1. Introduction

In this paper, we study the weak solutions to the incompressible 2D-MHD with power-law type nonlinear viscous fluid:

\[
\begin{aligned}
&u_t - \nabla \cdot S + (u \cdot \nabla)u + \nabla \pi = (b \cdot \nabla)b, \\
b_t - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0, & & \text{in } Q_T = \mathbb{R}^2 \times (0, T), \\
\text{div } u = 0, & & \text{div } b = 0,
\end{aligned}
\]

(1)

where \(u : \mathbb{R}^2 \times (0, T) \rightarrow \mathbb{R}^2\) is the flow velocity vector, \(b : \mathbb{R}^2 \times (0, T) \rightarrow \mathbb{R}^2\) is the magnetic vector, and \(\pi : \mathbb{R}^2 \times (0, T) \rightarrow \mathbb{R}\) is the total pressure. We consider the initial value problem of (1), which requires initial conditions:

\[
\begin{aligned}
u(x, 0) &= u_0(x), \\
b(x, 0) &= b_0(x) \quad x \in \mathbb{R}^2.
\end{aligned}
\]

(2)

We assume that the initial data \(u_0(x), b_0(x) \in L^2(\mathbb{R}^2)\) hold the incompressibility, i.e., \(\nabla u_0(x) = 0\) and \(\nabla b_0(x) = 0\), respectively. In this paper, we deal with \(S\) given as

\[
S = S(\nabla u) = (\mu_0 + \mu_1 |\nabla u|^2)^{\gamma - 2} \nabla u, \quad D_u = \frac{\nabla u + \nabla u^\top}{2},
\]

(3)

where \(\mu_0 \geq 0\) and \(\mu_1 > 0\) are constants (see, e.g., [1, 2]).

In modern industrial application, non-Newtonian fluids play an important role (see [3–5]). In particular, equation (1) is the simplest self-consistent model which describes the dynamics of electrically conducting liquid with involved rheological structure in a magnetic field.

Some examples of non-Newtonian fluids are coal-water, glues, soaps, etc. (see, e.g., [6]). One class of non-Newtonian fluids can be defined by \(S = \mu(|D|) D\) (\(D\) is the rate of the strain tensor, \(\mu(\cdot) > 0\) a real function). That is, the relation between the shear stress and the strain rate is nonlinear. In this paper, we study the case \(\mu(s) = \mu_0 + \mu_1 s^{q-2}\) which is called power law fluids. Commonly, the case of \(q > 2\) describes dilatant (or shear thickening) fluids whose viscosity increases with the rate of shear (see, e.g., [6]). On the other hand, pseudoplastic (or shear thinning) fluids correspond to the case of \(1 < q < 2\), where viscosity decreases with the increasing rate of shear (see, e.g., [1]).

In what follows, we review some known results related to our concerns. For incompressible Navier-Stokes equation for a non-Newtonian type, namely, \(b = 0\) in (1), the existence of weak solutions for \((3n + 2)/(n + 2) \leq q\) was first obtained in [7, 8], which is unique for \((n + 2)/2 \leq q\) for any dimension.
Later, the existence of weak solutions was investigated for $2n/(n + 2) < q$ in [10, 11]. On the other hand, in the case of $q = 2$, that is, $S(Du) = Du$ and $n = 3$, numerous results are known. Among them, we only mention that Ferreira and Villamizar-Roa [12] showed well-posedness, time decay, and stability for 3D magnetohydrodynamic equations.

In [13, 14], Samokhin first studied a nonstationary system of equations describing the motion of the Ostwald-de Waale media type and showed a unique existence of a generalized solution for $q \geq 1 + (2n/(n + 2))$ to the problem based on the Faedo-Galerkin method and the monotone operator method. Later on, Gunzburger et al. in [15] proved the global unique solvability of the initial boundary value problem for the modified Navier-Stokes equations coupled with the Maxwell equations. Here, the authors use the strain tension containing the diffusion operator; that is, they do not deal with the degenerate power law fluids. Recently, Razafimandimby [16] proved the existence of weak solutions for $q \in (1, (2n + 6)/(n + 2))$ to this model of bipolar type.

