APPLICATION OF THE WEIL REPRESENTATION: DIAGONALIZATION OF THE DISCRETE FOURIER TRANSFORM

SHAMGAR GUREVICH AND RONNY HADANI

Abstract. We survey a new application of the Weil representation to construct a canonical basis Φ of eigenvectors for the discrete Fourier transform (DFT). The transition matrix Θ from the standard basis to Φ defines a novel transform which we call the discrete oscillator transform (DOT for short). In addition, we describe a fast algorithm for computing Θ in certain cases.

0. Introduction

The discrete Fourier transform (DFT) is probably one of the most important operators in modern science. It is omnipresent in various fields of discrete mathematics and engineering, including combinatorics, number theory, computer science and, last but probably not least, digital signal processing. Formally, the DFT is a family \( \{ F_N \} \) of unitary operators, where each \( F_N \) acts on the Hilbert space \( \mathcal{H}_N = \mathbb{C}(\mathbb{Z}/N\mathbb{Z}) \) by the formula

\[
F_N [f] (w) = \frac{1}{\sqrt{N}} \sum_{t \in \mathbb{Z}/N\mathbb{Z}} e^{\frac{2\pi i}{N} wt} f(t).
\]

Although so widely used, the spectral properties of the DFT remains to some extent still mysterious. For example, the calculation of the multiplicities of its eigenvalues, which was first carried out by Gauss, is quite involved and requires multiple number theoretic manipulations [1].

A primary motivation for studying the eigenvectors of the DFT comes from digital signal processing. Here, a function is considered in two basic realizations: The time realization and the frequency realization. Each realization, yields information on different attributes of the function. The DFT operator acts as a dictionary between these two realizations

\[
\text{Time } F_N \rightleftharpoons \text{Frequency.}
\]

From this point of view, it is natural to look for a diagonalization basis, namely, a basis of eigenvectors (eigen modes) for \( F_N \). In this regard, the main conceptual difficulty comes from the fact that the diagonalization problem is ill-defined, since, \( F_N \) is an operator of order 4, i.e., \( F_N^4 = Id \), which means that it has at most four eigenvalues \( \pm 1, \pm i \), therefore each appears with large multiplicity (assuming \( N \gg 4 \)).

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An interesting approach to the resolution of this difficulty, motivated from results in continuous Fourier analysis, was developed by Grünbaum in [8]. In that approach, a tridiagonal operator $S_N$ which commutes with $F_N$ and admits a simple spectrum is introduced. This enabled him to give a basis of eigenfunctions for the DFT. Specifically, $S_N$ appears as a certain discrete analogue of the differential operator $D = \partial^2_r - t^2$ which commutes with the continuous Fourier transform. Other approaches can be found in [5, 17].

0.1. Main results presented in this survey. In this survey we describe a representation theoretic approach to the diagonalization problem of the DFT in the case when $N = p$ is an odd prime number. Our approach, puts to the forefront the Weil representation [19] of the finite symplectic group $Sp = SL_2(F_p)$ as the fundamental object underlying harmonic analysis in the finite setting (see [6, 12, 13, 14, 15, 18] for other recent applications of this point of view).

Specifically, we exhibit a canonical basis $\Phi_p$ of eigenvectors for the DFT. We also describe the transition matrix $\Theta_p$ from the standard basis to $\Phi_p$, which we call the discrete oscillator transform (DOT for short). In addition, in the case $p \equiv 1 \pmod{4}$, we describe a fast algorithm for computing $\Theta_p$ (FOT for short). More precisely, we explain how in this case one can reduce the DOT to a composition of transforms with existing fast algorithms.

It is our general feeling that the Weil representation yields a transparent explanation to many classical results in finite harmonic analysis. In particular, we describe an alternative method for calculating the multiplicities of the eigenvalues for the DFT, a method we believe is more suggestive than the classical calculations.

The rest of the introduction is devoted to a more detailed account of the main ideas and results of this survey.

0.2. Symmetries of the DFT. Let us fix an odd prime number $p$ and for the rest of the introduction suppress the subscript $p$ from all notations.

