Lectures on supergeometry

G. Sardanashvily

Department of Theoretical Physics, Moscow State University, Moscow, Russia

Abstract

Elements of supergeometry are an ingredient in many contemporary classical and quantum field models involving odd fields. For instance, this is the case of SUSY field theory, BRST theory, supergravity. Addressing to theoreticians, these Lectures aim to summarize the relevant material on supergeometry of modules over graded commutative rings, graded manifolds and supermanifolds.

Contents

1. Graded tensor calculus - 2, 2. Graded differential calculus and connections - 6, 3. Geometry of graded manifolds - 11, 4. Superfunctions - 18, 5. Supermanifolds - 22, 6. DeWitt supermanifolds - 25, 7. Supervector bundles - 26, 8. Superconnections - 29, 9. Principal superconnections - 31, 10. Supermetric - 36, 11. Graded principal bundles - 40.

Supergeometry is phrased in terms of $\mathbb{Z}_2$-graded modules and sheaves over $\mathbb{Z}_2$-graded commutative algebras. Their algebraic properties naturally generalize those of modules and sheaves over commutative algebras, but supergeometry is not a particular case of non-commutative geometry because of a different definition of graded derivations. In these Lectures, we address supergeometry of modules over graded commutative rings (Lecture 2), graded manifolds (Lectures 3 and 11) and supermanifolds.

It should be emphasized from the beginning that graded manifolds are not supermanifolds, though every graded manifold determines a DeWitt $H\infty$-supermanifold, and vice versa (see Theorem 6.2 below). Both graded manifolds and supermanifolds are phrased in terms of sheaves of graded commutative algebras. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves of supervector spaces. Note that there are different types of supermanifolds; these are $H\infty$, $G\infty$, $GH\infty$, $G$, and DeWitt supermanifolds. For instance, supervector bundles are defined in the category of $G$-supermanifolds.
1 Graded tensor calculus

Unless otherwise stated, by a graded structure throughout the Lectures is meant a \( \mathbb{Z}_2 \)-graded structure, and the symbol \([\cdot]\) stands for the \( \mathbb{Z}_2 \)-graded parity.

Let us recall some basic notions of the graded tensor calculus [2, 9].

An algebra \( \mathcal{A} \) is called graded if it is endowed with a grading automorphism \( \gamma \) such that \( \gamma^2 = \text{Id} \). A graded algebra seen as a \( \mathbb{Z} \)-module falls into the direct sum \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) of two \( \mathbb{Z} \)-modules \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) of even and odd elements such that

\[
\gamma(a) = (-1)^i a, \quad a \in \mathcal{A}_i, \quad i = 0, 1.
\]

One calls \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) the even and odd parts of \( \mathcal{A} \), respectively. In particular, if \( \gamma = \text{Id} \), then \( \mathcal{A} = \mathcal{A}_0 \).

Since

\[
\gamma(aa') = \gamma(a)\gamma(a'),
\]

we have

\[
[aa'] = ([a] + [a']) \text{mod 2}
\]

where \( a \in \mathcal{A}_{[a]}, a' \in \mathcal{A}_{[a']} \). It follows that \( \mathcal{A}_0 \) is a subalgebra of \( \mathcal{A} \) and \( \mathcal{A}_1 \) is an \( \mathcal{A}_0 \)-module.

If \( \mathcal{A} \) is a graded ring, then \( [1] = 0 \).

A graded algebra \( \mathcal{A} \) is said to be graded commutative if

\[
aa' = (-1)^{[a][a']} a'a,
\]

where \( a \) and \( a' \) are arbitrary homogeneous elements of \( \mathcal{A} \), i.e., they are either even or odd.

Given a graded algebra \( \mathcal{A} \), a left graded \( \mathcal{A} \)-module \( Q \) is a left \( \mathcal{A} \)-module provided with the grading automorphism \( \gamma \) such that

\[
\gamma(aq) = \gamma(a)\gamma(q), \quad a \in \mathcal{A}, \quad q \in Q,
\]

i.e.,

\[
[aq] = ([a] + [q]) \text{mod 2}.
\]

A graded module \( Q \) is split into the direct sum \( Q = Q_0 \oplus Q_1 \) of two \( \mathcal{A}_0 \)-modules \( Q_0 \) and \( Q_1 \) of even and odd elements. Similarly, right graded modules are defined.

If \( \mathcal{K} \) is a graded commutative ring, a graded \( \mathcal{K} \)-module can be provided with a graded \( \mathcal{K} \)-bimodule structure by letting

\[
qa = (-1)^{[a][q]} aq, \quad a \in \mathcal{K}, \quad q \in Q.
\]

A graded \( \mathcal{K} \)-module is called free if it has a basis generated by homogeneous elements. This basis is said to be of type \((n, m)\) if it contains \( n \) even and \( m \) odd elements.
In particular, by a (real) graded vector space $B = B_0 \oplus B_1$ is meant a graded $\mathbb{R}$-module. A graded vector space is said to be $(n,m)$-dimensional if $B_0 = \mathbb{R}^n$ and $B_1 = \mathbb{R}^m$.

The following are standard constructions of new graded modules from old ones.

- The direct sum of graded modules over the same graded commutative ring and a graded factor module are defined just as those of modules over a commutative ring.

- The tensor product $P \otimes Q$ of graded $K$-modules $P$ and $Q$ is an additive group generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying the relations

$$
(p + p') \otimes q = p \otimes q + p' \otimes q,
$$

$$
p \otimes (q + q') = p \otimes q + p \otimes q',
$$

$$
ap \otimes q = (-1)^{|p||a|} p a \otimes q = (-1)^{|p||a|}p \otimes aq =
$$

$$
(-1)^{|(p + q)|} |a| p \otimes qa, \quad a \in K.
$$

The set $\text{Hom}_K(P, Q)$ of graded morphisms of a graded $K$-module $P$ to a graded $K$-module $Q$ is naturally a graded $K$-module. The graded $K$-module $P^* = \text{Hom}_K(P, K)$ is called the dual of a graded $K$-module $P$.

A graded commutative $K$-ring $A$ is a graded commutative ring which is also a graded $K$-module. A graded commutative $\mathbb{R}$-ring is said to be of rank $N$ if it is a free algebra generated by the unit $1$ and $N$ odd elements. A graded commutative Banach ring $A$ is a graded commutative $\mathbb{R}$-ring which is a real Banach algebra whose norm obeys the additional condition

$$
\|a_0 + a_1\| = \|a_0\| + \|a_1\|, \quad a_0 \in A_0, \quad a_1 \in A_1.
$$
Let $V$ be a real vector space. Let $\Lambda = \wedge V$ be its ($\mathbb{N}$-graded) exterior algebra provided with the $\mathbb{Z}_2$-graded structure

$\Lambda = \Lambda_0 \oplus \Lambda_1$, \quad $\Lambda_0 = \mathbb{R} \bigoplus_{k=1}^{2k} \wedge V$, \quad $\Lambda_1 = \bigoplus_{k=1}^{2k-1} \wedge V$. \quad (1.1)$

It is a graded commutative $\mathbb{R}$-ring, called the Grassmann algebra. A Grassmann algebra, seen as an additive group, admits the decomposition

$\Lambda = \mathbb{R} \oplus R = \mathbb{R} \oplus R_0 \oplus R_1 = \mathbb{R} \oplus (\Lambda_1)^2 \oplus \Lambda_1$, \quad (1.2)$

where $R$ is the ideal of nilpotents of $\Lambda$. The corresponding projections $\sigma : \Lambda \rightarrow \mathbb{R}$ and $s : \Lambda \rightarrow R$ are called the body and soul maps, respectively.

**Remark 1.1.** Let us note that there is a different definition of a Grassmann algebra [14] which is equivalent to the above one only in the case of an infinite-dimensional vector space $V$ [9]. Let us mention the Arens–Michael algebras of Grassmann origin [7] which are most general graded commutative algebras, suitable for superanalysis (see Remark 5.3 below).

Hereafter, we restrict our consideration to Grassmann algebras of finite rank. Given a basis $\{c^i\}$ for the vector space $V$, the elements of the Grassmann algebra $\Lambda$ (1.1) take the form

$a = \sum_{k=0}^{\infty} \sum_{(i_1, \ldots, i_k)} a_{i_1 \ldots i_k} c^{i_1} \cdots c^{i_k}$, \quad (1.3)$

where the second sum runs through all the tuples $(i_1 \cdots i_k)$ such that no two of them are permutations of each other. The Grassmann algebra $\Lambda$ becomes a graded commutative Banach ring if its elements (1.3) are endowed with the norm

$\|a\| = \sum_{k=0}^{\infty} \sum_{(i_1, \ldots, i_k)} |a_{i_1 \ldots i_k}|.$

Let $B$ be a graded vector space. Given a Grassmann algebra $\Lambda$ of rank $N$, it can be brought into a graded $\Lambda$-module

$\Lambda B = (\Lambda B)_0 \oplus (\Lambda B)_1 = (\Lambda_0 \otimes B_0 \oplus \Lambda_1 \otimes B_1) \oplus (\Lambda_1 \otimes B_0 \oplus \Lambda_0 \otimes B_1)$,

called a superspace. The superspace

$B^{n\bar{m}} = [(\bigoplus^n \Lambda_0) \oplus (\bigoplus^m \Lambda_1)] \oplus [(\bigoplus^n \Lambda_1) \oplus (\bigoplus^m \Lambda_0)]$ \quad (1.4)$

is said to be $(n, m)$-dimensional. The graded $\Lambda_0$-module

$B^{n\bar{m}} = (\bigoplus^n \Lambda_0) \oplus (\bigoplus^m \Lambda_1)$

is called an $(n, m)$-dimensional supervector space.
Whenever referring to a topology on a supervector space $B^{n,m}$, we will mean the Euclidean topology on a $2^{N-1}[n + m]$-dimensional real vector space.

Given a superspace $B^{n|m}$ over a Grassmann algebra $\Lambda$, a $\Lambda$-module endomorphism of $B^{n|m}$ is represented by an $(n + m) \times (n + m)$ matrix

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$$

with entries in $\Lambda$. It is called a supermatrix. One says that a supermatrix $L$ is

- even if $L_1$ and $L_4$ have even entries, while $L_2$ and $L_3$ have the odd ones;
- odd if $L_1$ and $L_4$ have odd entries, while $L_2$ and $L_3$ have the even ones.

Endowed with this gradation, the set of supermatrices (1.5) is a graded $\Lambda$-ring. Unless otherwise stated, by supermatrices are meant homogeneous ones.

The familiar notion of a trace is extended to supermatrices (1.5) as the supertrace

$$\text{Str } L = \text{Tr } L_1 - (-1)^{|L|}\text{Tr } L_4.$$  

For instance, $\text{Str}(1) = n - m$.

A supertransposition $L^{st}$ of a supermatrix $L$ is the supermatrix

$$L^{st} = \begin{pmatrix} L_1^t & (-1)^{|L|}L_3^t \\ (-1)^{|L|}L_2^t & L_4^t \end{pmatrix},$$

where $L^t$ denotes the ordinary matrix transposition. There are the relations

$$\text{Str}(L^{st}) = \text{Str } L,$$

$$\text{Str}(LL')^{st} = (-1)^{|L||L'|}L^{st}L^{st},$$

$$\text{Str}(LL') = (-1)^{|L||L'|}\text{Str}(L'L) \quad \text{or} \quad \text{Str}([L,L']) = 0.$$  \hfill (1.6)

In order to extend the notion of a determinant to supermatrices, let us consider invertible supermatrices $L$ (1.5). They are never odd. One can show that an even supermatrix $L$ is invertible if and only if either the matrices $L_1$ and $L_4$ are invertible or the real matrix $\sigma(L)$ is invertible, where $\sigma$ is the body morphism.

Invertible supermatrices constitute a group $GL(n|m;\Lambda)$, called the general linear graded group. Then a superdeterminant of $L \in GL(n|m;\Lambda)$ is defined as

$$\text{Sdet } L = \text{det}(L_1 - L_2L_4^{-1}L_3)(\text{det } L_4^{-1}).$$

It satisfies the relations

$$\text{Sdet}(LL') = (\text{Sdet } L)(\text{Sdet } L'),$$

$$\text{Sdet}(L^{st}) = \text{Sdet } L,$$

$$\text{Sdet}(\exp(L)) = \exp(\text{Sdet } (L)).$$
Let $\mathcal{K}$ be a graded commutative ring. A graded commutative (non-associative) $\mathcal{K}$-algebra $\mathfrak{g}$ is called a Lie $\mathcal{K}$-superalgebra if its product, called the superbracket and denoted by $[\cdot,\cdot]$, obeys the relations

$$
[\varepsilon,\varepsilon'] = -(-1)^{[\varepsilon][\varepsilon'][\varepsilon',\varepsilon]}
$$

Obviously, the even part $\mathfrak{g}_0$ of a Lie $\mathcal{K}$-superalgebra $\mathfrak{g}$ is a Lie $\mathcal{K}_0$-algebra. A graded $\mathcal{K}$-module $P$ is called a $\mathfrak{g}$-module if it is provided with a $\mathcal{K}$-bilinear map

$$
(\varepsilon p) = ([\varepsilon] + [p]) \mod 2,
$$

$$
[\varepsilon,\varepsilon']p = (\varepsilon \circ \varepsilon' - (-1)^{[\varepsilon][\varepsilon']}\varepsilon' \circ \varepsilon)p.
$$

## 2 Graded differential calculus and connections

Linear differential operators and connections on graded modules over graded commutative rings are defined similarly to those in commutative geometry [12, 16, 26].

Let $\mathcal{K}$ be a graded commutative ring and $\mathcal{A}$ a graded commutative $\mathcal{K}$-ring. Let $P$ and $Q$ be graded $\mathcal{A}$-modules. The graded $\mathcal{K}$-module Hom$_\mathcal{K}(P,Q)$ of graded $\mathcal{K}$-module homomorphisms $\Phi : P \rightarrow Q$ can be endowed with the two graded $\mathcal{A}$-module structures

$$
(a\Phi)(p) = a\Phi(p), \quad (\Phi \cdot a)(p) = \Phi(ap), \quad a \in \mathcal{A}, \quad p \in P,
$$

(2.1)
called $\mathcal{A}$- and $\mathcal{A}^\ast$-module structures, respectively. Let us put

$$
\delta_a \Phi = a\Phi - (-1)^{[a][\Phi]}\Phi \cdot a, \quad a \in \mathcal{A}.
$$

(2.2)

An element $\Delta \in$ Hom$_\mathcal{K}(P,Q)$ is said to be a $Q$-valued graded differential operator of order $s$ on $P$ if

$$
\delta_a \circ \cdots \circ \delta_{a_s} \Delta = 0
$$

for any tuple of $s + 1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. The set Diff$_s(P,Q)$ of these operators inherits the graded module structures (2.1).

In particular, zero order graded differential operators obey the condition

$$
\delta_a \Delta(p) = a\Delta(p) - (-1)^{[a][\Delta]}\Delta(ap) = 0, \quad a \in \mathcal{A}, \quad p \in P,
$$

i.e., they coincide with graded $\mathcal{A}$-module morphisms $P \rightarrow Q$. A first order graded differential operator $\Delta$ satisfies the condition

$$
\delta_a \circ \delta_b \Delta(p) = ab\Delta(p) - (-1)^{[b][\Delta][a]}b\Delta(ap) - (-1)^{[b][\Delta]}a\Delta(bp) +
$$

$$
(-1)^{[b][\Delta] + ([\Delta] + [b][a])} = 0, \quad a, b \in \mathcal{A}, \quad p \in P.
$$
For instance, let $P = A$. Any zero order $Q$-valued graded differential operator $\Delta$ on $A$ is defined by its value $\Delta(1)$. Then there is a graded $A$-module isomorphism

$$\text{Diff}_0(A, Q) = Q$$

via the association

$$Q \ni q \mapsto \Delta_q \in \text{Diff}_0(A, Q),$$

where $\Delta_q$ is given by the equality $\Delta_q(1) = q$. A first order $Q$-valued graded differential operator $\Delta$ on $A$ fulfills the condition

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b) - (-1)^{([b]+[a][\Delta])}ab\Delta(1), \quad a, b \in A.$$ 

It is called a $Q$-valued graded derivation of $A$ if $\Delta(1) = 0$, i.e., the graded Leibniz rule

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b), \quad a, b \in A,$$ 

holds. One obtains at once that any first order graded differential operator on $A$ falls into the sum

$$\Delta(a) = \Delta(1)a + [\Delta(a) - \Delta(1)a]$$

of a zero order graded differential operator $\Delta(1)a$ and a graded derivation $\Delta(a) - \Delta(1)a$. If $\partial$ is a graded derivation of $A$, then $a\partial$ is so for any $a \in A$. Hence, graded derivations of $A$ constitute a graded $A$-module $\mathfrak{d}(A, Q)$, called the graded derivation module.

