Gaussian martingale inequality applies to random functions and maxima of empirical processes

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Abstract

We obtain a Bernstein type Gaussian concentration inequality for martingales. Our inequality improves Azuma-Hoeffding’s inequality for moderate deviations $x$. Following the work of McDiarmid \cite{24}, Talagrand \cite{32} and Boucheron, Lugosi and Massart \cite{5, 6}, we show that our result can be applied to the concentration of random functions, Erdős-Rényi random graph, and maxima of empirical processes. Several interesting Gaussian concentration inequalities have been obtained.

Keywords: martingales; concentration inequalities; Azuma-Hoeffding’s inequality; McDiarmid’s inequality; Freedman’s inequality; Self-bounding functions; Rademacher averages; maxima of empirical processes

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1. Introduction

Concentration inequalities for tail probabilities of random variables play an important role both in theoretical study and applications. Several methods have been introduced for obtaining such inequalities, including Bernstein’s method \cite{3} (see also Hoeffding \cite{19}), Talagrand’s induction method \cite{32, 33}, and the powerful so-called “entropy method”, based on logarithmic Sobolev inequalities, developed by Ledoux \cite{22}. The martingale method is also a useful way to derive concentration inequalities; see McDiarmid \cite{24, 25} for an excellent survey and also Bercu, Delyon and Rio \cite{2} for a recent monograph. For instance, when the martingale differences are bounded, Azuma-Hoeffding’s inequality is one of the classical results.

Assume that we are given a finite sequence of centered real-valued random variables $(X_i)_{i=1,...,n}$. Let $S_k = \sum_{i=1}^k X_i$ be the partial sums of $(X_i)_{i=1,...,n}$. If $(X_i)_{i=1,...,n}$ are independent and satisfy $|X_i| \leq a_i$ for some constants $a_i$, $i = 1, ..., n$, Azuma-Hoeffding’s inequality \cite{19} implies that, for
any positive $x$,\[ P\left(S_n \geq x\right) \leq \exp\left\{-\frac{x^2}{2\sum_{i=1}^{n} a_i^2}\right\}. \tag{1.1} \]

Denote by $\text{Var}(X)$ the variance of $X$. Notice that the factor $\sum_{i=1}^{n} a_i^2$ is an upper bound of the variance of $S_n$, that is $\text{Var}(S_n) \leq \sum_{i=1}^{n} a_i^2$. Since $\text{Var}(S_n) = \sum_{i=1}^{n} \text{Var}(X_i)$, it is easy to see that Azuma-Hoeffding’s inequality is Gaussian provided that $\text{Var}(X_i) = a_i^2$ for all $i = 1, ..., n$. Otherwise, $\text{Var}(S_n) < \sum_{i=1}^{n} a_i^2$ which means Azuma-Hoeffding’s inequality is sub-Gaussian and not tight enough. The following Bernstein inequality is significantly better than Azuma-Hoeffding’s inequality. Bernstein [3] proved that, for any positive $x$,
\[ P\left(S_n \geq x\right) \leq \exp\left\{-\frac{x^2}{2\text{Var}(S_n) + \frac{1}{3}a x}\right\}, \tag{1.2} \]
where $a = \max\{a_i, i = 1, ..., n\}$. It is obvious that Bernstein’s bound is smaller than Azuma-Hoeffding’s bound for all $0 \leq x \leq \frac{3}{a}(\sum_{i=1}^{n} a_i^2 - \text{Var}(S_n))$. Moreover, Bernstein’s bound (1.2) is a Gaussian bound for moderate deviations $x$, in contrast to (1.1) which is sub-Gaussian. That is why Bernstein’s inequality is significantly better than Azuma-Hoeffding’s inequality for moderate deviations $x$.

In the sequel, consider the martingale case. Assume that $(X_i, F_i)_{i=1, ..., n}$ is a sequence of real-valued martingale differences, which means $(S_k, F_k)_{k=1, ..., n}$ is a martingale. Assume $|X_i| \leq a_i$ for all $i = 1, ..., n$. Inequality (1.1) also holds true; see [1, 19]. This martingale version of Azuma-Hoeffding’s inequality has many interesting applications; see McDiarmid [24, 25] for concentration of functions of independent random variables and see Devroye [12] for nonparametric estimations. Notice that for bounded martingale differences $|X_i| \leq a_i$ for all $i = 1, ..., n$, the term $\sum_{i=1}^{n} a_i^2$ is also the upper bound of the variance of $S_n$. Thus we also would like to replace it by a smaller one. Freedman [18] (see also van de Geer [34], de la Peña [14] and [16, 17] for closely related results) has obtained the following inequality for martingales: for any positive $x$,
\[ P\left(S_n \geq x\right) \leq \exp\left\{-\frac{x^2}{2\left\|\langle S\rangle_n\right\|_{\infty} + \frac{1}{3}a x}\right\}, \tag{1.3} \]
where $a = \max\{a_i, i = 1, ..., n\}$ and $\langle S\rangle_n = \sum_{i=1}^{n} E[X_i^2 | F_{i-1}]$ is the predictable quadratic variance of $S_n$. Notice that $\langle S\rangle_n$ can be substantially smaller than $\sum_{i=1}^{n} a_i^2$. Thus Freedman’s inequality provides a sharper bound than Azuma-Hoeffding’s inequality for moderate deviations $x$. However, Freedman’s inequality does not generality provide a Gaussian bound. The reason is that $\left\|\langle S\rangle_n\right\|_{\infty}$ is usually larger than the variance of $S_n$ and Freedman’s inequality can be much worse than its “Gaussian” version which should make $\text{Var}(S_n)$ appear instead of $\left\|\langle S\rangle_n\right\|_{\infty}$.

In this paper we would like to establish a “Gaussian” version of Bernstein’s inequality for martingales. We prove that if the martingale differences satisfy the following Bernstein type condition
\[ E[|X_i|^k] \leq \frac{1}{2} \frac{k! a_i^{k-2}}{(k-1)^{k/2}} E[X_i^2], \quad k \geq 2, \tag{1.4} \]
for some positive constant $a$, then, by inequality (2.12), for any positive $x$,
\[
P(S_n \geq x) \leq \exp \left\{ -\frac{x^2}{2(\text{Var}(S_n) + ax\sqrt{n})} \right\}.
\] (1.5)

We will show that both the bounded difference $|X_i| \leq a$ and the normal random random variables satisfy condition (1.4). It is interesting to see that our inequality (1.5) is a Gaussian bound. Thus (1.5) has certain advantages over Azuma-Hoeffding’s inequality. Moreover, we also prove that the constant $a$ in (1.5) cannot be replaced by positive constants $b_n$, with $b_n \to 0$ as $n \to \infty$, under the stated condition. In this sense, inequality (1.4) can be regarded as a true martingale version of Bernstein’s inequality.

Assume that $\text{Var}(S_n)/n$ is bounded. Inequality (1.5) implies that
\[
P(S_n \geq x\sqrt{n}) = O\left(\exp\{-C_1 x\}\right), \quad x \to \infty,
\] (1.6)
for some positive constant $C_1$ which does not depend on $x$. The last equality is also the best possible in the sense that there exists a class of stationary martingale differences such that (1.4) holds and
\[
P(S_n \geq x\sqrt{n}) \geq \exp\{-C_2 x\}, \quad x \to \infty,
\] (1.7)
for some positive constant $C_2$ which does not depend on $x$.

Inequality (1.5) is user-friendly. Following the work of McDiarmid [24], we apply inequality (1.5) to Lipshitz functions of independent random variables. Following the work of Boucheron, Lugosi and Massart [5, 6, 8], and McDiarmid and Reed [26], we apply our inequalities to self-bounding functions of independent random variables, Vapnik-Chervonenkis entropies, Rademacher averages, and counting small subgraphs in random graphs. At last but not least, applying our inequalities to the maxima of empirical processes, we provide a partial positive answer to Massart’s question [23] about the best constants in Talagrand’s inequality [32]; see (5.68) for details. Note that our inequality is a Gaussian bound. To the best of our knowledge, such type bounds for the maxima of empirical processes have not yet been obtained before. The comparisons among our inequality with the known inequalities in literature show that our concentration inequalities have certain advantage for small and moderate deviations $x$.

For the methodology, our method is based on Doob’s martingale decomposition, Taylor’s expansion and Rio’s inequality. It is similar to the method developed by Rio [29] for establishing deviation inequality for martingales. However, Rio’s deviation inequality [29] does not provided a Gaussian bound which plays an important role in this paper.

