How can we derive Fourier’s Law from quantum mechanics?

Exact master equation analysis

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We derive the macroscopic Fourier’s Law of heat conduction from the exact gain-loss time convolutionless quantum master equation under three assumptions for the interaction kernel. To second order in the interaction, we show that the first two assumptions are natural results of the long time limit. The third assumption can be satisfied by a family of interactions consisting an exchange effect. The pure exchange model directly leads to energy diffusion in a weakly coupled spin-1/2 chain.

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Introduction.— Fourier’s law, connecting the rate of heat flow within a body to the temperature profile along the flow path, is an empirical law based on observation. Despite its fundamental nature, a derivation of this law from first principles is still missing [1]. In classical systems, extensive numerical simulations [2] and rigorous derivations [1, 3, 4] manifested the applicability of Fourier’s law for specifically designed nonlinear systems. It is still a great challenge to derive this macroscopic law from microscopic quantum principles. Quantum simulations of heat flow in spin chains pointed out on the validity of Fourier’s law, in connection with the onset of quantum chaos [5]. More recent studies have focused on the derivation of this law from Schrödinger dynamics [6, 7, 8]. By using the Hilbert space average method, Michel et al. demonstrated the emergence of heat diffusion within a quasi degenerate level local Hamiltonian using the truncated Dyson series for the short-time displacement operator [6]. Using this approach, the transition from diffusive to ballistic dynamics at different length scales was further explored [8]. These works successfully manifested the onset of Fourier’s law for a specific class of quantum models. The rigorous derivation of this law from quantum dynamics for general Hamiltonians is still a great challenge.

In this letter we employ the most general non-perturbative microscopic master equation of the gain-loss form to derive quantum-mechanically the Fourier’s law. Unlike the derivation in Ref. [2], applicable for a class of modular designed models, we use the most general local Hamiltonian. The diffusive behavior is obtained by making three assumptions on the interacting kernel in the microscopic master equation. These assumptions can be traced down to the structure of coupling between local sites. At weak coupling, going to the long time limit, we furthermore demonstrate that the first and second assumptions are independent of the specific form of the interaction, and the exchange interaction plays the decisive role in order for the third assumption to hold.

Master equation.— Consider a system with $N$ local units (particles), each has the same $M$ (could be infinite) dimensional eigenspace spanned by bases $\{|n\}\}$, possibly coupled to their own bathes. The eigenspace of all $N$ particles is spanned by bases $\{|m\}=|n_1\rangle_1|n_2\rangle_2|n_3\rangle_3\ldots|n_N\rangle_N\}$. The most general Hamiltonian of the system and its bath is

$$H = H_0 + \lambda H_I,$$

where $H_0 = H_S + H_B$ and $H_S = \sum_{i=1}^{N} \sum_{n} \epsilon_n(i) |n_i \rangle \langle n_i |$ is the system Hamiltonian. We assume that the energy spectra of the sites are identical, $\epsilon_n(i) = \epsilon_n(j) \geq 0$, and the ground state energy is set as zero. $H_B$ is the bath Hamiltonian, consisting degrees of freedom other than included in the system. The bath may act locally on each site, $H_B = \sum_{i=1}^{N} H_B(i)$, where $H_B(i)$ couples to the $i^{th}$ local site. The second term $H_I$ includes interactions between system particles, where $\lambda$ characterizes the strength of these interactions. It can be generally written as $H_I = \sum |n\rangle \langle m| B_{nm}$, where $B_{nm}$’s are the matrix elements of either system or bath operators. The dynamics obeys the Liouville equation $\frac{\partial}{\partial t} \rho_{tot}(t) = -i [H_I(t), \rho_{tot}(t)] = \lambda [H_I(t), \rho_{tot}(t)] = \lambda \mathcal{L}(t) \rho_{tot}(t)$, where $\rho_{tot}$ is the density matrix of the total system, and we work in the interaction representation $H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}$. The Liouville superoperator $\mathcal{L}(t)$ is defined by this equation. The super projection operation of interest, $\rho(t) = \mathcal{P}\rho_{tot}(t)$, defines the relevant part of the total density matrix for the open system. This part exactly satisfies the time-local master equation

