Quillen cohomology of enriched operads

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Abstract

A modern insight due to Quillen, which is further developed by Lurie, asserts that many cohomology theories of interest are particular cases of a single construction, which allows one to define cohomology groups in an abstract setting using only intrinsic properties of the category (or \(\infty\)-category) at hand. This universal cohomology theory is known as Quillen cohomology. This paper is devoted to the study of Quillen cohomology of enriched operads. Our main result provides an explicit formula for computing Quillen cohomology of enriched operads, based on a procedure of taking certain infinitesimal models of their cotangent complexes. Inspired by an idea of Lurie, we propose the construction of twisted arrow \(\infty\)-categories of simplicial operads. We then show that the cotangent complex of a simplicial operad can be represented as a spectrum valued functor on its twisted arrow \(\infty\)-category. As an illustration, we prove that Quillen cohomology of any little cubes operad, with certain coefficients, vanishes.

Keywords and phrases: operadic tangent category, operadic infinitesimal bimodule, cotangent complexes of enriched operads, twisted arrow \(\infty\)-category, simplicial operad.

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1. Introduction

A widespread idea in the domain of homotopy theory is to study a given object of interest by associating to it various kinds of cohomology group. From generalized cohomology theories for spaces and various Ext groups in homological algebra, through group cohomology, sheaf cohomology, Hochschild cohomology and André-Quillen cohomology, such invariants vary from fairly useful to utterly indispensable. A modern insight due to Quillen [1], which is further developed by Lurie [2], asserts that all these cohomology theories are particular cases of a single universal construction, which allows one to define cohomology groups in an abstract setting using only intrinsic properties of the category (or ∞-category) at hand. This universal cohomology theory is known as Quillen cohomology.

In Quillen’s approach, cohomology of an object of interest is classified by its derived abelianization. Recall that an abelian group object in a given category \( \mathbf{M} \) which admits finite products and a terminal object \( *_{\mathbf{M}} \) is an object \( A \) equipped with two maps \( *_{\mathbf{M}} \rightarrow A \) and \( A \times A \rightarrow A \) satisfying the classical axioms of an abelian group. Suppose further that \( \mathbf{M} \) is endowed with a model category structure. For each object \( X \in \mathbf{M} \), consider the over category \( \mathbf{M}_{/X} \). The category of abelian group objects in \( \mathbf{M}_{/X} \), \( \text{Ab}(\mathbf{M}_{/X}) \), possibly inherits a model structure transferred from that of \( \mathbf{M} \). In this situation, the free-forgetful adjunction \( \mathcal{F} : \mathbf{M}_{/X} \xrightarrow{\simeq} \text{Ab}(\mathbf{M}_{/X}) : \mathcal{U} \) forms a Quillen adjunction. The cotangent complex of \( X \), denoted by \( L_X \), is then defined to be \( L_X := \mathcal{L}\mathcal{F}(X) \) the derived abelianization of \( X \). Given an object \( M \in \text{Ab}(\mathbf{M}_{/X}) \), the \( n \)’th Quillen cohomology group of \( X \) with coefficients in \( M \) is formulated as

\[
H^n_Q(X, M) = \pi_0 \text{Map}_{\text{Ab}(\mathbf{M}_{/X})}^h(\mathcal{U}\mathcal{F}(X), \Sigma^n M).
\]

Example 1.0.1. Let \( k \) be a commutative ring of characteristic 0 and let \( \mathcal{P} \) be an operad enriched over \( \text{dg} \ k \)-modules. For each \( \mathcal{P} \)-algebra \( A \), it was known that the free-forgetful adjunction \( (\text{Alg}_\mathcal{P})_{/A} \xrightarrow{\simeq} \text{Ab}((\text{Alg}_\mathcal{P})_{/A}) \) is canonically (homotopically) equivalent to the adjunction

\[
\Omega^\mathcal{A}(-) : (\text{Alg}_\mathcal{P})_{/A} \xrightarrow{\simeq} \text{Mod}_A^\mathcal{P} : A \otimes (-)
\]

in which \( \text{Mod}_A^\mathcal{P} \) refers to the category of \( A \)-modules over \( \mathcal{P} \), the left adjoint takes each object \( B \in (\text{Alg}_\mathcal{P})_{/A} \) to \( \Omega^\mathcal{A}(B) \) the module of Kähler differentials of \( B \) over \( A \) and the right adjoint takes each object \( M \in \text{Mod}_A^\mathcal{P} \) to \( A \otimes M \) the square-zero extension of \( A \) by \( M \). Therefore, after sending coefficients into \( \text{Mod}_A^\mathcal{P} \), the \( n \)’th Quillen cohomology group of \( A \) with coefficients in some \( M \in \text{Mod}_A^R \) is formulated as

\[
H^n_Q(A, M) = \pi_0 \text{Map}_{\text{Mod}_A^\mathcal{P}}^h(\Omega^A(A^\text{cof}), M[n])
\]

where \( A^\text{cof} \) is a cofibrant resolution of \( A \) in \( \text{Alg}_\mathcal{P} \) and \( M[n] \) refers to the \( n \)-suspension of \( M \) in \( \text{Mod}_A^\mathcal{P} \). Furthermore, the latter group is in fact isomorphic to the zeroth homology of the complex of \( A \)-derivations from \( A^\text{cof} \) to \( M[n] \). (See, e.g., [7, 5]).

The Quillen’s approach has certain limitations, despite its success. Indeed, there is not a known traditional criterion assuring the existence of the transferred model structure on abelian group objects in a given model category and moreover, even when realized, this model category structure is not invariant under Quillen equivalences.
Lurie [2] developed the work of Quillen to establish the cotangent complex formalism, in the \( \infty \)-categorical framework, by extending the notion of abelianization to that of **stabilization**, which itself is inspired by the classical theory of spectra. Let \( \mathcal{C} \) be a presentable \( \infty \)-category and let \( X \) be an object of \( \mathcal{C} \). Consider the over \( \infty \)-category \( \mathcal{C}/X \). Conceptually, the **stabilization of \( \mathcal{C}/X \)** is the \( \infty \)-categorical limit of the tower

\[
\cdots \xrightarrow{\Omega} (\mathcal{C}/X)_* \xrightarrow{\Omega} (\mathcal{C}/X)_* \xrightarrow{\Omega} (\mathcal{C}/X)_*,
\]

where \( \Omega \) refers to the desuspension functor on \( (\mathcal{C}/X)_* \), the pointed \( \infty \)-category associated to \( \mathcal{C}/X \). As in [2], we will refer to the stabilization of \( \mathcal{C}/X \) as the **tangent category to \( \mathcal{C} \) at \( X \)** and denote it by \( \mathcal{T}_X \mathcal{C} \). By construction, \( \mathcal{T}_X \mathcal{C} \) is automatically a **stable \( \infty \)-category**. Moreover, the presentability of \( \mathcal{C} \) implies that the canonical functor \( \mathcal{T}_X \mathcal{C} \to \mathcal{C}/X \) admits a left adjoint, the **suspension spectrum functor**, written as \( \Sigma : \mathcal{C}/X \to \mathcal{T}_X \mathcal{C} \). By this way, Lurie defined the **cotangent complex of \( X \)** to be \( L_X := \Sigma \infty^X(X) \). By having that notion of cotangent complex, the \( n \)'th Quillen cohomology group of \( X \) with coefficients in a given object \( M \in \mathcal{T}_X \mathcal{C} \) is now formulated as

\[
H^n_Q(X, M) = \pi_0 \text{Map}_{\mathcal{T}_X \mathcal{C}}(L_X, M[n]).
\]

We refer the readers to [5] for a discussion on the naturality of the evolution from Quillen’s approach to Lurie’s, and also a comparison between them.

For necessary computations in abstract homotopy theory, model category (or a bit more generally, **semi model category** (cf., e.g., [8, 9, 10]) ) seems to be the most favorable environment, as far as we know. Fortunately, the mentioned Lurie’s definitions were completely translated into the model categorical language, thanks to the recent works of Y. Harpaz, J. Nuiten and M. Prasma [4, 5]. Based on their works, we will give necessary concepts relevant to Quillen cohomology theory in Section 4. In particular, following the setting given in [4], tangent (model) categories come after a procedure of taking left Bousfield localizations of model categories of interest. The obstacle is that taking left Bousfield localization usually requires the left properness. The recent result of Batanin and White [11] allows one to take left Bousfield localizations, in the semi model categorical framework, without left properness. Inspired by this result, under our settings, tangent categories exist, but only as semi model categories, which are basically convenient as well as (full) model categories.

The main purpose of this paper is to formulate Quillen cohomology of operads enriched over a general symmetric monoidal model category, which we will refer to as the **base category**. Our work generalizes the study of Quillen cohomology of enriched categories given in [5]. Given a base category \( S \), we denote by \( \text{Op}_{S}(S) \) the category of **S-enriched \( C \)-colored operads** with \( C \) being some fixed set of colors, yet the one we really care about is the category of **S-enriched operads** (with non-fixed sets of colors), which will be denoted by \( \text{Op}(S) \). Under some suitable conditions, \( \text{Op}(S) \) admits the **canonical model structure**, according to Caviglia’s [12]. In particular, when \( S \) is the category of simplicial sets, \( \text{Set}_\Delta \), equipped with the standard (Kan-Quillen) model structure, the canonical model structure on \( \text{Op}(\text{Set}_\Delta) \) agrees with the Cisinski-Moerdijk model structure, which was known to be a model for the theory of \( \infty \)-operads (cf. [13]).

Given an \( S \)-enriched \( C \)-colored operad \( P \), one can then consider \( P \) as either an object of \( \text{Op}_{S}(S) \) or an object of \( \text{Op}(S) \). As emphasized above, we are mostly concentrated in the latter case. Therefore, by **(resp. reduced) Quillen cohomology of \( P \)** we shall mean the Quillen cohomology of \( P \) when regarded as an object of **(resp. \( \text{Op}_{S}(S) \)) \text{Op}(S) \). Some attention was
given in the literature to reduced Quillen cohomology of operads, at least in the context of dg operads (in which case the resultant is described in terms of derivations, similarly as in Example 1.0.1); while the problems of formulating Quillen cohomology of operads and investigating its applications, which are essentially more valuable, have not yet been considered in the literature.

It has been widely acknowledged that Quillen cohomology theory keeps a key role in the study of obstruction theory (cf., e.g., [14, 15, 16]). We hope that the present paper may open the way to the study of obstruction theory of simplicial operads. We would like to leave this for future work. On other hand, the relation between Quillen cohomology and deformation theory will be outlined in our next paper, which is also devoted to the study of Quillen cohomology of dg operads and now in the process of being finalized.

We shall now summarize our main results with respect to some historical backgrounds. Suppose given a sufficiently nice base category $\mathcal{C}$, the category of $\mathcal{C}$-enriched categories with objects in $C := \text{Ob}(\mathcal{C})$. Recall that the category of $\mathcal{C}$-bimodules, $\text{BMod}(\mathcal{C})$, is the same the category of $\mathcal{C}$-enriched functors $\mathcal{C}^{\text{op}} \otimes \mathcal{C} \rightarrow \mathcal{C}$. There is a sequence of the obvious Quillen adjunctions $\text{BMod}(\mathcal{C})_{\mathcal{C}_f} \rightleftarrows \text{Cat}^{\mathcal{C}}(\mathcal{C})_{\mathcal{C}_f} \rightleftarrows \text{Cat}(\mathcal{C})_{\mathcal{C}_f}$.

**Theorem 1.0.2.** (Y. Harpaz, M. Prasma and J. Nuiten [5]) The above sequence induces a sequence of Quillen equivalences connecting the associated tangent categories:

$$\mathcal{T}_\mathcal{C} \text{BMod}(\mathcal{C}) \xrightarrow{\eta} \mathcal{T}_\mathcal{C} \text{Cat}^{\mathcal{C}}(\mathcal{C}) \xrightarrow{\eta} \mathcal{T}_\mathcal{C} \text{Cat}(\mathcal{C}) \quad (1.0.1)$$

Let us now fix $\mathcal{P}$ to be a fibrant and $\Sigma$-cofibrant $C$-colored operad in $\mathcal{C}$. We let $\text{BMod}(\mathcal{P})$ and $\text{IbMod}(\mathcal{P})$ respectively denote the categories of $\mathcal{P}$-bimodules and of infinitesimal $\mathcal{P}$-bimodules. Then there is a sequence of the obvious Quillen adjunctions of the form

$$\text{IbMod}(\mathcal{P})_{\mathcal{P}_f} \rightleftarrows \text{BMod}(\mathcal{P})_{\mathcal{P}_f} \rightleftarrows \text{Op}_C(\mathcal{C})_{\mathcal{P}_f} \rightleftarrows \text{Op}(\mathcal{C})_{\mathcal{P}_f}.$$  

**Theorem 1.0.3.** (5.0.3, 5.0.5) The above sequence induces a sequence of Quillen equivalences connecting the associated tangent categories:

$$\mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \xrightarrow{\eta} \mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}) \xrightarrow{\eta} \mathcal{T}_\mathcal{P} \text{Op}_C(\mathcal{C}) \xrightarrow{\eta} \mathcal{T}_\mathcal{P} \text{Op}(\mathcal{C}).$$

When $\mathcal{S}$ is in addition stable containing a strict zero object $0$, this sequence is prolonged to a sequence of Quillen equivalences of the form

$$\text{IbMod}(\mathcal{P})_{\mathcal{P}_f} \xrightarrow{\eta} \text{BMod}(\mathcal{P})_{\mathcal{P}_f} \xrightarrow{\eta} \text{Op}_C(\mathcal{C})_{\mathcal{P}_f} \xrightarrow{\eta} \text{Op}(\mathcal{C})_{\mathcal{P}_f},$$

where the functor $\ker : \text{IbMod}(\mathcal{P})_{\mathcal{P}_f} \rightarrow \text{IbMod}(\mathcal{P})$ is given by sending $\mathcal{P} \rightarrow M \rightarrow \mathcal{P}$ to $M \times_{\mathcal{P}} 0$.

We then compute the derived image of the cotangent complex $L_{\mathcal{P}} \in \mathcal{T}_\mathcal{P} \text{Op}(\mathcal{C})$ of $\mathcal{P}$ under the right Quillen equivalence $\mathcal{T}_\mathcal{P} \text{Op}(\mathcal{C}) \xrightarrow{\eta} \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P})$. In the first step, our treatment is inspired by the following:

**Theorem 1.0.4.** (Y. Harpaz, M. Prasma and J. Nuiten [5]) Let $\mathcal{C}$ be a fibrant $\mathcal{S}$-enriched category. Under the right Quillen equivalence $\mathcal{T}_\mathcal{C} \text{Cat}(\mathcal{C}) \xrightarrow{\eta} \mathcal{T}_\mathcal{C} \text{BMod}(\mathcal{C})$, the cotangent complex $L_{\mathcal{C}} \in \mathcal{T}_\mathcal{C} \text{Cat}(\mathcal{C})$ is identified to $L_{\mathcal{C}}^b[-1] \in \mathcal{T}_\mathcal{C} \text{BMod}(\mathcal{C})$, i.e., the desuspension of $L_{\mathcal{C}}^b \in \mathcal{T}_\mathcal{C} \text{BMod}(\mathcal{C})$ the cotangent complex of $\mathcal{C}$ when regarded as a bimodule over itself.
We prove that an analogue of this statement holds for the right Quillen equivalence \( \mathcal{T}_\mathcal{P} \text{Op}(\mathcal{S}) \to \mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}) \) (cf. Proposition 6.2.1). However, the approach as in the loc.cit fails when extending to our context. In particular, for our approach, the category \( \mathcal{S} \) is technically required to satisfy the extra condition (S8) 6.1.2, which is inspired by the work of Dwyer and Hess [17], Section 5. After having proved that, it remains to compute the derived image of the cotangent complex of \( \mathcal{P} \) (when regarded as a bimodule over itself) under the right Quillen equivalence \( \mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}) \to \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \). The resultant will be denoted by \( \mathbb{L}_\mathcal{P} \). Furthermore, when \( \mathcal{S} \) is in addition stable containing a strict zero object \( 0 \), we on compute the derived image of \( \mathbb{L}_\mathcal{P} \) under the right Quillen equivalence \( \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \to \text{IbMod}(\mathcal{P}) \), which is denoted by \( L_\mathcal{P} \). By having these “infinitesimal models” of \( L_\mathcal{P} \), we obtain the central result of the paper stated as below.

Let \( S^n \) denote the pointed \( n \)-sphere in \( \mathcal{S} \). Then, let \( S^n_C \) denote the \( C \)-collection which has \( S^n_C(c;c) = S^n \) for every \( c \in C \) and agrees with \( \varnothing_\mathcal{S} \) on the other levels.

**Theorem 1.0.5.** (6.2.8, 6.2.9) Suppose that \( \mathcal{S} \) additionally satisfies the condition (S8) 6.1.2. The \( n \)-th Quillen cohomology group of \( \mathcal{P} \) with coefficients in a given fibrant object \( M \in \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \) is formulated as

\[
H^n_Q(\mathcal{P}, M) \cong \pi_0 \text{Map}^h_{\mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P})}(\mathbb{L}_\mathcal{P}, M[n + 1])
\]

in which \( \mathbb{L}_\mathcal{P} \in \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \) is the prespectrum with \( (\mathbb{L}_\mathcal{P})_{n,n} = \mathcal{P} \circ S^n_C, n \geq 0 \). In particular, for each \( C \)-sequence \( \mathcal{C} = (c_1, \ldots, c_n; c) \), we have that \( (\mathbb{L}_\mathcal{P})_{n,n}(\mathcal{C}) = \mathcal{P}(\mathcal{C}) \otimes (S^n)^{\otimes m} \).

If \( \mathcal{S} \) is in addition stable containing a strict zero object \( 0 \) then the \( n \)-th Quillen cohomology group of \( \mathcal{P} \) with coefficients in a given fibrant object \( M \in \text{IbMod}(\mathcal{P}) \) is formulated as

\[
H^n_Q(\mathcal{P}, M) \cong \pi_0 \text{Map}^h_{\text{IbMod}(\mathcal{P})}(L_\mathcal{P}, M[n + 1])
\]

where \( L_\mathcal{P} \in \text{IbMod}(\mathcal{P}) \) is given on each level as \( L_\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathcal{C}) \otimes \text{hocolim}_n \Omega^n[(S^n)^{\otimes m} \times_{1\mathbb{S}} 0] \).

Moreover, we find a connection between Quillen cohomology and reduced Quillen cohomology of \( \mathcal{P} \), expressed as follows.

**Theorem 1.0.6.** (6.3.3) Suppose that \( \mathcal{S} \) additionally satisfies the condition (S8) 6.1.2. Given a fibrant object \( M \in \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \), there is a long exact sequence of abelian groups of the form

\[
\cdots \to H^{n-1}_Q(\mathcal{P}, M) \to H^n_{Q,r}(\mathcal{P}, M) \to H^n_0(\mathcal{P}, M) \to H^n_1(\mathcal{P}, M) \to \cdots
\]

where \( H^n_{Q,r}(\mathcal{P}, \mathcal{P}) \) refers to Quillen cohomology group of \( \mathcal{P} \) when regarded as a right module over itself and \( H^n_{Q,r}(\mathcal{P}, \mathcal{P}) \) refers to reduced Quillen cohomology group of \( \mathcal{P} \).

Turning to the context of simplicial operads, our main result extends the following:

**Theorem 1.0.7.** (Y. Harpaz, M. Prasma and J. Nuiten [5]) Let \( \mathcal{C} \) be a fibrant simplicial category. There is an equivalence of \( \infty \)-categories \( \mathcal{T}_\mathcal{C} \text{Cat}(\text{Set}_\Delta) \simeq \text{Fun}(\text{Tw}(\mathcal{C}), \text{Spectra}) \) with \( \text{Spectra} \) being the \( \infty \)-category of spectra and \( \text{Tw}(\mathcal{C}) \) being the twisted arrow \( \infty \)-category of \( \mathcal{C} \). Furthermore, the cotangent complex \( L_\mathcal{C} \in \mathcal{T}_\mathcal{C} \text{Cat}(\text{Set}_\Delta) \) is then identified to the constant functor \( \text{Tw}(\mathcal{C}) \to \text{Spectra} \) on \( \mathbb{S}[-1] \), i.e., the desuspension of the sphere spectrum. Consequently, the \( n \)-th Quillen cohomology group of \( \mathcal{C} \) with coefficients in a given functor \( \mathcal{F} : \text{Tw}(\mathcal{C}) \to \text{Spectra} \) is computed as \( H^n_Q(\mathcal{C}, \mathcal{F}) = \pi_{-n-1} \lim \mathcal{F} \).

The construction of twisted arrow \( \infty \)-categories (of \( \infty \)-categories) \( \text{Tw}(\mathcal{C}) : \text{Cat}_\infty \to \text{Cat}_\infty \) was originally introduced by Lurie [2], §5.2. We extend that to the construction of twisted
arrow $\infty$-categories of (fibrant) simplicial operads. Let $\mathcal{P}$ be a fibrant simplicial operad. In particular, objects of Tw$(\mathcal{P})$ are precisely the operations of $\mathcal{P}$ (i.e., the vertices of spaces of operations of $\mathcal{P}$). For example, Tw$(\mathcal{Com})$ is equivalent to Fin$^n_\mathbb{P}$ the (opposite) category of finite pointed sets, while Tw$(\mathcal{Ass})$ is equivalent to the simplex category $\Delta$ (cf. Proposition 7.1.11).

**Theorem 1.0.8. (7.2.1)** Let $\mathcal{P}$ be a fibrant and $\Sigma$-cofibrant simplicial operad. Then there is an equivalence of $\infty$-categories

$$\mathcal{I}_\mathcal{P} \text{Op}(\text{Set}_\Delta)_\infty \rightarrow \text{Fun}(\text{Tw}(\mathcal{P}), \text{Spectra}).$$

Moreover, under this equivalence, the cotangent complex $L_\mathcal{P} \in \mathcal{I}_\mathcal{P} \text{Op}(\text{Set}_\Delta)_\infty$ is identified to $\mathcal{I}_\mathcal{P}[-1]$ the desuspension of the functor $\mathcal{I}_\mathcal{P}: \text{Tw}(\mathcal{P}) \rightarrow \text{Spectra}$ (7.2.3), which is given on objects by sending each operation $\mu \in \mathcal{P}$ of arity $m$ to $\mathcal{I}_\mathcal{P}(\mu) = S^m$, i.e., the $m$-fold product of the sphere spectrum. Consequently, the $n$’th Quillen cohomology group of $\mathcal{P}$ with coefficients in a given functor $\mathcal{F}: \text{Tw}(\mathcal{P}) \rightarrow \text{Spectra}$ is formulated as

$$H_Q^n(\mathcal{P}; \mathcal{F}) = \pi_0 \text{Map}_{\text{Fun}(\text{Tw}(\mathcal{P}), \text{Spectra})}(\mathcal{I}_\mathcal{P}, \mathcal{F}[n + 1]).$$

A simplicial operad is said to be unital if all its unary (= 0-ary) spaces of operations are a singleton and furthermore, is said to be unitaly homotopy connected if it is unital with weakly contractible 1-ary spaces of operations. The following result, in particular, shows that Quillen cohomology of any little cubes operad with constant coefficients in any little cubes operad with constant coefficients vanishes.

**Corollary 1.0.9. (7.2.4)** Suppose that $\mathcal{P}$ is a fibrant, $\Sigma$-cofibrant and unitaly homotopy connected simplicial operad. Let $\mathcal{F}_0: \text{Tw}(\mathcal{P}) \rightarrow \text{Spectra}$ be a constant functor. Then Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{F}_0$ vanishes.

Let $R$ be a commutative ring. In the last section, we are interested in Quillen cohomology of operads enriched over sMod$(R)$ the category of simplicial $R$-modules. Let $\mathcal{P} \in \text{Op}(\text{sMod}_R)$ be a $\Sigma$-cofibrant $C$-colored operad. We first make use of the second part of Theorem 1.0.5 to give an explicit description of $L_\mathcal{P} \in \text{IbMod}(\mathcal{P})$. In particular, we find that, as a $C$-collection, $L_\mathcal{P}$ is isomorphic to $\mathcal{P} \circ (1) \mathcal{J}_C$, i.e., the infinitesimal composite product of $\mathcal{P}$ with the initial $C$-colored operad $\mathcal{J}_C$ (cf. Computations 8.0.1).

The free functor $R\{-\}: \text{Set}_\Delta \rightarrow \text{sMod}_R$ lifts to a functor $R\{-\}: \text{Op}(\text{Set}_\Delta) \rightarrow \text{Op}(\text{sMod}_R)$ between operads. The following result, in particular, shows that Quillen cohomology of the $R$-linearization of any little cubes operad with coefficients in itself vanishes.

**Theorem 1.0.10. (8.0.3)** Given a simplicial operad $\mathcal{Q}$, Quillen cohomology of $R\{\mathcal{Q}\}$ with coefficients in itself vanishes whenever $\mathcal{Q}$ is fibrant, $\Sigma$-cofibrant and unitaly homotopy connected.

**Organization of the paper.** In section 2, we recall briefly some necessary facts relevant to homotopy theory of enriched operads. In section 3, we set up several conditions on the base category $\mathcal{S}$, which we work with in the sections 5 and 6. Section 4 is devoted to the needed concepts relevant to Quillen cohomology theory. Our work in section 5 is mostly devoted to the proof of Theorem 1.0.3. In section 6, we first set up an extra condition on the base category $\mathcal{S}$ and by the way, provide several illustrations for this condition. The ultimate goal of this section is to prove Theorem 1.0.5, along with Theorem 1.0.6. In section 7, we first discuss on the construction of twisted arrow $\infty$-categories of simplicial operads, and then explain how this
construction classifies Quillen cohomology of simplicial operads. Finally, in section 8, we study Quillen cohomology of operads enriched over simplicial $R$-modules.

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### 2. Notations and preliminary results

Let $S$ be a symmetric monoidal category. Given a set $C$, regarded as the set of **colors**, we denote by $\text{Seq}(C)$ the set $\{(c_1,\ldots,c_n;c) : c_1, c \in C, n \geq 0\}$ and refer to it as the collection of $C$-**sequences**.

**Definition 2.0.1.** A **symmetric $C$-collection** (also called a $C$-**symmetric sequence**) in $S$ is a collection $M$ of objects in $S$, indexed by $\text{Seq}(C)$, equipped with a **symmetric action** whose data consists of the maps of the form $\sigma^* : M(c_1,\ldots,c_n;c) \rightarrow M(c_{\sigma(1)},\ldots,c_{\sigma(n)};c)$ with $\sigma \in \Sigma_n$. The category of symmetric $C$-collections will be denoted by $\text{Coll}_C(S)$.

The well known **operadic composite product** $- \circ - : \text{Coll}_S(S) \times \text{Coll}_S(S) \rightarrow \text{Coll}_S(S)$ endows $\text{Coll}_S(S)$ with a monoidal structure. The monoidal unit is then denoted by $\mathcal{J}_C$, with $\mathcal{J}_C(c;c) = 1_S$ for every $c \in C$ and agreeing with $\varnothing_S$ on the other levels. (See, e.g., [6, 18, 19]).

A **symmetric $S$-enriched $C$-colored operad** is by definition a monoid in the monoidal category $(\text{Coll}_S(S),- \circ -,\mathcal{J}_C)$ (cf. [21, 19]). We denote by $\text{Op}_C(S)$ the category of such objects.

Given $\mathcal{P} \in \text{Op}_C(S)$, each object $\mathcal{P}(c_1,\ldots,c_n;c)$ will be called a **space of $n$-ary operations** of $\mathcal{P}$. Recall that the collection of 1-ary operations of $\mathcal{P}$, denoted by $\mathcal{P}_1$, inherits an obvious $S$-enriched category structure. We shall refer to $\mathcal{P}_1$ as the **underlying category** of $\mathcal{P}$.

The notion of a **nonsymmetric $C$-colored operad** (resp. $C$-**collection**) is the same as that of a symmetric $C$-colored operad (resp. $C$-collection) when one forgets the symmetric action. The natural passage from nonsymmetric to symmetric context is performed by the **symmetrization functor** (see, e.g., [[19], Section 20.1]). Since we are almost concentrated in the symmetric context, from now on we always omit the word “symmetric” when mentioning an object of $\text{Op}_C(S)$ (or $\text{Coll}_C(S)$).

**Definition 2.0.2.** Let $\mathcal{P}$ be an $S$-enriched $C$-colored operad. A **$\mathcal{P}$-algebra** is an object $A \in S^{\times C}$ equipped, for each $(c_1,\ldots,c_n;c)$, with a $\mathcal{P}$-action map

$$\mathcal{P}(c_1,\ldots,c_n;c) \otimes A(c_1) \otimes \cdots \otimes A(c_n) \rightarrow A(c)$$

factoring through the tensor product over $\Sigma_n$. These maps must satisfy the essential axioms of associativity and unitality. We denote by $\text{Alg}_\mathcal{P}(S)$ the category of $\mathcal{P}$-algebras.

Let $A$ be a $\mathcal{P}$-algebra. As in [18], we denote by $\text{Env}(\mathcal{P},A) \in \text{Op}_C(S)$ the **enveloping operad** associated to the pair $(\mathcal{P},A)$ (see also [22]). The main interest in this construction is that there is a canonical isomorphism $\text{Alg}_{\text{Env}(\mathcal{P},A)}(S) \simeq \text{Alg}_\mathcal{P}(S)_A$ between the categories of $\text{Env}(\mathcal{P},A)$-algebras and of $\mathcal{P}$-algebras under $A$. Moreover, according to [[22], Theorem 1.10], the structure of an $A$-module over $\mathcal{P}$ is equivalent to that of an $S$-enriched functor $\text{Env}(\mathcal{P},A)_1 \rightarrow S$. More explicitly, an **$A$-module over** $\mathcal{P}$ is an object $M \in S^{\times C}$ equipped, for each sequence $(c_1,\ldots,c_n;c)$, with a mixed $(\mathcal{P},A)$-action map of the form

$$\mathcal{P}(c_1,\ldots,c_n;c) \otimes \bigotimes_{i \in \{1,\ldots,n\}-\{k\}} A(c_i) \otimes M(c_k) \rightarrow M(c)$$
factoring through the tensor product over $\Sigma_n$. These maps must satisfy the essential axioms of associativity and unitality.

