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Abstract

We recall that the Minkowskian geometry possesses basic units of space and time which are invariant under the Poincaré symmetry. We then show that, by comparison, the Riemannian geometry possesses space-time units which are not invariant under the symmetries of the Riemannian line element, thus causing evident physical ambiguities. We therefore introduce a novel formulation of general relativity in the isominkowskian geometry which is an axiom-preserving lifting of the conventional Minkowskian geometry but which nevertheless admits all possible Riemannian metrics thanks to a (positive-definite) $4 \times 4$ generalization of the basic unit. We construct the universal symmetry of the isominkowskian line elements called isopoincaré symmetry, prove that it is locally isomorphic to the conventional Poincaré symmetry, and show that, in this way, conventional Riemannian metrics and related field equations can be expressed with respect to invariant generalized units of space and time. We then show that the isominkowskian geometry and related isopoincaré symmetry permit: I) A classical geometric unification of the general and special relativities for matter into a formulation called isospecial relativity in which the former occurs for generalized units while admitting the latter as a particular case for conventional units; II) A novel operator formulation of gravity for matter based on the abstract axioms of relativistic quantum mechanics, thus showing hope for a possible resolution of the ambiguities in current theories of quantum gravity; and III) A novel classical and operator formulation of antimatter which is an antiautomorphic image of the preceding formulations for matter constructed via a map called isoduality. The experimental validity of the classical isominkowskian formulation of gravity for matter is derived from the preservation of the conventional Einstein’s field equations except for inessential multiplicative terms. The experimental verification of the operator isominkowskian formulation of gravity
for matter is derived from the preservation of conventional quantum mechanical laws, with gravitational effects being notoriously very small as compared to those of other interactions of the particle word. The validity of the isodual formulations for antimatter is inferred from its compatibility with available experimental data. The results of this paper have been made possible thanks to the recent achievement of sufficient maturity for mathematical content in memoir [3f], axiomatic consistency in memoir [3g] and generalized symmetry principles in memoir [7a]. Further studies, such as the formulation of an isotopic grand unified theory inclusive of gravitation, are presented in the forthcoming paper [10].

1 Introduction

As it is well known, the special relativity [1] constitutes one of the most majestic scientific achievements of this century for mathematical beauty, axiomatic consistency and unambiguous experimental verifications. By comparison, despite equally historical advances through this century, the general relativity [2] still remains afflicted by basic unresolved problematic aspects. In this note we therefore initiate studies aimed at a geometric unification of the general and special relativity via the abstract axioms of the special rather than of the general relativity.

Our central methodological tools for the characterization of matter are the so-called isotopies [3] which, for the case at hand, are characterized by the lifting of the unit of relativistic theories $I = \text{Diag.}(1,1,1,1)$ into a well behaved and positive–definite but otherwise arbitrary $4 \times 4$ matrix $\hat{I}(x, \dot{x}, \ddot{x}, ...)$ = $1/\hat{T}$ with associated lifting of the conventional associative product $A \times B$ among generic quantities $A, B$ into the isoproduct $\hat{A} \times \hat{B} = A \times \hat{T} \times B$ under which $\hat{I}$ is the correct left and right unit of the new theory.

For consistency the entire mathematical and physical structures of the original theories, must be reconstructed with respect to the above generalized unit and product, yielding the so-called isonumbers, isospaces, isoalgebras, isogeometries, etc. [3f]. It is easy to see that, for positive–definite generalized unit $\hat{I}$, all isotopic structures are locally isomorphic to the original ones and, in this sense, all isotopies are axiom-preserving. We should therefore indicate from the outset that the isotopies do not produce new theories, but merely new realizations of existing theories, which is the main line of study of this paper.

The isotopic structures which are particularly significant for this paper are: the isominkowskian spaces [4a]; the isolorentz [4a] and isopoincaré symmetries [4e,4f]; and the isospecial relativity [4], which is the axiom-preserving formulation of the conventional relativity on isominkowskian spaces under the isopoincaré symmetry (see [4h] for a general presentation).

The methods for the characterization of antimatter are the so-called isodualities [4b,5] which are characterized by the anti-automorphic map of all quantities $A$ for matter into their anti-Hermitean forms $-A^\dagger$, thus implying negative-definite units $-\hat{I}$. Again, for consistency the isodual map must be applied to all mathematical and physical formulations for matter, yielding isodual isonumbers, isodual isospaces, isodual isoalgebras, etc. The isodual struc-
tures which are particularly important for this paper are: the \isodual isominkowski\ spaces, the \isodual isopoincaré symmetry, and the \isodual isospecial relativity\ (see [4h] for a general presentation).

Independent reviews and developments can be found in monographs [6], papers [7] and literature quoted therein. Ref. [6e] provides a comprehensive bibliography up to 1984, while a subsequent bibliographical (and technical) survey is available in monograph [6d].

The main lines of the \classical geometric unification of the special and general relativities\ were first submitted in ref. [4d] as a natural consequence of the isopoincaré symmetry. The main lines of the \operator geometric unification of the special and general relativities\ were submitted for the first time at the VII Marcel Grossmann Meeting on General Relativity held at Stanford University in July 1994 [8a].

The above studies still lacked a rigorous form-invariant character because they were based on the \conventional\ differential calculus which has resulted to be noninvariant, and thus inapplicable under isotopies.

In this paper we present for the first time the fully form-invariant formulations of:

I) The classical geometric unification of the general and special relativities for matter into the isospecial relativity in which the former occurs for generalized units while admitting the latter as a particular case for conventional units;

II) The operator formulation of gravity for matter based on the abstract axioms of relativistic quantum mechanics, thus showing hope for a possible resolution of the ambiguities in current theories of quantum gravity; and

III) The classical and operator isodual formulations of antimatter.

The experimental validity of the classical isominkowskian formulation of gravity for matter is derived from the preservation of the conventional Einstein’s field equations except for inessential multiplicative terms. The experimental validity of the operator isominkowskian formulation of gravity for matter is derived from the preservation of conventional quantum mechanical laws, with gravitational effects being notoriously very small as compared to those of other interactions of the particle word. The validity of the isodual formulations for antimatter is inferred from its compatibility with available experimental data.

The above results have been made possible by the recent achievement of sufficient maturity for: mathematical content in memoir [3f] including the isotopies and isodualities of differential calculus and their applications to algebras, geometries and analytic mechanics; general axiomatic consistency in the physical formulations of both isotopic and isodual theories in memoir [3g]; and generalized symmetry principles for isotopic and isodual theories in memoir [7a]. Further studies, such as the formulation of a grand unified theory with an axiomatically consistent inclusion of gravitation are presented in the forthcoming paper [10]. An introductory outline of the main mathematical and physical aspects of this paper is available in Pages 18, 19 of Web Site [7u].

A primary motivation for this paper is the following:

**Theorem 1** The basic unit of all (nowhere degenerate, real valued and symmetric) geometries with non–null curvature over conventional fields is not invariant under the symmetry
of their line element in both classical and quantum formulations.

**Proof.** Let \( E = E(x, \delta, R) \) and \( \mathbb{R} = \mathbb{R}(y, g, R) \) be \( n \)-dimensional Euclidean and Riemannian spaces, respectively, with the same signature \((+ , + , \ldots , +)\), basic unit \( I = \text{diag}.(1,1,\ldots,1)\), metrics \( \delta = (\delta_{ij} = \text{Diag}.(1,1,\ldots,1) \) and \( g(y) = (g_{ij}) = g^t \), and local coordinates \( x = \{x^k\}, y = \{y^k\}, i, j, k = 1, 2, \ldots , n \), over the field \( R = \mathbb{R}(n, +, \times) \) of real numbers \( n \) with conventional sum \(+\) and multiplication \(\times\).

The transformation \( x \rightarrow y(x) \) for which the Euclidean metric is mapped into the Riemannian metric,

\[
\delta_{ij} \rightarrow g_{ij}(y) = \frac{\partial y^r}{\partial x^i} \delta_{rs} \frac{\partial y^s}{\partial x^j}, \tag{1.1}
\]

is necessarily *noncanonical* for non-null curvature. Therefore, the symmetries of the Riemannian line elements \( y^2 = y^r g_{xy} \) are necessarily *noncanonical*. As such, these symmetries do not generally preserve the basic unit \( I \) at the classical level because, by definition, noncanonical transforms do not leave invariant the fundamental canonical (symplectic) tensor

\[
(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{1.2}
\]

i.e., they are of the type in phase space with local coordinates \( b = \{x, p\} \rightarrow b'(b) = \{x', p'\} \)

\[
\omega_{\mu\nu} \rightarrow \omega'_{\mu\nu} = \frac{\partial b^\alpha}{\partial y^\rho} \omega_{\alpha\beta} \frac{\partial b^\beta}{\partial y^\nu} = \omega'_{\mu\nu}(a') \neq \omega_{\mu\nu} \tag{1.3}
\]

But the canonical tensor represents the fundamental space units of the theory, and this establishes the inability of classical conventional geometries with non-null curvature to have invariant basic units.

The symmetries of the same line element in operator formulation are then necessarily *nonunitary* for consistency (see Sect. 5), and this proves the lack of invariance of the basic unit also for operator theories. The same proof evidently applies for indefinite signatures \((+, +, \ldots, -, -\ldots)\) (see later on for the \((3+1)\)-dimensional case). *q.e.d.*

To understand the implications of the above theorem, recall that the basic unit of the \((3+1)\)-dimensional Riemannian geometry is given by \( I = \text{Diag}.((1,1,1,1)) \), where the first three components \((1,1,1)\) represent the space units (say 1 cm) in dimensionless form, and the fourth component represents the time unit (say 1 sec) also in dimensionless form.

Theorem 1 establishes that *curvature implies the lack of invariance of the fundamental space–time units of the theory*, thus implying evident problematic aspects in the comparison
of the theory with experimental data. In fact, one of the fundamental conditions for the applicability of the measurement theory is precisely the invariance of the basic unit.

As a result, Theorem 1 provides a new perspective of the various problematic aspects voiced on Einstein’s gravitation during this century (see, e.g., Ref. [5h] for an outline with references) because it indicates that they are not necessarily due to Einstein’s field equations but rather to their referral to a geometry in which the basic units of space and time are not invariant. In fact, following Theorem 1, the same problematic aspects can be proved to persist for all possible modifications—enlargement of Einstein field equations.

Equivalently, Theorem 1 establishes that Riemannian spaces are a noncanonical deformation of the Minkowskian spaces (because the former are obtainable via noncanonical transformations of the latter) and, as such, they suffer of all drawbacks of noncanonical theories when formulated on conventional spaces over conventional fields.

In the next section we present a formulation of classical gravitation for matter which preserves Einstein’s gravitational field equations unchanged and merely reformulates them in a new geometry, the isominkowskian geometry, with generalized yet invariant basic units under its universal isopoincarè symmetry. The operator and antimatter profiles are studied in subsequent section. To render this paper self-sufficient as well as for notational purposes, each section contains an outline of the new methods used therein.

2 Classical isominkowskian unification of the special and general relativities for matter

As it is well known, there cannot be really new physical advances without new mathematics. It turn, there cannot be really new mathematics without new numbers. Still in turn, the only possibility of identifying new numbers known to the author is via the generalization of their basic unit $I = +1 \rightarrow \hat{I}$ called lifting (first proposed in [3a,3b]), where $\hat{I}$ is in general a well behaved, $n \times n$ matrix with an unrestricted functional dependence of their elements on coordinates $x$, their derivatives of arbitrary order and any needed additional quantity,

$$I = +1 \rightarrow \hat{I} = \hat{I}(x, \dot{x}, \ddot{x}, ...) \tag{2.1}$$

The fundamental quantities of this paper are therefore given by new numbers with arbitrary units studied in details in ref.s [3e,3f].

A mere inspection of lifting (2.1) reveals a significant broadening of the conventional numbers and, therefore, of the mathematical and physical theories built on them. In fact, we have the following primary classification [3f]: 1) Ordinary numbers occurring for $\hat{I} = +1$; 2) isonumbers occurring for Hermitean generalized units $\hat{I} = \hat{I}^\dagger$; 3) genonumbers occurring for nonhermitean generalized unit $\hat{I} \neq \hat{I}^\dagger$; and 4) hypernumbers occurring for generalized units given by an ordered set of Hermitean quantities. The latter three classes then admit subclasses depending on whether the (real part of the) generalized unit is positive–definite,
singular, etc. The numbers, isonumbers, genonumbers and hypernumbers are used for the description of matter in condition of progressively increasing complexity (e.g., reversible, nonreversible, etc.), with the most general possible hypernumbers being studied for the description of matter in its most complex conditions (e.g., the DNA code).

Moreover, the conventional numbers and each of the above generalizations admit antiautomorphic images $\hat{1} \rightarrow \hat{I}^d = -\hat{I}^\dagger$ called isoduality [4b,3e,3f] which are used for the description of antimatter also in physical conditions of progressively increasing complexity [5].

It is evident that in a field of such a diversity we are forced for brevity to restrict our studies to the first lifting, those of isotopic type and its isodual. In this section we shall therefore study a classical representation of matter based on the isonumbers, the corresponding isodual representation of antimatter is studied in the next section. Corresponding operator images are studied in subsequent sections. The genotopic and hyperstructural extensions of isotopic formulation of this paper are contemplated for study in subsequent works.

Let $F = F(a,+,\times)$ be a field of numbers (i.e., real numbers $a = n \in R$, complex numbers $a = c \in C$ or quaternions $a = q \in Q$) with conventional sum $a + b$ and product $a \times b = ab$ and corresponding additive unit $0$ and multiplicative unit $I$.

The isofields are rings $\hat{F} = \hat{F}(\hat{a},+,\hat{\times})$ with elements $\hat{a} = a \times \hat{I}, a \in F$ called isonumbers, where $\hat{I}$ a positive–definite quantity (e.g., a matrix) generally outside the original set $F$ equipped with:

i) the isosum $\hat{a} + \hat{b} = (a + b) \times \hat{I}$ with conventional additive unit $\hat{0} \equiv 0$, $\hat{a} + \hat{0} \equiv \hat{0} + \hat{a} \equiv \hat{a} \in \hat{F}$ (the preservation of the additive unit $0$ is indicated by preserving the symbol $+$ unchanged in $\hat{F}(\hat{a},+,\hat{\times})$); and

ii) the isoproduct

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b} = (a \times b) \times \hat{I} \in \hat{F},$$

(2.2)

under which the quantity $\hat{I} = \hat{T}^{-1}$ is the correct new left and right unit of $\hat{F}$

$$\hat{I} \hat{\times} \hat{a} \equiv \hat{a} \hat{\times} \hat{I} \equiv \hat{A}, \forall \hat{a} \in \hat{F}, \hat{I} = \hat{T}^{-1},$$

(2.3)

(the change of the multiplicative unit is indicated with the new symbol $\hat{\times}$ in $\hat{F}(\hat{a},+,\hat{\times})$). When the above conditions are verified, the $\hat{I}$ is called the isounit and $\hat{T}$ is called the isotopic element.

The fundamental mechanism of the isotopies responsible for their axiom–preserving character is that of lifting the multiplicative unit $I \rightarrow \hat{I}$ while jointly the product is lifted by the inverse amount, $\times \rightarrow \hat{\times} = \times \hat{T} \times, \hat{I} = \hat{T}^{-1}$. This implies the following

**Lemma 1 [3e]:** Isofields satisfy all the axioms of a field.
Despite the local isomorphism \( \hat{F} \approx F \), the lifting \( F \to \hat{F} \) is not trivial because it requires a corresponding lifting of all operations of \( F \). For instance, the conventional square of a number \( n^2 = n \times n \) has no meaning for \( \hat{F} \) and must be lifted into the isosquare \( \hat{n}^2 = \hat{n} \times \hat{n} \). Along similar lines we have the isopowers \( \hat{n}^m = \hat{n} \times \hat{n} \times \ldots \times \hat{n} \) (\( m \) times); the isosquare root \( \hat{n}^{1/2} = n^{1/2} \times \hat{I}^{1/2} \); the isoquotient \( \hat{n} \div m = (\hat{n} \div \hat{m}) \times \hat{I} \); the isol lavor \( |\hat{a}| = |a| \times \hat{I} \), where \( |a| \) is the conventional norm; etc. (see [3e] for brevity for all details). The nontriviality of the lifting \( F \to \hat{F} \) is then illustrated by the fact that numbers which are not prime for \( I = +1 \) may become prime for other units [3e].