$$\frac{\partial}{\partial t} \rho(t) = \mathcal{K}(t) \rho(t),$$

given that the initial state is in the relevant subspace, $\mathcal{P} \rho_{tot}(0) = \rho_{tot}(0)$. The time-convolutionless generator $\mathcal{K}(t)$ is in general an extremely complicated object, calculated using perturbative expansions [9]. Though the time-local master equation [2] is less well known than the Nakajima-Zwanzig equation [10], it is easy to show that these forms are equivalent [12]. In order to project the diagonal part of the total density $\rho_{tot}(t)$ we
use the following projection

\[ \mathcal{P}\rho_{tot} = \sum_{\mathbf{n}} |\mathbf{n}\rangle \langle \mathbf{n}| \mathcal{P} \rho_{tot} \langle \mathbf{n}| \mathbf{n}\rangle \mathcal{P} \rho_{tot} \mathbf{n} B, \quad (3) \]

where \( \rho_{B} \) is the bath thermal equilibrium density matrix, and the trace is taken over both system and bath states. Note that \( \mathcal{P} \rho_{tot} \mathcal{P} \mathbf{n} = \rho_{n} \) defines the probability to find the system in state \( \mathbf{n} \). The time-local master equation can be exactly written as\[ \frac{d\mathcal{P}\rho_{n}}{dt} = \sum_{\mathbf{m}} W_{nm}(t) \mathcal{P}\rho_{m} - \sum_{\mathbf{m}} W_{mn}(t) \mathcal{P}\rho_{n}, \quad (4) \]

where the rates, or interacting kernels \( W_{nm}(t) \), are complicated functionals of the interaction \( H_{I}(t) \). Assuming that \( \mathcal{P}\mathcal{L}(t)\mathcal{P} = 0 \), which is true for closed systems\[11\] and for many open systems\[12\], to second order in \( PL \)

\[ \sum_{\mathbf{m}} W_{nm}(t) \mathcal{P}\rho_{m} - \sum_{\mathbf{m}} W_{mn}(t) \mathcal{P}\rho_{n} \]

Incorporating Eqs. (6), (7) and (8) into (4) we obtain

\[ \frac{d\mathcal{P}\rho_{n}}{dt} = w(i, t)[P_{n}(i + 1) - P_{n}(i)] - \]

\[ w(i - 1, t)[P_{n}(i) - P_{n}(i - 1)], \quad (9) \]

where for convenience we use the short notation \( P_{n}(i) \). In deriving\[12\] we have also utilized the symmetry \( W(i, i + 1) = W(i + 1, i) \) which holds exactly for a closed system at second order [see text after Eq. (5)]. Next we write an equation of motion for the internal energy at each site, \( u(i) = \sum \epsilon_{n} P_{n}(i) \),

\[ \frac{du(i)}{dt} = w(i, t)(u(i + 1) - u(i)) \]

\[-w(i - 1, t)(u(i) - u(i - 1)). \quad (10) \]

The continuous version of this equation is

\[ \frac{\partial u(x, t)}{\partial t} = a^{2} \frac{\partial}{\partial x} \left( w(x, t) \frac{\partial u(x, t)}{\partial x} \right), \quad (11) \]

where \( a \) is the distance between neighboring sites. Generalization of this derivation to a three dimensional simple cubic lattice is straightforward, leading to Eq. (11) with \( r \) and \( \nabla \) replacing \( \frac{\partial}{\partial x} \) and \( x \). Applying the continuity equation for the energy density, \( \frac{\partial u(x, t)}{\partial t} = -\nabla \cdot \mathbf{J}(r, t) \), \( \mathbf{J} \) is the heat current, we exactly obtain the Fourier’s law

\[ \mathbf{J}(r, t) = -w(r, t) a^{2} \nabla u(r, t) = -\kappa(r, t) \nabla T(r, t). \quad (12) \]

Here \( T(r, t) \) denotes the temperature profile and \( \kappa(r, t) = w(r, t) a^{2} C_{V} \) with \( C_{V} = \partial u/\partial T \) as the specific heat. The heat conductivity \( \kappa \) is essentially time-dependent due to the explicit time dependence of the microscopic rates \( w(r, t) \). In the long time, or Markovian limit, the microscopic rates and the temperature profile become constants, leading to a time-independent relation. Equation\[12\] is the main result of our paper. We emphasize that it was derived from the exact master equation\[2\] for a generic local Hamiltonian.
An exact model in the second order.—We present next a model Hamiltonian that exactly satisfies the three assumptions leading to Fourier’s law in the Markovian limit. We assume a closed system \((H_B\) is neglected), and employ a pure exchange interaction form

\[ H_I = \sum_{i=1}^{N-1} J_{i,i+1} \mathcal{E}^{(i,i+1)}. \tag{13} \]

