SUMS OF FOUR SQUAREFUL NUMBERS

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ABSTRACT. We find an asymptotic formula for the number of primitive vectors \((z_1, \ldots, z_4) \in (\mathbb{Z}^\times)^4\) such that \(z_1, \ldots, z_4\) are all squareful and bounded by \(B\), and \(z_1 + \cdots + z_4 = 0\). Our result agrees in the power of \(B\) and \(\log B\) with the Campana–Manin conjecture of Pieropan, Smeets, Tanimoto and Várilly-Alvarado.

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1. INTRODUCTION

The notion of Campana points, first discussed by Campana in [7] and [8], and Abramovich in [1], is receiving increasing attention in the field of arithmetic geometry. Campana points associated to an orbifold \((X, D)\) can be viewed as rational points on \(X\) that are integral with respect to a weighted boundary divisor \(D\), and thus provide a way to interpolate between integral and rational points.

The quantitative study of the arithmetic of Campana orbifolds was kick-started by the discussions in [1] and [22]. A motivating example for the development of the theory was given in [22], where Poonen poses the problem of finding the number of coprime integers \(z_0, z_1\) such that \(z_0, z_1\) and \(z_0 + z_1\) are all squareful and bounded by \(B\) (We recall that a nonzero integer \(z\) is \(m\)-full if for any prime \(p|z\), we have \(p^m|z\), and squareful if it is 2-full). This corresponds

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to counting Campana points on the orbifold $\mathbb{P}^1(D)$, where $D$ is the divisor $\frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. Since this problem seems too difficult at the moment, in [28], Van Valckenborgh considers the higher-dimensional analogue $\mathbb{P}^k(D)$, where $D = \sum_{i=1}^{k} \frac{1}{2}D_i$ for hyperplanes $D_1, \ldots, D_k$. This leads to the study of the asymptotic size $N_k(B)$ of the set

$$N_k(B) = \#(\mathcal{M}_k(B) \setminus \mathcal{T})$$

as $B \to \infty$. The main result in [28, Theorem 1] is that for any $k \geq 5$, we have $N_k(B) = cB^{k/2-1} + O(B^{k/2-1-\delta})$ for some constants $c, \delta > 0$. The remaining cases of interest are $k = 3, 4$. In this paper, we treat the case $k = 4$. We define

$$\mathcal{T} = \{(z_1, \ldots, z_4) \in \mathbb{Z}^4_{\text{prim}} : z_1 \cdots z_4 = 0\},$$

and consider the counting problem

$$N(B) = \#(\mathcal{M}_4(B) \setminus \mathcal{T}).$$

Our main result is the following theorem.

**Theorem 1.1.** For any $\epsilon > 0$ we have

$$N(B) = cB + O(B^{734/735+\epsilon}),$$

for an absolute constant $c > 0$ given explicitly in (5.9) and Lemma 5.5.

Interest in the quantitative arithmetic of Campana orbifolds has been further sparked by the recent work [21], in which Pieropan, Smeets, Tanimoto and Váríly-Alvarado formulate a Manin-type conjecture for Fano Campana orbifolds. The authors establish their conjecture in the special case of vector group compactifications, using the height zeta function method developed by Chambert-Loir and Tschinkel in [9] and [10].

In Section 2, we discuss the compatibility of Theorem 1.1 with the prediction from [21, Conjecture 1.1], which is henceforth referred to as the *Campana–Manin conjecture*. We find that our result agrees with the Campana–Manin conjecture in the power of $B$ and $\log B$, and the set $\mathcal{T}$ that we remove corresponds to a thin set of Campana points. The leading constant $c$ from Theorem 1.1 is discussed in Section 5. However, in the light of our recent work [25], we do not expect the leading constant to agree with the prediction from the Campana–Manin conjecture without the removal of further thin sets.

A number of other instances of the Campana–Manin conjecture have been studied in addition to the results mentioned above. In [6], Browning and Yamagishi consider the orbifold $\mathbb{P}^n(D)$ for $D = \sum_{i=0}^{n+1} (1 - \frac{1}{m_i})D_i$, where $D_0, \ldots, D_n$ are coordinate hyperplanes, and $D_{n+1}$ is a general hyperplane.
Their main result is an asymptotic formula for the number of points on this orbifold under the assumptions that $m_i \geq 2$ for all $i$, and that there exists some $j \in \{0, \ldots, n+1\}$ such that
\[
\sum_{0 \leq i \leq n+1 \atop i \neq j} \frac{1}{m_i(m_i+1)} \geq 1.
\]

In [20], Pieropan and Schindler establish the Campana–Manin conjecture for complete smooth split toric varieties satisfying an additional technical assumption, by developing a very general version of the hyperbola method. In [29], Xiao treats the case of biequivariant compactifications of the Heisenberg group over $\mathbb{Q}$, using the height zeta function method. Finally, in [27], Streeter studies $m$-full values of norm forms by considering the orbifold $(\mathbb{P}^{d-1}_K, (1-\frac{1}{m})\mathbb{V}(N_{E/K}))$, where $K$ is a number field, $\mathbb{V}(N_{E/K})$ the divisor cut out by a norm form associated to a degree-$d$ Galois extension $E/K$, and $m \geq 2$ is an integer which is coprime to $d$ if $d$ is not prime.

We now summarise the proof of Theorem 1.1. The approach is broadly similar to previous results by Browning and Yamagishi in [6] and Van Valkenborgh in [28]. Every nonzero squareful integer $z$ can be written uniquely in the form $z = y^3x^2$, where $y$ is a nonzero square-free integer and $x$ is a positive integer. For a fixed choice of $y = (y_1, \ldots, y_4)$, the equation $z_1 + \cdots + z_4 = 0$ defines a quadric $Q_y$ cut out by the polynomial $F_y^3(x) = \sum_{i=1}^{4} y_i^3x_i^2$. The condition $z_1 \cdots z_4 = \square$ from (1.2) is equivalent to the condition $y_1 \cdots y_4 = \square$. Therefore
\[
N(B) = \frac{1}{16} \sum_{y \in (\mathbb{Z}_{\neq 0})^4 \atop y_1, \ldots, y_4 \text{ square-free}} N_y(B),
\]

where
\[
N_y(B) = \# \left\{ x \in (\mathbb{Z}_{\neq 0})^4 : F_y^3(x) = 0, \gcd(x_1y_1, \ldots, x_4y_4) = 1, \max_{1 \leq i \leq 4} |y_i^3x_i^2| \leq B \right\}.
\]

The factor $1/16$ comes from the fact that for all $i \in \{1, \ldots, 4\}$, there are two choices for the sign of $x_i$ corresponding to the same choice of $z_i$.

In Section 4 we study the closely related counting problem
\[
N_a(B) = \# \left\{ x \in (\mathbb{Z}_{\neq 0})^4 : F_a(x) = 0, \max_{1 \leq i \leq 4} |a_ix_i^2| \leq B \right\},
\]

where we have removed the coprimality condition and replaced $y$ with an arbitrary vector $a = (a_1, \ldots, a_4) \in (\mathbb{Z}_{\neq 0})^4$.

**Remark 1.2.** Rational points on non-singular quadric surfaces are known to conform to the classical Manin’s conjecture [13]. Let $A = a_1 \cdots a_4$. It can be
shown that the Picard group of the quadric $F_a(x) = 0$ has rank 1 when $A \neq \Box$ and rank 2 when $A = \Box$. Thus Manin’s conjecture predicts that

$$N_a(B) \sim \begin{cases} c_a B, & \text{if } A \neq \Box \\ c'_a B \log B, & \text{if } A = \Box, \end{cases} \quad (1.7)$$

for constants $c_a, c'_a$ (depending on $a$) which can be expressed as a product of local densities. The extra factor of $\log B$ in the case $A = \Box$ explains why it is necessary to remove the set $\mathcal{F}$ in the formulation of Theorem 1.1.

In order to estimate $N_a(B)$, we apply a version of the circle method introduced by Duke, Friedlander and Iwaniec in [12], and refined by Heath-Brown in [15]. The more classical forms of the circle method used in [6] and [28] are not sufficient for our purposes, but the version in [15] is particularly well-suited to dealing with quadratic forms. In fact, in [15], Heath-Brown proves a vast generalisation of (1.7) by finding an asymptotic formula for the quantity

$$N_w(Q, B) = \sum_{x \in \mathbb{Z}^n : Q(x) = 0} w(B^{-1}x),$$

for any $n \geq 3$, any non-singular indefinite quadratic form $Q \in \mathbb{Z}[x_1, \ldots, x_n]$, and any $w : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ belonging to a suitably defined class of smooth weight functions. The following theorem is the main result from Section 4. It features the quantities

$$\Delta = \prod_{i=1}^4 \gcd(a_i, \prod_{j \neq i} a_j), \quad A = a_1 \cdots a_4,$$

as well as the singular series $\mathcal{S}_a$ and the singular integral $\sigma_\infty(\epsilon)$, defined respectively in (4.1) and (4.2). Note that $\sigma_\infty(\epsilon)$ only depends on $\epsilon_i = \text{sgn}(a_i)$.

**Theorem 1.3.** Let $a \in (\mathbb{Z}^\times)^4$ be such that $A \neq \Box$ and $|A| \leq B^{4/7}$. Then

$$N_a(B) = \frac{\mathcal{S}_a \sigma_\infty(\epsilon) B}{|A|^{1/2}} + O\left(\frac{B^{11/42 + \epsilon} \Delta^{1/3}}{|A|^{11/24}}\right), \quad (1.8)$$

where the implied constant depends only on $\epsilon$.

To complete the proof of Theorem 1.1 in Section 5 we apply Theorem 1.3 together with an inclusion-exclusion argument in order to reinsert the coprimality condition and obtain an estimate for $N_y(B)$. Returning to (1.3), we can take a sum over these estimates for $N_y(B)$ in the range $\max_i |y_i| \leq D$, where $D$ is a small power of $B$. The contribution from the remaining range $\max_i |y_i| > D$ is studied in Section 3 using an elementary argument, and forms part of the error term in Theorem 1.1.
It is important in the proof of Theorem 1.1 that the dependence of the estimate in Theorem 1.3 on the coefficients $a_1, \ldots, a_4$ is made completely explicit. Several authors have obtained estimates of this type for quadratic forms. In [2, Theorem 2], Browning applies the machinery from [15] to find such an estimate for the counting problem corresponding to $N_a(B)$, but in $n \geq 5$ variables. Subsequently in [4, Theorem 4.1], Browning and Heath-Brown carried this out in four variables. This latter result is nearly sufficient for our purposes, but Theorem 1.3 represents a refinement in which we drop the assumptions made in [4, Theorem 4.1] that $A$ is nearly square-free and all the coefficients $a_1, \ldots, a_4$ are roughly the same size. Finally, we mention that using other techniques from the geometry of numbers, Comtat provides in [11, Theorem 1.2] a completely uniform estimate for the number of zeros $(x_1, \ldots, x_n) \in \mathbb{Z}_{\text{prim}}^n$ of a non-singular quadratic form $Q$ in $n \geq 3$ variables which lie in an arbitrary box $|x_i| \leq B_i$ for $i \in \{1, \ldots, n\}$. However, the resulting bound $N_a(B) \leq B/|A|^{1/4}$ does not have a good enough dependence on $A$ to be useful for our purposes.

Notation. We take $\mathbb{N} = \mathbb{Z}_{\geq 1}$. We denote by $(\mathbb{Z}_{\neq 0})_{\text{prim}}^n$ the set of vectors $(a_1, \ldots, a_n)$ such that $a_1, \ldots, a_n$ are nonzero integers and $\gcd(a_1, \ldots, a_n) = 1$. For a prime $p$, we let $\nu_p$ denote the $p$-adic valuation. We write $e(\cdot)$ for the function $e^{2\pi i \cdot}$ and $e_q(\cdot)$ for the function $e^{2\pi i \cdot}/q$. We use boldface letters to denote vectors with four components, for example $z = (z_1, \ldots, z_4)$. For a vector $v$, we define $|v| = \max_{1 \leq i \leq 4} |v_i|$. All implied constants will be allowed to depend on $\epsilon$, but nothing else unless otherwise stated. Moreover, $\epsilon$ will denote a small positive number which for convenience we allow to take different values at different points in the argument.

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2. The Campana–Manin conjecture

In this section we recall from [21, Section 3] the definition of Campana points and the statement of the Campana–Manin conjecture. We demonstrate that the set $\mathcal{S}$ removed in the definition of $N(B)$ is a thin set of Campana points, and that the power of $B$ and $\log B$ in Theorem 1.1 is consistent with the prediction from the Campana–Manin conjecture.

Definition 2.1. Let $F$ be a field. A Campana orbifold is a pair $(X, D)$, where $X$ is a smooth variety over $F$ and

$$D = \sum_{\alpha \in \mathcal{A}} e_\alpha D_\alpha$$
is an effective Weil $\mathbb{Q}$-divisor of $X$ over $F$ (where the $D_\alpha$ are prime divisors) such that

(1) For all $\alpha \in \mathcal{A}$, either $\epsilon_\alpha = 1$ or $\epsilon_\alpha$ takes the form $1 - 1/m_\alpha$ for some $m_\alpha \in \mathbb{Z}_{\geq 2}$.

(2) The support $D_{\text{red}} = \sum_{\alpha \in \mathcal{A}} D_\alpha$ of $D$ has strict normal crossings on $X$.

We say that a Campana orbifold is klt if $\epsilon_\alpha \neq 1$ for all $\alpha \in \mathcal{A}$.

Let $(X, D)$ be a Campana orbifold. Campana points will be defined as points $P \in X(F)$ satisfying certain conditions. These conditions are dependent on a finite set $S$ of places of $F$ containing all archimedean places, and a choice of good integral model of $(X, D)$ over $\mathcal{O}_F$. This model is defined to be a pair $(\mathcal{X}, \mathcal{D})$, where $\mathcal{X}$ is a flat, proper model of $X$ over $\mathcal{O}_F$, with $\mathcal{X}$ regular, and

$$\mathcal{D} = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha \mathcal{D}_\alpha,$$

where $\mathcal{D}_\alpha$ denotes the Zariski closure of $D_\alpha$ in $\mathcal{X}$.

**Definition 2.2.** Let $P \in (X \setminus D_{\text{red}})(F)$. For a place $v \notin S$, let $\mathcal{P}_v$ denote the induced point in $\mathcal{X}(\mathcal{O}_v)$ obtained via the valuative criterion for properness as stated in [14, Theorem II.4.7]. For $\alpha \in \mathcal{A}$, we define the intersection multiplicity $n_v(\mathcal{D}_\alpha, P)$ of $\mathcal{D}_\alpha$ and $P$ at $v$ to be the colength of the ideal $\mathcal{P}_v^* \mathcal{D}_\alpha$ in $\mathcal{O}_v$. Then the intersection number of $P$ and $\mathcal{D}$ at $v$ is defined to be

$$n_v(\mathcal{D}, P) = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha n_v(\mathcal{D}_\alpha, P).$$

**Definition 2.3.** Let $(X, D)$ be a Campana orbifold with a good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_F$. A point $P \in (X \setminus D_{\text{red}})(F)$ is a Campana $\mathcal{O}_F$-point of $(\mathcal{X}, \mathcal{D})$ if for all $v \notin S$ and all $\alpha \in \mathcal{A}$, we have

(1) If $\epsilon_\alpha = 1$, then $n_v(\mathcal{D}_\alpha, P) = 0$.

(2) If $\epsilon_\alpha \neq 1$, so that $\epsilon_\alpha = 1 - 1/m_\alpha$ for some $m_\alpha \in \mathbb{Z}_{\geq 2}$, then either $n_v(\mathcal{D}_\alpha, P) = 0$ or $n_v(\mathcal{D}_\alpha, P) \geq m_\alpha$.

We denote the set of Campana $\mathcal{O}_F$-points of $(\mathcal{X}, \mathcal{D})$ by $(\mathcal{X}, \mathcal{D})(\mathcal{O}_F)$.

