On Dimensional Degression in $AdS_d$

A. Yu. Artsukevich and M. A. Vasiliev

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, Leninsky prospect 53, 119991, Moscow, Russia

Abstract

We analyze the pattern of fields in $d+1$ dimensional anti-de Sitter space in terms of those in $d$ dimensional anti-de Sitter space. The procedure, which is neither dimensional reduction nor dimensional compactification, is called dimensional degression. The analysis is performed group-theoretically for all totally symmetric bosonic and fermionic representations of the anti-de Sitter algebra. The field-theoretical analysis is done for a massive scalar field in $AdS_{d+d'}$ and massless spin one-half, spin one, and spin two fields in $AdS_{d+1}$. The mass spectra of the resulting towers of fields in $AdS_d$ are found. For the scalar field case, the obtained results extend to the shadow sector those obtained by Metsaev in [1] by a different method.
Contents

1 Introduction 3

2 Group-theoretical analysis 4

3 Field-theoretical degression of one dimension 8
   3.1 Scalar field 8
   3.2 Massless Dirac field 12
      3.2.1 Odd $d$ 13
      3.2.2 Even $d$ 15
   3.3 Massless spin one 16
   3.4 Massless spin two 17

4 Scalar field degression over several dimensions 21

5 Conclusion 24

6 Appendix 25
1 Introduction

It is well-known that a single particle in \(d + d'\) dimensions gives rise to towers of Kaluza-Klein modes in \(d\) dimensions via dimensional compactification \([2, 3, 4]\). The Kaluza-Klein compactification works when \(R^{d+d'}\) is replaced by \(R^d \times M^{d'}\) with some compact manifold \(M^{d'}\). Then, the expansion of a solution of, say, Klein-Gordon equation in \(R^{d} \times M^{d'}\) in the form \(\phi(x, y) = \sum_n \phi_n(x) \psi_n(y)\) where \(\psi_n(y)\) are eigenfunctions of the Laplace operator on \(M^{d'}\) gives a tower of Kaluza-Klein modes that starts with zero modes \(\psi_{n_0}(x)\). The Kaluza-Klein mass gap scales in the inverse size of the compact manifold \(M^{d'}\). The mass spectrum is determined by the spectrum of the Laplace operator on \(M^{d'}\).

For the text book \(d' = 1\) example of a circle of radius \(R\), i.e., \(M^1 = S^1\),

\[
\psi_n(y) = e^{\frac{i n y}{R}}, \quad n \in \mathbb{Z}, \; y \sim y + 2\pi R.
\]

The spectrum of \(d\)-dimensional particles resulting from a particle of mass \(m\) in \(d + 1\) dimensions is given by the well-known formula

\[
m_n^2 = m^2 + \frac{n^2}{R^2}.
\]

From here it follows in particular that the spectrum becomes continuous in the decompactifying limit \(R \rightarrow \infty\), that makes it difficult to interpret a particle in \(R^{d+d'}\) in terms of particles in \(R^d\).

If one starts with the anti-de Sitter space \(AdS_{d+d'}\) instead of \(R^{d+d'}\), the situation changes in two respects. The bad news is that it is not clear what should be an \(AdS\) analog of \(R^d \times M^{d'}\) since \(AdS_{d+d'}\) is not a product of two manifolds. The good news is however that the spectrum of states in \(AdS_{d+d'}\) is discrete. As a result, a module of the \(AdS_{d+d'}\) algebra \(\mathfrak{o}(d+d'-1, 2)\) that describes states of one or another particle in \(AdS_{d+d'}\) reduces to a countable number of irreducible modules of \(\mathfrak{o}(d-1, 2) \oplus \mathfrak{o}(d') \subset \mathfrak{o}(d+d'-1, 2)\) that describe particles in \(AdS_d\) which form some \(\mathfrak{o}(d')\) multiplets. The spectrum remains discrete with the mass gap scale given by the inverse \(AdS\) radius \(\rho^{-1}\). That this should happen is fairly clear from the structure of unitary lowest weight \(\mathfrak{o}(d, 2)\)-modules (see, e.g., \([5, 6, 7]\)). Since the phenomenon is neither reduction (all degrees of freedom are kept intact) nor compactification (nothing is compactified) we call it dimensional depression. Note that our approach differs from that of \([8]\) (see also \([9]\)) where the radial reduction of the flat to curved space was carried out.

The main aim of this paper is to check how the dimensional degression occurs in terms of fields. We analyze the problem in full generality for the example of a scalar field in \(AdS_{d+d'}\), showing that the spectrum of scalar field modes indeed matches the pattern of the dimensionally reduced unitary modules. We also demonstrate that our results reproduce those obtained previously by Metsaev \([1]\) by a different method based on a specific Ansatz. In addition, our approach reproduces the shadow unitary states that exist for sufficiently low energy eigenvalues with negative \(m^2\).

The obtained results may have several applications. First of all, this is the first step towards the analysis of the dimensional compactification-like effects in higher-spin (HS) theories in \(AdS_d\). This analysis may be useful, in particular, for understanding a mechanism of spontaneous breakdown of HS gauge symmetries. Indeed, we show that the dimensional degression gives rise to the gauge invariant (i.e., Stueckelberg) formulation of massive fields. As such it is analogous to the standard torus compactification mechanism used to derive massive HS theories from the massless ones in one higher dimension \([10, 11]\).
In other words, the degression mechanism in AdS may help to uncover the structure of the Higgs sector in HS gauge theory. In particular, it would be interesting to see how the gauge invariant formulation of massive HS fields in AdS results from dimensional degression.

The layout of the rest of the paper is as follows: in Section 2 relevant facts on the unitary $o(d,2)$-modules are recalled and the analysis of the branching of the unitary $o(d,2)$-modules into $o(d-1,2)$-modules is performed (detailed proofs are given in Appendix). In Section 3 the field-theoretical analysis of the dimensional degression of $AdS_{d+1}$ to $AdS_d$ is done for the case of a spin zero field of arbitrary mass and massless fields of spins one-half, one, and two. The generalization to several extra coordinates is given in Section 4 for a scalar field. Conclusions and perspectives are discussed in Conclusion.

2 Group-theoretical analysis

The generators $T^{AB} = -T^{BA}$ ($A, B = -1, 0, \ldots, d$) of the symmetry algebra $o(d,2)$ of $AdS_{d+1}$ have the commutation relations

$$[T^{AB}, T^{CD}] = \eta^{BC}T^{AD} - \eta^{AC}T^{BD} - \eta^{BD}T^{AC} + \eta^{AD}T^{BC},$$

where $\eta^{AB}$ is the invariant symmetric form of $o(d,2)$. We use the mostly minus convention with $\eta^{-1-1} = \eta^{00} = 1$ and $\eta^{ab} = -\delta^{ab}$ for the space-like values of $A = a = 1 \ldots d$. The $AdS_{d+1}$ energy operator is

$$E = iT^{-10}.$$ 

The noncompact generators of $o(d,2)$ are

$$T^{\pm a} = iT^{0a} \mp T^{-1a},$$

$$[E, T^{\pm a}] = \pm T^{\pm a}, \quad [T^{-a}, T^{+b}] = 2(\delta^{ab}E + T^{ab}).$$

The compact generators $T^{ab}$ of $o(d)$ commute with $E$. The generators $T^{AB}$ are anti-Hermitian, $(T^{AB})^\dagger = -T^{AB}$. Hence,

$$E^\dagger = E, \quad (T^{\pm a})^\dagger = T^{\mp a}, \quad (T^{ab})^\dagger = -T^{ab}.$$

An irreducible bounded energy unitary $o(d,2)$-module $\mathcal{H}(E_0, s)_{o(d,2)}$ is characterized by the eigenvalue $E_0$ of $E$ and the weight $s = (s_1, s_2, \ldots, s_{[d/2]})$ of $o(d)$ which refer to the lowest energy (vacuum) states $|E_0, s\rangle$ of $\mathcal{H}(E_0, s)_{o(d,2)}$ that satisfy $T^{-a}|E_0, s\rangle = 0$ and form a finite dimensional module of $o(d) \oplus o(2) \subset o(d,2)$. $E_0$ characterizes the energy of the ground state of a field (related to its mass) and $s$ characterizes the spin of a field.

Recall that in the complex case of $o(d|\mathbb{C})$, all $s_i$ are integers in the bosonic case and half-integers in the fermionic case that are all non-negative except $s_{\frac{d}{2}}$ for even $d$, which may be positive, negative or zero. Two sign possibilities for $s_{\frac{d}{2}}$ correspond to selfdual and anti-selfdual representations $o(d|\mathbb{C})$ (equivalently, left and right chiral spinor representations in the fermionic case). The structure of irreducible representations of the real algebra $o(d|\mathbb{R})$ is more subtle because the (anti)selfduality or chirality conditions may or may not be compatible with the reality (Majorana) conditions. We will still use the same spin notations as in the complex case, assuming that the chirality or self-duality conditions are imposed on the doubled set of tensors. The resulting module may be
irreducible if the chirality or self-duality conditions are incompatible with the reality condition or double-reducible otherwise. In practice, this convention is not essential for the analysis of bosonic fields in this paper because we mostly consider the case with \( s = 0 \). In the fermionic case, however, it corresponds to the consideration of the left or right spinors, neglecting the \( d \)-dependent analysis of Majorana conditions. Such a convention just matches the field theoretical analysis of this paper that will be performed in terms of Dirac spinors.

The requirement of unitarity results in the constraints on the energy \( E_0 \) and the spin \( s \). For scalar and spinor they are [22, 23, 24]

\[
E_0 \geq E_0(0) = \frac{d - 2}{2} \quad \text{if } s = 0, \quad E_0 \geq E_0(\frac{1}{2}) = \frac{d - 1}{2} \quad \text{if } s = \frac{1}{2}.
\] (1)

As shown by Metsaev [25, 26], for a generalized spin \( s \) with \( s_1 = \ldots = s_p > s_{p+1} \geq \ldots \geq s_{[d/2]} \) and \( s_1 \geq 1 \), the unitarity constraint is

\[
E_0 \geq E_0(s) = s_p - p - 1 + d.
\] (2)

Let \( D(E_0, s)_{o(d,2)} \) be a generalized Verma module induced from some irreducible \( o(d) \oplus o(2) \) vacuum module \( |E_0, s\rangle \) annihilated by \( T^{-a} \). It is spanned by the states

\[
T^{+a_1} \ldots T^{+a_M} |E_0, s\rangle,
\] (3)

with various \( M \). For the unitary case \( E_0 > E_0(s) \), \( D(E_0, s)_{o(d,2)} = \mathcal{H}(E_0, s)_{o(d,2)} \). At the boundary of the unitarity region \( E_0 = E_0(s) \), \( D(E_0, s)_{o(d,2)} \) contains a singular submodule \( S \) of null states. This has to be factored out to obtain a unitary \( o(d,2) \)-module \( \mathcal{H}(E_0, s)_{o(d,2)} = D(E_0, s)_{o(d,2)}/S \).

In this paper, we consider only the case of totally symmetric representations with \( s = (s, 0, \ldots, 0) \) or \( s^\pm = (s, \frac{1}{2}, \ldots, \pm \frac{1}{2}) \) if \( d \) is even and \( s = (s, \frac{1}{2}, \ldots, \frac{1}{2}) \) if \( d \) is odd. The respective unitary modules are denoted by \( \mathcal{H}(E_0, s)_{o(d,2)} \).

The branching rules for the unitary lowest weight modules associated with fields in \( AdS_{d+1} \) are summarized by the following Theorems.

