ESSENTIAL SETS FOR RANDOM OPERATORS
CONSTRUCTED FROM ARRATIA FLOW

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Abstract. In this paper we consider a strong random operator $T_t$ which describes shift of functions from $L_2(\mathbb{R})$ along an Arratia flow. We find a compact set in $L_2(\mathbb{R})$ that doesn’t disappear under $T_t$, and estimate its Kolmogorov widths.

1. Introduction. Arratia flow and random operators

In this paper we consider random operators in $L_2(\mathbb{R})$ which describe shifts of functions along an Arratia flow [1]. Let’s recall the definition.

Definition 1.1 ([1]). A family of random processes $\{x(u, s), u \in \mathbb{R}, s \geq 0\}$ is called an Arratia flow if

1) for each $u \in \mathbb{R}$ $x(u, \cdot)$ is a Wiener process with respect to the joint filtration such that $x(u, 0) = u$;
2) for any $u_1 \leq u_2$ and $t \geq 0$

$$x(u_1, t) \leq x(u_2, t) \text{ a.s.}$$

3) the joint characteristics are

$$d < x(u_1, \cdot), x(u_2, \cdot) > (t) = \Pi_{x(u_1, t) = x(u_2, t)} dt.$$

In the informal language, Arratia flow is a family of Wiener processes started from each point of $\mathbb{R}$, which move independently up to the meeting, coalesce, and move together. It was proved in [2, 3] that for any $a, b \in \mathbb{R}$ and $t > 0$ the set $x([a; b], t)$ is finite a.s. Since Arratia flow has a right-continuous modification [4], $x(\cdot, t) : \mathbb{R} \to \mathbb{R}$ is a step function for any time $t > 0$. Hence, for any $a, b \in \mathbb{R}$ and $t > 0$ with probability one there exists a random point $y \in \mathbb{R}$ for which

$$\lambda\{u \in [a; b] : x(u, t) = y\} > 0,$$

where $\lambda$ is Lebesgue measure on $\mathbb{R}$. Since $x(\cdot, t)$ is right-continuous step function then for a fixed countable set $A$

$$P\{x(\mathbb{R}, t) \cap A \neq \emptyset\} = P\{x(\mathbb{Q}, t) \cap A \neq \emptyset\} \leq \sum_{u \in \mathbb{Q}} P\{x(u, t) \in A\} = 0. \quad (1.2)$$

Since for any $a < b$ the difference $\frac{x(b, \cdot) - x(a, \cdot)}{\sqrt{2}}$ is a Wiener processes until the collision happens, and $\frac{x(b, 0) - x(a, 0)}{\sqrt{2}} = \frac{b-a}{\sqrt{2}}$, then one can find the distribution

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of the time of coalescence \( \tau_{a,b} = \inf \{ s \geq 0 \mid x(a,s) = x(b,s) \} \) of the processes \( x(a,\cdot), x(b,\cdot) \), i.e. for any \( t \geq 0 \)

\[
P\{ \tau_{a,b} \leq t \} = P\{ x(a,t) = x(b,t) \} = \sqrt{\frac{2}{\pi}} \int_{\frac{b-a}{\sqrt{2t}}}^{+\infty} e^{-\frac{u^2}{2t}} dv. \quad (1.3)
\]

Let’s notice that for a fixed time \( t > 0 \) and an Arratia flow \( X = \{ x(u,s), u \in \mathbb{R}, s \in [0;t] \} \) there exists an Arratia flow \( Y = \{ y(u,r), u \in \mathbb{R}, r \in [0;t] \} \) such that trajectories of \( X \) and \( Y = \{ y(u,t-r), u \in \mathbb{R}, r \in [0;t] \} \) don’t cross \([1,5]\). \( Y \) is called a conjugated (or dual) Arratia flow. It was proved in \([6]\) the following change of variable formula for an Arratia flow.

**Theorem 1.2** ([6]). For any time \( t > 0 \) and nonnegative measurable function \( h : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \int_{\mathbb{R}} h(u)du < +\infty \)

\[
\int_{\mathbb{R}} h(x(u,t))du = \int_{\mathbb{R}} h(u)dy(u,t) \quad a.s., \quad (1.4)
\]

where the last integral is in sense of Lebesgue-Stieltjes.

