COSMOLOGICAL “GROUND STATE” WAVE FUNCTIONS IN GRAVITY
AND ELECTROMAGNETISM

Michael P. Ryan, Jr.
Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
A. Postal 70-543, México 04510 D.F.
MEXICO

ABSTRACT

The coincidence of quantum cosmology solutions generated by solving a Euclidean version of the Hamilton-Jacobi equation for gravity and by using the complex canonical transformation of the Ashtekar variables is discussed. An examination of similar solutions for the free electromagnetic field shows that this coincidence is an artifact of the homogeneity of the cosmological space.

1. Introduction

Jerzy Plebaniński is one of the pioneers in complex methods applied to general relativity, especially in the field of self-dual solutions. It is one of the ironies that seem to abound in physics that the germ of what are now called the Ashtekar variables is contained in a 1977 paper of Jerzy’s. If the Hamiltonian form of the self-dual action presented there had been constructed, we might today be using the name “Plebaniński variables” instead of Ashtekar variables. In any case, everyone who works, no matter how briefly or superficially, in complex relativity will find himself acknowledging Jerzy’s work.

It is particularly fitting that those of us who work in relativity in Mexico acknowledge Jerzy’s contribution in reviving the study of gravitation in this country (One must say “revive” because almost fifty years ago Marcos Moshinsky, Alberto Barajas and Carlos Graef were making original contributions to the field2.) and creating a climate in which the study of gravitation was considered an interesting and valuable endeavor.

The present article considers the application of the Ashtekar variables3 to finding exact solutions in quantum gravity, especially minisuperspace or quantum cosmology solutions. The study of quantum minisuperspaces has a relatively long history since they were introduced by Charles Misner in the late sixties, and they have often been used as models for the quantization of gravity as well as being interpreted, especially by Stephen Hawking and his collaborators, as giving information about early epochs of the Universe. Here I will focus on quantum cosmologies as
models for the use of Ashtekar variables in quantum gravity.

A number of people have used the Ashtekar variables to find solutions to quantum cosmologies, including Ashtekar himself, Pullin\(^4\), Obregon\(^5\), and myself and Moncrief\(^6\). As will be mentioned below, one usually splits the cosmological minisuperspaces into a non-dynamical, time-independent part and a finite or infinite set of time-dependent variables that define the state space of the minisuperspace. The constraints of gravity (ADM or Ashtekar) can then be expressed in terms of these variables and a quantum cosmology can be constructed by turning the variables and their conjugate momenta into operators and applying them to a state function that is a function of some of them. In the ADM formalism the operator corresponding to the Hamiltonian constraint leads to the Wheeler-DeWitt equation and in the Ashtekar formalism the equivalent constraints lead to what has been called the Ashtekar-Wheeler-DeWitt equation\(^6\). The Ashtekar-Wheeler-DeWitt equation is closer in spirit to the Bargmann\(^7\) formalism in ordinary quantum mechanics, so the state function that is obtained as a solution must be interpreted in a different way from the usual ADM wave function in that a norm over these functions must be constructed or they must be mapped (perhaps by means of a kernel such as that constructed by Bargmann for the harmonic oscillator\(^7\)) into an ordinary ADM wave function.

The existence of such mappings allows one to find solutions to the Ashtekar-Wheeler-DeWitt equation and then map them to solutions of the usual Wheeler-DeWitt equation. For a simple-minded mapping this was done in Ref. \([6]\), where a constant solution to the Ashtekar-Wheeler-DeWitt equation for a diagonal Bianchi Type IX cosmological model was mapped to a “ground state” solution of the Wheeler-DeWitt equation for a certain factor ordering that had the form \(k \times e^{-S}\), \(k = \text{const.}\), where \(S\) was a solution of the Einstein-Hamilton-Jacobi equation\(^8\). This technique was extended to a supersymmetric theory by Obregon, Pullin, and myself\(^5\).

This procedure might seem to be a “magic bullet” that would allow one to find solutions of the \(e^{-S}\) form for many metrics in quantum gravity, not only the quantum cosmologies. I plan to show here that while the solutions to the Wheeler-DeWitt equation obtained by mapping constant solutions of the Ashtekar-Wheeler-DeWitt equation always exist, they are not necessarily “ground state” solutions in the sense of Ref. \([6]\). I will give a few examples and study the Maxwell analogue of this procedure where the difference between the mapped “Ashtekar” solutions and the \(e^{-S}\) solution is more obvious.

The plan of the paper is as follows. In Sec. 2 I will give a sketch of the Bargmann representation for the harmonic oscillator, along with a modified representation that is closer in spirit to the Ashtekar representation. In both cases I will give mappings between the Bargmann or “Ashtekar” state and states in the ordinary coordinate representation. In Sec. 3 I will give several quantum cosmology solutions obtained by the mapping procedure, some of which are more useful than
Section 4 is devoted to the electromagnetic analogue of the procedure and Sec. 5 contains conclusions.

### 2. The Bargmann Representation for the Harmonic Oscillator

In 1961 Bargmann studied in great detail a representation introduced originally by Fock in the quantum mechanics of the harmonic oscillator. The basic idea is the simple-minded one of taking the annihilation and creation operators $\hat{a} = (1/\sqrt{2})(q - ip)$, $\hat{a}^\dagger = (1/\sqrt{2})(q + ip)$ for the harmonic oscillator and achieving the commutation relation $[\hat{a}^\dagger, \hat{a}] = 1$ by realizing $\hat{a}^\dagger$ as $\partial/\partial a$. While this concept is seductive, $\hat{a}$ and $\hat{a}^\dagger$ are not unitary operators and the space of functions $F(a)$ over the complex numbers $a = (1/\sqrt{2})(q - ip)$ upon which the operator $\partial/\partial a$ is supposed to act is not a Hilbert space. Bargmann’s contribution was to make sense of this idea by means of a careful analysis of the problem. An important part of his contribution was to construct a norm on the function space $F(a)$ and to give a mapping from $F(a)$ to the ordinary Hilbert space of the harmonic-oscillator coordinate wave functions.

