On the applications of Hardy class functions in scattering theory

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Abstract
This paper is a response to an article [26] recently published in Journal of Physics A: Mathematical and Theoretical. The article claims that the theory of resonances and decaying states based on certain rigged Hilbert spaces of Hardy functions is physically untenable. In this paper we show that all of the key conclusions of [26] are the result of either the errors in mathematical reasoning or an inadequate understanding of the literature on the subject.

1 Introduction
A recently published work [26] carries out an analysis of time asymmetric quantum theory (TAQT) of resonance scattering and decay. This theory is anchored in a pair of rigged Hilbert spaces of Hardy functions. The primary claim of [26] is that Hardy functions are incompatible with quantum mechanics.

The author of [26] arrives at this conclusion by solving the Lippmann-Schwinger equations for a spherical shell potential. Specifically, the author constructs a class of solutions and argues that they are not of Hardy class
by claiming that they diverge at infinity. However, the author makes a rudimentary mathematical error here. What the author actually establishes is an upper bound for the wavefunctions, $|\varphi(E)| \leq f(E)$, where the upper bound $f(E)$ has an exponential blow up when $E$ approaches infinity in the complex plane. The author confuses this exponential blow up of the upper bound with that of the wavefunction itself and it is this rudimentary mistake that leads the author to conclude that the solutions obtained in [26] are not of Hardy class.

Equally importantly, even if the author had handled the problem with care and found a class of solutions that are not of Hardy class, the existence of such solutions certainly does not allow one to conclude that there exist no Hardy solutions to the particular potential studied, let alone all other potentials. Therefore, even if the study of [26] had been conducted with proper mathematical rigor, it could not have led a conclusion about the veracity of TAQT.

The author of [26] also quotes fragments of statements from the literature which, when viewed with a diffused focus and out of context, lead to hasty, erroneous conclusions. For instance, the author’s conclusion that only the zero function has an asymmetric, semigroup time evolution is the result of such a misrepresentation of the literature.

In this paper, we will explicitly point out the mathematical errors in [26] and also show how a pair of rigged Hilbert spaces of Hardy functions may be constructed for the spherical shell potential using the techniques of [26] itself. We will also explicate the context and the true content of some of the statements by the proponents of TAQT that the author uses to support the claims of [26].

2 Rigged Hilbert Spaces of Hardy Functions

In this section, we present a brief review of the essential mathematical features of time asymmetric quantum theory. Our intent here is to give just enough detail so that claims of [26] can be juxtaposed with the mathematical framework of TAQT. More complete and systematic presentations of both mathematical and physical features of TAQT can be found, for instance, in [11, 12, 18]. These expand on the ideas of the earliest publications on the subject [7, 6, 20].

Time asymmetric quantum theory is based on a pair of rigged Hilbert
spaces constructed from Hardy class functions:

\[ S \cap \mathcal{H}_+^2 \mid_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+, dE) \subset \left( S \cap \mathcal{H}_+^2 \mid_{\mathbb{R}^+} \right)^* \quad (2.1^+) \]

\[ S \cap \mathcal{H}_-^2 \mid_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+, dE) \subset \left( S \cap \mathcal{H}_-^2 \mid_{\mathbb{R}^+} \right)^* \quad (2.1^-) \]

In (2.1), the spaces \( \mathcal{H}_\pm^2 \) are Hardy class functions on the upper and lower half complex planes, respectively. The Hilbert space \( L^2(\mathbb{R}^+, dE) \) is the norm completion of wavefunctions in the energy representation. \( S \) is the Schwartz space on \( \mathbb{R} \) (smooth functions that vanish at infinity faster than the inverse of any polynomial). The symbol \( \mid_{\mathbb{R}^+} \) indicates restrictions of functions of \( S \cap \mathcal{H}_\pm^2 \), whose domain is the entire real line, to the positive semi-axis, \( \mathbb{R}^+ \).

The dual spaces \( \left( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \right)^\ast \) consist of continuous antilinear functionals on \( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \).

The spaces \( S \cap \mathcal{H}_\pm^2 \) can be obtained as the Fourier transforms of \( S(\mathbb{R}^\pm) \), the spaces of Schwartz functions with supports in the half-line \( \mathbb{R}^\pm \). That is, \( S \cap \mathcal{H}_\pm^2 = \mathcal{F}[S(\mathbb{R}^\pm)] \). This identity is an immediate implication of a Paley–Wiener theorem [24, 25], which states that the Fourier transformation \( \mathcal{F} \) is a unitary mapping between the Hilbert spaces \( L^2(\mathbb{R}^\pm) \) and the Hardy spaces \( \mathcal{H}_\pm^2 \), \( \mathcal{F}[L^2(\mathbb{R}^\pm)] = \mathcal{H}_\pm^2 \), and the fact that \( \mathcal{F} \) is a homeomorphism on the Schwartz space \( S \). Since \( S(\mathbb{R}^\pm) \) are closed subspaces with respect to the usual nuclear Fréchet topology of \( S \), it follows that \( S \cap \mathcal{H}_\pm^2 \) are also nuclear Fréchet spaces.

Using a theorem of van Winter [37], it can be shown that there exist one-to-one and onto mappings \( \theta_\pm \) between the functions of \( S \cap \mathcal{H}_\pm^2 \) and their restrictions to \( \mathbb{R}^+ \), \( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \) (see [11] and Appendix of [14]). These mappings can be used to transport the topology of the spaces \( S \cap \mathcal{H}_\pm^2 \) to the spaces \( \left( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \right)^\ast \). Since \( S \cap \mathcal{H}_\pm^2 \) are nuclear Fréchet spaces, the spaces \( \left( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \right)^\ast \) are also nuclear Fréchet spaces under this transported topology. Furthermore, the denseness of \( S \cap \mathcal{H}_\pm^2 \) in \( \mathcal{H}_\pm^2 \) and a result of van Winter that asserts the denseness of the restrictions \( \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \) in \( L^2(\mathbb{R}^+) \) can be used to prove that the spaces \( \left( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \right)^\ast \) are dense in the Hilbert space \( L^2(\mathbb{R}^+, dE) \). Therefore, if we denote by \( \left( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \right)^\ast \) the spaces of continuous antilinear functionals on \( \left( S \cap \mathcal{H}_\pm^2 \mid_{\mathbb{R}^+} \right) \), then (2.1) are a pair of rigged Hilbert spaces.