In this paper, we will prove the global-in-time existence and uniqueness of the weak solutions for the incompressible 2D-MHD with power law-type nonlinear viscous fluid (1)–(2) under a condition on the range of $q$. Our results are based on the standard Galerkin method and some uniform estimates.

Denote $M^2_{\text{sym}}$ by the vector space of all symmetric $2 \times 2$ matrices $\zeta = (\zeta_{ij})_{i,j=1,2}$. Let $S = |Du|^{q-2}Du$ and $1 \leq q < \infty$. The deviatoric stress tensor $S = (S_{ij})$, $i, j = 1, 2$, satisfies the following conditions:

(i) $S : Q_T \times M^2_{\text{sym}} \rightarrow M^2_{\text{sym}}$ is a Carathéodory function

(ii) Symmetry: $S_{ij} = S_{ji}$

(iii) Polynomial growth:

$$|S_{ij}(\xi)| \leq \left( \mu_0 + \mu_1 |\xi|^{q-2} \right) |\xi|.$$

(iv) Coercivity condition: there exists $c_1 > 1$ such that

$$\left( \mu_0 + \mu_1 |\xi|^{q-2} \right) |\eta|^2 \leq \frac{\partial S_{ij}}{\partial \xi_{kl}} \eta_{kl} \eta_{ij} \leq c_1 \left( \mu_0 + \mu_1 |\xi|^{q-2} \right) |\eta|^2.$$

(v) Strict monotonicity: for all $\xi, \eta \in M^2_{\text{sym}}(\xi \neq \eta), S(\xi) - S(\eta): (\xi - \eta) > 0$

By the weak solution of the incompressible 2D-MHD with power law-type nonlinear viscous fluid, we mean solutions satisfying the following definitions:

**Definition 1** (weak solution). Let $\mu_0 \geq 0, \mu_1 > 0, q < 1$. Suppose that $u_{\omega_0}, b_0 \in L^2(\mathbb{R}^2)$. We say that $(u, b)$ is a weak solution of the incompressible 2D-MHD with power law-type nonlinear viscous fluid (1)–(2) if $u$ and $b$ satisfy the following:

$$u \in L^\infty([0, T) ; L^2(\mathbb{R}^2)) \cap L^2([0, T) ; W^{1, q}(\mathbb{R}^2)),$$

$$b \in L^\infty([0, T) ; L^2(\mathbb{R}^2)) \cap L^2([0, T) ; H^1(\mathbb{R}^2)).$$

(i) $(u, b)$ satisfies (1) in the sense of distribution; that is,

$$\int_0^T \int_{\mathbb{R}^2} \left( \frac{\partial \phi}{\partial t} + (u \cdot \nabla) \phi \right) u \, dx \, dt + \int_0^T \int_{\mathbb{R}^2} S(Du) : \nabla \phi \, dx \, dt = \int_{\mathbb{R}^2} u_0 \phi(x, 0) \, dx + \int_0^T \int_{\mathbb{R}^2} (b \cdot \nabla) \phi \, b \, dx \, dt,$$

$$\int_0^T \int_{\mathbb{R}^2} \left( \frac{\partial \phi}{\partial t} + \Delta \phi + (u \cdot \nabla) \phi \right) b \, dx \, dt = \int_{\mathbb{R}^2} b_0 \phi(x, 0) \, dx + \int_0^T \int_{\mathbb{R}^2} (b \cdot \nabla) \phi \, u \, dx \, dt,$$

for all $\phi \in C_0^\infty((\mathbb{R}^2 \times [0, T))$ with $\text{div } \phi = 0$, and

$$\int_{\mathbb{R}^2} u \cdot \nabla \psi \, dx = 0, \quad \int_{\mathbb{R}^2} b \cdot \nabla \psi \, dx = 0,$$

for every $\psi \in C_0^\infty(\mathbb{R}^2)$.

