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Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications

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Abstract: A plethora of sufficient convergence criteria has been provided for single-step iterative methods to solve Banach space valued operator equations. However, an interesting question remains unanswered: is it possible to provide unified convergence criteria for single-step iterative methods, which are weaker than earlier ones without additional hypotheses? The answer is yes. Moreover, we provide only one sufficient convergence criterion suitable for single-step methods. In particular, we also give a finer convergence analysis. Numerical experiments involving boundary value problems and Hammerstein-like integral equations complete this paper.

Keywords: single step methods; Banach space; convergence criterion

1. Introduction

Numerous applications from mathematics, economics, engineering, physics, chemistry, biology, and medicine, to mention a few, can be modeled as follows:

\[ F(x) = 0, \]  

with operator \( F : \Omega \subseteq T_1 \to T_2 \) acting between \( T_1 \) and \( T_2 \), which are Banach spaces, whereas \( \Omega \) is nonempty. That is why determining a solution denoted by \( x^* \) of Equation (1) is of extreme importance. However, this task is difficult in general. Ideally, one desires \( x^* \) to be in closed form, but this task is only accomplished in some instances. Practitioners and researchers resort to mostly iterative methods, generating a sequence approximating \( x \) under certain conditions on the initial data. The most popular single step methods are as follows:

**Newton’s** [1,2]

\[ x_{m+1} = x_m - F'(x_m)^{-1} F(x_m). \]  

**Secant** [3]

\[ x_{m+1} = x_m - [x_m, x_{m-1}; F]^{-1} F(x_m), \]  

where \([\cdot, \cdot; F] : \Omega \times \Omega \to L(T_1, T_2)\).

**Steffensen’s-like** [4]

\[ x_{m+1} = x_m - [x_m + \lambda_1 F(x_m), x_m + \lambda_2 F(x_m); F]^{-1} F(x_m), \]  

for \( T_1 = T_2 \) and \( \lambda_1, \lambda_2 \) being parameters.

**Newton’s-type** [5–8]

\[ x_{m+1} = x_m - A_m^{-1} F(x_m), \]  

where \( A_m = A(x_m), A : \Omega \to L(T_1, T_2) \).

**Stirling’s** [9]

\[ x_{m+1} = x_m - G'(x_m)^{-1} G(x_m), \]  

where \( T_1 = T_2 \) and \( G(y) = y - F(y) \) are used to find fixed points of equation \( x = G(x) \).
where $c$ (Section 2). The numerical experiments can be found in Section 3. Conclusions appear in Section 4.

Q2 Can the estimates on $x$ be made tighter? Otherwise, we compute more iterates than we should to reach a predetermined error tolerance.

Q3 Can the location of solution be more precise?

Q4 Is there a uniform way of studying single-step methods?

Q5 Are there uniform convergence criteria for single-step methods?

The novelty of our paper is that we answer positively to all these questions (Q), without additional conditions.

In order to deal with single-step methods, we first consider the following iteration:

$$t_{m+2} = \psi(t_{m+1}, t_m, t_{m-1}) \quad \text{for each } m = 0, 1, 2, \ldots,$$

where $\psi: [0, \infty) \times [0, \infty) \times [0, \infty)$ is a function related to the initial data. The task of choosing $\psi$ so that sequence $\{t_n\}$ is majorizing for all methods listed previously is very difficult in general.

We define a special case of sequences given by (9) as follows:

$$t_{m+2} = \frac{d_1 t_{m+1} + d_2 t_{m-1} + d_3 t_{m-3} + d_4 t_{m-5} + d_5 t_{m-7} + d_6}{\bar{t}_{m+1} + \bar{t}_{m-1} + \bar{t}_{m-3} + \bar{t}_{m-5} + \bar{t}_{m-7} + \bar{t}_{m-9}}.$$ (10)

for each $m = 1, 2, \ldots$, where $a, b, c, d, e, f$ are nonnegative parameters. We shall show that all majorizing sequences used to study the preceding methods are specializations of $\{t_m\}$ given by (10).

Similarly, in the case of local convergence we show all preceding methods can be studied using the estimate as follows:

$$e_{m+1} \leq \lambda_m e_m,$$ (11)

where $c_1, c_2, c_3, d_1, d_2, d_3$ are nonnegative parameters, $e_m = \|x_m - x^*\|$ and $\lambda_m = \frac{c_1 d_1 + c_2 d_2 + c_3 d_3}{d_1 d_2 d_3}.$

We suppose from now on that $\{t_m\}$ is a majorizing sequence for $\{x_n\}$. Recall that an increasing real sequence $\{t_m\}$ is majorizing for a sequence $\{x_n\}$ in a Banach space $T_1$ if $\|x_{m+1} - x_m\| \leq t_{m+1} - t_m$ for each $m = 0, 1, 2, \ldots$ [11]. Additional conditions are needed to show that $F(\rho) = 0$, where $\rho := \lim_{m \to \infty} x_m$.

The paper contains also the semi-local as well as the local convergence of method (10) in Section 2. The numerical experiments can be found in Section 3. Conclusions appear in Section 4.
2. Majorizing Sequences and Convergence Analysis

In this section, we use majorizing sequence (10) to deal first with the semi-local convergence analysis for sequence \( \{x_n\} \).

We provide very general sufficient criteria for the convergence of sequence (10).