Bargmann defined the norm on $F(a)$ for an $n$-dimensional phase space. It is sufficient to show his result for a two-dimensional phase space $(q, p)$ since the overall structure is the same. He defines the inner product of two functions $\psi_1(a)$ and $\psi_2(a)$ in $F(a)$ by means of a kernel $\rho(q, p)$ as

$$
(\psi_1, \psi_2) = \int \psi_1^\star \psi_2 \rho(q, p) \, dq dp.
$$

He then determines $\rho$ by demanding that $\partial/\partial a$ be the adjoint of $a$ with respect to this inner product, which leads to a set of partial differential equations for $\rho$. He finds

$$
\rho = c \exp(-a^\star a) = c \exp(-[p^2 + q^2]).
$$

He next gives the mapping from the Hilbert space of the ordinary harmonic oscillator wave functions, also in terms of a kernel $A(a, q)$. This mapping from a wave function $\psi(q)$ to a wave function $\Psi(a)$ in $F(a)$ has the form

$$
\Psi(a) = \int A(a, q) \psi(q) \, dq.
$$

Demanding that $\hat{a} = (1/\sqrt{2})(q - \partial/\partial q)$ and $\hat{a}^\dagger = (1/\sqrt{2})(q + \partial/\partial q)$ map into $a\Psi(a)$ and $\partial \Psi(a)/\partial a$, respectively, again implies a set of partial differential equations for $A$. These equations have as a solution

$$
A(a, q) = \pi^{-1/4} \exp[-\frac{1}{2}(a^2 + q^2) - \sqrt{2}aq],
$$

If one now writes the time-independent harmonic-oscillator wave equation in terms of the operators $a$ and $\partial/\partial a$, one finds that, for the usual factor ordering,

$$
\left( a \frac{\partial}{\partial a} + \frac{1}{2} \right) \Psi(a) = E \Psi(a).
$$

3
Here we see the advantages of “Bargmannization”, since this equation is much easier to solve than the usual Schrödinger equation for the harmonic oscillator. The general solution is \( \Psi(a) = caE^{-\frac{1}{2}} \), \( c = \text{const.} \) Bargmann gives the simplest orthonormal (with respect to the inner product \([2.1]\)) set of eigenstates as \( u_n = a^n/\sqrt{n!} \). These states correspond to energy eigenstates with \( E = n + \frac{1}{2} \).

Bargmann also shows that the kernel \((2.3)\) maps the eigenstates \( u_n \) to the usual harmonic oscillator energy eigenstates \( \phi_n = [2n!\sqrt{\pi}]^{-1/2}H_n(q)e^{-q^2/2} \). Of particular interest is the ground state. The ground state in the Bargmann representation is \( u_0 = 1 \). This constant wave function maps to \( \phi_0(q) = (\sqrt{\pi})^{-1/2}e^{-q^2/2} \).

The Ashtekar variables are not exact analogues of the Bargmann variables for the harmonic oscillator. They are closer in spirit to the following transformation

\[
Q = q, \quad P = q + ip. \tag{2.5}
\]

As before, we would like to achieve the commutation relations \([\hat{P}, \hat{Q}] = 1\) by realizing \( \hat{P} \) as \( \partial/\partial Q \). Again we can follow Bargmann’s procedure, finding a kernel \( \rho(Q) \) that gives an inner product on the function space \( \mathcal{F}(Q) \), and another kernel \( A(q, Q) \) that maps the ordinary harmonic-oscillator wave functions \( \psi(q) \) to the “Ashtekar-like” functions \( \Psi(Q) \). The kernel \( \rho \) can be found by requiring that \( \hat{P}^{\dagger} = \hat{q} - i\hat{p} = -\hat{P} + 2\hat{Q} \) be the adjoint of \( \partial/\partial Q \), that is, if \( (\psi_1, \psi_2) = \int \psi_1^*(Q)\rho(Q)\psi_2(Q) \) \( dQ \),

\[
(\psi_1, \partial\psi_2/\partial Q) = ([-\hat{P} + 2Q]\psi_1, \psi_2), \tag{2.6}
\]
or,

\[
\int \rho(Q)\psi_1^* \frac{\partial\psi_2}{\partial Q} dQ = \int \left( -\frac{\partial\rho}{\partial Q}\psi_1^*\psi_2 - \rho \frac{\partial\psi_1^*}{\partial Q}\psi_2 \right) dQ
\]

\[
= \int \rho(Q) \left( -\frac{\partial\psi_1^*}{\partial Q}\psi_2 + 2Q\psi_1^*\psi_2 \right). \tag{2.7}
\]

These relations imply \(-\partial\rho/\partial Q = 2Q\rho\), or \( \rho = e^{-Q^2} \).

For the kernel \( A(q, Q) \) use Bargmann’s condition that \( Q\Psi \) be the transform of \( q\psi \) and that \( \partial\Psi/\partial Q \) be the transform of \((q + i\hat{p})\psi \). These two conditions are

\[
Q\Psi = \int QA(q, Q)\psi(q) dq = \int A(q, Q)q\psi(q), \tag{2.8}
\]

\[
\frac{\partial\Psi}{\partial Q} = \int \frac{\partial A}{\partial Q} \psi dq = \int A \left( \frac{\partial\psi}{\partial q} + q\psi \right) dq = \int \left( -\frac{\partial A}{\partial q} + qA \right) \psi dq. \tag{2.9}
\]

The first of these conditions can be satisfied if \( A(q, Q) = f(q)\delta(q - Q) \) [or \( f(Q)\delta(q - Q) \)], and the second gives \( A(q, Q) = ce^{-q^2/2}\delta(q - Q), c = \text{const.} \).
If one now writes the Hamiltonian for the harmonic oscillator in terms of \( \hat{P} \) and \( \hat{Q} \) we find (for a convenient factor ordering)

\[
\hat{H}\Psi = -\frac{1}{2} \hat{P}^2 \Psi + \hat{Q} \hat{P} \Psi + \frac{1}{2} \Psi. \tag{2.10}
\]

