ON THE HYBRID CONTROL OF METRIC ENTROPY FOR DOMINATED SPLITTINGS

XUFENG GUO\(^1\,\,^3\), GANG LIAO\(^2\,\,*\), WENXIANG SUN\(^3\) and DAWEI YANG\(^4\)

\(^1\,\,^3\) School of Mathematical Sciences, Peking University
Beijing 100871, China
\(^2\,\,^4\) School of Mathematical Sciences
Center for Dynamical Systems and Differential Equations
Soochow University, Suzhou 215006, China

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Abstract. Let \(f\) be a \(C^1\) diffeomorphism on a compact Riemannian manifold without boundary and \(\mu\) an ergodic \(f\)-invariant measure whose Oseledets splitting admits domination. We give a hybrid estimate from above for the metric entropy of \(\mu\) in terms of Lyapunov exponents and volume growth. Furthermore, for any \(C^1\) diffeomorphism away from tangencies, its topological entropy is bounded by the volume growth.

1. Introduction. In dynamical systems, the notion of entropy plays a crucial role in measuring the complexity of evolutions. A system with positive entropy is considered to be chaotic. In general it is a difficult task to calculate the value of entropy due to its technical definitions. Thus estimates of entropy in various settings interest us frequently.

Let \(M\) be a compact Riemannian manifold without boundary and \(f : M \to M\) a \(C^1\) diffeomorphism. Denote by \(\mathcal{M}_{\text{erg}}(M, f)\) the set of all ergodic \(f\)-invariant measures. The Oseledets theorem [13] states that for any \(\mu \in \mathcal{M}_{\text{erg}}(M, f)\), there exist real numbers \(\lambda_1(\mu) < \cdots < \lambda_r(\mu)(\mu) < 0 \leq \lambda_{r(\mu)+1}(\mu) < \cdots < \lambda_{\tau(\mu)}(\mu)\) and a splitting \(T_x M = \bigoplus_{1 \leq i \leq \tau(\mu)} E_i(x)\) for \(\mu\)-a.e., \(x\), such that

\[
\lim_{n \to \pm \infty} \frac{1}{n} \log \|D_x f^n|_{E_i(x)}\| = \lambda_i(\mu), \quad 1 \leq i \leq \tau(\mu).
\]

Those \(\lambda_i(\mu)\) are called the Lyapunov exponents and the corresponding splitting is the Oseledets splitting. Denote \(E^- = E_1 \oplus \cdots \oplus E_{r(\mu)}\) and \(E^+ = E_{r(\mu)+1} \oplus \cdots \oplus E_{\tau(\mu)}\). From Ruelle (or Margulis-Ruelle) inequality [16], the sum of positive Lyapunov exponents can control the metric entropy \(h_\mu(f)\) of \(\mu\):

\[
h_\mu(f) \leq \sum_{E_i \subset E^+} \lambda_i(\mu) \dim E_i, \quad (1)
\]

The attainment of the upper bound in (1) has been studied broadly when \(\mu\) is absolutely continuous relative to Lebesgue measure [14, 9, 18].

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* Author to whom any correspondence should be addressed.
If \( f \) is further a \( C^{1+\alpha} \) diffeomorphism, in 1980, Przytycki \([15]\) gave another estimate from above of \( h_\mu(f) \) in terms of the volume growth:

\[
    h_\mu(f) \leq \limsup_{n \to \infty} \frac{1}{n} \log \int_M \|(D_x f^n)^\wedge \dim E^+\| d\text{Leb}(x),
\]

(2)

where \( (D_x f)^\wedge \) is the map on the \( i \)-th exterior algebra of the tangent space \( T_x M \) induced by \( D_x f \). Together with the well known variational principle, (2) in fact gives rise to an upper bound of the topological entropy of \( f \):

\[
    h_{\text{top}}(f) \leq \limsup_{n \to \infty} \frac{1}{n} \log \int_M \|(D_x f^n)^\wedge\| d\text{Leb}(x),
\]

(3)

where \( (D_x f)^\wedge = \max_{1 \leq i \leq \dim M} (D_x f)^\wedge i \) is the map on the exterior algebra of the tangent space \( T_x M \) induced by \( D_x f \). Besides, Newhouse \([12]\) has extended (3) to all \( C^{1+\alpha} \) maps and Kozlovski \([8]\) established the inverse part of (3) for all \( C^\infty \) maps.

