Characterization of cycle domains via Kobayashi hyperbolicity

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Abstract

A real form $G$ of a complex semisimple Lie group $G^C$ has only finitely many orbits in any given $G^C$-flag manifold $Z = G^C/Q$. The complex geometry of these orbits is of interest, e.g., for the associated representation theory. The open orbits $D$ generally possess only the constant holomorphic functions, and the relevant associated geometric objects are certain positive-dimensional compact complex submanifolds of $D$ which, with very few well-understood exceptions, are parameterized by the Wolf cycle domains $\Omega_W(D)$ in $G^C/K^C$, where $K$ is a maximal compact subgroup of $G$. Thus, for the various domains $D$ in the various ambient spaces $Z$, it is possible to compare the cycle spaces $\Omega_W(D)$.

The main result here is that, with the few exceptions mentioned above, for a fixed real form $G$ all of the cycle spaces $\Omega_W(D)$ are the same. They are equal to a universal domain $\Omega_AG$ which is natural from the point of view of group actions and which, in essence, can be explicitly computed.

The essential technical result is that if $\hat{\Omega}$ is a $G$-invariant Stein domain which contains $\Omega_AG$ and which is Kobayashi hyperbolic, then $\hat{\Omega} = \Omega_AG$. The equality of the cycle domains follows from the fact that every $\Omega_W(D)$ is itself Stein, is hyperbolic, and contains $\Omega_AG$.

1 Introduction

Let $G$ be a non-compact real semi-simple Lie group which is embedded in its complexification $G^C$ and consider the associated $G$-action on a $G^C$-flag manifold $Z = G^C/Q$. It is known that $G$ has only finitely many orbits in $Z$; in particular, there exit open $G$-orbits $D$. In each such open orbit every maximal compact subgroup $K$ of $G$ has exactly one orbit $C_0$ which is a complex submanifold ([W]).

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Let $q := \dim C_0$, regard $C_0$ as a point in the space $C^q(Z)$ of $q$-dimensional compact cycles in $Z$ and let $\Omega := G^q \cdot C_0$ be the orbit in $C^q(Z)$. Define the Wolf cycle space $\Omega_W(D)$ to be the connected component of $\Omega \cap C^q(D)$ which contains the base cycle $C_0$.

Since the above mentioned basic paper ([W1]) there has been a great deal of work aimed at describing these cycle spaces. Even in situations where good matrix models are available this is not a simple matter. In fact an exact description of $\Omega_W(D)$ has only been given in very special situations (see e.g. [BLZ], [BGW], [DZ], [HS], [HW1], [N], [PR], [W1], [WZ2]). In concrete cases of complex geometric relevance, such as the period domain $D$ for marked $K3$-surfaces, only partial information was available (see [Hr], [H]). Recent progress, which is outlined below, only yielded qualitative information, such as the holomorphic convexity of $\Omega_W(D)$ (W2, HW2).

The results in the present paper change the situation: Except for a few exceptional cases, which we discuss below, for a given group $G$ the Wolf cycle spaces $\Omega_W(D)$ are all the same and are equal to a domain $\Omega_{AG}$ which can be explicitly described. The main new ingredients here involve a combination of complex geometric and combinatorial techniques for studying $G$-invariant, Kobayashi hyperbolic, Stein domains in the associated cycle space $\Omega$.

As indicated above, we shall now outline a number of recent developments and then state our main results. In order to avoid cumbersome statements involving products, it will be assumed that $G$ is simple. This is no loss of generality.

In ([W2]) Wolf shows that if the orbit $\Omega = G^q \cdot C_0$ in the cycle space is compact, then it is either of Hermitian type or just a point. In the former case $G$ is of Hermitian type and the cycle space $\Omega_W(D)$ is the dual bounded symmetric domain $\mathcal{B} \subset \Omega$ (or its complex conjugate $\overline{\mathcal{B}}$). The later case occurs if the real form $G_0$ acts transitively on $Z$ (see [W2], [O2] for a classification).

Here we do not consider the above described well-understood special cases but investigate only the generic case, where $\Omega$ is non-compact. In this case it is a complex-affine variety such that the connected component of the isotropy group $G^q_{C_0}$ is exactly the complexification $K^q$.

We replace $\Omega = G^q/G^q_{C_0}$ by the finite covering space $G^q/K^q$ (which in many cases is the same as $\Omega$), choose a base point $x_0$ with isotropy $K^q$ and regard $\Omega_W(D)$ as the "orbit of $x_0$" of the connected component of $\{g \in G^q : g(C_0) \subset D\}$ which contains the identity. In this way it is possible to compare all cycle domains for a fixed group $G$.

One of the main motivating factors for studying the complex geometry of the cycle domains $\Omega_W(D)$ is that $\mathcal{O}(\Omega_W(D))$, or, more generally, spaces of sections of holomorphic vector bundles over $\Omega_W(D)$, provides a rich source of $G$-representations.

As a general principle one relates representation theory and complex analysis by starting with a smooth manifold $M$ equipped, e.g., with a proper
$G$-action and $G$-equivariantly complexify $M$ to obtain a holomorphic $G^\mathfrak{t}$-action on a Stein manifold $M^\mathfrak{t}$ and a $G$-invariant basis of Stein neighborhoods of $M$ where the $G$-action is still proper, properness being useful for constructing invariant metrics, volume forms, etc.. This type of complexification exists in general (see [He],[Kr],[HHK]), but a concrete description of, e.g., maximal $G$-invariant Stein neighborhoods where the $G$-action remains proper is a difficult matter in the general setting.

Here we consider the special case of $M = G/K$ a Riemannian symmetric space of non-positive curvature (and $G$ semisimple) embedded in $M^\mathfrak{t} = G^\mathfrak{t}/K^\mathfrak{t}$ as an orbit of the same base point $x_0$ as was chosen above in the discussion of cycle spaces. In this setting Akhiezer and Gindikin ([AG]) define a neighborhood $\Omega_{AG}$ of $M$ in $M^\mathfrak{t}$ which is quite natural from the point of view of proper actions.

For this we fix a bit of the notation which will be used throughout this paper. Let $\theta$ be a Cartan involution on $\mathfrak{g}$ which restricts to a Cartan involution on $\mathfrak{g}$ such that $\text{Fix}(\theta|\mathfrak{g}) = \mathfrak{t}$ is the Lie algebra of the given maximal compact subgroup $K$. The anti-holomorphic involution $\sigma: \mathfrak{g}^\mathfrak{F} \to \mathfrak{g}^\mathfrak{F}$ which defines $\mathfrak{g}$ commutes with $\theta$ as well as with the holomorphic extension $\tau$ of $\theta|\mathfrak{g}$ to $\mathfrak{g}^\mathfrak{F}$.

Let $u$ be the fixed point set of $\theta$ in $\mathfrak{g}^\mathfrak{F}$, $U$ be the associated maximal compact subgroup of $G^\mathfrak{F}$ and define $\Sigma$ to be the connected component containing $x_0$ of $\{x \in U \cdot x_0 : G_z \text{ is compact}\}$. Set $\Omega_{AG} := G \cdot \Sigma$. It is shown in ([AG]) that $\Omega_{AG}$ is an open neighborhood of $M := G \cdot x_0 = G/K$ in $G^\mathfrak{F}/K^\mathfrak{F}$ on which the action of $G$ is proper.

To cut down on the size of $\Sigma$, one considers a maximal Abelian subalgebra $a$ in $\mathfrak{p}$ (where $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$) and notes that $G \cdot (\exp(a) \cap \Sigma) \cdot x_0 = \Omega_{AG}$. In fact there is an explicitly defined neighborhood $\omega_{AG}$ of $0 \in a$ such that $i \omega_{AG}$ is mapped diffeomorphically onto its images $\exp(i \omega_{AG})$ and $\exp(i \omega_{AG}) \cdot x_0$ and $\Omega_{AG} = G \cdot \exp(i \omega_{AG}) \cdot x_0$. The set $\omega_{AG}$ is defined by the set of roots $\Phi(a)$ of the adjoint representation of $a$ on $\mathfrak{g}$: It is the connected component containing $0 \in a$ of the set which is obtained from $a$ by removing the root hyperplanes $\{\mu = \pi/2\}$ for all $\mu \in \Phi(a)$. It is convex and is invariant under the action of the Weyl group $W(a)$ of the symmetric space $G/K$. Modulo $W(a)$, the set $\exp(i \omega_{AG}) \cdot x_0$ is an geometric slice for the $G$-action on $\Omega_{AG}$; (see [AG] for details).

A further viewpoint arises in the study of Schubert incidence varieties in which $\Omega := G^\mathfrak{t}/K^\mathfrak{t}$ is again regarded as a space of cycles ([JS],[H],[HW], [HW2]). For this, a Borel subgroup $B$ of $G^\mathfrak{F}$ which contains the factor $AN$ of an Iwasawa-decomposition $G = KAN$ is called a Borel-Iwasawa subgroup and a $B$-Schubert variety $S \subset Z = G^\mathfrak{F}/Q$ an Iwasawa-Schubert variety, i.e., the closure of some orbit of a Borel-Iwasawa subgroup.

Given a boundary point $p \in \partial(D)$, the incidence variety approach which was exemplified in ([JS]) and developed to its completion in ([H]) and ([HW2]), yields an Iwasawa-Schubert variety $Y$ with coding $Y = q + 1$ contained in the complement $Z \setminus D$ with $p \in Y$. The incidence hypersurface $H_Y := \{C \in \Omega : C \cap Y \neq \emptyset\}$ lies in the complement of $\Omega_{W}(D)$.
and in particular touches its boundary at the cycles which contain $p$. If $H = H_Y$ is moved by $g \in G$, then of course $g(H) \cap \Omega_W(D) = \emptyset$ as well. Thus, for any hypersurface $H$ in $\Omega$ which is invariant by some Borel-Iwasawa subgroup $B$ it is appropriate to consider the domain $\Omega_B$ which is the connected component containing the base point $x_0$ of $\Omega \setminus E(H)$, where $E(H)$ is the closure of the union of the hypersurfaces $g(H)$, $g \in G$.

Since $\Omega$ is a spherical homogeneous space, a Borel subgroup $B$ has only finitely many orbits in $\Omega$. In particular, there are only finitely many $B$-invariant hypersurfaces $H_1, \ldots, H_m$. For a given Iwasawa-Borel subgroup let $I(D) := \{ j : H_j \subset \Omega \setminus \Omega_W(D) \}$ and $I := \{ 1, \ldots, m \}$ the full index set.

The Schubert domain $\Omega_S(D)$ is defined to be the connected component containing $\Omega_W(D)$ of the intersection $\bigcap_{j \in I(D)} \Omega_H$, and the universal Iwasawa domain $\Omega_I$ to be the connected component containing the base point $x_0$ of $\bigcap_{j \in I} \Omega_H$. Of course all of these domains are $G$-invariant and Stein. The above mentioned existence of Iwasawa-Schubert varieties $Y$ at boundary points of $D$ implies that $\Omega_W(D) = \Omega_S(D)$, one of the main results in ([HW2]). On the analytic side the domain $\Omega_I$ plays a role in matters of holomorphic extension of certain spherical functions on $G/K$ (see [Fa], [Ola], [OO], [KS]).

It is known that $\Omega_I = \Omega_{AG}$ (see §3). Hence, excluding the few exceptional cases mentioned above, before the present paper it was known that $\Omega_{AG} = \Omega_I \subset \Omega_S(D) = \Omega_W(D)$ for all open $G$-orbits $D$. It should be underlined that apriori, for a fixed real form $G$ but arbitrary open $G$-orbits $D$ in the various projective $G^\mathbb{C}$-homogeneous manifolds $Z$, the cycle spaces $\Omega_W(D)$ might all be different.

Here, with the few well-understood exceptions mentioned above, we prove the following equality.

**Theorem** (see 5.2.6): $\Omega_{AG} = \Omega_S(D)$ for any open $G$-orbit $D$ in any $G^\mathbb{C}$-flag manifold.

Thus, $\Omega_W(D)$ is either a point, a Hermitian bounded symmetric space or (the generic case) $\Omega_W(D) = \Omega_{AG}$.

