Four-dimensional Painlevé-type difference equations

Hiroshi Kawakami*

College of Science and Engineering, Aoyama gakuin university, 5-10-1 Fuchinobe, Chuo-ku, Sagamihara-shi, Kanagawa 252-5258, Japan.

Abstract

We focus on Fuchsian equations with four accessory parameters and three singular points. We see that the Fuchsian equations admit a “degeneration scheme” in some sense, which is expected to give rise to a degeneration scheme of discrete isomonodromic deformation equations with four-dimensional phase space. We compute an example of discrete isomonodromic deformation equations of a certain Fuchsian equation.

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1 Introduction

The Painlevé equations are second order non-linear ordinary differential equations discovered by Painlevé and Gambier in an attempt to find new special functions. Recently, there have been many studies concerning generalizations of the Painlevé equations based on various aspects of them, such as the affine Weyl group symmetry, the spaces of initial conditions, and so on. Then it is natural to ask: How can we describe those generalizations in a unified way? The important fact is that they can also be obtained as compatibility conditions of linear differential equations. In other words, they can be regarded as isomonodromic deformation equations of some linear differential equations. Thus it is important to give a good description of the whole set of isomonodromic deformation equations.

An isomonodromic deformation is a deformation of a linear differential equation which keeps its “monodromy data” unchanged. Isomonodromic deformations fall into two classes: the continuous deformation and the discrete deformation. When we consider the continuous isomonodromic deformation of a linear equation, we can choose positions of its singular points and coefficients of its HTL canonical forms (see Definition 1) except the residue matrices as deformation parameters.

*kawakami@gem.aoyama.ac.jp
Then we obtain a system of non-linear differential equations satisfied by the coefficients of the linear equation. In the course of a continuous isomonodromic deformation, the residue matrices of the HTL forms (so-called “exponents of formal monodromy”) stay constant. However, we can consider a discrete change of the exponents, which does not change its monodromy data. Such a discrete deformation of a linear equation is called a Schlesinger transformation, which is expressed as a system of non-linear difference equations. See \[10\] \[11\] for details on isomonodromic deformations. In what follows we use the term Painlevé-type equations synonymously with isomonodromic deformation equations.

It is well-known that Painlevé-type differential equations can be written in Hamiltonian form. In particular, the dimensions of the phase spaces of Painlevé-type differential equations are even numbers. Recently, as a first step toward a comprehensive understanding of isomonodromic deformation equations, a classification of the Painlevé-type differential equations with four-dimensional phase space was obtained \[12\] \[13\] \[14\] \[15\]. The study of four-dimensional Painlevé-type equations owes much to the classification of Fuchsian equations with four accessory parameters by Oshima \[20\].

**Theorem 1.1** (Oshima). *Any irreducible Fuchsian equation with four accessory parameters can be transformed into one of the following 13 equations by a finite iteration of additions and middle convolutions.*

\[
\begin{array}{|c|c|c|c|}
\hline
\# \text{sing. pt.} & 11, 11, 11, 11 & 11, 11, 11, 11, 11 & 11, 11, 11, 11, 11, 11 \\
4 & 21, 21, 111, 111 & 31, 22, 22, 1111 & 22, 22, 22, 211 \\
3 & 211, 1111, 1111, 11111 & 221, 221, 11111, 11111 & 32, 11111, 11111 \\
& 222, 222, 2211 & 33, 2211, 111111 & 44, 2222, 22221 \\
& 44, 332, 1111111 & 55, 3331, 22222 & 66, 444, 2222211 \\
\hline
\end{array}
\]

The tuples of integers in the above table are called spectral types of Fuchsian equations, which represent multiplicities of characteristic exponents (see Section 2.2). We note that the number of accessory parameters of a linear equation coincide with the dimension of the phase space of the corresponding Painlevé-type equation. Fuchsian equations have only the position of singular points as continuous deformation parameters. If a Fuchsian equation has \(N\) singular points on the Riemann sphere, then three of the \(N\) points can be mapped to \(0, 1, \infty\) via the Möbius transformation. Thus the number of essential deformation parameters of the Fuchsian equation is \(N - 3\). The Painlevé-type differential equation corresponding to the Fuchsian equation of spectral type \(11, 11, 11, 11, 11\) is the Garnier system in two variables. The Painlevé-type equations corresponding to the Fuchsian equations with four singular points in the above table were clarified by Sakai \[22\]: the Fuji-Suzuki system \[6\] \[25\] (corresponding to \(21, 21, 111, 111\)), the Sasano system \[5\] \[23\] (corresponding to \(31, 22, 22, 1111\)), and the sixth matrix Painlevé system \[2\] \[22\] (corresponding to \(22, 22, 22, 211\)). In \[12\] \[13\] \[14\] \[15\], the degeneration scheme of the four-dimensional Painlevé-type differential equations was obtained starting from the four Painlevé-type equations. This provides a classification of four-dimensional Painlevé-type differential equations.
Symbols such as $H^\text{Mat}$ stand for Hamiltonians for four-dimensional Painlevé-type equations.

On the other hand, Painlevé equations with three singular points admit only trivial continuous isomonodromic deformations, but admit non-trivial discrete isomonodromic deformations. In the case of two accessory parameters, discrete isomonodromic deformations of Painlevé equations with three singular points yield the additive difference Painlevé equations that do not correspond to $H^\text{Mat}$. However, $H_{\text{Gar}}^\text{IV}$ stand for Hamiltonians for four-dimensional Painlevé-type equations.
Bäcklund transformations of differential Painlevé equations [1, 3, 4]. In this paper, we focus on the Fuchsian equations with three singular points and four accessory parameters. By considering the discrete isomonodromic deformation of these Fuchsian equations, we can obtain four-dimensional Painlevé-type difference equations.

The organization of this paper is as follows. In Section 2, we briefly review some notions related to linear differential equations. In Section 3, we show that the Fuchsian equations with three singular points and four accessory parameters admit “degeneration scheme” in some sense, and thereby the above degeneration scheme can be extended further upstream. In Section 4, we compute Schlesinger transformations of the linear equation of spectral type 211,111,111 as an example. The system of difference equations thus obtained can be regarded as a discrete analogue of the Garnier system in two variables.

2 Preliminaries

2.1 HTL canonical forms

Let $A(x)$ be a matrix-valued function in $x$. The transformation

$$A(x) \mapsto P[A(x)] := PA(x)P^{-1} + \frac{dP}{dx}P^{-1}$$

by an invertible matrix $P = P(x)$ is called the gauge transformation. This corresponds to the change of the dependent variable $Z = PY$ of a system of linear differential equations

$$\frac{dY}{dx} = A(x)Y.$$

We consider a system of linear differential equations with rational function coefficients

$$\frac{dY}{dx} = \left( \sum_{\nu=1}^{n} \sum_{k=0}^{r_{\nu}} \frac{A^{(k)}_{\nu}}{(x-u_{\nu})^{k+1}} + \sum_{k=1}^{r_{\infty}} A^{(k)}_{\infty} x^{k-1} \right) Y, \quad A_{\infty}^{(k)} \in M(m, \mathbb{C}). \quad (2.1)$$

Note that the residue matrix $A^{(0)}_{\infty}$ at $x = \infty$ is given by

$$A^{(0)}_{\infty} = -\sum_{\nu=1}^{n} A^{(0)}_{\nu}.$$

The system can be transformed into the “canonical form” at each singular point.

