CONVERGENCE OF EIGENVECTOR EMPIRICAL SPECTRAL DISTRIBUTION OF SAMPLE COVARIANCE MATRICES

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The eigenvector empirical spectral distribution (VESD) is a useful tool in studying the limiting behavior of eigenvalues and eigenvectors of covariance matrices. In this paper, we study the convergence rate of the VESD of sample covariance matrices to the Marčenko-Pastur (MP) distribution. Consider sample covariance matrices of the form $XX^*$, where $X = (x_{ij})$ is an $M \times N$ random matrix whose entries are independent (but not necessarily identically distributed) random variables with mean zero and variance $N^{-1}$. We show that the Kolmogorov distance between the expected VESD and the MP distribution is bounded by $N^{-1+\epsilon}$ for any fixed $\epsilon > 0$, provided that the entries $\sqrt{N}x_{ij}$ have uniformly bounded 6th moment and that the dimension ratio $N/M$ converges to some constant $d \neq 1$. This result improves the previous one obtained in [33], which gives the convergence rate $O(N^{-1/2})$ assuming i.i.d. $X$ entries, bounded 10th moment and $d > 1$. Moreover, we also prove that under the finite 8th moment condition, the convergence rate of the VESD is $O(N^{-1/2+\epsilon})$ almost surely for any fixed $\epsilon > 0$, which improves the previous bound $O(N^{-1/4+\epsilon})$ in [33].

1. Introduction. Let $X = (x_{ij})$ be an $M \times N$ real or complex data matrix whose entries are independent (but not necessarily identically distributed) random variables satisfying

\begin{align}
\mathbb{E}x_{ij} &= 0, \quad \mathbb{E}|x_{ij}|^2 = N^{-1}, \quad 1 \leq i \leq M, \quad 1 \leq j \leq N, \\
\text{and in addition,} \\
\mathbb{E}x_{ij}^2 &= 0, \quad \text{if } x_{ij} \text{ is complex.}
\end{align}

Then $XX^*$ gives a class of $M \times M$ sample covariance matrices. Its empirical spectral distribution (ESD) is defined as

\[ F_{XX^*}^{(M)}(x) := \frac{1}{M} \sum_{i=1}^{M} 1\{\lambda_i(XX^*) \leq x\}; \]

∗Supported by NSF Career Grant DMS-1552192 and Sloan fellowship.

MSC 2010 subject classifications: Primary 15B52, 62E20; secondary 62H99

Keywords and phrases: Sample covariance matrix, Empirical spectral distribution, Eigenvector empirical spectral distribution, Marčenko-Pastur distribution
where \( \lambda_1(XX^*) \geq \ldots \geq \lambda_M(XX^*) \) are the eigenvalues of \( XX^* \) and \( 1_\{\cdot\} \) denotes the conventional indicator function. Similarly, one can also consider \( X^*X \) and its ESD

\[
F_{X^*X}^{(N)}(x) := \frac{1}{N} \sum_{i=1}^{N} 1_{\{\lambda_i(X^*X) \leq x\}}.
\]

Sample covariance matrices are fundamental objects in modern multivariate statistics, where the advance of technology has led to high dimensional data such that \( M \) is comparable to or even larger than \( N \). These large dimensional covariance matrices have many applications in various fields, such as statistics \([7, 17, 18, 19]\), economics \([25]\) and population genetics \([26]\). Define the aspect ratio

\[
d_N := \frac{N}{M}.
\]

We are interested in the regime where \( \lim_{N \to \infty} d_N = d \in (0, \infty) \), i.e. \( M \) and \( N \) are proportional to each other. In this case, it is well-known that \( F_{XX}^{(M)} \) converges weakly to the Marčenko-Pastur (MP) law \( F_d(x) \) \([24]\), which has a

\[
(1 - d)^{+} \text{ mass at } x = 0 \text{ and has a density}
\]

\[
\rho_{1c}(x) = (1 - d)^{+} \delta_0 + \frac{d}{2\pi} \frac{\sqrt{[(\lambda_+ - x)(x - \lambda_-)]^+}}{x}, \quad \lambda_{\pm} = (1 \pm d^{-1/2})^2,
\]

in the interval \([\lambda_-, \lambda_+]\) (where \( \lambda_+ \) and \( \lambda_- \) are often referred to as the soft edge and hard edge, respectively). For \( z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im} \, z > 0\} \), the Stieltjes transform \( m_{1c}(d, z) \) of \( F_d \) satisfies the following self-consistent equation

\[
m_{1c}(d, z) + \frac{1}{z - (1 - d^{-1}) + zd^{-1}m_{1c}(d, z)} = 0,
\]

and has the closed form expression

\[
m_{1c}(d, z) = \frac{1 - d^{-1} - z + i \sqrt{(\lambda_+ - z)(z - \lambda_-)}}{2zd^{-1}}.
\]

Since \( XX^* \) and \( X^*X \) share the same nonzero eigenvalues, it is easy to see that \( F_{X^*X}^{(N)} \) also converges weakly to a deterministic law, whose Stieltjes transform \( m_{2c}(z) \) satisfies

\[
m_{2c}(d, z) = \frac{d^{-1} - 1}{z} + d^{-1}m_{1c}(d, z).
\]

In applications of spectral analysis of large dimensional random matrices, one of the important problems is the convergence rate of the ESD.
that the Kolmogorov distance between two distributions $F_1$ and $F_2$ is defined as
\[ \|F_1 - F_2\| := \sup_x |F_1(x) - F_2(x)|. \]
Then we use $\|F_{XX^*}^{(M)} - F_{d_N}^{(M)}\|$ to measure the convergence rate of $F_{XX^*}^{(M)}$. The convergence rate of the ESD of sample covariance matrices was first studied in [1] using Berry-Esseen type inequalities for the difference of two distributions in terms of their Stieltjes transforms. The Berry-Esseen type inequalities were later improved in [16] to show that the convergence rate is $O(N^{-1/2})$ in probability under finite 8th moment condition. A sharper bound was obtained in [27], where the authors proved that $\|F_{XX^*}^{(M)} - F_{d_N}^{(M)}\| = O(N^{-1}(\log N)^{O(\log \log N)})$ in probability under the sub-exponential decay assumption.

The properties of eigenvectors of large dimensional random matrices are much harder to study. However, great progress has been made in this direction; see [13, 3] for the delocalization and isotropic delocalization of eigenvectors, [20, 30] for the universality of eigenvectors, [5] for the local quantum unique ergodicity of eigenvectors, [4] for the eigenvectors of principal components, [28, 29, 2, 32, 33] for the asymptotical Haar property of the eigenmatrix based on the VESD (see (1.8) below), to name a few. Note that some of these results are proved for Wigner matrices, but their generalizations to sample covariance matrices usually are straightforward.

1.1. Main results. This paper is concerned with the eigenvector empirical spectral distribution (VESD) of sample covariance matrices, which we shall now define. Suppose $XX^*$ has the spectral decomposition
\[ XX^* = \sum_{k=1}^{M} \lambda_k(XX^*) \xi_k \xi_k^*, \]
where $\xi_k$ are the eigenvectors. We define the VESD of $XX^*$ as
\[ F_{XX^*}^{(M)}(v, x) = \sum_{i=1}^{M} |\langle \xi_i, v \rangle|^2 1_{\{\lambda_i(XX^*) \leq x\}}, \]
where $v$ is a deterministic unit vector in $\mathbb{C}^M$. It was proved in [2, 3] that for any fixed $v$, $F_{XX^*}^{(M)}(v, x)$ converges weakly to the MP law as $N \to \infty$. Compared with ESD, much less has been known about the convergence rate of VESD. The best result so far was obtained in [33], where the authors proved that if $d_N > 1$ and the entries of $X$ are i.i.d. random variables, then
\[
\| \mathbb{E} F_{XX^*}^{(M)}(\cdot) - F_{d_N}(\cdot) \| = O(N^{-1/2}) \]
under the finite 10th moment assumption, and \[
\| F_{XX^*}^{(M)}(\cdot) - F_{d_N}(\cdot) \| = O(N^{-1/4+\epsilon}) \]
almost surely under the finite 8th moment assumption. However, we find that both of these bounds are far away from being optimal, and can be improved using a different method. This is the main purpose of this paper. As demonstrated in [28, 29, 33], the convergence of the VESD for any fixed unit vector \( \mathbf{v} \) can be used to characterize the asymptotical Haar property of the eigenmatrix of \( XX^* \). Thus we expect that a better bound for the convergence rate will lead to a better understanding of the Haar properties of the eigenvectors of large sample covariance matrices.

Now we are ready to state the main result of this paper. For a reason that will be clear later (when we prove Corollary 1.2), we consider slightly more general random matrices \( X = (x_{ij}) \). More specifically, we define the following conditions: there exist constants \( C_0, c_0 > 0 \) such that for all \( 1 \leq i \leq M \) and \( 1 \leq j \leq N \),

\[
|E x_{ij}| \leq C_0 N^{-2-c_0},
\]
\[
|E |x_{ij}|^2 - N^{-1}| \leq C_0 N^{-2-c_0},
\]
\[
E |x_{ij}|^4 \leq C_0 N^{-2},
\]
\[
|E x_{ij}^2| \leq C_0 N^{-2-c_0}, \text{ if } x_{ij} \text{ is complex.}
\]

**Theorem 1.1.** Let \( X = (x_{ij}) \) be an \( M \times N \) random matrix whose entries are independent random variables satisfying (1.9), (1.10), (1.11) and (1.12). Suppose there exist constants \( C_1, \phi > 0 \) such that

\[
\max_{1 \leq i \leq M, 1 \leq j \leq N} |x_{ij}| \leq C_1 N^{-\phi}.
\]

Suppose \( d_N \to d \) for some constant \( d \neq 1 \). Then for any fixed (small) \( \epsilon > 0 \) and (large) \( D > 0 \), we have

\[
\| \mathbb{E} F_{XX^*}^{(M)}(\cdot) - F_{d_N}(\cdot) \| \leq N^{-1+\epsilon},
\]

and

\[
P \left( \| F_{XX^*}^{(M)}(\cdot) - F_{d_N}(\cdot) \| \geq N^\epsilon \left( N^{-2\phi} + N^{-1/2} \right) \right) \leq N^{-D},
\]

for all deterministic unit vectors \( \mathbf{v} \in \mathbb{C}^M \), provided that \( N \) is large enough.

As an immediate corollary, we have the following result.
COROLLARY 1.2. Let $X = (x_{ij})$ be an $M \times N$ random matrix whose entries are independent random variables satisfying (1.1) and (1.2). Suppose there exist constants $a, A > 0$ such that

\begin{equation}
\max_{1 \leq i \leq M, 1 \leq j \leq N} \mathbb{E} |\sqrt{N}x_{ij}|^a \leq A \tag{1.16}
\end{equation}

for all $N$. Suppose $d_N \to d$ for some constant $d \neq 1$. Then if $a \geq 6$, we have

\begin{equation}
\|\mathbb{E} F_{XX}(v, \cdot) - F_{dN}(\cdot)\| \leq N^{-1+\epsilon} \tag{1.17}
\end{equation}

for sufficiently large $N$. Moreover, if $a \geq 8$, we have

\begin{equation}
\mathbb{P}\left( \limsup_{N \to \infty} N^{1/2-\epsilon} \|F_{XX}(v, \cdot) - F_{dN}(\cdot)\| \leq 1 \right) = 1. \tag{1.18}
\end{equation}

**Proof.** In the proof, we fix $a > 4$ and choose a constant $\phi > 0$ small enough such that $(N^{1/2-\phi})^a \geq N^{2+\omega}$ for some constant $\omega > 0$. Then we introduce the following truncation

\[ \tilde{X} := 1_{\Omega}X, \quad \Omega := \left\{ |x_{ij}| \leq N^{-\phi} \text{ for all } 1 \leq i \leq M, 1 \leq j \leq N \right\}. \]

By the moment condition (1.16), we have

\begin{equation}
\mathbb{P}(\tilde{X} \neq X) = O(N^{2-a/2+a\phi}). \tag{1.19}
\end{equation}

Moreover, we have

\begin{equation}
\mathbb{P}(\tilde{X} \neq X \text{ i.o.}) = \lim_{k \to \infty} \mathbb{P}\left( \bigcup_{N=k}^\infty \bigcup_{i=1}^M \bigcup_{j=1}^N \left\{ |x_{ij}| \geq N^{-\phi} \right\} \right) = \lim_{k \to \infty} \mathbb{P}\left( \bigcup_{t=k}^\infty \bigcup_{N \in [2^t, 2^{t+1})} \bigcup_{i=1}^M \bigcup_{j=1}^N \left\{ |x_{ij}| \geq N^{-\phi} \right\} \right) \leq C \lim_{k \to \infty} \sum_{t=k}^\infty (2^{t+1})^2 \left( 2^{(t/2-\phi)} \right)^{-a} \leq C \lim_{k \to \infty} \sum_{t=k}^\infty 2^{-\omega t} = 0, \tag{1.20}
\end{equation}

i.e. $\tilde{X} = X$ almost surely as $N \to \infty$. Here in the above derivation, we regard $M \equiv N/d_N$ as a function depending on $N$, which, by the given condition on $d_N$, satisfies $M = O(N)$ for large enough $N$.

Using (1.16) and integration by parts, we can get that

\[ \mathbb{E} |x_{ij}| 1_{|x_{ij}| > N^{-\phi}} = O(N^{-2-\omega/2}), \quad \mathbb{E} |x_{ij}|^2 1_{|x_{ij}| > N^{-\phi}} = O(N^{-2-\omega/2}). \]
which imply that
\[ |\mathbb{E} \tilde{x}_{ij}| = O(N^{-2-\omega/2}), \quad |\mathbb{E} \tilde{x}_{ij}|^2 = N^{-1} + O(N^{-2-\omega/2}), \]
and
\[ |\mathbb{E} \tilde{x}_{ij}^2| = O(N^{-2-\omega/2}), \quad \text{if } x_{ij} \text{ is complex.} \]
Moreover, we trivially have
\[ |\mathbb{E} \tilde{x}_{ij}|^4 \leq |\mathbb{E} x_{ij}|^4 = O(N^{-2}). \]
Hence \( \tilde{X} \) is a random matrix satisfying the assumptions in Theorem 1.1.

Then using (1.14) and (1.19) with \( a = 6 \) and \( \phi = \epsilon/6 \), we conclude (1.17); using (1.15) and (1.20) with \( \phi = (1-\epsilon)/4 \) and \( a = 8 \), we conclude (1.18).

**Remark 1.3.** The estimates (1.17) and (1.18) improve the bounds obtained in [33] (and relax the moment assumptions as well). We believe that the convergence rates in (1.17) and (1.18) are close to optimal due to the following reasons. It was proved in [2] that for an analytic function \( f \),
\[
\sqrt{N} \int f(x) d \left( F_{XX^*}^{(M)}(v, x) - F_{d_N}(x) \right) \to \mathcal{N}(0, \sigma_f),
\]
where \( \mathcal{N}(0, \sigma_f) \) denotes the Gaussian distribution with mean zero and variance \( \sigma_f \). This shows that the fluctuation of \( F_{XX^*}^{(M)}(x) \) is of order \( N^{-1/2} \) and suggests the bound in (1.18). Taking expectation of (1.21), one can see that the order of \( |\mathbb{E} F_{XX^*}^{(M)}(v, x) - F_{d_N}(x)| \) should be even smaller. On the other hand, the fluctuation of the eigenvalues of \( XX^* \) on the microscopic scale \( N^{-1} \) will lead to an error of order at least \( N^{-1} \). This shows that the bound (1.17) is very close to being optimal.

**Remark 1.4.** In [33], the authors can only handle the \( M < N \) (i.e. \( d_N > 1 \)) case, while our proof will work for both the \( d > 1 \) and \( d < 1 \) cases. However, in the case with \( d_N \to 1 \), we will encounter some difficulties near the hard edge \( \lambda_- \), which converges to 0 as \( N \to \infty \) by (1.3). However, we can still prove weaker versions of (1.14) and (1.15) by restricting ourself to the region away from 0. For instance, we have for any fixed \( \tau > 0 \),
\[
\sup_{x \geq \tau} |\mathbb{E} F_{XX^*}^{(M)}(v, \cdot) - F_{d_N}(\cdot)| \leq N^{-1+\epsilon}
\]
under the assumptions in Theorem 1.1. Similarly, the bound in (1.15) also holds if we only take the sup over \( x \geq \tau \).
1.2. Main ideas. A basic tool for the proof is the Stieltjes transform. For any $z = E + i\eta \in \mathbb{C}_+$, we define the resolvent of $XX^*$ as

$$G(X, z) := (XX^* - z)^{-1}.$$ 

Then the Stieltjes transform of $F^{(M)}_{XX^*}(v, \cdot)$ is equal to $\langle v, G(X, z)v \rangle$, and we have the asymptotic estimate

$$\langle v, G(X, z)v \rangle \approx m_{1c}(d_N, z)$$

for any fixed $\eta > 0$, when $N$ is large. By taking the imaginary part, it is easy to see that a control of the Stieltjes transform $\langle v, G(X, z)v \rangle$ yields a control of the VESD on a small scale of order $\eta$ around $E$. An isotropic local law is an estimate of the form (1.22) for all $\eta \gg N^{-1}$. Such isotropic local law was first established in [21, 3] for sample covariance matrices and generalized Wigner matrices, assuming the matrix entries have arbitrarily high moments.

