Fuzzy $CP^{(n|m)}$ as a quantum superspace

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ABSTRACT

The ‘Chern-Simons Quantum Mechanics’ of a particle on $CP^{(n|m)}$ is shown to yield the fuzzy descriptions of these superspaces, for which we construct the non-(anti)commuting position operators. For a particle on the supersphere $CP^{(1|1)} \cong SU(2|1)/U(1|1)$, the particle’s wave-function at fuzziness level $2s$ is shown to be a degenerate irrep of $SU(2|1)$ describing a supermultiplet of $SU(2)$ spins $(s - \frac{1}{2}, s)$. 

* To appear in Symmetries in Gravity and Field Theory, proceedings of Conferencia Homenaje en el 60 cumpleaños de José Adolfo de Azcárraga, June 9-11, 2003, Salamanca, Spain.

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1 Introduction

Symplectic geometry provides a general framework for classical mechanics in which classical observables are functions on a symplectic manifold $M$ (phase space) and their time dependence is determined by the choice of one of these observables as the Hamiltonian. In the quantum theory, the choice of an observable as the Hamiltonian is again what determines the time evolution of all other observables, but the nature of these observables is again independent of the choice of Hamiltonian. If one wishes to focus on the nature of the correspondence between quantum and classical observables it is therefore possible, and convenient, to ignore the Hamiltonian. In this case the classical action is determined by the symplectic 2-form $F$ on $M$. As this form is closed we can write it locally as $F = dA$ for some abelian gauge potential $A$ on $M$. The action is then

$$I = s \int_w A$$

where the integral is over some 1-cycle $w$ (the worldline) in $M$, and $s$ is a ‘coupling constant’. It would be possible to absorb $s$ into the definition of $A$ but any multiple of $F$ determines a volume form on $M$ and hence, for compact $M$, a total volume. We normalize $F$ by choosing $F/(4\pi\hbar)$ to give $M$ a unit total volume. The ‘actual’ volume of phase space is thus proportional to an appropriate power of $s$. Note that although $M$ has a definite volume, it has no specified metric. The above action defines a ‘topological’ sigma-model with ‘target space’ $M$ that has been called ‘Chern-Simons-Quantum-Mechanics’ (CSQM) [1, 2]. We shall follow this usage.

The symmetry group $G$ of the CSQM action (1.1) is the infinite-dimensional group of symplectic diffeomorphisms of $M$. To every basis vector in the Lie algebra $\mathcal{G}$ of $G$ there is an associated ‘moment map’ from $M$ to $\mathcal{G}^*$, the vector space dual to $\mathcal{G}$. These are the Noether charges, and the Poisson bracket turns the vector space they span into a Lie algebra, which is either homomorphic to $\mathcal{G}$ or to a central extension of it. For present purposes, the important fact about the space $\mathcal{G}^*$ is that it carries an irreducible representation $R$ of $G$. This representation decomposes into an infinite sum of irreps of the maximal finite-dimensional subgroup $G_0$ of $G$. Consider, for example, $M = S^2$. In this case $G_0 = SO(3)$, and the representation of $SO(3)$ carried by $\mathcal{G}^*$ is

$$R = 3 \oplus 5 \oplus 7 \oplus \cdots \quad (1.2)$$

This is the $SO(3)$ representation content in the expansion of a non-trivial function on the 2-sphere in spherical harmonics. The subspace of $\mathcal{G}^*$ spanned by the triplet of Noether charges is the co-algebra of $G_0$ on which $G_0$ acts via its co-adjoint action. The co-adjoint orbits of $G_0$ in this space are 2-spheres homothetic to $M$.

In the quantum theory the Noether charges become Hermitian operators acting on a Hilbert space. They again span a Lie algebra, which we will call $\mathcal{G}_s^*$, but the Lie bracket is now $-i$ times the commutator. This Lie algebra must be finite-dimensional if $M$ is compact because a compact phase space implies a finite-dimensional Hilbert space and hence a finite number of linearly-independent observables. For present purposes, the important fact about $\mathcal{G}_s^*$ is that it carries a representation $R_s$ of $G_0$
that is a truncation of the infinite sum of irreps carried by $G^\ast$. For example, if $M = S^2$ then one can show that $2s$ must be an integer and the Hilbert space is a $(2s + 1)$-dimensional carrier space for the spin $s$ irrep of $SU(2)$ [3]. Note that the dimension of the Hilbert space grows linearly with $s$ as one would expect from the fact that $\hbar$ appears in $I/\hbar$ only through the combination $s/\hbar$. The non trivial quantum observables in this case are the Hermitian $(2s + 1) \times (2s + 1)$ matrices, excepting the identity matrix. They span a vector space of dimension $2s(2s + 1)$ on which $SU(2)$, and hence $SO(3) \cong SU(2)/\mathbb{Z}_2$, acts reducibly according to the representation

$$R_s = 3 \oplus \cdots \oplus (4s + 1).$$

(1.3)

The subspace of $G^\ast_s$ on which the triplet of $SU(2)$ acts irreducibly plays a special role because the hermitian operators that span it are position operators for a 'fuzzy' sphere at fuzziness level $2s$ [4]. These operators generate the associative algebra $\mathcal{A}_{(2s+1)}$ of complex $(2s + 1) \times (2s + 1)$ matrices, which may be considered as a finite-dimensional, but non-commutative, approximation to the infinite-dimensional commutative algebra (with respect to ordinary multiplication) of functions on $S^2$.

If one views the 2-sphere as the space $\mathbb{C}P^1 \cong SU(2)/U(1)$ then the generalization to $\mathbb{C}P^n \cong SU(n + 1)/U(n)$ suggests itself. These are examples of Kähler manifolds, which provide a convenient supply of symplectic manifolds because the Kähler 2-form is a symplectic form. In this case, the gauge potential $A$ is determined, up to a gauge transformation, in terms of a Kähler potential $K$ via the formula

$$A = -idz^i \frac{\partial K}{\partial \bar{z}^i} + c.c. \quad (1.4)$$

where $z^i (i = 1, \ldots, n)$ are $n$ complex local coordinates. For $\mathbb{C}P^n$ we may choose

$$K = \log (1 + \bar{z} \cdot z),$$

(1.5)

and the corresponding Kähler 2-form is

$$F = -\frac{2id\bar{z} \wedge d\bar{z}^i}{(1 + \bar{z} \cdot z)} + \frac{2i(d\bar{z} \cdot z) \wedge (\bar{z} \cdot dz)}{(1 + \bar{z} \cdot z)^2}. \quad (1.6)$$

This 2-form has a manifest, linearly realized, $U(n)$ invariance, but it is also invariant under the non-linear (and analytic) infinitesimal 'translations'

$$\delta z^i = a^i + (\bar{a} \cdot z) z^i \quad (1.7)$$

for $n$ complex parameters $a^i$, as follows from the fact that

$$\delta A = d(\bar{a} \cdot z + a \cdot \bar{z}). \quad (1.8)$$

These transformations close to yield a realization of the Lie algebra $su(n + 1)$ that exponentiates to a transitive action of $SU(n + 1)$ on $\mathbb{C}P^n$. 