The system (2.1) has singularity at $x = u_{\nu}$ ($\nu = 1, \ldots, n$) and $x = \infty =: u_{0}$. Set $z = x - u_{\nu}$ ($\nu = 1, \ldots, n$) or $z = 1/x$. We consider the system around $z = 0$:

$$\frac{dY}{dz} = \left( \frac{A_{0}}{z^{r+1}} + \frac{A_{1}}{z^{r}} + \cdots + A_{r+1} + A_{r+2}z + \cdots \right) Y.$$

Let $\mathcal{P}_{z} = \bigcup_{p>0} \mathbb{C}((z^{1/p}))$ be the field of Puiseux series where $\mathbb{C}((t))$ is the field of formal Laurent series in $t$. 

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**Definition 1** (HTL canonical form). An element
\[
\frac{T_0}{z^{l_0}} + \frac{T_1}{z^{l_1}} + \cdots + \frac{T_{s-1}}{z^{l_{s-1}}} + \frac{\Theta}{z}
\]
in \(M(m, \mathcal{P}_z)\) satisfying the following conditions:

- \(l_j\) is a rational number, \(l_0 > l_1 > \cdots > l_{s-1} > l_s = 1\),
- \(T_0, \ldots, T_{s-1}\) are commuting diagonalizable matrices,
- \(\Theta\) is a (not necessarily diagonalizable) matrix that commutes with all \(T_j\)’s

is called an **HTL canonical form**, or **HTL form** for short. \(\square\)

**Theorem 2.1** (Hukuhara [9], Turrittin [26], Levelt [17]). For any
\[
A(z) = \frac{A_0}{z^{r+1}} + \frac{A_1}{z^r} + \cdots \in M(m, \mathbb{C}(z)),
\]
there exists \(P \in \text{GL}(m, \mathbb{P}_z)\) such that \(P[A(z)]\) is an HTL form
\[
\frac{T_0}{z^{l_0}} + \frac{T_1}{z^{l_1}} + \cdots + \frac{T_{s-1}}{z^{l_{s-1}}} + \frac{\Theta}{z}.
\]
Here \(l_0, \ldots, l_{s-1}\) are uniquely determined only by \(A(z)\).

If
\[
\tilde{T}_0 + \frac{\tilde{T}_1}{z^{l_1}} + \cdots + \frac{\tilde{T}_{s-1}}{z^{l_{s-1}}} + \tilde{\Theta}
\]
is another HTL form of the same \(A(z)\), then there exist \(g \in \text{GL}(m, \mathbb{C})\) and \(k \in \mathbb{Z}_{\geq 1}\) such that
\[
\tilde{T}_j = gT_jg^{-1}, \quad \exp(2\pi ik\tilde{\Theta}) = g\exp(2\pi ik\Theta)g^{-1}
\]
hold.

The number \(l_0 - 1\) is called the **Poincaré rank** of the singular point. If there is a rational number \(l_j \in \mathbb{Q}\ \setminus \mathbb{Z}\), the singular point is called a **ramified** irregular singular point. A linear system is said to be of **ramified type** if the system has a ramified irregular singular point. We consider linear systems of unramified type below.

### 2.2 Riemann schemes and spectral types

Consider an HTL form
\[
\frac{T_0}{z^{b+1}} + \frac{T_1}{z^b} + \cdots + \frac{T_{b-1}}{z^2} + \frac{\Theta}{z} (b \in \mathbb{Z}_{\geq 0}).
\]
Here we assume \(T_j\)’s and \(\Theta\) are in Jordan canonical form. When \(\Theta\) is a diagonal matrix, we denote this HTL form by

\[
\begin{pmatrix}
x = u_i
\end{pmatrix}
\begin{pmatrix}
\theta_0^0 & \theta_1^0 & \cdots & \theta_1^{b-1} & \theta_0^1 & \cdots & \theta_1^1 & \cdots & \theta_m^0 & \theta_m^1 & \cdots & \theta_m^{b-1}
\end{pmatrix}
\]
where \( T_j = \text{diag}(t^j_1, \ldots, t^j_m) \), \( \Theta = \text{diag}(\theta_1, \ldots, \theta_m) \). We sometimes identify each column \((t^j_1, \ldots, t^j_m)\) with the matrix \( T_j \) itself. Then we write

\[
x = u_i \begin{array}{cccc}
T_0 & T_1 & \cdots & T_{b-1} & \Theta
\end{array}.
\]

In this paper we call \( t^j_i \)'s and \( \theta_j \)'s exponents, and we also call \( \theta_j \)'s characteristic exponents (this terminology might be different from the usual one). The table of the HTL forms represented by the above formula at all singular points is called the Riemann scheme of a linear system. In this paper we consider HTL forms whose residue matrices are diagonalizable. We also assume that any two eigenvalues of \( \Theta \) do not differ by a non-zero integer.

When we are not interested in the values of the exponents, a Riemann scheme is represented by a spectral type. The spectral type of an unramified linear system is described by the “refining sequence of partitions”. In the following we look at Riemann schemes and spectral types for particular cases. For a general description of spectral types, see [12, 15].

When \( r_\nu = 0 \) for all singular points in (2.1), the Riemann scheme is merely a table of the eigenvalues of all \( A^{(0)}_\nu \)'s. The spectral type is a tuple of partitions of \( m \) which represents the multiplicities of the eigenvalues.

**Example 2.2.** The Riemann scheme of a linear system

\[
\frac{dY}{dx} = \left( \frac{A_1^{(0)}}{x-u_1} + \frac{A_2^{(0)}}{x-u_2} + \frac{A_3^{(0)}}{x-u_3} \right) Y
\]

with the condition

\[
A_1^{(0)} \sim \text{diag}(\theta^1_1, \theta^1_1, \theta^1_1, \theta^1_2), \quad A_2^{(0)} \sim \text{diag}(\theta^2_1, \theta^2_1, \theta^2_2, \theta^2_2), \quad A_3^{(0)} \sim \text{diag}(\theta^3_1, \theta^3_1, \theta^3_2, \theta^3_2),
\]

and \( A_\infty^{(0)} = -(A_1^{(0)} + A_2^{(0)} + A_3^{(0)}) \sim \text{diag}(\theta^\infty_1, \theta^\infty_2, \theta^\infty_3, \theta^\infty_4) \) is

\[
\begin{pmatrix}
\theta^1_1 & \theta^2_1 & \theta^3_1 & \theta^\infty_1 \\
\theta^1_1 & \theta^2_1 & \theta^3_1 & \theta^\infty_2 \\
\theta^1_1 & \theta^2_1 & \theta^3_2 & \theta^\infty_3 \\
\theta^1_2 & \theta^2_2 & \theta^3_2 & \theta^\infty_4
\end{pmatrix}.
\]

The spectral type of the system is 31, 22, 22, 1111. \quad \Box

The next case is that one of the \( r_\nu \)'s is equal to one and the others are zero. For example, we consider the case when \( r_\infty = 1 \) and \( r_\nu = 0 \) (\( \nu = 1, \ldots, n \))

\[
\frac{dY}{dx} = \left( \sum_{\nu=1}^{n} \frac{A_\nu^{(0)}}{x-u_\nu} + A_\infty^{(1)} \right) Y, \quad A^{(k)}_\nu \in M(m, \mathbb{C}). \quad (2.2)
\]

