Commensurate and Incommensurate $O(n)$ Spin Systems: Novel Even-Odd Effects, A Generalized Mermin-Wagner-Coleman Theorem, and Ground States

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We examine $n$ component spin systems with arbitrary two spin interactions (of unspecified range) within a general framework to highlight some new subtleties present in incommensurate systems. We determine the ground states of all translationally invariant $O(n \geq 2)$ systems and prove that barring commensurability effects they are always spiral-like systems while their odd $n$ counterparts exhibit an exponential decay of correlations. We illustrate that many frustrated incommensurate continuous spin systems display smectic-like thermodynamics. We report on a generalized Mermin-Wagner-Coleman theorem for all two dimensional systems (of arbitrary range) with analytic kernels in momentum space. A new relation between generalization Mermin-Wagner-Coleman bounds and dynamics is further reported. We suggest a link between a generalized Mermin-Wagner-Coleman theorem to divergent decoherence (or bandwidth) in a quantum context. A generalization of the Peierls bound for commensurate systems with long range interactions is also discussed. We conclude with a discussion of $O(n)$ spin dynamics in the general case.

I. INTRODUCTION

In this article (constituting a portion of my thesis [1]), we aim to unveil some general properties of $O(n)$ spin systems having two-spin interactions. We investigate Ising and $O(n)$ ground states, derive generalized Mermin-Wagner inequalities and illustrate how Peierls’ bounds may be derived in some long range systems. Unlike the very rich ground state structures possible for incommensurate scalar systems (bubbles, dots, Wigner crystals), we prove here that for incommensurate continuous spin systems on a lattice, spiral states are the only ground states. Perhaps most noteworthy, we perform a thermal stability analysis whose results coincide with a generalization Mermin-Wagner-Coleman inequalities. In the general case, we discover a new intriguing odd-even $n$ effect. Soft spin analysis suggests that algebraic long range order is possible in certain frustrated incommensurate even $n$ systems while their odd $n$ counterparts exhibit an exponential decay of correlations. Odd $n$ incommensurate systems are generally more disordered than their even $n$ counterparts. In particular, we illustrate that many frustrated incommensurate continuous spin systems have smectic-like thermodynamics.

The outline of this article is as follows: After introducing, in section(II), the terms and notations that will be used throughout the work, we provide, in section (III) frustrated toy models which are rich enough to illustrate some of the general features that we aim to highlight. These models will be employed for illustrative purposes only. The contents of the present publication are not limited to these systems. All stated in the current article applies to all translationally invariant systems (whether these are the standard nearest neighbor Heisenberg model or the more complex frustrated models that we introduce in section (III)). Notwithstanding their pedagogical purposes in the current context, the toy models presented in section(III) and myriad variants have been a subject of much research in recent years, see e.g. [2]. The physics of incommensurate orders along with, in some instances, competing short- and long-range interactions is a rich and complex topic. These appear, amongst others, in the high temperature superconductors [3], [4], [5], [6], [7], manganates and nickelates [8], [9], quantum Hall systems [10], [11], [12] chemical and magnetic mixtures, crumpled membranes [2], and some theories of structural glasses [13], [14]. In many instances, there appear to be a large number of possible ground states exhibiting nanoscale phase separation, including stripes, labyrinths, patches, and dot arrays. Most new results reported in this article pertain to incommensurate spin structures.

In section(IV) we discuss the ground states of Ising spin models and show what patterns one should expect in general. Once the ground states will be touched on, we will head on to show how Peierls bounds may be established for many systems having infinite range interactions if the ground states are simple. In section(V) we shortly review mean field solutions of the general two spin Ising models. A generalization of the standard Peierls bound to other systems with long range interactions is provided in section(V). When fused with an additional $Z_2$ symmetry (which is not present in many of the models that
we discuss), Peierls’ bounds suffice to prove the existence of long range order.

Henceforth, the bulk of this article focuses on continuous spins whose number of components \( n \geq 2 \). In section(VII) we prove that, sans special commensurability effects, the ground states of all \( O(n \geq 2) \) will typically have a spiral like structure. Section(VIII) details an exceedingly simple spin wave stiffness analysis to gauge the effect of thermal fluctuations on the various \( O(n \geq 2) \) ground states. In section(IX) we will discuss thermal fluctuations within the framework of “soft-spin” XY model. We will see that the normalization constraint gives a Dirac like equation. In the aftermath, the fluctuation spectrum will be seen to match with that derived in section(VIII). We will show possible links to smectic like behavior in three dimensions. Next, we go one step further to study the fully constrained “hard-spin” \( O(2) \) and \( O(3) \) models and show (in section X) that all translationally invariant systems in two dimensions with an analytic rotationally symmetric interaction kernel never develop spontaneous magnetization. At the end of the section our analysis will match that of sections(VIII) and (IX) We extend the Mermin-Wagner-Coleman bounds, in section(XI), to high dimensions to show that intricacies occur if a certain high dimensional integral will be seen to diverge. In the low temperature limit, the integrand of this integral will, once again, match that derived by the much more naive spin stiffness and soft spin fluctuation analysis of earlier sections. In section(XII) we will examine the “soft-spin” version of Heisenberg spins. We will see that it might be naively expected that the spin fluctuations in odd \( n \) spin systems are larger than in those with an even number of spin components. The origin of this “odd” even-odd effect is that for odd \( n \) systems, one of the spin components is unpaired and may exhibit less inhibited fluctuations. Next, in section(XIII), we carry out the spin fluctuation analysis for four component soft spins to see that their spectra coincides with that predicted in the earlier spin stiffness analysis. We compute the correlation functions for this system and find that even \( n \) incommensurate systems may display algebraic long range order in situations where their odd \( n \) counterparts of the same system exhibit exponential decay of correlations at all finite temperatures (insofar as we may discern from a perturbative analysis of soft spin models).

We match the finite \( n \) analysis with its large \( n \) counterpart in section(XIV). We show that in the limit of large \( n \) both odd and even component spin systems behave in the same manner. Essentially, they all tend towards an “odd” behavior. We also report on a “holographic” like effect present in some frustrated systems. In these, the ground state entropy is shown to scale with the surface area of the system. In section(XVI), we compute the critical temperature of all translationally invariant \( O(n \geq 2) \) spin models within mean field theory. In section(XVII) we briefly remark that much of analysis is unchanged for arbitrary non-translationally invariant two spin interactions. We conclude with a discussion of \( O(n) \) spin dynamics. A central theme which will be repeatedly touched on throughout the paper is the possibility of non-trivial ground state manifolds. If the system is degenerate the effective topology of the low temperature phase of the system may be classified in momentum (or other basis). In such instances the low temperature behavior of the systems will be exceedingly rich.

II. DEFINITIONS

We consider simple classical spin models of the type

\[
H = \frac{1}{2} \sum_{\vec{x},\vec{y}} V(\vec{x},\vec{y}) [\vec{S}(\vec{x}) \cdot \vec{S}(\vec{y})].
\]  

(1)

Here, the sites \( \vec{x} \) and \( \vec{y} \) lie on a (generally hypercubic) lattice of size \( N \). The spins \( \{ \vec{S}(\vec{x}) \} \) are normalized and have \( n \) components, \( \sum_{i=1}^{n} \vec{S}^2(\vec{x}) = 1 \), at all lattice sites \( \vec{x} \). We will primarily focus on translationally invariant interactions \( V(\vec{x},\vec{y}) = V(\vec{x} - \vec{y}) \). We employ the non-symmetrical Fourier basis convention \( \{ \vec{k} \} = \sum_{\vec{x}} F(\vec{x}) e^{-i\vec{k} \cdot \vec{x} }; \quad F(\vec{x}) = \frac{1}{N} \sum_{\vec{k}} F(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \) wherein the Hamiltonian is diagonal and reads

\[
H = \frac{1}{2N} \sum_{\vec{k}} v(\vec{k}) |\vec{S}(\vec{k})|^2
\]  

(2)

where \( v(\vec{k}) \) and \( \vec{S}(\vec{k}) \) are the Fourier transforms of \( V(\vec{x}) \) and \( \vec{S}(\vec{x}) \). More generally, for some of the properties that we will illustrate, one could consider any arbitrary real two spin interactions \( \{ \vec{x} | V | \vec{y} \} \) which would be diagonalized in another basis \( \{ \vec{u} \} \) instead of the Fourier basis. For simplicity, we will set the lattice constant to unity- i.e. on a hypercubic lattice (of side \( L \)) with periodic boundary conditions the wave-vector components \( k_l = \frac{2\pi n_l}{L} \) where \( n_l \) is an integer (and the real space coordinates \( x_l \) are integers). Throughout, we employ \( \vec{k} \) to denote reciprocal lattice vectors and \( \Delta(\vec{k}) \) as a shorthand for the lattice lattice Laplacian:

\[
\Delta(\vec{k}) = 2 \sum_{l=1}^{d} (1 - \cos k_l).
\]  

(3)

In some of the frustrated systems that we will soon consider, \( v(\vec{k}) \) may be written explicitly as the sum of several terms: those favoring homogeneous states \( \vec{k} \to 0 \), and those favoring zero wavelength \( \vec{k} \to \infty \) (or \( \vec{k} \to (\pi, \pi, ..., \pi) \) on a lattice.) As a result of this competition, modulated structures arise on an intermediate scale.
III. TOY MODELS

Although we will keep the discussion very general, it might be useful to have a few explicit applications in mind. There is a lot of physical intuition which underlies the upcoming models. Unfortunately, here they will merely serve as nontrivial toy models on which we will be able to exercise our newly gained intuition. The systems to be presented are frustrated: not all two spin interactions can be simultaneously satisfied. They entail competing interactions. Such systems appear, amongst others, in the high temperature superconductors, CMR materials, quantum Hall bars, chemical and magnetic mixtures, crumpled membranes, and some theories of structural glasses [2]. Toy models in these issues can be addressed are useful. In the current context, we choose these examples as they highlight subtleties typically absent in the more standard spin models. In the continuum limit, any incommensurate rotationally symmetric model (with an interaction \( v(\vec{k}) \) which is analytic about its minima and having non-vanishing second derivatives) will share much the same physics as the two specific models discussed below. When any incommensurate spin system possesses a rotational symmetry, the minimizing manifold of \( v(\vec{k}) \) is a \((d-1)\) dimensional shell of radius \( q > 0 \).

The Coulomb Frustrated Ferromagnet

We now introduce the “Coulomb Frustrated Ferromagnet”. This is a toy model of a doped Mott insulator [4], [6], of phase separation in high Landau level Quantum Hall systems [11], [12] and of certain amphiphilic systems. It has been argued that within the Mott insulator the tendency, of holes, to phase separate at low doping is frustrated, in part, by electrostatic repulsion [4]. In three dimensions, a simple spin Hamiltonian [5] which represents these competing interactions is

\[
H_{\text{Mott}} = - \sum_{\langle \vec{x}, \vec{y} \rangle} S(\vec{x})S(\vec{y}) + \frac{Q}{8\pi} \sum_{\vec{x} \neq \vec{y}} \frac{S(\vec{x})S(\vec{y})}{|\vec{x} - \vec{y}|^d}
\]

\[
= \frac{1}{2N} \sum_{\vec{k}} |\Delta(\vec{k}) + \sum_{\vec{K} \neq \vec{0}} |\vec{k} - \vec{K}|^{-2}||S(\vec{k})|^2.
\]

In the second line we employed the Poisson summation formula. Here \( \{\vec{K}\} \) is the set of all reciprocal lattice vectors as introduced in Section(II). Here, \( S(\vec{x}) = \rho(\vec{x}) \) is a coarse grained scalar variable which represents the local density of mobile holes. Each site \( \vec{x} \) represents a small region of space in which \( S(\vec{x}) > 0 \), and \( S(\vec{x}) < 0 \) correspond to hole-rich and hole-poor phases respectively. In this Hamiltonian, the first “ferromagnetic” term represents the short-range (nearest-neighbor) tendency of the holes to phase-separate and form a hole-rich “metallic” phase, whereas the frustrating effect of the electrostatic repulsion between holes is present in the second term. Nonlinear terms in the full Hamiltonian typically fix the locally preferred values of \( S(\vec{x}) \). One may consider \( d \neq 3 \)-dimensional variants wherein the spins lie on a hyper-cubic lattice, and the Coulomb kernel in \( H_0 \) is replaced by \( \frac{Q}{2\pi} \ln |\vec{x} - \vec{y}|^2 \) (or by \( \frac{Q}{16\pi} \ln |\vec{x} - \vec{y}| \) in two dimensions [15]) where \( \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \). Here the competition between both terms, when \( Q < 1 \) favors states with wave-numbers \( \sim Q^{1/4} \). The introduction of the Coulomb interaction is manifestly non-perturbative: it is long range. Moreover, the previous ferromagnetic ground state becomes, tout a’ coup, infinite in energy. We will, for the most part, focus on the continuum limit of this Hamiltonian where the kernel becomes

\[
v^\text{cont}_{\text{Mott}}(\vec{k}) = Qk^{-2} + k^2(1 + \sum_{\vec{K} \neq \vec{0}} \frac{3}{|\vec{k}|^4})
\]

After rescaling, this may also be regarded as the small \( \vec{k} \) limit of the more general

\[
v_Q(\vec{k}) = \Delta(\vec{k}) + Q|\Delta(\vec{k})|^{-1} + A|\Delta(\vec{k})|^2
\]

\[
+ \lambda \sum_{i \neq j} (1 - \cos k_i)(1 - \cos k_j) + O(k^6).
\]

The constants \( A \) and \( \lambda \) are pinned down if we identify \( v_Q(\vec{k}) = v_{\text{Mott}}(\vec{k}) \). Here, we will modify them in order to streamline the quintessential physics of this system. First, we set \( A = 0 \): In the continuum limit this term is not large nor does it lift the “cubic rotational symmetry” [16] present to lower order. Next, we allow \( \lambda \) to vary in order to turn on and off “cubic rotational symmetry” breaking effects [16]. The kernel \( v_Q(\vec{k}) \) may be regarded as \( v_{\text{Mott}} \) augmented by all possible next to nearest neighbor interactions. As our Hamiltonian respects the hypercubic point symmetry group, by surveying all possible values of \( \lambda \) we should be able to make general statements regarding the possible phases (within the planes) of real doped Mott insulators. In dimension \( d = 2 \), whenever \( \lambda > 0 \), the minimizing wave-vectors will lie along the cubic axis and “horizontal” order will be expected. When \( \lambda < 0 \) the minimizing wave-vectors lie along the principal diagonals and diagonal order is expected. At large values of \( Q \), when the continuum limit no longer applies, trivial extensions of these minimizing modes are encountered where one or more of the wave vector components is set to \( \pi \). The work reported in this paper focuses on spin systems on a lattice. The situation becomes far richer in continuum field theories. In high level Quantum Hall systems, calculations on a similar model indicate [12] that the system undergoes a sequence of transitions. At very low excess filling fractions, \( \nu - N < 1/N \) (with \( N \) the integer part of the filling \( \nu \)), a Wigner crystal of the cyclotron orbit centers is formed. As \( \nu \) is further increased, transitions between bubbles of increasing size occur. Ultimately, at a sufficiently large filling \( \nu \), stripes appear. Similar textures are observed in chemical mixtures [2] which may be described by similar models [17]. We will prove, however, that this series of
transitions between various textures does not occur for multi-component spins on a lattice. On a lattice, barring special commensurate points that we detail below (e.g. the standard uniform and Neel phases and other, more intricate, commensurate states), only spiral (stripe-like) states may appear.

If the ferromagnetic system were frustrated by a general long range kernel of the form \( V(|\vec{x} - \vec{y}|) \sim |\vec{x} - \vec{y}|^{-p} \) we could replace the \( [\Delta(\vec{k})]^{-1} \) in \( v_Q(\vec{k}) \) by the more general \([\Delta(\vec{k})]^{(p-d)/2}\). For instance, in a spin-lattice version of the Quantum Hall problem, \( p = 1 \) and \( d = 2 \). Here, in the continuum limit, the minimizing modes are \( q \sim Q^{1/2+d-p} \) and as the reader will later be able easily verify all our upcoming analysis can be reproduced for any generic long range frustrating interaction with identical conclusions.

In Fig.(1), we schematically depict the manifold \( (M) \) of the minimizing modes in \( \vec{k} \) space. When no symmetry breaking terms (\( \lambda = 0 \)) are present, in the continuum limit it is \( M \) is the surface of sphere of radius \( Q^{1/4} \). If \( \lambda \neq 0 \) this degeneracy will be lifted: only a finite number of modes will minimize the energy. When \( \lambda > 0 \) there will be \( 2d \) minimizing modes (denoted by the big \( X \) in the figure) along the coordinate axes. In the up and coming we will focus mainly on \( \lambda \geq 0 \). When \( \lambda < 0 \), a moment’s reflection reveals that there will be \( 2d \) minimizing modes along the diagonals, i.e. parallel to \( (\pm 1, \pm 1, \pm 1) \) (and in this case, they will have a modulus which differs from \( Q^{1/4} \)). Unless explicitly stated otherwise, we will set \( \lambda = 0 \) for calculational convenience and when a finite \( \lambda \) is invoked it will be made positive (to avoid the \( \lambda \) dependence of \( |q| \) incurred when the former is negative). At times, we will present results for \( v_Q(\vec{k}) \) at sizable \( Q \), even though the model was motivated as a good caricature of \( v_{\text{Mott}} \) only in the continuum limit (at small wave-vectors \( q \sim Q^{1/4} \)).

**Membranes**

In several fluctuating membrane systems, the affinity of the molecular constituents (say A and B) for regions of different local curvature frustrates phase separation [2]. Let us define \( S(\vec{x}) \) to be the difference between the A and B densities at \( \vec{x} \). In the continuum, the energy of the system contains a contribution,

\[
H_{\text{mix}} = \frac{b}{2} \int d^2x |\nabla S|^2
\]  

reflecting the demixing of A and B species. Instead of considering long-range frustrating interactions, we now allow for out-of-plane (bending) distortions of the sheet. Specifically, we assume that the two molecular constituents display an affinity for regions of different local curvature of the sheet. This tendency can be modeled by introducing a coupling term between the local composition \( S(\vec{x}) \) and the curvature of the sheet. If the distortions remain small, we may write

\[
H_c = \int d^2x \left[ \frac{1}{2} |\nabla h(\vec{x})|^2 + \frac{\kappa}{2} |\nabla^2 h(\vec{x})|^2 + \Lambda S(\vec{x}) |\nabla^2 h(\vec{x})|^2 \right] 
\equiv \int d^2x H_c,
\]

where \( h(\vec{x}) \) represents the height profile of the sheet (relative to a flat reference state), \( \sigma \) is its surface tension, and \( \kappa \) is its bending modulus; \( \Lambda \), the coefficient of the last term in the expression measures the strength of the coupling of the local curvature \( \nabla^2 h \) and the local composition \( \phi \), which we have included here to lowest (bilinear) order. This coupling term reflects the different affinities of the molecular constituents A (\( S = 1 \) corresponds to pure A composition) and B (\( S = 0 \) corresponds to pure B composition) for, respectively, convex \( (\nabla^2 h > 0) \) and concave \( (\nabla^2 h < 0) \) regions of the interface. The variational Eqs. for the total energy \( H = H_\phi + H_c \), with respect to the membrane shape \( \{h(\vec{x})\} \) read

\[
-\sigma \nabla^2 h + \kappa \nabla^2 (\nabla^2 h) + \Lambda \nabla^2 S = 0.
\]

If \( |\kappa \nabla^2 (\nabla^2 h)| \ll \min \{|\sigma \nabla^2 h|, |\Lambda \nabla^2 S|\} \) then

\[
\Lambda S(\vec{x}) = \sigma h(\vec{x}) + g(\vec{x}),
\]

with \( g(\vec{x}) \) a harmonic function satisfying boundary conditions. In \( \mathcal{H}_c \), after an integration by parts,

\[
\Lambda S \nabla^2 h \approx \Lambda S \frac{\Lambda}{\sigma} \nabla^2 S \to -\frac{\Lambda^2}{\sigma} (\nabla S)^2.
\]

\[
H_{\text{mix}} + H_c \approx \int d^2x \left[ \frac{1}{2} b' |\nabla S|^2 + \frac{\Lambda^2 \kappa}{2\sigma^2} |\nabla^2 S|^2 \right],
\]

where \( b' \equiv b - \frac{\Lambda^2}{\sigma} \). This effective Hamiltonian, which is a function of \( S \) alone, reads

\[
H = \int d^2k \; v_{\text{membrane}}(\vec{k}) |S(\vec{k})|^2.
\]
where $v_{\text{membrane}}(\vec{k}) = \frac{\mu}{2} k^2 + \frac{\Delta^2}{2} k^4$ is the 2D Fourier transform. A negative $\vec{b}$ obtained when $b < \Delta^2/\sigma$, signals the onset of a curvature instability of the sheet. This instability generates a pattern of domains that differ in composition as well as in local curvature and thus assume convex or concave shapes. The characteristic domain size corresponds to the existence of the minimum of the free energy at a non-zero wave number. The modulation length $d \sim q^{-1} \simeq \sqrt{(\Delta^2/\sigma^2)}/|\vec{b}|$ with $q = |\vec{q}|$ the minimizing wavenumber modulus of $v_{\text{membrane}}(\vec{k})$. After scaling, this model may be regarded as the continuum version of the frustrated short range kernel
\[ v_s(\vec{k}) = z\Delta^2(\vec{k}) - \Delta(\vec{k}), \tag{13} \]
where $z = -\Delta^2/2|\vec{b}'|$ on the lattice. The real-space lattice Laplacian
\[ \langle \vec{x} | \Delta | \vec{y} \rangle = \left\{ \begin{array}{ll} 2d & \text{for } \vec{x} = \vec{y} \\ -1 & \text{for } |\vec{x} - \vec{y}| = 1. \end{array} \right. \tag{14} \]
All matrix elements $\langle \vec{x} | \Delta^2 | \vec{y} \rangle = 0$ for $|\vec{x} - \vec{y}| > R = \text{Range}$. Our system is of Range = 2. Explicitly,
\[ \langle \vec{x} | \Delta^2 | \vec{y} \rangle = \begin{cases} 2d(2d + 2) & \text{for } \vec{x} = \vec{y} \\ -4d & \text{for } |\vec{x} - \vec{y}| = 1 \\ 2 & \text{for } (\vec{x} - \vec{y}) = (\pm \hat{\epsilon}_x \pm \hat{\epsilon}_y) \text{ where } \ell \neq \ell' \\ 1 & \text{for } \pm 2\hat{\epsilon}_\ell \text{ separation}. \end{cases} \tag{15} \]
We will extend the investigation of this model over a broader range of parameters than suggested by its initial physical motivation. In the continuum limit, theories with high order derivative terms generally give rise to $v(\vec{k}) = P(k^2)$, where $P$ is some polynomial. Although $v_s(\vec{k})$ and its likes are artificial on the lattice, their continuum limit is quite generic. Later on we will show that if $P(k^2)$ attains its global minima at finite $|\vec{k}|$, then thermal instabilities can incur extremely low values of $T_c$.