If $Q = A$, the graded derivation module $\mathfrak{d}A$ is also a Lie superalgebra over the graded commutative ring $K$ with respect to the superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']}u' \circ u, \quad u, u' \in A.$$ 

We have the graded $A$-module decomposition

$$\text{Diff}_1(A) = A \oplus \mathfrak{d}A.$$ 

Let us turn now to jets of graded modules. Given a graded $A$-module $P$, let us consider the tensor product $A \otimes_K P$ of graded $K$-modules $A$ and $P$. We put

$$\delta^b(a \otimes p) = (ba) \otimes p - (-1)^{[a][b]}a \otimes (bp), \quad p \in P, \quad a, b \in A.$$ 

The $k$-order graded jet module $\mathcal{J}^k(P)$ of the module $P$ is defined as the quotient of the graded $K$-module $A \otimes_K P$ by its submodule generated by elements of type

$$\delta^{b_0} \circ \cdots \circ \delta^{b_k}(a \otimes p).$$

In particular, the first order graded jet module $\mathcal{J}^1(P)$ consists of elements $a \otimes_1 p$ modulo the relations

$$ab \otimes_1 p - (-1)^{[a][b]}b \otimes_1 (ap) - a \otimes_1 (bp) + 1 \otimes_1 (abp) = 0.$$ 

(2.7)
For any $h \in \operatorname{Hom}_\mathcal{A} (\mathcal{A} \otimes P, Q)$, the equality

$$
\delta_b (h(a \otimes p)) = (-1)^{[b][b]} h(\delta^b(a \otimes p))
$$

holds. One then can show that any $Q$-valued graded differential operator $\Delta$ of order $k$ on a graded $\mathcal{A}$-module $P$ factorizes uniquely

$$
\Delta : P \xrightarrow{\delta^k} J^k (P) \xrightarrow{\cdot} Q
$$

through the morphism

$$
J^k : p \ni p \mapsto 1 \otimes_k p \in J^k (P)
$$

and some homomorphism $\delta^\Delta : J^k (P) \rightarrow Q$. Accordingly, the assignment $\Delta \mapsto \delta^\Delta$ defines an isomorphism

$$
\operatorname{Diff}_s (P, Q) = \operatorname{Hom}_\mathcal{A} (J^s (P), Q). \quad (2.8)
$$

Let us focus on the first order graded jet module $J^1$ of $\mathcal{A}$ consisting of the elements $a \otimes_1 b$, $a, b \in \mathcal{A}$, subject to the relations

$$
ab \otimes_1 1 - (-1)^{[a][b]} b \otimes_1 a - a \otimes_1 b + 1 \otimes_1 (ab) = 0. \quad (2.9)
$$

It is endowed with the $\mathcal{A}$- and $\mathcal{A}^*$-module structures

$$
c(a \otimes_1 b) = (ca) \otimes_1 b, \quad c \cdot (a \otimes_1 b) = a \otimes_1 (cb).
$$

There are canonical $\mathcal{A}$- and $\mathcal{A}^*$-module monomorphisms

$$
i_1 : \mathcal{A} \ni a \mapsto a \otimes_1 1 \in J^1, \\
J^1 : \mathcal{A} \ni a \mapsto 1 \otimes_1 a \in J^1,
$$

such that $J^1$, seen as a graded $\mathcal{A}$-module, is generated by the elements $J^1 a$, $a \in \mathcal{A}$. With these monomorphisms, we have the canonical $\mathcal{A}$-module splitting

$$
J^1 = i_1 (\mathcal{A}) \oplus \mathcal{O}^1, \quad (2.10)
$$

$$
a J^1 (b) = a \otimes_1 b = ab \otimes_1 1 + a(1 \otimes_1 b - b \otimes_1 1),
$$

where the graded $\mathcal{A}$-module $\mathcal{O}^1$ is generated by the elements

$$
1 \otimes_1 b - b \otimes_1 1, \quad b \in \mathcal{A}.
$$

Let us consider the corresponding $\mathcal{A}$-module epimorphism

$$
h^1 : J^1 \ni 1 \otimes_1 b \mapsto 1 \otimes_1 b - b \otimes_1 1 \in \mathcal{O}^1 \quad (2.11)
$$

and the composition

$$
d = h^1 \circ J^1 : \mathcal{A} \ni b \mapsto 1 \otimes_1 b - b \otimes_1 1 \in \mathcal{O}^1. \quad (2.12)
$$
The equality
\[ d(ab) = a \otimes_1 b + b \otimes_1 a - ab \otimes_1 1 - ba \otimes_1 1 = (-1)^{[a][b]} bda + a \]
shows that \( d \) (2.12) is an even \( \mathcal{O}^1 \)-valued derivation of \( \mathcal{A} \). Seen as a graded \( \mathcal{A} \)-module, \( \mathcal{O}^1 \) is generated by the elements \( da \) for all \( a \in \mathcal{A} \).

In view of the splittings (2.5) and (2.10), the isomorphism (2.8) reduces to the isomorphism
\[ \mathfrak{d} \mathcal{A} = \mathcal{O}^{1s} = \text{Hom}_A(\mathcal{O}^1, \mathcal{A}) \] (2.13)
of \( \mathfrak{d} \mathcal{A} \) to the dual \( \mathcal{O}^{1s} \) of the graded \( \mathcal{A} \)-module \( \mathcal{O}^1 \). It is given by the duality relations
\[ \mathfrak{d} \mathcal{A} \ni u \leftrightarrow \phi_u \in \mathcal{O}^{1s}, \quad \phi_u(da) = u(a), \quad a \in \mathcal{A}. \] (2.14)
Using this fact, let us construct a differential calculus over a graded commutative \( K \)-ring \( \mathcal{A} \).

Let us consider the bigraded exterior algebra \( \mathcal{O}^* \) of a graded module \( \mathcal{O}^1 \). It consists of finite linear combinations of monomials of the form
\[ \phi = a_0 da_1 \wedge \cdots \wedge da_k, \quad a_i \in \mathcal{A}, \] (2.15)
whose product obeys the juxtaposition rule
\[ (a_0 da_1) \wedge (b_0 db_1) = a_0 d(a_1 b_0) \wedge db_1 - a_0 a_1 db_0 \wedge db_1 \]
and the bigraded commutative relations
\[ \phi \wedge \phi' = (-1)^{[\phi][\phi']+[\phi'][\phi]} \phi' \wedge \phi. \] (2.16)
In order to make \( \mathcal{O}^* \) to a differential algebra, let us define the coboundary operator \( d : \mathcal{O}^1 \to \mathcal{O}^2 \) by the rule
\[ d\phi(u, u') = -u'(u(\phi)) + (-1)^{[u][u']} u(u'(\phi)) + [u', u](\phi), \]
where \( u, u' \in \mathfrak{d} \mathcal{A}, \phi \in \mathcal{O}^1, \) and \( u, u' \) are both graded derivatives of \( \mathcal{A} \) and \( \mathcal{A} \)-valued forms on \( \mathcal{O}^1 \). It is readily observed that, by virtue of the relation (2.14), \( (d \circ d)(a) = 0 \) for all \( a \in \mathcal{A} \). Then \( d \) is extended to the bigraded exterior algebra \( \mathcal{O}^* \) if its action on monomials (2.15) is defined as
\[ d(a_0 da_1 \wedge \cdots \wedge da_k) = da_0 \wedge da_1 \wedge \cdots \wedge da_k. \]
This operator is nilpotent and fulfills the familiar relations
\[ d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{[\phi][\phi']} \phi \wedge d\phi'. \] (2.17)
It makes \( \mathcal{O}^* \) into a differential bigraded algebra, called a graded differential calculus over a graded commutative \( K \)-ring \( \mathcal{A} \).
Furthermore, one can extend the duality relation (2.14) to the graded interior product of \( u \in \mathfrak{dA} \) with any monomial \( \phi \) (2.15) by the rules
\[
\begin{align*}
  u \lrcorner (bda) &= (-1)^{|u||b|} u(a), \\
  u \lrcorner (\phi \wedge \phi') &= (u \lrcorner \phi) \wedge \phi' + (-1)^{|\phi||\phi'|} \phi \wedge (u \lrcorner \phi').
\end{align*}
\]

As a consequence, any graded derivation \( u \in \mathfrak{dA} \) of \( \mathfrak{A} \) yields a derivation
\[
\begin{align*}
  L_u \phi &= u \lrcorner d\phi + d(u \lrcorner \phi), \quad \phi \in \mathfrak{O}^*, \quad u \in \mathfrak{dA}, \\
  L_u(\phi \wedge \phi') &= L_u(\phi) \wedge \phi' + (-1)^{|u||\phi|} \phi \wedge L_u(\phi'),
\end{align*}
\]
of the bigraded algebra \( \mathfrak{O}^* \) called the graded Lie derivative of \( \mathfrak{O}^* \).

**Remark 2.1.** Since \( \mathfrak{dA} \) is a Lie \( \mathcal{K} \)-superalgebra, let us consider the Chevalley–Eilenberg complex \( \mathcal{C}^*[\mathfrak{dA}; \mathfrak{A}] \) where the graded commutative ring \( \mathfrak{A} \) is regarded as a \( \mathfrak{dA} \)-module [11]. It is the complex
\[
0 \to \mathfrak{A} \xrightarrow{\delta^0} \mathcal{C}^1[\mathfrak{dA}; \mathfrak{A}] \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^k} \mathcal{C}^k[\mathfrak{dA}; \mathfrak{A}] \to \cdots \tag{2.20}
\]
where
\[
\mathcal{C}^k[\mathfrak{dA}; \mathfrak{A}] = \text{Hom}_\mathcal{K}(\wedge^k \mathfrak{dA}, \mathfrak{A})
\]
are \( \mathfrak{dA} \)-modules of \( \mathcal{K} \)-linear graded morphisms of the graded exterior products \( \wedge^k \mathfrak{dA} \) of the \( \mathcal{K} \)-module \( \mathfrak{dA} \) to \( \mathfrak{A} \). Let us bring homogeneous elements of \( \wedge^k \mathfrak{dA} \) into the form
\[
\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \varepsilon_{r+1} \wedge \cdots \wedge \varepsilon_k, \quad \varepsilon_1 \in \mathfrak{dA}_0, \quad \varepsilon_j \in \mathfrak{dA}_1.
\]

Then the coboundary operators of the complex (2.20) are given by the expression
\[
\begin{align*}
\delta^{r+s-1} & \cdot c(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_s) = \\
& \sum_{i=1}^{r} (-1)^{i-1} \varepsilon_i c(\varepsilon_1 \wedge \cdots \hat{\varepsilon_i} \cdots \wedge \varepsilon_r \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_s) + \\
& \sum_{j=1}^{s} (-1)^{r} \varepsilon_i c(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \varepsilon_1 \wedge \cdots \hat{\varepsilon_j} \cdots \wedge \varepsilon_s) + \\
& \sum_{1 \leq i < j \leq r} (-1)^{i+j} c(\varepsilon_i, \varepsilon_j) \wedge \varepsilon_1 \wedge \cdots \varepsilon_i \cdots \varepsilon_j \cdots \varepsilon_r \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_s) + \\
& \sum_{1 \leq i < j \leq s} c(\varepsilon_i, \varepsilon_j) \wedge \varepsilon_1 \wedge \cdots \varepsilon_r \wedge \varepsilon_1 \wedge \cdots \hat{\varepsilon_i} \cdots \hat{\varepsilon_j} \cdots \varepsilon_s) + \\
& \sum_{1 \leq i < r, 1 \leq j \leq s} (-1)^{i+r+1} c(\varepsilon_i, \varepsilon_j) \wedge \varepsilon_1 \wedge \cdots \varepsilon_i \cdots \varepsilon_j \cdots \varepsilon_r \wedge \varepsilon_1 \wedge \cdots \varepsilon_j \cdots \wedge \varepsilon_s).
\end{align*}
\]

The subcomplex \( \mathfrak{O}^*[\mathfrak{dA}] \) of the complex (2.20) of \( \mathfrak{A} \)-linear morphisms is the graded Chevalley–Eilenberg differential calculus over a graded commutative \( \mathcal{K} \)-ring \( \mathfrak{A} \). Then
one can show that the above mentioned graded differential calculus $O^*$ is a subcomplex of the Chevalley–Eilenberg one $O^*[dA]$. ⋄

Following the construction of a connection in commutative geometry [12, 16, 26], one comes to the notion of a connection on modules over a graded commutative $\mathbb{R}$-ring $A$.

**Definition 2.1.** A connection on a graded $A$-module $P$ is an $A$-module morphism

$$dA \ni u \mapsto \nabla_u \in \text{Diff}_1(P,P) \quad (2.22)$$

such that the first order differential operators $\nabla_u$ obey the Leibniz rule

$$\nabla_u(ap) = u(a)p + (-1)^{|a||u|}a\nabla_u(p), \quad a \in A, \quad p \in P. \quad (2.23)$$

**Definition 2.2.** Let $P$ in Definition 2.1 be a graded commutative $A$-ring and $dP$ the derivation module of $P$ as a graded commutative $K$-ring. A connection on a graded commutative $A$-ring $P$ is a $A$-module morphism

$$dA \ni u \mapsto \nabla_u \in dP, \quad (2.24)$$

which is a connection on $P$ as an $A$-module, i.e., obeys the Leibniz rule (2.23). □

### 3 Geometry of graded manifolds

By a graded manifold of dimension $(n, m)$ is meant a local-ringed space $(Z, \mathfrak{A})$ where $Z$ is an $n$-dimensional smooth manifold $Z$ and $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is a sheaf of graded commutative algebras of rank $m$ such that [2]:

(i) there is the exact sequence of sheaves

$$0 \to \mathcal{R} \to \mathfrak{A} \xrightarrow{\sigma} C^\infty_Z \to 0, \quad \mathcal{R} = \mathfrak{A}_1 + (\mathfrak{A}_1)^2, \quad (3.1)$$

where $C^\infty_Z$ is the sheaf of smooth real functions on $Z$;

(ii) $\mathcal{R}/\mathcal{R}^2$ is a locally free sheaf of $C^\infty_Z$-modules of finite rank (with respect to pointwise operations), and the sheaf $\mathfrak{A}$ is locally isomorphic to the the exterior product $\wedge_{C^\infty_Z}(\mathcal{R}/\mathcal{R}^2)$.

The sheaf $\mathfrak{A}$ is called a structure sheaf of the graded manifold $(Z, \mathfrak{A})$, while the manifold $Z$ is said to be a body of $(Z, \mathfrak{A})$. Sections of the sheaf $\mathfrak{A}$ are called graded functions. They make up a graded commutative $C^\infty(Z)$-ring $\mathfrak{A}(Z)$.

A graded manifold $(Z, \mathfrak{A})$ has the following local structure. Given a point $z \in Z$, there exists its open neighborhood $U$, called a splitting domain, such that

$$\mathfrak{A}(U) \cong C^\infty(U) \otimes \wedge_{\mathbb{R}^m}. \quad (3.2)$$

It means that the restriction $\mathfrak{A}|_U$ of the structure sheaf $\mathfrak{A}$ to $U$ is isomorphic to the sheaf $C^\infty_U \otimes \wedge_{\mathbb{R}^m}$ of sections of some exterior bundle

$$\wedge E^*_U = U \times \wedge_{\mathbb{R}^m} \to U.$$
The well-known Batchelor’s theorem [2] states that such a structure of graded manifolds is global.

**Theorem 3.1.** Let \((Z, \mathcal{A})\) be a graded manifold. There exists a vector bundle \(E \to Z\) with an \(m\)-dimensional typical fibre \(V\) such that the structure sheaf \(\mathcal{A}\) of \((Z, \mathcal{A})\) is isomorphic to the structure sheaf \(\mathcal{A}_E\) of sections of the exterior bundle \(\wedge E^*\), whose typical fibre is the Grassmann algebra \(\wedge V^*\).

**Proof.** The local sheaves \(C^\infty_U \otimes \wedge \mathbb{R}^m\) are glued into the global structure sheaf \(\mathcal{A}\) of the graded manifold \((Z, \mathcal{A})\) by means of transition functions, which are assembled into a cocycle of the sheaf \(\text{Aut}(\wedge \mathbb{R}^m)^\infty\) of smooth mappings from \(Z\) to \(\text{Aut}(\wedge \mathbb{R}^m)^\infty\). The proof is based on the bijection between the cohomology sets \(H^1(Z; \text{Aut}(\wedge \mathbb{R}^m)^\infty)\) and \(H^1(Z; GL(m, \mathbb{R}^m)^\infty)\). □

It should be emphasized that Batchelor’s isomorphism in Theorem 3.1 fails to be canonical. At the same time, there are many physical models where a vector bundle \(E\) is introduced from the beginning. In this case, it suffices to consider the structure sheaf \(\mathcal{A}_E\) of the exterior bundle \(\wedge E^*\). We agree to call the pair \((Z, \mathcal{A}_E)\) a simple graded manifold. Its automorphisms are restricted to those induced by automorphisms of the vector bundle \(E \to Z\), called the characteristic vector bundle of the simple graded manifold \((Z, \mathcal{A}_E)\). Accordingly, the structure module

\[ \mathcal{A}_E(Z) = \wedge E^*(Z) \]

of the sheaf \(\mathcal{A}_E\) (and of the exterior bundle \(\wedge E^*\)) is said to be the structure module of the simple graded manifold \((Z, \mathcal{A}_E)\).

Combining Batchelor Theorem 3.1 and classical Serre–Swan theorem [12, 26], we come to the following Serre–Swan theorem for graded manifolds.

