Friedmann’s equations in all dimensions and Chebyshev’s theorem

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Abstract. This short but systematic work demonstrates a link between Chebyshev’s theorem and the explicit integration in cosmological time $t$ and conformal time $\eta$ of the Friedmann equations in all dimensions and with an arbitrary cosmological constant $\Lambda$. More precisely, it is shown that for spatially flat universes an explicit integration in $t$ may always be carried out, and that, in the non-flat situation and when $\Lambda$ is zero and the ratio $w$ of the pressure and energy density in the barotropic equation of state of the perfect-fluid universe is rational, an explicit integration may be carried out if and only if the dimension $n$ of space and $w$ obey some specific relations among an infinite family. The situation for explicit integration in $\eta$ is complementary to that in $t$. More precisely, it is shown in the flat-universe case with $\Lambda \neq 0$ that an explicit integration in $\eta$ can be carried out if and only if $w$ and $n$ obey similar relations among a well-defined family which we specify, and that, when $\Lambda = 0$, an explicit integration can always be carried out whether the space is flat, closed, or open. We also show that our method may be used to study more realistic cosmological situations when the equation of state is nonlinear.

Keywords: string theory and cosmology, alternatives to inflation, cosmological applications of theories with extra dimensions

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### 1 Introduction

The Friedmann equations play a fundamental role in modern cosmology, in particular for studies of inflationary universes. One of the main tools for simplification of these equations, in order to obtain an exact integration, is through the use of conformal-time reparametrization [1–10], although qualitative investigations are often more conveniently carried out using cosmological time [11–16]. Particularly noteworthy is the classification scheme of Harrison [11] which contains many explicit formulas for the scale factor as a function of cosmic time. However, the range of possible equations of state treated is rather limited; attention is restricted to the three-dimensional case and the paper is not directly concerned with finding all cases which can be integrated in finite terms. Here we show that, in view of Chebyshev’s theorem, the answer to this question: whether the Friedmann equations may be integrated explicitly in cosmological time becomes immediately transparent, at least for two important situations in all dimensions: spatially flat universes with an arbitrary cosmological constant and spatially curved universes with vanishing cosmological constant. More precisely, we show that, in the former situation, an explicit integration\(^1\) may be carried out when the ratio \(w\) of the pressure and energy density in the barotropic equation of state of the perfect-fluid universe is arbitrary, and in the latter situation and when \(w\) is rational, an integration may be carried out if and only if \(w\) assumes specific values among an infinite sequence.

Even though nowadays, numerical solutions of ordinary differential equations are readily available on laptops and other electronic devices, it is still the case that, for a rapid overview of the properties of the solutions and their detailed behavior, it is often more efficient to examine explicit formulas when they are available. Hence this paper.

Here is an outline of the rest of the paper. In section 2 we recall Friedmann’s equations and Chebyshev’s theorem and discuss the applicability of the latter to the former. In section 3 we show that, in view of Chebyshev’s theorem, an explicit integration of Friedmann’s equations in cosmological time may always be carried out in all spatial dimensions and for any value of the ratio \(w\) of the pressure and energy density in the equation of state of the perfect-fluid universe. In section 4 we tackle the non-flat situations assuming the cosmological constant vanishes and we identify an infinite family of values of \(w\) that permit an explicit integration in cosmological time. In section 5 we consider the problem in conformal time and

\(^1\)By an explicit integration we mean what is often referred to as an integration in finite terms or an analytic integration or an integration in terms of elementary functions. From now on we shall drop the adjective explicit and integration should be taken to mean in finite terms.
determine all the cases when an explicit integration can be carried out. This situation is also shown to complement that in cosmological time.

