Critical properties of the $N$-color London model

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The critical properties of $N$-color London model are studied in $d = 2 + 1$ dimensions. The model is dualized to a theory of $N$ vortex fields interacting through a Coulomb and a screened potential. The model with $N = 2$ shows two anomalies in the specific heat. From the critical exponents $\alpha$ and $\nu$, the mass of the gauge field, and the vortex correlation functions, we conclude that one anomaly corresponds to an inverted 3D $xy$ fixed point, while the other corresponds to a 3D $xy$ fixed point. There are $N$ fixed points, namely one corresponding to an inverted 3D $xy$ fixed point, and $N − 1$ corresponding to neutral 3D $xy$ fixed points. This represents a novel type of quantum fluid, where superfluid modes arise out of charged condensates.

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Ginzburg-Landau (GL) theories with several complex scalar matter fields minimally coupled to one gauge field are of interest in a wide variety of systems, such as multi-component (color) superconductors, metallic phases of light atoms such as hydrogen [1, 2], and as effective theories for easy-plane quantum antiferromagnets [3, 4, 5]. The model also is highly relevant in particle physics where it is called two-Higgs doublet model [6]. In metallic hydrogen the scalar fields represent Cooper pairs of electrons and protons, which excludes the possibility of inter-color pair tunneling, i.e. there is no Josephson coupling between different components of the condensate. The same two-color action in $(2 + 1)$ dimensions, where the matter fields originate in a bosonic representation of spin operators, is claimed to be the critical sector of a field theory separating a Néel state and a paramagnetic (valence bond ordered) state of a two dimensional quantum antiferromagnet at zero temperature with an easy-plane anisotropy present [7, 8]. This happens because, although the effective description of the antiferromagnet involves an a priori compact gauge field, it must be supplemented by Berry-phase terms in order to properly describe $S = 1/2$ spin systems [7, 8]. Berry-phases cancel the effects of monopoles at the critical point [9]. In this paper, we point out novel physics of the quantum fluid that arises out of an $N$-color charged condensate when no intercolor Josephson coupling is present.

For a detailed analysis of the phase transitions in such a generalized GL model, we study an $N$ component GL theory in $(2 + 1)$ dimensions with no Josephson coupling term. The model is defined by $N$ complex scalar fields $\{\Psi^{(a)}(r) | a = 1 \ldots N\}$ coupled through the charge $e$ to a fluctuating gauge field $A(r)$, with Hamiltonian

$$H = \sum_{a=1}^{N} \left[ \frac{\left| \nabla - i e A \right|}{2 M^{(a)}} \right]^{2} + V\left(\{\Psi^{(a)}\}\right) + \frac{1}{2} \left(\nabla \times A\right)^{2}$$

where $M^{(a)}$ is the $a$-component condensate mass. The potential $V\left(\{\Psi^{(a)}(r)\}\right)$ is assumed to be only a function of $|\Psi^{(a)}(r)|^{2}$. The model is studied in the phase-only (London) approximation $ \Psi^{(a)}(r) = |\Psi^{(a)}| \exp[i\theta^{(a)}(r)]$ and is discretized on a lattice with spacing $a = 1$. In the Villain approximation the partition function reads

$$Z = \int_{-\infty}^{\infty} DA \prod_{\gamma=1}^{N} \int_{-\pi}^{\pi} D\theta^{(\gamma)} \prod_{\eta=1}^{N} \sum_{\mathbf{n}^{(\eta)}} \exp\left(-S\right)$$

$$S = \sum_{\mathbf{r}} \left[ \sum_{a=1}^{N} \frac{\beta}{2 M^{(a)}} \left(\Delta \theta^{(a)} - e A + 2 \pi \mathbf{n}^{(a)}\right)^{2} + \frac{\beta}{2} \left(\nabla \times A\right)^{2} \right]$$