The axiomatic consistency of the emerging new structure is established by the local isomorphism between the conventional field and its isotopic image. In particular, it should be noted that the isounit preserves all axiomatic properties of the original unit, e.g.,

\[
\hat{I}^m = \hat{I} \times \hat{I} \times \cdots \hat{I} \equiv \hat{I}, \quad \hat{I}^2 \equiv \hat{I}; \quad \hat{I} \div \hat{I} \equiv \hat{I}, \quad \hat{|I|} \equiv \hat{I}, \text{ etc.}
\]

The assumption of the isonumbers as the fundamental numbers requires a simple, yet unique and unambiguous reconstruction of contemporary mathematical and physical theories into forms admitting of \( \hat{I} \), rather than \( I \), as the correct left and right unit.

Recall that the conventional metric spaces of contemporary physics are based on conventional fields of number, the first and most important implication of the lifting \( I \to \hat{I} \) is therefore the necessity to construct, for evident reason of consistency, corresponding liftings of metric spaces.

Let \( M = M(x, \eta, R) \) be a conventional Minkowski space [1] with coordinates \( x = \{x^\mu\} = \{r, c_o, \tau\}, \mu = 1, 2, 3, 4 \), where \( c_o \) is the speed of light in vacuum, with basic unit \( I = \text{Diag}(+1, +1, +1, +1) \) and metric \( \eta = \text{Diag}(+1, +1, +1, -1) \) over a field \( R = R(n, +x) \) of real numbers \( n \) equipped with the conventional sum + and product \times, additive unit 0 and multiplicative unit \( I \).

The lifting \( R(m, +, \times) \to \hat{R}(\hat{n}, +, \hat{\times}) \) then requires the corresponding lifting of the Minkowski into the isominkowski spaces \( \hat{M} = \hat{M}(x, \eta, R) \to \hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \) first submitted in [4a] which are characterized by the isocoordinates \( \hat{x} = x \times \hat{I} \) on \( \hat{R} \), the isometric \( \hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I} = \hat{T}_{\mu}^\alpha(x, \hat{x}, \hat{x}, \ldots) \eta_{\alpha\nu} \times \hat{I} \) where \( \hat{I} = \hat{T}^{-1} \) is now in general a \( 4 \times 4 \) positive–definite matrix, and the isoseparation is hereon expressed for diagonal isounits

\[
\hat{M}(\hat{x}, \hat{\eta}, \hat{R}) : \hat{x} = x \times \hat{I}, e\hat{I}a = \hat{T}(x, \hat{x}, \hat{x}, \ldots) \times \hat{\eta}, \quad \hat{I} = \hat{T}^{-1}, \quad (2.5)
\]

\[
(\hat{x} - \hat{y})^2 = (\hat{x}^\mu - \hat{y}^\mu) \times \hat{N}_{\mu\nu} \times (\hat{x}^\nu - \hat{y}^\nu) = [(x^\mu - y^\mu) \times \hat{\eta}_{\mu\nu} \times (x^\nu - y^\nu)] \times \hat{I} =
\]

\[
= [(x^1 - y^1) \times T_{11}(x, \ldots) \times (x^1 - y^1) + (x^2 - y^2) \times T_{22}(x, \ldots) \times (x^2 - y^2) +
+ (x^3 - y^3) \times T_{33}(x, \ldots) \times (x^3 - y^3) - (x^4 - y^4) \times T_{44}(x, \ldots) \times (x^4 - y^4)] \times (2.6)
\]

\[
\hat{T} = \text{Diag}(T_{11}, \hat{T}_{22}, \hat{T}_{33}, \hat{T}_{44}), \hat{T}_{\mu\nu} > 0, \quad x, y \in M, \hat{I} \not\in M. \quad (2.7)
\]
Note that all scalars of $M$ must become isoscalar to have meaning for $\hat{M}$, i.e., they must have the structure of the isonumbers $\hat{n} = n \times \hat{I}$. This condition requires the re-definition $x \to \hat{x} = x \times \hat{I}, \hat{\eta}_{\mu\nu} \to \hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}$, etc.

Note however the practical redundancy of using isocoordinates $\hat{x} = x \times \hat{I}$. In fact, we can write $\hat{x}^2 = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} = x^2$. For simplicity we shall hereon use the conventional coordinates. Note also the redundancy of using the full isoscalar form $\hat{N}$ of the isometric because the reduced form $\hat{\eta}$ with ordinary elements $\hat{\eta}_{\mu\nu}$ in $R$ is sufficient. The understanding is that the full isotopic formulations are needed for mathematical consistency.

We shall hereon assume the convention, rather familiar in the literature of the isotopies, that all quantities with a “hat” are computed in isospaces over isofields, and the corresponding quantities without a ”hat” are computed on conventional spaces over conventional fields.

Note the necessary condition that isospaces and isofields have the same isounit. This condition is absent in the conventional Minkowski space where the unit of the space is the unit matrix $I = \text{diag.}(1,1,1,1)$ while that of the underlying field is the number $I = +1$. Nevertheless, the latter can be trivially reformulated with the common unit matrix $I$, by achieving in this way the form admitted as a particular case by the covering isospaces

$$M(x,\eta,R) : x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R$$

Also, one should keep in mind for future needs the following

$$\text{Basic – Isoinvariant} = (\text{length})^2 \times (\text{unit})^2.$$  \hspace{1cm} (2.9)

A fundamental property of the infinite family of generalized spaces (8) is that the lifting of the basic unit $I \to \hat{I}$ while the metric is lifted of the inverse amount, $\eta \to \hat{\eta} = \hat{T} \times \eta, \hat{I} = \hat{T}^{-1}$, implies the preservation of all original axioms, and we have the following:

**Lemma 2 [4]:** The isominkowski spaces $\hat{M}(\hat{x},\hat{\eta},\hat{R})$ over the isofields $\hat{R}(\hat{n},+,\times)$ with a common positive–definite isounit $\hat{I}$ preserve all original axioms of the Minkowski space $M(x,\eta,R)$ over the reals $R(n,+,\times)$.

The nontriviality of the lifting is that of gaining an unrestricted functional dependence of the metric $\hat{e}_{\alpha} = \hat{\eta}(x,\dot{x},\ddot{x})$ under the conventional Minkowskian axioms.

The local isomorphism $\hat{M} \approx M$ holds to such an extent that the isominkowski and Minkowski spaces coincide at the abstract, realization–free level by conception and construction. Thus, the isominkowski spaces are not new spaces, but merely ”new realizations” of the original abstract Minkowskian axioms. In particular, the maximal causal speeds of both spaces coincides and it is given by $c_0$ as we shall see.
Lemma 2 illustrates again the "axiom-preserving" character of the isotopies indicated in Sect.1, this time at the level of metric spaces. It should be stressed that the "isotopies" are inequivalent to the various forms of "deformations" of the current literature for several reasons, such as: the former are axiom-preserving while the latter are not; the former are defined over generalized fields while the latter are not; etc. To avoid confusion, readers are discouraged from using the term "deformations" (of given structure into a nonisomorphic form) when referring to the "isotopies" (of the same structures into axiom-preserving isomorphic forms).

The isominkowskian geometry was first proposed in ref. [4a] (see ref.s [4g,4h] for comprehensive studies and ref.s [6,7] for independent works). These studies were however incomplete because based on the conventional differential calculus which has resulted to be inapplicable under isotopies. The foundations of the isominkowskian geometry formulated via the isodifferential calculus of ref. [3f] are introduced here apparently for the first time, with more detailed studies presented elsewhere [9].

Stated in a nutshell, the isominkowskian geometry is a symbiotic union of the Minkowskian and Riemannian geometries along the following main properties:

I) Isoflatness. It is easy to see that the isominkowskian geometry is flat in isospace over isofields, a property called isoflatness [4g]. This is due to the fact that curvature, which is represented by the factor $\hat{T}$ in the isometric $\eta = \hat{T} \times \eta$, is referred to its own inverse as unit. In fact, the new geometry permits the definition in isospace of straight line and intersecting angles, although in a predictable generalized form, which is not possible in the Riemannian geometry, thus confirming the preservation of the Minkowskian axioms.

In particular, isoflatness allows the reconstruction in isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ of the trigonometric, hyperbolic and other functions for a metric with an arbitrary functional dependence, which we cannot possibly review here (see [4g] for brevity).

More generally, the isominkowskian geometry is based on the new isofunctional analysis in which the isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ is turned into an isomanifold thanks to Kadeisvili’s isocontinuity [7p] and Tsagas–Sourlas isotopology [7q]. Within such a setting, the isogeometry must be solely elaborated with all special isofunctions, isotransforms and isodistributions, etc. (see [4g,4h] for details).

II) Pseudocurvature. In view of the arbitrary functional dependence of the isometric $\hat{\eta} = \hat{\eta}(x, \dot{x}, \ddot{x})$, the isominkowskian geometry can also be considered as being curved, but only when projected in the original space $M$ over the original field $R$, a property is called pseudocurvature which is of Riemannian (rather than Minkowskian) character.

This illustrates the symbiotic capacity of the isominkowskian geometry of unifying the main characteristics of both, the Minkowskian and Riemannian geometries. In turn, such character is evidently at the foundation of the proposed isominkowskian unification of the
special and general relativities.

To outline the pseudocurved character, consider the isominkowskian manifolds \( \hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \) equipped with Kadeisvil’s isocontinuity and Tsagas-Sourlas isotopology. The isodifferential calculus on \( \hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \) can be defined via the following notions of isodifferentials and isoderivatives [3f]

\[
\hat{d}x^\mu = \hat{I}_\mu^\nu \times dx^\nu, \quad \hat{\partial}_\mu = \hat{\partial}/\hat{\partial}x^\mu = \hat{T}_\mu^\nu \times \partial_\nu = \hat{T}_\mu^\mu \times \partial/\partial x^\mu, \quad (2.10)
\]

and other axiom-preserving properties here omitted for brevity, where we have ignored for notational simplicity the isoquotient and related factorization of the isounit. In this way, \([\hat{\partial}_\mu/\hat{\partial}x^\nu] \hat{x} F = [\hat{\partial}_\mu/\hat{\partial}x^\nu] \times F\).

Since the isometric \( \hat{\eta} \) has an explicit dependence on \( x \), the isominkowskian geometry does indeed allow the introduction of the following isoconnection, called isochristoffel’s symbols

\[
\hat{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}(\hat{\partial}_\alpha \hat{\eta}_{\beta\gamma} + \hat{\partial}_\gamma \hat{\eta}_{\alpha\beta} - \hat{\partial}_\beta \hat{\eta}_{\alpha\gamma}) = \hat{\Gamma}^\gamma_{\beta\alpha}, \quad (2.12)
\]

\[
\hat{\Gamma}_\alpha^\beta_\gamma = \hat{\eta}_\beta^\rho \times \hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}_{\gamma\alpha}^\beta, \quad \hat{\eta}_\beta^\rho = [\hat{\eta}_{\mu\nu}]^{-1}\beta^\rho. \quad (2.13)
\]

The isocovariant differential of a vector field can then be defined by

\[
\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}^\beta_\alpha_\gamma \hat{x}^\alpha \times \hat{d}\hat{x}^\gamma, \quad (2.14)
\]

where isoproduct can be reduced to ordinary ones because of the cancellation of \( \hat{I} \) and \( \hat{T} \), with corresponding isocovariant derivative

\[
\hat{X}_\gamma^\beta = \hat{\partial}_\gamma \hat{X}^\beta + \hat{\Gamma}_\alpha^\beta_\gamma \hat{X}^\alpha, \quad (2.15)
\]

The isotopy of the proof of the conventional Riemannian case [11], pp. 80–81, yields the following:

**Lemma 3 (Isoricci Lemma):** Under the assumed conditions, the isocovariant derivatives of all isometrics of the isominkowskian spaces are identically null,

\[
\hat{\eta}_{\alpha\beta\gamma} = 0, \quad \alpha, \beta, \gamma = 1, 2, 3, 4. \quad (2.16)
\]
The novelty is illustrated by the fact that the Christoffel’s symbols, the covariant derivative and the Ricci Lemma persist under: 1) an arbitrary dependence of the metric \( \hat{\eta} = \hat{\eta}(x, \dot{x}, \ddot{x}, ...), \) rather than the current restriction to the Riemann dependence only, \( g = g(x); \) 2) under the Minkowskian, rather than Riemannian axioms; and 3) with null curvature in isospace over isofields.

It should be noted that the above properties were studied in [3f] for the isoriemannian geometry, and the above isominkowskian reformulation is submitted here for the first time.

We now introduce on \( \hat{M}(\hat{x}, \hat{\eta}, \hat{R}); \) the following isocurvature tensor, isoricci tensor, and isocurvature isoscalar

\[
\hat{R}^\beta_{\alpha\gamma\delta} = \hat{\partial}\hat{\Gamma}^\beta_{\alpha\gamma} - \hat{\partial}\hat{\Gamma}^\beta_{\alpha\delta} + \hat{\Gamma}^\beta_{\rho\delta}\hat{\times}\hat{\Gamma}^\rho_{\alpha\gamma} - \hat{\Gamma}^\beta_{\rho\gamma}\hat{\times}\hat{\Gamma}^\rho_{\alpha\delta}, \tag{2.17}
\]

\[
\hat{R}_{\mu\nu} = \hat{R}^\beta_{\mu\nu\beta}, \tag{2.18}
\]

\[
\hat{R} = \hat{N}^{\alpha\beta}\times\hat{R}_{\alpha\beta}. \tag{2.19}
\]

Einstein’s field equations on isominkowskian spaces can then be written

\[
\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{\times}\hat{N}_{\mu\nu}\hat{\times}\hat{R} = \hat{k}\hat{\times}\hat{\tau}_{\mu\nu}, \tag{2.20}
\]

where \( \hat{\tau}_{\mu\nu} \) is the source tensor on \( \hat{M}(\hat{x}, \hat{\eta}, \hat{R}); \) \( \hat{k} = k \times \hat{I} \) and \( k \) the usual constant.

Despite apparent differences, it should be indicated that Eq.s (2.20) coincide numerically with Einstein’s equations. In fact, the isoderivative \( \hat{\partial}_\mu = \hat{T}^\alpha_\mu \times \partial_\alpha \) deviates from the conventional derivative \( \partial_\mu \) by the isotopic factor \( \hat{T} \) (here assumed as being diagonal). But its numerical value must be referred to \( \hat{I} = \hat{T}^{-1} \), rather than \( I \), thus preserving the original value of \( \partial_\mu \).

Similarly, the isochristoffel’s symbols (2.12) deviate from the conventional symbols by the same factor \( \hat{T} \) (because \( \hat{\eta} \equiv g \)). But these symbols must be referred to the isounit \( \hat{I} \), thus preserving conventional values. A similar situation occurs for the isocurvature tensor (2.17) because the factor \( \hat{T} \) from the covariant isoderivative \( \hat{\partial} \) is compensated by the factor \( \hat{I} \) originating from the contravariant index \( \beta \), with similar results holding for the remaining quantities. Possible residual terms are inessential because common factors to both sides of Eq.s (2.20).

A more detailed study of Eq.s (2.4) and related isominkowskian geometry is presented in ref. [9], including the use of the forgotten Freud identity of the Riemannian geometry in isotopic form.
III) Isosymmetries. A primary reason for introducing the isominkowskian spaces and related geometry is that they permit the construction of the universal symmetry of the line element (2.6) under an unrestricted functional dependence of the isometric \( \hat{\eta}(x, \dot{x}, \ddot{x}, ...) \) [4], while such a possibility is precluded in the Riemannian geometry.

Under the above isotopic reformulations, the symmetries of the isoinvariant (2.6) can be explicitly computed and are given by: the isorotational symmetry \( \hat{O}(3) \) [4b] for the space component of isoinvariant (2.5); the isolorentz symmetry \( \hat{L}(3.1) \) [4a]; the isotopic \( \hat{S}U(2) \) spin symmetry [4c]; the isopoincaré symmetry \( \hat{P}(3.1) = \hat{L}(3.1) \times T(3.1) \) [4d]; and the isospinorial isopoincaré symmetry \( \hat{R}(3.1) = SL(2, \hat{C}) \times \hat{T}(3.1) \) [4e].

These isosymmetries are constructed via the isotopies of Lie’s theory including the isotopies of enveloping algebras, Lie algebras, Lie groups transformation and representation theories, etc., originally proposed by the author in memoir [3a], developed in various works [3,4,5], studied by a number of independent researchers [6,7] and today called Lie–Santilli isotheory [6,7]. The isosymmetries \( \hat{O}(3), \hat{S}U(2), \hat{L}(3.1), \hat{P}(3.1) \) and \( \hat{P}(3.1) \) are essentially the conventional symmetries \( O(3), SU(2), L(3.1), P(3.1) \) and \( P(3.1) \) reconstructed for arbitrary generalized units \( \hat{I}(x, \dot{x}, \ddot{x}, ...) \) of the class admitted (4×4-dimensional, real-valued, symmetric and positive-definite matrices with the same dimension of the representation considered).

