ON SCHREIER VARIETIES OF RACKS

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Abstract

We prove that a subrack of a free rack is free and suggest a method to prove a similar statement about involutory racks.

Key words: free rack, quandle, Schreier varieties, Nilsen-Schreier theorem.

Introduction

A rack is a set equipped with a two binary operations \((R, \triangleright, \triangleright^{-1})\) such that the following equalities hold for every \(x, y, z \in R\):

\[ R1 \quad (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z), \]
\[ \quad (x \triangleright^{-1} y) \triangleright^{-1} z = (x \triangleright^{-1} z) \triangleright^{-1} (y \triangleright z); \]
\[ R2 \quad (x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y. \]

R1 states simply that the map \(x \rightarrow x \triangleright y\) is an endomorphism of \(Q\) for every \(y \in Q\). R2 implies that every such map is an automorphism. A rack does not need to be associative or to have an identity.

A rack with \(x \triangleright x = x\) is called a quandle. A rack in which \(\triangleright = \triangleright^{-1}\) is called involutory. Involutory quandles have been studied extensively under different names (symmetric sets, symmetric groupoids, see [1]).

Any group \(G\) provides an example of a quandle \(\text{Conj} G\) with \(x \triangleright y = y^{-1}xy\). \(\text{Conj}\) can be considered as a functor from the category of quandles to the category of groups. There exists a left adjoined functor to \(\text{Conj}\) which we denote by \(\text{Adconj}\). \(\text{Adconj} Q\) is the universal group in which to represent the quandle \(Q\) as a set closed under conjugation. We call it the associated group of the quandle.

One strong motivation for studying quandles and racks is provided by knot theory. There is a natural construction of a quandle \(Q(K)\) for any knot \(K\) using its diagram. It is called the knot quandle or the fundamental quandle of the knot (see [2] for details). This construction gives a full invariant of knots and other invariants can be derived from it (see [3], [4]). For example, the fundamental group of a knot is obtained as the associated group of its fundamental quandle.

A variety is a class of algebraic structures of the same type satisfying a set of identities [5]. A variety of algebras in which subalgebras of free algebras are free is called a Schreier variety.
So, by the Nielsen-Schreier theorem the variety of groups is Schreier. Schreier varieties of linear algebras have been studied in [6, 7].

V. Bardakov, M. Singh and M. Singh in [8, Problem 6.12] raised the question about an analogue of Nielsen–Schreier theorem for quandles: is it true that any subquandle of a free quandle is free. It was answered affirmatively in [9]. In this work we generalise this result to racks and propose a way to extend it to involutory racks:

**Theorem.** Any subrack of a free rack is free.

### Algebraic representation of racks and quandles

Free rack on $X$ is a rack that satisfies the universal property: given any function $\rho: X \to R$, where $R$ is an arbitrary rack, there exists a unique homomorphism $\overline{\rho}: FR(X) \to R$, such that $\overline{\rho} \circ \varphi = \rho$, making the following diagram commute (here $\varphi: X \to FR(X)$ is an embedding of $X$ into $FR(X)$):

$$
\begin{array}{ccc}
X & \xrightarrow{\rho} & FR(X) \\
\downarrow{\phi} & & \downarrow{\rho} \\
R & \xleftarrow{\varphi} & FR(X)
\end{array}
$$

We will use the following construction of a free rack on $X$ [10]. On the set $X \times F(X)$, where $F(X)$ is a free group, generated by $X$, we will define $\triangleright$ as follows:

$$(a, u) \triangleright (b, v) = (a, uv^{-1}bv)$$

$$(a, u) \triangleright^{-1} (b, v) = (a, uv^{-1}b^{-1}v)$$

for all $a, b \in X$, $u, v \in F(X)$.

A free quandle $FQ(X)$ on $X$ is a union of conjugacy classes of elements of $X$ in $F(X)$ with the operation defined the following way: $x \triangleright y = x^y = y^{-1}xy \forall x, y \in FQ(X)$ [2].

A free involutory quandle is a union of conjugacy classes of elements of $X$ in $\langle X | x^2 = 1 \forall x \in X \rangle$.

### Proof

For convenience we will denote $((r_0 \triangleright^e r_1) \triangleright^e \cdots \triangleright^e r_n) = r_0 \triangleright^e r_1 \triangleright^e \cdots \triangleright^e r_n$. We will also write $r^n, n \in \mathbb{Z}$ instead of $r \triangleright^e \cdots \triangleright^e r$, where $e = \text{sign}(n)$.

Consider $f: FR(X) \to FQ(X)$, $f((x, w)) = w^{-1}xw = x^w$. It is clear that $f$ is a rack homomorphism. We will consider an arbitrary subrack $R \subset FR(X)$ and show that it is free. Since $f$ is a homomorphism, the image of $R$ is a subquandle $Q \subset FQ(X)$, and thus is free.

