The development of V. V. Strygin’s ideas in numerical mathematics in the works of Samara mathematicians

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Abstract. The article reviews the most important results in the field of numerical mathematics obtained by Professor V. V. Strygin’s disciples and the disciples of his disciples in 2000–2019 in the city of Samara. We present the results on spline interpolation of functions with a boundary layer, methods and algorithms for a posteriori adaptation of computational grids for singularly perturbed boundary value problems and the results on applied wavelet analysis based on spline wavelets.

1. Introduction
In 2019 we celebrate Professor Vadim Vasilievich Strygin’s 80th anniversary. He achieved a lot of important outstanding results in the field of functional analysis, differential equations, theoretical mechanics. This article is dedicated to the development of his ideas in the field of computational mathematics in his Samara-based pupils’ works in the period of 2000–2019. The research started in Voronezh by I. A. Blatov and V. V. Strygin, continued by I. A. Blatov and his disciples starting from 2000 after I. A. Blatov moved to Samara in 1999. These studies developed in three main areas: 1. Methods of spline functions in the problems with boundary layer; 2. Numerical methods of solution of singularly perturbed boundary value problems; 3. Semi-orthogonal spline wavelets and their applications. This article provides an overview of the most important results obtained in these areas.

2. Methods of spline functions for boundary layer problems
Methods of spline functions for solving singularly perturbed equations had been investigated by V. V. Strygin and his pupils in Voronezh in the last decades of the last century. The results of these studies were recorded in the monograph [1]. However the behavior of interpolation splines minimum defect for functions with a boundary layer at strongly non-uniform grids remained unexplored. Corresponding results were obtained by I. A. Blatov, E. V. Kitaeva (Samara) and A. I. Zadorin (Omsk branch of the Institute of Mathematics of SB RAS) in the series of works [2–7]. This series of works was recognized as the most significant result of the Institute of
Mathematics of SB RAS for 2017–2018 in numerical mathematics. We shall give a brief review of the main results of this series.

2.1. Interpolation of functions with large gradients in the boundary layer by cubic splines

Let us assume that the interpolated function \( u(x) \) is represented as

\[
    u(x) = q(x) + \Phi(x), \quad x \in [0, 1],
\]

where \(|q^{(j)}(x)| \leq C_1, |\Phi^{(j)}(x)| \leq \frac{C_1}{\varepsilon^j} e^{-\alpha x / \varepsilon}, 0 \leq j \leq 4\), here, the functions \( q(x) \) and \( \Phi(x) \) are not given in explicit form, \( \alpha > 0, \varepsilon > 0 \), and \( C_1 \) is a constant independent of \( \varepsilon \).

Let us study the problem of cubic spline interpolation of function (1).

Let us assume that \( N — is a positive integer, \Delta — partition of \([0, 1]\) with nodes \( x_n, n = 0, 1, \ldots, N \). Let us assume that \( g_3(x, u) \in S(\Delta, 3, 1) \) is an interpolation cubic spline on \( \Delta \), determined by the conditions

\[
    g_3(x_n, u) = u(x_n), \quad 0 \leq n \leq N, \quad g'_3(0, u) = u'(0), \quad g'_3(1, u) = u'(1).
\]

Let us consider a mesh \( \Omega \) with nodes \( x_n, n = 0, 1, \ldots, N \), and steps \( h_n = h = \frac{N}{N+1}, n = 1, \ldots, N \), \( h_n = H = \frac{1}{2} \sigma^3, n = N/2 + 1, \ldots, N \). Let \( \sigma = \min \left\{ \frac{1}{2}, \frac{4 \ln N}{3} \right\} \). We assume below that \( \sigma < 1/2 \). Further on for brevity sake we denote it as \( g_3(x) = g_3(x, \Phi), g_3(x) \in S(\Omega, 3, 1) \).

**Theorem 1** There are constants \( C_2, C_3 \), such that, for \( N^{-1} \leq C_3 \varepsilon \) the following estimates hold:

\[
    || \Phi(x) - g_3(x, \Phi) ||_{C[0,1]} \leq C_2 N^{-4} \ln^4 N.
\]

**Theorem 2** There are constants \( C_4, C_5 \) and \( \beta > 0 \), independent of \( \varepsilon, N \), such that, for \( \varepsilon \leq C_4 N^{-1} \) we have the estimates

\[
    || g_3(x) - \Phi(x) ||_{C[x_n, x_{n+1}]} \leq C_5 \left\{ \begin{array}{l}
    N^{-4} \ln^4 N, 0 \leq n \leq N/2 - 1, \\
    \varepsilon^{-1} N^{-5} \exp \left( - \beta(n - N/2) \right), N/2 \leq n \leq N - 1.
    \end{array} \right.
\]

The following theorem shows that estimates (4) cannot be improved.