2 Friedmann’s equations and Chebyshev’s theorem

Consider an \((n + 1)\)-dimensional homogeneous and isotropic Lorentzian spacetime with the metric

\[
\begin{align*}
\text{d}s^2 &= g_{\mu\nu}\text{d}x^\mu\text{d}x^\nu = -\text{d}t^2 + a^2(t)g_{ij}\text{d}x^i\text{d}x^j, \quad i, j = 1, \ldots, n, \tag{2.1}
\end{align*}
\]

where \(t\) is the cosmological (or cosmic) time and \(g_{ij}\) is the metric of an \(n\)-dimensional Riemannian manifold \(M\) of constant scalar curvature characterized by an indicator, \(k = -1, 0, 1\), so that \(M\) is an \(n\)-hyperboloid, the flat space \(\mathbb{R}^n\), or an \(n\)-sphere, with the respective metric

\[
\begin{align*}
g_{ij}\text{d}x^i\text{d}x^j &= \frac{1}{1 - kr^2}\text{d}r^2 + r^2\text{d}\Omega^2_{n-1}, \tag{2.2}
\end{align*}
\]

where \(r > 0\) is the radial variable and \(\text{d}\Omega^2_{n-1}\) denotes the canonical metric of the unit sphere \(S^{n-1}\) in \(\mathbb{R}^n\). Inserting the metric (2.1)–(2.2) into the Einstein equations

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_n T_{\mu\nu}, \tag{2.3}
\]

where \(G_{\mu\nu}\) is the Einstein tensor, \(G_n\) the universal gravitational constant in \(n\) dimensions, and \(\Lambda\) the cosmological constant, the speed of light is set to unity, and \(T_{\mu\nu}\) is the energy-momentum tensor of an ideal cosmological fluid given by

\[
T^\nu_{\mu} = \text{diag}\{-\rho_m, p_m, \ldots, p_m\}, \tag{2.4}
\]

with \(\rho_m\) and \(p_m\) the \(t\)-dependent matter energy density and pressure, we arrive at the Friedmann equations

\[
\begin{align*}
H^2 &= \frac{16\pi G_n}{n(n-1)}\rho - \frac{k}{a^2}, \tag{2.5}
\end{align*}
\]

\[
\dot{H} = -\frac{8\pi G_n}{n-1}(\rho + p) + \frac{k}{a^2}, \tag{2.6}
\]

in which

\[
H = \frac{\dot{a}}{a}, \tag{2.7}
\]

denotes the usual Hubble ‘constant’, \(\dot{f} = \frac{df}{dt}\), and \(\rho, p\) are the effective energy density and pressure related to \(\rho_m, p_m\) through

\[
\rho = \rho_m + \frac{\Lambda}{8\pi G_n}, \quad p = p_m - \frac{\Lambda}{8\pi G_n}. \tag{2.8}
\]

On the other hand, recall that, with (2.1) and (2.4) and (2.8), the energy-conservation law, \(\nabla_\nu T^{\mu\nu} = 0\), takes the form

\[
\dot{\rho}_m + n(\rho_m + p_m)H = 0. \tag{2.9}
\]

It is readily seen that (2.5) and (2.9) imply (2.6). In other words, the full cosmological governing equations consist of (2.5) and (2.9) only.\(^2\)

\(^2\)In the cosmological literature (2.5) is often referred to as the Friedmann equation and (2.6) somewhat anachronistically as the Raychaudhuri equation. In this paper by the Friedmann equations we mean the pair (2.5), (2.9), supplemented with an equation of state, \(p_m = p_m(\rho_m)\).
To proceed further we assume a so-called barotropic equation of state,\(^3\)

\[
p_m = w \rho_m,
\]

(2.10)

where \(w\) is a constant so that \(w = 0\) leads to a vanishing pressure, \(p_m = 0\), corresponding to the dust model, and \(w = \frac{1}{3}\), the radiation-dominated model.

