Screening of the Coulomb potential in superstrong magnetic field: atomic levels and spontaneous production of positrons

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Abstract

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1 Introduction

Loop corrections modify the Coulomb potential: electron loop insertion into the photon propagator leads to the Uehling–Serber correction to the electric potential of point-like nuclei [1]. Though it leads to an important phenomenon contributing to the Lamb shift of the energies of atomic electrons numerically the shift being of the order of $\alpha^5 m_e$ is small (we are using Gauss system of units, where $\alpha = e^2 = 1/137$ and in all formulas $\hbar = c = 1$ is implied).

Analogous correction in the case of external magnetic field qualitatively change the behavior of atomic energies: in particular the energy of the ground level remains finite in the limit $B \to \infty$; also spontaneous production of positrons becomes possible only for nuclei with $Z \geq 52$. Without taking radiative corrections into account in the limit of infinite magnetic field energy of ground atomic level tends to minus infinity and point-like nucleus with any $Z$ becomes critical at large enough $B$. At magnetic fields $B > (Ze^2)^2 m_e^2 / e$ the characteristic size of the electron wave function in the transverse to the magnetic field direction $a_H = 1/\sqrt{eB}$ (the so-called Landau radius) becomes smaller than Bohr radius $a_B = 1/(Ze^2 m_e)$ making the Coulomb problem essentially one-dimensional. Singularity of the Coulomb potential in $d = 1$ is stronger than in $d = 3$. In $d = 1$ the energy of the ground level is unbounded from below: the “fall to the center” phenomenon occurs. In the case of external magnetic field the singularity is cured by the finite value of $a_H$: at $|z| \lesssim a_H$ the Coulomb problem remains three dimensional. This is the reason why ground level goes down when $B$ grows. At superstrong magnetic fields $B \gtrsim 3\pi m_e^2 / e^3$ radiative corrections screen the Coulomb potential at short distances $|z| \lesssim 1/m_e$ and the freezing of a ground state energy occurs: it remains finite at $B \to \infty$ ([2], [3], [4]). For $Z \geq 52$ the value of freezing energy is below $-m_e$, so the ground level enters lower continuum when $B$ increases and spontaneous production of $e^+e^-$ pair
from vacuum becomes energetically possible and thus takes place. In this process electron occupies ground level while positron is emitted to infinity. For $Z < 52$ freezing energy is above $-m_e$ and spontaneous positron production does not occur [5].

There exists the direct correspondence between radiative corrections to the Coulomb potential in $d = 3$ case in strong magnetic field $B > B_0 \equiv m_e^2/e$ and radiative corrections to the Coulomb potential in $d = 1$ QED. That is why we start our presentation (in Section 2) from the analysis of the Coulomb potential in $D = 2$ QED of massive fermions. When these fermions are light, $g^2 > m^2$, the exponential screening of the Coulomb potential at short distances occurs. In the limit $m \to 0$ (massless $D = 2$ QED, the so-called Schwinger model) this exponential screening occurs at all distances because photon gets mass, $m_\gamma = 2g [6]$. In Section 3 we analyze radiative corrections to the Coulomb potential in $D = 4$ QED in external magnetic field. The role of the coupling constant $g^2$ here plays the product $e^3B$, and for $e^3B > m^2$ the screening of the Coulomb potential occurs as well. In Section 4 the structure of atomic levels on which the lowest Landau level (LLL) in the presence of atomic nucleus splits is determined. In Section 5 the Dirac equation for hydrogen-like ion at superstrong magnetic field will be derived and effect of screening will be studied for $Z = 1$. In Section 6 the influence of the screening of the Coulomb potential on the values of critical nuclei charges is discussed. In Section 7 the obtained results are summarized.

Let us finish the Introduction discussing the numerical values of magnetic fields we are dealing with in these lectures. The magnetic field at which the Bohr radius of a hydrogen atom becomes equal to Landau radius is $B_a = e^3 m_e^2 \approx 2 \cdot 10^9$ gauss, which is much larger than a magnetic field ever made artificially on Earth: $B_{\text{lab}} \approx 3 \cdot 10^7$ gauss. An interest to the atomic spectrum in the magnetic fields $B > B_a$ was triggered by the experiments with semiconductors, where electron-hole bound system called exciton is formed. Both effective charge and mass of electrons in semiconductors are much lower than in vacuum making $B_a$ in kilogauss scale reachable.

The so-called Schwinger magnetic field $B_0 = m_e^2/e \approx 4.4 \cdot 10^{13}$ gauss and magnetic field at which the screening of the Coulomb potential occurs $B \approx 3\pi m_e^2 / e^3 \approx 6 \cdot 10^{16}$ gauss should be compared with the magnetic fields at pulsars $\sim 10^{13}$ gauss and magnetars $\sim 10^{15}$ gauss. Although the application of the results obtained in the condensed matter physics (say graphene, where the mass of charge carrier can be arbitrary low while the value of charge approach one) can not be excluded, our main interest in the problem considered is purely theoretical.

## 2 The Coulomb potential in $D = 2$ QED of massive fermions

Summing up diagrams shown in Fig. 1 we get the following formula for the potential of point-like charge:

$$\Phi(k) \equiv A_0(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)}, \quad \Pi_{\mu\nu} \equiv (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\Pi(k^2),$$

where $\Pi(k^2)$ is the one-loop expression for the photon polarization operator. It can be obtained from the textbook expression for the polarization operator calculated with the help
of dimensional regularization \([7]\) in the limit \(D \to 2\):

\[
\Pi(k^2) = 4g^2 \left[ \frac{1}{\sqrt{t(1+t)}} \ln(\sqrt{1+t} + \sqrt{t}) - 1 \right] \equiv -4g^2 P(t) ,
\]

(2)

where \(t \equiv -k^2/4m^2\), \([g] = \text{mass}\).

\[\begin{array}{c}
\text{Diagram 1} + \text{Diagram 2} + \ldots
\end{array}\]

Fig. 1. \textit{Modification of the Coulomb potential due to the dressing of the photon propagator.}

Let us note that \(\Pi(k^2)\) is finite though corresponding integral \(\sim \int d^2p/p^2\) is divergent in ultraviolet. The point is that the trace of gamma matrices which multiplies divergent integral is zero. In dimensional regularization the trace is proportional to \(D - 2\) while ultraviolet divergency of integral over virtual momentum produces the factor \(1/(D - 2)\) and the product of these two factors is finite.

In order to obtain an expression for the Coulomb potential in the coordinate representation we take \(k = (0, k\parallel)\) and make the Fourier transformation:

\[
\Phi(z) = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik\parallel z}dk\parallel/2\pi}{k^2 + 4g^2 P(k^2/4m^2)} .
\]

(3)

The potential energy for the charges \(+g\) and \(-g\) is

\[
V(z) = -g\Phi(z) .
\]

(4)

The integral in (3) cannot be expressed through elementary functions. However it appears possible to find an interpolating formula for \(P(t)\) which has good accuracy and is simple enough for the Fourier transformation to be performed analytically.