**Theorem 3** Let \( \Phi(x) = e^{-x/\varepsilon} \). Then there are constants \( C_4, C_6, \beta_1 > 0 \), independent of \( \varepsilon, N \), such that, for \( \varepsilon \leq C_4 N^{-1} \) we have the lower estimates

\[
    || g_3(x) - \Phi(x) ||_{C[x_n, x_{n+1}]} \geq C_6 N^{-5} \varepsilon^{-1} e^{-\beta_1(n-N/2)}, \quad N/2 \leq n \leq N - 1.
\]

Let us construct a modified interpolation spline. Let \( \bar{x}_{N/2} = \frac{x_{N/2} + x_{N/2+1}}{2}, \bar{x}_n = x_n, n \in [0, N/2 - 1] \cup [N/2 + 1, N] \). Let \( gm_3(x, u) \) — be a cubic interpolation spline determined by the conditions \( gm_3(\bar{x}_n, u) = u(\bar{x}_n), n \in [0, N], \quad gm'_3(0, u) = u'(0), \quad gm'_3(1, u) = u'(1) \). The only difference of \( gm_3(x, u) \) from \( g_3(x, u) \) is that the interpolation node \( x_{N/2} \) is replaced by \( \bar{x}_{N/2} \). The knots of the spline remain the same and coincide with the nodes of \( \Omega \).

**Theorem 4** There are constants \( \gamma_0, C > 0 \), independent of \( \varepsilon, N \) such that, for \( \varepsilon \ln N \leq \gamma_0 \) we have the estimate

\[
    || u(x) - gm_3(x, u) ||_{C[0,1]} \leq C N^{-4} \ln^4 N.
\]

**Remark 1** The condition \( \varepsilon \ln N \leq \gamma_0 \) holds if \( \varepsilon \leq C N^{-1} \). Therefore, by 1, 4 the application of the interpolation spline \( gm_3(x, u) \) for \( \varepsilon = O(N^{-1}) \) and the interpolation spline \( g_3(x, u) \) for \( N^{-1} = O(\varepsilon) \) yields estimates of form (3) and (6) that are uniform in \( N \) and \( \varepsilon \).
Theorem 5

henceforth that \( \sigma < \) the conditions:

\[ g \]

Let us assume that \( \Omega \) — is a uniform grid on the interval \([0, \bar{\Omega}]\), defined by the conditions: \( g_2(x_n, u) = u(x_n), \ 0 \leq n \leq N, \ g_2'(0, u) = u'(0), \ g_2'(1, u) = u'(1) \). Let as denote by \( g_2(x, u) \in S(\bar{\Omega}, 2, 1) \) — the parabolic interpolating spline on \( \bar{\Omega} \), determined by the conditions: \( g_2(x_n, u) = u(x_n), \ 0 \leq n \leq N, \ g_2'(0, u) = u'(0), \ g_2'(1, u) = u'(1) \). Let us assume henceforth that \( \sigma < 1/2 \).

Theorem 5

There are constants \( C, C_3, \) such that, for \( N^{-1} \leq C_3 \varepsilon \) the following estimate holds:

\[
\|u(x) - g_2(x, u)\|_{C[0,1]} \leq CN^{-3} \ln^3 N. \tag{8}
\]

Theorem 6

There are constants \( C_4, C_5 \) and \( \beta > 0 \), independent of \( \varepsilon, N \), such that the estimates

\[
\|g_2(x, \Phi) - \Phi(x)\|_{C[x_n, x_{n+1}]} \leq C_5 \begin{cases} 
N^{-3} \ln^3 N, & 0 \leq n \leq N/2 - 1, \\
\frac{1}{N^4 \varepsilon} e^{-\beta (n-N/2)}, & N/2 \leq n \leq N - 1 
\end{cases} \tag{9}
\]

hold for \( \varepsilon \leq C_4 N^{-1} \).

The following theorem shows that the second estimate in (9) is unimprovable.

Theorem 7

Let us assume that \( \Phi(x) = e^{-x/\varepsilon} \). Then there are \( C_6, C_7, \beta_1 > 0 \), independent of \( \varepsilon, N \), such that the estimates below will be valid

\[
\|g_2(x, \Phi) - \Phi(x)\|_{C[x_n, x_{n+1}]} \geq \frac{C_7}{N^4 \varepsilon} e^{-\beta_1 (n-N/2)}, \quad \frac{N}{2} \leq n \leq N - 1 \tag{10}
\]

hold for \( \varepsilon \leq C_6 N^{-1} \).

2.3. Exponential spline interpolation for functions with large gradient in the boundary layer

Let us assume that \( \Omega \) — is a uniform grid on the interval \([0, 1]\) with nodes \( x_n, n = 0, 1, \ldots, N \) and step \( h \). We assume that the function \( u(x) \) is specified at the nodes of the grid \( \Omega \), \( u_n = u(x_n), \ n = 0, 1, \ldots, N \). We define the space of \( L \)-splines taking into account the boundary layer component of the exponential type in the interpolated function:

\[
SL(\Omega, 3, 1) = \{S(x) \in C^2[0,1] : S(x) = a_n + b_n x + c_n x^2 + d_n e^{-x/\varepsilon}, \ x \in [x_n, x_{n+1}], \ 0 \leq n \leq N - 1\}. \]