**Theorem 1** For an integer spin \( s \), the \( o(d,2) \)-module \( D(E_0, s)_{o(d,2)} \) with \( E_0 \geq E_0(s) \) has the following branching into \( o(d - 1, 2) \)-modules

\[
D(E_0, s)_{o(d,2)} \simeq \sum_{q = 0}^{s} \sum_{k = 0}^{\infty} \oplus D(E_0 + k, q)_{o(d - 1, 2)}.
\] (4)

**Theorem 2** For a half-integer spin \( s \) and \( E_0 \geq E_0(s) \), the \( o(d, 2) \)-module \( D(E_0, s^\pm)_{o(d,2)} \) for even \( d \) and \( D(E_0, s)_{o(d,2)} \) for odd \( d \) have, respectively, the following branchings into \( o(d - 1, 2) \)-modules

\[
D(E_0, s^\pm)_{o(d,2)} \simeq \sum_{q = \frac{1}{2}}^{s} \sum_{k = 0}^{\infty} \oplus D(E_0 + k, q)_{o(d - 1, 2)},
\] (5)

\[
D(E_0, s)_{o(d,2)} \simeq \sum_{q^+ = \frac{1}{2}}^{s^+} \sum_{k = 0}^{\infty} \oplus D(E_0 + k, q^+)_{o(d - 1, 2)} \oplus \sum_{q^- = \frac{1}{2}}^{s^-} \sum_{k = 0}^{\infty} \oplus D(E_0 + k, q^-)_{o(d - 1, 2)}.
\] (6)
For the proofs see Appendix.

At the boundary energy \( D(E_0(s), s)_{o(d,2)} \) contains a singular submodule \( S \) which is

\[
S(s) \simeq D(d + s - 1, s - 1)_{o(d,2)}
\]

for \( s \geq 1 \) and

\[
S(0) \simeq D\left(\frac{d + 2}{2}, 0\right)_{o(d,2)}, \quad S\left(\frac{1}{2}\right) \simeq D\left(\frac{d + 1}{2}, \frac{1}{2}\right)_{o(d,2)}
\]

for \( s = 0 \) or \( 1/2 \). It is important that \( S \) itself can be endowed with an \( o(d,2) \) invariant positive-definite form (which, however, cannot be extended to an invariant form on \( D(E_0(s), s)_{o(d,2)} \)), hence forming a unitary \( o(d,2) \)-module. This allows us to apply Theorems 1 and 2 to its branching as well. For integer spins we obtain

\[
S(s) \simeq \sum_{q=0}^{s-1} \sum_{k=1}^{\infty} \oplus D\left(k + d + s - 2, q\right)_{o(d-1,2)},
\]

\[
S(0) \simeq \sum_{k=2}^{\infty} \oplus D\left(\frac{d - 2}{2} + k, 0\right)_{o(d-1,2)}
\]

for any \( d \). For half-integer spins

\[
S(s) \simeq \sum_{q=1}^{s-1} \sum_{k=1}^{\infty} \oplus D\left(k + d + s - 2, q\right)_{o(d-1,2)},
\]

\[
S\left(\frac{1}{2}\right) \simeq \sum_{k=1}^{\infty} \oplus D\left(k + \frac{d - 1}{2}, \frac{1}{2}\right)_{o(d-1,2)}
\]

for even \( d \) and

\[
S(s) \simeq \sum_{q=\frac{1}{2}}^{s-1} \sum_{k=1}^{\infty} \oplus D\left(k + d + s - 2, q^+\right)_{o(d-1,2)} \oplus \sum_{q=-\frac{1}{2}}^{s-1} \sum_{k=1}^{\infty} \oplus D\left(k + d + s - 2, q^-\right)_{o(d-1,2)},
\]

\[
S\left(\frac{1}{2}\right) \simeq \sum_{k=1}^{\infty} \oplus D\left(k + \frac{d - 1}{2}, \frac{1}{2}\right)_{o(d-1,2)} \oplus \sum_{k=1}^{\infty} \oplus D\left(k + \frac{d - 1}{2}, -\frac{1}{2}\right)_{o(d-1,2)}
\]

for odd \( d \).

This gives

**Corollary 1** The unitary \( o(d,2) \)-module \( \mathcal{H}(d+s-2, s)_{o(d,2)} \) with integer spin \( s = 1, 2, \ldots \) has the following branching into \( o(d - 1, 2) \) unitary modules

\[
\mathcal{H}(d+s-2, s)_{o(d,2)} \simeq \sum_{k=0}^{\infty} \oplus D\left(k + d + s - 2, s\right)_{o(d-1,2)} \oplus \sum_{q=0}^{s-1} \oplus D\left(d + s - 2, q\right)_{o(d-1,2)}.
\]
Corollary 2 The unitary $o(d,2)$-module $\mathcal{H}(d+s-2, s^\pm)_{o(d,2)}$ for even $d$ and $\mathcal{H}(d+s-2, s)_{o(d,2)}$ for odd $d$ with half-integer spin $s = \frac{3}{2}, \frac{5}{2}, \ldots$ have the following branching into the $o(d-1,2)$ unitary modules

$$
\mathcal{H}(d+s-2, s^\pm)_{o(d,2)} \simeq \sum_{k=0}^{\infty} \oplus D(k + d + s - 2, s)_{o(d-1,2)} \oplus \sum_{q=\frac{1}{2}}^{s-1} \oplus D(d + s - 2, q)_{o(d-1,2)},
$$

$$
\mathcal{H}(d+s-2, s)_{o(d,2)} \simeq \sum_{k=0}^{\infty} \oplus D(k + d + s - 2, s^+)_{o(d-1,2)} \oplus \sum_{k=0}^{\infty} \oplus D(k + d + s - 2, s^-)_{o(d-1,2)}
$$

$$
\oplus \sum_{q^+=\frac{1}{2}}^{s^+-1} \oplus D(d + s - 2, q^+)_{o(d-1,2)} \oplus \sum_{q^-=\frac{1}{2}}^{s^-1} \oplus D(d + s - 2, q^-)_{o(d-1,2)}.
$$

Corollary 3 $o(d,2)$ singleton modules have the following branchings into $o(d-1,2)$ unitary modules

$$
\mathcal{H}\left(\frac{d-2}{2}, 0\right)_{o(d,2)} \simeq D\left(\frac{d-2}{2}, 0\right)_{o(d-1,2)} \oplus D\left(\frac{d}{2}, 0\right)_{o(d-1,2)},
$$

for any $d$,

$$
\mathcal{H}\left(\frac{d-1}{2}, \frac{1}{2}\right)_{o(d,2)} \simeq D\left(\frac{d-1}{2}, \frac{1}{2}\right)_{o(d-1,2)}
$$

for even $d$ and

$$
\mathcal{H}\left(\frac{d-1}{2}, \frac{1}{2}\right)_{o(d,2)} \simeq D\left(\frac{d-1}{2}, \frac{1}{2}\right)_{o(d-1,2)} \oplus D\left(\frac{d-1}{2}, \frac{1}{2}\right)_{o(d-1,2)}
$$

for odd $d$.

These results have the following field-theoretical interpretation. The dimensional degression of one dimension for the spin $s$ field with the energy $E_0$ above the unitarity bound should give rise to the set of fields with all spins $0, 1, \ldots, s$ for the bosonic case and $\frac{1}{2}, \frac{3}{2}, \ldots, s$ for the fermionic one. In the latter case each spin is doubled if the dimension of $AdS_{d+1}$ is even. The energy spectrum is $E_0 + n, n = 0, 1, \ldots$ for every spin. Note that the dimensional degression of a massless scalar or Dirac field in $AdS_{d+1}$ does not produce a massless field in $AdS_d$. Similarly, a spin $s \geq 1$ massless field in $d+1$ dimensions has the lowest energy $E_0 = d + s - 2$. Its degression by one dimension gives a set of spin $s$ fields with energies $E_n = n + d + s - 2, n = 0, 1, \ldots$ plus a set of fields with spins $0, 1, \ldots, s - 1$ or $\frac{1}{2}, \frac{3}{2}, \ldots, s - 1$ and the energy $d + s - 2$. In the fermion case each field is doubled if the dimension of $AdS_{d+1}$ is even. All of these fields are massive in $d$ dimensions. Dimensional degression of a scalar singleton gives massless scalar fields. The dimensional degression of a fermion singleton yields one massless Dirac field if the original space is odd-dimensional or two otherwise. These results agree with the well-known fact that singletons are conformal fields in one lower dimension \cite{27, 28, 29, 23, 7}.

The obtained results make it easy to analyze the dimensional degression over several dimensions. Let us consider the example of the scalar representation. The branching of an $o(d + d' - 1, 2)$-module $D(E_0, 0)_{o(d+d'-1,2)}$ as $o(d - 1, 2)$-modules yields

$$
D(E_0, 0)_{o(d+d'-1,2)} \simeq \sum_{k=0}^{\infty} \oplus \left( \frac{k + d' - 1}{d' - 1} \right) D(E_0 + k, 0)_{o(d-1,2)}. \quad (13)
$$
Thus, the dimensional degression of a scalar field in AdS\(_{d+d'}\) over \(d'\) coordinates gives the set of scalar fields of the energies \(E_0 + k\) and the multiplicities \(\binom{k+d'-1}{d'-1}\), where \(k = 0, 1, \ldots\). The degeneracy manifests that the scalar fields of energy \(E_0 + k\) carry the module

\[
D(E_0 + k, 0)_{o(d-1,2)} \oplus \bigoplus_{l=0}^{[K]} D(k - 2l)
\]

of \(o(d-1,2) \oplus o(d') \subset o(d + d' - 1, 2)\), where \(D(k - 2l)\) is an \(o(d')\)-module of weight \((k - 2l, 0, \ldots)\). The binomial coefficients in (13) are just the dimensions of \(\bigoplus_{l=0}^{[K]} D(k - 2l)\) \(o(d')\)-modules.

### 3 Field-theoretical degression of one dimension

The aim of this section is to show how the results of the group-theoretical analysis of Section 2 are reproduced in the field-theoretical models.

Let us describe a \((d+1)\)-dimensional anti-de Sitter space as a hyperboloid

\[
y^2 - y_0^2 + \cdots - y_d^2 = 1
\]

in \(\mathbb{R}^{d+2}\). We choose the following coordinates in AdS\(_{d+1}\)

\[
y_{-1} = \cosh(z)\tilde{y}_{-1}, \quad y_0 = \cosh(z)\tilde{y}_0, \quad \ldots, \quad y_{d-1} = \cosh(z)\tilde{y}_{d-1}, \quad y_d = -\sinh(z),
\]

where \(z \in (-\infty, +\infty)\) and \(\tilde{y}_{-1}, \ldots, \tilde{y}_{d-1}\) satisfy

\[
\tilde{y}_{-1}^2 + \tilde{y}_0^2 + \cdots - \tilde{y}_{d-1}^2 = 1,
\]

i.e., describe AdS\(_d\). The line element of AdS\(_{d+1}\) has the form

\[
ds^2_{AdS_{d+1}} = \cosh^2(z)ds^2_{AdS_d} - dz^2,
\]

where \(ds^2_{AdS_d}\) is the line element of AdS\(_d\). The warp factor \(\cosh^2(z)\) manifests that AdS\(_{d+1}\) is not a direct product of AdS\(_d\) with some other manifold.

We use the following index conventions

\[
\alpha, \beta, \ldots = 0, 1, \ldots, d, \quad \mu, \nu, \ldots = 0, 1, \ldots, d-1, \quad \bullet \equiv d,
\]

i.e., lower case Greek letters \(\alpha, \beta \ldots\) correspond to coordinates of AdS\(_{d+1}\), lower case Greek letters \(\mu, \nu \ldots\) correspond to coordinates of AdS\(_d\), and \(\bullet\) denotes the extra coordinate of AdS\(_{d+1}\) compared to AdS\(_d\).

Let us comment on the relation between mass and energy in AdS\(_d\). We refer to the term \(m^2\) in the equation

\[
(\Box_{AdS_d} + m^2)\psi_{\mu_1 \ldots \mu_d}(x) = 0,
\]

where \(\Box_{AdS_d} \equiv D^\mu D_\mu\) is the Laplace-Beltrami operator, and \(D_\mu\) is the AdS\(_d\) covariant derivative, as the mass square of the symmetric bosonic field \(\psi_{\mu_1 \ldots \mu_d}(x)\). The relation with the lowest energy is \([30]\)

\[
m^2 = E(E - d + 1) - s.
\]
Equation (17) yields two solutions for the energies for the same mass. In the case of spin 1, 2, ..., only the highest energy obeys the unitary condition (2). In the scalar field case there are two options. If $m^2 > -(d-3)(d+1)/4$, only the highest energy satisfies the unitary condition (1). If $-(d-1)^2/4 \leq m^2 \leq -(d-3)(d+1)/4$, both solutions for the energy obey the unitarity condition. Thus, in the latter case, the scalar field equation (16) admits two types of normalizable solutions. Note that they are not mutually orthogonal with respect to the standard norm [31, 32] (see also Subsection 3.1).