In this paper we consider random operators \( T_t, t > 0 \), in \( L_2(\mathbb{R}) \) which are defined as follows

\[(T_tf)(u) = f(x(u,t)),\]

where \( f \in L_2(\mathbb{R}) \) and \( u \in \mathbb{R} \). It was proved in \([7]\) that \( T_t \) is a strong random operator \([8]\) in \( L_2(\mathbb{R}) \), but, as it was shown in \([6]\), is not a bounded one. Really, for the point \( y \) from (1.1) one can introduce a sequence of the intervals \( A_i = [r_i; p_i] \) such that \( y \in A_i \) for any \( i \geq 1 \) and \( p_i - r_i \rightarrow 0, i \rightarrow \infty \). Thus, for any \( i \geq 1 \)

\[
\| T_t 1_{A_i} \|_{L_2(\mathbb{R})}^2 \geq \lambda \{ u \in [a;b] : x(u,t) = y \} > 0,
\]

which can’t be true if \( T_t \) was a bounded random operator. Hence, the image of a compact set under \( T_t \) may not be a random compact set. Moreover, as it was mentioned in \([9]\), image of a compact set under strong random operator may not exist. However, in \([6]\) it was presented a family of compact sets in \( L_2(\mathbb{R}) \) which images under \( T_t \) exists and are random compact sets. In this paper we consider a compact set of this type, and investigate the change of its Kolmogorov widths \([10]\) under \( T_t \).

2. \( T_t \)-essential functions

If support of function \( f \in L_2(\mathbb{R}) \) is bounded, \( \text{supp } f \subset [a;b] \), then \( T_tf \) equals to 0 with positive probability. Really, by (1.4), one can check that

\[
P \left\{ \int_{-\infty}^{+\infty} f^2(x(u,t))du = 0 \right\} \geq P \{ x(\mathbb{R},t) \cap [a;b] = \emptyset \} =
\]

\[
= P \left\{ \int_{-\infty}^{+\infty} 1_{[a;b]}(x(u,t))du = 0 \right\} = P \left\{ \int_{-\infty}^{+\infty} 1_{[a;b]}(u)dy(u,t) = 0 \right\},
\]
Thus, by (1.3),
\[ P \left\{ \int_{-\infty}^{+\infty} \mathbb{I}_{[a,b]}(u) dy(u,t) = 0 \right\} = P \left\{ y(b,t) = y(a,t) \right\} > 0, \]
then \( P \left\{ \| T_t f \|_{L_2(\mathbb{R})} = 0 \right\} > 0. \) This leads to the following definition.

**Definition 2.1.** For a fixed \( t > 0 \) a function \( f \in L_2(\mathbb{R}) \) is said to be a \( T_t \)-essential if
\[ P \left\{ \| T_t f \|_{L_2(\mathbb{R})} > 0 \right\} = 1. \]

**Example 2.2.** Let \( f \in L_2(\mathbb{R}) \) be an analytic function which doesn’t equal totally to zero. Denote the set of its zeroes \( Z_f = \{ u \in \mathbb{R} \mid f(u) = 0 \} \). Then, by (1.2), \( P \{ x(\mathbb{R},t) \cap Z_f = \emptyset \} = 1 \), so \( f \) is a \( T_t \)-essential for any \( t > 0 \).

Let us notice that if \( t_1 \neq t_2 \) then \( T_{t_1} \)-essential function may not be a \( T_{t_2} \)-essential. To introduce a \( T_1 \)-essential and not \( T_2 \)-essential function lets consider an increasing sequence \( \{ u_k \}_{k=0}^{\infty} \) such that \( u_0 = 0, u_1 = 1 \) and for any \( n \in \mathbb{N} \)
\[ u_{2n+1} - u_{2n} = \frac{1}{2^n}, \quad u_{2n} = u_{2n-1} + 2n(\ln 2)^\frac{1}{2}. \]

**Theorem 2.3.** The function \( f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n};u_{2n+1}]} \) is a \( T_1 \)-essential, and is not a \( T_2 \)-essential.

**Proof.** To prove that \( f \) is not a \( T_2 \) essential we show that \( P \{ \| T_2 f \|_{L_2(\mathbb{R})} > 0 \} < 1. \)
Since \( [u_{2k};u_{2k+1}] \cap [u_{2j};u_{2j+1}] = \emptyset \) for any \( k \neq j \) then, by (1.4),
\[ P \left\{ \| T_2 f \|_{L_2(\mathbb{R})}^2 > 0 \right\} = P \left\{ \sum_{n=0}^{+\infty} \mathbb{I}_{[u_{2n};u_{2n+1}]}(x(u,2))^2 \ du > 0 \right\} = \]
\[ = P \left\{ \sum_{n=0}^{+\infty} \mathbb{I}_{[u_{2n};u_{2n+1}]}(x(u,2)) \ du > 0 \right\} = \]
\[ = P \left\{ \sum_{n=0}^{\infty} (y(u_{2n+1},2) - y(u_{2n},2)) > 0 \right\} = \]
\[ = P \left\{ \exists n \geq 0 : y(u_{2n+1},2) \neq y(u_{2n},2) \right\} \leq \sum_{n=0}^{\infty} P \left\{ y(u_{2n+1},2) \neq y(u_{2n},2) \right\}. \]

