (p \mathbb{M} + (1 - p) \mathbb{M}') = p \Gamma (\mathbb{M}) + (1 - p) \Gamma (\mathbb{M}') \) for all mea-
surements \( \mathbb{M}, \mathbb{M}' \in \mathcal{M}(d,n) \).

B. Resource monotones

In this subsection, we will recall some resource monotones
including generalized robustness and weight of states, their
channel versions, and the projective robustness of states.

The generalized robustness and weight of states are defined as follows [42, 43]:

\[
R_\sigma (\rho) = \inf \{ \lambda \mid \rho \leq \lambda \sigma, \sigma \in \mathcal{F} \},
W_\sigma (\rho) = \sup \{ \mu \mid \rho \geq \mu \sigma, \sigma \in \mathcal{F} \}.
\] (1)

Using the corresponding Choi matrices, the quantum channel
versions of both measures have been obtained \cite{22, 35, 56–59}
\begin{equation}
R_\mathcal{O}(E) = \inf\{ \lambda | \mathcal{F}_E \leq \lambda \mathcal{F}_M, M \in \mathcal{O} \},
\end{equation}
\begin{equation}
W_\mathcal{O}(E) = \sup\{ \mu | \mathcal{F}_E \geq \lambda \mathcal{F}_M, M \in \mathcal{O} \}.
\end{equation}

Next, we introduce the non-logarithmic variant of the max-relative entropy $D_{\max}$ \cite{43}, which is defined as
\begin{equation}
R_{\max}(\rho \parallel \sigma) = 2^{D_{\max}(\rho \parallel \sigma)} = \inf\{ \lambda | \rho \leq \lambda \sigma \}.
\end{equation}

For any channel $E : \mathcal{F} : A \rightarrow B$, the channel version of $R_{\max}$ is defined as \cite{59, 62}
\begin{equation}
R_{\max}(E \parallel \mathcal{F}) = \max_{\psi_{RA}} R_{\max}(\text{id} \otimes E(\psi_{RA}) \parallel \text{id} \otimes \mathcal{F}(\psi_{RA})) = R_{\max}(\mathcal{F}_E \parallel \mathcal{F}_M),
\end{equation}
\begin{equation}
= R_{\max}(\mathcal{F}_E \parallel \mathcal{F}_M) = \inf\{ \lambda : E \leq \lambda F \},
\end{equation}
where the inequality $E \leq \lambda F$ is understood as completelypositive (CP) ordering of super operators, i.e., $\lambda F - E$ is a CP map. Therefore, the generalized robustness and weight of quantum channels can be equivalently expressed as \cite{35}
\begin{equation}
R_{\mathcal{O}}(E) = \min_{M \in \mathcal{O}} R_{\max}(\mathcal{F}_E \parallel \mathcal{F}_M),
\end{equation}
\begin{equation}
W_{\mathcal{O}}(E)^{-1} = \min_{M \in \mathcal{O}} R_{\max}(\mathcal{F}_M \parallel \mathcal{F}_E).
\end{equation}

Recently, the projective robustness of quantum states based on the Hilbert projective metric is defined as \cite{40}
\begin{equation}
\Omega_{\mathcal{O}}(\sigma) = \min_{\rho \in \mathcal{F}} R_{\max}(\rho \parallel \sigma) R_{\max}(\sigma \parallel \rho),
\end{equation}
\begin{equation}
\Omega_{\mathcal{O}}(\rho) < \infty \text{ if and only if there exists a free state } \sigma \in \mathcal{F} \text{ such that } \text{supp} \rho = \text{supp} \sigma. \text{ Moreover, Eq.(6) has the following equivalent forms,}
\end{equation}
\begin{equation}
\Omega_{\mathcal{O}}(\rho) = \inf\{ \lambda | \rho \leq \lambda \sigma, \sigma \leq \mu \rho, \sigma \in \mathcal{F} \}
\end{equation}
\begin{equation}
= \inf\{ \gamma | \rho \leq \gamma \sigma, \sigma \in \text{cone}(\mathcal{F}) \}
\end{equation}
\begin{equation}
= \sup \left\{ \left( A, \rho \right), \left( B, \sigma \right) \right\} \leq 1, \forall \sigma \in \mathcal{F}, A, B \geq 0 \right\},
\end{equation}
where cone(\mathcal{F}) = \{ \lambda \sigma | \lambda \in \mathbb{R}_+, \sigma \in \mathcal{F} \}. Note that the closedness of \mathcal{F} ensures that the infimum is achieved as long as it is finite.

C. State discrimination and exclusion tasks

\textit{State discrimination task:} Let us consider a scenario in which an ensemble of quantum states is given as $\{ p_i, \sigma_i \}$, then we perform a chosen POVM measurement $\{ M_i \}$ to distinguish which of the states was sent. For a given state ensemble, and measurement, the average probability of successfully discriminating the states is given by
\begin{equation}
\rho_{\text{succ}} (\{ p_i, \sigma_i \}, \{ M_i \}) = \sum_i p_i \langle M_i, \sigma_i \rangle,
\end{equation}
where $\sum_i p_i = 1$ with $p_i \geq 0$ and $\sum_i M_i = I$ with operators $M_i \geq 0$. In this standard state discrimination task, our aim is to choose a measurement \{ $M_i$ \} to maximize Eq.(8). The generalized robustness of measurements is known to quantify the advantage that any resourceful measurement can provide over all free measurements \cite{27, 51}.

\textit{State exclusion task:} \cite{63} The state exclusion task can be seen as being the opposite of state discrimination. Our goal is to perform a given measurement \{ $M_i$ \} and outputs a guess $g$ of a state that was not sent. Then, Eq.(8) is understood as the average probability of guessing incorrectly, and our goal to minimize this quantity. In state exclusion, it is the weight of measurements which quantifies the advantage \cite{57, 59}.

III. PROJECTIVE ROBUSTNESS OF CHANNELS

Quantum channels as the extension of quantum states play a significant role in quantum information processing. In fact, they also can be seen as resourceful, and their operational characterization is currently an active research area \cite{19–22, 35, 36, 56, 59}. In this section, we will introduce the projective robustness of channels to quantify the resource amount contained a given channel.

\textbf{Definition 1.} For a given channel $E$, its projective robustness is defined as
\begin{equation}
\Omega_{\mathcal{O}}(E) = \min_{M \in \mathcal{O}} R_{\max}(E \parallel M) R_{\max}(M \parallel E)
\end{equation}
\begin{equation}
= \min_{M \in \mathcal{O}} R_{\max}(\mathcal{F}_E \parallel \mathcal{F}_M) R_{\max}(\mathcal{F}_M \parallel \mathcal{F}_E).
\end{equation}

The first equality in Eq.(9) can be equivalently understood as the optimization over free channels $M$ such that $\lambda M - E$ and $\mu E - M$ are CP maps.

