Lissajous-toric knots

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Abstract

A point in the \((N,q)\)-torus knot in \(\mathbb{R}^3\) goes \(q\) times along a vertical circle while this circle rotates \(N\) times around the vertical axis. In the Lissajous-toric knot \(K(N,q,p)\), the point goes along a vertical Lissajous curve (parametrized by \(t \mapsto (\sin(qt + \phi), \cos(pt + \psi))\)) while this curve rotates \(N\) times around the vertical axis. Such a knot has a natural braid representation \(B_{N,q,p}\) which we investigate here. If \(\gcd(q, p) = 1\), \(K(N,q,p)\) is ribbon; if \(\gcd(q, p) = d > 1\), \(B_{N,q,p}\) is the \(d\)-th power of a braid which closes in a ribbon knot. We give an upper bound for the 4-genus of \(K(N,q,p)\) in the spirit of the genus of torus knots; we also give examples of \(K(N,q,p)\)'s which are trivial knots.

1 Introduction

We study a class of knots generalizing torus knots, which we call Lissajous-toric: a torus knot is generated by a circle rotating around an axis and a Lissajous-toric knot is generated by a Lissajous curve rotating around the axis. There are several ways of describing them.

1.1 Lissajous-toric knots: various points of view

1.1.1 A description in \(\mathbb{R}^3\)

We recall the description of the \((N,q)\)-torus knot in \(\mathbb{R}^3\) endowed with an orthonormal frame \(Oxyz\) (see for exemple \([\text{Cr}]\) 1.5). If \(\Gamma\) is the circle of radius 1 centered at \((0,2,0)\) in the \(yz\) plane, a point travelling along the knot goes \(q\) times around \(\Gamma\) while \(\Gamma\) is rotated \(N\) times around the axis \(Oz\). In the case of the Lissajous-toric knots, we replace the vertical circle by a vertical Lissajous curve: we take three integers \(N, q, p\) with \((N,q) = (N,p) = \)
1 and a real number $\phi$, and we define a knot $K(N, q, p, \phi)$ as follows. Consider the curve $C_{q,p,\phi}$ given in a vertical plane by

$$t \in [0, 2\pi] \longrightarrow \mathbb{R}^3,$$

$$: t \mapsto \left(0, 2 + \sin(qt), \cos(p(t + \phi))\right)$$

and rotate $C_{q,p,\phi}$ is rotated $N$ times around the axis generated by $(0, 0, 1)$. In Cartesian coordinates, we write the knot as

\begin{align*}
\begin{cases}
x = (2 + \sin(qt)) \cos(Nt) \\
y = (2 + \sin(qt)) \sin(Nt) \\
z = \cos(p(t + \phi))
\end{cases}
\end{align*}

Thus $K(N, q, p, \phi)$ is a closed $N$-braid which we can write in the 3-cylinder $S^1 \times \mathbb{R}^2$ as follows:

$$e^{it} \mapsto \left(e^{Nwt}, \sin(qt), \cos(p(t + \phi))\right)$$

Note the similarity with the $(N, q)$-torus knot which is written in the 3-sphere or the 3-cylinder as

$$e^{it} \mapsto \left(\frac{1}{\sqrt{2}}e^{Nwt}, \frac{1}{\sqrt{2}}e^{qit}\right)$$

1.1.2 A description in the 3-dimensional cylinder

We write (*) above in cylindrical coordinates:

\begin{align*}
\begin{cases}
\theta = Nt \\
\rho = 2 + \sin(qt) \\
z = \cos(p(t + \phi))
\end{cases}
\end{align*}

1.1.3 Billiard curve in a solid torus

Just as Billiard curves are equivalent to Lissajous knots (cf. [JP]) Lissajous toric knots are equivalent to billiard curves in a square solid torus, namely a cube where the top and bottom have been identified; C. Lamm introduced them in [La 1], see also the related [L-O]. Such billiard curves are
parametrized similarly to Lissajous toric knots; the trigonometric functions are replaced by \emph{saw-tooth functions} of the type \(g(t) := 2|t - [t] - \frac{1}{2}|\) and \(h(t) := t - [t]\).

\begin{equation}
C(N, p, q, \phi) : \left([0, 2\pi] \rightarrow [-1, 1]^3 \right) \quad \quad t \mapsto (g(N,t), g(p.t + \phi), h(q.t)) \quad (3)
\end{equation}

C. Lamm noticed that these billiard curves in a solid torus do not depend on the phase up to mirror transformation and stated that, if \(p\) and \(q\) are mutually prime, the knot \(K(N, q, p)\) is ribbon.

### 1.1.4 Singularity knots of minimal surfaces

We first encountered the \(K(N, q, p, \phi)\)'s in [S-V] when we studied the singularities of minimal disks in \(\mathbb{R}^4\); having noticed that their knot types do not depend on the phase \(\phi\) up to mirror transformation, we dropped the \(\phi\) in the notation.

We consider a minimal, i.e. conformal harmonic, map \(F : \mathbb{D} \rightarrow \mathbb{R}^4\) where \(\mathbb{D}\) is the unit disk in \(\mathbb{C}\), with \(dF(0) = 0\), i.e. \(F\) has a branch point at 0. If moreover \(F\) is a topological embedding, we can copy Milnor’s construction of algebraic knots ([Mi]) and take the intersection of \(F(\mathbb{D})\) with a small sphere centered at \(F(0)\): we obtain a \emph{minimal knot}. Complex curves are a special case of minimal surfaces and the germ \(z \mapsto (z^N, z^q)\) yields the \((N, q)\)-torus
knot. In [S-V] the knots $K(N,q,p,\phi)$’s came from germs of singularities of the type
\[ z \mapsto (Re(z^N), Im(z^N), Im(z^q), Re(e^{i\phi}z^p)) \] \hspace{1cm} (4)
with
\[ N < p, q \] \hspace{1cm} (5)
In [S-V] we called the $K(N,q,p)$’s simple minimal knots; in the present paper we drop the assumption (5) and study these knots per se; Lissajous-toric is a more appropriate name for the general case.

1.2 Contents of the paper

In [S-V] we defined a braid $B_{N,q,p}$ naturally associated to the knot $K(N,q,p)$; we describe it here in much greater detail. We view $B_{N,q,p}$ as a collection of graphs of $N$ functions from $[\eta, 1 + \eta]$ to $\mathbb{R}^2$; the purpose of the small positive number $\eta$ is to avoid crossing points at the endpoints of the interval.

![Figure 2: Braid shadow of $B(5,q,p)$](image)

We prove in §4.1 below

**Proposition 1.** Let $d = \gcd(p, q)$, $\tilde{q} = \frac{q}{d}$, $\tilde{p} = \frac{p}{d}$; then
\[ B_{N,q,p} = B_{N,\tilde{q},\tilde{p}}^d \] \hspace{1cm} (6)

Since $\tilde{q}$ and $\tilde{p}$ are mutually prime and since the knot type does not change if we interchange $p$ and $q$, we make the
Assumption 1. The numbers $p$ and $q$ are mutually prime and $q$ is odd.

In §2, we construct two braids $\alpha_{N,q,p}$ and $\beta_{N,q,p}$ of the form

$$\alpha_{N,q,p} = \prod_{2 \leq 2k \leq N-1} \sigma^\pm_{2k} \quad \beta_{N,q,p} = \prod_{1 \leq 2k+1 \leq N-1} \sigma^\pm_{2k+1} \quad (7)$$

where the $\sigma_i$’s are the standard generators of the braid group $B_N$ and the exponents $\pm 1$ of the $\sigma_i$’s appearing in $\alpha$ and $\beta$ are given by simple formulae in $N, q, p$.

We will state below the Main Theorem which expresses the braid $B_{N,q,p}$ as a product of the braids $\alpha_{N,q,p}$, $\alpha_{N,q,p}^{-1}$, $\beta_{N,q,p}$ and $\beta_{N,q,p}^{-1}$ as follows :

$$B_{N,q,p} = Q_{N,q,p} \alpha_{N,q,p} Q_{N,q,p}^{-1} \beta_{N,q,p} \quad (8)$$

where the $N$-braid $Q_{N,q,p}$ is also a product of $\alpha_{N,q,p}^{\pm 1}$’s and $\beta_{N,q,p}^{\pm 1}$’s. We illustrate the Main Theorem in §3 by going through the examples we gave in [S-V] and we prove it in §4.