Its Riemann scheme can be obtained as follows. Assume that \( A_\infty^{(1)} \) is a diagonal matrix

\[
A_\infty^{(1)} = a_1 I_{m_1} \oplus \cdots \oplus a_k I_{m_k}, \quad a_i \neq a_j \quad (i \neq j), \quad m_1 + \cdots + m_k = m,
\]
and the matrices $A^{(0)}_\nu$'s are similar to diagonal matrices $\Theta_\nu$'s respectively. Partition the matrix $A^{(0)}_\infty = - \sum_{\nu=1}^n A^{(0)}_\nu$ into submatrices according to $A^{(1)}_\infty$

$$
\begin{pmatrix}
\Theta_1^\infty & A_{12} & \cdots & A_{1k}
A_{21} & \Theta_2^\infty & \cdots & A_{2k}
\vdots & \vdots & \ddots & \vdots
A_{k1} & A_{k2} & \cdots & \Theta_k^\infty
\end{pmatrix}
$$

where $A_{ij}$ is an $m_i \times m_j$ matrix. Here, using the conjugate action of $\text{Stab}(A^{(1)}_\infty) = \{ g \in \text{GL}(m, \mathbb{C}) \mid gA^{(1)}_\infty g^{-1} = A^{(1)}_\infty \}$, we can choose $\Theta_j^\infty$'s so that they are diagonal. Then the Riemann scheme of (2.2) is given by

$$
\begin{pmatrix}
x = u_1 & \cdots & x = u_n
\Theta_1 & \cdots & \Theta_n
\end{pmatrix}
\begin{aligned}
&\Theta_1^\infty \quad \Theta_2^\infty \quad \cdots \quad \Theta_k^\infty
\end{aligned}
$$

Let $\lambda_\nu$ be the partition of $m$ which represents the multiplicities of the diagonal entries of $\Theta_\nu$. Similarly, let $\mu_j$ be the partition of $m_j$ determined by the diagonal entries of $\Theta_j^\infty$. Then the spectral type of (2.2) is

$$
\lambda_1, \ldots, \lambda_n, (\mu_1)(\mu_2) \cdots (\mu_k).
$$

Sometimes $(\mu_1)(\mu_2) \cdots (\mu_k)$ is written as a pair of two partitions $[\lambda, \mu]$. Here $\lambda = m_1, \ldots, m_k$, and $\mu$ is the partition of $m$ obtained by arranging $\mu_1, \ldots, \mu_k$.

### 2.3 Laplace transform

It is known that for any rational function matrix with $r_\infty = 0$

$$
A(x) = \sum_{\nu=1}^n \sum_{k=0}^{r_\nu} \frac{A^{(k)}_\nu}{(x-u_\nu)^{k+1}}, \quad A^{(k)}_\nu \in M(m, \mathbb{C}),
$$

there exist a natural number $n$, an $n \times n$ matrix $T$, an $m \times n$ matrix $Q$, and an $n \times m$ matrix $P$ such that the following holds

$$
A(x) = Q(xI - T)^{-1}P.
$$

Moreover the quadruple $(n, T, Q, P)$ is essentially unique provided that $n$ is minimal (see [27]). In the following we often write a scalar matrix as $k$ instead of $kI$. Thus we can write a system with $r_\infty = 1$ in the form

$$
dY = (Q(x - T)^{-1}P + S) \ Y.
$$

Using this expression, we can show that a system with $r_\infty = 1$ behaves quite symmetric under the Laplace transform. In fact, by applying the Laplace transform $x \mapsto -\frac{d}{dx}$, $\frac{d}{dx} \mapsto x$, we obtain the following system [28]:

$$
dY = (-P(x - S)^{-1}Q - T) \ Y.
$$
When the matrices $S$ and $T$ are both diagonalizable, the correspondence of the Riemann schemes of (2.5) and (2.6) is as follows.

Let the rank of the system (2.5) be $m$ and the size of $T$ be $n$. We set

$$S = s_1 I_{m_1} \oplus \cdots \oplus s_k I_{m_k}, \quad T = t_1 I_{n_1} \oplus \cdots \oplus t_l I_{n_l}.$$ 

We assume $m \leq n$ (otherwise, consider the opposite correspondence of the following). Under this assumption, $m_j \leq n$ holds. Moreover, with appropriate choice of $Q$ and $P$, $n_j$ gives the rank of the residue matrix of the coefficient matrix of (2.5) at $x = t_j$. Thus we can assume $n_j \leq m$ without loss of generality. We partition $Q$ and $P$ as follows:

$$Q = \begin{pmatrix} Q_1 & \cdots & Q_l \end{pmatrix}, \quad P = \begin{pmatrix} P_1 \\ \vdots \\ P_l \end{pmatrix} = \begin{pmatrix} P^1 & \cdots & P^k \end{pmatrix}.$$ 

Here $Q_j$ is an $m \times n_j$ matrix, $P_j$ is an $n_j \times m$ matrix, $Q^j$ is an $m_j \times n$ matrix, and $P^j$ is an $n \times m_j$ matrix. We assume that $Q_j P_j$ and $-Q^j P^j$ are similar to diagonal matrices $\Theta_j$ and $K_j$ respectively.

Then the Riemann scheme of (2.5) is

$$\begin{pmatrix} x = t_1 & \ldots & x = t_l & x = \infty \\ \Theta_1 & \cdots & \Theta_l & \begin{pmatrix} -s_1 I_{m_1} & K_1 \\ \vdots & \vdots \\ -s_k I_{m_k} & K_k \end{pmatrix} \end{pmatrix},$$

and the Riemann scheme of (2.6) is

$$\begin{pmatrix} x = s_1 & \ldots & x = s_k & x = \infty \\ \tilde{K}_1 & \cdots & \tilde{K}_k & \begin{pmatrix} t_1 I_{n_1} & \tilde{\Theta}_1 \\ \vdots & \vdots \\ t_l I_{n_l} & \tilde{\Theta}_l \end{pmatrix} \end{pmatrix},$$

Here $\tilde{\Theta}_j$ denotes an $n_j \times n_j$ diagonal matrix obtained by eliminating $m - n_j$ zeros from $\Theta_j$, and $\tilde{K}_j$ denotes an $n \times n$ diagonal matrix obtained by adding $n - m_j$ zeros to $K_j$.

### 2.4 Schlesinger transformations

The Schlesinger transformation [24] was originally introduced as a discrete deformation of a Fuchsian system. This discrete deformation corresponds to shifting the characteristic exponents by integers. For a detailed description of Schlesinger transformations, see [11]. We will deal with Schlesinger transformations of a certain Fuchsian system in Section 4.