Generally, when a (diagonalizable) linear operator $A$ has eigenvalues admitting large multiplicities, a philosophy for resolving the multiplicity is to look for a group $G = G_A \subset GL(\mathcal{H})$ of "hidden" symmetries consisting of operators which commute with $A$. Alas, usually the problem of computing the group $G$ is formidable and, in fact, equivalent to the problem of diagonalizing $A$. If the operator $A$ arises "naturally", there is a chance that the group $G$ can be effectively described. In favorable situations, $G$ is commutative and large enough so that all degeneracies are resolved and the spaces of common eigenvectors with respect to $G$ are one-dimensional. The basis of common eigenvectors with respect to $G$ establishes a distinguish choice of eigenvectors for $A$. Philosophically, we can say that it is more correct to consider from start the group $G$ instead of the single operator $A$.

Interestingly, the DFT operator $F$ admits a natural group of symmetries $G_F$, which, in addition, can be effectively described using the Weil representation. For the sake of the introduction, it is enough to know that the Weil representation in this setting is a unitary representation $\rho : Sp \to U(\mathcal{H})$ and the key observation is that $F$ is proportional to a single operator $\rho(w)$. The group $G_F$ is the image under $\rho$ of the centralizer subgroup $T_w$ of $w$ in $Sp$.

0.3. The algebraic torus associated to the DFT. The subgroup $T_w$ can be computed explicitly and is of a very "nice" type, it consists of rational points of a maximal algebraic torus in $Sp$ which concretely means that it is maximal
commutative subgroup in $Sp$, consisting of elements which are diagonalizable over some field extension. Restricting the Weil representation to the subgroup $T_w$ yields a collection $G_F = \{ \rho(g) : g \in T_w \}$ of commuting operators, each acts unitarily on the Hilbert space $H$ and commutes with $F$. This, in turn, yields a decomposition, stable under Fourier transform, into character spaces

$$H = \bigoplus H_\chi,$$

where $\chi$ runs in the set of (complex valued) characters of $T_w$, namely, $\phi \in H_\chi$ if and only if $\rho(g) \phi = \chi(g) \phi$. The main technical statement of this survey, Theorem 3, roughly says that $\dim H_\chi = 1$ for every $\chi$ which appears in (0.1).

0.4. **The oscillator transform.** Choosing a unit representative $\phi_\chi \in H_\chi$ for every $\chi$, gives the canonical basis $\Phi = \{ \phi_\chi \}$ of eigenvectors for $F$. The oscillator transform $\Theta$ sends a function $f \in H$ to the coefficients in the unique expansion

$$f = \sum a_\chi \phi_\chi.$$

The fine behavior of $F$ and $\Theta$ is governed by the (split type) structure of $T_w$, which changes depending on the value of the prime $p$ modulo 4. This has several consequences. In particular, it gives a transparent explanation to the precise way the multiplicities of the eigenvalues of $F$ depend on the prime $p$. Another, algorithmic, consequence is related to the existence of a fast algorithm for computing $\Theta$.

0.5. **Remarks.**

0.5.1. **Field extension.** All the results in this survey were stated for the basic finite field $F_p$, for the reason of making the terminology more accessible. In fact, all the results can be stated and proved in essentially the same way for any field extension $F_q$, $q = p^n$. One just need to replace $p$ by $q$ in all appropriate places. The only non-trivial statement in this respect is Corollary 1.

0.5.2. **Properties of eigenvectors.** The character vectors $\phi_\chi$ satisfy many interesting properties and are objects of study in their own right. A comprehensive treatment of this aspect of the theory appears in [13, 15].

0.5.3. **Proofs.** Complete proofs for the statements that appear in this survey will be given in [11].

0.6. **Structure of the survey.** The survey consists of four main parts sections except of the introduction.

- In Section [1] we start with the definition of the finite Heisenberg group and the Heisenberg representation. Then we introduce the Weil representation of the finite symplectic group. First it is described in abstract terms and then more explicitly invoking the idea of invariant presentation of an operator. We proceed to discuss the theory of tori in the one-dimensional Weil representation.
- In Section [2] we explain how to associate to a maximal torus $T \subset SL_2$ a transform $\Theta_T$ that we call the oscillator transform. We describe a fast algorithm for computing $\Theta_T$ in the case $T$ is a split torus. Then we apply the theory to the specific torus associated with the DFT operator.
• In Section 3, we describe another application of the formalism developed in Section 1, i.e., a treatment of the multiplicity problem for the DFT, from the representation theoretic perspective.
• In Appendix A, consisting of an explicit formula for the particular oscillator transform $\Theta_T^w$ associated with the DFT.