Thus, by (1.3),
\[ \sum_{n=0}^{\infty} P \left\{ y(u_{2n+1},2) \neq y(u_{2n},2) \right\} = \sum_{n=0}^{\infty} \frac{1}{4\pi} \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} e^{-v^2} dv \leq \frac{1}{\sqrt{\pi}} < 1. \]
Consequently, the function \( f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n};u_{2n+1}]} \) is not a \( T_2 \)-essential. To prove that \( f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n};u_{2n+1}]} \) is a \( T_1 \)-essential one can show the following estimation.
Lemma 2.4. Let \( \{w(u_n, \cdot)\}_{n=0}^{\infty} \) be a family of independent Wiener processes on \([0; 1]\) such that \( w(u_n, 0) = u_n \). Then for any \( n \in \mathbb{N} \)

\[
\mathbb{P} \left\{ \max_{s \in [0;1]} \max_{j=0,2n-1} w(u_j, s) \geq \min_{s \in [0;1]} w(u_{2n}, s) \right\} < \frac{1}{2n^2 \sqrt{\pi \ln 2}}.
\]

Proof. Let \( w_1, w_2 \) be an independent Wiener processes on \([0; 1]\) started from point 0, i.e. \( w_1(0) = w_2(0) = 0 \). It can be noticed that

\[
\mathbb{P} \left\{ \max_{s \in [0;1]} \max_{j=0,2n-1} w(u_j, s) \geq \min_{s \in [0;1]} w(u_{2n}, s) \right\} = \mathbb{P} \left\{ \exists j = 0, 2n - 1 : \max_{s \in [0;1]} w(u_j, s) - \min_{s \in [0;1]} w(u_{2n}, s) \geq 0 \right\} \leq \sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0;1]} w(u_j, s) - \min_{s \in [0;1]} w(u_{2n}, s) \geq 0 \right\} \leq \sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0;1]} w_1(s) - \min_{s \in [0;1]} w_2(s) \geq u_{2n} - u_j \right\}.
\]

From the fact that \( \{u_n\}_{n=0}^{\infty} \) is an increasing sequence we can estimate the last expression and complete the proof

\[
\sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0;1]} w_1(s) - \min_{s \in [0;1]} w_2(s) \geq u_{2n} - u_j \right\} \leq \frac{1}{\sqrt{\pi}} \sum_{j=0}^{2n-1} \frac{1}{u_{2n} - u_j} e^{-\frac{(u_{2n} - u_j)^2}{4}} \leq \frac{2n - 1}{\sqrt{\pi (u_{2n} - u_{2n-1})}} e^{-\frac{(u_{2n} - u_{2n-1})^2}{4}} \leq \frac{1}{2n^2 \sqrt{\pi \ln 2}}.
\]

\( \square \)

Let prove that the function \( f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}, u_{2n+1}]} \) is a \( T_1 \)-essential. Using reasoning from the first part of the proof it can be checked that for considered function \( f \) the following equality holds

\[
\mathbb{P} \{ \| T_1 f \|_{L_2(\mathbb{R})} > 0 \} = \mathbb{P} \left\{ \sum_{n=0}^{\infty} (y(u_{2n+1}, 1) - y(u_{2n}, 1)) > 0 \right\}.
\]

Let prove that

\[
\mathbb{P} \left\{ \limsup_{n \to \infty} (y(u_{2n+1}, 1) - y(u_{2n}, 1)) \geq 1 \right\} = 1. \quad (2.1)
\]

Let build a new processes \( \{ \tilde{y}(u_n, \cdot)\}_{n=0}^{\infty} \) such that \( \{\tilde{y}(u_n, \cdot)\}_{n=0}^{\infty} \) and \( \{y(u_n, \cdot)\}_{n=0}^{\infty} \) have the same distributions in \( \mathcal{C} ([0; 1]) \) in the following way [2]. Let \( \{w(u_n, \cdot)\}_{n=0}^{\infty} \) be a given family of Wiener processes on \([0; 1]\), \( w(u_n, 0) = u_n \). Let denote collision time of \( f, g \in \mathcal{C} ([0; 1]) \) by \( \tau[f, g] := \inf \{ t \mid f(t) = g(t) \} \). Put \( \tilde{y}(u_0, \cdot) := w(u_0, \cdot) \). Then for any \( n \in \mathbb{N}, s \in [0; 1] \) one can define