Now we briefly describe the ideas for the proof of Theorem 1.1. Following the approach in [27] (which is used to prove the convergence rate of ESD), the main idea is that the estimates (1.14) and (1.15) follow from an appropriate isotropic local law for $G(X, z)$ up to the optimal scale $\eta \gg N^{-1}$ (see Section 3). In fact, a generalization of the proof in [3] gives roughly the following estimate (see Theorem 2.8): for any fixed $\epsilon > 0$,

$$|\langle v, G(X, z)v \rangle - m_{1c}(d_N, z)| \leq N^{-\epsilon} \left( N^{-2\phi} + (N\eta)^{-1/2} \right)$$

with extremely high probability for all $\text{Im } z \geq N^{-1+\epsilon}$. Then this estimate will imply (1.15). However, to conclude (1.14), we need a much stronger bound for the expected resolvent, i.e., for any fixed $\epsilon > 0$,

$$|\mathbb{E}\langle v, G(X, z)v \rangle - m_{1c}(d_N, z)| \leq \frac{N^\epsilon}{N\eta}$$

for all $\text{Im } z \geq N^{-1+\epsilon}$. The improvement of the weak bound (1.23) to the almost optimal one in (1.24) constitutes the main novelty of this paper.

A key observation is that (see Section 4.2), after taking the expectation the leading order term in $\Delta m(v, X) := \langle v, G(X, z)v \rangle - m_{1c}(d_N, z)$ vanishes, and hence make $\mathbb{E}\Delta m(v, X)$ to be one order smaller than the bound in (1.23). In other words, we have

$$|\mathbb{E}\langle v, G(X, z)v \rangle - m_{1c}(d_N, z)| \leq N^\epsilon \left( N^{-4\phi} + (N\eta)^{-1} \right).$$
This already gives the estimate (1.24) if $\phi \geq 1/4$. For $X$ satisfying (1.13) for some $\phi < 1/4$, we shall construct another random matrix $\tilde{X}$ which can well approximate $X$ but has bounded entries, i.e. $\max_{i,j} \sqrt{N} |x_{ij}| = O(1)$ (see Lemma 5.1). Then the resolvent of $\tilde{X}\tilde{X}^*$ satisfies (1.24) by taking $\phi = 1/2$ in (1.25). On the other hand, with a resolvent comparison method developed in [23], we will show that the difference between $\mathbb{E}(v, G(X, z)v)$ and $\mathbb{E}(v, G(\tilde{X}, z)v)$ is of order $(N\eta)^{-1}$; see Section 5. This concludes (1.24).

Remark 1.5. It is possible to generalize our proof to more general random matrix models. For example, one may consider sample covariance matrices of the form $Q := (TX)(TX)^*$ ($T$ is a general deterministic rectangular matrix), generalized Wigner matrices (i.e. Wigner ensembles whose entries have non-identical variances) and deformed Wigner matrices of the form $H + A$ ($H$ is a Wigner matrix and $A$ is a deterministic Hermitian matrix). The convergence of VESD of these models will be studied in future works. In particular, we expect that our proof applied to the Wigner matrices can improve the results obtained in [32].

Remark 1.6. For definiteness, we will focus on real sample covariance matrices during the proof. However, our proof also applies, after minor changes, to the complex case if we include the extra assumption (1.2) or (1.12). Also, we will only use $d_N$ (instead of $d$) in the rest of this paper. Correspondingly, we will use the quantities $\rho_{1c}(N), m_{1,2c}(N)$ and $\lambda_{\pm}(N)$, which are obtained by replacing $d$ with $d_N$ in (1.3)-(1.6). For simplicity, we shall always omit the superscript and still call them $\rho_{1c}, m_{1,2c}$ and $\lambda_{\pm}$ in the proof.

The rest of this paper is organized as follows. In Section 2, we introduce the notations and collect some tools that will be used in proving Theorem 1.1. The most important results in this section are Theorem 2.8 and Theorem 2.9, which give the isotropic local law for the resolvent $G(X, z)$. In Section 3, we prove Theorem 1.1 using Theorem 2.8 and Theorem 2.9. Finally, the Theorem 2.8 and Theorem 2.9 are proved in Section 4 and Section 5, respectively.

1.3. Conventions. The fundamental large parameter is $N$, and we regard $M \equiv M_N$ as depending on $N$. All quantities that are not explicitly constant may depend on $N$, and we usually omit the argument $N$ from our notations.

We use $C$ to denote a generic large positive constant, which may depend on some fixed parameters and whose value may change from one line to the next. Similarly, we use $c, \epsilon, \phi, \tau$, etc. to denote generic small positive constants. For two quantities $a_N$ and $b_N$ depending on $N$, the notation $a_N = \ldots$
$O(b_N)$ means that $|a_N| \leq C|b_N|$ for some constant $C > 0$, and $a_N = o(b_N)$ means that $|a_N| \leq c_N|b_N|$ for some positive sequence $\{c_N\}$ with $c_N \to 0$ as $N \to \infty$. We also use the notation $a_N \sim b_N$ if $a_N = O(b_N)$ and $b_N = O(a_N)$. For a matrix $A$, we use $\|A\| := \|A\|_{l^2 \to l^2}$ to denote its operator norm. For a vector $v = (v_i)_{i=1}^n \in \mathbb{C}^n$, $\|v\| \equiv \|v\|_2$ stands for the Euclidean norm of $v$, while $|v| \equiv \|v\|_1$ stands for the $l^1$-norm. We denote the inner product in $\mathbb{C}^n$ by $\langle v, w \rangle = \sum_{i=1}^n \bar{v}_i w_i$.

2. Main Tools.

2.1. Resolvents and local Marčenko-Pastur law. Our study of sample covariance matrices can be performed in a simple and unified fashion using the following $(N + M) \times (N + M)$ self-adjoint matrix $H$, which is a linear function of $X$.

\begin{equation}
H := \begin{pmatrix}
0 & X \\
X^* & 0
\end{pmatrix}.
\end{equation}

For $z \in \mathbb{C}_+$, we define the resolvent for $H$:

\begin{equation}
G(X, z) := \left( -I_{M \times M} X - zI_{N \times N} \right)^{-1},
\end{equation}

and the resolvents (or the Green functions) for $XX^*$ and $X^*X$:

\begin{equation}
\mathcal{G}_1(X, z) := (XX^* - z)^{-1}, \\
\mathcal{G}_2(X, z) := (X^*X - z)^{-1}.
\end{equation}

The Stieltjes transform of the ESD of $XX^*$ is given by

\[ m_1(X, z) := \int \frac{1}{x - z} dF_{XX^*}^{(M)}(x) = \frac{1}{M} \text{Tr} \mathcal{G}_1(X, z). \]

Similarly, we also define $m_2(X, z) := N^{-1} \text{Tr} \mathcal{G}_2(X, z)$. During the proof, we often omit the arguments $X, z$ from our notations.

**Remark 2.2.** Since the nonzero eigenvalues of $X^*X$ and $XX^*$ are identical and $XX^*$ has $M - N$ more (or $N - M$ less) zero eigenvalues, we have

\[ F_{XX^*}^{(M)} = d_N F_{X^*X}^{(N)} + (1 - d_N) \mathbf{1}_{[0, \infty)}, \]

which implies that (see also (1.6))

\begin{equation}
m_2(z) = \frac{d_N^{-1} - 1}{z} + d_N^{-1} m_1(z).
\end{equation}
For simplicity of notations, we define the index sets
\[ I_1 := \{1, \ldots, M\}, \quad I_2 := \{M + 1, \ldots, M + N\}, \quad I := I_1 \cup I_2. \]
We will consistently use the latin letters \( i, j \in I_1 \), greek letters \( \mu, \nu \in I_2 \), and \( a, b \in I \). Then we label the indices of \( X \) according to
\[ X = (X_{i\mu} : i \in I_1, \mu \in I_2). \]

Using Schur complement formula, it is easy to check that
\[ G = \begin{pmatrix} zG_1 & G_1 X \\ X^* G_1 & G_2 \end{pmatrix} = \begin{pmatrix} zG_1 & XG_2 \\ G_2 X^* & G_2 \end{pmatrix}. \]
Thus a control of \( G \) yields a control of the resolvents \( G_1 \) and \( G_2 \). Moreover, we have
\[ m_1 = \frac{1}{Mz} \sum_{i \in I_1} G_{ii}, \quad m_2 = \frac{1}{N} \sum_{\mu \in I_2} G_{\mu\mu}. \]

We will consistently use the notation \( E + i\eta \) for the spectral parameter \( z \). In the following proof, we always assume that \( z \) lies in the spectral domain
\[ D(\zeta, N) := \{ z \in \mathbb{C}_+ : \max(\zeta, \lambda_-/2) \leq E \leq 2\lambda_+, N^{-1+\zeta} \leq \eta \leq \zeta^{-1} \}, \]
for some small constant \( \zeta > 0 \), unless otherwise indicated. Note that if \( d_N \to d \) for some constant \( d \neq 1 \), then by (1.3) we have \( \lambda_- \sim 1 \) when \( N \) is sufficiently large. Thus we can always take \( \zeta \) to be sufficiently small such that \( \zeta \leq \lambda_-/2 \). We define the distance to the spectral edges as
\[ \kappa := \min\{|E - \lambda_+|, |E - \lambda_-|\}. \]
The next lemma gives some basic properties of \( m_{1,2c} \), which can be proved through direct calculations using (1.5) and (1.6).

**Lemma 2.3.** For \( z \in D \), we have
\[ |m_{1,2c}(z)| \sim 1, \quad \text{Im} m_{1,2c}(z) \sim \begin{cases} \eta/\sqrt{\kappa + \eta}, & \text{if } E \notin [\lambda_-, \lambda_+] \\ \sqrt{\kappa + \eta}, & \text{if } E \in [\lambda_-, \lambda_+] \end{cases}. \]

We will use the following notion of stochastic domination, which was first introduced in [9] and subsequently used in many works on random matrix theory, such as [3, 4, 10, 11, 22]. It simplifies the presentation of the results and their proofs by systematizing statements of the form “\( \xi \) is bounded by \( \zeta \) with high probability up to a small power of \( N \)”. 

Definition 2.4 (Stochastic domination). (i) Let
\[ \xi = \{\xi^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)}\}, \quad \zeta = \{\zeta^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)}\} \]
be two families of nonnegative random variables, where \(U^{(N)}\) is a possibly \(N\)-dependent parameter set. We say \(\xi\) is stochastically dominated by \(\zeta\), uniformly in \(u\), if for any (small) \(\epsilon > 0\) and (large) \(D > 0\),
\[ \sup_{u \in U^{(N)}} \mathbb{P}\left[\xi^{(N)}(u) > N^{\epsilon} \zeta^{(N)}(u)\right] \leq N^{-D} \]
for large enough \(N \geq N_0(\epsilon, D)\). Throughout this paper the stochastic domination will always be uniform in all parameters that are not explicitly fixed (such as matrix indices, deterministic vectors, and spectral parameter \(z \in \mathbb{D}\)). Note that \(N_0(\epsilon, D)\) may depend on quantities that are explicitly constant, such as \(d, C_1\) and \(\phi\) in Theorem 1.1.

(ii) If \(\xi\) is stochastically dominated by \(\zeta\), uniformly in \(u\), we use the notation \(\xi \prec \zeta\). Moreover, if for some complex family \(\xi\) we have \(|\xi| \prec \zeta\), we also write \(\xi \prec \zeta\) or \(\xi = O \prec (\zeta)\).

(iii) We say that an event \(\Xi\) holds with high probability if \(1 - 1(\Xi) \prec 0\).

The following lemma collects basic properties of stochastic domination \(\prec\), which will be used tacitly throughout the proof.

Lemma 2.5 (Lemma 3.2 in [3]). Let \(\xi\) and \(\zeta\) be families of nonnegative random variables.

(i) Suppose that \(\xi(u, v) \prec \zeta(u, v)\) uniformly in \(u \in U\) and \(v \in V\). If \(|V| \leq NC\) for some constant \(C\), then
\[ \sum_{v \in V} \xi(u, v) \prec \sum_{v \in V} \zeta(u, v) \]
uniformly in \(u\).

(ii) If \(\xi_1(u) \prec \zeta_1(u)\) uniformly in \(u \in U\) and \(\xi_2(u) \prec \zeta_2(u)\) uniformly in \(u \in U\), then
\[ \xi_1(u)\xi_2(u) \prec \zeta_1(u)\zeta_2(u) \]
uniformly in \(u \in U\).

(iii) Suppose that \(\Psi(u) \geq N^{-C}\) is deterministic and \(\xi(u)\) satisfies \(E\xi(u)^2 \leq NC\) for all \(u\). Then if \(\xi(u) \prec \Psi(u)\) uniformly in \(u\), we have
\[ E\xi(u) \prec \Psi(u) \]
uniformly in \(u\).
Definition 2.6 (Bounded support condition). We say a family of random matrices $X$ satisfy the bounded support condition with $q$, if

$$\max_{i \in I_1, \mu \in I_2} |X_{i\mu}| < q.$$  \hspace{1cm} (2.9)

Here $q \equiv q(N)$ is deterministic and usually satisfies $N^{-1/2} \leq q \leq N^{-\phi}$ for some (small) constant $\phi > 0$. Whenever (2.9) holds, we say that $X$ has support $q$.

Remark 2.7. If the entries of $X$ satisfy (1.13), then $X$ trivially satisfies the bounded support condition with $q = N^{-\phi}$. If we assume that $\sqrt{N}X_{i\mu}$ has arbitrarily high moments, i.e. for any $p \in \mathbb{N}$ there is a constant $C_p$ such that

$$\max_{i,\mu} \mathbb{E}|\sqrt{N}X_{i\mu}|^p \leq C_p.$$  \hspace{1cm} (2.10)

Then by Markov’s inequality, $X$ has support $N^{-1/2}$.

We define the deterministic limit

$$\Pi(z) := \begin{pmatrix} zm_{1c}(z)I_{M \times M} & 0 \\ 0 & m_{2c}(z)I_{N \times N} \end{pmatrix},$$  \hspace{1cm} (2.11)

and the control parameter

$$\Psi(z) := \sqrt{\frac{\text{Im} (m_{1c} + m_{2c})}{N\eta}} + \frac{1}{N\eta}.$$  \hspace{1cm} (2.12)

Note that by (2.8), we always have

$$\Psi \gtrsim N^{-1/2}, \quad \Psi^2 \lesssim (N\eta)^{-1},$$  \hspace{1cm} (2.13)

for $z \in D$. Now we are ready to state the local laws for the resolvent $G(X, z)$.

Theorem 2.8 (Local MP law). Let $X$ be an $M \times N$ real random matrix whose entries are independent random variables satisfying (1.9), (1.10), (1.11) and the bounded support condition (2.9) with $q \leq N^{-\phi}$ for some constant $\phi > 0$. Then the following estimates hold for all $z \in D$:

(1) the averaged local law:

$$|m_1(X, z) - m_{1c}(z)| \lesssim \frac{1}{N\eta};$$  \hspace{1cm} (2.14)
(2) the isotropic local law: for all deterministic unit vectors $u, v \in \mathbb{C}^I$,

\begin{equation}
|\langle u, G(X, z)v \rangle - \langle u, \Pi(z)v \rangle| \prec q + \Psi(z);
\end{equation}

(3) for all deterministic unit vector $v \in \mathbb{C}^{I_1}$,

\begin{equation}
|\langle v, G_1(X, z)v \rangle - m_{1c}| \prec q^2 + \sqrt{\frac{1}{N\eta}},
\end{equation}

and

\begin{equation}
|\mathbb{E}\langle v, G_1(X, z)v \rangle - m_{1c}(z)| \prec q^4 + \frac{1}{N\eta}.
\end{equation}

All of the above estimates are uniform in the spectral parameter $z$ and the deterministic vectors $u, v$.

The proof for Theorem 2.8 will be given in Section 4. Here we make some comments on the above estimates.