A Schlesinger transformation of a Fuchsian system is realized as the compatibility condition of the Fuchsian system and a system of linear difference equations:

$$\begin{cases} \frac{dY}{dx} = A(x)Y, & A(x) = \sum_{\nu=1}^{n} \frac{A_{\nu}}{x - u_{\nu}}, \\ Y = R(x)Y, & \end{cases}$$
where the Fuchsian system is usually normalized so that the residue matrix at \( x = \infty \) is diagonal. In Section \( \mathbb{II} \) however, we will adopt a different gauge from the usual one. Here \( R = R(x) \) is a matrix whose entries are rational functions in \( x \) and chosen so that the system of differential equations satisfied by \( \overline{Y} \)

\[
\frac{d\overline{Y}}{dx} = R[A(x)]\overline{Y}
\]

is a Fuchsian system with the same position of singular points as, and the similar gauge to, the original system. The matrix \( R(x) \) is called the \textit{multiplier} of this transformation (or deformation). Comparing \( \overline{A}(x) := R[A(x)] \) with the original \( A(x) \), we have a system of difference equations satisfied by the entries of \( A^{\nu} \)’s.

### 3 Four-dimensional Painlevé-type difference equations

As mentioned in Section \( \mathbb{II} \) there are nine Fuchsian equations which have four accessory parameters and three singular points. They have only trivial continuous isomonodromic deformations but admit non-trivial discrete isomonodromic deformations. Therefore these nine Fuchsian equations can yield four-dimensional \textit{additive difference} Painlevé-type equations.

In this section we will see that the nine Fuchsian equations admit a “degeneration scheme” in some sense, which is expected to give rise to a degeneration scheme of corresponding Painlevé-type difference equations. To see this, we introduce an equivalence relation of spectral types.

Let \( S_1 \) and \( S_2 \) be spectral types. Then \( S_1 \) is said to be equivalent to \( S_2 \) if a linear system of spectral type \( S_2 \) is obtained by a finite number of applications of a Möbius transformation, the Laplace transform, and an addition from a linear system of spectral type \( S_1 \). Here by “addition” we mean a shift of a coefficient matrix of (2.1) by a scalar matrix

\[
A^{(k)}_{\nu} \mapsto A^{(k)}_{\nu} + \alpha I_m,
\]

which can be realized by a scalar gauge transformation. We denote the equivalence class of a spectral type \( S \) by \([S]\). Moreover, we write \([S_1] \rightarrow [S_2]\) if there exist linear systems \( E_1 \) and \( E_2 \) whose spectral types are equivalent to \( S_1 \) and \( S_2 \) respectively such that \( E_2 \) is obtained from \( E_1 \) by means of a confluence of singular points or a degeneration of an HTL form. For a detailed description of these two kinds of degenerations of linear systems, see [12, 13, 14, 15].

**Example 3.1.** We take the linear system of spectral type \( 32, 11111, 11111 \) as an example. The confluence of singular points represented by \( 32 \) and \( 11111 \) leads to a non-Fuchsian system of spectral type \( 11111, (111)(11) \). Applying a Möbius transformation if necessary, we can write the non-Fuchsian system in the form

\[
\frac{dY}{dx} = \left( \frac{QP}{x} + S \right) Y,
\]

which is a system (2.5) with \( T = O \). Moreover, we can assume that the Riemann scheme of (3.1)
is of the form

\[
\begin{pmatrix}
  x = 0 & x = \infty \\
  0 & 0 & \beta_1 \\
  \alpha_1 & 0 & \beta_2 \\
  \alpha_2 & 0 & \beta_3 \\
  \alpha_3 & -1 & \beta_4 \\
  \alpha_4 & -1 & \beta_5
\end{pmatrix},
\]

namely, we can adjust an exponent of maximum multiplicity at each singular point (except characteristic exponents at \(x = \infty\)) to zero using additions. Then, applying the Laplace transform to the non-Fuchsian system (3.1), we have a Fuchsian system of spectral type 211, 1111, 1111 whose Riemann scheme is given by

\[
\begin{pmatrix}
  x = 0 & x = 1 & x = \infty \\
  0 & 0 & \alpha_1 \\
  \beta_1 & 0 & \alpha_2 \\
  \beta_2 & \beta_4 & \alpha_3 \\
  \beta_3 & \beta_5 & \alpha_4
\end{pmatrix}.
\]

The above relation of linear systems is illustrated in the following diagram:

\[
32, 11111, 11111 \longrightarrow (111)(11), 11111 \quad \text{equivalent} \quad 211, 1111, 1111
\]

This shows [32, 11111, 11111] \(\rightarrow\) [211, 1111, 1111].

**Remark 1.** The classification of Fuchsian systems with two accessory parameters is due to Kostov [16]:

\[
\#\text{sing.pt.} = 4 \quad 11, 11, 11, 11 \quad 3 \quad 111, 111, 111 \quad 22, 1111, 1111 \quad 33, 222, 111111
\]

The equivalence classes of the spectral types with three singular points in the above table admit the following degeneration scheme

\[
[33, 222, 111111] \rightarrow [22, 1111, 1111] \rightarrow [111, 111, 111].
\]

The difference Painlevé equations of type \(A_2^{(1)*}, A_1^{(1)*},\) and \(A_0^{(1)**}\) can be derived from Fuchsian systems of these three spectral types

\[
111, 111, 111, \quad 22, 1111, 1111, \quad 33, 222, 111111,
\]

respectively [1, 3, 4, 21].

By direct calculation, we have the following
**Theorem 3.2.** The equivalence classes of the Fuchsian systems with four accessory parameters and three singular points admit the following degeneration scheme.

Notice that the Fuchsian system of spectral type 55,3331,2222 does not admit confluences of singularities since any one of the three partitions is not a refinement of the others (see [15]).

The reason why we consider such an equivalence relation is that continuous isomonodromic deformation equations are invariant under the Laplace transform. A typical example is the Har- nad duality [8]. We expect that an analogous statement also holds for discrete isomonodromic deformations.

**Conjecture 3.3.** Discrete isomonodromic deformation equations are invariant under the Laplace transform.

If the conjecture is true, then linear systems which are mutually equivalent give the same Painlevé-type difference equation since Möbius transformations and additions clearly do not change discrete isomonodromic deformation equations. Therefore it is expected that Theorem 3.2 gives the degeneration scheme of four-dimensional Painlevé-type difference equations. Then the above scheme shows that the three streams starting from the Garnier system in two variables, the Fuji-Suzuki system, and the Sasano system come from the same source.

To write down a Painlevé-type equation explicitly, we have to parametrize linear systems which realize a given spectral type. The following lemma might be useful for that purpose.

**Lemma 3.4.** Consider a linear system

$$\frac{dY}{dx} = \left( \frac{A_0^{(1)}}{x^2} + \frac{A_0^{(0)}}{x} + \tilde{A}(x) \right) Y$$

where $\tilde{A}(x)$ is rational in $x$ and holomorphic at $x = 0$. Suppose that $A_0^{(1)}$ is diagonalizable and the spectral type at $x = 0$ is $[\lambda, \mu]$. Then there exists a linear system which has regular singular points at $x = 0, -\varepsilon$ with spectral type $\lambda, \mu$ respectively and tends to the system (3.2) as $\varepsilon \to 0$. 

Proof. By a change of the dependent variable, we can assume that \( A_0^{(1)} \) is diagonal:

\[
A_0^{(1)} = t_1 I_{m_1} \oplus \cdots \oplus t_k I_{m_k}.
\]

Further, using the action of \( \text{Stab}(A_0^{(1)}) \), we can assume that \( A_0^{(0)} \) has the form

\[
A_0^{(0)} = \begin{pmatrix}
\Theta_1 & A_{12} & \cdots & A_{1k} \\
A_{21} & \Theta_2 & \cdots & A_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & \cdots & \Theta_k
\end{pmatrix}
\]  

(3.3)

where \( \Theta_j \) is an \( m_j \times m_j \) diagonal matrix, \( A_{ij} \) is an \( m_i \times m_j \) matrix.