Since \( \hat{I} \) is positive-definite by assumption, the above isosymmetries are isomorphic to the original symmetries by conception and constructions [4]. As such, they are not ”new symmetries”, and merely constitute the most general known nonlinear, nonlocal and noncanonical realizations of the conventional symmetries.

For classical realizations of the above isosymmetries we have to refer the interested reader to monographs [4g,4h]. An outline of their operator realization is presented in Sect. 7.

4) Isospecial relativity. The preceding formalism is reduced to primitive laws under the name of isospecial relativity [4] which is the isotopic image of the special relativity [1] realized on isominkowskian spaces \( \hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \). As such, we do not have a ”new” relativity, but merely the most general known realization of the axioms of the conventional relativity. In fact, the special and isospecial relativity coincides at the abstract level by conception and construction, to such an extent to have the same light cone with the same maximal causal speed \( c_0 \) (see below).

5) Isoanalitic mechanics. Finally, the preceding formalism is complemented with step–by–step–isotopies of conventional Lagrangian and Hamiltonian mechanics called isoanalitic mechanics, which begins with a basic action in isospace and includes the isotopies of equations by Lagrange’s, Hamilton’s, Hamilton–Jacobi, etc. (see [3f] for brevity).
Despite all these similarities between conventional and isotopic structures, one should keep in mind that the conventional Minkowski geometry, the Poincaré symmetry and the special relativity are strictly linear, local–differential and Hamiltonian. On the contrary, their isotopic images are generally nonlinear (in any desired variable), nonlocal–integral (i.e., they admit integral terms under the Tsagas–Sourlas integro–differential topology [5q], provided that they are all embedded in the generalized unit $\hat{I}$), and nonlagrangian (in the sense of admitting terms simply beyond any representational capability of a (first-order) lagrangian (see later on for details).

Also, the isotopic structures have been proved by Aringazin [7r] to be directly universal, that is, applying for all infinitely possible, well behaved, signature preserving generalizations of the Minkowski metric (universality), directly in the x–frame of the observer (direct universality). As a result, any conventional deformation of M can be identically reformulated via the isotopic formalism, and the isotopic, axiom–preserving representation of deformed Minkowski spaces holds even when not desired, thus implying the existence of various applications and experimental verifications [7r].

The isominkowskian geometry, the isopoincaré symmetry and the related isospecial relativity were originally introduced [4] for a direct representation of interior dynamical problems, e.g., electromagnetic waves propagating within inhomogeneous and isotropic physical media with a locally varying speed $c = c_0/n$ as occurring, e. g., in our atmosphere. In fact, the first physical application of the isotopic line element (2.6) is that of directly representing in its fourth component the local speed of light $c = c_0/n$ with $\hat{T}^{\perp} = n_4 = n$, while the remaining components $\hat{T}_{kk} = n_k$ represent the anisotropy of the medium considered, and its inhomogeneity is represented, e. g., via a dependence of the quantities $n_\mu$ in the local density.

It is evident that the conventional formulation of the special relativity in Minkowski space $M$ is inapplicable to (and not ”violated” by) locally varying speeds $c = c(x,...)$. The use of the isominkowskian space permits the regaining of the validity of the special relativity because in the latter case the speed of light is deformed by the amount $c_0 \to c = c_0/n_4$ while the related unit is deformed by the inverse amount, $I_{44}^{\perp} = 1 \to \hat{I}_{44}^{\perp} = n_4$, thus implying the constant value $c_0$ in isospace.

In this way the special relativity is rendered ”directly universal” under the isotopies, that is, it is rendered applicable for all possible local speeds of light. Equivalently, we can say that the speed of light $c_0$ is a ”universal constant” only in isospace $\hat{M}$, while its projection in our space–time $M$ acquires the local value $c$.

Recall the known problematic aspects of the conventional formulation of the special relativity when applied to physical media, e.g., for the propagation of electromagnetic waves in water where $c = c_0/n_4 < c_0$ and electrons can propagate with speeds greater than the local speed of light (Cherenkov light), thus implying evident problems of causality. If one assumes as the maximal causal speed in water to be the speed of light in vacuum, the principle of causality is indeed salvaged, but there is the loss of the relativistic law of addition of speeds.
because the sum of two speeds of lights in water does not yield the local speed of light or that in vacuum, \(v_{tot} = 2c/(1 + c/c_0) \neq c\) and \(c_0, c = c_0/n\). All the above problematic aspects are resolved by the isospecial relativity in isospace with a number of additional preliminary applications and verifications (see [4–7] for brevity).

The isospecial relativity was also introduced [4] for an invariant description of extended–deformable particles moving within physical media under unrestricted external forces. This objective is achieved via the realization of the isounit in the diagonal form (for spheroidal ellipsoidal shapes)

\[ \hat{I} = \text{Diag.}(n_1^2, n_2^2, n_3^2, n_4^2) \times \hat{\Gamma}(x, \dot{x}, \ddot{x}, ...), \]  

(2.21)

where: a) the quantities \(n_1^2, n_2^2, n_3^2, n_4^2\) provides a geometrical representation of the extended, nonspherical and deformable shape under the volume preserving condition \(n_1^2 \times n_2^2 \times n_3^2 \times n_4^2 = 1\); the quantity \(n_4^2\) provides a geometric representation of the density of the medium in which motion occurs, such as the (square of the) index of refraction; and the quantity \(\hat{\Gamma}(x, \dot{x}, \ddot{x}, ...\) represents nonlinear, nonlocal integral and nonhamiltonian interactions; all this in a manifestly form–invariant way because the isounit \(\hat{I}\) is the basic invariant of the isopoincare symmetry [4]. These features have permitted additional applications and verifications with effects due to deformations of shape [4–7].

The application of the isospecial relativity submitted in this paper is basically different than the above ones. In fact, in this paper deals with the \textit{classical isotopic formulation of the general relativity for matter via the isominkowskian geometry}. It should be stress–front that it would be unreasonable to expect in this first introductory paper a comprehensive treatment at all possible epistemological, geometric, operator, quantum field field theoretical, experimental and other aspects. The rather limited objective of this study is to identify the essential axiomatic foundations of the isospecial relativity in its application to gravity, point out its plausibility for resolving at least some of rather old problematic aspects of the conventional formulation of gravity, and indicate its experimental validity.

In summary, the representation of gravity via the isospecial relativity is based on the restriction of the isominkowskian metric to represent identically any given Riemannian metric, and in the isotopic factorization of the \textit{conventional} Minkowski metric, relativity is characterized by the isominkowskian geometry whose isometric \(\hat{\eta}\) is assumed to coincide with the Riemannian one, \(\hat{\eta} \equiv g(x)\), and we shall write

\[ \hat{M}(\dot{x}, \ddot{x}, \hat{R}) : \hat{\eta} \equiv g(x) = \hat{T}_{gr}(x) \times \eta, \hat{I}_{gr}(x) = [\hat{T}_{gr}(x)]^{-1} > 0. \]  

(2.22)

Note that, since the Riemannian geometry is locally Minkowskian, the \(4 \times 4\) matrix \(\hat{T}_{gr}(x)\) in the isominkowskian factorization

\[ g(x) = \hat{t}_{gr}(x) \times \eta, \]  

(2.23)

is always \textit{positive–definite}. In this case \(\hat{T}_{gr}\) and \(\hat{I}_{gr}\) are called the \textit{gravitational isotopic element and isounit}, respectively. As an illustration, the gravitational isotopic element in
the isominkowskian decomposition of the Schwarzschild’s metric is given by

\[
\hat{T}_{\text{gr}} = \text{Diag.}((1 - M/r)^{-1}, (1 - M/r)^{-1}, (1 - M/r)^{-1}, (1 - M/r)^{-1}),
\]  

(2.24)

and recovers the conventional value \( I = \text{Diag.}(1, 1, 1, 1) \) for \( r \to \infty \). The isominkowskian formulation of any other model is then straightforward.

The primary motivations supporting the isominkowskian formulation of gravity are the following:

A) The formulation possesses an invariant generalized basic unit \( \hat{I}_{\text{gr}} \), thus resolving the problematic aspects caused by Theorem 1 (Sect. 1).

B) The formulation permits, apparently for the first time, the geometric unification of the special and general relativities into one single theory, the isospecial relativity [4], characterized by infinitely many possible, generalized, positive–definite units. The selection of the special or general relativity is then done via the the assumed specific value of the unit, the value \( \hat{I} \equiv I \) recovering the special relativity identically, while the more general value \( \hat{I} = \hat{I}_{\text{gr}}(x) = [\hat{T}_{\text{gr}}(x)]^{-1} \) implies the selection of the general relativity (see below on for a broader dependence).

C) The formulation permits the achievement, also for the first time, of a unique and universal symmetry of gravitation, Santilli’s isopoincaré symmetry \( \hat{P} \) [3–7]. In fact, as pointed out our earlier, \( \hat{P}(3.1) \) is the universal symmetry of all infinitely possible line elements (2.10) which admit as particular cases the Riemannian forms. This resolves the historical difference between the general and the special relativity whereby the latter is indeed equipped with a universal symmetry, the Poincaré symmetry, while the former is not. By no means, the availability of a universal symmetry for gravitation is a mere mathematical curiosity, because it carries the same rigid physical guidelines as provided for the special relativity, which are lacking in the current formulation of gravitation.

D) The formulation permits the resolution of known ambiguities in the compatibility of the general and special relativites. Recall that the generators of any symmetry are the total conserved quantities and they remain unchanged under isotopies [4]. Then, the total conservation laws of general relativity in isominkoewskian formulation are established by mere visual observation of the generators of the its universal symmetry, the isopoincaré symmetry, thus avoiding complex and controversial calculations. Moreover, the compatibility of relativistic and gravitational conservation laws is established by the mere visual observation that these generators are the same for both relativities, thus resolving additional known controversies on their claimed lack of compatibility, and similar occurrences hold for other aspects (see [4h] for the preservation of weight at the relativistic limit via the use of the forgotten Freud identity). Therefore, the lack of the rigid guidelines of a universal symmetry in gravitation appears to be the origin of some of the ambiguities here considered.

E) The geometric unification of the general and special relativities constitutes the foundation of a novel operator version of gravity studied in Sect. 7, which is as axiomatically consistent as relativistic quantum mechanics, thus avoiding known problematic aspects of conventional forms of quantum gravity.
F) The formulation permits the introduction of a novel unification of gravitational and electroweak interactions based on the embedding of the part truly representing curvature, the gravitational unit $I_{\text{gr}}(x) = [T_{\text{gr}}(x)]^{-1} > 0$, in the unit of unified gauge theories, whose studies were initiated by Gasperini [70] (see also the review in App. A of monograph [6a]) and pointed out in Ref. [4h].

G) The formulation permits a novel relativistic and gravitational treatment of antimatter at both classical and quantum levels [5] studied in the next section, which avoids the problematic aspects of the Riemannian representation of antimatter outlined below.

H) The formulation permits a direct geometric representation of interior relativistic and gravitational problems, for instance, a direct representation of the locally varying speed of light $c = c_0/n_4$ via the metric of the isominkowskian geometry with $T_{44} = n_4^{-3} \times g_{44}$. This latter possibility is permitted by the unrestricted functional dependence of the isometric which, when restricted to a sole $x$–dependence, characterizes exterior gravitational problems in the homogeneous and isotropic vacuum, while the use of a more general dependence characterizes interior gravitational problems within inhomogeneous and anisotropic physical media (with the understanding that the background space remains homogeneous and isotropic).

The axiomatic consistency of the isominkowskian formulation of gravity is assured by the axiom–preserving character of the isotopies.

The plausibility of the proposed theory is illustrated by a comparison of properties A)–H) above with the corresponding features of the conventional formulation of gravity.

The experimental verification of the isominkowskian formulation of gravity is established by the fact that Eq.s (2.20) coincide numerically with the conventional Einstein equations, as indicated earlier. The isominkowskian formulation of gravity therefore possesses the same verifications of the conventional Riemannian formulation.

3 Classical isodual isominkowskian unification of the special and general relativities for antimatter.

The current classical representation of antimatter is afflicted by a number of problematic aspects at the classical level, as well as operator levels. The mathematically correct map between matter and antimatter must be antiautomorphic (or, more generally, anti–isomorphic), as it is the case for the charge conjugation in quantum mechanics. The contemporary Riemannian representation of antimatter via the simple change of the sign of the charge and magnetic moments is therefore insufficient.

Also, current theoretical physics admits only one type of quantization, the conventional one of matter, and there is no separate quantization for antimatter. Therefore, the operator image of the current, classical, gravitational representation of antimatter is not the correct charge conjugate antiparticle, but merely a particle with the change of the sign of the charge and magnetic moments.

Moreover, the only energy–momentum tensor available in the riemannian geometry is the
conventional one with positive-definite energy. Such structure is manifestly incompatible
with the negative-energy solutions of relativistic field equations.

Because of these and other problematic aspects, Santilli [5] introduced in 1985 a new
antiautomorphic map, called isoduality, which can be applied to the entire formulations of
matter, beginning at the classical level and then continuing at the operator level.

The fundamental isodual map is that of the isounit

\[ \hat{I} > 0 \rightarrow \hat{I}^d = -\hat{I}^\dagger = -\hat{I} < 0. \]  

(3.1)

This requires the reconstruction of the entire isotopic formalism in such a way to admit \( \hat{I}^d \),
rather than \( \hat{I} \), as the correct left and right unit.

The most important quantities are, again, new numbers. In fact, isoduality must be first
applied to the basic isofields \( \hat{F}(\hat{a}, +, \times) \) of isoreals \( \hat{a} = \hat{n} \), isocomplex \( \hat{a} = \hat{c} \) or isoquaternions
\( \hat{a} = \hat{q} \), yielding the isodual isofields \( \hat{F}^d(\hat{a}^d, +, \times^d) \) (see [3e] for comprehensive studies), which
are rings of elements called isodual numbers

\[
\hat{a}^d = a^\dagger \times \hat{I}^d = -a^\dagger \times \hat{I} = -\hat{a}^\dagger, \quad (\hat{a}^d)^d \equiv \hat{a}.
\]  

(3.2)

(\text{where} \dagger \text{denotes Hermitean conjugation and} \overline{\text{complex conjugation), equipped with the}}
\text{isodual sum} \hat{a}^d + \hat{b}^d = (a + b)^\dagger \times \hat{I}^d, \text{and the isodual isoproduct}

\[
\hat{a}^d \times^d \hat{b}^d = \hat{a}^d \times \hat{T}^d \times \hat{b}^d, \quad \hat{T}^d = -\hat{T}^\dagger = -\hat{T},
\]  

(3.4)

under which \( \hat{I}^d = (\hat{T}^d)^{-1} \) is the correct left and right unit of \( \hat{F}^d \),

\[
\hat{I}^d \times^d \hat{a}^d \equiv \hat{a}^d \times^d \hat{I}^d \equiv \hat{a}^d, \quad \forall \hat{a}^d \in \hat{F}^d,
\]  

(3.5)

in which case (only) \( \hat{I}^d \) is called the isodual isounit and \( \hat{T}^d \) called the isodual isotopic element.

All operations of an isofield are then subjected to a simple, yet significant isodual map
here left to the interested reader [3e]. A quantity is called isoselfdual when it is invariant
under isoduality. This is the case for the imaginary unit because \( i^d = -\bar{i} \equiv i \) as well as other
quantities we shall identify later on.

A property of isodual isofields most important for this paper is that isofields and isodual
isofields are antiautomorphic with respect to each others, exactly as desired. Another
important property is that isodual isofields have a negative-definite norm

\[
\uparrow \hat{a}^d \uparrow^d = |a| \times \hat{I}^d = -|a| \times \hat{I} < 0.
\]  

(3.6)

As a consequence, all physical characteristics which are positive for matter become negative
for antimatter under isodual representation, as originally assumed at the discovery of antiparticles (Stueckelberg and others). This implies negative mass, negative energy, negative
(magnitude of the) angular momentum, motion backward in time, change of the sign of charges and magnetic moments, etc.

The novelty is that these negative characteristics are now defined with respect to negative units, thus rendering inapplicable existing arguments against negative energy. As a matter of fact, the referral of the negative–energy solutions to negative units permits the resolution of their un–physical behavior which historically motivated the ”hole theory” in second quantization [5].

Similarly, the isodual representation renders inapplicable existing argument against motion backward in time. In fact, motion backward in time referred to a negative unit of time is exactly as causal as motion forward in time referred to a positive unit of time (for these and other aspects, the interested reader may consult [4h,5]).