\textbf{Theorem 1.} The projective robustness of channels $\Omega_{\mathcal{O}}(E)$ satisfies the following properties:
(i) $\Omega_{\mathcal{O}}(E)$ is finite if and only if there exists a free channel $M \in \mathcal{O}$ such that $\text{supp}(\mathcal{F}_M) = \text{supp}(\mathcal{F}_E)$.
(ii) $\Omega_{\mathcal{O}}(kE) = \Omega_{\mathcal{O}}(E)$ for any $k > 0$.
(iii) When $\mathcal{O}$ is convex, $\Omega_{\mathcal{O}}$ is quasiconvex: for any $t \in [0, 1]$, it holds that
\begin{equation}
\Omega_{\mathcal{O}}(tE + (1-t)F) \leq \max(\Omega_{\mathcal{O}}(E), \Omega_{\mathcal{O}}(F)).
\end{equation}
(iv) For any superchannel $\Theta \in \mathcal{S}$, it holds that
\begin{equation}
\Omega_{\mathcal{O}}(\Theta(E)) \leq \Omega_{\mathcal{O}}(E).
\end{equation}
(v) When $\mathcal{O}$ is a compact convex set, $\Omega_{\mathcal{O}}$ can be computed as the optimal value of a conic linear optimization problem:
\begin{equation}
\Omega_{\mathcal{O}}(E) = \inf\{ \gamma | E \in \mathcal{F}_E \leq \gamma \mathcal{F}_E, \mathcal{F}_M \in \text{cone}(\mathcal{O}') \}
\end{equation}
\begin{equation}
= \sup \left\{ \left( A, J_{\mathcal{F}_E} \right), \left( B, J_{\mathcal{F}_M} \right) \right\} \leq 1, \forall J_{\mathcal{F}_E} \in \mathcal{O}', A, B \geq 0 \right\},
\end{equation}
where $\mathcal{O}'$ represent the set of Choi matrices corresponding to channels in $\mathcal{O}$. The cone(\mathcal{O}') is the closed convex cone
generated by the set $\Omega^J$, and $\text{cone}(\Omega^J)^*$ = \{X | (X, J_M) \geq 0, \forall J_M \in \Omega^J\}. Note that the compactness of $\Omega$ leads to the fact that the infimum is achieved as long as it is finite.

(vi) $\Omega_\Theta$ is lower semicontinuous, i.e. $\Omega_\Theta(E) = \lim_{n \to \infty} \inf_{\Theta(E_n)}$ for any sequence $\{E_n\} \to E$.

(vii) It can be bounded as

$$R_\Theta(E)W_\Theta(E)^{-1} \leq \Omega_\Theta(E) \leq R_\Theta(E)R_{\max}(\mathcal{M}_W^\| \| E),$$

where $\mathcal{M}_W^\|$ is an optimal channel such that $R_\Theta(E) = R_{\max}(\mathcal{M}_W^\| \| E)$, and similarly $\mathcal{M}_W^\|$ is an optimal channel such that $W_\Theta(E) = R_{\max}(\mathcal{M}_W^\| \| E)^{-1}$, whenever such channels exist.

Proof: (i) It immediately is verified by the fact that the quantity $R_{\max}(\rho \| \sigma)$ is finite if and only if $\text{supp} \rho \subseteq \text{supp} \sigma$.

(ii) It follows from the fact that $R_{\max}(\lambda \rho \| \mu M) = \lambda \mu^{-1} R_{\max}(\rho \| M)$ which can be easily verified by the definition of $R_{\max}$.

(iii) If either $\Omega_\Theta(E)$ or $\Omega_\Theta(\mathcal{F})$ is infinite, then the relation is trivial, so assume otherwise. Let $M$ be an optimal channel such that $J_E \leq \lambda J_M$ and $J_M \leq \mu \gamma J_M$ with $\lambda \mu = \Omega_\Theta(E)$, and analogously let $M'$ be an optimal channel such that $J_{E'} \leq \lambda' J_{M'}$ and $J_{M'} \leq \mu' \gamma' J_{M'}$ then

$$J_E + (1-t)J_E' \leq t \lambda J_M + (1-t) \lambda' J_{M'} = \frac{t \lambda M + (1-t) \lambda' M}{t + (1-t)}.$$

where $\frac{t \lambda M + (1-t) \lambda' M}{t + (1-t)} \in \Omega^J$ by convexity of $\Omega$. Then

$$\frac{t \lambda J_M + (1-t) \lambda' J_{M'}}{t + (1-t)} \leq \frac{t \gamma \mu J_E + (1-t) \gamma \mu' J_{E'}}{t + (1-t)} \leq \max\{\mu \gamma M, \mu' \gamma' M\}$$

$$J_E + (1-t)J_E', \quad \mu \gamma M, \mu' \gamma' M.$$

It is easy to see that this Choi matrices is a feasible solution for $\Omega_\Theta((t + (1-t)\mathcal{F}))$, then

$$\Omega_\Theta((t + (1-t)\mathcal{F})) \leq [t + (1-t), \lambda, \lambda'] \max[\mu \gamma M, \mu' \gamma' M]$$

$$\text{max} \{\mu \gamma M, \mu' \gamma' M\}.$$

It is shown that Eq.(10) holds.

(iv) If there is no $M \in \Omega$ such that $J_E = \sup \mathcal{J}_M$, then $\Omega_\Theta(E) = \infty$ and the result is trivial, so we shall assume otherwise. Then let $M \in \Omega$ be any channel such that $J_E \leq \lambda J_M$ and $J_M \leq \mu \gamma J_M$ with $\lambda \mu = \Omega_\Theta(E)$. By the definition of $\Theta \in \mathbb{S}$, we have that $\Theta(M) = M' \in \Omega$. Since $\Theta$ preserves positivity, it holds that $J_E(\Theta^{-1}) \leq \lambda J_E(\Theta^{-1}) = \lambda' J_M$ and $J_M = \Theta(M) \leq \mu \gamma \Theta(M)$, so $M'$ is a valid feasible solution for $\Omega_\Theta(\Theta(D))$, which concludes the proof.

(v) From the positivity of $R_{\max}(\| \| \Omega)$ for any channel $E$ and $M$, we have

$$\Omega_\Theta(E) = \min_{M \in \Omega} \{\inf\{\lambda | J_E \leq \lambda J_M \}, \inf\{\mu | J_M \leq \mu J_E \}\}$$

$$\text{inf}\{\mu | J_E \leq \lambda J_M, J_M \leq \mu J_E, M \in \Omega\}.$$  

Observe that any feasible solution to the problem

$$\inf\{\gamma | J_E \leq J_M \leq \gamma J_E, J_M \in \text{cone}(\Omega^J)\}$$

(18)

with objective function value $\gamma \mu$. Conversely, any feasible solution $\{J_M, \lambda, \mu\}$ to Eq.(17) gives a feasible solution to Eq.(18) as $J_M = \lambda J_M, \gamma = \mu$. Thus, the two problems are equivalent.

Writing the Lagrangian as

$$L(\gamma, \lambda ; A, B) = \gamma - \langle A, J_M + J_E \rangle - \langle B, C \rangle$$

(20)

$$\gamma(1 - (A, J_E)) + (B - A - C, J_M) + (A, J_E).$$

Optimizing over the Lagrange multipliers $A, B \geq 0$ and $C \in \text{cone}(\Omega^J)^*$, the corresponding dual form of primal problem Eq.(18) is written as

$$\sup\{\langle A, J_E \rangle | \langle B, J_E \rangle = 1, B \in \text{cone}(\Omega^J)^*\} = \sup\{\langle A, J_M \rangle | \langle B, J_M \rangle \leq 1, \forall J_M \in \Omega, A, B \geq 0\}.$$  

(21)

(22)

If we take $B = I$ and $A = \epsilon E$ for $0 < \epsilon < 1$, it is obvious that $A$ and $B$ are strictly feasible for the dual. Thus, it follows from Slater’s theorem [64] that strong duality holds, the optimal value of the primal problem Eq.(18) is equal to that of dual problem Eq.(21).

