On Correlation Functions for Non-critical Strings

with $c \leq 1$ but $d \geq 1$

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Abstract

We construct a Goulian-Li-type continuation in the number of insertions of the cosmological constant operator which is no longer restricted to one dimensional target space. The method is applied to the calculation of the three-point and a special four-point correlation function. Various aspects of the emerging analytical structure are discussed.

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1 Introduction

Till now non-critical string theories have been successfully constructed for target space dimensions $d \leq 1$ only. Besides the lattice approach along conventional lines (see e.g. [1]) a considerable amount of attention has been devoted to conformal matter theories with central charge $c \leq 1$ coupled to gravity in the formulation as matrix models [2]. Attempts to go beyond the $c = 1$ barrier in this scheme have been initiated recently [3].

On the other hand following refs. [4, 5, 6, 7] one can, at least for the lowest genera, reproduce the matrix model results for correlation functions in a continuum formulation by manipulations with the defining functional integrals. The central point in this procedure is a continuation in the number of insertions of the so called cosmological or screening operator. The technique is connected with an extensive use of kinematical peculiarities in $d = 1$ target space dimension. Although we cannot circumvent the restriction $c \leq 1$, we will be able to avoid the dependence on the special $d = 1$ kinematics. The present letter gives a short presentation of our continuation procedure for $d \geq 1$, the proof of equivalence with refs. [4, 5, 6, 7] for $d = 1$ and some discussion of the emerging analytical structure of our amplitudes.

2 Formulation of the model

The action of the combined matter and Liouville system is

$$S = \frac{1}{8\pi} \int d^2 z \sqrt{g(z)} \left( g^{mn}(z) \partial_m X^\mu(z) \partial_n X_\mu + iR(z) P^\mu X_\mu(z) \right) + \frac{1}{8\pi} \int d^2 z \sqrt{g(z)} \left( g^{mn}(z) \partial_m \Phi(z) \partial_n \Phi + QR(z) \Phi(z) + \mu^2 e^{\alpha \Phi(z)} \right).$$ (1)

Then the central charge $c$ of the matter theory is given by

$$c = d - 3P^2$$ (2)

for a $d$-dimensional target space ($\mu = 1, \ldots, d$). $Q$ and $\alpha$ are fixed by the requirement of Weyl invariance of the quantum field theory described by (1) and by the locality requirement for $exp(\alpha \Phi)$ [8, 9, 10]

$$Q^2 = \frac{25 - c}{3} = \frac{25 - d}{3} + P^2 ,$$ (3)

$$\alpha_{\pm} = \frac{Q}{2} \pm \frac{\sqrt{Q^2 - 8}}{2} , \quad \alpha \equiv \alpha_{-} .$$ (4)

We are interested in correlation functions of gravitational dressed local vertex operators [8, 9, 10]

$$V_k = e^{i k X(z)} e^{\beta \Phi(z)}$$ (5)

\[\text{We assume throughout this paper } d < 25\]
with

$$\beta = \frac{Q}{2} - \sqrt{m^2 + (k - P/2)^2}$$  \hspace{1cm} (6)$$

where

$$m^2 \equiv \frac{1 - d}{12}. \hspace{1cm} (7)$$

Real $\alpha$ requires $c \leq 1$. This can be realized in target space dimensions $d > 1 \ (m^2 < 0)$ for large enough $P^2$. $\beta$ is real for most of the $k$, but there is an exceptional compact domain $E$ whose boundary is given by $\beta = Q/2$ and which is characterized by

$$m^2 + k_\perp^2 < 0, \hspace{1cm} -\sqrt{-m^2 - k_\perp^2} < k_\parallel - \frac{|P|}{2} < +\sqrt{-m^2 - k_\perp^2} \hspace{1cm} (8)$$

with $k_\parallel, k_\perp$ denoting the components of $d$-dimensional momentum $k$ parallel and orthogonal to $P$. We start out with $k \notin E$ and continue afterwards.