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1. The Heisenberg and Weil representations

1.1. The Heisenberg group. Let $(V, \omega)$ be a two-dimensional symplectic vector space over the finite field $\mathbb{F}_p$. The reader should think of $V$ as $\mathbb{F}_p \times \mathbb{F}_p$ with the standard form $\omega((t, w), (t', w')) = tw' - wt'$. Considering $V$ as an abelian group, it admits a non-trivial central extension called the Heisenberg group. Concretely, the group $H$ can be presented as the set $H = V \times \mathbb{F}_p$ with the multiplication given by

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\omega(v, v')).$$

The center of $H$ is $Z = Z(H) = \{(0, z) : z \in \mathbb{F}_p\}$. The symplectic group $Sp = Sp(V, \omega)$, which in this case is isomorphic to $SL_2(\mathbb{F}_p)$, acts by automorphism of $H$ through its action on the $V$-coordinate.

1.2. The Heisenberg representation. One of the most important attributes of the group $H$ is that it admits a special family of irreducible representations. The precise statement goes as follows. Let $\psi : Z \to \mathbb{C}^\times$ be a non-trivial character of the center. For example in this survey we take $\psi(z) = e^{2\pi iz}$. It is not hard to show

**Theorem 1** (Stone-von Neuman). There exists a unique (up to isomorphism) irreducible unitary representation $(\pi, H, \mathcal{H})$ with the center acting by $\psi$, i.e., $\pi|_Z = \psi \cdot Id_{\mathcal{H}}$.

The representation $\pi$ which appears in the above theorem will be called the Heisenberg representation.

1.2.1. Standard realization of the Heisenberg representation. The Heisenberg representation $(\pi, H, \mathcal{H})$ can be realized as follows: $\mathcal{H}$ is the Hilbert space $\mathbb{C}(\mathbb{F}_p)$ of complex valued functions on the finite line, with the standard Hermitian product. The action $\pi$ is given by

$\pi(t, 0)[f](x) = f(x + t);
\pi(0, w)[f](x) = \psi(wx) f(x);
\pi(z)[f](x) = \psi(z) f(x),$

for every $f \in \mathcal{H}$, $x, t, w \in \mathbb{F}_p$ and $z \in Z$.

We call this explicit realization the standard realization.
1.3. The Weil representation. A direct consequence of Theorem [1] is the existence of a projective representation \( \tilde{\rho} : Sp \rightarrow PGL(H) \). The construction of \( \tilde{\rho} \) out of the Heisenberg representation \( \pi \) is due to Weil [19] and it goes as follows. Considering the Heisenberg representation \( \pi \) and an element \( g \in Sp \), one can define a new representation \( \pi^g \) acting on the same Hilbert space via \( \pi^g(h) = \pi(g(h)) \). Clearly, both \( \pi \) and \( \pi^g \) have the same central character \( \psi \), hence, by Theorem [1], they are isomorphic. Since the space \( \text{Hom}_H(\pi, \pi^g) \) is one-dimensional, choosing for every \( g \in Sp \) a non-zero representative \( \tilde{\rho}(g) \in \text{Hom}_H(\pi, \pi^g) \) gives the required projective Weil representation. In more concrete terms, the projective representation \( \tilde{\rho} \) is characterized by the formula

\[
\tilde{\rho}(g) \pi(h) \tilde{\rho}(g^{-1}) = \pi(g(h)),
\]

for every \( g \in Sp \) and \( h \in H \). A more delicate statement is that there exists a unique lifting of \( \tilde{\rho} \) into a linear representation.

**Theorem 2.** The projective Weil representation uniquely lifts to a linear representation

\[
\rho : SL_2(\mathbb{F}_p) \longrightarrow GL(H),
\]

that satisfies equation (1.1).

The existence of a linearization \( \rho \) follows from a known fact [2] that any projective representation of \( SL_2(\mathbb{F}_p) \) can be linearized to an honest representation. The uniqueness follows from the well known fact that the group \( SL_2(\mathbb{F}_p) \), \( p \neq 3 \), is perfect, i.e., it has no non-trivial characters.