**Theorem 2.** Let $2 < q < \infty$ and $\mu_0 \geq 0$ and $\mu_1 > 0$. Assume that $u_{\omega_0}, b_0 \in L^2(\mathbb{R}^2)$. A weak solution $(u, b)$ of the incompressible 2D-MHD with power law-type nonlinear viscous fluid (1)–(2) exists. In particular, in the case $\mu_0 > 0$ and $\mu_1 > 0$, the weak solution $(u, b)$ is unique. Moreover, we obtain the following decay rate of the weak solution:

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \leq C(1 + t)^{-1/2}.$$

2. Preliminaries

In this section, we introduce the notation. Let $I$ be a finite time interval. For $1 \leq a < \infty$, we denote by $W^{k,a}(\mathbb{R}^2)$ the usual Sobolev spaces, namely, $W^{k,a}(\mathbb{R}^2) = \{ f \in L^a(\mathbb{R}^2) : D^a f \in L^a(\mathbb{R}^2), 0 \leq |a| \leq k \}$. The set of the $q$-th power Lebesgue integrable functions on $\mathbb{R}^2$ is denoted by $L^q(\mathbb{R}^2)$, and $L^q_{loc}(\mathbb{R}^2)$ indicates the set of the locally $q$-th power Lebesgue integrable functions defined on $\mathbb{R}^2$. For a function $f(x, t), \Theta \subset \mathbb{R}^2$, and $J \subset I$, we denote $\|f\|_{L^q(\Theta \times J)} = \|f\|_{L^q(\Theta \times J)}$. For vector fields $u, v$, we write $(u, v)_{j=1,2,3}$ as $u \otimes v$. We denote $A : B = a_{ij} b_{ij}$ for $3 \times 3$ matrices $A = (a_{ij}), B = (b_{ij})$. The letter $C$ is used to represent a generic constant, which may change from line to line.
Before looking for a solution for the system (1), we give a lemma.

Lemma 3. Let \((u, b)\) be a solution to the initial value problem of (1)–(2) with the initial data \(u_0, b_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)\). Then, we have for \(2 \leq p\)

\[
|\dot{u}(\xi, t)| + |\dot{b}(\xi, t)| \leq C(\dot{\phi}_0(\xi) + |\dot{\phi}_0(\xi)| + |\xi|) \int_0^t \left(\|u(s)\|_{L^p(\mathbb{R}^2)}^p + \|\dot{b}(s)\|_{L^p(\mathbb{R}^2)}^p\right)ds
\]

\[
+ C|\xi| \left(\int_0^t \|u(s)\|_{L^p(\mathbb{R}^2)}^{p(p-1)}ds\right)^{1/(p-1)},
\]

(10)

where \(C\) depends only on the \((L^2 \cap L^1)(\mathbb{R}^2)\)-norm of \(u_0\) and \(b_0\).

Proof. The proof is easily checked. Indeed, it is almost the same as that in [17] replacing (2.5) in [17] by

\[
\int_0^t \|\nabla u(s)\|_{L^p(\mathbb{R}^2)}^{p-1}ds \leq C \left(\int_0^t \|u(s)\|_{L^p(\mathbb{R}^2)}^{1/(p-1)}\right)^{1/(p-1)}
\]

\[
\cdot \left(\int_0^t \|\nabla u(s)\|_{L^p(\mathbb{R}^2)}^{p-1}ds\right)^{(p-2)/(p-1)}
\]

\[
\leq C \left(\int_0^t \|u(s)\|_{L^p(\mathbb{R}^2)}^{p/(p-1)}ds\right)^{1/(p-1)}, \quad p > 2.
\]

(11)

3. Proof of Theorem 2

In this paper, we assume that \(\mu_0 = 0\) and \(\mu_1 = 1\) for convenience. Let

\[
V_q = \left\{ \phi \in D'(\mathbb{R}^2)^2 : \nabla \cdot \phi = 0 \right\},
\]

(12)

with \(\|\phi\|_{V_q} = \|D\phi\|_{L^2(\mathbb{R}^2)}\). Now, we will construct the existence of a weak solution to the system (1) via the standard Galerkin method. For this, first of all, we need to find a countable dense subset of the space \(\{\phi \in \mathcal{D}(\mathbb{R}^2) : \nabla \phi = 0\}\) in \(W^{2,2}(\mathbb{R}^2) \cap V_q\) in Lemma 3.10 of [18].

