ARISTOTELIAN ASSERTORIC SYLLOGISTIC

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To Raouf Doss Who introduced modern logic to Egypt

Abstract. Aristotelian assertoric syllogistic, which is currently of growing interest, has attracted the attention of the founders of modern logic, who approached it in several (semantical and syntactical) ways. Further approaches were introduced later on. These approaches (with few exceptions) are here discussed, developed and interrelated.

Among other things, different facets of soundness, completeness, decidability and independence are investigated. Specifically arithmetization (Leibniz), algebraization (Leibniz and Boole), and Venn models (Euler and Venn) are closely examined. All proofs are simple. In particular there is no recourse to maximal nor minimal conditions (with only one, dispensable, exception), which makes the long awaited deciphering of the enigmatic Leibniz characteristic numbers possible. The problem was how to look at matters from the right perspective.

Introduction. Aristotelian assertoric syllogistic (henceforth AAS), which is currently of growing interest (Glashoff 2005), has attracted the attention of the founders of modern logic. Leibniz, Boole, De Morgan, Venn, Peirce, Frege, Hilbert, Russell and Gödel all dealt with it. For some of them it was the starting point (cf. Boole 1948).

Modern treatment of AAS started closer to what may be currently called the semantical or model theoretic approach. This was threefold: arithmetical, algebraic and model theoretic.

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algebraic, and diagramatic (or set theoretic). The first trend was developed by Leibniz (Łukasiewicz 1998, pp. 126-9; Kneale and Kneale 1966, pp. 337-8; Glashoff 2002; and Sotirov 2015). The second was developed by Leibniz (Kneale and Kneale 1966, pp. 338-45; Lenzen 2004) and after about two centuries was again developed by Boole (1948), without mentioning the work of Leibniz. The last trend was developed by Euler, then by Venn (Venn 1880).

With the rise of proof theory late nineteenth century, six syntactical formalizations of AAS were developed:

(i) Monadic first order formalization which goes back to Frege (1967, on p.28 the square of logical opposition may be found). This formalization is adopted by Hilbert and Ackermann (1950, pp. 44-54).

(ii) Sentential formalization which goes back to Peirce (Bellucci and Pietarinen 2016, p. 226) and is adopted by Gödel (Adzic and Dosen 2016, p. 479). The most elaborate study of this formalization is that of Łukasiewicz (1998).

(iii) Dyadic first order formalization which goes back to Shepherdson (1956). The novel idea of regarding categorical sentences (or propositions) as binary relational sentences (or propositions) is due to De Morgan (Valencia 2004, pp. 506-7).

(iv) Natural deduction formalization which goes back to Corcoran (1972) and Smiley (1973).

(v) First order many-sorted formalization which goes back to Smiley (1962).

(vi) A recent formalization based on Hilbert’s epsilon and tau quantifiers (Pasquali and Retoré 2016).

All of the above will be considered below with only two exceptions. The first is the many-sorted formalization ((v) above), for it is a variant of the monadic first order formalization mentioned in (i) above; moreover it was,
apparently, abandoned even by its own author (cf. Smiley 1973). The formalization based on Hilbert’s epsilon and tau quantifiers ((vi) above) will take us far off the current mainstream of logic. So it will be the second exception and will not be further considered here, though it may have intrinsic merit especially for those who are interested in formalizing natural languages.

With only one exception, the modern syntactical formalizations of AAS degraded it to the rank of a secondary logic, a subordinate or a subsidiary sublogic of a superior fundamental or principal primary logic. In contrast, the natural deduction formalization ((iv) above, cf. Bocheński (1968, pp. 3, 31, 42, 49, 52)) rehabilitates it to a full-fledged primary logic, as was probably designed by its founder: Aristotle, and as was taken for granted for over two millennia. Accordingly, this formalization will be the focus of this article. Through completeness we shall see that as far as the basic sentences (to be defined in 1.6 below) are concerned other formalizations add nothing new.

In the sequel we deal -from modern standpoints- with AAS, not with medieval nor traditional syllogistic. In contrast to Boole (1948), Glashoff (2007), Hilbert and Ackermann (1950), Russinoff (1999), Shepherdson (1956) and Sotirov (1999), term negation (or complementation, to use a modern term; cf. Bocheński (1968, p. 50)) is not here permitted. Also, in contrast to Hilbert and Ackermann (1950), Łukasiewicz (1998), Shumann (2006), Shepherdson (1956) and Sotirov (1999), Boolean combinations of categorical sentences are not here permitted. So (the extensional aspect of) AAS will be just the logic of, or the fragment of set theory which deals with, inclusion (universal affirmative sentences) and exclusion (universal negative sentences) and their contradictories (particular sentences). However, some exceptions may be appropriate as will be clear, or clarified, at the proper places.

Among other things, different facets of soundness, completeness, decidability, and independence are investigated. Particularly arithmetization (Leibniz), algebraization (Leibniz and Boole) and Venn models (Euler and Venn) are closely examined.

All proofs given here are simple. In contrast to Corcoran (1972), Glashoff (2010), Martin (1997), Shepherdson (1956), Smiley (1973) and Smith (1983), our proofs have no recourse to maximal nor minimal principles nor conditions (with only one exception, which is indirect and may be dispensed with). This makes the long awaited deciphering of the enigmatic Leibniz characteristic numbers possible. The problem was how to look at matters from the right perspective.

To specify, in section 3 below we provide a polynomial time algorithm to
decide for any finite set of categorical sentences whether it is consistent and, if it is, to assign a Leibniz model (to be defined below) to it. This settles positively problem 2 of Glashoff (2002) for finite sets. The general case is discussed in section 2.

I hope that the simplicity of this exposition of AAS will help to re-incorporate it into the mainstream of mathematical logic.

After this introduction, the structure of the rest of the article is as follows:

1. Formalizations of AAS
2. Semantics of AAS
3. Decidability
4. Basic equivalence of the four formalizations
5. Venn soundness and completeness
6. Direct way to Venn models
7. Variations on NF(C)
8. Direct completion of direct deduction
9. Models of NF(C) revisited
10. Decidability revisited
11. Sorites
12. Independence
13. Algebraic semantics of AAS, a prelude
14. Algebraic interpretation of NF(C)
15. Annihilators: Embedding the partial into a total
16. Back to algebraic interpretation
17. Leibniz and Boole
18. Inadequacy: bounds of AAS

Acknowledgements
Appendix

1. **Formalizations of AAS.** Formalizations of AAS differ with regard to permitting the subject and the predicate of a formal symbolic categorical sentence (henceforth categorical sentence) to be the same. Smith (1983) and Glashoff (2010) follow Corcoran (1972) in not permitting sameness; as accommodating sameness “would entail rather more deviation from the Aristotelian text” says Corcoran (1972, p. 696).

On the other hand Smiley (1973) left the door open for permitting sameness, noting (p. 144) that “the variables he [Aristotle] uses for the major, middle and minor terms are all distinct from one another [...]” though when it
comes to substituting actual terms in the resulting forms we are of course at liberty to replace different variables by the same term (64a1).” Consequently, it seems that Aristotle excluded sameness, for technical-not philosophical-reasons. This is, possibly, why Łukasiewicz (1998) adopted sameness (see pp. 77, 88); while Martin (1997) simultaneously considered two systems, one of them is permitting sameness and the other is not.

In conformity with the current mainstream of mathematical logic, sameness is here permitted. Excluding sameness, and other variations, will be considered in section 7 below.

1.1. Monadic first order formalization of AAS. The language here is a standard first order language, with or without equality, whose set $P$ of non-logical constants has at least three elements, and all of its elements are unary relational symbols. In the sequel “$P$”, “$Q$” and “$R$” will be metalinguistic variables ranging over the elements of $P$. With abuse of notation, “$P$” will denote this language too.

ABBREVIATIONS 1.1.

- “$APQ$” is an abbreviation for “$\forall x (Px \to Qx)$”
- “$EPQ$” is an abbreviation for “$\forall x (Px \to \neg Qx)$”
- “$IPQ$” is an abbreviation for “$\exists x (Px \land Qx)$”
- “$OPQ$” is an abbreviation for “$\exists x (Px \land \neg Qx)$”

DEFINITION 1.2. $MF(P)$ is the theory based on $P$ with only one non-logical axiom schema, namely, $APQ \rightarrow IQP$ (which is equivalent to the schema $\exists x Px$).

PROPOSITION 1.3. The following are theorem schemata of $MF(P)$:

1. $EPQ \leftrightarrow \neg IPQ$
2. $OPQ \leftrightarrow \neg APQ$
3. $APP$
4. $APQ \rightarrow IQP$
5. $EPQ \rightarrow EQP$
6. $APQ \land AQR \rightarrow APR$
7. $APQ \land EQR \rightarrow ERP$

Proof. Routine. □
1.2. Sentential formalization of AAS. The symbols “A”, “E”, “I” and “O” were made use of in section 1.1, in this section they will be made use of differently. This abuse of notation is benign as long as the intended denotation is clear from the context, so it will be here permitted. Such abuses of notation may be permitted later on without further notice.

Let \( J \) be a set (whose elements are to correspond to categorical constants) having at least three elements, let \( A, E, I \) and \( O \) be four injective functions of pairwise disjoint ranges, each of domain \( J \times J \), and let \( AS(J) \) be the union of their ranges. In the sequel “\( i \)”, “\( j \)” and “\( k \)” will be metalinguistic variables ranging over the elements of \( J \).

The language here is a standard sentential language whose set of sentential symbols is \( AS(J) \). With abuse of notation “J” will denote this language too.

**DEFINITION 1.4.** \( SF(J) \) is the theory based on \( J \) with the following non-logical axiom schemata:

1. \( Eij \leftrightarrow \neg Iij \)
2. \( Oij \leftrightarrow \neg Aij \)
3. \( Aii \)
4. \( Aij \rightarrow Iji \)
5. \( Eij \rightarrow Eji \)
6. \( Aij \land Ajk \rightarrow Aik \)
7. \( Aij \land Ejk \rightarrow Eik \)

The proof machinery is modus ponens together with any standard set of sentential logical axiom schemata.

**REMARK 1.5.** There are two kinds of substitution: sentences for sentences and indices for indices. Each may be permitted, under some conditions, as a derived rule of inference (cf. Łukasiewicz 1998, p. 88; see section 1.5 below).

1.3. Dyadic first order formalization of AAS. The language here is a standard first order language, with or without, equality whose non-logical constants are four binary relation symbols “A”, “E”, “I” and “O”, together with a set \( C \) of individual (or categorical) constants having at least three elements. In the sequel “\( a \)”, “\( b \)”, and “\( c \)” will be metalinguistic variables ranging over the elements of \( C \). With abuse of notation “\( C \)” will denote this language too.

**DEFINITION 1.6.** \( DF(C) \) is the theory based on \( C \) whose non-logical axioms are the universal closures of:
1. $Exy \leftrightarrow Ixy$
2. $Oxy \leftrightarrow Axy$
3. $Axx$
4. $Axy \rightarrow Iyx$
5. $Exy \rightarrow Eyx$
6. $Axy \wedge Ayz \rightarrow Axz$
7. $Axy \wedge Eyz \rightarrow Exz$

1.4. Natural deduction formalization of AAS. The language here is a sublanguage of the language defined in 1.3. The alphabet is the four binary relation symbols $A, E, I$ and $O$, together with a set $C$ of individual (or categorical) constants having at least three elements. The sentences are the equality free atomic sentences of 1.3., viz. a sentence is a string $Yab$ where $Y \in \{A, E, I, O\}$ and $a, b \in C$. By abuse of notation “$C$” will denote this language too, and the set of all sentences will be denoted by “$S(C)$”. In the sequel “$\alpha$”, “$\beta$”, “$\gamma$”, “$\delta$”, “$\sigma$” and “$\rho$” will be metalinguistic variables ranging over the elements of $S(C)$.

Sentences starting with $A$ or $E$ are called universal, those starting with $I$ or $O$ are called particular. Also, sentences starting with $A$ or $I$ are called affirmative, those starting with $E$ or $O$ are called negative. For $W \in \{A, E, I, O\}$, sentences starting with $W$ are called $W$-sentences.

**DEFINITION 1.7.** $NF(C)$ is the logical system based on the language $C$ with the following deduction rules (or enrichments thereof, see sections 8 and 11 below):

0. \[ Aaa \quad (A\text{-}Id) \]
1. \[ Aab \quad Iba \quad (Apc) \]
2. \[ Eab \quad Eba \quad (Ec) \]
3. \[ Aab, Abc \quad Aac \quad (Barbara) \]
4. \[ Aab, Ebc \quad Eac \quad (Celarent). \]

“Barbara” and “Celarent” are, respectively, the medieval names of the rules 3 and 4; “$A\text{-}Id$”, “$Apc$” and “$Ec$” are, respectively, abbreviations for “$A$-identity”, “$A$-partial conversion” and “$E$-conversion”. For simplicity, we may write “rules” instead of “deduction rules”.

**DEFINITION 1.8.** A direct deduction (or d-deduction) of $\sigma(\in S(C))$ from $\Gamma(\subseteq S(C))$ is a sequence $< \rho_i >_{i \in \mathbb{N}^+}$ such that $\rho_{k-1} = \sigma$ and for each $i \in k$, $\sigma_i \in \Gamma$ or is the consequent of some rule of $NF(C)$ whose antecedents are previous terms of the sequence. In this case we write $\Gamma \vdash d \sigma$, and $\sigma$ is said
to be a direct consequence (or theorem) of $\Gamma$. Also $< \rho_i >_{i \in k}$ is said to be a direct (or $d$-) deduction from $\Gamma$. From now on, the rules 0-4 given above will be called also “$d$-rules”.

Regarding the current mainstream of mathematical logic, this definition is a typical definition. In contrast, corresponding definitions given in Corcoran (1972), Glashoff (2010), Martin (1997), Smiley (1973) and Smith (1983) are atypical, each has its own peculiarity.

To get closer to the Aristotelian tradition, a more restricted definition of direct deduction is presented in section 11 below, and its relationship to the above one is investigated there.

As usual, the contradictory $\widehat{\sigma}$ of $\sigma (\in S(C))$ is defined as follows:

$$\widehat{Aab} = Oab \quad \widehat{Eab} = Iab \quad \widehat{Iab} = Eab \quad \widehat{Oab} = Aab$$

so $\widehat{\sigma} = \sigma$.

A set $\Gamma (\subseteq S(C))$ is said to be $d$-inconsistent (or $d$-contradictory) if $\Gamma \vdash \sigma$ and $\Gamma \vdash \widehat{\sigma}$, for some $\sigma \in S(C)$; otherwise $\Gamma$ is said to be $d$-consistent.

DEFINITION 1.9. The general (or $g$-) deduction relation $\vdash^g (\subseteq \wp(S(C)) \times S(C))$ is defined as follows:

$\Gamma \vdash^g \sigma$ iff $\Gamma \cup \{\widehat{\sigma}\}$ is $d$-inconsistent.

For $\Gamma \subseteq S(C)$, “$\Gamma$ is $g$-inconsistent (or $g$-contradictory)” and “$\Gamma$ is $g$-consistent” may be defined along the above lines, replacing “$d$” by “$g$”. Obviously $\vdash \subseteq \vdash^g$, so if $\Gamma (\subseteq S(C))$ is $d$-inconsistent, it is $g$-inconsistent.

For $e \in \{d, g\}$ and $\Gamma \subseteq S(C)$, “$\Gamma^e$” will denote the closure of $\Gamma$ under $\vdash^e$, i.e.

$\Gamma^e = \text{the smallest } \Delta \subseteq S(C) \text{ such that } \Gamma \subseteq \Delta \text{ and for every } \sigma \in S(C), \sigma \in \Delta \text{ whenever } \Delta \vdash^e \sigma$.

LEMMA 1.10. Let $\Sigma, \Sigma'$ be subsets of $S(C)$ such that for every $\sigma \in \Sigma$, $\Sigma' \vdash^d \sigma$. For every $d$-deduction $< \rho_i >_{i \in k}$ from $\Sigma$, there are a $k'(\geq k)$, a $d$-deduction $< \rho'_i >_{i \in k'}$ from $\Sigma'$ and a strictly increasing function $f : k \to k'$ such that $f(k-1) = k' - 1$ and for every $i \in k$, $\rho_i = \rho'_{f(i)}$. Hence for every $\alpha \in S(C)$, $\Sigma' \vdash^d \alpha$ whenever $\Sigma \vdash^d \alpha$.

8
Proof. By induction on $\ell$, the number of times of making use in $< \rho_i >_{i \in k}$ of assumptions from $\Sigma$.

Basis: $\ell = 0$; take $k' = k$, $< \rho_i ' >_{i \in k'} = < \rho_i >_{i \in k}$ and $f$ the identity function on $k$.

Induction step: assume the required for $\ell = m$. Let $\ell = m + 1$ and let $j(\in k)$ be the last line in which an assumption from $\Sigma$ is made use of. The case $j = 0$ is easier than the case $j > 0$, so we shall deal only with the latter.

By the induction hypothesis, there are a $j' \geq j$, a deduction $< \alpha_i >_{i \in j'}$, from $\Sigma$ and a strictly increasing function $g : j \to j'$ such that $g(j - 1) = j' - 1$ and for every $i \in j$, $\rho_i = \alpha_g(i)$.

Let $< \beta_i >_{i \in m}$ be a deduction of $\rho_j$ from $\Sigma'$. Put:

$k' = j' + m + k - j - 1$,

$\gamma_i = \rho_{i+j+1}$ for $i \in k - j - 1$,

$< \rho_i ' >_{i \in k'} = < \alpha_i >_{i \in j'} < \beta_i >_{i \in m} < \gamma_i >_{i \in k - j - 1}$,

where “$\bowtie$” is the concatenation operation symbol.

Evidently $k' \geq k$. The completion of the proof is now easy. $\square$

Parts 3 and 4 of the next proposition are, respectively, reformulations of lemmata M$_1$ and M$_2$ of Corcoran (1972).

**PROPOSITION 1.11.** Let $\Gamma \cup \{ \sigma \} \subseteq S(C)$, $\Gamma' = \{ \rho \in S(C) : \Gamma \stackrel{g}{\vdash} \rho \}$, $U \in \{ A, E \}$, $W \in \{ I, O \}$, $e \in \{ d, g \}$ and $a, b, c \in C$, then:

1. $\Gamma \stackrel{d}{=} = \{ \rho \in S(C) : \Gamma \stackrel{d}{\vdash} \rho \}$.

2. If $\Gamma \cup \{ Wab \} \vdash \sigma$ and $\sigma \neq Wab$, then $\Gamma \vdash \sigma$.

3. If $\Gamma$ is $d$-consistent, then $\Gamma \stackrel{g}{\vdash} Uab$ iff $\Gamma \stackrel{d}{\vdash} Uab$.

4. $\Gamma$ is $d$-consistent iff it is $g$-consistent.

5. If $\rho \in \Gamma'$ and $\Gamma, \rho \vdash \sigma$, then $\Gamma \vdash \sigma$.

6. $\Gamma'^e = \Gamma'$, hence $\Gamma^g = \Gamma'$.

7. $\Gamma^e = \Gamma^e$; hence, for every $d$-rule $r$, if each antecedent of $r$ belongs to $\Gamma^e$, then so also does its consequent.
Proof.

1. The required is a corollary of the above lemma.
2. Generalizing upon the metalinguistic variable “$\sigma$”, the resulting sentence may be proved by course of values induction on the length of the $d$-deduction from $\Gamma \cup \{Wab\}$, noticing that $Wab$ is not a premise of any rule of $NF(C)$.
3. Let $\Gamma \models ^g Uab$, then for some $\alpha \in S(C)$, $\Gamma, \widehat{Uab} \overset{d}{\models} \alpha, \widehat{\alpha}$. So, by part 2, if $\Gamma$ is $d$-consistent then $\widehat{Uab} \in \{\alpha, \widehat{\alpha}\}$, then $Uab \in \{\alpha, \widehat{\alpha}\}$. So, by part 2 again, $\Gamma \overset{d}{\models} Uab$. The other direction is obvious.
4. Let $\Gamma$ be $d$-consistent and $g$-inconsistent, then there is a universal $\alpha \in S(C)$ such that $\Gamma \models ^g \alpha, \widehat{\alpha}$, then by part 3, $\Gamma \overset{d}{\models} \alpha$. Also there is $\beta \in S(C)$ such that $\Gamma, \alpha \overset{d}{\models} \beta, \widehat{\beta}$, so, by lemma 1.10., $\Gamma \overset{d}{\models} \beta, \widehat{\beta}$, hence $\Gamma$ is $d$-inconsistent. Consequently, if $\Gamma$ is $g$-inconsistent it is $d$-inconsistent. The other direction is obvious.
5. Obvious if $\Gamma$ is $d$-inconsistent, so let $\Gamma$ be $d$-consistent and let $\rho \in \Gamma'$ and $\Gamma, \rho \models ^g \sigma$. There is $\alpha \in S(C)$ such that $\Gamma, \rho, \widehat{\sigma} \overset{d}{\models} \alpha, \widehat{\alpha}$. If $\rho$ is universal, then by part 3 and lemma 1.10, $\Gamma, \widehat{\sigma} \overset{d}{\models} \alpha, \widehat{\alpha}$. Also, if $\rho$ is particular and $\rho \notin \{\alpha, \widehat{\alpha}\}$, then by part 2, $\Gamma, \widehat{\sigma} \overset{d}{\models} \alpha, \widehat{\alpha}$. In both cases $\Gamma \models ^g \sigma$. In the remaining case $\rho$ must be particular and $\Gamma, \rho, \widehat{\sigma} \overset{d}{\models} \rho, \widehat{\rho}$, then by part 2, $\Gamma, \widehat{\sigma} \overset{d}{\models} \widehat{\rho}$. But there is $\beta \in S(C)$ such that $\Gamma, \rho \overset{d}{\models} \beta, \widehat{\beta}$, then $\Gamma, \widehat{\sigma} \overset{d}{\models} \beta, \widehat{\beta}$, hence $\Gamma \models ^g \sigma$, which completes the proof.
6. By induction, part 5 may be generalized to: for every finite $\Delta \subseteq \Gamma'$, $\Gamma \models ^g \sigma$ whenever $\Gamma \cup \Delta \models ^g \sigma$. From this it readily follows that $\Gamma \models ^g \sigma$ whenever $\Gamma' \models ^g \sigma$, hence the result.
7. By part 1 and lemma 1.10, $\Gamma^{dd} = \Gamma^d$, and by part 6, $\Gamma^{gg} = \Gamma^g = \Gamma' = \Gamma$. To prove the last clause, let $r$ be a $d$-rule. If each antecedent of $r$ belongs to $\Gamma^e$, then its consequent belongs to $\Gamma^{ee} = \Gamma^e$. □

In view of part 4 of the above proposition, for $e \in \{d, g\}$, the prefix “$e$-” may be deleted from “$e$-consistent”, “$e$-inconsistent” and “$e$-contradictory”.

10
1.5. **Equality / equivalence.** For $\Gamma \subseteq S(C)$, $e \in \{d, g\}$ and $a, b \in C$, $\Gamma \vdash^e Aab, Aba$ is equivalent to each of:

1. $\Gamma \vdash^e Aac$ iff $\Gamma \vdash^e Abc$ all $c \in C$,
2. $\Gamma \vdash^e Aca$ iff $\Gamma \vdash^e Acb$ all $c \in C$.

Thus, $\Gamma \vdash^e Aab, Aba$ imply the substitutability of $a, b$ for each other in universal positive sentences. This will be generalized below to all universal sentences, respectively all sentences, for $e = d$, respectively $e = g$. The last generalization is the essence of equality (congruence or equivalence, depending on the situation). It holds, in the respective appropriate forms, for the other formalizations as is shown in the following:

**THEOREM 1.12.**

1. Let $P, Q \in P$, then:
   
   $MF(P) \vdash (APQ \land AQP) \rightarrow (\varphi \leftrightarrow \varphi')$ for every form $\varphi$ of $P$, where $\varphi'$ is a form obtained from $\varphi$ by substituting some occurrences of “$P(x)$” in $\varphi$ by “$Q(x)$”, or vice versa.

2. Let $i, j \in J$, then:
   
   $SF(J) \vdash (Aij \land Aji) \rightarrow (\alpha \leftrightarrow \alpha')$ for every sentence $\alpha$ of $J$, where $\alpha'$ is a sentence obtained from $\alpha$ by substituting some occurrences of “$i$” in $\alpha$ by “$j$”, or vice versa.