In the present paper, we plan to give a hybrid estimate of the metric entropy for systems with dominated splittings, replacing the Hölder condition of the derivative in the estimate of the volume growth. Here, a dominated splitting \( T_\Lambda M = \tilde{E}_1 \oplus_\prec \cdots \oplus_\prec \tilde{E}_i \) over a compact \( f \)-invariant set \( \Lambda \) is defined if there exists \( L \in \mathbb{N} \) such that \( \|(D_x f^L v)\| \leq (1/2)\|(D_x f^L w)\| \) for any \( x \in \Lambda, v \in \tilde{E}_i(x), w \in \tilde{E}_j(x) \) with \( \|v\| = \|w\| = 1 \) and \( 1 \leq i < j \leq L \). Noting that the domination on any \( f \)-invariant set can be naturally extended to its closure, we consider dominated splittings over the support \( \text{supp}(\mu) \) with respect to every \( \mu \in \mathcal{M}_{\text{erg}}(M, f) \).

**Theorem 1.1.** Let \( f \) be a \( C^1 \) diffeomorphism on a compact Riemannian manifold \( M \) without boundary and \( \mu \in \mathcal{M}_{\text{erg}}(M, f) \). If there exists a dominated splitting \( T_{\text{supp}(\mu)} M = E^- \oplus_\prec E_1 \oplus_\prec E_2 \) over \( \text{supp}(\mu) \), where \( E^- \) is the sum of Oseledets spaces with negative exponents, then

\[
    h_\mu(f) \leq \sum_{E_i \subset E^+} \lambda_i(\text{Leb}) \cdot \dim E_i + \limsup_{n \to \infty} \frac{1}{n} \log \int_M \|(D_x f^n)^\wedge \dim E_2\| d\text{Leb}(x).
\]

Usually, there is no necessary connection between Lyapunov exponent and volume growth except when \( \mu = \text{Leb} \) and ergodic, by the concavity of \( \log t \), one has

\[
    \sum_{E_i \subset E^+} \lambda_i(\text{Leb}) \cdot \dim E_i = \limsup_{n \to \infty} \frac{1}{n} \int_M \log \|(D_x f^n)^\wedge\| d\text{Leb}(x)
\]

\[
    \leq \limsup_{n \to \infty} \frac{1}{n} \log \int_M \|(D_x f^n)^\wedge\| d\text{Leb}(x).
\]

Therefore, for a \( C^1 \) generic \( f \) in \( \text{Diff}_{\text{Leb}}^1(M) \) which denotes the space of volume-preserving \( C^1 \) diffeomorphisms on \( M \), by the domination property along orbits of \( \text{Leb} \)-almost all points in Theorem 1 of \([3]\) and the ergodicity statement in Theorem A of \([2]\), the Oseledets splitting of \( \text{Leb} \) is dominated, which combing with the Pesin entropy formula in the domination situation \([18]\), implies

\[
    h_{\text{Leb}}(f) \leq \limsup_{n \to \infty} \frac{1}{n} \log \int_M \|(D_x f^n)^\wedge\| d\text{Leb}(x).
\]

Some results in the setting of \( C^1 \) diffeomorphisms with domination corresponding to the \( C^{1+\alpha} \) Pesin theory have been obtained, for example, the stable manifold theorem \([1]\) and Pesin entropy formula \([18]\). However, in some cases, it also may happen that the results are not parallel: in \([10]\), it is proved that the entropy map...
is upper semi-continuous in the setting of $C^1$ diffeomorphisms with domination but not upper semi-continuous in the setting of $C^{1+\alpha}$. Concerning the hybrid control of metric entropy, assuming a partially hyperbolic splitting $TM = E^c \oplus E^u$ over the whole manifold $M$, Saghin [17] showed that the metric entropy with respect to an ergodic measure $\mu$ is bounded from above by the volume growth on $E^u$ plus the maximum $\lambda_{E^u}$, between zero and the maximal Lyapunov exponent $\lambda_{E^c}$ on $E^c$ multiplied by $\dim E^c$. Here we further obtain the hybrid control of metric entropy from the partially hyperbolic splitting over $M$ to the dominated splitting over the support.

**Theorem 1.2.** Let $f$ be a $C^1$ diffeomorphism on a compact Riemannian manifold $M$ without boundary and $\mu \in \mathcal{M}_{\text{erg}}(M,f)$. If there exists a dominated splitting $T_{\text{supp} (\mu)} M = E^c \oplus F$ over $\text{supp}(\mu)$ and the Lyapunov exponents of $\mu$ are nonnegative on $F$, then

$$h_\mu(f) \leq \lambda^+_E(\mu) \dim E + \limsup_{n \to \infty} \frac{1}{n} \log \int_M \|(D_x f^n)^\wedge\| \text{dLeb}(x).$$

The upper bound in Theorem 1.2 consists of two parts: the Lyapunov exponent and the volume growth. To deduce the term on Lyapunov exponent, we don’t use the methods in the proof of Ruelle inequality and Pesin entropy formula but directly estimate the average maximal expanding rate on the $E$-bundle. For the estimate on volume growth, we use the domination to obtain invariant $C^1$ plaques with uniform size, which inherit the dynamical properties of tangent bundles (i.e., derivatives).