Inserting (2.10) into (2.9), we have

\[
\dot{\rho}_m + n(1 + w) \rho_m \frac{\dot{a}}{a} = 0,
\]

(2.11)

which can be integrated to yield

\[
\rho_m = \rho_0 a^{-n(1+w)},
\]

(2.12)

where \(\rho_0 > 0\) is an integration constant \([19]\). Using (2.12) in (2.8), we arrive at the relation \([17, 19]\)

\[
\rho = \rho_0 a^{-n(1+w)} + \frac{\Lambda}{8\pi G_n},
\]

(2.13)

From (2.5) and (2.13), we get the following equation of motion for the scale factor \(a\):

\[
\dot{a}^2 = \frac{16\pi G_n \rho_0}{n(n-1)} a^{-n(1+w)+2} + \frac{2\Lambda}{n(n-1)} a^2 - k.
\]

(2.14)

To integrate (2.14), we recall Chebyshev’s theorem \([20, 21]\): for rational numbers \(p, q, r\) \((r \neq 0)\) and nonzero real numbers \(\alpha, \beta\), the integral

\[
I = \int x^p (\alpha + \beta x^r)^q \, dx
\]

(2.15)

is elementary if and only if at least one of the quantities

\[
p + \frac{1}{r}, \quad q, \quad p + \frac{1}{r} + q,
\]

(2.16)

is an integer. In fact, the integral (2.15) may be rewritten as \([22]\)

\[
I = \frac{1}{r} \alpha^{p+1} + q \beta^{p+1} \, B_y \left( \frac{1+p}{r}, q - 1 \right)
\]

\[
= \frac{1}{p+1} \alpha^{p+1} + q \beta^{p+1} y^{1+r} F \left( \frac{p+1}{r}, 2 - q, \frac{1+p+r}{r}; y \right),
\]

(2.17)

where \(y = \frac{\beta}{\alpha} x^r\), and \(B_y \left( \frac{1+p}{r}, q - 1 \right)\) and \(F \left( \frac{p+1}{r}, 2 - q, \frac{1+p+r}{r}; y \right)\) are the incomplete beta function and hyper-geometric function, respectively. The result then follows from the classification of elementary cases of the hyper-geometric function.

Consequently, when \(k = 0\) or \(\Lambda = 0\), and \(w\) is rational, the Chebyshev theorem enables us to know that, for exactly what values of \(n\) and \(w\), the equation (2.14) may be integrated.

\(^3\)Later we will relax this assumption slightly by considering a mixture of two barotropic fluids with different values of \(w\), but set the cosmological constant to zero.
3 Spatially flat case

We first focus on the spatially flat situation $k = 0$, which is known to be most relevant for cosmology [23–28], and rewrite equation (2.14) as

$$
\dot{a} = \pm \sqrt{c_0 a^{-n(1+w) + 2} + \Lambda_0 a^2}, \quad c_0 = \frac{16\pi G_n \rho_0}{n(n-1)}, \quad \Lambda_0 = \frac{2\Lambda}{n(n-1)}.
$$

(This equation, when $n = 3$, arises in the study of holographic cosmology [19] [see particularly the words under (15) in [19]], which initiated the present systematic approach.) Formerly, (3.1) reads

$$
\pm \int a^{-1} \left(c_0 a^{-n(1+w)} + \Lambda_0\right)^{-\frac{1}{2}} da = t + C.
$$

It is clear that the integral on the left-hand side of (3.2) satisfies the integrability condition stated in the Chebyshev theorem for any $n$ and any rational $w$.

We have just seen that (3.1) can be integrated directly for any rational $w$. This indicates that (3.1) might be integrated for any real $w$ as well, not necessarily rational. Indeed, we show that this is true. To do so we apply $a > 0$ and get from (3.1) the equation

