CHARACTERIZING REGULAR LAGRANGIANS BY LEFSCHETZ FIBRATION

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Abstract. In [2] Eliashberg, Ganatra and Lazarev have introduced the notion of Regular Lagrangians in Weinstein cobordisms and also predicted that Regular Lagrangians can be characterized by the existence of Weinstein Lefschetz Fibrations. In this paper we present a proof of this fact. Although a method has been suggested in [2], but we have done it differently.

1. INTRODUCTION

In [2] Eliashberg, Ganatra and Lazarev have introduced the notion of Regular Lagrangians in Weinstein cobordisms and also predicted that Regular Lagrangians can be characterized by the existence of Weinstein Lefschetz Fibrations. In this paper we present a proof of this fact.

Let us first define the notion of Regular Lagrangians. But in order to do so we need to understand the notion of Liouville cobordisms.

A Liouville cobordism between contact manifolds $(\partial_\pm W, \xi_\pm)$ is an even dimensional cobordism (say of dimension $2n$) $(W, \partial_- W, \partial_+ W)$, where $W$ is equipped with a symplectic form $\omega$ and an expanding Liouville vector field $X$ which is outward pointing along $\partial_+ W$ and inward pointing along $\partial_- W$, moreover the contact structure induced by the Liouville form $\lambda = \iota(X) \omega$ on $\partial_\pm W$ coincides with the given ones.

A Liouville cobordism $(W, \partial_- W, \partial_+ W)$ is called a Weinstein cobordism if there exists a defining Morse function $\phi : W \to \mathbb{R}$ for which $X$ is gradient-like and $(\omega, X, \phi)$ is called a Weinstein cobordism structure on $W$. Equivalently if the Liouville form $\lambda$ is given then $\omega$ and $X$ can be recovered, so we may say $(\lambda, \phi)$ to be a Weinstein structure. A Weinstein cobordism $(W, \partial_- W, \partial_+ W)$ with $\partial_- W = \Phi(empty)$ is called a Weinstein domain.

Key words and phrases. Regular Lagrangians, Weinstein cobordism, Lefschetz Fibration.
In what follows we shall consider exact Lagrangian subcobordism with Legendrian boundary \((L, \partial_-L, \partial_+L) \subset (W, \partial_-W, \partial_+W)\) in Weinstein cobordism with contact boundary.

Now consider \((L, \partial_-L, \partial_+L) \subset (W, \partial_-W, \partial_+W)\) an exact Lagrangian cobordism in a Liouville cobordism and let \(N(L)\) be a small tubular neighborhood of \(L \subset W\). Define \(\partial^{int}N(L) = \partial N(L) \cap \text{Int}(W)\), \(\partial^{ext}N(L) = \partial N(L) \cap \partial W\) and \(W_L = W^{-} - N(L)\). As \(L\) is exact we can make \(X \not\pitchfork \partial N(L)\) and hence \((W_L, \omega|_{W_L})\) has a natural Liouville cobordism structure with sutured boundary \(\partial_+W_L \cup \partial_-W_L\), where

\[
\partial_+W_L = \partial W - \partial^{ext}N(L), \quad \partial_-W_L = \partial^{int}N(L)
\]

**Definition 1.1.** Let \(L \subset (W, \omega', X', \phi')\) be an exact Lagrangian cobordism in a Weinstein cobordism. \(L\) is called regular if the Weinstein structure is homotopic to a Weinstein cobordism structure \((W, \omega, X, \phi)\) through a homotopy of Weinstein structures for which \(L\) remains Lagrangian and \(X\) is tangent to \(L\). This is equivalent to the condition that the Liouville form \(\lambda := \iota(X)\omega\) has the property that \(\lambda|_L = 0\).

Such a Weinstein structure is called tangent Weinstein structure to the regular Lagrangian \(L\). It follows that the critical points of \(\phi|_L\) are global critical points for \(W\).

It turns out that for a regular Lagrangian \(L\) and any Weinstein structure tangent to \(L\) can further be adjusted so that the following happens.

The Weinstein cobordism structure \((W, \omega, X, \phi)\) tangent to \(L\) is called adjusted to \(L\) if there exists a regular value \(c \in \mathbb{R}\) of \(\phi\) such that

1. all critical points of \(\phi\) in the sublevel set \(\{\phi \leq c\}\) lies on \(L\) and the indeces of these critical points for \(\phi\) and \(\phi|_L\) coincides;

2. there are no critical points of \(\phi\) on \(L \cap \{\phi \geq c\}\).

So the handlebody presentation has the following description. First the handles corresponding to the critical points of \(\phi|_L\) are attached and then the remaining ones are attached so that the attaching spheres of these remaining handle attachments do not intersect \(\partial_+L\).

It turns out that...
(1) If $\partial_-L$ is empty then the Weinstein structure on $W(L) = \{\phi \leq c\}$ is equivalent (radially) to the disjoint union of the trivial cobordism over $\partial_-W$, i.e, $\partial_-W \times [0,1]$ and the canonical Weinstein structure on $T^*L$.

(2) If $\partial_-L$ is non-empty then the handlebody presentation of $L$ given by $\phi|_L$ and the gradient-like vector field $X|_L \in T L$ defines a Weinstein cobordism structure on the cotangent disk bundle $N$ of $L$ with $\partial_-N = \partial N \setminus \text{Int}_\partial N$ and $\partial_+N = \partial N$. Then $W(L)$ is obtained by attaching a generalized handle, i.e, by attaching the Weinstein cobordism $N$ to the trivial cobordism $\partial_-W \times [0,1]$ over $\partial_-W$ by identifying $\partial_-L$ with a neighborhood of a Legendrian sphere in $\partial_-W \times \{1\}$.

So in any case the underlying symplectic structure on $W(L)$ near $L$ is given by the symplectic structure on $T^*L$ but the Weinstein structure differs depending on whether $\partial_-L$ is empty or not.

**Remark 1.2.** Since we shall be considering Weinstein Lefschetz fibrations on $\mathbb{C}$, i.e, we shall be considering Weinstein domains instead of Weinstein cobordisms. So we shall only consider the option (1) from the above list, i.e, $\partial_-L$ empty as $\partial_-L \subset \partial_-W = \emptyset$. So in our case the Weinstein structure on $N(L)$ is $T^*L$.

Next we need to define the notion of Weinstein Lefschetz fibration.

**Definition 1.3.** (4) A Weinstein Lefschetz fibration is a tuple

$$W^{2n} = ((W_0, \lambda_0, \phi_0); L_1, ..., L_m)$$

where $W_0^{2n-2}$ is a Weinstein domain and $L_j$’s are exact parametrized Lagrangian spheres in $W_0^{2n-2}$, where parametrized means diffeomorphisms $S^{n-1} \to L$ up to precomposition with elements of $O(n)$. Its total space $|W|$ is given as follows. Consider the Weinstein manifold

$$(W_0 \times \mathbb{C}, \lambda_0 - J^*d(\frac{1}{2}|z|^2), \phi_0 + |z|^2)$$

The corresponding Liouville vector field is given by $X_{\lambda_0} + \frac{1}{2}(x\partial_x + y\partial_y)$. Take Legendrian lifts $\Lambda_j \subset (W_0 \times S^1, \lambda_0 + Nd\theta)$ of $L_j \subset W_0$ such that $\Lambda_j$ projects to small interval around $\frac{2\pi i}{m} \in S^1$. We take $N$ so large that these intervals remains disjoint. The embedding $S^1 \to \mathbb{R}$
as circle of radius $N^{1/2}$ pulls back the Liouville form $-J^* d \left( \frac{1}{2} |z|^2 \right)$ to the contact form $Nd\theta$. So $\Lambda_j$ could be thought as a Legendrian at the level set $\{|z| = N^{1/2}\} \subset W_0 \times \mathbb{C}$. Then the restriction of downward Liouville flow defines a map $\Lambda_j \times \mathbb{R}_{\geq 0} \to W_0 \times \mathbb{C}$ and which intersects the set $\{\phi_0 + |z|^2 = 0\}$ along a Legendrian $\Lambda'_j$. Again $N$ is taken so large that the projection of $\{\phi_0 + |z|^2 \leq 0\}$ to $\mathbb{C}$ is contained inside the disc of radius $N^{1/2}$. Now the total space $|W|$ is defined as the result of attaching Weinstein handles (5) to the Weinstein domain $\{\phi_0 + |z|^2 = 0\}$ along the Legendrians $\Lambda'_j$.

