Renormalization group for multiparticle production in (2+1) dimensions around the threshold

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Abstract

Multiparticle production in (2+1) dimensions is investigated. We show that in a small region around the threshold the perturbation theory becomes unapplicable due to infrared divergencies in a class of Feynman graphs with rescattering in final state. We develop a technique based on renormalization group for summing up leading logarithms and apply it to the $\phi^4$ models with and without symmetry breaking and to the $O(N) \phi^4$ theory.
1. Multiparticle production in the conventional $\phi^4$ scalar theory is attracting considerable interest. Though the initial observation \cite{1,2} was rather simple, only a few quantitative results has been obtained so far. Most of them concern the multiparticle threshold at which, for instance, formulas for the amplitude of production of $n$ particle by an initial virtual one has been derived at the tree \cite{3-5} and one-loop levels \cite{6-8}. The tree amplitude exhibits a factorial dependence on the number of created bosons $n$. The correction from the first loop is of order $\lambda n^2$ in comparison with the tree result (where $\lambda$ is the coupling constant) and one can expect the breakdown of the perturbation theory at large $n$. There have been several attempts to take into account contributions from all orders of the perturbation theory \cite{9,10}. In particular, the amplitude has been calculated at leading order in $1/N$ in the $O(N)$ symmetric scalar theory at large $N$ \cite{10}. However, it remains unclear whether the $1/N$ expansion is applicable at large number of final particles $n$.

In this paper we consider the multiparticle production in $(2+1)$ dimensions, where beside the above–mentioned inapplicability of the perturbation theory at large $n$ there is another problem related with the infrared divergence of a certain kind of loop graphs which breaks the conventional pertubative expansion in a region close enough to the threshold. The graphs of this kind contains loops related to the rescattering among final soft particles. It is a peculiar feature of $(2+1)$ dimensions that right at the threshold, rescattering graphs diverge logarithmically, so even at small number of final particles $n$ calculation of the amplitude around the threshold requires a nontrivial summing an infinite set of graphs.

We find a suitable technique to do such kind of summing which is a modification of the conventional renormalization group (RG). This technique allow us to sum up leading logarithms from all orders of the perturbation theory. By this technique we obtain the amplitude $1 \to n$ in the $\phi^4$ models with both broken and unbroken reflection symmetry, and in the $O(N)$ $\phi^4$ theory. We check the leading in $1/N$ amplitude in the latter model and show explicitly that the $1/N$ expansion breaks down at large number of final particles.

2. Let us consider the $\phi^4$ theory in $(2+1)$ dimensions,
\[ L = \frac{1}{2}(\partial_{\mu}\phi)^2 - \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4, \]

where, for convenience, we suppose that the mass of the particle is equal to unity. This is the theory of interacting relativistic bosons.

To describe bosons in the low–energy limit, one writes the following effective lagrangian in term of a non–relativistic bosonic field \( \Psi \),

\[ L_{\text{eff}} = \Psi^\dagger i\partial_0 \Psi - \frac{1}{2}(\partial_i \Psi^\dagger)(\partial_i \Psi) - g\Psi^\dagger \Psi^\dagger \Psi \Psi, \]

(1)

where \( g \) is some, yet to be determined, effective coupling.

Non–relativistic bosons, interacting via a delta–like potential (as in eq. (1)), have been known for a long time as an example of a non–relativistic system with dimensional transmutation \([11]\). In fact, from dimension counting in (2+1)d (in non–relativistic kinematics the relation between the dimensions of energy and momentum is \([E] = [p]^2\)) it is implied that \( \Psi^\dagger \Psi^\dagger \Psi \Psi \) is an marginal operator and the coupling constant \( g \) is dimensionless. Apparently, there is no scale parameter in the theory described by the lagrangian (1). However, this scale exists and is simply the boson mass (which is unity in our notation), which plays the role of the “ultraviolet” cutoff in the effective theory.