**Theorem 3.2.** Let \(Z\) be a smooth manifold. A graded commutative \(C^\infty(Z)\)-algebra \(\mathcal{A}\) is isomorphic to the structure ring of a graded manifold with a body \(Z\) iff it is the exterior algebra of some projective \(C^\infty(Z)\)-module of finite rank. □

Given a simple graded manifold \((Z, \mathcal{A}_E)\), every trivialization chart \((U; z^A, y^a)\) of the vector bundle \(E \to Z\) is a splitting domain of \((Z, \mathcal{A}_E)\). Graded functions on such a chart are \(\Lambda\)-valued functions

\[ f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1...a_k}(z)c^{a_1} \cdots c^{a_k}, \quad (3.3) \]

where \(f_{a_1...a_k}(z)\) are smooth functions on \(U\) and \(\{c^a\}\) is the fibre basis for \(E^*\). In particular, the sheaf epimorphism \(\sigma\) in (3.1) is induced by the body map of \(\Lambda\). We agree to call \(\{z^A, c^a\}\) the local basis for the graded manifold \((Z, \mathcal{A}_E)\). Transition functions

\[ y^{a} = \rho_{b}^{a}(z^{A}) y^{b} \]

of bundle coordinates on \(E \to Z\) induce the corresponding transformation

\[ c^{a} = \rho_{b}^{a}(z^{A}) c^{b} \quad (3.4) \]
of the associated local basis for the graded manifold \((Z, \mathfrak{A}_E)\) and the according coordinate transformation law of graded functions (3.3).

**Remark 3.1.** Although graded functions are locally represented by \(A\)-valued functions (3.3), they are not \(A\)-valued functions on a manifold \(Z\) because of the transformation law (3.4). ●

**Remark 3.2.** Let us note that general automorphisms of a graded manifold take the form

\[ c^a = \rho^a(z^A, c^b), \tag{3.5} \]

where \(\rho^a(z^A, c^b)\) are local graded functions. Considering simple graded manifolds, we actually restrict the class of graded manifold transformations (3.5) to the linear ones (3.4), compatible with given Batchelor’s isomorphism. ●

Let \(E \rightarrow Z\) and \(E' \rightarrow Z\) be vector bundles and \(\Phi : E \rightarrow E'\) their bundle morphism over a morphism \(\zeta : Z \rightarrow Z'\). Then every section \(s^*\) of the dual bundle \(E'^* \rightarrow Z'\) defines the pull-back section \(\Phi^* s^*\) of the dual bundle \(E^* \rightarrow Z\) by the law

\[ v_z \mid \Phi^* s^*(z) = \Phi(v_z) \mid s^*(\zeta(z)), \quad v_z \in E_z. \]

It follows that a linear bundle morphism \(\Phi\) yields a morphism

\[ S\Phi : (Z, \mathfrak{A}_E) \rightarrow (Z', \mathfrak{A}_{E'}) \tag{3.6} \]

of simple graded manifolds seen as local-ringed spaces [2, 16, 26]. This is the pair \((\zeta, \zeta \circ \Phi^*)\) of the morphism \(\zeta\) of the body manifolds and the composition of the pull-back

\[ \mathfrak{A}_{E'} \ni f \mapsto \Phi^* f \in \mathfrak{A}_E \]

of graded functions and the direct image \(\zeta_*\) of the sheaf \(\mathfrak{A}_E\) onto \(Z'\). Relative to local bases \((z^A, c^a)\) and \((z'^A, c'^a)\) for \((Z, \mathfrak{A}_E)\) and \((Z', \mathfrak{A}_{E'})\) respectively, the morphism (3.6) reads

\[ S\Phi(z) = \zeta(z), \quad S\Phi(c'^a) = \Phi_b^a(z)c^b. \]

Accordingly, the pull-back onto \(Z\) of graded exterior forms on \(Z'\) is defined.

Given a graded manifold \((Z, \mathfrak{A})\), by the sheaf \(\mathfrak{d}\mathfrak{A}\) of graded derivations of \(\mathfrak{A}\) is meant a subsheaf of endomorphisms of the structure sheaf \(\mathfrak{A}\) such that any section \(u\) of \(\mathfrak{d}\mathfrak{A}\) over an open subset \(U \subset Z\) is a graded derivation of the graded algebra \(\mathfrak{A}(U)\). Conversely, one can show that, given open sets \(U' \subset U\), there is a surjection of the derivation modules

\[ \mathfrak{d}(\mathfrak{A}(U)) \rightarrow \mathfrak{d}(\mathfrak{A}(U')) \]

[2]. It follows that any graded derivation of the local graded algebra \(\mathfrak{A}(U)\) is also a local section over \(U\) of the sheaf \(\mathfrak{d}\mathfrak{A}\). Sections of \(\mathfrak{d}\mathfrak{A}\) are called graded vector fields on the graded manifold \((Z, \mathfrak{A})\). They make up the graded derivation module \(\mathfrak{d}\mathfrak{A}(Z)\) of the
graded commutative \( \mathbb{R} \)-ring \( \mathcal{A}(Z) \). This module is a Lie superalgebra with respect to the superbracket (2.4).

In comparison with general theory of graded manifolds, an essential simplification is that graded vector fields on a simple graded manifold \((Z, \mathcal{A}_E)\) can be seen as sections of a vector bundle as follows.

Due to the vertical splitting

\[ VE \cong E \times E, \]

the vertical tangent bundle \( VE \) of \( E \to Z \) can be provided with the fibre bases \( \{ \partial/\partial c^a \} \), which are the duals of the bases \( \{ c^a \} \). These are the fibre bases for

\[ \text{pr}_2 VE \cong E. \]

Then graded vector fields on a trivialization chart \((U; z^A, y^a)\) of \( E \) read

\[ u = u^A \partial_A + u^a \frac{\partial}{\partial c^a}, \quad (3.7) \]

where \( u^A, u^a \) are local graded functions on \( U \). In particular,

\[ \frac{\partial}{\partial c^a} \circ \frac{\partial}{\partial c^b} = - \frac{\partial}{\partial c^b} \circ \frac{\partial}{\partial c^a}, \quad \partial_A \circ \frac{\partial}{\partial c^a} = \frac{\partial}{\partial c^a} \circ \partial_A. \]

The derivations (3.7) act on graded functions \( f \in \mathcal{A}_E(U) \) (3.3) by the rule

\[ u(f_{a...b} c^a \cdots c^b) = u^A \partial_A (f_{a...b}) c^a \cdots c^b + u^k f_{a...b} \frac{\partial}{\partial c^k} \{(c^a \cdots c^b). \quad (3.8) \]

This rule implies the corresponding coordinate transformation law

\[ u'^A = u^A, \quad u'^a = \rho_j^a u^j + u^A \partial_A (\rho_j^a) c^j \]

of graded vector fields. It follows that graded vector fields (3.7) can be represented by sections of the vector bundle \( \mathcal{V}_E \to Z \) which is locally isomorphic to the vector bundle

\[ \mathcal{V}_E|_U \approx \wedge^* E^* \otimes (E \oplus TZ)|_U, \quad (3.9) \]

and is characterized by an atlas of bundle coordinates

\[ (z^A, z_{a_1...a_k}, v^i_{b_1...b_k}), \quad k = 0, \ldots, m, \]

possessing the transition functions

\[ z'^A_{i_1...i_k} = \rho^{-1}_{a_1} \cdots \rho^{-1}_{a_k} z^A_{i_1...i_k}, \]

\[ v'^{i}_{j_1...j_k} = \rho^{-1}_{b_1} \cdots \rho^{-1}_{b_k} \left[ \rho_j^{i_j} v^j_{b_1...b_k} + \frac{k!}{(k-1)!} z^A_{b_1...b_{k-1}} \partial_A \rho^{i}_b \right], \]
which fulfil the cocycle condition. Thus, the derivation module $\mathfrak{d}\mathcal{A}_E(Z)$ is isomorphic to the structure module $\mathcal{V}_E(Z)$ of global sections of the vector bundle $\mathcal{V}_E \to Z$.

There is the exact sequence

$$0 \to \bigwedge^* E^* \otimes \frac{E}{Z} \to \mathcal{V}_E \to \bigwedge^* T\frac{E}{Z} \to 0$$

of vector bundles over $Z$. Its splitting

$$\tilde{\gamma} : z^A \partial_A \mapsto z^A (\partial_A + \tilde{\gamma}^a_A \frac{\partial}{\partial c^a})$$

transforms every vector field $\tau$ on $Z$ into the graded vector field

$$\tau = \tau^A \partial_A \mapsto \nabla_\tau = \tau^A (\partial_A + \tilde{\gamma}^a_A \frac{\partial}{\partial c^a}),$$

which is a graded derivation of the graded commutative $\mathbb{R}$-ring $\mathcal{A}_E(Z)$ satisfying the Leibniz rule

$$\nabla_\tau (sf) = \tau (ds) f + s \nabla_\tau (f), \quad f \in \mathcal{A}_E(Z), \quad s \in C^\infty(Z).$$

It follows that the splitting (3.11) of the exact sequence (3.10) yields a connection on the graded commutative $C^\infty(Z)$-ring $\mathcal{A}_E(Z)$ in accordance with Definition 2.2. It is called a graded connection on the simple graded manifold $(Z, \mathcal{A}_E)$. In particular, this connection provides the corresponding horizontal splitting

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a} = u^A (\partial_A + \tilde{\gamma}^a_A \frac{\partial}{\partial c^a}) + (u^a - u^A \tilde{\gamma}^a_A) \frac{\partial}{\partial c^a}$$

of graded vector fields.

**Remark 3.3.** By virtue of the isomorphism (3.2), any connection $\tilde{\gamma}$ on a graded manifold $(Z, \mathcal{A})$, restricted to a splitting domain $U$, takes the form (3.11). Given two splitting domains $U$ and $U'$ of $(Z, \mathcal{A})$ with the transition functions (3.5), the connection components $\tilde{\gamma}^a_A$ obey the transformation law

$$\tilde{\gamma}'_A^a = \rho_b^a (z) \tilde{\gamma}^b_A + \partial_A \rho^a + \partial_A \rho^a.$$

If $U$ and $U'$ are the trivialization charts of the same vector bundle $E$ in Theorem 3.1 together with the transition functions (3.4), the transformation law (3.13) takes the form

$$\tilde{\gamma}'_A^a = \rho_b^a (z) \tilde{\gamma}^b_A + \partial_A \rho^a (z) c^b.$$

**Remark 3.4.** It should be emphasized that the above notion of a graded connection is a connection on the graded commutative ring $\mathcal{A}_E(Z)$ seen as a $C^\infty(Z)$-module. It differs from that of a connection on a graded fibre bundle $(Z, \mathcal{A}) \to (X, \mathcal{B})$ in [1]. The latter is
a connection on a graded \(\mathcal{B}(X)\)-module represented by a section of the jet graded bundle \(J^1(Z/X) \to (Z, \mathfrak{A})\) of sections of the graded fibre bundle \((Z, \mathfrak{A}) \to (X, \mathcal{B})\) [22].

**Example 3.5.** Every linear connection

\[
\gamma = dz^A \otimes (\partial_A + \gamma_A^a b^b \partial_a)
\]
on the vector bundle \(E \to Z\) yields the graded connection

\[
\gamma_S = dz^A \otimes (\partial_A + \gamma_A^a b^b \frac{\partial}{\partial c^a})
\] (3.15)
on the simple graded manifold \((Z, \mathfrak{A}_E)\). In view of Remark 3.3, \(\gamma_S\) is also a graded connection on the graded manifold

\[(Z, \mathfrak{A}) \cong (Z, \mathfrak{A}_E),\]
but its linear form (3.15) is not maintained under the transformation law (3.13).

The curvature of the graded connection \(\nabla_\tau\) (3.12) is defined by the familiar expression

\[
R(\tau, \tau') = [\nabla_\tau, \nabla_{\tau'}] - \nabla_{[\tau, \tau']},
\]
\[
R(\tau, \tau') = \tau^A \tau'^B R_{AB}^a \frac{\partial}{\partial c^a} : \mathfrak{A}_E(Z) \to \mathfrak{A}_E(Z),
\]
\[
R_{AB}^a = \partial_A \tilde{\gamma}_B^a - \partial_B \tilde{\gamma}_A^a + \tilde{\gamma}_k^a \frac{\partial}{\partial c^k} \tilde{\gamma}_B^a - \tilde{\gamma}_k^a \frac{\partial}{\partial c^k} \tilde{\gamma}_A^a.
\] (3.16)
It can also be written in the form

\[
R : \mathfrak{A}_E(Z) \to \mathcal{O}^2(Z) \otimes \mathfrak{A}_E(Z),
\]
\[
R = \frac{1}{2} R_{AB}^a dz^A \wedge dz^B \otimes \frac{\partial}{\partial c^a}.
\] (3.17)

Let now \(\mathcal{V}_E \to Z\) be a vector bundle which is the pointwise \(\wedge E^*\)-dual of the vector bundle \(\mathcal{V}_E \to Z\). It is locally isomorphic to the vector bundle

\[
\mathcal{V}_E^*|_U \cong \wedge E^* \otimes (E^* \oplus T^*Z)|_U.
\] (3.18)
With respect to the dual bases \(\{dz^A\}\) for \(T^*Z\) and \(\{dc^b\}\) for \(pr_2 V^* E \cong E^*\), sections of the vector bundle \(\mathcal{V}_E^*\) take the coordinate form

\[
\phi = \phi_A dz^A + \phi_a dc^a,
\]

\[
\phi_A = \phi_A + \rho^{-1} \partial_A (\rho_j^a) \phi_b c^j.
\]
They are regarded as graded exterior one-forms on the graded manifold \((Z, \mathfrak{A}_E)\), and make up the \(\mathfrak{A}_E(Z)\)-dual
\[
C^1_E = \mathfrak{d}\mathfrak{A}_E(Z)^* 
\]
of the derivation module
\[
\mathfrak{d}\mathfrak{A}_E(Z) = \mathcal{V}_E(Z). 
\]
Conversely,
\[
\mathfrak{d}\mathfrak{A}_E(Z) = (C^1_E)^*. 
\]
The duality morphism is given by the graded interior product
\[
u \rfloor \phi = u^A \phi_A + (-1)^{|\phi_A|} u^a \phi_a. \tag{3.19}
\]
In particular, the dual of the exact sequence (3.10) is the exact sequence
\[
0 \rightarrow \wedge^* Z \otimes T^* Z \rightarrow \mathcal{V}_E^* \rightarrow \wedge^* Z \otimes E^* \rightarrow 0. \tag{3.20}
\]
Any graded connection \(\tilde{\gamma}\) (3.11) yields the splitting of the exact sequence (3.20), and determines the corresponding decomposition of graded one-forms
\[
\phi = \phi_A dz^A + \phi_a dc^a = (\phi_A + \phi_a \tilde{\gamma}^a_A) dz^A + \phi_a (dc^a - \tilde{\gamma}^a_A dz^A). 
\]
Higher degree graded exterior forms are defined as sections of the exterior bundle \(k \wedge Z \mathcal{V}_E^*\). They make up a bigraded algebra \(\mathcal{C}^*_E\) which is isomorphic to the bigraded exterior algebra of the graded module \(C^1_E\) over \(\mathcal{A}(Z)\). This algebra is locally generated by graded forms \(dz^A, dc^i\) such that
\[
dz^A \wedge dc^i = -dc^i \wedge dz^A, \quad dc^i \wedge dc^j = dc^j \wedge dc^i. \tag{3.21}
\]
The graded exterior differential \(d\) of graded functions is introduced by the condition \(u \rfloor df = u(f)\) for an arbitrary graded vector field \(u\), and is extended uniquely to graded exterior forms by the rule (2.17). It is given by the coordinate expression
\[
d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \frac{\partial}{\partial c^a} \phi, 
\]
where the derivatives \(\partial_A, \partial/\partial c^a\) act on coefficients of graded exterior forms by the formula (3.8), and they are graded commutative with the graded forms \(dz^A\) and \(dc^a\). The formulae (2.16) – (2.19) hold.
The graded exterior differential \(d\) makes \(\mathcal{C}^*_E\) into a bigraded differential algebra whose de Rham complex reads
\[
0 \rightarrow \mathbb{R} \rightarrow \mathfrak{A}_E(Z) \xrightarrow{d} \mathcal{C}^1_E \xrightarrow{d} \cdots \mathcal{C}_E^k \xrightarrow{d} \cdots . \tag{3.22}
\]
Its cohomology $H^*_{GR}(Z)$ is called the graded de Rham cohomology of the graded manifold $(Z, \mathfrak{A}_E)$. One can compute this cohomology with the aid of the abstract de Rham theorem. Let $\mathfrak{D}^k\mathfrak{A}_E$ denote the sheaf of germs of graded $k$-forms on $(Z, \mathfrak{A}_E)$. Its structure module is $C^k_E$. These sheaves make up the complex

$$0 \rightarrow \mathbb{R} \rightarrow \mathfrak{A}_E \xrightarrow{d} \mathfrak{D}^1\mathfrak{A}_E \xrightarrow{d} \cdots \mathfrak{D}^k\mathfrak{A}_E \xrightarrow{d} \cdots.$$  \hspace{1cm} (3.23)

Its members $\mathfrak{D}^k\mathfrak{A}_E$ are sheaves of $C^\infty_Z$-modules on $Z$ and, consequently, are fine and acyclic. Furthermore, the Poincaré lemma for graded exterior forms holds [2]. It follows that the complex (3.23) is a fine resolution of the constant sheaf $\mathbb{R}$ on the manifold $Z$. Then, by virtue of the abstract de Rham theorem [12, 26], there is an isomorphism

$$H^*_{GR}(Z) = H^*(Z; \mathbb{R}) = H^*(Z)$$  \hspace{1cm} (3.24)

of the graded de Rham cohomology $H^*_{GR}(Z)$ to the de Rham cohomology $H^*(Z)$ of the smooth manifold $Z$. Moreover, the cohomology isomorphism (3.24) accompanies the cochain monomorphism $\mathcal{O}^*(Z) \rightarrow C^*_E$ of the de Rham complex $\mathcal{O}^*(Z)$ of smooth exterior forms on $Z$ to the graded de Rham complex (3.22). Hence, any closed graded exterior form is split into a sum $\phi = d\sigma + \varphi$ of an exact graded exterior form $d\sigma \in \mathcal{O}^*\mathfrak{A}_E$ and a closed exterior form $\varphi \in \mathcal{O}^*(Z)$ on $Z$.  