**Construction 2.0.3.** Let $\varphi: C \rightarrow D$ be a map of sets. There is a **changing-colors functor** $\varphi^*: \text{Op}_D(S) \rightarrow \text{Op}_C(S)$ taking $Q \in \text{Op}_D(S)$ to $\varphi^*Q$ defined on each level as

$$\varphi^*Q(c_1, \ldots, c_n; c) := Q(\varphi(c_1), \ldots, \varphi(c_n); \varphi(c)).$$

**Definition 2.0.4.** The **category of $S$-enriched operads**, denoted by $\text{Op}(S)$, is the one whose objects are the pairs $(C, \mathcal{P})$ with $C \in \text{Sets}$ and $\mathcal{P} \in \text{Op}_C(S)$, and whose morphisms are the pairs $(\varphi, f): (C, \mathcal{P}) \rightarrow (D, \Omega)$ with $\varphi \in \text{Hom}_{\text{Sets}}(C, D)$ and $f \in \text{Hom}_{\text{Op}_C(S)}(\mathcal{P}, \varphi^*\Omega)$.

### 2.1 Operadic transferred model structures

Let us assume further that $S$ is a **symmetric monoidal model category** (see Hovey’s [23]). We first discuss on the model structure on $\text{Coll}_C(S)$. Due to its “linearity”, this category can be reformulated as a category of $S$-valued enriched functors on a certain discrete groupoid (cf., e.g., [19]). Consequently, under some suitable conditions (for instance, $S$ is cofibrantly generated with cofibrant unit), $\text{Coll}_C(S)$ admits the projective model structure whose weak equivalences (resp. fibrations) are the levelwise weak equivalences (resp. fibrations).

The (projective) **transferred model structure** on the category $\text{Op}_C(S)$ (or $\text{Alg}_S(S)$ with $\mathcal{P}$ being some operad) is the one whose weak equivalences (resp. fibrations) are precisely the levelwise weak equivalences (resp. fibrations). Fortunately, for each set $C$, there exists an **operad of $C$-colored operads**, denoted by $\mathcal{O}_C$, whose set of colors is $\text{Seq}(C)$ and whose algebras are precisely the $C$-colored operads (cf. [24], Section 3)). In particular, one just needs to consider the transferred model structure on algebras. According to the literature, one knows a criterion assuring the existence of that. Here are several settings.

**Definition 2.1.1.** A **symmetric monoidal fibrant replacement functor** on $S$ is a symmetric monoidal functor $R: S \rightarrow S$ together with a monoidal natural transformation $\varphi: \text{Id} \rightarrow R$ such that for each object $X \in S$, the map $\varphi_X: X \rightarrow R(X)$ exhibits $R(X)$ as a fibrant replacement of $X$.

**Definition 2.1.2.** A **functorial path data** on $S$ is a symmetric monoidal functor $\mathcal{P}: S \rightarrow S$ together with monoidal natural transformations $s: \text{Id} \rightarrow \mathcal{P}$ and $d_0, d_1: \mathcal{P} \rightarrow \text{Id}$ such that the composite map $X \xrightarrow{s_X} \mathcal{P}(X) \xrightarrow{(d_0, d_1)} X \times X$ exhibits $\mathcal{P}(X)$ as a path object for $X$.

The same arguments as in the proof of [25], Theorem 3.11] verify the following.

**Proposition 2.1.3.** Suppose that $S$ is a strongly cofibrantly generated symmetric monoidal model category. If $S$ admits both a symmetric monoidal fibrant replacement functor and a functorial path data then the transferred model structure on $\text{Alg}_S(S)$ exists for every operad $\mathcal{P}$. In particular, the category $\text{Op}_C(S)$ then admits the transferred model structure.

**Examples 2.1.4.** Typical examples for this statement include:
- the **Cartesian monoidal category of simplicial sets**, $(\text{Set}_\Delta, \times)$, equipped with the standard (Kan-Quillen) model structure,
- the **Cartesian monoidal category of topological spaces**, $(\text{Top}, \times)$, equipped with the classical Quillen model structure,
- the monoidal category of simplicial $R$-modules, $(s\text{Mod}_R, \otimes)$, with $R$ being any commutative ring, equipped with the standard model structure transferred from that of $\text{Set}_{\Delta}$, and
- the monoidal category of dg $\mathbb{k}$-modules, $(\mathcal{C}(\mathbb{k}), \otimes)$, with $\mathbb{k}$ being any commutative ring of characteristic 0, equipped with the projective model structure.

A given $C$-colored operad is said to be $\Sigma$-cofibrant if its underlying $C$-collection is cofibrant in $\text{Coll}_C(\mathbb{S})$. Suppose that the mentioned model category structures exist. When the unit $1_{\mathbb{S}}$ is cofibrant, every cofibrant operad is $\Sigma$-cofibrant (cf., e.g., $[12]$, Corollary C.9), while every cofibrant $C$-collection is levelwise cofibrant.

2.2 Dwyer-Kan and canonical model structures on enriched operads

Let $\mathbb{S}$ be a monoidal model category and let $\text{Cat}(\mathbb{S})$ denote the category of (small) $\mathbb{S}$-enriched categories. For each $\mathfrak{C} \in \text{Cat}(\mathbb{S})$, the homotopy category of $\mathfrak{C}$, denoted by $\text{Ho}(\mathfrak{C})$, is the category whose objects are the same as those of $\mathfrak{C}$ and whose hom-set $\text{Hom}_{\text{Ho}(\mathfrak{C})}(x, y)$, with $x, y \in \text{Ob}(\mathfrak{C})$, is defined to be $\text{Hom}_{\text{Ho}(\mathfrak{C})}(x, y) := \text{Hom}_{\text{Ho}(\mathbb{S})}(1_{\mathbb{S}}, \text{Map}_{\mathfrak{C}}(x, y))$.

**Definition 2.2.1.** A map $f : \mathfrak{C} \rightarrow \mathfrak{D}$ in $\text{Cat}(\mathbb{S})$ is called a Dwyer-Kan equivalence if it is a levelwise weak equivalence and such that the induced functor $\text{Ho}(f) : \text{Ho}(\mathfrak{C}) \rightarrow \text{Ho}(\mathfrak{D})$ between homotopy categories is essentially surjective.

**Dwyer-Kan model structure on** $\text{Cat}(\mathbb{S})$ is then the one whose weak equivalences are the Dwyer-Kan equivalences and whose trivial fibrations are the levelwise trivial fibrations surjective on objects (see, e.g., $[26], [3]$). On other hand, **canonical model structure on** $\text{Cat}(\mathbb{S})$, as introduced by Berger-Moerdijk $[20]$, is the one whose fibrant objects are the levelwise fibrant categories and whose trivial fibrations are the same as those of the Dwyer-Kan model structure.

By extending the two above, G. Caviglia $[12]$ established both the Dwyer-Kan and canonical model structures on $\text{Op}(\mathbb{S})$. Suppose further that $\mathbb{S}$ is a symmetric monoidal model category.

**Definition 2.2.2.** Dwyer-Kan model structure on $\text{Op}(\mathbb{S})$ is the one whose weak equivalences are the Dwyer-Kan equivalences (i.e., the maps whose underlying maps in $\text{Cat}(\mathbb{S})$ are Dwyer-Kan equivalences), and whose trivial fibrations are the levelwise trivial fibrations surjective on colors.

**Definition 2.2.3.** Canonical model structure on $\text{Op}(\mathbb{S})$ is the one whose fibrant objects are the levelwise fibrant operads and whose trivial fibrations are the same as those of the Dwyer-Kan model structure.

**Remark 2.2.4.** As originally introduced by G. Caviglia, a given map in $\text{Op}(\mathbb{S})$ is a fibration (resp. weak equivalence) with respect to the canonical model structure if and only if it is a levelwise fibration (resp. weak equivalence) and such that its underlying map in $\text{Cat}(\mathbb{S})$ is a fibration (resp. weak equivalence) with respect to that model structure.

Following up his work, we give a set of conditions on the base category $\mathbb{S}$ assuring the existence of the canonical model structure.

**Proposition 2.2.5.** (Caviglia, $[12]$) Let $\mathbb{S}$ be a combinatorial symmetric monoidal model category satisfying that:

(S1) the class of weak equivalences is closed under filtered colimits,
(S2) $\mathbb{S}$ admits a symmetric monoidal fibrant replacement functor and a functorial path data,
(S3) the monoidal unit is cofibrant, and
(S4) the model structure is right proper.

Then $\text{Op}(S)$ admits the canonical model structure, which is as well right proper and combinatorial. Moreover, this model structure coincides, then, with the Dwyer-Kan model structure.

**Proof.** See around Theorem 4.22(1) of loc.cit, along with noting that the condition (S2) ensures the existence of the transferred model structure on $\text{Op}_C(S)$ for every $C$, by Proposition 2.1.3. □

**Remark 2.2.6.** Under the same assumptions as in Proposition 2.2.5, the canonical model structure on $\text{Cat}(S)$ automatically exists and coincides with the Dwyer-Kan model structure.

### 3. Conventions

We first recall from [4] the following definition, which itself is inspired by [2], Definition 6.1.1.6.

**Definition 3.0.1.** A model category $M$ is said to be **differentiable** if the derived colimit functor $L \text{colim} : M^\Delta \to M$ preserves finite homotopy limits. Furthermore, a Quillen adjunction $\mathcal{L} : M \rightleftarrows N : \mathcal{R}$ is said to be **differentiable** if both $M$ and $N$ are differentiable and the right derived functor $\mathcal{R}$ preserves sequential homotopy colimits.

**Conventions 3.0.2.** In the sections 5 and 6, we will work on the base category $S$ which is supposed to be a combinatorial symmetric monoidal model category such that the domains of generating cofibrations are cofibrant, satisfies the conditions (S1)-(S4) of Proposition 2.2.5 and in addition, satisfies that

- (S5) $S$ is differentiable,
- (S6) the unit $1_S$ is **homotopy compact** in the sense that the functor $\pi_0 \text{Map}_S^h(1_S, -)$ sends filtered homotopy colimits to colimits of sets,
- (S7) $S$ satisfies the Lurie’s **invertibility hypothesis** [3], Definition A.3.2.12.

In particular, we will work on the canonical model structure on $\text{Op}(S)$ ($\text{Cat}(S)$), which coincides with the Dwyer-Kan model structure by Proposition 2.2.5.

**Remark 3.0.3.** Requiring the domains of generating cofibrations to be cofibrant is necessary for the existences of various types of operadic tangent category (cf. Section 4), which come after a procedure of taking left Bousfield localizations without left properness, inspired by the main result of [11].

**Remark 3.0.4.** The condition (S6), together with (S5), ensures that $\text{Cat}(S)$ is differentiable. The reader who prefers can replace (S6) simply by the differentiability of $\text{Cat}(S)$.

**Remark 3.0.5.** The condition (S7) allows us to inherit [5], Proposition 3.2.1 for the work of Section 6. Briefly, the invertibility hypothesis requires that, for any $\mathcal{C} \in \text{Cat}(S)$ containing a morphism $f$, localizing $\mathcal{C}$ at $f$ does not change the homotopy type of $\mathcal{C}$ whenever $f$ is already an isomorphism in $\text{Ho}(\mathcal{C})$. This condition is in fact pretty popular in practice. According to [27], if $S$ is already a combinatorial monoidal model category satisfying (S1) and such that every object of $S$ is cofibrant then $S$ satisfies the invertibility hypothesis. It also holds for dg modules over a commutative ring by [28], Corollary 8.7, and for any simplicial monoidal model category, according to [29], Theorem 0.9.
Example 3.0.6. Typical categories for Conventions 3.0.2 are again the ones of Examples 2.1.4.

4. Tangent categories and Quillen cohomology

This section is based on the works of [4, 5]. Tangent category comes after a procedure of taking stabilization of a model category. Under our settings, stabilizations exist only as semi model categories. Despite this, the needed results from those papers remain valid. We then get the notion of cotangent complex, which plays a central role in the Quillen cohomology theory.

**Definition 4.0.1.** A model category $\mathcal{M}$ is said to be **weakly pointed** if it contains a **weak zero object**, i.e., an object which is both homotopy initial and terminal.

Let $\mathcal{M}$ be a weakly pointed model category and let $X$ be an $(\mathbb{N} \times \mathbb{N})$-diagram in $\mathcal{M}$. The diagonal squares of $X$ are of the form

$$
\begin{array}{ccc}
X_{n,n} & \rightarrow & X_{n,n+1} \\
\downarrow & & \downarrow \\
X_{n+1,n} & \rightarrow & X_{n+1,n+1}
\end{array}
$$

**Definition 4.0.2.** An $(\mathbb{N} \times \mathbb{N})$-diagram in $\mathcal{M}$ is called

1. a **prespectrum** if all its off-diagonal entries are weak zero objects in $\mathcal{M}$,
2. an $\Omega$-**spectrum** if it is a prespectrum and all its diagonal squares are homotopy Cartesian,
3. a **suspension spectrum** if it is a prespectrum and all its diagonal squares are homotopy coCartesian.

The projective model category of $(\mathbb{N} \times \mathbb{N})$-diagrams in $\mathcal{M}$ will be denoted by $\mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}$.

**Definition 4.0.3.** ([4], Definition 2.1.2) Let $\mathcal{M}$ be a weakly pointed model category. A map $f : X \rightarrow Y$ in $\mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}$ is said to be a **stable equivalence** if for every $\Omega$-spectrum $Z$ the induced map between derived mapping spaces

$$
\text{Map}^h_{\mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}}(Y, Z) \rightarrow \text{Map}^h_{\mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}}(X, Z)
$$

is a homotopy equivalence. Note that a stable equivalence between $\Omega$-spectra is always a levelwise equivalence.

Following [4, Lemma 2.1.6], the $\Omega$-spectra in $\mathcal{M}$ can be characterized as the **local objects** against a certain set of maps. Inspired by Definition 2.1.3 of the loc.cit, we give the following definition, which is valid due to [11], Theorem 4.2.

**Definition 4.0.4.** Let $\mathcal{M}$ be a weakly pointed combinatorial model category such that the domains of generating cofibrations are cofibrant. **Stabilization** of $\mathcal{M}$, denoted by $\text{Sp}(\mathcal{M})$, is defined to be the left Bousfield localization of $\mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}$ with $\Omega$-spectra as the local objects. Explicitly, $\text{Sp}(\mathcal{M})$ is a **cofibrantly generated semi model category** (cf., e.g., [8, 11]) whose

- weak equivalences are the stable equivalences, and whose
- (generating) cofibrations are the same as those of $\mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}$.

In particular, fibrant objects of $\text{Sp}(\mathcal{M})$ are precisely the levelwise fibrant $\Omega$-spectra.
Remark 4.0.5. When $M$ is in addition left proper then the stabilization $Sp(M)$ exists as a (full) model category. In fact, we do not require the left properness throughout the paper.

Definition 4.0.6. ([4]) A (semi) model category $M$ is called stable if the following equivalent conditions hold:

1. The underlying $\infty$-category $M_\infty$ of $M$ (cf. [30, 31]) is stable in the sense of [2].

2. $M$ is weakly pointed and such that a square in $M$ is homotopy coCartesian if and only if it is homotopy Cartesian.

3. $M$ is weakly pointed and such that the adjunction $\Sigma : \text{Ho}(M) \xleftarrow{\sim} \text{Ho}(M) : \Omega$ of suspension-desuspension functors is an adjoint equivalence.

Facts 4.0.7. (Y. Harpaz, J. Nuiten and M. Prasma [4]) Let $M$ and $N$ be two weakly pointed combinatorial model categories such that the domains of their generating cofibrations are cofibrant.

1. There is a Quillen adjunction $\Sigma^\infty M \xleftarrow{\sim} Sp(M) : \Omega^\infty$ where $\Omega^\infty(X) = X_{0,0}$ and $\Sigma^\infty(X)$ is the constant diagram with value $X$.

2. The induced functor $(\Omega^\infty)_* : Sp(M)_\infty \to M_\infty$ exhibits $Sp(M)_\infty$ as the stabilization of $M_\infty$ in the sense of [2].

3. The stabilization $Sp(M)$ is stable. Furthermore, if $M$ is already stable then the adjunction $\Sigma^\infty M \xleftarrow{\sim} Sp(M) : \Omega^\infty$ is a Quillen equivalence.

4. A Quillen adjunction $F : M \xleftarrow{\sim} N : G$ lifts to a Quillen adjunction between stabilizations $F^{\text{Sp}} : Sp(M) \xleftarrow{\sim} Sp(N) : G^{\text{Sp}}$. Moreover, if $F \dashv G$ is a Quillen equivalence then so is $F^{\text{Sp}} \dashv G^{\text{Sp}}$.

Notation 4.0.8. Let $c$ be any category containing an object $X$. We will denote by $c_{X/} := (c/X)_*$, the pointed category associated to the over category $c/X$. More precisely, objects of $c_{X/}$ are the diagrams $X \xrightarrow{f} A \xrightarrow{g} X$ in $c$ such that $gf = \text{Id}_X$.

A morphism $f : X \to Y$ in $c$ gives rise to a canonical adjunction $f^* : c_{X/} \xleftarrow{\sim} c_{Y/} : f^*$ in which $f^*(X \to A) = A[Y/ X]$ while $f^*(Y \to B) = B \times_Y X$.

It can be shown that if $M$ is a combinatorial model category such that the domains of generating cofibrations are cofibrant then so is the transferred model structure on $M_{/A}$ (see Hirschhorn’s [32]). This makes the following definition valid.

Definition 4.0.9. Let $M$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let $A$ be an object of $M$. The tangent category to $M$ at $A$, denoted by $\mathcal{T}_A M$, is defined to be the stabilization of $M_{/A}$, i.e., $\mathcal{T}_A M := Sp(M_{/A})$.

There is a Quillen adjunction $\Sigma^\infty : M_{/A} \xleftarrow{\sim} \mathcal{T}_A M : \Omega^\infty$ given as the composition:

$M_{/A} \xrightarrow{\text{forgetful}} M_{/A} \xleftarrow{\text{constant}} \mathcal{T}_A M$.

Namely, for each $B \in M_{/A}$, then $\Sigma^\infty_B(B) = \Sigma^\infty(A \to A \cup B \to A)$ the constant diagram with value $A \cup B$; and for each $X \in \mathcal{T}_A M$, $\Omega^\infty(X) = [X_{0,0} \to A]$.

Definition 4.0.10. Let $M$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let $A$ be an object of $M$. The cotangent complex of $A$, denoted by $L_A$, is defined to be the derived suspension spectrum of $A$, i.e., $L_A := \Sigma^\infty(A) \in \mathcal{T}_A M$. 

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By Facts 4.0.7(iv), a given map $f : A \to B$ in $\mathcal{M}$ gives rise to a Quillen adjunction between tangent categories $f_{!}^{\text{Sp}} : \mathcal{T}_{A}\mathcal{M} = \text{Sp}(\mathcal{M}_{A/}) \rightleftarrows \text{Sp}(\mathcal{M}_{B/}) = \mathcal{T}_{B}\mathcal{M} : f_{!}^{\text{Sp}}$. Moreover, there is a commutative square of left Quillen functors

\[
\begin{array}{c}
\mathcal{M}_{/A} \xrightarrow{f_{!}} \mathcal{M}_{/B} \\
\downarrow \Sigma_{!}^{\infty} \quad \quad \downarrow \Sigma_{!}^{\infty} \\
\mathcal{T}_{A}\mathcal{M} \xrightarrow{f_{!}^{\text{Sp}}} \mathcal{T}_{B}\mathcal{M}
\end{array}
\quad (4.0.1)
\]

**Definition 4.0.11.** Let $\mathcal{M}$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let $f : A \to B$ be a map in $\mathcal{M}$. We will denote by $L_{B/A} := \text{hcofib}[\Sigma_{!}^{\infty}(f) \to L_{B}]$ the homotopy cofiber of the map $\Sigma_{!}^{\infty}(f) \to L_{B}$ in $\mathcal{T}_{B}\mathcal{M}$ and refer to $L_{B/A}$ as the relative cotangent complex of $f$.

Notice that the map $\Sigma_{!}^{\infty}(f) \to L_{B}$ can be identified to $f_{!}^{\text{Sp}}(L_{A}) \to L_{B}$, due to the commutativity of the square (4.0.1).

Finally, the most important definition in this paper is as follows:

**Definition 4.0.12.** ([5], Definition 2.2.1) Let $\mathcal{M}$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let $X$ be a fibrant object of $\mathcal{M}$. Suppose given a fibrant object $M \in \mathcal{T}_{X}\mathcal{M}$, regarded as the $\Omega$-spectrum of coefficients. For each $n \in \mathbb{Z}$, the $n$'th Quillen cohomology group of $X$ with coefficients in $M$ is defined to be

\[
H_{Q}^{n}(X, M) := \pi_{n}\text{Map}^{\text{h}}_{\mathcal{T}_{X}\mathcal{M}}(L_{X}, M[n])
\]

where $M[n] := \Sigma^{n}M$, i.e, the $n$-suspension of $M$ in $\mathcal{T}_{X}\mathcal{M}$.

**Remark 4.0.13.** Quillen cohomology is a homotopy invariant. Indeed, it is not hard to show that a given weak equivalence $f : X \xrightarrow{\sim} Y$ between fibrant objects induces, for each $n$, an isomorphism $H_{Q}^{n}(Y, M) \xrightarrow{\sim} H_{Q}^{n}(X, f_{!}^{\text{Sp}}M)$ between Quillen cohomology groups.

## 5. Operadic tangent categories

In this section, we work on the base category $\mathcal{S}$ of Conventions 3.0.2. In particular, we will work on the canonical model structure on $\text{Op}(\mathcal{S})$ ($\text{Cat}(\mathcal{S})$).

Let $\mathcal{P}$ be a $C$-colored operad in $\mathcal{S}$. We let $\text{LMod}(\mathcal{P})$ (resp. $\text{RMod}(\mathcal{P})$) denote the category of left (resp. right) $\mathcal{P}$-modules. Besides that, we let $\text{BMod}(\mathcal{P})$ and $\text{IbMod}(\mathcal{P})$ respectively denote the categories of $\mathcal{P}$-bimodules and of infinitesimal $\mathcal{P}$-bimodules. Let us revisit these quickly.

Operadic left module (resp. right module, bimodule) is the usual notion of left module (resp. right module, bimodule) over an operad when regarding operads as monoids in the monoidal category of symmetric sequences. These notions admit infinitesimal versions given as follows (cf., e.g., [33] for more details).

**Definition 5.0.1.** (i) An infinitesimal left $\mathcal{P}$-module is a $C$-collection $M$ equipped with the $\Sigma_{i}$-equivariant maps of the form

\[
ocite{5}
\begin{alignat}{2}
\alpha_{i}^{j} : \mathcal{P}(c_{1}, \cdots, c_{n}; c) & \otimes M(d_{1}, \cdots, d_{m}; c_{i}) & \to M(c_{1}, \cdots, c_{i-1}, d_{1}, \cdots, d_{m}, c_{i+1}, \cdots, c_{n}; c)
\end{alignat}
\quad (5.0.1)
\]

satisfying the classical axioms of associativity and unitality for left modules.
(ii) Dually, an **infinitesimal right \( P \)-module** is a \( C \)-collection \( M \) equipped with the \( \Sigma_+ \)-equivariant maps of the form

\[
o : M(c_1, ..., c_n; c) \otimes P(d_1, ..., d_m; c_i) \rightarrow M(c_1, ..., c_{i-1}, d_1, ..., d_m, c_{i+1}, ..., c_n; c)
\]

(5.0.2)

satisfying the classical axioms of associativity and unitality for right modules.

(iii) An **infinitesimal \( P \)-bimodule** is a \( C \)-collection \( M \) equipped with both an infinitesimal left and an infinitesimal right \( P \)-module structure which are subject to the essential compatibility for bimodules.

In Section 5.2 below, we give the definition of infinitesimal \( P \)-bimodule in diagrammatical style, which has certain advantages over the above definition.

**Remark 5.0.2.** The structure of an infinitesimal right \( P \)-module is equivalent to that of a right \( P \)-module structure.

To state our main theorem in this section, we first need to illustrate several links between mentioned operadic categories. Observe first that, as well as every type of monoid, there is a **restriction functor** \( \text{Op}_C(\mathcal{S})_{\mathcal{P}} \rightarrow \text{BMod}(\mathcal{P})_{\mathcal{P}'} \), which admits a left adjoint, usually, called the **induction functor**. On other hand, the **partial composition** in \( \mathcal{P} \) endows it with the structure of an infinitesimal bimodule over itself. Let \( M \) be a \( \mathcal{P} \)-bimodule under \( \mathcal{P} \). Then, \( M \) inherits a canonical infinitesimal \( \mathcal{P} \)-bimodule structure (under \( \mathcal{P} \)) induced by inserting the unit operations of \( \mathcal{P} \) into \( M \). This procedure determines a **restriction functor** \( \text{BMod}(\mathcal{P})_{\mathcal{P}'} \rightarrow \text{IbMod}(\mathcal{P})_{\mathcal{P}'} \), which admits as its left adjoint the **induction functor**, again. Moreover, there is an adjunction

\[
\mathcal{L}_P : \text{Op}_C(\mathcal{S})_{\mathcal{P}} \rightleftarrows \text{Op}(\mathcal{S})_{\mathcal{P}'} : \mathcal{R}_P
\]

(5.0.3)

where the left functor is the obvious inclusion and the right functor is given by the restriction of colors. Namely, let \( \mathcal{P} \rightarrow \mathcal{Q} \) be an object in \( \text{Op}(\mathcal{S})_{\mathcal{P}'} \), then \( \mathcal{R}_P(\mathcal{Q}) \) is defined on each level as

\[
\mathcal{R}_P(\mathcal{Q})(c_1, ..., c_n; c) := \mathcal{Q}(f(c_1), ..., f(c_n); f(c)).
\]

In conclusion, we obtain a sequence of adjunctions of the **induction-restriction** functors

\[
\mathcal{I}_P : \text{IbMod}(\mathcal{P})_{\mathcal{P}'} \rightleftarrows \text{BMod}(\mathcal{P})_{\mathcal{P}'} \rightleftarrows \text{Op}_C(\mathcal{S})_{\mathcal{P}} \rightleftarrows \text{Op}(\mathcal{S})_{\mathcal{P}'};
\]

(5.0.4)

which then lifts to a sequence of adjunctions connecting the associated tangent categories

\[
\mathcal{T}_P : \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_P : \text{BMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_P : \text{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_P : \text{Op}(\mathcal{S})
\]

(5.0.5)

The main result of this section is stated as:

**Theorem 5.0.3.** The adjunctions in the sequence (5.0.5) are all Quillen equivalences provided that \( \mathcal{P} \) is fibrant and \( \Sigma \)-cofibrant.

**Proof.** The categories \( \text{IbMod}(\mathcal{P}) \) and \( \text{BMod}(\mathcal{P}) \) admit the transferred model structures, respectively, by Remark 5.2.7 and Proposition 5.3.1. All the restriction functors are clearly preserve fibrations and weak equivalences, so all the adjunctions of (5.0.4) are Quillen adjunctions. It implies all the adjunctions of (5.0.5) are indeed Quillen adjunctions (cf. Facts 4.0.7(iv)). Proposition 5.1.4 proves that the adjunction \( \mathcal{T}_P : \text{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_P : \text{Op}(\mathcal{S}) \) is a Quillen equivalence when \( \mathcal{P} \) is fibrant. On other hand, when \( \mathcal{P} \) is fibrant and \( \Sigma \)-cofibrant, the adjunction \( \mathcal{T}_P : \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_P : \text{BMod}(\mathcal{P}) \) is a Quillen equivalence by Proposition 5.2.10, while the adjunction \( \mathcal{T}_P : \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_P : \text{Op}(\mathcal{S}) \) is one (when \( \mathcal{P} \) is levelwise cofibrant) by Proposition 5.3.6. These facts, along with the 2-out-of-3 property, prove the theorem.
Besides that we are interested in the case in which $\mathcal{S}$ is stable (cf. Definition 4.0.6). The following is an analogue of [[6], Lemma 2.2.3].

**Lemma 5.0.4.** Suppose that $\mathcal{S}$ is in addition stable containing a strict zero object 0 and let $M \in \text{IbMod}(\mathcal{P})$ be a levelwise cofibrant infinitesimal $\mathcal{P}$-bimodule. Then the adjunction

$$(-) \cup M : \text{IbMod}(\mathcal{P}) \rightleftarrows \text{IbMod}(\mathcal{P})_{M/\text{Ib}} : \ker$$

is a Quillen equivalence, where the functor $\ker$ is defined by sending $M \to P \to M$ to $P \times_M 0$.

**Proof.** Using the same arguments as in the loc.cit, along with noting the facts that (co)limits in IbMod($\mathcal{P}$) are taken levelwise (cf. Proposition 5.2.6) and that the base category $\mathcal{S}$ is, by convention, right proper; the proof is then straightforward.

