GEOMETRIC INVARIANTS OF SPACES WITH ISOLATED FLATS

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Abstract. We study those groups that act properly discontinuously, cocompactly, and isometrically on CAT(0) spaces with isolated flats and the Relative Fellow Traveller Property. The groups in question include word hyperbolic CAT(0) groups as well as geometrically finite Kleinian groups and numerous 2–dimensional CAT(0) groups. For such a group we show that there is an intrinsic notion of a quasiconvex subgroup which is equivalent to the inclusion being a quasi-isometric embedding. We also show that the visual boundary of the CAT(0) space is actually an invariant of the group. More generally, we show that each quasiconvex subgroup of such a group has a canonical limit set which is independent of the choice of overgroup.

The main results in this article were established by Gromov and Short in the word hyperbolic setting and do not extend to arbitrary CAT(0) groups.

1. Introduction

A group is word hyperbolic if it admits a geometric action (i.e., properly discontinuous, cocompact, and isometric) on a $\delta$–hyperbolic space. Numerous geometric features of such an action have been shown to be invariants of the group, in particular the visual boundary ([Gro87]) and the set of quasiconvex subgroups ([Sho91]). A quasiconvex subgroup of a word hyperbolic group is again a word hyperbolic group, and its boundary equivariantly embeds into the boundary of the larger group as a limit set ([Gro87]). These results do not extend from the negatively curved setting to arbitrary nonpositively curved groups.

The goal of this article is to show that these results do, in fact, hold for a special class of nonpositively curved groups, namely those which act geometrically on CAT(0) spaces with isolated flats and the Relative Fellow Traveller Property, which were introduced by the author in [Hru04]. The groups in question include the fundamental group of any compact nonpositively curved 2–complex whose 2–cells are regular Euclidean hexagons as well as all geometrically finite Kleinian groups.

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In the $\delta$–hyperbolic setting Gromov established a Fellow Traveller Property, which states that quasigeodesics with common endpoints track close together. This fundamental property is a key component of the proofs of the main results in the word hyperbolic setting.

The Relative Fellow Traveller Property is a generalization of the Fellow Traveller Property in which pairs of quasigeodesics fellow travel “relative to flats” in a sense that we make precise in Section 5. The Relative Fellow Traveller Property is useful in conjunction with the isolated flats property, defined in Section 3, which roughly states that flat Euclidean subspaces are “disjoint at infinity.” Morally speaking, the CAT(0) spaces with isolated flats are the CAT(0) spaces that are closest to being $\delta$-hyperbolic, while still containing flat subspaces.

Although the notions of isolated flats and the Relative Fellow Traveller Property were explicitly introduced by the author in [Hru04], the ideas were implicit in earlier work of Kapovich–Leeb ([KL95]), Wise ([Wis96, Wis98]), and Epstein ([ECH+92, Chapter 11]), and have also been studied by Kleiner.

Let $\rho: G \to \text{Isom}(X)$ be a geometric group action. A subgroup $H$ of $G$ is quasiconvex with respect to $\rho$ if the orbit $Hx$ is a quasiconvex subspace of $X$ for some basepoint $x$. In the word hyperbolic setting quasiconvexity does not depend on the choice of geometric action $\rho$ or even on the choice of space $X$. In fact, quasiconvexity of $H$ is equivalent to the intrinsic property that $H \hookrightarrow G$ is a quasi-isometric embedding. This result does not extend to the general CAT(0) setting (see Section 2). Nevertheless, in the presence of isolated flats and the Relative Fellow Traveller Property, we have the following theorem.

**Theorem 1.1** (Quasiconvex $\iff$ undistorted). Let $\rho$ be a geometric action of a group $G$ on a CAT(0) space $X$, where $X$ has isolated flats and the Relative Fellow Traveller Property, and let $H \leq G$ be any finitely generated subgroup. Then $H$ is quasiconvex with respect to $\rho$ if and only if the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding.

The boundary of a (complete) CAT(0) space is the space of all geodesic rays emanating from a fixed basepoint, endowed with the compact-open topology. In the word hyperbolic setting, the boundary is a group invariant in the sense that, if a group acts geometrically on two different $\delta$–hyperbolic spaces then the spaces have the same boundary. Furthermore, as mentioned above, a quasiconvex subgroup of a word hyperbolic group is again word hyperbolic, and there is an equivariant embedding of the boundary of the subgroup into the boundary of the supergroup as a limit set.

In principle, the same is true in the presence of both isolated flats and the Relative Fellow Traveller Property. However, the statement is more subtle since it is currently unknown whether a quasiconvex subgroup of a CAT(0) group is itself CAT(0). (Recall that a group is CAT(0) if it admits a geometric action on a CAT(0) space.)
Theorem 1.2 (Boundary of a quasiconvex subgroup is well-defined). Let \( \rho_1 \) and \( \rho_2 \) be geometric actions of groups \( G_1 \) and \( G_2 \) on CAT(0) spaces \( X_1 \) and \( X_2 \) each having isolated flats and the Relative Fellow Traveller Property. For each \( i \), let \( H_i \leq G_i \) be a quasiconvex subgroup with respect to \( \rho_i \). Then any isomorphism \( \eta: H_1 \to H_2 \) induces an \( \eta \)-equivariant homeomorphism \( \Lambda H_1 \to \Lambda H_2 \).

Roughly speaking the idea is that, if a group \( H \) is a quasiconvex subgroup of two different groups \( G_1 \) and \( G_2 \), then the limit set of \( H \) is the same in the boundary of both groups. If \( H \) is itself a CAT(0) group, then this limit set must also be the boundary of \( H \).

An immediate corollary is that the boundary is a group invariant for CAT(0) groups with isolated flats and the Relative Fellow Traveller Property. Croke and Kleiner showed that this corollary does not extend to the general CAT(0) setting ([CK00]). In fact Wilson has shown that the Croke–Kleiner construction produces a continuous family of homeomorphic 2–complexes whose universal covers all have topologically distinct boundaries ([Wil]).

In order to prove the main theorems, we first prove some basic algebraic facts due to Wise (personal communication) about geometric actions on CAT(0) spaces with isolated flats in Section 3. In particular, we show that maximal flats correspond to maximal virtually abelian subgroups of rank at least two in Theorem 3.7. We also establish that these virtually abelian subgroups lie in only finitely many conjugacy classes (Theorem 3.9). One notable corollary to this analysis is the following.

Theorem 1.3 (\( \mathbb{Z} \times \mathbb{Z} \) subgroups). Suppose \( G \) acts geometrically on a CAT(0) space with isolated flats. Then either \( G \) is word hyperbolic or \( G \) contains a \( \mathbb{Z} \times \mathbb{Z} \) subgroup.

We prove Theorem 1.3 in Section 4. The proof is a more detailed version of the techniques used to prove Theorem 3.7.

Then in Section 6 we examine the geometry of quasiflats, i.e., the quasi-isometrically embedded Euclidean subspaces of dimension at least two. In particular we prove the following theorem, which generalizes a lemma proved by Schwartz in his study of quasi-isometric rigidity of nonuniform lattices in rank one symmetric spaces ([Sch95]).

Theorem 1.4 (Quasiflats are close to flats). Let \( X \) be a CAT(0) space with isolated flats and the Relative Fellow Traveller Property. Given constants \( \lambda \) and \( \epsilon \), there is a constant \( D = D(\lambda, \epsilon, X) \) such that each \( (\lambda, \epsilon) \)-quasiflat lies in a \( D \)-neighborhood of some flat \( F \).

Theorem 1.4 and the Flat Torus Theorem are key components in the proof of the following theorem, which improves the Relative Fellow Traveller Property to a genuine Fellow Traveller Property under an equivariance assumption.
Theorem 1.5 (Equivariant Fellow Traveller Property). Let $G$ act geometrically on two CAT(0) spaces $X$ and $Y$. Suppose further that $X$ has isolated flats and the Relative Fellow Traveller Property. Then any $G$–equivariant quasi-isometry $X \to Y$ maps geodesics in $X$ uniformly close to geodesics in $Y$.