3. Let $a, b \in C$, then:
   
   $DF(C) \vdash (Aab \land Aba) \rightarrow (\varphi \leftrightarrow \varphi')$ for every form $\varphi$ of $C$, where $\varphi'$ is a form obtained from $\varphi$ by substituting some occurrences of “$a$” in $\varphi$ by “$b$”, or vice versa; provided -for languages with equality- no substitution takes place in a form or a subform of the form $t = t'$, where $t$ and $t'$ are terms.

4. Let $\Gamma \subseteq S(C)$, $e \in \{d, g\}$ and $a, b \in C$, and let $\Gamma \vdash^e Aab, Aba$, then for all $c \in C$:
   
   $\Gamma \vdash^e Y ac$ iff $\Gamma \vdash^e Y bc$ and $\Gamma \vdash^e Y ca$ iff $\Gamma \vdash^e Y cb$, where $Y \in \{A, E\}, \{AE, I, O\}$ for $e = d, g$ respectively.

**Proof.** The first three parts may be proved by the standard methods developed in the respective formal systems.
For the last part, part 7 of proposition 1.11 secures the required for \( e \in \{d, g\} \) and \( Y \in \{A, E\} \).

It remains to consider the cases where \( e = g \) and \( Y \in \{I, O\} \). If \( \Gamma \) is inconsistent the required follows by the definition of \( \vdash_g \), so let \( \Gamma \) be consistent. Assume \( \Gamma \vdash_g Aab, Aba \) and \( \Gamma \vdash_g Ica \), then by part 3 of proposition 1.11, \( \Gamma \vdash_d Aab, Aba \), hence \( \Gamma, Ecb \vdash Eca \). But there is \( \alpha \in S(C) \) such that \( \Gamma, Eca \vdash \alpha, \hat{\alpha} \), consequently \( \Gamma, Ecb \vdash \alpha, \hat{\alpha} \) and, by definition, \( \Gamma \vdash_g Icb \). The other cases are similar or easier. \( \square \)

1.6. Basic sentences. In each of the four formalizations \( MF(P) \), \( SF(J) \), \( DF(C) \) and \( NF(C) \) the sentences to be made use of in the Aristotelian syllogistic will be called basic (or categorical) sentences. The sets of basic sentences will be denoted, respectively, by \( "BM(P)" \), \( "BS(J)" \), \( "BD(C)" \) and \( "BN(C)" \). That is:

\[
BM(P) = \{YPQ : Y \in \{A, E, I, O\} \text{ and } P, Q \in P\}.
\]

\[
BS(J) = AS(J) (= \text{the set of all atomic sentences of } J).
\]

\[
BD(C) = \text{the set of all (equality free) atomic sentences of } C.
\]

\[
BN(C) = S(C) (= BD(C)).
\]

1.7. Interpretation. Let \( h : C \to J \), with abuse of notation (no confusion will ensue) we define another function \( h : BN(C) \to BS(J) \) by \( h(Yab) = Yhahb \), for \( Y \in \{A, E, I, O\} \) and \( a, b \in C \). As usual, for \( \Gamma \subseteq BN(C) \), the image of \( \Gamma \) under \( h \) is denoted by \( "h(\Gamma)" \); also we may write \( "Yh^ab" \), \( "\Gamma^h" \) for \( "h(Yab)" \), \( "h(\Gamma)" \) respectively. The function \( h \) is said to be an interpretation of \( BN(C) \) in \( BS(J) \). Similarly \( BM(P) \), \( BS(J) \) and \( BN(C) (= BD(C)) \) may be interpreted in each other.

PROPOSITION 1.13. Let \( \Gamma \cup \{\sigma\} \subseteq BN(C) \), let \( h \) and \( H \) be interpretations of \( BN(C) \) in \( BS(J) \) and \( BM(P) \) respectively, and let \( < \Gamma', \sigma', \hat{\sigma}', T > \in \{ < \Gamma, \sigma, \hat{\sigma}, DF(C) >, < \Gamma^h, \sigma^h, \hat{\sigma}^h, SF(J) >, < \Gamma^H, \sigma^H, \hat{\sigma}^H, MF(P) > \} \). Then:

1. \( T \vdash \hat{\sigma}' \leftrightarrow \gamma \sigma' \),

2. \( T \cup \Gamma' \vdash \sigma' \) whenever \( \Gamma \vdash \sigma \).

Proof.

1. Easy.
2. By proposition 1.3 for $T = MF(P)$, and by the definitions for the other cases. □

PROPOSITION 1.14. Let $T \in \{MF(P), DF(C)\}$, let $h$ be an interpretation of $BS(J)$ in the set of basic sentences of $T$, and let $\Gamma \cup \{\sigma\} \subseteq BS(J)$. Then: $T \cup \Gamma h \vdash \sigma h$ whenever $SF(J) \cup \Gamma \vdash \sigma$.

Proof. The interpretations of the axioms of $SF(J)$ are theorems of $T$, and the proof machinery of $T$ is not weaker than that of $SF(J)$. □

To investigate the converses of proposition 1.14 and part 2 of proposition 1.13, we first go to:

2. Semantics of AAS. The theories $MF(P)$ and $DF(C)$ are first order, and the theory $SF(J)$ is sentential; so each has its usual class of models with respect to which it is sound and complete.

2.1. Models of $MF(P)$. A model $\mathfrak{B}$ of $MF(P)$ is an ordered pair $< B, \mu >$ where $B$ is a non-empty set and $\mu$ maps $P$ into $\wp(B) - \{\phi\}$. $B$ is called the universe, or the base, of $\mathfrak{B}$ and may be denoted also by $\mathfrak{B}$. $\mathfrak{B}$ is a model of $MF(P)$ if it satisfies its axioms.

2.2. Models of $SF(J)$. A model $\mathfrak{B}$ of $SF(J)$ is a mapping form $AS(J)$ into $2 (= \{0, 1\})$, which satisfies all the axioms of $SF(J)$. For $\sigma \in S(J)$, $\mathfrak{B} \models \sigma$ means that $\sigma$ takes the value 1 under the usual extension of $\mathfrak{B}$.

2.3. Models of $DF(C)$. A structure $\mathfrak{B}$ of the dyadic language $C$ (or a $DF(C)$-structure $\mathfrak{B}$) is a 6-tuple $< B, A^*, E^*, I^*, O^*, \mu >$ where $B$ is a non-empty set and $A^*, E^*, I^*$ and $O^*$ are binary relations on $B$ corresponding to the relation symbols “$A$”,”$E$”,”$I$” and “$O$” respectively, and $\mu$ is a mapping of $C$ into $B$. $B$ is called the universe, or the base, of $\mathfrak{B}$ and may be denoted also by $\mathfrak{B}$. $\mathfrak{B}$ is a model of $DF(C)$ if it satisfies its axioms.

Since, by axioms 1 and 2, $E^* = I^{sc}$ and $O^* = A^{sc}$ (where “$c$” denotes the complement with respect to $B \times B$) we may -by abuse of notation-say that $< B, A^*, I^*, \mu >$ is a model of $DF(C)$ whenever the expansion $< B, A^*, I^{sc}, I^*, A^{sc}, \mu >$ is a model of $DF(C)$.

PROPOSITION 2.1. Let $B \neq \phi$, $\mu : C \rightarrow B$ and $R_1, R_2 \subseteq B \times B$. Then $< B, R_1, R_2, \mu >$ is a model of $DF(C)$ iff:

1. $R_1$ is reflexive and transitive (i.e. $R_1$ is a pre-ordering on $B$),
2. $R_2$ is symmetric,
3. $R_1 \subseteq R_2$,
4. $R_2 | R_1 \subseteq R_2$, where $R_2 | R_1$ is the relative product of $R_2$ and $R_1$.

Proof. $< B, R_1, R_2, R_2^c, \mu >$ satisfies axioms 1-6 iff conditions 1-3 above are satisfied.

Axiom 7 is equivalent to $\forall x \forall z [\exists y (Axy \land Eyz) \rightarrow Exz]$. So axiom 7 is satisfied iff $R_1 | R_2^c \subseteq R_2^c$ which, in the presence of condition 2, is equivalent to condition 4. □

2.4. Models of $NF(C)$. The structures in which $NF(C)$ may be interpreted (henceforth $NF(C)$-structures) are exactly the $DF(C)$-structures.

For $\Gamma \cup \{ \sigma \} \subseteq BN(C)$ we write $\Gamma \models_{BN} \sigma$ to mean that $\mathfrak{B} \models \sigma$ whenever $\mathfrak{B} \models \Gamma$, where $\mathfrak{B}$ is an $NF(C)$-structure and $\mathfrak{B} \models \sigma$, $\mathfrak{B} \models \Gamma$ are defined as usual.

Two $NF(C)$-structures $\mathfrak{B}$, $\mathfrak{B}'$ are said to be $BN(C)$-equivalent, basically equivalent, or (for short) $B$-eq if for every $\sigma \in BN(C)$, $\mathfrak{B} \models \sigma$ iff $\mathfrak{B}' \models \sigma$; in this case we may say also that $\mathfrak{B}$ is $B$-eq to $\mathfrak{B}'$. This notion may be extended in an obvious way to the other formalization of $AAS$.

DEFINITION 2.2. An $NF(C)$-structure $\mathfrak{B}$ is said to be a direct model (or, for short, a $d$-model) if for every $\Gamma \cup \{ \sigma \} \subseteq BN(C)$, $\Gamma \models_{BN} \sigma$ whenever $\Gamma \models \sigma$. The proof of the following is straightforward.

PROPOSITION 2.3. An $NF(C)$-structure $< B, A^*, E^*, I^*, O^*, \mu >$ is a $d$-model iff all of the rules of inference of $NF(C)$ are valid in it (in the sense that if the antecedents are true in it, then so also is the consequent), iff:

1. $A^*\mu$ is reflexive on $\mu(C)$ and transitive (equivalently, $A^*\mu$ is a pre-ordering on $\mu(C)$),
2. $A^*\mu \subseteq I^*\mu$,
3. $E^*\mu$ is symmetric,
4. $A^*\mu | E^*\mu \subseteq E^*\mu$,

where, for a set $X$, $X\mu = X \cap (\mu(C) \times \mu(C))$; and for a binary relation $R$, $\tilde{R}$ is its converse. □
DEFINITION 2.4. The canonical structure $\mathfrak{B}_\Gamma$ corresponding to $\Gamma(\subseteq BN(C))$ is the $NF(C)$-structure $< B, A^*, E^*, I^*, O^*, \mu >$ satisfying:

1. $B = C$,
2. $\mu = I_C$, where for a set $X$, $I_X$ is the identity function on $X$,
3. for every $Y \in \{A, E, I, O\}$, $Y^* = \{< a, b > : Yab \in \Gamma \}$.

An $NF(C)$-structure is said to be canonical if it is equal to $\mathfrak{B}_\Gamma$, for some $\Gamma \subseteq BN(C)$.

The basic property of $\mathfrak{B}_\Gamma$ is:

$\mathfrak{B}_\Gamma \models \sigma$ iff $\sigma \in \Gamma$ all $\sigma \in BN(C)$.

Every $NF(C)$-structure $\mathfrak{B} = < B, A^*, E^*, I^*, O^*, \mu >$ in which $\mu = I_C$ is an extension of a canonical structure; namely the canonical structure corresponding to $\{\sigma \in BN(C) : \mathfrak{B} \models \sigma \}$.

LEMMA 2.5. For $\Gamma \subseteq BN(C)$, $\mathfrak{B}_\Gamma^d$ is a $d$-model (of $\Gamma^d$ hence of $\Gamma$).

Proof. Let $\Delta \cup \{\sigma\} \subseteq BN(C)$ and let $\Delta \models^d \sigma$. If $\mathfrak{B}_\Gamma^d \models \Delta$ then $\Delta \subseteq \Gamma^d$, hence $\sigma \in \Gamma^d$, consequently $\mathfrak{B}_\Gamma^d \models \sigma$. □

THEOREM 2.6. (Direct soundness and completeness). Direct deduction is sound and complete with respect to the class of all direct models. That is, for every $\Gamma \cup \{\sigma\} \subseteq BN(C)$,

$$\Gamma \models^d \sigma \iff \Gamma^d \models \sigma$$

where “$\Gamma \models^d \sigma$” means that $\Gamma \models \sigma$ for every $d$-model $\mathfrak{B}$.

Proof. Soundness is immediate by the definition. To prove completeness assume $\Gamma \models^d \sigma$ then, in particular, $\Gamma \models_{\mathfrak{B}^d} \sigma$. But $\mathfrak{B}_\Gamma^d \models \Gamma$, then $\mathfrak{B}_\Gamma^d \models \sigma$, consequently $\sigma \in \Gamma^d$, hence $\Gamma \models^d \sigma$. □

DEFINITION 2.7. (General models). An $NF(C)$-structure $\mathfrak{B}$ is said to be a general model (or, for short, a $g$-model) if for every $\Gamma \cup \{\sigma\} \subseteq BN(C)$, $\Gamma \models_{\mathfrak{B}} \sigma$ whenever $\Gamma \models^g \sigma$.

LEMMA 2.8. For $\Gamma \subseteq BN(C)$, $\mathfrak{B}_\Gamma^g$ is a $g$-model (of $\Gamma^g$ hence of $\Gamma$).


Proof. Replace “d” by “g” in the proof of lemma 2.5.

THEOREM 2.9. (General soundness and completeness). General deduction is sound and complete with respect to the class of all general models. That is, for every \( \Gamma \cup \{ \sigma \} \subseteq BN(C) \),

\[
\Gamma \vdash^g \sigma \quad \text{iff} \quad \Gamma \models^g \sigma
\]

where “\( \Gamma \models^g \sigma \)" means that \( \Gamma \models \sigma \) for every g-model \( \mathfrak{B} \).

Proof. Replace “d” by “g” in the proof of lemma 2.6.

THEOREM 2.10. (NF\((C)\)-compactness). For every \( e \in \{d, g\} \), for every \( \Gamma \cup \{ \sigma \} \subseteq BN(C) \),

\[
\Gamma \vdash^e \sigma \quad \text{iff} \quad \text{for some finite } \Gamma_1 \subseteq \Gamma, \quad \Gamma_1 \vdash^e \sigma.
\]

Proof. By e-soundness and e-completeness.

In theorem 9.4 below the g-models will be fully characterized. Now we confine ourselves to the following:

REMARKS and definitions 2.11.

1. Every g-model is a d-model (obvious) but not vice versa. For, let \( \Gamma = \{ Eaa \} \) for some \( a \in C \), then \( \mathfrak{B}_{\Gamma^d} \) is a d-model but not a g-model.

2. Every model of \( DF(C) \) is obviously a g-model (hence a d-model) but not vice versa. For let \( \Gamma = \{ Oaa \} \) for some \( a \in C \), then \( \mathfrak{B}_{\Gamma^g} \) is a g-model but not a model of \( DF(C) \).

3. For every g-model \( \mathfrak{B} = \langle B, A^*, E^*, I^*, O^*, \mu \rangle \) in which \( \mu \) is surjective, \( A^* = E^* = I^* = O^* = B \times B \) iff \( (A^* \cap O^* \neq \emptyset \text{ or } E^* \cap I^* \neq \emptyset) \) iff for some \( \sigma \in BN(C) \), \( \mathfrak{B} \models \sigma, \hat{\sigma} \). Such models are called full models.

If \( A^* \cup O^* = B \times B = E^* \cup I^* \) (respectively \( A^* \cap O^* = \emptyset = E^* \cap I^* \)), \( \mathfrak{B} \) is said to be complete (respectively consistent). Thus \( \mathfrak{B} \) is not full iff it is consistent. \( \mathfrak{B}_{\hat{\sigma}} \) is an example of a g-model in which \( \mu \) is bijective, while it is not complete.
These notions may be generalized to all $NF(C)$-structures, $\mu$ does not have to be surjective.

4. Let $\mathfrak{B} = < B, A^*, E^*, I^*, O^*, \mu >$ be an $NF(C)$-structure in which $\mu$ is surjective. Then $\mathfrak{B}$ is not complete iff for some $\sigma \in BN(C)$, $(\mathfrak{B} \not\models \sigma$ and $\mathfrak{B} \not\models \widehat{\sigma})$ iff for some $\sigma \in BN(C)$, $\mathfrak{B} \not\models \sigma$ and for all $\rho \in BN(C)$, $\{\widehat{\sigma}\} \models_{\mathfrak{B}} \rho, \widehat{\rho}$.

5. For $\Gamma \subseteq BN(C)$, $\Gamma$ is consistent iff it has a consistent $d$-model.

6. Direct deduction is sound with respect to any class of models with respect to which general deduction is sound.

2.5. Order models and Venn models. To the best of my knowledge Shepherdson (1956) was the first to make use of a version of order models; the ordering was pre-ordering (reflexive and transitive, but not necessarily antisymmetric) and the context was the semantics of a version of $DF(C)$. In the context of the semantics of $NF(C)$, or versions thereof, versions of order models were made use of in Martin (1997) and in Glashoff (2002). The former required a model to be some variation on a lower semi-lattice with a smallest element, the latter relaxed these conditions; none of them mentioned that Shepherdson (1956) made use of order models.

Following Shepherdson (1956), let $R_1$ be a pre-ordering on a non-empty set $B$; and following Glashoff (2002), put:

$$R_2 = \{< x, y > \in B \times B : \{x, y\} \text{ has an } R_1\text{-lower bound}\}.$$ 

Let $\mu$ be a function from $C$ to $B$, then $< B, R_1, R_2, \mu >$ is a model of $DF(C)$, hence a $g$-model and a $d$-model. Such models are said to be order models. If $R_1$ is a partial ordering (equivalently, antisymmetric) the order model will also be called partial (or antisymmetric). $< B, R_1 >$ is said to be the order structure underlying the order model $< B, R_1, R_2, \mu >$. Notice that if $< B, R_1, R'_2, \mu >$ is a model of $DF(C)$, then $R_2 \subseteq R'_2$.

A concrete order model (henceforth c.o.m, and c.o.m.s for the plural) is an order model in which $B$ is a collection of non-empty sets and $R_1$ is $\subseteq$, so the c.o.m.s are partial. If $R'$ is defined on such a $B$ by $x R' y$ iff $x \cap y \neq \phi$ then for every $\mu : C \rightarrow B$, $< B, \subseteq, R', \mu >$ is a model of $DF(C)$. Such models are said to be Venn models.

In an order model $\mathfrak{B} = < B, R_1, R_2, \mu >$, $R_2$ is determined by $B$ and $R_1$, so we may write -for short- “$\mathfrak{B} = < B, R_1 >$”. For similar reasons we may
write “< B, µ >” to denote the Venn model < B, ⊆, R′, µ >; the c.o.m with
the same B and µ is denoted by “< B, ⊆, µ >”.

Let B = { {1, 2}, {2, 3} }, then for every µ : C → B, < B, ⊆, µ > is a c.o.m
but not a Venn model and < B, µ > is a Venn model but not a c.o.m. This
is not always the case, for if the universe is the set of all non-empty subsets
of a non-empty set, then the model is both a Venn model and a c.o.m. Every
Venn model is embeddable in such a model which is B-eq to it. A Venn
model with universe B is a c.o.m iff for every b, b′ ∈ B there is c ∈ B such
that c ⊆ b ∩ b′ whenever b ∩ b′ ≠ φ.

Let C, C′ ∈ { the class of all Venn models, the class of all c.o.m.s, the class
of all partial order models, the class of all order models }, then for every B ∈ C
there is B′ ∈ C which is B-eq to it, hence ⊨ e = ⊨ e′ (with the usual meaning).
This is a corollary of the above discussion and the following observation.

Let B = < B, R1, R2, µ > be an order model. Define the function ’ from
B to φ(B) by b′ = R1[b], where for a binary relation ρ, ρ[y] = \{ x | x ∈ Domain
ρ : xρy \}. Let B′ be the range of ’ and define the function µ′ from C to B′
by µ′(c) = µ(c)′. Then B′ = < B′, ⊆, µ′ > is a c.o.m which is a Venn model
and ’ is a homomorphism from B onto B′. It is an isomorphism iff R1 is
antisymmetric. In all cases B and B′ are B-eq.

2.6. Models and interpretations.

2.6.1. MF(P) and SF(J). Let f be an interpretation of BM(P) in
BS(J) and let B be a model of SF(J). Put:

\[ \mu : P \rightarrow \wp(J) - \{ \phi \} \]
\[ \mu(Q) = \{ j \in J : B \models Ajf(Q) \} \]

then B′ = < J, µ > is a model of MF(P). It is easy to see that:

1. For every positive universal \( \alpha \in BM(P) \), B′ \( \models \alpha \) iff B \( \models \alpha^f \).

2. For every positive particular \( \alpha \in BM(P) \), B′ \( \models \alpha \) only if B \( \models \alpha^f \). The
other direction holds iff for every i, j ∈ Range f there is k ∈ J such that
both B \models Aki and B \models Akj whenever B \models Iij. In this case:

B′ \models \alpha \iff B \models \alpha^f \text{ for every } \alpha \in BM(P).
On the other hand, let \( h \) be an interpretation of \( BS(J) \) in \( BM(P) \) and let \( \mathfrak{B} \) be a model of \( MF(P) \). Put:

\[
\mathfrak{B}' : BS(J) \to 2 \\
\mathfrak{B}'(\alpha) = 1 \quad \text{iff} \quad \mathfrak{B} \models \alpha^h,
\]

then \( \mathfrak{B}' \) is a model of \( SF(J) \).

### 2.6.2. \( SF(J) \) and \( DF(C) \)

Let \( f \) be an interpretation of \( BS(J) \) in \( BD(C) \) and let \( \mathfrak{B} \) be a model of \( DF(C) \). Put:

\[
\mathfrak{B}' : BS(J) \to 2 \\
\mathfrak{B}'(\alpha) = 1 \quad \text{iff} \quad \mathfrak{B} \models \alpha^f,
\]

then \( \mathfrak{B}' \) is a model of \( SF(J) \).

On the other hand, let \( h \) be an interpretation of \( BD(C) \) in \( BS(J) \) and let \( \mathfrak{B} \) be a model of \( SF(J) \). Define:

\[
R_1 = \{ <i,j> \in J \times J : \mathfrak{B}(A_{ij}) = 1 \}, \\
R_2 = \{ <i,j> \in J \times J : \mathfrak{B}(I_{ij}) = 1 \},
\]

then \( \mathfrak{B}' = < J, R_1, R_2, h > \) is a model of \( DF(C) \) and for every \( \alpha \in BD(C) \)

\[
\mathfrak{B}' \models \alpha \quad \text{iff} \quad \mathfrak{B} \models \alpha^h.
\]

### 2.6.3. \( DF(C) \) and \( MF(P) \)

Let \( f \) be an interpretation of \( BD(C) \) in \( BM(P) \) and let \( \mathfrak{B} = < B, \mu > \) be a model of \( MF(P) \), then \( \mathfrak{B}' = < \wp(B) - \{ \phi \}, \mu \circ f > \) is a Venn model of \( DF(C) \) and for every \( \alpha \in BD(C) \), \( \mathfrak{B}' \models \alpha \) iff \( \mathfrak{B} \models \alpha^f \).

On the other hand, let \( h \) be an interpretation of \( BM(P) \) in \( BD(C) \) and let \( \mathfrak{B} = < B, R_1, R_2, \mu > \) be a model of \( DF(C) \). Put:

\[
\mu' : P \to \wp(B) - \{ \phi \} \\
\mu'(Q) = R_i[\mu h(Q)],
\]

then \( \mathfrak{B}' = < B, \mu' > \) is a model of \( MF(P) \). It is easy to see that:
1. For every positive universal \( \alpha \in BM(P) \), \( \mathcal{B}' \models \alpha \) iff \( \mathcal{B} \models \alpha^h \).