The time-independent Schrödinger equation becomes

\[
-\frac{1}{2} \frac{\partial^2 \Psi}{\partial Q^2} + Q \frac{\partial \Psi}{\partial Q} + \frac{1}{2} \Psi = E \Psi, \tag{2.11}
\]

which, for \( E = n + \frac{1}{2} \) is Hermite’s equation with solutions \( \Psi = H_n(Q) \), where the \( H_n \) are the Hermite polynomials. It is now obvious that \( \rho \) is just the usual convergence factor for the Hermite polynomials and \( A(q, Q) \) maps the harmonic oscillator wave functions to the Hermite polynomials. Notice that the (unnormalized) ground state “Ashtekar” wave function \( \Psi_0(Q) \) is just one, and that it is the mapping of \( \psi_0 = e^{-q^2/2} \).

The “Bargmannization” given here is relatively trivial, and the mapping between \( \psi \) and \( \Psi \) can be achieved by treating the transformation \( Q = q, P = q + ip \) as a simple complex canonical transformation (similar to the real canonical transformations that Anderson has been using to good effect lately in quantum gravity). The action

\[
I = \int [p \dot{q} - \frac{1}{2} (p^2 + q^2)] \, dt \tag{2.12}
\]

becomes

\[
I = \int [(P/i) \dot{Q} + i(q^2/2) - \frac{1}{2} \{P^2 + 2PQ\}] \, dt \tag{2.13}
\]

The \( (P/i) \dot{Q} \) term allows one to realize \( -i \hat{P} \) as \( -i \partial/\partial Q \), which gives the previous form of \( \hat{P} \), and the generating function \( G(q) \) of this transformation is \( iq^2/2 \). The usual mapping from \( \Psi(Q) \) to \( \psi(q) \) is \( e^{iG} \Psi(Q) = \psi(q) = e^{-q^2/2} \Psi(Q) \), which was what was found before. Again notice that for the ground state \( \Psi(Q) = 1 \), we recover the usual ground state \( \psi(q) = e^{-q^2/2} \).

There is one last element that is needed in order to discuss the construction of ground state wave functions which is the use of the “Euclidean” Hamilton-Jacobi equation (i.e. with the sign of the potential term reversed). For the harmonic oscillator discussed above this equation is

\[
\frac{1}{2} \left( \frac{\partial S}{\partial q} \right)^2 - \frac{1}{2} q^2 = \frac{\partial S}{\partial t}. \tag{2.14}
\]

In order to find a ground state one begins by taking \( \partial S/\partial t = 0 \) and solving (2.14) for \( S \). Here the solution is

\[
S = \pm q^2/2 + c', \quad c' = \text{const.} \tag{2.15}
\]
The ground state wave function is then \( \psi(q) = e^{-S} \) for the plus sign in (2.15), or \( \psi = ce^{-q^2/2} \), the same function that was found in the “Ashtekar” case. The main question is how general this coincidence of the ground states is, especially for gravity. One important feature of the Hamilton-Jacobi procedure that will be seen to be important below is that a partial differential equation such as (2.14) can have a fairly large number of solutions, even though in this simple case the only freedom one has is the choice of \( c' \).

In order to see what happens in quantum gravity I will give a fairly brief discussion of the application of the two procedures discussed above to quantum cosmologies.

3. “Ground State” Wave Functions in Quantum Cosmology

A reasonable template for quantum cosmology is the diagonal Bianchi IX model,

\[
ds^2 = -dt^2 + e^{2\alpha(t)} \epsilon_{ij}^{2\beta(t)} \sigma^i \sigma^j,
\]

where \( \beta_{ij}(t) = \text{diag}\{\beta_+(t) + \sqrt{3}\beta_-(t), \beta_+(t) - \sqrt{3}\beta_-(t), -2\beta_+(t)\} \) and the \( \sigma^i \) are invariant one-forms on the three-sphere that obey \( d\sigma^i = \frac{1}{2} \epsilon^{ijk} \sigma^j \wedge \sigma^k \). The Ashtekar formalism for this (and other Bianchi models) is discussed in Refs. [4][5][6]. I will give a quick precis of the discussion in Ref.[6] (which was based on an article by Friedman and Jack\textsuperscript{11}). The \( \sigma^i \) have the form \( \sigma^i \equiv \hat{e}^i_a(x^b) dx^a \), where the \( \hat{e}^i_a \) are known non-dynamical functions of the space coordinates. The orthonormal one-forms for the metric (3.1) are \( \omega^i_j = e^\alpha e^{\beta ij} \hat{e}^j_a dx^a \equiv e^i_a dx^a \). The spin connections \( \Gamma_a^i = \frac{1}{2} \epsilon^{ijk} \Gamma^j_{ka} \), where \( \nabla_a e^b_j = \Gamma^i_{ja} e^b_i \), are

\[
\Gamma_a^i = \left( \frac{1}{2} e^{2\beta} e^\beta_{in} - e^{3\beta} e^m_n \right) \hat{e}^n_a.
\]

The Ashtekar variables \( A_a^i \) are

\[
A_a^i = \Gamma_a^i + iK_a^i,
\]

where the \( K_a^i \) are \( e^{ib} K_{ba}, K_{ab} \) the second fundamental form of \( g_{ab} = g_{ij} \hat{e}^i_a \hat{e}^j_b \). It is easy to show that \( K_a^i = P_{in}(t) \hat{e}^n_a \), so

\[
A_a^i = [G_{in}(t) + iP_{in}(t)] \hat{e}^n_a \equiv \tau_{in} \hat{e}^n_a.
\]

The densitized basis elements \( \hat{e}^n_a \) are \( \sqrt{\hbar} e^{3a} \epsilon^{ik} \hat{e}^a_k \), where \( \hbar = \det\hat{h}_{ab}, \hat{h}_{ab} = \hat{e}^a_i \hat{e}^b_i \).