The asymptotics of \(P(t)\) are:

\[
P(t) = \begin{cases} 
\frac{2t}{3} & \text{if } t \ll 1 \\
1 & \text{if } t \gg 1 .
\end{cases}
\]

(5)

Let us take as an interpolating formula for \(P(t)\) the following expression:

\[
\mathcal{P}(t) = \frac{2t}{3 + 2t} .
\]

(6)

The accuracy of this approximation is not worse than 10\% for the whole interval of \(t\) variation, \(0 < t < \infty\). Substituting an interpolating formula in (3) we get:

\[
\Phi = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik\parallel z}dk\parallel/2\pi}{k^2 + 4g^2(2k^2/3m^2)/(3 + k^2/2m^2)} = \\
= \frac{4\pi g}{1 + 2g^2/3m^2} \int_{-\infty}^{\infty} \left[ \frac{1}{k^2} + \frac{2g^2/3m^2}{k^2 + 6m^2 + 4g^2} \right] \frac{e^{ik\parallel z}dk\parallel}{2\pi} = \\
= \frac{4\pi g}{1 + 2g^2/3m^2} \left[ -\frac{1}{2} |z| + \frac{g^2/3m^2}{\sqrt{6m^2 + 4g^2}} \exp(-\sqrt{6m^2 + 4g^2}|z|) \right] .
\]

(7)
In the case of heavy fermions \((m \gg g)\) the potential is given by the tree level expression; the corrections are suppressed as \(g^2/m^2\).

In the case of light fermions \((m \ll g)\):

\[
\Phi(z) \bigg|_{m \ll g} = \begin{cases} 
\pi e^{-2gz} & , \; z \ll \frac{1}{g} \ln \left( \frac{2g}{m} \right) \\
-2\pi g \left( \frac{2m^2}{g^2} \right) |z| & , \; z \gg \frac{1}{g} \ln \left( \frac{2g}{m} \right)
\end{cases}
\]

(8)

\(m = 0\) corresponds to Schwinger model; photon gets mass.

Light fermions make transition from \(m > g\) to \(m = 0\) continuous. In the case of light fermions the Coulomb potential in \(D = 2\) QED is screened at distances \(|z| \gtrsim 1/(2g)\).

In Fig. 2 the potential energy for \(g = 0.5\), \(m = 0.1\) is shown. It is normalized to \(V(0) = 0\).

![Potential energy of the charges +g and −g in D = 2. The solid curve corresponds to P; the dashed curve corresponds to P.](image)

3 The Coulomb potential in \(D = 4\) QED in superstrong magnetic field

In order to find the potential of a point-like charge we need an expression for photon polarization operator in the external magnetic field \(B\). Long ago an expression for the electron propagator in constant and homogeneous external magnetic field was found by Schwinger [8] as a parametric integral. For \(B \gg B_0 \equiv m_e^2/e\) integration can be easily performed and compact expression for \(G(k)\) follows. Using it one obtains an analytic expression for a photon polarization operator (see for example [9]).

To understand the reason for great simplification of the expression for the electron propagator in the limit \(B \gg B_0\) one should start from the spectral representation of the propagator. The solutions of Dirac equation in the homogeneous constant in time \(B\) are known, so
one can write the spectral representation of the electron Green function. The denominators contain \(k^2 - m^2 - 2neB\), and for \(B >> m^2/e\) and \(k_\parallel << eB\) in sum over levels the lowest Landau level (LLL, \(n = 0\)) dominates. In the coordinate representation the transverse part of LLL wave function is: \(\Psi \sim \exp((-x^2 - y^2)eB)\) which in the momentum representation gives \(\Psi \sim \exp((-k_x^2 - k_y^2)/eB)\) (we suppose that \(B\) is directed along the \(z\) axis).

Substituting the electron Green functions we get the expression for the polarization operator in superstrong \(B\).

For \(B >> B_0\), \(k_\parallel << eB\) the following expression is valid [10]:

\[
\Pi_{\mu\nu} \sim e^2eB \int \frac{dq_x dq_y}{eB} \exp\left(-\frac{q_x^2 + q_y^2}{eB}\right) * \\
* \exp\left(-\frac{(q + k)^2}{eB}\right) dq_0 dq_z \gamma_\mu \frac{1}{q_{0,z}} (1 - i\gamma_1 \gamma_2) \gamma_\nu * \\
* \frac{1}{q_{0,z} + k_{0,z} - m} \left(1 - i\gamma_1 \gamma_2\right) = e^3 B * \exp\left(-\frac{k_\parallel^2}{2eB}\right) \Pi^{(2)}_{\mu\nu} (k_\parallel \equiv k_z).
\]

With the help of it the following result was obtained in [4]:

\[
\Phi(k) = \frac{4\pi e}{k_\parallel^2 + k_\perp^2 + \frac{2eB}{\pi} \exp\left(-\frac{k_\parallel^2}{2eB}\right) P\left(\frac{k_\parallel^2}{4m_e^2}\right)},
\]

\[
\Phi(z) = 4\pi e \int \frac{e^{ik_\parallel z} dk_\parallel d^2 k_\perp (2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2eB}{\pi} \exp\left(-\frac{k_\perp^2}{2eB}\right) (k_\parallel^2/2m_e^2)/(3 + k_\parallel^2/2m_e^2)} = \\
= \frac{e}{|z|} \left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3B+6m_e^2}|z|}\right].
\]

For the magnetic fields \(B \ll 3\pi m^2/e^3\) the potential is Coulomb up to small power suppressed terms:

\[
\Phi(z) \Bigg| e^3B \ll m_e^2 = \frac{e}{|z|} \left[1 + O\left(\frac{e^3B}{m_e^2}\right)\right]
\]

in full accordance with the \(D = 2\) case with the substitution \(e^3B \rightarrow g^2\).

In the opposite case of the superstrong magnetic fields \(B >> 3\pi m_e^2/e^3\) we get:

\[
\Phi(z) = \begin{cases} 
\frac{e}{|z|} e^{-\sqrt{(2/\pi)e^3B}|z|}, & |z| \leq \frac{1}{\sqrt{(2/\pi)e^3B}} \\
\frac{1}{m_e} \ln \left(\sqrt{\frac{e^3B}{3\pi m_e^2}}\right), & \frac{1}{m_e} > |z| > \frac{1}{\sqrt{(2/\pi)e^3B}} \\
\frac{e}{|z|}, & |z| > \frac{1}{m_e}
\end{cases}
\]

\[
\tilde{V}(z) = -e \Phi(z).
\]

The Coulomb potential is screened at short distances \(1/m_e > |z| > 1/\sqrt{e^3B} \equiv a_H/e\).