If we assume (1.1) (instead of (1.9) and (1.10)) and $q = N^{-1/2}$, then (2.14) and (2.15) have been proved in [3]. If we have (1.1) and $q \leq N^{-\phi}$, then it was proved in Lemma 3.11 and Theorem 3.14 of [8] that the averaged local law (2.14) and the entrywise local law

\begin{equation}
\max_{a,b \in I} |G_{ab}(X, z) - \Pi_{ab}(z)| \prec q + \Psi(z)
\end{equation}

hold uniformly in $z \in D$. With (2.18) and the moment assumption (1.11), one can repeat the arguments in [3, Section 5] or [31, Section 5] to get the isotropic local law (2.15). The main novelties of this Theorem are the bounds (2.16) and (2.17). The bound (2.16) is relatively easier to prove. In fact, if we only consider the upper left and lower right blocks of $G(X, z)$, we can get the following version of the entrywise law:

\begin{equation}
\max_{r=1,2, a,b \in I_r} |G_{ab}(X, z) - \Pi_{ab}(z)| \prec q^2 + \frac{1}{N\eta},
\end{equation}

which can be proved easily with (2.18) (see Appendix A). Then with (2.19) and (1.11), we can apply the arguments in [3, Section 5] to conclude the isotropic local law (2.16) (see Appendix B).

On the other hand, the improvement from (2.16) to (2.17) is more crucial, and is the main reason why we can improve the bound in [33] to the almost optimal one in (1.14). In fact, the leading order term of $\langle v, G_1v \rangle - m_{1c}$ vanishes after taking expectation, and hence leads to a bound that is one
order smaller than the one in (2.16). The proof of (2.17) will be given in Sections 4.2-4.4, which constitutes the main novelty of this paper.

Finally, if the variance assumption in (1.1) is relaxed to the one in (1.10), we can repeat the previous arguments to get the desired estimates (2.14)-(2.17). In fact, it is easy to check that the $O(N^{-2-\alpha})$ term leads to a negligible error at each step, and the whole proof remains unchanged. The relaxation of the mean zero assumption in (1.1) to (1.9) is a little more involved, which will be handled with a centralization argument in Section 4.1.

If $q = N^{-1/4+\epsilon}$ for some sufficiently small constant $\epsilon > 0$, then (2.16) and (2.17) already give that

$$|\langle v, G_1(X, z)v \rangle - m_{1c}(z)| < \sqrt{\frac{1}{N\eta}}, \quad |\mathbb{E}\langle v, G_1(X, z)v \rangle - m_{1c}(z)| < \frac{1}{N\eta},$$

which is sufficient to conclude Theorem 1.1. However, we observe that the above bound on $|\mathbb{E}\langle v, G_1(X, z)v \rangle - m_{1c}(z)|$ is still valid under a much weaker support assumption. More specifically, we have the following theorem. Its proof will be given in Section 5. The main strategy is a resolvent comparison method that was developed in [23].

**Theorem 2.9.** Let $X$ be an $M \times N$ real random matrix satisfying the assumptions in Theorem 2.8. Then we have

$$(2.20) \quad |\mathbb{E}\langle v, G_1(X, z)v \rangle - m_{1c}(z)| < \frac{1}{N\eta},$$

uniformly in $z \in \mathcal{D}$ and any deterministic unit vector $v \in \mathbb{C}^{I_1}$.

We define the classical location $\gamma_j$ of the $j$-th eigenvalue of $XX^*$ as

$$\int_{\gamma_j}^{+\infty} \rho_{1c}(x)dx = \frac{j}{M}, \quad 1 \leq j \leq K,$$

where $\rho_{1c}$ is defined in (1.3) and $K := \min\{M, N\}$. As a corollary of (2.14), we have the following rigidity of eigenvalues of $XX^*$. For its proof, one can refer to the arguments in [15, Section 5], [11, Section 7] and [27, Section 8].

**Theorem 2.10 (Rigidity of eigenvalues).** Suppose (2.14) holds and $\lambda_\alpha \geq c$ for some constant $c > 0$. Then we have

$$(2.21) \quad |\lambda_j(XX^*) - \gamma_j| \prec (\min\{j, K + 1 - j\})^{-1/3}N^{-2/3}, \quad 1 \leq j \leq K.$$
2.2. \textit{Resolvent estimates.} In this subsection, we collect some useful identities from linear algebra and some simple estimates that follow from Theorem 2.8.

\textbf{Definition 2.11 (Minors).} For $T \subseteq \mathcal{I}$, we define the minor $H^{(T)} := (H_{ab} : a, b \in \mathcal{I} \setminus T)$ obtained by removing all rows and columns of $H$ indexed by $a \in T$. Note that we keep the names of indices when defining $H^{(T)}$, i.e. $H^{(T)}_{ab} := 1_{\{a, b \in T\}} H_{ab}$. Correspondingly, we define the Green function

$$G^{(T)} := (H^{(T)})^{-1} = \begin{pmatrix} zG^{(T)}_{11} & G^{(T)}_{12} \\ G^{(T)}_{21} & G^{(T)}_{22} \end{pmatrix} = \begin{pmatrix} zG^{(T)}_{11} & XG^{(T)}_{12} \\ G^{(T)}_{21} & X^*G^{(T)}_{22} \end{pmatrix},$$

and the partial traces

$$m_1^{(T)} := \frac{1}{M} \text{Tr} G_1^{(T)} = \frac{1}{M} \sum_{i \in \mathcal{I}_1} G^{(T)}_{ii}, \quad m_2^{(T)} := \frac{1}{N} \text{Tr} G_2^{(T)} = \frac{1}{N} \sum_{\mu \in \mathcal{I}_2} G^{(T)}_{\mu\mu}.$$ 

We will abbreviate $(\{a\}) \equiv (a)$ and $(\{a, b\}) \equiv (ab)$ in the proof.

\textbf{Lemma 2.12 (Resolvent identities).} \begin{enumerate}[(i)] 
\item For $i \in \mathcal{I}_1$ and $\mu \in \mathcal{I}_2$, we have

$$G^{(T)}_{ii} = -1 - \left( XG^{(i)}_{\mu\mu} \right)_{ii}, \quad G^{(T)}_{\mu\mu} = -z - \left( X^*G^{(\mu)}_{\mu\mu} \right)_{\mu\mu}. \quad (2.22)$$

\item For $i \neq j \in \mathcal{I}_1$ and $\mu \neq \nu \in \mathcal{I}_2$, we have

$$G_{ij} = G^{(i)}_{ij} G^{(i)}_{jj} \left( XG^{(j)}_{\mu\mu} \right)_{ij}, \quad (2.23)$$

and

$$G_{\mu\nu} = G^{(\mu)}_{\mu\nu} G^{(\mu)}_{\nu\nu} \left( X^*G^{(\mu\mu)}_{\mu\nu} \right)_{\mu\nu}. \quad (2.24)$$

\item For $a \in \mathcal{I}$ and $b, c \in \mathcal{I} \setminus \{a\}$,

$$G_{bc} = G^{(a)}_{bc} G^{(b)}_{ca} G^{(a)}_{aa} = \frac{1}{G_{bb}} = \frac{1}{G^{(b)}_{bb}} - \frac{G^{(a)}_{ab}}{G^{(a)}_{bb} G^{(a)}_{aa}}. \quad (2.25)$$

\item All of the above identities hold for $G^{(T)}$ instead of $G$ for $T \subseteq \mathcal{I}$.
\end{enumerate}

\textbf{Proof.} The above identities can be proved using Schur complement formula. The reader can refer to e.g. [3, Lemmas 3.6 and 3.8] or [22, Lemma 4.4].
Lemma 2.13. Suppose $\Phi(z)$ is a deterministic function on $\mathbb{D}$ satisfying $N^{-1/2} \leq \Phi(z) \leq N^{-c}$ for some constant $c > 0$. Suppose $|G_{ab}(z) - \Pi_{ab}(z)| \prec \Phi(z)$ uniformly in $z \in \mathbb{D}$. Fix an $l \in \mathbb{N}$. Then for any $T \subseteq \mathcal{I}$ with $|T| \leq l$, we have

$$\left| G_{ab}(z) - G_{ab}(T)(z) \right| \prec \Phi^2(z), \quad a, b \in \mathcal{I} \setminus T,$$

and

$$|m_1(z) - m_1(T)(z)| + |m_2(z) - m_2(T)(z)| \prec \Phi^2(z),$$

uniformly in $z \in \mathbb{D}$.

Proof. The bound (2.26) can be proved by repeatedly applying the first resolvent expansion in (2.25) with respect to the indices in $T$ and using the entrywise local law. The bound (2.27) is a trivial consequence of (2.26). □

For $v, w \in \mathbb{C}^\mathcal{I}$, $a \in \mathcal{I}$ and any $I \times I$ matrix $A$, we abbreviate

$$A_{vw} := \langle v, Aw \rangle, \quad A_{va} := \langle v, Ae_a \rangle, \quad A_{aw} := \langle e_a, Aw \rangle,$$

where $e_a$ denotes the standard unit vector in the coordinate direction $a$. We shall call them the generalized matrix entries. We sometimes identify vectors $v \in \mathbb{C}^\mathcal{I}_1$ and $w \in \mathbb{C}^\mathcal{I}_2$ with their natural embeddings $egin{pmatrix} v \\ 0 \end{pmatrix}$ and $egin{pmatrix} 0 \\ w \end{pmatrix}$ in $\mathbb{C}^\mathcal{I}$. The exact meanings will be clear from the context.

Lemma 2.14. For any $M \times N$ matrix $Y$, the following estimates hold for $G \equiv G(Y, z)$ and any $z \in \mathbb{D}$. There exists a constant $C > 0$ such that

$$\|G\| \leq C \eta^{-1}, \quad \|\partial_z G\| \leq C \eta^{-2}.$$

Moreover, for $v \in \mathbb{C}^\mathcal{I}_1$ and $w \in \mathbb{C}^\mathcal{I}_2$, we have the following identities

$$\sum_{i \in \mathcal{I}_1} |G_{vi}|^2 = \sum_{i \in \mathcal{I}_1} |G_{iv}|^2 = \frac{|z|^2}{\eta} \text{Im} \left( \frac{G_{vv}}{z} \right),$$

$$\sum_{i \in \mathcal{I}_1} |G_{wi}|^2 = \sum_{i \in \mathcal{I}_1} |G_{iw}|^2 = G_{ww} + \bar{z} \eta \text{Im} G_{ww},$$

$$\sum_{\mu \in \mathcal{I}_2} |G_{\mu v}|^2 = \sum_{\mu \in \mathcal{I}_2} |G_{\mu v}|^2 = \frac{\text{Im} G_{ww}}{\eta},$$

$$\sum_{\mu \in \mathcal{I}_2} |G_{v\mu}|^2 = \sum_{\mu \in \mathcal{I}_2} |G_{v\mu}|^2 = \frac{G_{vv}}{z} + \frac{\bar{z}}{\eta} \text{Im} \left( \frac{G_{vv}}{z} \right).$$

These estimates remain true for $G(T)$ instead of $G$ for any $T \subseteq \mathcal{I}$.
Proof. These estimates and identities can be proved through simple calculations using (2.5) and the spectral decomposition of $G$. The reader can also refer to, for example, [22, Lemma 4.6], [31, Lemma 3.5] and [8, Lemma A.3].

Suppose (2.15) holds. Then using (2.30)-(2.33) and (2.26), it is easy to verify that

$$\max \left\{ \sum_i |G_{vi}^{(T)}|^2, \sum_i |G_{i\nu}^{(T)}|^2, \sum_{\mu} |G_{\mu\nu}^{(T)}|^2 \right\} < \eta^{-1},$$

for any deterministic unit vector $v \in \mathbb{C}^I$ and $T \subseteq I$ with fixed length.

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1 using Theorems 2.8-2.10. The following arguments have been used in previous papers to control the Kolmogorov distance between the ESD of a random matrix and the limiting law. For example, the reader can refer to [14, Lemma 6.1] and [27, Lemma 8.1]. By the remark below (2.6), we can choose the constant $\zeta > 0$ such that $\lambda - / 2 > \zeta$ for all sufficiently large $N$.

Proof of (1.14). The key inputs of the proof are the bounds (2.20) and (2.21). Suppose $\langle v, G_1(X, z)v \rangle$ is the Stieltjes transform of $\hat{\rho}_v$. Then we define

$$\hat{n}_v(E) := \int 1_{[0,E]}(x)\hat{\rho}_v dx, \quad n_c(E) := \int 1_{[0,E]}(x)\rho_{1c} dx,$$

and $\rho_v := E\hat{\rho}_v, n_v := E\hat{n}_v$. Hence we would like to bound

$$\|F_{X,X'}(v, \cdot) - F_{d_N}(\cdot)\| = \sup_E |n_v(E) - n_c(E)|.$$

For simplicity, we denote $\Delta \rho := \rho_v - \rho_{1c}$ and its Stieltjes transform by

$$\Delta m(z) := \mathbb{E}\langle v, G_1(X, z)v \rangle - m_{1c}(z).$$

Let $\chi(y)$ be a smooth cutoff function with support in $[-1,1]$, with $\chi(y) = 1$ for $|y| \leq 1/2$ and with bounded derivatives. Fix $\eta_0 = N^{-1+\zeta}$ and $3\lambda_- / 2 \leq E_1 < E_2 \leq 3\lambda_+ / 2$. Let $f \equiv f_{E_1,E_2,\eta}$ be a smooth function supported in $[E_1 - \eta, E_2 + \eta]$ such that $f(x) = 1$ if $x \in [E_1 + \eta, E_2 - \eta]$, and $|f'| \leq C\eta_0^{-1}, |f''| \leq C\eta_0^{-2}$ if $|x - E_i| \leq \eta_0$. Using the Helffer-Sjöstrand calculus (see e.g. [6]), we have

$$f(E) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{iyf''(x)\chi(y) + i(f(x) + iyf'(x))\chi'(y)}{E - x - iy} dx dy.$$
Then we obtain that
\[
\left| \int f(E) \Delta \rho(E) dE \right| 
\]
(3.2) \[\leq C \int_{\mathbb{R}^2} \left( |f(x)| + |y| |f'(x)| \right) |x'(y)||\Delta m(x + iy)| dxdy \]
(3.3) \[+ C \sum_i \left| \int_{|y| \leq \eta_0} \int_{|x - E_i| \leq \eta_0} y f''(x) \chi(y) \text{Im} \Delta m(x + iy) dxdy \right| \]
(3.4) \[+ C \sum_i \left| \int_{|y| \geq \eta_0} \int_{|x - E_i| \leq \eta_0} y f''(x) \chi(y) \Delta m(x + iy) dxdy \right|. \]
By (2.20) with \( \eta = \eta_0 \), we have
\[
\eta_0 \text{Im} \mathbb{E}(\mathbf{v}, \mathcal{G}_1(X, E + i\eta_0)\mathbf{v}) \prec N^{-1+\zeta}. \]
Since \( \eta \text{Im} \mathbb{E}(\mathbf{v}, \mathcal{G}_1(X, E + i\eta)\mathbf{v}) \) and \( \eta \text{Im} m_{1c}(E + \eta) \) are increasing with \( \eta \), we obtain that
\[
\eta |\text{Im} \Delta m(E + \eta)| \prec N^{-1+\zeta} \quad \text{for all } 0 \leq \eta \leq \eta_0. \]
Moreover, since \( G(X, z)^* = G(X, \bar{z}) \), the estimates (2.20) and (3.6) also hold for \( z \in \mathbb{C}_- \).

