On The Symmetries Of Topological Quantum Field Theories

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Abstract  We display properties of the general formalism which associates to any given gauge symmetry a topological action and a system of topological BRST and anti-BRST equations. We emphasize the distinction between the antighosts of the geometrical BRST equations and the antighosts occurring in field theory. We propose a transmutation mechanism between these objects. We illustrate our general presentation by examples.
1 Introduction

It is now well known that any given classical gauge symmetry can be put in correspondence with a BRST and anti-BRST differential algebra [1]. This permits an elegant and systematic construction of quantum field theories with a Hilbert space describing the same number of gauge invariant degrees of freedom as the associated classical gauge theories. In this approach, the understanding of the statistics of various ghost and Lagrange multiplier fields is straightforward. The selection of physical observables, as well as the identification and the control of anomalies, amount to the algebraic problem of solving the cohomology of the BRST operator. This general and aesthetical procedure is very powerful. As an example, it permits one to handle in a strikingly unified way systems with degenerate gauge symmetries, that is systems which contain ghost of ghost fields at the quantum level with possibly higher order ghost interactions, and also systems with global zero modes. It contains as a limiting case the Faddeev-Popov method.

In this paper, I will present properties concerning the generalization of this ”principle of BRST symmetry” for topological quantum field theories [3]. The motivation is two-fold. Firstly, starting from a gauge symmetry, one ends up with a beautiful differential algebra, which includes as a ”hard kernel” the usual BRST and anti-BRST transformations laws of the gauge symmetry. Secondly, this provides a general framework to quantize lagrangians which can be written locally as pure divergences. Indeed, inspired by Witten ideas [2], physicists have realized that it makes sense to quantize such lagrangians although they do not generate classical equations of motion. The idea is to introduce an equal number of commuting and anticommuting fields, in order to get a configuration space with an effective number of degrees of freedom almost equal to zero, and to define the propagation of modes by BRST invariant gauge fixing terms with appropriate compensations between fermionic and bosonic degrees of freedom. Eventually, it becomes possible to compute non trivial topological information by path integrations of certain operators, which count various well weighted combinations of contributions of zero mode of Dirac-like operators. Many examples show that the general formalism presented in this paper is appropriate for such quantization schemes [3].
We will also introduce a mechanism which involves more fields than the usual BRST formalism, with the following motivation. In quantum field theory, the ghost fields have a clear geometrical interpretation, essentially because their quantum numbers are the same as those of the infinitesimal gauge transformations. The antighosts do not have such a natural interpretation. Rather, they play the role of Lagrange multipliers for the BRST variations of gauge functions and their quantum numbers are thus generally different from those of the ghosts, in a gauge dependent way. On the other hand, in the geometrical construction of the BRST and anti-BRST symmetry, one finds that geometrical antighosts can be introduced on an equal footing as ghosts. This distinction between the geometrical antighosts and the field theory antighosts can be seen as an unpleasant feature, or even as a contradiction. Our point of view is that it is the signal that the BRST formalism should be improved to permit a transmutation mechanism between these fields in a framework where a full ghost-antighost and BRST anti-BRST symmetry is pre-existing before the construction of gauge-fixed BRST invariant lagrangians. We thus propose in this paper to enlarge the set of fields of the BRST formalism: in addition to the geometrical ghosts and antighosts which are the building blocks of the gauge symmetry BRST and anti-BRST differential algebra, we introduce the field theory antighosts as elements of pairs with trivial BRST and anti-BRST cohomology. Then, the derivation of field theory actions which have possible asymmetries in their ghost and antighost dependences is done by suitable gauge-fixing terms which eliminate effectively some of the fields by supersymmetric compensations. This mechanism, where one starts from a configuration space with a full symmetry between the ghosts and the antighosts, can occur in ordinary gauge theories as well as in topological field theories.

One could even speculate on further developments based on the possibility of introducing ex-nihilo new fields which count locally as zero degrees of freedom in a way which is compatible with a classical gauge symmetry. Their BRST invariant coupling to ordinary gauge invariant non-topological models could be a hint to define new type of order parameters and to determine to which phase these systems belong, for instance a confined or deconfined phase in Yang-Mills theories, through spontaneous breaking of either the ghost number conservation symmetry or the topological BRST symmetry. It
could have applications also in the interpretation of the Gribov problem in the BRST formalism \[1, 2\].

The paper is organized as follows: we recall in a first section the BRST and anti-BRST formalism for general non-topological field gauge symmetries and explain in this case the transmutation mechanism. We give as an example the bosonic string theory, expressed in the conformal gauge (more precisely in gauges where the conformally invariant part of the metric is set equal to a background value). Then, we consider the case of topological field theories with an inner gauge symmetry for which we show that the BRST anti-BRST formalism and the transmutation mechanism can be also applied, in the antighost as well as in the antighost for antighost sectors. We emphasize the possibility of using the anti-BRST symmetry as a useful tool to select interesting gauge choices for these theories and give the Donaldson-Witten theory as an example.

2 BRST and anti-BRST formalism associated to a gauge theory, and antighost transmutation mechanism

Let us consider a system of fields \(\varphi^i(x)\), \(1 \leq i \leq N\) which undergo the following infinitesimal gauge transformations

\[
\delta_\epsilon \varphi^i(x) = R^i_\alpha(\varphi^i(x))\epsilon^\alpha(x)
\]  

(2.1)

The local parameters \(\epsilon^\alpha(x)\), \(1 \leq \alpha \leq r < N\), can be commuting and/or anticommuting, and the \(R^i_\alpha\) are functions of the fields \(\varphi\). \(x\) denotes the space-time variable. We assume the consistency of these gauge transformations, that is their closure relation and (graded) Jacobi identity

\[
[\delta_\epsilon, \delta_\epsilon'] \varphi^i = f^\alpha_{\beta\gamma} R^i_\alpha \epsilon^\beta \epsilon'^\gamma
\]  

(2.2)

\[
([\delta_\epsilon, \delta_\epsilon'], \delta_\epsilon') + cyclic and/or anticyclic permutations) \varphi^i = 0
\]  

(2.3)
The $f^\alpha_{\beta\gamma}$'s can be functions of the fields $\varphi$'s, since this situation can occur in physics.