**Theorem 5.0.5.** Suppose that $\mathcal{S}$ is in addition stable containing a strict zero object 0 and that $\mathcal{P}$ is fibrant and $\Sigma$-cofibrant. The sequence (5.0.5) is then prolonged to a sequence of Quillen equivalences of the form

$$\text{IbMod}(\mathcal{P}) \overset{(-) \cup \mathcal{P}}{\xrightarrow{\Sigma}} \text{IbMod}(\mathcal{P})_{\mathcal{P}/\mathcal{P}} \overset{\Sigma_{\mathcal{P}}}{\xrightarrow{\Omega}} \mathcal{P} \text{IbMod}(\mathcal{P}) \overset{\rho_{\mathcal{P}}}{\xrightarrow{\mathcal{P}}} \mathcal{P} \text{BMod}(\mathcal{P}) \overset{\rho_{\mathcal{P}}}{\xrightarrow{\mathcal{P}}} \mathcal{P} \text{Op}_{\mathcal{C}}(\mathcal{S}) \overset{\rho_{\mathcal{P}}}{\xrightarrow{\mathcal{P}}} \mathcal{P} \text{Op}(\mathcal{S})$$

(5.0.6)

**Proof.** The category IbMod($\mathcal{P}$) is stable since $\mathcal{S}$ is already so (cf. Remark 5.2.7), and hence the category IbMod($\mathcal{P}$)$_{\mathcal{P}/\mathcal{P}}$ is stable as well. Thus, by Facts 4.0.7, the adjunction

$$\Sigma^\infty : \text{IbMod}(\mathcal{P})_{\mathcal{P}/\mathcal{P}} \rightleftarrows \mathcal{P} \text{IbMod}(\mathcal{P}) : \Omega^\infty$$

is a Quillen equivalence. The statement hence follows by Lemma 5.0.4 and Theorem 5.0.3.

The below subsections are mostly devoted to the proof of Theorem 5.0.3. These may require making use of the **Comparison theorem** [6], which we now recall. Let $\mathcal{M}$ be a symmetric monoidal model category and let $\mathcal{O}$ be an $\mathcal{M}$-enriched operad. We denote by $\mathcal{O}_{e_1}$ the operad obtained from $\mathcal{O}$ by removing the operations of arity $\geq 1$ and by $\mathcal{O}_0$ the collection of unary (= 0-ary) operations of $\mathcal{O}$. Note that $\mathcal{O}_0$ inherits the obvious structure of an $\mathcal{O}$-algebra, then, becomes an initial object in the category $\text{Alg}_{\mathcal{O}}(\mathcal{M})$.

**Definition 5.0.6.** The operad $\mathcal{O}$ is said to be **admissible** if the transferred model structure on $\text{Alg}_{\mathcal{O}}(\mathcal{M})$ exists. Furthermore, $\mathcal{O}$ is called **stably (resp. semi) admissible** if it is admissible and the stabilization $\text{Sp}(\text{Alg}_{\mathcal{O}}^{\text{aug}}(\mathcal{M}))$ exists as a (resp. semi) model category, where $\text{Alg}_{\mathcal{O}}^{\text{aug}}(\mathcal{M}) := \text{Alg}_{\mathcal{O}}(\mathcal{M})_{\mathcal{O}_0/\mathcal{O}_0}$ the **augmented category** associated to the category $\text{Alg}_{\mathcal{O}}(\mathcal{M}) (= \text{Alg}_{\mathcal{O}}(\mathcal{M})_{\mathcal{O}_0})$.

Note that there is a canonical isomorphism $\text{Alg}_{\mathcal{O}_{e_1}}(\mathcal{M}) \cong \text{Alg}_{\mathcal{O}_1}(\mathcal{M})_{\mathcal{O}_0}$. The inclusion of operads $\varphi : \mathcal{O}_{e_1} \to \mathcal{O}$ induces a Quillen adjunction

$$\varphi_{\text{aug}}^* : \text{Alg}_{\mathcal{O}_{e_1}}^{\text{aug}}(\mathcal{M}) = \text{Alg}_{\mathcal{O}_1}(\mathcal{M})_{\mathcal{O}_0/\mathcal{O}_0} \rightleftarrows \text{Alg}_{\mathcal{O}}(\mathcal{M})_{\mathcal{O}_0/\mathcal{O}_0} = \text{Alg}_{\mathcal{O}}^{\text{aug}}(\mathcal{M}) : \varphi_{\text{aug}}^*.$$

**Theorem 5.0.7.** (**Comparison theorem**) [Y. Harpaz, J. Nuiten and M. Prasma [[6]]] Let $\mathcal{M}$ be a differentiable, left proper and combinatorial symmetric monoidal model category and let $\mathcal{O}$ be a $\Sigma$-cofibrant stably admissible operad in $\mathcal{M}$. Assume either $\mathcal{M}$ is right proper or $\mathcal{O}_0$ is fibrant. Then the induced Quillen adjunction between stabilizations

$$\varphi_{\text{Sp}}^* : \text{Sp}(\text{Alg}_{\mathcal{O}_{e_1}}^{\text{aug}}(\mathcal{M})) \rightleftarrows \text{Sp}(\text{Alg}_{\mathcal{O}}^{\text{aug}}(\mathcal{M})) : \varphi_{\text{Sp}}^*$$

is a Quillen equivalence.
Remark 5.0.8. In fact, many model categories of interest are not left proper (where Op_C(S) and Op(S) are typical examples) and as a sequel, their stabilizations do not exist as (full) model categories (cf. Remark 4.0.5). In the loc.cit, the authors were aware of this fact, and made sure to include Corollary 4.1.4 saying that the restriction functor $\varphi^*_\text{aug} : \text{Alg}_{O}^\text{aug}(M) \rightarrow \text{Alg}_{\text{aug}O}(M)$, under the same assumptions as in Proposition 5.0.7 except the left properness of M, induces an equivalence of relative categories after taking stabilizations

$$\varphi^*_{\text{Sp}} : \text{Sp}(\text{Alg}_{O}^\text{aug}(M)) \xrightarrow{\simeq} \text{Sp}(\text{Alg}_{\text{aug}O}(M)).$$

In particular, when the stabilizations exist as semi model categories then $\varphi^*_{\text{Sp}} \circ \varphi^*_{\text{Sp}}$ is indeed a Quillen equivalence. So keep in mind that the statement of the Comparison theorem remains valid when $\mathcal{P}$ is just stably semi admissible.

5.1 The first Quillen equivalence

The Quillen adjunction $L_{\mathcal{P}} : \text{Op}_C(S)_{/\mathcal{P}} \xleftarrow{\simeq} \text{Op}(S)_{/\mathcal{P}} : R_{\mathcal{P}}$ (5.0.3) lifts to a Quillen adjunction between the associated tangent categories

$$L^\text{Sp}_{/\mathcal{P}} : \text{Op}_C(S) \xleftarrow{\simeq} \text{T}_{\mathcal{P}} \text{Op}(S) : R^\text{Sp}_{/\mathcal{P}}.$$

Our goal in this subsection is to prove that this adjunction is a Quillen equivalence when provided that $\mathcal{P}$ is fibrant, yet let us start with the following simple observations.

Observations 5.1.1. (i) A given map between $C$-colored operads is a weak equivalence (resp. trivial fibration, cofibration) in $\text{Op}_C(S)$ if and only if it is a weak equivalence (resp. trivial fibration, cofibration) in $\text{Op}(S)$.

(ii) A given $C$-colored operad is cofibrant (resp. fibrant) as an object of $\text{Op}_C(S)$ if and only if it is cofibrant (resp. fibrant) as an object of $\text{Op}(S)$.

(iii) A given cofibrant resolution $\gamma_{\text{col}} : \mathcal{P} \rightarrow \mathcal{P}$ of $\mathcal{P}$ when regarded as an object of $\text{Op}_C(S)$ is also a cofibrant resolution of $\mathcal{P}$ when regarded as an object of $\text{Op}(S)$.

Proof. Let $f : \mathcal{P} \rightarrow \varnothing$ be a map between $C$-colored operads.

(i) If $f$ is a weak equivalence in $\text{Op}(S)$ then it is in particular a levelwise weak equivalence, and hence is a weak equivalence in $\text{Op}_C(S)$. Conversely, suppose that $f$ is a weak equivalence in $\text{Op}_C(S)$, i.e., a levelwise weak equivalence. Since $f$ is the identity on colors, the induced map $\text{Ho}(f_{1})$ is obviously essentially surjective (cf. Definition 2.2.1). Thus, by definition, $f$ is indeed a Dwyer-Kan equivalence, i.e., a weak equivalence in $\text{Op}(S)$. The claim about trivial fibrations immediately follows by definition. This claim implies that if $f$ is a cofibration in $\text{Op}(S)$ then it is one in $\text{Op}_C(S)$. The converse direction follows by the fact that the inclusion $L_{\mathcal{P}} : \text{Op}_C(S)_{/\mathcal{P}} \rightarrow \text{Op}(S)_{/\mathcal{P}}$ is a left Quillen functor.

(ii) The claim about the fibrancy immediately follows by definition. Now, if $\mathcal{P}$ is cofibrant as an object of $\text{Op}(S)$ then it is so as an object of $\text{Op}_C(S)$, by the claim about trivial fibrations of (i), again. For the converse direction, by the last claim of (i), it suffices to show that the initial $C$-colored operad $I_{C}$ is also cofibrant as an object of $\text{Op}(S)$. Notice that a map in $\text{Op}(S)$, from $I_{C}$ to a given operad $\varnothing$, is fully characterized by a map from $C$ to the set of colors of $\varnothing$. The claim hence follows by the fact that any trivial fibration in $\text{Op}(S)$ has underlying map between colors being surjective.

(iii) This follows by the two above.
In particular, by (ii), we usually say a certain \( C \)-colored operad is (co)fibrant without indicating precisely it is (co)fibrant as an object of \( \text{Op}_C(\mathcal{S}) \) or \( \text{Op}(\mathcal{S}) \).

The main tool for proving that the adjunction \( \mathcal{L}_p^{\text{Sp}} : \text{Op}_C(\mathcal{S}) \overset{\sim}{\longrightarrow} \text{Op}(\mathcal{S}) \overset{\mathcal{R}_p^{\text{Sp}}}{\longrightarrow} \) is a Quillen equivalence will be [[4], Corollary 2.4.9]. To be able to use this tool, we have to show that the induced Quillen adjunction

\[
\mathcal{L}_p^{\text{aug}} : \text{Op}_C(\mathcal{S})_{/\mathcal{P}} \overset{\sim}{\longrightarrow} \text{Op}(\mathcal{S})_{/\mathcal{P}} : \mathcal{R}_p^{\text{aug}}
\]

is differentiable (cf. Definition 3.0.1).

**Remark 5.1.2.** By convention, the base category \( \mathcal{S} \) is differentiable and has the class of weak equivalences being closed under sequential colimits. Thus, the (underived) colimit functor \( \text{colim} : \mathcal{S}^\mathbb{N} \to \mathcal{S} \) already preserves homotopy Cartesian squares and homotopy terminal objects. An analogue does hold for the functor \( \text{colim} : \text{Cat}(\mathcal{S})^\mathbb{N} \to \text{Cat}(\mathcal{S}) \), due to Remark 3.0.4 and [[5], Lemma 3.1.10] (saying that weak equivalences in \( \text{Cat}(\mathcal{S}) \) are closed under sequential colimits).

**Lemma 5.1.3.** The Quillen adjunction \( \mathcal{L}_p : \text{Op}_C(\mathcal{S})_{/\mathcal{P}} \overset{\sim}{\longrightarrow} \text{Op}(\mathcal{S})_{/\mathcal{P}} : \mathcal{R}_p \) is differentiable. Consequently, the induced Quillen adjunction \( \mathcal{L}_p^{\text{aug}} : \text{Op}_C(\mathcal{S})_{/\mathcal{P}} \overset{\sim}{\longrightarrow} \text{Op}(\mathcal{S})_{/\mathcal{P}} : \mathcal{R}_p^{\text{aug}} \) is differentiable as well.

**Proof.** Firstly, we claim that weak equivalences in \( \text{Op}(\mathcal{S}) \) are closed under sequential colimits. To this end, we first observe that sequential colimits of enriched operads are taken levelwise, similarly as those of enriched categories. The claim hence follows by Remark 2.2.4, together with the fact that weak equivalences in \( \mathcal{S} \) (or \( \text{Cat}(\mathcal{S}) \)) are already closed under sequential colimits.

Next, we claim that a given square in \( \text{Op}(\mathcal{S}) \) is homotopy Cartesian if and only if the following two conditions hold

(i) the induced squares of spaces of operations are homotopy Cartesian in \( \mathcal{S} \), and

(ii) the induced square of underlying categories is homotopy Cartesian in \( \text{Cat}(\mathcal{S}) \).

Notice that this statement is already correct when we forget the word “homotopy”. The claim hence follows just by Remark 2.2.4. On other hand, it is not hard to show that an object of \( \text{Op}(\mathcal{S}) \) is homotopy terminal if and only if it has spaces of operations being homotopy terminal in \( \mathcal{S} \) and has underlying category being so as an object of \( \text{Cat}(\mathcal{S}) \).

We now prove that \( \text{Op}(\mathcal{S}) \) is differentiable. By the first paragraph, it suffices to verify that the (underived) colimit functor \( \text{colim} : \text{Op}(\mathcal{S})^\mathbb{N} \to \text{Op}(\mathcal{S}) \) preserves homotopy Cartesian squares and homotopy terminal objects. This follows by the second paragraph, along with Remark 5.1.2.

We are now in position to prove the main result of this subsection.

**Proposition 5.1.4.** The adjunction \( \mathcal{L}_p^{\text{Sp}} : \mathcal{J}_p \text{Op}_C(\mathcal{S}) \overset{\sim}{\longrightarrow} \mathcal{J}_p \text{Op}(\mathcal{S}) : \mathcal{R}_p^{\text{Sp}} \) is a Quillen equivalence when provided that \( \mathcal{P} \) is fibrant.

**Proof.** Let \( \mathcal{Q} \in \text{Op}_C(\mathcal{S})_{/\mathcal{P}} \) be a fibrant object, exhibited by a diagram \( \mathcal{P} \to \mathcal{Q} \to \mathcal{P} \) in \( \text{Op}_C(\mathcal{S}) \) such that the second map is a fibration. The same arguments as in the proof of [[5], Lemma 3.1.13] show that the map between the homotopy pullbacks \( \mathcal{P} \times^h_{\mathcal{L}_p \mathcal{R}_p(\mathcal{Q})} \mathcal{P} \to \mathcal{P} \times^h_\mathcal{Q} \mathcal{P} \) is a weak
equivalence in $\mathcal{O}(\mathcal{S})$. In particular, the induced map $N_{\mathcal{Sp}}^{\mathcal{Sp}}(\mathcal{Q}) \to \Omega \mathcal{Q}$ is a weak equivalence in $\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}$. This fact, together with Lemma 5.1.3, allows us to apply [[4, Corollary 2.4.9]] to deduce that the derived counit of the Quillen adjunction $L_{\mathcal{Sp}}^{\mathcal{Sp}} \dashv R_{\mathcal{Sp}}^{\mathcal{Sp}}$ is a stable equivalence for every fibrant $\Omega$-spectrum.

It remains to show that the derived unit of $L_{\mathcal{Sp}}^{\mathcal{Sp}} \dashv R_{\mathcal{Sp}}^{\mathcal{Sp}}$ is a stable equivalence for any cofibrant object. In fact, we will show that this holds for the larger class of levelwise cofibrant objects. Since $R_{\mathcal{Sp}}^{\mathcal{Sp}} \circ L_{\mathcal{Sp}}^{\mathcal{Sp}}$ is isomorphic to the identity functor and since $R_{\mathcal{Sp}}^{\mathcal{Sp}}$ preserves weak equivalences, the derived unit of $L_{\mathcal{Sp}}^{\mathcal{Sp}} \dashv R_{\mathcal{Sp}}^{\mathcal{Sp}}$ is a weak equivalence. By the first part of [[4, Corollary 2.4.9]], the derived unit of $L_{\mathcal{Sp}}^{\mathcal{Sp}} \dashv R_{\mathcal{Sp}}^{\mathcal{Sp}}$ is a stable equivalence for any levelwise cofibrant prespectrum. But every levelwise cofibrant object in $\mathcal{P} \mathcal{O}(\mathcal{S})$ is stably equivalent to a levelwise cofibrant prespectrum (see [4, Remark 2.3.6]), it therefore suffices to show that $L_{\mathcal{Sp}}^{\mathcal{Sp}}$ preserves stable equivalences. Let us see how it goes.

Our treatment is motivated by the following observation: (*) For every $\mathcal{Q} \in \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}$ and $\mathcal{R} \in \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}$ (exhibited by a diagram $\mathcal{P} \xrightarrow{f} \mathcal{R} \to \mathcal{P}$ in $\mathcal{O}(\mathcal{S})$), there is a natural homotopy equivalence of derived mapping spaces
\begin{equation}
\text{Map}^h_{\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(\mathcal{Q}, \mathcal{R}) \simeq \text{Map}^h_{\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(\mathcal{Q}, f^*\mathcal{R})
\tag{5.1.1}
\end{equation}
where $f^* : \mathcal{O}(\mathcal{S}) \to \mathcal{O}(\mathcal{S})$ is the changing-colors functor (see Section 2). To prove (*), let us assume without loss of generality that $\mathcal{Q}$ is cofibrant and $\mathcal{R}$ is fibrant. Take $\mathcal{Q}^\bullet$ to be a cosimplicial resolution of $\mathcal{Q}$ in $\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}$. Since the inclusion $\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}} \to \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}$ is a left Quillen functor, $\mathcal{Q}^\bullet$ itself is a cosimplicial resolution of $\mathcal{Q}$ in $\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}$. Then we find the desired equivalence as follows
\begin{equation}
\text{Map}^h_{\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(\mathcal{Q}, \mathcal{R}) \simeq \text{Hom}_{\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(\mathcal{Q}^\bullet, \mathcal{R}) \simeq \text{Hom}_{\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(\mathcal{Q}^\bullet, f^*\mathcal{R}) \simeq \text{Map}^h_{\mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(\mathcal{Q}, f^*\mathcal{R}).
\end{equation}

Now, let $Y \sim Y'$ be a stable equivalence in $\mathcal{P} \mathcal{O}(\mathcal{S})$. By definition, we need to show that for every fibrant $\Omega$-spectrum $Z \in \mathcal{P} \mathcal{O}(\mathcal{S})$ exhibited by the diagrams $\mathcal{P} \xrightarrow{f_{n}} Z_{n,n} \xrightarrow{g_{n}} \mathcal{P}$ $(n \geq 0)$, the induced map
\begin{equation}
\text{Map}^h_{\mathcal{P} \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(Y', Z) \to \text{Map}^h_{\mathcal{P} \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(Y, Z)
\tag{5.1.2}
\end{equation}
is a homotopy equivalence (see Definition 4.0.4). It is very similar to (5.1.1) to show that there are natural homotopy equivalences
\begin{equation}
\text{Map}^h_{\mathcal{P} \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(Y, Z) \simeq \text{Map}^h_{\mathcal{P} \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(Y, f^*Z)
\tag{5.1.3}
\end{equation}
\begin{equation}
\text{Map}^h_{\mathcal{P} \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(Y', Z) \simeq \text{Map}^h_{\mathcal{P} \mathcal{O}(\mathcal{S})_{\mathcal{P}/\mathcal{P}}}(Y', f^*Z)
\tag{5.1.4}
\end{equation}
in which $f^*Z$ refers to the prespectrum with $(f^*Z)_{n,n} = f^*_n Z_{n,n}$. (The $f^*_n$’s are again the changing-colors functors). We are showing that $f^*Z$ is an $\Omega$-spectrum. To this end, we have to verify that for every $n$ the following square
\begin{equation}
\begin{array}{ccc}
f^*_n Z_{n,n} & \xrightarrow{f_{n+1}} & \mathcal{P} \\
\downarrow & & \downarrow \\
\mathcal{P} & \to & f^*_{n+1} Z_{n+1,n+1}
\end{array}
\tag{5.1.5}
\end{equation}
is homotopy Cartesian in $\mathcal{O}(\mathcal{S})$. Note that this square agrees with the image through $f^*_n$ of
the square

\[
\begin{array}{ccc}
Z_{n,n} & \longrightarrow & g_n^*P \\
\downarrow & & \downarrow \\
g_n^*P & \longrightarrow & g_n^*f_{n+1}^*Z_{n+1,n+1}
\end{array}
\]

of operads with the fixed set of colors being that of \(Z_{n,n}\). By the assumption that \(Z\) is an \(\Omega\)-spectrum and by the second paragraph of the proof of Lemma 5.1.3, the latter square is homotopy Cartesian. It implies that the square (5.1.5) is indeed homotopy Cartesian, and hence \(f^*Z\) is an \(\Omega\)-spectrum, as expected. Finally, the naturalities of the equivalences (5.1.3) and (5.1.4) tell us that the map (5.1.2) is weakly equivalent to the map

\[
\text{Map}^h_{\text{Op}(\mathcal{S})\text{Col}(\mathcal{S})}(Y', f^*Z) \longrightarrow \text{Map}^h_{\text{Op}(\mathcal{S})\text{Col}(\mathcal{S})}(Y, f^*Z),
\]

which is indeed an equivalence because the map \(Y \xrightarrow{\nu} Y'\) is a stable equivalence in \(T_{\mathcal{P}}\text{Op}(\mathcal{S})\) and \(f^*Z\) is an \(\Omega\)-spectrum. So the proof is completed.

\[\square\]

5.2 Operadic infinitesimal bimodules - The second Quillen equivalence

We will construct an \(\mathcal{S}\)-enriched category encoding infinitesimal \(\mathcal{P}\)-bimodules (an uncolored version of this construction can be found in [33], § 2). We then prove that \(T_{\mathcal{P}}\text{Op}(\mathcal{S})\) is Quillen equivalent to \(T_{\mathcal{P}}\text{IbMod}(\mathcal{P})\).

But, as promised, we shall first give the definition of infinitesimal \(\mathcal{P}\)-bimodule in diagrammatical style reformulating the one given in Definition 5.0.1. To this end, let us start with the notion of infinitesimal composite product:

\[-\circ_{(1)} - : \text{Coll}_C(\mathcal{S}) \times \text{Coll}_C(\mathcal{S}) \longrightarrow \text{Coll}_C(\mathcal{S}),\]

which can be thought of as the “right linearization” of the well known operadic composite product \(-\circ - : \text{Coll}_C(\mathcal{S}) \times \text{Coll}_C(\mathcal{S}) \longrightarrow \text{Coll}_C(\mathcal{S})\). Formally, given two \(C\)-collections \(M\) and \(N\), \(M \circ_{(1)} N\) is by definition the sub-collection of \(M \circ (J_{\mathcal{C}} \cup N)\) which is linear in \(N\). To be precise, looking at the explicit formula of \(M \circ (J_{\mathcal{C}} \cup N)\) (cf., e.g., [6]), on each level, \((M \circ_{(1)} N)(c_1, \ldots, c_n; c)\) is the sub-object of \(M \circ (J_{\mathcal{C}} \cup N)(c_1, \ldots, c_n; c)\) consisting of the multi-tensor products which contain one and only one factor in \(N\). The readers can find out about this construction, in terms of single-colored dg operads, in [7], Section 6.1.

Now, observe that, for each \(M \in \text{Coll}_C(\mathcal{S})\), there is a natural inclusion

\[\mathcal{P} \circ_{(1)} (\mathcal{P} \circ_{(1)} M) \longrightarrow (\mathcal{P} \circ_{(1)} \mathcal{P}) \circ_{(1)} M.\]

On other hand, the partial composition in \(\mathcal{P}\) gives a map \(\mu_{(1)} : \mathcal{P} \circ_{(1)} \mathcal{P} \longrightarrow \mathcal{P}\). The following is equivalent to Definition 5.0.1(i).

**Definition 5.2.1.** An **infinitesimal left \(\mathcal{P}\)-module** is a \(C\)-collection \(M\) equipped with a map \(\mathcal{P} \circ_{(1)} M \longrightarrow M\) satisfying the classical axioms of associativity and unitality for left modules, which are depicted as the commutativities of the following diagrams
Dually, an infinitesimal right \( P \)-module is a \( C \)-collection \( M \) equipped with an action map \( M \circ_{(1)} P \to M \) satisfying the classical axioms of associativity and unitality for right modules. However, it can be observed that the structure of an infinitesimal right \( P \)-module is the same as that of a right \( P \)-module.

Now, notice that for each \( M \in \text{Coll}_C(S) \), there is a natural inclusion

\[
(P \circ_{(1)} M) \circ P \to (P \circ P) \circ_{(1)} (M \circ P).
\]

The following is equivalent to Definition 5.0.1(iii).

**Definition 5.2.2.** An infinitesimal \( P \)-bimodule is a \( C \)-collection \( M \) endowed with an infinitesimal left \( P \)-module structure, exhibited by a map \( P \circ_{(1)} M \to M \) and with a right \( P \)-module structure, exhibited by a map \( M \circ P \to M \) which are subject to the essential compatibility, depicted as the commutativity of the following diagram

\[
\begin{array}{ccc}
(P \circ_{(1)} M) \circ P & \xrightarrow{\delta} & (P \circ P) \circ_{(1)} (M \circ P) \\
(P \circ P) \circ_{(1)} (M \circ P) & & M \circ P \\
(P \circ_{(1)} M) & \xrightarrow{\delta} & M \circ P \\
\end{array}
\]

**Remark 5.2.3.** In the above diagram, the \( C \)-collection \( (P \circ_{(1)} M) \circ P \) does not present the free infinitesimal \( P \)-bimodule generated by \( M \). (This does not even admit a canonical infinitesimal \( P \)-bimodule structure). To find out the exact one, we factor the free functor \( \text{Coll}_C(S) \to \text{IbMod}(P) \) as \( \text{Coll}_C(S) \xrightarrow{F_1} \text{RMod}(P) \xrightarrow{F_2} \text{IbMod}(P) \) where \( F_1 \) (\( F_2 \)) refers to the left adjoint of the associated forgetful functor. Observe now that \( F_1 \simeq (-) \circ P \), while \( F_2 \simeq P \circ_{(1)} (-) \). In conclusion, the functor \( F^b \) is given as \( F^b = P \circ_{(1)} (- \circ P) \). On other hand, the free infinitesimal left \( P \)-module functor is simply \( P \circ_{(1)} (-) \).

We shall now turn to our main interests in this subsection.

**Notation 5.2.4.** We will write \( \text{Fin}_* \) standing for the category whose objects are finite pointed sets \( \langle m \rangle := \{0,1,\ldots,m\} \) (with 0 as the basepoint), \( m \geq 0 \), and whose morphisms are basepoint-preserving maps.
CONSTRUCTION 5.2.5. We now establish an $S$-enriched category, $\mathbf{Ib}^\mathcal{P}$, which encodes infinitesimal $\mathcal{P}$-bimodules. The set of objects of $\mathbf{Ib}^\mathcal{P}$ is $\text{Seq}(C)$, while its mapping spaces are defined as follows. For each map $(m) \xrightarrow{f} (n)$ in $\text{Fin}_*$, we denote by

$$\text{Map}^f_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c), (d_1, \ldots, d_m; d)) := \mathcal{P}(c, \{d_j\}_{j \in f^{-1}(0)}; d) \otimes \mathcal{P}(\{d_j\}_{j \in f^{-1}(1)}; c_i)$$

in which, for each $k \in \{0, \ldots, n\}$, the elements of $\{d_j\}_{j \in f^{-1}(k)}$ are put in the natural ascending order of $j \in \{1, \ldots, m\}$. Then, we define

$$\text{Map}_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c), (d_1, \ldots, d_m; d)) := \bigsqcup_{(m) \xrightarrow{f} (n)} \text{Map}^f_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c), (d_1, \ldots, d_m; d)).$$

Observe that

$$\text{Map}^{\text{Id}(n)}_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c), (c_1, \ldots, c_n; c)) = \mathcal{P}(c; c) \otimes \mathcal{P}(c_1; c_1) \otimes \cdots \otimes \mathcal{P}(c_n; c_n).$$

This suggests that we can define the unit morphisms of $\mathbf{Ib}^\mathcal{P}$, canonically, via the unit operations of $\mathcal{P}$. Moreover, the categorical structure maps of $\mathbf{Ib}^\mathcal{P}$ are canonically defined via the composition in $\mathcal{P}$, along with the symmetric action on $\mathcal{P}$.

PROPOSITION 5.2.6. There is a canonical isomorphism

$$\text{IbMod}(\mathcal{P}) \cong \text{Fun}(\mathbf{Ib}^\mathcal{P}, S)$$

between the category of infinitesimal $\mathcal{P}$-bimodules and the category of $S$-valued enriched functors on $\mathbf{Ib}^\mathcal{P}$.

Proof. (1) Let $M : \mathbf{Ib}^\mathcal{P} \rightarrow S$ be an enriched functor given on objects by

$$M = \{M(c_1, \ldots, c_n; c)\}_{(c_1, \ldots, c_n; c) \in \text{Seq}(C)}.$$

It is very natural, by the construction of $\mathbf{Ib}^\mathcal{P}$, to establish the canonical infinitesimal $\mathcal{P}$-bimodule structure on such data of $M$. For instance, the symmetric action on $M$ is defined as follows. Each permutation $\alpha \in \Sigma_n$ determines a map $(n) \xrightarrow{\alpha} (n)$. Now, note that the component of

$$\text{Map}^\mathcal{P}_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c), (c_{\alpha(1)}, \ldots, c_{\alpha(n)}; c))$$

indexed by $\alpha$ is

$$\text{Map}^\mathcal{P}_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c), (c_{\alpha(1)}, \ldots, c_{\alpha(n)}; c)) = \mathcal{P}(c; c) \otimes \mathcal{P}(c_1; c_1) \otimes \cdots \otimes \mathcal{P}(c_n; c_n).$$

In particular, the enriched functor structure map of $M$

$$\text{Map}^\mathcal{P}_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c, (c_{\alpha(1)}, \ldots, c_{\alpha(n)}; c)) \otimes M(c_1, \ldots, c_n; c) \rightarrow M(c_{\alpha(1)}, \ldots, c_{\alpha(n)}; c)$$

has a component given as

$$\mathcal{P}(c; c) \otimes \mathcal{P}(c_1; c_1) \otimes \cdots \otimes \mathcal{P}(c_n; c_n) \otimes M(c_1, \ldots, c_n; c) \rightarrow M(c_{\alpha(1)}, \ldots, c_{\alpha(n)}; c).$$

Now, the evaluation at the unit operations $\text{id}_c, \text{id}_{c_1}, \ldots, \text{id}_{c_n}$ of $\mathcal{P}$ determines the symmetric action of typical form: $M(c_1, \ldots, c_n; c) \xrightarrow{\text{Id}_{\mathcal{P}}} M(c_{\alpha(1)}, \ldots, c_{\alpha(n)}; c)$.