Then a basis $S_Q$ exists, such that $Q = \langle S_Q \rangle$. Any element of $Q$ can be represented as $q_0 \triangleright^e q_1 \triangleright^e q_2 \cdots \triangleright^e q_n$, where $q_i \in S_Q$.

The preimage of $x^w$ in $FQ(X)$ is the subrack $\{(x, x^nw) | n \in \mathbb{Z}\}$, generated by any one of its elements. To prove this, assume that $(x, w_1)$ and $(y, w_2)$ are such that $f((x, w_1)) = x^{w_1} = x^{w_2} = x^{w_3}$.
Since $x$ and $y$ belong to the same conjugacy class in $FR(X)$, $x = y$. Now $x^{w_1} = x^{w_2}$ implies $x = x^{w_1w_2^{-1}}$, which is possible only if $w_1w_2^{-1} = x^n$ for some $n \in \mathbb{Z}$.

Note that $f^{-1}(x^w) \subset R$.

From each preimage of $q_i = x^w \in S_Q$ choose $r_i = (x, w')$, where $w'$ does not start with a power of $x$. Note that every $r_i$ is unique, otherwise $q_i$ are not independent from each other and do not form a basis of $Q$. We will denote this set of $r_i$ by $S_R$ and show that it generates $R$ freely.

For $r = (x, w) \in R$ consider $f(r) = x^w \in Q$. Since $r$ is contained in the preimage of $x^w$, it can be represented as

$$(q_1 \triangleright^1 q_2 \triangleright^2 \ldots \triangleright^{n-1} q_m)^n, \quad q_i \in S_Q, \quad q_1 \neq q_2.$$  

Using the equality $(r \triangleright^t l)^k = r^k \triangleright^t l$, which holds in every rack, we obtain

$$r = q_1^n \triangleright^1 q_2 \triangleright^2 \ldots \triangleright^{n-1} q_m$$

Now let us show that this representation is unique. Assume that $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_m$ are such that

$$k, l \neq 0, x_0 \neq x_1, y_0 \neq y_1, \quad x_k \triangleright^{x_1} x_1 \triangleright^{x_2} x_2 \ldots \triangleright^{x_n} x_n = y_0 \triangleright^{y_1} y_1 \triangleright^{y_2} y_2 \ldots \triangleright^{y_m} y_m$$

Denote $f(x_i)$ and $f(y_j)$ by $\overline{x_i}$ and $\overline{y_j}$ respectively. Applying $f$ to both sides gives us

$$x_0 \triangleright^{x_1} x_1 \triangleright^{x_2} x_2 \ldots \triangleright^{x_n} x_n = y_0 \triangleright^{y_1} y_1 \triangleright^{y_2} y_2 \ldots \triangleright^{y_m} y_m$$

Since this is an equation on basis elements in $Q$, we have $n = m$, $\epsilon_i = \xi_i$ and $\overline{x_i} = \overline{y_i}$ for every $i$. The mapping $f$ is injective on elements of $S_R$, which means that $\overline{x_i} = \overline{y_i}$ implies $x_i = y_i$. Now all $x_i$, where $i \geq 1$, can be cancelled out. What is left is

$$x_0^k = x_0^l.$$ 

In $FR(X)$ this is possible only if $k = l$. This concludes the proof.

A similar proof can be carried out with a construction of free involutory racks as the proof above does not change with $\triangleright = \triangleright^{-1}$, given that the variety of involutory quandles is Schreier:

**Theorem.** Every subrack of a free involutory rack is a free involutory rack.

**References**

[1] D. Stanovský, *The origins of involutory quandles*, 2015.

[2] D. Joyce, “A classifying invariant of knots, the knot quandle,” *Journal of Pure and Applied Algebra*, vol. 23, no. 1, pp. 37–65, 1982, ISSN: 0022-4049.

[3] S. Kamada, “Knot invariants derived from quandles and racks,” 2002.

[4] S. Nelson, “A polynomial invariant of finite quandles,” 2007.

[5] G. Birkhoff, “On the structure of abstract algebras,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 31, no. 4, pp. 433–454, 1935.
[6] U. U. Umirbaev, “Schreier varieties of algebras,” Algebra and Logic, vol. 33, no. 3, pp. 180–193, May 1, 1994, ISSN: 1573-8302.

[7] V. Dotsenko and U. Umirbaev, An effective criterion for nielsen-schreier varieties, 2022.

[8] V. Bardakov, M. Singh, and M. Singh, “Free quandles and knot quandles are residually finite,” Proceedings of the American Mathematical Society, vol. 147, no. 8, pp. 3621–3633, Apr. 2019.

[9] S. O. Ivanov, G. Kadantsev, and K. Kuznetsov, Subquandles of free quandles, 2019.

[10] R. Fenn and C. Rourke, “Racks and links in codimension two,” Journal of Knot Theory and Its Ramifications, vol. 01, no. 4, pp. 343–406, Dec. 1, 1992, Publisher: World Scientific Publishing Co., ISSN: 0218-2165.

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