Generalised Brownian bridges: examples

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Abstract

We observe that the probability distribution of the Brownian motion with drift 
\(-\frac{c}{1-t}\) where \(c \neq 1\) is singular with respect to that of the classical Brownian 
bridge measure on \([0, 1]\), while their Cameron-Martin spaces are equal set-wise if 
and only if \(c > \frac{1}{2}\), providing also examples of exponential martingales on \([0, 1]\) 
not extendable to a continuous martingale on \([0, 1]\). Other examples of generalised 
Brownian bridges are also studied.

Key words. Brownian bridges, time-dependent singular drifts, Gaussian measures, 
equivalence, Cameron-martin Spaces.

1 Introduction

For an \(L^2\) analysis on the loop space over a manifold pinned at \(x_0\) and \(y_0\), it is standard 
to use the Brownian bridge measure, the latter is a Brownian motion from \(x_0\) conditioned 
to reach \(y_0\) at 1 and is also given by the solution to the stochastic differential 
equation (SDE) \(dz_t = \sigma dx_t + \nabla \log p_{1-t}(x_0, z_t)dt\) with initial value \(x_0\), where \(x_t\) is 
a Brownian motion and \(p_{t}(x, y)\) denotes the heat kernel. This measure is notoriously 
difficult to understand for it would involve precise analysis of the gradient and the Hes-
sian of the logarithm of the heat kernel. For example the class of manifolds for which 
the Poincaré inequality are known to hold for its Brownian bridge measure are limited: 
they are \(\mathbb{R}^n\) [15], the hyperbolic space [10], and a class of asymptotically flat manifolds [1]. 
On the other hand an integration by parts formula was shown to hold on a manifold 
with a pole for the probability measure induced by the semi-classical Brownian bridge [18]. 
The latter solves an SDE with a time-dependent gradient drift which differs from 
\(\nabla \log k_{1-t}(x_0, \cdot)\), in general, but appears to be easier to treat. Hence it is interesting 
to know whether the two measures are equivalent [18]. We also note that the heat ker-
nels measure on the loop space over a simply connected compact Lie group and the 
Brownian bridge measure are proven to be equivalent [2].

The equivalence of two measures on the loop space are subtle. The purpose of this 
article is to give simple examples of generalised bridge measures, which we introduce 
shortly, that are not equivalent. The first class of examples are the probability measure 
induced by the solution of the stochastic differential equation \(dz_t = dB_t - \frac{c}{1-t}z_t \) dt,
where \((B_t)\) is a Brownian motion on \(\mathbb{R}^n\). They induce a family of Gaussian measures, \(\nu^{(c)}\), on \(C_0([0,1];\mathbb{R}^n)\), the loop space of continuous paths from \([0,1]\) to \(\mathbb{R}^n\) pinned at 0. Gaussian measures are quasi-invariant under translation by a vector from its Cameron-Martin space and they are determined by their Cameron-Martin space and their covariance operators. The Cameron-Martin space for the Wiener measure is \(H_0\), the space of finite energy, and that for the Brownian bridge measure \(\nu^{(1)}\) is its sub-space \(H_{0,0} = \{h : [0,1] \to \mathbb{R}^n : h \in H_0, h(1) = 0\}\).

We show that \(\nu^{(c)}\) is singular with respect to \(\nu^{(1)}\) unless \(c = 1\), while their Cameron-Martin spaces are the same, as sets, for all \(c > \frac{1}{2}\). We also note that the Cameron-Martin space of the Gaussian measure given by the SDE \(dz_t = dB_t - \frac{\alpha}{2} dt\), where \(\alpha > 1\), is not the same as \(H_{0,0}\). Finally we give examples of generalised ‘Brownian bridge measures’ which are equivalent to \(\nu^{(1)}\).