**Example 2.4.** Campana points are related to $m$-full values of polynomials. We consider a smooth projective variety $X \subseteq \mathbb{P}^n$ over $\mathbb{Q}$, and a divisor

$$D = \sum_{i=1}^k \left(1 - \frac{1}{m_i}\right) D_i,$$

where $m_i \geq 2$ are integers, and $D_i$ are prime divisors on $X$ cut out by polynomial equations $f_i = 0$. Choosing the obvious good integral model $(\mathcal{X}, \mathcal{D})$, a rational point $z \in (X \setminus \bigcup_{i=1}^k D_i)(\mathbb{Q})$, represented by $(z_0, \ldots, z_n) \in \mathbb{Z}_{\text{prim}}^{n+1}$ is
a Campana $\mathbb{Z}$-point of $(\mathcal{X}, \mathcal{D})$ if and only if $f_i(z_0, \ldots, z_n)$ is $m_i$-full for all $i \in \{0, \ldots, k\}$.

**Definition 2.5.** For an irreducible variety $X$ over $F$, a subset $A \subseteq X(F)$ is **type I** if $A$ is a proper closed subvariety of $X$, and **type II** if it can be written in the form $A = \varphi(V(F))$, where $V$ is an integral projective variety with $\dim(V) = \dim(X)$ and $\varphi : V \to X$ is a generically surjective morphism of degree at least 2. A **thin** set of $X(F)$ is a subset of a finite union of type I and type II sets. In [21, Definition 3.7], a thin set of Campana $\mathcal{O}_{F,S}$-points is defined to be the intersection of a thin set of $X(F)$ with the set of Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{F,S})$.

We now come to the statement of the Campana–Manin conjecture given in [21, Conjecture 1.1]. Let $K$ be a number field, and let $(X, D)$ be a Campana orbifold over $\mathcal{O}_K$ with a good integral model $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_{K,S}$. Let $(\mathcal{L}, \| \cdot \|)$ be an adelically metrized big and nef line bundle on $X$ and $[L]$ the associated divisor class. We let $H_{\mathcal{L}} : X(K) \to \mathbb{R}_{\geq 0}$ denote the corresponding height function, as defined in [19, Section 1]. We recall that the effective cone $\Lambda_{\text{eff}}$ of a variety $X$ is defined as

$$\Lambda_{\text{eff}} = \{ [D] \in \text{Pic}(X) : [D] \geq 0 \} \otimes_{\mathbb{Z}} \mathbb{R}.$$  

**Definition 2.6.** Let $[K_X]$ denote the canonical divisor class. Given the above data, we define

$$a = \inf \{ t \in \mathbb{R} : t[L] + [K_X] + [D] \in \Lambda_{\text{eff}} \},$$

and we define $b$ to be the codimension of the minimal supported face of $\Lambda_{\text{eff}}$ which contains $a[L] + [K_X] + [D]$.

**Conjecture 2.7** (Pieropan, Smeets, Tanimoto, Várilly-Alvarado). **Suppose that** $(X, D)$ is a klt Campana orbifold, such that $-(K_X + D)$ is ample (in this case we say that the orbifold is Fano). **Assume that the set of Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})$ is not itself thin. Then there is a thin set $\mathcal{T}$ of Campana $\mathcal{O}_{K,S}$-points such that**

$$\# \{ P \in (\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S}) \setminus \mathcal{T} : H_{\mathcal{L}}(P) \leq B \} \sim cB^a(\log B)^{b-1},$$

as $B \to \infty$, where $a, b$ are as in Definition 2.6 and $c > 0$ is an explicit constant described in [21, Section 3.3].

**Remark 2.8.** The hypothesis that the Campana points themselves are not thin is discussed by Nakahara and Streeter in [18]. The authors establish in [18, Theorem 1.1] a connection between thin sets of Campana points and weak approximation, in the spirit of Serre’s arguments in [24, Theorem 3.5.7]. Combining this with [18, Corollary 1.4], it can be shown that this hypothesis holds for the orbifold we consider below.
We now put the counting problem \( N(B) \) from (1.3) into the context of Conjecture 2.7. The Campana orbifold under consideration is a special case of Example (2.4). We consider the variety \( X = \mathbb{P}^2 \) over \( \mathbb{Q} \). Below we will use \( z_1, z_2, z_3 \) to denote a representative of the point \( [z_1 : z_2 : z_3] \in \mathbb{P}^2(\mathbb{Q}) \) such that \( (z_1, z_2, z_3) \in \mathbb{Z}^3_{\text{prim}} \). Let \( D \) be the divisor on \( X \) given by
\[
D = \sum_{i=1}^{4} \frac{1}{2} D_i,
\]
where
\[
D_i = \begin{cases} 
\{z_i = 0\}, & \text{if } 1 \leq i \leq 3, \\
\{z_1 + z_2 + z_3 = 0\}, & \text{if } i = 4.
\end{cases}
\]
The support of \( D \) has strict normal crossings on \( \mathbb{P}^2 \), and so \( (\mathbb{P}^2, D) \) is a Campana orbifold. Let \((\mathcal{X}, \mathcal{D})\) denote the obvious smooth proper model of \((\mathbb{P}^2, D)\) over \( \mathbb{Z} \). Here we have chosen the set of bad places \( S \) to consist only of the archimedean place \( | \cdot |_\infty \).

Let \( z \in (\mathbb{P}^2 \setminus D_{\text{red}})(\mathbb{Q}) \). The intersection multiplicity of \( z \) and \( D_i \) at a prime \( p \) is then the \( p \)-adic valuation \( \nu_p(z_i) \), when \( i \in \{1, 2, 3\} \), and \( \nu_p(z_1 + z_2 + z_3) \) when \( i = 4 \). The condition for a point \( z \in (X \setminus D_{\text{red}})(\mathbb{Q}) \) to be Campana \( \mathbb{Z} \)-point of \((\mathcal{X}, \mathcal{D})\) is therefore that \( z_1, z_2, z_3 \) and \( z_1 + z_2 + z_3 \) are all squareful.

For convenience we introduce a new variable \( z_4 \) which is defined by the equation \( z_4 = z_1 + z_2 + z_3 \). We choose the ample line bundle \( \mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1) \), and the generating set \( \{z_1, z_2, z_3, z_4\} \) for the global sections of \( \mathcal{L} \). This choice of generating set gives rise to the height function
\[
H : \mathbb{P}^2 \to \mathbb{R}_{\geq 0}
\]
\[
z = [z_1 : z_2 : z_3] \mapsto \max(|z_1|, |z_2|, |z_3|, |z_4|).
\]
We note that \( (z_1, z_2, z_3) \in \mathbb{Z}^3_{\text{prim}} \) if and only if \( (z_1, \ldots, z_4) \in \mathbb{Z}^4_{\text{prim}} \). Therefore, recalling the definition of \( \mathcal{M}_4(B) \) from (1.1), there is a 2-1 map
\[
\varphi : \mathcal{M}_4(B) \to \{z \in (\mathbb{P}^2, \mathcal{D})(\mathbb{Z}) : H(z) \leq B\}
\]
\[
(z_1, \ldots, z_4) \mapsto [z_1 : z_2 : z_3]. \tag{2.2}
\]

We now compute the constants \( a \) and \( b \) from Conjecture 2.7 in this example. We let \([L]\) denote the hyperplane class corresponding to the line bundle \( \mathcal{L} \), and \([K_{\mathbb{P}^2}]\) the canonical divisor class. We recall that \( \text{Pic} \mathbb{P}^2 \cong \mathbb{Z} \), with the isomorphism given by the degree. We have \( \text{deg}([L]) = 1 \), \( \text{deg}([K_{\mathbb{P}^2}]) = -3 \) and \( \text{deg}([D]) = 4 \cdot 1/2 = 2 \). Therefore
\[
a = \inf \{t \in \mathbb{R} : t[L] + [K_{\mathbb{P}^2}] + [D] \in \Lambda_{\text{eff}}\}
\]
\[
= \inf \{t \in \mathbb{R} : t - 3 + 2 \geq 0\}
\]
\[
= 1.
\]
The minimal supported face of $\Lambda_{\text{eff}}$ which contains $a[L] + [K_{P^2}] + [D] = [0]$ is $\{0\}$, which has codimension one in $\Lambda_{\text{eff}}$, so $b = 1$. Conjecture $2.7$ therefore states that there is a thin set of Campana points $Z = \{\text{for some constant } c > 0\}$, which has codimension one in $\Lambda_{\text{eff}}$. Theorem $1.1$ states that this result holds true when $\mathcal{Z} = \varphi(\mathcal{T})$, where $\mathcal{T}$ is defined in $(1.2)$, except possibly for a different value for the leading constant. To conclude this section, we show that $\varphi(\mathcal{T})$ is indeed thin.

**Lemma 2.9.** Let $\mathcal{T}$ and $\varphi$ be defined as in $(1.2)$ and $(2.2)$ respectively. Then $\varphi(\mathcal{T}) \cap (\mathbb{P}^2, \mathcal{D})(\mathbb{Z})$ is a thin set of Campana points in $(\mathbb{P}^2, \mathcal{D})(\mathbb{Z})$.

**Proof.** It suffices to show that $\varphi(\mathcal{T})$ is a thin set in $\mathbb{P}^2(\mathbb{Q})$. By abuse of notation we view $\mathcal{T}$ as a subset of $\mathbb{P}^3(\mathbb{Q})$ via the map $(z_1, \ldots, z_4) \mapsto [z_1 : \cdots : z_4]$. We begin by showing that $\mathcal{T}$ is a thin subset $\mathbb{P}^3(\mathbb{Q})$. Consider the weighted projective space $\mathbb{P}_Q(2, 1, 1, 1)$ with variables $t, z_1, \ldots, z_4$ (we refer the reader to $[17]$ for the basic definitions pertaining to weighted projective spaces). We have an embedding

$$
\nu : \mathbb{P}(2, 1, 1, 1) \hookrightarrow \mathbb{P}^{10}
$$

$$(t : z_1 : \cdots : z_4) \mapsto [t : z_1^2 : z_1z_2 : \cdots : z_4^2],$$

which on the $z_i$-variables is the Veronese embedding of degree 2. The polynomial $f(t, z_1, \ldots, z_4) = t^2 - z_1z_2z_3z_4$ is weighted homogeneous of degree 4, and so defines a subvariety $V$ of $\mathbb{P}(2, 1, 1, 1, 1)$. Let $Y$ denote the image of $\nu$ and write $t, y_{11}, \ldots, y_{44}$ for variables on $\mathbb{P}^{10}$. Then $\nu(V)$ is a hypersurface of $Y$ defined by the equation $t^2 = y_{12}y_{34}$. From this we see that $V$ is integral, projective and of dimension 3.

Consider the morphism $\pi : V \rightarrow \mathbb{P}^3$ defined by $[t : z_1 : \cdots : z_4] \mapsto [z_1 : \cdots : z_4]$. This is étale of degree 2 on the open subset $V'$ of $V$ defined by $z_1 \cdots z_4 \neq 0$. The set $W \subseteq \mathbb{P}^3(\mathbb{Q})$ defined by the equation $z_1 \cdots z_4 = 0$ is a type I thin set, and $\mathcal{T} = \pi(V'(\mathbb{Q})) \cup W$, so we deduce that $\mathcal{T}$ is thin in $\mathbb{P}^3$. We can now show that $\varphi(\mathcal{T})$ is thin in $\mathbb{P}_Q^2$. To do this, we intersect $\mathbb{P}^3, V'$ and $W$ with the hyperplane $H$ defined by the equation $z_1 + z_2 + z_3 + z_4 = 0$. The map $\pi$ is ramified along the set $W$, which is a union of hyperplanes $H_i$ given by $\{z_i = 0\}$. Since the intersection of $H$ with the $H_i$ is smooth and transversal, it follows from $[23]$ Section 9.4 that $\pi((V'(\mathbb{Q}) \cap H(\mathbb{Q})) \cup (W \cap H(\mathbb{Q})))$ is thin in $H(\mathbb{Q})$. The image of this set under the obvious isomorphism $H \cong \mathbb{P}^2$ sending $[z_1 : \cdots : z_4]$ to $[z_1 : z_2 : z_3]$ is precisely $\varphi(\mathcal{T})$, so we conclude that $\varphi(\mathcal{T})$ is a thin set in $\mathbb{P}_Q^2$. \qed
3. DEALING WITH THE LARGE COEFFICIENTS

Given a nonzero squareful number \( z_i \), we let \( x_i, y_i \) denote the unique integers such that \( x_i \in \mathbb{N}, y_i \) is square-free, and \( z_i = y_i^3 x_i^2 \). We will also use the notation \( Y = y_1 \cdots y_4 \). For \( B, D \geq 1 \), we define

\[
M(B, D) = \# \left\{ z \in (\mathbb{Z} \neq 0)_{\operatorname{prim}}^4 : \sum_{i=1}^4 z_i = 0, z_i \text{ squareful for all } i, |z| \leq B, |Y| \geq D \right\}.
\]

The aim of this section is to prove the following upper bound.

**Proposition 3.1.** We have \( M(B, D) = O(B^{1+\epsilon} D^{-1/12}) \).

The key result in the proof of Proposition 3.1 is the following upper bound for the quantity

\[
N(X, Y) = \# \left\{ x, y \in (\mathbb{Z} \neq 0)^4 : y_1, \ldots, y_4 \text{ square-free}, \sum_{i=1}^4 x_i^2 y_i^3 = 0, |x_i| \leq X_i, |y_i| \leq Y_i \text{ for all } i \right\}.
\]

**Proposition 3.2.** We have

\[
N(X, Y) = O((X_1 \cdots X_4)^{1/2+\epsilon} (Y_1 \cdots Y_4)^{2/3+\epsilon}).
\]

We explain how to deduce Proposition 3.1 from Proposition 3.2. We define

\[
M_1(B; R) = \# \left\{ z \in (\mathbb{Z} \neq 0)^4 : \sum_{i=1}^4 z_i = 0, z_i \text{ squareful for all } i, |z_i| \leq B, R_i \leq |y_i| < 2R_i \text{ for all } i \right\}.
\]

Then

\[
M(B, D) \ll \sum_{\substack{R \text{ dyadic} \ R_1, \ldots, R_4 \geq D}} M_1(B; R), \tag{3.1}
\]

We observe that the conditions \( |z| \leq B \) and \( R_i \leq |y_i| \) imply that \( |x_i|^2 \leq B/R_i^3 \). Consequently,

\[
M_1(B; R) \leq N \left( \left( \sqrt{\frac{B}{R_1^3}}, \ldots, \sqrt{\frac{B}{R_4^3}} \right), (2R_1, \ldots, 2R_4) \right).
\]

Applying Proposition 3.2 we obtain

\[
M_1(B; R) \ll B^{1+\epsilon} (R_1 \cdots R_4)^{-1/12}.
\]

We conclude from (3.1) that

\[
M(B, D) \ll B^{1+\epsilon} \sum_{\substack{R \text{ dyadic} \ R_1, \ldots, R_4 \geq D}} (R_1 \cdots R_4)^{-1/12} \ll B^{1+\epsilon} D^{-1/12},
\]

as claimed in Proposition 3.1.
Proof of Proposition 3.2. For \(k \in \{1, \ldots, 4\}\), we define
\[
S_k(\alpha) = \sum_{x_k \in \mathbb{Z} \neq 0} \sum_{y_k \in \mathbb{Z} \neq 0} |x_k| \leq X_k \sum_{y_k \text{ square-free}} e(x_k^2 y_k^3).
\]
Then
\[
N(X, Y) = \int_0^1 \prod_{k=1}^4 S_k(\alpha) d\alpha.
\]
By Hölder's inequality, we have
\[
N(X, Y) \leq \left( \prod_{k=1}^4 \int_0^1 |S_k(\alpha)|^4 d\alpha \right)^{1/4}.
\] (3.2)
We fix \(k \in \{1, \ldots, 4\}\), and to ease notation we write \(X_k = X, Y_k = Y\). Then
\[
\int_0^1 |S_k(\alpha)|^4 d\alpha = N(X, Y),
\] (3.3)
where
\[
N(X, Y) = \# \left\{ x, y \in (\mathbb{Z} \neq 0)^4 : \begin{array}{l}
y_1, \ldots, y_4 \text{ square-free,} \\
x_1^2 y_1^3 + x_2^2 y_2^3 = x_3^2 y_3^3 + x_4^2 y_4^3 \\
x \leq X, |y| \leq Y \end{array} \right\}.
\]
In view of (3.2), the task now is to show that
\[
N(X, Y) = O(X^{2+\epsilon} Y^{8/3+\epsilon}).
\] (3.4)
Throughout the remainder of the argument, we make repeated use of the trivial estimate for the divisor function, namely that the number of divisors of a positive integer \(d\) is \(O(d^\epsilon)\).