where $\mathbf{n}^{(a)}(\mathbf{r})$ are integer vector fields ensuring $2\pi$ periodicity, and the lattice position index vector $\mathbf{r}$ of the fields is suppressed. The symbol $\Delta$ denotes the lattice difference operator and $\beta = 1/T$ is the inverse temperature. Here, we stress the importance of keeping track of the $2\pi$ periodicity of the individual phases. The kinetic energy terms are linearized by introducing $N$ auxiliary fields $\mathbf{v}^{(a)}$. Integration over all $\theta^{(a)}$ produces the local constraints $\Delta \cdot \mathbf{v}^{(a)} = 0$, which are fulfilled by the replacement $\mathbf{v}^{(a)} \rightarrow \Delta \times \mathbf{h}^{(a)}$. We recognize $\mathbf{h}^{(a)}$ as the dual gauge fields of the theory. By fixing the gauge $h_{z}^{(a)} = 0$ and performing a partial integration we may introduce the vortex fields $\mathbf{m}^{(a)} = \Delta \times \mathbf{n}^{(a)}$. We integrate out the gauge field $A$ and get a theory in the dual gauge fields $\mathbf{h}^{(a)}$ and the vortex fields $\mathbf{m}^{(a)}$ where $\Delta \cdot \mathbf{m}^{(a)} = 0$

$$S = \sum_{\mathbf{r}} \left[ 2 \pi i \sum_{a=1}^{N} \mathbf{m}^{(a)} \cdot \mathbf{h}^{(a)} + \sum_{a=1}^{N} \frac{\left(\Delta \times \mathbf{h}^{(a)}\right)^{2}}{2 \beta |\psi^{(a)}|^{2}} + \frac{e^{2}}{2 \beta} \left(\sum_{a=1}^{N} \mathbf{h}^{(a)}\right)^{2} \right]$$

where $|\psi^{(a)}|^{2} = |\Psi^{(a)}|^{2}/M^{(a)}$. Note how the algebraic sum of the dual photon fields is massive. This differs from the case $N = 1$, where $e$ produces one massive dual photon with bare mass $e^{2}/2$, and the model describes a
vortex field $\mathbf{m}$ interacting through a massive dual gauge field $\mathbf{h}$. However, when $N \geq 2$, since $\Delta \cdot \mathbf{m}^{(\alpha)} = 0$, a gauge transformation $\mathbf{h}^{(\alpha)} \rightarrow \mathbf{h}^{(\alpha)} + \Delta \mathbf{m}^{(\alpha)}$ for $\alpha = 1 \ldots N$ leaves the action invariant if one of the gauge fields, say $\mathbf{h}^{(n)}$ compensates the sum in the last term in (3) with $\Delta q^{(n)} = -\sum_{\alpha \neq n} \Delta q^{(\alpha)}$.

Integrating out the dual gauge fields we get a generalized theory of $N$ interacting vortex fields

$$Z = \sum_{\mathbf{m}^{(1)}} \cdots \sum_{\mathbf{m}^{(N)}} \delta_{\mathbf{m}, \mathbf{m}^{(0)}} \cdots \delta_{\mathbf{m}, \mathbf{m}^{(N),0}} \times e^{-S_V},$$

$$S_V = \sum_{r, r', \alpha, \eta} \mathbf{m}^{(\alpha)}(r) D^{(\alpha, \eta)}(r - r') \mathbf{m}^{(\eta)}(r'),$$

(4)

where $\delta_{\alpha,\beta}$ is the Kronecker-delta, and the vortex interaction potential $D^{(\alpha, \eta)}(\mathbf{r})$ is the inverse discrete Fourier transform of $\bar{D}^{(\alpha, \eta)}(\mathbf{q})$, where

$$\bar{D}^{(\alpha, \eta)}(\mathbf{q}) = \frac{\lambda^{(\alpha)}}{|\mathbf{Q}_q|^2 + m_0^2} + \frac{\delta_{\alpha,\eta} - \lambda^{(\eta)}}{|\mathbf{Q}_q|^2},$$

(5)

$$\lambda^{(\alpha)} = |\psi^{(\alpha)}|^2/\psi^2$$

and $\psi^2 = \sum_{\alpha = 1}^N |\psi^{(\alpha)}|^2$. Here, $m_0^2 = e^2 \psi^2$ is the square of the bare inverse screening length in the intervortex interaction, and $|\mathbf{Q}_q|^2$ is the Fourier representation of the lattice Laplace operator. The first term of the vortex interaction potential $\bar{h}$ is a Yukawa screened potential, while the second term mediates long range Coulomb interaction between vortex fields. If $N = 1$ the latter cancels out exactly and we are left with the well studied vortex theory of the GL model which has a charged fixed point for $\epsilon \neq 0$. For $N \geq 2$ we find a theory of vortex loops of $N$ colors interacting through long range Coulomb interaction. If $N$ grows to infinity, $\psi^2 \rightarrow \infty$ and the vortex fields interact via a diagonal unscreened $N \times N$ Coulomb matrix. This reflects the inability of one single gauge field $\mathbf{A}$ to screen a large number of vortex species. The case $N \geq 2$ has features with no counterpart in the case $N = 1$ and, namely neutral superfluid modes arising out of charged condensates.

