Three Self-Consistent Kinematics in (1+1)D Special-Relativity

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Abstract

When introducing special relativity, an elegant connection to familiar rules governing Galilean constant acceleration can be made, by describing first the discovery at high speeds that the clocks (as well as odometers) of different travelers may proceed at different rates. One may then show how to parameterize any given interval of constant acceleration with either: Newtonian (low-velocity approximation) time, inertial relativistic (unaccelerated observer) time, or traveler proper (accelerated observer) time, by defining separate velocities for each of these three kinematics as well. Kinematic invariance remains intact for proper acceleration since \( ma_o = dE/dx \). This approach allows students to solve relativistic constant acceleration problems \textit{with the Newtonian equations}! It also points up the self-contained and special nature of the accelerated-observer kinematic, with its frame-invariant time, 4-vector velocities which in traveler terms exceed Newtonian values and the speed of light, and of course relativistic momentum conservation.

03.30.+p, 01.40.Gm, 01.55.+b
I. INTRODUCTION

Paradigm changes in physics are often accompanied by a transformation of language and a hardening of distinctions. Thus the emphasis on acceleration as the second time derivative of displacement in teaching Newtonian dynamics has hardened\textsuperscript{1,2}, even as the concept of relativistic mass and inertial frame acceleration is being slowly replaced by Lorentz covariant definitions of force and acceleration in the teaching of special relativity\textsuperscript{3–5}. The intent of this note is to assist the second process by showing that: (i) introduction of three relativistic time-parameterizations, including the Newtonian one, provides a natural bridge for introductory physics students to some deep truths about relativistic space-time BEFORE the comparatively abstract concept of Lorentz transforms need be introduced, and (ii) there are two relativistic alternatives to the popular inertial-frame perspective, which perspective sometimes makes velocity-dependent mass a tempting simplification\textsuperscript{2,4}, questions the special-relativistic treatability of acceleration\textsuperscript{6}, and implies that travelers as well as spectators are bound by inertial velocity light-speed limits\textsuperscript{7}. We also show why students here have said that "without the 82 nano-roddenberry speed-limit, and with the right vehicle, it’s amazing how little of my time it would take for me to get from point A to point B!"

II. OBSERVATIONS

We consider in this paper only one-dimensional constant acceleration, and only observers using the same inertial frame spatial coordinate with which to measure change of position $\Delta x$. We further use the Lorentz-invariant proper acceleration $a_o$ for describing traveler motion, so that the relativistic work-energy theorem gives us $\Delta E \equiv mc^2 = ma_o \Delta x$, independent of the time-parameterization in use. Here $m$ is the rest mass of our accelerated object, relativistic energy factor $\gamma$ is defined dynamically as the accelerated object’s total energy in units of the rest energy $mc^2$, and $c$ is the speed of light.
Consider first the task of assigning to each point on the world line of an accelerated traveler 3 time parameters, namely \( b \), \( t \) and \( \tau \), referring respectively to inertial, Newtonian, and traveler time-maps (or kinematics). We denote the velocities corresponding to these time maps, respectively, as \( w \equiv dx/db \), \( w \equiv dx/dt \), and \( u \equiv dx/d\tau \). The time maps can be most simply specified, using the work-energy theorem above, by determining the dependence of \( \gamma \) on the velocities \( w \), \( v \) and \( u \).

From the Lorentz expression for \( \gamma \) in terms of relativistic inertial velocity \( w \), the \( u = \gamma w \) relationship between inertial and 4-vector velocities, and the familiar Newtonian expression for kinetic energy \( K = \frac{1}{2}mv^2 \), one can show that \( \gamma \equiv E/mc^2 = \frac{mc^2+K}{mc^2} = 1/\sqrt{1-w^2/c^2} = 1+\frac{1}{2}v^2/c^2 = \sqrt{1+u^2/c^2} = db/d\tau \). Substitution into the work-energy equation, \( \Delta x = \Delta \gamma c^2/a_o \), yields a velocity-distance equation, like the familiar Newtonian \( \Delta x = \Delta (\frac{1}{2}v^2)/a_o \), for each of the 3 kinematics.

Taking the time derivative of this displacement equation *in each of the three kinematics* yields a relationship between the velocities and time and acceleration. After defining for the traveler kinematic the *hyperbolic velocity angle* \( \eta \equiv \text{asinh}(u/c) \) which is additive under Lorentz transform, and recalling for the inertial kinematic that 4-velocity \( u \) can be expressed in terms of inertial velocity \( w \), these velocity-time equations can be put into the form \( a_o = \Delta u/\Delta b = \Delta v/\Delta t = c\Delta \eta/\Delta \tau \). For a fuller comparison, the equations used for treating constant acceleration, force, and momentum conservation in introductory physics are summarized in this way in Table I, while the exact equations for application of all three kinematics to constant acceleration problems are listed in Table II. Note that, except for a correction to the momentum equation needed at high velocity, namely that momentum \( p \) becomes \( mv\sqrt{1+\frac{1}{4}v^2/c^2} \), the introductory physics equations for one-dimensional motion remain exact at arbitrarily high speed, even if other velocities and times have to be calculated to provide information on the kinematic variables (e.g. physical clocks) of interest.