Now, we consider Galerkin approximate solutions \(u^m(t) = \sum_{i=1}^m g_i^m(t)\phi_j\) and \(b^m(t) = \sum_{i=1}^m h_i^m(t)\psi_j\), where the \(\phi_j, \psi_j\) are the eigenfunctions which are chosen by using Lemma 3.10 of [18].

\[
(u^m - \nabla \cdot S(Du)) + (u^m \cdot \nabla)u^m - (b^m \cdot \nabla)b^m, = 0,
\]

(13)

\[
(b^m - \Delta b^m + (u^m \cdot \nabla)b^m - (b^m \cdot \nabla)u^m, \psi) = 0,
\]

for \(\phi \in \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}\) and \(\psi \in \text{span}\{\psi_1, \psi_2, \ldots, \psi_m\}\). The initial conditions were

\[
u^m(0, x) = \sum_{i=1}^m a_i\phi_i(x), \quad b^m(0, x) = \sum_{i=1}^m c_i\psi_i(x),
\]

(15)

where \(a_i = \int_{\mathbb{R}^2} u^m(x, 0) \cdot \phi_i(x)\) and \(c_i = \int_{\mathbb{R}^2} b^m(x, 0) \cdot \psi_i(x)\). Indeed, the functions \(g^m(t)\) and \(h^m(t)\) satisfy the following ordinary differential equations as follows:

\[
g^m(t) + \lambda_m g^m(t) + (g^m(t))^{p-1} \int_{\mathbb{R}^2} \nabla \phi \cdot \nabla \psi \phi = 0,
\]

(16)

\[
h^m(t) + \lambda_m h^m(t) + g^m(t)h^m(t) \int_{\mathbb{R}^2} \nabla \psi \cdot \nabla \phi \psi = 0.
\]

(17)

By the Carathéodory theorem (see [19], Theorem 3.4 in Appendix), there exist \(T_m\) such that equation (16) has unique solutions on \([0, T_m]\). Now set \(T_m = T, T < \infty\).

Proof of Theorem 2. For a proof of existence for a weak solution, we assume that \(\mu_0 = 0\) because it is easier for \(\mu_0 > 0\).

Part A: existence

Multiplying equation (13) by \(u^m\) and equation (14) by \(b^m\) and summing up the equations, we have

\[
\|u^m(T)\|_{L^2(\mathbb{R}^2)}^2 + \|b^m(T)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u^m\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla b^m\|_{L^2(\mathbb{R}^2)}^2
\]

\[
\leq \|u^m(0)\|_{L^2(\mathbb{R}^2)}^2 + \|b^m(0)\|_{L^2(\mathbb{R}^2)}^2,
\]

(18)

where we use the divergence free condition, Korn’s inequality, and vector identity for the magnetic vector field \(b\). For the distributive time derivative \(du^m/dt\), we have \((du^m/dt) \in L^4(0, T; (W^{1,4}(\mathbb{R}^2))^2)

