Convergence of level sets in total variation denoising without source condition

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Abstract

We present some results of geometric convergence of level sets for solutions of total variation denoising as the regularization parameter tends to zero. The common feature among them is that they make use of explicit constructions of variational mean curvatures for general sets of finite perimeter. Consequently, no additional regularity of the level sets of the ideal data is assumed, but other restrictions on it or on the noise are required.

1 Introduction and main results

We aim to provide a precise analysis of the generalized Rudin-Osher-Fatemi denoising scheme based on total variation minimization in the low noise regime, in general dimension and with no source condition assumptions. More precisely, given a real function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \), some ideal data \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) with compact support, a perturbation \( w \), as well as a regularization parameter \( \alpha > 0 \), we consider minimizers of

\[
\inf_{u \in BV(\mathbb{R}^d)} \int_{\mathbb{R}^d} \psi(u - f - w) + \alpha \text{TV}(u).
\]

We make the following assumptions on the function \( \psi \) appearing in the data term and its Fenchel conjugate \( \psi^* \):

\[
\begin{align*}
\psi & \text{ is strictly convex and even with } \psi(0) = 0, \psi(t) > 0 \text{ for } t \neq 0, \\
|\psi(s)| & \leq C|s|^{d/(d-1)} \text{ for some } C > 0, \text{ and } \psi^* \text{ is uniformly convex.}
\end{align*}
\]

(A)

If \( 1 < p \leq 2 \) the functions \( t \mapsto |t|^p/p \) satisfy these convexity properties [9, Example 5.3.10], so in particular the case \( p = d/(d-1) \) satisfies all the conditions of Assumption (A).

Remark 1.1. \( \psi^* \) being uniformly convex implies that \( \psi \) is differentiable with \( \psi' \) uniformly continuous [9, Thm. 5.3.17, Prop. 4.2.14], in particular \( \psi \in C^1(\mathbb{R}) \). Moreover, strict convexity of \( \psi \) implies that \( \psi^* \) is also differentiable [9, Thm. 5.3.7]. We will use both of these properties in the sequel.

We study the regime in which \( \alpha \) and \( w \) tend to zero simultaneously, for which under natural assumptions it is easy to prove (see Proposition 1.5 below) that the unique minimizers \( u_{\alpha,w} \) of (1) converge to \( f \) in the strong \( L^1_{\text{loc}} \) topology. In particular, if along a sequence the solutions \( u_{\alpha_n,w_n} \) have a common compact support, we have, using Fubini’s theorem [24, Thm. 2], for a.e. \( s > 0 \) that

\[
|\{u_{\alpha_n,w_n} > s\} \Delta \{f > s\}| \rightarrow 0.
\]

2020 Mathematics Subject Classification: 49Q20, 53A10, 68U10, 49Q05.

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Moreover, this can in some cases be improved to Hausdorff convergence, which can be interpreted as geometric uniform convergence. This type of convergence has been proved in [15] for classical ROF denoising in the plane, and in [24, 23] for linear inverse problems, bounded domains, Banach space measurements and general dimensions. All of these results (with the exception on when explicit dual certificates are known [15, Sec. 8]) assume a source condition of the type \( \partial TV(f) \neq \emptyset \). On the one hand this condition guarantees, in particular, that the level sets of the minimizers \( u_{\alpha,w} \) satisfy uniform density estimates independent of \( \alpha \) and \( w \), as long as these are related through a certain condition in the parameter choice. On the other, this source condition excludes cases of interest where geometric convergence is expected, like the case when \( f \) is the indicatrix of a planar polygon [15, Sec. 3.3].

Our main goal is to obtain this improved mode of convergence in (2) while assuming as little regularity of \( \{ f > s \} \) as possible, and this is achieved in two different situations. The first is when \( f \) is the characteristic function of a bounded finite perimeter set, and admitting noisy measurements with a natural parameter choice. The second concerns a generic class of BV functions in which “flat regions are controlled” and including piecewise constant functions, but with noiseless measurements. The techniques used have as a central point the variational mean curvatures for general finite perimeter sets introduced in [8, 6] which, through comparison arguments, are used as a lower integrability replacement for the missing dual certificates for \( f \).

### 1.1 Main results

**Theorem 1.2.** Assume that \( f = 1_D \), the indicatrix of a bounded finite perimeter set \( D \subset \mathbb{R}^d \), and that the sequences \( \alpha_n \to 0 \) and \( w_n \) are such that

\[
\frac{\| w_n \|_{L^{d/(d-1)}(\mathbb{R}^d)}}{\alpha_n} \leq C_{\psi,d} \frac{m_{\psi^*}(\Theta_d)}{\Theta_d},
\]

where \( \Theta_d \) denotes the isoperimetric constant in \( \mathbb{R}^d \) and \( m_{\psi^*} \) is the largest modulus of uniform convexity for \( \psi^* \). Then we have, up to a not relabelled subsequence, the convergence

\[
d_H(\partial\{u_{\alpha_n,w_n} > s\}, \partial D) \to 0 \text{ for a.e. } s \in (0,1).
\]

**Theorem 1.3.** Denote \( E^s_\alpha = \{ u_{\alpha,0} > s \} \) and \( E^s_0 = \{ f > s \} \) if \( s > 0 \), and \( E^s_\alpha = \{ u_{\alpha,0} < s \} \) and \( E^s_0 = \{ f < s \} \) for \( s < 0 \). Let \( f \) and \( s \) satisfy that \( |\partial E^s_\alpha \Delta E^0_\alpha| \to 0 \), and also that

\[
\lim_{\nu \to 0^+} d_H \left( E^s_\alpha \cap E^{s+\nu}_0, \mathbb{R}^d \setminus E^{s+\nu}_0 \right) = 0.
\]

Then, in absence of noise \( (w = 0) \) we have the Hausdorff convergence \( d_H(\partial E^s_\alpha, \partial E^s_0) \to 0 \).

**Remark 1.4.** Any piecewise constant function satisfies trivially (4) at all the values that it does not attain, and therefore the conclusions of Theorem 1.3 are valid. Moreover, it also holds for almost every \( s \) under the source condition of [15, 23], a fact we prove in Corollary 5.9.

### 1.2 Structure of the paper

We start with some preliminary results in Subsection 1.3. Section 2 is dedicated to auxiliary results about convergence in Hausdorff distance of bounded subsets of \( \mathbb{R}^d \). In Section 3 we are concerned with variational mean curvatures, with the main goal of pinning down the construction of such a curvature on the outside of any finite perimeter set, and also stating basic comparison results. Section 4 aims at the proof of Theorem 1.2 using density estimates that degenerate as the set \( D \) is approached and dual stability estimates with respect to the noise, which are themselves proved in Appendices A and B. Then, Section 5 is devoted to the proof of Theorem 1.3 by approximation of finite perimeter sets with the level sets of their optimal variational mean curvatures. Finally, in Section 6 we explore whether it is possible to recover uniform density estimates without the source condition; it turns out that this is possible for indicatrices of some planar polygons.
1.3 Preliminaries

The total variation $TV$ appearing in (1) is the norm of the distributional derivative as a Radon measure

$$TV(u) := |Du|(\mathbb{R}^d) = \sup \left\{ \int_{\mathbb{R}^d} u \, \text{div} \, z \, dx \mid z \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d), \|z\|_{L^\infty(\mathbb{R}^d)} \leq 1 \right\}.$$ 

Correspondingly we say that a function $u : \mathbb{R}^d \to \mathbb{R}$ is of bounded variation whenever it belongs to

$$BV(\mathbb{R}^d) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^d) \mid TV(u) < +\infty \right\},$$

where we remark that we only require such functions to be locally summable. Likewise, the space $BV_{\text{loc}}(\mathbb{R}^d)$ consists of those functions $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ for which $|Du|(K) < +\infty$ for each compact set $K$. A set $E$ is called of finite perimeter whenever its indicatrix $1_E$ is of bounded variation, and the perimeter is defined as

$$\text{Per}(E) := TV(1_E).$$

Since this notion is invariant with respect to zero Lebesgue measure modification of $E$, we need a notion of boundary which satisfies this invariance as well. For this purpose, we can take a representative of $E$ for which the topological boundary equals the support of the derivative of $1_E$, which can be described [30, Prop. 12.19] as

$$\partial E = \text{Supp} \, D1_E = \left\{ x \in \mathbb{R}^d \mid 0 < \frac{|E \cap B(x,r)|}{|B(x,r)|} < 1 \text{ for all } r > 0 \right\},$$

and this choice will be assumed in all what follows. Notice that we might have $|\partial E| > 0$ (see [30, Example 12.25] for an example), so particular care is needed when combining topological and measure-theoretic arguments for this boundary. The topological interior of a set $E$ will be denoted by $\hat{E}$, and by $E^{(1)}$ its subset of points of full density in $E$. Moreover, convex hulls are denoted as Conv $E$.

From general properties of the space $BV(\mathbb{R}^d)$ one can deduce the following basic result on existence and convergence of minimizers for (1):

**Proposition 1.5.** Assuming (A) and $\int \psi(w) < +\infty$, the minimization problem (1) admits a unique solution $u_{\alpha, w}$. Furthermore, if $\alpha_n \to 0$ and $w_n$ are such that

$$\frac{1}{\alpha_n} \int \psi(w_n) \leq C,$$

then $u_{\alpha_n, w_n} \to f$ weakly in $L^{d/(d-1)}$ and strongly in $L^q_{\text{loc}}$ for $1 \leq q < d/(d-1)$.

**Proof.** Let $u_k$ be minimizing sequence for (1). Discarding some elements of the sequence if necessary and using the symmetry of $\psi$ we have the estimate

$$\frac{1}{\alpha} \int \psi(u_k - f - w) + TV(u_k) \leq \frac{1}{\alpha} \int \psi(w) + TV(f). \quad (6)$$

On the other hand we also have the Sobolev inequality [4, Thm. 3.47]

$$\|u_k - c_k\|_{L^{d/(d-1)}} \leq C \, TV(u_k)$$

for some constants $c_k \in \mathbb{R}$. Since $f$ is compactly supported and $\psi(t) > 0$ for $t \neq 0$, we have that (6) and $\int \psi(w)$ being finite imply $c_k = 0$ for all $n$. Therefore, using weak-* compactness in BV [4, Thm. 3.23] and weak compactness in $L^{d/(d-1)}$ we can extract a limit $u_{\alpha, w}$ in those topologies. Moreover, we have lower semicontinuity of $TV$ with respect to $L^1_{\text{loc}}$ convergence [4, Rem. 3.5], while positivity and convexity of $\psi$ implies that the first term of (1) is also lower semicontinuous with respect to weak $L^{d/(d-1)}$ convergence [18, Thm. 3.20], so $u_{\alpha, w}$ must be a minimizer of (1).
In view of (6) and (5), one can apply the same compactness arguments to \( u_{\alpha_n, w_n} \) to obtain a subsequence converging weakly in \( L^{d/(d-1)} \) and strongly in \( L^q_{\text{loc}} \). Moreover since \( \psi \) is strictly convex, \( \psi(0) = 0 \) and \( \psi(t) > 0 \) if \( t > 0 \) it must be increasing on \([0, +\infty)\), so (5) implies that \( w_n \to 0 \) in measure, which in turn implies that [21, Thm. 2.30] up to possibly taking a further subsequence also \( u_n(x) \to 0 \) for a.e. \( x \). Finally (6) also gives that \( \int \psi(u_{\alpha_n, w_n} - f - w_n) \to 0 \), so the limit must be \( f \). Since for any subsequence we are able to find a further subsequence converging to the fixed limit \( f \), the whole sequence \( u_{\alpha_n, w_n} \) must converge to it.

We recall the fundamental fact that for any \( E, F \) with finite perimeter, we have

\[
\text{Per}(E \cap F) + \text{Per}(E \cup F) \leq \text{Per}(E) + \text{Per}(F),
\]

and the isoperimetric inequality [4, Thm. 3.46]

\[
\text{Per}(F) \geq \Theta_d \min \left( |F|^{(d-1)/d}, |\mathbb{R}^d \setminus F|^{(d-1)/d} \right), \quad \text{with } \Theta_d = \frac{\text{Per}(B(0, 1))}{|B(0, 1)|^{(d-1)/d}}.
\]

Many of our results rely on studying in detail the minimization of (1) with \( f = 1_D \) and \( w = 0 \), for which the minimizer \( u_\alpha := u_{\alpha, 0} \) has level sets \( E_s := \{ u > s \} \) that minimize for a.e. \( s \in (0, 1) \)

\[
E \mapsto \text{Per}(E) + \frac{1}{\alpha} \int_E \psi'(s - f(x)) \, dx = \text{Per}(E) - \frac{\psi'(1 - s)}{\alpha} |E \cap D| + \frac{\psi'(s)}{\alpha} |E \setminus D|,
\]

as can be seen from (56), the coarea formula for BV functions [4, Thm. 3.40] and the general layer cake formula [29, Thm. 1.13]. More generally we have:

**Proposition 1.6.** Let \( u \) minimize (1). Then for \( s \in \mathbb{R} \) its upper level sets \( E^s := \{ u > s \} \) minimize, among sets of finite mass, the functional

\[
E \mapsto \text{Per}(E) + \frac{1}{\alpha} \int_E \psi'(s - f),
\]

and moreover we have

\[
\text{Per}(E^s) = \frac{1}{\alpha} \int_{E^s} \psi' (f - s).
\]

For the lower level sets \( \{ u < s \} \) analogous statements hold by changing the sign on the integral terms.

**Proof.** The proof of the first statement can be found in [25, Prop. 2.3.14]. The second is proven in [15, Prop. 3].

**Remark 1.7.** Note that if \( s < 0 \) it is often convenient to work with the lower level sets \( \{ u < s \} \) since then \( |\{ u < s \}| < +\infty \), in which case the integral terms change sign. This will be useful in some results below.

## 2 Density estimates and Hausdorff convergence

We begin with some auxiliary results about convergence in the Hausdorff distance, defined for sets \( E, F \subset \mathbb{R}^d \) as

\[
d_H(E, F) := \max \left\{ \sup_{x \in E} \text{dist}(x, F), \sup_{y \in F} \text{dist}(y, E) \right\},
\]

and its relation with \( L^1 \) convergence when density estimates are available, which will be used in the proof of the main results.
Definition 2.1. Let \( \{E_\gamma\}_\gamma \) be a family of finite perimeter sets of uniformly bounded measure, that is, there is \( M > 0 \) such that \( |E_\gamma| < M \) for all \( \gamma \). If there are constants \( r_0 > 0 \) and \( C \in (0, 1) \) such that for all \( \gamma \) and all \( x \in \partial E_\gamma \) we have for all \( r < r_0 \) that

\[
\frac{|E_\gamma \cap B(x, r)|}{|B(x, r)|} \geq C,
\]

we say that this family satisfies uniform inner density estimates with constant \( C \) at scale \( r_0 \). Similarly, if instead we have for \( r \leq r_0 \)

\[
\frac{|B(x, r) \setminus E_\gamma|}{|B(x, r)|} \geq C
\]

we say that this family satisfies uniform outer density estimates, again with constant \( C \) at scale \( r_0 \). When speaking of uniform density estimates, we understand that both estimates hold with the same constants.

First, in [24, 23] the following result is claimed, although with some flaws in its presentation:

**Proposition 2.2.** Assume we have \( \{E_n\}_n, E_0 \) are subsets of \( \mathbb{R}^d \) satisfying uniform inner density estimates with some scale \( r_0 \) and constant \( C \), and such that \( |E_n \Delta E_0| \to 0 \). Then \( d_H(E_n, E_0) \to 0 \).

**Proof.** First, we notice that if we have the estimate

\[
|E_n \cap B(x, r)| \geq C|B(x, r)| \quad \text{for} \quad x \in \partial E_n \quad \text{and} \quad r \leq r_0,
\]

then we also have

\[
|E_n \cap B(y, \tilde{r})| \geq \frac{C}{2^d}|B(y, \tilde{r})| \quad \text{for} \quad y \in \overline{E_n}, \quad \text{and} \quad \tilde{r} \leq 2r_0.
\]

To see this, first set \( r = \tilde{r}/2 \). Then, if \( \text{dist}(y, \partial E_n) \geq r \), the whole ball \( B(y, r) \subset E_n \), so that \( |E_n \cap B(y, \tilde{r})| \geq |B(y, r)| = |B(y, \tilde{r})|/2^d \) and (12) holds. If \( 0 \leq \text{dist}(y, \partial E) < r \), then there is at least one boundary point \( x_y \in \partial E_n \) for which \( B(x_y, r) \subset B(y, \tilde{r}) \), and applying (11) to \( x_y \) and \( r \) we get (12).

