Effects of tidal forces and heat flow on the late evolution of the universe

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Abstract. Based upon the intrinsic symmetries approach to inhomogeneous cosmologies we propose an exact solution to Einstein’s field equations where the spatial sections are flat and the source is a non-perfect fluid such that dissipative terms are related to spatial gradients of the energy density. It is shown through the calculation of the luminosity distance as a function of the redshift that the presence of such inhomogeneities may contribute to the acceleration of the expansion of the observable universe and account for part of the dark matter sector. As we compare to the standard ΛCDM cosmological model, this fact is another evidence that inhomogeneities beyond small perturbations should be taken into account in the evolution of the universe.
1 Introduction

Several attempts to describe our actual clumped universe have been made by adopting fewer symmetries than the ones present in the standard cosmological model, described by the Friedmann-Lemaître-Robertson-Walker (FLRW) class of geometries, which is invariant under a six-parameter group of isometries and whose surfaces of transitivity are three-dimensional spacelike hypersurfaces of constant curvature [1]. A number of inhomogeneous geometries have been constructed following this former idea in order to give a more realistic view of cosmology [2, 3]. Special attention is given to the models dealing with the averaging and backreaction issues [4–6], and the exact Lemaître-Tolman (LT) and Szekeres models [7].

A particularly interesting technique to generate inhomogeneous cosmologies is the one developed by Szafron and Collins [8–10] in which exact solutions to the Einstein’s field equations (EFE) are found through the so-called intrinsic symmetries approach, where only submanifolds of the whole space-time admit certain group of isometries. For instance, when the fluid flow is irrotational, one could impose symmetries on the hypersurfaces orthogonal to the fluid seen as three-dimensional manifolds and still the full space-time would possess no symmetries whatsoever.

Following this procedure, exact solutions were studied considering simple matter contents, as vacuum [11], irrotational dust [12, 13] and perfect fluids [14, 15]. In [11], flatness of three-dimensional hypersurfaces (that could be either timelike or spacelike) was imposed together with the condition of a traceless extrinsic curvature, for simplicity. In [12] irrotational dust metrics were considered by demanding that the hypersurfaces orthogonal to the fluid flow had constant curvature, and the authors found that the resulting solutions are either contained in the Szekeres dust solutions [16] or are homogeneous of certain Bianchi types. Also in [13] an irrotational dust solution was considered and a maximal group of isometries, that is, a six-parameter group of motions, was imposed on the spacelike hypersurfaces orthogonal to the fluid velocity, and the conclusion was that all the irrotational dust solutions of EFE with flat spatial geometry were either Bianchi I or were subfamilies of the Szekeres geometry.

In this work we propose an explicit inhomogeneous solution of the EFE by restricting the hypersurfaces orthogonal to the irrotational fluid flow to be flat and considering the matter content to be general, in principle. We find that we only need to determine the energy density function in order to completely specify the geometry and the matter distribution. As
we specify the equations of state for the anisotropic stresses and the pressure, we show how the inhomogeneities in the model yield an accelerated expansion. Compared to the FLRW model, this leads to a behavior usually attributed to the dark energy component.

We begin by presenting in the next section the inhomogeneous solution in detail. Then we move, in Sec. 3, to the analysis of the luminosity distance and the calculation of the deceleration parameter in the approximation of small values of the redshift. We then proceed to put bounds, in Sec. 4, on the parameters of our model based on the observed value for the deceleration parameter. Concluding remarks are presented in Sec. 5 and, for completeness, we compute the Newman-Penrose invariants of the geometry in the Appendix. We adopt conventions as in Ref. [1], except that Greek indices $\alpha, \beta, \gamma \ldots$ run from 0 to 3 and Latin indices $i, j, k \ldots$ run from 1 to 3 (the three spatial directions). Geometric units are assumed such that $c = \kappa = 1$, where $c$ is the speed of light and $\kappa$ is the Einstein constant.

2 The solution

The properties of the metric we propose here are inspired by the observed homogeneity and isotropy of the universe encoded in the cosmological principle, which is usually put forward by adopting the FLRW model with flat spatial curvature scalar. However, this hypothesis can be relaxed by imposing that just the hypersurfaces at constant time $t$ are maximally symmetric without requiring that their corresponding isometries are also symmetries of the whole space-time. With this in mind, the simplest manner to implement this scenario, in the flat case, is through the infinitesimal line element

$$ds^2 = -e^{f(t,x,y,z)} dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (2.1)$$

where $f(t,x,y,z)$ is an arbitrary function (at least of class $C^2$) and $a(t)$ is the scale factor. With no further assumptions, this is an algebraically general metric of Petrov type I (see the Appendix for details). Furthermore, from a recent discussion in the literature [17–19]—in which it was elucidated that linear perturbations of homogeneous and isotropic models lead to different results when compared to a linearized inhomegeneous solutions—we can state that the exact solution of EFE derived here is indeed the simplest generalization of the flat standard model in terms of intrinsic symmetries.