### 1.3.1. Invariant presentation of the Weil representation. Let us denote by \( C(H, \psi) \) the space of (complex valued) functions on \( H \) which are \( \psi \)-equivariant with respect to the action of the center, namely, a function \( f \in C(H, \psi) \) satisfies \( f(zh) = \psi(z) f(h) \) for every \( z \in Z, h \in H \). Given an operator \( A \in \text{End}(H) \), it can be written in a unique way as \( A = \pi(K_A) \), where \( K_A \in C(H, \psi^{-1}) \) and \( \pi \) denotes the extended action \( \pi(K_A) = \sum_{h \in H} K_A(h) \pi(h) \). The function \( K_A \) is called the kernel of \( A \) and it is given by the matrix coefficient

\[
K_A(h) = \frac{1}{\text{dim}H} Tr(A \pi(h^{-1})).
\]

In the context of the Heisenberg representation, formula (1.2) is usually referred to as the Weyl transform [20].

Using the Weyl transform one is able to give an explicit description of the Weil representation. The idea [9] is to write each operator \( \rho(g), g \in Sp \), in terms of its kernel function \( K_g = K_{\rho(g)} \in C(H, \psi^{-1}) \). The following formula is taken from [9]

\[
K_g(v, z) = \frac{1}{\text{dim}H} \sigma(-\det(\kappa(g) + I)) \psi\left(\frac{1}{4} \omega(\kappa(g) v, v) + z\right)
\]

for every \( g \in Sp \) such that \( g - I \) is invertible, where \( \sigma \) denotes the unique quadratic character (Legendre character) of the multiplicative group \( \mathbb{F}_p^* \) and \( \kappa \) is the Cayley transform \( \kappa(g) = \frac{g - I}{g + I} \).

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1. Uniquely, except in the case the finite field is \( \mathbb{F}_3 \). For the canonical choice in the latter case see [9].

2. A more direct proof of the linearity of the Weil representation exists in [9] [19].
Remark 1. Sketch of the proof of (1.3) (see details in [9]). First, one can easily show that
\[ K_g(v,z) = \frac{1}{\dim H} \mu_g \cdot \psi \left( \frac{1}{4} \omega(\kappa(g)v, v) + z \right), \]
for \( g \in Sp \) with \( g-I \) invertible, and for some \( \mu_g \in \mathbb{C} \). Second, for \( \rho \) to be a linear representation the kernel \( K_g : H \to \mathbb{C} \) should satisfy the multiplicativity property
\[ K_{g_1 \cdot g_2} = K_{g_1} \ast K_{g_2}, \]
where the \( \ast \) operation denotes convolution with respect to the Heisenberg group action. Finally, a direct calculation reveals that \( \mu_g \) must be equal to \( \sigma( - \det(\kappa(g) + I)) \) for (1.4) to hold. Hence, we obtain (1.3).

1.4. The theory of tori. A maximal (algebraic) torus in \( Sp \) is a maximal commutative subgroup which becomes diagonalizable over some field extension. There exists two conjugacy classes of maximal (algebraic) tori in \( Sp \). The first class consists of those tori which are diagonalizable already over \( \mathbb{F}_p \) or equivalently those are the tori that are conjugated to the standard diagonal torus
\[ A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p \right\}. \]

A torus in this class is called a split torus. The second class consists of those tori which become diagonalizable over a quadratic extension \( \mathbb{F}_{p^2} \) or equivalently those are tori which are not conjugated to \( A \). A torus in this class is called a non-split torus (sometimes it is called inert torus).

Example 1 (Example of a non-split torus). It might be suggestive to explain further the notion of non-split torus by exploring, first, the analogous notion in the more familiar setting of the field \( \mathbb{R} \). Here, the standard example of a maximal non-split torus is the circle group \( SO(2) \subset SL_2(\mathbb{R}) \). Indeed, it is a maximal commutative subgroup which becomes diagonalizable when considered over the extension field \( \mathbb{C} \) of complex numbers. The above analogy suggests a way to construct an example of a maximal non-split torus in the finite field setting as well.