## 4 Superfunctions

By analogy with smooth manifolds, supermanifolds are constructed by gluing together of open subsets of supervector spaces $B^{n,m}$ with the aid of transition superfunctions [2, 16]. Therefore, let us start with the notion of a superfunction.

Though there are different classes of superfunctions, they can be introduced in the same manner as follows.

Let

$$B^{n,m} = \Lambda^0_n \oplus \Lambda^m_1, \quad n, m \geq 0,$$

be a supervector space, where $\Lambda$ is a Grassmann algebra of rank $0 < N \geq m$. Let

$$\sigma^{n,m} : B^{n,m} \rightarrow \mathbb{R}^n, \quad s : B^{n,m} \rightarrow R^{n,m} = R_0^n \oplus R_1^m$$

be the corresponding body and soul maps (see the decomposition (1.2)). Then any element $q \in B^{n,m}$ is uniquely split as

$$q = (x, y) = (\sigma(x^i) + s(x^i))e^0_i + y^je^1_j,$$  \hspace{1cm} (4.1)

where $\{e^0_i, e^1_j\}$ is a basis for $B^{n,m}$ and $\sigma(x^i) \in \mathbb{R}$, $s(x^i) \in R_0$, $y^j \in R_1$.  

18
Let $\Lambda'$ be another Grassmann algebra of rank $0 \leq N' \leq N$ which is treated as a subalgebra of $\Lambda$, i.e., the basis $\{e^a\}, \ a = 1, \ldots, N'$, for $\Lambda'$ is a subset of the basis $\{e^i\}, \ i = 1, \ldots, N$, for $\Lambda$. Given an open subset $U \subset \mathbb{R}^n$, let us consider a $\Lambda'$-valued function

$$f(z) = \sum_{k=0}^{N'} \frac{1}{k!} f_{a_1 \ldots a_k}(z) c^{a_1} \ldots c^{a_k} \quad (4.2)$$

on $U$ with smooth coefficients $f_{a_1 \ldots a_k}(z), \ z \in U$. It is a graded function on $U$. Its prolongation to $(\sigma^{n,0})^{-1}(U) \subset B^{n,0}$ is defined as the formal Taylor series

$$f(x) = \sum_{k=0}^{N'} \frac{1}{k!} \left[ \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^p f_{a_1 \ldots a_k}}{\partial z^{i_1} \ldots \partial z^{i_p}}(\sigma(x)) s(x^{i_1}) \ldots s(x^{i_p}) \right] c^{a_1} \ldots c^{a_k}. \quad (4.3)$$

Then a superfunction $F(q)$ on $(\sigma^{n,m})^{-1}(U) \subset B^{n,m}$ is given by a sum

$$F(q) = F(x, y) = \sum_{r=0}^{m} \frac{1}{r!} f_{j_1 \ldots j_r}(x) y^{j_1} \ldots y^{j_r}, \quad (4.4)$$

where $f_{j_1 \ldots j_r}(x)$ are functions (4.3). However, the representation of a superfunction $F(x, y)$ by the sum (4.4) need not be unique.

The germs of superfunctions (4.4) constitute the sheaf $\mathcal{S}_{N'}$ of graded commutative $\Lambda'$-algebras on $B^{n,m}$, but it is not a sheaf of $C_{\mathbb{R}^{m}}$-modules since superfunctions are expressed in Taylor series.

Using the representation (4.4), one can define derivatives of superfunctions as follows. Let $f(x)$ be a superfunction on $B^{n,0}$. Since $f$, by definition, is the Taylor series (4.3), its partial derivative along an even coordinate $x^i$ is defined in a natural way as

$$\partial_i f(x) = (\partial_i f)(\sigma(x), s(x)) = \sum_{k=0}^{N'} \frac{1}{k!} \left[ \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^p f_{a_1 \ldots a_k}}{\partial z^{i_1} \ldots \partial z^{i_p}}(\sigma(x)) s(x^{i_1}) \ldots s(x^{i_p}) \right] c^{a_1} \ldots c^{a_k}. \quad (4.5)$$

This even derivative is extended to superfunctions $F$ on $B^{n,m}$ in spite of the fact that the representation (4.4) is not necessarily unique. However, the definition of odd derivatives of superfunctions is more intricate.

Let $\mathcal{S}^0_{N'} \subset \mathcal{S}_{N'}$ be the subsheaf of superfunctions $F(x, y) = f(x)$ (4.3) independent of the odd arguments $y^j$. Let $\wedge \mathbb{R}^m$ be a Grassmann algebra generated by $(a^1, \ldots, a^m)$.
The expression (4.4) implies that, for any open subset $U \subset B^{n,m}$, there exists the sheaf morphism

$$\lambda : \mathcal{G}^0_{N'} \otimes \Lambda R^m \to \mathcal{G}_{N'},$$

(4.6)

$$\lambda(x, y) : \sum_{r=0}^{m} \frac{1}{r!} f_{j_1...j_r}(x) \otimes (a^{j_1} \cdots a^{j_r}) \to \sum_{r=0}^{m} \frac{1}{r!} f_{j_1...j_r}(x)y^{j_1} \cdots y^{j_r},$$

(4.7)

over $B^{n,m}$. Clearly, the morphism $\lambda$ (4.6) is an epimorphism. One can show that this epimorphism is injective and, consequently, is an isomorphism if and only if

$$N - N' \geq m$$

(4.8)

[2]. Roughly speaking, in this case, there exists a tuple of elements $y^{j_1}, \ldots, y^{j_r} \in \Lambda$ for each superfunction $f$ such that

$$\lambda(f \otimes (a^{j_1} \cdots a^{j_r})) \neq 0$$

at the point $(x, y^{j_1}, \ldots, y^{j_m})$ of $B^{n,m}$.

If the condition (4.8) holds, the representation of each superfunction $F(x, y)$ by the sum (4.4) is unique, and it is an image of some section $f \otimes a$ of the sheaf $\mathcal{G}^0_{N'} \otimes \Lambda R^m$ with respect to the morphism $\lambda$ (4.7). Then an odd derivative of $F$ is defined as

$$\frac{\partial}{\partial y^{j}}(\lambda(f \otimes y)) = \lambda(f \otimes \frac{\partial}{\partial a^{j}}(a)).$$

This definition is consistent only if $\lambda$ is an isomorphism, i.e., the relation (4.8) holds. If otherwise, there exists a non-vanishing element $f \otimes a$ such that

$$\lambda(f \otimes a) = 0,$$

whereas

$$\lambda(f \otimes \partial_j(a)) \neq 0.$$

For instance, if

$$N - N' = m - 1,$$

such an element is

$$f \otimes a = c^1 \cdots c^{N'} \otimes (a^1 \cdots a^m).$$

In order to classify superfunctions, we follow the terminology of [2, 19, 20].
• If $N' = N$, one deals with $G^\infty$-superfunctions, introduced in [18]. In this case, the inequality (4.8) is not satisfied, unless $m = 0$.
  
• If the condition (4.8) holds, $\mathcal{G}_{N'} = \mathcal{G}\mathcal{H}_{N'}$ is the sheaf of $GH^\infty$-superfunctions.
  
• In particular, if $N' = 0$, the condition (4.8) is satisfied, and $\mathcal{G}_{N'} = \mathcal{H}^\infty$ is the sheaf of $H^\infty$-superfunctions

\[
F(x, y) = \sum_{r=0}^{m} \frac{1}{r!} \left[ \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^p f_{j_1 \ldots j_r}}{\partial z_1 \ldots \partial z_p}(x) s(x^{i_1}) \ldots s(x^{i_p}) \right] y^{j_1} \ldots y^{j_r},
\]  

(4.9)

where $f_{j_1 \ldots j_r}$ are real functions [4, 10].

Superfunctions of the above three types are called smooth superfunctions. The fourth type of superfunctions is the following.

Given the sheaf $\mathcal{G}\mathcal{H}_{N'}$ of $GH^\infty$-superfunctions on a supervector space $B^{n,m}$, let us define the sheaf of graded commutative $\Lambda$-algebras $\mathcal{G}_{N'} \cong \mathcal{G}\mathcal{H}_{N'} \otimes \Lambda$, where $\Lambda$ is regarded as a graded commutative $\Lambda'$-algebra. The sheaf $\mathcal{G}_{N'}$ (4.10) possesses the following important properties [2].

• There is the evaluation morphism

\[
\delta : \mathcal{G}_{N'} \ni F \otimes a \mapsto Fa \in C^\Lambda_{B^{n,m}},
\]

(4.11)

where

\[
C^\Lambda_{B^{n,m}} \cong C^0_{B^{n,m}} \otimes \Lambda
\]

is the sheaf of continuous $\Lambda$-valued functions on $B^{n,m}$. Its image is isomorphic to the sheaf $\mathcal{G}^\infty$ of $G^\infty$-superfunctions on $B^{n,m}$.

• For any two integers $N'$ and $N''$ satisfying the condition (4.8), there exists the canonical isomorphism between the sheaves $\mathcal{G}_{N'}$ and $\mathcal{G}_{N''}$. Therefore, one can define the canonical sheaf $\mathcal{G}_{n,m}$ of graded commutative $\Lambda$-algebras on the supervector space $B^{n,m}$ whose sections can be seen as tensor products $F \otimes a$ of $H^\infty$-superfunctions $F$ (4.9) and elements $a \in \Lambda$. They are called $G$-superfunctions.

• The sheaf $\mathcal{D}\mathcal{G}_{n,m}$ of graded derivations of the sheaf $\mathcal{G}_{n,m}$ is a locally free sheaf of $\mathcal{G}_{n,m}$-modules of rank $(n, m)$. On any open set $U \subset B^{n,m}$, the $\mathcal{G}_{n,m}(U)$-module $\mathcal{D}\mathcal{G}_{n,m}(U)$ is generated by the derivations $\partial/\partial x^i, \partial/\partial y^j$ which act on $\mathcal{G}_{n,m}(U)$ by the rule

\[
\frac{\partial}{\partial x^i}(F \otimes a) = \frac{\partial F}{\partial x^i} \otimes a, \quad \frac{\partial}{\partial y^j}(F \otimes a) = \frac{\partial F}{\partial y^j} \otimes a.
\]

(4.12)

These properties of $G$-superfunctions make $G$-supermanifolds most suitable for differential geometric constructions.
5 Supermanifolds

A paracompact topological space $M$ is said to be an $(n, m)$-dimensional smooth supermanifold if it admits an atlas

$$\Psi = \{U_\zeta, \phi_\zeta\}, \quad \phi_\zeta : U_\zeta \to B^{n,m},$$

such that the transition functions $\phi_\zeta \circ \phi_\xi^{-1}$ are supersmooth. Obviously, a smooth supermanifold of dimension $(n, m)$ is also a real smooth manifold of dimension $2^{N-1}(n + m)$. If transition superfunctions are $H^{\infty}$-, $G^{\infty}$- or $GH^{\infty}$-superfunctions, one deals with $H^{\infty}$-, $G^{\infty}$- or $GH^{\infty}$-supermanifolds, respectively. This definition is equivalent to the following one.

**Definition 5.1.** A smooth supermanifold is a graded local-ringed space $(M, \mathcal{S})$ which is locally isomorphic to $(B^{n,m}, S)$, where $S$ is one of the sheaves of smooth superfunctions on $B^{n,m}$. The sheaf $S$ is called the structure sheaf of a smooth supermanifold. □

In accordance with Definition 5.1, by a morphism of smooth supermanifolds is meant their morphism $(\varphi, \Phi)$ as graded local-ringed spaces, where $\Phi$ is an even graded morphism. In particular, every morphism $\varphi : M \to M'$ yields the smooth supermanifold morphism $(\varphi, \Phi = \varphi^*)$.

Smooth supermanifolds however are effected by serious inconsistencies as follows. Since odd derivatives of $G^{\infty}$-superfunctions are ill defined, the sheaf of derivations of the sheaf of $G^{\infty}$-superfunctions is not locally free. Nevertheless, any $G$-supermanifold has an underlying $G^{\infty}$-supermanifold.

In the case of $GH^{\infty}$-supermanifolds (including $H^{\infty}$-ones), spaces of values of $GH^{\infty}$-superfunctions at different points are not mutually isomorphic because the Grassmann algebra $\Lambda$ is not a free module with respect to its subalgebra $\Lambda'$. By these reasons, we turn to $G$-supermanifolds. Their definition repeats Definition 5.1.

**Definition 5.2.** An $(n, m)$-dimensional $G$-supermanifold is a graded local-ringed space $(M, G_M)$, satisfying the following conditions:

- $M$ is a paracompact topological space;
- $(M, G_M)$ is locally isomorphic to $(B^{n,m}, \mathcal{G}_{n,m})$;
- there exists a morphism of sheaves of graded commutative $\Lambda$-algebras $\delta : G_M \to C^\Lambda_M$, where

$$C^\Lambda_M \cong C^0_M \otimes \Lambda$$

is sheaf of continuous $\Lambda$-valued functions on $M$, and $\delta$ is locally isomorphic to the evaluation morphism (4.11). □

**Example 5.1.** The triple $(B^{n,m}, \mathcal{G}_{n,m}, \delta)$, where $\delta$ is the evaluation morphism (4.11), is called the standard $G$-supermanifold. For any open subset $U \subset B^{n,m}$, the space $\mathcal{G}_{n,m}(U)$ can be provided with the topology which makes it into a graded Fréchet algebra. Then
there are isometrical isomorphisms
\[ G_{n,m}(U) \cong \mathcal{H}^\infty(U) \otimes \Lambda \cong C^\infty(\sigma_n^m(U)) \otimes \Lambda \otimes \Lambda^m \cong \]
\[ C^\infty(\sigma_n^m(U)) \otimes \Lambda^N \cong (5.1). \]

**Remark 5.2.** Any GH_{n,m}-supermanifold \((M, GH_{M}^\infty)\) with the structure sheaf \(GH_{M}^\infty\) is naturally extended to the \(G\)-supermanifold \((M, GH_{m}^\infty \otimes \Lambda)\). Every \(G\)-supermanifold defines an underlying \(G^\infty\)-supermanifold \((M, \delta(G_M))\), where \(\delta(G_M) = G_{M}^\infty\) is the sheaf of \(G^\infty\)-superfunctions on \(M\). 

As in the case of smooth supermanifolds, the underlying space \(M\) of a \(G\)-supermanifold \((M, G_{M})\) is provided with the structure of a real smooth manifold of dimension \(2^{N-1}(n + m)\), and morphisms of \(G\)-supermanifolds are smooth morphisms of the underlying smooth manifolds. However, it may happen that non-isomorphic \(G\)-supermanifold have isomorphic underlying smooth manifolds.

**Remark 5.3.** Let us present briefly the axiomatic approach to supermanifolds which enables one to obtain all the previously known types of supermanifolds in terms of \(R^\infty\)-supermanifolds \([2, 3, 7]\). This approach to supermanifolds refines that in \([21]\). The \(R^\infty\)-supermanifolds are introduced over the above mentioned Arens–Michael algebras of Grassmann origin \([7]\), but we here omit the topological side of their definition, though just the topological properties differ \(R^\infty\)-supermanifolds from \(R\)-supermanifolds in \([21]\).

Let \(\Lambda\) be a graded commutative algebra of the above mentioned type (for the sake of simplicity, the reader can think of \(\Lambda\) as being a Grassmann algebra). A superspace over \(\Lambda\) is a triple \((M, R^\infty, \delta)\), where \(M\) is a paracompact topological space, \(R^\infty\) is a sheaf of graded commutative \(\Lambda\)-algebras, and \(\delta : R^\infty \to C_M^\Lambda\) is an evaluation morphism to the sheaf \(C_M^\Lambda\) of continuous \(\Lambda\)-valued functions on \(M\). Sections of \(R^\infty\) are called \(R^\infty\)-superfunctions. Let \(M_q\) denote the ideal of the stalk \(R^\infty_q\), \(q \in M\), formed by the germs of \(R^\infty\)-superfunctions \(f\) vanishing at a point \(q\), i.e., such that \(\delta(f)(q) = 0\). An \(R^\infty\)-supermanifold of dimension \((n, m)\) is a superspace \((M, R^\infty, \delta)\) satisfying the following four axioms \([7]\).

**Axiom 1.** The graded \(R^\infty\)-dual \((\delta R^\infty)^*\) of the sheaf of derivations is a locally free sheaf of graded \(R^\infty\)-modules of rank \((n, m)\). Every point \(q \in M\) has an open neighborhood \(U\) with sections \(x^1, \ldots, x^n \in R^\infty(U)_0\), \(y^1, \ldots, y^m \in R^\infty(U)_1\) such that \(\{dx^i, dy^j\}\) is a graded basis for \((\delta R^\infty)^*(U)\) over \(R^\infty(U)\).

**Axiom 2.** Given the above mentioned coordinate chart, the assignment
\[q \mapsto (\delta(x^i), \delta(y^j))\]
determines a homeomorphism of \(U\) onto an open subset of \(B^{n,m}\).