Here \(\mathcal{E}^{(i,i+1)}\) is the permutation operator and \(J_{i,i+1}\) are nearest-neighbor coupling constants (superexchange for spin system), taken as constants \(J_{i,i+1} = J\) and set to one in the following discussion. In the two-level case \((M = 2)\), since \(\mathcal{E}^{(i,i+1)} = \frac{1}{2}(1 + \sigma_i \cdot \sigma_{i+1})\), Eq. 13 is simply the Heisenberg spin-1/2 exchange interaction model. The correlation function in \((15)\) is given by

\[ D_{nm}(\tau) = \sum_{i=1}^{N-1} \delta_{n_i,m_{i+1}} \delta_{n_{i+1},m_i} \prod_{j \neq i,i+1} \delta_{n_j,m_j}, \tag{14} \]

when \(n \neq m\), satisfying the first assumption. Furthermore, the second assumption is fulfilled here exactly as the matrix \(W\) has the exact form of Eq. \((7)\) with \(w(i,t) = 2\lambda^2 t\). In order to check the validity of the second-order approximation, we compare this result with the exact solution for a three unit systems, \(N = 3\) and \(M = 2\). Let the initial state be \(|1\rangle |0\rangle |0\rangle_3\). We can get an exact solution for the dynamics for the interaction \((13)\), \(dP_i(2)/dt = w(t)P_i(1) + P_i(3) - 2P_i(2)\), where \(w(t) = \frac{\sqrt{\lambda}}{2 \cos^{\frac{\pi}{2}}(\sqrt{\lambda} t)}\). For weak coupling (small \(\lambda\)), \(w(t) = 2\lambda^2 t\), which is the same as the second order result. While this model satisfies all three assumptions, in the long time limit it does not lead to Fourier’s law.

However, the energy difference between spin states of site \(i\) may slightly differ from that of site \(i + 1\), \(\Delta_{n_i,n_{i+1}}(i) = |\epsilon_{n_i}(i) - \epsilon_{n_{i+1}}(i)| = |\epsilon_{n_i}(i + 1) - \epsilon_{n_{i+1}}(i + 1)| \neq 0\). This effect may originate from thermal fluctuations due to the existence of local baths at each site, which do not contribute to the diffusive behavior. In this case, \(w_{n_i,n_{i+1}}(i,t) = 2\lambda^2 \sin(\Delta_{n_i,n_{i+1}}(i))/\Delta_{n_i,n_{i+1}}(i)\). At short times, if the difference \(\Delta_{n_i,n_{i+1}}(i)\) is small, one again finds \(w(t) \propto t\). In contrast, at long times \(w_{n_i,n_{i+1}}(i) \rightarrow 2\pi \lambda^2 \delta(\Delta_{n_i,n_{i+1}}(i))\). When the energy spectrum of each spin state forms a band that is dense enough \([12]\), the Fermi’s Golden rule is obtained, \(w_{n_i,n_{i+1}}(i) = 2\pi \lambda^2 \Gamma(0)\), where \(\Gamma(0)\) is the density of states at zero detuning.

Thus, this simple exchange model exactly reproduces the long time Fourier’s law \([12]\) with \(\kappa = 2\pi \lambda^2 a^2 C_V \Gamma(0)\). This result manifests that the local sites do not need to acquire exactly identical spectra. It also provides a microscopic explanation of the validity of Fourier’s law in spin chains with Heisenberg-type interactions \([16]\).

Long Time (Markovian) Limit.—Next we show that at second order in the interacting kernel the first two assumptions [Eqs. \((6) - (7)\)] are model-independent in the long time limit, and the third assumption is valid for family of interactions with exchange effect. We consider the most general interaction \(V(i,i+1)\) for a one-dimensional system. It is easy to show that

\[ B_{nm} = \sum_{i=1}^{N-1} V_{n_i,n_{i+1},m_i,m_{i+1}}(i,i+1) \prod_{l \neq i,i+1} \delta_{n_l,m_l}. \tag{15} \]