\[
\tilde{y}(u_n, s) := w(u_n, s) \mathbb{I}\{ s < \tau[w(u_n, \cdot), \tilde{y}(u_{n-1}, \cdot)] \} +
\]
Thus, Proof. According to (2.1) it is sufficient to note that the function $\tilde{g}(u_n, \cdot) \in L_2(\mathbb{R})$ for any $t > 0$. Consequently, for any $t > 0$ the following relations are true

$$P\{ \exists N \in \mathbb{N}: \forall n \geq N \quad \tilde{g}(u_n, t) = w(u_n, t),$$

$$\tilde{g}(u_{n+1}, t) = w(u_{n+1}, t) \mathbb{I}\{ t < \tau[w(u_n, \cdot), w(u_{n+1}, \cdot)] \} +$$

$$+ w(u_n, t) \mathbb{I}\{ t \geq \tau[w(u_n, \cdot), w(u_{n+1}, \cdot)] \} = 1. \quad (2.2)$$

Thus,

$$P\{ \exists N \in \mathbb{N}: \forall n \geq N \quad \tilde{g}(u_{n+1}, t) - \tilde{g}(u_n, t) = w(u_{n+1}, t) - w(u_n, t) \} = 1.$$  

For the considered sequence $\{u_n\}_{n=0}^{\infty}$ and any $n \in \mathbb{N}$ the following inequality holds

$$P\{ w(u_{n+1}, t) - w(u_n, t) \geq 1 \} = \int_1^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{(v-\frac{1}{2})^2}{4}} dv \geq \frac{1}{\sqrt{4\pi}} \int_1^\infty e^{-\frac{v^2}{4}} dv.$$  

Therefore, by the Borel-Cantelli lemma and (2.2),

$$P\{ \limsup_{n \to \infty} (\tilde{g}(u_{n+1}, t) - \tilde{g}(u_n, t)) \geq 1 \} = 1.$$  

\[\square\]

Using observation from Example 2.2 one can introduce a family of $T_1$-essential functions for all $t > 0$.

For any $\varepsilon > 0$ let us consider an integral operator $K_\varepsilon$ in $L_2(\mathbb{R})$ with the kernel

$$k_\varepsilon(v_1, v_2) = \int_\mathbb{R} p_\varepsilon(u-v_1)p_\varepsilon(u-v_2)dy(u, t), \quad (2.3)$$

where $v_1, v_2 \in \mathbb{R}$, and $p_\varepsilon(u) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{u^2}{2\varepsilon}}$. By the change of variables formula for an Arratia flow [6],

$$(K_\varepsilon f, f) = \int_\mathbb{R} (f \ast p_\varepsilon)^2(x(u, t))du. \quad (2.4)$$

**Lemma 2.5.** For any $\varepsilon > 0$ and nonzero function $f \in L_2(\mathbb{R})$

$$P\{ (K_\varepsilon f, f) \neq 0 \} = 1.$$  

**Proof.** According to (2.1) it is sufficient to note that $f \ast p_\varepsilon$ is an analytic function. Consequently, for any $t > 0$ the following relations are true

$$P\{ (K_1 f, f) > 0 \} = P\{ ||T_t(f \ast p_1)||_{L_2(\mathbb{R})} > 0 \} = P\{ x(\mathbb{R}, t) \cap Z_{f \ast p_1} = \emptyset \} = 1.$$  

\[\square\]

According to the last theorem and (2.4), for any $\varepsilon > 0$ and nonzero $f \in L_2(\mathbb{R})$ the function $f \ast p_\varepsilon$ is a $T_1$-essential for each $t > 0$. 

ESSENTIAL SETS FOR RANDOM OPERATORS CONSTRUCTED FROM ARRATIA FLOW 5
3. On change of compact sets under strong random operator generated by an Arratia flow

As it was noticed in the introduction any function with bounded support isn’t a \( T_t \)-essential. Consequently, if \( K \subseteq L_2(\mathbb{R}) \) is a compact set of functions with uniformly bounded supports such that \( T_t(K) \) is well-defined, then the image \( T_t(K) \) equals to \( \{0\} \) with positive probability. It was shown in [6] that \( T_t \) may also change the geometry of \( K \) even in the case of a compact set \( K \) for which \( T_t(K) \neq \{0\} \) a.s. For example, the image \( T_t(K) \) of a convergent sequence and its limiting point may not have limiting points. In this section we build a compact set \( K \) for which \( T_t(K) \neq \{0\} \) a.s. and investigate the change of its Kolmogorov-widths in \( L_2(\mathbb{R}) \) under random operator \( T_t \).