The Cameron-Martin [9] theorem states that the Wiener measure on the Wiener space \(C_0([0,1];\mathbb{R})\) is quasi-invariant under the linear transformation \(x \mapsto x + h\) if and only if \(h\) is a Cameron-Martin vector, i.e. \(h\) belongs to the Sobolev space \(H_0\). By quasi-invariance we mean that the pushed forward Wiener measure is equivalent to the Wiener measure (i.e. their null sets are preserved). This is related to the popular Girsanov transform for martingales, for Brownian motions and for stochastic differential equations. Following this, Woodward gave a sufficient condition on \(L\) to ensure the linear transformation on \(C_0([0,1];\mathbb{R}), x(\cdot) \mapsto x(\cdot) + \int_0^1 L(\cdot, s)dx(s)\), to be one to one and onto, where the integral is the Wiener integral. In [30], Shepp gave a necessary and sufficient condition for the stochastic process \(x(t) = (h(t))^{-\frac{1}{2}} W(h(t))\), where \(h\) is an increasing function with \(h(0) = 0\), to be equivalent to the Wiener process \(W(t)\). This generalises a result of Segal [29] and others on non-stochastic transformations and uses a representation of Hitsuda [16]. See also Varberg [31]. For further discussions on Gaussian measures see [7].

Coming back to Brownian bridges, there are other recent related studies. Lupu, Pitman and Tang [21,26] studied the probability distributions of Brownian bridges under Vervaat transformation, using the first time at which the minimum of the Brownian motion is attained, in terms of path decompositions and Brownian excursions. They were inspired by the study of the quartile functions, see also [8]. One of their main questions is a Skorohod embedding type problem for the Brownian bridge: is \(W(\tau + \cdot) - W(\tau)\) a Brownian bridge for some random time \(\tau\)? This problem belongs also to the domain of shift coupling, see [3]. See also [5,6,33,32,22,4].

2 Generalised Brownian bridges and examples

We define a generalised Brownian bridge process, say \((z_t, t \in [0,1])\), between \(x_0\) and \(y_0\) with terminal time 1, to be a stochastic process satisfying the following conditions:

1. \([z_t, t \in [0,1]]\) is a Markov process with infinitesimal generator of the form:
\[
\frac{1}{2} \Delta f(t, z_t) - f(t, z_t) \frac{\partial}{\partial z} f(t, z_t) + \int_0^{z_t} \frac{\partial^2}{\partial z^2} f(t, z_t) \, dz,
\]
where \(f\) is a suitably smooth real valued function on \([0,1]\) with \(\lim_{t \to 1} f(t) = \infty\) and \(r\) is a distance function on the state space;

2. \(\lim_{t \to 1} z_t = y_0\) a.e..

Their probability distributions on the space of continuous paths are called generalised Brownian bridge measures. See also [19,18] for other types of Brownian bridges. In the rest of the section we take the space to be \(\mathbb{R}^n\) and the end points to be \(x_0 = y_0 = 0\).
A function $G : [0, 1] \times [0, 1) \to \mathbb{R}_+$ is said to be an approximation to the identity if the following holds:

1. $\lim_{t \to 1} \int_0^t G(s, t) ds = 1$,
2. $\lim_{t \to 0} \int_0^t G(s, t) ds = 0$ for any $t_0 < 1$.

If $G$ is an approximation to the identity and if $\sigma : [0, 1] \to \mathbb{R}$ is a continuous function, then $\lim_{t \to 1} \left( \int_0^t G(s, t) \sigma(s) ds - \sigma(t) \right) = 0$.

**Proposition 1.** Let $f : [0, 1) \to \mathbb{R}$ be a function such that for any $t_0 \in (0, 1)$.

$$\lim_{t \to 1} \int_0^t f(s) ds = \infty, \quad \int_0^{t_0} f(s) ds < \infty.$$  

Then $G_f(s, t) := f(s) e^{-\int_0^t f(r) dr}$ is an approximation to the identity. It follows that the solution to the SDE $dy_t = dB_t - f(t) y_t dt$ on $\mathbb{R}^n$, where $t < 1$ and $y_0 = 0$, is a generalised Brownian bridge from 0 to 0.

**Proof.** Since $\int_0^t G_f(s, t) ds = e^{-\int_0^t f(r) dr} \left( e^{\int_0^t f(s) ds} - 1 \right)$, and by the assumption that $\lim_{t \to 1} e^{-\int_0^t f(s) ds} = 0$, $G_f$ is indeed an approximation to the identity. The solution to the SDE is explicit and given by the formula

$$y(t) = B_t - e^{-\int_0^t f(r) dr} \int_0^t B_s \frac{d}{ds} e^{\int_0^s f(r) dr} ds \to 0 = B_t - \int_0^t G_f(s, t) B_s ds,$$

proving the proposition.