The above vortex system may be formulated as a field theory, introducing $N$ complex matter fields $\phi^{(\alpha)}$ for each vortex species, minimally coupled to the dual gauge fields $\mathbf{h}^{(\alpha)}$. This generalizes the dual theory for $N = 1$ pioneered in [15]. The theory reads (see also [2])

$$S_{\text{dual}} = \sum_r \left[ \sum_{\alpha = 1}^N \left( m_0^2 |\phi^{(\alpha)}|^2 + (|\Delta - \mathbf{i} \mathbf{h}^{(\alpha)})|\phi^{(\alpha)}|^2 \right) \right.\right.$$  

$$+ \frac{(\Delta \times \mathbf{h}^{(\alpha)})^2}{2\beta |\psi^{(\alpha)}|^2} \right] + e^2 \left( \sum_{\alpha = 1}^N \mathbf{h}^{(\alpha)} \right)^2 \right) \right.\right.$$  

$$+ \sum_{\alpha, \eta} g^{(\alpha, \eta)} |\phi^{(\alpha)}|^2 |\phi^{(\eta)}|^2 \right].$$

(6)

Here, we have added chemical potential (core-energy) terms for the vortices, as well as steric short-range repulsion interactions between vortex elements. In the $N = 1$ case, a RG treatment of the mass term of the dual gauge field yields $\partial^2 / \partial \ln l = e^2$, and hence this term scales up, suppressing the dual gauge field. Correspondingly, for $N \geq 2$, this suppresses $\sum_{\alpha} \mathbf{h}^{(\alpha)}$, but not each individual dual gauge field. For the particular case $N = 2$, assuming the same to hold, we end up with a gauge theory of two complex matter fields coupled minimally to one massless gauge field, which was also precisely the starting point. Thus the theory is self-dual for $N = 2$ [14]. For $N = 1$, it is known that a charged theory in $d = 2 + 1$ dualizes into a $|\phi|^4$ theory and vice versa [11]. The vortex tangle of the 3Dxy model is incompressible and the dual theory is a massless gauge theory such that $\langle \phi \rangle \neq 0$ is prohibited. For $e \neq 0$, the dual theory has global symmetry, and vortex condensation and $\langle \phi \rangle \neq 0$ is possible [11].

For $N = 2$, Monte Carlo (MC) simulations have been carried out for the action [11] with parameters $|\psi^{(1)}|^2 = 1/2$, $|\psi^{(2)}|^2 = 1$, $e^2 = 1/4$, and $m_0^2 = 3/8$. Here, $|\psi^{(1)}|^2$ and $|\psi^{(2)}|^2$ have been chosen to have well-separated bare energy scales associated with the twist of the two types of phases, and $m_0$ has been chosen to be of the order of the inverse lattice spacing in the problem to avoid difficult finite-size effects. One MC update consists of inserting elementary vortex loops of random direction and species according to the Metropolis algorithm.

We observe two anomalies in the specific heat at $T_{c1}$ and $T_{c2}$ where $T_{c1} \leq T_{c2}$. We find $T_{c1}$ and $T_{c2}$ from scaling of the second moment of the action $(\langle S_V - \langle S_V \rangle \rangle^2)$ to be $T_{c1} = 1.4(6)$ and $T_{c2} = 2.7(8)$. To check the criticality of these anomalies we have calculated the critical exponents $\alpha$ and $\nu$ by applying finite size scaling (FSS) of $M_3 = \langle (S_V - \langle S_V \rangle \rangle^2 \rangle$ [14]. The peak to peak value of this quantity scales with system size $L$ as $L^{1+\alpha}/\nu$, the width between the peaks scales as $L^{-1/\nu}$. The advantage of this is that asymptotically correct behavior is reached for practical system sizes. The FSS plots for system sizes $L = 4, 6, 8, 10, 12, 14, 16, 20, 24$ are shown in Fig. [11]. From the scaling we conclude that both anomalies are in fact critical points, and we obtain $\alpha = -0.02 \pm 0.02$ and $\nu = 0.67 \pm 0.01$ for $T_{c1}$ and $\alpha = -0.03 \pm 0.02$ and $\nu = 0.67 \pm 0.01$ for $T_{c2}$. These values are consistent with those of the 3Dxy and the inverted 3Dxy universality classes found with high precision to be $\alpha = -0.0148(6)$ and $\nu = 0.67155(3)$ [15].