As shown in [1], one can construct two graded differential operators $s$ and $\bar{s}$, called the BRST and anti-BRST operators associated to the gauge symmetry defined in [2,3]. $s$ and $\bar{s}$ act on an enlarged set of fields $\varphi^i$, $c^\alpha$, $\bar{c}^\alpha$ and $B^\alpha$ with the property

$$s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$$
$$sd + ds = \bar{s}d + \bar{s}d = 0 \tag{2.4}$$

$d = dx^\mu \partial_\mu$ is the exterior derivative. The ghosts and antighosts $c^\alpha$ and $\bar{c}^\alpha$ have the same quantum numbers as the local parameters $\epsilon^\alpha(x)$ but the opposite statistics, while the fields $B^\alpha$ have the same quantum numbers and statistics as the $\epsilon^\alpha(x)$. One assigns ghost numbers 0, 1, $-1$ and 0 to $\varphi^i$, $c^\alpha$, $\bar{c}^\alpha$ and $B^\alpha$ respectively. One defines the total grading of any given product of fields as the sum of their form degrees and ghost numbers. In the next section, devoted to topological field theories, we will refine this grading by a splitting along ghost and antighost directions.

It is convenient to set the basic fields on the following diagram

$$\begin{array}{ccc}
\varphi^i \\
c^\alpha & \bar{c}^\alpha \\
B^\alpha
\end{array} \tag{2.5}$$

$s$ and $\bar{s}$ are defined as follows

$$s\varphi^i = R^i_\alpha c^\alpha \quad \bar{s}\varphi^i = R^i_\alpha \bar{c}^\alpha$$
$$sc^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \quad \bar{s}\bar{c}^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} \bar{c}^\beta \bar{c}^\gamma \tag{2.6}$$

$$s\bar{c}^\alpha + \bar{s}c^\alpha + f^\alpha_{\beta\gamma} c^\beta \bar{c}^\gamma = 0 \tag{2.7}$$

One assumes that $s$ and $\bar{s}$ commute with the space derivative $\partial_\mu$, that is

$$sd + ds = \bar{s}d + \bar{s}d = 0 \tag{2.8}$$
Equations 2.2 and 2.3 imply relations between the $f^{\alpha}_{\beta\gamma}$ and the $R^i_\alpha$ which are exactly what is needed to prove the property that

$$s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$$  \hspace{1cm} (2.9)

on all fields $\varphi^i$, $c^\alpha$ and $\bar{c}^\alpha$, even in cases where the $f^{\alpha}_{\beta\gamma}$ are field dependent [1].

Notice that there is yet no $B$ dependence in the equations, and that $s\bar{c}^\alpha$ and $\bar{s}B^\alpha$ are undetermined. The introduction of $B$ through the following definition, raises this degeneracy, while maintaining automatically \[2.9\]

$$s\bar{c}^\alpha = B^\alpha \quad \bar{s}c^\alpha = -B^\alpha - f^{\alpha}_{\beta\gamma}B^\beta\bar{c}^\gamma$$

$$sB^\alpha = 0 \quad \bar{s}B^\alpha = -\bar{s}\left(f^{\alpha}_{\beta\gamma}B^\beta\bar{c}^\gamma\right)$$  \hspace{1cm} (2.10)

One expects that the cohomology of the operations $s$ (resp. $\bar{s}$) with zero or positive (resp. negative) ghost number only involves functions of the fields $\varphi$ and $c$ (resp. $\bar{c}$). This remark would be important if one were to investigate the classification of possible anomalies of the gauge symmetry.

After having introduced the graded differential operators $s$ and $\bar{s}$ through this construction, (which can be quite easily generalized for symmetries requiring ghosts of ghosts), one can solve the problem of gauge fixing a lagrangian $L_{cl}(\varphi)$ invariant under the symmetry \[2.1\]. The principle is to add to $L_{cl}(\varphi)$ a BRST invariant term which is BRST-exact and induces a propagation of the "longitudinal" modes, that is the modes which can be gauged away and have therefore no classical dynamics induced by $L_{cl}(\varphi)$. The quantum lagrangian is therefore

$$L_{cl}(\varphi) \rightarrow L_Q(\varphi, c, \bar{c}, B) = L_{cl}(\varphi) + s(K_{-1}) + \bar{s}(K_0)$$  \hspace{1cm} (2.11)

$K_{-1}$ and $K_0$ should be well chosen local functions of all fields: after expansion of $s(K_{-1}) + \bar{s}(K_0)$ one must get terms which are not gauge invariant and which determine a local propagation of the longitudinal degrees of freedom of the fields. There is less arbitrariness in their choices if one requires that the action has ghost number zero, in which case $K_{-1}$ and $K_0$ have ghost number -1 and 0 respectively, and also that it
is invariant under relevant global symmetries, such as the Lorentz invariance in particle models. Power counting argument can be used too, and some gauge choices can possibly present interesting accidental symmetries.

The BRST invariance of the action implies Ward identities which can be used to determine the whole quantum theory. Moreover they allow one to classify the possible anomalies as the solutions of consistency equations, and to determine their influence on the theory. The determination of the cohomology of the BRST Hamiltonian charge is a clear way of selecting the physical observables: they must commute with this charge and not be BRST exact. It also permits one to separate the Hilbert space of the theory into physical and unphysical sectors. In practice, these very general features must of course be verified according to the computation rules specific to the model that one considers. In particular, one must carefully determine the BRST charge and ghost number charge of the vacuum, in order to justify the above definition of the observables.

The way of thinking which leads one to the BRST formalism and can be called ”principle of BRST symmetry” might appear as too abstract. However, all along the road, the physicist keeps a rather natural guide-line: the introduction of ghosts as propagating fields is a necessity to compensate the propagation of unphysical, i.e.gauge dependent, components of the classical field in a covariant way. Then, the BRST symmetry can be thought of as a natural definition of the gauge symmetry for the enlarged set of fields including the ghosts. Once these ideas becomes intuitive ones, one may raise at the level of a principle the requirement of BRST symmetry.

In this geometrical set-up, the following observation is however troublesome. Due to the freedom of choosing different gauges, the quantum numbers of the antighosts and Lagrange multipliers that one uses in quantum field theory do not generally coincide with the quantum numbers of the gauge symmetry parameters. More precisely, if one denotes by $\lambda^A$ the lagrange multipliers of the gauge functions, then the indices $A$ and $a$ run over the same number of independent values, but they generally describe different representation spaces. Therefore the Lagrange multipliers $\lambda^A$ should not be identified with the fields $B^a$ in 2.10, although the latter enter so naturally in the geometrical con-
struction of the BRST symmetry. This situation occurs for instance in the quantization of the bosonic string in the conformal gauge, for which \( \alpha \) denotes a 2-dimensional vector field while \( A \) denotes a 2-dimensional quadratic differential. In contrast, the example of the Yang-Mills in Lorentz type gauges is simplest: \( \lambda \) has in this case the same quantum numbers as the infinitesimal parameters of a Yang-Mills transformation, and can thus be identified with the geometrical field \( B \).