2
The $SU(n+1)$ invariance of $F$ implies the existence, via Noether’s theorem, of $n(n+2)$ functions that span the Lie algebra $su(n+1)$ with respect to the Poisson bracket, which is

$$\{F,G\}_{PB} = \frac{i}{2s} (1 + \bar{z} \cdot z) \left( \delta^i_j + \bar{z}^i \bar{z}^j \right) \left( \frac{\partial F}{\partial z^i} \frac{\partial G}{\partial \bar{z}^j} - \frac{\partial F}{\partial \bar{z}^j} \frac{\partial G}{\partial z^i} \right)$$

(1.9)

for any two functions $F, G$ on $\mathbb{C}P^n$. The vector space $\mathcal{G}^*$ of all non-trivial functions on $\mathbb{C}P^n$ is an infinite-dimensional Lie algebra with respect to this bracket, with $su(n+1)$ as its maximal finite-dimensional subalgebra. Its $SU(n+1)$ representation content is

$$R = n(n+2) \oplus \frac{1}{4} n(n+4)(n+1)^2 \oplus \cdots.$$  

(1.10)

This is a generalization of (1.2). In the quantum theory this series is truncated to

$$R_s = n(n+2) \oplus \frac{1}{4} n(n+4)(n+1)^2 \oplus \cdots \oplus \frac{(n+2s-1)!^2(n+4s)}{(n-1)!n!(2s)!^2},$$

(1.11)

which generalizes (1.3). The $n(n+2)$ hermitian matrices form an irrep of $SU(n+1)$ of dimension $(n+2s)!/n!(2s)!$. These generate an algebra of complex matrices that defines the fuzzy versions of $\mathbb{C}P^n$ [5, 6, 7, 8, 9].

Just as a fuzzy symplectic manifold can be obtained from a CSQM model model with a smooth symplectic manifold as its target space, so a fuzzy orthosymplectic supermanifold can be obtained from a CSQM model with an orthosymplectic target superspace. Again, Kähler supermanifolds provide a convenient supply of orthosymplectic ones, the simplest being the superspaces $\mathbb{C}^{(n|m)}$, for which the real and imaginary parts of the $m$ complex Grassmann-odd coordinates become elements of a Clifford algebra in the quantum theory. This idea can be traced back to the 1959 papers of Martin on the mechanics of Grassmann variables [10]. It also underlies the 1981 ‘quantum superspace’ proposal of Brink and Schwarz [11]. A systematic study of non-(anti)commutative deformations of Minkowski superspace has been undertaken in recent years (e.g. [12, 13]) and recent work on a string theory realization (e.g. [14, 15, 16]) has initiated a revival of interest in non-(anti)commutative superspaces.

In this paper we consider the class of coset superspaces

$$\mathbb{C}P^{(n|m)} \cong SU(n+1|m)/U(n|m)$$

(1.12)

which all have $\mathbb{C}P^n$ ‘body’ and are both homogeneous and symmetric. They are also Kähler; in complex coordinates

$$Z^M = (z^i, \xi^\alpha) \quad (i = 1, \ldots, n; \alpha = 1, \ldots, m),$$

(1.13)

with complex conjugates $\bar{Z}_M = (\bar{z}^i, \bar{\xi}^\alpha)$, the Kähler potential is

$$K = \log \left( 1 + \bar{Z}_M Z^M \right)$$

(1.14)
Note that $K$ is manifestly $U(n|m)$ invariant. The corresponding Kähler metric $ds^2 = dZ^M dZ_N \partial^N \partial_M K$ (which, however, is not used by the CSQM model) is invariant under the larger supergroup $SU(n+1|m)$, which is the maximal finite-dimensional sub-supergroup of the invariance supergroup of the Kähler 2-form. The principal new results of this paper concern the definition and study of the fuzzy, non(anti)commutative, versions of these superspaces. As we do this using CSQM methods, fuzzy $CP^{(n|m)}$ emerges as a ‘quantum superspace’ in the sense of Brink and Schwarz. One of our results is a construction of the non-(anti)commuting position operators that generate the super-matrix algebra that defines the fuzzy supergeometry of $CP^{(n|m)}$. As we shall see, there is no real distinction between position and momentum operators in CSQM, but the position operator interpretation is natural in the semi-classical, large $s$, limit.

By an appropriate rescaling, in which one focuses only on the local properties of the target superspace, one can recover $\mathbb{C}^{(n|m)}$ from $CP^{(n|m)}$, as we show in the final section of this paper. Here we should raise a point that might have occurred to the alert reader. Darboux’s theorem states that any two symplectic manifolds of the same dimension are locally diffeomorphic, and the same applies to orthosymplectic superspaces as we shall illustrate in the final section. Thus, if we are concerned only with local properties then a coordinate transformation suffices to take us from the CSQM model with target space $CP^{(n|m)}$ to one with target space $\mathbb{C}^{(n|m)}$; there is no need to take a limit! The natural Kähler metric on $CP^{(n|m)}$ is, of course, quite different from the flat Kähler metric on $\mathbb{C}^{(n|m)}$, but this is just a reflection of the topological nature of the CSQM action: it does not require a choice of metric on the target space. Under these circumstances one might wonder why we do not begin by choosing local coordinates for which the symplectic 2-form, and hence the CSQM action, takes the simplest form. We will answer this point in more detail later, but the main reason is that the global properties of $CP^{(n|m)}$ are most simply taken into account by viewing it as a homogeneous coset superspace for the supergroup $SU(n+1|m)$. An additional advantage is that the $SU(n+1|m)$ transformations are then analytic. This feature is actually essential to the applicability of our method, which was applied to the $n = 0$ case in [17]. The method has a history of relevance to this conference, and we take the opportunity to comment on it. We then illustrate it with the $CP^n$ case before turning to $CP^{(n|m)}$.

