MOTIVIC INVARIANTS OF BIRATIONAL MAPS

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ABSTRACT. We construct invariants of birational maps with values in the Kontsevich–Tschinkel group and in the truncated Grothendieck groups of varieties. These invariants are morphisms of groupoids and are well-suited to investigating the structure of the Grothendieck ring and L-equivalence. Building on known constructions of L-equivalence, we prove new unexpected results about Cremona groups.

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1. INTRODUCTION

Let \( k \) be a field. The Cremona groups \( \text{Cr}_n(k) = \text{Bir}(\mathbb{P}_k^n) \) have been actively studied in birational geometry since the 19th century. The aim of this work is to contribute to the study of Cremona groups through a motivic viewpoint and bring new methods and results to light.

1.1. Generating sets of Cremona groups.

We address the problem about the existence of simple generating sets of Cremona groups. For the Cremona group of complex projective plane, it has a well-known generating set due to M. Noether and Castelnuovo.

Theorem 1.1 (Noether–Castelnuovo [56, 16]). For any algebraically closed field \( k \), the Cremona group \( \text{Cr}_2(k) \) is generated by \( \text{PGL}_3(k) \) and the Cremona involution

\[
[X : Y : Z] \mapsto [YZ : XZ : XY].
\]
When \( n \geq 3 \), the Cremona groups \( \text{Cr}_n(k) \) are much less well understood. In terms of the size of generating sets, Hudson and Pan showed that for an algebraically closed field \( k \) of characteristic zero, \( \text{Cr}_n(k) \) is never generated by \( \text{PGL}_{n+1}(k) \) together with any set of transformations of bounded degree or countably many elements when \( k \) is uncountable [58]. It is therefore an intricate question in higher dimension whether there exist, group-theoretically or geometrically, simple transformations which generate the entire \( \text{Cr}_n(k) \). See e.g. [9, 1.C] for a historical account of this problem, which has been already mentioned in the 1895 lectures by Enriques. Currently, no explicit set of generators of Cremona groups in dimension \( n \geq 3 \) is known.

Among the birational automorphisms, regularizable maps form one of the most explicit classes of elements of \( \text{Cr}_n(k) \). Recall that an element \( \phi \in \text{Cr}_n(k) \) is called regularizable if there exists a birational map \( \alpha : \mathbb{P}^n \rightarrow X \) to some variety \( X \) such that \( \alpha \circ \phi \circ \alpha^{-1} \in \text{Bir}(X) \) is a regular automorphism. See Definition 4.1 for a more general definition of pseudo-regularizable maps. Regular automorphisms of \( \mathbb{P}^n \) are regularizable, and so are elements of \( \text{Aut}(U) \subset \text{Cr}_n(k) \) for every open \( U \subset \mathbb{P}^n \). Furthermore, all elements of \( \text{Cr}_n(k) \) of finite order are regularizable by Weil’s regularization theorem (see e.g. [17, Theorem 1.7]). Thus Theorem 1.1 implies that \( \text{Cr}_2(\mathbb{C}) \) is generated by regularizable elements.

In higher dimensions, our main result shows that the contrary holds in most situations and accordingly, any set of generators of \( \text{Cr}_n(k) \) is necessarily quite complicated.

**Theorem 1.2.** In each of the following cases, the Cremona group \( \text{Cr}_n(k) \) is not generated by pseudo-regularizable elements (which include \( \text{PGL}_{n+1}(k) \) and all elements of finite order):

1. \( n = 3 \), and all number fields \( k \), or all function fields \( k \) over a number field, over a finite field or over an algebraically closed field;
2. \( n \geq 4 \), and all fields \( k \subset \mathbb{C} \);
3. \( n \geq 5 \), and all infinite fields \( k \).

We refer to Theorem 4.4 for a more general statement regarding \( \text{Bir}(X) \) for classes of varieties \( X \) birational to \( \mathbb{P}^n \times W \) where \( W \) is a (separably) rationally connected variety. Theorem 1.2 disproves a conjecture by Cheltsov in 2004, which says that birational automorphism groups are generated by regularizable elements [17, Conjecture 1.12]. It also gives a negative answer to a more specific question of Dolgachev [22, Question on p.1] on whether Cremona groups are generated by involutions. We will explain below why certain types of fields appear in Theorem 1.2. For now let us note that even though any field extension \( k \subset k' \) induces an embedding \( \text{Cr}_n(k) \subset \text{Cr}_n(k') \), there is no straightforward group-theoretic comparison between \( \text{Cr}_n(k) \) and \( \text{Cr}_n(k') \) (in particular \( \text{Cr}_n(\overline{k}) \)) in terms of their generating sets.

### 1.2. Motivic invariants.

Our method of proof relies on a motivic invariant of birational maps. Here, by ‘motivic’ we mean in the sense of the Grothendieck ring of varieties, which is one the original constructions of the ring of motives due to Grothendieck, and the Burnside ring of Kontsevich–Tschinkel [39].
For an \( n \)-dimensional variety \( X \) over \( k \), this is a group homomorphism
\[
c : \text{Bir}(X) \to \mathbb{Z}[\text{Bir}_{n-1}/k].
\]
Here \( \text{Bir}_{n-1}/k \) is the set of \((n-1)\)-dimensional birational equivalence classes of varieties over \( k \). The invariant \( c \) records the birational classes of the exceptional divisors created and contracted by the birational map. We prove that in the situations described in Theorem 1.2 for \( X = \mathbb{P}^n \), the invariant is non-trivial, while it vanishes on all pseudo-regularizable elements. Furthermore, using the explicit non-vanishing of the invariant \( c \) we can construct infinitely many homomorphisms \( \text{Cr}_n(k) \to \mathbb{Z} \) for various fields \( k \) and \( n \geq 3 \), see Corollary 4.5. This is in contrast to the homomorphisms \( \text{Cr}_n(k) \to \mathbb{Z}/2 \) constructed in [9].

Our invariant \( c \) has an enhancement \( \tilde{c} : \text{Bir}(X) \to \mathcal{K}_0(\text{Var}^{\leq n-1}/k) \), taking values in the truncated Grothendieck group of varieties in such a way that \( c \) is the composition
\[
\text{Bir}(X) \xrightarrow{\tilde{c}} \mathcal{K}_0(\text{Var}^{\leq n-1}/k) \to \mathbb{Z}[\text{Bir}_{n-1}/k],
\]
where the second homomorphism is the projection to the top dimensional birational classes. We develop the invariant \( \tilde{c} \) because some proofs are more transparent in this context, and also because it provides full control over the Grothendieck ring of varieties, as it determines how \( \mathcal{K}_0(\text{Var}^{\leq n-1}/k) \) is mapped to \( \mathcal{K}_0(\text{Var}^{\leq n}/k) \) (see Proposition 2.10). None of these constructions require any form of the resolution of singularities, and we work over an arbitrary field \( k \).

Once the invariants \( \tilde{c} \) and \( c \) are constructed and their properties settled in Section 2, in Section 3 we introduce and construct examples of the so-called L-links (L stands for the Lefschetz class \( L \) in the Grothendieck ring and L-equivalence [42]); these links provide a geometric input suitable for L-equivalence constructions and for showing the non-vanishing of \( c(\text{Bir}(X)) \). To construct non-trivial L-links, we rely on the geometric constructions of L-equivalence involving genus one curves [53, 19, 63], K3 surfaces [27], and Calabi-Yau threefolds [31]. Even though these constructions already exist in the literature when \( k = \mathbb{C} \), much effort goes into generalizing them and checking that such L-links \( \phi \) are motivically non-trivial, namely \( c(\phi) \neq 0 \), over a more general field \( k \) as in Theorem 1.2.

For instance, links involving genus one curves become motivically trivial over \( \mathbb{C} \), but not over small fields as in Theorem 1.2(1); roughly speaking, Theorem 1.2(1) works for these fields because they have infinitely many Galois extensions of fixed degree. To prove Theorem 1.2(2), we use the construction [27], taking as input a general complex K3 surface of degree 12 and Picard rank one, together with some rational points on it. We extend the construction to \( \mathbb{Q} \) nevertheless, representing K3 surfaces of degree 12 as hyperplane sections of a rational Fano threefold of degree 12 defined over \( \mathbb{Q} \), and using Terasoma’s argument to prove the existence of K3 surfaces over \( \mathbb{Q} \) of Picard rank one [67] involving Lefschetz pencils. Finally to prove Theorem 1.2(3), we need to show that the L-links with Calabi-Yau threefolds as centers [31]
exist and can be nontrivial over any infinite field \( k \). We have to deal with Calabi-Yau varieties which are possibly uniruled, and standard arguments in characteristic zero involving MRC fibrations do not apply. Instead, we rely on a weaker statement in the spirit of MRC fibrations for separably rationally connected varieties, which we prove in Appendix A. We also include Theorem A.1, due to Burt Totaro, showing that birational varieties with nef canonical class and Picard number one are isomorphic, in arbitrary characteristic, which was previously known in characteristic zero or in positive characteristic in dimension up to three.

1.3. Relation to other work.

1.3.1. \( L \)-equivalence.

This paper was motivated by understanding the geometric meaning of the information encoded by the Grothendieck ring of varieties and its variants, which have played a spectacular role in the recent results on the specialization of stable rationality [55] and rationality [39]. In particular we want to understand \( L \)-equivalence [12], [27], [42], [31]; our concept of \( L \)-links formalizes the existing geometric constructions leading to \( L \)-equivalence.

This is a continuation of our work [48] where jointly with Susanna Zimmermann we have studied the invariant \( c(\phi) \) for surfaces \( X \) over perfect fields using Minimal Model Program. We showed that \( c(\text{Bir}(X)) = 0 \) and as a consequence, birational maps between surfaces do not produce non-trivial \( L \)-equivalence.

On the other hand, non-trivial \( L \)-equivalence already exists from dimension 3 and on, as we will explain a construction in Subsection 3.3 originating from curves in rational threefolds. Namely, by Theorem 3.7, we have \( L([C] - [C']) = 0 \) for a pair of genus one curves. This improves the exponent of \( L \) in [63] from \( L^4 \) to the minimal possible.

1.3.2. Studying birational maps through the blow up centers.

Making an invariant \( c \) from the blow up centers or the exceptional divisors is similar in spirit to the filtration on the Cremona groups by the genus of the curves blown up by the maps of threefolds, considered and studied by Frumkin [25], Lamy [43], Bernardara [7] and Blanc–Cheltsov–Duncan–Prokhorov [8]. In fact similar considerations also appear in the proof of the Hudson–Pan theorem [58], which relies on the existence of a large set of possible types of exceptional divisors for Cremona transformations. Comparing to the above works, our approach has the key feature that the invariant \( c \) is a homomorphism, which in a way refines these filtrations.

The relationship between birational maps and the structure of the Grothendieck ring of varieties also appears in the work of Zakharevich in terms of the differentials in a certain spectral sequence [68] which converges to the higher K-theory groups lying over the Grothendieck ring of varieties. Our invariant \( c(\phi) \) coincides with the first differential from that spectral sequence [68, Lemma 3.2], and the invariant \( \tilde{c}(\phi) \) is a certain enhancement of that map.

On the other hand, relating the Grothendieck ring of varieties and the Cremona groups is implicitly suggested in [27]. Part of the motivation for our work was to understand to which
extent the blow up centers of a rationality construction $\mathbb{P}^n \dasharrow X$ are determined by $X$ itself, which was asked in [27, Introduction]. The intrinsic ambiguity in these centers leads to the non-vanishing of $c(\text{Bir}(X))$, which makes our applications to the Cremona groups possible. Along the way we answer a question of Hassett and Tschinkel about the realization of non-isomorphic genus one curves as factorization centers for rational threefolds over non-closed fields [28, Remark 23 (2)], by Theorem 3.7 which was already mentioned above.