Now we end this section with the main theorem of this paper.

**Theorem 1.4.** Let $L \subset W^{2n}$ be a regular Lagrangian, then there exists a Weinstein Lefschetz fibration over $\mathbb{C}$ which projects $L$ to a ray in $\mathbb{R} \subset \mathbb{C}$.

**Remark 1.5.** Observe that we have only stated the 'only if' part, the 'if' part is obvious.

**Method:** Our method is to improvise the Donaldson’s construction from [1] in order to accomplish the following.

We shall construct a section $s$ of the line bundle $E^k$ satisfying the Donaldson’s transversality condition so that the zero set of $s$, i.e., $Z(s)$ turns out to be symplectic submanifold of $W$ for large $k$, moreover the stable manifold of a zero on $L$ of the Hamiltonian vector field $X_{Im \log s}$ correspondin to the Hamiltonian function $Im \log s = Arg(s)$, is diffeomorphic to $L$ and is very close in the $C^0$-sense to $L$. Let us call this stable manifold $L'$. Then with help of $h$-principle technique and 2.1 we construct a symplectomorphism which is equal to the symplectomorphism given by 2.1 sending $L$ to $L'$ on a tubular $\tilde{N}(L) \subset N(L)$ (assuming $L' \subset \tilde{N}(L)$) and out side $N(L)$ is equal to the identity. Then we simply pull-back the weinstein structure by this symplectomorphism as well as $Z(s)$.

**2. Preliminaries**

First we need some notations. Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be a point in $\mathbb{C}^n$. Define

1. $\frac{\partial}{\partial z_j} \equiv \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \ j = 1, \ldots, n.$

2. $\frac{\partial}{\partial \bar{z}_j} \equiv \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right), \ j = 1, \ldots, n.$

3. $dz_j \equiv dx_j + idy_j, \ j = 1, \ldots, n.$
(4) $d\bar{z}_j \equiv dx_j - idy_j, \ j = 1, ..., n$

For $0 \leq p, q \leq n$, the differential form

$$\omega = \sum_{|\alpha|=p, |\beta|=q} \omega_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

is said to be of type or bidegree $(p, q)$, where $\alpha, \beta$ are multi-indices. Any differential form is the sum of the terms of the form $\omega_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$. For

$$\omega = \sum_{|\alpha|=p, |\beta|=q} \omega_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

define

(1) $\partial \omega = \sum_{j=1}^n \sum_{\alpha,\beta} \partial \omega_{\alpha\beta} \frac{\partial z_j}{\partial \bar{z}_j} dz_j \wedge dz^\alpha \wedge d\bar{z}^\beta$

(2) $\bar{\partial} \omega = \sum_{j=1}^n \sum_{\alpha,\beta} \frac{\partial \omega_{\alpha\beta}}{\partial z_j} d\bar{z}_j \wedge dz^\alpha \wedge d\bar{z}^\beta$

Now we shall state a theorem from [3] which we need for the proof of 1.4.

**Theorem 2.1.** (Weinstein Neighborhood Theorem, [3]) Any isotropic and, in particular, Lagrangian immersion $L \to W$ extends to an isosymplectic immersion $U \to W$, where $U$ is a tubular neighborhood of the zero-section in the cotangent bundle $T^*L$ endowed with the canonical symplectic structure.

3. $h$-Principle

This section does not have any new result, we just recall some facts from the theory of $h$-principle which we shall need in our proof.

Let $X \to M$ be any fiber bundle and let $X^{(r)}$ be the space of $r$-jets of germs of sections of $X \to M$ and $j^r f : M \to X^{(r)}$ be the $r$-jet extension map of the section $f : M \to X$. A section $F : M \to X^{(r)}$ is called holonomic if there exists a section $f : M \to X$ such that $F = j^r f$. In the following we use the notation $Op(A)$ to denote a small open neighborhood of $A \subset M$ which is unspecified.

Let $\mathcal{R}$ be a subset of $X^{(r)}$. Then $\mathcal{R}$ is called a differential relation of order $r$. $\mathcal{R}$ is said to satisfy $h$-principle if any section $F : M \to \mathcal{R} \subset X^{(r)}$ can be homotopped to a holonomic section $\tilde{F} : M \to \mathcal{R} \subset X^{(r)}$ through sections whose images are contained in $\mathcal{R}$. Put differently, if the space of sections of $X^{(r)}$ landing into $\mathcal{R}$ is denoted by $Sec\mathcal{R}$ and the space of holonomic sections of $X^{(r)}$ landing into $\mathcal{R}$ is denoted by $Hol\mathcal{R}$ then $\mathcal{R}$ satisfies $h$-principle.
if the inclusion map $\text{Hol}\mathcal{R} \to \text{Sec}\mathcal{R}$ induces a epimorphism at 0-th homotopy group $\pi_0$. $\mathcal{R}$ satisfies parametric $h$-principle if $\pi_k(\text{Sec}\mathcal{R}, \text{Hol}\mathcal{R}) = 0$ for all $k \geq 0$.

Let $(V, d\alpha_V)$ and $(W, d\alpha_W)$ be two exact symplectic manifolds with $\text{dim}(V) \leq \text{dim}(W)$. $\mathcal{R}_{\text{isosymp}}$ be the relation of isosymplectic immersions from $V$ to $W$. Let $A \subset V$ be a polyhedron of positive codimension.

**Theorem 3.1.** (3) All forms of local $h$-principle holds for the inclusion

$$\text{Hol}_{\text{Op}A} \mathcal{R}_{\text{isosymp}} \to \text{Sec}_{\text{Op}A} \mathcal{R}_{\text{isosymp}}$$

on $\text{Op}A$.

**Proof.** Direct consequence of 16.4.2 of [3].

4. **Donaldson’s Construction with a Preassigned Singularity**

In this section we follow the construction in [1] with some changes in order to construct a sequence of sections $s_k : W^{2n} \to E^k$ which will have a non-degenerate singularity on $L \subset W$, where $L, W$ are as in [2,4] and $E$ is a hermitian line bundle on $W$ and $E^k = E^{\otimes k}$.

Now we shall recall some parts of the local theory as in [1].

A complex structure on a real vector space $V$ is a decomposition of $V^* \otimes \mathbb{C}$ as a complex conjugate subspaces, the complex linear and anti-linear functionals. If a reference complex structure is fixed then $V$ becomes $\mathbb{C}^n$ and in this case the conjugate subspaces turns out to be just one forms of type $(1,0)$ and $(0,1)$, i.e,

$$\text{Hom}_R(\mathbb{C}^n, \mathbb{C}) = \Lambda^{1,0} \oplus \Lambda^{0,1}$$

If $J$ is another complex structure then $\Lambda^{1,0}_J$ is the graph of a complex linear map

$$\mu : \Lambda^{1,0} \to \Lambda^{0,1}$$

and in this case $\Lambda^{0,1}_J$ is the graph of $\bar{\mu} : \Lambda^{0,1} \to \Lambda^{1,0}$. The condition that $\Lambda^{1,0}_J \cap \Lambda^{0,1}_J$ is that $(1 - \mu \bar{\mu})$ is invertible. For an almost-complex structure $J$ the corresponding $\mu$ will be a bundle map i.e., it depends on the base point smoothly and we shall denote it by $\mu_z$ where $z$ is the base point. In this case

$$\bar{\partial}_J(f) = \bar{\partial}f - \mu(\partial f),$$
where $\partial$, $\bar{\partial}$ are the ordinary operators defined by the standard complex structure on $\mathbb{C}^n$.