In Fig. 3 the plot of a modified by the superstrong magnetic field Coulomb potential as well as its short- and long-distance asymptotics are presented.
Let us find the 3-dimensional shape of the screened Coulomb potential. The behavior of the potential in the transverse plane ($z = 0$) can be found analytically in the limit $B \gg 3\pi m_e^2/e^3$ from the general expression:

$$\Phi(z, \rho) = 4\pi e \int \frac{e^{i\vec{k}_\perp \cdot \rho + ik_\parallel z} d\vec{k}_\parallel d^2k_\perp}{k_\parallel^2 + k_\perp^2 + 2e^3B/\pi \exp(-k_\perp^2/2eB)} e^{3\pi m_e^2/e}.$$  

(15)

neglecting exponent in the denominator, which is valid for $\rho \gtrsim a_B$:

$$\Phi(0, \rho) = \begin{cases} \frac{\epsilon}{\rho} \exp\left(-\sqrt{\frac{2}{\pi}} e^3 B \rho\right), & \rho < \frac{1}{\sqrt{(2/\pi) e^3 B}} \ln \sqrt{\frac{e^3 B}{3\pi m_e^2}} \\ \sqrt{\frac{3\pi m_e^2}{e^3 B}} \rho, & \frac{1}{\sqrt{(2/\pi) e^3 B}} \ln \sqrt{\frac{e^3 B}{3\pi m_e^2}} < \rho \end{cases}$$  

(16)

and the Coulomb potential is screened at large $\rho$ in complete analogy with the $D = 2$ case.

For $|z| \gg 1/m_e$ in the integral (15) the values $|k_\parallel| \ll m_e$ dominate and we get:

$$\Phi(\rho, z) \bigg|_{z \gg 1/m_e} = \frac{e}{\sqrt{z^2 + (1 + \frac{e^3 B}{3\pi m_e^2}) \rho^2}}.$$  

(17)
Fig. 4. The equipotential lines at $B/B_0 = 10^4$. The dashed line corresponds to $\sqrt{z^2 + \rho^2} = \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln \left( \frac{e^3 B}{3\pi m_e^2} \right)$.

In Fig. 4 the equipotential lines are shown. The behavior of the screened Coulomb potential in the transverse plane was found numerically in [2].

Finally for $3\pi m^2/e^3 \gg B > B_0$ expanding (17) we get:

$$\Phi(r) = \frac{e}{r} \left\{ 1 - \frac{\alpha}{6\pi} \left( \frac{B}{B_0} \right) \sin^2 \theta \right\}, \quad \theta = \vec{r} \cdot \vec{B}, \quad r \equiv \sqrt{\rho^2 + z^2} \gg 1/m_e,$$

which coincides with the result obtained in [10] where the expression for the photon polarization operator at $B > B_0$ was obtained as well.

The expression for the screened Coulomb potential was obtained from the one-loop contribution to the photon polarization operator in the external magnetic field $B \gg B_0$. In momentum space it looks like:

$$\Phi(k||, k_0 = k_\perp = 0) = \frac{4\pi e}{k^2|| + \frac{2e^3 B}{\pi} \frac{k^2||}{k^2|| + 6m^2_e}},$$

If higher loops contain the terms $\sim e^3 B (e^3 B/k^2||)^{n-1}$ they will drastically change the shape of the potential in the coordinate space.

To calculate the radiative corrections one should use the electron propagator in an external homogeneous magnetic field $B$. The spectral representation of the electron propagator is a sum over Landau levels and for $B \gg B_0$ the contribution of the lowest level dominates [10 11]:

$$G(k) = e^{-k^2_\perp/eB} \frac{1}{1 - i\gamma_1 \gamma_2} \frac{k^2_0 + m_e}{k^2_{0,3} - m^2_e},$$
where the projector \((1-i\gamma_1\gamma_2)\) selects the virtual electron state with its spin oriented opposite to the direction of the magnetic field \(\vec{B} = (0, 0, B)\). The contributions of the excited Landau levels to \(G\) yield a term in the denominator proportional to \(eB\) and they produce a correction of order \(e^2 \equiv \alpha\) in the denominator of (19).

Two kinds of terms contribute to the polarization operator at the two-loop level. First, there are terms in the electron propagators which represent the contributions of higher Landau levels. Just like in the one-loop case they produce corrections suppressed as \(e^2\) in the denominator of (19), i.e. terms of the order \(e^5B\) which can be safely neglected in comparison with the leading \(\sim e^3B\) term. Second, there is the contribution from the leading term in the electron propagator, given by (20). Let us consider the simplest diagram: the photon dressing of the electron propagator. Neglecting the electron mass we get:

\[
\gamma_\mu (1 - i\gamma_1\gamma_2) \hat{k}_{0,3}\gamma_\mu = -2[\hat{k}_{0,3} - i\hat{k}_{0,3}\gamma_1\gamma_2] = -2\hat{k}_{0,3}(1 + i\gamma_1\gamma_2),
\]

which gives zero when multiplying external propagator of electron, since \((1 + i\gamma_1\gamma_2)(1 - i\gamma_1\gamma_2) = 0\). This result is a manifestation of the following well-known fact: in massless QED in \(D = 2\) (Schwinger model) all loop diagrams are zero except the one-loop term in the photon polarization operator (see for example [12]). That is why the contributions of the second kind are of the order of \(\alpha(e^3B) \left( m_e^2k^2/(k^2 + m_e^2)^2 \right)\) and they are not important.

The generalization of the above arguments to higher loops is straightforward. Let us note that absence of higher loop corrections to polarization operator in Schwinger model is related to the absence of renormalization of axial anomaly by higher loops. In \(D = 2\) anomaly is given by correlator of two currents and axial current is proportional to vector current (see for example [13]).

### 4 Atomic levels in superstrong \(B\)

We are interested in the spectrum of a hydrogen-like ion in a very strong magnetic field \(B\). We will write all formulas for hydrogen since their generalization for \(Z > 1\) is straightforward.

In the absence of magnetic field the spatial size of the wave function of the ground state atomic electron is characterized by the Bohr radius \(a_B = 1/(m_e e^2)\), its energy equals \(E_0 = -m_e e^4/2 \equiv -Ry\), where \(Ry\) is the Rydberg constant. The transverse (with respect to \(B\)) size of the ground state of the electron wave function in an external magnetic field \(B\) is characterized by the Landau radius \(a_H = 1/\sqrt{eB}\). The Larmour frequency of the electron precession is \(\omega_L = eB/m_e\). For a magnetic field \(B_a = e^3m_e^2 = 2.35 \cdot 10^9\) gauss called “atomic magnetic field”, these sizes and energies are close to each other: \(a_B = a_H, E_0 \sim \omega_L\). We wish to study the spectrum of the hydrogen atom in magnetic fields much larger than \(B_a\). In this case the motion of the electron is mainly controlled by the magnetic field: it makes many oscillations in this field before it makes one in the Coulomb field of the nucleus. This is the condition for applicability of the adiabatic approximation, used for this problem for the first time in [14].

