The Congruence Subgroup Problem for finitely generated Nilpotent Groups

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Abstract

The congruence subgroup problem for a finitely generated group $\Gamma$ and $G \leq \text{Aut}(\Gamma)$ asks whether the map $\hat{G} \rightarrow \text{Aut}(\hat{\Gamma})$ is injective, or more generally, what is its kernel $C(G, \Gamma)$? Here $\hat{X}$ denotes the profinite completion of $X$. In the case $G = \text{Aut}(\Gamma)$ we denote $C(\Gamma) = C(\text{Aut}(\Gamma), \Gamma)$.

Let $\Gamma$ be a finitely generated group, $\bar{\Gamma} = \Gamma/\{\Gamma, \Gamma\}$, and $\Gamma^* = \bar{\Gamma}/\text{tor}(\bar{\Gamma}) \cong \mathbb{Z}^{(d)}$. Denote

$$\text{Aut}^*(\Gamma) = \text{Im}(\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma^*)) \leq GL_d(\mathbb{Z}).$$

In this paper we show that when $\Gamma$ is nilpotent, there is a canonical isomorphism $C(\Gamma) \cong C(\text{Aut}^*(\Gamma), \Gamma^*)$. In other words, $C(\Gamma)$ is completely determined by the solution to the classical congruence subgroup problem for the arithmetic group $\text{Aut}^*(\Gamma)$.

In particular, in the case where $\Gamma = \Psi_{n,c}$ is a finitely generated free nilpotent group of class $c$ on $n$ elements, we get that $C(\Psi_{n,c}) = C(\mathbb{Z}^{(n)}) = \{e\}$ whenever $n \geq 3$, and $C(\Psi_{2,c}) = C(\mathbb{Z}^{(2)}) = \hat{F}_c = \text{the free profinite group on countable number of generators}.$

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1 Introduction

Let $G \leq GL_n(\mathbb{Z})$. The classical congruence subgroup problem (CSP) asks whether every finite index subgroup of $G$ contains a principal congruence subgroup, i.e. a subgroup of the form $G(m) = \ker(G \rightarrow GL_n(\mathbb{Z}/m\mathbb{Z}))$ for some $0 \neq m \in \mathbb{Z}$. Equivalently, it asks whether the natural map $\hat{G} \rightarrow GL_n(\hat{\mathbb{Z}})$ is injective, where $\hat{G}$ and $\hat{\mathbb{Z}}$ are the profinite completions of the group $G$ and the ring $\mathbb{Z}$, respectively. More generally, the CSP asks what is the kernel of this map. It is a classical 19th century result that for $G = GL_n(\mathbb{Z})$ the answer
is negative when \( n = 2 \). Moreover (but not so classical, cf. [Mel], [L1]), the kernel in this case is \( \hat{F}_\omega \) - the free profinite group on a countable number of generators. On the other hand, it was proved in the sixties by Mennicke [Men] and Bass-Lazard-Serre [BLS] that for \( n \geq 3 \) the map is injective, and the kernel is therefore trivial. This breakthrough led to a rich theory which studied the CSP for many other arithmetic groups. It has been solved for many arithmetic groups, but not yet for all. See [Rag] and [Rap] for surveys.

By the observation \( GL_n(\mathbb{Z}) \cong Aut(\mathbb{Z}^n) \), the CSP can be generalized as follows: Let \( \Gamma \) be a group and \( G \leq Aut(\Gamma) \). For a finite index characteristic subgroup \( M \leq \Gamma \) denote

\[
G(M) = \ker(G \to Aut(\Gamma/M)).
\]

Such a \( G(M) \) is called a “principal congruence subgroup” and a finite index subgroup of \( G \) which contains \( G(M) \) for some \( M \) is called a “congruence subgroup”.

One can see that the CSP is equivalent to the question: Is the congruence map \( \hat{G} = \lim\leftarrow G/U \to \lim\leftarrow G(G(M)) \) injective? Here, \( U \) ranges over all finite index normal subgroups of \( G \), and \( M \) ranges over all finite index characteristic subgroups of \( \Gamma \). When \( \Gamma \) is finitely generated, it has only finitely many subgroups of a given index \( m \), and thus, the characteristic subgroups \( M_m = \cap \{ \Delta \leq \Gamma \mid [\Gamma : \Delta] = m \} \) are of finite index in \( \Gamma \). Hence, one can write \( \Gamma = \lim\leftarrow m \in \mathbb{N} \Gamma/M_m \) and have

\[
\lim\leftarrow G(G(M)) = \lim\leftarrow m \in \mathbb{N} G(G(M_m)) \leq \lim\leftarrow m \in \mathbb{N} Aut(\Gamma/M_m) = Aut(\hat{\Gamma}).
\]

Therefore, when \( \Gamma \) is finitely generated, the CSP is equivalent to the question: Is the congruence map \( \hat{G} \to Aut(\hat{\Gamma}) \) injective? More generally, the CSP asks what is the kernel \( C(G, \Gamma) \) of this map. For \( G = Aut(\Gamma) \) we will also use the simpler notation \( C(\Gamma) = C(Aut(\Gamma), \Gamma) \).

The classical CSP results mentioned above can therefore be reformulated as \( C(\mathbb{Z}^{(2)}) = \hat{F}_\omega \) while \( C(\mathbb{Z}^{(n)}) = \{ e \} \) for \( n \geq 3 \). Recently, it was proved that when \( \Gamma = \Phi_n \) is the free metabelian group on \( n \) generators, we have: \( C(\Phi_2) = \hat{F}_\omega \), \( C(\Phi_3) \supseteq \hat{F}_\omega \), and for every \( n \geq 4 \), \( C(\Phi_n) \) is abelian (see [Be1, Be2, Be3, BL]). I.e. while in the free abelian case there is a dichotomy between \( n = 2 \) and \( n \geq 3 \), in the free metabelian case we have dichotomy between \( n = 2, 3 \) and \( n \geq 4 \).

The goal of this paper is to show that contrary to the above metabelian cases, when \( \Gamma \) is a finitely generated nilpotent group, the CSP for \( \Gamma \) is completely determined by the CSP for abelian groups. Let us put things more precise: Let

\[1\] When we write \( Aut(\hat{\Gamma}) \) we basically mean to consider the group of continuous automorphisms of \( \hat{\Gamma} \). However, by the celebrated theorem of Nikolov and Segal which asserts that every finite index subgroup of a finitely generated profinite group is open [NS], whenever \( \Gamma \) is finitely generated, the group of continuous automorphisms of \( \hat{\Gamma} \) is equal to the group of automorphisms of \( \hat{\Gamma} \).
Γ be a finitely generated group, \( \bar{\Gamma} = \Gamma/\Gamma, \Gamma \) and \( \Gamma^* = \bar{\Gamma}/\text{tor} (\bar{\Gamma}) \), so \( \Gamma^* \cong \mathbb{Z}^{(d)} \) for some \( d \). Denote

\[
\text{Aut}^* (\Gamma) = \text{Im} (\text{Aut} (\Gamma) \to \text{Aut}(\Gamma^*)) \leq GL_d (\mathbb{Z}).
\]

When \( \Gamma \) is nilpotent, the group \( \text{Aut}^* (\Gamma) \) is known to be an arithmetic subgroup of \( GL_d (\mathbb{Z}) \), and every arithmetic subgroup of \( GL_d (\mathbb{Z}) \) is obtained like that for some nilpotent group \( \Gamma \) (\([\text{Ba}] [\text{BG}] [\text{BP}]\)).

The canonical map \( \text{Aut}(\Gamma) \to \text{Aut}^* (\Gamma) \) induces a map

\[
C(\Gamma) \to C(\text{Aut}^* (\Gamma), \Gamma^*).
\]

Here is the main theorem of the paper:

**Theorem 1.1.** Let \( \Gamma \) be a finitely generated nilpotent group. Then, the canonical map \( C(\Gamma) \to C(\text{Aut}^* (\Gamma), \Gamma^*) \) is an isomorphism.