For $\rho < 1$ define $\delta_\rho : \mathbb{C}^n \to \mathbb{C}^n$ by $\delta_\rho(z) = \rho z$. Consider the almost-complex structure $\tilde{J} = \delta_\rho^* J$ defined on $\rho^{-1}\Omega$ where $\Omega$ is the open neighborhood of zero in $\mathbb{C}^n$ where $J$ was defined. The $\tilde{\mu}$ corresponding to $\tilde{J}$ is given by

$$\tilde{\mu}_z = \mu_{\rho z}$$

In this case we get the following estimate

$$|\tilde{\mu}_z| \leq C \rho |z|, \quad |\nabla \tilde{\mu}| \leq C \rho,$$

where $C$ is a constant related to the Nijenhuis tensor and its derivative. Throughout the paper we use the convention that $C$ represents a positive constant which changes from line to line.

Now consider the Kähler form $\omega_0 = i \sum_{j=1}^{n} dz_j d\bar{z}_j$ on $\mathbb{C}^n$. Set $A = \frac{1}{4} (\sum_{j=1}^{n} z_j d\bar{z}_j - \bar{z}_j dz_j)$ then $\omega_0 = i dA$.

If $\xi \to \mathbb{C}^n$ be a line bundle with connection having curvature $\omega_0$ there is a trivialization of $\xi$ with respect to which the connection matrix turns out to be $A$. $A$ defines a coupled $\bar{\partial}$-operator $\bar{\partial}_A$,

$$\bar{\partial}_A(f) = \bar{\partial} f + A^{0,1} f,$$

where $A^{0,1}$ is the $(0,1)$ component of $A$. Similarly we define $\partial_A$. Now observe that

1. $\bar{\partial}_A(e^{-(Re \sum z_j^2)/4} + i) = \frac{e^{-(Re \sum z_j^2)/4}}{4} \sum_j (z_j - \bar{z}_j) d\bar{z}_j + \frac{1}{4} \sum_j z_j d\bar{z}_j$

2. $\partial_A(e^{-(Re \sum z_j^2)/4} + i) = -\frac{e^{-(Re \sum z_j^2)/4}}{4} \sum_j (z_j + \bar{z}_j) dz_j - \frac{1}{4} \sum_j \bar{z}_j dz_j$

We replace $\omega_0$ by $k\omega_0$, for a positive integer $k$. The replacement of $\xi$ by $\xi^k$ is same as the dilation with scalar factor $k^{-1/2}$ on $\mathbb{C}^n$.

Now let $(W,\omega)$ be a Liouville manifold with a compatible complex structure $J$ and a complex line bundle $E \to W$ with a $U(1)$ connection having curvature $-i\omega$. Let $g$ be the Riemannian metric determined by $J$ and $\omega$. Then $g_k = kg$ is the one determined by $J$ and $k\omega$. 
We consider the cubical coordinate chart which is longer in the $y$-coordinate directions, i.e., for $R \gg r$ set

$$I^n_{x,R} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x_j| < r, |y_j| < R\}$$

In fact we take $R = k^{-1/2}$ and $r = k^{-1}$. Consider the Darboux chart $\chi_{p,k} : I^{2n}_{k^{-1/2} + \epsilon_k} \to W$ around $p \in W$, where $\epsilon_k = e^{-k}$. We may further assume that all derivatives of $\chi_p$ are bounded and derivative of $\chi_p$ at zero is complex linear with respect to $J$ on $T_pW$. Therefore $\chi^*J$ represents a structure determined by a bundle map $\mu$ over $I^{2n}_{k^{-1/2} + \epsilon_k}$ as explained above. The derivatives of $\mu$ satisfies bounds independent of $p \in W$. So now given $k$ we consider the new chart

$$\tilde{\chi}_{p,k} = \chi_{p,k} \circ \delta_{k^{-1/2}} : k^{1/2}I^{2n}_{k^{-1/2} + \epsilon_k} = \tilde{I}^{2n}_{k^{-1/2} + \epsilon_k, 1 + \epsilon_k} \to W$$

Let $\tilde{\mu}$ be the bundle map representing the almost-complex structure on this new chart. So $\tilde{\mu}$ satisfies the following bounds

$$|\tilde{\mu}_z| \leq Ck^{-1/2}|z|,$$
$$|\nabla\tilde{\mu}_z| \leq Ck^{-1/2}.$$  

$C$ does not depend on $p$. Observe that $\tilde{\chi}^*(-ik\omega) = -i\omega_0$ where $-ik\omega$ is the curvature on $E^k$. So $\tilde{\chi}$ admits a connection preserving bundle map which we shall also denote by $\tilde{\chi}$. So $\tilde{\chi} : \xi \to E^k$ is the bundle map. So one can think of $\sigma = e^{-(\text{Re}\Sigma \zeta_j^2)/4} + i$ as local sections of $E^k$ around $p \in W$.

Let $\tilde{\partial}_{A,j}$ denote the $\tilde{\partial}$-operator defined by $\tilde{J}$ on $k^{1/2}I^{2n}_{k^{-1/2} + \epsilon_k, 1 + \epsilon_k}$ with connection matrix $A$. Then

$$\tilde{\partial}_{A,j}(f) = (\tilde{\partial}f + A^{0,1}f) + \tilde{\mu}(\partial f + A^{1,0}f).$$

If we replace $f$ by $e^{-(\text{Re} \Sigma \zeta_j^2)/4} + i$ we get the following estimate

$$|\tilde{\partial}_{A,j}(e^{-(\text{Re} \Sigma \zeta_j^2)/4} + i)| \leq \frac{|z|}{4} (2e^{-(\text{Re} \Sigma \zeta_j^2)/4} + 1) + Ck^{-1/2}|z|^2 (2e^{-(\text{Re} \Sigma \zeta_j^2)/4} + 1)$$

Furthermore we want the estimate on the derivative. So observe

$$\nabla = \tilde{\partial}_{A,j} + \partial_{A,j} = \tilde{\partial}_A + \partial_A + \tilde{\mu}\partial_A + \tilde{\mu}\tilde{\partial}_A$$

The only term in $\tilde{\partial}_{A,j}$ for which $\tilde{\mu}$ is not a factor is

$$\tilde{\partial}_A(e^{-(\text{Re} \Sigma \zeta_j^2)/4} + i) = \frac{e^{-(\text{Re} \Sigma \zeta_j^2)/4}}{4} \Sigma_j (z_j - \bar{z}_j) d\bar{z}_j + \frac{i}{4} \Sigma_j z_j d\bar{z}_j$$
So let us compute \((\overline{\partial}_A + \partial_A)(\overline{\partial}_A(e^{-(Re\Sigma z_j^2)/4} + i))\). So

\[
(\overline{\partial}_A + \partial_A)(\overline{\partial}_A(e^{-(Re\Sigma z_j^2)/4} + i)) = -\frac{e^{-(Re\Sigma z_j^2)/4}}{4} \sum_{i,j}(z_i + \bar{z}_i)(z_j - \bar{z}_j)dz_i \wedge d\bar{z}_j \\
+ \frac{i}{4} \sum_{i,j}(z_i z_j d\bar{z}_i \wedge d\bar{z}_j - \bar{z}_i z_j dz_i \wedge d\bar{z}_j) \\
+ \frac{e^{-(Re\Sigma z_j^2)/4}}{4} \sum_{i,j}dz_i \wedge d\bar{z}_j + \frac{i}{4} \sum_{i,j}dz_i \wedge d\bar{z}_j
\]

So the estimate we need is (using \(|z_j| \leq |z|\))

\[
|\nabla \overline{\partial}_A, j(e^{-(Re\Sigma z_j^2)/4} + i)| \leq \frac{e^{-(Re\Sigma z_j^2)/4}}{4} \left(\sum (2(n) + 1)|z|^2 + n + \frac{1}{8} [(2(n) + 1)|z|^2 + 2n] \right) \\
+ Ck^{-1/2} [e^{-(Re\Sigma z_j^2)/4} P^3(|z|)] + Ck^{-1/2} Q^3(|z|)
\]

where \(P^3, Q^3\) are cubic polynomial with constant term zero.