With these facts, let us assume that there is \( \delta > 0 \) such that \( d_H(E_n, E_0) > \delta \) for infinitely many \( n \), and derive a contradiction. Reducing \( \delta \) if necessary, we can assume that \( \delta \leq 2r_0 \). In view of the definition (8) we must then have a subsequence \( n_k \) for which either \( \sup_{x \in E_{n_k}} \text{dist}(x, E_0) > \delta \) or \( \sup_{x \in E_0} \text{dist}(x, E_{n_k}) > \delta \). For the first case, we then have a sequence \( x_{n_k} \in E_{n_k} \) for which \( \text{dist}(x, E_0) > \delta \). Then (12) applied to \( E_{n_k} \) and with \( \tilde{r} = \delta \) gives

\[
|E_{n_k} \Delta E_0| \geq |E_{n_k} \setminus E_0| \geq |E_{n_k} \cap B(x_{n_k}, \delta)| \geq \frac{C}{2^d} \delta^d |B(0, 1)|,
\]

a contradiction with \( |E_n \Delta E_0| \to 0 \). For the second case, we obtain \( x_{n_k} \in E_0 \) for which \( \text{dist}(x_{n_k}, E_{n_k}) > \delta \). In this case, we use again (12) for \( E_0 \) to end up as before with

\[
|E_{n_k} \Delta E_0| \geq |E_0 \setminus E_{n_k}| \geq |E_0 \cap B(x_{n_k}, \delta)| \geq \frac{C}{2^d} \delta^d |B(0, 1)|,
\]

again a contradiction. \( \square \)

In Proposition 2.2 we only used the inner density estimates. However, for level sets of total variation minimizers and imaging applications one is mostly interested in convergence of their boundaries. The latter is not implied by the convergence of the sets themselves, even under the other modes of convergence assumed, as demonstrated in the following example. We will see later that to obtain convergence of the boundaries, the outer density estimates also need to be used.
Example 2.3. Consider the unit square $E_0 := (0, 1)^2$ and a sequence obtained by removing from it thin triangles:

$$E_n := (0, 1)^2 \setminus \text{Conv} \left( \left\{ \left( \frac{1}{2} - \frac{1}{n+2}, 0 \right), \left( \frac{1}{2} + \frac{1}{n+2}, 0 \right), \left( \frac{1}{2}, \frac{1}{2} \right) \right\} \right),$$

which admits uniform inner density estimates, but with the outer densities not being uniform at $(1/2, 1/2)$. We have $|E_n \Delta E_0| \to 0$ and $D1_{E_n} \xrightarrow{n \to \infty} D1_{E_0}$. To see the latter, just notice that $D1_{E_n} = \nu_{E_n} \mathcal{H}^1 \ll \partial^* E_n$, and that the non-vanishing sides of the triangle converge to a vertical segment, but with opposite orientation. Moreover, since $E_n \subset E_0$ also

$$d_H(E_n, E_0) = \sup_{x \in E_0} \text{dist}(x, E_n) \leq \frac{1}{n+2} \to 0,$$

but $d_H(\partial E_n, \partial E_0) = \frac{1}{2}$.

Remark 2.4. In general, the Hausdorff distances $d_H(E, F)$ and $d_H(\partial E, \partial F)$ are not related. In [34, Thm. 14] it is proven that these are equal for bounded closed convex sets, and in [34, Examples 6 and 13] examples are given for pairs of planar sets where both possible strict inequalities hold.

Under $L^1$ convergence, the Hausdorff convergence of boundaries is in fact stronger:

**Proposition 2.5.** Assume that $\{E_n\}_n, E_0$ are subsets of $\mathbb{R}^d$ such that we have the convergences

$$|E_n \Delta E_0| \to 0 \text{ and } d_H(\partial E_n, \partial E_0) \to 0.$$

Then also $d_H(E_n, E_0) \to 0$.

**Proof.** Assume that the hypotheses are satisfied but $d_H(E_n, E_0) \not\to 0$. Then there is $\delta > 0$ with $d_H(E_{n_k}, E_0) > \delta$ for some subsequence $n_k$. Removing leading terms if needed, we can assume that

$$d_H(\partial E_{n_k}, \partial E_0) < \frac{\delta}{2}. \quad (13)$$

Now, we have

$$d_H(E_{n_k}, E_0) = \max \left( \sup_{x \in E_{n_k}} \text{dist}(x, E_0), \sup_{x \in E_0} \text{dist}(x, E_{n_k}) \right) > \delta,$$

so at least one of the arguments in the supremum must be larger than $\delta$ for infinitely many $k$. Assume that it is the first one, and relabel the subsequence $n_k$ so that

$$\sup_{x \in E_{n_k}} \text{dist}(x, E_0) > \delta,$$

implying that there is a sequence $x_{n_k} \in E_{n_k}$ for which $\text{dist}(x_{n_k}, E_0) > \delta$. In consequence for all $y \in E_0$, and in particular for all $y \in \partial E_0$, we have $|x_{n_k} - y| \geq \delta$. Therefore, for each $k$ we must have $\text{dist}(x_{n_k}, \partial E_{n_k}) \geq \delta/2$, since otherwise (13) and the triangle inequality would lead to a contradiction. We have then

$$B \left( x_{n_k}, \frac{\delta}{3} \right) \subset E_{n_k} \text{ and } d( x_{n_k}, E_0 ) > \delta, \text{ so } B \left( x_{n_k}, \frac{\delta}{3} \right) \subset E_{n_k} \setminus E_0,$$

a contradiction with $|E_n \Delta E_0| \to 0$. The other case is dealt with in an entirely similar way. \qed

**Proposition 2.6.** Let $E, F \subset \mathbb{R}^d$. Then

$$d_H(\partial E, \partial F) \leq \max \left( d_H(E, F), d_H(E^c, F^c) \right). \quad (14)$$
We now turn our attention to the weak notion of mean curvature for boundaries which will be

Assume that

Theorem 2.7.

Assume we have

Theorem 2.8.

which also applies to the cases treated in [24, 23]:

we can apply Proposition 2.2 for

E

Proposition 2.9.

Assume we have

Proposition 2.6 gives then the conclusion.

and that by taking complements the roles of (9) and (10) are reversed. Therefore, using both
density estimates:

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Combining Propositions 2.5 and 2.6 we obtain

\[ d_\mathcal{H}(E, F) = \inf \{ d \geq 0 \mid E \cap B \subset \partial B \subset F \} \]

(15)

for the dilations \( U_r A = \{ \text{dist} (\cdot, A) \leq r \} \). Now, if the inequality to be proved failed, denoting

\[ r = \max \{ d_\mathcal{H}(E, F), d_\mathcal{H}(E^c, F^c) \} \]

we would have that either \( U_r \partial F \setminus \partial E \neq \emptyset \) or \( U_r \partial E \setminus \partial F \neq \emptyset \). Without loss of generality assume

that the first case holds, so that there is \( x \in \partial E \) for which \( \text{dist}(x, \partial F) > r \). If \( x \in F^c \), by the properties of the boundary we can produce \( \hat{x} \in E \setminus F \) with \( \text{dist}(\hat{x}, \partial F) > r \), which since \( \hat{x} \in F^c \) also implies

\[ \text{dist}(\hat{x}, F) = \text{dist}(\hat{x}, \partial F) > r, \]

contradicting \( r \geq d_\mathcal{H}(E, F) \). Similarly, if \( x \in \{ \text{dist}(\cdot, \partial F) > r \} \cap F \), we can find \( \hat{x} \in E^c \) with \( \text{dist}(\hat{x}, \partial F) > r \) as well, and as before since \( \hat{x} \in F \setminus E \) we have

\[ \text{dist}(\hat{x}, F^c) = \text{dist}(\hat{x}, \partial F) > r, \]

a contradiction with \( r \geq d_\mathcal{H}(E^c, F^c) \).

Combining Propositions 2.5 and 2.6 we obtain

**Theorem 2.7.** Assume that \( \{ E_n \}_n, E_0 \) are subsets of \( \mathbb{R}^d \) such that \( |E_n \triangle E_0| \to 0. \) Then

\[ d_\mathcal{H}(\partial E_n, \partial E_0) \to 0 \text{ if and only if } d_\mathcal{H}(E_n, E_0) \to 0 \text{ and } d_\mathcal{H}(E_n^c, E_0^c) \to 0 \text{ simultaneously.} \]

We can conclude Hausdorff convergence of the boundaries without the need of derivatives, by using both density estimates:

**Theorem 2.8.** Assume we have \( \{ E_n \}_n, E_0 \) finite perimeter sets satisfying uniform density estimates with some scale \( r_0 \) and constant \( C \), and such that \( |E_n \triangle E_0| \to 0. \) Then

\[ d_\mathcal{H}(\partial E_n, \partial E_0) \to 0. \]

**Proof.** We notice that

\[
E_n \triangle E_0 = (E_n \setminus E_0) \cup (E_0 \setminus E_n) = (E_n \cap E_0^c) \cup (E_0 \cap E_n^c) \\
= (E_0^c \setminus E_n^c) \cup (E_n^c \setminus E_0^c) = E_0^c \setminus E_n^c,
\]

and that by taking complements the roles of (9) and (10) are reversed. Therefore, using both
we can apply Proposition 2.2 for \( E_n \) and for \( E_n^c \) so that \( d_\mathcal{H}(E_n, E_0) \to 0 \) and \( d_\mathcal{H}(E_n^c, E_0^c) \to 0 \). Proposition 2.6 gives then the conclusion.

As a direct consequence we get the following result, proved but not explicitly stated in [15],
which also applies to the cases treated in [24, 23]:

**Proposition 2.9.** Assume we have \( \{ E_n \}_n, E_0 \) finite perimeter sets satisfying uniform density estimates with some scale \( r_0 \) and constant \( C \), and such that the characteristic functions \( 1_{E_n} \rightharpoonup 1_E \)
\( \text{in BV. Then } d_\mathcal{H}(\partial E_n, \partial E_0) \to 0. \)

### 3 A few results on variational mean curvatures

We now turn our attention to the weak notion of mean curvature for boundaries which will be
our main tool to describe the behaviour of level sets of minimizers of (1):
Definition 3.1. We say that a set $A \subset \mathbb{R}^d$ has a variational mean curvature $\kappa : \mathbb{R}^d \to \mathbb{R}$ if it minimizes, among $E \subset \mathbb{R}^d$, the functional

$$E \mapsto \text{Per}(E) - \int_E \kappa.$$  \hspace{1cm} (16)

If the set $A$ has a smooth boundary and $\kappa$ is continuous, this minimization property implies that the restriction of $\kappa$ to the boundary $\partial A$ is, up to a constant factor, the usual mean curvature of $\partial A$. To see this, just notice [30, Rem. 17.6] that if $\partial A$ is $C^2$, the first variation of the perimeter along the flow generated by a vector field $V \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ is

$$\int_{\partial A} \text{div}_A V \, d\mathcal{H}^{d-1} = \int_{\partial A} (d - 1) H_A \cdot \nu A \, d\mathcal{H}^{d-1},$$  \hspace{1cm} (17)

for $\text{div}_A$ the surface divergence, $\nu A$ the outward normal vector and $H_A$ the usual mean curvature of $\partial A$, while that of the integral term in (16) for continuous $\kappa$ amounts to

$$-\int_{\partial A} \kappa \cdot \nu A \, d\mathcal{H}^{d-1},$$

from which we conclude by noticing that $A$ is a minimizer of (16) and $T$ is arbitrary. Analogously, if we had $u \in C^2$ a minimizer of (1) with $w = 0$ and $f$ continuous, using the implicit function theorem and Proposition 1.6, we would find for $E^s := \{u > s\}$ that $H_{E^s} = -\psi'(s - f)/\alpha$, and if additionally $\nabla u(x) \neq 0$ for all $x$ then taking the first variation of $\text{TV}(u)$, which under this assumption is differentiable and equals $\int |\nabla u|$, leads to

$$-\frac{1}{\alpha} \psi'(u(x) - f(x)) = (d - 1) H_{E^s}(x) = \text{div} \left( \frac{\nabla u(x)}{|\nabla u(x)|} \right) \text{ for } x \in \partial E^s, \text{ so } u(x) = s.$$  

We recall that there is a natural weak notion of mean curvature based on (17), the distributional mean curvature, which can be defined not just for boundaries of finite perimeter sets but also for most notions of non-regular surfaces (e.g. varifolds). The distributional and variational mean curvatures are equivalent in the very regular case just described, but it is not quite clear whether they do on less regular cases where both are available; some positive results are given in [7].

From the definition one sees that variational mean curvatures for a given set, as functions defined in $\mathbb{R}^d$, contain “too much information” and one can not expect them to be unique. In fact, if $\kappa$ is a variational curvature for $A$, any other function $\kappa'$ with $\kappa' \geq \kappa$ on $A$ and $\kappa' \leq \kappa$ on $\mathbb{R}^d \setminus A$ is another variational mean curvature for $A$ as well.

Remark 3.2. Using the coarea and layer-cake formulas as for Proposition 1.6, it is straightforward to check that if we have $v \in \partial \text{TV}(f)$ for some $f \in L^{d/(d-1)}$ and $v \in L^d$, almost all of the upper level sets of $f$ are minimizers of (16), making $v$ a variational curvature for all of them.

We make extensive use of the following basic but fundamental comparison lemma for variational mean curvatures:

Lemma 3.3. Assume that the finite perimeter sets $E_1$ and $E_2$ admit variational mean curvatures $\kappa_1$ and $\kappa_2$ respectively, and such that $\kappa_1 < \kappa_2$ in $E_1 \setminus E_2$. Then $|E_1 \setminus E_2| = 0$, that is $E_1 \subseteq E_2$ up to Lebesgue measure zero.

Proof. We can write

$$\text{Per}(E_1) - \int_{E_1} \kappa_1 \leq \text{Per}(E_1 \cap E_2) - \int_{E_1 \cap E_2} \kappa_1,$$

$$\text{Per}(E_2) - \int_{E_2} \kappa_2 \leq \text{Per}(E_1 \cup E_2) - \int_{E_1 \cup E_2} \kappa_2.$$
Then one has
\[ u_r \text{ with equality if and only if} \]
which implies the result.

We will repeatedly use the previous lemma to compare with balls:

**Example 3.4.** For \( x_0 \in \mathbb{R}^d \) and \( r > 0 \), any function \( v_B(x_0, r) \in L^1(\mathbb{R}^d) \) with
\[
v_{B(x_0, r)} = \frac{d}{r} \text{ in } B(x_0, r), \quad v_{B(x_0, r)} < 0 \text{ in } \mathbb{R}^d \setminus B(x_0, r), \quad \text{and} \quad \int_{\mathbb{R}^d \setminus B(x_0, r)} v_{B(x_0, r)} = -\text{Per}(B(x_0, r))
\]
is a variational mean curvature for \( B(x_0, r) \). To check this, first we notice that
\[
\text{Per}(B(x_0, r)) - \int_{B(x_0, r)} v_{B(x_0, r)}(x) \, dx = 0.
\]
Moreover, for any other finite perimeter set with \( |E| < \infty \), we have by the isoperimetric inequality that for arbitrary \( y \in \mathbb{R}^d \)
\[
\text{Per}(B(y, r_E)) \leq \text{Per}(E), \quad \text{with} \quad r_E := \left( \frac{|E|}{|B(0, 1)|} \right)^{1/d},
\]
and clearly \( |B(y, r_E) \cap B(x_0, r)| \) is maximized by picking \( y = x_0 \). If \( r_E > r \) then
\[
\text{Per}(B(x_0, r_E)) > \text{Per}(B(x_0, r_0)) \quad \text{but} \quad \int_{B(x_0, r_E)} v_{B(x_0, r)}(x) \, dx < \int_{B(x_0, r)} v_{B(x_0, r)}(x) \, dx,
\]
so \( E \) could not be a minimizer. If \( r_E \leq r \), then
\[
\text{Per}(B(x_0, r_E)) = \left( \frac{r_E}{r} \right)^{d-1} \text{Per}(B(x_0, r)) = \left( \frac{r_E}{r} \right)^{d-1} \int_{B(x_0, r)} v_{B(x_0, r)}(x) \, dx
\]
\[
= \frac{r}{r_E} \int_{B(x_0, r_E)} v_{B(x_0, r)}(x) \, dx \leq \int_{B(x_0, r_E)} v_{B(x_0, r)}(x) \, dx,
\]
with equality if and only if \( r_E = r \). The case in which \( |\mathbb{R}^d \setminus E| < +\infty \), in which case \( E \) must be
for the form \( \mathbb{R}^d \setminus B(y, r_E) \) for some \( r_E > 0 \), is handled with similar computations once we notice
that condition (18) prevents the full space \( \mathbb{R}^d \) from having negative energy.