The spectrum of a Dirac electron in a pure magnetic field is well known [15]; it admits a continuum of energy levels due to the free motion along the field:

\[
\varepsilon_n^2 = m_e^2 + p_z^2 + (2n + 1 + \sigma_z)eB, \tag{22}
\]

where \(n = 0, 1, 2, \ldots; \sigma_z = \pm 1\) is the spin projection of the electron on \(z\) axis multiplied by two. For magnetic fields larger than \(B_0 = m_e^2/e\), the electrons are relativistic with only one
exception: electrons belonging to the lowest Landau level (LLL, $n = 0$, $\sigma_z = -1$) can be non-relativistic.

In what follows we will study the spectrum of electrons from LLL in the Coulomb field of the proton modified by the superstrong $B$. The solution can be found in [16] of the Schrödinger equation for an electron in a constant in time homogeneous magnetic field $B$ in the gauge in which $\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$ in cylindrical coordinates ($\rho, z$). The electron energies are:

$$E_{p,n_\rho m \sigma_z} = \left( n_\rho + \frac{|m| + m + 1 + \sigma_z}{2} \right) \frac{eB}{m_e} + \frac{p_z^2}{2m_e},$$

where $n_\rho = 0, 1, 2, \ldots$ is the number of nodal surfaces, $m = 0, \pm 1, \pm 2, \ldots$ is the electron orbital momentum projection on the $z$ axis (direction of the magnetic field) and $\sigma_z = \pm 1$. According to [16], the LLL wave functions are:

$$R_{0m}(\rho) = \left[ \pi (2a_H^2)^{1/2} \right]^{1/2} \rho^{|m|} e^{i m \varphi - \rho^2/(4a_H)} ,$$

$$\rho = |\vec{\rho}|, \quad \int |R_{0m}(\rho)|^2 d^2 \rho = 1, \quad m = 0, -1, -2, \ldots$$

We should now take into account the electric potential of the atomic nucleus located at $\vec{\rho} = z = 0$. For $a_H \ll a_B$ the adiabatic approximation can be used and the wave function should be looked for in the following form:

$$\Psi_{n0m(-1)} = R_{0m}(\rho)\chi_n(z) ,$$

where $\chi_n(z)$ is the solution of a Schrödinger equation for an electron motion along the direction of the magnetic field:

$$\left[ -\frac{1}{2m_e} \frac{d^2}{dz^2} + U_{\text{eff}}(z) \right] \chi_n(z) = E_n \chi_n(z) .$$

Without screening the effective potential is given by the following formula:

$$U_{\text{eff}}(z) = -e^2 \int \frac{|R_{0m}(\rho)|^2}{\sqrt{\rho^2 + z^2}} d^2 \rho ,$$

which becomes the Coulomb potential for $|z| \gg a_H$

$$U_{\text{eff}}(z) \bigg|_{z \gg a_H} = -\frac{e^2}{|z|}$$

and is regular at $z = 0$

$$U_{\text{eff}}(0) \sim -\frac{e^2}{|a_H|} .$$

To take screening into account we must use (11) to modify (27) (see below). Since $U_{\text{eff}}(z) = U_{\text{eff}}(-z)$, the wave functions are odd or even under reflection $z \rightarrow -z$; the ground states (for $m = 0, -1, -2, \ldots$) are described by even wave functions.

In Fig. 5 the different scales important in the consideration of the hydrogen atom in strong magnetic field are shown.
To calculate the ground state of hydrogen atom in [17] the shallow-well approximation is used:

\[ E_{sw} = -2m_e \left[ \int_{a_H}^{a_B} U(z)dz \right]^2 = -(m_em^4/2)ln^2(B/(m^2_em^3)) \]  (30)

Let us derive this formula. The starting point is the one-dimensional Schrödinger equation:

\[ -\frac{1}{2\mu} \frac{d^2}{dz^2} \chi(z) + U(z)\chi(z) = E_0\chi(z) \]  (31)

Neglecting \( E_0 \) in comparison with \( U \) and integrating (31) we get:

\[ \chi'(a) = 2\mu \int_{0}^{a} U(z)\chi(z)dz , \]  (32)

where we assume \( U(z) = U(-z) \), that is why \( \chi \) is even.

The next assumptions are: 1. the finite range of the potential energy: \( U(z) \neq 0 \) for \( a > z > -a \); 2. \( \chi \) undergoes very small variations inside the well. Since outside the well \( \chi(z) \sim e^{-\sqrt{2\mu|E_0|}z} \), we readily obtain:

\[ |E_0| = 2\mu \left[ \int_{0}^{a} U(z)dz \right]^2 . \]  (33)
For
\[ \mu |U|a^2 \ll 1 \tag{34} \]
(condition for the potential to form a shallow well which means that the absolute value of the energy of ground level is much smaller than the absolute value of the potential in the well) we get that, indeed, \( |E_0| \ll |U| \) and that the variation of \( \chi \) inside the well is small, \( \Delta \chi / \chi \sim \mu |U|a^2 \ll 1 \).

Concerning the one-dimensional Coulomb potential, it satisfies this condition only for \( a \ll 1/(m_e e^2) \equiv a_B \). This explains why the accuracy of \( \log^2 \) formula (30) is very poor.

Much more accurate equation for atomic energies in strong magnetic field was derived by B.M.Karnakov and V.S.Popov [18]. It provides a several percent accuracy for the energies of EVEN states for \( H > 10^3 \ (H \equiv B/(m_e^2 e^3)) \).

Main idea is to integrate Shrödinger equation with effective potential from \( x = 0 \) till \( x = z \), where \( a_H < z < a_B \) and to equate obtained expression for \( \chi'(z)/\chi(z) \) to the logarithmic derivative of Whittaker function - the solution of Shrödinger equation with Coulomb potential, which exponentially decreases at \( z >> a_B \). In this way in [18] the following equation was obtained:

\[
2 \ln \left( \frac{z}{a_H} \right) + \ln 2 - \psi(1 + |m|) + O(a_H/z) = 2 \ln \left( \frac{z}{a_B} \right) + \lambda + 2 \ln \lambda + 2\psi \left( 1 - \frac{1}{\lambda} \right) + 4\gamma + 2 \ln 2 + O(z/a_B), \tag{35}
\]

\[ E = -(m_e e^4/2)\lambda^2, \tag{36} \]

where \( \psi(x) \) is the logarithmic derivative of the gamma function.

The energies of the ODD states are:

\[ E_{\text{odd}} = -\frac{m_e e^4}{2n^2} + O \left( \frac{m_e^2 e^3}{B} \right), \quad n = 1, 2, \ldots. \tag{37} \]

So, for superstrong magnetic fields \( B \sim m_e^2 e^3 \) the deviations of odd states energies from the Balmer series are negligible.

From (33) we get an equation for \( \lambda \):

\[ \ln(H) = \lambda + 2 \ln \lambda + 2\psi \left( 1 - \frac{1}{\lambda} \right) + \ln 2 + 4\gamma + \psi(1 + |m|), \tag{38} \]

where \( \psi(x) \) has simple poles at \( x = 0, -1, -2, \ldots \). So to reproduce large number at left hand side \( \lambda \) should be large (ground level) or follow Balmer series (excited levels).