In the long time (Markovian) limit Eq. \((10)\) reduces to

\[ W_{nm}(t \rightarrow \infty) = 2\pi \lambda^2 |B_{nm}|^2 \delta(E_n - E_m), \tag{16} \]

using \(\frac{\sin((E_n - E_m)t)}{(E_n - E_m)t} \rightarrow \pi \delta(E_n - E_m)\) when \(t \rightarrow \infty\). Assuming the bound states are non-degenerate, a somewhat tedious calculation shows that the first assumption is satisfied, and the off-diagonal matrix elements are given by

\[ W_{n_i,n_{i+1},m_i,m_{i+1}}(i,i+1; t) = 2\pi \lambda^2 |V_{n_i,n_{i+1},m_i,m_{i+1}}(i,i+1)|^2 \delta(\Delta_{n_i,n_{i+1},m_i,m_{i+1}}), \tag{17} \]

where \(\Delta_{n_i,n_{i+1},m_i,m_{i+1}} = \epsilon_{n_i} + \epsilon_{n_{i+1}} - \epsilon_{m_i} - \epsilon_{m_{i+1}}\). Since \(\epsilon_n \geq 0\), the delta function in Eq. \((17)\) implies that \(\epsilon_n = \epsilon_{m_{i+1}}\) and \(\epsilon_{n_{i+1}} = \epsilon_{m_i}\), provided that the energy spectra are anharmonic. Therefore, in the non-degenerate case, only transitions between the quantum states \(n_i = m_{i+1}\) and \(n_{i+1} = m_i\) exist, in accordance with the second assumption. In contrast, when the energy spectra are strictly harmonic, the delta function can be satisfied for large number of combinations, \(\epsilon_n = \epsilon_{m_{i+1}}\) and \(\epsilon_{n_{i+1}} = \epsilon_{m_i}\), provided that the energy difference between subunit states (equal for all \(N\) and \(j\) is an integer). In this case the second assumption does not hold, and we cannot derive Eq. \((10)\) and the subsequent result \([12]\). Thus, interestingly, in order to derive Fourier’s law, the system spectrum should be anharmonic, in accordance with classical results \([1, 2]\).

At second order, the first two assumptions are therefore the results of the Markovian limit. As discussed above, we can expect that there is a dense band structure around each level \(\epsilon_n\), originating from bath fluctuations or quantum tunneling between sites. Defining the density of states at the energy difference around \(\Delta\) as \(\Gamma(\Delta)\), we obtain \([13]\)

\[ w_{n_i,n_{i+1}}(i) = 2\pi \lambda^2 \Gamma(0) |V_{n_i,n_{i+1},m_i,m_{i+1}}(i,i+1)|^2, \tag{18} \]

which manifests that the third assumption is valid, depending on the form of specific Hamiltonians. We calculate next the microscopic rates for different types of interactions. Assuming translational symmetry in the system, we need only discuss a pair of sites, for instance between site \(1\) and \(2\), \(V_{n_1,n_2,n_2,n_1}(1,2) = (n_1 n_2) \langle V(1,2) | n_2 n_1 \rangle\).

In general, particles at each site may be electrons or atoms. To simplify, we consider the case with one particle at each site. Our first example is the short-range delta
interaction, \( V(1, 2) = g \delta(r_{12}) \), where \( r_{12} = |x_2 - x_1| \), and \( x_1(x_2) \) is the coordinate of the first (second) particle. Physically, this interaction describes particles that move almost independently in the site interior, while collisions, leading to energy exchange between particles, occur at the edge points \( x_1 = x_2 \). This picture is appropriate for describing phonon collisions in solids [17]. For this type of interaction the matrix elements become

\[
V_{n_1n_2\gamma_{n_1n_2}}(1, 2) = g \int dx |\phi_{n_1}(x)|^2 |\phi_{n_2}(x)|^2. \tag{19}
\]

This integral is almost independent of quantum numbers \( n_1 \) and \( n_2 \) for many systems [18]. Therefore, the third assumption is generally valid for short-range interactions. For example, in one-dimensional infinite square well (width \( d \)) with the wave function \( \phi_n(x) = \sqrt{2/\pi} \sin n \pi x/d \), the matrix element \( V_{n_1n_2\gamma_{n_1n_2}}(1, 2) = 1/d \) is completely independent of quantum numbers, assuming \( d > a \) so that two particles could collide.