3 Three point function

The three point function has been calculated in \cite{5, 7} for positive integer $s$

$$\alpha s = Q - \sum_{j=1}^{3} \beta_j \hspace{1cm} (9)$$

$$A_3 = \delta \left( \sum_j k_j - P \right) \frac{\Gamma(-s)}{\alpha} \frac{\Gamma(1 + s)}{\Gamma(-\frac{\alpha^2}{2})} \left( \frac{\mu^2}{8} \frac{\Gamma(1 + \frac{\alpha^2}{2})}{\Gamma(-\frac{\alpha^2}{2})} \right)^s \times$$

$$\exp \left\{ f\left( -\frac{\alpha^2}{2}, -\frac{\alpha^2}{2} |s \right) - f\left( 1 + \frac{\alpha^2}{2}, \frac{\alpha^2}{2} |s \right) + \sum_{j=1}^{3} \left( f\left( 1 - \alpha \beta_j, -\frac{\alpha^2}{2} |s \right) - f\left( \alpha \beta_j, \frac{\alpha^2}{2} |s \right) \right) \right\}. \hspace{1cm} (10)$$

The infinity due to the pole in $\Gamma(-s)$ is understood as a signal of the logarithmic modification of the $\mu$-scaling law for positive integer $s$ \cite{7, 11}. We introduced $f(a, b|s)$ as

$$f(a, b|s) = \sum_{j=0}^{s-1} \log \Gamma(a + bj). \hspace{1cm} (11)$$

This function fulfills the relations

$$f(a, b|s + 1) = f(a, b|s) + \log \Gamma(a + bs), \hspace{1cm} (12)$$

$$f(a + 1, b|s) = f(a, b|s) + s \log b + \log \Gamma\left( \frac{a}{b} + s \right) - \log \Gamma\left( \frac{a}{b} \right), \hspace{1cm} (13)$$

\footnote{If compared to \cite{11} we use a slightly changed normalization: $Q^2 + P^2 \rightarrow Q^2, \beta = \frac{2a}{b}, \alpha = \frac{2a}{\beta}$.}
\[ f(a + b, b | s) = f(a, b | s) + \log \Gamma(a + bs) - \log \Gamma(a) , \quad (14) \]
\[ f(a + b(s - 1), -b | s) = f(a, b | s) , \quad (15) \]
\[ f(2a, 2b | s) = f(a, b | s) + f(a + \frac{1}{2}, b | s) + (s(s - 1)b + s(2a - 1)) \log 2 - \frac{s}{2} \log \pi , \quad (16) \]
\[ f(a, 0 | s) = s \log \Gamma(a) , \quad (17) \]
\[ f(a, \frac{1}{s} | s) = \frac{1}{2}(s - 1) \log 2\pi + \left( \frac{1}{2} - s a \right) \log s + \log \Gamma(sa). \quad (18) \]

We have found a continuation of \( f \) to arbitrary \( s \), which is given by the following integral representation

\[ f(a, b | s) = \int_0^\infty \frac{dt}{t} \left( s(a - 1)e^{-t} + b \frac{s(s - 1)e^{-t} - s e^{-t}}{2} \right. \]
\[ \left. + \frac{(1 - e^{-tbs})e^{-at}}{(1 - e^{-t})(1 - e^{-t})} \right). \quad (19) \]

By careful manipulations with this integral representation one can show, that our continuation fulfills all the functional relations (12-18) also for generic complex \( a, b, s \). The integral representation (19) is convergent for positive real parts of \( a, b, s \). Together with the functional relations this fixes the complete analytical structure of \( f(a, b | s) \). Due to (15) we can restrict ourselves to \( Reb > 0 \). Then we find via (12) and (13) that \( \exp(f(a, b | s)) \) has poles at

\[ a = -bj - l \quad \text{(poles)} \quad (20) \]

and zeros at

\[ a + bs = -bj - l \quad \text{(zeros)} \quad , \quad (21) \]

with \( Reb > 0; \ j, l = 0, 1, 2, ... \). The order of poles and zeros is determined by the number of different realizations of the r.h.s. of eqs. (20) and (21), respectively. Using (13) we get from (10)