The second line of Eq.(21) follows since any feasible solution to this program can be rescaled as $A \mapsto b A, B \mapsto b B$ to give a feasible solution to the dual, and vice versa. Here, we implicitly constrain ourselves to $B$ such that $(B, J_M) \neq 0$ and $(B, J_M) \neq 0, \forall J_M \in \Omega^J$; this can always be achieved by taking $B$ as a small multiple of the identity.

(vi) Verifying lower semicontinuity of $\Omega_\Theta$ is equivalent to showing that the sublevel sets $\{E | \Omega_\Theta(E) \leq \gamma\}$ are closed for all $\gamma \in \mathbb{R}$ [65]. Consider then a sequence $E_n \to E$ such that $\Omega_\Theta(E_n) \leq \gamma \forall n$ for some $\gamma$, where we can take $\gamma \geq 1$ to avoid trivial cases. By (v), this entails that there exists $\tilde{J}_M \in \text{cone}(\Omega^J)$ such that $J_{E_n} \leq \tilde{J}_M \leq \gamma J_{E_n}$ for each $n$. Since $\{\tilde{J}_M\}$ forms a bounded sequence, by the Bolzano-Weierstrass theorem we can assume that it converges as $\{\tilde{J}_M\} \to \tilde{J}_M$, up to passing to a subsequence. The closedness of $\text{cone}(\Omega^J)$ ensures that $\tilde{J}_M \in \text{cone}(\Omega^J)$. The convergent sequences $\{\tilde{J}_M - J_{E_n}\}$ and $\{\gamma J_{E_n} - \tilde{J}_M\}$ then must converge to positive semi-definite operators by the closedness of the positive semidefinite cone. This gives $J_{E_n} \leq \tilde{J}_M \leq \gamma J_{E_n}$, showing that the sublevel set of $\Omega_\Theta$ is closed. Since $\gamma$ was arbitrary, the desired property is proved.

(vii) The lower bound is obtained by noting that

$$\Omega_\Theta(E) = \min_{M \in \Theta \Omega} R_{\max}(\mathcal{M}_W^\| \| E)$$

$$\geq \min_{M \in \Theta \Omega} R_{\max}(\mathcal{M}_W^\| \| E) \left( \begin{array}{c} \min_{M \in \Theta \Omega} R_{\max}(\mathcal{M}_W^\| \| E) \end{array} \right)$$

$$= R_\Theta(E)W_\Theta(E)^{-1}.$$
The upper bounds follow by using $M_R$ and $M_w$ as feasible solutions in the definition of $\Omega_C$.

**Proposition 1.** For any replacement channel $R_\omega : L(A) \to L(B)$ defined as $R_\omega(\cdot) = \text{Tr}(\cdot)\omega$ with some fixed $\omega \in \mathcal{D}(B)$, it holds that

$$\Omega_C(R_\omega) \geq \Omega_\varepsilon(\omega).$$

(23)

If the class of operations $\circ$ contains all replacement channels $R_\tau$ with $\tau \in \mathbb{F}$, then equality holds in the above.

**Proof.** Taking any dual feasible solution $A, B$ for $\Omega_\varepsilon(\omega)$ and any $\tau \in \mathbb{F}$, we can see that $\langle \tau^T \otimes A, \mathcal{J}_M \rangle = \langle A, M(\tau) \rangle$, $\langle \tau^T \otimes B, \mathcal{J}_M \rangle = \langle B, M(\tau) \rangle$. Since

$$\frac{\langle \tau^T \otimes A, \mathcal{J}_M \rangle}{\langle \tau^T \otimes B, \mathcal{J}_M \rangle} = \frac{\langle A, M(\tau) \rangle}{\langle B, M(\tau) \rangle} \leq 1$$

(24)

for any $M \in \mathcal{O}$ using the Choi-Jamiolkowski isomorphism, it implies that $\tau^T \otimes A$ and $\tau^T \otimes B$ are dual feasible to $\Omega_C(R_\omega)$. This immediately gives that

$$\Omega_C(R_\omega) \geq \sup \left\{ \frac{\langle \tau^T \otimes A, \mathcal{J}_{R_\omega} \rangle}{\langle \tau^T \otimes B, \mathcal{J}_{R_\omega} \rangle} \mid \langle A, \sigma \rangle \leq 1 \forall \sigma \in \mathcal{F}, A, B \geq 0 \right\}$$

$$= \sup \left\{ \frac{\langle A, \omega \rangle}{\langle B, \omega \rangle} \mid \langle A, \sigma \rangle \leq 1 \forall \sigma \in \mathcal{F}, A, B \geq 0 \right\}$$

$$= \Omega_\varepsilon(\omega).$$

(25)

Now, assume that $R_\tau \in \mathcal{O}$ for any $\sigma \in \mathbb{F}$. From the definition of the projective robustness of states, let $\sigma \in \mathcal{F}$ be an optimal state such that $\omega \leq \lambda \sigma$ and $\sigma \leq \mu \omega$ with $\lambda \mu = \Omega_\varepsilon(\omega)$. Moreover, $I \otimes \omega \leq I \otimes (\lambda \sigma) = \lambda J_{R_\omega}$ and $I \otimes \sigma \leq I \otimes (\mu \omega) = \mu J_{R_\omega}$, which imply $\Omega_C(R_\omega) \leq \Omega_\varepsilon(\omega)$. $\Box$

The above proposition explicitly shows the relation between the projective robustness of states and its channel version. As a consequence of part (iv) of Theorem 1 and Proposition 1, we have the following corollary.

**Corollary 1.** Let $\circ$ be any class of free operations, which satisfies $R_\tau \in \mathcal{O}$ for any $\sigma \in \mathbb{F}$. Let $N : L(A) \to L(B)$ be any channel such that:

(i) there exists a free superchannel $\Gamma \in \mathbb{S}$ and a state $\omega$ such that $\Gamma(R_\omega) = N$,

(ii) there exists a free superchannel $\Theta \in \mathbb{S}$ such that $\Theta(N) = R_\omega$.

Then

$$\Omega_C(N) = \Omega_\varepsilon(\omega).$$

(26)

By the projective robustness of channels, Corollary 1 again claims a universal property given in Ref. [38], which states that if some channels can be reversibly interconverted with state resources through free superchannels, then we regard these two types of resources as equivalent.

In this section, we study the properties of the projective robustness of channels in detail, which will lay a good foundation for further research to manipulate channel resources.

### A. Channel resource distillation

Resource distillation is one of the most important operational tasks in resource theory. The dynamical resource distillation can be generally understood as converting a noisy resource channel $\mathcal{E} : L(A) \to L(B)$ into some target channels $\mathcal{T} : L(C) \to L(D)$, which are viewed as "pure" or "perfect" resources. Note that we choose some resourceful channels converting any pure state into the pure state as target channel $\mathcal{T}$. Our purpose is to understand when the transformation $\Theta(\mathcal{E}) = \mathcal{T}$ can be achieved by using the free superchannel $\Theta \in \mathbb{S}$.