**Definition 3.1 ([10]).** The Kolmogorov \( n \)-width of a set \( C \subseteq H \) in a Hilbert space \( H \) is given by

\[
d_n(C) = \inf_{\dim L \leq n} \sup_{f \in C} \inf_{g \in L} \|f - g\|_H,
\]

where \( L \) is a subspace of \( H \).

We consider the following compact set in \( L_2(\mathbb{R}) \)

\[
K = \{ f \in W_2^2(\mathbb{R}) \mid \int_{\mathbb{R}} f^2(u)(1 + |u|^3)\,du + \int_{\mathbb{R}} (f'(u))^2(1 + |u|)^7\,du \leq 1 \}.
\]

(3.1)

Estimations on its Kolmogorov-widths in \( L_2(\mathbb{R}) \) are presented in the next lemma.

**Lemma 3.2.** There exist positive constants \( C_1, C_2 \) such that for any \( n \in \mathbb{N} \)

\[
\frac{C_1}{n} \leq d_n(K) \leq \frac{C_2}{n^{10}}.
\]

**Proof.** Let \( n \in \mathbb{N} \) be fixed. To estimate \( d_n(K) \) from above one can consider the partition \( \{u_k\}_{k=0}^{n} \subseteq [0, n^{10}] \) into \( n \) segments \( \{[u_k; u_{k+1}], k = 0, n-1\} \) with equal lengths. Let’s show that for the \( n \)-dimensional subspace

\[
L_n = LS\{1_{[u_k; u_{k+1}]} \mid k = 0, n-1\}
\]

\[
\sup_{f \in K} \inf_{g \in L_n} \|f - g\|_{L_2(\mathbb{R})} \leq \frac{C_2}{n^{10}}.
\]

If \( f \in K \) then

\[
\int_{|u|>c} f^2(u)(1 + |u|^3)\,du \leq 1.
\]

Thus, for any \( C > 0 \)

\[
\int_{|u|>c} f^2(u)\,du \leq \frac{1}{(1 + C)^3} \int_{|u|>c} f^2(u)(1 + |u|^3)\,du \leq \frac{1}{C^3}.
\]

So, for the function \( g_f = \sum_{k=0}^{n-1} f(u_k)1_{[u_k; u_{k+1}]}, u \in L_n \) the following estimation is true

\[
\|f - g_f\|^2_{L_2(\mathbb{R})} \leq \frac{1}{n^{10}} + \int_{|u| \leq n^{10}} (f(u) - g_f(u))^2\,du.
\]

By the Cauchy inequality, for \( f \in K \) and \( u \in [u_k; u_{k+1}] \)

\[
\left( \int_{u_k}^{u} f'(v)\,dv \right)^2 \leq \int_{u_k}^{u} \frac{dv}{(1 + |v|)^7} \leq u - u_k.
\]
Consequently,
\[ \int_{|u| \leq n^{\frac{1}{m}}} (f(u) - g_f(u))^2 \, du = \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left( \int_{u_k}^{u} f'(v) \, dv \right)^2 \, du \leq \frac{1}{2} \sum_{k=0}^{n-1} (u_{k+1} - u_k)^2 = \frac{2}{n^2}, \]
and the upper estimation for \( d_n(K) \) holds with the constant \( C_2 = 3^\frac{1}{m} \).

To get a lower estimation for \( d_n(K) \) we use the theorem about \( n \)-width of \((n+1)\)-dimensional ball [10]. Let \( \{u_k\}_{k=0}^{2(n+1)} \) be a partition of \([0; 1]\) into \(2(n+1)\) segments \( \{|u_k; u_{k+1}|, k = 0, 2n + 1\} \) with equal lengths. Consider \((n+1)\)-dimensional space \( L_{n+1} = LS\{f_k, k = 0, n\} \), where the functions \( f_k, k = 0, n \), are defined as follows

\[
\begin{align*}
    f_k &= \begin{cases} 
    0, & u \notin [u_{2k}; u_{2k+1}], \\
    1, & u \in [u_{2k} + \frac{1}{6(n+1)}; u_{2k} + \frac{2}{6(n+1)}], \\
    6(n+1)(u - u_{2k}), & u \in [u_{2k}; u_{2k} + \frac{1}{6(n+1)}], \\
    -6(n+1)(u - u_{2k+1}), & u \in [u_{2k} + \frac{2}{6(n+1)}; u_{2k+1}].
    \end{cases} 
\end{align*}
\]  
(3.2)

