An effective Hamiltonian for 2D black hole Physics

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In another application of the methods of Henneaux, Teitelboim, and Vergara developed for diffeomorphisms invariant models, the CGHS theory of 2D black holes is focused in order to obtain the true degrees of freedom, the simplectic structure and the effective Hamiltonian that rules the dynamics in reduced phase-space.

I. INTRODUCTION

The so-called “zero-Hamiltonian problem” (ZH problem) is present in diffeomorphisms invariant models. Being more specific, we can say that in theories of gravitation the canonical hamiltonian density is a linear combination of constraints; therefore, after complete gauge fixing, it reduces to a strongly zero quantity. The ZH problem was analysed by Henneaux, Teitelboim and Vergara: the idea was to construct an extension on the original action that is invariant under gauge transformations not vanishing at the end-points; the boundary conditions were then modified through the gauge generators. The extension mentioned above is related to the physical (effective) Hamiltonian of the theory that is going to rule the dynamics of the physical degrees of freedom. An alternative approach was proposed by Fulop, Gitman and Tyutin: the main point here is that one works in the reduced phase-space. Once determined the simplectic structure, after complete gauge fixing, an special time-dependent canonical transformation is performed, obtaining the generator of dynamics for the physical variables. A direct application of these techniques was done in the 2D induced gravity model of Polyakov; here it was possible to obtain the physical Hamiltonian and dynamics of the true degrees of freedom in a systematic way; without the complications found when other methods are used. The ZH problem was also focused in the 2D black hole theory, using dilatonic gravity models (in particular, the CGHS model). Here a different approach was considered: the basic idea was that the physical Hamiltonian must be a proper quantity. Although the results were consistent the techniques used were not systematical, using several arbitrary assumptions related to the particular model under analysis. In this work, as an application of the methods of Henneaux et al. and Gitman et al., we focus the 2D CGHS gravity model and show that it is possible to recognize its true degrees of freedom and obtain the correspondent reduced phase-space physics in a step-by-step procedure; the key point is the calculation of the effective hamiltonian density using those methods. The manuscript is structured as follows. In the second section we make a brief description, as a review, of the techniques used to analyse the ZH problem in the induced gravity case. In the third section we follow the Henneaux et al. method to study the CGHS model, working in the conformal gauge as a concrete example. In the final section we display our conclusions.

II. THE ZH PROBLEM IN INDUCED 2D GRAVITY

The presence of the ZH problem in field theories is a consequence of diffeomorphisms invariance. As is well known, in those systems the extended hamiltonian density \( H_E \) is a linear combination of constraints

\[
H_E = H_0 + \lambda^a G_a \approx 0 ,
\]

where \( G_a \) represent the first-class constraints. Therefore, the Hamiltonian is a strongly zero quantity after the (complete) gauge fixing procedure, leaving no generator of dynamics in reduced phase-space. Henneaux, Teitelboim and Vergara proposed to perform an extension on the action that takes into account end-point contributions. The action for the paths obeying these open boundary conditions (the gauge parameters \( \epsilon^a \) do not vanish at the end points) is

\[
S = \int_{\tau_1}^{\tau_2} (pq - H_0 - \lambda^a G_a)d\tau - [P_i \frac{\partial G}{\partial P_i} - G]_{\tau_1}^{\tau_2} ,
\]

with \( G \equiv \epsilon^a G_a \). The corresponding generating function \( (M) \) is related to the gauge \( (\text{Diff}) \) generator. We have in fact

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\[ M = P_i \frac{\partial G}{\partial P_i} - G. \quad (3) \]

