Controllability for Problems with Mixed Constraints

A. V. Arutyunov* and S. E. Zhukovskiy**

1 Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, 117997 Russia
   e-mail: *arutyunov@cs.msu.ru, **s-e-zhuk@yandex.ru

Received November 4, 2021; revised November 4, 2021; accepted February 7, 2022

INTRODUCTION

For the controlled system
\[ \dot{x} = F(x, u, t), \quad x(t_0) = x_0, \]  
let mixed and geometric constraints have, respectively, the form
\[ G^1(x, u, t) = 0, \]
\[ G^2(x, u, t) \leq 0 \]
and
\[ u(x, t) \in U. \]

Here \( x \in \mathbb{R}^n \) is the state variable, \( u \in \mathbb{R}^m \) is the control vector, \( t \geq t_0 \) is time, the mapping
\( F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \)
given and is continuous in the first and second variables, measurable in the third variable, and bounded in a neighborhood of the fixed point \((x_0, u_0, t_0)\), and the given set \( U \subset \mathbb{R}^m \) is nonempty, closed, and convex. The vector functions \( G^l : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^s, l = 1, 2, \)
are continuous; relations (2) are satisfied coordinatewise.

By a control we mean a continuous function \( u \) of the variables \( x \) and \( t \) defined in some neighborhood of the point \((t_0, x_0)\) and such that \( u(x, t) \in U \).

We say that system (1)–(3) is locally solvable at a point \((x_0, u_0, t_0)\) if there exists a neighborhood \( \Omega \) of the point \((x_0, t_0)\), a continuous mapping \( \bar{u} : \Omega \to U \), a number \( \tau > 0 \), and an absolutely continuous function \( \bar{x} : [t_0, t_0 + \tau) \to \mathbb{R}^n \)
such that \( \bar{u}(x_0, t_0) = u_0 \), for all \( t \in [t_0, t_0 + \tau] \) one has the relations
\[ G^1(\bar{x}(t), \bar{u}(\bar{x}(t), t), t) = 0, \]
\[ G^2(\bar{x}(t), \bar{u}(\bar{x}(t), t), t) \leq 0, \]
and the function \( \bar{x}(\cdot) \) is a solution of the Cauchy problem
\[ \dot{x} = F(\bar{x}, \bar{u}(\bar{x}, t), t), \quad \bar{x}(t_0) = x_0; \]
i.e., \( \dot{x}(t) = F(x(t), u(x(t), t), t) \) for a.a. \( t \in [t_0, t_0 + \tau] \) and \( \bar{x}(t_0) = x_0 \). The indicated function \( \bar{u}(\cdot) \)
is conventionally called an feasible positional control and the pair \((\bar{x}(\cdot), \bar{u}(\cdot))\) is called a feasible process.

Our goal is to obtain sufficient local solvability conditions for system (1)–(3) both in terms of the first derivative of the mappings \( G^1 \) and \( G^2 \) and in terms of the second derivative of these mappings.
1. FIRST-ORDER REGULARITY

In this section, we give conditions for the regularity of mixed constraints in terms of the first derivative. Assume that

\[ G^1(x_0, u_0, t_0) = 0, \]
\[ G^2(x_0, u_0, t_0) \leq 0. \]

By \( I \) we denote the set of all those indices \( i \in \{1, \ldots, s_2\} \) for which the \( i \)th component \( G^2_i(x_0, u_0, t_0) \) of the vector \( G^2(x_0, u_0, t_0) \) is zero. For a vector function \( G^l \), \( l = 1, 2 \), by \( G^l_u \) we denote the matrix of its first partial derivatives with respect to the components of the vector \( u \), and for an arbitrary \( \xi \in \mathbb{R}^m \), by \( (G^l_u(x_0, u_0, t_0)\xi)_i \) we denote the \( i \)th component of the vector \( G^2_u(x_0, u_0, t_0)\xi \in \mathbb{R}^{s_2} \).

**Theorem 1.** Let the mappings \( F \) and \( G^l \), \( l = 1, 2 \), be continuous, let the mappings \( G^l \) be strictly differentiable with respect to \( u \) at the point \((x_0, u_0, t_0)\) uniformly in \( x \), and let the following assumptions be true:

**(R1)** \( 0 \in \text{int} G^1_u(x_0, u_0, t_0)(U - \{u_0\}) \).

**(R2)** There exists a vector \( \xi \in U - \{u_0\} \) such that \( G^1_u(x_0, u_0, t_0)\xi = 0 \) and

\[ (G^2_u(x_0, u_0, t_0)\xi)_i < 0 \quad \text{for all} \quad i \in I. \]

Then system (1)–(3) is locally solvable at the point \((x_0, u_0, t_0)\).