We begin by considering the trivial cases \(x_i = \pm x_j\) for some \(i \neq j\). Without loss of generality, suppose \(i = 1, j = 2\). We obtain
\[
x_1^2 (y_1^3 + y_2^3) = -(x_3^2 y_3^3 + x_4^2 y_4^3).
\] (3.5)
If both sides of (3.5) are zero, then \(y_1 = -y_2\), since we are assuming \(x_1 \neq 0\). Hence there are \(O(X Y)\) choices for \(x_1, y_1, y_2\). On the right hand side, since \(y_3, y_4\) are square-free, it follows that \(x_3 = x_4\) and \(y_3 = -y_4\), hence there are \(O(X Y)\) choices for \(x_3, x_4, y_3, y_4\). This gives a total of \(O(X^2 Y^2)\) solutions. If both sides of the equation are nonzero, then there are \(O(X^2 Y^2)\) choices for \(x_3, x_4, y_3, y_4\), and for any such choice, there are \(O(X Y^\epsilon)\) choices for \(x_1, y_1, y_2\) by the trivial estimate for the divisor function. Overall, in the case \(x_i = \pm x_j\), we conclude that there are \(O(X^{2+\epsilon} Y^{2+\epsilon})\) solutions, which is satisfactory for establishing (3.4). From now on we assume \(x_i \neq \pm x_j\) for all \(i \neq j\).
Returning to the integral representation of $N(X, Y)$ from (3.3), we can apply the Cauchy–Schwarz inequality in two different ways to $|S_k(\alpha)|^2$. This gives the inequalities

$$|S_k(\alpha)|^2 \leq X \sum_{|x| \leq X} \sum_{y_1, y_2 \text{ square-free}} e(\alpha x^2(y_1^3 - y_2^3)),$$

$$|S_k(\alpha)|^2 \leq Y \sum_{|y| \leq Y} \sum_{x, |x_1|, |x_2| \leq X} e(\alpha y^3(x_1^3 - x_2^3)).$$

Applying these inequalities once each to $|S_k(\alpha)|^4$, we obtain

$$N(X, Y) \leq XY L'(X, Y),$$

(3.6)

where

$$L'(X, Y) = \# \left\{ \left( x, x_1, x_2, y, y_1, y_2 \right) \in (\mathbb{Z}_{\neq 0})^6 : \right. \left\{ \begin{array}{l}
\left| x \right|, \left| x_1 \right|, \left| x_2 \right| \leq X, \\
|y|, |y_1|, |y_2| \leq Y \text{ and } y, y_1, y_2 \text{ square-free,} \\
x^2(y_1^3 - y_2^3) = y^3(x_1^3 - x_2^3). \end{array} \right. \right\}.$$  

It will be convenient to work with the quantity $L(X, Y)$ defined to be $L'(X, Y)$ but with the additional assumption $U/2 < |x| \leq U$, which we denote by $|x| \sim U$. We will then perform a sum over dyadic intervals for $U \leq 2X$ at the end of the argument. We also need to extract from $L(X, Y)$ the greatest common divisor $k = \gcd(x, y)$. Note that since $y$ is square-free, so is $k$. Hence

$$L(X, Y) = \sum_{k \leq \min(U, Y)} \mu^2(k) L_k(X, Y),$$

where

$$L_k(X, Y) = \# \left\{ \left( x, x_1, x_2, y, y_1, y_2 \right) \in (\mathbb{Z}_{\neq 0})^6 : \right. \left\{ \begin{array}{l}
\left| x \right| \sim U/k, \left| x_1 \right|, \left| x_2 \right| \leq X, \\
|y| \leq Y/k, |y_1|, |y_2| \leq Y \text{ and } y, y_1, y_2 \text{ square-free,} \\
(x, y) = 1, \\
x^2(y_1^3 - y_2^3) = ky^3(x_1^3 - x_2^3). \end{array} \right. \right\}.$$  

We now obtain two different estimates for $L_k(X, Y)$, depending on the size of $k$. Our first estimate is suitable for large values of $k$.

We have $ky^3(x_1^3 - x_2^3) \neq 0$, since $y \neq 0$, and the cases $x_1 = \pm x_2$ have already been dealt with above. Consequently, for any fixed $x, y_1, y_2, k$, there are $O((XY)^4)$ choices for $y, x_1, x_2$ such that $x^2(y_1^3 - y_2^3) = ky^3(x_1^3 - x_2^3)$, by the trivial estimate for the divisor function. Therefore

$$L_k(X, Y) \ll (XY)^4 A(k),$$  

(3.7)
where

\[ A(k) = \# \{ \lvert x \rvert \sim Y : a \leq k \}, \]

Now \( k \mid x^2(y_1^3 - y_2^3) \) implies that there exists integers \( d, r \) with \( dr = k \), such that \( d \mid x^2 \) and \( r \mid (y_1^3 - y_2^3) \). Observe that since \( k \) is square-free, \( d \mid x^2 \) if and only if \( d \mid x \). Defining

\[ N(r) = \# \{ y_1, y_2 \leq Y : r \mid (y_1^3 - y_2^3) \}, \]

we therefore have

\[ A(k) \leq \sum_{d, r} N(r) \ll U \sum_{d, r} \frac{N(r)}{dk}. \tag{3.8} \]

Note that \( r \leq k \leq Y \), so \( N(r) \ll Y^2 \varrho(r)/r^2 \), where we have defined

\[ \varrho(r) = \# \{ \eta_1, \eta_2 \pmod{r} : \eta_1^3 \equiv \eta_2^3 \}. \]

It is easy to see that \( \varrho(r) \ll r^{1+\epsilon} \). Indeed, for a prime \( p \) and a fixed \( \eta_1 \pmod{p} \), there are at most 3 choices for \( \eta_2 \pmod{p} \) such that \( \eta_1^3 \equiv \eta_2^3 \pmod{p} \), and so \( \varrho(p) \ll 3p \). Since \( r \) is square-free and \( \varrho \) is multiplicative, it follows from the Chinese Remainder theorem that

\[ \varrho(r) = \prod_{p \mid r} \varrho(p) \ll \prod_{p \mid r} 3p \ll r^{1+\epsilon}. \]

Applying this bound on \( \varrho(r) \), we conclude that \( N(r) \ll Y^2/r^{1-\epsilon} \ll Y^{2+\epsilon}/r \), and hence from (3.7) and (3.8) we obtain

\[ L_k(X, Y) \ll \frac{(XY)^\epsilon Y^2+\epsilon}{k} \sum_{d, r} \frac{Y^2+\epsilon}{dr} \ll \frac{X^\epsilon Y^{2+\epsilon}}{k^2}. \tag{3.9} \]

We now find a different estimate for when \( k \) is small. If \( (x, y) = 1 \) and \( x^2(y_1^3 - y_2^3) = ky^3(x_1^3 - x_2^3) \), then \( x^2 \mid k(x_1^3 - x_2^3) \), so we may define the integer

\[ u = k(x_1^3 - x_2^3)/x^2. \]

Note \( u \neq 0 \) by the assumption \( x_1 \neq \pm x_2 \). We have

\[ u \ll kX^2/(U/k)^2 = k^3X^2/U^2. \]

Therefore

\[ L_k(X, Y) \ll \sum_{u \ll k^3X^2/U^2} \sum_{|x| \sim U/k} \sum_{|y| \ll Y/k} 1. \tag{3.10} \]

For a fixed \( n = uy^3 \), there are \( O(Y^{1+\epsilon}) \) solutions \( y_1, y_2 \) to \( y_1^3 - y_2^3 = n \), using the trivial estimate for the divisor function, and so the inner sum of (3.10) is
\(O(Y^{1+\epsilon}/k)\). Similarly, for a given \(|x| \sim U/k\), there are \(O(X^\epsilon)\) choices for \(x_1, x_2\) in the middle sum. Hence

\[
L_k(X,Y) \ll (XY)^\epsilon \sum_{u \ll k^2X^2} \frac{UY}{k^2} \ll \frac{kX^{2+\epsilon}Y^{1+\epsilon}}{U}.
\]

Combining this with the estimate in (3.9), we obtain

\[
L_k(X,Y) \ll X^\epsilon Y^{1+\epsilon} \min \left( \frac{UY}{k^2}, \frac{kX^2}{U} \right)
\]

\[
\ll X^\epsilon Y^{1+\epsilon} \left( \frac{UY}{k^2} \right)^{2/3} \left( \frac{kX^2}{U} \right)^{1/3}
\]

\[
= \frac{X^{2/3+\epsilon}Y^{5/3+\epsilon}U^{1/3}}{k}.
\]

Finally, we take a sum over \(k \leq Y\) and dyadic intervals \(U \leq 2X\) to obtain

\[
L(X,Y) \ll X^{2/3+\epsilon}Y^{5/3+\epsilon} \sum_{U \text{ dyadic}} \sum_{k \leq Y} \frac{U^{1/3}}{k}
\]

\[
\ll X^{1+\epsilon}Y^{5/3+\epsilon}.
\]

Recalling (3.6), we have established (3.4), which from (3.2) gives the required bound for \(N(X,Y)\).

\[\square\]

4. The circle method

In this section, we will use the circle method from [15] to count zeros of diagonal quadratic forms in four variables, the main goal being the proof of Theorem 1.3. Before proceeding with the proof, we collect together some of the notation which we will use throughout this section. Much of the notation depends on a vector \(a = (a_1, \ldots, a_4) \in (\mathbb{Z}_{\neq 0})^4\), which remains fixed throughout this section.

- \(F_a(x)\) denotes the quadratic form \(\sum_{i=1}^4 a_i x_i^2\).
- \(P = (P_1, \ldots, P_4)\), where \(P_i = \sqrt{\frac{2}{|a_i|}}\) for \(i \in \{1, \ldots, 4\}\).
- \(N_a(B) = \# \{x \in (\mathbb{Z}_{\neq 0})^4 : F_a(x) = 0, |x_i| \leq P_i\}\), as defined in (1.6).
- \(A = a_1 \cdots a_4\).
- \(\Delta = \Delta(a) = \prod_{i=1}^4 \text{gcd} \left( a_i, \prod_{j \neq i} a_j \right)\).
- \(\epsilon = (\epsilon_1, \ldots, \epsilon_4) \in \{\pm 1\}^4\), where \(\epsilon_i = \text{sgn}(a_i) = a_i/|a_i|\) is the sign of \(a_i\).
- \(G(x)\) is the quadratic form \(\sum_{i=1}^4 \epsilon_i x_i^2 = F_\epsilon(x)\).
- \(\eta \in (0, 1/4)\) is a small real parameter depending only on \(a\) and \(B\).
• $w : \mathbb{R}^4 \to \mathbb{R}_{\geq 0}$ is an infinitely differentiable smooth weight function, which has compact support and satisfies $w(\mathbf{x}) = 0$ for $|\mathbf{x}| \leq \eta$.
• $w_1, w_2$ are particular choices of such a smooth weight function, defined explicitly in (4.7).
• $Q = B^{1/2}$, and for a positive integer $q$, we write $r = q/Q$.
• $\mathbf{c}$ denotes a vector in $\mathbb{Z}^4$, and we define $\mathbf{v} = (v_1, \ldots, v_4)$ by $v_i = q^{-1}P_i \mathbf{c}_i$.
• $C_i = \eta^{-1}B^\epsilon|a_i|^{1/2}$ for $i \in \{1, \ldots, 4\}$.

**Definition 4.1.** The singular integral $\sigma_\infty(\epsilon)$ and the singular series $G_a$ associated to $F_a$ are given respectively by the equations

$$\sigma_\infty(\epsilon) = \int_{-\infty}^{\infty} \int_{[-1,1]^4} e(-\theta G(\mathbf{x})) \, d\mathbf{x} \, d\theta, \quad (4.1)$$

$$G_a = \sum_{q=1}^{\infty} q^{-4} \sum_{k \mod q} \sum_{b \mod q} e_q(kF_a(b)). \quad (4.2)$$

For convenience we record again here the statement of Theorem 1.3.

**Theorem 4.2.** Let $a \in (\mathbb{Z} \neq 0)^4$ be such that $A \neq \square$ and $|A| \leq B^{4/7}$. Then

$$N_a(B) = \frac{\mathcal{G}_a(\epsilon)B}{|A|^{1/2}} + O\left(\frac{B^{11/42+\epsilon} \Delta^{1/3}}{|A|^{1/24}}\right), \quad (4.3)$$

The circle method from [15] uses of smooth weight functions $w : \mathbb{R}^4 \to \mathbb{R}_{\geq 0}$, which we will take to approximate the characteristic function on $[-1,1]^4$. We introduce a smoothly weighted version of $N_a(B)$ given by

$$N_{w,a}(B) = \sum_{\mathbf{x} \in (\mathbb{Z} \neq 0)^4 \atop F_a(\mathbf{x}) = 0} w(P_1^{-1}x_1, \ldots, P_4^{-1}x_4). \quad (4.4)$$

We also introduce a weighted singular integral

$$\sigma_\infty(w) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} w(\mathbf{x})e(-\theta G(\mathbf{x})) \, d\mathbf{x} \, d\theta. \quad (4.5)$$

We will need to assume that $w$ is infinitely differentiable, has compact support, and vanishes in a neighborhood of the origin ($w(\mathbf{x}) = 0$ for $|\mathbf{x}| \leq \eta$). Whilst the arguments in this section can be applied for any such choice of $w$, with implied constants depending on $w$, in order to get the power saving stated in Theorem 1.1 we will need to keep track of the dependence on $w$ in our estimates. Therefore, for a given $\eta > 0$, we fix a particular choice of smooth weight functions $w_1, w_2 : \mathbb{R}^4 \to \mathbb{R}_{\geq 0}$. We recall the standard bump function...
ψ : \mathbb{R}^4 \to \mathbb{R}_{\geq 0} is defined by

\[
\psi(x) = \begin{cases} 
\exp(- (1 - |x|^2)^{-1}), & \text{if } |x| < 1, \\
0, & \text{otherwise.}
\end{cases}
\] (4.6)

We then define \( w_1, w_2 \) to be

\[
w_1(x) = \begin{cases} 
0, & \text{if } |x| \leq \eta, \\
e^\psi\left(\frac{|x|}{\eta} - 2\right), & \text{if } \eta < |x| \leq 2\eta, \\
1, & \text{if } 2\eta < |x| \leq 1 - \eta, \\
e^\psi\left(\frac{|x|}{\eta} - \frac{1 - \eta}{\eta}\right), & \text{if } 1 - \eta < |x| \leq 1, \\
0, & \text{if } |x| > 1,
\end{cases}
\] (4.7)

\[
w_2(x) = \begin{cases} 
0, & \text{if } |x| \leq \eta, \\
e^\psi\left(\frac{|x|}{\eta} - 2\right), & \text{if } \eta < |x| \leq 2\eta, \\
1, & \text{if } 2\eta < |x| \leq 1, \\
e^\psi\left(\frac{|x|}{\eta} - \frac{1}{\eta}\right), & \text{if } 1 < |x| \leq 1 + \eta, \\
0, & \text{if } |x| > 1 + \eta.
\end{cases}
\]

For any integers \( j_1, \ldots, j_4 \geq 0 \) with \( j_1 + \cdots + j_4 \leq N \), and for any \( i \in \{1, 2\} \), we have

\[
\frac{\partial^{j_1 + \cdots + j_4}}{\partial x_1^{j_1} \cdots \partial x_4^{j_4}} w_i(x) \ll N \eta^{-N}. \quad (4.8)
\]

We now state a smoothly weighted version of Theorem 4.2.

