Existence of solutions for a nonlocal type problem in fractional Orlicz Sobolev spaces

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Abstract
In this paper, we investigate the existence of weak solution for a fractional type problems driven by a nonlocal operator of elliptic type in a fractional Orlicz–Sobolev space, with homogeneous Dirichlet boundary conditions. We first extend the fractional Sobolev spaces $W^{s,p}$ to include the general case $W^{s,L_A}$, where $A$ is an N-function and $s \in (0, 1)$. We are concerned with some qualitative properties of the space $W^{s,L_A}$ (completeness, reflexivity and separability). Moreover, we prove a continuous and compact embedding theorem of these spaces into Lebesgue spaces.

Keywords Fractional Orlicz–Sobolev spaces · Fractional $a$-Laplace operator · Direct method in calculus of variations

Mathematics Subject Classification 35R11 · 46E30 · 58E05 · 35J60

1 Introduction
In this paper, we establish the existence of a weak solution for the following Dirichlet type equation

\[(P_a) \begin{cases} (-\Delta)^s_a u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}\]

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where \( \Omega \) is an open bounded subset in \( \mathbb{R}^N \) with Lipschitz boundary \( \partial \Omega \), \( 0 < s < 1 \), \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function and \((-\Delta)^s_a\) is the fractional \(a\)-Laplacian operator defined as

\[
(-\Delta)^s_a u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{dy}{|x - y|^{N+s}},
\]

where \( a : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) which will be specified later.

The study of nonlinear elliptic equations involving quasilinear homogeneous type operators is based on the theory of Sobolev spaces and fractional Sobolev spaces \( W^{s,p}(\Omega) \) in order to find weak solutions. In certain equations, precisely in the case of nonhomogeneous differential operators, when trying to relax some conditions on these operators (as growth conditions), the problem can not be formulated with classical Lebesgue and Sobolev spaces. Hence, the adequate functional spaces is the so-called Orlicz–Sobolev spaces. These spaces consists of functions that have weak derivatives and satisfy certain integrability conditions. Many properties of Orlicz–Sobolev spaces were proved in [1,23,27,37]. For this, many researchers have studied the existence of solutions for the eigenvalue problems involving nonhomogeneous operators in the divergence form through Orlicz–Sobolev spaces by using variational methods and critical point theory, monotone operator methods, fixed point theory and degree theory (see, for instance, [2,6,10,11,19]).

The problem \((P_a)\) involves the fractional \(a\)-Laplacian operator, the most appropriate functional framework for dealing with this problem is the fractional Orlicz Sobolev space introduced by Salort et al [13], namely a fractional Sobolev space constructed from an Orlicz space at the place of \( L^p(\Omega) \). As we know, the Orlicz spaces represent a generalization of classical Lebesgue spaces in which the role usually played by the convex function \( t^p/p \) is assumed by a more general convex function \( A(t) \); they have been extensively studied in the monograph of Krasnosel’skii and Rutickii [27] as well as in Luxemburg’s doctoral thesis [30]. If the role played by \( L^p(\Omega) \) in the definition of fractional Sobolev spaces \( W^{s,p}(\Omega) \) is assigned to an Orlicz \( L_A(\Omega) \) space, the resulting space \( W^s L_A(\Omega) \) is exactly a fractional Orlicz–Sobolev space. Many properties of fractional Sobolev spaces have been extended to fractional Orlicz–Sobolev spaces (see Sect. 3).

In applied PDE, fractional spaces and the corresponding nonlocal equations, are now experiencing impressive applications in different subjects, such as, among others, the thin obstacle problem [32], finance [20], phase transitions [3,14], stratified materials [18], crystal dislocation [9], soft thin films [28], semipermeable membranes and flame propagation [15], conservation laws [8], ultra-relativistic limits of quantum mechanics [25], quasi-geostrophic flows [17], multiple scattering [24], minimal surfaces [16], materials science [7], water waves [41], gradient potential theory [34] and singular set of minima of variational functionals [33]. See also [39] for further motivation.
When \( a(t) = t^{p-1} \), problem \((P_a)\) reduces to the fractional \( p\)-Laplacian problem

\[
(P_p) \quad \begin{cases}
    (-\Delta)^s_p u = f(x, u) & \text{in } \Omega \\
    u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \((-\Delta)^s_p\) is the fractional \( p\)-Laplacian operator which, up to normalization, may be defined as

\[
(-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy.
\]

One typical feature of problem \((P_p)\) is the nonlocality, in the sense that the value of \((-\Delta)^s_p u(x)\) at any point \( x \in \Omega \) depends not only on \( \Omega \), but actually on the entire space \( \mathbb{R}^N \). In recent years, problem \((P_p)\) has been studied in many papers, we refer to \([4,5,26,29]\), in which the authors have used different methods to get existence of solutions for \((P_p)\).

This paper is organized as follows: in the second section, we recall some properties of Orlicz–Sobolev and fractional Sobolev spaces. The third section is devoted to prove some properties of fractional Orlicz–Sobolev spaces. Finally, using the direct method in calculus variations, we obtain the existence of a weak solution of problem \((P_a)\).

## 2 Some preliminaries results

First, we briefly recall the definitions and some elementary properties of the Orlicz–Sobolev spaces. We refer the reader to \([1,27,37]\) for further reference and for some of the proofs of the results in this section.

### 2.1 Orlicz–Sobolev Spaces

We start by recalling the definition of the well-known N-functions.

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). Let \( A : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be an N-function, that is, \( A \) is continuous, convex, with \( A(t) > 0 \) for \( t > 0 \), \( \frac{A(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{A(t)}{t} \to \infty \) as \( t \to \infty \). Equivalently, \( A \) admits the representation \( A(t) = \int_0^t a(s)ds \) where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) is non-decreasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) \( \forall t > 0 \) and \( a(t) \to \infty \) as \( t \to \infty \). The conjugate N-function of \( A \) is defined by \( A(t) = \int_0^t \overline{a}(s)ds \), where \( \overline{a} : \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( \overline{a}(t) = \sup \{ s : a(s) \leq t \} \). Evidently we have

\[
st \leq A(t) + \overline{A}(s),
\]

which is known as Young inequality. Equality holds in (1) if and only if either \( t = \overline{a}(s) \) or \( s = a(t) \).
We will extend these N-functions into even functions on all $\mathbb{R}$. The N-function $A$ is said to satisfy the global $\Delta_2$-condition if, for some $k > 0$,

$$A(2t) \leq kA(t), \forall t \geq 0.$$  

When this inequality holds only for $t \geq t_0 > 0$, $A$ is said to satisfy the $\Delta_2$-condition near infinity.

We call the pair $(A, \Omega)$ is $\Delta$-regular if either :

- $A$ satisfies a global $\Delta_2$-condition, or
- $A$ satisfies a $\Delta_2$-condition near infinity and $\Omega$ has finite volume.

Throughout this paper, we assume that

$$1 < p^- := \inf_{s > 0} \frac{sa(s)}{A(s)} < p^+ := \sup_{s > 0} \frac{sa(s)}{A(s)} < +\infty.$$  \hspace{1cm} (2)

which assures that $A$ satisfies the global $\Delta_2$-condition (see [31, Proposition 2.3]).

**Lemma 1** (see. [13, Lemma 2.9]). Let $A$ be an N-function which satisfies the global $\Delta_2$-condition. Then we have,

$$\overline{A}(a(t)) \leq cA(t) \text{ for all } t \geq 0$$  \hspace{1cm} (3)

where $c > 0$.

**Definition 1** Let $A, B$ be two N-function. $A$ is stronger (resp essentially stronger) than $B$, $A \succ B$ (resp $A \succ\succ B$) in symbols, if

$$B(x) \leq A(ax), \quad x \geq x_0 \geq 0,$$

for some (resp for each) $a > 0$ and $x_0$ (depending on $a$).

**Remark 1** (see. [1, Section 8.5]). $A \succ\succ B$ is equivalent to the condition

$$\lim_{x \to \infty} \frac{B(\lambda x)}{A(x)} = 0,$$

for all $\lambda > 0$.