We determine the interpolation \( L \)-spline \( S(x; u) \in SL(\Omega, 3, 1) \) of the function \( u(x) \) from the condition

\[
S(x_n; u) = u(x_n), \ 0 \leq n \leq N, \ S''(0; u) = u''(0), \ S''(1; u) = u''(1). \tag{11}
\]
Theorem 8 For any function \( u(x) \in C_b[0,1] \) for all \( h = 1/N \in (0,1], \varepsilon \in (0, +\infty) \) there exists a unique interpolation spline \( S(x; u) \in SL(\Omega, 3, 1) \), satisfying conditions (11) with error estimates
\[
\|S(x; u) - u(x)\|_{C[0,1]} \leq C \min \left\{ \frac{1}{h}, 1 + \frac{h}{\varepsilon} \right\} \inf_{v \in SL(\Omega, 3, 1)} \|u - v\|_{C,h}.
\] (12)

If the function \( u(x, \varepsilon) \) has form (1), then we have the estimates
\[
\|S(x; u) - u(x, \varepsilon)\|_{C[0,1]} \leq C \min \left\{ \frac{h^3}{\varepsilon^3}, \frac{h^4}{\varepsilon} \right\}, \varepsilon \in (0, 1],
\] (13)

Remark 2 Estimates (13) imply the third-order, uniform with respect to \( \varepsilon \in (0, +\infty) \) convergence of interpolation process (11) for functions of the form (1).

Theorem 9 There is a constant \( C_1 > 0 \) such that, for any \( \varepsilon \in (0, 1], h = 1/N \in (0, 1] \) there exists a function \( u(x) \in C_b[0,1] \) such that \( u''(0) = u''(1) = 0 \), and
\[
\|S(x; u) - u(x)\|_{C[0,1]} \geq C_1 \min \left\{ \frac{1}{h}, 1 + \frac{h}{\varepsilon} \right\} \inf_{v \in SL(\Omega, 3, 1)} \|u - v\|_{C[0,1]}
\] (14)

and a function \( u_1(x) \in C^4[0,1] \) such that
\[
\|S(x; u_1) - u_1(x)\|_{C[0,1]} \geq C_1 \min \left\{ \frac{h^3}{\varepsilon^3}, \frac{h^4}{\varepsilon} \right\} \|u_1\|_{C[0,1]}.
\] (15)

2.3.1. Corollaries for polinomial splines Let us obtain some implications of Theorems 8 and 9 for parabolic and cubic splines.

Let us assume that \( S_3(x; u) \in S(\Omega, 3, 1) \) — is the interpolation cubic spline of a function \( u(x) \), determined from conditions (2), and \( S_2(x; u) \in S(\Omega, 2, 1) \) — be the interpolation parabolic spline determined from the conditions
\[
S_2(x_n; u) = u(x_n), 0 \leq n \leq N, \quad S_2'(1-0; u) = u''(1).
\] (16)

Theorem 10 For any function \( u(x) \in C_b[0,1] \) the interpolation splines \( S_2(x; u) \) and \( S_3(x; u) \) exist, are unique, and satisfy the formulae
\[
\lim_{\varepsilon \to +\infty} \|S(x; u) - S_3(x; u)\|_{C[0,1]} = 0,
\] (17)
\[
\lim_{\varepsilon \to 0^+} \|S(x; u) - S_2(x; u)\|_{C[0,1]} = 0.
\] (18)

Corollary 1 It is true that
\[
\|S_3(x; u) - u(x)\|_{C[0,1]} \leq C \inf_{v \in S(\Omega, 3, 1)} \|u - v\|_{C,h},
\] (19)
\[
\|S_2(x; u) - u(x)\|_{C[0,1]} \leq \frac{C}{h} \inf_{v \in S(\Omega, 2, 1)} \|u - v\|_{C,h}.
\] (20)

If \( u(x) \in C^4[0,1] \), then we have the error estimates
\[
\|S_3(x; u) - u(x)\|_{C[0,1]} \leq Ch^4,
\] (21)
\[
\|S_2(x; u) - u(x)\|_{C[0,1]} \leq Ch^3.
\] (22)
Remark 3 In the theorem 10 the limit calculated for a fixed $h = 1/N$. This result also holds for $\varepsilon/h \rightarrow +\infty$.

Corollary 2 Under the conditions and notation of theorem 10 the formula

$$
\lim_{\varepsilon \in (0,1], \varepsilon/h \rightarrow +\infty} \| S(x; u) - S_B(x; u) \|_{C[0,1]} = 0
$$

is valid.

Theorem 11 There is a constant $C_1 > 0$ such that, for any $h = 1/N \in (0,1]$ there exists a function $u(x) \in C_h[0,1]$, such that $u'(0) = u''(1) = 0$, and

$$
\| S_2(x; u) - u(x) \|_{C[0,1]} \geq \frac{C_1}{h} \inf_{v \in SL(0,3,1)} \| u - v \|_{C[0,1]}.
$$

3. Convergence of the grid adaptation algorithms for singular perturbed boundary value problems

In Voronezh (1985–1999) V. V. Strygin and his pupils developed the projection-grid approximate solution methods of solutions of singularly perturbed boundary value problems [1]. After moving to Samara I. A. Blatov and his disciples continued research in this direction in terms of development and justification for such tasks as adaptation algorithms for moving computational grids. Despite the large number of works dedicated to posterior estimates errors on moving grids, there was no theory allowing to prove the convergence of the grids to the limit partition and get error estimates on this partition. Thanks to the Galerkin projection method, developed by I. A. Blatov guided by V. V. Strygin, such a device was created and successfully used in the works of I. A. Blatov and his students in Samara [8–10]. This section briefly sets out the fundamental results in this direction.