1.3.3. Non-simplicity of the Cremona groups and constructions of non-trivial homomorphisms.

There has been important progress concerning the non-simplicity of the Cremona groups over the last decade; see [13] and [9, Introduction] for an excellent overview of the history and recent results. First of all, in 2012 Cantat and Lamy proved that for any algebraically closed field $k$, $\text{Cr}_2(k)$ is not a simple group [15]. This result was later extended to $\text{Cr}_2(k)$ for any field $k$ by Lonjou [50]. Using the Minimal Model Program and Birkar’s boundedness, Blanc–Lamy–Zimmermann [9] recently constructed infinitely many homomorphisms $\text{Cr}_n(k) \to \mathbb{Z}/2$ ($n \geq 3$) when $k$ is a subfield of $\mathbb{C}$ (see also [11]). Their constructions originate from Sarkisov links and relations between them, and are of quite different nature to the homomorphisms $\text{Cr}_n(k) \to \mathbb{Z}$ we obtain in Corollary 4.5 through the motivic invariants.

1.3.4. Generation of Cremona groups by involutions.

As a consequence of Theorem 1.1, $\text{Cr}_2(\mathbb{C})$ is generated by involutions. The question whether same is true in higher dimension was asked by Dolgachev during his series of lectures in Toulouse in 2016. Motivated by this question, Deserti has proved that $\text{PGL}_{n+1}(\mathbb{C})$ is generated by involutions [22, Proposition C]. Lamy and Schneider have proved that Cremona groups $\text{Cr}_2(k)$ over a perfect field $k$ are generated by involutions [44], thus providing a positive answer to Dolgachev’s question for $n = 2$. On the other hand, Blanc, Schneider and Yasinsky announced that birational automorphisms of Severi–Brauer surfaces are not generated by elements of finite order [10], extending previous results by Shramov [64]. As a consequence of their result, they are able to give a different proof that $\text{Cr}_n(\mathbb{C})$ for $n \geq 4$ is not generated by $\text{PGL}_{n+1}(\mathbb{C})$ and elements of finite order.

1.4. Notation and conventions.

Unless specified otherwise, we work over an arbitrary field $k$. By a variety over $k$, we mean an irreducible and reduced separated scheme of finite type over $k$. We will use various Grothendieck groups and rings, which we summarize for the convenience of the reader:

(a) $\mathbb{Z}[\text{Bir}/k]$ (resp. $\mathbb{Z}[\text{Bir}_n/k]$) is the free abelian group generated by birational isomorphism classes of varieties (resp. varieties of dimension $n$) over $k$. By definition $\mathbb{Z}[\text{Bir}/k] = \bigoplus_{n \geq 0} \mathbb{Z}[\text{Bir}_n/k]$. This is the graded Burnside ring of Kontsevich–Tschinkel [39];

(b) $K_0(\text{Var}/k)$ (resp. $K_0(\text{Var}^{\leq n}/k)$) is the Grothendieck ring (resp. group) of varieties (resp. varieties of dimension $\leq n$) over $k$ modulo cut and paste relations; see Subsection 2.3 for the precise definitions;
(c) $K_0(PPAV/k)$ is the Grothendieck group of the additive category of principally polarized abelian varieties over $k$ modulo relations $[A \times B] = [A] + [B]$;

(d) for a profinite group $G$, $K_0(\text{Rep}(G, \mathbb{Q}_\ell))$ is the Grothendieck group of the abelian category of finite-dimensional continuous $G$-representations with coefficients in $\mathbb{Q}_\ell$. See [48, Section 2.1] for the details of this construction.

Among these, (a) and (b) are central for this paper and are explained in detail and related to each other in Subsection 2.3. The Grothendieck groups (c) and (d) will be used for some technical purposes as “motivic realizations” of varieties and birational classes, see in particular (2.5), (2.6).

2. The motivic invariants

2.1. Invariant $c(\phi)$ with values in $\mathbb{Z}[\text{Bir}_{n-1}/k]$.

Let $k$ be a field and let $X$ and $Y$ be two varieties over $k$. A birational map $\phi : X \dashrightarrow Y$ is a morphism $U \to Y$ over $k$ defined on a Zariski dense open subscheme $U \subset X$ which is isomorphic onto its image. Two birational maps $\phi, \psi : X \dashrightarrow Y$ are identical if there exists an open $U \subset X$ such that $\phi|_U = \psi|_U$ as morphisms. We can compose birational maps and each birational map $\phi$ has an inverse $\phi^{-1}$, thereby defining a groupoid $\text{Bir}/k$ of birational classes consisting of $k$-varieties as objects and birational maps as morphisms.

We say that $X$ and $Y$ are birational if there is a birational map between them. We say that $X$ and $Y$ are stably birational if $X \times \mathbb{P}^m_k$ and $Y \times \mathbb{P}^m_k$ are birational for some $k, \ell \geq 0$. The set of birational equivalence classes of $n$-dimensional $k$-varieties is denoted by $\text{Bir}_n/k$. The set of birational maps from $X$ to $Y$ is denoted by $\text{Bir}(X, Y)$.

Let $\phi : X \dashrightarrow Y$ be a birational map between $n$-dimensional $k$-varieties. We say that $\phi$ is an isomorphism at a point $x \in X$ if there exists a Zariski open $U \subset X$ containing $x$ such that $\phi|_U = \psi|_U$ as morphisms. We define the exceptional set as

$\text{Ex}(\phi) := \{ x \in X \mid \phi \text{ is not an isomorphism at } x \} \subset X$.

According to the definition, $\text{Ex}(\phi)$ is a closed subset containing the indeterminacy locus of $\phi$.

By definition, $\phi$ establishes an isomorphism

$$X \setminus \text{Ex}(\phi) \simeq Y \setminus \text{Ex}(\phi^{-1})$$

**Definition 2.1.** For every birational map $\phi : X \dashrightarrow Y$ of $k$-varieties, we set

$$c(\phi) := \sum_{y \in \text{Ex}(\phi)^{-1}, \text{codim}_Y y = 1} [k(y)] - \sum_{x \in \text{Ex}(\phi), \text{codim}_X x = 1} [k(x)] \in \mathbb{Z}[\text{Bir}_{n-1}/k]$$

where $\mathbb{Z}[\text{Bir}_{n-1}/k]$ is the free abelian group generated by $\text{Bir}_{n-1}/k$, and where the function fields $k(y), k(x)$ are identified with the corresponding birational classes.
Since both $\text{Ex}(\phi) \subset X$ and $\text{Ex}(\phi^{-1}) \subset Y$ are Zariski closed, the sums in the definition of $c(\phi)$ are finite. The notation $c$ stands for centers of the exceptional divisors [38, Definition 2.24], as they are encoded in (2.2) in a certain way (see e.g. Proposition 2.3).

A crucial property of $c$ is that it defines a homomorphism from the groupoid $\overline{\text{Bir}}/k$ of birational classes to $\mathbb{Z}[\text{Bir}/k]$.

**Lemma 2.2.** Given birational maps $\phi : X \to Y$ and $\psi : Y \to Z$, we have

$$c(\psi \circ \phi) = c(\phi) + c(\psi).$$

Lemma 2.2 has a natural proof based on motivic cut-and-paste, and we postpone the proof until Subsection 2.4, see Theorem 2.8.

Note that we work over arbitrary fields. In particular, the definition of $c$ and the proof of Lemma 2.2 do not rely on any version of the resolution of singularities. Over fields of characteristic zero or for surfaces over arbitrary fields we understand birational maps better, thanks to Hironaka’s resolution of singularities (see e.g. [37]) and Weak Factorization [1], [66, Lemma 54.17.2]. The latter asserts that every birational map $\phi : X \to Y$ between smooth complete varieties over $k$ admits a factorization

$$\xymatrix{ X & Y_1 \ar[d]_{p_1} \ar[l]_{q_1} & Y_2 \ar[d]_{p_2} \ar[l]_{q_2} & \cdots & Y_{m-1} \ar[d]_{p_{m-1}} \ar[l]_{q_{m-1}} & Y_m \ar[d]_{p_m} \ar[l]_{q_m} & Y \ar[l]_{q_m} }$$

where each $p_i$ (resp. $q_i$) is a sequence of blow ups (resp. blow downs) along smooth irreducible closed subvarieties of codimension $\geq 2$ or an isomorphism (which we understand as a blow up with empty center).

**Proposition 2.3.** If a birational map $\phi : X \to Y$ between $n$-dimensional smooth complete $k$-varieties admits a factorization (2.3), then

$$c(\phi) = \sum_{Z \in \mathcal{W}} [\mathbb{P}^{n-\dim(Z)-1} \times Z] - \sum_{T \in \mathcal{D}} [\mathbb{P}^{n-\dim(T)-1} \times T] \in \mathbb{Z}[\text{Bir}_{n-1}/k]$$

where $\mathcal{W}$ (resp. $\mathcal{D}$) is the set of blow up centers of $p_i$ (resp. $q_i$).

In particular, the alternating sum in (2.4) is well-defined, namely it is independent of the choice of weak factorization of $\phi$.

In (2.4) we use the convention that $[\emptyset] = 0$.

**Proof.** Proposition 2.3 follows easily from Lemma 2.2 and the definition of $c$, because the exceptional divisors of the $p_i$’s are of the form $\mathbb{P}^{n-\dim(Z)-1} \times Z$, for $Z \in \mathcal{W}$ and similarly for the $q_i$’s. \qed

We can relate $c(\phi)$ to the Galois action on the exceptional divisors. Let $G_k = \text{Gal}(k^{\text{sep}}/k)$ and let $K_0(\text{Rep}(G_k, \mathbb{Q}_\ell))$ be the Grothendieck ring of continuous finite-dimensional $\ell$-adic representations of $G_k$; here $\ell$ is a fixed prime number different from $\text{char}(k)$. Define the group
homomorphism

\[ \sigma : \mathbb{Z}[^{\text{Bir/}}/{k}] \rightarrow K_0(\text{Rep}(G_{{}^k}, \mathbb{Q}_\ell)) \]

by sending the class of a \( {}^k \)-irreducible variety \( X \) to the permutation representation on the irreducible components of \( X_{{}^{k}\text{sep}} \) with coefficients in \( \mathbb{Q}_\ell \).

For a smooth projective variety \( X/{}^k \), we consider the Néron-Severi group with \( \mathbb{Q}_\ell \)-coefficients

\[ N(X) := \text{NS}(X_{{}^{k}\text{sep}}) \otimes \mathbb{Q}_\ell. \]

Then \( N(X) \) is a finite-dimensional \( \mathbb{Q}_\ell \)-vector space [36, Theorem II.4.5] with a continuous Galois action.

**Proposition 2.4.** Let \( k \) be a field of characteristic zero. For every birational map \( \phi : X \to Y \) between smooth projective varieties, we have

\[ \sigma(c(\phi)) = [N(Y)] - [N(X)]. \]

In particular, if \( \phi \in \text{Bir}(X) \), then \( \sigma(c(\phi)) = 0 \).

**Proof.** By the Weak Factorization Theorem, the statement is reduced to the case where \( \phi : X \to Y \) is a single blow up with a smooth \( k \)-connected center \( Z \) of codimension \( \geq 2 \). In this case the result follows from the isomorphism of Galois representations \( N(X) \simeq N(Y) \oplus \mathbb{Q}_\ell[E] \), where \( \mathbb{Q}_\ell[E] \) stands for the permutation representation on the geometric irreducible components of \( E \), because \( \sigma(c(\phi)) = \sigma(-[E]) = -[\mathbb{Q}_\ell[E]] \). \qed

In the rest of this section, we first explain for which kinds of varieties \( c \) is identically zero, then enhance \( c(\phi) \) to an invariant \( \tilde{c}(\phi) \) which takes values in the Grothendieck groups of varieties truncated by the dimension. We will explain how these invariants control the structure of the Grothendieck ring viewed as the colimit of its truncations. The reader who is interested in examples and geometric applications can skip these parts and go directly to Section 3.