Let us identify the symplectic plane \( V = \mathbb{F}_p \times \mathbb{F}_p \) with the quadratic extension \( \mathbb{F}_{p^2} \). Under this identification, \( \mathbb{F}_{p^2} \) acts on \( V \) and for every \( g \in \mathbb{F}_{p^2} \) we have \( \omega(gu, gv) = \det(g) \omega(u,v) \), which implies that the group
\[ T_{ns} = \left\{ g \in \mathbb{F}_p^\times : \det(g) = 1 \right\} \]
naturally lies in \( Sp \). The group \( T_{ns} \) is an example of a non-split torus which the reader might think of as the "finite circle".

1.4.1. Decompositions with respect to a maximal torus. Restricting the Weil representation to a maximal torus \( T \subset Sp \) yields a decomposition
\[ H = \bigoplus_\chi H_\chi, \]
where \( \chi \) runs in the set \( T^\vee \) of complex valued characters of the torus \( T \). More concretely, choosing a generator\(^3\) \( t \in T \), the decomposition (1.3) naturally corresponds to the eigenspaces of the linear operator \( \rho(t) \). The decomposition (1.3) depends on the split type of \( T \). Let \( \sigma_T \) denote the unique non-trivial character of \( T \) of order 2.

Theorem 3. If \( T \) is a split torus, we have

\(^3\)A maximal torus \( T \) in \( SL_2(\mathbb{F}_p) \) is a cyclic group, thus there exists a generator.
If $T$ is a non-split torus, we have
\[
\dim H_\chi = \begin{cases} 
1 & \chi \neq \sigma_T, \\
0 & \chi = \sigma_T.
\end{cases}
\]

2. The discrete oscillator transform

Let us fix a maximal torus $T$. Every vector $f \in H$ can be written uniquely as a direct sum $f = \sum f_\chi$ with $f_\chi \in H_\chi$ and $\chi$ runs in $I = \text{Spec}_T(H)$ - the spectral support of $H$ with respect to $T$ consisting of all characters $\chi \in T^\vee$ such that $\dim H_\chi \neq 0$. Let us choose, in addition, a collection of unit vectors $\phi_\chi \in H_\chi$, $\chi \in I$, and let $\phi = \sum \phi_\chi$. We define the transform $\Theta_T = \Theta_{T,\phi} : H \to \mathbb{C}(I)$ by
\[
\Theta_T[f](\chi) = \langle f, \phi_\chi \rangle.
\]
We will call the transform $\Theta_T$ the discrete oscillator transform (DOT for short) with respect to the torus $T$ and the test vector $\phi$.

**Remark 2.** We note that in the case $T$ is a non-split torus, $\Theta_T$ maps $H$ isomorphically to $\mathbb{C}(I)$. In the case $T$ is a split torus, $\Theta_T$ has a kernel consisting of $f \in H$ such that $\langle f, \phi_\sigma_T \rangle = 0$.

2.1. The oscillator transform (integral form). Let $\mathcal{M}_T : \mathbb{C}(T) \to \mathbb{C}(T^\vee)$ denote the Mellin transform
\[
\mathcal{M}_T[f](\chi) = \frac{1}{\#T} \sum_{g \in T} \overline{\chi}(g) f(g),
\]
for $f \in \mathbb{C}(T)$, where $\overline{\chi}$ denotes the complex conjugate of $\chi$. Let us denote by $m_T : H \to \mathbb{C}(T)$ the matrix coefficient $m_T[f](g) = \langle f, \rho(g^{-1}) \phi \rangle$ for $f \in H$.

**Lemma 1.** We have
\[
\Theta_T = \mathcal{M}_T \circ m_T.
\]

2.2. Fast oscillator transforms. In practice, it is desirable to have a "fast" algorithm for computing the oscillator transform (FOT for short). We work in the following setting. The vector $f$ is considered in the standard realization $H = \mathbb{C}(F_p)$ (see [1,2,4]). In this context the oscillator transform gives the transition matrix between the basis of delta functions and the basis $\{\phi_\chi\}$ of character vectors. We will show that when $T$ is a split torus and for an appropriate choice of $\phi$, the oscillator transform can be computed in $O(p \log(p))$ arithmetic operations. Principally, what we will show is that the computation reduces to an application of DFT followed by an application of the standard Mellin transform, both transforms admit a fast algorithm [4].