Let us consider the boundary value problems

$$
L_\varepsilon u_\varepsilon \equiv -\varepsilon u''_\varepsilon + p(t, \varepsilon) u'_\varepsilon + q(t, \varepsilon) u_\varepsilon = f(t), u_\varepsilon(0) = u_\varepsilon(1) = 0,
$$

$$
M_\varepsilon v_\varepsilon \equiv -\varepsilon v''_\varepsilon + (p(t, \varepsilon)v'_\varepsilon) + q(t, \varepsilon)v_\varepsilon = g(t), v_\varepsilon(0) = v_\varepsilon(1) = 0.
$$

Assumption 1 We assume that $p(t, \varepsilon) \in C^4[0,1]$, $q(t, \varepsilon) \in C^2[0,1]$, $f(t) \in C[0,1]$, $g(t) \in C[0,1]$, and

$$
|q^{(i)}(t, \varepsilon)| \leq C(1 + \varepsilon^{-i}\exp(p_0(t - 1)/\varepsilon))), i = 0, 1, 2, 0 < C_1 \leq p_0 \leq C_2,
$$

$$
p(t, \varepsilon) \geq p_0 > 0, |p^{(i)}(t, \varepsilon)| \leq C(1 + \varepsilon^{-i}\exp(p_0(t - 1)/\varepsilon))), i = 0, 1, 2, 3, 4.
$$

In [1] it was proved that the problems (25)–(26) have unique solutions with boundary layer at the right end of the segment $[0,1]$. We proceed to the definition of the numerical method. Let $\phi_\varepsilon = 1 - (2\varepsilon/p_0)\ln\varepsilon, \psi_\varepsilon = \phi_\varepsilon + 2(1 - \varepsilon)/p_0$.

$$
\chi(y) = \begin{cases} 
    y, & y \in [0, \phi_\varepsilon], \\
    1 + \frac{2\varepsilon}{p_0} \ln\{(p_0/2)|y - \phi_\varepsilon + (2/p_0)\varepsilon|\}, & y \in [\phi_\varepsilon, \psi_\varepsilon].
\end{cases}
$$

Let us assume that $n$ — is a positive integer, $\phi_\varepsilon = \tau_n$.

In $[\phi_\varepsilon, \psi_\varepsilon]$ we define $\tau_i = \tau_{i-1} + (\psi_\varepsilon - \phi_\varepsilon)/n, i = n + 1, n + 2, \ldots, 2n + 1, \tau_0 = 0$. The points of $\Delta$ have the form

$$
t_i = \chi(\tau_i), \ i = 0, \ldots, 2n.
$$
Theorem 12

in the case of Shishkin mesh.

in the case of Bakhvalov mesh and together with second derivatives, then a place for the problem (26).

\[ u \] solution

There are numbers \( p \), \( q \).

Now we define the numerical method. For a problem (25) we seek the function \( u_n(t) \) in \( E \) such that

\[
-\varepsilon u_n'(t_i + 0) + \varepsilon u_n'(t_{i-1} + 0) + (pu_n' + qu_n, f_i) = (f, f_i), \quad i = 1, 2, \ldots, n,
\]

\[
\varepsilon u_n'(t_n + 0) + (\varepsilon u_n', f_{n+1}') + (pu_n' + qu_n, f_{n+1}) = (f, f_{n+1}),
\]

\[
(\varepsilon u_n', f_i') + (pu_n' + qu_n, f_i) = (f, f_i), \quad i = n + 2, n + 3, \ldots, 2n - 1.
\]

Here \((, )\) is the scalar inner product in \( L_2[0, 1] \). Similarly we consider the problem (26).

**Theorem 12** There are numbers \( \varepsilon_0, n_0 \) such that for all \( \varepsilon \in (0, \varepsilon_0], n \geq n_0 \) there is unique solution \( u_n(t) \) of the problem (33)–(35) on Bakhvalov and Shishkin meshes. Similar result has a place for the problem (26).

If \( p(t, \varepsilon), q(t, \varepsilon), f(t) \) are sufficiently smooth functions, bounded in \( C[0, 1] \) uniformly in \( \varepsilon \) together with second derivatives, then

\[
\| u_n - u_{\varepsilon} \|_{C[0, 1]} \leq C_1/n^2, \quad \| v_n - v_{\varepsilon} \|_{C[0, 1]} \leq C_1/n^2
\]

in the case of Bakhvalov mesh and

\[
\| u_n - u_{\varepsilon} \|_{C[0, 1]} \leq C_1(\ln n/n)^2, \quad \| v_n - v_{\varepsilon} \|_{C[0, 1]} \leq C_1(\ln n/n)^2
\]

in the case of Shishkin mesh.
3.1. Adaptation of the grid and main result
Let us now consider the grid adaptation algorithm in the case of an unknown boundary of a boundary layer.