IV. ISING GROUND STATES

In most sections to follow, the bulk of our reported results pertain to continuous spins. We now, however, present a short overview of discrete spin systems in order present new results and to illustrate how Ising systems may be addressed within a similar momentum space framework. Towards the end of this section, will establish the existence of a new “holographic” ground state entropy in certain incommensurate Ising systems wherein the ground state entropy scales as the surface area of the system.

In an “Ising” system $S(\vec{x}) = \pm 1$ at all lattice sites $\vec{x}$. Stated alternatively, the scalar $(n = 1)$ spins satisfy a normalization constraints $\{S^2(\vec{x}) = 1\}$ at all $N$ lattice sites $\vec{x}$. Henceforth, we will adopt the latter point of view. The set of minimizing wave-vectors $\vec{q}$, $v(\vec{q} \in M) \equiv \min_{\vec{k}}(v(\vec{k}))$ defines a manifold $M$. If the local normalization constraints are swept aside then it is clear that the ground states are superpositions of sinusoidal waves with wave-vectors $\vec{q} \in M$. One would expect this to be true, in spirit, also in the highly constrained Ising case, if $v(\vec{k})$ is sharply dipped at its global minima. If, in a non-rigorous setting, we “digitize” a particular plane wave $S(\vec{x}) = \text{sign}(\cos(\vec{q}_i \cdot \hat{\vec{x}}))$ and compare it with the exact (numerical) ground state, then we will find encouraging agreement in certain cases. For instance, this gives reasonable accord when $H = H^{Mott}$ [1,5,6]. This Hamiltonian (with some twists) was investigated in [4] on a square ($d = 2$) lattice. In the continuum limit (i.e. if the lattice is thrown away) we might naively anticipate a huge ground state degeneracy- a “digitized plane wave” for each wave vector $\vec{q}$ lying on the $(d - 1)$ dimensional manifold $\{M_Q : q^d = Q\}$ This large degeneracy might give rise to a loss of stability against thermal fluctuations. Striped phases (i.e. “digitized plane waves”) were found in virtually all of the parameter range [1,5,6]. Only for a very small range of parameters were more complicated periodic structures found.

An intuitive feeling can be gained by considering a one dimensional pattern such as
\[ + + - - - - - - - - ... \tag{16} \]
This pattern is a pure mode $S_{\text{period}=4}(x) = \sqrt{2} \cos[\frac{\pi}{2} x - \frac{\pi}{2}]$. With this observation at hand, a double checkerboard pattern such as
\[ + + - - + + - - + + - - ... \]
\[ - - + + - - + + - - + + - - ... \]
extending in all directions in the plane is given by
\[ S(\vec{x}) = 2 \cos(\frac{\pi}{2} x_1 - \frac{\pi}{4}) \cos(\frac{\pi}{2} x_2 - \frac{\pi}{4}) = \cos(\frac{\pi}{2} (x_1 + x_2) - \frac{\pi}{2}) + \cos(\frac{\pi}{2} (x_1 - x_2)). \tag{17} \]
Such a $4 \times 4 \times 4$ periodic pattern in three dimensions would include the eight modes $\frac{\pi}{2}(\pm \pi, \pm \pi, \pm \pi)$. This example illustrates an simple premise. A system composed of a periodic building block whose dimensions $p_1 \times p_2 \times p_3$, is given by
\[ S(\vec{x}) = S_{p_1}(x_1)S_{p_2}(x_2)S_{p_3}(x_3). \tag{19} \]
If a configuration $S_p(x)$ contains the modes $\{k^m_p\}$ with amplitudes $\{S_p(k^m_p)\}$, then Fourier transforming the periodic configuration $S(\vec{x})$ one will find the modes $\{\pm k^m_{p_1}, \pm k^m_{p_2}, \pm k^m_{p_3}\}$ appearing with a weight $\sim |S_p(k_1) \times S_p(k_2) \times S_p(k_3)|^2$. For high values of the periods $p$, the weight gets scattered over a large set of wave-vectors. If $v(\vec{k})$ has sharp minima, such states will not
be favored. The system will prefer to generate patterns s.t. in all directions \( i \) albeit one \( p_i = 1 \) (or perhaps 2). For a \( p_1 \times p_2 \times p_3 \) repetitive pattern, the discrete Fourier Transform will be nonzero for only \( \prod_{i=1}^{3} p_i \) values of \( \vec{k} \). This trivial observation suggests the phase diagram obtained by [6] in two dimensions. The intuition is obvious. We have derived [5], rigorously, the ground states in only several regions of its parameter space (those corresponding to ordering with half a reciprocal lattice vector), and on a few special surfaces (corresponding to ordering with a quarter of a reciprocal lattice vector). In all of these cases the Ising states may be expressed as superpositions of the lowest energy modes \( \exp[i\vec{q}_n \cdot \vec{x}] \). Lately, a beautiful extension was carried out by [18]. Another nice work, focusing on scalar fields in the continuum is [19]. In the current publication, however, we are concerned only with spins on a lattice. We now ask whether commensurate lock-in is to be expected. The energy of the Ising “digitized plane wave” on an \( L \times L \times L \) lattice where \( \vec{q} = (q_1, 0, 0) \) with \( q_1 = 2\pi/m \), with even \( m \), reads

\[
E = \frac{1}{2N} \sum_{\vec{k}} v(\vec{k})|S(\vec{k})|^2 = \frac{8}{m} \sum_{j=1,3,\ldots,m-1} \frac{v(\vec{k} = (\frac{2\pi j}{m}, 0, 0))}{\exp[2\pi i j/m] - 1^2} = \frac{2}{m} \sum_{j=1,3,\ldots,m-1} \frac{v(\vec{k} = (\frac{2\pi j}{m}, 0, 0))}{\sin^2(\pi j/m)}. \tag{20}
\]

The lowest energy state amongst all states of the form considered is a possible candidate for the ground state.

Unlike the previous paragraphs and those to follow, we now try to present the reader with an intuitive feeling. The following paragraph is strictly non-rigorous. For the particular model long-range introduced in Eq.(6), it seems that for small values of \( Q \), it might be worthwhile to have an incommensurate phase. This is, in a sense, obvious- all low energy modes are of very small wave-number \( q \) and hence not of low commensurability. In a stripe phase having of size \( 2\pi|\vec{q}_{\text{ground}}|^{-1} \), the energy

\[
E = \frac{1}{N} \sum_{n=0}^{\infty} 16v(\vec{k} = (2n + 1)\vec{q}_{\text{ground}}) \frac{(2n + 1)^2\pi^2}{(2n + 1)^2\pi^2} \tag{21}
\]

In the example of Eq.(6) for \( q_{\text{ground}} \ll 1 \), the higher harmonics \( \vec{k} = (2n + 1)\vec{q} \), do not entail high energies. For large values of \( Q \), \( q_{\text{ground}} \simeq O(1) \), and \( v(\vec{k} = (2n + 1)\vec{q}_{\text{ground}}) \) can be very large if \((2n + 1)\vec{q}_{\text{ground}} \) approaches a reciprocal lattice vector \( \vec{K} \). Under these circumstances it will pay off to have a commensurate structure; for a \( u_1 \times u_2 \times \ldots \times u_d \) repetitive block only the modes \( \vec{k} = 2\pi(\frac{u_1}{q_1}, \frac{u_2}{q_2}, \ldots, \frac{u_d}{q_d}) \) will be populated (i.e. have a non-vanishing \( |S(\vec{k})|^2 \)) - the ferromagnetic point (a reciprocal lattice point) will not be approached arbitrarily close- if that is not true weight will be smeared over energetic modes. Generically, we will not be expect commensurate lock-in in lim \( q \rightarrow 0 \) for any theory with a frustrating long range interaction. Although we have considered only striped phases (which have previously argued are the only ones generically expected), it is clear that this argument may be reproduced for more exotic configurations (such as bubbles, dots, cylinders etc.) As an aside, by replacing Fourier lattice sums by integrals, we note that in the continuum limit of \( H_{\text{Mott}} \), the ground state wave-vector in three dimensions, is immediately seen to be \( q_{\text{ground}} \sim Q^{1/3} \) [20] - not the value suggested by dimensional analysis (if the lattice constant units are ignored) - the minimizing wavenumber \( q \sim Q^{1/4} \). We will prove that for continuous spins \( q_{\text{ground}} = q \). Similar results are obtained for the full Hamiltonian \( H_{\text{Mott}} \) (not only its continuum limit) and related systems, e.g. [18], [17], [21].

For finite range interactions it is easy to prove, by covering the system with large maximally overlapping blocks, that there will be a regime about \( \vec{q} = 0 \), for which we will find the ferromagnetic ground state. A polynomial in \( \Delta(\vec{k}) \) will have its minima at \( \Delta(\vec{k}) = \text{const} \), i.e. on a (d-1) dimensional hypersurface(s) in \( \vec{k} \)-space or at the (anti)ferromagnetic point. The kernel \( v_\ell(\vec{k}) = i\Delta^2 - \Delta \) has its minima \( (z > 0) \) at \( \vec{q} \) in \( M_z : \Delta(\vec{q}) = \min(\frac{1}{\Delta}, 4d) \). For \( z > \frac{1}{\Delta} \) : \( M_z \) is (d - 1) dimensional. We may divide the lattice into all maximally over-lapping \( 5 \times 5 \times \ldots \times 5 \) hyper-cubes centered about each site of the lattice.

\[
\text{Energy} = \frac{1}{5 \times 6^{d-1}} \sum_{\text{hypercubes}} \epsilon(\text{hypercube}) \tag{22}
\]

and evaluate the energies \( \epsilon \) of all \( 5 \times 5 \times \ldots \times 5 \) Ising configurations. Of all \( 2^{5d} \) configurations the Néel state will have the lowest energy for a sliver about \( z = \frac{1}{\Delta} \). Analogously for \( z > z_{\text{top}} \gg 1 \), by explicit evaluation, the ground state will be ferromagnetic. Contour arguments can be employed and a finite lower bound on \( T_c \) generated.

If such a system satisfies periodic boundary conditions along the two square lattice diagonals \( \vec{e}_\pm : \) defined by \( x_1 \mp x_2 = \text{const} \), (with \( x_i \) the Cartesian coordinates), then an exponentially large number of Ising ground states can be constructed whenever \( z = \frac{1}{\Delta} \). To illustrate this, we note that when \( z = \frac{1}{\Delta} \), all minimizing modes lie on

\[
\vec{q} \in M_{z=\frac{1}{\Delta}} : |q_1 \pm q_2| = \pi. \tag{23}
\]

By prescribing an arbitrary spin configuration along \( x_- \) and fixing \( S(x_-, x_+) = S(x_-, 0)(-1)^{x_-} : \)

\[
S(\vec{k}) = \sum_{x_-} S(x_-, 0) \exp(ik_+x_-) \sum_{x_+} (-1)^{x_+} \exp(ik_+x_+) \tag{24}
\]
vanishes for $|k_+| = |k_1 + k_2| \neq \pi$. Similarly, by taking the transpose of these configurations we can generate patterns having $S(\vec{k}) = 0$ unless $|k_-| = |k_1 - k_2| = \pi$. The ground state degeneracy is bounded from below by the number of independent spin configurations that can be fashioned along $x_+$ or $x_-, (2^L + 1 - 2)$ where $L$ is the length of the system along the $x_\pm$ axis. The number of $\vec{q}$ values, commensurate with the diagonal periodic boundary conditions, lying on $M_{\vec{q}} = \frac{\pi}{2}$ is $(4L - 2)$. We have thus proved that the ground state entropy in this two dimensional system is, at least, linear in the perimeter of the system. In a later section, we will later demonstrate that in the large $n$ limit, such “holographic” like entropies do generically arise whenever the minimizing manifold is (d-1) dimensional. The Ising ground state degeneracies are bounded from above by those of the spherical model (as any Ising configuration is a viable configuration in the spherical model). As the ground state entropy for our model is bounded both from above and from below by the perimeter of the system, the ground state entropy rigorously scales as the system size $L$.

Similarly, in dimensions $d > 2$, we can set $(d-2)$ of the $\vec{q}$ components to zero. There are $d(d-1)/2$ cross-sections of the d-dimensional $M_{\vec{q}} = 0$, all looking like the the two-dimensional $M$ just discussed (i.e. $|q_1 \pm q_2| = \pi$). The real-space ground state degeneracy is bounded from below by $d(d-1)(2^L - 1)$ (along the $(d-2)$ zero-mode directions the ground state spin configurations display no flip). If we regard each diagonal row of spins as a “super-spin” then we will see that flipping any “super-spin” entails no energy cost. This is reminiscent to a nearest neighbor Ising chain where the energy cost for flipping a spin is dwarfed by comparison to the (logarithmically) extensive entropy. We might expect that here, too, ordering might be somewhat inhibited. In two dimensions, 

$$\lim_{z \to \infty} M_z: \vec{q}^2 = 1,$$

(25)

the “average” number of allowed $\vec{q} \in M_z$ values $\ll O(L)$ (and similarly for the onset $lim_{q \to 0} M_z : (\vec{q} - (\pm \pi, \pm \pi))^2 = (\frac{1}{8q} - \frac{1}{2\pi})^2$). For a hypercubic lattice of size $L_1 \times L_2 \times \ldots \times L_d$ in $d > 2$ many discrete reciprocal points will give rise to the same value of $\Delta(\vec{k}) \sim \vec{k}^2$. The proof is trivial: if all $L_i = L$, then the number of possible $\vec{k}^2$ values is bounded by $dL^2$, whereas there are $L^d \ k$-values. Therefore, on “average”, the number of $k$ points lying on $M_z$, or more precisely lying the closest to $M_z$, s.t. $|\Delta(\vec{k}) - \frac{1}{2\pi}|$ is min, is, at least, $O(L^{d-2})$.

Of these, $\frac{d^d}{4^d(n_d)!}$ wave-vectors, with $n_i$ (and $z$) denoting the number of identical components (and the number of zero components) of a certain $\vec{k}_i$ nearest to $M_z$, are related to $\vec{k}$ by symmetry. As we have stated previously, in the continuum limit $(q \to 0)$ any short range kernel (including this one) will have a uniform (ferromagnetic) ground state. However the impossibility of constructing ground states that contain only “good” Fourier modes $S(\vec{k} \in M)$ when $M$ shrinks to a curved surface enclosing the origin is more general and will proved in the next section. For this short range model, even for $z \neq \frac{1}{d}$ a huge ground state degeneracy is expected. A “plane wave” might correspond to each wave-vector $\vec{q}$ (or commensurate wave-vectors nearby) lying on the $(d-1)$ dimensional manifold $M$. As we shall prove later on, even in high dimensions, and even if the interactions are long ranged, in the continuum limit it will not be possible to construct Ising states in which $S(\vec{k} \in M) = 0$ unless the minimizing manifold $M$ contains flat non-curved segments (or more generally intersects a plane at many points).

V. A UNIVERSAL PEIERLS BOUND

In this section, we illustrate how a Peierls’ bound can be obtained in a host of systems with two spin interactions. Although we specialize to translationally invariant two spin interaction in this section, as we will show argue later, the technique that we outline below can be extended to non-translationally invariant systems as well. In Peierls’ elegant proof of long range order in the two and higher dimensional Ising models at sufficiently low temperatures, one of the necessary ingredients is the Peierls’ bound. This bound amounts to the demonstration that having one ground state domain surrounded by another costs an energy which is, at least, linear in the length of their interface. In this section, we prove that such a bound is quite generic to many long range systems. Albeit suggestive, the bound, on its own, does not suffice to demonstrate long range order. The unfamiliar reader may peruse [22]. If a real hermitian kernel $v(\vec{k})$ attains its minima in only a finite number of commensurate reciprocal lattice points $\{\vec{q}_i\}$, then a Peierls bound can, in some instances, be proven for an infinite range model: When possible this is suggestive of a finite $T_c$. For instance, the bound for a (lattice) Coulomb gas (with the kernel
solving the discrete Laplace equation on the lattice) is trivially generated.

\[ v_c(\vec{k}) - v_c(\vec{q}) = e/\Delta(\vec{k}) - e/\Delta(\vec{q}) = \pi, \pi, \pi \]
\[ \geq -A(\Delta(\vec{k}) - \Delta(\vec{q}) = \pi, \pi, \pi)), \]

with \( A = 16d^2 \). Here, the Hamiltonian

\[ H = \frac{1}{2N} \sum_k v_c(\vec{k}) |S(\vec{k})|^2 = \frac{1}{2N} \sum_k \beta \Delta(k)|S(\vec{k})|^2 \]

(27)
depicts a lattice Coulomb interaction (wherein the spins portray charges). The right hand side of Eq.(26) is the kernel of an antiferromagnet. Both system share the same ground states. For a given configuration the energy penalty for the Coulomb gas

\[ \Delta E_c = 1/(2N) \sum_k |v_c(\vec{k}) - v_c(\vec{q})| |S(\vec{k})|^2 \]

(28)
is bounded from below by the corresponding penalty in an antiferromagnet of strength \( A \). In \( d = 2 \) the contour penalty of the antiferromagnet is \( 2A|\Gamma| \), \(|\Gamma| \equiv \text{length of the contour } \Gamma \). A similar trick may frequently be employed when the minimizing wave-vectors attain other commensurate values. It relies on comparison to a short range kernel for which a Peierls bound is trivial. This can be extended quite generally. All translationally invariant system with commensurate minimizing wave-vectors \((\vec{q} = (0,0,\ldots,0), (\pi,0,\ldots,0), \ldots)\) at which the minimum of \( v(\vec{k}) \) at \( \vec{k} = \vec{q} \) is quadratic may be bounded by a kernel of a system having nearest neighbor ferromagnetic and antiferromagnetic bonds for which the Peierls bound is trivial (linear in the perimeter of the domain wall \( \Gamma \)).