**Theorem 1.** Suppose that for each \( m = 0, 1, 2, \ldots \) and \( b_0, b_6 \in [0, 1) \),

\[
b_1(t_1 - t_0) + b_2(t_0 - t_{-1}) + b_3t_1 + b_4t_0 + b_5t_{-1} + b_6 < 1,
\]

and

\[
b_1(t_{m+1} - t_m) + b_2(t_m - t_{m-1}) + b_3t_{m+1} + b_4t_m + b_5t_{m-1} + b_6 < 1.
\]

(12)

Then, sequence \( \{t_m\} \) developed by (10) exists, is nondecreasing, bounded from above by

\[
t^{**} = \frac{1 - b_6}{b_1 + b_2 + b_3 + b_4 + b_5}
\]

and converges to its unique least upper bounds denoted by \( t^* \), which satisfies

\[
t_1 \leq t^* \leq t^{**}.
\]

**Proof.** Using the definition of sequence \( \{t_k\} \) we see that 0 \( \leq t_k \leq t_{k+1} \) holds for each \( k = 0, 1, 2, \ldots \). Moreover, by condition (12), \( t_{k+1} < t^{**} \). So, sequence \( \{t_k\} \) converges to \( t^* \). \( \square \)

**Remark 1.** Condition (12) can be satisfied only in some special cases. Next, we provide stronger conditions, which can easily be verified.

It is convenient for the following convergence analysis to develop real functions, parameters and sequences. Define functions on the interval \([0, 1)\) for \( \mu = t_2 - t_1 \) by the following:

\[
f_k(t) = a_3 \mu b_1 + a_2 \mu b_2 + a_3 t_1
\]

\[
+ a_3(1 + t + \ldots + t^{k-1}) + a_4 t_1 + a_4(1 + t + \ldots + t^{k-2})
\]

\[
a_3 t_1 + a_3(1 + t + \ldots + t^{k-3}) + a_4 + b_1 \mu
\]

\[
+ b_2 \mu b_1 + b_3 t_1 + b_3 \mu(1 + t + \ldots + t^k)
\]

\[
+ b_4 t_1 + b_4(1 + t + \ldots + t^{k-1}) + b_5 t_1
\]

\[
+ b_5(1 + t^2 + \ldots + t^{k-2}) + b_6 t - t,
\]

(13)

\[
h(t) = (b_1 + b_3) t^3 + (a_1 + a_3 - b_1 + b_2 + b_4) t^2
\]

\[
+ (a_2 - a_1 + a_4 - b_2 + b_5) t + a_5 - a_2,
\]

(14)

\[
g(t) = a_3 t_1 + a_3 \mu t_1 + a_4 t_1 + a_4 \frac{\mu}{t_1}
\]

\[
+ a_4 t_1 + a_4 \frac{\mu}{t_1} + a_6 + b_3 t_1 t + \frac{b_4 t_1}{t_1} + b_4 t_1
\]

\[
+ \frac{b_4 t_1}{t_1} t + b_5 t + b_5 \mu t + b_6 t - t,
\]

(15)

and sequence

\[
\delta_k = \frac{a_1 (t_{k-1} - t_k) + a_2 (t_k - t_{k-1}) + a_3 t_{k-1} + a_4 t_k + a_5 t_{k-1} + a_6}{1 - (b_1 (t_{k+1} - t_k) + b_2 (t_k - t_{k-1}) + b_3 t_{k-1} + b_4 t_k + b_5 t_{k-1} + b_6)}.
\]

(16)

Suppose that equations

\[
h(t) = 0
\]

(17)

and

\[
g(t) = 0
\]

(18)

have minimal solutions \( \delta \) and \( \lambda \), respectively in the interval \((0, 1)\) satisfying the following:

\[
0 \leq \delta \leq \delta < \lambda.
\]

(19)
Notice that
\[ f_{i+1}(t) = f_i(t) + \mu h(t) t^{i-2} \] (20)
and
\[ f_{i+1}(\delta) = f_i(\delta). \] (21)

Indeed, by the definition of sequence \( \{f_i\} \) and function \( h \), we obtain in turn by adding and subtracting \( f_i(t) \) (in the definition of \( f_{i+1}(t) \)) the following:

\[
\begin{align*}
f_{i+1}(t) &= f_i(t) + a_1 \mu t^i - a_1 \mu t^{i-1} + a_2 \mu t^{i-1} - a_2 \mu t^{i-2} + a_3 \mu t^i \\
&
+ a_4 \mu t^{i-1} + a_5 \mu t^{i-2} + b_1 \mu t^{i+1} - b_1 \mu t^i \\
&
+ b_2 \mu t^i - b_2 \mu t^{i-1} + b_3 \mu t^{i+1} + b_4 \mu t^i + b_5 \mu t^{i-1} \\
&= f_i(t) + h(t) \mu t^{i-2}.
\end{align*}
\]

In particular, by the definition of \( \delta \) and (20) we obtain (21) since \( h(\delta) = 0 \).

**Remark 2.** Functions \( h \) and \( g \) appear in the proof of Theorem 1. The former is related to two consecutive functions \( f_i \) and \( f_{i+1} \) (see (20)). Then, (21) is true if (17) holds for \( t = \delta \). The latter relates to the limit of these recurrent functions \( f_i \) and is independent of \( i \). This function \( g \) then should satisfy (27) and that happens if (18) holds. Condition \( 0 \leq \delta_1 \leq \delta \) (see (19)) is needed to show that (22) holds for \( i = 1 \), which will imply the following:

\[ t_3 - t_2 \leq \delta(t_2 - t_1) \]

and the induction for \( 0 \leq \delta_i \leq \delta \) can begin. The condition \( \delta < \lambda \) is needed to show (29).

Next, we show the convergence of sequence \( \{t_n\} \) under conditions (17)–(19).