However, the practical physical system cannot perfectly transform the channels, i.e., the manipulation of channels always leads to the possibility of error of the transformation. Thus, the core task of resource distillation is to provide a threshold for such error to characterize capabilities and limitations of distillation protocols manipulating quantum resources. To accomplish this task, we start by introducing the worst-case fidelity between the two channels $\mathcal{E}, \mathcal{F} : L(A) \to L(B)$, which is defined as follows [60, 61]

$$F(\mathcal{E}, \mathcal{F}) : = \min_{\rho \in \mathcal{E}} F(\text{id} \otimes \mathcal{E}(\rho), \text{id} \otimes \mathcal{F}(\rho))$$

$$= \min_{\rho \in \mathcal{E}} F(\text{id} \otimes \mathcal{E}(\rho), \text{id} \otimes \mathcal{F}(\rho))$$

where $F(\rho, \sigma) = \|\sqrt{\mathcal{F}} \sqrt{\mathcal{G}}\|_1^2$ is the fidelity, and in the second line the optimization is over all pure input states. Thus, our goal is to achieve a free superchannel satisfying $F(\Theta(E), T) \geq 1 - \varepsilon$ for some small error $\varepsilon > 0$. To quantify how closely the target channel is to a free channel, the fidelity-based measure of the overlap with free channels is defined as

$$F_C(T) = \max_{M \in \mathcal{C}} F(T, M)$$

$$= \max_{M \in \mathcal{C}} \min_{\rho \in \mathcal{E}} F(\text{id} \otimes T(\rho), \text{id} \otimes M(\rho))$$

$$= \min_{\rho \in \mathcal{E}} \min_{\rho \in \mathcal{E}} F(\text{id} \otimes T(\rho), \text{id} \otimes M(\rho)).$$

In the state case, we can write

$$F_\varepsilon(\phi) = \max_{\sigma \in \mathbb{F}} F(\phi, \sigma) = \max_{\sigma \in \mathbb{F}} F(\phi, \sigma).$$

The following result builds a bound on distillation error in channel deterministic transformations.

**Theorem 2.** Let $\mathcal{T} \in \mathcal{CPTP}$ and suppose that $\text{id} \otimes T(\rho)$ is pure for any pure state $\rho$. If there exists a free superchannel $\Theta \in \mathbb{S}$ such that $\Theta(T) = N$ with $F(N, T) \geq 1 - \varepsilon$, then

$$\varepsilon \geq \left( \frac{F_C(T)}{F_C(T) + 1} \right)^{-1}.$$

(30)

**Proof.** The relation is trivial when $\Omega_C(N) = \infty$, so assume otherwise. Then $\Omega_C(N) < \infty$, so let $M \in \mathcal{O}$ be a free channel such that $\mathcal{J}_N \leq \lambda \mathcal{J}_M$ and $\mathcal{J}_M \leq \mu \mathcal{J}_N$ with $\lambda \mu = \Omega_C(N)$. For any $\phi$, we have

$$1 - \varepsilon \leq \langle \text{id} \otimes N(\phi), \text{id} \otimes T(\phi) \rangle$$

$$\leq \lambda \langle \text{id} \otimes M(\phi), \text{id} \otimes T(\phi) \rangle$$

$$\leq \lambda F_\varepsilon(\phi)(T).$$

(31)
\[
\Omega_\otimes(\psi) = \max_{M \in \mathbb{C}} (\text{id} \otimes M(\psi), \text{id} \otimes T(\psi)). \quad \text{From the second and third lines of Eq.}(31), \text{we have}
\]
\[
1 - F^{\psi}_\otimes(T) \leq 1 - (\text{id} \otimes M(\psi), \text{id} \otimes T(\psi))
\]
\[
= \langle \text{id} \otimes \Theta(M)(\psi), 1 - \text{id} \otimes T(\psi) \rangle
\]
\[
\leq \mu \langle \text{id} \otimes N(\psi), 1 - \text{id} \otimes T(\psi) \rangle
\]
\[
\leq \mu \epsilon.
\]

Due to the arbitrariness of \(\psi\), for Eqs.(31) and (32), it immediately holds that
\[
1 - \epsilon \leq \lambda F^{\psi}_\otimes(T)
\]
and
\[
1 - F^{\psi}_\otimes(T) \leq \mu \epsilon.
\]
From Eqs.(33) and (34),
\[
\Omega_\otimes(\psi) \geq \Omega_\otimes(N) = \lambda \mu \geq \frac{(1 - \epsilon)(1 - F^{\psi}_\otimes(T))}{\epsilon F^{\psi}_\otimes(T)},
\]
where we use the monotonicity of \(\Omega_\otimes\). Rearranging, we can immediately get Eq.(30). \(\square\)

The bound above provides a general error threshold that cannot be exceeded by any deterministic protocol in any channel resources.

**Remark 1.** Particularly, it is easy to see that the above theorem still holds if we take \(T\) as unitary channel \(U = U \cdot U^\dagger\) or the replacement channel \(R = \text{Tr}(\cdot)\phi\) for some resourceful pure state \(\phi\). Moreover, when the input and target channels are the preparation channels \(P_\psi\) and \(P_\psi\), respectively, we have
\[
\epsilon \geq \left(\frac{F_{\psi}(\Omega_{\psi}(\rho)) + 1}{\epsilon F_{\psi}(\Omega_{\psi}(\rho))}\right)^{-1},
\]
which is consistent with the result of deterministic state resource distillation in Ref.\[40\].

Notice that part (iv) of Theorem 1 provides a necessary condition for the deterministic transformation of single-channel. Next, we will provide a necessary condition for the deterministic transformation from multiple channels to single-channel in the most general protocol, i.e., \(\Omega_\otimes\) satisfies sub- or super-multiplicativity under any free quantum process.

**Sub- or supermultiplicativity.** To construct the most general protocol to manipulate multiple channels (including parallel, sequential, or adaptive, with or without a definite causal order), the set of free quantum processes was defined as follows \[33\]
\[
\mathcal{S}_n = \{ T | T(M_1, \ldots, M_n) \in \mathcal{O}, \forall M_1, \ldots, M_n \in \mathcal{O}\}, \quad (36)
\]
where the \(n\)-channel quantum process \(T\) is an \(n\)-linear map which transforms \(n\)-channels to a single channel and we only require \(T(N_1, \ldots, N_n) \in \text{CPTP}\) for any \(N_1, \ldots, N_n \in \text{CPTP}\). Obviously, superchannel is a special case of quantum process \(T\) where ones take a single channel as the input.

**Theorem 3.** Let \((E_1, \ldots, E_n)\) be a collection of \(n\) channels. For any free protocol \(T \in \mathcal{S}_n\), it holds that
\[
\Omega_\otimes(\psi(E_1, \ldots, E_n)) \leq \prod_i \Omega_\otimes(\psi_i).
\]

In particular, let the \(n\)-tuple \((E_1, \ldots, E_n) =: E^{\otimes n}\) represent an application of \(n\) copies of the same channel, we have
\[
\Omega_\otimes(\psi(E^{\otimes n})) \leq \Omega_\otimes(\psi)\otimes \mathcal{F}.
\]

**Proof.** For each \(E_i\), we assume that \(\Omega_\otimes(E_i)\) is finite. Otherwise, the result is trivial. Let \(M_i \in \mathcal{O}\) be an optimal channel such that \(\mathcal{F}_E \leq \lambda_i M_i\) and \(M_i \leq \lambda_i M_i\) with \(\lambda_i \mu_i = \Omega_\otimes(E_i)\). Using the \(n\)-linearity of the transformation \(T\), we have
\[
T(E_1, \ldots, E_n) = T(E_1 - \lambda_1 M_1, E_2, \ldots, E_n) + T(\lambda_1 M_1, E_2, \ldots, E_n) + T(E_1, \lambda_1 M_2, E_3, \ldots, E_n) + \ldots
\]
\[
+ T(E_1, \ldots, \lambda_1 M_{n-1}, E_n - \lambda_n M_n) + T(\lambda_1 M_1, \ldots, \lambda_n M_n).
\]

