Dimensional confinement and Yang-Baxter integrability in cold-atom systems via optical lattices

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Abstract. Using two-dimensional optical lattices, it has recently become possible to split up clouds of ultra cold atoms into collections of parallel one-dimensional tubes in which the dynamics is restricted to the direction along their axis. Since the Feshbach-resonance interactions among atoms are short ranged, they can be approximated as contact interactions. It is thus possible to realize optical-trap versions of some classic integrable systems first studied by Lieb, Gaudin, Yang and others. Such systems can display exotic physics like spin-charge separation and unusual paired states. We focus on the study of systems of fermions in one-dimension. At finite temperatures, regimes such as the spin-coherent and the spin-incoherent Luttinger liquids can be realized by tuning the inter-atomic interaction strength and trap parameters. We identify the noise correlation of density fluctuations as a robust observable to discriminate between these two regimes. We also address the effects of spin imbalance for the case of attractive interactions, and study the one-dimensional analog of spatially modulated (\textsc{fflo}) superconducting states. Our focus is on how the temperature affects the density profiles that are being measured in the experiments. The theoretical study of temperature effects in these systems is of experimental relevance for the problem of thermometry.

1. Introduction
Recent developments in atomic and optical physics allow experimentalists to trap and cool down clouds of atomic gases beyond the limits of quantum degeneracy. This capability combined with the use of Feshbach resonances \cite{1} and optical lattices \cite{2}, allows to engineer systems in which the interactions and the dimensionality are controllable. Thus, new opportunities open up to realize experimentally model systems with interesting quantum dynamics. One such class of systems of particular importance are integrable one-dimensional models. These models are special because an infinite number of conserved charges (or symmetries) allows to find closed non-perturbative solutions with non-trivial ground states and excitations.

The experiments that achieve the dimensional reduction to 1d, rely on multiple laser beams used to create a 2d optical lattice, say in the \textit{x-y} plane, resulting in a configuration consisting of a square array of tubes as depicted in Fig. 1. Each tube constitutes an elongated quasi-1d trap if the tunneling between the tubes is negligible and the transverse trapping frequency is large compared to all other scales (including the atom-atom interactions). Further, if the transverse confinement is strong, then a single transverse band can be considered (\textit{i.e.} no extra band quantum numbers are required). Such controlled 1d-trap arrays provided the first successful demonstrations of the 1d interacting boson gas or Tonks-Girardeau gas \cite{3, 4} and played a key
Figure 1. Sketch of the 3d cigar-shaped cloud (light blue halo) and the array of 1d tubes formed by a 2d optical lattice potential.

role in the early theoretical proposals for the realization of the fermionic counterparts using cold-atom gases \([5, 6]\).

In our work, we addressed in-depth the case of fermionic 1d interacting gases. These are also integrable systems and were first discussed as such by Lieb, Gaudin, Yang and others \([7, 8]\). The main complication that arises in connecting the idealized theories with the realities of the cold-atom systems is due to the presence of the confining (or trapping) potential. The resulting inhomogeneous realizations of the models are no longer integrable and have to be treated in an approximate way that exploits the detailed knowledge available for the homogeneous cases.

The most studied example is the case of two-component fermions with repulsive interactions. At large densities and in the low-energy limit, the homogeneous system is usually referred to as Luttinger liquid and can be conveniently studied using bosonization. We developed a scheme to study the inhomogeneous system by combining a non-linear conformal mapping and finite-size bosonization \([9, 10]\). Our results stress the qualitative differences that arise when the fermion densities are not large and the system can be considered, instead, a realization of the so called spin-incoherent Luttinger liquid \([11]\) (in our context, spin refers to the extra label used to distinguish between the two types of fermions – in practice, two different hyperfine states of the atoms). We showed that correlations of density fluctuations are able to qualitatively distinguish between different regimes and provide an ideal probe in the cold-atom context. While the off-diagonal correlators are positive for the (usual) spin-coherent Luttinger liquid, they are negative for its spin-incoherent counterpart. A result that does not depend on the details of the trap and should be easy to access experimentally.