The paper is organized as follows. In Section 2, we present a Bernstein type condition for martingales and give a Gaussian version of Freedman’s inequality under the stated condition. The applications are discussed in Sections 3, 4 and 5.
2. Concentration inequalities for martingales

Assume that we are given a sequence of real-valued martingale differences \((X_i, F_i)_{i=0,\ldots,n}\), defined on some probability space \((\Omega, \mathcal{F}, P)\), where \(X_0 = 0\) and \(\emptyset, \Omega = \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \mathcal{F}\) are increasing \(\sigma\)-fields. So, by definition, we have \(E[X_i | F_{i-1}] = 0\), \(i \in [1, n]\). For any \(j \in [1, n]\), denote by

\[ S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad \langle S \rangle_n = \sum_{i=1}^{n} E[X_i^2 | F_{i-1}] . \]

Hence \((S_i, F_i)_{i=1,\ldots,n}\) is a martingale. Throughout the paper, \(\text{Var}(S)\) stands for the variance of a random variable \(S\). It is well known that \(\text{Var}(S_n) = \sum_{i=1}^{n} E[X_i^2]\). For simplicity of notation, denote

\[ \sigma^2 = \frac{1}{n} \text{Var}(S_n). \]

2.1. Gaussian version of Freedman’s inequality

In our main result, we shall make use of the following Bernstein type condition: there exists an \(\epsilon\), may depending on \(n\), such that

\[ \sum_{i=1}^{n} E[|X_i|^k] \leq \frac{1}{2} k! \epsilon^{k-2} \sum_{i=1}^{n} E[X_i^2] \quad \text{for all} \quad k \geq 2. \quad (2.8) \]

It is worth noting that the bounded martingale differences \(|X_i| \leq a\) satisfy condition \((2.8)\) with \(\epsilon = 2^{3/2}a/3\); see Theorem 2.3. Moreover, the normal random variables also satisfy condition \((2.8)\); see \((2.27)\). An equivalent condition of \((2.8)\) is given by the following theorem.

Proposition 2.1. Condition \((2.8)\) is equivalent to the following one: there exists a \(\rho\), such that

\[ \sum_{i=1}^{n} E[(X_i)^k] \leq \frac{1}{2} k! \rho^{k-2} \sum_{i=1}^{n} E[X_i^2], \quad \text{for all} \quad k \geq 2. \quad (2.9) \]

Proof. It is obvious that \((2.8)\) implies \((2.9)\) with \(\rho = \epsilon\). Next, we prove \((2.9)\) implies \((2.8)\). For even number \(k = 2l, l \geq 1\), it is obvious that \((2.9)\) implies \((2.8)\) with \(\epsilon = \rho\). Thus we only need to prove the case of \(k = 2l + 1, l \geq 1\). By Cauchy-Schwarz’s inequality, it is easy to see that

\[ \sum_{i=1}^{n} E[|X_i|^k] \leq \sum_{i=1}^{n} \left( E[|X_i|^{2l}] \right)^{1/2} \left( E[|X_i|^{2(l+1)}] \right)^{1/2} . \]

By Cauchy-Schwarz’s inequality again, it follows that

\[ \sum_{i=1}^{n} \left( E[|X_i|^{2l}] \right)^{1/2} \left( E[|X_i|^{2(l+1)}] \right)^{1/2} \leq \left( \sum_{i=1}^{n} E[|X_i|^{2l}] \right)^{1/2} \left( \sum_{i=1}^{n} E[|X_i|^{2(l+1)}] \right)^{1/2} . \]
Thus
\[ \sum_{i=1}^{n} E[|X_i|^k] \leq \left( \sum_{i=1}^{n} E[|X_i|^{2l}] \right)^{1/2} \left( \sum_{i=1}^{n} E[|X_i|^{2(l+1)}] \right)^{1/2}. \]

Notice that $2l$ and $2(l+1)$ are even integers. Using (2.9), we obtain
\[
\sum_{i=1}^{n} E[|X_i|^k] \leq \frac{1}{2} \rho^{2l-1} \left( \frac{(2l)!}{(2l-1)l(2l+1)^{l+1}} \right)^{1/2} \sum_{i=1}^{n} E[X_i^2] = \frac{k! \rho^{k-2} f(l)}{2 (k-1)^{k/2}} \sum_{i=1}^{n} E[X_i^2],
\]
where
\[ f(l) = \left( \frac{(2l)! (2l+2)}{(2l-1)l(2l+1)^{l+2}} \right)^{1/2}. \]

Since $f(l) \leq f(1) = (32/27)^{1/2}$ for all $l \geq 1$, we have
\[
\sum_{i=1}^{n} E[|X_i|^k] \leq \frac{k! \rho^{k-2} f(1)}{2 (k-1)^{k/2}} \sum_{i=1}^{n} E[X_i^2] \leq \frac{k! (f(1) \rho)^{k-2}}{2 (k-1)^{k/2}} \sum_{i=1}^{n} E[X_i^2]
\]
for $k = 2l + 1, l \geq 1$. Thus (2.9) implies (2.8) with $\epsilon = (32/27)^{1/2} \rho$. □

In the following theorem, we give a Gaussian version of Freedman’s inequality. Our inequality is similar to Bernstein’s inequality.

**Theorem 2.1.** Assume condition (2.8). Then, for any $0 \leq t < \epsilon^{-1}$,
\[ E \left[ \exp \left\{ t \frac{S_n}{\sqrt{n}} \right\} \right] \leq \exp \left\{ \frac{t^2 \sigma^2}{2 (1 - t \epsilon)} \right\}, \tag{2.10} \]
and, for any positive $x$,
\[ P \left( \max_{1 \leq k \leq n} S_k \geq x \sqrt{\text{Var}(S_n)} \right) \leq \exp \left\{ - \frac{x^2}{1 + \sqrt{1 + 2 x \epsilon / \sigma + x \epsilon / \sigma}} \right\}, \tag{2.11} \]
\[ \leq \exp \left\{ - \frac{x^2}{2(1 + x \epsilon / \sigma)} \right\}. \tag{2.12} \]

Moreover, the same inequalities hold when replacing $S_k$ by $-S_k$.

**Remark 2.1.** Let us make some comments on Theorem 2.1.

1. Compared with Freedman’s inequality (1.3), our inequalities (2.11) and (2.12) are expressed in terms of $\sigma^2$ instead of $\|\langle S \rangle_n\|_\infty/n$. Moreover, our inequalities are valid for the martingales with unbounded differences.
2. Burkholder \[10\] proved that, for any \(p > 1\),
\[
||S_n||_p \leq C_p \left((X_1^2 + X_2^2 + ... + X_n^2)^{1/2}\right)_p.
\] (2.13)

In his paper in Astérisque, Burkholder \[11\] obtained (2.13) with \(C_p = p - 1\) for \(p > 2\). He also proved that this constant is optimal. From (2.13), for any \(p > 2\) the following Marcinkiewicz-Zygmund type inequality holds:
\[
||S_n||_p^2 \leq c_p (||X_1||_p^2 + ||X_2||_p^2 + ... + ||X_n||_p^2)
\] (2.14)

with \(c_p = (p - 1)^2\). Rio \[29\] recently proved that (2.14) holds with \(c_p = p - 1\), and that this constant cannot be improved.

3. Under the condition \(\mathbb{E}[\exp\{X_k^2\}] \leq 1 + C\) for any positive \(k\) and some positive \(C\), Rio (cf. Corollary 2.2 of \[29\]) has obtained the following result: for any positive \(x\),
\[
\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \geq x\sqrt{ne/2}\right) \leq \frac{C}{(2e)^{1/2}} \left(\cosh(x) - 1\right)^{-1}.
\] (2.15)

This bound has an exponentially decaying rate similar to that of (2.11) as \(x \to \infty\), but it is not a Gaussian bound.

4. In certain cases, our inequality (1.5) is better than Rio’s inequality (2.15). For instance, assume that \(|X_k| \leq \sqrt{k}\) for all \(k \geq 1\), and that \(\text{Var}(X_k), k \geq 1\), are uniformly bounded. Thus \(\mathbb{E}[\exp\{X_k^2\}] \leq e^n\) for any positive \(k\). Rio’s inequality (2.15) implies that, for any positive \(x\),
\[
\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \geq x\sqrt{ne/2}\right) \leq \frac{e^n - 1}{(2e)^{1/2}} \left(\cosh(x) - 1\right)^{-1}.
\] (2.16)

Our result implies that, for any positive \(x\),
\[
\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \geq x\sqrt{n}\right) \leq 2 \exp\left\{\frac{-x^2}{2(\sigma^2 + 2^{3/2}x/3)}\right\}
\]
which is significantly better than (2.16).