The topological nature of the CSQM model with target space $CP^{(n|m)}$ means that there is no dynamics; one can consider the model ‘solved’ when the nature of the Hilbert space is known. For the models with target superspace $CP^{(n|m)}$, this essentially amounts to a determination of the $SU(n+1|m)$ representation content of the Hilbert space. This was worked out for the special case of $n = 0$ in [17]. Here we consider the $n = 1$ case; this is of particular interest because the $CP^{(1|m)}$ supermanifolds all have a $CP^1 \cong S^2$ ‘body’ and can therefore be viewed as ‘$m$-extended superspheres’. The simplest of these, with real dimension $(2|2)$ is

$$CP^{(1|1)} \cong SU(2|1)/U(1|1).$$

We call this the ‘supersphere’ because it is the super-extension of the sphere to a
homogeneous symmetric Kähler supermanifold with $SU(2)/U(1) \cong S^2$ `body’ of minimal total dimension\(^1\). We will show that the ‘Hilbert’ space of a particle on a supersphere at fuzziness level $2s$ is a degenerate irrep of $SU(2|1)$ [24] that decomposes with respect to $SU(2)$ into a supermultiplet of $SU(2)$ spins $(s - \frac{1}{2}, s)$.

2 Analytic quantization of Kähler models

In local complex coordinates $z^i$ for Kähler target space $M$, the Lagrangian of the CSQM model with action (1.1) is

$$L = sz^i A_i + c.c.$$  \hfill (2.1)

where $A_i$ are the complex components of the one-form $A$, given in terms of the Kähler potential $K$ by (1.4). One could pass to the quantum theory by the usual Poisson bracket to commutator prescription, as outlined in the introduction, but here we wish to promote another method with some advantages that will hopefully become apparent as we proceed. This method takes as its starting point the alternative Lagrangian

$$L = \left[ p_i \dot{z}^i + \ell^i \varphi_i \right] + c.c.$$  \hfill (2.2)

where $p_i$ are complex momentum variables canonically conjugate to the position space variables $z^i$, and $\ell^i$ are complex Lagrange multipliers that impose the complex constraints\(^2\) $\varphi_i \approx 0$, with

$$\varphi_i = p_i - s A_i.$$  \hfill (2.3)

Solving the constraints for $p_i$ returns us to the original Lagrangian (2.1), with phase space $M$, but the phase space of our alternative Lagrangian is, nominally, the cotangent bundle of $M$, so Poisson brackets of position variables now vanish. However, the constraints are `second-class’, in Dirac’s terminology, and the standard way of dealing with this is to replace Poisson brackets by Dirac brackets. As these are, by construction, equivalent to the original Poisson brackets derived from (2.1), it might seem that nothing has been gained.

However, there is an alternative way of dealing with second-class constraints in these models. It arises from the observation that the complex functions $\varphi_i$ are in involution, as are their complex conjugates; it is only when we consider the two sets together that the constraints become second class. This suggests the possibility of regarding the constraints $\varphi_i \approx 0$ as gauge fixing conditions for gauge invariances generated by the functions $\bar{\varphi}^i$. If we now step back to the un-gauge-fixed theory then

\(^1\)The term ‘supersphere’ has been used previously for the coset superspace $USp(1|2)/U(1)$ [18, 19, 20, 21, 22]. Although this superspace is often stated to have real dimension (2|2), its ‘reality’ is defined with respect to a ‘pseudoconjugation’; see e.g. [23] for details. With respect to standard complex conjugation, it actually has real dimension (2|4) since spinors of $USp(2) \cong SU(2)$ span a vector space of dimension 4 over the reals.

\(^2\)The symbol $\approx$ indicates ‘weak’ equality in Dirac’s sense.
Dirac tells us that we should first quantize *without constraint* by setting
\[ p_i = -i \frac{\partial}{\partial z^i}, \quad \bar{p}^i = -i \frac{\partial}{\partial \bar{z}_i}, \] (2.4)
which means that that the classical constraint functions become the quantum operators
\[ \varphi_i = -i \left[ \frac{\partial}{\partial z^i} - s \frac{\partial K}{\partial z^i} \right], \quad \bar{\varphi}^i = -i \left[ \frac{\partial}{\partial \bar{z}_i} + s \frac{\partial K}{\partial \bar{z}_i} \right]. \] (2.5)
We now ignore the \( \varphi_i \) constraints and treat their complex conjugates as first class by imposing the physical-state conditions
\[ \bar{\varphi}^i |\Psi\rangle = 0, \quad (i = 1, \ldots, n). \] (2.6)
This restricts physical wavefunctions to take the form
\[ \Psi(z, \bar{z}) = e^{-sK(z, \bar{z})} \Phi(z) \] (2.7)
for holomorphic function \( \Phi \) of the \( n \) complex coordinates of \( M \), which we shall call the ‘reduced’ wavefunction. As the Kähler potential \( K \) is not globally defined (in general), this result shows that \( \Psi \) is not a true function.\(^3\)

A Hilbert space norm for \( \Psi \) will take the form
\[ ||\Psi||^2 = \int d\mu |\Psi|^2 = \int d\mu e^{-2sK} |\Phi(z)|^2 \] (2.8)
where the integral is over \( M \) and \( d\mu \) is a measure on \( M \). A natural measure is provided by the volume form determined by the appropriate exterior power of the Kähler form \( F \). In particular, this defines a \( G_0 \)-invariant norm. Given this norm, one must then further restrict the physical states to be normalizable, and this condition ensures that the physical Hilbert space is finite-dimensional.

For fermionic constraints, this alternative method of dealing with second-class constraints can be traced back to the 1976 papers of Casalbuoni [26] and papers in the early 1980s of Azcárraga et al. [27, 28] and Lusanna [29]. A clear statement of it can be found, again for fermionic constraints, in a 1986 paper of de Azcárraga and Lukierski [30], who called it ‘Gupta-Bleuler’ quantization by analogy with the procedure of that name for covariant quantization of electrodynamics.\(^4\) It was also called Gupta-Bleuler quantization in the 1991 book of Balachandran et al. [31], where it is explained for particle mechanics models with bosonic constraints. The justification for this method that we have sketched above arose in independent work

\(^3\)It is instead, see e.g. [25], a section of a line bundle over \( M \) with curvature 2-form \( sF/(2\pi) \), which must be integral for consistency. Given our normalization of \( F \), this requires requires \( 2s \) to be an integer.