### 2.2. Vanishing results for \( c(\phi) \).

Given a birational automorphism \( \phi : X \to X \) of variety \( X \), there are situations where \( c(\phi) = 0 \) is automatic. First of all, by definition this is always the case when \( \phi \) is an isomorphism, or just a pseudo-isomorphism (isomorphism in codimension one). This applies to smooth projective curves and to all smooth projective varieties \( X \) with \( K_X \) is nef (e.g. Calabi–Yau varieties), by Theorem A.1.

In dimension two, the following vanishing result was proven in [48].

**Theorem 2.5.** [48, Theorem 3.4] If \( k \) is a perfect field and \( X/{}^k \) is a surface, then \( c(\text{Bir}(X)) = 0 \).

In dimension 3, the vanishing \( c(\text{Bir}(X)) = 0 \) still holds in certain cases, mostly depending on the base field \( k \). The proof relies on the existence of the principally polarized intermediate Jacobian defined for rationally connected threefolds over non-algebraically closed fields [2], [3], [6].

**Proposition 2.6.** Let \( k \) be a field of characteristic zero, and \( X \) be a smooth projective rationally connected threefold. The image \( c(\text{Bir}(X)) \) is contained in the subgroup of \( \mathbb{Z}[\text{Bir}_2/{}^k] \) generated by
classes of the form $[\mathbb{P}^1 \times C]$ where $C$ is an irreducible smooth curve whose geometric irreducible components have genus zero or one. If $k$ is algebraically closed, then $c(\text{Bir}(X)) = 0$.

See Theorem 3.7 for genus one curves appearing in the image of $c(\text{Bir}(\mathbb{P}^3))$ over non-algebraically closed fields.

**Proof.** By the Weak Factorization theorem and Proposition 2.3, the image $c(\text{Bir}(X))$ is contained in the subgroup of $Z[\text{Bir}_2/k]$ generated by classes $[\mathbb{P}^1 \times C]$ for smooth irreducible curves $C$ (if the blow up center is a closed point $Z$, the corresponding term is $[\mathbb{P}^2 \times Z] = [\mathbb{P}^1 \times \mathbb{P}^1_Z]$).

Now consider the assignment

$$[X] \mapsto \begin{cases} J(C), & \text{if } X \text{ is birational to a ruled surface over } C \\ 0, & \text{otherwise} \end{cases}$$

for every birational class $[X] \in \text{Bir}_2/k$ of irreducible surfaces, where $C$ is taken to be an irreducible smooth projective curve, and $J(C)$ is the Jacobian of $C$. This defines a group homomorphism

$$j : Z[\text{Bir}_2/k] \to K_0(\text{PPAV}/k),$$

where $K_0(\text{PPAV}/k)$ is the Grothendieck group of principally polarized abelian varieties.

Let us prove that for any $\phi \in \text{Bir}(X,Y)$ where $X$ and $Y$ are smooth projective rationally connected threefolds, we have

$$j(c(\phi)) = [J(Y)] - [J(X)],$$

where $J(X)$ is the intermediate Jacobian of $X$ [6]. Once again by Weak Factorization and Lemma 2.2, it suffices to check (2.7) for a single blow up along an irreducible center. This follows from the blow up formulas [6, Lemma 2.10] that $J(\text{Bl}_C(X)) \simeq J(X) \times J(C)$ when $C \subset X$ is a smooth projective curve, and $J(\text{Bl}_P(X)) \simeq J(X)$ for a point $P \in X$. In particular, for any $\phi \in \text{Bir}(X)$ we have

$$c(\phi) \in \text{Ker}(j) \cap ([\mathbb{P}^1] \cdot Z[\text{Bir}_1/k]).$$

For every irreducible smooth projective curve $C$ over $k$, the Jacobian $J(C)$ is indecomposable over $k$ as a principally polarized abelian variety. Indeed, since $C$ is irreducible, the $\text{Gal}(\mathbb{F}/k)$-action on the geometric irreducible components $C_1, \ldots, C_r$, and therefore on the factors $J(C_{\mathbb{F}}) \simeq J(C_1) \times \cdots \times J(C_r)$ of the decomposition of $J(C_{\mathbb{F}})$ is transitive. Since each $J(C_i)$ is indecomposable, it follows from the uniqueness of the decomposition of principally polarized abelian varieties into irreducible ones (see [32, Theorem 3.3] or [21]) that $J(C)$ is indecomposable.

Since principally polarized abelian varieties admit unique decompositions into indecomposable ones, the Grothendieck group of principally polarized abelian varieties $K_0(\text{PPAV}/k)$ is isomorphic to the free abelian group generated by indecomposable ones. From Serre’s Torelli
theorem [47, Appendix], it follows that $J(C)$, as a principally polarized abelian variety determines $C$ when all geometric components of $C$ have genus $\geq 2$. Thus the kernel
\[ \text{Ker}(\mathbb{Z}[\text{Bir}_1/\mathbb{k}] \xrightarrow{[\mathbb{P}^1]} \mathbb{Z}[\text{Bir}_2/\mathbb{k}] \xrightarrow{i} \mathbb{K}_0(\text{PPAV}/\mathbb{k})) \]
is contained in the subgroup generated by birational classes of curves with geometric irreducible components of genus zero or genus one. This proves the first statement of Proposition 2.6.

Finally, if $\mathbb{k}$ is algebraically closed, then the Torelli theorem holds for genus one curves as well. Thus for any $\phi \in \text{Bir}(X)$, necessarily $c(\phi) = m[\mathbb{P}^1 \times \mathbb{P}^1]$ for some $m \in \mathbb{Z}$. It follows from Proposition 2.4 that $m = 0$. □

2.3. Truncated Grothendieck groups $K_0(\text{Var}^{\leq n}/\mathbb{k})$.

Let $\mathbb{k}$ be a field. Recall that $\text{Bir}_n/\mathbb{k}$ (resp. $\text{Bir}/\mathbb{k}$) denotes the set of birational isomorphism classes of $n$-dimensional varieties (resp. all varieties), and we have the free abelian group
\[ \mathbb{Z}[\text{Bir}/\mathbb{k}] = \bigoplus_{n \geq 0} \mathbb{Z}[\text{Bir}_n/\mathbb{k}]. \]
Similarly, let $\mathbb{Z}[\text{StBir}/\mathbb{k}]$ denote the free abelian group generated by the stable birational classes. We have a surjective homomorphism $\mathbb{Z}[\text{Bir}/\mathbb{k}] \twoheadrightarrow \mathbb{Z}[\text{StBir}/\mathbb{k}]$ sending each birational class to the corresponding stable birational class. Furthermore, if $\text{char}(\mathbb{k}) = 0$, we have the Larsen-Lunts isomorphism [46]
\[ K_0(\text{Var}/\mathbb{k}) \quad \xrightarrow{\sim} \quad \mathbb{Z}[\text{StBir}/\mathbb{k}], \]
where $K_0(\text{Var}/\mathbb{k})$ is the Grothendieck ring of varieties over $\mathbb{k}$ and $\mathbb{L} = [\mathbb{A}^1_\mathbb{k}]$. There is no natural homomorphism between the groups in the upper row, even in characteristic zero.

We write $K_0(\text{Var}^{\leq n}/\mathbb{k})$ for the Grothendieck group of varieties of dimension $\leq n$: the generators of $K_0(\text{Var}^{\leq n}/\mathbb{k})$ are classes $[X]_n$ of schemes of finite type $X/\mathbb{k}$ with $\dim(X) \leq n$ and the relations are generated by
\[ [X]_n = [U]_n + [Z]_n \]
for every open $U \subset X$ with closed complement $Z$. We have a sequence of natural maps
\[ \cdots \rightarrow K_0(\text{Var}^{\leq n-1}/\mathbb{k}) \xrightarrow{i_{n-1}} K_0(\text{Var}^{\leq n}/\mathbb{k}) \rightarrow K_0(\text{Var}^{\leq n+1}/\mathbb{k}) \rightarrow \cdots. \]
The colimit of the direct system defined by (2.9) is the underlying group of the Grothendieck ring of varieties
\[ K_0(\text{Var}/\mathbb{k}) = \text{colim}_n K_0(\text{Var}^{\leq n}/\mathbb{k}). \]

In this and the next subsections we will express the kernel and the cokernel of
\[ K_0(\text{Var}^{\leq n-1}/\mathbb{k}) \xrightarrow{i_{n-1}} K_0(\text{Var}^{\leq n}/\mathbb{k}) \]
for each \( n \), in terms of information contained in the groupoid of birational classes. For the cokernel, we have the exact sequence
\[
(2.10) \quad K_0(\text{Var}^{\leq n-1}/k) \xrightarrow{\iota_{n-1}} K_0(\text{Var}^{\leq n}/k) \xrightarrow{\pi_n} \mathbb{Z}[\text{Bir}_n/k] \to 0,
\]
where we define \( \pi_n \) on the classes of finite type schemes of dimension \( \leq n \) by
\[
(2.11) \quad \pi_n([X]_n) = \begin{cases} [X_1] + \cdots + [X_r], & \text{dim}(X) = n \\ 0, & \text{dim}(X) < n \end{cases},
\]
where \( X_1, \ldots, X_r \) are the irreducible components of \( X \) of dimension \( n \). The maps \( \iota_{n-1} \) are not injective in general; in Proposition 2.10 we describe their kernels, see also Remark 2.11 where we explain which ones are injective.

The following lemma is the starting point for studying the Grothendieck groups inductively. See Lemma 2.9 for a reformulation of the statement using the motivic invariant \( \tilde{c} \).

**Lemma 2.7.** Let \( \text{Var}^n/k \) denote the set of isomorphism classes of \( n \)-dimensional \( k \)-varieties. Then the natural homomorphism
\[
(2.12) \quad K_0(\text{Var}^{\leq n-1}/k) \oplus \mathbb{Z}[\text{Var}^n/k] \to K_0(\text{Var}^{\leq n}/k)
\]
is surjective and its kernel admits the following presentation:
\[
(2.13) \quad \langle ([X \setminus U]_{n-1}, -[X] + [U]) \rangle_{U \subset X},
\]
for all \( n \)-dimensional varieties \( X \) and open subsets \( U \subset X \).

Recall that by our conventions, varieties are assumed to be irreducible, while \( K_0(\text{Var}^{\leq n}/k) \) is defined with \( k \)-schemes of finite type as generators. Lemma 2.7 shows in particular that this distinction is unimportant, and \( K_0(\text{Var}^{\leq n}/k) \) could have been defined with irreducible schemes of finite type as generators.