One of the notable aspects of the experiments with cold atoms is that one can, unlike for other systems, turn the interactions to be attractive, rather than repulsive. For 1d systems, this is interesting since it allows to realize the one-dimensional analog of superfluid/superconducting phases. Combined with the additional experimental ability of controlling at will the population balance of the two fermionic species, one can realize exotic polarized superfluid phases. One such state is the Ferrell-Fulde-Larkin-Ovchinikov state (FFLO) in which fermions pair up with non-zero center-of-mass momentum \([12, 13]\). This modified superconducting states are not seen in usual materials, but are thought to be relevant in the context of heavy fermions \([14]\) and organic conductors \([15]\). Not only that, they are also probably crucial to the understanding of the expected superfluid states of nuclear and quark matter \([16]\).

In the 1d case, the scattering is factorizable in the Yang-Baxter sense and these models can be studied using Bethe Ansatz. A number of recent papers have looked at the homogeneous case \([17]\), (there exists also some bosonization-based calculations \([18]\), but only numerical studies of the inhomogeneous case were available \([19, 20]\). In our presentation during the symposium, we reported about our recent contribution in which we extended the integrability-based calculations to the inhomogeneous case, and did so at finite temperatures using the so called Thermodynamic Bethe Ansatz (TBA) equations \([21]\).
2. Model

We start from the Hamiltonian of a single 1d tube in a harmonic trap

$$H = -\frac{\hbar^2}{2m} \sum_i \frac{\partial^2}{\partial z_i^2} + g_{1d} \sum_{i<j} \delta(z_i - z_j) + \frac{m\omega^2}{2} \sum_i z_i^2,$$  \hspace{1cm} (1)

where $m$ is the mass of the atoms, $\omega_z$ is the trap frequency and $g_{1d} = \frac{2\hbar^2 a_{3d}}{ma_{1}\sqrt{1-C a_{3d}/a_{1}}} \frac{1}{T}$ [with $C = |\zeta(1/2)|/\sqrt{2} \approx 1.0326$] is the effective interaction strength between particles, which can be tuned using a Feshbach resonance. The parameter $a_{3d}$ is the 3d scattering length and $a_{1}$ is the transverse oscillator length. If we measure lengths in units of the harmonic oscillator length $(\hbar/m\omega_z)^{1/2}$ and energies in units of $\hbar\omega_z/2$, the resulting dimensionless Hamiltonian is

$$H = -\sum_i \frac{\partial^2}{\partial z_i^2} - 4\Delta \sum_i \delta(z_i - z_j) + \sum_i z_i^2,$$  \hspace{1cm} (2)

where $\Delta = -\frac{g_{1d}}{\hbar \omega_z} \frac{\bar{n}}{k} > 0$ (attractive interactions). We start from the solution of the uniform system and take the effects of the trap into account via a local density, Thomas-Fermi, approximation (widely used in studying trapped gases, though its violation has been observed in highly elongated 3d clouds [22] due to interface interaction effects [23, 24]). For the case of attractive fermions in 1d, one has smooth crossovers and comparisons with Quantum Monte Carlo calculations [19] indicate the Thomas-Fermi scheme gives good results for density profiles as soon as the densities are not too small.

3. TBA analysis and results

The corresponding TBA equations are given by:

$$f_u - f_a = G \ast f_b - G \ast f_{s1},$$

$$f_b - f_o = 2[k^2 - \Delta^2 - \mu(z)]/T + K_1 \ast f_u + K_2 \ast f_b,$$

$$f_{sn} - f_{sn} = \delta_{n,1} G \ast f_u + G \ast (f_{s(n+1)} + \delta_{n,1} f_{sn-1}),$$

$$\lim_{n\rightarrow\infty} (K_{n+1} \ast f_{sn} - K_n \ast f_{sn+1}) = -2\hbar/T,$$

as reported elsewhere [21]. These are integral equations that can be solved numerically for different values of the temperature and interaction strength (cf. Ref. [25]). Density profiles obtained combining the solutions of the TBA and a finite-temperature Thomas-Fermi approximation are plotted in Fig. 2. The two plots illustrate two qualitatively different behaviors that are realized depending on the degree of global polarization, $P = (N_{\uparrow} - N_{\downarrow})/(N_{\uparrow} + N_{\down\downarrow})$. In one case, the edge of the cloud is a balanced paired superfluid while in the other is a fully polarized Fermi gas. In both cases, the inner core of the cloud is formed by a partially polarized 1d-superfluid with FFLO-like features. We have shown elsewhere that the different regions remain identifiable up to temperatures of about 10% of the Fermi temperature [21].