The next isodual map must be applied to the basic carrier spaces, yielding the isodual isominkowskian spaces [4d]

\[
\hat{M}^d(\hat{x}^d, \hat{\eta}^d, \hat{R}^d) : \hat{x}^d = -\hat{x}, \hat{\eta}^d = -\hat{\eta}, \hat{I}^d = -\hat{I} = (\hat{T}^d)^{-1} = -\hat{T}^{-1} < 0, \quad (3.7)
\]

\[
(\hat{x}^d - \hat{y}^d)^{\hat{2}d} = (\hat{x}^{\mu d} - \hat{y}^{\mu d}) \times \hat{N}^d_{\mu \nu} \times (\hat{x}^{\nu d} - \hat{y}^{\nu d}) =
\]

\[
= \left[ (\hat{x}^{\mu} - \hat{y}^{\mu}) \times \hat{\eta}_{\mu \nu} \times (\hat{x}^{\nu} - \hat{y}^{\nu}) \right] \times \hat{I}^d =
\]

\[
= \left[ -(\hat{x}^1 - \hat{y}^1) \times T_{11}(x, ...) \times (\hat{x}^1 - \hat{y}^1) - \right.
\]

\[
- (\hat{x}^2 - \hat{y}^2) \times T_{22}(x, ...) \times (x^2 - y^2) -
\]

\[
- (\hat{x}^3 - \hat{y}^3) T_{33}(x, ...) \times (x^3 - y^3) +
\]

\[
+ (\hat{x}^4 - \hat{y}^4) \times T_{44}(x, ...) \times (x^4 - y^4) \right] \times \hat{I}^d, \quad (3.8)
\]

\[
\hat{T}^d = \text{Diag.}(-\hat{T}_{11}, -\hat{T}_{22}, -\hat{T}_{33}, -\hat{T}_{44}), \hat{T}_{\mu \nu} > 0, \quad (3.9)
\]

with all remaining properties conjugated with respect to \(\hat{M}(\hat{x}, \hat{\eta}, \hat{R})\) here omitted for brevity.

The next isodual map is that of the underlying calculus, yielding the isodual differential calculus [3f] which is based on the rules

\[
\hat{d}^{\hat{d} \hat{z}^{\mu d}} = \hat{I}^{d \nu}_{\mu} \times d^{\hat{x}^{d \nu}} \equiv \hat{d}^{\hat{\phi}^{\hat{d} \hat{\phi}^{d \mu}}} = -\hat{T}^{-1} \hat{\partial}^{\hat{x}^{\mu}}, \text{etc.} \quad (3.10)
\]

The next conjugation is that of the applicable geometry, yielding the isodual isominkowskian geometry [3e,4g] which can be constructed via the above rules and the following isodualities

\[
\hat{\Gamma}^{\hat{d} \hat{d} \gamma}_{\alpha \beta} = -\hat{\Gamma}^{\alpha \beta \gamma}, \hat{D}^{\hat{d} \hat{X}^{d \mu}} = \hat{D}^{\hat{\phi}^{d \mu}} \hat{X}^{\hat{d} \mu} = -\hat{X}^{\hat{\beta} \mu}, \quad (3.11)
\]

\[
\hat{R}^{\hat{d} \hat{d} \gamma \delta} = -\hat{R}^{\alpha \gamma \delta}, \hat{R}^{\hat{d} \mu \nu} = -\hat{R}^{\mu \nu}, \hat{R}^{\hat{d}} = -\hat{R}, \text{etc.} \quad (3.12)
\]

The isodual isofield equations then read
\[ \hat{G}^d_{\mu\nu} = \hat{R}^d_{\mu\nu} - \frac{1}{2} \hat{d} \times \hat{N}^d_{\mu\nu} \times \hat{d} \hat{R}^d = \hat{k}^d \times \hat{d} \hat{r}^d_{\mu\nu}, \]  

and they result to be the negative image of Eq.s. (2.20). Other aspects are studied in the forthcoming paper [9].

A property of the isodual isominkowskian geometry most important for this paper is that the isodual energy–momentum tensor is negative–definite, exactly as needed for overall consistency in any theory of antimatter.

The next isoduality is that of the basic isosymmetries studied in [4d,4e], yielding the isodual isorotation, isodual isolorentz and the isodual isopoincaré symmetry. The next conjugation is that of the isorelativity, yielding the isodual isospecial relativity, and of analytic mechanics, yielding the isodual isoanalytic mechanics which are not outlined for brevity [4d,4h].

In order to apply the above results to a unified treatment of antimatter, the reader should be aware that all preceding formulations admit as particular cases the isodual numbers, isodual Minkowskian geometry, isodual Poincaré symmetry and isodual special relativity, namely, they admit hitherto unknown antiautomorphic images of conventional theories which are here assumed for the relativistic characterization of antimatter in vacuum.

The classical isominkowskian reformulation of general relativity for antimatter is then given by the isodual isospecial relativity under the particular realization of the isodual isotopic element and isodual isounit

\[ \hat{M}^d(x^d, \hat{\eta}^d, \hat{R}^d) : \hat{\eta}^d(x^d) = \hat{T}^d_{gr}(x^d) \times \eta, \hat{I}^d_{gr}(x) = [\hat{T}^d_{gr}(x^d)]^{-1}, \]  

which admits as particular case the isodual special relativity for \( \hat{I}^d_{gr} = I^d = -\hat{I} \).

The axiomatic consistency of the above classical, isodual, relativistic and gravitational representation of antimatter appears to be established beyond reasonable doubts. Its plausibility is established by its resolution of the problematic aspects of conventional formulations (see also next sections). The physical validity of the classical isodual theory is established by the verification of the sole classical experimental data on antimatter available at this writing, those under electromagnetic interactions, because those under gravitational interactions are still unknown.

We can therefore conclude these introductory considerations on the unification of the special and general relativity for antimatter by indicating that the emerging novelty warrants additional studies in the field. In fact, the isodual theory permits a mathematically correct study of the gravitational field of astrophysical bodies made up of antimatter and the initiation of studies for the future experimental resolution whether a far away star or quasar is made up of matter or of antimatter [5].
4 Problematic aspects of quantum gravity

We now pass to the study of the unification of the special and general relativities at the operator level. Again, we have to insist that it would be unreasonable to expect a comprehensive treatment in this introductory paper of all operator and field theoretical aspects, because they are so many to discourage even a partial outline.

The remaining scope of this paper is merely that of identify the essential operator foundations of the isospecial relativity, and establish its plausibility as compared to other operator forms of gravity at the simplest possible level of ”first quantization”. In any reasonable conduction of research, quantum field theoretical aspects can only be considered after the identification of hitherto unknown isogravitational grounds in first quantization. By keeping in mind that the physical validity of the conventional quantum treatment of gravity is still debated after three quarters of a centuries of studies, the reader should not expect the resolution of the physical validity of the isoolerator treatment of gravity in its first presentation.

The basic open problems in the operator version of general relativity are the following:

A) On one side, relativistic quantum mechanics (RQM) needs a meaningful Hamiltonian while, on the other side, Einstein’s gravitational in vacuum has a null Hamiltonian.

B) There is the need of an operator gravity which is as axiomatically consistent as the conventional RQM, i.e., invariant under its own time evolution with physical quantities which are Hermitean–observable at all times, with unique and invariant numerical predictions, etc..

C) Recent studies on interior gravitational problems of black holes (see, e. g., the studies by Ellis et all. [7s] and references quoted therein) have indicated that operator gravity should be a nonunitary image of conventional quantum theories, as needed, e. g., for a representation of irreversibility.

The isominkowskian reformulation of general relativity permits a new operator version of gravity for matter here called operator isotopic gravity, or operator isogravity (OIG) for short, which is based on an axiom–preserving lifting of the unit of RQM from the trivial value $I = \text{Diag.}(1,1,1,1)$ to the gravitational value $\hat{I}_{gr}(x)$. As such OIG requires no Hamiltonian at all, thus resolving the first historical problem. The axiomatic consistency of the proposed OIG is guaranteed by the preservation of the abstract axioms of RQM, only realized in a more general way, including form–invariance, Hermiticity of observables at all times, etc., thus resolving the second problem. Finally, the proposed OIG is a rather natural nonunitary image of conventional RQM, thus verifying the third condition.

The isodualites then permit an antiautomorphic isodual operator isogravity (IOIG) which is based on the negative–definite gravitational units of the preceding section.

A preliminary comparison of the OIG with the conventional theory currently used, called quantum gravity (QG) [12] is in order. Even though both theories are of operator character, OIG and the conventional QG are inequivalent, as illustrated by the fact that, e.g., they are defined on inequivalent Hilbert spaces and fields. In any case, the term ”quantum” would be inappropriate under a nonunitary structure as requested by condition C) and, for this and
other reasons we have preferred the generic term "operator".

As we shall see, the term "quantum" is also questionable for "quantum gravity" because this latter theory too is outside the equivalence classes of RQM. In short, this paper presents evidence supporting the fact that a "quantum" version of gravity, that is a version obeying conventional quantum mechanics, cannot exist, as one can anticipate from the noncanonical structure of the Riemannian geometry (Theorem 1 of Sect.1).

The identification of the following problematic aspects of QM [3g,13] is in order, not only because necessary to appraise the plausibility of OIG on a comparative basis, but also because they do not appear to be well known in the specialized literature in the field:

I) QG does not possess an invariant basic unit, as established by theorem 1. Also, the time evolution of QG is necessarily nonunitary (otherwise QG would be a trivial element of the equivalence class of RQM), thus confirming the lack of invariance of the basic unit,

\[ I \rightarrow I' = U \times I \times U^\dagger \neq I. \] (4.1)

Therefore, all forms of QG gravity with a nonunitary time evolution (hereon assumed) lack unambiguous applications to experimental verifications;

II) QG does not preserve Hermiticity in time when formulated on a conventional Hilbert space over a conventional field. In fact, under a nonunitary transform, the familiar associative modular action of the Schrödinger’s representation \( H \times |\psi> \), where \( H \) is an operator Hermitean at the initial time, becomes

\[
U \times H \times |\psi> = U \times H \times U^\dagger \times (U \times U^\dagger)^{-1} \times U \times |\psi>= \hat{H} \times \hat{T} \times |\hat{\psi}>,
\]

\[ U \times U^\dagger \neq I, \quad \hat{T} = (U \times U^\dagger)^{-1}, \quad |\psi>= U \times |\psi>, \quad \hat{H} = U \times H \times U^\dagger. \] (4.2)

By nothing that \( \hat{T} \) is Hermitean, \( \hat{T} = (U \times U^\dagger)^{-1} = \hat{T}^\dagger \), the initial condition of Hermiticity of \( H \), \( <\psi| \times H \times |\psi> = <| \times \hat{H}^\dagger \times |\psi> \), when applied to the conventional Hilbert space \( \mathcal{H} \) with states \( |\hat{\psi}>, |\hat{\phi}> \), etc, requires the action of the transformed operator (??) on a conventional inner product, resulting in the expressions

\[
<\hat{\psi}| \times \{\hat{H} \times \hat{T} \times |\hat{\psi}>\} = \{<\hat{\psi}| \times \hat{T} \times \hat{H}^\dagger\} \times |\hat{\psi}> , \text{i.e., } \hat{H}^\dagger = \hat{T}^{-1} \times \hat{H} \times \hat{T} \neq \hat{H}. \] (4.3)

As such, Hermiticity is not preserved under the time evolution of nonunitary theories when formulated on conventional space \( \mathcal{H} \) over conventional fields \( C \), because of the lack of general commutativity of \( \hat{T} \) and \( \hat{H} \). Consequently, QG does not admit physically acceptable observables, an occurrence first indicated by Lopez [13a].

III) QG does not admit invariant physical laws and numerical predictions. These are an evident consequence of the nonunitarity of the time evolution and do not require further elaboration.

An objectives of this paper is to see whether our OIG permits the resolution of at least some of the above problematic aspects. At any rate, the lack of resolution until now of the
above problematic aspects does warrant the study of structurally novel operator formulations of gravity.

5 Isotopic completion of relativistic quantum mechanics and its isodual

We now outline for the reader’s convenience the operator isotopies, first proposed in ref. [3b] of 1978 and then studied by numerous authors and known under the name of relativistic hadronic mechanics (RHM). This outline is recommendable because axiomatic maturity of the new mechanics has been reached only recently in memoir [3g], following the achievement of mathematical maturity in Ref. [3f].

The operator isotopies are nowadays defined as maps of any given linear, local and unitary structure into its most general possible nonlinear, nonlocal and nonunitary forms, which are however capable of restoring linearity, locality and unitarity on suitable generalized spaces over generalized fields.

The fundamental isotopy is the lifting of the conventional (3 + 1)-dimensional unit of relativistic quantum mechanics (RQM) as for classical isothories

\[ I \rightarrow \hat{I}(x, \dot{x}, \ddot{x}, \psi, \partial \psi, ...) > 0, \]  

with the additional dependence on the wavefunctions and their derivatives.

Jointly the conventional associative product among generic operators \( A, B, A \times B = AB \) (e.g., the elements of an enveloping algebra \( \xi \)) must be lifted into the form

\[ A \times B \rightarrow A \hat{\times} B = A \times \hat{T} \times B, \]  

where \( \hat{T} \) is fixed for all elements of \( \xi \). Under the conditional \( \hat{I} = \hat{T}^{-1}, \hat{I} \) results to be the correct (left and right) generalized unit of the new theory,

\[ \hat{I} \hat{\times} A = A \hat{\times} \hat{I} \equiv A, \forall A \in \xi. \]  

The lifting \( A \times B \rightarrow A \hat{\times} B \) is called isotopic in the sense that it preserves the all original axioms, including associativity, \( A \hat{\times} (B \hat{\times} C) = (A \hat{\times} b) \hat{\times} C \).

The most direct way to construct operator isotopic methods is via nonunitary transforms of RQM, called, isotopic completion of relativistic quantum mechanics [3g]. Let \( U \) characterize a conventional nonunitary transformation on a conventional Hilbert space \( H \) over a conventional field \( C \) of complex numbers. The deviation of the transform from \( I \) is assumed to be precisely equal to the isounit of the new theory, \( U \times U^\dagger = \hat{I} = \hat{I}^\dagger \neq I \). For the case of the canonical commutation rules we have

\[ U \times U^\dagger = \hat{I} = \hat{I}^\dagger \neq I, \hat{T} = (U \times U^\dagger)^{-1} = \hat{I}^{-1} = \hat{T}^\dagger, \]

22
\[ \bar{x}_\mu = U \times x_\mu \times U^\dagger, \quad \bar{p}_\nu = U \times p_\nu \times U^\dagger, \quad (5.5) \]
\[ U \times [x_\mu, p_\nu] \times U^\dagger = \bar{x}_\mu \times \hat{T} \times \bar{p}_\nu - \bar{p}_\nu \times \hat{T} \times x_\mu = \]
\[ = i \times \eta_{\mu\nu} \times U \times \hat{I} \times U^\dagger = i \times \eta_{\mu\nu} \times \hat{I}, \quad (5.7) \]

where one should note that the isounit and the isotopic element have the correct Hermiticity property, and the emerging new commutation rules have precisely the needed isotopic character of RHM.

It is easy to see that the above isotopic theory suffers of essentially the same problematic aspects of QG indicated in Sect. 4. In fact, the above theory is not form–invariant under additional nonunitary transforms when treated with the conventional mathematics of RQM. In particular, nonunitary transforms do not preserve the original unit \( \hat{I} \), thus preventing unambiguous applications to measurements. Moreover, such a theory does not preserve Hermiticity at all times, thus preventing the unambiguous representation of observable. Finally, it is easy to see that such a theory does not possess invariant numerical predictions, because of the lack of invariance of the special functions needed in the data elaboration.

A resolution of these problem requires the construction of the isotopies of the entire structure of RQM [3f,3g] without any exception known to this author. In fact, the characterization of RHM via the use in part of isotopic structures and in part of conventional quantum structures, is afflicted by rather fundamental inconsistencies which often remain undetected by the non–expert in the field. Equivalently we can say that the problematic aspects are resolved by the application of the nonunitary transform to the totality of the conventional formalism, including numbers, fields, metric and Hilbert spaces, algebras, geometries, etc.

In this way, RQM, RHM is characterized by the following primary structures:

A) the lifting of the field \( C = C(c, +, \times) \) of complex numbers into the isofields \( \hat{C}(\hat{c}, +, \hat{\times}) \) of isocomplex numbers of Sect.2 with the isofield of isoreal numbers \( \hat{R}(\hat{n}, +, \hat{\times}) \) as a particular case.