We will need the following estimates.

**Lemma 1.** If $h \in H$ then $\frac{h(1) - h(u)}{1 - u} \in L^2$. If $f \in L^2([0, 1] \times \mathbb{R}^n)$ and $c \in (\frac{1}{2}, 1]$, set

$$g(x) = \frac{1}{(1 - x)^{1 - c}} \int_0^x \frac{f(y)}{(1 - y)^c} dy.$$  

Then $\|g\|_{L^2} \leq \frac{2}{2c - 1} \|f\|_{L^2}$.

**Proof.** If $h \in H$, it is well known that $\frac{h(1) - h(u)}{1 - u} \in L^2$. In fact

$$\int_0^1 \frac{|h(s) - h(1)|^2}{(1 - s)^2} ds = \frac{|h(t) - h(1)|^2}{1 - t} - \int_0^t \frac{(h(s) - h(1), 2h(s))}{1 - s} ds$$

$$\leq \frac{|h(t) - h(1)|^2}{1 - t} + 2 \int_0^t |h(s)|^2 ds + \frac{1}{2} \int_0^t \frac{|h(s) - h(1)|^2}{(1 - s)^2} ds.$$
Let $f \in L^2$, then
\[
\left( \|g\|_{L^2} \right)^2 = \int_0^1 \int_0^1 \frac{1}{(1-x)^2-2c} \int_0^x \int_0^z (f(y), f(z))^2 (1-y)^c (1-z)^c dy \, dz \, dx \\
= \int_0^1 \int_0^1 (f(y), f(z))^2 (1-y)^c (1-z)^c \int_{y \wedge z}^1 \frac{1}{(1-x)^2-2c} \, dx \, dy \, dz \\
= \frac{1}{2c-1} \int_0^1 \int_0^1 (f(y), f(z))^2 (1-y)^c (1-z)^c (1-y \wedge z)^{2c-1} dy \, dz \\
= \frac{2}{2c-1} \int_0^1 (g(z), f(z)) dz.
\]

The required estimate follows from the Cauchy-Schwartz inequality. \square

For a real valued function $f$ on $[0, 1)$ we define
\[
\Phi(f)(t) = e^{\int_0^t f(s) ds}, \quad t < 1.
\]

**Example 2.** For $\alpha > 1$ set $f_\alpha(t) = \frac{1}{1-t^{1/\alpha}}$ and let $(y_t^n)$ be the solution to the following SDE on $\mathbb{R}^n$: $dy_t = dB_t - f_\alpha(t) y_t dt$ where $t < 1$, and $y_0 = 0$. Then $y_t^n$ is a generalised Brownian bridge. Its probability distribution is singular with respect to the Brownian bridge measure on the loop space $C_{0,0} \mathbb{R}^n$. They have different Cameron-Martin spaces.

**Proof.** Since $(y_t^n, t \leq 1)$ is a Gaussian process, by the Feldman-Hajek theorem it is either equivalent to the Brownian bridge measure or singular to it. For two Gaussian measure to be equivalent it is a necessary condition that their Cameron-Martin spaces agree. The Cameron-Martin space for $(y_t^n)$ is
\[
H^{(\alpha)} = \left\{ k : [0, 1] \to \mathbb{R}^n : k(t) = \Phi(-f_\alpha)(t) \int_0^t \Phi(f_\alpha)(s) \frac{d}{ds} h(s) ds, h \in H \right\}.
\]

This follows from the following fact. Let $\mu$ be a measure on a Banach space $E$ and let $T$ be a linear map from $E$ to another Banach space $\tilde{E}$. We denote by $\nu = T_\ast \mu$ the pushed forward measure. Then the Cameron-Martin space of $\nu$ is the image of the Cameron-Martin space of $\mu$ by $T$. Take $\mu$ to be the Wiener measure, $E = L^2([0, 1]; \mathbb{R}^n)$ and $T$ the map
\[
T(\sigma)(t) = \Phi(-f_\alpha)(t) \int_0^t \Phi(f_\alpha)(s) d\sigma(s).
\]