Assume $T$ is a split torus. Since all split tori are conjugated to one another, there exists, in particular, an element $s \in Sp$ conjugating $T$ with the standard diagonal torus $A$. In more details, we have a homomorphism of groups $Ad_s : T \to A$ sending $g \in T$ to $Ad_s(g) = sgs^{-1} \in A$. Dually, we have a homomorphism $Ad_s^\vee : A^\vee \to T^\vee$ between the corresponding groups of characters.

The main idea is to relate the oscillator transform with respect to $T$ with the oscillator transform with respect to $A$. The relation is specified in the following simple lemma.
Lemma 2. We have

\[(Ad^\vee)^* \circ \Theta_{T,\phi} = \Theta_{A,\rho(s)\phi} \circ \rho(s).\]

Remark 3. Roughly speaking, \[(2.1)\] means that (up to a “reparametrization” of \(T^\vee\) by \(A^\vee\) using \(Ad^\vee_s\)) the oscillator transform of a vector \(f \in \mathcal{H}\) with respect to the torus \(T\) is the same as the oscillator transform of the vector \(\rho(s)f\) with respect to the diagonal torus \(A\).

In order to finish the construction of the fast algorithm we need to recall some well-known facts about the Weil representation in the standard realization.

First, we recall that the group \(Sp\) admits a Bruhat decomposition \(Sp = B \cup BwB\), where \(B\) denote the Borel subgroup of lower triangular matrices and \(w\) denotes the Weyl element

\[w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\]

Furthermore, the Borel subgroup \(B\) can be written as a product \(B = AU = UA\), where \(A\) is the standard diagonal torus and \(U\) is the standard unipotent group.

Therefore, we can write the Bruhat decomposition also as \(Sp = UA \cup UwUA\).

Second, we give an explicit description of the operators in the Weil representation, associated with different types of elements in \(Sp\). The operators are specified up to a unitary scalar.

- The standard torus \(A\) acts by (normalized) scaling: An element \(g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\), acts by

  \[S_a[f](x) = \sigma(a) f(a^{-1}x).\]

- The group \(U\) of unipotent matrices acts by quadratic characters (chirps):

  An element \(g = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\), acts by

  \[M_b[f](x) = \psi(-\frac{b}{2}x^2)f(x).\]

- The Weyl element \(w\) acts by discrete Fourier transform

  \[F[f](y) = \frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p} \psi(yx)f(x).\]

The above formulas can be easily verified using identity \[(1.1)\]. Hence, we conclude that every operator \(\rho(g), g \in Sp\), can be written either in the form \(\rho(g) = M_b \circ S_a\) or in the form \(\rho(g) = M_{b_2} \circ F \circ M_{b_1} \circ S_a\). In particular, given a function \(f \in \mathbb{C}(\mathbb{F}_p)\), applying formula \[(2.1)\] with \(\phi = \rho(s)^{-1}\delta_1\) yields

\[(2.3) \quad \Theta_{T,\phi}[f](Ad^\vee_s(\chi)) = \frac{1}{p-1} \sum_{a \in \mathbb{F}_p^\times} \sigma(a) \mathcal{T}(a) \rho(s)[f](a),\]

for every \(\chi \in A^\vee\), where \(s\) is the specific element that conjugates \(T\) to \(A\).

In conclusion, formula \[(2.3)\] implies that \(\Theta_{T,\phi}[f]\) can be computed by, first, applying the (DFT type) operator \(\rho(s)\) to \(f\) and then applying Mellin transform
APPLICATION OF THE WEIL REPRESENTATION: DIAGONALIZATION OF THE DFT

Problem 1. Does there exists a fast algorithm for computing the oscillator transform associated to a non-split torus?

2.3. Diagonalization of the discrete Fourier transform. In this subsection we apply the previous development in order to exhibit a canonical basis of eigenvectors for the DFT. We will show that the DFT can be naturally identified (up to a normalization scalar) with an operator $\rho(w)$ in the Weil representation, where $w$ is an element in a maximal torus $T_w \subset Sp$. We take $w \in Sp$ to be the Weyl element $\langle 2.2 \rangle$.