So, the CSP for nilpotent groups is completely reduced to the classic CSP. In particular, in the free cases, we have:

**Corollary 1.2.** Let \( \Gamma = \Psi_{n,c} \) be the free nilpotent group of class \( c \) on \( n \) elements. Then

\[
C(\Gamma) \cong C(\text{Aut}^* (\Gamma), \Gamma^*) = C(GL_n (\mathbb{Z}), \mathbb{Z}(n)) = C(\mathbb{Z}(n)).
\]

In particular:

- For \( n = 2 \) one has \( C(\Psi_{2,c}) \cong C(\mathbb{Z}(2)) \cong \hat{F}_\omega \).
- For \( n \geq 3 \) one has \( C(\Psi_{n,c}) \cong C(\mathbb{Z}(n)) \cong \{e\} \).

**Remark 1.3.** As mentioned above, every arithmetic subgroup \( D \) of \( GL_d (\mathbb{Z}) \) can appear as \( \text{Aut}^* (\Gamma) \) for a suitable nilpotent \( \Gamma \). The possible congruence kernels for such arithmetic groups are not fully known as the classical CSP is not yet fully solved. But these include, besides the trivial groups and \( \hat{F}_\omega \) mentioned above, also finite cyclic groups (when \( D \) is the restriction of scalars from suitable number fields) as well as infinite abelian groups of finite exponent (if \( D \) is an arithmetic group of a non simply connected group).

Here is the main line of the proof. For a finitely generated group \( \Gamma \) consider the commutative exact diagram

\[
1 \to IA^* (\Gamma) \to \text{Aut} (\Gamma) \to \text{Aut}^* (\Gamma) \to 1
\]

\[
1 \to IA^* (\hat{\Gamma}) \to \text{Aut}(\hat{\Gamma}) \to \text{Aut}^* (\hat{\Gamma}) \to 1
\]

when we define \( IA^* (\Gamma) = \ker (\text{Aut}(\Gamma) \to \text{Aut}(\Gamma^*)) \) and \( \text{Aut}^* (\hat{\Gamma}), IA^* (\hat{\Gamma}) \) defined to be the image and the kernel of the natural map \( \text{Aut}(\hat{\Gamma}) \to \text{Aut}(\hat{\Gamma}^*) = GL_d (\hat{\mathbb{Z}}) \), respectively. This diagram gives rise to the commutative exact diagram (see Lemma 2.1 in \([\text{BER}]\))

\[
1 \to IA^* (\hat{\Gamma}) \to \text{Aut}(\hat{\Gamma}) \to \text{Aut}^* (\hat{\Gamma}) \to 1
\]

(1.1)
Notice that \( C(Aut^*(\Gamma), \Gamma^*) = \ker(Aut^*(\Gamma) \to Aut^*(\hat{\Gamma})) \). We prove the following theorem:

**Theorem 1.4.** Let \( \Gamma \) be a finitely generated nilpotent group. Then:

1. For any \( G \leq IA^*(\Gamma) \), the natural map \( G \to IA^*(\hat{\Gamma}) \) is an embedding. In other words
\[
C(G, \Gamma) = \{e\}
\]
so we have an affirmative solution to the CSP for any \( G \leq IA^*(\Gamma) \).

2. The group \( IA^*(\Gamma) \) is dense in \( IA^*(\hat{\Gamma}) \).

Notice that from the first part of Theorem 1.4 we obtain that in particular \( C(IA^*(\Gamma), \Gamma) = \{e\} \) for any finitely generated nilpotent group \( \Gamma \). This is not true in general (cf. [Be1, Be2, Be3, BL] for free metabelian groups). In some sense, the second part of Theorem 1.4 means that the map \( IA^*(\Gamma) \to IA^*(\hat{\Gamma}) \) satisfies a "strong approximation" property. This is not true in general either (compare [L2] for free groups). From the two parts of Theorem 1.4 we obtain the following corollary, which also implies Theorem 1.1:

**Corollary 1.5.** Let \( \Gamma \) be a finitely generated nilpotent group. Then
\[
\hat{IA}^*(\Gamma) \cong IA^*(\hat{\Gamma}).
\]

Corollary 1.5, together with chasing diagram (1.1), imply Theorem 1.4. Corollary 1.5 is a form of combination of congruence subgroup property as well as strong approximation for the group \( IA^*(\Gamma) \). Indeed, its proof boils down to these results for a suitable \( \mathbb{Q} \)-unipotent group. But the reduction is slightly delicate: in [3] it is shown that the proof of Corollary 1.5 (or Theorem 1.4) can be reduced to the case when \( \Gamma \) is torsion free. In [2] we treat the torsion free case, by reducing it first from \( \Gamma \) to \( \Delta \), when \( \Delta \) is the "lattice hull" of \( \Gamma \). This \( \Delta \) contains \( \Gamma \) as a finite index subgroup and it is contained in its Mal’cev completion \( R \). It enjoys the property that \( \log(\Delta) \) is a \( \mathbb{Z} \)-lattice of the Lie algebra \( L \) of \( R \). This fact enables us to give \( IA^*(\Delta) \) the structure of the \( \mathbb{Z} \)-points of a suitable unipotent group for which the \( \mathbb{Z} \)-points are exactly \( IA^*(\hat{\Delta}) \). Hence, the classical CSP and strong approximation for this unipotent group imply Corollary 1.5.

In [4] we sketch another proof to Corollary 1.5 which is more direct, in the case where \( \Gamma = \Psi_{n,c} \) is a finitely generated free nilpotent group.

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2 The case of Torsion Free Nilpotent Groups

In this section we are going to prove Theorem 1.4 in the case where $\Gamma$ is torsion free. So let $\Gamma$ be a finitely generated torsion free nilpotent group. For our convenience, we will follow the approach presented in [S], and consider $\Gamma$ as a subgroup of $Tr_1(n, \mathbb{Z})$, the group of $n \times n$ upper triangular matrices over $\mathbb{Z}$ with 1-s on the diagonal, for some $n$ (see Chapter 5 therein). Recall the one-to-one correspondence given by the maps

$$\log : Tr_1(n, \mathbb{Q}) \to Tr_0(n, \mathbb{Q})$$
$$\exp : Tr_0(n, \mathbb{Q}) \to Tr_1(n, \mathbb{Q})$$

where $Tr_0(n, \mathbb{Q})$ is the Lie algebra of $n \times n$ upper triangular matrices with 0-s on the diagonal. Let $L$ be the Lie subalgebras of $Tr_0(n, \mathbb{Q})$ spanned by $\log(\Gamma)$.

The following is well known and can be found in [S] as well (Chapter 6):

**Theorem 2.1.** There exists a unique (up to isomorphism) group $R$, called the radicable hull of $\Gamma$, or Mal’cev completion of $\Gamma$ with the following properties:

- $\Gamma$ is a subgroup of $R$.
- For every $a \in R$ and $m \in \mathbb{N}$ there exists $b \in R$ such that $b^m = a$.
- For every $a \in R$ there exists $m \in \mathbb{N}$ such that $a^m \in \Gamma$.
- The group $R$ can be identified with $\exp(L) \leq Tr_1(n, \mathbb{Q})$.

The connection between the group operation of $R$ and the Lie algebra operation of $L$ is given through the Baker-Campbell-Hausdorff (BCH) formula. One can use it in order to prove the following lemma ([BG], Lemma 2.1):

**Lemma 2.2.** Under the correspondence between the underlying sets of $R$ and $L$ one has $R' = L'$. This equality gives a natural group isomorphism between $R/R'$ and the additive group $L/L'$.

One can use the BCH formula in order to prove that $L'$ is the $\mathbb{Q}$-span of $\log(\Gamma')$, and hence $R'$ can be identified with the radicable hull of $\Gamma'$. Denote $\delta(\Gamma) = \ker(\Gamma \to \Gamma^* \simeq \mathbb{Z}^d)$. Then, as any element of $\delta(\Gamma)$ has some power in $\Gamma'$, we have $\log(\delta(\Gamma)) \subseteq L'$, and hence $L'$ is also the $\mathbb{Q}$-span of $\log(\delta(\Gamma))$, and $R'$ is also the radicable hull of $\delta(\Gamma)$. One gets from this that ([BG], Lemma 2.2):

**Lemma 2.3.** We have $\Gamma \cap R' = \delta(\Gamma)$ and $\dim_{\mathbb{Q}}(R/R') = \text{rank}_{\mathbb{Z}}(\Gamma/\delta(\Gamma)) = d$.