The Hilbert space \( L^2(\mathbb{R}^+, dE) \) is thus the starting point of the construction of the rigged Hilbert spaces (2.1). This Hilbert space is what provides the realization of the system in the energy representation. To be specific, let
\( \mathcal{H} \) be an abstract Hilbert space in which there exists a self-adjoint Hamiltonian \( H \). Let \( H \) have a non-degenerate, absolutely continuous spectrum that coincides with the positive real line, \( \mathbb{R}^+ \), and let the point and singularly continuous spectra of \( H \) be empty. (In fact, it is not necessary to assume that the point spectrum of \( H \) be empty \[11\].) Then, the spectral representation theorem [31] says that there exists a unitary operator \( W \), not necessarily unique, from \( \mathcal{H} \) to \( L^2(\mathbb{R}^+, dE) \) such that \( WHW^{-1} \) is the multiplication operator \( E \) in \( L^2(\mathbb{R}^+, dE) \). This means that for any \( \varphi(E) \in L^2(\mathbb{R}^+, dE) \) with \( E\varphi(E) \in L^2(\mathbb{R}^+, dE) \), one has

\[
(WHW^{-1}\varphi)(E) = (E\varphi)(E) = E\varphi(E) \tag{2.2}
\]

The operator \( W \) that furnishes the spectral representation of \( \mathcal{H} \) is not necessarily unique, and in the construction of the rigged Hilbert spaces (2.1), two such operators \( W_\pm \) have been used \[11, 18\]. The existence the operators \( W_\pm \) follows from the spectral representation theorem and the conditions on the interaction potential that ensures the existence of Møller wave operators. In particular, let the Hamiltonian that determines the time evolution of the system be of the form \( H = H_0 + V \), where \( H_0 \) is an operator that governs some suitable “free dynamics” and \( V \), the “interaction potential” that represents a perturbation to \( H_0 \). As is customary, let us assume that both \( H \) and \( H_0 \) have empty point and singularly continuous spectra and that their absolutely continuous spectra are non-degenerate and coincide with the positive real line, \( \mathbb{R}^+ \). This assumption leads to notational simplicity but does not limit the generality of the results.

Under certain conditions on the interaction potential \( V \) [32], there exist unitary Møller wave operators \( \Omega_\pm \) on \( \mathcal{H} \):

\[
\Omega_\pm = \text{s-lim}_{t \to \mp \infty} e^{iHt} e^{-iH_0t} \tag{2.3}
\]

where the limit s-lim is the strong limit with respect to the norm topology of \( \mathcal{H} \), the abstract Hilbert space in which the operators \( H \) and \( H_0 \) are defined. The Møller operators \( \Omega_\pm \) fulfill the following intertwining relation:

\[
H\Omega_\pm = \Omega_\pm H_0 \tag{2.4}
\]

Now, by the spectral representation theorem, there exists a unitary operator \( U \) from \( \mathcal{H} \) to \( L^2(\mathbb{R}^+, dE) \) such that the free Hamiltonian \( H_0 \) is mapped to
the multiplication operator, $UH_0U^{-1} = \mathcal{E}$, in $L^2(\mathbb{R}^+, dE)$. That is, for all $\varphi \in L^2(\mathbb{R}^+, dE)$ such that $\mathcal{E}\varphi \in L^2(\mathbb{R}^+)$,

$$
(UH_0U^{-1}\varphi)(E) = (\mathcal{E}\varphi)(E) = E\varphi(E)
$$

(2.5)

From (2.4) and (2.5), it follows that the operators $W_\pm := U\Omega_\pm^{-1}$ provide unitary mappings between $\mathcal{H}$ and $L^2(\mathbb{R}^+, dE)$ such that the exact Hamiltonian $H$ is mapped to the multiplication operator $\mathcal{E}$ in $L^2(\mathbb{R}^+, dE)$. Thus, for all $\varphi \in L^2(\mathbb{R}^+, dE)$ such that $E\varphi \in L^2(\mathbb{R}^+, dE)$,

$$
(W_\pm HW_\pm^{-1}\varphi)(E) = (\mathcal{E}\varphi)(E) = E\varphi(E)
$$

(2.6)

By taking the inverse image of (2.1+) under $W_+$ and of (2.1-) under $W_-$, we can obtain two rigged Hilbert spaces whose spectral realizations are given by (2.1). Defining $\Phi_\pm := W_\pm^{-1}(S \cap \mathcal{H}_2^+\rvert_{\mathbb{R}^+})$ and letting $\Phi_\pm^\times$ be the (anti)dual spaces of $\Phi_\pm$, we then have a pair of abstract rigged Hilbert spaces

$$
\Phi_\pm \subset \mathcal{H} \subset \Phi_\pm^\times
$$

(2.7)

The construction can be summarized by the following diagram:

\[
\begin{array}{ccc}
\Phi_\pm & \subset & \mathcal{H} & \subset & \Phi_\pm^\times \\
\downarrow W_\pm & & \downarrow W_\pm & & \downarrow W_\pm^\times \\
S \cap \mathcal{H}_2^\pm\rvert_{\mathbb{R}^+} & \subset & L^2(\mathbb{R}^+, dE) & \subset & (S \cap \mathcal{H}_2^\pm\rvert_{\mathbb{R}^+})^\times \\
\downarrow \theta_\pm^{-1} & & \downarrow (\theta_\pm^{-1})^\times & & \\
S \cap \mathcal{H}_2^\pm & \subset & L^2(\mathbb{R}, dE) & \subset & (S \cap \mathcal{H}_2^\pm)^\times
\end{array}
\]