**Theorem 4.3.** Suppose \( \eta \in (0, 1/4) \) and \( a \in (\mathbb{Z}_{\neq 0})^4 \). Let \( w \) be one of the weights \( w_1 \) or \( w_2 \) from (4.7). Suppose that \( A \neq \square \). Then

\[
N_{w,a}(B) = \frac{\mathcal{G}_a \sigma_{\infty}(w) B}{|A|^{1/2}} + O \left( \frac{\eta^{-6} B^{5/6+\epsilon} \Delta^{1/3}}{|A|^{5/24}} \right) + O(\eta^{-7} B^{1/2+\epsilon}). \quad (4.9)
\]

We explain how Theorem 4.2 can be deduced from Theorem 4.3 by applying the methods from [3, Section 5.3] and [5, Section 2.4], together with the upper bound \( \mathcal{G}_a \ll |A|^\epsilon \Delta^{1/4} \) for the singular series, which we prove in Lemma 4.15 at the end of this section.

Let \( 1_{[-1,1]^4} \) denote the characteristic function on \( [-1, 1]^4 \), and for an integer \( j \geq 0 \), define \( w_2^{(j)}(x) = w_2((2\eta)^{-j} x) \). Then for any \( x \in \mathbb{R}^4 \), we have

\[
w_1(x) \leq 1_{[-1,1]^4}(x) \leq \sum_{j=0}^\infty w_2^{(j)}(x),
\]
and consequently,

$$N_{w_1,a}(B) \leq N_a(B) \leq \sum_{j=0}^{\infty} N_{w_2^{(j)},a}(B) = \sum_{j=0}^{\infty} N_{w_2,a}((2\eta)^j B).$$

The assumption $|A| \leq B^{4/7}$ in the statement of Theorem 4.2 implies that the error term $\eta^{-7}B^{1/2+\epsilon}$ is dominated by the other error term in (4.9), provided that $\eta \gg B^{-5/14+\epsilon}$. For any $\eta \in (0, 1/4)$ satisfying this condition, it follows from Theorem 4.3 that

$$\sum_{j=0}^{\infty} N_{w_2,a}((2\eta)^j B) = \frac{(1 + O(\eta))\mathfrak{g}_a \sigma_\infty(w_2)B}{|A|^{1/2}} + O\left(\frac{\eta^{-6}B^{5/6+\epsilon}\Delta^{1/3}}{|A|^{5/24}}\right).$$

As explained in [5, Lemma 2.9], for $i \in \{1, 2\}$, we have

$$|\sigma_\infty(w_i) - \sigma_\infty(\epsilon)| \ll \eta \sigma_\infty(\epsilon) \ll \eta,$$

from which we deduce that

$$N_a(B) = \frac{(1 + O(\eta))\mathfrak{g}_a \sigma_\infty(\epsilon)B}{|A|^{1/2}} + O\left(\frac{\eta^{-6}B^{5/6+\epsilon}\Delta^{1/3}}{|A|^{5/24}}\right).$$

We choose

$$\eta = \frac{1}{5}B^{-1/42}|A|^{1/24}.$$

Clearly $\eta \gg B^{-5/14+\epsilon}$. Moreover, using the assumption $|A| \leq B^{4/7}$, we have that $\eta \in (0, 1/4)$, as was required in order to apply Theorem 4.3. Theorem 4.2 now follows from the estimate $\mathfrak{g}_a \ll |A|^{\epsilon}\Delta^{1/4}$ found in Lemma 4.15.

We commence with the proof of Theorem 4.3. It will be convenient to make the assumptions

$$\eta^{-1}, |a| \ll B^R \quad (4.10)$$

for some fixed $R \geq 0$, so that quantities bounded by an arbitrarily small power of $\eta^{-1}$ or $|A|$ are also $\ll B^\epsilon$ for any $\epsilon > 0$. If these assumptions do not hold, then the statement of Theorem 4.3 is trivial. Employing the machinery from [15, Theorem 2], we have

$$N_{w,a}(B) = \frac{C_q}{Q_2} \sum_{c \in \mathbb{Z}^4} \sum_{q=1}^{\infty} q^{-4}S_q,a(c)I_q,a(c), \quad (4.11)$$
where \( C_Q = 1 + O_N(Q^{-N}) \), and \( Q = B^{1/2} \). Here \( I_{q,a}(c) \) and \( S_{q,a}(c) \) are exponential sums and integrals defined analogously to \([15]\), via

\[
S_{q,a}(c) = \sum_{k \mod q} \sum_{b \mod q} e_q(kF_a(b) + b \cdot c),
\]

\[I_{q,a}(c) = \int_{\mathbb{R}^4} w(P_1^{-1}x_1, \ldots, P_4^{-1}x_4)h \left( \frac{q}{Q}, \frac{F_a(x)}{Q^2} \right) e_q(-c \cdot x) \, dx.\]  

(4.12)

The smooth function \( h : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) is as defined in \([15, \text{Section 3}]\) and is given by

\[
h(x, y) = \sum_{j=1}^{\infty} (x_j)^{-1} (u(x_j) - u(|y|/x_j)),
\]

(4.13)

where \( u = c\psi(4x - 3) \) is a linear transformation of the standard bump function (the one-dimensional analogue of (4.6)), scaled by a constant \( c \) so that its integral over \( \mathbb{R} \) is equal to 1. As observed in \([15, \text{Section 3}]\), it is straightforward to check from (4.13) that \( h(x, y) \ll x^{-1} \) and \( h(x, y) = 0 \) whenever \( x > \max(1, 2|y|) \).

We note that by changing variables \( P_i^{-1}x_i \) into \( x_i \), we can rewrite \( I_{q,a}(c) \) as

\[
I_{q,a}(c) = \int_{\mathbb{R}^4} w(x)h(r, G(x))e(-v \cdot x) \, dx.
\]

(4.14)

4.1. **The main term.** The main term for \( N_{w,a}(B) \) comes from the case \( c = 0 \) in (4.11). We define

\[
M(B) = \frac{1}{Q^2} \sum_{q=1}^{\infty} q^{-4} S_{q,a}(0) I_{q,a}(0).
\]

(4.15)

We have

\[
\mathfrak{S}_a = \sum_{q=1}^{\infty} q^{-4} S_{q,a}(0),
\]

(4.16)

and from (4.14),

\[
I_{q,a}(0) = \int_{\mathbb{R}^4} w(x)w_0(G(x))h(r, G(x)) \, dx.
\]  

We let \( w_0 \) denote a smooth weight function supported on \([-17, 17]\) and taking the value 1 in \([-16, 16]\). Since \( \text{supp}(w) \subseteq [-1 - \eta, 1 + \eta] \subset [-2, 2] \), we have \( w_0(G(x)) = 1 \) whenever \( x \in \text{supp}(w) \). We obtain

\[
I_{q,a}(0) = \int_{\mathbb{R}^4} w(x)w_0(G(x))h(r, G(x)) \, dx.
\]
The arguments at the beginning of [4, Section 4.3] can be applied to obtain
\[
I_{q,n}(0) = P_1 \cdots P_4 \left( \lim_{\delta \to 0} \int_{-\infty}^{\infty} \left( \frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 J(w; \theta) L(\theta) \, d\theta \right),
\]
where
\[
J(w; \theta) = \int_{\mathbb{R}^4} w(x) e(-\theta G(x)) \, dx,
\]
\[
L(\theta) = \int_{-\infty}^{\infty} w_0(t) h(r, t) e(\theta t) \, dt.
\]
We would like to compare the bracketed expression in (4.17) with the weighted singular integral \( \sigma_\infty(w) \) from (4.5), which by definition equals \( \int_{-\infty}^{\infty} J(w; \theta) d\theta \).

We apply [4, Lemma 4.11] to estimate \( L(\theta) \). In addition to \( N \), the implied constants in [4, Lemma 4.11] depend only on \( \text{supp}(w_0) \). However, we note that we have chosen \( w_0 \) independently of \( w \), and hence independently of \( \eta \).

Therefore for any \( q \leq Q \) and \( N \geq 1 \), we have
\[
L(\theta) = 1 + O_N(\{1 + |\theta|^N\} r^N).
\]

Our next task is to bound \( J(w; \theta) \). In order to achieve this, we need to work with more general smooth weight functions \( \tilde{w} : \mathbb{R}^4 \to \mathbb{R}_{\geq 0} \) which belong to a class \( S \) defined by the following properties.

**Definition 4.4.** We say \( \tilde{w} \in S \) if
1. \( \tilde{w} \) is smooth,
2. \( \tilde{w} \) is supported on \([-2, 2]^4 \),
3. \( \tilde{w}(x) = 0 \) for \( |x| \leq \eta \),
4. \( |\tilde{w}(x)| \ll 1 \) for all \( x \in \mathbb{R}^4 \),
5. The derivatives of \( \tilde{w}(x) \) are bounded as in (4.8).

Since \( w \) is equal to \( w_1 \) or \( w_2 \), we note in particular that \( w \in S \).

**Lemma 4.5.** For any \( \tilde{w} \in S \) and any \( M \in \mathbb{Z}_{\geq 0} \), we have \( J(\tilde{w}; \theta) \ll_M |\eta^2 \theta|^{-M} \).

**Proof.** The case \( M = 0 \) is trivial since the integrand in the definition of \( J(\tilde{w}; \theta) \) is bounded and has compact support. For \( M \geq 1 \), we follow a similar argument to [15, Lemma 10], by applying integration by parts and induction on \( M \). We have
\[
J(\tilde{w}; \theta) = \int_{\mathbb{R}^4} \tilde{w}(x) e(-\theta G(x)) \, dx \ll \max_{1 \leq j \leq 4} \int_{|x| = \max_i |x_i|} \tilde{w}(x) e(-\theta G(x)) \, dx.
\]
Without loss of generality, we may assume that \( j = 1 \). We note that
\[
e(-\theta G(x)) = \frac{1}{-4\pi i \epsilon_1 \theta x_1} \frac{\partial}{\partial x_1} e(-\theta G(x)).
\]
We perform integration by parts with respect to \( x_1 \) to obtain
\[
J(\tilde{w}; \theta) \ll \left| \int_{x \in \text{supp } w} \frac{\partial}{\partial x_1} \left( \frac{\tilde{w}(x)}{4\pi i \theta x_1} \right) e(-\theta G(x)) dx \right|.
\]

Since \( |x_1| = \max_i |x_i| \), we have \( |x_1|^{-1} = |x|^{-1} \leq \eta^{-1} \) for any \( x \in \text{supp } w \). Therefore by the assumptions on the size of \( \tilde{w} \) and its derivatives, we see that
\[
\frac{\partial}{\partial x_1} \left( \frac{\tilde{w}(x)}{4\pi i \theta x_1} \right) = \tilde{w}(1)(4\pi i \theta \eta)^{-1}
\]
for some \( \tilde{w}(1) \in S \). Hence
\[
J(\tilde{w}; \theta) \ll |\eta^2 \theta|^{-1} |J(\tilde{w}(1); \theta)|.
\]
Proceeding by induction, we obtain
\[
J(\tilde{w}; \theta) \ll M |\eta^2 \theta|^{-M} |J(\tilde{w}^{(M)}; \theta)|
\]
for some \( \tilde{w}^{(M)} \in S \), from which we deduce the required result by applying the trivial bound \( J(\tilde{w}(1); \theta) \ll 1 \).

From Lemma 4.5, for any integer \( M \geq 1 \), we have
\[
J(w; \theta) \ll_M (1 + |\eta^2 \theta|^{-M} \leq \eta^{-2M} (1 + |\theta|)^{-M}.
\]
Combining this with (4.17), (4.20) and the fact that
\[
\lim_{\delta \to 0} \int_{-\infty}^{\infty} \left( \frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 J(\theta; w) d\theta = \int_{-\infty}^{\infty} J(\theta; w) d\theta = \sigma_\infty(w),
\]
we obtain
\[
|\{(P_1 \cdots P_4)^{-1} I_{q,a}(0) - \sigma_\infty(w)| \ll_{M,N} \int_{-\infty}^{\infty} \frac{r^N \{1 + |\theta|\}^N}{\eta^{2M} \{1 + |\theta|\}^M} d\theta.
\]
In order to ensure that the integral converges, we make the choice \( M = N + 2 \), and we conclude that
\[
I_{q,a}(0) = P_1 \cdots P_4 \{ \sigma_\infty(w) + O_N(\eta^{-2N-4}r^N) \},
\]
for any integer \( N \geq 1 \).

Let \( R = B^{1/2+\epsilon} \eta^2 \). We note that if \( q \leq R \), then the error term in (4.21) can be made smaller than any negative power of \( B \) by appropriate choice of \( N \) (depending on \( \epsilon \)), due to the assumption in (4.10).

We now split the main term up. We have
\[
M(B) = \frac{P_1 \cdots P_4 \sigma_\infty(w)}{B} \sum_{q \leq R} q^{-4} S_{q,a}(0) + O(T(R) + 1),
\]
where
\[
T(R) = \frac{1}{Q^2} \sum_{q > R} q^{-4} S_{q,a}(0) I_{q,a}(0).
\]
We will estimate $T(R)$ using partial summation. The following lemma is similar to [4] Lemma 4.3, and gives bounds for $I_{q,a}(c)$ and its derivative. Only the case $c = 0$ is needed in the study of the main term, but the case $c \neq 0$ will be useful later.

**Lemma 4.6.** Let $c \in \mathbb{Z}^4$ and $k \in \{0, 1\}$. Then

1. If $k = 0$ or $c = 0$ then $q^k \frac{\partial^k I_{q,a}(c)}{\partial q^k} \ll P_1 \cdots P_4$,
2. If $k = 1$ and $c \neq 0$ then $q^k \frac{\partial^k I_{q,a}(c)}{\partial q^k} \ll q^{-1} P_1 \cdots P_4$.

**Proof.** Suppose that $c = 0$. Recalling (4.14) and the notation $r = q/Q$, it is clear that

$$q^k \frac{\partial^k I_{q,a}(0)}{\partial q^k} = r^k P_1 \cdots P_4 \int_{\mathbb{R}^4} w(x) \frac{\partial^k h(r, G(x))}{\partial r^k} \, dx.$$  

(4.24)

From [15] Lemma 5 with $m = k, n = 0$ and $N = 2$, we have

$$\frac{\partial^k h(r, G(x))}{\partial r^k} \ll r^{-1-k} \left( r^2 + \min \left\{ 1, \frac{r^2}{|G(x)|^2} \right\} \right),$$

and so

$$(P_1 \cdots P_4)^{-1} \frac{\partial^k I_{q,a}(c)}{\partial q^k} \ll r + r^{-1} \int_{x \in \text{supp} w, |G(x)| \leq r} \, dx + r \int_{x \in \text{supp} w, |G(x)| > r} \frac{1}{|G(x)|^2} \, dx.$$  

(4.25)

We need to show that the right hand side of (4.25) is $O(1)$. We can assume that the first term $r$ is $O(1)$, since $h(r, G(x)) = 0$ when $r > \max(1, |G(x)|)$, and $G(x) \ll 1$ for $x \in \text{supp} w$, and so $I_{q,a}(0) = 0$ unless $r \ll 1$. In order to estimate the integrals in (4.25), we consider for $z > 0$ the Lebesgue measure $m(z)$ of the set

$$S(z) = \{ x \in \text{supp} w : |G(x)| \leq z \}.$$

We can find distinct indices $i, j$ such that $\epsilon_i = \epsilon_j$. Without loss of generality we may assume $i = 1, j = 2$, and in addition that $\epsilon_1 = \epsilon_2 = 1$. For a fixed choice of $x_3, x_4$, we define $c = \epsilon_3 x_3^2 + \epsilon_4 x_4^2$, so that if $x \in S(z)$ then $x_1^2 + x_2^2 \in [-c - z, -c + z]$. The measure of the set of pairs $x_1, x_2$ satisfying this condition is $O(z)$. For $x \in \text{supp} w$, we require $|x_3|, |x_4| \ll 1$, and hence $m(z) = O(z)$. From this we see that the first integral in (4.25) is $O(r)$, and by breaking into dyadic intervals, the second integral is $O(r^{-1})$. Therefore the right hand side of (4.25) is $O(1)$, as required.