VI. ISING WEISS MEAN-FIELD THEORY

We now estimate the critical temperature for all incommensurate spin system in which within mean-field. If, in these systems, when \( T < T_c \), the on-site magnetization \( \langle S(\vec{x}) \rangle = s \text{ sign}(\cos(\vec{q} \cdot \vec{x})) \) (as suggested by some of the examples discussed hitherto), then the sum

\[ \sum_{\vec{y}} \langle S(\vec{y}) \rangle V(\vec{x} = 0, \vec{y}) = \frac{1}{N} \sum_k \langle S(\vec{k}) \rangle v(-\vec{k}) \]
\[ = s \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)\pi} [v(\vec{k} = (2n+1)\vec{q}) + v(\vec{k} = -(2n+1)\vec{q})], \]

where we assumed \( q_0 = \frac{\lambda}{u_i} \) with \( u_i \gg 1 \) for all \( d \) components in replacing a discrete Fourier transform sum by an integral. The self-consistency equation

\[ s = \langle S(\vec{x} = 0) \rangle = -\tanh[\beta \sum_{\vec{y}} V(\vec{x} = 0, \vec{y}) \langle S(\vec{y}) \rangle] \]

(29)
yields

\[ \beta_c^{-1} = \left( \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} v(\vec{k} = (2n+1)\vec{q}) \right). \]

(30)

The non-trivial solution (with a non-zero magnetization \( s \)) should be self-consistent at all lattice sites (not only at \( \vec{x} = 0 \)). Self consistency at other values of \( \vec{x} \) will lead to other lower bounds on \( T_c \) which will read

\[ \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} v(\vec{x} = (2n+1)\vec{q}) \cos[(2n+1)\vec{q} \cdot \vec{x}]. \]

(31)

VII. O(N ≥ 2) GROUND STATES

Henceforth, the bulk of the article focuses on continuous spin systems having \( n \geq 2 \) components. In an \( O(n) \) spin system, the spins \( \{\vec{S}(\vec{x})\} \) have \( n \) components and are all normalized to unity- \( \vec{S}(\vec{x}) = 1 \).

A common intuitive picture held by many is that in complex systems of the type presented in Section(III) is the complex ground states can arise. For instance, in real systems, nanoscale phase separation in the models of Section(III) can proceed through a certain class of viable ground states. Wigner crystals and arrays of dots are anticipated at large \( q \). As \( q \) is decreased, the spatial size of the dots increases, gradually leading to a state approaching macroscopic phase separation. Phase diagrams exhibiting devil staircases are not uncommon in scalar systems. Albeit their intuitive appeal and occurrence in many continuum systems having scalar order parameters \([2, 5, 6, 19]\), ground states such as these do not generically occur in any \( O(n \geq 2) \) system on a cubic lattice. In the up and coming we will prove that, barring special commensurability effects, any translationally invariant \( O(n \geq 2) \) system will generically display only spiral (stripe like) or screw like \([24]\) ground states. For a discussion of the occurrence of dots, Wigner crystals etc. in the scalar case, the reader is referred to \([2, 5, 6, 19]\).

Configurations such as these were detected numerically (\([4]\)) and analytically for the scalar spherical model (\([5]\)) for the spin models that we discuss. We will explicitly prove, for the first time, that these do not generically occur in incommensurate continuous finite \( n \) spin systems by deriving the relations to be satisfied by the ground state configurations containing a number of pairs of minimizing wave-numbers \( \{\pm \vec{q}_i\} \). We will illustrate that special Wigner crystalline like patterns may occur for special commensurate wave-numbers alone. We explain how these can arise in the spherical limit in Section(XIV).

We first note that our previous ansatz \( S(\vec{x}) = \text{sign}(\cos(\vec{q} \cdot \vec{x})) \) is readily fortified in the \( O(n \geq 2) \) scenario: here there is no need to “digitize”- in the spiral.
\[ S_1(\vec{x}) = \cos(\vec{q} \cdot \vec{x}), \quad S_2(\vec{x}) = \sin(\vec{q} \cdot \vec{x}), \quad S_{i>2}(\vec{x}) = 0 \tag{32} \]

The only non-zero Fourier components are \( \vec{S}(\vec{q}), \vec{S}(-\vec{q}) \). In plain terms, this state can be constructed with the minimizing wave-vectors only. It follows that any ground state \( g \) must be of the form

\[ S_i(\vec{x}) = \sum_m a_i^m \cos(\vec{q}_m \cdot \vec{x} + \phi_i^m). \tag{33} \]

We now turn to the normalization of the spins at all sites,

\[ 1(\vec{x}) = \sum_{i=1}^n S_i^2(\vec{x}) = \frac{1}{2} \sum_{m, m'} \sum_{i=1}^n a_i^m a_i^{m'} \left[ \cos(\phi_i^m + \phi_i^{m'}) \cos([\vec{q}_m + \vec{q}_{m'}] \cdot \vec{x}) - \sin(\phi_i^m + \phi_i^{m'}) \sin([\vec{q}_m + \vec{q}_{m'}] \cdot \vec{x}) \right. \\
- \cos(\phi_i^m - \phi_i^{m'}) \cos([\vec{q}_m - \vec{q}_{m'}] \cdot \vec{x}) + \sin(\phi_i^m - \phi_i^{m'}) \sin([\vec{q}_m - \vec{q}_{m'}] \cdot \vec{x}) \right]. \tag{34} \]

If \( 1(\vec{x}) = 1 \) is to hold identically for all sites \( \vec{x} \), then all non-zero Fourier components must vanish. For the \( \{\cos(\vec{A} \cdot \vec{x})\} \) components:

\[ 0 = \left[ \sum_{\vec{q}_m + \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n a_i^m a_i^{m'} \cos(\phi_i^m + \phi_i^{m'}) \right. \]
\[ + \left. \sum_{\vec{q}_m - \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n a_i^m a_i^{m'} \cos(\phi_i^m - \phi_i^{m'}) \right], \tag{35} \]

and a similar relation is to be satisfied by the \( \{\sin(\vec{A} \cdot \vec{x})\} \) components:

\[ 0 = \left[ \sum_{\vec{q}_m + \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n a_i^m a_i^{m'} \sin(\phi_i^m + \phi_i^{m'}) \right. \]
\[ + \left. \sum_{\vec{q}_m - \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n a_i^m a_i^{m'} \sin(\phi_i^m - \phi_i^{m'}) \right], \tag{36} \]

\[ a_i^m \cos \phi_i^m \equiv v_i^m, \quad a_i^m \sin \phi_i^m \equiv u_i^m. \tag{37} \]

The \( \{\cos(\vec{A} \cdot \vec{x})\} \) and \( \{\sin(\vec{A} \cdot \vec{x})\} \) conditions read

\[ 0 = \left[ \sum_{\vec{q}_m + \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n [v_i^m v_i^{m'} - u_i^m u_i^{m'}] \right. \]
\[ + \left. \sum_{\vec{q}_m - \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n [v_i^m u_i^{m'} + u_i^m v_i^{m'}] \right], \]

\[ 0 = \left[ \sum_{\vec{q}_m + \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n [v_i^m u_i^{m'} - u_i^m v_i^{m'}] \right. \]
\[ + \left. \sum_{\vec{q}_m - \vec{q}_{m'} = \vec{A}} \sum_{i=1}^n [v_i^m v_i^{m'} + u_i^m u_i^{m'}] \right]. \tag{38} \]

Let us now consider the case of two pairs of minimizing modes. For two pairs of wave-vectors \( \pm \vec{q}_1 \) and \( \pm \vec{q}_2 \) minimizing \( v(\vec{k}) \), both not equal half a reciprocal lattice vector: \( (0,0,\ldots), (\pi,0,\ldots,0), (0,\pi,0,\ldots,0), \ldots, (\pi,\pi,0,\ldots,0), \ldots, (\pi,\pi,\ldots,\pi) \), the vector \( \vec{A} \) (up to an irrelevant sign) may attain four non-zero values: \( \vec{A} = 2\vec{q}_1, 2\vec{q}_2, \vec{q}_1 \pm \vec{q}_2 \).

When \( \vec{A} = \vec{q}_1 - \vec{q}_2 \), the conditions are

\[ 0 = \sum_{i=1}^n [v_i^1 v_i^2 - u_i^1 u_i^2], \quad 0 = \sum_{i=1}^n [v_i^1 u_i^2 + u_i^1 v_i^2]. \tag{39} \]

When \( \vec{A} = \vec{q}_1 + \vec{q}_2 \), these conditions read

\[ 0 = \sum_{i=1}^n [v_i^1 v_i^2 + u_i^1 u_i^2], \quad 0 = \sum_{i=1}^n [v_i^1 u_i^2 - u_i^1 v_i^2]. \tag{40} \]

For \( \vec{A} = 2\vec{q}_\alpha \ (\alpha = 1, 2) \):

\[ 0 = \sum_{i=1}^n [v_i^\alpha v_i^\alpha - u_i^\alpha u_i^\alpha], \quad 0 = 2 \sum_{i=1}^n u_i^\alpha v_i^\alpha. \tag{41} \]

Next, we define

\[ \vec{U}_\alpha \equiv (u_{i=1}^\alpha, u_{i=2}^\alpha, \ldots, u_{i=n}^\alpha) \]
\[ \vec{V}_\alpha \equiv (v_{i=1}^\alpha, v_{i=2}^\alpha, \ldots, v_{i=n}^\alpha). \tag{42} \]

The previous conditions for \( \vec{A} = \vec{q}_1 \pm \vec{q}_2 \) imply that

\[ \vec{V}^1 \cdot \vec{V}^2 = \vec{U}^1 \cdot \vec{U}^2 = 0 \]
\[ \vec{V}^1 \cdot \vec{V}^2 = \vec{U}^1 \cdot \vec{U}^2 = 0 \]
\[ \vec{U}^1 \cdot \vec{V}^1 = \vec{U}^2 \cdot \vec{V}^2 = 0. \tag{43} \]

The four vectors \( \{\vec{U}^1, \vec{U}^2, \vec{V}^1, \vec{V}^2\} \) are all mutually orthogonal. The number of spin components \( n \geq 4 \). Two additional demands that follow are

\[ \vec{V}^\alpha \cdot \vec{V}^\alpha = \vec{U}^\alpha \cdot \vec{U}^\alpha \]
\[ \sum_{\alpha=1}^2 [\vec{V}^\alpha \cdot \vec{V}^\alpha + \vec{U}^\alpha \cdot \vec{U}^\alpha] = 2 \sum_{\alpha=1}^2 \vec{V}^\alpha \cdot \vec{V}^\alpha = 2. \tag{44} \]

The last equation is the normalization condition- the statement that the coefficient of \( \cos(\vec{A} \cdot \vec{x}) \), when \( \vec{A} = 0 \), is equal to 1. For the case of a single pair of wave-vectors, \( \pm \vec{q}_1 \), \( \vec{A} = 2\vec{q}_1 \), and the sole conditions are encapsulated in the last of equations(42) and in equation (43). A moment’s reflection reveals that this only allows for a spiral in the plane defined by \( \vec{U}^1 \) and \( \vec{V}^1 \). When \( n < 4 \) there are no configurations which satisfy \( \vec{S}^2(\vec{x}) = 1 \) identically for all sites \( \vec{x} \) [excluding those having \( 2(\vec{q}_1 + \vec{q}_2) \) = \( \vec{A} \) equal to a reciprocal lattice vector] that are a superposition of exactly two modes. For instance, with the convention that a “double checkerboard state” is a Neel state of \( 2 \times 2 \) blocks \( [6] \) \( p_1 = p_2 = 2 \) in the two dimensional planar version of Eq.(19), we note that a (double checkerboard
state along the $i = 1$ axis) $\otimes$ (a spiral in the 23 plane) has pairs $(i, j)$ is a configuration in which $\vec{A} = 2(\vec{q}_{i} + \vec{q}_{j})$ is a reciprocal lattice vector. As the number of minimizing modes $\{\vec{q}_{m}\}$ increases, some of the conditions may degenerate into one, e.g. if $(\vec{q}_{i} + \vec{q}_{j}) = (\vec{q}_{3} - \vec{q}_{2})$ (i.e. the modes are collinear). This degeneracy is the second route that might allow for Ising configurations which are superpositions of several “good”/minimum energy modes $\exp(i\vec{q} \cdot \vec{x})$. The highly degenerate Ising ground states that we have constructed previously can fall under either one of these categories. If neither one of these situations occurs, Ising states cannot be superpositions of several minimum energy modes: we will be left with too many equations of constraints with too few degrees of freedom.

For three pairs of minimizing modes, none of which is a half a reciprocal lattice vector, $\{\pm \vec{q}_{m}\}_{m=1}^{3}$ with

$$\vec{q}_{w} \pm \vec{q}_{t} \neq \vec{q}_{c} \pm \vec{q}_{f} \neq 2\vec{q}_{w}$$

for all $w \neq t$, and $r \neq s$, conditions similar to those that previously written for $\vec{A} = \vec{q}_{1} \pm \vec{q}_{2}$, now hold for all $(\vec{q}_{w} \pm \vec{q}_{t})$.

$$\vec{U}_{\alpha} \cdot \vec{U}_{\beta} = \vec{V}_{\alpha} \cdot \vec{V}_{\beta} = \vec{V}_{\alpha}^{2} = \vec{U}_{\alpha} \cdot \vec{V}_{\beta} = 0$$

The relation $\vec{U}_{\alpha} \cdot \vec{V}_{\alpha} = 0$ ($\alpha = \beta$ in the last Eq above) is enforced by setting $\vec{A} = 2\vec{q}_{o}$. Thus, when exactly three pairs of minimizing wave-vectors satisfying the equation are present, the vectors $\{\vec{U}_{\alpha}, \vec{V}_{\alpha}\}$ define a 6-dimensional space, and hence $n \geq 6$. For $p$ pairs of minimizing wave-vectors, $n$ must be at least $2p$-dimensional. This bound is saturated when $\vec{S}$ is a (spiral state in the 12 plane) $\otimes$ (a spiral in the 34 plane) $\otimes$...$\otimes$ (a spiral in the 2p plane), i.e.

$$(a_{1}\cos(\vec{q}_{1} \cdot \vec{x} + \phi_{1}), a_{1}\sin(\vec{q}_{1} \cdot \vec{x} + \phi_{1}), \ldots, a_{p}\cos(\vec{q}_{p} \cdot \vec{x} + \phi_{p}), a_{p}\sin(\vec{q}_{p} \cdot \vec{x} + \phi_{p}))$$

with $\sum_{\alpha=1}^{p} a_{\alpha}^{2} = 1$. When wave-vectors with $\vec{q}_{w} \pm \vec{q}_{t} = \vec{q}_{c} \pm \vec{q}_{f}$ or $\vec{q}_{w} \pm \vec{q}_{t} = 2\vec{q}_{w}$ are present, pairs of conditions degenerate into single linear combinations. Thus far we assumed that for all $i$ and $j$, $\vec{A} = 2(\vec{q}_{i} + \vec{q}_{j})$ is not a reciprocal lattice vector, s.t. $\sin(\vec{A} \cdot \vec{x})$ is not identically zero at all $\vec{x} \in Z^{d}$. We term the such a $p = 2$ configuration a bi-spiral. It is simple to see by counting the number of degrees of freedom for $n = 4$, that the bi-spirals overwhelm states having only one mode $\pm \vec{q}_{1}$. This is a simple instance of a general trend: High $p$ states are statistically preferred. Moreover, as we shall see later, they are more stable against thermal fluctuations.

In summary, we outlined a way to determine all $O(n \geq 2)$ ground states, whether commensurate or incommensurate, for a given kernel $V(\vec{x}, \vec{g}) = V(\vec{x} - \vec{g})$. Whenever $n \geq 2$, any ground state configuration can be decomposed into Fourier components, $\vec{S}^{\vec{g}}(\vec{x}) = \sum_{i=1}^{\mid M \mid} \left\{ \cos[\vec{q}_{i} \cdot \vec{x}] + \sin[\vec{q}_{i} \cdot \vec{x}] \right\}$. The vectors $\vec{q}_{i}$ are chosen from the set of wave vectors which minimize $v(\vec{k})$. Here, the modulus $\mid M \mid$ is the number of minimizing modes (the “measure” of the modes on the minimizing surface $M$. As long as these wave-vectors $\vec{q}_{i}$ which minimize $v(\vec{k})$ are “non-degenerate”, in the sense that the sum of any pair of wave vectors, $\vec{q}_{i} \pm \vec{q}_{j}$ is not equal to the sum of any other pair of wave vectors, and “incommensurate” in the sense that for all $i$ and $j$, $2(\vec{q}_{i} + \vec{q}_{j})$ is not equal to a reciprocal lattice vector, then that the condition $\vec{S}^{2}(\vec{x}) = 1$ can be satisfied only if $\mid M \mid \leq n/2$. (In our toy model of the doped Mott insulator, these conditions are always satisfied for $Q < 4$.) Thus, for $n \leq 3$ only simple spiral ($\mid M \mid = 1$) ground-states are permitted, while for $n = 4$, a double spiral saturates the bound. Thus, generically, for $2 \leq n \leq 4$ all ground states will be spirals containing only one mode. The reader should bear in mind that in the usual short range ferromagnetic case, the ground states are globally $SO(n)$ symmetric and are labeled by only $(n - 1)$ continuous parameters. Here, for each minimizing mode there are $(2n - 3)$ continuous internal degrees of freedom labeling all possible spiral ground states. For $n > 2$ this guarantees a much higher degeneracy than that of the usual ferromagnetic ground state. If there are many minimizing modes (e.g. if the minimizing manifold $M$ were endowed with $SO(d-1)$ symmetry) then the ground state degeneracy is even larger! When $n \geq 4$, there are (generically) even many more ground states (poly-spirals). These poly-spiral states have a degeneracies larger than those of simple spiral. Their degeneracy

$$g = p(2n - 2p - 1)|M|^{p},$$

where $\mid M \mid$ is the number of minimizing modes. We just proved that if frustrating interactions cause the ground states to be modulated then the associated ground state degeneracy (for $n > 2$) is much larger by comparison to the usual ferromagnetic ground states.

### VIII. SPIN STIFFNESS

In this section, we focus for the sake of concreteness alone, on the Coulomb Frustrated Ferromagnet. A similar analysis, with different results, may be reproduced for all other interactions. When $Q > 0$ in the in the continuum (small $k$ limit) of the Coulomb Frustrated Ferromagnet of Eq.(6), with all $O(k^{4})$ and higher terms ignored, the minimizing modes of $v_{Q}(\vec{k})$ lie on the surface of a sphere $\{M_{Q} : \vec{q}^{2} = \sqrt{Q}\}$. As $Q \rightarrow 0$, this surface $M_{Q}$ shrinks and shrinks yet is still a $(d - 1)$ dimensional surface of a sphere. When $Q = 0$, the minimizing manifold evaporates into a single point $\vec{q} = 0$. This sudden
change in the dimensionality has profound consequences. As we shall see shortly, it lends itself to suggest (quite strongly) that order is inhibited for a Heisenberg \((n = 3)\) realization of our model. Before doing so, let us indeed convince ourselves, on an intuitive level, that the large degeneracy in \(\vec{k}\)-space brought about by the frustration gives rise to a reduced spin stiffness. We now assume an ordered spiral state of momentum \(\vec{q}\) on a cubic lattice (after all, we proved that in the general incommensurate case, these are the only ground states), and we examine the energy cost of varying the modulation wave-vector in a direction parallel and transverse to \(\vec{q}\). The exact origin of the spiral ground is completely irrelevant (whether it is the the Coulomb Frustrated Model, the membrane model or other very different models)- all that matters is that the interaction kernel attains its minima at fine incommensurate momenta \(|\vec{q}|\) which cover a spherical shell of radius \(q\). For general rotationally symmetric incommensurate systems, \(Q\) may be regarded as a convenient shorthand, \(Q = q^4\). The upshot of the below, rather trivial, calculation is that rotationally invariant incommensurate continuous spin systems are very susceptible to transverse perturbations- a transverse twist may incur no energy penalty in the thermodynamic limit.

**A. Longitudinal Spin Stiffness**

We first examine the energy cost of a longitudinal twist to find that for highly incommensurate frustrated systems with shells of minimizing modes, the energy cost for such a perturbation is just the same as in a nearest neighbor three dimensional XY ferromagnet. In a longitudinal twist

\[
\delta \tilde{S}(\vec{x}) = \cos\left(\frac{2\pi x}{L} + qx\right)\hat{e}_1 + \sin\left(\frac{2\pi x}{L} + qx\right)\hat{e}_2 + \delta \tilde{S}
\]

whisk \(\vec{k} = (\frac{2\pi}{L} + q)\hat{e}_1\). The energy cost of this twisted state relative to the ground state is

\[
\Delta E[\delta \tilde{S}(\vec{x})] = \frac{1}{2N} \sum_{\vec{k'}} [v(\vec{k'}) - v(\vec{q})][\tilde{S}(\vec{k'})]^2
\]

Ignoring \(\delta \tilde{S}\) contributions,

\[
\Delta E = \frac{N}{2} [v(\vec{k}) - v(\vec{q})] = \frac{N}{2\sqrt{Q}}\left(\frac{2\pi}{L} + q\right)^2 - q^2)^2 \approx \frac{8\pi^2 N}{L^2}.
\]

It is readily seen that this energy gain exactly coincides with the energy gain incurred by a uniform longitudinal twist in the three dimensional nearest neighbor ferromagnetic XY system, \(\delta \tilde{S}(\vec{x}) = \cos\left(\frac{2\pi x}{L}\right)\hat{e}_1 + \sin\left(\frac{2\pi x}{L}\right)\hat{e}_2\).