Finally, the Hamiltonian constraint, \( \mathcal{H} + i\nabla_a G^a = -\hat{e}^{ia} e^j_b F_{ij}^a \), where

\[
F_{ij}^a = (\epsilon^{ijk} \tau_{kn} \epsilon_{n\ell m} + \tau_{jm} \tau_{i\ell} - \tau_{im} \tau_{j\ell}) \hat{e}^m_a \hat{e}^\ell_b,
\]
becomes (after dropping the non-dynamical known functions $\hat{c}_a$ and $\sqrt{\hbar}$)
\[
\mathcal{H} + i\nabla_a G^a = e^{3\alpha}(2e^\alpha e^\beta \tau_{ij} - e^{-2\alpha} e^{\gamma} e^{-\gamma} \tau_{ijs} \tau_{ips} + e^{-2\alpha} e^{-\gamma} e^{-\gamma} \tau_{ips} \tau_{ips}). \tag{3.6}
\]

It is now possible to convert $\mathcal{H} + i\nabla_a G^a$ into an operator and apply it to a state function $\Psi_A$. It has become customary in Ashtekar variable studies to use the “momentum” representation where the $A^i_a$ are taken to be coordinates, but I will use the “coordinate” representation of Friedman and Jack where $A^i_a = \frac{1}{2} \delta / \delta \tilde{e}^a_i$ and the $\tilde{e}^a_i = e^{2\alpha} e^{\beta \gamma} \beta$ are the “coordinates”. The $\tilde{e}^a_i$ are

\[
\tilde{e}_1^1 = \equiv \mu, \quad \tilde{e}_2^2 = \equiv \nu, \quad \tilde{e}_3^3 = \equiv \lambda, \tag{3.7}
\]

where $\mu = e^{2\alpha} e^{-\beta + \sqrt{3} \beta}$, $\nu = e^{2\alpha} e^{-\beta + \sqrt{3} \beta}$, $\lambda = e^{2\alpha} e^{2\beta}$. The $\tau_{ij}$ now become

\[
(1/2) \partial / \partial \tilde{e}_i^j, \text{ so }
\tau_{11} \rightarrow \frac{1}{2} \partial / \partial \mu, \quad \tau_{22} \rightarrow \frac{1}{2} \partial / \partial \nu, \quad \tau_{33} \rightarrow \frac{1}{2} \partial / \partial \lambda. \tag{3.8}
\]

The state function $\Psi_A$ will now be $\Psi_A(\mu, \nu, \lambda)$, and the Hamiltonian constraint operator applied to $\Psi_A$ gives the Ashtekar-Wheeler-DeWitt equation:

\[
\nu \lambda \frac{\partial \Psi_A}{\partial \mu} + \mu \lambda \frac{\partial \Psi_A}{\partial \nu} + \mu \lambda \frac{\partial \Psi_A}{\partial \lambda} + \frac{1}{2 \sqrt{\mu \nu \lambda}} \left[ \mu \nu \frac{\partial^2 \Psi_A}{\partial \mu \partial \nu} + \nu \lambda \frac{\partial^2 \Psi_A}{\partial \nu \partial \lambda} + \mu \lambda \frac{\partial^2 \Psi_A}{\partial \mu \partial \lambda} \right] = 0, \tag{3.9}
\]

where the factors have been ordered so that the derivatives stand to the right.

This equation is the equivalent of the Schrödinger equation (2.11), where the dreibein variables $\mu, \nu, \lambda$ play the role of the coordinate variable $Q$ there. As for Eq. (2.11), there should be a “ground state” Ashtekar wave function equivalent to $H_0(Q) = 1$, that is, $\Psi_A = \text{const}$. There should also exist a mapping, constructed either by means of a Bargmann-like kernel or by the canonical transformation procedure. Notice that the variable $A^i_a = \Gamma^i_a + iK^i_a$ plays the role of $P = q + ip$ in (2.5), since the $\Gamma^i_a$ are combinations of the “coordinate” variables $\tilde{e}^a_i$ (and, of course, their spatial derivatives) and the extrinsic curvature, $K^i_a$, is basically their conjugate momenta. One would expect a mapping of the form $\psi_{ADM} = e^{iG} \Psi_A$, where $\psi_{ADM}$ is a solution to the ordinary Wheeler-DeWitt equation in ADM variables and $\Psi_A$ is a solution to the Ashtekar-Wheeler-DeWitt equation (3.9). Such a mapping was found by Kodama$^{12}$ and has the form

\[
G = \pm 2i \int \tilde{e}^a_i \Gamma^i_a d^3x. \tag{3.10}
\]

Notice that I have used $\tilde{e}^a_i$ as the coordinate variables and $A^i_a$ as the $q+ip$ momentum variables. This means that $G$ is appropriate only for this choice, and that if one
were to take $A_i^a$ as a coordinate variable, as is often done, the $G$ given here would no longer be appropriate\textsuperscript{13}. In Ref. [6], $G$ was calculated for the diagonal Bianchi Type IX model and found to be

$$G = i16\pi^2 e^{2\alpha} [e^{-4\beta+} + 2e^{2\beta+} \cosh(2\sqrt{3}\beta_-)].$$  \hfill (3.11)