Theorem 4. We have

$$F = C \cdot \rho(w),$$

where $C = i^{\frac{p-1}{2}}$.

Corollary 1. In case one considers a field extension of the form $\mathbb{F}_q$, $q = p^d$ and the additive character $\psi_q : \mathbb{F}_q \to \mathbb{C}^\times$, given by $\psi_q(x) = \psi(q \cdot x)$. In this case, the Fourier transform $F$ associated with $\psi_q$ and the operator $\rho(w)$ coming from the Weil representation of $SL_2(\mathbb{F}_q)$, are related by

$$F = C \cdot \rho(w),$$

with $C = (-1)^{d-1}i^{\frac{p-1}{2}}$.

Theorem 4 implies that the diagonalization problems of the operators $F$ and $\rho(w)$ are equivalent. The second problem can be approached using representation theory, which is what we are going to do next.

Let us denote by $T_w$ the centralizer of $w$ in $Sp$, namely $T_w$ consists of all elements $g \in Sp$ such that $gw = wg$, in particular we have that $w \in T_w$. Proposition 1. The group $T_w$ is a maximal torus. Moreover the split type of $T_w$ depends on the prime $p$ in the following way: $T_w$ is a split torus when $p \equiv 1 \pmod{4}$ and is a non-split torus when $p \equiv 3 \pmod{4}$.

Proposition 1 has several consequences. First consequence is that choosing a unit character vector $\phi_\chi \in \mathcal{H}_\chi$ for every $\chi \in \text{Spec}_{T_w}(\mathcal{H})$ gives a canonical (up to normalizing unitary constants) choice of eigenvectors for the DFT. A second, more mysterious consequence is that although the formula of the DFT is uniform in $p$, its qualitative behavior changes dramatically between the cases when $p \equiv 1 \pmod{4}$ and $p \equiv 3 \pmod{4}$. This is manifested in the structure of the group of symmetries: In the first case, the group of symmetries is a split torus consisting of $p-1$ elements and in the second case it is a non-split torus consisting of $p+1$ elements. It also seems that the structure of the symmetry group is important from the algorithmic perspective, in the case $p \equiv 1 \pmod{4}$ we built a fast algorithm for computing $\Theta = \Theta_{T_w}$, while in the case $p \equiv 3 \pmod{4}$ the existence of such an algorithm remains open (see Problem 1).

For the convenience of the reader, we enclose in Appendix A an explicit formula for $\Theta_{T_w}$ in the case $p \equiv 1 \pmod{4}$.

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4In the case $T_w$ is a split torus there is a slight ambiguity in the choice of a character vector with respect to $\sigma_{T_w}$. This ambiguity can be resolved by further investigation which we will not discuss here.
3. Multiplicities of eigenvalues of the DFT

Considering the group $T_w$ we can give a transparent computation of the eigenvalues multiplicities for the operator $\rho(w)$. First we note that, since $w$ is an element of order 4, the eigenvalues of $\rho(w)$ lies in the set $\{\pm 1, \pm i\}$. For $\lambda \in \{\pm 1, \pm i\}$, let $m_\lambda$ denote the multiplicity of the eigenvalue $\lambda$. We observe that

$$m_\lambda = \bigoplus_{\chi \in I_\lambda} \dim H_\chi,$$

where $I_\lambda$ consists of all characters $\chi \in \text{Spec}_{T_w}(H)$ such that $\chi(w) = \lambda$. The result now follows easily from Theorem 3, applied to the torus $T_w$. We treat separately the split and non-split cases.

- **Assume $T_w$ is a split torus**, which happens when $p = 1 \pmod{4}$, namely, $p = 4l + 1$, $l \in \mathbb{N}$. Since $\dim H_\chi = 1$ for $\chi \neq \sigma_T(w)$ it follows that $m_{\pm 1} = \frac{p-1}{4} = l$. We are left to determine the values of $m_{\pm 1}$, which depend on whether $\sigma_T(w) = w^{\frac{p}{2} \pm 1}$ is 1 or −1. Since $w$ is an element of order 4 in $T_w$ we get that

$$\sigma_T(w) = \begin{cases} 1, & p \equiv 1 \pmod{8}, \\ -1, & p \equiv 5 \pmod{8}, \end{cases}$$

which implies that when $p \equiv 1 \pmod{8}$ then $m_1 = l + 1$ and $m_{-1} = l$, and when $p \equiv 5 \pmod{8}$ then $m_1 = l$ and $m_{-1} = l + 1$.