When $c \neq 0$, we first note that for $k = 0$, we can reduce to the case above by applying the trivial estimate to the extra exponential factor $\epsilon(-v \cdot x)$ appearing in the integral defining $I_{q,a}(c)$. It remains only to deal with the case $c \neq 0, k = 1$. Define the vector $v \in \mathbb{R}^4$ by $v_i = q^{-1} P_i c_i$. Similarly to (4.24),
we obtain
\[(P_1 \cdots P_4)^{-1} q \frac{\partial I_{q,a}(c)}{\partial q} \ll r \int_{\mathbb{R}^4} w(x) \frac{\partial \{h(r,G(x))e(-\mathbf{v} \cdot \mathbf{x})\}}{\partial r} \, dx\]
\[= r \int_{\mathbb{R}^4} w(x) \left( \frac{\partial h(r,G(x))}{\partial r} e(-\mathbf{v} \cdot \mathbf{x}) + h(r,G(x)) \frac{-2\pi i \mathbf{v} \cdot \mathbf{x}}{r} e(-\mathbf{v} \cdot \mathbf{x}) \right) \, dx.\]

The first term can again be dealt with using the trivial estimate for \(e(-\mathbf{v} \cdot \mathbf{x})\) and the argument for the case \(c = 0\) given above. In order to estimate the second term, as in [15, Lemma 14], we apply the divergence theorem
\[\int_{\mathbb{R}^4} \nabla \cdot (w(x)h(r,G(x))e(-\mathbf{v} \cdot \mathbf{x})x) \, dx = 0\]
in order to remove the additional factor of \(2\pi i \mathbf{v} \cdot \mathbf{x}\). This yields
\[\int_{\mathbb{R}^4} w(x)h(r,G(x))e(-\mathbf{v} \cdot \mathbf{x})(2\pi i \mathbf{v} \cdot \mathbf{x}) \, dx = \int_{\mathbb{R}^4} \tilde{w}(x)h(r,G(x))e(-\mathbf{v} \cdot \mathbf{x}) \, dx\]
\[+ \int_{\mathbb{R}^4} 4w(x)h(r,G(x))e(-\mathbf{v} \cdot \mathbf{x}) \, dx + \int_{\mathbb{R}^4} 2w(x)G(x)e(-\mathbf{v} \cdot \mathbf{x}) \frac{\partial h(r,G(x))}{\partial G(x)} \, dx,\]
(4.26)

where \(\tilde{w}(x) = (x \cdot \nabla)w(x)\). The second integral on the right hand side of (4.26) can now be treated in the same way as the easier cases \(k = 0\) or \(c = 0\) using the trivial estimate for \(e(-\mathbf{v} \cdot \mathbf{x})\). The third integral can similarly be estimated by noting that \(G(x) \ll 1\) for \(x \in \text{supp}(w)\), and applying [15, Lemma 5] with \(n = 1, m = 0, N = 0\). For the first integral, we note that \(\tilde{w}(x)\) is uniformly bounded by \(\eta^{-1}\). Hence the above arguments can be applied to the first integral as well, but with an extra factor of \(\eta^{-1}\). \(\square\)

We define
\[\Sigma(x; c) = \sum_{q \leq x} q^{-3} S_{q,a}(c).\]
(4.27)

Moreover, we let \(F^*_a\) denote the dual form of \(F_a\), i.e., the quadratic form given by the equation
\[F^*_a(c) = a_2a_3a_4c_1^2 + a_1a_3a_4c_2^2 + a_1a_2a_4c_3^2 + a_1a_2a_3c_4^2.\]

We also define
\[\Delta_a(c) = \prod_{i=1}^{4} \gcd \left( \gcd(a_i,c_i), \prod_{j \neq i} \gcd(a_j,c_j) \right).\]
(4.28)

In particular we have \(\Delta_a(c) = \Delta\) when \(c = 0\).

The following lemma is a slight modification of [4, Lemma 4.9].
Lemma 4.7. Suppose \( A \neq \square \). Let \( c \in \mathbb{Z}^4 \) be such that \( F^*_a(c) = 0 \). Then
\[
\Sigma(x; c) \ll A^{3/16+\epsilon} \Delta_b(a)^{3/8} x^{1/2+\epsilon}.
\]

Proof. We use the fact that \( S_{q,a}(c) \) is multiplicative in \( q \), as proved by Heath-Brown in the discussion following [15, Lemma 28]. This allows us to write
\[
\Sigma(x; c) = \sum_{q_2 \leq x \quad q_2 | (2A)_{\infty}} q_2^{-3} S_{q_2,a}(c) \sum_{q_1 \leq x/q_2 \quad \gcd(q_1, 2A) = 1} q_1^{-3} S_{q_1,a}(c), \tag{4.29}
\]
where the notation \( q_2 | (2A)_{\infty} \) means that every prime dividing \( q_2 \) also divides \( 2A \). By [4, Lemma 4.6], we have
\[
\sum_{q_1 \leq x \quad q_1 \leq x/q_2 \quad \gcd(q_1, 2A) = 1} q_1^{-3} S_{q_1,a}(c) = \sum_{q_1 \leq x/q_2 \quad \gcd(q_1, 2A) = 1} \left( \frac{A}{q_1} \right) \varphi(q_1),
\]
where \( \varphi \) denotes the Euler totient function. Applying the Burgess bound as in the proof of [4, Lemma 4.9], we obtain
\[
\sum_{q_1 \leq x/q_2 \quad \gcd(q_1, 2A) = 1} q_1^{-3} S_{q_1,a}(c) \ll \frac{|A|^{3/16+\epsilon} x^{1/2}}{q_2^{1/2}}.
\]
Returning to (4.29), we have
\[
\Sigma(x; c) \ll |A|^{3/16+\epsilon} x^{1/2} \sum_{q_2 \leq x \quad q_2 | (2A)_{\infty}} \frac{|S_{q_2,a}(c)|}{q_2^{7/2}}. \tag{4.30}
\]

Now [4, Lemma 4.5] states that \( S_{q,a}(c) \ll q^3 \prod_{i=1}^{4} \gcd(q, a_i, c_i) \). Moreover, there are \( O(x^{\epsilon} A^4) \) choices for \( q_2 \leq x \) with \( q_2 | (2A)_{\infty} \). Therefore
\[
\sum_{q_2 \leq x \quad q_2 | (2A)_{\infty}} \frac{|S_{q_2,a}(c)|}{q_2^{7/2}} \ll x^{\epsilon} A^4 \sup_{q_{(2A)_{\infty}}} \left( \prod_{i=1}^{4} \gcd(q, a_i, c_i) \right)^{1/2} \cdot \tag{4.31}
\]
Let \( p \) be a prime. We define
\[
L_p = \nu_p \left( \prod_{i=1}^{4} \frac{\gcd(q, a_i, c_i)}{q} \right), \quad k_p = \nu_p(q).
\]
For an index \( i = 1, \ldots, 4 \), we let \( m_{i,p} \) denote the \( i \)th smallest element of \( \nu_p(\gcd(a_1, c_1)), \ldots, \nu_p(\gcd(a_4, c_4)) \). Then
\[
L_p = \sum_{i=1}^{4} (\min(k_p, m_{i,p}) - k_p).
\]
It can be checked that the maximum possible value of $L_p$ is attained by choosing $k_p = m_3, p$, and so $L_p \leq m_1, p + m_2, p + m_3, p$. We have $L_p \leq 0$ unless $p|\Delta_c(a)$. Hence the supremum in (4.31) is bounded by

$$\prod_{p|\Delta_c(a)} p^{(m_1, p + m_2, p + m_3, p)/2}.$$ 

For any prime $p$, we have

$$\nu_p(\Delta_c(a)) = m_1, p + m_2, p + m_3, p + \min(m_1, p + m_2, p + m_3, p, m_4, p) \geq \frac{4}{3}(m_2, p + m_3, p),$$

and hence the supremum in (4.31) is bounded by $\Delta_c(a)^{1/3}$. Recalling (4.30) we obtain the desired estimate for $\Sigma(x; c)$. □

**Proposition 4.8.** Suppose $A \neq \square$, and let $M(B)$ be as defined in (4.15). Then

$$M(B) = \frac{G_a \sigma_\infty(w) B}{|A|^{1/2}} + O\left(\frac{\eta^{-2}B^{3/4+\epsilon}\Delta^{3/8}}{|A|^{5/16}}\right).$$

**Proof.** We recall the estimate for $M(B)$ from (4.22) and the definition of $T(R)$ from (4.23). We may restrict the sum to the range $R < q \ll Q$, since $h(r, G(x)) = 0$ unless $q \ll Q$. Using Lemma 4.7 with $x \ll Q$ and $c = 0$, together with Lemma 4.6 and partial summation, we obtain

$$T(R) = \frac{-I_R(0)}{Q^2 R} \sum(R; 0) - \frac{1}{Q^2} \int_R^Q \sum(x; 0) \frac{\partial}{\partial x} \left(\frac{I_x, a(0)}{x}\right) dx$$

$$\ll \frac{P_1 \cdots P_4}{BR} \sup_{R \ll q \ll Q} |\sum(t; 0)|$$

$$\ll \frac{P_1 \cdots P_4}{BR} |A|^{3/16+\epsilon} \Delta^{3/8} B^{1/4+\epsilon}$$

$$\ll \frac{\eta^{-2}B^{3/4+\epsilon}\Delta^{3/8}}{|A|^{5/16}},$$

which is the error term claimed in the proposition. Finally, the same error term is also obtained when we extend the sum $\sum_{q \leq R} q^{-4} S_{q, a}(0)$ in (4.22) to the singular series $G_a = \sum_{q=1}^\infty q^{-4} S_{q, a}(0)$, as can be seen by applying Lemma 4.7 and partial summation in a very similar manner to above. □

**4.2. The error term.** We now study the error term coming from the case $c \neq 0$. We begin with a lemma which is similar to [4, Lemma 4.2 (i)].

**Lemma 4.9.** For any nonzero $c \in \mathbb{Z}^4$ and any integer $N \geq 0$, we have

$$I_{q, a}(c) \ll N \frac{P_1 \cdots P_4}{r} \left(1 + \frac{r}{\eta}\right)^N \min_{1 \leq i \leq 4} \left(\frac{|a_i|^{1/2}}{|c_i|}\right)^N.$$
Proof. We apply integration by parts $N$ times to the integral in (4.14), where we differentiate $f(x) := w(x)h(r, G(x))$ and integrate $g(x) := e(v \cdot x)$. Each integral of $g(x)$ with respect to $x_i$ introduces a factor of $(2\pi iq^{-1}P_i c_i)^{-1}$.

We recall from (4.8) that for any $i \in \{1, \ldots, 4\}$ and any $N \geq 1$,

$$\left| \frac{\partial^N}{\partial x_i^N} w(x) \right| \ll_N \eta^{-N}.$$ 

Using the product rule, we have

$$\left| \frac{\partial^N}{\partial x_i^N} f(x) \right| \ll_N \max_{0 \leq j \leq N} \left| \eta^{-N} \frac{\partial^j}{\partial x_i^j} h(r, G(x)) \right|.$$ 

Clearly

$$\left| \frac{\partial^j}{\partial x_i^j} G(x) \right| \ll 1$$

for all $x \in \text{supp } w$. By [4, Equation (4.10)], we have

$$\frac{\partial^j h(r, y)}{\partial y^j} \ll_j r^{-1-j}.$$ 

(4.32)

Hence by the chain rule, we have

$$\left| \frac{\partial^N}{\partial x_i^N} f(x) \right| \ll_N \max_{0 \leq j \leq N} r^{-1-j} \eta^{-N} = r^{-1} \min(r, \eta)^{-N} \leq r^{-1} \left( \frac{1}{r} + \frac{1}{\eta} \right)^N.$$ 

(4.33)

After performing integration by parts $N$ times, we apply (4.33) to all derivatives of $f(x)$ that appear, and the trivial estimate $|e(-v \cdot x)| \leq 1$, to obtain for any $i \in \{1, \ldots, 4\}$,

$$I_{q,a}(c) \ll_N \frac{P_1 \cdots P_4}{r(q^{-1}|P_i c_i|)^N} \left( \frac{1}{r} + \frac{1}{\eta} \right)^N.$$ 

Recalling that $P_i = (B/|a_i|)^{1/2}, Q = B^{1/2}, r = q/Q$, and choosing the index $i$ such that $|P_i c_i|$ is maximised, the last expression rearranges to give the desired estimate. \hfill \square

Remark 4.10. Lemma 4.9 allows us to assume $|c_i| \ll \eta^{-1/2}B = C_i$ for all $i \in \{1, \ldots, 4\}$, since outside this range the estimate in Lemma 4.9 can be made smaller than any power of $B$ by an appropriate choice of $N$.

The following lemma is a variant of the first derivative test and is based on [15, Lemma 10].

Lemma 4.11. Let $f(x) = \theta G(x) - v \cdot x$, where $v \in \mathbb{R}^4$ and $\theta \in \mathbb{R}$ are such that $|v| \geq 5|\theta|$. Then for any integer $N \geq 0$ and any $w$ in the class of smooth
integrand appearing in the definition of \( I \) does not have compact support. We define the smooth weight \( w \) where

\[
\int_{\mathbb{R}^4} w(\mathbf{x}) e(f(\mathbf{x})) d\mathbf{x} \ll_N (\eta|\mathbf{v}|)^{-N}.
\]

Proof. The case \( N = 0 \) is trivial. We define \( f_j(\mathbf{x}) = \partial f(\mathbf{x})/\partial x_j = \pm 2\theta x_j - v_i \). By the assumption \(|\mathbf{v}| \geq 5|\theta|\), there is some index \( j \in \{1, \ldots, 4\} \) such that \(|f_j(\mathbf{x})| \gg |\mathbf{v}| \) for any \( \mathbf{x} \in [-2, 2]^4 \). Without loss of generality, we may assume \( j = 1 \).

We write

\[
w(\mathbf{x})e(f(\mathbf{x})) = \frac{w(\mathbf{x})}{2\pi i f_1(\mathbf{x})} \partial e(f(\mathbf{x}))
\]

and integrate by parts with respect to \( x_1 \) to obtain

\[
\int_{\mathbb{R}^4} w(\mathbf{x}) e(f(\mathbf{x})) d\mathbf{x} = -(2\pi i |\mathbf{v}|)^{-1} \int_{\mathbb{R}^4} e(f(\mathbf{x})) \tilde{w}(\mathbf{x}) d\mathbf{x},
\]

where

\[
\tilde{w}(\mathbf{x}) = \frac{\partial}{\partial x_1} \left( \frac{|\mathbf{v}| w(\mathbf{x})}{f_1(\mathbf{x})} \right) = \frac{\partial}{\partial x_1} (\eta w(\mathbf{x})) \frac{|\mathbf{v}|}{f_1(\mathbf{x})} + \eta w(\mathbf{x}) \frac{\partial}{\partial x_1} \left( \frac{|\mathbf{v}|}{f_1(\mathbf{x})} \right).
\]

It remains to show that \( \tilde{w} \) belongs to the class of smooth weight functions \( S \) from Definition 4.4 since the result then follows by induction on \( N \). But this is immediate from the observations that for any \( \mathbf{x} \in [-2, 2]^4 \),

\[
\left| \frac{|\mathbf{v}|}{f_1(\mathbf{x})} \right| \ll 1, \quad \left| \frac{\partial}{\partial x_1} \left( \frac{|\mathbf{v}|}{f_1(\mathbf{x})} \right) \right| \ll 1, \quad \text{and} \quad \frac{\partial}{\partial x_1}(\eta w) \in S. \quad \square
\]

We require one further estimate for \( I_{\eta,a}(\mathbf{c}) \), which involves the second derivative test.