In this model, the fluid flow will be described by the normalized, irrotational and shear-free vector field $u^\mu = e^{-f/2} \delta^\mu_0$ and the flat 3-hypersurfaces, with metric given by $h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu}$, are orthogonal to it. The energy-momentum tensor can be generally decomposed by this field as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu}, \quad (2.2)$$

where $\rho$ is the energy density, $p$ is the isotropic pressure, $q_\mu$ is the heat flow, $\pi_{\mu\nu}$ is the traceless symmetric anisotropic pressure tensor and $q_\nu u^\nu = 0$ and $\pi_{\mu\nu} u^\nu = 0$.

In terms of this matter content, the $0-0$ component of the EFE $G_{\mu\nu} = T_{\mu\nu}$ gives

$$e^{f(t,x,y,z)} = \frac{3H^2(t)}{\rho(t,x,y,z)}, \quad (2.3)$$

where $H = \dot{a}/a$ and dot means derivative with respect to the $t$-coordinate. Similarly to [20], we can rewrite the line element using the scale factor as the time coordinate and introduce an
explicit dependence of the metric with respect to the energy density through relation (2.3), obtaining
\[ ds^2 = -\frac{3}{a^2 \rho(a,x,y,z)} da^2 + a^2 (dx^2 + dy^2 + dz^2). \] (2.4)

In this new coordinate system, the 0−0 component of Einstein’s equations is an identity. The other components lead to
\[ \frac{1}{\sqrt{3} \rho} \partial_i \rho = q_i \] (2.5)
\[ \sqrt{\rho} \left[ \nabla^2 \left( \rho^{-\frac{1}{2}} \right) - \partial_{ij} \left( \rho^{-\frac{1}{2}} \right) \right] - \frac{a^3}{3} \partial_j \rho - a^2 \rho = a^2 p + \pi_{ij}, \quad \text{for } i = j, \] (2.6)
\[ \frac{1}{4} \left( 2\rho \partial_{ij} \rho - 3 \partial_i \rho \partial_j \rho \right) = \pi_{ij}, \quad \text{for } i \neq j, \] (2.7)

where \( \partial_i \equiv \partial / \partial x^i \), \( \partial_{ij} \equiv \partial^2 / \partial x^i \partial x^j \) and \( \nabla^2 := \delta^{ij} \partial_{ij} \) is the 3-dimensional Euclidean Laplacian. Note that the energy density in the equations above comes from the geometry through EFE. Once it is somehow provided, all the other fluid components are determined. Indeed, Eqs. (2.5)-(2.7) can be seen as a set of physically meaningful equations of state for a viscous fluid that is compatible with Einstein’s equations. The physical meaning of this assertion will become clearer afterwards.

Concerning the non-trivial kinematic quantities associated to the fluid four-velocity \( u^\mu \), we find that the expansion coefficient is given by
\[ \vartheta = u^\mu \partial_\mu = \sqrt{3} \rho, \] (2.8)
while the acceleration vector \( a_\mu = u_\mu \eta^\nu \) is expressed as
\[ a_i = -\frac{1}{2} \partial_i \ln \rho. \] (2.9)

Note that the expansion is not homogeneous and the acceleration is nonzero due to the spatial dependence of the energy density. Thus, we see that we are not dealing with a geodesic congruence, even though it is irrotational and shear-free. Due to the non-null acceleration, the evolution of the deviation vector along the fluid flow lines, \( \eta^\mu \), after substituting the EFE, becomes
\[ \frac{d^2 \eta^\mu}{d\lambda^2} = \left[ \frac{1}{2} \pi^\mu_\alpha - E^\mu_\alpha - \frac{(\rho + 3 p)}{6} h^\mu_\alpha \right] \eta^\alpha + a^\mu_\alpha \eta^\alpha, \] (2.10)
where \( E^\mu_\nu = C^\mu_\nu\alpha\beta u^\alpha u^\beta \) denotes the electric part of the Weyl tensor \( C^\mu_\nu\alpha\beta \).

The phenomenological and nonrelativistic equation of state for the anisotropic pressure \( (\pi_{\mu\nu} = -\xi \sigma_{\mu\nu}) \) is not suitable here, since the matter content is described in terms of shear-free comoving observers. In this way, if viscosity is still present in the cosmological fluid in the form of an anisotropic pressure, then it could be related to dissipation of the gravitational potential energy, in particular, through tidal forces [21]. Thus, we have to appeal to an extension of the phenomenological approach by allowing \( \pi_{\mu\nu} \) to be a function of the electric part of the Weyl tensor too. In general cases, the equation of state for the anisotropic pressure should be
\[ \pi_{\mu\nu} = \pi_{\mu\nu}(\sigma_{\alpha\beta}, E_{\alpha\beta}). \]
For the sake of compatibility with the evolution of the kinematical quantities, in particular to the shear evolution (see details in [1, 22]), there is a natural and unique equation of state given by