By the positivity of \(T\), we have
\[
0 \leq T(E_1, \ldots, E_n) - T(E_1, \ldots, E_n)
\]
\[
= \prod_i \lambda_i \cdot T(M_1, \ldots, M_n) - T(E_1, \ldots, E_n)
\]
\[
= \prod_i \lambda_i \cdot M - T(E_1, \ldots, E_n)
\]
and
\[
0 \leq T(E_1, \ldots, E_n) - T(M_1, \ldots, M_n)
\]
\[
= \prod_i \mu_i \cdot T(E_1, \ldots, E_n) - T(M_1, \ldots, M_n)
\]
\[
= \prod_i \mu_i \cdot T(E_1, \ldots, E_n) - M
\]
for \(\mathcal{T}(M_1, \ldots, M_n) = M' \in \mathcal{O}\). From Eqs.(40) and (41), it follows that \(\prod_i \Omega_\otimes(E_i) \leq \prod_i \lambda_i \mu_i\) is a feasible optimal value for \(\Omega_\otimes(T(E_1, \ldots, E_n))\), which leads to Eq.(37). \(\square\)

Since the tensor product and composition are two special cases of free quantum process, the following corollary is immediately concluded.

**Corollary 2.** If the resource theory is well-behaved under tensor product, i.e., \(M, M' \in \mathcal{O} \Rightarrow M \otimes M' \in \mathcal{O}\), then
\[
\Omega_\otimes(E \otimes \mathcal{F}) \leq \Omega_\otimes(E)\otimes \mathcal{F}.
\]
If the resource theory is well-behaved under composition, i.e., $\mathcal{M}, \mathcal{M}' \in \mathcal{O} \Rightarrow \mathcal{M} \circ \mathcal{M}' \in \mathcal{O}$, then
\begin{equation}
\Omega_{\mathcal{O}}(\mathcal{E} \circ \mathcal{T}) \leq \Omega_{\mathcal{O}}(\mathcal{E}) \Omega_{\mathcal{O}}(\mathcal{T}).
\end{equation}

The universal bound of Theorem 2 and the sub- or super-multicopy of $\Omega_{\mathcal{O}}$ can provide limitations on the overhead required in the most general many-copy channel transformation protocol.

**Theorem 4.** For any distillation protocol $\mathcal{T} \in \mathcal{S}(n)$ which transforms $n$ uses of a channel $\mathcal{E}$ to some target resourceful channel $\mathcal{T}$ up to accuracy $\epsilon > 0$, it necessarily holds that
\begin{equation}
n \geq \log_{\Omega_{\mathcal{O}}(\mathcal{E})}(1 - \epsilon)(1 - F_{\mathcal{O}}(\mathcal{T}))
\end{equation}
where the last inequality follows from any feasible protocol $\mathcal{T}$.

**Proof.** It immediately follows from Eqs. (30) and (37). \hfill \Box

**Remark 2.** Particularly, it is easy to see that the above theorem still holds if we take $\mathcal{T}$ as unitary channel $\mathcal{U} = U \cdot U^\dagger$ or the replacement channel $\mathcal{R}_\mathcal{E} = \text{Tr}(\cdot \phi)$ for some resourceful pure state $\phi$. Moreover, when the input and target channels are the preparation channels $\mathcal{P}_\rho$ and $\mathcal{P}_\phi$, respectively, this result reduces to the result of determinist state distillation overheads in Ref. [27], i.e., $n \geq \log_{\Omega_{\mathcal{O}}(\mathcal{E})}(1 - \epsilon)(1 - F_{\mathcal{O}}(\mathcal{T}))$, which characterize the number of copies of a state needed to perform the distillation $\mathcal{M}(\rho^m) \rightarrow \phi$ up to error $\epsilon$.

**B. Interpretation in state discrimination**

In this section, we will show a final application of $\Omega_{\mathcal{O}}$ in a variant of state discrimination task. Let us consider a scenario in which an ensemble of quantum states is given by $\{p_i, \sigma_i\}$ with $p_i \geq 0$ and $\sum_i p_i = 1$, and a given bipartite channel $\mathcal{E}$ applied to every state. By measuring the output of this process with a chosen POVM $\{M_i\}$, we aim to distinguish which of the states was sent. The average probability of successfully discriminating the states is given by
\begin{equation}
p_{\text{succ}}(\{p_i, \sigma_i\}, \{M_i\}, \mathbb{I} \otimes \Lambda) = \sum_i p_i (\mathbb{I} \otimes \Lambda(\sigma_i), M_i).
\end{equation}

When we consider state discrimination, our aim is to choose a measurement to maximize Eq. (45). The $\mathcal{R}_\mathcal{E}$ is known to quantify the advantage that a given channel can provide over all free channels [27]. When we consider state exclusion, our goal is to do our best to determine which states were not sent. Then, Eq. (45) is understood as the average probability of guessing incorrectly, and our aim is to minimize this quantity. In state exclusion, it is the $\mathcal{W}_\mathcal{O}$ which quantifies the advantage [38][39]. Next, we can show that $\Omega_{\mathcal{O}}(\Lambda)$ quantifies the advantage when state discrimination and exclusion tasks are considered at the same time.

**Theorem 5.** The projective robustness of channels quantifies the maximal advantage that a given channel $\Lambda$ gives over all free channels $\mathcal{E} \in \mathcal{O}$ in simultaneous discrimination and exclusion of a fixed state ensemble, as quantified by the ratio
\begin{equation}
p_{\text{succ}}(\{p_i, \sigma_i\}, \{M_i\}, \mathbb{I} \otimes \Lambda) / p_{\text{succ}}(\{p_i, \sigma_i\}, \{N_i\}, \mathbb{I} \otimes \Lambda),
\end{equation}
where we aim to perform state discrimination with measurement $\{M_i\}$ and state exclusion with measurement $\{N_j\}$.

Specifically,
\begin{equation}
\sup_{\{p_i, \sigma_i\}, \{M_i\}, \{N_i\}} \frac{p_{\text{succ}}(\{p_i, \sigma_i\}, \{M_i\}, \mathbb{I} \otimes \Lambda)}{p_{\text{succ}}(\{p_i, \sigma_i\}, \{N_i\}, \mathbb{I} \otimes \Lambda)} = \Omega_{\mathcal{O}}(\Lambda),
\end{equation}
where the maximization is over all finite state ensembles $\{p_i, \sigma_i\}_{i=1}^n$ and all POVMs $\{M_i\}_{i=1}^n$, $\{N_i\}_{i=1}^n$ for which the expression in Eq. (47) is well defined.