As an immediate corollary, we get an alternate proof of the group invariance of quasiconvexity without using the techniques of Section 4. We also obtain a direct proof of the invariance of the boundary. To prove the more general result of Theorem 1.2 requires a more detailed analysis which combines techniques from the proofs of Theorems 1.1 and 1.5. We undertake this analysis in Section 5.

In [Hru04], the author shows that among proper, compact CAT(0) 2–complexes the isolated flats property is equivalent to the Relative Fellow Traveller Property. Furthermore, Wise has shown that these equivalent properties are satisfied if and only if the 2–complex does not contain an isometrically embedded triplane (see [Hru04] for a proof). A triplane is the space obtained by gluing three Euclidean halfplanes together along their boundary line. For instance the universal cover of a compact nonpositively curved 2–complex whose 2–cells are all regular Euclidean hexagons cannot contain a triplane, and hence has isolated flats and the Relative Fellow Traveller Property.

Every geometrically finite Kleinian group acts geometrically on a truncated version of the convex hull of the limit set. It is a well-known folk theorem that this “truncated convex hull” is a CAT(0) space. In Section 9 we prove the following result.

Theorem 1.6. The truncated convex hull associated to a geometrically finite subgroup $\Gamma \leq \text{Isom}(\mathbb{H}^n)$ has isolated flats and the Relative Fellow Traveller Property.

The Relative Fellow Traveller Property is established using a technical result of Epstein ([ECH+92, 11.3.1]).

Based on the evidence above, it seems likely that the isolated flats property and the Relative Fellow Traveller Property are equivalent for arbitrary (proper and cocompact) CAT(0) spaces.

The author has been informed that Kleiner has unpublished work from 1997 related to the paper [CK02] which also proves some of the results in this article. In particular he showed that equivariant quasi-isometries between CAT(0) spaces with isolated flats map geodesics to within uniform Hausdorff distance of geodesics, which implies that the spaces have a well-defined boundary.

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Dani originally conjectured several of the main results of this article and had established the results of Section 3 in the two dimensional setting before I started this project. In fact a variant of Theorem 5.7 appeared as Proposition 4.0.4 in Dani’s own thesis [Wis96]. I thank him for his vision and encouragement.

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2. Geometric preliminaries

This section is a review of some basic geometric facts that we will need throughout this article. It also serves to establish notation. A good reference for the facts discussed here is [BH99].

We use two distinct metrics on the set of subsets of a metric space $X$. If $A$ and $B$ are subsets of $X$, the distance between $A$ and $B$ is defined by

$$d(A, B) = \inf \{ d(a, b) \mid a \in A, b \in B \}.$$ 

The Hausdorff distance between $A$ and $B$ is

$$d_H(A, B) = \inf \{ \varepsilon \mid A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A) \},$$

where $N_\varepsilon(C)$ denotes the $\varepsilon$–neighborhood of $C$.

2.1. Quasi-isometries.

**Definition 2.1** (Quasi-isometry). Let $X$ and $Y$ be metric spaces. A $(\lambda, \varepsilon)$–quasi-isometric embedding of $X$ into $Y$ is a function $f : X \to Y$ satisfying

$$\frac{1}{\lambda} d(a, b) - \varepsilon \leq d(f(a), f(b)) \leq \lambda d(a, b) + \varepsilon$$

for all $a, b \in X$. If, in addition, every point of $Y$ lies in the $\varepsilon$–neighborhood of the image of $f$, then $f$ is a $(\lambda, \varepsilon)$–quasi-isometry and $X$ and $Y$ are quasi-isometric.

Every quasi-isometry $f$ has a quasi-inverse $g$ with the property that the maps $f \circ g$ and $g \circ f$ are each within a bounded distance of the identity.

A geodesic in a metric space $X$ is an isometric embedding $I \to X$, where $I \subseteq \mathbb{R}$ is an interval. A metric space is geodesic if every pair of points is connected by a geodesic. A group action is geometric if it is properly discontinuous, cocompact, and isometric. The following well-known result was discovered by Efroimovich and Švarc ([Efr53, Sva55]) and rediscovered by Milnor ([Mil68]).
Theorem 2.2. Suppose a group $G$ acts geometrically on a geodesic space $X$. Then the map $G \to X$ given by $g \mapsto g(x_0)$ for some $x_0 \in X$ is a quasi-isometry.

2.2. CAT(0) spaces.

Definition 2.3. Given a geodesic triangle $\Delta$ in $X$, a comparison triangle for $\Delta$ is a triangle in the Euclidean plane with the same edge lengths as $\Delta$. A geodesic space is CAT(0) if distances between points on any geodesic triangle $\Delta$ are less than or equal to the distances between the corresponding points on a comparison triangle for $\Delta$.

Theorem 2.4 (Convexity of the CAT(0) metric). Let $\gamma$ and $\gamma'$ be geodesic segments in a CAT(0) space, each parametrized from 0 to 1 proportional to arclength. Then for each $t \in [0,1]$ we have

$$d(\gamma(t), \gamma'(t)) \leq (1-t) d(\gamma(0), \gamma'(0)) + t d(\gamma(1), \gamma'(1)).$$

Theorem 2.5 (Orthogonal projection). Let $C$ be a complete, convex subspace of a CAT(0) space $X$. Then there exists a unique map $\pi : X \to C$, called the orthogonal projection of $X$ onto $C$, satisfying the following properties.

1. For each $x \in X$, we have $d(x, \pi(x)) = d(x, C)$.
2. For every $x, y \in X$ we have $d(x, y) \geq d(\pi(x), \pi(y))$.

Recall that an isometry $g$ of a metric space $X$ is semisimple if some point of $X$ is moved a minimal distance by $g$. If $G$ acts geometrically on a metric space $X$, then every element of $G$ is a semisimple isometry of $X$ (see [BH99, II.6.10(2)]).

Theorem 2.6 (Flat Torus Theorem). Let $A$ be a free abelian group of rank $k$ acting properly discontinuously by semisimple isometries on a CAT(0) space $X$. Then $A$ stabilizes some $k$–flat $F$, and the action of $A$ on $F$ is by Euclidean translations with quotient a $k$–torus.

2.3. Quasiconvexity.

Definition 2.7 (Quasiconvex subspace). A subspace $Y$ of a geodesic metric space $X$ is $\nu$–quasiconvex, for $\nu \geq 0$, if every geodesic in $X$ connecting two points of $Y$ lies inside a $\nu$–neighborhood of $Y$. The subspace $Y$ is quasiconvex if there exists a nonnegative constant $\nu$ such that $Y$ is $\nu$–quasiconvex.

Definition 2.8 (Quasiconvex subgroup). Let $\rho : G \to \text{Isom} X$ be a geometric action of the group $G$ on the CAT(0) space $X$. A subgroup $H$ of $G$ is quasiconvex with respect to $\rho$ if there is a point $x_0 \in X$ such that the orbit $Hx_0$ is a quasiconvex subspace of $X$.

Theorem 2.9. Quasiconvex subgroups are finitely generated and quasi-isometrically embedded. Furthermore, if two subgroups are quasiconvex with respect to the same action, then their intersection is again quasiconvex.
In the general CAT(0) setting, quasiconvexity depends on the choice of action, as illustrated in the following example.

**Example 2.10.** Consider the group

\[ G = F_2 \times \mathbb{Z} = \langle a, b \rangle \times \langle t \rangle, \]

and let \( \rho: G \to \text{Isom}(X) \) be the natural action of \( G \) on the universal cover of the presentation 2–complex, metrized as a product of two trees. Since \( \rho \) respects this product decomposition, the direct factor \( H = \langle a, b \rangle \) is quasiconvex with respect to \( \rho \).