3.1 Scalar field

The free wave equation of a massive scalar field in $AdS_{d+1}$ is

$$\Box_{AdS_{d+1}} + M^2)\Phi(x, z) = 0,$$

(18)

where $x$ are local coordinates of $AdS_d$. In the coordinates (14) the Laplace-Beltrami operator $\Box_{AdS_{d+1}} = \frac{1}{\sqrt{|g|}} \partial_\alpha g^{\alpha\beta} \sqrt{|g|} \partial_\beta$ is

$$\Box_{AdS_{d+1}} = \frac{1}{\cosh^2(z)} \Box_{AdS_d} - \frac{1}{\cosh^d(z)} \partial_z \cosh^d(z) \partial_z,$$

(19)

where $\partial_z \equiv \partial/\partial z$.

Recall, that the space of solutions is endowed with the inner product defined as the electric charge

$$(\phi, \psi) = (\phi^+, \psi^+) = i \int_\Sigma d\sigma^\alpha \phi^+ \nabla^\alpha \psi^+ = i \int_{t=\text{const}} g^{00} \sqrt{|g_{AdS_{d+1}}|} d^d x \phi^+ \nabla^0 \psi^+$$

(20)

of the positive frequency parts of two real solutions $\phi$ and $\psi$ of the wave equation. $\Sigma$ is a space-like surface that can be chosen to be a surface of constant time $t$. Since the electric charge conserves, so defined inner product is formally independent of the integration surface choice. The conditions of finiteness and actual conservation for the norm impose certain boundary conditions that restrict the energies of solutions.

Let us look for solutions of the equation (18) in the form

$$\Phi(x, z) = \sum_{N=0}^{\infty} \phi_N(x) P_N(z),$$

(21)

where $P_N(z)$ form an orthogonal complete set with respect to the norm

$$\int_{-\infty}^{\infty} dz \cosh^{d-2}(z) P_N(z) P_M(z) = \delta_{NM}$$

(22)

inherited from the norm (20) in the $z$ sector. The conservation of the norm (20) requires solutions to satisfy the boundary conditions

$$\cosh^d(z) \left( P_N(z) \partial_z P_M(z) - P_M(z) \partial_z P_N(z) \right) \bigg|_{z \to \pm \infty} = 0.$$

(23)

Substitution of the expansion (21) into Eq. (18) yields a tower of massive scalar fields in $AdS_d$

$$(\Box_{AdS_d} + m^2_N)\phi_N(x) = 0$$

(24)
along with the following equation on $P_N(z)$

$$\partial_z^2 P_N(z) + d \tanh(z) \partial_z P_N(z) + \left[ \frac{m_N^2}{\cosh^2(z)} - M^2 \right] P_N(z) = 0. \quad (25)$$

By the change of variables $\tan(\varphi) = \sinh(z)$ and $p_N(\varphi) = \cos^{-\frac{d-1}{2}}(\varphi) P_N(\varphi)$ with the coordinate $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the equation (25) and norm (22) are mapped to the one-dimensional quantum mechanics on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with the energy $E = m_N^2 + \frac{(d-1)^2}{4}$ and the Pöschl-Teller potential [33] $U(\varphi) = \cos^{-2}(\varphi) \left( M^2 + \frac{d^2-1}{4} \right)$

$$\partial^2_\varphi p_N + \left[ m_N^2 + \frac{(d-1)^2}{4} - \cos^{-2}(\varphi) \left( M^2 + \frac{d^2-1}{4} \right) \right] p_N = 0, \quad (26)$$

$$\frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi p_N(\varphi) p_M(\varphi) = \delta_{NM}. \quad (27)$$

The orthogonality and boundary conditions (22) and (23) constrain solutions to the “irregular” class

$$P_N^{irr}(z) = N_N^{-\kappa} \cosh^{-\frac{d-2\kappa}{2}}(z) \, _2F_1\left(N - 2\kappa + 1, -N; 1; \frac{1 + \tanh(z)}{2}\right), \quad (28)$$

$$M^2 \in \left[ -\frac{d^2}{4}, -\frac{d^2}{4} + 1 \right)$$

and “regular” class

$$P_N^{reg}(z) = N_N^{\kappa} \cosh^{-\frac{d+2\kappa}{2}}(z) \, _2F_1\left(N + 2\kappa + 1, -N; 1 + \kappa; \frac{1 + \tanh(z)}{2}\right), \quad (29)$$

$$M^2 \in \left[ -\frac{d^2}{4}, +\infty \right),$$

where $_2F_1(a, b; c; x)$ is the hypergeometric function (see e.g. [34]),

$$\kappa \equiv \sqrt{\frac{d^2}{4} + M^2}$$

and the normalization factor

$$N_N^{\kappa} = \sqrt{\frac{2\kappa + 2n + 1}{2^{\kappa+\frac{1}{2}} \Gamma(\kappa+1)}} \sqrt{\frac{\Gamma(2\kappa+n+1)}{n!}} \quad (30)$$

is fixed by (22).

For the “irregular” solutions the lowest energies and the related mass spectrum (17) are given by the formulas

$$E_N = N + \frac{d - 2\kappa}{2}, \quad (31)$$

$$m_N^2 = \left(N - \kappa + \frac{1}{2}\right)^2 - \frac{(d-1)^2}{4}.$$
The “regular” solutions have the energies
\[ E_N = N + \frac{d + 2\kappa}{2} \] (32)
and the masses
\[ m_N^2 = \left( N + \kappa + \frac{1}{2} \right)^2 - \frac{(d - 1)^2}{4}. \]

Let us discuss the normalization conditions for “regular” and “irregular” solutions in more detail. First of all, we observe that the integral (22) absolutely converges in the union of “regular” and “irregular” spaces. So, the key role is played by the boundary conditions (23). As is obvious from (29), the regular solutions decrease as \( \exp(-d + 2\kappa|z|) \) at \( z \rightarrow \pm\infty \) \((N = 0, 1, \ldots)\) so that each of the terms in (23) vanishes. For the “irregular” class (28) the leading contribution to each term in (23) increases at infinity as \( \exp(2\kappa|z|) \), but it cancels between the two terms both at \( z \rightarrow +\infty \) and at \( z \rightarrow -\infty \). The subleading terms behave as \( \exp(2(\kappa - 1)|z|) \) and vanish for \( \kappa < 1 \), which is just the domain where the “irregular” solutions form a unitary module. If a “regular” solution is paired with an “irregular” one, the condition (23) is not satisfied because different terms in (23) tend to different constants that cannot cancel out. As a result, the \( o(d, 2) \)-invariant norm (20) is well-defined for the spaces of “regular” and “irregular” solutions separately, but not for their union. This implies in particular that the “regular” and “irregular” solutions do not possess definite mutual orthogonality properties.

Using that both \( P^\text{irr}_N \) and \( P^\text{reg}_N \) form orthonormal bases, we can perform the dimensional degression at the action level for each class. Indeed, plugging (19) and (21) into the scalar field action in \( \text{AdS}_{d+1} \)
\[ S_{\text{AdS}_{d+1}} = \frac{1}{2} \int d\mu_{d+1} \Phi(\Box_{\text{AdS}_{d+1}} + M^2)\Phi, \]
we observe that
\[ S_{\text{AdS}_{d+1}} = \sum_{N=0}^{\infty} S^\text{AdS}_N, \]
where
\[ S^\text{AdS}_N = \frac{1}{2} \int d\mu_d \phi_N(\Box_{\text{AdS}_d} + m_N^2)\phi_N \]
is the action of a scalar field of the mass \( m_N^2 \) in \( \text{AdS}_d \). Here \( d\mu_d \equiv \sqrt{|g_{\text{AdS}_d}|}d^d x \) is the \( \text{AdS}_d \)-invariant volume element.

These results are consistent with the group-theoretical analysis of Section 2. In accordance with Theorem 1 for a fixed energy \( E_0 \), the dimensional degression gives the set of scalar fields with the energy spectrum \( E_0 + k, k = 0, 1, \ldots \). The appearance of two types of solutions, namely, “regular” and “irregular” ones, results from the ambiguity in the relation between the square mass and the lowest energy in (17). Note that the dimensional degression of a massless scalar field over one coordinate produces no massless fields.

Let us now compare our results with those of Metsaev [1] who analyzed analogous problem for the conformal mass case \( m^2 = -\frac{d^2 - 1}{4} \) where the free wave equation is conformal invariant. In [1], the following Fourier expansion was used
\[ \Phi(x, X, Z) = \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{in\tau} \phi_n(x, z), \] (33)
where $x, X$ and $Z$ are Poincaré coordinates of $\text{AdS}_{d+1}$

$$ds^2_{\text{AdS}_{d+1}} = \frac{1}{Z^2}(dx^0 dx_a + dX^2 + dZ^2), \quad Z > 0$$

(in the rest of this section we use the mostly plus metric convention of [1]) and the coordinates $z$ and $\varphi$ are defined by

$$Z = z \sin \varphi, \quad X = z \cos \varphi,$$

with $\varphi \in (0, \pi)$ and $z > 0$. Plugging the Fourier expansion into the scalar field equation gives the mass spectrum

$$m_n^2 = n^2 - \frac{(d - 1)^2}{4}$$

and the energy spectrum

$$E_n = n + \frac{d - 1}{2}, \quad n = 0, 1, \ldots$$

To establish the precise correspondence with our results, we should however use the Fourier expansion

$$\Phi(x, X, Z) = \frac{1}{\sqrt{z}} \sum_{n=0}^{\infty} \psi_n \cos(n \varphi) \quad \text{or} \quad \Phi(x, X, Z) = \frac{1}{\sqrt{z}} \sum_{n=1}^{\infty} \xi_n \sin(n \varphi).$$

This yields the energy spectra

$$E_{\psi_n} = n + \frac{d - 1}{2}, \quad E_{\xi_{n+1}} = n + \frac{d + 1}{2}, \quad n = 0, 1, \ldots$$

The relationship between $\psi_n, \xi_n$ and $\phi_n$ is

$$\phi_n = \begin{cases} \psi_n - i \xi_n & n = 1, 2, \ldots, \\ \psi_0 & n = 0, \\ \psi_n + i \xi_n & n = -1, -2, \ldots. \end{cases}$$

We see that the expansion (33) of [1] mixes the two branches (34). Note that the states described by (33) do not form an orthogonal set with respect to the invariant norm (20) because the classes of “regular” and “irregular” solutions do not possess definite orthogonality properties with respect to the norm (20).

### 3.2 Massless Dirac field

The action of Dirac spinor field $\tilde{\Psi}$ of mass $M$ in $\text{AdS}_{d+1}$ is

$$S = \int d\mu_{d+1} \tilde{\Psi} \left(i \Gamma^\alpha E_{\alpha}^a \nabla_a - M\right) \tilde{\Psi},$$

where

$$\{\Gamma^\alpha, \Gamma^\beta\} = 2\eta^{\alpha \beta}, \quad \eta^{\alpha \beta} = (+, -, \ldots, -).$$

The covariant derivative is

$$\nabla_a = \partial_a - \frac{i}{2} \Omega_{\alpha \beta}^a M_{\alpha \beta}, \quad M_{\alpha \beta} = \frac{i}{4} [\Gamma_\alpha, \Gamma_\beta].$$
$E^\alpha_\mu$ and $\Omega^{\alpha\beta}_\Omega$ are inverse vielbein and spin-connection of $AdS_{d+1}$.