After an integration by parts, we see that $T(\sigma)(t) = \sigma(t) - \tilde{T}(\sigma)(t)$ where
\[
\tilde{T}(\sigma)(t) = \Phi(-f_\alpha)(t) \int_0^t \frac{d}{ds} \left[ \Phi(f_\alpha)(s) \right] \sigma(s) ds.
\]

Since $\Phi(-f_\alpha(t)) \to 0$ as $t \to 1$, $\Phi(-f_\alpha)(t) \frac{d}{ds} \left[ \Phi(f_\alpha)(s) \right]$ is an approximation of the identity and $\lim_{\alpha \to 1} y_t^n = 0$ for $\alpha \geq 1$.

For $\alpha = 1$ it is easy to see that $H^{(1)} = H_{0,0}$, where $H_{0,0} = \{ h \in H : h(1) = 0 \}$. Suppose that $k$ and $h$ are related by the formula $k(t) = (1-t) \int_0^t \frac{h(s)}{1-s} ds$. If $k \in H_{0,0}$
then $\hat{h}(s) = \hat{k}(s) + \frac{k(s)}{1-s} \in L^2([0,1])$; if $h \in H$ then $\dot{h}(t) = \hat{h}(t) - \int_0^t \frac{\dot{h}(s)}{1-s} ds$ belongs to $L^2([0,1])$. Both by Lemma [1] for $\alpha > 1$ consider the inverse map $T^{-1}$:

$$h = T^{-1}(k) = \Phi(-f_0) \frac{d}{dt} \left[ \Phi(f_0)k \right].$$

For $k \in H_{0,0}$, $\dot{h}(t) = \dot{k}(t) - \Phi(-f_0) \frac{d}{dt} \left[ \Phi(f_0) \right] k(t)$ belongs to $L^2$ if and only if the second term, $\Phi(-f_0) \frac{d}{dt} \left[ \Phi(f_0) \right] k(t) = \frac{k(t)}{(1-t)^\alpha}$, does. It is possible to find $k \in H_{0,0}$ such that $\frac{k(t)}{(1-t)^\alpha}$ does not belong to $L^2$, e.g. take $k(t)$ to be of the order $(1-t)^{\frac{1}{2}+\epsilon}$ where $\epsilon < \alpha - 1$. This means that $H^{(\alpha)} \neq H$, and the generalised Brownian bridge measure is not equivalent to the Brownian bridge measure. □

**Remark 3.** Let $(M_t^c, t < 1)$ denote the exponential martingale, in the Girsanov transform from the Brownian bridge to the generalised Brownian bridge. It cannot be extended to a martingale on $[0,1]$, see Theorem 3.1 of [28]. In view of the use of strict local martingales in the study for financial bubbles, see [12, 23, 24, 25, 13, 14, 20], this type of exponential martingales might be interesting in mathematical finance. The same can be said of the class of, somewhat surprising, examples below.

**Example 4.** Let $c > 0$ be a real number and let $(z^c(t))$ be the solution to the SDE

$$dz_t = dB_t - \frac{c}{1-t} z_t dt$$

with $z_0 = 0$. The solution will be called the generalised Brownian bridge with parameter $c$. Then

(a) $(z^c_t)$ is a generalised Brownian bridge.

(b) Denote by $\nu^c$ its probability distribution on the loop space. Then its Cameron-Martin space $H^{(c)}$ agrees with $H^{(1)}$ as a set if and only if $c > \frac{1}{2}$.

(c) The generalised Brownian bridge measures $\nu^{(1)}$ and $\nu^{(c)}$ are mutually singular unless $c = 1$.