2. For every positive particular \( \alpha \in BM(P) \), \( \mathcal{B}' \models \alpha \) only if \( \mathcal{B} \models \alpha^h \). The other direction holds if \( \mathcal{B} \) is an order model, in this case:

\[ \mathcal{B}' \models \alpha \quad \text{iff} \quad \mathcal{B} \models \alpha^h \quad \text{for every } \alpha \in BM(P). \]

2.7. Leibniz models. Let \( \eta \) be the partial ordering defined on the set \( \mathbb{N} \) of natural numbers by \( m \eta n \) iff \( m \) is a multiple of \( n \), and define the partial ordering \( R \) on \( \mathbb{N} \times \mathbb{N} \) by \( < m_1, n_1 > R < m_2, n_2 > \) iff \( m_1 \eta m_2 \) and \( n_1 \eta n_2 \). Denote the binary operations of the greatest common divisor and the least common multiple on \( \mathbb{N} \) by \( \hat{\land} \) and \( \hat{\lor} \) respectively, and put:

\[ B = \{ < m, n > \in \mathbb{N} \times \mathbb{N} : m \hat{\land} n = 1 \}. \]

The restriction of \( R \) on \( B \), to be also denoted by "\( R' \)", partially orders \( B \). So, for every \( \mu : C \rightarrow B \), \( < B, R, \mu > \) is an order model of \( DF(C) \), hence a \( g \)-model and a \( d \)-model. Such models are called Leibniz models, for they were first introduced -in a different setting- by him in 1679, as may be learned from Łukasiewicz (1998, pp. 126-9), Kneale and Kneale (1966, pp. 337-8) and Glashoff (2002). Leibniz practically defines \( A^* \) to be \( R \), but he sets \( I^* < m_1, n_1 > < m_2, n_2 > \) iff \( m_1 \hat{\land} n_2 = 1 = n_1 \hat{\land} m_2 \). To show that this gives rise to an order model as defined in 2.5 above, notice that \( \{ < m_1, n_1 >, < m_2, n_2 > \} \) has an \( R \)-lower bound iff there is \( < m, n > \in B \) such that \( < m_3, n_3 > R < m_1, n_1 > \) and \( < m_3, n_3 > R < m_2, n_2 > \), which is equivalent to \( m_3 \eta (m_1 \hat{\lor} m_2) \) and \( n_3 \eta (n_1 \hat{\lor} n_2) \). But \( m_3 \hat{\land} n_3 = 1 \), so the condition is equivalent to \( (m_1 \hat{\lor} m_2) \hat{\land} (n_1 \hat{\lor} n_2) = 1 \). The l.h.s. = \( (m_1 \hat{\land} n_1) \hat{\lor} (m_1 \hat{\land} n_2) \hat{\lor} (m_2 \hat{\land} n_1) \hat{\lor} (m_2 \hat{\land} n_2) \). But \( m_1 \hat{\land} n_1 = 1 = m_2 \hat{\land} n_2 \), so the condition is equivalent to \( (m_1 \hat{\land} n_2) \hat{\lor} (m_2 \hat{\land} n_1) = 1 \) which is equivalent to Leibniz condition. Via reductio ad absurdum Glashoff (2002) gave a different proof of the same result.

Every Leibniz model is isomorphic to a Venn model. The converse is not true, for \( B \) is denumerable while there are non-denumerable Venn models.

2.7.1. Assigning Leibniz models. For \( \Gamma \subseteq BN(C) \) put:
\[ C_{\Gamma} = \{ c \in C : c \text{ occurs in some element of } \Gamma \text{ having two distinct categor-} \]
ical constants}.

\( \Gamma \) will be called essentially finite if \( C_{\Gamma} \) is finite. This notion may be general-ized to subsets of \( BM(\mathcal{P}) \) and \( BS(\mathcal{J}) \).

**Lemma 2.12.** \( C_{\Gamma^d} = C_{\Gamma} \).

**Proof.** By proposition 1.11 and induction on the length of the deduction. \( \square \)

**Theorem 2.13.** To each consistent essentially finite \( \Gamma \subseteq BN(C) \) a Leibniz model of \( \Gamma \) may be assigned (cf. Glashoff 2010, Lemma 3.4).

**Proof.** Let \( < B, R > \) be the order structure underlying the Leibniz models. Put \( \ell = |C_{\Gamma}| \) and let \( < c_i >_{i \in \ell} \) be an injective enumeration of \( C_{\Gamma} \), \( < p_i >_{i \in \ell} \) be an injective \( \ell \)-sequence of primes and \( b \in B \). Define \( \mu : C \rightarrow B \) as follows:

\( \mu(c) = b \) if \( c \in C - C_{\Gamma} \),

and for \( i \in \ell \), \( \mu(c_i) = < m_i, n_i > \) where

\[
m_i = \prod_{j \in \ell} p_j;
\]

\[
n_i = \prod_{j \in \ell} p_j;
\]

\( m_i \) and \( n_i \) are square free finite products (the empty product is equal to 1).

By the consistency of \( \Gamma \), \( m_i \wedge n_i = 1 \) for all \( i \in \ell \). Therefore \( \mathcal{B} = < B, R, \mu > \) is a Leibniz model.

To show that \( \mathcal{B} \models \Gamma \):

1. Let \( i, k \in \ell \), then \( Ac_i c_k \in \Gamma^d \) only if \( (\forall j \in \ell)[(Ac_k c_j \in \Gamma^d \rightarrow Ac_k c_j \in \Gamma^d) \land (E c_k c_j \in \Gamma^d \rightarrow E c_k c_j \in \Gamma^d)] \) only if \( < m_i, n_i > R < m_k, n_k > \) only if \( m_i \eta p_k \) only if \( Ac_i c_k \in \Gamma^d \). So \( Ac_i c_k \in \Gamma^d \iff \mathcal{B} \models Ac_i c_k \). Consequently, for every \( c, c' \in C \), \( \mathcal{B} \models Acc' \) if \( Acc' \in \Gamma \).

Moreover if for some \( c, c' \in C \), \( Occ' \in \Gamma \), then by the consistency of \( \Gamma \) there are \( i, k \in \ell \) such that \( i \neq k \) and \( c = c_i \), \( c' = c_k \).

Again by the consistency of \( \Gamma \), \( Ac_i c_k \not\in \Gamma^d \), hence \( \mathcal{B} \not\models Ac_i c_k \), consequently \( \mathcal{B} \models Oc_i c_k \).
2. Let \( i, k \in \ell \) be such that \( i \neq k \), then \( Ec_ic_k \in \Gamma^d \) only if \( n_i \eta p_k \) only if \( n_i \land m_k \neq 1 \) only if \( \mathfrak{B} \not\models Ic_ic_k \) only if \( \exists j \in \ell [Ac_i c_j, Ec_k c_j \in \Gamma^d \lor Ac_k c_j, Ec_i c_j \in \Gamma^d] \) only if \( Ec_i c_k \in \Gamma^d \). From this and the consistency of \( \Gamma \) it follows that for every \( c, c' \in C \), \( \mathfrak{B} \models Ecc' \), \( \mathfrak{B} \models Icc' \) whenever \( Ecc' \in \Gamma \), \( Icc' \in \Gamma \) respectively. 

2.7.2. Leibniz soundness and completeness. For \( e \in \{d, g\} \), \( e \)-deduction is sound with respect to the set of all Leibniz models (to be denoted, henceforth, by “\( L \)”) as they are order models. Regarding completeness, for \( \Gamma \cup \{\sigma\} \subseteq BN(C) \) put:

\[ \Gamma \models \sigma \quad \text{iff} \quad \Gamma \models_{\mathfrak{B}} \sigma \text{ for every } \mathfrak{B} \in L. \]

THEOREM 2.14. If \( \Gamma \) is essentially finite, then \( \Gamma \models^g \sigma \) whenever \( \Gamma \models \sigma \).

Proof. Obvious if \( \Gamma \) is inconsistent. Let \( \Gamma \) be consistent and \( \Gamma \models \sigma \), then by theorem 2.13 \( \Gamma \cup \{\hat{\sigma}\} \) is inconsistent, from which the result follows. \( \square \)

REMARKS 2.15.

1. From Łukasiewicz (1998, pp. 126-9) it follows that:

\[ SF(J) \vdash \alpha \quad \text{iff} \quad \models_{L'} \alpha \]

where \( \alpha \) is any sentence (not necessarily basic) of the language \( J \), and \( L' \) is the obvious adaptation of \( L \) to \( J \). Consequently, for every \( \Gamma \cup \{\alpha\} \subseteq BS(J) \):

\[ SF(J) \cup \Gamma \vdash \alpha \quad \text{only if} \quad \Gamma \models_{L'} \alpha, \]

the other direction holds if \( \Gamma \) is essentially finite.

2. In the above remark, as well as in theorem 2.14, only square free Leibniz models (with the obvious definition) may be taken into consideration.

2.7.3. Generalization. Theorem 2.13 cannot be unconditionally generalized to infinite \( C_{\Gamma} \). For, let \( < c_i >_{i \in \mathbb{N}} \) be an injective enumeration of some denumerable subset of \( C \). Put:

\[ \Gamma = \{Ac_i c_{i+1} : i \in \mathbb{N}\} \cup \{Oc_{i+1} c_i : i \in \mathbb{N}\} \]
then $\Gamma$ is consistent but has no Leibniz model, though it has a Venn model. The following theorem gives a sufficient condition for $\Gamma$ to have a Leibniz model if $C_\Gamma$ is denumerable.

**Theorem 2.16.** Let $C_\Gamma$ be denumerable and let $< c_i >_{i \in \mathbb{N}}$ be an injective enumeration of it. Then $\Gamma$ has a Leibniz model if it is consistent and for every $i \in \mathbb{N}$, $\{q \in \mathbb{N} : Ac_i c_q \in \Gamma_d\}$ is finite.

**Proof.** Along the lines of the proof of theorem 2.13 with the following modifications. Let $< p_i >_{i \in \mathbb{N}}$ be an injective enumeration of the primes, put:

$$m_i = \prod_{Ac_i c_j \in \Gamma^d} p_j, \quad n_i = \prod_{Ec_i c_j \in \Gamma^d} p_j, \quad j < \max\{q \in \mathbb{N} : Ac_i c_q \in \Gamma_d\}$$

To see that the condition of the above theorem is essentially necessary, define the equivalence relation $\sim_{\Gamma}$ on $C_\Gamma$ by $a \sim_{\Gamma} b$ iff $Aab, Aba \in \Gamma_d$ (cf. section 1.5 above).

**Theorem 2.17.** If $\Gamma$ has a Leibniz model then there is a consistent extension $\Gamma'$ of $\Gamma$ such that $C_{\Gamma'} = C_\Gamma$, $C_{\Gamma''} / \sim_{\Gamma''}$ is countable and for every $a \in C_{\Gamma'}$, with at most two exceptions, $Q_a(= \{c \in C_{\Gamma'} : Aac \in \Gamma_d\} / \sim_{\Gamma''})$ is finite.

**Proof.** Let $\mathfrak{B}(= < B, R, \mu >)$ be a Leibniz model of $\Gamma$. Put:

$$\Gamma' = \Gamma \cup \{\sigma \in BN(C_\Gamma) : \mathfrak{B} \models \sigma\},$$

then $C_{\Gamma'} = C_\Gamma$ and $C_{\Gamma''} / \sim_{\Gamma''}$ is countable.

If for some $a \in C_{\Gamma'}$, $Q_a$ is infinite, then $\mu(a) \in \{< 0, 1 >, < 1, 0 >\}$ from which the last part of the theorem follows.

**Remarks 2.18.**

1. In the underlying order structure of a Leibniz model $\mathfrak{B} = < B, R, \mu >$, $< 1, 1 >$ is the greatest element and $< 0, 1 >, < 1, 0 >$ are the only minimal elements. Let $a, c \in C$. If $\mu(a) = < 1, 1 >$ then $\mathfrak{B} \models Ac a$. Also assuming that
2. There would be no exceptions in the above theorem had \( \mathbb{N} \) been replaced by \( \mathbb{N}^+ \) in the definition of Leibniz models, which is equivalent to excluding \( <0,1> \) and \( <1,0> \) from the universe of Leibniz models.

3. Noticing that \( c \sim_\Gamma c' \) forces \( c, c' \) to be assigned the same value in any Leibniz model of \( \Gamma \), with a slight modification of its proof, theorem 2.16 may be strengthened as follows:

\( \Gamma \) has a Leibniz model if there is a consistent extension \( \Gamma' \) of \( \Gamma \) such that:

1. \( C_{\Gamma'} = C_{\Gamma} \).
2. \( C_{\Gamma'/\sim_{\Gamma'}} \) is countable.
3. For every \( a \in C_{\Gamma}, \{ c/\sim_{\Gamma}; Aac \in \Gamma' \} \) is finite.

4. The above strengthening is very close to be the converse of theorem 2.17. As a matter of fact, it is its converse had \( \mathbb{N} \) been replaced by \( \mathbb{N}^+ \) in the definition of Leibniz models.

5. The completeness theorem 2.14 may be generalized in line with the above generalizations.

2.7.4. Logico-philosophical discussion of Leibniz models. “It is strange that his [Leibniz’s] philosophic intuitions, which guided him in his research, yielded such a sound result.” says Łukasiewicz (1998, p. 126). Hopefully the above reasoning would make matters less strange.

Following is a further discussion taking into consideration the Liebnizian correlation between prime and composite numbers on one hand and atomic and composite sentences, propositions, concepts or attributes on the other hand (cf. Glashoff 2002, 2010).

If the primes \( p_1, p_2 \) correspond, respectively, to the atomic sentences \( p'_1, p'_2 \), it is natural to let the composite number \( p_1 p_2 \) correspond to the composite sentence \( p'_1 \land p'_2 \). The difficulty here is that \( p_1^2 \), which is not equal to a prime, would correspond to the sentence \( p'_1 \land p'_1 \), which is equivalent to an atomic sentence; as conjunction of sentences is idempotent, while multiplication of numbers is not. Obviously this difficulty will not arise for square free numbers.

Notice that in the definitions of \( \mu : C \rightarrow B \) given above, the values
assigned by $\mu$ to the elements of $C_T$ are always ordered pairs of square free numbers. Extending this property to all elements of $C$, after relaxing it to permit $<0,1>,<1,0>$ also to be taken as values, gives rise to what will be called essentially square free Leibniz models.

To investigate the relationship between the Leibniz models and the essentially square free Leibniz models, let $<q_{ij}>_{i,j \in \mathbb{N}}$ be an injective double sequence of primes. Map the $k^{th}$ power of the $i^{th}$ prime $p_i$ on $\prod_{j \in k} q_{ij}$. This mapping may be extended in the obvious way to an injection $\nu$ from $\mathbb{N}$ to $\mathbb{N}$ such that $\nu(0) = 0$ and for $n \geq 1$, $\nu(n)$ is square free (being the empty product of primes, $\nu(1) = 1$). The mapping $\nu$ may be further extended, in the obvious way, to $\mathbb{N} \times \mathbb{N}$, the extension also will be denoted by “$\nu$”. It may be easily seen that $\nu(B) \subseteq B$ and that $<m_1,n_1>R<m_2,n_2>$ iff $\nu(<m_1,n_1>)R\nu(<m_2,n_2>)$ for every $<m_1,n_1>,<m_2,n_2> \in B$. So for every Leibniz model $\mathfrak{B}(=<B,R,\mu>)$, $\nu$ is a monomorphism from $\mathfrak{B}$ into $\mathfrak{B}^{\nu}(=\langle B,R,\nu \mu \rangle)$ which is essentially square free and is basically equivalent to $\mathfrak{B}$. Moreover if in $\mathfrak{B}^{\nu}$, $B$ is replaced by $\nu(B)$ and $R$ by $R \cap (\nu(B) \times \nu(B))$, then $\nu$ will be an isomorphism. Such models will be called proper Leibniz models. Since every Leibniz model is isomorphic to a proper Leibniz model, attention may be confined to the latter.

Let $<q'_{ij}>_{i,j \in \mathbb{N}}$ be an injective double sequence of atomic sentences in some sentential language. For $<m, n> \in \nu(B)$ put:

$$\lambda(<m, n>) = \bigwedge_{m \eta q_{ij}} q'_{ij} \land \bigwedge_{n \eta q_{ij}} \neg q'_{ij}.$$  

As $1 \eta p$ for no prime $p$, $\bigwedge_{m \eta q_{ij}} q'_{ij} = \bigwedge_{n \eta q_{ij}} \neg q'_{ij}$ = the empty conjunction, which is always true. So $\lambda(<1,1>)$ is always true.

On the other hand, $0 \eta p$ for every prime $p$, so

$$\lambda(<0,1>) = \bigwedge_{i,j \in \mathbb{N}} q_{ij} \quad \text{and} \quad \lambda(<1,0>) = \bigwedge_{i,j \in \mathbb{N}} \neg q_{ij}.$$  

These are the only infinitary sentences to be considered.

It may be easily seen that for every proper Leibniz model $\mathfrak{B}$, $\mathfrak{B} \models Ac_1c_2$ iff $\lambda \mu c_1 \rightarrow \lambda \mu c_2$ is a tautology, and $\mathfrak{B} \models Ec_1c_2$ iff $\lambda \mu c_1 \land \lambda \mu c_2$ is a contradiction.

As a matter of fact $<0,1>,<1,0>$ and $<1,1>$ are not indispensable as elements of the universe of proper Leibniz models. To keep them or not
is a philosophical choice. Rejecting them is probably more compatible with the Aristotelian legacy.

Following Boole (1948, p. 49), to each \( m, n \in \nu(B) \) the set \( \theta(<m, n>) \) of all truth assignments which satisfy \( \lambda(<m, n>) \) may be appropriated. For every proper Leibniz model (hence for every Leibniz model) \( \mathfrak{B} \), \( \theta \) induces an isomorphism of \( \mathfrak{B} \) onto a Venn model which is a concrete order model.

3. Decidability.

REMARKS 3.1. Let \( a, b, c \in C \) and \( \Gamma \subseteq BN(C) \).

1. If \( \{Ecc, Occ\} \cap \Gamma \neq \emptyset \), \( \Gamma \) may be easily seen to be contradictory. In such a case \( \Gamma \) is said to be plainly contradictory.

2. \( \Gamma \vdash Oab \) iff \( Oab \in \Gamma \).

3. If \( \Gamma \vdash Ecc \) then \( Ecc \in \Gamma \) or \( c \in C_\Gamma \).

4. If \( \Gamma \) is not plainly contradictory, then \( \Gamma \) is contradictory iff there are \( \sigma, \hat{\sigma} \in (\Gamma^d \cap BN(C_\Gamma)) \).

5. \( \Gamma^d \cap BN(C_\Gamma) = (\Gamma \cap BN(C_\Gamma))^d \cap BN(C_\Gamma) \).

6. In a different context, Glashoff (2005) presents an algorithm which may be regarded as a prelude to the one given below. Roughly speaking, it amounts -in our terminology- to: For a finite \( \Gamma(\subseteq BN(C)) \), \( \Gamma^d \cap BN(C_\Gamma) \) may be obtained from \( \Gamma \) in finitely many steps.

THEOREM 3.2. There is a polynomial (of degree 8) time algorithm to decide for any essentially finite \( \Gamma(\subseteq BN(C)) \) which is not plainly contradictory whether it is contradictory, and to assign a Leibniz model to it if it is not.

Proof. Let \( \Gamma \) satisfy the conditions of the theorem, then \( BN(C_\Gamma) \) is finite. Put \( \Gamma' = \Gamma \cap BN(C_\Gamma) \) and \( \Delta = \Gamma^d \cap BN(C_\Gamma) \).

The input of the algorithm is \( \Gamma' \) structured as a list \( <\gamma_i>_{i \in n} \) where \( n = |\Gamma'| \), and for every \( i \in n \), \( \gamma_i = <\gamma_{ij}>_{j \in \mathbb{N}} \) where \( \gamma_{io} \in \{A, E, I, O\} \) and...
γ₁, γ₂ ∈ CΓ. CΓ may be obtained from Γ′ or supplied as a secondary input. |CΓ| ≤ 2n and |BN(CΓ)| ≤ 16n².

The next step is to extract for each Y ∈ \{A, E, I, O\}, Γ′_Y (the set of all elements of Γ′ starting with Y) which may be done through a simple scanning procedure in a linear time. Then construct Δ_Y (with the obvious meaning) for each Y ∈ \{A, E, I, O\}.

Notice that Δ_O = Γ′_O and Δ_A is needed to construct each of Δ_E and Δ_I. To construct Δ_A start with the list Γ′_A. At most 16n⁴ comparisons are needed to determine all the possible applicabilities of Barbara. And for each possible applicability at most 4n² comparisons are needed to check whether the consequent is already there. If not, append it.

It is needed to repeat this process at most 4n² times to cover all the required applications of Barbara. In addition, for each c ∈ CΓ at most 4n² comparisons are needed to check whether Acc is listed; if not, append it. It is easy to see that this completes the construction of Δ_A.

By simple variations on the above procedure Δ_E may be constructed. Constructing Δ_I is much simpler.

Γ is contradictory iff σ, ˆσ ∈ Δ for some σ, which needs at most 32n⁴ comparisons to check.

If Γ is consistent assign to it a Leibniz model along the lines of the proof of theorem 2.13 (in the appendix a polynomial (in n of degree 6) time algorithm will be presented to generate the first n primes).

The total running time is bounded above by a polynomial (in n) of degree 8. □

4. Basic equivalence of the four formalizations. Let Γ ∪ \{σ\} ⊆ BN(C), let h and H be bijective interpretations of BN(C) in BS(J) and BM(P) respectively, and let < Γ′, σ′, T > ∈ \{< Γ, σ, DF(C) >, < Γ^h, σ^h, SF(J) >, < Γ^H, σ^H, MF(P) >\}.

THEOREM 4.1.

1. Γ is consistent iff Γ′ ∪ T is.

2. Γ ⊩^g σ iff Γ′ ∪ T ⊩ σ′.

Proof. From proposition 1.13, if of (1) and only if of (2) follow.
The other two directions for \(< \Gamma', \sigma', T > = < \Gamma^h, \sigma^h, SF(J) >\) follow from the corresponding directions for \(< \Gamma', \sigma', T > = < \Gamma, \sigma, DF(C) >\), this is a consequence of proposition 1.14. So it remains to prove these two other directions for \(< \Gamma', \sigma', T > \in \{ < \Gamma, \sigma, DF(C) >, < \Gamma^H, \sigma^H, MF(P) > \}\).

Only if of (1): Assume \( \Gamma \) is consistent. Let \( \Delta \) be a finite subset of \( \Gamma \), then by theorem 2.13 it has a Leibniz model, \( \mathfrak{B} \) say. \( \mathfrak{B} \) is a model of \( \Delta \cup DF(C) \); from this the consistency of \( \Gamma \cup DF(C) \) follows. Since \( \Delta \) may be assumed to be the inverse image of some finite \( \Delta' \subseteq \Gamma^H \), then by subsection 2.6.3, \( \mathfrak{B} \) induces a model of \( \Delta' \cup MF(P) \). From this the consistency of \( \Gamma^H \cup MF(P) \) follows.

If of (2): Let \( \Gamma' \cup T \vdash \sigma' \), then \( \Gamma' \cup \{ \hat{\sigma}' \} \cup T \) is inconsistent. By part (1), \( \Gamma \cup \{ \hat{\sigma} \} \) is inconsistent, hence \( \Gamma \nvDash \sigma \). \( \square \)

REMARKS 4.2.

1. As far as the basic sentences are concerned, the four formalizations are equivalent in the sense expressed by part 2 of the above theorem; so it may be said, for brevity, that they are basically equivalent.

2. In the above theorem, the only if direction of (1) and the if direction of (2) may be directly proved for \(< \Gamma', \sigma', T > = < \Gamma^h, \sigma^h, SF(J) >\).

5. Venn soundness and completeness. Let \( \Gamma \cup \{ \sigma \} \subseteq BN(C) \).

DEFINITION 5.1. \( \Gamma \nvDash \sigma \) iff \( \Gamma \nvDash \sigma \) for every Venn model \( \mathfrak{B} \).

THEOREM 5.2. (Venn soundness and completeness). General deduction is sound and complete with respect to the class of Venn models. That is \( \Gamma \nvDash \sigma \) iff \( \Gamma \nvDash \sigma \) (2 of remarks 2.11). This guarantees soundness.

Proof. Every Venn model is a \( DF(C) \) model (subsection 2.5), then a \( g \)-model (2 of remarks 2.11). To prove completeness, let \( \Gamma \nvDash \sigma \), then \( \Gamma \cup \{ \hat{\sigma} \} \) is consistent then, by theorem 4.1, \( (\Gamma \cup \{ \hat{\sigma} \})^H \cup MF(P) \) is consistent where \( H \) is a bijective interpretation of \( BN(C) \) in \( BM(P) \), for some appropriate \( P \). By well known results in first order logic, \( (\Gamma \cup \{ \hat{\sigma} \})^H \cup MF(P) \) has a model. By subsection 2.6.3 and 2 of remarks 2.11, \( \Gamma \cup \{ \hat{\sigma} \} \) has a Venn model, hence \( \Gamma \nvDash \sigma \). \( \square \)
Alternatively theorem 6.3 below may be made use of to directly show that \( \Gamma \cup \{ \tilde{\sigma} \} \) has a Venn model.