For the sake of simplicity let us consider only the massless case of $M = 0$. Let us note that for the even-dimensional $AdS_{d+1}$ a Weyl spinor field does not form a $o(d, 2)$-module. In this case it is convenient to choose the following coordinates

$$d^2_{AdS_{d+1}} = \cos^2(\varphi) \left( ds^2_{AdS_d} - d\varphi^2 \right), \quad \varphi \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$$

related to the coordinates $\left(14\right)$ via tan($\varphi$) = sinh($z$). In these coordinates, the vielbein and the spin-connection of $AdS_{d+1}$ can be expressed via the vielbein $e^\mu_\mu$ and the spin-connection $\omega^{\mu\nu}_\mu$ of $AdS_d$ as follows

$$E^\mu_\mu = \cos^{-1}(\varphi) e^\mu_\mu, \quad E^\bullet_\mu = \cos^{-1}(\varphi),$$

$$E^\mu_\bullet = 0, \quad E^\bullet_\mu = 0,$$

$$\Omega^{\mu\nu}_\mu = \omega^{\mu\nu}_\mu, \quad \Omega^{\mu\nu}_\bullet = -\tan(\varphi) e^\mu_\bullet,$$

Also substituting

$$\bar{\Psi}(x) = \cos^d(\varphi)\Psi(x),$$

the Dirac action in coordinates $\left(38\right)$ takes the form

$$S = \int d\mu_d \int d\varphi \bar{\Psi} \left( i\Gamma^\mu e^\mu_\mu D_\mu + i\Gamma^\bullet \partial \varphi \right) \Psi,$$

where $D_\mu = \partial_\mu - \frac{i}{2} \omega^{\mu\nu}_\mu M_{\nu\nu}$ is the covariant derivative in $d$ dimensions.

The norm on the space of solutions is defined in the standard way

$$(\Psi_1, \Psi_2) = \int_{t=\text{const}} \sqrt{|g_{AdS_{d+1}}|} d^{d+1}x \ E^\alpha_\mu \bar{\Psi}_1 \Gamma^\alpha \Psi_2 = \int_{t=\text{const}} \sqrt{|g_{AdS_d}|} d^{d-1}x \ e^\alpha_\mu \frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \ \bar{\Psi}_1 \Gamma^\alpha \Psi_2.$$  \hfill (42)

The norm conservation condition requires that solutions satisfy the following boundary conditions at $\varphi = \pm \frac{\pi}{2}$

$$\left. \int \sqrt{|g_{AdS_d}|} d^{d-1}x \ \bar{\Psi}_1 \Gamma^\bullet \Psi_2 \right|_{\varphi = \pm \frac{\pi}{2}} = 0.$$  \hfill (43)

The analysis for even and odd $d + 1$ is different.

### 3.2.1 Odd $d$

For odd $d$ we use the following representation of gamma matrices $\Gamma^\alpha$ in $d+1$ dimensions

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^1, \quad \Gamma^\bullet = 1 \otimes i\sigma^2,$$

This is because the spinor representation of $o(d, 2)$ beside generators $T^{\alpha\beta} = \frac{1}{4} [\gamma^\alpha, \gamma^\beta]$ corresponding its Lorentz subalgebra $o(d, 1)$ involves $T^{\alpha^{-1}} = \frac{1}{2} \gamma^\alpha$ that do not commute with the Weyl projectors.
where \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \) and \( \sigma^1, \sigma^2, \sigma^3 \) are Pauli matrices. Let us expand the field \( \Psi \) as follows

\[
\Psi(x, \varphi) = \sum_n (\psi^n_+ (x) \otimes \chi^n_+ (\varphi) + \psi^n_- (x) \otimes \chi^n_- (\varphi)),
\]

(45)

where \( \psi^n_\pm (x) \) are 0(d-1,2) Weyl spinors. The substitution of this expansion into the action (41) gives the sum of Dirac actions in \( d \) dimensions

\[
S = S^+ + S^- , \quad S^\pm = \sum_n \int d\mu_d \overline{\psi}^\pm_n \left( i\gamma^\mu e^\mu \mu D^\mu \mp m_n \right) \psi^\pm_n ,
\]

(46)

where \( m_n \) are eigenvalues of the mass operator

\[
\hat{m} = \begin{pmatrix} i\partial_\varphi & 0 \\ 0 & -i\partial_\varphi \end{pmatrix}
\]

(47)

\( \hat{m} \chi^\pm_n = \pm m_n \chi^\pm_n \). The eigenvectors \( \chi_n \) can be represented as

\[
\chi^+_n (\varphi) = \frac{1}{\sqrt{\pi}} \left( \cos \alpha e^{-im_n \varphi} \right), \quad \chi^-_n (\varphi) = \frac{1}{\sqrt{\pi}} \left( -\sin \alpha e^{im_n \varphi} \right),
\]

(48)

where \( \alpha \in [0, 2\pi) \).

For any \( \Psi^\pm_n (x, \varphi) = \psi^n_\pm (x) \otimes \chi^n_\pm (\varphi) \) the scalar product (42) is

\[
(\Psi^\pm_n, \Psi^\pm_{n'}) = \int_{t=\text{const}} \sqrt{|g_{\text{AdS}}|} d^{d-1}x \frac{\partial}{\partial \varphi} \overline{\psi}^\pm_n \gamma^\mu \psi^\pm_{n'} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \ (\chi^\pm_n)^\dagger \chi^\pm_{n'}
\]

(49)

\[
= \delta_{nn'} \int_{t=\text{const}} \sqrt{|g_{\text{AdS}}|} d^{d-1}x \frac{\partial}{\partial \varphi} \overline{\psi}^\pm_n \gamma^\mu \psi^\pm_{n'}
\]

(50)

and

\[
(\Psi^\pm_n, \overline{\Psi}^\mp_{n'}) = 0 ,
\]

provided that the mass spectrum is

\[
m_n = 2n + a , \quad n \in \mathbb{Z} , \quad a \in [0, 2) .
\]

The boundary conditions (43) determine the values of \( a = \frac{1}{2} \) and \( \alpha = \frac{\pi}{4} \). Therefore, we have the spectrum

\[
m_n = 2n + a , \quad n \in \mathbb{Z}.
\]

(51)

These results are in agreement with the group-theoretical analysis of Section 2. Namely, the energy of massless Dirac field in AdS\(_{d+1}\) is \( E_0 = \frac{d}{4} \). The mass parameter in AdS\(_d\) is \( m_n = E_0 - \frac{d+1}{2} \). By Theorem 2 the degression of a massless spin 1/2 field results in two mass spectra both defined by the formula \( m_n = n + \frac{1}{2} \), \( n = 0, 1, \ldots \). To relate they with (46) and (51), we rewrite the spectrum (51) as \( m_n = 2n + \frac{1}{2} \) and \( -m_n = 2n + 1 + \frac{1}{2} \), \( n = 0, 1, \ldots \).
3.2.2 Even \(d\)

Let us now consider the case of even \(d\). We expand Weyl \(o(d,2)\) spinor \(\Psi\) as follows

\[
\Psi = \sum_n \left( \psi_n^+ (x) f_n^+ (\varphi) + \psi_n^- (x) f_n^- (\varphi) \right),
\]

where \(f_n^\pm (\varphi)\) are complex functions, \(\psi_n^\pm \equiv \Pi^\pm \psi_n\) and \(\Pi^\pm = \frac{1}{2} (1 \pm i \Gamma^\bullet)\) are projectors, i.e.,

\[
i \Gamma^\bullet \psi_n^\pm = \pm \psi_n^\pm.
\]

Substituting the expansion (52) into the action (41), we demand that the resulting action be of the form

\[
S = \sum_n \int d\mu \overline{\psi}_n \left( i \Gamma^\mu \epsilon^\mu_{\alpha} D_\alpha - m_n \right) \psi_n.
\]

This gives the equations

\[
\partial_\varphi f_n^+ = -m_n f_n^- , \quad \partial_\varphi f_n^- = m_n f_n^+.
\]

along with the normalizing conditions

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi (f_n^\pm)^* f_n^\pm = \delta_{nn'}.
\]

From the conditions (56) it follows that, for any \(\Psi_n (x, \varphi) = \psi_n^+ (x) f_n^+ (\varphi) + \psi_n^- (x) f_n^- (\varphi)\), the scalar product (42) is

\[
(\Psi_n , \Psi_{n'}) = \delta_{nn'} \int_{t=\text{const}} \sqrt{|g_{\text{AdS}d}|} d^{d-1}x \epsilon^0_{\alpha} \overline{\psi}_n \Gamma^\alpha \psi_{n'}.
\]

The boundary condition (43) requires

\[
f_n^+(f_n^-)^* |_{\varphi = \pm \frac{\pi}{2}} = 0.
\]

The complete orthonormal sets of functions \(f_n^\pm\) that satisfy (55), (56), (58) are

\[
f_n^+ = \frac{1}{\sqrt{2\pi}} \left( i e^{im_n \varphi} + e^{-im_n \varphi} \right), \quad f_n^- = \frac{1}{\sqrt{2\pi}} \left( i e^{-im_n \varphi} + e^{im_n \varphi} \right)
\]

with the mass spectrum

\[
m_n = 2n + 1, \quad n \in \mathbb{Z}
\]

Taking into account that a sign of \(m_n\) does not matter we observe that, in agreement with Section 2 this mass spectrum is equivalent to

\[
m_n = n + 1, \quad n = 0,1,2,\ldots.
\]
3.3 Massless spin one

The Maxwell action of a vector massless field $h^\alpha$ in $AdS_{d+1}$ is

$$S = -\frac{1}{4} \int d\mu_{d+1} F_{\alpha \beta} F^{\alpha \beta},$$

where $F_{\alpha \beta} = \partial_\alpha h_\beta - \partial_\beta h_\alpha$. It is invariant under the spin one gauge transformation

$$\delta h_\alpha = \partial_\alpha \xi.$$

Using index conventions (15), the vector field $h_\alpha$ in $d+1$ dimensions decomposes into a vector $h^\mu$ and a scalar $h^\bullet$ in $d$ dimensions. In the coordinates (14), the action takes the form

$$S = \int d\mu \int dz \cosh^d(z) \left\{ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \cosh^2(z) h^\mu \left[ \partial_z^2 + (d + 2 \tan(z) \partial_z + 2d - \frac{2(d - 1)}{\cosh^2(z)} \right] h_\mu + \frac{1}{2} \cosh^2(z) \partial_\mu h^\bullet \partial^\mu h^\bullet + \frac{1}{2} \cosh^2(z) \partial_\mu h^\bullet \partial_\nu \cosh^2(z) h^\nu \right\}.$$

The gauge transformations of the $d$-dimensional fields are

$$\delta h^\mu = \partial^\mu \xi,$$
$$\delta h^\bullet = -\partial_z \cosh^2(\xi).$$

Let us expand the vector and scalar fields in $d$ dimensions as

$$h^\mu = \sum_{n=0}^{\infty} a^\mu_n(x) P^1_n(z), \quad h^\bullet = \sum_{n=0}^{\infty} \phi_n(x) R^1_n(z),$$

where both the set of functions $P^1_n(z)$ and the set $R^1_n(z)$ form orthonormal bases with respect to appropriate norms whose precise form will be specified below. Eq. (65) suggests the expansion for the gauge parameter

$$\xi = \sum_{n=0}^{\infty} \xi_n(x) P^1_n(z).$$

The basis functions can be chosen in such a way that the substitution of the expansions (66) into the action gives

$$S = \sum_{n=0}^{\infty} S_n,$$

$$S_n = \int d\mu \left\{ -\frac{1}{4} f_{\mu \nu} f^{\mu \nu} + \frac{m_n^2}{2} a^\mu_n a_\mu n + m_n a^\mu_n \partial_\mu \phi_{n+1} + \frac{1}{2} \partial_\mu \phi_n \partial^\mu \phi_n \right\},$$

where $f_{\mu \nu} = \partial_\mu a_{\nu \mu} - \partial_\nu a_{\mu \mu}$ and the masses are $m_n^2 = (n+1)(n+d-2)$. The gauge transformations take the form

$$\delta a^\mu_n = \partial^\mu \xi_n, \quad \delta \phi_n = -m_{n-1} \xi_{n-1}, \quad n = 0, 1, 2, \ldots$$
To obtain the action (67) the following properties have to be satisfied by \( P_n^1(z) \) and \( R_n^1(z) \). The diagonalization of the spin-one part (63) demands \( P_n^1(z) \) to solve the equation

\[
\partial^2_z P_n^1(z) + (d + 2) \tanh(z) \partial_z P_n^1(z) + \left[ \frac{m_n^2 - 2(d - 1)}{\cosh^2(z)} + 2d \right] P_n^1(z) = 0.
\] (69)