Observe that \( H' = \langle a, bt \rangle \) is not quasiconvex with respect to \( \rho \). For if it were, then \( H \cap H' \) would be finitely generated. But \( H \cap H' \) is the subgroup of all elements in the free group \( H \) for which the exponent sum of \( b \) is zero, which is not finitely generated.

The map \( \phi \in \text{Aut}(G) \) given by

\[ a \mapsto a \quad b \mapsto bt \quad t \mapsto t \]

sends \( H \) to \( H' \). So the \( \rho \) action of \( H' \) is the same as the \( \rho \circ \phi \) action of \( H \).

In other words, \( H \) is not quasiconvex with respect to \( \rho \circ \phi \).

### 3. Isolated flats

In this section, we define the notion of a CAT(0) space with *isolated flats*. We prove some algebraic facts about groups which act geometrically on CAT(0) spaces with isolated flats. In particular, we show in Theorem 3.7 that maximal flats are in one-to-one correspondence with maximal free abelian subgroups of rank at least two. We also prove Theorem 3.9 which states that such groups have finitely many conjugacy classes of maximal virtually abelian subgroups of rank at least two.

The results in this section were proved by Wise in the 2–dimensional setting (personal communication). The proofs here are straightforward generalizations of those given by Wise. In fact, a variant of Theorem 3.7 appears as Proposition 4.0.4 in [Wis96].

**Definition 3.1 (Flats).** A flat in a CAT(0) space \( X \) is an isometric embedding of Euclidean space \( \mathbb{R}^k \) into \( X \) for some \( k \geq 2 \). A \( k \)–flat is a flat of dimension \( k \).

**Definition 3.2 (Isolated flats).** A CAT(0) space \( X \) has *isolated flats* if it contains a family \( \mathcal{F} \) of flats with the following properties.

1. (Maximal) There is a constant \( B \) such that every flat \( F \) in \( X \) is contained in a \( B \)–neighborhood of some flat \( F' \in \mathcal{F} \).
2. (Isolated) There is a function \( \psi: \mathbb{R}_+ \to \mathbb{R}_+ \) such that for every pair of distinct flats \( F_1, F_2 \in \mathcal{F} \) and for every \( k \geq 0 \), the intersection \( N_k(F_1) \cap N_k(F_2) \) of \( k \)–neighborhoods of \( F_1 \) and \( F_2 \) has diameter at most \( \psi(k) \).
3. (Equivariant) The set of flats \( \mathcal{F} \) is invariant under the action of \( \text{Isom}(X) \).
Observe that \(\delta\)-hyperbolic \(\text{CAT}(0)\) spaces vacuously satisfy the isolated flats property since such spaces do not contain flats.

We note the following immediate consequence of isolated flats, which will be useful in the sequel.

**Lemma 3.3.** Let \(X\) be a \(\text{CAT}(0)\) space with isolated flats. For every \(k \geq 0\), each flat disc \(D\) in \(X\) of radius at least \(\psi(k)\) lies in a \(k\)-neighborhood of at most one flat \(F \in F\). □

The following proposition shows that in any proper \(\text{CAT}(0)\) space with isolated flats, the family \(F\) is locally finite.

**Proposition 3.4 (Locally finite).** Let \(X\) be a proper \(\text{CAT}(0)\) space with isolated flats. Then only finitely many flats from the family \(F\) intersect any compact set \(K \subseteq X\).

**Proof.** It suffices to show that only finitely many flats in \(F\) intersect any metric ball \(B(x_0, r)\). By Lemma 3.3 we can choose a constant \(R\) sufficiently large that every flat disc \(D\) in \(X\) of radius at least \(R\) lies in a 1-neighborhood of at most one flat \(F \in F\). Let \(\{F_i\}\) be the collection of all flats \(F_i \in F\) which intersect the ball \(B(x_0, r)\). Let \(p_i\) be the point in \(F_i\) closest to \(x_0\), and let \(D_i\) be the closed disc of radius \(R\) in \(F_i\) centered at \(p_i\). Note that every such disc lies inside the closed ball \(B(x_0, r + R)\), which is compact since \(X\) is proper.

Suppose by way of contradiction that the collection \(\{F_i\}\) is infinite. Passing to a subsequence if necessary, we may assume that the sequence of discs \(\{D_i\}\) converges in the Hausdorff metric (see, for instance, [BH99 I.5.31]). In particular, it is a Cauchy sequence with respect to the metric \(d_H\), so some pair \(D_i, D_j\) with \(i \neq j\) has \(d_H(D_i, D_j) < 1\). But then \(D_i\) lies inside a 1-neighborhood of the distinct flats \(F_i\) and \(F_j\), contradicting our choice of \(R\). □

**Corollary 3.5.** Suppose a group \(G\) acts geometrically on a \(\text{CAT}(0)\) space \(X\) with isolated flats. Then \(G\) contains only finitely many conjugacy classes of stabilizers of flats \(F \in F\).

**Proof.** Fix a compact set \(K\) in \(X\) whose \(G\)-translates cover \(X\). Every flat \(F \in F\) intersects \(g(K)\) for some \(g \in G\). So the flat \(g^{-1}(F)\) intersects \(K\) and has a stabilizer conjugate to the stabilizer of \(F\). Since only finitely many flats in \(F\) intersect \(K\), we see that there are only finitely many conjugacy classes of flat stabilizers. □

**Definition 3.6 (Periodic).** Suppose a group \(G\) acts geometrically on a metric space \(X\). A \(k\)-flat \(F\) in \(X\) is periodic if there is a free abelian subgroup \(A \leq G\) of rank \(k\) that acts by translations on \(F\) with quotient a \(k\)-torus.

The following theorem is, in a sense, a converse to the Flat Plane Theorem in the context of isolated flats.
Theorem 3.7 (Flats are periodic). Suppose a group $G$ acts geometrically on a CAT(0) space $X$ which has isolated flats. Then every $F \in \mathcal{F}$ is periodic.

Proof. Since $G$ acts cocompactly, the quotient $G \backslash X$ has a bounded diameter $r$. Choose a $k$–flat $F \in \mathcal{F}$, and let $\{g_j \mid j \in \mathbb{N}\}$ be a minimal set of group elements such that every point of $F$ lies within a distance $r$ of some $g_j(x)$, where $x$ is a fixed basepoint in $X$. Then the flat $F_j = g_j^{-1}(F)$ intersects $B(x, r)$. Since $\mathcal{F}$ is invariant under isometries of $X$, each flat $F_j$ is an element of $\mathcal{F}$. But only finitely many flats in $\mathcal{F}$ intersect this ball. So the collection $\{F_j \mid j \in \mathbb{N}\}$ is finite.

Let $G_F$ denote the stabilizer of $F$. Notice that if two flats $F_i$ and $F_j$ coincide, then $g_j g_i^{-1}$ is an element of $G_F$. For each $j$, let $x_j$ denote the point in $F'$ closest to $g_j(x)$. Then the $x_j$ lie in only finitely many different $G_F$–orbits. So for some $j$ the $G_F$–orbit of $x_j$ is infinite and does not lie inside a bounded neighborhood of any hyperplane of $F'$. Since $G_F$ acts properly discontinuously by isometries on the Euclidean space $F$, it follows that $G_F$ has a free abelian subgroup $A$ of finite index and finite rank (see Corollary 4.1.13 of [Thu97]). Then by Theorem 2.6 there is an $m$–flat $F_A$ in $F$ stabilized by $G_F$ on which $A$ acts by translations, where $m$ is equal to the rank of $A$. So each orbit under $G_F$ lies within a bounded distance of $F_A$. (Although the specific bound depends on the choice of orbit.) Since $F$ contains a $G_F$–orbit which does not lie in a bounded neighborhood of any hyperplane, we must have $m = k$, in other words, $F_A = F$. Since $A$ acts freely and cocompactly on $F$ by translations, the quotient $A \backslash F$ is a $k$–torus as desired. □

The following algebraic result is an immediate consequence of Theorem 3.7.