\(^4\)Noting that the Lorentz gauge condition cannot be consistently imposed as a physical state condition, Gupta and Bleuler suggested that it be separated into its positive and negative frequency parts (of which \( \varphi \) and \( \bar{\varphi} \) are analogs) and that the positive frequency part be imposed as the physical state condition.
on general models with bosonic second-class constraints that can be separated into
two sets of real constraints, each in involution \([32, 33]\). In this context the method
has become known as the method of ‘gauge-unfixing’. When, in a recent paper with
Pashnev \([17]\), we advocated the holomorphic/anti-holomorphic variant of this gauge-
unfixing procedure we were initially unaware of its earlier use and chose to call it the
method of ‘analytic quantization’ because the physical state conditions are essentially
analyticity conditions.

Let us illustrate the method for \(M = \mathbb{C}P^n\). Given the Kähler potential of (1.5),
we deduce that physical wavefunctions have the form

\[
\Psi = (1 + \bar{z} \cdot z)^{-s} \Phi(z)
\]

(2.9)

for holomorphic reduced wavefunction \(\Phi\). The Kähler 2-form (1.6) determines the
\(SU(n+1)\)-invariant measure

\[
d\mu = \prod_{i=1}^{n} (dz_i d\bar{z}_i) (1 + \bar{z} \cdot z)^{-(n+1)},
\]

(2.10)

so the Hilbert space norm is

\[
||\Psi||^2 = \prod_{i=1}^{n} \int dz_i d\bar{z}_i (1 + \bar{z} \cdot z)^{-(n+1+2s)} |\Phi(z)|^2
\]

(2.11)

where the integral is over all values of \(z\). Normalizability of \(\Psi\) thus requires \(\Phi\) to be a
polynomial of maximum degree \(2s\), from which it follows that \(2s\) must be a positive
integer. The \((n+2s)!/[n!(2s)!]\) coefficients of \(\Phi\) span the irrep of \(SU(n+1)\) formed
from the \((2s)\)-fold symmetric tensor product of the fundamental \((n+1)\) irrep.

3 Position operators for fuzzy \(\mathbb{C}P^n\)

Within the ‘analytic’, or ‘Gupta-Bleuler’, method of quantization, we have a ‘large’
Hilbert space in which position operators commute, so that the target space \(M\) retains
its status as a smooth, classical, configuration space. However, these ‘naive’ position
operators do not act on the physical subspace of Hilbert space because they take
physical states into unphysical, or un-normalized, states. Thus, the question arises
of how to construct the physical, non-commuting, position space operators. We will
first address this issue for \(M = \mathbb{C}P^n\). In doing so it is useful to first consider how
\(\mathbb{C}P^n\) can be embedded in a Euclidean space.

Any group acts naturally on the dual of its Lie algebra, and the orbits of this
coadjoint action are symplectic manifolds. Let \(t_A\) span the Lie algebra \(su(n+1)\),
such that

\[
t_At_B = \frac{1}{n+1} \delta_{AB} + \frac{1}{\sqrt{2}} \left( d_{ABC}^C + if_{ABC}^C \right) t_C
\]

(3.1)

where \(f_{ABC}^C\) are the structure constants; \(d_{ABC}^D \equiv d_{ABC}^D \delta_{CD}\) is the totally symmetric
third-rank invariant tensor of \(SU(n+1)\). The Kronecker delta \(\delta_{AB}\) is a Euclidean
metric, in cartesian coordinates $X^A$, on the $n(n+2)$-dimensional vector space dual to $su(n+1)$. The submanifold of this space defined by the constraints [8]

$$X^B X^A \delta_{AB} = \frac{n}{n+1}, \quad X^B X^A d_{AB}^C = \frac{\sqrt{2}(n-1)}{(n+1)} X^C$$

(3.2)
is a co-adjoint orbit of $SU(n+1)$ that is isomorphic to $CP^n$. Functions on $CP^n$ are therefore functions of the cartesian coordinates $X^A$ subject to these constraints. Thus, any (scalar) function $\phi$ has an expansion that starts as

$$\phi = \phi_0 + \phi_A X^A + \phi_{AB} X^A X^B + \ldots.$$  

(3.3)

The second term is the $n(n+2)$ (adjoint) irrep of $SU(n+1)$, so these functions span its Lie algebra with respect to the Poisson bracket (1.9). The third term might appear to contain all irreps appearing in the symmetric product of two adjoints, but only the $\frac{1}{2}n(n+4)(n+1)^2$ irrep survives the constraints (3.2).

We now turn to the quantum theory. The ‘naive’ position operators are just multiplication by the complex coordinates $z^i$ and $\bar{z}_i$. However, these operators do not act on the physical Hilbert space. Most obviously, multiplication by $\bar{z}_i$ is not physical because this operation fails to commute with $\bar{\phi}^i$. Multiplication by $z^i$ does commute with $\bar{\phi}^i$, and takes a holomorphic reduced wavefunction $\Phi$ to the new holomorphic reduced wavefunction $z^i \Phi$, but if $\Phi$ is a polynomial of maximal degree then $z^i \Phi$ will not be normalizable. Thus, multiplication by $z^i$ is not physical either. We therefore need to modify the ‘naive’ position operators. As $\hbar$ occurs only through the combination $\hbar/s$, and we choose units such that $\hbar = 1$, the dimensionless number $1/s$ effectively plays the role of $\hbar$. We thus expect quantum corrections to be of order $1/s$. As the classical coordinates commute, we thus seek physical position operators $\hat{z}_i$ of the form

$$\hat{z}_i = z^i + O(1/s).$$

(3.4)

The quantum correction must be such that these operators, and their hermitian conjugates (defined via the Hilbert space norm) commute weakly with the operators $\bar{\phi}_i$ that annihilate physical states. This condition has the unique solution:

$$\hat{z}_i = z^i - \frac{1}{s} \left[ \frac{\partial}{\partial \bar{z}_i} + z^i \left( \bar{z} \cdot \frac{\partial}{\partial z} \right) \right], \quad \hat{\bar{z}}_i = \bar{z}_i + \frac{1}{s} \left[ \frac{\partial}{\partial z^i} + \bar{z}_i \left( z \cdot \frac{\partial}{\partial \bar{z}} \right) \right].$$