The diagonalization condition for the cross term (64) gives the equation for \( R_n^1(z) \)

\[
\partial^2_z R_n^1(z) + d \tanh(z) \partial_z R_n^1(z) + \left[ \frac{M_n^2 - d + 2}{\cosh^2(z)} + 2(d - 2) \right] R_n^1(z) = 0,
\] (70)

where \( M_n^2 = n(n + d - 3) \). The normalization conditions on \( P_n^1(z) \) and \( R_n^1(z) \) result from the integration of the coefficients in front of the kinetic terms of the action over the degression coordinate \( z \)

\[
\int_{-\infty}^{\infty} dz \cosh^d(z) P_n^1(z) P_m^1(z) = \delta_{nm}, \quad \int_{-\infty}^{\infty} dz \cosh^{d-2}(z) R_n^1(z) R_m^1(z) = \delta_{nm}.
\] (71)

The solutions of the equations (69), (70) and (71) are

\[
P_n^1(z) = N_n^\frac{d-2}{2} \cosh^{-d}(z) _2F_1 \left( -n, n + d - 1; \frac{d}{2}; \frac{1 + \tanh(z)}{2} \right),
\]

\[
R_n^1(z) = N_n^\frac{d+2}{2} \cosh^{d+2}(z) _2F_1 \left( -n, n + d - 3; \frac{d-2}{2}; \frac{1 + \tanh(z)}{2} \right),
\]

where \( N_n^\frac{d}{2} \) is defined in (30). The \( P_n^1(z) \) and \( R_n^1(z) \) form orthonormal complete sets with respect to the corresponding norms. Note that \( P_n^1(z) \) and \( R_n^1(z) \) are related by

\[
\frac{1}{\cosh^{d-2}(z)} \partial_z \left( \cosh^{d-2}(z) R_n^1(z) \right) = -m_n P_n^1(z).
\]

The gauge transformation laws (65) for the scalar fields \( \phi_n \) with \( n = 1, 2 \ldots \) imply that all of them are Stueckelberg, i.e., can be gauged fixed to zero. As a result, the dimensional degression of the massless spin one field gives the tower of massive spin one fields with the energies

\[
E_n = n + d - 1, \quad n = 0, 1, \ldots
\] (72)

and just one scalar field with energy \( E = d - 1 \). This precisely matches the pattern of Theorem 1. Let us note again that all fields resulting from the dimensional degression of a massless spin one are massive.

### 3.4 Massless spin two

A massless field of spin two \( h^{\alpha_1 \alpha_2} \) in \( AdS_{d+1} \) is described by the action (35)

\[
S = \int d\mu_{d+1} \left\{ \frac{1}{2} \nabla^\alpha h_{\beta_1 \beta_2} \nabla_\alpha h^{\beta_1 \beta_2} - \nabla_\alpha h^{\alpha \beta} \nabla^\gamma h_{\gamma \beta} + \nabla_\alpha h^{\alpha \beta} \nabla_\beta h^\eta - \frac{1}{2} \nabla^\alpha h^\eta \nabla_\alpha h^\eta + h_{\alpha_1 \alpha_2} h^{\alpha_1 \alpha_2} + \frac{d-2}{2} h^2 \right\},
\] (73)
where
\[ h' = g_{\alpha\beta}h^{\alpha\beta}. \]
The action (73) is invariant under the gauge transformation
\[ \delta h^{\alpha\beta} = \nabla^{(\alpha}\xi^{\beta)}, \]
where \( \nabla^{\alpha} \) is the covariant derivative in \( AdS_{d+1} \).

Using the index conventions (15) the field \( h^{\alpha_1...\alpha_2} \) in \( d+1 \) dimensions decomposes into a spin two field \( h^{\mu_1\mu_2} \), a vector \( h^{\mu\nu} \), and a scalar \( h^{\bullet\bullet} \) in \( d \) dimensions. In the coordinates (14) the action (73) takes the form
\[ S = S_{22} + S'_{22} + S_{11} + S_{21} + S'_{21} + S_0, \]
where
\[ S_{22} = \int d\mu_d \int dz \cosh^d(z) \left\{ \frac{1}{2} \cosh^2(z) D^\mu h_{\sigma_1\sigma_2} D_\mu h^{\sigma_1\sigma_2} - \cosh^2(z) D^\mu h_{\mu\nu} D_\nu h^{\mu\nu} \right\} + \frac{1}{2} \cosh^4(z) h_{\sigma_1\sigma_2} \left( \partial_z^2 + (d + 4) \tanh(z) \partial_z + 2d \tanh^2(z) + 4 \right) h^{\sigma_1\sigma_2}, \]
\[ S'_{22} = \int d\mu_d \int dz \cosh^{d+2}(z) \left\{ -\frac{1}{2} D^\mu h' D_\mu h' + \frac{d-1}{d} h' h' - \frac{1}{2} \cosh^2(z) h' \left( \partial_z^2 + (d + 4) \tanh(z) \partial_z + 2d \tanh^2(z) + 4 \right) h' \right\}, \]
\[ S_{11} = \int d\mu_d \int dz \cosh^d(z) \left\{ -D^\mu h^{\bullet} D_\mu h^{\bullet} + D_\mu h^{\mu\nu} D_\nu h^{\mu\nu} + (d - 1) h^{\bullet\bullet} h^{\bullet\bullet} \right\}, \]
\[ S_{21} = -2 \int d\mu_d \int dz \cosh^{d+2}(z) D^\mu h_{\mu\nu} \frac{1}{\cosh^d(z)} \partial_z \cosh^d(z) h^{\mu\nu}, \]
\[ S'_{21} = 2 \int d\mu_d \int dz \cosh^d(z) D_\mu h^{\bullet\bullet} \partial_z \cosh^d(z) h', \]
\[ S_0 = \int d\mu_d \int dz \cosh^d(z) \left\{ h_{\bullet\bullet} D_\mu D_\nu h^{\mu\nu} + D^\mu h' D_\mu h^{\bullet\bullet} + \frac{d(d-1)}{2} \tanh^2(z) h_{\bullet\bullet}^2 \right. \]
\[ \left. + (d - 1) h_{\bullet\bullet} \tanh^2(z) \partial_z \cosh^2(z) \coth(z) h' + 2(d - 1) \tanh(z) h_{\bullet\bullet} D_\mu h^{\bullet\bullet} \right\}. \]
The gauge transformations take the form
\[ \delta h^{\mu\nu} = \frac{1}{2} \left( D^\mu \xi^\nu + D^\nu \xi^\mu \right) + \tanh(z) \xi g^{\mu\nu}, \]
\[ \delta h^{\mu\bullet} = -\frac{1}{2} \partial_z \cosh^2(z) \xi^\mu + \frac{1}{2} D^\mu \xi, \]
\[ \delta h^{\bullet\bullet} = -\partial_z \cosh^2(z) \xi. \]

A new feature of the spin two case compared to spins zero and one is that there are two types of scalar field modes. One comes from the scalar component \( h_{\bullet\bullet} \) while another
one comes from the trace part of $h^{\mu \nu}$ which had no analogue for $s = 0$ or 1. This results
in the ambiguity in field redefinitions that mix $h_{\bullet \bullet}$ with $h^{\mu \nu}$. To obtain the action in the
form of infinite sum of actions that describe finite subsystems of fields, the field variables
should be chosen as follows

$$
\tilde{h}^{\mu \nu} = h^{\mu \nu} - \frac{1}{d-2} g^{\mu \nu} \cosh^{-2}(z) h_{\bullet \bullet},
$$
(86)

$$
\tilde{h}_{\bullet \bullet} = h_{\bullet \bullet}.
$$

In the sequel, we shall discard tilde over the new fields $\tilde{h}^{\mu \nu}$, $\tilde{h}_{\bullet \bullet}$ denoting them as $h^{\mu \nu}$, $h_{\bullet \bullet}$.

The redefinition (86) only affects $S_0$ (82) as the only one containing the field $h_{\bullet \bullet}$. It gets the form

$$
\tilde{S}_0 = \int d\mu_d \int dz \cosh^d(z) \left\{ \frac{d-1}{2(d-2)} \cosh^{-2}(z) D_\mu h_{\bullet \bullet} D^\mu h_{\bullet \bullet} + \\
+ \frac{d}{2(d-2)} h_{\bullet \bullet} (-\partial_\xi^2 + (d-4) \tanh(z) \partial_z + (d-1)(d-2) \tanh^2(z) - d + 2) h_{\bullet \bullet} - \\
- \frac{d-1}{d-2} \cosh^2(z) h' \left( \partial_\xi^2 + 2(d-1) \tanh(z) \partial_z + (d-1)(d-2) \tanh^2(z) + d - 2 \right) h_{\bullet \bullet} + \\
+ \frac{2}{d-2} D_\mu h_{\bullet \bullet} \frac{1}{\cosh^{d-2}(z)} \partial_z \cosh^{d-2}(z) h_{\bullet \bullet} \right\}.
$$

In terms of new fields, the gauge transformations (83), (84), (85) read as

$$
\delta h^{\mu \nu} = \frac{1}{2} (D^\mu \xi^\nu + D^\nu \xi^\mu) + \frac{1}{d-2} g^{\mu \nu} \cosh^{-d}(z) \partial_z \cosh^d(z) \xi,
$$
(87)

$$
\delta h_{\bullet \bullet} = -\frac{1}{2} \partial_z \cosh^2(z) \xi + \frac{1}{2} D^\mu \xi,
$$
(88)

$$
\delta h_{\bullet \bullet} = -\partial_z \cosh^2(z) \xi.
$$
(89)

Let us expand the tensor, vector, and scalar fields as follows

$$
h^{\mu_1 \mu_2} = \sum_{n=0}^{\infty} \phi_n^{\mu_1 \mu_2}(x) P^2_n(z), \quad h^{\mu \bullet} = \sum_{n=0}^{\infty} \phi_n^{\mu}(x) R^2_n(z), \quad h_{\bullet \bullet} = \sum_{n=0}^{\infty} \phi_n(x) Q^2_n(z),
$$

where the sets of functions $\{ P^2_n(z) \}$, $\{ R^2_n(z) \}$, and $\{ Q^2_n(z) \}$ form orthogonal bases with
respect to appropriate norms whose specific form will be given later on. The gauge
transformations (87) and (88) suggest analogous expansion for the gauge parameters

$$
\xi^\mu = \sum_{n=0}^{\infty} \xi_n^\mu P^2_n(z), \quad \xi = \sum_{n=0}^{\infty} \xi_n R^2_n(z).
$$