**Theorem 1.1** is applicable for any $C^1$ diffeomorphism $f$ away from tangencies and any $\mu \in \mathcal{M}_{\text{erg}}(M,f)$ ([11, 4, 20]): the Oseledets bundle with negative (positive) exponents and the Oseledets bundle with nonnegative (nonpositive) exponents are dominated by Proposition 3.4 of [11], hence the term on Lyapunov exponent can be taken trivially in Theorem 1.1. Together with the variational principle, we actually get the $C^{1+\alpha}$ theorem of Przytycki [15] in the setting away from tangencies.

**Corollary 1.3.** Let $f$ be a $C^1$ diffeomorphism on a compact Riemannian manifold $M$ without boundary and away from tangencies, then

$$h_{\text{top}}(f) \leq \limsup_{n \to \infty} \frac{1}{n} \log \int_M \|(D_x f^n)^\wedge\| \text{dLeb}(x).$$

2. **Dominated Oseledets splitting.** Let $\mu \in \mathcal{M}_{\text{erg}}(M,f)$ and $T_x M = \bigoplus_{1 \leq i \leq \tau(\mu)} E_i(x)$ be its Oseledets splitting, which exists for $\mu$-a.e., $x \in M$. We first prove Theorem 1.1. Denote $\lambda = \sum_{E_i \subset F_i} \lambda_i(\mu) \dim E_i$ for $N \in \mathbb{N}$, $\eta > 0$, define

$$Q_{N,\eta} = \left\{ x : \|D_x f^n|_{E^c(x)}\| \leq e^{n\eta}, \quad \|(D_x f^n|_{F_i(x)})^\wedge\| \leq e^{n\hat{\lambda} + n\eta}, \right\},$$

$$m(D_x f^n|_{E^c(x)}) \geq e^{-n\eta}, \quad \forall n \geq N,$$

where the minimal norm $m(A)$ for a linear map $A$ is defined by $\inf_{\|v\| = 1} \|Av\|$. By Oseledets theorem, for any $\eta > 0$,

$$\lim_{N \to \infty} \mu(Q_{N,\eta}) = 1.$$ 

For any sufficiently small $\varepsilon > 0$, we can take $N$ large such that $\mu(Q_{N,\eta}) > 1 - \varepsilon/4$. Let $\Lambda = Q_{N,\eta}$ in the following discussions.

Note that $E^c = F_1 \oplus F_2$. By the domination $T_{\text{supp}(\mu)} M = E^- \oplus E^+$, we are able to construct continuous families of $C^1$ plaques: $\{W_2^* : x \in \text{supp}(\mu)\}$, $* \in \{\text{E}^{-}, \text{E}^{+}\}$
by [6]. Precisely, there is some $\beta > 0$ such that $\angle(E^-(x), E^+(x)) \geq \beta$ for every $x \in \text{supp}(\mu)$, where $\angle(E_1, E_2) = \inf \{ \frac{\|v \wedge w\|}{\|v\|\|w\|} : 0 \neq v \in E_1, 0 \neq w \in E_2 \}$ denotes the angle between $E_1$ and $E_2$, $v \wedge w$ is the wedge product of two vectors $v, w$. Then for any $\gamma \in (0, \beta/4)$ (the value of $\gamma$ will be specified later), there exist $0 < r_1 < r_0$ such that for any $x \in \text{supp}(\mu)$, there exist $C^1$ submanifolds $W^+_x$ with the following properties:

(i) almost tangency: $T_x W^+_x = \ast(x)$ and $T_y W^+_x$ lies in a cone of width $\gamma$ of $\ast(x)$ for $y \in W^+_x(x, r_0)$, where $V(z, \rho)$ denotes the ball on a submanifold $V$ centered at $z$ with radius $\rho$;

(ii) local invariance: $f^\pm W^+_x(x, r_1) \subset W^+_x(f^\pm(x), r_0)$.