In the rest of the paper, we drop the Assumption 1 and study the topology of the knot. In §5.1, we prove a theorem stated by Lamm

**Theorem 1.** If $p$ and $q$ are mutually prime, the knot $K(N, q, p)$ is ribbon.

**Corollary 1.** If $d = \gcd(p, q) > 1$, the knot $K(N, q, p)$ is periodic and its braid is the $d$-th power of a braid which closes in a ribbon knot.

**Theorem 2.** If $d = \gcd(p, q)$, the four-genus of $K(N, q, p)$ verifies

$$g_4(K(N, q, p)) \leq \frac{(N-1)(d-1)}{2}. \quad (9)$$

**Remark 1.** The right-hand side of $(9)$ is the genus of the $K(N, d)$-torus knot (cf. [K-M]).

**Remark 2.** The inequality $(9)$ can be strict: for example the knot $K(3, 5, 10)$ is $10_{123}$ which is slice.

There is one case where we know that $(9)$ is an equality:
Proposition 2. Let $N, q, p$ be positive integers with $(N, q) = (N, p) = 1$, $d = \text{gcd}(q, p)$ and let

\[ \tilde{p} = \frac{p}{d} \quad \tilde{q} = \frac{q}{d} \quad (10) \]

If

\[ \tilde{p} + \tilde{q} \equiv 0 \pmod{2N} \quad \text{or} \quad \tilde{p} - \tilde{q} \equiv 0 \pmod{2N} \quad (11) \]

the knot $K(N, q, p)$ is represented by a quasipositive braid and its 4-genus is

\[ g_4(K(N, q, p)) = \frac{(N - 1)(d - 1)}{2} \quad (12) \]

Finally, replacing $t$ by $t + \pi$ in the expression of $K(N, q, p)$ given in §1.1.1 yields

Proposition 3. If $p$ and $q$ have different parities (and thus $N$ is odd), then $K(N, q, p)$ is preserved by the involution

\[ (x, y, z) \mapsto (-x, -y, -z). \]

Hence it is positive strongly amphicheiral.

Some of the $K(N, q, p)$’s are actually trivial knots; in [6] show:

Proposition 4. If $N$ and $q$ are mutually prime, the knots $K(N, q, q + N)$, $K(N, q, 1)$, $K(N, q, 2Nq + 1)$ and $K(N, q, 2Nq - 1)$ are trivial.

Can we get all the trivial $K(N, q, p)$’s this way? We did computer simulations using the braid software from the Liverpool knot group ([br]) and KnotPlot ([KP]): they told us that in some cases (the $K(4, 5, .)$’s for example) the answer is yes but in most cases the answer is no (see the lists of Jones polynomials at the end of the paper).

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2 The structure of a simple minimal braid

2.1 Overview

Here is an informal description of the contents of the Main Theorem.

There are $2q$ values of $t$ in $[\eta, 1 + \eta]$ (we call them crossing values) above which two or more of the $N$ graphs forming $B_{N,q,p}$ meet (at crossing points) and the data of all these crossing points make up the braid (see Figure 2); above each of the crossing values $t$’s, the generators of the braid group $B_N$ describing the corresponding crossing points are all even (i.e. of the form $\sigma_{2k}^{\pm 1}$) or all odd (i.e. of the form $\sigma_{2k+1}^{\pm 1}$).

The set of crossing points of $B_{N,q,p}$ above a crossing value $t$ can be represented by one of the braids: $\alpha_{N,q,p}$, $\alpha_{N,q,p}^{-1}$, $\beta_{N,q,p}$ or $\beta_{N,q,p}^{-1}$ which were introduced in formula (7); thus $B_{N,q,p}$ is a product of the $\alpha_{N,q,p}$’s and $\beta_{N,q,p}$’s and of their inverses.

We order the $2q$ crossing values $t_1 < t_2 < ... < t_q < ... < t_{2q}$. Going from $t_k$ to $t_{k+1}$ changes $\alpha_{N,q,p}^{\pm 1}$ into $\beta_{N,q,p}^{\pm 1}$ or vice-versa. A formula gives us the exponent $+1$ or $-1$ of the $\alpha_{N,q,p}$ or $\beta_{N,q,p}$ above a given crossing point $t_k$ in terms only of $N, q, p$ and $k$.

Finally we notice that, if we have an $\alpha_{N,q,p}$ (resp. $\alpha_{N,q,p}^{-1}$, $\beta_{N,q,p}$, $\beta_{N,q,p}^{-1}$) for $t_k$ (with $k \neq q$), we have a $\alpha_{N,q,p}^{-1}$ (resp. $\alpha_{N,q,p}$, $\beta_{N,q,p}^{-1}$, $\beta_{N,q,p}$) for $t_{2q-k}$: this explains the presence of $Q_{N,q,p}$ and $Q_{N,q,p}^{-1}$ in the product (8).

2.2 Statement of the structure theorem

Main Theorem. Let $N, p, q$ be three integers such that $q$ is odd and $(p, q) = (N, q) = (N, p) = 1$; and let $A, B$ two integers such that

$$2NA + Bq = 1 \quad (13)$$

For $i \in \{1, ..., N - 1\}$, we let

$$\epsilon_{N,q,p}(i) = (-1)^{\left[\frac{2pi}{N}\right]} \quad (14)$$

where $[\ ]$ denotes the integral part and we define

$$\alpha_{N,q,p} = \prod_{2 \leq 2i \leq N-1} \sigma_{2i}^{\epsilon_{N,q,p}(2i)} \quad \beta_{N,q,p} = \prod_{1 \leq 2i+1 \leq N-1} \sigma_{2i+1}^{\epsilon_{N,q,p}(2i+1)} \quad (15)$$
For \( k \in \{1, \ldots, 2q\} \), \( k \neq q, k \neq 2q \), we let

\[
\lambda_{N,q,p}(k) = (-1)^{\left\lfloor \frac{2Apk}{q} \right\rfloor}
\]  

(16)

Up to mirror transformation, the knot \( K(N,q,p) \) is represented by the braid

\[
B_{N,q,p} = \alpha_{N,q,p}^{\lambda(1)} \beta_{N,q,p}^{\lambda(2)} \cdots \alpha_{N,q,p}^{\lambda(q-2)} \beta_{N,q,p}^{\lambda(q-1)} \alpha_{N,q,p}^{-\lambda(q-1)} \beta_{N,q,p}^{-\lambda(q-2)} \cdots \alpha_{N,q,p}^{-\lambda(2)} \beta_{N,q,p}^{-\lambda(1)}
\]

(17)

The \( k \)-th factor in this expression corresponds to the \( k \)-th crossing value \( t_k \).

Notice that the arithmetic formulae (14) and (16) can be written in terms of the Conway sign:

Definition 1. ([Co]) If \( m \) and \( n \) are two integers, \( n \) is said to be positive (resp. negative) modulo \( m \) if \( n \) is congruent to an integer inside \((0, \frac{m}{2})\) (resp. \((0, -\frac{m}{2})\)).

3 Illustrations and examples

In this section we go through the examples featured in [S-V] and we write their braid using the terminology of the Main Theorem.

We define three permutations of the crossing values \( t_k \), \( k \in \{1, \ldots, 2q\} \) and their corresponding action on the blocks \( \alpha \) and \( \beta \):

\[
T(k) = k + q, S(k) = 2q - k, R(k) = q - k
\]

\[
T : \alpha \mapsto \beta, \beta \mapsto \alpha
\]

\[
S : \alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}
\]

\[
R : \alpha \mapsto \beta^{-1}, \beta \mapsto \alpha^{-1}
\]

(18)

Since it is clear in each case of the following list what the \( N,q,p \) are, we dropped the indices \( N,q,p \).

- \( N = 3, q = 4, p = 5 \): square knot \( 3_1 \# 3_1 \)

\[
Q \sigma_2^{-1} \sigma_1^{-1} \quad \text{where} \quad Q = \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_1
\]

- \( N = 3, q = 4, p = 7 \): trivial knot

\[
Q \sigma_2 Q^{-1} \sigma_1 \quad \text{where} \quad Q = (\sigma_2 \sigma_1^{-1} \sigma_2^{-1})^2
\]
• $N = 3, q = 4, p = 10$: figure eight knot
  \[ B_{3,4,10} = B_{3,2,5}^2 = (Q\sigma_2Q^{-1}\sigma_1^{-1})^2 \quad \text{where} \quad Q = \sigma_2\sigma_1\sigma_2\sigma_1 \]

• $N = 3, q = 5, p = 7$: 10_{155}
  \[ Q\sigma_2^{-1}Q^{-1}\sigma_1^{-1} \quad \text{where} \quad Q = \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 \]
  Note that this knot verifies the assumptions of Theorem 2.