Our main results are consequences of considerations involving the Kobayashi pseudo-metric. In particular, building on work from ([H]), we show that the domains $\Omega_S(D)$ are Kobayashi hyperbolic (see §4). It is clear from the definitions that they are $G$-invariant and Stein. The essential point can then be formulated as follows

**Theorem** (see 5.2.3): If $\tilde{\Omega}$ is a $G$-invariant, Kobayashi hyperbolic, Stein domain containing $\Omega_{AG}$, then $\tilde{\Omega} = \Omega_{AG}$.

The equality $\Omega_{AG} = \Omega_S(D) = \Omega_W(D)$ is a direct corollary of this theorem.

2 **Equivalence of $\Omega_{AG}$ and $\Omega_I$**

Here we regard $\Omega_I$ and $\Omega_{AG}$ as being set-up at the chosen base point $x_0$ in $G^\mathbb{C}/K^\mathbb{C}$ and show that they are in fact equal. As one would expect,
this involves proving two inclusions. Partial results have been obtained by various authors from various points of view (see [B] and [KS]). In the sequel we give the complex analytic proof of the inclusion $\Omega_{AG} \subset \Omega_I$ from ([H]) and an elementary proof of the other inclusion which in certain aspects follows that in ([H]).

For the inclusion $\Omega_{AG} \subset \Omega_I$ the following observation on the critical set of plurisubharmonic function is of use.

**Lemma 2.0.1.** Let $\rho$ be a strictly plurisubharmonic function on a $n$-dimensional complex manifold $X$ and $M$ be a real submanifold of $X$ with $M \subset \{d\rho = 0\}$. Then $M$ is isotropic with respect to the symplectic form $dd^c\rho$; in particular it is at most $n$-dimensional and is totally real.

**Proof.** If $\lambda := dd^c\rho$ and $i : M \hookrightarrow X$ is the natural injection, then $i^*(\lambda) = 0$. Therefore $i^*(dd^c\rho) = di^*(\lambda) = 0$. \qed

The inclusion $\Omega_{AG} \subset \Omega_I$ follows from the existence of a $G$-invariant strictly plurisubharmonic function on $\Omega_{AG}$.

**Proposition 2.0.2.** If $\Omega_{AG}$ and $\Omega_I$ are set-up at the base point $x_0$, then $\Omega_{AG} \subset \Omega_I$.

**Proof.** Let $\Omega_{adpt}$ be the maximal domain of existence of the adapted complex structure in the tangent bundle $G \times_K \mathfrak{p}$ of the Riemannian symmetric space $G/K$. It is known that the polar map $\Omega_{adpt} \rightarrow G^\mathfrak{t}/K^\mathfrak{t}$, $[(g, \xi)] \mapsto g \exp(i\xi)$, is a biholomorphic map onto $\Omega_{AG}$ ([BHH]).

The square of the norm function on the tangent bundle of $G/K$ which is defined by the invariant metric is strictly plurisubharmonic in the adapted structure. Transporting it via the polar map, we have a $G$-invariant strictly plurisubharmonic function $\rho : \Omega_{AG} \rightarrow \mathbb{R} \geq 0$. Note that $\{\rho = 0\} = \{d\rho = 0\}$ is the totally real orbit $G \cdot x_0$.

Now let $B$ be an Borel-Iwasawa subgroup of $G^\mathfrak{t}$. If $\Omega_I$ would not contain $\Omega_{AG}$, there would exit a $B$-invariant hypersurface $H$ in $G^\mathfrak{t}/K^\mathfrak{t}$ having non-empty intersection with $\Omega_{AG}$. This cannot happen for the following reason: Recall that $B$ contains an Iwasawa-component $AN$. Observe also that, since $\Omega_{AG}$ can be identified with a domain in the tangent bundle of $G/K$, the action of $AN$ on $\Omega_{AG}$ is free and the $AN$-orbits are parameterized by $\Sigma := K \exp(i\omega_{AG}) \cdot x_0$. In particular, $\dim_{\mathbb{R}} AN \cdot y = \dim_{\mathbb{C}} \Omega_{AG}$. Therefore, since $\dim_{\mathbb{C}} (H \cap \Omega_{AG}) = \dim_{\mathbb{C}} \Omega_{AG} - 1$, the $AN$-orbits in $H \cap \Omega_{AG}$ are not totally real.

Now, on the other hand $AN \cdot x_0 = G \cdot x_0 \cong G/K$ is a totally real submanifold of $\Omega_{AG}$. Therefore, for $x \in \exp(i\omega_{AG}) \cdot x_0$ near $x_0$, so is the orbit $AN \cdot x$. Consequently, if $\rho(x)$ is sufficiently small, then $AN \cdot x$ is still totally real. Let $r_0$ be the smallest value of $\rho(y)$, for $y \in H \cap \Sigma$. It follows that $\rho(y) > 0$. Now, the restriction of $\rho$ to the hypersurface $X := A^\mathfrak{t}N^\mathfrak{t} \cdot y \cap \Omega_{AG}$ has a minimum at least along $M = AN \cdot y$. But this is contrary to the above Lemma, because $AN \cdot y$ is not totally real. \qed
The inclusion $\Omega_I \subset \Omega_{AG}$ follows directly from our main theorem 5.2.6.

Here, we give another completely elementary proof.

**Theorem 2.0.3.** $\Omega_{AG} = \Omega_I$.

**Proof.** It is only necessary to prove the opposite inclusion to the one above. For this, assume that $\Omega_I$ is not contained in $\Omega_{AG}$. Then there exists a sequence $\{z_n\} \subset \Omega_{AG} \cap \Omega_I$ with $z_n \to z \in \text{bd}(\Omega_{AG}) \cap \Omega_I$. From the definition of $\Omega_{AG}$, it follows that there exist $\{g_m\} \subset G$ and $\{w_m\} \subset \exp(\omega_{AG})$ such that $g_m(w_m) = z_m$.

Write $g_m = k_m a_m n_m$ in a $KAN$-decomposition of $G$. Since $\{k_m\}$ is contained in the compact group $K$, it may be assumed that $k_m \to k$; therefore that $g_m = a_m n_m$.

Since $\omega_{AG}$ is relatively compact in $a$, it may also be assumed that $w_m \to w \in \text{cl}(\exp(\omega_{AG}))$. Thus $w_m = A_m x_0$, where $\{A_m\} \subset \exp(\omega_{AG})$ and $A_m \to A$. Write $a_m n_m(w_m) = a_m n_m A_m x_0 = a_m n_m A_m n_m x_0$, where $a_m = a_m A_m$ and $n_m = A_m^{-1} n_m A_m$ are elements of $A^f$ and $N^f$, respectively.

Now $\{z_m\}$ and the limit $z$ are contained in $\Omega_I$ which is in turn contained in $A^f N^f \cdot x_0$. Furthermore, $A^f N^f$ acts freely on this orbit. Thus $a_m \to a \in A^f$ and $n_m \to n \in N^f$ with $a \cdot x_0 = z$. Since $A_m \to A$, it follows that $a_m \to a \in A$ and $n_m \to n \in N$ with $a_n w = z$. Since $z \notin \Omega_{AG}$, it follows that $w \in \text{bd}(\exp(\omega_{AG}))$, and $z \in \Omega_I$ implies that $w \in \Omega_I$.

On the other hand, since $w \in \text{bd}(\exp(\omega_{AG}))$, the isotropy group $G_w$ is non-compact. But $\Omega_I$ is Kobayashi hyperbolic ([3]). Therefore the $G$-action on $\Omega_I$ is proper (see e.g. [4]) and consequently $w \notin \Omega_I$, which is a contradiction. □

We note that the standard definition of $\Omega_I$ looks somewhat different from the one above (see e.g. [5]). However, the two definitions are, in fact, equivalent (see [4W2]).

## 3 Spectral properties of $\Omega_{AG}$

### 3.1 Linearization

The map $\eta : G^f \to \text{Aut}_{\mathfrak{g}^f}(\mathfrak{g}^f)$, $x \mapsto \sigma \circ \text{Ad}(x) \circ \tau \circ \text{Ad}(x^{-1})$, provides a suitable linearization of the setting at hand (see [6] for other applications of $\eta$). In this section basic properties of $\eta$ are summarized.

Let $G^f$ act on $\text{Aut}_{\mathfrak{g}^f}(\mathfrak{g}^f)$ by $h \cdot \varphi := \text{Ad}(h) \circ \varphi \circ \text{Ad}(h^{-1})$.

**Lemma 3.1.1.** (G-equivariance) For $h \in G$ it follows that $\eta(h \cdot x) = h \cdot \eta(x)$ for all $x \in G^f$.

**Proof.** By definition $\eta(h \cdot x) = \sigma \text{Ad}(h) \text{Ad}(x) \tau \text{Ad}(x^{-1}) \text{Ad}(h^{-1})$. Since $h \in G$, it follows that $\sigma$ and $\text{Ad}(h)$ commute and the desired result is immediate. □

The normalizer of $K^f$ in $G^f$ is denoted by $N^f := N_G(K^f)$. It is indeed the complexification of $N := N_U(K)$. 
Lemma 3.1.2. \((N^\mathfrak{g}-\text{invariance})\) The map \(\eta\) factors through a \(G\)-equivariant embedding of \(G^\mathfrak{g}/N^\mathfrak{g}\):

\[
\eta(x) = \eta(y) \iff y = xg^{-1} \text{ for some } g \in N^\mathfrak{g}.
\]

Proof. We may write \(y = xg^{-1}\) for some \(g \in G^\mathfrak{g}\). Thus it must be shown that \(\eta(x) = \eta(xg^{-1})\) if and only if \(g \in N^\mathfrak{g}\). But \(\eta(x) = \eta(xg^{-1})\) is equivalent to \(\text{Ad}(g)\tau = \tau \text{Ad}(g)\), which, in turn, is equivalent to the fact that \(\text{Ad}(g)\) stabilizes the complexified Cartan decomposition \(\mathfrak{g}^\mathfrak{g} = (\mathfrak{g}^\mathfrak{g})^\tau \oplus (\mathfrak{g}^\mathfrak{g})^{-\tau} = \mathfrak{t}^\mathfrak{g} \oplus \mathfrak{p}^\mathfrak{g}\).

Now, if \(\text{Ad}(g)\) stabilizes \(\mathfrak{t}^\mathfrak{g} \oplus \mathfrak{p}^\mathfrak{g}\), then \(\text{Ad}(g)(\mathfrak{t}^\mathfrak{g}) = \mathfrak{t}^\mathfrak{g}\), i.e., \(g \in N^\mathfrak{g}\). On the other hand, given any \(g \in N^\mathfrak{g}\), it follows \(\text{Ad}(g)(\mathfrak{p}^\mathfrak{g}) = \mathfrak{p}^\mathfrak{g}\), because \(\mathfrak{p}^\mathfrak{g}\) is the orthogonal complement of \(\mathfrak{t}^\mathfrak{g}\) with respect to the Killing form of \(\mathfrak{g}^\mathfrak{g}\).

Note that \(N^\mathfrak{g}/K^\mathfrak{g}\) is a finite Abelian group (see \([\text{Fe}]\) for a classification). Consequentially, up to finite covers, \(\eta\) is an embedding of the basic space \(G^\mathfrak{g}/K^\mathfrak{g}\).

The involutions \(\sigma\) and \(\tau\) are regarded as acting on \(\text{Aut}_R(\mathfrak{g}^\mathfrak{g})\) by conjugation. On \(\text{Im}(\eta)\) their behavior is particularly simple.

Lemma 3.1.3. (Action of the basic involutions) For all \(x \in G^\mathfrak{g}\) it follows that

1. \(\eta(\tau(x)) = \tau(\eta(x))\)
2. \(\sigma(\eta(x)) = \eta(x)^{-1}\)

In particular \(\text{Im}(\eta)\) is both \(\sigma\)- and \(\tau\)-invariant.

Proof. Let \(\varphi_* : \mathfrak{g}^\mathfrak{g} \to \mathfrak{g}^\mathfrak{g}\) denote the differential of \(\varphi : G^\mathfrak{g} \to G^\mathfrak{g}\) and \(\text{Int}(x) : G^\mathfrak{g} \to C^\mathfrak{g}\) be defined by \(\text{Int}(x)(z) := zxx^{-1}\). The first statement follows directly from the facts that \(\sigma\) and \(\tau\) commute and

\[
\tau \text{Ad}(x)\tau = (\tau \text{Int}(x)\tau)_* = \text{Int}(\tau(x))_* = \text{Ad}(\tau(x)).
\]

For the second statement note that \(\sigma\eta(x) = \text{Ad}(x)\tau \text{Ad}(x^{-1})\), and thus \(\eta(x)\sigma\eta(x) = \sigma\).