Corollary 3.8. Suppose a group $G$ acts geometrically on a CAT(0) space $X$ with isolated flats. Then $X$ contains a $k$–flat if and only if $G$ contains a subgroup isomorphic to $\mathbb{Z}^k$. □

Taken together, Corollary 3.5 and Theorem 3.7 have the following algebraic consequence.

Theorem 3.9. If $G$ acts geometrically on a CAT(0) space $X$ with isolated flats, then $G$ contains only finitely many conjugacy classes of maximal virtually abelian subgroups of rank at least two.

Proof. By Corollary 3.5, it suffices to show that the set of all stabilizers of flats in $\mathcal{F}$ is the same as the set $\mathcal{A}$ of all maximal virtually abelian subgroups of rank at least two of $G$.

By the Flat Torus Theorem, each $A \in \mathcal{A}$ stabilizes a flat $E$. But $E$ lies in a tubular neighborhood of a unique flat $F \in \mathcal{F}$. The equivariance of $\mathcal{F}$ shows that $A$ is contained in the virtually abelian group stabilizing $F$, so in fact $A = \text{Stab}(F)$. Conversely, for each $F \in \mathcal{F}$, Theorem 3.7 gives
Stab(F) ⊆ A = Stab(F') for some A ∈ A and F' ∈ F. By isolated flats, we must have F = F'. □

4. Invariance of quasiconvexity

A key step in the proof of Theorem 3.7 is a generalization of the proof of Theorem 3.7. In the earlier proof, we considered a flat coarsely covered by orbit points g_i(x_0). We used elements of the form g_jg_i^{-1} to generate a large virtually abelian group stabilizing the given flat.

In this section, we prove the following lemma involving a curve in a flat, which is coarsely covered by orbit points h_i(x_0) in a subgroup H. We will use elements of the form h_jh_i^{-1} to generate a virtually abelian subgroup of H that densely fills a subflat coarsely containing the given curve.

**Lemma 4.1.** Suppose a group G acts geometrically on a CAT(0) space X with isolated flats. For each constant µ > 0, there is a positive constant L = L(µ) having the following property. Let α: [0, 1] → F be any path in a flat F ∈ F, and let H be a subgroup of G. Suppose Im(α) lies in a µ−neighborhood of some orbit Hx under H. Then:

1. Im(α) lies in an L−neighborhood of a flat subspace ⃗F of F on which a free abelian subgroup ⃗B ≤ H acts cocompactly by translations.
2. For each y ∈ ⃗F the orbit ⃗By is L−dense in ⃗F.
3. The geodesic segment γ connecting α(0) and α(1) lies in an L−neighborhood of Hx.

The proof of Lemma 4.1 uses the following elementary lemma, whose proof we leave as an exercise.

**Lemma 4.2.** Suppose a group G acts by isometries on a metric space X. If some connected set C in X lies in a κ−neighborhood of the union of n distinct G−orbits, then C lies in a (2kn)−neighborhood of a single orbit. □

**Proof of Lemma 4.1.** Given a path α: [0, 1] → F in some flat F ∈ F, choose a minimal set \{ h_i \mid i ∈ I \} of elements of H so that every point of Im(α) lies within a distance µ of some h_i(x). For convenience, replace the points h_i(x) with points inside F as follows. For each i ∈ I, let x_i = π(h_i(x)), where π: X → F is the orthogonal projection onto F. Since projections do not increase distances, every point of Im(α) lies within a distance µ of some x_i.

For each i ∈ I, the flat h_i^{-1}(F) is an element of F intersecting the ball B(x, µ), and h_i^{-1}(x_i) is the closest point in this flat to the basepoint x. Let N = N(µ) be the number of flats in F which intersect this ball, which we know to be finite by Proposition 3.3. Then the set \{ h_i^{-1}(x_i) \mid i ∈ I \} contains at most N elements. If two flats h_i^{-1}(F) and h_j^{-1}(F) coincide, then h_jh_i^{-1} stabilizes F and maps x_i to x_j. If we let H_F denote the elements of H which stabilize F, then the points x_i lie in at most N distinct H_F−orbits.

By Theorem 3.7, we know that G_F, the subgroup of G stabilizing F, has a free abelian finite index subgroup A_F which acts on F by Euclidean...
translations. But there are only finitely many conjugacy classes of the stabilizers \( G_F \) for \( F \in \mathcal{F} \) by Corollary 3.3. So there is a universal bound \( M \) on the index \([G_F : A_F]\). Consequently, \( H_F \) also has a free abelian subgroup \( B \) of index at most \( M \) which acts by translations on \( F \). So the points \( \{x_i\} \) lie in at most \( MN \) distinct \( B \)-orbits. Since \( \text{Im}(\alpha) \) is connected, Lemma 4.2 shows that \( \text{Im}(\alpha) \) lies in a \( \mu' \)-neighborhood of a single \( B \)-orbit, where \( \mu' = 2\mu MN \).

Choose a point \( y \) in \( F \) and a collection \( \{b_j \mid j \in J\} \) in \( B \) so that
\[
d(b_j(y), b_{j+1}(y)) < 2\mu'
\]
and \( \text{Im}(\alpha) \) lies in a \( \mu' \)-neighborhood of \( \bigcup_j \{b_j(y)\} \). Let \( \hat{B} \) be the subgroup of \( B \) generated by all elements of the form \( b_{j+1}b_j^{-1} \) for \( j \in J \). Then \( \hat{B} \) stabilizes some \( k \)-flat \( \hat{F} \) containing \( y \), where \( k \) is the rank of \( \hat{B} \).

Since the orbit \( \hat{B}y \) lies entirely within \( \hat{F} \), it follows that \( \text{Im}(\alpha) \) lies inside a \( \mu' \)-neighborhood of \( \hat{F} \). Furthermore, since \( \hat{B} \) is generated by elements with translation length at most \( 2\mu' \), the \( k \)-flat \( \hat{F} \) lies in a \( 2k\mu' \)-neighborhood of \( \hat{B}y \). Since the rank \( k \) of \( \hat{B} \) is bounded by the dimension of the largest flat in \( X \), we see that \( \text{Im}(\alpha) \) lies within a uniformly bounded neighborhood of the orbit \( \hat{B}y \).

As the \( \mu' \)-neighborhood of \( \hat{F} \) is convex, the geodesic \( \gamma \) connecting the endpoints of \( \alpha \) also lies uniformly close to \( \hat{B}y \), so \( \text{Im}(\gamma) \) lies within a uniform neighborhood of the full orbit \( H\gamma \). Since each endpoint of \( \gamma \) also lies within a distance \( \mu \) of the original orbit \( Hx \), we see that the Hausdorff distance between the orbits \( Hx \) and \( H\gamma \) is uniformly bounded. So \( \text{Im}(\gamma) \) lies in a uniformly bounded neighborhood of \( Hx \) as desired. \( \square \)

**Lemma 4.3.** Let \( X \) be a CAT(0) space with isolated flats and the Relative Fellow Traveller Property. Fix constants \( \mu, \lambda, \) and \( \epsilon \). Then there exists a positive constant \( \nu \) so that the following property holds.

Suppose a group \( G \) acts geometrically on \( X \). Let \( \alpha : [0, 1] \to X \) be a \((\lambda, \epsilon)\)-quasigeodesic, and let \( \gamma \) be the geodesic connecting \( \alpha(0) \) and \( \alpha(1) \). Let \( H \leq G \) be a subgroup such that \( \text{Im}(\alpha) \) lies in a \( \mu \)-neighborhood of \( Hx \) for some basepoint \( x \in X \). Then \( \text{Im}(\gamma) \) lies in a \( \nu \)-neighborhood of \( Hx \).