To reconcile the beauty of the geometrical construction of the BRST and anti-BRST symmetry with the necessary freedom of the choice of the gauge functions, it is therefore tempting enough to try to find a way of transmuting the pair \( \bar{c}^\alpha, B^\alpha \) into another pair \( \bar{\kappa}^A, \lambda^A \). As we will see shortly, a natural way of doing this is the introduction of \( \lambda^A \) as part of a quartet of fields which count as a whole for zero degrees of freedom and undergo the BRST and anti-BRST symmetry in a way which is cohomologically trivial.

We thus add to the field spectrum 2.5 the following field quartet

\[
L^A, \quad \eta^A, \quad \bar{\kappa}^A, \quad \lambda^A
\]  

One assigns ghost numbers 0, 1, \(-1\) and 0 to \( L^A, \eta^A, \bar{\kappa}^A \) and \( \lambda^A \) respectively, and one defines

\[
sL^A = \eta^A, \quad \bar{s}L^A = \bar{\kappa}^A
\]

\[
s\eta^A = 0, \quad \bar{s}\kappa^A = 0
\]

\[
s\bar{\kappa}^A + \bar{s}\eta^A = 0
\]  

These equations can be written under the following form, which is useful in view of a comparison with the transformation laws of a topological field theory

\[
(s + \bar{s})L^A = \eta^A + \bar{\kappa}^A
\]

\[
(s + \bar{s})(\eta^A + \bar{\kappa}^A) = 0
\]  

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To determine $s\pi^A$ and $\overline{s}\eta^A$, while maintaining 2.9, we set

$$s\pi^A = \lambda^A \quad \overline{s}\eta^A = -\lambda^A$$

$$s\lambda^A = 0 \quad \overline{s}\lambda^A = 0 \quad (2.15)$$

If the index $A$ has a geometrical meaning, structure functions $f^A_{B\gamma}$ analogous to the $f^B_{\beta\gamma}$ should exist, with a Jacobi identity of the type 2.3. Then, it is geometrically meaningful to redefine $\eta^A \to \eta^A - f^A_{B\gamma}c^\gamma L^B$, $\pi^A \to \pi^A - f^A_{B\gamma}\overline{c}^\gamma L^B$ and $\lambda^A \to \lambda^A - f^A_{B\gamma}c^\gamma \pi^B$. This amounts to the system

$$(s + \overline{s})L^A + f^A_{B\gamma}(c^\gamma + \overline{c}^\gamma)L^B = \eta^A + \pi^A$$

$$\quad (s + \overline{s})(\eta^A + \pi^A) + f^A_{B\gamma}(c^\gamma + \overline{c}^\gamma)(\eta^A + \pi^A) = 0 \quad (2.16)$$

We will check the usefulness of these redefinitions on the bosonic string example.

To substitute the pairs $\sigma, B$ into the pairs $\pi, \lambda$ we will use the form of the BRST symmetry for the fields in 2.12 which permits, as we will see, the decoupling of certain fields from the lagrangian without changing the effective number of degrees of freedom of the theory. To explain the mechanism it is sufficient to consider the definition of the BRST symmetry in 2.13 rather than in 2.16. We assume the existence of possibly field dependent operators $O_{A\alpha}$ which relate the representation spaces denoted respectively by the indices $A$ and $\alpha$. Then, the possibility of a BRST invariant transmutation relies on the following identity

$$\int [d\varphi][dc][d\overline{\pi}][dB][dL][d\eta][d\overline{\lambda}][d\lambda] \exp S[\varphi, c, \pi, \lambda] + \int dx s(c^\alpha O_{A\alpha}L^A)$$

$$\sim \int [d\varphi][dc][d\overline{\pi}][d\lambda] \exp S[\varphi, c, \pi, \lambda] \quad (2.17)$$

Indeed, the result of the path integration of $\exp \int dx s(c^\alpha O_{A\alpha}L^A)$ over the fields $B, L, \pi, \eta$ is formally equal to one, as a ratio of equal determinants, since one has

$$s(c^\alpha O_{A\alpha}L^A) = (B^\alpha - c^\beta sO_{B\beta}O^{-1aB})O_{A\alpha}L^A - \overline{c}^\alpha O_{A\alpha}\eta^A \quad (2.18)$$

The effect of the BRST invariant gauge fixing term $s(c^\alpha O_{A\alpha}L^A)$ is thus an effective decoupling of the modes contained in the fields $\pi, B, L, \eta$. The remaining action $S[\varphi, c, \pi, \lambda]$. 

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is generally dissymmetric in the ghosts and antighosts since it involves only sub-sectors of the quartets of fields defined in [2.5] and [2.12].

Other possibilities exist: instead of \( s(\bar{\tau}^a O_{Aa} L^A) \), one could consider expressions of the type \( s\bar{\tau}(\varphi^i O_{\alpha} L^A) \) which give the anti-BRST invariance in an automatic way. Once again, let us stress that in the context of field theory, the validity of [2.17] should be verified case by case, according to the computation rules of the models that one wishes to explore.

The advantage of maintaining the anti-BRST symmetry is that this symmetry can be used as a tool to examine the possible background gauge invariances of the model. Let us suppose indeed that \( S[\varphi, c, \kappa, \lambda] \) is \( s \) and \( \bar{s} \) invariant. After the transmutation, the remaining fields undergo the following transformations

\[
\begin{align*}
{s\varphi^i} &= R^i_{\alpha} c^\alpha \\
{s\kappa^A} &= -\frac{1}{2} f^A_{\beta\gamma} c^\beta c^\gamma \\
{s\lambda^A} &= -f^A_{B\gamma} c^\gamma \kappa^B \\
{s\mu^z} &= \partial_z c^z + c^z \partial_z \mu^z - \mu^z \partial_z c^z \\
{s\kappa^z} &= c^z \partial_z c^z \\
{s\lambda^z} &= -f^A_{B\gamma} c^\gamma \lambda^B \\
{s\mu^z} &= 0 \\
{s\kappa^z} &= -B^z - c^z \partial_z \kappa^z - \kappa^z \partial_z c^z \\
{s\lambda^z} &= -B^z + B^z \partial_z \kappa^z + B^z \partial_z c^z
\end{align*}
\]

The action is independent on \( \bar{\tau}^a \) which, from the definition of \( \bar{s} \), can be understood as the ghostified parameter of a background symmetry.