We have seen that \(\eta\) is a \(G\)-equivariant map which induces a finite equivariant map \(\eta : G^\mathfrak{g}/K^\mathfrak{g} \to \text{Aut}_R(\mathfrak{g}^\mathfrak{g})\). We will shortly see that the image \(\eta(G^\mathfrak{g}/K^\mathfrak{g})\) is also closed in \(\text{Aut}_R(\mathfrak{g}^\mathfrak{g})\). Hence, for a characterization of \(G\)-orbits in \(G^\mathfrak{g}/K^\mathfrak{g}\) and their topological properties we may identify \(G^\mathfrak{g}/K^\mathfrak{g}\) with its image in \(\text{Aut}_R(\mathfrak{g}^\mathfrak{g})\) on which \(G\) acts by conjugation.

The following special case of a general result on conjugacy classes (see \([\text{Hu}]\) p. 117 and \([\text{Bir}]\)) is of basic use.

Lemma 3.1.4. Let \(V\) be a finite-dimensional \(R\)-vector space, \(H\) a closed reductive algebraic subgroup of \(\text{GL}_R(V)\) and \(s \in \text{GL}_R(V)\) an element which normalizes \(H\). Regard \(H\) as acting on \(\text{GL}_R(V)\) by conjugation. Then, for a semisimple \(s\) the orbit \(Hs\) is closed.

Corollary 3.1.5. The image \(\text{Im}(\eta)\) is closed in \(\text{Aut}_R(\mathfrak{g}^\mathfrak{g})\).
Proof. It is enough to show that \( G^\mathfrak{f} \cdot \tau = \{ \text{Ad}(g) \tau \text{Ad}(g^{-1}) : g \in G^\mathfrak{f} \} \) is closed in \( \text{Aut}_\mathbb{R}(\mathfrak{g}^\mathfrak{f}) \). Since \( \tau \) is semi-simple and normalizes \( G^\mathfrak{f} \) in this representation, this follows from Lemma 3.1.4. \( \square \)

3.2 Jordan Decomposition

Here \( x \) denotes an arbitrary element of \( G^\mathfrak{f} \) and \( su = us = \eta(x) \) is its Jordan decomposition in \( \text{GL}_{\mathbb{R}}(\mathfrak{g}^\mathfrak{f}) \). Since \( \text{Aut}_{\mathbb{R}}(\mathfrak{g}^\mathfrak{f}) \) is algebraic, \( s, u \in \text{Aut}_{\mathbb{R}}(\mathfrak{g}^\mathfrak{f}) \) as well. If \( \eta(x) = us \) is not semi-simple, i.e., \( u \neq 1 \), consider \( \xi = \log(u) \in \text{End}_\mathbb{R}(\mathfrak{g}^\mathfrak{f}) \). Since \( \xi \) is nilpotent, \( t \mapsto \exp(t\xi) \) is an algebraic map and \( \exp(\mathbb{R}\xi) \subset \text{Aut}_{\mathbb{R}}(\mathfrak{g}^\mathfrak{f}) \). It follows that \( \exp(\xi) \in \text{Aut}_{\mathbb{R}}(\mathfrak{g}^\mathfrak{f}) \) for all \( t \in \mathbb{R} \). In particular, \( u \) is in the connected component \( \text{Aut}_{\mathbb{R}}(\mathfrak{g}^\mathfrak{f})^0 \), and \( \xi \) is a derivation: \( \xi = \text{ad}(N) \) for some nilpotent \( N \in \mathfrak{g}^\mathfrak{f} \). Finally, \( u = \text{Ad}(\exp(N)) = \exp(\text{ad}(N)) \).

Given an element \( z \in \text{Aut}_{\mathbb{R}}(\mathfrak{g}^\mathfrak{f}) \), let \( (\mathfrak{g}^\mathfrak{f})^z = \{ X \in \mathfrak{g}^\mathfrak{f} : z(X) = X \} \) denote the subalgebra of fixed points. Observe also that if \( \nu : \mathfrak{g}^\mathfrak{f} \to \mathfrak{g}^\mathfrak{f} \) is any involution such that \( \nu(z) = z \) or \( \nu(z) = z^{-1} \), then the subalgebra \( (\mathfrak{g}^\mathfrak{f})^z \) is \( \nu \)-stable. For \( z \) semisimple the subalgebra \( (\mathfrak{g}^\mathfrak{f})^z \) is reductive.

**Proposition 3.2.1. (Lifting of the Jordan decomposition)** For \( x \in G^\mathfrak{f} \) with Jordan decomposition \( \eta(x) = u \cdot s \) there exists a nilpotent element \( N \in (\mathfrak{g}^\mathfrak{f})^x \cap i\mathfrak{g} \) such that

1. \( u = \text{Ad}(\exp(N)) \)
2. \( \eta(\exp(\frac{1}{2}N) \cdot x) = s \).

**Proof.** Let \( N \in \mathfrak{g}^\mathfrak{f} \) be the element with \( u = \text{Ad}(\exp(N)) \) as explained above. First we show that \( N \in (\mathfrak{g}^\mathfrak{f})^x \cap i\mathfrak{g} \). From Lemma 3.1.2 it follows that \( \sigma(\eta(x)) = \sigma(us) = s^{-1}u^{-1} \). This implies \( \sigma(u) = u^{-1} \) or, equivalently, \( \sigma(N) = -N \), i.e., \( N \in i\mathfrak{g} \). Secondly the statement that \( \text{Ad}(\exp(N)) \) commutes with \( s \) is equivalent to \( se^{\text{ad}(N)}s^{-1} = e^{\text{ad}(N)} \) which is the same as \( s(N) = N \) in the semi-simple case. Thus \( N \in (\mathfrak{g}^\mathfrak{f})^x \cap i\mathfrak{g} \).

Finally, since \( N \in (\mathfrak{g}^\mathfrak{f})^x \), it follows that \( \text{Ad}(\exp(N)) \) commutes with \( s \) for all \( t \in \mathbb{R} \). Having also in mind that \( \sigma(N) = -N \), it follows that

\[
\eta(\exp(\frac{1}{2}N) \cdot x) = \sigma(\text{Ad}(\exp(\frac{1}{2}N) \cdot x) \cdot \text{Ad}(x^{-1}) \cdot \text{Ad}(x^{-1}) \cdot \text{Ad}(\exp(-\frac{1}{2}N)) = \\
= \text{Ad}(\exp(-\frac{1}{2}N) \cdot \sigma \cdot \text{Ad}(x) \cdot \text{Ad}(x) \cdot \text{Ad}(\exp(-\frac{1}{2}N)) = \\
= \text{Ad}(\exp(-\frac{1}{2}N) \cdot s \cdot \text{Ad}(\exp(-\frac{1}{2}N) = \\
= s \cdot \text{Ad}(\exp(-\frac{1}{2}N) \cdot u \cdot \text{Ad}(\exp(-\frac{1}{2}N) = s.
\]

Observe now that since \( \sigma(s) = s^{-1} \), \( (\mathfrak{g}^\mathfrak{f})^x \) is a \( \sigma \)-stable reductive subalgebra. Let \( (\mathfrak{g}^\mathfrak{f})^x = \mathfrak{h} \oplus \mathfrak{q} \) be its \( \sigma \)-eigenspace decomposition. We now build an appropriate \( \mathfrak{s}_\mathfrak{t}_2 \)-triple \( (E, H, F) \) around \( N = E \) in \( (\mathfrak{g}^\mathfrak{f})^x \).

**Lemma 3.2.2.** Let \( E \in (\mathfrak{g}^\mathfrak{f})^x \cap i\mathfrak{g} \) be an arbitrary non-trivial nilpotent element. There exists an \( \mathfrak{s}_\mathfrak{t}_2 \)-triple \( (E, H, F) \) in \( (\mathfrak{g}^\mathfrak{f})^x \), i.e., \( [E, F] = H \), \( [H, E] = 2E \) and \( [H, F] = -2F \) such that \( E, F \in \mathfrak{q} \) and \( H \in \mathfrak{h} \).
Proof. Since $(\mathfrak{g}^\ell)^*$ is reductive, there exists a $\mathfrak{sl}_2$-triple $(E,H,F)$ in $(\mathfrak{g}^\ell)^*$ by the theorem of Jacobson-Morozov. It can be chosen to be $\sigma$-compatible.

To see this, split $H = H^u + H^s$ with respect to the $\sigma$-eigenspace decomposition of $(\mathfrak{g}^\ell)^*$. Since $[H,E] = 2E$ and $\sigma(E) = -E$, it follows that $[H^{-\sigma},E] = 0$. Hence, we may assume that $H = H^s$ (see [Bou], Chpt. VIII, §11, Lemme 6). Observe further that in this case $[E,F] = [E,(F)^{-\sigma}] = H$ and $[H,(F)^{-\sigma}] = (F)^{-\sigma}$. The desired result follows then from the uniqueness of the third element $F$ in a $\mathfrak{sl}_2$-triple.

Now we have all the ingredients which are needed to give a complete characterization of the closed orbits in $\text{Im}(\eta)$:

Proposition 3.2.3. (Closed orbits) If $\eta(x) = u s$ is the Jordan decomposition, then the orbit $G.\eta(x) = G.(su)$ contains the closed orbit $G.s$ in its closure $G.\eta(x)$. In particular, $G.\eta(x)$ is closed if and only if $\eta(x)$ is semi-simple and $s \in \text{Im}(\eta)$.

Proof. Let $u = \text{Ad}(\exp N)$ with $N$ as in Prop. 3.2.2. Hence, by Lemma 3.2.2 there is a $\mathfrak{sl}_2$-triple $(N,H,F)$ ($E = N$) such that $[H,N] = 2tN$, i.e., $\exp(NH)(N) = e^{2tN}$ for every $t \in \mathbb{R}$. Note also that $\exp(\mathbb{R}H) \subset G \cap \exp(\mathfrak{g}^\ell)^*$ by construction of the $\mathfrak{sl}_2$-triple. It follows that
\[
\eta(\exp tH \cdot x) = \exp tH.(u s) = \text{Ad}(\exp tH) \cdot u s = \text{Ad}(\exp tH) \cdot s = \text{Ad}(\exp tH) \text{Ad}(\exp N) \text{Ad}(\exp tH) \cdot s = \text{Ad}(\exp e^{2tN}) \cdot s.
\]

For $t \to -\infty$ it follows $\exp tH.(u s) = \text{Ad}(\exp e^{2tN}) \cdot s \to s$. Hence, the closed orbit $G.s$ lies in the closure of $G.(u s)$. In particular $G.(u s)$ is non-closed if $u \neq 1$, i.e., if $\eta(x)$ is not semisimple. This, together with Lemma 3.1.4 implies that $G.\eta(x)$ is closed if and only if $\eta(x)$ is semisimple. Recall that the image $\text{Im}(\eta)$ is closed. This forces $s \in \text{Im}(\eta)$ and the proof is now complete.

3.3 Elliptic elements and closed orbits

Every non-zero complex number $z$ has the unique decomposition $r \cdot e^{i\phi}$ into the hyperbolic part $r > 0$ and elliptic part $e^{i\phi}$. This generalizes for an arbitrary semisimple element $s \in \text{GL}(\mathfrak{g}^\ell)$: By decomposing its eigenvalues one obtains the unique decomposition $s = s_{\text{ell}} s_{\text{hyp}} = s_{\text{hyp}} s_{\text{ell}}$. An element $x \in G^\ell$ is said to be elliptic if and only if $\eta(x) = s$ is semi-simple with eigenvalues lying in the unit circle. It should be remarked that $x$ itself may in such a case not be a semisimple element of the group $G^\ell$, e.g., $K^\ell$ contains unipotent elements.

Let $\Omega_{\text{ell}} \subset G^\ell$ be the set of elliptic elements. This set is invariant by the right-action of $K^\ell$, and therefore by choosing the same base point $x_0$ as in the case of $\Omega_{\text{reg}}$, by abuse of notation we also regard $\Omega_{\text{ell}}$ as a subset of $G^\ell/K^\ell$. We reiterate that, since the map $\eta$ is not a group morphism, the classical notion of an elliptic element in $G^\ell$ differs from the above definition.
Lemma 3.3.1. For $U$ the maximal compact subgroup of $G^\mathfrak{k}$ defined by $\theta$ it follows that $U \subset \Omega_{\text{ell}}$.