Proof of Theorem 2.1. By Taylor’s expansion of \(e^x\) and \(\mathbb{E}[S_n] = 0\), we have, for all \(t \geq 0\),
\[
\mathbb{E}\left[\exp\left\{t \frac{S_n}{\sqrt{n}}\right\}\right] = 1 + \sum_{k=2}^{\infty} \frac{t^k}{k!} \mathbb{E}\left[\left(\frac{S_n}{\sqrt{n}}\right)^k\right].
\] (2.17)

Using the following Rio inequality (see Theorem 2.1 of \[29\]): for any \(p \geq 2\),
\[
\left(\mathbb{E}[||S_n||_p^2]\right)^{2/p} \leq (p - 1)\left(\sum_{i=1}^{n} \left(\mathbb{E}[||X_i||_p^2]\right)^{2/p}\right),
\] (2.18)
we get, for all \( k \geq 2 \),

\[
E[|S_n|^k] \leq (k - 1)^{k/2} \left( \sum_{i=1}^{n} (E[|X_i|^k])^{2/k} \right)^{k/2}.
\]  \hfill (2.19)

Hence, by the inequality

\[
(a_1 + a_2 + ..., + a_n)^p \leq n^{p-1}(a_1^p + a_2^p + ..., + a_n^p),
\]  \hfill (2.20)

inequality (2.19) implies that, for all \( k \geq 2 \),

\[
E[|S_n|^k] \leq (k - 1)^{k/2} n^{k/2-1} \sum_{i=1}^{n} E[|X_i|^k].
\]  \hfill (2.21)

Applying the last inequality to (2.17), we obtain

\[
E\left[ \exp \left\{ t \frac{S_n}{\sqrt{n}} \right\} \right] \leq 1 + \sum_{k=2}^{\infty} \left( \frac{t^k}{k!} (k - 1)^{k/2} n^{-1} \sum_{i=1}^{n} E[|X_i|^k] \right).
\]  \hfill (2.22)

Hence condition (2.8) implies that, for all \( 0 \leq t < \epsilon^{-1} \),

\[
E\left[ \exp \left\{ t \frac{S_n}{\sqrt{n}} \right\} \right] \leq 1 + \sum_{k=2}^{\infty} \frac{\sigma^2}{2} t^k k^{-2} = 1 + \frac{t^2 \sigma^2}{2 (1 - t \epsilon)}.
\]  \hfill (2.23)

By the inequality \( 1 + x \leq e^x \), it follows that, for all \( 0 \leq t < \epsilon^{-1} \),

\[
E\left[ \exp \left\{ t \frac{S_n}{\sqrt{n}} \right\} \right] \leq \exp \left\{ \frac{t^2 \sigma^2}{2 (1 - t \epsilon)} \right\},
\]

which gives the desired inequality (2.10). Applying Doob’s maximal inequality to the nonnegative submartingale \( \exp\{tS_k/(\sqrt{n}\sigma)\}, F_k\}_{1 \leq k \leq n}, 0 \leq t < \epsilon^{-1} \), it is easy to see that, for any \( x \geq 0 \),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq x \sqrt{n}\sigma \right) \leq P\left( \max_{1 \leq k \leq n} \exp \left\{ t \frac{S_k}{\sqrt{n}\sigma} \right\} \geq \exp \left\{ t x \right\} \right)
\]

\[
\leq \exp \left\{ -t x \right\} E\left[ \exp \left\{ t \frac{S_n}{\sqrt{n}\sigma} \right\} \right].
\]

Hence

\[
P\left( \max_{1 \leq k \leq n} S_k \geq x \sqrt{n}\sigma \right) \leq \inf_{0 \leq t \leq \epsilon^{-1}} \exp \left\{ -t x + \frac{t^2}{2 (1 - t \epsilon/\sigma)} \right\}
\]

\[
= \exp \left\{ -\frac{x^2}{1 + \sqrt{1 + 2 x \epsilon/\sigma + x \epsilon/\sigma}} \right\}.
\]
which gives (2.11). Using the inequality \(\sqrt{1 + 2x\epsilon/\sigma} \leq 1 + x\epsilon/\sigma\), we get (2.12) from (2.11). □

From (2.12), it is interesting to see that for small \(x\), i.e. \(0 \leq x = o(\sigma/\epsilon)\), the tail probabilities on partial maximum martingales \(P\left(\max_{1 \leq k \leq n} S_k \geq x\sqrt{\text{Var}(S_n)}\right)\) have the exponential Gaussian bound \(\exp\left\{-x^2/2\right\}\). We wonder the range \(0 \leq x = o(\sigma/\epsilon)\) can be extended to a larger one under the stated condition. In other words, could we establish the following inequality under condition (2.2)

There exist some positive constants \(b_n\), with \(b_n \to 0\) as \(n \to \infty\), such that, for any positive \(x\),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\sqrt{\text{Var}(S_n)}\right) \leq \exp\left\{-\frac{x^2}{2\left(1 + x b_n\epsilon/\sigma\right)}\right\}
\]

(2.24)

When \((X_i)_{i \geq 1}\) are i.i.d., Bernstein’s inequality shows that we can take \(b_n = 1/\sqrt{n}\). However, for martingale differences, the following theorem gives a negative answer to question (2.24).

**Theorem 2.2.** There exists a stationary sequence of martingale differences \((X_i, F_i)_{i \geq 1}\) satisfying condition (2.8) and, for all \(x \to \infty\),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\sqrt{\text{Var}(S_n)}\right) \geq \exp\left\{-C x^2\right\}
\]

(2.25)

where \(C\) is a positive constant and does not depend on \(n\) and \(x\).

Assume that there exist two positive constants \(a, b\) such that \(\sigma/\epsilon \in [a, b]\) uniformly for all \(n \geq 1\), which holds for stationary martingale difference sequence. The inequality (2.24) implies that, for \(x_n \to \infty\) and \(x_n = o(b_n^{-1})\) as \(n \to \infty\),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x_n\sqrt{\text{Var}(S_n)}\right) \leq \exp\left\{-C x_n^2\right\}
\]

(2.26)

where \(C\) is a positive constant and does not depend on \(n\). For \(x_n \to \infty\) and \(x_n = o(b_n^{-1})\) as \(n \to \infty\), it is obvious that \(\exp\left\{-C_1 x_n^2\right\} < \{ - C_2 x_n \}\), and thus (2.26) contradicts (2.25). Hence, we cannot establish (2.24) under condition (2.8) even for stationary martingale difference sequences.

**Proof of Theorem 2.2.** We proceed as in Fan et al. [17]. Let \(X\) be a standard normal random variable. By Stirling’s formula

\[n! = \sqrt{2\pi n} n^n e^{-n} e^{1/12n}, \quad 0 \leq \theta_n \leq 1,\]

it is easy to verify that, for all \(k \geq 3\),

\[
E[|X|^k] \leq (k - 1)!! \leq \sqrt{k!}
\]

\[
\leq \frac{k!}{2} \sqrt{\frac{1}{k - 1}} \left(1 + \frac{1}{k - 1}\right)^{\frac{k}{2} - \frac{1}{2}} \left(\frac{2}{e^{1/2}}\right)^{k - 2}
\]

\[
\leq \frac{k!}{2} \left(\frac{2}{e^{1/2}}\right)^{k - 2}.
\]

(2.27)
Assume that \((\xi_i)_{i \geq 1}\) are Rademacher random variables independent of \(X\), i.e. \(P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}\). Set \(X_i = X\xi_i\) and \(\mathcal{F}_i = \sigma(X, (\xi_k)_{k=1,\ldots,i})\). Then by Theorem 2.1, \((X_i; \mathcal{F}_i)_{i \geq 1}\) is a stationary sequence of martingale differences and satisfies (2.8) with \(\sigma = 1\) and \(\epsilon = 2/\sqrt{e}\). It is easy to see that, for any positive \(x\),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\sqrt{n}\right) \geq P\left(S_n \geq x\sqrt{n}\right) \geq P\left(\sum_{i=1}^{n} \xi_i \geq \sqrt{nx}\right) P\left(X \geq \sqrt{x}\right).
\]

Since for \(x\) large enough,

\[
P\left(\sum_{i=1}^{n} \xi_i \geq \sqrt{nx}\right) \geq \exp\left\{-\frac{(\sqrt{nx})^2}{n}\right\} = e^{-x},
\]

we get

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\sqrt{n}\right) \geq \frac{1}{\sqrt{2\pi}(1 + \sqrt{x})} \exp\left\{-x - \frac{1}{2}x\right\} \geq \exp\left\{-2x\right\},
\]

which gives (2.25). This ends the proof of Theorem 2.2.