**Proof.** Following the proof of Example [2] we define:

$$T(\sigma)(t) = (1-t)^c \int_0^t \frac{1}{1-s} \sigma(s) ds = \sigma(t) - c(1-t)^c \int_0^t \frac{\sigma(s)}{(1-s)^{c+1}} ds,$$

the first integral being a stochastic integral. The Cameron-Martin space of the Gaussian distribution of $(z^c_t)$ is

$$H^{(c)} = \left\{ k : [0,1] \to \mathbb{R}^n : k(t) = (1-t)^c \int_0^t \frac{\dot{h}(s)}{(1-s)^c} ds, h \in H \right\}.$$

If $k \in H_{0,0} \equiv H^{(1)}$, let $h(t) := T^{-1}(k)(t) = (1-t)^c \frac{d}{dt} [(1-t)^{-c} k(t)]$. Since $k(1) = 0$, $\frac{k(t)}{1-t} \in L^2$ by Lemma [1] consequently, $\dot{h}(t) = \dot{k}(t) + c \frac{k(t)}{1-t}$ belongs to $L^2$. Hence $T^{-1}$ maps $H_{0,0}$ to $H^{(c)}$.

Let $h \in H$. It is clear that $T(h)(0) = 0$ and $T(h)(1) = \lim_{t \to 1} (1-t)^c \int_0^t \frac{\dot{h}(s)}{(1-s)^c} ds = 0$. We only need to prove the second term, in the following formula

$$\frac{d}{dt} T(h)(t) = \dot{h}(t) - c(1-t)^{c-1} \int_0^t \frac{\dot{h}(s)}{(1-s)^c} ds,$$
belongs to $L^2$. For $c \in (\frac{1}{2}, 1]$ this follows from Lemma [1]. Let $c > 1$. We observe that
$$\int_0^1 (1 - t)^{2c-2}dt = \frac{1}{2c-1}(1 - u \vee v)^{2c-1},$$
and $(1 - t)^{2c-1}$ makes sense on $[0, 1]$, so we have the estimate below.
\begin{align*}
\int_0^1 (1 - t)^{2c-2} \left[ \int_0^t \frac{\hat{h}(s)}{(1 - s)^c}ds \right]^2 dt &= \int_0^1 \int_0^1 \frac{\langle \hat{h}(u), \hat{h}(v) \rangle}{(1 - u)^c(1 - v)^c} (1 - u \vee v)^{2c-1}dudv \\
&= \frac{2}{2c-1} \int_0^1 \int_u^1 \left\langle \frac{\hat{h}(v)}{1 - v}, \hat{h}(u) (1 - u)^{-c-1} \right\rangle dv du \leq \frac{2}{2c-1} \int_0^1 \int_0^1 \frac{\hat{h}(v)}{1 - v} dv |\hat{h}(u)| du,
\end{align*}
which is finite by Cauchy-Schwarz inequality and by Lemma [1]. Hence $H^{(c)}$ is contained in $H_{0,0}$ for $c > \frac{1}{2}$.

If $c \leq \frac{1}{2}$, for $\int_0^1 (1 - t)^{2c-2} \left[ \int_0^t \frac{\hat{h}(s)}{(1 - s)^c}ds \right]^2 dt$ to be finite it is necessary that
$$\lim_{t \to 0} \int_0^t \frac{\hat{h}(s)}{(1 - s)^c} ds = 0,$$
a condition not satisfied by every $h \in H$. We have proved that $H^{(c)} = H_{0,0}$ for and only for $c > \frac{1}{2}$. In particular the measure of the generalised bridge process with parameter $c \leq \frac{1}{2}$ is singular with respect to the bridge measure.

We need a more subtle argument to show that the generalised bridge process with parameter $c > \frac{1}{2}$ is singular with respect to the bridge measure. Since their Cameron-Martin spaces agree as a set we will need to study their covariance operators. Set
\begin{align*}
(R\phi)(t) &= \int_0^1 (s \wedge t - st) \phi(s)ds, \\
Q_c(s,t) &= \int_0^{s \wedge t} (1 - t)^c(1 - s)^c (1 - r)^{2c}dr.
\end{align*}
Let $R_c : L^2([0,1]; \mathbb{R}^n) \to L^2([0,1]; \mathbb{R}^n)$ denote the covariance operator of the distribution of $(z_t^{(c)})$:
\begin{align*}
(R_c\phi)(t) &= \int_0^1 \phi(s)Q_c(s,t) ds = \frac{(1 - t)^c}{2c - 1} \int_0^1 \phi(s)(1 - s)^c [([1 - (s \wedge t)^{1-2c}] - 1)] ds.
\end{align*}
In particular $R$ is the covariance operator for the Brownian bridge. By the Feldman-Hajek theorem the two probability measures are equivalent if and only if $R^{-\frac{1}{2}} R_c R^{-\frac{1}{2}}$ is the sum of an identity map and a Hilbert-Schmidt operator, where $R^{1/2}$ is the square root defined by functional calculus [1].