**Theorem 2.** Under conditions (17)–(19), the conclusions of Theorem 1 hold for sequence \( \{t_k\} \) but \( t^{**} \) is replaced by \( s = t_1 + \frac{\mu}{1-\delta} \).

**Proof.** We shall show by induction

\[ 0 \leq \delta_i \leq \delta. \] (22)

Item (22) holds for \( i = 1 \) by (19). Then, the definition of sequence \( \{t_i\} \) and (22) give

\[
0 < t_3 - t_2 \leq \delta(t_2 - t_1) \quad \Rightarrow \quad t_3 \leq t_2 + \delta(t_2 - t_1) = t_2 + (1 + \delta)(t_2 - t_1) - (t_2 - t_1) \\
\leq t_1 + \frac{1-\delta^2}{1-\delta}(t_2 - t_1) < s.
\]

Suppose (22) holds. Then, we have the following:

\[ 0 < t_{i+2} - t_{i+1} \leq \delta^i(t_2 - t_1) \] (23)

and

\[ t_{i+2} \leq t_1 + \frac{(1-\delta^{i+1})(t_2 - t_1)}{1-\delta} < s. \] (24)

Item (22) holds, if \( \delta_{i+1} \leq \delta \) (25)
(since $\delta_{i+1} \geq 0$). Evidently, item (25) holds, by (23) and (24), if we have the following:

$$a_1\delta^{i-1} + a_2\delta^{i-2} + a_3(1 + \frac{(1-\delta)\mu}{1-\sigma}) + a_4(t_1 + \frac{(1-\delta-1)\mu}{1-\sigma})$$
$$+ a_5(1 + \frac{(1-\delta-2)\mu}{1-\sigma}) + a_6 + b_1\delta + b_2\delta^{i-1}$$
$$+ b_3(t_1 + \frac{(1-\delta)\mu}{1-\sigma}) + b_4\delta(t_1 + \frac{(1-\delta-1)\mu}{1-\sigma})$$
$$+ b_5\delta(t_1 + \frac{(1-\delta-2)\mu}{1-\sigma}) + b_6\delta - \delta \leq 0$$

or

$$f_{i+1}(\delta) \leq 0$$

or

$$f_i(\delta) \leq 0$$

by the definition of $f_i$, (20) and (21). In view of (21), one obtains the following:

$$g(t) = f_{\infty}(t) := \lim_{i \to \infty} f_i(t).$$

So, we can show instead of (27) that the following holds:

$$h(\delta) \leq 0,$$

which is true by the definition of $\lambda$ and (19). Hence, the induction for (22) is completed. Then, items (23) and (24) hold. Consequently, sequence $\{t_k\}$ converges to $t^*$. □

Remark 3. The conditions of Theorem 2 imply condition (12) of Theorem 1 but not necessarily vice versa.

Next, we specialize $a_j, b_j, a_j, b_j$ in some interesting cases, justifying the already stated advantages.

Case 1: Newton’s method. Let us abbreviate what is known. Suppose the following conditions (C) hold:

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$

$$\|F'(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell_1\|v_2 - v_1\|$$

for each $v_1, v_2 \in \Omega,$

$$H_1 = \ell_1\eta \leq \frac{1}{2}$$

and

$$U[x_0, u^*] \subset \Omega,$$

where $u^* = \frac{1-\sqrt{1-2\ell_1\eta}}{\ell_1}.$

Next, we present the celebrated Newton–Kantorovich theorem (NKT) [10].

Theorem 3. Suppose the conditions (C) hold. Then, Newton’s method converges to a unique solution $x^*$ of equation $F(x) = 0$ in $U(x_0, u^*) \cap \Omega,$ and

$$\|x_{k+1} - x_k\| \leq \frac{\ell_1\|x_k - x_{k-1}\|^2}{2(1 - \ell_1\|x_k - x_0\|)} \leq \frac{\ell_1(u_k - u_{k-1})^2}{2(1 - \ell_1u_k)} = u_{k+1} - u_k$$

and

$$\|x_k - x^*\| \leq u^* - u_k,$$

where $u_0 = \eta$ and

$$u_{k+1} = u_k + \frac{\ell_1(u_k - u_{k-1})^2}{2(1 - \ell_1u_k)}$$

for each $k = 0, 1, 2, \ldots.$
Let us see what we obtain under our conditions. Suppose the following conditions (A) hold:

\[ \|F'(x_0)^{-1}F(x_0)\| \leq \eta, \]
\[ \|F'(x_0)^{-1}(F'(v) - F'(x_0))\| \leq \ell_0 \|v - x_0\| \text{ for each } v \in \Omega. \]

Set \( U = \Omega \cap U(x_0, \frac{1}{\ell_0}). \)

\[ \|F'(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell \|v_2 - v_1\| \text{ for each } v_1, v_2 \in U, \]
\[ H = \tilde{\ell} \eta \leq \frac{1}{2} \]
and
\[ U[x_0, t^*] \subset \Omega, \]
where \( \tilde{\ell} = \frac{1}{4}(4\ell_0 + \sqrt{\ell_0 \ell + 8\ell_0^2 + \sqrt{\ell_0 \ell}}). \)

**Remark 4.** Notice that \( \ell = \ell(\Omega, \ell_0) \) but \( \ell_1 = \ell_1(\Omega). \) Hence, \( U \) is used to define \( \ell. \) It is important to see that in practice the computation of Lipschitz constant \( \ell_1 \) requires that of center Lipschitz constant \( \ell_0 \) and that of restricted Lipschitz constant \( \ell \) as special cases. Hence, the conditions involving \( \ell_0 \) and \( \ell \) are not additional to the one involving \( \ell_1. \) Moreover, they are also weaker. This is also verified in the numerical section. In other words, the condition involving \( \ell_1 \) implies the other two but not necessarily vice versa.