For studying dynamics in neighborhoods, we take $C^1$ local foliations $\mathcal{F}_x$ on $B(x, r_0)$ for every $x \in \text{supp}(\mu)$, with leaves whose tangent bundles lie in a cone of width $\gamma$ of $W^+_x$. (Note that we don’t require invariance of $\mathcal{F}$, because invariant foliations may not be $C^1$ smooth [5]). Given a foliation $\mathcal{F}$ and a point $y$ in the domain, we denote by $\mathcal{F}(y)$ the leaf through $y$. We have that $\mathcal{F}_x$ is uniformly absolutely continuous: there exist continuous functions, $p_x : W^E_x \to \mathbb{R}$, $q_x : B(x, r_0) \to \mathbb{R}$ such that for any integrable function $\phi : M \to \mathbb{R}$,

$$\int_{B(x, r_0)} \phi(y) d\text{Leb}(y) = \int_{W^E_x \cap B(x, r_0)} p_x(y) \int_{\mathcal{F}_x(y)} q_x(z) \phi(z) d\text{Leb}_{\mathcal{F}_x(y)}(z) d\text{Leb}_{W^E_x}(y).$$

In the above, by the compactness of $\text{supp}(\mu)$, we can choose a constant $a_0 > 1$ such that $a_0^{-1} < p_x, q_x < a_0$ for any $x \in \text{supp}(\mu)$, and for any ball of radius $\rho$ inside $\mathcal{F}_x$ (or $W^E_x$) with respect to the induced metric in $\mathcal{F}_x$ (or $W^E_x$) has $l$-volume between $a_0^{-1} \rho^l$ and $a_0 \rho^l$, where $l = \dim E^+$ (or $\dim E^-$). Moreover, for any $\rho_1, \rho_2 \in (0, r_0)$, any submanifold $V$ containing $y \in W^E_x(x, \rho_1)$ with tangent bundles in $\gamma$-cone of $E^+(x)$, it holds that the ball $B_V(y, \rho_2)$ in $V$ with center $y$ and radius $\rho_2$ is contained in $B(x, \rho_1 + \rho_2)$.

To estimate local dynamical growth, we adopt the concept of Bowen ball. For each $x \in M$, $n \in \mathbb{N}$, $\rho > 0$, define $n$-step Bowen ball by $B_n(x, \rho, f) = \{ y \in M : d(f^i(x), f^i(y)) \leq \rho, 0 \leq i < n \}$, and $B_\infty(x, \rho, f) = \{ y \in M : d(f^i(x), f^i(y)) \leq \rho, i \geq 0 \}$. By the invariance property (ii) we could also define Bowen ball along $W^+_x$ for any $x \in \text{supp}(\mu)$ and $x \leq r_1$ by $W^+_x(n, \rho, f) = \{ y \in W^+_x : d_{W^+_x}(f^i(y), f^i(x)) \leq \rho, 0 \leq i < n \}$, where $d_{W^+_x}$ is the induced metric in $W^+_x$ for any $z \in \text{supp}(\mu)$.

3. Local dynamical growth. Let $e^{-\alpha_1} = \inf_{x \in M} \{ m(D_x f), m(D_x f^{-1}) \} = 1/ \sup_{x \in \mathcal{M}} \{ \| D_x f \|, \| D_x f^{-1} \| \}$. Denote $g = f^N$, then $m(Dg) \geq e^{-N\alpha_1}$. As $g$ is $C^1$, we could take $0 < \gamma_1 < \gamma_2$, $0 < r_1 < r_0$ small enough and find some $b_0 \in (0, r_1)$ such that for any $d(y, z) \leq b_0$, any unit vectors $v_1^{(1)}, \ldots, v_k^{(1)} \in T_y \mathcal{M}$ and $v_1^{(2)}, \ldots, v_k^{(2)} \in T_z \mathcal{M}$ ($1 \leq i \leq \dim M$) with $\angle(v_1^{(k)}, v_i^{(k)}) \geq \gamma_2$, $\angle(v_j^{(1)}, v_i^{(2)}) \leq \gamma_1$, $1 \leq i_1 \neq i_2 \leq i$, $k = 1, 2, 1 \leq j \leq i$, the following holds:

$$\frac{\| (D_y g)^{\lambda_1}(v_1^{(1)}, \ldots, v_k^{(1)}) \|}{\| (D_z g)^{\lambda_1}(v_1^{(2)}, \ldots, v_k^{(2)}) \|} \leq e^{N\eta},$$

(4)
We let the $\gamma$ from the construction of the local invariant plaques and the local foliations be in $(0, \gamma_1)$. Observing $\mu$ may be not ergodic for $g$, we give a lemma to deal with this problem.