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $K_A(\Omega)$ (resp. the Orlicz space $L_A(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions $u$ on $\Omega$ such that

$$\int_{\Omega} A(|u(x)|)dx < \infty \quad \text{(resp. } \int_{\Omega} A(\lambda |u(x)|)dx < \infty \text{ for some } \lambda > 0).$$  \hspace{1cm} (4)
is a Banach space under the Luxemburg norm
\[
\|u\|_A = \inf \left\{ \lambda > 0 : \int_{\Omega} A \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\},
\]  

and \( K_A(\Omega) \) is a convex subset of \( L_A(\Omega) \). The closure in \( L_A(\Omega) \) of the set of bounded measurable functions on \( \Omega \) with compact support in \( \overline{\Omega} \) is denoted by \( E_A(\Omega) \). The equality \( E_A(\Omega) = L_A(\Omega) \) holds if and only if \( (A, \Omega) \) is \( \Delta \)-regular.

Using the Young’s inequality, it is possible to prove a Hölder type inequality, that is,
\[
\left| \int_{\Omega} u v \, dx \right| \leq 2 \|u\|_A \|v\|_{\overline{A}} \quad \text{for all } u \in L_A(\Omega) \text{ and all } v \in L_{\overline{A}}(\Omega).
\]  

**Proposition 1** (see. [31, Proposition 2.1]) Let \( A \) be an \( N \)-function, assume condition (2) is satisfied. Then the following relations holds true
\[
\|u\|_A^p \leq \int_{\Omega} A(|u|) \, dx \leq \|u\|_A^{p+}, \quad \forall u \in L_A(\Omega) \text{ with } \|u\|_A > 1,
\]
\[
\|u\|_A^{p+} \leq \int_{\Omega} A(|u|) \, dx \leq \|u\|_A^{p-}, \quad \forall u \in L_A(\Omega) \text{ with } \|u\|_A < 1.
\]

**Theorem 1** (see. [1, Theorem 8.23]) Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) which has a finite volume, and suppose \( A, B \) two \( N \)-function such that \( B \ll A \). Then any bounded subset \( S \) of \( L_A(\Omega) \) which is precompact in \( L^1(\Omega) \), is also precompact in \( L_B(\Omega) \).

### 2.2 Fractional Sobolev spaces

This subsection is devoted to the definition of fractional Sobolev spaces, and we recall some result of regarding continuous and compact embedding of these spaces. We refer the reader to [21,22] for further reference and for some of the proofs of these results.

We start by fixing the fractional exponent \( s \in (0, 1) \). For any \( p \in [1, \infty) \), we define the fractional Sobolev space \( W^{s,p}(\Omega) \) as follows:
\[
W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{N+s/p}} \in L^p(\Omega \times \Omega) \right\},
\]
that is, an intermediary Banach space, endowed with its natural norm
\[
\|u\|_{s,p} = \left( \int_{\Omega} |u|^p \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+N}} \, dxdy \right)^{1/p}.
\]
Theorem 2 (see: [21, Corollary 4.53], [22]) Let $s \in (0, 1)$, $p \in [1, +\infty)$ and let $\Omega$ be an open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. Then there exists a constant $C = C(N, s, p, \Omega)$ such that, for all $f \in W^{s,p}(\Omega)$, we have

$$\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)}$$

for all $q \in [p, p_s^*]$, that is,

$$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ for all } q \in [p, p_s^*],$$

where

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } N > sp \\ \infty & \text{if } N \leq sp. \end{cases}$$

If, in addition, $\Omega$ is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*]$.

Theorem 3 (see: [21, Theorem 4.58]). Let $s \in (0, 1)$, $p \in [1, +\infty)$ and let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. Then

- if $sp < N$, then the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, p_s^*]$;
- if $sp = N$, then the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, \infty)$;
- if $sp > N$, then the embedding $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.

Theorem 4 (see: [40, Theorem 1.2]) Suppose that $X$ is a reflexive Banach space with norm $\|\cdot\|$ and let $V \subset X$ be a weakly closed subset of $X$. Suppose $E : V \to \mathbb{R} \cup \{+\infty\}$ is coercive and (sequentially) weakly lower semi-continuous on $V$ with respect to $X$, that is, suppose the following conditions are fulfilled:

- $E(u) \to \infty$ as $\|u\| \to \infty$, $u \in V$.
- For any $u \in V$, any sequence $\{u_n\}$ in $V$ such that $u_n \rightharpoonup u$ weakly in $X$ there holds:

$$E(u) \leq \liminf_{n \to \infty} E(u_n).$$

Then $E$ is bounded from below on $V$ and attains its infimum in $V$.

3 Variational framework

Now, we define the fractional Orlicz–Sobolev spaces introduced [13], and we will present some important results of them.

Definition 2 Let $A$ be an N-function. For a given domain $\Omega$ in $\mathbb{R}^N$ and $0 < s < 1$, we define the fractional Orlicz–Sobolev space $W^s L_A(\Omega)$ as follows:

$$W^s L_A(\Omega) = \left\{ u \in L_A(\Omega) : \int_{\Omega} \int_{\Omega} A \left( \frac{\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} < \infty \text{ for some } \lambda > 0 \right\}.$$
This space is equipped with the norm,

\[ ||u||_{s,A} = ||u||_A + [u]_{s,A}, \]  

(7)

where \([\cdot]_{s,A}\) is the Gagliardo seminorm, defined by

\[ [u]_{s,A} = \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dx dy}{|x - y|^N} \leq 1 \right\}. \]  

(8)

**Definition 3** Let \(A\) be an \(N\)-function. For a given domain \(\Omega\) in \(\mathbb{R}^N\) and \(0 < s < 1\), we define the space \(W^s E_A(\Omega)\) as follows:

\[ W^s E_A(\Omega) = \left\{ u \in E_A(\Omega) : \int_{\Omega} \int_{\Omega} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} < \infty \right\}. \]  

(9)

**Remark 2**
- \(W^s E_A(\Omega) \subset W^s L_A(\Omega)\).
- \(W^s E_A(\Omega)\) coincides with \(W^s L_A(\Omega)\) if and only if \((A, \Omega)\) is \(\Delta\)-regular.
- If \(1 < p < \infty\) and \(A_p(t) = t^p\), then \(W^s L_{A_p}(\Omega) = W^s E_{A_p}(\Omega) = W^{s,p}(\Omega)\).

To simplify the notation, we put

\[ h_{x,y}(u) := \frac{u(x) - u(y)}{|x - y|^s}. \]

**Theorem 5** (see: [13, Proposition 2.11]) Let \(\Omega\) be an open subset of \(\mathbb{R}^N\), and let \(s \in (0, 1)\). The space \(W^s L_A(\Omega)\) is a Banach space with respect to the norm (7), and a separable (resp. reflexive) space if and only if \((A, \Omega)\) is \(\Delta\)-regular (resp. \((A, \Omega)\) and \((\overline{A}, \Omega)\) are \(\Delta\)-regular). Furthermore if \((A, \Omega)\) is \(\Delta\)-regular and \(A(\sqrt{t})\) is convex, then the space \(W^s L_A(\Omega)\) is uniformly convex.

Let \(\tilde{W}^s L_A(\Omega)\) denote the closure of \(C_0^\infty(\Omega)\) in the norm \(||\cdot||_{s,A}\) defined in (7). Then we have the following result.

**Theorem 6** (Generalized Poincaré inequality). Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^N\), and let \(s \in (0, 1)\). Let \(A\) be an \(N\)-function. Then there exists a positive constant \(\mu\) such that,

\[ ||u||_A \leq \mu [u]_{s,A} \]  

for all \(u \in \tilde{W}^s L_A(\Omega)\).

Therefore, if \(\Omega\) is bounded and \(A\) be an \(N\)-function, then \([\cdot]_{s,A}\) is a norm of \(\tilde{W}^s L_A(\Omega)\) equivalent to \(||\cdot||_{s,A}\).