The following can be found in [S], Chapter 6:

**Proposition 2.4.** There exists a unique minimal intermediate subgroup $\Gamma \leq \Delta \leq R$, called the lattice hull of $\Gamma$, that its image in $L$, namely $\log(\Delta)$, is a lattice. I.e. $\log(\Delta)$ is a free $\mathbb{Z}$-module that spans $L$ over $\mathbb{Q}$. One has $[\Delta : \Gamma] < \infty$.

**Remark 2.5.** Notice that $R$ is also the radicable hull of $\Delta$. 

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Given a set \( X \subseteq L \) denote \( N_{\text{Aut}(L)}(X) = \{ g \in \text{Aut}(L) \mid g(X) = X \} \). The following can also be found in [S], Chapter 6:

**Theorem 2.6.** The correspondence between \( R \) and \( L \) induces an isomorphism

\[
\text{Aut}(R) \simeq \text{Aut}(L) \leq GL_k(\mathbb{Q})
\]

where \( k = \dim(L) \). Under this isomorphism, one can identify

\[
\text{Aut}(\Gamma) \cong N_{\text{Aut}(L)}(\log(\Gamma)) \leq \text{Aut}(L)
\]

From Proposition 2.4, it follows that \( \text{Aut}(\Gamma) \leq \text{Aut}(\Delta) \leq \text{Aut}(R) \simeq \text{Aut}(L) \).

I.e. any automorphism of \( \Gamma \) can be uniquely extended to an automorphism of \( \Delta \), and any of the latter can be uniquely extended to an automorphism of \( R \).

Theorem 2.6 and Lemma 2.3 imply:

**Proposition 2.7.** Under the above notation we can identify

\[
\text{IA}^*(\Gamma) = \text{Aut}(\Gamma) \cap \ker(\text{Aut}(R) \to \text{Aut}(R/R')) \\
\cong N_{\text{Aut}(L)}(\log(\Gamma)) \cap \ker(\text{Aut}(L) \to \text{Aut}(L/L'))
\]

\[
\text{IA}^*(\Delta) = \text{Aut}(\Delta) \cap \ker(\text{Aut}(R) \to \text{Aut}(R/R')) \\
\cong N_{\text{Aut}(L)}(\log(\Delta)) \cap \ker(\text{Aut}(L) \to \text{Aut}(L/L')).
\]

An immediate corollary of Proposition 2.4 and Theorem 2.6 is that \( \text{IA}^*(\Gamma) \) is naturally embedded in \( \text{IA}^*(\Delta) \). We would like now to show that the same property is valid also for the profinite completions of \( \Gamma \) and \( \Delta \). Namely:

**Proposition 2.8.** Any automorphism of \( \hat{\Gamma} \) can be uniquely extended to an automorphism of \( \hat{\Delta} \). In particular, \( \text{IA}^*(\hat{\Gamma}) \) is naturally embedded as a finite index subgroup of \( \text{IA}^*(\hat{\Delta}) \).

In order to prove Proposition 2.8, we are going to show that one can describe the relation between \( \text{IA}^*(\hat{\Gamma}) \) and \( \text{IA}^*(\hat{\Delta}) \) in a very similar way to the description of the relation between \( \text{IA}^*(\Gamma) \) and \( \text{IA}^*(\Delta) \) above. Before we do that, let us present an immediate consequence of Proposition 2.8:

**Proposition 2.9.** Let \( \Gamma \) be a finitely generated torsion free nilpotent group, and let \( \Delta \) be the lattice hull of \( \Gamma \). Let \( G \leq \text{IA}^*(\Gamma) \leq \text{IA}^*(\Delta) \). Then:

1. If \( \hat{G} \to \text{IA}^*(\hat{\Gamma}) \) is injective, then \( \hat{G} \to \text{IA}^*(\hat{\Gamma}) \) is injective.

2. If \( \text{IA}^*(\Delta) \) is dense in \( \text{IA}^*(\hat{\Delta}) \), then \( \text{IA}^*(\Gamma) \) is dense in \( \text{IA}^*(\hat{\Gamma}) \).

**Proof.** The first statement is an immediate corollary of Proposition 2.8. The second part also follows immediately from Proposition 2.8 since \( \text{IA}^*(\Delta) \cap \text{IA}^*(\hat{\Gamma}) = \text{IA}^*(\Gamma) \).
Proposition 2.9 shows us that in order to prove Theorem 1.4 for finitely generated torsion free nilpotent group $\Gamma$, it is enough to show it for its lattice hull $\Delta$. We turn now to describe the relation between $Aut(\hat{\Gamma})$ and $Aut(\hat{\Delta})$. The description is going to give more than just a proof to Proposition 2.8, and we are going to use it also toward the rest of the section.

Let $\Gamma_p$ be the pro-$p$ completion of $\Gamma$. As $\Gamma$ is nilpotent, we have $\hat{\Gamma} = \prod_p \Gamma_p$. In addition, as $\Gamma$ is finitely generated and unipotent, it is arithmetic ([S], Chapter 6). Hence, by the affirmative solution to the congruence subgroup problem for arithmetic soluble groups (see [C, P, PS]), we can view $\hat{\Gamma}$ as the closure of $\Gamma$ under the map $\Gamma \hookrightarrow T_{r_1}(n, \mathbb{Z}) \to T_{r_1}(n, \hat{\mathbb{Z}}) = \prod_p T_{r_1}(n, \mathbb{Z}_p)$.

As $T_{r_1}(n, \mathbb{Z}_p)$ is a pro-$p$ group, and $\hat{\Gamma} = \prod_p \Gamma_p$, it follows that we can identify $\Gamma_p$ with the closure of $\Gamma$ under the map $\Gamma \hookrightarrow T_{r_1}(n, \mathbb{Z}) \to \prod_p T_{r_1}(n, \mathbb{Z}_p) \to T_{r_1}(n, \mathbb{Z}_p)$.

Extending $log : T_{r_1}(n, \mathbb{Q}_p) \to T_{r_0}(n, \mathbb{Q}_p)$ and $exp : T_{r_0}(n, \mathbb{Q}_p) \to T_{r_1}(n, \mathbb{Q}_p)$, log and exp are continuous with relation to the topology induced by $\mathbb{Q}_p$. We define $L_p$ to be the $\mathbb{Q}_p$-span of $log(\Gamma_p)$ and $R_p = exp(L_p)$.

**Lemma 2.10.** The set $L_p$ is a $\mathbb{Q}_p$-Lie algebra.

**Proof.** The BCH clearly gives $L_p$ a structure of a $\mathbb{Q}$-Lie algebra, just like it gives $L$. We just need to explain why $L_p$ is closed under multiplication of scalars from $\mathbb{Q}_p$. By definition, it is enough to show that it is closed under multiplication of scalars from $\mathbb{Z}_p$. So let $g \in \Gamma_p$ and let $m = \lim_{i \to \infty} m_i \in \mathbb{Z}_p$ for some $m_i \in \mathbb{Z}$. Let $\rho_g$ be the natural homomorphism $\rho_g : \mathbb{Z}_p \to \Gamma_p$ defined by sending the generator of $\mathbb{Z}_p$ to $g$. Then, as log is continuous we have

$$log(\rho_g(m)) = log(\rho_g(\lim_{i \to \infty} m_i)) = \lim_{i \to \infty} log(\rho_g(m_i)) = \lim_{i \to \infty} log(g^{m_i}) = \lim_{i \to \infty} (m_i \cdot log(g)) = (\lim_{i \to \infty} m_i) \cdot log(g) = m \cdot log(g).$$

It follows that the set $log(\Gamma_p)$ is closed under multiplication by elements from $\mathbb{Z}_p$, and so is $L_p$. \qed

The proof of the following is similar to the proof of the corresponding properties of the Mal’cev completion in Theorem 2.4.