**B. Transverse Spin Stiffness**

When subjected to a transverse twist, the system responds with only a quartic restoring potential. For a transverse twist,

\[
\delta \tilde{S}(\vec{x}) = \cos\left(\frac{2\pi x}{L} + qy\right)\hat{e}_1 + \sin\left(\frac{2\pi x}{L} + qy\right)\hat{e}_2 + \delta \tilde{S}
\]

we find that as the difference of the energy kernels, \(v(\vec{k}) = \vec{q} \hat{e}_2 + 2\pi \hat{e}_1) - v(\vec{q}) = \frac{8\pi^2 N}{L^2\sqrt{Q}}\), the energy gain, ignoring \(\delta \tilde{S}\), is \(\Delta E = \frac{8\pi^4 N}{L^4 Q}\). This energy penalty for a transverse twist vanishes as \(L \to \infty\) in \(d = 3\)- there is a complete loss of stiffness against transverse fluctuations. As \(N = L^d\), this energy penalty scales as \(O(\frac{1}{L})\). We note, in passing, that, to this order, this energy penalty matches with that of an XY chain,

\[
\Delta E \sim \frac{2\pi^2}{L^2} N = \frac{2\pi^2}{L} = O(\frac{1}{L}).
\]

An effective reduction in dimensionality seems to occur- the system effectively behaves as a one dimensional system when exposed to transverse perturbations.

In the general case, for all incommensurate systems, this simple spin wave analysis yields a response to a twist \(\delta\) with an effective kernel \(E_{\text{low}}(\delta) = v(\vec{q} + \delta) - v(\vec{q})\). For systems with a spherical shell of incommensurate minimizing modes, \(E_{\text{low}} \approx A_1 + \delta_1^2\) for momentum deviations transverse to the minimizing \(\vec{q}\), while \(E_{\text{low}} \approx A_1 + \delta_1^2\) for deviations parallel to \(\vec{q}\). Such a fluctuation spectrum is reminiscent to that obtained in smectic liquid crystals. It will be noted, however, that at this stage of approximation, \(E_{\text{low}} \neq A_1 + \delta_1^2 + A_1 + \delta_1^2\), for general deviations \(\delta\): the precise fluctuation spectrum is much softer than that of a smectic liquid crystal (e.g. \(E_{\text{low}}\) vanishes for \(\delta\) connecting \(\vec{q}\) with another vector on the minimizing manifold).

**IX. THERMAL FLUCTUATIONS OF ANY TRANSLATIONALLY INVARIANT XY MODEL**

In the previous section, we examined the response of an arbitrary incommensurate system having a shell of minimizing modes to various twists by simply examining the energy difference between a ground state spiral and a spiral of another wave-number. We found that the system was very unstable to transverse twists. We now go one step further and completely treat the “soft-spin” version of the classical XY model with arbitrary incommensuration (not only those with a shell of minimizing modes). Towards the end of this section, we investigate what our general results imply in the case of rotationally symmetric incommensurate systems (which have a shell of minimizing modes) where we will find a link to smectic liquid crystal thermodynamics.
In this general theory, we include the non-linear interaction \( H_1 \),
\[
H_{soft} = H_0 + u \sum_x [S^2(\vec{x}) - 1]^2 \equiv H_0 + H_1 \tag{52}
\]
with small \( u > 0 \), and forget about the normalization conditions \( |\langle \vec{S}(\vec{x}) \rangle| = 1 \) at all lattice sites \( \vec{x} \). (The normalized, “hard-spin”, version can be viewed as the \( u \to \infty \) limit of the soft-spin model.) As we proved previously, the only generic ground states (for both hard- and soft-spin models) when the spins \( \vec{S}(\vec{x}) \) have two (and also three) components are spirals
\[
S^1_{\text{ground-state}}(\vec{x}) = \cos(\vec{q} \cdot \vec{x}); \quad S^2_{\text{ground-state}}(\vec{x}) = \sin(\vec{q} \cdot \vec{x}).
\]
These are also the lowest energy eigenstates of \( H_{soft} \).

We will shortly expand \( H_{soft} \) about these ground states, keeping only the lowest order (quadratic) terms in the fluctuations \( \delta S \). The quadratic term in \( \{\delta S_i(\vec{k})\} \) stemming from \( H_{soft} \) is the bilinear \( \Sigma (\delta S)^+ \cdot M(\delta S) \) where
\[
(\delta S)^+ = (\delta S_1(-\vec{k}_1), \delta S_2(-\vec{k}_1), \delta S_1(-\vec{k}_2) \delta S_2(-\vec{k}_2), \delta S_1(-\vec{k}_3), \delta S_2(-\vec{k}_3), \ldots, \delta S_1(-\vec{k}_N), \delta S_2(-\vec{k}_N))
\]
and the matrix \( M \) reads
\[
\begin{pmatrix}
4 & 0 & 1 & i & 0 & 0 & 0 & 0 \\
0 & 4 & 1 & -i & 0 & 0 & 0 & 0 \\
1 & i & 4 & 0 & 1 & i & 0 & 0 \\
-i & -1 & 0 & 4 & i & -1 & 0 & 0 \\
1 & -i & 4 & 0 & 1 & i & 0 & 0 \\
-i & -1 & 0 & 4 & i & -1 & 0 & 0 \\
0 & 0 & 1 & -i & 4 & 0 & 0 & 0 \\
0 & 0 & i & -1 & 4 & 0 & 0 & 0
\end{pmatrix}
\]
The sub-matrices are \((2 \times 2)\) matrices in the internal spin indices. The off diagonal blocks are separated from the diagonal ones by wave-vectors \((\pm 2\vec{q})\). Note that \( (\vec{k}|M|\vec{k}') = M(\vec{k} - \vec{k}') \). Making a unitary (symmetric Fourier) transformation to the real space basis: \(|\vec{x}\rangle \equiv N^{-1/2} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{x}} |\vec{k}\rangle \), the matrix \( M \) becomes block diagonal, \( |\vec{x}\rangle M|\vec{x}'\rangle = \tilde{M}(\vec{x}) \delta_{\vec{x}, \vec{x}'} \). Diagonalizing in the internal spin basis, we find that
\[
\lambda_{\pm} = 6, 2. \tag{54}
\]
These eigenvalues may be regarded, in the usual ferromagnetic case (the limit \( q = 0 \)) as a two step (state) potential barrier separating the two polarizations. I.e., the normalization constraint of the XY spins (embodied in \( H_1 \)) gives rise to an effective binding interaction. As we shall later see, when the number of spin components \( n \) is odd, one spin component will remain unpaired. \( H_1 \) literally “couples” the spin polarizations. Employing Eq.(54), the corrected fluctuation spectrum \( \{\psi_m\}_m^N \) (to quadratic order) satisfies a Dirac like equation
\[
[U^+ v(-i\partial_x) U + 2u \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}] U^+ |\psi_m(\vec{x})\rangle = E_m U^+ |\psi_m(\vec{x})\rangle. \tag{55}
\]
Here,
\[
U = \begin{pmatrix}
\frac{\sin(2\vec{q} \cdot \vec{x})}{2\cos(\vec{q} \cdot \vec{x})} & \frac{1}{2} \\
-\frac{1-\cos(2\vec{q} \cdot \vec{x})}{2\sin(\vec{q} \cdot \vec{x})} & \frac{1}{2}
\end{pmatrix}.
\tag{56}
\]
Alternatively, expanding in the fluctuations \( \delta S(\vec{k}) \), leads to bilinear \( (\delta S)^+ \cdot \mathcal{H}(\delta S) \) where
\[
\mathcal{H}_{\vec{k}, \vec{k}} = \begin{pmatrix}
v(\vec{k}) + 8u & 0 \\
0 & v(\vec{k}) + 8u
\end{pmatrix}
\] along the diagonal, and \( \mathcal{H}_{\vec{k}, \vec{k}+2\vec{q}} = 2u(\sigma_3 \pm i\sigma_1) \) off the diagonal. Next, we perform a unitary transformation \( exp(i\vec{q} \sigma_1) \mathcal{H}_{\vec{k}, \vec{k}+2\vec{q}} \) \( exp(-i\vec{q} \sigma_1) = 2u\sigma^\pm \), while \( \mathcal{H}_{\vec{k}, \vec{k}} \) of Eq.(57) is unchanged. Till now, all that we stated, held for arbitrarily large \( u \) - our only error was neglecting \( O((\delta S)^2) \) terms by comparison to \( O((\delta S)^3) \). Note that the main difficulty with the approach taken till now was the coupling between \( \vec{k} \) and \( \vec{k} \pm 2\vec{q} \). i.e. \( \vec{k} \) is coupled to \( \vec{k} \pm 2\vec{q} \), while \( \vec{k} + 2\vec{q} \) is coupled to \( \vec{k} + 4\vec{q} \) and \( \vec{k} \). so on. Unless \( \vec{q} \) is of low commensurability an exact solution to this problem is impossible. Equations with similar structure appear in very different arenas (e.g. two dimensional Bloch electrons in a magnetic field \([27], [28]\)) and such systems are often addressed via a continued fraction representation or by mapping the problem onto a Harper like Equation. To make rapid progress and to elucidate the similarity between incommensurate systems and smectic liquid crystals, here we will address our problem is a very direct and short fashion. Let us assume that \( u \) is small. In this case the lowest eigenstates of the fluctuation matrix will contain only a superposition of the low lying \( \vec{k} \) states (i.e. those close to the \((d - 1)\) dimensional \( M(q > 0) \)). If \( \vec{k}_1 = \vec{q} + \vec{\delta} \) is close to \( M \), then the only important modes in the sequence \( \{S_i(\vec{k} = \vec{k}_1 + 2n\vec{q})\} \) are \( \vec{k}_1 \) and \( \vec{k}_2 = \vec{k}_1 + 2\vec{q} = -\vec{q} + \vec{\delta} \). The sub-matrix in the relevant sector reads
\[
\begin{pmatrix}
v(\vec{k}_1) + 8u & 0 & 0 & 4u \\
0 & v(\vec{k}_1) + 8u & 0 & 0 \\
0 & 0 & v(\vec{k}_2) + 8u & 0 \\
4u & 0 & 0 & v(\vec{k}_2) + 8u
\end{pmatrix}
\]
The lowest eigenvalue reads
\[
E_{low} = \frac{1}{2}[v(\vec{k}_1) + v(\vec{k}_2)] + 4u - \frac{1}{2} \sqrt{[v(\vec{k}_1) - v(\vec{k}_2)]^2 + 64u^2}. \tag{58}
\]

Equivalently, this can be determined from the direct computation of the determinant to $O(u^2)$: to obtain $O(u^2)$ contributions we need to swerve off the diagonal twice.

$$
det \mathcal{H} = \prod_{i=1}^{N} [v(\vec{k}_i) + 8u]^2 - (4u)^2 \sum_j [v(\vec{k}_j) + 8u] \times [v(\vec{k}_j + 2\vec{q}) + 8u] \prod_{\vec{k}_i \neq \vec{k}_j, \vec{k}_j + 2\vec{q}} [v(\vec{k}_i) + 8u]^2
$$

The fluctuation spectrum is trivially determined by replacing $v(\vec{k})$ by $|v(\vec{k}) - E|$ in the determinant and setting it to zero. To this order we re-derive $E_{\text{low}}$. Higher order terms in the determinant may be trivially computed. The partition function is $Z = \text{const} \cdot |\det \mathcal{H}|^{-1/2}$. For any $u$, no matter how small, there exists a neighborhood of wavevectors $\vec{k}$ near $\vec{q}$ such that $|v(\vec{k}) - v(\vec{k} + 2\vec{q})| \ll u$ and as before we may re-expand the characteristic equation for these low lying modes, solve a simple quadratic equation, expand in the components of $(\vec{k} - \vec{q})$ and obtain a simple dispersion relation.

Till now, our results, held for arbitrary incommensurate XY systems. We now investigate what transpires when the minimizing modes form a continuous shell $\text{M}$ (e.g. all incommensurate systems with a rotational symmetry that assures, as $q \neq 0$, a shell of radius $q$ of minimizing modes about the origin). For concreteness, if $\tilde{\delta} = \delta_\perp + \delta_{||} \vec{e}_{||}$, with $\perp$, $||$ denoting directions orthogonal, and parallel to $\vec{n} \perp \text{M}$, then by expanding Eq.(58) to lowest orders for the example presented in Eq.(6)

$$
E_{\text{low}} = A_\perp \delta_\perp^2 + A_{||} \delta_{||}^2,
$$

where

$$
A_{||} = \frac{1}{2} \frac{d^2v(|\vec{k}|)}{dk^2} |_{|k|=q}; \quad A_\perp = \frac{A_{||}}{4q^2}
$$

Examining the fluctuations about an ordered state, we find that the effect of thermal fluctuations is reduced by comparison to the dispersion relation in the large $n$ model and spin wave stiffness analysis. Nonetheless, at this level of analysis, order is still inhibited, much unlike many models wherein “order out of disorder” occurs [25] by entropic stabilization about certain viable ground states. This dispersion is akin to the fluctuation spectrum of the smectic liquid crystals [44] which is well known to give rise to algebraic decay of correlations at low temperatures. In our case, $\vec{q} \neq 0$ and thus the correlations should have an oscillatory prefactor. To be more concrete we find that, for general rotationally symmetric incommensurate spin systems having a minimizing shell of modes at $|q| = q > 0$, the correlator (in cylindrical coordinates) is

$$
G(\vec{x}) = \langle \vec{S}(0) \cdot \vec{S}(\vec{x}) \rangle \simeq \frac{4d}{x_\perp^2} \exp[-2\eta \gamma - \eta E_1(\frac{x_\perp^2}{2})] \times \cos[q x_\perp],
$$

where $\eta = \frac{K_\perp q}{d}$ and $d = \frac{2\pi}{n}$ where $\Lambda$ is the ultra violet momentum cutoff, $\gamma$ is Euler’s const.,

$$
E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} (-n)^n \frac{z^n}{n(n!)}
$$

is the exponential integral. For the uninitiated reader, we present this standard derivation in the appendix.

The thermal fluctuations $\int d^d k / E_{\text{low}}(\vec{k})$ diverge as $-\ln |\vec{r}|$ with $\gamma = 0$ a lower cutoff on $|k - q|$ [29]. Such a logarithmic divergence is also encountered in ferromagnetic two dimensional $O(2)$ system if it were exposed to the same analysis. Indeed, both models share similar characteristics albeit having different physical dimensionality. Thus far, our analysis was performed on incommensurate structures with minimizing shells in $\vec{k}$-space. For frustrated systems having special commensurabilities, intricacies may arise (some are explicitly detailed in [26]).

**Relations to Smectic Liquid Crystals**

We now fortify our earlier conclusion regarding a tight relation between incommensurate continuous spin systems and smectic liquid crystals. For incommensurate spin systems with minima on a $(d-1)$ dimensional shell in $\vec{k}$ space, the resulting spectra attained by both the Dirac like equation and the simple more naive spin stiffness analysis, matches that of smectic liquid crystals. Smectic ordering involves a breaking of rotational symmetry and of translational symmetry along one direction alone. Let $\vec{n}$ be the displacement of the smectic along the $z$-direction. When the wave-vector $\vec{k}$ is along the $z$-direction the displacement is longitudinal, and the energy is of the elastic form $\frac{1}{2}B_k^2 (\vec{u}(\vec{k}))^2$, where $B$ is the compressibility for the smectic layers. When $\vec{k}$ is normal to $z$, the displacement is transverse to the layer separation. No second order in $\vec{k}_\perp$ the displacement costs no energy. Here, the restoring force is associated with a director splay distortion leading to an elastic energy $\frac{1}{2}K_\perp (\vec{u}(\vec{k}))^2$ with $K$ the splay constant. In, e.g. Smectic $A$, as $\vec{n}$ is normal to the layers, $\vec{B} = -\vec{n}$ and an elastic energy $\frac{1}{2}K_{\vec{k}\perp} (\vec{u}(\vec{k}))^2$. Thus in smectic liquid crystals, the kernel

$$
v(\vec{k}) = B_k^2 + K\vec{k}\perp^4,
$$

This is much alike our dispersion for incommensurate phases apart from a shift: in the incommensurate phases the deviations were about a finite wavenumber $\vec{q}$ whereas in liquid crystal, the uniform $\vec{k} = 0$ is the ground state. As seen in the form that we obtained for $G(\vec{x})$, the penetration depth $\lambda = \sqrt{K/B}$ determines the decay of an undulation distortion (splay director distortion) imposed at the surface of the smectic.

In conclusion, if $H_{\text{soft}}$ is indeed soft ($u \ll 1$) then, in rotationally symmetric incommensurate soft spin XY systems, quasi long range (algebraic) order may be observed at low temperatures. These systems essentially may smectic thermodynamics.
We will now generalize the Mermin-Wagner-Coleman theorem [30, 31]: All continuous spin systems with translationally invariant two-spin interactions in two dimensions with a rotationally symmetric twice differentiable [33] Fourier transformed kernel $g(k)$ show no spontaneous symmetry breaking at finite temperatures $T > 0$. This is true for all systems, irrespective of the range of the interaction or of its nature. As will be made clear shortly, our analysis holds for both commensurate and incommensurate systems.

### A. The Classical Case

Our approach is the standard one. We will keep it more general instead of specializing to anti/ferromagnetic order or to interactions of one special sort. With the notations introduced in Section(II), we investigate $n$ component spins on a lattice. An applied magnetic field

$$\tilde{h}(\vec{x}) = h \cos(q \cdot \vec{x}) \hat{e}_\alpha$$

(64)

causes the spins to take on their ground state values.

If $n = \alpha = 2$ the unique spiral ground state ($\tilde{S}^\alpha(\vec{x})$), to which a low temperature system would collapse to under the influence of such a perturbation is

$$S^\alpha(\vec{x}) = \sin(q \cdot \vec{x}).$$

(65)

When $n = 3$ the ground state is not unique:

$$S^\alpha_{<n}(\vec{x}) = \frac{1}{n-1} \sum_{i=1}^{n-1} r_i^2 = 1$$

(66)

and a magnetic field may be applied along two directions, with all the ensuing steps trivially modified. With the magnetic field applied

$$H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} \sum_{i=1}^{n} V(\vec{x} - \vec{y}) S_i(\vec{x}) S_i(\vec{y}) - \sum_{\vec{x}} h_n(\vec{x}) S_n(\vec{x}).$$

(67)

Note that the knowledge of the ground state is not imperative in providing the forthcoming proof [32] (whether a spiral or any other state). Our analysis thus holds for both commensurate and incommensurate systems. We exploit the standard rotational invariance of the measure

$$\int d\mu \cdot = Z^{-1} \int \prod_{\vec{x}} d^n S(\vec{x}) \delta(S^2(\vec{x}) - 1) e^{-\beta H}.$$  

(68)

This is not applicable to many other spin models (e.g. Dzyaloshinskii-Moriya interactions, orbital Jahn-Teller and Kugel-Khoskii models). The generators of rotation in the $[\alpha \beta]$ plane at a lattice site $\vec{x}$ are

$$L^\alpha_{\vec{x}} \equiv S_\alpha(\vec{x}) \frac{\partial}{\partial S^\beta(\vec{x})} - S_\beta(\vec{x}) \frac{\partial}{\partial S^\alpha(\vec{x})}.$$  

(69)

For any single spin,

$$0 = \frac{d}{d\theta} \int d^n S(\delta(S^2 - 1))$$

$$f(S_1, ..., S_\alpha \cos \theta + S_\beta \sin \theta, ..., S_\beta \cos \theta - S_\alpha \sin \alpha, ..., S_n).$$

(70)

It follows that

$$0 = \int d^n S(\delta(S^2 - 1)) L^\alpha \beta f(S)$$

(71)

In the up and coming, $\perp$ will denote the the projection along the $\beta$ direction. We define the operators

$$\tilde{A}(\vec{k}) = \sum_{\vec{x}} \exp[i\vec{k} \cdot \vec{x}] S_L(\vec{x}),$$

$$\tilde{B}(\vec{k}) = \sum_{\vec{x}} \exp[i(\vec{k} + \vec{q}) \cdot \vec{x}] \tilde{L}_x(\beta H),$$

(72)

where $\tilde{L}_x = (L^i_x)_{i=1}^{n-1}$ with $L^i_x = L^\alpha_{\vec{x}} = \perp$.