**Proof of Theorem 6** Since \(\tilde{W}^s L_A(\Omega)\) is the closure of \(C_0^\infty(\Omega)\) in \(W^s L_A(\Omega)\), then it is enough to prove that there exists a positive constant \(\mu\) such that,

\[ ||u||_A \leq \mu [u]_{s,A} \]  

for all \(u \in C_0^\infty(\Omega)\).
Indeed, let \( u \in C_0^\infty(\Omega) \) and \( B_R \subset \mathbb{R}^N \setminus \Omega \), that is, the ball of radius \( R \) in the complement of \( \Omega \). Then for all \( x \in \Omega \), \( y \in B_R \) and all \( \lambda > 0 \) we have,

\[
A \left( \frac{|u(x)|}{\lambda} \right) = A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{s}} \right) \frac{|x - y|^N}{|x - y|^N},
\]

this implies that,

\[
A \left( \frac{|u(x)|}{\lambda} \right) \leq A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{s}} \right) \frac{\text{diam}(\Omega \cup B_R)^N}{|x - y|^N},
\]

we suppose \( \alpha = \text{diam}(\Omega \cup B_R)^s \), we get

\[
A \left( \frac{|u(x)|}{\alpha\lambda} \right) \leq A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{s}} \right) \frac{\text{diam}(\Omega \cup B_R)^N}{|x - y|^N},
\]

therefore

\[
|B_R| A \left( \frac{|u(x)|}{\alpha\lambda} \right) \leq \text{diam}(\Omega \cup B_R)^N \int_{B_R} A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{s}} \right) \frac{dy}{|x - y|^N},
\]

then

\[
\int_{\Omega} A \left( \frac{|u(x)|}{\alpha\lambda} \right) dx \leq \frac{\text{diam}(\Omega \cup B_R)^N}{|B_R|} \int_{\Omega} \int_{B_R} A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{s}} \right) \frac{dxdy}{|x - y|^N},
\]

so,

\[
||u||_A \leq \mu[u], \quad \text{for all} \ u \in C_0^\infty(\Omega),
\]

where \( \mu = \frac{\text{diam}(\Omega \cup B_R)^N \alpha}{|B_R|} \). By passing to the limit, the desired result is obtained.

\[\square\]

**Corollary 1** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), and let \( s \in (0, 1) \). Let \( A \) be an \( N \)-function. We define the space \( W_0^s L_A(\Omega) \) as follows:

\[
W_0^s L_A(\Omega) = \left\{ u \in W^s L_A(\mathbb{R}^N) : u = 0 \text{ a.e in } \mathbb{R}^N \setminus \Omega \right\},
\]

Then there exists a positive constant \( \mu \) such that,

\[
||u||_A \leq \mu[u], \quad \text{for all} \ u \in W_0^s L_A(\Omega).
\]

Proof of this corollary is similar to proof of Theorem 6.

**Theorem 7** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), let \( 0 < s < 1 \) and let \( A \) be an \( N \)-function. Then,
\[ C_0^2(\Omega) \subset W_0^s L_A(\Omega). \]

**Lemma 2** Let \( A \) be an \( N \)-function. Then
\[
\frac{\delta(x)}{|x|^s} \in L_A(\mathbb{R}^N, |x|^{-N} \, dx) \text{ with } \delta(x) = \min\{1, |x|\}.
\]

**Proof of Lemma 2** We put \( \Omega_1 = \{ x \in \mathbb{R}^N : |x| > 1 \} \) and \( \Omega_2 = \{ x \in \mathbb{R}^N : |x| \leq 1 \} \). Then, we have
\[
\int_{\mathbb{R}^N} A\left(\frac{\delta(x)}{|x|^s}\right) \frac{dx}{|x|^N} = \int_{\Omega_1} A\left(\frac{\delta(x)}{|x|^s}\right) \frac{dx}{|x|^N} + \int_{\Omega_2} A\left(\frac{\delta(x)}{|x|^s}\right) \frac{dx}{|x|^N} = \int_{\Omega_1} A\left(\frac{1}{|x|^s}\right) \frac{dx}{|x|^N} + \int_{\Omega_2} A\left(\frac{|x|}{|x|^s}\right) \frac{dx}{|x|^N} \leq A(1) \int_{\Omega_1} \frac{dx}{|x|^{N+s}} + A(1) \int_{\Omega_2} \frac{dx}{|x|^{N+s-1}},
\]

note that the last integrals are finite since \( N + s > N \) and \( N + s - 1 < N \) respectively. Therefore
\[
\int_{\mathbb{R}^N} A\left(\frac{\delta(x)}{|x|^s}\right) \frac{dx}{|x|^N} < \infty.
\]

The proof of Lemma 2 is complete. \( \square \)

**Proof of Theorem 7** Let \( u \in C_0^2(\Omega) \), we only need that to check that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A\left(\frac{\lambda|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^N} < \infty \text{ for some } \lambda > 0.
\]

Indeed, since \( u \) vanishes outside \( \Omega \), we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^N} = \int_{\Omega} \int_{\Omega} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^N} + 2 \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^N} \leq 2 \int_{\Omega} \int_{\mathbb{R}^N} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^N}.
\]

Now we notice that
\[
|u(x) - u(y)| \leq ||\nabla u||_{L^\infty(\mathbb{R}^N)} |x - y| \text{ and } |u(x) - u(y)| \leq 2||u||_{L^\infty(\mathbb{R}^N)}.
\]

Accordingly, we get
\[
|u(x) - u(y)| \leq 2||u||_{C^1(\mathbb{R}^N)} \min\{1, |x - y|\} := \alpha \delta(x - y),
\]

\( \square \) Birkhäuser
with $\alpha = 2\|u\|_{C^1(\mathbb{R}^N)}$ and since $\frac{\delta(x)}{|x|^s} \in L_A(\mathbb{R}^N, |x|^{-N} \, dx)$. There exists $\lambda > 0$, such that,
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A \left( \frac{\lambda |u(x) - u(y)|}{\alpha |x - y|^s} \right) \frac{dxdy}{|x - y|^N} \leq 2 \int_{\Omega} \int_{\mathbb{R}^N} A \left( \frac{\delta(x - y)}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}
\]
\[
= 2|\Omega| \int_{\mathbb{R}^N} A \left( \frac{\lambda \delta(\xi)}{|\xi|^s} \right) \frac{d\xi}{|\xi|^N} < \infty,
\]
this implies that $u \in W_0^s L_A(\Omega)$.

\[ \square \]

**Remark 3** A trivial consequence of Theorem 7, $W^s L_A(\Omega)$ is non-empty.

**Proposition 2** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $A$ be an $N$-function. Assume condition (2) is satisfied, then the following relations hold true
\[
[u]_{s,A}^{p^+} \leq \phi(u) \leq [u]_{s,A}^{p^-} \text{ for all } u \in W^s L_A(\Omega) \text{ with } [u]_{s,A} > 1,
\]
\[
[u]_{s,A}^{p^-} \leq \phi(u) \leq [u]_{s,A}^{p^+} \text{ for all } u \in W^s L_A(\Omega) \text{ with } [u]_{s,A} < 1,
\]
where $\phi(u) = \int_{\Omega} \int_{\Omega} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N}$.