One would expect the ADM wave function $\psi_{ADM} = e^{iG}\Psi_A$, where the solution $\Psi_A = 1$ is taken, to be a “ground state” wave function for the diagonal Bianchi IX model. In fact, in Ref. [6] we showed that there is a close connection between this “Ashtekar” solution and one constructed from the “Euclidean” Einstein-Hamilton-Jacobi equation for the diagonal Bianchi IX models,

$$\left( \frac{\partial S}{\partial \alpha} \right)^2 - \left( \frac{\partial S}{\partial \beta_+} \right)^2 - \left( \frac{\partial S}{\partial \beta_-} \right)^2 + e^{4\alpha} V(\beta_\pm) = 0,$$  \hfill (3.12)

where the “three-curvature” term $e^{4\alpha} V(\beta_\pm)$ has the opposite sign from the term in the usual Einstein-Hamilton-Jacobi equation and the potential $V(\beta_\pm)$ is the usual $V(\beta_\pm) = \frac{1}{3} e^{-8\beta+} - \frac{4}{3} e^{-2\beta+} \cosh(2\sqrt{3}\beta_-) + \frac{2}{3} e^{4\beta+} \cosh(4\sqrt{3}\beta_-) - 1$. One solution to (3.12) is $S = \frac{1}{6} e^{2\alpha} [e^{-4\beta+} + 2e^{2\beta+} \cosh(2\sqrt{3}\beta_-)]$, so $e^{-S}$ is exp$[(3/8\pi^2)iG]$. Because (3.12) is nonlinear, the constant factor in the exponent by which these two expressions differ cannot be removed by a simple rescaling of $S$. However, Eq. (3.1) implies a choice of normalization constant in the definition of the three metric, that is, Eq. (3.1) should read $ds^2 = -dt^2 + R_0^2 \exp(2\alpha) \exp(2\beta_{ij}) \sigma^i \sigma^j$, and $R_0$ was taken to be one. By choosing $R_0$ properly, the two expressions for the “ground state” can be made to coincide. Note, however, that these two wave functions do differ, even if the difference is trivial and removable.

It seems that we have discovered a “magic bullet” that can generate “ground state” wave functions, since the expression for $G$ in Eq. (3.10) is constructive and can be calculated for any metric, so $e^{iG}$ can be generated easily, while the equivalent of Eq. (3.12) is a nonlinear functional differential equation that must be solved. Unfortunately, as I will show below, $e^{iG}$ does not always correspond to interesting “ground state” solutions. In the next section of the article I will look at the electromagnetic analogue of the gravitational problem.

The $e^{iG}$ “ground state” for the Bianchi Type I models is easy to calculate since the spatial metric is

$$ds^2 = e^{2\alpha} e^{2\beta}_{ab} dx^a dx^b,$$  \hfill (3.15)

the dreibein vectors $\hat{e}^i_a$ are just $\hat{e}^i_a = \delta^i_a$, and the orthonormal vectors are $e^a e^b_{ij} \delta^i_a$. Since both the metric $h_{ab} = e^a e^b = e^{2\alpha} e^{2\beta}_{ab}$ and the $e^a$ themselves are constant on $t = \text{const.}$ hypersurfaces, $\nabla_a e^b_j = 0$ and $\Gamma^i_a = 0$. This means that $\hat{e}^i_a \Gamma^i_a = 0$ and $G = 0$. The $e^{iG}$ solution is just equal to one. The Einstein-Hamilton-Jacobi equation in this case is

$$\left( \frac{\partial S}{\partial \alpha} \right)^2 - \left( \frac{\partial S}{\partial \beta_+} \right)^2 - \left( \frac{\partial S}{\partial \beta_-} \right)^2 = 0,$$  \hfill (3.16)
and $S = 0$ is, of course, a solution, so $e^{-S} = 1$ gives the same state function as above, but whether or not this solution is a “ground state” is a moot point. There is, however, another system for which the $e^{iG}$ state probably has even less claim to be a “ground state” than in the Bianchi I case. This system is the polarized Gowdy metric\(^{14}\)
\[
ds^2 = e^{\frac{-\lambda}{2}}(-e^{4t}dt^2 + d\theta^2) + e^{2t}(e^\beta d\sigma^2 + e^{-\beta}d\delta^2),
\]  
where $\beta = \beta(\theta, t)$, $\lambda = \lambda(\theta, t)$, $0 \leq \theta, \sigma, \delta < 2\pi$. The Hamiltonian formulation and the quantization of this model were first discussed by Misner\(^{15}\) and Berger\(^{16}\). A full discussion of the “ground state” problem will be given elsewhere\(^{17}\), but I will give a quick sketch of the results to show where the problem is with taking $e^{iG}$ to be the “ground state”. The Hamiltonian constraint for this metric is
\[
H = H + \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[ \frac{1}{2} p_\beta^2 + \frac{1}{2} e^{4t}(\beta')^2 \right] = 0,
\]  
where $t$ is $\partial/\partial \theta$, $p_\beta$ is the momentum conjugate to $\beta$, and $H = (1/2\pi) \int_0^{2\pi} d\theta p_t$, where $p_t$ is the momentum conjugate to $t$. It is obvious that a gauge has been chosen where the metric component $t$ is an internal time, so $H$ is the ADM Hamiltonian of the problem. Since $H$ plays the role of the Hamiltonian, the “Euclidean” Einstein-Hamilton-Jacobi equation is
\[
\frac{\partial S}{\partial t} + \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[ \frac{1}{2} \left( \frac{\delta S}{\delta \beta} \right)^2 - \frac{1}{2} e^{4t}(\beta')^2 \right] = 0.
\]  
If one expands $\beta$ and $p_\beta$ in real Fourier series,
\[
\beta = q_0 + \sqrt{2} \sum_{n=1}^{\infty} [q_n(t) \cos n\theta + q_{-n}(t) \sin n\theta],
\]
\[
p_\beta = p_0 + \sqrt{2} \sum_{n=1}^{\infty} [p_n(t) \cos n\theta + p_{-n}(t) \sin n\theta],
\]  
Misner and Berger have shown that the Wheeler-DeWitt equation is
\[
\frac{i}{\partial t} \psi = \sum_{n=-\infty}^{\infty} \frac{1}{2} \left( -\frac{\partial^2 \psi}{\partial q_n^2} + n^2 e^{4t} q_n^2 \psi \right).
\]  
The “ground state” solution of Berger\(^ {16}\) is
\[
\psi_{GS} = \exp(-\sum_n A_n(t) q_n^2),
\]
where the \( A_n(t) \) are given by Berger. The relation of this ground state to the \( e^{-S} \) state will be considered in Ref. [17].