REMARKS 5.3.

1. In view of subsection 2.5, the above theorem entails that general deduction is sound and complete with respect to each of the classes of order, partial order, and concrete order models.

2. Direct ways to Venn models on one hand, and to order and partial order models on the other hand, will be presented in sections 6 and 9 respectively.

3. For the Venn soundness and completeness of Łukasiewicz’s system, Sheperdson (1956) may be consulted.

6. Direct way to Venn models. Let \( \Gamma \subseteq BN(C) \), put \( D = \{ <a, b> \in C \times C : \{Iab, Iba\} \cap \Gamma^d \neq \phi \} \), \( B = \phi(D) - \{ \phi \} \). Define the function \( \mu \) from \( C \) to \( B \) by:
\[
\mu(c) = \{ <a, b> \in D : \{Aac, Abc\} \cap \Gamma^d \neq \phi \}.
\]
Then \( <B, \mu> \) is a Venn model (which is a concrete order model), denote it by “\( \mathfrak{B}^\Gamma \)”.  

LEMMA 6.1. For every \( c, c' \in C \), the following are equivalent:

1. \( Acc' \in \Gamma^d \),
2. \( \mu(c) \subseteq \mu(c') \) (which is equivalent to \( \mathfrak{B}^\Gamma \models Acc' \)),
3. \( <c, c'> \in \mu(c') \).

Proof. Straightforward. \( \square \)

LEMMA 6.2. Let \( c, c' \in C \), consider:

1. \( <c, c'> \in D \),
2. \( <c, c'> \in \mu(c) \cap \mu(c') \),
3. \( \mu(c) \cap \mu(c') \neq \phi \) (which is equivalent to \( \mathfrak{B}^\Gamma \models Icc', Ic'c \)).
4. \( \{ Ecc', Ec'c \} \cap \Gamma^d = \phi \), then 1 is equivalent to 2 which implies 3 which, for consistent \( \Gamma \), implies 4.

Proof. The first two parts are easy to see. For the last part assume \( \mu(c) \cap \mu(c') \neq \phi \neq \{ Ecc', Ec'c \} \cap \Gamma^d \), then \( Ecc' \in \Gamma^d \) and there are \( a, b \in C \) such that \( (Iab \in \Gamma^d \) or \( Iba \in \Gamma^d \)), \( (Aac \in \Gamma^d \) or \( Abc \in \Gamma^d \)) and \( (Aac' \in \Gamma^d \) or \( Abc' \in \Gamma^d \)). So there are eight cases to consider. We deal only with the case \( Iab, Aac, Abc' \in \Gamma^d \); the other cases are similar or easier. In this case \( Aac, Ecc' \in \Gamma^d \), then \( Ecc' \in \Gamma^d \) and there are \( a, b \in C \) such that \( (Iab \in \Gamma^d \) or \( Iba \in \Gamma^d \)), \( (Aac \in \Gamma^d \) or \( Abc \in \Gamma^d \)) and \( (Aac' \in \Gamma^d \) or \( Abc' \in \Gamma^d \)). So there are eight cases to consider. We deal only with the case \( Iab, Aac, Abc' \in \Gamma^d \); the other cases are similar or easier. In this case \( Aac, Ecc' \in \Gamma^d \), then \( Ecc' \in \Gamma^d \) and there are \( a, b \in C \) such that \( (Iab \in \Gamma^d \) or \( Iba \in \Gamma^d \)), \( (Aac \in \Gamma^d \) or \( Abc \in \Gamma^d \)) and \( (Aac' \in \Gamma^d \) or \( Abc' \in \Gamma^d \)).

THEOREM 6.3. (Existence of Venn models). Let \( \Gamma \subseteq BN(C) \) be consistent, then \( \mathfrak{B}^\Gamma \) is a Venn model (which is a concrete order model) of \( \Gamma \).

Proof. Let \( c, c' \in C \). By lemma 6.1, \( Acc' \in \Gamma^d \) iff \( \mathfrak{B}^\Gamma \models Acc' \). From this it follows that for \( Y \in \{ A, O \}, \mathfrak{B}^\Gamma \models Y cc' \) if \( Y cc' \in \Gamma^d \).

Moreover, if \( Icc' \in \Gamma^d \) then \( < c, c' > \in D \) then, by lemma 6.2, \( \mathfrak{B}^\Gamma \models Icc' \). Finally, if \( Ecc' \in \Gamma^d \) then, by lemma 6.2, \( \mu(c) \cap \mu(c') = \phi \) then \( \mathfrak{B}^\Gamma \models Ecc' \). \( \square \)

Lemma 6.1 syntactically characterizes \( \{ Acc' \in BN(C) : \mathfrak{B}^\Gamma \models Acc' \} \), hence it syntactically characterizes \( \{ Occ' \in BN(C) : \mathfrak{B}^\Gamma \models Occ' \} \). The following syntactical characterization of \( \{ Icc' \in BN(C) : \mathfrak{B}^\Gamma \models Icc' \} \), hence of \( \{ Ecc' \in BN(C) : \mathfrak{B}^\Gamma \models Ecc' \} \),

\[ \mathfrak{B}^\Gamma \models Icc' \quad \text{iff} \quad \Gamma^d \vdash Icc' \]

is an immediate consequence of lemma 8.3 below; the definition of “\( \vdash \)” may be found at the beginning of section 8 below.

Slightly modifying the above construction, light may be shed on the role played by the Venn models among the models of \( DF(C) \).

THEOREM 6.4. For every \( DF(C) \) model \( \mathfrak{B} =< B, R_1, R_2, \mu > \) there is a Venn model \( \mathfrak{B}' =< B', \mu' > \) and surjection \( h : B \to B' \) such that:

1. \( \mu' = h \mu \) and for every \( b_1, b_2 \in B : \)

\( b_1 R_1 b_2 \) iff \( h(b_1) \subseteq h(b_2) \), \( b_1 R_2 b_2 \) iff \( h(b_1) \cap h(b_2) \neq \phi \),
2. \( \mathfrak{B} \) and \( \mathfrak{B}' \) are basically equivalent,

3. \( h \) is an isomorphism iff \( R_1 \) is antisymmetric.

Proof. Put:

\[
\begin{align*}
  h & : B \to \wp(\wp(B)) \\
  h(b) & = \{ \{b_1, b_2\} \in \wp(B) : (b_1 R_2 b_2 \text{ or } b_2 R_2 b_1) \\
  & \quad \text{and } (b_1 R_1 b \text{ or } b_2 R_1 b) \}, \\
  B' = h(B) , \quad \mu' = h\mu.
\end{align*}
\]

The rest of the proof is easy. \( \square \)

7. Variations on \( NF(C) \). As was promised in section 1, we follow in subsection 7.1 the long standing tradition of not permitting the subject and the predicate of a categorical sentence to be the same. The resulting formalization, \( WF(C) \), and its relationship to \( NF(C) \) are discussed.

In subsection 7.2 the standpoint that \( Acc' \) requires that all \( c \) are \( c' \) but not vice versa, will be considered.

7.1. Weak natural deduction formalization of AAS. The alphabet of the logical system \( WF(C) \), the weak natural deduction formalization of AAS, is the same as the alphabet of \( NF(C) \). The set \( W(C) \) of sentences of \( WF(C) \) is defined as follows:

\[
W(C) = S(C) - \{ Y cc : Y \in \{ A, E, I, O \} \text{ and } c \in C \}.
\]

In accordance with subsection 1.6, the set \( BW(C) \) of basic sentences of \( WF(C) \) is \( W(C) \) itself.

The rules of inference of \( WF(C) \) are those of \( NF(C) \) after dropping the first one (\( \text{Aaa} \)). The weak direct and general deduction relations are respectively denoted by "\( \vdash \)" and "\( \vdash' \)" and are defined along the lines of definitions 1.8 and 1.9 respectively. The definition of the other notions introduced in the theory of \( NF(C) \) may be modified in the obvious way to render the corresponding definitions for the theory of \( WF(C) \).

The theory of \( WF(C) \) may be obtained from that of \( NF(C) \) by making the obvious modifications. The key observations are the following, where
\[ \Gamma \cup \{ \sigma \} \subseteq BW(C). \]

**Proposition 7.1.**
\[ \Gamma^{wd} \vdash \sigma \iff \Gamma^{d} \vdash \sigma. \]

**Proof.** The only if direction is obvious. To prove the other direction let \( \langle \rho_i \rangle_{i \in n} \) be a \( d \)-deduction of \( \sigma \) from \( \Gamma \). We show by induction that for every \( i \in n \), \( \Gamma^{wd} \vdash \rho_i \) if \( \rho_i \in BW(C) \).

Distinguish between four cases:

1. \( \rho_i = \text{Occ}' \), then \( \rho_i \in \Gamma \), then \( \Gamma^{wd} \vdash \rho_i \).
2. \( \rho_i = \text{Acc}' \), then the result follows by the induction hypothesis.
3. \( \rho_i = \text{Icc}' \), then \( \text{Icc}' \in \Gamma \) or \( \rho_j = \text{Ac}' \) for some \( j < i \). From this and part 2 the result follows by the induction hypothesis.
4. \( \sigma = \text{Ecc}' \), then the result follows by the induction hypothesis noting that if \( \text{Ecc}' \) is obtained via applying \( \text{Acc}', \text{Ec}' \) then the first occurrence of \( \text{Ec}' \) in the deduction must be obtained via \( \text{Ac}', \text{Ec}' \) for some \( c' \neq c \). By part 2 and the induction hypothesis \( \Gamma \vdash \text{Acc}', \text{Ac}', \text{Ec}', \text{Ec}' \), hence \( \Gamma^{wd} \vdash \text{Ecc}' \).

**Proposition 7.2.**
\( \Gamma \) is \( wd \)-consistent iff \( \Gamma \) is \( d \)-consistent (hence \( \Gamma^{wd} \vdash \sigma \iff \Gamma^{d} \vdash \sigma \)).

**Proof.** The if direction easily follows from proposition 7.1. To prove the other direction assume that \( \Gamma \) is \( d \)-inconsistent, then \( \Gamma^{d} \vdash \text{Ycc}', \text{Ycc}' \) for some \( Y \in \{ A, E, I, O \} \) and some \( c, c' \in C \). If \( c \neq c' \) the result follows by the previous proposition. Else, distinguish between two cases:

1. \( \text{Occ} \in \{ \text{Ycc}, \text{Ycc}' \} \), then \( \text{Occ} \in \Gamma \) which is not permitted.
2. \( \text{Ecc} \in \{ \text{Ycc}, \text{Ycc}' \} \). In this case there is a \( d \)-deduction of \( \text{Ecc} \) from \( \Gamma \). The rule made use of to justify the first occurrence of \( \text{Ecc} \) in this deduction must be \( \text{Acc}', \text{Ec}' \) for some \( c'' \neq c \). By the previous proposition
\[\Gamma \vdash \text{I}c''c, \text{E}c''c, \text{hence} \Gamma \text{ is } wd\text{-inconsistent.} \]

COROLLARY 7.3. \(\Gamma\) is \(wd\)-consistent iff \(\Gamma\) is \(d\)-consistent iff \(\Gamma\) is \(g\)-consistent iff \(\Gamma\) is \(wg\)-consistent.

Proof. \(\Gamma\) is \(wd\)-consistent only if \(\Gamma\) is \(d\)-consistent only if \(\Gamma\) is \(g\)-consistent only if \(\Gamma\) is \(wg\)-consistent only if \(\Gamma\) is \(wd\)-consistent. \(\square\)

REMARK 7.4. From propositions 7.1 and 7.2 it follows that the results concerning Leibniz soundness and completeness (subsection 2.7.2) and Venn and order soundness and completeness (section 5) apply to \(WF(C)\) after replacing \(d, g\) and \(BN(C)\) by \(wd, wg\) and \(BW(C)\) respectively.

7.2. Proper natural deduction formalization of AAS. AAS may be interpreted to require \(Acc'\) to hold iff all \(c\) are \(c'\) but not vice versa. That is, extensionally, the denotation of “\(c'\)” is required to be a proper subclass of the denotation of “\(c\)”.

To satisfy this requirement introduce the logical system \(PF(C)\), the proper natural deduction formalization of AAS, based on the same language as the system \(NF(C)\). So the set \(P(C)\) of sentences of \(PF(C)\) is the same as \(S(C)\). In accordance with subsection 1.6 the set \(BP(C)\) of basic sentences of \(PF(C)\) is \(P(C)\) itself. For “\(Occ''\)” to remain to be the contradictory of “\(Acc''\)”, it must be interpreted as some \(c\) are not \(c'\) or (all \(c\) are \(c'\) and vice versa). The rules of inference of \(PF(C)\) are to be obtained from those of \(NF(C)\) by dropping the first one and augmenting the remaining ones by \(\overrightarrow{Icc}(I-Id)\) and \(\overrightarrow{Occ}(O-Id)\).

The proper direct and general deduction relations are respectively denoted by “\(\vdash^d\)” and “\(\vdash^g\)” and are defined along the lines of definitions 1.8 and 1.9 respectively. The definitions of the other notions introduced in the theory of \(NF(C)\) may be modified in the obvious way to render the corresponding definitions for the theory of \(PF(C)\).

PROPOSITION 7.5. For \(\Gamma \cup \{\sigma\} \subseteq BW(C)\):

1. \(\Gamma \vdash^d \sigma \text{ iff } \Gamma \vdash^d \sigma \text{ (iff } \Gamma \vdash^d \sigma)\).
2. If $\Gamma$ is pd-consistent then it is wd-consistent (equivalently d-consistent), but not always vice versa.

3. If $\Gamma \vdash \sigma$ (equivalently $\Gamma \vdash \sigma$) then $\Gamma \vdash \sigma$, but not always vice versa.

Proof.

1. The proof of part 1 is similar to that of proposition 7.1.

2. That $\Gamma$ is wd-consistent if it is pd-consistent easily follows from part 1. To see that the other direction does not always hold consider $\{\text{Acc}', \text{Ac}'c\}$ for some $c, c' \in C$ such that $c \neq c'$; this proves part 2.

3. Part 3 is a direct consequence of part 2. □

PROPOSITION 7.6. $\Delta(\subseteq BP(C))$ is pd-consistent iff it is pg-consistent.

Proof. Along the lines of the proof of part 4 of proposition 1.11. □

Order models, Leibniz models, and Venn models are not $e(\in \{\text{pd}, \text{pg}\})$-models, so it does not make sense to ask whether $e$ is sound or complete with respect to any of these classes. However, with some modifications, to be shown below, everything goes as expected.

Let $\mathfrak{B} =< B, A^*, E^*, I^*, O^*, \mu >$ be a d-model of $\Gamma(\subseteq BW(C))$ such that $\mu$ is injective on $C_G$ and $A^*\mu$ is antisymmetric, and let $A^p, O^p$ be subsets of $B \times B$ such that $A^*\mu - I^p(c) \subseteq A^p \subseteq A^*\mu$ and $O^*\mu \cup I^p(c) \subseteq O^p$. Put:

$$\mathfrak{B}^p =< B, A^p, E^*, I^*, O^p, \mu > .$$

PROPOSITION 7.7. $\mathfrak{B}^p$ is a pd-model of $\Gamma$.

Proof. Assume $\mathfrak{B} \models \Gamma$. To show that $\mathfrak{B}^p \models \Gamma$, let $\gamma \in \Gamma$ then $\gamma = Ycc'$ for some $Y \in \{A, E, I, O\}$ and some $c, c' \in C$ such that $c \neq c'$, hence $\mu(c) \neq \mu(c')$. If $Y = A$ then $< \mu(c), \mu(c') > \in A^*\mu - I^p(c) \subseteq A^p$. The other cases are obvious.

To show that $\mathfrak{B}^p$ is a pd-model, assume $\mathfrak{B}^p \models \text{Acc}', \text{Ac}'c''$. If not $\mathfrak{B}^p \models \text{Acc}''$ then $\mu(c) = \mu(c'')$, then $< \mu(c), \mu(c') >, < \mu(c'), \mu(c) > \in A^*\mu$, then $\mu(c') = \mu(c) = \mu(c'')$ which is absurd. So Barbara is valid. The other rules
are easier to deal with.

Accordingly, it is legitimate to adopt in the sequel the following modifications:

\[ A^p = A^* - I_{\mu(C)} \quad , \quad O^p = O^* \cup I_{\mu(C)}. \]

THEOREM 7.8. Let \( \Delta(\subseteq BP(C)) \) be \( pd \)-consistent, then it has a modified Venn model which is a modified c.o.m and which is also a \( pg \)-model. If, in addition, \( \Delta \) is essentially finite then it has also a modified Leibniz model which is a \( pg \)-model.

Proof. Put \( \Gamma = \Delta \cap BW(C) \), then \( \Gamma \) is \( pd \)-consistent, hence it is \( d \)-consistent, hence \( B^\Gamma \) is a \( d \)-model of \( \Gamma \) in which \( A^* \) is antisymmetric.

To show that \( \mu \) is injective let \( c, c' \in C \) be such that \( c \neq c' \) and \( \mu(c) = \mu(c') \), then \( \Gamma \vdash Acc, Ac'c \), then \( \Gamma \vdash Acc', Ac'c \), then \( \Gamma \vdash pd Acc, Ac'c \), then \( \Gamma \vdash pd Acc \) which contradicts that \( \Gamma \) is \( pd \)-consistent.

Therefore \( B^{\Gamma^p} \) is a modified Venn model (which is also a modified c.o.m) of \( \Gamma \). By proposition 7.7 it is a \( pd \)-model of \( \Gamma \), from which it may be easily seen that it is a \( pg \)-model of \( \Delta \). The proof of the additional result in case \( \Delta \) is essentially finite is almost the same. The only major difference is that \( \mu \) may not be injective. But its restriction to \( C_\Delta \) is injective, which is sufficient for our purpose. \( \square \)

REMARK 7.9. The last theorem shows that remark 7.4 applies to \( PF(C) \) after making the obvious modifications.

8. Direct completion of direct deduction. In this section the five rules of inference given in definition 1.7 are augmented by five more rules, in order that \( \Gamma^g \) may be directly obtained from \( \Gamma \) in case \( \Gamma \) is consistent (cf. Glashoff (2005) where related problems are dealt with by brute force via a computer program). The additional five rules are:

5. \[
\frac{I_{ab}}{I_{ba}} \quad (Ic)
\]

6. \[
\frac{I_{ab},A_{bc}}{I_{ac}} \quad (Darii)
\]

7. \[
\frac{I_{ab},E_{bc}}{O_{ac}} \quad (Ferio)
\]

8. \[
\frac{O_{ab},A_{ch}}{O_{ac}} \quad (Baroco)
\]
Taking the ten rules of inference into consideration, the $d'$-deduction relation $\Gamma^d \vdash \sigma$ may be defined along the lines of the definition of $\Gamma^d \vdash \sigma$. Likewise, all other definitions involving $\Gamma^d \vdash \sigma$ may be modified in an obvious way to give corresponding definitions involving $\Gamma^d \vdash \sigma$.

**PROPOSITION 8.1.**

1. $\Gamma^d = \{ \sigma \in S(C) : \Gamma \vdash \sigma \}$.
2. $C_{\Gamma^d} = C_{\Gamma}$. 

**Proof.** Along the lines of the proofs of the corresponding results for $\Gamma^d$: Part 1 of proposition 1.11 and lemma 2.12, respectively. □

The next definition and parts 1,2 of the next lemma are essentially due to Smith (1983).

**DEFINITION 8.2.** Let $a, b, a', b' \in C$. An $a\mathbf{-}b$ chain is a sequence $< c_i >_{i \in n} \in \mathbb{N}^+$, such that $c_0 = a$ and $c_{n-1} = b$. This chain is said to be a $\Gamma$-chain, or a chain in $\Gamma$, if $\{ Ac_i c_{i+1} : i \in n-1 \} \subseteq \Gamma$; it is said to be an $< a', b' >$ chain if there is $i \in n-1$ such that $c_i = a'$ and $c_{i+1} = b'$.

**LEMMA 8.3.** For $a, b \in C$ and $\Gamma \subseteq BN(C)$:

1. $\Gamma \vdash Aab$ iff $\Gamma^d \vdash Aab$ iff there is an $a\mathbf{-}b$ chain in $\Gamma$.

2. $\Gamma \vdash Eab$ iff $\Gamma^d \vdash Eab$ iff there is $Ea'b' \in BN(C)$ such that $\{ Ea'b', Eb'a' \} \cap \Gamma \neq \phi$ and $\Gamma \vdash Aa', Ab'$.

3. $\Gamma \vdash Iab$ iff $Iab \in \Gamma$ or $\Gamma \vdash Aa$.

3'. $\Gamma \vdash Iab$ iff for some $a', b' \in C$, $\Gamma \vdash Aa'a, Aa'b$ or $\{ Ia'b', Ib'a' \} \cap \Gamma \neq \phi$ and $\Gamma \vdash Aa'a, Ab'b$. 

36
4. $\Gamma \vdash Oab$ iff $Oab \in \Gamma$.

4'. $\Gamma \vdash Oab$ iff for some $a', b' \in C$, $\Gamma \vdash Ia'a, Ea'b$ or $Oa'b' \in \Gamma$ and $\Gamma \vdash Aa'a, Abb'$.

Proof.

1. It is easy to show that the first statement implies the second. By induction it may be shown that the second statement implies the third. Again by induction it may be shown that the third statement implies the first.

2. It is easy to show that the first statement implies the second and that the third implies the first. By induction it may be shown that the second statement implies the third.

Parts 3 and 4 are easy. In each of the parts 3' and 4' one direction is easy, the other may be shown by induction. $\square$

PROPOSITION 8.4. For $\Gamma \cup \{\sigma\} \subseteq BN(C)$:

1. If $\Gamma \vdash \sigma$ then $\Gamma \vdash^d \sigma$.

2. If $\Gamma \vdash^d \sigma$ then $\Gamma \vdash^g \sigma$.

3. $\Gamma$ is $g$-consistent iff $\Gamma$ is $d'$-consistent iff $\Gamma$ is $d$-consistent.

(So for $e \in \{d, d', g\}$ the prefix “$e$-” may be deleted from “$e$-consistent”, “$e$-inconsistent” and “$e$-contradictory”).

Proof. Part 1 is obvious, and part 3 is an easy consequence of parts 1 and 2 above and part 4 of proposition 1.11.

Part 2 is immediate if $\Gamma$ is $d$-inconsistent. To complete the proof assume that $\Gamma$ is $d$-consistent and proceed by course of values induction. Let $\Gamma \vdash \sigma$ and let $< \rho_i>_{i \in \mathbb{N}}$ be a $d'$-deduction of $\sigma$ from $\Gamma$. If the annotation of $\rho_{n-1}$ ($= \sigma$) is that it belongs to $\Gamma$ or that it is the consequent of a $d$-rule whose premises are previous sentences, the result easily follows.

It remains to assume that the annotation of $\rho_{n-1}$ is that it is the conse-
quent of a new rule. The completion of the proof depends on the specific rule in use. Following is a proof in the case of Darii. The other cases are similar or easier.

Let \( \rho_{n-1} = Iac \) and let its annotation be that it follows from \( Iab, Abc \) by Darii. By the induction hypothesis \( \Gamma \vdash_g Iab, Abc \). By part 3 of proposition 1.11, \( \Gamma \vdash_d Abc \), and by the definition of \( \vdash_g \), there is \( \eta \in BN(C) \) such that \( \{ \eta, \hat{\eta} \} = \{ Ic'c'', E\hat{c}'c'' \} \). By lemma 8.3, \( \Gamma \vdash_d Ic'c'' \) and there is \( Eab' \in BN(C) \) such that \( \{ Eab', Eb'a' \} \cap (\Gamma \cup \{ Eab \}) \neq \phi \) and \( \Gamma \cup \{ Eab \} \vdash_d Ac'a', Ac''b' \), hence \( \Gamma \vdash_d Ac'a', Ac''b' \). In view of the \( d \)-consistency of \( \Gamma \), lemma 8.3 implies that \( Eab \in \{ Eab', Eb'a' \} \). Let \( Eab = Eab' \) (the other case is similar), then \( \Gamma \vdash_d Ac'a, Ac''b \). But \( \Gamma \vdash_d Abc \), then \( \Gamma, Eac \vdash_d E\hat{c}'c'', \) hence \( \Gamma \vdash_g Iac \). \( \square \)

In view of the \( g \)-deduction completeness with respect to the class of Venn models, part 2 of the above proposition is an immediate consequence of:

**Proposition 8.5.** The \( d' \)-deduction is sound with respect to the class of Venn models (hence with respect to the class of order models).