**Proof** First we show that $\phi(u) \leq [u]_{s,A}^{p^+}$ for all $u \in W^s L_A(\Omega)$ with $[u]_{s,A} > 1$. Indeed, since $p^+ \geq \frac{ta(t)}{A(t)}$ for all $t > 0$ it follows that for letting $\sigma > 1$ we have
\[
\log(A(\sigma t)) - \log(A(t)) = \int_t^{\sigma t} \frac{a(\tau)}{A(\tau)} \, d\tau \leq \int_t^{\sigma t} \frac{p^+}{\tau} \, d\tau = \log(\sigma p^+).
\]
Thus, we deduce
\[
A(\sigma t) \leq \sigma p^+ A(t) \text{ for all } t > 0 \text{ and } \sigma > 1.
\]

Let now $u \in W^s L_A(\Omega)$ with $[u]_{s,A} > 1$. Using the definition of the Luxemburg norm and the relation (12), we deduce
\[
\int_{\Omega} \int_{\Omega} A \left( |h_{x,y}(u)| \right) \frac{dxdy}{|x - y|^N} = \int_{\Omega} \int_{\Omega} A \left( [u]_{s,A} |h_{x,y}(u)| \right) \frac{dxdy}{|x - y|^N}
\]
\[
\leq [u]_{s,A}^{p^+} \int_{\Omega} \int_{\Omega} A \left( \frac{|h_{x,y}(u)|}{[u]_{s,A}} \right) \frac{dxdy}{|x - y|^N}
\]
\[
\leq [u]_{s,A}^{p^+}.
\]

Now, we show that $\phi(u) \geq [u]_{s,A}^{p^-}$ for all $u \in W^s L_A(\Omega)$ with $[u]_{s,A} > 1$. Similar techniques as those used in the proof of relation (12), we deduce
\[
A(\sigma t) \geq \sigma p^- A(t) \text{ for all } t > 0 \text{ and } \sigma > 1.
\]
Let \( u \in W^s L_A(\Omega) \) with \([u]_{s,A} > 1\), we consider \( \beta \in (1, [u]_{s,A}) \), since \( \beta < [u]_{s,A} \), so by definition of Luxemburg norm, it follows that

\[
\int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{\beta} \right) \frac{dxdy}{|x-y|^N} > 1,
\]

the above consideration implies that

\[
\int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{\beta} \right) \frac{dxdy}{|x-y|^N} = \int_\Omega \int_\Omega A \left( \frac{\beta |h_{x,y}(u)|}{\beta} \right) \frac{dxdy}{|x-y|^N} \\
\geq \beta^{p^-} \int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{\beta} \right) \frac{dxdy}{|x-y|^N} \\
\geq \beta^{p^-},
\]

letting \( \beta < [u]_{s,A} \), we deduce that relation (10) hold true.

Next, we show that \( \phi(u) \leq [u]^{p^-}_{s,A} \) for all \( u \in W^s L_A(\Omega) \) with \([u]_{s,A} < 1\). Similar techniques as those used in the proof of relation (12) and (13), we have

\[
A(t) \leq \tau^{p^-} A \left( \frac{t}{\tau} \right) \text{ for all } t > 0 \text{ and } \tau \in (0, 1). \tag{14}
\]

Let \( u \in W^s L_A(\Omega) \) with \([u]_{s,A} < 1\). Using the definition of the Luxemburg norm and the relation (14), we deduce

\[
\int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{|x-y|^N} \right) \frac{dxdy}{|x-y|^N} \leq [u]^{p^-}_{s,A} \int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{[u]_{s,A}} \right) \frac{dxdy}{|x-y|^N} \\
\leq [u]^{p^-}_{s,A}.
\]

Finally, we show that \( \phi(u) \geq [u]^{p^+}_{s,A} \) for all \( u \in W^s L_A(\Omega) \) with \([u]_{s,A} < 1\). Similar techniques as those used in the proof of relation (12) and (13), we have

\[
A(t) \geq \tau^{p^+} A \left( \frac{t}{\tau} \right) \text{ for all } t > 0 \text{ and } \tau \in (0, 1). \tag{15}
\]

Let \( u \in W^s L_A(\Omega) \) with \([u]_{s,A} < 1\) and \( \beta \in (0, [u]_{s,A}) \), so by (15) we find

\[
\int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{|x-y|^N} \right) \frac{dxdy}{|x-y|^N} \geq \beta^{p^+} \int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{\beta} \right) \frac{dxdy}{|x-y|^N}. \tag{16}
\]

We define \( v(x) = \frac{u(x)}{\beta} \) for all \( x \in \Omega \), we have \([v]_{s,A} = \frac{[u]_{s,A}}{\beta} > 1\). Using the relation (10) we find

\[
\int_\Omega \int_\Omega A \left( \frac{|h_{x,y}(u)|}{\beta} \right) \frac{dxdy}{|x-y|^N} = \int_\Omega \int_\Omega A \left( |h_{x,y}(v)| \right) \frac{dxdy}{|x-y|^N} > [v]^{p^-}_{s,A} > 1. \tag{17}
\]
by (16) and (17) we obtain
\[
\int_{\Omega} \int_{\Omega} A \left( |h_{x,y}(u)| \right) \frac{dxdy}{|x-y|^N} \geq \beta^p.
\]

Letting $\beta \nearrow [u]_{x,A}$, we deduce that relation (11) hold true. □

### 3.1 Some embeddings results

The embeddings results obtained in the fractional Sobolev space $W^{s,p}(\Omega)$ can also be formulated for the fractional Orlicz–Sobolev spaces. For this results, we follow the approach of Donaldson and Trudinger in [23].

Let $0 < s < 1$ and let $A$ be a given N-function, satisfying the following conditions:

\[
\int_0^1 \frac{A^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau < \infty, \quad (18)
\]

\[
\int_1^\infty \frac{A^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau = \infty. \quad (19)
\]

For instance if $A(t) = \frac{t^p}{p}$, then (18) holds precisely when $sp < N$.

If (19) is satisfied, we define the inverse Sobolev conjugate N-function of $A$ as follows,

\[
A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau. \quad (20)
\]

**Remark 4** Function $A_*$ is not essentially weaker than the pure power $t^{(p^-)_s}$, where

\[
(p^-)_s = \begin{cases} 
\frac{Np^-}{N-sp^-} & \text{if } N > sp^- \\
\infty & \text{if } N \leq sp^-.
\end{cases}
\]

Indeed, if we consider for example

\[
a(t) = \log(1 + |t|)|t|^{p-2}t
\]

where $p > 1$, we have

\[
A(t) = \frac{1}{p} \log(1 + |t|)|t|^p - \frac{1}{p} \int_0^|t| \frac{\tau^p}{1+\tau} d\tau.
\]

By example 2 in [19, p. 243], it follows that $A$ is an N-function and $p^- = p$. On the other hand, $A_*$ is essentially stronger than the pure power $t^{(p^-)_s} = t^{p_s}$, i.e.

\[
\lim_{t \to \infty} \frac{t^{p^*_s}}{A_*(kt)} = 0 \text{ for all } k > 0.
\]
By [1, p. 231], the above result will be satisfied if only if
\[
\lim_{t \to \infty} A_{s}^{-1}(t) \frac{1}{t^{p_s}} = 0. \tag{21}
\]
To prove (21), we use L’Hôpital’s rule. Then we have
\[
\limsup_{t \to \infty} A_{s}^{-1}(t) \frac{1}{t^{p_s}} \leq p_{s}^{*} \limsup_{t \to \infty} A_{s}^{-1}(t) + \frac{s}{N},
\]
setting \( y = A_{s}^{-1}(t) \) we obtain
\[
\limsup_{t \to \infty} A_{s}^{-1}(t) \frac{1}{t^{p_s}} \leq p_{s}^{*} \limsup_{y \to \infty} y \frac{A_{s}(y)}{y^{N + sp_{s}} - p} \left( 1 - \frac{\int_{0}^{y} \frac{\tau^{p}}{1 + \tau} d\tau}{\log(1 + y)} \right)^{-1}. \tag{22}
\]
Since
\[
\frac{Np_{s}^{*}}{N + sp_{s}^{*}} - p = \frac{-Np + p_{s}^{*}(N - sp)}{N + sp_{s}^{*}} = \frac{-Np + Np}{N + sp_{s}^{*}} = 0.
\]
Then
\[
\limsup_{y \to \infty} y^{N + sp_{s}^{*}} A_{s}(y) = \limsup_{y \to \infty} \frac{1}{y \log(1 + y)} \left( 1 - \frac{\int_{0}^{y} \frac{\tau^{p}}{1 + \tau} d\tau}{\log(1 + y)} \right)^{-1}. \tag{22}
\]
Next, by a simple calculation, we get
\[
0 \leq \limsup_{y \to \infty} \frac{\int_{0}^{y} \frac{\tau^{p}}{1 + \tau} d\tau}{\log(1 + y) y^{p}} \leq \limsup_{y \to \infty} \int_{0}^{y} \frac{\tau^{p}}{1 + \tau} d\tau \leq \limsup_{y \to \infty} \frac{1}{p} y^{p} = 0,
\]
thus
\[
\limsup_{y \to \infty} \int_{0}^{y} \frac{\tau^{p}}{1 + \tau} d\tau = 0. \tag{23}
\]
From (23) and (22) we see that the condition (21) is satisfied.