To characterize these phase transitions further, we consider $G_A(q) = \langle A_q \cdot A_{-q} \rangle$ and $G_{2\chi h}(q) = \langle (\sum_\alpha \mathbf{h}^{(\alpha)}_q) \cdot (\sum_\alpha \mathbf{h}^{(\alpha)}_{-q}) \rangle$, expressed in terms of $G^+(q) = \langle |\sum_\alpha |\psi^{(\alpha)}|^2 \mathbf{m}^{(\alpha)}_q|^2 \rangle$ as

$$G_A(q) = \frac{2/\beta}{|Q_q|^2 + m_0^2} \left( 1 + \frac{2\pi^2 m_0^2}{|Q_q|^2} \frac{G^+(q)}{m_0^2 + m_q^2} \right),$$

$$G_{2\chi h}(q) = \frac{2\beta \psi^2}{|Q_q|^2 + m_0^2} \left( 1 - \frac{2\pi^2 \beta}{\psi^2} \frac{G^+(q)}{|Q_q|^2 + m_0^2} \right).$$

(7)

The masses of $A$ and $\sum_\alpha \mathbf{h}^{(\alpha)}$ are defined by $m_A^2 = \cdots$
which has recently been verified numerically \cite{11,16}.

The scaling of the width between the peaks $\Delta \beta$ labeled (\( \bullet \)) and (\( \times \)) for $T_{c1}$ and $T_{c2}$ respectively. The lines are power law fits to the data for $L > 6$ used to extract $\alpha$ and $\nu$.

\[
\lim_{q \to 0} 2G_A(q)^{-1/2} \text{ and } m_{\Sigma h}^2 = \lim_{q \to 0} 2\beta \nu^2 G_{\Sigma h}(q)^{-1}.
\]

We briefly review the case $N = 1$ \cite{11}. The dual field theory of the neutral fixed point ($m_0^2 = 0$) is a charged theory describing an incompressible vortex tangle. The leading behavior of the vortex correlator is $\lim_{q \to 0} 2\pi^2 \beta G^{(+)}(q) \sim |1 - C_3(T)|q^2$, $q^2 - C_3(T)q^{2-\eta_h}$, and $q^2 + C_4(T)q^4$ for $T < T_c$, $T = T_c$, and $T > T_c$ respectively. For $T < T_c$ we have $m_{\Sigma h}^2 = 0$ ($N = 1$), however for $T > T_c$ the $1/q^2$ terms in $G_{\Sigma h}(q)$ cancel out exactly and this mass attains an expectation value. At the charged fixed point ($m_0^2 \neq 0$) of the GL model, the effective field theory of the vortices is a neutral theory. The vortex tangle is incompressible with a scaling ansatz for the vortex correlator $\lim_{q \to 0} G^{(+)}(q) \sim q^2$, $q^{2-\eta_A}$, and $c(T)$ for $T < T_c$, $T = T_c$, and $T > T_c$, respectively. Consequently, from \cite{7}, the mass $m_A$ drops to zero at $T_c$ and the mass of the dual gauge field $m_{\Sigma h}$ is finite for all temperatures and has a kink at $T_c$. Renormalization group arguments yield $\eta_A = 4 - d$ where $d$ is the dimensionality \cite{11,12}, which has recently been verified numerically \cite{11,16}.

\[
\text{FIG. 1: The FSS of the peak to peak value of the third moment $\Delta M_3$ labeled (\( \square \)) and (+) for $T_{c1}$ and $T_{c2}$ respectively. The scaling of the width between the peaks $\Delta \beta$ labeled (\( \bullet \)) and (\( \times \)) for $T_{c1}$ and $T_{c2}$ respectively. The lines are power law fits to the data for $L > 6$ used to extract $\alpha$ and $\nu$.
}

\[
\text{FIG. 2: } G^{(+)}(q) \text{ for } N = 2, L = 32. \text{ For } T = 2.86 > T_{c2}, T = 2.76 \approx T_{c2}, \text{ and } T = 2.63 < T_{c2}, \lim_{q \to 0} G^{(+)}(q) \sim c(T), \sim q, \text{ and } \sim q^2, \text{ respectively.}
\]