(3.5)

The effect of the physical position operators on the reduced wavefunction is

$$\hat{z}_i : \Phi \to s^{-1} u^i \Phi, \quad \hat{\bar{z}}_i : \Phi \to s^{-1} \bar{u}_i \Phi$$

(3.6)

where

$$u^i = z^i \left( 2s - z \cdot \frac{\partial}{\partial z} \right), \quad \bar{u}_i = \frac{\partial}{\partial \bar{z}_i}.$$  

(3.7)

from which it follows immediately that a normalized reduced wavefunction is taken to another normalized reduced wavefunction. Acting on the coordinates $z^i$, the operators $u, \bar{u}$ generate the infinitesimal non-linear translations (1.7). In other words,
the operators \( s\hat{z}^i \) are momentum operators for a particle on \( CP^n \). There is no real distinction between the position and momentum operators. For large \( s \) the position operator interpretation is the natural one, because of (3.4), whereas the momentum operator interpretation is natural for small \( s \). Thus, CSQM models exhibit a kind of duality between the semi-classical and ultra-quantum regimes in which position and momentum are interchanged.

Note that \( \hat{z}^i \) and \( \hat{\bar{z}}^j \) commute, but

\[
[\hat{z}^i, \hat{\bar{z}}^j] = \frac{1}{s^2} \left[ \hat{J}_{ij} + \left( \frac{n+1}{n} \right) \delta^i_j \hat{J}_0 \right] \tag{3.8}
\]

where

\[
\hat{J}_{ij} = z^i \partial / \partial z^j - \bar{z}_j \partial / \partial \bar{z}_i - \frac{1}{n} \left( z \cdot \partial / \partial z - \bar{z} \cdot \partial / \partial \bar{z} \right) \delta^i_j \tag{3.9}
\]

is the generator of \( SU(n) \), and

\[
\hat{J}_0 = z \cdot \partial / \partial z - \bar{z} \cdot \partial / \partial \bar{z} - 2s \frac{n}{n+1} \tag{3.10}
\]

is the \( U(1) \) generator. As expected, the physical quantum position operators are non-commutative, but the non-commutativity disappears in the classical limit \( s \to \infty \). The shift in the \( U(1) \) charge by a term proportional to \( s \) is due to the fact that the Lagrangian is not invariant under (1.7) but changes by a total derivative. This implies that the Hilbert space will carry a projective representation of \( SU(n+1) \) rather than a true representation\(^5\).

The commutation relation (3.8) has an analog for the operators \( u, \bar{u} \), acting on reduced wavefunctions. Specifically,

\[
[u^i_j, \bar{u}^j_k] = \hat{J}^i_j + \left( \frac{n+1}{n} \right) \delta^i_j \hat{J}_0 \tag{3.11}
\]

where

\[
\hat{J}^i_j = z^i \partial / \partial z^j - \frac{1}{n} \delta^i_j \left( z \cdot \partial / \partial z - \bar{z} \cdot \partial / \partial \bar{z} \right), \quad \hat{J}_0 = z \cdot \partial / \partial z - 2s \frac{n}{n+1} \tag{3.12}
\]

are the \( SU(n) \) and \( U(1) \) operators acting on reduced wavefunctions. The commutation relations of these operators with \( u, \bar{u} \) imply that the latter have \( U(1) \) charges 1, −1, and transform as the \( n, \bar{n} \) of \( SU(n) \), respectively. The same conclusion results, with more effort, from the commutation relations of \( \hat{J}^i_j \) and \( \hat{J}_0 \) with \( \hat{z}^i \) and \( \hat{\bar{z}}^i \). Thus, whether they act on the ‘full’ Hilbert space or the reduced Hilbert space, the physical position operators of a fuzzy \( CP^n \) are proportional to the ladder operators of a representation of \( SU(n+1) \).

This result has a classical counterpart that is related to our earlier description of \( CP^n \) as a co-adjoint orbit of \( SU(n+1) \). The functions on phase space corresponding

\[^{5}\text{We pass over the associated subtleties here, but note that this has been a major theme of Jose-Adolfo de Azcárraga’s work; we refer the reader to his bool with J-M Izquierdo [34].}\]
to the physical position operators $\hat{z}^i$ are

$$z^i - \frac{i}{s} \left[ \hat{p}^i + z^i (z \cdot p) \right] = \frac{2z^i}{1 + \bar{z} \cdot z} - \frac{i}{s} \left[ \hat{\varphi}^i + z^i (z \cdot \varphi) \right] \approx \frac{2z^i}{1 + \bar{z} \cdot z}.$$  

(3.13)

The functions corresponding to the rescaled quantum position operators $s \hat{z}^i$ are therefore

$$w^i = 2s \left( 1 + \bar{z} \cdot z \right)^{-1} z^i.$$  

(3.14)

The Poisson bracket (1.9) of $w^i$ with $w^j$ is zero but

$$i \{ w^i, \bar{w}_j \}_{PB} = J^i_j + \left( \frac{n + 1}{n} \right) \delta^i_j J_0$$  

(3.15)

where

$$J^i_j = \frac{2s}{1 + \bar{z} \cdot z} \left( z^i \bar{z}^j - \frac{1}{n} \delta^i_j \bar{z} \cdot z \right), \quad J_0 = \frac{2s}{n + 1} \left( \bar{z} \cdot z - \frac{n}{1 + \bar{z} \cdot z} \right).$$  

(3.16)

These are the classical generators of $U(n)$, with Poisson brackets

$$i \{ J^i_j, J^k_l \}_{PB} = (J^i_l \delta^k_j - J^k_l \delta^i_j), \quad \{ J^i_j, J^k_0 \}_{PB} = 0.$$  

(3.17)

In addition,

$$i \{ J^i_j, w^k \} = w^i \delta^k_j - \frac{1}{n} \delta^i_j w^k, \quad i \{ J_0, w^i \} = w^i,$$

$$i \{ J^i_j, \bar{w}_k \} = -\bar{w}_j \delta^i_k + \frac{1}{n} \delta^i_j \bar{w}_k, \quad i \{ J_0, \bar{w}_i \} = -\bar{w}_i.$$  

(3.18)

The functions ($w, \bar{w}, J, J_0$) thus span a Lie algebra, with the Poisson bracket as the algebra product. This is the Lie algebra of $SU(n+1)$. It follows that ($w, \bar{w}, J, J_0$) are linear combinations of the real embedding space coordinates $X^A$ that we introduced previously. The precise relation can be found by comparing the identity

$$w^i \bar{w}_i + \frac{1}{2} J^i_j J^j_i + \frac{n + 1}{2n} J_0^2 = \frac{2n}{n + 1} s^2$$  

(3.19)

with the first equation in (3.2).