\[ A_3 = \delta(\sum_j k_j - P) \frac{\Gamma(-s)}{\alpha} \Gamma(1 + s) \left( \frac{\mu^2 \Gamma(1 + \frac{\alpha^2}{2})}{8 \Gamma(-\frac{\alpha^2}{2})} \right)^s \prod_{i=0}^{3} F_i , \quad (22) \]

where we introduced

\[ \overline{\beta}_i \equiv 1/2(\beta_j + \beta_k - \beta_i) = 1/2(Q - \alpha s) - \beta_i \ , \quad (i,j,k) = \text{perm}(1,2,3) \quad , \quad (23) \]

\[ F_i = \exp\{f(\alpha \overline{\beta}_i, \alpha^2/2 | s) - f(\alpha \beta_i, \alpha^2/2 | s)\} \quad (24) \]

and

\[ \alpha \beta_0 = 1 + \frac{\alpha^2}{2}, \quad \alpha \overline{\beta}_0 = -\frac{\alpha^2 s}{2}. \quad (25) \]

Eqs. (20) and (21) determine the poles and zeros of \( F_i \) as follows

\[ 2\beta_i = \alpha_+ l_i + \alpha_- (j_i - s) \quad \text{(poles)}, \quad (26) \]
\[ 2 \beta_i = \alpha_+ l_i + \alpha_- j_i \quad \text{(zeros).} \] (27)

The integers \( j_i, l_i \) have to be both \( \leq 0 \) or both \( > 0 \), i.e.

\[ (j_i - 1/2) (l_i - 1/2) > 0. \] (28)

For general \( s \) poles and zeros are located at different values of \( \beta_i \), but if \( \alpha_- s = \alpha_- m + \alpha_+ n \) with integer \( m, n \) a lot of poles and zeros cancel.

Of special interest for our later discussion is integer \( s > 0 \). In this case the remaining pole-zero pattern is given by

\[ 2 \beta_i = \alpha_+ (l_i + 1) - \alpha_- j_i \quad \text{(poles)}, \] (29)

\[ 2 \beta_i = -\alpha_+ l_i - \alpha_- j_i \quad \text{(zeros)}, \] (30)

with \( \text{integer } s \geq 0; \; j_i = 0, 1, \ldots, s - 1; \; l_i = 0, 1, \ldots \infty. \) (31)

In contrast to the general case the pole positions in \( \beta_i \) are bounded from below.

Such boundedness is achieved also for \( s > 0 \) with

\[ \alpha_- s = \alpha_+ n + \alpha_- ([s] - m); \; 1 \leq n \leq m \leq [s]. \] (32)

Special care is needed for \( F_0 \). Due to (25) for all \( s \) we are just sitting on the \( j_0 = l_0 = 1 \) zero of (27). On the other hand, for positive integer \( s \), where we start our continuation, due to pole-zero cancellation \( F_0 \) is finite. A finite \( F_0 \) emerges also in the more general situation

\[ \alpha_- s = \alpha_+ (l_0 - 1) + \alpha_- (j_0 - 1), \] (33)

with \( l_0, j_0 \) fulfilling (28).

In all cases where pole-zero cancellation does not guarantee finite \( F_0 \) we treat it using (13) as

\[ F_0 = -\left(\frac{\alpha^2}{2}\right)^s \frac{\pi}{\Gamma(1 + s)} \exp \left[ f(1 - \frac{\alpha^2}{2} s, \frac{\alpha^2}{2} |s|) - f(1 + \frac{\alpha^2}{2}, \frac{\alpha^2}{2} |s|) \right], \] (34)

where a factor \( \Gamma(0) \sin \pi s \) has been set equal to 1. Of course, this procedure still has to be justified by comparing the \( s \)-asymptotics of the resulting expression with that of the original functional integral. (For \( d = 1 \) see ref [12].)

The analytic structure of \( A_3 \) beyond the trivial \( s \)-dependence in \( F_0 \) and the first factors in (22) is determined by that of \( \prod_{i=1}^{3} F_i \). Eliminating \( s \) in (26) by using (23) we get finally

\[ 2 \beta_i = -\alpha_- (j_i - 1) - \alpha_+ (l_i - 1) \quad \text{(poles)}, \] (35)

\[ 2 \beta_i = \alpha_- j_i + \alpha_+ l_i \quad \text{(zeros)}, \] (36)

with \( j_i, l_i \) respecting (28).