To illustrate these general formula, let us consider the bosonic string theory. Using the Beltrami differential parametrization, the classical string lagrangian is

\[
L_{\text{string}} = \frac{1}{2\pi} \left( \partial \bar{z} - \mu \partial \bar{z} \right) \left( \partial z - \mu \partial z \right) X - \mu \bar{z} \partial \bar{z} \partial \bar{z} X
\]

(2.20)

The role of \( \varphi \) is played by the Beltrami differential \( \mu \). The BRST equations in the \( 2 - D \) gravity sector are

\[
\begin{align*}
{s\mu^z} &= \partial_z c^z + c^z \partial_z \mu^z - \mu^z \partial_z c^z \\
{s\kappa^z} &= c^z \partial_z c^z \\
{s\lambda^z} &= B^z \\
{s\mu^z} &= 0 \\
{s\kappa^z} &= -B^z - c^z \partial_z \kappa^z - \kappa^z \partial_z c^z \\
{s\lambda^z} &= -B^z + B^z \partial_z \kappa^z + B^z \partial_z c^z
\end{align*}
\]
One has of course the mirror equation in the antiholomorphic sector, by changing $z \rightarrow \bar{z}$ and $\bar{z} \rightarrow z$. To define the theory in the gauge where the Beltrami differential $\mu_z$ is equal to a background value $\mu_{0\bar{z}}$, we use our general formalism and add to the string action the following BRST invariant term

$$L_{gf} = s(\bar{z}(\mu_{\bar{z}z} - \mu_{0\bar{z}})) + c.c \quad (2.23)$$

This yields the usual result of the conventional BRST quantization. Indeed, an easy computation gives firstly

$$L_{gf} = s(\bar{z}(\mu_{\bar{z}z} - \mu_{0\bar{z}}) - L_{zz}(\partial_\bar{z}\bar{c}^z + \bar{c}^z \partial_\bar{z}\mu_{\bar{z}z} - \mu_{\bar{z}z} \partial_\bar{z}\bar{c}^z)) \quad (2.24)$$

and then

$$L_{gf} = \lambda_{zz}'(\mu_{\bar{z}z} - \mu_{0\bar{z}}) - b_{zz}(\partial_\bar{z}\bar{c}^z + c^z \partial_\bar{z}\mu_{\bar{z}z} - \mu_{\bar{z}z} \partial_\bar{z}\bar{c}^z)$$

$$+ \eta_{zz}'(\partial_{\bar{z}} - \mu_{\bar{z}z} \partial_\bar{z} + \partial_\bar{z}\mu_{\bar{z}z})\bar{c}^z$$

$$- L_{zz}'(\partial_\bar{z} - \mu_{\bar{z}z} \partial_\bar{z} + \partial_\bar{z}\mu_{\bar{z}z})B^z + c.c \quad (2.25)$$

We have done simple field redefinitions $\lambda_{zz}' = \lambda_{zz} + \ldots$, $\eta_{zz}' = \eta_{zz} + \ldots$, $L_{zz}' = L_{zz} + \ldots$, $b_{zz} = \pi_{zz} + \ldots$. The two last of $L_{gf}$ compensate each other in the path integration by $N=2$ supersymmetry. Curiously, they can be understood as a particular case of a bidimensional topological gravity.

The two remaining terms are the known gauge fixing terms of the bosonic string theory, so we have in illustration of the identity $[2.17]$ with

$$S \sim \int dzd\bar{z}(\frac{1}{1 - \mu_{\bar{z}z}\mu_{\bar{z}z}}(\partial_\bar{z} - \mu_{\bar{z}z}\partial_\bar{z})X(\partial_\bar{z} - \mu_{\bar{z}z}\partial_\bar{z})X$$

$$- b_{zz}(\partial_\bar{z}\bar{c}^z + c^z \partial_\bar{z}\mu_{\bar{z}z} - \mu_{\bar{z}z} \partial_\bar{z}\bar{c}^z) - c.c) \quad (2.26)$$
Moreover, the BRST anti-BRST symmetry operators $s$ and $\bar{s}$ which leave invariant this action and satisfy $s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$ are

\[
\begin{align*}
 s\mu^z &= \partial_z c^z + c^z \partial_z \mu^z - \mu^z \partial_z c^z \\
 s\bar{c}^z &= c^z \partial_z \bar{c}^z \\
 s\bar{c}^z &= -c^z \partial_z \bar{c}^z - \bar{c}^z \partial_z c^z \\
 s\bar{b} = 0 &= -\bar{c}^z \partial_z b^z - 2b^z \partial_z \bar{c}^z 
\end{align*}
\] 

(2.27)

$\bar{s}$ can be interpreted as the ghostified form of the background diffeomorphism symmetry, with the spectator $2-D$ vector field antighost, and can be used as in [6] to deduce the "Virasoro Ward identity" of the conformal theory.

For cases where the gauge function has the same quantum numbers as the parameters of the theory, (for instance the Yang-Mills theory with a Lorentz gauge function), there is no need to introduce the quartet of fields 2.12. Otherwise, these fields can be fully decoupled by mean of the following action

\[
s(\bar{\kappa}^A L^A) = \lambda^A L^A - \bar{\kappa}^A \eta^A
\] 

(2.28)

3 Topological BRST and anti-BRST equations associated to a gauge theory

We will now consider the case of topological field theories. We consider the same classical fields as in the previous section, but assume that they are submitted to arbitrary local fields transformations. Since the geometrical idea is to describe such generalized gauge transformations modulo the usual gauge transformations 2.1, we introduce the following infinitesimal transformations

\[
\delta_\epsilon \varphi^i = \epsilon^i(x) + R^i_\alpha \epsilon^\alpha (x)
\] 

(3.1)

The local parameters $\epsilon^\alpha (x)$ have the same meaning as in 2.1 while the $\epsilon^i(x)$ have the same quantum numbers as the fields $\varphi^i$'s.
One has obviously a "gauge invariance" in the space of transformation parameters

\[ \epsilon^\alpha(x) \rightarrow \epsilon^\alpha(x) + \omega^\alpha(x) \]

\[ \epsilon^i(x) \rightarrow \epsilon^i(x) - R^i_\alpha \omega^\alpha(x) \quad (3.2) \]

The symmetry (3.1) is an invariance of topological terms, if any, which can be expressed as functionals of the \( \varphi \)'s. It represents the largest gauge symmetry acting on the \( \varphi \)'s.