$$
\frac{d}{dt} \ln a = \pm \sqrt{c_0 a^{-n(1+w)} + \Lambda_0},
$$

or equivalently,

$$
\dot{u} = \pm \sqrt{c_0 e^{-n(1+w)u} + \Lambda_0}, \quad u = \ln a.
$$

Set

$$
\sqrt{c_0 e^{-n(1+w)u} + \Lambda_0} = v.
$$

Then

$$
u = \frac{\ln c_0}{n(1+w)} - \frac{1}{n(1+w)} \ln(v^2 - \Lambda_0).$$

Inserting (3.6) into (3.4), we find

$$
\dot{v} = \mp \frac{1}{2} n(1+w)(v^2 - \Lambda_0),
$$

whose integration gives rise to the expressions

$$
v(t) = \begin{cases} 
\frac{v_0 \left(1 + \frac{1}{2} n(1+w)v_0 t \right)^{-1}}{v_0 \left(1 + C_0 e^{\mp(n(1+w)} \sqrt{-\Lambda_0} t \right) \left(1 - C_0 e^{\pm(n(1+w)} \sqrt{-\Lambda_0} t \right)^{-1}}, & \Lambda_0 = 0; \\
\sqrt{-\Lambda_0} \tan \left(\mp \frac{1}{2} n(1+w) \sqrt{-\Lambda_0} t + \arctan \frac{v_0}{\sqrt{-\Lambda_0}}\right), & \Lambda_0 < 0,
\end{cases}
$$

where $v_0 = v(0)$. Hence, in terms of $v$, we obtain the time-dependence of the scale factor $a$:

$$
a^{n(1+w)}(t) = \frac{8\pi G_n \rho_0}{\frac{8\pi G_n \rho_0}{2n(n-1)v^2(t) - \Lambda}},
$$

(3.9)
We now demonstrate how to use the analytic solutions we have obtained to study cosmology. For definiteness, we assume

\[ w > -1 \]  

in the equation of state in our subsequent discussion. We are particularly interested in solutions satisfying

\[ a(0) = 0, \]  

which represents a universe evolving from a single point singularity.

In view of (3.8), (3.9), and (3.11), we get

\[
a^{n(1+w)}(t) = \begin{cases} 
4\pi G_n \rho_0 \left( \frac{n}{n-1} \right) (1+w)^2 t^2, & \Lambda = 0; \\
\frac{8\pi G_n \rho_0}{\Lambda} \sinh^2 \left( \sqrt{\frac{n\Lambda}{2(n-1)}} (1+w) t \right), & \Lambda > 0.
\end{cases} \]  

(3.12)

Both cases lead to an expanding universe.

We now consider the case when \( \Lambda < 0 \) separately and rewrite (3.9) as

\[
a^{n(1+w)}(t) = \frac{8\pi G_n \rho_0}{(-\Lambda)} \cos^2 \left( \sqrt{\frac{n(-\Lambda)}{2(n-1)}} (1+w) t \mp \sqrt{\frac{n(n-1)}{-2\Lambda}} v_0 \right). \]  

(3.13)

If we require \( a(0) = 0 \), then (3.13) leads to the conclusion

\[
a^{n(1+w)}(t) = \frac{8\pi G_n \rho_0}{(-\Lambda)} \sin^2 \sqrt{\frac{n(-\Lambda)}{2(n-1)}} (1+w) t, \]  

(3.14)

which gives rise to a periodic universe.

The above solutions cover those obtained in the case \( n = 3 \) presented in [3, 29].

**Application to cosmic jerk.** Needlessly to say, with the formulas (3.12) and (3.14), various quantities of cosmological interest may be calculated exactly. For example, the quantity

\[
Q = a^2 \left( \frac{da}{dt} \right)^{-3} \frac{d^3 a}{dt^3} 
\]

is listed [30] as one of four cosmological scalars known as jerk. In order to facilitate the computation, we may represent the scale factor as \( a(t) = cf^s(t) \), where \( c, s > 0 \) are constants, and insert it into (3.15) to obtain

\[
Q = \frac{(s-1)(s-2)}{s^2} + 3(s-1) \frac{f \ddot{f}}{(f')^2} + \frac{1}{s^2} \frac{f^2 \dot{f}^3}{(f')^3}. 
\]  

(3.16)