**Proof.** First, assume that \( G^2(x_0, u_0, t_0) = 0 \). For arbitrary

\[ x \in \mathbb{R}^n, \quad v = (\lambda, \nu, y)^T \in (0, +\infty) \times \mathbb{R}^m \times \mathbb{R}^{s_2}, \]

and \( t \in \mathbb{R} \) we set

\[ G(x, t, v) := \begin{pmatrix} G^1(x, u_0 + \nu/\lambda, t) \\ G^2(x, u_0 + \nu/\lambda, t) + y \end{pmatrix}. \]

Define a set \( K \) by the formula

\[ K := \{(\lambda, \lambda \nu, y)^T : \lambda \geq 0, \nu \in U - \{u_0\}, y \in \mathbb{R}^{s_2}_+\}, \]

where \( \mathbb{R}^{s_2}_+ \) is the cone in \( \mathbb{R}^{s_2} \) formed by vectors with nonnegative components.

Let us show that the set \( K \) is a convex closed cone.

For arbitrary \( \mu \geq 0 \) and \( (\lambda, \lambda \nu, y)^T \in K \), we have \( \mu (\lambda, \lambda \nu, y)^T = (\mu \lambda, \mu \lambda \nu, \mu y)^T \in K \), because, first, \( \mu \lambda \geq 0 \), second, \( \nu \in U - \{u_0\} \), and third, \( \mu y \in \mathbb{R}^{s_2}_+ \). Hence the set \( K \) is a cone.

For arbitrary \( \mu \in [0, 1], (\lambda, \lambda \nu, y)^T \in K, \) and \( (\lambda', \lambda' \nu', y')^T \in K, \) we have

\[ \mu (\lambda, \lambda \nu, y)^T + (1 - \mu) (\lambda', \lambda' \nu', y')^T \]

\[ = \left( \mu \lambda + (1 - \mu) \lambda', \mu \lambda + (1 - \mu) \lambda' \nu + \frac{(1 - \mu) \lambda'}{\mu \lambda + (1 - \mu) \lambda'}, \mu y + (1 - \mu) y' \right)^T \in K. \]

Here the inclusion follows from the inequality \( \mu \lambda + (1 - \mu) \lambda' \geq 0 \) and the inclusions

\[ \frac{\mu \lambda}{\mu \lambda + (1 - \mu) \lambda'} u + \frac{(1 - \mu) \lambda'}{\mu \lambda + (1 - \mu) \lambda'} u' \in U - \{u_0\} \quad \text{and} \quad \mu y + (1 - \mu) y' \in \mathbb{R}^{s_2}_+, \]

which follow from the convexity of the sets \( U - \{u_0\} \) and \( \mathbb{R}^{s_2}_+ \), respectively. Thus, the cone \( K \) is convex.

Let us take an arbitrary sequence \{\((\lambda_j, \lambda_j \nu_j, y_j)^T\)\} \( \subset K \) converging to some point \((\lambda, z, y)^T\). Since \( \{y_j\} \subset \mathbb{R}^{s_2}_+ \) and \( \{y_j\} \to y \), we have \( y \in \mathbb{R}^{s_2}_+ \). Consider two cases: \( \lambda = 0 \) and \( \lambda > 0 \). Let \( \lambda = 0 \).
Then \( z = \lambda \lim_{j \to \infty} \nu_j = 0 \) and hence \((\lambda, z, y)^T \in K\). Let \( \lambda > 0 \). Then, since \( \{\nu_j\} \to z/\lambda, \{\nu_j\} \subset U - \{u_0\}, \) and the set \( U - \{u_0\} \) is closed, we have \( z/\lambda \in U - \{u_0\} \) and hence \((\lambda, z, y)^T = (\lambda, \lambda z/\lambda, y)^T \in K\). Consequently, the set \( K \) is closed.