We show that if \( c = \frac{2^3(5+2^9\cdot3^2)}{5} \) then the ball \( B_{n+1} = \{f \in L_{n+1} \mid \|f\|_{L_2(R)} \leq \frac{1}{\sqrt{n}}\} \) is a subset of \( K \). Since \( \|f_k\|_{L_2(R)} = \frac{5}{18(n+1)}, k = 0, n \), then for any \( f \in B_{n+1} \) such that \( f = \sum_{k=0}^{n} c_k f_k \) the following relation holds \( \sum_{k=0}^{n} c_k^2 \leq \frac{36}{5cn} \). Thus, according to (3.2),
\[
\int_{R} f^2(u)(1 + |u|)^2 \, du + \int_{R} (f'(u))^2 (1 + |u|)^7 \, du \leq
\]
\[
\leq 2^3\|f\|_{L_2(R)}^2 + 2^7 \cdot \sum_{k=0}^{n} c_k^2 \left( \int_{u_{2k}}^{u_{2k+1} + \frac{1}{3n+1}} (6(n+1))^2 \, du + \int_{u_{2k} + \frac{1}{3n+1}}^{u_{2k+1}} (6(n+1))^2 \, du \right) \leq
\]
\[
\leq \frac{2^3}{cn^2} + 2^{10} \cdot 3n \cdot \frac{36}{5cn} \leq \frac{1}{c} \cdot \frac{2^3(5+2^9\cdot3^2)}{5} = 1.
\]
Consequently, \( B_{n+1} \subset K \) and \( d_n(K) \geq d_n(B_{n+1}) \). Due to the theorem about \( n \)-width of \((n+1)\)-dimensional ball, \( d_n(B_{n+1}) = \frac{1}{\sqrt{n} \cdot \sqrt{c}} \) [10]. So the lower estimation for \( d_n(K) \) holds with \( C_1 := \sqrt{c} \).

To show that estimations from above for the Kolmogorov-widths of considered compact set \( K \) don’t change under \( T_1 \) one may use the same idea as in Lemma 2.

**Theorem 3.3.** There exists \( \Omega \) of probability one such that for any \( \omega \in \Omega \) and \( n \in \mathbb{N} \)
\[
d_n(T_1^\omega(K)) \leq C(\omega) \frac{1}{n^{10}}, \quad (3.3)
\]
where the constant \( C(\omega) > 0 \) doesn’t depend on \( n \).
Proof. For a fixed $n \in \mathbb{N}$ let us consider a partition $\{u_k\}_{k=0}^n$ of $[-n^\frac{1}{2}; n^\frac{1}{2}]$ into $n$ segments with equal length. To prove (3.3) it’s sufficient to show the following inequality for the linear space $L^\omega_n = LS\{ T^\omega_t \mathbb{I}_{[u_k;u_{k+1}]}, \ k = 0, n-1 \}$ with dimension at most $n$

$$\sup_{h_1 \in T^\omega_t(K)} \inf_{h_2 \in L^\omega_n} \|h_1 - h_2\|_{L_2(\mathbb{R})} \leq \frac{C(\omega)}{n^\frac{1}{2}}.$$ 

According to the change of variable formula for an Arratia flow, one can check the equality for any $f \in K$

$$\left\|T^\omega_t f - T^\omega_t \left(\sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k;u_{k+1}]}\right)\right\|^2_{L_2(\mathbb{R})} = \int_{|u| > n^\frac{1}{2}} f^2(u) dy(u,t,\omega) +$$

$$+ \int_{|u| \leq n^\frac{1}{2}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k;u_{k+1}]}(u)\right)^2 dy(u,t,\omega).$$

To estimate from above the last integral let us notice that

$$\int_{|u| \leq n^\frac{1}{2}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k;u_{k+1}]}(u)\right)^2 dy(u,t,\omega) \leq$$

$$\leq \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^{u} |f'(v)| \, dv\right)^2 dy(u,t,\omega).$$

Due to (3.1), for any $f \in K$ and $u \in [u_k;u_{k+1}]

$$\left(\int_{u_k}^{u} |f'(v)| \, dv\right)^2 \leq \int_{u_k}^{u} \frac{dv}{(1+|v|)^t} \leq u_{k+1} - u_k.$$

Thus,

$$\sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^{u} |f'(v)| \, dv\right)^2 dy(u,t,\omega) \leq$$

$$\leq \sum_{k=0}^{n-1} (u_{k+1} - u_k) \int_{u_k}^{u_{k+1}} dy(u,t,\omega) = \frac{2}{n^{\frac{1}{2}}}(y(n^\frac{1}{2},t,\omega) - y(-n^\frac{1}{2},t,\omega)).$$