**Axiom 3.** For every \(q \in M\), the ideal \(\mathcal{M}_q\) is finitely generated.

**Axiom 4.** For every open subset \(U \subset M\), the topological algebra \(R^\infty(U)\) is Hausdorff and complete.
An $R$-supermanifold over a graded commutative Banach algebra satisfying Axiom 4 is an $\mathbb{R}^\infty$-supermanifold. The standard $G$-supermanifold in Example 5.1 is an $\mathbb{R}^\infty$-supermanifold. Moreover, in the case of a finite Grassmann algebra $\Lambda$, the category of $\mathbb{R}^\infty$-supermanifolds and the category of $G$-supermanifolds are equivalent. 

Let $(M,G_M)$ be a $G$-supermanifold. As was mentioned above, it satisfies the Axioms 1-4. Sections $u$ of the sheaf $\mathfrak{d}G_M$ of graded derivations are called supervector fields on the $G$-supermanifold $(M,G_M)$, while sections $\phi$ of the dual sheaf $\mathfrak{d}G_M^*$ are one-superforms on $(M,G_M)$. Given a coordinate chart $(q^i) = (x^i, y^j)$ on $U \subset M$, supervector fields and one-superforms read

$$u = u^i \partial_i, \quad \phi = \phi_i dq^i,$$

where coefficients $u^i$ and $\phi_i$ are $G$-superfunctions on $U$. The graded differential calculus in supervector fields and superforms obeys the standard formulae (2.4), (2.16), (2.17) and (2.18).

Let us consider cohomology of $G$-supermanifolds. Given a $G$-supermanifold $(M,G_M)$, let

$$\mathfrak{D}_A^k \Lambda = \mathfrak{D}_A^k \otimes \Lambda$$

be the sheaves of smooth $\Lambda$-valued exterior forms on $M$. These sheaves are fine, and they constitute the fine resolution

$$0 \to \Lambda \to C^\infty_M \otimes \Lambda \to \mathfrak{D}_M^1 \otimes \Lambda \to \cdots$$

of the constant sheaf $\Lambda$ on $M$. We have the corresponding de Rham complex

$$0 \to \Lambda \to C^\infty_A(M) \to \mathfrak{D}_A^1(M) \to \cdots$$

of $\Lambda$-valued exterior forms on $M$. By virtue of the abstract de Rham theorem [12, 26], the cohomology $H^*_A(M)$ of this complex is isomorphic to the sheaf cohomology $H^*(M; \Lambda)$ of $M$ with coefficients in the constant sheaf $\Lambda$ and, consequently, is related to the de Rham cohomology as follows:

$$H^*_A(M) = H^*(M; \Lambda) = H^*(M) \otimes \Lambda. \quad (5.2)$$

Thus, the cohomology groups of $\Lambda$-valued exterior forms do not provide us with information on the $G$-supermanifold structure of $M$.

Let us turn to cohomology of superforms on a $G$-supermanifold $(M,G_M)$. The sheaves $\wedge^k \mathfrak{d}G_M^*$ of superforms constitute the complex

$$0 \to \Lambda \to G^*_M \to \mathfrak{d}^*G_M \to \cdots. \quad (5.3)$$

The Poincaré lemma for superforms is proved to hold [2, 6], and this complex is exact. However, the structure sheaf $G_M$ need not be acyclic, and the exact sequence (5.3) fails.
to be a resolution of the constant sheaf \( \Lambda \) on \( M \) in general. Therefore, the cohomology \( H^*_S(M) \) of the de Rham complex of superforms are not equal to cohomology \( H^*(M; \Lambda) \) of \( M \) with coefficients in the constant sheaf \( \Lambda \), and need not be related to the de Rham cohomology \( H^*_c(M) \) of the smooth manifold \( M \). In particular, cohomology \( H^*_S(M) \) is not a topological invariant, but it is invariant under \( G \)-isomorphisms of \( G \)-supermanifolds.

**Proposition 5.3.** The structure sheaf \( G_{n,m} \) of the standard \( G \)-supermanifold \( (B^{n,m}, G_{n,m}) \) is acyclic, i.e.,

\[
H^{k>0}(B^{n,m}; G_{n,m}) = 0.
\]

The proof is based on the isomorphism (5.1) and some cohomological constructions [2, 7].

### 6 DeWitt supermanifolds

There exists a particular class of supermanifolds, called DeWitt supermanifolds. Their notion implies that a supervector space \( B^{n,m} \) is provided with the DeWitt topology, which is coarser than the Euclidean one. This is the coarsest topology such that the body map

\[
\sigma^{n,m} : B^{n,m} \to \mathbb{R}^n
\]

is continuous. The open sets in the DeWitt topology are of the form \( V \times \mathcal{R}^{n,m} \), where \( V \) are open sets in \( \mathbb{R}^n \). Clearly, this topology is not Hausdorff.

**Definition 6.1.** A smooth supermanifold (resp. a \( G \)-supermanifold) is said to be a DeWitt supermanifold if it admits an atlas such that the local morphisms \( \phi_\xi : U_\xi \to B^{n,m} \) in Definition 5.1 (resp. Definition 5.2) are continuous with respect to the DeWitt topology, i.e., \( \phi_\xi(U_\xi) \subset B^{n,m} \) are open in this topology. □

Given an atlas \( (U_\xi, \phi_\xi) \) of a DeWitt supermanifold in accordance with Definition 6.1, it is readily observed that its transition functions \( \phi_\xi \circ \phi_\xi^{-1} \) must preserve the fibration \( \sigma^{n,m} \) (6.1) whose fibre \( (\sigma^{n,m})^{-1}(z) \) over \( z \in \mathbb{R}^n \) is equipped with the coarsest topology, where only \( \emptyset \) and \( (\sigma^{n,m})^{-1}(z) \) are open sets. This fact leads to the following.

Every DeWitt supermanifold is a locally trivial topological fibre bundle \( \sigma_M : M \to Z_M \) over an \( n \)-dimensional smooth manifold \( Z_M \) with the typical fibre \( R^{n,m} = \sigma(B^{n,m}) \). The base \( Z_M \) of this fibre bundle is said to be a body manifold of a DeWitt supermanifold, while the surjection \( \sigma_M \) is called a body map.

There is the above mentioned correspondence between the graded manifolds and the DeWitt \( H^{\infty} \)-supermanifolds. It is based on the following facts.

(i) The structure sheaf \( \mathfrak{A} \) of a graded manifold \( (Z, \mathfrak{A}) \) is locally isomorphic to the sheaf \( C^\infty_U \otimes \wedge \mathbb{R}^m \).
(ii) Given a DeWitt $H^\infty$-supermanifold $(M, H^\infty_M)$, the direct image $\sigma_*(H^\infty_M)$ of its structure sheaf onto the body manifold $Z_M$ is locally isomorphic to the sheaf $C^\infty_u \otimes \Lambda \mathbb{R}^m$. The expression (4.9) provides this isomorphism in an explicit form.

(iii) The graded local-ringed spaces $(M, H^\infty_M)$ and $(Z_M, \sigma_*(H^\infty_M))$ determine the same element of the cohomology set $H^1(Z_M; \text{Aut}(\Lambda \mathbb{R}^m)^\infty)$.

Thus, we come to the following statement [2, 4].

**Theorem 6.2.** Given a DeWitt $H^\infty$-supermanifold $(M, H^\infty_M)$, the associated pair $(Z_M, \sigma_*(H^\infty_M))$ is a graded manifold. Conversely, for any graded manifold $(Z, A)$, there exists a DeWitt $H^\infty$-supermanifold, whose body manifold is $Z$ and whose structure sheaf $A$ is isomorphic to $\sigma_*(H^\infty_M)$. □

Then by virtue of Batchelor’s Theorem 3.1 and Theorem 6.2, there is one-to-one correspondence between the isomorphism classes of DeWitt $H^\infty$-supermanifolds of odd rank $m$ with a body manifold $Z$ and the equivalence classes of $m$-dimensional vector bundles over $Z$. This result is extended to DeWitt $GH^\infty$, $C^\infty$- and $G$-supermanifolds because their isomorphism classes are in one-to-one correspondence with isomorphism classes of DeWitt $H^\infty$-supermanifolds [2].

Let us say something more on DeWitt $G$-supermanifolds.

**Proposition 6.3.** The structure sheaf $G_M$ of a DeWitt $G$-supermanifold is acyclic, and so is any locally free sheaf of graded $G_M$-modules [2, 7]. □

**Proposition 6.4.** There is an isomorphism of the cohomology $H^*_S(M)$ of superforms on a DeWitt $G$-supermanifold to the cohomology (5.2) of $\Lambda$-valued exterior forms on its body manifold $Z_M$, i.e., $H^*_S(M) = H^*(Z_M) \otimes \Lambda$ [17]. □

These results are based on the fact that the structure sheaf $G_M$ on $M$, provided with the DeWitt topology, is fine. However, this does not imply automatically that $G_M$ is acyclic since the DeWitt topology is not paracompact. Nevertheless, it follows that the image $\sigma_*(G_M)$ of $G_M$ on the body manifold $Z_M$ is fine and acyclic. Then Proposition 5.3 lead to Proposition 6.3. In particular, the sheaves of superforms on a DeWitt $G$-supermanifold are acyclic. Then they constitute the resolution of the constant sheaf $\Lambda$ on $M$, and we obtain isomorphisms

$$H^*_S(M) = H^*(M; \Lambda) = H^*(M) \otimes \Lambda.$$ 

Since the typical fibre of the fibre bundle $M \to Z_M$ is contractible, then $H^*(M) = H^*(Z_M)$ such that the isomorphism in Proposition 6.4 takes place.

## 7 Supervector bundles

Supervector bundles are considered in the category of $G$-supermanifolds. We start with the definition of the product of two $G$-supermanifolds seen as a trivial supervector bundle.
Let \((B^{n,m}, \mathcal{G}_{n,m})\) and \((B^{r,s}, \mathcal{G}_{r,s})\) be two standard \(G\)-supermanifolds in Example 5.2. Given open sets \(U \subset B^{n,m}\) and \(V \subset B^{r,s}\), we consider the presheaf
\[
U \times V \to \mathcal{G}_{n,m}(U) \hat{\otimes} \mathcal{G}_{r,s}(V),
\] (7.1)
where \(\hat{\otimes}\) denotes the tensor product of modules completed in Grothendieck’s topology. Due to the isomorphism (5.1), it is readily observed that the structure sheaf \(\mathcal{G}_{n+r,m+s}\) of the standard \(G\)-supermanifold on \(B^{n+r,m+s}\) is isomorphic to that, defined by the presheaf (7.1). This construction is generalized to arbitrary \(G\)-supermanifolds as follows.

Let \((M, G_M)\) and \((M', G_{M'})\) be two \(G\)-supermanifolds of dimensions \((n, m)\) and \((r, s)\), respectively. Their product
\[
(M, G_M) \times (M', G_{M'})
\]
is defined as the graded local-ringed space \((M \times M', G_M \hat{\otimes} G_{M'})\), where \(G_M \hat{\otimes} G_{M'}\) is the sheaf determined by the presheaf
\[
U \times U' \to G_M(U) \hat{\otimes} G_{M'}(U'),
\]
\[
\delta : G_M(U) \hat{\otimes} G_{M'}(U') \to C_{\sigma(U)}^\infty \otimes C_{\sigma(U')}^\infty = C_{\sigma(U) \times \sigma(U')}^\infty,
\]
for any open subsets \(U \subset M\) and \(U' \subset M'\). This product is a \(G\)-supermanifold of dimension \((n + r, m + s)\) [2]. Furthermore, there is the epimorphism
\[
pr_1 : (M, G_M) \times (M', G_{M'}) \to (M, G_M).
\]
One may define its section over an open subset \(U \subset M\) as the \(G\)-supermanifold morphism
\[
s_U : (U, G_M|_U) \to (M, G_M) \times (M', G_{M'})
\]
such that \(pr_1 \circ s_U\) is the identity morphism of \((U, G_M|_U)\). Sections \(s_U\) over all open subsets \(U \subset M\) determine a sheaf on \(M\). This sheaf should be provided with a suitable graded commutative \(G_M\)-structure.

For this purpose, let us consider the product
\[
(M, G_M) \times (B^{r|s}, \mathcal{G}_{r|s}),
\] (7.2)
where \(B^{r|s}\) is the superspace (1.4). It is called a product \(G\)-supermanifold. Since the \(\Lambda_0\)-modules \(B^{r|s}\) and \(B^{r+s,r+s}\) are isomorphic, \(B^{r|s}\) has a natural structure of an \((r + s, r + s)\) -dimensional \(G\)-supermanifold. Because \(B^{r|s}\) is a free graded \(\Lambda\)-module of the type \((r, s)\), the sheaf \(S_M^{r|s}\) of sections of the fibration
\[
(M, G_M) \times (B^{r|s}, \mathcal{G}_{r|s}) \to (M, G_M)
\] (7.3)
has the structure of the sheaf of free graded \(G_M\)-modules of rank \((r, s)\). Conversely, given a \(G\)-supermanifold \((M, G_M)\) and a sheaf \(S\) of free graded \(G_M\)-modules of rank \((r, s)\) on
there exists a product $G$-supermanifold (7.2) such that $S$ is isomorphic to the sheaf of sections of the fibration (7.3).

Let us turn now to the notion of a supervector bundle over $G$-supermanifolds. Similarly to smooth vector bundles [12, 26], one can require of the category of supervector bundles over $G$-supermanifolds to be equivalent to the category of locally free sheaves of graded modules on $G$-supermanifolds. Therefore, we can restrict ourselves to locally trivial supervector bundles with the standard fibre $B^{r|s}$.

**Definition 7.1.** A supervector bundle over a $G$-supermanifold $(M,G_M)$ with the standard fibre $(B^{r|s},G_{r|s})$ is defined as a pair $((Y,G_Y),\pi)$ of a $G$-supermanifold $(Y,G_Y)$ and a $G$-epimorphism

$$\pi : (Y,G_Y) \to (M,G_M) \quad (7.4)$$

such that $M$ admits an atlas $\{(U_\zeta,\psi_\zeta)\}$ of local $G$-isomorphisms

$$\psi_\zeta : (\pi^{-1}(U_\zeta),G_Y|_{\pi^{-1}(U_\zeta)}) \to (U_\zeta,G_M|_{U_\zeta}) \times (B^{r|s},G_{r|s}).$$

\[ \square \]

It is clear that sections of the supervector bundle (7.4) constitute a locally free sheaf of graded $G_M$-modules. The converse of this fact is the following [2].

**Theorem 7.2.** For any locally free sheaf $S$ of graded $G_M$-modules of rank $(r,s)$ on a $G$-supermanifold $(M,G_M)$, there exists a supervector bundle over $(M,G_M)$ such that $S$ is isomorphic to the structure sheaf of its sections. \[ \square \]

The fibre $Y_q$, $q \in M$, of the supervector bundle in Theorem 7.2 is the quotient

$$S_q/\mathcal{M}_q \cong S_{Mq}^{r|s}/(\mathcal{M}_q \cdot S_{Mq}^{r|s}) \cong B^{r|s}$$

of the stalk $S_q$ by the submodule $\mathcal{M}_q$ of the germs $s \in S_q$ whose evaluation $\delta(f)(q)$ vanishes. This fibre is a graded $\Lambda$-module isomorphic to $B^{r|s}$, and is provided with the structure of the standard $G$-supermanifold.

**Remark 7.1.** The proof of Theorem 7.2 is based on the fact that, given the transition functions $\rho_{\zeta\xi}$ of the sheaf $S$, their evaluations

$$g_{\zeta\xi} = \delta(\rho_{\zeta\xi}) \quad (7.5)$$

define the morphisms

$$U_\zeta \cap U_\xi \to GL(r|s;\Lambda),$$

and they are assembled into a cocycle of $G^\infty$-morphisms from $M$ to the general linear graded group $GL(r|s;\Lambda)$. Thus, we come to the notion of a $G^\infty$-vector bundle. Its definition is a repetition of Definition 7.1 if one replaces $G$-supermanifolds and $G$-morphisms with the $G^\infty$-ones. Moreover, the $G^\infty$-supermanifold underlying a supervector bundle (see Remark 5.2) is a $G^\infty$-supervector bundle, whose transition functions $g_{\zeta\xi}$ are related
to those of the supervector bundle by the evaluation morphisms (7.5), and are \( GL(r|s; \Lambda) \)-valued transition functions.

Since the category of supervector bundles over a \( G \)-supermanifold \( (M, G_M) \) is equivalent to the category of locally free sheaves of graded \( G_M \)-modules, one can define the usual operations of direct sum, tensor product, etc. of supervector bundles.

Let us note that any supervector bundle admits the canonical global zero section. Any section of the supervector bundle \( \pi \) (7.4), restricted to its trivialization chart \( (U, G_M |_U) \times (B^{|s}; G_{r|s}) \), (7.6) is represented by a sum \( s = s^a(q) \epsilon_a \), where \( \{ \epsilon_a \} \) is the basis for the graded \( \Lambda \)-module \( B^{|s} \), while \( s^a(q) \) are \( G \)-superfunctions on \( U \). Given another trivialization chart \( U' \) of \( \pi \), a transition function

\[
s^b(q) \epsilon'_b = s^a(q) h^b_a(q) \epsilon_b, \quad q \in U \cap U',
\]

is given by the \((r+s) \times (r+s)\) matrix \( h \) whose entries \( h^b_a(q) \) are \( G \)-superfunctions on \( U \cap U' \). One can think of this matrix as being a section of the supervector bundle over \( U \cap U' \) with the above mentioned group \( GL(r|s; \Lambda) \) as a typical fibre.