Furthermore, Lemma 3.3 combines with the strict convexity of \( \psi \) to give a comparison principle for denoised solutions:

**Proposition 3.5.** Let \( f \leq g \) and \( u^f_{0, a}, u^g_{0, a} \) be the corresponding minimizers of (1) with \( w = 0 \). Then one has \( u^f_{0, 0} \leq u^g_{0, 0} \).

**Proof.** To simplify the notation we drop the subindices that remain arbitrary, but fixed, in what follows. By Proposition 1.6, one can see that the level sets \( \{u^f \geq s\} \) and \( \{u^g \geq s\} \) are the
maximal minimizers among \( E \) of respectively
\[
\text{Per}(E) + \frac{1}{\alpha} \int_E \psi'(s - f), \quad \text{and} \quad \text{Per}(E) + \frac{1}{\alpha} \int_E \psi'(s - g).
\]
Since \( \psi \) is strictly convex, we then have, for \( s' < s \),

\[
\psi'(s - f) > \psi'(s' - g),
\]

which implies by Lemma 3.3 that \(|\{u^g \geq s'\} \setminus \{u^f \geq s\}| = 0\). Since \( s' < s \) was arbitrary and these sets are nested with respect to \( s' \), we infer

\[
|\{u^g \geq s\} \setminus \{u^f \geq s\}| = \left| \bigcap_n \left\{ u^g \geq s - \frac{1}{n} \right\} \setminus \{u^f \geq s\} \right| = 0.
\]

Denoting the set

\[
A := \bigcup_{s \in \mathbb{R}} \{u^g \geq s\} \setminus \{u^f \geq s\} = \bigcup_{s \in \mathbb{R}} \{u^g \geq s\} \cap \{u^f < s\},
\]

we would like to see that \(|A| = 0\) so that \( u^g \geq u^f \) almost everywhere. We cannot immediately conclude since the union is over an uncountable index set. To proceed, define

\[
A_Q := \bigcup_{r \in \mathbb{Q}} \{u^g \geq r\} \cap \{u^f < r\}
\]

with \(|A_Q| = 0\), and let \( x \in A \setminus A_Q \). Then there is some \( s_0 \in \mathbb{R} \) for which both \( u^g(x) \geq s_0 \) and \( u^f(x) < s_0 \) hold. However, for all \( r \in \mathbb{Q} \) we have either \( u^g(x) < r \) or \( u^f(x) \geq r \). Let \( \{r_n\}_{n \in \mathbb{Q}} \subset \mathbb{Q} \) with \( r_n < s_0 \) and \( r_n \to s_0 \). If we had that \( u^g(x) < r_n \) for some \( n \), then \( u^g(x) < r_n < s_0 \), a contradiction. So we must have \( u^f(x) \geq r_n \) for all \( n \), implying that \( u^f(x) \geq s_0 \), which is again a contradiction. Therefore \( A = A_Q \) and we conclude.

### 3.1 Construction of variational mean curvatures for bounded sets

A natural question is whether a variational mean curvature can be found for a given set. The following crucial result proven in [8, 6] provides a positive answer:

**Theorem 3.6.** Let \( D \) be a bounded set with finite perimeter. Then, \( D \) has variational mean curvatures in \( L^1(\mathbb{R}^d) \). In addition, there exists a variational mean curvature \( \kappa_D \) for \( D \) which minimizes all the \( L^p(D) \) norms among such curvatures.

The construction of \( \kappa_D \) in [8, 6] involves an arbitrary positive function \( g \in L^1(\mathbb{R}^d) \) and minimizers of the problems

\[
\min_{E \subset D} \text{Per}(E) - \lambda \int_E g, \quad \text{and} \quad (19)
\]

\[
\min_{F \subset \mathbb{R}^d \setminus D} \text{Per}(F) - \lambda \int_F g. \quad (20)
\]

Namely, for \( x \in D \) one defines

\[
\kappa_D(x) := \inf \left\{ \lambda g(x) \mid \lambda > 0 \text{ and } x \in E^\lambda, \text{ for } E^\lambda \text{ any minimizer of (19)} \right\}, \quad (21)
\]

and for \( x \in \mathbb{R}^d \setminus D \)

\[
\kappa_D(x) := -\inf \left\{ \lambda g(x) \mid \lambda > 0 \text{ and } x \in F^\lambda, \text{ for } F^\lambda \text{ any minimizer of (20)} \right\}. \quad (22)
\]

By definition \( \kappa_D > 0 \) in \( D \) and \( \kappa_D < 0 \) in \( \mathbb{R}^d \setminus D \), consistent with the lack of uniqueness for variational mean curvatures described above. For completeness, we check that \( \kappa_D \) is well defined:
Proposition 3.7. The problems (19) and (20) admit at least one minimizer. Moreover if for every compact set $K \subset \mathbb{R}^d$ one can find $c_K$ such that
\[
g(x) \geq c_K > 0 \text{ for a.e. } x \in K, \tag{23}\]
then for almost every $x \in D$, we have that $x \in E^{\lambda_x}$ for some $\lambda_x > 0$ and $E^{\lambda_x}$ a minimizer of (19), and similarly for a.e. $x \in \mathbb{R}^d \setminus \overline{D}$ and a corresponding minimizer of (20).

Proof. Let us focus first on minimizers of (20), for which we consider the equivalent problem for the complement
\[
\min_{E \subset F} \text{Per}(E) + \lambda \int_{E} g.
\]
Let $\{E_n\}_n$ be a minimizing sequence for this problem. The objective is nonnegative, so comparing with any fixed nonempty set we have an upper bound for $\text{Per}(E_n)$. Since $1_{E_n}(x) \in \{0, 1\}$ and hence bounded in $L^1_{\text{loc}}$, we can then apply compactness in $\text{BV}_{\text{loc}}$ [4, Thm. 3.23] to obtain $v \in \text{BV}_{\text{loc}}(\mathbb{R}^d)$ such that $1_{E_n} \to v$ in $L^1_{\text{loc}}$. Since $\text{Per}(E_n) = |D1_{E_n}|(\mathbb{R}^d) \leq C$ we have in fact that $v \in \text{BV}(\mathbb{R}^d)$, and since the convergence is in $L^1_{\text{loc}}$ strong, there must be a finite perimeter set $E_0$ for which $v = 1_{E_0}$. Furthermore, by the lower semicontinuity of the total variation [4, Rem. 3.5] with respect to $L^1_{\text{loc}}$ convergence and since $g > 0$, we have
\[
\text{Per}(E_0) \leq \liminf_n \text{Per}(E_n), \quad \text{and} \quad \int_{E_0} g \leq \liminf_n \int_{E_0} g,
\]
so $\mathbb{R}^d \setminus E_0$ is a minimizer of (20). For (19) one proceeds similarly, with the difference that the constraint $E \subset D$ and the fact that $D$ is bounded allow to obtain full $L^1$ convergence of a minimizing sequence $\{E_n\}_n$, so that $\int_{E_n} g$ in fact converges.

To see the second part, we treat the inside and outside problems separately. First, notice that $D$ is admissible in (19), so we have that
\[
\text{Per}(E^\lambda) - \lambda \int_{E^\lambda} g \leq \text{Per}(D) - \lambda \int_D g,
\]
or equivalently
\[
\lambda \left( \int_D g - \int_{E^\lambda} g \right) \leq - \text{Per}(F^\lambda) + \text{Per}(D) \leq \text{Per}(D),
\]
where since $g > 0$ and $E^\lambda \subset D$ the left hand side is positive, and using (23) for $\overline{D}$ we get
\[
|D \setminus E^\lambda| \leq \frac{1}{c_D} \left( \int_D g - \int_{E^\lambda} g \right) \xrightarrow{\lambda \to \infty} 0,
\]
so for a.e. $x \in D$ we must have $x \in E^{\lambda_x}$ for some $\lambda_x$. Similarly $\mathbb{R}^d \setminus D$ is admissible in (20), so using $\text{Per}(\mathbb{R}^d \setminus D) = \text{Per}(D)$ we have for $F^\lambda$ any minimizer of (20) the bound
\[
\lambda \left( \int_{\mathbb{R}^d \setminus D} g - \int_{F^\lambda} g \right) \leq \text{Per}(D).
\]
This time, to be able to use (23) we would need to see that $(\mathbb{R}^d \setminus D) \setminus F^\lambda = \mathbb{R}^d \setminus F^\lambda$ is bounded, which is not a priori obvious. For large enough $\lambda$ we prove in Lemma 3.8 below that $\mathbb{R}^d \setminus F^\lambda$ is indeed bounded, allowing us to conclude. \hfill $\square$

Lemma 3.8. Assume that for every compact set $K \subset \mathbb{R}^d$ one can find $c_K$ such that (23) holds, and that $D \subset B(0, 1)$. Then there is some $\lambda_1$ such that if $\lambda > \lambda_1$ all minimizers $F^\lambda$ of (20) satisfy $\mathbb{R}^d \setminus F^\lambda \subset \overline{B}(0, \lambda)$, and in particular $|\mathbb{R}^d \setminus F^\lambda| < +\infty$. Moreover in that case $F^\lambda \cap B(0, 2)$ is also a minimizer of
\[
\min_{F \subset B(0, 2) \setminus D} \text{Per}(F; B(0, 2)) - \lambda \int_F g. \tag{24}
\]
Proof. Let us define the compact set
\[ K_1 := B(0, 2) \setminus B(0, 1) = \bigcup_{x \in \partial B(0, 3/2)} B(x, 1/2). \]
Using Lemma 3.3, Example 3.4, this expression and the condition on \( g \) we see that if
\[ \lambda > \frac{2d}{cK_2} =: \lambda_1, \]
then \( K_1 = B(0, 2) \setminus B(0, 1) \subset \partial F^\lambda \).
Since \( \text{Supp} D_1 F^\lambda = \partial F^\lambda \) and the above implies \( \partial F^\lambda \cap (B(0, 2) \setminus B(0, 1)) = \emptyset \), we have for any \( 0 < \delta < 1/2 \) the decomposition
\[ \text{Per}(F^\lambda) = \text{Per}(F^\lambda; B(0, 1 + \delta)) + \text{Per}(F^\lambda; \mathbb{R}^d \setminus B(0, 2 - \delta)). \]
But since \( g > 0 \) this means that
\[ \text{Per}(F^\lambda \cup (\mathbb{R}^d \setminus B(0, 3/2))) \leq \text{Per}(F^\lambda) \text{ and } \int_{F^\lambda \cup (\mathbb{R}^d \setminus B(0, 3/2))} g \geq \int_{F^\lambda} g, \]
so necessarily \( \mathbb{R}^d \setminus B(0, 3/2) \subset F^\lambda \) for all \( \lambda > \lambda_1 \) as well, hence \( \mathbb{R}^d \setminus B(0, 1) \subset F^\lambda \). These considerations also directly prove that \( F^\lambda \cap B(0, 2) \) minimizes (24).

The curvatures arising from this construction are in fact not independent of the choice of the density \( g \), as is shown in Proposition 3.9 below. As has been noted in previous works [6, 23], since we work with bounded \( D \), this ambiguity can be mitigated by choosing \( g(x) = 1 \) for all \( x \in D \). However \( g \in L^1(\mathbb{R}^d \setminus D) \) is required to make sense of the unbounded problem (20), and there is no canonical choice for it outside of \( D \). Moreover, as opposed to most other works using this variational mean curvature, we plan to make explicit use of \( \kappa_D \) on \( \mathbb{R}^d \setminus D \) and the corresponding minimizers of (20).

**Proposition 3.9.** For any bounded \( D \) and any positive \( g \in L^1(\mathbb{R}^d) \), there exists some \( \lambda_g > 0 \) such that if \( \lambda \leq \lambda_g \), the only minimizer of (20) is the empty set, and in consequence \( \kappa_D(x) \geq -\lambda g(x) \) for a.e. \( x \). Moreover, if additionally \( |\mathbb{R}^d \setminus F^\lambda| < +\infty \) for all \( \lambda \) then
\[ \kappa_D(x) = -\lambda_g g(x) \text{ for a.e. } x \in \mathbb{R}^d \setminus \text{Conv } D. \]

**Proof.** Let us define
\[ G_D := \arg\min_{D \subset E} \text{Per}(E), \]
which exists by the same compactness arguments as in Proposition 3.7 (if \( d = 2 \) then in fact \( G_D = \text{Conv } D \) [20]). Then any minimizer \( F \neq \emptyset \) of (20) must have \( \text{Per}(F) \geq \text{Per}(G_D) \), so that
\[ \text{Per}(F) - \lambda \int_F g \geq \text{Per}(G_D) - \lambda \int_{\mathbb{R}^d \setminus D} g. \]
But whenever
\[ \lambda < \lambda_c := \frac{\text{Per}(G_D)}{\int_{\mathbb{R}^d \setminus D} g} \]
we have that the right hand side of (25) is positive, making \( F \) is a worse competitor than the empty set. We can then define
\[ \lambda_g := \sup \left\{ \lambda > 0 \bigg| \inf_{F \subset \mathbb{R}^d \setminus D} \text{Per}(F) - \lambda \int_F g = 0 \right\} \geq \lambda_c > 0, \]

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and notice that $\lambda_g < +\infty$ by Proposition 3.7.

To prove the second part, notice that having $\kappa_D(x) < -\lambda_g g(x)$ means that $x \notin F^\lambda \varepsilon$ for some $\varepsilon > 0$, or equivalently, that $x$ belongs to the minimal minimizer $E^\lambda \varepsilon$ of

$$\min_{D \subseteq E} \text{Per}(E) + (\lambda_g + \varepsilon) \int_E g.$$  

However since by assumption we have $|E^\lambda + \varepsilon| = |\mathbb{R}^d \setminus F^\lambda + \varepsilon| < \infty$, taking its intersection with a convex set cannot increase the perimeter (see [12, Lem. 3.5] for a proof in the general setting) and we get

$$\text{Per}(E^\lambda + \varepsilon) + (\lambda_g + \varepsilon) \int_{E^\lambda + \varepsilon} g \geq \text{Per}(E^\lambda + \varepsilon \cap \text{Conv } D) + (\lambda_g + \varepsilon) \int_{E^\lambda + \varepsilon \cap \text{Conv } D} g,$$

and the inequality would be strict if $|E^\lambda + \varepsilon \setminus \text{Conv } D| > 0$, so necessarily

$$\left| \left\{ x \in \mathbb{R}^d \setminus D \mid \kappa_D(x) < -\lambda_g g(x) \right\} \right| = \left| \bigcap_{\varepsilon > 0} E^\lambda + \varepsilon \setminus \text{Conv } D \right| = 0.$$ 

We introduce now a concrete choice of density $g$ which, although it cannot eliminate this phenomenon, at least allows for a purely geometric description of minimizers for $\lambda$ large enough.

**Definition 3.10.** Assume $D \subseteq B(0,1)$. For any $R > 1$ we define $g_R$ by

$$g_R(x) := \begin{cases} 1 & \text{if } 0 \leq |x| \leq R, \\ g_f & \text{if } |x| > R, \end{cases}$$

with some $g_f \in L^1(\mathbb{R}^d \setminus B(0,R))$ with $0 < g_f \leq 1$ and satisfying (23).

Since we will make extensive use of minimizers of (19) and (20) with this particular choice of density, we introduce some notation for them.