(2.8)

The rigged Hilbert spaces (2.1) or, equivalently, (2.7), have the following important properties:

2.1. The multiplication operator $\mathcal{E}$ is reduced by both spaces $S \cap \mathcal{H}^2_\pm\rvert_{\mathbb{R}^+}$, i.e.,

$$
\mathcal{E} \left(S \cap \mathcal{H}^2_\pm\rvert_{\mathbb{R}^+}\right) \subset S \cap \mathcal{H}^2_\pm\rvert_{\mathbb{R}^+}
$$

(2.9)

Further, as an operator in $L^2(\mathbb{R}^+, dE)$ defined with domain $S \cap \mathcal{H}_2^\pm\rvert_{\mathbb{R}^+}$ or $S \cap \mathcal{H}^2_\pm\rvert_{\mathbb{R}^+}$, $\mathcal{E}$ is essentially self-adjoint. Therefore, the nuclear spectral
theorem, the precise mathematical statement for Dirac’s continuous basis vector expansion, holds for $\mathcal{E}$.

By the Stone-von Neumann theorem, the operator $-i\mathcal{E}$ generates the one parameter unitary group $U(t) = e^{-i\mathcal{E}t}$ in $L^2(\mathbb{R}^+,dE)$.

As an operator in $\mathcal{S} \cap \mathcal{H}^2_{\pm} \vert_{\mathbb{R}^+}$, $\mathcal{E}$ is continuous with respect to the nuclear Fréchet topology of these spaces [11, 18]. Furthermore, the operators $\pm i\mathcal{E}$ generate two differentiable one parameter semigroups $U_{\pm}(t) = e^{\pm it\mathcal{E}}$, $t \geq 0$, in $\mathcal{S} \cap \mathcal{H}^2_{\pm} \vert_{\mathbb{R}^+}$.

Since the spaces $\mathcal{S} \cap \mathcal{H}^2_{\pm} \vert_{\mathbb{R}^+}$ of (2.1) and $\Phi_{\pm}$ of (2.7) are homeomorphic, the above conclusions for the operator $\mathcal{E}$ in $\mathcal{S} \cap \mathcal{H}^2_{\pm} \vert_{\mathbb{R}^+}$ also hold for the operator $H$ in $\Phi_{\pm}$. In particular, as an operator in the Hilbert space $\mathcal{H}$, $H$ is essentially self-adjoint in the domains $\Phi_{\pm}$ and is reduced by them. As an operator in $\Phi_{\pm}$, $H$ is continuous and the operators $\pm iH$ generate two one parameter semigroups $e^{\pm itH}$ in $\Phi_{\pm}$. Specifically, let us use the mappings $W_{\pm} : \Phi_{\pm} \rightarrow \mathcal{S} \cap \mathcal{H}^2_{\pm} \vert_{\mathbb{R}^+}$ as defined above for $\phi_{\pm} \in \Phi_{\pm}$ and write $W_{\pm} \phi_{\pm} = \varphi_{\pm}$. Then, for any $\phi_{\pm} \in \Phi_{\pm}$

$$W_{\pm} e^{\pm itH} \phi_{\pm} = W_{\pm} e^{\pm itH} W_{\pm}^{-1} W_{\pm} \phi_{\pm} = e^{\pm it\mathcal{E}} \varphi_{\pm} \quad (2.10)$$

Since the inclusion $e^{\pm it\mathcal{E}} \left( \mathcal{S} \cap \mathcal{H}^2_{\pm} \vert_{\mathbb{R}^+} \right) \subset \mathcal{S} \cap \mathcal{H}^2_{\pm} \vert_{\mathbb{R}^+}$ holds if and only if $t \geq 0$, it follows from (2.10)

$$e^{\pm itH} (\Phi_{\pm}) \subset \Phi_{\pm} \quad \text{if and only if } t \geq 0 \quad (2.11)$$

That is, the operators $\pm iH$ generate two one parameter semigroups $e^{\pm itH}$, $t \geq 0$, in $\Phi_{\pm}$.

2.2. The nuclear spectral theorem holds for $H$ in $\Phi_{\pm}$. That is, there exists vectors $|E^-\rangle \in \Phi_+^*$ and $|E^+\rangle \in \Phi_-^*$, $E \in \mathbb{R}^+ = \text{spectr}um(H)$, such that for any $\psi^- \in \Phi_+$ and for any $\phi^+ \in \Phi_-$,

$$|\psi^-\rangle = \int_0^\infty dE |E^-\rangle \langle -E| \psi^-\rangle$$

$$|\phi^+\rangle = \int_0^\infty dE |E^+\rangle \langle +E| \phi^+\rangle \quad (2.12)$$
The vectors \(|E^\pm\rangle\) are eigenvectors of \(H^x\)

\[ H^x |E^\pm\rangle = E |E^\pm\rangle \]  

(2.13)

Furthermore, by construction of the spaces by means of the unitary operators \(V_\pm := U\Omega_{\pm}^{-1}\), it follows that

\[ |E^\pm\rangle = \Omega_\pm^x |E\rangle, \quad \text{where} \quad H_0^x |E\rangle = E |E\rangle \]  

(2.14)

In analogy to the heuristic eigenkets of the Lippmann-Schwinger equations, we call the basis vectors \(|E^\pm\rangle\) that fulfill (2.12) – (2.14) Lippmann-Schwinger kets.