**Example 7.2.** Given a \( G \)-supermanifold \( (M, G_M) \), let us consider the locally free sheaf \( \mathcal{D}G_M \) of graded derivations of \( G_M \). In accordance with Theorem 7.2, there is a supervector bundle \( T(M, G_M) \), called supertangent bundle, whose structure sheaf is isomorphic to \( \mathcal{D}G_M \). If \( (q^1, \ldots, q^{m+n}) \) and \( (q'^1, \ldots, q'^{m+n}) \) are two coordinate charts on \( M \), the Jacobian matrix

\[
h^i_j = \frac{\partial q^i}{\partial q'^j}, \quad i, j = 1, \ldots, n + m,
\]

(see the prescription (4.12)) provides the transition morphisms for \( T(M, G_M) \).

It should be emphasized that the underlying \( G^\infty \)-vector bundle of the supertangent bundle \( T(M, G_M) \), called \( G^\infty \)-supertangent bundle, has the transition functions \( \delta(h^i_j) \) which cannot be written as the Jacobian matrices since the derivatives of \( G^\infty \)-superfunctions with respect to odd arguments are ill defined and the sheaf \( \mathcal{D}G_M^\infty \) is not locally free.

8 Superconnections

Given a supervector bundle \( \pi \) (7.4) with the structure sheaf \( S \), one can introduce a connection on this supervector bundle as a splitting of the the exact sequence of sheaves

\[
0 \to \mathcal{D}G_M^* \otimes S \to (G_M \oplus \mathcal{D}G_M^*) \otimes S \to S \to 0
\]

[16]. Its splitting is an even sheaf morphism

\[
\nabla : S \to \mathcal{D}G_M^* \otimes S
\]

(8.2)
satisfying the Leibniz rule
\[ \nabla(fs) = df \otimes s + f \nabla(s), \quad f \in G_M(U), \quad s \in S(U), \] (8.3)
for any open subset \( U \subset M \). The sheaf morphism (8.2) is called a superconnection on the supervector bundle \( \pi \) (7.4). Its curvature is given by the expression
\[ R = \nabla^2 : S \rightarrow \bigwedge^2 \mathcal{G}_M^* \otimes S. \] (8.4)

The exact sequence (8.1) need not be split. One can apply the criterion in Section 1.8 in order to study the existence of a superconnection on supervector bundles. Namely, the exact sequence (8.1) leads to the exact sequence of sheaves
\[ 0 \rightarrow \text{Hom}(S, \mathcal{G}_M^* \otimes S) \rightarrow \text{Hom}(S, (G_M \oplus \mathcal{G}_M^*) \otimes S) \rightarrow \text{Hom}(S, S) \rightarrow 0 \]
and to the corresponding exact sequence of the cohomology groups
\[ 0 \rightarrow H^0(M; \text{Hom}(S, \mathcal{G}_M^* \otimes S)) \rightarrow H^0(M; \text{Hom}(S, (G_M \oplus \mathcal{G}_M^*) \otimes S)) \rightarrow H^0(M; \text{Hom}(S, \mathcal{G}_M^* \otimes S)) \rightarrow \cdots. \]

The exact sequence (8.1) defines the Atiyah class
\[ \text{At}(\pi) \in H^1(M; \text{Hom}(S, \mathcal{G}_M^* \otimes S)) \]
of the supervector bundle \( \pi \) (7.4). If the Atiyah class vanishes, a superconnection on this supervector bundle exists. In particular, a superconnection exists if the cohomology set \( H^1(M; \text{Hom}(S, \mathcal{G}_M^* \otimes S)) \) is trivial. In contrast with the sheaf of smooth functions, the structure sheaf \( G_M \) of a \( G \)-supermanifold is not acyclic in general, cohomology \( H^*(M; \text{Hom}(S, \mathcal{G}_M^* \otimes S)) \) is not trivial, and a supervector bundle need not admit a superconnection.

**Example 8.1.** In accordance with Proposition 5.3, the structure sheaf of the standard \( G \)-supermanifold \((B^{n,m}, \mathcal{G}_{n,m})\) is acyclic, and the trivial supervector bundle
\[ (B^{n,m}, \mathcal{G}_{n,m}) \times (B^r|_s, \mathcal{G}_r|_s) \rightarrow (B^{n,m}, \mathcal{G}_{n,m}) \] (8.5)
has obviously a superconnection, e.g., the trivial superconnection. •

**Example 8.2.** By virtue of Proposition 6.3, the structure sheaf of a DeWitt \( G \)-supermanifold \((M, G_M)\) is acyclic, and so is the sheaf \( \text{Hom}(S, \mathcal{G}_M^* \otimes S) \). It follows that any supervector bundle over a DeWitt \( G \)-supermanifold admits a superconnection. •

Example 8.1 enables one to obtain a local coordinate expression for a superconnection on a supervector bundle \( \pi \) (7.4), whose typical fibre is \( B^r|_s \) and whose base is a \( G \)-supermanifold locally isomorphic to the standard \( G \)-supermanifold \((B^{n,m}, \mathcal{G}_{n,m})\). Let \( U \subset M \) (7.6) be a trivialization chart of this supervector bundle such that every section \( s \) of \( \pi|_U \) is represented by a sum \( s^a(q)\epsilon_a \), while the sheaf of one-superforms \( \mathcal{G}_M|_U \) has a local
basis \( \{ dq^i \} \). Then a superconnection \( \nabla \) (8.2) restricted to this trivialization chart can be given by a collection of coefficients \( \nabla_i^{a_b} \):

\[
\nabla(\epsilon_a) = dq^i \otimes (\nabla_i^{b_a} \epsilon_b),
\]

which are \( G \)-superfunctions on \( U \). Bearing in mind the Leibniz rule (8.3), one can compute the coefficients of the curvature form (8.4) of the superconnection (8.6). We have

\[
R(\epsilon_a) = \frac{1}{2} dq^i \wedge dq^j \otimes R_{ij}^{b_a} \epsilon_b,
\]

\[
R_{ij}^{a_b} = (-1)^{[i][j]} \partial_i \nabla_j^{a_b} - \partial_j \nabla_i^{a_b} + (-1)^{[i][j] + [a][b]} \nabla_j a \nabla_i b - (-1)^{[j][a] + [k]} \nabla_i k \nabla_j b.
\]

In a similar way, one can obtain the transformation law of the superconnection coefficients (8.6) under the transition morphisms (7.7). In particular, any trivial supervector bundle admits the trivial superconnection \( \nabla_i^{b_a} = 0 \).

9 Principal superconnections

In contrast with a supervector bundle, the structure sheaf \( G_P \) of a principal superbundle \( (P,G_P) \to (M,G_M) \) is not a sheaf of locally free \( G_M \)-modules in general. Therefore, the above technique of connections on modules and sheaves is not applied to principal superconnections in a straightforward way. Principal superconnections are introduced on principal superbundles by analogy with principal connections on smooth principal bundles [2]. For the sake of simplicity, let us denote \( G \)-supermanifolds \( (M,G_M) \) and their morphisms

\[
(\varphi : M \to N, \quad \Phi : G_N \to \varphi_*(M))
\]

by \( \widehat{M} \) and \( \widehat{\varphi} \), respectively. Given a point \( q \in M \), by \( \widehat{q} = (q, \Lambda) \) is meant the trivial \( G \)-supermanifold of dimension \( (0, 0) \). We will start with the notion of a \( G \)-Lie supergroup \( \widehat{H} \). The relations between \( G \)-\( GH^\infty \)- and \( G^\infty \)-Lie supergroups follow the relations between the corresponding classes of superfunctions.

**Definition 9.1.** A \( G \)-supermanifold \( \widehat{H} = (H, \mathcal{H}) \) is said to be a \( G \)-Lie supergroup if there exist the following \( G \)-supermanifold morphisms:

- a multiplication \( \widehat{m} : \widehat{H} \times \widehat{H} \to \widehat{H} \),
- a unit \( \widehat{e} : \widehat{e} \to \widehat{H} \),
- an inverse \( \widehat{S} : \widehat{H} \to \widehat{H} \),




together with the natural identifications

\[
\widehat{e} \times \widehat{H} = \widehat{H} \times \widehat{e} = \widehat{H},
\]
which satisfy the associativity
\[ \hat{m} \circ (\text{Id} \times \hat{m}) = \hat{m} \circ (\hat{m} \times \text{Id}) : \hat{H} \times \hat{H} \times \hat{H} \to \hat{H} \times \hat{H} \to \hat{H}, \]
the unit property
\[ (\hat{m} \circ (\hat{e} \times \text{Id}))((\hat{e} \times \hat{H}) = (\hat{m} \circ (\text{Id} \times \hat{e}))(\hat{H} \times \hat{e}) = \text{Id} H, \]
and the inverse property
\[ (\hat{m} \circ (\hat{S}, \text{Id}))((\hat{H}) = (\hat{m} \circ (\text{Id}, \hat{S}))(\hat{H}) = \hat{e}(\hat{e}). \]

Given a point \( g \in H \), let us denote by \( \hat{g} : \hat{e} \to \hat{H} \) the \( G \)-supermanifold morphism whose range in \( H \) is \( g \). Then one can introduce the notions of the left translation \( \hat{L}_g \) and the right translation \( \hat{R}_g \) as the \( G \)-supermanifold morphisms
\[
\hat{L}_g : \hat{H} = \hat{e} \times \hat{H} \xrightarrow{\hat{g} \times \text{Id}} \hat{H} \times \hat{H} \xrightarrow{\hat{m}} \hat{H},
\hat{R}_g : \hat{H} = \hat{H} \times \hat{e} \xrightarrow{\text{Id} \times \hat{g}} \hat{H} \times \hat{H} \xrightarrow{\hat{m}} \hat{H}.
\]

**Remark 9.1.** Given a \( G \)-Lie supergroup \( \hat{H} \), the underlying smooth manifold \( H \) is provided with the structure of a real Lie group of dimension \( 2^{N-1}(n+m) \), called the underlying Lie group. In particular, the actions on the underlying Lie group \( H \), corresponding to the left and right translations by \( \hat{g} \), are ordinary left and right translations by \( g \).

Let us reformulate the group axioms in Definition 9.1 in terms of the structure sheaf \( H \) of the \( G \)-Lie supergroup \((H, \mathcal{H})\). We observe that \( \mathcal{H} \) has properties of a sheaf of graded Hopf algebras as follows.

If \((H, \mathcal{H})\) is a \( G \)-Lie supergroup, the structure sheaf \( \mathcal{H} \) is provided with the sheaf morphisms:
- a comultiplication \( \hat{m}^* : \mathcal{H} \to m_*(\mathcal{H} \hat{\otimes} \mathcal{H}) \),
- a counit \( \hat{e}^* : \mathcal{H} \to e_*(\Lambda) \),
- a coinverse \( \hat{S}^* : \mathcal{H} \to s_* \mathcal{H} \).

Let us denote
\[ k = m \circ (\text{Id} \times m) = m \circ (m \times \text{Id}) : H \times H \times H \to H. \]

Then the group axioms in Definition 9.1 are equivalent to the relations
\[
(\text{Id} \otimes \hat{m}^*)(\mathcal{H}) = ((\hat{m}^* \otimes \text{Id}) \circ \hat{m}^*)(\mathcal{H}) = k_*(\mathcal{H} \hat{\otimes} \mathcal{H} \hat{\otimes} \mathcal{H}),
(\hat{m}^* \circ (\text{Id} \otimes \hat{e}^*))(\mathcal{H} \hat{\otimes} e_*(\Lambda)) = (\hat{m}^* \circ (\hat{e}^* \otimes \text{Id})) \circ (e_*(\Lambda) \hat{\otimes} \mathcal{H}) = \text{Id} \mathcal{H},
(\text{Id} \cdot \hat{S}^*) \circ \hat{m}^* = (\hat{S}^* \cdot \text{Id}) \circ \hat{m}^* = \hat{e}^*. \]
Comparing these relations with the axioms of a Hopf algebra in Section 10.2, one can think of the structure sheaf of a $G$-Lie group as being a sheaf of graded topological Hopf algebras.

**Example 9.2.** The general linear graded group $GL(n|m; \Lambda)$ is endowed with the natural structure of an $\hat{H}^{\infty}$-supermanifold of dimension $(n^2 + m^2, 2nm)$. The matrix multiplication gives the $\hat{H}^{\infty}$-morphism

$$m : GL(n|m; \Lambda) \times GL(n|m; \Lambda) \to GL(n|m; \Lambda)$$

such that $F(g, g') \mapsto F(gg')$. It follows that $GL(n|m; \Lambda)$ is an $\hat{H}^{\infty}$-Lie supergroup. It is trivially extended to the $G$-Lie supergroup $\hat{GL}(n \ | \ m; \Lambda)$, called the general linear supergroup.

A Lie superalgebra $\mathfrak{h}$ of a $G$-Lie supergroup $\hat{H}$ is defined as an algebra of left-invariant supervector fields on $\hat{H}$. Let us recall that a supervector field $u$ on a $G$-supermanifold $\hat{H}$ is a derivation of its structure sheaf $\mathcal{H}$. It is called left-invariant if

$$(\text{Id} \otimes u) \circ \hat{m}^* = \hat{m}^* \circ u.$$ 

If $u$ and $u'$ are left-invariant supervector fields, so are $[u, u']$ and $au + a'u'$, $a, a' \in \Lambda$. Hence, left-invariant supervector fields constitute a Lie superalgebra. The Lie superalgebra $\mathfrak{h}$ can be identified with the supertangent space $T_e(\hat{H})$. Moreover, there is the sheaf isomorphism

$$\mathcal{H} \otimes \mathfrak{h} = \mathcal{O}\mathcal{H}, \quad (9.1)$$

i.e., the sheaf of supervector fields on a $G$-Lie supergroup $\hat{H}$ is the globally free sheaf of graded $\mathcal{H}$-modules of rank $(n, m)$, generated by left-invariant supervector fields. The Lie superalgebra of right-invariant supervector fields on $\hat{H}$ is introduced in a similar way.

Let us consider the right action of a $G$-Lie supergroup $\hat{H}$ on a $G$-supermanifold $\hat{P}$. This is a $G$-morphism

$$\hat{\rho} : \hat{P} \times \hat{H} \to \hat{P}$$

such that

$$\hat{\rho} \circ (\hat{\rho} \times \text{Id}) = \hat{\rho} \circ (\text{Id} \times \hat{m}) : \hat{P} \times \hat{H} \times \hat{H} \to \hat{P},$$

$$\hat{\rho} \circ (\text{Id} \times \hat{\varepsilon})(\hat{P} \times \hat{e}) = \text{Id} \hat{P}.$$ 

The left action of $\hat{H}$ on $\hat{P}$ is defined similarly.

**Example 9.3.** Obviously, a $G$-Lie supergroup acts on itself both on the left and on the right by the multiplication morphism $\hat{m}$.

The general linear supergroup $\hat{GL}(n|m; \Lambda)$ acts linearly on the standard supermanifold $B^{n|m}$ on the left by the matrix multiplication which is a $G$-morphism.

Let $\hat{P}$ and $\hat{P}'$ be $G$-supermanifolds that are acted on by the same $G$-Lie supergroup $\hat{H}$. A $G$-supermanifold morphism $\varphi : \hat{P} \to \hat{P}'$ is said to be $\hat{H}$-invariant if

$$\varphi \circ \hat{\rho} = \hat{\rho}' \circ (\varphi \times \text{Id}) : \hat{P} \times \hat{H} \to \hat{P}'.$$
Definition 9.2. A quotient of an action of a $G$-Lie supergroup on a $G$-submanifold $\hat{P}$ is a pair $(\hat{M}, \hat{\pi})$ of a $G$-supermanifold $\hat{M}$ and a $G$-supermanifold morphism $\hat{\pi} : \hat{P} \to \hat{M}$ such that:

(i) there is the equality
$$\hat{\pi} \circ \hat{\rho} = \hat{\pi} \circ \hat{\rho}_1 : \hat{P} \times \hat{H} \to \hat{M},$$

(ii) for every morphism $\hat{\varphi} : \hat{P} \to \hat{M}'$ such that $\hat{\varphi} \circ \hat{\rho} = \hat{\varphi} \circ \hat{\rho}_1$, there is a unique morphism $\hat{g} : \hat{M} \to \hat{M}'$ with $\hat{\varphi} = \hat{g} \circ \hat{\pi}$. □

The quotient $(\hat{M}, \hat{\pi})$ does not necessarily exists. If it exists, there is a monomorphism of the structure sheaf $G_M$ of $\hat{M}$ into the direct image $\pi^* G_P$. Since the $G$-Lie group $\hat{H}$ acts trivially on $\hat{M}$, the range of this monomorphism is a subsheaf of $\pi^* G_P$, invariant under the action of $\hat{H}$. Moreover, there is an isomorphism

$$G_M \cong (\pi^* G_P)^H$$

(9.3)

between $G_M$ and the subsheaf of $G_P$ of $\hat{H}$-invariant sections. The latter is generateted by sections of $G_P$ on $\pi^{-1}(U)$, $U \subset \hat{M}$, which are $\hat{H}$-invariant as $G$-morphisms $\hat{U} \to \Lambda$, where one takes the trivial action of $\hat{H}$ on $\Lambda$.