(2) Conversely, let $M$ be an infinitesimal $\mathcal{P}$-bimodule. We want to see how $M$ admits a canonical enriched functor structure $\mathbf{Ib}^\mathcal{P} \rightarrow S$. We have to define the maps of the form

$$\text{Map}^\mathcal{P}_{\mathbf{Ib}^\mathcal{P}}((c_1, \ldots, c_n; c), (d_1, \ldots, d_m; d)) \otimes M(c_1, \ldots, c_n; c) \rightarrow M(d_1, \ldots, d_m; d).$$

This map must consist of, for each $(m) \xrightarrow{f} (n)$, a component map of the form

$$\mathcal{P}(c, \{d_j\}_{j \in f^{-1}(0)}; d) \otimes \mathcal{P}(\{d_j\}_{j \in f^{-1}(1)}; c_i) \otimes M(c_1, \ldots, c_n; c) \rightarrow M(d_1, \ldots, d_m; d).$$
The latter can be naturally defined using the (two sided) infinitesimal \( \mathcal{P} \)-action on \( M \), along with the symmetric action on \( M \) (we will revisit this in Notation 7.1.6).

The explicit verifications are elephantine, but not complicated.

**Remark 5.2.7.** As a consequence, the category \( \text{IbMod}(\mathcal{P}) \) admits the canonical transferred model structure, which coincides with the projective model structure on \( \text{Fun}(\text{Ib}^{\mathcal{P}}, S) \). In particular, \( \text{IbMod}(\mathcal{P}) \) is stable as long as \( S \) is stable (see [6], Remark 2.2.2).

**Construction 5.2.8.** There is also an enriched category encoding the category of right \( \mathcal{P} \)-modules, defined as follows. Let \( \text{Fin} \) denote the smallest skeleton of the category of finite sets whose set of objects consists of \( \emptyset := \emptyset \) and \( m := \{1,\ldots,m\} \) for \( m \geq 1 \). Let us denote by \( \text{R}^{\mathcal{P}} \) the category whose set of objects is \( \text{Seq}(C) \) and whose mapping objects are given as

\[
\text{Map}_{\text{R}^{\mathcal{P}}}((c_1,\ldots,c_n;c),(d_1,\ldots,d_m;d)) := \bigsqcup_{m^*} \mathcal{P}(c;d) \otimes \bigotimes_{i=1,\ldots,n} \mathcal{P}\{d_j\}_{j \in f^{-1}(i)};c_i\).
\]

It can then be verified that \( \text{RMod}^{\mathcal{P}} \cong \text{Fun}(\text{R}^{\mathcal{P}}, S) \), similarly as in the proof of Proposition 5.2.6. In particular, the category \( \text{RMod}^{\mathcal{P}} \) admits the transferred model structure, which is stable whenever \( S \) is so.

**Proposition 5.2.9.** Suppose that \( \mathcal{P} \) is a cofibrant \( C \)-colored operad. Then the adjunction \( \text{IbMod}(\mathcal{P})_{/\mathcal{P}} \rightleftarrows \mathcal{P}_{/\mathcal{P}} \) induces a Quillen equivalence between the associated tangent categories \( \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}} \mathcal{P}_{/\mathcal{P}} \). Consequently, the adjunction \( \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}} \mathcal{P}_{/\mathcal{P}} \) is a Quillen equivalence provided that \( \mathcal{P} \) is in addition fibrant.

**Proof.** We regard \( \mathcal{P} \) as an algebra over \( \mathcal{O}_\mathcal{C} \) the operad of \( C \)-colored operads. Then we get a canonical isomorphism \( \text{Alg}_{\text{Env}(\mathcal{O}_\mathcal{C},\mathcal{P})}^{\mathcal{P}}(S) \cong \mathcal{P}_{/\mathcal{P}} \mathcal{O}^{\mathcal{P}}_{\mathcal{C}}(S) \) between the categories of algebras over the enveloping operad \( \text{Env}(\mathcal{O}_\mathcal{C},\mathcal{P}) \) and of \( C \)-colored operads under \( \mathcal{P} \). On other hand, the same arguments as in the proof of [[24], Proposition 3.5] show that the structure of an infinitesimal \( \mathcal{P} \)-bimodule is equivalent to that of a \( \mathcal{P} \)-module over \( \mathcal{O}_\mathcal{C} \). So we have a canonical isomorphism of categories \( \text{Alg}_{\text{Env}(\mathcal{O}_\mathcal{C},\mathcal{P})}^{\mathcal{P}}(S) \cong \text{IbMod}(\mathcal{P}) \) (see Section 2).

We are now applying the Comparison theorem 5.0.7 (along with noting Remark 5.0.8) to the operad \( \text{Env}(\mathcal{O}_\mathcal{C},\mathcal{P}) \). By construction, \( \mathcal{O}_\mathcal{C} \) is clearly \( \Sigma \)-cofibrant. Moreover, since \( \mathcal{P} \) is cofibrant, it implies that \( \text{Env}(\mathcal{O}_\mathcal{C},\mathcal{P}) \) is \( \Sigma \)-cofibrant as well (cf. [[18], Lemma 6.1]). This, together with the combinatoriality of \( S \), makes the Comparison theorem work in our data. The first paragraph shows that the functor

\[
\text{Alg}_{\text{Env}(\mathcal{O}_\mathcal{C},\mathcal{P})}^{\mathcal{P}}(S) \longrightarrow \text{Alg}_{\text{Env}(\mathcal{O}_\mathcal{C},\mathcal{P})}^{\mathcal{P}}(S)
\]

turns out to coincide with the left Quillen functor \( \text{IbMod}(\mathcal{P})_{/\mathcal{P}} \longrightarrow \text{Op}_{\mathcal{C}}(\mathcal{P})_{/\mathcal{P}} \). So the adjunction \( \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}} \mathcal{P}_{/\mathcal{P}} \) is indeed a Quillen equivalence. Finally, by combining the latter fact with Proposition 5.1.4, we deduce that the adjunction \( \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}} \mathcal{P}_{/\mathcal{P}} \) is a Quillen equivalence when provided that \( \mathcal{P} \) is bifibrant (i.e., both fibrant and cofibrant).

The cofibrancy of \( \mathcal{P} \) as required in this proposition is very strict and should be refined in order that it can work in the larger class of \( \Sigma \)-cofibrant operads.

**Proposition 5.2.10.** The second statement of Proposition 5.2.9 is already correct when \( \mathcal{P} \) is fibrant and \( \Sigma \)-cofibrant.
Proof. By Observations 5.1.1, we can take \( f : Q \xrightarrow{\sim} P \) to be a bifibrant resolution of \( P \) in \( \text{Op}(\mathcal{S}) \) such that \( f \) is a map in \( \text{Op}_C(\mathcal{S}) \). By Proposition 5.2.9, we have a Quillen equivalence \( \mathcal{T}_\mathcal{S} \text{IbMod}(\mathcal{Q}) \xrightarrow{\sim} \mathcal{T}_\mathcal{S} \text{Op}(\mathcal{S}) \). Thus, by the naturality, it suffices to prove that the induced adjunctions \( \text{IbMod}(\mathcal{Q})_{\mathcal{Q}/\mathcal{Q}} \rightleftarrows \text{IbMod}(\mathcal{P})_{\mathcal{P}/\mathcal{P}} \) and \( \text{Op}(\mathcal{S})_{\mathcal{Q}/\mathcal{Q}} \rightleftarrows \text{Op}(\mathcal{S})_{\mathcal{P}/\mathcal{P}} \) are a Quillen equivalence.

Let us start with the second one. We first show that the adjunction \( f : \text{Op}(\mathcal{S})_{\mathcal{Q}/\mathcal{Q}} \rightleftarrows \text{Op}(\mathcal{S})_{\mathcal{P}/\mathcal{P}} : f^* \) is a Quillen equivalence. Since \( f^* \) creates weak equivalences, it suffices to verify that, for any cofibration \( Q \rightarrow R \) in \( \text{Op}(\mathcal{S}) \), the induced map \( R \rightarrow R \cup_{\mathcal{Q}} P \) is a weak equivalence. This immediately follows by the relative left properness of \( \text{Op}(\mathcal{S}) \) (cf. [12], Theorem 6.7). Now we want to prove that \( f^* : \text{Op}(\mathcal{S})_{\mathcal{Q}/\mathcal{Q}} \rightleftarrows \text{Op}(\mathcal{S})_{\mathcal{P}/\mathcal{P}} : f^* \) is a Quillen equivalence. Our treatment makes use of the following observation, which can be readily verified using definition:

\[(*) \text{ Suppose given a Quillen equivalence } F : M \xrightarrow{\sim} N : G \text{ between right proper model categories. Let } \alpha : F(A) \xrightarrow{\sim} B \text{ be a weak equivalence in } N \text{ with } A \in M \text{ being cofibrant and with } B \in N \text{ being fibrant. Then the induced adjunction } F_{\alpha} : M_{/A} \rightleftarrows N_{/B} : G_{\alpha} \text{ is a Quillen equivalence.}\]

By applying this \((*)\) to the data of \( M = \text{Op}(\mathcal{S})_{\mathcal{Q}/\mathcal{Q}} \), \( N = \text{Op}(\mathcal{S})_{\mathcal{P}/\mathcal{P}} \), \( A = \text{Id}_\mathcal{Q} \), and \( B = \text{Id}_\mathcal{P} \), we deduce that the adjunction \( f_{\text{aug}}^* : \text{Op}(\mathcal{S})_{\mathcal{Q}/\mathcal{Q}} \rightleftarrows \text{Op}(\mathcal{S})_{\mathcal{P}/\mathcal{P}} : f_{\text{aug}}^* \) is indeed a Quillen equivalence.

It remains to prove that the adjunction \( \text{IbMod}(\mathcal{Q})_{\mathcal{Q}/\mathcal{Q}} \rightleftarrows \text{IbMod}(\mathcal{P})_{\mathcal{P}/\mathcal{P}} \) is a Quillen equivalence. Since both \( \mathcal{Q} \) and \( \mathcal{P} \) are levelwise cofibrant, the map \( f : \mathcal{Q} \xrightarrow{\sim} \mathcal{P} \) induces a weak equivalence \( \text{Ib}^{\mathcal{Q}} \xrightarrow{\sim} \text{Ib}^{\mathcal{P}} \) of \( \mathcal{S} \)-enriched categories (see Construction 5.2.5). So the induced adjunction

\[\text{IbMod}(\mathcal{Q}) = \text{Fun}(\text{Ib}^{\mathcal{Q}}, \mathcal{S}) \rightleftarrows \text{IbMod}(\mathcal{P}) = \text{Fun}(\text{Ib}^{\mathcal{P}}, \mathcal{S})\]

is a Quillen equivalence. The claim can then be verified in the same fashion as above.

\[\square\]

5.3 Operadic bimodules - The third Quillen equivalence

Let \( \mathcal{P} \) be an \( \mathcal{S} \)-enriched \( C \)-colored operad. We shall construct an operad whose algebras are precisely \( \mathcal{P} \)-bimodules under \( \mathcal{P} \). In light of this construction, we may prove that \( \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \) is Quillen equivalent to \( \mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}) \).

Proposition 5.3.1. The (projective) transferred model structures on \( \text{BMod}(\mathcal{P}) \) and \( \text{LMod}(\mathcal{P}) \) exist.

Proof. By convention, \( \mathcal{S} \) admits a symmetric monoidal fibrant replacement functor and a functorial path data. The proof is then straightforward, as well as the proof of Proposition 2.1.3. \[\square\]

The following will be helpful for our work in Section 6.

Lemma 5.3.2. The forgetful functor \( \text{U} : \text{BMod}(\mathcal{P}) \rightarrow \text{LMod}(\mathcal{P}) \) is a left Quillen functor provided that \( \mathcal{P} \) is \( \Sigma \)-cofibrant.

Proof. The functor \( \text{U} \) is given by forgetting the right \( \mathcal{P} \)-action, which is certainly linear. It implies that \( \text{U} \) preserves colimits. The adjoint functor theorem, along with the combinatoriality of \( \mathcal{S} \), hence shows that \( \text{U} \) is indeed a left adjoint functor. Since \( \text{U} \) preserves weak equivalences, the proof will be completed after showing that it preserves cofibrations. To this end, we first prove that every cofibration in \( \text{RMod}(\mathcal{P}) \) has underlying map in \( \text{Coll}_C(\mathcal{S}) \) being a cofibration as well. Observe that the model structure on \( \text{RMod}(\mathcal{P}) \) admits a set of generating cofibrations given as \( \{ i \circ \mathcal{P} : M \circ \mathcal{P} \rightarrow N \circ \mathcal{P} \}_i \), where the map \( i : M \rightarrow N \) ranges over the set of those of \( \text{Coll}_C(\mathcal{S}) \). Since
the forgetful functor $\text{RMod}(\mathcal{P}) \rightarrow \text{Coll}_C(\mathcal{S})$ is a left adjoint functor, it suffices to show that each map $i \circ \mathcal{P} : M \circ \mathcal{P} \rightarrow N \circ \mathcal{P}$ is a cofibration in $\text{Coll}_C(\mathcal{S})$. Let $\emptyset_C$ denote the initial $C$-collection, which agrees with $\emptyset_S$ on every level. Then, factor $i \circ \mathcal{P}$ as $M \circ \mathcal{P} \rightarrow N \circ \emptyset_C \sqcup_M M \circ \mathcal{P} \rightarrow N \circ \mathcal{P}$.

In this composition, the first map is a cofibration because the map $i \circ \emptyset_C : M \circ \emptyset_C \rightarrow N \circ \emptyset_C$, which agrees with the underlying map of $i$ between the collections of level 0, is one. The second map is also a cofibration by [[8], Lemma 11.5.1], along with the $\Sigma$-cofibrancy of $\mathcal{P}$. Thus, $i \circ \mathcal{P}$ is indeed a cofibration. It can be shown that the model structure on $\text{BMod}(\mathcal{P})$ admits a set of generating cofibrations given as $\{\mathcal{P} \circ j : \mathcal{P} \circ K \rightarrow \mathcal{P} \circ L\}_{j}$ where the map $j : K \rightarrow L$ ranges over the set of those of $\text{RMod}(\mathcal{P})$. It therefore suffices to show that each map $\mathcal{P} \circ j : \mathcal{P} \circ K \rightarrow \mathcal{P} \circ L$ is a cofibration in $\text{LMod}(\mathcal{P})$. This is now clear since $\mathcal{P} \circ (-)$ presents the free left $\mathcal{P}$-module functor $\text{Coll}_C(\mathcal{S}) \rightarrow \text{LMod}(\mathcal{P})$, which is a left Quillen functor, and since $j$ is a cofibration in $\text{Coll}_C(\mathcal{S})$, as indicated above. 

\[ \Box \]

**Remark 5.3.3.** As well as every type of monoid, for every $\mathcal{P}$-bimodule $M$, there is a canonical isomorphism

\[
\text{Hom}_{\text{BMod}(\mathcal{P})}(\mathcal{P}, M) \cong \text{Hom}_{\text{Coll}_C(\mathcal{S})}(\mathcal{I}_C, M)
\]

in which $\text{Hom}_{\text{Coll}_C(\mathcal{S})}(\mathcal{I}_C, M) \subseteq \text{Hom}_{\text{Coll}_C(\mathcal{S})}(\mathcal{I}_C, M)$ is the subset consisting of those $\varepsilon : \mathcal{I}_C \rightarrow M$ making the following diagram commutative

\[
\begin{array}{ccc}
\mathcal{P} \circ \mathcal{I}_C = \mathcal{I}_C \circ \mathcal{P} & \xrightarrow{\varepsilon \circ \mathcal{P}} & M \circ \mathcal{P} \\
\varepsilon \circ \mathcal{P} & \downarrow & \downarrow M \circ \mathcal{P} \\
\mathcal{P} \circ M & \xrightarrow{\varepsilon} & M
\end{array}
\]

**Construction 5.3.4.** The $\mathcal{S}$-enriched operad which encodes $\mathcal{P}$-bimodules under $\mathcal{P}$ will be denoted by $\mathbb{B}^{\mathcal{P}/}$. Its set of colors is again $\text{Seq}(C)$. The unary operations of $\mathbb{B}^{\mathcal{P}/}$ agree with $\mathcal{P}$, i.e.,

\[
\mathbb{B}^{\mathcal{P}/}(\varepsilon_1, \ldots, c_n ; c) := \mathcal{P}(c_1, \ldots, c_n ; c),
\]

while its 1-ary operations coincide with those of $\mathbb{B}^{\mathcal{P}}$ (see Construction 5.2.5), i.e.,

\[
\mathbb{B}^{\mathcal{P}/}(\varepsilon_1, \ldots, c_n ; c) ; (d_1, \ldots, d_m ; d) = \bigcup_{\langle m \rangle \rightarrow \langle n \rangle} \left[ \mathcal{P}(c_1, \ldots, c_n) \otimes \bigotimes_{i=1, \ldots, n} \mathcal{P}\left(\{d_j\}_{j \in f^{-1}(i)} ; c_i\right) \right]
\]

where $f$ ranges over the set $\text{Hom}_{\text{Fin}}((\langle m \rangle), (\langle n \rangle))$. Then we may extend the above formula to obtain the spaces of operations of higher arities. For instance, a typical space of binary (= 2-ary) operations of $\mathbb{B}^{\mathcal{P}/}$ should be written as

\[
\mathbb{B}^{\mathcal{P}/}(\varepsilon_1, \ldots, c_r ; c_1, \ldots, c_{r+s} ; c') ; (d_1, \ldots, d_m ; d)
\]

\[
= \bigcup_{\langle m \rangle \rightarrow \langle r+s \rangle} \left[ \mathcal{P}(c, c') , \{d_j\}_{j \in f^{-1}(0)} ; d \right] \otimes \bigotimes_{i=1, \ldots, r+s} \mathcal{P}(\{d_j\}_{j \in f^{-1}(i)} ; c_i)
\]

The composition of $\mathbb{B}^{\mathcal{P}/}$ is canonically defined via the composition of $\mathcal{P}$ while its unit operations are exactly those of $\mathbb{B}^{\mathcal{P}}$. We just defined $\mathbb{B}^{\mathcal{P}/}$ as a nonsymmetric operad. The operad we will care about, still denoted by $\mathbb{B}^{\mathcal{P}/}$, is in fact the symmetrization of the previous one.
Proposition 5.3.5. There is a canonical isomorphism
\[ \text{BMod}(\mathcal{P})_{\mathcal{P}/} \cong \text{Alg}_{\mathcal{B}_{\mathcal{P}/}}(\mathcal{S}) \]
between the category of \( \mathcal{P} \)-bimodules under \( \mathcal{P} \) and the category of algebras over \( \mathcal{B}^{\mathcal{P}/} \).

Proof. (1) Let \( M \) be a \( \mathcal{B}^{\mathcal{P}/} \)-algebra. First, note that since the underlying category of \( \mathcal{B}^{\mathcal{P}/} \) agrees with \( \text{Ib}^\mathcal{P} \), \( M \) inherits a canonical right \( \mathcal{P} \)-module structure (cf. Proposition 5.2.6).

Now, let us see how \( M \) comes equipped with a left \( \mathcal{P} \)-action. For simplicity, we only establish the action maps of the form
\[ \mathcal{P}(c, d; e) \otimes M(c_1, \ldots, c_n; c) \otimes M(d_1, \ldots, d_m; d) \rightarrow M(c_1, \ldots, c_n, d_1, \ldots, d_m; e) \]  \hspace{1cm} (5.3.1)
To this end, observe first that the \( \mathcal{B}^{\mathcal{P}/} \)-algebra structure map of \( M \) of the form
\[ \mathcal{B}^{\mathcal{P}/}((c_1, \ldots, c_n; c), (d_1, \ldots, d_m; d); (c_1, \ldots, c_n, d_1, \ldots, d_m; e)) \otimes \]
\[ \otimes M(c_1, \ldots, c_n; c) \otimes M(d_1, \ldots, d_m; d) \rightarrow M(c_1, \ldots, c_n, d_1, \ldots, d_m; e) \]
has a component given as
\[ \mathcal{P}(c, d; e) \otimes \mathcal{P}(c_1; c_1) \otimes \cdots \otimes \mathcal{P}(c_n; c_n) \otimes \mathcal{P}(d_1; d_1) \otimes \cdots \otimes \mathcal{P}(d_m; d_m) \otimes \]
\[ \otimes M(c_1, \ldots, c_n; c) \otimes M(d_1, \ldots, d_m; d) \rightarrow M(c_1, \ldots, c_n, d_1, \ldots, d_m; e) \]
The evaluation at the unit operations \( \text{id}_{c_1}, \ldots, \text{id}_{c_n}, \text{id}_{d_1}, \ldots, \text{id}_{d_m} \) of \( \mathcal{P} \) to the latter gives us the action map (5.3.1), as desired.

Finally, the action of the unary operations of \( \mathcal{B}^{\mathcal{P}/} \) on \( M \) gives us a canonical map \( \mathcal{P} \rightarrow M \).

(2) Conversely, let \( M \) be a \( \mathcal{P} \)-bimodule under \( \mathcal{P} \). We let the composite map \( \varepsilon : \mathcal{I}_C \rightarrow \mathcal{P} \rightarrow M \) exhibit the images of the unit operations of \( \mathcal{P} \) in \( M \). In order to establish the canonical \( \mathcal{B}^{\mathcal{P}/} \)-algebra structure on \( M \), one will need to make use of the \( \mathcal{P} \)-bimodule structure of \( M \), along with some suitable involvement of \( \varepsilon \).

The explicit verifications are elephantine, but not complicated (with a help of Remark 5.3.3 at some points).

\[ \square \]

Proposition 5.3.6. The adjunction \( \text{IbMod}(\mathcal{P})_{\mathcal{P}/} \rightleftarrows \text{BMod}(\mathcal{P})_{\mathcal{P}/} \) induces a Quillen equivalence of the associated tangent categories \( \mathcal{T}_x \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_x \text{BMod}(\mathcal{P}) \) whenever \( \mathcal{P} \) is levelwise cofibrant.

Proof. Suppose that \( \mathcal{P} \) is levelwise cofibrant. Since \( \mathcal{B}^{\mathcal{P}/} \) is the symmetrization of a levelwise cofibrant nonsymmetric operad, it is automatically \( \Sigma \)-cofibrant. We are now applying the Comparison theorem 5.0.7, along with noting Remark 5.0.8, to the operad \( \mathcal{B}^{\mathcal{P}/} \). The keypoint is that the adjunction \( \text{Alg}^\text{aug}_{\mathcal{B}_{\mathcal{P}/}}(\mathcal{S}) \rightleftarrows \text{Alg}^\text{aug}_{\mathcal{B}^{\mathcal{P}/}}(\mathcal{S}) \) which arises from the inclusion \( (\mathcal{B}^{\mathcal{P}/})_\Sigma \rightarrow \mathcal{B}^{\mathcal{P}/} \) is the same as the adjunction of induction-restriction functors:
\[ \text{IbMod}(\mathcal{P})_{\mathcal{P}/\mathcal{P}} \rightleftarrows \text{BMod}(\mathcal{P})_{\mathcal{P}/\mathcal{P}}. \]
Thus, the adjunction \( \mathcal{T}_x \text{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_x \text{BMod}(\mathcal{P}) \) is indeed a Quillen equivalence.

\[ \square \]
6. Quillen cohomology of enriched operads

Again, in this section, we work on the base category $S$ of Conventions 3.0.2. In particular, we will work on the canonical model structure on $\mathcal{O}p(S)$ (also, $\mathcal{C}at(S)$). This section contains the central result of the paper, which gives an explicit formula for computing Quillen cohomology of $S$-enriched operads. Let $\mathcal{P}$ be an $S$-enriched $C$-colored operad.

Notations 6.0.1. We let $L_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}} \mathcal{O}p(S)$ and $L^{b}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}} \mathcal{B}Mod(\mathcal{P})$, respectively, denote the cotangent complexes of $\mathcal{P}$ when regarded as an object of $\mathcal{O}p(S)$ and $\mathcal{B}Mod(\mathcal{P})$. Besides that, we denote by $L^{red}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}} \mathcal{O}p_{C}(S)$ the cotangent complex of $\mathcal{P}$ when regarded as an object of $\mathcal{O}p_{C}(S)$ and refer to it as the reduced cotangent complex of $\mathcal{P}$.

Conventions 6.0.2. From now on, by (resp. reduced) Quillen cohomology of $\mathcal{P}$ we will mean the Quillen cohomology of $\mathcal{P}$ when regarded as an object of (resp. $\mathcal{O}p_{C}(S)$) $\mathcal{O}p(S)$, which is classified by (resp. $L^{red}_{\mathcal{P}}$) $L_{\mathcal{P}}$.

By Theorem 5.0.3, when $\mathcal{P}$ is fibrant and $\Sigma$-cofibrant, we have a sequence of Quillen equivalences connecting the tangent categories:

$$\mathcal{T}_{\mathcal{P}} \mathcal{I}b\mathcal{M}od(\mathcal{P}) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \mathcal{B}Mod(\mathcal{P}) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \mathcal{O}p_{C}(S) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \mathcal{O}p(S).$$

Notations 6.0.3. Two composed adjunctions taken from the above sequence will be denoted as $\mathcal{T}^{\mathcal{P}}_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}} \mathcal{I}b\mathcal{M}od(\mathcal{P}) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \mathcal{O}p_{C}(S) : \mathcal{U}^{\mathcal{P}}_{\mathcal{P}}$ and $\mathcal{T}^{\mathcal{P}}_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}} \mathcal{B}Mod(\mathcal{P}) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \mathcal{O}p(S) : \mathcal{U}^{\mathcal{P}}_{\mathcal{P}}$.

In order to get the desired formula of Quillen cohomology of $\mathcal{P}$, we compute the derived image of $L_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}} \mathcal{O}p(S)$ under the composed right Quillen equivalence

$$\mathcal{U}^{\mathcal{P}}_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}} \mathcal{O}p(S) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \mathcal{B}Mod(\mathcal{P}) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \mathcal{I}b\mathcal{M}od(\mathcal{P}).$$

As the first step, we prove that the derived image of $L^{b}_{\mathcal{P}}$ in $\mathcal{T}_{\mathcal{P}} \mathcal{B}Mod(\mathcal{P})$ is weakly equivalent to $L^{b}_{\mathcal{P}}$, up to a shift (see Notations 6.0.1). Our work therefore extends [5], Proposition 3.2.1, but in a different approach. For our approach, the base category $S$ is technically required to satisfy an extra condition. By the way, we shall give several illustrations for this condition. After having done that first step, it remains to compute the derived image of $L^{b}_{\mathcal{P}}$ in $\mathcal{T}_{\mathcal{P}} \mathcal{I}b\mathcal{M}od(\mathcal{P})$.

In the remainder, we prove that there is a long exact sequence relating Quillen cohomology and reduced Quillen cohomology of $\mathcal{P}$.

6.1 An extra condition

Notation 6.1.1. We will denote by $\mathcal{B}Mod(\mathcal{P})^{\ast} := \mathcal{B}Mod(\mathcal{P})_{\mathcal{P} \mathcal{P} \mathcal{P}}$ the category of $\mathcal{P}$-bimodules under $\mathcal{P} \circ \mathcal{P}$ (which presents the free $\mathcal{P}$-bimodule generated by $\mathcal{I}c_{\mathcal{P}}$), and refer to it as the category of pointed $\mathcal{P}$-bimodules. Observe that the composition $\mu : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ exhibits $\mathcal{P}$ itself as a pointed $\mathcal{P}$-bimodule.

Let $f + g : \mathcal{P} \cup \mathcal{P} \longrightarrow \mathcal{Q}$ be a map in $\mathcal{O}p_{C}(S)$. Then $\mathcal{Q}$ inherits a $\mathcal{P}$-bimodule structure with the left (resp. right) $\mathcal{P}$-action induced by $f$ (resp. $g$). In particular, there is a restriction functor $\mathcal{O}p_{C}(S)_{\mathcal{P} \mathcal{P} \mathcal{P}} \longrightarrow \mathcal{B}Mod(\mathcal{P})^{\ast}$, which admits a left adjoint denoted by $E : \mathcal{B}Mod(\mathcal{P})^{\ast} \longrightarrow \mathcal{O}p_{C}(S)_{\mathcal{P} \mathcal{P} \mathcal{P}}$. Observe then that $E$ sends $\mathcal{P} \in \mathcal{B}Mod(\mathcal{P})^{\ast}$ to itself $\mathcal{P} \in \mathcal{O}p_{C}(S)_{\mathcal{P} \mathcal{P} \mathcal{P}}$ equipped with the fold map $\text{Id}_{\mathcal{P}} + \text{Id}_{\mathcal{P}} : \mathcal{P} \cup \mathcal{P} \longrightarrow \mathcal{P}$. Dwyer and Hess ([17], section 5) proved that, in the context of nonsymmetric simplicial operads, the left derived functor of $E$ sends $\mathcal{P}$ to itself $\mathcal{P}$. Inspired by their work, we set up an extra condition on the base category $S$ as follows.
By construction, there is a unique map $L$ built up by taking diag

other sends the

For this, we will follow C. Rezk’s

C simplicial

Condition

Proposition

C

The proof first requires constructing a nice cofibrant resolution of $\mathcal{P}$ as an object in $\text{BMod}(\mathcal{P})^*$. For this, we will follow C. Rezk’s [34, § 3.7.2]. (However, note that the operadic model structures considered in the loc. cit are different from ours, so it should be used carefully). Let $\mathcal{M}$ be a simplicial model category. The diagonal (or realization) functor $\text{diag}: \mathcal{M}^\Delta^{op} \rightarrow \mathcal{M}$ is by definition the left adjoint to the functor $\mathcal{M} \rightarrow \mathcal{M}^\Delta^{op}$ taking each $X \in \mathcal{M}$ to the simplicial object $[n] \mapsto X^\Delta^n$. For each $Y_* \in \mathcal{M}^\Delta^{op}$, one defines the latching object $L_n Y_*$ as the coequalizer in $\mathcal{M}$ of the form

\[
\bigsqcup_{0 \leq i < j \leq n} Y_{n-1} \rightrightarrows \bigsqcup_{0 \leq k \leq n} Y_n \rightarrow L_n Y_*
\]

in which one of the two maps sends the $(i,j)$ summand to the $j$’th summand by $s_i$ while the other sends the $(i,j)$ summand to the $i$’th summand by $s_{j-1}$. By convention, one puts $L_1 Y_* := \emptyset$. By construction, there is a unique map $L_n Y_* \rightarrow Y_{n+1}$ factoring the map $s_k: Y_n \rightarrow Y_{n+1}$ for every $k = 0, \ldots, n$. One then establishes a filtration $\text{diag}(Y_*) = \text{colim}_n \text{diag}_n Y_*$ of $\text{diag}(Y_*)$, inductively, built up by taking $\text{diag}_0 Y_* := Y_0$ and, for each $n \geq 1$, taking the pushout:

\[
d_n Y_* \longrightarrow \Delta^n \otimes Y_n \\
\text{diag}_{n-1} Y_* \longrightarrow \text{diag}_n Y_*
\]

in which $d_n Y_* := \Delta^n \otimes L_{n-1} Y_* \bigsqcup \partial \Delta^n \otimes L_{n-1} Y_* \partial \Delta^n \otimes Y_n$. As a consequence, if for every $n \geq 0$ the latching map $L_{n-1} Y_* \rightarrow Y_n$ is a cofibration then $\text{diag}(Y_*)$ is cofibrant. More generally, we have the following observation.