**Proof.** Let us assume that $\Omega_{\mathcal{O}}(\Lambda)$ is finite and take $\mathcal{J}_\Lambda \leq \lambda \mathcal{J}_\mathcal{E}$ and $\mathcal{J}_\mathcal{E} \leq \mu \mathcal{J}_\Lambda$ such that $\Omega_{\mathcal{O}}(\Lambda) = \mu$. Let $\{p_i, \sigma_i\}$ be any state ensemble, and $\{M_i\}_{i=1}^n$, $\{N_i\}_{i=1}^n$ be any feasible POVMs. Then
\begin{equation}
p_{\text{succ}}(\{p_i, \sigma_i\}, \{M_i\}) \mathbb{I} \otimes \Lambda) / p_{\text{succ}}(\{p_i, \sigma_i\}, \{N_i\}) \mathbb{I} \otimes \Lambda) \leq \lambda \mu \max_{\mathcal{E} \in \mathcal{O}} p_{\text{succ}}(\{p_i, \sigma_i\}, \{M_i\}) \mathbb{I} \otimes \Xi) / p_{\text{succ}}(\{p_i, \sigma_i\}, \{N_i\}) \mathbb{I} \otimes \Xi),
\end{equation}
where the first inequality follows by the complete positivity of $\Lambda$ and linearity of $p_{\text{succ}}$. This implies that $\Omega_{\mathcal{O}}(\Lambda)$ always upper bounds the left hand side(LHS) of Eq.(47). Next, we choose two POVMs $\{M_i\}_{i=1}^n = \{A, \mathbb{I} \otimes \Lambda\}$ and $\{N_i\}_{i=1}^n = \{B, \mathbb{I} \otimes \Lambda\}$ where $A$ and $B$ are any dual feasible solutions for $\Omega_{\mathcal{O}}(\Lambda)$, and define the state ensemble $\{p_i, \sigma_i\}_{i=1}^n$ as $p_i = 1$, $\sigma_i = \Phi^+ \Phi^* |$ and $\mathbb{I} \otimes \Lambda(\Phi^*) | \mathbb{I} \otimes \Lambda) = \{A, \mathcal{J}_\Lambda\} \min \{B, \mathcal{J}_\mathcal{E}\} / \{B, \mathcal{J}_\mathcal{E}\}$ (49)

\begin{equation}
\frac{p_{\text{succ}}(\{p_i, \sigma_i\}, \{M_i\}) \mathbb{I} \otimes \Lambda)}{p_{\text{succ}}(\{p_i, \sigma_i\}, \{N_i\}) \mathbb{I} \otimes \Lambda)} = \frac{(A, \mathcal{J}_\Lambda)(B, \mathcal{J}_\mathcal{E})}{(B, \mathcal{J}_\mathcal{E})(A, \mathcal{J}_\Lambda)}.
\end{equation}

where the last inequality follows from any feasible $A, B$ satisfying $\{A, \mathcal{J}_\mathcal{E}\} \leq \{B, \mathcal{J}_\mathcal{E}\}, \forall \mathcal{E} \in \mathcal{O}$. Here we have restricted operators $A, B$ such that $\{A, \mathcal{J}_\mathcal{E}\} \neq 0 \neq \{B, \mathcal{J}_\mathcal{E}\}$ and $\{A, \mathcal{J}_\mathcal{E}\} \neq 0 \neq \{B, \mathcal{J}_\mathcal{E}\}, \forall \mathcal{E} \in \mathcal{O}$ to ensure that Eq.(47) is well defined. Since the supremum over all feasible $A, B$ is precisely $\Omega_{\mathcal{O}}(\Lambda)$, this shows that the projective robustness of channels is a lower bound on the LHS of Eq.(47). The proof is completed. \hfill \Box

Note that the above theorem applies to any convex resource theory of channels, providing a general operational meaning for the projective robustness of channels.

**IV. PROJECTIVE ROBUSTNESS OF MEASUREMENTS**

It is well-known that any meaningful information process involves a measurement at the end, so it is quite natural
and necessary to study the quantification and manipulation of quantum measurement resources [14–18, 27, 31, 37, 39]. In this section, we will introduce a new measurement resource monotone with a clear operational interpretation. To accomplish this goal, we start by recalling the generalized robustness.

Definition 2. For a given measurement \( M \), its projective robustness with respect to \( M \) is defined as

\[
R_M(p;M) = \inf \{ \lambda | M_i \leq \lambda F_i, \forall i \} \in M(d, n), \quad \{ F_i \} \in M_F \},
\]

\[
W_M(p;M) = \inf \{ w | M_i = w N_i + (1 - w) F_i, \forall i \} \in M(d, n), \quad \{ F_i \} \in M_F \}.
\]

(50)

In fact, it is easy to verify that they are equivalent to the following forms:

\[
R_M(p;M) = \inf \{ \lambda | M_i \leq \lambda F_i, \forall i \} \in M_F \},
\]

\[
W_M(p;M) = \sup \{ \mu | M_i \geq \mu F_i, \forall i \} \in M_F \}.
\]

(51)

where inequality \( M_i \leq \lambda N_i \) is understood as ordering of positive semidefinite operators.

Theorem 6. The projective robustness of channels \( \Omega_{M_F}(p;M) \) satisfies the following properties:

(i) \( \Omega_{M_F}(p;M) \) is finite if and only if there exists a free measurement \( N \in M_F \) such that \( \text{supp}(M_i) = \text{supp}(N_i) \) for all \( i \).

(ii) \( \Omega_{M_F}(p;M) = \tilde{\Omega}_{M_F}(p;M) \) for any \( k > 0 \).

(iii) When \( M_F \) is a convex set, \( \Omega_{M_F} \) is quasi-convex: for any \( t \in [0, 1] \), it holds that

\[
\Omega_{M_F}(p_t M + (1 - t)N) \leq \max \{ \Omega_{M_F}(p_M), \Omega_{M_F}(N) \}.
\]

(53)

(iv) \( \Omega_{M_F}(\Gamma(M)) \leq \Omega_{M_F}(p;M) \) for any free operation \( \Gamma \in O_F \), i.e. \( \Gamma(M_F) \subseteq M_F \).

(v) When \( M_F \) is a compact convex set, \( \Omega_{M_F} \) can be computed as the optimal value of a conic linear optimization problem:

\[
\Omega_{M_F}(p;M) = \inf \gamma \in \mathbb{R}_+ \left\{ M_i \leq N_i \leq \gamma M_i, \{ N_i \} \in \text{cone}(M_F^\circ) \right\}
\]

\[
\sup \left\{ \sum_i \langle A_i, M_i \rangle \left| \sum_i \langle B_i, M_i \rangle = 1, A_i, B_i \geq 0, \forall i, \right. \right\}
\]

\[
\left. \left\{ B_i - A_i \right\} \in \text{cone}(M_F^\circ) \right\}.
\]

(54)

\[
\Omega_{M_F}(p;M) = \inf \left\{ M_i \leq N_i \leq \gamma M_i, \{ N_i \} \in \text{cone}(M_F^\circ) \right\}
\]

Theorem 7. The projective robustness of channels quantifies the maximal advantage that a given measurement \( M = \{ M_i \} \) gives over all free measurements \( N \in M_F \) in simultaneous discrimination of a fixed state ensemble \( \{ p_i, \sigma_i \} \) and exclusion of a fixed state ensemble \( \{ q_i, \tau_i \} \), as quantified by the ratio

\[
p_{\text{succ}}(\{ p_i, \sigma_i \}, \{ M_i \}) \leq \frac{p_{\text{succ}}(\{ q_i, \tau_i \}, \{ M_i \})}{p_{\text{succ}}(\{ q_i, \tau_i \}, \{ M_i \})} \leq \frac{p_{\text{succ}}(\{ p_i, \sigma_i \}, \{ N_i \})}{p_{\text{succ}}(\{ q_i, \tau_i \}, \{ N_i \})}.
\]

(57)

where we aim to perform state discrimination and exclusion with the same measurement \( M \).

Specifically,

\[
\max_{\gamma \in \mathbb{R}_+} \frac{p_{\text{succ}}(\{ p_i, \sigma_i \}, \{ M_i \})}{p_{\text{succ}}(\{ q_i, \tau_i \}, \{ M_i \})} = \Omega_{M_F}(p;M),
\]

(58)

where the maximization is over all finite state ensembles \( \{ p_i, \sigma_i \}_{i=1}^n \) and \( \{ q_i, \tau_i \}_{i=1}^n \) for which the expression in Eq.(57) is well defined.