Here, \(q^*\) is the conjugate of \(p\), and \((W^{1,4}(\mathbb{R}^2))^*\) is the dual space for \(W^{1,4}(\mathbb{R}^2)\). Indeed, for \(\phi \in L^4(0, T; (W^{1,4}(\mathbb{R}^2))) \cap L^2(0, T; (W^{1,2}(\mathbb{R}^2)))\) with \(\nabla \phi = 0,

\[
\int_{\mathbb{R}^2} \frac{du^m}{dt} \cdot \phi dx dt = \int_{\mathbb{R}^2} \frac{Du^m}{dt} \cdot \phi dx dt = \int_{\mathbb{R}^2} \frac{Du^m}{dt} \cdot \phi dx dt + \int_{\mathbb{R}^2} \frac{Du^m}{dt} \cdot \phi dx dt
\]

\[
= -\int_{\mathbb{R}^2} \frac{Du^m}{dt} \cdot \phi dx dt - \int_{\mathbb{R}^2} (u^m \cdot \phi) \phi dx dt - \int_{\mathbb{R}^2} (b^m \cdot b^m) \phi dx dt
\]

(19)

(i) Estimate of \(J_1\): using Hölder’s inequality and the energy estimate (18), we have

\[
\text{(i) Estimate of } J_1: \text{ using Hölder’s inequality and the energy estimate (18), we have}
\]

\[
\]
(i) Estimate of $\mathcal{J}_2$: since $u^m$ belongs to $L^{2q}(0, T; L^{2q})$, we have
\[
|\mathcal{J}_2| \leq \|u^m \otimes u^m\|_{L^1(0, T; L^{2q})} \leq \|u^m\|_{L^q(0, T; L^{2q})} \leq C(\|u_0\|_{L^2(\mathbb{R}^2)} + \|b_0\|_{L^q(\mathbb{R}^2)}) \leq C.
\]
(20)

(ii) Estimate of $\mathcal{J}_3$: we combine (20), (21), and (22) to get
\[
\frac{du^m}{dt} \in L^q(0, T; (W^{1,q})^*) + L^q(0, T; (W^{1,4})^*) + L^2(0, T; (W^{1,2})^*).
\]
(23)

To obtain the distributive time derivative $db^m/dt$, using the similar argument above, we have
\[
\frac{db^m}{dt} \in L^2(0, T; (W^{1,2})^*) + L^2(0, T; (W^{1,4})^*).
\]
(24)

Indeed, for $\phi \in L^2(0, T; W^{1,2}) \cap L^2(0, T; W^{1,4})$ with $\nabla \phi = 0$,
\[
\int_0^T \int_{\mathbb{R}^2} \frac{du^m}{dt} \cdot \phi dx dt = -\int_0^T \int_{\mathbb{R}^2} \nabla b^m \cdot \nabla \phi dx dt = \mathcal{J}_1 + \mathcal{J}_2.
\]
(25)

(iv) Estimate of $\mathcal{J}_1$: using Hölder's inequality and the estimate (18), we have
\[
|\mathcal{J}_1| \leq \|\nabla b^m\|_{L^q(0, T; L^{2q})} \|\nabla \phi\|_{L^q(0, T; L^{2q})} \leq C.
\]
(26)

(v) Estimate of $\mathcal{J}_2$: using Hölder's inequality, we have
\[
|\mathcal{J}_2| \leq C\|u^m\|_{L^{2q}(0, T; L^{2q})} \|b^m\|_{L^q(0, T; L^{4q})} \|\nabla \phi\|_{L^q(0, T; L^{2q})} \leq C.
\]
(27)

Due to the energy estimate (18) and time derivative class for $u^m$ and $b^m$, we can choose subsequences $u^{m_k}$ and $b^{m_k}$ such that
\[
\begin{align*}
\lim_{k \to \infty} u^{m_k} & \rightarrow u \text{ weakly in } L^\infty(0, T; L^q(\mathbb{R}^2)) \cap L^q(0, T; W^{1,q}(\mathbb{R}^2)), \\
\lim_{k \to \infty} b^{m_k} & \rightarrow b \text{ weakly in } L^\infty(0, T; L^q(\mathbb{R}^2)) \cap L^q(0, T; W^{1,2}(\mathbb{R}^2)), \\
\partial_t u^{m_k} & \rightarrow \partial_t u \text{ weakly in } L^q(0, T; (W^{1,q})^*) \\
& \quad + L^q(0, T; (W^{1,4})^*), \\
\partial_t b^{m_k} & \rightarrow \partial_t b \text{ weakly in } L^2(0, T; (W^{1,2})^*) + L^2(0, T; (W^{1,4})^*),
\end{align*}
\]
(28)