To build its associated graded differential BRST and anti-BRST operations, we must introduce more fields than in (2.3). We consider the following set of fields

\[ \phi^{(0,0)} \]

\[ \psi^{(1,0)} \quad \beta^{(1,1)} \quad \overline{\psi}^{(0,1)} \]

\[ \Phi^{(2,0)} \quad \eta^{(1,0)} \quad L^{(1,1)} \quad \overline{\eta}^{(0,1)} \quad \overline{\Phi}^{(0,2)} \]

\[ \epsilon^{(1,0)} \quad B^{(1,1)} \quad \overline{c}^{(0,1)} \quad \Phi^{(0,2)} \]

\[ \eta^{(1,0)} \quad L^{(1,1)} \quad \overline{\eta}^{(0,1)} \quad \Phi^{(0,2)} \]

\[ c^{(1,0)} \quad B^{(1,1)} \quad \overline{c}^{(0,1)} \]

\[ \epsilon^{(1,0)} \quad B^{(1,1)} \quad \overline{c}^{(0,1)} \]

\[ \phi^{(0,0)} \]

Notice that we have improved the ghost number attributions of fields: the integer indices \( g \) and \( \overline{g} \) count respectively the ghost and antighost numbers of a field \( X^g,\overline{g} \) and determine a bi-grading. The knowledge of the sum of the form degree and of \( g - \overline{g} \) is however sufficient to determine the commutation properties: if this quantity is even (resp. odd), \( X^g,\overline{g} \) has the physical (resp. unphysical) spin-statistics relation.

The topological BRST and anti-BRST operators \( s \) and \( \overline{s} \) associated to (3.1) are defined as follows (we use the notation that \( X,\overline{X} = \frac{\delta X}{\delta \phi^i} \)):

\[ s\phi^i = \psi^i + R^i_\alpha c^\alpha \]

\[ s\psi^i = \Phi^\alpha \]

\[ s\epsilon^\alpha = \Phi^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \]

\[ s\Phi^\alpha = f^\alpha_{\beta\gamma} c^\beta \]

\[ s\psi^i = \Phi^\alpha \overline{c}^\alpha - R^i_\alpha \Phi^\alpha \psi^j \]

\[ s\psi^i = \Phi^\alpha \overline{c}^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} \psi^j c^\beta c^\gamma \]

\[ s\Phi^\alpha = f^\alpha_{\beta\gamma} c^\beta + \frac{1}{2} f^\alpha_{\beta\gamma} \psi^j c^\beta c^\gamma \]

\[ (3.4) \]

and
\[s\bar{c}^\alpha + \bar{s}c^\alpha + f^\alpha_{\beta\gamma} c^\beta \gamma + L^\alpha = 0\]

\[s\bar{\Psi}^j + \bar{s}\Psi^j + R^i_\alpha L^\alpha + R^j_\alpha \bar{\Sigma}^j \alpha c^\alpha + R^j_\alpha \Psi^j \bar{c}^\alpha = 0\]

\[s\Phi^\alpha + \bar{s}L^\alpha + f^\alpha_{\beta\gamma} c^\beta \bar{\Phi}^\gamma + f^\alpha_{\beta\gamma} c^\beta L^\gamma + \frac{1}{2} f^\alpha_{\beta\gamma} \Psi^i \bar{\Psi}^i \bar{c}^\beta c^\gamma = 0\]

\[s\Phi^\alpha + sL^\alpha + f^\alpha_{\beta\gamma} c^\beta F^\gamma + f^\alpha_{\beta\gamma} c^\beta L^\gamma + \frac{1}{2} f^\alpha_{\beta\gamma} \Psi^i \bar{\Psi}^i \bar{c}^\beta c^\gamma = 0\] (3.5)

As for the pure gauge transformation BRST algebra, the closure and Jacobi relations 2.2 and 2.3 of the gauge symmetry enforce the fundamental nilpotency property \(s^2 = \bar{s}s + ss = s^2 = 0\) on all fields. The topological BRST equations yield a consistent separation between arbitrary fields deformations and the pure gauge transformations 2.1: the latter can be seen as a meaningful truncation of the latter. Observe that the idea of having this kind of field spectrum for a topological symmetry with a BRST and anti-BRST symmetry as in 3.4 was first introduced in the particular context of the Yang-Mills theory [7].

The geometrical equations 3.5 mix ghost and antighost parts. Indeed, these equations could be obtained directly from the sole definition of the operation \(s\) on \(\varphi, c, \Psi, \Phi\) by changing \(s \rightarrow s + \bar{s}, c \rightarrow c + \bar{c}, \Psi \rightarrow \Psi + \bar{\Psi}, \Phi \rightarrow \Phi + L + \bar{\Phi}\). To solve the degeneracy in 3.3, one uses the ”auxiliary” fields \(B, \Psi, \eta\) and \(\bar{\eta}\) in 3.3.

Firstly, one defines

\[s\bar{c}^\alpha = B^\alpha \quad sB^\alpha = 0\]

\[s\bar{\Psi}^i = B^i \quad sB^i = 0\]

\[s\Phi^\alpha = \bar{\eta}^\alpha \quad s\bar{\eta}^\alpha = 0\]

\[sL^\alpha = \eta^\alpha \quad s\eta^\alpha = 0\] (3.6)

Then, the rest of the equations is defined as follows: one obtains the action of \(\bar{s}\) on \(c, \Psi, L\) and \(\Phi\) by combining the last equation and 3.3 by imposing that \(s^2 = 0\) on these fields, one finds afterwards the action of \(\bar{s}\) on the auxiliary fields \(B, \Psi, \eta\) and \(\bar{\eta}\); finally,
one obtains the property \((d + s + \overline{s})^2 = 0\), which means that \(d\) anticommutes with \(s\) and \(\overline{s}\), and that \(s^2 = s\overline{s} + \overline{s}s = \overline{s}^2 = 0\) on all fields.

It is impossible to construct a local action invariant under the transformations 3.1 which would be only function of the classical fields \(\varphi\) and would generate equations of motions. Indeed, the gauge symmetry is so large that the only possibility would be to consider lagrangians which are locally pure derivatives, which means that the action is a topological invariant. As a matter of fact, to construct a topological field theory associated to this huge symmetry, one must define the path integral by using fields which count as a whole as zero degrees of freedom, which is the number of physical local degrees of freedom left locally by the symmetry 3.1. Therefore, it is natural to introduce the set of fields 3.2 as fundamental fields, and to define the theory by the gauge functions associated to a lagrangian of the following form

\[
L_Q = d\omega + s(\overline{K}_{-1}) + s\overline{s}(K_0)
\]  

(3.7)

In other words, our principle is to postulate the BRST invariance associated to the symmetry 3.1.

At this point, it is clear that we must face the fact that, as in ordinary gauge theories, interesting gauge functions can take their values in various representation spaces, so that some of the geometrical fields displayed in 3.3 must be transmuted into field theory objects with different quantum numbers. We will therefore generalize the idea explained in the previous section and couple the theory to cohomologically trivial pairs of additional pairs.