**Lemma 3.1.** For any $\varepsilon_1, \varepsilon_2 \in (0, 1)$ and $\Lambda \subset M$ with $\mu(\Lambda) > 1 - \varepsilon_1 \varepsilon_2$, there exists $\Lambda_1$ with $\mu(\Lambda_1) > (1 - \varepsilon_1)(1 - \varepsilon_2)$ such that for any $x \in \Lambda_1$ we have

$$\lim_{n \to +\infty} \frac{\#\{0 \leq i < n \mid g^i x \in \Lambda \}}{n} \geq 1 - \varepsilon_1.$$

**Proof.** As $\mu$ is ergodic for $f$, $\mu$ can be decomposed as

$$\mu = \frac{1}{t} (\mu_1 + \cdots + \mu_t),$$

where $t \in \mathbb{N}^+$ divides $N$ and each $\mu_i$ is an ergodic $g$-invariant measure such that $\mu_{i+1} = f_* \mu_i$ for each $i \pmod{t}$. By Birkhoff ergodic theorem, $\exists \Gamma_i$ with $\mu_i(\Gamma_i) = 1$, such that

$$\lim_{n \to +\infty} \frac{\#\{0 \leq i < n \mid g^i x \in \Lambda \}}{n} = \mu_i(\Lambda),$$

for any $x \in \Gamma_i$ and $i = 1, \cdots, t$. Denote $\Phi(N) = \{i : \mu_i(\Lambda) \geq 1 - \varepsilon_1\}$. Then

$$1 - \varepsilon_1 \varepsilon_2 < \mu(\Lambda) = \frac{1}{t} (\mu_1(\Lambda) + \cdots + \mu_t(\Lambda)) \leq \frac{1}{t} ((t - \#\Phi(N))(1 - \varepsilon_1) + \#\Phi(N) \cdot 1),$$

which implies that

$$\#\Phi(N) \geq t(1 - \varepsilon_2).$$

Take $\Lambda_1 = \cup_{i \in \Phi(N)} \Gamma_i$. We have

$$\mu(\Lambda_1) \geq \frac{\#\Phi(N)}{t} (1 - \varepsilon_1) \geq (1 - \varepsilon_1)(1 - \varepsilon_2).$$

Applying Lemma 3.1 for $\varepsilon_1 = \varepsilon_2 = \varepsilon^{1/2}$, we get $\Lambda_1$ with $\mu(\Lambda_1) \geq (1 - \varepsilon^{1/2})^2$ and $\mu_{i}(\Lambda_1) \geq 1 - \varepsilon^{1/2}$ for $i \in \Phi(N) = \{i : \mu_i(\Lambda) \geq 1 - \varepsilon^{1/2}\}$, where $\mu = (1/t) \sum_{1 \leq i \leq t} \mu_i$ and each $\mu_i$ is $g$-ergodic. By Egorov’s theorem, there exists $\Lambda_2 \subset \Lambda_1$ with $\mu_1(\Lambda_2) > 1 - 2\varepsilon^{1/2}$ for $i \in \Phi(N)$, and $\exists \mathcal{E} \in \mathbb{N}$, such that for any $x \in \Lambda_2$ and $n \geq S$, we have

$$\frac{\#\{0 \leq i < n \mid g^i(x) \in \Lambda \}}{n} \geq 1 - 2\varepsilon^{1/2}.$$

**Proposition 3.2.** For any $x \in \Lambda_2$, $n \geq S$, $\rho \in (0, b_0)$,

$$W_x^{E^-}(x, \rho e^{-2nN((1-2\varepsilon^{1/2})\eta+\varepsilon^{1/2})}) \subset W_x^{E^-}(n, \rho, g).$$

**Proof.** For any $y \in W_x^{E^-}(x, \rho e^{-2nN((1-2\varepsilon^{1/2})\eta+\varepsilon^{1/2})})$, let $\sigma$ be the extreme path connecting $x$ and $y$ in $W_x^{E^-}$. Then for any $0 \leq i < n$,

$$d_{W_x^{E^-}}(g^i(x), g^i(y)) \leq \int_{\sigma} \|Dg^i|_{Tz_{\sigma}}\|d\text{Leb}_{\sigma}(z) \leq \int_{\sigma} \|Dg|_{Tz_{\sigma}}\|\cdots\|Dg|_{Tz_{\sigma}}\|d\text{Leb}_{\sigma}(z).$$