By \( G_v \) we denote the matrix of the first partial derivatives of the vector function \( G \) with respect to the components of the vector \( v \). Denote also \( v_0 := (1, 0, 0)^T \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^2 \) and

\[ K := K + \text{span}\{v_0\} \quad \text{and} \quad C := G_v(x_0, t_0, v_0)K. \]

Let us show that the Robinson regularity condition in the variable \( v \) is satisfied at the point \( v_0 \) for the mapping \( G \) and the cone \( K \); i.e., \( C = \mathbb{R}^{s_1} \times \mathbb{R}^{s_2} \). Set \( A_l = G''_{v_l}(x_0, u_0, t_0), l = 1, 2 \). We have

\[ G_v(x_0, t_0, v_0) = \begin{pmatrix} 0 & A_1 & 0 \\ 0 & A_2 & E \end{pmatrix} \]

(here \( E : \mathbb{R}^{s_2} \to \mathbb{R}^{s_2} \) is the identity operator) and

\[ K = \{ (\mu, \lambda \nu, y)^T : \mu \in \mathbb{R}, \ \lambda \geq 0, \ \nu \in U - \{u_0\}, \ y \in \mathbb{R}^{s_2}_+ \}. \]

Let us prove the last equality, we take arbitrary \((y_1, y_2)^T \in \mathbb{R}^{s_1} \times \mathbb{R}^{s_2} \) and show that the system

\[
\begin{align*}
\lambda A_1 \nu &= y_1, \\
y + \lambda A_2 \nu &= y_2
\end{align*}
\]

has a solution \((\lambda, \nu, y)^T \in \mathbb{R} \times (U - \{u_0\}) \times \mathbb{R}^{s_2}_+ \). By virtue of assumption (R1), there exist \( \gamma > 0 \) and \( \bar{\nu} \in U - \{u_0\} \) such that \( \gamma A_1 \bar{\nu} = y_1 \). Consequently,

\[
\gamma A_1 (\tau \bar{\nu})/\tau = y_1 \quad \text{for all} \quad \tau \in (0, 1).
\]

By virtue of assumption (R2), there exists a vector \( \bar{\xi} \in U \) such that \( A_1 \bar{\xi} = 0 \) and each component of the vector \( A_2 \bar{\xi} \) is negative. Then there exists a \( \tau \in (0, 1) \) for which

\[
\gamma A_2 (1 - \tau) \bar{\xi} / \tau \leq y_2 - \gamma A_2 \bar{\nu}
\]

componentwise. Consequently, there exists a \( y \in \mathbb{R}^{s_2}_+ \) such that

\[
y + \gamma A_2 (\tau \bar{\nu} + (1 - \tau) \bar{\xi}) / \tau = y_2.
\]

Set \( \lambda := \gamma / \tau \) and \( \nu := \tau \bar{\nu} + (1 - \tau) \bar{\xi} \). Then \( \lambda \geq 0, \nu \in U - \{u_0\}, \ y \in \mathbb{R}^{s_2}_+ \), and relations (10) are satisfied.

Thus, for the mapping \( G \) we have the assumptions of the classical implicit function theorem (see, e.g., [1, 5]). Therefore, there exists a neighborhood \( \Omega \subset \mathbb{R}^n \times \mathbb{R} \) of the point \( (x_0, t_0) \) and continuous mappings \( \lambda : \Omega \to (0, +\infty), \ \bar{\nu} : \Omega \to \mathbb{R}^m \), and \( \bar{y} : \Omega \to \mathbb{R}^{s_2}_+ \) such that

\[
G^1(x, u_0 + \bar{\nu}(x, t)/\bar{\lambda}(x, t), t) = 0,
G^2(x, u_0 + \bar{\nu}(x, t)/\bar{\lambda}(x, t), t) = -\bar{y}(x, t),
\]

where \((\bar{\lambda}(x, t), \bar{\nu}(x, t), \bar{y}(x, t)) \in K, (x, t) \in \Omega \). Set \( \bar{u}(x, t) := u_0 + \bar{\nu}(x, t)/\bar{\lambda}(x, t), (x, t) \in \Omega \). Then

\[
G^1(x, \bar{u}(x, t), t) = 0, \quad G^2(x, \bar{u}(x, t), t) \leq 0, \quad \bar{u}(x, t) \in U \quad \text{for all} \quad (x, t) \in \Omega,
\]
and the function \( \bar{u} \) is continuous. Consequently, the function \((x, t) \mapsto F(x, \bar{u}(x, t), t), (x, t) \in \Omega, \) is continuous in \( x \), measurable in \( t \), and bounded in some neighborhood of the point \((x_0, t_0)\). Hence (see, e.g., [2, p. 214 of the Russian translation]) there exists a solution \( \bar{x}(\cdot) \) of the Cauchy problem (4) defined on \([t_0, t_0 + \tau)\) for some \( \tau > 0 \).