We now define the cut-off function \(\beta_k : \mathbb{C}^n \to \mathbb{R}\) as a \(C^\infty\)-function with \(\beta_k = s_k\) on \(I_k^{n-1/2-\epsilon_k,1-\epsilon_k}\) and \(\beta_k = 0\) outside \(I_k^{n-1/2+\epsilon_k,1+\epsilon_k}\), where \(s_k\) is a positive real sequence so small such that derivative \(\nabla \beta_k\) satisfies \(|\nabla \beta_k| \leq Ck^{-1}|z|^2\) and hence \(|\nabla \nabla \beta_k| \leq Ck^{-1}|z|\). So we have

\[
|\overline{\partial}_A, j(\beta_k f)| \leq |\beta_k| |\overline{\partial}_A, j f| + |\nabla \beta_k||f|
\]

Similarly

\[
|\nabla (\overline{\partial}_A, j(\beta_k f))| \leq |\nabla \nabla \beta_k||f| + |\nabla \beta_k||\nabla A f| + |\beta_k||\nabla (\overline{\partial}_A, j f)| + |\nabla \beta_k||\overline{\partial}_A, j f|
\]

So the cut-off function improves the estimates as follows.
We may choose nested sets 

\[ \Lambda_s \subset \bigcup_{i} \text{neighborhood of } p \]

Later we shall translate the origins of these coordinate charts except one (the first one) such that there is only one singularity on \( L \).

Now consider the terms of the right hand sides of the estimates for

\[ |\tilde{\partial}_{A_j} \beta_k (e^{-(Re \Sigma z_j^2)/4} + i)| \]
with the understanding that outside of the neighborhoods of \( p_i \)'s they vanish as \( \beta_k \) vanishes outside these neighborhoods. We also note that we can replace \( s_k \) by \( k^{-1/2}s_k \) for large \( k \).

Our final section \( s \) will be

\[
s = s_w = \sum_1^M w_i \sigma_i, \quad \sigma_i = \sigma_{p_i}, \quad |w_i| \leq 1, \quad w = (w_1, ..., w_M)
\]

Obviously it depends on \( k \). Later we shall make some specific choice for \( w_i \)'s. So we have the following

**Theorem 4.1.** For any choice of co-efficient vector \( w \) with \(|w_i| \leq 1 \) and a very small choice of \( s_k \) for large \( k \), the section \( s = s_w \) satisfies

\[
|s| \leq C \\
|\partial_E s| \leq C(k^{-1/2} + k^{-1}) \\
|\nabla_W \partial_E s| \leq C(k^{-1/2} + k^{-1})
\]

Now we shall recall a result from [1].

**Theorem 4.2.** ([1]) If \( a', a'' : \mathbb{C}^n \to \mathbb{C} \) are respectively complex linear and anti-linear maps and if \(|a''| < |a'|\), then the subspace \( \ker(a' + a'') \subset \mathbb{C}^n \) is symplectic.

In view of 4.1 and 4.2 if we prove the following result then we shall prove that the zero locus of \( s \) is a symplectic submanifold as in [1].

**Theorem 4.3.** There is an \( \epsilon > 0 \) such that for all large \( k \) we can choose \( w \) with \(|w_i| \leq 1 \), so that \( s = s_w \) satisfies the transversality condition

\[
|\partial s| > \epsilon
\]

on the zero locus of \( s \) i.e \( Z(s) \).

**Remark 4.4.** We shall further need that \( w_i = c + ic \), or \( -c + ic \) for \( c > 0 \) and shall also see that such a choice is possible in this context.

Now we shall prove 4.3 which similar to [1] (not identical though).

**Theorem 4.5.** ([1]) Given any \( D > 0 \) there is a number \( N = N(D) \), independent of \( k \), such that for any large \( k \) we can choose a collection of centers \( p_i \) satisfying the conditions of the proof of 4.2 in addition with the property that there is a partition of the set \( \Lambda = \{1, ..., M\} \) into \( N \) disjoint subsets \( \Lambda = \Lambda_1 \cup ... \cup \Lambda_N \) such that for each \( \alpha \)

\[
d(p_i, p_j) \geq D \text{ if } p_i, p_j \in \Lambda_\alpha
\]
So now we assume that $D$ is fixed and $p_i$’s accordingly as well as the partition $\{\Lambda_\alpha\}$. The choice of $D$ will be made at the end of this section. We think of this partition $\{\Lambda_\alpha\}$ as the colouring of the cubes $I^{2n}_{k^{-1/2+\epsilon_k,1+\epsilon_k}}$.

We shall adjust now the co-efficients $w_i$’s belonging to the same $\Lambda_\alpha$ starting with $\Lambda_1$ with the co-efficient vector $w_0 = (w_1, 0, \ldots, 0)$, $w_1 = c + ic$, $c > 0$ where $p_1 \in L$. Here it is different from [1]. In the next stage we shall change the co-efficients $w_i$’s belonging to $\Lambda_2$ leaving the others unchanged and proceed this way. Thus the total number $N$ of stages in this construction depends on $D$ but not on $k$.

Now we can focus on the adjustment criterion for $w_i$’s. The goal is to achieve the controlled transversality over the cubes belonging to $\Lambda_\alpha$ at the $\alpha$-th stage. We set

$$W_\alpha = \bigcup_{i \in \Lambda_\alpha, \beta \leq \alpha} I^{2n}_i$$

where $I^{2n}_i$ is the $i$-th cube $I^{2n}_{k^{-1/2+\epsilon_k,1+\epsilon_k}}$. So $\Phi(\text{empty}) = W_0 \subset \ldots \subset W_N = W$. We want that for suitable $\epsilon$, the section $s_\alpha$ satisfies $|\partial s_\alpha| > \epsilon$ on $Z(s_\alpha) \cap W_\alpha$. So the criterion can be summarized as

1. The change in the co-efficient belonging to $\Lambda_\alpha$ must achieve controlled transversality over the cubes belonging to $\Lambda_\alpha$.

2. The change in the co-efficients belonging to $\Lambda_\alpha$ must not destroy the controlled transversality that has already been achieved over the cubes belonging to $\Lambda_\beta$ for $\beta < \alpha$.

So the precise form of transversality we need is the following

**Definition 4.6.** (1) Let $f : U \to \mathbb{C}$ be a smooth map on an open set $U \subset \mathbb{C}$ and let $w \in \mathbb{C}$. For $\eta > 0$ we say that $f$ is $\eta$-transverse to $w$ over $U$ if for any $z \in U$ such that $|f(z) - w| \leq \eta$ the derivative satisfies $|(\partial f)_z| \geq \eta$.

$4.6$ is stable under $C^1$-perturbations, moreover if $f$ is $\eta$-transverse to $w$ and $g : U \to \mathbb{C}$ is such that $|f - g|_{C^1} \leq \delta$ then $g$ is $\eta - \delta$-transverse to $w$.

Now we consider any section $s$ of $E^k \to W$. On a fixed chart around say $p_i$, $E^k \to W$ has a standard trivialization given by $\sigma_i$. We write $s = f_i \sigma_i$, for a function $f_i$ defined on the cube
\( \mathcal{I}_{2n}^{k-1/2+\epsilon, 1+\epsilon_k} \). We say that the section \( s \) is \( \eta \)-transverse on the chart if \( f_i \) is \( \eta \)-transverse to 0 on the cube \( \mathcal{I}_{k-1/2, 1}^{2n} \).

**Lemma 4.7.** If \( s = s_w \) is the section of \( E^k \) as described in [4.4] and \( f_i \)'s the corresponding function then

1. \( |f_i|_{C^1(\mathcal{I}_{k-1/2, 1}^{2n})} \leq C \)

2. \( |\partial f_i|_{C^1(\mathcal{I}_{k-1/2, 1}^{2n})} \leq C(k^{-1/2} + k^{-1}) \)

3. If \( |\partial f_i| > \epsilon \) on \( f_i^{-1}(0) \cap \mathcal{I}_{k-1/2, 1}^{2n} \), then for large \( k \), \( |\partial E| > C^{-1} \epsilon \) on the intersection of \( Z(s) \) and the chart.

In view of [4.1], the proofs of [4.7] is same as the proofs of Lemma-18 from [1].

