A Spin Pump Characterized by Entanglement Chern Numbers

Takahiro Fukui$^1$ and Yasuhiro Hatsugai$^2$

$^1$Department of Physics, Ibaraki University, Mito 310-8512, Japan and
$^2$Division of Physics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan

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We study a spin pump on a two-leg ladder chain of the Rice–Mele model. To characterize the spin pump, we propose the Chern number for the many-body ground state of the entanglement Hamiltonian, which is referred to as the entanglement Chern number. We show that this model has two phases distinguished by the entanglement Chern numbers. These two phases can be experimentally verified in cold atoms.

Many-body ground states of symmetry-protected topological phases are characterized by bulk topological invariants as well as edge (surface) states. This feature is known as the bulk-edge correspondence. Two-dimensional systems with no specific symmetries (Class A) are classified by the Chern number (CN). Correspondingly, the numbers of edge states with chirality $\pm$ at a specific boundary, $n_\pm$, are constrained to be $c = n_+ - n_-$. Generically, for a given $c$, a model with $n_+ = 0$ or $n_- = 0$ can be a minimum model. Although a more generic number of edge states may be possible, pair annihilation of edge states with opposite chiralities makes the minimal realization by chiral edge states most stable. The symmetry-protected topological phases are exceptions to this rule. For example, time-reversal (TR) symmetry inducing the Kramers degeneracy allows unusual edge states typically with $n_\pm = 1$ even when $c = 0$. These are characterized by the bulk $Z_2$ invariant.

A weak topological insulator is another type of nontrivial system. This material shows two or zero Dirac surface states depending on the surfaces. Two surface Dirac states appear to be unstable at first sight but have stability even against disorder. A similar phase has been found also in two dimensions, which has a vanishing CN but shows boundary-dependent edge states. This phase is also referred to as a weak topological phase. This phase also has stability since it is characterized by the entanglement CN (ECN).

The ECN has been introduced recently as the CN of the many-body ground state of the entanglement Hamiltonian (EH). As shown in Ref. [23], the ECN can serve as an alternative to the $Z_2$ invariant. Indeed, the global phase diagram of the Kane–Mele model is reproduced by the ECN. Since the ECN is defined even without TR symmetry, the phase diagram in the presence of a magnetic field has also been discussed. On this basis, in this paper we study a spin pump, i.e., a one-dimensional analog of the topological insulator, which is attracting much current interest owing to the recent experimental success of the charge pump. We propose a simple spin pump model that is experimentally more feasible than the Fu–Kane model at the cost of TR symmetry.

Even without TR symmetry, the ground state of the model we propose has a vanishing CN. According to the conventional topological classification scheme, $c = 0$ states with no specific symmetries are topologically trivial. As stated in the von Neumann–Wigner theorem, any bulk ground state may be adiabatically connected by the inclusion of infinitesimal symmetry-breaking terms. However, this is too restrictive when one considers the physical implication of a gapped topological system. If the mother state is a symmetry-protected topological state, daughter states with reduced symmetries still preserve edge states as the low-energy modes. Then, by focusing on the edge states as the key feature of the nontrivial bulk, the state is still topologically nontrivial.

For such daughter states, no matter how small symmetry-breaking perturbations are, the $Z_2$ invariant is no longer well-defined, whereas the CN is still zero as long as the perturbations are sufficiently small. The ground state of the Kane–Mele model with a weak magnetic field is one example. Nevertheless, the ECN suggests a nontrivial ground state even for such a system. Thus, we expect that the ECN may be helpful in describing such an almost symmetry-protected topological state. Unfortunately, there is still a lack of rigorous mathematical foundation for the ECN; thus, we need to study more examples other than the Kane–Mele model.

A spin pump is suitable for this purpose since we expect experimental verification in cold atom systems. Knowing the stability of topological states under the relaxation of the symmetry restriction is of both theoretical and experimental interest.