when $k \to \infty$. From the class of $u^{m_k}$ and $b^{m_k}$ in the convergence above and by the Aubin-Lions lemma (e.g., [20], Lemma 3.1), we have
\[
\begin{align*}
u^{m_k} & \rightarrow u \text{ strongly in } L^p_{loc}(\mathbb{R}^2 \times I), p \in [1, 2q), \\
b^{m_k} & \rightarrow b \text{ strongly in } L^{\tilde{p}}_{loc}(\mathbb{R}^2 \times I), \tilde{p} \in [1, 4).
\end{align*}
\]
(29)

Thus, we have
\[
\begin{align*}
\lim_{k \to \infty} u^{m_k} & \rightarrow u \text{ strongly in } L^2_{loc}(\mathbb{R}^2 \times I), \\
\lim_{k \to \infty} b^{m_k} & \rightarrow b \text{ strongly in } L^2_{loc}(\mathbb{R}^2 \times I),
\end{align*}
\]
(30)

as $k \to \infty$. So then, due to the weak and strong convergence above, it is possible to pass to the limit in the nonlinear terms (see, e.g., [21]). Moreover, $S(Du^{m_k})$ is uniformly bounded in $L^q(\mathbb{R}^2 \times (0, T))$, and so $S(Du) \rightarrow A$ in this class. Hence, we will check $A = S(Du)$ which is shown by monotonicity trick (see [13], pp. 635-636). For this, we note that for $q \geq 2$,
\[
\int_0^T (u \cdot \nabla u) \cdot u dx + \int_0^T (u \cdot b) \cdot b dx \leq C(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)
\]
(32)

From the energy equality, we have for $0 \leq s \leq T$
\[
\frac{1}{2} (\|u(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2) + \int_0^s \|\nabla b\|_{L^2}^2 dt + \int_0^s A \cdot Du dx dt
\]
\[
= \frac{1}{2} (\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2).
\]
Define
\[
X^m = \int_0^t (S(Du^m) - S(D\phi), Du^m - D\phi) dt + \frac{1}{2} \|u(s)\|_{L^2}^2, \phi \in L^1(0, T; W^{1,q}_0).
\]

(33)

Here, \( W^{1,q}_0 := \{ v \in W^{1,q}(\mathbb{R}^2) : \nabla v = 0 \}. \) So, due to the property of the monotone operator \( S \) and the semicontinuity of the norm, we obtain
\[
\lim\inf_{m \to \infty} X^m_s \geq \frac{1}{2} \| u(s) \|_2^2,
\]
and also
\[
\lim_{m \to \infty} X^m_s = \int_0^s (b \cdot \nabla b) \cdot dt - \int_0^s (A, D\phi) dt - \int_0^s (S(D\phi), Du - D\phi) dt.
\]

(35)

Then, due to the equality (32), we have
\[
\int_0^t (A - S(D\phi)) \cdot (Du - D\phi) dt \geq 0, \text{ a.e.s } [0, T].
\]

(36)

Putting \( \phi = u - \lambda w \) for \( \lambda > 0 \) and \( w \in L^4(0, T; W^{1,q}_0) \), we obtain
\[
\int_0^t (A - S(Du - \lambda w)) \cdot \nabla dt \geq 0.
\]

(37)

As \( \lambda \to 0 \), we deduce
\[
\int_0^t (A - S(Du)) \cdot \nabla dt \geq 0,
\]
which means that \( A = S(Du) \) for a.e. \( s \in [0, T] \). Hence, the proof of existence for weak solutions is completed.