**Lemma 4.8.** If \( s = s_w \) be as in [4.4], for any \( \alpha \), let \( w' \) be another vector which agrees with \( w \) except for the co-efficients belonging to \( \Lambda_\alpha \) and suppose that \( |w'_j - w_j| \leq \delta \) for \( j \in \Lambda_\alpha \). Set \( s' = s_{w'} \) and let \( f_i, f'_i \) be the functions representing these sections, then for large \( D \)

1. For any \( i \), \( |f'_i - f_i|_{C^1(\mathcal{I}_{k-1/2, 1}^{2n})} \leq \delta \)

2. If \( i \in \Lambda_\alpha \) and \( w'_i = w_i + \theta_i \) then \( |f'_i - f_i - \theta_i|_{C^1(\mathcal{I}_{k-1/2, 1}^{2n})} = 0 \)

**Proof.** We write \( s = \Sigma_j w_j \sigma_j \). So on \( i \)-th smaller cube \( \mathcal{I}_{k-1/2, 1}^{2n} \) our equation \( s = f_i \sigma_i \) gives us \( \Sigma_{j \neq i} w_j \sigma_j + w_i \sigma_i = f_i \sigma_i \) and similarly \( \Sigma_{j \neq i} w'_j \sigma_j + w'_i \sigma_i = f_i \sigma_i \). So we get

\[
\begin{align*}
    f'_i - f_i - \theta_i &= w'_i - w_i - \theta_i + \Sigma_{j \neq i} \frac{\sigma_j}{\sigma_i} (w'_j - w_j) \\
    \Rightarrow |f'_i - f_i - \theta_i| &\leq \Sigma_{j \neq i} |w'_j - w_j| \left| \frac{\sigma_j}{\sigma_i} \right|, \text{ As } w'_i - w_i - \theta_i = 0
\end{align*}
\]

Similarly \( |f'_i - f_i| \leq \delta + \Sigma_{j \neq i} |w'_j - w_j| \left| \frac{\sigma_j}{\sigma_i} \right| \). Now considering the derivative we get

\[
\begin{align*}
    \nabla (f'_i - f_i - \theta_i) &= \Sigma_{j \neq i} (w'_j - w_j) \nabla \left( \frac{\sigma_j}{\sigma_i} \right) \\
    &= \Sigma_{j \neq i} (w'_j - w_j) \left[ \frac{1}{\sigma_i} \nabla \sigma_j - \frac{\sigma_i}{\sigma_j} \nabla \sigma_i \right] \\
    \Rightarrow |\nabla (f'_i - f_i - \theta_i)| &\leq \Sigma_{j \neq i} |w'_j - w_j| \left[ \left| \nabla \sigma_j \right| + \left| \frac{\sigma_i}{\sigma_j} \right| \left| \nabla \sigma_i \right| \right]
\end{align*}
\]

Now as \( w'_j = w_j \) for \( j \notin \Lambda_\alpha \), the summations \( \Sigma_{j \neq i} \) in the above estimates can be replaced by \( \Sigma_{j \in \Lambda_\alpha} \). So when \( D \) is large \( |\sigma_j| = 0 = |\nabla \sigma_j| \) on the chart \( \chi_i(\mathcal{I}_{k-1/2, 1}^{2n}) \). \( \square \)

Now define \( Q_\rho(\delta) = \log(\delta^{-1})^{-\rho}, \delta > 0 \). The next result (similar to Theorem-20 of [1]) gives us the suitable \( w \) in order to get the transversality condition. The result is also known
as Quantitative Transversality theorem. We shall however show that we can make specific choice for \( w_j \)'s in our context which will be needed in the proof of [1,4].

**Theorem 4.9.** ([1]) For \( \sigma > 0 \), let \( H_\sigma \) denote the set of functions \( f \) on \( I_{k^{-1/2},1} \) such that

1. \(|f|_{C^0(I_{k^{-1/2},1})} \leq 1 \)

2. \(|\partial f|_{C^0(I_{k^{-1/2},1})} \leq \sigma \)

Then there is an integer \( p \), depending only on the dimension \( n \), such that for any \( \delta \) with \( 0 < \delta < 1/2 \), if \( \sigma \leq Q_p(\delta) \delta \), then for any \( f \in H_\sigma \) there is a \( w \in \mathbb{C} \) with \(|w| \leq \delta \) such that \( f \) is \( Q(\delta) \delta \)-transverse to \( w \) over \( I_{k^{-1/2},1} \). Moreover \( w \) can be taken to be \(-(a+ib), a > 0 \) and \(-(a+ib), b = -a > 0 \). The choice between these two options will be clarified later.

We shall prove 4.9 at the end of this section. For now let us continue with our construction.

Let us assume that we have reached the \( \alpha - 1 \)-th stage of our construction and we want to reach the \( \alpha \)-th stage. So we have chosen the \( w^{\alpha-1} \) so that \( s_{\alpha-1} \) is \( \eta_{\alpha-1} \)-transverse over \( W_{\alpha-1} \subset W \) for some positive \( \eta_{\alpha-1} \) with \( 0 < \eta_{\alpha-1} < \rho \). We will now choose \( w_i^\alpha \), \( i \in \Lambda_\alpha \). We need \(|w_i^\alpha - w_i^{\alpha-1}| \leq \delta_\alpha \). We shall set \( \delta_\alpha > 0 \) shortly. With this choice of \( w_i^\alpha \), [4.8] gives us that \( s^\alpha \) is \( \eta_{\alpha-1} - \delta_\alpha \)-transverse over \( W_{\alpha-1} \). So if we set \( \delta_\alpha = \frac{1}{2} \eta_{\alpha-1} \) we can conclude that \( s^\alpha \) is still \( \frac{1}{2} \eta_{\alpha-1} \)-transverse on \( W_{\alpha-1} \). Now we shall consider the cubes \( I_i^\alpha, i \in \Lambda_\alpha \). Here the section \( s^{\alpha-1} \) is represented by the function \( f_i = f_i^{\alpha-1} = s_i^{\alpha-1} \). By [4.7] \( f_i \) is bounded by a fixed constant \( C \) over \( I_{k^{-1/2},1} \) and \( C^{-1} f_i \in H_\sigma \) for \( \sigma = C^{-1} (k^{-1/2} + k^{-1}) \). Now we apply [4.9] \( C^{-1} f_i \) for suitably small \( \rho \) and \( \delta = C^{-1} \delta_\alpha \) as long as

\[
k^{-1/2} + k^{-1} \leq C \delta_\alpha Q_p(\delta_\alpha)
\]

And thus we get \( v_i, i \in \Lambda_\alpha \) with \(|v_i| \leq \delta_\alpha \) such that \( f_i \) is \( Q_p(\delta_\alpha) \delta_\alpha \)-transverse to \( v_i \) and equivalently \( f_i - v_i \) is \( Q_p(\delta_\alpha) \delta_\alpha \)-transverse to 0. Observe that \( f_i - v_i \) represents \( s_{w^\alpha} \) where \( w^\alpha_j = w_j^{\alpha-1}, i \neq j \) and \( w^\alpha_i = w_i^{\alpha-1} - v_i \). In view of [4.8] for large \( D \) we can make all the changes to the \( w_i^{\alpha-1}, i \in \Lambda_\alpha \) simultaneously as explained above.

Now we shall see the proof of [4.9] which is same as the proof of Theorem-20 of [1]. Although we shall outline the entire proof for completeness but all we need to show that we can make the choice for \( w_i \)'s in our context. The choice is different from [1].
Let \( P : \mathbb{R}^n \to \mathbb{R} \) be a polynomial function of degree \( d \). Let \( S \subset \mathbb{R}^n \) be the subset
\[
S = \{ x \in \mathbb{R}^n : |x| \leq 1, \ P(x) \leq 1 \}
\]
Similarly set \( S(\theta) = \{ x : |x| \leq 1, \ P(x) \leq 1 + \theta \} \).

**Proposition 4.10.** \footnote{[1]} There are constants \( C, \nu, \) depending only on the dimension, such that for any polynomial \( P \) there are arbitrarily small positive \( \theta \) so that \( S \) can be decomposed into pieces
\[
S = S_1 \cup \ldots \cup S_A
\]
where \( A \leq Cd^\nu \), in such a way that any pair of points in the same piece \( S_r \) can be joined by a path in \( S(\theta) \) of length less than \( Cd^\nu \).