To begin with, we review the basic model of a charge pump, described by the (spinless) Hamiltonian, where

$$ H(t) = \sum_{i,j} c_i^\dagger \mathcal{H}_{ij}(t) c_j, $$

$$ \mathcal{H}_{ij}(t) = \frac{t_0 + (-)^i \delta(t)}{2} (\delta_{i+1,j} + \delta_{i,j+1}) + (-)^j \Delta(t) \delta_{i,j} $$

(1)

with time-dependent parameters $\delta(t) = \delta_0 \cos(2\pi t/T)$ and $\Delta(t) = \Delta_0 \sin(2\pi t/T)$. Here, we regard $t$ as an external parameter that controls the pumping. Below, this model is referred to as the Rice–Mele model as in Ref. [23]. The half-filled ground state of this model has the CN $c = 1$, implying that after one period $T$, all the Wannier states located at the sites adiabatically move...
together to their neighbors \[11, 26, 30, 31\]. Recently, a topological charge pump has been experimentally observed in ultracold atoms \[24, 27\].

FIG. 1: Rice–Mele ladder chain. Each chain is labeled by the pseudo-spin $\uparrow, \downarrow$, and the charges are pumped towards the directions indicated by the horizontal arrows. Nonequivalent sites in the unit cell are labeled by $a = \pm$. The vertical solid lines and the dashed lines denote interchain couplings associated with a “magnetic field” $H_{\text{mf}}$ and a “spin-orbit coupling” $H_{\text{so}}$, respectively.

To study a spin pump, let us consider two Rice–Mele chains forming a two-leg ladder. The time-dependent parameters are defined such that charges on two chains are pumped towards opposite directions if interchain couplings are switched off. Let us distinguish the sites on the two chains by a pseudo-spin $\sigma = \uparrow, \downarrow$, as illustrated in Fig. 1. The fermion operator is then denoted as $c_{j\sigma}$. The Hamiltonian is

$$H(t) = H_0(t) + H_{\text{mf}},$$

$$H_0(t) = \sum_{i,j} \sum_{\sigma} c_{i\sigma}^\dagger H_{ij}(t) c_{j\sigma},$$

where $t_\uparrow = \theta/2 + t$ and $t_\downarrow = \theta/2 - t$ with a relative pumping phase $\theta$, and $H_{\text{mf}}$ is an interchain coupling defined by

$$H_{\text{mf}} = \sum_{j} \sum_{\sigma,\sigma'} c_{j\sigma}'(\mathbf{h} \cdot \mathbf{\sigma})_{\sigma\sigma'} c_{j\sigma}'.$$

Note that $H_0(t)$ is invariant under TR: $\mathcal{T} H_0(t) \mathcal{T}^{-1} = H_0(-t)$, where the TR transformation is defined by $\mathcal{T} c_{ij} \mathcal{T}^{-1} = i\sigma_2 K c_{ij}$. Since $H_{\text{mf}}$ breaks TR symmetry, it serves as a magnetic field for spins.

If the magnetic field given by Eq. (3) is replaced by the spin-orbit coupling

$$H_{\text{so}} = \sum_{j} \sum_{\sigma,\sigma'} c_{j\sigma}'(ie \cdot \mathbf{\sigma})_{\sigma\sigma'} c_{j+1\sigma'} + h.c.,$$

where $\mathbf{e} = (e_x, e_y, e_z)$ is the set of real parameters, the model simply becomes the Fu-Kane model \[11\] for the $Z_2$ spin pump when $\theta = 0$. Even with a finite $\theta$, the model $H_0(t) + H_{\text{so}}$ is invariant under TR. The half-filled ground states of these models with $H_{\text{mf}}$ and/or $H_{\text{so}}$ have the vanishing CN $c = 0$.