We define an upper triangular matrix \( A_0 \) and a lower triangular matrix \( A_1 \) as follows:

\[
A_0 = \begin{pmatrix}
\rho_1 I_{m_1} & A_{12} & \cdots & A_{1k} \\
O & \rho_2 I_{m_2} & \cdots & A_{2k} \\
\vdots & \ddots & \ddots & \vdots \\
O & O & \cdots & \rho_k I_{m_k}
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
S_1 & O & \cdots & O \\
A_{21} & S_2 & \cdots & O \\
\vdots & \ddots & \ddots & \vdots \\
A_{k1} & A_{k2} & \cdots & S_k
\end{pmatrix}.
\]

Here \( \rho_j = t_j / \varepsilon \) and \( S_j = \Theta_j - (t_j / \varepsilon)I_{m_j} \). Then we consider the following system

\[
\frac{dY}{dx} = \left( \frac{A_0}{x} + \frac{A_1}{x + \varepsilon} + \tilde{A}(x) \right) Y.
\]

(3.4)

Noting that

\[
\frac{A_0}{x} + \frac{A_1}{x + \varepsilon} = \frac{\varepsilon A_0}{x(x + \varepsilon)} + \frac{A_0 + A_1}{x + \varepsilon},
\]

we see that the system (3.4) is a desired one since it tends to (3.2) as \( \varepsilon \) tends to zero. \( \square \)

Remark 2. This upper-lower triangular gauge appears naturally in the study of integrable systems. For example, concerning isomonodromic deformations, see [7, 18, 22]. \( \square \)

Example 3.5. We can parametrize linear systems of spectral type 111, 111, 111 using a parametrization of 11, 11, 11-systems. Linear systems of spectral type 11, 11, 11 are parametrized as follows [22] (here the choice of the gauge is slightly different from that in [22], and we omit the gauge parameter):

\[
\frac{dY}{dx} = Q(x - T)^{-1} PY
\]

(3.5)

where

\[
T = \text{diag}(t, 1, 0), \quad Q = \begin{pmatrix}
1 & 1 & 1 \\
tp & pq - \theta_2^\infty & 0 \\
\theta^t + pq & -q/t & 0 \\
\theta^1 + \theta_2^\infty - pq & 1 & \theta^0 \\
\theta^0 & -q/t & 0 \\
\theta^0 & -q/t & 0
\end{pmatrix}, \quad P = \begin{pmatrix}
1 & 1 & 1 \\
tp & pq - \theta_2^\infty & 0 \\
\theta^t + pq & -q/t & 0 \\
\theta^1 + \theta_2^\infty - pq & 1 & \theta^0 \\
\theta^0 & -q/t & 0 \\
\theta^0 & -q/t & 0
\end{pmatrix}.
\]
Its Riemann scheme is
\[
\begin{pmatrix}
x = 0 & x = 1 & x = t & x = \infty \\
0 & 0 & 0 & \theta_1^\infty \\
\theta_0 & \theta_1 & \theta_1 & \theta_2^\infty \\
\end{pmatrix}
\]

The Painlevé-type differential equation corresponding to (3.5) is the sixth Painlevé equation. Applying the Laplace transform and a Möbius transformation \((x \mapsto 1/x)\) to (3.5), we have
\[
\frac{dY}{dx} = \left( \frac{T}{x^2} + \frac{PQ}{x} \right) Y.
\]
(3.6)

It is easy to see that the spectral type of the system (3.6) is \((1)(1)(1), 111\).

By virtue of Lemma 3.4, we can construct a Fuchsian system which gives the system (3.6) through the confluence procedure:
\[
\frac{dY}{dx} = \left( \frac{A_0}{x} + \frac{A_1}{x+\varepsilon} \right) Y
\]
(3.7)

where
\[
A_0 = \begin{pmatrix}
\rho_t & (PQ)_{12} & (PQ)_{13} \\
0 & \rho_1 & (PQ)_{23} \\
0 & 0 & 0
\end{pmatrix},
A_1 = \begin{pmatrix}
\sigma_t & 0 & 0 \\
(PQ)_{21} & \sigma_1 & 0 \\
(PQ)_{31} & (PQ)_{32} & \sigma_0
\end{pmatrix}.
\]

Then the Riemann scheme of the system (3.7) is
\[
\begin{pmatrix}
x = 0 & x = -\varepsilon & x = \infty \\
\rho_t & \sigma_t & \theta_1^\infty \\
\rho_1 & \sigma_1 & \theta_2^\infty \\
0 & \sigma_0 & 0
\end{pmatrix},
\]
(3.8)

and the spectral type of which is \(111, 111, 111\). The relation of the exponents between (3.6) and (3.7) is given by
\[
\rho_t = \frac{t}{\varepsilon}, \quad \rho_1 = \frac{1}{\varepsilon}, \quad \sigma_t = \theta_1^t - \frac{t}{\varepsilon}, \quad \sigma_1 = \theta_1^1 - \frac{1}{\varepsilon}, \quad \sigma_0 = \theta^0.
\]

Conversely, we can show by direct calculation that a linear system with the Riemann scheme (3.5) can be written as (3.7). Thus (3.7) gives a parametrization of linear systems of spectral type \(111, 111, 111\).

Similarly, using the parametrization of \(111, 111, 111\)-systems, we can parametrize linear systems of spectral type \(22, 1111, 1111\), and using the parametrization of \(22, 1111, 1111\), we can obtain a parametrization of linear systems of spectral type \(33, 222, 111111\).

4 Discrete analogue of the Garnier system

By considering Schlesinger transformations of Fuchsian equations which appear in Theorem 1.1 (especially the ones with three singular points), we can obtain four-dimensional Painlevé-type
difference equations. In this section, as an example, we calculate Schlesinger transformations of the Fuchsian system of spectral type 211,1111,1111. The confluence of two singular points of this Fuchsian system represented by 1111 and 1111 leads to the spectral type (1)(1)(1)(1), 211. This is equivalent to 11,11,11,11,11, which corresponds to the Garnier system in two variables. On the other hand, the confluence of 211 and 1111 leads to (11)(1)(1) 1111. This is equivalent to 21 21 111 1111, which corresponds to the Fuji-Suzuki system. Here we consider the Schlesinger transformations corresponding to discretizations of the two time evolutions of the Garnier system.

First we note the following proposition, which can be verified by direct calculation.