We define an operators $A : L^2([0,1]; \mathbb{R}^n) \to L^2([0,1]; \mathbb{R}^n)$ by the formula:
$$\langle Af \rangle(t) = \int_0^t f(s)ds - t \int_0^1 f(s)ds.$$ 
We also define $A^* : L^2([0,1]; \mathbb{R}^n) \to L^2([0,1]; \mathbb{R}^n)$ by the formula
$$\langle A^*\phi \rangle(t) = \int_0^1 \phi(s)ds - \int_0^t s\phi(s)ds.$$ 
The image of $A$ is the set of $L^2$ functions with the boundary condition $\phi(0) = \phi(1) = 0$ while that of $A^*$ is $E = \{ \int \in L^2([0,1]; \mathbb{R}^n) : \int_0^1 f(s)ds = 0 \}$. Denote by $(A^*)^{-1}$ the left inverse of $A^*$, then $(A^*)^{-1} = -\frac{\partial}{\partial t}$ without any boundary condition. Furthermore $A^* f = f$ when restricted to $H_{0,0}$. We observe that the image of $R_c$ is contained in the Cameron-Martin space $H^{(c)} = H_{0,0}$ (it is also dense there). Restricted to $H_{0,0}$, the right inverse of $A$, which we denote by $A^{-1}$, is $\frac{\partial}{\partial t}$. 

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It is easy to see that \( R \frac{d}{dt} \phi = -A\phi, \quad -\frac{d}{dt} R \frac{d}{dt} \phi = \phi - \int_0^1 \phi(s) ds \) and \( R = AA^* \). Observe that \( R \) is invertible on \( E, A \) and \( R^{1/2} \) differ by a unitary operator. To check whether \( R^{-\frac{1}{2}} R_c R^{-\frac{1}{2}} \) is the sum of an identity map and a Hilbert-Schmidt operator, it is sufficient to prove that \( A^{-1} R_c (A^*)^{-1} \) is the sum of an identity map and a Hilbert-Schmidt operator. Indeed, if \( A^{-1} R_c (A^*)^{-1} = I + K \), where \( K \) is a Hilbert-Schmidt operator, then \( R^{-\frac{1}{2}} R_c R^{-\frac{1}{2}} = I + (R^{-\frac{1}{2}} A) K (R^{-\frac{1}{2}} A)^* \).

Let \( \phi : [0, 1] \to \mathbb{R}^n \) be a test function, then
\[
- (R_c (A^*)^{-1} \phi)(t) = (R_c \phi)(t) = \int_0^1 \int_0 s \wedge t \frac{(1 - s)^c (1 - t)^c}{(1 - r)^{2c}} dr \phi(s) ds
\]
\[
= (1 - t)^c \left[ \int_0^1 \int_0 s \wedge t \frac{dr}{(1 - r)^{2c}} (1 - s)^c \phi(s) ds + \int_0^1 (1 - s)^c \phi(s) ds \int_0^1 \frac{r}{(1 - r)^{2c}} dr \right].
\]

We then apply the operator \( A^{-1} \):
\[
\frac{d}{dt} (R_c (A^*)^{-1} \phi)(t) = - \left[ \frac{1}{(1 - t)^c} \int_0^1 (1 - s)^c \phi(s) ds - c - 1 \right] (R_c \phi)(t)
\]
\[
= \phi(t) - c \int_0^1 \frac{(1 - s)^c - 1}{(1 - t)^c} \phi(s) ds + c^2 \int_0^1 \int_0 s \wedge t \frac{dr}{(1 - r)^{2c}} (1 - s)^c \phi(s) ds.
\]