4 Fuzzy $CP^{(n|m)}$

We now turn to the case of $CP^{(n|m)}$, for which the Kähler potential is given in (1.14). The corresponding CSQM Lagrangian is

$$L = -is \left( 1 + \bar{Z} \cdot Z \right)^{-1} \bar{Z} \cdot \dot{Z} + c.c.$$  

(4.1)

---

6This defines a transformation of the coordinates within the ‘large’ phase space. See [35] for a discussion of this point in non-commutative quantum mechanics, motivated by results in [36, 37].
and the Kähler 2-form is
\[ F = -\frac{2idZ^M \wedge d\bar{Z}_M}{(1 + Z \cdot Z)} + \frac{2i(d\bar{Z}^M \cdot Z)(\bar{Z} \cdot dZ)}{(1 + Z \cdot Z)^2}. \] (4.2)

Apart from the manifest $U(n|m)$ invariance, this 2-form is also invariant under the infinitesimal, and analytic, transformation
\[ \delta Z^M = a^M + (\bar{a} \cdot Z)Z^M \] (4.3)

where $a^M = (a^i, \epsilon^\alpha)$ for commuting parameters $a^i$ and anticommuting parameters $\epsilon^\alpha$. These transformations close to yield a realization of the Lie superalgebra $su(n+1|m)$ that exponentiates to the action of $SU(n+1|m)$ on $CP^{(n|m)}$. We can therefore identify these spaces as the coset superspaces $SU(n+1|m)/U(n|m)$, as claimed in the introduction.

To make this identification precise, we should clarify how the superalgebra $u(n|m)$ is embedded in $su(n+1|m)$. Let $Q^a_\alpha (a, b = 1, \ldots n + 1; \alpha, \beta = 1, \ldots m)$ be the odd $su(n+1|m)$ charges. Their anticommutation relations are
\[ \{Q^a_\alpha, Q^b_\beta\} = \delta^\alpha_\beta I^a_\alpha + \delta^\alpha_\beta T^a_\beta + \delta^a_\beta \delta^\beta_\alpha B \quad (I^a_\alpha = T^a_\alpha = 0) \] (4.4)

where $I^\alpha_\beta, T^a_\beta$ and $B$ are the generators of the mutually commuting $su(m), su(n+1)$ and $u(1)$ algebras, which form the even sub-algebra of $su(n+1|m)$. The odd charges have the following commutation relations with the $su(m)$ generators:
\[ [I^\alpha_\beta, Q^a_\gamma] = \delta^\alpha_\gamma Q^a_\beta - \frac{1}{m} \delta^\alpha_\beta Q^a_\gamma, \quad [I^\alpha_\beta, \bar{Q}^\gamma_a] = -\delta^\gamma_\beta \bar{Q}^\alpha_a + \frac{1}{m} \delta^\alpha_\beta \bar{Q}^\gamma_a. \] (4.5)

Their commutation relations with the $su(n+1)$ generators $T^a_\beta$ are similar, but with $1/m \to 1/n + 1$. The Jacobi identities uniquely fix the commutation relations with the $u(1)$ generator to be
\[ [B, Q^a_\beta] = \left(\frac{1}{m} - \frac{1}{n+1}\right) Q^a_\beta, \quad [B, \bar{Q}^\beta_a] = - \left(\frac{1}{m} - \frac{1}{n+1}\right) \bar{Q}^\beta_a. \] (4.6)

Now let us see how the subalgebra $u(n|m)$ of $su(n+1|m)$ is singled out. Let
\[ Q^a_\beta = (Q^a_\beta, S_\beta), \quad \bar{Q}^\beta_a = (\bar{Q}^\beta_a, \bar{S}^\beta_a) \quad (i = 1, \ldots n). \] (4.7)

Then the basic anticommutator of $su(n|m)$ is
\[ \{Q^i_\beta, Q^a_\gamma\} = \delta^i_\beta I^a_\gamma + \delta^i_\gamma T^i_\beta + \delta^i_\beta \delta^\gamma_\beta \tilde{B} \] (4.8)

where
\[ \tilde{T}^i_j = T^i_j + \frac{1}{n} \delta^i_j J, \quad \tilde{B} = \left( B - \frac{1}{n} J \right), \quad J \equiv T^{n+1}_{n+1}. \] (4.9)

The new ‘internal’ $U(1)$ charge $\tilde{B} \subset su(n|m)$ has the following commutation relations with the odd charges:
\[ [\tilde{B}, Q^i_\alpha] = \left(\frac{1}{m} - \frac{1}{n}\right) Q^i_\alpha. \] (4.10)
We could define $u(n|m)$ by adding the generator $B$ to those of $su(n|m)$, with which it forms a semi-direct sum, but this would clearly not lead to a coset superspace with body $SU(n + 1)/U(n) \cong CP^n$. Instead we add the generator $J = T^{n+1}_{n+1}$, which has the following comutation relations with the odd charges:

$$[J, Q^i_\alpha] = -\frac{1}{n+1} Q^i_\alpha.$$ (4.11)

This $u(n|m)$ subgroup is again a semi-direct sum of $u(1)$ with $su(n|m)$, and it yields a coset superspace with $CP^n$ body.

Superfields on $CP^{(n|m)}$ can be viewed as unitary irreps of $SU(n+1|m)$ induced from an unitary representation of its $U(n|m)$ subgroup, but we may limit ourselves to representations of $U(n|m)$ that are $SU(n|m)$ singlets with non-zero $U(1)$ charge $J$. Since the ‘matrix’ part of $\tilde{B}$ is trivial for such $SU(n|m)$ singlets, the natural $U(1)$ charge labelling the superfields is $J \subset su(n+1)$ (its ‘matrix’ part); as we shall see, this charge is just $2s$.