Noting that the frequency of $\{x, \cdots, f^n x\}$ in $\Lambda$ is at least $1 - 2\varepsilon^{1/2}$ and the deviation of $Dg$ with respect to the distance in $W^{E^-}$ is at most $\varepsilon^{N\mu}$ by (4), it follows that

$$d_{W_x^{E^-}}(g^i(x), g^i(y)) < \rho, \quad 0 \leq i < n.$$
Next we estimate the growth of Bowen balls along $\mathcal{F}$:

**Proposition 3.3.** For any $x \in \Lambda_2$, $n \geq S$, $\rho \in (0, b_0)$, $y \in W^E_x(n, \rho, g)$, there exists a connected subset $\mathcal{R}_x(y, n, \rho, g) \subset F_x(y)$ containing $x$ and satisfying

(i) $\mathcal{R}_x(y, n, \rho, g) \subset B_n(x, 2\rho, g)$;

(ii) $\text{vol}_{\dim E^+}(g^{(n-1)}(\mathcal{R}_x(y, n, \rho, g))) \geq a_0^{-1}(\rho e^{-2nN((1-2e^{1/2})\eta+\varepsilon'/2, a_1)})^{\dim E^+}$, where $\text{vol}_{\dim E^+}$ is the $E^+$-volume.

**Proof.** First, take $R_0$ as a ball in $F_x(y)$ centered at $y$ with radius $\rho$. By domination, $g(R_0)$ is a submanifold with tangent bundles in $\gamma$-cone of $E^+(g(x))$. By induction, we suppose $R_i$ has been defined, $0 \leq i < n - 1$. If $g^i(x) \in \Lambda = Q_{N, \eta}$, then $g$ has expanding rate $\geq e^{-2N\eta}$ on $R_i$, which implies we can choose a ball $R_{i+1}$ in $g(R_i)$ centered at $g^{i+1}(y)$ whose radius is $\rho e^{-2N\eta}$ times the radius of $R_i$; otherwise, choose $R_{i+1}$ as the ball in $g(R_i)$ centered at $g^{i+1}(y)$ whose radius is $\rho e^{-Na_1}$ times the radius of $R_i$. Continue the process until $R_{n-1}$ is defined. As the frequency in $\Lambda$ is no less than $1 - 2\varepsilon^{1/2}$, we deduce the radius of $R_{n-1}$ is at least $\rho e^{-2(1-2e^{1/2})nN\eta-2\varepsilon^{1/2}nA_1}$. Let $\mathcal{R}_x(y, n, \rho, g) = g^{-(n-1)}(R_{n-1})$. Then by domination, for any $i \in [0, n)$, $g^i(\mathcal{R}_x(y, n, \rho, g))$ belongs to $R_i$ whose tangent bundles lie in $\gamma$-cone of $E^+(g^i(x))$. Hence, $g^i(\mathcal{R}_x(y, n, \rho, g)) \subset B(g^i(x), 2\rho)$ and we have

$$
\text{vol}_{\dim E^+}(g^{(n-1)}(\mathcal{R}_x(y, n, \rho, g))) \geq a_0^{-1}(\rho e^{-2(1-2e^{1/2})nN\eta-2\varepsilon^{1/2}nA_1})^{\dim E^+}
$$

$$
= a_0^{-1}(\rho e^{-2nN((1-2e^{1/2})\eta+\varepsilon'/2, a_1)})^{\dim E^+}.
$$

$\square$

4. Estimates of entropy. Recall $\mu = (1/t) \sum_{1 \leq i \leq t} \mu_i$ with all $\mu_i$ are g-ergodic. We adopt Katok’s definition of metric entropy [7]. For $1 \leq i \leq t$, $n > N$, $\delta \in (0, b_0)$, denote by $\omega_i(n, \delta, g)$ the minimal number of $(n, \delta)$-Bowen balls $B_n(x, \delta, g)$ whose union covers a set of $\mu_i$-measure of at least $1/2$. Then

$$
h_{\mu_i}(g) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{\log \omega_i(n, \delta, g)}{n} = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{\log \omega_i(n, \delta, g)}{n}.
$$

Take $\Upsilon(n, \delta, g)$ to be an $(n, \delta)$-separated set of $\Lambda_2$ for $g$ with maximal cardinality. Hence it is also an $(n, \delta)$-spanning set of $\Lambda_2$, which implies

$$
\mu_i(\bigcup_{x \in \Upsilon(n, \delta, g)} B_n(x, \delta, g)) \geq \mu_i(\Lambda_2) > 1 - 2\varepsilon^{1/2} > \frac{1}{2}, \quad \forall i \in \Phi(N).
$$

Thus $\omega_i(n, \delta, g) \leq \# \Upsilon(n, \delta, g)$ for $i \in \Phi(N)$.