Our second example is the long-range interaction. We choose the general form \( V(1, 2) = V(r_{12})(\lambda_1 + \lambda_2 \mathcal{E}(1, 2)) \), where the \( \lambda \)'s are constants and \( \mathcal{E}(1, 2) \) is the spacial exchange operator [19]. Considering the first order contribution to this potential, \( V(1, 2) = |x_2 - x_1| (\lambda_1 + \lambda_2 \mathcal{E}(1, 2)) \), the matrix elements \( V_{n_1n_2\gamma_{n_1n_2}}(1, 2) \) become the sum of the direct (\( D \)) and exchange (\( E \)) terms,

\[
V_{n_1n_2\gamma_{n_1n_2}}^D(1, 2) = \lambda_1 \int dx_1 dx_2 \phi_{n_1}^*(x_1) \phi_{n_2}^*(x_2) |x_2 - x_1| \phi_{n_1}(x_2) \phi_{n_2}(x_1),
\]

\[
V_{n_1n_2\gamma_{n_1n_2}}^E(1, 2) = \lambda_2 \int dx_1 dx_2 |\phi_{n_1}(x_1)|^2 |\phi_{n_2}(x_2)|^2 |x_2 - x_1|. \tag{20}
\]

In what follows we take \( \lambda_1 = \lambda_2 = 1 \). If the overlap between the wave functions of two particles is zero, the direct integral diminishes because of the orthogonality of two states, and the exchange integral (\( x_2 > x_1 \)) is

\[
V_{n_1n_2\gamma_{n_1n_2}}^E(1, 2) = a + \langle n_2 | x_2' | n_2' \rangle - \langle n_1 | x_1 | n_1 \rangle, \quad x_2' = x_2 - a \text{ is the relative coordinate of particle 2. If the wavefunction } \phi_n(x) \text{ has a well defined parity, } \langle n | x | n \rangle = 0 \text{ for the relative coordinate of each particle, implying that } V_{n_1n_2\gamma_{n_1n_2}}(1, 2) = a \text{ is a constant. This satisfies the third assumption. Note that the result is independent of the details of the local Hamiltonian.}

If the overlap is not zero, yet small, the direct integral is expected to be small. Considering again the one-dimensional infinite square well, when \( a/d < 1 \), two particles will overlap. Numerical calculations show that for quantum numbers \( n_1, n_2 \) ranging from 1 to 20 the direct integrals are less than one tenth of the exchange integrals and \( V_{n_1n_2\gamma_{n_1n_2}}(1, 2) \) is almost a constant, very close to \( a \) (derivation around 3% when \( 0.4 < a/d < 1 \)). The bigger the values of \( a/d \) is, the smaller the ratio between the direct and the exchange integrals is.

We also calculate the interaction \( V(1, 2) \) for the harmonic potential \( V(r_{12}) = r_{12}^2 \). We find that when \( a/d \) varies from 0.7 to 0.95 the exchange integral is almost a constant with deviations from 14% to 4%, and the ratio between the direct and exchange integrals is 0.14-0.1.

The temperature dependence of the heat conductivity \( \kappa \) results from the interplay between the specific heat and the interaction kernel. The specific heat, defined per unit, can be easily calculated for different models, e.g., for a spin chain of \( \omega_0 \) spacing, \( C_V = \frac{\omega_0^2}{4 \pi} e^{\omega_0/T} (e^{\omega_0/T} + 1)^{-2} \). In contrast, our approach does not directly bring in the temperature dependence of the kernel \( W_{mn} \). Physically, since \( V_{n_1n_2\gamma_{n_1n_2}}(1, 2) \) is a function of the intersite separation, the lattice vibrations can modify it, thus introduce temperature dependent transition rates. This effect can be successfully included by phenomenologically introducing a temperature dependent interaction [20, 21]. Here we adopt the simple form \( V_T(r_{12}) = V(r_{12}) \gamma(r_{12}, T) \), where \( T \) may be the average temperature of site 1 and 2, and \( \gamma(r_{12}, T) = \exp(-\alpha f(T)r_{12}) \). In the case of interacting ions [20], \( \alpha \) is inversely proportional to the Fermi velocity and \( f(T) = T \). When \( f(T) \rightarrow \infty \), \( V_T(r_{12}) \rightarrow -\frac{\gamma}{2}\alpha f(T) V(0) \delta(r_{12}) \). Therefore, for high temperatures \( \kappa \propto f(T)^2 \), as the lattice specific heat typically saturates. For the ionic solid of Ref. [20] one obtains \( \kappa \propto 1/T^2 \), in agreement with standard expectations [17].

Summary.— We have presented here a microscopic quantum derivation of Fourier’s law of heat conduction that is not limited to specifically designed models. The derivation relies on three assumptions that are satisfied in weak coupling and at long times for a family of exchange interaction potentials. Our analysis naturally implies that energy diffusion cannot emerge in harmonic models, in agreement with the behavior of classical systems.

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