The proof decomposes into two parts. First we prove the \[4.9\] for holomorphic \( f \) and then use some estimates to prove the result for approximately holomorphic \( f \).

**Lemma 4.11.** \footnote{[1]} Let \( f : I_{k-1/2,1}^{2n} \to \mathbb{C} \) be a holomorphic map with \( |f(z)| \leq 1 \). Then for any \( \epsilon \) satisfying \( 0 < \epsilon < 1/2 \) there is a complex polynomial function \( g \) on \( \mathbb{C}^n \) of degree less than \( C \log \epsilon^{-1} \) such that \( |f(z) - g(z)|, |\partial f - \partial g| \leq \epsilon \) in the interior region.

Now we shall prove \[4.9\] for holomorphic \( f \). So let us assume that \( f \) is as in \[4.11\] and we use \[4.11\] to approximate \( f \) by a polynomial \( g \) of degree \( d \leq C \log \epsilon^{-1} \). Define
\[
S_f = \{ z \in I_{k-1/2,1}^{2n} \cup \mathbb{C}^n : |\partial f| \leq \epsilon \}
\]
\[
S^g = \{ z \in I_{k-1/2,1}^{2n} \cup \mathbb{C}^n : |\partial g| \leq 2\epsilon \}
\]
Obviously \( S_f \subset S^g \) and therefore \( f(S_f) \subset f(S^g) \subset N_\epsilon(g(S^g)) \) where \( N_\epsilon(g(S^g)) \) is the \( \epsilon \)-neighborhood of \( g(S^g) \). Observe that \( S^g \) is a semi-algebraic set of the kind in \[4.10\] where we have taken \( P = (2\epsilon)^{-2}|\partial g|^2 \) whose degree is \( 2(d - 1) \). So \[4.10\] gives a decomposition of \( S^g \) into \( A \) pieces. Let \( z_1, z_2 \) are in the same piece of \( S^g \) then integrating the derivative of \( g \) over a path of length less than \( Cd^\nu \) in \( P(\theta) \), for suitable \( \theta \), joining \( z_1 \) to \( z_2 \), we have
\[
|g(z_1) - g(z_2)| \leq 4\epsilon Cd^\nu.
\]
Therefore \( g(S^g) \) is contained in the union of \( A \) discs in \( \mathbb{C} \) each of radius \( 4\epsilon Cd^\nu \). So \( f(S^g) \) and hence \( f(S_f) \) is contained in the union of slightly larger discs of radius \( \epsilon(4Cd^\nu + 1) \).

The \( \epsilon \)-transversality between \( f \) and \( w \) is that \( w \) lies outside the \( \epsilon \)-neighborhood of \( f(S_f) \) and the neighborhood is contained in the union of \( A \) discs of radius \( \epsilon(4Cd^\nu + 2) \). The total
area in $\mathbb{C}$ of these discs is at most $A\pi(4Cd^\nu + 2)\epsilon^2$. If we choose $\rho$ such that the area of the half disc
$$\Omega = \{ w \in \mathbb{C} : |w| \leq \rho, \text{Im}(w) \leq 0 \}$$
is bigger than the total area covered by these discs, there is a $w \in \Omega$ not contained in the $\epsilon$-neighborhood of $f(S^f)$. The condition on $\rho$ is
$$\frac{1}{2} \pi \rho^2 > A\pi(4Cd^\nu + 2)\epsilon^2$$
which is $\rho > (2A)^{1/2}(4Cd^\nu + 2)\epsilon$. Observe that $A$ are bounded by powers of degree of $P$, hence of the degree $d$ of $g$ which is bounded by a power of $\log(\epsilon)^{-1}$.

So over all, for arbitrarily small $\epsilon$ there exists a $w \in \mathbb{C}$ such that $f$ is $\epsilon$-transverse to $w$ and $|w| \leq C\epsilon \log(\epsilon^{-1})^p$ for some $p$. Now one needs to re-arrange the parameters. For fixed $C, p$, the function $h$ given by $h(\epsilon) = C\epsilon \log(\epsilon^{-1})^p$ is increasing for for small $\epsilon$ and tends to 0 as $\epsilon$ approaches 0. If $\eta = h(\epsilon)$ then
$$\eta \log(\eta^{-1})^p = C\epsilon \left( \log \epsilon^{-1} - p \log \log \epsilon^{-1} - \log C \right)^p < 2C\epsilon, \text{ (if } \epsilon \text{ small)}$$
By inverting $h$, we conclude that for small $\eta$ there is a $w$ with $|w| \leq \eta$ such that $f$ is $\frac{1}{2\pi C} \eta (\log \eta^{-1})^p$-transverse to $w$, and by increasing $p$ and assuming $\eta$ to be small, we can replace the factor $\frac{1}{2\pi C}$ by 1.

In the above we considered holomorphic $f$. Now we shall consider approximately holomorphic $f$. Although we have not shown that we can make the specific choice for $w$ in our context, but that we shall show this at the end of the proof.

Using Hörmander’s weighted $L^2$ space method [6] one gets the following

**Lemma 4.12.** ([1]) For each $r < 1$ there is a constant $K = K(r)$ such that if $f$ is any smooth complex valued function on $I_{k^{-1/2},1}^{2n}$, then there is a holomorphic function $\tilde{f}$ on $rI_{k^{-1/2},1}^{2n}$ such that
$$|f - \tilde{f}|_{C^1(I_{k^{-1/2},1}^{2n})} \leq K(|\bar{\partial}f|_{C^0(I_{k^{-1/2},1}^{2n})} + |\bar{\nabla}\bar{\partial}f|_{C^0(I_{k^{-1/2},1}^{2n})})$$

Now if $f \in \mathcal{H}_\sigma$, we approximate $f$ by a holomorphic function $\tilde{f}$ on a slightly smaller region with $|f - \tilde{f}|_{C^1} \leq C\sigma$. Then use the above arguments for $\tilde{f}$. 
Now we show how we can make the specific choice for \( w \) in the half disc
\[
\{ w \in \mathbb{C} : |w| \leq \rho, \quad Im(w) \leq 0 \}
\]
We must show that \( f - v_i \) is \( \varepsilon \)-transverse to 0 for some small \( \varepsilon > 0 \). Any coordinate chart intersect only finitely many adjacent charts and the number depends on the dimension \( n \). We only consider one such intersection. Let \( \phi = (\phi_1, \ldots, \phi_n) \) be the transition function. So we take
\[
f = \frac{\beta'[(ce^{-(Re\Sigma\phi_j^2)/4} - d) + i(de^{-(Re\Sigma\phi_j^2)/4} + c)]}{\beta(e^{-(Re\Sigma\phi_j^2)/4} + i)}
\]
where either \( c = d > 0 \) or \( d = -c > 0 \). Here \( \beta \) corresponds to the chart under consideration and \( \beta' \) corresponds to the adjacent chart. The sign of \( c, d \) changes as we are adding \( -\sigma \) because \( w' = w_1^\sigma - v_i \) and at the beginning we set \( w_0 = (w_1, 0, \ldots, 0) \) where \( w_1 = c + \sigma, \ c > 0 \). Simplifying we get
\[
f = \frac{\beta'[(ce^{-(Re\Sigma\phi_j^2)/4} - d) + i(de^{-(Re\Sigma\phi_j^2)/4} + c)]}{\beta(e^{-(Re\Sigma\phi_j^2)/4} + i)}
\]
\[
\times \left[ \{c(1 + e^{-(Re\Sigma\phi_j^2)/4} - d) + d(e^{-(Re\Sigma\phi_j^2)/4} - e^{-(Re^2\Sigma\phi_j^2)/4}) \}
\]
\[
+ i\{c(e^{-(Re\Sigma\phi_j^2)/4} - e^{-(Re\Sigma\phi_j^2)/4}) + d(1 + e^{-Re^2\Sigma\phi_j^2/4} - e^{-Re^2\Sigma\phi_j^2/4}) \} \right]
\]
Now observe that for large \( k \), i.e., for range of \( |x_j| \) being very small \( (Assuming \ z_j = x_j + iy_j) \), the imaginary part of \( f \) for both \( c = d > 0 \) and \( d = -c > 0 \) is non-negative. So the imaginary part of \( f - v_i \) is bigger than \( a \) if we take \( v_i = -(a + ia) \), \( a > 0 \) and is bigger than \( b \) if we take \( v_i = -(a + ib) \), \( b = -a > 0 \) which concludes the argument. For more intersection we may need to take larger \( k \).