**Proof.** By the Relative Fellow Traveller Property we know that \( \alpha \) and \( \gamma \) track \( \delta \)-close relative to some sequence of flats in \( \mathcal{F} \). The result is clear for any subsegment of \( \gamma \) which lies within a \( \delta \)-neighborhood of \( \text{Im}(\alpha) \). So we only need to verify the result for pieces of \( \gamma \) which wander away from \( \alpha \).

Let \( \xi \) be a subpath of \( \alpha \) whose endpoints are within a distance \( \delta \) of \( \text{Im}(\gamma) \) and which stays in a \( \delta \)-neighborhood of some flat \( F \in \mathcal{F} \). Letting \( \pi : X \to F \) denote the orthogonal projection, the Hausdorff distance between \( \text{Im}(\xi) \) and \( \pi(\text{Im}(\xi)) \) is at most \( \delta \), so \( \text{Im}(\pi \circ \xi) \) lies in a \( (\delta + \mu) \)-neighborhood of \( Hx \).

Applying Lemma 4.4 to the curve \( \pi \circ \xi \), we get a constant \( L \) so that the geodesic \( \eta \) connecting the endpoints of \( \pi \circ \xi \) lies in an \( L \)-neighborhood of \( Hx \). The result follows from the observation that the endpoints of \( \eta \) are within a distance \( 2\delta \) of \( \text{Im}(\gamma) \). \( \square \)
At this point, the proof of Theorem 1.1 is nearly immediate.

Proof of Theorem 1.1. The direction \((\Rightarrow)\) is Theorem 2.9. For \((\Leftarrow)\), suppose \(H \Emb Q G\) is a quasi-isometric embedding. Then there is an \(H\)–equivariant \((\lambda, \epsilon)\)–quasi-isometric embedding \(\phi: Y \to X\) where \(Y\) is some Cayley graph for \(H\). Let \(x = \phi(1)\).

Note that \(H\) is a \((1/2)\)–quasiconvex subspace of \(Y\). Let \(\gamma\) be the geodesic in \(Y\) joining two arbitrary points \(h_1\) and \(h_2\) of \(H\). Then \(\phi \circ \gamma\) is a \((\lambda, \epsilon)\)–quasigeodesic joining \(h_1(x)\) and \(h_2(x)\) and lying in a \((\lambda/2 + \epsilon)\)–neighborhood of \(Hx\). It now follows from Lemma 4.3 that there is a constant \(\nu = \nu(\lambda, \epsilon)\) so that the geodesic \([h_1(x), h_2(x)]\) lies in a \(\nu\)–neighborhood of \(Hx\). Thus \(H\) is a \(\nu\)–quasiconvex subgroup. \(\square\)

5. The Relative Fellow Traveller Property

Roughly speaking, the idea is that the two paths alternate between tracking close together and travelling near a common flat as illustrated in Figure 1.

Definition 5.1 (Fellow travelling relative to flats). A pair of paths

\[
\alpha: [0, a] \to X \quad \text{and} \quad \alpha': [0, a'] \to X
\]

in a space \(L\)–fellow travel relative to a sequence of flats \((F_1, \ldots, F_n)\) if there are partitions

\[
0 = t_0 \leq s_0 \leq t_1 \leq s_1 \leq \cdots \leq t_n \leq s_n = a
\]

and

\[
0 = t_0' \leq s_0' \leq t_1' \leq s_1' \leq \cdots \leq t_n' \leq s_n' = a'
\]

so that for \(0 \leq i \leq n\) the Hausdorff distance between the sets \(\alpha([t_i, s_i])\) and \(\alpha'([t_i', s_i'])\) is at most \(L\), while for \(1 \leq i \leq n\) the sets \(\alpha([s_{i-1}, t_i])\) and \(\alpha'([s_{i-1}', t_i'])\) lie in an \(L\)–neighborhood of the flat \(F_i\).

We will frequently say that paths \(L\)–fellow travel relative to flats if they \(L\)–fellow travel relative to some sequence of flats.

Definition 5.2 (Relative Fellow Traveller Property). A space \(X\) satisfies the Relative Fellow Traveller Property if for each choice of constants \(\lambda\) and \(\epsilon\) there is a constant \(L = L(\lambda, \epsilon, X)\) such that \((\lambda, \epsilon)\)–quasigeodesics in \(X\) with common endpoints \(L\)–fellow travel relative to flats.
6. Quasiflats

This section consists of the proof of Theorem 1.4 that quasiflats are close to flats.

**Proof.** Let \( Q : \mathbb{E}^k \rightarrow X \) be a \((\lambda, \epsilon)\)-quasiflat in \( X \). Observe that in Euclidean space, any two consecutive sides of a square form a \((\sqrt{2}, 0)\)-quasigeodesic. So the image under \( Q \) of any square \( S \) in \( \mathbb{E}^k \) can be considered as a pair of \((\lambda', \epsilon')\)-quasigeodesics \( \alpha_S \) and \( \beta_S \) in \( X \) with common endpoints, where \( \lambda' \) and \( \epsilon' \) depend only on \( \lambda \) and \( \epsilon \).

By the Relative Fellow Traveller Property, there is a constant \( L = L(\lambda', \epsilon') \) such that \( \alpha_S \) and \( \beta_S \) \( L \)-fellow travel relative to flats. But if \( S \) is sufficiently large, \( \alpha_S \) and \( \beta_S \) separate by more than a distance \( L \) except near their endpoints. So they must lie within an \( L \)-neighborhood of some flat \( F_S \). By isolated flats, this flat lies in a \( B \)-neighborhood of some flat \( F'_S \in F \).

Furthermore, if two sufficiently large squares \( S_1 \) and \( S_2 \) are within a Hausdorff distance 1 of each other, then the resulting flats \( F'_S \) and \( F'_{S_2} \) must coincide by isolated flats.

To complete the proof, notice that for any given constant \( C \), one can easily construct a family of squares \( \{S_i\} \) in \( \mathbb{E}^k \) each of side length \( C \) with the following two properties.

1. \( \mathbb{E}^k \) is the union of all the squares \( S_i \).
2. Any two squares \( S \) and \( S' \) can be connected by a finite chain

\[ S = S_0, S_1, \ldots, S_\ell = S' \]

so that the Hausdorff distance between any two consecutive squares \( S_i, S_{i+1} \) is at most 1. \( \square \)

7. Equivariant Quasi-isometries

The goal if this section is to prove Theorem 1.5 which states that equivariant images of geodesics lie uniformly close to geodesics.

**Definition 7.1** (Quasi-equivariance). Suppose a group \( G \) acts by isometries on two metric spaces \( X \) and \( Y \). A map \( f : X \rightarrow Y \) is \( \epsilon \)-quasi-equivariant if for each \( g \in G \), the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
| & \downarrow{g} & | \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

commutes up to a distance \( \epsilon \). In other words for each \( g \in G \) and \( x \in X \), the distance \( d(g(f(x)), f(g(x))) \) is less than \( \epsilon \).

In the sequel, we prove several results about equivariant quasi-isometries. In fact, each of these results also holds for quasi-equivariant quasi-isometries.
with only trivial modifications to the proofs (such as introducing extra additive constants). We will usually suppress mention of quasi-equivariance in order to simplify matters slightly.

The proof of Theorem 1.5 uses the following result, which states that isolated flats and the Relative Fellow Traveller Property pull back under equivariant quasi-isometries. In particular, an equivariant quasi-isometry induces a one-to-one correspondence between the distinguished families of flats for the two spaces.

**Proposition 7.2.** Suppose a group $G$ acts geometrically on CAT(0) spaces $X$ and $Y$. Suppose further that
- $X$ has isolated flats with respect to the family of flats $\mathcal{F}_X$, and
- $X$ has the Relative Fellow Traveller Property.

Let $\phi: Y \to X$ be a $G$–equivariant quasi-isometry. Then

1. $Y$ has isolated flats with respect to a family of flats $\mathcal{F}_Y$,
2. $\phi$ maps the flats of $\mathcal{F}_Y$ uniformly close to the flats of $\mathcal{F}_X$, inducing a one-to-one correspondence between $\mathcal{F}_Y$ and $\mathcal{F}_X$, and
3. $Y$ has the Relative Fellow Traveller Property.