2.2. Case for bounded differences

When the martingale differences are bounded, Theorem 2.1 implies the following corollary which is even better than the classical Azuma-Hoeffding inequality for moderate deviations \(x\).

**Theorem 2.3.** Assume that \(|X_i| \leq a\) for some constant \(a\) and all \(i \in [1, n]\). Then the inequalities (2.10), (2.11) and (2.12) hold with \(\epsilon = 2^{3/2}a/3\). Moreover, the same inequalities hold when replacing \(S_k\) by \(-S_k\).

**Proof.** Define the function

\[f(k) = \frac{1}{2} k! \frac{1}{(k-1)^{k/2}}\]

for \(k \geq 3\). Note that the function \(g(k) = f(k)(f(3))^{2-k}\) is increasing in \(k \geq 3\) and \(g(3) = 1\). Thus \(f(3)^{k-2} \leq f(k)\) for any \(k \geq 3\). If \(|X_i| \leq a\), then, for any \(k \geq 3\),

\[
\sum_{i=1}^{n} \mathbb{E}[|X_i|^k] \leq (f(3))^{k-2} \left(\frac{a}{f(3)}\right)^{k-2} \sum_{i=1}^{n} \mathbb{E}[X_i^2] \leq f(k) \left(\frac{a}{f(3)}\right)^{k-2} \sum_{i=1}^{n} \mathbb{E}[X_i^2].
\]

Hence condition (2.8) holds with \(\epsilon = a/f(3) = 2^{3/2}a/3\). By Theorem 2.1, we obtain the desired inequalities. \[\square\]

Next, we examine the general \(\mathcal{F}_n\)-measurable random variables.
Theorem 2.4. Let $S_n$ be a random variable, and let $\emptyset = \emptyset \subset F_0 \subset F_1 \subset \ldots \subset F_n$ be a sequence of increasing $\sigma$-fields. Assume that $S_n$ is $F_n$-measurable. Assume that there exists a sequence of $F_i$-measurable random variables $T_i$ such that

$$T_{i-1} \leq \mathbb{E}[S_n | F_i] \leq T_i - 1 + M_i, \quad i = 1, \ldots, n,$$  \hspace{1cm} (2.29)

where $M$ is a constant. Then, for any positive $x$,

$$\mathbb{P}\left( S_n - \mathbb{E}[S_n] \geq x \right) \leq \exp\left\{ -\frac{x^2}{2(\text{Var}(S_n) + 2^{3/2}M\sqrt{n}x/3)} \right\}.$$  \hspace{1cm} (2.30)

Moreover, the same inequality holds when replacing $S_n$ by $-S_n$.

Comparing with Azuma-Hoeffding’s inequality, our inequality (2.30) has certain advantages. Assume that

$$T_{i-1} \leq \mathbb{E}[S_n | F_i] \leq T_i - 1 + M_i, \quad i = 1, \ldots, n,$$  \hspace{1cm} (2.31)

where $T_i$ are $F_i$-measurable random variables and $M_i$ are some constants. Under condition (2.31), it is well known that $\text{Var}(S_n) \leq \frac{1}{4} \sum_{i=1}^{n} M_i^2$; see McDiarmid [24]. The classical Azuma-Hoeffding inequality states that, for any positive $x$,

$$\mathbb{P}\left( S_n - \mathbb{E}[S_n] \geq x \right) \leq \exp\left\{ -\frac{2x^2}{\sum_{i=1}^{n} M_i^2} \right\}.$$  \hspace{1cm} (2.32)

First, it is easy to see that inequality (2.30) is a Gaussian bound, while Azuma-Hoeffding’s inequality does not share this feature. Second, inequality (2.30) is better than Azuma-Hoeffding’s inequality for all $x$ in the range

$$0 \leq x \leq \frac{3}{2^{3/2}M\sqrt{n}} \left( \frac{1}{4} \sum_{i=1}^{n} M_i^2 - \text{Var}(S_n) \right),$$

where $M = \max\{M_i : i = 1, \ldots, n\}$. To illustrate the last range, consider the case that $\text{Var}(S_n)$ is in order of $n$ and $M_i = i^\alpha$ for some $\alpha > 0$ and all $i \geq 1$. Our inequality (2.30) with $M = n^\alpha$ improves Azuma-Hoeffding’s inequality (2.32) for all

$$0 \leq x \leq \frac{3}{2^{3/2}n^{\alpha + 1/2}} \left( \frac{1}{4} \sum_{i=1}^{n} i^{2\alpha} - \text{Var}(S_n) \right).$$

Notice that for the right hand side of the last inequalities, it holds

$$\frac{1}{n^{\alpha + 1/2}} \left( \frac{1}{4} \sum_{i=1}^{n} i^{2\alpha} - \text{Var}(S_n) \right) \rightarrow \frac{1}{8\alpha + 4} n^{\alpha + 1/2}, \quad n \rightarrow \infty.
Recall that $\text{Var}(S_n)$ is in order of $n$. Thus inequality (1.5) improves Azuma-Hoeffding’s inequality (1.1) for all standard $x$ in a range $0 \leq x = O(n^{\alpha}), n \to \infty$. This range is quite large. Here and after, call $x$ is standard if we refer to the tail probabilities for the standardized sums, i.e. $P(S_n/\sqrt{\text{Var}(S_n)} \geq x)$.

**Proof of Theorem 2.4.** Let $S_n = \sum_{i=1}^{n} X_i$ be Doob’s martingale decomposition of $S_n$, where

$$X_i = E[S_n | \mathcal{F}_i] - E[S_n | \mathcal{F}_{i-1}].$$

Thus (2.29) implies that

$$T_{i-1} - E[S_n | \mathcal{F}_{i-1}] \leq X_i \leq T_{i-1} + M - E[S_n | \mathcal{F}_{i-1}]. \quad (2.33)$$

By the fact $E[X_i | \mathcal{F}_{i-1}] = 0$, it follows that

$$T_{i-1} - E[S_n | \mathcal{F}_{i-1}] \leq 0 \quad \text{and} \quad T_{i-1} + M - E[S_n | \mathcal{F}_{i-1}] \geq 0.$$

Thus we have

$$T_{i-1} + M - E[S_n | \mathcal{F}_{i-1}] \leq M \quad \text{and} \quad -M \leq T_{i-1} - E[S_n | \mathcal{F}_{i-1}].$$

Returning to (2.33), we get $|X_i| \leq M$. Then Theorem 2.4 follows from Theorem 2.3. \hfill \square

3. Concentration for functions of independent random variables

In this section we give some applications of our inequalities to functions of independent random variables. Let $(\xi_i)_{i=1,...,n}$ be a sequence of independent random variables with values in some complete separable metric space $\mathcal{X}$. Let $f$ be a function from $\mathcal{X}^n$ to $\mathbb{R}$. We are interested in the concentration for the random function $f(\xi_1,...,\xi_n)$.

3.1. McDiarmid type inequality

Assume that $(\mathcal{X}, d_i), i = 1,...,n$, are separable metric spaces with positive finite diameters $M_i$, i.e. $d_i(\cdot,\cdot) \leq M_i$, where $d_i$ are some distances on $\mathcal{X}$. Let $f$ be a separately Lipschitz function such that

$$|f(x_1,...,x_n) - f(y_1,...,y_n)| \leq d_1(x_1, y_1) + ... + d_n(x_n, y_n). \quad (3.34)$$

Set

$$Z_n = f(\xi_1,...,\xi_n). \quad (3.35)$$

McDiarmid [24] has obtained the following concentration inequality: for any positive $t$,

$$P\left(Z_n - E[Z_n] \geq t\sqrt{n}\right) \leq \exp\left\{-\frac{2nt^2}{T_n^2}\right\}, \quad (3.36)$$
where
\[ T_n^2 = \sum_{i=1}^{n} M_i^2. \]

He also showed that \( \text{Var}(Z_n) \leq \frac{1}{4} T_n^2 \). McDiarmid’s inequality can be regarded as a generalization of Azuma-Hoeffding’s inequality in the functional setting. Recently, Rio [30] has obtained the following improvement on McDiarmid’s inequality: for any \( t \in [0, 1] \),
\[
P( Z_n - \mathbb{E}[Z_n] \geq t D_n ) \leq (1 - t)^{(2-t)D_n^2/T_n^2},
\]
where
\[ D_n = \sum_{i=1}^{n} M_i. \]

As pointed out by Rio, his bound (3.37) is less than McDiarmid’s bound (3.36). Moreover, Rio’s bound has an interesting feature, that is \( P(Z_n - \mathbb{E}[Z_n] \geq t) = 0 \) when \( t > D_n \), which coincides with the property \( Z_n - \mathbb{E}[Z_n] \leq D_n \). Rio’s inequality (3.37) can be rewritten in the following form: for any \( t \in [0, D_n/\sqrt{n}] \),
\[
P( Z_n - \mathbb{E}[Z_n] \geq t \sqrt{n} ) \leq \exp \left\{ -\frac{t^2}{1 + \frac{1}{2} \sqrt{1 + 2^{5/2} M_t/\sigma + \frac{3}{2} M_t/\sigma}} \right\} \quad (3.38)
\]
Notice that for all \( 0 \leq t = o(D_n/\sqrt{n}) \), Rio’s bound (3.38) is of type \( \exp \left\{ -\frac{2nt^2}{T_n^2} (1 + o(1)) \right\} \), which is similar to McDiarmid’s bound (3.36).