**Definition 3.11.** Let $\lambda > 0$ and $D \subset B(0,1)$ of finite perimeter. We denote by $D^\lambda$ the maximal (in the sense of inclusion) minimizer of (19) with density $g_2$, that is, of

$$\min_{E \subseteq D} \text{Per}(E) - \lambda |E|.$$

We also define $D^{-\lambda}$ as

$$D^{-\lambda} := \mathbb{R}^d \setminus F^\lambda,$$
where $F^\lambda$ is the maximal minimizer of (20) with density $g_2$. By Lemma 3.8, whenever $\lambda > 2d$ we have that $F^\lambda$ can be determined from (24), which since $g_2 \equiv 1$ on $B(0, 2)$ turns into

$$\min_{F \subset B(0,2) \setminus D} \text{Per}(F; B(0,2)) - \lambda |F|.$$  

Moreover, let us note that $D^{-\lambda}$ can also be found directly as the minimal minimizer of

$$\min_{E \supset D} \text{Per}(E) + \lambda \int_E g_2.$$  

**Remark 3.12.** We have chosen $D \subset B(0,1)$ but other bounded sets can be treated by rescaling. If for any set $E$ we consider the rescaling $qE$ with we have

$$\text{Per}(qE) - \lambda \int_{qE} g_R(x) \, dx = q^{d-1} \text{Per}(E) - q^d \lambda \int_E g_R/q(y) \, dy,$$

so the minimization problem is equivalent to that with $\lambda$ replaced by $q\lambda$ and $R$ replaced by $R/q$.

The choice of signs in the notation is motivated by (21) and (22), and by the fact that the free boundaries of $D^\lambda$ and $D^{-\lambda}$ have curvature $\lambda$ and $-\lambda$ respectively, with respect to their outer normals.

**Remark 3.13.** From now on, whenever we use the variational mean curvature $\kappa_D$ for some $D \subset B(0,1)$, we well always assume that the density used is $g_2$, as in Definition 3.11 above.

Speaking about maximal/minimal minimizer is possible since being a minimizer of either of these two problems is stable by intersection and union:

**Proposition 3.14.** Let $E_1$ and $E_2$ be two minimizers of (19). Then, $E_1 \cap E_2$ and $E_1 \cup E_2$ are also minimizers of (19). The same is true for minimizers of (20).

**Proof.** One can write, using the minimality of $E_1$ and $E_2$ and noting that $E_1 \cap E_2$ as well as $E_1 \cup E_2$ are admissible for that problem,

$$\text{Per}(E_1 \cap E_2) - \lambda \int_{E_1 \cap E_2} g \geq \text{Per}(E_1) - \lambda \int_{E_1} g,$$

and

$$\text{Per}(E_1 \cup E_2) - \lambda \int_{E_1 \cup E_2} g \geq \text{Per}(E_2) - \lambda \int_{E_2} g.$$  

Summing these inequalities and noticing that the volume terms exactly compensate, we obtain

$$\text{Per}(E_1 \cap E_2) + \text{Per}(E_1 \cup E_2) \geq \text{Per}(E_1) + \text{Per}(E_2).$$

Since the reverse inequality is also true by (7), we must have an equality, which also implies that the two first inequalities are equalities and the expected conclusion.

We will see in later sections that for large values of $\lambda$, the sets $D^\lambda$ and $D^{-\lambda}$ provide us with an approximation of $D$ in Hausdorff distance from the inside and outside respectively, motivating the notation. Moreover, they also determine the curvature $\kappa_D$ through (21) and (22).
3.2 Bounds and examples of variational mean curvatures

Lemma 3.15. Assume that \(x_0, r\) are such that \(B(x_0, r) \subseteq D\) up to measure zero, that is, \(|B(x_0, r) \setminus D| = 0\). Then the optimal variational mean curvature \(\kappa_D\) of \(D\) satisfies

\[
\kappa_D|_{B(x_0, r)} \leq \frac{d}{r}.
\]  

(27)

In consequence, for any interior point \(x \in \hat{D}\), we have

\[
\kappa_D(x) \leq \frac{d}{\text{dist}(x, \partial D)}.
\]  

(28)

Similarly, for \(x \in \mathbb{R}^d \setminus \overline{D}\) we have \(-\kappa_D(x) \leq d/\text{dist}(x, \partial D)\). Therefore, for any \(K \subset \mathbb{R}^d\) we have

\[
\|\kappa_D\|_{L^\infty(K)} \leq \frac{d}{\text{dist}(K, \partial D)},
\]  

(29)

where \(\text{dist}(K, \partial D) := \inf_{x \in K} \text{dist}(x, D)\).

Proof. By the definition of \(D^\lambda\) as the maximal solution of (26) and that of \(\kappa_D\) in (21), \(x \in D^\lambda\) implies \(\kappa_D(x) \leq \lambda\). On the other hand, by Lemma 3.3 and since by Example 3.4 we know that \((d/r)B(x_0, r)\) is a variational mean curvature for \(B(x_0, r)\), we have that \(\lambda > d/r\) implies \(B(x_0, r) \subseteq D^\lambda\), so for \(x \in B(x_0, r)\) we get \(\kappa_D(x) \leq \lambda\) for every \(\lambda > d/r\), which is (27). To see (28), just notice that since \(x \in \hat{D}\), we have that \(B(x, r) \subset D\) for each \(r < \text{dist}(x, \partial D)\).

If \(x \in \mathbb{R}^d \setminus \overline{D}\), we can proceed similarly using \(F^\lambda = \mathbb{R}^d \setminus D^{-\lambda}\) and its variational problem (20). These two cases prove (29), since the bound is trivial when \(\text{dist}(K, \partial D) = 0\).

Lemma 3.16. Let \(D \subset \mathbb{R}^d\) be bounded. Denote by

\[
h(D) := \min_{E \subseteq D} \frac{\text{Per}(E)}{|E|}
\]

the Cheeger constant of \(D\), the minimum being attained at Cheeger sets of \(D\). Then \(\kappa_D(x) \geq h(D)\) for \(x \in D\), with equality for \(x \in C_D\), the maximal Cheeger set.

Proof. We again consider the problem

\[
\min_{E \subseteq D} \text{Per}(E) - \lambda|E|,
\]  

(30)

with \(D^\lambda\) its maximal solution. Assume \(\lambda > 0\) is such that \(|D^\lambda| \neq 0\), then by comparing with the empty set we have

\[
\text{Per}(D^\lambda) - \lambda|D^\lambda| \leq 0,
\]

which implies

\[
\lambda \geq \frac{\text{Per}(D^\lambda)}{|D^\lambda|} \geq \min_{E \subseteq D} \frac{\text{Per}(E)}{|E|} = h(D),
\]

proving that \(\kappa_D(x) \geq h(D)\) for all \(x \in D\). Similarly, considering (30) for \(\lambda = h(D)\) we get

\[
\text{Per}(D^{h(D)}) - h(D)|D^{h(D)}| \leq 0,
\]

so that \(D^{h(D)}\) is a Cheeger set, and in fact by maximality \(D^{h(D)} = C_D\), which in turn implies \(\kappa_D(x) = h(D)\) for \(x \in C_D\). \(\square\)
Proposition 3.17. Let \( S = (0, 1) \times (0, 1) \) be the unit square in \( \mathbb{R}^2 \). Then denoting by \( Q^1 := (0, 1/2) \times (0, 1/2) \) the lower left quadrant we have

\[
\kappa_S(x) = \begin{cases} 
  h(S) = 1/r_S := 2 + \sqrt{\pi} & \text{if } x \in C_S, \\
  (x_1 + x_2 + \sqrt{2x_1x_2})^{-1} & \text{if } x \in (S \setminus C_S) \cap Q^1,
\end{cases}
\]

and similarly for the other quadrants. The Cheeger set \( C \) is given by

\[
C_S = \{ x \in S \mid d(x, (r_S, 1-r_S) \times (r_S, 1-r_S)) < r_S \}.
\]

Proof. If \( x \in C_S \), the value of \( r_S \) can be found for example in [26, Thm. 3]. Without loss of generality we can assume that \( x = (x_1, x_2) \) is in \( (S \setminus C_S) \cap Q^1 \). Now, \( x \) belongs to the circle centered at \( (R(x), R(x)) \) with radius \( R(x) \), that is

\[
(x_1 - R(x))^2 + (x_2 - R(x))^2 = R(x)^2.
\]

This is a quadratic equation for \( R(x) \) that we can solve to find \( \kappa_S(x) = 1/R(x) \). \( \square \)

Remark 3.18. It is proved in [32, Thm. 3.32(i)] that for a convex planar set \( E \) and \( \lambda \) small enough, \( E^\lambda \) can be written as a union of balls of radius \( 1/\lambda \). In this situation, the proof of [26, Thm. 1] building up on the previously mentioned result implies in particular that

\[
E^\lambda = [E]^1/\lambda + \frac{1}{\lambda} B(0, 1),
\]

where \( [E]^1/\lambda \subset E \) is the \( 1/\lambda \)-offset of \( E \) in the direction of its inner normal, so that \( E^\lambda \) is obtained by a “rolling ball” procedure. Furthermore, it has been recently proven [27] that the same characterization holds for more general planar sets, namely Jordan domains with no necks.

4 Convergence for indicatrices with noise

In this section we prove Theorem 1.2 on Hausdorff convergence of level sets for denoising the indicatrix of a bounded finite perimeter set \( D \subset B(0, 1) \), with the variational mean curvature \( \kappa_D \in L^1(\mathbb{R}^d) \) constructed with the choices of Definition 3.11, and with noise and parameter choice controlled by (3). To this end, let \( u_{N,0} \) be the precise representative of the minimizer of (1) with \( f = 1_D \) and \( w = 0 \), that is, with no noise added. We denote by

\[
\kappa := u_{N,0} = \frac{1}{\alpha} \psi'(1_D - u_{N,0})
\]

the corresponding variational mean curvature associated to \( u_{N,0} \) through duality in Proposition A.1, and by \( E_{\alpha} \) its upper level sets. The definition of \( \kappa_D \) also provides us with a natural precise representative for it; we will implicitly use these precise representatives in the rest of the section.

We implement the local strategy of [15, Thm. 2], which requires that \( v_{N,0} \) is \( L^d \)-equiintegrable only on \( K_\delta = \{ \text{dist}(\cdot, \partial D) \geq \delta \} \) for each \( \delta > 0 \). This equiintegrability is in turn a consequence of Lemma 3.15 and Proposition 4.1 below, which combine to give the bound

\[
\|v_{N,0}\|_{L^\infty(K_\delta)} \leq \frac{d}{\delta}.
\]

Proposition 4.1. The noiseless dual variable \( \kappa \) satisfies \( |\kappa| \leq |\kappa_D| \) almost everywhere.

To prove this proposition, we will use the following lemmas:

Lemma 4.2. The denoised solution \( u_{N,0} \) with \( w = 0 \) satisfies \( u_{N,0} = 0 \) a.e. outside of \( \text{Conv } D \).
Proof. Denote \( u := u_{\alpha, 0} \), and assume for the sake of contradiction that

\[ |\{ u \neq 0 \} \setminus \text{Conv } D | > 0. \]

This implies, defining \( u_c := u 1_{\text{Conv } D} \), that

\[
\int_{\mathbb{R}^d} \psi(u - 1_D) = \int_{\text{Conv } D} \psi(u - 1_D) + \int_{\mathbb{R}^d \setminus \text{Conv } D} \psi(u) > \int_{\text{Conv } D} \psi(u - 1_D) = \int_{\mathbb{R}^d} \psi(u_c - 1_D). 
\]

Moreover we can write the coarea formula for \( u \) as

\[
\text{TV}(u) = \int_0^{+\infty} \text{Per}(\{ u > s \}) \, ds + \int_{-\infty}^0 \text{Per}(\{ u < s \}) \, ds 
\geq \int_0^{+\infty} \text{Per}(\{ u > s \} \cap \text{Conv } D) \, ds + \int_{-\infty}^0 \text{Per}(\{ u < s \} \cap \text{Conv } D) \, ds 
= \int_0^{+\infty} \text{Per}(\{ u_c > s \}) \, ds + \int_{-\infty}^0 \text{Per}(\{ u_c < s \}) \, ds = \text{TV}(u_c),
\]

where we have used that the level sets \( \{ u > s \} \) for \( s > 0 \) and \( \{ u < s \} \) for \( s < 0 \) must have finite mass since \( u \in L^{d/(d-1)} \), and the convexity of \( \text{Conv } D \). These two inequalities mean that \( u \) could not be a minimizer.

\[ \square \]

Lemma 4.3. Let \( 0 < s \leq 1 \). Then, for \( 0 < \lambda < \psi'(1-s)/\alpha \), one has \( D^\lambda \subset E^s_\alpha \) whereas for \( 0 > -\lambda > -\psi'(s)/\alpha \), one has \( E^s_\alpha \subset D^{-\lambda} \).

Proof. First, notice that the problems satisfied by \( D^\lambda \) and \( D^{-\lambda} \) are of obstacle type, so that as in [24, Lem. 9], one can lift the obstacle constraint and conclude that \( D^\lambda \) minimizes

\[ E \mapsto \text{Per}(E) - \int_E \kappa^\lambda_D \quad \text{with} \quad \kappa^\lambda_D = \lambda 1_D + \kappa_D 1_{\mathbb{R}^d \setminus D} \tag{33} \]

whereas \( D^{-\lambda} \) minimizes

\[ E \mapsto \text{Per}(E) - \int_E \kappa^\lambda_o \quad \text{with} \quad \kappa^\lambda_o = -\lambda g_R 1_{\mathbb{R}^d \setminus D} - \kappa_D 1_D. \]

Therefore, we can write

\[ \text{Per}(E^s_\alpha \cap D^\lambda) - \int_{E^s_\alpha \cap D^\lambda} \kappa^\lambda_D \geq \text{Per}(D^\lambda) - \int_{D^\lambda} \kappa^\lambda_D \]

and

\[ \text{Per}(E^s_\alpha \cup D^\lambda) - \int_{E^s_\alpha \cup D^\lambda} \kappa_D \geq \text{Per}(E^s_\alpha) - \int_{E^s_\alpha} \kappa_D. \]

Summing these two inequalities, we obtain

\[ \int_{D^\lambda \setminus E^s_\alpha} \kappa_D^\lambda \geq \int_{D^\lambda \setminus E^s_\alpha} \kappa_D \]

that rewrites, since \( D^\lambda \subset D \) and using the definition of \( \kappa^\lambda_D \) in (33),

\[ \int_{D^\lambda \setminus E^s_\alpha} \left( \lambda - \frac{\psi'(1-u_\alpha)}{\alpha} \right) \geq 0. \tag{34} \]

Now, on \( (E^s_\alpha)^c \) by definition \( u_\alpha \leq s \) which, since \( \psi' \) is strictly increasing, implies \( \lambda - \psi'(1-u_\alpha)/\alpha \leq \lambda - \psi'(1-s)/\alpha \). Therefore, for \( \lambda < \psi'(1-s)/\alpha \), (34) can hold only if \( |D^\lambda \setminus E^s_\alpha| = 0 \), that is \( D^\lambda \subset E^s_\alpha \) a.e.
Similarly, we write

$$\text{Per}(E_\alpha^s \cup D^{-\lambda}) - \int_{E_\alpha^s \cup D^{-\lambda}} \kappa_\alpha^\lambda \geq \text{Per}(D^{-\lambda}) - \int_{D^{-\lambda}} \kappa_\alpha^\lambda,$$

$$\text{Per}(E_\alpha^s \cap D^{-\lambda}) - \int_{E_\alpha^s \cap D^{-\lambda}} \kappa_\alpha \geq \text{Per}(E_\alpha^s) - \int_{E_\alpha^s} \kappa_\alpha,$$

to sum these inequalities and obtain (using \((D^{-\lambda})^c \subset D^c\))

$$0 \geq \int_{E_\alpha^s \setminus D^{-\lambda}} \left( \frac{\psi'(s)}{\alpha} + \lambda g_R \right).$$

Hence, since \(0 < g_R \leq 1\) on \(\text{Supp} \, u_\alpha\), as soon as \(-\lambda > -\psi'(s)/\alpha\), we have \(E_\alpha^s \subset D^{-\lambda}\) a.e. \(\square\)

**Proof of Proposition 4.1.** First we take \(x \in D\) and define \(s := u_{\alpha,0}(x)\), implying \(\kappa_\alpha(x) = \psi'(1-s)/\alpha\) and \(\kappa_D(x) \geq 0\), and assume that for some \(\varepsilon > 0\)

$$\kappa_\alpha(x) = \frac{\psi'(1-s)}{\alpha} \geq \kappa_D(x) + \varepsilon,$$  \hspace{1cm} (35)

to then use Lemma 4.3 to derive a contradiction. By definition of the level sets we have that for all \(\delta > 0\),

$$x \in E_{\alpha}^{-s-\delta}, \text{ and } x \notin E_{\alpha}^{s+\delta}.$$  \hspace{1cm} (36)

This, combined with Lemma 4.3 implies that \(x \notin D^\lambda\), whenever \(0 < \lambda < \psi'(1-s-\delta)/\alpha\). On the other hand, the construction of \(\kappa_D\) and (35) give \(x \in D^\lambda\) for all \(\lambda \geq \psi'(1-s)/\alpha - \varepsilon > 0\), where for the last inequality we have used (35). Choosing \(\delta\) such that

$$\psi'(1-s) - \psi'(1-s-\delta) \leq \alpha \varepsilon,$$

which is possible since \(\psi \in C^1(\mathbb{R})\), these two statements are contradictory and therefore we must have \(\kappa_\alpha(x) \leq \kappa_D(x)\) for all \(x \in D\).