**Proof.** It is clear from definitions that (2.12) is surjective and that its kernel contains (2.13), so that we have a well-defined surjective homomorphism
\[
(2.14) \quad \frac{K_0(\text{Var}^{\leq n-1}/k) \oplus \mathbb{Z}[\text{Var}^n/k]}{\langle ([X \setminus U]_{n-1}, -[X] + [U]) \rangle_{U \subset X}} \xrightarrow{\beta} K_0(\text{Var}^{\leq n}/k).
\]
To show that (2.14) is also injective, we construct the inverse homomorphism \( \alpha \) of \( \beta \). First we define \( \alpha(X) \) for a scheme \( X/k \) of finite type of dimension \( \leq n \). Since the class \([X]_n\) only depends on the reduced structure, we can assume that \( X \) is reduced. We stratify \( X \) into irreducible varieties which are locally closed in \( X \) and let \( X_1, \ldots, X_r \) be its \( n \)-dimensional strata \( (r = 0 \text{ if dim}(X) < n) \). The element
\[
\alpha(X) := \left([X \setminus \bigcup_{i=1}^r X_i]_{n-1}, \sum_{i=1}^r [X_i]\right)
\]
is well-defined (that is, independent of the choice of stratification of \( X \)) when considered in the left-hand side of (2.14). We verify that \( \alpha(X) = \alpha(V) + \alpha(X \setminus V) \) for every open subscheme
$V \subset X$, so that $\alpha$ defines a homomorphism from $K_0(\text{Var}^{\leq n}/k)$. By construction $\alpha$ and $\beta$ are mutually inverse. \hfill \square 

2.4. Invariant $\tilde{c}(\phi)$ with values in $K_0(\text{Var}^{\leq n-1}/k)$.

The next result implies Lemma 2.2.

**Theorem 2.8.** There exists a unique assignment

$$ \tilde{c} : \text{Bir}(X, Y) \to K_0(\text{Var}^{\leq n-1}/k) $$

defined on the birational maps between $n$-dimensional varieties over $k$ which satisfies the following two properties.

1. If $i : U \hookrightarrow X$ is the inclusion of an open subset then $\tilde{c}(i) = [X \setminus U]_{n-1}$.
2. For every $\phi \in \text{Bir}(X, Y)$, $\psi \in \text{Bir}(Y, Z)$, we have $\tilde{c}(\psi \circ \phi) = \tilde{c}(\phi) + \tilde{c}(\psi)$.

Furthermore, we have $c(\phi) = \pi_{n-1}(\tilde{c}(\phi))$, where $c(\phi)$ is defined by (2.2) and $\pi_{n-1}$ is defined by (2.11).

**Proof of Theorem 2.8 and Lemma 2.2.** Let $\phi \in \text{Bir}(X, Y)$, then there is an open subset $i : U \subset X$ such that $j = \phi|_U$ is an open embedding. We have $\phi = j \circ i^{-1}$ so that properties (1) and (2) determine $\tilde{c}(\phi)$ to be

$$(2.15) \quad \tilde{c}(\phi) = \tilde{c}(j) - \tilde{c}(i) = [Y \setminus j(U)]_{n-1} - [X \setminus U]_{n-1}$$

This shows that $\tilde{c}$ is unique.

To prove the existence, we first check that (2.15) is well-defined, that is $\tilde{c}(\phi)$ is independent of the choice of $U \subset X$. Let $i' : U' \subset X$ be another open subset such that $j' = \phi|_{U'}$ is an open embedding. Passing to the intersection of $U$ and $U'$ we may assume that $U' \subset U$. Using the defining relations of $K_0(\text{Var}^{\leq n-1}/k)$ we obtain that

$$\tilde{c}(j') - \tilde{c}(i') = [Y \setminus j'(U')]_{n-1} - [X \setminus U']_{n-1} = [Y \setminus j(U)]_{n-1} - [X \setminus U]_{n-1} = \tilde{c}(j) - \tilde{c}(i)$$

hence $\tilde{c}$ is well-defined.

To check (2), let $\phi \in \text{Bir}(X, Y)$ and $\psi \in \text{Bir}(Y, Z)$, and choose $U \subset X$ such that both $j = \phi|_U$ and $j' = \psi|_{j(U)}$ are open embeddings. Then

$$\tilde{c}(\psi) + \tilde{c}(\phi) = [Z \setminus j'(j(U))]_{n-1} - [Y \setminus j(U)]_{n-1} + [Y \setminus j(U)]_{n-1} - [X \setminus U]_{n-1} = \tilde{c}(\psi \circ \phi).$$

For the final claim, by (2.1) we can factorize $\phi$ through the inclusions

$$X \leftarrow X \setminus \text{Ex}(\phi) \simeq Y \setminus \text{Ex}(\phi^{-1}) \hookrightarrow Y,$$

so that by (1) and (2) we have $\tilde{c}(\phi) = [\text{Ex}(\phi^{-1})]_{n-1} - [\text{Ex}(\phi)]_{n-1}$. It follows from the definitions that $\pi_{n-1}(\tilde{c}(\phi)) = c(\phi). \hfill \square$

2.5. Invariant $\tilde{c}$ and the structure of the Grothendieck rings.

The material of this section makes explicit the first page of the spectral sequences of [68, Section 3] and generalizes [48, §3.2].
When we put $Y = X$ in Theorem 2.8, we obtain a homomorphism
\begin{equation}
\tilde{c}|_{\text{Bir}(X)} : \text{Bir}(X) \to K_0(\text{Var}^{\leq n-1}/k),
\end{equation}
which, as we will see, is in general nonzero. We note that it is crucial that we consider $K_0(\text{Var}^{\leq n-1}/k)$, not the full Grothendieck ring as the target of $\tilde{c}|_{\text{Bir}(X)}$. Indeed, consider the homomorphism
\begin{equation}
K_0(\text{Var}^{\leq n-1}/k) \overset{\iota_{n-1}}{\to} K_0(\text{Var}^{\leq n}/k).
\end{equation}
The following lemma shows in particular that $\iota_{n-1} \circ \tilde{c}|_{\text{Bir}(X)} = 0$.

**Lemma 2.9.** The kernel of (2.12) admits the following presentation:
\begin{equation}
\langle \ (\tilde{c}(\phi), -[Y] + [X]) \rangle_{\phi \in \text{Bir}(X,Y)}
\end{equation}
for all birational isomorphisms $\phi$ between all irreducible $n$-dimensional varieties $X, Y$.

**Proof.** The subgroup (2.13) can be rewritten as
\begin{equation}
\langle \ (\tilde{c}(j : U \to X), -[X] + [U]) \rangle_{U \subset X}
\end{equation}
where $U \subset X$ runs over open subsets of all irreducible $n$-dimensional varieties $X$. It remains to show that the subgroups (2.19) and (2.18) coincide. It is clear that (2.19) is a subgroup of (2.18). Conversely, every element of (2.18) can be written as a combination of elements from (2.19) after decomposing $\phi$ as a composition of open embeddings and their inverses. \hfill \Box

**Proposition 2.10.** We have
\begin{equation}
\text{Ker}(\iota_{n-1}) = \sum_{X \in \text{Bir}_n/k} \tilde{c}(\text{Bir}(X))
\end{equation}
so that there is an exact sequence
\begin{equation}
0 \to \text{Im}(\tilde{c})_n \to K_0(\text{Var}^{\leq n-1}/k) \overset{\iota_{n-1}}{\to} K_0(\text{Var}^{\leq n}/k) \overset{\pi_n}{\to} \mathbb{Z}[\text{Bir}_n/k] \to 0,
\end{equation}
where we write $\text{Im}(\tilde{c})_n$ for the right hand side of (2.20).

**Proof.** $\text{Ker}(\iota_{n-1})$ coincides with kernel of (2.12) intersected with $K_0(\text{Var}^{n-1}/k)$. Hence by Lemma 2.9 every element in $\text{Ker}(\iota_{n-1})$ has the form
\begin{equation}
\sum_{i=1}^r \tilde{c}(\phi_i)
\end{equation}
with $\phi_i : X_i \dashrightarrow Y_i$ and $\sum_{i=1}^r ([X_i] - [Y_i]) = 0 \in \mathbb{Z}[\text{Var}^n/k]$. Since the latter is a free abelian group on the isomorphism classes of $n$-dimensional irreducible varieties, there is a permutation $\sigma \in S_r$ such that $Y_i \simeq X_{\sigma(i)}$. Let $i \in \{1, \ldots, r\}$ and let $\ell$ be the length of $\sigma$-orbit of $i$. Then we have a composition
\begin{equation}
\psi_i : X_i \phi_i \phi_{\sigma(i)} \phi_{\sigma^2(i)} \phi_{\sigma^3(i)} \cdots \phi_{\sigma^{\ell-1}(i)} \to X_{\sigma^i(i)} = X_i.
\end{equation}
This allows to rewrite (2.22) as a sum of $\tilde{c}(\psi_i)$, with $\psi_i \in \text{Bir}(X_i)$, and $i$ running over a set of representative classes for orbits of $\sigma$ on $\{1, \ldots, r\}$. □

**Remark 2.11.** Using Proposition 2.10 and Theorem 2.5, one can prove that $\iota_0$ and $\iota_1$ are injective for all perfect fields $k$ [48, Corollary 3.10].

On the other hand when $n \geq 2$, geometric constructions in the following section shows in an explicit way that $\iota_n$ is not injective over various fields $k$; see Theorems 3.7, 3.11, 3.15, and Theorem 4.4.

3. **Geometric constructions**

3.1. **Strong birational isomorphisms.**

Recall that two varieties $X$ and $Y$ are isomorphic in codimension one if there exists a birational map $\gamma : X \dashrightarrow Y$ such that $\text{Ex}(\gamma)$ and $\text{Ex}(\gamma^{-1})$ have codimension $\geq 2$. We introduce a motivic version of this notion:

**Definition 3.1.** We say that two varieties $X$ and $Y$ are strongly birational if there exists $\gamma \in \text{Bir}(X,Y)$ such that $c(\gamma) = 0$. We say that $X$ is strongly rational if $X$ is strongly birational to $\mathbb{P}^n$.

Note that strong birationality is an equivalence relation by the additivity of $c$ (Lemma 2.2). Furthermore, isomorphism in codimension one implies strong birationality.

When $k$ is a field of characteristic zero, it follows from Proposition 2.4 that strongly birational varieties have equal ranks of their geometric Néron–Severi groups. In particular, a strongly rational variety has geometric Néron–Severi group of rank one.

The following lemma provides some examples of strongly rational varieties.

**Lemma 3.2.** If $X$ is a smooth projective variety containing an irreducible rational divisor $D \subset X$ such that $U := X \setminus D$ is isomorphic to $\mathbb{A}^n$, then $X$ is strongly rational. In particular, if $G$ is a split reductive algebraic group over a field $k$, then every homogeneous space $X = G/P$ of Picard rank $\rho(X) = 1$ is strongly rational.

**Proof.** Consider the birational map $\phi : X \dashrightarrow \mathbb{P}^n$ induced by the isomorphism $U \simeq \mathbb{A}^n$. We have

$$\phi : X \leftrightarrow U \simeq \mathbb{A}^n \leftrightarrow \mathbb{P}^n,$$

so $c(\phi) = [\mathbb{P}^{n-1}] - [D] = 0$. Thus $X$ is strongly rational.

Now we prove the second statement. A homogeneous space $X$ admits the Bruhat stratification into affine spaces [34, Proposition 1.3]. It has a dense open stratum isomorphic to $\mathbb{A}^n$ and the number of codimension one strata $\mathbb{A}^{n-1}$ in the Bruhat stratification is equal to the Picard rank $\rho(X) = 1$ by [26, Example 1.9.1]. The closure of the $\mathbb{A}^{n-1}$ stratum is therefore the complement of the dense stratum $\mathbb{A}^n$. It follows from the first part of the lemma that $X$ is strongly rational. □
3.2. L-links.

Let $X, Y$ be irreducible geometrically reduced varieties.

**Definition 3.3.** An L-link with centers $X$ and $Y$ is a diagram

$$
\begin{array}{ccc}
X \xrightarrow{\text{closed}} & T & \xleftarrow{\text{closed}} Y \\
\downarrow \psi & & \downarrow \psi \\
\text{Bl}_X & & \text{Bl}_Y
\end{array}
$$

of blow ups with centers $X$ and $Y$, subject to the following conditions:

(L1) $\mathcal{X}$ and $\mathcal{X}'$ are varieties smooth in codimension one, such that $[\mathcal{X}] = [\mathcal{X}']$ in $K_0(\text{Var}/k)$.