We compute the last term,
\[
\frac{c}{1 - t} (R_c \phi)(t) = - c(1 - t)^{c-1} \left[ \int_0^1 \frac{\phi(s)}{(1 - s)^c} ds \right] + c^2 \int_0^1 \int_0 s \wedge t \frac{(1 - t)^{c-1}(1 - s)^c - 1}{(1 - r)^{2c}} dr \phi(s) ds.
\]

Thus,
\[
(A^{-1} R_c (A^*)^{-1} \phi)(t) = \frac{d}{dt} (R_c (A^*)^{-1} \phi)(t) = \phi(t) + \int_0^1 q_c(s, t) \phi(s) ds,
\]
where
\[
q_c(s, t) = -c \frac{(1 - s \vee t)^{c-1}}{(1 - s \wedge t)^c} + c^2 (1 - t)^{c-1}(1 - s)^c - 1 \int_0^1 \frac{dr}{(1 - r)^{2c}}.
\]

For \( c = 1 \), \( q_1(s, t) = -1 \) and \( \int_0^1 q_1(s, t) \phi(s) ds \) vanishes on \( E \). For \( c \neq \frac{1}{2} \),
\[
q_c(s, t) = -c(1 - c) \frac{(1 - s \vee t)^{c-1}}{(1 - s \wedge t)^c} - \frac{c^2}{2c - 1} (1 - t)^{c-1}(1 - s)^c.
\]

This is not an \( L^2 \) function: the second term on the right hand side is \( L^2 \) integrable for \( c > \frac{1}{2} \) while the first term has a logarithmic singularity unless \( c = 1 \). We have proved that the generalised Brownian bridge bridge process with parameter \( c \) and the classical Brownian bridge are mutually singular for any \( c \in (0, 1) \cup (1, \infty) \).

Finally we observe a perturbative result.

**Example 5.** Let \( \delta \in (0, 1/2) \) and \( c \) be constants satisfying \( c^2 < \frac{1}{4}(1 - 2\delta) \). We consider the equation
\[
dx = dB_t - \frac{c}{2} x dt + \frac{f(t, x)}{(1-t)^{\delta}} dt
\]
where \( f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies the bound \( |f(t, x)|^2 \leq c|x|^2 + c \). Suppose that the SDE is well posed and is conservative. Then the probability distribution of \( (x_t) \), which we denote by \( \nu_2 \), is equivalent to \( \nu^{(i)} \).

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Proof. Let \( z_t \) solves \( dz_t = dB_t - \frac{z_t}{2} dt \). Since \( B_t - t B_1 \) is a representation of the Brownian bridge, we see that \( E \exp(a \sup_{0 \leq t \leq 1} |z_t|^2) \) is finite if \( 4a < \frac{1}{2} \). This follows from Fernique’s theorem. (Integrability of exponential of Bessel bridges were studied in [17, 27].) For \( t < 1 \),

\[
\frac{d\nu_2}{d\nu^{(1)}} = \exp \left( \int_0^t \left\langle dB_s, \frac{f(s, z_s)}{(1-s)^{\frac{1}{2}}} \right\rangle - \frac{1}{2} \int_0^t \frac{|f(s, z_s)|^2}{(1-s)^{2\beta}} ds \right).
\]

By the assumption,

\[
\frac{1}{2} \int_0^t \frac{|f(s, z_s)|^2}{(1-s)^{2\beta}} ds \leq (1 + \sup_{s \in [0,1]} |z_s|^2) \frac{c^2}{2(1-2\delta)}.
\]

which is exponentially integrable. Let \( N_t = \int_0^t \left\langle dB_s, \frac{f(s, z_s)}{(1-s)^{\frac{1}{2}}} \right\rangle \). We invoke the Novikov criterion to conclude that the exponential martingale of \( N_t \) is uniformly integrable on \([0,1)\) and converges in \( L^1 \) as \( t \) approaches 1. It follows that \( \nu_2 \) is absolutely continuous with respect to \( \nu^{(1)} \). Since \( (N_t - \frac{1}{2} \langle N, N \rangle_t, t \in [0,1]) \) is \( L^2 \) bounded, it converges in \( L^2 \) and so has a finite limit. Thus \( \lim_{t \to 1} G_t \neq 0 \) and the two measures are equivalent. 

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