Now, if \(x \in \mathbb{R}^d \setminus \overline{D}\), by Lemma 4.2 we can assume \(x \in \text{Conv} \, D \setminus \overline{D}\), since otherwise we would have \(\kappa_\alpha(x) = 0\) and the inequality is trivially satisfied. This implies in particular that \(g_R(x) = 1\).

We have \(\kappa_\alpha(x) = \psi'(-s)/\alpha\) and \(\kappa_D(x) \leq 0\), and failure of the statement means that for some \(\varepsilon > 0\) we have

$$\kappa_\alpha(x) = \frac{\psi'(-s)}{\alpha} \leq \kappa_D(x) - \varepsilon.$$

As before, for any \(\delta\) (36) holds and Lemma 4.3 then implies that \(x \in D^{-\lambda}\) as soon as \(0 > -\lambda > \psi'(-s + \delta)/\alpha\). However the definition of \(\kappa_D\) implies \(x \notin D^{-\lambda}\) if \(-\lambda < \kappa_D(x) \leq \psi'(-s)/\alpha + \varepsilon < 0\), so that if \(\delta\) is such that \(\psi'(-s + \delta) - \psi'(-s) \leq \alpha \varepsilon\) we again derive a contradiction. \(\square\)

**Remark 4.4.** It might seem slightly surprising that even though the construction of \(\kappa_D\) depends on the density \(g\), as has been seen in Proposition 3.9, we can still obtain the inequality \(|\kappa_\alpha| \leq |\kappa_D|\). There are two reasons for this. First, we were able to bound the support of \(u_\alpha\) in Lemma 4.2, allowing to avoid the unintuitive behaviour of \(\kappa_D\) for small negative values and far away from \(D\). Second, with our particular choice we have \(g_2 = 1\) in \(\text{Conv} \, D\), so we can still obtain the desired comparison without distorting the values of \(\kappa_\alpha\) in question.

As a consequence of (32) we can obtain uniform density estimates for \(E_{\alpha,w}^s\) outside of \(K_\delta\) with scale \(r_\delta\) and constant \(C_\delta\) possibly degenerating as \(\delta \to 0\). These are proved in Proposition B.1 of Appendix B. A further consequence of these density estimates is the following compact support result.
Proposition 4.5. Assume a parameter choice such that
\[
\|v_{\alpha,w} - v_{\alpha,0}\|_{L^d(\mathbb{R}^d)} \leq C_0 < \Theta_d,
\] (37)
holds and \(f = 1_D\) for \(D\) bounded. We then have that there is \(R > 0\) such that
\[
\text{Supp } u_{\alpha,w} \subset B(0, R),
\]
with \(R\) depending on \(C_0\) but not on the specific \(\alpha\) and \(w\).

Proof. Denote \(E := E_{\alpha,w}^s\) where \(\alpha, w\) are fixed. Since \(\text{Per}(E) = \int_E v_{\alpha,w}\) by Proposition 1.6, using (37), the isoperimetric inequality and Proposition 4.1 we get
\[
\text{Per}(E) \leq \left|\int_E (v_{\alpha,w} - v_{\alpha,0})\right| + \int_E |v_{\alpha,0}| \leq C_0 |E|^{(d-1)/d} + \int_E |v_{\alpha,0}| \leq \Theta_d^{-1} C_0 \text{Per}(E) + \|\kappa_D\|_{L^1},
\]
which gives a uniform bound for \(\text{Per}(E)\), and by the isoperimetric inequality also for \(|E|\).

Moreover, we have that by (32) the hypotheses of Proposition B.1 are satisfied with \(K = \{\text{dist}(\cdot, \partial D) \geq 1\}\) and
\[
r_{K, \varepsilon} = \frac{\varepsilon^{1/d}}{d|B(0, 1)|^{1/d}},
\]
so the \(E\) satisfy uniform density estimates at some scale \(r_K\) and with constant \(C_K\) outside the bounded set \(K\), which combined with the mass bound implies also a uniform bound for \(\text{diam}(E)\).

To see the last claim, consider points \(x_n \in \partial E \setminus K, n = 1, \ldots, N\) with \(|x_i - x_j| > r_0\) for \(i \neq j\). The inner density estimate implies \(|E \cap B(x_n, r)| > C|B(x_n, r_0)|\) combined with the uniform bound for \(|E|\) gives a uniform upper bound for \(N\), hence also for \(\text{diam}(E)\).

We are now ready to prove the main result of this section.

4.1 Proof of Theorem 1.2

Proof of Theorem 1.2. First, we notice that the definition of the Hausdorff distance reads
\[
d_H(\partial E_{\alpha,w}^s, \partial D) = \max \left(\sup_{x \in \partial E_{\alpha,w}^s} \text{dist}(x, \partial D), \sup_{x \in \partial D} \text{dist}(x, \partial E_{\alpha,w}^s)\right).
\]
Therefore, we need to prove the two statements
\[
\sup_{x \in \partial D} \text{dist}(x, \partial E_{\alpha,w}^s) \to 0, \quad \text{and} \quad \sup_{x \in \partial E_{\alpha,w}^s} \text{dist}(x, \partial D) \to 0.
\] (38) (39)

Let us start with (38), for which the argument follows closely that of [15, Prop. 9], which we reproduce for completeness. Since the parameter choice (3) implies in particular condition (5), arguing as in the proof of Proposition 1.5 we have, up to a subsequence, that \(D u_{\alpha_n,w_n} \rightharpoonup D1_D\).

Now the coarea formula, as formulated for example in [5, Thm. 10.3.3], tells us that we can slice the above measures (and not just their total variations) so that for a.e. \(s \in (0, 1)\) we in fact have
\[
D1_{E_{\alpha_n,w_n}^s} \rightharpoonup * D1_D.
\] (40)

Now let \(x \in \text{Supp } D1_D\), then for any \(r > 0\) using (40) we get
\[
0 < |D1_D|(B(x, r)) \leq \lim inf_n |D1_{E_{\alpha_n,w_n}^s}|(B(x, r)),
\]

which implies that \( \limsup_n \text{dist}(x, \text{Supp } D1_{E_{\alpha_n, w_n}^s}) \leq r \). Since \( r > 0 \) was arbitrary we conclude \( \text{dist}(x, \text{Supp } D1_{E_{\alpha_n, w_n}^s}) \to 0 \), and in particular

\[
\sup_{x \in \text{Supp } D1_D} \text{dist}(x, \text{Supp } D1_{E_{\alpha_n, w_n}^s}) \to 0.
\]

Finally, by [30, Prop. 12.19], there are representatives for \( E_{\alpha_n, w_n}^s \) and \( D \) for which

\[
\partial E_{\alpha_n, w_n}^s = \text{Supp } D1_{E_{\alpha_n, w_n}^s}, \quad \partial D = \text{Supp } D1_D,
\]

completing the proof of (38).

To prove (39) we assume it does not hold to reach a contradiction. One the one hand, we notice that using the parameter choice (3) and Proposition A.2, we can apply Proposition 4.5 to see that the \( u_{\alpha_n, w_n} \) have a common compact support. This, combined with the convergence in \( L^1_{\text{loc}} \) of \( u_{\alpha_n, w_n} \) proved in Proposition 1.5, implies that \( |E_{\alpha_n, w_n}^s \Delta D| \to 0 \). On the other, if (39) fails, from Proposition 2.6 and its proof we see that we must have either

\[
\sup_{x \in E_{\alpha_n, w_n}^s} \text{dist}(x, D) \not\to 0, \quad \text{or} \quad \sup_{x \in \mathbb{R}^d \setminus E_{\alpha_n, w_n}^s} \text{dist}(x, \mathbb{R}^d \setminus D) \not\to 0.
\]

Assume the first is true, so that there is \( \delta > 0 \) and a subsequence \( x_n \in E_{\alpha_n, w_n}^s \) for which \( \text{dist}(x_n, D) > \delta \). In that case, by the inner density estimates proved above, we have that

\[
|E_{\alpha_n, w_n}^s \Delta D| \geq |E_{\alpha_n, w_n}^s \setminus D| \geq |B(x_n, \delta) \cap E_{\alpha_n, w_n}^s| \geq C_\delta |B(0, \delta)|,
\]

a contradiction. If the second case of (41) was true, we instead use the outer density estimate to again contradict the \( L^1 \) convergence.

\[\square\]

5 Convergence for generic BV functions without noise

This section is aimed at the proof of Theorem 1.3 and simplified versions of it for piecewise constant functions. We are concerned with the noiseless situation, that is, we assume \( w = 0 \) throughout. Moreover, we always assume \( \text{Supp } f \subset B(0, 1) \) which, arguing as in Lemma 4.2, implies

\[
\text{Supp } u_{\alpha, 0} \subset \text{Conv } (\text{Supp } f) \subset \overline{B(0, 1)}.
\]

5.1 Approximation with the level sets of \( \kappa_D \)

The key to the results of this section will be to know that we can approximate any \( D \subset B(0, 1) \) with the sets \( D^\lambda \) and \( D^{-\lambda} \) as \( \lambda \to \infty \) arising from the choices of Definition 3.11. First we note that this approximation happens in mass:

**Lemma 5.1.** For every bounded finite perimeter set \( D \subset \mathbb{R}^d \), we have as \( \lambda \to +\infty \) that

\[
|D \setminus D^\lambda| \to 0, \quad \text{and} \quad |D^{-\lambda} \setminus D| \to 0.
\]

**Proof.** It is contained in the proof of Proposition 3.7. For the inside approximants \( D^\lambda \), the result is proven also in [33, Thm. 2.3(ii)]. \[\square\]

Moreover, this two-sided approximation also holds in Hausdorff distance of the corresponding boundaries:

**Lemma 5.2.** For every bounded finite perimeter set \( D \subset \mathbb{R}^d \) and every \( \varepsilon > 0 \) there exists \( \lambda_\varepsilon > 0 \) such that \( D^{\lambda_\varepsilon} \subset D \subset D^{-\lambda_\varepsilon} \), \( d_H(\partial D^{\lambda_\varepsilon}, \partial D) \leq \varepsilon \) and \( d_H(\partial D^{-\lambda_\varepsilon}, \partial D) \leq \varepsilon \).
Proof. The interior approximation is proved in [33, Thm. 2.3(iv)]; we reproduce their argument here, and see that it can also be applied for the exterior approximation with \( D^{-\lambda} \).

To start, let \( x \in D \) with \( \text{dist}(x, \partial D) > \varepsilon \). Then we have that \( B(x, \varepsilon) \subset D \), and as in the proof of Lemma 3.15 we must have \( B(x, \varepsilon) \subset D^\lambda \) for all \( \lambda > \varepsilon/d \), in particular \( x \in D^\lambda \setminus \partial D^\lambda \), implying

\[
\sup_{x \in \partial D^\lambda} \text{dist}(x, \partial D) \leq \varepsilon \text{ for all } \lambda > \varepsilon/d.
\]

For the other term of the Hausdorff distance, the strategy is to cover \( \partial D \) with finitely many balls \( B(x_j, \varepsilon) \) with \( j = 1, \ldots, N_\varepsilon \), which is possible since \( \partial D \) is bounded. Then, since Lemma 5.1 implies that \( |D \setminus D^\lambda| \to 0 \) as \( \lambda \to \infty \) we can choose \( \lambda_\varepsilon \) such that \( |D^\lambda \cap B(x_j, \varepsilon)| > 0 \) for all \( j \). Since these balls cover \( \partial D \), we have that

\[
\sup_{x \in \partial D^\lambda} \text{dist}(x, \partial D^\lambda) \leq \varepsilon \text{ for all } \lambda > \lambda_\varepsilon.
\]

Since it was only used that \( \partial D = \partial(\mathbb{R}^d \setminus D) \) is bounded, we can proceed in the same way for the approximation with \( D^{-\lambda} \). For the first part, it suffices to notice that by definition \( F^\lambda = \mathbb{R}^d \setminus D^{-\lambda} \) are minimizers of (20), so if \( x \in \mathbb{R}^d \setminus D \) with \( \text{dist}(x, \partial D) > \varepsilon \) we must also have \( B(x, \varepsilon) \subset F^\lambda = \mathbb{R}^d \setminus D^{-\lambda} \) for all \( \lambda > \varepsilon/d \). Moreover, we have \( |D^{-\lambda} \setminus D| \to 0 \) by the second part of Lemma 5.1.

Corollary 5.3. For every bounded finite perimeter set \( D \subset \mathbb{R}^d \) and every \( \varepsilon > 0 \) there exists \( \lambda_\varepsilon > 0 \) such that

\[
\text{dist}(D^\lambda, D) \leq \varepsilon \text{ and } \text{dist}(D^{-\lambda_\varepsilon}, D) \leq \varepsilon,
\]

and also

\[
\text{dist}(\mathbb{R}^d \setminus D^\lambda, \mathbb{R}^d \setminus D) \leq \varepsilon \text{ and } \text{dist}(\mathbb{R}^d \setminus D^{-\lambda_\varepsilon}, \mathbb{R}^d \setminus D) \leq \varepsilon.
\]

Proof. It follows by Lemmas 5.1 and 5.2 combined with Theorem 2.7. Note that the latter theorem is not quantitative and we could get different values of \( \lambda_\varepsilon \) from it for the different convergences, but we can then just use the largest one.

Example 5.4. A result like Lemma 5.2 can only hold for bounded sets. As a counterexample, consider \( D \) defined by

\[
D := \bigcup_{j=0}^{\infty} B\left((j,0), \frac{1}{2^{j+1}}\right).
\]

Clearly we have \( |D| \leq \infty \) and \( \text{Per}(D) \leq \infty \), but \( D^\lambda \) must be a union of finitely many balls, so \( d_H(\partial D, \partial D^\lambda) = \infty \) for all \( \lambda > 0 \).

First, we see that to obtain Hausdorff convergence of the level sets in the noiseless case, we do not need to use density estimates; for indicatrices it is enough to combine Lemmas 4.3 and 5.2:

Proposition 5.5. Let \( f = 1_D \) with \( D \subset B(0,1) \), \( w = 0 \) and denote by \( u_{\alpha,0} \) the corresponding minimizers of (1). Then for almost every \( s \), the boundary \( \partial E_\alpha^s \) of the level set \( E_\alpha^s = \{ u_{\alpha,0} > s \} \) converges in Hausdorff distance to \( \partial E_0^s = \partial \{ f > s \} \) as \( \alpha \to 0 \).