In the following theorem, we extend Theorem 2.3 to the functional setting.

**Theorem 3.1.** Let \( Z_n \) be defined by (3.35). Denote
\[ \sigma^2 = \frac{1}{n} \text{Var}(Z_n) \quad \text{and} \quad M = \max\{M_i : 1 \leq i \leq n\}. \]
Then, for any positive \( t \),
\[
P( Z_n - \mathbb{E}[Z_n] \geq t \sqrt{\text{Var}(Z_n)} ) \leq \exp \left\{ -\frac{t^2}{1 + \sqrt{1 + 2^{5/2} M_t/\sigma + \frac{3}{2} M_t/\sigma}} \right\} \quad (3.39)
\]
\[
\leq \exp \left\{ -\frac{t^2}{2(1 + 2^{5/2} M_t/\sigma)} \right\} \quad (3.40)
\]
Moreover, the same inequalities hold when replacing \( Z_n \) by \(-Z_n\).

**Proof.** Denote \( F_i = \sigma\{\xi_j, 1 \leq j \leq i\} \). Let \( Z_n - \mathbb{E}[Z_n] = S_n \) be Doob’s martingale decomposition of \( Z_n \), where \( X_i = \mathbb{E}[Z_n|F_i] - \mathbb{E}[Z_n|F_{i-1}] \). Let \( (\xi'_j)_{i=1,...,n} \) be an independent copy of the random variables \( (\xi_i)_{i=1,...,n} \). Then it is easy to see that
\[
X_i = \mathbb{E}[f(\xi_1, ..., \xi_i, \xi'_i+1, ..., \xi_n) - f(\xi_1, ..., \xi_{i-1}, \xi'_i, ..., \xi_n)|F_i] \\
\leq \mathbb{E}[d_i(\xi_i, \xi'_i)|F_i] \\
\leq M. \quad (3.41)
\]
Similarly, we have $X_i \geq -M$. Thus $|X_i| \leq M$. Then the inequalities (3.39) and (3.40) follow by Theorem 2.3. This completes the proof of theorem. □

Comparing to the inequalities of McDiarmid (3.36) and Rio (3.38), our inequalities (3.39) and (3.40) have the following two interesting features. The first feature is that for $0 \leq t = o(1)$, our bounds have the exponential Gaussian form $\exp \{ -t^2/2 \}$, in contrast to the bounds of McDiarmid and Rio which do not share this property. The second feature is that our inequalities are better than (3.36) and (3.38) in the range $0 \leq t = O(n^{\alpha}), n \to \infty$. Notice that for all $0 \leq t = o(D_n/\sqrt{n})$, Rio’s bound (3.38) is similar to McDiarmid’s bound.

Notice that (3.40) is better than McDiarmid’s inequality for moderate deviations $t$, and that McDiarmid’s inequality is better than (3.40) for $t$ large enough. Therefore, it is natural to minimize the two bounds. Then we derive the following corollary.

**Corollary 3.1.** Assume conditions of Theorem 3.1. Then, for any positive $t$,

$$
\Pr \left( Z_n - \mathbb{E}[Z_n] \geq t \sqrt{n} \right) \leq \exp \left\{ -\frac{t^2}{2 \min \{ \sigma^2 + \frac{2^{3/2}}{3} Mt, \frac{1}{4n} T_n^2 \}} \right\}. \tag{3.42}
$$

Moreover, the same inequality holds when replacing $Z_n$ by $-Z_n$.

**Remark 3.1.** For the sake of simplicity we restrict ourselves to the slightly weaker bound (3.42), although (3.42) can be improved by minimizing Rio’s bound (3.37) and the bound (3.39).

In many applications, we would like to obtain information about the variance $\text{Var}(Z_n)$. Thus we collect the following three estimations of $\text{Var}(Z_n)$, where the last two estimations of $\text{Var}(Z_n)$ can be found in Boucheron, Lugosi and Bousquet [7].

1. Let $(ξ'_i)_{i=1,...,n}$ be an independent copy of the random variables $(ξ_i)_{i=1,...,n}$. Write

$$
Z'_i = f(ξ_1, ..., ξ_{i-1}, ξ_i, ξ_{i+1}, ..., ξ_n).
$$

The Efron-Stein inequality (cf. Efron-Stein [15] and Steele [31]) states that

$$
\text{Var}(Z_n) \leq \frac{1}{2} \sum_{i=1}^n \mathbb{E}[(Z_n - Z'_i)^2]. \tag{3.43}
$$

In particular, under condition (3.34), it implies that $\text{Var}(Z_n) \leq \frac{1}{2} \sum_{i=1}^n \mathbb{E}[d_i^2(ξ_i, ξ'_i)].$

2. Write $E_i$ for the expected value without respect to the variable $ξ_i$, that is

$$
E_i = \mathbb{E}[\cdot | ξ_1, ..., ξ_{i-1}, ξ_{i+1}, ..., ξ_n].
$$

Then

$$
\text{Var}(Z_n) \leq \sum_{i=1}^n \mathbb{E}[(Z_n - E_i[Z_n])^2].
$$
3. Write
\[ \tilde{Z}_i = f_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n) \]
for arbitrary measurable function \( f_i : \mathcal{X}^{n-1} \to \mathbb{R} \) of \( n - 1 \) variables. Then
\[ \text{Var}(Z_n) \leq \sum_{i=1}^{n} \mathbb{E}[(Z_n - \tilde{Z}_i)^2]. \] (3.44)

3.2. Concentration for self-bounding functions

Let \( \xi \in \mathcal{X}^n \). Denote \( \xi^{(i)} = (\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n) \in \mathcal{X}^{n-1} \), obtained by dropping the \( i \)-th component of \( \xi \). For each \( i \leq n \), denote by \( f_i \) a measurable function from \( \mathcal{X}^{n-1} \) to \( \mathbb{R} \). Set
\[ Z = f(\xi_1, \ldots, \xi_n). \] (3.45)

We have the following concentration inequalities for \( Z \) around its expected value.

**Theorem 3.2.** Define \( Z \) by (3.45). Assume that
\[ 0 \leq f(\xi) - f_i(\xi^{(i)}) \leq 1 \] (3.46)
for all \( i = 1, \ldots, n \) and all \( \xi \in \mathcal{X}^n \). Denote \( \sigma^2 = \frac{1}{n} \text{Var}(Z) \). Then, for any positive \( t \),
\[ \mathbb{P}\left( Z - \mathbb{E}[Z] \geq t\sqrt{\text{Var}(Z)} \right) \leq \exp\left\{ -\frac{t^2}{1 + \sqrt{1 + \frac{2^{5/2}}{3} t/\sigma + \frac{2^{3/2}}{3} t/\sigma}} \right\} \] (3.47)
\[ \leq \exp\left\{ -\frac{t^2}{2(1 + \frac{2^{3/2}}{3} t/\sigma)} \right\}. \] (3.48)

Moreover, the same inequalities hold when replacing \( Z \) by \( -Z \).