By the Schwarz inequality,

$$|\sum_{i=\alpha, \beta} A^*_i B_i | \leq \sum_{i=\alpha, \beta} |A^*_i A_i| \sum_{i=\alpha, \beta} |B^*_i B_i|.$$  

(73)

We will let $i = \alpha, \beta$ in the sum span a two element subset of the $n$ spin components. For any functional $C$:

$$\tilde{L}_x(e^{-\beta H} C) = e^{-\beta H} \{\tilde{L}_x(C) + C \tilde{L}_x(-\beta H)\}.$$  

(74)

Invoking Eq.(71),

$$0 = \int \prod_{\vec{x}} d^n S(\vec{x}) \delta(S^2(\vec{x}) - 1) \tilde{L}_x[e^{-\beta H} C].$$

(75)

Employing the last two equations in tow,

$$\langle CB(\vec{p}) \rangle = \langle \sum_{\vec{x}} \exp[i\vec{p} \cdot \vec{x}] \tilde{L}_x(C) \rangle.$$  

(76)

It is readily seen that

$$\sum_{i=\alpha, \beta} \langle L^i_x(L^i_y(\beta H)) \rangle = \beta \sum_{i=\alpha, \beta} \langle S_i(\vec{x}) S_i(\vec{y}) \rangle - h(\vec{x}) S_n(\vec{x}),$$  

(77)

and

$$\langle \tilde{B}(\vec{k}) \tilde{B}(\vec{k}) \rangle = \beta \sum_{\vec{x}, \vec{y}} \{ (\cos(\vec{k} + \vec{q}) \cdot (\vec{x} - \vec{y}) - 1) \}$$

$$\left[ \sum_{i=\alpha, \beta} \langle S_i(\vec{x}) S_i(\vec{y}) \rangle V(\vec{x} - \vec{y}) \right]$$

$$- h(\vec{x}) \langle S_n(\vec{x}) \rangle \geq 0,$$  

(78)
Henceforth, for simplicity, we specialize to \( n = 2 \). Fourier expanding the interaction kernel

\[
V(\vec{x} - \vec{y}) = \frac{1}{N} \sum_i v(i) e^{i \vec{k} \cdot (\vec{x} - \vec{y})},
\]

(79)

and substituting

\[
\langle \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}) \rangle = \frac{1}{N^2} \sum_{\vec{u}} \langle |\vec{S}(\vec{u})|^2 \rangle e^{i \vec{u} \cdot (\vec{x} - \vec{y})},
\]

(80)

we obtain

\[
0 \leq \langle \vec{B}(\vec{k})^* \cdot \vec{B}(\vec{k}) \rangle \equiv \beta \Delta_2^{(2)} E - \beta h \vec{q} \langle S_n(-\vec{q}) \rangle = \frac{\beta}{2N} \sum_{\vec{u}} \left[ v(\vec{u} + \vec{k}) + v(\vec{u} - \vec{k}) - 2v(\vec{u}) \right] \langle |\vec{S}(\vec{u})|^2 \rangle - \beta h \vec{q} \langle S_n(-\vec{q}) \rangle
\]

(81)

where \( \Delta_2^{(2)} E \) measures the finite difference of the internal energy with respect to a boost of momentum \( \vec{k} \).

\[
\langle \vec{A}(\vec{k})^* \cdot \vec{A}(\vec{k}) \rangle = \sum_{\vec{x}, \vec{y}} \langle \vec{S}_{\perp}(\vec{x}) \cdot \vec{S}_{\perp}(\vec{y}) \rangle \exp[i \vec{k} \cdot (\vec{x} - \vec{y})].
\]

(82)

\[
\langle \vec{A}(\vec{k})^* \cdot \vec{B}(\vec{k}) \rangle = \sum_{i, \vec{x}} L_i^j \langle \vec{S}_{\perp}(\vec{x}) \rangle \exp[i(\vec{k} + \vec{Q}) \cdot \vec{x}] = m_q, \]

where \( m_q \equiv \langle S_n(\vec{q}) \rangle \) and, as noted earlier, \( \perp \) refers to the \( i = 1 \) spin direction orthogonal to \( i = n = 2 \). Note that with our convention for the Fourier transformations, a macroscopically modulated state of wave-vector \( \vec{q} \), the magnetization \( m_q = \mathcal{O}(N) \) as is the energy difference in Eq.(81). Trivially rewriting the Schwarz inequality and summing over all momenta \( \vec{k} \),

\[
\sum_{\vec{k}} \frac{\langle \vec{A}(\vec{k})^* \cdot \vec{B}(\vec{k}) \rangle}{\langle \vec{B}(\vec{k})^* \cdot \vec{B}(\vec{k}) \rangle} \leq \frac{\beta}{\pi^2} \sum_{\vec{k}} \langle |\vec{S}(\vec{u})|^2 \rangle \left[ v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u}) \right] \langle |\vec{S}(\vec{u})|^2 \rangle - \beta h \vec{q} \langle S_n(-\vec{q}) \rangle
\]

(83)

which explicitly reads

\[
2N|m_q|^{\beta} \sum_{\vec{k}} \left( \langle |\vec{S}(\vec{u})|^2 \rangle \right)^{\frac{1}{2}} \left[ v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) + 2|\vec{u}| \langle |\vec{S}(\vec{u})|^2 \rangle \right]^{-1} \leq \frac{\beta}{\pi^2} \sum_{\vec{k}, \vec{x}, \vec{y}} \langle \vec{S}_{\perp}(\vec{x}) \cdot \vec{S}_{\perp}(\vec{y}) \rangle e^{i \vec{k} \cdot (\vec{x} - \vec{y})} = N \sum_{\vec{x}} \langle \vec{S}_{\perp}^2(\vec{x}) \rangle.
\]

(84)

Explicitly, as the integral \( \int |\vec{k}| > \delta \frac{d^4 k}{(2\pi)^4} \ldots \) is non-negative (as \( \langle \vec{B}(\vec{k})^* \cdot \vec{B}(\vec{k}) \rangle \geq 0 \) the denominator in Eq.(84) is positive for each individual value of \( \vec{k} \), and as \( \langle \vec{S}_{\perp}^2(\vec{x}) \rangle \leq 1 \), we obtain in the thermodynamic limit

\[
\frac{2}{\beta} |m_q|^2 \int |\vec{k}| < \delta \frac{d^4 k}{(2\pi)^4} \left[ \int \frac{d^4 u}{(2\pi)^4} \langle |\vec{S}(\vec{u})|^2 \rangle \right] \left[ v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u}) + 2|\vec{u}| \langle |\vec{S}(\vec{u})|^2 \rangle \right]^{-1} \leq 1.
\]

(85)

Taking \( \delta \) to be small we may bound from above (for each value of \( \vec{k} \)) the positive denominator in the square brackets and consequently

\[
\int |\vec{k}| < \delta \frac{d^4 k}{(2\pi)^4} \left[ \int \frac{d^4 u}{(2\pi)^4} \langle |\vec{S}(\vec{u})|^2 \rangle \right] \left[ v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u}) + 2|\vec{u}| \langle |\vec{S}(\vec{u})|^2 \rangle \right]^{-1} \leq 1.
\]

(86)

with \( \lambda_\ell \) chosen to be the largest principal eigenvalue of the \( d \times d \) matrix \( \partial_i \partial_j [v(\vec{u})] \), and \( A_1 \) a constant. For a twice differentiable \( v(\vec{u}) \), and for \( |\vec{k}| \leq \delta \) where \( \delta \) is finite, \( (v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u}) \leq A_1 \lambda_\ell k^2 \leq B_1 k^2 \) \( \sum_{\vec{k}} \langle \vec{A}(\vec{k})^* \cdot \vec{B}(\vec{k}) \rangle \) diverges making it possible to satisfy Eq.(85) at finite temperatures, when the external magnetic field \( h \rightarrow 0 \) only if the magnetization \( m_q = 0 \). If finite size effects are restored, in a system of size \( N = L \times L \), where the surface cutoff in the integral is \( \mathcal{O}(\frac{L}{10}, \frac{L}{10}, \ldots) \), the latter integral diverges as \( \mathcal{O}(\ln N) \). This implies that the upper bound on \( |m_q| \) scales as \( \mathcal{O}(N/\sqrt{\ln N}) \), much lower than the \( \mathcal{O}(N) \) required for finite on-site magnetization. For further details see [32]. If there are \( M \leq 2 \) pairs of minimizing modes and \( 2p_1 + 1 \geq n \geq 2p \) (with an integer \( p \)) then we may apply an infinitesimal symmetry breaking magnetic field along, at most, \( \min\{p, M\} \) independent spin directions (\( \alpha \)). Employing the spin rotational invariance within each plane \( [\alpha, \beta] \) associated with any individual mode \( \vec{k} \) (au lieu of a specific \( \vec{q} \)) we may produce a bound similar that in Eq.(85) wherein \( \langle |\vec{S}(\vec{u})|^2 \rangle \) will be replaced by \( \sum_{i=\alpha, \beta} \langle |S_i(\vec{u})|^2 \rangle \) and \( m_q \rightarrow m_{\alpha, \beta}(\vec{k}) \).

B. The Quantum Case

The finite temperature behavior of a quantum system is, in many respects, similar to that of a classical system.
The quantum system is also invariant under rotations with $S^2(\vec{x}) = S(S + 1)$. Alternatively, one could directly tackle the $n = 3$ quantum case by applying the Bogoliubov inequality

$$\frac{\beta}{2}\langle\{A, A^1\}\rangle * \langle[[[C, H], C^1]\rangle \geq |\langle[[C, A]\rangle|^2$$  \hspace{1cm} (88)

with $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ the commutator and anticommutator respectively. From this inequality, it follows that $\langle[[[C, H], C^1]\rangle$ is positive definite. In particular, for any six operators $\{A_1, A_2, A_3, C_1, C_2, C_3\}$,

$$\frac{\beta}{2}\left(\sum_{a=1}^3\langle\{A_a, A_a^1\}\rangle\right)\langle[[[C_a, H], C_a^1]\rangle \geq \sum_{a} |\langle[[C_a, A]\rangle|^2$$  \hspace{1cm} (89)

Setting $A_1 = S_2(\vec{q} - \vec{k})$ and $C_1 = S_1(\vec{k})$ we will once again obtain Eq(85) with the classical spins replaced by their quantum counterparts.

Rather explicitly, employing

$$[S^a(\vec{k}), S^b(\vec{k}')] = i\epsilon^{abc} S_c(\vec{k} + \vec{k}')$$  \hspace{1cm} (90)

we find for the Hamiltonian of Eq.(67) (with $n = 3$),

$$\langle[[[C_1, H], C_1^1]\rangle = \frac{1}{2N} \sum_{\vec{k}'} (S_2(\vec{k'})S_2(-\vec{k'}) + S_3(\vec{k'})S_3(-\vec{k'}))$$

$$\times [v(\vec{k} + \vec{k}') - 2v(\vec{k}) + v(\vec{k'} - \vec{k})] + \frac{1}{N} \sum_{\vec{k}'} h_3(\vec{k'})S_3(-\vec{k'}).$$

Similarly,

$$\langle[[[C_1, A]\rangle]^2 = \langle|S_3(\vec{q})|^2\rangle,$$  \hspace{1cm} (91)

the squared magnetization along the $z$ (or 3) direction for a mode $\vec{q}$, and

$$\sum_{\vec{k}} \langle\{A_1, A_1^1\}\rangle = 2 \sum_{\vec{k}} \langle S_2(\vec{k})S_2(-\vec{k})\rangle$$

$$= 2 \langle|S_2(\vec{x} = 0)|^2\rangle.$$  \hspace{1cm} (92)

Next, let us cyclically set, $A_2 = S_3(\vec{q} - \vec{k}), A_2 = S_2(\vec{k}), A_3 = S_1(\vec{q} - \vec{k})$, and $C_3 = S_3(\vec{k})$. The commutators associated with these operators are all identically the same apart from a uniform cyclic permutation of all spin components involved. Trivially rewriting the symmetrized Bogoliubov inequality Eq.(89) and summing over all modes $\vec{k}$,

$$\frac{\beta}{2}\sum_{\vec{k},a}\langle\{A_a, A_a^1\}\rangle \geq \sum_{\vec{k}} \sum_{a} |\langle[[C_a, A]\rangle|^2$$

Replacing the $\vec{k}$ sums by integrals in the thermodynamic limit, and employing the positivity of $|\langle[[[C_a, H], C_a^1]\rangle$ follows from the Bogoliubov inequality for each individual value of $\vec{k}$, we find the quantum analogue of Eq.(85),

$$\frac{1}{2\beta} |m_{a\vec{q}}|^2 \int |\vec{k}| < \delta \frac{d^d\vec{k}}{(2\pi)^d} \left[ \int \frac{d^d\vec{u}}{(2\pi)^d} \langle|\vec{S}(\vec{u})|^2\rangle (v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u})) + |h||m_{a\vec{q}}|^2 \right] \leq S(S + 1).$$  \hspace{1cm} (94)

Apart from the simple scaling factor of $4S(S + 1) = 2 \sum_{a} \langle[\{A_a, A_a^1\}\rangle$ by comparison to the classical case, there is no difference between this inequality and its classical counterpart in the zero field limit. We symmetrized the Bogoliubov inequality (Eq.(89)) in order to avoid the appearance of only two transverse spin components in the Fourier weights $\langle|S_a(\vec{q})|^2\rangle$ so as to give the resulting Mermin-Wagner inequality a transparent physical meaning associated with the energy boost differences $\Delta E^{(2)}$ which are symmetric in all spin indices. From here onward the discussion can proceed as in the classical case.

XI. MERMIN-WAGNER BOUNDS IN HIGH DIMENSIONS

In any dimension, Eq.(85) reads in the limit $\hbar \rightarrow 0$

$$2|m_{a\vec{q}}|^2 T \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\Delta E^{(2)}} \leq 1,$$  \hspace{1cm} (95)

with the shorthand defined by Eq.(81). By parity invariance and noting that $\{\vec{S}(\vec{x})\}$ are real,

$$\sum_{\vec{a}} v(\vec{k} + \vec{u}) \langle|\vec{S}(\vec{u})|^2\rangle = \sum_{\vec{a}} v(\vec{k} - \vec{u}) \langle|\vec{S}(\vec{u})|^2\rangle.$$  \hspace{1cm} (96)

Thus the denominator of Eq.(95) reads $\Delta E^{(2)} = [2(E_{\vec{k}} - E_0)]$ where $E_{\vec{k}}$ is the internal energy of system after undergoing a boost of momentum $\vec{k}$ and $E_0$ denotes the internal energy of the un-boosted system. In some instances when the dispersion relation about an assumed zero temperature ground state is inserted into Eq.(95), we will find that the integral in Eq.(95) diverges: At arbitrarily low temperatures we cannot assume the zero temperature ground state with the natural dispersion relation $\Delta E$ for fluctuations about it. A case in point is the dispersion relation for the Coulomb Frustrated Ferromagnet. The denominator in Eq.(85) is a finite temperature extension of the $T = 0$ dispersion relations $\Delta E^{(2)}$. In general, at zero temperature,

$$\langle|\vec{S}(\vec{k})|^2\rangle = \frac{N^2}{2} |\delta_{\vec{k},\vec{q}} + \delta_{\vec{k},-\vec{q}}|$$  \hspace{1cm} (97)
and the integral in Eq.(95) becomes

\[ \int \frac{d^d k}{(2\pi)^d} \frac{1}{v(\vec{k} + \vec{q}) + v(\vec{k} - \vec{q}) - 2v(\vec{q})}. \]  \tag{98} \]

For the incommensurate models, the reader will recognize this as none other than the integral obtained by other methods via the Dirac like analysis in Section(IX). Specifically, the denominator of Eq.(98) is \( E_{\text{low}} \) of Eq.(58) wherein \( \vec{k}_1 = \vec{q} + \vec{k} \) and \( \vec{k}_2 = -\vec{q} + \vec{k} \) (which transformed into Eq.(59) (with \( \delta \) now portrayed by \( \vec{k} \)) for general rotationally symmetric incommensurate systems.

Whenever Eq.(98) diverges, an assumption of an almost ordered ground state at arbitrarily low temperatures \( (T = 0^+) \) with the ensuing zero temperature dispersion relation \( \Delta_k^{(2)} = 2(E_{\vec{k}} - E_0) \) about that ordered state is flawed: Eq.(95) is strongly violated. This poses no problem for most canonical \( d > 2 \) dimensional models where the integral in Eq.(98) is finite. For all three dimensional incommensurate systems with a rotational symmetry that secures, as \( q \neq 0 \), a shell of radius \( q \) of minimizing modes about the origin (e.g. the three dimensional Coulomb Frustrated Ferromagnet and other high dimensional systems), the divergence of this integral hints possible non-trivialities. The decoherence time scale (or more precisely, bandwidth time scale \[34]\] \( \tau \)) may diverge if we assume a zero temperature dispersion of fluctuations about an ideal crystal. A divergent decoherence time scale suggests the absence of broken translational symmetry. Formally,

\[ \int \frac{d^d k}{(2\pi)^d} \tau_k = \tau(\vec{r} = 0). \]  \tag{99} \]

In order for magnetization \( m_q = \mathcal{O}(N) \) to arise, the average of the inverse boost energy over all of \( \vec{k} \) space

\[ \tau \equiv \frac{1}{\Delta_k^{(2)}} \leq \mathcal{O}(T^{-1}), \]  \tag{100} \]

at all \( 0 < T < T_c \). In most \( d > 2 \) systems this is trivially satisfied with the average bounded by a constant at zero temperature. This average diverges at \( T = 0 \) whenever the integral of Eq.(98) does. There are two possibilities:

(i) The system is ordered at all temperatures \( T < T_c \) in which case, the thermodynamic average of Eq.(100) is finite for all \( T > 0 \) and \( \tau \) is non-analytic at \( T = 0 \).

(ii) The system is disordered at all finite temperatures and orders classically only at \( T = 0 \).

The first possibility ((i)) was argued for by a non-rigorous yet elegant diagrammatic analysis by Brazovskii \[36\] long ago: Thermal fluctuations, on their own, may enhance (or generate) cubic terms fortifying (or triggering weak) first order transitions. This cannot be ruled out by the rigorous Mermin-Wagner inequalities that we derived. As reiterated, if the fluctuation integral diverges then we may not obtain the low temperature \( T = 0^+ \) dispersion by assuming a nearly perfectly ordered state. This does not rigorously preclude order. Order, if it exists at arbitrarily low temperatures must display non-trivial excitations spectra about it having a vital explicit thermal dependence. If such a possibility arises, it might be inmaterial if the system is permanently frozen into a glass before reaching an equilibrium thermodynamic transition [13]. In both cases, \( \tau(T) \) is non-analytic at \( T = 0 \). The integral of Eqs.(98,100) has a suggestive physical interpretation. If the quantum spin system is subjected to a boost of momentum \( \vec{k} \), then

\[ 1/\Delta_k^{(2)} E \equiv \tau_k \]  \tag{101} \]

is the characteristic lifetime of the excited state. The average in Eqs.(100) is the characteristic relaxation (or decoherence) time of the system averaged over magnons of all possible momenta. Whenever the average characteristic relaxation time

\[ \langle \tau_k \rangle = \int \frac{d^d k}{(2\pi)^d} \left[ \int \frac{d^d u}{(2\pi)^d} \right] \left[ \langle |\vec{S}(\vec{u})|^2 \rangle (v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u})) \right]^{-1} \]  \tag{102} \]

diverges then by our generalized inequality

\[ 2|m_q|^2 T \langle \tau_k \rangle = 2|m_q|^2 T \tau_{\tau=0} \leq 1 \]  \tag{103} \]

the system does not order in such a way that the fluctuation dispersion about any viable ground state is valid. We emphasize that, even in case (ii), the Mermin-Wagner inequality allows for algebraic long range order. Indeed, we have found by our primitive \( n = 2 \) analysis (assuming a fluctuation spectrum about a \( T = 0 \) ordered state), that although the average magnetization vanishes at all \( T > 0 \), the correlations decay algebraically (see Eq.(61) and the appendix where this correlator is explicitly evaluated). As fortold, the main difference between the results of our rigorous analysis here and that of the soft spin model in section(IX) is that the soft spin analysis relied on a comparison to the ground state so that no explicit thermal dependence of the mode occupancies \( \langle |\vec{S}(\vec{k})|^2 \rangle \) was allowed. As fortold, if the mode occupancies have a non-trivial thermal dependence which cannot be neglected even at arbitrarily low \( T = 0^+ \) temperatures, ordering may occur.