$$\pi_{\mu\nu} = -2E_{\mu\nu}, \quad (2.11)$$

indicating that the viscosity of the fluid and the tidal forces acting on it are indeed related. Remarkably, the terms involving the acceleration vector and its covariant derivatives in the evolution equation for the shear tensor are proportional to the electric part of the Weyl tensor, showing the consistency of the equation of state aforementioned. It should also be noticed that such relation is not the one derived in the existing literature (see [23–26]). In our case, the spatial components of $E_{\mu\nu}$ can be written down as

$$E_{ij} = \sqrt{\rho} \left[ \frac{1}{2} \partial_{ij} \left( \rho^{-\frac{1}{3}} \right) - \frac{1}{6} \delta_{ij} \nabla^2 \left( \rho^{-\frac{1}{3}} \right) \right]. \quad (2.12)$$

Comparison of Eq. (2.12) with Eq. (2.7), using the equation of state (2.11), gives that the off-diagonal spatial components of Einstein’s equations are identically satisfied. Thus, the remaining diagonal spatial components of the Einstein tensor are all equal to

$$\frac{\nabla^2 \left( \rho^{-\frac{1}{3}} \right)}{\rho^{-\frac{1}{3}}} = \frac{a^2}{2} \left( a \frac{\partial \rho}{\partial a} + 3\gamma \rho \right), \quad (2.13)$$

where we have assumed a barotropic equation of state of the form $p = (\gamma - 1)\rho$ in which $\gamma$ is considered a function of the scale factor.

In the sequence we consider the energy density as a product of a function depending only on $a$ and a function depending only on the spatial coordinates in the convenient following manner

$$\rho(a,x,y,z) = \frac{\epsilon(a)}{[\chi(x,y,z)]^2}, \quad (2.14)$$

resulting, for the time dependent part,

$$a^3 \frac{d\epsilon}{da} + 3\gamma a^2 \epsilon = -2\kappa, \quad (2.15)$$

and for the spatial dependent part

$$\chi \nabla^2 \chi = -\kappa, \quad (2.16)$$

where $\kappa$ is a constant of separability and the minus sign was chosen for later convenience. First, let us analyze the dependence of the energy density with respect to the scale factor. Eq. (2.15) can be directly integrated in the case of $\gamma$ constant, yielding

$$\gamma = \text{const.} \quad \Rightarrow \quad \epsilon(a) = \epsilon_1 \left( \frac{a_0}{a} \right)^{3\gamma} - \frac{2\kappa}{(3\gamma - 2) a^2}, \quad (2.17)$$

where $\epsilon_1 = \epsilon_0 + \frac{2\kappa}{(3\gamma - 2) a_0^2}$ and $\epsilon_0 = \epsilon(a = a_0)$. The case $\gamma = 2/3$ has to be solved separately, resulting in

$$\gamma = 2/3 \quad \Rightarrow \quad \epsilon(a) = \epsilon_0 \left( \frac{a_0}{a} \right)^2 - \frac{2\kappa}{a^2} \ln \left( \frac{a}{a_0} \right). \quad (2.18)$$

\footnote{It must be emphasized that the above condition is a consequence of the shear-free and flat spatial geometry, no other relation being allowed.}
The first term on the right hand side of Eq. (2.17) is the same as in the flat FLRW models and the second term (proportional to $a^{-2}$) mimics a spatial curvature term for $\kappa \neq 0$ and $\gamma \neq 2/3$ which plays an important role at late times in the evolution of the universe, as we shall see. It should also be noticed that this solution is invariant under the transformation $a \rightarrow \lambda a$ and $\kappa \rightarrow \lambda^2 \kappa$. In an expanding universe model there is a combination of the constants such that the energy density goes to zero at a finite value of the scale factor given by

$$\frac{a}{a_0} = \left[ \frac{2\kappa}{(3\gamma - 2)\epsilon_1 a_0^2} \right]^{\frac{1}{(3\gamma - 2)\epsilon_1}}, \quad \text{for} \quad \gamma \neq \frac{2}{3} \quad \text{and} \quad \frac{a}{a_0} = \exp \left( \frac{\epsilon_0 a_0^2}{2\kappa} \right), \quad \text{for} \quad \gamma = \frac{2}{3}.$$

(2.19)

When this happens the metric becomes singular in the same manner as the closed FLRW case.

Therefore, by assuming that the relation in Eq. (2.11) is valid and assuming a separation of variables of the form (2.14) with a linear equation of state for the isotropic pressure, the geometry is then completely determined by solutions of Eq. (2.16) with appropriate boundary conditions. In the literature, this is well-known and corresponds to the static version of a nonlinear Klein-Gordon equation whose potential is inversely proportional to $\chi$ [27].