Next, we present our extended version of the Newton–Kantorovich Theorem 3.

**Theorem 4.** Suppose the conditions (A) hold. Then, Newton’s method converges to a unique solution \( x^* \) of equation \( F(x) = 0 \) in \( U(x_0, \frac{1}{\ell_0}) \cap \Omega, \) and the following:

\[ \|x_1 - x_0\| \leq t_1 - t_0, \]
\[ \|x_2 - x_1\| \leq \frac{\ell_0 \|x_1 - x_0\|^2}{2(1 - \ell_0 \|x_1 - x_0\|)} \leq \frac{\ell_0(t_1 - t_0)^2}{2(1 - \ell_0 t_1)} = t_2 - t_1, \]
\[ \|x_{k+2} - x_{k+1}\| \leq \frac{\ell \|x_{k+1} - x_k\|^2}{2(1 - \ell_0 \|x_{k+1} - x_0\|)} \leq \frac{\ell(t_{k+1} - t_k)^2}{2(1 - \ell_0 t_{k+1})} \]
for each \( k = 1, 2, \ldots. \)

**Proof.** Simply choose \( t_{-1} = 0, t_0 = 0, a_1 = \frac{\ell_0}{2}, a_2 = \ldots = a_6 = 0, b_1 = \ell_0, \)
\( b_2 = \ldots = b_6 = 0, a_1 = \frac{\ell_0}{2}, a_2 = 0 \ldots = a_6 = 0, b_0 = \ell_0 \text{ and } b_2 = \ldots = b_6 = 0. \)
Then, (19) reduces to (31). In particular, we use the following estimates:

\[ \|F'(x_0)^{-1}(F'(x_{i+1}) - F'(x_0))\| \leq \ell_0 \|x_{i+1} - x_0\| \leq \ell_0(t_{i+1} - t_0) \leq \ell_0 t_{i+1} < 1, \]

so \( F'(x_{i+1})^{-1} \in L(T_2, T_1) \) by the Banach perturbation lemma on invertible linear operators [10] and the following:

\[ \|F'(x_{i+1})^{-1}F'(x_0)\| \leq \frac{1}{1 - \ell_0 \|x_{i+1} - x_0\|.} \]

Then, since

\[ F(x_{i+1}) = F(x_{i+1}) - F(x_i) - F'(x_i)(x_{i+1} - x_i) \]
\[ = \int_0^1 (F'(x_i + \tau(x_{i+1} - x_i)) - F'(x_i))d\tau(x_{i+1} - x_i), \]
we obtain the following:

\[ \|x_{i+2} - x_{i+1}\| \leq \frac{\ell \|x_{i+1} - x_i\|^2}{2(1 - \ell_0 \|x_{i+1} - x_i\|)} \leq t_{i+2} - t_{i+1}, \]

where we also used the following:

\[ \|x_{i+1} - x_0\| \leq \sum_{m=1}^i \|x_m - x_{m-1}\| \leq \sum_{m=1}^{i+1} (t_m - t_{m-1}) \leq t_{m+1} - t_0 = t_m \leq t^*, \]

and

\[ \|x_i + \tau(x_{i+1} - x_i) - x_0\| \leq t_i + \tau(t_{i+1} - t_i) \leq t^* \]

for each \( \tau \in [0,1] \). So, the sequence \( \{t_k\} \) is majorizing for \( \{x_k\} \). Then, sequence \( \{x_k\} \) is fundamental in \( T_1 \), which is a Banach space, so \( \lim_{k \to \infty} x_k = x^* \in U[x_0, t^*] \), which solves Equation (1), since

\[ \|F'(x_0)^{-1}F(x_{i+1})\| \leq \frac{\ell}{2} \|x_{i+1} - x_i\|^2 \leq \frac{\ell}{2} (t_{i+1} - t_i)^2 \to 0 \]

as \( i \to \infty \). Then, we conclude that \( F(x^*) = 0 \) since \( F \) is a continuous operator, where

\[ \ell = \left\{ \begin{array}{ll} \ell_0, & i = 0 \\ \ell_i, & i = 1, 2, \ldots \end{array} \right. \]

Let \( x_i \in U(x_0, \frac{1}{\ell_0}) \cap \Omega \) with \( F(x_i) = 0 \). Set \( M = \int_0^1 F'(x_i + \tau(x^* - x_i))d\tau \). Using the center Lipschitz condition, we have the following:

\[ \|F'(x_0)^{-1}(M - F'(x_0))\| \leq \ell_0 \int_0^1 [(1 - \tau)\|x_i - x_0\| + \tau\|x^* - x_0\|]d\tau < \ell_0 \frac{1}{\ell_0} = 1 \]

so \( x^* = x_i \) follows since \( M^{-1} \) exists and \( M(x^* - x_i) = F(x^*) - F(x_i) = 0 \).

**Remark 5.** (a) We have by the definition of \( U \)

\[ U \subset \Omega, \quad (32) \]

so

\[ \ell \leq \ell_1, \quad \ell_0 \leq \ell_1 \quad (33) \]

and

\[ \ell \leq \ell_1. \quad (34) \]

Hence, we have

\[ H_1 \leq \frac{1}{2} \Rightarrow H \leq \frac{1}{2} \quad (35) \]

\[ t_{k+1} - t_k \leq u_{k+1} - u_k \quad (36) \]

and

\[ t^* \leq u^*. \quad (37) \]

Estimates (35)–(37) justify the benefits as stated previously. In the numerical section, we provide examples where (32)–(34) are strict, and (31) holds but not (30).