**Proposition 4.1.** Let $A_0$ and $A_1$ be the following square matrices:

$$A_0 = \begin{pmatrix} \lambda & t_0 m^{-1} \\ b & B \end{pmatrix}, \quad A_1 = \begin{pmatrix} \mu & t_c \\ 0 m^{-1} & C \end{pmatrix}$$

where $\lambda, \mu \in \mathbb{C}$, $b, c \in \mathbb{C}^{m-1}$, $B, C \in M(m-1, \mathbb{C})$, with $B$ being upper triangular and $C$ lower triangular. We have denoted by $0_k$ the zero vector $(0, \ldots, 0) ^T$. Assume that $\lambda$ (resp. $\mu$) is not an eigenvalue of $B$ (resp. $C$). Define the square matrix $G$ as

$$G = \begin{pmatrix} l_1^{-1} & t_0 \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} 1 & t_c(\mu - C)^{-1} \\ (\lambda - B)^{-1} b & I_{m-1} \end{pmatrix}.$$ 

Here $l_1 \in \mathbb{C}^\times$, $U_{22} \in \text{GL}(m-1, \mathbb{C})$ are determined by the following LU decomposition

$$\begin{pmatrix} 1 & t_c(\mu - C)^{-1} \\ 0 & I_{m-1} \end{pmatrix} \begin{pmatrix} 1 & t_0 \\ (\lambda - B)^{-1} b & I_{m-1} \end{pmatrix} = \begin{pmatrix} l_1 & t_0 \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} l & l \end{pmatrix} \begin{pmatrix} u_1 & t_u \\ 0 & U_{22} \end{pmatrix},$$

(4.1)

where $U_{22}$ is upper triangular and $L_{22}$ lower triangular. Then $G A_0 G^{-1}$ is upper triangular and $G A_1 G^{-1}$ is lower triangular. More explicitly, we have

$$G A_0 G^{-1} = \begin{pmatrix} \lambda & -l_1^{-1} t_c(\mu - C)^{-1}(\lambda - B)U_{22}^{-1} \\ 0 m^{-1} & U_{22} B U_{22}^{-1} \end{pmatrix},$$

$$G A_1 G^{-1} = \begin{pmatrix} \mu & t_0 m^{-1} \\ -u_1^{-1} L_{22}^{-1}(\mu - C)(\lambda - B)^{-1} b & L_{22}^{-1} C L_{22} \end{pmatrix}.$$ 

**Remark 3.** We do not assume a particular normalization of the LU decomposition (4.1), thus $l_1, L_{22},$ etc., are not unique (see Remark [4]).

We parametrize linear systems of spectral type 211,1111,1111. Here we utilize a parametrization of linear systems of spectral type 11,11,11,11,11 [22],

$$\frac{dY}{dx} = Q(x - T)^{-1} P Y$$

(4.2)
where
\[ T = \text{diag}(t_1, t_2, 1, 0), \quad U = \text{diag}(1, u), \quad W = \text{diag}(w_1, w_2, w_3, w_4), \]
\[ Q = U^{-1} \hat{Q} W, \quad \hat{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_1 p_1 & t_2 p_2 & p_1 q_1 + p_2 q_2 - \theta_2^\infty & 0 \end{pmatrix}, \]
\[ P = W^{-1} \hat{P} U, \quad \hat{P} = \begin{pmatrix} \theta^t_1 + p_1 q_1 & -q_1/t_1 \\ \theta^t_2 + p_2 q_2 & -q_2/t_2 \\ \theta^0 + \theta_2^\infty - p_1 q_1 - p_2 q_2 & 1 \\ \theta_1^t & \theta_2^t & \theta_1^\infty & \theta_2^\infty \end{pmatrix}. \]

Its Riemann scheme is
\[ \begin{pmatrix} x = 0 \\ x = 1 \\ x = t_1 \\ x = t_2 \\ x = \infty \end{pmatrix}, \]
\[ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \theta^t_1 \\ \theta^t_2 \\ \theta^t_1 \theta_1^{\infty} \theta_2^{\infty} \end{pmatrix}. \]

Applying the Laplace transform and a Möbius transformation \( (x \mapsto 1/x) \) to (4.2), we obtain
\[
\frac{dY}{dx} = \left( \frac{T}{x^2} + \frac{PQ}{x} \right) Y. \tag{4.3}
\]

Then the Riemann scheme of (4.3) is
\[ \begin{pmatrix} x = 0 \\ x = \infty \\ t_1 \theta^t_1 \theta_1^\infty \\ t_2 \theta^t_2 \theta_2^\infty \\ 1 \theta^t_1 \theta_1^{\infty} \theta_2^{\infty} \\ 0 \theta^0 \theta^0 \theta_1^{\infty} \theta_2^{\infty} \end{pmatrix}. \]

The system (4.2) (and (4.3)) has two deformation parameters \( t_1, t_2 \). The (continuous) isomonodromic deformation of these systems is governed by the Garnier system in two variables:
\[
\frac{\partial q_i}{\partial t_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial t_j} = -\frac{\partial H_i}{\partial q_i} \quad (i, j = 1, 2)
\]

where the Hamiltonians are given by
\[
t_i(t_i - 1)H_i \left( \begin{array}{c} \theta^0, \theta^1 \\ \theta_1^t, \theta_2^t, \theta_1^{\infty}, \theta_2^{\infty} \\ t_1, t_2, q_1, p_1 \end{array} \right) = t_i(t_i - 1)H_{VI} \left( \begin{array}{c} \theta_1^{t_i}, \theta_2^{t_i} \\ \theta^0 + \theta_1^{t_i + 1} \\ t_i q_i, p_i \end{array} \right) + (2q_ip_i + q_{i+1}p_{i+1} - \theta^1 - 2\theta_2^{\infty})q_1 q_2 p_{i+1}
\]
\[
- \frac{1}{t_i - t_{i+1}} \{ t_i(t_i - 1)(p_i q_i + \theta^t_i)p_i q_{i+1} + t_i(t_{i+1} - 1)(2p_i q_i + \theta^t_i)p_{i+1} q_{i+1} 
\]
\[
+ t_{i+1}(t_{i+1} - 1)(p_{i+1} q_{i+1} + \theta^{t+1}_i(p_{i+1} - p_i))q_i \} \quad (i \in \mathbb{Z}/2\mathbb{Z}).
\]

Here \( H_{VI} \) stands for the Hamiltonian for the sixth Painlevé equation (see [15]).
By virtue of Lemma 3.4, there exists a Fuchsian system which gives the system (4.3) through the conflunce procedure:

\[
\frac{dY}{dx} = \left( \frac{A_0}{x} + \frac{A_1}{x + \varepsilon} \right) Y
\]  

(4.4)

where

\[
A_0 = \begin{pmatrix}
\rho_{t_1} & (PQ)_{12} & (PQ)_{13} & (PQ)_{14} \\
0 & \rho_{t_2} & (PQ)_{23} & (PQ)_{24} \\
0 & 0 & \rho_1 & (PQ)_{34} \\
0 & 0 & 0 & 0
\end{pmatrix},
A_1 = \begin{pmatrix}
\sigma_1 & 0 & 0 & 0 \\
(PQ)_{21} & \sigma_2 & 0 & 0 \\
(PQ)_{31} & (PQ)_{32} & \sigma_3 & 0 \\
(PQ)_{41} & (PQ)_{42} & (PQ)_{43} & \sigma_4
\end{pmatrix}.
\]

Its Riemann scheme is

\[
\begin{pmatrix}
x = 0 \\
x = -\varepsilon \\
x = \infty
\end{pmatrix}
\begin{pmatrix}
\rho_{t_1} \\
\sigma_1 \\
\theta_1^\infty
\end{pmatrix}
\begin{pmatrix}
\rho_{t_2} \\
\sigma_2 \\
\theta_2^\infty
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\sigma_3 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
\sigma_4 \\
0
\end{pmatrix}.
\]

This gives a parametrization of systems of spectral type 211,1111,1111. The relation of the exponents between (4.3) and (4.4) is given by

\[
\rho_{t_j} = \frac{t_j}{\varepsilon} (j = 1, 2), \quad \rho_1 = \frac{1}{\varepsilon}, \quad \sigma_j = \theta_{t_j} - \frac{t_j}{\varepsilon} (j = 1, 2), \quad \sigma_3 = \theta_1 - \frac{1}{\varepsilon}, \quad \sigma_4 = \theta^0.
\]

Now we consider the Schlesinger transformations $S_1$ and $S_2$

\[
S_1 : \rho_{t_1} \mapsto \rho_{t_1} + 1, \quad \sigma_1 \mapsto \sigma_1 - 1, \quad S_2 : \rho_{t_2} \mapsto \rho_{t_2} + 1, \quad \sigma_2 \mapsto \sigma_2 - 1.
\]

Thus $S_j$ corresponds to the shift $t_j \mapsto t_j + \varepsilon$.