Observe the elements of $\{B(x, n, \delta/2, g) : x \in \Upsilon(n, \delta, g)\}$ are disjoint with each other. Considering the estimate of volume growth, by the domination $T_{\supp(\mu)} M = E^\ast \oplus F_1 \oplus F_2$, combing (4) and the choice of $\Lambda_2 \subset \Lambda_1 \subset \Lambda = Q_{N, \eta}$, we have

$$
\int_M \|D_x g^{(n-1)}(\cdot) \wedge \dim F_2\|d\text{Leb}(x)
$$

$$
\geq \int_{\bigcup_{x \in \Upsilon(n, \delta, g)} B_n(x, \delta/2, g)} \|D_x g^{(n-1)}(\cdot) \wedge \dim F_2\|d\text{Leb}(x)
$$

$$
\geq a_0^{-2} e^{-nN\eta} \sum_{x \in \Upsilon(n, \delta, g)}
$$
By Jacobs theorem (see Theorem 8.4 of [19]), we obtain that

\[ \int_{W_E^+ \cap B_n(x, \delta/2, g)} \int_{F_x(y) \cap B_n(x, \delta/2, g)} \| (D_x f^n)^{\dim F_2} \| d\text{Leb}_{F_x(y)}(z) d\text{Leb}_{W_E^+}(y) \geq a_2 \int_{\mathcal{T}(n, \delta, g)} \sum_{x \in \mathcal{T}(n, \delta, g)} \| (D_x f^n)^{\dim F_2} \| d\text{Leb}_{F_x(y)}(z) d\text{Leb}_{W_E^+}(y) \]

\[ \int_{W_E^+ \cap B_n(x, \delta/2, g)} \int_{F_x(y) \cap B_n(x, \delta/2, g)} \| (D_x g^{n-1})^{\dim F_1} \| d\text{Leb}_{F_x(y)}(z) d\text{Leb}_{W_E^+}(y) \]

(6)

Applying Propositions 3.3 and 3.2,

\[ \sum_{x \in \mathcal{T}(n, \delta, g)} \int_{W_E^+ \cap B_n(x, \delta/2, g)} \| (D_x f^n)^{\dim F_2} \| d\text{Leb}_{F_x(y)}(z) d\text{Leb}_{W_E^+}(y) \]

\[ \geq a_0 \int_{\mathcal{T}(n, \delta, g)} \sum_{x \in \mathcal{T}(n, \delta, g)} \int_{W_E^+ \cap B_n(x, \delta/2, g)} \| (D_x f^n)^{\dim F_2} \| d\text{Leb}_{F_x(y)}(z) d\text{Leb}_{W_E^+}(y) \]

\[ \geq a_0 \int_{\mathcal{T}(n, \delta, g)} \sum_{x \in \mathcal{T}(n, \delta, g)} \int_{W_E^+ \cap B_n(x, \delta/2, g)} \| (D_x g^{n-1})^{\dim F_1} \| d\text{Leb}_{F_x(y)}(z) d\text{Leb}_{W_E^+}(y) \]

\[ \geq a_0^2 \int_{\mathcal{T}(n, \delta, g)} \sum_{x \in \mathcal{T}(n, \delta, g)} \int_{W_E^+ \cap B_n(x, \delta/2, g)} \| (D_x g^{n-1})^{\dim F_1} \| d\text{Leb}_{F_x(y)}(z) d\text{Leb}_{W_E^+}(y) \]

where \( a_2 \) is a constant depending on the uniform angles of dominated splittings

Take logarithm, divide by \( a_2 \), and let \( t \to 0 \), in the above inequalities (5)(6), then we finally have

\[ \lim_{n \to \infty} \frac{1}{n} \log \int_M \| (D_x f^n)^{\dim F_2} \| d\text{Leb}(x) \]

\[ \geq \lim_{n \to \infty} \frac{1}{n} \log \int_M \| (D_x g^{n-1})^{\dim F_2} \| d\text{Leb}(x) \]

\[ \geq \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{Y}(n, \delta, g) - \lambda \]

\[ \geq h_{\mu_1}(g)/N - \lambda. \]

Recall that \( \Phi(N) = \{ 1 \leq i \leq t : \mu_i(\Lambda) > 1 - \varepsilon^{1/2} \} \) and \( (1 - \varepsilon^{1/2})t \leq \# \Phi(N) \leq t \).