**Proof.** Routine. \( \square \)

**Remark 8.6.** The converse of part 2 of proposition 8.4 does not always hold. For if \( \Gamma \) is inconsistent then \( C_{g} = C \), while it is easy to find an inconsistent \( \Gamma \) such that \( C_{d} = C \neq C \). Also the weaker statement: \( \Gamma^d \cap BN(C) = \Gamma^g \cap BN(C) \), does not always hold. A counter example is \( \Gamma = \{ Aab, Oab \} \).

The consistency of \( \Gamma \) solves the problem as the following theorem shows (cf. Smith 1983).

**Theorem 8.7.** For consistent \( \Gamma \), \( \Gamma^{d} = \Gamma^{g} \).

**Proof.** The inclusion of \( \Gamma^d \) in \( \Gamma^g \) is guaranteed by part 2 of proposition 8.4. For the other direction assume that \( \Gamma \) is consistent and \( \Gamma \vdash_g \sigma \). If \( \sigma \) is universal the result follows by part 3 of proposition 1.11 and part 1 of proposition
8.4. So it remains to deal with the particulars. The consistency of \( \Gamma \) restricts what to be considered to the following:

Case 1. \( \sigma \) is \( Iab \) for some \( a, b \in C \). By the method made use of in the proof of part 2 proposition 8.4, consideration may be restricted to the following subcase only. There are \( c, c' \in C \) such that \( \Gamma \vdash \tfrac{d}{d'} Icc', Aca, Ac'b \), which implies that \( \Gamma \vdash Iab \).

Case 2. \( \sigma \) is \( Oab \) for some \( a, b \in C \). In this case \( \Gamma, Aab \vdash \rho, \hat{\rho} \) for some \( \rho \in BN(C) \). Distinguish between two subcases.

Subcase 2.1. For some \( c, c' \in C \), \( \{\rho, \hat{\rho}\} = \{Aca', Oc\} \). Then \( \Gamma \vdash Aca, Abc', O\) which implies that \( \Gamma \vdash Oab \).

Subcase 2.2. For some \( c, c' \in C \), \( \{\rho, \hat{\rho}\} = \{Ecc, Icc'\} \). This subcase may be divided into the following three subsubcases.

Subsubcase 2.2.1. \( \Gamma \vdash Ecc', \Gamma \nvDash Icc' \). Then \( \Gamma \vdash Ac'a, Abc, Ecc' \) which implies that \( \Gamma \vdash Oab \).

Subsubcase 2.2.2. \( \Gamma \nvDash Ecc' \) and \( \Gamma \vdash Icc' \). Then there is \( Ea'b' \in BN(C) \) such that \( \Gamma \vdash Ea'b' \) and \( \Gamma, Aab \vdash Aca', Ac'b' \); while \( \Gamma \nvDash Aca' \) or \( \Gamma \nvDash Ac'b' \), but -by the consistency of \( \Gamma \) - not both.

This subsubcase may be further divided into two subsubsubcases.

Subsubsubcase 2.2.2.1. \( \Gamma \nvDash Aca' \) but \( \Gamma \vdash Ac'b' \). Then \( \Gamma \vdash Icc', Ea'b', Ac'b', Aca, Aba' \) from which \( \Gamma \vdash Oab \) follows.

Subsubcase 2.2.2.2. \( \Gamma \vdash Aca' \) but \( \Gamma \nvDash Ac'b' \). Similar to subsubcase 2.2.2.1.

Subsubcase 2.2.3. \( \Gamma \nvDash Ecc' \) and \( \Gamma \nvDash Icc' \). Then there is \( Ea'b' \in BN(C) \) such that \( \Gamma \vdash Ac'a, Abc, Ea'b' \) and \( \Gamma, Aab \vdash Aca', Ac'b' \), while \( \Gamma \nvDash Aca' \) or \( \Gamma \nvDash Ac'b' \). But the consistency of \( \Gamma \) implies that \( \Gamma \vdash Ac'b' \), then \( \Gamma \vdash Aca' \), then \( \Gamma \vdash Aca, Aba' \). In particular, \( \Gamma \vdash Aca, Ac'b', Aba', Ea'b' \), hence the result. \( \square \)
REMARK 8.8. In a different context, Smith (1983):

1. Excluded subcase 2.1 under the claim that it is impossible that $\Gamma \vdash OCC'$. 
2. Subsubcase 2.2.3 was deemed to be impossible.

9. Models of $NF(C)$ revisited. An $NF(C)$-structure $\mathfrak{B}$ is said to be a $d'$-model if for every $\Gamma \cup \{\sigma\} \subseteq BN(C)$, $\Gamma \vdash_{\mathfrak{B}} \sigma$ whenever $\Gamma \vdash \sigma$.

An immediate consequence of this definition is:

PROPOSITION 9.1. An $NF(C)$-structure $\mathfrak{B} =< B, A^*, E^*, I^*, O^*, \mu >$ is a $d'$-model iff it is a $d$-model (hence satisfying conditions 1-4 of proposition 2.3) and:

5. $(I^*\mu | A^*\mu) \subseteq I^*\mu \subseteq I^*\mu$.
6. $(I^*\mu | E^*\mu) \cup (O^*\mu | A^*\mu) \cup (A^*\mu | O^*\mu) \subseteq O^*\mu$. □

Along the lines of the proofs of lemma 2.5, theorem 2.6 and theorem 2.10, the following may be proved:

THEOREM 9.2. For every $\Gamma \cup \{\sigma\} \subseteq BN(C)$:

1. $\mathfrak{B}_{\Gamma^{d'}}$ is a $d'$-model (of $\Gamma^{d'}$, hence of $\Gamma$).

2. $d'$-deduction is sound and complete with respect to the class of $d'$-models. That is $\Gamma \vdash \sigma$ iff $\Gamma \vdash_{d'} \sigma$.

3. $\Gamma^{d'} \vdash_{d'} \sigma$ iff $\Gamma_1^{d'} \vdash_{d'} \sigma$ for some finite $\Gamma_1 \subseteq \Gamma$.
   (This is called $d'$-compactness). □

REMARK 9.3. All remarks given in remarks and definitions 2.11 hold with "$d'$" replacing "$d$". All proofs of the original versions essentially go through; the only exception is the first remark, whose modified version may be proved by part 2 of proposition 8.4.

THEOREM 9.4. An $NF(C)$-structure $\mathfrak{B}(=< B, A^*, E^*, I^*, O^*, \mu >)$ is a $g$-model iff it is a $d'$-model and:

40
1. \( A^* \cap O^* = \emptyset = E^* \cap I^* \), or

2. \( A^* = E^* = I^* = O^* = \mu(C) \times \mu(C) \).

**Proof.** Only if: By part 2 of proposition 8.4 and an obvious generalization of part 3 of remarks and definitions 2.11.

If: Every \( NF(C) \)-structure which satisfies condition 2 is a \( g \)-model. So, assume that \( B \) is a \( d' \)-model which satisfies condition 1. To see that it is a \( g \)-model, let \( \Gamma \cup \{ \sigma \} \subseteq BN(C) \), \( \Gamma \vdash^g \sigma \) and \( B \models \Gamma \). By remark 9.3, \( \Gamma \) is consistent, hence by theorem 8.7, \( \Gamma \vdash \sigma \), hence \( B \models \sigma \). \[ \square \]

Theorem 9.4 fully characterizes the class of \( g \)-models, as was promised after the proof of theorem 2.10.

**DEFINITIONS and remarks 9.5.**

1. For an \( NF(C) \)-structure \( B \) and a relation symbol \( W \in \{ A, E, I, O \} \), define \( Bt^W B \) (the basic \( W \)-theory of \( B \)), \( Bt^+ B \) (the basic positive theory of \( B \)), \( Bt^- B \) (the basic negative theory of \( B \)) and \( Bt B \) (the basic theory of \( B \)) as follows:

\[
Bt^W B = \{ Wab \in BN(C) : B \models Wab \}.
\]

\[
Bt^+ B = Bt^A B \cup Bt^I B.
\]

\[
Bt^- B = Bt^E B \cup Bt^O B.
\]

\[
Bt B = Bt^+ B \cup Bt^- B.
\]

So two \( NF(C) \)-structures are \( B \)-equivalent iff they have the same basic theory.

For \( i \in 2 \) let \( B_i (=< B_i, A_i, E_i, I_i, O_i, \mu_i >) \) be an \( NF(C) \)-structure.

2. \( B_o \) is said to be a substructure of \( B_1 \) and \( B_1 \) is said to be a superstructure of \( B_o \) if \( B_o \subseteq B_1, \mu_o = \mu_1 \) and for every \( W \in \{ A, E, I, O \} \),
$W_o = W_1 \cap (B_o \times B_o)$. If, moreover, $B_o = \text{Range } \mu_1 (= \text{Range } \mu_o)$, $\mathfrak{B}_o$ is said to be a core substructure of $\mathfrak{B}_1$. Obviously each $NF(C)$-structure has a unique core substructure, to be called its core substructure. $\mathfrak{B}_o$ is a core substructure of some $NF(C)$-structure iff it is the core substructure of itself iff $B_o = \text{Range } \mu_o$. In this case $\mathfrak{B}_o$ is said to be a core structure. Obviously every canonical structure is a core structure.

$\mathfrak{B}_o$, $\mathfrak{B}_1$ have the same core substructure iff $\mu_o = \mu_1$ and $Bt\mathfrak{B}_o = Bt\mathfrak{B}_1$.

3. If $\mathfrak{B}_o$ is a substructure of $\mathfrak{B}_1$ then they have the same core substructure and the three structures have the same basic theory. Hence for $e \in \{d, d', g\}$ if one of them is an $e$-model, so also are the other two.

In this case $\mathfrak{B}_o$ is said to be an $e$-submodel of $\mathfrak{B}_1$, and $\mathfrak{B}_1$ is said to be an $e$-supermodel of $\mathfrak{B}_o$; and the core substructure is said also to be the core $e$-submodel. If a core structure is an $e$-model, it is said to be a core $e$-model.

4. $\mathfrak{B}_o$ is said to be a positive semisubstructure of $\mathfrak{B}_1$ and $\mathfrak{B}_1$ is said to be a positive semisuperstructure of $\mathfrak{B}_o$ if $B_o \subseteq B_1$, $\mu_o = \mu_1$ and:

$$W_o = W_1 \cap (B_o \times B_o) \quad \text{for every } W \in \{A, I\},$$

$$W_o \subseteq W_1 \cap (B_o \times B_o) \quad \text{for every } W \in \{E, O\}.$$ 

In this case $Bt^+\mathfrak{B}_o = Bt^+\mathfrak{B}_1$ and $Bt^-\mathfrak{B}_o \subseteq Bt^-\mathfrak{B}_1$. For each $e \in \{d, d', g\}$ if, in addition, $\mathfrak{B}_o$ and $\mathfrak{B}_1$ are both $e$-models, it is said also that $\mathfrak{B}_o$ is a positive $e$-semisubmodel of $\mathfrak{B}_1$ and $\mathfrak{B}_1$ is a positive $e$-semisupermodel of $\mathfrak{B}_o$.

**THEOREM 9.6.** For each $e \in \{d, d', g\}$ if $fB_o$ (defined as above) is a consistent core $e$-model then there is an order model $\mathfrak{B}_1(= < B_1, A_1, \mu_1 >)$ such that:

1. $\mathfrak{B}_1$ is a positive $e$-semisupermodel of $\mathfrak{B}_o$.
2. If $A_o$ is a partial ordering, then so also is $A_1$.
3. If $\mathfrak{B}_o$ is complete, then it is the core $e$-submodel of $\mathfrak{B}_1$.

For $e = d$, the above holds after weakening part 1 to become:

1'. $B_o \subseteq B_1, \mu_o = \mu_1, Bt^{A}\mathfrak{B}_o \subseteq Bt^{A}\mathfrak{B}_1$, and $A_o = A_1 \cap (B_o \times B_o)$; hence $Bt^{A}\mathfrak{B}_o = Bt^{A}\mathfrak{B}_1$.

**Proof.** Let $e \in \{d, d', g\}$ and let $\mathfrak{B}_o$ be a consistent core $e$-model. Put:
Let $B' = \{(a_o, a_1) \subseteq B_o : < a_o, a_1 > \in I_o \text{ or } < a_1, a_o > \in I_o, \text{ and } \{a_o, a_1\} \text{ has no } A_o \text{-lower bound}\}$,
\[ B_1 = B_o \cup B' \text{ (} B_o, B' \text{ may be assumed disjoint)} \]
\[ A_1 = A_o \cup I_{B'} \cup \{ < a_o, a_1 > : < a_o, a_2 > \in A_o \text{ or } < a_1, a_2 > \in A_o \} \]
\[ \mu_1 = \mu_o \]

$A_1$ is reflexive on $B_1$ since $A_o$ is reflexive on $B_o$. To prove the transitivity of $A_1$, let $< b_o, b_1 >, < b_1, b_2 > \in A_1$. If $< b_o, b_1 >$ or $< b_1, b_2 >$ belongs to $I_{B'}$, then $< b_o, b_2 > \in A_1$, else $< b_1, b_2 > \in A_o$. If $< b_o, b_1 > \in A_o$ then $< b_o, b_2 > \in A_1$. It remains to consider the case where $b_o = \{a_o, a_1\}$ for some $\{a_o, a_1\} \in B'$ such that $< a_o, b_1 > \in A_o$ or $< a_1, b_1 > \in A_o$, in both cases $< b_o, b_2 > \in A_1$. So $A_1$ is transitive. Hence $< B_1, A_1, \mu_1 >$ is an order model, which is to be denoted by “$\mathfrak{B}_1$.”

To prove part 2 it suffices to notice that if $< b_o, b_1 >, < b_1, b_o > \in A_1$ then they both belong to $A_o$ or both belong to $I_{B'}$.

To prove parts 1, 1’ notice that $B_o \subseteq B_1$ and, by the disjointness of $B_o, B'$, $A_o = A_1 \cap (B_o \times B_o)$. Let $< a_o, a_1 > \in I_o$. If $\{a_o, a_1\}$ has an $A_o$-lower bound then it is an $A_1$-lower bound, else the element $\{a_o, a_1\} \in B_1$ is an $A_1$-lower bound of the subset $\{a_o, a_1\} \subseteq B_1$. In both cases $< a_o, a_1 > \in I_1$, hence $I_o \subseteq I_1 \cap (B_o \times B_o)$.

At this point the proof forks into two branches:

(i) Assume $e \in \{d', g\}$ and let $< a_o, a_1 > \in I_1 \cap (B_o \times B_o)$. To show that $< a_o, a_1 > \in I_o$ several cases have to be considered, following is one of them, the others are similar or easier.

There is $< a_2, a_3 > \in I_o$ such that $< a_2, a_o >, < a_3, a_1 > \in A_o$. Since $B_o = \text{Range } \mu_o$ then, by theorem 9.4 and part 5 of proposition 9.1, $< a_o, a_1 > \in I_o$. So $I_1 \cap (B_o \times B_o) \subseteq I_o$. Hence $I_o = I_1 \cap (B_o \times B_o)$.

That $E_o \subseteq E_1 \cap (B_o \times B_o)$ and $O_o \subseteq O_1 \cap (B_o \times B_o)$ is guaranteed by the consistency of $\mathfrak{B}_o$. This completes the proof of 1.

(ii) The other branch is $e = d$. To show that $E_o \subseteq E_1$ assume that there is $< a_o, a_1 > \in (E_o - E_1)$, then $< a_o, a_1 > \in I_1$, then $\{a_o, a_1\}$ has an $A_1$-lower bound. To show that this is absurd, several cases have to be considered;
following is one of them, the others are easier or similar.

There is \(<a_2, a_3 > \in I_o\) such that \(<a_3, a_o >, <a_2, a_1 > \in A_o\). Since \(B_o = \text{Range } \mu_o\) then, by parts 3, 4 of proposition 2.3, \(<a_2, a_3 > \in E_o\) which contradicts the consistency of \(B_o\).

That \(O_o \subseteq O_1\) is guaranteed by the consistency of \(B_o\), since \(A_o = A_1 \cap (B_o \times B_o)\). This completes the proof of 1’ and ends the forkation.

For \(e \in \{d, d', g\}\), if \(B_o\) is complete then “\(\subseteq\)” may be replaced by “\(=\)” at the appropriate places, which proves part 3. \(\square\)

Taking the relationship between the \(e\)-models \((e \in \{d, d', g\}\)) and their respective core \(e\)-submodels into consideration, a weaker result, which holds for a wider class of \(e\)-models, immediately follows:

**COROLLARY 9.7.** For \(e \in \{d, d', g\}\), if \(B\) is an \(e\)-model whose core \(e\)-submodel is consistent, then there is an order model \(B'\) such that
\(BtB \subseteq BtB'\). Moreover,
\[Bt^+B = Bt^+B' \quad \text{if } e \in \{d', g\},\]
\[Bt^AB = Bt^AB' \quad \text{if } e = d.\] \(\square\)

In view of the last part of subsection 2.5, the above corollary may be immediately strengthened as follows:

**COROLLARY 9.8.** In the above corollary “an order model” may be replaced by “a partial order model which is a c.o.m and a Venn model at the same time”. \(\square\)

Part 4 of theorem 1.12 may be extended to the case \(e = d'\), to get a result similar to that obtained there for the case \(e = g\); the result obtained (there) for the case \(e = d\) is weaker. Call the collection of these three results “syntactical congruence”.

Syntactical congruence together with the definitions of core \(e\)-models \((e \in \{d, d', g\}\)) yield semantical congruence as formulated by parts 1 and 2 of the next theorem. Part 3 of the same theorem (whose proof is straightforward) strengthens the conclusion of part 2, under some additional condition.
Alternatively, semantical congruence may be directly proved by the characterizations of $e$-models ($e \in \{d, d', g\}$) given in propositions 2.3 and 9.1 and theorem 9.4.

**THEOREM 9.9.** Let $e \in \{d, d', g\}$ and let $\mathcal{B} = < B, A^*, E^*, I^*, O^*, \mu >$ be a core $e$-model (consistent or not). Put $\sim = A^* \cap \check{A}^*$, then:

1. $\sim$ is a congruence relation on $< B, A^*, E^*, \mu >$ and $A^*/\sim$ is a partial ordering on $B/\sim$.

Moreover, for $e \in \{d', g\}$:

2. $\sim$ is a congruence relation on $\mathcal{B}$. The mapping $b \mapsto b/\sim$ is an epimorphism from $\mathcal{B}$ onto $\mathcal{B}/\sim$. Hence $\mathcal{B}/\sim$ is a core $e$-model which is basically equivalent to $\mathcal{B}$.

3. If $\mathcal{B}$ is, in addition, an order model, then $\mathcal{B}/\sim$ is also a partial order model. □

For $e \in \{d', g\}$, semantical congruence makes it possible to replace “$\mathcal{B}_o$” in theorem 9.6 by “$\mathcal{B}_o/\sim$”. This provides, for $e \in \{d', g\}$, an alternative proof of a weaker form of corollary 9.8, where the partial order model may be neither concrete nor Venn.

The corresponding weaker result for the case $e = d$ may likewise be obtained, but the alternative proof is a bit more involved.

**REMARKS and definitions 9.10.**

1. Theorem 9.6 (or corollary 9.7) and corollary 9.8 (or its weaker forms) provide, respectively, direct ways to order models and partial order models for consistent $\Gamma(\subseteq BN(C))$. Simply in each of them let the core $e$-model be the canonical structure $\mathcal{B}_{\Gamma e}(e \in \{d, d', g\})$. In the case of corollary 9.8 the partial order model may be required to be a concrete order model and a Venn model at the same time.

2. Let $e \in \{d, d', g\}$ and let $\mathcal{C}$ be a class of $NF(C)$-structures, then:

   1. $e$ is said to be $\mathcal{C}$-strongly semantically complete if for every $\Gamma \subseteq BN(C)$ there is $\mathcal{B} \in \mathcal{C}$ such that $Bt\mathcal{B} = \Gamma^e$.

   2. $e$ is said to be $\mathcal{C}$-syntactically complete if for every $\Gamma \cup \{\sigma\} \subseteq BN(C)$, $\Gamma \vdash e \sigma$ whenever $\Gamma \vdash e \sigma$. 45
3. $e$ is said to be $\mathfrak{C}$-consistently syntactically complete if for every $e$-consistent $\Gamma \subseteq BN(C)$ and every $\sigma \in BN(C)$, $\Gamma \vdash e \sigma$ whenever $\Gamma \vDash e \sigma$.

4. $e$ is said to be $\mathfrak{C}$-consistently semantically complete if every $e$-consistent $\Gamma \subseteq BN(C)$ has a model in $\mathfrak{C}$.

For $i \in \{1, 2, 3\}$, the condition given in clause $i$ implies the condition given in clause $i + 1$.

3. Put:

- $Or$ = the class of all order models,
- $Po$ = the class of all partial order models,
- $Le$ = the set of all Leibniz models,
- $Co$ = the class of all concrete order models,
- $Ve$ = the class of all Venn models.

And for $e \in \{d, d', g\}$ put:

- $Be = \{\mathfrak{B}_T^e : \Gamma \subseteq BN(C)\}$.

Also put:

- $M = \{Or, Po, Le, Co, Ve\} \cup \{Be : e \in \{d, d', g\}\}$.

4. $Le \cup Co \subseteq Po \subseteq Or$, $Bg \subseteq Bd' \subseteq Bd$.

5. Every element of $\bigcup M$ is a $d$-model.
- Every element of $(\bigcup M - Bd) \cup Bd'$ is a $d'$-model.
- Every element of $(\bigcup M - Bd) \cup Bg$ is a $g$-model.

6. For $e \in \{d, d', g\}$, $e$ is $Be$-strongly semantically complete.
- For $\mathfrak{C} \in M - \{Le\}$, $d$ (respectively $d'$, $g$) is $\mathfrak{C}$-consistently semantically (respectively consistently syntactically, syntactically) complete. If $C$ is finite, the exclusion of $Le$ may be dropped.

7. For $e \in \{d, d', g\}$ and $\Gamma \subseteq BN(C)$, $\Gamma$ is said to be $e$-syntactically complete if for every $\sigma \in BN(C)$, $\Gamma \vdash e \sigma$ or $\Gamma \vdash \hat{e} \sigma$.

8. For $e \in \{d, d', g\}$ and $\mathfrak{C} \in M - \{Le\}$, if $\Gamma$ is consistent and $e$-syntactically complete then there is $\mathfrak{B} \in \mathfrak{C}$ such that $Bt\mathfrak{B} = \Gamma^e$. If, moreover, $C_{\Gamma}$ is finite, the exclusion of $Le$ may be dropped.
10. Decidability revisited.

THEOREM 10.1. For each $e \in \{d, d', g\}$ there is a polynomial (of degree at most 8) time algorithm to decide for any $< \Gamma, \sigma > \in \varphi(BN(C)) \times BN(C)$ whether $\Gamma^e \vdash \sigma$, provided that $\Gamma$ is essentially finite and:

$$\Gamma \cap (\{Ecc : c \in (C - C_T)\} \cup \{Occ : c \in (C - C_T)\}) = \phi.$$  

Proof. For $e = d$ a proof may be obtained by slightly modifying the appropriate parts of the proof of theorem 3.2.

In view of lemma 8.3, a proof for the case $e = d'$ may be obtained along the same lines as above.

In view of remarks 3.1, the first part of this theorem may be made use of to determine whether $\Gamma$ is inconsistent. If yes, $\Gamma^g \vdash \sigma$; else $\Gamma^g \vdash \sigma$ iff $\Gamma^{d'} \vdash \sigma$, by theorem 8.7. 

\[\square\]

11. Sorites. Soriteses are well known in Aristotelian syllogistic (see Hurley, P. J. 1982, p. 201; Rosenthal, M. and Yudin, R (eds.) 1967, p. 423; also cf. Boger, G. 1998, pp. 197-8; Smiley, T.J. 1973, pp. 139-40).

The notion of a sorites may be explicated as follows.

DEFINITION 11.1. Let $e \in \{d, d'\}$ and let $\Gamma \subseteq BN(C)$. An annotation of an $e$-deduction $< \sigma_i >_{i \in k}$ from $\Gamma$ is said to be an $e$-sorites annotation if the following conditions are satisfied:

1. $\sigma_i \neq \sigma_j$ whenever $i \neq j$ ($i, j \in k$).