(L2) $\mathcal{X}$ and $\mathcal{X}'$ are strongly birational.

(L3) The exceptional divisors of $\text{Bl}_X$ and $\text{Bl}_Y$ are irreducible.

We say that the L-link $\psi$ (3.1) is smooth (resp. projective) if $X, Y, \mathcal{X}, \mathcal{X}', T$ are smooth (resp. projective). An L-link $\psi$ is called motivically non-trivial if $c(\psi) \neq 0$. For instance, this is the case when the link is smooth and $X$ is not stably birational to $Y$.

**Example 3.4.** Special Cremona transformations [19] give rise to examples of L-links with $\mathcal{X} = \mathcal{X}' = \mathbb{P}^n$ and $\gamma = \text{Id}$.

L-links are closely related to L-equivalence. Recall that smooth projective connected varieties $X, Y$ are called L-equivalent if $[X] - [Y] = 0$ in $K_0(\text{Var}/k)$ for some $d \geq 0$.

**Lemma 3.5.** Given an L-link (3.1), let $m = \text{codim}_X(\mathcal{X}), m' = \text{codim}_Y(\mathcal{X}')$.

1. If the link is smooth, we have $L([\mathbb{P}^{m-2}][X] - [\mathbb{P}^{m'-2}][Y]) = 0$ in $K_0(\text{Var}/k)$.

2. Let $\phi = \gamma^{-1}\psi \in \text{Bir}(\mathcal{X})$, where $\gamma$ comes from Definition 3.1. Then

$$
c(\phi) = c(\psi) = [\mathbb{P}^{m-1} \times X] - [\mathbb{P}^{m'-1} \times Y]
$$

in $\mathbb{Z}[\text{Bir}/k]$. In particular, if the link (3.1) is motivically non-trivial (e.g. if $X$ and $Y$ are not stably birational), then $c(\phi) \neq 0$.

**Proof.** (1) follows by computing the class of $[T]$ in two ways using the two blow up morphisms and (L1). For (2), we have $c(\phi) = c(\psi)$ by (L2). Since $\mathcal{X}$ is smooth in codimension one and $X$ is geometrically reduced, there exists a closed subscheme $Z \subset \mathcal{X}$ of codimension two such that both $\mathcal{X}\setminus Z$ and $X\setminus Z$ are smooth. As the exceptional divisor $E$ of $\text{Bl}_X$ is irreducible by (L3), it follows that $E$ is birational to $\mathbb{P}^{m-1} \times X$. Similarly, the exceptional divisor of $\text{Bl}_Y$ is birational to $\mathbb{P}^{m'-1} \times X$. Thus by Lemma 2.2, we have $c(\phi) = c(\psi) = [\mathbb{P}^{m-1} \times X] - [\mathbb{P}^{m'-1} \times Y]$. \qed

3.3. Elliptic curves.

Let $C$ be a smooth projective connected curve and let $k \in \mathbb{Z}$. Let $J^k(C)$ denote the Jacobian of divisors of degree $k$ on $C$. Each $J^k(C)$ is a torsor under the algebraic group $J^0(C)$. If $C$ is a curve of genus one, then $J^k(C)$ are twisted forms of $C$ which are not necessarily isomorphic to it. We concentrate on curves $C$ of genus one and degree five for which the only interesting Jacobian is $J^2(C)$ [63].
Proposition 3.6. Let $C$ be a genus one curve having a divisor of degree 5, and let $C' = J^2(C)$. We have the following smooth projective L-link:

\[ (3.2) \]

Here $Q^3$ is a smooth quadric threefold.

Proof. By Riemann-Roch, the linear system of a divisor of degree 5 on $C$ gives rise to a degree 5 embedding $C \subset \mathbb{P}^4$, and it is well-known that $C$ is the scheme-theoretic intersection of the quadrics containing it, see e.g. [23, Proposition 4.2(ii)]. Thus the map given by the linear system $|2H - C|$ is resolved by blowing up $C$, and we obtain the Crauder–Katz quadro-cubic transformation [19, Theorem 2.2(ii)]

\[ (3.3) \]

where $S$ is the surface parameterizing the secant lines of $C$. Thus $S$ is isomorphic to $\text{Sym}^2(C)$ (cf Proof of [19, Theorem 3.3(A)]) and is embedded as a quintic ruled surface $\text{Sym}^2(C) \to J^2(C)$ into $\mathbb{P}^4$.

To obtain the L-link (3.2) we take a hyperplane section of (3.3) as follows. Take any smooth quadric $Q^3 \subset \mathbb{P}^4$ containing $C$ and restrict $\psi'$ onto $Q^3$. The image $\psi'(Q^3)$ is a hyperplane $\mathbb{P}^3$, and we obtain a birational map $\psi : Q^3 \dasharrow \mathbb{P}^3$ and a diagram (3.2) with $C' = S \cap \mathbb{P}^3$. This curve is a section of the projection $S \to J^2(C)$, hence $C' \simeq J^2(C)$, and $C' \subset \mathbb{P}^3$ is a degree five genus one curve.

We claim that $Q^3$ contains $k$-lines. If $k$ is a finite field, then this is true for any smooth quadric in $\mathbb{P}^4$, as the quadratic form in 5 variables over a finite field is isomorphic to a direct sum of two copies of the hyperbolic plane and a one-dimensional form, by the Chevalley–Warning theorem. Let us assume that $k$ is infinite. Under the map $\psi$, general lines on $Q^3$ correspond to general conics in $\mathbb{P}^3$ which are five-secant to $C'$. Such conics are parameterized by planes $\mathbb{P}^2 \subset \mathbb{P}^3$, and taking a general plane $\mathbb{P}^2 \subset \mathbb{P}^3$ defined over $k$ we get a $k$-rational line on $Q^3$.

Finally (3.2) satisfies the definition of L-link because $[Q^3] = [\mathbb{P}^3]$ as $Q^3$ contains a line [42, Example 2.8], and $Q^3$ is strongly rational (see e.g. Lemma 3.2).

Theorem 3.7. Let $k$ be a field of one of the following types:

- a number field,
- a function field over a field $\mathbb{F}$, where $\mathbb{F}$ can be
  - an algebraically closed field,
  - a number field,
  - a finite field.

\[ \square \]
Then we can choose a curve $C$ such that (3.2) is motivically non-trivial. Furthermore, the set $I$ of isomorphism classes of such curves has the cardinality of $k$.

Here by a function field, we mean a function field of a positive-dimensional $F$-variety.

Before we prove the theorem, we need a preliminary result on the arithmetic of elliptic curves.

**Lemma 3.8.** [45, 18] For a field $k$ as in Theorem 3.7, there exist infinitely many genus one curves $C$ without rational points but having a divisor of degree 5, and such that $j(J^0(C)) \neq 1728$. The set $I$ of isomorphism classes of such curves has the cardinality of $k$.

**Proof.** If $k$ is an infinite finitely generated field over $\mathbb{Q}$ or over a finite field, then infinitely many such curves exist by [18, Theorem 1.10]; the condition $j(J^0(C)) \neq 1728$ is not explicitly checked in [18], however the elliptic curves they use to produce torsors indeed satisfy this condition [18, Section 4].

The remaining case is a function field $k$ over an algebraically closed field $F$. We apply [45, Theorem 7]. First of all since $k$ is Hilbertian [24, Proposition 12.3.3, Theorem 13.4.2], it has infinitely many cyclic extensions of degree 5 which are linearly disjoint in $k$ [24, Corollary 16.2.7]. In particular, these cyclic extensions are non-isomorphic. Next we take any ordinary elliptic curve $E/F$ with $j(E) \neq 1728$ in $F$. Since $F$ is algebraically closed, $E(\mathbb{F})$ has non-trivial 5-torsion. The base change $E_k$ of $E$ to $k$ has also non-trivial 5-torsion and $j(E_k) \neq 1728$ in $k$. The group $E(k)/5E(k)$ is finite by [18, Remark 1.3, Proposition 1.4]. We thus conclude by [45, Theorem 7] that there exists infinitely many $E_k$-torsors of index 5.

Note that among the fields we consider, $k$ is uncountable only if $k$ is the function field over an algebraically closed field $F$. In this case $F$ and $k$ have the same cardinality. As the set of elliptic curves $E/F$ considered above has the same cardinality as $F$, the last statement follows. 

**Proof of Theorem 3.7.** Take the curve $C$ defined by Lemma 3.8 and set $C' = J^2(C)$. By [63, Lemma 2.7] we have $C \not\cong C'$ (the cited result assumes char$(k) = 0$, however the proof works without this assumption since there is no $\sigma \in \text{Aut}(E)$ of order 4 when $j \neq 1728$ for any field [65, Proposition A.1.2]). As $C$ and $C'$ are non isomorphic smooth curves, they are not birational. Hence the L-link (3.2) is motivically non-trivial.

**Remark 3.9.** The L-link (3.2) appears in the classification of Fano threefolds of rank two by Mori–Mukai [53, 2] on page 117. It is a very interesting question whether there exist other nontrivial smooth projective L-links between strongly rational varieties in dimension 3. The ultimate question in this direction is to fully describe the image $c(\text{Bir}(\mathbb{P}^3))$ using the Sarkisov link decomposition.

**Remark 3.10.** Let $X \to B$ be an elliptic surface with a multisection of degree five over $k = \mathbb{C}$. Let $Y = J^2(X/B)$ be the relative Jacobian of degree two divisors. Then spreading out the construction of Theorem 3.7, one can show that $[X] - [Y] \in \text{Im}(c_{\mathbb{P}^3 \times B})$. See [63] for a detailed analysis when $X$, $Y$ are not birational in the case of elliptic K3 surfaces.
3.4. K3 surfaces of degree 12.

**Theorem 3.11.** Let $\mathbb{k} \subset \mathbb{C}$. There exist motivically non-trivial L-links

\[
\begin{array}{c}
\text{Bl}_{S_0} \quad T \\
S_0 \quad \dashrightarrow \mathbb{P}^4 \quad \dashrightarrow \mathbb{P}^4 \\
\text{Bl}_{S'_0} \end{array}
\]

where $S_0$ and $S'_0$ are projective surfaces such that the minimal models $S$ and $S'$ of their normalizations are non-isomorphic K3 surfaces of geometric Picard rank one and degree 12, and $S'_C$ is the unique Fourier–Mukai partner of $S_C$. Furthermore, the set $I$ of isomorphism classes of the K3 surfaces $S$ occurring in these L-links has the cardinality of $\mathbb{k}$.

Theorem 3.11, over $\mathbb{C}$, is the main result of Hassett and Lai [27]. We start by summarizing their construction, and then we explain how to extend it to an arbitrary subfield $\mathbb{k} \subset \mathbb{C}$ using Lefschetz pencils.

**Theorem 3.12 ([27, Theorem 2.1, Theorem 3.1]).** Let $(S, H)$ be a general polarized complex K3 surface with $H^2 = 12$ and let $\Sigma = \{x_1, x_2, x_3\} \subset S$ be a general triple of points. The linear system $|H|$ defines an embedding $S \hookrightarrow \mathbb{P}^7$; let $S_0 \subset \mathbb{P}^4$ be the proper transform of $S$ under the projection $\mathbb{P}^7 \rightarrow \mathbb{P}^4$. We have the following:

1. $S_0$ has three transverse double points, and the normalization of $S_0$ is $\text{Bl}_\Sigma S$.
2. The linear system of quartics containing $S_0$ cuts out $S_0$ as a scheme, and defines a birational map $\psi : \mathbb{P}^4 \rightarrow \mathbb{P}^4$.
3. The inverse $\psi^{-1} : \mathbb{P}^4 \rightarrow \mathbb{P}^4$ is constructed the same way, starting from another K3 surface $S'$ of degree 12 and three points $\Sigma' = \{x'_1, x'_2, x'_3\}$ on $S'$.
4. We have $\text{Bl}_{S_0} \mathbb{P}^4 \simeq \text{Bl}_{S_0'} \mathbb{P}^4$ and it resolves $\psi$, where $S'_0$ the proper transform of $S'$ under the projection $\mathbb{P}^7 \rightarrow \mathbb{P}^4$ from the plane generated by $\Sigma'$.