In the \textit{induced} 2D gravity case \cite{4} the action is given by
\[ S = \int d^2x \sqrt{-\bar{g}} (\varphi \nabla^\mu \nabla_\mu \varphi - \alpha R \varphi), \quad (4) \]
where \( \varphi(x) \) is an auxiliary field and \( R \) is the 2D scalar curvature. For this model the generating function \( M \) is \cite{5}
\[ M = \int dy \ m = \int dy \left[ P_i \frac{\partial G}{\partial P_i} - G \right], \quad (5) \]
with
\[ m = \frac{\sqrt{-\bar{g}}}{g_{11}} e^0 \left[ \frac{\varphi^2}{2} + \frac{2}{\alpha^2} (g_{11} \pi_{11})^2 - \frac{\alpha \partial_1 g_{11}}{2g_{11}} \partial_1 \varphi 
\right.
\[ + \alpha \partial_1^2 \varphi + \frac{2}{\alpha} g_{11} \pi_{11} \pi_\varphi \right] + (\epsilon^1 + \frac{g_{11}}{g_{11}} e^0) \left[ -2g_{11} \partial_1 \pi_{11} - 2 \partial_1 g_{11} \pi_{11} \right], \quad (6) \]
being a function of the gravitational field components \( g_{\mu\nu} \), the auxiliary field \( \varphi \) and their conjugated momenta. It is also possible to obtain a non-zero \textit{effective} Hamiltonian \( \bar{H} \), in reduced phase-space, using the technique proposed in \cite{3}. After complete gauge fixing a canonical transformation is performed, whose generator \( F \) is determined by the form of the gauge fixing constraints \cite{3}:
\[ \bar{H} = \left[ H_E + \frac{\partial F}{\partial \tau} \right] \bigg|_{\text{fixed}}, \quad (7) \]
We obtain the last equality after (complete) gauge fixing, meaning that \( \bar{H} \) is the Hamiltonian for the new variables \( \bar{Q} \), in reduced phase-space. The equations of motion will be
\[ \dot{\bar{Q}} = \{ \bar{Q}, \bar{H} \}_D \quad \dot{\bar{P}} = \{ \bar{P}, \bar{H} \}_D, \quad (8) \]
where \( \text{D} \) denotes the Dirac bracket \cite{1} operation. In the 2D induced gravity case we found \cite{3}
\[ \{ g_{11}(x), \pi_{11}(y) \}_D = \delta(x - y). \quad (9) \]
Although this is the canonical bracket relation the gravitational field and the corresponding momentum are not independent quantities in this case \cite{3}. For the \textit{effective} hamiltonian density we obtained \cite{3}
\[ H_{\text{eff}} = g_{11} + \alpha \left( 1 - \frac{1 + g_{11}}{2g_{11}} \right) \pi_{11} \partial_1 g_{11}, \quad (10) \]
this density rules the dynamics of the physical gravitational field \( g_{11} \) in reduced phase-space.

\section*{III. THE 2D BLACK HOLE}

2D black hole physics can be described using the CGHS model of \textit{dilatonic} gravity \cite{8}. The corresponding action can be written in the following form
\[ S = \int d^2x \sqrt{-\bar{g}} (\eta R - \lambda), \quad (11) \]
the cosmological constant is \( \lambda; \eta \) is related to the dilaton field \( \varphi \) through \( \eta = e^{-2\varphi} \); in the following we will call \( \eta \) the dilaton field for simplification. The 2D scalar curvature \( R \) is constructed out of the metric \( g_{\mu\nu} \) as usual, while the physical gravitational field is represented by the \( \bar{g}_{\mu\nu} = \frac{g_{\mu\nu}}{\eta} \) components. Being a gravitation theory the local gauge transformations are the 2D diffeomorphisms. In fact, The dilaton and gravitational fields transform as
where
\[ \delta \eta = \epsilon^\nu \partial_\nu \eta \]  
\[ \delta g^{\mu \nu} = \partial_\mu g^{\nu \rho} - g^{\mu \rho} \partial_\nu + g^{\nu \rho} \partial_\mu \epsilon^\rho = 0 . \]  

Using the physical gravitational field components we can put the action (4) in a more convenient form
\[ \bar{I}_2 = \int d^2 x \sqrt{-g} e^{-2\varphi} \left( R + \frac{4}{\alpha} g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi - \Lambda \right) . \]  