For an Arratia flow $\{y(u,s), u \in \mathbb{R}, s \in [0; t]\}$ the following relation is true [11]

$$\lim_{|u| \to +\infty} \frac{|y(u,t)|}{|u|} = 1 \ a.s.$$ 

Consequently, for any $\omega \in \hat{\Omega} = \{\omega' \in \Omega | \lim_{|u| \to +\infty} \frac{|y(u,t,\omega')|}{|u|} = 1\}$ the estimation holds

$$\int_{|u| \leq n^\frac{1}{2}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k;u_{k+1}]}(u)\right)^2 dy(u,t,\omega) \leq \frac{4c(\omega)}{n^\frac{1}{2}} \quad (3.4)$$

with the constant

$$c(\omega) := \sup_{|u| \geq 1} \frac{|y(u,t,\omega)|}{|u|}. \quad (3.5)$$
ESSENTIAL SETS FOR RANDOM OPERATORS CONSTRUCTED FROM ARRATIA FLOW 9

Let's prove that for any $\omega \in \tilde{\Omega}$ there exists a constant $\tilde{c}(\omega)$ such that

$$\int_{|u| > n^{\frac{1}{5}}} f^2(u) dy(u, t, \omega) \leq \frac{\tilde{c}(\omega)}{n^{\frac{3}{5}}}.$$ 

It can be noticed that $\int_{|u| > n^{\frac{1}{5}}} f^2(u) dy(u, t) \leq \frac{1}{n^{\frac{3}{5}}} \int_{|u| > n^{\frac{1}{5}}} f^2(u)(1 + |u|)^3 dy(u, t)$.

Denote by $\{\theta_j\}_{j=1}^\infty$ a sequence of jump points of the function $y(\cdot, t)$ on $\mathbb{R}_+$. Thus, one may show

$$\int_{u > n^{\frac{1}{5}}} f^2(u)(1 + u)^3 dy(u, t) = \sum_{\theta_i \geq n^{\frac{1}{5}}} f^2(\theta_i)(1 + \theta_i)^3 \Delta y(\theta_i, t) =$$

$$= \sum_{k=1}^\infty \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i)(1 + \theta_i)^3 \Delta y(\theta_i, t) \leq$$

$$\leq \sum_{k=1}^\infty (2 + k)^3 \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i) \Delta y(\theta_i, t).$$

According to the Cauchy inequality and (3.1), for any $u \in \mathbb{R}_+$ the following relations hold

$$f^2(u) \leq \int_u^{+\infty} (f'(v))^2 (1 + v)^2 dv \cdot \frac{dv}{(1 + v)^2} \leq \frac{1}{6u^6}.$$ 

Consequently, due to (3.5), the inequalities are true

$$\sum_{k=1}^\infty (2 + k)^3 \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i) \Delta y(\theta_i, t) \leq$$

$$\leq \sum_{k=1}^\infty (2 + k)^3 \frac{1}{6k^6}(y(k + 1, t) - y(k, t)) \leq \frac{16c}{3} \sum_{k=1}^\infty \frac{1}{k^2}.$$ 

Hence, for any $\omega \in \tilde{\Omega}$ there exists the constant $C_1(\omega) = \frac{16c(\omega)}{3}$ such that

$$\int_{u > n^{\frac{1}{5}}} f^2(u) dy(u, t, \omega) \leq \frac{C_1(\omega)}{n^{\frac{3}{5}}}.$$ 

Similarly, it can be proved that $\int_{u < -n^{\frac{1}{5}}} f^2(u) dy(u, t, \omega) \leq \frac{C_1(\omega)}{n^{\frac{3}{5}}}$. According to this and (3.4), for any $\omega \in \tilde{\Omega}$ an upper estimation for $d_n(T_t^\omega(K))$ is true. $\square$

The functions from Lemma 2 that were used to build the $(n + 1)$-dimensional subspace are not $T_t$ essential for any $t > 0$. Thus, the image of this subspace under the random operator $T_t$ may be equal to $\{0\}$ with positive probability. So, one can ask about existence of a finite-dimensional subspace such that for any $t > 0$ its image under $T_t$ is a linear subspace with the same dimension.
4. On subspace that preserves dimension under random operator generated by an Arratia flow

In this section for any \( t > 0 \) and \( n \in \mathbb{N} \) we present a family \( \{ g_k, k = \overline{0, n} \} \) of linearly independent \( T_t \)-essential functions such that its images under \( T_t \) are linearly independent. Such family generates a subspace which preserves dimension under random operator generated by an Arratia flow. It can be used to get a lower estimation of \( d_n(T_t(K)) \).