Furthermore, if $\text{Pic}(S) = \mathbb{Z} \cdot H$, then $S'$ is the unique Fourier–Mukai partner of $S$ such that $S \not\cong S'$. Finally, if $S \hookrightarrow \mathbb{P}^7$ and $\Sigma \subset S$ are defined over a subfield $\mathbb{k} \subset \mathbb{C}$, then both $\psi$ and the pair $(S', \Sigma')$ are defined over $\mathbb{k}$.

**Proof.** The main statement follows from [27, §1.1, Theorem 2.1]. The statement about $S \not\cong S'$ follows from [27, Theorem 3.1], together with the derived invariance of Picard number for K3 surfaces and [57, Proposition 1.10]. Finally, if $S \hookrightarrow \mathbb{P}^7$ and $\Sigma \subset S$ are defined over $\mathbb{k}$, then $\psi$ is defined over $\mathbb{k}$ by construction. As $S'_0$ is the fundamental locus of $\psi^{-1}$, $S'_0$ is defined over $\mathbb{k}$ as well. Since $S'$ is the minimal model of the normalization of $S'_0$, and $\Sigma' \subset S'$ is the image of the $(-1)$-curves, we conclude that $(S'_0, \Sigma')$ is defined over $\mathbb{k}$. \hfill $\square$

Let $\mathbb{k} \subset \mathbb{C}$. Let $S \subset \mathbb{P}^7$ be a K3 surface of degree 12 defined over $\mathbb{k}$ and $\Sigma \subset S$ a smooth subscheme of length three. We say that $(S, \Sigma)$ is HL-admissible if Theorem 3.12(1-4) holds for $S_C$ and $\Sigma_C$. 
It is a construction going back to Mukai that general K3 surfaces of degree 12 can be obtained as linear sections of a Grassmannian $\text{OG}_+(5, 10) \subset \mathbb{P}^{15}$, see [27, 2.1]. Consider a general 3-dimensional linear section $V \subset \mathbb{P}^8$ over $\mathbb{Q}$ of this Grassmannian. As $\text{OG}_+(5, 10)$, being a rational homogeneous variety, has dense $\mathbb{Q}$-points and since the linear section cutting off $V$ is general, we can assume that $V$ is smooth and has a $\mathbb{Q}$-point. This implies that $V$ is $\mathbb{Q}$-rational [41, Theorem 1.1].

Let $\mathbb{P}^6 \subset \mathbb{P}^8$ be a linear subspace and $\Sigma = \{x_1, x_2, x_3\} \subset V \cap \mathbb{P}^6$. We call the pair $(\mathbb{P}^6, \Sigma)$ **HL-admissible** if the following conditions are satisfied:

(a) The pencil $S_t$, $t \in \mathbb{P}^1$, defined by intersecting $V$ with hyperplanes containing $\mathbb{P}^6$ is a Lefschetz pencil.

(b) For one (hence for general) $t$, the pair $(S_t, \Sigma)$ is HL-admissible.

**Lemma 3.13.** For a general $V$ as above, there exists an HL-admissible pair $(\mathbb{P}^6, \Sigma)$ defined over $\mathbb{Q}$.

**Proof.** As both conditions (a) and (b) are nonempty Zariski open conditions, HL-admissible pairs form a Zariski dense open subset $U$ in the incidence variety

$$\mathcal{X} := \left\{ (x_1, x_2, x_3; L) \in V^3 \times \text{Gr}(\mathbb{P}^6, \mathbb{P}^8) \mid x_1, x_2, x_3 \in V \cap L \right\}.$$ 

It is easy to see that $\mathcal{X}$ is a $\mathbb{Q}$-rational variety, hence the open subset $U \subset \mathcal{X}$ has (dense) $\mathbb{Q}$-rational points. □

**Proposition 3.14.** There exist infinitely many polarized K3 surfaces $(S, H)$ of degree 12 over $\mathbb{Q}$ together with three $\mathbb{Q}$-points $\Sigma = \{x_1, x_2, x_3\} \subset S(\mathbb{Q})$ such that

1. $\text{Pic}(S_\mathbb{C}) \simeq \mathbb{Z}$;
2. $(S; \Sigma)$ is HL-admissible.

The same conclusion holds if $\mathbb{Q}$ is replaced by any subfield $k$ of $\mathbb{C}$, and the isomorphism classes of such K3 surfaces $S$ form a set $I$ which has the cardinality of $k$.

**Proof.** The Lefschetz pencil $\{S_t\}$ given by the HL-admissible pair from Lemma 3.13 is defined over $\mathbb{Q}$, hence Terasoma’s argument in [67, Step 2 and Step 3 in §3] proving [67, Theorem 1] applies verbatim and shows that there exist infinitely many K3 surfaces among $\{S_t\}$ defined over $\mathbb{Q}$ (and thus over any subfield $k \subset \mathbb{C}$) with geometric Picard rank 1. If $k$ is uncountable, then since “having geometric Picard rank 1” is a very general property, the set of K3 surfaces over $k$ among $\{S_t\}$ for which (1) holds is also uncountable.

We claim that the Lefschetz pencil $\{S_t\}$ is non-isotrivial. This is because the monodromy action on $H^2(S_t, \mathbb{C}, \mathbb{Q})$ is nontrivial by the invariant cycle theorem, and $\text{Aut}(S_t, \mathbb{C}) = \{\text{Id}\}$ if $\rho(S_t, \mathbb{C}) = 1$ [30, Corollary 15.2.12], hence monodromy does not act by holomorphic transformations. Thus $\{S_t\}$ contains infinitely many isomorphism classes of K3 surfaces for any $k \subset \mathbb{C}$, uncountably many if $k$ is uncountable. Finally, since the subset of K3 surfaces $S_t$ in $\{S_t\}$ for which $(S_t; \Sigma)$ is HL-admissible is open, Proposition 3.14 follows. □
Proof of Theorem 3.11. Let $S$ be a K3 surface over $\mathbb{Q}$ as in Proposition 3.14 and let $S'$ be as in Theorem 3.12. In particular, $S_C \not\cong S'_C$ and $S'_C$ is a Fourier-Mukai partner of $S_C$.

Let $\psi \in \text{Bir}(\mathbb{P}^4)$ be the map as in Theorem 3.12. By Theorem 3.12, both $\psi$ and the K3 surface $S'$ are defined over $\mathbb{Q}$. Since $S_0$ has at worst transversal double points as singularities, the exceptional divisor of $\text{Bl}_{S_0} : T \to \mathbb{P}^4$ is birational to $S_0 \times \mathbb{P}^1$, which is further birational to $S \times \mathbb{P}^1$ by Theorem 3.12(1). Similarly, the exceptional divisor of $\text{Bl}_{S'_0} : T' \to \mathbb{P}^4$ is birational to $S'_0 \times \mathbb{P}^1$. Thus,

$$c(\psi_k) = [S_k \times \mathbb{P}^1_k] - [S'_k \times \mathbb{P}^1_k] \in \mathbb{Z}[\text{Bir}/\mathbb{Q}].$$

Since $S_k$ and $S'_k$ are non-isomorphic K3 surfaces because $S_C \not\cong S'_C$, they are not stably birational, see e.g. Corollary A.5. Hence $c(\psi_k) \neq 0$, so $\psi_k$ is motivically non-trivial. □

3.5. K-trivial threefolds.

Theorem 3.15. Let $k$ be an infinite field. There exist motivically non-trivial smooth projective $L$-links:

$$
\begin{array}{c}
Z \\
\downarrow \text{Bl}_Z \\
X \\
\downarrow \psi \\
X' \\
\downarrow \text{Bl}_{Z'} \\
Z'
\end{array}
$$

where $X$ and $X'$ are five-dimensional $G_2$-Grassmannians, and $Z$ and $Z'$ are K-trivial threefolds of Picard rank 1.

Here, a K-trivial variety is a smooth projective variety $X$ with $\omega_X \simeq \mathcal{O}_X$.

The construction of Theorem 3.15 is given by Ito–Miura–Okawa–Ueda [31] over a field of characteristic zero. We explain that the construction extends to an arbitrary infinite field. The proof of Theorem 3.15 occupies the rest of the section.

Let $G$ be the connected and simply connected split simple group scheme of type $G_2$ over $\mathbb{Z}$. Let $B \subset G$ be the Borel subgroup, and $P, P' \subset G$ the two maximal parabolic subgroups of $G$ containing $B$. The quotients $G/B$, $G/P$, and $G/P'$ are smooth and projective over $\mathbb{Z}$ by [52, Theorem 13.33] and [5, Corollary XXII.5.8.5]. The natural projection $p : G/B \to G/P$ is also smooth. Since $p_\mathbb{Q} : G_\mathbb{Q}/B_\mathbb{Q} \to G_\mathbb{Q}/P_\mathbb{Q}$ is a $\mathbb{P}^1$-fibration, so is $p : G/B \to G/P$. Likewise, the other projection $p' : G/B \to G/P'$ is also a smooth $\mathbb{P}^1$-fibration.

Our assumptions on $G$ imply that $X^*(G) = 0$ and Pic($G$) = 0 [60, Remark VII.1.7.a)], where $X^*(G)$ is the character group of $G$. It follows that

$$\text{Pic}(G/P) \simeq X^*(P) \simeq \mathbb{Z}$$

by [60, Proposition VII.1.5]. Similarly, $\text{Pic}(G/P') \simeq \mathbb{Z}$. Let $M$ and $M'$ denote the ample generators of Pic($G/P$) and Pic($G/P'$) respectively. The tensor product

$$L := p^*M \otimes p'^*M'$$
is then ample as well. Since \( \deg(L_{k|F_k}) = 1 \) when \( \text{char}(k) = 0 \) where \( F_k \simeq \mathbb{P}^1_k \) is a fiber of \( p_k : G_k/B_k \to G_k/P_k \) (see e.g., [31]), the same holds true for any field \( k \). It follows that \( E := p_k L \) is locally free of rank 2 and \( G/B \simeq \mathbb{P}(E) \) over \( G/P \).

Now let \( k \) be an infinite field and let \( \sigma \in H^0(G_k/B_k, L_k) \) be a general section. Let \( X_k = Z(\sigma) \subset G_k/B_k \) and let \( Z_k := Z(p_k \sigma) \subset G_k/P_k \) with \( p_k \sigma \) regarded as a section of \( E_k = (p_k L)_k \).