Using a new set of canonical transformations it is possible to write the metric components in terms of the shift vector \( N \) and the lapse function \( n \). We have
\[ \phi = \frac{1}{4\alpha} e^{-2\varphi} , \quad \bar{g}_{\mu \nu} = \frac{1}{4\alpha \phi} \bar{g}_{\mu \nu} . \]  

With these new fields we furnish the primary (first class) constraints of the theory namely \( \pi_N = 0 \) and \( \pi_n = 0 \). In turn, the consistency-in-time condition of these quantities give two secondary constraints
\[ \omega_1 = \frac{1}{2} \left( \partial_1 \phi^2 - \frac{4}{\alpha^2} (g \pi_g)^2 - \frac{4}{\alpha} (g \pi_g) \pi_\phi - \alpha \frac{\partial_1 g}{g} + 2 \alpha \bar{\partial}_1^2 \phi + \alpha^2 \beta g \right) \]  
\[ \omega_2 = \pi_\phi \partial_1 \phi - 2 g \partial_1 \pi_g - \pi_g \partial_1 g , \]  
and the canonical hamiltonian density is, as expected, a combination of these secondary (first class) constraints
\[ H_e = \frac{N}{g} \omega_1 + n \omega_2 . \]  

In an analogous procedure to the one used in [3] we choose as gauge fixing conditions
\[ \Gamma_5 = \pi_g - f(t) \quad \Gamma_6 = \partial_1 \phi - 1 \]  
where \( f(t) \) is an arbitrary function of time. To obtain a more convenient form of the Dirac matrix we use the following linear combinations
\[ \Lambda_1 = \omega_1 + \Gamma_5 \quad \Lambda_2 = \omega_1 - \Gamma_5 \]  
whose Poisson brackets are
\[ \{ \Lambda_1(x), \Lambda_1(y) \} = -2 \alpha \partial_x \delta(x - y) . \]  

The Dirac brackets for the physical degrees of freedom can be obtained in a two-steps procedure. First we fix the \([\pi_n, \pi_N] \) sector using the conformal gauge fixing condition [6]; this is straightforward. In a second step we take the sector formed by \( \omega_1, \omega_2, \Lambda_1 \) and \( \Lambda_2 \). We obtain as fundamental Dirac bracket
\[ \{ g(x), \pi_g(y) \}_D = \delta(x - y) , \]  

analogously to the induced gravity case; the complete simplectic structure of reduced phase-space follows from this relation. To find the effective Hamiltonian in reduced phase-space we perform, as was explained in section II, a time-dependent canonical transformation. In the gravity sector we have
\[ \Pi_g = \pi_g - f(t) , \quad G = g . \]  

The new Lagrangian density reads
\[ \bar{L} = L + \partial_\mu F^\mu , \]  
where \( F^\mu \) is the generator of the canonical transformation. The correct equations of motion are obtained when
\[ F^0 = \Pi g , \quad F^1 = \alpha \left( 1 - \frac{1 + g}{2g} \right) g' \Pi_g . \]  

Going back to the “physical” variables of action (14) we finally obtain that the effective hamiltonian density is
\[ H_{eff} = \frac{1}{2\pi} \left( e^{-2\varphi} [2 \partial_1 \varphi + \lambda] \right) . \]  

This is the ADM hamiltonian density, in conformal gauge, in agreement with [3]; the dilaton appears as the fundamental field (in fact, \( g \) is related to the dilaton through the secondary constraints \( \omega_1 \) and \( \omega_2 \)). The hamiltonian density (27) replaces the original (equation (19), that is strongly zero after complete gauge fixing) for the reproduction of the black hole equations, using the simplectic structure of the reduced phase-space.

3
IV. CONCLUSIONS

Using the methods developed by Henneaux, Teitelboim and Vergara; Gitman and Tyutin, we have obtained a solution to the ZH problem for the 2D dilatonic gravity model, getting the reduced phase-space Physics whose hamiltonian density is in agreement with the one found in [6] (for conformal gauge fixing). The expressions are the result of a totally systematic approach; contrary to what is usually found in the literature when considering diffeomorphisms invariant models.

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