Solutions of the elliptic partial differential equation (2.16) can be found in the implicit form

$$c_1 x + c_2 y + c_3 z + c_4 + c_5 - \sqrt{c_1^2 + c_2^2 + c_3^2} \int \frac{d\chi}{\sqrt{-2\kappa \ln \chi - (c_1^2 + c_2^2 + c_3^2) c_4}} = 0.$$

(2.20)

However, this general expression is not needed for our purposes here, as we shall see in the next section.

### 3 Redshift and luminosity distance

We perform now an analysis of the luminosity distance in this geometry, restricting ourselves afterwards in the regime of small values of the redshift. With this in mind, we start by computing the geodesic equations for the metric (2.1), which are given by

$$\frac{dk^0}{d\lambda} + H(k^0)^2 + 2 \frac{\chi'}{\chi} k^0 = 0,$$

(3.1)

$$\frac{d\vec{k}}{d\lambda} + \frac{\chi \nabla \chi}{a^2} (k^0)^2 + 2H k^0 \vec{k} = 0,$$

(3.2)

where $k^0 = dt/d\lambda$, $\vec{k} = \left( \frac{dx}{d\lambda}, \frac{dy}{d\lambda}, \frac{dz}{d\lambda} \right)$ and $\chi' = \vec{k} \cdot \nabla \chi$ with $\lambda$ as the affine parameter of the geodesics and where dot ($\cdot$) stands for the usual Euclidean scalar product. The line element provides the constraint

$$b = -\chi^2 (k^0)^2 + a^2(t) (k \cdot \vec{k}),$$

(3.3)

where $b = -1, 0, +1$ if the geodesic is time-like, light-like or space-like, respectively. In particular, to find the corrected expression for the redshift, we only need to study the null geodesics. Thus, substitution of Eq. (3.3) into Eq. (3.1) yields

$$k^0 = \frac{E}{a\chi^2},$$

(3.4)
where $E$ is an integration constant.

According to [1], the redshift definition is

$$1 + z = \frac{(-u_\mu k^\mu)}{(-u_\mu k^\mu)}_{\epsilon},$$

(3.5)
in which the subscript $\epsilon$ indicates the spacetime event where the photon was emitted and the subscript $o$ indicates the spacetime event where the photon was observed. The spatial dependence of $z$ through $\chi$ makes the redshift expression completely different from the one in FLRW models. In this way, the equation for $z$ becomes

$$1 + z(\lambda) = \frac{a_0 \chi_0}{a(\lambda) \chi(\lambda)},$$

(3.6)

where $a_0$ is the scale factor today and $\chi_0$ is the inverse square root of the spatial energy density at the location in which the photon is observed. In virtue of the freedom in the parametrization of the null geodesics, we can choose $(-u_\mu k^\mu)|_o = 1$. It implies here that $E = a_0 \chi_0$.

Now we proceed to calculate the luminosity distance. First, we consider the Sachs equations [28, 29]

$$\frac{d^2D_A}{d\lambda^2} + \left(\Sigma^2 + \frac{1}{2}R_{\mu\nu}k^\mu k^\nu\right)D_A = 0,$$

(3.7)

$$\frac{d\Sigma}{d\lambda} + 2\left(\frac{d\ln D_A}{d\lambda}\right)\Sigma = C_{\alpha\beta\mu\nu}m^\alpha k^\beta m^\mu k^\nu$$

(3.8)

for the angular diameter distance $D_A$ and the shear $\Sigma$ of the null congruence, where $m^\mu$ is a space-like vector orthogonal to $k^\mu$. As we set the initial conditions for the observer

$$D_A(0) = 0 \text{ and } \left.\frac{dD_A}{dz}\right|_0 = \frac{1}{H_0},$$

(3.9)

where $H_0$ is the Hubble parameter at the instant of time in which the photon is observed, we see that for $z = 0$ we have $\frac{d^2D_A}{d\lambda^2} = 0$, which implies,

$$\left(\frac{dz}{d\lambda}\right)^2 \frac{d^2D_A}{dz^2} = -\left(\frac{dz}{d\lambda}\right)^2 \frac{1}{H_0} \text{ as } z = 0.$$

(3.10)

In a certain way, the analysis of $D_A(z)$ up to terms of order $z^2$ around $z = 0$ resembles the Dyer-Roeder approach for studying light propagation in inhomogeneous backgrounds [30], where the optical shear $\Sigma$ and $R_{\mu\nu}k^\mu k^\nu$ are set to zero along every null curve, that is, $\frac{d^2D_A}{d\lambda^2} = 0$ along the entire light path. But it should be clear that we have $\frac{d^2D_A}{d\lambda^2} \neq 0$ for $z \neq 0$, and this changes everything. As we compare to their work, we should expect that the function $D_A(z)$ contains information about the local inhomogeneities such that the introduction of a single phenomenological constant adapted to the FLRW space-time could not describe.