5. Main Theorem

In this section we prove \[1,2\]. As in the previous section we consider \( s \) for large values of \( k \). So \( Z(s) \) is a symplectic submanifold and hence \( s^{-1}(z) \) for \( z \) close to 0 is also symplectic. This forms a tubular neighborhood of \( Z(s) \).

Now we consider the \( p_1 \in L \) the center of the first chart in the construction of \( s \) as in the previous section. Obviously it is a singularity for \( s \) and now we’ll figure out the stable manifold corresponding to the vector field \( X_{Im \log s} \), the Hamiltonian vector field corresponding to \( Im \log s = \text{Arg}(s) \). But before we do so let us show that it is a hyperbolic zero for \( X_{Im \log s} \).

**Proposition 5.1.** \( p_1 \in L \subset W \) is a hyperbolic zero for \( X_{Im \log s} = X_{\theta} \), where \( \theta = \text{Arg}(s) \) and the tangent space to the stable manifold is given by \( \{x_j = y_j : j = 1, \ldots, n\} \) at \( p_1 \). Moreover on the set \( \beta_1 = \text{constant} \neq 0 \) the stable manifold is given by \( \{x_j = y_j : j = 1, \ldots, n\} \).
Proof. On the set $\beta_1 = \text{constant} \neq 0$, $s$ is given by $s = \beta_1(a + ib)(e^{-\frac{(Re\Sigma z_j^2)}{4} + i})$. So we get

\[ \tan \theta = \frac{be^{-\frac{(Re\Sigma z_j^2)}{4} + a}}{ae^{-\frac{(Re\Sigma z_j^2)}{4} - b}} = \frac{A}{B} \]

\[ \Rightarrow \sec^2 \theta d\theta = \frac{1}{(1/B)[(1/2)be^{-\frac{(Re\Sigma z_j^2)}{4}} \Sigma_j(y_jdy_j - x_jdx_j)]} \]

\[ \Rightarrow d\theta = \frac{B}{2(A^2 + B^2)} [ae^{-\frac{(Re\Sigma z_j^2)}{4} \Sigma_j(y_jdy_j - x_jdx_j)}] \]

The second line is achieved from the first by differentiating and the third line is achieved by multiplying by $\cos^2 \theta$. Note that because we are differentiating both sides of an equation we get $d_A = d$.

Now let $X_\theta = \Sigma_j(a_j\partial_{x_j} + b\partial_{y_j})$ and observe that the symplectic structure on the tubular neighborhood of $L \subset W$ is represented by $\omega_{st} = \Sigma_j dx_j \wedge dy_j$ on coordinate charts. Hence Hamiltonian equation $d\theta = \iota_{X_\theta} \omega_{st}$ gives us $d\theta = \Sigma_j(a_jdy_j - b_jdx_j)$. Solving we get

\[ a_j = \frac{y_je^{-\frac{(Re\Sigma z_j^2)}{4}}}{2(A^2 + B^2)} [Bb - Aa] \]

Similarly

\[ b_j = \frac{x_je^{-\frac{(Re\Sigma z_j^2)}{4}}}{2(A^2 + B^2)} [Bb - Aa] \]

Now we compute the partial derivatives of $a_j$ and $b_j$ at zero. Observe that zero in this coordinates corresponds to $p_1$. So we get

\[ (\partial_{x_j}a_j)|_0 = 0, \quad (\partial_{y_j}a_j)|_0 = \frac{Bb - Aa}{2(A^2 + B^2)}|_0 = -\frac{1}{4} \]

Similarly

\[ (\partial_{y_j}b_j)|_0 = 0, \quad (\partial_{x_j}b_j)|_0 = \frac{Bb - Aa}{2(A^2 + B^2)}|_0 = -\frac{1}{4} \]

Define $F(z) = (a_1, b_1, ..., a_n, b_n)$, then $DF|_0$ has all eigenvalues equal to $\pm \frac{1}{4}$. The tangent of the stable manifold at zero corresponds to $-\frac{1}{4}$ which is given by $\{x_j = y_j : j = 1, ..., n\}$.

Now let $(x(t), y(t)) = (x_1(t), y_1(t), ..., x_n(t), y_n(t))$ be the flow of $X_\theta$. So

\[ \partial_t x_j(t) = a_j(x(t), y(t)) \quad \text{and} \quad \partial_t y_j(t) = b_j(x(t), y(t)) \]

Multiplying the first equation $\partial_t x_j(t) = a_j(x(t), y(t))$ by $x_j(t)$ and the second equation $\partial_t y_j(t) = b_j(x(t), y(t))$ by $y_j(t)$ and integrating we get (suppressing the $t$ all along below
Again as in 5.1 let
\[
\frac{1}{2} (x_j^2 - y_j^2) = 0
\]
which proves the remaining statement.

Now we shall figure out the stable manifold globally. But for this we need to figure out the flow of the vector field \(X_0\). Let us consider \(s\) on a chart adjacent to the one containing \(p_1\) and intersecting \(L\). Recall that on these charts \(L\) is given by \(\{y_j = 0, \ j = 1, \ldots, n\}\). Now on this chart \(\tan \theta\) is given by
\[
\tan \theta = \frac{\beta_1 e^{-\frac{(Re \Sigma z_j^2)}{4}} + \beta_1 a + \beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}} + \beta_2 c}{\beta_1 e^{-\frac{(Re \Sigma z_j^2)}{4}} - \beta_1 b + \beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}} - \beta_2 d}
\]
where \(\beta_2\) corresponds to the chart containing \(p_1\) and \(\beta_1\) corresponds to the chart adjacent to it and \(\phi = (\phi_1, \ldots, \phi_n)\) is the transition function. Obviously \(c = d > 0\) but the choice of \(a, b\) will be made clear shortly between the two options given by 5.9. Now we proceed as in the proof of 5.1.

\[
\tan \theta = \frac{\beta_1 e^{-\frac{(Re \Sigma z_j^2)}{4}} + \beta_1 a + \beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}} + \beta_2 c}{\beta_1 e^{-\frac{(Re \Sigma z_j^2)}{4}} - \beta_1 b + \beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}} - \beta_2 d} = \frac{A}{B}
\]

\[
\Rightarrow \sec^2 \theta d\theta = (1/B)[(1/2)b\beta_1 e^{-\frac{(Re \Sigma z_j^2)}{4}}\Sigma_j (y_j dy_j - x_j dx_j) + de^{-\frac{(Re \Sigma \phi_j^2)}{4}}\Sigma_j (\partial_j \beta_1 dx_j + \partial_j \beta_1 dy_j) + \Sigma_j \partial_j (\beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}}) dx_j + \Sigma_j \partial_j (\beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}}) dy_j]
\]

\[
\Rightarrow d\theta = \frac{B}{A + B^2}[(1/2)b\beta_1 e^{-\frac{(Re \Sigma z_j^2)}{4}}\Sigma_j (y_j dy_j - x_j dx_j) + de^{-\frac{(Re \Sigma \phi_j^2)}{4}}\Sigma_j (\partial_j \beta_1 dx_j + \partial_j \beta_1 dy_j) + \Sigma_j \partial_j (\beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}}) dx_j + \Sigma_j \partial_j (\beta_2 de^{-\frac{(Re \Sigma \phi_j^2)}{4}}) dy_j]
\]

Again as in 5.1 let \(X_0 = \Sigma_j (a_j \partial_j + b \partial_j)\). Then solving the Hamiltonian equation we get
Similarly we get $b$. Now let $(x(t), y(t)) = (x_1(t), y_1(t), ..., x_n(t), y_n(t))$ be the flow of $X_\theta$. So

$$\partial_t x_j(t) = a_j(x(t), y(t)) \text{ and } \partial_t y_j(t) = b_j(x(t), y(t))$$