A function \( f \) is called \((a, b)\)-self-bounding, introduced by McDiarmid and Reed [26], if condition (3.46) holds and, moreover, for some \( a > 0, b \geq 0 \) and all \( \xi \in \mathcal{X}^n \),
\[ \sum_{i=1}^{n} \left( f(\xi) - f_i(\xi^{(i)}) \right) \leq af(\xi) + b. \] (3.49)

In particular, \((1, 0)\)-self-bounding function is known as self-bounding function; see Boucheron, Lugosi and Massart [8]. For any \((a, b)\)-self-bounding function \( f \), McDiarmid and Reed [26] proved that, for any positive \( t \),
\[ \mathbb{P}\left( Z - \mathbb{E}[Z] \geq t \right) \leq \exp\left\{ -\frac{t^2}{2(a\mathbb{E}[Z] + b + ct)} \right\}, \] (3.50)
where \( c = a \) if \( Z = f(\xi_1, \ldots, \xi_n) \), and \( c = 1/3 \) if \( Z = -f(\xi_1, \ldots, \xi_n) \). See also Boucheron, Lugosi and Massart [8] for the self-bounding functions.
It is worth noting that (3.48) does not assume condition (3.49). Hence, our inequality (3.48) extends the inequality of McDiarmid and Reed [26]. Moreover, for \((a, b)\)-self-bounding function, it holds \(\text{Var}(Z) \leq a \mathbb{E}[Z] + b\). Indeed, by (3.44), (3.46) and (3.49), it is easy to see that

\[
\text{Var}(Z) \leq \mathbb{E} \left[ \sum_{i=1}^{n} (Z - \tilde{Z}_i)^2 \right] \leq \mathbb{E} \left[ \sum_{i=1}^{n} (Z - \tilde{Z}_i) \right] \leq \mathbb{E}[aZ + b],
\]

where \(\tilde{Z}_i = f_i(\xi^{(i)})\). Thus, our bound (3.48) is less than the bound of McDiarmid and Reed (3.50) for standard \(0 \leq t = O\left(\sqrt{\frac{n \text{Var}(Z)}{\text{Var}(Z)}} \right)\) as \(n \to \infty\). The last range is large provided that \(n/\text{Var}(Z) \to \infty\).

**Proof of Theorem 3.2.** Let \((\mathcal{F}_i)_{i=1,\ldots,n}\) be the natural filtration of the random variables \((\xi_i)_{i=1,\ldots,n}\). Denote \(Z = f(\xi)\), where \(\xi = (\xi_1, \ldots, \xi_n)\). Let \(Z - \mathbb{E}[Z] = S_n\) be Doob’s martingale decomposition of \(Z\), where \(X_i = \mathbb{E}[Z|\mathcal{F}_i] - \mathbb{E}[Z|\mathcal{F}_{i-1}]\). By condition (3.46), it is easy to see that

\[
X_i \leq \mathbb{E}[1 + f_i(\xi^{(i)})|\mathcal{F}_i] - \mathbb{E}[f(\xi)|\mathcal{F}_{i-1}]
\]

\[
= \mathbb{E}[1 + f_i(\xi^{(i)}) - f(\xi)|\mathcal{F}_{i-1}]
\]

\[
\leq 1.
\]

Similarly, we have

\[
X_i \geq \mathbb{E}[f_i(\xi^{(i)})|\mathcal{F}_i] - \mathbb{E}[f(\xi)|\mathcal{F}_{i-1}]
\]

\[
= \mathbb{E}[f_i(\xi^{(i)}) - f(\xi)|\mathcal{F}_{i-1}]
\]

\[
\geq -1.
\]

Thus \(|X_i| \leq 1\). Then the inequalities (3.47) and (3.48) follow from Theorem 2.3. When \(Z = -f(\xi)\), the proof is similar to the case of \(Z = f(\xi)\). \(\square\)

3.3. Vapnik-Chervonenkis entropies

Let \(\mathcal{A}\) be an arbitrary collection of subsets of \(\mathcal{X}\), and let \(x = (x_1, \ldots, x_n)\) be a vector of \(n\) points of \(\mathcal{X}\). Define the trace of \(\mathcal{A}\) on \(x\) by

\[
\text{tr}(x) = \{A \cap \{x_1, \ldots, x_n\} : A \in \mathcal{A}\}.
\]

The *shatter coefficient* of \(\mathcal{A}\) in \(x\) is defined by \(T(x) = |\text{tr}(x)|\), namely the size of the trace. \(T(x)\) is the number of different subsets of the \(n\)-point set \(\{x_1, \ldots, x_n\}\) generated by intersecting it with elements of \(\mathcal{A}\). The Vapnik-Chervonenkis entropy (VC entropy) is defined as

\[
H(x) = \log_2 T(x),
\]

with the convention that \(\log_2 0 = -1\). Note that

\[
0 \leq H(x) - H(x^{(i)}) \leq 1.
\]
The VC entropy $H(x)$ is of particular interest, as it plays a key role in some applications in pattern recognition and machine learning; see Devroye, Györfi and Lugosi [13] and Vapnik [35]. Denote the random VC entropy by

$$H = H(\xi_1, \ldots, \xi_n).$$

Boucheron, Lugosi and Massart [5] have obtained the following concentration inequalities: for any positive $t$,

$$P \left( H - E[H] \geq t \right) \leq \exp \left\{ -\frac{t^2}{2(E[H] + \frac{1}{3}t)} \right\}$$

(3.53)

and

$$P \left( H - E[H] \leq -t \right) \leq \exp \left\{ -\frac{t^2}{2E[H]} \right\}.$$ (3.54)

Moreover, they also proved that, for any $x \in X^n$,

$$\sum_{i=1}^{n} \left( H(x) - H(x^{(i)}) \right) \leq H(x);$$

see Lemma 1 of [5]. The last inequality and (3.52) together implies that $H$ is self-bounding.

Using (3.48) and (3.51), we have the following result similar to (3.53) and (3.54).

**Theorem 3.3.** The random VC entropy satisfies

$$\text{Var}(H) \leq E[H],$$

(3.55)

and for any positive $t$,

$$P \left( H - E[H] \geq t \right) \leq \exp \left\{ -\frac{t^2}{2(\text{Var}(H) + \frac{23/2}{3}t\sqrt{n})} \right\}.$$ (3.56)

Moreover, the last inequality holds when replacing $H$ by $-H$.

By a simple calculation, it is easy to see that (3.56) improves the inequalities (3.53) and (3.54) for all $t$ in a range $0 \leq t = O\left(\frac{1}{\sqrt{n}}(E[H] - \text{Var}(H))\right)$ as $n \to \infty$.

### 3.4. Rademacher averages

As another application of Theorem 3.2, we consider Rademacher averages which play an important role in the theory of probability in Banach spaces; see Ledoux and Talagrand [21]. Let $B$ denote a separable Banach space, and let $X_1, \ldots, X_n$ be independent and identically distributed...
bounded $B$–valued random variables. Without loss of generality, assume that $\|X_1\| \leq 1$ almost surely. The quantity of interest is the conditional Rademacher average

$$R = \mathbb{E} \left[ \left\| \sum_{i=1}^{n} \varepsilon_i X_i \right\| X_1, ..., X_n \right],$$

where the $\varepsilon_i$ are independent Rademacher random variables.

Boucheron, Lugosi and Massart [6] proved that the inequalities (3.53) and (3.54) hold when replacing $H$ by $R$. (cf. Theorem 16). They also proved that $f(x) = \mathbb{E}[\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|]$ is self-bounding. Using (3.51) and (3.48) again, we obtain the following concentration inequalities for $R$, which refines the inequalities of Boucheron, Lugosi and Massart [6] for all $t$ in the range $0 \leq t = O\left(\frac{1}{\sqrt{n}}(\mathbb{E}[R] - \text{Var}(R))\right)$.

**Theorem 3.4.** The conditional Rademacher average $R$ satisfies

$$\text{Var}(R) \leq \mathbb{E}[R]$$

and, for any positive $t$,

$$\mathbb{P}\left( R - \mathbb{E}[R] \geq t \right) \leq \exp \left\{ - \frac{t^2}{2(\mathbb{E}[R] + \frac{23/2}{3} t \sqrt{n})} \right\}. \quad (3.58)$$

Moreover, the last inequality holds when replacing $R$ by $-R$.

4. Counting small subgraphs in random graphs

Consider the Erdős-Rényi $G(n, p)$ model of a random graph. Such a graph has $n$ vertices and for each pair $(u, v)$ of vertices an edge is inserted between $u$ and $v$ with probability $p$, independently. We write $m = \binom{n}{2}$, and denote the indicator variables of the $m$ edges by $Y_1, ..., Y_m$.