Thus, in view of (3.16) and the expressions given in (3.12) and (3.14), we may insert

\[
f(t) = \begin{cases} 
t, & \Lambda = 0; \\
\sinh \left( \sqrt{\frac{n\Lambda}{2(n-1)}} (1+w) t \right), & \Lambda > 0; \\
\sin \left( \sqrt{\frac{n(-\Lambda)}{2(n-1)}} (1+w) t \right), & \Lambda < 0.
\end{cases} 
\]  

(3.17)
into (3.16) to obtain

\[
Q = \begin{cases} 
\frac{(s-1)(s-2)}{s^2}, & \Lambda = 0, \\
\frac{(s-1)(s-2)}{s^2} + \frac{(3s-2)}{s^2} \tanh^2 \sqrt{\frac{n\Lambda}{2(n-1)}} (1 + w)t, & \Lambda > 0, \\
\frac{(s-1)(s-2)}{s^2} - \frac{(3s-2)}{s^2} \tanh^2 \sqrt{\frac{n(-\Lambda)}{2(n-1)}} (1 + w)t, & \Lambda < 0,
\end{cases}
\tag{3.18}
\]

where

\[
s = \frac{2}{n(1 + w)}. \tag{3.19}
\]

From (3.18), we see clearly that the only \(t\)-independent jerk \(Q\), uniformly valid for any cosmological constant \(\Lambda\), is when \(s = \frac{2}{3}\), thus \(Q \equiv 1\), which leads to the condition

\[
n(1 + w) = 3. \tag{3.20}
\]

This condition spells out a relation between the spatial dimension and the ratio of the pressure and energy density in the equation of state. In particular, when \(n = 3\), we must have the dust universe, \(w = 0\).

It is a striking fact that our present universe (for which \(n = 3\) is well approximated as being spatially flat and with matter content given by pressure free matter, \(w = 0\), and a positive cosmological constant is completely characterised by the statement \(Q = 1\), i.e that “the cosmic jerk is unity”. From this point of view the cosmological constant is just one of the three integration constants of the third-order ordinary differential equation obtained by equating the right-hand side of (3.15) to one. Since (3.20) implies that

\[
w = \frac{3}{n} - 1, \tag{3.21}
\]

we see that in dimensions higher than 3 the pressure is negative and decreases monotonically to a cosmological constant or dark energy equation of state, i.e. \(w = -1\), as \(n\) tends to infinity.

### 4 Zero cosmological constant case

We now turn our attention to the situation \(\Lambda = 0\) in (2.14) so that the equation reads

\[
a^2 = \frac{16\pi G_n \rho_0}{n(n-1)} a^{-n(1+w)+2} - k. \tag{4.1}
\]

In order to apply Chebyshev’s theorem, we now assume that \(w\) is rational. Thus we see that the question whether (4.1) may be integrated in cosmological time is equivalent to whether

\[
I = \int a^{\frac{1}{2}n(1+w)-1} \left( -ka^{-n(1+w)-2} + \sigma \right)^{-\frac{1}{2}} da
= \frac{2}{n(1 + w)} \int (-k u^\gamma + \sigma)^{-\frac{1}{2}} du, \quad u = a^{\frac{1}{2}n(1+w)}, \quad \sigma = \frac{16\pi G_n \rho_0}{n(n-1)}, \tag{4.2}
\]

is an elementary function of \(u\), where

\[
\gamma = 2 \left( 1 - \frac{2}{n(1 + w)} \right). \tag{4.3}
\]
By (4.3), we see that (4.2) is not elementary unless \( \frac{1}{7} \) or \( \frac{2-\gamma}{2\gamma} \) is an integer.\(^4\) That is, (4.1) may only be integrated directly in cosmological time when \( w \) satisfies one of the following:

\[
\begin{align*}
\gamma & = \frac{4N}{n(2N - 1)} - 1, & N = 0, \pm 1, \pm 2, \ldots; \\
\gamma & = \frac{2}{n} + \frac{1}{nN} - 1, & N = \pm 1, \pm 2, \ldots.
\end{align*}
\]