2.3. The functions in \(S \cap \mathcal{H}_\pm^2|_{\mathbb{R}^+}\), dense subspaces of \(L^2(\mathbb{R}^+, dE)\), are supported on \(\mathbb{R}^+\), the spectrum of the physical Hamiltonian \(H\). However, it is a property of Hardy class functions [37] that their values on the entire real line as well as the relevant open complex half plane are determined by their values on the positive semi-axis \(\mathbb{R}^+\). This property allows the domains of the functions of \(S \cap \mathcal{H}_\pm^2|_{\mathbb{R}^+}\) to be extended to the entire open half planes \(\mathbb{C}^\pm\), as well as to the negative semi-axis \(\mathbb{R}^-\). As a result, Gamow vectors can be defined as generalized eigenvectors \(|z_R^-\rangle\) of the Hamiltonian with complex eigenvalue \(z_R\), the complex resonance pole position of the \(S\)-matrix.

2.4. As a consequence of paragraph 2.1, the Gamow vectors \(|z_R^-\rangle\), the Lippmann-Schwinger vectors \(|E^\pm\rangle\), the prepared state vectors \(\phi^+\) and observable vectors \(\psi^-\) all have asymmetric, semigroup time evolutions. It is in this sense that the rigged Hilbert space theory of Hardy functions is a time asymmetric quantum theory of resonance scattering and decay.

Notice that the construction of the rigged Hilbert spaces (2.1) and (2.7) holds for any potential \(V\) that satisfies the following two conditions:

1. The full Hamiltonian \(H = H_0 + V\) is essentially self-adjoint and has an absolutely continuous spectrum bounded from below.

2. The Møller wave operators exist and are asymptotically complete.

Therefore, the assertion that ‘nobody has found a potential to which such a theory applies’ in the article [26] is absurd. In fact, as we will see in Section 3, the author of [26] has provided an example of a potential which is completely consistent with the Hardy space theory.
3 Spherical shell potential

All of the conclusions of [26] about time asymmetric quantum theory are drawn from the single example of scattering off a spherical shell potential. We will show here that this potential is in fact perfectly consistent with TAQT and that the conclusions of [26] are the consequences of rudimentary mathematical errors.

Let us first summarize the solutions of the Lippmann-Schwinger equations obtained in [26] for the spherical shell potential. The analysis of [26] begins with the radial Schrödinger equation for $l = 0$,

$$\left(-\frac{d^2}{dr^2} + V(r)\right)\chi^\pm(r, E) = E\chi^\pm(r, E) \quad (3.1)$$

where the potential $V(r)$ is given by

$$V(r) = \begin{cases} 
\infty & r < 0 \\
0 & 0 \leq r \leq a \\
1 & a \leq r \leq b \\
0 & b < r 
\end{cases} \quad (3.2)$$

The Lippmann-Schwinger functions are the solutions of the radial Lippmann-Schwinger equations

$$\langle r | E^\pm \rangle = \langle r | E \rangle + \frac{1}{E - H_0 \pm i0} V | E^\pm \rangle \quad (3.3)$$

For $l = 0$, the solutions of equation (3.3) can be obtained as solutions of the radial Schrödinger equation

$$\left(-\frac{d^2}{dr^2} + V(r)\right)\langle r | E^\pm \rangle = E\langle r | E^\pm \rangle \quad (3.4)$$

subject to the following boundary conditions

$$\langle r | E^\pm \rangle = 0 \text{ if } r \leq 0. \quad (r)$$

$$\langle r | E^\pm \rangle \text{ is continuous at } r = a, b. \quad (r)$$

$$\frac{d}{dr} \langle r | E^\pm \rangle \text{ is continuous at } r = a, b. \quad (r)$$

$$\langle r | E^\pm \rangle \sim e^{-ikr} - S(E) e^{ikr} \text{ as } r \to \infty. \quad (3.5)$$
where
\[ k := \sqrt{\frac{2mE}{\hbar^2}} \text{ and } E \in [0, \infty) \] (3.6)

The analyticity properties of \( \langle r|E^\pm \rangle \) are studied in [27]. The notation
\( \langle r|E^\pm \rangle \) suggests that both \( |r\rangle \) and \( |E^\pm \rangle \) are functionals on some suitably
defined vector spaces (e.g., test function spaces) and that \( \langle r|E^\pm \rangle \) is an integral
kernel. Thus, with the choice of test function space
\[ \Phi := S(R^+/\{a,b\}) \] (3.7)
i.e., the subspace of Schwartz functions supported on the positive semiaxis
\( R^+ \) that vanish at the points \( a \) and \( b \), the kets \( |E^\pm \rangle \) are defined as antilinear
functionals by means of the following formula:
\[ \varphi^\pm(E) := \langle E^\pm|\varphi^\pm \rangle = \int_0^\infty dr \varphi^\pm(r) \langle E^\pm|r \rangle = \int_0^\infty dr \varphi^\pm(r) \langle r|E^\pm \rangle \] (3.8)

These functions \( \varphi^\pm \) are claimed to have analytic extensions into the complex
plane [26]. These extensions in turn follow from the analytic extensions of
\( |E^\pm \rangle \) so that
\[ \varphi^+(z) = \int_0^\infty dr \varphi^+(r) \langle z^+|r \rangle \] (3.9)

Using the following inequality for \( |\langle z^+|r \rangle| \) attributed to [38],
\[ |\langle z^+|r \rangle| \leq C \frac{|z|^{1/4}r}{1 + |z|^{1/2}r} \] (3.10)

where \( C \) is a positive constant, and another inequality for Jost functions \( J^-(z) \) drawn from [27],
\[ \left| \frac{1}{J^-(z)} \right| \leq C, \quad z \in \mathbb{C},^+ \] (3.11)

the author of [26] concludes that for an infinitely differentiable \( \varphi^+(r) \) with
behavior
\[ \varphi^+(r) \sim e^{-r^2/\pi} \text{ as } r \to \infty, \] (3.12a)
the function $\varphi^+(z)$ blows up exponentially on $\mathbb{C}$:

$$
\varphi^+(z) \sim e^{\frac{\Im(z)b}{b}} \text{ as } z \to \infty.
$$

In (3.12), $a$ and $b$ are two real numbers fulfilling $\frac{1}{a} + \frac{1}{b} = 1$. Furthermore, if $\varphi^+(r)$ is an infinitely differentiable function with compact support, the author obtains the inequality [26]

$$
|\varphi^+(z)| \leq C \frac{|z|^{1/4} A}{1 + |z|^{1/2} A} e^{\Im\sqrt{z}} A
$$

where $A$ and $C$ are positive constants. From this the author of [26] concludes that “when $\varphi^+(r) \in C_0^\infty$, $\varphi^+(z)$ blows up exponentially in the infinite arc of $\mathbb{C}$:

If $|\varphi^+(r)| = 0$ when $r > A$, then $|\varphi^+(z)| \sim e^{A\Im\sqrt{z}}$ as $z \to \infty$ (3.14)

This conclusion is simply wrong. Even if the mathematics leading to (3.13) and (3.10) is correct, all that one can conclude from these inequalities is that a particular upper bound on $\varphi^+(z)$ blows up for $z \to \infty$. The inequalities (3.13) and (3.10) do not say anything at all about the behavior of $\varphi^+(z)$ for $z \to \infty$ except that $|\varphi^+(z)|$ is smaller than or equal to a function that increases without bound for $z \to \infty$. Though trivial, it needs to be pointed out that it is a divergent lower bound for $|\varphi^+(z)|$ that proves the divergence of $\varphi^+(z)$ at infinity. It should also be pointed out that even if one succeeds in finding a basis for $L^2(\mathbb{R}^+, dE)$ that does not belong to Hardy spaces, this does not mean that one has ruled out the Hardy solutions. The spaces $\mathcal{S} \cap \mathcal{H}_2^\pm|_{\mathbb{R}^+}$ are dense subspaces of $L^2(\mathbb{R}^+, dE)$, and so one always has Hardy functions if one works with a Hilbert space.

In fact, while a better bound that shows the finiteness of $|\varphi^+(z)|$ at infinity may not be found easily, it is easy to show that a rigged Hilbert space of Hardy functions can be constructed from the solutions of (3.4) for the spherical shell potential. To that end, consider the Hilbert spaces $L^2(\mathbb{R}^+, dr)$ and $L^2(\mathbb{R}^+, dE)$. Both are the same Hilbert space of square integrable functions on the positive semiaxis, although we use a different notation for each because the former is the space of radial wavefunctions and the latter contains the same states in the energy representation. The mappings $U_\pm$

$$
U_\pm : L^2(\mathbb{R}^+, dr) \hookrightarrow L^2(\mathbb{R}^+, dE)
$$

(3.15)
given by
\[
(U \pm f)(E) := \hat{f}^\pm(E) = \int_{0}^{\infty} f(r) \langle r|E^\pm \rangle dr, \quad f \in L^2(\mathbb{R}^+, dr)
\]  
(3.16)
are unitary. The inverse operators \( U_{\pm}^{-1} \) of (3.15) are given by
\[
(U_{\pm}^{-1} \hat{f}^\pm)(r) = f(r) = \int_{0}^{\infty} \hat{f}^\pm(E) \langle r|E^\pm \rangle dE
\]  
(3.17)
Here, \( \langle r|E^\pm \rangle \) are the solutions of (3.4).

Now, let \( \hat{\varphi}^+ \in \mathcal{S} \cap \mathcal{H}^2_{\mathbb{R}^+} \). It is obvious that \( \hat{\varphi}^+ \in L^2(\mathbb{R}^+, dE) \) and therefore we can use it in the integrand of (3.17) to obtain a function \( \varphi^+ = U_{+}^{-1} \hat{\varphi}^+ \in L^2(\mathbb{R}^+, dr) \). Since \( U_+ \) is unitary, the use of \( \varphi^+ \) in (3.16) returns the original function \( \hat{\varphi}^+ \in \mathcal{S} \cap \mathcal{H}^2_{\mathbb{R}^+} \).

What this simple argument shows is that the same procedure used in [26] for constructing wave functions in the energy representation can be used to construct a rigged Hilbert space of Hardy functions. If, as claimed in [27], \( U_{\pm} \) are unitary and the Hamiltonian is mapped by \( U_{\pm} \) to the multiplication operator in \( L^2(\mathbb{R}^+, dE) \), then all of the conditions enumerated in Section 2, in particular (2.7)–(2.14), are fulfilled for the spherical shell potential used in [26] and thus a rigged Hilbert space of Hardy functions can be built for this potential. In this sense, albeit inadvertently, [26] has given us an explicit method of constructing the spaces \( \Phi_{\pm} \) in the position representation that leads to the rigged Hilbert spaces (2.1) (or, equivalently, (2.7)) for the spherical shell potential (3.2): choose \( \Phi_{\pm} := U_{\pm}^{-1} \left[ \mathcal{S} \cap \mathcal{H}^2_{\mathbb{R}^+} \right] \), where \( U_{\pm} \) are the transformations defined by (3.16) and (3.17). The method runs closely parallel to the general procedure outlined in (2.8) for constructing rigged Hilbert spaces of Hardy functions. The only remaining problem here is to identify the form and properties of those functions in \( L^2(\mathbb{R}^+, dr) \) that are in one-to-one correspondence with the subspaces \( \mathcal{S} \cap \mathcal{H}^2_{\mathbb{R}^+} \) by way of the unitary transformations \( U_{\pm} \). While this requires further mathematical investigations, the main features of the rigged Hilbert spaces of Hardy class functions and the properties of the resulting theory of resonance scattering and decay for the spherical shell potential have been exemplified. In other words, the example studied in [26] answers its own complaint “nobody has found a potential to which such a [time asymmetric quantum] theory applies” in the affirmative. The article [26] misses this conclusion as a result of mathematical mistakes.
4 On other aspects of the critique

The main weakness of [26] is the rather rudimentary mathematical mistakes that we have outlined in Section 3. In view of these mathematical errors, the crux of the case of [26] against rigged Hilbert spaces of Hardy functions falls apart. However, the article is also riddled with erroneous remarks as well as citations attributed to the proponents of TAQT. In the following subsections, we will point out some of the more glaring errors of diffused focus of [26] with respect to citing from the literature and errors of reasoning with respect to drawing strong physical conclusions from mathematical fallacies.