Proof. Let us assume that \( \Omega_{M_F}(p;M) \) is finite and take \( M_i \leq \lambda N_i \) and \( N_i \leq \mu M_i \) such that \( \Omega_{M_F}(p;M) = \lambda \mu \). Let \( \{ p_i, \sigma_i \} \) and \( \{ q_i, \tau_i \} \) be any state ensembles, and \( \{ M_i \}_{i=1}^n \) be a given POVM measurement. Then

\[
p_{\text{succ}}(\{ p_i, \sigma_i \}, \{ M_i \}) \leq \lambda p_{\text{succ}}(\{ p_i, \sigma_i \}, \{ N_i \}) \leq \lambda p_{\text{succ}}(\{ q_i, \tau_i \}, \{ N_i \}) \leq \lambda \mu \max_{\gamma \in \mathbb{R}_+} \frac{p_{\text{succ}}(\{ p_i, \sigma_i \}, \{ N_i \})}{p_{\text{succ}}(\{ q_i, \tau_i \}, \{ N_i \})},
\]

where the first inequality follows by the linearity of \( p_{\text{succ}} \). This implies that \( \Omega_{M_F}(p;M) \) always upper bounds the LHS of Eq.(57).

\[
\end{proof}
Consider any dual feasible solutions \( \{A_i\} \) and \( \{B_i\} \) for \( \Omega_m (\mathbb{M}) \) in dual problem Eq.(54). We define the state ensembles \( \{p_i, \sigma_i\} \) and \( \{q_i, \tau_i\} \) as

\[
\begin{align*}
\{p_i, \sigma_i\} &= \begin{cases} 
\left\{ \frac{\langle A_i \rangle}{\sum_i \langle A_i \rangle}, A_i \right\}, & \text{if } \langle A_i \rangle > 0, \\
\{0, \sigma\}, & \text{if } \langle A_i \rangle = 0,
\end{cases} \\
\{q_i, \tau_i\} &= \begin{cases} 
\left\{ \frac{\langle B_i \rangle}{\sum_i \langle B_i \rangle}, B_i \right\}, & \text{if } \langle B_i \rangle > 0, \\
\{0, \sigma\}, & \text{if } \langle B_i \rangle = 0,
\end{cases}
\end{align*}
\tag{59}
\]

where \( \sigma \) is an arbitrary state, and

where the last inequality is since any feasible \( A_i, B_i \) satisfy \( \sum_i \langle N_i, A_i \rangle \leq \sum_i \langle N_i, B_i \rangle \), \( \forall \{N_i\} \in M_F \). Here we have constrained ourselves to the operators \( A_i, B_i \) such that \( \sum_i \langle M_i, A_i \rangle \neq 0 \neq \sum_i \langle M_i, B_i \rangle \) and \( \sum_i \langle N_i, A_i \rangle \neq 0 \neq \sum_i \langle N_i, B_i \rangle \), \( \forall \{N_i\} \in M_F \) to ensure that Eq.(57) is well defined. Since the supremum over all feasible \( A_i, B_i \) is precisely \( \Omega_m (\mathbb{M}) \), this establishes the projective robustness of channels as a lower bound on the LHS of Eq.(57). The proof is completed. \( \Box \)

Note that the above theorem applies to any convex resource theory of measurements, thus providing a general operational meaning for the projective robustness of measurements.

In Ref. [45], it is shown that any incompatible measurement gives an advantage over all compatible measurements in quantum state discrimination. It is easy to see that the set of all compatible measurements forms a convex resource theory of measurements. Next, we will discuss the projective robustness of measurement incompatibility.

**Measurement incompatibility.** The set of POVMs \( \{M_{\text{dx}}\} \), where \( M_{\text{dx}} \) is the POVM element with outcome \( a \) for the measurement labeled by setting \( x \), is called compatible (or jointly measurable) [16] if there exists a parent measurement \( \{N_i\} \) and a conditional probability distribution \( q(a \mid x, i) \) such that \( M_{\text{dx}} = \sum_{a, x} q(a \mid x, i) N_i \), where \( a = 1, \ldots, n \) and \( x = 1, \ldots, m \). Let \( \mathcal{M} \) denote all sets of POVMs, and \( \mathcal{C} \) denote the set of compatible POVMs. For any set of POVMs \( \{M_{\text{dx}}\} \), we can directly define the projective robustness of measurement incompatibility as

\[
\Omega_{\mathbb{C}} \left( \{M_{\text{dx}}\} \right) = \text{inf} \{ \mu \mid M_{\text{dx}} \leq \lambda N_{\text{dx}}, N_{\text{dx}} \leq \mu M_{\text{dx}}, \forall a, x, N_{\text{dx}} \in \mathcal{C} \}.
\tag{62}
\]

In Ref. [36], a specific theory of channel based on the theory of measurement incompatibility was formulated by taking \( \Omega_{\mathbb{C}} \) to be the set of channels representing the sets of POVMs as

\[
\Omega_{\mathbb{C}} := \left\{ E_{\{M_{dx}\}} \mid \{M_{dx}\} \in \mathcal{M} \right\}.
\tag{63}
\]

where we defined

\[
E_{\{M_{dx}\}} (\sigma \otimes \rho) := \sum_{x,a} \langle x | \sigma | x \rangle \langle M_{dx} | x \rangle |a\rangle \langle a| \tag{64}
\]

where \( \{x\} \) and \( \{a\} \) are orthonormal bases representing classical variables for measurement settings and measurement outcomes, respectively. Naturally, the set of free channels \( \mathcal{O} \subseteq \mathcal{O}_{\mathbb{C}} \) was defined using the set of compatible POVMs \( \mathcal{C} \)

\[
\mathcal{O} := \left\{ E_{\{M_{dx}\}} \right\} \{M_{dx}\} \in \mathcal{C} \}. \tag{65}
\]

**Proposition 2.** Let \( \mathcal{O}_{\mathbb{C}} \) be the set of bipartite channels defined in (63) and \( \mathcal{O} \) be the set of free channels that represents the compatible POVMs. Then, it holds that

\[
\Omega_{\mathbb{C}} (\{M_{dx}\}) = \Omega_{\mathbb{C}} (E_{\{M_{dx}\}}).
\tag{66}
\]