Let us denote the morphism in the equality (9.2) by $\vartheta$. It is readily observed that the invariant sections of $G_P(\pi^{-1}(U))$ are exactly the elements which have the same image under the morphisms

$$\hat{\rho}^* : G_P(\pi^{-1}(U)) \to (\mathcal{H} \hat{\otimes} G_P)(\vartheta^{-1}(U)),$$

$$\hat{\rho}_1^* : G_P(\pi^{-1}(U)) \to (\mathcal{H} \hat{\otimes} G_P)(\vartheta^{-1}(U)).$$

Then the isomorphism (9.3) leads to the exact sequence of sheaves of $\Lambda$-modules on $M$

$$0 \longrightarrow G_M \overset{\hat{\pi}^*}{\longrightarrow} \pi^* G_P \overset{\hat{\rho}^* - \hat{\rho}_1^*}{\longrightarrow} \vartheta^* (G_M \hat{\otimes} \mathcal{H}).$$

(9.4)

Definition 9.3. A principal superbundle of a $G$-Lie supergroup $\hat{H}$ is defined as a locally trivial quotient $\pi : \hat{P} \to \hat{M}$, i.e., there exists an open covering $\{U_\zeta\}$ of $M$ together with $\hat{H}$-invariant isomorphisms

$$\hat{\psi}_\zeta : \hat{P} |_{\hat{U}_\zeta} \to \hat{U}_\zeta \times \hat{H},$$

where $\hat{H}$ acts on

$$\hat{U}_\zeta \times \hat{H} \to \hat{U}_\zeta$$

(9.5)

by the right multiplication. □

Remark 9.4. In fact, we need only the condition (i) in Definition 9.2 of the action of $\hat{H}$ on $\hat{P}$ and the condition of local triviality of $\hat{P}$. •
In an equivalent way, one can think of a principal superbundle as being glued together of trivial principal superbundles (9.5) by $\hat{H}$-invariant transition functions

$$\hat{\phi}_{\zeta\xi} : \hat{U}_{\zeta\xi} \times \hat{H} \rightarrow \hat{U}_{\zeta\xi} \times \hat{H}, \quad U_{\zeta\xi} = U_\zeta \cap U_\xi,$$

which fulfill the cocycle condition.

As in the case of smooth principal bundles, the following two types of supervector fields on a principal superbundle are introduced.

**Definition 9.4.** A supervector field $u$ on a principal superbundle $\hat{P}$ is said to be invariant if

$$\hat{\rho}^* \circ u = (u \otimes \text{Id}) \circ u : G_P \rightarrow \rho_*(G_P \hat{\otimes} \mathcal{H}).$$

Fundamental supervector fields generate the sheaf $\mathcal{V}G_P$ of $G_P$-modules of vertical supervector field on the principal superbundle $\hat{P}$, i.e., $u \circ \pi^* = 0$. Moreover, there is an isomorphism of sheaves of $G_P$-modules

$$G_P \otimes \mathfrak{h} \ni F \otimes v \mapsto F\hat{v} \in \mathcal{V}G_P,$$

which is similar to the isomorphism (9.1).

Let us consider the sheaf

$$(\pi_* \mathcal{V}G_P)^H = \pi_*(\mathcal{V}G_P) \cap \mathcal{d}^H(\pi_* G_P)$$

on $M$ whose sections are vertical $\hat{H}$-invariant supervector fields.

**Proposition 9.6.** [2]. There is the exact sequence of sheaves of $G_M$-modules

$$0 \rightarrow (\pi_* \mathcal{V}G_P)^H \rightarrow \mathcal{d}^H(\pi_* G_P) \rightarrow \mathcal{d}G_M \rightarrow 0.$$  \hspace{1cm} (9.6)

The exact sequence (9.6) is similar to the exact sequence of sheaves of $C^\infty$-modules

$$0 \rightarrow (V_G P)_X \rightarrow (T_G P)_X \rightarrow \mathcal{d}C^\infty_X \rightarrow 0$$

in the case of smooth principal bundles. Accordingly, we come to the following definition of a superconnection on a principal superbundle.
Definition 9.7. A superconnection on a principal superbundle \( \hat{\pi} : \hat{P} \to \hat{M} \) (or simply a principal superconnection) is defined as a splitting

\[
\nabla : \mathfrak{d}G_M \to \mathfrak{d}^H(\pi_*G_P)
\]

of the exact sequence (9.6). \( \square \)

In contrast with principal connections on smooth principal bundles, principal superconnections on a \( \hat{H} \)-principal superbundle need not exist.

A principal superconnection can be described in terms of a \( \mathfrak{h} \)-valued one-superform \( \omega : \mathfrak{d}G_P \to \mathfrak{G}_P \), on \( \hat{P} \) called a superconnection form. Indeed, every splitting \( \nabla \) (9.7) defines the morphism of \( \mathfrak{G}_P \)-modules

\[
\tilde{\pi}^*(\mathfrak{d}G_M) \to \tilde{\pi}^*(\mathfrak{d}^H(\pi_*G_P)) \cong \mathfrak{d}G_P
\]

which splits the exact sequence

\[
0 \to \mathcal{V}G_P \to \mathfrak{d}G_P \to \tilde{\pi}^*(\mathfrak{d}G_M) \to 0.
\]

Therefore, there exists the exact sequence

\[
0 \to \tilde{\pi}^*(\mathfrak{d}G_M)\mathcal{V}G_P \to \mathfrak{d}G_P \overset{\omega}{\to} \mathcal{V}G_P \to 0.
\]

Let us note that, by analogy with associated smooth bundles, one can introduce associated superbundles and superconnections on these superbundles. In particular, every supervector bundle of fibre dimension \((r, s)\) is a superbundle associated with \( \hat{GL}(r|s; \Lambda) \)-principal superbundle \([2]\).

### 10 Supermetric

In gauge theory on a principal bundle \( P \to X \) with a structure Lie group \( G \) reduced to its subgroup \( H \), the corresponding global section of the quotient bundle \( P/H \to X \) is regarded as a classical Higgs field \([12, 23]\), e.g., a gravitational field in gauge gravitation theory \([12, 13, 24]\).

Let \( \pi : P \to X \) be a principal smooth bundle with a structure Lie group \( G \). Let \( H \) be a closed (consequently, Lie) subgroup of \( G \). Then \( G \to G/H \) is an \( H \)-principal fiber bundle and, by the well known theorem, \( P \) is split into the composite fiber bundle

\[
P \overset{\pi_H}{\to} P/H \to X,
\]

where \( P \to P/H \) is an \( H \)-principal bundle and \( P/H \to X \) is a \( P \)-associated bundle with the typical fiber \( G/H \). One says that the structure group \( G \) of a principal bundle \( P \) is reducible to \( H \) if there exists an \( H \)-principal subbundle of \( P \). The necessary and sufficient
conditions of the reduction of a structure group are stated by the well known theorem [12, 23].

**Theorem 10.1.** There is one-to-one correspondence \( P^h = \pi^{-1}_H(h(X)) \) between the reduced \( H \)-principal subbundles \( P^h \) of \( P \) and the global sections \( h \) of the quotient bundle \( P/H \to X \). \( \square \)

As was mentioned above, sections of \( P/H \to X \) are treated in gauge theory as classical Higgs fields. For instance, let \( P = LX \) be the \( GL(n, \mathbb{R}) \)-principal bundle of linear frames in the tangent bundle \( TX \) of \( X \) (\( n = \dim X \)). If \( H = O(k, n-k) \), then a global section of the quotient bundle \( LX/O(k, n-k) \) is a pseudo-Riemannian metric on \( X \).

Our goal is the following extension of Theorem 10.1 to principal superbundles [25].

**Theorem 10.2.** Let \( \hat{P} \to \hat{M} \) be a principal \( G \)-superbundle with a structure \( G \)-Lie supergroup \( \hat{G} \), and let \( \hat{H} \) be a closed \( G \)-Lie supersubgroup of \( \hat{G} \) such that \( \hat{G} \to \hat{G}/\hat{H} \) is a principal superbundle. There is one-to-one correspondence between the principal \( G \)-superubundles of \( \hat{P} \) with the structure \( G \)-Lie supergroup \( \hat{H} \) and the global sections of the quotient superbundle \( \hat{P}/\hat{H} \to \hat{M} \) with the typical fiber \( \hat{G}/\hat{H} \). \( \square \)

In order to proof Theorem 10.2, it suffices to show that the morphisms

\[
\hat{P} \longrightarrow \hat{P}/\hat{H} \longrightarrow \hat{M}
\]

form a composite \( G \)-superbundle. A key point is that underlying spaces of \( G \)-supermanifolds are smooth real manifolds, but possessing very particular transition functions and morphisms. Therefore, the condition of local triviality of the quotient \( \hat{G} \to \hat{G}/\hat{H} \) is rather strong. However, it is satisfied in the most interesting case for applications when \( \hat{G} \) is a supermatrix group and \( \hat{H} \) is its Cartan supersubgroup. For instance, let \( \hat{P} = \hat{L}M \) be a principal superbundle of graded frames in the tangent superspaces over a supermanifold \( \hat{M} \) of even-odd dimension \( (n, 2m) \). If its structure general linear supergroup \( \hat{G} = \hat{GL}(n|2m; \Lambda) \) is reduced to the orthogonal-symplectic supersubgroup \( \hat{H} = \hat{OSp}(n|m; \Lambda) \), one can think of the corresponding global section of the quotient bundle \( \hat{L}M/\hat{H} \to \hat{M} \) as being a supermetric on \( \hat{M} \). Note that a Riemannian supermetric on graded manifolds has been considered in a different way [29].

**Proof.** Let \( \hat{\pi} : \hat{P} \to \hat{P}/\hat{G} \) be a principal superbundle with a structure \( G \)-Lie group \( \hat{G} \). Let \( \hat{i} : \hat{H} \to \hat{G} \) be a closed \( G \)-Lie supersubgroup of \( \hat{G} \), i.e., \( i : H \to G \) is a closed Lie subgroup of the Lie group \( G \). Since \( H \) is a closed subgroup of \( G \), the latter is an \( H \)-principal fiber bundle \( G \to G/H \) [28]. However, \( G/H \) need not possesses a \( G \)-supermanifold structure. Let us assume that the action

\[
\hat{\rho} : \hat{G} \times \hat{H} \longrightarrow \hat{G} \times \hat{G} \xrightarrow{\hat{m}} \hat{G}
\]

of \( \hat{H} \) on \( \hat{G} \) by right multiplications defines the quotient

\[
\hat{\zeta} : \hat{G} \to \hat{G}/\hat{H}
\]
which is a principal superbundle with the structure $G$-Lie supergroup $\hat{H}$. In this case, the $G$-Lie supergroup $\hat{G}$ acts on the quotient supermanifold $\hat{G}/\hat{H}$ on the left by the law

$$\hat{\sigma} : \hat{G} \times \hat{G}/\hat{H} = \hat{G} \times \hat{\zeta}(\hat{G}) \to (\hat{\zeta} \circ \hat{m})(\hat{G} \times \hat{G}).$$

Given this action of $\hat{G}$ on $\hat{G}/\hat{H}$, we have a $\hat{P}$-associated superbundle

$$\hat{\Sigma} = (\hat{P} \times \hat{G}/\hat{H})/\hat{G} \overset{\pi_{\hat{\Sigma}}}{\longrightarrow} \hat{M}$$

with the typical fiber $\hat{G}/\hat{H}$. Since

$$\hat{P}/\hat{H} = ((\hat{P} \times \hat{G})/\hat{G})/\hat{H} = (\hat{P} \times \hat{G}/\hat{H})/\hat{G},$$

the superbundle $\hat{\Sigma}$ (10.4) is the quotient $(\hat{P}/\hat{H}, \hat{\pi}_H)$ of $\hat{P}$ with respect to the right action

$$\hat{\rho} \circ (\text{Id} \times \hat{\iota}) : \hat{P} \times \hat{H} \longrightarrow \hat{P} \times \hat{G} \longrightarrow \hat{P}$$

of the $G$-Lie supergroup $\hat{H}$. Let us show that this quotient $\hat{\pi}_H : \hat{P} \to \hat{P}/\hat{H}$ is a principal superbundle with the structure supergroup $\hat{H}$. Note that, by virtue of the well-known theorem [28], the underlying space $P$ of $\hat{P}$ is an $H$-principal bundle $\pi_H : P \to P/H$. Let $\{V_\kappa, \hat{\Psi}_\kappa\}$ be an atlas of trivializations

$$\hat{\Psi}_\kappa : (\zeta^{-1}(V_\kappa), \mathcal{G}_G|_{\zeta^{-1}(V_\kappa)}) \to (V_\kappa, \mathcal{G}_{G/H}|_{V_\kappa}) \times \hat{H},$$

of the $\hat{H}$-principal bundle $\hat{G} \to \hat{G}/\hat{H}$, and let $\{U_\alpha, \hat{\psi}_\alpha\}$ be an atlas of trivializations

$$\hat{\psi}_\alpha : (\pi^{-1}(U_\alpha), \mathcal{G}_P|_{\pi^{-1}(U_\alpha)}) \to (U_\alpha, \mathcal{G}_M|_{U_\alpha}) \times \hat{G}$$

of the $\hat{G}$-principal superbundle $\hat{P} \to \hat{M}$. Then we have the $G$-isomorphisms

$$\hat{\psi}_{\alpha\kappa} = (\text{Id} \times \hat{\Psi}_\kappa) \circ \hat{\psi}_\alpha : (\psi^{-1}_\alpha(U_\alpha \times \zeta^{-1}(V_\kappa)), \mathcal{G}_P|_{\psi^{-1}_\alpha(U_\alpha \times \zeta^{-1}(V_\kappa))}) \to (U_\alpha, \mathcal{G}_M|_{U_\alpha}) \times (V_\kappa, \mathcal{G}_{G/H}|_{V_\kappa}) \times \hat{H} = (U_\alpha \times V_\kappa, \mathcal{G}_M|_{U_\alpha} \hat{\otimes} \mathcal{G}_{G/H}|_{V_\kappa}) \times \hat{H}. \quad (10.5)$$

For any $U_\alpha$, there exists a well-defined morphism

$$\hat{\psi}_\alpha : (\pi^{-1}(U_\alpha), \mathcal{G}_P|_{U_\alpha}) \to (U_\alpha \times G/H, \mathcal{G}_M|_{U_\alpha} \hat{\otimes} \mathcal{G}_{G/H}) \times \hat{H} = (U_\alpha, \mathcal{G}_M|_{U_\alpha}) \times \hat{G}/\hat{H} \times \hat{H}$$

such that

$$\hat{\psi}_\alpha|_{\psi^{-1}_\alpha(U_\alpha \times \zeta^{-1}(V_\kappa))} = \hat{\psi}_{\alpha\kappa}.$$  

Let $\{U_\alpha, \hat{\varphi}_\alpha\}$ be an atlas of trivializations

$$\hat{\varphi}_\alpha : (\pi^{-1}(U_\alpha), \mathcal{G}_{\Sigma}|_{\pi^{-1}_\Sigma(U_\alpha)}) \to (U_\alpha, \mathcal{G}_M|_{U_\alpha}) \times \hat{G}/\hat{H}$$

38
of the $\hat{P}$-associated superbundle $\hat{P}/\hat{H} \rightarrow \hat{M}$. Then the morphisms

$$(\hat{\varphi}^{-1}_a \times \text{Id}) \circ \hat{\Psi}_a : (\pi^{-1}(U_a), G_P|_{U_a}) \rightarrow (\pi^{-1}_\Sigma(U_a), G_{\Sigma}|_{\pi^{-1}_\Sigma(U_a)}) \times \hat{H}$$

make up an atlas $\{\pi^{-1}_\Sigma(U_a), (\hat{\varphi}^{-1}_a \times \text{Id}) \circ \hat{\Psi}_a\}$ of trivializations of the $\hat{H}$-principal super-bundle $\hat{P} \rightarrow \hat{P}/\hat{H}$. As a consequence, we obtain the composite superbundle (10.2). Now, let $\hat{\iota}_h : \hat{P}_h \rightarrow \hat{P}$ be an $\hat{H}$-principal supersubbundle of the principal superbundle $\hat{P} \rightarrow \hat{M}$. Then there exists a global section $\hat{h}$ of the superbundle $\hat{\Sigma} \rightarrow \hat{M}$ such that the image of $\hat{P}_h$ with respect to the morphism $\hat{\pi}_H \circ \hat{\iota}_h$ coincides with the range of the section $\hat{h}$. Conversely, given a global section $\hat{h}$ of the superbundle $\hat{\Sigma} \rightarrow \hat{M}$, the inverse image $\hat{\pi}_H^{-1}(\hat{h}(\hat{M}))$ is an $\hat{H}$-principal supersubbundle of $\hat{P} \rightarrow \hat{M}$. QED

Let us show that, as was mentioned above, the condition of Theorem 10.2 hold if $\hat{H}$ is the Cartan supersubgroup of a supermatrix group $\hat{G}$, i.e., $\hat{G}$ is a $G$-Lie supersubgroup of some general linear supergroup $G\Lambda(n|m;\Lambda)$.

Recall that a Lie superalgebra $\hat{\mathfrak{g}}$ of an $(n, m)$-dimensional $G$-Lie supergroup $\hat{G}$ is defined as a $\Lambda$-algebra of left-invariant supervector fields on $\hat{G}$, i.e., derivations of its structure sheaf $G_G$. A supervector field $u$ is called left-invariant if

$$(\text{Id} \otimes u) \circ \hat{m}^* = \hat{m}^* \circ u.$$ 

Left-invariant supervector fields on $\hat{G}$ make up a Lie $\Lambda$-superalgebra. Being a superspace $B^{n|m}$, a Lie superalgebra is provided with a structure of the standard $G$-supermanifold $B^{n+m,n+m}$. Its even part $\hat{\mathfrak{g}}_0 = \hat{B}^{n,m}$ is a Lie $\Lambda_0$-algebra.