4.1. After an inaccurate, extended and partially misleading paraphrasing of the section 3 of [10], the author of [26] writes on page 9262:

"Then, as a mathematical statement for ‘no preparations for $t > 0'$, one takes

$$|\langle E|\varphi_{\text{in}}(t)\rangle| = |\langle^+E|\varphi^+(t)\rangle| = 0, \quad t > 0 \quad (4.3) \text{ of } [26]$$

for all energies, which implies

$$0 = \int_{-\infty}^{\infty} dE \langle E|\varphi_{\text{in}}(t)\rangle = \int_{-\infty}^{\infty} dE \langle^+E|\varphi^+(t)\rangle = \int_{-\infty}^{\infty} dE \langle^+E|e^{-itH}\varphi^+\rangle \quad (4.4) \text{ of } [26]$$

or

$$0 = \int_{-\infty}^{\infty} dE e^{-itE} \varphi^+(E) \equiv \tilde{\varphi}^+(t), \quad \text{for } t > 0 \quad (4.5) \text{ of } [26]"$$

Then, the author of [26] goes on to conclude:

"The first flaw lies in assumption (4.3). From such an assumption, it follows that

$$0 = \langle E^+|\varphi^+(t)\rangle = e^{-itE} \varphi^+(E) \quad (4.9) \text{ of } [26]$$

for all energies. Hence,

$$0 = \varphi^+(E) \quad (4.10) \text{ of } [26]$$

for all energies, which can happen only when $\varphi^+$ is identically 0. Thus, the preparation-registration arrow of time holds only in the meaningless case of the zero wavefunction."
In contrast, in the original article [10], almost the same words are used to justify another mathematical statement:

“As the mathematical statement for “no preparations for \( t > 0 \)” we therefore write (the slightly weaker condition)

\[
0 = \int dE \langle \mathcal{E} | \phi^{\text{in}}(t) \rangle = \int dE \langle \mathcal{E}^+ | \phi^+(t) \rangle = \int dE \langle \mathcal{E}^+ | e^{-itH} | \phi^+ \rangle
\]

or

\[
0 = \int_{-\infty}^{\infty} dE \langle \mathcal{E}^+ | e^{-itE} \rangle \equiv \mathcal{F}(t) \quad \text{for} \quad t > 0 \quad (3.4) \text{of [10]}
\]

That is, the authors of [10] never use the ‘assumption’ (4.3) of [26] anywhere in their derivation. In particular, they do not use the vanishing of the integrand to conclude that integral (3.4) of [10] vanishes, as the author of [26] (mis)leads his readers into inferring. It is (3.4) of [10] (or (4.4) of [26]) that the authors take as the mathematical statement for “no preparations for \( t \geq 0 \)” and not (4.3) of [26].

While it is certain that the assumption \( \langle \mathcal{E}, \eta | \phi^{\text{in}}(t) \rangle = 0 \) for \( t > 0 \) (or, \( \langle \mathcal{E}^+, \eta | \phi^+(t) \rangle = 0 \) for \( t > 0 \)) implies \( \phi^{\text{in}} = 0 \) (or \( \phi^+ = 0 \)), it is no less certain that (3.4) of [10] does not imply (4.10) of [26]. According to the Paley-Wiener theorems [24, 25], the mathematical statement (3.4) of [10] for “no preparations for \( t \geq 0 \)” is equivalent to the hypothesis that \( \langle \mathcal{E} | \phi^{\text{in}} \rangle = \langle \mathcal{E}^+ | \phi^+ \rangle \) is a Hardy function in the open lower half plane \( \{ z : z \in \mathbb{C}; \text{Im} \, z < 0 \} \). This equivalence has been extensively discussed in many other references, including [14, 8, 13, 12, 15, 16].

It appears at a first glance that an objection can be made, as the author of [26] does, to the fact that integration over energy in (3.4) of [10] extends from \(-\infty\) to \(+\infty\) rather than from \(0\) to \(\infty\), the physical spectrum of energy. However, we point out here that (3.4) of [10] applies to the space \( H_2^- \) (more precisely, the subspace \( S \cap H_2^- \)) and not the physical space \( S \cap H_2^- |_{\mathbb{R}^+} \). In particular, the vanishing of the integral (3.4) of [10] establishes, by way of Paley-Wiener theorems, that the operator \(-i\mathcal{E}\) generates a one parameter semigroup \( e^{-i\mathcal{E}t}, t \geq 0 \), in \( H_2^- \). From this and the topological properties of the space \( S \cap H_2^- \) it follows that \( e^{-i\mathcal{E}t} \) is a differentiable semigroup in \( S \cap H_2^- \). Now, by van Winter’s theorem [37] it finally follows that \( e^{-i\mathcal{E}t} \) is a differentiable semigroup
in the space $S \cap \mathcal{H}_2^2|_{\mathbb{R}^+}$. It is this space $S \cap \mathcal{H}_2^2|_{\mathbb{R}^+}$ consisting of functions defined over the physical energy spectrum $[0, \infty)$ that we take as the space of ‘in’ wavefunctions and the time evolution of these wavefunctions is given by a Hamiltonian generated differentiable semigroup. A similar computation shows that $e^{iE}t, t \geq 0$, is a differentiable semigroup defined in $S \cap \mathcal{H}_2^2|_{\mathbb{R}^+}$, the space of functions that represent the measured ‘out’ states (more properly called out-observables).