The idea is choose the functions $P^2_n(z)$, $R^2_n(z)$, and $Q^2_n(z)$ such that the resulting
action acquire the form of the sum of actions of spin two, spin one, and spin zero fields
plus some lower-derivative cross-terms that mix different fields. The resulting action
takes the form

$$
S = S_2 + S_{11} + S_{21} + S'_{21} + \tilde{S}_0,
$$
(90)
where

\[
S_2 = \sum_{n=0}^{\infty} \int d\mu_d \left\{ \frac{1}{2} D_{\mu} \phi_{n+1} D_{\mu} \phi_{n+1} - D_{\mu} \phi_{n+2} D_{\mu} \phi_{n+2} + D_{\mu} \phi_{n+1} D_{\mu} \phi_{n+1} \right\},
\]

\[
S_{11} = \sum_{n=0}^{\infty} \int d\mu_d \left\{ -D_{\mu} \phi_{n+1} D_{\mu} \phi_{n+1} + D_{\mu} \phi_{n+2} D_{\mu} \phi_{n+2} + (d-1) \phi_{n+1} \phi_{n+2} \right\},
\]

\[
S_{21} = \sum_{n=0}^{\infty} \int d\mu_d \phi_{n+1} D_{\mu} \phi_{n+2},
\]

\[
S'_{21} = \sum_{n=0}^{\infty} 2\sqrt{(n+1)(n+2)} \int d\mu_d \phi_{n+1}^2,
\]

\[
\tilde{S}_0 = \frac{d-1}{d-2} \sum_{n=0}^{\infty} \int d\mu_d \left\{ \frac{1}{2} D_{\mu} \phi_{n+1} D_{\mu} \phi_{n+1} + \frac{d (n+1)(n+2) - 2}{d-2} \phi_{n+1}^2 \right\}
\]

\[
- \sqrt{(n+1)(n+2)(n+2)}(n+1)(n+2) \phi_{n+1} \phi_{n+2} - 2 \sqrt{(n+1)(n+2)} \phi_{n+1} D_{\mu} \phi_{n+2} \}
\]

The gauge transformations are

\[
\delta \phi_{\mu} = -\frac{1}{2} (D_{\mu} \xi_{n+1} + D_{\mu} \xi_{n}) + \frac{\sqrt{(n+1)(n+2)}}{d-2} \xi_{n+1} \phi_{\mu},
\]

\[
\delta \phi_{\mu} = -\frac{1}{2} \sqrt{n(n+d-1)} \xi_{n+1} + \frac{1}{2} D_{\mu} \xi_{n},
\]

\[
\delta \phi_{\mu} = -\sqrt{n(n+d-3)} \xi_{n-1}.
\]

To bring the action \( \text{(91)} \) to the desired form the following conditions on the basis functions have to be imposed. The diagonalization condition of the spin-two part \( \text{(75)} \) gives the following equation on \( P^2_n(z) \)

\[
\partial_z P^2_n(z) + (d+4) \tanh(z) \partial_z P^2_n(z) + \left[ \frac{m_n^2 - 2d}{\cosh^2(z)} + 2d + 4 \right] P^2_n(z) = 0,
\]

where

\[
m_n^2 = n^2 + (d+1)n + d - 2
\]

is the mass square parameter in the action. The diagonalization condition for the vector field part \( \text{(79)} \) of the action and gauge symmetry \( \text{(89)} \) gives the equations on \( R^2_n(z) \) and \( Q^2_n(z) \)

\[
\partial_z R^2_n(z) + (d+2) \tanh(z) \partial_z R^2_n(z) + \left[ \frac{M^2_{Rn} - d}{\cosh^2(z)} + 2d \right] R^2_n(z) = 0,
\]

\[
\partial_z Q^2_n(z) + d \tanh(z) \partial_z Q^2_n(z) + \left[ \frac{M^2_{Qn} - d + 2}{\cosh^2(z)} + 2(d-2) \right] Q^2_n(z) = 0,
\]

where

\[
M^2_{Rn} = n(n+d-1),
\]

\[
M^2_{Qn} = n(n+d-3), \quad n = 0, 1, \ldots
\]
The normalization conditions on $P_n^2(z), R_n^2(z)$, and $Q_n^2(z)$ are obtained from the integral over the degression coordinate $z$ in the corresponding kinetic terms of the action

$$\int_{-\infty}^{\infty} dz \cosh^{d+2}(z) P_n^2(z) P_m^2(z) = \delta_{nm}, \quad \int_{-\infty}^{\infty} dz \cosh^d(z) R_n^2(z) R_m^2(z) = \delta_{nm}, \quad (100)$$

$$\int_{-\infty}^{\infty} dz \cosh^{d-2}(z) Q_n^2(z) Q_m^2(z) = \delta_{nm}. \quad (101)$$

The solutions of equations (94), (96), (97) that satisfy the normalizability conditions (100), (101) are

$$P_n^2(z) = N_n^d \cosh^{-d-2}(z) F_1 \left( -n, n + d + 1; \frac{d + 2}{2}; \frac{1 + \tanh(z)}{2} \right), \quad (102)$$

$$R_n^2(z) = N_n^{d-2} \cosh^{-d}(z) F_1 \left( -n, n + d - 1; \frac{d}{2}; \frac{1 + \tanh(z)}{2} \right), \quad (103)$$

$$Q_n^2(z) = N_n^{d-4} \cosh^{-d+2}(z) F_1 \left( -n, n + d - 3; \frac{d - 2}{2}; \frac{1 + \tanh(z)}{2} \right), \quad (104)$$

where $N_n^d$ is defined by (30). The following relationships between $P_n^2$, $R_n^2$, and $Q_n^2$ hold

$$\cosh^{-d}(z) \partial_z \left( \cosh^d(z) R_n^2(z) \right) = -M_{R_n} R_{n-1}^2(z),$$

$$\partial_z \left( \cosh^d(z) P_n^2(z) \right) = M_{R_{n+1}} R_{n+1}^2(z),$$

$$\partial_z \left( \cosh^d(z) R_n^2(z) \right) = M_{Q_n} Q_{n+1}^2(z),$$

$$\cosh^{-d+2}(z) \partial_z \left( \cosh^{d-2}(z) Q_n^2(z) \right) = -M_{Q_n} R_{n-1}^2(z).$$

We observe that all gauge symmetries (91), (92) and (93) are Stueckelberg. Upon gauge fixing the Stueckelberg fields to zero, we find that the spectrum of fields resulting from the dimensional degression contains a tower spin two fields with masses and energies

$$m_n^2 = n^2 + (d + 1)n + d - 2, \quad E_n = n + d, \quad n = 0, 1, \ldots$$

a single vector field $\phi_0^d$ and a single scalar field $\phi_0$ with masses $d - 1$, and $d$, respectively, both having the energy $E_0 = d$. This precisely matches Theorem 1.

4 Scalar field degression over several dimensions

Let $AdS_{d'+d}$ with $d' \geq 2$ be realized as a hyperboloid

$$y_{-1}^2 + y_0^2 - y_1^2 - \ldots - y_{d-1}^2 - y_d^2 - \ldots - y_{d+d'-1}^2 = 1$$

in $\mathbb{R}^{d+d'+1}$. We choose the following coordinates

$$y_{-1} = \cosh(z) \bar{y}_{-1}, \quad y_0 = \cosh(z) \bar{y}_0, \quad \ldots, \quad y_{d-1} = \cosh(z) \bar{y}_{d-1},$$

$$y_d = \sinh(z) \bar{y}_d, \quad \ldots, \quad y_{d+d'-1} = \sinh(z) \bar{y}_{d+d'-1},$$

21
where \( z \in [0, \infty) \), \( \tilde{y}_1, \ldots, \tilde{y}_{d-1} \) describe a hyperboloid
\[
\tilde{y}_1^2 + \tilde{y}_0^2 - \tilde{y}_1^2 - \ldots - \tilde{y}_{d-1}^2 = 1
\]
in a \( \mathbb{R}^{d+1} \), while \( \tilde{y}_d, \ldots, \tilde{y}_{d+d'-1} \) describe \( S^{d'-1} \)
\[
\tilde{y}_d^2 + \ldots + \tilde{y}_{d+d'-1}^2 = 1.
\]
The line element of \( AdS_{d+d'} \) is
\[
ds^2_{AdS_{d+d'}} = \cosh^2(z)ds^2_{AdS_d} - dz^2 - \sinh^2(z)ds^2_{S^{d'-1}}.
\]
In this coordinates, the Laplace-Beltrami operator is
\[
\Box_{AdS_{d+d'}} = \frac{1}{\cosh^2(z)} \Box_{AdS_d}
- \frac{1}{\cosh^d(z) \sinh^{d'-1}(z)} \partial_z \cosh^d(z) \sinh^{d'-1}(z) \partial_z
- \frac{1}{\sinh^2(z)} \Box_{S^{d'-1}},
\]
where \( \Box_{AdS_d} \) and \( \Box_{S^{d'-1}} \) are the Laplace-Beltrami operators on \( AdS_d \) and \( S^{d'-1} \), respectively.

The free scalar field equation in \( AdS_{d+d'} \) is
\[
(\Box_{AdS_{d+d'}} + M^2)\Phi(x, z, u) = 0,
\]
where \( x \) are local coordinates of \( AdS_d \) and \( u \) are those of \( S^{d'-1} \). Let us look for its solutions in the form
\[
\Phi(x, z, u) = \sum_{N,L=0}^{\infty} \sum K \phi_{NLK}(x)P_{NL}(z)Y_{LK}(u),
\]
where \( Y_{LK}(u) \) are spherical functions on \( S^{d'-1} \), \( K = (k_1, \ldots, \pm k_{d'-2}) \) so that \( L \geq k_1 \geq \ldots \geq k_{d'-2} \geq 0 \) and
\[
\Box_{S^{d'-1}}Y_{LK}(u) = -L(L + d' - 2)Y_{LK}(u).
\]
The functions \( P_{NL}(z) \) form an orthogonal complete set with respect to the norm
\[
\int_0^\infty dz \cosh^{d-2}(z) \sinh^{d'-1}(z)P_{NL}(z)P_{NL'}(z) = \delta_{NN'},
\]
where \( \delta_{NN'} \) is the Kronecker delta.

Inherited from (20). Requiring \( P_{NL}(z) \) to satisfy the equation
\[
\partial_z^2 P_{NL}(z) + (d \tanh(z) + (d'-1) \coth(z)) \partial_z P_{NL}(z)
+ \left( \frac{m_{NL}^2}{\cosh^2(z)} - \frac{l(l + d' - 2)}{\sinh^2(z)} - M^2 \right) P_{NL}(z) = 0
\] and plugging the expansion (106) into Eq. (105), we obtain the tower of massive scalar fields
\[
(\Box_{AdS_d} + m_{NL}^2)\phi_{NLK}(x) = 0.
\]
The requirement of finiteness of the norm (107) constrains solutions $P_{NL}(z)$ to the “irregular” ones

$$P_{NL}^{\text{irr}}(z) = N^{-\kappa}_{NL} \cosh^{\Lambda}(z) \sinh^{2N+L}(z)_{2} F_{1} \left( -N, -N - L - \frac{d + d'}{2} + 1 - \kappa; \frac{1}{\sinh^{2} z} \right),$$

$$\Lambda = -2N - L - \frac{d + d' - 1}{2} + \kappa, \quad M^2 \in \left[ -\frac{(d + d' - 1)^2}{4}, -\frac{(d + d' - 1)^2}{4} + 1 \right]$$

and “regular” solutions

$$P_{NL}^{\text{reg}}(z) = N^{\kappa}_{NL} \cosh^{\Lambda'}(z) \sinh^{2N+L}(z)_{2} F_{1} \left( -N, -N - L - \frac{d + d'}{2} + 1 + \kappa; -\frac{1}{\sinh^{2} z} \right),$$

$$\Lambda' = -2N - L - \frac{d + d' - 1}{2} - \kappa, \quad M^2 \in \left[ -\frac{(d + d' - 1)^2}{4}, +\infty \right),$$

where $\kappa \equiv \sqrt{M^2 + \frac{(d + d' - 1)^2}{4}}$ and

$$N^{\kappa}_{NL} = \Gamma(1 + \kappa) \sqrt{\frac{N! \Gamma(N + L + \frac{d'}{2})}{2 (2N + L + \frac{d'}{2} + \kappa) \Gamma(\kappa + N + 1) \Gamma(N + L + \frac{d'}{2} + \kappa)}}$$

is fixed by (107).