Curiously, the author does not know of a proof that isolated flats by itself is a group invariant. Namely, if a group acts geometrically on two CAT(0) spaces and one has isolated flats, must the other also have isolated flats? It seems likely that the answer is yes.

**Proof.** We first construct the family $\mathcal{F}_Y$ of flats in $Y$. By Theorem 3.7, each $k$–flat $F \in \mathcal{F}_X$ is stabilized by a free abelian subgroup $A \leq G$ of rank $k$. By the Flat Torus Theorem, $A$ also stabilizes some $k$–flat $F'$ in $Y$. Let $\psi: X \to Y$ be a $G$–equivariant quasi-inverse for $\phi$. Since $\psi(F)$ and $F'$ are each stabilized by $A$, it is easy to see that the Hausdorff distance between them is finite. So $\phi$ sends $\psi(F)$ and $F'$ to a pair of quasiflats which are each within a finite Hausdorff distance of $F$. Applying Theorem 1.4 to the space $X$ produces a uniform constant $B$ which bounds this Hausdorff distance. Consequently, the Hausdorff distance between $\psi(F)$ and $F'$ is bounded by some other uniform constant $B'$ depending only on the spaces $X$ and $Y$ and the constants associated to the maps $\phi$ and $\psi$. If we let the family $\mathcal{F}_Y$ contain one flat $F' \subseteq Y$ for each flat $F \in \mathcal{F}_X$ as constructed above, then (2) follows immediately.

We now verify that $Y$ has isolated flats with respect to the family $\mathcal{F}_Y$. Choose a flat $E \subseteq Y$. By Theorem 1.4 the quasiflat $\phi(E)$ lies within a $D$–neighborhood of some flat $F \in \mathcal{F}_X$. The argument in the previous paragraph shows the existence of a constant $D'$ such that $E$ lies in a $D'$–neighborhood of some flat in $\mathcal{F}_Y$. Similarly because $X$ has isolated flats it is easy to produce a function $\eta: \mathbb{R}_+ \to \mathbb{R}_+$ such that for any two flats $E_1, E_2 \in \mathcal{F}_Y$ and any constant $C$, the intersection $\mathcal{N}_C(E_1) \cap \mathcal{N}_C(E_2)$ has diameter bounded by $\eta(C)$. We have now established (1).
Finally, let $\alpha$ and $\alpha'$ be a pair of quasigeodesics in $Y$ with common endpoints. Then $\phi \circ \alpha$ and $\phi \circ \alpha'$ are a pair of quasigeodesics in $X$ with common endpoints. By the Relative Fellow Traveller Property for $X$, these quasigeodesics in $X$ fellow travel relative to some sequence of maximal flats. Notice that for any pair of subpaths of $\phi \circ \alpha$ and $\phi \circ \alpha'$ which are Hausdorff close, the corresponding subpaths of $\alpha$ and $\alpha'$ are also Hausdorff close. On the other hand, given a pair of subpaths $\xi$ and $\xi'$ which travel far apart but whose endpoints are close, there is some maximal $k$–flat $F$ such that $\xi$ and $\xi'$ both lie close to $F$. So in $Y$ the corresponding subpaths of $\alpha$ and $\alpha'$ lie close to the quasiflat $\psi(F)$, which is Hausdorff close to some $k$–flat $F'$. It follows that $\alpha$ and $\alpha'$ fellow travel relative to some sequence of flats in $Y$, establishing $\ref{lem:relft}$.

Before proving Theorem $\ref{thm:relft}$ we consider the following special case in which the spaces in question are isometric to Euclidean space. This case turns out to be quite easy, since the given quasi-isometry is then close to an affine map.

**Lemma 7.3.** Let $\rho_1$ and $\rho_2$ be geometric actions of $\mathbb{Z}^n$ on spaces $F_1$ and $F_2$ each isometric to Euclidean space $\mathbb{E}^n$, and let $\phi: F_1 \to F_2$ be a $\mathbb{Z}^n$–equivariant $\langle \lambda, \epsilon \rangle$–quasi-isometry. Choose $\mu$ so that $F_1$ is contained in a $\mu$–neighborhood of each orbit. Then the image of a geodesic under $\phi$ lies within a Hausdorff distance $L$ of a geodesic, where $L$ depends only on $\lambda$, $\epsilon$, and $\mu$.

**Proof.** First choose a basepoint $x \in F_1$, and notice that $\phi$ maps the orbit of $x$ to the orbit of $\phi(x)$. There is an affine map $\psi: F_1 \to F_2$ which agrees with $\phi$ on the orbit of $x$. But $F_1$ is contained in a $\mu$–neighborhood of this orbit. So the sup-norm distance between $\psi$ and $\phi$ is bounded in terms of $\mu$, $\lambda$, and $\epsilon$. But affine maps of Euclidean spaces send lines to lines. Therefore, $\phi$ sends each geodesic to within a Hausdorff distance $L$ of a geodesic, where $L = L(\lambda, \epsilon, \mu)$, as desired. $\square$

**Proof of Theorem $\ref{thm:relft}$.** Proposition $\ref{prop:relft}$ shows that $Y$ has the Relative Fellow Traveller Property and that $\phi$ maps flats in $X$ uniformly close to periodic flats in $Y$. Pick a quasi-inverse $\psi: Y \to X$ for $\phi$, and choose a geodesic segment $\alpha$ in $X$. Let $\gamma$ be the geodesic in $Y$ connecting the endpoints of $\phi \circ \alpha$.

It follows from the proof of Proposition $\ref{prop:relft}$ that the quasigeodesics $\alpha$ and $\psi \circ \gamma$ fellow travel relative to some sequence of flats $(E_1, \ldots, E_n)$ in $F_X$, while the quasigeodesics $\phi \circ \alpha$ and $\gamma$ fellow travel relative to the sequence $(F_1, \ldots, F_n)$ in $F_Y$, where $F_i$ is a flat parallel to the quasiflat $\phi(E_i)$.

Let $\xi$ and $\xi'$ be subsegments of $\alpha$ and $\psi \circ \gamma$ which stay far apart except near their endpoints. Then the images of $\xi$ and $\xi'$ lie near some flat $E_i$ in $X$, while the images of $\phi \circ \xi$ and $\phi \circ \xi'$ lie near the flat $F_i$. Recall that $\xi$ is a geodesic parallel to $E_i$, while $\phi \circ \xi'$ is a geodesic parallel to $F_i$.

But $\phi$ composed with the orthogonal projection $\pi_i: Y \to F_i$ gives a quasi-isometry $q: E_i \to F_i$, which is quasi-equivariant with respect to a maximal
free abelian subgroup $A \leq G$ that stabilizes both flats. By Lemma 7.3, this quasi-isometry $q$ maps geodesics $\epsilon$-close to geodesics for some constant $\epsilon$ depending on our choice of flat. However, without loss of generality we may assume that the collections $\mathcal{F}_X$ and $\mathcal{F}_Y$ are $G$-equivariant, so that each consists of only finitely many $G$–orbits of flats. Therefore, the constants guaranteed by Lemma 7.3 are uniformly bounded. It now follows that $\phi \circ \xi$ lies uniformly close to $\phi \circ \xi'$. Hence, $\phi \circ \alpha$ lies close to $\gamma$ as desired.

The case where $\alpha$ is either a geodesic ray or line follows easily by a standard argument. □

8. Limit sets of quasiconvex subgroups

In this section, we combine the techniques developed separately in Sections 4 and 7 to prove Theorem 1.2.

Definition 8.1 (Limit Set). Suppose a group $H$ acts by isometries on a CAT(0) space $X$. Then the limit set $\Lambda H$ is the set of accumulation points in $\partial X$ of any orbit $Hx$. Since any two orbits $Hx_1$ and $Hx_2$ accumulate on the same set, the choice of basepoint is irrelevant.