**Theorem 8** Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If (18), (19) and (2) hold, then

$$W^s L_A(\Omega) \hookrightarrow L_{A_s}(\Omega).$$

The proof will be carried out in a several lemmas. The first of these establishes an estimate for the Sobolev conjugate $N$-function $A^*$, defined by (20).

**Lemma 3** Let $s \in (0, 1)$, let $A$ be an $N$-function satisfying (18) and (19), and suppose that, for some $p$ such that $1 \leq p < N$, the function $B$ defined by $B(t) = A\left(t^{\frac{1}{p}}\right)$ is an $N$-function. Let $A_s$ be defined by (20). Then for all $s' \in (0, s)$, the following conclusions may be drawn.

1. $[A_s(t)]^{\frac{N-s'}{N}}$ is an $N$-function, in particular, $A_s$ is an $N$-function.
2. For every $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ such that for every $t$,

$$[A_s(t)]^{\frac{N-s'}{N}} \leq \frac{1}{2\varepsilon} A_s(t) + \frac{K_\varepsilon}{\varepsilon} t.$$

**Proof of lemma 3** (1) Let $Q(t) = [A_s(t)]^{\frac{N-s'}{N}}$, Noting that $B^{-1}(t) = [A^{-1}(t)]^p$, we get

$$(Q^{-1})'(t) = \frac{d}{dt} A_s^{-1}(t^{\frac{N}{N-s'}})$$

$$= \frac{N}{N-s'} t^{\frac{N}{N-s'}} - 1 A_s^{-1}(t^{\frac{N}{N-s'}}) \left[ t^{\frac{N}{N-s'}} \right]^{\frac{N-s'}{N}}$$

$$= \frac{N}{N-s'} A_s^{-1}(t^{\frac{N}{N-s'}}) \left[ t^{\frac{N}{N-s'}} \right]^{\frac{N-s'}{N}}$$

$$= \frac{N}{N-s'} \left[ B^{-1}(t^{\frac{N}{N-s'}}) \right]^{\frac{1}{p}}$$

where $\mu = \frac{N + s - s'}{N-s'} - \frac{N}{N-s'} = \frac{N(p-1) + (s-s')p}{(N-s')p} \geq 0$. Being the inverse of an $N$-function, $B^{-1}$ satisfies

$$\lim_{t \to 0^+} \frac{B^{-1}(t)}{t} = \infty$$

and

$$\lim_{t \to \infty} \frac{B^{-1}(t)}{t} = 0.$$
Moreover, \( B^{-1} \) is concave, so for \( 0 < r < \sigma \) we have, \( \frac{B^{-1}(r)}{B^{-1}(\sigma)} > \frac{r}{\sigma} \). Hence, if \( 0 < t_1 < t_2 \), then we get,

\[
\frac{(Q^{-1})'(t_1)}{(Q^{-1})'(t_2)} \geq \left( \frac{t_2}{t_1} \right)^{-\mu} > 1.
\]

It follows that \((Q^{-1})'\) is positive and decreases monotonically from \( \infty \) to 0 as \( t \) increases from 0 to \( \infty \), so that \( Q \) is an N-function.

(2) Let \( g(t) = \frac{A_*(t)}{t} \) and \( h(t) = \frac{[A_*(t)]^{N-s'}}{t} \). It readily checked that \( h \) is bounded on finite intervals and \( \lim_{t \to \infty} \frac{g(t)}{h(t)} = \infty \), then for all \( \varepsilon > 0 \) there exists \( t_0 > 0 \) such that for every \( t \geq t_0, h(t) \leq \frac{g(t)}{2\varepsilon} \). We pose \( K_{\varepsilon} = \varepsilon \sup_{0 \leq t \leq t_0} h(t) \), then

\[
[A_*(t)]^{\frac{N-s'}{N}} \leq \frac{1}{2\varepsilon} A_*(t) + \frac{K_{\varepsilon}}{\varepsilon} t.
\]

\[ \square \]

**Lemma 4** Let \( \Omega \) be an open subset of \( \mathbb{R}^N \), and \( 0 < s < 1 \). Let \( f \) satisfies a Lipschitz-condition on \( \mathbb{R} \) and \( f(0) = 0 \), then,

1. For every \( u \in W^{s,1}_{loc}(\Omega), g \in W^{s,1}_{loc}(\Omega) \) where \( g(x) = f(|u(x)|) \).
2. For every \( u \in W^s L_A(\Omega), g \in W^s L_A(\Omega) \) where \( g(x) = f(|u(x)|) \).

In particular, for every \( u \in W^{s,1}(\Omega), g \in W^{s,1}(\Omega) \) where \( g(x) = f(|u(x)|) \).

**Proof of lemma 4** (1) Let \( K \) be a compact subset of \( \Omega \), follows that \( 1_K g \in W^{s,1}(\Omega) \).

Since \( f(0) = 0 \), then we have,

\[
\int_\Omega |1_K(x)g(x)|dx = \int_\Omega |1_K(x)(f(u(x)) - f(0))|dx \\
\leq C \int_\Omega |1_K(x)u(x)|dx < \infty,
\]

where \( C \) is the Lipschitz constant of \( f \). On the other hand,

\[
\int_\Omega \int_\Omega \frac{|1_K(x)g(x) - 1_K(y)g(y)|}{|x - y|^{N+s}} |dxdy = \int_K \int_K \frac{|g(x) - g(y)|}{|x - y|^{N+s}} |dxdy
\]

\[ + 2 \int_{\Omega \setminus K} \int_K \frac{|1_K(x)g(x)|}{|x - y|^{N+s}} |dxdy
\]

\[ + \int_{\Omega \setminus K} \int_{\Omega \setminus K} \frac{|1_K(x)g(x) - 1_K(y)g(y)|}{|x - y|^{N+s}} |dxdy. \tag{26}
\]
where the third term in the right hand-side of (26) is null, and since \( f \) satisfies the Lipschitz-condition, then,

\[
\int_{\Omega} \int_{\Omega} \left| \mathbb{1}_K(x)g(x) - \mathbb{1}_K(y)g(y) \right| \frac{dx dy}{|x - y|^{N+s}} \leq C \int_{\Omega} \int_{K} \frac{|u(x) - u(y)|}{|x - y|^{N+s}} \frac{dx dy}{|x - y|^{N+s}} + 2 \int_{\Omega \setminus K} \int_{K} \frac{|g(y)|}{|x - y|^{N+s}} \frac{dx dy}{|x - y|^{N+s}},
\]

where the first term in the right hand-side of (27) is finite since \( u \in W_{loc}^{s,1}(\Omega) \) and

\[
2 \int_{K} \int_{\Omega \setminus K} \frac{|g(x)|}{|x - y|^{N+s}} \frac{dx dy}{|x - y|^{N+s}} \leq 2 \int_{K} |g(x)| \int_{\Omega \setminus K} \frac{1}{d(y, \partial K)^{N+s}} \frac{dy}{|x - y|^{N+s}} < \infty.
\]

Note that due to the fact that \( K \) is a compact subset, then \( dis(y, \partial K)^{N+s} > 0 \) for all \( y \in \mathbb{R}^N \setminus K \) and we have \( N + s > N \). Therefore,

\[
\int_{\Omega} \int_{\Omega} \left| \mathbb{1}_K(x)g(x) - \mathbb{1}_K(y)g(y) \right| \frac{dx dy}{|x - y|^{N+s}} < \infty.
\]