**Assumption 2** For the adaptation algorithms we assume that \( p(t, \varepsilon) = p(t) \), \( q(t, \varepsilon) = q(t) \), \( p_0 = p(1) \). We also suppose that \( u_0(1) \neq 0 \), \( v_0(1) \neq 0 \), where \( u_0(t) \), \( v_0(t) \) are solutions of Cauchy problems:
\[
p(t)u_0' + q(t)u_0 = f(t), \quad u_0(0) = 0, \quad (p(t)v_0) + q(t)v_0 = f(t), \quad v_0(0) = 0.
\]

In this case the main term of the asymptotic expansion in the boundary layer is given by \(-u_0(1)e^{p_0 t}t^{1/2}\).

**Definition 1** We say that \( \phi = \phi(\varepsilon, n) \) is n-boundary of boundary layer if \( \max_{t \in [0, \varepsilon]} e^{p_0 t} t^{1/2} \leq 1/\varepsilon^2 \).

**Definition 2** The number \( \tilde{\phi} = \tilde{\phi}(\varepsilon, n) = \sup_{\phi} \phi(\varepsilon, n) \) is exact n-boundary of the boundary layer,
\[
\tilde{\phi} = 1 - \frac{2}{p_0} \varepsilon \ln n.
\]

We assume that we know the location of the boundary layer (a neighborhood of \( t = 1 \)), but it is unknown the exact n-boundary (parameter \( p_0 \)). We propose the following algorithm to find \( p_0 \).

- **Step 1.** We fix large enough \( p_0 \geq p_0 \). Let \( k = 0 \).
- **Step 2.** We define mesh \( \Delta_{n,p^k} \) as Bakhvalov or Shishkin mesh, when we use \( p^k \) instead \( p_0 \).
- **Step 3.** We find solution \( u^{n,k}_{n,p^k}(t) \) on the mesh \( \Delta_{n,p^k} \).
- **Step 4.** We define \( p^{k+1} = p^k - \tau_k \), where \( \tau_k \) is chosen so that \( t_{n+1,k} - t_{n+1,k+1} = \varepsilon \ln n \) in the case of Bakhvalov mesh and \( t_{n,k} - t_{n,k+1} = \varepsilon \ln n \) in the case of Shishkin mesh.
- **Step 5.** We find solution \( u^{n,k+1}_{n,p^{k+1}}(t) \) on the mesh \( \Delta_{n,p^{k+1}} \).
- **Step 6.** We calculate \( \mu_k = \| u^{n,k+1}_{n,p^{k+1}}(t) - u^{n,p^k}_{n,p^k}(t) \|_{C[t_{n+1,k+1}, t_{n+1,k}]} \), where \( t_{n+1,k} \) is the node of the mesh \( \Delta_{n,p^k} \).
- **Step 7.** If \( k = 0 \), then \( k := k + 1 \) and goto 2, else goto 8.
- **Step 8.** If \( \mu_k > \frac{\ln n}{n^2} \) in the case of Bakhvalov mesh and \( \mu_k > \frac{\ln^3 n}{n^2} \) in the case of Shishkin mesh, then \( k := k + 1 \) and goto 2, else \( \tilde{\phi} \approx t_{n+1,k+1} \), the end.

**Theorem 13** There are \( \varepsilon_0 > 0 \), natural \( n_0 \), \( C_1, C_2, C_3 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \), \( n \geq n_0 \) algorithm 1-8 finishes its work with \( k < C_1 \frac{\ln n}{\ln \ln n} \) steps, and the following error estimates will be correct
\[
|\tilde{\phi} - t_{n+1,k+1}| \leq C_2 \varepsilon \ln n, \quad |\tilde{\phi} - t_{n,k+1}| \leq C_2 \varepsilon \ln n, \quad (39)
\]
\[
\| u^{n,p^{k+1}}_{n,p^{k+1}}(t) - u^\varepsilon(t) \|_{C[0,1]} \leq C_3 \frac{\ln n}{n^2}, \quad \| u^{n,p^{k+1}}_{n,p^{k+1}}(t) - u^\varepsilon(t) \|_{C[0,1]} \leq C_3 \frac{\ln^3 n}{n^2} \quad (40)
\]
in the case of Bakhvalov and Shishkin meshes respectively.

**Corollary 3** Theorem 13 is formulated for the problem (25). Similar result holds for the problem (26).
4. Semi-orthogonal spline wavelets and their application

In Voronezh, I. A. Blatov and V. V. Strygin developed a theory of estimates of the error of solutions of finite method elements of singularly perturbed boundary value problems. This theory was based on the estimate of the norms of orthogonal in $L_2[\Omega]$ projectors on spaces of finite element type in the sense of the norm spaces of continuous functions. It required study matrix structures that are inverse to sparse matrices, particularly, Gram matrices. Further on, these studies took shape as an independent direction. In 1999, I. A. Blatov maintained his doctoral thesis on “Operators with Pseudosparse Matrices and Their Applications”.