The vortex correlator for $N = 2$ is sampled in real space and $G^{(+)}(q)$ is found by discrete Fourier transformation, it is shown in Fig. \( \square \). At $T = T_{c1}$ the leading behavior is $G^{(+)}(q) \sim q^2$ on both sides of $T_{c1}$. Consequently, due to \cite{7}, $m_A$ and $m_{\Sigma h}$ are finite in this regime. This shows that the vortex tangle is incompressible and that the anomalous scaling dimension $\eta_A = 0$, which corresponds to a neutral fixed point. Below $T_{c2}$ the dominant behavior is $G^{(+)}(q) \sim q^2$ whereas $G^{(+)}(q) \sim c(T)$ above $T_{c2}$. At $T = T_{c2}$, $G^{(+)}(q) \sim q$ indicating $\eta_A = 1$. Accordingly, $m_A$ is finite below $T_{c2}$ and zero for $T \geq T_{c2}$.

For $T \lesssim T_{c2}$, $m_A$ scales according to $G_A(q)^{-1} \propto m_A^2 + Cq^{2-\eta_A} + O(q^4)$ for small $q$ where $\delta > 2 - \eta_A$ \cite{10}, with a corresponding Ansatz for $G_{\Sigma h}(q)$. For each coupling we fit $G_A(q)^{-1}$ data from system sizes $L = 8, 12, 20, 32$ to the Ansatz. The results for $m_A$ (and $m_{\Sigma h}$, found similarly), are given in Fig. \( \square \). The system exhibits Higgs mechanism at $T = T_{c2}$ when $m_A$ drops to zero. Furthermore $m_A$ has a kink at $T_{c1}$ due to ordering of $\theta^{(1)}$. The anomalies in $m_A$ coincide precisely with $T_{c1}$ and $T_{c2}$ determined from scaling of $\langle (S_V - \langle S_V \rangle)^2 \rangle$. Note also how $m_{\Sigma h}$ changes abruptly at $T_{c2}$. This is due to a sudden change in screening of $\sum_{n=1}^{N} m(n)$ by the vortex-loop proliferation at $T = T_{c2}$, giving an abrupt increase in $m_{\Sigma h}$, analogously to what happens for $N = 1, e \neq 0$ \cite{11}. Above $T_{c2}$, $A$ is massless, giving a compressible vor- text tangle which accesses configurational entropy better than an incompressible one. Below $T_{c2}$, $A$ is massive and merely renormalizes $|\Psi|^4$ terms in Eq. \cite{11}. The theory is effectively a $|\Psi|^4$ theory in this regime. Thus, the remaining proliferated vortex species originating in the matter fields with lower bare stiffnesses form vortex tangles as if they originated in a neutral superfluid. For the general $N$ case, a Higgs mass is generated at the highest critical temperature, after which $A$ merely renormalizes the $|\Psi|^4$ term, such that the Higgs fixed point is followed by $N - 1$ neutral fixed points as the temperature is lowered.

\[
\text{FIG. 3: The mass } m_A (\bullet) \text{ and } 1 - m_A/m_{\Sigma h} (\times) \text{ found from Eq. } \square. \text{ Two non-analyticities can be seen in } m_A \text{ at } T_{c1} \text{ and } T_{c2}, \text{ corresponding to a neutral fixed point and a charged Higgs fixed point, respectively. An abrupt increase in } m_{\Sigma h} \text{ due to vortex condensation is located at } T_{c2}.
\]
We now discuss the vortex mode \( \mathbf{m}^{(1)} - \mathbf{m}^{(2)} \), demonstrating that it should be identified as a superfluid mode in the system. Its properties are controlled by \( G_{\Delta h}(q) \equiv \langle |h_q^{(1)} - h_q^{(2)}|^2 \rangle \). A dual Higgs phenomenon for \( N = 2, T = T_{c1} \) involving \( G_{\Delta h}(q) \) may be demonstrated as follows. Introducing \( G^{(-)}(q) = \langle |m_q^{(1)} - m_q^{(2)}|^2 \rangle \) and \( G^{(m)}(q) = \langle (m_q^{(1)} - m_q^{(2)}) \cdot (\sum_{\alpha=1}^{2} \psi^{(\alpha)} q^{(\alpha)} m_{\lambda q}^{(\alpha)}) \rangle \) we find, in the notation used in Eqs. (9) and (7),