Lemma 6.1.4. Let $X_* \rightarrow Y_*$ be a map of simplicial objects in $\mathcal{M}$. Suppose that for every $n \geq 0$ the (relative) latching map

\[
X_n \bigsqcup_{L_{n-1} X_*} L_n Y_* \rightarrow Y_n
\]

is a cofibration. Then the induced map $\text{diag}(X_*) \longrightarrow \text{diag}(Y_*)$ is a cofibration as well.

Proof. By the filtrations of $\text{diag}(X_*)$ and $\text{diag}(Y_*)$ mentioned above, the map $\text{diag}(X_*) \longrightarrow \text{diag}(Y_*)$ is a cofibration as soon as, for every $n \geq 0$, the map $\text{diag}_n X_* \longrightarrow \text{diag}_n Y_*$ is one. Note first that, when $n = 0$, the latching map (6.1.2) coincides with the map $\text{diag}_0 X_* \longrightarrow \text{diag}_0 Y_*$. Let us assume by induction that the map $\text{diag}_{n-1} X_* \longrightarrow \text{diag}_{n-1} Y_*$ is a cofibration. Then, factor the map $\text{diag}_n X_* \longrightarrow \text{diag}_n Y_*$ as

\[
\text{diag}_n X_* \longrightarrow \text{diag}_n X_* \bigsqcup_{\text{diag}_{n-1} X_*} \text{diag}_{n-1} Y_* \longrightarrow \text{diag}_n Y_.*
\]
By the inductive assumption, the first map in this composition is a cofibration. Hence, it remains to show that the map \( \varphi \) is a cofibration. Let us denote by \( L_{n-1}(X_\bullet, Y_\bullet) := X_n \sqcup_{L_{n-1}X_n} L_{n-1}Y_\bullet \). We can then form a canonical map

\[
\partial \Delta^n \otimes L_{n-1}(X_\bullet, Y_\bullet) \to \partial \Delta^n \otimes Y_n \to \Delta^n \otimes Y_n,
\]

which is a cofibration by the pushout-product axiom. Unwinding computation, this map turns out to be isomorphic to the canonical map

\[
d_n Y_\bullet \to \Delta^n \otimes X_n \to \Delta^n \otimes Y_n. \tag{6.1.3}
\]

Now, consider the following commutative cube

\[
\begin{array}{ccc}
d_n X_\bullet & \to & \Delta^n \otimes X_n \\
\downarrow & & \downarrow \\
d_n Y_\bullet & \to & \Delta^n \otimes Y_n \\
\end{array}
\]

\[
\begin{array}{ccc}
diag_{n-1} X_\bullet & \to & diag_n X_\bullet \\
\downarrow & & \downarrow \\
diag_{n-1} Y_\bullet & \to & diag_n Y_\bullet \\
\end{array}
\]

whose the front and back sides are coCartesian squares. By applying the pasting law of pushouts iteratively, we find that the map \( \varphi \) turns out to be cobase change of the map (6.1.3), which is a cofibration as indicated there. We therefore get the conclusion.

The category of simplicial \( C \)-collections admits a canonical simplicial model structure. It implies that the category \( \text{BMod}(\mathcal{P}) \) admits a canonical simplicial model structure, by the transferred principle in simplicial version (see \([34], \text{Propositions 3.1.5, 3.2.8}\)). One constructs the Hochschild resolution of \( \mathcal{P} \) as follows.

**Construction 6.1.5.** Let \( H_\bullet \mathcal{P} : \Delta^{op} \to \text{BMod}(\mathcal{P}) \) be the simplicial object of \( \mathcal{P} \)-bimodules with \( H_n \mathcal{P} := \mathcal{P}^{(n+2)} \), the face map \( d_i : H_n \mathcal{P} \to H_{n-1} \mathcal{P} \) given by using the composition \( \mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P} \) to combine the factors \( i+1 \) and \( i+2 \) in \( H_n \mathcal{P} \) and with the degeneracy map \( s_i \) given by inserting the unit operations of \( \mathcal{P} \) between the factors \( i+1 \) and \( i+2 \). The realization \( \text{diag}(H_\bullet \mathcal{P}) \in \text{BMod}(\mathcal{P}) \) has \( n \)-simplices being those of \( H_n \mathcal{P} \). The map \( \mu \) induces a canonical map of simplicial objects \( H_\bullet \mathcal{P} \to \mathcal{P} \). The augmentation map \( \psi : \text{diag}(H_\bullet \mathcal{P}) \to \text{diag}(\mathcal{P}) = \mathcal{P} \) is then a weak equivalence by \([34], \text{Corollary 3.7.6}\), (this even comes with a contracting homotopy). The map \( \psi \) now exhibits \( \text{diag}(H_\bullet \mathcal{P}) \) as the **Hochschild resolution of \( \mathcal{P} \in \text{BMod}(\mathcal{P}) \)**.

On other hand, since \( \mathcal{P} \circ \mathcal{P} = H_0 \mathcal{P} \), there is a unique map of simplicial objects \( \mathcal{P} \circ \mathcal{P} \to H_\bullet \mathcal{P} \), which is the identity on degree 0. Now, the diagonal functor gives a map \( \rho : \mathcal{P} \circ \mathcal{P} \to \text{diag}(H_\bullet \mathcal{P}) \) of \( \mathcal{P} \)-bimodules, satisfying that the composition \( \mathcal{P} \circ \mathcal{P} \to \text{diag}(H_\bullet \mathcal{P}) \) agrees with \( \mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P} \).

**Lemma 6.1.6.** Suppose that \( \mathcal{P} \) is a \( \Sigma \)-cofibrant simplicial operad. The map \( \psi \) indeed exhibits \( \text{diag}(H_\bullet \mathcal{P}) \) as a cofibrant resolution of \( \mathcal{P} \) regarded as a bimodule over itself. Moreover, the map \( \rho : \mathcal{P} \circ \mathcal{P} \to \text{diag}(H_\bullet \mathcal{P}) \) is a cofibration of \( \mathcal{P} \)-bimodules. In particular, \( \text{diag}(H_\bullet \mathcal{P}) \) is also a cofibration resolution of \( \mathcal{P} \) when regarded as a pointed \( \mathcal{P} \)-bimodule.

**Proof.** The first statement is an analogue of \([34], \text{Corollary 3.7.6}\). It suffices to show that the latching map \( L_{n-1}H_n \mathcal{P} \to H_n \mathcal{P} \) is a cofibration for every \( n \geq 0 \). This can be done by an inductive argument, assisted by \([34], \text{Lemma 3.7.8}\), which says that there exist certain maps \( k_n \)'s of
symmetric sequences such that the map \( L_{n-1} H_\bullet P \rightarrow H_n P \) is isomorphic to the free \( \mathcal{P} \)-bimodule map generated by \( k_n \) and moreover, the map \( k_{n+1} \) is the pushout-product of \( k_n \) with the unit map \( \mathcal{I}_C \rightarrow \mathcal{P} \). Besides that, one will need the fact that, given two maps \( f \) and \( g \) of symmetric sequences in a sufficiently nice symmetric monoidal model category, the pushout-product of \( f \) with \( g \) is a cofibration as soon as both of them are one and, in addition, the domain of \( g \) is cofibrant (cf. ([8], Lemma 11.5.1)).

To prove that the map \( \rho \) is a cofibration, we make use of Lemma 6.1.4. Since \( \mathcal{P} \circ \mathcal{P} \) is considered as a constant simplicial object, the latching map (6.1.2) is simply \( L_{n-1} H_\bullet P \rightarrow H_n P \) when \( n \geq 1 \) and the identity map \( \text{Id}_{\mathcal{P} \circ \mathcal{P}} \) when \( n = 0 \). But the first map is a cofibration by the above paragraph. The proof is hence completed. \( \square \)

**Remark 6.1.7.** Let \( \mathcal{P} \) be any simplicial operad and let \( A \) be any \( \mathcal{P} \)-algebra. Hochschild resolution of \( A \) is the realization of the simplicial \( \mathcal{P} \)-algebra \( H_\bullet^\mathcal{P} A \) with \( H_n^\mathcal{P} A = \mathcal{P}^{(n+1)} \circ A \). The augmentation map \( \text{diag}(H_\bullet^\mathcal{P} A) \rightarrow A \) is a weak equivalence by ([34], Corollary 3.7.4), and indeed exhibits \( \text{diag}(H_\bullet^\mathcal{P} A) \) as a cofibrant resolution of \( A \). To see this, one repeats the arguments as in the above proof, along with noting the fact that, in the monoidal category of simplicial symmetric sequences, pushout-product of any two injections is an injection (cf. ([34], Proposition 3.4.5)).

**Proof of Proposition 6.1.3.** In the first step, we follow the arguments of ([17], section 5). By applying the functor \( E \) to \( H_\bullet \mathcal{P} \) degreewise, one obtains a simplicial object, \( \mathcal{E}H_\bullet \mathcal{P} \), of operads under \( \mathcal{P} \cup \mathcal{P} \). The realization \( \text{diag}(\mathcal{E}H_\bullet \mathcal{P}) \) is then an operad under \( \mathcal{P} \cup \mathcal{P} \). Since \( \mathcal{E}(\mathcal{P}) \cong \mathcal{P} \) in \( \text{Op}_C(\text{Set}_\Delta)_{\mathcal{P},\mathcal{P}} \), there is a canonical map \( \varphi_\mathcal{P} : \text{diag}(\mathcal{E}H_\bullet \mathcal{P}) \rightarrow \mathcal{P} \) of operads under \( \mathcal{P} \cup \mathcal{P} \). One observes that there is a canonical isomorphism \( \text{diag}(\mathcal{E}H_\bullet \mathcal{P}) \cong E(\text{diag}(H_\bullet \mathcal{P})) \) of operads under \( \mathcal{P} \cup \mathcal{P} \) and over \( \mathcal{P} \) (cf. ([17], Proposition 5.3)). Since \( E(\text{diag}(H_\bullet \mathcal{P})) \) is already a model for \( L \mathcal{E}(\mathcal{P}) \) by Lemma 6.1.6, it just remains to show that the map \( \varphi_\mathcal{P} : \text{diag}(\mathcal{E}H_\bullet \mathcal{P}) \rightarrow \mathcal{P} \) is a weak equivalence of operads. By the diagonal principle, \( \varphi_\mathcal{P} \) is a weak equivalence as soon as the map \( \mathcal{E}H_n \mathcal{P} \rightarrow \mathcal{P} \) is one, for every \( n \geq 0 \). Moreover, one finds that \( \mathcal{E}H_n \mathcal{P} \cong \mathcal{P} \cup \mathcal{P} \) of \( \mathcal{P} \)-algebras under \( \mathcal{P} \cup \mathcal{P} \) where

\[
\mathcal{J}_n : \text{Coll}_C(\text{Set}_\Delta)_{\mathcal{P}} \rightarrow \text{Op}_C(\text{Set}_\Delta)
\]

refers to the free-operad functor on pointed \( C \)-collections. This tells us that if \( \varphi_\mathcal{Q} \) is a weak equivalence between cofibrant operads then \( \varphi_\mathcal{Q} \) is a weak equivalence if and only if \( \varphi_\mathcal{P} \) is one. Applying the diagonal principle in the other direction, we get that \( \varphi_\mathcal{P} \) is a weak equivalence as soon as, for every \( n \geq 0 \), the map \( \varphi_{\mathcal{P}^c} \) is one (where \( \mathcal{P} \) is the operad of \( n \)-simplices of \( \mathcal{P} \)).

Now, consider \( \mathcal{P} \) as an \( \text{Op}_C \)-algebra with \( \text{Op}_C \) being the operad of simplicial \( C \)-colored operads. The above remark suggests that we can make use of \( \mathcal{P}^c := \text{diag}(H_\bullet^{\text{Op}_C} \mathcal{P}) \) as (another) cofibrant model for \( \mathcal{P} \). By the first paragraph, it suffices to verify that the map \( \varphi_\mathcal{P} \) is a weak equivalence. Again, the first paragraph tells us that it will suffice to show that, for every \( n \), the map \( \varphi_{\mathcal{P}^c} \) is a weak equivalence. More precisely, we have that

\[
\mathcal{P}^c = (\text{Op}_C)^{(n+1)} \circ \mathcal{P} = (\text{Op}_C)^{(n+1)} \circ \mathcal{P}^c.
\]

(The second identification is because of the fact that \( \text{Op}_C \) is a discrete operad). In particular, \( \mathcal{P}^c \) is a discrete free \( \text{Op}_C \)-algebra. But, unwinding definition, a free \( \text{Op}_C \)-algebra is the same as the free operad generated by a free symmetric sequence (i.e., symmetrization of a nonsymmetric sequence).
By the second paragraph, we can assume without loss of generality that \( \mathcal{P} \) is the free operad generated by a discrete free symmetric sequence. Alternatively, \( \mathcal{P} \) is the symmetrization of a discrete free nonsymmetric operad. Back to the first paragraph, we therefore have to show that the map \( \varphi : \text{diag}(EH_* \mathcal{P}) = EH_* \mathcal{P} \to \mathcal{P} \) is a weak equivalence of operads. In the case where \( \mathcal{P} \) is a discrete free nonsymmetric (simplicial) operad, the map \( \varphi \) was proved to be a weak equivalence by Dwyer and Hess (see the proof of [17, Proposition 5.4]). Thus, for our last duty, their arguments can be repeated, without any difference.

\[ \square \]

**Remark 6.1.8.** Hochschild resolutions work in the context of simplicial \( R \)-modules under a slightly different setting. Let \( \mathcal{P} \in \text{Op}_C(\text{sMod}_R) \) be given. The composition \( \mathcal{P} \circ \mathcal{P} \to \text{diag}(H_* \mathcal{P}) \otimes \mathcal{P} \) will exhibit \( \text{diag}(H_* \mathcal{P}) \) as a cofibrant resolution of \( \mathcal{P} \in \text{BMod}(\mathcal{P})^* \) when provided that the unit map \( I_C \to \mathcal{P} \) is a cofibration of symmetric sequences, (the proof is very similar to that of Lemma 6.1.6). On other hand, let \( A \) be a \( \mathcal{P} \)-algebra. The augmentation map \( \text{diag}(H_* \mathcal{P}) A \to A \) will exhibit \( \text{diag}(H_* \mathcal{P}) A \) as a cofibrant resolution of \( A \) when provided that the unit map \( I_C \to \mathcal{P} \) is a levelwise cofibration and that \( A \) is levelwise cofibrant. For the proof, one repeats the arguments as in the proof of Lemma 6.1.6, along with using ([34], Proposition 3.4.5).

**Proposition 6.1.9.** The category \( \text{sMod}_R \) satisfies the condition (S8) 6.1.2. Namely, for every cofibrant object \( \mathcal{P} \in \text{Op}_C(\text{sMod}_R) \), the left derived functor \( LE : \text{BMod}(\mathcal{P})^* \to \text{Op}_C(\text{sMod}_R)_{\mathcal{P}, \mathcal{P}/} \) of \( E \) sends \( \mathcal{P} \) to itself \( \mathcal{P} \).

**Proof.** By the above remark, we can make use of \( \text{diag}(H_* \mathcal{P}) \) as a cofibrant resolution of \( \mathcal{P} \in \text{BMod}(\mathcal{P})^* \) and \( \text{diag}(H^{O_C} \mathcal{P}) \) as (another) cofibrant model for \( \mathcal{P} \in \text{Op}_C(\text{sMod}_R) \) (since the operad \( O_C \) is discrete, its unit map is in particular a levelwise cofibration). We then pick up the first two paragraphs of the proof of Proposition 6.1.3. It hence suffices to prove that, when provided that \( \mathcal{P} \) is the free operad generated by a discrete free symmetric sequence, the map \( \varphi : \text{diag}(EH_* \mathcal{P}) \to \mathcal{P} \) is a weak equivalence. The latter is in fact equivalent to our original problem: proving that \( LE \mathcal{P} \cong \mathcal{P} \). Let us see how it goes.

The free-forgetful adjunction \( R\{-\} : \text{Set}_\Delta \rightleftarrows \text{sMod}_R : \mathcal{U} \) lifts to a Quillen adjunction

\[
R\{-\} : \text{Op}_C(\text{Set}_\Delta) \rightleftarrows \text{Op}_C(\text{sMod}_R) : \mathcal{U}
\]

between operads. By the assumption on \( \mathcal{P} \), there exists a simplicial operad \( Q \in \text{Op}_C(\text{Set}_\Delta) \) such that \( R\{Q\} = \mathcal{P} \) and \( Q \) is the free operad generated by a discrete free symmetric sequence. On other hand, the functor \( R\{-\} \) does lift to a left Quillen functor \( R^b\{-\} : \text{BMod}(Q)^* \to \text{BMod}(\mathcal{P})^* \), which fits into the following commutative square of left Quillen functors

\[
\begin{array}{ccc}
\text{BMod}(Q)^* & \xrightarrow{E} & \text{Op}_C(\text{Set}_\Delta)_{Q, Q/} \\
R^b\{-\} \downarrow & & \downarrow R\{-\} \\
\text{BMod}(\mathcal{P})^* & \xrightarrow{E} & \text{Op}_C(\text{sMod}_R)_{\mathcal{P}, \mathcal{P}/}
\end{array}
\]

Note that both the functors \( R^b\{-\} \) and \( R\{-\} \) preserve weak equivalences. Now, observe that \( LE(\mathcal{P}) \cong LE(LR^b\{Q\}) \cong LR\{LE(Q)\} \) in \( \text{Ho}(\text{Op}_C(\text{sMod}_R)_{\mathcal{P}, \mathcal{P}/}) \). On other hand, by Proposition 6.1.3, \( LE(Q) \cong Q \), and hence \( LE(\mathcal{P}) \cong LR\{Q\} \cong \mathcal{P} \), as expected. \( \square \)
QUILLEN COHOMOLOGY OF ENRICHED OPERADS

**Proposition 6.1.10.** The category of topological spaces $\text{Top}$ satisfies the condition (S8) 6.1.2. Namely, for every cofibrant object $\mathcal{P} \in \text{Op}_C(\text{Top})$, the left derived functor $L\mathcal{E} : \text{BMod}(\mathcal{P})^* \to \text{Op}_C(\text{Top})_{\mathcal{P}, \mathcal{P}_f}$ of $\mathcal{E}$ sends $\mathcal{P}$ to itself $\mathcal{P}$.

**Proof.** The adjunction $|-| \to \text{Sing}$ of the realization and singular functors is a monoidal Quillen equivalence (in Hovey’s sense [23]). It hence lifts to a Quillen equivalence $|{-}| : \text{Op}_C(\text{Set}_\Delta) \cong \cong \text{Op}_C(\text{Top}) : \text{Sing}$ between simplicial and topological operads. The proof is now straightforward, due to Proposition 6.1.3. □

### 6.2 Cotangent complexes and Quillen cohomology of enriched operads

Let $\mathcal{P}$ be an $S$-enriched $C$-colored operad. As we discussed at the beginning of this section, we first wish to prove the following.

**Proposition 6.2.1.** Suppose that $S$ additionally satisfies the condition (S8) 6.1.2 and that $\mathcal{P}$ is fibrant and $\Sigma$-cofibrant. Then the left Quillen equivalence $\mathcal{F}_b^\mathcal{P} : \mathcal{P}_f \text{BMod}(\mathcal{P}) \xrightarrow{\simeq} \mathcal{P}_f \text{Op}(S)$ identifies $L_{\mathcal{P}}^b$ to $L_{\mathcal{P}}[1]$ (see Notations 6.0.1). Alternatively, the right Quillen equivalence $\mathcal{U}_b^\mathcal{P} : \mathcal{P}_f \text{Op}(S) \xrightarrow{\simeq} \mathcal{P}_f \text{BMod}(\mathcal{P})$ identifies $L_{\mathcal{P}}$ to $L_{\mathcal{P}}[1]$.

**Lemma 6.2.2.** Let $\mathcal{C} \in \text{Op}(S)$ be a fibrant operad concentrated in arity 1. Then the left Quillen equivalence $\mathcal{F}_b^\mathcal{C} : \mathcal{P}_f \text{BMod}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{P}_f \text{Op}(\mathcal{C})$ sends $L_{\mathcal{C}}^b$ to $L_{\mathcal{C}}[1]$.

**Proof.** We also regard $\mathcal{C}$ as an $S$-enriched category. The proof is then straightforward by observing that the category $\text{Cat}(S)$ is already a “neighborhood” of $\mathcal{C}$ in $\text{Op}(S)$. This idea is expressed as follows. There is a commutative square of left Quillen functors

$$
\begin{array}{ccc}
\mathcal{F}_{\text{Map}} \text{Fun}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}, S) & \xrightarrow{\mathcal{F}_b^\mathcal{C}} & \mathcal{F}_{\text{BMod}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathcal{F}_{\text{Cat}} \text{Cat}(S) & \xrightarrow{\mathcal{F}_b^\mathcal{C}} & \mathcal{F}_{\text{Op}}(S) \\
\end{array}
$$

In this square, $\text{Fun}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}, S)$ is the same as the category of $\mathcal{C}$-bimodules, and $\text{Map}_\mathcal{C} : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \to S, (x, y) \mapsto \text{Map}_\mathcal{C}(x, y)$ is nothing but $\mathcal{C}$ viewed as a bimodule over itself. Moreover, the left vertical functor is the left Quillen equivalence appearing in Theorem 1.0.2, which sends $L_{\text{Map}_\mathcal{C}} \in \mathcal{F}_{\text{Map}} \text{Fun}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}, S)$ to $L_{\mathcal{C}}[1] \in \mathcal{F}_{\text{Cat}} \text{Cat}(S)$ by Theorem 1.0.4; while the horizontal functors are the obvious embedding functors, which clearly preserve cotangent complexes. The commutativity of the square hence proves the lemma. □

We mentioned $\text{BMod}(\mathcal{P})^*$ the category of pointed $\mathcal{P}$-bimodules in Notation 6.1.1. There is a canonical isomorphism of categories

$$
\text{BMod}(\mathcal{P})^*_\mathcal{P} /_{\mathcal{P}} \xrightarrow{\text{def}} (\text{BMod}(\mathcal{P})_{\mathcal{P} \otimes \mathcal{P}})_{\mathcal{P} /_{\mathcal{P}}} \cong \text{BMod}(\mathcal{P})_{\mathcal{P} /_{\mathcal{P}}},
$$

which identifies the transferred model structures on both sides. In particular, we get a Quillen equivalence of the associated tangent categories $\mathcal{F}_\mathcal{P} \text{BMod}(\mathcal{P})^* \simeq \mathcal{F}_\mathcal{P} \text{BMod}(\mathcal{P})$. The following is inspired by [[5], Proposition 2.2.10].

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LEMMA 6.2.3. Suppose that $\mathcal{P}$ is $\Sigma$-cofibrant. The Quillen equivalence $\mathcal{I}_P \mathbf{BMod}(\mathcal{P})^* \simeq \mathcal{I}_P \mathbf{BMod}(\mathcal{P})$ identifies the cotangent complex of $\mathcal{P}$ when considering $\mathcal{P}$ as a pointed bimodule, denoted by $L^b_{\mathcal{P}} \in \mathcal{I}_P \mathbf{BMod}(\mathcal{P})^*$, to the relative cotangent complex of the composition $\mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ regarded as a map in $\mathbf{BMod}(\mathcal{P})$, denoted by $L^b_{\mathcal{P} \circ \mathcal{P}^*} \in \mathcal{I}_P \mathbf{BMod}(\mathcal{P})$ (see Definition 4.0.11).

\textbf{Proof.} Take a factorization $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}^\cof \xrightarrow{\simeq} \mathcal{P}$ of the map $\mu$ exhibiting $\mathcal{P}^\cof$ as a cofibrant resolution of $\mathcal{P} \in \mathbf{BMod}(\mathcal{P})^*$. In particular, the map $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}^\cof$ is a cofibration in $\mathbf{BMod}(\mathcal{P})$.

By definition, $L^b_{\mathcal{P}} = \Sigma^\infty(\mathcal{P} \cup \mathcal{P}^\cof)$ with the coproduct taken in $\mathbf{BMod}(\mathcal{P})^*$. Note that the underlying $\mathcal{P}$-bimodule of $\mathcal{P} \cup \mathcal{P}^\cof \in \mathbf{BMod}(\mathcal{P})^*$ is the pushout $\mathcal{P} \cup_{\mathcal{P}^\cof} \mathcal{P}$ in $\mathbf{BMod}(\mathcal{P})$. Consider the following coCartesian square in $\mathbf{BMod}(\mathcal{P})$

$$
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \mathcal{P} \cup \mathcal{P}^\cof \\
\downarrow & & \downarrow \\
\mathcal{P} & \longrightarrow & \mathcal{P} \cup_{\mathcal{P}^\cof} \mathcal{P}
\end{array}
$$

(All the coproducts are now taken in $\mathbf{BMod}(\mathcal{P})$). Considering this as a coCartesian square of left $\mathcal{P}$-modules, it has the vertices being all cofibrant and has the top horizontal map being a cofibration, (these follow from Lemma 5.3.2). So it is homotopy coCartesian when regarded as a square in $\mathbf{LMod}(\mathcal{P})$. Using Lemma 5.3.2 again, we deduce that it is a homotopy coCartesian square of $\mathcal{P}$-bimodules. Consequently, it is also homotopy coCartesian when regarded as a square in $\mathbf{BMod}(\mathcal{P})_{\mathcal{P}^\perp \mathcal{P}}$. Now, by applying the functor $\Sigma^\infty : \mathbf{BMod}(\mathcal{P})_{\mathcal{P}^\perp \mathcal{P}} \to \mathcal{I}_P \mathbf{BMod}(\mathcal{P})$ to this square, we obtain a homotopy cofiber sequence in $\mathcal{I}_P \mathbf{BMod}(\mathcal{P})$ of the form

$$
\Sigma^\infty(\mathcal{P} \cup (\mathcal{P} \circ \mathcal{P})) \longrightarrow \Sigma^\infty(\mathcal{P} \cup \mathcal{P}^\cof) \longrightarrow \Sigma^\infty(\mathcal{P} \cup_{\mathcal{P}^\cof} \mathcal{P}).
$$

In this sequence, the first term is a model for $L^b_{\mathcal{P}}(\mu)$ (i.e., the derived image of $\mu$ under the left Quillen functor $\Sigma^\infty : \mathbf{BMod}(\mathcal{P})_{\mathcal{P}^\perp \mathcal{P}} \to \mathcal{I}_P \mathbf{BMod}(\mathcal{P})$), the second term is nothing but the cotangent complex $L^b_{\mathcal{P}}$, and the other models $L^b_{\mathcal{P}}^*$ as discussed above. Thus, by definition of relative cotangent complex, $L^b_{\mathcal{P}}^*$ is weakly equivalent to $L^b_{\mathcal{P} \circ \mathcal{P}^*}$, as desired.

\textbf{Proof of Proposition 6.2.1.} We can take $f : \mathcal{Q} \xrightarrow{\simeq} \mathcal{P}$ to be a bifibrant resolution of $\mathcal{P}$ in $\mathbf{Op}(\mathcal{S})$ such that $f$ is a map in $\mathbf{Op}(\mathcal{S})$ (cf. Observations 5.1.1). The map $f$ gives rise to a commutative square of left Quillen equivalences

$$
\begin{array}{ccc}
\mathcal{I}_\mathcal{Q} \mathbf{BMod}(\mathcal{Q}) & \xrightarrow{\simeq} & \mathcal{I}_\mathcal{Q} \mathbf{Op}(\mathcal{S}) \\
\downarrow & & \downarrow \\
\mathcal{I}_\mathcal{P} \mathbf{BMod}(\mathcal{P}) & \xrightarrow{\simeq} & \mathcal{I}_\mathcal{P} \mathbf{Op}(\mathcal{S})
\end{array}
$$

(see the proof of Proposition 5.2.10). It is then not hard to prove that the vertical functors preserve cotangent complexes. Therefore, if the statement holds for $\mathcal{Q}$ then it holds for $\mathcal{P}$ as well. So we can assume without loss of generality that $\mathcal{P}$ is bifibrant.