To make contact between the effective lagrangian and the initial Lorentz–invariant one, one compares formulas for the amplitude of elastic scattering of two bosons computed in both theories. This results the following relation between \( g \) and \( \lambda \),

\[ g = 3\lambda/8. \]

(2)

Considering \( g \) as the bare coupling, or the coupling at the scale of ultraviolet cutoff, one can question about the evolution of \( g \) as a function of the momentum scale. For this purpose we introduce the running coupling constant \( g(t) \), which has the physical meaning of the strength of the interaction between bosons at the momentum scale \( p = e^{-t}, (t = \ln(1/p)) \). There is only one Feynman diagram (shown in fig. [1]) that makes contribution to the corresponding beta function. Simple calculations yield the following RG equation,
\[ \frac{dg(t)}{dt} = -\frac{g^2(t)}{\pi}, \]

which has the solution

\[ g(t) = \frac{g}{1 + \frac{2}{\pi} t} = \frac{3\lambda}{8} \left( 1 + \frac{3\lambda}{8\pi} \right)^{-1}, \]

where we have made use of the initial condition on \( g(t) \), \( g(0) = g \), and \( g \) is defined by eq. (4). From eq. (3) one sees that the intensity of the interaction between bosons decreases as the momenta of particles tend to zero. Later, we will demonstrate that this property holds for a more general case of \( O(N) \phi^4 \) model, but in the theory with symmetry breaking the behavior of \( g(t) \) is diametrically opposite.

The coupling constant changes considerably from its initial value only when the momentum scale is exponentially small on \( \lambda, t = \ln(1/p) \sim 1/\lambda \). So, the renormalization group is suitable for considering the regime \( \lambda \to 0, \lambda t \sim 1 \). All further considerations will be done for this particular regime.

In fact the flow of the effective coupling, eq. (3), can be obtained by a more trivial method by direct summing bubble graphs: in non–relativistic theories only these graphs contribute to the elastic scattering \( 2 \to 2 \). However, in calculation of the amplitude of multiparticle production around the threshold more complicated diagrams are involved and the problem cannot be reduced to the summing of bubble graphs. For example, for the \( 1 \to 3 \) process the one– and two–loop diagrams that make contribution in our interesting regime is presented in fig. 2. To deal with these processes the RG technique is required. On the diagramatic language, the renormalization group corresponds to summing leading logarithms, i.e. terms proportional to \( (\lambda t)^n \) from the whole perturbative series.

Let us try to describe the production of \( n \) final particles in term of the non–relativistic bosonic creation operator \( \Psi \). The only relevant candidate is \( A_n \Psi^{\dagger n} \). The following relation should take place,

\[ \langle n|\phi|0\rangle = \langle n|A_n \Psi^{\dagger n}|0\rangle_{\text{eff}}, \]
where the matrix element in the l.h.s. is written in the initial $\phi^4$ model and the r.h.s. is understood as a matrix element in the effective non-relativistic theory. Form this equation one finds that $A_n$ is equal to the $1 \rightarrow n$ amplitude when the spatial momenta of final particles is small, but not exponentially small (so the logarithms do not appear in loops). Let us for simplicity consider small enough $n$ where $A_n$ coincides with the tree $1 \rightarrow n$ amplitude [3–5],

$$A_n = A_n^{\text{tree}} = n! \left( \frac{\lambda}{8} \right)^{(n-1)/2}.$$  \hspace{1cm} (4)

When characteristic scale of momenta of final particles is exponential, the r.h.s. of eq. (4) is substantially renormalized by loops. One can treat this renormalization by introducing the running vertex $A_n(t)$ and solving the corresponding RG equation,

$$\frac{dA_n(t)}{dt} = -\frac{n(n-1)}{2} \frac{1}{\pi} g(t) A_n(t).$$  \hspace{1cm} (5)

Graphically, this equation is represented in fig. 3. Substituting the function $g(t)$ found in (3) to eq. (5), the function $A(t)$ can be found easily,

$$A_n(t) = A_n \left( 1 + \frac{g(0)}{\pi} t \right) \frac{n(n-1)}{\pi} = A_n^{\text{tree}} \left( 1 + \frac{3\lambda}{8\pi} t \right) \frac{n(n-1)}{\pi}.$$  \hspace{1cm} (6)

So, we have found that the RG technique allows us to calculate the $1 \rightarrow n$ amplitude $A_n(t)$ in a region close to the threshold where $\lambda t \sim 1$. Note that exactly at the threshold, i.e. at $t = +\infty$, the amplitude vanishes.