Multiplying the first equation $\partial_t x_j(t) = a_j(x(t), y(t))$ by $x_j(t)$ and the second equation $\partial_t y_j(t) = b_j(x(t), y(t))$ by $y_j(t)$ and integrating we get (suppressing the $t$ all along below for convenient notations)

$$\frac{1}{2}(x_j^2 - y_j^2) = \int \left[ \frac{bB}{A^2 + B^2} e^{-\frac{\beta_1}{2}}(x_j \partial_{y_j} y_j - y_j \partial_{x_j} x_j)(\beta_1) + \frac{d}{A^2 + B^2} e^{-\frac{\beta_2}{2}}(x_j \partial_{y_j} y_j - y_j \partial_{x_j} x_j)(\beta_2) + \frac{aB + bA}{A^2 + B^2} e^{-\frac{\beta_1}{2}}(x_j \partial_{y_j} y_j - y_j \partial_{x_j} x_j)(\beta_1) \right] dt$$

First we simplify the integrands. Consider the first term. So

$$Bb - Aa = -\beta_1(a^2 + b^2) + \beta_2(bc - ad) e^{-\frac{\beta_1}{2}} - \beta_2(bd + ac)$$

Now consider the forth term. So

$$aB + bA = \beta_1(a^2 + b^2) e^{-\frac{\beta_1}{2}} + \beta_2(ac + bd) e^{-\frac{\beta_1}{2}} + \beta_2(bc - ad)$$

Hence

$$(Bb - Aa) e^{-\frac{\beta_1}{2}} + (aB + bA) = \beta_2(bc - ad)(e^{-\frac{\beta_1}{2}} - e^{-\frac{\beta_2}{2}}) + \beta_2(ac + bd)(e^{-\frac{\beta_1}{2}} - e^{-\frac{\beta_2}{2}}) + 2bc\beta_2$$

The last line achieved by putting $c = d > 0$ and $b = -a > 0$. Similarly considering the second term we get

$$Bd - Ac = \beta_1(ad - bc) e^{-\frac{\beta_1}{2}} - \beta_1(bd + ac) - \beta_2(c^2 + d^2)$$

$$= -2bc\beta_1 e^{-\frac{\beta_1}{2}} - 2\beta_2c^2$$
Again the last line is achieved by putting $c = d > 0$ and $b = -a > 0$. Now consider the third term.

\[
Bc + Ad = \beta_1(ac + bd)e^{-(Re\Sigma z_j^2)/4} - \beta_1(bc - ad) + \beta_2(c^2 + d^2)e^{-(Re\Sigma \phi_j^2)/4} = -2bc\beta_1 + 2\beta_2c^2e^{-(Re\Sigma \phi_j^2)/4}, \quad (putting \ c = d > 0, \ b = -a > 0)
\]

Observe that inside the chart, i.e, when $\beta_2 = 0$ and $\beta_1 = \text{constant} \neq 0$, all the integrands in the equation of the flow vanishes. So the equation becomes $x_j^2 - y_j^2 = \text{constant}$. We shall now make the constant on the right side in the resulting equation of the flow negative for large values of $k$ and for suitable translation of the origin 0 in this chart.

\[
\begin{align*}
\text{Stable Manifold} & \quad PQ \\
\cdot & \quad o' \\
\cdot & \quad o' \\
. & \quad Q \quad \text{Stable Manifold} \quad P
\end{align*}
\]

In the above picture first we consider the right side cube, i.e, $P$. The integrands are non-zero only on the shaded region in the picture. So we try to determine the signs of the integrands in this region.

Obviously $A^2 + B^2 \geq 0$. So consider the sum of first and forth term as above, i.e, the equation

\[
(Bb - Aa)e^{-(Re\Sigma z_j^2)/4} + (aB + bA) = 2bc\beta_2(e^{-(Re\Sigma \phi_j^2)/4}e^{-(Re\Sigma z_j^2)/4} + 1)
\]

Obviously it is bigger than or equal to zero. So let us determine the sign of

\[
(x_j\partial_{y_j} - y_j\partial_{x_j})(\beta_1)
\]

Now observe that for large $k$ the range of $|x_j|$ is very small. So the sign of $-y_j\partial_{x_j}(\beta_1)$ determines the sign if we translate the origin as shown in the picture above. Observe that on the shaded region $\beta_1$ with respect to $x_j$ is increasing. Hence $\partial_{x_j}\beta_1 \geq 0$. So $-y_j\partial_{x_j}(\beta_1) < 0$ as
on the intersection of the stable manifold and the shaded region, $y_j > 0$. So this integrand is negative on the shaded region.

Now consider the second and the third integrands. The second integrand contains a factor

$$
(x_j \partial y_j - y_j \partial x_j)(\beta_2 e^{-(Re \Sigma \phi_j^2)/4}) = e^{-(Re \Sigma \phi_j^2)/4}(x_j \partial y_j - y_j \partial x_j)(\beta_2) + \beta_2(x_j \partial y_j - y_j \partial x_j)(e^{-(Re \Sigma \phi_j^2)/4})
$$

Observe that the third integrand contains a factor $(x_j \partial y_j - y_j \partial x_j)(\beta_2)$. So we compute

$$
(Bd - Ac)(e^{-(Re \Sigma \phi_j^2)/4}) + (Bc + Ad) = -2bc\beta_1(e^{-(Re \Sigma \phi_j^2)/4}e^{-(Re \Sigma \phi_j^2)/4} + 1) < 0
$$

Now as above the sign is determined by $-y_j \partial x_j \beta_2$. But this time we have $\beta_2$ instead of $\beta_1$, which is decreasing on the shaded region. So $-y_j \partial x_j \beta_2 > 0$. Hence the sign is negative.

But we are still left with one term, namely the one containing the factor

$$
\beta_2(x_j \partial y_j - y_j \partial x_j)(e^{-(Re \Sigma \phi_j^2)/4})
$$

from the second integrand. Now we know that $Bd - Ac < 0$. As $e^{-(Re \Sigma \phi_j^2)/4}$ is decreasing with respect to $x_j$, so if we take $k$ large and shift the origin in the $y_j$ component down enough we can make this integrand negative and hence we are done with the cube $P$.

We can do similar thing for the cube $Q$ and the global stable manifold is achieved by observing that in the above argument the role of $(a, b)$ and $(c, d)$ can be interchanged.

Observe that this stable manifold is diffeomorphic to the regular Lagrangian $L \subset W$. We call the stable manifold $L'$.

Now according to 2.1 the diffeomorphism $L \to L'$ can be extended to an isosymplectic immersion $\chi : \hat{N}(L) \to N(L')$, where $\hat{N}(L) \subset N(L)$ is such that $L' \subset \hat{N}(L)$. Now using one-parametric version of 3.1 for $A = L$ and using the fact that $Sp_{2n}$ path-connected, we can construct a symplectomorphism $\Xi : W \to W$, which is equal to $\chi$ on $\hat{N}(L)$ and is equal to $id$ (identity) outside of $N(L)$ and which sends $L \to L'$.

Define $W_0 = (s \circ \Xi)^{-1}(0)$ and the Weinstein structure is the pull-back Weinstein structure by this symplectomorphism $\Xi$. Let us now define the Lagrangian sphere required in 1.3. Under parallel transport along the radial paths in $D^2 \subset \mathbb{C}$ with respect to the symplectic
connection induced by $\omega$, the critical points of $s$ sweeps out a Lagrangian disc called **Lefschetz thimble** ([8]) which is the stable manifold of the Hamiltonian vector field $X_{\text{Im}\log s}$. The fiber over 0 of a Lefschetz thimble is an exact Lagrangian sphere called the **vanishing cycle**. Let $L'_1$ be the vanishing cycle corresponding to the singularity $p_1$ and let $\Xi^{-1}(L'_1) = L_1 \subset W_0$. Then the required Weinstein Lefschetz fibration is $((W_0, \lambda_0, \phi_0); L_1)$.

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