Now we bound the terms (3.2), (3.3) and (3.4). Using (2.20) and that the support of \( \chi' \) is in \( 1 \geq |y| \geq 1/2 \), the term (3.2) is estimated by
\[
\int_{\mathbb{R}^2} \left( |f(x)| + |y| |f'(x)| \right) |x'(y)||\Delta m(x + iy)| dxdy \prec N^{-1}. \]
Using \( |f''| \leq C\eta_0^{-2} \) and (3.6), we can bound the terms in (3.3) by
\[
\left| \int_{|y| \leq \eta_0} \int_{|x - E_i| \leq \eta_0} y f''(x) \chi(y) \text{Im} \Delta m(x + iy) dxdy \right| \prec N^{-1+\zeta}. \]
Finally, we integrate the term (3.4) by parts first in \( x \), and then in \( y \) (and use the Cauchy-Riemann equation \( \partial \text{Im}(\Delta m)/\partial x = -\partial \text{Re}(\Delta m)/\partial y \)) to get that
\[
\int_{y \geq \eta_0} \int_{|x - E_i| \leq \eta_0} y f''(x) \chi(y) \text{Im} \Delta m(x + iy) dxdy 
\]
(3.9) \[= - \int_{|x - E_i| \leq \eta_0} \eta_0 \chi'(\eta_0) f'(x) \text{Re} \Delta m(x + i\eta_0) dx \]
(3.10) \[- \int_{y \geq \eta_0} \int_{|x - E_i| \leq \eta_0} (y \chi'(y) + \chi(y)) f'(x) \text{Re} \Delta m(x + iy) dxdy. \]
We bound the term in (3.9) by $O_{g}(N^{-1})$ using (2.20) and $|f'| \leq C\eta_0^{-1}$. The first term in (3.10) can be estimated by $O_{g}(N^{-1})$ as in (3.7). For the second term in (3.10), we again use (2.20) and $|f'| \leq C\eta_0^{-1}$ to get that

$$\left| \int_{y \geq \eta_0} \int_{|x-E_i| \leq \eta_0} \chi(y)f'(x)\Re \Delta m(x+iy)dxdy \right| < \int_{\eta_0}^{1} \frac{1}{Ny} dy < N^{-1}.$$ 

Combining the above estimates, we obtain that

$$\left| \int_{y \geq \eta_0} \int_{|x-E_i| \leq \eta_0} yf''(x)\chi(y)\Im \Delta m(x+iy)dxdy \right| < N^{-1}.$$ 

Obviously, the same estimate also holds for the $y \leq -\eta_0$ part. Together with (3.7) and (3.8), we conclude that

$$\left(3.11\right) \quad \left| \int f(E)\Delta \rho(E)dE \right| < N^{-1+\zeta}.$$ 

For any interval $I := [E - \eta_0, E + \eta_0]$ with $E \in [\lambda_+/2, 2\lambda_+]$, we have

$$\hat{n}_{\nu}(E + \eta_0) - \hat{n}_{\nu}(E - \eta_0) = \sum_{\lambda_k \in (E - \eta_0, E + \eta_0)} |\langle \xi_k, \nu \rangle|^2 \leq 2\eta_0 \sum_{k=1}^{M} \frac{|\langle \xi_k, \nu \rangle|^2\eta_0}{(\lambda_k - E)^2 + \eta_0^2} = 2\eta_0 \Im \langle \nu, G_1(X, E + i\eta_0)\nu \rangle,$$

where we used the spectral decomposition

$$G_1(X, E + i\eta) = \sum_{k=1}^{M} \frac{\xi_k\xi_k^*}{\lambda_k - E - i\eta},$$

which follows from (1.7). Then by (3.5) and Lemma 2.5, we get that

$$\left(3.13\right) \quad n_{\nu}(E + \eta_0) - n_{\nu}(E - \eta_0) < N^{-1+\zeta}.$$ 

On the other hand, we trivially have

$$\left(3.14\right) \quad n_{c}(E + \eta_0) - n_{c}(E - \eta_0) \leq C\eta_0 = CN^{-1+\zeta}$$

since $\rho_{1c}(x)$ is bounded for $x$ away from 0.

Now we set $E_2 = 3\lambda_+/2$. With (3.11), (3.13) and (3.14), we get that for any $E \in [\lambda_+/4, E_2]$, 

$$\left(3.15\right) \quad |(n_{\nu}(E_2) - n_{\nu}(E)) - (n_{c}(E_2) - n_{c}(E))| < N^{-1+\zeta}.$$
Note that by (2.21), the eigenvalues of $XX^*$ are inside $\{0\} \cup [3\lambda_-/4, E_2]$ with high probability. Hence we have with high probability,

$$\hat{n}_v(E_2) = n_c(E_2) = 1, \quad \hat{n}_v(3\lambda_-/4) = \hat{n}_v(0).$$

Together with (3.15), we get that

$$\sup_{E \geq 0} |n_v(E) - n_c(E)| \prec N^{-1+\zeta}. \tag{3.17}$$

This concludes (1.14) since $\zeta$ can be arbitrarily small.

**Proof of (1.15).** The proof for (1.15) is similar except that we shall use the estimate (2.16) instead of (2.20). By (2.16), we have

$$|\langle v, G_1(X, z)v \rangle - m_1c(z)| \prec N^{-2\phi} + (N\eta)^{-1/2} \tag{3.18}$$

uniformly in $z \in D$. Then we would like to bound (recall (3.1))

$$\|F^{(M)}_{XX^*}(v, \cdot) - F_{dN}(\cdot)\| = \sup_E |\hat{n}_v(E) - n_c(E)|,$$

where $\hat{n}_v$ is defined in (3.1). We denote

$$\Delta \hat{\rho} := \hat{\rho}_v - \rho_{1c}, \quad \Delta \hat{m} := \langle v, G_1(X, z)v \rangle - m_1c(z).$$

Then for $f_{E_1, E_2, \eta_0}$ defined in the previous proof, we can repeat the Helffer-Sjöstrand argument with the estimate (3.18) to get that

$$\sup_{E_1, E_2} \left| \int f_{E_1, E_2, \eta_0}(E)\Delta \hat{\rho}(E)dE \right| \prec N^{-2\phi} + N^{-1/2}, \tag{3.19}$$

which, together with (3.12) and (3.16), implies

$$\sup_{E \geq 0} |\hat{n}_v(E) - n_c(E)| \prec N^{-2\phi} + N^{-1/2}.$$ 

This concludes (1.15) by the Definition 2.4.

4. **Proof of Theorem 2.8.**
4.1. Centralization. For $X$ satisfying the assumptions in Theorem 2.8, we write $X = X_1 + B$, where $X_1 := X - EX$ is a random matrix satisfying (1.10), (1.11) and

\[(4.1) \quad E(X_1)_{i\mu} = 0, \quad i \in I_1, \mu \in I_2, \]

and $B := EX$ is a deterministic matrix such that

\[(4.2) \quad \max_{i,\mu} |B_{i\mu}| \leq C_0 N^{-2-c_0}. \]

**Lemma 4.1.** If Theorem 2.8 holds for $X_1$, then it also holds for $X$.

**Proof.** For $z \in D$, we have

\[(4.3) \quad G(X, z) := \begin{pmatrix} -I_{M \times M} & X_1 + B \\ X_1^* + B^* & -zI_{N \times N} \end{pmatrix}^{-1} = (G_1^{-1} + V)^{-1}, \]

where we abbreviate $G_1(z) := G(X_1, z)$ and $V := \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$. By our assumption, (2.15) holds for $G_1$. Then we expand $G$ using the resolvent expansion

\[(4.4) \quad G = G_1 - G_1 V G_1 + (G_1 V)^2 G_1 - (G_1 V)^3 G. \]

For any unit vectors $v, u \in \mathbb{C}^I$, we have

\[(4.5) \quad |\langle v, G_1 V G_1 u \rangle| \leq \sum_{b \in I} \sum_{a \in I} |(G_1)_{va} V_{ab}| |(G_1)_{bu}| \]

\[\leq \max_b \left( \sum_{a \in I} |V_{ab}|^2 \right)^{1/2} \sum_{b \in I} |(G_1)_{bu}| \]

\[\leq N^{-1-c_0} \left( \sum_{b \in I} |(G_1)_{bu}|^2 \right)^{1/2} \leq N^{-1-c_0} \eta^{-1/2}, \]

where in the second step we used (2.15) for $G_1$, in the third step the Cauchy-Schwarz inequality and (4.2), and in the last step (2.34). With a similar argument, we obtain that

\[(4.6) \quad |\langle v, (G_1 V)^2 G_1 u \rangle| \leq N^{-2-2c_0} \eta^{-1}. \]

Combining this estimate with the rough bound (2.29) for $G$, we get that

\[(4.7) \quad |\langle v, (G_1 V)^3 G u \rangle| = \left| \sum_{a,b} ((G_1 V)^2 G_1)_{va} V_{ab} G_{bu} \right| \]

\[\leq \left( N^{-2-2c_0} \eta^{-1} \right)^{1/2} \sum_a \left( \sum_b |V_{ab}|^2 \right)^{1/2} \leq C N^{-3/2-3c_0} \eta^{-1}, \]
where we used $\eta \geq N^{-1}$ for $z \in D$. Plugging the estimates (4.5)-(4.7) into (4.4), we conclude that
\begin{equation}
|\langle v, Gu \rangle - \langle v, G_1 u \rangle| \prec N^{-1-\omega/4} \eta^{-1/2} \leq (N\eta)^{-1}
\end{equation}
for all deterministic unit vectors $v, u \in \mathbb{C}^I$. We can then easily conclude the lemma with this estimate. \hfill \Box

Thus in the following proof, we can assume that the entries of $X$ are centered without loss of generality. According to the comments below Theorem 2.8, we can repeat the proof in [8] to get (2.14) and the entrywise local law (2.18). Then combining (2.18), the moment assumption (1.11) and the arguments in [3, Section 5], we can obtain (2.15) (see also the proof for Lemma 4.3 in Appendix B). The bound (2.16) follows from Lemma 4.1 and the next two lemmas.

**Lemma 4.2.** Let $X$ be an $M \times N$ real random matrix whose entries are independent random variables satisfying (4.1), (1.10), (1.11) and the bounded support condition (2.9) with $q \leq N^{-\phi}$ for some constant $\phi > 0$. Then if (2.14) and (2.18) holds, the local law (2.19) also holds for all $z \in D$.

**Lemma 4.3.** Suppose the assumptions in Lemma 4.2 hold for $X$. Suppose $\Phi(z)$ is a deterministic function on $D$ satisfying $c_0(N^{-1/2} + q^2) \leq \Phi(z) \leq N^{-c_0}$ for some constant $c_0 > 0$. If we have
\begin{equation}
\max_{a,b \in \mathcal{I}} |G_{ab}(z) - \Pi_{ab}(z)|^2 \prec \Phi, \quad \max_{r=1,2} \max_{a,b \in \mathcal{I}_r} |G_{ab}(z) - \Pi_{ab}(z)| \prec \Phi
\end{equation}
for all $z \in D$, then
\begin{equation}
|\langle v, G_1 (X, z) v \rangle - m_{1c}| \prec \Phi(z)
\end{equation}
uniformly in all $z \in D$ and all deterministic unit vector $v \in \mathbb{C}^{I_1}$.

We will give the proof of Lemma 4.2 and Lemma 4.3 in appendix. In the rest of this section, we focus on proving our main estimate (2.17). For simplicity, we denote $\Phi := q^2 + (N\eta)^{-1/2}$ in the proof below. Also, by Lemma 4.1, we can assume that the entries of $X$ are centered.

**4.2. Sketch of the proof for (2.17).** We want to estimate $|\mathbb{E}\langle v, G_1 v \rangle - m_{1c}|$ for any deterministic unit vector $v \in \mathbb{C}^{I_1}$. Note that (2.16) gives the a priori bound
\begin{equation}
\left| \sum_{i,j} \bar{v}_i v_j \mathbb{E}(G_1)_{ij} - m_{1c} \right| \prec \Phi.
\end{equation}
We will show that after taking expectation, the leading order term in \((G_1)_{ij} - m_{1c} \delta_{ij}\) vanishes and gives the improved estimate (2.17). We deal with the diagonal and off-diagonal parts separately:

\[
\sum_i |v_i|^2 [E(G_1)_{ii} - m_{1c}(z)], \quad \sum_{i \neq j} \bar{v}_i v_j E(G_1)_{ij}.
\]

For any \(T \subseteq I\), we define the \(Z\) variables

\[
Z_i^{(T)} := (1 - E_i)(G^{(T)})^{-1}_{ii} = m_2^{(Ti)} - (XG^{(Ti)}X^*)_{ii}, \quad i \notin T,
\]

where \(E_i[\cdot] := E[\cdot|H^{(i)}]\), i.e. it is the partial expectation in the randomness of the \(i\)-th row and column of \(H\), and we used (2.22) in the second step. If \(T = \emptyset\), we abbreviate \(Z_i \equiv Z_i^{(\emptyset)}\). By (A.10), we have \(|Z_i| \prec \Phi\). Then using (2.22) we get that

\[
E G_{ii} - zm_{1c} = \frac{1}{-1 - m_{2c} - (m_2^{(i)} - m_{2c}) + Z_i} - zm_{1c}
\]

\[
= \frac{1}{-1 - m_{2c}} - zm_{1c} - \frac{1}{(1 + m_{2c})^2}EZ_i + O_{\prec} \left( \frac{\Phi^2}{N}\right) = O_{\prec} \left( \frac{\Phi^2}{N}\right),
\]

where in the second step we used the bound for \(Z_i\), (2.14) and (2.27), and in the third step we used (1.4), (1.6) and \(E Z_i = 0\). Thus we can bound the diagonal part by

\[
\sum_i |v_i|^2 [E(G_1)_{ii} - m_{1c}(z)] = \frac{1}{z} \sum_i |v_i|^2 [E G_{ii} - zm_{1c}(z)] \prec q^4 + \frac{1}{N\eta}
\]

for \(z \in D\) (recall that \(|z| \geq E \sim 1\) by (2.6)).

For the off-diagonal part, we claim that for all \(i \neq j \in \mathcal{I}_1\),

\[
E(G_1)_{ij} \prec N^{-1} \Phi^2.
\]

Then using (4.13) and \(\|v\|_1 \leq \sqrt{M}\), we obtain that

\[
\left| \sum_{i \neq j} \bar{v}_i v_j E(G_1)_{ij} \right| \prec \|v\|_1^2 N^{-1} \Phi^2 \leq C \left( q^4 + \frac{1}{N\eta} \right),
\]

Together with (4.12), this concludes (2.17).

To prove (4.13), we follow the arguments in [3, Section 5] and [31, Section 5]. We illustrate the basic idea with some simplified calculations. Using the
resolvent identities (2.23) and (2.25), we get
\begin{equation}
\mathbb{E}G_{ij} = \mathbb{E}G_{ii}G_{jj}^{(ij)} \left( XG^{(ij)}X^* \right)_{ij}
\end{equation}
(4.14)
\begin{equation}
= \mathbb{E}G_{ii}^{(j)} G_{jj}^{(i)} \left( XG^{(ij)}X^* \right)_{ij} + \mathbb{E}\frac{G_{ij}G_{ji}}{G_{jj}} G_{jj}^{(i)} \left( XG^{(ij)}X^* \right)_{ij}.
\end{equation}

We now focus on the first term. Applying (2.22) gives that
\begin{equation}
\mathbb{E}G_{ii}^{(j)} G_{jj}^{(i)} \left( XG^{(ij)}X^* \right)_{ij} = \mathbb{E} \frac{\left( XG^{(ij)}X^* \right)_{ij}}{\left[ 1 + (XG^{(ij)}X^*)_{ii} \right] \left[ 1 + (XG^{(ij)}X^*)_{jj} \right]}
\end{equation}
(4.15)
\begin{equation}
= \mathbb{E} \frac{\left( XG^{(ij)}X^* \right)_{ij}}{\left[ (1 + m_{2c}) - \epsilon_i \right] \left[ (1 + m_{2c}) - \epsilon_j \right]},
\end{equation}
where we have \(|(1 + m_{2c})^{-1}| = |zm_{1c}| \sim 1\) and
\begin{equation}
\epsilon_i := m_{2c} - (XG^{(ij)}X^*)_{ii} = m_{2c} - m_{2}^{(ij)} + Z_{i}^{(ij)} \prec \Phi
\end{equation}
(4.16) by (2.16), (2.27) (with \(\Phi = q + \Psi\)) and (A.10). We now expand the fractions in (4.15) in order to take the expectation. Note that the \(G^{(ij)}\) entries are independent of the \(X\) entries in the \(i,j\)-th rows and columns. Thus to attain a nonzero expectation, each \(X\) entry must appear at least twice in the expression. Due to this reason, the leading and next-to-leading order terms in the expansion vanish. The “real” leading order term is
\begin{equation}
\frac{\epsilon_i \epsilon_j \left( XG^{(ij)}X^* \right)_{ij}}{(1 + m_{2c})^4} = \frac{1}{(1 + m_{2c})^4} \mathbb{E}(XG^{(ij)}X^*)_{ii} (XG^{(ij)}X^*)_{jj} (XG^{(ij)}X^*)_{ij}
\end{equation}
(4.17)
\begin{equation}
= \frac{1}{(1 + m_{2c})^4} \sum_{\mu, \nu} C_{\mu, \nu} \mathbb{E}G^{(ij)}_{\mu \mu} G^{(ij)}_{\nu \nu}
\end{equation}
\begin{equation}
= \frac{1}{(1 + m_{2c})^4} \sum_{\mu \neq \nu} \frac{C_{\mu, \nu} m_{2c}^2 \mathbb{E}G^{(ij)}_{\mu \nu}}{N^3} + O_{\prec}(N^{-1}\Phi^2),
\end{equation}
where the constants \(C_{\mu, \nu}\) depend on the moments of \(X_{i\mu}\) and \(X_{j\nu}\) (recall (1.11)). Here in the last step, we used \(|G^{(ij)}_{\mu \mu} - m_{2c}| \prec \Phi\) (by (2.16) and (2.26)), and bounded the \(\mu = \nu\) terms by \(O_{\prec}(N^{-2}) = O_{\prec}(N^{-1}\Phi^2)\). Now applying (2.24) to \(G^{(ij)}_{\mu \nu}\), we get that
\begin{equation}
\mathbb{E}G^{(ij)}_{\mu \nu} = \mathbb{E}G^{(ij)}_{\mu \nu} G^{(ij\mu)} (X^* G^{(ij\mu)} X)_{\mu \nu}
\end{equation}
(4.18)
\begin{equation}
= \mathbb{E} m_{2c}^2 (X^* G^{(ij\mu)} X)_{\mu \nu} + O_{\prec}(\Phi^2) = O_{\prec}(\Phi^2),
\end{equation}
where in the second step we used $|G_{ij}^{(ij)} - m_{2c}| + |G_{ij}^{(ij\mu)} - m_{2c}| \ll \Phi$ and

\[
\left(X^* G^{(ij\mu\nu)} X\right)_{\mu\nu} = G_{ij}^{(ij)} \left(G_{ij\mu}^{(ij\mu)} G_{ij\nu}^{(ij\nu)}\right)^{-1} \ll \Phi,
\]

which follow easily from (2.16) and (2.26), and in the last step the leading order term vanishes since the two $X$ entries are independent for $\mu \neq \nu$. Then by (4.18), the terms in (4.17) are bounded by $O_{\prec}(N^{-1}\Phi^2)$.