Lemma 4.12. Let \( \mathbf{c} \neq 0 \). Then

\[
I_{\eta,a}(\mathbf{c}) \ll \eta^{-4} B^{3/2+\varepsilon} q \min_{1 \leq i \leq 4} \left( \frac{|a_i|^{1/2}}{|c_i|} \right).
\]

Proof. We would like to apply Fourier inversion to \( w(\mathbf{x})h(r, G(\mathbf{x}))e(-\mathbf{v} \cdot \mathbf{x}) \), the integrand appearing in the definition of \( I_{\eta,a}(\mathbf{c}) \). The function \( \mathbf{x} \mapsto h(r, G(\mathbf{x})) \) does not have compact support. We define the smooth weight \( w_0 : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \) by

\[
w_0(x) = \psi(x/17),
\]

where \( \psi(x) \) is the standard bump function in one variable. Then we define the smooth weight \( w_3 : \mathbb{R}^4 \rightarrow \mathbb{R}_{\geq 0} \) by

\[
w_3(\mathbf{x}) = \begin{cases} \frac{w(\mathbf{x})}{w_0(G(\mathbf{x}))}, & \text{if } w_0(G(\mathbf{x})) \neq 0, \\ 0, & \text{otherwise.} \end{cases}
\]

Note that \( w \) is supported on \([-2, 2]^4\), and in this region \( G(\mathbf{x}) \) takes values in \([-16, 16]\). Therefore \( w_0(G(\mathbf{x})) \gg 1 \) for all \( \mathbf{x} \in \text{supp}(w) \). We deduce that
supp(w_3) = supp(w) and w(x) = w_3(x)w_0(G(x)) for all x ∈ ℝ^4. Moreover, since all the derivatives of w_0(G(x)) are O(1) for any x ∈ supp(w), we have w_3 ∈ S. Applying Fourier inversion, we obtain

\[ I_{q,a}(c) = P_1 \cdots P_4 \int_{-\infty}^{\infty} p(\theta) \int_{\mathbb{R}^4} w_3(x)e(\theta G(x) - v \cdot x) \, dx \, d\theta, \quad (4.34) \]

where

\[ p(\theta) = \int_{-\infty}^{\infty} w_0(k)h(r,k)e(-\theta k) \, dk. \]

We have the estimate \( p(\theta) \ll 1 \). Indeed, taking \( m = n = 0 \) and \( N = 2 \) in [15 lemma 5] we see that

\[ h(r,k) \ll r^{-1}(r^2 + \min(1, (r/|k|)^2)), \]

and so

\[ p(\theta) \ll r^{-1} \left( \int_{|k|<r} 1 \, dk + \int_{r \leq |k| \leq 17} \left( \frac{r}{|k|} \right)^2 \, dk \right) \ll r^{-1}(r + r^2 \cdot r^{-1}) \ll 1. \]

To deal with the inner integral in (4.34), which we denote by \( I(\theta; v) \), we divide into the cases 5|\theta| \leq |v| and 5|\theta| \geq |v|). In the former case, we apply Lemma [11] with \( N = 2 \) to obtain \( I(\theta; v) \ll (|v|)^{-2} \). The contribution to \( I_{q,a}(c) \) is

\[ (|v|)^{-2}P_1 \cdots P_4 \int_{5|\theta| \leq |v|} 1 \, d\theta \ll |v|^{-2}P_1 \cdots P_4. \]

In the case 5|\theta| \geq |v|, we use the arguments from [16] Lemma 3.2 to obtain

\[ I(\theta; v) \ll \left\{ \int_{\mathbb{R}^4} |\widehat{w}_3(y)| \, dy \right\} \sup_{y \in \mathbb{R}^4} \left| \prod_{i=1}^{4} \int_{[-2,2]} e(\theta \epsilon_i x_i^2 + x_i(y_i \pm v_i)) \, dx_i \right|, \quad (4.35) \]

where \( \widehat{w}_3 \) denotes the Fourier transform of \( w_3 \). The first factor on the right hand side of (4.35) is the \( L^1 \)-norm of \( \widehat{w}_3 \), which we denote by \( \|\widehat{w}_3\|_{L^1} \). The \( v_i \)'s can be absorbed into the supremum over \( y \), and so it remains to study

\[ \int_{[-2,2]} e(\pm \theta x_i^2 - y x_i) \, dx_i. \]

We can write the integrand as \( e(\pm \theta \Phi(x_i)) \), where \( \Phi(x_i) = x_i^2 + x_i y_i \theta^{-1} \). Then \( |\Phi''(x_i)| \geq 1 \) throughout the interval \([-2,2]\), and so we can apply the second derivative test as found in [26] Ch. 8, Proposition 2.3] to bound this integral by \(|\theta|^{-1/2}\). Therefore

\[ I(\theta; v) \ll \|\widehat{w}_3\|_{L^1} |\theta|^{-2}, \]
Returning to (4.34), the contribution to $I_{q,a}(c)$ from the range $5|\theta| \geq |v|$ is bounded by

\[ \|\hat{w}_3\|_{L_1} P_1 \cdots P_4 \int_{|\theta| > |v|} |\theta|^{-2} d\theta \ll \|\hat{w}_3\|_{L_1} P_1 \cdots P_4 |v|^{-1}. \]

Since

\[ \frac{P_1 \cdots P_4}{|v|} = \frac{B^{3/2} q}{|A|^{1/2}} \min_{1 \leq i \leq 4} \left( \frac{|a_i|^{1/2}}{|c_i|} \right), \]

it remains only to show that $\|\hat{w}_3\|_{L^1} \ll \eta^{-4}$. We have

\[ \|\hat{w}_3\|_{L^1} = \int_{|x| < \eta^{-1}} |\hat{w}_3(y)| \, dy + \int_{|x| > \eta^{-1}} |\hat{w}_3(y)| \, dy. \]

The first integral is trivially $O(\eta^{-4})$. For the second integral, we have

\[ \int_{|x| > \eta^{-1}} |\hat{w}_3(y)| \, dy \ll \int_{|y| = \max_i |y_i|} |\hat{w}_3(y)| \, dy. \]

We can now apply integration by parts five times with respect to $x_1$ to obtain

\[ \int_{|y| > \eta^{-1}} |\hat{w}_3(y)| \, dy \ll \int_{|y| = \max_i |y_i|} \left( \int_{\mathbb{R}^4} \frac{\partial^5 w_3(x)}{\partial x_1^5} e(-x \cdot y) (-2\pi i y_1)^5 \, dx \right) \, dy \]

\[ \ll \eta^{-5} \int_{|y| = \max_i |y_i|} y_1^{-5} \, dy \ll \eta^{-5} \int_{y_1 > \eta^{-1}} y_1^{-2} \, dy_1 \ll \eta^{-4}. \]

We recall the notation $C_i = \eta^{-1} B^i |a_i|^{1/2}$ from Remark 4.10. We are now ready to estimate the quantity

\[ E_a(B) = \sum_{c \in \mathbb{Z}^4 \setminus \{0\}} \sum_{|c_i| \leq C_i} \frac{1}{Q^2} \sum_{q=1}^{\infty} q^{-4} S_{q,a}(c) I_{q,a}(c). \quad (4.36) \]

Note that $I_{q,a}(c)$ vanishes for $q \gg Q$ by the properties of the function $h$, and so we may restrict the $q$-sum to a sum over $q \ll Q$. We recall the notation $\Sigma(x; c) = \sum_{q \leq x} S_{q,a}(c)$. We would like to use Lemma 4.12 together with the bound for $\Sigma(x; c)$ from [1] Lemma 4.7 to estimate the inner sum of (4.36). Unfortunately, [1] Lemma 4.7 requires the dual form $F^*_a(c)$ not to vanish. This was always true in [1] Section 4 due to the assumptions made in the setup, but we must treat this case separately. To this end, we let $E_a(B) = E_{a,1}(B) + E_{a,2}(B)$, where in $E_{a,1}(B)$ we add the restriction $F^*_a(c) \neq 0$
to the $c$-sum and in $E_{a,2}(B)$ we sum over the remaining values of $c$ where $F^*_a(c) = 0$.

4.2.1. Analysis of $E_{a,1}(B)$. Using Lemma 4.12, we have

$$E_{a,1}(B) \ll \frac{\eta^{-4}B^{1/2+\epsilon}}{|A|^{1/2}} \sum_{c \in \mathbb{Z}\setminus\{0\}} \min_{0 < |c| < C_i} \left( \frac{|a_i|^{1/2}}{|c_i|} \right) \sum_{q \ll Q} q^{-3}S_{q,a}(c).$$

For a fixed choice of $c$, let $j(c) \in \{1, \ldots, 4\}$ denote the index where $|a_i|^{1/2}|c_i|^{-1}$ is minimized. Note in particular that $c_{j(c)} = 0$. We have

$$E_{a,1}(B) \ll \frac{\eta^{-4}B^{1/2+\epsilon}}{|A|^{1/2}} \sum_{j=1}^{4} |a_j|^{1/2} \sum_{c \in \mathbb{Z}\setminus\{0\}} \frac{1}{|c_j|} \sum_{q \ll Q} q^{-3}S_{q,a}(c).$$

An application of \cite{4} Lemma 4.7 with $x = Q$ yields

$$\sum_{c \in \mathbb{Z}\setminus\{0\}} \frac{1}{|c_j|} \sum_{q \ll Q} q^{-3}S_{q,a}(c) \ll B^\epsilon \sum_{c \in \mathbb{Z}\setminus\{0\}} \frac{1}{|c_j|} \prod_{i=1}^{4} \gcd(a_i, c_j)^{1/2}.$$ 

For $i \neq j$, we have

$$\sum_{0 < |c| < C_i} \gcd(a_i, c)^{1/2} \ll C_i B^\epsilon,$$

and for $i = j$, we have

$$\sum_{0 < |c_j| < C_j} \frac{\gcd(a_j, c_j)^{1/2}}{|c_j|} \ll B^\epsilon.$$ 

Hence

$$E_{a,1}(B) \ll \frac{\eta^{-4}B^{1/2+\epsilon}}{|A|^{1/2}} \sum_{j=1}^{4} |a_j|^{1/2} \prod_{i \neq j} C_i \ll \eta^{-7}B^{1/2+\epsilon}.$$
4.2.2. Analysis of $E_{a,2}(B)$. We write $E_{a,2}(B) = E_{a,2}^{(1)}(B) + E_{a,2}^{(2)}(B)$, where

$$E_{a,2}^{(1)}(B) = \frac{1}{B} \sum_{c \in \mathbb{Z}^4 \setminus \{0\}} \sum_{q \leq M} q^{-4} S_{q,a}(c) I_{q,a}(c),$$

$$E_{a,2}^{(2)}(B) = \frac{1}{B} \sum_{c \in \mathbb{Z}^4 \setminus \{0\}} \sum_{M \leq q \leq Q} q^{-4} S_{q,a}(c) I_{q,a}(c),$$

For a real parameter $M$ to be determined later. To bound $E_{a,2}^{(1)}(B)$, we apply Lemma 4.12 to obtain

$$E_{2,a}(P) \ll \eta^{-4} B^{1/2+\epsilon} \sum_{j=1}^{4} \sum_{c \in \mathbb{Z}^4 \setminus \{0\}} \frac{|a_j|^{1/2}}{|c_j|} \sum_{q \leq M} q^{-3} |S_{q,a}(c)|.$$

By [4, Lemma 4.5], we have $|S_{q,a}(c)| \ll q^3 \prod_{i=1}^{4} \gcd(a_i, c_i)^{1/2}$, and hence summing trivially over $q$ and proceeding as before, we obtain

$$E_{a,2}^{(1)}(B) \ll \eta^{-7} B^{1/2+\epsilon} M.$$  

To bound $E_{a,2}^{(2)}(B)$ we require some cancellation from the sum over $q$. To achieve this, we use summation by parts, as in the treatment of the main term in Proposition 4.8 and then apply Lemma 4.7 to obtain a better estimate for the exponential sums. We also perform the $q$-sum over $R \leq q \leq 2R$ and later take a dyadic sum over $M \leq R \ll Q$. Summation by parts yields

$$\sum_{R \leq q \leq 2R} q^{-4} S_{q,a}(c) I_{q,a}(c) \ll -\Sigma(R; c) \frac{I_{R,a}(c)}{R} - \int_{R}^{2R} \Sigma(x; c) \frac{\partial}{\partial x} \left( \frac{I_{x,a}(c)}{x} \right) dx.$$

Therefore, from Lemma 4.16 we obtain

$$\frac{1}{B} \sum_{R \leq q \leq 2R} q^{-4} S_{q,a}(c) I_{q,a}(c) \ll \frac{\eta^{-1} P_1 \cdots P_4}{BR} \sup_{R \leq x \leq 2R} |\Sigma(x; c)|.$$  

(4.38)
We recall the definition of $\Delta_c(a)$ from (4.28). Using the estimate for $|\Sigma(x; c)|$ from Lemma 4.7, we have

$$E_{a_2}^{(2)}(B) \ll \eta^{-1} P_1 \cdots P_4 |A|^{3/16} \sum_{R \text{ dyadic}} R^{-1/2+\epsilon} \sum_{c \in \mathbb{Z}^4 \setminus \{0\}} R^{\frac{1}{2+\epsilon}} \sum_{|c| \ll C} \Delta_c(a)^{3/8}$$

$$\ll \eta^{-1} B^{1+\epsilon} \sum_{c \in \mathbb{Z}^4 \setminus \{0\}} \Delta_c(a)^{3/8}. \quad (4.39)$$

It is possible to obtain the asymptotic formula from Theorem 1.1 (albeit with a smaller power saving) by bounding the sum in (4.39) trivially by $C_1 \cdots C_4 \Delta^{3/8}$. However, in order to obtain the error terms claimed in Theorem 4.2, we now make use of the constraint $F^*_a(c) = 0$ in the $c$-sum to obtain a more refined estimate. In the following two lemmas, we adopt the notation that $vw = (v_1 w_1, \ldots, v_4 w_4)$ for vectors $v, w \in \mathbb{Z}^4$.

**Lemma 4.13.** For any $d, e \in (\mathbb{Z} \neq 0)^4$, we have

$$\Delta(de) \leq (d_1 \cdots d_4)^2 \Delta(e).$$

**Proof.** Fix a prime $p$, and define $\delta_i = \nu_p(d_i), \varepsilon_i = \nu_p(e_i)$ for $i \in \{1, \ldots, 4\}$. Without loss of generality, we may assume that $\varepsilon_1 \leq \cdots \leq \varepsilon_4$. Then

$$\nu_p(\Delta(de)) = \sum_{i=1}^4 \min \left( \delta_i + \varepsilon_i, \sum_{j \neq i} (\delta_j + \varepsilon_j) \right)$$

$$\leq \sum_{i=1}^3 (\delta_i + \varepsilon_i) + \min \left( \delta_4 + \varepsilon_4, \sum_{i=1}^3 (\delta_i + \varepsilon_i) \right),$$

and

$$\nu_p(\Delta(e)) = \sum_{i=1}^3 \varepsilon_i + \min \left( \varepsilon_4, \sum_{i=1}^3 \varepsilon_i \right).$$

Therefore

$$\nu_p(\Delta(de)) - \nu_p(\Delta(e)) \leq \begin{cases} \sum_{i=1}^3 2\delta_i, & \text{if } \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq \varepsilon_4, \\ \sum_{i=1}^4 \delta_i, & \text{otherwise}. \end{cases}$$

$$\leq \nu_p((d_1 \cdots d_4)^2).$$

Taking a product over all primes $p$ completes the proof of the lemma. \qed
Lemma 4.14. We have

\[
\sum_{c \in \mathbb{Z}^4 \setminus \{0\}, \quad |c_i| \leq C_i, \quad F^*_a(c) = 0} \Delta_{c}(a)^{3/8} \ll \eta^{-3} B^\varepsilon \Delta^{1/2}.
\]

Proof. For a nonzero integer \(m\), we define \(\text{sqf}(m)\) to be the smallest positive integer \(r\) such that \(|m|/r\) is a square. For \(i \in \{1, \ldots, 4\}\), we decompose \(a_i\) into a product \(a_i = e_i n_i k_i^2\), where

\[
e_i = \gcd\left(a_i, \prod_{j \neq i} a_j\right), \quad n_i = \text{sqf}\left(a_i/e_i\right), \quad k_i^2 = a_i/(e_in_i).
\]

By definition, we have \(\prod_{i=1}^4 e_i = \Delta\). For any \(i \in \{1, \ldots, 4\}\), it is clear that

\[
F^*_a(c) = 0 \implies a_i|c_i^2 \prod_{j \neq i} a_j \implies n_i k_i^2 | c_i^2 \implies n_i k_i | c_i. \quad (4.40)
\]