B) the lifting of the conventional Minkowski space \( M = M(x, \eta, R) \) into the isominkowskian space of Sect. 2 plus the enlarged functional dependence of isotopic element and, thus of the isometric,

\[ \hat{M}(\hat{x}, \hat{\eta}, \hat{R}) : \hat{x} = x \times \hat{I}, \hat{\eta} = \hat{T}(x, \hat{x}, \psi, \partial \psi, \partial \partial \psi, ...) \times \eta, \hat{I} = \hat{T}^{-1}. \quad (5.8) \]

C) the lifting of the conventional Hilbert space \( \mathcal{H} \) with states \( |\psi>, |\phi>, ... \) and inner product \( < \phi|\psi> \in C(c, +, \times) \) into the isohilbert space \( \hat{\mathcal{H}} \) with isostates \( |\hat{\psi}>, |\hat{\phi}>, ..., \) isoinner product and isorenormalization

\[ < \hat{\phi} \uparrow \hat{\psi} >= < \hat{\phi}| \times \hat{T} \times |\hat{\psi} > \times \hat{I} \in \hat{C}(\hat{c}, +, \hat{\times}), \quad (5.9) \]
\[ < \hat{\psi}| \times \hat{T} \times |\hat{\psi} > = I, \quad (5.10) \]

\[ 23 \]
D) the lifting of theory of linear operators in Hilbert space into the corresponding theory on $\hat{H}$, including the lifting of the familiar eigenvalue equations $H \times |\psi> = E_0 \times |\psi>$ into the isoschrödinger equation

$$H \hat{\times} |\psi> = H \times \hat{T} \times |\psi> = \hat{E} \hat{\times} |\psi> = E \times \hat{T} \times |\psi> \equiv E \times |\hat{\psi}>, \quad E \neq E_0, \quad (5.11)$$

indicating that the final numbers of the theory are conventional; the proof that the iso-eigenvalues of an isohermitean operators are real, etc.

E) the isodifferential calculus on $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ of Sect. 2,

$$\hat{dx}^\mu = \hat{I}^\mu_\nu \times dx^\nu, \quad \hat{dx}^\mu/\partial x^\nu = \hat{T}^\mu_\nu \times \partial x^\nu,$$

$$\hat{dx}^\mu/\partial x^\nu = \hat{\eta}^\mu_\nu \times \partial x^\nu, \quad \hat{dx}^\mu/\partial \hat{x}^\nu = \hat{\eta}^\mu_\nu; \quad (5.12)$$

$$\hat{x}^\mu/\partial \hat{x}^\nu = \delta^\mu_\nu,$$

$$\hat{x}^\mu/\partial x^\nu = \hat{\eta}^\mu_\nu,$$

$$\hat{\eta}^\mu_\nu; \quad (5.13)$$

F) the lifting of Heisenberg’s equation, the equation on the linear momentum and the fundamental commutation rules into the following isoheisenberg equation, isolinear momentum and fundamental isocommutation rules, first formulated in an axiomatically complete and correct form in memoir [3g]

$$i \times \hat{d}/\hat{dt} = [\hat{A}, \hat{H}] = \hat{\Delta} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A}, \quad (5.15)$$

$$\hat{p}_\mu \hat{\times} |\psi> = \hat{p}_\mu \times \hat{T} \times |\psi> = -i \times \hat{\partial}_\mu |\psi> = -i \times \hat{T}^\mu_\nu \times \partial x^\nu |\psi>, \quad (5.16)$$

$$[x_\mu, \hat{p}_\nu] \hat{\times} |\psi> = (\hat{x}_\mu \times \hat{T} \times \hat{p}_\mu \times \hat{\delta}_\nu \times \hat{T} \times \hat{x}_\mu \times \hat{T} \times |\psi> = i \times \hat{\eta}^\nu_\mu \times |\psi>, \quad (5.17)$$

G) the lifting of expectation values $< A >=< \psi | \times A | \psi > / < \psi | \psi >$ into the isoexpectation values

$$< \hat{A} >=< \hat{\psi} | \times \hat{T} \times A \times \hat{T} \times |\hat{\psi}> / < \hat{\psi} | \times \hat{T} \times |\hat{\psi}> ; \quad (5.18)$$

and the compatible liftings of the remaining aspects of RQM [3g].

H) The preceding formalism is completed with the isotopies of the naive or symplectic quantization, which establish the unique and unambiguous derivability of RHM from the corresponding classical isoanalitic mechanics (Sect. 2) (see [3f,4h] for brevity).

The following comments are in order. First, it is easy to see that the theory is highly nonlinear (hereon referred to as nonlinearity in the wavefunction), e.g., iso-eigenvalue equations (5.11) can be written explicitly

$$\hat{H} \hat{\times} |\psi> = \hat{H} \times \hat{T}(x, p, \psi, \partial \psi, ...) \times |\psi> = E \times |\hat{\psi}>, \quad (5.19)$$

Nevertheless, the theory does satisfy the conditions of linearity in isospace,
\[ \hat{A} \hat{\times} (a \hat{\times} \hat{x} + b \hat{\times} \hat{y}) = a \hat{\times} \hat{A} \hat{\times} \hat{x} + b \times \hat{A} \hat{\times} \hat{y}, \quad \hat{A} \in \xi, \quad a, b \in \hat{R}(\hat{n}, +, \hat{x}), \quad \hat{x}, \hat{y} \in \hat{M}(\hat{x}, \hat{\eta}, \hat{R}). \]

and it is called isolinear.

Also, all nonlinear theories can be identically reformulated in a isolinear form in which all nonlinear terms are embedded in the isotopic element, e.g.,

\[ H(x, p, \psi, \ldots) \times |\psi > \equiv H_0(x, p) \times \hat{T}(\psi, \ldots) \times |\psi > = H_0 \hat{x} |y >. \]  

This resolves the loss of the superposition principle which is inherent in all nonlinear theories, with consequential loss of consistent treatment of composite systems [3g,13].

Second, the theory verifies the conditions of locality in isospace over isofields whenever all nonlocal terms are embedded in the isounit, and it is then called isolocal. Also, all nonlocal theories can be identically rewritten in an isolocal form.

Finally, the theory readily reconstructs unitarity in isospace, and it is called isounitary. In fact, nonunitary transforms of the same "magnitude" \( \hat{I} \) (i.e., such that \( W \times W^\dagger = \hat{I} \)) can always be written [3g]

\[ W = \hat{W} \times \hat{T}^{\frac{1}{2}}, \]  

and be, therefore, turned into the isounitary transforms on \( \hat{H} \),

\[ \hat{W} \hat{x} \hat{W}^\dagger = \hat{U} \hat{x} \hat{U} = \hat{I}, \]  

Thus, all nonunitary theories can be identically rewritten in an isounitary form.

As an incidental comment, one should note that the admission of nonunitary transforms with "magnitude" \( U \times U^\dagger = \hat{I} \) different than \( \hat{I} \) would imply the transition to a different physical system. The transformation theory of RHM is therefore restricted for each system considered to the selected \( \hat{I} \), in exactly the same way as conventional RQM restricts the admitted transforms to those with conventional "magnitude" \( I \) only.

In view of the above properties, RHM is form–invariant and resolves the physical problematic aspects of other nonunitary theories indicated in Sect. 2. In fact, we have the following properties:

i) RHM possesses an invariant isounit. In fact, \( \hat{I} \) is numerically preserved under isounitary transforms and it is preserved in time,

\[ \hat{I} \rightarrow \hat{I}' = \hat{U} \hat{x} \hat{I} \hat{x} \hat{U}^\dagger \equiv \hat{I}, \]  

\[ i \times d\hat{I}/dt = [\hat{I}, \hat{H}] = \hat{I} \hat{x} H - H \hat{x} \hat{I} = H - H \equiv 0; \]  

with consequential unambiguous application of the theory to measurements;

ii) RHM preserves Hermiticity and observability at all times. In fact, the condition of Hermiticity on \( \hat{H} \) over \( \hat{C}(\hat{c}, +, \hat{x}) \) now reads
\[ \{ \langle \hat{\psi} | \times \hat{T} \times \hat{H} \rangle \times | \hat{\psi} > \} \times | \hat{\psi} > = \langle \hat{\psi} | \times \{ \hat{H} \times \hat{T} \times | \hat{\psi} > \}, \]  
(5.26)

\[ \hat{H} \hat{\dagger} \equiv \hat{T}^{-1} \times \hat{T} \times \hat{H} \times \hat{T}^{-1} \times \hat{T} \equiv \hat{H} \hat{\dagger} = \hat{H}, \]  
(5.27)

and, as such, it coincides with the Hermiticity on \( \mathcal{H} \) over \( \mathcal{C}(c, +\times) \). Therefore, all observables of RQM remain observables for RHM.

iii) RHM possesses invariant numerical predictions, physical laws and special functions. This is due to the invariance of the isounit and of the isoassociative product, \( \hat{U} \times (\hat{A} \times \hat{B}) \times \hat{U} \hat{\dagger} = \bar{A} \times \bar{B} \); the invariance of the fundamental isocommutation rules,

\[
\hat{U} \times (\hat{x}_\mu \times \hat{p}_\nu - \hat{p}_\nu \times \hat{x}_\mu) \times \hat{U} \hat{\dagger} | \hat{\psi} > = (\hat{x}_\mu \times \hat{p}_\nu - \hat{p}_\nu \times \hat{x}_\mu) \times | \hat{\psi} > = i \times \eta_{\mu\nu} \times | \hat{\psi} >, \\
| \bar{\psi} > = \hat{U} \times | \hat{\psi} >, \quad \bar{x}_\mu = \hat{U} \times \hat{x}_\mu \times \hat{U} \hat{\dagger}, \quad \bar{p}_{\mu\nu} = \hat{U} \times \hat{p}_{\mu\nu} \times \hat{U} \hat{\dagger}. 
\]  
(5.28)

and other aspects (see [3h,3g] for all details).

Even though evidently not unique, RHM is directly universal in the sense of admitting all infinitely possible, well behaved, nonlinear, integro–differential signature–preserving deformations of the Minkowski metric \( \eta = \hat{T} \times \eta \) (universality) directly in the frame of the observer (direct universality).

We also point out that RHM is a "completion" of RQM much along the celebrated argument by Einstein, Podolsky and Rosen [14a] and, for this reason, it is also called isotopic completion of RQM. In fact, Eq.s (5.11) constitute an explicit and concrete realization of the theory of "hidden variables" \( \lambda \) [14b] actually realized as "hidden operators", \( \lambda = \lambda(x, \dot{x}, \psi, \partial \psi, ...) = \hat{T} \). It should be indicated that the celebrated von Neumann theorem [14c] and Bell’s inequalities [14d] do not apply to RHM, trivially, because of its essential nonunitary structure (see [4h], App.4.C for details).

The mechanics is called "hadronic" because it was originally recommended for the study of the structure and interactions of hadrons [3b] with nonlinear, nonlocal and nonunitary internal effects, as well as for all interactions of particles with appreciable overlapping of the wavepackets, irrespective of whether the charges are point–like or not. The application to hadrons remains the main objective of RHM, as outlined in Web Site [7t], Page 19, Sect. V.

In this paper we shall study the particularization of RHM for OIG, thus opening intriguing possible relationships between what are today called "strong interactions" and gravitation planned for study elsewhere. At this point we merely note that all verifications of RHM [4h,4i,5d] may eventually result to be verifications of OIG because of the admission of the latter as a particular case.

The isodual relativistic hadronic mechanics (IRHM) is the antiautomorphic image of RHM characterized by the application of the isodual map to each and every quantity and operator, including:

a) the isodual isofields \( \hat{C}^d(\hat{c}^d, +, \hat{x}^d) \) and \( \hat{R}^d(\hat{r}^d, +, \hat{x}^d) \) with fundamental unit, elements and product.
\[ \hat{I}^d = -\hat{I} = (\hat{T}^d)^{-1} < 0, \quad \hat{c}^d = c \times \hat{I}^d, \quad \hat{n}^d = n \times \hat{I}^d, \quad \hat{a} \times \hat{b} = \hat{a} \times \hat{T}^d \times \hat{b}, \] (5.29)

b) the isodual isominkowski space \( \hat{M}^d(\hat{x}^d, \hat{n}^d, \hat{R}^d) \) of Sect. 3;

c) the isodual isohilbert space \( \hat{H}^d \) characterized by the following isodual isostates, isodual isoinner product and isodual isonormalization:

\[ |\hat{\psi}^d> = -|\hat{\psi}^d| = -<\hat{\psi}|, \quad d < \hat{\phi}| \times \hat{T}^d \times |\hat{\psi}^d > \times \hat{T}^d \in \hat{O}^d, \quad d < \hat{\phi}| \times \hat{T}^d \times |\hat{\psi}^d > = 1, \] (5.30)

d) the isodual isoassociative operator algebra characterized by the unit, elements and product:

\[ \hat{c}^d : \hat{I}^d, \quad \hat{X}^d = -\hat{X}, \quad \hat{X}^d \times \hat{X}^d = \hat{X} \times \hat{T}^d \hat{X}^d, \] (5.31)

e) the isodual isoeigenvalues equations:

\[ \hat{H}^d \times |\hat{\psi}^d > = \hat{E}^d \times |\hat{\psi}^d > = E^d \times |\hat{\psi}^d >, \] (5.32)

with the correct negative eigenvalues \( E^d = -E \); and

f) the isodual dynamical equations:

\[ i^d \times \hat{\partial}^d |\hat{\psi}^d > = \hat{\partial}^d \times \hat{d} |\hat{\psi}^d > = \hat{E}^d \times |\hat{\psi}^d >, \] (5.33)

\[ i^d \times \hat{\partial}^d \hat{A}^d ^/d^t \hat{d} = [\hat{A}, \hat{\partial}^d] = \hat{A}^d \times \hat{d} \hat{H}^d - \hat{H}^d \times \hat{d} \hat{A}^d, \] (5.34)

\[ \hat{p}^d \times \hat{\partial}^d \hat{d} |\hat{\psi}^d > = -i^d \times \hat{d} \partial^d \hat{d} |\hat{\psi}^d >, \] (5.35)

\[ [\hat{p}_i, \hat{r}_j]^d = -\delta^d_i, \quad [\hat{p}_i, \hat{p}_j]^d = [\hat{r}_i, \hat{r}_j]^d = 0, \] (5.36)

where we have used the isoselfduality of the imaginary unit (Sect. 3)

g) the isodual naive or symplectic isoquantization, which establishes the unique derivability of the preceding formalism from the isodual isoanalytic mechanics of Sect. 3 (see [3f,3g,4h] for brevity).

The proof of the equivalence of isoduality and charge conjugation is presented in ref. [5a]. Note that the above isodual relativistic hadronic mechanics admits as a particular case the isodual relativistic quantum mechanics. The mechanics emerging from our study and their unique interconnecting maps can therefore be summarized as follows:

The most convincing evidence in favor of the isodual representation of antiparticles can be seen in the structure of the conventional Dirac equation (see Sect. 7). Here we mention that the axiomatic consistency of both, the isotopies and isodualities, can be also established via the following new invariance laws of the conventional inner product of Hilbert spaces presented apparently for the first time in Ref. [3g]:

27
1) The *isoselfscalarity*, expressing the invariance under the charge of the unit, here expressed for isounit independent from the integration variables

\[ <\psi| \times |\psi> = <\psi| \times \hat{T} \times |\psi> \times \hat{T}^{-1} = <\psi| \Dagger \psi> \in \hat{C}, \quad (5.37) \]

2) the *isoselfduality*, which expressed the invariance of the same inner product under the antiautomorphic isodual map

\[ <\psi| \times \hat{T} \times |\psi> \equiv \hat{d} <\psi| \times \hat{T}^d \times |\psi>^d \times \hat{d}, \quad (5.38) \]

Evidently invariance (5.37) expresses the preservation of the abstract quantum axioms under changes of the basic unit (isotopy), thus establishing the transition from RQM to RHM. Invariance (5.37) establishes that the same laws for particles are also valid for antiparticles under their antiautomorphic interpretation (isoduality).

In summary, the isominkowskian formalism of Sect. 2 does indeed admit a unique and unambiguous operator counterpart which *cannot be* "quantum", but which is nevertheless characterized by the abstract quantum axioms only in a more general realization.

The isodual isominkowskian formalism of Sect. 3 does indeed admit a unique operator counterpart which is the antiautomorphic image of the preceding one, thus being particularly suited for the representation of antiparticles. Also, the isodual operator formalism originates from a *new quantization specifically built for the quantization of antimatter into antiparticles* [3g]. This appears to resolve the long standing impasse of the theoretical representation of antimatter caused by the uniqueness of quantization *vis-a-vis* the duality of matter and antimatter, and opens up a new horizon of possibilities.