In general, after the expansion of the two fractions in (4.15), we get a summation of terms of the form

\[
A_{m,n} := \mathbb{E}_{\epsilon_{i}^{n_{1}}} \epsilon_{j}^{n_{2}} (XG^{(ij)} X^*)_{ij}, \quad i \neq j,
\]

up to some constant coefficients of order 1. Since $|\epsilon_{i,j}| \ll \Phi \lesssim N^{-\zeta/2}$ for $z \in D$ (we can take $\zeta$ small enough such that $N^{-\zeta/2} \geq q^2$), we only need to include the terms with $m + n \leq 2 + 2/\zeta$ and the tail will be smaller than $N^{-1}\Phi^2$. Note that in $A_{m,n}$, the $X_{i\ast}$ entries, $X_{j\ast}$ entries and $G^{(ij)}$ entries are mutually independent. Moreover, both the number of $X_{i\ast}$ entries and the number of $X_{j\ast}$ entries are odd. Thus to attain a nonzero expectation, we must pair the $X$ entries such that there are two products of the forms $X_{n_{1}i\mu}$ and $X_{n_{2}j\nu}$ for some $n_{1}, n_{2} \geq 3$. As a result, we lose $(n_{1} - 2)/2 + (n_{2} - 2)/2 \geq 1$ free indices, which contributes an $N^{-1}$ factor. On the other hand, for the product of $G$ entries, we have three cases: (1) if there are at least 2 off-diagonal $G$ entries, then we bound them with $O_{\prec}(\Phi^2)$; (2) if there is only 1 off-diagonal $G$ entry, then we can use the trick in (4.17) and the bound (4.18); (3) if there is no off-diagonal $G$ entry, then we lose one more free index and get an extra $N^{-1}$ factor. This gives the estimate (4.13) for the term in (4.15).

For the second term in (4.14), we again use (2.22), (2.23) and (2.25) to expand the $G_{ij}, G_{ji}$ and $G_{jj}^{-1}$ entries. Our goal is to expand all the $G$ entries into polynomials of the terms

\[
S_{kl} := (XG^{(ij)} X^*)_{kl}, \quad k, l \in \{i, j\},
\]

so that the $X$ entries and $G^{(ij)}$ entries are independent in the resulting expression. In particular, the maximally expanded terms (see (4.20)) can be expanded into $S_{kl}$ variables directly through (2.22) and (2.23). However, non-maximally expanded terms are also created along the expansions in (2.23) and (2.25). Then we need to further expand these newly appeared terms. In general, this process will not terminate. However, we will show in Lemma 4.7 that after sufficiently many expansions, the resulting expression either has enough off-diagonal terms, or is maximally expanded. In the former case, it suffices to bound each off-diagonal term by $O_{\prec}(\Phi)$. In the latter
case, the expression will only consist of $S_{kl}$ variables. Following the argument in the previous paragraph, the expectation over the $X$ entries produces an $N^{-1}$ factor, while the expectation over the $G$ entries produces a $\Phi^2$ factor.

In the rest of this section, we will give a rigorous proof based on the above arguments.

4.3. Resolvent expansion. To perform the resolvent expansion in a systematic way, we introduce the following notions of string and string operator. Recall the definition of $S_{kl}$ in (4.19).

**Definition 4.4 (Strings).** Let $\mathcal{A}$ be the alphabet containing all symbols that will appear during the expansion:

$$\mathcal{A} = \{ G_{kl}, G_{kk}^{-1}, S_{kl} \text{ with } k, l \in \{ i, j \} \} \cup \left\{ G_{ii}^{(j)}, G_{jj}^{(i)}, (G_{ii}^{(j)})^{-1}, (G_{jj}^{(i)})^{-1} \right\}.$$

We define a string $s$ to be a concatenation of the symbols from $\mathcal{A}$, and we use $\llbracket s \rrbracket$ to denote the random variable represented by $s$. We denote an empty string by $\emptyset$ with value $\llbracket \emptyset \rrbracket = 0$.

**Remark 4.5.** It is important to distinguish the difference between a string $s$ and its value $\llbracket s \rrbracket$. For example, “$G_{ij}$” and “$G_{ii}G_{jj}^{(i)}S_{ij}$” are different strings, but they represent the same random variable by (2.23).

We shall call the following symbols the maximally expanded symbols:

$$\mathcal{A}_{\text{max}} = \{ G_{ij}, G_{ji}, G_{ii}^{(j)}, G_{jj}^{(i)}, (G_{ii}^{(j)})^{-1}, (G_{jj}^{(i)})^{-1}, S_{ii}, S_{jj}, S_{ij}, S_{ji} \}.$$ (4.20)

A string $s$ is said to be maximally expanded if all of its symbols are in $\mathcal{A}_{\text{max}}$. We shall call $G_{ij}, G_{ji}, S_{ij}, S_{ji}$ the off-diagonal symbols and all the other symbols in $\mathcal{A}$ diagonal. By the local law (2.16) and (2.26), we have $[a_o] \prec \Phi$ if $a_o$ is an off-diagonal symbol (note that $S_{ij} = G_{ij}/(G_{ii}G_{jj}^{(i)}) \prec \Phi$ by (2.23)) and $[a_d] \prec 1$ if $a_d$ is a diagonal symbol. We use $F_{n-\text{max}}(s)$ and $F_{\text{off}}(s)$ to denote the number of non-maximally expanded symbols and the number of off-diagonal symbols, respectively.

**Definition 4.6 (String operators).** Let $k \neq l \in \{ i, j \}$.

(i) We define an operator $\tau_0$ acting on a string $s$ in the following sense. Find the first $G_{kk}$ or $G_{kk}^{-1}$ in $s$. If $G_{kk}$ is found, replace it with $G_{kk}^{(l)}$; if $G_{kk}^{-1}$ is found, replace it with $(G_{kk}^{(l)})^{-1}$; if neither is found, set $\tau_0(s) = s$ and we say that $\tau_0$ is trivial for $s$. 

(ii) We define an operator $\tau_1$ acting on a string $s$ in the following sense. Find the first $G_{kk}$ or $G_{kk}^{-1}$ in $s$. If $G_{kk}$ is found, replace it with $G_{kk}G_{lk}(G_{ll})^{-1}$; if $G_{kk}^{-1}$ is found, replace it with $-G_{kl}G_{lk}(G_{kk})^{-1}(G_{ll})^{-1}$; if neither is found, set $\tau_1(s) = \emptyset$ and we say that $\tau_1$ is null for $s$.

(iii) Define an operator $\rho$ acting on a string $s$ in the following sense. Replace each $G_{kl}$ in $s$ with $G_{kk}G_{lk}^{(k)}S_{kl}$.

By Lemma 2.12, it is clear that for any string $s$,

$$[\tau_0(s)] + [\tau_1(s)] = [s], \quad [\rho(s)] = [s].$$

Moreover, a string $s$ is trivial under $\tau_0$ and null under $\tau_1$ if and only if $s$ is maximally expanded. Given a string $s$, we abbreviate $s_0 := \tau_0(s)$ and $s_1 := \rho(\tau_1(s))$. For any sequence $w = a_1a_2\ldots a_m$ with $a_i \in \{0, 1\}$, we denote

$$s_w := \rho^{a_m}a_m\ldots \rho^{a_2}a_2\rho^{a_1}a_1(s), \quad \text{where } \rho^0 := 1.$$ 

Then by (4.21) we have

$$\sum_{|w|=m} [s_w] = [s],$$

where the summation is over all binary sequences $w$ with length $|w| = m$.

**Lemma 4.7.** Consider the string $s = \llbracket G_{ij}G_{jj}^{(i)} S_{ij} \rrbracket$. Let $w$ be any binary sequence with $|w| = 4l_0$ and such that $s_w \neq \emptyset$. Then either $F_{\text{off}}(s_w) \geq 2l_0$ or $s_w$ is maximally expanded.

**Proof.** It suffices to show that any nonempty string $s_w$ with $F_{\text{off}}(s_w) < 2l_0$ is maximally expanded.

By Definition 4.6, a nontrivial $\tau_0$ reduces the number of non-maximally expanded symbols by 1, and keeps the number of off-diagonal symbols the same; a $\rho\tau_1$ increases the number of non-maximally expanded symbols by 2 or 3, and increases the number of off-diagonal symbols by 2. Hence $F_{\text{off}}(s_w) < 2l_0$ implies that there are at most $(l_0 - 1)$’s in $w$. These $\rho\tau_1$ operators increase $F_{\text{off}}$ at most by $3(l_0 - 1)$ in total. On the other hand, there are at least $3l_0$ 0’s in $w$, which is sufficient to eliminate all the non-maximally expanded symbols, whose number is at most $3(l_0 - 1) + 1 = 3l_0 - 2$ in total (note that $F_{\text{off}}(s) = 1$ for the initial string).

Now we choose $l_0 = 1 + 1/\zeta$. Then we have

$$\sum_{|w|=4l_0} [s_w] \cdot 1(F_{\text{off}}(s_w) \geq 2l_0) < 2^{4l_0} \phi^{2l_0} \ll N^{-1}\phi^2.$$
using $\Phi = O(N^{-\epsilon/2})$. By Lemma 4.7, we see that to prove (4.13), it suffices to show that

\[(4.23) \quad |E[\mathbf{s}_w]| \prec N^{-1}\Phi^2\]

for any maximally expanded string $\mathbf{s}_w$ with $|w| = 4l_0$.

Note that the maximally expanded string $\mathbf{s}_w$ thus obtained consists only of the symbols

$$G^{(l)}_{kk}, (G^{(l)}_{kk})^{-1}, S_{kl}, \quad \text{with} \ k \neq l \in \{i, j\}.$$

By (2.22), we can replace $(G^{(l)}_{kk})^{-1}$ with

\[(4.24) \quad (G^{(l)}_{kk})^{-1} = -1 - S_{kk}.\]

Note that $|S_{kk} - m_{2c}| \prec \Phi$ by (4.16). Then we can expand $G^{(l)}_{kk}$ as

\[(4.25) \quad G^{(l)}_{kk} = \frac{1}{-1 - m_{2c} + (m_{2c} - S_{kk})} = \frac{-1}{1 + m_{2c}} \sum_{k=0}^{2l_0} \left( \frac{m_{2c} - S_{kk}}{1 + m_{2c}} \right)^k + O(N^{-1}\Phi^2).\]

We apply the expansions (4.24) and (4.25) to the $G$ symbols in $\mathbf{s}_w$, disregard the sufficiently small tails, and denote the resulting polynomial (in terms of the symbols $S_{kl}$) by $P_w$. Then $P_w$ can be written as a finite sum of maximally expanded strings (or monomials) consisting of the $S_{kl}$ symbols. Moreover, the number of such monomials depends only on $l_0$. Hence it suffices to show that for any such monomial $M_w$, we have

\[(4.26) \quad |E[M_w]| \prec N^{-1}\Phi^2.\]

Let $N_i$ ($N_j$) be the number of times that $i$ ($j$) appears as a (lower) index of the $S$ symbols in $M_w$. We have $N_i = N_j = 3$ for the initial string $\mathbf{s} = "G_{ii}G^{(l)}_{jj}S_{ij}"$. From Definition 4.6, it is easy to see that the operators $\tau_0, \tau_1$ and $\rho$ do not change the parity of $N_i$ and $N_j$. The expansions (4.24) and (4.25) also do not change the parity of $N_i$ and $N_j$. This leads to the following key observation:

\[(4.27) \quad \text{both } N_i \text{ and } N_j \text{ are odd in } M_w.\]
4.4. A graphical proof. In this subsection, we finish the proof of (4.26). Suppose \( M_w = C(z)(S_{ii})^{m_1}(S_{jj})^{m_2}(S_{ij})^{m_3}(S_{ji})^{m_4} \), where \( C(z) \) denotes a deterministic function of order 1 for all \( z \in D \). Then we write

\[
[M_w] \sim \sum_{\mu^*, \nu^* \in \mathcal{I}_2} \prod_{a=1}^{m_1} X_{i\mu_a(1)} G_{\mu_a(1) \nu_a(1)}(1) X^*_{\nu_a(1)i} \prod_{b=1}^{m_2} X_{j\mu_b(2)} G_{\mu_b(2) \nu_b(2)}(2) X^*_{\nu_b(2)j} \prod_{c=1}^{m_3} X_{i\mu_c(3)} G_{\mu_c(3) \nu_c(3)}(3) X^*_{\nu_c(3)i} \prod_{d=1}^{m_4} X_{j\mu_d(4)} G_{\mu_d(4) \nu_d(4)}(4) X^*_{\nu_d(4)j}.
\]

(4.28)

To avoid the heavy expressions, we introduce the following graphical notations. We use a connected graph \((V, E)\) to represent the string \( M_w \), where the vertex set \( V \) consists of the indices in (4.28) and the edge set \( E \) consists of the \( X \) and \( G \) variables. The indices \( i, j \) are represented by the black vertices in the graph, while the \( \mu, \nu \) indices are represented by the white vertices. The \( X \) edges are represented by the zig-zag lines and the \( G \) edges are represented by the straight lines. One can refer to Fig. 1 for an example of such a graph.

![Graphical representation of \( M_w \)](image)

**Fig 1.** The resulting graph after expanding \( S_{ii}(S_{ij})^3(S_{jj})^2 \).

We organize the summation in (4.28) in the following way. We first partition the white vertices into blocks by requiring that any pair of white vertices take the same value if they are in the same block, and take different values otherwise. Then we do the summation over the white blocks which take values in \( \mathcal{I}_2 \). Finally, we sum over all possible partitions. Note that the number of different partitions depends only on the total number of \( S \) variables in \( M_w \), which in turn depends only on \( l_0 \).

Fix a partition \( \Gamma \) of the white vertices. We denote its blocks by \( b_1, \ldots, b_k \), where \( k \) gives the number of distinct blocks in \( \Gamma \). We denote by \( n_i^j (n_j^i) \) the number of white vertices in \( b_l \) that are connected to the vertex \( i \) (\( j \)). Let
$G(\Gamma)$ be the product of all the $G$ edges in the graph. Then we have

\begin{equation}
[M_w] \sim \sum_{\Gamma} \sum_{b_1,\ldots,b_k}^* G(\Gamma) \prod_{l=1}^k (X_{ib_l})^{n_i^l} (X_{jb_l})^{n_j^l},
\end{equation}

where $\sum^*$ denotes the summation subject to the condition that $b_1,\ldots,b_k$ take distinct values. Note that $k$, $b_i$, $n_i^l$ and $n_j^l$ all depend on $\Gamma$, and we have omitted the $\Gamma$ dependence for simplicity of notations.

From (4.28), it is easy to see that the $X$ edges are independent of $G(\Gamma)$. Thus taking expectation of (4.29) gives that

\begin{equation}
|E[M_w]| \leq C \sum_{\Gamma} \sum_{b_1,\ldots,b_k}^* |E[G(\Gamma)]| \prod_{l=1}^k |E[X_{ib_l}]|^{n_i^l} |E[X_{jb_l}]|^{n_j^l} 1(n_i^l \neq 1, n_j^l \neq 1),
\end{equation}

Note that we must have $n_i^l + n_j^l \geq 2$ for $1 \leq l \leq k$, because we only consider nonempty blocks. On the other hand, if all $n_i^l$ are even, then $N_i = \sum_{l=1}^k n_i^l$ must be even, which contradicts (4.27). Hence we can find some $1 \leq l_1 \leq k$ such that $n_i^{l_1}$ is odd and $n_i^{l_1} \geq 3$. Similarly, we can also find some $1 \leq l_2 \leq k$ such that $n_j^{l_2}$ is odd and $n_j^{l_2} \geq 3$. We abbreviate $\hat{n}_i^l := n_i^l \wedge 3$ and $\hat{n}_j^l := n_j^l \wedge 3$.