In contrast to the \( d = 1 \) case, which we are going to discuss in the next section, for general \( d > 1 \) the pole structure does not factorize into leg poles. An attempt to interpret the pole structure will be made in section 5.
4 The case $d = 1$

For $d > 1$ the three $\beta_i$'s are independent. For $d = 1$ there remains no angular degree of freedom in $k$-space. The defining equation (36) reduces to

$$\beta_i = \frac{Q}{2} - (k_i - \frac{P}{2})\epsilon_i; \quad \epsilon_i = \text{sign}(k_i - \frac{P}{2}) \quad .$$

(37)

Then momentum conservation $\sum k_i = P$ induces constraints among the $\beta_i$'s [7]. The $\epsilon_i$ must be either all equal to one another but opposite to $\text{sign}(P)$ or of the type $(++-)$ or $(-+-)$: One finds in the first case for $i=1,2,3$

$$2\beta_i = -2\beta_i + \frac{3Q + P}{2} \quad .$$

(38)

For $(+++)$ one gets instead

$$2\beta_1 = 2\beta_2 - \frac{Q + P}{2}, \quad 2\beta_2 = 2\beta_1 - \frac{Q + P}{2}, \quad 2\beta_3 = \frac{Q + P}{2} \quad .$$

(39)

Finally, for the case $(-+-)$ one has to replace in (39) $P$ by $-P$.

In all cases the position of any pole or zero of $\prod_{i=1}^{3} F_i$ becomes a function of a single $\beta_i$. To be more explicit, we consider the situation studied in detail in [7], i.e. $(++-)\) and $P < 0$. Then $2\alpha = Q + P$ and using eqs. (13),(14),(24) and (34) we get from (22)

$$A_3 = -\pi \delta(\sum k_i - P)\frac{1}{\alpha}\left(-\frac{\alpha^2}{16}\Delta(\frac{\alpha^2}{2})\right)^s \prod_{i=1}^{3} \Delta(m_i) \quad ,$$

(40)

with $\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \quad m_i = (\beta_i^2 - k_i^2)/2$. This coincides with [7] up to a trivial factor $\pi(-\frac{\alpha^2}{2})^s$.

5 Four point function

The calculation of the general 4-point function for $d > 1$ requires integrals not available in [7, 13]. Therefore, as a first step, we consider

$$A_4(k_1, k_2, k_3, k_4 = 0) = \frac{d}{d\mu^2}A_3(k_1, k_2, k_3) = \frac{s}{\mu^2}A_3(k_1, k_2, k_3).$$

(41)

This seems to be still sufficient to explore the spectrum of the theory by analyzing the singularities e.g. in the (1,2)-channel. Furthermore, since [7] together with the momentum conservation implies for any permutation $(i, j, l)$ of (1,2,3)

$$2k_ik_j = (\beta_l - \frac{Q}{2})^2 - (\beta_i - \frac{Q}{2})^2 - (\beta_j - \frac{Q}{2})^2 + m^2 + \frac{P^2}{4} \quad .$$

(42)
any candidates for Mandelstam variables of the special 4-point function (41) can be expressed in terms of $\beta_1, \beta_2, \beta_3$. If one defines with a $d$-dimensional picture in mind

$$t_{12} = (k_1 + k_2 - P/2)^2 = (k_3 - P/2)^2 = t_{34}$$

etc., the pole positions (35) are determined by

$$\sqrt{t_{23} + m^2} - \sqrt{t_{13} + m^2} - \sqrt{t_{12} + m^2} = \alpha_- (j - \frac{1}{2}) + \alpha_+ (l - \frac{1}{2}).$$

(44)

To get a more familiar picture we now use the correspondence to $(d + 1)$-dimensional critical strings [7]. For $\mu^2 = 0$ the amplitudes are equal to that of a $(d + 1)$-dimensional critical string theory with a conservation law

$$\sum k_i = P, \quad \sum \beta_i = Q$$

and eq. (45) can be interpreted as a mass shell condition

$$(\beta - \frac{Q}{2})^2 - (k - \frac{P}{2})^2 = m^2.$$  

(45)