**Lemma 2.11.** The set $R_p$ is a group containing $\Gamma, R$ and $\Gamma_p$. Moreover, $R_p$ is a Mal’cev completion of $\Gamma_p$ in the sense that it satisfies the following properties:

- For every $a \in R_p$ and $m \in \mathbb{N}$ there exists $b \in R_p$ such that $b^m = a$.
- For every $a \in R_p$ there exists $m \in \mathbb{N}$ such that $a^m \in \Gamma_p$.

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Also, similarly to Lemma 2.2 one can use the BCH formula in order to show:

**Lemma 2.12.** Under the correspondence between $R_p$ and $L_p$ one has $R'_p = L'_p$. This equality gives a natural group homomorphism between $R_p/R'_p$ and the additive group $L_p/L'_p$.

One can use the BCH formula in order to prove that $L'_p$ is the $\mathbb{Q}$-span of $\log(\Gamma'_p)$, and hence $R'_p$ is a Mal’cev completion of $\Gamma'_p$ in the sense of Lemma 2.11. Denoting $\delta(\Gamma_p) = \ker(\Gamma_p \to (\Gamma^*)_p \simeq \mathbb{Z}_p^d)$ where $d = \text{rank}_{\mathbb{Z}}(\Gamma^*)$, we have also a similar property as in Lemma 2.3.

**Lemma 2.13.** One has $\Gamma_p \cap R'_p = \delta(\Gamma_p)$ and

$$\dim_{\mathbb{Q}_p}(R_p/R'_p) = \text{rank}_{\mathbb{Z}_p}(\Gamma_p/\delta(\Gamma_p)) = d.$$

**Proof.** We first prove that $\delta(\Gamma_p) = \ker(\Gamma_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p))$, where $\tilde{\Gamma}_p = \Gamma'_p/\Gamma_p$. As $(\Gamma^*)_p$ is a torsion free abelian quotient of $\Gamma_p$, it follows that $\delta(\Gamma_p) \supseteq \ker(\Gamma_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p))$. On the other hand, the map $\Gamma \to \Gamma_p$ induces a map $\Gamma^* \to \Gamma_p/\text{tor}(\tilde{\Gamma}_p)$. Now, as $\Gamma_p$ is a finitely generated pro-$p$ group, $\Gamma_p$ is closed in $\Gamma$ (see Proposition 1.19 in [DDSM]). Hence, $\Gamma_p \to \tilde{\Gamma}_p$ is a continuous homomorphism, and hence, $\Gamma_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p)$ is continuous as well. Thus, we can say that the image of $\Gamma^*$ under the map $\Gamma^* \to \Gamma_p/\text{tor}(\tilde{\Gamma}_p)$, is dense in $\Gamma_p/\text{tor}(\tilde{\Gamma}_p)$. Hence, we have a surjective continuous homomorphism $(\Gamma^*)_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p)$. It follows that $\delta(\Gamma_p) \subseteq \ker(\Gamma_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p))$, so $\delta(\Gamma_p) = \ker(\Gamma_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p))$ as required.

Now, as $R_p$ contains $\Gamma_p$, we have $\Gamma_p/(\Gamma_p \cap R'_p) \leq R_p/R'_p$. It follows that $\Gamma_p/(\Gamma_p \cap R'_p)$ is a torsion free abelian quotient of $\Gamma_p$. Hence

$$\delta(\Gamma_p) = \ker(\Gamma_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p)) \subseteq \Gamma_p \cap R'_p.$$

On the other hand, as every element of $R'_p$ has a power in $\Gamma'_p$, it follows that we also have $\Gamma_p \cap R'_p \leq \ker(\Gamma_p \to \tilde{\Gamma}_p/\text{tor}(\tilde{\Gamma}_p)) = \delta(\Gamma_p)$, as required.

The equality $\Gamma_p \cap R'_p = \delta(\Gamma_p)$ implies that $\Gamma_p/\delta(\Gamma_p) \leq R_p/R'_p$. Thus $\Gamma_p/\delta(\Gamma_p)$ is a free $\mathbb{Z}_p$-module that spans the vector space $R_p/R'_p$ over $\mathbb{Q}_p$.

Hence $\dim_{\mathbb{Q}_p}(R_p/R'_p) = \text{rank}_{\mathbb{Z}_p}(\Gamma_p/\delta(\Gamma_p)) = d$, as required.

The following corollary will be needed later:

**Corollary 2.14.** We have $\dim_{\mathbb{Q}}(L) = \dim_{\mathbb{Q}_p}(L_p)$.

**Proof.** We saw that

$$\dim_{\mathbb{Q}_p}(L_p/L'_p) = \dim_{\mathbb{Q}_p}(R_p/R'_p) = \text{rank}_{\mathbb{Z}_p}(\Gamma_p/\delta(\Gamma_p)) = d = \text{rank}_{\mathbb{Z}}(\Gamma/\delta(\Gamma)) = \dim_{\mathbb{Q}}(R/R') = \dim_{\mathbb{Q}}(L/L').$$

By the fact that the commutator subgroup of a finitely generated nilpotent group is finitely generated, and $L'$ is the $\mathbb{Q}$-span of $\log(\Gamma')$ one has

$$\text{rank}_{\mathbb{Z}_p}((\Gamma')_p/\delta((\Gamma')_p)) = \text{rank}_{\mathbb{Z}}((\Gamma')/\delta(\Gamma')) = \dim_{\mathbb{Q}}(L'/L'').$$
Using the CSP for arithmetic soluble groups, we can identify $(\Gamma')_p$ with the closure of $\Gamma'$ in $\Gamma_p \leq Tr_1(n, Z_p)$. As explained in the proof of Lemma 2.15 the latter can be identified with $(\Gamma'')_p$, and hence $(\Gamma')_p = (\Gamma'')_p$. Hence, $L'_p$, which is the $Q$-span of $\log((\Gamma'')_p)$, is actually the $Q$-span of $\log((\Gamma')_p)$. It follows that
\[
\dim Q_p(L'_p/L''_p) = \text{rank}_Q((\Gamma')_p/\delta((\Gamma'')_p)) = \dim Q(L'/L'').
\]
Continuing like that, we obtain that $\dim Q(L^{(i)}/L^{(i+1)}) = \dim Q_p(L^{(i)}_p/L^{(i+1)}_p)$ for any $i$, where $L^{(i)}, L^{(i)}_p$ are the $i$-th derivatives of $L, L_p$ respectively. Therefore
\[
\dim Q(L) = \sum_i \dim Q(L^{(i)}/L^{(i+1)}) = \sum_i \dim Q_p(L^{(i)}_p/L^{(i+1)}_p) = \dim Q_p(L_p)
\]
as required. 

**Remark 2.15.** We presented a proof for Corollary 2.14 based on the previous line of discussion, but the knowledgeable reader can also deduce it by recalling that $\dim Q(L)$ is equal to $h(\Gamma)$ - the Hirsch length of $\Gamma$, and $h(\Gamma)$ is equal to $\dim(\Gamma_p)$ - the dimension of the pro-$p$ completion of $\Gamma$, and the later is equal to $\dim Q_p(L_p)$ as $L_p$ is the Lie algebra of $\Gamma_p$.

Recall $\Delta$, the lattice hull of $\Gamma$. For our convenience, without loss of generality, we can assume that $\Gamma \leq \Delta \leq Tr_1(n, Z)$ (see Lemma 2 in Chapter 6 of [S]). Hence, $\Delta_p$ can be identified with the closure of $\Delta$ in $Tr_1(n, Z_p)$.

**Lemma 2.16.** One has $\log(\Delta_p) = Z_p \log(\Delta) \subseteq L_p$, where $Z_p \log(\Delta)$ is the $Z_p$-lattice of $L_p$ spanned by $\log(\Delta)$. In particular, $\log(\Delta_p)$ is a $Z_p$-lattice in $L_p$ and $\Gamma_p \leq \Delta_p \leq R_p$.