The lowest energy and mass spectra are

$$E_{NL} = 2N + L + \frac{d + d' - 1 - 2\kappa}{2}, \quad (111)$$

$$m_{NL}^2 = \left( 2N + L - \kappa + \frac{d'}{2} \right)^2 - \frac{(d - 1)^2}{4}$$

for the “irregular” solutions and

$$E_{NL} = 2N + L + \frac{d + d' - 1 + 2\kappa}{2}, \quad (113)$$

$$m_{NL}^2 = \left( 2N + L + \kappa + \frac{d'}{2} \right)^2 - \frac{(d - 1)^2}{4}$$

for the “regular” ones. Recall that the relation of the mass and the energy is given by (17).

These results can also be lifted to the action level. Substituting the expansion (106) into the action

$$S^{\text{AdS}_{d+d'}} = \frac{1}{2} \int d\mu_{d+d'} \Phi(\Box_{\text{AdS}_{d+d'}} + M^2) \Phi,$$

we find that

$$S^{\text{AdS}_{d+d'}} = \sum_{N=0,L=0}^{\infty} \sum_{K} S_{NLK}^{\text{AdS}_{d}},$$

where

$$S_{NLK}^{\text{AdS}_{d}} = \frac{1}{2} \int d\mu_{d} \phi_{NLK}(\Box_{\text{AdS}_{d}} + m_{NL}^2) \phi_{NLK}.$$
These results are similar to those of the dimensional degression over one coordinate. Recall that the appearance of “irregular” and “regular” solutions results from the ambiguity in the relation between mass and energy. Fixing a particular vacuum energy $E_0$ value, we see that the dimensional degression over several coordinates gives rise to a set of scalar fields of energies $E_0 + k$, $k = 0, 1, \ldots$ Each energy level has the multiplicity $13$ in agreement with the group-theoretical analysis of Section 2.

As was pointed out by Metsaev [1], the dimensional degression over two coordinates produces a massless scalar field from a massless one. Namely, there are two energies $E_0 = \frac{d + d'}{2}$ and $E_0 = \frac{d + d' - 2}{2}$ corresponding to the massless scalar field in $AdS_{d+d'}$. If $d' = 2$ the second energy becomes $E_0 = \frac{d}{2}$, i.e., the energy of the massless scalar field in $AdS_d$. Obviously, for spin $s \geq \frac{1}{2}$ this phenomenon does not take place because there is only one energy corresponding to a massless field. Thus, except for the scalar field the dimensional degression of a massless field never gives a massless field at least if the original field corresponds to some symmetric representations of the $AdS$ algebra.

Let us compare our results with those of Metsaev [1]. In the case of dimensional degression of a massless scalar field over several coordinates the mass spectrum (112) for “irregular” solutions at $N = 0$ corresponds to the spectrum given by the formula (47) in [1]. The mass spectrum (114) for “regular” solutions corresponds to the spectrum given by formula (54) of [1] but with some disagreement in the restrictions (55) of [1] on the possible values of the parameters $L$ (denoted in [1] as $l$) and $\kappa$. The difference is that the dimensional degression of the scalar field with the mass in the interval $-(d + d' - 1)^2/4 \leq M^2 < -(d + d' - 1)^2/4 + 1$ gives two mass spectra (112) and (114) compatible with unitarity. The spectrum associated with “irregular” solutions is most likely ruled out in [1] by too strong boundary conditions.

5 Conclusion

In this paper we extend the previously known results on the dimensional degression of a massless scalar in anti-de Sitter space to the cases of massive scalar (including shadow sector) and massless fields of spins one-half, one, and two. Our field-theoretical analysis matches the group-theoretical analysis of the branching of unitary $o(d + d', 2)$–modules into $o(d, 2) \oplus o(d')$–modules also performed in this paper. It is shown that dimensional degression of a massless field of spin $s \geq \frac{1}{2}$ in $d + 1$ dimension produces an infinite set of massive fields in lower dimension. The spectrum is discrete, being quantized in values of the inverse $AdS$ radius. For this reason, in the $AdS$ case, it is not necessary to compactify space-time to obtain discrete spectrum in lower dimension. (In fact it is not even clear what could be an analog of the torus compactification in the $AdS$ case.)

Our results may have several applications. First of all this is the first step towards the analysis of the dimensional compactification-like effects in HS theories, that have $AdS$ rather than Minkowski space-time as their most symmetric vacuum. This analysis is interesting, in particular, for better understanding of possible mechanisms of spontaneous breakdown of HS gauge theories. Indeed, we have shown that the dimensional degression gives rise to the gauge invariant (i.e., Stueckelberg) formulation of massive fields. As such it is just analogous to the standard torus compactification mechanism used to derive massive HS theories from the massless ones in one higher dimension [10, 11].

In other words, the degression mechanism in $AdS$ uncovers the structure of necessary Higgs fields in HS gauge theory. Of course our results for spin one and two are not new
in this respect. The gauge invariant formulation of symmetric fields has been now well understood in the formalism of symmetric tensors\cite{13, 14, 15, 16}. However, to make it appropriate for the analysis of nonlinear HS theories we should better understand how the higgsing works in terms of the HS frame-like formalism of\cite{36, 37, 38, 39} which is at the moment the only working one at the full interacting level (see\cite{42, 43, 44} for more detail and references on nonlinear HS theories). Moreover, an extension of our analysis to the case of mixed symmetry massless fields in $AdS_d$\cite{45} will allow us to achieve a gauge invariant formulation of massive fields of general symmetry type in $AdS_d$.

**Acknowledgments**

The authors acknowledge with gratitude the collaboration of O. Shaynkman at the early stage of this work. We are grateful to R. R. Metsaev for useful discussions and to V. I. Ritus for useful remarks. This research was supported in part by INTAS Grant No 05-1000008-7928, RFBR Grant No 08-02-00963, LSS No 1615.2008.2 The work of M.V. was partially supported by the Alexander von Humboldt Foundation Grant PHYS0167. The work of A.A. was partially supported by the grant of Dynasty Foundation.

6 Appendix

**Theorem 1** For an integer spin $s$, the $o(d,2)$-module $D(E_0,s)_{o(d,2)}$ with $E_0 \geq E_0(s)$ has the following branching into $o(d-1,2)$-modules

$$D(E_0,s)_{o(d,2)} \cong \sum_{q=0}^{s} \sum_{k=0}^{\infty} \oplus D(E_0+k,q)_{o(d-1,2)} .$$

(115)

**Proof.**

An $o(d,2)$-module $D(E_0,s)_{o(d,2)}$ is spanned by vectors\cite{3}

$$T^{+a_1} \ldots T^{+a_M} |E_0,s\rangle$$

with various $M$ and $a = 1, \ldots, d$. The irreducible vacuum $o(d) \oplus o(2)$-module $|E_0,s\rangle$, where $E_0$ and $s = (s,0,\ldots,0)$ are the weights for $o(2)$ and $o(d)$, respectively, decomposes as an $o(d-1) \oplus o(2)$-module into

$$\sum_{q=0}^{s} \oplus |E_0,q\rangle .$$

Using the index decomposition $n = 1, \ldots, d-1$, $\bullet = d$ and denoting $t^{+} = T^{+\bullet}$, the module $D(E_0,s)_{o(d,2)}$ is spanned by the vectors\cite{4}

$$T^{+n_1} \ldots T^{+n_N} (t^{+})^k |E_0,q\rangle ,$$

(116)

where $N,k = 0,1,\ldots,\infty$, $q = 0,\ldots,s$.\footnote{See also recent papers\cite{40} and\cite{41} that extend the frame-like formalism to the cases of reducible sets of massless fields and massive fields, respectively.}
Let $W_{kq}$ be a vector space spanned by

$$T^{+n_1} \ldots T^{+n_N} (t^+)^k \left| E_0, q \right> , \quad N = 0, 1, \ldots$$

From the definitions of $W_{kq}$ it follows that

$$T^{-n} W_{kq} \subset W_{kq} \oplus W_{k-1,q-1} \oplus W_{k-1,q+1} \oplus W_{k-2,q} , \quad (117)$$

$$T^{+n} W_{kq} \subset W_{kq} , \quad T^{mn} W_{kq} \subset W_{kq} , \quad EW_{kq} \subset W_{kq} . \quad (118)$$

Let $V_{ij}$ be vector spaces defined recurrently as

$$V_{ij} = \sum_{q=0}^{j} \oplus W_{iq} \oplus V_{i-1,s} ,$$

where $i = 0, 1, \ldots, \infty$, $j = 0, 1, \ldots, s$, $V_{-1j} = \{0\}$. According to this definition,

$$V_{ij} \subset V_{i+1,j} , \quad V_{ij} \subset V_{i+1,j} . \quad (119)$$

From (117) and (118) it follows that $V_{ij}$ are $o(d-1,2)$–modules. As a result we obtain the filtration of the $o(d-1,2)$–module $D (E_0, s)_{o(d,2)}$ by the $o(d-1,2)$–modules $V_{ij}$

$$\{0\} \subset V_{00} \subset V_{01} \subset \ldots \subset V_{0s} \subset V_{10} \subset \ldots = D (E_0, s)_{o(d,2)} .$$

Obviously, the composition factors form the following $o(d-1,2)$–modules

$$V_{ij}/V_{ij-1} \simeq D (E_0 + i,j)_{o(d-1,2)} , \quad j \neq 0 ,$$

$$V_{i0}/V_{i-1s} \simeq D (E_0 + i,0)_{o(d-1,2)} ,$$

which are irreducible because the inequalities (1) and (2) are satisfied in $d$ dimensions as a consequence of those in $d+1$ dimensions. Here we use the unitarity of $D (E_0, s)_{o(d,2)}$ with $E_0 \geq E_0 (s)$ as an $o(d-1,2)$ module. Clearly, once the inequality $E_0 > E_0 (s)$ holds, $D (E_0, s)_{o(d,2)}$ is a unitary $o(d,2)$–module and hence also a unitary $o(d-1,2)$–module. Taking into account that a reducible unitary module is fully reducible (i.e. it decomposes into direct sum of irreducible submodules), we conclude that (115) is true.

The case where the energy $E_0$ belongs to the boundary of the unitary region $E_0 = E_0 (s)$ is considered analogously, eventhough, in this case, $D (E_0 (s), s)_{o(d,2)}$ is not a unitary $o(d,2)$–module because it contains null states that have zero $o(d,2)$ invariant norm. The key observation is that the energies of the lowest energy states of the $o(d-1,2)$–modules contained in the boundary $o(d,2)$–modules remain inside the unitarity region for $o(d-1,2)$. As a result, all $o(d-1,2)$–modules remain irreducible on the boundary of the unitarity region for $o(d,2)$. (Note, that this implies that one can introduce an $o(d-1,2)$ invariant norm such that $D (E_0 (s), s)_{o(d,2)}$ be a unitary $o(d-1,2)$–module.)

\textbf{Theorem 2} For a half-integer spin $s$ and $E_0 \geq E_0 (s)$, the $o(d,2)$–module $D (E_0, s^\pm)_{o(d,2)}$ for even $d$ and $D (E_0, s)_{o(d,2)}$ for odd $d$ have, respectively, the following branchings into $o(d-1,2)$–modules

$$D (E_0, s^\pm)_{o(d,2)} \simeq \sum_{q=\frac{1}{2}}^{s} \sum_{k=0}^{\infty} \oplus D (E_0 + k, q)_{o(d-1,2)} , \quad (120)$$

$\blacksquare$
\[
D(E_0, s)_{o(d,2)} \cong \sum_{q^+ = \frac{1}{2}}^{s^+} \sum_{k=0}^{\infty} \oplus D(E_0 + k, q^+)_{o(d-1,2)} \oplus \sum_{q^- = \frac{1}{2}}^{s^-} \sum_{k=0}^{\infty} \oplus D(E_0 + k, q^-)_{o(d-1,2)}.
\]

(121)

**Proof.**

A unitary \(o(d, 2)\)-module \(D(E_0, s^\pm)_{o(d,2)}\) if \(d\) is even and \(D(E_0, s)_{o(d,2)}\) if \(d\) is odd is spanned by vectors

\[
T^{+a_1} \cdots T^{+a_M} |E_0, s^\pm \rangle \quad \text{or} \quad T^{+a_1} \cdots T^{+a_M} |E_0, s\rangle
\]

with various \(M\). The irreducible vacuum \(o(d) \oplus o(2)\)-modules \(|E_0, s^\pm\rangle\) and \(|E_0, s\rangle\) decompose as an \(o(d - 1) \oplus o(2)\)-modules into

\[
\sum_{q = \frac{1}{2}}^{s} \oplus |E_0, q\rangle,
\]

and

\[
\sum_{q^+ = \frac{1}{2}}^{s^+} \oplus |E_0, q^+\rangle \oplus \sum_{q^- = \frac{1}{2}}^{s^-} \oplus |E_0, q^-\rangle.
\]

For simplicity we consider the case of odd \(d\). The case of even \(d\) can be obtained analogously by discarding \(\pm\) labels.