If a point $p$ is the limit of a sequence in $Hx$ that lies within a finite Hausdorff distance of some geodesic ray, then $p$ is a conical limit point of $H$. The conical limit set $\Lambda_c H$ is the set of all conical limit points of $H$.

Clearly $\Lambda_c H$ lies inside $\Lambda H$. Furthermore, since $\Lambda H$ and $\Lambda_c H$ are $H$–invariant, the action of $H$ on $\partial X$ restricts to an action of $H$ on both $\Lambda H$ and $\Lambda_c H$.

The key idea in the proof of Theorem 1.2 is contained in the following lemma, which generalizes the Equivariant Fellow Traveller Property of Theorem 1.5. We will see that the isomorphism $\eta: H_1 \to H_2$ induces a map which sends geodesics in $X_1$ that lie near the orbit of $H_1$ to geodesics in $X_2$ that lie near the orbit of $H_2$.

Lemma 8.2. Let $\rho_1$ and $\rho_2$ be geometric actions of groups $G_1$ and $G_2$ on CAT(0) spaces $X_1$ and $X_2$ each with isolated flats and the Relative Fellow Traveller Property. Let $H_i \leq G_i$ be a subgroup which is $\nu$–quasiconvex with respect to $\rho_i$ for some constant $\nu$. Let $Y_i = N_\nu(H_ix_i)$, where $x_i$ is a basepoint in $X_i$. Then any isomorphism $\eta: H_1 \to H_2$ induces a quasi-isometry $f: Y_1 \to Y_2$ which sends geodesic segments uniformly close to geodesic segments.

Proof. The map $h \mapsto h(x_i)$ gives a quasi-isometry $\phi_i: H_i \to Y_i$ with quasi-inverse $\psi_i: Y_i \to H_i$. Since an isomorphism between two finitely generated groups is a quasi-isometry with respect to any choice of word metrics, the map $f: Y_1 \to Y_2$ given by $f = \phi_2 \circ \eta \circ \psi_1$ is a $(\lambda, \epsilon)$–quasi-isometry for some constants $\lambda$ and $\epsilon$. We may also assume that $f$ is $\epsilon$–quasi-equivariant with
Figure 2. The quasi-isometry $f$ maps the pair of quasigeodesics $\alpha_1$ and $\beta_1$ in $Y_1$ close to the pair $\alpha_2$ and $\beta_2$ in $Y_2$. respect to the isomorphism $\eta$; in other words, for each $h \in H_1$ the diagram

$$
\begin{array}{c}
Y_1 \xrightarrow{f} Y_2 \\
\downarrow h \\
Y_1 \xrightarrow{f} Y_2
\end{array}
$$

commutes up to a distance $\epsilon$. The map $f$ has an $\epsilon$–quasi-inverse $g : Y_2 \to Y_1$ which is also $\epsilon$–quasi-equivariant.

We need to see that $f$ maps geodesics uniformly close to geodesics. It is enough to prove the result for geodesic segments connecting two points in the orbit $H_1x_1$. So let $\alpha_1 = [h(x_1), k(x_1)]$ for $h, k \in H_1$. Then $f \circ \alpha_1$ is a $(\lambda, \epsilon)$–quasigeodesic segment in $Y_2$. Perturbing $f \circ \alpha_1$ produces a continuous $(\lambda', \epsilon')$–quasigeodesic $\alpha_2$ with endpoints $\eta(h)(x_2)$ and $\eta(k)(x_2)$ such that the Hausdorff distance $d_H(f \circ \alpha_1, \alpha_2)$ is bounded in terms of $\lambda$ and $\epsilon$ and such that the constants $\lambda'$ and $\epsilon'$ depend only on $\lambda$ and $\epsilon$.

Since $H_2x_2$ is a $\nu$–quasiconvex subspace of $X_2$, the geodesic segment $\beta_2$ joining the endpoints of $\alpha_2$ lies inside $Y_2$. As before, $g \circ \beta_2$ is within a bounded Hausdorff distance of a continuous $(\lambda', \epsilon')$–quasigeodesic $\beta_1$ with the same endpoints as $\alpha_1$, as illustrated in Figure 2.

Let $F_i$ denote the distinguished family of isolated flats in $X_i$. The paths $\alpha_1$ and $\beta_1$ are $(\lambda', \epsilon')$–quasigeodesics in $X_1$ with common endpoints. So by the Relative Fellow Traveller Property, they $\delta$–fellow travel relative to the flats in $F_1$ for some $\delta = \delta(X_1, \lambda', \epsilon')$. Let $\xi$ and $\xi'$ be subsegments of $\alpha_1$ and $\beta_1$ which stay far apart except near their endpoints. Then $\xi$ and $\xi'$ both lie in a $\delta$–neighborhood of some flat $F_1 \in F_1$. By Lemma 4.1, there is a constant $L = L(X_1, \delta)$ such that $\xi$ and $\xi'$ lie in the $L$–neighborhood of some subflat $F_1 \subseteq F_1$ on which a free abelian subgroup $A_1 \leq H_1$ of rank at least
two acts cocompactly by translations. Furthermore, we may assume that
for each \( y \in \hat{F}_1 \), the orbit \( A_1(y) \) is \( L \)-dense in \( \hat{F}_1 \).

Since each point of \( \xi \) lies within a uniformly bounded neighborhood of the
orbit \( H_1 x_1 \) and also within an \( L \)-neighborhood of \( \hat{F}_1 \), there is some \( b \in H_1 \)
so that \( b(x_1) \) lies in a uniformly bounded neighborhood of \( \hat{F}_1 \). So \( \hat{F}_1 \) lies
in a bounded neighborhood of the orbit \( A_1b(x_1) \). In particular, there is a
quasi-isometry Furthermore, we may assume that for each \( y \in \hat{F}_1 \), the orbit
\( A_1(y) \) is \( L \)-dense in \( \hat{F}_1 \).

Since each point of \( \xi \) lies within a uniformly bounded neighborhood of the
orbit \( H_1 x_1 \) and also within an \( L \)-neighborhood of \( \hat{F}_1 \), there is some \( b \in H_1 \)
so that \( b(x_1) \) lies in a uniformly bounded neighborhood of \( \hat{F}_1 \). So \( \hat{F}_1 \) lies
in a bounded neighborhood of the orbit \( A_1 b(x_1) \). In particular, there is a
quasi-isometry \( \hat{F}_1 \rightarrow A_1 b(x_1) \) which moves points by at most a uniformly
bounded amount. So the composition of quasi-isometric embeddings
\( \hat{F}_1 \rightarrow A_1 b(x_1) \rightarrow Y_1 \rightarrow Y_2 \rightarrow X_2 \)
is a quasiflat \( Q : \hat{F}_1 \rightarrow X_2 \). Furthermore, if we let \( A_2 = \eta(A_1) \) then the
map \( Q \) is quasi-equivariant with respect to the isomorphism \( \eta : A_1 \rightarrow A_2 \).

By Theorem 13 there is a universal constant \( D \) so that the quasiflat \( Q \)
lies in a \( D \)-neighborhood of some flat \( F_2 \in \mathcal{F}_2 \) which is stabilized by \( A_2 \).
Since \( Q \) is quasi-equivariant and \( A_2 \) acts on \( F_2 \) by translations, \( Q \) must lie
inside a uniformly bounded neighborhood of some subflat \( \hat{F}_2 \subset F_2 \) on which
\( A_2 \) acts cocompactly.