When screening is taken into account an expression for effective potential transforms into

\[
\tilde{U}_{\text{eff}}(z) = -e^2 \int \frac{|R_{0m}(\vec{r})|^2}{\sqrt{\rho^2 + z^2}} d^2 \rho \left[ 1 - e^{-\sqrt{6m_e^2}z} + e^{-\sqrt{2/\pi}(e^3B+6m_e^2)z} \right], \tag{39}
\]

Screening modifies the Coulomb potential at the distances \( |z| < 1/m_e \) and since at these distances \( m_e |U|a^2 = m_e e^2 a < e^2 \ll 1 \), the approach leading to (35) still works.

The modified Karnakov - Popov equation, which takes screening into account looks like:

\[
\ln \left( \frac{H}{1 + \frac{e^6}{3\pi}H} \right) = \lambda + 2 \ln \lambda + 2\psi \left( 1 - \frac{1}{\lambda} \right) + \ln 2 + 4\gamma + \psi(1 + |m|). \tag{40}
\]
We see that at $H \equiv B/(m_e^2 e^3) \approx 3\pi/e^6, B \approx 3\pi m_e^2/e^3$ freezing of the energies occur: left hand side of (40) approach constant when $B$ further grows. In particular, for a ground state at $B \gg 3\pi m_e^2/e^3$ we obtain: $\lambda_0 = 11.2, E_0 = -1.7$ keV.

Energy levels on which LLL is splitted in the hydrogen atom at $B \gg 3\pi m_e^2/e^3$ are shown on Fig. 6.

Fig.6. *Spectrum of hydrogen levels in the limit $B \gg 3\pi m_e^2/e^3$. Energies are given in rydberg units, $Ry \equiv 13.6$ eV.*

## 5 Dirac equation with a screened Coulomb potential, $Z = 1$

In the previous Section the spectrum of energies on which the lowest Landau level (LLL) splits in the proton electric field was found by solving the corresponding Schrödinger equation. Since the ground state energy of hydrogen in the limit of infinite $B$ equals $E_0 = -1.7$ keV, the use of the nonrelativistic Schrödinger equation is at least selfconsistent. However, the size $a_H$ of the electron wave function for $B > m_e^2/e^3$ in the direction transverse to the magnetic field is much smaller than the electron Compton wavelength, $a_H \equiv 1/\sqrt{eB} < e/m_e \ll 1/m_e$, which makes the nonrelativistic approach a bit suspicious ($a_H = 1/m_e$ for $B = B_0$). That is why in this Section we will study the ground state energy of the electron in a hydrogen-like ion.
in the presence of an external magnetic field by analyzing the Dirac equation. Without taking screening into account this problem was considered in paper [19] (see also [20], were results obtained in [19] were reproduced), soon after it was found that a hydrogen-like ion becomes critical at \( Z \approx 170\): the electron ground level sinks into the lower continuum (\( \varepsilon_0 < -m_e \)) and the vacuum becomes unstable by spontaneous \( e^+e^- \) pairs production. These results were obtained by solving the Dirac equation for an electron moving in the field of a nucleus of finite radius. That the phenomenon of criticality can be studied only in the framework of the Dirac equation is an additional motivation for us to go from Schrödinger to Dirac.

From the numerical solution of the Dirac equation for the ground electron level of a hydrogen atom in the Coulomb potential we will find that the corrections to the nonrelativistic and the vacuum becomes unstable by spontaneous \( e^+e^- \) pairs production. These results were obtained by solving the Dirac equation for an electron moving in the field of a nucleus of finite radius. That the phenomenon of criticality can be studied only in the framework of the Dirac equation is an additional motivation for us to go from Schrödinger to Dirac.

Let us parametrize bispinor which describes electron wave function in the following way:

\[
\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \varphi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.
\]

Substituting \( \Psi \) in the Dirac equation for the electron in an external electromagnetic field we obtain:

\[
\begin{cases}
(\varepsilon - m - e\varphi) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + (-i\vec{\sigma} \frac{\partial}{\partial \varphi} + e\vec{A}\vec{\sigma}) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \\
-(i\vec{\sigma} \frac{\partial}{\partial \varphi} - e\vec{A}\vec{\sigma}) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + (\varepsilon + m - e\varphi) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0
\end{cases}
\]

(42)

Taking vector potential which describes constant magnetic field \( \vec{B} \) directed along \( z \) axis in the form \( \vec{A} = (-\frac{1}{2}By, \frac{1}{2}Bz, 0) \), we get:

\[
e\vec{A}\vec{\sigma} = -\frac{e}{2}B \begin{pmatrix} 0 & y + ix \\ y - ix & 0 \end{pmatrix} = -\frac{i}{2}eB\rho \begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix},
\]

where \( \rho = \sqrt{x^2 + y^2}, \theta \equiv \arctan(y/x) \). Analogously we obtain:

\[
-i\vec{\sigma} \frac{\partial}{\partial \varphi} = -i \left( e^{i\theta} \frac{\partial}{\partial \rho} + i e^{i\theta} \frac{\partial}{\partial \varphi} + e^{-i\theta} \frac{\partial}{\partial \rho} - \frac{ie^{-i\theta}}{\rho} \frac{\partial}{\partial \varphi} \right).
\]

(44)

Substituting two last expressions in the Dirac equation we get:

\[
\begin{cases}
(\varepsilon - m - e\varphi) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - i \left( e^{i\theta} \left( -\frac{1}{2}eB\rho + \frac{\partial}{\partial \rho} + i \frac{\partial}{\partial \varphi} \right) + e^{-i\theta} \left( \frac{1}{2}eB\rho + \frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \varphi} \right) \right) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \\
(\varepsilon + m - e\varphi) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - i \left( e^{i\theta} \left( -\frac{1}{2}eB\rho + \frac{\partial}{\partial \rho} + i \frac{\partial}{\partial \varphi} \right) + e^{-i\theta} \left( \frac{1}{2}eB\rho + \frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \varphi} \right) \right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0
\end{cases}
\]

(45)

Axial symmetry of electromagnetic field allows to determine \( \theta \) dependence of the functions \( c_i \) and \( b_i \):

\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1(\rho, z) e^{i(M-1/2)\theta} \\ c_2(\rho, z) e^{i(M+1/2)\theta} \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1(\rho, z) e^{i(M-1/2)\theta} \\ b_2(\rho, z) e^{i(M+1/2)\theta} \end{pmatrix},
\]

(46)

where \( M = \pm 1/2, \pm 3/2, \ldots \) is the projection of electron angular momentum on \( z \) axis. Substituting (46) in (45) we get four linear equations for four unknown functions \( c_i \) and \( b_i \).
(here and below $c_1 \equiv c_1(\rho, z)$, $b_1 \equiv b_1(\rho, z)$ ...):

\[
\begin{align*}
(\varepsilon - m - e\varphi)c_1 + i(-b_{1z} - b_{2\rho} - \frac{M + 1/2}{\rho} b_2 - \frac{eB\rho}{2} b_2) &= 0 \\
(\varepsilon - m - e\varphi)c_2 + i(-b_{1\rho} + \frac{M - 1/2}{\rho} b_1 + \frac{eB\rho}{2} b_2) &= 0 \\
(\varepsilon + m - e\varphi)b_1 + i(-c_{1z} - c_{2\rho} - \frac{M + 1/2}{\rho} c_2 - \frac{eB\rho}{2} c_2) &= 0 \\
(\varepsilon + m - e\varphi)b_2 + i(-c_{1\rho} + \frac{M - 1/2}{\rho} c_1 + \frac{eB\rho}{2} c_1 + c_{2z}) &= 0 ,
\end{align*}
\]