5.2 Convergence for piecewise constant data

The approach of Proposition 5.5, by comparison using Lemma 4.3, can be extended to piecewise constant data. We assume that \( f \in \text{BV}(\mathbb{R}^d) \) attains exactly \( n \) nonzero values \( 0 < f_1 < \ldots < f_n \), so that

\[
f = \sum_{\ell=1}^{n} f_j \chi_{\Omega_j} \text{ for } \Omega_j \subset B(0,1) \text{ with } \Omega_j \cap \Omega_k = \emptyset \text{ if } j \neq k.
\]
Assuming \( s \in (f_k, f_{k+1}) \) for \( k \in \{1, \ldots, n-1 \} \), we want to prove that

\[
\partial E^s_\alpha \xrightarrow{d_H} \partial U_k = \partial \{ f > s \}, \quad \text{for } U_k := \bigcup_{\ell=k+1}^n \Omega_\ell.
\]

Considering the set \( U^\lambda_\kappa \) defined as in Lemma 5.2 to approximate \( U_k \) from the inside with \( d_H(\partial U_k, \partial U^\lambda_\kappa) < \varepsilon \), we have by minimality of \( E^s_\alpha \) that

\[
\Per(E^s_\alpha) + \frac{1}{\alpha} \sum_{\ell=1}^n \left| \Omega_\ell \cap E^s_\alpha |\psi'(s - f_\ell) \right| \leq \Per(E^s_\alpha \cup U^\lambda_\kappa) + \frac{1}{\alpha} \sum_{\ell=1}^n \left| \Omega_\ell \cap \left( E^s_\alpha \cup U^\lambda_\kappa \right) \right| |\psi'(s - f_\ell)|,
\]

which since \( U^\lambda_\kappa \subset U_k = \bigcup_{\ell=k+1}^n \Omega_\ell \) brings us to

\[
\Per(E^s_\alpha) \leq \Per(E^s_\alpha \cup U^\lambda_\kappa) + \frac{1}{\alpha} \sum_{\ell=1}^n \left| \Omega_\ell \cap \left( U^\lambda_\kappa \setminus E^s_\alpha \right) \right| |\psi'(s - f_\ell)| \leq \Per(E^s_\alpha \cup U^\lambda_\kappa) + \frac{1}{\alpha} |\lambda^\alpha_k \setminus E^s_\alpha| |\psi'(s - f_{k+1})|.
\]

Since \( U^\lambda_\kappa \) has a variational mean curvature \( \kappa^\lambda_{i,k} \) (defined as in (33) of Lemma 4.3) we have

\[
\Per(U^\lambda_\kappa) - \int_{U^\lambda_\kappa} \kappa^\lambda_{i,k} \leq \Per(E^s_\alpha \cap U^\lambda_\kappa) - \int_{E^s_\alpha \cap U^\lambda_\kappa} \kappa^\lambda_{i,k},
\]

and summing we end up with

\[
- \int_{U^\lambda_\kappa \setminus E^s_\alpha} \kappa^\lambda_{i,k} \leq \frac{\psi'(s - f_{k+1})}{\alpha} \left| U^\lambda_\kappa \setminus E^s_\alpha \right|,
\]

which since \( \kappa^\lambda_{i,k} = \lambda_\epsilon > 0 \) in \( B(0,1) \supset U_k \supset U^\lambda_\kappa \) and \( \psi \) has even symmetry implies \( |U^\lambda_\kappa \setminus E^s_\alpha| = 0 \) when \( \alpha \leq \psi'(f_{k+1})/\lambda_\epsilon \), which can always be attained by choosing \( \alpha \) small enough since \( s < f_{k+1} \).

We also have an outside approximation \( U_k^{-\lambda_\kappa} \supset U_k \) for which \( d_H(\partial U_k, \partial U_k^{-\lambda_\kappa}) < \varepsilon \), possibly reducing \( \lambda_\kappa \) in compared to the value used in the previous paragraph. By minimality of \( E^s_\alpha \) we can write

\[
\Per(E^s_\alpha) + \frac{1}{\alpha} \sum_{\ell=1}^n \left| \Omega_\ell \cap E^s_\alpha |\psi'(s - f_\ell) \right| \leq \Per(E^s_\alpha \cap U_k^{-\lambda_\kappa}) + \frac{1}{\alpha} \sum_{\ell=1}^n \left| \Omega_\ell \cap \left( E^s_\alpha \cap U_k^{-\lambda_\kappa} \right) \right| |\psi'(s - f_\ell)|,
\]

which using the definition of \( U_k \) and \( U_k \subset U_k^{-\lambda_\kappa} \) leads to

\[
\Per(E^s_\alpha) + \frac{1}{\alpha} \left| E^s_\alpha \setminus U_k^{-\lambda_\kappa} \right| \psi'(s - f_k) = \Per(E^s_\alpha) + \frac{1}{\alpha} \sum_{\ell=1}^k \left| \Omega_\ell \cap E^s_\alpha \right| |\psi'(s - f_\ell)| \leq \Per(E^s_\alpha) + \frac{1}{\alpha} \sum_{\ell=1}^k \left| \Omega_\ell \cap E^s_\alpha \right| |\psi'(s - f_\ell)| \leq \Per(E^s_\alpha \cup U_k^{-\lambda_\kappa}).
\]

Denoting by \( \kappa^\lambda_{o,k} \) the corresponding variational mean curvature for \( U_k^{-\lambda_\kappa} \), we also have

\[
\Per(U_k^{-\lambda_\kappa}) - \int_{U_k^{-\lambda_\kappa}} \kappa^\lambda_{o,k} \leq \Per(E^s_\alpha \cup U_k^{-\lambda_\kappa}) - \int_{E^s_\alpha \cup U_k^{-\lambda_\kappa}} \kappa^\lambda_{o,k},
\]

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and summing we get
\[ \frac{\psi'(s - f_k)}{\alpha} \left| E^s_{\alpha} \setminus U^s_\lambda \right| \leq \int_{E^s_{\alpha} \setminus U^s_\lambda} \kappa^s_{\lambda, i, j}, \]
which implies \( |E^s_{\alpha} \setminus U^s_\lambda| = 0 \) when \( \alpha \leq \psi'(s - f_k)/\lambda_s \), which is possible by reducing \( \alpha \) because \( s > f_k \).

Therefore, we end up with \( U^s_\lambda \subset E^s_{\alpha} \subset U^{-s}_\lambda \) a.e. and by the triangle inequality for the Hausdorff distance (which can be easily proved from characterization (15), see [13, Prop. 7.3.3]) also \( d_H(\partial U^s_\lambda, \partial U^{-s}_\lambda) < 2\varepsilon \), providing the result as \( \varepsilon \) (and hence \( \lambda_s \) and \( \alpha \)) converge to zero.

### 5.3 Denoising of a generic BV function. Proof of Theorem 1.3

Let now \( f \) be a generic \( BV \) function supported in \( B(0, 1) \) and let \( u_\alpha \) the minimizer of (1) with \( w = 0 \). One wants to reproduce the construction of Section 5.1 for every level set of \( u_\alpha \). We call \( E^s_{\alpha} := \{ f > s \} \) for \( s > 0 \) and \( E^s_{\alpha} := \{ f < s \} \) for \( s < 0 \) the level sets of \( f \) and similarly \( E^s_{\alpha} = \{ u_\alpha > s \} \) for \( s > 0 \), \( E^s_{\alpha} = \{ u_\alpha < s \} \) for \( s < 0 \) the ones of \( u_\alpha \). The sets \( E^s_{\alpha} \) minimize
\[ E \mapsto \text{Per}(E) + \frac{\text{sign}(s)}{\alpha} \int_E \psi'(s - f). \]

In comparison with Section 5.2, since the number of attained values is not finite anymore, we cannot take a uniform approximation parameter for all level sets at once. Moreover, in contrast to the situation in Sections 4 and 5.2, the functions involved may take negative values, but by using lower level sets for \( s < 0 \) we ensure that these are also contained in \( B(0, 1) \). With this in view, we have for each \( s \) an approximation \( (E^s_{\alpha})^{\lambda_{\varepsilon, s}} \) (we will write \( E^{\lambda_{\varepsilon, s}} \) to make the notation slightly lighter) of \( E^s_{\alpha} \) from inside with \( d_H(\partial E^{\lambda_{\varepsilon, s}}, \partial E^s_{\alpha}) < \varepsilon \) and curvature \( \kappa^{\lambda_{\varepsilon, s}} \) bounded above by \( \lambda_{\varepsilon, s} \). Similarly we denote by \( E^{-s}^{\lambda_{\varepsilon, s}} \) the approximation \( (E^s_{\alpha})^{-\lambda_{\varepsilon, s}} \) of \( E^s_{\alpha} \) from outside with \( d_H(\partial E^{-\lambda_{\varepsilon, s}}, \partial E^s_{\alpha}) < \varepsilon \) and curvature \( \kappa_{\lambda_{\varepsilon, s}} \) bounded below by \( -\lambda_{\varepsilon, s} \) on \( B(0, 1) \). Note that as opposed to the piecewise constant case where we had only finitely many inequalities to satisfy, we cannot take \( \lambda_{\varepsilon, s} \) independent of \( s \).

We first prove

**Lemma 5.6.** Let \( \delta > 0 \). Then, and \( \alpha \) small enough (depending on \( s \), \( \delta \) and \( \varepsilon \),
\[ |E^{\lambda_{\varepsilon, s}}_{s + \delta} \setminus E^s_{\alpha}| = 0 \text{ and } |E^s_{\alpha} \setminus E^{-\lambda_{\varepsilon, s} - \delta}_{s - \delta}| = 0 \text{ for } s > 0, \]
and analogously
\[ |E^{\lambda_{\varepsilon, s}}_{s - \delta} \setminus E^s_{\alpha}| = 0 \text{ and } |E^s_{\alpha} \setminus E^{-\lambda_{\varepsilon, s} + \delta}_{s + \delta}| = 0 \text{ for } s < 0. \]

**Proof.** We assume that \( s > 0 \), since the case \( s < 0 \) follows in a completely analogous way after noticing that using lower level sets induces a change of sign in (43) as well as a change in the direction of inclusions with respect to \( s \).

Therefore, let \( s > 0 \) and \( \delta > 0 \) be fixed. Using the minimality of \( E^s_{\alpha} \) in (43), one can write
\[ \text{Per}(E^s_{\alpha}) + \int_{E^s_{\alpha}} \frac{\psi'(s - f)}{\alpha} \leq \text{Per}(E^s_{\alpha} \cup E^{\lambda_{\varepsilon, s} + \delta}_{s + \delta}) + \int_{E^s_{\alpha} \cup E^{\lambda_{\varepsilon, s} + \delta}_{s + \delta}} \frac{\psi'(s - f)}{\alpha}. \]

On the other hand, by definition of \( \kappa^{\lambda_{\varepsilon, s} + \delta} \), one has
\[ \text{Per}(E_{s + \delta}^{\lambda_{\varepsilon, s} + \delta}) - \int_{E^{\lambda_{\varepsilon, s} + \delta}_{s + \delta}} \kappa^{\lambda_{\varepsilon, s} + \delta} \leq \text{Per}(E_{s + \delta}^{\lambda_{\varepsilon, s} + \delta} \cap E^s_{\alpha}) - \int_{E_{s + \delta}^{\lambda_{\varepsilon, s} + \delta} \cap E^s_{\alpha}} \kappa^{\lambda_{\varepsilon, s} + \delta}. \]

Summing these two inequalities and using (7), we get
\[ \int_{E^{\lambda_{\varepsilon, s} + \delta}_{s + \delta} \setminus E^s_{\alpha}} \kappa^{\lambda_{\varepsilon, s} + \delta} \leq \int_{E^{\lambda_{\varepsilon, s} + \delta}_{s + \delta} \setminus E^s_{\alpha}} \frac{\psi'(s - f)}{\alpha}. \]
Now, recall that $E^s_{s+\delta} \subset E^s_0$, meaning that $f \geq s + \delta$ on this set. Hence $s - f \leq -\delta$ and, since $\psi'$ is increasing and $\psi$ is even, (44) implies

$$\int_{E^s_{s+\delta} \setminus E^s_\alpha} \left( -\kappa_{s+\delta} \lambda_{s+\delta} + \frac{\psi'(\delta)}{\alpha} \right) \leq 0.$$ 

Since $\kappa_{s+\delta} \lambda_{s+\delta} \leq \lambda_{s+\delta}$, as soon as $\alpha \leq \psi'(\delta)/\lambda_{s+\delta}$, which is always possible since $\psi'(\delta) > 0$ by strict monotonicity, one must have $|E^s_{s+\delta} \setminus E^s_\alpha| = 0$.

Similarly, the equality $|E^s_{s-\delta} \setminus E^s_\alpha| = 0$ is obtained writing

$$\text{Per}(E^s_\alpha) + \int_{E^s_\alpha} \frac{\psi'(s-f)}{\alpha} \leq \text{Per}(E^s_{s-\delta} \cap E^s_\alpha) + \int_{E^s_{s-\delta} \cap E^s_\alpha} \frac{\psi'(s-f)}{\alpha}$$

and

$$\text{Per}(E^s_{s-\delta} \setminus E^s_\alpha) \leq \text{Per}(E^s_{s-\delta} \cup E^s_\alpha) - \int_{E^s_{s-\delta} \cup E^s_\alpha} \frac{-\lambda_{s-\delta}}{\kappa_{s-\delta}}.$$

Summing these inequalities we obtain

$$\int_{E^s_\alpha \setminus E^s_{s-\delta}} \frac{\psi'(s-f)}{\alpha} \leq \int_{E^s_\alpha \setminus E^s_{s-\delta}} \frac{-\lambda_{s-\delta}}{\kappa_{s-\delta}}.$$

Now, the complement of $E^s_{s-\delta}$ contains the complement of $E^s_0$, therefore on this set, one has $f \leq s - \delta$, which implies

$$\int_{E^s_\alpha \setminus E^s_{s-\delta}} \left( \frac{\psi'(\delta)}{\alpha} + \frac{-\lambda_{s-\delta}}{\kappa_{s-\delta}} \right) \leq 0,$$

which, together with $\kappa_{s-\delta} \lambda_{s-\delta} \geq -\lambda_{s-\delta}$ on $B(0, 1)$ and (42) forces the expected equality as soon as $\alpha \leq \psi'(\delta)/\lambda_{s-\delta}$. \qed

We can now prove the main result of this section:

Proof of Theorem 1.3. The proof strongly relies on Lemma 5.6, and once again we assume without loss of generality that $s > 0$ is fixed. Let $\eta > 0$ and $\varepsilon = \eta/2$. First, we show that one can find $\alpha_0$ such that $d_H(E^s_\alpha, E^s_0) \leq \eta$ for every $\alpha \leq \alpha_0$. Using assumption (4), there exists $\delta > 0$ such that

$$d_H(E^s_{s+\delta}, E^s_0) \leq \varepsilon. \quad (45)$$

Then, Lemma 5.6 ensures the existence of $\alpha_0$ such that for $\alpha \leq \alpha_0$, we have to measure zero $E^s_{s+\delta} \subset E^s_\alpha \subset E^s_{s-\delta}$. Now, we just have to note that

$$d_H(E^s_\alpha, E^s_0) = \max \left\{ \sup_{x \in E^s_\alpha} \text{dist}(x, E^s_0), \sup_{x \in E^s_0} \text{dist}(x, E^s_\alpha) \right\}.$$

Note that

$$E^s_{s+\delta} \subset E^s_\alpha \Rightarrow \sup_{x \in E^s_\alpha} \text{dist}(x, E^s_0) \leq \sup_{x \in E^s_0} \text{dist}(x, E^s_{s+\delta})$$

and

$$E^s_\alpha \subset E^s_{s-\delta} \Rightarrow \sup_{x \in E^s_\alpha} \text{dist}(x, E^s_0) \leq \sup_{x \in E^s_{s-\delta}} \text{dist}(x, E^s_0).$$

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But the triangle inequality for the Hausdorff distance, (45), the Hausdorff convergence of Lemma 5.2 and Theorem 2.7 imply
\[
\sup_{x \in E_0^s} \operatorname{dist}(x, E_{s+\delta}^{\alpha,s+\delta}) \leq \mathcal{H}(E_0^s, E_{s+\delta}^{\alpha,s+\delta}) \leq \mathcal{H}(E_0^s, E_{s+\delta}^\alpha) + \mathcal{H}(E_{s+\delta}^{\alpha,s+\delta}) \leq 2\varepsilon = \eta,
\]
and
\[
\sup_{x \in E_{s-\delta}^{\alpha,s-\delta}} \operatorname{dist}(x, E_0^s) \leq \mathcal{H}(E_0^s, E_{s-\delta}^{\alpha,s-\delta}) \leq \mathcal{H}(E_0^s, E_{s-\delta}^\alpha) + \mathcal{H}(E_{s-\delta}^{\alpha,s-\delta}) \leq 2\varepsilon = \eta.
\]
We therefore conclude
\[
\mathcal{H}(E_0^s, E_\alpha^s) \leq 2\varepsilon = \eta \text{ for } \alpha \leq \alpha_0,
\]
as claimed.