After I. A. Blatov moved to Samara the theory methods of pseudosparse matrices developed in combination with the methods of applied wavelet analysis. We considered semi-orthogonal spline wavelets for which the forward and inverse fast wavelet transform algorithms were developed. Using the theory of pseudosparse matrices we studied Gram matrices of the basis wavelets and various functions of these matrices. N. V. Rogova, E. K. Yakovlev and Yu. A. Gerasimov maintained PhD theses on the use of semi-orthogonal spline wavelets for modeling thin-wire antennas, spacecraft orientation problems and decorrelation time series, respectively. The most important results of these studies were published in [11–15]. Present section briefly presents the results on semi-orthogonal spline wavelets that formed the basis of the above studies.

4.1. Spline wavelets

Let $[a, b] = [0, 1]$, $m$ be a positive integer, and $n_0$ be an integer such that $2^{n_0} < 2m - 1 < 2^{n_0+1}$. Consider the family $\Delta = \{\Delta_n, n = n_0, n_0 + 1, \ldots\}$ of partitions $[0, 1]$ with the constant step $h = h_n = 1/2^n$. On each partition $\Delta_n$ consider a space of splines $L_n = S(\Delta_n, m - 1, 1)$. Then, for each $k \geq n_0$ the space $S(\Delta_k, m - 1, 1)$ can be represented as the direct sum $L_k = L_{n_0} \oplus W_{n_0+1} \oplus W_{n_0+2} \oplus \ldots \oplus W_k$, where $W_k$ denotes the orthogonal complement of $L_{k-1}$ with respect to $L_k$. The desired wavelet basis is constructed in the form of the union of the basis in $L_{n_0}$ and all the bases in $W_n$, $n_0 < n \leq k$.

First, we construct a basis in the orthogonal complement $W_n$ of $L_{n-1}$ with respect to $L_n$. Fix $n \geq n_0 + 1$. Assume, if necessary, that each partition $\Delta_n$ can be continued with the same step to the entire number line to form nodes $x_i^0, -\infty < i < +\infty$. The normalized $B$-splines on $\Delta_n$ are denoted by $N_{m-1,j,n}$.

Let $i \geq 0$ be a fixed integer such that $i + 2m - 1 \leq 2^{n-1}$, i.e., the interval $[x_i^{n-1}, x_{i+2m-1}^{n-1}]$ lies entirely in $[0, 1]$. A function $\psi_{i,n}(x) \in W_n$ is sought in the form

$$\psi_{i,n}(x) = \sum_{j=2i}^{2i+3m-2} \alpha_j N_{m-1,j,n}. \quad (41)$$

Let $(u, v) = \int_0^1 u(x)v(x)dx$ denote the inner product in the space $L_2[0, 1]$. For $\psi_{i,n} \in W_n$, it is sufficient that

$$(\psi_{i,n}, N_{m-1,k,n-1}) = 0, \quad k = i - m + 1, i - m + 2, \ldots, i + 2m - 2, \quad (42)$$

since the other orthogonality conditions hold automatically because the supports are disjoint.

Substituting representation (41) into (42), we obtain a homogeneous system of $3m - 2$ equations with $3m - 1$ unknowns:

$$\sum_{j=2i}^{2i+3m-2} \alpha_j (N_{m-1,j,n}, N_{m-1,k,n-1}) = 0, \quad k = i - m + 1, i - m + 2, \ldots, i + 2m - 2, \quad (43)$$
which always has a nontrivial solution. Finding this nontrivial solution yields the desired set of coefficients and a function $\psi_{i,n}(x)$ of form (41).

An algorithm for constructing $\psi_{i,n}(x)$, is reduced to the computation of the coefficients $\alpha_j$.

1. Set $i = 0$, $n = n_0 + 1$.
2. Compute the matrix of SLAE (43), finding each inner product $(N_{m-1,j,n}, N_{m-1,k,n-1})$ as the sum of integrals over the partial segments constituting the support $(N_{m-1,j,n} N_{m-1,k,n-1})$. Each such integral is found with the help of a Gaussian quadrature formula that is exact for polynomials of degree $2m - 2$.
3. Set $\alpha_{3m-2} = 1$.
4. Solve SLAE (43) of $3m - 2$ equations with $3m - 2$ unknowns $\alpha_0, \alpha_1, \ldots, \alpha_{3m-3}$ by applying Gaussian elimination with pivoting to obtain a function $\psi_{0,n_0+1}(x)$ of form (41).

**Remark 4** We have

$$\psi_{i,n}(x) = \psi_{0,n_0+1} \left( 2^{n-n_0-1}x - \frac{b-a}{2^{n_0}} \right), \quad n = n_0 + 1, \ldots, k, \quad i = 0, 1, \ldots, 2^{n-1} - 2m + 1,$$

i.e., the collection of constructed wavelet functions is obtained by shifting and scaling the only function $\psi_{0,n_0+1}(x)$.

Representation (41) and the properties of $B$-splines imply that $\text{supp}(\psi_{i,n}) \subset [x_{2i}, x_{2i+4m-2}]$, i.e., this support contains $4m - 2$ adjacent partial segments.

A question arises as to whether a wavelet with a shorter support can be constructed. The following theorem shows that this is not the case.