We divide up the sum over \(c\) according to how many of the coordinates \(c_1, \ldots, c_4\) are equal to zero. At most two of the coordinates can be zero, due to the assumptions \(c \neq 0\) and \(F^*_a(c) = 0\). Up to reordering the indices, we obtain

\[
\sum_{c \in \mathbb{Z}^4 \setminus \{0\}, \quad |c_i| \leq C_i, \quad F^*_a(c) = 0} \Delta_{c}(a)^{3/8} \ll T_0 + T_1 + T_2, \quad (4.41)
\]

where

\[
T_0 = \sum_{c \in (\mathbb{Z}_{\neq 0})^4, \quad |c_i| \leq C_i, \quad F^*_a(c) = 0} \Delta_{c}(a)^{3/8}, \quad T_1 = \sum_{c \in (\mathbb{Z}_{\neq 0})^3, \quad |c_i| \leq C_i, \quad F^*_a((c,0)) = 0} \Delta_{(c,0)}(a)^{3/8}, \quad T_2 = \sum_{c \in (\mathbb{Z}_{\neq 0})^2, \quad |c_i| \leq C_i, \quad F^*_a((c,0,0)) = 0} \Delta_{(c,0,0)}(a)^{3/8}
\]

We begin by studying \(T_0\). Given that \(F^*_a(c) = 0\), we only need to take the sum over \(c_1, c_2\) and \(c_3\), because this implicitly determines (up to sign) the value of
c_4. Together with (4.40), this implies that

\[ T_0 \ll \sum_{c_1, c_2, c_3 \in \mathbb{Z}_{\not\equiv 0}} \Delta_c(a)^{3/8} \]

\[ \leq \sum_{|c_i| \leq C_i} \sum_{d \in \mathbb{Z}_{\not\equiv 0}^4} \Delta(dnk)^{3/8} \]

\[ \leq \left( \prod_{i=1}^{3} \frac{C_i}{n_i k_i} \right) \sum_{d \in \mathbb{Z}_{\not\equiv 0}^4} \Delta(dnk)^{3/8} \prod_{i=1}^{4} d_i^{-1/4}, \quad (4.43) \]

where in the last line we have applied Lemma 4.13. However, \( \Delta(nk) = 1 \) by the definition of \( e_1, \ldots, e_4 \). There are \( O(A^\varepsilon) \) choices for \( d \) in the sum in (4.43), and each summand is bounded by 1. Therefore

\[ T_0 \ll \left( \prod_{i=1}^{3} \frac{C_i}{n_i k_i} \right) A^\varepsilon. \quad (4.44) \]

Returning to (4.44) and recalling that \( C_i = \eta^{-1}B^\varepsilon a_i^{1/2} = \eta^{-1}B^\varepsilon (e_i n_i)^{1/2} k_i \), we conclude that

\[ T_0 \ll \eta^{-3}B^\varepsilon \left( \prod_{i=1}^{3} \frac{|a_i|^{1/2}}{n_i k_i} \right) \]

\[ \leq \eta^{-3}B^\varepsilon \Delta^{1/2}. \]

The approach for estimating \( T_1 \) and \( T_2 \) is very similar, so we will focus on the main differences. It is sufficient to only use the divisibility conditions from (4.40) in these cases. For \( T_1 \), all the sums in the above argument will be over
vectors indexed by \( \{1, 2, 3\} \), because we have fixed \( c_4 = 0 \). We have

\[
T_1 \ll \left( \prod_{i=1}^{3} \frac{C_i}{n_i k_i} \right) A^\epsilon \Delta(n_1 k_1, n_2 k_2, n_3 k_3, a_4)^{3/8}
\]

\[
= \left( \prod_{i=1}^{3} \frac{C_i}{n_i k_i} \right) A^\epsilon
\]

\[
\ll \eta^{-3} B^\epsilon \Delta^{1/2}.
\]

For \( T_2 \), we only take sums over vectors indexed by \( \{1, 2\} \), since we have fixed \( c_3 = c_4 = 0 \). We have

\[
T_2 \ll \left( \prod_{i=1}^{2} \frac{C_i}{n_i k_i} \right) A^\epsilon \Delta(n_1 k_1, n_2 k_2, a_3, a_4)^{3/8}
\]

\[
= \left( \prod_{i=1}^{2} \frac{C_i}{n_i k_i} \right) A^\epsilon (a_3, a_4)^{3/4}
\]

\[
\ll \eta^{-2} B^\epsilon (e_1 e_2)^{1/2} (e_3 e_4)^{3/8}
\]

\[
\ll \eta^{-2} B^\epsilon \Delta^{1/2}.
\]

Thus each of \( T_0, T_1, T_2 \) is bounded by \( \eta^{-3} B^\epsilon \Delta^{1/2} \), as required. \( \Box \)

Recalling (4.39), we therefore have

\[
E_{2, a}(B) \ll \eta^{-4} B^{1+\epsilon} \Delta^{1/2} / M^{1/2} |A|^{5/16}.
\]

To combine with the error term \( E_{1, a}(B) \), we make the choice

\[
M = \eta B^{1/3} \Delta^{1/3} |A|^{-5/24}.
\]

This yields the estimate

\[
E_a(B) \ll \frac{\eta^{-6} B^{5/6+\epsilon} \Delta^{1/3} + \eta^{-7} B^{1+2+\epsilon}}{|A|^{5/24}}. \quad (4.45)
\]

This estimate is larger than the error term from Proposition 4.8. Combining with Proposition 4.8, this completes the proof of Theorem 4.3.

The final ingredient in the proof of Theorem 4.2 is an estimate for the singular series \( \mathcal{S}_a \).

**Lemma 4.15.** Let \( a \in (\mathbb{Z}_{\neq 0})^4 \) and assume that \( A \neq \Box \). Then

\[
|\mathcal{S}_a| \ll |A|^\epsilon \Delta^{1/4}.
\]
Proof. Beginning with the definition of the singular series, we have
\[
\mathfrak{G}_a = \sum_{q=1}^{\infty} q^{-4} S_{q,a}(0) = \sum_{q<T} q^{-4} S_{q,a}(0) + \sum_{q>T} q^{-4} S_{q,a}(0)
\]
for any \( T \geq 1 \). For the sum over \( q > T \), we apply partial summation and Lemma 4.7. We have
\[
\sum_{q>T} q^{-4} S_{q,a}(0) = \sum_{q > T} q \int_{T}^{\infty} S(x;0) \frac{\partial}{\partial x}(x^{-1}) \, dx \ll T^{-1/2+\epsilon}|A|^{3/16+\epsilon} \Delta^{3/8}.
\]
By choosing \( T = |A|^2 \), we can ensure that the contribution from this region is negligible.

For the remaining sum over \( q \leq T \), we follow a similar argument to the proof of Lemma 4.7. Using the multiplicativity of \( S_{q,a}(0) \) in \( q \), we have
\[
\sum_{q \leq T} q^{-4} S_{q,a}(0) \ll \left( \sum_{q_1 \leq T} \frac{\varphi(q_1)}{q_1^2} \right) \left( \sum_{q_2 \leq T/q_1} q_2^{-4} S_{q_2,a}(0) \right)
\]
\[
\ll T^\epsilon |A|^{\epsilon} \max_{q | (2A)^\infty} \left( \prod_{i=1}^{4} \gcd(q,a_i)^{1/2} \right)^2,
\]
where in the second line we have estimated the sum over \( q_1 \) trivially, and the sum over \( q_2 \) by applying Lemma 4.5. We let \( m_{i,p} \) denote the \( i \)th smallest element of \( \nu_p(a_1), \ldots, \nu_p(a_4) \). We define
\[
L_p = \nu_p \left( \frac{\prod_{i=1}^{4} \gcd(q,a_i)}{q} \right), \quad k_p = \nu_p(q).
\]
Then
\[
L_p = \sum_{j=1}^{4} \min(k_p, m_{i,p}) - 2k_p.
\]
The maximum possible value of \( L_p \) is attained by choosing \( k_p = m_{2,p} \), and so \( L_p \leq m_{1,p} + 2m_{2,p} \). We have
\[
\nu_p(\Delta) = m_{1,p} + m_{2,p} + m_{3,p} + \min(m_{1,p} + m_{2,p} + m_{3,p} + m_{4,p}) \geq 2(m_{1,p} + m_{2,p}),
\]
and so \( L_p \leq \nu_p(\Delta)/2 \). Taking a product over \( p | 2A \) completes the proof of the lemma. \( \Box \)
5. Proof of Theorem 1.1

We recall that the quantity $N_a(B)$ studied in Section 4 did not involve any primitivity conditions. In order to insert the condition $\gcd(z_1, \ldots, z_4) = 1$, we apply an inclusion-exclusion argument which is a special case of [6, Section 3].

We begin by fixing some notation which we will use throughout this section. Let $z_1, \ldots, z_4$ denote nonzero squareful numbers, and $x_1, \ldots, x_4 \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_4 \in \mathbb{Z}_{\neq 0}$ the unique integers such that $z_i = x_i^2 \gamma_i^3$ and $\gamma_i$ is square-free for all $i$. Let $A = a_1 \cdots a_4, R = r_1 \cdots r_4, S = s_1 \cdots s_4$ and $Y = y_1 \cdots y_4$.

For vectors $v, w \in \mathbb{N}^4$, we write $v \mid w$ to mean $v_i \mid w_i$ for all $i = 1, \ldots, 4$. For an integer $m$, we write $v \mid m$ for $v_i \mid m$ for all $i = 1, \ldots, 4$, and $\gcd(m, v)$ for $(\gcd(m, v_1), \ldots, \gcd(m, v_4))$. We also define $v^{[m]}$ to be the vector in $\mathbb{N}^4$ with $i$th coordinate $v_i^{[m]} = \prod_{p | m} P_p^{v_p(v_i)}$, where $v_p$ denotes the $p$-adic valuation.

We recall from Proposition 3.1 that

$$N(B) = N(D, B) + O(B^{1+\epsilon} D^{-1/12}), \quad (5.1)$$

where $N(B)$ is defined in (1.3) and $N(D, B)$ is defined in the same way but with the additional constraint $|Y| \leq D$. For $r, s \in \mathbb{N}^4$ and $s_0 \in \mathbb{N}$, we define

$$\mathcal{N}(B; r, s, s_0) = \left\{ z \in (\mathbb{Z}_{\neq 0})^4 : |z| \leq B, |Y| \leq D, Y \neq \emptyset, z_1 + \cdots + z_4 = 0, r \mid y, s \mid x, s_0 \mid x \right\}. \quad (5.2)$$

**Definition 5.1.** Given $r, s \in \mathbb{N}^4$ and $s_0 \in \mathbb{N}$, we define $\omega(r, s, s_0)$ as follows:

1. $\omega(r, s, s_0) = \mu(s_0) \prod_p \omega(r^{[p]}, s^{[p]}, 1)$. In particular, $\omega(1, 1, s_0) = \mu(s_0)$.
2. If one of $r_1, \ldots, r_4, s_1, \ldots, s_4$ is not square-free, then $\omega(r, s, s_0) = 0$.
3. If $\gcd(s_1, \ldots, s_4) > 1$ then $\omega(r, s, s_0) = 0$.
4. If $\gcd(s_0, s) \neq 1$, then $\omega(r, s, s_0) = 0$.
5. If $p | RS$ but $p \nmid r_i s_i$ for some $i$, then $\omega(r^{[p]}, s^{[p]}, 1) = 0$.
6. If $p | r_i s_i$ for every $i$, $\gcd(s_1, \ldots, s_4) = 1$, $\gcd(s_0, s) = 1$, and $r_i, s_i$ are square-free for every $i$, then $k := \nu_p(RS)$ satisfies $4 \leq k \leq 7$. Define $\omega(r^{[p]}, s^{[p]}, 1) = (-1)^{k+1}$.

The motivation for this choice of $\omega$ comes from the following lemma.

**Lemma 5.2.** We have

$$N(D, B) = \sum_{r, s \in \mathbb{N}^4} \omega(r, s, s_0) \# \mathcal{N}(B; r, s, s_0). \quad (5.2)$$

**Proof.** We write the right hand side of (5.2) as

$$\sum_{z \in \mathcal{N}(B; 1, 1, 1)} \sum_{r, s, s_0 \in \mathbb{N}} \omega(r, s, s_0). \quad (5.3)$$
We would like to show that the inner sum is the indicator function for the condition \( \gcd(z_1, \ldots, z_4) = 1 \). If \( \gcd(z_1, \ldots, z_4) = 1 \), then by property (5) of Definition 5.1, the only nonzero term in the inner sum of (5.3) is the term \( \omega(1, 1, 1) = 1 \). From now on, we suppose that \( \gcd(z_1, \ldots, z_4) > 1 \). By properties (1) and (2) from Definition 5.1, it suffices to show that for any prime \( p \) dividing \( \gcd(z_1, \ldots, z_4) \), we have

\[
\sum_{\substack{r,s \in \mathbb{N}^4, s_0 \in \mathbb{N} \\
 r \mid (p,y), s \mid (p,x), s_0 \mid (p,x) \}} \omega(r, s, s_0) = 0. \tag{5.4}
\]

The condition \( s_0 \mid (p, x) \) implies that \( s_0 = 1 \) or \( s_0 = p \). If \( s_0 = p \) and \( \omega(r, s, s_0) \neq 0 \), then by (4) and (5) of Definition 5.1, we have \( s = 1 \) and \( r = 1 \) or \( r = (p, p, p, p) \). Therefore the left hand side of (5.4) becomes

\[
\mu(p)(\omega(1, 1, 1) + \omega((p, p, p, p), 1, 1)) = -(1 - 1) = 0.
\]

Now suppose that \( s_0 = 1 \). Let \( k_0 \) denote the number of the number of \( y_1, \ldots, y_4, x_1, \ldots, x_4 \) that are a multiple of \( p \), and let \( k \) denote the number of \( r_1, \ldots, r_4, s_1, \ldots, s_4 \) that are a multiple of \( p \). If \( \omega(r, s, 1) \neq 0 \), then we have \( k \leq k_0 \) and \( k_0 = 4, 5 \) or 6. Indeed, if \( p \) divides seven of \( y_1, \ldots, y_4, x_1, \ldots, x_4 \) then it must also divide the eighth, which contradicts condition (3) from Definition 5.1, and so \( k_0 = 7, 8 \) are not possible. We go through the cases \( k_0 = 4, 5, 6 \) in turn. For convenience we write \( \omega(r, s, 1) = \omega(k) \) and \( \omega(1, 1, 1) = \omega(0) = 1 \).