Using the index decomposition \(n = 1, \ldots, d - 1, \bullet = d\) and denoting \(t^+ \equiv T^{+\bullet}\), the module \(D(E_0, s)_{o(d,2)}\) is spanned by the vectors

\[
T^{+n_1} \cdots T^{+n_N} (t^+)^k |E_0, q^\pm\rangle,
\]

(122)

where \(N, k = 0, 1, \ldots, \infty\), \(q^\pm = \frac{1}{2}, \ldots, s^\pm\).

Let \(W_{kq}^\pm\) be a vector space spanned by

\[
T^{+n_1} \cdots T^{+n_N} (t^+)^k |E_0, q^\pm\rangle, \quad N = 0, 1, \ldots
\]

From the definition of \(W_{kq}^\pm\) it follows that

\[
T^{-n} W_{kq}^\pm \subset W_{kq}^\pm \oplus W_{k-1,q-1}^\pm \oplus W_{k-1,q+1}^\pm \oplus W_{k-2,q}^\pm \oplus W_{k-1,q}^\pm,
\]

(123)

\[
T^{+n} W_{kq}^\pm \subset W_{kq}^\pm, \quad T^{nm} W_{kq}^\pm \subset W_{kq}^\pm, \quad EW_{kq}^\pm \subset W_{kq}^\pm.
\]

(124)

Let \(V_{ij}\) be vector spaces defined recurrently as

\[
V_{ij} = \sum_{q=-s}^{j} \oplus U_{iq} \oplus V_{i-1,s},
\]

where \(j = -s, \ldots, s\) and

\[
U_{ij} = \begin{cases} W_{ij}^+ & \text{if } j = \frac{1}{2}, \ldots, s \\ W_{i-j}^- & \text{if } j = -s, \ldots, -\frac{1}{2} \end{cases}
\]

According to this definition,

\[
V_{ij} \subset V_{ij+1}, \quad V_{ij} \subset V_{i+1,j}.
\]

(125)
From (123) and (124) it follows that \( V_{ij} \) are \( o(d-1,2) \)-modules. As a result we obtain the filtration of the \( o(d-1,2) \)-module \( D(E_0, s)_{o(d,2)} \) by the \( o(d-1,2) \)-modules \( V_{ij} \)

\[
\{0\} \subset V_{-s} \subset \ldots \subset V_{0-\frac{1}{2}} \subset \ldots \subset V_{0s} \subset V_{1-s} \subset \ldots = D(E_0, s)_{o(d,2)}.
\]

Obviously, the composition factors form the following \( o(d-1,2) \)-modules

\[
\begin{align*}
V_{ij}/V_{ij-1} &\simeq D(E_0 + i, j^+)_{o(d-1,2)} , & j = \frac{1}{2}, \ldots, s , \\
V_{ij}/V_{ij-1} &\simeq D(E_0 + i, j^-)_{o(d-1,2)} , & j = -s + 1, \ldots, -\frac{1}{2} , \\
V_{i-s}/V_{i-1s} &\simeq D(E_0 + i, s^-)_{o(d-1,2)} ,
\end{align*}
\]

which are irreducible because the inequalities (11), (12) are satisfied in \( d \) dimensions as a consequence of those in \( d + 1 \) dimensions.

\( D(E_0, s)_{o(d,2)} \) with \( E_0 \geq E_0(8) \) is a unitary \( o(d-1,2) \)-module by the same reason as in the bosonic case. Taking into account that a reducible unitary module is fully reducible (i.e. it decomposes into direct sum of irreducible submodules), we obtain (120) and (121).

References

[1] R. R. Metsaev, “Massive fields in AdS(3) and compactification in AdS spacetime,” Nucl. Phys. Proc. Suppl. 102, 100 (2001), hep-th/0103088.

[2] T. Kaluza, “On The Problem Of Unity In Physics,” Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys. ) 1921, 966 (1921).

[3] O. Klein, “Quantum theory and five-dimensional theory of relativity,” Z. Phys. 37, 895 (1926) [Surveys High Energ. Phys. 5, 241 (1986)].

[4] M. J. Duff, B. E. W. Nilsson and C. N. Pope, “Kaluza-Klein Supergravity,” Phys. Rept. 130, 1 (1986).

[5] H. Nicolai, “Representations Of Supersymmetry In Anti-De Sitter Space,” in: Supersymmetry and supergravity, ’84, Ed. B. de Witt, P. Fayet and P. van Nieuwenhuizen (World Scientific, Singapore, 1984)

[6] B. de Wit and I. Herger, “Anti-de Sitter supersymmetry,” Lect. Notes Phys. 541 79 (2000), arXiv:hep-th/9908005.

[7] M. A. Vasiliev, “Higher spin superalgebras in any dimension and their representations,” JHEP 0412, 046 (2004), hep-th/0404124.

[8] T. Biswas and W. Siegel, “Radial dimensional reduction: (Anti) de Sitter theories from flat,” JHEP 0207 005 (2002), arXiv:hep-th/0203115.

[9] D. Francia, J. Mourad and A. Sagnotti, “(A)dS exchanges and partially-massless higher spins,” arXiv:0803.3832 [hep-th].

[10] S. D. Rindani and M. Sivakumar, “Gauge - Invariant Description Of Massive Higher - Spin Particles By Dimensional Reduction,” Phys. Rev. D 32 3238 (1985).

[11] S. D. Rindani, D. Sahdev and M. Sivakumar, “Dimensional reduction of symmetric higher spin actions. 1. Bosons,” Mod. Phys. Lett. A 4 265 (1989).
[12] C. Aragone, S. Deser, and Z. Yang, “Massive Higher Spin from Dimensional Reduction of Gauge Fields”, Ann. of Phys. 179, 76 (1987).
[13] Yu. M. Zinoviev, “On massive high spin particles in (A)dS,” arXiv:hep-th/0108192.
[14] I. L. Buchbinder, V. A. Krykhtin and P. M. Lavrov, “Gauge invariant Lagrangian formulation of higher spin massive bosonic field theory in AdS space,” Nucl. Phys. B 762, 344 (2007), arXiv:hep-th/0608005.
[15] R. R. Metsaev, “Gauge invariant formulation of massive totally symmetric fermionic fields in (A)dS space,” Phys. Lett. B 643, 205 (2006), arXiv:hep-th/0609029.
[16] I. L. Buchbinder, V. A. Krykhtin, L. L. Ryskina and H. Takata, “Gauge invariant Lagrangian construction for massive higher spin fermionic fields,” Phys. Lett. B 641, 386 (2006), arXiv:hep-th/0603212.
[17] L. P. S. Singh and C. R. Hagen, “Lagrangian formulation for arbitrary spin. 1. The boson case,” Phys. Rev. D 9, 898 (1974).
[18] L. P. S. Singh and C. R. Hagen, “Lagrangian formulation for arbitrary spin. 2. The fermion case,” Phys. Rev. D 9, 910 (1974).
[19] R. R. Metsaev, “Massive totally symmetric fields in AdS(d),” Phys. Lett. B 590, 95 (2004), arXiv:hep-th/0312297.
[20] I. L. Buchbinder and V. A. Krykhtin, “Gauge invariant Lagrangian construction for massive bosonic higher spin fields in D dimensions,” Nucl. Phys. B 727, 537 (2005), arXiv:hep-th/0505092.
[21] D. Francia, “Geometric Lagrangians for massive higher-spin fields,” Nucl. Phys. B 796, 77 (2008), arXiv:0710.5378 [hep-th].
[22] S. Ferrara and C. Fronsdal, “Conformal Maxwell theory as a singleton field theory on AdS(5), IIB three branes and duality,” Class. Quant. Grav. 15, 2153 (1998), arXiv:hep-th/9712239.
[23] M. Laoues, “Some properties of massless particles in arbitrary dimensions,” Rev. Math. Phys. 10, 1079 (1998), arXiv:hep-th/9806101.
[24] S. Ferrara and C. Fronsdal, “Conformal fields in higher dimensions,” arXiv:hep-th/0006009.
[25] R. R. Metsaev, “Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time,” Phys. Lett. B 354, 78 (1995).
[26] R. R. Metsaev, “Fermionic fields in the d-dimensional anti-de Sitter spacetime”, Phys. Lett. B 419 (1998), hep-th/9802097.
[27] P. A. M. Dirac, “A Remarkable Representation Of The 3+2 De Sitter Group,” J. Math. Phys. 4, 901 (1963).
[28] M. Flato and C. Fronsdal, “One Massless Particle Equals Two Dirac Singletons,” Lett. Math. Phys. 2, 421 (1978).
[29] M. Flato and C. Fronsdal, “On Dis And Racs,” Phys. Lett. B 97, 236 (1980).
[30] R. R. Metsaev, “Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space,” arXiv:hep-th/9810231.
[31] P. Breitenlohner and D. Z. Freedman, “Stability In Gauged Extended Supergravity,” Annals Phys. 144, 249 (1982).
[32] V. Balasubramanian, P. Kraus and A. E. Lawrence, “Bulk vs. boundary dynamics in anti-de Sitter spacetime,” Phys. Rev. D 59, 046003 (1999), arXiv:hep-th/9805171.

[33] S. Flugge, “Practical Quantum Mechanics”, vol.1, Mir, Moscow, 1974.

[34] I. S. Gradshteyn and I. M. Ryzhik, “Table of Integrals, Series and Products”, Moscow 1963.

[35] C. Fronsdal, “Singletons And Massless, Integral Spin Fields On De Sitter Space,” Phys. Rev. D 20, 848 (1979).

[36] V. E. Lopatin and M. A. Vasiliev, “Free Massless Bosonic Fields Of Arbitrary Spin In D-Dimensional de Sitter Space,” Mod. Phys. Lett. A 3 257 (1988).

[37] M. A. Vasiliev, “Free Massless Fermionic Fields Of Arbitrary Spin In D-Dimensional De Sitter Space,” Nucl. Phys. B 301 26 (1988).

[38] M. A. Vasiliev, “Cubic interactions of bosonic higher spin gauge fields in AdS(5),” Nucl. Phys. B 616, 106 (2001) [Erratum-ibid. B 652, 407 (2003)] arXiv:hep-th/0106200.

[39] E. D. Skvortsov and M. A. Vasiliev, “Geometric formulation for partially massless fields,” Nucl. Phys. B 756, 117 (2006), arXiv:hep-th/0601095.

[40] D. Sorokin and M. A. Vasiliev, “Reducible higher-spin multiplets in flat and AdS spaces and their geometric frame-like formulation” arXiv:0807.0206 [hep-th].

[41] Yu. M. Zinoviev, “Frame-like gauge invariant formulation for massive high spin particles” arXiv:0808.1778 [hep-th].

[42] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, “Nonlinear Higher Spin Theories in Various Dimensions,” arXiv:hep-th/0503128.

[43] A. Sagnotti, E. Sezgin and P. Sundell, “On higher spins with a strong $Sp(2,\mathbb{R})$ condition,” Proceedings of the First Solvay Workshop on Higher-Spin Gauge Theories (Brussels, May 2004), hep-th/0501156.

[44] D. Sorokin, “Introduction to the classical theory of higher spins,” hep-th/0405069.

[45] K. B. Alkalaev, O. V. Shaynkman and M. A. Vasiliev, “On the frame-like formulation of mixed-symmetry massless fields in (A)dS(d),” arXiv:hep-th/0311164