Following the steps presented in Ref. [31], we consider Taylor expansions for $a$ and $\chi$ up to the second order in $\lambda$, as follows

$$a(\lambda) = a_0 \left(1 + a_1 H_0 \lambda + \frac{1}{2} a_2 H_0^2 \lambda^2\right) + O(\lambda^3)$$

(3.11)
\[
\chi(\lambda) = \chi_0 \left( 1 + \chi_1 H_0 \lambda + \frac{1}{2} \chi_2 H_0^2 \lambda^2 \right) + O(\lambda^3). \tag{3.12}
\]

Using Eqs. (3.1) and (3.4), we see that the parameters \(a_1\) and \(a_2\) are related to the constants \(\chi_i\) for \(i = 0, 1, 2\) through

\[
a_1 = \frac{1}{\chi_0} \quad \text{and} \quad a_2 = -\frac{1 + q_0 + 2\chi_0 \chi_1}{\chi_0^2}, \tag{3.13}
\]

where \(q(t) = -\ddot{a}/aH^2\) is the FLRW deceleration parameter and \(q_0\) its value in the moment of the observation.

At this order, we are able to compute the contribution of the inhomogeneity to the deceleration parameter and then try, in a first moment, to put bounds on such contributions based on observations. Thus, differentiating Eq. (3.6) with respect to \(\lambda\) and rewriting the outcome in terms of \(z\) yields

\[
z = 0 \Rightarrow \frac{dz}{d\lambda} = -H_0 (a_1 + \chi_1) \quad \text{and} \quad \frac{d^2z}{d\lambda^2} = H_0^2 \left[ 2(a_1^2 + a_1 \chi_1 + \chi_1^2) - \chi_2 - a_2 \right]. \tag{3.14}
\]

As we use the relation \(D_L = (1 + z)^2 D_A\) between the luminosity and angular distances [32], the Taylor expansion of order \(z^2\) is expressed as [33]

\[
D = \frac{D^{(1)}}{H_0} z + \frac{D^{(2)}}{2H_0} z^2, \tag{3.15}
\]

where the coefficients are linked through

\[
\begin{align*}
D^{(1)}_L &= D^{(1)}_A = 1, \\
D^{(2)}_L &= D^{(2)}_A + 4 \equiv 1 - q_{\text{eff}} \tag{3.16}
\end{align*}
\]

with \(q_{\text{eff}}\) the effective (observed) deceleration parameter. We get from the equations (3.10) and (3.14)

\[
D^{(2)}_A = -\frac{2(a_1^2 + a_1 \chi_1 + \chi_1^2) - \chi_2 - a_2}{(a_1 + \chi_1)^2}, \tag{3.17}
\]

resulting for the effective deceleration parameter the expression

\[
q_{\text{eff}} = \frac{q_0 + 2\chi_1 - 4\chi_0 \chi_1 - \chi_0^2 \chi_1^2 - \chi_0^2 \chi_2}{(1 + \chi_0 \chi_1)^2}. \tag{3.18}
\]

Note that \(q_{\text{eff}}\) reduces to the usual FLRW value \(q_0\) when we neglect the inhomogeneous terms \(\chi_1\) and \(\chi_2\), as expected.

In the next section we attempt to give a first approximation of the expressions found above relating the inhomogeneities to the cosmological parameters in order to verify whether they can fit observations.
4 Accelerated expansion from inhomogeneities

We now provide a first analysis showing how our model can fit the data regarding the observations of the accelerated expansion of the universe. To that end, we consider the physical interpretation of the parameters composing \( q_{\text{eff}} \), where we let \((t_0, x_0, y_0, z_0)\) represent the event of observation at \( z = 0 \) and set \( f(t_0, x_0, y_0, z_0) = 0 \), \( \rho_0 := \rho(t_0, x_0, y_0, z_0) \) and \( \epsilon_0 = \epsilon(a_0) = \rho_0 \). This implies that

\[
\chi_0 = 1, \quad E = a_0 \quad \text{and} \quad \epsilon_0 = 3H_0^2.
\]

(4.1)

We now use \( \gamma_0 = \gamma(a_0) \) and the relation \( \epsilon = 3H^2 \) in the equation (2.15) to determine \( q_0 \) as

\[
q_0 = 3 \frac{\gamma_0}{2} - 1 + \frac{\kappa}{3a_0^2 H_0^2}.
\]

(4.2)

From Eqs. (3.3) and (3.4) we have for the norm of the wave vector \( \|\vec{k}\| = \sqrt{\vec{k} \cdot \vec{k}} \)

\[
\|\vec{k}\| = \frac{1}{a_0} \left( \frac{a_0}{a} \right)^2 \sqrt{\frac{\rho}{\epsilon}}.
\]

(4.3)

As we use the definitions of \( \chi \) and \( \chi_1 \), we obtain

\[
\chi_1 = \left( \frac{1}{H \chi} \frac{d\chi}{d\lambda} \right)_{z=0} = - \left( \frac{1}{2H \rho} \vec{k} \cdot \vec{\nabla} \rho \right)_{z=0}.
\]

(4.4)