The ground state of the model with the spin-orbit coupling allows two edge states at each boundary. The level crossing at the TR invariant $t = T/2$ is the Kramers degeneracy, which is preserved even with spin-nonconserving terms. By the bulk-edge correspondence, it is directly related to the $Z_2$ topological invariant of the bulk \[11\]. On the other hand, for the model with $H_{\text{mf}}$, it is clear that the bulk $Z_2$ invariant cannot be defined. Nevertheless, the ground state can still be nontrivial, since as long as the TR-breaking terms are sufficiently small and the bulk gap remains open, the edge states cannot disappear immediately even though the Kramers degeneracy is lifted. The edge states in this situation again imply the existence of some nontrivial bulk quantity according to the bulk-edge correspondence. We expect that the ECN can be used to characterize the bulk.

In addition, the ECN is useful even for a topological insulator with TR symmetry \[22\]. In what follows, we therefore rely on the ECN to study the ground state with or without TR symmetry on an equal footing. To this end, we here introduce the notion of the ECN. The Hamiltonian in Eq. (2) is Fourier-transformed to

$$H(t) = \sum_k \sum_{\alpha,\beta} c_{\alpha}^\dagger(k) h_{\alpha\beta}(k,t)c_{\beta}(k),$$

where $\alpha = \sigma \sigma'$ denotes the spin $\sigma$ and bipartite site index $a = \pm$ in the unit cell, as depicted in Fig. 1. Let $|G(t)\rangle$ be the half-filled ground state of the Hamiltonian in Eq. (4). It is divided into each momentum $k$ as $|G(t)\rangle = \prod_k |G(k,t)\rangle$. If the state $|G(k,t)\rangle$ is decomposed into spin sectors, it generically becomes the sum of the tensor products of the form

$$|G(k,t)\rangle = \sum_{i,j} D_{ij}|\Psi_{i\uparrow}(k,t)\rangle \otimes |\Psi_{i\downarrow}(k,t)\rangle,$$

where $|\Psi_{\sigma i}(k,t)\rangle$ is an orthonormal basis state in the spin-$\sigma$ sector. The singular value decomposition of the matrix $D$ leads to the diagonal form

$$|G(k,t)\rangle = \sum_i \lambda_i(k,t)|\tilde{\Psi}_{i\uparrow}(k,t)\rangle \otimes |\tilde{\Psi}_{i\downarrow}(k,t)\rangle.$$

The normalization of $|G(k,t)\rangle$ requires $\sum_i \lambda_i^2 = 1$; thus, we can choose $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. When $\lambda_1 \sim 1$ and the other $\lambda_i \sim 0$ ($i \geq 2$), the most dominant tensor-product state

$$|G_{\text{de}}(k,t)\rangle \equiv |\tilde{\Psi}_{1\uparrow}(k,t)\rangle \otimes |\tilde{\Psi}_{1\downarrow}(k,t)\rangle$$

is simply the most disentangled state in $|G(k,t)\rangle$ associated with the spin partition. The remaining terms associated with $\lambda_i$ ($i \geq 2$) are referred to as the residual entanglement (RE).

Let $\rho(t) = |G(t)\rangle\langle G(t)|$ be the density matrix of the ground state. Then, it is also divided into $\rho(t) = \prod_k \rho(k,t) = \prod_k |G(k,t)\rangle\langle G(k,t)|$. Tracing out the wavefunctions $|\Psi_{-\sigma i}(k,t)\rangle$ associated with spin $-\sigma$ in Eq. (11), we obtain the reduced density matrix $\rho_{-\sigma}$ associated with
spin $\sigma$

$$\rho_{\sigma}(k, t) = \text{tr}_{-\sigma} \rho(k, t) = \sum_i \lambda_i^2 |\tilde{\Phi}_{\sigma i}(k, t)) \langle \tilde{\Phi}_{\sigma i}(k, t)|. \quad (9)$$