Let $\hat{G}$ be a matrix $G$-Lie supergroup. Then there is an exponential map

$$\xi(J) = \exp(J) = \sum_k \frac{1}{k!} J^k$$

of some open neighbourhood of the origin of the Lie algebra $\hat{\mathfrak{g}}_0$ onto an open neighbourhood $U$ of the unit of $\hat{G}$. This map is an $H^\infty$-morphism, which is trivially extended to a $G$-morphism.

Let $\hat{H}$ be a Cartan supersubgroup of $\hat{G}$, i.e., the even part $\hat{\mathfrak{h}}_0$ of the Lie superalgebra $\hat{\mathfrak{h}}$ of $\text{wh}H$ is a Cartan subalgebra of the Lie algebra $\hat{G}_0$, i.e.,

$$\hat{G}_0 = \hat{\mathfrak{f}}_0 + \hat{\mathfrak{h}}_0, \quad [\hat{\mathfrak{f}}_0, \hat{\mathfrak{f}}_0] \subset \hat{\mathfrak{h}}_0, \quad [\hat{\mathfrak{f}}_0, \hat{\mathfrak{h}}_0] \subset \hat{\mathfrak{f}}_0.$$ 

Then there exists an open neighbourhood, say again $\hat{U}$, of the unit of $\hat{G}$ such that any element $g$ of $\hat{U}$ is uniquely brought into the form

$$g = \exp(F) \exp(I), \quad F \in \hat{\mathfrak{f}}_0, \quad I \in \hat{\mathfrak{h}}_0.$$ 

Then the open set $\hat{U}_H = \hat{\mathfrak{m}}(\hat{U} \times \hat{H})$ is $G$-isomorphic to the direct product $\xi(\xi^{-1}(U) \cap \hat{\mathfrak{h}}_0) \times \hat{H}$. This product provides a trivialization of an open neighbourhood of the unit of $\hat{G}$. Acting
on this trivialization by left translations $\hat{L}_g$, $g \in \hat{G}$, one obtains an atlas of a principal superbundle $\hat{G} \to \hat{H}$.

For instance, let us consider a superspace $B^{n|2m}$, coordinated by $(x^a, y^i, \overline{y}^\jmath)$, and the general linear supergroup $\hat{GL}(n|2m; \Lambda)$ of its automorphisms. Let $B^{n|2m}$ be provided with the $\Lambda$-valued bilinear form

$$\omega = \sum_{i=1}^{n} (x^i x'^i) + \sum_{j=1}^{m} (y^i \overline{y}^\jmath - \overline{y}^\jmath y'^i).$$

(10.6)

The supermatrices (1.5) preserving this bilinear form make up the orthogonal-symplectic supergroup $\hat{OSp}(n|m; \Lambda)$ [11]. It is a Cartan subgroup of $\hat{GL}(n|2m; \Lambda)$. Then one can think of the quotient $\hat{GL}(n|2m; \Lambda)/\hat{OSp}(n|m; \Lambda)$ as being a supermanifold of $\Lambda$-valued bilinear forms on $B^{n|2m}$ which are brought into the form (10.6) by general linear supertransformations.

Let $\hat{M}$ be $G$-supermanifold of dimension $(n, 2m)$ and $T\hat{M}$ its tangent superbundle. Let $L\hat{M}$ be an associated principal superbundle. Let us assume that its structure supergroup $\hat{GL}(n|2m; \Lambda)$ is reduced to the supersubgroup $\hat{OSp}(n|m; \Lambda)$. Then by virtue of Theorem 10.2, there exists a global section $h$ of the quotient $L\hat{M}/\hat{OSp}(n|m; \Lambda) \to \hat{M}$ which can be regarded as a supermetric on a supermanifold $\hat{M}$.

Note that, bearing in mind physical applications, one can treat the bilinear form (10.6) as *sui generis* superextension of the Euclidean metric on the body $\mathbb{R}^n = \sigma(B^{n|m})$ of the superspace $B^{n|m}$. However, the body of a supermanifold is ill-defined in general [8].

## 11 Graded principal bundles

Graded principal bundles and connections on these bundles can be studied similarly to principal superbundles and principal superconnections, though the theory of graded principal bundles preceded that of principal superbundles [1, 15]. Therefore, we will touch on only a few elements of the graded bundle technique (see, e.g. [27] for a detailed exposition).

Let $(Z, \mathfrak{A})$ be a graded manifold of dimension $(n, m)$. A useful object in the graded manifold theory, not mentioned above, is the finite dual $\mathfrak{A}(Z)^\circ$ of the algebra $\mathfrak{A}(Z)$ which consists of elements $a$ of the dual $\mathfrak{A}(Z)^*$ vanishing on an ideal of $\mathfrak{A}(Z)$ of finite codimension. This is a graded commutative coalgebra with the comultiplication

$$(\Delta^\circ(a))(f \otimes f') = a(ff'), \quad f, f' \in \mathfrak{A}(Z),$$

and the counit

$$\epsilon^\circ(a) = a(1_\mathfrak{A}).$$

In particular, $\mathfrak{A}(Z)^\circ$ includes the evaluation elements $\delta_z$ such that

$$\delta_z(f) = (\sigma(f))(z).$$
Given an evaluation element $\delta_z$, elements $u \in \mathfrak{A}(Z)^\circ$ are called primitive elements with respect to $\delta_z$ if they obey the relation
\[
\Delta^\circ(v) = u \otimes \delta_z + \delta_z \otimes u.
\] (11.1)
These elements are derivations of $\mathfrak{A}(Z)$ at $z$, i.e.,
\[
u(ff') = (uf)(\delta_z f') + (-1)^{|u||f|}(\delta_z f)(uf').
\]

**Definition 11.1.** A graded Lie group $(G, \mathcal{G})$ is defined as a graded manifold such that $G$ is an ordinary Lie group, the algebra $\mathcal{G}(G)$ is a graded Hopf algebra $(\Delta, \epsilon, S)$, and the algebra epimorphism $\sigma : \mathcal{G}(G) \to C^\infty(G)$ is a morphism of graded Hopf algebras. \(\Box\)

One can show that $\mathcal{G}(G)^\circ$ is also equipped with the structure of a Hopf algebra with the multiplication law
\[
a \ast b = (a \otimes b) \circ \Delta, \quad a, b \in \mathcal{G}(G)^\circ.
\] (11.2)
With respect to this multiplication, the evaluation elements $\delta_g, \mathcal{G} \in G$, constitute a group $\delta_g \ast \delta_{g'} = \delta_{gg'}$ isomorphic to $G$. Therefore, they are also called group-like elements. It is readily observed that the set of primitive elements of $\mathcal{G}(G)^\circ$ with respect to $\delta_e$, i.e., the tangent space $T_e(G, \mathcal{G})$ is a Lie superalgebra with respect to the multiplication (11.2). It is called the Lie superalgebra $\mathfrak{g}$ of the graded Lie group $(G, \mathcal{G})$.

One says that a graded Lie group $(G, \mathcal{G})$ acts on a graded manifold $(Z, \mathfrak{A})$ on the right if there exists a morphism
\[
(\varphi, \Phi) : (Z, \mathfrak{A}) \times (G, \mathcal{G}) \to (Z, \mathfrak{A})
\] such that the corresponding algebra morphism
\[
\Phi : \mathfrak{A}(Z) \to \mathfrak{A}(Z) \otimes \mathcal{G}(G)
\]
defines a structure of a right $\mathcal{G}(G)$-comodule on $\mathfrak{A}(Z)$, i.e.,
\[
(\text{Id} \otimes \Delta) \circ \Phi = (\Phi \otimes \text{Id}) \circ \Phi, \quad (\text{Id} \otimes \epsilon) \circ \Phi = \text{Id}.
\]
For a right action $(\varphi, \Phi)$ and for each element $a \in \mathcal{G}(G)^\circ$, one can introduce the linear map
\[
\Phi_a = (\text{Id} \otimes a) \circ \Phi : \mathfrak{A}(Z) \to \mathfrak{A}(Z).
\] (11.3)
In particular, if $a$ is a primitive element with respect to $\delta_e$, then $\Phi_a \in \partial \mathfrak{A}(Z)$.

Let us consider a right action of $(G, \mathcal{G})$ on itself. If $\Phi = \Delta$ and $a = \delta_g$ is a group-like element, then $\Phi_a$ (11.3) is a homogeneous graded algebra isomorphism of degree zero which corresponds to the right translation $G \to Gg$. If $a \in \mathfrak{g}$, then $\Phi_a$ is a derivation of
Given a basis \( \{ u_i \} \) for \( \mathfrak{g} \), the derivations \( \Phi_{u_i} \) constitute the global basis for \( \mathfrak{g}(G) \), i.e., \( \mathfrak{g}(G) \) is a free left \( \mathcal{G}(G) \)-module. In particular, there is the decomposition

\[
\mathcal{G}(G) = \mathcal{G}'(G) \oplus_R \mathcal{G}''(G),
\]

\[
\mathcal{G}'(G) = \{ f \in \mathcal{G}(G) : \Phi_u(f) = 0, \ u \in \mathfrak{g}_0 \},
\]

\[
\mathcal{G}''(G) = \{ f \in \mathcal{G}(G) : \Phi_u(f) = 0, \ u \in \mathfrak{g}_1 \}.
\]

Since \( \mathcal{G}'(G) \cong C^\infty(G) \), one finds that every graded Lie group \( (G, \mathcal{G}) \) is the sheaf of sections of some trivial exterior bundle \( G \times \mathfrak{g}_1^* \to G \) \[1, 5, 15\].

Let us turn now to the notion of a graded principal bundle. A right action \((\varphi, \Phi)\) of \((G, \mathcal{G})\) on \((Z, \mathfrak{A})\) is called free if, for each \( z \in Z \), the morphism

\[
\Phi_z : \mathfrak{A}(Z) \to \mathcal{G}(G)
\]

is such that the dual morphism

\[
\Phi_z^* : \mathcal{G}(G)^\circ \to \mathfrak{A}(Z)^\circ
\]

is injective.

A right action \((\varphi, \Phi)\) of \((G, \mathcal{G})\) on \((Z, \mathfrak{A})\) is called regular if the morphism

\[
(\varphi \times \text{pr}_1) \circ \Delta : (Z, \mathfrak{A}) \times (G, \mathcal{G}) \to (Z, \mathfrak{A}) \times (Z, \mathfrak{A})
\]

defines a closed graded submanifold of \((Z, \mathfrak{A}) \times (Z, \mathfrak{A})\).

**Remark 11.1.** Let us note that \((Z', \mathfrak{A}')\) is said to be a graded submanifold of \((Z, \mathfrak{A})\) if there exists a morphism \((Z', \mathfrak{A}') \to (Z, \mathfrak{A})\) such that the corresponding morphism \(\mathfrak{A}'(Z')^\circ \to \mathfrak{A}(Z)^\circ\) is an inclusion. A graded submanifold is called closed if \(\dim (Z', \mathfrak{A}') < \dim (Z, \mathfrak{A})\). ●

Then we come to the following variant of the well-known theorem on the quotient of a graded manifold \[1, 27\].

**Theorem 11.2.** A right action \((\varphi, \Phi)\) of \((G, \mathcal{G})\) on \((Z, \mathfrak{A})\) is regular if and only if the quotient \((Z/G, \mathfrak{A}/G)\) is a graded manifold, i.e., there exists an epimorphism of graded manifolds \((Z, \mathfrak{A}) \to (Z/G, \mathfrak{A}/G)\) compatible with the surjection \(Z \to Z/G\). \(\Box\)

In view of this Theorem, a graded principal bundle \((P, \mathfrak{A})\) can be defined as a locally trivial submersion

\[
(P, \mathfrak{A}) \to (P/G, \mathfrak{A}/G)
\]

with respect to the right regular free action of \((G, \mathcal{G})\) on \((P, \mathfrak{A})\). In an equivalent way, one can say that a graded principal bundle is a graded manifold \((P, \mathfrak{A})\) together with a free right action of a graded Lie group \((G, \mathcal{G})\) on \((P, \mathfrak{A})\) such that the quotient \((P/G, \mathfrak{A}/G)\) is a graded manifold and the natural surjection

\[
(P, \mathfrak{A}) \to (P/G, \mathfrak{A}/G)
\]
is a submersion. Obviously, \( P \to P/G \) is an ordinary \( G \)-principal bundle.

A graded principal connection on a graded \((G, \mathcal{G})\)-principal bundle \((P, \mathfrak{A}) \to (X, \mathfrak{B})\) can be introduced similarly to a superconnection on a principal superbundle. This is defined as a \((G, \mathcal{G})\)-invariant splitting of the sheaf \( \mathfrak{dA} \), and is represented by a \( \mathfrak{g} \)-valued graded connection form on \((P, \mathfrak{A})\) [27].

**Remark 11.2.** In an alternative way, one can define graded connections on a graded bundle \((Z, \mathfrak{A}) \to (X, \mathfrak{B})\) as sections \( \Gamma \) of the jet graded bundle

\[
J^1(Z/X) \to (Z, \mathfrak{A})
\]

of sections of \((Z, \mathfrak{A}) \to (X, \mathfrak{B})\) [1], which is also a graded manifold [22]. In the case of a \((G, \mathcal{G})\)-principal graded bundle, these sections \( \Gamma \) are required to be \((G, \mathcal{G})\)-equivariant.

### References

[1] A.Almorox, Supergauge theories in graded manifolds, In: *Differential Geometric Methods in Mathematical Physics*, Lect. Notes in Math. **1251** (Springer, Berlin, 1987), p.114.

[2] C.Bartocci, U.Bruzzo and D.Hernández Ruipérez, *The Geometry of Supermanifolds* (Kluwer Academic Publ., Dordrecht, 1991).

[3] C.Bartocci, U.Bruzzo, D.Hernández Ruipérez and V.Pestov, Foundations of supermanifold theory: the axiomatic approach, *Diff. Geom. Appl.* **3** (1993) 135.

[4] M.Batchelor, Two approaches to supermanifolds, *Trans. Amer. Math. Soc.* **258** (1980) 257.

[5] C.Boyer and O. Sánchez Valenzuela, Lie supergroup action on supermanifolds, *Trans. Amer. Math. Soc.* **323** (1991) 151.

[6] U.Bruzzo, Supermanifolds, supermanifold cohomology, and super vector bundles, In: *Differential Geometric Methods in Theoretical Physics* (Kluwer, Dordrecht, 1988), p. 417.

[7] U.Bruzzo and V.Pestov, On the structure of DeWitt supermanifolds, *J. Geom. Phys.* **30** (1999) 147.

[8] R.Catenacci, C.Reina and P.Teoflatto, On the body of supermanifolds *J. Math. Phys.* **26** (1985) 671.

[9] R.Cianci, *Introduction to Supermanifolds* (Bibliopolis, Naples, 1990).

[10] B.DeWitt, *Supermanifolds* (Cambridge Univ. Press, Cambridge, 1984).
[11] D.Fuks, *Cohomology of Infinite-Dimensional Lie Algebras* (Consultants Bureau, New York, 1986).

[12] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *Advanced Classical Field Theory* (World Scientific, Singapore, 2009).

[13] D.Ivanenko and G.Sardanashvily, The gauge treatment of gravity *Phys. Rep.* **94** (1983) 1.

[14] A.Jadczyk and K.Pilch, *Superspaces and Supersymmetries*, *Commun. Math. Phys.* **78** (1981) 391.

[15] B.Kostant, Graded manifolds, graded Lie theory, and prequantization, In: *Differential Geometric Methods in Mathematical Physics*, Lect. Notes in Math. **570** (Springer, Berlin, 1977) p. 177.

[16] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

[17] J.Rabin, Supermanifold cohomology and the Wess–Zumino term of the covariant superstring action, *Commun. Math. Phys.* **108** (1987) 375.

[18] A.Rogers, A global theory of supermanifolds, *J. Math. Phys.* **21** (1980) 1352.

[19] A.Rogers, Graded manifolds, supermanifolds and infinite-dimensional Grassmann algebras, *Commun. Math. Phys.* **105** (1986) 375.

[20] A.Rogers, *Supermanifolds: Theory and Applications* (World Scientific, Singapore, 2007).

[21] M.Rothstein, The axioms of supermanifolds and a new structure arising from them, *Trans. Amer. Math. Soc.* **297** (1986) 159.

[22] D.Ruípez and J.Masqué, Global variational calculus on graded manifolds, *J. Math. Pures et Appl.* **63** (1984) 283; **64** (1985) 87.

[23] G.Sardanashvily, Geometry of classical Higgs fields *Int. J. Geom. Methods Mod. Phys.* **3** (2006) 139; *E-print arXiv*: hep-th/0510168.

[24] G.Sardanashvily, Gauge gravitation theory from geometric viewpoint, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) No. 1, Preface; *E-print arXiv*: gr-qc/0512115.

[25] G.Sardanashvily, Supermetrics on supermanifolds, *Int. J. Geom. Methods Mod. Phys.* **5** (2008) 271; *E-print arXiv*: hep-th/0510168.
[26] G. Sardanashvily, Fibre bundles, jet manifolds and Lagrangian theory. Lectures for theoreticians, *E-print arXiv: 0908.1886*.

[27] T. Stavracou, Theory of connections on graded principal bundles, *Rev. Math. Phys.* **10** (1998) 47.

[28] N. Steenrod, *The Topology of Fibre Bundles* (Princeton Univ. Press, Princeton, 1972).

[29] M. Zirnbauer, Riemannian symmetric superspaces and their origin in random-matrix theory *J. Math. Phys.* **37** (1996) 4986.