In the case of \( G^2(x_0, u_0, t_0) \neq 0 \), the proof is carried out in a similar way with the replacement of \( G^2 \) by the mapping obtained by crossing out the \( i \)th components \( G^{i,2}, i \notin I, \) of the vector function \( G^2 \). The proof of the theorem is complete.

**Remark 1.** If there are no equality-type mixed constraints in problem (1)–(3), then assumptions (R1) and (R2) take the form

(R2') There exists a vector \( \xi \in U - \{u_0\} \) such that

\[
(G^2_u(x_0, u_0, t_0))_i < 0 \quad \text{for all } i \in I.
\]

Note also that a sufficient (but not necessary) condition for (R2') is the condition of linear independence of the vectors \( (G^2_u(x_0, u_0, t_0))_i, i \in I. \)

Consider the controlled system (1), (2) with the control

\[ u(t) \in U. \]

(11)

This system is said to be **locally solvable** at a point \((t_0, x_0)\) if there exists a \( \tau > 0 \), an absolutely continuous function \( \bar{x} : [t_0, t_0 + \tau) \to \mathbb{R}^n \), and a measurable essentially bounded function \( \bar{u} : [t_0, t_0 + \tau) \to \mathbb{R}^m \) such that

\[
G^1(\bar{x}(t), \bar{u}(t), t) = 0,
\]

\[
G^2(\bar{x}(t), \bar{u}(t), t) \leq 0
\]

for a.a. \( t \) and the function \( x = \bar{x}(\cdot) \) is a solution of the Cauchy problem

\[
\dot{x} = F(x, \bar{u}(t), t), \quad x(t_0) = x_0.
\]

The function \( \bar{u}(\cdot) \) is conventionally called a **feasible program control**.

The local solvability of system (1)–(3) at a point \((x_0, u_0, t_0)\) implies that there exists a feasible program control \( \bar{u}(\cdot) \) of system (1), (2), (11). Indeed, if \( \bar{u}(\cdot) \) is a feasible positional control and \( \bar{x}(\cdot) \) is the corresponding trajectory, then \( \bar{u}(t) \equiv \bar{u}(t, \bar{x}(t)) \) is a feasible program control. The converse statement, generally speaking, is not true. Let us illustrate the above with an example.

**Example 1.** Define a mapping \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) by the formula

\[
\Psi(u) := \frac{1}{|u|} \left( \frac{u_1^2 - u_2^2}{2u_1u_2} \right), \quad u = (u_1, u_2)^T \in \mathbb{R}^2, \quad u \neq 0, \quad \Psi(0) = 0.
\]

Set \( x_0 := 0, \ t_0 := 0, \) and \( u_0 := 0. \) Consider the controlled system

\[
\dot{x} = u, \quad x(0) = 0, \quad \Psi(u) = x;
\]

i.e., \( n = m = 2, \ G^1(x, u, t) = \Psi(u) - x, \) and \( G(x, u, t) \equiv 0. \)

This system has the feasible program control \( \bar{u}(t) \equiv 0. \) However, there is no feasible positional control \( \bar{u} \) for the system under consideration. Indeed, if some function \( \bar{u} \) is a feasible positional control, then it is continuous, \( \bar{u}(0, 0) = 0, \) and \( \Psi(\bar{u}(x, t)) \equiv x; \) i.e., \( \bar{u}(\cdot, 0) \) is a continuous right inverse of \( \Psi \) in a neighborhood of zero. The latter contradicts Example 2 in [3], where it was shown that there exists no continuous right inverse mapping of \( \Psi \) in a neighborhood of zero.

In conclusion, we note that the controlled system in Example 1 satisfies the assumptions of the theorem on the existence of feasible program controls [4, Theorem 2]. The assumptions in [4, Theorem 2] are weaker than those in Theorem 1. Example 1 shows that under the assumptions in [4, Theorem 2] system (1)–(3) may fail to have a feasible program control in a neighborhood of a given point.
2. SECOND-ORDER REGULARITY

Let us study the solvability of the controlled system (1)-(3) for the case in which conditions (R1) and (R2) are violated.

Set

\[ V := \ker G^1_u(x_0, u_0, t_0) \cap \{ v \in \mathbb{R}^m : G^2_u(x_0, u_0, t_0)v \leq 0 \}. \]

By \( G^l_{uu}, l = 1, 2 \), we denote the matrix of second partial derivatives of the vector function \( G \) with respect to the components of the vector \( u \).