To display the formalism, we will firstly consider the cases for which the gauge functions for the "longitudinal" part of the fields \(\varphi\) have the same quantum numbers as the gauge symmetry parameters. In this situation, there is no need of transmutation of the geometrical pair \(\overline{c}^\alpha, B^\alpha\). On the other hand, to gauge-fix the remaining \(N - r\) "transverse" part of the fields \(\varphi\), one must get rid of the pairs \(\overline{\Psi}, B^i, L^\alpha\) and \(\eta^\alpha\), and replace them by a pair \(\overline{\kappa}^M, \lambda^M\), where the index \(M\) runs between 1 and \(N - r\), to get eventually a gauge fixing term of the form \(\lambda^M G_M(\varphi) + \ldots\).
For this purpose, we introduce cohomologically trivial pairs with the relevant quantum numbers (denoted by the indices \( M, N, \ldots \))

\[
\begin{align*}
L^{M(0,0)} \\
\eta^{M(0,0)} \\
\kappa^{M(0,0)} \\
\lambda^{M(0,0)} (3.8)
\end{align*}
\]

Notice that the effective number of degrees of freedom carried by the fields \( \bar{\Psi}, B^i, L^\alpha, \eta^a \) and \( L^M, \eta^M \) counts overall for zero.

To obtain a BRST invariant elimination of the fields \( \bar{\Psi}, B^i \), we include in the action a term of the following form

\[
s \left( \bar{\Psi}^i (O_{iM} L^M + O_{i\alpha} L^\alpha) \right) = -\bar{\Psi}^i (O_{iM} \eta^M + O_{i\alpha} \eta^\alpha) \\
B^i - \bar{\Psi}^j (sO_{jN}) O^{-1N_i} O_{iM} L^A + (B^i - \bar{\Psi}^j (sO_{j\beta}) O^{-1\beta_i} O_{i\alpha} L^\alpha (3.9)
\]

Indeed, if one chooses appropriately the transfer operators \( O_{iA} \) and \( O_{i\alpha} \), one expects that the fields \( \bar{\Psi}, B^i \) and \( L^A, L^\alpha, \eta^A, \eta^\alpha \) will decouple from the theory through supersymmetric compensations analogous to those which justify the identity 2.17.

Let us stress that the presence of the "medium" geometrical ghost of ghost \( L^\alpha \) is necessary to ensure automatically the relation \((s + \overline{s})^2 = 0\). Its elimination from the spectrum is thus generally not compatible with the existence of an operation \( \overline{s} \) which would anticommute with \( s \). However, when the structure functions \( f_{\beta\gamma} \) are field independent, which covers the case of all the gauge symmetries with a Lie algebra structure, a consistent \( \overline{s} \) operation still exists. Indeed, let us write the form of the BRST anti-BRST algebra which remains after the elimination of the fields \( \bar{\Psi}, L \) and \( \eta \):

\[
\begin{align*}
s \phi^i &= \Psi^i + R^i_\alpha c^\alpha \\
s \phi^\alpha &= \Phi^\alpha - \frac{1}{2} f_{\beta\gamma} c^\beta c^\gamma \\
s \phi^i &= \Psi^i - R^i_\alpha c^\alpha \\
s \phi^\alpha &= \Phi^\alpha + \frac{1}{2} f_{\beta\gamma} \Psi^i c^\beta c^\gamma \\
s \phi^\alpha &= -f_{\beta\gamma} \phi^\beta \Phi^\gamma
\end{align*}
\]
\[ s\Phi^\alpha = \eta^\alpha - f^\alpha_{\beta\gamma} c^\beta \Phi^\gamma \quad \overline{s}\Phi^\alpha = f^\alpha_{\beta\gamma} \Phi^\gamma c^\beta \]

\[ s\eta^\alpha = f^\alpha_{\beta\gamma} \Phi^\beta \Phi^\gamma - f^\alpha_{\beta\gamma} c^\beta \eta^\gamma \quad \overline{s}\eta^\alpha = -f^\alpha_{\beta\gamma} c^\beta \eta^\gamma \]  
(3.10)

\[ sc^\alpha + \overline{s}c^\alpha + f^\alpha_{\beta\gamma} c^\beta c = 0 \]

(3.11)

\[ s\lambda^M = \lambda^M - f^M_{N\alpha} c^\alpha \kappa^N \quad \overline{s}\kappa^M = -f^M_{N\alpha} c^\alpha \kappa^N \]

\[ s\lambda^M = -f^M_{N\alpha} \Phi^\alpha \kappa^N - f^M_{N\alpha} c^\alpha \kappa^N - f^M_{N\alpha} \psi^j \Phi^\alpha \lambda^N \quad \overline{s}\lambda^M = -f^M_{N\alpha} c^\alpha \lambda^N \]  
(3.12)

One can verify that the property \( s^2 = \overline{s}^2 = 0 \) is still satisfied. However, the property that \( s \) and \( \overline{s} \) anticommute, that is \( s\overline{s} + \overline{s}s = 0 \), is generally broken by terms proportional to the field derivatives of structure functions \( f^\alpha_{\beta\gamma} \) and \( f^M_{N\alpha} \).

To go further and enforce the anticommutation relation between \( s \) and \( \overline{s} \), we restrict therefore ourself to cases where these structure functions are independent of the fields. Then, one can adopt a convenient bracket anti-bracket notation, and write the BRST and anti-BRST equations as follows

\[ s\varphi = \Psi + Rc \quad \overline{s}\varphi = R\overline{c} \]

\[ sc = \Phi - \frac{1}{2} [c, c] \quad \overline{s}c = -\frac{1}{2} [\overline{c}, \overline{c}] \]

\[ s\Psi = -R\Phi - [c, \Psi] \quad \overline{s}\Psi = [\overline{c}, \Psi] \]

\[ s\Phi = -[c, \Phi] \quad \overline{s}\Phi = -[\overline{c}, \Phi] \]

\[ s\overline{\Phi} = \overline{\eta} - [c, \overline{\Phi}] \quad \overline{s}\overline{\Phi} = [\overline{c}, \overline{\Phi}] \]

\[ s\overline{\eta} = [\Phi, \overline{\Phi}] - [c, \overline{\eta}] \quad \overline{s}\overline{\eta} = -[\overline{c}, \overline{\eta}] \]  
(3.13)

\[ sc + \overline{s}c + [c, \overline{c}] = 0 \]