From the above discussions, we see that

\begin{equation}
\frac{1}{2} \sum_{l=1}^k (\hat{n}_i^l + \hat{n}_j^l) \geq \frac{1}{2} \sum_{l \neq l_1,l_2} (\hat{n}_i^l + \hat{n}_j^l) + \frac{3}{2} + \frac{3}{2} \geq (k - 2) + 3 = k + 1.
\end{equation}

Now using the moment assumption (1.11), we can bound (4.30) by

\begin{equation}
|E[M_w]| \leq C \sum_{\Gamma} \sum_{b_1,\ldots,b_k}^* |E[G(\Gamma)]| N^{-\frac{1}{2}} \sum_{l=1}^k (\hat{n}_i^l + \hat{n}_j^l)/2.
\end{equation}

Next we deal with $|E[G(\Gamma)]|$. We consider the following 3 cases separately:

(1) there are at least 2 off-diagonal $G$-edges in $G(\Gamma)$;
(2) there is only 1 off-diagonal $G$-edge in $G(\Gamma)$;
(3) there is no off-diagonal $G$-edge in $G(\Gamma)$.

In case (1), we trivially have $|E[G(\Gamma)]| < \Phi^2$, because the diagonal edges are of order $O_\prec(1)$, while the off-diagonal edges are of order $O_\prec(\Phi)$.
In case (2), we use the same trick as in (4.17). Let the off-diagonal $G$-edge be $G_{\alpha\beta}^{(ij)}$. For each diagonal $G_{\alpha\alpha}^{(ij)}$, we replace it with

$$(G_{\alpha\alpha}^{(ij)} - m_{2c}) + m_{2c} = m_{2c} + O(\Phi).$$

Plugging these expansions into $EG(\Gamma)$, we obtain that

$$|EG(\Gamma)| \prec \Phi^2 + \left|EG_{\mu\nu}^{(ij)}\right| \prec \Phi^2,$$

where we used (4.18) in the second step.

Finally, in case (3), we have $|EG(\Gamma)| \prec 1$. Moreover, $n^i_l + n^j_l$ is even for any $1 \leq l \leq k$. Take $1 \leq l_1, l_2 \leq k$ such that $n^i_{l_1}, n^j_{l_2}$ are odd and $n^i_{l_1}, n^j_{l_2} \geq 3$. If $l_1 \neq l_2$, then we must have $\hat{n}^i_{l_1} + \hat{n}^j_{l_1} \geq 4$, $\hat{n}^i_{l_2} + \hat{n}^j_{l_2} \geq 4$, and hence

$$\frac{1}{2} \sum_{l=1}^{k} \left(\hat{n}^i_l + \hat{n}^j_l\right) \geq \frac{1}{2} \sum_{l \neq l_1, l_2} (\hat{n}^i_l + \hat{n}^j_l) + 4 \geq k + 2.$$

Otherwise, if $l_1 = l_2$, then

$$\frac{1}{2} \sum_{l=1}^{k} \left(\hat{n}^i_l + \hat{n}^j_l\right) \geq \frac{1}{2} \sum_{l \neq l_1} (\hat{n}^i_l + \hat{n}^j_l) + 3 \geq k + 2.$$

Now applying the above estimates and (4.31) to (4.32), we obtain that

$$\mathbb{E}[M_{ij}] \prec \sum_{\Gamma \text{ in Case (1), (2)}} \Phi^2 N^{k - \sum_{l=1}^{k} (\hat{n}^i_l + \hat{n}^j_l)/2} + \sum_{\Gamma \text{ in Case (3)}} N^{k - \sum_{l=1}^{k} (\hat{n}^i_l + \hat{n}^j_l)/2} \leq C(N^{-1} \Phi^2 + N^{-2}) \leq CN^{-1} \Phi^2.$$

This concludes the proof of (4.26), and hence finishes the proof of (4.13).

5. Proof of Theorem 2.9.

5.1. Basic notations. Without loss of generality, by (4.8), we can assume

$$\mathbb{E}X_{i\mu} = 0, \ i \in I_1, \ \mu \in I_2,$$

in the following proof. Then given $X$ satisfying the assumptions in Theorem 2.9 and (5.1), we first construct another random matrix $\tilde{X}$ whose entries have the same first four moments as those of $X$ but have size of order $N^{-1/2}$. 
Lemma 5.1 (Lemma 5.1 of [23]). Suppose $X$ satisfies the assumptions in Theorem 2.8 and (5.1). Then there exists another matrix $\tilde{X} = (\tilde{X}_{i\mu})$ such that $\mathbb{P}(\max_{i,\mu}|\tilde{X}_{i\mu}| \leq CN^{-1/2}) = 1$ for some constant $C > 0$ and the first four moments of the entries of $X$ and $\tilde{X}$ match, i.e.

\begin{equation}
\mathbb{E}X^k_{i\mu} = \mathbb{E}\tilde{X}^k_{i\mu}, \quad k = 1, 2, 3, 4.
\end{equation}

Taking $q = N^{-1/2}$ in (2.17), we see that (2.20) holds for $G_1(\tilde{X}, z)$. Then due to (5.2), we expect that $G(X, z)$ has “similar” properties as $G(\tilde{X}, z)$, so that (2.20) also holds for $G_1(X, z)$. This will be proved through a resolvent comparison approach that is developed in [23, Sections 6] and [8, Section 6]. More specifically, we will apply the Lindeberg replacement strategy, i.e., we change $\tilde{X}$ to $X$ entry by entry and show that the error (due to the resolvent expansion) appeared at each step is negligible. In this subsection, we introduce some notations that will simplify the presentation of our proof.

Fix a bijective ordering map $\Phi$ on the index set of $X$,

$$
\Phi : \{(i, \mu) : i \in \mathcal{I}_1, \mu \in \mathcal{I}_2\} \to \{1, \ldots, \gamma_{\max} = MN\}.
$$

For any $1 \leq \gamma \leq \gamma_{\max}$, we define the matrix $X^\gamma = (X^\gamma_{i\mu})$ such that $X^\gamma_{i\mu} = X_{i\mu}$ if $\Phi(i, \mu) \leq \gamma$, and $X^\gamma_{i\mu} = \tilde{X}_{i\mu}$ otherwise. Note that $X^0 = \tilde{X}$, $X^{\gamma_{\max}} = X$, and $X^\gamma$ satisfies the bounded support condition with $q \leq N^{-\phi}$ for all $0 \leq \gamma \leq \gamma_{\max}$. Correspondingly, we define

\begin{equation}
H^\gamma := \begin{pmatrix}
0 & X^\gamma \\
(X^\gamma)^* & 0
\end{pmatrix}, \quad G^\gamma := \left(\begin{array}{cc}
-I_{M \times M} & X^\gamma \\
(X^\gamma)^* & -zI_{N \times N}
\end{array}\right)^{-1}.
\end{equation}

Note that $H^\gamma$ and $H^{\gamma-1}$ differ only at $(i, \mu)$ and $(\mu, i)$ elements, where $\Phi(i, \mu) = \gamma$. Then we define two $\mathcal{I} \times \mathcal{I}$ matrices $V$ and $W$ by

$$
V_{ab} = (\delta_{ai}\delta_{b\mu} + \delta_{a\mu}\delta_{bi})X_{i\mu}, \quad W_{ab} = (\delta_{ai}\delta_{b\mu} + \delta_{a\mu}\delta_{bi})\tilde{X}_{i\mu},
$$

such that $H^\gamma$ and $H^{\gamma-1}$ can be written as

\begin{equation}
H^\gamma = Q + V, \quad H^{\gamma-1} = Q + W,
\end{equation}

for some $\mathcal{I} \times \mathcal{I}$ matrix $Q$ satisfying $Q_{i\mu} = Q_{\mu i} = 0$.

For simplicity, for any $1 \leq \gamma \leq \gamma_{\max}$, we denote the resolvents by

\begin{equation}
S^\gamma := G^\gamma, \quad T^\gamma := G^{\gamma-1}, \quad R^\gamma := \left(\begin{array}{cc}
Q & I_{M \times M} \\
0 & zI_{N \times N}
\end{array}\right)^{-1}.
\end{equation}
We often omit the superscript if $\gamma$ is fixed. By (5.4), we can write

\[
S = \left( Q - \begin{pmatrix} I_{M \times M} & 0 \\ 0 & zI_{N \times N} \end{pmatrix} \right)^{-1} + V = (1 + RV)^{-1}R.
\]

Thus we can expand $S$ using the resolvent expansion

\[
S = R - RVR + (RV)^2 R + \ldots + (-1)^m (RV)^m R + (-1)^{m+1} (RV)^{m+1} S.
\]

On the other hand, we can also expand $R$ in terms of $S$:

\[
R = (1 - SV)^{-1} S = S + SVS + (SV)^2 S + \ldots + (SV)^m S + (SV)^{m+1} R.
\]

We can get similar expansions for $T$ and $R$ by replacing $V$, $S$ with $W$, $T$ in (5.7) and (5.8).

By the bounded support conditions for $X$ and $\tilde{X}$, we have

\[
\max_{a,b \in \mathcal{I}} |V_{ab}| = |X_{ii\mu}| < N^{-\phi}, \quad \max_{a,b \in \mathcal{I}} |W_{ab}| = |\tilde{X}_{ii\mu}| \leq CN^{-1/2}.
\]

Also, note that $S$, $R$, $T$ satisfy the following deterministic bound by (2.29):

\[
\sup_{z \in \mathcal{D}} \max_{\gamma} \max \{ \|S\|, \|T\|, \|R\| \} \leq \sup_{z \in \mathcal{D}} (C\eta^{-1}) \leq N.
\]

Then using expansion (5.8) in terms of $T, W$ with $m = 3$, the isotropic local law (2.15) for $T$, and the bound (5.10) for $R$, we can get that for any fixed unit vectors $u, v \in \mathbb{C}^I$, $|R_{uv}| = O(1)$ with high probability. Thus there exists a uniform constant $C_1 > 0$ such that with high probability,

\[
\sup_{z \in \mathcal{D}} \max_{\gamma} \sup_{\text{deterministic unit } u, v} \max \{ |S^\gamma_{uv}|, |T^\gamma_{uv}|, |R^\gamma_{uv}| \} \leq C_1.
\]

From the definitions of $V$ and $W$, one can see that it is helpful to introduce the following notations to simplify the expressions.

**Definition 5.2 (Matrix operators $*_\gamma$).** For $\mathcal{I} \times \mathcal{I}$ matrices $A$ and $B$, we define $A *_{\gamma} B$ as

\[
(A *_{\gamma} B)_{ab} = A_{ai} B_{ib} + A_{a\mu} B_{ib}, \quad \Phi(i, \mu) = \gamma.
\]

We denote the $m$-th power of $A$ under $*_\gamma$-product by $A^{*_{\gamma}m}$, i.e.,

\[
A^{*_{\gamma}m} := A *_{\gamma} A *_{\gamma} A *_{\gamma} \ldots *_{\gamma} A.
\]
Definition 5.3 ($\mathcal{P}_{\gamma,k}$ and $\mathcal{P}_{\gamma,k}$). For $k \in \mathbb{N}$, $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$ and $\gamma = \Phi(i, \mu)$, we define

$$
\mathcal{P}_{\gamma,k}^\gamma_G := \prod_{t=1}^s G_{uv_t}^{\gamma_{(k_t+1)}}, \quad \mathcal{P}_{\gamma,k}^{\gamma} \left( \prod_{t=1}^s G_{uv_t} \right) := \prod_{t=1}^s \mathcal{P}_{\gamma,k_t} G_{uv_t}.
$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are products of resolvent entries as above, then we define

$$
\mathcal{P}_{\gamma,k}(\mathcal{G}_1 + \mathcal{G}_2) := \mathcal{P}_{\gamma,k} \mathcal{G}_1 + \mathcal{P}_{\gamma,k} \mathcal{G}_2.
$$

Note that $\mathcal{P}_{\gamma,k}^\gamma$ and $\mathcal{P}_{\gamma,k}^\gamma$ are not linear operators, but just notations we use for simplification.

Using Definition 5.3, we may write, for example,

$$
\mathcal{P}_{\gamma,k}^{\gamma} \left( \prod_{t=1}^s G_{uv_t}^{\gamma_{(k_t+1)}} \right) := \prod_{t=1}^s S_{uv_t}^{\gamma_{(k_t+1)}}, \quad \mathcal{P}_{\gamma,k}^{\gamma} \left( \prod_{t=1}^s G_{uv_t}^{\gamma_{(k_t+1)}} \right) := \prod_{t=1}^s \mathcal{P}_{\gamma,k_t}^{\gamma_{(k_t+1)}}.
$$

For $k, s \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^{s+1}$, it is easy to verify that

$$
G^{s+\gamma} G^{s+\gamma} = G^{s+\gamma+k}, \quad \mathcal{P}_{\gamma,k}^{\gamma} (\mathcal{G}_{uv}^{\gamma,s} \mathcal{G}_{uv}) = \mathcal{P}_{\gamma,s+|\mathbf{k}|} G_{uv},
$$

where $|\mathbf{k}| = \sum_{t=1}^s k_t$. For the second equality, note that $\mathcal{P}_{\gamma,s} G_{uv}$ is a sum of the products of the $G$ entries, where each product contains $s + 1$ entries.

5.2. Proof of (2.20). As mentioned in the last subsection, we will prove (2.20) with the resolvent comparison method. The basic idea is that we expand $S$ and $T$ in terms of $R$ by repeatedly applying the expansions (5.7) and (5.8), and then compare the resulting expressions. The main terms will cancel since $X_{i\mu}$ and $\tilde{X}_{i\mu}$ have the same first four moments, and the error terms are small since $X_{i\mu}$ and $\tilde{X}_{i\mu}$ have support bounded by $N^{-\phi}$.

The proof of the following Lemma 5.4 is almost the same as the one for [23, Lemma 6.5]. In fact, we can copy their arguments almost verbatim, except for some notational differences. Hence we omit the details. In the following expressions, for any $\mathbf{k} = (k_1, \ldots, k_p) \in \mathbb{N}^p$, we use $|\mathbf{k}| = \sum k_t$ to denote its $l^1$-norm.

Lemma 5.4. Assume $z \in \mathcal{D}$ and $\gamma = \Phi(i, \mu)$. Fix any $p \in \mathbb{N}$ and $r > 0$. Then for $S, R$ in (5.5), we have

$$
\mathbb{E} \prod_{t=1}^p S_{uv_t} = \sum_{0 \leq k \leq 4} A_k \mathbb{E} \left[ (-X_{i\mu})^k \right] + \sum_{5 \leq |\mathbf{k}| \leq r/\Phi, \mathbf{k} \in \mathbb{N}^p} A_k \mathbb{E} \mathcal{P}_{\gamma,k} \prod_{t=1}^p S_{uv_t} + O_{\prec}(N^{-r}),
$$

(5.17)
where $A_k$, $0 \leq k \leq 4$, depend only on $R$, $A_k$'s are independent of $(u_t, v_t)$, $1 \leq t \leq s$, and we have the bound

\[(5.18)\]
$$|A_k| \prec N^{-|k|/10 - 2}.$$  

It is obvious that a result similar to Lemma 5.4 also holds for the product of $T$ entries. As in (5.17), we define the notation $A^{\gamma,a}$, $a = 0, 1$ as follows:

\[(5.19)\]
$$\begin{align*}
\mathbb{E} \prod_{t=1}^{p} S_{u_t v_t} &= \sum_{0 \leq k \leq 4} A_k \mathbb{E} \left[ (-X_{i\mu})^k \right] \\
&\quad + \sum_{5 \leq |k| \leq r/\phi, k \in \mathbb{N}^p} A_k^{\gamma,0} \mathbb{E} \mathcal{P}_{\gamma,k} \prod_{t=1}^{p} S_{u_t v_t} + O_\prec (N^{-r}), \\
\mathbb{E} \prod_{t=1}^{p} T_{u_t v_t} &= \sum_{0 \leq k \leq 4} A_k \mathbb{E} \left[ (-\tilde{X}_{i\mu})^k \right] \\
&\quad + \sum_{5 \leq |k| \leq r/\phi, k \in \mathbb{N}^p} A_k^{\gamma,1} \mathbb{E} \mathcal{P}_{\gamma,k} \prod_{t=1}^{p} T_{u_t v_t} + O_\prec (N^{-r}).
\end{align*}$$

Since $A_k$, $0 \leq k \leq 4$, depend only on $R$ and $X_{i\mu}, \tilde{X}_{i\mu}$ have the same first four moments, we get from (5.19) and (5.20) that

\[(5.21)\]
$$\begin{align*}
\mathbb{E} \prod_{t=1}^{p} G_{u_t v_t} - \mathbb{E} \prod_{t=1}^{p} \tilde{G}_{u_t v_t} &= \sum_{\gamma=1}^{\gamma_{\max}} \left( \mathbb{E} \prod_{t=1}^{p} G^{\gamma}_{u_t v_t} - \mathbb{E} \prod_{t=1}^{p} G^{\gamma-1}_{u_t v_t} \right) \\
&= \sum_{\gamma=1}^{\gamma_{\max}} \sum_{k \in \mathbb{N}^p} \left( A_k^{\gamma,0} \mathbb{E} \mathcal{P}_{\gamma,k} \prod_{t=1}^{p} G^{\gamma}_{u_t v_t} - A_k^{\gamma,1} \mathbb{E} \mathcal{P}_{\gamma,k} \prod_{t=1}^{p} G^{\gamma-1}_{u_t v_t} \right) \\
&\quad + O_\prec (N^{-r+2}).
\end{align*}$$

where we abbreviate $G := G(X, z)$ and $	ilde{G} := G(\tilde{X}, z)$.