The multiplier $R_1$ for $S_1$ is given as follows:

\[
R_1 = G_1 \left( I_4 + \frac{-\varepsilon E_1}{x + \varepsilon} \right), \quad G_1 = \hat{G}_1 W, \quad E_1 = \text{diag}(1,0,0,0),
\]

\[
\hat{G}_1 = \begin{pmatrix}
l_1^{-1} & \hat{0}_3 \\
\hat{0}_3 & U_1
\end{pmatrix}
\begin{pmatrix}
1 \\
-(\rho_{t_1} + 1 - \hat{B})^{-1} \hat{b}
\end{pmatrix}
\begin{pmatrix}
\hat{c}(\sigma_1 - 1 - \hat{C})^{-1} \\
I_3
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
\hat{b} \\
\hat{c}
\end{pmatrix} = \begin{pmatrix}
(P\hat{Q})_{21}, (P\hat{Q})_{31}, (P\hat{Q})_{41} \\
(P\hat{Q})_{12}, (P\hat{Q})_{13}, (P\hat{Q})_{14}
\end{pmatrix}, \quad \begin{pmatrix}
\hat{c}
\end{pmatrix} = \begin{pmatrix}
\hat{c}
\end{pmatrix} = \begin{pmatrix}
(P\hat{Q})_{23}, (P\hat{Q})_{24} \\
(P\hat{Q})_{32}, (P\hat{Q})_{34} \\
(P\hat{Q})_{42}, (P\hat{Q})_{43}
\end{pmatrix},
\]

and $l_1 \in \mathbb{C}^\times, U_1 \in \text{GL}(3, \mathbb{C})$ are determined by the LU decomposition

\[
\begin{pmatrix}
1 & \hat{c}(\sigma_1 - 1 - \hat{C})^{-1} \\
0 & I_3
\end{pmatrix}
\begin{pmatrix}
1 & \hat{0} \\
I_3
\end{pmatrix}
\begin{pmatrix}
\hat{b} \\
I_3
\end{pmatrix} = \begin{pmatrix}
l_1 & \hat{0} \\
l_1 & L_1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_1 \hat{u}_1
\end{pmatrix}.
\]
We define the matrices $\overline{A}_0$, $\overline{A}_1$ by

$$\overline{A}_0 \frac{x}{x + \varepsilon} + \overline{A}_1 \frac{x}{x + \varepsilon} := R_1 \left[ \frac{A_0}{x} + \frac{A_1}{x + \varepsilon} \right] = G_1 \left( \frac{\hat{A}_0}{x} + \frac{\hat{A}_1}{x + \varepsilon} \right) G_1^{-1},$$

where

$$\hat{A}_0 = W^{-1} \left( \rho_{t_1} + 1 \frac{\hat{b}}{\hat{B}} \right) W, \quad \hat{A}_1 = W^{-1} \left( \sigma_1 - 1 \frac{\hat{c}}{\hat{C}} \right) W.$$

Notice that the desired shift of characteristic exponents is achieved. Obviously we have $\hat{A}_0 + \hat{A}_1 = A_0 + A_1$. Proposition 4.1 guarantees that $\overline{A}_0$ and $\overline{A}_1$ are again upper triangular and lower triangular, respectively. Thus, by comparing $\overline{A}_0$, $\overline{A}_1$ with $A_0$, $A_1$, we can obtain the discrete time evolution of $q_i, p_i, w_i$'s (which we denote by $\overline{q}_i, \overline{p}_i, \overline{w}_i$) along the $S_1$-direction. To see this, we consider the time evolution of $PQ$:

$$\overline{PQ} = \overline{A}_0 + \overline{A}_1 = G_1(\hat{A}_0 + \hat{A}_1)G_1^{-1} = G_1(A_0 + A_1)G_1^{-1} = \hat{G}_1 \hat{P} \hat{Q} \hat{G}_1^{-1}.$$ 

We set $M_1 = \left( m_{ij}^{(1)} \right) := \hat{G}_1 \hat{P} \hat{Q} \hat{G}_1^{-1}$. Thus the difference equations satisfied by $q_i, p_i, w_i$ are determined by the equation

$$\overline{PQ} = M_1.$$  \hfill (4.5)

Multiplying (4.5) from the left by $\overline{t} \overline{w} = (\overline{w}_1, \overline{w}_2, \overline{w}_3, \overline{w}_4)$, we see that $\overline{w}$ satisfy

$$\left( \theta_1^\infty + \overline{t} M_1 \overline{w} \right) = 0.$$

Since $\text{rank}(\theta_1^\infty + \overline{t} M_1) = 3$, $\overline{w}$ is determined up to a scalar multiple. From the (1, 4) and (2, 4) entries of the equation (4.5) we then obtain

$$\overline{p}_1 \overline{q}_1 = \frac{\overline{w}_1}{\overline{w}_4} m_{14}^{(1)} - \rho_{t_1} - \sigma_1, \quad \overline{p}_2 \overline{q}_2 = \frac{\overline{w}_2}{\overline{w}_4} m_{24}^{(1)} - \rho_{t_2} - \sigma_2.$$

From the equation (4.6) and (3, 1), (3, 2) entries of (4.5) we obtain the discrete time evolutions of $p_1$ and $p_2$:

$$\frac{\rho_{t_1} + 1}{\rho_1} \overline{p}_1 = \frac{\overline{w}_3}{\overline{w}_1} m_{31}^{(1)} + \frac{\overline{w}_1}{\overline{w}_4} m_{14}^{(1)} + \frac{\overline{w}_2}{\overline{w}_4} m_{24}^{(1)} + \sigma_4 + \theta_1^\infty,$$

$$\frac{\rho_{t_2}}{\rho_1} \overline{p}_2 = \frac{\overline{w}_3}{\overline{w}_2} m_{32}^{(1)} + \frac{\overline{w}_1}{\overline{w}_4} m_{14}^{(1)} + \frac{\overline{w}_2}{\overline{w}_4} m_{24}^{(1)} + \sigma_4 + \theta_1^\infty.$$