It should be stressed again that we have indicated in this section the axiomatic consistency and plausibility of the operator isotopic treatment of particles (generally intended
for particles in interior conditions) and the isodual isotopic representation of antiparticles (generally intended for antiparticles in interior conditions).

Preliminary studies indicate encouraging possibilities of experimental verifications under the conditions in which the formulations are applicable, although such physical validity can only be established as a result of collegian and predictably protracted investigations.

6 The universal isopoincaré symmetry and its isodual

Any appraisal of OIG requires at least a minimal knowledge of the operator form of its universal symmetry, the isopoincaré symmetry [4], specifically realized for gravity, which is studied in this section following the achievement of sufficient maturity in isosymmetries by Kadeisvili in memoir [7a].

The isopoincaré symmetry (also called in the literature the Santilli’s isopoincaré symmetry [4,5]) is the universal symmetry of isoline element (2.6). Therefore, the basic invariant quantity is not (length)$^2$, but the broader structure (length)$^2$×(unit)$^2$.

The isopoincaré symmetry can be constructed via the step–by–step application of the isotopies of enveloping associative algebras, Lie algebras, Lie groups, transformation and representation theory, etc called Lie–Santilli isotheory [6–7] and consists in the reconstruction of the conventional symmetry $P(3.1)$ for the generalized unit $\hat{I} = \hat{T}^{-1}$. Since $\hat{I} > 0$, one can see form the inception that $\hat{P}(3.1) \approx P(3.1)$.

Evidently we cannot review here the rigorous construction of the isosymmetry $\hat{P}(3.1)$ and we have to limit ourselves for brevity to identify its essential aspects. The lifting $P(3.1) \to \hat{P}(3.1)$ is constructed by preserving the conventional generators of the Poincaré symmetry

$$X = \{X_k\} = \{M_{\mu\nu}, x_\alpha\}, \quad M_{\mu\nu} = x_\mu \times p_\nu - x_\nu \times p_\mu, \quad k = 1, 2, ..., 10, \quad \mu, \nu = 1, 2, 3, 4,$$

and the conventional parameters

$$w = w_k = (\theta, v), a \in R,$$

although they are now formulated in isospaces over isofields, and by submitting to an isotopies the operations constructed on them.

In fact the above quantities represent physical characteristics such as energy, linear momentum, angles, velocities, etc., which are not affected by short range interactions, the latter being represented by generalized operations among conventional physical quantities.

The connected component of the isopoincaré symmetry is given by $\hat{P}_0(3.1) = \hat{S}\hat{O}(3.1) \times \hat{T}(3.1)$, where $\hat{S}\hat{O}(3.1)$ is the connected isolorentz group first introduced in [4a] and $\hat{T}(3.1)$ is the group of isotranslations [3d], and it is characterized by the isotransforms on $\hat{M}(\hat{x}, \hat{y}, \hat{R})$,

$$x' = \hat{A}(\hat{w}) \hat{x} = \hat{A}(\hat{w}) \times \hat{T} \times x = \hat{A}(w) \times x, \quad \hat{A} = \hat{A} \times \hat{I},$$

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where the first form is the mathematically correct one, the last form being used for computational simplicity.

The above isotransforms can be expressed via the isoeponention in \( \hat{\xi} \)

\[
\hat{A}(\hat{w}) = e^{i \times X \times w} = \hat{I} + (i \times X \times w)/1! + (i \times X \times w)\hat{x}(i \times X \times w)/2! + \ldots
\]

characterized by the isotopic Poincaré–Birkhoff–Witt Theorem \([3a,3d]\) and reducible to the conventional exponentiation for computational simplicity

\[
\hat{A}(\hat{w}) = e^{i \times X \times w} = \{e^{i \times X \times T \times w}\} \times \hat{I} = \tilde{A}(w) \times \hat{I}.
\]

The (connected component of the) isopoincaré group can therefore be written as (or defined by) \([4]\)

\[
\hat{P}_0(3.1) : \hat{A}(\hat{w}) = \Pi_k e^{i \times X \times \hat{w}} = (\Pi_k e^{i \times X \times \hat{T} \times w})\hat{I} = \tilde{A}(w) \times \hat{I},
\]

(6.7)

The preservation of the original dimension is ensured by the isotopic Baker–Campbell–Hausdorff Theorem \([3a,3d]\). It is easy to see that structure (6.7) forms a connected Lie–Santilli transformation isogroup with isogroup laws

\[
\hat{A}(\hat{w}) \hat{\times} \hat{A}(\hat{w'}) = \hat{A}(\hat{w'}) \hat{\times} \hat{A}(\hat{w}) = \hat{A}(\hat{w} + \hat{w'}), \quad \hat{A}(\hat{w}) \hat{\times} \hat{A}(-\hat{w}) = \hat{A}(0) = \hat{I} = \hat{T}^{-1}.
\]

(6.8)

Note that the use of the original Poincaré transform \( x' = A(w) \times x \) would now violate linearity in isospace, besides not yielding the desired symmetry of isoseparation (2.6).

The isotopy of the discrete transforms is elementary \([4e]\), and reducible to the forms

\[
\hat{\pi} \hat{\times} x = \pi \times x = (-r, x^4), \quad \hat{\tau} \hat{\times} x = \tau \times x = (r, -x^4),
\]

where \( \hat{\pi} = \pi \times \hat{I} \), and \( \pi, \tau \) are the conventional inversion operators.

To identify the isoalgebra \( \hat{p}_0(3.1) \) of \( \hat{P}_0(3.1) \) we use the isodifferential calculus \([3f]\) on \( \hat{M} \) outlined earlier which yields the iso-commutation rules \([4]\)

\[
[\hat{M}_{\mu \nu}, \hat{M}_{\alpha \beta}] = i \times (\hat{\eta}_{\alpha \nu} \times \hat{M}_{\mu \beta} - \hat{\eta}_{\alpha \beta} \times \hat{M}_{\mu \nu} - \hat{\eta}_{\mu \beta} \times \hat{M}_{\alpha \nu} + \hat{\eta}_{\mu \nu} \times \hat{M}_{\alpha \beta}), \quad (6.10)
\]

\[
[\hat{M}_{\mu \nu}, \hat{p}_{\alpha}] = i \times (\hat{\eta}_{\mu \alpha} \hat{p}_{\nu} - \hat{\eta}_{\nu \alpha} \hat{p}_{\mu}), \quad (6.11)
\]

\[
[\hat{p}_{\alpha}, \hat{p}_{\beta}] = 0, \quad (6.12)
\]

where \([A, B]\) is the Lie–Santilli product which satisfies the Lie axioms in isospace, as one can verify.

The isocasimir invariants are given by

\[
C^{(0)} = \hat{I}(x, \dot{x}, \psi, \partial \psi, \ldots) = \hat{T}^{-1}.
\]
The local isomorphism \( \hat{p}_0(3.1) \approx p_0(3.1) \) is ensured by the positive–definiteness of \( \hat{T} \). Alternatively, the use of the generators in the form \( M^\mu_{\nu} = x^\mu \ast p_\nu - x^\nu \ast p_\mu \) yields the conventional structure constants under a generalized Lie product, as one can verify. The above local isomorphism is sufficient, per sé, to guarantee the axiomatic consistency of RHM.

The space components \( S\hat{O}(3) \), called isorotations and first introduced in [4b], can be computed from isoexportations (6.7) and the space components \( \hat{T}_{kk} \) of the isotopic element in diagonal form, \( \hat{T} = \text{diag.}(T_{\mu\nu}) \), yielding the explicit form in the \( (x,y) \)-plane

\[
x' = x \times \cos(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3) - \hat{y} \times \hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \sin(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3),
\]

\[
y' = \hat{x} \times \hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \sim (\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3) + \hat{y} \cos(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3),
\]

(see [3h] for general isorotations in all three Euler angles).

As one can verify, isotransforms (6.14) leave invariant all infinitely possible ellipsoidal deformations of the sphere \( x \times x + y \times y + z \times z = r \) in the Euclidean space \( E(r, \delta, R) \), \( r = x, y, z, \delta = \text{diag.}(1, 1, 1) \),

\[
r^t \times \delta \times r = x \times \hat{T}_{11} \times x + y \times \hat{T}_{22} \times y + z \times \hat{T}_{33} \times z = r.
\]

In the isoeuclidean spaces

\[
\hat{E}(\hat{r}, \hat{\delta}, \hat{R}), \hat{r} = \{\hat{r}^k\}, \hat{\delta} = \hat{T}_s \times \delta, \hat{T}_s = \text{diag.}(\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}), \hat{I}_s = \hat{T}^{-1}_s,
\]

ellipsoid (6.13) become perfect spheres \( r^\hat{2} = (r^t \times \hat{\delta} \times r) \times \hat{I} \) called isospheres [4g].

In fact, the lifting of the semi axes \( 1_k \rightarrow \hat{T}_{kk} \) while the related units are lifted of the inverse amounts \( 1_k \rightarrow \hat{T}_{kk}^{-1} \), preserves the perfect sphericity. This isohericity is the geometric origin of the isomorphism \( \hat{O}(3) \approx O(3) \), as well as of the preservation of the rotational invariance for the ellipsoidal deformations of sphere [4b].

The connected isolorentz symmetry \( SO(3,1) \) (also called in the literature Santilli’s isolorentz symmetry [6,7], is given by the superposition of the isorotations and the isoboosts first introduced in [4a] which can be written in the (3,4)-plane

\[
x^1 t = x^1, \quad x^2 t = x^2,
\]

\[
x^3 t = x^3 \times \sinh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) - x^4 \times \hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times \cosh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) = \hat{\gamma} \times (x^3 - \hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times \beta \times x^4),
\]

\[
x^4 t = -x^3 \times \hat{T}_{33}^{\frac{1}{2}} \times c_0^{-1} \times \hat{T}_{44}^{\frac{1}{2}} \times \sinh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) + x^4 \times \cosh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) = \hat{\gamma} \times (x^4 - \hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times \beta \times x^3),
\]

(6.17)
where

\[
\hat{\beta} = (v_k \times \hat{T}_{kk} \times v_k/c_0 \times \hat{T}_{44} \times c_0)^{\frac{1}{2}},
\]
(6.18)

\[
\hat{\gamma} = (1 - \hat{\beta}^2)^{-\frac{1}{2}}.
\]
(6.19)

Note that the above isotransforms are nonlinear in \(x, \dot{x}, \psi, \partial \psi, \ldots\), precisely as desired, and are formally similar to the Lorentz transforms, as expected from their isotopic character. This also confirms the local isomorphism \(SO(3.1) \approx SO(3.1)\) [4].

The Lorentz–Santilli isotransforms characterize the so-called isolight cone [4], i.e., the perfect cone in isospace \(\hat{M}(\hat{x}, \hat{\eta}, \hat{R})\). In a way similar to the isosphere, we have the deformation of the light cone axes \(1_\mu \rightarrow \hat{T}_\mu\mu\) while the corresponding units are deformed of the inverse amount \(1_\mu \rightarrow \hat{T}_\mu\mu^{-1}\), thus preserving the original perfect cone character. Such a preservation is then the geometric foundation of the local isomorphism \(\hat{SO}(3.1) \approx SO(3.1)\).

The isotopy of the light cone is so strong that even the characteristic angle of the cone remains the conventional one, i.e., the maximal causal speed in isospace \(\hat{M}(\hat{x}, \hat{\eta}, \hat{R})\) remains the speed of light \(c_0\) in vacuum [4] (it should be noted that the proof of this property requires, for consistency, the use of the isotrigonometric and isohyperbolic functions we cannot review here for brevity).

The isotranslations in the coordinates can be written [4d]

\[
x' = (e^{i\pi x\alpha}) \hat{x} = x + a \times A(x, \ldots), \quad p' = (\hat{e}^p x\alpha) \hat{p} = p,
\]
(6.20)

\[
A_\mu = \hat{T}_\mu\mu^{1/2} + a^{\alpha}[\hat{T}_{\mu\mu}^{1/2}, \hat{p}_\alpha]1! + \ldots
\]
(6.21)

It is generally believed that the conventional, ten-parameter, Poincaré symmetry is the broadest possible linear symmetry of the conventional separation on Minkowski spaces \(M(x, \eta, R)\)

\[
(x - y)^2 = [(x^\mu - y^\mu) \times \eta_{\mu\nu} \times (x^{\nu} - y^{\nu})] \times I \in R(n, +, \times).
\]
(6.22)

The isotopies have identified a new symmetry, called isoselfscalarity, first identified in memoir [3g], which which is given by the lifting of the trivial unit \(I = \text{diag}(1, 1, 1, 1)\) with a new parameter \(n\) independent from the integration variables, under which we have the new invariance

\[
I \rightarrow \hat{I} = n^2 \times I, \quad \eta \rightarrow \hat{\eta} = n^{-2} \times \eta, \quad n \neq 0,
\]

\[
(x - y)^2 = [(x^\mu - y^\mu) \times \eta_{\mu\nu} \times (x^{\nu} - y^{\nu})] \times I \equiv
\]

\[
\equiv [(x^\mu - y^\mu) \times (n^{-2} \times \eta_{\mu\nu}) \times (x^{\nu} - y^{\nu})] \times (n^2 \times I)
\]

\[
= [(x^\mu - y^\mu) \times \hat{\eta}_{\mu\nu} \times (x^{\nu} - y^{\nu})] \times \hat{I} = (x - y)^2.
\]
(6.23)

As a result, the most general possible invariance of the Minkowskian line element for positive-definite units has eleven, rather than ten dimensions.
Note that the invariant for the first form of the line element is the conventional Poincaré symmetry, while the invariance of the latter form is a bona–fide isopoincaré symmetry because the isotopic element \( \hat{T} = n^2 \) enters into the arguments of the isorotation (6.14) and isoboosts (6.17). As such, the above symmetry is nontrivial.

A second, hitherto unknown invariance is characterized by the isodual map [5d]

\[
I \rightarrow I^d = -I, \quad \eta \rightarrow \eta^d = -\eta, \quad x \rightarrow x^d = -x,
\]

\[
(x - y)^2 = [(x^\mu - y^\mu) \times \eta_{\mu\nu} \times (x^\nu - y^\nu)] \times I \equiv \\
[(x^\mu - y^\mu) \times (-\eta_{\mu\nu}) \times (x^\nu - y^\nu)] \times (-I) = \\
= [(x^\mu - y^\mu)^d \times \eta^d_{\mu\nu} \times (x^\nu - y^\nu)^d] \times I^d \equiv (x - y)^{d2d}.
\] (6.24)

called by this author isoselfduality, which is at the foundation of the isodual representation of antimatter [5].

The above invariance evidently assures the plausibility of the isodual treatment of antimatter also at the geometric level, because the same Minkowskian invariant holds for both conventional and isodual systems.

Isoselfduality (6.24) establishes the existence and applicability of the isodual Poincaré symmetry \( \hat{P}^d(3.1) [4d] \), which can be easily constructed from the isodual rules of the preceding section, and it is hereon assumed as known.

We here define as the restricted isopoincaré transforms that constituted by isorotations, isolorentz boosts, isotranslations, isoinversions and isoselfscalar transformations when all parameters are constants, otherwise we have the general isopoincaré transforms, with corresponding definitions under isodualities.

The most salient difference between the special and general isotransforms is that the former preserve the inertial character of the frames, while the latter identify a broader class of noninertial frames.

Note that isodual parameters are independent of the conventional ones. As a result, the general invariance of isoseparation (??) has 22–dimensions with structure

\[
\hat{S}_{\text{dot}} = \hat{P}(3.1)_{i\bar{j}} \times \hat{P}^d(3.1)_{\bar{i}d}
\]

and the same result holds for the symmetry of the conventional Minkowskian separation as a particular case.

The preceding analysis establishes the following property [4,7a]

**Theorem 2:** The general isopoincaré group is the universal isolinear invariance of all infinitely possible, well behaved isoseparations (2.4) on the isominkowski space over isoreal fields with a common isounit.
The verification of the above theorem is trivial and can be done by just plotting the isotransforms in isoinvariant. Note that there is nothing to compute, because the theory provides the solution for the general invariance of the isoseparation for all infinitely possible isometrics in the diagonal form \( \tilde{\eta} = \tilde{T} \times \eta = \text{diag}(T_{\mu\mu} \times \eta_{\mu\mu}) \). One merely plots the isotopic elements \( T_{\mu\mu} \) in the isotransforms. The invariance of the isoseparation is then guaranteed by the isotopic methods.