**XII. CLASSICAL \( O(N = 3) \) FLUCTUATIONS**

We now return to the more naive “soft-spin” (see section (IX) for the soft spin XY) \( O(3) \) models in order to witness an intriguing even-odd binding-unbinding (or pairing - unpairing) effect that could have otherwise been
missed. We will see that $H_1$ plays the role of a pairing interaction. We will see the incommensurate classical even n soft-spin systems are more "gapped" than their odd n counterparts. When, in three dimensions, incommensurate continuous spin systems are endowed with a spatial rotational symmetry which secures a finite shell of minimizing modes, odd n systems may be disordered while their even n counterparts may exhibit, by the same soft-spin analysis, critical algebraic long range order. This even/odd alteration is reminiscent of the appearance/non-appearance of a gap for quantum spin $s$ chains with even/odd values of $(2s + 1)$ [37], [38] and the physics for an even/odd number of legs in an $s = 1/2$ quantum spin ladder [39] where such even/odd discrepancies are triggered by Berry phase terms. As before, we will begin our analysis for general incommensurate soft spin models and only at the end investigate what occurs for rotationally symmetric systems with a shell of minimizing modes. As we proved, for an $n = 3$ system the generic ground states are simple spirals. If we rotate the helical ground-state to the $1 - 2$ plane, the single quadratic term in $\delta S_{i = 3}(\vec{x})$ is $\sum \bar{\delta} S_{i = 3}(\vec{x})$. The eigenvalues $\lambda_{i \geq 3} = 2 = \lambda_{\min}$ (see Eq. (54)). Here, all that follows holds for arbitrarily large $u$ - the only approximation that we are making is neglecting $O(\delta S^3)$ terms by comparison to quadratic terms: i.e. assuming that $\delta S(\vec{x}) < 1$. Unlike the above treatment of the XY spins, no small $u$ is necessary in order to make headway on the Heisenberg problem. For $n = 3$ we find that the fluctuation eigenstates of the are the products of an eigenstate of $\mathcal{H}$ within the plane of the spiral ground state and a fluctuation eigenstate of $v(\vec{k})$ along the direction orthogonal to the spiral plane. Written formally, to quadratic order, the fluctuation eigenstates are $|\psi_m\rangle \equiv |\delta S_3(\vec{k})\rangle$. Fluctuations along the $i = 3$ axis are orthogonal (in a geometrical and formal sense) to the ground-state plane. This is expected as fluctuations in any hyper-plane perpendicular to the [12] plane do not change, to lowest order, the norm of the spin. $|\delta S_3(\vec{k})\rangle$ is literally the "odd" man out. As foretold, this is a general occurrence. Whenever the number of spin components is odd, one unpaired spin component is unaffected by the interaction enforcing the spin normalization constraint.

Within the $i = 3$ subspace $|\vec{k}, M|\vec{k}\rangle = 2\delta_{i = 3}^{\vec{k}, \vec{k}'}$ and our previous analysis follows. The dispersion $E_{k(3)} = |v(\vec{k}) + 2\frac{\lambda}{u_{\lambda = 3}}|$ does not have a higher minimum than with $E_m$ [40]. Both $E_{k(3)}$ and $E_m$ share the same value of $\lambda$ and consequently the $\delta S_{i = 3}(\vec{x})$ fluctuations are minimized at wave-vectors $\vec{k} \in M_Q$ such that $v(\vec{k}) = v(\vec{q}) = \min_{\vec{q}} v(\vec{k})$. As $|\psi_m\rangle$ is a normalized superposition of $|\delta S_3(\vec{k})\rangle$ modes (the latter spin vectors being in the $1 - 2$ plane) and $v(\vec{k})$ is diagonal, and attains its minimum at $\vec{k} \in M_Q$:

$$\min_{m} \{E_m\} \geq \min_{\vec{k} \in M_{Q}} \{E_{\vec{k}(3)}\}. \quad (104)$$

Here, we invoked the trivial inequality

$$\min\langle \psi | [H_0 + H_1]| \psi \rangle \geq \min_{\phi} \langle \phi | H_0 | \phi \rangle + \min_{\xi} \langle \xi | H_1 | \xi \rangle. \quad (105)$$

Thus, there exist Goldstone modes corresponding to $\delta S_{i = 3}$ fluctuations, and one must adjust additive constants such that $\min_{\xi} \{E_{\vec{k}(3)}\} = 0$. The resulting fluctuation integral reads

$$\langle \delta S_{i = 3}(\vec{x} = 0)^2 \rangle = k_B T \int \frac{d^2k}{(2\pi)^2} \frac{1}{v(\vec{k}) - v(\vec{q})}, \quad (106)$$

where we invoke $\langle \delta S_{i = 3}(\vec{k})\delta S_{i = 3}(\vec{q}) \rangle = \delta_{\vec{k} + \vec{q}, \vec{0}} \langle \delta S_{i = 3}(\vec{k})^2 \rangle$ by translational invariance, and employed equipartition. Till now we investigated general incommensurate systems, we now specialize to rotationally symmetric systems. As pointed out earlier, when the incommensurate spin system possesses a rotational symmetry, the minimizing manifold $M$ is a $(d - 1)$ dimensional shell of radius $q > 0$. The fluctuation integral receives divergent contributions from the low energy modes nearby. By quadratic expansion about the minimum along $\vec{n} \perp M$, a divergent one-dimensional integral for the bounded $\langle \delta S^2(\vec{x} = 0) \rangle$ signals that quadratic fluctuation analysis calculation is inconsistent. We are led to the conclusion that higher order constraining terms are imperative: We cannot throw away cubic and quartic spin fluctuation terms ($\delta S^3, \delta S^4(\vec{x})$) relative to the quadratic $(\delta S^2(\vec{x}))$ terms (notwithstanding the fact that all of these terms appear with $O(u)$ prefactors irrespective of how large $u$ is). The spin fluctuations $\delta S_{i = 3}(\vec{x})$ are of order unity at all finite temperatures and $T_c(q > 0) \approx 0$. Note that this is "almost a theorem". Here we do not demand that $u$ be small (only $\delta S$). This is an important point. $|\psi_m\rangle$ is an eigenstate for arbitrarily large $u$. To quadratic order in the fluctuations, the minimum belongs to $|\delta S_3\rangle$ (or is degenerate with it). The divergent fluctuations here signal that $O(\delta S^3) = O(\delta S^4)$. Thus assuming that $\delta S \lesssim \sqrt{J/0}$ (with $J = 1$ the exchange constant) we reach a paradox. Thus, if the integral in Eq. (106) diverges for an $O(3)$ model then at all finite $T$: $\delta S \geq \sqrt{J/0}$. Unlike the case for $n = 2$, there is no algebraic long range order. Computing the correlator, we find that $G(\vec{x})$ decays exponentially at all temperatures (much unlike the algebraic long range order seen in Eq. (61)).

When the incommensurability $q$ vanishes, the minimizing manifold shrinks to a point - the number of nearby low energy modes is small and our fluctuation integral converges in $d > 2$. This is in accord with the well known finite temperature phase transition of the nearest-neighbor Heisenberg ferromagnet: $T_c(q = 0) = O(1)$ (or when dimensions are fully restored- it is of the order of the exchange constant). In many rotationally symmetric incommensurate spin systems (such as the Coulomb Frustrated Ferromagnet parameterized by the Coulomb strength $Q = q^2$), a discontinuity in $T_c$ may occur as $M_Q \rightarrow M_{Q = 0} \equiv \langle \vec{q} = 0 \rangle$. We anticipate that small lattice corrections ($\lambda \neq 0$) in $u_{Q(\vec{k})}$ to yield insignifi-
cant modifications to $T_c(q)$: one way to intuit this is to estimate $T_c$ by the temperature at which the fluctuations, as computed within the quadratic Hamiltonian $\langle \delta S^2(\vec{x} = 0) \rangle = O(1)$.

To summarize, we argued that in (essentially hard spin) Heisenberg realizations of sufficiently frustrated incommensurate models, no conventional long range order is possible in the continuum limit at finite temperatures. More specifically, it is suggested that if the integral $\int \frac{d^d k}{v(\vec{k}) - v(\vec{q})}$ diverges then no long range order is possible at finite temperature, unless an expansion about the ground state is void (the thermal dependence of the higher energy mode occupancies cannot be neglected even at $T = 0^+$). If lattice effects are mild then $T_c(q > 0)$ is expected to be small. In the unfrustrated ($q = 0$) Heisenberg ferromagnet in $d = 3$: $T_c = O(1)$. Fusing these facts, a discontinuity in $T_c$ in all rotationally symmetric incommensurate spin systems:

$$\delta T_c = \lim_{q \to 0} |T_c(0) - T_c(q)|$$

(107)

is suggested by soft-spin analysis. Such a discontinuity will occur as $Q \to 0$ in the Coulomb frustrated ferromagnet.

XIII. $O(N \geq 4)$ FLUCTUATIONS

We will now find that, on the level of lowest order spin wave fluctuation analysis, algebraic long range order might be possible for $n = 4$ component spins even when their $n = 3$ counterparts may exhibit exponential decay of correlations at all finite temperatures for the same Hamiltonian. The fluctuation analysis of any $O(n > 2)$ system about a spiral ground state is qualitatively similar to that of the Heisenberg system. As noted previously, poly-spiral states will tend to dominate at large $n$. The reader can convince him/herself that for even $n$ with a $p < n/2$ poly-spiral ground state and for all odd $n$ the fluctuations will give rise to a leading order $\epsilon^{-1}$ divergence. The reasoning is simple: the poly-spiral states extend along an even number of axis. If $n$ is odd then there will be at least one internal spin direction $i$ along which $S_i^q = 0$ and our analysis of the Heisenberg model can be reproduced. The lowest eigenenergy associated with the fluctuations $\langle \psi_m \rangle$ in the $(2p)$ dimensional space spanned by the ground state is higher than the lowest eigen-energy for fluctuations along an orthogonal direction. The term of constraint is positive definite, $\langle \psi | H_{sof f} | \psi \rangle \geq 0$. If $|S| < 1$, this implies that the quadratic term in $\delta S(\vec{x})$ stemming from $H_1$ is non-negative definite. For $S_i^q(\vec{x}) = 0$, this quadratic term in $\delta S_i(\vec{x})$ is zero. The eigenvalue $\lambda_{min} = \lambda_{-} = 2$ corresponds to the zero contribution in $O(\delta S^2(\vec{x}))$ from $H_1$. Once again, $\min_{m} \{E_m \} \geq E_1^p_{\xi \in M_{d'}}$, from the simple

$$\min_{\psi} \langle \psi | [H_0 + H_{sof f}] | \psi \rangle \geq \min_{\phi} \langle \phi | H_0 | \phi \rangle + \min_{\xi} \langle \xi | H_{sof f} | \xi \rangle.$$  

(108)

The fluctuations of even component spin about a $p = n/2$ poly-spiral ground state are more complicated. As before, coupling between different modes occurs. In this case they are more numerous. For $n = 4$, the fluctuation energy, to quadratic order, about a bi-spiral reads $\delta H = \frac{4u^2}{N} \sum_{\vec{k}, \vec{q}} \{ \sum_{\vec{k}, \vec{q}} [v(\vec{k}) - v(\vec{q})] \delta \langle \vec{S}(\vec{k}) \rangle^2 + O(\xi) \}$. The reader can convince him/herself $\lim_{\vec{q} \to 0} |T_c(0) - T_c(q)|$. The fluctuations of even component spin about a $p = n/2$ poly-spiral ground state are more complicated. As before, coupling between different modes occurs. In this case they are more numerous. For $n = 4$, the fluctuation energy, to quadratic order, about a bi-spiral reads $\delta H = \frac{4u^2}{N} \sum_{\vec{k}, \vec{q}} \{ \sum_{\vec{k}, \vec{q}} [v(\vec{k}) - v(\vec{q})] \delta \langle \vec{S}(\vec{k}) \rangle^2 + O(\xi) \}$. The reader can convince him/herself $\lim_{\vec{q} \to 0} |T_c(0) - T_c(q)|$. The fluctuations of even component spin about a $p = n/2$ poly-spiral ground state are more complicated. As before, coupling between different modes occurs. In this case they are more numerous. For $n = 4$, the fluctuation energy, to quadratic order, about a bi-spiral reads $\delta H = \frac{4u^2}{N} \sum_{\vec{k}, \vec{q}} \{ \sum_{\vec{k}, \vec{q}} [v(\vec{k}) - v(\vec{q})] \delta \langle \vec{S}(\vec{k}) \rangle^2 + O(\xi) \}$. The reader can convince him/herself $\lim_{\vec{q} \to 0} |T_c(0) - T_c(q)|$. The fluctuations of even component spin about a $p = n/2$ poly-spiral ground state are more complicated. As before, coupling between different modes occurs. In this case they are more numerous. For $n = 4$, the fluctuation energy, to quadratic order, about a bi-spiral reads $\delta H = \frac{4u^2}{N} \sum_{\vec{k}, \vec{q}} \{ \sum_{\vec{k}, \vec{q}} [v(\vec{k}) - v(\vec{q})] \delta \langle \vec{S}(\vec{k}) \rangle^2 + O(\xi) \}$. The reader can convince him/herself $\lim_{\vec{q} \to 0} |T_c(0) - T_c(q)|$.
\[ \times \prod_{i \neq j} [(v(k_i) - E)^2] \]  

(109)

where we have shifted \( E \) by a constant. Notice that decoupling trivially occurs - terms of the form \( [v(k) - E)]|v(k') - E| \) where the modes \( k - k' = q_1 \pm q_2 \), cancel. The four coupled polarizations break into two pairs and that \( H_1 \) plays the role of a pairing interaction. Schematically, for a high dimensional minimizing manifold \( M \), with low lying states i.e. for the terms containing \( k_1 = q_1 + \delta_1 \) and \( k_2 = q_2 + \delta_2 \):

\[ E = E_{\text{min}} + a_i^2[A_i|k_i|_1 + A_\perp \delta_{\perp,i}^1,1] + a_j^2[A_j|k_j|_2 + A_\perp \delta_{\perp,j}^1,2] \]  

(110)

with \( \delta_{\perp,m} \) parallel and perpendicular to the minimizing manifold \( M \) at \( \vec{q}_m \), trivially satisfies \( \det[\mathcal{H} - E] = 0 \).

This dispersion relation agrees, once again, with the result derived from the spin wave stiffness analysis

\[ \Delta E = \frac{1}{2N} \sum_k (v(\vec{k}) - v(\vec{q}))|\vec{S}(\vec{k})|^2 \]  

(111)

and by expansion of \( \Delta H \) for different sorts of twists, the dispersion relations of the two spiral simply lumped together. When \( d = 3 \), as in the \( O(2) \) case \( (p = 1) \) this dispersion gives rise (in the Gaussian approximation) to diverging logarithmic fluctuations: \( O(|\ln \epsilon|) \). Applying equipartition, the Gaussian spin fluctuations in the \([2i - 1, 2i]\) plane:

\[ \Delta S^2_{[2i-1,2i]}(\vec{x} = 0) > \Delta S^2_{\text{low}} \left[ [2i-1,2i]\right](\vec{x} = 0) \]

\[ = k_BT \int \frac{d^d k}{(2\pi)^d} \frac{1}{a_i^2[A_i|k_i|_1 + A_\perp \delta_{\perp,i}^1,1]} \]  

(112)

As the spin fluctuations in the \([12]\) and \([34]\) plane decouple, we may easily compute the correlation functions to find that they are a sum of two decoupled pieces of the form (Eq.(61)) found earlier for XY systems. For general rotationally symmetric incommensurate spin systems having a shell of minimizing modes of radius \( q > 0 \) in momentum space, fluctuations about a certain polypiramide state explicitly lead to the soft-spin

\[ G(\vec{x}) \approx \sum_{i=1,2} \frac{4a_i^2d^2 \cos(qx_{i||})}{x_{i\perp}^2} \exp[-2\eta_\gamma - \eta E_1(x_{i||}Q^{1/4} \frac{1}{2x_{i||}})], \]

where \( x_{i||} \) denotes a spatial coordinate in a direction parallel to \( \vec{q}_i \) etc. As before, \( \eta = \frac{k_BT}{4d}Q^{1/4} \), \( d = \frac{\pi}{\Lambda} \) where \( \Lambda \) is the ultraviolet momentum cutoff, \( \gamma \) is Euler's const., \( q = Q^{1/4} \), and \( E_1(z) \) is the exponential integral.

For all odd \( n \) and for all even \( n \) with \( p < n/2 \) there will be divergent fluctuations similar to those encountered for the \( O(3) \) model, no algebraic long range order is found.

In conclusion, if frustrating interactions cause the ground states to be modulated then the associated ground state degeneracy (for \( n > 2 \)) is much larger by comparison to the usual ferromagnetic ground states. For even \( n \) we have found that, generically, the a three dimensional system will not have long range order when \( M \) is two dimensional. When \( n \) is odd the system will never show long range order if \( M \) is \((d-1)\) or \((d-2)\) dimensional.

\[ \langle \Delta \vec{S}^2(\vec{x} = 0) \rangle \geq (n - 2|\mathcal{M}|) \int \frac{d^d k}{(2\pi)^d} \frac{k_BT}{v(\vec{k}) - v(\vec{q})}. \]  

(113)

This contribution is monotonically increasing in \( n \); Within our scheme, \( T_c \) is finite and may be estimated by the temperature at which the fluctuations are of order unity. By tweaking the symmetry breaking terms to smaller and smaller values, the fluctuation integral becomes larger and larger. For instance, if take \( \lambda \ll 1 \) in \( v_Q(\vec{k}) \) then the integral is very large and \( T_c \) extremely low (in can be made arbitrarily low). Thus as the system will be cooled from high temperatures, it might first undergo a Kosterlitz-Thouless like transition at \( T_{KT} \) to an algebraically ordered state and develop true long range order at critical temperatures \( T_c < T_{KT} \).

### XIV. The Large N Limit- Fluctuation Spectrum, Ground States, Ground State Entropy, and \( T_c \)

So far we have seen that the classical even \( n \) systems are more “gapped” than their odd counterparts. This is reminiscent of the appearance/non-appearance of a gap for quantum spin \( s \) chains with even/odd values of \((2s + 1) \) [37], [38] and the physics for an even/odd number of legs in an \( s = 1/2 \) quantum spin ladder [39] where such even/odd discrepancies are triggered by Berry phase terms. There is no problem in the classical large \( n \) (or spherical model) limit (just as there is none in the quite different quantum spin systems). In this limit, wherein a single normalization constraint is imposed

\[ \sum_{\vec{x}} S^2(\vec{x}) = N, \]  

(114)

the effective number of spin components \( n \) is of the order of the number of sites in the system \( N \). The span of the system \( N \) (the number of Fourier modes allowed within the Brillouin zone) is always larger than the number of minimizing modes \( \{q_i\} \). In such a case we will be left with a divergence as in equation (113) due to the many
unpaired spin components. Within the spherical model, which is easily solvable the fluctuation integral exactly marks the value of the inverse critical temperature

$$\frac{1}{k_B T_c} = \int_{B.Z.} \frac{d^d k}{(2\pi)^d} \frac{1}{v(\vec{k}) - v(\vec{q})}. \quad (115)$$

Thus, $T_c = 0$ if the latter integral diverges and our circle of ideas nicely closes on itself. In terms of ground state degeneracies, in the large $n$ limit, any configuration

$$S^g(\vec{x}) = \sum_{\vec{q} \in M} S(\vec{q}) e^{i\vec{q} \cdot \vec{x}} \quad (116)$$

satisfying the “reality” condition $(S(\vec{x}) = S^*(\vec{x}))$ in momentum space $S^*(\vec{q}) = S(-\vec{q})$, and global normalization (Eq.(114))

$$\sum_{\vec{q} \in M} |S(\vec{q})|^2 = N^2 \quad (117)$$

is a ground state. The proof is the same as before: any configuration $S^g(\vec{x})$ having as its non-vanishing Fourier components only those momenta $\vec{q}$ that minimize the interaction kernel $v(\vec{k})$, is a ground state. Thus, with the ground state degeneracy denoted by $g$,

$$\ln(g) \sim |M|. \quad (118)$$

The ground state entropy is given by the number of minimizing modes in $\vec{k}$ space- the size of the manifold $M$ spanned by these modes. As the spherical model is the least restricted of all $O(n)$ variants, this serves as an upper bound on finite $n$, $O(n)$ ground state degeneracies. The astute reader will note that the Ising ground state constructed in section (IV) for the kernel $v_z = z \Delta^2 - \Delta$ with $z = 1/8$ on the square lattice indeed do saturate the spherical bound. In the situation that the minimizing modes lie on $(d-1)$ dimensional manifold $M$, the zero temperature entropy is sub-extensive yet very large, scaling with the surface area of the system- $\ln(g) \sim L^{d-1}$. We note that arbitrary periodic patterns (including amongst many others Wigner crystals) will be found in the spherical re-incarnation of many spin models [42].