\[
G_{\Delta h}(q) = \frac{8\beta \lambda^2 \psi^2}{|Q_q|^2} \left\{ 1 - 2\pi^2 \beta \lambda^2 \psi^2 G^{(-)}(q) \right\} + \left( \lambda^1 - \lambda^2 \right)^2 G_{\Sigma h}(q). \tag{8}
\]

The \( G^{(-)}(q) \) correlation function is always \( \sim q^2, q \to 0 \), but has a nonanalytic coefficient of \( q^2 \), determined by the helicity modulus \( \Upsilon \) of the neutral mode \( \mathbf{m}^{(1)} - \mathbf{m}^{(2)} \). When \( \Upsilon \) vanishes at \( T_{c1} \) through a disordering of \( \theta^{(1)} \), thus destroying the superfluid neutral mode, the first and second term in the bracket cancel, which in turn cancels the \( 1/q^2 \) term in \( G_{\Delta h}(q) \). This produces a dual Higgs mass \( m_{\Delta h} \) defined by \( G_{\Delta h}(q) \sim 1/(q^2 + m_{\Delta h}^2) \) for \( T > T_{c1} \). The remaining terms in Eq. (8) contribute to determining the actual value of \( m_{\Delta h} \). Thus, while \( h^{(1)} - h^{(2)} \) is always massive, cf. Eq. (9), \( h^{(1)} - h^{(2)} \) is massless below \( T_{c1} \) and massive above \( T_{c1} \). Therefore \( h^{(1)} - h^{(2)} \) plays the role of a gauge degree of freedom, providing a dual counterpart to \( A \) in Eq. (1). This is evident when \( |\psi^{(1)}|^2 = |\psi^{(2)}|^2 \). Then Eq. (8) for \( N = 2 \), \( \epsilon = 0 \) has the same form as the dual gauge field correlator for the case \( N = 1, e = 0 \), which exhibits a dual Higgs phenomenon [11]. Thus, for \( N = 2, e \neq 0, m^{(1)} - m^{(2)} \) behaves as vortices for \( N = 1, e = 0 \), i.e. it is a superfluid mode arising out of superconducting condensates. A nonzero \( m_{\Delta h} \) is produced by disordering \( \theta^{(1)} \) at \( T_{c1} \) while a nonzero \( m_A \) is destroyed by disordering \( \theta^{(2)} \) at \( T_{c2} \).

We have analysed the \( N \)-color London model Eq. (2) in vortex representation Eqs. (3) and (4). The dual theory is given by Eqs. (8) and (9). For \( N = 2 \), we have performed large scale Monte Carlo simulations computing \emph{i)} critical exponents \( \alpha \) and \( \nu \), \emph{ii)} gauge field and dual gauge field correlators, \emph{iii)} the corresponding masses, and \emph{iv)} critical couplings using FSS. For \( \psi^{(1)} = \psi^{(2)} \) we find one neutral low-temperature critical point at \( T_{c1} \), and one charged critical point at \( T_{c2} > T_{c1} \). For general \( N \), a Higgs mass \( m_A \) is generated at the highest critical temperature, followed by \( N - 1 \) neutral fixed points as the temperature is lowered.

These results apply to electronic and protonic condensates in liquid metallic hydrogen under extreme pressure. Estimates exist for \( T_{c2} \) for such systems, \( T_{c2} \approx 160 \text{K} \), and hence \( T_{c1} \approx 0.1 \text{K} \). Hence, in addition to the emergence of the Meissner effect at \( T_{c2} \) and a corresponding divergence in the magnetic penetration length \( \lambda \sim 1 - T/T_{c2}^{-1/2-N_A} \) [17], there will also be a novel effect, namely a low-temperature anomaly in the magnetic penetration length \( \lambda \sim 1 - T/T_{c1}^{-1/2-N_A} \) [17].

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