As in the proof of Theorem 13 projecting \( Q \) onto \( \hat{F}_2 \) gives an equivariant
quasi-isometry \( \hat{F}_1 \rightarrow \hat{F}_2 \). Such a map sends geodesics uniformly close to geodesics by Lemma 14. Since \( \xi \) is a geodesic segment in the \( L \)-neighborhood of \( \hat{F}_1 \), its image \( f \circ \xi \) in \( X_2 \) lies close to a geodesic. But the endpoints of
\( f \circ \xi \) are close to the geodesic \( \beta_2 \). It now follows that \( f \) maps the entire
geodesic \( \alpha_1 \) into a uniformly bounded neighborhood of \( \beta_2 \). The uniform
bound in question depends only on our original choice of quasi-isometric embeddings \( H_i \rightarrow X_i \) and on the given isomorphism \( \eta : H_1 \rightarrow H_2 \).

Proof of Theorem 1.2 Fix a basepoint \( X_i \in X_i \), and consider the quasi-isometric embedding \( H_i \rightarrow X_i \) given by \( h \mapsto h(x_i) \). By Lemma 8.2 any
isomorphism \( \eta : H_1 \rightarrow H_2 \) induces a quasi-isometry \( f \) from
\( Y_1 = \mathcal{N}_{2\nu}(H_1(x_i)) \) to \( Y_2 = \mathcal{N}_{2\nu}(H_2(x_2)) \)
which sends geodesic segments uniformly close to geodesics. It follows easily
that \( f \) maps geodesic rays uniformly close to geodesic rays. Thus we have a
one-to-one correspondence between rays in \( Y_1 \) and rays in \( Y_2 \). To complete
the proof we need to see that every point of \( \lambda H_i \) can be represented by a
ray in \( Y_1 \).

Consider a sequence \( \{ h_j(x_i) \} \) limiting to a point of \( \partial X_i \) as \( j \rightarrow \infty \).
Extract a subsequence so that the segments \( [h_j(x_i), h_j(x_i)] \) converge point-
wise to a geodesic ray \( c \) based at \( h_1(x_i) \). By quasiconvexity, each segment
[h_1(x_i), h_2(x_i)] lies inside the \( \nu \)-neighborhood of the orbit \( H_i(x_i) \). So the limiting ray \( c \) lies inside \( Y_i \). Therefore, every point of \( \Lambda H_i \) is represented by a geodesic ray inside \( Y_i \) based at the point \( h_1(x_i) \).

\[ \square \]

9. Geometrically finite groups

This section is devoted to proving Theorem 1.6. We begin by considering the finite volume case.

A **truncated hyperbolic space** is a subspace of \( \mathbb{H}^n \) obtained by removing a collection of disjoint open horoballs and endowing the resulting subset with the induced length metric. Every truncated hyperbolic space is a complete CAT(0) space ([BH99, II.11.27]). A discrete subgroup \( \Gamma \leq \text{Isom}(\mathbb{H}^n) \) with finite covolume acts cocompactly on a truncated space obtained by removing a \( \Gamma \)-equivariant family of horoballs centered at the parabolic fixed points of \( \Gamma \) ([GR70], see also [Thu97, §4.5]).

**Proposition 9.1.** Let \( X \subset \mathbb{H}^n \) be any truncated hyperbolic space. Then \( X \) has isolated flats.

**Proof.** We may assume \( n \geq 3 \), since otherwise \( X \) is \( \delta \)-hyperbolic. For every deleted horoball, the bounding horosphere is isometric to \( \mathbb{E}^{n-1} \). Since \( X \) is locally isometric to \( \mathbb{H}^n \) away from these flats, these horospheres are the only flats in \( X \).

To verify that \( X \) has isolated flats, we need to bound the diameter \( D(k) \) of the intersection of \( k \)-neighborhoods of any two distinct flats. Notice that a tubular neighborhood of a horoball is again a horoball. Furthermore the diameter of the intersection decreases monotonically as a function of the distance between the two flats. So it suffices to consider the case where the two horoballs are tangent at a single point, in which case it is clear that the diameter obtained is finite and depends only on \( k \). \[ \square \]

A geometrically finite group \( \Gamma \leq \text{Isom}(\mathbb{H}^n) \) acts geometrically on a **truncated convex hull** obtained as follows. Let \( \Lambda \) be the limit set of \( \Gamma \) in \( \partial \mathbb{H}^n \), and let \( \text{Hull}(\Lambda) \subseteq \mathbb{H}^n \) be the hyperbolic convex hull of \( \Lambda \). If \( \Gamma \) is geometrically finite, then there is a \( \Gamma \)-equivariant collection of disjoint open horoballs, with union \( U \), centered at the parabolic fixed points of \( \Gamma \), such that the action of \( \Gamma \) on the truncated convex hull \( Y = \text{Hull}(\Lambda) \cap (\mathbb{H}^n - U) \) is properly discontinuous and cocompact ([Bow93]).

If the horoballs in \( U \) are chosen sufficiently small, then the truncated convex hull is a convex subspace of the truncated hyperbolic space, and hence is CAT(0). This fact seems to be well-known, though the only explicit reference the author has found in the literature is Exercise II.11.37(2) of [BH99]. We direct the reader towards the lemma in [CS92, §1.7], which is a key step in proving this exercise. This lemma is proved by Culler–Shalen only in the three-dimensional setting, but the generalization to higher dimensions is straightforward. Henceforth, we assume that all truncated convex hulls are CAT(0).
Proposition 9.2 (Geometrically finite hyperbolic). Let $\Gamma$ be any geometrically finite subgroup of $\text{Isom}(\mathbb{H}^n)$. Then the associated truncated convex hull $Y$ has isolated flats.

Proof. As in the proof of Proposition 9.1, the only flats of $Y$ are contained in the bounding horospheres. The stabilizer in $\Gamma$ of each horosphere $S$ is virtually abelian of rank $k$ for some $k < n$. The intersection of $S$ with $\text{Hull}(\Lambda)$ is isometric to a product $F \times Z$ with $F$ isometric to $\mathbb{E}^k$ and $Z$ a compact convex subset of $\mathbb{E}^{n-k-1}$. Let $z \in Z$ be the circumcenter of $Z$ (see [BH99, II.2.7]). Define $F$ to be the set of all flats $F \times \{z\}$ whose stabilizer in $\Gamma$ has rank at least two. By construction, $F$ is invariant under $\text{Isom}(Y)$. Since the horospheres $S$ are isolated, it follows that $F$ is isolated as well. Furthermore, since $\Gamma$ has only finitely many conjugacy classes of maximal parabolic subgroups, it is easy to see that each flat in $Y$ lies in a universally bounded neighborhood of some flat in $F$. □

In order to prove Theorem 1.6 all that remains is to establish the Relative Fellow Traveller Property for the truncated convex hull, which follows easily from the following result due to Epstein ([ECH+92, Theorem 11.3.1]).

Theorem 9.3 (Quasigeodesics outside horoballs). Let $\lambda \geq 1$ and $\epsilon \geq 0$ be fixed real constants. Then there is a positive real number $\ell$, depending only on $k$ and $\epsilon$, with the following property. Let $r > 3\ell$. Let $U$ be a union of disjoint horoballs in $\mathbb{H}^n$, such that any two components of $U$ are a distance at least $r$ apart, and let $X$ be the truncated space $\mathbb{H}^n - U$. Let $\alpha : [a, b] \to X$ be a $(\lambda, \epsilon)$-quasigeodesic in $X$. Let $\phi$ be the hyperbolic geodesic from $\alpha(a)$ to $\alpha(b)$. Then the union of the $\ell$-neighborhood of $U$ and the $\ell$-neighborhood of $\phi$ contains the image of $\alpha$.

Proof of Theorem 1.6. Consider the truncated hyperbolic space $X = \mathbb{H}^n - U$ and truncated convex hull $Y = \text{Hull}(\Lambda) \cap X$ associated to $\Gamma$. Shrink the horoballs in $U$ equivariantly so that they satisfy the hypothesis of Theorem 9.3.

Since $Y$ is convex in $X$, quasigeodesics in $Y$ are also quasigeodesics in $X$. So the result follows from the fact that, for each bounding horosphere $S$, the intersection $S \cap Y$ lies uniformly close to either a flat or a geodesic line. □

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