Similarly, let \( Z'_k := Z(p'_k \sigma) \subset G_k/P'_k \). Since \( L_k \) is very ample [59, Theorem 3] and \( k \) is infinite, \( X_k \) is smooth by Bertini’s theorem [33, Théorème 6.3], [54, Theorem 1.10]. Consider the commutative diagram

\[
\begin{array}{ccc}
X_k & \xrightarrow{\tau} & G_k/B_k \\
\downarrow \tau' & & \downarrow \psi \\
G_k/P_k & \xrightarrow{\sigma} & G_k/P'_k
\end{array}
\]

which defines the maps \( \tau \) and \( \tau' \). By construction, \( \tau \) maps \( X_k \setminus \tau^{-1}(Z_k) \) isomorphically onto its image and \( \tau^{-1}(Z_k) \to Z_k \) is a \( \mathbb{P}^1 \)-fibration. By [20, Theorem 2], \( \tau \) is thus a sequence of blow ups along smooth centers of codimension 2 lying above \( Z_k \). Since \( X_k \) has Picard rank \( \rho(X_k) = \rho(G_k/B_k) = 2 \) by Grothendieck–Lefschetz’ theorem, necessarily \( \tau \) is the blow up of \( G_k/P_k \) along \( Z_k \) and \( Z_k \) is connected. Similarly, \( \tau' \) is the blow up of \( G_k/P'_k \) along \( Z'_k \), and \( Z'_k \) is connected. We thus have a birational map

\[ \psi := \tau' \circ \tau^{-1} : G_k/P_k \dashrightarrow G_k/P'_k. \]

By construction, \( \psi \) is regular away from \( Z_k \) and \( \psi^{-1} \) is regular away from \( Z'_k \).

**Lemma 3.16.** \( Z_k, Z'_k \) are K-trivial threefolds, which are not stably birational to each other.

**Proof.** When \( \text{char}(k) = 0 \), Lemma 3.16 is proven in [31]. Assume that \( \text{char}(k) = p > 0 \). Let \( R \) be an integral local ring of characteristic zero with residue field \( k \) (see [62, Satz 20] or [4, Theorem 2.10] for the existence). Since \( H^j(G_k/B_k, L_k) = 0 \) for all \( j > 0 \) [59, Theorem 2] (and for any field \( k \)), we have

\[ H^0(G_k/B_k, L_k) \simeq H^0(G_R/B_R, L_R) \otimes_R k \]

by Grauert’s base change. Thus the section \( \sigma \in H^0(G_k/B_k, L_k) \) defining \( Z_k \) can be lifted to a section \( \sigma_K \in H^0(G_K/B_K, L_K) \) where \( K := \text{Frac}(R) \), which is of characteristic zero. This defines a lifting of \( Z_k \) to a smooth projective subvariety \( Z_K \subset G_K/P_K \). Since \( \omega_{Z_K} \) is trivial, so is \( \omega_{Z'_k} \). The same argument shows that \( \omega_{Z'_k} \) is trivial.

To show that \( Z_k, Z'_k \) are not stably birational, we may assume \( k = \overline{k} \). First we show that \( Z_k \) and \( Z'_k \) are not isomorphic. We have seen that \( \text{rk NS}(X_k) = 2 \). As \( X_k \to Z_k \) and \( X_k \to Z'_k \) are \( \mathbb{P}^1 \)-bundles, necessarily \( \text{rk NS}(Z_k) = \text{rk NS}(Z'_k) = 1 \). Let \( H \) (resp. \( H' \)) be the ample generator
of NS($Z_k$) (resp. NS($Z'_k$)). Let $a, a' \in \mathbb{Z}_{>0}$ such that

$$c_1(M_k|Z_k) = aH \quad \text{and} \quad c_1(M'_k|Z'_k) = a'H'.$$

As

$$c_1(M_k|Z_k)^3 = c_1(M_K)^3 \cdot Z_k = c_1(M'_k|Z'_k)^3 \quad \text{and} \quad c_1(M'_k|Z'_k)^3 = c_1(M'_K)^3 \cdot Z'_k,$$

are distinct and equal to 14 or 42 by [31], necessarily $a = a' = 1$. Hence $H^3 \neq H'^3$, so $Z_k$ is not isomorphic to $Z'_k$. It follows from Corollary A.5 that $Z_k$ is not stably birational to $Z'_k$.

**Proof of Theorem 3.15.** We set $X = G_k/P_k$ and $X' = G_k/P'_k$. Both $X$ and $X'$ have a Bruhat decomposition with a single cell in every dimension so that $[X] = [P_5] = [X']$. Furthermore, $X$ and $X'$ are strongly birational, because they are both strongly rational by Lemma 3.2. Thus diagram (3.6) gives us the L-link (3.5). Finally, since $Z_k$ and $Z'_k$ are not (stably) birational by Lemma 3.16, the L-link (3.5) is motivically non-trivial. □

**Remark 3.17.** In the proof of Theorem 3.15, the only place where we need the assumption that $k$ is infinite is to guarantee the existence of $\sigma \in H^0(G_k/B_k, L_k)$ such that the vanishing loci $Z_k \subset G_k/P_k$ and $Z'_k \subset G_k/P'_k$ of $p^*\sigma$ and $p'^*\sigma$ are smooth. Based on generic smoothness, the same conclusion of Theorem 3.15 thus holds for all but finitely many finite fields $k$.

### 4. Applications to Cremona groups

Let $X/k$ be a variety.

**Definition 4.1.** A birational automorphism $\phi \in \text{Bir}(X)$ is pseudo-regularizable if $\phi = \alpha^{-1} \circ \gamma \circ \alpha$ where

- $\alpha : X \dasharrow X'$ is a birational map (we do not assume $X'$ is smooth nor projective);
- $\gamma \in \text{Bir}(X')$ is a pseudo-automorphism, namely an isomorphism in codimension one.

If moreover the map $\gamma$ is biregular, we call $\phi$ a regularizable map.

Pseudo-regularizable maps have been studied in [14, 51] in the context of the actions of Cremona groups on various combinatorial complexes.

**Example 4.2.**

1. Regular automorphisms, or more generally birational maps $\phi \in \text{Bir}(X)$ which have an invariant dense open subset $U \subset X$, are regularizable.
2. Finite order elements of $\text{Bir}(X)$ are regularizable. Indeed, given such an element $\phi \in \text{Bir}(X)$, the complement

$$U := X \setminus \bigcup_{k=1}^{\text{ord}(\phi)-1} \text{Ex}(\phi^k),$$

is an invariant dense open subset, and we apply (1) to conclude. By Weil’s regularization theorem (see also [40]), they are even projectively regularizable, namely there exists a regularization $\alpha : X \dasharrow X'$ with projective $X'$.  

Let us write $\text{Bir}(X)^{p-reg}$ for the normal subgroup generated by pseudo-regularizable elements.

**Lemma 4.3.** We have

$$\text{Bir}(X)^{p-reg} \subset \text{Ker}(c|_{\text{Bir}(X)}).$$

**Proof.** It suffices to show that for any pseudo-regularizable element $\phi \in \text{Bir}(X)^{p-reg}$ we have $c(\phi) = 0$. By definition, for any such element we have $\phi = \alpha^{-1} \circ \gamma \circ \alpha$ where $\gamma$ is a pseudo-automorphism. Thus $c(\gamma) = 0$ and

$$c(\phi) = -c(\alpha) + c(\gamma) + c(\alpha) = 0.$$

□

**Theorem 4.4.** Assume that $X/\mathbb{k}$ is a variety that satisfies one of the following conditions:

(a) $\mathbb{k}$ is a number field, or a function field over a number field, over a finite field, or over an algebraically closed field, and $X$ is birational to $\mathbb{P}^3 \times W$ for a separably rationally connected variety $W$ (e.g. $X = \mathbb{P}^n$, $n \geq 3$).

(b) $\mathbb{k} \subset \mathbb{C}$ is any subfield and $X$ is birational to $\mathbb{P}^4 \times W$ for a rationally connected variety $W$ (e.g. $X = \mathbb{P}^n$, $n \geq 4$).

(c) $\mathbb{k}$ is any infinite field and $X$ is birational to $\mathbb{P}^5 \times W$ for a separably rationally connected variety $W$ (e.g. $X = \mathbb{P}^n$, $n \geq 5$).

We have $c|_{\text{Bir}(X)} \neq 0$. As a consequence, $\text{Bir}(X)$ is not generated by pseudo-regularizable elements.

In particular, $\text{Bir}(X)$ is not generated by any collection of elements from Example 4.2.

**Proof.** By Lemma 4.3, it suffices to explain that $c|_{\text{Bir}(X)}$ is nonzero for $\mathbb{k}$ and $X$ as in the statement of the theorem.

Let us first assume that $W = \text{Spec}(\mathbb{k})$. In case (a) (resp. (b) and (c)) we use the motivically non-trivial L-link constructed in Proposition 3.6 (resp. Theorem 3.11 and Theorem 3.15), which together with Lemma 3.5 gives a map $\phi \in \text{Bir}(\mathbb{P}^3)$ (resp. Bir($\mathbb{P}^4$), Bir($\mathbb{P}^5$)) such that

$$c(\phi) = [Z \times \mathbb{P}^1] - [Z' \times \mathbb{P}^1] \neq 0 \in \mathbb{Z}[\text{Bir}/\mathbb{k}].$$

where $Z$ and $Z'$ are non-isomorphic $K$-trivial varieties of Picard number 1. For an arbitrary $W$ as in Theorem 4.4, it then follows from Corollary A.5 that

$$c(\phi \times \text{Id}_W) = [Z \times \mathbb{P}^1 \times W] - [Z' \times \mathbb{P}^1 \times W] \neq 0 \in \mathbb{Z}[\text{Bir}/\mathbb{k}].$$

This proves the non-vanishing of $c|_{\text{Bir}(X)}$, and Theorem 4.4 follows. □

**Corollary 4.5.** Let $X/\mathbb{k}$ be as in cases (a) or (b) in Theorem 4.4. Then we have a surjective homomorphism $\text{Bir}(X) \to A$ where $A = \bigoplus J \mathbb{Z}$ is a free abelian group with $J$ a set of the same cardinality as $\mathbb{k}$. In particular, $A$ is a direct summand of $\text{Bir}(X)^{ab}$. 
Proof. We use the same L-links which appear in the proof of Theorem 4.4. By Theorem 3.7 in case (a), and Theorem 3.11 in case (b), these L-links are parameterized by a set $I$ having the cardinality of $\mathbb{k}$.

Let $X_i$ and $Y_i$ be the centers of these links. Since the $X_i$’s are mutually non-isomorphic $K$-trivial varieties of Picard rank one, their birational types are all distinct (see e.g. Theorem A.1). Furthermore, $X_i$ and $Y_i$ are in a symmetric relation with each other. Namely, in case (a), the relation is $Y_i \simeq J^2(X_i)$, so that $X_i \simeq J^2(Y_i)$, because we work with index 5 genus one curves, see [63, 2.2]. In case (b), the relation is: $Y_i, C$ is the unique nontrivial Fourier–Mukai partner of $X_i, C$ [27, 3.1]. Thus in either case,

$$J := \{ \{ X_i, Y_i \} \subset I \mid i \in I \}$$

forms a partition of $I$ and has the same cardinality as $\mathbb{k}$.

We let $A := \bigoplus_{\{ X_i, Y_i \} \in J} Z \left([X_i \times \mathbb{P}^1 \times W] - [Y_i \times \mathbb{P}^1 \times W]\right) \subset Z[\text{Bir}/\mathbb{k}]$.

By construction this a direct summand of $Z[\text{Bir}/\mathbb{k}]$, and let $\pi : Z[\text{Bir}/\mathbb{k}] \to A$ be the projection homomorphism. For each $i$ there exists $\phi \in \text{Bir}(X)$ such that

$$\pi(c(\phi)) = [X_i \times \mathbb{P}^1 \times W] - [Y_i \times \mathbb{P}^1 \times W].$$

Thus the composition $\pi \circ c : \text{Bir}(X) \to A$ is surjective.

The statement about the abelianization follows because $\text{Bir}(X)^{\text{ab}}$ admits a surjective homomorphism onto the free $\mathbb{Z}$-module $A$. This homomorphism admits a splitting, hence it is a projection onto a direct summand. \hfill \square

Remark 4.6. (i) After the results of this paper were announced, Blanc, Schneider and Yasinsky have proved that for all $n \geq 4$, $\text{Cr}_n(\mathbb{C})$ admits a surjective homomorphism onto an uncountable free product of $Z$ [10, Theorem B] (which is a stronger statement than what the Corollary 4.5 says for $X = \mathbb{P}^n$ and $\mathbb{k} = \mathbb{C}$), with a completely different method.