2. For $i \in k - 1$, $\sigma_i$ is involved in the annotation of another sentence in the following and only in the following way.

2.1. If $1 \leq i \leq k - 3$ then exactly one of the following holds:

2.1.1. $\sigma_{i+1}$ is annotated as the consequent of $\sigma_i$ by some $e$-rule with one premise.

2.1.2. $\sigma_{i+1}$ is annotated as the consequent of $\sigma_{i-1}, \sigma_i$ or $\sigma_i, \sigma_{i-1}$ by some $e$-rule with two premises.

2.1.3. $\sigma_{i+2}$ is annotated as the consequent of $\sigma_i, \sigma_{i+1}$ or $\sigma_{i+1}, \sigma_i$ by some $e$-rule with two premises.

2.2. If $1 \leq i = k - 2$ then exactly one of 2.1.1 and 2.1.2 holds.

2.3. If $i = 0$ then exactly one of the following holds:

2.3.1. $k = 2$ and 2.1.1 holds.
2.3.2. \( k > 2 \) and exactly one of 2.1.1 and 2.1.3 holds.

An \( e \)-sorites from \( \Gamma \) is an \( e \)-deduction from \( \Gamma \) which admits a sorites annotation. An \( e \)-sorites of \( \sigma (\in BN(C)) \) from \( \Gamma \) is an \( e \)-deduction of \( \sigma \) from \( \Gamma \) which is an \( e \)-sorites. In case there is such a sorites, we write “\( \Gamma \vdash e \sigma \)”.

Condition 2 of the above definition entails that, with the exception of the last sentence, every sentence occurring in an \( e \)-sorites from \( \Gamma \) is made use of exactly once as a premise of some application of some \( e \)-rule, and in this (hence in each) application the premise or the premises immediately precede the consequent.

For \( e \in \{d, d'\} \) there is, obviously, a set \( \Gamma \subseteq BN(C) \) and an \( e \)-deduction from \( \Gamma \) which is not an \( e \)-sorites from \( \Gamma \). So the best we may hope for is to find an \( e \)-sorites of \( \sigma \) from \( \Gamma \), for every \( \Gamma \cup \{\sigma\} \subseteq BN(C) \) such that \( \Gamma \vdash e \sigma \). Even this is not always attainable.

Let \( \Gamma_o = \{Aca, Ebc\} \) and \( \Gamma_1 = \{Ax, Ebx, Ica\} \), then for \( i \in 2 \), \( \Gamma_i \) is consistent and \( \Gamma_i \vdash Oab \), but not \( \Gamma_i \vdash Oab \). For \( i = 0 \), adding the rule \( \frac{Eab}{Oab} \) (\( E \)-sub) as an additional rule of inference will solve the problem. Same holds for \( i = 1 \) if, instead, \( \frac{Ibc, Ebc}{Oac} \) (Ferison) is added.

**11.1. Further extension of direct deduction.** Taking into consideration the following two rules of inference.

10. \( E \)-sub

11. Ferison

in addition to the ten rules of inference of \( d' \), the \( d'' \)-deduction relation “\( \vdash d'' \)” may be defined along the lines of the definition of “\( \vdash d' \)”. Likewise all the other definitions involving “\( d' \)” may be modified in an obvious way to give corresponding definitions involving “\( d'' \)”.

**PROPOSITION 11.2.** For \( \Gamma \cup \{\sigma\} \subseteq BN(C) \),

\[
\Gamma \vdash d' \sigma \quad \text{iff} \quad \Gamma \vdash d'' \sigma.
\]

**Proof.** One direction is obvious, the other is easy. \( \square \)

**DEFINITION/remark 11.3.** The \( d'' \)-models may be defined along the lines of the definition of the \( d' \)-models.
THEOREM 11.4. Let $\Gamma \cup \{\sigma\} \subseteq BN(C)$ and $e \in \{d, d', d''\}$ then:

\[
\Gamma^{es} \vdash \sigma \quad \text{whenever} \quad \Gamma^e \vdash \sigma
\]

provided one of the following conditions holds:

1. $\sigma$ is affirmative,
2. $\sigma$ is universal negative and $\Gamma$ is consistent,
3. $\sigma$ is particular negative, $e = d$ or $\Gamma$ is consistent and $e = d''$.

The other direction unconditionally holds, so the two sides are equivalent if $\Gamma$ is consistent and $e \in \{d, d''\}$.

Proof. Assume $\Gamma^e \vdash \sigma$. Distinguish between the following cases.

1. $\sigma = Aab$, for some $a, b \in C$. By proposition 11.2 and lemma 8.3 there is an $a$-$b$ chain in $\Gamma$, $< c_i >_{i \in n}$ say. We may assume that this chain is injective. If $n = 1$, then there is an $e$-sorites of $\sigma$ from $\Gamma$ of length 1. Else $n \geq 2$; define $< \rho_i >_{i \in 2n-3}$ as follows:

\[
\rho_{2j} = Ac_{c_j+1} = Aac_{j+1} \quad j \in n - 1,
\]

\[
\rho_{2j+1} = Ac_{j+1}c_{j+2} \quad j \in n - 2.
\]

Then $< \rho_i >_{i \in 2n-3}$ is an $e$-sorites of $\sigma$ from $\Gamma$.

2. $\sigma = Iab$, for some $a, b \in C$. If $e = d$, the result is an easy consequence of part 1 of this proof and lemma 8.3. Else, by proposition 11.2 we may assume that $e = d'$. By lemma 8.3 it suffices to deal with the following three subcases (for some $a', b' \in C$):

2.1. $Ia'b' \in \Gamma$ and $\Gamma^d \vdash Aa'a, Ab'b$,

2.2. $Ib'a' \in \Gamma$ and $\Gamma^d \vdash Aa'a, Ab'b$,

2.3. $\Gamma^d \vdash Aa'a, Aa'b$.

Assume 2.1 (the other two subcases are not harder), then there are an $a'$-$a$ chain and a $b'$-$b$ chain in $\Gamma$, let them be, respectively $< c_i >_{i \in k}$ and $< c'_j >_{j \in l}$. We may assume that the ranges of these two chains are disjoint, otherwise this subcase will be reduced to subcase 2.3. Also we may assume that each of these two chains is injective. The following is a $d'$ (hence a $d''$)-sorites of $\sigma$ from $\Gamma$: $Ab'c'_1, Ac'_1c'_2, Ab'c'_2, ..., Ab'c'_{l-2}, Ac'_{l-2}b, Ab'b, Ia'b', Ia'b, Iba', Aa'c_1, Ibc_1, ...,$
3. \( \sigma = Eab \), for some \( a, b \in C \) and \( \Gamma \) is consistent. By proposition 11.2 and lemma 8.3 there are \( a', b' \in C \) such that \( \{Ea'b', Eb'a'\} \cap \Gamma \neq \emptyset \) and there are an \( a-a' \) chain and a \( b-b' \) chain in \( \Gamma \), let them be, respectively, \( < c_i \>_{i \in k} \) and \( < \ell_j \>_{j \in \ell} \). By the consistency of \( \Gamma \), the ranges of the two chains are disjoint. Moreover, we may assume that each of them is injective. Let \( Ea'b' \in \Gamma \) (the other case is not harder), then the following is an \( e \)-sorites of \( \sigma \) from \( \Gamma \): \( Ea'b', Ac_{k-2}a', Ec_{k-2}b', \ldots, Ec_1b', Aac_1, Eab', Eb'a, Ac_1b', Ec_1a, \ldots, Ec_1a, Abc_1, Eba, Eab \).

By lemma 8.3 it suffices to deal with the following two subcases.

4. \( \sigma = Oab \), for some \( a, b \in C \). If \( e = d \), then there is a one line \( e \)-sorites of \( \sigma \) from \( \Gamma \). Else assume that \( \Gamma \) is consistent and \( e = d'' \), by proposition 11.2 and lemma 8.3 it suffices to deal with the following two subcases.

4.1. There are \( a', b' \in C \) such that \( Oa'b' \in \Gamma \) and \( \Gamma \vdash Aa'a, Abb' \). Making use of Bocardo and Baroco it may be shown, along the lines of part 3 of this proof, that there is a \( d' \) (hence a \( d'' \))-sorites of \( \sigma \) from \( \Gamma \).

4.2. There is \( c \in C \) such that \( \Gamma \vdash Ica, Ecb \). As in part 3 of this proof, there are \( c', b' \in C \) such that:

\[
\{Ec'b', Eb'c'\} \cap \Gamma \neq \emptyset \quad \text{and} \quad \Gamma \vdash Acc', Abb'
\]

By lemma 8.3 it suffices to deal with the following two subsubcases.

4.2.1. For some \( c'' \in C, \Gamma \vdash Aec''c, Ac''a \). By this and \( (*) \), \( \Gamma \vdash Abb', Ac''c', Ac''a \). So there are \( b', c''-c' \) and \( c''-a \) injective \( \Gamma \)-chains; let them be \( < x_i >_{i \in k} \), \( < y_i >_{i \in l} \) and \( < z_i >_{i \in m} \) respectively.

By the consistency of \( \Gamma \), the range of \( < x_i >_{i \in k} \) and the union of the ranges of \( < y_i >_{i \in l} \) and \( < z_i >_{i \in m} \) are disjoint. Assume that the ranges of \( < y_i >_{i \in l} \) and \( < z_i >_{i \in m} \) have \( c'' \) only in common (the other case is similar).

If \( Ec'b' \in \Gamma \), then there is a \( d' \) (hence a \( d'' \))-sorites of \( \sigma \) from \( \Gamma \). Else \( Eb'c' \in \Gamma \) and the following is a \( d'' \)-sorites of \( \sigma \) from \( \Gamma \): \( Abx_1, Ax_1x_2, Abx_2, \ldots, Abx_{k-2}, Ax_{k-2}b', Abb', Eb'c', Ebc', Ec'b, Ay_{-2}c', Ey_{-2}b, \ldots, Ey_i b, Ac''y_1, Ec''b, Oc''b \) (here \( E \)-sub is made use of), \( Ac''z_1, Oz_1b, Az_1z_2, Oz_2b, \ldots, Oz_m-2b, Az_m-2a, Oab \).

4.2.2. \( \{Ic''a', Ia'c''\} \cap \Gamma \neq \emptyset \) and \( \Gamma \vdash Aec''c, Aa'a \), for some \( c'', a' \in C \). By
this and (*), $\Gamma \vdash \text{Abb}'$, $\text{Ac}''c'$, $\text{Aa}'a$. So there are $b-b', c''-c'$ and $a'-a$ injective $\Gamma$-chains; let them be $< x_i >_{i \in k}$, $< y_i >_{i \in l}$ and $< z_i >_{i \in m}$ respectively.

By the consistency of $\Gamma$, the range of $< x_i >_{i \in k}$ and the union of the ranges of $< y_i >_{i \in l}$ and $< z_i >_{i \in m}$ are disjoint. If the ranges of $< y_i >_{i \in l}$ and $< z_i >_{i \in m}$ are not disjoint, this case will be reduced to the above case; so assume that they are disjoint.

If $Ia'c'' \in \Gamma$ then there is a $d'$ (hence a $d''$)-sorites of $\sigma$ from $\Gamma$. Else $Ic''a' \in \Gamma$, assume $E\text{b}'c' \in \Gamma$ (the other case is similar), then the following is a $d''$-sorites of $\sigma$ from $\Gamma$. $\text{Ab}x_1, \text{Ax}_1x_2, \text{Ab}x_2, ..., \text{Abb}', E\text{b}'c', E\text{bc}', E\text{cb}, A\text{y}_{l-2}c', E\text{y}_{l-2}b, ..., E\text{c}''b, Ic''a', Oa'b$ (here Ferison is made use of), $\text{Ab}'z_1, O\text{z}_1b, ..., O\text{ab}$. □

PROPOSITION 11.5. If $\Gamma(\subseteq BN(C))$ is inconsistent then it is ds-inconsistent, in the sense that there is $\sigma \in BN(C)$ such that $\Gamma \vdash \sigma, \tilde{\sigma}$.

Proof. Let $\Gamma$ be inconsistent, then there is a universal $\rho \in BN(C)$ such that $\Gamma \vdash_d \rho, \tilde{\rho}$. Distinguish between two cases:

1. $\rho = Aab$, for some $a, b \in C$. In this case the result is a direct consequence of theorem 11.4.

2. $\rho = Eab$, for some $a, b \in C$. As in part 3 of the proof of theorem 11.4, there are $a', b' \in C$ such that $\{Ea'b', E\text{b}'a'\} \cap \Gamma \neq \phi$ and there are injective $a'a', b'b'$ chains in $\Gamma$; let them be, respectively, $< c_i >_{i \in k}$ and $< c'_j >_{j \in l}$.

If the ranges of these chains are disjoint, the result follows by theorem 11.4 and the methods made use of in its proof. Else there is $c'' \in \{c_i : i \in k\} \cap \{c'_j : j \in l\}$. Then there are injective $c''-a', c''-b'$ chains in $\Gamma$. Along the lines of the proof of theorem 11.4 it may be shown that $\Gamma \vdash E\text{a}'c'', I\text{a}'c''$ (and $\Gamma \vdash E\text{b}'c'', I\text{b}'c''$). □

To show that the consistency condition in each of the parts 2,3 of theorem 11.4 cannot be completely dispensed with, we prove:

PROPOSITION 11.6. Let $\Gamma \subseteq BN(C), c \in \{d, d', d''\}$ and $a, a', b, b', c, c' \in C$; and assume that $c \neq c'$.

1. If every $a-a'$ chain in $\Gamma$ is a $< c, c' >$ chain, then $\text{Acc}'$ occurs as an assumption in every $e$-deduction of $A\text{a}a'$ from $\Gamma$; moreover it is made use of as a premise in the deduction if it is different from $A\text{a}a'$. 51
In parts 2 and 3 below, \( Ebb' \) is assumed to be the only universal negative sentence in \( \Gamma \).

2. If every \( a-b \) chain and every \( a-b' \) chain in \( \Gamma \) is a \( <c,c'> \) chain, then \( Acc' \) occurs as an assumption and is made use of as a premise in every \( e \)-deduction of \( Eaa' \) and every \( e \)-deduction of \( Ea'a \) from \( \Gamma \).

3. If every \( a-b \) chain, every \( a-b' \) chain, every \( a'-b \) chain and every \( a'-b' \) chain in \( \Gamma \) is a \( <c,c'> \) chain, then for every injective \( e \)-deduction \( <\sigma_i>_{i\in k} \) of \( Eaa' \) from \( \Gamma \) there is \( j \in k \) such that \( \sigma_j \) is made use of as a premise at least twice.

In parts 4 and 5 below, \( Obb' \) is assumed to be the only negative sentence in \( \Gamma \).

4. If every \( b-a \) chain or every \( a'-b' \) chain in \( \Gamma \) is a \( <c,c'> \) chain, then \( Acc' \) occurs as an assumption and is made use of as a premise in every \( e \)-deduction of \( Oaa' \) from \( \Gamma \).

5. If every \( b-a \) chain and every \( a'-b' \) chain in \( \Gamma \) is a \( <c,c'> \) chain, then for every injective \( e \)-deduction \( <\sigma_i>_{i\in k} \) of \( Oaa' \) from \( \Gamma \) there is \( j \in k \) such that \( \sigma_j \) is made use of as a premise at least twice.

Proof. Generalize the first part to become:

For every \( u, u' \in C \), if every \( u-u' \) chain in \( \Gamma \) is a \( <c,c'> \) chain, then \( Acc' \) occurs as an assumption in every \( e \)-deduction of \( Auu' \) from \( \Gamma \); moreover, it is made use of as a premise in the deduction if it is different from \( Auu' \).

The stronger statement may be easily proved by course of values induction on the length of the \( e \)-deduction.

Parts 2 and 4 may be proved similarly.

Again generalize part 3 to become:

For every \( u, u' \in C \) if every \( u-b \) chain, every \( u-b' \) chain, every \( u'-b \) chain and every \( u'-b' \) chain in \( \Gamma \) is a \( <c,c'> \) chain, then for every injective \( e \)-deduction \( <\sigma_i>_{i\in k} \) of \( Euu' \) from \( \Gamma \) there is \( j \in k \) such that \( \sigma_j \) is made use of as a premise at least twice.

The stronger statement may be proved by course of values induction on
as follows. Assume the required for \( r < k \) and let \( \sigma_i >_{i \in k} \) be an e-deduction of \( E_{uu'} \) from \( \Gamma \). Since \( \{b, b'\}, \{u, u'\} \) are disjoint and \( \sigma_{k-1} = E_{uu'} \), then there are only two cases to consider:

1. For some \( l < k - 1 \), \( \sigma_l = Eu'u \), in this case the result is immediate by the induction hypothesis.

2. For some \( l, m < k - 1 \) and some \( \nu \in C \) it is the case that \( l < m \), \( \{\sigma_l, \sigma_m\} = \{A_{uv}, E_{vu'}\} \) and \( \sigma_{k-1} \) is obtained from them as the conclusion of applying the rule \( \frac{A_{uv}, E_{vu'}}{Eu'u} \).

If \( \sigma_l \) is made use of as a premise in a step whose conclusion is \( \sigma_j \) for some \( j < k - 1 \), the result is immediate. Also if there is some \( u-v \) chain in \( \Gamma \) which is not a \( <c, c'\) chain, the result follows by the induction hypothesis.

So it remains to assume that every \( u-v \) chain in \( \Gamma \) is a \( <c, c'\) chain and for every \( j < k - 1 \), \( \sigma_i \) is not made use of as a premise in the step which gives rise to \( \sigma_j \). Put:

\[
\Delta_o = \{\sigma_i\},
\]
\[
\Delta_{j+1} = \Delta_j \cup \{\sigma_i : i \in l \text{ and } \sigma_i \text{ is made use of as a premise in a step whose conclusion is in } \Delta_j\}.
\]

Then for some \( n \), \( \Delta_{n+1} = \Delta_n \). Hence \( \sigma_i >_{i \in l+1, \sigma_i \in \Delta_n} \) is an e-deduction of \( \sigma_i \) from \( \Gamma \), so by parts 1,2 above \( Acc' \in \Delta_n \).

Assume, towards a contradiction, that for every \( i \in k \), \( \sigma_i \) is made use of as a premise at most once. Then \( \sigma_i >_{i \in m+1, \sigma_i \notin \Delta_n} \) is an e-deduction of \( \sigma_m \) from \( \Gamma - \Delta_n \). Again by parts 1,2 above, \( Acc' \notin \Delta_n \). Hence the result.

Part 5 may be proved similarly. \(\square\)

EXAMPLES 11.7. To see that the consistency condition in each of the parts 2,3 of theorem 11.4 cannot be completely dispensed with, put:

\[
\Gamma_o = \{Aac, Aa'c, Acc', Ac'b, Ac'b', Ebb'\} \quad , \quad \sigma_o = Eaa'
\]
\[
\Gamma_1 = \{Aa'c, Abc, Acc', Ac'a, Ac'b', Obb'\} \quad , \quad \sigma_1 = Oaa' .
\]

For \( i \in 2 \), \( \Gamma_i \vdash \sigma_i \) (in fact \( \Gamma_o \vdash \sigma_o \)), but by the above proposition \( \Gamma_i \nvdash \sigma_i \).

This example still works even if \( d'' \) is augmented by all of the Aristotelian syllogisms.

To see that consistency is not always necessary, just notice that whether \( \Gamma \) is consistent or not, \( \Gamma \vdash \sigma \) whenever \( \sigma \in \Gamma \).
Following are basic properties of sorites.

**DEFINITIONS** and remarks 11.8. Let \( e \in \{d, d', d''\} \), \( \Gamma \subseteq BN(C) \) and \( k \in \mathbb{N}^+ \), and let \( \langle \sigma_i \rangle_{i \in \mathbb{N}} \) be an \( e \)-sorites of \( \sigma_{k-1} \) from \( \Gamma \) according to some annotation.

1. Two annotations of an \( e \)-deduction from \( \Gamma \) are said to be essentially the same if the only difference between them is interchanging “assumption” (i.e. the corresponding sentence belongs to \( \Gamma \)) and “A-Id” in some places. An \( e \)-deduction from \( \Gamma \) is said to have essentially one, or unique, annotation (of some sort) if all of its annotations (of this sort) are essentially the same.

2. For \( 1 \leq \ell \leq k \), \( \langle \sigma_i \rangle_{i \in \ell} \) is an \( e \)-sorites from \( \Gamma \) according to the restriction of the given annotation iff \( \ell = 1 \) or the annotation of \( \sigma_{\ell-1} \) is neither “A-Id” nor “assumption”.

3. Let \( 1 \leq \ell < k \), then for at least one \( j \in \{\ell, \ell+1\} \), \( \langle \sigma_i \rangle_{i \in j} \) is an \( e \)-sorites from \( \Gamma \) according to the restriction of the given annotation.

4. For \( e \in \{d, d'\} \), every \( e \)-sorites from \( \Gamma \) has essentially one sorites annotation. This does not apply to \( d'' \), for the \( d'' \)-deduction \( < Ixx, Exy, Oxy > \) has two \( d'' \)-sorites annotations which are not essentially the same. Only one of them is a \( d' \)-sorites annotation.

5. Ferison and \( E \)-sub are the only \( d'' \)-rules which are not \( d' \)-rules. In every \( d'' \)-sorites annotation at most one of them is made use of, at most once.

6. In each \( d'' \)-sorites at most one triple of the form \( < Ixx, Exy, Oxy > \) or \( < Exy, Ixx, Oxy > \) occurs, at most once.

If no such triple occurs, the sorities will have an essentially unique \( d'' \)-sorites annotation. Else all of its \( d'' \)-sorites annotations are essentially the same, with the only exception that an occurrence of “\( Oxy \)” may be annotated as the consequence of the preceding two sentences by Ferio (which is a \( d' \)-rule) in some of them and by Ferison (which is not) in the others.

**12. Independence.**

**DEFINITIONS** 12.1. Let \( e \) be a deduction system and let \( r \) be a rule of \( e \). The deduction system obtained from \( e \) by excluding \( r \) will be denoted by \( "e_r" \).

1. \( r \) is said to be derivable in \( e \) if \( \Gamma \vdash \sigma \) whenever \( \Gamma \) is a set of antecedents of an instance of \( r \), and \( \sigma \) is the corresponding conclusion. Otherwise \( r \) is said to be independent in \( e \).
2. $e$ is said to be independent if each of its rules is independent in it.
3. $r$ is said to be weakly independent in $e$ if for some set $\Delta \cup \{\rho\}$ of sentences, $\Delta \vdash \rho$ while $\Delta \not\vdash \rho$.
4. $e$ is said to be weakly independent if each of its rules is weakly independent in it.

REMARKS 12.2.

1. Independence implies weak independence.
2. For $e \in \{d, d', d''\}$, each rule $r$ of $e$ is independent in $e$ iff it is weakly independent in $e$, hence $e$ is independent iff it is weakly independent.
3. Each of $d, d'$ is independent (cf. Glashoff (2005) where similar results are obtained via brute force computation).
4. The independence of each of $ds$ and $d's$ is an immediate consequence of the independence of each of $d$ and $d'$ respectively.

THEOREM 12.3.

1. $E$-sub, Ferio and Ferison are derivable in $d'' s$, hence in $d''$. Each of the other rules of $d''$ is independent in $d''$, hence in $d'' s$.
2. $d''$ is not independent, hence not weakly independent.
3. $d'' s$ is weakly independent; however, it is not independent.

Proof.
1. The sequence Aaa, Iaa, Eab, Oab shows that $E$-sub is derivable in $d'' s$. The corresponding proofs for Ferio and Ferison are not harder.

Put $r = Bocardo$ and let $a, b, c$ be three pairwise distinct elements of $C$.
Put $\Gamma = \{Aba, Obc\}$ and $\sigma = Oac$. It is easy to see that if $\Gamma \vdash \rho$ then $\rho \in \{Aba, Obc, Iab, Iba\} \cup \{Axx : x \in C\} \cup \{Ixx : x \in C\}$. From this the independence of $r$ in $d''$ follows. Similarly the other required results may be obtained.

2. By part 1 above and part 2 of remarks 12.2.

3. Part 1 above shows that $d'' s$ is not independent. It shows also that to prove the weak independence of $d'' s$ it suffices to deal with $E$-sub, Ferio and
Ferson only.