• $N = 3, q = 5, p = 10$: 10_{123}
  \[ B_{3,5,10} = B_{3,1,2}^5 = (\sigma_2^{-1}\sigma_1)^5 \]

• $N = 3, q = 7, p = 8$: 5_1\#\bar{5}_1
  \[ Q\sigma_2^{-1}Q^{-1}\sigma_1 \quad \text{where} \quad Q = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1 \]

• $N = 3, q = 7, p = 19$: 14N11995
  \[ Q\sigma_2Q^{-1}\sigma_1 \quad \text{where} \quad Q = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1 \]

• $N = 4, q = 5, p = 7$: 5_2\#\bar{5}_2
  \[ Q\alpha Q^{-1}\beta \quad \text{where} \quad \alpha = \sigma_2^{-1} \quad \beta = \sigma_1\sigma_3 \quad Q = \alpha^{-1}\beta^{-1}\alpha\beta \]

• $N = 4, q = 5, p = 13$: 9_{46}
  \[ Q\alpha Q^{-1}\beta \quad \text{where} \quad \alpha = \sigma_2 \quad \beta = \sigma_1\sigma_3 \quad Q = \alpha\beta\alpha^{-1}\beta^{-1} \]

• $N = 5, q = 6, p = 22$: 7_7
  \[ B_{5,6,22} = B_{5,3,11}^2 = (Q\alpha Q^{-1}\beta)^2 \quad \text{where} \quad \alpha = \sigma_2\sigma_4^{-1} \quad \beta = \sigma_1^{-1}\sigma_3 \quad Q = \alpha^{-1}\beta \]
Figure 3: $B_{4,5,13}$, $A = 4$, $\alpha = \sigma_2$, $\beta = \sigma_1\sigma_3$, $\lambda(k) = (-1)^{\frac{2k}{3}}$.

Figure 4: $B_{5,6,22} = B^2_{5,3,11}$, $\alpha = \sigma_2\sigma_4^{-1}$, $\beta = \sigma_1^{-1}\sigma_3$. 

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4 Proof of the Main Theorem on the simple minimal braid $B_{N,q,p}$

We recall some facts from [S-V]. We endow $\mathbb{R}^3$ with coordinates $(t, y, z)$: the braid is the collection of the graphs in $\mathbb{R}^3$ of the functions $\psi_k$ for $k = 1, \ldots, N$:

$$\psi_k = (\psi_k^{(1)}, \psi_k^{(2)}): [\eta, 1 + \eta] \rightarrow \mathbb{R}^2$$

$$t \mapsto (y, z) = (\psi_k^{(1)}(t), \psi_k^{(2)}(t)) = \left(\sin \frac{2\pi q}{N}(t + k), \cos \frac{2\pi p}{N}(t + k + \phi)\right) \quad (19)$$

### Figure 5: Graph of $\{\psi_k\}_{k=1,2,3}$ of knot $K(3, 7, 5)$

4.1 Periodic braids: proof of Proposition [1]

We now prove Proposition [1] (stated in [1.2]). We divide the interval $[\eta, 1 + \eta]$ into $d$ intervals

$$I_n = \left[\frac{n}{d} + \eta, \frac{n + 1}{d} + \eta\right], \quad n = 0, \ldots, d - 1.$$
After a change of variables \( t \mapsto s = dt \), we see that the braid above an interval \( I_n \) consists in the collection of graphs of the functions \([d\eta, 1 + d\eta] \rightarrow \mathbb{R}^2\)

\[
s \mapsto \left( \sin \frac{2\pi \tilde{q}}{N}(s + dk), \cos \frac{2\pi \tilde{p}}{N}(s + dk + d\phi) \right)
\]

(20)

Since \((N,d) = 1\), the map \( k \mapsto kd \pmod{N} \) induces a permutation of \( \{1, \ldots, N - 1\} \); hence the piece of \( B_{N,q,p} \) above \( I_n \) is the collection of graphs above \([\eta, 1 + \eta] \) of the functions

\[
s \mapsto \left( \sin \frac{2\pi \tilde{q}}{N}(s + k), \cos \frac{2\pi \tilde{p}}{N}(s + k + d\phi) \right)
\]

i.e. it is the braid \( B_{N,\tilde{q},\tilde{p}} \), representing the knot \( K(N, \tilde{q}, \tilde{p}, d\phi) \); this proves Proposition 1.

\[\square\]

### 4.2 Crossing values and crossing points of the braid

The **braid shadow** is the projection of the braid onto the first two components \((t, y)\) of \(\mathbb{R}^2\) i.e. the collection of the graphs of the \(\psi^{(1)}_k\)’s.

A **crossing point** \( P \) of the braid is the data of two different integers, \( k, l \) with \( 0 \leq k, l \leq N - 1 \) and a number \( t \in [\eta, 1 + \eta] \) called a **crossing value** such that

\[
\psi^{(1)}_k(t) = \psi^{(1)}_l(t) \quad \text{i.e.} \quad \sin \left( \frac{2\pi}{N} q(t + k) \right) = \sin \left( \frac{2\pi}{N} q(t + l) \right).
\]

There is a total of \((N - 1)q\) crossing points, as in the case of the \((N, q)\) torus knot (where \( q = p \)).

A straightforward computation (cf. [S-V]) shows that, for a crossing point \( P \) between the \(k\)-th and \(l\)-th strands of \( B_{N,q,p} \), the corresponding crossing value \( t \) verifies for some integer \( m \)

\[
t = -\frac{k + l}{2} + \frac{N}{4q}(2m + 1)
\]

(21)

The sign \( \Sigma(P) \) of a crossing point \( P \) is

\[
\Sigma(P) = \text{sign of } \left( \psi^{(2)}_k(t) - \psi^{(2)}_l(t) \right) \left( \psi^{(1)*}_l(t) - \psi^{(1)*}_k(t) \right)
\]

(22)

In [S-V], we computed this \( \Sigma(P) \) as:

\[
\Sigma(P) = (-1)^m(-1)^{\frac{p+1}{4} + \frac{2\phi}{N}}(-1)^{\left\lfloor \frac{k-l}{N} \right\rfloor}(-1)^{\left\lfloor \frac{k+l}{N} \right\rfloor}
\]

(23)

where \([ \ ]\) denotes the integral part.
4.2.1 Determination of the crossing points above a given crossing value

Let \( t \) be a crossing value of \( B_{N,q,p} \). We look for the \( y \)'s such that \((t,y)\) is a crossing point of the braid shadow.

We derive from (21) the existence of at least one ordered pair of integers \((m,s)\) such that

\[
t = -\frac{s}{2} + \frac{N}{4q}(2m + 1).
\]

(with \( s = k + l \)). There can be several \((m,s)\)'s verifying (24) for the same crossing value \( t \); however

\[
1 \leq s = k + l \leq 2N - 3
\]

which implies that there are at most two possible \((m,s)\)'s (Lemma 1 below).

We will see later that for \( t \) and \((m,s)\) given, a crossing point \((t,y)\) of the braid shadow above \( t \) will be given by the data of \( \tilde{d} = k - l \).

Lemma 1. If \( t \) is a crossing value, one of the following two cases occurs:

- **1st case.** There is exactly one ordered pair \((m,s)\), \( 1 \leq s \leq 2N - 3 \) verifying (24); we denote it \((m(t),s(t))\) and \( s(t) \) is either \( N - 2 \), \( N - 1 \) or \( N \).

- **2nd case.** There exist exactly two \((m,s)\)'s satisfying (24) with \( 1 \leq s \leq 2N - 3 \); we denote them \((m(t),s(t))\) and \((m(t) + q,s(t) + N)\).