(2) Let \( u \in W^s L_A(\Omega) \) then there exists \( \lambda > 0 \) such that,

\[
\int_{\Omega} \int_{\Omega} M \left( \frac{\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} < \infty.
\]

Let \( C > 0 \) denotes the Lipschitz constant of \( f \). If \( |C| \leq 1 \), then we have,

\[
\int_{\Omega} \int_{\Omega} A \left( \frac{\lambda|g(x) - g(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} = \int_{\Omega} \int_{\Omega} A \left( \frac{\lambda|f \circ u(x) - f \circ u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N}
\]

\[
\leq \int_{\Omega} \int_{\Omega} M \left( \frac{|C|\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N}
\]

\[
\leq \int_{\Omega} \int_{\Omega} A \left( \frac{\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} < \infty.
\]

If \( |C| > 1 \) for \( \lambda_1 = \frac{\lambda}{|C|} \) we get,

\[
\int_{\Omega} \int_{\Omega} A \left( \frac{\lambda_1|g(x) - g(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} = \int_{\Omega} \int_{\Omega} A \left( \frac{\lambda_1|f \circ u(x) - f \circ u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N}
\]

\[
\leq \int_{\Omega} \int_{\Omega} A \left( \frac{|C|\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N}
\]

\[
\leq \int_{\Omega} \int_{\Omega} A \left( \frac{\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} < \infty,
\]

this implies that \( g \in W^s L_A(\Omega) \). \( \square \)

**Lemma 5** (see: [12, Proposition 2.9]) Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) and let \( 0 < s' < s < 1 \). Assume condition (2) is satisfied, then the space \( W^s L_A(\Omega) \) is continuously embedded in \( W^{s',q}(\Omega) \) for all \( q \in [1, p^-] \).
Remark 5 If $s = s'$, the conclusion of Lemma 5 will be incorrect. Indeed, if we take

$$A(t) = \frac{t^p}{p},$$

then, we have $p^- = p$ and the embedding $W^{s,p}(\Omega)$ into $W^{s,q}(\Omega)$ for all $q \in [1, p)$ is not satisfied (see [35]).

Proof of theorem 8 Let $0 < s' < s < 1$, $\sigma(t) = \left[ A_*(t) \right]^\frac{N-s'}{N}$ and $u \in W^s L_A(\Omega)$, we suppose for the moment that $u$ is bounded on $\Omega$ and not equal to zero in $L_A(\Omega)$, then

$$\int_\Omega A_* \left( \frac{|u(x)|}{k} \right) dx$$

decreases continuously from infinity to zero as $\lambda$ increases from zero to infinity and according, assumes the value unity for some positive value $k$ of $\lambda$, thus

$$\int_\Omega A_* \left( \frac{|u(x)|}{k} \right) dx = 1 , k = ||u||_{A_*}. \quad (28)$$

Let $f(x) = \sigma \left( \frac{u(x)}{k} \right)$. By Lemma 5 $u \in W^{s',1}(\Omega)$, and $\sigma$ is Lipschitz (see Lemma 3), so that by Lemma 4 we have $f \in W^{s',1}(\Omega)$, and since $N > s'$, then by Theorem 2, one has,

$$W^{s',1}(\Omega) \hookrightarrow L^{\frac{N}{N-s'}}(\Omega).$$

So

$$||f||_{L^{\frac{N}{N-s'}}} \leq k_1 \left( ||f||_{L^1} + [f]_{s',1} \right),$$

and by (28),

$$1 = \left( \int_\Omega A_* \left( \frac{|u(x)|}{k} \right) dx \right)^{\frac{N-s'}{N}} = ||f||_{L^{\frac{N}{N-s'}}},$$

this implies that,

$$1 \leq k_1 \left( ||f||_{L^1} + [f]_{s',1} \right)$$

$$= k_1 \left( \int_\Omega \sigma \left( \frac{u(x)}{k} \right) dx + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x - y|^{N+s'}} dxdy \right)$$

$$= k_1 \left( \int_\Omega \sigma \left( \frac{u(x)}{k} \right) dx + \int_\Omega \int_\Omega \frac{|\sigma \left( \frac{u(x)}{k} \right) - \sigma \left( \frac{u(y)}{k} \right)|}{|x - y|^{N+s'}} dxdy \right) \quad (29)$$

$$= k_1 I_1 + k_1 I_2.$$
By (25) we have for \( \varepsilon = k_1 \),

\[
k_1 I_1 \leq \frac{1}{2} \int_\Omega A_\varepsilon \left( \frac{|u(x)|}{k} \right) dx + \frac{k_\varepsilon}{k} \int \Omega |u(x)| dx \leq \frac{1}{2} + \frac{k'_\varepsilon}{k} ||u||_A, \tag{30}
\]

where \( k'_\varepsilon = 2k_\varepsilon ||1||_A \) since \( \Omega \) has a finite volume.

On the other hand, since \( \sigma \) is Lipschitz, then there exists \( C > 0 \) such that,

\[
k_1 I_2 \leq \frac{C}{k} \int \int \frac{|u(x) - u(y)|}{|x - y|^{N+s'}} dxdy.
\]

But by Lemma 5, we have

\[
\int \int \frac{|u(x) - u(y)|}{|x - y|^{N+s'}} dxdy \leq C'[u]_{s,A}, \tag{31}
\]

and

\[
k_1 I_2 \leq \frac{C}{k} C'[u]_{s,A}. \tag{32}
\]

We pose \( k_3 = Ck_1 C' \). Combining (30)-(32) we obtain

\[
1 \leq \frac{1}{2} + \frac{k'_\varepsilon}{k} ||u||_A + \frac{k_3}{k} [u]_{s,A},
\]

this implies that,

\[
\frac{k}{2} \leq k'_\varepsilon ||u||_A + k_3 [u]_{s,A}.
\]

So we obtain,

\[
||u||_{A_s} \leq k_4 ||u||_{s,A},
\]

where \( k_4 = \max \{ 2k'_\varepsilon, 2k_3 \} \).

If \( u \in W^s L_A(\Omega) \) arbitrary, we define

\[
u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq n, \\ n \text{ sgn } u(x) & \text{if } |u(x)| > n, \end{cases}
\]

\( \{u_n\} \) is bounded and by Lemma 4 it is belongs to \( W^s L_A(\Omega) \). Moreover

\[
||u_n||_{A_s} \leq k_4 ||u_n||_{s,A} \leq k_4 ||u||_{s,A}.
\]
Let $\lim_{n \to \infty} ||u_n||_{A_*} = k$, then $k \leq k_4 ||u||_{s,A}$. By Fatou's Lemma we get,

$$\int_{\Omega} A_\ast \left( \frac{|u(x)|}{k} \right) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} A_\ast \left( \frac{|u_n(x)|}{||u_n||_{A_*}} \right) \, dx < 1,$$

so $u \in L_{A_*}(\Omega)$ and $||u||_{A_*} \leq k$. \hfill \Box

**Theorem 9** Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and $C^{0,1}$-regularity with bounded boundary. If (18), (19) and (2) hold, then

$$W^s L_A(\Omega) \hookrightarrow L_B(\Omega),$$

is compact for all $B \ll A_\ast$.

**Proof** By Lemma 5, we have,

$$W^s L_A(\Omega) \hookrightarrow W^{s',1}_1(\Omega) \hookrightarrow L^1(\Omega).$$

The latter embedding being compact by Theorem 3. A bounded subset $S$ of $W^s L_A(\Omega)$ is also a bounded subset of $L_{A_*}(\Omega)$ and precompact in $L^1(\Omega)$, hence by Theorem 1 it is precompact in $L_B(\Omega)$. \hfill \Box

By combining Lemma 5 and Theorem 3, we obtain the following results.