We now turn to the quantum theory of the CSQM model with $CP^{(n|m)}$ as target space. As for the $CP^n$ case, we expect the quantum position operators to take the form

$$\hat{Z}^M = Z^M + O(1/s)$$ (4.12)

with the quantum corrections such that both the operators $\hat{Z}^M$, and their Hermitian conjugates with respect to the norm (4.17), act on the physical subspace of ‘Hilbert space’. There is again a unique solution to this problem, and one finds that

$$\hat{Z}^M = Z^M - \frac{1}{s} \left[ \frac{\partial}{\partial Z^M} + Z^M \left( Z \cdot \frac{\partial}{\partial Z} \right) \right] \approx s^{-1} W^M$$

$$\hat{Z}_M = \bar{Z}_M + \frac{1}{s} \left[ \frac{\partial}{\partial \bar{Z}^M} + \bar{Z}_M \left( \bar{Z} \cdot \frac{\partial}{\partial \bar{Z}} \right) \right] \approx s^{-1} \bar{W}_M$$ (4.13)

where

$$W^M = 2s \left( 1 + \bar{Z} \cdot Z \right)^{-1} Z^M.$$ (4.14)

The classical functions $W^M$ and their complex conjugates have Poisson brackets that close on the functions generating the manifest $U(n|m)$ symmetry of the Lagrangian, and together these span the Lie superalgebra $su(n+1|m)$ with respect to the Poisson bracket determined by the othosymplectic 2-form $F$. In the quantum theory, the rescaled quantum position operators $s\hat{Z}^M$ and $s\hat{Z}_M$ act on the reduced wavefunction via the operators

$$U^M = Z^M \left( 2s - Z \cdot \frac{\partial}{\partial Z} \right), \quad U_M = \frac{\partial}{\partial Z^M},$$ (4.15)

which generalize the corresponding operators (3.7) of the $CP^n$ case. These operators close on $su(n+1|m)$. Thus the physical position operators of a fuzzy $CP^{(n|m)}$ are
proportional to the ladder operators of a representation of \( SU(n+1|m) \), a result that precisely parallels the one we found for \( CP^n \).

We now turn to the nature of the physical states, and their representation content. Proceeding exactly as in the \( CP^n \) case, one finds that physical wavefunctions take the form

\[
\Psi = \left[ 1 + \bar{Z} \cdot Z \right]^{-s} \Phi(Z) \tag{4.16}
\]

for holomorphic superfield \( \Phi(Z) \). The ‘Hilbert space norm’ is

\[
||\Psi||^2 = \int d\mu_0 \left[ 1 + \bar{Z} \cdot Z \right]^{m-n-1} |\Psi|^2 = \int d\mu_0 \left[ 1 + \bar{Z} \cdot Z \right]^{m-n-1-2s} |\Phi|^2 \tag{4.17}
\]

where (allowing for an arbitrary normalization factor \( N \))

\[
d\mu_0 = N \prod_{i=1}^n dz_i \bar{d}z_i \prod_{\alpha=1}^m \frac{\partial}{\partial \xi_\alpha} \frac{\partial}{\partial \bar{\xi}_\alpha}. \tag{4.18}
\]

If \( n \neq 0 \) then normalizability of \( \Psi \) requires \( \Phi \) to be a polynomial in \( Z \) of maximal degree \( 2s \). The case \( n = 0 \) is special because all coordinates are Grassmann-odd. There is no quantization condition on \( 2s \) in this case but special features emerge for particular integer values of \( 2s \). Generically, the ‘Hilbert’ space transforms as a reducible but not fully reducible representation of \( SU(1|m) \) but for certain integer values of \( 2s \) there are zero norm states, and this requires a redefinition of the physical ‘Hilbert’ space as an equivalence class of states modulo the addition of zero norm states. This redefined physical subspace is an \( SU(1|m) \) singlet when \( 2s = m - 1 \). When \( 2s = m - 2 \) (assuming \( m > 2 \)) the redefined physical ‘Hilbert’ space is \((m+1)\)-dimensional and transforms as the fundamental irrep of \( SU(1|m) \). We refer to \([17]\) for details\(^7\).

We will pass over the issue of the \( SU(n+1|m) \) content of ‘Hilbert’ space for the general \( n \neq 0 \) case, and instead concentrate on \( n = 1 \). As mentioned in the introduction, this case corresponds to a particle on the \( m \)-extended supersphere. We begin with \( m = 1 \); a particle on the (simple) supersphere. In this case, the ‘Hilbert’ space norm (4.17) is

\[
||\Psi||^2 = \int d\mu_0 \left[ 1 + \bar{Z} \cdot Z \right]^{-(1+2s)} |\Phi|^2 \tag{4.19}
\]

where \( \Phi(z, \xi) \) is the ‘reduced’ holomorphic wavefunction corresponding to \( \Psi \). It has the superfield expansion

\[
\Phi = \phi_0(z) + \xi \phi_1(z) \tag{4.20}
\]

\(^7\)The supergroup called \( SU(1|m) \) here was called \( SU(m|1) \) in \([17]\), and we deviate from that reference in a few other minor respects too. Note that the parameter \( \alpha \) of \([17]\) can be chosen to equal the parameter \( \gamma \) which should then be identified with the parameter \( 2s \) of this paper.
where $\phi_0$ and $\phi_1$ are two holomorphic functions of opposite Grassmann parity. After performing the Berezin integrals over $\xi$ and $\bar{\xi}$, one finds that

$$
||\Psi||^2 \propto \left[ \int_{S^2} \frac{dzd\bar{z}}{(1 + \bar{z}z)^{2s+1}} |\phi_1|^2 - (1 + 2s) \int_{S^2} \frac{dzd\bar{z}}{(1 + \bar{z}z)^{2s+2}} |\phi_0|^2 \right].
$$

(4.21)

The relative minus sign between the two integrals does not imply that the norm is indefinite because we can choose the overall normalization such that the offending integral is the one with the nilpotent integrand. Leaving aside this issue, normalizability of the second integral implies that $\phi_0(z)$ is a polynomial of maximum degree $2s$ and hence that its coefficients transform as spin $s$ under $SU(2)$, while normalizability of the first integral implies that $\phi_1(z)$ is a polynomial in $z$ of maximum degree $(2s-1)$, and hence that its coefficients transform as spin $s - \frac{1}{2}$ under $SU(2)$. If we wish to respect the spin-statistics connection\(^8\) then we should choose $\Phi$ to have Grassmann parity $(-1)^{2s}$. We then have a ‘Hilbert’ space that is a supermultiplet with spins $(s - \frac{1}{2}, s)$ carrying a $2s \oplus (2s + 1)$ representation of $SU(2)$; this is the decomposition into $SU(2)$ irreps of the ‘degenerate’ irrep of $SU(2|1)$ of total dimension $4s + 1$ [24].