Proof. We let \((m(t),s(t))\) be the ordered pair such that \( s(t) \) is the smallest \( s \) for the \((m,s)\)'s verifying (24) and (25). If \((m_1,s_1)\) and \((m_2,s_2)\) both verify (24) for the same \( t \), we have

\[
q(s_1 - s_2) = N(m_2 - m_1)
\]
Since $q$ and $N$ are mutually prime, it follows that, for some integer $a$,

$$ s_2 = s_1 + aN \quad m_2 = m_1 + aq \quad (27) $$

Since $s(t)$ is the smallest one, it verifies

$$ s(t) \leq N. \quad (28) $$

If we are in the 1st case of the Lemma 1, i.e. a single $(m, s)$, we derive from (27) that $s(t) + N$ does not verify (25), i.e.

$$ s(t) + N > 2N - 3 \quad (29) $$

Putting together (28) and (29), we get

$$ N - 2 \leq s(t) \leq N $$

which concludes the proof of the 1st case.

The 2nd case is clear: since $s(t) + 2N > 2N - 3$, $s(t) + N$ is the only other integer $s$ in $[1, 2N - 3]$ which can appear in (24); the corresponding $m$ is $m(t) + q$.

Lemma 1 told us which $s$’s and $m$ occur for crossing points $(t, y)$ above a crossing value $t$: we now find the $k, l$’s such that $s = k + l$ and derive the $\sigma_i^\pm$’s corresponding to the $(t, y)$’s.

**Definition 2.** Let $P = (t, y)$ be a crossing point; we denote by $i(P) \in \{1, ..., N - 1\}$ the corresponding generator subscript, i.e. $P$ is represented by $\sigma_{i(P)}$ or $\sigma_{i(P)}^{-1}$.

**Lemma 2.** Let $t$ be a crossing value of $B_{N,q,p}$.

1. The point $P = (t, y)$ is a crossing point of the braid shadow if and only if

$$ y = (-1)^{m(t)} \cos\left(\frac{qd}{N} \pi\right) \quad (30) $$

where $d$ is any integer in $[1, ..., N - 1]$ of the same parity as $s(t)$

2. To determine $i(P)$, we do the Euclidean division of $qd$ by $2N$

$$ qd = 2Nn + w \quad (31) $$

with $n \geq 0, -N < w < N$. 

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(a) If \( m(t) \) is even,
\[
i(P) = i\left(t, \cos\left(\frac{q\theta}{N} \pi\right)\right) = |w|
\] (32)

(b) If \( m(t) \) is odd,
\[
i(P) = i\left(t, -\cos\left(\frac{q\theta}{N} \pi\right)\right) = N - |w|
\] (33)

Proof. Proof of 1: we treat separately the two cases of Lemma 1.

1. 1st case: a single ordered pair \((m, s)\).
   Let \( k, l \) such that \( k + l = s(t) \) and assume that \( l < k \); note that \( \delta = k - l \) has the parity of \( s(t) \).
   
   (a) If \( s(t) = N - 1 \), the smallest possible value for \( l \) is 0 and \( k - l \) runs through all integers \( \delta \), \( 1 \leq \delta \leq N - 1 \) with the parity of \( s(t) = N - 1 \).

   (b) If \( s(t) = N - 2 \) (resp. \( s(t) = N \)), the smallest value for \( l \) is 0 (resp. 1) and \( \delta \) runs through the integers in \([1, N - 2]\) with the parity of \( N - 2 \) or \( N \); since \( N - 1 \) has parity opposite to \( N \) and \( N - 2 \), we can actually assume \( \delta \) in \([1, N - 1]\).

To derive (30), we plug (24) into
\[
y = \sin\left(\frac{2\pi}{N} q(t + k)\right)
\] (34)

2. 2nd case: two ordered pairs: \((m(t), s(t))\) and \((m(t) + q, s(t) + N)\).

   (a) We first consider the \( k, l \)'s such that \( l < k \) and \( k + l = s(t) \). As above, \( k - l \) runs through the integers \( \delta \) with the parity of \( s(t) \) and such that
\[
1 \leq \delta \leq s(t)
\] (35)

   (b) If \( k + l = s(t) + N \), we look at \( \delta = k - l \)'s with \( l < k \):
\[
l = s(t) + N - k \geq s(t) + N - (N - 1) = s(t) + 1
\]

   hence \( \delta = s(t) + N - 2l \leq s(t) + N - 2s(t) - 2 = N - s(t) - 2 \) and
\[ 1 \leq d \leq N - s(t) - 2 \] (36)

Moreover every integer in \( d \in [1, N - s(t) - 2] \) with the parity of \( s(t) + N \) is a legitimate \( d \), i.e. there exist \( k, l \) in \( \{1, ..., N - 1\} \) with \( d = k - l \) and \( k + l = s(t) + N \); for example \( N - s(t) - 2 = (N - 1) - (s(t) + 1) \).

Using (34), we derive the \( y \)-coordinate of the crossing point:

\[ y = (-1)^{m(t)}(-1)^q \cos(\frac{q \delta}{N}) = (-1)^{m(t)} \cos(\frac{q \tilde{d}}{N}) \] (37)

where \( \tilde{\delta} = N - \delta \); if \( d \) verifies (36), then

\[ s(t) + 2 \leq \tilde{\delta} \leq N - 1 \] (38)

Since \( d \) has the parity of \( s(t) + N \), \( \tilde{\delta} \) has the parity of \( s(t) \).

Putting together the intervals (35) and (38) concludes the proof of the 2nd case.

**Proof of 2.** We derive from (31) that

\[ \cos(\frac{\pi q \delta}{N}) = \cos(\frac{\pi w}{N}) = \cos(\frac{|w|}{N}) \]

so 2 (a) of Lemma 2 follows from the fact that the function \( \cos \) is decreasing on \((0, \pi)\):

\[ \cos \frac{\pi}{N} > \cos \frac{2\pi}{N} > \cdots > \cos \frac{(N - 1)\pi}{N} \]

and 2 (b) of Lemma 2 follows from

\[ -\cos(\frac{\pi q \delta}{N}) = -\cos(\frac{\pi w}{N}) = -\cos(\frac{|w|}{N}) = \cos(\frac{N - |w|}{N}) \].

To see how many crossing points \((t, y)\) occur above \( t \), i.e. how many values (30) takes for a given \( t \), we notice the following.

- If \( u, v \in \{1, ..., N - 1\} \) and \( \cos(\frac{\pi u}{N}) = \cos(\frac{\pi v}{N}) \), then \( u = v \).
- If \( u, v \in \{1, ..., N - 1\} \) and \( \cos(\frac{\pi u}{N}) = -\cos(\frac{\pi v}{N}) \), then \( u = N - v \).
So the sets \( \{ \cos(\pi \frac{u}{N}) / 1 \leq u \leq N - 1 \} \) and \( \{ - \cos(\pi \frac{u}{N}) / 1 \leq u \leq N - 1 \} \) are identical. We derive from Lemma 2

**Corollary 2.** Let \( t \) be a crossing value. The indices \( i(P) \)'s for the crossing points \( P \)'s above \( t \) are all the \( i \)'s in \( \{1,...,N-1\} \) and with

1. the parity of \( \delta \) and \( s(t) \) if \( m(t) \) is even
2. the parity of \( \delta + N \) and \( s(t) + N \) if \( m(t) \) is odd.

### 4.2.2 The sign of the crossing points

We now compute the sign of the crossing points described in Lemma 2.

**Lemma 3.** The sign of the crossing point corresponding to the \((t,y)\) appearing in (30) is given by

\[
\Sigma(t, y) = (-1)^{m(t)} \sigma \left( \frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N} \right) \sigma \left( q \frac{N - \tilde{d}}{N} \right) \sigma \left( \frac{p\tilde{d}}{N} \right)
\]

where \( \sigma(r) \) is the parity of the integer part:

\[
\sigma(r) = (-1)^{[r]}
\]

**Proof.** We recall (Lemma 1) that \( k + l = s(t) \) or \( k + l = s(t) + N \). If \( k + l = s(t) \) and \( \delta = k - l \), then (39) is just the formula (23) for the sign of a crossing point. So we assume that \( k + l = s(t) + N \): we have seen above that \( y \) is given by (37) with \( k - l = N - \tilde{d} \) and we write (23) for the sign of the crossing point

\[
(-1)^{m(t)+q} \sigma \left( q \left( \frac{N - \tilde{d}}{N} \right) \right) \sigma \left( \frac{m(t)}{q} \right) \sigma \left( \frac{p}{2q} + \frac{2p\phi}{N} \right)
\]

\[
= (-1)^{m(t)} \sigma \left( q \frac{\tilde{d}}{N} \right) \sigma \left( \frac{p\tilde{d}}{N} \right) \sigma \left( \frac{m(t)}{q} \right) \sigma \left( \frac{p}{2q} + \frac{2p\phi}{N} \right)
\]

\[\square\]