By Jacobs theorem (see Theorem 8.4 of [19]), we obtain that

\[ h_{\mu}(f) = \frac{1}{N} \sum_{i=1}^{t} h_{\mu_i}(g) = \frac{1}{N} \sum_{i \in \Phi(N)} h_{\mu_i}(g) + \frac{1}{N} \sum_{i \notin \Phi(N)} h_{\mu_i}(g) \]

\[ \leq \frac{1}{t} \#(\Phi(N)) \cdot \left( \limsup_{n \to \infty} \frac{1}{n} \log \int_M \| (D_x f^n)^{\dim F_2} \| d\text{Leb}(x) + \hat{\lambda} \right) + (t - \#(\Phi(N)) \cdot a_1^{\dim M}) \]
Since $\varepsilon$ is arbitrary, we conclude

$$h_\mu(f) \leq \limsup_{n \to +\infty} \frac{1}{n} \log \int_M \|D_x f^n\|^{\dim F} \|dLeb(x)\ + \hat{\lambda}.$$ 

This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** This can be obtained by a modification of the proof of Theorem 1.1. For any $N \in \mathbb{N}$, $\eta > 0$, define

$$Q_{N, \eta} = \left\{ x : \|D_x f^n|_{E(x)}\| \leq e^{n(\lambda^+_E(\mu) + \eta)}, \ n(D_x f^n|_{F(x)}) \geq e^{-n\eta}, \ \forall n \geq N \right\}.$$ 

For any $\varepsilon > 0$, we can take $N$ large such that $\mu(Q_{N, \eta}) > 1 - \varepsilon/4$ and let $\Lambda = Q_{N, \eta}$. Let $g = f^N$, then $\mu$ can be decomposed as $(1/t) \sum_{1 \leq i \leq t} \mu_i$ with all $\mu_i$ are $g$-ergodic. Denote $\Phi(N) = \{1 \leq i \leq t : \mu_i(\Lambda) > 1 - \varepsilon/2\}$. Replace $E^-$ and $E^+$ by $E$ and $F$, respectively, in the proof of Theorem 1.1. It follows that for $i \in \Phi(N)$ and sufficiently large $n$,

$$\int_{W^E \cap B_n(x, \delta/2, g)} dLeb_{W^E}(y) \geq a_0^{-1}(\delta^2/4)^{nN((1-2\varepsilon/2)(\lambda^+_E(\mu) + 2\eta) + 2\varepsilon/2a_1)\dim E},$$

which is used in (6). Moreover,

$$\int_M \|D_x g^{(n-1)}\|^{\dim F} \|dLeb(x)\$$

$$\geq \int_{U \in \Theta(n, \delta, g)B_n(x, \delta/2, g)} \|D_x g^{(n-1)}\|^{\dim F} \|dLeb(z)\$$

$$\geq a_0^{-2} \sum_{x \in \Theta(n, \delta, g)} \int_{W^E \cap B_n(x, \delta/2, g)} \|D_x g^{(n-1)}|_{T_{F_x}(y)}\|^{\dim F} \|dLeb_{F_x}(y)\|dLeb_{W^E}(y)$$

$$\geq a_0^{-2} \sum_{x \in \Theta(n, \delta, g)} \int_{W^E \cap B_n(x, \delta/2, g)} \text{vol}_{\dim F}(g^{(n-1)}(\mathcal{R}_x(y, n, \delta/4, g)))dLeb_{W^E}(y)$$

$$\geq a_0^{-3}(\delta^2/4)^{2nN((1-2\varepsilon/2)\eta + 2\varepsilon/2a_1)\dim E} \sum_{x \in \Theta(n, \delta, g)} \int_{W^E \cap B_n(x, \delta/2, g)} dLeb_{W^E}(y).$$

Take logarithm, divide by $n$, take limit superior and let $\delta, \varepsilon, \eta \to 0$, then we get

$$\limsup_{n \to +\infty} \frac{1}{n} \log \int_M \|D_x f^n\|^{\dim F} \|dLeb(x)\ \geq \ h_\mu(g)/N - \lambda^+_E(\mu) \dim E,$$

which, together with Jacobs theorem, consequently gives rise to

$$h_\mu(f) \leq \limsup_{n \to +\infty} \frac{1}{n} \log \int_M \|D_x f^n\|^{\dim F} \|dLeb(x)\ + \lambda^+_E(\mu) \dim E.$$ 

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E-mail address: guoxu342@pku.edu.cn
E-mail address: lg@suda.edu.cn
E-mail address: sunwx@math.pku.edu.cn
E-mail address: yangdw@suda.edu.cn