(3.14)

\[ s\overline{\kappa} = \lambda - [c, \overline{\kappa}] \quad \overline{s}\overline{\kappa} = -[\overline{c}, \overline{\kappa}] \]

\[ s\lambda = [\Phi, \overline{\kappa}] - [c, \lambda] \quad \overline{s}\lambda = -[\overline{c}, \lambda] \]  
(3.15)
Notice that, in the ghost sector, one can summarize all equations as follows

\[ (s + \bar{s})(c + \bar{c}) + \frac{1}{2}[c + \bar{c}, c + \bar{c}] = \Phi \]

\[ (s + \bar{s})\kappa + [c + \bar{c}, \kappa] = \lambda \]

\[ (s + \bar{s})\overline{\Phi} + [c + \bar{c}, \overline{\Phi}] = \overline{\eta} \]

with the "Bianchi identities"

\[ (s + \bar{s})\Phi + [c + \bar{c}, \Phi] = 0 \]

\[ (s + \bar{s})\overline{\eta} + [c + \bar{c}, \overline{\eta}] = [\Phi, \overline{\eta}] \]

\[ (s + \bar{s})\lambda + [c + \bar{c}, \lambda] = [\Phi, \lambda] \]

(3.16)

In this way of writing the BRST equation, the geometrical interpretation of the ghost of ghost $\Phi$ as the component of a curvature is almost obvious, since we have generally

\[ [S, S] = \Phi \]

(3.18)

where $S = s + [c, \ ]$.

One may wonder what is the role left to the anti-BRST symmetry after the elimination of the geometrical fields which break the symmetry between $s$ and $\bar{s}$. As a matter of fact, the last equations show that \( \bar{s} \) can be interpreted as a spectator field which "ghostifies" a background symmetry, in a way which generalizes the case of non-topological field theories. This permits the attribution of well-defined quantum numbers to all fields, including the topological ghosts and antighosts.

Let us now show that the anti-BRST symmetry permits one to distinguish between the gauge fixing of the transverse and longitudinal modes. On the one hand, one observes that since the $\bar{s}$ transformations of the pairs $(\pi^M, \lambda^M)$ and $(\Phi^\alpha, \eta^\alpha)$ are analogous to ordinary gauge transformations, having a $\bar{s}$-invariant gauge fixing lagrangian of the following form

\[ s(\bar{s} G_M(\varphi) + \overline{\Phi}^i F_{\alpha i}(\varphi) \Psi^i) \]

(3.19)

implies that the gauge functions $G_M(\varphi)$ and $F_{\alpha i}(\varphi)\Psi^i$ for the "transverse" part of the gauge field and for the degenerate topological ghost $\Psi^i$ are gauge covariant. One then
notices that, due to the elimination of the field $L$, the field functionals $\pi^M G_M(\varphi) + \Phi^\alpha \Phi_{Mi}(\varphi)\Psi^i$ is $\pi$-invariant without being $\bar{\pi}$-exact. In contrast, the gauge fixing part of the ”longitudinal” modes can always be done by terms of the type $s\bar{\pi}(\varphi^iO_{ij}f^j + \ldots)$.

This suggests therefore that the search of the gauge fixing functions in the transverse sector amounts to find the cohomology with ghost number -1 of the $\pi$ operation, while the gauge fixing terms for the longitudinal sector belong to the trivial part.

We can summarize these remarks by writing the gauge fixing terms under the following form

$$L_{GF} \sim s(\pi^M G_M(\varphi) + \Phi^\alpha F_{\alpha i}(\varphi)\Psi^i) + \frac{1}{2} s\bar{\pi}(\varphi^iO_{ij}f^j + \ldots) \quad (3.20)$$

It is nevertheless interesting enough that actions of the type 3.19 can be deduced from the action

$$s\bar{\pi}(L^M G_M(\varphi) + \bar{c}^\alpha F_{\alpha i}(\varphi)\Psi^i) \quad (3.21)$$

provided that one uses as constraints the algebraic equations of motion stemming from the lagrangian 3.9. The elimination of geometrical fields as $L$ and $\Psi$ has thus induced a non trivial cohomology for the $\pi$ operation, which we believe useful for the classification of ”interesting” gauge choices.

To make more explicit these observations, it is time to give an example. Let us chose the $4-D$ topological Yang-Mills theory, expressed in a Lorentz type gauge with self-duality gauge conditions [2] [8]. In this case, $\varphi$ stands for the Yang-mills field $A_\mu$, valued in a Lie algebra $G$ and $R$ is the covariant derivative $D_\mu$, $[ , ]$. The index $M$ means that $\pi$ and $\lambda$ are Lie algebra valued self-dual 2-forms. In what follows, products of two fields mean their trace in $G$, and $[X,Y]$ is the graded commutator of $X$ and $Y$.

The BRST and anti-BRST equations are

$$sA_\mu = \Psi_\mu + D_\mu c \quad \bar{s}A_\mu = D_\mu \bar{c}$$
$$sc = \Phi - \frac{1}{2}[c,c] \quad \bar{s}c = -\frac{1}{2}[\bar{c},\bar{c}]$$
$$s\Psi_\mu = -D_\mu \Phi - [c, \Psi_\mu] \quad \bar{s}\Psi_\mu = -[\bar{c}, \Psi_\mu]$$

19
\[
\begin{align*}
&s\Phi = -[c, \Phi] \quad s\Phi = -[\bar{c}, \Phi] \\
&s\bar{\Phi} = \eta - [c, \bar{\Phi}] \quad s\bar{\Phi} = [\bar{c}, \bar{\Phi}] \\
&s\bar{\eta} = [\Phi, \bar{\Phi}] - [c, \bar{\eta}] \quad s\bar{\eta} = -[\bar{c}, \bar{\eta}]
\end{align*}
\]
\[ (3.22) \]

\[
\begin{align*}
&s\mathcal{C} + s\mathcal{C} + [c, \bar{c}] = 0 \\
&s\mathcal{C} + s\mathcal{C} = 0
\end{align*}
\]
\[ (3.23) \]

\[
\begin{align*}
&s\kappa_{\mu\nu} = \lambda_{\mu\nu} - [c, \kappa_{\mu\nu}] \quad s\kappa_{\mu\nu} = -[\bar{c}, \kappa_{\mu\nu}] \\
&s\lambda_{\mu\nu} = [\Phi, \kappa_{\mu\nu}] - [c, \lambda_{\mu\nu}] \quad s\lambda_{\mu\nu} = -[\bar{c}, \lambda_{\mu\nu}]
\end{align*}
\]
\[ (3.24) \]

\[
\begin{align*}
&s\mathcal{C} = B \quad s\mathcal{C} = -B - [\bar{c}, c] \\
&s\mathcal{B} = 0 \quad s\mathcal{B} = -[\bar{c}, B]
\end{align*}
\]
\[ (3.25) \]