The Ashtekar “ground state” \( e^{iG} \) is readily calculated. If one takes the orthonormal basis \( \sigma^1 = e^{i/3 - t/4} d\theta \), \( \sigma^2 = e^{t + \beta/2} d\sigma \), \( \sigma^3 = e^{-t - \beta/2} d\delta \), the basis vectors \( e_i^a \) are diagonal (i.e. \( e_i^a = f_a \delta_i^a \)), while the only non-zero \( \Gamma^i_a \) are the non-diagonal terms \( \Gamma_3^2 \) and \( \Gamma_2^3 \), so \( e_1^a \Gamma_2^i = 0 \), and, as in the Bianchi I case the Ashtekar “ground state” is \( \Psi = 1 \). The is obviously unrelated to the state given in Eq. (3.22). Of course, the fact that \( G \) is not gauge invariant means that one can rotate the basis used here and make \( G \) into anything desired, but this seems a bit artificial if the Ashtekar “ground state” is to be something that comes from a constructive procedure that is easy to implement.

The fact that the Ashtekar variable procedure leads to “ground states” that are not what one would expect, and especially to states that seem to differ from those obtained from the Einstein-Hamilton-Jacobi procedure, means that there must be a fundamental difference between the two procedures. In the next section I will consider the simpler system of a free electromagnetic field, where the differences between the analogues of the two procedures is more easily understood, and the coincidence between them in certain cases is more understandable.

### 4. Electromagnetic “Ground State” Wave Functions

If one considers the usual action for the free electromagnetic field in Hamiltonian form,

\[
I = \frac{1}{16\pi} \int \left[ \pi^i \dot{A}_i - \left\{ \frac{1}{8} \pi^i \pi^i + 2(\nabla \times A)^2 \right\} - A_0 \pi_i \right] d^4 x, \tag{4.1}
\]

\( i = 1, 2, 3 \) and expands \( A_0, A_i \) and their conjugate momenta \( \pi^i \) in real Fourier series (using box normalization in a box of side \( L \)) as

\[
A_i = \frac{1}{L^{3/2}} \sum_n \left[ A_i^{(n)}(t) \cos \left( \frac{2\pi}{L} n \cdot x \right) + \tilde{A}_i^{(n)}(t) \sin \left( \frac{2\pi}{L} n \cdot x \right) \right],
\]

\[
\pi^i = \frac{1}{L^{3/2}} \sum_n \left[ P_i^{(n)}(t) \cos \left( \frac{2\pi}{L} n \cdot x \right) + \tilde{P}_i^{(n)}(t) \sin \left( \frac{2\pi}{L} n \cdot x \right) \right],
\]

\[
A_0 = \frac{1}{L^{3/2}} \sum_n \left[ A_0^{(n)}(t) \cos \left( \frac{2\pi}{L} n \cdot x \right) + \tilde{A}_0^{(n)}(t) \sin \left( \frac{2\pi}{L} n \cdot x \right) \right], \tag{4.2}
\]

the action reduces to

\[
I = \frac{1}{32\pi} \int \sum_n \left[ P_i^{(n)} \dot{A}_i^{(n)} + \tilde{P}_i^{(n)} \dot{\tilde{A}}_i^{(n)} \right] -
\]
\[\begin{align*}
-\left\{ \frac{1}{8} \delta^{i(n)} P^{i(n)} + \frac{1}{8} \delta^{i(n)} \tilde{P}^{i(n)} + \frac{8\pi^2}{L^2} (n \times A^{(n)})^2 + \right. \\
+ \frac{8\pi^2}{L^2} (n \times \tilde{A}^{(n)})^2 \} - A_0^{(n)} n \cdot P^{(n)} - \tilde{A}_0^{(n)} n \cdot \tilde{P}^{(n)} \right\} dt. \\
\end{align*}\] (4.3)

If the sign of the potential term in the above action is changed, the “Euclidean” Hamilton-Jacobi equation that corresponds to the action is

\[\frac{\partial S}{\partial t} = \sum_n \left[ \frac{1}{8} \frac{\partial S}{\partial A_i^{(n)}} \frac{\partial S}{\partial A_i^{(n)}} + \frac{1}{8} \frac{\partial S}{\partial \tilde{A}_i^{(n)}} \frac{\partial S}{\partial \tilde{A}_i^{(n)}} - \right. \]

\[- \frac{8\pi^2}{L^2} (n \times A^{(n)})^2 - \frac{8\pi^2}{L^2} (n \times \tilde{A}^{(n)})^2 \right\]. (4.4)

For \( \partial S/\partial t = 0 \) we can find two distinct solutions to this equation. For simplicity I will take the Coulomb gauge, \( n \cdot A^{(n)} = n \cdot \tilde{A}^{(n)} = 0 \). The first of these solutions is

\[S = - \sum_n \frac{8\pi}{L} |n| \{ |A^{(n)}|^2 + |\tilde{A}^{(n)}|^2 \}. \] (4.5)

The function \( e^S \) is the box equivalent of the Wheeler ground state functional for the free electromagnetic field\(^{18}\),

\[\psi = N \exp \left\{ - \int \int \frac{B(x_1) \cdot B(x_2)}{16\pi^3 h c r_{12}} d^3 x_1 d^3 x_2 \right\}. \] (4.6)

However, there is another solution for \( \partial S/\partial t = 0 \) that can be obtained by rewriting the right-hand side of (4.4), that is,

\[\sum_n \left\{ \left[ \frac{\partial S}{\partial A_i^{(n)}} + \frac{8\pi}{L} (n \times \tilde{A}^{(n)}) \right] \left[ \frac{\partial S}{\partial A_i^{(n)}} - \frac{8\pi}{L} (n \times \tilde{A}^{(n)}) \right] + \right. \]

\[+ \left[ \frac{\partial S}{\partial \tilde{A}_i^{(n)}} + \frac{8\pi}{L} (n \times A^{(n)}) \right] \left[ \frac{\partial S}{\partial \tilde{A}_i^{(n)}} - \frac{8\pi}{L} (n \times A^{(n)}) \right] \right\} = 0. \] (4.7)

This form of the equation shows that

\[S = \pm \frac{4\pi}{L} \sum_n \left[ (n \times \tilde{A}^{(n)}) \cdot A^{(n)} - \tilde{A}^{(n)} \cdot (n \times A^{(n)}) \right] \] (4.8)

is also a solution. This solution is

\[S = \mp 4 \int A \cdot B \, d^3 x = \pm 4 \int A \cdot (\nabla \times A) \, d^3 x, \] (4.9)
which is the Chern-Simons term for a \( U(1) \) gauge theory.