Similarly, using the second part of (4) we notice that Lemma 5.6 also provides us with the reverse inclusions for the complements
\[
\mathbb{R}^d \setminus E_{s+\delta}^{\alpha,s+\delta} \supset \mathbb{R}^d \setminus E_\alpha^s \supset \mathbb{R}^d \setminus E_{s-\delta}^{\alpha,s-\delta},
\]
so we find, possibly reducing \(\alpha_0\), that also
\[
\mathcal{H}(\mathbb{R}^d \setminus E_\alpha^s, \mathbb{R}^d \setminus E_0^s) \leq \eta \text{ for } \alpha \leq \alpha_0.
\]
Using inequality (14) of Proposition 2.6, convergence in Hausdorff distance of the sets \(E_\alpha^s\) and their complements as \(\alpha \to 0\) implies convergence of the boundaries. \(\square\)

**Remark 5.7.** We recall that \(|E_\alpha^s \Delta E_0^s| \to 0\) holds for a.e. \(s\) because of the strong \(L^1\) convergence \(u_\alpha \to u\), which is implied by (42) and the \(L^1_{\text{loc}}\) convergence proved in Proposition 1.5.

**Proposition 5.8.** Let \(f\) be such that the level sets \(E_0^s\) satisfy density estimates at some scale \(r_0\) and constant \(C\), independent of the level \(s\). Then (4) holds for a.e. \(s\).

**Proof.** By the assumption, we have that at any point \(x \in E_0^s\) we have the inner density estimate
\[
\frac{|E_0^s \cap B(x, r)|}{|B(x, r)|} \geq C, \quad (46)
\]
for \(r \leq r_0\) with \(r_0, C\) independent of \(x, r_0\) and \(s\). Moreover, since the \(E_0^s\) are decreasing in \(s\) we may assume \(\delta > 0\) in the limit, and to conclude \(\mathcal{H}(E_0^s, E_{s+\delta}^s)\) we just need to check
\[
\sup_{x \in E_0^s} \operatorname{dist}(x, E_{s+\delta}^s) \xrightarrow{\delta \to 0} 0, \quad (47)
\]
since the other term in the Hausdorff distance vanishes. However, if (47) were false, we can find \(\{\delta_i\}_i, \rho > 0\) and \(x_\delta \in E_0^s\) such that \(\operatorname{dist}(x_\delta, E_{s+\delta}^s) > \rho\). But using (46) for \(E_0^s\) and \(x_\delta\), and possibly reducing \(\rho\) so that \(\rho \leq r_0\) that
\[
|E_0^s \setminus E_{s+\delta}^s| \geq |E_0^s \cap B(x_\delta, \rho)| \geq C|B(x_\delta, \rho)| = C|B(0, \rho)|,
\]
which is a contradiction with \(|E_0^s \Delta E_{s+\delta}^s| \to 0\).

Moreover, we also have the outer density estimate
\[
\frac{|B(x, r) \setminus E_0^s|}{|B(x, r)|} \geq C, \quad (48)
\]
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again for \( r \leq r_0 \) and with \( r_0, C \) independent of \( x \) and \( s \). Since the sets \( \mathbb{R}^d \setminus E_0^s \) are increasing in \( \delta \), to conclude that \( d_H(\mathbb{R}^d \setminus E_0^s, \mathbb{R}^d \setminus E_0^s+\delta) \to 0 \) we must check

\[
\sup_{x \in \mathbb{R}^d \setminus E_0^s+\delta} \text{dist}(x, \mathbb{R}^d \setminus E_0^s) \xrightarrow{\delta \to 0} 0.
\]

If this does not hold, we can find \( \{\delta_i\}_i, \rho > 0 \) and \( x_{\delta_i} \in \mathbb{R}^d \setminus E_0^s+\delta_i \) such that \( \text{dist}(x_{\delta_i}, \mathbb{R}^d \setminus E_0^s) > \rho \). But using (48) for \( E_0^s+\delta_i \) and \( x_{\delta_i} \) and with \( \rho \leq r_0 \) we have

\[
\left| \left( \mathbb{R}^d \setminus E_0^s+\delta_i \right) \setminus \left( \mathbb{R}^d \setminus E_0^s \right) \right| = |E_0^s \setminus E_0^s+\delta_i| \geq |B(x_{\delta_i}, \rho) \setminus E_0^s+\delta_i| \\
\geq C|B(x_{\delta_i}, \rho)| = C|B(0, \rho)|,
\]

leading again to a contradiction. \( \square \)

The results of [15] or [23] then directly imply that this assumption is also valid when the source condition holds:

**Corollary 5.9.** Let \( f \) be such that

\[ \partial \text{TV}(f) \neq \emptyset. \]

Then the level sets \( E_0^s \) of \( f \) satisfy (4) for a.e. \( s \).

### 6 Can we have uniform density estimates at fixed scale?

In Section 4 we have proved Hausdorff convergence of level sets for denoising of \( 1_D + w \) by using density estimates at scales that converge to 0 as \( \alpha \to 0 \). However, as the next example shows, often more can be expected out of the denoised solutions:

**Example 6.1.** Consider for \( \ell_n > r_n \to 0 \), with \( S = (0,1)^2 \subset \mathbb{R}^2 \) and \( \psi(t) = t^2/2 \) the situation

\[
f = 1_S, w_n = 1_{B_n} \quad \text{with} \quad B_n := B((-\ell_n, -\ell_n), r_n), \quad \text{so that} \quad \text{dist}(S, B_n) = \ell_n - r_n.
\]

The nontrivial level sets of \( f+w_n \) are all \( S \cup B_n \), and we clearly have that \( d_H(\partial(S \cup B_n), \partial S) \to 0 \), but they contain a spurious connected component not seen in the limit. Moreover we notice that if \( \ell_n/r_n \to +\infty \) the sets \( S \cup B_n \) fail to satisfy uniform density estimates, since in that case

\[
\frac{|(S \cup B_n) \cap B(x_0, \ell_n - r_n)|}{|B(x_0, \ell_n - r_n)|} = \frac{|B_n|}{|B(x_0, \ell_n - r_n)|} \to 0.
\]

Now, again using \( \ell_n/r_n \to +\infty \) we have that \( \text{Per}(\text{Conv}(S \cup B_n)) > \text{Per}(S \cup B_n) \), so we have for the level sets of minimizers of (1) that \( E_{\alpha_n, w_n}^s \subset S \cup B_n \). Moreover, if \( s \) and \( \alpha_n \) are such that \( (1-s)/\alpha_n < 2/r_n \) we have that \( E_{\alpha_n, w_n}^s \cap B_n = \emptyset \), as can be seen from the computations done in Example 3.4. This implies, when using a linear parameter choice \( \alpha_n = C\|w_n\|_L^2 = C\sqrt{\pi} r_n \), that whenever \( C > \sqrt{\pi}/2 \) and for \( n \) large enough the effect of \( w_n \) is not seen in the solution. In that case it is easy to see that level sets admit uniform density estimates at fixed scale, since then \( E_{\alpha_n, w_n}^s = S^{s/\alpha_n} \) where the notation \( S^{s/\alpha_n} \) is understood in the sense of Definition 3.11, which are explicitly computed for the case of the square \( S \) in Proposition 3.17 and Remark 3.18.

Uniform density estimates along the sequence of level sets of minimizers provide not only Hausdorff convergence of the boundaries of level sets, but also prevent the appearance of spurious structures at smaller and smaller scales. For general sets of finite perimeter as in Section 4, since the limit is not regular, we cannot in general expect uniform density for the level sets approaching it.

On the opposite side, if we knew that \( \partial \text{TV}(1_D) \neq \emptyset \), uniform density estimates for the level sets are implied by the results of [15] for \( d = 2 \) and [23] for \( d > 2 \). However, this source condition
excludes large classes of sets $D$ where we would expect the level sets of minimizers to also satisfy uniform density estimates, in particular sets $D$ with general Lipschitz boundary and satisfying density estimates themselves, like the square in the example above. The question then arises of how to derive these estimates such cases. We are not able to give a complete answer, but we collect some observations here, specialized to the two-dimensional case and $\psi(t) = t^2/2$.

Examining the proof of the density estimates in Proposition B.1, to have a uniform scale at which the estimates hold it would be sufficient to have an inequality bounding the integral of $\kappa_D$ on small sets by a quantity strictly less than their perimeter. In particular, since connected components of $D$ inherit the curvature of the whole set, such an inequality implies that no arbitrarily small components can be created, a property which (by the dual stability of Proposition A.2) is also true for the denoised level sets with an adequate parameter choice. We formulate this property as the following assumption:

**Assumption 6.2.** There is a constant $0 < \xi_D < 1$ and a scale $r_0 > 0$ such that for any $A \subset \mathbb{R}^2$ admitting a variational mean curvature in $L^2(\mathbb{R}^2)$, $x \in \mathbb{R}^2$ and $0 < r \leq r_0$ the following inequalities hold:

$$\int_{A \cap B(x,r)} \kappa_D^+ \leq \xi_D \text{Per}(B(x,r) \cap A)$$

and

$$\int_{A \cap B(x,r)} \kappa_D^- \leq \xi_D \text{Per}(B(x,r) \cap A),$$

where $\kappa_D^+ = \max(\kappa_D, 0)$ and $\kappa_D^- = -\min(\kappa_D, 0)$.

Let us check that Assumption 6.2 holds for the square.

**Example 6.3.** Denote the unit square by $S \subset \mathbb{R}^2$ and the test set directly by $E := A \cap B(x,r)$, since we will not use its form or regularity explicitly. By definition $\kappa_S \geq 0$ in $S$, and since $S$ is convex, $\kappa_S = -\lambda g(x)$ for $x \in \mathbb{R}^2 \setminus S$. Let us start with the first case. It is enough to prove that

$$\int_E \kappa_S \leq \xi_S \text{Per}(E),$$

for $E \subset S$, $|E|$ small, and with $\xi_S < 1$. Assuming $|E| < |S \setminus C_S|/4$, and in view of the optimal curvature of the square (31) we have that

$$\int_E \kappa_S \leq \int_{(S \setminus E_{\Lambda}) \cap Q^1} \kappa_S,$$

for some $\Lambda$ such that $|(S \setminus E_{\Lambda}) \cap Q^1| = |E|$ and with $Q^1$ the lower left quadrant. Now, for each $\lambda > 0$

$$|(S \setminus E_{\Lambda}) \cap Q^1| = \frac{1}{\Lambda^2} - \frac{\pi}{4\Lambda^2} = \frac{4 - \pi}{4\Lambda^2}.$$

Therefore

$$\int_{(S \setminus E_{\Lambda}) \cap Q^1} \kappa_S = \Lambda |\{\kappa_S > \Lambda\}| + \int_{\Lambda}^{\infty} |\{\kappa_S > \lambda\}| \, d\lambda$$

$$= \Lambda |(S \setminus E_{\Lambda}) \cap Q^1| + \int_{\Lambda}^{\infty} |(S \setminus E_{\Lambda}) \cap Q^1| \, d\lambda$$

$$= \frac{4 - \pi}{2\Lambda}$$

and on the other hand, by the isoperimetric inequality

$$\text{Per}(E) \geq 2\sqrt{|E|} = 2 \sqrt{\pi \frac{4 - \pi}{4\Lambda^2}} = \frac{\sqrt{\pi(4 - \pi)}}{\Lambda},$$

(51)
and we get the first inequality of (49) with any \( \xi_S \) such that

\[
1 > \xi_S > \frac{4 - \pi}{2\sqrt{\pi(4 - \pi)}} \approx 0.26.
\]

To prove the second part of (49) we notice that whenever

\[
r \leq \frac{2\sqrt{\pi}}{\xi_S \lambda_g |B(0,1)|^{1/2}}
\]

for any \( F \subset B(x_0, r) \) we can write

\[
\frac{1}{\text{Per}(F)} \int_F \kappa_F = \frac{\lambda_g}{\text{Per}(F)} \int_{F \setminus S} g \leq \frac{\lambda_g}{\text{Per}(F)} \int_F g \leq \frac{\lambda_g |F|}{2\sqrt{\pi} |F|^{1/2}} = \frac{\lambda_g r |B(0,1)|^{1/2}}{2\sqrt{\pi}} \leq \xi_S,
\]

where we have used \( g \leq 1 \), the isoperimetric inequality and (52).

**Remark 6.4.** For a convex polygon \( P \), one could try to repeat the proof above around a vertex with angle \( 2\theta \), and \( \lambda > 0 \) large enough so that the contact points of a circle of radius \( 1/\lambda \) lie on the two edges the vertex belongs to. The analogous formulas to (50) and (51) are then

\[
\int_E \kappa_P \leq \frac{2}{\Lambda} \left( \frac{1}{\tan \theta} + \theta - \frac{\pi}{2} \right) \quad \text{and} \quad \text{Per}(E) \geq 2\sqrt{\pi |E|} = \frac{2\sqrt{\pi}}{\Lambda} \sqrt{\left( \frac{1}{\tan \theta} + \theta - \frac{\pi}{2} \right)}.
\]

However, the quotient of these to quantities is below 1 only for \( \theta \) larger than \( \approx 0.219 \). It is very likely this is a problem of the proof method, since the isoperimetric estimate used is far from sharp, and that in fact Assumption 6.2 holds for any polygon \( P \). Convexity is likely also not required, since a polygon cannot have arbitrarily thin necks, so by the results cited in Remark 3.18 inclusions of balls characterize \( P^\Lambda \), and we can also determine \( P^{\Lambda - \Lambda} \) analogously.

Now we check that Assumption 6.2 indeed implies uniform density estimates at a fixed scale. Our scheme will be to work first with the solutions corresponding to noiseless data, and then comparing them with the noisy ones using Proposition 4.1.

**Theorem 6.5.** Let \( w_n \in L^2(\mathbb{R}^2) \) and \( \alpha \to 0^+ \) satisfying the parameter choice

\[
\frac{\|w_n\|_{L^2_\alpha}}{\alpha_n} \to 0,
\]

and let \( u_{\alpha_n,w_n} \) denote the corresponding minimizers of (1) with \( f = 1_D + w_n \), where \( D \) satisfies Assumption 6.2. Then for a.e. \( s \in (0,1) \), the level sets

\[
E_{\alpha_n,w_n}^s := \{ u_{\alpha_n,w_n} > s \}
\]

satisfy uniform density estimates at scale \( r_0 \) and with constant \( C_0 \), that is

\[
\frac{|E_{\alpha,w}^s \cap B(x,r)|}{|B(x,r)|} \geq C_0, \quad \text{and} \quad \frac{|B(x,r) \setminus E_{\alpha,w}^s|}{|B(x,r)|} \geq C_0
\]

for all \( x \in \partial E_{\alpha,w}^s \) and \( 0 < r \leq r_0 \).
Proof. For such $x, r$ we have by the Cauchy-Schwartz inequality, Proposition 4.1 and since
\[ \text{sign}(v_{\alpha, n}) = \text{sign}(\kappa_D) \]
that
\[
\int_{E_{\alpha, w}^s \cap B(x, r)} v_{\alpha, w} \leq |E_{\alpha, w}^s \cap B(x, r)|^{1/2} \|v_{\alpha, w} - v_{\alpha, 0}\|_{L^2(\mathbb{R}^2)} + \int_{E_{\alpha, w}^s \cap B(x, r)} v_{\alpha, 0} \leq |E_{\alpha, w}^s \cap B(x, r)|^{1/2} \|v_{\alpha, w} - v_{\alpha, 0}\|_{L^2(\mathbb{R}^2)} + \int_{E_{\alpha, w}^s \cap B(x, r)} \kappa_D^+ \]
(54)
\[
\leq |E_{\alpha, w}^s \cap B(x, r)|^{1/2} \|v_{\alpha, w} - v_{\alpha, 0}\|_{L^2(\mathbb{R}^2)} + \xi_D \text{Per}(E_{\alpha, w}^s \cap B(x, r)) + \xi_D \text{Per}(E_{\alpha, w}^s \cap B(x, r))
\]
where we have used the first inequality in (49) for the penultimate step and for the last step the
parameter choice (53) which combined with the dual stability of Proposition A.2 allows $\eta > 0$ to
be chosen as small as needed. Plugging this in formula (64) of Appendix B, we obtain
\[
\text{Per}(E_{\alpha, w}^s \cap B(x, r)) (1 - \xi_D) - |E_{\alpha, w}^s \cap B(x, r)|^{1/2} \eta \leq 2 \text{Per} \left( B(x, r); (E_{\alpha, w}^s)^{(1)} \right).
\]
Using now the isoperimetric inequality, we get
\[
|E_{\alpha, w}^s \cap B(x, r)|^{1/2} \left( 2\sqrt{\pi} (1 - \xi_D) - \eta \right) \leq 2 \text{Per} \left( B(x, r); (E_{\alpha, w}^s)^{(1)} \right).
\]
Taking $\eta < 2\sqrt{\pi} (1 - \xi_D)$, we can derive the inner density estimate $|E_{\alpha, w}^s \cap B(x, r)| \geq C_0 |B(x, r)|$
by integrating this differential inequality up to $r_0$.