Now, the functor $E : \mathbf{BMod}(\mathcal{P})^* \to \mathbf{Op}(\mathcal{S})_{\mathcal{P}^\perp \mathcal{P}}$ lifts to a left Quillen functor

$$
\tilde{E} : \mathbf{BMod}(\mathcal{P})^* \to (\mathbf{Op}(\mathcal{S})_{\mathcal{P}^\perp \mathcal{P}})^{\mathcal{P}^\perp \mathcal{P}} \simeq \mathbf{Op}(\mathcal{S})_{\mathcal{P}^\perp \mathcal{P}},
$$

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which induces a left Quillen functor between tangent categories $E^{Sp}: \mathcal{T}_P \text{BMod}(\mathcal{P})^* \to \mathcal{T}_P \text{Op}_C(S)$. The condition (S8) 6.1.2 implies that $E$ sends $\mathcal{P} \cup \mathcal{P} \in \text{BMod}(\mathcal{P})_{\mathcal{P} \cup \mathcal{P}}^*$ to $\Sigma(\mathcal{P} \cup \mathcal{P}) = \mathcal{P} \cup \mathcal{P} \in \text{Op}_C(S)_{\mathcal{P} \cup \mathcal{P}}$ (where the letter “h” refers to homotopy coproduct or pushout). This implies that $E^{Sp}$ sends $L^b_{\mathcal{P} \cup \mathcal{P}} \in \mathcal{T}_P \text{BMod}(\mathcal{P})^*$ (given in the above lemma) to $L^\text{red}_{\mathcal{P} \cup \mathcal{P}}[1] \in \mathcal{T}_P \text{Op}_C(S)$ the suspension of the reduced cotangent complex of $\mathcal{P}$ (see Notations 6.0.1).

Next, observe that the functor $\mathcal{T}_P^b : \mathcal{T}_P \text{BMod}(\mathcal{P}) \to \mathcal{T}_P \text{Op}(S)$ agrees with the composition

$$\mathcal{T}_P \text{BMod}(\mathcal{P}) \simeq \mathcal{T}_P \text{BMod}(\mathcal{P}) \xrightarrow{E^{Sp}} \mathcal{T}_P \text{Op}_C(S) \xrightarrow{\pi} \mathcal{T}_P \text{Op}(S).$$

(This can be deduced by comparing their right adjoints, which are both simply restriction functors). In this composition, the first identification identifies $L^b_{\mathcal{P} \cup \mathcal{P}} \in \mathcal{T}_P \text{BMod}(\mathcal{P})$ to $L^b_{\mathcal{P} \cup \mathcal{P}} \in \mathcal{T}_P \text{BMod}(\mathcal{P})^*$, by Lemma 6.2.3. By the above paragraph, the second functor $E^{Sp}$ sends $L^b_{\mathcal{P} \cup \mathcal{P}}$ to $L^\text{red}_{\mathcal{P} \cup \mathcal{P}}[1]$. Furthermore, under the third equivalence $\mathcal{T}_P \text{Op}_C(S) \xrightarrow{\pi} \mathcal{T}_P \text{Op}(S)$, $L^\text{red}_{\mathcal{P} \cup \mathcal{P}} \in \mathcal{T}_P \text{Op}_C(S)$ is identified to the relative cotangent complex $L^\text{red}_{\mathcal{P} \cup \mathcal{P}} \in \mathcal{T}_P \text{Op}(S)$ of the unit map $\eta : \mathcal{C} \to \mathcal{P}$ (cf. Lemma 6.3.1). These facts together prove that the left Quillen equivalence $\mathcal{T}_P^b$ identifies $L^b_{\mathcal{P} \cup \mathcal{P}}$ to $L^\text{red}_{\mathcal{P} \cup \mathcal{P}}[1]$. Then, by definition of relative cotangent complex, we get a homotopy (co)fiber sequence in $\mathcal{T}_P \text{Op}(S)$ of the form

$$\mathcal{T}_P^b(\Sigma^\infty(\mu)) \to \mathcal{T}_P^b(L^b_{\mathcal{P} \cup \mathcal{P}}) \to \mathcal{T}_P^b(\Sigma^\infty(\mu)) \to \mathcal{T}_P^b(L^b_{\mathcal{P} \cup \mathcal{P}}) \to \mathcal{T}_P^b(\Sigma^\infty(\mu)) \to \cdots$$

This square factors as

$$\begin{array}{ccc}
\mathcal{T}_P^b(\Sigma^\infty(\mu)) & \to & \mathcal{T}_P^b(L^b_{\mathcal{P} \cup \mathcal{P}}) \\
\downarrow & & \downarrow \\
\Sigma^\infty(\eta)[1] & \to & \mathcal{T}_P^b(L^b_{\mathcal{P} \cup \mathcal{P}}) \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{T}_P^b(L^b_{\mathcal{P} \cup \mathcal{P}}) \\
\end{array}$$

In this diagram, the bottom and outer squares are homotopy (co)cartesian, and hence so is the top square. Consequently, the canonical map $\mathcal{T}_P^b(L^b_{\mathcal{P} \cup \mathcal{P}}) \to \mathcal{T}_P^b[1]$ is a weak equivalence if and only if $\mathcal{T}_P^b(\Sigma^\infty(\mu)) \to \mathcal{T}_P^b(\Sigma^\infty(\mu)) \to \cdots$ is one. So the proof will be completed after showing that the latter map is indeed a weak equivalence. Consider the square of left Quillen functors

$$\begin{array}{ccc}
\mathcal{T}_{\mathcal{J}_C} \text{BMod}(\mathcal{J}_C) & \xrightarrow{\mathcal{T}_{\mathcal{J}_C}^b} & \mathcal{T}_{\mathcal{J}_C} \text{Op}(S) \\
\downarrow & & \downarrow \eta \\
\mathcal{T}_P \text{BMod}(\mathcal{P}) & \xrightarrow{\mathcal{T}_P^b} & \mathcal{T}_P \text{Op}(S) \\
\end{array} \quad (6.2.3)$$

Let us start with the cotangent complex $L^b_{\mathcal{J}_C} \in \mathcal{T}_{\mathcal{J}_C} \text{BMod}(\mathcal{J}_C)$ of $\mathcal{J}_C \in \text{BMod}(\mathcal{J}_C)$. Firstly, $\mathcal{T}_{\mathcal{J}_C}^b$ sends $L^b_{\mathcal{J}_C}$ to $L^b_{\mathcal{J}_C}[1] \in \mathcal{T}_{\mathcal{J}_C} \text{Op}(S)$, by Lemma 6.2.2. On other hand, note that the functor $\text{BMod}(\mathcal{J}_C) \to \text{BMod}(\mathcal{P})$ coincides with the free $\mathcal{P}$-bimodule functor $\text{Coll}_{\mathcal{C}}(S) \to \text{BMod}(\mathcal{P})$, which in particular takes $\mathcal{J}_C$ to $\mathcal{P} \circ \mathcal{P}$. Due to this, we find that the left vertical functor sends $L^b_{\mathcal{J}_C} = \Sigma^\infty(\mathcal{J}_C \cup \mathcal{J}_C)$ to
\[\Sigma^\infty((\mathcal{P} \circ \mathcal{P}) \cup \mathcal{P}) \in \mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}),\] which is exactly a model for \(\Sigma^\infty(\mu)\). Therefore, the commutativity of (6.2.3) shows that \(\mathcal{T}_\mathcal{P}^b(\Sigma^\infty(\mu))\) is weakly equivalent to \(\eta(\mathcal{L}_{\mathcal{D}/\mathcal{C}}[1])\) in \(\mathcal{T}_\mathcal{P} \text{Op}(\mathcal{S})\). But \(\eta(\mathcal{L}_{\mathcal{D}/\mathcal{C}}[1])\) is the same as \(\Sigma^\infty(\eta)[1]\) (by the words after Definition 4.0.11). We hence obtain that \(\mathcal{T}_\mathcal{P}^b(\Sigma^\infty(\mu))\) is weakly equivalent to \(\Sigma^\infty(\eta)[1]\), as desired.

In the next step, we will compute the cotangent complex \(L_b^\mathcal{P} \in \mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P})\) on each level. Properly, \(L_b^\mathcal{P}\) is modeled as \(\Sigma^\infty(\mathcal{P} \cup \mathcal{P}^\text{cof})\) with \(\mathcal{P}^\text{cof}\) being a cofibrant resolution of \(\mathcal{P}\) in \(\text{BMod}(\mathcal{P})\). But the map \(\mathcal{P} \cup \mathcal{P}^\text{cof} \rightarrow \mathcal{P} \cup \mathcal{P}\) is a weak equivalence of \(\mathcal{P}\)-bimodules, due to Lemma 5.3.2, so we will exhibit \(\Sigma^\infty(\mathcal{P} \cup \mathcal{P})\) as a model for \(L_b^\mathcal{P}\). According to [4, Corollary 2.3.3], \(L_b^\mathcal{P} = \Sigma^\infty(\mathcal{P} \cup \mathcal{P})\) admits a suspension spectrum replacement simply given by fixing \(\mathcal{P} \cup \mathcal{P}\) as its value in the bidegree \((0,0)\), and hence the value at the bidegree \((n,n)\) is \(\Sigma^n(\mathcal{P} \cup \mathcal{P})\), i.e., the \(n\)-suspension of \(\mathcal{P} \cup \mathcal{P}\) in \(\text{BMod}(\mathcal{P})\). So \(L_b^\mathcal{P}\) is fully determined just by describing the \(\mathcal{P}\)-bimodule \(\Sigma^n(\mathcal{P} \cup \mathcal{P})\) for every \(n \geq 0\). In fact, we may compute the latter on each level. Let us see how it goes.

**Notation 6.2.4.** For each \(n \geq 0\), we denote by \(S^n := \Sigma^n(1_\mathcal{S} \cup 1_\mathcal{S}) \in \mathcal{S}\) with the suspension \(\Sigma(-)\) computed in \(S_{11/1_\mathcal{S}}\), and refer to \(S^n\) as the **pointed n-sphere** in \(\mathcal{S}\). Furthermore, we will write \(S^n_C\) standing for the \(C\)-collection which has \(S^n_C(c; c) = S^n\) for every \(c \in C\) and agrees with \(\varnothing_\mathcal{S}\) on the other levels.

**Computations 6.2.5.** By Lemma 5.3.2, the underlying left \(\mathcal{P}\)-module of \(\Sigma^n(\mathcal{P} \cup \mathcal{P})\) is nothing but \(\Sigma^n(\mathcal{P} \cup \mathcal{P}) \in \text{LMod}(\mathcal{P})|_{\mathcal{P}^\text{ff/P}}\). The good thing is that \(\mathcal{P}\) is free (generated by \(\mathcal{I}_C\)) as a left module over itself. Thanks to this, we may compute \(\Sigma^n(\mathcal{P} \cup \mathcal{P})\) on each level as follows. First, note that \(\mathcal{P} \cup \mathcal{P} \in \text{LMod}(\mathcal{P})\) is isomorphic to \(\mathcal{P} \circ S^0_\mathcal{C}\) the free left \(\mathcal{P}\)-module generated by \(S^0_\mathcal{C}\). We have further that

\[\Sigma(\mathcal{P} \cup \mathcal{P}) \simeq \mathcal{P} \circ \left(\mathcal{I}_C^h \bigcup \mathcal{I}_C^1\right) \simeq \mathcal{P} \circ S^1_\mathcal{C}.\]

Inductively, we find that \(\Sigma^n(\mathcal{P} \cup \mathcal{P}) \simeq \mathcal{P} \circ S^n_\mathcal{C}\) the free left \(\mathcal{P}\)-module generated by \(S^n_\mathcal{C}\). Now, for each \(C\)-sequence \(\overline{c} := (c_1, \ldots, c_m; c)\), we find that

\[\Sigma^n(\mathcal{P} \cup \mathcal{P}) (\overline{c}) \simeq \mathcal{P} \circ S^n_\mathcal{C} (\overline{c}) = \mathcal{P}(\overline{c}) \otimes (S^n)^{\otimes m}.\]

**Notations 6.2.6.** We denote by \(\tilde{L}_\mathcal{P} \in \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P})\) the derived image of \(L_b^\mathcal{P} \in \mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P})\) under the right Quillen equivalence \(\mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}) \xrightarrow{\Omega^\infty} \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P})\) (cf. Theorem 5.0.3). Furthermore, recall that, when \(\mathcal{S}\) is in addition stable containing a strict zero object \(0\), we have a sequence of right Quillen equivalences

\[\mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}) \xrightarrow{\Omega^\infty} \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P}) \xrightarrow{\ker} \text{IbMod}(\mathcal{P})\]

(cf. Theorem 5.0.5). In this situation, we will denote by \(\mathcal{L}_\mathcal{P} := \mathcal{R}(\ker \circ \Omega^\infty)(\tilde{L}_\mathcal{P})\) the derived image of \(\tilde{L}_\mathcal{P}\) under that composed right Quillen equivalence.

**Computations 6.2.7.** Let us compute \(\tilde{L}_\mathcal{P}\) and \(L_\mathcal{P}\).

(1) It is not difficult to show that the Quillen adjunction \(\text{IbMod}(\mathcal{P})|_{\mathcal{P}^\text{ff/P}} \xleftarrow{\Omega} \text{BMod}(\mathcal{P})|_{\mathcal{P}^\text{ff/P}}\) is differentiable (see Definition 3.0.1). The [4, Corollary 2.4.8] hence shows that the right Quillen equivalence \(\mathcal{T}_\mathcal{P} \text{BMod}(\mathcal{P}) \xrightarrow{\Omega} \mathcal{T}_\mathcal{P} \text{IbMod}(\mathcal{P})\) identifies \(L_b^\mathcal{P}\) (which is now identified to its suspension

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spectrum replacement) simply to its underlying prespectrum of infinitesimal $\mathcal{P}$-bimodules. In particular, this says that $\mathbb{L}_\mathcal{P}$ is the same as $L^b_{\mathcal{P}}$ on each level.

(2) Suppose that $S$ is in addition stable containing a strict zero object 0. Let us compute $L_{\mathcal{P}} = R(\ker o \Omega^\infty)(\mathbb{L}_\mathcal{P})$. By $[[4], \text{Remark 2.4.7}]$, we have $R\Omega^\infty(\mathbb{L}_\mathcal{P}) \simeq \hocolim_n \Omega^n(\mathbb{L}_\mathcal{P})_{n,n}$. Now, by Computations 6.2.5, we find that

$$R(\ker o \Omega^\infty)(\mathbb{L}_\mathcal{P}) \simeq (\hocolim_n \Omega^n(\mathcal{P} \circ S^0_C)) \simeq \hocolim_n \Omega^n[\mathbb{L}(\mathcal{P} \circ S^0_C) \times_{\mathcal{P}} 0].$$

More explicitly, for each $\mathcal{P} = (c_1, \ldots, c_m; c)$, we find that

$$R(\ker o \Omega^\infty)(\mathbb{L}_\mathcal{P}) (\mathcal{P}) \simeq \hocolim_n \Omega^n[\mathbb{L}(\mathcal{P}) \circ (S^n)^{\otimes m}] \simeq \mathcal{P}(\mathcal{P}) \circ \hocolim_n \Omega^n[(S^n)^{\otimes m} \times_{1,\mathcal{P}} 0]$$

in which the last desuspension $\Omega(-)$ is now computed in $\text{IbMod}(\mathcal{P})$.

**Proposition 6.2.8.** Suppose that $S$ additionally satisfies the condition $(S8)$ 6.1.2 and that $\mathcal{P}$ is fibrant and $\Sigma$-cofibrant. Then the right Quillen equivalence

$$\mathcal{U}^b_{\mathcal{P}} : \text{Op}(\mathcal{P}) \xrightarrow{\simeq} \text{IbMod}(\mathcal{P})$$

identifies the cotangent complex $L_{\mathcal{P}}$ to $\mathbb{L}_{\mathcal{P}}[-1] \in \text{IbMod}(\mathcal{P})$ in which $\mathbb{L}_{\mathcal{P}}$ is the prespectrum with $(\mathbb{L}_{\mathcal{P}})_{n,n} = \mathcal{P} \circ S^0_C$. In particular, on each level, $(\mathbb{L}_{\mathcal{P}})_{n,n} (\mathcal{P}) = \mathcal{P}(\mathcal{P}) \circ (S^n)^{\otimes m}$ with $\mathcal{P} := (c_1, \ldots, c_m; c)$. Furthermore, if $S$ is in addition stable containing a strict zero object 0, then, under the right Quillen equivalence

$$\mathcal{P} \text{Op}(\mathcal{P}) \xrightarrow{\simeq} \text{IbMod}(\mathcal{P}),$$

the cotangent complex $L_{\mathcal{P}}$ is identified to $L_{\mathcal{P}}[-1]$ with $L_{\mathcal{P}} \in \text{IbMod}(\mathcal{P})$ being given on each level as

$$L_{\mathcal{P}}(\mathcal{P}) = \mathcal{P}(\mathcal{P}) \circ \hocolim_n \Omega^n[(S^n)^{\otimes m} \times_{1,\mathcal{P}} 0].$$

**Proof.** First, write $\mathcal{U}^b_{\mathcal{P}}$ as the composite functor $\mathcal{P} \text{Op}(\mathcal{P}) \xrightarrow{\mathcal{U}^b_{\mathcal{P}}} \mathcal{P} \text{BMod}(\mathcal{P}) \xrightarrow{\simeq} \mathcal{P} \text{IbMod}(\mathcal{P})$. By Proposition 6.2.1 (and by Notations 6.2.6), we hence get that $R\mathcal{U}^b_{\mathcal{P}}(L_{\mathcal{P}}) \simeq \mathbb{L}_{\mathcal{P}}[-1]$. By Computations 6.2.7(1), $\mathbb{L}_{\mathcal{P}}$ agrees with $L^b_{\mathcal{P}}$ on each level. The description of the latter is included in Computations 6.2.5.

When $S$ is in addition stable containing a strict zero object, the mentioned right Quillen equivalence is the composition $\mathcal{P} \text{Op}(\mathcal{P}) \xrightarrow{\mathcal{U}^b_{\mathcal{P}}} \mathcal{P} \text{IbMod}(\mathcal{P}) \xrightarrow{\ker o \Omega^\infty} \text{IbMod}(\mathcal{P})$. The claim hence follows by combining the above paragraph with Computations 6.2.7(2).

By the definition of Quillen cohomology group 4.0.12, we give the following conclusion, which is the central result of the paper.

**Theorem 6.2.9.** Suppose given a fibrant object $M \in \mathcal{P} \text{IbMod}(\mathcal{P})$. Under the same assumptions as in Proposition 6.2.8, the $n$'th Quillen cohomology group of $\mathcal{P}$ with coefficients in $M$ is formulated as

$$H^n_Q(\mathcal{P}, M) \simeq \pi_0 \text{Map}^b_{\mathcal{P} \text{IbMod}(\mathcal{P})}(\mathbb{L}_{\mathcal{P}}[-1], M[n]) \simeq \pi_0 \text{Map}^b_{\mathcal{P} \text{IbMod}(\mathcal{P})}(\mathbb{L}_{\mathcal{P}}, M[n + 1]).$$

Furthermore, suppose that $S$ is in addition stable containing a strict zero object 0. For a given fibrant object $M \in \text{IbMod}(\mathcal{P})$, the $n$'th Quillen cohomology group of $\mathcal{P}$ with coefficients in $M$ is formulated as

$$H^n_Q(\mathcal{P}, M) \simeq \pi_0 \text{Map}^b_{\mathcal{P} \text{IbMod}(\mathcal{P})}(\mathbb{L}_{\mathcal{P}}[-1], M[n]) \simeq \pi_0 \text{Map}^b_{\mathcal{P} \text{IbMod}(\mathcal{P})}(\mathbb{L}_{\mathcal{P}}, M[n + 1]).$$
6.3 Long exact sequence relating Quillen cohomology and reduced Quillen cohomology

To find a connection between Quillen cohomology and reduced Quillen cohomology of \( \mathcal{P} \), we first survey how \( L_{\mathcal{P}} \) links to \( L_{\mathcal{P}}^{\text{red}} \) (see Notations 6.0.1). Recall that, when \( \mathcal{P} \) is fibrant, there is a Quillen equivalence \( L_{\mathcal{P}}^{\text{Sp}} : \mathcal{T}_{\varnothing} \mathcal{O} \mathcal{P}_{C}(S) \xrightarrow{\simeq} \mathcal{T}_{\varnothing} \mathcal{O} \mathcal{P}(S) : \mathcal{R}_{\mathcal{P}}^{\text{Sp}} \) (cf. Proposition 5.1.4).

**Lemma 6.3.1.** Suppose that \( \mathcal{P} \) is bifibrant. Then there is a weak equivalence \( L_{\mathcal{P}}^{\text{Sp}}(L_{\mathcal{P}}^{\text{red}}) \xrightarrow{\simeq} L_{\mathcal{P}/J_{C}} \) in \( \mathcal{T}_{\varnothing} \mathcal{O} \mathcal{P}(S) \) where \( L_{\mathcal{P}/J_{C}} \) is the relative cotangent complex of the unit map \( \eta : J_{C} \rightarrow \mathcal{P} \).

**Proof.** Note first that there are canonical isomorphisms of categories \( \mathcal{O} \mathcal{P}_{C}(S)_{/J_{\mathcal{P}}} \simeq (\mathcal{O} \mathcal{P}_{C}(S)_{/J_{\mathcal{P}}})_{\eta/\eta} \) and \( \mathcal{O} \mathcal{P}(S)_{/J_{\mathcal{P}}} \simeq (\mathcal{O} \mathcal{P}(S)_{/J_{\mathcal{P}}})_{\eta/\eta} \). These respectively induce the Quillen equivalences \( \mathcal{T}_{\varnothing} \mathcal{O} \mathcal{P}_{C}(S) \simeq \mathcal{T}_{\eta}(\mathcal{O} \mathcal{P}_{C}(S)_{/J_{\mathcal{P}}}) \) and \( \mathcal{T}_{\varnothing} \mathcal{O} \mathcal{P}(S) \simeq \mathcal{T}_{\eta}(\mathcal{O} \mathcal{P}(S)_{/J_{\mathcal{P}}}) \). Moreover, there is a commutative square of left Quillen equivalences of the form

\[
\begin{array}{ccc}
\mathcal{T}_{\varnothing}(\mathcal{O} \mathcal{P}_{C}(S)_{/J_{\mathcal{P}}}) & \xrightarrow{\simeq} & \mathcal{T}_{\varnothing}(\mathcal{O} \mathcal{P}(S)_{/J_{\mathcal{P}}}) \\
\downarrow & & \downarrow \\
\mathcal{T}_{\varnothing} \mathcal{O} \mathcal{P}_{C}(S) & \xrightarrow{\eta} & \mathcal{T}_{\varnothing} \mathcal{O} \mathcal{P}(S)
\end{array}
\]

We let \( L_{\eta}^{\text{red}} \in \mathcal{T}_{\eta}(\mathcal{O} \mathcal{P}_{C}(S)_{/J_{\mathcal{P}}}) \) and \( L_{\eta} \in \mathcal{T}_{\eta}(\mathcal{O} \mathcal{P}(S)_{/J_{\mathcal{P}}}) \) denote the cotangent complexes of \( \eta \) as an object of \( \mathcal{O} \mathcal{P}_{C}(S)_{/J_{\mathcal{P}}} \) and \( \mathcal{O} \mathcal{P}(S)_{/J_{\mathcal{P}}} \), respectively. Now, observe that the top horizontal functor sends \( L_{\eta}^{\text{red}} \) to \( L_{\eta} \), while the left vertical functor sends \( L_{\eta}^{\text{red}} \) to \( L_{\eta}^{\text{red}} \) because \( J_{C} \) is the initial object of \( \mathcal{O} \mathcal{P}_{C}(S) \). By the commutativity of the above square, it remains to show that the right vertical functor sends \( L_{\eta} \) to \( L_{\mathcal{P}/J_{C}} \in \mathcal{T}_{\mathcal{P}} \mathcal{O} \mathcal{P}(S) \). To do this, we pick up the arguments given in the proof of Lemma 6.2.3. We therefore have to show that the following coCartesian square in \( \mathcal{O} \mathcal{P}(S) \)

\[
\begin{array}{ccc}
\mathcal{P} \cup J_{C} & \xrightarrow{\simeq} & \mathcal{P} \cup \mathcal{P} \\
\downarrow & & \downarrow \\
\mathcal{P} & \rightarrow & \mathcal{P} \cup J_{C}
\end{array}
\]

is already homotopy coCartesian. This is clear, by the cofibrancy of \( \mathcal{P} \).

The unit map \( \eta : J_{C} \rightarrow \mathcal{P} \) gives rise to the Quillen adjunctions

\[
\eta_{ib}^{ib} : \mathcal{T}_{J_{C}} \mathcal{I} \mathcal{M} \mathcal{O} \mathcal{D}(J_{C}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}} \mathcal{I} \mathcal{M} \mathcal{O} \mathcal{D}(\mathcal{P}) : \eta_{ib}^{*}, \quad \eta_{op}^{op} : \mathcal{T}_{J_{C}} \mathcal{O} \mathcal{P}(S) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}} \mathcal{O} \mathcal{P}(S) : \eta_{op}^{*}.
\]

Moreover, there is commutative diagram of Quillen adjunctions of the form

\[
\begin{array}{ccc}
\mathcal{T}_{\mathcal{J}_{C}} \mathcal{I} \mathcal{M} \mathcal{O} \mathcal{D}(J_{C}) & \xrightarrow{T_{ib}^{ib}} & \mathcal{T}_{\mathcal{J}_{C}} \mathcal{O} \mathcal{P}(S) \\
\eta_{ib} & \downarrow & \eta_{ib}^{*} \\
\mathcal{T}_{\mathcal{P}} \mathcal{I} \mathcal{M} \mathcal{O} \mathcal{D}(\mathcal{P}) & \xleftarrow{T_{op}^{op}} & \mathcal{T}_{\mathcal{P}} \mathcal{O} \mathcal{P}(S)
\end{array}
\]

The following is an analogue of [[5], Corollary 3.2.9].
Lemma 6.3.2. Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 6.1.2 and that $\mathcal{P}$ is bifibrant. There is a (homotopy) cofiber sequence in $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$ of the form

$$\mathbb{R}U^b_{\mathcal{P},C} L^\text{red}_{\mathcal{P}} \longrightarrow \mathbb{L}\eta^b_{\mathcal{P}} (\mathbb{L}_{\mathcal{P}}) \longrightarrow \mathbb{L}_{\mathcal{P}}$$  \hspace{1cm} (6.3.1)

where $U^b_{\mathcal{P},C}$ is the right Quillen equivalence $U^b_{\mathcal{P},C} : \mathcal{T}_{\mathcal{P}} \text{Op}_C(\mathcal{S}) \rightarrow \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$ appearing in Theorem 5.0.3.

Proof. Proposition 6.2.8 proves the existences of weak equivalences $\theta_{\mathcal{C}} : \mathbb{L}_{\mathcal{C}}[-1] \xrightarrow{\sim} \mathbb{R}U^b_{\mathcal{P},C}(\mathbb{L}_{\mathcal{P}})$ and $\theta_{\mathcal{P}} : \mathbb{L}_{\mathcal{P}}[-1] \xrightarrow{\sim} \mathbb{R}U^b_{\mathcal{P},C}(\mathbb{L}_{\mathcal{P}})$ in $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{C})$ and $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$, respectively. Applying $\mathbb{L}\eta^\text{op}_{\mathcal{P}}$ to $\theta_{\mathcal{P}}^d : \mathbb{L}\mathcal{P}^{ib} (\mathbb{L}_{\mathcal{P}}) \xrightarrow{\sim} \mathbb{L}_{\mathcal{P}}[1]$ (i.e., the adjoint of $\theta_{\mathcal{P}}$), and taking then the adjoint of the resultant, we obtain a weak equivalence in $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$ of the form $\mathbb{L}\eta^b_{\mathcal{P}} (\mathbb{L}_{\mathcal{P}}) \xrightarrow{\sim} \mathbb{R}U^b_{\mathcal{P}} \mathbb{L}\eta^\text{op}_{\mathcal{P}}(\mathbb{L}_{\mathcal{P}}[1])$.

On other hand, by the definition of relative cotangent complex, there is a cofiber sequence in $\mathcal{T}_{\mathcal{P}} \text{Op}(\mathcal{S})$ of the form $\mathbb{L}\eta^\text{op}(\mathbb{L}_{\mathcal{P}}) \xrightarrow{\sim} \mathbb{L}_{\mathcal{P}}(\mathbb{L}_{\mathcal{P}})$, which shifts to a new cofiber sequence:

$$\mathbb{L}_{\mathcal{P}}[1] \rightarrow \mathbb{L}\eta^\text{op}(\mathbb{L}_{\mathcal{P}})[1] \rightarrow \mathbb{L}_{\mathcal{P}}[1].$$

By applying $\mathbb{R}U^b_{\mathcal{P}}$ to the latter and by the first paragraph, we get a cofiber sequence in $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$:

$$\mathbb{R}U^b_{\mathcal{P}}(\mathbb{L}_{\mathcal{P}}) \longrightarrow \mathbb{L}\eta^b_{\mathcal{P}} (\mathbb{L}_{\mathcal{P}}) \longrightarrow \mathbb{L}_{\mathcal{P}}.$$

Now, note that the functor $U^b_{\mathcal{P},C}$ is the same as the composition $U^b_{\mathcal{P},C} \circ U^\mathcal{S}_{\mathcal{P}}$. Lemma 6.3.1 hence shows that there is a weak equivalence $\mathbb{R}U^b_{\mathcal{P},C} L^\text{red}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{R}U^b_{\mathcal{P}}(\mathbb{L}_{\mathcal{P}})$ in $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$. So we get the desired cofiber sequence.

We end this section by the following theorem.