**Proof.** By a similar argument as in Lemma 2.10 for any element $m \in Z_p$ and $a \in \log(\Delta)$ we have $m \cdot a \in \log(\Delta_p)$. Hence, we will get $\log(\Delta_p) \supseteq Z_p \log(\Delta)$ once we show that for any $g, h \in \Delta_p$ there exists $k \in \Delta_p$ such that $\log(g) + \log(h) = \log(k)$. By assumption, the group $\Delta$ is dense in $\Delta_p$ where the topology on $\Delta_p$ coincides with the topology induced by $Tr_1(Q_p)$. Let $g_i, h_i, k_i \in \Delta$ be such that
\[
\lim_{i \to \infty} g_i = g \quad \lim_{i \to \infty} h_i = h \quad \log(k_i) = \log(g_i) + \log(h_i).
\]
We need to show that $\lim_{i \to \infty} k_i = k \in \Delta_p$ exists and $\log(g) + \log(h) = \log(k)$. As $\exp$ and $\log$ are continuous, we have
\[
k = \lim_{i \to \infty} k_i = \lim_{i \to \infty} \exp(\log(g_i) + \log(h_i))
= \exp(\lim_{i \to \infty} g_i + \log(h_i)) = \exp(\log(g) + \log(h)).
\]
As $\Delta_p$ is compact, we have $k \in \Delta_p$ where $\log(g) + \log(h) = \log(k)$. For the opposite inclusion: $\Delta$ is dense in $\Delta_p$, so $\log(\Delta)$ is dense in $\log(\Delta_p)$. As $Z_p \log(\Delta)$ is closed in $L_p$, it follows that $\log(\Delta_p) \subseteq Z_p \log(\Delta)$, as required. 

\[
\square
\]
Lemma 2.17. The group $\Delta_p$ is a unique minimal subgroup $\Gamma_p \leq \Delta_p \leq R_p$ that its image in $L_p$ is a $Z_p$-lattice.

Proof. Suppose that $\Delta_p$ satisfies $\Gamma_p \leq \Delta_p \leq R_p$, $\log(\Delta_p)$ is a $Z_p$-lattice, and $\log(\Delta_p) \not\subseteq \log(\Delta_p)$. It follows that $\Gamma \leq \Delta_p \cap \Delta \not\subseteq \Delta$ where $\log(\Delta_p \cap \Delta) = \log(\Delta_p) \cap \log(\Delta)$ is a lattice of $L$ that contains $\log(\Gamma)$. This is a contradiction to the minimality of $\Delta$. \qed

Now, using Lemma 2.11 and the property $\Gamma_p \leq \Delta_p \leq R_p$ (Lemma 2.10), and following the same steps for the proof of [S] to Theorem 2.6, we have:

Theorem 2.18. There are natural isomorphisms

$$\text{Aut}(R_p) \cong \text{Aut}(L_p)$$
$$\text{Aut}(\Gamma_p) \cong N_{\text{Aut}(L_p)}(\log(\Gamma_p)) \leq \text{Aut}(L_p)$$
$$\text{Aut}(\Delta_p) \cong N_{\text{Aut}(L_p)}(\log(\Delta_p)) \leq \text{Aut}(L_p).$$

By Lemmas 2.12, 2.13 and Theorem 2.18 we obtain:

Proposition 2.19. Denote $IA^*(\Gamma_p) = \ker(\text{Aut}(\Gamma_p) \to \text{Aut}((\Gamma')_p))$. Then

$$IA^*(\Gamma_p) \cong N_{\text{Aut}(L_p)}(\log(\Gamma_p)) \cap \ker(\text{Aut}(L_p) \to \text{Aut}(L_p/L'_p)).$$

Similarly, as $\Gamma_p \leq \Delta_p \leq R_p$ (Lemma 2.10), we also have

$$IA^*(\Delta_p) \cong N_{\text{Aut}(L_p)}(\log(\Delta_p)) \cap \ker(\text{Aut}(L_p) \to \text{Aut}(L_p/L'_p)).$$

We can now deduce Proposition 2.8.

Proof. (of Proposition 2.8) From Lemma 2.17 and Proposition 2.19 we get that for any prime $p$ one has $IA^*(\Gamma_p) \hookrightarrow IA^*(\Delta_p)$. Therefore

$$IA^*(\hat{\Gamma}) = \prod_p IA^*(\Gamma_p) \hookrightarrow \prod_p IA^*(\Delta_p) = IA^*(\hat{\Delta}).$$

Moreover, $IA^*(\hat{\Gamma}) = \{ \alpha \in IA^*(\hat{\Delta}) | \alpha(\hat{\Gamma}) = \hat{\Gamma} \}$, hence it is open in $IA^*(\hat{\Delta})$. \qed

As we mentioned previously, by Proposition 2.8 in order to prove Theorem 1.3 for $\Gamma$, it is enough to prove it for $\Delta$. So from now on $\log(\Delta)$ is a lattice that spans $L$, and hence $k = \dim_{\mathbb{Q}}(L) = \text{rank}_{\mathbb{Z}}(\log(\Delta))$. Our objective now is to construct a basis for $\log(\Delta)$ that with relation to it, we will be able to view $IA^*(\Delta)$ as the $\mathbb{Z}$-points of a $\mathbb{Q}$-algebraic group whose $\mathbb{Z}_p$-points are $IA^*(\Delta_p)$. Once we show this, we will see that Theorem 1.3 (1) follows from the classical CSP for the arithmetic group $IA^*(\Delta)$, and Theorem 1.3 (2) is the classical strong-approximation theorem for this group.

Let $g_1, \ldots, g_d \in \Delta$ be such that $\Delta^* = \Delta/\delta(\Delta) \simeq \mathbb{Z}^d$ is generated by the images $\tilde{g}_1, \ldots, \tilde{g}_d$. Then, $\hat{g}_1, \ldots, \hat{g}_d$ also generate $(\Delta^*)_p = \Delta_p/\delta(\Delta_p) \simeq \mathbb{Z}_p^d$ as a pro-$p$ group. Hence, by Lemmas 2.9 and 2.13 $l_1 = \log(g_1), \ldots, l_d = \log(g_d)$ generate $R/R' \cong L/L' \cong \mathbb{Q}^d$ over $\mathbb{Q}$ and generate $R_p/R'_p \cong L_p/L'_p \cong \mathbb{Q}_p^d$ over $\mathbb{Q}_p$. It follows that $1, \ldots, l_d$ are linearly independent over $\mathbb{Q}_p$ (and over $\mathbb{Q}$) and since $L$ and $L_p$ are nilpotent, they generate $L$ as a Lie algebra over $\mathbb{Q}$ and generate $L_p$ as a Lie algebra over $\mathbb{Q}_p$. 

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Lemma 2.20. Let \( L = \gamma_1(L), L' = \gamma_2(L), ..., 0 = \gamma_c(L) \) be the lower central series of \( L \). The set \( l_1, ..., l_d \) can be completed to a basis
\[
B = \{ l_1, ..., l_d, l_{d+1}, ..., l_k \}
\]
of the lattice \( \log(\Delta) \) such that:

1. \( B \) contains bases for \( \gamma_j(L)/\gamma_{j+1}(L) \) mod \( \gamma_{j+1}(L) \) for every \( j \).

2. \( l_{j+1} \) lies in the same term as \( l_j \) in the lower central series of \( L \), or deeper.

Proof. Now, \( l_1, ..., l_d \) lie in \( \gamma_1(L) - \gamma_2(L) \) and provide a basis for \( L/L' = \gamma_1(L)/\gamma_2(L) \) mod \( \gamma_2(L) \). Denote \( i_1 = d \).