The corresponding Mandelstam variables are

$$T_{ij}^0 = (\beta_i + \beta_j - \frac{Q}{2})^2 - (k_i + k_j - \frac{P}{2})^2.$$ 

Also for $\mu^2 \neq 0$ but integer $s_4 \geq 0$, with

$$\alpha s_4 = Q - \sum_{i=1}^{4} \beta_i,$$

(46)

a $(d + 1)$-dimensional interpretation is possible. The 4-point function then turns into a $(4 + s_4)$-point function with $k_5 = k_6 = \ldots = k_{4+s_4} = 0$. Motivated by these special cases we define in general

$$T_{ij} = (\beta_i + \beta_j - \frac{Q - \alpha s_4}{2})^2 - (k_i + k_j - \frac{P}{2})^2.$$ 

(47)

This ensures $T_{12} = T_{34}$ etc. and fulfills

$$T_{12} + T_{13} + T_{14} = 4m^2 - \frac{1}{4}(Q^2 - P^2 + \alpha s_4 (\alpha s_4 + 2Q)).$$ 

(48)

Applying this general definition to our special case (41) we get $(k_4 = 0, \beta_4 = \alpha, s_4 = s - 1)$

$$T_{i4} = m^2 - \alpha (1 + \frac{s_4}{2})(2\beta_4 + \frac{\alpha s_4}{2}); \quad i = 1, 2, 3.$$ 

(49)

This gives a transcription of the pole structure (35) of the 3-point function into poles in the Mandelstam variables (47) of the (special) 4-point function. At least under the
constraint of fixed $T_{14} + T_{24} + T_{34}$, i.e. fixed $s_4 = s - 1$ the poles (33) turn directly into poles in the individual Mandelstam variables. The resulting spectrum is for general $s$ unbounded in both directions.

For the particular case of fixed integer $s \geq 0$, due to the pole-zero cancellation leading to (29), (30) we find poles in e.g. $T_{12} = T_{34}$ only for

$$T_{34} = m^2 + (2 + s_4)(l - \frac{\alpha^2}{2}(j - \frac{s_4}{2})).$$

(50)

These poles can be seen to be due to the standard poles at $m^2 + 2l$, integer $l \geq 0$ in those channels of the $(4 + s_4)$-point amplitude which are constituted by the momenta $k_1, k_2$ and $j$ times $k = 0$. The pole spectrum in $T_{34}$ is bounded from below also for values of $s$ fulfilling (32). It is remarkable that than in addition $F_0$ is well defined without any modification.

6 Conclusions

Since the continuation from a discrete set of points is mathematically ambiguous our procedure still has to be proven to yield the correct solution for the physical problem under discussion. This can be done by comparing the $|s| \to \infty$ asymptotics with that of the original functional integral. The corresponding problem in one dimensional target space has been solved in ref.[12]. Unfortunately the technique of that paper cannot directly be applied here as it depends essentially on the discreteness of the momentum spectrum in minimal models.

Nevertheless we are confident to have presented a correct continuation procedure. First for $d = 1$ it reduces to that of refs. [3], [7], [12] and second, the resulting analytical structure is already completely determined by the functional equations for $f$ and the absence of singularities for $Re (a, b, s) \geq 0$.

The interpretation of the spectrum is still not completely clear. While the pole structure for integer $s \geq 0$ fits into the $(d + 1)$-dimensional critical string picture well known from the case of vanishing Liouville mass $\mu$, for general $s$ the pole-zero cancellation enabling just this interpretation is removed and an unacceptable spectrum emerges. This may either signal a hidden disease of our model or only a breakdown of its $(d + 1)$-dimensional interpretation. Favouring the last alternative one has to analyse eq. (44) and to think about mass shell conditions in $d$-dimensional space induced by quantization of the $\beta_i$ values due to the Liouville dynamics.

A last comment concerns the values of $s$ respecting (32) at which the spectrum is at least bounded from below. These noninteger values of $s$ belong to the set which can be screened by insertions of screening operators constructed with $\alpha_-$ and $\alpha_+$ [6].

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