In this section we consider the number of triangles in a random graph. A triangle is a set of three edges defined by vertices $u, v, w$ such that the edges are of the form $\{u, v\}, \{v, w\}$ and $\{w, u\}$. Let $Z$ denote the number of triangles in a random graph. Thus $Z$ can be expressed in the following form

$$Z = \sum_{(i,j,k) \in \ell} Y_i Y_j Y_k,$$

where $\ell$ contains all triples of edges which form a triangle. Note that

$$\mathbb{E}[Z] = \binom{n}{3} p^3 \sim \frac{n^3 p^3}{6}$$

and

$$\text{Var}(Z) = \binom{n}{3} (p^3 - p^6) + 2 \binom{n}{4} \binom{4}{2} (p^5 - p^6) \sim \frac{n^3}{6} (p^3 - p^6) + \frac{n^4}{2} (p^5 - p^6)$$

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as $n \to \infty$. Boucheron, Lugosi and Massart [6] offered the following exponential inequality for the upper tail probabilities of $Z$. Let $K > 1$. Then, for all $0 \leq t \leq (K^2 - 1)\mathbb{E}[Z]$,

$$
\mathbb{P}\left(\left| Z - \mathbb{E}[Z] \right| \geq t \right) \leq \exp \left\{ - \frac{t^2}{\left( K^2 + 1 \right)^2 \mathbb{E}[Z] (24np^2 + 24 \log n + 14t/(\sqrt{\mathbb{E}[Z]}))} \right\} \vee \left\{ - \frac{t^2}{12n\mathbb{E}[Z] + 6nt} \right\}.
$$

We offer the following exponential inequality for tail probabilities of $Z$.

**Theorem 4.1.** Let $Z$ denote the number of triangles in the random Erdős-Rényi graph. Then, for any positive $t$,

$$
\mathbb{P}\left(\left| Z - \mathbb{E}[Z] \right| \geq t \right) \leq \exp \left\{ - \frac{t^2}{2(\text{Var}(Z) + \frac{23/2}{3}(n-2)\sqrt{n}t)} \right\}.
$$

Moreover, the same inequality holds when replacing $Z$ by $-Z$.

**Proof.** Denote $Z = Z(Y_1, ..., Y_n)$. It is easy to see that

$$Z(Y_1, ..., Y_k, ..., Y_n) - Z(Y_1, ..., Y'_k, ..., Y_n) = \sum_{(i,j,k) \in \ell} Y_i Y_j (Y_k - Y'_k).$$

Denote by $Y_k$ the edge of the form $\{u, v\}$. Then $w$ belongs to the rest $n-2$ vertices, and $\sum_{(i,j,k) \in \ell} Y_i Y_j (Y_k - Y'_k)$ contains $n-2$ summands. Since $|Y_i Y_j (Y_k - Y'_k)| \leq 1$, we deduce that

$$\sup_k \left| Z(Y_1, ..., Y_k, ..., Y_n) - Z(Y_1, ..., Y'_k, ..., Y_n) \right| \leq n - 2.$$

Applying Theorem 3.1 to $Z$, we obtain (4.60).

To understand inequality (4.60), we summarize some of its consequences for different choices of $t$ and for different ranges of the parameter $p$. For different ranges of $t$, we obtain the following bounds: For all $0 \leq t = o(n^{3/2}p^3 + n^{5/2}p^5)$,

$$
\mathbb{P}\left( Z - \mathbb{E}[Z] \geq t \right) \leq \exp \left\{ - \frac{t^2}{2\text{Var}(Z)} \right\}.
$$

This is the “Gaussian” range. Notice that the denominator coincides with the variance. For all $n^{3/2}p^3 + n^{5/2}p^5 \leq t$,

$$
\mathbb{P}\left( Z - \mathbb{E}[Z] \geq t \right) \leq \exp \left\{ - \frac{3t^{3/2}}{2n^{7/2}(n-2)\sqrt{n}} \right\}.
$$

Comparing to inequality (4.59), our inequality has the following three features: First, our inequality holds for all $t \geq 0$ instead of the range $0 \leq t \leq (K^2 - 1)\mathbb{E}[Z]$. Second, our inequality (4.62) gives a sharper bound. Indeed, the bound (4.62) behaviors as $\exp \left\{ - t^2/(n^4p^5 + 3n^3p^3) \right\}$, while the bound (4.59) looks as $\exp \left\{ - t^2/(4(K^2 + 1)^2n^4p^5 + 2n^4p^3) \right\}$. Thus the bound (4.62) is less than the bound (4.59) for all $0 \leq t = o(n^{3/2}p^3 + n^{5/2}p^5)$. Third, we give an upper bound for $-Z$, that is an upper bound on the tail probabilities $\mathbb{P}(Z - \mathbb{E}[Z] \leq -t)$ for all $t > 0$, while inequality (4.59) usually does not provide a similar inequality for $-Z$.
5. Concentration for maxima of empirical processes

5.1. Talagrand’s inequality

Talagrand (cf. Theorem 1.4 of [32]) gave the following concentration inequalities for the maxima of empirical processes.

**Theorem 5.1.** Let \((\xi_i)_{i=1,...,n}\) be a sequence of independent random variables with values in a measurable space \(\mathcal{X}\). Consider a countable class \(\mathcal{F}\) of measurable functions on \(\mathcal{X}\). Consider the random variable

\[ Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(\xi_i). \]

Denote

\[ U = \sup_{f \in \mathcal{F}} \|f(\xi_i)\|_{\infty} \quad \text{and} \quad V = \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f^2(\xi_i) \right]. \]

Then, for any positive \(t\),

\[ \mathbb{P} \left( Z - \mathbb{E}[Z] \geq t \right) \leq K \exp \left\{ - \frac{t^2}{2(c_1 V + c_2 Ut)} \right\}, \tag{5.64} \]

where \(K, c_1\) and \(c_2\) are positive absolute constants. Moreover, the same inequality holds when replacing \(Z\) by \(-Z\).

Talagrand’s proof of Theorem 5.1 is rather intricate and does not lead to very attractive values for the constants \(K, c_1\) and \(c_2\). One year after Talagrand’s work [32], Ledoux [22] developed a new and much simpler method for establishing similar concentration inequalities of Talagrand. Ledoux’s method is known as “entropy method”. His method allows one to obtain explicit constants. In particular, Ledoux [22] showed that for an adequate positive constant \(C\), replacing \(V\) by \(V + C \mathbb{E}[Z]\), inequality (5.64) holds with \(K = 2, c_1 = 42\) and \(c_2 = 8\). Notice that, by a remark of Massart [23], Ledoux’s inequalities did not recover exactly Talagrand’s inequality (5.64) due to the difference of \(V\). Moreover, he did not provide the same bound for \(-Z\) as Talagrand’s inequalities in general. The entropy method has become very popular in recent years. Many interesting concentration inequalities have been established via this method; see Bobkov and Ledoux [4], Massart [23], Rio [27, 28], Bousquet [9], Boucheron et al. [6], Klein and Rio [20]. Similar to Ledoux’s inequality, such type concentration inequalities also do not hold for \(Z\) replacing by \(-Z\) in general.

5.2. Talagrand type concentration inequalities

In this section, we would like to give some concentration inequalities for the maxima of empirical processes. Our inequalities are similar to Talagrand’s inequality (5.64), and they hold also when replacing \(Z\) by \(-Z\). Moreover, our inequalities are Gaussian bounds. To the best of our knowledge, such type bounds for the maxima of empirical processes have not been obtained before.
Theorem 5.2. Let $(\xi_i)_{i=1,...,n}$ be a sequence of independent random variables with values in a measurable space $\mathcal{X}$. Consider a countable class $\mathcal{F}$ of measurable functions on $\mathcal{X}$. Assume that

$$U_i = \sup_{f \in \mathcal{F}} ||f(\xi_i)||_{\infty} < \infty.$$ 

Let $Z$ denote as one of the following formulas

$$\sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(\xi_i), \quad \sup_{f \in \mathcal{F}} |\sum_{i=1}^{n} f(\xi_i)|,$$

$$\sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(\xi_i) - \mathbb{E}[f(\xi_i)], \quad \text{and} \quad \sup_{f \in \mathcal{F}} |\sum_{i=1}^{n} f(\xi_i) - \mathbb{E}[f(\xi_i)]|.$$ 

Denote

$$U = \max\{U_i : i = 1, ..., n\} \quad \text{and} \quad \sigma^2 = \frac{1}{n} \mathbb{V}(Z).$$

Then, for any positive $t$,

$$\mathbb{P}\left(Z - \mathbb{E}[Z] \geq t\sqrt{\mathbb{V}(Z)}\right) \leq \exp\left\{- \frac{t^2}{1 + \sqrt{1 + \frac{2^{7/2}}{3} U t/\sigma + \frac{25/2}{3} U t/\sigma}} \right\}$$

(5.65)

$$\leq \exp\left\{- \frac{t^2}{2(1 + \frac{5^{7/2}}{3} U t/\sigma)} \right\}. \quad (5.66)$$

Moreover, the same inequalities hold when replacing $Z$ by $-Z$.