We stress that our results can also be used for thermometry. Indeed, the information from the particle-density trap profiles was used in the analysis of the experimental data obtained recently by the cold-atom experimental group at Rice University [26]. Those results show good agreement with the theory and the reported temperatures are about 10% of the Fermi temperature with some deviations towards larger values at the tails of the cloud (notice the experimental error bars also increase near the edges).

A more recent, and previously unreported, result is about the profiles for entropy density in the trap; these are shown in Fig. 3 for two different temperatures. One observes that the curves are peaked at the interface between the inner FFLO-like region and the outer cloud, and
Figure 2. Trap density profiles for $N = 200$ and $T = 0$. Two different global polarizations, $P = 4\%$ and $21\%$ are shown. The plots show spin-up ($n_{\uparrow}$, solid), spin-down ($n_{\downarrow}$, dashed) particle densities and their difference ($n_s$, dotted). The critical polarization at which all the densities go to zero together at the edge of the cloud is $P \approx 15\%$ (in that case the 1d-FFLO state spans the whole cloud). These behaviors were recently demonstrated experimentally [26]. (At the time of the qts6 symposium, only preliminary experimental data was available.)

then again near the edge of the outer cloud. One can understand the result in terms of atomic densities: when they become small, the phase space for arranging particles is larger (even if interactions are strong) and the entropy per particle is larger as a result. Similar observations were made before for lattice gases (bosonic and fermionic Hubbard models) and there have been suggestions of exploiting this effect for cooling schemes [27]. The idea is to somehow let the particles near the edge of the clouds fly out and take with them a comparatively large share of the entropy, thus lowering the total (and also average) entropy of the system. The process is different from the usual evaporative cooling technique in that the shape of the trap would be engineered so that there is no substantial reduction of the particle density at the trap center (this step would be experimentally challenging, but is supposed to produce big gains in terms of reduction of the system temperature as a fraction of the Fermi temperature).
4. Conclusions

To summarize, we have studied attractive spin-imbalanced ultracold atomic clouds of fermions in 1d. Combining the Bethe Ansatz method and the Thomas-Fermi approximation we have shown how to calculate trap profiles of different observables. We reported, in particular, on the particle density, magnetization density and entropy density profiles. Our finite-temperature calculations serve also to indicate how much one needs to cool in order to be able to observe exotic pairing states and shed light on the possibility of applying recently proposed cooling schemes to this 1d systems. During the symposium (and also in Ref. [21]) we discussed about ways of mapping out the ‘phase diagram’ of the trapped system by reading it out from the temperature profiles. Recent experiments did make use of those ideas but ultimately followed a different protocol for analyzing the data. The problem of dealing with a bundle of tubes, rather than a single one, introduces additional complications. The experiments used a method of reconstructing the total-number profiles per species across tubes by using an inverse Abel transform of their data after first projecting onto the plane perpendicular to the tube’s axis in order to improve the signal to noise ratio (following our advise). Then our analysis of a single tube was repeated and combined for many tubes in order to produce predictions for the whole bundle – rather than trying to extract single-tube information from the experiments.

A similar TBA analysis can be also performed for repulsive interactions between particles, where the coexistence of different Luttinger-liquid regimes has been proposed as discussed in the introduction [9]. The approach is well suited to explore interesting crossover effects, which are challenging to capture with other methods. Such systems would also be within the reach of the present generation of experiments by working on the other side of the Feshbach resonance. Conversely, our inhomogeneous bosonization scheme could be applied to the attractive case in order to answer questions about dynamics that are currently still open. Efforts in this direction are underway.

It is extremely useful that the experiments are able to realized models that are close to integrable ones. This not only has intellectual appeal in terms of motivating further studies of these unique models, but also helps to develop the experimental tools required to probe non-trivial strongly correlated quantum systems. Such tools could later be applied to a wider class of systems and, along the way, perhaps improve our understanding of the origins of integrability.

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