Theorem 6.3.3. Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 6.1.2 and that $\mathcal{P}$ is fibrant and $\Sigma$-cofibrant. Given a fibrant object $M \in \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$, there is a long exact sequence of abelian groups of the form

$$\cdots \rightarrow H^0_{\mathcal{Q}}(\mathcal{P}, M) \rightarrow H^0_{\mathcal{Q},r}(\mathcal{P}, M) \rightarrow H^0_{\mathcal{Q},\text{red}}(\mathcal{P}, M) \rightarrow H^0_{\mathcal{Q}}(\mathcal{P}, M) \rightarrow H^1_{\mathcal{Q},r}(\mathcal{P}, M) \rightarrow \cdots$$

where $H^*_{\mathcal{Q},r}(\mathcal{P}, -)$ refers to Quillen cohomology group of $\mathcal{P}$ when regarded as a right module over itself, while $H^*_{\mathcal{Q},\text{red}}(\mathcal{P}, -)$ refers to Quillen cohomology group of $\mathcal{P}$ and $H^*_{\mathcal{Q},\text{red}}(\mathcal{P}, -)$ refers to reduced Quillen cohomology group of $\mathcal{P}$ (cf. Conventions 6.0.2).

Proof. Firstly, since Quillen cohomology is homotopy invariant with respect to the class of fibrant objects (cf. Remark 4.0.13), we can assume without loss of generality that $\mathcal{P}$ is bifibrant. The cofiber sequence of Lemma 6.3.2 induces a fiber sequence of derived mapping spaces:

$$\text{Map}^b_{\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})}(\mathbb{L}_{\mathcal{P}}, M) \rightarrow \text{Map}^b_{\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})}(\mathbb{L}\eta^b_{\mathcal{P}}(\mathbb{L}_{\mathcal{P}}), M) \rightarrow \text{Map}^b_{\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})}(\mathbb{R}U^b_{\mathcal{P},C} L^\text{red}_{\mathcal{P}}, M).$$

In this sequence, by notation, $\mathbb{R}U^b_{\mathcal{P},C} L^\text{red}_{\mathcal{P}}$ classifies the reduced Quillen cohomology of $\mathcal{P}$, while $\mathbb{L}_{\mathcal{P}}$ classifies the Quillen cohomology of $\mathcal{P}$, by Theorem 6.2.9. That fiber sequence will hence give rise to the desired long exact sequence after showing that $\mathbb{L}\eta^b_{\mathcal{P}}(\mathbb{L}_{\mathcal{P}})$ classifies the Quillen cohomology of $\mathcal{P}$ when regarded as a right module over itself. To this end, we first consider the Quillen adjunction $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P}) \xrightarrow{\sim} \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$ which is induced by the free-forgetful adjunction $\text{IbMod}(\mathcal{P}) \xrightarrow{\sim} \text{IbMod}(\mathcal{P})$. We denote by $L_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$ the cotangent complex of $\mathcal{P}$ when regarded as a right module over itself. It therefore suffices to prove that the derived image of $L_{\mathcal{P}}$ in $\mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$ is weakly equivalent to $\mathbb{L}\eta^b_{\mathcal{P}}(\mathbb{L}_{\mathcal{P}})$. For this last claim, observe first that the functor $\eta^b_{\mathcal{P}}$ is the same as the functor $\mathcal{T}_{\mathcal{C}} \text{Coll}_C(\mathcal{S}) \rightarrow \mathcal{T}_{\mathcal{P}} \text{IbMod}(\mathcal{P})$ which is induced.
by the free functor $\text{Coll}_C(\mathcal{S}) \to \text{IbMod}(\mathcal{P})$. Moreover, under the identification $\mathcal{I}_C \text{IbMod}(\mathcal{I}_C) \simeq \mathcal{I}_C \text{Coll}_C(\mathcal{S})$, the object $\mathfrak{L}_{\mathcal{I}_C}$ is nothing but the cotangent complex of $\mathcal{I}_C$ when regarded as an object of $\text{Coll}_C(\mathcal{S})$, which has the derived image, under the left Quillen functor $\mathcal{I}_C \text{Coll}_C(\mathcal{S}) \to \mathcal{T}_\mathcal{P} \text{RMod}(\mathcal{P})$, being $\mathfrak{L}_{\mathcal{I}_C}$ obviously. The proof is hence completed.

\section{Quillen cohomology of simplicial operads}

Simplicial operads are precisely operads enriched over the Cartesian monoidal category of simplicial sets, $\text{Set}_\Delta$. This category comes equipped with the standard (Kan-Quillen) model structure, then, satisfies the conditions of Conventions 3.0.2 and also the extra condition (S8) 6.1.2 (cf. Proposition 6.1.3). We therefore inherit the results of Section 6 for the work of this section.

According to the work of Y. Harpaz, J. Nuiten and M. Prasma ([5]), the cotangent complex of a simplicial operad is indeed represented as a spectrum valued functor on its twisted arrow $\infty$-category (see Theorem 1.0.7). The construction of twisted arrow $\infty$-categories (of $\infty$-categories) $\text{Tw}(\cdot) : \text{Cat}_\infty \to \text{Cat}_\infty$ was originally introduced by Lurie [2], § 5.2. For a given fibrant simplicial category $\mathcal{C}$, the twisted arrow $\infty$-category of $\mathcal{C}$ is simply defined to be $\text{Tw}(\mathcal{C}) := \text{Tw}(\mathcal{N} \mathcal{C})$ with $\mathcal{N} \mathcal{C}$ being the simplicial nerve of $\mathcal{C}$. We shall extend the latter to the construction of twisted arrow $\infty$-categories of (fibrant) simplicial operads. We then assert that the cotangent complex of a simplicial operad is indeed represented as a spectrum valued functor on its twisted arrow $\infty$-category.

\subsection{Twisted arrow $\infty$-categories of simplicial operads}

Let $\mathcal{C}$ be a fibrant simplicial category. Following [2, Proposition 5.2.1.11], the unstraightening functor

$$\text{Un} : \text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Set}_\Delta) \xrightarrow{\simeq} (\text{Set}_\Delta^{\text{cov}})_{/N \mathcal{C}^{\text{op}} \times N \mathcal{C}},$$

(7.1.1)

from the projective model category of functors $\mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set}_\Delta$ to the covariant model category of simplicial sets over $N \mathcal{C}^{\text{op}} \times N \mathcal{C}$, identifies $\text{Map}_\mathcal{C}$ to $\text{Tw}(\mathcal{C})$ comming together with a canonical left fibration $\text{Tw}(\mathcal{C}) \to N \mathcal{C}^{\text{op}} \times N \mathcal{C}$. Here we should think of $\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Set}_\Delta)$ as the category of $\mathcal{C}$-bimodules. Then, $\text{Map}_\mathcal{C} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set}_\Delta$ is nothing but $\mathcal{C}$, viewed as a bimodule over itself. (The readers may review straightening and unstraightening functors in [3], §2.2.1).

Let $\mathcal{P}$ be a fibrant simplicial operad. Recall from Section 5.2 that there is a canonical isomorphism of categories $\text{IbMod}(\mathcal{P}) \simeq \text{Fun}(\mathcal{Ib}^{\mathcal{P}}, \text{Set}_\Delta)$. Thus, by the above words, it does make sense to define $\text{Tw}(\mathcal{P})$ as the image of $\mathcal{P}$ under the unstraightening functor

$$\text{Un} : \text{IbMod}(\mathcal{P}) = \text{Fun}(\mathcal{Ib}^{\mathcal{P}}, \text{Set}_\Delta) \xrightarrow{\simeq} (\text{Set}_\Delta^{\text{cov}})_{/N(\mathcal{Ib}^{\mathcal{P}})},$$

(7.1.2)

which is in fact an extension of (7.1.1).

An explicit formula of the unstraightening construction can be found in [35]. We are recalling its procedures with the opposite convention of loc.cit. Let $\mathcal{C}[-] : \text{Set}_\Delta \to \text{Cat}(\text{Set}_\Delta)$ denote the left adjoint to the simplicial nerve functor $\mathcal{N} : \text{Set}_\Delta \to \text{Cat}(\text{Set}_\Delta)$ (cf. [3]). One first defines for each $n \in \mathbb{N}$ a simplicial functor

$$\mathcal{D}_n : \mathcal{C}^{\Delta^n} \to \text{Set}_\Delta$$

given by sending each object $i \in \{0, \cdots, n\}$ to $\mathcal{D}_n(i) := \mathcal{N} P_r(i)$ the nerve of the poset $P_r(i) := \{S \mid \{i\} \subseteq S \subseteq \{0, \cdots, i\}\}$. The structure maps of simplicial functor is defined by applying the union
operation of subsets in an obvious way. Furthermore, for each map \( \delta : \Delta^n \to \Delta^n \), one defines a natural transformation

\[
D_\delta : D_{\Delta^n} \to D_{\Delta^n} \circ C[\delta]
\]

of the simplicial functors \( C[\Delta^m] \to \text{Set}_\Delta \) which is given at each object \( i \in C[\Delta^m] \) by the map \( D_\delta(i) : D_{\Delta^n}(i) \to D_{\Delta^n}(\delta i) \) induced by the map of posets \( P_r(i) \to P_r(\delta i) \), \( S \mapsto \delta(S) \).

**CONSTRUCTION 7.1.1.** The twisted arrow \( \infty \)-category of \( \mathcal{P} \), \( Tw(\mathcal{P}) \), is defined to be the image of \( \mathcal{P} \) through the functor \( \ln \) (7.1.2). More explicitly, the data of an \( n \)-simplex of \( Tw(\mathcal{P}) \) consists of

- an \( n \)-simplex \( f \in N(\mathcal{Ib}^p) \), i.e., a functor \( f : C[\Delta^n] \to \mathcal{Ib}^p \), and
- a natural transformation \( t : D_{\Delta^n} \to \mathcal{P} \circ f \) between simplicial functors \( C[\Delta^n] \to \text{Set}_\Delta \).

For each map \( \delta : \Delta^m \to \Delta^n \), the simplicial structure map \( Tw(\mathcal{P})_n \to Tw(\mathcal{P})_m \) is given by sending each pair \( (f, t) \in Tw(\mathcal{P})_n \) to the pair

\[
C[\Delta^m] \xrightarrow{C[\delta]} C[\Delta^n] \xrightarrow{f} \mathcal{Ib}^p, \quad D_{\Delta^m} \xrightarrow{D_\delta} D_{\Delta^n} \circ C[\delta] \xrightarrow{\text{total}} \mathcal{P} \circ f \circ C[\delta].
\]

**EXAMPLES 7.1.2.** When \( \mathcal{P} \) is discrete then \( Tw(\mathcal{P}) \) is isomorphic to the nerve of a certain ordinary category. In this situation, we will identify \( Tw(\mathcal{P}) \) to the corresponding category and refer to it as the twisted arrow category of \( \mathcal{P} \). For example, it is not hard to show that the twisted arrow category of the commutative operad, \( Tw(\text{Com}) \), is equivalent to \( \text{Fin}^{op}_\mathcal{C} \). We also prove in this subsection that the twisted arrow category of the associative operad, \( Tw(\text{Ass}) \), is equivalent to the simplex category \( \Delta \) (cf. Proposition 7.1.11).

**PROPOSITION 7.1.3.** The construction \( Tw(-) \) determines a homotopy invariant from fibrant simplicial operads to \( \infty \)-categories.

**Proof.** Let \( f : \mathcal{P} \to \mathcal{Q} \) be a map between fibrant simplicial operads. By the compatibility of the unstraightening functor with taking base change along \( f : \mathcal{P} \to \mathcal{Q} \), we obtain the induced map \( Tw(f) : Tw(\mathcal{P}) \to Tw(\mathcal{Q}) \) fitting into the following Cartesian square of \( \infty \)-categories

\[
\begin{array}{ccc}
Tw(\mathcal{P}) & \to & Tw(\mathcal{Q}) \\
\downarrow & & \downarrow \\
N(\mathcal{Ib}^p) & \to & N(\mathcal{Ib}^q)
\end{array}
\]

Note that this square is already homotopy Cartesian (with respect to the Joyal model structure), due to the fact that the right vertical map is a left fibration.

We are showing that the map \( Tw(f) : Tw(\mathcal{P}) \to Tw(\mathcal{Q}) \) is an equivalence when provided that \( f : \mathcal{P} \to \mathcal{Q} \) is a weak equivalence. We first show that the induced map \( \mathcal{Ib}^f : \mathcal{Ib}^p \to \mathcal{Ib}^q \) is a weak equivalence of simplicial categories (i.e., a Dwyer-Kan equivalence). It is clear by construction that \( \mathcal{Ib}^f \) is a levelwise weak equivalence. Hence it remains to show that \( \mathcal{Ib}^f \) is essentially surjective. Suppose given an object \( (d_1, \ldots, d_n; d_0) \) of \( \mathcal{Ib}^q \). Since the underlying simplicial functor \( f_1 : \mathcal{P}_1 \to \mathcal{Q}_1 \) of \( f \) is essentially surjective, for each \( i \in \{0, \ldots, n\} \), there exists an object \( c_i \) of \( \mathcal{P} \) together with an isomorphism \( \theta_i : f(c_i) \to d_i \) in the homotopy category of \( \mathcal{Q}_1 \). The morphisms \( \theta_i \)'s together form a morphism \( \theta : (f(c_1), \ldots, f(c_n); f(c_0)) \to (d_1, \ldots, d_n; d_0) \) in \( \mathcal{Ib}^q \). It can then be verified by definition that \( \theta \) is an isomorphism in the homotopy category of \( \mathcal{Ib}^q \). We just proved that
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$\text{Ib}^f$ is a weak equivalence (between fibrant simplicial categories). So the map $\text{N}(\text{Ib}^P) \to \text{N}(\text{Ib}^Q)$ is an equivalence of $\infty$-categories. This fact, together with the first paragraph, proves that the map $\text{Tw}(f) : \text{Tw}(P) \to \text{Tw}(Q)$ is indeed an equivalence.

To analyze the structure of $\text{Tw}(P)$, we will need the following notations.

**Notation 7.1.4.** We always denote the permutations by listing their values. For instance, $\sigma = [i_1, \ldots, i_n]$ refers to the permutation $\sigma \in \Sigma_n$ with $\sigma(k) = i_k$.

**Notations 7.1.5.** Let $f : \langle n \rangle \to \langle m \rangle$ be a map in $\text{Fin}_*$. For each $s \in \{0, 1, \ldots, m\}$, we let $[f^{-1}(s)]$ denote the increasing sequence of the elements of $f^{-1}(s)$, written as $[f^{-1}(s)] = [i_1^s < \cdots < i_k^s]$. (Of course, this could be empty.) Then $f$ can be represented by the sequence obtained by concatenating those sequences for $s = 1, \ldots, m$, written as:

$$[f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m)] = [i_1^1 < \cdots < i_{k_1}^1 \mid i_1^2 < \cdots < i_{k_2}^2 \mid \cdots \mid i_1^m < \cdots < i_{k_m}^m],$$

or alternatively, by the extended sequence $[f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m) \mid f^{-1}(0)^0]$ formed in the same manner as the previous one, in which $[f^{-1}(0)^0] := [f^{-1}(0)] \setminus \{0\}$. Moreover, we denote by $\sigma_f$ the permutation $[f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m) \mid f^{-1}(0)^0] \in \Sigma_n$.

**Notation 7.1.6.** Let $\overline{c} := (c_1, \ldots, c_m; c)$ and $\overline{d} := (d_1, \ldots, d_n;d)$ be two $C$-sequences and let $f : \langle n \rangle \to \langle m \rangle$ be a map in $\text{Fin}_*$. Given a vertex

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m) \in \text{Map}^f_{\text{Ib}^P}(\overline{c}, \overline{d})$$

with $\alpha_0 \in \mathcal{P}(c, \{d_j\}_{j \in f^{-1}(0)}; d)$ and $\alpha_i \in \mathcal{P}(\{d_j\}_{j \in f^{-1}(i)}; c_i)$ ($i = 1, \ldots, m$), we will denote by $\alpha^* : \mathcal{P}(\overline{c}) \to \mathcal{P}(\overline{d})$ the image of $\alpha$ under the map $\text{Map}^f_{\text{Ib}^P}(\overline{c}, \overline{d}) \to \text{Map}_{\text{Set}_\Delta}(\mathcal{P}(\overline{c}), \mathcal{P}(\overline{d}))$ which is part of the simplicial functor structure of $\mathcal{P} : \text{Ib}^P \to \text{Set}_\Delta$. By construction, for each simplex $\theta \in \mathcal{P}(\overline{c})$, we have that

$$\alpha^*(\theta) = (\alpha_0 \circ_1 \theta \circ (\alpha_1, \ldots, \alpha_m)) \sigma_f^{-1} \in \mathcal{P}(\overline{d})$$

the action of $\sigma_f^{-1} \in \Sigma_n$ on $\alpha_0 \circ_1 \theta \circ (\alpha_1, \ldots, \alpha_m)$ where $"(\circ_1) \circ"$ refers to the (partial) composition in $\mathcal{P}$. (See Construction 5.2.5.

Unwinding definition, we will see that $\text{Tw}(\mathcal{P})$ indeed looks like something obtained by twisting “multiarrows” of $\mathcal{P}$.

- Objects of $\text{Tw}(\mathcal{P})$ are precisely the operations of $\mathcal{P}$ (i.e., the vertices of the spaces of operations of $\mathcal{P}$).

- Let $\mu \in \mathcal{P}(c_1, \ldots, c_m; c)$ and $\nu \in \mathcal{P}(d_1, \ldots, d_n; d)$ be two operations of $\mathcal{P}$, the data of a morphism (edge) $\mu \to \nu$ in $\text{Tw}(\mathcal{P})$ consists of
  - a map $f : \langle n \rangle \to \langle m \rangle$ in $\text{Fin}_*$,
  - a tuple of operations $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m) \in \text{Map}^f_{\text{Ib}^P}(\overline{c}, \overline{d})$, and
  - a 1-simplex $h : \Delta^1 \to \mathcal{P}(d_1, \ldots, d_n; d)$ with $h(0) = \nu$ and $h(1) = \alpha^*(\mu)$, viewed as a homotopy from $\nu$ to $\alpha^*(\mu)$.

By convention, we will write $(f, \alpha, t) : \mu \to \nu$ standing for such a typical morphism. It is
convenient to depict this morphism as the following square

\[
\begin{array}{c}
(c_1, \ldots, c_m) \\ \downarrow \mu \\ (c)
\end{array}
\quad \begin{array}{c}
\alpha_n \\ \downarrow \nu
\end{array}
\begin{array}{c}
(d_1, \ldots, d_n) \\
\end{array}
\] (7.1.3)

which is “commutative up to a chosen homotopy”.

- In general, a \(k\)-simplex of \(\text{Tw}(\mathcal{P})\) is the one, formally, depicted as the composition of \(k\) squares of the type (7.1.3) equipped with a collection of homotopies, homotopies between homotopies, and so forth.

Despite the data of simplices of \(\text{Tw}(\mathcal{P})\) are elephantine, it is feasible to survey some its local properties. For instance, we can formulate an “equivalence” between two operations of \(\mathcal{P}\).

**Lemma 7.1.7.** A morphism \((f, \alpha, t) : \mu \rightarrow \nu\) from \(\mu \in \mathcal{P}(c_1, \ldots, c_m; c)\) to \(\nu \in \mathcal{P}(d_1, \ldots, d_n; d)\) is an equivalence in \(\text{Tw}(\mathcal{P})\) if and only if the following conditions hold:

1. \(f : (n) \rightarrow (m)\) is bijective.
2. There exist \(\beta \in \text{Map}_{\mathcal{P}}(\bar{d}, \bar{c})\), an edge \(\gamma : \mu \rightarrow \beta^*(\nu)\) and together with two edges

\[
\begin{array}{c}
[\alpha] = \text{Map}_{\mathcal{P}}^{id_{\mathcal{P}}}(\bar{c}, \bar{c}) \quad , \quad [h : id_{\mathcal{P}} \rightarrow \alpha \beta] = \text{Map}_{\mathcal{P}}^{id_{\mathcal{P}}}(\bar{d}, \bar{d})
\end{array}
\]

which are all subject to the existences of two 2-simplices of the forms

\[
\begin{array}{c}
\mu \xrightarrow{\beta^*(\nu)} \beta^*(\nu) \\
\alpha^*(\mu) \xrightarrow{\alpha^*(\nu)} \nu
\end{array}
\]

belonging to \(\mathcal{P}(c_1, \ldots, c_m; c)\) and \(\mathcal{P}(d_1, \ldots, d_n; d)\), respectively. In this item, \(id_{\mathcal{P}} : (id_{c_1}, id_{c_2}, \ldots, id_{c_m})\) referring to the identity morphism on \(\bar{c}\), while \(h^*\) is given as the composition

\[
\mathcal{P}(\bar{c}) \times \Delta^1 \xrightarrow{id_{\mathcal{P}} \times h} \mathcal{P}(\bar{c}) \times \text{Map}_{\mathcal{P}}^{id_{\mathcal{P}}}(\bar{c}, \bar{c}) \rightarrow \mathcal{P}(\bar{c}).
\]

**Proof.** Following Construction 7.1.1, we can get the explicit description of 2-simplices of \(\text{Tw}(\mathcal{P})\). The proof is then straightforward.

**Corollary 7.1.8.** Let \(\mu \in \mathcal{P}(c_1, \ldots, c_m; c)\) and \(\sigma \in \Sigma_m\) be given. Then there is a canonical equivalence \(\mu \rightarrow \mu^\sigma\) in \(\text{Tw}(\mathcal{P})\). Consequently, \(\text{Tw}(\mathcal{P})\) admits a skeleton whose objects are obtained by taking the quotient of the class of operations of \(\mathcal{P}\) under the relation “\(\mu \sim \mu^\sigma\)”.

**Proof.** We take a canonical edge \((f, \alpha, t) : \mu \rightarrow \mu^\sigma\) as follows. The map \(f : (m) \rightarrow (m)\) agrees with \(\sigma\) on \(\{1, \ldots, m\}\). Then, we take \(\alpha\) to be the tuple of unit operations

\[
\alpha := (id_{c_1}, id_{c_2}, \ldots, id_{c_m}) \in \text{Map}_{\mathcal{P}}(\bar{c}_1, \ldots, \bar{c}_m; c, (c_\sigma(1), \ldots, c_\sigma(m); c)).
\]

By construction, the induced map

\[
\alpha^* : \mathcal{P}(c_1, \ldots, c_m; c) \rightarrow \mathcal{P}(c_\sigma(1), \ldots, c_\sigma(m); c)
\]

is nothing but the map defined by the action of the permutation \(\sigma\) on \(\mathcal{P}\) (see the proof of Proposition 5.2.6). In particular, we get that \(\alpha^*(\mu) = \mu^\sigma\). Finally, we take \(t := s_0\mu^\sigma\). Using
Lemma 7.1.7, we can readily verify that the obtained edge \((f, \alpha, t)\) is indeed an equivalence in \(\Tw(\mathcal{P})\).

We find a somewhat large class of simplicial operads whose twisted arrow \(\infty\)-categories admit terminal objects.

**Definition 7.1.9.** A given simplicial operad is said to be **unital** if all its spaces of unary operations are a singleton. Furthermore, we will say that a given simplicial operad is **unitally homotopy connected** if it is unital with weakly contractible spaces of 1-ary operations.

Typical examples for this definition include the **little cubes operads** \(\mathcal{C}_n\), \(n = 0, \ldots, \infty\).

**Lemma 7.1.10.** Let \(\mathcal{P}\) be a fibrant and unital simplicial operad. For each color \(d\) of \(\mathcal{P}\), let \(\mu_d\) denote the unique operation in \(\mathcal{P}(d)\). Then \(\mu_d\) is a terminal object of \(\Tw(\mathcal{P})\) if and only if \(d\) is a (strongly) homotopy terminal object of the underlying simplicial category of \(\mathcal{P}\). Consequently, if \(\mathcal{P}\) is fibrant and unitally homotopy connected then \(\Tw(\mathcal{P})\) admits the terminal objects being precisely the unary operations of \(\mathcal{P}\).

**Proof.** The second condition in the statement is equivalent to saying that the space \(\mathcal{P}(c; d)\) is contractible for every color \(c\). In the other hand, by definition, \(\mu_d\) is a terminal object of \(\Tw(\mathcal{P})\) precisely if for every operation \(\mu \in \mathcal{P}\) the mapping space \(\text{Map}_{\Tw(\mathcal{P})}(\mu, \mu_d)\) is contractible. We will use \(\text{Hom}_{\Tw(\mathcal{P})}(\mu, \mu_d)\) the space of left morphisms from \(\mu\) to \(\mu_d\) as a model for \(\text{Map}_{\Tw(\mathcal{P})}(\mu, \mu_d)\).

Given any \(\mu \in \mathcal{P}(c_1, \ldots, c_n; c)\), it suffices to show that there is a homotopy equivalence of spaces:

\[
\mathcal{P}(c; d) \simeq \text{Hom}_{\Tw(\mathcal{P})}(\mu, \mu_d).
\]

Recall that an \(n\)-simplex of \(\text{Hom}_{\Tw(\mathcal{P})}(\mu, \mu_d)\) is an \((n + 1)\)-simplex \(z: \Delta^{n+1} \to \Tw(\mathcal{P})\) of \(\Tw(\mathcal{P})\) such that \(z(0) = \mu\) and \(d_0z\) agrees with the degeneracy of \(\mu_d\). Let \(\delta_0: \Delta^n \to \Delta^{n+1}\) denote the face of \(\Delta^{n+1}\) missing the 0'th vertex. By Construction 7.1.1, the data of such a simplex \(z\) consists of a pair \((f, t)\) with

- \(f: \mathcal{C}[\Delta^{n+1}] \to \mathcal{Ib}^\mathcal{P}\) being a simplicial functor such that \(f(0) = (c_1, \ldots, c_n; c)\) and the composition \(\mathcal{C}[\Delta^n] \xrightarrow{\mathcal{C}[\delta_0]} \mathcal{C}[\Delta^{n+1}] \xrightarrow{f} \mathcal{Ib}^\mathcal{P}\) coincides with the constant functor on \((d)\),

- \(t: \mathcal{O}_{\Delta^{n+1}} \to \mathcal{P} \circ f\) being a natural transformation between the simplicial functors \(\mathcal{C}[\Delta^{n+1}] \to \text{Set}_\Delta\).

Since \(\mathcal{P}(d)\) is a singleton, the data of \(t\) is reduced to the choice of a map \(t(0): \Delta^0 = \mathcal{O}_{\Delta^{n+1}}(0) \to \mathcal{P}(c_1, \ldots, c_n; c)\), which is always required to agree with the vertex \(\mu \in \mathcal{P}(c_1, \ldots, c_n; c)\). It implies that the data of \(z\) is fully enclosed in that of \(f\). Observe now that \(f\) is identified to an \((n + 1)\)-simplex \(f: \Delta^{n+1} \to \mathcal{N}(\mathcal{Ib}^\mathcal{P})\) of the nerve of \(\mathcal{Ib}^\mathcal{P}\) satisfying that \(f(0) = (c_1, \ldots, c_n; c)\) and that \(d_0f\) is the degeneracy of \((d)\). By this way, we get a canonical isomorphism

\[
\text{Hom}_{\Tw(\mathcal{P})}(\mu, \mu_d) \simeq \text{Hom}_{\mathcal{N}(\mathcal{Ib}^\mathcal{P})}( (c_1, \ldots, c_n; c), (d) ).
\]

The right hand side models the mapping space \(\text{Map}_{\mathcal{N}(\mathcal{Ib}^\mathcal{P})}( (c_1, \ldots, c_n; c), (d) )\), which is homotopy equivalent to \(\text{Map}_{\mathcal{Ib}^\mathcal{P}}( (c_1, \ldots, c_n; c), (d) )\) (see [3], § 2.2). Moreover, since \(\mathcal{P}\) is unital, the latter is isomorphic to \(\mathcal{P}(c; d)\). We hence obtain the expected canonical homotopy equivalence \(\mathcal{P}(c; d) \simeq \text{Hom}_{\Tw(\mathcal{P})}(\mu, \mu_d)\).

Finally, for more illustration, we prove the following:
PROPOSITION 7.1.11. There is a canonical equivalence \( \varphi : \Delta \xrightarrow{\sim} \mathrm{Tw}(\text{Ass}) \) between the simplex category and the twisted arrow category of the associative operad \( \text{Ass} \).

To this end, we first revisit some basic constructions.

CONSTRUCTION 7.1.12. There is a canonical functor \( \iota : \Delta \rightarrow \text{Fin}_{\ast}^{\text{op}} \) (which is essentially used to define underlying cosimplicial spaces of \textbf{gamma spaces}) defined by sending \( [n] \in \Delta \) to \( \langle n \rangle \in \text{Fin}_{\ast}^{\text{op}} \) and by giving the natural maps \( \iota_{m,n} : \mathrm{Hom}_{\Delta}([m],[n]) \rightarrow \text{Hom}_{\text{Fin}}(\langle n \rangle,\langle m \rangle) \) as follows. Given a map \( f : [m] \rightarrow [n] \), let us identify \( f \) with the increasing sequence \( [j_1 < \cdots < j_k] \) of its values together with a tuple \( (p_1,\ldots,p_k) \) with \( p_r \) being the cardinality of the fiber \( f^{-1}(j_r) \). The map \( \iota_{m,n}(f) : \langle n \rangle \rightarrow \langle m \rangle \) is given by listing its nonempty fibers as following:

\[
\iota_{m,n}(f)^{-1}(p_1) = \{ j_1 + 1, \ldots, j_2 \}, \quad \iota_{m,n}(f)^{-1}(p_1 + p_2) = \{ j_2 + 1, \ldots, j_3 \}, \ldots,
\]

\[
\iota_{m,n}(f)^{-1}(p_1 + p_2 + \cdots + p_{k-1}) = \{ j_{k-1} + 1, \ldots, j_k \}, \quad \iota_{m,n}(f)^{-1}(0) = \{ 0, 1, \ldots, n \} - \{ j_1 + 1, \ldots, j_k \}.
\]

Remark 7.1.13. It is straightforward to verify the following observations:

1. The image \( \text{Im}(\iota_{m,n}) \subseteq \text{Hom}_{\text{Fin}}(\langle n \rangle,\langle m \rangle) \) of \( \iota_{m,n} \) consists of precisely those maps \( g \) such that \( [g^{-1}(1) \mid g^{-1}(2) \mid \cdots \mid g^{-1}(m)] \) (cf. Notations 7.1.5) is either empty or forms a sequence of consecutive natural numbers.