For every \( j \geq 2 \) the group \( (\log(\Delta) \cap \gamma_j(L))/(\log(\Delta) \cap \gamma_{j+1}(L)) \) is a lattice in \( \gamma_j(L)/\gamma_{j+1}(L) \) and hence
\[
\text{rank}_\mathbb{Q}(\log(\Delta) \cap \gamma_j(L))/(\log(\Delta) \cap \gamma_{j+1}(L)) = \dim_\mathbb{Q}(\gamma_j(L))/\gamma_{j+1}(L).
\]
Let \( l_{i_{j-1}+1}, ..., l_{i_j} \in \log(\Delta) \cap \gamma_j(L) \) be a basis to
\[
(\log(\Delta) \cap \gamma_j(L))/(\log(\Delta) \cap \gamma_{j+1}(L)) \text{ mod } \log(\Delta) \cap \gamma_{j+1}(L).
\]
Then \( l_{i_{j-1}+1}, ..., l_{i_j} \) gives a basis for \( \gamma_j(L)/\gamma_{j+1}(L) \) mod \( \gamma_{j+1}(L) \) and they all lie in \( \gamma_j(L) - \gamma_{j+1}(L) \).

This procedure gives us \( l_1, ..., l_{i_{c-1}} \) that satisfy the two conditions in the lemma. Also, by the construction, \( l_1, ..., l_{i_{c-1}} \) span the abelian group \( \log(\Delta) \) and provide a basis for \( L \). It follows that \( i_{c-1} = k \) and that \( l_1, ..., l_k \) is the desired basis for \( \log(\Delta) \).

Let \( B \) be a basis for \( \log(\Delta) \) as in the above lemma. Clearly, \( B \) is also a basis for \( L \) as a vector space over \( \mathbb{Q} \). As \( L_p \) is generated as a Lie algebra by \( l_1, ..., l_d \) over \( \mathbb{Q}_p \), the set \( B \) also spans \( L_p \) over \( \mathbb{Q}_p \). As \( \dim_\mathbb{Q}(L) = \dim_\mathbb{Q}_p(L_p) \) (Proposition 2.14), it follows that \( B \) is also a basis for \( L_p \) as a vector space over \( \mathbb{Q}_p \). Hence, using the basis \( B \) we can identify
\[
\text{Aut}(L) = GL_k(\mathbb{Q}) \cap \text{Aut}(L_p) \leq GL_k(\mathbb{Q}_p).
\]
In addition, as \( \log(\Delta) \) is a lattice and \( \log(\Delta_p) = \mathbb{Z}_p \log(\Delta) \) (Lemma 2.10), using the basis \( B \) we can identify
\[
\text{Aut}(\Delta) = N_{\text{Aut}(L)}(\log(\Delta)) = GL_k(\mathbb{Z}) \cap \text{Aut}(L)
\]
\[
\text{Aut}(\Delta_p) = N_{\text{Aut}(L_p)}(\log(\Delta_p)) = GL_k(\mathbb{Z}_p) \cap \text{Aut}(L_p).
\]
Moreover, we can identify \( \text{Aut}(L) \) and \( \text{Aut}(L_p) \) with the groups
\[
\text{Aut}(L) = \{ A \in GL_k(\mathbb{Q}) \mid [Ax, Ay] - A[x, y] = 0 \ \forall x, y \in B \}
\]
\[
\text{Aut}(L_p) = \{ A \in GL_k(\mathbb{Q}_p) \mid [Ax, Ay] - A[x, y] = 0 \ \forall x, y \in B \}.
\]
Now, notice that \( \sigma \in \ker(\text{Aut}(L) \to \text{Aut}(L/L')) \) if and only if for every \( i = 1, ..., d \) we have \( \sigma(l_i) \in l_i + L' \). As \( l_1, ..., l_d \) generate \( L \) over \( \mathbb{Q} \), it follows that
for every \( \sigma \in \ker(\text{Aut}(L) \to \text{Aut}(L/L')) \) and every \( i \) (not only for \( i = 1, \ldots, d \)) we have

\[
\sigma(l_i) = l_i + n_i
\]

where \( n_i \) lies in a strictly deeper term in the lower central series of \( L \), than of the one that \( l_i \) lies in. Hence, by the construction in Lemma 2.20 with relation to \( B \) we have

\[
\ker(\text{Aut}(L) \to \text{Aut}(L/L')) = \text{Aut}(L) \cap \text{Tr}_1(k, Q) \leq \text{GL}_k(Q_p).
\]

It follows that \( IA^*(\Delta) \) can be identified with the arithmetic group of \( \mathbb{Z} \)-points of \( \text{Aut}(L) \cap \text{Tr}_1(k, Q) \). Similarly, we have

\[
\ker(\text{Aut}(L_p) \to \text{Aut}(L_p/L'_p)) = \text{Aut}(L_p) \cap \text{Tr}_1(k, Q_p) \leq \text{GL}_k(Q_p).
\]

and hence \( IA^*(\Delta_p) \) can be identified with the \( \mathbb{Z}_p \)-points of \( \text{Aut}(L_p) \cap \text{Tr}_1(k, Q_p) \).

By the description above, \( \text{Aut}(L) \cap \text{Tr}_1(k, Q) \) and \( \text{Aut}(L_p) \cap \text{Tr}_1(k, Q_p) \) are unipotent algebraic groups which are defined by the same equations, over \( Q \) and \( Q_p \) respectively.

Now, \( IA^*(\Delta) \) is a unipotent subgroup of \( \text{GL}_k(\mathbb{Z}) \). Hence, by the congruence subgroup property for unipotent arithmetic groups we have

\[
\hat{IA^*}(\Delta) = \prod_p(IA^*(\Delta))_p \hookrightarrow \prod_p \text{Tr}_1(k, \mathbb{Z}_p) = \text{Tr}_1(k, \mathbb{Z}) \quad \uparrow \quad \prod_p IA^*(\Delta_p) = IA^*(\hat{\Delta})
\]

and hence \( \hat{IA^*}(\Delta) \hookrightarrow IA^*(\hat{\Delta}) \) is injective. As \( IA^*(\Delta) \) is a finitely generated nilpotent group, for any \( G \leq IA^*(\Delta) \) we have \( \hat{G} \hookrightarrow \hat{IA^*}(\Delta) \), and hence \( \hat{G} \hookrightarrow IA^*(\hat{\Delta}) \). This gives the first part of Theorem 1.4 in the case of finitely generated torsion free nilpotent groups.

For the second statement of Theorem 1.4 notice that by the above description for \( IA^*(\Delta) \) and \( IA^*(\Delta_p) \), we obtain that \( IA^*(\Delta) \) is dense in \( IA^*(\Delta_p) \) by the strong approximation property for arithmetic unipotent groups (see [PR], Proposition 7.1, and the corollary afterward). Therefore, as each of the groups \( IA^*(\Delta_p) \) is a pro-\( p \) group, we obtain that \( IA^*(\Delta) \) is dense in

\[
\prod_p IA^*(\Delta_p) = IA^*(\hat{\Delta}).
\]

This completes the proof of Theorem 1.4 for finitely generated torsion free nilpotent groups.

### 3 The general case

The aim of this section is to prove Theorem 1.4 given its validity for finitely generated torsion free nilpotent group. Along the section, \( \Gamma \) is a finitely generated nilpotent group, and \( \Delta = \Gamma/\text{tor}(\Gamma) \). As a finitely generated nilpotent
group, $\Gamma$ is residually finite. Hence, we can think on $\Gamma$ as a subgroup of $\hat{\Gamma}$. One has $\text{tor}(\hat{\Gamma}) = \text{tor}(\Gamma)$ (see Lemma 2.3 in [KW], Corollary 7.5 in [P]). In other words, $\text{tor}(\Gamma)$ is a normal subgroup of $\hat{\Gamma}$, and

$$\hat{\Delta} = \Gamma/\text{tor}(\Gamma) = \hat{\Gamma}/\text{tor}(\Gamma)$$

is torsion free. In addition, we get that $\text{tor}(\Gamma)$ is characteristic, not only as a subgroup of $\Gamma$, but also as a subgroup of $\hat{\Gamma}$. Therefore, we have a map $IA^*(\hat{\Gamma}) \to IA^*(\hat{\Delta})$ and we obtain the commutative diagram

$$
\begin{array}{ccc}
IA^*(\Gamma) & \to & IA^*(\Delta) \\
\downarrow & & \downarrow \\
IA^*(\hat{\Gamma}) & \to & IA^*(\hat{\Delta})
\end{array}
$$

Denote $\tilde{K} = \ker(IA^*(\Gamma) \to IA^*(\Delta))$. Let $x_1, ..., x_n$ be a generating set for $\Gamma$. Then, every $\alpha \in \tilde{K}$ can be described by

$$x_i \mapsto x_i a_i$$

for some $a_i \in \text{tor}(\Gamma)$. As $\text{tor}(\Gamma)$ is finite, $\tilde{K}$ is also finite.