Put \( r = E \)-sub and let \( \Delta = \{ Eab \} \) and \( \rho = Oba \), for some distinct \( a, b \in C \). \( \Delta \vdash Oba \). To see that \( \Delta \nvdash Oba \) notice that the only \( d''s_r \) rules which yield an \( O \)-sentence are Ferio, Baroco, Bocardo, and Ferison. To obtain \( Oba \) by applying Baroco or Bocardo the \( O \)-sentence occurring as one of the antecedents -in the present case- will be the same as the conclusion, which is forbidden in sorites. To apply Ferio or Ferison, the antecedents -in the present case- must be \( Ibb \) and \( Eba \). But if \( \eta_i, \eta_{i+1}, \eta_{i+2} \) is a subsequence of a \( d''s_r \) sorites deduction from \( \Delta \) and the annotation of \( \eta_{i+2} \) is that it is obtained from \( \eta_i, \eta_{i+1} \) or \( \eta_i, \eta_{i+1}, \eta_i \) by some rule, then \( \eta_{i+1} \in \Delta \cup \{ Acc : c \in C \} \). So neither Ferio nor Ferison is applicable, hence \( \Delta \nvdash Oba \).

Next, put \( r = \) Ferio (Ferison) and let \( \Delta = \{ Eab, Icb \} \) (\( \{ Eab, Ibc \} \)) and \( \rho = Oca \) for some pairwise distinct \( a, b, c \in C \). By a slight modification of the above technique it may be shown that \( \Delta \vdash \rho \), but \( \Delta \nvdash \rho \). \( \square \)

12.1. Independence of \( g \) and variations thereof. \( g \) and \( d \) have the same deduction rules, but the notion of \( g \)-deduction is weaker than that of \( d \)-deduction. By definition 1.9, for \( \Gamma \cup \{ \sigma \} \subseteq BN(C) \), \( \Gamma \vdash g \sigma \) iff \( \Gamma \cup \{ \hat{\sigma} \} \) is \( d \)-inconsistent. Likewise for each rule \( r \) of \( g \) (equivalently of \( d \)) define the \( g_r \)-deduction relation “\( g_r \)” by: \( \Gamma \vdash g_r \sigma \) iff \( \Gamma \cup \{ \hat{\sigma} \} \) is \( d_r \)-inconsistent. From this and remarks 12.2 it easily follows that \( r \) is independent in \( g \) iff it is weakly independent in \( g \), hence \( g \) is independent iff it is weakly independent.

THEOREM 12.4. \( g \) is independent.

Proof. Let \( a, b, c \) be pairwise distinct elements of \( C \), and put \( r = Barbara \) and \( \Gamma = \{ Aab, ABC \} \). \( \Gamma \vdash \neg Aac \) iff \( \Gamma \cup \{ Oac \} \) is \( d_r \)-consistent. But the set of all \( d_r \)-consequences of \( \Gamma \cup \{ Oac \} \) is \( \{ Aab, ABC, Oac, Iba, Icb \} \cup \{ Axx : x \in C \} \cup \{ Ixx : x \in C \} \), hence \( \Gamma \cup \{ Oac \} \) is \( d_r \)-consistent. Consequently Barbara is independent in \( g \).

The proofs of the independence of the other rules are similar or easier. \( \square \)

To get closer to the usual deduction systems, we introduce two new deduction systems \( g' \), \( g'' \) and show that each of them is equivalent to \( g \) and discuss its independence.
12.1.1. First variation on $g$. The deduction system $g'$ is obtained by augmenting the system $d'$ by the rule: $\frac{\delta \sigma}{\sigma}$ (contradiction, Co for short).

Let $\Gamma \cup \{\sigma\} \subseteq BN(C)$. It is easy to see that if $\Gamma \vdash g' \sigma$ then there is a $g'$-deduction of $\sigma$ from $\Gamma$ in which Co is never made use of or it is made use of only at the last step; moreover, this applies to $g'_r$ for each rule $r$ of $d'$.

THEOREM 12.5. The following are equivalent:
1. $\Gamma \vdash g \sigma$,
2. $\Gamma \vdash g' \sigma$,
3. $\Gamma \vdash d' \sigma$ or $\Gamma$ is inconsistent,
4. $\Gamma \cup \{\bar{\sigma}\}$ is inconsistent.

Proof. Easy if $\Gamma$ is inconsistent; and in all cases parts 1 and 4 are equivalent by definition 1.9.

Assume $\Gamma$ is consistent. By theorem 8.7, parts 1 and 3 are equivalent, and by the definition of $g'$, part 3 implies part 2. Finally assume part 2, then there is a $g'$-deduction of $\sigma$ from $\Gamma$ in which Co is never made use of, this implies part 3. 

The following theorem settles the independence of $g'$.

THEOREM 12.6.

1. Every rule of $g'$ is independent in $g'$ iff it is weakly independent in $g'$, hence $g'$ is independent iff it is weakly independent.
2. $\Gamma \vdash g_{\text{co}} \sigma$ iff $\Gamma \vdash d' \sigma$.
3. For every rule $r$ of $d'$, $\Gamma \vdash g_{r} \sigma$ iff $\Gamma \vdash d'_{r} \sigma$ or $\Gamma$ is $d'_{r}$-inconsistent.
4. $g'$ is independent.

Proof. The proof of the first three parts is easy.

To prove the last part let $a, b \in C$, then $\frac{Aab, Oab}{Eab}$ is an instance of Co. But by lemma 8.3, $\{Aab, Oab\} \nvdash Eab$. So by part 2 above $\{Aab, Oab\} \nvdash g_{\text{co}} Eab$. Therefore Co is independent in $g'$. To complete the proof let $r$ be some other
rule of \( g' \), then \( r \) is a rule of \( d' \). By part 3 above the independence of \( r \) in \( g' \) may be proved by choosing a consistent set \( \Gamma \) of antecedents of \( r \) such that \( \Gamma \nvdash \sigma \), where \( \sigma \) is the corresponding conclusion; which is always possible. \( \Box \)

12.1.2. Second variation on \( g \). Though \( g' \) is closer than \( g \) to the contemporary deduction systems, it is not as close to the Aristotelian spirit as \( g \). Inspired by Gentzen-type sequent systems (cf. Kleene, S.C. 1967, p. 306) we introduce a second variation \( g'' \) on \( g \), which will hopefully be close enough to both modern and Aristotelian traditions. The deduction rules of \( g'' \) are:

\[ 0'. \frac{}{\Gamma \vdash Aaa} \quad (A\text{-Id'}) \]
\[ 1'. \frac{}{\Gamma \vdash Aab, \Delta \vdash Iba} \quad (Apd') \]
\[ 2'. \frac{}{\Gamma \vdash Eab, \Delta \vdash Eba} \quad (Epc') \]
\[ 3'. \frac{}{\Gamma \vdash Aab, \Delta \vdash Abc, \Gamma \cup \Delta \vdash Aac} \quad (Barbara') \]
\[ 4'. \frac{}{\Gamma \vdash Aab, \Delta \vdash Ebc, \Gamma \cup \Delta \vdash Eac} \quad (Celarent') \]
\[ 5'. \frac{}{\Gamma \vdash \eta} \quad (Ass) \]
\[ 6'. \frac{}{\Gamma \cup \{\hat{\rho}, \sigma\} \vdash \rho, \Delta \cup \{\hat{\rho}, \sigma\} \vdash \hat{\rho}} \quad (Raa) \]

where \( a, b, c \in C \), \( \Gamma \cup \Delta \cup \{\rho, \sigma\} \subseteq BN(C) \) and \( \eta \in \Gamma \). “Ass” and “Raa” are abbreviations for “Assumption” and “Reductio ad absurdum” respectively. “\( \vdash \)” is just a symbol, instead we could have made use of ordered pairs and write, e.g. “\( < \Gamma, \sigma > \)” in place of “\( \Gamma \vdash \sigma \)”.

**DEFINITION** and remarks 12.7. Let \( S \) be a set of sequents, i.e. \( S \subseteq \{\Gamma \vdash \sigma : \Gamma \cup \{\sigma\} \subseteq BN(C)\} \).

1. A \( g'' \)-deduction from \( S \) is a sequence \( < \Gamma_i \vdash \sigma_i >_{i \in k} \) of sequents, where \( k \in \mathbb{N} \) and for each \( i \in k \), \( \Gamma_i \vdash \sigma_i \in S \) or may be obtained from preceding terms of the sequence by some \( g'' \)-deduction rule.

If \( k \neq 0 \), \( < \Gamma_i \vdash \sigma_i >_{i \in k} \) is said to be a \( g'' \)-deduction of \( \Gamma_{k-1} \vdash \sigma_{k-1} \) from

58
S. In this case we write \( S'\vdash \Gamma_{k-1} \vdash \sigma_{k-1} \).

2. We write “\( \Gamma \vdash \sigma \)” for “\( \phi \vdash \Gamma \vdash \sigma \)”, “\( g''\)-deduction” for “\( g''\)-deduction from \( \phi \)” and “\( g''\)-deduction of \( \Gamma_{k-1} \vdash \sigma_{k-1} \) (or of \( \Delta \vdash \rho \))” for “\( g''\)-deduction of \( \Gamma_{k-1} \vdash \sigma_{k-1} \) (or of \( \Delta \vdash \rho \))” from \( \phi \).

3. The above definition and remark may be generalized to subsystems of \( g'' \).

4. The notions of derivability, independence and weak independence may be extended to \( g'' \) in the obvious way.

5. A deduction rule of \( g'' \) is independent in \( g'' \) iff it is weakly independent in \( g'' \). Hence \( g'' \) is independent iff it is weakly independent.

THEOREM 12.8. For every \( \Gamma \cup \{ \sigma \} \subseteq BN(C) \):

\[
\Gamma \vdash^g \sigma \quad \text{iff} \quad \Gamma \vdash^{g''} \sigma.
\]

**Proof.** Let \( \Gamma \vdash \sigma \) then, by definition 1.9, \( \Gamma \cup \{ \hat{\sigma} \} \vdash^d \rho, \hat{\rho} \) for some \( \rho \in BN(C) \). Let < \( \xi_i >_{i \in k} \) and < \( \eta_j >_{j \in l} \) be, respectively, \( d \)-deductions of \( \rho \) and \( \hat{\rho} \) from \( \Gamma \cup \{ \hat{\sigma} \} \). Then < \( \Gamma \cup \{ \hat{\sigma} \} \vdash \xi_i >_{i \in k} \) and < \( \Gamma \cup \{ \hat{\sigma} \} \vdash \eta_j >_{j \in l} \) are, respectively, \( g'' \)-deductions of \( \Gamma \cup \{ \hat{\sigma} \} \vdash \rho \) and \( \Gamma \cup \{ \hat{\sigma} \} \vdash \hat{\rho} \). To their concatenation (which is a \( g'' \)-deduction) add one more line to obtain \( \Gamma \vdash \sigma \) by Raa from lines \( k-1 \) and \( k+l-1 \). This proves the only if direction.

To prove the other direction let \( \Gamma \vdash^{g''} \sigma \), let < \( \Delta_i \vdash \rho_i >_{i \in k} \) be a \( g'' \)-deduction of \( \Gamma \vdash \sigma \), and assume that \( \Delta_i \vdash^{g} \rho_i \) for each \( i < k-1 \). To show that \( \Delta_{k-1} \vdash^{g} \rho_{k-1} \) we deal with as many cases as there are \( g'' \)-deduction rules. Following we consider Celarent' (4') and Raa (6'), the other cases are similar or easier.

Celarent: There are \( a, b, c \in C \) and \( j, l \in \mathbb{N} \) such that \( j < l < k-1 \), \( \Delta_{k-1} = \Delta_j \cup \Delta_l, \{ \rho_j, \rho_l \} = \{ Aab, Ebc \} \) and \( \rho_{k-1} = Eac \). By the above assumption \( \Delta_j \vdash^{g} \rho_j \) and \( \Delta_l \vdash^{g} \rho_l \). Hence \( \Delta_{k-1} = \Delta_j \cup \Delta_l \vdash^{g} \rho_j, \rho_l \). So by part 7 of proposition 1.11, \( \Delta_{k-1} \vdash^{g} \rho_{k-1} \).

Raa: There are \( \Sigma, \Sigma', \eta, j, l \) such that \( \Sigma \cup \Sigma' \cup \{ \eta \} \subseteq BN(C); j, l \in \mathbb{N} \); \( j < l < k-1 \), \( \Delta_j = \Sigma \cup \{ \rho_{k-1} \} \), \( \Delta_l = \Sigma' \cup \{ \hat{\rho}_{k-1} \} \), \( \{ \rho_j, \rho_l \} = \{ \eta, \hat{\eta} \} \).
and $\Delta_{k-1} = \Sigma \cup \Sigma'$. By the above assumption $\Delta_j \vdash \rho_j$ and $\Delta_l \vdash \rho_l$. Hence $\Delta_{k-1} \cup \{\hat{\rho}_{k-1}\} = (\Delta_j \cup \Delta_l) \vdash \rho_j, \rho_l$. So by part 4 of proposition 1.11 and the relevant definitions, $\Delta_{k-1} \vdash \rho_{k-1}$. □

Notice that in the $g''$-deductions $< \Gamma \cup \{\hat{\sigma}\} \vdash \xi_i >_{i \in k}$ and $< \Gamma \cup \{\hat{\sigma}\} \vdash \eta_j >_{j \in l}$ which occur in the proof of the only if direction of the above theorem, only the rules $0' - 5'$ are made use of. Moreover, if $\Gamma \vdash \sigma$, then there is a $g''$-deduction of $\Gamma \vdash \sigma$ in which Raa is never made use of.

This is essentially sufficient to prove the following:

**COROLLARY 12.9.** If $\Gamma \vdash \sigma$ then there is a $g''$-deduction of $\Gamma \vdash \sigma$ in which Raa is never made use of or is made use of only in the last step. □

This section is concluded by proving the independence of $g''$.

**THEOREM 12.10.** Let $\Gamma \cup \{\sigma\} \subseteq BN(C)$, $r$ be a $d$-deduction rule, and $r'$ be the corresponding $g''$-deduction rule.

1. $\Gamma \vdash \sigma \iff \Gamma \vdash \sigma$,
2. $\Gamma \vdash \sigma \iff \phi \vdash \sigma \iff \phi \vdash \sigma \iff \phi \vdash \sigma$,
3. $\Gamma \vdash \sigma \iff \Gamma \vdash \sigma$,
4. $g''$ is independent.

*Proof.* The proofs of the first and the third parts are along the lines of the proof of theorem 12.8 noting that proposition 1.11 still holds after replacing “$d$”, “$g$” by “$d_r$”, “$g_r$” respectively. For part 2 it is sufficient to notice that each of the four statements holds iff $\sigma$ is of the form $Ycc$ for some $c \in C$ and some $Y \in \{A, I\}$.

To prove the last part we consider three cases:

Rules $0' - 4'$: Let $r'$ be one of these rules and let $r$ be the corresponding $d$-rule. Since $r$ is independent in $g$, there is a set $\Gamma$ of antecedents of an instance of $r$ such that $\Gamma \not\vdash \sigma$, where $\sigma$ is the corresponding conclusion. Put $S = \{\Gamma \vdash \rho : \rho \in \Gamma\}$ then $S$ is a set of antecedents of $r'$ and $\Gamma \vdash \sigma$ is the
corresponding conclusion. By parts 2, 3 of definition and remarks 12.7 and part 1 above:

\[ S \models g'' \Gamma \vdash \sigma \text{ iff } \phi \models g'' \Gamma \vdash \sigma \text{ iff } \Gamma \vdash \sigma \text{ iff } \Gamma \vdash \sigma. \]

But \( \Gamma \not\vdash r' \), hence \( r' \) is independent in \( g'' \).

Rule Ass: For \( a, b \in C \), \( \phi \models \{Oab\} \vdash Oab \) while, by 2 above, \( \phi \not\models \{Oab\} \vdash Oab \). Hence Ass is independent in \( g'' \).

Rule Raa: For distinct elements \( a, b \) of \( C \) let \( \sigma = Iab \), \( \rho = Iba \), \( \Gamma = \{\rho\} \) and \( S = \{\Gamma \cup \{\hat{\sigma}\} \vdash \rho, \Gamma \cup \{\hat{\sigma}\} \vdash \hat{\rho}\} \) then \( S \) is a set of antecedents of Raa and \( \Gamma \vdash \sigma \) is the corresponding conclusion. By a slight modification of the proof given above for the rules \( 0' - 4' \) it may be shown that Raa is independent in \( g'' \). \( \square \)

13. Algebraic semantics of AAS, a prelude. The most well known attempt to algebraically interpret Aristotelian syllogistic is that of Boole (1948, first published 1847); however, it is not the first. More than a century and a half earlier, this area of research was pioneered by Leibniz (Kneale and Kneale 1966, pp. 338-45; Lenzen 2004). Following is a discussion of the subject in general; the works of Leibniz and Boole will be briefly discussed in section 17 below.

Regarding the central role played by order models in the semantics of \( NF(C) \), they will be our starting point for algebraization. Each underlying order structure of an order model will induce an algebra which may be expanded to make the interpretation of \( NF(C) \) possible.

The simplicity of order models stems from the fact that all relations are determined by only one of them, namely the interpretation of \( A \), which is compatible with the Aristotelian view that Barbara is the essential syllogism. Likewise, algebras defined in this section will each have one (partial) binary operation and no others.

DEFINITIONS and remarks 13.1.

1. Let \( B \) be a non-empty set and let \( \oplus \) be a function from a subset of \( B \times B \) to \( B \), then \( \oplus \) is said to be a partial binary operation on \( B \), and \( < B, \oplus > \) is said to be a partial algebra.
2. Let \( \leq \) be a binary relation on a set \( B \), the partial binary operation \( +_\leq \) induced by \( \leq \) on \( B \) is defined by:

\[
+_\leq : \leq \to B
\]

\[
a +_\leq b = a
\]

\( +_\leq \) is commutative (see 3.3 below) if \( \leq \) is antisymmetric. \( < B, +_\leq > \) is called the partial algebra induced by \( < B, \leq > \).

3. Let \( < B, \leq > \) be an order structure, then \( < B, +_\leq > \) satisfies:

1. Right associativity:

\[
(a +_\leq b) +_\leq c = a +_\leq (b +_\leq c)
\]

in the sense that for every \( a, b, c \in B \) if the rhs exists, so does the lhs and they are equal.

2. Idempotence:

\[
a +_\leq a = a \quad \text{all} \quad a \in B.
\]

If, moreover, \( \leq \) is antisymmetric, then \( < B, +_\leq > \) satisfies:

3. Commutativity:

\[
a +_\leq b = b +_\leq a
\]

in the sense that for every \( a, b \in B \), if both sides exist they are equal.

Honouring Leibniz, a partial algebra \( < B, \oplus > \) satisfying conditions 1 and 2 will be called a Leibniz algebra (LA for short). If, moreover, it satisfies condition 3 it will be called a commutative Leibniz algebra (CLA for short).

So, \( < B, +_\leq > \) is a LA if \( < B, \leq > \) is an order structure; moreover, it is a CLA if \( \leq \) is antisymmetric.

4. An idempotent partial algebra \( < B, \oplus > \) will be called a weak Leibniz algebra (WLA for short) if it satisfies:
1. Weak right associativity:

\[(a ⊕ b) ⊕ c = a ⊕ (b ⊕ c)\]

in the sense that for every \(a, b, c \in B\) if \((a ⊕ b)\) and the rhs both exist, then the lhs exists and equals the rhs.

If, moreover, \(< B, ⊕ >\) is commutative (in the sense of condition 3.3 above), it will be called a commutative weak Leibniz algebra (CWLA for short).

Obviously every LA (CLA) is a WLA (CWLA).

5. With abuse of notation, “LA”, “CLA”, “WLA” and “CWLA” will denote also the classes of all LAs, CLAs, WLAs and CWLAs respectively; what is intended will be clear from the context.

Abuses of notations such as this may take place later on without further notice.

6. Let \(< B, ⊕ >\) be a partial algebra, the binary relation \(≤_⊕\) induced by \(⊕\) on \(B\) is defined by:

\[≤_⊕ = \{ < a, b > ∈ B : < < a, b >, a > ∈ ⊕ \},\]

so \(a ≤_⊕ b\) iff \(a ⊕ b = a\) (in the sense that the lhs exists and equals the rhs). Obviously, \(≤_⊕\) is antisymmetric if \(⊕\) is commutative.

7. Let \(≤_≤\) and \(⊕\) be, respectively, a binary relation and a partial binary operation on a set \(B\), and let \(+≤, ≤_⊕, ≤_⊕+\) and \(+≤_⊕\) be as defined above. Then:

1. \(≤_⊕+ = ≤_≤\)
2. \(+≤_⊕ ⊆ ⊕\); moreover, \(+≤_⊕\) is commutative if \(⊕\) is.

8. Let \(< B, ⊕ >\) be a WLA, then \(< B, ≤_⊕ >\) is an order structure, called the order structure induced by \(< B, ⊕ >\). Moreover, \(≤_⊕\) is antisymmetric if \(⊕\) is commutative.
14. Algebraic interpretation of $NF(C)$.

DEFINITION 14.1. Let $\mathfrak{B} = < B, \oplus >$ be a WLA and let $\mu : C \rightarrow B$. The structure $\mathfrak{B}^\mu = < B, \oplus, \mu >$ is said to be a weak Leibniz structure (WLS for short). The reduct $\mathfrak{B}$ shall be called the WLA base of $\mathfrak{B}^\mu$.

Leibniz structures (LS for short), commutative Leibniz structures (CLS for short) and commutative weak Leibniz structures (CWLS for short) are defined analogously.

The following definition shows how $NF(C)$ may be interpreted in these structures. So they may, and will, be considered as $NF(C)$-structures and will be treated like other $NF(C)$-structures when dealing with semantics. In particular, all semantical notions (such as “$\mathfrak{B}^\mu$ is a (n algebraic) model of $\Gamma$” or “$\Gamma \models \sigma”, for $\Gamma \cup \{\sigma\} \subseteq BN(C)$ and $\mathfrak{C} \subseteq WLS$) will be assumed to be known.

DEFINITION 14.2. Let $\mathfrak{B}^\mu = < B, \oplus, \mu >$ be a WLS, and let $a, b \in C$, then:

1. $\mathfrak{B}^\mu \models Aab$ iff $\mu a \oplus \mu b$ exists and equals $\mu a$ (iff $<< \mu a, \mu b >, \mu a > \in \oplus$).

2. $\mathfrak{B}^\mu \models Iab$ iff the system of equations $x \oplus \mu a = x$, $x \oplus \mu b = x$ has a solution (iff the equation $x \oplus \mu a = x \oplus \mu b$ has a solution, iff the equation $x \oplus \mu a = y \oplus \mu b$ has a solution).

3. $\mathfrak{B}^\mu \models Eab$ iff $\mathfrak{B}^\mu \not\models Iab$.

4. $\mathfrak{B}^\mu \models Oab$ iff $\mathfrak{B}^\mu \not\models Aab$.

REMARKS 14.3.

1. The order (partial order) model $< B, \leq, \mu >$ is basically equivalent to the WLS (CLS), $< B, \leq, \mu >$.

2. The WLS (CLS), $< B, \oplus, \mu >$, is basically equivalent to the order (partial order) model $< B, \leq, \mu >$.

3. Consequently, every WLS (hence every LS, every CWLS and every CLS) is an $e$-model for $e \in \{d, d', d'', g\}$.
4. In the light of remarks and definitions 9.10 it may be easily seen that:

1. For \( e \in \{d, d', d'', g\} \), \( e \) is sound wrt WLS, hence wrt every subclass of it.

2. \( g \) is CLS-syntactically complete.

3. For \( e \in \{d', d'', g\} \), \( e \) is CLS-consistently syntactically complete.

4. For \( e \in \{d, d', d'', g\} \), \( e \) is CLS-consistently semantically complete.

In clauses 2-4, CLS may be replaced by any class intermediate between it and WLS.

15. Annihilators: Embedding the partial into a total. An annihilator of a (partial) binary operation \(*\) on a set \( B \) is an element \( b \in B \) such that:

\[
x \ast b = b = b \ast x \quad \text{all } x \in B
\]

Obviously \(*\) has at most one annihilator.

An annihilator algebra is an ordered triple \( \mathfrak{B} = < B, \ast, b > \) such that the reduct \( ^r \mathfrak{B} = < B, \ast > \) is a partial algebra, and \( b \) is an annihilator of \( \ast \).