**Theorem 2.** Let the mappings \( F \) and \( G^l, l = 1, 2 \), be continuous, and let the mappings \( G^l, l = 1, 2 \), be twice continuously differentiable with respect to \( u \) in a neighborhood of the point \((x_0, u_0, t_0)\). Let the following assumptions be satisfied:

(R3) There exists a vector \( h \in V \cap (U - \{u_0\}) \) such that

\[ -G^1_{uu}(x_0, u_0, t_0)[h, h] \in G^1_u(x_0, u_0, t_0) \text{ cone } (U - \{u_0\}), \]

\[ G^1_u(x_0, u_0, t_0) \text{ cone } (U - \{u_0\}) + G^1_{uu}(x_0, u_0, t_0)[h, \text{ cone } (U - \{u_0\}) \cap V ] = \mathbb{R}^s_+. \]

(R4) There exists a vector \( \xi \in V \cap (U - \{u_0\}) \) such that

\[ (G^2_u(x_0, u_0, t_0)\xi)_i < 0 \text{ for all } i \in I \text{ such that } (G^2_u(x_0, u_0, t_0)h)_i = 0. \]

Then the system is locally solvable at the point \((x_0, u_0, t_0)\).

**Proof.** We assume that \( G^2(x_0, u_0, t_0) = 0 \). If \( G^2(x_0, u_0, t_0) \neq 0 \), then the proof can be carried out in a similar way but with the replacement of \( G^2 \) by the function \( G(x, t, v) \) by equality (5). Let \( A_1, l = 1, 2 \), be the same matrices as in the proof of Theorem 1, and let \( Q_i := (A_{1i} h, h) + 2(A_{2i} h)_i \) for arbitrary \( l = 1, 2 \). Define the set \( K \) by formula (6). In the proof of Theorem 1, it is shown that \( K \) is a convex closed cone.

Let \( h \) be the vector in assumption (R3). Since \( -Q_1[h, h] \in A_1 \text{ cone } (U - \{u_0\}) \), it follows that there exists a number \( \gamma > 0 \) and a vector \( \bar{v} \in U - \{u_0\} \) such that

\[ -Q_1[h, h] = \gamma A_1 \bar{v}. \]

By assumption (R4), there exists a vector \( \xi \in U - \{u_0\} \) such that

\[ A_1 \xi = 0, \ A_2 \xi \leq 0, \text{ and } (A_2 \xi)_i < 0 \text{ for all } i \in I \text{ such that } (A_2 h)_i = 0. \]

Consequently, there exist numbers \( \theta > 0 \) and \( \tau \in (0, 1) \) and a vector \( y \in \mathbb{R}^s_{+} \) such that

\[ Q_2[h, h] - \gamma A_2 \bar{v} = y + 2\theta A_2 h + \frac{1 - \tau}{\tau} A_2 \xi. \]

As above, set \( v_0 := (1, 0, 0)^T \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^s \) and define sets \( K \) and \( C \) by equalities (7). Then we have the representations (8) and (9) and

\[ C = \left\{ \left( \begin{array}{c} \lambda A_1 \nu \\ y + \lambda A_2 \nu \end{array} \right) : \lambda \geq 0, \ \nu \in U - \{u_0\}, \ y \in \mathbb{R}^s_{+} \right\}. \]

**II.** Let us show that the mapping \( G \) is 2-regular at the point \((x_0, t_0, v_0)\) with respect to the cone \( K \) in the direction of the vector \( h_0 := (\theta, h, -A_2 h)^T \), i.e., that

\[ G(x_0, t_0, v_0)h_0 = 0, \]

\[ -G_{vu}(x_0, t_0, v_0)[h_0, h_0] \in C, \]

\[ G(x_0, t_0, v_0)K + G_{vu}(x_0, t_0, v_0)[h_0, K \cap \ker G(x_0, t_0, v_0)] = \mathbb{R}^s_+ \times \mathbb{R}^s. \]
The inclusion $h_0 \in \mathcal{K}$ follows from (9) and the definitions of the set $\mathcal{K}$ and the vector $h_0$. Equality (8) and assumption (R3) imply that

$$G_v(x_0, t_0, v_0)h_0 = \begin{pmatrix} 0 & A_1 & 0 \\ 0 & A_2 & E \end{pmatrix} \begin{pmatrix} \theta \\ h \\ -A_2 h \end{pmatrix} = \begin{pmatrix} A_1 h \\ A_2 h - A_2 h \end{pmatrix} = 0.$$  