Therefore, denoting the angle between \( \vec{\nabla} \rho \) and \( \vec{k} \) at the event of observation by \( \psi_0 \),

\[
\chi_1 = - \frac{1}{6a_0 H_0^3} \|\vec{\nabla} \rho\|_0 \cos \psi_0,
\]

(4.5)

where \( \|\vec{\nabla} \rho\|_0 = \|\vec{\nabla} \rho(t_0, x_0, y_0, z_0)\| \). The definition of \( \chi_2 \), in its turn, is

\[
\chi_2 = \left( \frac{1}{H^2 \chi} \frac{d^2\chi}{d\lambda^2} \right)_{z=0} = \frac{1}{H_0^2} \left( \frac{d\vec{k}}{d\lambda} \vec{\nabla} \chi + k^i k^j \partial_i \partial_j \chi \right)_{z=0}.
\]

(4.6)

If we use the equations (3.2), (3.4) and (4.5), we obtain

\[
\frac{1}{H_0^2} \left( \frac{d\vec{k}}{d\lambda} \cdot \vec{\nabla} \chi \right)_{z=0} = \frac{\|\vec{\nabla} \rho\|_0}{3a_0 H_0^3} \cos \psi_0 - \left( \frac{\|\vec{\nabla} \rho\|_0}{6a_0 H_0^3} \right)^2.
\]

(4.7)

As \( \Delta \chi \) is the trace of \( \partial_i \partial_j \chi \), we define \( B_{ij} \) to be its traceless part at the event of observation \((\delta^{ij}B_{ij} = 0)\). Therefore, from equation (2.16), we get

\[
\partial_i \partial_j \chi(t_0, x_0, y_0, z_0) = - \frac{\kappa}{3} \delta_{ij} + B_{ij}.
\]

(4.8)

Since the last term becomes diagonal by an Euclidean rotation, we can take it as the diagonal matrix \( B_{ij} = B \text{ diag} \{\sin b, \sin(b + 2\pi/3), \sin(b + 4\pi/3)\} \), where \( B \) measures the intensity of the anisotropy in the Hessian matrix of \( \chi \) at the observation point and \( b \) its angular distribution\(^2\). Defining the polar and azimuthal angles in the usual way, with \( k^1 = \|k\| \cos \varphi \),

\(^2\)This kind of treatment is common in Bianchi I spacetimes. See [34], for example.
\( k^2 = \|k\| \cos \theta \sin \varphi \) and \( k^3 = \|k\| \sin \theta \sin \varphi \), we substitute the equations (4.7) and (4.8) in (4.6) to get

\[
\chi_2 = -\frac{\kappa}{3 a_0^2 H_0^2} \nabla^2 \rho(t_0, x_0, y_0, z_0) \cos \psi_0 - \left( \frac{\|\nabla \rho\|_0}{6 a_0 H_0^2} \right)^2 + \frac{B}{a_0^2 H_0^2} A(\varphi, \theta, b),
\]

where

\[
A(\varphi, \theta, b) = \left( 1 - \frac{3}{2} \sin^2 \varphi \right) \sin b - \frac{\sqrt{3}}{2} \cos(2\theta) \sin^2 \varphi \cos b.
\]

Returning to the effective deceleration parameter in the equation (3.18), we get

\[
q_{\mathrm{eff}} = \frac{1}{2} \left( \frac{\Omega_M}{\Omega_I} - \frac{2}{3} \Omega_I \right).
\]

Clearly, \( \Omega_I = 0 \) suffices to recover the equivalent \( \Lambda \)CDM formula. If we assume \( q_{\mathrm{eff}} = -0.598 \) [35] for instance, we would have the constraint

\[
\Omega_\Lambda = 0.732 - \frac{2}{3} \Omega_I.
\]

Furthermore, this component might also diminish the contribution of the cold dark matter in \( \Omega_M \), for we would have

\[
\Omega_M = 0.268 - \frac{1}{3} \Omega_I.
\]

If we add to this equation the condition \( \Omega_M > 0 \), we get the inequality

\[
0 \leq \Omega_I < 0.804.
\]
5 Concluding Remarks

An exact solution of Einstein equations was derived which presents maximally symmetric sub-
manifolds corresponding to flat tri-dimensional spaces of constant time. The matter content
required for this is a viscous fluid where the dissipative terms are given through physically
reasonable equations of state. The set of equations can be split into time and space, where
the time evolution is driven by Friedmann equation with an effective curvature term ($\sim a^{-2}$)
while the spatial dependence is given by a sort of a nonlinear Klein-Gordon equation. In
comparison to the FLRW models, the expressions for the redshift and the luminosity distance
are more complicate due to the presence of inhomogeneties. Thus, for small redshifts, we solve
the Sachs equation perturbatively up to second order admitting a Taylor expansion of the
solution. In this way, we were able to compute the angular diameter and luminosity distances
and, consequently, we could find an equation for the cosmological deceleration parameter in
which is clear how inhomogeneities could contribute to a late accelerated expansion.