**Corollary 2** Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. Let $0 < s' < s < 1$ and let $A$ be an $N$-function satisfies the condition (2), we define

$$p_{s'}^* = \begin{cases} \frac{Np^-}{N - s'p^-} & \text{if } N > s'p^- \\ \infty & \text{if } N \leq s'p^- \end{cases}$$

- If $s'p^- < N$, then $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, p_{s'}^*)$ and the embedding $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, p_{s'}^*)$.
- If $s'p^- = N$, then $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, \infty]$ and the embedding $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, \infty]$.
- If $s'p^- > N$, then the embedding $W^s L_A(\Omega) \hookrightarrow L^{\infty}(\Omega)$ is compact.

**4 Mains results**

In this section, we prove existence of a weak solution for problem $(P_a)$ in fractional Orlicz Sobolev spaces, by means of the direct method in calculus of variations. For this, we suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

$\Box$ Birkhäuser
There exists a function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) which is non-decreasing, right continuous, with \( g(0) = 0, g(t) > 0, \forall t > 0 \) and \( g(t) \to \infty \) as \( t \to \infty \), there exist \( \theta_1, \theta_2 > 0 \), and an open bounded set \( \Omega_0 \subset \Omega \) such that

\[
\begin{align*}
(f_1) & \quad |f(x, t)| \leq \theta_1 (1 + g(|t|)) \text{ a.e. } (x, t) \in \Omega \times \mathbb{R}^N, \\
(f_2) & \quad |f(x, t)| \geq \theta_2 g(|t|) \text{ a.e. } (x, t) \in \Omega_0 \times \mathbb{R}^N.
\end{align*}
\]

By setting

\[
G(s) = \int_0^s g(t) dt, \quad \overline{G}(s) = \int_0^s \overline{g}(t) dt.
\]

where \( \overline{g}(t) = \sup \{ s : g(s) \leq t \} \), we obtain complementary \( N \)-functions which define corresponding Orlicz spaces \( L_G \) and \( L_{\overline{G}} \). We will also assume that

\[
\begin{align*}
(f_3) & \quad 1 < q^- = \inf_{s>0} \frac{tg(t)}{G(s)} < q^+ = \sup_{s>0} \frac{tg(t)}{G(s)} < +\infty. \\
(H_1) & \quad \lim_{t \to \infty} \frac{G(kt)}{A_+(t)} = 0 \text{ for all } k > 0.
\end{align*}
\]

**Remark 6** Condition \( (H_1) \) implies that \( G \prec \prec A_\ast \). So by Theorem 9, the following embedding:

\[
W_0^s L_A(\Omega) \hookrightarrow L_G(\Omega)
\]

is compact.

In this section, we work in closed space \( W_0^s L_A(\Omega) \) which can be equivalently renormed by setting \( ||.|| := [.]_{x,A} \).

**Definition 4** We say that \( u \in W_0^s L_A(\Omega) \) is a weak solution of problem \( (P_a) \) if

\[
(-\Delta)^s_a u, v = \int_\Omega \int_\Omega a(|h_{x,y}(u)|) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_{x,y}(v) \frac{dx dy}{|x-y|^N} = \int_\Omega f(x,u)vdx,
\]

for all \( v \in W_0^s L_A(\Omega) \).

**Theorem 10** Let \( A \) be an \( N \)-function satisfies \( (2), (18) \) and \( (19) \). Suppose that \( f \) satisfies \( (f_1) - (f_3) \) and \( (H_1) \). If \( p^- > q^+ \) then the problem \( (P_a) \) has a nontrivial weak solution in \( W_0^s L_A(\Omega) \).

For \( u \in W_0^s L_A(\Omega) \), we define

\[
\begin{align*}
J(u) &= \int_\Omega \int_\Omega A \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx dy}{|x-y|^N} \quad \text{and} \quad H(u) = \int_\Omega F(x,u)dx \\
I(u) &= J(u) - H(u)
\end{align*}
\]

where \( F(x,t) = \int_0^t f(x,t) dt \). Obviously the energy functional \( I : W_0^s L_A(\Omega) \rightarrow \mathbb{R} \) associated with problem \( (P_a) \) is well defined.
Remark 7 We note that the functional \( I : W_0^s L_A(\Omega) \rightarrow \mathbb{R} \) in (35) is well defined. Indeed, if \( u \in W_0^s L_A(\Omega) \), then by condition \((H_1)\), we have that \( u \in L_G(\Omega) \) and thus \( u \in L^1(\Omega) \). Hence, by the condition \((f_1)\),

\[
|F(x, u)| \leq \int_0^u |f(x, t)| dt = \theta_1(|u| + G(|u|))
\]

and thus,

\[
\int_\Omega |F(x, u)| dx < \infty.
\]

Lemma 6 Assume assumption \((f_1)\) is satisfied. Then the functional \( H \in C^1(\mathcal{W}_0^s L_A(\Omega), \mathbb{R}) \) and

\[
\langle H'(u), v \rangle = \int_\Omega f(x, u)v dx \text{ for all } u, v \in W_0^s L_A(\Omega).
\]

Proof First, observe that by Remark 7, \( H \) is well-defined on \( W_0^s L_A(\Omega) \). Usual arguments show that \( H \) is Gâteaux-differentiable on \( W_0^s L_A(\Omega) \) with the derivative is given by (36). Actually, let \( \{u_n\} \subset W_0^s L_A(\Omega) \) be a sequence converging strongly to \( u \in W_0^s L_A(\Omega) \). Since \( W_0^s L_A(\Omega) \) is compactly embedded in \( L_G(\Omega) \), then \( \{u_n\} \) converges strongly to \( u \) in \( L^q(\Omega) \). So there exist a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), and a function \( \bar{u} \in L_G(\Omega) \) such that \( \{u_n\} \) converges to \( u \) almost everywhere in \( \Omega \) and \( |u_n| \leq |\bar{u}| \) for all \( n \in \mathbb{N} \) and almost everywhere in \( \Omega \). By \((f_1)\), we have for all measurable functions \( u : \Omega \rightarrow \mathbb{R} \), the operator defined by \( u \mapsto f(., u(.) ) \) maps \( L_G(\Omega) \) into \( L_G(\Omega) \). Fix \( v \in W_0^s L_A(\Omega) \) with \( ||v|| \leq 1 \). By using the Hölder inequality and the embedding of \( W_0^s L_A(\Omega) \) into \( L_G(\Omega) \), we have

\[
||H'(u_n) - H'(u)|| = \left| \int_\Omega (f(x, u_n(x)) - f(x, u(x))) v(x) dx \right| \\
\leq ||f(x, u_n(x)) - f(x, u(x))||_G ||v||_G, \\
\leq c_1 ||f(x, u_n(x)) - f(x, u(x))||_G ||v||_G,
\]

for some \( c_1 > 0 \). Thus, passing to the supremum for \( ||v|| \leq 1 \), we get

\[
||H'(u_n) - H'(u)||_{(W_0^{s, p}(\Omega))^*} \leq c_1 ||f(., u_n(.) ) - f(., u(.) )||_G.
\]

Since \( f \) is a continuous function, then

\[
f(., u_n(.) ) - f(., u(.) ) \longrightarrow 0 \text{ as } n \rightarrow \infty
\]

and

\[
|f(x, u_n(x)) - f(x, u(x))| \leq \theta_1(2 + g(|\bar{u}(x)|) + g(|u(x)|).
\]
for almost everywhere $x \in \Omega$. Note that the majorant function in the previous relation is in $L_{\mathcal{F}}(\Omega)$. Hence, by applying the dominate convergence theorem we get that $\| f(x, u_n(x)) - f(x, u(x)) \|_{L_{\mathcal{F}}} \to 0$ as $n \to \infty$. This proves that $H'$ is continuous. □

**Lemma 7** The function $J \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ and

$$\{ J'(u), v \} = \int_\Omega \int_\Omega a\left( |h_{x,y}(u)| \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_{x,y}(v) \frac{dxdy}{|x - y|^N}$$

for all $u, v \in W_0^s L_A(\Omega)$. Moreover, for each $u \in W_0^s L_A(\Omega)$, $J'(u) \in (W_0^s L_A(\Omega))^*$.