Let us prove the inclusion (17). We have

$$G_{vv}(x_0, t_0, v_0)[\delta, \delta] = \begin{pmatrix} -2\delta_x A_1 \delta_v + Q_1[\delta_v, \delta_v] \\ -2\delta_x A_2 \delta_v + Q_2[\delta_v, \delta_v] \end{pmatrix} \quad \text{for each} \quad \delta = (\delta_x, \delta_u, \delta_v),$$  

$$G_{vv}(x_0, t_0, v_0)[h_0, h_0] = \begin{pmatrix} Q_1[h, h] \\ -2\theta A_2 h + Q_2[h, h] \end{pmatrix}. \quad \text{(19)}$$

Therefore,

$$-G_{vv}(x_0, t_0, v_0)[h_0, h_0] = -\begin{pmatrix} Q_1[h, h] \\ -2\theta A_2 h + Q_2[h, h] \end{pmatrix} = \begin{pmatrix} \gamma A_1(\tau \bar{\nu} + (1 - \tau)\xi)/\tau \\ y + \gamma A_2(\tau \bar{\nu} + (1 - \tau)\xi)/\tau \end{pmatrix} \in C.$$  

Here the second equality follows from the relations (14), $A_1 \xi = 0$, and (16). The inclusion (17) has been proved.

Let us prove relation (18). To this end, we take an arbitrary pair $(y_1, y_2) \in \mathbb{R}^{s_1} \times \mathbb{R}^{s_2}$ and show that the equation

$$G_v(x_0, t_0, v_0)\zeta + G_{vv}(x_0, t_0, v_0)[h_0, \zeta'] = (y_1, y_2)^T \quad \text{(20)}$$

has a solution $(\zeta, \zeta')$ such that $\zeta \in \mathcal{K}$ and $\zeta' \in \mathcal{K} \cap \ker G_v(x_0, t_0, v_0)$.

According to assumption (R3), there exist $\gamma > 0$, $\nu \in U - \{u_0\}$, $\lambda' > 0$, and $\nu' \in U - \{u_0\}$ such that

$$A_1 \nu' = 0, \quad A_2 \nu' \leq 0, \quad \gamma A_1 \nu + Q_1[h, \lambda' \nu'] = y_1. \quad \text{(21)}$$

By virtue of (15), there exist $\tau > 0$, $\mu' < 0$, and $y \in \mathbb{R}^{s_2}$ such that

$$\gamma \frac{1 - \tau}{\tau} A_2 \xi - \mu' A_2 h + y = y_2 - \gamma A_2 \nu + \theta \lambda' A_2 \nu' - Q_2[h, \lambda' \nu']. \quad \text{(22)}$$

Set

$$\zeta := (0, \gamma(\tau \nu + (1 - \tau)\xi)/\tau, y)^T,$$

$$\zeta' := (\mu', \lambda' \nu', -\lambda' A_2 \nu')^T. \quad \text{(23)}$$

Let us show that $\zeta \in \mathcal{K}$, $\zeta' \in \mathcal{K} \cap \ker G_v(x_0, t_0, v_0)$, and the pair $(\zeta, \zeta')$ is a solution of Eq. (20). The inclusion $\zeta \in \mathcal{K}$ follows from the convexity of the set $U - \{u_0\}$, the relations $\gamma > 0$, $\tau > 0$, and $y \in \mathbb{R}^{s_2}_+$, and the definition of the set $\mathcal{K}$. The inclusion $\zeta' \in \mathcal{K}'$ follows from the relations $\lambda' > 0$, $\nu' \in U - \{u_0\}$, and $A_2 \nu' \leq 0$. Moreover,

$$G_v(x_0, t_0, v_0)\zeta' = \begin{pmatrix} \lambda' A_1 \nu' \\ \lambda' A_2 \nu' - \lambda' A_2 \nu' \end{pmatrix} = 0,$$

because $A_1 \nu' = 0$. Consequently, $\zeta' \in \mathcal{K} \cap \ker G_v(x_0, t_0, v_0)$. Finally,