Proof. For $\theta$ the Cartan involution defining $u$, observe that $\hat{U} := \{ \varphi \in \text{Aut}_R(\mathfrak{g}^\mathfrak{k}) : \varphi \theta = \theta \varphi \}$ is a maximal compact subgroup of $\text{Aut}_R(\mathfrak{g}^\mathfrak{k})$ (with identity component $\text{Ad}(U)$).

Now $\theta$ commutes with every term in the definition of $\eta(u)$ for every $u \in U$. It follows that $\theta \eta(u) = \eta(u).$ Therefore $\eta(U)$ is contained in the compact group $\hat{U}$ and consequently $U \subset \Omega_{\text{ell}}$. \hfill $\square$

Proposition 3.3.2. (Elliptic elements) In the homogeneous space $G^\mathfrak{k}/K^\mathfrak{k}$ the set of elliptic elements is described as $\Omega_{\text{ell}} = G \cdot \exp(\mathfrak{a}) \cdot x_0$.

Proof. Observe that $\Omega_{\text{ell}}$ is $G$-invariant. Hence, the above Lemma implies that $G \cdot \exp(\mathfrak{a}) \cdot x_0 \subset \Omega_{\text{ell}}$.

Conversely, suppose $x$ is elliptic, i.e., $\eta(x)$ is contained in some maximal compact subgroup of $\text{Aut}_R(\mathfrak{g}^\mathfrak{k})$. Hence, there is a Cartan involution $\theta'' : \mathfrak{g}^\mathfrak{k} \rightarrow \mathfrak{g}^\mathfrak{k}$ which commutes with $\eta(x)$. We now make the usual adjustments so that, after replacing $x$ by an appropriate $G$-translate, $\eta(x)$ will commute with the given Cartan involution $\theta$.

For this, if $\theta''$ does not commute with $\sigma$, define the semisimple element $\rho := \sigma \theta'' \sigma \theta''$ which is diagonalizable with all positive eigenvalues over $\mathbb{R}$. It follows that $\rho^t$ is defined for all $t \in \mathbb{R}$, and $\theta' := \rho^t \theta'' \rho^{-1}$ commutes with $\sigma$ (see §[Hel], Chp. III, §7). By direct calculation one verifies that $\rho$, hence $\rho^t$, commutes with $\eta(x)$. Thus it follows that $\theta'$ and $\eta(x)$ commute.

Finally, since $\theta'$ and our original $\theta$ both commute with $\sigma$, there exists $h \in G$ such that $\text{Ad}(h) \theta' \text{Ad}(h^{-1}) = \theta$. Consequently, if $x$ is replaced by $h^{-1} \cdot x$, then we may assume that $\eta(x)$ and $\theta$ commute.

Now we will adjust $x$ so that it lies in $U$. With respect to the global Cartan decomposition of $G^\mathfrak{k}$ defined by $\theta$ write $x = u \exp(Z)$, i.e., $u \in U$ and $\theta(Z) = -Z$. We now show that in fact $\exp(Z) \in K^\mathfrak{k}$.

Since $\theta$ commutes with $\sigma$, $\tau$ and $u$ and anti-commutes with $Z$, we have

$$
\theta \eta(x) = \theta \cdot (\sigma \text{Ad}(u) \text{Ad}(\exp(Z)) \tau \text{Ad}(\exp(-Z)) \text{Ad}(u^{-1})) = \\
= \sigma \text{Ad}(u) \text{Ad}(\exp(-Z)) \tau \text{Ad}(\exp(Z)) \text{Ad}(u^{-1}) \cdot \theta.
$$

On the other hand

$$
\theta \eta(x) = \eta(x) \theta = \sigma \text{Ad}(u) \text{Ad}(\exp(Z)) \tau \text{Ad}(\exp(-Z)) \text{Ad}(u^{-1}) \cdot \theta.
$$

Combining these two equations, we obtain $\text{Ad}(\exp(Z)) \tau \text{Ad}(\exp(-Z)) = \text{Ad}(\exp(Z)) \tau \text{Ad}(\exp(Z))$ and consequently $\text{Ad}(\exp(2Z))$ commutes with $\tau$. Since the restriction $\text{Ad} : \exp(\mathfrak{a}) \rightarrow \text{Aut}(\mathfrak{g}^\mathfrak{k})$ is injective, it follows that $\tau(\exp(Z)) = \exp(Z)$, i.e., $\exp(Z) \in K^\mathfrak{k}$. Replacing $x$ by $x \exp(-Z)$, it follows that $x \cdot x_0 = x \exp(-Z) \cdot x_0$; hence, we may assume that $x \in U$.

Since $U = K \cdot \exp(\mathfrak{a}) \cdot K$, we may assume that $x \in K \exp(\mathfrak{a})$ and then translate it by left multiplication by an element of $K$ to reach the following
conclusion: If \( x \in G^\mathfrak{e} \) is elliptic, then there exists \( h \in G \) and \( l \in K^\mathfrak{e} \) with \( hx \in \exp \mathfrak{a} \) or, equivalently, there is \( h \in G \) with \( hx \cdot x_0 \in \exp(\mathfrak{a}) \cdot x_0 \). This proves the inclusion \( \Omega_{\text{ell}} \subseteq G \cdot \exp(\mathfrak{a}) \cdot x_0 \). 

The above proposition yields key information on the set \( \Omega_{\text{ell}} \) of closed \( G \)-orbits in \( G^\mathfrak{e}/K^\mathfrak{e} \). For a subset \( M \) of \( \Omega \) let \( c\ell(M) \) denote its topological closure.

**Corollary 3.3.3.**

\[
\Omega_{\text{ell}} \cap c\ell(\Omega_{AG}) = G \cdot c\ell(\exp(i\omega_{AG})x_0) = \Omega_{\text{ell}} \cap c\ell(\Omega_{AG}).
\]

**Proof.** By the above Proposition, \( \Omega_{AG} = G \cdot \exp(i\omega_{AG}) \cdot x_0 \subset \Omega_{\text{ell}} \). Thus, by continuity, if \( x \in c\ell(\Omega_{AG}) \), then the semisimple part of \( \eta(x) \) is elliptic. As a result, if \( x \in \Omega_{\text{ell}} \cap c\ell(\Omega_{AG}) \), then \( \eta(x) \) is semisimple by Prop \( \[ 2.2.3 \] \) and hence it is elliptic. It follows that \( x \in G \cdot \exp(\mathfrak{a}) \cdot x_0 \). But \( \exp(\mathfrak{a}) \cap c\ell(\Omega_{AG}) = c\ell(\exp(i\omega_{AG})) \) (”\( \supset \)” is obvious and ”\( \subset \)” follows by transversality of \( G \)-orbits at \( \exp(\mathfrak{a}) \cdot x_0 \) and therefore \( \Omega_{\text{ell}} \cap c\ell(\Omega_{AG}) \subset G \cdot c\ell(\exp(i\omega_{AG})) \).

As a consequence of the above Proposition and Lemma \( \[ 3.1.4 \] \) we see that \( G \cdot c\ell(\exp(i\omega_{AG})) \subset \Omega_{\text{ell}} \cap c\ell(\Omega_{AG}) \).

Finally, if \( x \in \Omega_{\text{ell}} \), then in particular \( \eta(x) \) is semisimple, and therefore \( G \cdot x \) is closed. Hence, \( \Omega_{\text{ell}} \cap c\ell(\Omega_{AG}) \subset \Omega_{\text{ell}} \cap c\ell(\Omega_{AG}) \). 

4 \( \mathbf{Q}_2 \)-slices

At a generic point \( y \in \text{bd}(\Omega_{AG}) \) we determine a 3-dimensional, \( \sigma \)-invariant, semi-simple subgroup \( \mathfrak{S}^\mathfrak{e} \) such that \( \mathfrak{S} = (\mathfrak{S}^\mathfrak{e})^\sigma = \text{Fix}(\sigma : \mathfrak{S}^\mathfrak{e} \to \mathfrak{S}^\mathfrak{e}) \) is a non-compact real form and such that the isotropy group \( \mathfrak{S}_y^\mathfrak{e} \) is either a maximal complex torus or its normalizer. Geometrically speaking, \( \mathfrak{Q}_2 = \mathfrak{S}^\mathfrak{e} \cdot y \) is either the 2-dimensional affine quadric, which can be realized by the diagonal action as the complement of the diagonal in \( \mathbb{P}_1(\mathfrak{q}) \times \mathbb{P}_1(\mathfrak{q}) \), or its (2-1)-quotient, which is defined by exchanging the factors and which can be realized as the complement of the (closed) 1-dimensional orbit of \( \text{SO}_3(\mathfrak{q}) \) in \( \mathbb{P}_2(\mathfrak{q}) \). By abuse of notation, we refer in both cases to \( \mathfrak{Q}_2 \cdot y \) as a 2-dimensional affine quadric.

The key property is that, up to the above mentioned possibility of a (2-1)-cover, the intersection \( \mathfrak{Q}_2 \cap \Omega_{AG} \) is the Akhiezer-Gindikin domain in \( \mathfrak{S}^\mathfrak{e}/K^\mathfrak{e} \) for the unit disk \( \mathcal{S}/K_{\mathfrak{S}} \).

For the sake of brevity we say that the orbit \( \mathfrak{S}^\mathfrak{e} \cdot y \) is a \( \mathfrak{Q}_2 \)-slice at \( y \) whenever it has all of the above properties.

4.1 Existence

Given a non-closed \( G \)-orbit \( G \cdot y \) in \( \text{bd}(\Omega_{AG}) \), we may apply Prop. \( \[ 2.2.1 \] \) to obtain a lifting of the semi-simple (elliptic) part of the Jordan decomposition of \( \eta(x) \). For an appropriate base point \( z \) this lifting can be chosen in \( \text{bd}(\exp(i\omega_{AG})) \). Recall that the action \( G^\mathfrak{e} \times \text{Aut}_R(\mathfrak{g}^\mathfrak{e}) \to \text{Aut}_R(\mathfrak{g}^\mathfrak{e}) \)
is given by conjugation (see 3.1). Note that the isotropy Lie algebra at \( \varphi \in \text{Aut}_E(\mathfrak{g}^E) \) is the totally real subalgebra of fixed points \((\mathfrak{g}^E)^\varphi = \{ Z \in \mathfrak{g}^E : \varphi(Z) = Z \} \).

**Lemma 4.1.1. (Optimal base point)** Every non-closed \( G \)-orbit \( G \cdot y \) in \( \text{bd}(\Omega_{AC}) \) contains a point \( z = \exp E \cdot \exp iA \cdot x_0 \) such that \( E \in (\mathfrak{g}^E)^{\eta(\exp iA)} \cap i\mathfrak{g} \) is a non-trivial nilpotent element.

**Proof.** Let \( \eta(y) = su \) be the Jordan decomposition and let \( N \in (\mathfrak{g}^E)^s \cap i\mathfrak{g} \) be as in Prop. 3.2.1. We then have

\[
\eta(\exp y) = \eta(\exp(-\frac{1}{2}N) \exp(\frac{1}{2}N) \cdot y) = \text{Ad}(\exp N) \circ \eta(\exp(\frac{1}{2}N) \cdot y) = u \cdot s.
\]

By Prop. 3.2.3 and Cor. 3.3.3 the semisimple element \( \eta(\exp(\frac{1}{2}N) \cdot y) \) is elliptic. Hence, Prop. 3.3.2 implies the existence of \( g \in G \) and \( A \in \text{bd}(\omega_{AC}) \) such that \( \exp \frac{1}{2}N \cdot y = g^{-1} \exp iA \cdot x_0 \).

Define now \( E := \text{Ad}(g)(-\frac{1}{2}N) \) and observe that \( g \cdot y = \exp E \exp iA \cdot x_0 \).

Finally, \( E \in (\mathfrak{g}^E)^{\eta \circ \text{Ad}(iA)} \) and the lemma is proved.

Recall that \((\mathfrak{g}^E)^{\eta(\exp iA)} \) is a \( \sigma \)-stable real reductive algebra. Let \((\mathfrak{g}^E)^{\eta(\exp iA)} = \mathfrak{h} \oplus \mathfrak{q} \) be the decomposition into \( \sigma \)-eigenspaces. In this notation, the nilpotent element \( E \) as in the above lemma belongs to \( \mathfrak{q} \).