Let’s fix any \( n \in \mathbb{N} \), and build a family of \((n + 1)\) linearly independent functions in the following way. Let \( \{ u_k \}_{k=0}^{2(n+1)} \) be a partition of \([0; n^{-1}]\) into \( 2(n+1) \) segments with equal lengths. For any \( k = \overline{0, n} \) define \( f_k \) by

\[
f_k = \begin{cases} 
0, & u \notin [u_{2k}; u_{2k+1}], \\
1, & u \in [u_{2k}; u_{2k} + \frac{n-2}{2(n+1)}] \\
\frac{n-2}{2(n+1)}(u - u_{2k}), & u \in [u_{2k} - \frac{n-2}{2(n+1)}; u_{2k} + \frac{n-2}{2(n+1)}], \\
\frac{n-2}{2(n+1)}(u - u_{2k+1}), & u \in [u_{2k} + \frac{n-2}{2(n+1)}; u_{2k+1}]. 
\end{cases} \tag{4.1}
\]

**Lemma 4.1.** There exists \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon \leq \varepsilon_0 \) the functions \( \{ f_k \ast p_\varepsilon, k = \overline{0, n} \} \) are linearly independent.

**Proof.** Since considered functions \( \{ f_k, k = \overline{0, n} \} \) are linearly independent then its Gram determinant doesn’t equal to 0, i.e. \( G(f_0, \ldots, f_n) \neq 0 \). For each \( k = \overline{0, n} \)

\[
f_k \ast p_\varepsilon \rightarrow f_k, \; \varepsilon \rightarrow 0.
\]

Hence, due to continuity of Gram determinant, one may notice that there exists \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon \leq \varepsilon_0 \)

\[
G(f_0 \ast p_\varepsilon, \ldots, f_n \ast p_\varepsilon) \neq 0,
\]

and the desired result is proved. \( \Box \)

**Theorem 4.2.** There exists a set \( \Omega_0 \) of probability one such that for any \( \omega \in \Omega_0 \) the functions \( T_t(f_0 \ast p_\varepsilon), \ldots, T_t(f_n \ast p_\varepsilon) \) are linearly independent.

**Proof.** Denote by \( K_\varepsilon \) the integral operator in \( L_2(\mathbb{R}) \) with the kernel \( k_\varepsilon \). To prove the statement of the theorem it’s enough to show that on some \( \Omega_0 \) of probability one the following inequality holds \( (K_\varepsilon f, f) > 0 \) for any nonzero \( f \in LS\{ f_0, \ldots, f_n \} \). Due to (1.4)

\[
(K_\varepsilon f, f) = \sum_\theta (f \ast p_\varepsilon)^2(\theta) \Delta y(\theta, t), \tag{4.2}
\]

where \( \theta \) is a point of jump of the function \( y(\cdot, t) \).

It was proved in [12] that there exists \( \Omega_0 \) of probability one such that for any \( \omega \in \Omega_0 \) a linear span of the functions \( \{ p_\varepsilon(\cdot - \theta(\omega)) | [0, 1] \} \theta(\omega) \) is dense in \( L_2([0, 1]) \). Thus, on the set \( \Omega_0 \) for any \( f \in LS\{ f_0, \ldots, f_n \} \subset L_2([0, 1]) \) one can find a random point \( \theta_f \) such that \( (f(\cdot), p_\varepsilon(\cdot - \theta_f)) \neq 0 \). Since \( y(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R} \) is nondecreasing then \( \Delta y(\theta, t) > 0 \) for any jump-point \( \theta \). Consequently, on the set \( \Omega_0 \)

\[
\sum_\theta (f \ast p_\varepsilon)^2(\theta) \Delta y(\theta, t) = \sum_\theta (f(\cdot), p_\varepsilon(\cdot - \theta))^2 \Delta y(\theta, t) \geq \]


\[
\sum_\theta (f \ast p_\varepsilon)^2(\theta) \Delta y(\theta, t) = \sum_\theta (f(\cdot), p_\varepsilon(\cdot - \theta))^2 \Delta y(\theta, t) \geq
\]

\[
\sum_\theta (f \ast p_\varepsilon)^2(\theta) \Delta y(\theta, t) = \sum_\theta (f(\cdot), p_\varepsilon(\cdot - \theta))^2 \Delta y(\theta, t) \geq
\]
which proves the theorem. □

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