Applying (5.21) with $p = 1$, $r = 3$ and fixed unit vector $u_t = v_t = v \in \mathbb{C}^{\mathbb{F}_1}$, we obtain that

\[(5.22)\]
$$\mathbb{E}(G - \tilde{G})_{vv} \leq \sum_{\gamma=1}^{\gamma_{\max}} \sum_{a=0,1} \sum_{1.5 \leq k \leq 3/\phi} |A_k^{\gamma,a}| \left| \mathbb{E} \mathcal{P}_{\gamma,k} G^{\gamma-a}_{vv} \right| + O_\prec (N^{-1}).$$

Using (5.11), (5.18) and Lemma 2.5, we can bound the sum in (5.22) by

\[(5.23)\]
$$\sum_{\gamma=1}^{\gamma_{\max}} \sum_{a=0,1} \sum_{1.5 \leq k \leq 3/\phi} |A_k^{\gamma,a}| \left| \mathbb{E} \mathcal{P}_{\gamma,k} G^{\gamma-a}_{vv} \right| \prec \sum_{5 \leq k \leq 3/\phi} N^{-k\phi/10} \prec N^{-\phi/2}. $$
To apply Lemma 2.5 (iii), we need a second moment bound for $|P_{\gamma,k}G_{\gamma,a}^\gamma|$, which follows easily from (5.10). Recall that $P_{\gamma,k}G_{\gamma,a}^\gamma$ is also a sum of the products of $G$ entries. Then applying (5.21) to $|E_P_{\gamma,k}G_{\gamma,a}^\gamma|$ and replacing $\gamma_{\text{max}}$ with $\gamma - a$, we obtain that

$$
|E_{\gamma,k}G_{\gamma,a}^\gamma| \leq |E_{\gamma,k}G_{\gamma,a}^0|
$$

(5.24)

$$
+ \sum_{\gamma'=1}^{\gamma-a} \sum_{a'=0,1}^{5|k'| \leq 3/\phi} \sum_{k' \in \mathbb{N}^1+k} |A_{\gamma'}^{\gamma,a'}| |E_P_{\gamma',k'}P_{\gamma,k}G_{\gamma,a}^\gamma| + O_\prec(N^{-1}).
$$

Together with (5.22) and (5.18), we get that

$$
|E(G - \tilde{G})_{\gamma,a}| \leq \sum_{\gamma,a} \sum_{k} |A_{\gamma,a}^{\gamma,a}| |E_P_{\gamma,k}G_{\gamma,a}^0|
$$

$$
+ \sum_{\gamma,a,a',k,k'} |A_{\gamma}^{\gamma,a}A_{\gamma'}^{\gamma,a'}| |E_P_{\gamma',k'}P_{\gamma,k}G_{\gamma,a}^\gamma| + O_\prec(N^{-1}).
$$

Again using (5.11), (5.18) and Lemma 2.5, we obtain that

$$
\sum_{\gamma,a,a',k,k'} |A_{\gamma}^{\gamma,a}A_{\gamma'}^{\gamma,a'}| |E_P_{\gamma',k'}P_{\gamma,k}G_{\gamma,a}^\gamma| \prec N^{-\phi},
$$

(5.25)

where we used that $k + |k'| \geq 10$. Repeating this process, we can make the remainder term smaller and smaller. At the end, we obtain that

$$
|E(G - \tilde{G})_{\gamma,a}| \leq \sum_{n=0}^{2/\phi} \sum_{a_1,\ldots,a_n} \prod_{j} A_{k_j}^{a_j} |E_P_{\gamma_n,a_n} \cdots P_{\gamma_1,a_1}G_{\gamma,a}^0|
$$

$$
+ O_\prec(N^{-1}),
$$

where

$$
k_1 \in \mathbb{N}^1, \ k_2 \in \mathbb{N}^1+|k_1|, \ k_3 \in \mathbb{N}^1+|k_1|+|k_2|, \quad \text{etc.}, \quad \text{and} \quad 5 \leq |k_i| \leq \frac{3}{\phi}.
$$

Using (5.18) and Lemma 2.5, we obtain that

$$
|E(G - \tilde{G})_{\gamma,a}| \prec \max_{k,n}(N^{-2}n)(N^{-\frac{4}{\phi}}) \sum_{i=1}^{n} |k_i| \sum_{\gamma_1,\ldots,\gamma_n} |E_P_{\gamma_n,a_n} \cdots P_{\gamma_1,a_1}G_{\gamma,a}^0| + N^{-1}.
$$

(5.27)

Now we finish the proof of (2.20) using the estimate (5.27) and the bound (2.20) for $G^0 = G(X, z)$. We see that it suffices to control the term

$$
P_{\gamma_n,a_n} \cdots P_{\gamma_1,a_1} \tilde{G}_{\gamma,a}
$$

(5.28)
for $k_1, \ldots, k_n$ satisfying (5.26). By definition of $P$, (5.28) is a sum of at most $C \sum |k_i|$ products of $G_{v\phi}, G_{b\phi}$ and $G_{a\phi}$ entries, where the total number of $G$ entries in each product is at most $\sum |k_i| + 1 = O(\phi^{-2})$. Due to the deterministic bound (5.10), (5.28) is always bounded by $N^{O(\phi^{-2})}$, and hence Lemma 2.5 (iii) can be applied.

For each product in (5.28), there are two $v$’s in the indices of $G$. These two $v$’s appear as $G_{v\phi}G_{b\phi}$ in the product, where $a, b$ come from some $\gamma_k$ and $\gamma_l$ ($1 \leq k, l \leq n$) via $P$. Thus after taking the average $N^{-2} \sum_{k} G_{\gamma_k}$ and $N^{-2} \sum_{l} G_{\gamma_l}$, the term $G_{v\phi}G_{b\phi}$ contributes a factor $O_<(\eta^{-1})$ by (2.34) and Cauchy-Schwarz inequality. For all other $G$ factors in the product with no $v$’s, we control them by $O_<(1)$ using (5.11). Thus for any fixed $\gamma_1, \ldots, \gamma_n$, $k_1, \ldots, k_n$, we have proved that

$$N^{-2n} \sum_{\gamma_1, \ldots, \gamma_n} |E \mathcal{P}_{\gamma_0, k_n} \cdots \mathcal{P}_{\gamma_1, k_1} \tilde{G}_{vv} | < \frac{1}{N \eta}.$$ 

Then using (5.27) and (2.20) for $\tilde{G}$, we obtain that

$$|E (G_1)_{vv} - m_1(z)| < |E (\tilde{G}_1)_{vv} - m_1(z)| + \frac{1}{N \eta} < \frac{1}{N \eta},$$

where we abbreviated $G_1 = z^{-1}G$ and $\tilde{G}_1 = z^{-1}\tilde{G}$. This concludes (2.20).

APPENDIX A: PROOF OF LEMMA 4.2

We only prove

$$\max_{i, j \in I} |G_{ij}(X, z) - \Pi_{ij}(z)| < q^2 + (N \eta)^{-1/2}. \quad (A.1)$$

The proof for (2.19) with $a, b \in I_2$ is exactly the same. First, we recall the following large deviation bounds proved in [12].

**Lemma A.1** (Lemma 3.8 of [12]). Let $(x_i), (y_i)$ be independent families of centered and independent random variables, and $(A_i), (B_{ij})$ be families of deterministic complex numbers. Suppose the entries $x_i$ and $y_j$ have variance $O(N^{-1})$ and satisfy (2.9) with $N^{-1/2} \leq q \leq N^{-\phi}$ for some fixed $\phi > 0$. Then for $K = O(N)$, we have the following bounds:

(A.2) \[ \left| \sum_{1 \leq i, j \leq K} x_i B_{ij} y_j \right| < q^2 B_d + q B_o + \frac{1}{N} \left( \sum_{i \neq j} |B_{ij}|^2 \right)^{1/2}, \]

(A.3) \[ \left| \sum_{1 \leq i \neq j \leq K} \bar{x}_i B_{ij} x_j \right| < q B_o + \frac{1}{N} \left( \sum_{i \neq j} |B_{ij}|^2 \right)^{1/2}, \]

(A.4) \[ \left| \sum_{1 \leq i \leq K} \left( |x_i|^2 - \mathbb{E} |x_i|^2 \right) B_{ii} \right| < q B_d, \]
where \( B_d := \max_i |B_{ii}| \) and \( B_o := \max_{i \neq j} |B_{ij}| \).

In fact, these bounds are stated in slightly stronger forms in [12] with a different notion for high probability events. Here we choose to present (A.2)-(A.4) in terms of the stochastic domination, which will be more convenient for our use. Moreover, if we assume the fourth moment of \( x_i \) is bounded for all \( i \) as in (1.11), then we have a better bound for the LHS of (A.4).

**Lemma A.2.** Suppose the assumptions in Lemma A.1 hold and \( x_i, 1 \leq i \leq K \), satisfy (1.11). Then we have

\[
\left| \sum_i (|x_i|^2 - \mathbb{E}|x_i|^2) B_{ii} \right| \prec \left( q^2 + N^{-1/2} \right) B_d.
\]

**Proof.** We abbreviate \( z_i := (|x_i|^2 - \mathbb{E}|x_i|^2) B_{ii}/B_d \). By Markov’s inequality, it suffices to prove that for any fixed \( p \in \mathbb{N} \),

\[
\mathbb{E} \left| \sum_i z_i \right|^{2p} \prec \left( q^2 + N^{-1/2} \right)^{2p}.
\]

Note that by the assumption, we have

\[
\mathbb{E} z_i = 0, \quad \mathbb{E} |z_i|^n < q^{2n-4} N^{-2} \text{ for fixed } n \geq 2.
\]

Now we expand the LHS of (A.6) to get

\[
\mathbb{E} \left| \sum_i z_i \right|^{2p} = \sum_{i_1, \ldots, i_{2p}} \mathbb{E} y_{i_1} \cdots y_{i_{2p}},
\]

where we denote \( y_{i_l} := z_{i_l} \) for \( 1 \leq l \leq p \) and \( y_{i_l} := \bar{z}_{i_l} \) for \( p + 1 \leq l \leq 2p \). To organize the summation over the indices \( i_1, \ldots, i_{2p} \), we look at the partitions \( \Gamma \) of the set of the labels \( \{1, \ldots, 2p\} \) according to the equivalence relation that \( k, l \) are in the same class if and only if \( i_k = i_l \). We use \( b_l, 1 \leq l \leq k \), to denote the equivalence classes of \( \Gamma \) and \( n_l \) to denote the size of \( b_l \). Obviously, \( k, b_l \) and \( n_l \) all depend on \( \Gamma \), but we will omit this dependence in the following expressions. Moreover, since the random variables are centered, we must have \( n_l \geq 2 \) for all \( l \) to attain a nonzero expectation. Hence we have

\[
\mathbb{E} \left| \sum_i z_i \right|^{2p} \leq \sum_{\Gamma} \sum_{b_1, \ldots, b_k} \mathbb{E} |y_{\mu_{b_1}}|^{n_1} \cdots \mathbb{E} |y_{\mu_{b_k}}|^{n_k},
\]

where \( \sum^* \) denotes the summation subject to the conditions that \( b_1, \ldots, b_k \) are all distinct, \( n_l \geq 2 \) for all \( l \), and \( \sum_{l=1}^k n_l = 2p \). Note that under these conditions, we trivially have \( k \leq p \).
Using \((A.7)\), we get
\[
\sum_{b_1, \ldots, b_k}^* \mathbb{E}|y_{b_1}^1 \ldots y_{b_k}^k|^n < \sum_{b_1, \ldots, b_k}^* (q^{2n_1-4} N^{-2}) \ldots (q^{2n_k-4} N^{-2}) = \sum_{b_1, \ldots, b_k}^* N^{-2k} q^{4p-4k} \leq C N^{-k} q^{4p-4k}.
\]
Since the number of partitions of \(\{1, \ldots, 2p\}\) is finite and depends only on \(p\), \((A.8)\) can be bounded by
\[
\mathbb{E}\left| \sum_i z_i^{2p} \right| < \max_{1 \leq k \leq p} N^{-k} q^{4p-4k} \leq q^{4p} + N^{-p},
\]
where in the last step, \(q^{4p}\) and \(N^{-p}\) can be obtained from the extreme cases \(k = 0\) and \(k = p\), respectively. This concludes \((A.6)\). \(\Box\)

Now using \((2.23)\) and \((A.2)\), we get that
\[
|G_{ij}| \sim \left| \sum_{\mu, \nu} X_{i\mu} G_{ij}^{(ij)} X_{j\nu}^* \right| \sim q^2 \max_{\mu} |G_{ij}^{(ij)}| + q \max_{\mu \neq \nu} |G_{ij}^{(ij)}| + \left( \frac{1}{N^2} \sum_{\mu \neq \nu} |G_{ij}^{(ij)}|^2 \right)^{1/2} \sim q^2 + (q + \Psi) + \left( \frac{1}{N^2} \right)^{1/2} \sim q^2 + (N\eta)^{-1/2},
\]
where we used \((2.18), (2.26)\) and the bound \((2.34)\). For the diagonal estimate, we need to control the \(Z\) variables defined in \((4.11)\). Using \((A.3)\) and \((A.5)\), we get that for any \(T \subset I\) with fixed length,
\[
|Z_i^{(T)}| = \left| \sum_\mu G_{ij}^{(T)} (|X_{i\mu}|^2 - \mathbb{E}|X_{i\mu}|^2) + \sum_{\mu \neq \nu} X_{i\mu} G_{ij}^{(T)} X_{j\nu}^* \right| \sim \left( q^2 + N^{-1/2} \right) + q \max_{\mu \neq \nu} |G_{ij}^{(T)}| + \left( \frac{1}{N^2} \sum_{\mu \neq \nu} |G_{ij}^{(T)}|^2 \right)^{1/2} \sim q^2 + (N\eta)^{-1/2},
\]
where we used \((2.18), (2.26)\) and \((2.34)\) again. Then using \((2.22)\), we get
\[
G_{ii} - zm_{1c} = \frac{1}{-1 - m_{2c} - (m_{2c}^{(i)} - m_{2c}) + Z_i} - zm_{1c} = \frac{1}{-1 - m_{2c} - zm_{1c} + O_\infty \left( q^2 + (N\eta)^{-1/2} \right)} = O_\infty \left( q^2 + (N\eta)^{-1/2} \right)
\]
where we used \((A.10), (2.14), (2.27)\) (with \(\Phi = q + \Psi\)) in the second step, and \((1.4), (1.6)\) in the third step. Together with \((A.9)\), this concludes \((A.1)\).
APPENDIX B: PROOF OF LEMMA 4.3

Note that by (4.9), we immediately get $\sum_i |v_i|^2 ((G_1)_{ii} - m_{1c}) < \Phi$. Hence it remains to show that

$$\sum_{i \neq j} \bar{v}_i v_j (G_1)_{ij} < \Phi.$$  

By Markov’s inequality and (2.5), it suffices to prove that

$$(B.1) \quad E \left| \sum_{i \neq j} \bar{v}_i v_j G_{ij} \right|^{2p} < \Phi^{2p}$$

for any fixed $p \in \mathbb{N}$. The proof of (B.1) is similar to the ones in [3, Section 5] and [31, Section 5]. The main difference is that in [3, 31], the matrix entries are assumed to have arbitrarily high moments, while we assume that the $X$ entries have finite third moment and support bounded by $q$ in our proof. In particular, for any fixed $n \geq 3$, we have

$$(B.2) \quad E|X_{i\mu}|^n < q^{n-3} N^{-3/2}, \quad i \in \mathcal{I}_1, \mu \in \mathcal{I}_2.$$  

Note that we have a stronger moment assumption in (1.11). However, the finite fourth moment condition will not be used in the proof below. We only need the weaker bound (B.2). Also we remark that some of the basic ideas have been illustrated in the proof for (2.17) in Section 4.

We first rewrite the product in (B.1) as

$$\left| \sum_{i \neq j} \bar{v}_i G_{ij} v_j \right|^{2p} = \sum_{i_k \neq j_k \in \mathcal{I}_1} \prod_{k=1}^{p} \bar{v}_{i_k} G_{i_k,j_k} v_{j_k} \cdot \prod_{k=p+1}^{2p} \bar{v}_{i_k} G_{i_k,j_k} v_{j_k} = \sum_{\Gamma} \sum_{b_1,\ldots,b_n} \prod_{k=1}^{p} \bar{v}_{\Gamma(i_k)} G_{\Gamma(i_k)\Gamma(j_k)} v_{\Gamma(j_k)} \cdot \prod_{k=p+1}^{2p} \bar{v}_{\Gamma(i_k)} G_{\Gamma(i_k)\Gamma(j_k)} v_{\Gamma(j_k)},$$

where (recall the notations in the proof for Lemma A.2) $\Gamma$ ranges over all partitions of the set of the labels $\{i_1,\ldots,i_{2p},j_1,\ldots,j_{2p}\}$ with the restriction that $i_k,j_k$ cannot be in the same equivalence class for all $k$, $\{b_1,\ldots,b_n\}$ is the set of equivalence classes for a fixed $\Gamma$, $\Gamma(\cdot)$ is regarded as a symbolic mapping from the set of labels to the set of equivalence classes, and $\sum^*$ denotes the summation subject to the condition that $b_1,\ldots,b_n$ all take distinct values and $\Gamma(i_k) \neq \Gamma(j_k)$ for all $k$.