We write the last two factors of (39) in terms of \( i(P) \):
Lemma 4. We let $P = (t, (-1)^{m(t)} \cos \frac{\pi q}{N})$ be a crossing point of $B_{N,q,p}$ with $m(t), d$ and $w$ as in 2) of Lemma 2. Then

$$\sigma\left(\frac{qd}{N}\right)\sigma\left(\frac{p\varnothing}{N}\right) = \begin{cases} \sigma\left(\frac{Bp(P)}{N}\right) & \text{if } p \text{ is odd} \\ (-1)^{m(t)}\sigma\left(\frac{Bp(P)}{N}\right) & \text{if } p \text{ is even} \end{cases}$$

(41)

Proof. It follows from (31) that

$$\sigma\left(\frac{qd}{N}\right) = \sigma\left(\frac{w}{N}\right).$$

(42)

We recall (13), namely $2NA + Bq = 1$, hence $d = 2NA\varnothing + Bq\varnothing$; putting this together with the Euclidean division in (31), we have

$$\varnothing = 2NA\varnothing + 2NnB + Bw$$

(43)

Thus

$$\sigma\left(\frac{p\varnothing}{N}\right) = \sigma\left(\frac{pBw}{N}\right).$$

(44)

Now

$$\sigma\left(\frac{w}{N}\right)\sigma\left(\frac{pBw}{N}\right) = \sigma\left(\frac{|w|}{N}\right)\sigma\left(\frac{pB|w|}{N}\right) = \sigma\left(\frac{pB|w|}{N}\right)$$

(45)

- If $m(t)$ is even, then $i(P) = |w|$ and the Lemma is proved.
- If $m(t)$ is odd, we use the fact that $B$ is odd to write

$$\sigma\left(\frac{pB|w|}{N}\right) = \sigma\left(\frac{pB(N - i(P))|w|}{N}\right) = \begin{cases} \sigma\left(\frac{pB_i(P)}{N}\right) & \text{if } p \text{ is odd} \\ -\sigma\left(\frac{pB_i(P)}{N}\right) & \text{if } p \text{ is even} \end{cases}$$

It follows from Lemma 4 that the sign $\Sigma(P)$ given in (39) of a crossing point $P$ of crossing value $t$ is ($\epsilon$ has been defined in (14) above)

- $\sigma\left(\frac{p^{m(t)}q}{q} + \frac{j}{2q} + \frac{2p\phi}{N}\right)\epsilon(N, q, p)(i)$ if $p$ is even

- $\sigma\left(\frac{p^{m(t)}q}{q} + \frac{j}{2q} + \frac{2p\phi}{N}\right)(-1)^{m(t)}\epsilon(N, q, p)(i)$ if $p$ is odd.

We recall (see the formulae (15) in the Main Theorem) that the $\pm 1$-exponent of a $\sigma_i$ in $\alpha_{N,q,p}$ or $\beta_{N,q,p}$ is $\epsilon(N, q, p)(i)$, hence
Lemma 5. The $\pm 1$-exponent of the $\alpha_{N,q,p}$ or $\beta_{N,q,p}$ corresponding to a crossing value $t$ in $B_{N,q,p}$ is

1. $\sigma\left(p \frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right)$ if $p$ is even

2. $(-1)^{m(k)}\sigma\left(p \frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right)$ if $p$ is odd.

The formulae in Lemma 5 depend on $m(t)$, where $t$ goes through the $2q$ crossing values. If $t$ is the $h$-th crossing value, for $h = 1, \ldots, 2q$, we want to have $m(t)$ directly as an expression in $h$ so we number the crossing values

$$t_1 < t_2 < \ldots < t_q < \ldots < t_{2q} \quad (46)$$

and for any integer $k$, with $1 \leq k \leq 2q$, we let

$$m(k) = m(t_k) \quad s(k) = s(t_k) \quad (47)$$

If $t_k$ and $t_{k+1}$ are two consecutive crossing values, we derive from (24)

$$t_{k+1} - t_k = \frac{1}{2q}\left[q(s(k) - s(k + 1)) + N(m(k + 1) - m(k))\right] \geq \frac{1}{2q} \quad (48)$$

Since there are $2q$ crossing values in $[\eta, 1+\eta]$, (48) is an equality and we have

$$q(s(k) - s(k + 1)) + N(m(k + 1) - m(k)) = 1 \quad (49)$$

We note in passing that, if we plug (49) into Proposition 2, we get the confirmation of the obvious fact

Lemma 6. The crossing points above $t_k$ and $t_{k+1}$ are represented by $\sigma_i^{\pm 1}$’s with $i$’s of opposite parities.

We now confront (49) with $2NA + Bq = 1$ and derive the existence of an integer $\nu_k$ such that

$$m(k + 1) - m(k) = \nu_k q + 2A.$$ 

Thus, for any $k$, there exists an integer $a_k$ such that

$$m(k) = m(1) + a_k q + 2(k - 1)A \quad (50)$$
\((-1)^{m(k)} = (-1)^{m(1)}(-1)^{a_k} \) \hspace{1cm} (51)

Define \( \phi_0 \) by

\[
q \frac{m(1)}{q} - \frac{2Ap}{q} + \frac{p}{2q} + \frac{2p \phi_0}{N} = 0
\] \hspace{1cm} (52)

We see (23) that \( \phi_0 \) is a critical phase, i.e. a phase for which the knot \( K(N, p, q, \phi) \) is singular. So we pick a phase

\[
\phi = \phi_0 + \xi
\] \hspace{1cm} (53)

where \( \xi \) is a very small positive number. Using (50), we rewrite

\[
\sigma \left( \frac{p m(t)}{q} + \frac{p}{2q} + \frac{2p \phi}{N} \right) = (-1)^{a_p} (-1) \frac{2Ap k + \xi}{q}
\] \hspace{1cm} (54)

It follows from Lemma 5 and equations (50), (54) that the exponent of the \( \alpha_{N,q,p} \) or \( \beta_{N,q,p} \) at the \( k \)-th crossing value is

\[
\begin{cases}
(-1)^{m(1)}(-1) \frac{2Ap k + \xi}{q} & \text{if } p \text{ is odd} \\
(-1) \frac{2Ap k + \xi}{q} & \text{if } p \text{ is even}
\end{cases}
\]

Since we are working up to mirror transformation, we assume

\[
(-1)^{m(1)} = 1.
\]

We now conclude: the expression \( \frac{2Ap k + \xi}{q} \) is equal to 1 for \( k = q, 2q \) and equal to \( \lambda(k) \) for the other \( k \)'s. Going back to the statement of the Main Theorem, this gives us the exponent for the \( k \)-th crossing values with \( k \leq q \) or \( k = 2q \). We settle the case of the \( k \)'s with \( q < k < 2q \) by noticing that

\[
\lambda(2q - k) = -\lambda(k).
\]

\( \square \)

5 The four-genus

5.1 Ribbon knots

A ribbon knot in \( S^3 \) bounds a disk in \( S^3 \) with only ribbon singularities; equivalently it bounds an embedded disk in \( B^4 \) with does not have local maxima for the distance to the origin of \( B^4 \). Thus a ribbon knot is slice, i.e. its 4-genus is zero; the long-standing Slice-Ribbon conjecture asks if the converse is true.
5.1.1 Proof of Theorem 1

Th. 1 has been stated by Lamm and also follows from his more general construction of ribbon symmetric unions ([La 2], [K-T]). His proof is fairly allusive so we felt it would be useful to give a more detailed proof.

We recall the well-known fact:

**Proposition 5.** Let $K$ be a knot in $\mathbb{R}^3$ which is symmetric with respect to a plane $P$ in $\mathbb{R}^3$; then it is ribbon.

**Proof.** We endow $\mathbb{R}^3$ with the frame $Oxyz$ and assume that $P$ is defined by the equation $x = 0$. By genericity arguments, we assume

1. $K$ meets $P$ at a finite number of points
2. outside of $P$, $K$ is never tangent to the direction of $Ox$.

Since $K$ has one component, it meets $P$ at exactly two points.

We let $K_+$ (resp. $K_-$) be the intersection of $K$ with the half-space of $\mathbb{R}^3$ defined by $z \geq 0$ (resp. $z \leq 0$): $K_+$ and $K_-$ are both diffeomorphic to a closed interval.