3. All the said above is equally applicable for the case of broken symmetry,

$$L = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{\lambda}{4} (\phi^2 - v^2)^2,$$

with the only exception that for calculating the bare coupling $g$ entering into the lagrangian of the effective theory from the initial model one must take into account not only the diagram with a four–boson vertex, but also diagrams with two three–boson vertices, i.e. those with exchange of a virtual particle in $s$-, $t$- or $u$-channels. In contrast with the case without symmetry breaking, the resulting amplitude is negative, which means the attractive character of the force between boson at low energies. One has
\[ g = -\frac{3\lambda}{2} \]

(we assume \( v^2 = (2\lambda)^{-1} \), so the mass of the boson is unity). Formula (3) implies that the strength of the interaction increases with \( t \),

\[ g(t) = \frac{g}{1 + \frac{3}{2}t} = -\frac{3\lambda}{2} \left( 1 - \frac{3\lambda}{2\pi}t \right)^{-1}. \tag{7} \]

Litterally, eq.(7) predict an infinite coupling constant at an exponentially small momentum scale of

\[ p_0 = \exp(-2\pi/3\lambda). \]

This fact is a direct analog of the Landau pole in field theories without asymptotic freedom. In reality, it is a manifestation of the existence of a two–particle bound state in our model (recall that at least one bound state exists in every, arbitrarily weak, two–dimensional attractive potential). One can show that in our case, the energy of the bound state is of order \( p_0^2 \).

The \( 1 \rightarrow n \) amplitude is now

\[ A_n(t) = A_n^{\text{tree}} \left( 1 - \frac{3\lambda}{2\pi}t \right)^{-\frac{n(n-1)}{2}}. \tag{8} \]

It is interesting to compare our result for the case of large \( n \), \( n \gg 1 \) to that obtained in ref. [9] for (2+1) dimension by a different method. In the case when \( \lambda t \) is small, \( \lambda t \ll 1 \), our formula in fact reproduces the result of [9]

\[ A_n = A_n^{\text{tree}} \exp \left( \frac{3\lambda}{4\pi n^2 t} \right). \]

However, if \( \lambda t \) is comparable with 1, the formula (8) does not coincide with that obtained in ref. [9]. We consider it as a counter–argument to the claim of ref. [9].

4. Application of the technique described above the multi–component \( \phi^4 \) model requires a slight modification. The lagrangian of the \( O(N) \) \( \phi^4 \) model,

\[ L = \frac{1}{2} (\partial_\mu \phi_a)(\partial_\mu \phi_a) - \frac{1}{2} \phi_a \phi_a - \frac{\lambda}{4} (\phi_a \phi_a)^2, \tag{9} \]
\(a = 1 \ldots N\) is the isospin index) contains one coupling constant, \(\lambda\). However, if one tries to write down the most general \(O(N)\) symmetric effective non-relativistic lagrangian, one sees that there may be two different effective couplings, \(g_1\) and \(g_2\), which correspond to the two possible structures of the potential term,

\[
L_{\text{eff}} = \Psi_a^\dagger i\partial_0 \Psi_a - \frac{1}{2} (\partial_0 \Psi_a^\dagger)(\partial_0 \Psi_a) - g_1 \Psi_a^\dagger \Psi_b \Psi_a \Psi_b - 2 g_2 \Psi_a^\dagger \Psi_b \Psi_a \Psi_b. \tag{10}
\]

\(g_1\) is the low-energy elastic scattering scattering amplitude, singlet in the \(s\)-channel, while \(g_2\) determines amplitudes, singlet in \(t\)- and \(u\)-channels. From the initial lagrangian (9) one obtains the bare value of the coupling constants,

\[
g_1 = g_2 = \frac{\lambda}{8}. \tag{11}
\]

The fact that the bare values of \(g_1\) and \(g_2\) are equal is the remnant of the crossing symmetry of our initial lagrangian (8). However, the evolution equations for the running coupling constants are rather different,

\[
\frac{dg_1(t)}{dt} = \frac{1}{\pi} [Ng_2^2(t) + 4g_1(t)g_2(t)], \tag{12}
\]

\[
\frac{dg_2(t)}{dt} = \frac{2}{\pi} g_2^2(t). \tag{13}
\]

In order to simplify the RG equations, let us introduce instead of \(g_1(t)\) a linear combination of the two couplings,

\[
g(t) = g_1(t) + \frac{2}{N} g_2(t).
\]