Part B: uniqueness

For this part, we consider the equation for \( v = u^1 - u^2, h = b^1 - b^2 \), and \( \pi = \pi^1 - \pi^2 \):
\[
\partial_t v - \nabla \cdot \left( (1 + |Du^1|)^{q-2} Du^1 \right) + \nabla \cdot \left( (1 + |Du^2|)^{q-2} Du^2 \right) + (u^1 \cdot \nabla) v + (v \cdot \nabla) u^2 - (b^1 \cdot \nabla) h - (h \cdot \nabla) b^2 + \nabla \pi = 0,
\]
\[
\partial_t h - \Delta h + (u^1 \cdot \nabla) h + (v \cdot \nabla) b^2 - (b^1 \cdot \nabla) v - (h \cdot \nabla) u^2 = 0,
\]

(39)

with \( \text{div } v = 0 \) and \( \text{div } h = 0 \). Testing \( v \) and \( h \) to the equations above, we have
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} (|v|^2 + |h|^2) dx + \min \{ C, 1 \} \int_{\mathbb{R}^2} (|\nabla u^1|^2 + |\nabla b^1|^2) \right) \leq 0,
\]

(42)

where \( C > 0 \) is a Korn-type constant. Applying Plancherel’s theorem to (42) yields
\[
\frac{d}{dt} \int_{\mathbb{R}^2} (|u^1|^2 + |b^1|^2) dx + \min \{ C, 1 \} \int_{\mathbb{R}^2} |\xi|^2 \cdot \left( |\mathcal{F}(u^1, \xi, t)|^2 + |\mathcal{F}(b^1, \xi, t)|^2 \right) \leq 0.
\]

(43)

Put \( \mathcal{F}(\xi, t) = |u^1(\xi, t)|^2 + |b^1(\xi, t)|^2 \). Let \( f(t) \) be a smooth function of \( t \) with \( f(0) = 1, f(t) > 0 \) and \( f'(t) > 0 \).

Set \( S(t) = \{ \xi \in \mathbb{R}^n : \min \{ C, 1 \} |f(t)| |\xi|^2 \leq f'(t) \} \). Then,
\[
2 \min \{ C, 1 \} f(t) \int_{\mathbb{R}^2} |\xi|^2 |\mathcal{F}(\xi, t)|^2 d\xi
\]
\[
\geq f'(t) \int_{\mathbb{R}^2} |\mathcal{F}(\xi, t)|^2 d\xi - f'(t) \int_{S(t)} |\mathcal{F}(\xi, t)|^2 d\xi.
\]

(44)

Since
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} |\mathcal{F}(\xi, t)|^2 d\xi \right) + 2 \min \{ C, 1 \} f(t) \int_{\mathbb{R}^2} |\xi|^2 |\mathcal{F}(\xi, t)|^2 d\xi
\]
\[
\leq f'(t) \int_{\mathbb{R}^2} |\mathcal{F}(\xi, t)|^2 d\xi.
\]

(45)
we have

\[
\frac{d}{dt} \left( f(t) \int_{\mathbb{R}} |u \wedge (\xi, t)|^2 d\xi \right) \leq f'(t) \int_{S(t)} |u \wedge (\xi, t)|^2 d\xi. \quad (46)
\]

Integrating in time, we get

\[
f(t) \int_{\mathbb{R}} \left( |u \wedge (\xi, t)|^2 + |b \wedge (\xi, t)|^2 \right) d\xi \\
\leq \int_{\mathbb{R}} \left( |u \wedge (\xi)|^2 + |b \wedge (\xi)|^2 \right) d\xi + C \int_0^t f'(s) \int_{S(s)} |\xi|^2 d\xi ds
\]

\[
\times \left( |u \wedge (\xi, s)|^2 + |b \wedge (\xi, s)|^2 \right) d\xi ds.
\]