Letting $S$ be the symmetry in $\mathbb{R}^3$ with respect to $P$, we let

$$\Phi = [0, 1] \times K_+ \longrightarrow \mathbb{R}^3$$

$$(t, X) \mapsto tX + (1 - t)S(X)$$ (55)

The self-intersections of $\Phi$ are given by the data of $t_1, t_2, X_1, X_2$ such that

$$t_1X_1 + (1 - t_1)S(X_1) = t_2X_2 + (1 - t_2)S(X_2)$$ (56)

We denote by $(x_i, y_i, z_i), i = 1, 2$ the coordinates of $X_i$. Since $S(x_i, y_i, z_i) = (-x_i, y_i, z_i)$, (56) implies that

$$y_1 = y_2, z_1 = z_2.$$

Thus the line segments $I_1 = X_1S(X_1)$ and $I_2 = X_2S(X_2)$ are both included in the line which is defined by the equations $y = y_1, z = z_1$. Moreover, one of them is included in the other one and we have a ribbon singularity. 

\[\square\]
The Main Theorem tells us that, if \( p \) and \( q \) are mutually prime, with \( q \) odd, \( B_{N,q,p} \) is as in Fig. 8.

If \( N \) is the number of strands, there are \( N - 1 \) half-twist tangles connecting \( Q \) and \( Q^{-1} \); we replace them by \( N - 1 \) tangles and get the \( N \)-component link \( L \) of Fig. 8 which is symmetric w.r.t. a plane.

Proposition 5 tells us that \( L \) bounds \( N \) ribbon disks \( D_1, \ldots, D_N \); and the same arguments show us that two of these disks only have ribbon-type intersection.

We now connect each \( D_i \) to \( D_{i+1} \) by a half-twisted band bounded by the half-twist tangle of Fig. 8. The resulting surface is a topological disk with only ribbon singularities.

5.2 General case: proof of Theorem 2

We use an idea by Brandenbursky and Kedra ([B-K]). If \( b \) is a \( N \)-braid, we denote by \( \hat{b} \) the link obtained by closing the braid \( b \). If \( b_1 \) and \( b_2 \) are \( N \)-two braids, [B-K] constructed a cobordism of Euler characteristic \( -N \) between the closure of the product \( \hat{b}_1 \hat{b}_2 \) and the disjoint union of the closures \( \hat{b}_1 \sqcup \hat{b}_2 \).

Letting \( \tilde{q} = \frac{q}{N} \), \( \tilde{p} = \frac{p}{N} \), we recall that

\[
B_{N,q,p} = B_{N,\tilde{q},\tilde{p}}^d.
\]

Applying [B-K]’s result \( d \) times, we derive a cobordism in \( \mathbb{B}^4 \) of Euler characteristic \( -N(d - 1) \) between \( \hat{B}_{N,\tilde{q},\tilde{p}} \) and \( \hat{B}_{N,\tilde{q},\tilde{p}} \sqcup \hat{B}_{N,\tilde{q},\tilde{p}} \sqcup \cdots \hat{B}_{N,\tilde{q},\tilde{p}} \). Since \( \hat{B}_{N,\tilde{q},\tilde{p}} \)
is a ribbon knot (Theorem 1), it bounds an embedded disk in $\mathbb{B}^4$. Thus $\hat{B}_{N,q,p}$ bounds a surface of Euler characteristic

$$-d(N-1) + d = 1 - (d-1)(N-1).$$

We recover the formula (9) for the genus and Theorem 2 is proved.

5.3 Quasipositive knots: proof of Proposition 2

Lee Rudolph (see [Ru] for details) defines a braid $\gamma \in B_N$ to be quasipositive if it is a product of conjugates $w\sigma_i w^{-1}$ of positive braid generators, i.e.

$$\gamma = w_1 \sigma_{i_1} w_1^{-1} w_2 \sigma_{i_2} w_2^{-1} \ldots w_k \sigma_{i_k} w_k^{-1}$$

(57)

Theorem 3. ([Ru]) If $\gamma$ is a closed quasipositive braid written as in (57) closing in a knot $\hat{\gamma}$, its four-genus verifies

$$1 - 2g_4(\hat{\gamma}) = n - k.$$

It is easy to check that under the assumptions of Proposition 2 the exponents of all the $\sigma_{2k}$’s and $\sigma_{2k+1}$’s appearing respectively in $\alpha_{N,q,p}$ and $\beta_{N,q,p}$ are all of the same sign. Since we are working up to mirror symmetry, we can assume all these exponents to be equal to 1; thus

$$\alpha_{N,q,p} = \prod_{2 \leq 2k \leq N-1} \sigma_{2k} \quad \beta_{N,q,p} = \prod_{1 \leq 2k+1 \leq N-1} \sigma_{2k+1}$$

(58)

Hence $B_{N,p,q} = (Q \alpha_{N,q,p} Q^{-1} \beta_{N,q,p})^d$ is a quasipositive braid; Theorem 3 tells us that $1 - 2g_4(K(N, q, p)) = N - d(N-1)$ and Proposition 2 follows.

6 Trivial knots: proof of Proposition 4

6.1 The knot $K(N, q, q+N)$ is trivial

We set

$$A = \prod_{1 \leq 2k \leq N} \sigma_{2k} \quad B = \prod_{1 \leq 2k+1 \leq N} \sigma_{2k+1}$$

(59)

Lemma 7.

$$B_{N,q,q+N} = A(BA)^{q-1} (B^{-1} A^{-1})^{q-1} B^{-1}$$

(60)
The main thing to note about this formula is that all the positive generators are on one side and all the negative generators are on the other side.

\textbf{Proof.} We use the Main Theorem. It assumes that \( q \) is odd but in the present case, if \( q \) is even, \( N \) has to be odd, hence \( q + N \) is odd and we switch \( q \) and \( q + N \) to apply the theorem. We compute

\[ \epsilon(i) = (-1)^i \quad \lambda(k) = (-1)^k \]  

Thus \( \alpha_{N,q,q+N} = \prod_{1 \leq 2k \leq N} \sigma_{2k} = A \) and \( \beta_{N,q,q+N} = \prod_{1 \leq 2k+1 \leq N} \sigma_{2k+1}^{-1} = B^{-1} \).

We conclude by noticing that

\[ (AB)^{\frac{q+1}{2}} A = A(BA)^{\frac{q+1}{2}} \]

\[ \square \]

We construct a trivial pure braid \( B_N \); we will show that \( B_{N,N+q,q} \) is the product of a power of \( B_N \) and of a piece of \( B_N \).

If \( N = 2k \) is even, we let

\[ B_N = (BA)^k(B^{-1}A^{-1})^k. \]  

(62)

If \( N = 2k + 1 \) is odd, we let

\[ B_N = (BA)^kBA^{-1}(B^{-1}A^{-1})^k. \]  

(63)

In both cases, we check that the corresponding permutation between the endpoints of the braid is the identity, thus \( B_N \) is a pure braid.

To prove that it is a trivial braid, we discuss when one strand of \( B_N \) is above another one; so let us fix some terminology.

We number the strands of \( B_N \): the \( j \)-th strand, \( 0 \leq j \leq N - 1 \) is the strand starting at the \((j + 1)\)-th point on the left (the points being counted from top to bottom).

We say that the \( j \)-th strand is above the \( k \)-th strand if, wherever there is a crossing point between these two strands, the \( j \)-th strand is above the \( k \)-th strand. As an exemple, in Fig. 9, the red strand is above all the other strands.
Lemma 8. If \( j, k \) are two integers with \( 0 \leq j < k \leq N - 1 \), the \( j \)-th strand of \( B_N \) is above the \( k \)-th strand. Thus \( B_N \) closes in \( N \) unlinked trivial links, i.e. \( B_N = 1 \).

The figure 9 illustrates the lemma.

Proof. We describe the strands of \( B_N \) in the braid shadow, i.e. their projection to the \( xy \). We endow the plane with a coordinate \( Oxy \) such that the \( j \)-th strand starts at \((0, -j)\) and ends at \((2N, -j)\). The upper left point has coordinates \((0, 0)\) (in Figure 9 it is the starting point of the red strand).

We say that a strand is ascending, denoted \( \uparrow \) (resp. descending, denoted \( \downarrow \)) if it has a +1 (resp. −1) slope. It is horizontal, denoted \( \rightarrow \), when the slope is 0.