A lagrangian which satisfies the above mentioned criteria of \( s \) and \( \bar{s} \) invariances, and is renormalizable by power counting is

\[
L = s(\bar{\Psi}_i L_0^i + \bar{\Psi}_0 L)
\]

\[
s\bar{s}\left(L^{\mu\nu}(F_{\mu\nu} + F_{\mu\nu}^\ast) + \frac{1}{2} \bar{\Psi}_\mu \Psi^\mu + \frac{1}{2} A^\mu A_\mu\right)
\]
\[ (3.26) \]

After elimination of the fields \( \bar{\Psi}_\mu, B_\mu, L_{\mu\nu}, L, \eta_{\mu\nu} \) and \( \eta \), the lagrangian has the form

\[
L \sim s\left(\bar{\kappa}_{\mu\nu}^\ast(F_{\mu\nu} + F_{\mu\nu}^\ast) + \frac{1}{2} \lambda_{\mu\nu} + \bar{\Phi} D_\mu \Psi^\mu\right) + s\bar{s}\left(\frac{1}{2} A^\mu A_\mu\right)
\]
\[ (3.27) \]

On this example, we check that the gauge fixing term for the transverse part of the gauge field belongs to the non trivial part of the cohomology with ghost number -1 of \( \bar{s} \), since \( \bar{\kappa}_{\mu\nu}^\ast(F_{\mu\nu} + F_{\mu\nu}^\ast + \frac{1}{2} \lambda_{\mu\nu} + \bar{\Phi} D_\mu \Psi^\mu) \) is \( \bar{s} \)-invariant but not \( \bar{s} \)-exact, while the gauge fixing term for the longitudinal part belongs to the trivial part, since it is \( \bar{s} \)-exact

\[
s\bar{s}\left(\frac{1}{2} A^\mu A_\mu\right) = s(A^\mu \partial_\mu = -\bar{s}(A^\mu \partial_\mu c) = A^\mu \partial_\mu B + D^\mu v \partial_\mu \bar{c}
\]
\[ (3.28) \]

A next level of complication of the transmutation mechanism is for theories of the type of the conformal 2D—topological gravity: in such cases one should transmute also the
antighost $\bar{\sigma}$. This could be done by a mere combination of the results of this section and of the previous section, and it is not worth giving the details. Rather, let us concentrate on a more exotic situation, although we are not aware of models for which it would occur, when the gauge fixing functions $F_{\Psi}(\varphi)\Psi^i$ of the degenerate topological ghosts $\Psi$ would have different quantum numbers than those of the secondary antighosts $\Phi^\alpha$. We will show that our idea of a transmutation could be applied in this case.

According to our point of view, we would need to introduce new fields to transmute the pairs $\Phi^\alpha, \bar{\eta}^\alpha$ into new pairs $X, \bar{N}$ with the relevant quantum numbers. After a little bit of experimentation, one finds that one must introduce the following set of fields, such that the addition of their degrees of freedom is once more effectively zero

\[
\begin{aligned}
\Gamma^{P(1,0)} & \quad \Lambda^{P(1,1)} & \quad \Gamma^{P(0,1)} \\
X^{P(2,0)} & \quad N^{P(1,0)} & \quad Y^{P(1,1)} & \quad \bar{N}^{P(0,1)} & \quad \bar{X}^{P(0,2)}
\end{aligned}
\] (3.29)

We are now familiar with the most direct way to get the BRST and anti-BRST equations for such fields. They are

\[
(s + \bar{s})(\Gamma + \bar{\Gamma}) = X + Y + \bar{X}
\]

\[
(s + \bar{s})(X + Y + \bar{X}) = 0
\] (3.30)

After expansion, one gets

\[
\begin{aligned}
s\Gamma &= X & \quad \bar{s}\Gamma &= -\Lambda + Y \\
s\bar{\Gamma} &= \Lambda & \quad \bar{s}\bar{\Gamma} &= \bar{X} \\
sX &= 0 & \quad \bar{s}X &= N \\
s\bar{X} &= \bar{N} & \quad \bar{s}X &= 0 \\
sY &= -N & \quad \bar{s}Y &= -\bar{N} \\
s\Lambda &= 0 & \quad \bar{s}\Lambda &= -\bar{N} \\
sN &= 0 & \quad \bar{s}N &= 0 \\
s\bar{N} &= 0 & \quad \bar{s}\bar{N} &= 0
\end{aligned}
\] (3.31)
Whenever the index $P$ has a geometrical meaning, one has the option of redefining $X + Y + \overline{X} + [c + \overline{c}, X + Y + \overline{X}]$.

The transmutation mechanism is now quite simple: one considers the following $s$-exact lagrangians

$$s(\Phi^\alpha O_{\alpha P} \Gamma^P) = \Phi^\alpha O_{\alpha P} X^P + (\eta^\alpha + \Phi^\beta sO_{\beta Q} O^{-1Q^\alpha})O_{\alpha P} \Gamma^P$$

$$s(Y^Q O_{QP} \Gamma^P) = Y^Q O_{QP} \Lambda^P + (N^Q + Y^R sO_{RS} O^{-1S^Q})O_{QP} \Gamma^P \quad (3.32)$$

These terms permit supersymmetric compensations between the fields $\Phi^\alpha, \eta^\alpha, X^P, \Gamma^P, Y^P, \Gamma^P, N^P$ and $\Lambda^P$. If the transition functions $O_{\alpha P}$ and $O_{QP}$ are well chosen, one can assume that these fields eventually decouple. Then, with the remaining fields $X^P$ and $\overline{N}^P$, one can perform the gauge fixing of the geometrical topological ghosts

$$s(\overline{X}^P F_{P_i}(\varphi) \Psi^i) = \overline{X}^P F_{P_i}(\varphi)(\Psi^i + R_i^\alpha c^\alpha)$$

$$+ (\overline{N}^P + \overline{X}^Q sF_{Qj} O^{-1jP})O_{P_i} \Psi^i \quad (3.33)$$

where the gauge functions $F_{P_i}(\varphi) \Psi^i$ are now chosen at will. This transmutation mechanism in the ghost of ghost sector is of course very analogous to the one in the primary ghost sector.

4 Conclusion

We have established general formula which associate to any given system of gauge transformation a system of topological BRST and anti-BRST equations. By coupling these equations to additional pairs of fields with trivial cohomology one can in principle handle all possible types of gauge fixing procedure in a formalism which respects a full symmetry between the ghosts and the antighosts. This idea of adding to a system new fields which count as a whole an effective number of degrees of freedom equal to zero, with some possibilities of transferring the degrees of freedom from one sector to the other could have further applications: our general formula could be useful to couple purely topological theories to ordinary gauge systems, with the purpose of of triggering phase transitions by
spontaneous breaking of either the ghost number conservation or the topological BRST symmetry.

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