**Proof.** Following the definition of \( \Omega_{\mathbb{C}} \),

\[
\Omega_{\mathbb{C}} \left( E_{\{M_{dx}\}} \right) = \text{inf} \{ \mu \mid E_{\{M_{dx}\}} \leq \gamma F, F \leq \mu E_{\{M_{dx}\}}, F \in \mathcal{O} \} = \text{inf} \{ \mu \mid E_{\{M_{dx}\}} \leq \lambda E_{\{F_{dx}\}}, E_{\{F_{dx}\}} \leq \mu E_{\{M_{dx}\}}, F_{dx} \in \mathcal{O} \} \leq \text{inf} \{ \mu \mid \forall \sigma, \rho, \sum_{x,a} \langle x | \sigma | x \rangle \langle \lambda F_{dx} - M_{dx}, \rho \rangle |a\rangle \langle a| \geq 0, \forall F_{dx} \in \mathcal{O} \} = \text{inf} \{ \mu \mid M_{dx} \leq \lambda F_{dx}, F_{dx} \leq \mu M_{dx} \forall a, x, F_{dx} \in \mathcal{O} \}, \tag{67}
\]

where the last equality is obtained by observing that in the expression of the third line, the fact that two inequalities hold for any \( \sigma \), as well as that \( |a\rangle \langle a| \) is a classical state, implies that \( \langle \lambda F_{dx} - M_{dx}, \rho \rangle \geq 0 \) and \( \langle \mu F_{dx} - F_{dx}, \rho \rangle \geq 0 \), \( \forall a, x \), and further imposing that this holds for any \( \rho \) gives \( \lambda F_{dx} - M_{dx} \geq 0 \) and \( \mu F_{dx} - F_{dx} \geq 0 \), \( \forall a, x \). On the other hand, \( \lambda F_{dx} - M_{dx} \geq 0 \) and \( \mu F_{dx} - F_{dx} \geq 0 \), \( \forall a, x \) imply the expression in the third line. Note that the last expression is precisely \( \Omega_{\mathbb{C}} (\{M_{dx}\}) \). The proof is completed. \( \Box \)

Proposition 2 shows that the projective robustness of measurement incompatibility coincides with the channel-based measure defined in our framework. This helps to further study resource manipulation of incompatible measurements.

**V. CONCLUSION**

In this work, we introduced projective robustness for channels and measurements in any convex quantum resource theory and discussed their operational interpretations in simultaneous discrimination and exclusion of states. First, we defined the projective robustness of channels in any convex dynamic resource theory and showed that it satisfies some good properties such as quasicom convexity, invariance under scaling, lower semicontinuity, and sub- or supermultiplicativity. Importantly, it can always be computed as a convex (conic) optimization
problem. Moreover, the projective robustness of channels provided the lower bounds on the error and many-copy distillation overhead in any deterministic channel resource distillation task. Meanwhile, the projective robustness of channels could be regarded as the maximal advantage provided by a given resourceful channel over all resourceless channels in simultaneous discrimination and exclusion of a fixed state ensemble by an application of the bipartite channel. Second, we defined the projective robustness of measurements in any convex resource theory of measurements. Similarly, we proved that it has some good properties and showed that it can exactly quantify the maximal advantage provided by a given resourceful measurement over all resourceless measurements in the simultaneous discrimination and exclusion of two fixed state ensembles. Finally, in a specific channel resource setting based on measurement incompatibility, we found an equivalence relation between the projective robustness for quantum channels and the projective robustness for measurement incompatibility.

Our results apply to general convex resource theories of channels and measurements. We hope that the work can deepen one’s comprehension of the quantification and manipulation of channel resources and effectively facilitate one’s capacity of exploiting dynamic resources.

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APPENDIX A: THE PROOF OF THEOREM 3

The proofs of (i)-(iii) are similar as the (i)-(iii) of Theorem 3.

(iv) If $\Omega_M(M) = \infty$, then the result is trivial, so we shall assume otherwise. Let $\{F_i\} \in M_f$ be a free measurement such that $M_i \leq \lambda F_i$ and $F_i \leq \mu M_i$ for any $i$ with $\Omega_M(M) = \lambda \mu$. Since $\Gamma$ is a free operation, we have $\{\Gamma(F_i)i\} \in M_f$, and $\Gamma(M_i) \leq \lambda \Gamma(F_i)$ and $\Gamma(F_i) \leq \mu \Gamma(M_i)$ for any $i$. So we can directly get $\Omega_M(M) \leq \Omega_M(M_f)$.

(v) We can see that

$$\Omega_M(M) = \min_{\lambda \in M_f} \left[ \inf_{\mu \in M_f} | M_i \leq \lambda N_i, \forall i | \inf_{\mu | N_i \leq \mu M_i, \forall i} | \mu | \right].$$

Let $\tilde{N}_i = \lambda N_i$ for all $i$. Observe that any feasible solution to the problem

$$\inf \left\{ \gamma \mid M_i \leq \tilde{N}_i \leq \gamma M_i, \tilde{N}_i \in \text{cone}(M_f) \right\}$$

(68)
gives a feasible solution to Eq.(68) as

$$N_i = \frac{\tilde{N}_i}{\sum_i \tilde{N}_i}, \lambda = \frac{\sum_i \langle \tilde{N}_i \rangle}{d}, \mu = \frac{\gamma}{\sum_i \langle \tilde{N}_i \rangle}. \quad (70)$$

with objective function value $\lambda \mu = \gamma$. Conversely, any feasible solution $\{N_i, A, \mu\}$ to Eq.(69) gives a feasible solution to Eq.(68) as $\tilde{N}_i = \lambda N_i$, $\forall i$ and $\gamma = \lambda \mu$. Thus, the two problems are equivalent.

Writing the Lagrangian as

$$L(\gamma, \tilde{N}_i; \{A_i\}, \{B_i\}, \{C_i\}) = \gamma - \sum_i \langle A_i, \tilde{N}_i - M_i \rangle - \sum_i \langle B_i, \gamma M_i - \tilde{N}_i \rangle - \sum_i \langle C_i, \tilde{N}_i \rangle = \gamma [1 - \sum_i \langle B_i, M_i \rangle] + \sum_i \langle A_i, M_i \rangle + \sum_i \langle B_i - A_i - C_i, \tilde{N}_i \rangle. \quad (71)$$

Optimizing over the Lagrange multipliers $A_i, B_i \geq 0$ for all $i$, and $\{C_i\} \in \text{cone}(M_f)^+$, the corresponding dual form of primal problem Eq.(69) is written as

$$\sup \left\{ \sum_i \langle A_i, M_i \rangle \left| \sum_i \langle B_i, M_i \rangle = 1, A_i, B_i \geq 0 \forall i, \{B_i - A_i\} \in \text{cone}(M_f)^+ \right\}$$

(72)

$$= \sup \left\{ \frac{\sum_i \langle A_i, M_i \rangle}{\sum_i \langle B_i, M_i \rangle} \left| A_i, B_i \geq 0 \forall i, \frac{\sum_i \langle A_i, N_i \rangle}{\sum_i \langle B_i, N_i \rangle} \leq 1, \forall \{N_i\} \in M_f \right\}. \quad (73)$$

If we take $B_i = I$ and $A_i = \epsilon I$ for all $i$ where $0 < \epsilon < 1$, it is obvious that $A_i$ and $B_i$ are strictly feasible for the dual. Thus, it follows from Slater’s theorem [64] that strong duality holds, the optimal value of the primal problem Eq.(69) is equal to that of dual problem Eq.(72).

The second line of Eq.(72) follows since any feasible solution to this program can be rescaled as $A_i \mapsto \frac{A_i}{\sum_i \langle A_i, M_i \rangle}$, $B_i \mapsto \frac{B_i}{\sum_i \langle B_i, M_i \rangle}$ to give a feasible solution to the dual, and vice versa. Here, we explicitly constrain ourselves to $B_i$ such that $\sum_i \langle B_i, M_i \rangle \neq 0$ and $\sum_i \langle B_i, N_i \rangle \neq 0 \forall \{N_i\} \in M_f$; this can always be ensured by taking $B_i$ as a small multiple of the identity.

(vi) The lower bound is obtained by noting that

$$\Omega_M(M) = \min_{\lambda \in M_f} R_{\max}(M) R_{\max}(N) R_{\max}(M) \quad (73)$$

The upper bounds follow by using $N_R^L$ and $N^W_R$ as feasible solutions in the definition of $\Omega_M$. 

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