**XV. PERMUTATIONAL SYMMETRY AND THE INTEGRABILITY OF THE SPHERICAL MODEL**

In many integrable models, the system possesses an extremely large symmetry in the thermodynamic limit. Perhaps closest in spirit to the spherical model which possesses an $N!$ fold permutational symmetry is a permutational like character and symmetry within integrable Bethe ansatz models where not only the total momentum is conserved but also the momentum of each individual particle does not change apart from permutations upon scattering. Although trivial, the permutational symmetry of the large $n$ model has often been overlooked. In fact, several authors have attempted (unsuccessfully) to find systems having a different geometry of minimizing manifolds in $\vec{k}$- space (spherical surfaces versus sheets etc.) yet with the same degeneracy that have different thermodynamics. This quest was unsuccessful for fundamental reasons. The classical spin spherical model (or $O(n \to \infty)$) partition function

$$Z = \text{const} \left( \prod_{\vec{k}} \frac{1}{\sqrt{\beta\hat{v}(\vec{k}) + \mu}} \right), \quad (119)$$

where the chemical potential $\mu$ satisfies

$$\beta = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\hat{v}(\vec{k}) + \mu}, \quad (120)$$

is invariant under permutations of $\{\hat{v}(\vec{k})\} \to \{\hat{v}(\vec{P}\vec{k})\}$. In the above, the permutations

$$\{\vec{k}_i\}_{i=1}^N \to \{\vec{P}\vec{k}\} \quad (121)$$

correspond to all possible shufflings of the $N$ wave-vectors $\vec{k}_i$. Although quite simple, this is not universally realized. Several authors have attempted to compute the critical exponents in the spherical limit (via an RG calculation) for systems having different minimizing manifolds yet all sharing the same relevant density of states. This quest was not very economical. As unrealized by these authors, by permutational symmetry these models are identical.

This simple invariance allows all $d$-dimensional translationally invariant systems to be mapped onto a 1-dimensional one. Let us design an effective one dimensional kernel $V_{eff}(k)$ by

$$\int \delta[\hat{v}(\vec{k}) - v]\, d^d k = |dV_{eff}/dk|^{-1}_{V_{eff}(k)=\hat{v}}. \quad (122)$$

The last relation secures that the density of states and consequently the partition function is preserved. For the two-dimensional nearest-neighbor ferromagnet:

$$|dV_{eff}/dk|^{-1} = \rho(V_{eff}) \quad (123)$$

and consequently

$$k(V_{eff}) = c_1 \int_0^{V_{eff}} F(\sin^{-1}\sqrt{\frac{2}{(3-u)u}} \cdot \sqrt{4u - u^2}) du, \quad (124)$$

where $F(t, s)$ is an incomplete elliptic integral of the first kind. Eq.(124) may be inverted and Fourier transformed
to find the effective one dimensional real space kernel \( V_{eff}(x) \). We have just mapped the two dimensional nearest neighbor ferromagnet onto a one dimensional system. In a similar fashion, within the spherical (or equivalently the \( O(n \rightarrow \infty) \)) limit all high dimensional problems may be mapped onto a translationally invariant one dimensional problem. It follows that the, large \( n \), critical exponents of the \( d \) dimensional nearest neighbor ferromagnet are the same as those of translationally invariant one dimensional system with longer range interactions. We have just shown that a two dimensional \( O(n \gg 1) \) system may have the same thermodynamics as a one dimensional system. By permutational symmetry, such a mapping may be performed for all systems irrespective of the dimensionality of the lattice or of the nature of the interaction (so long as it translationally invariant). This demonstrates once again that the notion of universality (with dependence only on the order parameter symmetry, dimensionality etc.) may apply only to the canonical interactions.

Permutational symmetry is broken to \( O(\beta^4) \) for finite \( n \). Upon performing a Hubbard-Stratonovich transformation, for a constraining term (e.g. \( \sum_x \ln[\cosh(\eta(x))] \) for \( O(1) \) spins) symmetric in \( \{\eta(k)\} \) to a given order, one may re-arrange the non-constraining term \( \sum_k \eta(k)\eta(k)^\dagger = \sum_k \bar{\eta(k)}\eta(P\bar{k})\eta(P\bar{k})^\dagger \) and relabel the dummy integration variables \( H[\{\eta(k)\}] \rightarrow H[\{\eta(P\bar{k})\}] \) to effect the constraining term augmented to a shuffled spectra \( \bar{\eta}(P\bar{k}) \).

\section{XVI. \( O(N \geq 2) \) Weiss Mean Field Theory of Any Translationally Invariant Theory.}

In this section the critical temperature is computed exactly for all translationally invariant incommensurate \( O(n \geq 2) \) systems. Unlike the situation for Ising spins, we proved that simple spiral states are the only ground states for incommensurate systems. Here, for \( O(n \geq 2) \) when \( T < T_c \):

\[
\langle \vec{S}(\vec{x}) \rangle = s \bar{S}_{\text{ground-state}}(\vec{x}). \tag{125}
\]

For the particular case

\[
\begin{align*}
S_1^{\text{ground-state}}(\vec{x}) &= \cos(\vec{q} \cdot \vec{x}) \\
S_2^{\text{ground-state}}(\vec{x}) &= \sin(\vec{q} \cdot \vec{x}) \\
S_3^{\text{ground-state}}(\vec{x}) &= 0
\end{align*}
\tag{126}
\]

Now only the \( \pm \vec{q} \) modes have finite weight. Repeating the previous steps of section(VI),

\[
\sum_{\vec{y}} V(\vec{x} = 0, \vec{y}) S_2^{\text{ground-state}}(\vec{y}) = 0 \tag{127}
\]

and \( |\langle \vec{S}(\vec{x} = 0) \rangle| = |\langle S_1(\vec{x} = 0) \rangle|e_1 \). We now define

\[
M[z] = -\frac{d}{dz} \ln[(2/z)^{(n/2-1)}I_{n/2-1}(z)], \tag{128}
\]

with \( [I_{n/2-1}(z)] \) a Bessel function. With this definition in hand, the mean-field equation reads

\[
|\langle S_1(\vec{x} = 0) \rangle| = s = M[| \sum_{\vec{y}} V(\vec{x}, \vec{y}) \langle S_1(\vec{y}) \rangle |].
\]

The onset of the non-zero solutions is at

\[
|\beta_c v(q)| = n. \tag{129}
\]

If \( V(\vec{x} = 0) = 0 \) (no on-site interaction), then \( \int d^d k v(\vec{k}) = 0 \), implying that \( v(q) < 0 \) and \( T_c > 0 \). Note that within the mean field approximation, \( T_c \) is a continuous function of all the parameters in the Hamiltonian. Here the ground state is symmetric with respect to all sites. The above is the exact value of \( T_c \) within Weiss mean field theory for the helical ground-states. For polynomials we will get \( p \) identical equations: both sides of the self consistency equations are multiplied by \( a_i^2 \) where \( a_i \) is the amplitude of the \( l-th \) spiral in the \([2l-1, 2l] \) plane. As \( v(q_m) = v(q) \), we will arrive at the same value of \( T_c \) as for the case of simple spirals.

\section{ XVII. Extensions to Arbitrary Two Spin Interactions}

Any real kernel \( V(\vec{x}, \vec{y}) \) may be symmetrized \( [V(\vec{x}, \vec{y}) + V(\vec{y}, \vec{x})]/2 \rightarrow V(\vec{x}, \vec{y}) \) to a hermitian form. Consequently, by a unitary transformation, it will become diagonal. The Fourier modes are the eigen-modes of \( V \) when it is translationally invariant. We may similarly envisage extensions to other, arbitrary, \( V(\vec{x}, \vec{y}) \) diagonalized in another complete orthogonal basis \( \{ \vec{u} \} \):

\[
\langle \vec{u}_i | V | \vec{u}_j \rangle = \delta_{ij} \langle \vec{u}_i | V | \vec{u}_i \rangle \tag{130}
\]

Many of the statements that we have made hitherto have a similar flavor in this more general case.

For instance, the large \( n \) fluctuation integrals are of the same form

\[
\int d^d u \frac{1}{v(\vec{u}) - v_{\text{min}}} \tag{131}
\]

with the wave-vector \( \vec{k} \) traded in for \( \vec{u} \). Here, \( v_{\text{min}} = \min_{
vec{a}} v(\vec{u}) \). Once again, one may examine the topology of the minimizing manifold in \( \vec{u} \) space. If the surface is \( (d - 1) \) dimensional and \( v(\vec{u}) \) is analytic in its environs then, for large \( n \), \( T_c = 0 \). More generally, in the large \( n \) limit, the partition function

\[
Z = \text{const} \left[ \prod_{\vec{a}} \left| \frac{1}{\sqrt{\beta v(\vec{u}) + \mu}} \right| \right], \tag{132}
\]

where the chemical potential \( \mu \) satisfies
\[ \beta N = \sum_{\vec{u}} \frac{1}{v(\vec{u}) + \mu} \] (133)

The topology of the ground state sector of O(n) models will once again be governed by a direct product of the topology of the minimizing manifold in \( \vec{u} \) space with the spherical manifold of the \( O(n) \) group. In the general case it will be dramatically rich.

We may similarly extend the Peierls bounds of section \( (V) \) to some infinite range interactions also in this case by contrasting the energy penalties in the now diagonalizing \( \vec{u} \) basis with those that occur for short range systems which are diagonal in Fourier space. For instance, we may examine spin configurations computed with the non-translationally invariant ("disordered") kernel \( V \) and compare the energies relative to those in a simple reference short range interaction which is diagonal in \( k - \text{space} \) (\( V_{\text{short}}(\vec{k}) \)) and has a minimum \( v_{\text{short \ min}} \equiv \min_{\vec{k}} \{v_{\text{short}}(\vec{k})\} \). The energy penalties for excitations in a system with the kernel \( V \),

\[ \Delta E_{\text{disordered}} = \frac{1}{2N} \sum_{\vec{u}} |\vec{S}(\vec{u})|^2 [|\vec{u}| V[\vec{u}] - v_{\text{min}}(\vec{u})] \]

\[ \geq \frac{1}{2N} \sum_{\vec{k}} |\vec{S}(\vec{k})|^2 [v_{\text{short}}(\vec{k}) - v_{\text{short \ min}}] \] (134)

if the kernels \( V \) and \( V_{\text{short}} \) share the same realizable lowest energy Ising eigenstate.

**XVIII. O(N) Spin Dynamics and Simulations**

We now investigate the dynamics in all (commensurate or incommensurate) continuous spin systems. It is seen that the equations of motion are relatively trivial in \( \vec{k} \) space. Potentially, this allows for the construction of new efficient algorithms. If the Hamiltonian

\[ H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} V(\vec{x}, \vec{y}) \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}), \] (135)

then the force, \( \vec{F}(\vec{z}) = -\frac{\partial H}{\partial \vec{S}(\vec{z})} \), on given spin at size \( \vec{z} \) is

\[ \vec{F}(\vec{z}) = -\frac{1}{2} \sum_{\vec{y}} \{ V(\vec{z}, \vec{y}) + V(\vec{y}, \vec{z}) \} \vec{S}(\vec{y}). \] (136)

We may symmetrize \( V(\vec{x}, \vec{y}) = V(\vec{y}, \vec{x}) \), as we have done repeatedly, without changing \( H \). For a more general two spin kernel \( V(\vec{x}, \vec{y}) \) which is not translationally invariant the Fourier space index \( \vec{k} \) should be replaced by the more general \( \vec{u} \). It is readily verified that

\[ \frac{d^2 \vec{S}(\vec{k})}{dt^2} = -\frac{1}{N} v(\vec{k}) \vec{S}(\vec{k}). \] (137)

Let an arbitrary constant \( A \) satisfy \( A > -\min_{\vec{k}} \{v(\vec{k})\} \).

The equations of motion in a shifted system with the interaction kernel \( (v(\vec{k}) + A) \) are

\[ \frac{d^2 \vec{S}(\vec{k})}{dt^2} = -(A + v(\vec{k})) \vec{S}(\vec{k}) \]

\[ \equiv -\omega_k^2 \vec{S}(\vec{k}), \] (138)

which may be trivially integrated. Adding the constant \( A \) merely shifts \( H \rightarrow H + A/2 \) with no change in the underlying physics. The solution to Eq. (138) is

\[ \vec{S}_{\text{un}}(\vec{k}, t) = \vec{S}(\vec{k}, 0) \cos \omega_k t + \frac{d\vec{S}(\vec{k}, t)}{dt}\big|_{t=0} \times \frac{1}{\omega_k} \sin \omega_k t, \]

\[ \vec{S}_{\text{un}}(\vec{k}, t+\delta t) = \vec{S}(\vec{k}, t) \cos \omega_k \delta t \]

\[ + \frac{\delta \vec{S}(\vec{k}, t)}{\delta t} \frac{1}{\omega_k} \sin \omega_k \delta t. \]

This suggests the following simple recursive algorithm: (i) At time \( t \), start off with initial values \( \{ \vec{S}(\vec{x}, t) \} \); (ii) Fourier transform to find \( \{ \vec{S}(\vec{k}, t) \} \); (iii) Integrate to find the un-normalized \( \{ \vec{S}_{\text{un}}(\vec{k}, t+\delta t) \} \); (iv) Fourier transform back to find the un-normalized real-space spins \( \{ \vec{S}_{\text{un}}(\vec{x}, t+\delta t) \} \); (v) Normalize the spins:

\[ \vec{S}(\vec{x}, t+\delta t) = \frac{\vec{S}_{\text{un}}(\vec{x}, t+\delta t)}{|\vec{S}_{\text{un}}(\vec{x}, t+\delta t)|}; \] (139)

(vi) Compute \( \{ \delta \vec{S}(\vec{x}, t+\delta t) \} = \{ [\vec{S}(\vec{x}, t+\delta t) - \vec{S}(\vec{x}, t)] \} \); (vii) Fourier transform to find \( \{ \delta \vec{S}(\vec{k}, t+\delta t) \} \); and finally (viii) Go back to (ii). Thus far, we neglected thermal effects. To take these into account, we could integrate these equations with a thermal noise term augmented to the restoring force

\[ \frac{d^2 \vec{S}(\vec{k})}{dt^2} = -\frac{1}{N} v(\vec{k}) \vec{S}(\vec{k}) + \vec{F}^{\text{noise}}(T, t). \] (140)

Expressed in this format, the execution of this algorithm for continuous \( O(n) \geq 2 \) spins seems easier than that for a discrete Ising system. Here the equations of motion may be integrated to produce arbitrarily small updates at all sites. A priori, this algorithm might be better than a real space more brute force approach whereby, effectively, the torque equations in angular variables (the spins \( \vec{S}(\vec{x}) \) are automatically normalized) are integrated [43].

**XIX. Conclusions**

In this article, we discussed \( O(n) \) spin models on a cubic lattice in mostly translationally invariant systems. These include frustrated systems that have been the subject of much attention in recent years. Here we outline a few of our results.
1) Certain frustrated Ising systems were shown to display a huge ground state degeneracy. The ground state entropy in these systems scaled as the surface area, illustrating a “holographic” like effect. Computing the energies of various contending states is trivial and easily allows us, on a non-rigorous footing, to see if lock-in effect may arise and what scaling might be anticipated.

2) The Peierls energy bound may be rigorously extended to many Ising systems in which the interaction kernel displays minima at a finite number of commensurate wave-vectors. These systems include some with arbitrarily long range interactions. This bound can also be extended in certain instances to systems displaying no translational invariance.

3) The ground states of all translationally invariant XY models were rigorously shown to be spirals (and only spirals) unless certain special commensurability conditions were met. No other ground state are generically possible. The trivial uniform and Neel states correspond to commensurate wave-number ordering (or spirals of infinite or two unit length pitch respectively).

4) The Weiss mean-field equations of the general $O(n)$ systems were looked at. The mean field transition temperature $T^{MF}$ for all $O(n \geq 2)$ models was exactly computed.

5) For $O(n \geq 3)$ spins, poly-spiral states are the dominant ones. These correspond to a hybrid of $p \leq \text{Int}[n/2]$ (with $\text{Int}[\cdot]$ denoting the integer part) spirals in different orthogonal planes. Once again, unless commensurability conditions amongst the minimizing modes are met, we can easily prove that these are the only viable ground states. It is also easy to demonstrate that the poly-spiral states with the largest viable $p$ (i.e. $p = \text{Int}[n/2]$) are statistically preferred. Moreover, they are more stable against thermal fluctuations.

6) We performed a non-rigorous spin stiffness analysis to test the stiffness of XY spins under different sorts of external twists. We found that certain frustrated systems display a smectic like low energy dispersion.

7) A thermal fluctuation analysis of soft XY spins for arbitrary interactions may be easily done. We found that, in the general case, a Dirac like equation arises for the two component “spinors”. In general, for incommensurate momenta, the different momenta are coupled. By truncating the equations for small $u$ (the soft spin limit), we regained the exact same energy dispersion attained by the simple minded spin-stiffness analysis. We found that in rotationally symmetric incommensurate spin systems, the resulting spectra matches that of smectic liquid crystals. Liquid crystalline properties are natural for many of the frustrated systems appearing when competing interactions are present.

8) We generalized the Mermin-Wagner-Coleman theorem to all translationally invariant $d \leq 2$ systems with a twice differentiable interaction kernel. Here we did not only focus on ferromagnetic or antiferromagnetic states. All possible orders were excluded. The connection of the denominator to Galilean boosts was noted. At zero temperature, the denominator of the integrand in the general inequality matches that attained by the Dirac like analysis.

9) The resulting generalized Mermin-Wagner-Coleman bounds may be examined in high dimensions ($d > 2$). We showed that if a certain integral diverges, the generalized Mermin-Wagner-Coleman bounds disallow us to think of the dispersion relation for spin waves at arbitrarily low temperatures $T = 0^+$ as that computed by assuming a perfectly ordered ground state. We also formally linked the integrand to a decoherence (or bandwidth) time scale [34] in the quantum case. We further noted that exactly such a divergence of “decoherence time” is linked to a non-rigorous replica derivation of glassy behavior in many systems. A large degeneracy and near degeneracy may be the underlying cause of two effects- a greater fragility of order and sluggish glassy dynamics.

10) The effect of thermal fluctuation in the most general translationally invariant $O(n = 3)$ (Heisenberg) systems was considered. We showed that, for incommensurate wave-numbers, the effects of fluctuations here are expected to be larger than in XY spins. The only approximation made here was that the spin fluctuations $\delta S$ about the only viable ground state (proved in an earlier section) were small. This in turn led to a paradox at finite temperature: assuming only small spin fluctuations led to a divergent thermal fluctuation disallowed by the normalization of the spin. The parameter $u$ enforcing the normalization of the spin via the additional term $H_1 = u \sum x (S^2(x) - 1)^2$, may be arbitrarily large in this treatment (we are not confined to only soft spins $u \ll 1$).

11) We extended our thermal fluctuation analysis to all translationally invariant $O(n \geq 4)$ systems. For even $n$, the determination of the spectrum led to the analysis of the Dirac like equation obtained for the XY model. For odd $n$, one component was left unpaired and the fluctuations were seen to be much larger. The integral appearing in the generalized Mermin-Wagner inequality is a strict lower bound on the fluctuations. Thus, spin systems with a non-trivial minimizing manifold in $k$ space, much like spin ladders, display an interesting even-odd effect. When we analyze the fluctuations of spins having an even number of components ($n$), we find precisely the Mermin-Wagner integral as the relevant thermal fluctuation integral. For a system of spins having an odd number of com-
ponents, we find a more divergent (for on-shell minima in Fourier space) fluctuation integral. Under the influence of the quartic $H_1$ term, spin (or field) components bunch up in pairs. For an odd number of spin components $n$, a lone unpaired spin component can give rise to more divergent fluctuations. Computing the correlators within truncated soft spin models, we found that, in rotationally symmetric incommensurate systems, weak algebraic long range order may persist in even $n$ systems, while being absent in odd $n$ systems of the same Hamiltonian. Much as in quantum spin ladders and chains, the energy spectrum is more “gapped” and inhibited for even spins. In the large $n$ limit, both tend to the “odd” $n$ spherical limit fluctuation integral which is none other than the spherical limit expression. The bottom line is that the bound we derive by a generalized Mermin-Wagner inequality is a strict lower bound on the fluctuations. For odd component spins $n$, the thermal fluctuations are much larger. This goes against some of the common held intuition (correct for conventional commensurate ordering at $q_0 = 0, (\pi, \pi, \ldots, \pi)$) that as $n$ is monotonically increased, the system orders at lower and lower temperatures as entropy effects become ever large. By contrast, we find that for incommensurate orders, the size of the thermal fluctuations is not monotonic in $n$ but rather exhibits novel even-odd alterations.