For the outer density, one proceeds in an analogous fashion the complements $\mathbb{R}^2 \setminus E_{\alpha, w}^s$, which
switches the sign of the curvature to $-v_{\alpha, w}$ and makes $\kappa_D^-$ play a role in (54) through the second
inequality in (49).

We notice that in the situation of Theorem 6.5, the Hausdorff convergence $d_H(\partial E_{\alpha, w}^s, \partial D) \to 0$
for $0 < s < 1$ follows then by Proposition 4.5 and Theorem 2.8. Moreover:

**Corollary 6.6.** Under the assumptions of Theorem 6.5 and for either $s < 0$ or $s > 1$, we
additionally have
\[
\limsup_{\alpha} \partial E_{\alpha, w}^s = \emptyset,
\]
where $\limsup_{\alpha} \partial E_{\alpha, w}^s$ is defined to be [31, Def. 4.1] the set of all limits of subsequences of points
in $\partial E_{\alpha, w}^s$.

Proof. If $s < 0$ or $s > 1$, by the convergence $u_{\alpha, w} \to 1_D$ in $L^q$ we have $|E_{\alpha, w}^s| \to 0$. Assume for
a contradiction that we had $x \in \limsup \partial E_{\alpha, w}^s$. Then we have a subsequence $x_{\alpha_n} \in \partial E_{\alpha, w}^s$ such
that $x_{\alpha_n} \to x$. Now, as in the proof of Theorem 1.2, by using the inner density estimate, which
now holds with constant $C$ and for scales $r \leq r_0$ uniformly both in $\alpha$ and the chosen points we
get
\[
|E_{\alpha_n, w_n}^s| \geq |B(x_{\alpha_n}, r_0) \cap E_{\alpha_n, w_n}^s| \geq C|B(0, r_0)|,
\]
which contradicts $|E_{\alpha_n, w_n}^s| \to 0$. \(\square\)

Observe that in the setting of Theorem 1.2 where the density estimates depend on the distance
to $\partial D$, the proof we have given for this corollary fails. Indeed, with such density estimates we
could only get that $\text{dist}(x, \partial D) \leq r$ for all $r > 0$ small enough and $x \in \limsup_{\alpha} \partial E_{\alpha, w}^s$, or
\[
\limsup_{\alpha} \partial E_{\alpha, w}^s \subset \partial D,
\]
which for $s < 0$ or $s > 1$ is not a satisfactory conclusion. We conclude with some further
observations about when inequality (49) could be expected to hold.
Remark 6.7. Although it is naturally of $L^2$ scaling for $\kappa_D$, Assumption 6.2 can be formulated in more general spaces with this scaling, giving some hope that it could hold for sets with Lipschitz boundary. For example, we would have (49) with $\xi_D < 1$ if we had $\|\kappa_D\|_{L^2,w} < 2\sqrt{\pi}$ for the weak $L^2$ norm. In fact, in the notation of [28, Def. 3.3], it is also enough to have $\|\kappa_D\|_{S(\mathbb{R}^2)} < 1$, and in [28, Thm. 3.7] it is shown that $S(\mathbb{R}^2)$ in fact coincides with the Morrey space $L^{1,1}$ (with different norms, a priori). The quantitative bounds are necessary, since the example in [28, Thm. 8.5] provides a set $D$ without density estimates, whose curvature $\kappa_D$ belongs to $L^{1,1}$.

Remark 6.8. We have by definition that
\[
\|\kappa_D\|_{L^2,w} = \sup_{\lambda} \lambda \left\{ \{\kappa_D\} \geq \lambda \right\}^{1/2}.
\]
Now, if $D$ is convex the construction of $\kappa_D$ implies that $\kappa_D \geq h(D)$ in $D$, for
\[
h(D) = \inf_{A \subset D} \frac{\text{Per}(A)}{|A|}
\]
the Cheeger constant of $D$, attained by the Cheeger set $C_D$, which is unique [3]. So with the isoperimetric inequality and that $C_D \subset D$ we have
\[
h(D) \{\kappa_D \geq h(D)\}^{1/2} = h(D)|D|^{1/2} = \frac{\text{Per}(C_D)}{|C_D|} |D|^{1/2} \geq 2\sqrt{\pi} \frac{|D|^{1/2}}{|C_D|^{1/2}} \geq 2\sqrt{\pi},
\]
with equality if and only if $D$ is a circle. This means that if one wants to use the language of weak norms, then it is necessary to restrict/truncate to small scales or large curvatures.

Remark 6.9. If $D$ is convex we have that $u_{\alpha,0} = (1 - \alpha\kappa_D)^+1_D$ (see [3, Prop. 2.2] or [14, Thm. 6]). This implies that one can construct a vector field $z \in L^\infty(\mathbb{R}^d)$ with $|z| \leq 1$ with divergence $\kappa_D$, and which coincides with the normal to $D$ on $\partial D$. The Green formula would provide us with (49) if $z$ was for example continuous in $\hat{D}$, since then cancellations of the flux would appear.

Remark 6.10. An inequality resembling (49) in Assumption 6.2 also appears in some works dealing with prescribed mean curvature surfaces in periodic media, like [17] and [22]. In that case, the setting is that of a bounded cell $Q$ and a potential $g \in L^d(Q)$ with $\int_Q g \leq (1-\delta)\text{Per}(E; Q)$ for fixed $\delta \in (0, 1)$ and all $E \subset Q$ is used. In fact, it is proved in [17, Prop. 4.1] using the results of [11] that in this case there is a continuous vector field $z \in C(Q; \mathbb{R}^d)$ with $|z| \leq 1$ for which $\text{div} z = g$, which is also incompatible with $g$ being the variational mean curvature of a nonsmooth set $D$, since in that case we would expect that $z|_{\partial D} = \nu_D$ [16, Thm. 3.7]. This, after Remark 6.8, is yet more evidence that (49) can only be expected for small $r$.

A Dual problem and its stability

Proposition A.1. Assume that $f, w \in L^{d/(d-1)}(\mathbb{R}^d)$. The Fenchel dual of (1) reads
\[
\sup_{v \in \partial TV(0)} \int_{\partial TV(0)} (v(f+w) - \frac{1}{\alpha} \int \psi^*(-\alpha v)), \tag{55}
\]
which has a unique maximizer $v_{\alpha,w}$ that satisfies the optimality condition
\[
v_{\alpha,w} = -\frac{1}{\alpha} \psi'(u_{\alpha,w} - f - w) \in \partial TV(u_{\alpha,w}), \tag{56}
\]
where $u_{\alpha,w}$ is the unique minimizer of (1).
Proof. Existence follows strong duality in Banach spaces \([10, \text{Thm. 4.4.3, p. 136}]\) applied to the space \(L^{d/(d-1)}(\mathbb{R}^d)\) with functions \(TV(\cdot), g(\cdot) = \frac{1}{\alpha} \int_{\mathbb{R}^d} \psi(\cdot - f - w)\) and the identity operator, while uniqueness is a direct consequence of strict convexity of \(\psi^*\).

To apply strong duality we need a qualification condition. Since \(g\) is up to a shift and a constant factor the functional \(\int \psi (f + w)\) and \(f, w \in L^{d/(d-1)}\), it is enough to check that \(u \mapsto \int \psi(u)\) is continuous on \(L^{d/(d-1)}\), so that \(g\) is in particular continuous at 0. By \([19, \text{Prop. IV.1.1}]\) continuity holds as soon as we can guarantee that \(\psi \circ u \in L^1\) for every \(u \in L^{d/(d-1)}\), which is directly implied by the inequality \(|\psi(t)| \leq C|t|^{d/(d-1)}\) included in Assumption (A).

The Fenchel conjugate of \(g\) reads

\[
g^*(v) = -\int v(f + w) + \frac{1}{\alpha} \int \psi^*(\alpha v).
\]

As already computed in \([24, \text{Thm. 1}]\), the conjugate of the total variation is \(TV^* = \chi_{\partial TV(0)}\), the indicator function of the convex set \(\partial TV(0)\). In this duality setting, we have \([19, \text{Eqs. I.(4.24), I.(4.25)}]\) the optimality conditions \(v_{\alpha,w} \in \partial TV(u_{\alpha,w})\) and \(-v_{\alpha,w} \in \partial g(u_{\alpha,w})\) as well, which are exactly (56).

Now, we would like to use assumption (A) to arrive at a stability result for the maximizers \(v_{\alpha,w}\) of (55).

**Proposition A.2.** We have the stability estimate

\[
\|v_{\alpha,w} - v_{\alpha,0}\|_{L^d(\mathbb{R}^d)} \leq \sigma_{\psi} \left( \frac{\|w\|_{L^{d/(d-1)}}}{\alpha} \right),
\]

where \(\sigma_{\psi}\) is the inverse of the function \(t \mapsto m_{\psi^*}(t)/t\), with \(m_{\psi^*}\) the largest modulus of uniform convexity for \(\psi^*\).

Proof. The computations are analogous to the ones in \([23, \text{Prop. 3.5, Prop. 3.6}]\), in turn originating from the methods in \([1, 2]\), adapted to the slightly different framework here. The main idea is, for the weak-* closed convex set \(K = \partial TV(0) \subset L^d\), to define a generalized projection \(\pi : L^{d/(d-1)} \to K\) by

\[
\pi(u) := \arg \min_{v \in K} \int \psi(u - vu + \psi^*(v)),
\]

and then noticing that the dual variable is obtained as

\[
v_{\alpha,w} = \frac{1}{\alpha} \pi(f + w),
\]

where we have used that \(\psi\) being even implies that \(\psi^*\) is also even.

Now, given any \(u \in L^{d/(d-1)}\) and \(v \in L^d\), differentiating the argument of the right hand side of (58) in direction \(\pi(u) - v\) and using minimality at \(\pi(u)\) we end up with

\[
\int (v - \pi(u)) (u - (\psi^*)' \circ \pi(u)) \geq 0.
\]

On the other hand, we have the uniform monotonicity inequality (for a proof, see \([23, \text{Lem. 1.2}]\))

\[
\int ((\pi(u_1) - \pi(u_2)) ((\psi^*)' \circ \pi(u_1) - (\psi^*)' \circ \pi(u_2)) \geq 2m_{\psi^*} (\|\pi(u_1) - \pi(u_2)\|_{L^{d/(d-1)}}),
\]

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for whose left hand side we have, using (60) twice and Hölder inequality, that

\[
\int (\pi(u_1) - \pi(u_2)) \left( (\psi^*)' \circ \pi(u_1) - (\psi^*)' \circ \pi(u_2) \right) \\
\leq \int (\pi(u_1) - \pi(u_2)) (u_1 - u_2) \\
+ \int (\pi(u_1) - \pi(u_2)) \left( (\psi^*)' \circ \pi(u_1) - u_1 \right) \\
- \int (\pi(u_1) - \pi(u_2)) \left( (\psi^*)' \circ \pi(u_2) - u_2 \right) \\
\leq \int (\pi(u_1) - \pi(u_2)) (u_1 - u_2) \\
\leq \|\pi(u_1) - \pi(u_2)\|_{L^d} \|u_1 - u_2\|_{L^d/(d-1)}.
\]

(62)

The combination of (62), (61) and (59) allows us then to conclude (57). As already noted in [23], the property (see [9, Fact 5.3.16]) \( m_{\psi^*}(ct) > c^2 m_{\psi^*}(t) \) for all \( c > 1 \) implies that \( t \mapsto m_{\psi^*}(t)/t \) is strictly increasing, therefore its inverse is well defined.

\[ \square \]

**B Density estimates for denoised level sets**

**Proposition B.1.** Let \( K \subset \mathbb{R}^d \) be a bounded set, assume that

\[ \|v_{\alpha,w} - v_{\alpha,0}\|_{L^4(\mathbb{R}^d)} \leq C_0 < \Theta_d, \]

which is possible by Proposition A.2. Furthermore, assume that for each \( \varepsilon > 0 \) there is \( r_{K,\varepsilon} > 0 \) such that for all \( x \in \mathbb{R}^d \setminus K \) and all \( \alpha \) we have the equi-integrability estimate

\[ \int_{B(x,r_{K,\varepsilon})} |v_{\alpha,0}|^d \leq \varepsilon. \]

(63)

Then the level sets \( E_{\alpha,w}^s \) satisfy uniform density estimates at some scale \( r_K \) and with constant \( C_K \) outside \( K \), that is

\[ \frac{|E_{\alpha,w}^s \cap B(x,r)|}{|B(x,r)|} \geq C_K, \quad \text{and} \quad \frac{|B(x,r) \setminus E_{\alpha,w}^s|}{|B(x,r)|} \geq C_K \]

for all \( x \in \partial E_{\alpha,w}^s \setminus K \) and \( 0 < r \leq r_K \).

**Proof.** Let \( x \in \partial E_{\alpha,w}^s \setminus K \). We start from the formula (the subscript \( (1) \) denoting points of full density)

\[ \text{Per}(E_{\alpha,w}^s \cap B(x,r)) - \int_{E_{\alpha,w}^s \cap B(x,r)} v_{\alpha,w} \leq 2 \text{Per} \left( B(x,r); (E_{\alpha,w}^s)^{(1)} \right), \]

(64)

which can be seen to hold (for a proof, see for example [24, Lem. 8]) for almost every \( r > 0 \) by repeated application of the precise formulas for perimeter of an intersection [30, Thm. 16.3], and noticing that substantial tangential contact can only happen on a set of radii of measure zero.

On the other hand we have, thanks to the Hölder inequality, the condition B.1 and local equiintegrability (63) that for \( 0 < r \leq r_{K,\varepsilon} \)

\[
\int_{E_{\alpha,w}^s \cap B(x,r)} v_{\alpha,w} \leq |E_{\alpha,w}^s \cap B(x,r)|^{(d-1)/d} \|v_{\alpha,w} - v_{\alpha,0}\|_{L^d(\mathbb{R}^d)} + \int_{E_{\alpha,w}^s \cap B(x,r)} |v_{\alpha,0}| \\
\leq |E_{\alpha,w}^s \cap B(x,r)|^{(d-1)/d} C_0 + \int_{E_{\alpha,w}^s \cap B(x,r)} |v_{\alpha,0}| \\
\leq |E_{\alpha,w}^s \cap B(x,r)|^{(d-1)/d} (C_0 + \varepsilon)
\]

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Plugging this in (64), we obtain
\[ \text{Per}(E_{\alpha, w}^s \cap B(x, r)) - |E_{\alpha, w}^s \cap B(x, r)|^{(d-1)/d}(C_0 + \varepsilon) \leq 2 \text{Per} \left( B(x, r); (E_{\alpha, w}^s)^{(1)} \right). \]

Using now the isoperimetric inequality, we get
\[ |E_{\alpha, w}^s \cap B(x, r)|^{(d-1)/d}(\Theta_d - C_0 - \varepsilon) \leq 2 \text{Per} \left( B(x, r); (E_{\alpha, w}^s)^{(1)} \right). \quad (65) \]

Taking some fixed \( \varepsilon_0 < \Theta_d - C_0 \), and since for a.e. \( r > 0 \)
\[ \text{Per} \left( B(x, r); (E_{\alpha, w}^s)^{(1)} \right) = \mathcal{H}^{d-1} \left( B(x, r) \cap (E_{\alpha, w}^s)^{(1)} \right) = \left. d \right|_{t=r} |E_{\alpha, w}^s \cap B(x, t)|, \]
we can derive the inner density estimate \( |E_{\alpha, w}^s \cap B(x, r)| \geq C_K |B(x, r)| \) by integrating the differential inequality (65) up to \( r_K := r_{K, \varepsilon_0} \). The outer density estimate follows analogously by considering the complement \( \mathbb{R}^d \setminus E_{\alpha, w}^s \), which admits the variational mean curvature \(-v_{\alpha, w}\).

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