Set \( f(t) = (1 + t)^2 \). From Lemma 3 with Young’s inequality and the energy estimate, we have

\[
(1 + t)^2 \int_{\mathbb{R}} \left( |u \wedge (\xi, t)|^2 + |b \wedge (\xi, t)|^2 \right) d\xi
\]

\[
\leq \int_{\mathbb{R}} \left( |u \wedge (\xi)|^2 + |b \wedge (\xi)|^2 \right) d\xi + C \int_0^t (1 + s) \int_{S(s)} |\xi|^2 d\xi ds
\]

\[
\times \left( |u \wedge (\xi, s)|^2 + |b \wedge (\xi, s)|^2 \right) d\xi ds + C \int_0^t (1 + s) \int_{S(s)} s|\xi|^2 d\xi ds
\]

\[
\times \left( \int_0^s |u(s)|^2_{L^2} + |b(s)|^2_{L^2} \right) d\xi ds
\]

\[
+ C \int_0^t (1 + s) \int_{S(s)} s|\xi|^2 \left( \int_0^s |u(s)|^2_{L^2(\mathbb{R}^d)} ds + C \right) d\xi ds
\]

\[
\leq \int_{\mathbb{R}} \left( |u \wedge (\xi)|^2 + |b \wedge (\xi)|^2 \right) d\xi + C \int_0^t (1 + s) \int_{S(s)} |\xi|^2 d\xi ds
\]

\[
\times \left( |u \wedge (\xi, s)|^2 + |b \wedge (\xi, s)|^2 \right) d\xi ds + C \int_0^t (1 + s) \int_{S(s)} \left( \int_0^s |u(s)|^2_{L^2(\mathbb{R}^d)} ds + C \right) d\xi ds
\]

\[
\leq C + C(1 + t) + \left( \int_0^t (|u(s)|^2_{L^2} + |b(s)|^2_{L^2}) ds \right). \quad (48)
\]

Thus, we get

\[
(1 + t) \int_{\mathbb{R}} \left( |u \wedge (\xi, t)|^2 + |b \wedge (\xi, t)|^2 \right) d\xi
\]

\[
\leq C + C \int_0^t (1 + s) \left( |u(s)|^2_{L^2} + |b(s)|^2_{L^2} \right) (1 + s)^{-1} ds.
\]

Applying Gronwall’s inequality, we immediately deduce that

\[
\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \leq C(1 + t)^{-1/2},
\]

thus, we finally obtain the desired result.

**Appendix**

Here, we mention the existence of unique strong solution for (1)–(2). Its proof is easily checked from the argument in [15] or [22]. And thus, we omit the proof.

**Definition A.1.** Let \( 2 < q < \infty \) and \( \mu_0 \geq 0 \) and \( \mu_1 > 0 \). Suppose that \( u_0 \in (W^{1,2} \cap W^{1,q})(\mathbb{R}^2) \) and \( b_0 \in W^{1,2}(\mathbb{R}^2) \). We say that a weak solution \((u, b)\) is a strong solution to the incompressible 2D-MHD equations of non-Newtonian fluids (1)–(2) if

\[
\forall u \in L^{\infty}(0, T; L^q \cap L^2(\mathbb{R}^2)),
\]

\[
b \in L^q(0, T; W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2)),
\]

\[
u_t, b_t \in L^q(0, T; W^{q,q'}(\mathbb{R}^2)), S(Du) \in L^q(0, T; W^{1,q'}(\mathbb{R}^2)).
\]

\[
\int_0^T \int_{\mathbb{R}^2} |Du|^{q-2} |D^2u|^2 dx dt < \infty.
\]

Here, \( q' \) means the Hölder conjugate of \( q \).

**Theorem A.2.** Let \( 2 < q < \infty \) and \( \mu_0 \geq 0 \) and \( \mu_1 > 0 \). Suppose that \( u_0 \in (W^{1,2} \cap W^{1,q})(\mathbb{R}^2) \) and \( b_0 \in W^{1,2}(\mathbb{R}^2) \). Then, there exists a strong solution \((u, b)\) of the incompressible 2D-MHD equations of non-Newtonian type (1)–(2) in the sense of Definition A.1.

**Data Availability**

This paper uses the method of theoretical analysis.

**Conflicts of Interest**

The author declares that he has no conflicts of interest.

**Acknowledgments**

Jae-Myoung Kim was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521).

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