(ii) For $n = 3$ we do not know if there exist nontrivial homomorphisms $\text{Cr}_3(\mathbb{C}) \to Z$ (or $\text{Cr}_3(\mathbb{Q}) \to Z$). Whether $\text{Cr}_3(\mathbb{C})$ is generated by involutions, or regularizable elements, is an outstanding open question about Cremona groups.

Appendix A. Stably birational geometry of K-trivial varieties

The following result is well-known in characteristic zero. We present the proof which works over arbitrary fields as well as a more general statement (see Remark A.2), both communicated to us by Burt Totaro.

Theorem A.1 (Totaro). Let $X_1$ and $X_2$ be smooth projective varieties over a field with a birational map $\phi : X_1 \dashrightarrow X_2$. Assume that $K_{X_1}$ and $K_{X_2}$ are nef. Then $\phi$ is a pseudo-isomorphism. If in addition $\text{NS}(X_1)_{\mathbb{Q}}$ or $\text{NS}(X_2)_{\mathbb{Q}}$ is one-dimensional, then $\phi$ is an isomorphism.
Proof. Let us show that $\phi$ is a pseudo-isomorphism, that is both $\phi$ and $\phi^{-1}$ do not have any exceptional divisors. We follow the proof of [38, Corollary 3.54], which works in any characteristic. Let $Y$ be the normalization of the closure of the graph of $\phi$, with birational morphisms $g_1 : Y \to X_1$ and $g_2 : Y \to X_2$. We have a linear equivalence of $\mathbb{Q}$-Weil divisors:

$$K_Y \sim_{\mathbb{Q}} g_1^*(K_{X_1}) + Z_i$$

for $i = 1, 2$, where $Z_i$ is an effective $\mathbb{Q}$-divisor whose support is the union of all exceptional divisors of $g_i$ (because $X_i$ are smooth, hence terminal [35, Claim 2.10.4]). So we have

$$g_1^*(K_{X_1}) - g_2^*(K_{X_2}) \sim_{\mathbb{Q}} Z_2 - Z_1.$$ 

Here $g_1^*(K_{X_1})$ and $g_2^*(K_{X_2})$ are nef. Applying the negativity lemma to $g_1 : Y \to X_1$ (resp. $g_2 : Y \to X_2$), we obtain that $Z_2 - Z_1$ (resp. $Z_1 - Z_2$) is effective [38, Lemma 3.39]. (The negativity lemma holds in any characteristic, because the proof uses resolution of singularities only in dimension 2.) Thus $Z_1 = Z_2$ and so $g_1$ and $g_2$ have the same exceptional divisors, hence $\phi$ is a pseudo-isomorphism.

Thus $\phi^*$ gives an isomorphism

$$\text{Pic}(X_2) \simeq \text{Pic}(X_1),$$

which takes divisors to effective divisors (but in general does not preserve ampleness). Furthermore, we have an induced isomorphism $\text{NS}(X_2) \simeq \text{NS}(X_1)$ [26, Example 19.1.6]. Now assume that $\text{NS}(X_1)_\mathbb{Q} \simeq \text{NS}(X_2)_\mathbb{Q} \simeq \mathbb{Q}$. Since ampleness is a numerical condition, in this case every nonzero effective divisor is ample, in particular $\phi^*$ takes ample divisors to ample divisors. Take any ample divisor $H_2 \in \text{Pic}(X_2)$ and let $H_1 = \phi^*(H_2)$. For every $m \geq 0$, $\phi$ induces an isomorphism $H^0(X_1, \mathcal{O}(mH_1)) \cong H^0(X_2, \mathcal{O}(mH_2))$. These are compatible with products, that is we have a ring isomorphism

$$\bigoplus_{m \geq 0} H^0(X_1, \mathcal{O}(mH_1)) \cong \bigoplus_{m \geq 0} H^0(X_2, \mathcal{O}(mH_2)).$$

By taking Proj of both sides, it follows that $\phi$ is in fact an isomorphism from $X_1$ to $X_2$. 

Remark A.2. The proof of Theorem A.1 goes through in the relative case: given proper morphisms of varieties over a field $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ with $X_1$ and $X_2$ smooth (or just $\mathbb{Q}$-Gorenstein terminal), assume that $K_{X_i}$ is nef over $S$ for $i = 1, 2$. Let $\phi : X_1 \dashrightarrow X_2$ be a birational map over $S$. Then $\phi$ is a pseudo-isomorphism.

We need to recall some results about separable maps in positive characteristic. A dominant map $\phi : X \dashrightarrow Y$ between $k$-varieties is called separable if the corresponding field extension $k(Y)/k(X)$ is separable [66, Tag 0301], that is $k(Y)$ is a finite separable extension of a purely transcendental extension of $k(X)$. Equivalently, $k(X)$ is geometrically reduced over $k(Y)$ [66, Tag 05DT]. Recall that a variety $X$ is called separably uniruled if there exists a separable dominant map

$$Y \times \mathbb{P}^1 \dashrightarrow X$$
for some variety \( Y \) with \( \dim Y = \dim X - 1 \) (see e.g. [36, Definition IV.1.1]). The following lemma should be well-known.

**Lemma A.3.** Let \( X \) be a smooth complete variety over an algebraically closed field \( k \). Suppose that there exist a variety \( Y \) and a separable dominant map 
\[
u : Y \times \mathbb{P}^1 \to X
\]
such that the proper transform of \( \{ y \} \times \mathbb{P}^1 \) is not a point for general \( y \in Y \). Then \( X \) is separably uniruled.

**Proof.** By [36, Theorem IV.1.9], it suffices to prove the existence of a morphism \( f : \mathbb{P}^1 \to X \) which is free (this means that \( f^*T_X \) is globally generated). This is shown in [61, Lemma 1.2]; for the convenience of the reader, we reproduce the proof. Since \( k \) is assumed to be algebraically closed (hence perfect), after shrinking \( Y \), we can assume that \( Y \) is smooth. Therefore \( \nu \) is a morphism outside of a locus of codimension two. As such a locus does not dominate \( Y \), up to further shrinking \( Y \), we can assume that \( \nu \) is a morphism.

As \( u \) is separable, by generic smoothness [66, Tag 056V] the tangent map \( du : T_{Y \times \mathbb{P}^1} \to u^*T_X \) is surjective at a general point \( (y, t) \in Y \times \mathbb{P}^1 \). Write \( \mathbb{P}^1_y := \{ y \} \times \mathbb{P}^1 \), then \( (du)|_{\mathbb{P}^1_y} \) is generically surjective. Thus we have a morphism of sheaves on \( \mathbb{P}^1 \)
\[
\mathcal{O}_{\mathbb{P}^1_y}(2) \oplus \mathcal{O}_{\mathbb{P}^1_y}^{\dim Y} \simeq T_{Y \times \mathbb{P}^1}|_{\mathbb{P}^1_y} \to (u^*T_X)|_{\mathbb{P}^1_y} \simeq \bigoplus_{i=1}^{\dim X} \mathcal{O}(a_i)
\]
with torsion cokernel and it follows immediately that all \( a_i \geq 0 \), so \( (u^*T_X)|_{\mathbb{P}^1_y} \) is globally generated and \( u|_{\mathbb{P}^1_y} : \mathbb{P}^1_y \to X \) is a free morphism. \( \square \)

Recall that a variety \( W \) is called separably rationally connected [36, IV.3.2.3] if there exist a variety \( S \) and a rational map \( u : S \times \mathbb{P}^1 \to W \) such that the two-point evaluation map
\[
(A.1) \quad u_2 = u \times_S u : S \times \mathbb{P}^1 \times \mathbb{P}^1 \to W \times W
\]
is dominant and separable. Note that this condition implies that \( u \) is dominant and separable.

**Lemma A.4.** Let \( Z \) and \( Z' \) be smooth projective varieties of the same dimension over an algebraically closed field \( k \). Let \( W \) be a separably rationally connected variety over \( k \). Suppose that both \( Z \) and \( Z' \) are not separably uniruled. Then for every birational map \( \phi : Z \times W \to Z' \times W \), there is a unique birational map \( \overline{\phi} : Z \to Z' \) making the diagram
\[
(A.2) \quad \begin{array}{ccc}
Z \times W & \xrightarrow{\phi} & Z' \times W \\
\downarrow \quad \overline{\phi} & & \downarrow \\
Z & \to & Z'
\end{array}
\]
commutative (here, the vertical arrows are the projections onto the first factors).
See [49, Theorem 2] for a similar result, where a more general statement was proven in characteristic zero by Liu and Sebag. Our proof follows a similar strategy to that of [49, Theorem 2], relying on the statement analogous to [49, Lemma 5].

**Proof.** Let
\[ p : Z \times W \to Z' \times W \to Z' \]
be the composition of \( \phi \) with the projection. Since \( W \) is separably rationally connected, there exist a variety \( S \) and a rational map \( u : S \times \mathbb{P}^1 \to W \) such that the map \( u_2 \) (A.1) is dominant and separable. Since \( u \) is also dominant and separable, so is the composition
\[ \Phi : Z \times S \times \mathbb{P}^1 \xrightarrow{id_S \times u} Z \times W \xrightarrow{p} Z'. \]

Since \( Z' \) is not separably uniruled, \( \Phi \) contracts \( \{(z, s)\} \times \mathbb{P}^1 \) for general \( (z, s) \in Z \times S \). In other words, \( p(z, -) : W \to Z' \) contracts general rational curves of \( W \) parameterized by \( S \) through \( u \). Since these curves connect general points of \( W \) because \( u_2 \) is dominant, \( p(z, -) \) contracts \( W \) for general \( z \in Z \).

Let \( \Gamma \subset (Z \times W) \times Z' \) be the closure of the graph of \( p \) and let \( \overline{\Gamma} \subset Z \times Z' \) be the image of \( \Gamma \). The fiber of \( \overline{\Gamma} \to Z \) over a general point \( z \in Z \) is the point \( p(\{z\} \times W) \). Moreover, \( \overline{\Gamma} \to Z \) has degree one, because the composition of projections
\[ (Z \times W) \times Z' \to Z \times Z' \to Z \]
induces the composition of surjective morphisms
\[ \Gamma_w := \Gamma \cap (Z \times \{w\} \times Z') \to \overline{\Gamma} \to Z, \]
which is birational for general \( w \in W \); this is because \( \Gamma_w \) is the graph of the rational map \( p|_{Z \times \{w\}} \). Therefore \( \overline{\Gamma} \) is the graph of a rational dominant map \( \overline{\phi} : Z \to Z' \), and it makes (A.2) commutative by construction. It is clear from (A.2) that \( \phi \) uniquely determines \( \overline{\phi} \).

To show that \( \overline{\phi} \) is birational, we apply the same construction to \( \phi^{-1} \), and by uniqueness both compositions of \( \overline{\phi} \) with \( \phi^{-1} \) are identity maps.

**Corollary A.5.** Let \( Z \) and \( Z' \) be K-trivial varieties of Picard number 1 over an algebraically closed field \( \mathbb{k} \). If \( Z \not\cong Z' \), then \( Z \times W \) is not birational to \( Z' \times W \) for any separably rationally connected variety \( W \) over \( \mathbb{k} \). In particular, \( Z \) is not stably birational to \( Z' \).

**Proof.** Since K-trivial varieties are not separably uniruled [36, Corollary IV.1.11], Theorem A.1 and Lemma A.4 show that \( Z \times W \) is not birational to \( Z' \times W \).[36] □

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