It is also parameterized as $\rho_{\sigma}(k, t) \propto \exp[-H_{\sigma}(k, t)]$, where the EH is defined by $H_{\sigma}(k, t) = \sum_{a,b} c_{\sigma a}^T(k) h_{\sigma,a}(k) c_{\sigma b}(k)$ in the case of non-interacting systems [32]. The entanglement spectrum (ES) is the spectrum of $h_{\sigma}(k, t)$; $\sum_{a,b} h_{\sigma,a}(k, t) \psi_{\sigma,b}(k, t) = \epsilon_{\sigma,a}(k, t) \psi_{\sigma,a}(k, t)$. To obtain the ES, it is convenient to utilize the projection operator to the ground state defined by

$$P_{\beta\alpha}(k, t) = \langle G(k, t)| c_{\beta i}^T(k) c_{\beta i}(k) |G(k, t) \rangle = \sum_{\text{occupied } n} \psi_{\beta n}(k, t) \psi_{\beta n}(k, t), \quad (10)$$

where $\psi_{\beta n}(k, t)$ is the wavefunction in the $n$th band of $h(k, t)$ in Eq. (4). By restricting $P_{\alpha\sigma}$ to the spin-$\sigma$ sector such that $\alpha = \sigma a$ and $\beta = \sigma b$, we have the projected two-point correlation function $P_{\sigma,ba}(k, t) = P_{\sigma,ba}(k, t)$. Remarkably, this can alternatively be written as

$$P_{\sigma,ba}(k, t) = \text{tr} c_{\sigma a}^T(k) c_{\sigma b}(k) \rho_{\sigma}(k, t) = \sum_{\mu} \psi_{\sigma,\mu}(k, t) \frac{1}{e^{\epsilon_{\sigma,\mu}(k, t)} + 1} \psi_{\sigma,\mu}^T(k, t). \quad (11)$$

Therefore, an eigenvalue of $P_{\sigma,ba}(k, t)$ is simply the distribution function associated with the ES, and $\psi_{\sigma,\mu}(k, t)$ is a simultaneous eigenfunction of $h_{\sigma}(k, t)$ and $P_{\sigma,ba}(k, t)$.

We now define the many-body ground state of the EH $h_{\sigma}$ as the state with the negative ES states fully occupied. This new ground state corresponds to the disentangled state $|\Psi_{G1}(k, t)\rangle$ in Eq. (4). Suppose that the ES is gapped. The gapped ES implies that the largest $\lambda_1$ term in Eq. (4) is unique and that the CN for the new ground state $|\Psi_{G1}(k, t)\rangle$ can be defined. It is practically computed by the use of the wavefunction $\psi_{\sigma,\mu}(k, t)$ [21, 22]. The CN thus defined will be denoted by $c_{\sigma}$ below. This is similar to the spin CN calculated under spin-dependent twisted boundary conditions [33, 34]. However, the computation of the CN is much simpler because it is carried out in the momentum space using the techniques developed in [35]. As long as the RE is sufficiently small, $c = c_{\uparrow} + c_{\downarrow}$ is expected.

The qualitative behavior of the model Eq. (4) with $H_{\text{mf}}$ is as follows. For simplicity, we restrict our discussion to the cases $h = (h_x, 0, 0)$. For a small value of $h_x < h_{c1}$, the one-particle spectrum of $h(k, t)$ in Eq. (5) has a finite energy gap, but it closes at $h_{c1}$, and when $h_{c2} < h_x$, a new gap opens again. In both gapped regions, the CN is $c = 0$, but they are distinguished by the ECN, as we show below.