Now, let $G \leq IA^*(\Gamma)$, and denote its image in $IA^*(\Delta)$ by $H$. Then $K = \ker(G \to H) \leq \tilde{K}$ is finite, and hence we obtain the following commutative exact diagram

$$
\begin{array}{ccc}
1 & \to & K \\
\downarrow & & \downarrow \\
& & IA^*(\hat{\Gamma}) \\
& & \downarrow \\
& & IA^*(\hat{\Delta})
\end{array}
$$

Notice that as $G \leq IA^*(\Gamma) \to IA^*(\hat{\Gamma})$, the map $K \to IA^*(\hat{\Gamma})$ is injective. Notice also that as $K$ is finite, we have $K = \tilde{K}$. Hence, moving to the profinite completion of the upper row, we get the commutative exact diagram

$$
\begin{array}{ccc}
K = \tilde{K} & \to & \hat{G} \\
\downarrow & & \downarrow \\
& & IA^*(\hat{\Gamma}) \\
& & \downarrow \\
& & IA^*(\hat{\Delta})
\end{array}
$$

It follows that $\tilde{K} \to IA^*(\hat{\Gamma})$ is injective and by [2] also $\hat{H} \to IA^*(\hat{\Delta})$ is injective. Hence, by diagram chasing, $\hat{G} \to IA^*(\hat{\Gamma})$ is also injective. This proves the first assertion of Theorem [4] in the general case. We move now to prove the second assertion.

As $\Gamma$ is residually finite, and $\text{tor}(\Gamma)$ is finite, there exists $t \in \mathbb{N}$ such that $\text{tor}(\Gamma) \cap \Gamma^t = \{e\}$ where $\Gamma^t$ is the normal subgroup

$$\Gamma^t = \langle g^t \mid g \in G \rangle \triangleleft \Gamma.$$

Fix this $t$. It follows that whenever $t|m$, we have $\text{tor}(\Gamma) \cap \Gamma^m = \{e\}$, and therefore $\text{tor}(\Gamma)$ is naturally embedded in $\Gamma/\Gamma^m$ for such $m$. As $\Gamma$ is nilpotent
and finitely generated, $\Gamma/\Gamma^m$ and $\Delta/\Delta^m$ are finite for any $m$ (see Corollary 3.3 in [P]), and hence one has

$$\hat{\Gamma} = \lim_{m \in \mathbb{N}} \Gamma/\Gamma^m = \lim_{t|m} \Gamma/\Gamma^m$$

$$\hat{\Delta} = \lim_{m \in \mathbb{N}} \Delta/\Delta^m = \lim_{t|m} \Delta/\Delta^m.$$ 

Now, let $\hat{\alpha} \in IA^*(\hat{\Gamma})$, and write

$$\hat{\alpha} = (\alpha_m)_{t|m} \in IA^*(\hat{\Gamma}) \leq \lim_{t|m} Aut(\Gamma/\Gamma^m)$$

where $\alpha_m \in Aut(\Gamma/\Gamma^m)$. In order to prove the second part of Theorem 1.4, namely that $IA^*(\Gamma)$ is dense in $IA^*(\hat{\Gamma})$, it is enough to show that:

**Proposition 3.1.** Let $m$ such that $t|m$, and let $\hat{\alpha}, \alpha_m$ be as above. Then, there exists $\alpha \in IA^*(\Gamma)$ such that $\alpha \to \alpha_m$ through the map $IA^*(\Gamma) \to Aut(\Gamma/\Gamma^m)$.

So fix $m$ such that $t|m$, and let $\hat{\beta} \in IA^*(\hat{\Delta})$ be the image of $\hat{\alpha}$ under the map $IA^*(\hat{\Gamma}) \to IA^*(\hat{\Delta})$. Write

$$\hat{\beta} = (\beta_m)_{t|m} \in IA^*(\hat{\Delta}) \leq \lim_{t|m} Aut(\Delta/\Delta^m)$$

where $\beta_m \in Aut(\Delta/\Delta^m)$ is the image of $\alpha_m \in Aut(\Gamma/\Gamma^m)$ under the map $Aut(\Gamma/\Gamma^m) \to Aut(\Delta/\Delta^m)$. By assumption, $IA^*(\Delta)$ is dense in $IA^*(\hat{\Delta})$, and hence there exists $\beta \in IA^*(\Gamma)$ such that $\beta \to \beta_m$. So we have the diagram

$$\begin{array}{ccc}
\beta & \to & IA^*(\Delta) \\
\downarrow & & \downarrow \\
\alpha_m & \to & \beta_m
\end{array}$$

under the diagram

$$\begin{array}{ccc}
Aut(\Gamma/\Gamma^m) & \to & Aut(\Delta/\Delta^m) \\
\downarrow & & \downarrow \\
P_1 & \to & Q
\end{array}$$

We want to use $\alpha_m$ and $\beta$ in order to construct $\alpha \in IA^*(\Gamma)$ such that $\alpha \to \alpha_m$.

Let us recall the following notion: let $P_1, P_2$ and $Q$ be groups with epimorphisms $\pi_i : P_i \to Q$, $i = 1, 2$. The group

$$U = P_1 \times_Q P_2 = \{(x, y) \in P_1 \times P_2 \mid \pi_1(x) = \pi_2(y)\}$$

is called the fiber product of $P_1, P_2$ along $Q$ (or the pullback). It is easy to check that as $\pi_i$ are surjective, we get the following commutative diagram of surjective maps

$$\begin{array}{ccc}
U & \to & P_1 \\
\downarrow & & \downarrow \\
P_2 & \to & Q
\end{array}$$

such that $P_1 \simeq U/(U \cap \{(e) \times P_2\})$ and $P_2 \simeq U/(U \cap (P_1 \times \{e\}))$. Regarding our context, as we have the commutative surjective diagram

$$\begin{array}{ccc}
\Gamma & \to & \Delta \\
\downarrow & & \downarrow \\
\Gamma/\Gamma^m & \to & \Delta/\Delta^m
\end{array}$$
we get a natural map $\Gamma \overset{\rho}{\rightarrow} \Delta \times_{\Delta^m} \Gamma/\Gamma^m$. Now, as we assume that $t|m$, we have 
\[ \ker(\Gamma \rightarrow \Gamma/\Gamma^m) \cap \ker(\Gamma \rightarrow \Delta) = \Gamma^m \cap \text{tor}(\Gamma) = \{e\} \]
and hence the map $\rho$ is injective. In addition 
\[ \ker(\Gamma \rightarrow \Delta/\Delta^m) = \Gamma^m \cdot \text{tor}(\Gamma) = \ker(\Gamma \rightarrow \Gamma/\Gamma^m) \cdot \ker(\Gamma \rightarrow \Delta) \]
and hence, one can check that it follows that $\rho$ is also surjective. Therefore, we can identify $\Gamma$ with $\Delta \times_{\Delta^m} \Gamma/\Gamma^m$, the fiber product of $\Delta$ and $\Gamma/\Gamma^m$ along $\Delta/\Delta^m$. The following lemma is elementary:

**Lemma 3.2.** Notation as above. Let $\sigma_i \in \text{Aut}(P_i)$ for $i = 1, 2$ be automorphisms preserving $\ker(\pi_i)$ and inducing $\bar{\sigma}_i \in \text{Aut}(Q)$. If $\bar{\sigma}_1 = \bar{\sigma}_2$, then there exists $\sigma \in \text{Aut}(U)$ such that $\sigma$ preserves $U \cap (P_1 \times \{e\})$ and $U \cap (\{e\} \times P_2)$ and induces $\sigma_1$ on $P_1$ and $\sigma_2$ on $P_2$.