- **Assume $T_w$ is a non-split torus**, which happens when $p \equiv 3 \pmod{4}$, namely, $p = 4l + 3$, $l \in \mathbb{N}$. Since $\dim H_\chi = 1$ for $\chi \neq \sigma_T(w)$ it follows that $m_{\pm 1} = \frac{p+1}{4} = l + 1$. The values of $m_{\pm 1}$ depend on whether $\sigma_T(w) = w^{\frac{p}{2} \pm 1}$ is 1 or −1. Since $w$ is an element of order 4 in $T_w$ we get that

$$\sigma_T(w) = \begin{cases} 1, & p \equiv 7 \pmod{8}, \\ -1, & p \equiv 3 \pmod{8}, \end{cases}$$

which implies that when $p \equiv 7 \pmod{8}$ then $m_1 = l$ and $m_{-1} = l + 1$, and when $p \equiv 3 \pmod{8}$ then $m_1 = l + 1$ and $m_{-1} = l$.

Summarizing, the multiplicities of the operator $\rho(w)$ are

| $p = 8k + 1$ | $2k + 1$ | $2k$ | $2k$ | $2k$ |
|-----------|------------|------|------|------|
| $p = 8k + 3$ | $2k$ | $2k + 1$ | $2k + 1$ | $2k + 1$ |
| $p = 8k + 5$ | $2k + 1$ | $2k + 2$ | $2k + 1$ | $2k + 1$ |
| $p = 8k + 7$ | $2k + 2$ | $2k + 2$ | $2k + 2$ | $2k + 2$ |

(3.1)

Considering now the DFT operator $F$. If we denote by $n_\mu$, $\mu \in \{\pm 1, \pm i\}$ the multiplicity of the eigenvalue $\mu$ of $F$ then the values of $n_\mu$ can be deduced from table 3.1 by invoking the relation $n_\mu = m_\lambda$ where $\lambda = i^{\frac{p-1}{2}} \cdot \mu$ (see Theorem 4).

Summarizing, the multiplicities of the DFT are

| $p = 4l + 1$ | $l + 1$ | $l$ | $l$ |
|------------|---------|-----|-----|
| $p = 4l + 3$ | $l + 1$ | $l + 1$ | $l + 1$ |

For a comprehensive treatment of the multiplicity problem from a more classical point of view see [1]. Other applications of Theorem 4 appear in [12].
APPLICATION OF THE WEIL REPRESENTATION: DIAGONALIZATION OF THE DFT

Appendix A. Explicit formulas for the oscillator transform

Looking at formula (2.3) of the oscillator transform $\Theta_{T_w}$, in the case $p \equiv 1 \pmod{4}$, we see that in order to have an explicit description we need to describe the operator $\rho(s)$, where $s \in Sp$ is an element which conjugates the torus $T_w$ to the standard torus $A$.

It is enough to describe $\rho(s)$ up to a unitary scalar, which is what we are going to do. Since, we assume that $p \equiv 1 \pmod{4}$, there exist an element $\epsilon \in \mathbb{F}_p^\times$, which satisfy $\epsilon^2 = -1$. Let

$$s = \left( \begin{array}{cc} \frac{1}{\epsilon} & \frac{\epsilon}{2} \\ \epsilon & 1 \end{array} \right).$$

Direct verification reveals that $sT_w s^{-1} = A$. Now, the element $s$ can be decomposed according to the Bruhat decomposition

$$s = \left( \begin{array}{cc} \frac{1}{\epsilon} & \frac{\epsilon}{2} \\ \epsilon & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ \frac{\epsilon}{2} & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \frac{\epsilon}{2} & 0 \end{array} \right),$$

which implies, using the explicit formulas which appears in Section 2.2, that $\rho(s) = C \cdot M_{\frac{1}{\epsilon}} \circ F \circ M_\frac{\epsilon}{2} \circ S_{\frac{\epsilon}{2}}$, where $C$ is some unitary scalar.

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