Let now an arbitrary non-closed orbit \( G \cdot E \exp iA \cdot x_0 \) be given. Fix a sl2-triple \((E, H, F)\) as in Lemma 3.2.2. Let \( S^E \) be the complex subgroup of \( G^E \) defined by this triple. Set \( e := iE, f := -iF \) and let \( S \) be the \( \sigma \)-invariant real form in \( S^E \). The Lie algebra of \( S \) is then the subalgebra generated by the sl2-triple \((e, H, f)\). Finally, let \( x_1 = \exp(iA) \cdot x_0 \) be the base point chosen as above in the closure of a given \( G \)-orbit.

**Lemma 4.1.2.** The connected component \((S^E_{x_1})^0 \) of the \( S^E \)-isotropy at \( x_1 \) is the 1-parameter subgroup \{\exp(zH) : z \in \mathfrak{h} \} \cong \mathfrak{h}^*.

**Proof.** Since the action \( S^E \times G^E/K^E \to G^E/K^E \) is affine-algebraic, the orbit \( S^E \cdot \exp iA \cdot x_0 = S^E \cdot x_1 \) is an affine variety. Then the isotropy at \( \exp iA \cdot x_0 \) is 1-dimensional or \( S^E \). Note that \( S^E \cdot x_1 \) cannot be a point, because by construction \( \exp E \cdot x_1 \neq x_1 \); therefore \( S^E_{x_1} \) is 1-dimensional.

We now show that \( \exp \mathbb{R} H \cdot x_1 = x_1 \), or equivalently, \( \exp tH \eta(x_1) = \eta(x_1) \). Define \( \varphi := \text{Ad}(\exp(iA)) \circ \text{Ad}(\exp(-iA)) \) and note that \( H \in (\mathfrak{g}^E)^\varphi \cap (\mathfrak{g}^E)^\eta = \mathfrak{h} \) yields

\[
\exp tH \eta(x_1) = \text{Ad}(\exp tH) \circ \sigma \varphi \circ \text{Ad}(\exp -tH) = \\
\text{Ad}(\exp tH) \text{Ad}(\exp -tH) \circ \sigma \varphi = \eta(x_1).
\]

It follows that \( \exp \mathbb{C} H \cdot x_1 = x_1 \). Since \( S^E_{x_1} \) is 1-dimensional and \( H \) semisimple, we deduce \((S^E_{x_1})^0 = \exp \mathbb{C} H \cong \mathfrak{h}^* \).
4.2 Genericity

Without going into a technical analysis of \( \text{bd}(\Omega_{AG}) \), we will construct \( Q_2 \)-slices only at its generic points. The purpose of this section is to introduce the appropriate notion of "generic" and prove that the set of such points is open and dense. The set of generic points is defined to be the complement of the union of small semi-algebraic sets \( C \) and \( E \) in \( \text{bd}(\Omega_{AG}) \). We begin with the definition of \( C \).

Let \( R := \text{bd}(\exp(\omega_{AG}) \cdot x_0) \) and recall that for \( y \in \text{bd}(\Omega_{AG}) \) the orbit \( G \cdot y \) is closed if and only if \( G \cdot y \cap R \neq \emptyset \). In fact \( R \) parameterizes the closed orbits in \( \text{bd}(\Omega_{AG}) \) up to the orbits of finite group. Recall also that \( R \) is naturally identified with \( \text{bd}(\omega_{AG}) \), which is the boundary of a convex polytope, and is defined by linear inequalities. Let \( E \) be the image in \( R \) of the lower-dimensional edges in \( \text{bd}(\omega_{AG}) \), i.e., the set of points which are contained in at least two root hyperplanes \( \{ \alpha = c_\alpha \} \).

Finally, let \( R_{\text{gen}} := R \setminus E \).

As we have seen in Cor. 3.3.3, the set of closed orbits in the boundary of \( \Omega_{AG} \) can be described as \( G \cdot \text{bd}(\exp(\omega_{AG})x_0) \). This is by definition the set \( C \).

**Lemma 4.2.1.** For \( x \in R \) it follows that \( \dim G \cdot x \leq \text{codim}_0 \text{bd}(\Omega) - 2 \).

**Proof.** Note that \( \text{bd}(\Omega) \) is connected and of codimension 1 in \( \Omega \). The \( G \)-isotropy group \( C_{K\!(\!K)}(a) \) at generic points of \( \exp(\omega_{AG})x_0 \) fixes this slice pointwise and therefore is contained in a maximal compact subgroup of the isotropy subgroup \( G_x \) of each of its boundary points. Since by definition \( G_x \) is non-compact, it follows that \( \dim G_x \) is larger than the dimension of the generic \( G \)-isotropy subgroup at points of \( \exp(\omega_{AG})x_0 \).

**Remark.** For \( x \in \text{bd}(\Omega)_{\text{gen}} \) the isotropy subgroup \( G_x \) is precisely calculated in [3]. This shows that \( \dim G \cdot x = \text{codim}_0 \text{bd}(\Omega) - m \), where \( m \) is at least 2. Thus, by semi-continuity we have the estimate \( \dim G \cdot x \leq \text{codim}_0 \text{bd}(\Omega) - m \) for all \( x \in \text{bd}(\Omega) \).

Now let \( X := \text{Im}(\eta) \subset \text{Aut}_R(\mathfrak{g}^\mathfrak{f}) \). It is a connected component of a real algebraic submanifold in \( \text{Aut}_R(\mathfrak{g}^\mathfrak{f}) \). The complexification \( X^\mathfrak{f} \) of \( X \) which is contained in the complexification \( \text{Aut}_\mathbb{C}(\mathfrak{g}^\mathfrak{f} \times \mathfrak{g}^\mathfrak{f}) \) of \( \text{Aut}_R(\mathfrak{g}^\mathfrak{f}) \) is biholomorphic to \( G^\mathfrak{f}/N^\mathfrak{f} \times G^\mathfrak{f}/N^\mathfrak{f} \), where \( N^\mathfrak{f} \) denotes the normalizer of \( K^\mathfrak{f} \) in \( G^\mathfrak{f} \). The complexification of the piecewise real analytic variety \( R \) is piecewise complex analytic subvariety \( \text{R}^\mathfrak{f} \) of \( X^\mathfrak{f} \) defined in a neighborhood of \( R \) in \( X^\mathfrak{f} \). Finally, let \( \pi : X^\mathfrak{f} \to X^\mathfrak{f}//G^\mathfrak{f} \) be the complex categorical quotient.

Recall that in every \( \pi \)-fiber there is a unique closed \( G^\mathfrak{f} \)-orbit. The closed \( G \)-orbits in \( X \) are components of the the real points of the closed \( G^\mathfrak{f} \)-orbits which are defined over \( \mathbb{R} \). For a more extensive discussion of the interplay between the real and complex points in complex varieties defined over \( \mathbb{R} \) see [Sch1]. [Sch2], [Hi].

Let \( k := \dim_{\mathbb{R}} \Omega - \dim R - m \) be the dimension of the generic \( G \)-orbits of points of \( R \) and let \( S_k \) be the closure in \( X^\mathfrak{f} \) of \( \{ z \in X^\mathfrak{f} : G^\mathfrak{f} \cdot z \text{ is closed and } k \text{-dimensional} \} \). Define \( C_k := S_k \cap \text{R}^\mathfrak{f} \). It follows that \( C_k \) is a piecewise complex analytic set of dimension \( k + \dim_{\mathbb{R}} \text{R}^\mathfrak{f} \).
Proposition 4.2.2. The set $G \cdot R = \{ x \in \text{bd}(\Omega AG) : G \cdot x \text{ is closed} \}$ is contained in a closed semi-algebraic subset $\mathcal{C}$ of codimension at least one in $\text{bd}(\Omega AG)$.

Proof. The set $\mathcal{C}$ is defined to be the intersection of the real points of $C_k$ with $\text{bd}(\Omega AG)$. The desired result follows from $\dim \mathcal{C}_k = k + \dim_R R^E$. □

Recall that $\pi$ denotes the categorical quotient map $\pi : X^\mathbb{C} \to X^\mathbb{C}/G^\mathbb{C}$. Define $\mathcal{E} := \eta^{-1}(\pi^{-1}(\pi(E))) \cap \text{bd}(\Omega AG)$. In particular it is a closed semi-algebraic subset of $\text{bd}(\Omega AG)$ which contains the set $\{ x \in \text{bd}(\Omega AG) : c(\ell(G \cdot x)) \cap E \neq \emptyset \}$.

Definition. A point $z \in \text{bd}(\Omega AG)$ is said to be generic if it is contained in the complement of $\mathcal{C} \cup \mathcal{E}$.

Let $\text{bd}_{\text{gen}}(\Omega AG)$ denote the set of generic boundary points.

Proposition 4.2.3. The set of generic points $\text{bd}_{\text{gen}}(\Omega AG)$ is open and dense in $\text{bd}(\Omega AG)$.

It has already been noted that $\mathcal{C}$ and $\mathcal{E}$ are closed. Since $\mathcal{C}$ is of codimension two, the complement of $\mathcal{C}$ is dense. Thus this proposition is an immediate consequence of the following fact.

Proposition 4.2.4. The saturation $\mathcal{E}$ is at least 1-codimensional in $\text{bd}(\Omega AG)$.

This in turn follows from a computation of the dimension of the fibers at points of $E$ of the above mentioned categorical quotient. For this it is convenient to use the Jordan decomposition $\eta(z) = u \cdot s$ for $z \in \Omega$ such that $x = \exp(\imath \cdot x_0)$ is in $c(\ell(G \cdot z))$.

As in Lemma 4.1.1 we choose an optimal base point such that $\eta(x) = s$ and $u = \text{Ad}(\exp(N))$ with $N \in q$, where $h \oplus q$ is the $\sigma$-decomposition of $\mathfrak{l} = (\mathfrak{g}^\mathbb{C})^\mathbb{C}$. Let $N_x$ be the the cone of nilpotent elements in $q$ and observe that the saturation $\mathcal{E}_x = \{ z \in \text{bd}(\Omega AG) : x \in c(\ell(G \cdot z)) \}$ is an $N_x$-bundle over the closed orbit $G \cdot x$. Thus it is necessary to estimate $\dim_R N_x$.

Recall that any two maximal toral Abelian subalgebras of $\mathfrak{q}^\mathbb{C}$ are conjugate and therefore the dimension $m$ of one such is an invariant. Since $\mathfrak{a}^\mathbb{C}$ is such an algebra, the following is quite useful (see [KR]).

Lemma 4.2.5. The complex codimension in $\mathfrak{q}^\mathbb{C}$ of every component of the nilpotent cone in $\mathfrak{q}^\mathbb{C}$ is $m$.

Proof of Prop. 4.2.4

We prove the estimate $\text{codim}_\mathbb{C} \mathcal{E}_x \geq \dim \mathfrak{a}$. For this observe that, since $G \cdot s$ is closed in $\text{Aut}_{\mathbb{R}}(\mathfrak{g}^\mathbb{C})$, an application of the Luna slice theorem for the (closed) complex orbit $G^\mathbb{C} \cdot s$ in the complexification of $\text{Im}(\eta) = G \times_{\mathbb{C}} q$ locally near $s$; in particular $\text{codim}_\mathbb{C} N_x = \text{codim}_\mathbb{C}(\mathcal{E}_x)$. The result follows from the above Lemma by noting that $\text{codim}_\mathbb{C} N_x$ is at most the complex codimension of the nilpotent cone in $\mathfrak{q}^\mathbb{C}$ and, as mentioned above, that $\dim \mathfrak{a} = m$. □
The group $S^\xi$ constructed above for a generic boundary point has the property that the intersection of the $S^\xi$-orbit, i.e., a 2-dimensional affine quadric $Q_2 \cong SL_2(\mathbb{C})/\mathbb{C}^*$ (or $\cong SL_2(\mathbb{C})/N(\mathbb{C}^*)$) with $\Omega_{AG}$ contains an Akhiezer-Gindikin domain $\Omega_{AG}^\xi \cong D \times \mathcal{D}$ of $Q_2$. To see this, we will conjugate $S^\xi$ by an element of $G$ in order to relate $S^\xi$ to the fixed Abelian Lie algebra $\mathfrak{a}$. This is carried out in the next section.