If \( k_0 = 4 \), there is one summand of (5.4) with \( k = 0 \), namely \( (r, s) = (1, 1) \), and one with \( k = 4 \), and all other terms are zero. Therefore the contribution to the sum in (5.4) from \( k_0 = 4 \) is \( \omega(0) + \omega(4) = 1 - 1 = 0 \). If \( k_0 = 5 \), then there is one summand in (5.4) with \( k = 0 \), two with \( k = 4 \) and one with \( k = 5 \), and so we obtain \( \omega(0) + 2\omega(4) + \omega(5) = 1 - 2 + 1 = 0 \). The relevant calculation for \( k = 6 \) is

\[
\omega(0) + 4\omega(4) + 4\omega(5) + \omega(6) = 1 - 4 + 4 - 1 = 0.
\]

We have therefore established that (5.4) holds. \( \square \)

We now use Theorem 1.3 to estimate \( \# \mathcal{N}(B; r, s, s_0) \) for each choice of \( r, s \in \mathbb{N}^4, s_0 \in \mathbb{N} \) satisfying \( \omega(r, s, s_0) \neq 0 \). We have

\[
\# \mathcal{N}(B; r, s, s_0) = \sum_{\epsilon \in \{\pm 1\}^4} \sum_{|Y| \leq D, Y \neq \square \atop \operatorname{sgn} y_i = \epsilon_i \atop r \mid Y} \mu^2(y_1) \cdots \mu^2(y_4) N_y(B; s, s_0),
\]

where

\[
N_y(B; s, s_0) = \# \left\{ x \in \mathbb{N}^4 : \sum_{i=1}^4 y_i^3 x_i^2 = 0, s | x, s_0 | x, \ | x_i | \leq \left( \frac{B}{|y_i|} \right)^{1/2} \text{ for all } i \right\}. \tag{5.5}
\]
We make use of condition (4) from Definition 5.1 which allows us to make a change of variables from $x_i$ to $s_0s_is_i$. In the notation of (1.6), we have $N_{\mathcal{V}}(B; s, s_0) = N_{s^2y^3}(B/s_0^2)$, where $s^2y^3$ denotes the vector $(s_1^2y_1^3, \ldots, s_2^2y_2^3)$. Assuming $Y \neq \square$, we have $S^2Y^3 \neq \square$. Moreover, we note that the conditions $\omega(r, s, s_0) \neq 0$ and $r|y$ imply that $S|Y^3$. (Indeed, if $\omega(r, s, s_0) \neq 0$, then by conditions (2) and (3) of Definition 5.1, $S$ must be fourth-power free. Since condition (5) implies that every prime dividing $S$ must also divide $R$, we deduce that $S|R^3$, and together with the assumption $r|y$, this implies that $S|Y^3$.) Therefore $|Y| \leq D$ implies that $|S^2Y^3| \leq D^9$. In particular, $|S^2Y^3| \leq B^{4/7}$ when $D \leq B^{1/16}$. Hence we may apply Theorem 4.2 with $a = s^2y^3$ to obtain for any $D \leq B^{1/16}$,

$$
\# \mathcal{N}(B; r, s, s_0) = \sum_{\epsilon \in \{\pm 1\}^4} \sum_{|Y| \leq D} \left( \frac{\mathcal{G}_{s^2y^3}(\epsilon)B}{S|Y|^{3/2}s_0^2} + O \left( \frac{B^{41/42+\epsilon}D^{1/3}}{|S^2Y^3|^{11/24}} \right) \right),
$$

(5.6)

where as in Section 4 we define

$$
\Delta = \Delta(s^2y^3) = \prod_{i=1}^4 \gcd \left( s_i^2y_i^3, \prod_{j \neq i} s_j^2y_j^3 \right).
$$

We begin by studying the main term from (5.6). We would like to replace the sum over $|Y| \leq D$ with a sum over all $y \in (\mathbb{Z}_{\neq 0})^4$ satisfying $\sgn y_i = \epsilon_i$ for all $i$. In order to estimate the error term in doing so, we appeal to Lemma 4.15.

We define

$$
E_1(D) = \sum_{r, s \in \mathbb{N}^4} \omega(r, s, s_0) \sum_{\epsilon \in \{\pm 1\}^4} \sum_{y \in (\mathbb{Z}_{\neq 0})^4} \frac{\mu^2(y_1) \cdots \mu^2(y_4) \mathcal{G}_{s^2y^3}(\epsilon)}{S|Y|^{3/2}s_0^2}.
$$

**Lemma 5.3.** For any $D \geq 1$, we have $E_1(D) = O(D^{-1/4+\epsilon}).$

**Proof.** From the observation $S|Y^3$ made above, and the fact that $\sigma_\infty(\epsilon) \ll 1$, we have

$$
E_1(D) \ll \sum_{\epsilon \in \{\pm 1\}^4} \sum_{y \in \mathbb{N}^4} |\mathcal{G}_{s^2y^3}| \frac{Y^{-3/2}}{S} \sum_{s_0 \in \mathbb{N}^4} \sum_{r \in \mathbb{Y}} |\omega(r, s, s_0)|.
$$

Since $|\omega(r, s, s_0)| \leq 1$, the sum over $s_0$ is convergent. The sum over $r$ only contributes $O(Y^\epsilon)$ by the trivial bound for the divisor function. Applying the estimate from Lemma 4.15, we have $|\mathcal{G}_{s^2y^3}| \ll (SY)^\epsilon \Delta(s^2y^3)^{1/4}$ whenever
$Y \neq \Box$. Therefore

$$E_1(D) \ll \sum_{y \in \mathbb{N}_4^4} Y^{-3/2+\varepsilon} \sum_{s \in \mathbb{N}_4^4} \frac{\Delta(s^2y^3)^{1/4}}{S^{1-\varepsilon}}. \quad (5.7)$$

From Lemma 4.13, we have $\Delta(s^2y^3) \leq S^4 \Delta(y^3)$. Continuing from (5.7), we obtain the estimate

$$E_1(D) \ll \sum_{y \in \mathbb{N}_4^4} \frac{\Delta(y^3)^{1/4}}{Y^{3/2-\varepsilon}} \sum_{s \in \mathbb{N}_4^4} S^\varepsilon \ll \sum_{y \in \mathbb{N}_4^4} \frac{\Delta(y^3)^{3/4}}{Y^{3/2-\varepsilon}}.$$

We note that $\Delta(y)$ is always squareful. Therefore, for any $R \geq 1$, there are $O(R^{1/2})$ possible values for $\Delta(y)$ in the range $[R, 2R]$. We define the quantity $M(y) = Y/\Delta(y)$. Given $M, \Delta \geq 1$, there are $O((M\Delta)^{\varepsilon})$ choices for $y \in \mathbb{N}_4^4$ satisfying $M = M(y), \Delta = \Delta(y)$. Breaking into dyadic intervals, we obtain

$$\sum_{y \in \mathbb{N}_4^4} \frac{\Delta(y)^{3/4}}{Y^{3/2-\varepsilon}} \ll \sum_{R_1, R_2 \text{ dyadic}} R_1^{1/2} R_2^{-1/4+\varepsilon} \max_{y \in \mathbb{N}_4^4} M(y)^{-3/2+\varepsilon} \Delta(y)^{-3/4+\varepsilon}$$

$$\ll \sum_{R_1, R_2 \text{ dyadic}} R_1^{-1/2+\varepsilon} R_2^{-1/4+\varepsilon} \ll D^{-1/4+\varepsilon},$$

as required. \hfill \Box

We now study the error term

$$E_2(D) = \sum_{r, s \in \mathbb{N}_4^4} \omega(r, s, s_0) \sum_{\varepsilon \in \{-1, 1\}^4} \sum_{|y_i| \leq D, Y \neq \Box \text{ sign } y_i = 1} \sum_{r|y, \text{ y square-free}} \frac{\Delta(s^2y^3)^{1/3}}{|S^2y^3|^{11/24}}. \quad (5.8)$$

**Lemma 5.4.** We have $E_2(D) \ll D^{11/8+\varepsilon}$.

**Proof.** We proceed in a similar fashion to the proof of Lemma 5.3. Let $M(y)$ be as defined in Lemma 5.3. Then
\[ E_2(D) \ll \sum_{y \in \mathbb{N}^4 \atop Y \leq D} \sum_{S \mid Y^3} \frac{\Delta(s^2y^3)^{1/3}}{(S^2Y^3)^{11/24 - \epsilon}} \]
\[ \ll \sum_{y \in \mathbb{N}^4 \atop Y \leq D} \frac{\Delta(y)}{Y^{11/8 - \epsilon}} \sum_{s \in \mathbb{N}^4 \atop S \mid Y^3} S^{5/12 + \epsilon} \]
\[ \ll \sum_{R_1, R_2 \text{ dyadic} \atop R_1, R_2 \leq D} R_1^{1/2} \max_{y \in \mathbb{N}^4 \atop R_1 \leq M(y) \leq 2R_1, R_2 \leq \Delta(y) \leq 2R_2} M(y)^{-1/8 + \epsilon} \Delta(y)^{7/8 + \epsilon} \]
\[ \ll \sum_{R_1, R_2 \text{ dyadic} \atop R_1, R_2 \leq D} R_1^{7/8 + \epsilon} R_2^{11/8 + \epsilon} \]
\[ \ll D^{11/8 + \epsilon}. \quad \square \]

We now conclude the proof of Theorem 1.1. Combining (5.1), Lemma 5.2 (5.6), Lemma 5.3 and Lemma 5.4, for any \( D \leq B^{1/16} \), we have
\[ N(B) = cB + O(B^{1+\epsilon} D^{-1/12}) + O(B^{41/42+\epsilon} D^{11/8}), \]
where
\[ c = \frac{1}{16} \sum_{\epsilon \in \{\pm 1\}^4} \sigma_\infty(\epsilon) \sum_{y \in (\mathbb{Z}_{\neq 0})^4 \atop \text{sgn}(y_i) = \epsilon_i \atop y \neq 0} \frac{\mu^2(y_1) \cdots \mu^2(y_4)}{|y|^{3/2}} \sum_{r, s, s_0 \in \mathbb{N}^4 \atop s_0 \in \mathbb{N} \atop \forall y} \frac{\varpi(r, s, s_0) \mathcal{G}_{s^2y^3}}{S^2 S_0^2}. \quad (5.9) \]
Making the choice \( D = B^{4/245} \), we obtain \( N(B) = cB + O(B^{734/735+\epsilon}). \)

5.1. **The leading constant.** The expression for the leading constant in (5.9) is analogous to [6, Equation (3.14)]. Similarly to [6, Equation (3.15)], we now demonstrate that the inner sum of (5.9) can be expressed as a product of local densities. We define for any prime \( p \) and any \( n \in \mathbb{N}^4 \),
\[ M_n(y, p) = \# \left\{ m \pmod{p^n} : \sum_{i=1}^4 y_i^3 m_i^2 \equiv 0 \pmod{p^n}, p \nmid m_j y_j \text{ for some } j \right\}. \]

**Lemma 5.5.** We have
\[ \sum_{r, s, s_0 \in \mathbb{N}^4 \atop s_0 \in \mathbb{N} \atop \forall y} \frac{\varpi(r, s, s_0) \mathcal{G}_{s^2y^3}}{S^2 S_0^2} = \prod_p \lim_{N \to \infty} \left( \frac{M_N(y, p)}{p^{3N}} \right). \]
Proof. We first express the singular series \( \mathfrak{G}_{s^2y^3} \) as a product of local densities. Since \( \mathfrak{G}_{s^2y^3} = \sum_{q=1}^{\infty} S_{q,s^2y^3}(0) \), and \( S_{q,s^2y^3}(0) \) is multiplicative in \( q \), we have

\[
\mathfrak{G}_{s^2y^3} = \prod_{p} \left( 1 + \sum_{n=1}^{\infty} p^{-4n} S_{p^n,s^2y^3}(0) \right).
\]

Moreover,

\[
S_{p^n,s^2y^3}(0) = \sum_{b \mod p^n} \sum_{0 \leq k < p^n, \gcd(k,p)=1} \epsilon_{p^n} \left( k \sum_{i=1}^{4} s_i^2y_i^3b_i^2 \right) = p^n N_{s,y}(p^n) - p^{n+3} N_{s,y}(p^{n-1}),
\]

where

\[
N_{s,y}(p^n) = \# \left\{ m \pmod{p^n} : \sum_{i=1}^{4} s_i^2y_i^3m_i^2 \equiv 0 \pmod{p^n} \right\}.
\]

Therefore

\[
\sum_{n=1}^{\infty} p^{-4n} S_{p^n,s^2y^3}(0) = \lim_{N \to \infty} \sum_{n=1}^{N} \left( p^{-3n} N_{s,y}(p^n) - p^{-3(n-1)} N_{s,y}(p^{n-1}) \right)
\]

\[
= \lim_{N \to \infty} \left( p^{-3N} N_{s,y}(p^N) \right) - 1,
\]

and hence

\[
\mathfrak{G}_{s^2y^3} = \prod_{p} \left( \lim_{N \to \infty} \frac{N_{s,y}(p^N)}{p^{3N}} \right). \tag{5.10}
\]

For the remainder of the proof, we will use a sum over \( r^{[p]}, s^{[p]}, s_0^{[p]} \) to denote a sum over all \( r, s \in \mathbb{N}^4 \), \( s_0 \in \mathbb{N} \), with \( (r, s, s_0) = (r^{[p]}, s^{[p]}, s_0^{[p]}) \) and \( r \mid y \). Recalling that \( \omega(r, s, s_0) \) is also multiplicative, we obtain

\[
\sum_{r, s, s_0 \in \mathbb{N}^4} \omega(r, s, s_0) \mathfrak{G}_{s^2y^3} = \prod_{p} \left( \sum_{r^{[p]}, s^{[p]}, s_0^{[p]}} \omega(r^{[p]}, s^{[p]}, s_0^{[p]}) \lim_{N \to \infty} \left( \frac{N_{s,y}(p^N)}{p^{3N}} \right) \right) \tag{5.11}
\]
To complete the proof, it suffices to show that for a fixed prime $p$, the factor on the right hand side of (5.11) is equal to $\lim_{N \to \infty} \left( p^{-3N} M_N(y, p) \right)$. We define

$$N_{s,s_0,y}(n) = \# \left\{ m \pmod{p^n} : \sum_{i=1}^{4} y_i^3 m_i^2 \equiv 0, s|m, s_0|m \right\}.$$ 

We claim that

$$M_n(y, p) = \sum_{r[p], s[p], s_0[p]} \omega(r, s, s_0) N_{s,s_0,y}(n). \quad (5.12)$$

To see this, we fix $m \pmod{p^n}$ such that $\sum_{i=1}^{4} y_i^3 m_i^2 \equiv 0$, and consider the quantity

$$\sum_{r[p], s[p], s_0[p]} \omega(r, s, s_0).$$

This expression has already been encountered in (5.4), and from the proof of Lemma 5.2 we see that it is equal to 1 if $p | y^2 m_i$ for some $i$, and zero otherwise. This establishes (5.12).

Changing variables from $m_i$ to $s_0 s_i m_i$, we have

$$N_{s,s_0,y}(n) = \frac{s_0^{4[p]} N_{s,y}(p^{n-2\mu(s_0)})}{S[p]}.$$ 

Therefore

$$\lim_{N \to \infty} \left( \frac{N_{s,s_0,y}(N)}{p^{3N}} \right) = \lim_{N \to \infty} \left( \frac{N_{s,y}(p^N)}{S[p] s_0^{2[p]} \prod_p} \right).$$

Combining this with (5.12), we deduce that $\lim_{N \to \infty} \left( p^{-3N} M_N(y, p) \right)$ matches the Euler factor from the right hand side of (5.11). □

We do not expect the leading constant $c$ from (5.9) to agree with the prediction from [21, Conjecture 1.1]. In [25], we study the counting problem $N_k(B)$ from (1.1) in the case $k = 3$. This corresponds to the orbifold $(\mathbb{P}^1, D)$, where $D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. The more detailed discussion around the case $k = 3$ in [25] is readily adapted to deal with the case $k = 4$ considered in this paper. Therefore, we content ourselves with giving a brief summary here.

We recall that $N_y(B)$ denotes the contribution to $N(B)$ from a fixed choice of $y \in (\mathbb{Z}_{\neq 0})^4$ satisfying $Y \neq \Box$ and $\mu^2(y_1) = \cdots = \mu^2(y_4) = 1$. We have shown that each $N_y(B)$ contributes a positive proportion to $N(B)$, namely $N_y(B) \sim c_y B$, where

$$c_y = \frac{\sigma_{\infty}(\epsilon)}{|Y|^{3/2}} \prod_p \left( \lim_{N \to \infty} \frac{M_N(y, p)}{p^{3N}} \right).$$
Moreover, Manin’s conjecture can be applied to the quadric $Q_y$ given by the equation $\sum_{i=1}^{4} y_i^3 x_i^2 = 0$, and the constant thus predicted is in agreement with $c_y$ (as was expected due to Remark 1.7). Hence the expression in (5.9) is naturally interpreted as a sum over $y$ of leading constants arising from Manin’s conjecture applied to the quadrics $Q_y$. This sum is not multiplicative in $y$, and it does not appear to be possible to express (5.9) as an Euler product. In contrast, the leading constant predicted in [21, Conjecture 1.1] for $N(B)$ is by definition an Euler product. In [25], we find that when $k = 3$, the analogous constant to (5.9) does not agree with [21, Conjecture 1.1] numerically, and it seems very likely that the same will hold true for $k = 4$.

A natural question that arises is whether thin sets could explain a discrepancy between $c$ and the constant predicted by [21, Conjecture 1.1]. However, in analogy to [25, Theorem 1.3], it can be shown that any constant in $(0, c]$ could be obtained by the removal of an appropriate thin set. Most of these thin sets have no clear geometric interpretation in relation to the original orbifold.

From this point of view, the definition of thin sets of Campana points from [21, Definition 3.7] seems too permissive, and $c$ seems to be the most natural choice of leading constant for the orbifold considered in this paper.

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