(b) The proof in Theorem 3 used the less precise estimate as follows:

\[ \|F'(x_{i+1})^{-1}F'(x_0)\| \leq \frac{1}{1 - \ell_1 \|x_{i+1} - x_0\|}. \]

Our modification leads to (31) instead of (30). Moreover, in [15] we showed Theorem 4 but using the following:

\[ H_2 = \ell_2 \eta \leq \frac{1}{2}. \]
where
\[ \ell_2 = \frac{1}{8} \left( 4\ell_0 + \sqrt{\ell_0\ell_1 + 8\ell_0^2} + \sqrt{\ell_0\ell_1} \right) \geq \ell, \]
so
\[ H_2 \leq \frac{1}{2} \Rightarrow H \leq \frac{1}{2}. \]

Hence, our results extend the ones in [15] too.

(c) Let us see how parameters \( \delta_1, \delta, \lambda \) and functions \( h, g \) look like in the case of Newton’s method. We obtain by (17)–(19) the following:
\[ \delta_1 = \frac{\ell\eta}{2(1 - \ell_0\eta)}, \delta = \frac{2\ell}{\ell + \sqrt{\ell^2 + 8\ell_0\ell}}, \lambda = 1 - \frac{1}{2} \left( \frac{\ell_0\eta}{1 - \ell_0\eta} \right)^2, \]
\[ h(t) = 2\ell_0t^2 + \ell t - \ell, \]
and
\[ g(t) = \left( \frac{\ell_0}{1 - t} + \ell_0\eta - 1 \right) t. \]

Notice that \( \delta, \lambda \) solve Equations (17) and (18), respectively. Then, if we solve inequality (19), we obtain (31).

Comments similar to the ones given in the previous five remarks can be made for the methods that follow in this Section.

**Case 2: Secant method [14]** Choose \( t_{-1} = 0, t_0 = \beta, t_1 = \beta + \eta, \bar{a}_1 = \bar{a}_2, \bar{b}_1 = \bar{b}_2, \bar{a}_3 = \bar{a}_4 = \bar{a}_5 = \bar{a}_6 = b_3 = b_4 = b_5 = 0, a_1 = a_2 \) and \( b_1 = b_2. \)

The nonzero parameters are again connected to the following:
\[ \|x_0 - x_{-1}\| \leq \beta, \| [x_0, x_{-1}; F]^{-1} F(x_0) \| \leq \eta \]
\[ \| [x_0, x_{-1}; F]^{-1} ( [v_1, v_2; F] - [x_0, x_{-1}; F] ) \| \leq \frac{\ell_0}{2} ( \| v_1 - x_0 \| + \| v_2 - x_{-1} \| ) \]
for each \( v_1, v_2 \in \Omega, \)
\[ \| [x_0, x_{-1}; F]^{-1} ( [v_1, v_2; F] - [z, w; F] ) \| \leq \frac{\ell}{2} ( \| v_1 - z \| + \| v_2 - w \| ) \]
for each \( v_1, v_2, z, w \in V, \) provided that
\[ [v_1, v_2; F] = \int_0^1 F' (v_2 + \tau (v_1 - v_2)) d\tau. \]

The standard condition used in connection to the secant method [14] is the following:
\[ \| [x_0, x_{-1}; F]^{-1} ( [v_1, v_2; F] - [z, w; F] ) \| \leq \frac{\ell_1}{2} ( \| v_1 - z \| + \| v_2 - w \| ) \]
for each \( v_1, v_2, z, w \in \Omega. \) Then, we have again the following:
\[ \ell \leq \ell_1 \]
and
\[ \ell_0 \leq \ell_1. \]

The old majorizing sequence \( \{ u_n \} \) [14] is defined by the following:
\[ u_{-1} = 0, u_0 = \beta, u_1 = \beta + \eta \]
\[ u_{k+2} = u_{k+1} + \ell_1 (u_{k+1} - u_{k-1}) (u_{k+1} - u_k) \]
\[ 2(1 - \frac{\ell}{\ell_0^2} (u_{k+1} + u_k + \beta)) \]
with the following estimates:

\[ ||x_{k+2} - x_{k+1}|| \leq \frac{\ell \|x_{k+1} - x_{k-1}\| \|x_{k+1} - x_k\|}{2(1 - \frac{\ell}{2} (\|x_{k+1} - x_0\| + \|x_k - x_0\| + \beta))} \leq u_{k+2} - u_{k+1}.\]

However, ours is as follows:

\[
\begin{align*}
t_{-1} &= 0, t_0 = \beta, t_1 = \beta + \eta \\
t_{k+2} &= t_{k+1} + \frac{\ell(t_{k+1} - t_{k-1})(t_{k+1} - t_k)}{2(1 - \frac{\ell}{2} (t_{k+1} + t_k + \beta))}
\end{align*}
\]

with corresponding estimates

\[ ||x_{k+2} - x_{k+1}|| \leq \frac{\ell \|x_{k+1} - x_{k-1}\| \|x_{k+1} - x_k\|}{2(1 - \frac{\ell}{2} (\|x_{k+1} - x_0\| + \|x_k - x_0\| + \beta))} \leq t_{k+2} - t_{k+1}
\]

which are tighter, where

\[ \ell = \begin{cases} \ell_0, & k = 0 \\ \ell, & k = 1, 2, \ldots \end{cases} \]

The old sufficient convergence criterion [14] is \(\beta + \sqrt{2\ell_1 \eta} \leq 1\) but the new one is (for \(\ell_0 = \ell\) \(\beta + \sqrt{2\ell_1 \eta} \leq 1\), which is weaker. Hence, we obtain the semi-local convergence of the secant method.