**Theorem 14** Let $p < m - 1$ and let the function $\psi_{i,n}(x)$ given by

$$\psi_{i,n}(x) = \sum_{j=2i}^{2i+m+2p} \alpha_j N_{m-1,j,n}$$

satisfy the conditions

$$(\psi_{i,n}, N_{m-1,k,n-1}) = 0, \quad k = i - m + 1, i - m + 2, \ldots, i + m + p - 1.$$

Then $\psi_{i,n}(x) \equiv 0$.

**Corollary 4** Set of functions are $\{\psi_{i,n}\}, \quad i = 0, 1, \ldots, 2^{n-1} - 2m + 1$ linearly independent on each interval $[x_j^{n-1}, x_{j+m-1}^{n-1}]$, $0 \leq j \leq 2^{n-1}$ (i.e. their restrictions to each such segment are linearly independent).

Thus, we have constructed a collection of semiorthogonal wavelets $\{\psi_{i,n}\}, \quad i = 0, 1, \ldots, 2^{n-1} - 2m + 1$. However, the dimension of the orthogonal complement $W_n$ is $2^{n-1}$, i.e. we need another $2(m-1)$ wavelet functions to obtain a basis in $W_n$. Let us construct them. For this purpose, we consider the functions $\psi_{i,n}(x)$ for $-2m + 2 \leq i \leq 2^{n-1} - 1$ on an extended partition $\Delta_n$.

The first group of $m - 1$ lacking wavelets is sought in the form

$$\tilde{\psi}_{i,n}(x) = \psi_{i,n}(x) - \sum_{j=-2m+2}^{-m} \alpha_j \psi_{j,n}(x), \quad -m + 1 \leq i \leq -1,$$

by applying the conditions

$$(\tilde{\psi}_{i,n}, N_{m-1,k,n-1}) = 0, \quad k = -m + 1, -m + 2, \ldots, -1.$$
where the inner product is understood in the sense of $L_2[a, b]$.

Substituting (47) into (48), yields a system of linear algebraic equations

$$
\sum_{j=-m+2}^{m} \alpha_j (\psi_{j,n}, N_{m-1,k,n-1}) = (\psi_{i,n}, N_{m-1,k,n-1}), \quad k = -m + 1, -m + 2, \ldots, -1
$$

(49) for determining $\alpha_j$. The matrix of system (49) is nonsingular; otherwise, there would exist a nontrivial solution of the corresponding homogeneous system, meaning that $\sum_{j=-m+2}^{m} \alpha_j \psi_{j,n}(x)$ is a wavelet function on $[a, b]$ with the support $[x_0^n, x_{m-2}^n]$, which is not possible by Theorem 1. Solving system (49), we find that function (47) is the desired wavelet function, since orthogonality is a wavelet function on $[a, b]$, independent of the previously constructed functions, which follows from the form of (47) and the corollary to Theorem 1.

Thus, we have constructed a collection $m - 1$ wavelet functions (47). They are linearly independent of the previously constructed functions, which follows from the form of (47) and the corollary to Theorem 1.

The second group of $m - 1$ lacking wavelets are determined using symmetry considerations:

$$
\tilde{\psi}_{i,n}(x) = \tilde{\psi}_{n_0+1,2n-1-2m+i+1} \left( 2^{n-n_0-1}(1-x) \right), \quad 2^n - 2m + 2 \leq i \leq 2^n - 2m - 1.
$$

(50)

Thus, we have constructed a collection of $m - 1$ wavelet functions (50). Together with functions (41) and (47) they form the desired basis in $W_n$, if $2^n - 2m > 2m - 1$.

As a basis in $L_{n_0}$ we use a collection of "truncated" $B$-splines

$$
\{ \varphi_{k,n_0} = N_{m-1,k,n_0}, -m + 1 \leq k \leq 2^{n_0-1} \}.
$$

(51)

Thus, the collection of functions (51) and (41), (47), (50) for $n_0 + 1 \leq n \leq k$ forms the desired wavelet basis in $L_k$.

**Remark 5** In this section, wavelet functions were constructed in the case of $[a, b] = [0, 1]$. Obviously, a wavelet system on an arbitrary interval $[a, b]$ can be obtained from the above-constructed one with the help of the linear substitution $x' = \frac{x-a}{b-a}$, treating the wavelets as functions of $x'$. For this reason, we assume in what follows that the wavelet system has been constructed for the arbitrary interval $[a, b]$.