### 4.3 The intersection property

To complete our task we conjugate the group $S^\xi$ obtained in § 4.1 above by an element $h$ in the isotropy group $G_{x_1}$ so that it can be easily seen that the resulting orbit $Q_2 = S^\xi \cdot x_1$ intersects $\Omega_{AG}$ in the Akhiezer-Gindikin domain of $Q_2$.

The following is a first step in this direction.

**Proposition 4.3.1.** Let $G \exp E \exp iA \cdot x_0 = G \cdot x_1$ be any non-closed orbit in $\text{bd}(\Omega_{AG})$ and $(E, H, F)$ a $\mathfrak{sl}_2$-triple in $(\mathfrak{g}^\xi)_{\text{adj}}(\exp iA)$ as in Lemma 2.2. Given $Z := E - F$, then there exists $h \in G_{x_1}$ so that $\text{Ad}(h)(Z) \in \mathfrak{i} \mathfrak{a}$.

This result is an immediate consequence of the following basic fact.

**Lemma 4.3.2.** Let $\mathfrak{l}$ be a real reductive Lie algebra, $\theta$ a Cartan involution and $\sigma$ a further involution which commutes with $\theta$. Let $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{p}$ be the eigenspace decomposition with respect to $\theta$ and $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{q}$ with respect to $\sigma$. Then, if $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ is a maximal Abelian subalgebra of $\mathfrak{q}$ and $\xi$ is a hyperbolic semisimple element of $\mathfrak{q}$, there exists $h \in \text{Int}(\mathfrak{h})$ such that $\text{Ad}(h)(\xi) \in \mathfrak{q}$.

**Proof.** Since $\xi$ is hyperbolic, we may assume that there is a Cartan involution $\theta' : \mathfrak{l} \to \mathfrak{l}$ such that $\theta'(\xi) = -\xi$ and $\theta' \sigma = \sigma \theta'$. Then there exists $h \in \text{Int}(\mathfrak{h})$ with $\text{Ad}(h)\theta' \text{Ad}(h^{-1}) = \theta$ (see [M1]) and $\text{Ad}(h)(\xi) \in \mathfrak{p} \cap \mathfrak{q}$.

To complete the proof, just note that $(\mathfrak{h} \cap \mathfrak{t}) \oplus (\mathfrak{p} \cap \mathfrak{q})$ is a Riemannian symmetric Lie algebra where any two maximal Abelian algebras in $\mathfrak{p} \cap \mathfrak{q}$ are conjugate by an element of $\text{Int}(\mathfrak{h} \cap \mathfrak{t})$. \qed

**Proof of Prop. 4.3.1.** Observe that $\text{ad}(Z)$ has only imaginary eigenvalues. Replacing $(\mathfrak{g}^\xi)_{\text{adj}}(\exp iA) = \mathfrak{h} \oplus \mathfrak{q}$ by the dual $\mathfrak{l} := \mathfrak{h} \oplus \mathfrak{q}$ and defining $\tilde{\sigma}$ and $\tilde{\theta}$ accordingly, we apply the above Lemma to $\xi := iZ$ and the Abelian Lie algebra $\mathfrak{a} \subset \tilde{\mathfrak{q}}$ to obtain $h \in \text{Int}(\mathfrak{h})$ with $\text{Ad}(h)(\xi) \in \mathfrak{a}$. Thus $\text{Ad}(h)(Z)$ has the required property $\text{Ad}(h)(Z) \in \mathfrak{i} \mathfrak{a}$. \qed

We now show that for $z \in \text{bd}_{\text{gen}}(\Omega_{AG})$ the group $S^\xi$ which is associated to the $\mathfrak{sl}_2$-triple constructed in the above proposition as produces a $Q_2$-slice. For a precise formulation it is convenient to let $\text{bd}_{\text{gen}}(\Omega_{AG}) := \text{bd}(\Omega_{AG}) \setminus E$, where $E$ is the union of the lower-dimensional strata as in § 1.2.

**Proposition 4.3.3.** For $z \in \text{bd}_{\text{gen}}(\Omega_{AG})$ and $x_1 = \exp iA \cdot x_0$ the associated point with $iA \in \text{bd}_{\text{gen}}(\Omega_{AG})$ it follows that the line $\mathbb{R}(E - F)$ is transversal to $\text{bd}_{\text{gen}}(\Omega_{AG})$ at $iA$ in $\mathfrak{i} \mathfrak{a}$. 

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Lemma 4.3.4. 

For this, recall the root decompositions of $g$ and $g^\mathfrak{f}$ with respect to $a$ or $a^\mathfrak{f}$, respectively: $g^\mathfrak{f} = C \Phi(a) \oplus a^\mathfrak{f} \oplus \bigoplus_{\Phi(a)} g^\mathfrak{f}_\lambda$. The behavior of this decomposition with respect to our involutions is the following: $\theta(g^\mathfrak{f}_\lambda) = g^\mathfrak{f}_{-\lambda}$ and $\pi(g^\mathfrak{f}_\lambda) = g^\mathfrak{f}_{-\lambda}$; furthermore, the root decomposition is $\sigma$-stable, i.e., $\sigma(g^\mathfrak{f}_\lambda) = g^\mathfrak{f}_\lambda$. Fix a $\tau$-stable basis of root vectors, i.e., select any basis $L^1_\lambda, ..., L^n_\lambda$ of $g_\lambda = (g^\mathfrak{f}_\lambda)^\mathfrak{s}$ and define $L^j_\lambda : = \tau(L^j_\lambda)$. Define $g[\lambda] : = g_\lambda \oplus g_{-\lambda}$. Introducing such a basis is that the complex subspaces $(g[\lambda])^{\mathfrak{r}}$, $(g[\lambda])^{-\mathfrak{r}}$ are $\text{Ad}(t)$-stable for any $t : = \exp iA$, $A \in a$.

Express $\text{Ad}(t)$ as a matrix with respect to the basis $X^j_{\lambda}, Y^j_{\lambda}$:

$$
\text{Ad}(t)ig|_{((X^j_{\lambda}), Y^j_{\lambda})} = \begin{pmatrix}
\cosh \lambda iA & \sinh \lambda iA \\
\sinh \lambda iA & \cosh \lambda iA
\end{pmatrix}
$$

Let $g^\mathfrak{f} = \mathfrak{f}^\mathfrak{f} \oplus p^\mathfrak{f}$ be the complexification of the Cartan decomposition of $g$. A simple calculation yields for $t = \exp iA$:

$$
\mathfrak{h} = g \cap \text{Ad}(t)(\mathfrak{f}^\mathfrak{f}) = C \Phi(a) \oplus \bigoplus_{\lambda(A) = 2\pi} g[\lambda]^\mathfrak{r} \oplus \bigoplus_{\lambda(A) = \frac{\pi}{2} + 2\pi} g[\lambda]^{-\mathfrak{r}}
$$

$$
\mathfrak{q} = i\mathfrak{g} \cap \text{Ad}(t)(p^\mathfrak{f}) = ia \oplus \bigoplus_{\lambda(A) = 2\pi} ig[\lambda]^\mathfrak{r} \oplus \bigoplus_{\lambda(A) = \frac{\pi}{2} + 2\pi} ig[\lambda]^{-\mathfrak{r}}.
$$

Let $A \in \text{bd}_{gen}(\omega_{AG})$ be boundary-generic, i.e., there is a single $\lambda \in \Phi(a)$ with

$$
\lambda(A) = \pm \frac{\pi}{2}, \quad \mu(A) \notin \frac{\pi}{2} \mathbb{Z} \quad \text{for all} \quad \mu \in \Phi(a) \setminus \{ \pm \lambda \}.
$$

The above general formulas imply that the centralizer subalgebra $(g^\mathfrak{f})^{\eta(\exp iA)}$ for such a boundary-generic point as above is given by

$$(g^\mathfrak{f})^{\eta(\exp iA)} = m \oplus g[\lambda]^{-\mathfrak{r}} \oplus ia \oplus ig[\lambda]^\mathfrak{r}.$$  

To complete the proof of the proposition it is then enough to show that for the selected $\mathfrak{sl}_2$-triple $(E, H, F) \in g^\mathfrak{f}^{\eta(\exp iA)}$ it follows that $E - F \in \mathbb{R} h_\lambda$, where $h_\lambda \in a$ is the coroot determined by the root $\lambda \in \Phi(a)$. This is the content of the following

**Lemma 4.3.4.** Let $A \in \{ \lambda = \pi/2 \} \cap \text{bd}_{gen}(\omega_{AG})$ be boundary-generic as above. Then $E - F \in \mathbb{R} h_\lambda$.

**Proof.** Let $1 : = (g^\mathfrak{f})^{\eta(\exp iA)} = \mathfrak{h} \oplus \mathfrak{q}$. Since $(\mathbb{R}[E, H, F])_{\mathbb{R}}$ is semisimple, it follows that $(\mathbb{R}[E, H, F])_{\mathbb{R}} \subset [1 : 1]$. Hence, since $B([g_\lambda : g_{-\lambda}], \{ \lambda = 0 \}) = 0$ ($B$ denotes the Killing form) we have

$$
[1 : 1] = [m \oplus g[\lambda]^{-\mathfrak{r}} \oplus ia \oplus ig[\lambda]^\mathfrak{r}] : m \oplus g[\lambda]^{-\mathfrak{r}} \oplus ia \oplus ig[\lambda]^\mathfrak{r} = m \oplus \mathbb{R} h_\lambda \oplus g[\lambda]^{-\mathfrak{r}} \oplus ig[\lambda]^\mathfrak{r}
$$

By Prop. 4.3.1 we have $E - F \in ia$. Finally, $E - F \in ia \cap [1 : 1] = \mathbb{R} h_\lambda$. □
Recall that the set \( \text{bd}_{\text{gen}}(\Omega_{AG}) = \text{bd}(\Omega_{AG}) \setminus (C \cup E) \) consists of certain non-closed orbits in the boundary of \( \Omega_{AG} \).

**Theorem 4.3.5.** On every \( G \)-orbit in \( \text{bd}_{\text{gen}}(\Omega_{AG}) \) there exists a point of the form \( z := \exp E \exp iA \cdot x_0, \ A \in \text{bd}_{\text{gen}}(\omega_{AG}) \), \( E \) nilpotent, and a corresponding 3-dimensional simple subgroup \( S^E \subset G^E \) such that

1. The 2-dimensional affine quadric \( S^E \cdot \exp iA \cdot x_0 =: S^E \cdot x_1 \) contains \( z \).

2. The intersection \( \Omega_{AG} \cap S^E \cdot x_1 \) contains an Akhiezer-Gindikin domain \( \Omega^S_{AG} \) of \( S^E, x_1 \), i.e., the orbit \( S^E \cdot x_1 \) is a \( Q_2 \)-slice.

**Proof.** Given a non-closed \( G \)-orbit in \( \text{bd}_{\text{gen}}(\Omega_{AG}) \) let \( z := \exp E \exp iA \cdot x_0 \) be an optimal base point as in Lemma [1.1]. By Prop. [4.3.1] we may choose an \( sl_2 \)-triple \( (E, H, F) \) in \( (g^E)^{\text{exp}(A)} \) such that \( E - F \in i a \). Let \( S^E \subset G^E \) be the complex subgroup with Lie algebra \( s^E := \langle [E, H, F] \rangle_S \). By construction \( S^E \cdot x_1 \) contains \( z \).

For a boundary-generic point \( x_1 \), with \( \lambda(A) = \pi/2 \) and \( \mu(A) \neq \pm \lambda \) for all \( \mu \neq \pm \lambda \) we already know by [1.3.4] that \( E - F \in \mathbb{R} h_{\lambda} \). Assume that \( h_{\lambda} \in a \) is the normalized coroot of \( \lambda \), i.e., \( \lambda(h_{\lambda}) = 2 \). Since \( \omega_{AG} \) is invariant under the Weyl group, the image \( A' \) of \( A \) under the reflection on \( \{ \lambda = 0 \} \) is also boundary-generic, and the intersection of \( A - \mathbb{R} h_{\lambda} \) with \( \omega_{AG} \) is the segment \( \{ A - th_{\lambda} : t \in (0, 1/2) \} \) with boundary points \( A \) and \( A' \).