Thus, the author of [26] is mistaken about both the physics and mathematics of [10]. In our view, the only objection that can be made against [10] is semantic in nature; namely, the condition (3.4) of [10] should not be called “slightly weaker” than the condition $\langle E, \eta^+|\phi^+(t) \rangle = 0$ for it is a mathematically different condition, equivalent to the Hardy hypothesis.

4.2. Section 5 of [26] is simply a reiteration of its Section 3. Here, the author of [26] claims again that the wavefunctions for the spherical shell potential cannot be of Hardy class because they are not square integrable when extended to negative energy values, in contrast to Hardy class functions which are square integrable over the whole real line. To “prove” the non-square integrability of the energy wavefunctions over the negative semi-axis, the author uses the inequality ((5.11), [26])

$$|\phi^+(E)| \leq C \frac{|E|^{1/4}}{1 + |E|^{1/2}} e^{A|E|^{1/2}} \quad (4.1)$$

and makes the now familiar conclusion that $|\phi^+(E)|$ blows up for $E \to -\infty$ simply because it is bounded from above by a divergent function. The entire Section 5 of [26] is written in a rather contrived manner. The notation $(\varphi^+, \varphi^+)_\text{Hardy}$ and equations (5.15) and (5.16) suggest that Hardy class functions are not square integrable and that TAQT suffers from these divergences. It is standard knowledge that Hardy class functions $\mathcal{H}_2^2$ are subspaces of the Hilbert space $L^2(\mathbb{R})$ [24] and thus equations (5.15) and (5.16) of [26] do not hold for Hardy functions, despite the author’s notation that suggests so.

The last paragraph of Section 5 of [26] is laden with mangled logic and mathematical fallacies. For instance, with the assertion “The problem is that, although there is nothing wrong with analytically continuing
from the physical spectrum into the negative real line of the second sheet, the resulting integrals are not convergent, as shown above”, the author of [26] lays the charge against the Hardy space theory with the ‘problem’ of divergence of his own wavefunctions, itself a non sequitur in view of (4.1). To reiterate, Hardy functions are perfectly square integrable over the whole real line. Therefore, the integral (5.13) of [26] converges if Hardy functions are used in the integrand. Likewise, if \( \phi^+ \) belongs to the space \( S \cap \mathcal{H}^2_{\pm} |_{\mathbb{R}^+} \), as required in TAQT, then the integral (5.14) is also convergent. Therefore, the divergences that the author refers to, in particular (5.13)-(5.16), first of all, are wrong in view of (4.1) and our explanation of the error just below it in the text, and second of all have nothing to do with Hardy functions. In other words, even if one accepts (5.13) and (5.14) of [26], these would apply to the wavefunctions of [26] and not to Hardy functions.

The author of [26] has certainly proven no mathematical statement about the semiboundedness of the Hamiltonian, other than simply stating his opinion that Hardy functions are not consistent with this requirement. Once again, let us restate what we stated in Section 2: The Hilbert space \( L^2(\mathbb{R}^+, dE) \) consists of square integrable functions defined over the positive real line \( \mathbb{R}^+ \). The spaces \( S \cap \mathcal{H}^2_{\pm} |_{\mathbb{R}^+} \) are subspaces of \( L^2(\mathbb{R}^+, dE) \), and thus the functions of \( S \cap \mathcal{H}^2_{\pm} |_{\mathbb{R}^+} \) have domains \([0, \infty)\). The Hamiltonian, which has the realization as the multiplicative operator in \( L^2(\mathbb{R}^+, dE) \) with the domain \( S \cap \mathcal{H}^2_{\pm} |_{\mathbb{R}^+} \) or \( S \cap \mathcal{H}^2_{\pm} |_{\mathbb{R}^+} \), is essentially self-adjoint and has a non-degenerate, absolutely continuous spectrum that coincides with the positive real line \( \mathbb{R}^+ \). These facts have been well established not only for the non-relativistic case considered in [26] but also for the relativistic case [15]. The author of [26] does nothing by way of showing where the proponents of TAQT have made mathematical errors. In fact, to quote the author of [26], “it is not that the math of Bohm-Gadella theory is wrong, it is rather that the math of the Bohm-Gadella theory is inconsistent with quantum mechanics”.

However, the assertion that Hardy hypothesis is inconsistent with the semiboundedness of the Hamiltonian ([26], page 9265, last paragraph) is inexorably a mathematical statement and it is not in the spirit of mathematical physics to make such a definitive pronouncement without even an attempt at a proof.

As a final remark on Section 5 of [26], let us make believe that the
author of [26] has not made mathematical mistakes and has found a set of wavefunctions that have the divergence properties described in [26]. Then the only conclusion that one can draw from the study of [26] is that there are solutions to the Schrödinger equation that are not of Hardy class. This does not provide a basis for determining if TAQT is mathematically or physically tenable. Nobody has ever asserted that all solutions of the Schrödinger equation are of Hardy class, in much the same way that nobody has ever asserted, for instance, that all solutions of the Schrödinger equation are square integrable. It is common knowledge that the Schrödinger equation must be solved subject to some physically meaningful boundary conditions, such as square integrability. In TAQT, the Hardy hypothesis constitutes such a boundary condition subject to which the quantum dynamical equations must be solved. Finding solutions which do not obey these boundary conditions means just that: there exist solutions that do not obey the boundary conditions of TAQT. The existence of other solutions is not tantamount to a counter-example that invalidates TAQT. In fact, the author of [26] does not even succeed in showing that there are non-Hardy solutions in his example, much less prove that there are no Hardy solutions.

In connection with Section 6 of [26], we wish to make the following remarks:

4.3. Hardy spaces do not appear to be essential for the construction of vectors with complex energy eigenvalues that represent resonances. In fact, what really matters for the construction of these vectors is the use of wavefunctions that, in the energy representation, admit suitable analytic extensions. The subspace of these wavefunctions should be dense in the whole Hilbert space of states.