Since the number of such partitions $\Gamma$ is finite and depends only on $p$, it suffices to show that for any fixed $\Gamma$,

$$(B.3) \quad E \sum_{b_1,\ldots,b_n} \prod_{k=1}^{p} \bar{v}_{\Gamma(i_k)} G_{\Gamma(i_k)\Gamma(j_k)} v_{\Gamma(j_k)} \cdot \prod_{k=p+1}^{2p} \bar{v}_{\Gamma(i_k)} G_{\Gamma(i_k)\Gamma(j_k)} v_{\Gamma(j_k)} < \Phi^{2p}.$$
We abbreviate

\[ P(b_1, ..., b_n) := \prod_{k=1}^{p} G_{\Gamma(i_k)\Gamma(j_k)} \cdot \prod_{k=p+1}^{2p} G_{\Gamma(i_k)\Gamma(j_k)}. \]

For simplicity, we shall omit the overline for complex conjugate in the following proof. In this way, we can avoid a lot of immaterial notational complexities that do not affect the proof.

For \( k = 1, ..., n \), we denote by \( \deg(b_k, P) \) the number of times that \( b_k \) appears as an index of the \( G \) entries in \( P \), i.e. \( \deg(b_k, P) := |\Gamma^{-1}(b_k)| \). We define \( h := \#\{1 \leq k \leq n : \deg(b_k, P) = 1\} \), i.e. \( h \) is the number of \( b_k \)'s that only appear once in the indices of \( P \). Without loss of generality, we assume these \( b_k \)'s are \( b_1, ..., b_h \). Then we have the following properties:

\[ \sum_{k=1}^{n} \deg(b_k, P) = 4p, \quad \text{and} \quad \deg(b_k, P) = 1, \quad \text{for} \quad k = 1, ..., h. \quad (B.4) \]

Now we claim that

\[ |EP| \prec N^{-h/2}\Phi^{2p}. \quad (B.5) \]

Note that by \( \|v\|_2 = 1 \) and Cauchy-Schwarz inequality, we have \( \sum_i |v_i| \leq \sqrt{M} \) and \( \sum_i |v_i|^n \leq 1 \) for \( n \geq 2 \). Then if \((B.5)\) holds, we can bound the left hand side of \((B.3)\) by

\[ N^{-h/2}\Phi^{2p} \prod_{k=1}^{n} \sum_{b_k} |v_{b_k}|^{\deg(b_k, P)} \leq N^{-h/2}\Phi^{2p}(\sqrt{M})^{h/2} \leq C\Phi^{2p}. \]

Hence it suffices to prove \((B.5)\).

We define the \( S \) variables as (one can compare them with \((4.19)\))

\[ S_{ij} := (XG^{(L)}X^*)_{ij}, \quad (B.6) \]

for \( i, j \in \mathcal{I}_1 \) and \( L := \{b_1, ..., b_h\} \). As in \((A.9)\) and \((A.10)\), we can verify that \( |S_{ij} - m_{2c}\delta_{ij}| \prec \Phi \) for \( i, j \in \mathcal{I}_1 \) using \((4.9)\), \((2.26)\) and Lemmas A.1-A.2. Then as in Section 4.3, we keep expanding the \( G \) entries in \( P \) using the resolvent expansions in Lemma 2.12, until each monomial in the expression either consists of \( S \) variables only or has sufficiently many off-diagonal terms.

The following lemma corresponds to the previous Lemma 4.7 and has been proved in [3, Lemma 5.9] and [31, Lemma 5.9].
Lemma B.1. After finitely many expansions, we can write \( P \) as

\[
P = \sum_{\alpha=1}^{A} c_{\alpha} Q_{\alpha} + O_{\prec}(N^{-h/2} \Phi^{2p}),
\]

where \( A \in \mathbb{N} \) depends only on \( p \) and \( c_0 \) (recall that \( \Phi(z) \leq N^{-c_0} \) by assumption), \( c_{\alpha}'s \) are constants of order \( O(1) \), and \( Q_{\alpha} \) are products of \( S \) variables only and the number of \( S \) variables in each product again depends only on \( p \) and \( c_0 \). Moreover, for \( k = 1, ..., n \) and \( \alpha = 1, ..., A \), we have that

\[
deg_{\alpha}(b_k, Q_{\alpha}) \geq \deg_{\alpha}(b_k, P),\quad \deg_{\alpha}(b_k, Q_{\alpha}) = \deg_{\alpha}(b_k, P) \mod 2,
\]

and the number of off-diagonal \( S \) variables in \( Q \) is at least \( 2p \). Here \( \deg_{\alpha}(b_k, Q_{\alpha}) \) denotes the number of times that \( b_k \) appears as an index of the off-diagonal \( S \) variables in \( Q_{\alpha} \) and \( \deg_{\alpha}(b_k, P) := \deg(b_k, P) \) (which is consistent with the previous definition since \( P \) only contains off-diagonal entries).

Now given the expansion in (B.7), we see that to conclude (B.5), it suffices to show that for any \( Q_{\alpha} \),

\[
|E Q_{\alpha}| \prec N^{-h/2} \Phi^{2p}.
\]

In the following proof, we fix one such \( Q \equiv Q_{\alpha} \) and write

\[
Q = \prod_{j=1}^{J} S_{b_{kj} b_{lj}} = \prod_{\mu_j, \nu_j \in I} \prod_{j=1}^{J} X_{b_{kj} \mu_j} G_{\mu_j \nu_j} X_{\nu_j b_{lj}}^* = \sum_{W} \sum_{w_1, ..., w_m} ^* \prod_{j=1}^{J} X_{b_{kj} W(\mu_j)} G_{W(\mu_j) W(\nu_j)} X_{b_{lj} W(\nu_j)} \]

where \( J \) is the number of \( S \)-variables in \( Q \), \( W \) ranges over all partitions of the set of the labels \( \{\mu_1, ..., \mu_J, \nu_1, ..., \nu_J\} \), \( \{w_1, ..., w_m\} \) denotes the set of distinct equivalence classes for a particular \( W \), \( W(\cdot) \) is regarded as a symbolic mapping from the set of labels to the set of equivalence classes, and \( \sum^* \) denotes the summation subject to the condition that \( w_1, ..., w_m \) all take distinct values. Note that the number of partitions depends only on \( J \). For a fixed partition \( W \), we denote

\[
R(w_1, ..., w_m; W) := \prod_{j=1}^{J} X_{b_{kj} W(\mu_j)} G_{W(\mu_j) W(\nu_j)} X_{b_{lj} W(\nu_j)}.
\]
Then to prove (B.9), it suffices to show that
\[(B.10) \quad |\mathbb{E}R(w_1, \ldots, w_m; W)| \prec N^{-m-h/2}\Phi^2 p.\]
for any partition \(W\).

To facilitate the proof, we introduce the graphical notations as in Section 4.4. We use a connected graph \((V, E)\) to represent \(R\), where the vertex set \(V\) consists of black vertices \(b_1, \ldots, b_n\) and white vertices \(w_1, \ldots, w_m\), and the edge set \(E\) consists of \((k, \alpha)\) edges representing \(X_{b_k w_\alpha}\) and \((\alpha, \beta)\) edges representing \(G_{w_\alpha w_\beta}\). We denote
\[e_{k\alpha} := \text{number of } (k, \alpha) \text{ edges in } R, \quad d_\alpha := \text{number of } (\alpha, \alpha) \text{ edges in } R.\]

Note that to attain a nonzero expectation, we must have
\[(B.11) \quad e_{k\alpha} = 0 \text{ or } e_{k\alpha} \geq 2 \text{ for all } k, \alpha.\]

We also define
\[e_{k\alpha}^{(o)} := \text{number of } (k, \alpha) \text{ edges that are from off-diagonal } S \text{ in } Q.\]

Then we have
\[(B.12) \quad \sum_{\alpha} e_{k\alpha}^{(o)} = \deg_o(b_k, Q)\]

By (B.4), (B.11) and the parity conservation due to (B.8), there exist edges \((1, \alpha_1), \ldots, (h, \alpha_h)\) such that \(e_{k\alpha_k}^{(o)}\) is odd and \(e_{k\alpha_k}^{(o)} \geq 3, 1 \leq k \leq h\). Let \(H := \{(1, \alpha_1), \ldots, (h, \alpha_h)\}\) be the set of these edges. Denote by \(F\) the set of \((k, \alpha)\) edge such that \(e_{k\alpha}^{(o)} \geq 2\) and \((k, \alpha) \notin H\). Denote
\[s_\alpha := \sum_{k=1}^n e_{k\alpha}, \quad h_{k\alpha} := 1_{(k, \alpha) \in H}, \quad h_\alpha := \sum_{k=1}^n h_{k\alpha}, \quad f_\alpha := \sum_{k=1}^n 1_{(k, \alpha) \in F}\]

for all \(k = 1, \ldots, n\) and \(\alpha = 1, \ldots, m\). By the above definitions, we have \(s_\alpha \geq 2\) and \(h_\alpha + f_\alpha > 0\) (since the classes \(w_\alpha\) are nontrivial), \(s_\alpha \geq 2d_\alpha\), and
\[(B.13) \quad \sum_{\alpha} h_{k\alpha} = 1(1 \leq k \leq h), \quad \sum_{\alpha} h_\alpha = h.\]

Note that there are \(\frac{1}{2} \sum_{k, \alpha} e_{k\alpha} - d_\alpha\) off-diagonal \(G\) edges in \(R\). Hence by (4.9) and (B.2), we have
\[
|\mathbb{E}R| \prec \prod_{\alpha=1}^m \left(\Phi^{-d_\alpha} \prod_{k=1}^n \Phi^{e_{k\alpha}} \mathbb{E}|X_{b_k w_\alpha}|^{e_{k\alpha}}\right)
\]
\[
\prec \prod_{\alpha=1}^m \Phi^{s_\alpha/2 - d_\alpha} \left(\prod_{(k, \alpha) \in H} q^{e_{k\alpha} - 3N^{-3/2}}\right) \left(\prod_{(k, \alpha) \in F} q^{e_{k\alpha} - 2N^{-1}}\right) =: \prod_{\alpha=1}^m R_\alpha.
\]

Now we consider the following four cases for \(R_\alpha\).
(i) \( d_\alpha = 0 \). In this case we have
\[
R_\alpha \preceq \Phi^{s_\alpha/2} \prod_{(k, \alpha) \in H} N^{-3/2} \prod_{(k, \alpha) \in F} N^{-1} = \Phi^{s_\alpha/2} (N^{-1})^{h_\alpha + f_\alpha} N^{-h_\alpha/2}
\]
\[
\preceq \Phi^{s_\alpha/2} N^{-1} N^{-h_\alpha/2} \preceq \Phi^{\sum_{k=1}^h h_{ka}/2 + \sum_{k=h+1}^n e_{ka}^{(o)}/2} N^{-1} N^{-h_\alpha/2}
\]
where in the third step we used \( h_l + f_l > 0 \), and in the fourth step we used
\[
s_\alpha \geq \sum_{k=1}^h e_{ka}^{(o)} \geq \sum_{k=1}^k h_{ka} + \sum_{k=h+1}^n e_{ka}^{(o)}.
\]
where we used that \( e_{ka}^{(o)} \geq h_{ka} \) for \( 1 \leq k \leq h \) (recall that if \((k, \alpha) \in H\), then \( e_{ka} \) is odd and hence one of the edges must come from the off-diagonal \( S \)).

(ii) \( d_\alpha \neq 0 \), \( h_\alpha = 1 \) and \( f_\alpha = 0 \). Then there is only one \( k \) such that \( e_{ka} > 0 \) and \( s_\alpha = e_{ka} \) is odd. Hence we have \( s_l/2 \geq d_l + 1/2 \) and we can bound \( R_\alpha \) as
\[
R_\alpha \preceq \Phi^{\frac{1}{2} s_\alpha - d_\alpha} (N^{-1})^{h_\alpha} + f_\alpha N^{-h_\alpha/2} \preceq \Phi^{1/2} N^{-1} N^{-h_\alpha/2}
\]
\[
= \Phi^{\sum_{k=1}^h h_{ka}/2 + \sum_{k=h+1}^n e_{ka}^{(o)}/2} N^{-1} N^{-h_\alpha/2},
\]
where in the last step we used
\[
1 = \sum_{k=1}^h h_{ka} + \sum_{k=h+1}^n e_{ka}^{(o)}
\]
since all the summands except one \( h_{ka} \) are 0.

(iii) \( d_\alpha \neq 0 \), \( h_\alpha = 0 \) and \( f_\alpha = 1 \). Then there is only one \( k \) such that \( e_{ka} > 0 \) and \( s_\alpha = e_{ka} \). Thus the \((\alpha, \alpha)\) edges are expanded from the diagonal \( S \) variables (otherwise \( \alpha \) must connect to at least two different \( k \)'s), which implies \( \frac{1}{2} s_\alpha - d_\alpha = \frac{1}{2} e_{ka}^{(o)} \). Then we can bound \( R_\alpha \) by
\[
R_\alpha \preceq \Phi^{\sum_{k=1}^h h_{ka}/2 + \sum_{k=h+1}^n e_{ka}^{(o)}/2} N^{-1} N^{-h_\alpha/2}
\]
where, as in Case (i), we used \( e_{ka}^{(o)} \geq h_{ka} \) for \( 1 \leq k \leq h \).
(iv) \( d_\alpha \neq 0 \) and \( h_\alpha + f_\alpha \geq 2 \). Then using \( s_\alpha \geq 2d_\alpha \), \( q < \Phi^{1/2} \) and \( N^{-1/2} < \Phi \), we get that

\[
R_\alpha < \prod_{(k,\alpha) \in H} \Phi^{e_{k\alpha}/2-3/2} N^{-3/2} \prod_{(k,\alpha) \in F} \Phi^{e_{k\alpha}/2-1} N^{-1} \\
< \prod_{(k,\alpha) \in H} \Phi^{e_{k\alpha}/2-1/2} N^{-1} \prod_{(k,\alpha) \in F} \Phi^{e_{k\alpha}/2} N^{-1/2} \\
= \Phi^{\left(s_\alpha - h_\alpha\right)/2} N^{-\left(h_\alpha + f_\alpha\right)/2} N^{-h_\alpha/2} \leq \Phi^{\left(s_\alpha - h_\alpha\right)/2} N^{-1} N^{-h_\alpha/2} \\
\leq \Phi^{\sum_{k=1}^{h} h_{k\alpha}/2 + \sum_{k=h+1}^{n} e_{k\alpha}^{(o)}/2} N^{-1} N^{-h_\alpha/2}
\]

where in the last step we used the definitions of \( s_\alpha \) and \( h_\alpha \), \( e_{k\alpha} \geq 2h_{k\alpha} \) for \( 1 \leq k \leq h \) (since \( e_{k\alpha} \geq 3 \) whenever \( h_{k\alpha} = 1 \)), and \( h_{k\alpha} = 0 \) for \( k \geq h + 1 \).

Combining the above four cases, we obtain that

\[
\|E_R\| = \prod_{\alpha=1}^{m} R_\alpha < N^{-m} N^{-\frac{1}{2} \sum_{\alpha=1}^{h} h_\alpha} \Phi^{\frac{1}{2} \sum_{k=1}^{h} h_{k\alpha}/2 + \sum_{k=h+1}^{n} e_{k\alpha}^{(o)}/2} 
\]

Recall that \( \sum_{\alpha} h_\alpha = h \). Then to prove (B.10), it remains to show that

\[
(B.14) \sum_{\alpha} \left( \sum_{k=1}^{h} h_{k\alpha} + \sum_{k=h+1}^{n} e_{k\alpha}^{(o)} \right) \geq 4p.
\]

For \( k = 1, \ldots, h \), using (B.13) and (B.4) we get that

\[
\sum_{\alpha=1}^{m} h_{k\alpha} = 1 = \text{deg}(b_k, P).
\]

For \( k = h + 1, \ldots, n \), using (B.12) and (B.8) we get that

\[
\sum_{\alpha=1}^{m} e_{k\alpha}^{(o)} = \text{deg}_o(b_k, Q) \geq \text{deg}(b_k, P).
\]

With (B.4), we then conclude (B.14), which finishes our proof.

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