Recall that \((e, H, f)\) with \( E = i e \) and \( F = -i f \) is a \( sl_2 \)-triple in \( s^E \) such that \( s := g \cap s^E = \langle [e, H, f] \rangle_{\mathbb{R}} \). Let \( S \) denote the corresponding subgroup in \( S^C \) (isomorphic to \( SL_2(\mathbb{R}) \) or \( PSL_2(\mathbb{R}) \)). The \( S \)-isotropy at all points \( \exp ((-\frac{\pi}{2}, 0)i h_{\lambda} + iA) \cdot x_0 \) is compact and it is non-compact at \( \exp iA \cdot x_0 \) and \( \exp iA' \cdot x_0 \). Hence, \( S : \exp ((-\frac{\pi}{2}, 0)i h_{\lambda} + iA) \cdot x_0 \) is an Akhiezer-Gindikin domain in \( S^E, x_1 \) which is contained in \( \Omega_{AG} \).

### 4.4 Domains of holomorphy

Let \( S^E = SL_2(\mathbb{C}), S = SL_2(\mathbb{R}) \) be embedded in \( S^E \) as the subgroup of matrices which have real entries and let \( K_S = SO_2(\mathbb{R}) \). To fix the notation, let \( D_0 \) and \( D_{\infty} \) be the open \( S \)-orbits in \( \mathbb{P}_1(\mathbb{C}) \). Further, choose \( \mathbb{C} \subset \mathbb{C}^1 = \mathbb{C} \cup \{ \infty \} \) in such a way that \( 0 \in D_0 \) and \( \infty \in D_{\infty} \) are the \( K_S \)-fixed points.

Now let \( S^E \) act diagonally on \( Z = \mathbb{C}^1 \times \mathbb{C}^1 \) and note that the open orbit \( \Omega \), which is the complement of the diagonal \( \text{diag}(\mathbb{C}^1) \) in \( Z \), is the complex symmetric space \( S^E/K_S^E \). Note that in \( \mathbb{C}^1 \times \mathbb{C}^1 \) there are 4 open \( SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \)–orbits: the bi-disks \( D_0 \times D_0 \) for any pair \((\alpha, \beta)\) from \( \{0, \infty\} \). As \( S \)-spaces, the domains \( D_0 \times D_{\infty} \) and \( D_{\infty} \times D_0 \) are equivariantly biholomorphic; further, they are actually subsets of \( \Omega \), and the Riemannian symmetric space \( S/K_S \) sits in each of them as the totally real \( S \)-orbit \( S \cdot (0, \infty) \) (or \( S \cdot (\infty, 0) \), respectively). Depending on which of these points is chosen as a reference point in \( \Omega \), both domains can be considered as the Akhiezer-Gindikin domain

\[
\Omega_{AG} = D_0 \times D_{\infty} = S \cdot \exp i \omega_{AG} \cdot (0, \infty) \quad D_{\infty} \times D_0 = S \cdot \exp i \omega_{AG} \cdot (\infty, 0)
\]
with $\omega_{AG} = (-\frac{\pi}{2}, \frac{\pi}{2})h_\alpha$ and $h_\alpha \in \mathfrak{a}$ is the normalized coroot (i.e., $\alpha(h_\alpha) = 2$).

Our main point here is to understand $S$-invariant Stein domains in $\Omega$ which properly contain $\Omega_{AG}$. By symmetry we may assume that such has non-empty intersection with $D_0 \times D_0$. Observe that $(D_0 \times D_0) \cap \Omega = D_0 \times D_0 \setminus \text{diag}(D_0)$. Furthermore, other than $\text{diag}(D_0)$, all $S$-orbits in $D_0 \times D_0$ are closed real hypersurfaces. For $D_0 \times D_0 \setminus \text{diag}(D_0)$ let $\Omega(p)$ be the domain bounded by $S \cdot p$ and $\text{diag}(D_0)$. We shall show that a function which is holomorphic in a neighborhood of $S \cdot p$ extends holomorphically to $\Omega(p)$.

For this, define $\Sigma := \{(-s, s) : 0 \leq s < 1\} \subset D_0 \times D_0$. It is a geometric slice for the $S$-action. We say that a (1-dimensional) complex curve $C \subset \mathfrak{g}_\mathbb{C}$ is a supporting curve for $\text{bd}(\Omega(p))$ at $p$ if $C \cap \text{cl}(\Omega(p)) = \{p\}$. Here, $\text{cl}(\Omega(p))$ denotes the topological closure in $D_0 \times D_0$.

**Proposition 4.4.1.** For every $p \in D_0 \times D_0 \setminus \text{diag}(D_0)$ there exists a supporting curve for $\text{bd}(\Omega(p))$ at $p$.

**Proof.** Recall that we consider $D_0$ embedded in $\mathfrak{g}_\mathbb{C}$ as the unit disc. It is enough to construct such a curve $C \subset \mathfrak{g}_\mathbb{C}$ at each point $p_\alpha = (-s, s) \in \Sigma$, $s \neq 0$. For this we define $C_\alpha := \{(-s + z, s + z) : z \in \mathbb{C}\}$. To prove $C_\alpha \cap \text{cl}(\Omega(p_\alpha)) = \{p_\alpha\}$ let $d$ be the Poincare metric of the unit disc $D_0$, considered as the function $d : D_0 \times D_0 \to \mathbb{R}_{\geq 0}$. Note that it is an $S$-invariant function on $D_0 \times D_0$. In fact the values of $d$ parameterize the $S$-orbits.

We now claim that $d(-s + z, s + z) \geq d(-s, s) = d(p_\alpha)$ for $z \in \mathbb{C}$ and $(-s + z, s + z) \in D_0 \times D_0$, with equality only for $z = 0$, i.e., $C_\alpha$ touches $\text{cl}(\Omega(p_\alpha))$ only at $p_\alpha$. To prove the above inequality, it is convenient to compare the Poincare length of the Euclidean segment $\text{seg}(z - s, z + s)$ in $D_0$ with the length of $\text{seg}(-s, s)$. Writing the corresponding integral for the length, it is clear, without explicit calculation, that $d(-s + x, s + x) > d(-s, s)$ for $z = x \in \mathbb{R} \setminus 0$. The same argument shows also that $d(-s + x + iy, s + x + iy) > d(-s + x, s + x)$ for all non-zero $y \in \mathbb{R}$ and the proposition is proved. $\square$

From the above construction it follows that the boundary hypersurfaces $S(p)$ are strongly pseudoconvex. Since then the smallest Stein domain containing a $S$-invariant neighborhood of $S(p)$ is $\Omega(p) \setminus \text{diag}(D_0)$, the following is immediate.

**Corollary 4.4.2.** For $p \in D_0 \times D_0 \setminus \text{diag}(D_0)$ every function $f$ which is holomorphic on some neighborhood of the orbit $S \cdot p$ extends holomorphically to $\Omega(p) \setminus \text{diag}(D_0)$. An analogous statement is valid for $p \in D_\infty \times D_\infty \setminus \text{diag}(D_\infty)$.

Observe that the set $\text{bd}_{\text{gen}}(D_0 \times D_\infty)$ of generic boundary points, which was introduced in section 4.2, consists of the two $S$-orbits $\text{bd}(D_0) \times D_\infty \cup D_0 \times \text{bd}(D_\infty)$. Let $z \in \text{bd}(D_0) \times D_\infty$ (or $z \in D_0 \times \text{bd}(D_\infty)$ be such a boundary point.
**Corollary 4.4.3.** Let $\hat{\Omega} \subset Q_2 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ be an $S$-invariant Stein domain which contains $D_0 \times D_\infty$ and the boundary point $z$. Then $\hat{\Omega}$ also contains $D_0 \times \mathbb{CP}^1 \setminus \text{diag}(\mathbb{CP}^1)$ (or $\mathbb{CP}^1 \times D_\infty \setminus \text{diag}(\mathbb{CP}^1)$, respectively).

**Proof.** Let $B$ be a ball around $z$ which is contained in $\hat{\Omega}$. For $p \in B(z) \cap D_\infty \times D_\infty$ sufficiently close to $z$ it follows that $S \cdot q \subset \hat{\Omega}$ for all $q \in B(z) \cap (D_\infty \times D_\infty)$. The result then follows from the previous Corollary.

If $\hat{\Omega}$ is as in the above Corollary, then the fibers of the projection of $\hat{\Omega} \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{P}^1$ can be regarded as non-constant holomorphic curves $f : \mathbb{C} \to \hat{\Omega}$. One says that a complex manifold $X$ is Brody hyperbolic if there are no such curves.

**Corollary 4.4.4.** If $\hat{\Omega}$ is as above, then $\hat{\Omega}$ is not Brody hyperbolic.

A complex manifold $X$ is said to be Kobayashi hyperbolic whenever the Kobayashi pseudo-metric is in fact a metric (see [K]). The pseudo-metric is defined in such a way that, if there exists a non-constant holomorphic curve $f : \mathbb{C} \to X$, then $X$ is not hyperbolic, i.e., Kobayashi hyperbolicity is a stronger condition than Brody hyperbolic. For an arbitrary semisimple group $G$ the domain $\Omega_{AG}$ is indeed Kobayashi hyperbolic ([H], see §5 for stronger results).

The following is our main application of the existence of $Q_2$-slices at generic points of $\text{bd}(\Omega_{AG})$.

**Theorem 4.4.5.** A $G$-invariant, Stein and Brody hyperbolic domain $\hat{\Omega}$ in $G^\mathfrak{k}/K$ which contains $\Omega_{AG}$ is equal to $\Omega_{AG}$.

**Proof.** Arguing by contraposition, if $\Omega_{AG}$ is strictly contained in a $G$-invariant Stein domain $\hat{\Omega}$, then by Thm. 4.3.3 there exists a $Q_2$-slice at a generic boundary point $z \in \text{bd}(\Omega_{AG}) \cap \hat{\Omega}$ with $Q_2 \cap \hat{\Omega}$ an $S$-invariant Stein domain properly containing the Akhiezer-Gindikin domain of $Q_2$. However, by Cor. 4.4.4 such a domain in $Q_2$ is not Brody hyperbolic.

5 Hyperbolicity and the characterization of cycle domains

In this section it is shown the Wolf cycle domains $\Omega_W(D)$ are Kobayashi hyperbolic. The above theorem then yields their characterization (see §5.2.6).

5.1 Families of hyperplanes

We start by proving a general result concerning families of hyperplanes in projective space and their intersections with locally closed subvarieties. Since such a subvariety is usually regarded as being embedded by sections of some line bundle, it is natural to regard the projective space as the
projectivization $\mathbb{P}(V^*)$ of the dual space and a hyperplane in $\mathbb{P}(V^*)$ as a point in $\mathbb{P}(V)$.

We will think of a subset $S \subset \mathbb{P}(V)$ as parameterizing a family of hyperplanes in $\mathbb{P}(V^*)$. A non-empty subset $S \subset \mathbb{P}(V)$ is said to have the normal crossing property if for every $k \in \mathbb{N}$ there exist $H_1, \ldots, H_k \in S$ so that for every subset $I \subset \{1, \ldots, k\}$ the intersection $\bigcap_{i \in I} H_i$ is $|I|$-codimensional. If $|I| \geq \dim \mathbb{P}(V)$, this means that the intersection is empty.

In the sequel $\langle S \rangle$ denotes the complex linear span of $S$ in $\mathbb{P}(V)$, i.e., the smallest plane in $\mathbb{P}(V)$ containing $S$.

**Proposition 5.1.1.** A locally closed, irreducible real analytic subset $S$ with $\langle S \rangle = \mathbb{P}(V)$ has the normal crossing property.

**Proof.** We proceed by induction over $k$. For $k = 1$ there is nothing to prove. Given a set $\{H_{s_1}, \ldots, H_{s_k}\}$ of hyperplanes with the normal crossing property and a subset $I \subset \{s_1, \ldots, s_k\}$, define

$$\Delta_I := \bigcap_{s \in I} H_s, \quad \mathcal{H}(I) := \{s \in S : H_s \supset \Delta_I\} \quad \mathcal{C}_k := \bigcup_{J \subset \{s_1, \ldots, s_k\} : \Delta_J \neq \emptyset} \mathcal{H}(J).$$

We wish to prove that $S \setminus \mathcal{C}_k \neq \emptyset$. For this, note that each $\mathcal{H}(I)$ is a real analytic subvariety of $S$. Hence, if $S = \mathcal{C}_k$, then $S = \mathcal{H}(J)$ for some $J$ with $\Delta_J \neq \emptyset$. However, $\{H \in \mathbb{P}(V^*) : H \supset \Delta_j\}$ is a proper, linear plane $\mathcal{L}(J)$ of $\mathbb{P}(V)$. Consequently, $S \subset \mathcal{L}(J)$, and this would contradict $(S) = \mathbb{P}(V)$. Therefore, there exists $s \in S \setminus \mathcal{C}_k$, or equivalently, $\{H_{s_1}, \ldots, H_{s_k}, H_s\}$ has the normal crossing property. \qed

It is known that if $H_1, \ldots, H_{2m+1}$ are hyperplanes having the normal crossing property, where $m = \dim \mathbb{P}(V)$, then $\mathbb{P}(V^*) \setminus \bigcup H_j$ is Kobayashi hyperbolic ([D], see also [K] p. 137).