That we can use different kinds of spaces of analytic functions can be shown as follows. Let us consider infinitely differentiable functions with compact support. By Paley-Wiener theorems, their Fourier transforms are entire analytic functions [34]. The values these functions on the positive semiaxis determine their values on the whole complex plane by the Principle of Analytic Continuation [28]. Moreover, the restrictions of these entire functions to $\mathbb{R}^+$ form a dense subspace of $L^2(\mathbb{R}^+, dE)$. This dense subspace can be used to obtain a class of resonance state
Hardy functions are used in TAQT for the following reasons:

(a) The intersection of Hardy functions with Schwartz functions fulfills an adequate set of mathematical conditions for a consistent theory of scattering and decay. In particular, as outlined in Section 2, the spaces \( S \cap \mathcal{H}_\pm^2 \big|_{\mathbb{R}^+} \) lead to the rigged Hilbert spaces (2.1). The decaying Gamow vectors can be properly defined as elements in the dual space \( (S \cap \mathcal{H}_\pm^2 \big|_{\mathbb{R}^+})^\times \).

(b) The rigged Hilbert spaces of Hardy functions (2.1) permit a time asymmetric formulation of resonance phenomena [4, 14, 8, 12, 13, 16, 3, 2, 15, 17].

(c) Functions that are both Hardy and Schwartz have very good behavior at infinity, both on the real line and on the appropriate complex semi-plane. This regularity property can be used, for instance, to give a workable formula for the background that is always present in all resonance processes as an integral over the negative semiaxis. A further advantage of Hardy functions is that by means of the Mellin transform the background integral can be written as an integral over the positive semiaxis [21].

4.4. It is inaccurate to state that the proponents of TAQT dispense with the principle of asymptotic completeness. Asymptotic completeness means the ‘in’ scattering states and ‘out’ scattering states inhabit the same Hilbert space, i.e., the ranges of the Møller operators are the same, \( \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} \). What TAQT establishes is that the ‘in’ and ‘out’ scattering vectors inhabit two different vector spaces \( \Phi_- \) and \( \Phi_+ \), each of which is dense in the same Hilbert space. Therefore, the Hilbert space completion of \( \Phi_+ \) and \( \Phi_- \) gives the conventional principle of asymptotic completeness and thus TAQT is not in contradiction with conditions that define the Møller operators or the \( S \)-operator. In fact, as we have seen in Section 2, Møller operators play a role in the construction of the rigged Hilbert spaces of Hardy functions (2.1), and asymptotic completeness has been used in the early publications on TAQT [20] as well as later review articles [11, 18]. TAQT does introduce a topological refinement to the principle of asymptotic completeness in that \( \Phi_+ \neq \Phi_- \) (both as dense subspaces of \( \mathcal{H} \) and as complete topological
vector spaces), and this is the content of the statements that the author attributes to the proponents of TAQT.

4.5. No serious questions exist about the construction of Gamow vectors by solving the Schrödinger equation subject to purely outgoing boundary conditions. In fact, there is a well-known procedure to obtain Gamow vectors this way [29, 18, 5].

4.6. To conclude this section, we want to comment on the last assertion of [26]: The major achievement of the Bohm-Gadella theory, namely time asymmetry and the rigorous construction of resonance states, can be achieved rigorously within standard quantum mechanics. The author does not give any references to support his assertion and it is not clear what the author means by ‘standard quantum mechanics.’ If by ‘standard quantum mechanics’ one means the use of a Hilbert space to represent the space of states, we are aware of two mathematically rigorous approaches to resonances. One is given by the Lax-Phillips theory [35, 36, 30]. The other is the dilation of analytic potentials or complex scaling [33]. The TAQT has some features that are found also in these formalisms but it has other strengths that make it rather unique.

The Lax-Phillips formalism is attractive and, in principle, it accommodates time asymmetry. It works in the context of the Hilbert space but the price to pay for retaining the Hilbert space is that Gamow vectors are eigenvectors of a non self-adjoint, dissipative operator with complex eigenvalues. In contrast, in TAQT (based on rigged Hilbert spaces of Hardy functions) Gamow vectors are eigenvectors of an extension (into the dual space $\Phi^+_\infty$) of a self-adjoint Hamiltonian. In both formalisms the eigenvalues are the same and coincide with the resonance poles of the analytic $S$-matrix. Furthermore, the two formalisms are closely connected as shown by Y. Strauss [36].

In the complex scaling formalism, the Hamiltonian is “dilated” by a complex parameter and the dilated Hamiltonian is no longer self-adjoint. The relevant part of this complex parameter is its imaginary part that behaves like an angle. The number of resonances defined as complex eigenvalues of the dilated Hamiltonian is directly related to the size of the imaginary part of the dilation parameter but the resonance parameters themselves do not depend on the value of the
dilation parameter. Moreover, the resonance eigenvalues of the dilated Hamiltonian coincide with the poles of the extended resolvent.

5 Concluding remarks.

We agree with the sentiment expressed by the title of [26]: if by ‘standard quantum mechanics’ the author means unitary evolutions and Hilbert spaces, indeed TAQT is not consistent with ‘standard quantum mechanics’. Aside from this, as we have shown in the foregoing response, [26] does not bear scrutiny as a critique on Time Asymmetric Quantum Theory for two reasons: first, the author makes mistakes that the main mathematical conclusions of [26] are false; second, even if there exist proper mathematical methods by which one can arrive at solutions to the Schrödinger equation for the spherical shell potential that are not of Hardy class, this is not equivalent to saying that there are no solutions to the Schrödinger equation that are of Hardy class for this potential or, much less, all other. Therefore, even if it had been conducted with proper mathematical care, the study of [26] could not have led to a conclusion about the veracity TAQT. While critical inquiry is crucial to the practice and philosophy of science, the one carried out in [26] falls short of the normative technical rigor and reasoning of theoretical physics that it does not inform.

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