(47)

where $b_{1z} \equiv \partial b_1/\partial z$, $b_{1\rho} \equiv \partial b_1/\partial \rho$, ... Ground energy state has $s_z = -1/2$, $l_z = 0$. Taking $M = -1/2$ we should look for solution of (47) with $c_1 = b_1 = 0$:

\[
\begin{align*}
\begin{cases}
b_{2\rho} + \frac{eB\rho}{2} b_2 &= 0 \\
c_{2\rho} + \frac{eB\rho}{2} c_2 &= 0 ,
\end{cases}
\end{align*}
\]

(48)

The dependence on $\rho$ is determined by (48):

\[
\begin{align*}
\begin{cases}
b_2(\rho, z) &= e^{-eB\rho^2/4}(-i)f(z) \\
c_2(\rho, z) &= e^{-eB\rho^2/4}g(z) .
\end{cases}
\end{align*}
\]

(50)

Substituting the last expressions in (49) and averaging over fast motion in transverse to the magnetic field plane we obtain two first order differential equations which describes electron motion along magnetic field in an effective potential $\bar{V}(z)$:

\[
\begin{align*}
g_z - (\varepsilon + m_e - \bar{V}) f &= 0,  \\
f_z + (\varepsilon - m_e - \bar{V}) g &= 0,
\end{align*}
\]

(51)

\[
\bar{V}(z) = -\frac{Ze^2}{a_H^2} \int_0^\infty \frac{\exp\left(-\frac{\rho^2}{2a_H^2}\right)}{\sqrt{\rho^2 + z^2}} \rho \, d\rho .
\]

(52)

At large distances $|z| \gg a_H$ the effective potential equals Coulomb, and the solutions of the equations (51) exponentially decreasing at $|z| \to \infty$ are linear combinations of the Whittaker functions. At short distances the equations (51) can be easily integrated for $|\bar{V}(z)| \gg |\varepsilon \pm m_e|$ (as far as $|\varepsilon| < m_e$ condition for $|\bar{V}(z)|$ will be for sure valid for $|\bar{V}(z)| > 2m_e$, which is equivalent to the following inequality: $z \ll Ze^2/(2m_e)$), where they looks like:

\[
\begin{align*}
g_z + \bar{V} f &= 0 ,  \\
f_z - \bar{V} g &= 0 .
\end{align*}
\]

(53)

The result of the integration is:

\[
\begin{align*}
g(z) &= B_1 \cos w(z) + B_2 \sin w(z) ,  \\
f(z) &= B_1 \sin w(z) - B_2 \cos w(z)
\end{align*}
\]

(54)
where

\[ w(z) = \int_{0}^{z} \bar{V}(z') dz' \tag{55} \]

and \( B_1, B_2 \) are normalization constants.

The functions \( g(z) \) and \( f(z) \) have opposite parities; for the ground state \( g(z) \) should be even, so \( B_2 = 0 \), and matching logarithmic derivatives at the point \( z_0 \) we obtain:

\[-\bar{V}(z_0) \tan w(z_0) = \frac{d}{dz} \ln g(z_0) , \tag{56}\]

\[ Ze^2/(2m_e) \gg z_0 \gg a_H \tag{57}\]

Substituting proper combination of the Whittaker functions for \( g(z) \) we obtain an algebraic equation for the ground state energy (it coincides with Eq. (22) in \[19\] in the limit \( R/a_H \ll 1 \), where \( R \) is the nucleus radius):

\[ Ze^2 \ln \left( 2\sqrt{m_e^2 - \varepsilon^2} / \sqrt{eB} \right) + \arg \Gamma \left( - \frac{Ze^2 \varepsilon}{\sqrt{m_e^2 - \varepsilon^2}} + iZe^2 \right) + \]

\[ + \arctan \left( \frac{m_e + \varepsilon}{m_e - \varepsilon} \right) - \arg \Gamma (1 + 2iZe^2) - \frac{Ze^2}{2} (\ln 2 + \gamma) = \frac{\pi}{2} + n\pi , \tag{58}\]

where \( \gamma = 0.5772... \) is the Euler constant, and the argument of the gamma function is given by

\[ \arg \Gamma (x + iy) = -\gamma y + \sum_{k=1}^{\infty} \left( \frac{y}{k} - \arctan \frac{y}{x + k - 1} \right) . \tag{59}\]

For the ground level at \( \varepsilon > 0 \) one should take \( n = 0 \), while for \( \varepsilon < 0 \) it should be changed to \( n = -1 \).

According to (58) when the magnetic field increases the ground state energy goes down and reaches the lower continuum.

A matching point exists only if \( B \gg 4m_e^2/(e(Ze^2)^2) \) (see (57)) and (58) is valid only for these values of the magnetic field.

Thus, without taking screening into account, from (58) we can obtain the dependence of the ground state energy of a hydrogen atom on the magnetic field for \( B \gg 4m_e^2/e^5 \). Screening modifies the Coulomb potential at distances smaller than the electron Compton wavelength, and from the condition \( |\bar{V}(1/m_e)| \gg 2m_e \) we get \( Ze^2 \gg 2 \). It means that at \( B > 3\pi m_e^2/e^3 \) the phenomena of screening does not allow to find analytically the ground state energy. In order to find the ground state energy at \( B \lesssim 4m_e^2/e^5 \) and to take screening into account the equations (51) were solved numerically. This system can be transformed into one second order differential equation for \( g(z) \). By substituting \( g(z) = (\varepsilon + m_e - \bar{V})^{1/2} \chi(z) \) a Schrödinger-like equation for the function \( \chi(z) \) was obtained in [19],

\[ \frac{d^2\chi}{dz^2} + 2m_e(E - U)\chi = 0 , \tag{60}\]

\[ ^1\text{This trick was exploited by V.S. Popov for the qualitative analysis of the phenomenon of critical charge} \[21\].\]
\begin{equation}
E = \frac{\varepsilon^2 - m_e^2}{2m_e}, \quad U = \frac{\varepsilon}{m_e} \bar{V} - \frac{1}{2m_e} \bar{V}^2 + \frac{\bar{V}''}{4m_e(\varepsilon + m_e - \bar{V})} + \frac{3/8(\bar{V}')^2}{m_e(\varepsilon + m_e - \bar{V})^2},
\end{equation}

where \( \varepsilon \) is the energy eigenvalue of the Dirac equation and \( \bar{V}(z) \) is given in (52). An equation (60) was integrated numerically. Let us note that, while for \( z \gg 1/m_e \) the last three terms in the expression for \( U \) are much smaller than the first one (the only one remaining in the nonrelativistic approximation), at \( z \lesssim 1/m_e \) the relativistic terms dominate and are very big for \( B \gg B_0 \) at \( z \sim a_H \) which makes numerical calculations very complicated.