If one takes \( e^{-S} \) for (4.6) and (4.9) one can generate two “ground state” quantum solutions. Which of these two corresponds to the \( e^{iG} \) Ashtekar solution of the previous section? It is possible to set up the analogue of the Ashtekar variables for the free electromagnetic field by taking the new variables \( \pi^{\prime i} = \pi^{i} + 4B^{i} \) and \( A^{\prime i} = A^{i} \). The mode parameters of \( \pi^{\prime i} \) are

\[
P^{\prime i}(n) = iP^{i}(n) + \frac{8\pi}{L}(n \times \vec{A}(n))^{i},
\]

\[
\tilde{P}^{\prime i}(n) = i\tilde{P}^{i}(n) - \frac{8\pi}{L}(n \times A(n))^{i}.
\] (4.10)

The Hamiltonian in terms of these variables (with all momenta standing to the right) is

\[
H = \sum_{n} \left[ -\frac{1}{8}P^{\prime i}(n)P^{\prime i}(n) + \frac{2\pi}{L}(n \times \vec{A}(n))^{i}P^{\prime i}(n) - \frac{1}{8}\tilde{P}^{\prime i}(n)\tilde{P}^{\prime i}(n) - \frac{2\pi}{L}(n \times A(n))^{i}\tilde{P}^{\prime i}(n) \right].
\] (4.11)

If the \( P^{\prime i}(n) \) and \( \tilde{P}^{\prime i}(n) \) are realized as \( \partial / \partial A^{(n)}_{i} \) and \( \partial / \partial \vec{A}^{(n)}_{i} \), \( \dot{H}\Psi(A^{(n)}_{i}, \vec{A}^{(n)}_{i}, t) = i\partial\Psi / \partial t \) has as a time-independent solution \( \Psi = \text{const.} \). The generating function of the transformation between \( P^{i}(n), A^{i}_{i}, \tilde{P}^{i}(n), \vec{A}^{(n)}_{i} \) and \( P^{\prime i}(n), A^{(n)}_{i}, \tilde{P}^{\prime i}(n), \vec{A}^{(n)}_{i} \) is calculated in exactly the same way as for the harmonic oscillator example in Sec. 2, and we have

\[
G(A^{(n)}_{i}, \vec{A}^{(n)}_{i}) = \frac{4\pi i}{L} \sum_{n} \left[ (n \times \vec{A}(n)) \cdot A_{i}(n) + \vec{A}(n) \cdot (n \times A_{i}(n)) \right]
\]

\[
= 4i \int A \cdot (\nabla \times A) \, d^{3}x.
\] (4.12)

So, \( e^{iG} \) is \( e^{-S} \) for the Chern-Simons \( S \) given by (4.9).

The two solutions (4.6) and (4.9) are both valid solutions for \( S \), but only (4.6) is the true ground state, while \( e^{-S} \) for the proper sign of (4.9) and the “Ashtekar” \( e^{iG} \) cannot reasonably be called a “ground state” of the free electromagnetic field. The electromagnetic “Ashtekar” analysis leads one to suspect that the Ashtekar “ground state” for the diagonal Bianchi IX model given in the last section is somehow the equivalent of (4.9), but, if so, the striking coincidence between it and the solution from the Einstein-Hamilton-Jacobi equation should be explained. Of course, it might be that the gravitational Ashtekar state and the “ground state” solution from the Einstein-Hamilton-Jacobi equation are simply the same, but a study of
the “cosmological” equivalents of the free electromagnetic field shows that the coincidence is most likely an artifact of homogeneity of the cosmological gravitational field.

By “cosmological” electromagnetic field I mean a vector potential defined as a spatially homogeneous vector field over a Bianchi-type three-space. That is, \( A = A_i(t)x^i \), where the \( \sigma^i = \xi_a^i dx^a \) are homogeneous one-forms on one of the nine Bianchi three-spaces. Here \( d\sigma^i = (1/2)C^i_{jk}\sigma^j \wedge \sigma^k \) [or \( (\xi^i_{a,b} - \xi^i_{b,a})\xi^a_j = C^i_{jk} \)], \( C^i_{jk} = \text{const.} \). The three-space metric is no longer Euclidean, so indices must be raised and lowered correctly and, in principle, covariant derivatives must be used, but the curl of \( A_a \) is unchanged, and \( g^{ac}g^{bd}(A_{c,d} - A_{d,c})(A_{a,b} - A_{b,a}) \) for example is \( C^i_{\ell m}C_{\ell m}^{kj}A_kA_l \). The action for the electromagnetic field is \( (\pi^i(t) = \pi^a_i, \pi^i = \dot{\pi}^a_i, \pi^i = 0) \)

\[
I = \frac{1}{16\pi} \int \{[\pi^i \dot{A}_i - (1/8)\pi^i \pi^i + C^i_{\ell m}C_{\ell m}^{kj}A_kA_l]\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \} dt. \tag{4.14}
\]

For all class A Bianchi models \( C^i_{jk} = \varepsilon_{ijk}m^i, m^{ij} = \text{const.} \), so \( C^i_{\ell m}C_{\ell m}^{ki} = 2m^{ks}m^{sk} \), and

\[
I = \frac{V}{16\pi} \int \{\pi^i \dot{A}_i - (1/8)\pi^i \pi^i + 2m^{is}m^{sk}A_iA_k\} dt, \tag{4.15}
\]

where \( V \equiv \int \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \) (and the space is artificially closed when necessary).