5. **Bubnov — Galerkin method for singular integral equations**

Consider a Fredholm integral equation of the second kind

$$
u(x) + \int_a^b K(x, y)u(y)dy = f(x),
$$

(52)

or of the first kind

$$
\int_a^b K(x, y)u(y)dy = f(x),
$$

(53)

where $f$ is a given function and $u$ is an unknown function. Assume that equations (52), (53) have unique solutions an arbitrary continuous $f(x)$, and the kernel satisfies the following estimates with a constant $C > 0$:

$$
\left| \frac{\partial^l K(x, y)}{\partial x^s \partial y^{l-s}} \right| \leq C \frac{1}{|x-y|^l}, \quad 0 \leq l \leq 4.
$$

(54)
Consider the Bubnov—Galerkin methods based on the constructed wavelet functions of degree \( m - 1 \) as applied to equations (52) and (53). Given a fixed positive integer \( k > n_0 \), a solution of equations (53) is sought in the form

\[
u = \sum_{j=-m+1}^{2^{n_0}-1} d_{0j} \phi_{j,n_0} + \sum_{i=1}^{k-n_0} \sum_{j=-m+1}^{2^{n_0+i}-m} c_{ij} \psi_{j,n_0+i}.
\]

(55)

Substituting (55) into the integral equation and taking the scalar product of the result and the basis functions, we obtain (in the case of (53))

\[
\int_a^b \int_a^b K(x,y) \left( \sum_{i,j=-m+1}^{2^{n_0}-1} d_{0ij} \phi_{j,n_0}(y) + \sum_{i=1}^{k-n_0} \sum_{j=-m+1}^{2^{n_0+i}-m} c_{ij} \psi_{j,n_0+i}(y) \right) \phi_{l,n_0}(x) dy dx =
\]

\[
= \int_a^b f(x) \phi_{l,n_0}(x) dx, \quad -m + 1 \leq l \leq 2^{n_0} - 1,
\]

(56)

\[
\int_a^b \int_a^b K(x,y) \left( \sum_{i,j=-m+1}^{2^{n_0}-1} d_{0ij} \phi_{j,n_0}(y) + \sum_{i=1}^{k-n_0} \sum_{j=-m+1}^{2^{n_0+i}-m} c_{ij} \psi_{j,n_0+i}(y) \right) \psi_{l,n}(x) dy dx =
\]

\[
= \int_a^b f(x) \psi_{l,n_0}(x) dx, \quad -m + 1 \leq l \leq 2^{n_0-1} - m, \quad n_0 + 1 \leq n \leq k.
\]

A similar system can be derived in the case of (52).

The collection of conditions (56) is a SLAE with a square matrix \( M \) of order \( 2^k + m - 1 \). It can be represented as \( M = \tilde{A} + A \), where \( \tilde{A} = diag\{ \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{k-n_0-1} \} \), \( \tilde{A}_s = \{ \tilde{a}_{ij}^{(s)} \} \),

\[
a_{ij}^{(1)} = \int_a^b \phi_{i,n_0}(x) \phi_{j,n_0}(x) dx, \quad a_{ij}^{(s)} = \int_a^b \psi_{i,s}(x) \psi_{j,s}(x) dx, \quad \text{i.e.} \quad \tilde{A} \quad \text{is the Gram matrix of basis functions:}
\]

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1k-n_0-1} \\
A_{21} & A_{22} & \cdots & A_{2k-n_0-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k-n_0-1} & A_{k-n_0-2} & \cdots & A_{k-n_0-1k-n_0-1}
\end{pmatrix}, \quad A_{pq} = \{ a_{ij}^{pq} \}
\]

(57)

\[
a_{ij}^{pq} = \int_a^b \int_a^b K(x,y) \psi_{i,p}(y) \psi_{j,v}(x) dy dx \quad (p \geq 2, \nu \geq 2),
\]

\[
a_{ij}^{1v} = \int_a^b \int_a^b K(x,y) \phi_{i,n_0}(y) \phi_{j,v}(x) dy dx, \quad a_{ij}^{11} = \int_a^b \int_a^b K(x,y) \phi_{i,n_0}(y) \phi_{j,n,0}(x) dy dx.
\]

For classical systems of functions in the Galerkin method, these numbers are mainly nonzero and are not small enough to neglect them and treat SLAE (56) as a sparse system. Below, we obtain asymptotically sharp estimates for these numbers in the case of spline wavelets under consideration.

**Remark 6** Here and below, the basis wavelet functions are multiplied by normalizing factors; i.e., the substitutions \( \psi_{i,n} \to 2^{n/2} \psi_{i,n} \), \( \phi_{i,n} \to 2^{n/2} \phi_{i,n} \), are made in the previously constructed functions so that their \( L_2[a,b] \) norms become \( O(1) \).

**Theorem 15** The elements of the matrix \( A \) satisfy the estimates

\[
| a_{ij}^{pp+} | \leq C 2^{-(p+3s/2)} \frac{1}{(1 + |i - \frac{s}{2}|)^m}; \quad | a_{ij}^{ps+p} | \leq C 2^{-(p+3s/2)} \frac{1}{(1 + |j - \frac{s}{2}|)^m}
\]

(58)

where \( C \) is a constant independent of \( p, s, k \).

The theorem implies that the matrix of system (56) is pseudosparse; i.e., many of its elements are small in absolute value.

Similar estimates of correlation matrices of self-similar discrete random processes made it possible to develop and justify efficient decorrelation algorithms for them [15].
6. Conclusion
The article gives an overview of the main results of I. A. Blatov and him pupils in the city of Samara, developing the ideas of Professor V. V. Strygin. These results were obtained in 2000–2019 in three directions: 1. Spline interpolation of functions with a boundary layer. 2. Methods and algorithms for a posteriori adaptation of computational grids for singularly perturbed boundary value problems. 3. Results on applied wavelet analysis based on spline wavelets.

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