In particular, in the special situations when \( n = 3 \), we have

\[
w = -1, -\frac{2}{3}, -\frac{5}{9}, -\frac{1}{2}, -\frac{7}{15}, -\frac{4}{9}, -\frac{3}{7}, \ldots, -\frac{2}{9}, -\frac{11}{15}, -\frac{1}{5}, -\frac{1}{6}, -\frac{1}{9}, 0, \frac{1}{3}, \ldots
\]

so that \(-\frac{1}{3}\) is the only limiting point. Solutions of the Friedmann equations in cosmological time corresponding to \( w = 0 \) (dust model) and \( w = \frac{1}{3} \) (radiation model) are discussed in [3], among others.

As an illustration, we integrate (4.1) for

\[
w = -\frac{5}{9},
\]

\((n = 3)\) so that (4.2) becomes

\[
I = \frac{3}{2} \left(-ku^{-1} + \sigma\right)^{-\frac{1}{2}} du, \quad u = a^\frac{2}{3}.
\]

When \( k = 1 \) (closed universe), we may use the substitution \( U = \sqrt{\sigma u - 1} \) to carry out the integration, which gives us the explicit solution

\[
a^\frac{1}{3} \sqrt{\sigma a^\frac{2}{3} - 1} - a_0^\frac{1}{3} \sqrt{\sigma a_0^\frac{2}{3} - 1} + \frac{1}{\sqrt{\sigma}} \ln \left( \frac{\sqrt{\sigma a^\frac{2}{3} - 1} + \sqrt{\sigma a_0^\frac{2}{3}}}{\sqrt{\sigma a_0^\frac{2}{3} - 1} + \sqrt{\sigma a_0^\frac{2}{3}} - \sqrt{\sigma}} \right) = \frac{2}{3} \sigma t,
\]

\( t \geq 0, \quad a_0 = a(0), \)

where \( a_0 \) satisfies the consistency condition \( \sigma a_0^\frac{2}{3} \geq 1 \) or

\[
a_0 \geq \left( \frac{3}{8\pi G_3 \rho_0} \right)^\frac{3}{2},
\]

which spells out a minimum size of the universe in terms of \( \rho_0 \) whose initial energy density in view of (2.12) is given by

\[
\rho_m(0) = \frac{64}{9} \pi^2 G_3^2 \rho_0^3.
\]

When \( k = -1 \) (open universe), we may likewise use the substitution \( U = \sqrt{\sigma u + 1} \) to obtain the solution

\[
a^\frac{1}{3} \sqrt{\sigma a^\frac{2}{3} + 1} - a_0^\frac{1}{3} \sqrt{\sigma a_0^\frac{2}{3} + 1} + \frac{1}{\sqrt{\sigma}} \ln \left( \frac{\sqrt{\sigma a^\frac{2}{3} + 1} + \sqrt{\sigma a_0^\frac{2}{3}}}{\sqrt{\sigma a_0^\frac{2}{3} + 1} + \sqrt{\sigma a_0^\frac{2}{3}} - \sqrt{\sigma}} \right) = \frac{2}{3} \sigma t,
\]

\( t \geq 0, \quad a_0 = a(0), \)

\(^4\)The case when \( \gamma = 0 \) or \( w = \frac{2-\gamma}{n\gamma} \) is trivial since it renders \( a(t) \) a linear function through (4.1).
where no restriction is imposed on the initial value of the scale factor \( a = a(t) \). In particular, if we adopt the big bang scenario, we can set \( a_0 = 0 \) to write down the special solution

\[
a^\frac{1}{3} \sqrt{\sigma a^2} + 1 - \frac{1}{\sqrt{\sigma}} \ln \left( \sqrt{\sigma a^2} + 1 + \sqrt{\sigma a^2} \right) = \frac{2}{3} \sigma t, \quad t \geq 0.
\] (4.13)