Note the need for consistency that the generalized unit must be the same for the isospace and for the isofield. This is not the case in conventional treatments where the unit of the space is the unit matrix \( I = \text{Diag.}(1,1,...) \) and the unit of the field is the number +1. Nevertheless, the latter treatment can be easily reformulated for the same unit I.

Note also from the above studies that the abstract identity of the Poincaré and isopoincaré symmetries implies that the special and isospecial relativity also coincide at the abstract level (and the same occurrence holds under isoduality). However, the special relativity has only one formulation, the conventional one. On the contrary, all isotopic structures, thus including the isospecial relativity, have two different formulations, one in isospace and the other in their projection in the conventional space.

All differences between the special and isospecial relativity solely occur when the latter is projected in the space–time of the former, because the isospecial relativity in isospace preserves all features of the conventional relativity, including the perfect light cone, the maximal causal speed \( c_0 \), etc. [4].

It should be also noted that the general isopoincaré symmetry does not restrict the value of \( n \) (except for the conditions \( n > 0 \)). Thus, the isospecial relativity predicts arbitrary causal speeds of light within homogeneous and isotropic media, because \( c \) can be smaller, equal or bigger than \( c_0 \).

The case of light speeds smaller than \( c_0 \) is established in homogeneous and isotopic physical media such as water. Speeds bigger than \( c_0 \) have been identified in a number of cases, such as photons tunneling tests [15a,15d], expulsion of matter in astrophysical explosions [15c,15d,15e], solutions of ordinary relativistic wave equations [15f] and other cases.

The isominkowskian space, the isopoincaré symmetry and the isospecial relativity restore the validity of the abstract axioms of the special relativity for arbitrary speeds of light, whether smaller or bigger than \( c_0 \), because they are all projected in isospace into the unique and universal speed \( c_0 \), thus rendering the special relativity truly ”universal”.

It is remarkable that the two novel invariances (5.37)–(5.38) and (6.23)–(6.24) have remained undetected throughout this century, to our best knowledge. This should not be surprising because their detection required the prior discovery of new numbers, the isonumbers with arbitrary positive units for invariances (5.37) and (6.23) and the isodual numbers with arbitrary negative units for invariances (5.38) and (6.24).
7 Operator isominkowskian gravity for matter and its isodual for antimatter

We are now sufficiently equipped to identify the foundations of OIG and appraise its plausibility on comparative grounds with QG [12]. A number of applications and verifications are presented in future works [4i]. OIG was first presented in ref. [8a] and preliminary studied in [8b,8c]. The basic dynamical equations of OIG in their axiomatically correct form are presented in this paper for the first time following the achievement of mathematical maturity in memoir [3f] and axiomatic maturity in memoir [3g].

We should indicate from the outset that the expectation of the existence of a "quantum" description of gravity is disproved by our studies. This is established by Theorem 1 which identifies the classical noncanonical structure of gravity with a necessary nonunitary counterpart at the operator level, under which no "quantum" law is expected to apply identically.

In fact, QG requires notorious departures from a true "quantum mechanical" setting and the isotopic reformulation of gravity does not escape from the same occurrence. The best that can be done on ground of our current knowledge is a formulation of gravity via the abstract axioms of quantum mechanics, only realized in a more general form.

Equivalently we can say that the main idea of OIG is to turn the notorious "nonunitary" structure of the operator description of gravity on a conventional Hilbert space, into an identical "isounitary" formulation on our isohilbert space, thus regaining in this way the abstract axioms of RQM.

Alternatively, our studies indicate that no formulation of gravity appears to be possible via the use of the quantum axioms in their simplest possible realization, that with the unit +1. On the contrary, if more general realizations are admitted, then realistic possibilities for basic advances emerge.

From the analysis of the preceding sections it is evident that RHM, with underlying isominkowskian geometry, isopoincaré symmetry and isospecial relativity, provides an operator characterization of gravity under the sole condition of restricting the isounit and isotopic element to the gravitational values

\[
\hat{T}_{\text{gr}}(x)_{\mu}^{\nu} = \eta_{\mu\alpha} \times g^{\alpha\nu}(x),
\]

\[
\hat{I}_{\text{gr}}(x) = (\hat{I}_{\text{gr}}(x)_\beta^\alpha) = (|\hat{T}_{\text{gr}}(x)_{\alpha}^{\beta}|^{-1})_{\nu}^{\nu}, \quad \eta \in M, \quad g \in \mathbb{R}.
\]

called gravitational isotopic element and gravitational isounit, respectively.

Conventional RQM represents systems via the assignment of the Hamiltonian \(\hat{H}\) and the tacit assumption of the simplest possible unit \(\hat{I}\). OIG requires the assignment of two quantities, the conventional Hamiltonian \(\hat{H}\) which represents conventional interactions, and the selection of the isounit \(\hat{I} = \hat{I}_{\text{gr}}(x) = [\hat{T}_{\text{gr}}(x)]^{-1}\) which represents the essential part of curvature, the isotopic element in our isominkowskian decomposition of the Riemannian
metric $g(x) = \hat{T}_{gr} \times \eta$. We assume the reader is now familiar with the mathematical structure of OIG which requires all products to be isotopic with isotopic element $\hat{T}_{gr}(x)$.

Recall that the fundamental notion of RQM is the Poincaré symmetry $P(3.1)$ in operator realization. By the same token, the ultimate and most fundamental notion of the operator theory herein submitted from which all properties and applications can be uniquely and unambiguously derived, is the gravitational Poincaré–Santilli isosymmetry in operator form $\hat{P}_{gr}(3.1)$, i.e., the isosymmetry of the preceding section constructed with respect to the gravitational isounit $\hat{I}_{gr}(x)$.

The main characteristics of the emerging operator gravity are the following:

**Property I:** OIG is based on the embedding of gravity in the isotopic lifting of Planck’s constant.

Recall that the Plank constant is the basic unit of RQM. The fundamental isotopy of OIG is then precisely that of the latter, and we shall write

$$\hat{h} = I \rightarrow \hat{I}_{gr}(r),$$

(7.3)

where $\hat{I}_{gr}(r)$ is the $3 \times 3$–dimensional space component of the $4 \times 4$–dimensional gravitational isounit $\hat{I}_{gr}(x)$.

The isotopic character of the lifting is readily established by the fact that $\hat{I}_{gr}(r)$ preserves all axiomatic properties of $\hat{h}$, e.g.,

$$\hat{I}_{gr}^n = \hat{I}_{gr} \times \hat{I}_{gr} \times \ldots \times \hat{I}_{gr} \equiv \hat{I}_{gr}, \quad \hat{I}_{gr}^r \equiv \hat{I}_{gr}, \quad \hat{I}_{gr}^s \equiv \hat{I}_{gr},$$

(7.4)

The fundamental dynamical equations of OIG are then based on lifting (5.2). Note that the conventional Schrödinger and Heisenberg’s equations can be written in the form

$$i \times \hat{\partial}_{\hat{t}}|\psi> = \hat{H}(\hat{t}, \hat{r}, \hat{p}) \times \hat{h}^{-1} \times |\psi> = E \times \hat{h}^{-1} \times |\psi>, \quad (7.5)$$

$$i \times d\hat{A}/d\hat{t} = \hat{A} \times \hat{h}^{-1} \times \hat{H} - \hat{H} \times \hat{h}^{-1} \times \hat{A}, \quad (7.6)$$

$$p_k \times \hat{h}^{-1} \times |\psi> = -i \times \hat{\partial}_k|\psi>, \quad (7.7)$$

$$p_i \times \hat{r}^j - \hat{r}^j \times \hat{h}^{-1} \times \hat{p}_i = -\delta^j_i \quad (7.8)$$

$$p_i \times \hat{h}^{-1} \times \hat{p}_j - p_i \times \hat{h}^{-1} \times \hat{p}_i \equiv \hat{r}^i \times \hat{h}^{-1} \times \hat{r}^j - \hat{r}^j \times \hat{h}^{-1} \times \hat{r}^i \equiv 0. \quad (7.9)$$

Then, the fundamental non–isorelativistic dynamical equations of OIG are given by

$$i \times \hat{\partial}_{\hat{t}}|\psi> = \hat{H}(\hat{t}, \hat{r}, \hat{p}) \times \hat{h} \times \hat{T}_{gr}(r) \times |\psi> = \hat{E} \times \hat{T}_{ho}(r) \times |\psi>, \quad (7.10)$$

$$i \times d\hat{A}/d\hat{t} = \hat{A} \times \hat{T}_{gr}(r) \times \hat{H} - \hat{H} \times \hat{T}_{gr}(r) \times \hat{A}, \quad (7.11)$$

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\[ p_k \times \hat{T}_{\text{gr}}(r) \times |\tilde{\psi}\rangle = -i \times \hat{\delta}_k |\tilde{\psi}\rangle, \quad (7.12) \]

\[ \hat{p}_i \times \hat{T}_{\text{gr}}(r) \times r^j - r^j \times \hat{T}_{\text{gr}}(r) \times p_i = -i \times \delta^i_j, \quad (7.13) \]

\[ p_i \times \hat{T}_{\text{gr}}(r) \times p_j - p_j \times \hat{T}_{\text{gr}}(r) \times p_i \equiv r^i \times \hat{T}_{\text{gr}}(r) \times r^j - r^j \times \hat{T}_{\text{gr}}(r) \times r^i \equiv 0, \quad (7.14) \]

where the isounit of the time isoderivatives is evidently the fourth component of \( \hat{I}_{\text{gr}}(x) \), under the proviso that the totality of quantities, operations and special functions and transforms are of isotopic type.

To be more specific on this fundamental point, the appraisal of OIG with conventional QG notions, such as the magnitude of the angular momentum \( J^2 = J_k \times J^k \) leads to a host of inconsistencies which are generally not detected by nonexpert in the field (e.g., violation of isolinearity). Similarly, data elaborations via ordinary trigonometric functions or with the familiar Dirac’s delta function have no meaning of any nature for OIG, because said conventional notions cannot be even defined in isospaces over isofields.

With a clear understanding to above requirements, we note that the gravitational isounit is indeed the fundamental invariant of OIG because it is numerically invariant under the transformation theory

\[ \hat{I}_{\text{gr}} \rightarrow \hat{I}'_{\text{gr}} = \hat{U} \hat{I}_{\text{gr}} \hat{U}^\dagger \equiv \hat{I}_{\text{gr}}, \quad (7.15) \]

and it is preserved under the time evolution

\[ i \times \hat{dI}_{\text{gr}} / dt = [\hat{I}_{\text{gr}}, \hat{H}] = \hat{H} - \hat{H} \equiv 0. \quad (7.16) \]

Moreover, from isorule \((??)\), the isoexpectation values of the space components of the gravitational isounit reproduce Plank’s constant \( \hbar = 1 \) identically,

\[ \langle \hat{I}_{\text{gr}} \rangle = \langle \hat{T}_{\text{gr}} \times \hat{T}_{\text{gr}}^{-1} \hat{T}_{\text{gr}} \times |\rangle \times \hat{T}_{\text{gr}} \times |\rangle \equiv \hbar = 1. = \hbar = 1. \quad (7.17) \]

This identifies the “hidden” character of OIG in conventional RQM and its character of being a “completion” of RQM much along the E–P–R argument \([14] \). After all, gravity is embedded in the unit of RQM. As such, the inclusion of gravity in RQM is so natural to creep in unnoticed.

Also, property \((7.17)\) establishes that one should not expect OIG to yield deviations from established quantum mechanical laws. This occurrence is made clearer by the fact that the uncertainties of the center–of–mass trajectories of a systems of particles obeying OIG are conventional. In fact, from isocommutation rules \((??)\) we have \( \hbar = 1 \) \([3g]\)

\[ \Delta r \times \Delta p \geq \frac{1}{2} <[\hat{r}, \hat{p}] > = \frac{1}{2}. \quad (7.18) \]

The preservation of the fundamental physical laws by our ”axioms– preserving” isotopies should be compared with the departures from the same laws implied by QG as well as by the
"axiom-violating" deformations, and illustrates again our insistence in avoiding the term "deformations" whenever dealing with "isotopies" (Sect. 2).

The presence of gravitational IN OUR OIG is established by numerous aspects, all verifying conventional quantum laws, such as deviations from conventional quantum eigenvalues, or the resolution of the paradox of quantum mechanics at gravitational singularities by the introduction of gravitational isoinvariant (or the resolution of the paradox of quantum mechanics at gravitational singularities by the introduction of gravitational isoinvariant.

The isorelativistic equations of OIG are uniquely identified by the isopoincaré symmetry via its isocasimir invariants (6.13) and related isorepresentation theory we cannot possibly study here for brevity (see the study of ref. [4h,7a]). The first difference is that now the gravitational isounit is given by the full $4 \times 4$–dimensional structure $\hat{I}_{gr}(x)$. The fundamental gravitational isocommutation rules are given by:

$$p_\mu \hat{x} |\hat{\psi} > = p_\mu \times \hat{T}_{gr} \times |\hat{\psi} > = -i \times \hat{\partial}_\mu |\hat{\psi} > = -i \hat{T}_{gr\mu} \partial_\nu |\hat{\psi} >,$$  \hspace{1cm} (7.19)

$$[x^\mu, p_\nu] \hat{x} |\hat{\psi} > = [\hat{x}^\mu \times \hat{T}_{gr}(x) \times \hat{p}_\nu - \hat{p}_\nu \times \hat{T}_{gr}(x) \times \hat{x}^\mu] \times \hat{T}_{gr}(x) \times |\hat{\psi} > = i \times \delta^\mu_\nu \times |\hat{\psi} >,$$ \hspace{1cm} (7.20)

The second–order isorelativistic equation of QIG is then given by the realization of the isoinvariant (??) plus the conventional minimal coupling rule to an external electromagnetic field with four-potential $\hat{A}_\mu(x)$

$$\{[\hat{p}_\mu + i \times e \times \hat{A}_\mu(x)] \hat{x} [\hat{p}^\mu + i \times e \times \hat{A}^\mu(x)] + \hat{m}^2 \} \hat{x} |\hat{\psi} > =$$  \hspace{1cm} (7.21)

$$\hat{r}_{gr}^\mu(x) \times [\hat{p}_\mu + i \times e \times \hat{A}_\mu(x)] \times [\hat{p}_\mu + i \times e \times \hat{A}_\mu(x)] +$$ \hspace{1cm} (7.22)

$$+ (m \times m) \times \hat{I}_{gr}(x) \times \hat{T}_{gr}(x) \times |\hat{\psi} > =$$  \hspace{1cm} (7.23)

$$= \{ \hat{I}_{gr}(x)^\mu_{\alpha} \times \hat{\eta}^{\alpha\nu} [-i \times \hat{T}_{gr}(x)^\nu_\gamma \times \partial_\gamma + i \times e \times \hat{A}_\mu(x)] \times$$ \hspace{1cm} (7.24)

$$\times [ -i \times \hat{T}_{gr}(x)^\delta_\delta \times \partial_\delta + i e \times \hat{A}_\nu(x)] + m^2 \} \times |\hat{\psi} > =$$  \hspace{1cm} (7.25)

$$\{ \eta^{\alpha\nu} \times [-i \times \partial_\rho + i e \times \hat{I}_{gr}(x)^\nu_\rho \times \hat{A}_\mu(x)] \times$$ \hspace{1cm} (7.26)

$$\times [-i \times \partial_\sigma + i e \hat{I}_{gr}(x)^\sigma_\nu \times \hat{A}_\sigma(x)] + m^2 \} \times |\hat{\psi} > = 0,$$ \hspace{1cm} (7.27)

where quantities with "hats" are computed in isospace and those without are conventional.

As one can see, the projection of dynamical equation (7.27) in the conventional space–time can be expressed via the conventional Klein–Gordon equation plus the "isorenormalization" of the electromagnetic potential $\hat{A}_\mu$ via the multiplication by the gravitational isounit $\hat{I}_{gr}(x)$. Conceivable physical applications of this new setting will be studied elsewhere.