**Proof.** For $(x, y) \in U$ define $\sigma(x, y) = (\sigma_1(x), \sigma_2(y))$. We claim that we have $(\sigma_1(x), \sigma_2(y)) \in H$. Indeed 
\[ \pi_1(\sigma_1(x)) = \bar{\sigma}_1(\pi_1(x)) = \bar{\sigma}_2(\pi_2(y)) = \bar{\sigma}_2(\pi_2(y)) = \pi_2(\sigma_2(y)). \]
Showing that $\sigma$ is a bijective homomorphism follows from the same properties for $\sigma_1, \sigma_2$. The other properties of $\sigma$ follow straightforward from the definition. □

Applying Lemma 3.2 on $\sigma_m \in \text{Aut}(\Gamma/\Gamma^m)$ and $\beta \in IA^*(\Delta)$ we obtain $\alpha \in \text{Aut}(\Gamma)$ that its projection is $\alpha_m$ through $\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/\Gamma^m)$. Hence, in order to finish the proof of Theorem 1.3 it remains to show that the $\alpha$ we got is not only in $\text{Aut}(\Gamma)$ but also in $IA^*(\Gamma)$.

**Lemma 3.3.** Recall that $\Gamma^* = \Gamma/\text{tor}(\Gamma)$ where $\Gamma = \Gamma/\Gamma'$, and denote $\Delta^* = \Delta/\text{tor}(\Delta)$ where $\Delta = \Delta/\Delta'$. Then, we have a canonical isomorphism $\Gamma^* \simeq \Delta^*.$

**Proof.** We have a natural projection 
\[ \Delta = \Gamma/\text{tor}(\Gamma) \rightarrow \Gamma/\text{tor}(\Gamma) = \Gamma^* \]
which gives rise to a map $\Delta = \Delta/\Delta' \rightarrow \Gamma^*$ and to a map $\Delta^* = \Delta/\text{tor}(\Delta) \rightarrow \Gamma^*$. Obviously, this map is the inverse of the natural map $\Gamma^* \rightarrow \Delta^*$. So $\Gamma^* \simeq \Delta^*$ as required. □

From this lemma we get that the preimage of $IA^*(\Delta)$ under the map $\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Delta)$ is $IA^*(\Gamma)$. Thus, as $\alpha$ is a preimage of $\beta$ through $\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Delta)$ and $\beta \in IA^*(\Delta)$ it follows that indeed $\alpha \in IA^*(\Gamma)$, as required.

## 4 The Free cases

In this section we sketch a more straightforward proof to Corollary 1.3 in the special case where $\Psi_c$ is a free nilpotent group on $n$ elements, a proof which does not refer neither to the CSP nor to the strong approximation for unipotent groups. Notice that as $\Psi'_c = \Psi_c/\Psi'_c = \mathbb{Z}^{(n)}$ and $\hat{\Psi}_c' = \Psi_c/\hat{\Psi}_c' = \hat{\mathbb{Z}}^{(n)}$, in this case $IA(\Psi_c) = IA^*(\Psi_c)$ and $IA(\Psi_c) = IA^*(\hat{\Psi}_c)$. Let us formulate the assertion:
**Theorem 4.1.** Let $\Psi_c$ be the free nilpotent group on $n$ elements. For every $c \in \mathbb{N}$, the map $IA(\Psi_c) \rightarrow IA(\hat{\Psi}_c)$ is an isomorphism.

**Proof.** We will prove it by induction on $c$. For $c = 1, 2$ the result is trivial, as $\Psi_1 = \{e\}$ and $\Psi_2 = \mathbb{Z}^{(n)}$, so

$$IA(\Psi_1) = IA(\hat{\Psi}_1) = IA(\Psi_2) = IA(\hat{\Psi}_2) = \{e\}.$$  

For the induction step we will use the (easy) facts that for every $c$, the natural map $Aut(\Psi_{c+1}) \rightarrow Aut(\Psi_c)$ is surjective (see [A]), and the natural map $Aut(\Psi_{c+1}) \rightarrow Aut(\hat{\Psi}_c)$ is also surjective (see Section 5.2 in [L2]). So let $c \geq 2$. Denote

$$A(\Psi_{c+1}) = ker(IA(\Psi_{c+1})) \rightarrow IA(\Psi_c)$$

$$A(\hat{\Psi}_{c+1}) = ker(IA(\hat{\Psi}_{c+1})) \rightarrow IA(\hat{\Psi}_c).$$

We have the commutative exact diagram

$$
\begin{array}{cccccc}
1 & \rightarrow & A(\Psi_{c+1}) & \rightarrow & IA(\Psi_{c+1}) & \rightarrow & IA(\Psi_c) & \rightarrow & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \rightarrow & A(\hat{\Psi}_{c+1}) & \rightarrow & IA(\hat{\Psi}_{c+1}) & \rightarrow & IA(\hat{\Psi}_c) & \rightarrow & 1
\end{array}
$$

which gives rise to the commutative exact diagram

$$
\begin{array}{cccccc}
A(\Psi_{c+1}) & \rightarrow & IA(\Psi_{c+1}) & \rightarrow & IA(\Psi_c) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
A(\hat{\Psi}_{c+1}) & \rightarrow & IA(\hat{\Psi}_{c+1}) & \rightarrow & IA(\hat{\Psi}_c) & \rightarrow & 1
\end{array}
$$

By the induction hypothesis and diagram chasing, it is enough to prove that the map $A(\Psi_{c+1}) \rightarrow A(\hat{\Psi}_{c+1})$ is an isomorphism.

Let $x_1, \ldots, x_n$ be a set of free generators for $\Psi_{c+1}$. Denote

$$Z(\Psi_{c+1})^{(n)} = Z(\Psi_{c+1}) \times \cdots \times Z(\Psi_{c+1})$$

where $Z(\Psi_{c+1})$ is the center of $\Psi_{c+1}$. Observe now that $A(\Psi_{c+1})$ can be viewed as a subgroup of $Z(\Psi_{c+1})^{(n)}$ in the following way: For $\alpha \in A(\Psi_{c+1})$, one can describe it by its action on the generators of $\Psi_{c+1}$ and write $\alpha(x_i) = x_i u_i$ for some $u_i \in Z(\Psi_{c+1})$. We claim that the map $\alpha \mapsto (u_1, \ldots, u_n) \in Z(\Psi_{c+1})^{(n)}$ is a natural injective homomorphism $A(\Psi_{c+1}) \rightarrow Z(\Psi_{c+1})^{(n)}$. Indeed, let $\alpha, \beta \in A(\Psi_{c+1})$ defined by $\alpha(x_i) = x_i u_i$ and $\beta(x_i) = x_i v_i$ for some $u_i, v_i \in Z(\Psi_{c+1})$. Now, as $u_i \in Z(\Psi_{c+1}) \subseteq \Psi'_{c+1}$ it can be written as a product of commutators of words on the generators $x_i$. As $u_i, v_i \in Z(\Psi_{c+1})$, it is easy to see that $(\beta \circ \alpha)(x_i) = x_i v_i u_i$. This proves the claim. In [A] it is proven that the above map $A(\Psi_{c+1}) \rightarrow Z(\Psi_{c+1})^{(n)}$ is also surjective, and hence, it is an isomorphism. I.e., for every choice of elements $u_1, \ldots, u_n \in Z(\Psi_{c+1})$, one can define an element $\alpha \in A(\Psi_{c+1})$ by defining $\alpha(x_i) = x_i u_i$ for the generators $x_1, \ldots, x_n$ of $\Psi_{c+1}$.  

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This is indeed an automorphism as \( \{ x_i u_i \}_{i=1}^{n} \) generate \( \Psi_{c+1} \), which is a free nilpotent group.

Using a similar approach, one can prove that \( A(\hat{\Psi}_{c+1}) \cong Z(\hat{\Psi}_{c+1})^{(n)} \) (see also [L2]) and thus

\[
A(\hat{\Psi}_{c+1}) \cong Z(\hat{\Psi}_{c+1})^{(n)} \cong Z(\hat{\Psi}_{c+1})^{(n)} \cong A(\hat{\Psi}_{c+1})
\]

as required. \( \square \)

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