Here we have shown, above all other possible conclusions, that models with nonlinear
effects of inhomogeneity (and possibly anisotropy) beyond small perturbations of FLRW must
be pushed forward and tested against the standard model. Just after such a tenacious scrutiny
we could conclude that the Universe is of a $\Lambda$CDM type, or composed just by ordinary matter
in an inhomogeneous and anisotropic way or somewhere in between.

A Newman-Penrose (NP) invariants

In the $(a, x, y, z)$ coordinate system, we define a null tetrad basis $(l^\mu, n^\nu, m^\mu, \bar{m}^\mu)$ in which
the vectors are given by

$$l^\mu = \left(a \sqrt{\frac{\rho}{3}}, 0, \frac{1}{a}, 0\right), \quad n^\mu = \left(\frac{a}{2} \sqrt{\frac{\rho}{3}}, 0, -\frac{1}{2a}, 0\right) \quad \text{and} \quad m^\mu = \left(0, i \sqrt{\frac{a}{2}}, 0, \frac{1}{\sqrt{2a}}\right)$$

and $\bar{m}^\mu$ is the complex conjugate of $m^\mu$. This basis satisfies the relations $l^\mu m_\mu = 0$, $n^\mu m_\mu = 0$,
$l^\mu n_\mu = -1$ and $m^\mu \bar{m}_\mu = 1$. In possession of this, a direct calculation provides the NP scalars
associated to the Weyl tensor as

$$\Psi_0 = \frac{(\partial_x + i \partial_y)^2 \chi}{2a^2 \chi},$$
$$\Psi_1 = \sqrt{2} \partial_x (\partial_x + i \partial_y) \chi / 4a^2 \chi,$$
$$\Psi_2 = \frac{\nabla^2 \chi}{8a^2 \chi} - \frac{(\partial_x + i \partial_y) (\partial_x - i \partial_y) \chi}{4a^2 \chi},$$
$$\Psi_3 = \sqrt{2} \partial_x (\partial_x - i \partial_y) \chi / 8a^2 \chi,$$
$$\Psi_4 = \frac{(\partial_x - i \partial_y)^2 \chi}{8a^2 \chi}. \quad (A.1)$$

Following the scheme presented in [36] we are led to conclude that this geometry is alge-
braically general, that is, a Petrov type I spacetime. For completeness, the other invariants
related to the Ricci tensor are found to be
\[ \Phi_{00} = \frac{(\partial_+ + i\partial_-)(\partial_+ - i\partial_-)}{2a^2\chi} \chi - \frac{1}{\chi} \left( \frac{\dot{a}}{a} - \frac{2a\ddot{a}_\chi}{a^2} - \frac{\dot{a}^2}{a^2} \right), \]

\[ \Phi_{01} = \frac{\sqrt{2}}{4a^2\chi} \frac{(2i\chi\partial_+ - \partial_+)(\partial_+ + i\partial_-)}{\chi}, \]

\[ \Phi_{02} = -\frac{(\partial_+ + i\partial_-)^2}{4a^2\chi} \chi = -2\Psi_0, \]

\[ \Phi_{11} = \frac{\partial_{yy}}{4a^2\chi} + \frac{1}{4\chi^2} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right), \]

\[ \Phi_{12} = \frac{\sqrt{2}}{8a^2\chi^2} \frac{(2i\chi\partial_+)(\partial_+ - i\partial_-)}{\chi}, \]

\[ \Phi_{22} = \frac{(\partial_+ + i\partial_-)(\partial_+ - i\partial_-)}{8a^2\chi} - \frac{1}{4\chi^2} \left( \frac{\ddot{a}}{a} + \frac{2a\ddot{a}_\chi}{a^2} - \frac{\dot{a}^2}{a^2} \right), \]

\[ R = \frac{6}{\chi} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) - \frac{2\nabla_2^2\chi}{a^2\chi}. \]

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References

[1] G. F. R. Ellis, R. Maartens and M.A.H. MacCallum, Relativistic Cosmology, Cambridge University Press (1973).

[2] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, Exact solutions to Einstein's field equations, 2nd edition, Cambridge University Press (2003).

[3] A. Krasinski, Inhomogeneous Cosmological Models, Cambridge University Press, New York (1997).

[4] D. L. Wiltshire, What is dust?–Physical foundations of the averaging problem in cosmology, Class. Quantum Grav. 28 164006 (2011).

[5] E. W. Kolb, Backreaction of inhomogeneities can mimic dark energy, Class. Quantum Grav. 28 164009 (2011).

[6] T. Buchert, On Average Properties of Inhomogeneous Fluids in General Relativity: Dust Cosmologies, Gen. Relat. Grav. 32 105 (2000).