**Corollary 5.1.2.** If $S$ is a locally closed, irreducible and generating real analytic subset of $\mathbb{P}(V)$, then there exist hyperplanes $H_1, \ldots, H_{2m+1} \in S$ so that the complement $\mathbb{P}(V^*) \setminus \bigcup H_j$ is Kobayashi hyperbolic.

Our main application of this result arises in the case where $S$ is an orbit of the real form at hand.

**Corollary 5.1.3.** Let $G^F$ be a reductive complex Lie group, $G$ a real form, $V^*$ an irreducible $G^F$-representation space and $S$ a $G$-orbit in $\mathbb{P}(V)$. Then there exist hyperplanes $H_1, \ldots, H_{2m+1} \in S$ so that $\mathbb{P}(V^*) \setminus \bigcup H_j$ is Kobayashi hyperbolic.

**Proof.** From the irreducibility of the representation $V^*$, it follows that $V$ is likewise irreducible and this, along with the identity principle, implies that for $(S) = \mathbb{P}(V)$. \qed
5.2 Hyperbolic domains in $G^\mathbb{C}/K^\mathbb{C}$

As we have seen above, hypersurfaces $H$ in $\Omega = G^\mathbb{C}/K^\mathbb{C}$ which are invariant under the action of an Iwasawa-Borel group $B$ play a key role in the study of $G$-invariant domains (see also [1], [IS], [HW1], and [HW2]). In the sequel we shall simply refer to such $H$ simply as a $B$-hypersurface.

Recall that if $H_1, \ldots, H_n$ are all the irreducible $B$-hypersurfaces in $G^\mathbb{C}/K^\mathbb{C}$ and if $\bigcup_{g \in G} g(H_i)$ is removed from $\Omega$, then the resulting set $\Omega$ is the Akhiezer-Gindikin domain $\Omega_G$ (see Theorem 2.0.3). In particular, the resulting domain is non-empty.

Now if $H$ is just one (possibly not irreducible) $B$-hypersurface, then the set $\bigcup_{g \in G} g(H) = \bigcup_{s \in K} g(H)$ is closed and its complement in $\Omega = G^\mathbb{C}/K^\mathbb{C}$ is open. Let $\Omega_H$ be the connected component of that open complement, containing the chosen base point $x_0$. It is likewise a non-empty $G$-invariant Stein domain in $\Omega = G^\mathbb{C}/K^\mathbb{C}$. Here we shall prove that, if $G$ is not Hermitian, any such $\Omega_H$ is Kobayashi hyperbolic. In the Hermitian case one easily describes the situation where $\Omega_H$ is not hyperbolic.

Let $H$ be given as above and let $L$ be the line bundle which it defines. Let $\sigma_H$ be the corresponding section, i.e., $\{\sigma_H = 0\} = H$. For convenience we may regard $L$ as an algebraic $G^\mathbb{C}$-bundle on a smooth, $G^\mathbb{C}$-equivariant projective compactification $X$ of $G^\mathbb{C}/K^\mathbb{C}$ to which the $B$-hypersurface $H$ extends.

Note that $\sigma_H$ is a $B$-eigenvector in $\Gamma(X, L)$. Let $V_H \subset \Gamma(X, L)$ be the irreducible $G^\mathbb{C}$-representation space which contains $\sigma_H$. Define $\varphi_H : \Omega \to \mathbb{P}(V_H^*)$ to be the canonically associated $G^\mathbb{C}$-equivariant meromorphic map.

**Lemma 5.2.1.** The map $\varphi_H|_{\Omega} : \Omega \to \mathbb{P}(V_H^*)$ is a regular morphism onto a quasi-projective $G^\mathbb{C}$-orbit $G^\mathbb{C}. v_0^* =: \Omega$.

**Proof.** By definition $\varphi_H$ is $G^\mathbb{C}$-equivariant; in particular its set $E$ of base points is $G^\mathbb{C}$-invariant. Since $\Omega$ is $G^\mathbb{C}$-homogeneous, $E = \emptyset$. \hfill \Box

From now on we replace $\varphi_H$ by its restriction to $\Omega$ and only discuss that map. By definition every section $s \in V_H$ is the pull-back $\varphi_H^*(s)$ of a hyperplane section. Thus, there is a uniquely defined $B$-hypersurface $\bar{H}$ in $\mathbb{P}(V_H^*)$ with $\varphi_H^{-1}(\bar{H}) = H$. Let $\Omega_{\bar{H}} \subset \mathbb{P}(V_H^*)$ be defined analogously to $\Omega_H$, i.e., $\Omega_{\bar{H}} = \mathbb{P}(V_H^*) \setminus \bigcup_{g \in G} g(H)$. Applying Cor. 5.1.3 to $\mathbb{P}(V_H^*)$ and $S := G \cdot H \subset \mathbb{P}(V_H^*)$, it follows that the domain $\Omega_{\bar{H}}$ is Kobayashi hyperbolic. Further, the connected component of $\varphi_H^{-1}(\Omega_{\bar{H}})$ which contains the base point $x_0$ is just the original domain $\Omega_H$.

If $\varphi$ has positive dimensional fibers, which indeed can happen in the Hermitian case, then, since the connected components of its fibers contain many holomorphic curves $f: \mathcal{U} \to \Omega$, it follows that $\Omega_H$ is not Kobayashi hyperbolic.

In the case of finite fibers, since preimages under locally biholomorphic maps of hyperbolic manifolds are hyperbolic, the opposite is true.
Theorem 5.2.2. If the $\varphi_H$-fibers are finite, then $\Omega_H$ is Kobayashi hyperbolic.

Corollary 5.2.3. If $G$ is not of Hermitian type, then $\Omega_H$ is Kobayashi hyperbolic.

Proof. If $G$ is not of Hermitian type, then $K^C$ is dimension theoretically maximal in $G^C$ and, since $\varphi_H$ is non-constant, it follows that it has finite fibers.

Theorem 5.2.4. The Wolf cycle domain $\Omega_W(D)$ of an open orbit $D$ of an arbitrary real form $G$ of an arbitrary complex semisimple group $G^C$ in an arbitrary flag manifold $Z = G^C/Q$ is Stein and Kobayashi hyperbolic.

Proof. In ([HW2]) it was shown that every Wolf cycle space $\Omega_W(D)$ is the intersection of certain of the $\Omega_H$. In the notation of ([HW2]) such an intersection is referred to as the associated Schubert domain $\Omega_S(D)$. Thus the cycle domains $\Omega_W(D)$ are Stein.

If $G$ is not of Hermitian type, then, since it is contained in $\Omega_H$ for certain $B$-hypersurfaces $H$, Cor. 5.2.3 implies that it is hyperbolic.

If $G$ is of Hermitian type, then $\Omega_W(D)$ is either the associated bounded symmetric domain $B$, its complex conjugate or, if $\Omega$ is non-compact, $B \times \overline{B}$ ([W2], [WZ1], [WZ2], [HW2]). Since bounded domains are hyperbolic, this completes the proof.

The following consequence of Cor. 4.4.4 is from our point of view the main technical result of this paper.

Theorem 5.2.5. If $\Omega$ is a $G$-invariant Stein domain in $G^C/K^C$ which properly contains $\Omega_{AG}$, then $\Omega$ is not Brody hyperbolic.

Proof. Let $z \in \partial(\Omega_{AG})$ be a generic boundary point in the sense of Prop. 4.2.3 which is also contained in $\Omega$. We may assume that $z$ is an optimal base point and that $x_1 = \exp(A)x_0$ is in its closure as in Lemma 4.1.1. Let $S^C$ be as in Theorem 4.3.3 so that $Q_2 = S^C, x_1$ is a $Q_2$-slice which contains $z$. Since $Q_2 \cap \Omega$ contains the Akhiezer-Gindikin domain of $Q_2$ ($D_0 \times D_\infty$ in the language of §4.4) as well as the boundary point $z$, it follows from Cor. 4.4.4 that $\Omega$ is not Brody hyperbolic.

We now give a characterization of all Wolf cycle domains, including the few exceptions mentioned above. For this recall that $D$ is an open $G$-orbit in $Z = G^C/Q$, $C_0$ the base cycle in $D$, and $G^C, C_0 = \Omega$ is the corresponding orbit in the cycle space $C^0(Z)$.

Theorem 5.2.6. If $\Omega$ is compact, then either $\Omega_W(D)$ consists of a single point or $G$ is Hermitian and $\Omega_W(D)$ is either the associated bounded symmetric domain $B$ or its complex conjugate $\overline{B}$. If $\Omega$ is non-compact, then, regarding $\Omega_W(D)$ as a domain $G^C/K^C$, it follows that $\Omega_W(D) = \Omega_S(D) = \Omega_I = \Omega_{AG}$ for every open $G$-orbit in every $G^C$-flag manifold $Z = G^C/Q$. 22
Proof. The exceptional case where $\Omega$ is compact is discussed in detail in the proof of Theorem 5.2.4 and therefore we restrict here to the non-compact case.

The statement $\Omega_W (D) = \Omega_Z (D)$ is proved in ([HW2]). In §2 above it is proved that $\Omega_{AG} = \Omega_I$. By definition $\Omega_S (D) \supset \Omega_I = \Omega_{AG}$. Since $\Omega_W (D)$ is Stein and hyperbolic (Theorem 5.2.4), by Theorem 5.2.5 it follows that $\Omega_W (D) = \Omega_{AG}$, and all equalities are forced.

As a final remark we would like to mention ([GM]) where cycle domains $C(\gamma)$ are introduced for every $G$-orbit $\gamma \in Orb_Z (G)$. If $\kappa$ is the dual $K^\mathbb{C}$-orbit to $\gamma$, then $C(\gamma)$ is the connected component containing the identity of $\{ g \in G^\mathbb{C} : g(\kappa) \cap \gamma \text{ is compact and non-empty} \}$. Of course these spaces can be regarded in $G^\mathbb{C}/K^\mathbb{C}$, and for open orbits $\gamma = D$ they are the same as the Wolf domains $\Omega_W (D)$.

Note that, by regarding the cycle domains as lying in the group $G^\mathbb{C}$ and going to the intersections, one avoids the special considerations in the Hermitian case. When the cycle spaces $\mathcal{B}$ and $\bar{\mathcal{B}}$ are regarded as subsets of $C^\mathbb{C}(Z)$, the intersection of the associated spaces in $G^\mathbb{C}/K^\mathbb{C}$ yields the domain $\mathcal{B} \times \bar{\mathcal{B}}$ in $G^\mathbb{C}/K^\mathbb{C}$.

Define

$$C_Z (G) := \bigcap_{\gamma \in Orb_Z (G)} C(\gamma).$$

By Proposition 8.1 of [GM], if $x$ is in every cycle domain $\Omega_W (\bar{D})$, where $\bar{D}$ is an open $G$-orbit in $\bar{Z} = G^\mathbb{C}/B$, then $x \in C_Z (G)$ for every $\bar{Z}$.

The following result was proved in ([GM]) for classical and exceptional Hermitian groups via a case-by-case argument.

**Corollary 5.2.7.** For $G$ an arbitrary real form of an arbitrary complex semisimple group $G^\mathbb{C}$ and $Z = G^\mathbb{C}/Q$ any $G^\mathbb{C}$-flag manifold, it follows that $C_Z (G) = \Omega_{AG}$.

**Proof.** Using the above mentioned Prop. 8.1, the remark in the Hermitian case and Theorem 5.2.6, it follows immediately that $\Omega_{AG} \subset C_Z (G)$. For the other inclusion, note that by definition $C_Z (G)$ is contained in the intersection of the cycle domains $\Omega_W (D)$ for $D$ an open $G$-orbit in $Z$. By Theorem 5.2.6 this intersection is again $\Omega_{AG}$. 

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