In Table 1 the results for the ground state energy of a hydrogen atom without screening are presented. The values of the magnetic field in units of \( B_0 \) are given in the first column, while in columns 2-5 the values of \( \lambda \) are given. By definition

\begin{equation}
E = \frac{\varepsilon^2 - m_e^2}{2m_e} \equiv -\frac{m_e e^4}{2} \lambda^2.
\end{equation}

From Table 1 we see that:

1. the results of the numerical integrations of the Schrödinger and Dirac equations coincide within four digits:
   - with the analytical Karnakov–Popov formula for the ground state energy (\( n_\rho = m = 0 \)) [18] in the case of the Schrödinger equation;
   - with formula (58) for \( Z = 1 \) in the case of the Dirac equation;

2. for the relativistic shift of energy the following estimate works:

\begin{equation}
E_{\text{Dirac}} - E_{\text{Schr}} \sim E_{\text{Schr}} \frac{E_{\text{Schr}}}{m_e},
\end{equation}

\begin{equation}
\delta \lambda \sim e^4\lambda^3/4.
\end{equation}

To take screening into account, the following formula for the effective potential should be used in (60) instead of (52):

\begin{equation}
\bar{V}(z) = -\frac{Ze^2}{a_H^2} \left[ 1 - e^{-\sqrt{6m_e^2}z} + e^{-\sqrt{(2/\pi)e^kB+6m_e^2}z} \right] \int_0^\infty \frac{e^{-\rho^2/2a_H^2}}{\sqrt{\rho^2 + z}} \rho d\rho,
\end{equation}

where \( Z = 1 \) for hydrogen.

The freezing of the ground state energy is due to a weaker singularity of the potential with screening (63) at \( z \to 0 \) for \( B \to \infty \) than that of the potential without screening (52). While the non-screened potential behaves like \( 1/z \) at small \( z \), the screened potential is proportional to \( \delta(z) \) because, when \( B \to \infty \), the width of the region where it behaves like \( 1/z \) shrinks to zero [2].

In Table 2 the results of the analytical formula for \( \lambda \) with the account of screening for the Schrödinger equation (10) are compared with the results of the numerical integration of the Dirac equation. We see that in the case of screening the relativistic shift of energy is also very small, and due to it the ground state energies become a little bit higher, just like without taking screening into account. The freezing of the ground state energy occurs at \( B/B_0 = 10^3 \div 10^4 \), when \( B \approx 3\pi m_e^2/e^3 \).

\footnote{Let us note that the definition of \( \lambda \) used in [19] differs from our: \( \lambda^{[19]} = e^2 \lambda \).}
Table 1: Values of $\lambda$ for $Z = 1$ without screening obtained from the Schrödinger and Dirac equations. They start to differ substantially at enormous values of the magnetic field.

| $B/B_0$ | KP-equation (Schrödinger) | Numerical results (Schrödinger) | Eq. (58) (Dirac) | Numerical results (Dirac) |
|---------|---------------------------|-------------------------------|-----------------|---------------------------|
| $10^0$  | 5.737                     | 5.735                        | 5.735           | 5.734                     |
| $10^1$  | 7.374                     | 7.374                        | 7.370           | 7.371                     |
| $10^2$  | 9.141                     | 9.141                        | 9.136           | 9.135                     |
| $10^3$  | 11.00                     | 11.00                        | 10.99           | 10.99                     |
| $10^4$  | 12.93                     | 12.93                        | 12.91           | 12.91                     |
| $10^5$  | 14.91                     | 14.91                        | 14.88           | 14.88                     |
| $10^6$  | 16.93                     | 16.93                        | 16.89           | 16.89                     |
| $10^7$  | 18.98                     | 18.98                        | 18.93           | 18.92                     |
| $10^8$  | 21.06                     | 21.05                        | 20.98           | 20.98                     |
| $10^9$  | 23.16                     | 23.15                        | 23.05           | 23.05                     |
| $10^{10}$| 25.27                     | 25.27                        | 25.14           | 25.13                     |
| $10^{11}$| 27.40                     | 27.40                        | 27.23           |                           |
| $10^{12}$| 29.54                     | 29.54                        | 29.33           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{15}$| 36.03                     | $\ldots$                     | 35.64           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{20}$| 46.99                     | $\ldots$                     | 46.11           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{25}$| 58.07                     | $\ldots$                     | 56.40           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{30}$| 69.22                     | $\ldots$                     | 66.38           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{35}$| 80.43                     | $\ldots$                     | 75.98           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{40}$| 91.67                     | $\ldots$                     | 85.10           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{45}$| 102.95                    | $\ldots$                     | 93.67           |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{50}$| 114.25                    | $\ldots$                     | 101.62          |                           |
| $\ldots$| $\ldots$                 | $\ldots$                     | $\ldots$       |                           |
| $10^{55}$| 125.57                    | $\ldots$                     | 108.89          |                           |

6  Screening versus critical nucleus charge

According to [19] nuclei with $Z \geq 40$ become critical in an external $B$ (for smaller $Z$ the values of $a_H$ at which the criticality is reached become smaller than the nucleus radius, the Coulomb potential diminishes and thus the ground level does not reach the lower continuum).

In Table 3 one can see the dependence of the ground state electron energy $\varepsilon_0$ on the external magnetic field for $Z = 40$. The numerical solutions of (50) are in good correspondence
Table 2: Values of $\lambda$ for $Z = 1$ with screening.

| $B/B_0$ | Eq. (40) (Schrödinger) | Numerical results (Schrödinger) | Numerical results (Dirac) |
|---------|------------------------|-------------------------------|--------------------------|
| $10^0$  | 5.7                    | 5.7                           | 5.7                      |
| $10^1$  | 7.4                    | 7.4                           | 7.4                      |
| $10^2$  | 9.1                    | 9.1                           | 9.1                      |
| $10^3$  | 10.5                   | 10.6                          | 10.6                     |
| $10^4$  | 11.1                   | 11.2                          | 11.2                     |
| $10^5$  | 11.2                   | 11.3                          | 11.3                     |
| $10^6$  | 11.2                   | 11.4                          | 11.3                     |
| $10^7$  | 11.2                   | 11.4                          | 11.3                     |
| $10^8$  | 11.2                   | 11.4                          | 11.3                     |

Table 3: Values of $\varepsilon_0/m_e$ for $Z = 40$.