**Theorem 5.** Under the preceding conditions secant method \(\{x_n\} \subset U[x_0, \ell]\) and \(\lim k \rightarrow \infty x_k = x^* \in U[x_0, \ell]\) with \(F(x^*) = 0\).

**Proof.** As in Theorem 4, we obtain the following:

\[ \| [x_{i+1}, x_i; F]^{-1} [x_0, x_{-1}; F] \| \leq \frac{1}{1 - \frac{\ell}{2} (\|x_{i+1} - x_0\| + \|x_i - x_{-1}\|)} \]

and

\[ ||x_{i+2} - x_{i+1}|| \leq \| [x_{i+1}, x_i; F]^{-1} [x_0, x_{-1}; F] \| \times \| [x_0, x_{-1}; F]^{-1} (\{x_{i+1}, x_i; F\} - \{x_i, x_{-1}; F\}) \| \]

\[ \leq \frac{\ell \|x_{i+1} - x_{i-1}\| \|x_{i+1} - x_i\|}{2(1 - \frac{\ell}{2} (\|x_{i+1} - x_0\| + \|x_i - x_0\| + \beta))} \leq t_{i+2} - t_{i+1} \]

(see also [14]). \(\square\)

**Case 3: Newton-type method** [8,16] Choose: \(t_{-1} = 0, t_0 = 0, t_1 = \eta, a_1 = \frac{\ell_0}{2}, a_2 = 0, a_3 = \ell_5, a_4 = a_5 = 0, a_6 = \ell_6, d_1 = \frac{\ell_0}{2}, d_2 = 0, a_3 = \ell_5, a_4 = a_5 = 0, a_6 = \ell_6, b_1 = \ell_2, b_2 = 0, b_3 = b_4 = b_5 = 0, b_6 = \ell_3, b_1 = \ell_2, b_2 = 0, b_3 = b_4 = b_5 = 0, b_6 = \ell_3.\)

The parameters are connected to the following:

\[ \|A(x_0)^{-1} F(x_0)\| \leq \eta, \]
\[ \|A(x_0)^{-1} (F(v) - F(x_0))\| \leq \ell_0 \|v - x_0\| \text{ for each } v \in \Omega \]

and

\[ \|A(x_0)^{-1} (A(v) - A(x_0))\| \leq \ell_2 \|v - x_0\| + \ell_3. \]
Set \( V_1 = \Omega \cap U[x_0, \frac{1-\ell_3}{\ell_2}], \ell_2 \neq 0, \ell_3 \in [0, 1). \)

\[
\|A(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell_4 \|v_2 - v_1\| \text{ for each } v_1, v_2 \in V_1
\]

and

\[
\|A(x_0)^{-1}(F'(v) - A(v))\| \leq \ell_5 \|v - x_0\| + \ell_6 \text{ for each } v \in V_1
\]

The conditions in [8,16] use the following:

\[
\|A(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell_7 \|v_2 - v_1\| \text{ for each } v_1, v_2 \in \Omega
\]

and

\[
\|A(x_0)^{-1}(F'(v) - A(v))\| \leq \ell_8 \|v - x_0\| + \ell_9 \text{ for each } v \in \Omega.
\]

We have the following:

\[
V_1 \subseteq \Omega,
\]

so

\[
\ell_4 \leq \ell_7,
\]

\[
\ell_5 \leq \ell_8
\]

and

\[
\ell_6 \leq \ell_9.
\]

The old majorizing sequence \( \{u_n\} \) [8,16] is defined for \( t_{-1} = 0, u_0 = 0, u_1 = \eta, \sigma_1 = \max\{\ell_7, \ell_8 + \ell_2\} \) by

\[
u_{i+1} = u_i + \frac{\sigma}{2} (u_i - u_{i-1}) + \ell_6 u_{i-1} + \ell_9 (u_i - u_{i-1})
\]

with the following estimates:

\[
\|x_{i+1} - x_i\| \leq \|A_i(x_i)^{-1}A(x_0)\| \left( \int_0^1 A(x_0)^{-1}(F'(x_i + \tau(x_{i-1} - x_i)) - F'(x_{i-1}))d\tau \right)
\]

\[
+ \|A(x_0)^{-1}(F'(x_{i-1}) - A(x_{i-1}))\| \|x_i - x_{i-1}\|
\]

\[
\leq \left( \frac{\sigma}{2} \|x_i - x_{i-1}\| + \ell_6 \|x_{i-1} - x_0\| + \ell_9 \right) \|x_i - x_{i-1}\|
\]

\[
\leq \frac{\sigma}{2} \|u_i - u_{i-1}\| + \ell_6 u_{i-1} + \ell_9 (u_i - u_{i-1})
\]

\[
= u_{i+1} - u_i.
\]

However, ours is for \( t_{-1} = 0, t_0 = 0, t_1 = \eta, \sigma = \max\{\ell_4, \ell_5 + \ell_2\} \)

\[
t_{i+1} = t_i + \frac{\sigma}{2} (t_i - t_{i-1}) + \ell_5 t_{i-1} + \ell_6 (t_i - t_{i-1})
\]

with the following estimates:

\[
\|x_{i+1} - x_i\| \leq \frac{\sigma}{2} \|x_i - x_{i-1}\| + \ell_6 \|x_{i-1} - x_0\| + \ell_9 \|x_i - x_{i-1}\|
\]

\[
\leq \frac{\sigma}{2} (t_i - t_{i-1}) + \ell_5 t_{i-1} + \ell_6 (t_i - t_{i-1})
\]

\[
= t_{i+1} - t_i.
\]
The old sufficient convergence criterion \([8,16]\) is the following:

\[ C_1 = \sigma_1 \eta \leq \frac{1}{2} (1 - (\ell_3 + \ell_9))^2, \quad \ell_3 + \ell_9 < 1. \]

The new one is the following:

\[ C = \sigma \eta \leq \frac{1}{2} (1 - (\ell_3 + \ell_6))^2, \quad \ell_3 + \ell_6 < 1. \]

However, \(\sigma \leq \sigma_1\), so again condition \(C\) is weaker than \(C_1\).