We first consider the small-$h_x$ region. Exactly at $h_x = 0$, two spin sectors are decoupled; thus, the pumped charges in one period $T$ are $\Delta q_1 = -\Delta q_2 = 1$, and therefore, the pumped spin $\Delta s_z = (\Delta q_1 - \Delta q_2)/2 = 1$ is quantized. This is ensured by the fact that the CNs for the up- and down-spin sectors of the ground state are 1 and $-1$, respectively. Once $H_{\text{mf}}$ is introduced, it is no longer possible to define such CNs, but the ECN can be, nevertheless, well-defined. [21, 22]. In the region $h_x < h_{c1}$, the ES associated with the spin partition is indeed gapped and the ECNs are computed as $(c_{\uparrow}, c_{\downarrow}) = (1, -1)$. Therefore, as far as the disentangled state $|G_{\text{de}}\rangle$ in Eq. (8) is concerned, topological spin pumping occurs and the pumped spin is quantized within this disentangled state. In passing, we mention that for the model with $H_{so}$, the ground state also has the same ECN $(1, -1)$. Thus, in terms of the ECN, both models belong to the same topological class.

However, the physical ground state $|G\rangle$ includes the effect of the RE. This is manifest especially in the spectrum of the edge states. We show in Fig. 2(a) the energy spectrum of a finite Rice–Mele ladder chain with $H_{\text{mf}}$. We can observe gapped edge states. Spectrum (b) is in sharp contrast to spectrum (a), in which TR invariance guarantees the Kramers’ degeneracy at the TR-invariant point $t = T/2$.

So far, we have argued that as long as the symmetry-breaking perturbation is small and the RE is also small, the dominant component of $|G\rangle$, namely, the disentangled state $|G_{\text{de}}\rangle$, can be topologically nontrivial. Here, TR invariance is unimportant. Correspondingly, the edge states exist in the bulk gap, even though they are gapped in the absence of TR symmetry. When the gapped edge states are due to the nontrivial ECN of the disentangled...
state, it may be possible to make them gapless without closing the bulk gap. Indeed, in the present case, if the edge states are shifted in $t$ by changing the relative pumping phase $\theta$, spectrum (c) with $H_{\text{mf}}$ becomes very similar to spectrum (d) with $H_{\text{sp}}$. Experimentally, it may be possible to observe the pumped charge at each chain and thus the pumped spin. The observables in the spin-$\sigma$ sector are affected by the RE, and therefore, the observed pumped spin is not quantized. However, it should be stressed that the experimental observation of a finite spin pump suggests a nontrivial ECN.

For a strong magnetic field, $h_x > h_{c2}$, the ECN associated with the spin is ill-defined, since the spins are strongly entangled and the ES of $P_\sigma$ in (14) becomes gapless. On the other hand, a new partition associated with $a = \pm$ may be useful, since for a large inter-chain coupling, particularly for a large $h_x$, local interchain dimerized states are expected. To see this, we introduce the bipartition with respect to $a = \pm$ and calculate the ES associated with the reduced density matrix $\rho_a$, where $a = \pm$, by tracing out $-a$. This can be obtained by the use of the correlation matrix $P_{\sigma\alpha}(k, t)$ by restricting $\alpha$ and $\beta$ to $\alpha a$ and $\alpha a'$, which is denoted by $P_{\alpha\sigma\alpha'} = P_{a\alpha a\sigma a'}$ with $a = \pm$ fixed. In the region $h > h_{c2}$, it turns out that such a partition gives a gapped ES but the ECNs are trivial, $(e_+, e_-) = (0, 0)$. Thus, we conclude that the two gapped phases $h_x < h_{c1}$ and $h_{c2} < h_x$ are distinct. This can also be seen from the spectrum in Fig. 3(a), in which no sign of edge states is observed. Contrary to this model, the model with $H_{\text{sp}}$ exhibits a bulk gap for a rather strong spin-orbit coupling, as shown in Fig. 3(b).

In summary, we have introduced the ECN and proposed a spin pump model that shows distinct phases characterized by the ECN. More detailed analysis based on the bulk-edge correspondence in a topological pumping [30, 37] is an interesting future issue. We expect that the concept of the ECN will give us a way of studying the stability of symmetry-protected topological phases against symmetry-breaking perturbations, which will also open up the possibility of the experimental realization of topological phenomena.

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