The "isolinearization" of the second–order equation is done via a simple isotopy of the conventional case (see [4h] for details). In its simplest possible realization for a diagonal isounit, such linearization leads to the following gravitational isodirac equation

$$(\hat{\gamma}^\mu \hat{x} \hat{p}_\mu + i \times \hat{m}) \hat{x} |\hat{\psi} > = [\eta_{\mu\nu}(x) \times \hat{\gamma}^\mu(x) \times \hat{T}_{gr}(x) \times \hat{p}^\nu - i \times m \times \hat{I}_{gr}(x) \times \hat{T}_{gr}(x) \times |\hat{\psi} > = 0,$$ \hspace{1cm} (7.28)
\[
\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = \hat{\gamma}^\mu \times \hat{T}_{gr}(x) \times \hat{\gamma}^\nu + \hat{\gamma}^\nu \times \hat{T}_{gr}(x) \times \hat{\gamma}^\mu = 2 \times \hat{\eta}^\mu{}_{\nu} \equiv 2g^\mu{}_{\nu}, \tag{7.29}
\]

\[
\hat{\gamma}^k = [\hat{T}_{kk}(x)]^{1/2} \times \gamma^k \times \hat{I}_{gr}(x) = [\hat{T}_{kk}(x)]^{1/2} \times \left( \begin{array}{cc} 0 & \sigma^k \\ \sigma^{dk} & 0 \end{array} \right) \times \hat{I}_{gr}(x) \tag{7.30}
\]

\[
\hat{\gamma}^4 = [\hat{T}_{44}(x)]^{1/2} \times \gamma^4 \times \hat{I}_{gr}(x) = [\hat{T}_{kk}(x)]^{1/2} \times \left( \begin{array}{cc} I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2} \end{array} \right) \times \hat{I}_{gr}(x) \tag{7.31}
\]

with a simple extension with the minimal coupling rule, where the \(\gamma\)'s are the conventional Dirac matrices, the \(\hat{\gamma}\)'s are the isodirac matrices, and the symbol \(d\) stands for isoduality, i.e., \(\sigma^d = -\sigma{}^\dagger = -\sigma\), \(I^d = -I\).

As one can see, the anti–isocommutators of the isogamma matrices yield (twice) the Riemannian metric \(g(x)\), thus confirming the representation of gravitation in the structure of Dirac’s equation with the conventional Riemannian metric \(g(x)\), as desired. As an example, we have the particular case for the iso–Dirac–Schwarzchild equation [2d]

\[
\hat{\gamma}_k = (1 - 2M/r)^{-1/2} \times \gamma_k \times \hat{I}_{gr}(x), \quad \hat{\gamma}_4 = (1 - 2M/r)^{1/2} \times \gamma_4 \times \hat{I}_{gr}(x). \tag{7.32}
\]

Similar isorelativistic gravitational equations can be easily constructed by the interested reader.

It is generally believed that the conventional Poincaré symmetry in its spinorial covering \(\mathcal{P}(3.1) = SL(2,c) \times T(3.1)\) is the general symmetry of the conventional Dirac equation. This belief can be disproved by the isodual mathematics [5]. In fact, we have the following

**Theorem 3:** The largest possible isolinear symmetry of the isodirac equation is given by the isospinorial isopoincaré symmetry in the following 22-dimensional isoselfdual form

\[
\hat{S}_{tot} = \hat{\mathcal{P}}(3.1) \times \hat{\mathcal{P}}^d(3.1) = [SL(2,C) \times \hat{T}(3.1)] \times [SL^d(2,C^d) \times \hat{T}^d(3.1)], \tag{7.33}
\]

**Prof.** The conventional gamma matrices are isoselfdual, i.e., invariant under isoduality, \(\gamma_\mu \equiv \gamma^d_\mu = -\gamma_\mu{}^\dagger\). A necessary condition for a Lie transformation group to be a symmetry of the conventional Dirac’s equation is therefore that it must also be isoselfdual for consistency. The isospinorial Poincaré symmetry \(\mathcal{P}(3.1)\) is not isoselfdual, \(\mathcal{P}^d(3.1) \neq \mathcal{P}(3.1)\), and therefore does not verify the indicated necessary condition. However the direct product \(S_{tot} = \mathcal{P}(3.1) \times \mathcal{P}^d(3.1)\) is isoselfdual, \(S_{tot} \equiv S^d_{tot}\), and therefore verifies this necessary condition. The sufficiency can be proved as in the conventional case. Since the isotopies are axiom–preserving, the above properties of the conventional Dirac equation persists under all its infinitely possible isotopies, including the gravitational particularization. Finally, the 22-dimensional character of the total symmetry originates from the independence of conventional and isodual parameters, as well as the inclusion of the isoselfscalar transforms and their isoduals of Theorem 2 of Sect. 6 q.e.d.
Theorem 3 can also be reached by an inspection of the conventional Dirac matrices in Eq.s (??) and (??). In fact the latter are centrally dependent on the negative $2 \times 2$–dimensional unit for the internal space of spin which we have merely rewritten in our isodual form $I_{2 \times 2}^d = -I_{2 \times 2} = -\text{Diag}(1,1)$. This illustrates that the birth of the isodual theories for antiparticles can be seen in the conventional Dirac’s equation because of the essential presence of a negative unit in the very structure of the gamma matrices.

The latter was not interpreted by Dirac as a bona–fide unit because he lacked the knowledge of the related new numbers with negative units, the isodual numbers [3e].

Similarly, the space part of the conventional Dirac matrices reveals the presence of the isodual Pauli matrices, only written in our formalism $\sigma^{kd} = -\sigma^k = -\sigma_k$. The latter occurrence has intriguing implications, such as:

a) It establishes the validity of the isodual representation of antimatter. In fact, the referral of the negative–energy solutions to negative units eliminates their un–physical behavior [5];

b) It establishes the insufficient character of current interpretation of $\mathcal{P}(3.1)$ as being the maximal linear symmetry of the Dirac equations, in favor of the isoselfdual form $\mathcal{P}(3.1) \times \mathcal{P}^d(3.1)$.

c) It disproves another popular belief according to which the spin in Dirac matrices is characterized by a $4 \times 4$–dimensional representation of SU(2). In fact, the treatment of spin is restored as being entirely characterized by the 2–dimensional Pauli’s representation for the case of particles, with the independent antiautomorphic isodual Pauli’s matrices for the characterization of the spin of antiparticles, again, along the isoselfndual structure $SU(2) \times SU^d(2) \in \mathcal{P}(3.1) \times \mathcal{P}^d(3.1)$. At any rate, SU(2) is not isoselfdual and, as such, it cannot consistently characterize the spin of Dirac’s equation. For additional studies along these lines we refer the interested reader to ref.s [4h,5].

Property II: OIG coincides at the abstract level with RQM.

In view of the positive–definiteness of the gravitational isounit $\hat{I}_{gr}(x)$ (originating from the local Minkowskian character of the (3+1)–dimensional Riemannian spaces, Sect. 1), at the abstract level we have the identity of $I$ and $\hat{I}_{gr}(x)$, $\mathcal{H}$ and $\hat{\mathcal{H}}$, $R(n, +, \times)$ and $\hat{R}(\hat{n}, +, \hat{\times})$, etc. The same holds for the dynamical equations, e.g., at the abstract level we have $(\gamma^{\mu} \times p_\mu \times i \times m) \hat{\times} > \equiv (\hat{\gamma}^{\mu} \hat{\times} \hat{p}_\mu + \hat{i} \hat{m}) \hat{\times} >$.

Needless to say, the above abstract identity of OIG and RQM guarantees the axiomatic consistency of OIG and, and such, it is simply invaluable for the resolution of the problematic aspects of QG (see below).

An inspection of the properties of the isopoincaré symmetry, particularly the isocommutativity of coordinates and momenta, Eq.s (??), established the following
Property III OIG is isoflat (i.e., it verifies the axiom of flatness in isospace).

By comparison, QG is curved in the sense that its coordinates and momenta do not commute. This difference is basic for the understanding of the differences of the two theories and their consequential comparative appraisals.

In essence, we can state that the isocommutativity of coordinates and momenta is a necessary condition for the unambiguous applicability of the theory to measurements (in view of Theorem 1).

The reader should recall the dual formulation of all isotopic theories, that in isospaces and its projection in ordinary spaces. Therefore, the isocommutativity of coordinates and momentum does not imply that the theory is ordinarly flat, in which case no representation of gravitation is evidently possible owing to the historical teaching of the Founders of the theory [2]. In fact, the projection of OIG in ordinary spaces over conventional fields recovers all conventional Riemannian characteristics.

The distinction between isoflatness and ordinary flatness is here merely indicated with technical treatments presented elsewhere for brevity [9].

An important implication of the above studies is that the isospecial relativity can indeed unify the special and general relativities in both their classical and quantum versions for the exterior gravitational problem of matter in vacuum, with isodual images for the exterior gravitational problem of antimatter. The relativities are unified for both classical and operator versions via the basic unit which can represent gravitation when assuming the gravitational form \( \hat{I}_{gr}(x) \) and admits as particular case the special relativity when assuming the trivial form \( I = \text{Diag.}(1,1,1,1) \), with isodual images for antimatter.

It should be noted that the isospecial relativity can also unify the special and general relativities for interior gravitational problems of matter within physical media in both their classical and operator versions, with isodual images for antimatter. The latter unification is permitted by the unrestricted functional dependence of the isounit which can represent interior gravitational conditions via realizations of the type

\[
\hat{I}_{gr,int} = K(x, \dot{x}, \ddot{x}, ...) \times \hat{I}_{gr}(x),
\]

where \( K \) is a positive–definite \( 4 \times 4 \) matrix representing arbitrary nonlinear and nonlocal internal effects, the interior relativistic conditions emerging for the simpler value

\[
\hat{I}_{rel,int} = K(x, \dot{x}, \ddot{x}, ...) \times I.
\]

The simplest possible realization of the internal gravitational isounit is given by [3g]

\[
\hat{I} = [\text{Diag.}(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2})] \times F(x, \dot{x}, ...) \times \hat{I}_{gr}(x).
\]

All conventional exterior gravitational models (e.g., Schwarzschild’s exterior metric [2]) can the be easily lifted to the above interior conditions.
The main physical result is the extension of general relativity to locally varying speeds of light under the preservation of its abstract axioms. In fact, the speed of electromagnetic waves in interior conditions is a rather complex function of local variables, such as density $\mu$, temperature $\tau$, frequency $\omega$, etc., $c = c(x, \mu, \tau, \omega, \ldots) = c_0/n(x, \mu, \tau, \omega, \ldots)$. The important point is that the above local speed can be directly represented, that is, represented via the geometric line element of the theory, rather than current indirect manipulations (e.g., the representation of the classical speed $c = c_0/n$ via the scattering of photons among molecules in second quantization).

As an example, it is well known that the representation of the locally varying character of the speed of light in interior conditions is not possible in the Schwarzschild’s geometry. By comparison, its representation becomes elementary under our isotopic extension to interior conditions, due to the lifting of the fourth component (here assuming $F = I$ for simplicity)

$$g_{44} \to \hat{g}_{44} = g_{44}/n^2_4.$$  \hspace{1cm} (7.37)

The space components $n^2_k$ then emerge as the space–time symmetrization of the index of refraction, or via simple application of the Lorentz–Santilli isoboosts.

The isominkowskian reformulation of gravity also permits a novel and physically more accurate representation of gravitational horizons and singularities via the zeros of the isotopic element or isounit, according to the rules [4h]

**Gravit. – Horizons**:

$$\begin{align*}
\text{Space – component} &:\quad \hat{I}_{gr,\text{int}} = \text{Diag}.(\hat{I}_{11}, \hat{I}_{22}, \hat{I}_{33}) = 0, \\
\text{Time – component} &:\quad \hat{T}_{gr,\text{int}} = \hat{T}_{44} = 0,
\end{align*}$$

**Gravit. – Singularities**:

$$\begin{align*}
\text{Time – component} &:\quad \hat{I}_{gr,\text{int}} = \hat{I}_{44} = 0, \\
\text{Space – component} &:\quad \hat{T}_{gr,\text{int}} = \text{Diag}.(\hat{t}_{11}, \hat{T}_{22}, \hat{T}_{33}) = 0.
\end{align*}$$ \hspace{1cm} (7.38)

To understand there rules, note first that they are verified by the Schwarzschild metric for which we have [4h]

**Gravitational – Horizons**: space – component of $g = (1 - 2M/r)\text{Diag}.(1, 1, 1) = 0$,

**Gravitational – Singularity**: time – component of $g = (1 - 2M/r)^{-1} = 0$.  \hspace{1cm} (7.39)

As a result, rules (7.38) contain conventional horizons and singularities.

However, gravitational horizons and singularities are some of the most significant cases of interior gravitational problems, thus having nonlinear, nonlocal and nonlagrangian effects indicated in Sect. 1 which are outside the representational capability of the Riemannian geometry. Rules (7.38) therefore permit novel studies on gravitational horizons and singularities with a more realistic inclusion of internal, velocity–dependent and nonlocal–integral effects which will be conducted elsewhere.

It should also be mentioned that gravitational horizons are today studied via the use of conventional light cones, which implies the assumption of light in vacuum $c_0$. But the
exterior of gravitational horizons is made up of hyperdense chromospheres in which the speed of electromagnetic waves is a locally varying quantity. The use of our isolight cone then permits more realistic calculations with actual speeds of light, also contemplated in future works [5e].

OIG also permits the resolution of the paradox of quantum mechanics for gravitational collapse, identified and resolved in ref. [5d]. Consider a conventional QM particle in the interior of a star. As such, the particle obeys conventional uncertainties. Suppose now that the star collapses all the way to a singularity. Then, the particle considered must be located at the singularity. But the star is a classical object. The location of its singularity can therefore be classically determined with the desired precision. The paradox of quantum mechanics here considered then follows, because the QM particle should have uncertainties at a point which is classically determined.

It seems evident that no known resolution of this paradox exists in ordinary RQM. On the contrary, OIG can indeed resolve the paradox. In fact, the isoexpectation value of the commutation rules reads

\[
< [\hat{r}, \hat{p}] > = \frac{< | \times \hat{T}_{gr} \times (\hat{r} \times \hat{T}_{gr} \times p - \hat{p} \times \hat{T}_{gr} \times \hat{r}) \times \hat{T}_{gr} \times | >}{< | \times \hat{T}_{gr} \times | >} \tag{7.40}
\]

where the numerator is of the 4–th order in \( \hat{T}_{gr} \) while the denominator is of the 1–st order. From rule (5.30) it is then evident that

\[
\text{Lim}(\Delta r \times \Delta p)_{\hat{T}_{gr} \rightarrow 0} = \frac{1}{2} \text{Lim} < [\hat{r}, \hat{p}] >_{\hat{T}_{gr} \rightarrow 0} \equiv 0, \tag{7.41}
\]

namely, the isouncertainties acquire their classical deterministic value at the limit of gravitational collapse to a singularity, which is the main result in of ref. [5d] (other results in the same papers should be reformulate with the subsequent isodifferential calculus to avieve invariance, as done in this study).

The above findings provide additional elements of plausibility for OIG and indicate that the ”completion” of RQM for the inclusion of gravitation along the historical E–P–R teaching [14] is deeper that just the the achievement of an explicit operator realization of the theory of "hidden variables", because it implies rather subtle revisions of current studies on the relationship between classical and operator theories tacitly restricted to the simplest possible unit \( \hbar = 1 \).

As an example, recall that OIG is an image of RQN under nonunitary transforms which, as such, are known not to preserve eigenvalues and boundary values. It then follows that the image of Bell’s inequalities which is applicable to OIG is necessarily given by a nonunitary image of the conventional forms [14d]. This evidently implies the alteration of the numerical value of the conventional upper boundary of the inequalities which, as such, can become indeed compatible with conventional classical counterparts for the interior problem only (see [4h], App. 4.C for details, including the construction of the nonunitary image of
Pauli’s matrices). Note the restriction, again, of the latter arguments to the interior classical and operator problems. In fact, the exterior classical problem remains the conventional Hamiltonian–unitary one.

In summary, we can state that, subject to further studies and independent scrutinies, OIG appears to resolve the problematic aspects of QG, i.e., the lack of invariant unit with consequential ambiguous applicability to measurements, the lack of conservation of Hermiticity in time with consequential lack of physically acceptable observables, the lack of invariance of the numerical predictions, physical laws and special functions with consequential lack of consistent physical applications.

Moreover, QG is known to have difficulties in achieving a formulation comparable to ordinary RQM because of serious technical problems, e.g., in the construction of PCT symmetry. These difficulties are removed ab initio in OIG precisely thanks to its isotopic character, that is, the inherent capability of preserving all original characteristics, as illustrated above.

Besides the transparent axiomatic advantages of OIG as compared to QG, the plausibility of QIG is further established by preservation of conventional quantum laws, such as the conventional uncertainties for the center–of–mass trajectories, Eq.s (7.18).

The latter property provide evident experimental support for our isominkowskian formulation of gravity, this time, at the operator level, because it indicates its compatibility with existing, quantum mechanical experimental data, a property which does not appear to be verified by QG. The understanding is that contributions at the particle level due to gravity are notoriously very small as compared to those due to other interactions, as established by isotopic elements of the Schwarzschild’s type (2.24) where M is now the mass of the particle considered. These very small contributions can therefore be verified via suitable tests only after achieving the necessary technology.

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