As in introductory physics, armed with a displacement and a velocity equation for our kinematic and any three of the 5 variables (initial/final velocity, time elapsed, distance traveled, and constant acceleration), we can solve for the other two. Equation *pairs* for
dealing with this task for each of the 10 types of initial condition definable within each of the 3 kinematics are listed in Table III. If the 3 independent variables for a particular problem include velocities from different kinematics, they can be easily converted to the kinematic of choice using the expressions above for $\gamma$ and the fact that $u = v\sqrt{\frac{\gamma+1}{2}} = w\gamma$.

If time-elapsed values from more than one kinematic are included in the initial inputs, then the equality above between velocity-time expressions can be used, at least numerically, to find a solution within the kinematic of choice. Of course, once the solution for a particular kinematic is in hand, these same equations allow the 4 velocities and 2 times associated with the other kinematics to be calculated as well. Thus for three input variables one now gets 8 quantities of potential interest, instead of only two!

III. DISCUSSION

All of the foregoing requires no knowledge of Lorentz transforms, since the inertial frame for measuring distance traveled was fixed. Of course, once such transformations are understood, it is easy to show other things. For example, since the spatial component of the velocity 4-vector is $u = c \sinh \eta$, while its time component is $c\gamma = c \cosh \eta$, it is easy to show that $a_o$ is the magnitude of the 4-vector acceleration (i.e. the proper time $\tau$ derivative of the 4-vector velocity), and hence is Lorentz invariant as well as independent of kinematic.

We’ve also found it to be useful (and enjoyable) in practice to associate different units with the velocities and times in different kinematics. In relativistic problems we are often dealing with distances on the scale of lightyears, and time on the scale of years, so we often speak of inertial time $\Delta b$ in "iyears", Newtonian time in year or whatever MKS units are handy, and traveler time $\Delta \tau$ in "tyears". Likewise, we usually speak of inertial velocities $w$ in fractional units of the speed of light (e.g. $0.5c$), while leaving Newtonian velocities in whatever units fit the problem.

Traveler velocities $u$ are most interesting of all, since they exceed the speed of light even more enthusiastically than Newtonian velocities do (cf. Fig. 1). The favorite strategy
here, among students I’ve discussed this with, has been to define one lightyear per year of traveler velocity as 1 roddenberry or \([rb]\), in tribute to Gene Roddenberry, the producer of the original StarTrek television series whose offspring continue to ignore the light speed limit as well as the way that space time is put together at high speed. The approach described here might help a bit, in that ”warp speeds” (which still require that one somehow drops out of our space-time if one is to effect such rapid travel without inertial effects to passengers and ”temporal effects” to their colleagues at home\(7\)) at least can now be mapped to traveler-frame speeds, rather than to inertial-frame speeds greater than the speed of light (which given the fabric of spacetime make no sense at all).

We illustrate with an example. Describe timeelapsed values and final velocities for a space dragster accelerating from rest at \(a_o = 1[g] \approx 1.03[ly/yr^2]\) over a distance of 4 lightyears \([ly]\). Using only the Newtonian equations for constant acceleration, one can infer (Table III, Row 6, Column 2) that our dragster will reach a final Newtonian velocity of \(v_f = 2.86[ly/yr]\) in the Newtonian time of \(\Delta t = 2.79[yr]\). The solution is thus put in hand, and only simple conversion to other kinematics (cf. Table II) remains! One obtains inertial time \(\Delta b = 4.87[iyr]\) using \(u = v\sqrt{1 + \frac{1}{4}v^2/c^2}\) and \(\Delta u = a_o\Delta b\). This also gives final traveler speed \(u_f = 5.03[rb]\). One obtains traveler time \(\Delta \tau = 2.25[tyr]\) given that \(\eta = \text{asinh}[u/c]\) and \(c\Delta \eta = a_o\Delta \tau\). Finally, given \(w = u/\gamma\) where \(\gamma = 1 + \frac{1}{2}v^2/c^2\), it follows that the final inertial velocity \(w_f = 0.98[c]\).

This is only one example. The rather simple equations of Newtonian constant acceleration in one dimension allow one to solve a wide array of types of problems, to which we can now add many relativistic problems as well. Interactive worksheets and example problems for all of the 30 cases listed in Table 3 are being assembled on the web at \texttt{http://newton.umsl.edu/\~{}run}\.

In conclusion, this three-kinematic approach allows one to look at and solve some relativistic constant-acceleration problems in a new way. In particular, it puts both the Newtonian equations for constant acceleration and the Traveler/Proper Time equations into a new light as fully alternative relativistic kinematics, even if the former remains self-consistent.
only in one spatial dimension. Moreover, it seems to provide more new answers per problem (i.e. 6), than there are new concepts required for a post-introductory physics student to apply and understand them.

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