The subreduct \( ^r \mathfrak{B} \) of \( \mathfrak{B} \) is the ordered pair \( < B', \ast' > \), where:

\[
B' = B - \{b\} \quad , \quad \ast' = \ast \cap (B' \times B') \times B'
\]

Here, and in the sequel, \( B' \) is assumed to be non-empty.

DEFINITIONS 15.1.

1. An annihilator Leibniz algebra (ALA for short) is an annihilator algebra whose subreduct is a LA.

Annihilator commutative Leibniz algebras (ACLA for short), annihilator weak Leibniz algebras (AWLA for short) and annihilator commutative weak Leibniz algebras (ACWLA for short) are defined analogously.

2. A Leibniz algebra with annihilator (LAA for short) is an annihilator algebra whose reduct is a LA.

Commutative Leibniz algebras with annihilators (CLAA for short), weak Leibniz algebras with annihilators (WLAA for short) and commutative weak
Leibniz algebras with annihilators (CWLAA for short) are defined analogously.

As usual, an algebra or a structure based on an algebra is said to be total if each of its operations is total. "TLA" will stand for "total Leibniz algebra", "TLS" will stand for "total Leibniz structure" and similarly for the other cases.

REMARKS 15.2.

1. The subreduct of an annihilator algebra is a LA (respectively CLA, WLA or CWLA) if the reduct is.
   Hence LAA ⊆ ALA, and similarly for the other cases.

2. Let $B = \langle B, \ast, b \rangle$ be a total annihilator algebra whose subreduct also is total. Then $B$ is LAA iff it is ALA; "LA" may be replaced by "CLA", "WLA" or "CWLA".

3. TCLAA = TCWLAA = ICSGA (idempotent commutative semigroups with annihilators)
   = OSLA (operational semilattices with annihilators)

   The order structures induced by these algebras are lower semilattices with smallest elements.

   In the above equations “C”, the last “A” or both, may be dropped everywhere (the corresponding parenthetic clause is to be modified accordingly).

   The following definition designates to each partial algebra a total annihilator algebra in which it may be embedded.

DEFINITION and remarks 15.3.

1. For $i \in 2$, let $\mathcal{B}_i (= \langle B_i, \ast_i \rangle)$ be a partial algebra. A bijection $f$ from $B_0$ to $B_1$ is said to be an isomorphism from $\mathcal{B}_0$ to $\mathcal{B}_1$ if for every $x, y, z \in B_0$:

   $\langle \langle x, y \rangle, z \rangle \in \ast_0$ iff $\langle \langle f x, f y \rangle, f z \rangle \in \ast_1$.

   $\mathcal{B}_0$ and $\mathcal{B}_1$ are said to be isomorphic if there is an isomorphism from one of them to the other.
2. If two partial algebras are isomorphic and one of the partial binary operations has an annihilator, then its image is an annihilator of the other.

3. Two annihilator algebras are said to be isomorphic if their reducts are.

4. Two total annihilator algebras are isomorphic iff their subreducts are.

5. Every partial algebra $\mathfrak{B} = \langle B, \ast \rangle$ is the subreduct of some total annihilator algebra. For, let $0 \notin B$ and $0_B = B \cup \{0\}$.
   
   Put:
   
   $0^* : \quad 0_B \times 0_B \to 0_B$

   $x \ 0^* \ y = \begin{cases} x \ast y & \text{if } <x, y> \in \text{Domain}, \\ 0 & \text{otherwise} \end{cases}$

   Then $0\mathfrak{B} = \langle 0B, 0^*, 0 \rangle$ is a total annihilator algebra, and $\mathfrak{B}$ is its subreduct.

6. Every total annihilator algebra whose subreduct is isomorphic to $\mathfrak{B}$, is isomorphic to $0\mathfrak{B}$. This warrants calling $0\mathfrak{B}$ the total annihilator algebra induced by $\mathfrak{B}$.

   The identity map on $B$ is an embedding of $\mathfrak{B}$ into $0\mathfrak{B}$.

7. $0\mathfrak{B}$ is a TALA iff $\mathfrak{B}$ is a LA. “LA” may be replaced by “CLA”, “WLA” or “CWLA”.

16. **Back to algebraic interpretation.** Let $\mathfrak{B} = \langle B, \ast, 0 \rangle$ be a WLAA, then its reduct $\mathfrak{rB} = \langle B, \ast \rangle$ is a WLA. So $\langle B, \ast, \mu \rangle$ is a WLS, for every $\mu : C \to B$. Obviously, for all $a, b \in C$, $Iab$ is satisfied in this structure. Hence none of $d, d', d''$ nor $g$ is consistently semantically complete wrt any class of such structures, though every one of them is sound wrt each of these classes. Evidently expanding the structure to $\langle B, \ast, 0, \mu \rangle$ will not solve the problem.

   As a matter of fact, the annihilator is the source of the difficulty, and we may get around it by not permitting the annihilator to be assigned as a value corresponding to any element of $C$, nor accepting it as a solution of any of the relevant equations below. An additional advantage of this approach is to be able to consider the more general AWLA.
DEFINITION 16.1. (non-annihilator interpretation of $NF(C)$ in annihilator algebras)

1. Let $\mathfrak{B}(= < B, *, 0 >)$ be an AWLA and let $\mu : C \to B'(= B - \{0\})$. The structure $\mathfrak{B}^\mu(= < B, *, 0, \mu >)$ is called an annihilator weak Leibniz structure (AWLS for short). The reduct $\mathfrak{B}$ of $\mathfrak{B}^\mu$ is called the AWLA base of $\mathfrak{B}^\mu$.

The structures based on the other algebras (total or not) are defined, and their names are abbreviated, analogously.

2. For each $a, b \in C$:
   1. $\mathfrak{B}^\mu \models Aab$ iff $\mu a \ast \mu b$ exists and equals $\mu a$ (equivalently $<< \mu a, \mu b >, \mu a >\in \ast$).
   2. $\mathfrak{B}^\mu \models Iab$ iff the system of equations $x \ast \mu a = x$,
      $x \ast \mu b = x$ has a solution different from 0
      (iff the equation $x \ast \mu a = x \ast \mu b$
      has a solution which makes $x \ast \mu b \neq 0$,
      iff the equation $x \ast \mu a = y \ast \mu b$
      has a solution which makes $y \ast \mu b \neq 0$).
   3. $\mathfrak{B}^\mu \models Eab$ iff $\mathfrak{B}^\mu \not\models Iab$.
   4. $\mathfrak{B}^\mu \models Oab$ iff $\mathfrak{B}^\mu \not\models Aab$.

This shows how $NF(C)$ may be interpreted in the structures defined in part 1. So they may, and will, be considered as $NF(C)$-structures and will be treated like other $NF(C)$-structures when dealing with semantics. In particular, all semantical notions (such as “$\mathfrak{B}^\mu$ is a (total algebraic) model of $\Gamma$” or “$\Gamma \models e \sigma$”, for $\Gamma \cup \{\sigma\} \subseteq BN(C)$ and $e \subseteq AWLS$) will be assumed to be known.

REMARKS 16.2.

1. $\mathfrak{B}^\mu$ and $\ast \mathfrak{B}^\mu$ are basically equivalent, hence every AWLS is an $e$-model for $e \in \{d, d', d'', g\}$.

2. In part 4 of remarks 14.3, “WLS” and “CLS” may be, respectively, replaced by “TAWLS” and “TACLs”.

3. If $\mathfrak{B}^\mu$ is a TCLSA (equivalently TCWLSA), the provisions given in part 2 of definitions 16.1 may be simplified in the obvious way; in particular, the second provision will be equivalent to $\mu a \ast \mu b \neq 0$. 

68
To investigate the relationship between TCLSA and the Venn models we make use of a (n intermediate) subclass of TCLSA, namely the subclass of those TCLSA based on OSLA which are reducts of Boolean algebras.

These reducts will be called Boolean-Leibniz algebras with annihilators, BLAA for short. As usual BLSA is a Boolean-Leibniz structure with annihilator, i.e. a LSA based on a BLAA.

PROPOSITION 16.3. Every TCLAA may be embedded in a BLAA.

Proof. Let \( \mathfrak{B}(=\langle B, \ast, 0 \rangle) \) be a TCLAA. The mapping:

\[
\begin{align*}
  f &: B \to \wp(B) \\
  f(b) &= \{x \in B : x \ast b = x \} - \{0\}
\end{align*}
\]

is an embedding of \( \mathfrak{B} \) in the BLAA: \( \langle \wp (B), \cap, \wp \rangle \). \( \square \)

\( \langle \wp (B), \cap, \wp \rangle \) will be called the BLAA corresponding to \( \mathfrak{B} \) and will be denoted by “\( Bl(\mathfrak{B}) \)”. For \( \mu : C \to B, \langle \wp (B), \cap, \wp, f \mu \rangle \) is a BLSA; it will be called the BLSA corresponding to, the TCLSA, \( \mathfrak{B}^\mu \) and will be denoted by “\( Bl(\mathfrak{B}^\mu) \)”. The relevant definitions and part 3 of remarks 16.2 show that \( \mathfrak{B}^\mu \) and \( Bl(\mathfrak{B}^\mu) \) are basically equivalent.

THEOREM 16.4. Every TCLSA is basically equivalent to a Venn model. And every Venn model is basically equivalent to a BLSA (hence to a TCLSA); moreover, the BLSA may be assumed to be based on a concrete BLAA whose universe is a power set.

Proof. Let \( \mathfrak{B}^\mu(=\langle B, \ast, 0, \mu \rangle) \) be a TCLSA, then \( Bl(\mathfrak{B}^\mu) \) is a BLSA and \( \mathfrak{B}' = \langle \wp (B) - \{\wp\}, f \mu \rangle \) is a Venn model. They all are basically equivalent.

On the other hand, let \( \mathfrak{B}(=\langle B, \mu \rangle) \) be a Venn model, then \( \langle \wp (\bigcup B), \cap, \wp, f \mu \rangle \) is a BLSA which is basically equivalent to it. \( \square \)

COROLLARY 16.5. In part 4 of remarks 14.3, “WLS” and “CLS” may be, respectively, replaced by “TWLSA” and “BLSA” (either the superclass TCLSA, or the subclass consisting of those elements each of which is based on a concrete BLAA whose universe is a power set, may replace BLSA). \( \square \)
17. Leibniz and Boole. The calculus de continentibus et contentis, or the calculus of identity and inclusion—which is an algebraic treatment of concepts—was developed by Leibniz during 1679-90 (Kneale and Kneale 1966, p. 337). As may be gathered from a passage of the same reference (pp. 340-3), or from a translation of an original text of Leibniz (Lewis 1960, pp. 297-305), this calculus is the theory of operational semilattices (OSL for short) with applications to concepts; commutativity and idempotence are explicitly stated, while associativity is implicitly taken for granted (the aforementioned passage is abbreviated with some slight changes from the aforementioned translation (Kneale and Kneale 1966, p. 343); notice that the edition of Lewis’ book referred to in Kneale and Kneale (1966) is earlier than the one referred to above).

Kneale and Kneale (1966)’s assessment of this calculus is unfavorable. It asserts (p. 337) that Leibniz “intended, no doubt, to produce something wider than traditional logic. […] But […] he never succeeded in producing a calculus which covered even the whole theory of syllogism.”. On p. 345 this assertion is elaborated “What he [Leibniz] produced was certainly much less than he hoped to produce. For the last scheme [the calculus de continentibus et contentis], lacking as it does any provision for negation or for consideration of conjunction and disjunction together, is still a fragment. So far from including all Aristotle’s syllogistic theory as a part, it contains no principle of syllogism except the first […]”.

Likewise, Lenzen (2004)’s assessment of the calculus de continentibus et contentis is unfavorable. It asserts (p. 28) that this calculus “remains a very weak and uninteresting system […]; thus it shall no longer be considered here.”.

On the contrary, we have shown that neither negation (of terms) nor any additional operations are needed to algebraically interpret AAS. It suffices to require the OSL to possess an annihilator, i.e. to be OSLA. For the structures based on the OSLA are the TCLSA and, by corollary 16.5, AAS is both sound and complete with respect to them.

According to Kneale and Kneale (1966, p. 339) it may be seen that Leibniz practically introduced annihilators when he interpreted \( \text{Eab} \) as \( \text{ab} (\mu_a \ast \mu_b, \text{in our terminology}) \) is nothing.

Lenzen (2004) goes even further. It (pp. 2-3) asserts that Leibniz developed stronger calculi, the most important of them (p.3) “is L1, the full algebra of concepts […], L1 is deductively equivalent or isomorphic to the ordinary algebra of sets. Since Leibniz happened to provide a complete set of axioms
for L1, he “discovered” the Boolean algebra 160 years before Boole.

Moreover, Lenzen (2004) asserts that Leibniz succeeded in making use of his logical theory to derive the basic laws of Aristotelian syllogism (p. 55). In particular, the Aristotelian inferences may be derived as theorems of L1, or the stronger calculus L2 (p. 56); a detailed discussion of the subject may be found in Lenzen (2004, §8, pp. 55-73). Indeed, as we have shown, AAS does not need all of this.

Boole did more than just algebraically interpreting AAS. In addition to annihilators, which are sufficient for dealing with Aristotelian syllogisms (which involve no term negation), he introduced complementation (which corresponds to term negation) and a second binary operation. This is possibly to:

1. be able to interpret all the Aristotelian categorical sentences into equations (cf. Boole 1948, pp. 20-5),
2. deal with medieval categorical sentences which may involve term negation (cf. Boole 1948, pp. 20, 27-47), or
3. deal with hypotheticals (cf. Boole 1948, pp. 48-59).

In addition to establishing the Aristotelian syllogistic rules, Boole (1948) established some non-Aristotelian ones. For example (p. 37) $\frac{E_{xy}O_{yz}}{O_{x'z}}$, where “$x'$” denotes “not-$x$”.

Boole (1948) did not address the question of completeness, neither did he consider consequences of more than two premises. However, it discussed (pp. 76-81) a general scheme to solve arbitrarily finite systems of simultaneous equations in arbitrarily finitely many variables; applying, in particular, Lagrange’s method of indeterminate multipliers. This discussion took place after making (p. 18) the confounding assertion “[...] all the processes of common algebra are applicable to the present [Boolean] system.”.

For one more confounding assertion see below.

**18. Inadequacy: bounds of AAs.** Calling the symbols of its system “elective symbols” (p. 16), Boole (1948) makes (p. 59) another confounding assertion: “Every Proposition which language can express may be represented by elective symbols, and the laws of combination of those symbols are in all cases the same; but in one class of instances the symbols have reference to collections of objects, in the other, to the truths of constituent Proposition.”. This, probably, amounts -in modern language- to asserting: Every proposition which language can express is equivalent to a sentential combination of categorical sentences (SCCS for short).
SCCS should be taken seriously, since a stronger assertion has dominated human thought over more than two millennia: Every argument can be put in a syllogistic form. Even Bertrand Russell (1967, p. 198) asserts “Of course it would be possible to re-write mathematical arguments in syllogistic form, but this would be very artificial and would not make them any more cogent.”.

Concerning these assertions, it is worthwhile to bring to the fore what Bocheński (1968) calls attention to. On p. 63 it observes that Aristotle “says explicitly that not all logical entailment is “Syllogistic”.”. Moreover it observes on the same page that Aristotle declares that some logical entailments cannot be reduced to syllogisms. So it may be concluded that Aristotle himself contradicts the aforementioned assertions of Boole and Russell, which makes making them deeply confounding, and makes it more urgent for historians of thought to investigate the matter.

Understanding SCCS depends on understanding the notion of categorical sentences. If term negation is permitted, the sentences will be called “Boolean categorical sentences” and the corresponding assertion will be denoted by “SCBCS”. Otherwise, the sentences will be called “Aristotelian categorical sentences” and the corresponding assertion will be denoted by “SCACS”.

Hilbert and Ackermann (1950) formalizes the Boolean categorical sentences (pp. 44-8) and informally refutes SCBCS (pp. 55-6).

To formally discuss SCACS (making use only of the methods developed above and the well known results of sentential logic) augment the alphabet of the language \( C \) of the natural deduction formalization defined in section 1.4 above, by a ternary relation symbol \( E' \), and add \( E'abc \) \((a, b, c \in C)\) to the set of sentences based on \( C \). Denote the new set of sentences by “\( BN'(C) \)”.

Intuitively, we like \( E'abc \) to mean that no \( a \) which is \( b \), is \( c \). This may be formalized as follows:

Interpret \( BN'(C) \) in a WLS \( \mathfrak{B}^\mu =\langle B, *, \mu \rangle \) by adding the following provision to the provisions of definition 14.2.

5. \( \mathfrak{B}^\mu \vdash E'abc \) iff the system of equations: \( x \ast \mu a = x \), \( x \ast \mu b = x \) and \( x \ast \mu c = x \) has no solution.

In an AWLS \( \mathfrak{B}^\mu =\langle B, *, 0, \mu \rangle \), \( BN'(C) \) is interpreted by adding the following provision to the provisions of part 2 of definitions 16.1.

5. \( \mathfrak{B}^\mu \vdash E'abc \) iff 0 is the only solution of the system of equations: \( x \ast \mu a = x \), \( x \ast \mu b = x \) and \( x \ast \mu c = x \).

Recall that 0 is not in the range of \( \mu \); also notice that if \( \mathfrak{B}^\mu \) is a TCLSA,
then this provision is equivalent to

\[ 5'. \mu a \ast \mu b \ast \mu c = 0. \]

The other syntactical and semantical notions remain the same, or to be appropriately modified in the obvious way.

Let \( \Gamma_0, \Gamma_1 \subseteq BN'(C) \) and let \( \mathcal{D} \subseteq WLS \cup AWLS \). \( \Gamma_0 \) is said to \( \mathcal{D} \)-imply \( \Gamma_1 \) (symbolically \( \Gamma_0 \models_{\mathcal{D}} \Gamma_1 \)) if for every \( D \in \mathcal{D} \), \( D \models \Gamma_1 \) whenever \( D \models \Gamma_0 \). \( \Gamma_0 \) is said to be \( \mathcal{D} \)-equivalent to \( \Gamma_1 \), or \( \Gamma_0, \Gamma_1 \) are \( \mathcal{D} \)-equivalent, if each of them \( \mathcal{D} \)-implies the other. \( \Gamma_0 \) is said to be \( \mathcal{D} \)-valid if \( \phi \models_{\mathcal{D}} \Gamma_0 \), it is said to be \( \mathcal{D} \)-consistent if \( \mathcal{D} \models \Gamma_0 \) for some \( D \in \mathcal{D} \).

The above notions may be generalized, in the obvious way, to sets of sentential combinations of elements of \( BN'(C) \). If \( \Gamma_0 \) or \( \Gamma_1 \) is a singleton, it may be replaced by its unique element, e.g. \( \{ \rho \}\models_{\mathcal{D}} \{ \sigma \} \).

In what follows \( c_0, c_1 \) and \( c_2 \) are assumed to be pairwise distinct elements of \( C \). For every \( \mathcal{D} \subseteq WLS \cup AWLS \), \( Ec_0c_1 \mathcal{D} \)-implies \( Ec_0c_1c_2 \). The converse depends on \( \mathcal{D} \). In particular it does not hold for \( \mathcal{D} = BLSA \). As a matter of fact we have the following:

**THEOREM 18.1.** Let \( \sigma \) be a sentential combination of elements of \( BN(C) \), then:

1. \( Ec_0c_1c_2 \) is not \( BLSA \)-equivalent to \( \sigma \), hence
2. \( Ec_0c_1c_2 \) is not deductively equivalent to \( \sigma \) (i.e. one of them does not deductively entail the other), for each deductive system which is sound with respect to \( BLSA \).

To prove this, we first prove:

**LEMMA 18.2.** Put:

\[ \Gamma_0 = \{ Ic_0c_1, Ic_1c_2, Ic_2c_0 \} \quad \text{and} \quad \Gamma_1 = \Gamma_0 \cup \{ Ec_0c_1c_2 \} \]

then:

1. \( \Gamma_1 \) is \( BLSA \)-consistent.
2. \( \Gamma_1 \) is not \( BLSA \)-implied by any \( BLSA \)-consistent \( \Gamma \subseteq BN(C) \).

**Proof.** Part 1 is easy. To see part 2, assume that there is a subset \( \Gamma \subseteq BN(C) \) which is both \( BLSA \)-consistent and \( BLSA \)-implies \( \Gamma_1 \). Then there is \( \mathfrak{B}^\mu \in BLSA \) which is a model of \( \Gamma \cup \Gamma_1 \). By theorem 16.4 it may be assumed that \( \mathfrak{B}^\mu =< \phi(B), \cap, \phi, \mu > \) for some \( B \).
Let $B' = B \cup \{ a \}$ for some $a \notin B$ and let $\mathfrak{B}^{\mu'} = \langle \varphi(B'), \cap, \phi, \mu' \rangle$ where for every $c \in C$, 

$$
\mu'(c) = \begin{cases} 
\mu(c) \cup \{ a \} & \text{if } \mathfrak{B}^{\mu} \models Ac_i c \text{ for some } i \in 3, \\
\mu(c) & \text{otherwise.}
\end{cases}
$$

$\mathfrak{B}^{\mu'}$ is a BLSA which is basically equivalent to $\mathfrak{B}^{\mu}$, hence it is a model of $\Gamma$; but it is not a model of $\Gamma_1$. From this the required follows. $\square$

**Proof of theorem 18.1.** Assume that $E'c_0c_1c_2$ is BLSA-equivalent to a sentential combination of elements of $BN(C)$, $\sigma$ say. Then $E'c_0c_1c_2 \land Ic_0c_1 \land Ic_1c_2 \land Ic_2c_0$ ($\rho$ for short) is BLSA-equivalent to $\sigma \land Ic_0c_1 \land Ic_1c_2 \land Ic_2c_0$ ($\sigma_1$ for short) which also is a sentential combination of elements of $BN(C)$.

By sentential logic, $\sigma_1$ may be assumed to be a disjunction of conjunctions of elements of $BN(C)$ and their negations. Since $\rho$ is BLSA-consistent and the negation of any element of $BN(C)$ is BLSA-equivalent to some element of $BN(C)$, $\sigma_1$ may further be assumed to be a non-empty disjunction of BLSA-consistent conjunctions of elements of $BN(C)$. Consequently $\rho$ is BLSA-implied by each of these conjunctions, which contradicts part 2 of lemma 18.2. From this the required follows. $\square$

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**Appendix.** The following algorithm, to generate the first $n(>0)$ primes, may not be efficient, but it is simple, and its running time (see below) makes it sufficient for our purposes.

**Input:** $n$ (positive integer)  
**Output:** $p$ (the strictly increasing list of the first $n$ primes)  
**Procedure:**  
Declare $i, j, k, m$ natural number parameters;  
$p_0 \leftarrow 2$;
If $n = 1$ go to ***
Else $p_1 \leftarrow 3$, $i \leftarrow 1$, $m \leftarrow 2$
End If;
For $i < n - 1$ do
  $k \leftarrow p_i + 2$, $m \leftarrow mp_i$
For $k \leq m + 1$ do
  $j \leftarrow 0$
For $j \leq i$ do
  If $p_j | k$ go to *
  Else $j \leftarrow j + 1$
End If;
Repeat
End For3;
* If $j > i$ go to **
  Else $k \leftarrow k + 2$
End If;
Repeat
End For2;
** $i \leftarrow i + 1$
  $p_i \leftarrow k$
Repeat
End For1;
*** Print $p$;
End Algorithm.

The termination of this algorithm is guaranteed by the respective upper bounds stipulated at the beginnings of the three For loops. The correctness is guaranteed by the well known fact which goes back to Euclid’s Elements:

$p_{i+1} \leq 1 + \prod_{j=0}^{i} p_j$, together with the simple fact that $p_{i+1}$ is the first (odd) integer greater than $p_i$, which is not a multiple of any of $p_0, ..., p_i$.

To estimate the running time, notice that (Landau 1958, p. 91) for large $n$, $p_n < n^2$. For such $n$ the For1 loop is iterated at most $n$ times, for each iteration the For2 loop is iterated at most $n^2$ times, and for each of these iterations the For3 loop is iterated at most $n$ times. All the steps of the algorithm are simple assignment or comparison steps, the only exception is the test $p_i | k$ which needs at most $k (\leq n^2)$ simple steps. So the total running time is a polynomial in $n$, of degree at most $1 + 2 + 1 + 2 = 6$. 

75
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77
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