| $B/B_0$ | Eq. (58) (Dirac) | Numerical results (Dirac) | Numerical results with screening (Dirac) |
|---------|------------------|---------------------------|------------------------------------------|
| $10^0$  | 0.819            | 0.850                     | 0.850                                    |
| $10^1$  | 0.653            | 0.667                     | 0.667                                    |
| $10^2$  | 0.336            | 0.339                     | 0.346                                    |
| $10^3$  | -0.158           | -0.159                    | -0.0765                                  |
| $10^4$  | -0.758           | -0.759                    | -0.376                                   |
| $2 \cdot 10^4$ | -0.926 | -0.927                    | -0.423                                   |
| ...     | at $B/B_0 \approx 2.85 \cdot 10^4$, $\varepsilon_0 = -m_e$ | ...                          | ...                                      |
| $10^5$  | -                |                           | -0.488                                   |
| $10^6$  | -                |                           | -0.524                                   |
| $10^7$  | -                |                           | -0.535                                   |
| $10^8$  | -                |                           | -0.538                                   |

Table 4: Values of freezing ground state energies for different $Z$ from the Schrödinger and the Dirac equations. In order to find the freezing energies we must take $B/B_0 \gg 3\pi/e^2$. In numerical calculations we took $B/B_0 = 10^8$.

| $Z$ | $(E_{fr}^{z})_{\text{Schr}}^{\text{numerical}}$, keV | $(\varepsilon_{0}^{fr} - m_e)_{\text{Dirac}}^{\text{numerical}}$, keV |
|-----|-----------------------------------------------------|-------------------------------------------------|
| 1   | -1.7                                                | -1.7                                            |
| 10  | -88                                                 | -87                                             |
| 20  | -288                                                | -273                                            |
| 30  | -582                                                | -519                                            |
| 40  | -966                                                | -787                                            |
| 49  | -                                                   | -1003                                           |

with the values of $\varepsilon_0$ obtained from (58). The numerical results with screening are shown in the last column; we see that freezing occurs in the relativistic domain $\varepsilon_0 \approx -m_e/2$ and the
ground level never reaches lower continuum, \( \varepsilon_0 > -m_e \).

In Table 4 we compare freezing energies for different \( Z \) obtained numerically from the nonrelativistic Schrödinger equation and from the Dirac equation. We see that for \( Z > 20 \) the freezing occurs in the relativistic regime, where the Schrödinger equation should not be used. Let us stress that the value of \( B \) at which the freezing occurs does not depend on \( Z \).

From (58) we obtain in the limiting case \( \varepsilon \to -m_e \) an equation which defines the value of the magnetic field at which a nucleus with charge \( Z \) becomes critical without taking screening into account (it coincides with Eq. (32) from [19]):

\[
\frac{B}{B_0} = 2(Z_{cr}e^2)^2 \exp \left( -\gamma + \frac{\pi - 2 \arg \Gamma(1 + 2iZ_{cr}e^2)}{Z_{cr}e^2} \right).
\]

This equation is used to calculate the numbers in the second column of Table 5.

From Table 5 we see that with the account of screening only the atoms with \( Z > \sim 52 \) become supercritical at the values of \( B/B_0 \) shown in the fourth column. Because of screening a larger \( B \) is needed for a nucleus to become supercritical and the nuclei with \( Z < 52 \) never reach supercriticality. This phenomenon is illustrated in Fig. 7.

| \( Z_{cr} \) | Eq. (64) | Numerical results without screening | Numerical results with screening |
|------------|-----------|----------------------------------|-------------------------------|
| 90         | 118       | 116                              | 122                           |
| 85         | 157       | 154                              | 164                           |
| 80         | 213       | 210                              | 229                           |
| 75         | 301       | 297                              | 335                           |
| 70         | 444       | 438                              | 527                           |
| 65         | 689       | 681                              | 923                           |
| 60         | 1144      | 1133                             | 1964                          |
| 55         | 2068      | 2053                             | 6830                          |
| 54         | 2357      | 2340                             | 10172                         |
| 53         | 2699      | 2681                             | 17012                         |
| 52         | 3107      | 3087                             | 35135                         |
| 51         | 3594      | 3572                             | \( 1.20 \cdot 10^5 \)         |
| 50         | 4181      | 4157                             | \( 1.14 \cdot 10^7 \)         |
| 45         | 9826      | 9787                             | —                             |
| 40         | 28478     | 28408                            | —                             |
| 35         | \( 1.12 \cdot 10^5 \) | \( 1.12 \cdot 10^5 \) | —                             |
| 30         | \( 6.99 \cdot 10^5 \) | \( 6.98 \cdot 10^5 \) | —                             |
| 25         | \( 9.27 \cdot 10^6 \) | \( 9.27 \cdot 10^6 \) | —                             |
Fig. 7. The values of $B^Z_{cr}$: a) without screening according to (64), dashed (green) line; b) numerical results with screening, solid (blue) line. The dotted (black) line corresponds to the field at which $a_H$ becomes smaller than the size of the nucleus.

From Tables 1, 3, and 5 we see that (58) is very good in describing the dependence of the energy on the magnetic field; at least a numerical integration produces almost identical results. In Table 6 we demonstrate several cases where the accuracy of (58) is not that good. It happens at low $B/B_0$ since the matching condition $B > 4m^2_e/(e(Ze^2)^2)$ fails and when $\varepsilon_0$ is relativistic. However, $B$ should not be too low to make the adiabaticity condition $a_B \gg a_H$, or $B \gg (Ze^2)^2m^2/e$ applicable.

Table 6: Values of $\varepsilon_0/m_e$ at $B/B_0 = 5$.

| $Z$ | Eq. (58) (Dirac) | Numerical results (Dirac) |
|-----|----------------|--------------------------|
| 90  | 0.2050         | 0.2512                   |
| 80  | 0.3096         | 0.3539                   |
| 70  | 0.4139         | 0.4542                   |
| 60  | 0.5171         | 0.5516                   |
| 50  | 0.6185         | 0.6454                   |
| 40  | 0.7165         | 0.7349                   |
| 30  | 0.8086         | 0.8188                   |
| 20  | 0.8914         | 0.8952                   |
| 10  | 0.9596         | 0.9601                   |
| 1   | 0.998745       | 0.998745                 |

Textbooks [22] contain detailed consideration of the phenomenon of critical charge.
7 Conclusions

An analytical formula for the Coulomb potential $\Phi(z)$ in a superstrong magnetic field has been derived. It reproduces the results of the numerical calculations made in [2] with good accuracy. Using it, an algebraic formula for the energy spectrum of the levels of a hydrogen atom originating from the lowest Landau level in a superstrong $B$ has been obtained. The energies start to deviate from those obtained without taking the screening of the Coulomb potential into account at $B \gtrsim 3\pi m_e^2/e^3 \approx 6 \cdot 10^{16}$ gauss and the energy of ground state in the limit $B \to \infty$ remains finite.

A magnetic field plays a double role in the critical charge phenomenon. By squeezing the electron wave function and putting it in the domain of a stronger Coulomb potential it diminishes the value of the critical charge substantially [19]. However, for nuclei with $Z < 52$ to become critical such a strong $B$ is needed that the screening of the Coulomb potential occurs and acts in the opposite direction: the electron ground state energy freezes and the nucleus remains subcritical in spite of growing $B$.

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