Hence, we obtain the semilocal convergence of the Newton-type method.

**Theorem 6.** Under the preceding conditions Newton-type method \(\{x_n\} \subset U[x_0, t^*]\) and \(\lim_{k \to \infty} x_k = x^* \in U[x_0, t^*]\) with \(F(x^*) = 0\).

**Proof.** It follows from the aforementioned estimates (see also \([8,16]\)). Hence, again the results are extended.

Similar benefits are derived in the local convergence case.

Suppose the conditions (B) hold:

Suppose the conditions (B) hold:

\(x^* \in \Omega\) is a simple solution of equation \(F(x) = 0\),

\[ \|F'(x^*)^{-1}(F'(v_2) - F'(v_1))\| \leq L_1 \|v_2 - v_1\| \quad \text{for each } v_1, v_2 \in \Omega, \]

and

\[ U[x^*, r] \subset \Omega, \]

where \(r = \frac{2}{L_1\ell_0}\). Then, we have the following local convergence result arrived at independently by Rheinboldt \([17]\) and Traub \([18]\). \(\square\)

**Theorem 7.** Suppose that the conditions (B) hold. Then, Newton’s method converges to \(x^*\) so that the following holds:

\[ \|x_{k+1} - x^*\| \leq \frac{L_1 \|x_k - x^*\|^2}{2(1 - L_1 \|x_k - x^*\|)} \]

for each \(k = 0, 1, 2, \ldots\), provided that \(x_0 \in U(x^*, r)\).

In our case, we consider the conditions (D): \(x^* \in \Omega\) is a simple solution of equation \(F(x) = 0\).

\[ \|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \leq L_0 \|v - x^*\| \quad \text{for each } v \in \Omega. \]

Set \(V = \Omega \cap U(x^*, \frac{1}{L_0 t_0})\).

\[ \|F'(x^*)^{-1}(F'(v_2) - F'(v_1))\| \leq L \|v_2 - v_1\| \quad \text{for each } v_1, v_2 \in V. \]

\(U(x^*, R) \subset \Omega\), where \(R = \frac{2}{2L_0 t_0 + L_0}\).

**Theorem 8.** Suppose that the conditions (D) hold. Then, Newton’s method converges to \(x^*\) so the following holds:

\[ \|x_{k+1} - x^*\| \leq \frac{\hat{L} \|x_k - x^*\|^2}{2(1 - L_0 \|x_k - x^*\|)} \]

for each \(k = 0, 1, 2, \ldots\), provided that \(x_0 \in U(x^*, R)\), where \(\hat{L} = \begin{cases} L_0, & k = 0 \\ L, & k = 1, 2, \ldots \end{cases}\).

**Proof.** Choose \(c_1 = \frac{L_0}{2}, c_2 = c_3 = d_2 = d_3 = 0\) in \((11)\). Then, we obtain the following:
\[ \|x_{i+1} - x^*\| = \|x_i - x^* - F'(x^*)^{-1}F(x_i)\| \leq \|F'(x_i) - F'(x^*)\| \int_0^1 \|F'(x^*)^{-1}(F'(x^* + \tau(x_i - x^*)) - F'(x_i))\| d\tau(x_i - x^*) \leq \frac{\bar{L}\|x_i - x^*\|^2}{2(1 - L_0\|x_i - x^*\|)}. \]

\[ \square \]

**Remark 6.** We have again the following:

\[ V \subset \Omega, \]

so

\[ L_0 \leq L_1, \]
\[ L \leq L_1, \]
\[ \bar{L} \leq \bar{L}_1, \]
\[ r \leq R, \]
\[ \lambda_n \leq \lambda^1_n, \]

where \( \lambda_k = \frac{\bar{L}\|x_k - x^*\|}{2(1 - L_0\|x_k - x^*\|)} \) and \( \lambda^1_k = \frac{L_1\|x_k - x^*\|}{2(1 - L_1\|x_k - x^*\|)} \) (see also the numerical section).

The same benefits can be obtained for the other single-step methods. Moreover, our idea can similarly be extended to multi-step and multi-point methods [4,5,13,19–37].

3. Numerical Experiments

We conduct some experiments showing that the old convergence criteria are not verified, but ours are. Hence, there is no assurance that the methods converge under the old conditions. However, under our approach, convergence can be established.

**Example 1.** Define function as the following:

\[ f(x) = \theta_0 x + \theta_1 + \theta_2 \sin \theta_3 x, \quad x_0 = 0, \]

where \( \theta_j, j = 0, 1, 2, 3 \) are parameters. Then, clearly for \( \theta_3 \) large and \( \theta_2 \) small, \( \frac{\ell_0}{\ell_1} \) can be small (arbitrarily). Notice that as \( \frac{\ell_0}{\ell_1} \to 0, \frac{H}{\eta} \to 0 \) too. So, the utilization of Newton’s method is extended numerous (infinitely many) times under the data (\( \Omega, F, x_0, \ell_0, \ell, \eta \)).