Proof. Denote by $Z = Z_n(\xi_1, ..., \xi_n)$. When $Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(\xi_i)$, by the fact $U_i = \sup_{f \in \mathcal{F}} ||f(\xi_i)||_{\infty}$, it is easy to see that

$$Z_n(\xi_1, ..., \xi_k, ..., \xi_n) - Z_n(\xi_1, ..., \xi'_k, ..., \xi_n) \geq Z - \sup_{f \in \mathcal{F}} \left( \sum_{i=1}^{n} f(\xi_i) + f(\xi'_k) - f(\xi_k) \right)$$

$$\geq Z - Z - \sup_{f \in \mathcal{F}} \left( f(\xi'_k) - f(\xi_k) \right)$$

$$\geq -2 U_k,$$

where $(\xi'_k)_{k=1,...,n}$ is an independent copy of $(\xi_k)_{k=1,...,n}$. The last inequality holds also for

$$Z_n(\xi_1, ..., \xi'_k, ..., \xi_n) - Z_n(\xi_1, ..., \xi_k, ..., \xi_n).$$

Thus

$$\sup_{\xi_1, ..., \xi_n} \left| Z_n(\xi_1, ..., \xi_k, ..., \xi_n) - Z_n(\xi_1, ..., \xi'_k, ..., \xi_n) \right| \leq 2 U_k. \quad (5.67)$$
Applying Theorem 2.3 to $Z$, we obtain (5.65) and (5.66) for $Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(\xi_i)$. When $Z = \sup_{f \in \mathcal{F}} |\sum_{i=1}^{n} f(\xi_i)|$, by the fact $\sup_{f \in \mathcal{F}} \|f(\xi_k) - f(\xi_k)\|_\infty \leq 2U_k$, again, the inequalities (5.65) and (5.66) also follow by a similar argument. When $Z$ denotes either

$$\sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(\xi_i) - \mathbb{E}[f(\xi_i)] \quad \text{or} \quad \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} f(\xi_i) - \mathbb{E}[f(\xi_i)] \right|,$$

by the fact

$$\sup_{f \in \mathcal{F}} \left| f(\xi_i) - \mathbb{E}[f(\xi_i)] - f(\xi_i') + \mathbb{E}[f(\xi_i')] \right| = \sup_{f \in \mathcal{F}} \left| f(\xi_i) - f(\xi_i') \right|_\infty \leq 2U_i,$$

inequality (5.67) holds true. Then the inequalities (5.65) and (5.66) follow by a similar argument again. \qed

**Remark 5.1.** *Inspiring the proof of Theorem 5.2, it is easy to see that if denote $Z$ as one of the following formulas

$$\sup_{(f_1, \ldots, f_n) \in \mathcal{F}^n} \sum_{i=1}^{n} f_i(\xi_i), \quad \sup_{(f_1, \ldots, f_n) \in \mathcal{F}^n} \left| \sum_{i=1}^{n} f_i(\xi_i) \right|, \quad \sup_{(f_1, \ldots, f_n) \in \mathcal{F}^n} \sum_{i=1}^{n} f_i(\xi_i) - \mathbb{E}[f_i(\xi_i)] \quad \text{and} \quad \sup_{(f_1, \ldots, f_n) \in \mathcal{F}^n} \left| \sum_{i=1}^{n} f_i(\xi_i) - \mathbb{E}[f_i(\xi_i)] \right|,$$

then the inequalities (5.65) and (5.66) also hold true.*

Notice that (5.66) can be rewritten in the following form: for any positive $x$,

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq x) \leq \exp \left\{ -\frac{x^2}{2(\mathbb{V}(Z) + \frac{2^{5/2}}{3}Ux \sqrt{n})} \right\}. \quad (5.68)$$

The last inequality shows that we can take $K = 1, c_1 = 1, V = \mathbb{V}(Z)$ and $c_2 = \frac{2^{5/2}}{3} \sqrt{n}$ in Talagrand’s inequality (5.64). This provides a partial positive answer to Massart’s question [23] about the best constants in Talagrand’s inequality. Moreover, our bound (5.68) is less than Talagrand’s bound (5.64) for $x$ in the range

$$0 \leq x = O \left( \frac{1}{U \sqrt{n}} \left( \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f^2(\xi_i) \right] - \mathbb{V}(Z) \right) \right).$$

When $\mathbb{V}(Z)/\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f^2(\xi_i) \right] \to 0$ as $n \to \infty$, this range is large. To illustrate it, consider the case that $\mathbb{V}(Z)$ is in order of $n^\alpha, 0 < \alpha < 1$, and $\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f^2(\xi_i) \right]$ is in order of $n$ as $n \to \infty$, then our inequalities is significantly better than Talagrand’s inequality.
for standard $t$ in the range $0 \leq t = o(n^{(1-\alpha)/2})$. Recall that a $t$ is called standard if we refer to the tail probabilities $P\left(\frac{(Z - E[Z])}{\sqrt{\text{Var}(Z)}} \geq t\right)$.

Here we would like to give some comparisons between our results and the inequalities of Klein and Rio [20]. Assume that $E[f(\xi_i)] = 0$ for all $i \in [1, n]$ and all $f \in F$. Klein and Rio [20] have obtained the following inequality. Let $Z = \sup_{f \in F} \sum_{i=1}^{n} f(\xi_i)$ and let $U = 1$. Then, for any positive $x$,

$$P\left(Z - E[Z] \geq x\right) \leq \exp\left\{-\frac{x^2}{2v + 3x}\right\}$$

and

$$P\left(Z - E[Z] \leq -x\right) \leq \exp\left\{-\frac{x^2}{2v + 2x}\right\},$$

where

$$v = \sup_{f \in F} \text{Var}\left(\sum_{i=1}^{n} f(\xi_i)\right) + 2E[Z]$$

and

$$\sup_{f \in F} \text{Var}\left(\sum_{i=1}^{n} f(\xi_i)\right) \leq E\left[\sup_{f \in F} \sum_{i=1}^{n} f^2(\xi_i)\right].$$

Moreover, Klein and Rio (cf. Corollary 1.1 of [20]) have pointed out that

$$\text{Var}(Z) \leq v.$$  \hspace{1cm} (5.71)

Comparing to the results of Klein and Rio [20], our inequalities have the following three interesting features. First, our inequalities (5.65) and (5.66) do not need the assumption that $(\xi_i)_{i=1,\ldots,n}$ are centered with respect to $f$, i.e. $E[f(\xi_i)] = 0$ for all $i \in [1, n]$ and all $f \in F$. Second, it is easy to see that our inequalities (5.65) and (5.66) are Gaussian type bounds, while the inequality of Klein and Rio [20] does not share this feature due to the fact $\text{Var}(Z) \leq v$. Third, by (5.68), we find that our bound is better than Klein-Rio’s bound for $x$ in the range $0 \leq x = \frac{3}{2n^2\sqrt{n}}(v - \text{Var}(Z))$. For instance, if $v$ is in order of $n$ and $\text{Var}(Z)$ is in order of $n^\alpha$, $0 < \alpha < 1$, as $n \rightarrow \infty$, then for standard $t$ in the range $0 \leq t = o(n^{(1-\alpha)/2})$ our bound is better than Klein-Rio’s.

Inequality (5.66) can also be rewritten in the following form: for any positive $t$,

$$P\left(Z - E[Z] \geq \sqrt{n}(\sqrt{c^2U^2t^2 + 2t\sigma^2} + cUt)\right) \leq \exp\left\{-t\right\},$$

where $c = 2^{5/2}/3$. Since

$$\sqrt{c^2U^2t^2 + 2t\sigma^2} \leq \sigma\sqrt{2t} + cUt,$$

inequality (5.72) implies the following bound: for any positive $t$,

$$P\left(Z - E[Z] \geq \sqrt{n}(\sigma\sqrt{2t} + 2cUt)\right) \leq \exp\left\{-t\right\}.$$
Such type bound can be found in Theorem 3 of Massart [23]. When $Z$ denotes either

$$\sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} f(\xi_i) \right| \quad \text{or} \quad \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} f(\xi_i) - \mathbb{E}[f(\xi_i)] \right|,$$

Massart has established the following inequality: for positive $t$ and a positive $\varepsilon$,

$$\mathbb{P}\left(Z - \mathbb{E}[Z] \geq \varepsilon \mathbb{E}[Z] + \rho \sqrt{2 \kappa t} + \kappa(\varepsilon) Ut\right) \leq \exp\left\{-t\right\},$$

where $\rho^2 = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \text{Var}(f(\xi_i)), \kappa = 4$ and $\kappa(\varepsilon) = 2.5 + 32 \varepsilon^{-1}$.

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