2. Let \( \mathrm{Hom}_{\Delta}^{\text{const}}([m],[n]) \subseteq \mathrm{Hom}_{\Delta}([m],[n]) \) denote the subset consisting of the constant maps and let \( \mathrm{Hom}_{\Delta}^{\text{op}}([m],[n]) \) denote the complement of the previous one in \( \mathrm{Hom}_{\Delta}([m],[n]) \). Likewise, we denote by \( \text{const}_0 \in \text{Hom}_{\text{Fin}}(\langle n \rangle,\langle m \rangle) \) the unique constant map (with value \( 0 \in \langle m \rangle \)) and let \( \text{Im}^{\circ}(\iota_{m,n}) \) be the complement \( \text{Im}(\iota_{m,n}) - \{ \text{const}_0 \} \). Observe then that the restriction of \( \iota_{m,n} \) to \( \mathrm{Hom}_{\Delta}^{\text{op}}([m],[n]) \) induces a natural bijection \( \iota_{m,n} : \mathrm{Hom}_{\Delta}^{\text{op}}([m],[n]) \xrightarrow{\sim} \text{Im}^{\circ}(\iota_{m,n}) \).

Recall that the associative operad \( \text{Ass} \) is the single-colored operad whose set of \( n \)-ary operations is \( \text{Ass}(n) = \Sigma_n, \ n \geq 0 \), equipped with the canonical right action of \( \Sigma_n \) on itself. The composition is defined by concatenating linear orders. It is more convenient for us to keep using the notation of typical form \( \text{Ass}(c_1,\ldots,c_n; c) \) standing for the set of \( n \)-ary operations, and keep in mind that all the colors which appear below are the only one \( \ast \).

Proof of Proposition 7.1.11. By Corollary 7.1.8, \( \text{Tw}(\text{Ass}) \) admits a skeleton whose objects are the trivial permutations \( \mu_n : = [1,\cdots,n] \in \Sigma_n, \ n \geq 0 \). We define \( \varphi \) on objects by sending \( [n] \in \Delta \) to \( \mu_n \). It remains to establish the natural isomorphisms of the form

\[
\varphi_{m,n} : \mathrm{Hom}_{\Delta}([m],[n]) \xrightarrow{\sim} \mathrm{Hom}_{\text{Tw}(\text{Ass})}(\mu_m,\mu_n).
\]

Let us analyze the right hand side. By definition, we write

\[
\mathrm{Hom}_{\text{Tw}(\text{Ass})}(\mu_m,\mu_n) = \bigsqcup_{\langle n \rangle \subseteq \{ m \}} A_f
\]

in which \( A_f \subseteq \mathrm{Hom}_{\text{Fin}_{\ast}}^{\text{op}}(\langle c_1,\cdots,c_m; c \rangle,\langle d_1,\cdots,d_n; d \rangle) \) denotes the subset consisting of those \( \alpha = (\alpha_0,\alpha_1,\cdots,\alpha_m) \) such that \( \alpha^\ast(\mu_m) = \mu_n \) (see Notation 7.1.6). Unwinding definition, the latter is equivalent to the equation

\[
\alpha_0 \circ_1 \mu_m \circ (\alpha_1,\cdots,\alpha_m) = [f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m) \mid f^{-1}(0)^\circ]
\]  \hspace{1cm} (7.1.4)

Observe that there is a unique choice of \( (\alpha_1,\cdots,\alpha_m) \) such that this equation possibly admits solutions, being precisely \( (\mu_{h(1)},\cdots,\mu_{h(n)}) \), i.e., the tuple of trivial permutations with \( h(i) \) referring to the arity of \( \alpha_i \). Thus, by comparing the two sides of this equation after substituting
(α₁, ⋯, αₘ) = (µ₁(1), ⋯, µₘ(m)) back to it, we realize that the set \(A_f\) is nonempty only if the sequence \([f⁻¹(1) \mid f⁻¹(2) \mid ⋯ \mid f⁻¹(m)]\), whenever it is nonempty, forms a sequence of consecutive natural numbers. But this condition is equivalent to saying that \(f \in \text{Im}(t_{m,n})\) (see Remark 7.1.13), so we can rewrite

\[
\text{Hom}_{\text{Tw}(\text{Ass})}(µ_m, µ_n) = \bigsqcup_{f \in \text{Im}(t_{m,n})} A_f.
\]

As in Remark 7.1.13, we write \(\text{Im}(t_{m,n}) = \{\text{const}_0\} \sqcup \text{Im}_0(t_{m,n})\). Unwinding computation, we have that

- when \(f \in \text{Im}_0^*(t_{m,n})\), there is also a unique choice of \(α_0\) solving the equation (7.1.4), and
- when \(f = \text{const}_0\), there are \((n + 1)\) choices of \(α_0 \in Σ_{n+1}\) solving (7.1.4) precisely given as

\[
α_0^i := [i, 1, ⋯, i - 1, i + 1, ⋯, n - 1], \quad i = 1, ⋯, n + 1.
\]

So we can rewrite once more

\[
\text{Hom}_{\text{Tw}(\text{Ass})}(µ_m, µ_n) = \{α_0^i\}_{i=1}^{n+1} \sqcup \text{Im}_0(t_{m,n}).
\]

Finally, we find the desired bijection \(φ_{m,n}\), naturally, separated into two components as following:

\[
φ_{m,n} = φ_{m,n}^\text{const} \sqcup t_{m,n}^\circ : \text{Hom}_Δ^\text{const}([m], [n]) \sqcup \text{Hom}_Δ^\circ([m], [n]) \xrightarrow{φ} \{α_0^i\}_{i=1}^{n+1} \sqcup \text{Im}_0(t_{m,n})
\]

in which the first component \(φ_{m,n}^\text{const}\) sends each constant map \([m] \to [n]\) with value \(i\) to \(α_0^i\), while the second component \(t_{m,n}^\circ\) is the natural bijection mentioned in Remark 7.1.13.

### 7.2 Main statements

Suppose that \(P\) is a fibrant and \(Σ\)-cofibrant simplicial operad. We shall now prove that the cotangent complex \(L_P ∈ \mathcal{T}_P \text{Op}(\text{Set}_Δ)\) can be represented as a spectrum valued functor on \(\text{Tw}(P)\). Our treatment is inspired by the work of [5], §3.3.

As discussed in the previous subsection, \(\text{Tw}(P)\) is defined to be the image of \(P\) through the unstraightening functor

\[
\text{un} : \text{IbMod}(\mathcal{P}) = \text{Fun}(\text{Ib}^\mathcal{P}, \text{Set}_Δ) \xrightarrow{φ} (\text{Set}_Δ^\text{cov})/N(\text{Ib}^\mathcal{P}).
\]

In particular, there is a canonical left fibration \(\text{Tw}(P) \to N(\text{Ib}^\mathcal{P})\).

As the starting point, we observe that the functor \(\text{un}\) induces a right Quillen equivalence (denoted by)

\[
\text{un}_P : \text{IbMod}(\mathcal{P})_{//P} \xrightarrow{φ} (\text{Set}_Δ^\text{cov})/\text{Tw}(\mathcal{P})
\]

where the right hand side refers to the **pointed model category** associated to the covariant model category \((\text{Set}_Δ^\text{cov})/\text{Tw}(\mathcal{P})\).

Now, observe that the straightening functor \((\text{Set}_Δ^\text{cov})/\text{Tw}(\mathcal{P}) \xrightarrow{φ} \text{Fun}(\mathcal{C}[\text{Tw}(\mathcal{P})], \text{Set}_Δ)\) lifts to a left Quillen equivalence (denoted by)

\[
\text{St}_P : (\text{Set}_Δ^\text{cov})/\text{Tw}(\mathcal{P}) \xrightarrow{φ} \text{Fun}(\mathcal{C}[\text{Tw}(\mathcal{P})], (\text{Set}_Δ)_*)
\]

(7.2.1)

where \((\text{Set}_Δ)_*\) denotes the pointed model category associated to \(\text{Set}_Δ\). The latter induces a left Quillen equivalence of stabilizations

\[
(\text{St}_P)^{\text{sp}} : \text{Sp}((\text{Set}_Δ^\text{cov})/\text{Tw}(\mathcal{P})) \xrightarrow{φ} \text{Fun}(\mathcal{C}[\text{Tw}(\mathcal{P})], \text{Spectra})
\]
where Spectra refers to the stable model category of spectra.

We now obtain a sequence of right or left Quillen equivalences

\[
\mathcal{F}_P \operatorname{Op}(\mathcal{D}) \xrightarrow{\mathcal{F}_P \operatorname{IbMod}(\mathcal{P})} \operatorname{Sp}((\mathcal{Set}^\text{cov})_{\mathcal{T}w}(\mathcal{P})/\mathcal{T}w(\mathcal{P})) \xrightarrow{\mathcal{F}_P \operatorname{Fun}(\mathcal{C}[\mathcal{T}w(\mathcal{P})], \text{Spectra})} \mathcal{F}_P(\mathcal{P} \circ \mathcal{S}^n_C) = \mathcal{F}_P(\mathcal{P} \circ \mathcal{S}^n_C)
\]

(7.2.2)

Let us compute the derived image of \(L \mathcal{P} \in \mathcal{F}_P \operatorname{Op}(\mathcal{D})\) through this composition. We computed that \(\mathcal{R}(\mathcal{F}_P(\mathcal{P})) \mathcal{T}w(\mathcal{P}) \simeq \mathcal{F}_P \mathcal{L} \mathcal{P}\) (cf. Proposition 6.2.8). Namely, \(\mathcal{L} \mathcal{P} \in \mathcal{F}_P \operatorname{IbMod}(\mathcal{P})\) is the prespectrum with \((\mathcal{L} \mathcal{P})_{n,n} = \mathcal{P} \circ \mathcal{S}^n_C\) where \(\mathcal{S}^n_C\) is the \(C\)-collection with \(\mathcal{S}^n_C(c; c) = S^n\) for every color \(c\) and agreeing with \(\emptyset\) on the other levels. In our settings, we will refer to \(S^n\) as a Kan replacement of the \(n\)-sphere so that \(\mathcal{P} \circ \mathcal{S}^n_C\) is fibrant. Furthermore, the right derived functor \(\mathcal{R}(\mathcal{F}_P(\mathcal{P}))\) \(\operatorname{Sp}\) simply sends \(\mathcal{L} \mathcal{P}\) to the prespectrum

\[
\mathcal{L}(\mathcal{P} \circ \mathcal{S}^n_C) \in \operatorname{Sp}((\mathcal{Set}^\text{cov})_{\mathcal{T}w}(\mathcal{P})/\mathcal{T}w(\mathcal{P}))
\]

with \(\mathcal{L}(\mathcal{P} \circ \mathcal{S}^n_C)_{n,n} := \mathcal{L}(\mathcal{P} \circ \mathcal{S}^n_C)\). Finally, let us denote by

\[
\mathcal{F}_P := \mathcal{L}(\mathcal{F}_P(\mathcal{P})) \in \operatorname{Fun}(\mathcal{C}[\mathcal{T}w(\mathcal{P})], \text{Spectra})
\]

(7.2.3)

the derived image of \(\mathcal{L}(\mathcal{P} \circ \mathcal{S}^n_C)\) through \(\mathcal{F}_P(\mathcal{P})\), which is exactly the derived image of \(L \mathcal{P}[1] \in \mathcal{F}_P \operatorname{Op}(\mathcal{D})\) under the composed functor \((7.2.2)\).

The explicit description of \(\mathcal{F}_P\) is somewhat complicated, yet we can get it on objects as follows. Observe that there is a canonical equivalence of \(\infty\)-categories

\[
\operatorname{Fun}(\mathcal{C}[\mathcal{T}w(\mathcal{P})], \text{Spectra}) \simeq \operatorname{Fun}(\mathcal{T}w(\mathcal{P}), \text{Spectra})
\]

where \text{Spectra} is the \(\infty\)-category of spectra. So we will regard \(\mathcal{F}_P\) as an \(\infty\)-functor \(\mathcal{T}w(\mathcal{P}) \to \text{Spectra}\). Since the map \(\mathcal{L}(\mathcal{P} \circ \mathcal{S}^n_C) \to \mathcal{L}(\mathcal{P})\) is a left fibration, for each operation \(\mu \in \mathcal{P}(c_1, \ldots, c_m; c)\), \(\mathcal{F}_P(\mu)\) is the prespectrum with:

\[
\mathcal{F}_P(\mu)_{n,n} = \mathcal{L}(\mathcal{P} \circ \mathcal{S}^n_C) \times_{\mathcal{L}(\mathcal{P})} \{\mu\}.
\]

Recall that, on each level, \((\mathcal{P} \circ \mathcal{S}^n_C)(c_1, \ldots, c_m; c) = \mathcal{P}(c_1, \ldots, c_m; c) \times (S^n)^{\times m}\). It can then be computed that \(\mathcal{F}_P(\mu)_{n,n} = (S^n)^{\times m}\) (see also [3], Remark 2.2.2.11).

We summarize the above steps in the following:

**Theorem 7.2.1.** Let \(\mathcal{P}\) be a fibrant and \(\Sigma\)-cofibrant simplicial operad. There is an equivalence of \(\infty\)-categories

\[
\mathcal{F}_P \operatorname{Op}(\mathcal{D}) \simeq \operatorname{Fun}(\mathcal{T}w(\mathcal{P}), \text{Spectra}).
\]

Moreover, under this equivalence, the cotangent complex \(L \mathcal{P}\) is identified to \(\mathcal{F}_P[-1]\) the desuspension of the functors \(\mathcal{F}_P : \mathcal{T}w(\mathcal{P}) \to \text{Spectra} (7.2.3)\), which is given on objects by sending each operation \(\mu \in \mathcal{P}\) of arity \(m\) to \(\mathcal{F}_P(\mu) = S^{\times m}\), i.e., the \(m\)-fold product of the sphere spectrum. Consequently, for a given functor \(\mathcal{F} : \mathcal{T}w(\mathcal{P}) \to \text{Spectra}\), the \(n\)th Quillen cohomology group of \(\mathcal{P}\) with coefficients in \(\mathcal{F}\) is computed by

\[
H_{\mathcal{Q}}(\mathcal{P}; \mathcal{F}) = \pi_0 \operatorname{Map}_{\operatorname{Fun}(\mathcal{T}w(\mathcal{P}), \text{Spectra})}(\mathcal{F}_P, \mathcal{F}[n+1]).
\]

**Example 7.2.2.** By Proposition 7.1.3, the twisted arrow \(\infty\)-category of the little cubes operad \(\mathcal{C}_\infty\) is equivalent to that of \(\mathcal{C}_\infty\), and hence is equivalent to \(\operatorname{Fin}_\infty^{op}\). So the tangent category \(\mathcal{F}_\infty \operatorname{Op}(\mathcal{D})\) is (up to a zig-zag of Quillen equivalences) equivalent to \(\operatorname{Fun}(\operatorname{Fin}_\infty^{op}, \text{Spectra})\) endowed with the projective model structure. The functor \(\mathcal{F}_\infty : \operatorname{Fin}_\infty'^{op} \to \text{Spectra}\) takes each object


\[
\begin{align*}
\end{align*}
\]
(m) to $\mathbb{S}^m$. Moreover, for each map $f : (m) \to (n)$ in Fin$_*$, the map $\mathcal{T}_{\mathbb{S}}(f) : \mathbb{S}^n \to \mathbb{S}^m$ is given by, for each $i \in \{1, \ldots, n\}$, copying the $i$'th factor to the factors of position $j \in f^{-1}(i)$ when this fiber is nonempty or collapsing that factor to the zero spectrum otherwise.

**Example 7.2.3.** Following Proposition 7.1.11, the twisted arrow category of the associative operad Ass is equivalent to $\Delta$. So the tangent category $\mathcal{T}_{\text{Ass}} \text{Op}(\text{Set}_\Delta)$ is (up to a zig-zag of Quillen equivalences) equivalent to Fun($\Delta$, Spectra) endowed with the projective model structure. The functor $\mathcal{F}_{\text{Ass}} : \Delta \to \text{Spectra}$ takes each object $[m]$ to $\mathbb{S}^m$. Moreover, each map $g : [n] \to [m]$ in $\Delta$ is “indexed” by a unique map in Fin$_*$ given as $\iota(g) : (m) \to (n)$ (see Construction 7.1.12), and hence it induces a canonical map $\mathbb{S}^n \to \mathbb{S}^m$, just as in the above example.

We end this section by the following result, which in particular shows that Quillen cohomology of any little cubes operad with **constant coefficients** vanishes.

**Corollary 7.2.4.** Suppose that $\mathcal{P}$ is fibrant, $\Sigma$-cofibrant and unitally homotopy connected (cf. Definition 7.1.9). Let $\mathcal{T}_0 : \text{Tw}(\mathcal{P}) \to \text{Spectra}$ be a constant functor. Then Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{T}_0$ vanishes.

**Proof.** By Theorem 7.2.1 and by the assumption that $\mathcal{T}_0$ is a constant functor, Quillen cohomo-

logy of $\mathcal{P}$ with coefficients in $\mathcal{T}_0$ is computed as

$$H_\mathcal{Q}^*(\mathcal{P}; \mathcal{T}_0) = \text{Map}_{\text{Spectra}}(\text{colim}\mathcal{T}_\mathcal{P}, \mathcal{T}_0[\bullet + 1]).$$

By Lemma 7.1.10, $\text{Tw}(\mathcal{P})$ admits the terminal objects being precisely the unary operations of $\mathcal{P}$. It implies that $\text{colim}\mathcal{T}_\mathcal{P}$ is weakly equivalent to $\mathcal{T}_\mathcal{P}(\mu_0)$ with $\mu_0 \in \mathcal{P}$ being an arbitrary unary operation. But $\mathcal{T}_\mathcal{P}(\mu_0)$ is just the zero spectrum, and hence $H_\mathcal{Q}^*(\mathcal{P}; \mathcal{T}_0)$ vanishes as desired. $\square$

### 8. Quillen Cohomology of Operads in Simplicial Modules

Let $R$ be a commutative ring. The monoidal category of simplicial $R$-modules, $\text{sMod}(R)$, equipped with the standard model structure transferred from that of simplicial sets, satisfies the conditions of Conventions 3.0.2 and also the extra condition (S8) 6.1.2 (cf. Proposition 6.1.9). Moreover, this category is stable containing a strict zero object. So we inherit the results of Section 6. Note that every operad in $\text{sMod}(R)$ is fibrant.

Let $\mathcal{P}$ be a $\Sigma$-cofibrant $C$-colored operad in $\text{sMod}(R)$. The second part of Proposition 6.2.8 tells us that the tangent category $\mathcal{T}_\mathcal{P} \text{Op}(\text{sMod}(R))$ is homotopically identified to $\text{IbMod}(\mathcal{P})$ and moreover, under this identification, the cotangent complex $L_\mathcal{P}$ is weakly equivalent to $L_\mathcal{P}[-1] \in \text{IbMod}(\mathcal{P})$.

**Computations 8.0.1.** Let us give an explicit description of $L_\mathcal{P}$. As indicated there, $L_\mathcal{P}$ is given at each $(c_1, \ldots, c_m; c) \in \text{Seq}(C)$ as

$$L_\mathcal{P}(c_1, \ldots, c_m; c) = \mathcal{P}(c_1, \ldots, c_m; c) \otimes \text{hocolim}_n \Omega^n[(S^n)^{\otimes m} \times_R 0]$$

where the desuspension $\Omega(-)$ is taken in $\text{sMod}(R)$ and $S^n = \Sigma^n(R \sqcup R)$ with the suspension $\Sigma(-)$ being taken in $\text{sMod}(R)/R/R$. To compute that homotopy colimit, we will push it into the category of **connective dg $R$-modules** $\mathcal{C}_{\mathbb{S}0}(R)$, via the **normalized complex functor** $N : \text{sMod}(R) \xrightarrow{n} \mathcal{C}_{\mathbb{S}0}(R)$. Notice the facts that $N$ is not monoidal, but instead lax monoidal and that its lax monoidal structure maps are weak equivalences. For each $M \in \mathcal{C}_{\mathbb{S}0}(R)$, we write $M[k]$
standing for the complex which equals to \( M \) concentrated in degree \( k \geq 0 \). Now, observe that \( N(S^n) \simeq R[0] \oplus R[n] \). Then, we compute that

\[
\text{hocolim}_n \Omega^n [(R[0] \oplus R[n])^m \times R[0] 0] \simeq \text{hocolim}_n \Omega^n [(R[0] \oplus \bigoplus_{k=1}^m (C^k_m R)[kn]) \times R[0] 0]
\]

\[
\simeq \text{hocolim}_n \Omega^n \bigoplus_{k=1}^m (C^k_m R)[kn] \simeq \text{hocolim}_n \bigoplus_{k=1}^m (C^k_m R)[(k-1)n] \simeq \bigoplus_{k=0}^{m-1} \text{hocolim}_n (C^{k+1}_m R)[kn].
\]

It remains to compute \( \text{hocolim}_n (C^{k+1}_m R)[kn] \) for each \( k = 0, \ldots, m-1 \). When \( k = 0 \), the resultant is \( \text{hocolim}_n (mR)[0] \simeq (mR)[0] \). When \( k \geq 1 \), we have that \( \text{hocolim}_n (C^{k+1}_m R)[kn] \simeq 0 \), (this follows from the fact that the homology functor commutes with filtered colimits). So we find that

\[
\text{hocolim}_n \Omega^n [(R[0] \oplus R[n])^m \times R[0] 0] \simeq (mR)[0],
\]

and hence we deduce that \( L_{\mathcal{P}}(c_1, \ldots, c_m; c) \simeq \mathcal{P}(c_1, \ldots, c_m; c)^m \), i.e., the \( m \)-fold coproduct of \( \mathcal{P}(c_1, \ldots, c_m; c) \).

Unwinding verification, we find that, as a \( C \)-collection, \( L_{\mathcal{P}} \) is isomorphic to \( \mathcal{P} \circ (\_)^1 \mathcal{J}_C \) (cf. Section 5.2). A typical element of \( \mathcal{P} \circ (\_)^1 \mathcal{J}_C(c_1, \ldots, c_m; c) \) is of the form \((\mu, \text{id}_{c_i})\) with \( \mu \in \mathcal{P}(c_1, \ldots, c_m; c) \) and \( \text{id}_{c_i} \) referring to the element \( 1_R \in \mathcal{J}_C(c_i; c_i) = R \). The infinitesimal \( \mathcal{P} \)-bimodule structure of \( L_{\mathcal{P}} = \mathcal{P} \circ (\_)^1 \mathcal{J}_C \) is given as follows. As an infinitesimal left \( \mathcal{P} \)-module, \( L_{\mathcal{P}} \) is free generated by \( \mathcal{J}_C \) (cf. Remark 5.2.3); while for each \( \lambda \in \mathcal{P}(d_1, \ldots, d_n; c_j) \), the infinitesimal right action of \( \lambda \) on \((\mu, \text{id}_{c_i})\) is defined as

\[
(\mu, \text{id}_{c_i}) \circ^r \lambda := \begin{cases} 
(\mu \circ \lambda, \text{id}_{c_i}) & \text{if } j \neq i \\
(\mu \circ \lambda, \text{id}_{d_i}) + \cdots + (\mu \circ \lambda, \text{id}_{d_n}) & \text{if } j = i.
\end{cases}
\]

By having that description of \( L_{\mathcal{P}} \), we get the following conclusion.

**Theorem 8.0.2.** Suppose that \( \mathcal{P} \in \text{Op}(s\text{Mod}(R)) \) is a \( \Sigma \)-cofibrant operad. For each \( M \in \text{IbMod}(\mathcal{P}) \), the \( n \)’th Quillen cohomology group of \( \mathcal{P} \) with coefficients in \( M \) is computed as

\[
H^n_{\mathcal{P}}(\mathcal{P}; M) = \pi_0 \text{Map}^b_{\text{IbMod}(\mathcal{P})}(L_{\mathcal{P}}, M[n+1]).
\]

In the remainder, we are interested in the class of operads in \( s\text{Mod}(R) \) which are linearizations of unitally homotopy connected simplicial operads (cf. Definition 7.1.9).

The free-forgetful adjunction \( R\{ - \} : \text{Set}_{\Delta} \xrightarrow{\cong} \text{sMod}_R : \mathcal{U} \) induces the Quillen adjunctions \( R\{ - \} : \text{Op}(\text{Set}_{\Delta}) \xrightarrow{\cong} \text{Op}(s\text{Mod}_R) : \mathcal{U} \) and \( R\{ - \} : \text{IbMod}(\mathcal{P}) \xrightarrow{\cong} \text{IbMod}(R\{\mathcal{P}\}) : \mathcal{U} \). For each simplicial operad \( \mathcal{P} \), the operad \( R\{\mathcal{P}\} \in \text{Op}(s\text{Mod}_R) \) will be called the \( R \)-linearization of \( \mathcal{P} \). Consider the induced Quillen adjunction between tangent categories:

\[
R\{ - \}^{\mathcal{P}} : \mathcal{T}_R \text{IbMod}(\mathcal{P}) \xrightarrow{\cong} \mathcal{T}_{R\{\mathcal{P}\}} \text{IbMod}(R\{\mathcal{P}\}) : \mathcal{U}^{\mathcal{P}}.
\]

Let us denote by \( L^b_{R\{\mathcal{P}\}} \in \mathcal{T}_{R\{\mathcal{P}\}} \text{IbMod}(R\{\mathcal{P}\}) \) the cotangent complex of \( R\{\mathcal{P}\} \) when regarded as an object in \( \text{IbMod}(R\{\mathcal{P}\}) \). Furthermore, we let \( \mathcal{T}_{R\{\mathcal{P}\}} : \text{Tw}(\mathcal{P}) \rightarrow \text{Spectra} \) denote the derived image of \( L^b_{R\{\mathcal{P}\}} \) under the composition

\[
\mathcal{T}_{R\{\mathcal{P}\}} \text{IbMod}(R\{\mathcal{P}\}) \xrightarrow{L^b_{\mathcal{P}}} \mathcal{T}_R \text{IbMod}(\mathcal{P}) \xrightarrow{\mathcal{U}^{\mathcal{P}}} \text{Fun}(\text{Tw}(\mathcal{P}), \text{Spectra})
\]

in which the second functor is the equivalence indicated in Section 7.2.
Theorem 8.0.3. Quillen cohomology of $R\{\mathcal{P}\}$ with coefficients in itself $R\{\mathcal{P}\} \in \text{IbMod}(R\{\mathcal{P}\})$ vanishes whenever $\mathcal{P}$ is fibrant, $\Sigma$-cofibrant and unitally homotopy connected. In particular, the statement holds for the $R$-linearizations of the little cubes operads.

Proof. By adjunction, there is a canonical isomorphism

$$H^*_Q(\mathcal{P}; \mathcal{F}_{R(\mathcal{P})}) \cong H^*_Q(R\{\mathcal{P}\}; R\{\mathcal{P}\})$$

between Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{F}_{R(\mathcal{P})}$ and Quillen cohomology of $R\{\mathcal{P}\}$ with coefficients in itself. Hence, the statement will be verified by Corollary 7.2.4 after showing that $\mathcal{F}_{R(\mathcal{P})}$ is a constant functor. Let us see how it goes.

The cotangent complex $L_{ib}^b(\mathcal{P})$, regarded as a suspension spectrum (see around Computations 6.2.5), is given at each bidegree $(n, n)$ as $(L_{ib}^b(\mathcal{P}))[n, n] = R(\mathcal{P} \times S^n)$, i.e., the infinitesimal $R\{\mathcal{P}\}$-bimodule with $R(\mathcal{P} \times S^n)(\tau) := R(\mathcal{P}(\tau) \times S^n)$ for every $\tau \in \text{Seq}(C)$. First, argue that since $L_{ib}^b(\mathcal{P})$ is a fibrant $\Omega$-spectrum, its image $\mathcal{U}^{\mathcal{P}}(L_{ib}^b(\mathcal{P})) \in \mathcal{F}_P \text{IbMod}(\mathcal{P})$ has already the right type. By construction, $\mathcal{U}^{\mathcal{P}}(L_{ib}^b(\mathcal{P}))$ is the $\Omega$-spectrum whose value at each bidegree $(n, n)$ is taken as the pullback in $\text{IbMod}(\mathcal{P})$:

$$\mathcal{U}^{\mathcal{P}}(L_{ib}^b(\mathcal{P}))[n, n] \longleftarrow R(\mathcal{P} \times S^n) \quad (8.0.1)$$

It can be shown that, for each operation $\mu \in \mathcal{P}(\tau)$, the $\Omega$-spectrum $\mathcal{F}_{R(\mathcal{P})}(\mu)$ is given at each bidegree $(n, n)$ as the fiber in $\text{Set}_\Delta$:

$$\mathcal{F}_{R(\mathcal{P})}(\mu)[n, n] = \mathcal{U}^{\mathcal{P}}(L_{ib}^b(\mathcal{P}))[n, n, \tau] \times_{\mathcal{F}_P(\mathcal{P})} \{\mu\},$$

(this is very similar to the computations given in Section 7.2). By the Cartesian square (8.0.1), $\mathcal{F}_{R(\mathcal{P})}(\mu)[n, n]$ is the same as the fiber $R(\mathcal{P}(\tau) \times S^n) \times_{R(\mathcal{P}(\tau))} \{\mu\}$. Certainly, the latter does not depend on the choice of $\tau$ or $\mu$ whenever $\mathcal{P}(\tau)$ is nonempty. The resultant is simply given as the fiber

$$R(S^n) \times_{R(\Delta^n)} \Delta^n = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in R(S^n) \mid \lambda_1 + \cdots + \lambda_r = 1\}.$$

Let us denote by $R(S)_1 := (R(S^n) \times_{R(\Delta^n)} \Delta^n)_{n\geq 0}$ the obtained $\Omega$-spectrum. In conclusion, we just computed that $\mathcal{F}_{R(\mathcal{P})} : \text{Tw}(\mathcal{P}) \rightarrow \text{Spectra}$ is the constant functor (with value $R(S)_1$), as expected.

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