$$G_v(x_0, t_0, v_0)\zeta + G_{vv}(x_0, t_0, v_0)[h_0, \zeta']$$

$$= \begin{pmatrix} \gamma A_1(\tau \nu + (1 - \tau)\xi)/\tau \\ \gamma A_2(\tau \nu + (1 - \tau)\xi)/\tau + y \end{pmatrix} + \begin{pmatrix} -\theta \lambda' A_1 \nu' - \mu' A_1 h + \lambda' Q_1[h, \nu'] \\ -\theta \lambda' A_2 \nu' - \mu' A_2 h + \lambda' Q_2[h, \nu'] \end{pmatrix}$$

$$= \begin{pmatrix} (\gamma A_1 + \theta \lambda' A_1)\nu' - \mu' A_1 h + \lambda' Q_1[h, \nu'] \\ (\gamma A_2 + \theta \lambda' A_2)\nu' - \mu' A_2 h + \lambda' Q_2[h, \nu'] \end{pmatrix} \in C.$$
Here the first equality follows from relations (9), (19), and (23); the second, from the fact that $A_1\xi = 0$, $A_1\nu' = 0$, and $A_1h = 0$; and the third, from the equalities in (21) and (22). Consequently, the pair $(\zeta, \zeta')$ is a solution of Eq. (20). Relation (18) has been proved.

**III.** The mapping $G$ is 2-regular at the point $(x_0, t_0, v_0)$ with respect to the cone $K$ in the direction $h_0$. Consequently, the assumptions of the implicit function theorem are satisfied for $G$ in a neighborhood of the abnormal point [1, Theorem 3]. Therefore, there exists a neighborhood $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ of the point $(x_0, t_0)$ and continuous mappings $\bar{\lambda} : \Omega \to (0, +\infty)$, $\nu : \Omega \to \mathbb{R}^m$, and $\bar{y} : \Omega \to \mathbb{R}^+_2$ such that

$$G^1(x, u_0 + \bar{v}(x, t)/\bar{\lambda}(x, t), t) = 0,$$

$$G^2(x, u_0 + \bar{v}(x, t)/\bar{\lambda}(x, t), t) = -\bar{y}(x, t),$$

where $(\bar{\lambda}(x, t), \bar{\nu}(x, t), \bar{y}(x, t)) \in K$, $(x, t) \in \Omega$. Set $\bar{u}(x, t) := u_0 + \bar{v}(x, t)/\bar{\lambda}(x, t)$, $(x, t) \in \Omega$. Then

$$G^1(x, \bar{u}(x, t), t) = 0, \quad G^2(x, \bar{u}(x, t), t) \leq 0, \quad \bar{u}(x, t) \in U \quad \text{for all} \quad (x, t) \in \Omega,$$

and the function $\bar{u}$ is continuous. Consequently, the function $(x, t) \mapsto F(x, \bar{u}(x, t), t)$, $(x, t) \in \Omega$, is continuous. Hence (see, e.g., [2, p. 214 of the Russian translation]) there exists a solution $\bar{x}(\cdot)$ of the Cauchy problem (4) defined on $[t_0, t_0 + \tau)$ for some $\tau > 0$. The proof of the theorem is complete.

**Remark 2.** If there are no inequality-type mixed constraints in problem (1)–(3), then assumption (R3) is sufficient for the local solvability of system (1)–(3) at the point $(x_0, u_0, t_0)$.

**Remark 3.** Theorem 2 gives sufficient conditions for the local solvability of the controlled system also in the case where the regularity conditions (R1) and (R2) are violated. If, however, conditions (R1) and (R2) are satisfied, then conditions (R3) and (R4) are satisfied as well. Let us show this.

Let assumptions (R1) and (R2) be satisfied. Set $h := 0$. Then

$$-G^1_{uu}(x_0, u_0, t_0)[h, h] = 0, \quad G^1_{uu}(x_0, u_0, t_0)[h, v] = 0 \quad \text{for each} \quad v \in \mathbb{R}^m.$$

In addition, it follows from assumption (R1) that

$$G^1_u(x_0, u_0, t_0) \text{ cone } (U - \{u_0\}) = \mathbb{R}^s_1.$$

Therefore,

$$-G^1_{uu}(x_0, u_0, t_0)[h, h] = 0 \in G^1_u(x_0, u_0, t_0) \text{ cone } (U - \{u_0\});$$

$$G^1_u(x_0, u_0, t_0) \text{ cone } (U - \{u_0\}) + G^1_{uu}(x_0, u_0, t_0) \left[ h, \text{ cone } (U - \{u_0\}) \cap \mathcal{V} \right]$$

$$= G^1_u(x_0, u_0, t_0) \text{ cone } (U - \{u_0\}) + \{0\} = \mathbb{R}^s_1.$$

Thus, assumption (R3) holds true.