Thus the time evolutions of $q_1$ and $q_2$ are determined by

$$q_i = \frac{\overline{p}_i \overline{q}_i}{\overline{p}_i} \quad (i = 1, 2),$$

that is,

$$\frac{\rho_{t_1} + 1}{\rho_1} q_1 = \frac{\overline{w}_3}{\overline{w}_1} m_{31}^{(1)} + \frac{\overline{w}_1}{\overline{w}_4} m_{14}^{(1)} + \frac{\overline{w}_2}{\overline{w}_4} m_{24}^{(1)} + \sigma_4 + \theta_1^\infty,$$

$$\frac{\rho_1}{\rho_2} q_2 = \frac{\overline{w}_3}{\overline{w}_2} m_{32}^{(1)} + \frac{\overline{w}_1}{\overline{w}_4} m_{14}^{(1)} + \frac{\overline{w}_2}{\overline{w}_4} m_{24}^{(1)} + \sigma_4 + \theta_1^\infty.$$
Remark 4. The matrix $M_1$ is determined up to conjugation by a diagonal matrix $D$ (see Proposition 4.2 below). Difference equations satisfied by $w_i$’s depend on this $D$. However, the equations satisfied by $q_i$ and $p_i$ are determined independent of $D$, since $\mathfrak{m}_i$’s appear in the expressions of $\mathfrak{q}_i$, $\mathfrak{p}_i$ in the form of $m_i^{(1)}/\mathfrak{m}_j$. Therefore $\mathfrak{q}_i$, $\mathfrak{p}_i$ are well-defined. $\square$

Proposition 4.2. Let $A_0$ (resp. $A_1$) $\in M(m, \mathbb{C})$ be an upper (resp. lower) triangular matrix with $m$ distinct eigenvalues. Suppose that $g \in \text{GL}(m, \mathbb{C})$ satisfy the following properties: 1) $gA_0g^{-1}$ is upper triangular, 2) $gA_1g^{-1}$ is lower triangular, 3) the order of the diagonal entries of $gA_sg^{-1}$ is the same as $A_*$ ($\ast = 0, 1$). Then $g$ is diagonal.

Next, we consider the multiplier $R_2$ for $S_2$. Let $R_{20}$, $\tilde{A}_0$, and $\tilde{A}_1$ be the following matrices:

$$R_{20} = \left( I_4 + \frac{-\varepsilon E_2}{x + \varepsilon} \right) \left\{ \begin{pmatrix} \rho \tau_1 & (PQ)_{12} \\ -(PQ)_{21} & \sigma_1 - \sigma_2 \end{pmatrix} \right\} + I_2, \quad E_2 = \text{diag}(0, 1, 0, 0),$$

$$\frac{\tilde{A}_0}{x} + \frac{\tilde{A}_1}{x + \varepsilon} := R_{20} \left[ \frac{A_0}{x} + \frac{A_1}{x + \varepsilon} \right].$$

Define $\hat{b}, \hat{c} \in \mathbb{C}^2$ and $\hat{B}, \hat{C} \in M(2, \mathbb{C})$ by

$$\tilde{A}_0 = W^{-1} \begin{pmatrix} \rho \tau_1 & * & ** \\ 0 & \rho \tau_1 + 1 & \hat{t} \hat{0}_2 \\ 0 & \hat{b} & \hat{B} \end{pmatrix} W, \quad \tilde{A}_1 = W^{-1} \begin{pmatrix} \sigma_1 & 0 & \hat{t} \hat{0}_2 \\ * & \sigma_2 - 1 & \hat{c} \hat{c} \hat{c} \\ * & \hat{0}_2 & \hat{C} \end{pmatrix} W.$$

The multiplier $R_2$ is given by

$$R_2 = G_2 R_{20}, \quad G_2 = \left( (1) \oplus \hat{G}_2 \right) W,$$

$$\hat{G}_2 = \begin{pmatrix} l_2^{-1} \hat{0}_2 & \hat{t} \hat{0}_2 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} 1 & \hat{c}(\sigma_2 - 1 - \hat{C})^{-1} \\ - (\rho \tau_1 + 1 - \hat{B})^{-1} \hat{b} & I_2 \end{pmatrix},$$

where $l_2 \in \mathbb{C}^\times$ and $U_2 \in \text{GL}(2, \mathbb{C})$ are determined by

$$\begin{pmatrix} 1 & \hat{c}(\sigma_2 - 1 - \hat{C})^{-1} \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} 1 & \hat{t} \hat{0}_2 \\ (\rho \tau_1 + 1 - \hat{B})^{-1} \hat{b} & I_2 \end{pmatrix} = \begin{pmatrix} l_2 & \hat{t} \hat{0}_2 \\ l_2 & L_2 \end{pmatrix} \begin{pmatrix} \hat{w}_2 & \hat{t} \hat{u}_2 \\ 0 & U_2 \end{pmatrix}. $$

We define $\overline{A}_0$ and $\overline{A}_1$ by

$$\frac{\overline{A}_0}{x} + \frac{\overline{A}_1}{x + \varepsilon} := R_2 \left[ \frac{A_0}{x} + \frac{A_1}{x + \varepsilon} \right] = G_2 \left( \frac{\tilde{A}_0}{x} + \frac{\tilde{A}_1}{x + \varepsilon} \right) G_2^{-1}.$$
we obtain the following

\[
\begin{align*}
\rho_1 \frac{\rho_1}{\rho_1} p_1 &= \frac{w_3}{w_1} m_{31} + \frac{w_1}{w_4} m_{14} + \frac{w_2}{w_4} m_{24} + \sigma_4 + \theta_1^\infty, \\
\rho_1 \frac{\rho_1}{\rho_1} p_2 &= \frac{w_3}{w_2} m_{32} + \frac{w_1}{w_4} m_{14} + \frac{w_2}{w_4} m_{24} + \sigma_4 + \theta_1^\infty, \\
\rho_1 \frac{\rho_1}{\rho_1} q_1 &= \frac{w_3}{w_1} m_{31} + \frac{w_1}{w_4} m_{14} + \frac{w_2}{w_4} m_{24} + \sigma_4 + \theta_1^\infty, \\
\rho_1 \frac{\rho_1}{\rho_1} q_2 &= \frac{w_3}{w_2} m_{32} + \frac{w_1}{w_4} m_{14} + \frac{w_2}{w_4} m_{24} + \sigma_4 + \theta_1^\infty.
\end{align*}
\]

Note that these time evolutions are rather complicated when written explicitly in terms of \( q_i, p_i \)’s.

There remains some ambiguity regarding time evolutions of \( w_i \)’s. Although this ambiguity is resolved by the compatibility condition of the \( S_1 \)-direction and the \( S_2 \)-direction, we do not go into the details.

**Remark 5.** A construction in this section can be generalized to the Garnier system in \( N \)-variables. Discrete analogues of the Garnier systems were also considered in [19], where they were derived from \( 2 \times 2 \) linear systems. The relationship between the two constructions is not well understood at this time.

**Remark 6.** The following Schlesinger transformation of (4.4):

\[
\theta_1^\infty \mapsto \theta_1^\infty + 1, \quad \sigma_1 \mapsto \sigma_1 - 1
\]

gives a discrete analogue of the Fuji-Suzuki system.

**Remark 7.** For any Painlevé-type differential equation corresponding to a Fuchsian system, in the same way as in Section[4] we can construct a system of difference equations which can be regarded as a discrete analogue of the Painlevé-type differential equation.

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