[7] K. Bolejko, M.-N. Célérier and A. Krasinski, Inhomogeneous cosmological models: exact solutions and their applications, Class. Quantum Grav. 28 164002 (2011). ArXiv:1102.1449v2 [astro-ph.CO]

[8] C. B. Collins and D. A. Szafron, A new approach to inhomogeneous cosmologies: Intrinsic Symmetries. I., J. Math. Phys. 20, 2347 (1979).

[9] D. A. Szafron and C. B. Collins, A new approach to inhomogeneous cosmologies: Intrinsic Symmetries. II. Conformally flat slices and an invariant classification, J. Math. Phys. 20, 2354 (1979).

[10] C. B. Collins and D. A. Szafron, A new approach to inhomogeneous cosmologies: Intrinsic Symmetries. III. Conformally flat slices and their analysis, J. Math. Phys. 20, 2362 (1979).

[11] T. Wolf, About vacuum solutions to Einstein's field equations with flat three-dimensional hypersurfaces, J. Math. Phys. 27, 2354 (1986).

[12] C. Bona and P. Palou, Dust metrics with comoving constant curvature slices, J. Math. Phys. 33, 705 (1992).
[13] C. F. Sopuerta, Cosmological models with flat spatial geometry, Class. Quantum Grav. 17, 4685 (2000).
[14] F. Argueso and J. L. Sanz, Some exact inhomogeneous solutions of Einstein’s equations with symmetries on the hypersurface $t = \text{const}$, J. Math. Phys. 26, 3118 (1985).
[15] T. Wolf, A class of perfect fluid metrics with flat three-dimensional hypersurfaces, J. Math. Phys. 27, 2340 (1986).
[16] P. Szekeres, A class of inhomogeneous cosmological models, Commun. Math. Phys. 41, 55 (1975).
[17] S. R. Green and R. M. Wald, How well is our universe described by an FLRW model?, Class. Quantum Grav. 31 (2014) 234003.
[18] T. Buchert, M. Carfora, G.F.R. Ellis, E.W. Kolb, M.A.H. MacCallum, J.J. Ostrowski, S. Rasanen, B.F. Roukema, L. Andersson, A.A. Coley, D.L. Wiltshire, Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?, Class. Quantum Grav. 32 (2015) 215021.
[19] S. R. Green, R. M. Wald, Comments on Backreaction, arXiv:1506.06452 [gr-qc].
[20] L. G. Gomes, On the local form of static plane symmetric spacetimes in the presence of matter, Class. Quantum Grav. 32 185010 (2015).
[21] A. P. Douce, Thermodynamics of the Earth and Planets Cambridge University Press (2011).
[22] M. Novello, E. Bittencourt and J. M. Salim, The Quasi-Maxwellian Equations of General Relativity: Applications to Perturbation Theory, Braz. J. Phys. 44 832 (2014).
[23] Mimoso J P and Crawford P 1993 Shear-free anisotropic cosmological models Class. Quantum Grav. 10 315.
[24] D.J. McManus and A.A. Coley, Shear-free irrotational, geodesic, anisotropic fluid cosmologies, Class. Quant. Grav. 11 (1994) 2645 [gr-qc/9405035].
[25] E. Bittencourt, J.M. Salim and G.B. dos Santos, Magnetic fields and the Weyl tensor in the early universe, Gen. Rel. Grav. 46 (2014) 1790 [arXiv:1402.3121].
[26] G.B. Santos, E. Bittencourt and J.M. Salim, Scalar perturbations in a Friedmann-like metric with non-null Weyl tensor JCAP06(2015)013.
[27] A. D. Polyanin ans V. F. Zaitsev, Handbook of nonlinear partial differential equations, CRC Press (2011).
[28] K. Bolejko and P. G. Ferreira, Ricci focusing, shearing, and the expansion rate in an almost homogeneous Universe, JCAP05(2012)003.
[29] E. Bentivegna, T. Clifton, J. Durk, M. Korzyński and K. Rosquist, Black-hole lattices as cosmological models, Class. Quantum Grav. 35 175004 (2018).
[30] C. C. Dyer and R. C. Roeder, The Distance-Redshift Relation for Universes with no Intergalactic Medium, Astrophys. J. 174 L115.
[31] Mattia Villani, Taylor expansion of luminosity distance in Szekeres cosmological models: effects of local structure evolution on cosmographic parameters, JCAP06(2014)015.
[32] I. M. H. Etherington, On the definition of distance in general relativity, Philosophical Magazine ser. 7 15, 761 (1933).
[33] Matt Visser, Jerk, snap and the cosmological equation of state, 2004 Class. Quantum Grav. 21 2603.
[34] E. Bittencourt, L. Gomes and R. Klippert, Bianchi-I cosmology from causal thermodynamics, Class. Quantum Grav. 34 045010 (2017).
[35] J. Lu, L. Xu and M. Liu, 
Constraints on kinematic models from the latest observational data, Phys. Lett. B 699 (2011).

[36] M. A. Acevedo M., M. Enciso-Aguilar, J. López- Bonilla, 
Petrov classification of the conformal tensor. EJTP 9 (2006) 79â–82.