Of course, the solutions (4.9) and (4.12) may collectively and explicitly be recast in the form of an elegant single formula:

\[
a^\frac{1}{3} \sqrt{\sigma a^2} - k - a^\frac{1}{3} \sqrt{\sigma a^0} - k + k \ln \left( \frac{\sqrt{\sigma a^2} - k + \sqrt{\sigma a^2}}{\sqrt{\sigma a^0} - k + \sqrt{\sigma a^0}} \right) = \frac{2}{3} \sigma t, \quad t \geq 0, \quad a_0 = a(0), \quad k = \pm 1.
\] (4.14)

We see that, in both closed and open situations, \( k = \pm 1, \) respectively, the universe grows following a power law of the type \( a(t) = O(t^{\frac{3}{2}}) \) for all large time so that a greater Newton’s constant or initial energy density gives rise to a greater growth rate.

**Application: dust and radiation.** To see how we may apply Chebyshev’s theorem to the study of some realistic models in cosmology, we consider \( \Lambda = 0, k = 0 \) and insert

\[
\rho = \frac{\rho_m}{a^n} + \frac{\rho_r}{a^{n+1}},
\] (4.15)

where \( \rho_m > 0, \rho_r > 0 \) are constants, in (2.5). This corresponds to the mixture of pressure-free matter, \( \rho = \rho_m/a^n \), and radiation matter, \( \rho = \rho_r/a^{n+1} \), valid in the universe before decoupling and after reheating. Hence we have

\[
a^2 = \frac{16 \pi G_n}{n(n-1)} \left( \frac{\rho_m}{a^{n-2}} + \frac{\rho_r}{a^{n-1}} \right),
\] (4.16)

which leads us to the integral

\[
\int a^{\frac{1-n}{2}} \sqrt{\rho_m a + \rho_r} \, da.
\] (4.17)

It is clear that the Chebyshev criterion is satisfied for any integer \( n \geq 1 \). In other words, the model can be explicitly integrated in any dimensions.

For example, when \( n = 3 \), we readily see that the solution with the initial condition \( a(0) = 0 \) is given by

\[
(\rho_m a + \rho_r)^{\frac{3}{2}} - 3 \rho_r (\rho_m a + \rho_r)^{\frac{1}{2}} + 2 \rho_r^{\frac{3}{2}} = \sqrt{6 \pi G_3} \rho_m^2 t, \quad t \geq 0.
\] (4.18)

### 5 Integration in conformal time

It is also interesting to investigate the issue of integrating the Friedmann equations in terms of conformal time \( \eta \) which is related to the cosmological time \( t \) by the scale factor: \( dt = da \eta \), since many papers employ conformal time for which Chebyshev’s theorem also applies. In this situation, it is convenient to use \( a' \) to denote \( \frac{da}{d\eta} \). Then \( a' = a \dot{a} \) and (2.14) becomes

\[
(a')^2 = \frac{16 \pi G_n \rho_0}{n(n-1)} a^{-n(1+w)+4} + \frac{2\Lambda}{n(n-1)} a^4 - ka^2.
\] (5.1)
In order to apply Chebyshev’s theorem, we assume that \( w \) is a rational number and consider \( k = 0 \) and \( \Lambda = 0 \) separately. When \( k = 0 \), the conformal time version of (3.1) reads

\[
a' = \pm a^2 \sqrt{c_0 a^{-n(1+w)} + \Lambda_0},
\]

(5.2)

whose integration is

\[
\pm \int a^{-2} \left( c_0 a^{-n(1+w)} + \Lambda_0 \right)^{-\frac{1}{2}} da = \eta + C.
\]

(5.3)

Consequently Chebyshev’s theorem indicates that, when \( \Lambda \neq 0 \), the left-hand side of (5.3) is elementary if and only if

\[
\frac{1}{n(1+w)} - \frac{1}{2} \text{ is an integer (again we exclude the trivial case } w = -1 \text{), or more explicitly, } w \text{ satisfies one of the following conditions:}
\]

\[
w = -1 + \frac{1}{nN}, \quad N = \pm 1, \pm 2, \ldots; \quad (5.4)
\]

\[
w = -1 + \frac{1}{n \left( N + \frac{1}{2} \right)}, \quad N = 0, \pm 1, \pm 2, \ldots. \quad (5.5)
\]