By virtue of assumption (R2), there exists a vector $\xi \in U - \{u_0\}$ such that $G^1_u(x_0, u_0, t_0)\xi = 0$, and we have

$$(G^2_u(x_0, u_0, t_0)\xi)_i < 0 \quad \text{for all} \quad i \in I.$$

Hence $\xi \in \mathcal{V} \cap (U - \{u_0\})$, and consequently, assumption (R4) is true as well. Let us give an example of a controlled system to which Theorem 2 applies and Theorem 1 does not.

**Example 2.** Set $G^1(x, u, t) := u_1^2 - u_2^2 - t|x|$, $F(x, u, t) = u$, $x \in \mathbb{R}^2$, $u = (u_1, u_2)^T \in \mathbb{R}^2$, $t \in \mathbb{R}$, $U := \mathbb{R}^2$, $x_0 = 0$, $u_0 = 0$, and $t_0 = 0$. Then system (1)–(3) acquires the form

$$\dot{x} = u, \quad x(0) = 0, \quad u_1^2 - u_2^2 - t|x| = 0.$$
For this system, assumption (R1) is violated, because

\[ G^1_u(x_0, u_0, t_0) = 0. \]

Let us show that assumption (R3) is satisfied.

Set \( h := (1,1)^T \). We have \( V = \mathbb{R}^2 \), \( \text{cone}(U - \{u_0\}) = \mathbb{R}^2 \), and

\[ G^1_{uu}(x_0, u_0, t_0)[u, v] = 2u_1v_1 - 2u_2v_2 \quad \text{for all} \quad u = (u_1, u_2)^T \in \mathbb{R}^2 \quad \text{and} \quad v = (v_1, v_2)^T \in \mathbb{R}^2. \]

Consequently,

\[ -G^1_{uu}(x_0, u_0, t_0)[h, h] = 0 \in \{0\} = G^1_u(x_0, u_0, t_0) \text{cone}(U - \{u_0\}), \]

\[ G^1_u(x_0, u_0, t_0) \text{cone}(U - \{u_0\}) + G^1_{uu}(x_0, u_0, t_0)\left[h, \text{cone}(U - \{u_0\}) \cap V\right] = G^1_{uu}(x_0, u_0, t_0)[h, \mathbb{R}^2] = \{2v_1 - 2v_2 : (v_1, v_2)^T \in \mathbb{R}^2\} = \mathbb{R}. \]

Hence assumption (R3) is satisfied in the example under consideration. Therefore, the system is solvable at the point \((x_0, u_0, t_0)\) by Theorem 2 (see Remark 2).

In conclusion, let us give a simple example illustrating the fact that the regularity conditions for (R3) and (R4) are essential. Consider the controlled system

\[ \dot{x} = u, \quad x(0) = 0, \quad u^2 + t^2 \leq 0. \]

It is obvious that this system is not solvable at the point \((0,0,0)\) and assumptions (R3) and (R4) are violated for this system.

**FUNDING**

The results in Sec. 1 were obtained with the financial support of the Russian Science Foundation, project no. 22-11-00042; the results in Sec. 2 were obtained with the financial support of a grant from the President of the Russian Federation, project no. MD-2658.2021.1.1.

**OPEN ACCESS**

This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this license, visit [http://creativecommons.org/licenses/by/4.0/](http://creativecommons.org/licenses/by/4.0/).

**REFERENCES**

1. Arutyunov, A.V., Implicit function theorem without a priori assumptions about normality, *Comput. Math. Math. Phys.*, 2006, vol. 46, no. 2, pp. 195–205.
2. Warga, J., *Optimal Control of Differential and Functional Equations*, New York–London: Academic Press, 1972. Translated under the title: *Optimal’noe upravlenie differentsial’nymi i funktsional’nymi uravneniyami*, Moscow: Nauka, 1977.
3. Arutyunov, A.V. and Zhukovskiy, S.E., Existence of backward maps and their properties, *Tr. MIAN*, 2010, vol. 271, pp. 18–28.
4. Arutyunov, A.V. and Zhukovskii, S.E., Local solvability of control systems with mixed constraints, *Differ. Equations*, 2010, vol. 46, no. 11, pp. 1561–1570.
5. Alekseev, V.M., Tikhomirov, V.M., and Fomin, S.V., *Optimal’noe upravlenie* (Optimal Control), Moscow: Nauka, 1979.