**Example 2.** Let \( T_1 = T_2 = \mathbb{R}, x_0 = 1 \) and \( \Omega = U[1, 1 - q] \) for \( q \in (0, \frac{1}{2}) \). Let function \( f \) on \( \Omega \) as the following:

\[ f(s) = s^3 - q. \]

We consider case 1 of Newton’s method. Then, we obtain \( \ell_0 = 3 - q, \ell = \ell_1 = 2(2 - q) \) and \( \eta = \frac{3}{2} (1 - q) \). However, then, \( H_1 > \frac{1}{2} \) for all \( q \in (0, \frac{1}{2}) \). So, the Newton–Kantorovich theorem cannot assure convergence. However, we have \( H \leq \frac{1}{2} \) for all \( q \in I = [0.4271907643, \frac{1}{2}] \). Hence, our result guarantees convergence to \( x^* = \sqrt[3]{q} \) as long as \( q \in I \).

**Example 3.** Let \( T_1 = T_2 = S([0, 1]) \) the domain of functions given on \([0, 1]\) which are continuous. We consider the norm-max. Choose \( \Omega = U(0, d), d > 1 \). Define \( F \) on \( \Omega \) by the following:

\[ F(x)(s) = x(s) - w(s) - \xi \int_0^1 K(s, t)x^3(t)dt, \]  

(38)
Comparison table of criteria (30) and (31).

Table 1.

| d       | ξ*         | 2H3        | 2H         |
|---------|------------|------------|------------|
| 2.09899 | 0.9976613778 | 1.007515200 | 0.9639223786 |
| 2.19897 | 0.9831766058 | 1.055056000 | 0.9678118280 |
| 2.29597 | 0.9698185659 | 1.102065600 | 0.9715205068 |
| 3.095467| 0.87963113211 | 1.485824160 | 1.000082409 |

Example 4. Let $T_1$, $T_2$ and $Ω$ be as in the Example 3. It is well known that the boundary value problem [2]

$$φ(0) = 0, φ(1) = 1,$$

$$φ'' = -φ - λφ^2$$

can be given as a Hammerstein-like nonlinear integral equation as follows:

$$φ(s) = s + \int_0^1 K(s, t)(φ^3(t) + λφ^2(t))dt$$

where $λ$ is a parameter. Then, define $F : Ω \longrightarrow T_2$ by the following:

$$[F(x)](s) = x(s) - s - \int_0^1 K(s, t)φ^2(t)dt.$$ 

Choose $φ_0(s) = s$ and $Ω = U(φ_0, r_0)$. Then, clearly $U(φ_0, r_0) \subset U(0, r_0 + 1)$, since $∥φ_0∥ = 1$. Suppose $2λ < 5$. Then, by conditions (C) they are satisfied for the following:

$$\ell_0 = 2λ + 3r_0 + 6, \quad \ell_1 = \ell = \frac{λ + 6r_0 + 3}{4}$$

and $η = \frac{1+λ}{5-2λ}$. Notice that $ℓ_0 < ℓ_1$.

The rest of the examples are given for the local convergence study of Newton’s method.
Example 5. Let $T_1 = T_2 = \mathbb{R}^3$, $\Omega = U[0, 1]$ and $x^* = (0, 0, 0)^T$. Define mapping $E$ on $\Omega$ for

$$
\lambda = (\lambda_1, \lambda_2, \lambda_3)^T
$$

as

$$
E(\lambda) = (e^{\lambda_1} - 1, \frac{e - 1}{2} \lambda_2^2 + \lambda_1, \lambda_3)^T.
$$

Then, conditions (B) and (D) hold, provided that $L_0 = e - 1$, $L = e^{\frac{1}{10}}$ and $L_1 = e$, since $F'(x^*)^{-1} = F'(x^*) = \text{diag}\{1, 1, 1\}$. Notice that

$$
L_0 < L < L_1
$$

and

$$
r = 0.24 < R = 0.38.
$$

Hence, our radius of convergence is larger.

Example 6. Let $T_1$, $T_2$ and $\Omega$ be as in Example 3. Define $F$ on $\Omega$ as

$$
F(\varphi_1)(x) = \varphi_1(x) - \int_0^1 x \varphi_1(j) dj.
$$

By this definition, we obtain the following:

$$
F'(\varphi_1)(\psi_1)(x) = \varphi_1(x) - 3 \int_0^1 x \varphi_1(j)^2 \psi_1(j) dj
$$

for all $\psi_1 \in \Omega$. So, we can choose $\ell_0 = 1.5$, $\ell = \ell_1 = 3$. However, then, we again obtain the following:

$$
r = \frac{2}{3} < R = \frac{1}{3}.
$$

4. Conclusions

We have provided a single sufficient criterion for the semi-local convergence of single step methods. Upon specializing the parameters involved, we showed that although our majorizing sequence is more general than earlier ones, the convergence criteria are weaker (i.e., the utility of the methods is extended), the upper error estimates are more accurate (i.e., at least as few iterates are required to achieve a predecided error tolerance), and we have, at most, an as-small ball containing the solution. These benefits are obtained without additional hypotheses. According to our new technique, we locate a more accurate domain than the earlier ones containing the iterates, leading to a more accurate Lipschitz condition (at least as small).

Our theoretical results are further justified using numerical experiments. In the future, we plan to extend these results by replacing the Lipschitz constants by generalized functions along the same lines [2,12,13].

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