12) The large $n$ limit, which is exactly solvable, gives a fluctuation integral ($1/T_c$) similar to that for the odd $n$ case. We further rigorously illustrate how the ground state entropy scales as the size of the manifold of minimizing modes in $k$ space. In particular, for many frustrated models, this leads to an entropy which increases as the surface area of the system, leading to a “holographic” like effect.

13) We illustrated how we may think about the topology of the minimizing manifold and use other concepts that we introduced hitherto also when the interaction is not translationally invariant, as in disordered systems. In these cases we considered the new coordinate $\vec{u}$ parameterizing the eigenstates of the interaction kernel. The large $n$ analysis proceeds as before. We illustrated how Peierls’ bounds may be constructed in certain systems with long range interactions.

14) The dynamics of $O(n)$ spins in $k$ space was considered in the general case, and new algorithm was suggested. The dynamics in $k$ space is no less easily captured than in most standard methods that work only in real space. No torques or forces need be considered. The evolution of the spin in $k$ space under the influence of the two spin interactions is trivial.

XX. APPENDIX

Here we follow the beautiful treatment of [44]. Within the (hard spin) fully constrained XY model:

$$G(\vec{x} - \vec{y}) = \langle \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}) \rangle = \langle \cos(\theta(\vec{x}) - \theta(\vec{y})) \rangle.$$  

(141)

Here $\theta(\vec{x}) = \vec{q} \cdot \vec{x} + \Delta \theta(\vec{x})$, i.e. $\Delta \theta$ denotes the phase fluctuations about our spiral ground state and

$$G(\vec{x} - \vec{y}) = \cos(\vec{q} \cdot (\vec{x} - \vec{y})) \langle e^{i(\Delta \theta(\vec{x}) - \Delta \theta(\vec{y}))} \rangle.$$  

(142)

In our harmonic approximation $\{\delta \theta(\vec{x})\}$ are random Gaussian variables and only the first term in the cumulant expansion is non-vanishing. The correlator

$$G(\vec{x}) = \exp\left[-\frac{1}{2} \langle ([\Delta \theta(\vec{x}) - \Delta \theta(0)]^2) \rangle \right]$$

$$= \exp\left[k_B T \int \frac{d^d k}{(2\pi)^d} \frac{1 - \cos \vec{q} \cdot \vec{x}}{A_{||}(\delta_{||}^2 + A_{\perp} \delta_{\perp}^2)} \right].$$  

(143)

Now let us shift variables $\vec{k} \rightarrow \vec{k} - \vec{q} = \delta$, and for purposes of convergence explicitly introduce an upper bound on $k_{\perp}$: $0 < k_{\perp} < \Lambda$

$$I(\vec{x}_{||}, x_{\perp}) = \int \frac{1 - \cos(\vec{q} \cdot \vec{x})}{A_{||} k_{||}^2 + A_{\perp} k_{\perp}^2}.$$  

(144)

This may be computed by first integrating over $k_{||}$ employing

$$\int_{-\infty}^{\infty} \frac{1 - \cos(a b - x)}{x^2 + c^2} dx = \frac{\pi}{c} \left[1 - e^{-ac} \cos(ab)\right],$$  

(145)

to obtain

$$\frac{1}{A_{||}} \int_0^\infty \frac{1 - \cos(k_{||} x_{||} + k_{\perp} \cdot \vec{x}_{\perp})}{k_{||}^2 + (A_{\perp} k_{\perp}^2 / A_{||})} dk_{||}$$

$$= \frac{1}{2} \frac{\pi}{k_{\perp}^2} \sqrt{\frac{1}{A_{||} A_{\perp}}} \left[1 - \exp\left(-\sqrt{\frac{A_{\perp}}{A_{||}}} k_{\perp}^2 x_{||}\right) \cos(k_{\perp} \cdot \vec{x}_{\perp})\right].$$  

(146)

If $\phi$ denotes the angle between $k_{\perp}$ and $\vec{x}_{\perp}$ then

$$\int_0^{2\pi} \left[1 - \exp\left(-\sqrt{\frac{A_{\perp}}{A_{||}}} k_{\perp}^2 x_{||}\right) \cos(k_{\perp} \cdot \vec{x}_{\perp})\right] d\phi$$

$$= 2\pi - \exp\left(-\sqrt{\frac{A_{\perp}}{A_{||}}} k_{\perp}^2 x_{||}\right) \int_0^{2\pi} \cos(k_{\perp} x_{\perp} \cos \phi) d\phi.$$  

(147)

As

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \cos \phi) d\phi,$$  

(148)

the integral of Eq.(144),

$$I(\vec{x}) = \frac{1}{2A_{||}} \frac{\pi 2\pi}{k_{\perp}} \int_0^A 1 - \exp\left(-\sqrt{\frac{A_{||}}{A_{\perp}}} k_{\perp}^2 x_{||}\right) J_0(k_{\perp} x_{\perp})$$

$$\sqrt{\frac{A_{\perp}}{A_{||}}} k_{\perp}^2.$$  

We may now insert the series expansion of $J_0(x)$ and integrate term by term. The Bessel function
Comparing to the series for the exponential integral

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-n)^n z^n}{n(n!)}$$  \hspace{1cm} (150)

($\gamma$ is Euler’s constant), we find that

$$G(\vec{x}) \sim \frac{4d^2}{x_1^4} \exp[-2\eta \gamma - \eta E_1\left(\frac{x_1^2 q}{4x_1 \sqrt{\frac{4}{A^2}}}\right)] \times \cos[\xi x_1]|$$  \hspace{1cm} (151)

where $\eta = \frac{k_B T}{8\pi} \sqrt{\frac{\Delta_{Q}}{A}} = \frac{k_B T}{10\pi} \xi q$ (in the last equality Eq. (60) is invoked), and $d = \frac{2\pi}{\Lambda}$, with $\Lambda$ the ultra violet momentum cutoff.

**XXI. ACKNOWLEDGMENTS**

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[1] Z. Nussinov, Thesis, UCLA (1999)
[2] M. Seul and D. Andelman, Science 267, 476 (1995)
[3] J. M. Tranquada et al., Nature 375, 561 (1995)
[4] V. J. Emery and S. A. Kivelson, Physica C 209, 597 (1993)
[5] L. Chayes, V. J. Emery, S. A. Kivelson, Z. Nussinov, G. Tarjus, Physica A 225, 129 (1996)
[6] U. Löw et al., Phys. Rev. Lett. 72, 1918 (1994)
[7] E. W. Carlson, V. J. Emery, S. A. Kivelson, D. Orgad cond-mat/0206271, Review chapter to appear in ‘The Physics of Conventional and Unconventional Superconductors’ ed. by K. H. Bennemann and J. B. Ketterson (Springer-Verlag)
[8] S- W. Cheong et al., Phys. Rev. Lett. 67, 1791 (1991)
[9] D. I. Golosov, Phys. Rev. B, vol. 67, 064404 (2003) (cond-mat/0206257)
[10] M. P. Lilly, K. B. Cooper, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West Phys. Rev. Lett. 82, 394 (1999) (cond-mat/9808227)
[11] E. Fradkin, S. A. Kivelson Phys. Rev. B 59, 8065 (1999) (cond-mat/9810151)
[12] M. M. Fogler, cond-mat/0111001, p. 98-138, in High Magnetic Fields: Applications in Condensed Matter Physics and Spectroscopy, ed. by C. Berthier, L.-P. Levy, G. Martinez (Springer-Verlag, Berlin, 2002)
[13] Z. Nussinov, cond-mat/0209029
[14] D. Kivelson, S. A. Kivelson, X. Zhao, Z. Nussinov, and G. Tarjus (Physica A 219, 27 (1995))
[15] We define the Coulomb kernel as that solving the Poisson equation in $d$ dimensions. With this definition in hand, the 2-d Coulomb kernel depends logarithmically on the separation between the two particles (or spins). Physical ‘two dimensional systems’ can be of many different types, e.g.: (a) charges confined to a single 2-d sheet, which interact via 3-d (1/r) Coulomb interactions in empty space; (b) the same, but with the sheet embedded in some surrounding dielectric or conducting medium which modifies the Coulomb interaction (adding screening); (c) a stack of 2-d sheets, where the carriers are confined to a single sheet but can interact Coulombically with carriers on all sheets; (d) an array of infinitely long wires, with uniform charging along the wires, (e) the interaction between magnetic vortices and superficially (f) the interaction between dislocation and disclination lines (or points in this two dimensional context) in a solid which, amongst other things, carry an additional direction dependent tensorial character. Type (a) might be a model for the quantum Hall effect; (b) for a quantum Hall effect with conducting gate; (c) for holes in a cuprate superconductor; and (d) might be applied to a stripe array, looked at sideways, if the stripes are all long and straight. The Coulomb interactions that we are addressing are of the form of systems (d), (e), and (f).
[16] By “cubic rotational symmetry”, we allude to the symmetry transformations $\vec{k} \rightarrow \vec{k}'$ that leave the hyper-cubic lattice Laplacian $\Delta(\vec{k})$ invariant. In the continuum limit $|\vec{k}| \ll 1$, the Lattice Laplacian $\Delta(\vec{k}) \rightarrow k^2$ and the “cubic rotational symmetry” amounts to the standard rotational symmetry, i.e. the set of transformations that conserve $k^2 = (\vec{k}')^2$.
[17] T. Ohta and K. Kawasaki, Macromolecules 19, 2621 (1986)
[18] M. Grousson et al., Phys. Rev. E 62, 7781 (2000)
[19] C. B. Muratov, cond-mat/0205070
[20] It is sufficient to consider a “stripe” state of period $\ell$ (even) in a linear spin chain with the coupling, say, given by Eq. (6) in which for simplicity only the first two terms are kept. If $S_i = +1$ for $i = 0, 1, \ldots, \frac{\ell}{2} - 1$, and $S_i = -1$ for $i = \frac{\ell}{2}, \frac{\ell}{2} + 1, \ldots, \ell - 1$, then after a simple calculation we get $|S_i|^2 = 8(1 - \cos k)$ for the square of the Fourier-transform of $S_i$ with $k = 2\pi j/l$ and $j$ odd, and $0$ for $j$ even. The energy of one period for such a configuration consists of two parts: the part due to the short-range ferromagnetic coupling, computed directly to be equal to $4$, and the part due to the long-range interaction, which is $\frac{Q^2}{\pi^2} \sum k \Delta^{-1}(k)|S_k|^2$. Assuming that $\ell \gg 1$, we can expand the cosine in the Taylor series and extend the summation in $k$ to infinity. Then, the energy per spin becomes

$$\frac{E}{\ell} = \frac{4}{N} + \frac{Q^2}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{4}{\ell} + \frac{Q^2}{96},$$

where we employ $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^4}{96}$. Minimizing this expression with respect to $\ell$ we obtain the optimal period
of the stripe in a viable ground state, $\ell_{\text{viable}} = 4\sqrt{3}Q^{-1/3}$.

Note that this value does not match that suggested by dimensional analysis (the length $q^{-1} = Q^{-1/4}$). For $O(n \geq 2)$ spins, the ground state $\vec{q}$ always exactly match modes minimizing the interaction kernel $v(\vec{k})$. In Ising systems, however, this is not the case.

[21] R. Choksi, J. Nonlinear Sci., 11, 223 (2001)
[22] R. Peierls, Proc. Cambridge Phil. Soc., 32, 477 (1936)
A textbook detailing, amongst many other things, the Peierls argument and the Mermin-Wagner theorem is also J. Glimm and A. Jaffe, “Quantum Physics: A functional Integral Point of View”, Springer Verlag, second edition (1987)

[23] P. Bak, Rep. Prog. Phys., 45, 587 (1982), and references therein
[24] T. Nagamiya, K. Nagata, and Y. Kitano, Prog. Theor. Phys., 27, 1253 (1962)
[25] J. Villain, R. Bidaux, J. P. Carton, and R. Conte, Journal de Physique 41, 1263 (1980).

[26] To concretely highlight the subtleties in various frustrated incommensurate systems, note that, for example, $Q = 16$ in the Coulomb Frustrated Ferromagnet (Eq.(6)), $\vec{q} = (\pi, \pi, 0) \in M_Q$, and $2\vec{q} \equiv 0 (mod\ 2\pi)$. The fluctuation matrix is diagonal in the $\vec{k}$ basis, and the fluctuations are divergent at finite temperatures. An identical situation occurs for $\vec{q} = (\pi, \pi, 0) \in M_{Q=64}$. When $Q = 4$, $\vec{q} = (\pi, 0, 0) \in M_Q$, and it is easy to show that determining the eigenvalue spectrum degenerates into a problem in two parameters $(\Delta_{2}, k_1)$ where $\Delta_2 \equiv 2 \sum_{i=1}^{n} (\frac{1}{2} - \cos k_1 )$ (s.t. $\Delta(\vec{k} + \vec{q}) = \Delta_2 + 2 \cos k_1$, $\Delta(\vec{k}) = \Delta_2 - 2 \cos k_1$). The fluctuation integral about the chosen ground state exhibits a $(d - 2)$ dimensional minimizing manifold (parameterized, in our case, by $(\Delta_2, k_1)$). Note that, at higher order commensurabilities, the dimensionality of the minimizing manifold is low. In fact, the interaction will no longer be diagonalized in $\vec{k}$-space.
[27] D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976)
[28] M. Y. Azbel, Zh. Eksp. Teor. Fiz. 46, 929 (1964) [Sov. Phys.- JETP 19, 634 (1964)]
[29] In the usual case $\vec{q} = 0$ and the quadratic fluctuations diverge as $\ln L$ as $L$ (the size of the system) sets a lower bound: $k_1 \geq 2\pi/L$ in the integral.

[30] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966)
[31] S. Coleman, Comm. Math. Phys. 31, 259 (1973)

[32] Note that even if we had not known the ground states we could still prove that there is no magnetization. All one would need to do is to apply an infinitesimal magnetic field $\vec{h}(\vec{x}) = h \vec{S}_{\text{ground-state}}(\vec{x})$ (152)

with as yet an unknown ground state $\vec{S}_{\text{ground-state}}(\vec{x})$. Replacing any appearance of the minimizing wave-vector $\vec{q}$ in the Schwarz inequality and the definition of $B(\vec{k})$ by the more general wave-vector $\vec{\ell}$ and setting $h \rightarrow 0^+$ one would arrive at the conclusion that $m_\ell = 0$ for each mode $\vec{\ell}$. Consequently,

$$\langle S_n(\vec{x}) \rangle = \frac{1}{N} \sum_\ell e^{i\vec{q}_\ell \cdot \vec{x}} m_\ell = 0$$ (153)

Stated alternatively, by Parseval’s theorem,

$$\sum_\ell \langle \vec{S}(\vec{x}) \rangle^2 = \frac{1}{N^2} \sum_\ell \langle \vec{S}(\vec{l}) \rangle \langle \vec{S}(\vec{-l}) \rangle.$$ (154)

Taking careful note of the system size $(N)$ dependence in Eq.(85),

$$\sum_\ell \langle \vec{S}(\vec{x}) \rangle^2 = O\left(\frac{N}{\ln N}\right)$$ (155)

and thus the average value of $|\langle \vec{S}(\vec{x}) \rangle|$ at any given site $\vec{x}$ diminishes as $O(1/\ln N)$ which vanishes in the thermodynamic $(N \rightarrow \infty)$ limit.

[33] More precisely, everything to be claimed holds for the more general class of functions $v(\vec{k})$ for which

$$\max_{\vec{v} \in B. \vec{x}} \left[ v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{0}) \right] \leq B_1k^2$$ (156)

is obeyed for all momenta $\vec{k}$ with some finite positive constant $B_1$.

[34] R. Moessner, Can. J. Phys. 79, 1283 (2001)
[35] C. Itzykson and J-M. Drouffe “Statistical field theory”, Cambridge University Press (1989)
[36] S. Brazovskii, Sov. Phys. JETP 41, 85(1975); It should be noted that in the Appendix of this work and in other related diagrammatic treatments, it was hinted that spiral (“one dimensional periodic”) order may be inhibited. As we proved in this article, spiral states are the only canonical ground states in incommensurate spin systems. Thus, the Brazovkii mechanism cannot account for order of continuous spins in any rotationally symmetric $O(n \geq 2)$ incommensurate system.

[37] F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983); F. D. M. Haldane, Phys. Lett. 93A, 464 (1983); I. Affleck, Nucl. Phys. B 257, 397 (1985).
[38] E. Fradkin, "Field Theories of Condensed Matter Systems", Addison Wesley, 1991, and references therein
[39] D. C. Johnston et al., Phys. Rev. B, 35, 219 (1987); Z. Hiroi et al., J. Solid State Chem. 95, 230 (1995); M. Azuma et al., Phys. Rev. Letts., 73, 3463 (1994)

[40] Looking at the real space form of the constraining term it is readily seen that $|\psi H_1 | \psi \rangle \geq 0$. If $|\delta S| \ll 1$ this implies that the quadratic term in $\delta S_2(\vec{x})$ originating from $H_1$ is non-negative definite. For $S_{\text{ground}}(\vec{x}) = 0$, this quadratic term in $\delta S_3(\vec{x})$ vanishes. The lowest eigenvalue $\lambda_{\text{min}} = 2$ corresponds to this vanishing $O(\delta S^2)$ contribution from $H_1$. An analogous occurs in the fluctuation analysis about a poly-spiral state.

[41] For a vanishing lower cutoff $\epsilon$ on $|\vec{k} - \vec{q}|$, the fluctuations of an $n - \text{component}$ spin $(d = 3)$ within the Coulomb Frustrated Ferromagnet about a helical ground state are given by

$$\frac{\langle (\Delta S)^2 \rangle}{k_B T} = \frac{(n - 2) \sqrt{Q}}{4\pi^2 \epsilon} - \frac{1}{16\pi} Q^{1/4} \ln |\epsilon|.$$ (157)

More generally, in $d > 2$ dimensions, the leading order infrared contribution reads
\[
\frac{(n - 2) Q^{(d-1)/4}}{2^d \pi^{d/2} \Gamma(d/2)} - \frac{\epsilon^{d-3} Q^{1/4}}{2^{d+1} \pi^{(d-1)/2} (d - 3) \Gamma(d/2)}.
\] (158)

[42] Any given finite periodic spin configuration (including any Wigner crystal), contains only a finite set of wave-numbers \(\{\vec{q}_i\}\). An interaction kernel \(v(\vec{k})\) may always be fashioned such that its minima occur at \(\vec{k} = \vec{q}_i\). For those with interest in real charged systems, we note that as in the standard lattice gas mapping, any charge only problem may be translated onto a scalar spin problem via the generalized \(S_i = c(p_i - \bar{p})\) with \(S_i\) the spin at site \(i\), \(c\) a fixed constant (conventionally \(c = 2\) for Ising systems), \(p_i\) the charge at the given site, and \(\bar{p}\) the average charge density over the entire system (the background charge taken to be a half in the standard Ising lattice gas problem). Within the spherical model, only global normalization of the spins is required and the charge density may be arbitrary at any given site. As the Hamiltonian is quadratic in \(\{S_i\}\), the average \(\langle S \rangle = 0\), self-consistently implying that the average charge density \(\langle \rho \rangle = \bar{p}\).

[43] In real space, the equations of motion trivially read
\[
\frac{d^2 \phi(\vec{x})}{dt^2} = \sum_\vec{y} V(\vec{x} - \vec{y}) \sin[\phi(\vec{x}) - \phi(\vec{y})].
\] (159)

for an O(2) system. For the three-component spin system, with \(\theta\) and \(\phi\) the longitudinal and azimuthal angles,
\[
\sin^2 \theta(\vec{x}) \frac{d^2 \phi(\vec{x})}{dt^2} + \sin 2\theta(\vec{x}) \frac{d\phi(\vec{x})}{dt} \frac{d\theta(\vec{x})}{dt} = \sum_\vec{y} V(\vec{x} - \vec{y}) \sin \theta(\vec{x}) \times \sin \theta(\vec{y}) \sin[\phi(\vec{x}) - \phi(\vec{y})];
\]
\[
\frac{d^2 \theta(\vec{x})}{dt^2} = \frac{1}{2} \sin 2\theta(\vec{x}) \left( \frac{d\phi(\vec{x})}{dt} \right)^2
\]
\[
+ \sum_\vec{y} V(\vec{x} - \vec{y}) \left[ \sin \theta(\vec{x}) \cos \theta(\vec{y}) - \cos \theta(\vec{x}) \sin \theta(\vec{y}) \cos[\phi(\vec{x}) - \phi(\vec{y})] \right].
\] (160)

[44] J. Als- Nielsen et al., Phys. Rev. B 22, 312 (1980)