When $m > 1$ we have a particle on the $m$-extended supersphere. An analysis along the same lines leads to an $m$-extended supermultiplet of $SU(2)$ irreps with multiplicities given by binomial coefficients, which is reminiscent of particle supermultiplets of the super-Poincaré group. For example, when $m = 2$ and $2s = 1$ we get the supermultiplet of spins $(0, \frac{1}{2}, \frac{1}{2}, 1)$, which could be viewed as a non-relativistic analogue of the vector supermultiplet of $N = 2$ super-Poincaré supersymmetry. The supergroups $SU(2|m)$ can thus be viewed as non-relativistic supersymmetry groups, their degenerate representations yielding supermultiplets of $SU(2)$ spins. In the simplest, $m = 1, 2s = 1$, case we have a supermultiplet of spins $(0, 1/2)$, described by a Grassmann-even scalar wavefunction and a Grassmann-odd Pauli-spinor wavefunction respectively.

## 5 The local limit

Let us define new coordinates $V^M$ for $CP^{(n|m)}$ by setting

$$
Z^M = \left(1 - \bar{V} \cdot V\right)^{-\frac{1}{2}} V^M .
$$

(5.1)

One may verify, by substitution, that

$$
F = -2i d\bar{V}_M \wedge dV^M ,
$$

(5.2)

and hence that the Lagrangian in the new coordinates is

$$
L = -is \bar{V} \cdot \dot{V} + c.c .
$$

(5.3)

\(^8\)This is not a mathematical necessity here because the spins are non-relativistic.
This illustrates the extension to orthosymplectic supergeometry of Darboux’s theorem in symplectic geometry, which states that all symplectic manifolds of the same dimension are \textit{locally} equivalent. In this case the equivalence is between \( CP^{(n|m)} \), with its Kähler 2-form as the symplectic form, and \( \mathbb{C}^{(n|m)} \) with its orthosymplectic 2-form determined by the natural complex structure. The superspace \( CP^{(n|m)} \) differs \textit{globally} from \( \mathbb{C}^{(n|m)} \) by the restriction to the ‘superball’ in \( \mathbb{C}^{(n|m)} \) with the ‘unit sphere’ \( \bar{V} \cdot V = 1 \) as its boundary; note that this boundary corresponds to infinity in the \( Z \) coordinates.

The natural complex structure on \( CP^{(n|m)} \) differs from the one implicit in the choice of \( Z \) coordinates on \( CP^{(n|m)} \) because \( Z \) is not a holomorphic function of \( V \). This has the consequence that a hermitian metric in the \( Z \) coordinates will not be hermitian in the \( V \) coordinates, and vice-versa. Thus, the hermitian (Kähler) metric \( g \) that one may naturally associate with the Kähler 2-form \( F \) via the formula

\[
F = -2i d\bar{Z}^M g_{MN} \wedge dZ_N \quad (5.4)
\]

is not diffeomorphic to the metric that this formula yields in the \( V \) coordinates, because \( g \) is not hermitian in these coordinates. The Kähler metric in the \( V \) coordinates defined by the analogous formula is a \textit{flat} metric on \( \mathbb{C}^{(n|m)} \). This illustrates the topological nature of CSQM: the action does not involve a choice of metric on the target space \( M \). However, the quantum theory does involve a choice of complex structure (in our approach this choice is implicit in the separation of the second class constraints into holomorphic and anti-holomorphic pairs) and once a complex structure has been chosen then it is preserved only by holomorphic diffeomorphisms. The holomorphic diffeomorphisms that leave \( F \) invariant are holomorphic symplectic diffeomorphisms, generated by holomorphic functions on \( M \).

One reason that we chose to consider the CSQM model for target space \( CP^{(n|m)} \) in the \( Z \) coordinates is that the transformations (4.3) are analytic in these coordinates. They are not analytic in the \( V \) coordinates. Thus, while one can always put the CSQM action in the form (5.3) by a change of coordinates, one cannot do this with a \textit{holomorphic} change of coordinates. This makes the extension of Darboux’s theorem to the quantum theory problematic; not surprisingly in view of the non-local features of quantum mechanics\(^9\). Nevertheless, one can still focus on the local properties of the target space by introducing rescaled coordinates \( \tilde{Z}^M \) such that

\[
Z^M = \frac{1}{\sqrt{s}} \tilde{Z}^M. \quad (5.5)
\]

In the limit that \( s \to \infty \) we have

\[
s \log \left(1 + \bar{\tilde{Z}} \cdot \tilde{Z} \right) \to \bar{\tilde{Z}} \cdot \tilde{Z}, \quad (5.6)
\]

which is the Kähler potential for \( \mathbb{C}^{(n|m)} \). Physical wavefunctions now take the coherent state form

\[
\Psi = e^{-\tilde{Z} \cdot \tilde{Z}} \Phi(\tilde{Z}). \quad (5.7)
\]

\(^9\)For example, in the path-integral approach one must consider amplitudes for all paths in phase space, so the global features of phase space can affect local physics.
The rescaled physical position operators are now
\[ \hat{Z}^M = Z^M - \frac{\partial}{\partial \bar{Z}^M}, \quad \hat{\bar{Z}}_M = \bar{Z}_M + \frac{\partial}{\partial Z^M}, \] (5.8)
for which the non-zero (anti)commutation relations are
\[ [\hat{Z}_M, \bar{Z}^N] = 2\delta^N_M. \] (5.9)
In other words, the physical position operators obey the (anti)commutation relations of the non-(anti)commutative \( C(n|m) \).

Acknowledgements. We thank José-Adolfo de Azcárraga and Joaquim Gomis for helpful discussions. E.I. thanks the Department of Physics of the University of Padua for warm hospitality during the final stages of this study. L.M. thanks the theory group at JINR for hospitality and partial financial support, and the organisers of the Workshop on Branes and Generalized Dynamics (Argonne, October 20-24, 2003) for the opportunity to present some of the results of this paper. The work of E.I. was supported in part by grants DFG No.436 RUS 113/669, RFBR-DFG 02-02-04002, INTAS 00-00254, RFBR 03-02-17440 and a grant of the Heisenberg-Landau program. L.M. was supported in part by the National Science Foundation under grant PHY-9870101.

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