We describe here the \( k \)-strands for \( k \) odd (the case of an even \( k \) is similar): it goes up and down as follows

1. \( \downarrow \) from \((0, -k)\) to \((N - 1 - k, -(N - 1))\)
2. \( \rightarrow \) from \((N - 1 - k, -(N - 1))\) to \((N - k, -(N - 1))\)
3. \( \uparrow \) from \((N - k, -(N - 1))\) to \((2N - 1 - k, 0)\)
4. \( \rightarrow \) from \((2N - 1 - k, 0)\) to \((2N - k, 0)\)
5. \( \downarrow \) from \((2N - k, 0)\) to \((2N, -k)\)

Assume now that the \( k \)-th strand is above the \( j \)-th strand at a crossing point \((x, y)\). Assuming that \( j \) is odd (the even case is similar), one of the following two cases occurs.
1. \(0 \leq x \leq N\) and the \(k\)-th (resp. \(j\)-th) strand is \(\searrow\) (resp. \(\nearrow\)). Then \(S_k\) is as 1. above and \(S_j\) is as 3.

2. \(N \leq x \leq 2N\) and the \(k\)-th (resp. \(j\)-th) strand \(\nearrow\) (resp. \(\searrow\)). Then \(S_k\) is as 3. above and \(S_j\) is as 5.

In both cases it is easy to check that \(k < j\).

We conclude the proof of the proposition in the case when \(N\) is even; the odd case is similar. We derive from Lemma 8 that for an \(n > k\),

\[
A(BA)^n(B^{-1}A^{-1})^nB = A(BA)^{n-k}(BA)^k(B^{-1}A^{-1})^k(B^{-1}A^{-1})^{n-k}B
= A(BA)^{n-k}(B^{-1}A^{-1})^{n-k}B.
\]

Thus, if \(b\) is the remainder of the division of \(\frac{q-1}{2}\) by \(k\), we have

\[
B_{N,q,q+N} = A(BA)^b(B^{-1}A^{-1})^bB \quad (64)
\]

The braid (64) is a piece of the braid \((BA)^k(B^{-1}A^{-1})^k\) where the \(i\)-th strand is above the \(j\)-th strands, for \(j > i\). Thus the same is true for (64) which closes therefore in a trivial knot.

6.2 The knot \(K(N,1,p)\) is trivial

This follows from the Main Theorem. We can also prove it directly by computing the crossing points and their sign: we see that every \(\sigma_i^\pm\) appears once and only once in the braid \(B_{N,1,p}\) and so braid represents a trivial knot.

6.3 The other knots of Proposition 4

We have now seen two cases where \(K(N,q,p)\) is trivial. We know that \(K(N,q,p)\) and \(K(N,p,q)\) are isotopic; and \(K(N,q,k)\) and \(K(N,q,2qN + k)\) (resp. \(K(N,q,2qN - k)\)) are isotopic (resp. mirror image of one another). Thus we can get more examples of trivial knots, e.g. \(K(3,5,29)\) and \(K(3,5,31)\).
7 Lists of Jones polynomials

Jones polynomial for $q = 11$ and

$p = 1 : 1$
$p = 2 : 1$
$p = 3 : \frac{1}{t^3}, \frac{1}{t^2} - t - t^2 - t^3$
$p = 4 : 1$
$p = 5 : 1$
$p = 7 : 21 + \frac{1}{t^6} - \frac{1}{t^5} + \frac{2}{t^4} - \frac{3}{t^3} + \frac{4}{t^2} - \frac{5}{t} - 14 t - 8 t^2 - 4 t^3 - t^4$
$p = 8 : 1$
$p = 10 : 7 - \frac{1}{t^7} + \frac{1}{t^6} + \frac{2}{t^5} - \frac{3}{t^4} + \frac{4}{t^3} - \frac{5}{t^2} - 5 t - 5 t^2 - 4 t^3 - 3 t^4 - 2 t^5 - t^6 - t^7$
$p = 11 : t^{10} - t^{12} - t^{22}$
$p = 13 : 32 + \frac{1}{t^{10}} - \frac{2}{t^9} - \frac{5}{t^8} + \frac{10}{t^7} + \frac{16}{t^6} + \frac{24}{t^5} + \frac{31}{t^4} + \frac{36}{t^3} + \frac{36}{t^2} - 24 t - 16 t^2 - 10 t^3 + 5 t^4 - 2 t^5 - t^6$
$p = 14 : 1$
$p = 16 : 193 - \frac{1}{t^{10}} + \frac{5}{t^9} - \frac{14}{t^8} + \frac{31}{t^7} + \frac{56}{t^6} - \frac{89}{t^5} + \frac{126}{t^4} + \frac{159}{t^3} - \frac{183}{t^2} - 183 t - 159 t^2 - 126 t^3 - 89 t^4 - 56 t^5 - 31 t^6 - 14 t^7 - 5 t^8 - t^9$
$p = 17 : 1$
$p = 19 : 825 + \frac{1}{t^{12}} - \frac{8}{t^{11}} + \frac{30}{t^{10}} - \frac{78}{t^9} + \frac{165}{t^8} - \frac{296}{t^7} + \frac{465}{t^6} - \frac{652}{t^5} + \frac{824}{t^4} - \frac{946}{t^3} - \frac{990}{t^2} + \frac{946}{t} - 652 t - 465 t^2 - 296 t^3 - 165 t^4 - 78 t^5 - 30 t^6 - 8 t^7 + t^8$
$p = 20 : 1$
$p = 22 : 4863 + \frac{1}{t^{11}} - \frac{11}{t^{10}} + \frac{55}{t^9} + \frac{176}{t^8} + \frac{429}{t^7} + \frac{869}{t^6} + \frac{1518}{t^5} + \frac{2343}{t^4} + \frac{3245}{t^3} + \frac{4070}{t^2} + 4652 t - 4070 t^2 - 3245 t^3 - 2343 t^4 - 1518 t^5 - 869 t^6 - 429 t^7 - 176 t^8 - 55 t^9 - 11 t^{10} + t^{11}$
$p = 23 : 1$

$p = 25 : 825 + \frac{1}{t^{12}} - \frac{8}{t^{11}} + \frac{30}{t^{10}} - \frac{78}{t^9} + \frac{165}{t^8} - \frac{296}{t^7} + \frac{465}{t^6} - \frac{652}{t^5} + \frac{824}{t^4} - \frac{946}{t^3} - \frac{990}{t^2} + \frac{946}{t} - 652 t - 465 t^2 - 296 t^3 - 165 t^4 - 78 t^5 - 30 t^6 - 8 t^7 + t^8$
$p = 26 : 3 - \frac{1}{t^3} + \frac{1}{t^2} - t - t^2 - t^3$
$p = 28 : 193 - \frac{1}{t^{10}} + \frac{5}{t^9} - \frac{14}{t^8} + \frac{31}{t^7} + \frac{56}{t^6} - \frac{89}{t^5} + \frac{126}{t^4} + \frac{159}{t^3} - \frac{183}{t^2} - 183 t - 159 t^2 - 126 t^3 - 89 t^4 - 56 t^5 - 31 t^6 - 14 t^7 - 5 t^8 - t^9$
$p = 29 : 21 - \frac{1}{t^4} + \frac{4}{t^3} - \frac{8}{t^2} + \frac{14}{t} - 24 t - 26 t^2 - 24 t^3 - 20 t^4 - 14 t^5 - 8 t^6 - 4 t^7 - t^8$
$p = 31 : 32 - \frac{1}{t^3} + \frac{2}{t^2} + \frac{5}{t} + 10 t + 16 t^2 + 24 t^3 + 31 t^4 + 36 t^5 + 36 t^6 - 24 t^7 - 16 t^8 - 10 t^9 + 5 t^{10} - 2 t^{11} - t^{12}$
$p = 32 : 7 - \frac{1}{t^3} + \frac{2}{t^2} + \frac{3}{t} + \frac{4}{t^2} - \frac{5}{t} - 5 t - 5 t^2 - 4 t^3 - 3 t^4 - 2 t^5 - t^6 - t^7$