As a simple illustration, we choose \( N = 1 \) in (5.4) so that \( w = \frac{1}{n} - 1 \). Then (5.2) becomes

\[
a' = \pm a^2 \sqrt{c_0 a^{-1} + \Lambda_0},
\]

(5.6)

which can be integrated to yield the solution

\[
\frac{c_0}{a} + \Lambda_0 = \frac{c_0^2}{4} (\eta + C)^2,
\]

(5.7)

where \( C \) is an integrating constant.

On the other hand, however, when \( \Lambda = 0 \), the Friedmann equation is

\[
a' = \pm a \sqrt{c_0 a^{-n(1+w)+2} - k},
\]

(5.8)

which in view of Chebyshev’s theorem can be integrated in terms of elementary functions when \( w \) is any rational number. In fact, as before, (5.8) may actually be integrated to yield its exact solution expressed in elementary functions for any \( w \):

\[
\pm \left( 1 - \frac{1}{2} n(1+w) \right) \eta + C = \begin{cases} 
-\frac{1}{v}, & k = 0; \\
\arctan v, & k = 1; \\
\frac{1}{2} \ln \left| \frac{v-1}{v+1} \right|, & k = -1,
\end{cases}
\]

(5.9)

where \( C \) is an integrating constant and

\[
v = \sqrt{c_0 a^{-n(1+w)+2} - k} \quad \text{or} \quad a^{-n(1+w)+2} = \frac{1}{c_0} (v^2 + k).
\]

(5.10)

To obtain inflationary solutions, we assume

\[
n(1+w) > 2,
\]

(5.11)
which contains the dust and radiation matter situations since \( n \geq 3 \). We assume the initial condition \( a(0) = 0 \). When \( k = 0 \), (5.9) gives us the solution
\[
a^{n(1+w)-2}(\eta) = \frac{4\pi G_n \rho_0}{n(n-1)}(n(1+w) - 2)^2 \eta^2, \quad \eta \geq 0.
\]
(5.12)

When \( k = 1 \), (5.9) renders the solution
\[
a^{n(1+w)-2}(\eta) = \frac{16\pi G_n \rho_0}{n(n-1)} \sin^2 \left( \frac{1}{2} (n[1 + w] - 2) \eta \right), \quad \eta \geq 0,
\]
(5.13)

which gives rise to a periodic universe. When \( k = -1 \), (5.9) yields the solution
\[
a^{n(1+w)-2}(\eta) = \frac{16\pi G_n \rho_0}{n(n-1)} \sinh^2 \left( \frac{1}{2} (n[1 + w] - 2) \eta \right), \quad \eta \geq 0,
\]
(5.14)

which leads to an inflationary universe. It is interesting to note that the closed universe situation here \( (k = 1, \Lambda = 0) \), in conformal time, is comparable to the flat universe with a negative cosmological constant \( (k = 0, \Lambda < 0) \), in cosmological time, and the open universe situation here \( (k = -1, \Lambda = 0) \), in conformal time, is comparable to the flat universe with a positive cosmological constant \( (k = 0, \Lambda > 0) \), also in cosmological time.

It is immediate to check that, when \( k = 0 \) and \( \Lambda = 0 \), the solutions (3.12) and (5.12) are the same since the cosmological time \( t \) and conformal time \( \eta \) are related through the equation
\[
r^{n(1+w)-2} = 4\pi G_n \rho_0 (1+w)^2 \left( \frac{n}{n-1} \right) \left( 1 - \frac{2}{n(1+w)} \right)^{n(1+w)} \eta^{n(1+w)}. \]
(5.15)

Thus, in particular, we see that, regarding integrability of the Friedmann equation in terms of elementary functions, the conclusions in the cosmological and conformal time situations in view of Chebyshev’s theorem are elegantly complementary.

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