From the Hamiltonian \((1/8)\pi^i \pi^i + 2m^{is}m^{sk}A_iA_k\), the Euclidean Hamilton-Jacobi equation is

\[
\frac{\partial S}{\partial t} = \frac{1}{8} \frac{\partial S}{\partial A_i} \frac{\partial S}{\partial A_i} - 2m^{is}m^{sk}A_iA_k. \tag{4.16}
\]

For \( \partial S/\partial t = 0 \) the “ground state” solution is \( S = \pm 2m^{ij}A_iA_j \). Also for \( \partial S/\partial t = 0 \) it is possible to factor Eq. (4.16) as was done in Eq. (4.7) as

\[
\left( \frac{\partial S}{\partial A_i} + 4m^{is}A_s \right) \left( \frac{\partial S}{\partial A_i} - 4m^{ik}A_k \right) = 0. \tag{4.17}
\]

This equation has two solutions which are exactly the same \( S = \pm 2m^{ij}A_iA_j \) as the “ground state” solution given above. One can also make the Ashtekar transformation \( \pi'^i = i\pi^i + 4m^{is}A_s, A'_i = A_i \), and the Hamiltonian reduces to

\[
-\frac{1}{8}\pi'^i \pi'^i + m^{is}A'_s \pi'^i. \tag{4.18}
\]

If in the quantum theory one realizes \( \hat{\pi}'^i \) as \( \partial/\partial A_i \), \( \hat{H}\Psi = 0 \) has as a solution \( \Psi = \text{const.} \). It is easy to show that the generating function of the transformation is \( G = i(V/16\pi)(2m^{is}A_iA_s) \).

In this cosmological case the three methods of generating a “ground state” wave function, \( e^{-S} \) for the “Wheeler” \( S \), the Chern-Simons \( S \), and the “Ashtekar” \( e^{iG} \) give essentially the same function. Notice that \( G \) has the overall factor \( V/16\pi \),
which is a distinguishing feature noted in Ref. [6]. Of course, the three space can be rescaled in such a way that \( V = 16\pi \), as was done there, so the coincidence between the three functions becomes exact. Notice also that for the Bianchi I space all three wave functions are constant.

It is easy to see that the reason the three different methods lead to the same wave function is that the curl of \( \mathbf{A} \) is a constant matrix times \( \mathbf{A} \) in this “cosmological” case. In general, \( \nabla \times \mathbf{A} \) can be quite different from \( \mathbf{A} \) itself. For a constant \( \mathbf{B} \) field in flat space, \( \mathbf{A} = \mathbf{B} \times \mathbf{r} \), which is perpendicular to \( \mathbf{B} \), so \( \mathbf{A} \cdot \mathbf{B} = 0 \). However, for a homogeneous field over a Class A Bianchi space (with the exception of the Bianchi I space), the linear relation between \( \nabla \times \mathbf{A} \) and \( \mathbf{A} \) means that the Chern-Simons (and “Ashtekar”) wave functions are equal to the “Wheeler” ground state. Unfortunately, this is fortuitous and the Chern-Simons state cannot, in general, be the ground state of the free quantum electromagnetic field.

5. Conclusions

In Ref. [6] the Ashtekar variable approach was used to generate a wave function of the Bianchi-type IX universe that was the same as one constructed from a solution to the “Euclidean” Einstein-Hamilton-Jacobi equation. In the present article it was shown that the coincidence of these two solutions in that case was something of an accident, since the Ashtekar solution is closer in spirit to one constructed from the Chern-Simons term in the Maxwell field. In general one cannot expect these two solutions to coincide, and the polarized Gowdy model is an example where they do not.

Notice, however, that for the free electromagnetic field the Chern-Simons term generates an acceptable quantum solution for the field, whether or not it can be called a true “ground state”. In the gravitational case it should be possible to generate these solutions in a number of cases which will give a set of perhaps useful solutions to the Wheeler-DeWitt equation. From the analysis of the “cosmological” electromagnetic fields, one might expect that the Ashtekar procedure will lead to correct “ground state” solutions, at least in the case of Bianchi quantum cosmologies, and perhaps in other cases as well.

Acknowledgements

This article is based in part on discussions with V. Moncrief in preparation for a more extended article.

References

1. J. Plebański, J. Math. Phys. 18 (1977) 2511.
2. See, for example, A. Barajas, G. D. Birkhoff, C. Graef, and N. Vallarta, Phys. Rev. 66 (1944) 54; M. Moshinsky, Phys. Rev. 80 (1950) 514.

3. See, for example, A. Ashtekar, New Perspectives in Quantum Gravity (Bibliopolis, Naples, 1988).

4. A. Ashtekar and J. Pullin, Ann. Israel Phys. Soc. 9 (1990) 66.

5. O. Obregon, J. Pullin, and M. Ryan, To appear, Phys. Rev. D.

6. V. Moncrief and M. Ryan, Phys. Rev. D44 (1991) 2375.

7. V. Bargmann, Comm. Pure App. Math. 14 (1961) 187.

8. C. Misner, K. Thorne, and J. Wheeler, Gravitation (Freeman, San Francisco, 1973).

9. V. Fock, Z. Physik 49 (1928) 339.

10. A. Anderson, Private communication.

11. J. Friedman and I. Jack, Phys. Rev. D37 (1988) 3495.

12. H. Kodama, Prog. Theor. Phys. 80 (1988) 1024; Phys. Rev. D36 (1987) 1587.

13. R. Capovilla, Private communication.

14. R. Gowdy, Phys. Rev. Lett. 27 (1971) 826; Ann. Phys. (N. Y.) 83 (1974) 203.

15. C. Misner, Phys. Rev. D8 (1973) 3271.

16. B. Berger, Ann. Phys. (N. Y.) 83 (1974) 458; Phys. Rev. D11 (1975) 2770.

17. V. Moncrief and M. Ryan, In preparation.

18. J. Wheeler, Geometrodynamics (Academic Press, New York, 1962).