The RG equation for \(g(t)\) is simpler than that of \(g_1(t)\), eq. (12),

\[
\frac{dg(t)}{dt} = \frac{N}{\pi} g^2(t). \tag{14}
\]

The solution to the RG equations, eqs. (13) and (14), which satisfies the initial condition (11), can be found,

\[
g_2(t) = \frac{\lambda}{8} \left(1 + \frac{\lambda}{4\pi} t\right)^{-1}.
\]
\[ g(t) = \left(1 + \frac{2}{N}\right) \frac{\lambda}{8} \left(1 + (N + 2) \frac{\lambda}{8\pi} t\right). \]

In analogy with the simple $\phi^4$ case, the production of $n$ soft bosons from an initial particle with isospin $a$ can be described by an effective operator $A_n \Psi^\dagger_a (\Psi^\dagger_b \Psi^\dagger_b)^{(n-1)/2}$ ($n$ must be odd).

After some calculations we obtain the following RG equation for $A_n(t)$,

\[
\frac{dA_n(t)}{dt} = \frac{1}{\pi} \left[ \frac{n-1}{2} (N + n - 1) g(t) + (n - 1)^2 \left(1 - \frac{1}{N}\right) g_2(t) \right] A_n(t). \tag{15}
\]

Having substituted the formulas for $g_2(t)$ and $g(t)$ to eq. (15) and solved it, one obtains the dependence of the $1 \to n$ amplitude on the logarithm of the characteristic momentum of final particles, $\ln(1/p) = t$

\[
A_n(t) = A_n^{\text{tree}} \left(1 + (N + 2) \frac{\lambda}{8\pi} t\right)^{-\frac{n-1}{2N(N+n-1)}} \left(1 + \frac{\lambda}{4\pi} \right)^{-\frac{(n-1)^2}{2N(N-1)}}. \tag{16}
\]

Recall that the regime where we are working is $\lambda \to 0$, $\lambda t \sim 1$, provided other parameters as $n$ and $N$ are fixed.

Let us consider the large $N$ limit and show that the result of ref. [10] can be reobtained in (2+1) dimensions by our technique. Consider the case when $n$ much smaller than both the number of boson flavors $N$ and the inverse coupling constant, eq.(16) reduces to a much simpler formula,

\[
A_n(t) = A_n^{\text{tree}} \left[1 + \frac{N\lambda}{8\pi} t\right]^{-(n-1)/2}. \tag{17}
\]

Remember that the coupling constant enters into the tree amplitude by the factor of $\lambda^{(n-1)/2}$, one can rewrite eq.(17) into the following form,

\[
A_n(t) \sim n!\lambda_R^{(n-1)/2}, \tag{18}
\]

where

\[
\lambda_R = \frac{\lambda}{1 + \frac{N\lambda}{8\pi} t}
\]

is just the renormalized singlet scattering amplitude. One sees that for a small number of final particles, $n \ll N$, leading order in $1/N$ result is given just by the tree–level formula
where the coupling constant is replaced by the renormalized one. This is precisely the result
of [10] in the particular case of (2+1) dimensions.

However, formula (18) has been deduced provided that \( n \ll N \). If the number of final
particles is comparable with \( N \), the effect of loops is obviously not a simple renormalization
of the coupling constant. One can find from (10) that the correction to the leading order
(17) is proportional to \( n^2/N \). When the number of final particles becomes comparable to
the number of their sorts, the 1/N expansion becomes totally unreliable. One can expect
that the breakdown of the 1/N expansion is not a peculiar feature of (2+1) dimensions but
holds also in the realistic (3+1) dimensional theory.

6. So, we see that the renormalization group is a powerful mean for investigating the multi-
particles amplitudes in (2+1) dimensional scalar field theory at and around the threshold.
The exact formula for the amplitude, if ever be found, must incorporate the information
obtained here from the renormalization group equations.

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FIGURES

FIG. 1. The only graph contributing to the lowest–order beta function in the effective non–relativistic theory.

FIG. 2. One– and two–loops Feynman graphs that contribute to the $1 \to 3$ amplitude in the interesting regime.

FIG. 3. A graph that make contribution to RG equation for $1 \to n$ amplitude. There are $n(n-1)/2$ diagrams of this type, corresponding to different possibilities to choose of a pair of final particle to rescatter.
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Fig. 1

Fig. 2
Fig. 3