**Proof** First, it is easy to see that

$$\{ J'(u), v \} = \int_\Omega \int_\Omega a\left( |h_{x,y}(u)| \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_{x,y}(v) \frac{dxdy}{|x - y|^N} \quad (37)$$

for all $u, v \in W_0^s L_A(\Omega)$. It follows from (37) that $J'(u) \in (W_0^s L_A(\Omega))^*$ for each $u \in W_0^s L_A(\Omega)$.

Next, we prove that $J \in C^1(W_0^s L_A(\Omega), \mathbb{R})$. Let $\{ u_n \} \subset W_0^s L_A(\Omega)$ with $u_n \to u$ strongly in $W_0^s L_A(\Omega)$, then $h_{x,y}(u_n) \to h_{x,y}(u)$ in $L_A(\Omega \times \Omega, d\mu)$, so by dominated convergence theorem, there exist a subsequence $h_{x,y}(u_{n_k})$ and a function $h$ in $L_A(\Omega \times \Omega, d\mu)$ such that

$$U_{x,y}(u_{n_k}) := \alpha( |h_{x,y}(u_{n_k})| ) \frac{u_{n_k}(x) - u_{n_k}(y)}{|u_{n_k}(x) - u_{n_k}(y)|} \to U_{x,y}(u) := \alpha( |h_{x,y}(u)| ) \frac{u(x) - u(y)}{|u(x) - u(y)|}$$

and

$$|U_{x,y}(u_{n_k})| \leq |\alpha(|h|)|,$$

for almost every $(x, y)$ in $\Omega \times \Omega$. By (3), $|\alpha(h)| \in L_A^p(\Omega \times \Omega, d\mu)$, so for $v \in W_0^s L_A(\Omega)$ we have $h_{x,y}(v) \in L_A(\Omega \times \Omega, d\mu)$ and by Hölder’s inequality,

$$\left| \int_\Omega \int_\Omega (U_{x,y}(u_{n_k}) - U_{x,y}(u)) h_{x,y}(v) \frac{dxdy}{|x - y|^N} \right| \leq 2 \left\| U_{x,y}(u_{n_k}) - U_{x,y}(u) \right\|_{L_A^p} \left\| v \right\|_{L_A^q} \frac{1}{|x - y|^N} \left| x - y \right|^{q - 1} \left| x - y \right|^{q - 1}$$

Then by dominated convergence theorem we obtain that

$$\sup_{\| v \| \leq 1} \left| \int_\Omega \int_\Omega (U_{x,y}(u_{n_k}) - U_{x,y}(u)) h_{x,y}(v) \frac{dxdy}{|x - y|^N} \right| \to 0.$$

The proof of Lemma 7, is complete. □
Combining Lemma 6 and Lemma 7, we get $I \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ and

$$\langle I'(u), v \rangle = \int_{\Omega} \int_{\Omega} a\left(|h_{x,y}(u)|\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_{x,y}(v) \frac{dxdy}{|x-y|^N} - \int_{\Omega} f(x,u)vdx$$

for all $u, v \in W_0^s L_A(\Omega)$.

**Lemma 8** Let $(f_1)$ be satisfied. Then the functional $I \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ is weakly lower semicontinuous.

**Proof** First, note that the map :

$$u \mapsto \int_{\Omega} \int_{\Omega} A\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N}$$

is lower semicontinuous for the weak topology of $W_0^s L_A(\Omega)$. Indeed, by Lemma 7, we have $J \in C^1(W_0^s L_A(\Omega), \mathbb{R}^N)$ and

$$\langle J'(u), v \rangle = \int_{\Omega} \int_{\Omega} a\left(|h_{x,y}(u)|\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_{x,y}(v) \frac{dxdy}{|x-y|^N}$$

for all $u, v \in W_0^s L_A(\Omega)$. On the other hand, since $A$ is convex so $J$ is also convex. Now, let $\{u_n\} \subset W_0^s L_A(\Omega)$ with $u_n \rightharpoonup u$ weakly in $W_0^s L_A(\Omega)$. Then by convexity of $J$, we have

$$J(u_n) - J(u) \geq \langle J'(u), u_n - u \rangle,$$

and hence, we obtain $J(u) \leq \liminf J(u_n)$, that is, the map

$$u \mapsto \int_{\Omega} \int_{\Omega} A\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N}$$

is lower semicontinuous.

Let $u_n \rightharpoonup u$ weakly in $W_0^s L_A(\Omega)$, so by Theorem 9, $u_n \longrightarrow u$ in $L^G(\Omega)$. Without loss of generality, we assume that $u_n \longrightarrow u$ a.e. in $\Omega$. By assumption $(f_1)$ and similar to the proof of Lemma 6, we obtain

$$\lim_{n \to \infty} \int_{\Omega} F(x,u_n)dx = \int_{\Omega} F(x,u)dx.$$

Thus, the functional $H$ is weakly continuous. Further, we get that $I$ is weakly lower semicontinuous. \qed
Proof of Theorem 10} Let \( u \in W_0^S L_A(\Omega) \) with \(||u||| > 1\). From assumption \((f_1)\) and Proposition 2, we have

\[
I(u) = \int_{\Omega} \int_{\Omega} A \left( \frac{|u(x) - u(y)|}{|x - y|^{s}} \right) \frac{dxdy}{|x - y|^{N}} - \int_{\Omega} F(x, u)dx \\
\geq \int_{\Omega} \int_{\Omega} A \left( \frac{|u(x) - u(y)|}{|x - y|^{s}} \right) \frac{dxdy}{|x - y|^{N}} - \theta_1 \int_{\Omega} G(||u||)dx - \theta_1 \int_{\Omega} |u|dx \\
\geq ||u||^p - \theta_1 C ||u||^{q^+} - \theta_1 C ||u||,
\]

since \( p^- > q^+ > 1 \), so we have \( I(u) \to \infty \) as \(||u||| \to \infty\), by Lemma 8 \( I \) is weakly lower semi-continuous on \( W_0^S L_A(\Omega) \), then by Theorem 4 functional \( I \) has a minimum point \( u_0 \) in \( W_0^S L_A(\Omega) \) and \( u_0 \) is a weak solution of problem \((P_0)\).

Next we need to verify that \( u_0 \) is nontrivial. Let \( x_0 \in \Omega_0 \), \( 0 < R < 1 \) satisfy \( B_{2R}(x_0) \subset \Omega_0 \), where \( B_{2R}(x_0) \) is the ball of radius \( 2R \) with center at the point \( x_0 \) in \( \mathbb{R}^N \). Let \( \varphi \in C_0^\infty(B_{2R}(x_0)) \) satisfies \( 0 \leq \varphi \leq 1 \) and \( \varphi \equiv 1 \) in \( B_{2R}(x_0) \). Theorem 7 implies that \(||\varphi||| < \infty \). Then for \( 0 < t < 1 \), by \((f_2)\), we have

\[
I(t\varphi) = \int_{\Omega} \int_{\Omega} A \left( \frac{|t\varphi(x) - t\varphi(y)|}{|x - y|^{s}} \right) \frac{dxdy}{|x - y|^{N}} - \int_{\Omega} F(x, t\varphi)dx \\
\leq ||t\varphi||^p - \theta_2 \int_{\Omega_0} G(||t\varphi||)dx \\
\leq ||t\varphi||^p - \theta_2 ||t\varphi||^{q^+}_G \\
\leq t^p - ||\varphi||^p - \theta_2 t^{q^+_G} ||\varphi||^{q^+_G}.
\]

Since \( p^- > q^+ \) and \(||\varphi|||^{q^+_G} > 0 \), we have \( I(t_0\varphi) < 0 \) for \( t_0 \in (0, t) \) sufficiently small. Hence, the critical point \( u_0 \) of functional \( I \) satisfies \( I(u_0) \leq I(t_0\varphi) < 0 = I(0) \), that is \( u_0 \neq 0 \).

\[\square\]

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