Figure 10: List of Jones polynomials of knots $K(3, 11, p)$
Jones polynomial for \( q = 11 \) and

\[
p = 1: 1
\]

\[
p = 3: 1 - 2 t + 2 t^2 - 2 t^3 + t^4 - t^5
\]

\[
p = 5: 1 - 1 + 5 - 12 + 18 - 22 + 23 - 20 + 15 - 8 + 2 t - 3 t^2 - t^3
\]

\[
p = 7: 1
\]

\[
p = 9: 166 - 6 + 23 - 30 + 78 - 72 + 27 - 58 + 126 + 48 t^2 - 331 t^3 - 346 t^4 - 307 t^5 - 228 t^6 - 138 t^7 - 66 t^8 - 23 t^9 - 6 t^{10}
\]

\[
p = 11: t^{14} - t^{15} + t^{16} - t^{17} + t^{18} - t^{19} + t^{20} - t^{21} + t^{22} - t^{23} + t^{24}
\]

\[
p = 13: -317 - 1 + 7 + 31 + 102 + 272 + 598 + 1040 + 1566 + 1952 + 2108 + 1938 + 1474 + 832 + 102 + 786 + 370 - 576 t^2 - 436 t^3 - 260 t^4 - 192 t^5 + 31 t^6 - 7 t^7 - t^8
\]

\[
p = 15: 1
\]

\[
p = 17: 924 + 1 - 8 + 34 + 106 + 256 + 690 + 763 + 943 + 949 + 616 + 45 + 28 + 3 t - 17 + 2 t^2 + 3 t^3 - 182 t + 2513 t^3 - 2839 t^4 - 2744 t^5 - 30 t^6 + 2 t^7 - 3 t^8 + 4 t^9 - 3 t^{10}
\]

\[
p = 19: 2355 + 1 - 2 + t + 2 t^2 - 2 t^3 - t^4
\]

\[
p = 21: -1784 - 1 + 8 + 42 + 141 + 305 + 896 + 1003 + 3154 + 4829 + 6873 + 7605 + 7792 + 6845 + 4956 + 2449
\]

\[
\quad + 2719 t + 2779 t^2 + 2244 t^3 + 1491 t^4 + 824 t^5 + 377 t^6 + 141 t^7 - 42 t^8 + 9 t^9 - t^{10}
\]

\[
p = 23: 1
\]

\[
p = 25: 7200 + 1 - 10 + 53 + 199 + 591 + 1450 + 2981 + 5144 + 7542 + 9109 + 8943 + 6066 + 324 - 1696 t - 21101 t^2 - 2428 t^3 - 25994 t^4 - 20751 t^5 - 15816 t^6 - 10638 t^7 - 4308 t^8 - 3289 t^9 - 1552 t^{10} - 595 t^{11} - 199 t^{12} - 53 t^{13} - 16 t^{14} + t^{15}
\]

\[
p = 27: 1 + 3 + 3 + 2 - 8 t - 15 t^2 - 20 t^3 - 23 t^4 - 12 t^5 - 10 t^6 - 12 t^7 - t^8 - t^9
\]

\[
p = 29: -19300 - 1 + 11 + 65 + 269 + 974 + 2737 + 5578 + 11540 + 21180 + 24536 + 50381 + 64383 + 73381
\]

\[
\quad + 28232 t + 28251 t^2 + 24438 t^3 + 17168 t^4 + 10334 t^5 + 5318 t^6 + 2339 t^7 - 872 t^8 - 269 t^9 - 65 t^{10} + 11 t^{11} - t^{12}
\]

\[
p = 31: 166 + 1 - 6 + 23 + 66 + 138 + 228 + 307 + 346 + 331 + 264
\]

\[
\quad + 58 - 27 t^2 - 72 t^3 - 74 t^4 - 50 t^5 - 22 t^6 - 6 t^7 - t^8
\]

\[
p = 33: -48510 - 1 + 21 + 209 + 1287 + 5866 + 18313 - 107635 t - 205908 t^2 + 347017 t^3 - 527775 t^4 - 777275 t^5 - 998424 t^6 - 1041146 t^7 - 1118705 t^8 - 1120034 t^9 + 1051138 t^{10} - 925980 t^{11} - 76078 t^{12} - 63374 t^{13} - 447986 t^{14} + 124944 t^{15} - 259022 t^{16} - 130558 t^{17} - 76168 t^{18} - 41174 t^{19} - 24666 t^{20} - 9142 t^{21} - 3631 t^{22} - 1224 t^{23} - 353 t^{24} - 78 t^{25} + 12 t^{26} + t^{27}
\]

\[
p = 35: -317 + 1 - 2 + 31 + 102 + 240 + 456 + 574 + 570 + 182 t - 832 t^2 - 1474 t^3 - 1938 t^4 - 2108 t^5 - 1952 t^6 - 1546 t^7 - 1046 t^8 + 584 t^9 - 272 t^{10} - 102 t^{11} - 31 t^{12} - t^{13} + t^{14}
\]

\[
p = 37: -19300 + 1 - 11 + 65 + 269 + 974 + 2737 + 5578 + 11540 + 21180 + 24536 + 50381 + 64383 + 73381
\]

\[
\quad - 28232 t - 28251 t^2 - 24438 t^3 - 17168 t^4 - 10334 t^5 + 5318 t^6 + 2339 t^7 + 872 t^8 - 269 t^9 + 65 t^{10} - 11 t^{11} - t^{12}
\]

\[
p = 39: -19300 + 1 - 8 + 34 + 106 + 272 + 506 + 1071 + 1667 + 2305 + 2744 + 2829 + 2513 + 1820 - 45 t + 416 t^2 - 949 t^3 + 743 t^4 + 685 t^5 + 480 t^6 + 256 t^7 - 106 t^8 - 34 t^9 - t^{10} + t^{11} + t^{12}
\]

\[
p = 41: -924 + 1 - 10 + 53 + 199 + 591 + 1450 + 2981 + 5144 + 7542 + 9109 + 8943 + 6066 + 324 - 1696 t - 21101 t^2 - 2428 t^3 - 25994 t^4 - 20751 t^5 - 15816 t^6 - 10638 t^7 - 4308 t^8 - 3289 t^9 - 1552 t^{10} - 595 t^{11} - 199 t^{12} - 53 t^{13} - 16 t^{14} + t^{15}
\]

\[
p = 43: -1784 - 1 + 9 + 42 + 141 + 377 + 824 + 1491 + 2244 + 2779 + 2719 - 38 t - 2449 t^2 - 4916 t^3 - 6843 t^4 - 7792 t^5 - 7605 t^6 - 6471 t^7 - 4829 t^8 - 3156 t^9 - 1803 t^{10} - 896 t^{11} - 385 t^{12} - 141 t^{13} - 42 t^{14} - 9 t^{15} - t^{16}
\]

Figure 11: List of Jones polynomials of knots \( K(4,11,p) \)
8 Appendix

We give a better proof of the following fact from [S-V]:

**Proposition 6.** Let $\phi_1$ and $\phi_2$ two real numbers. The knots $K(N, q, p, \phi_1)$ and $K(N, q, p, \phi_2)$ defined in [1] are either isotopic or mirror image of one another.

**Proof.** Without loss of generality, we assume $\phi_1 < \phi_2$.

If there is no critical phase (i.e. a phase for which the knot is singular) between $\phi_1$ and $\phi_2$, the two knots are isotopic.

In [S-V] we showed that the difference between two critical phases is of the form

$$\frac{N}{2}(\frac{m}{p} + \frac{n}{q})$$

for two integers $m, n$.

Thus it is enough to prove that, for a given $\phi_3$, and integers $m$ and $n$, the knots $K(N, q, p, \phi_3)$ and $K(N, q, p, \phi_3 + \frac{N}{2}(\frac{m}{p} + \frac{n}{q}))$ are the same or mirror images of one another.

Consider the parametrization of $K(N, q, p, \phi)$ given in [19]; we change its variable by setting

$$s = t + \frac{Nn}{2q}$$

and we rewrite the expression in (19)

$$\left( \sin \frac{2\pi q}{N} (t + k), \cos \frac{2\pi p}{N} (t + k + \phi_3 + \frac{N}{2}(\frac{m}{p} + \frac{n}{q})) \right)$$

$$= \left( (-1)^n \sin \frac{2\pi q}{N} (s + k), (-1)^m \cos \frac{2\pi p}{N} (s + k + \phi_3) \right)$$

Thus, if $m$ and $n$ have the same (resp. opposite) parities, the two knots are isotopic (resp. mirror image of one another). 


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