Family Switching Formula and the $-n$ Exceptional Rational Curves

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1 preliminary

In this paper, we prove the switching formula for Family Seiberg-Witten invariants. Besides its role in the family Seiberg-Witten theory, the formula is also one of the main ingredients in the proof of the Göttsche and the Göttsche-Yau-Zaslow conjecture\cite{Liu1}.

Recall that in the ordinary Seiberg-Witten theory, a simple formula similar to the blow up formula was proved by R. Stern and R. Fintushel\cite{FS} regarding the change of Seiberg-Witten invariants for the two different $spin^c$ structures which differ by a multiple of classes with self intersection number $= -n$. Our formula generalizes theirs to the family Seiberg-Witten theory. It turns out that through our generalization, the Gromov-Taubes aspect of R. Stern and R. Fintushel’s formula can be fully explored.

In an earlier paper\cite{Liu3}, the current author has derived the family blowup formula for $-1$ spheres. The family blowup formula plays a rather crucial role in understanding the enumerating question upon the number of nodal curves on an algebraic surface. In this paper, we generalize the approach to prove the $-n$ sphere switching formula.

In the following, we list all the few main theorems proved in the paper.

The following theorem is the family switching formula of $-n$ spheres.

**Main Theorem 1** Let $\pi : X \to B$, $\pi : C \to B$, $\mathcal{L}$ and $\mathcal{L}_k$ be the fiber bundle of four-manifolds, the relative $S^2$ fiber bundle with fiberwise self-intersection number $-n$, the two fiberwise $spin^c$ structures.

Assume that the family moduli space expected dimensions are non-negative, then $\exists V \to B$, a complex virtual vector bundle over $B$ called the relative obstruction virtual bundle and the following family switching formula relates the family invariants of $spin^c$ structures $\mathcal{L}_k$ and $\mathcal{L}$.

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For the notations and the structure of \( V \), please consult section 3 and the statement of theorem 2 on page 5.

The following theorem for algebraic family Seiberg-Witten invariants in section 4 is the algebraic analogue of the switching formula in which we allow the \( \mathbb{P}^1 \) fibration to have singular fibers.

**Main Theorem 2** Let \( \pi : X \to B \) be an algebraic fiber bundle of algebraic surfaces over a proper and smooth algebraic manifold \( B \). Let \( C \) be a \((1,1)\) class of \( X \) which restricts to \((1,1)\) classes of the fibers.

Let \( C \subset X \) be a rational curve fibration over \( B \) with smooth generic fibers and self-intersection number \(-n\).

Under these assumptions \( AF(i) - AF(iv) \). (see page 22 for the details), the pure algebraic family invariants of \( C \) and the mixed family invariant of \( C \) are related by the following formula,

\[
AF_{SW}(X \to B(1, C + kPD(C))) = \sum_{0 \leq i < \infty} AF_{SW}(c_{i}(V_{1 \to k}), C),
\]

where the relative obstruction virtual sheaf \( V_{1 \to k} \) can be identified with the virtual sheaf

\[
R^{1} \pi_{*}O_{kC}(D_{C} + kC) - R^{0} \pi_{*}O_{kC}(D_{C} + kC).
\]

In section 5 and section 6 we apply the ideas embedded in the proof of the family switching formula to some concrete example on the restriction of the universal family \( X = \mathbb{M}_{n+1} \times \mathbb{M}_{n} \to Y(\Gamma) = B \). In section 5 we compare the canonical algebraic Kuranishi models of two classes \( C - \mathbb{M}(E)E - \sum_{e_{i} \cdot (C - \mathbb{M}(E)E) < 0} e_{i} \) and \( C - \mathbb{M}(E)E \). In section 6 we apply the technique of localized top Chern classes and identify certain localized contribution of \( AF_{SW}(\mathbb{M}_{n+1} \times \mathbb{M}_{n} \to \mathbb{M}_{n} \times \mathbb{T}(\mathbb{M}(1, C - \mathbb{M}(E)E) - \sum_{e_{i} \cdot (C - \mathbb{M}(E)E) < 0} e_{i}).

**Main Theorem 3** Under the Simplifying Assumption, the integer \( j_{Y_{1}} \cdot Z_{Y_{1}}(s_{\text{canon}}) \cap c_{1}(\mathcal{H})^{\dim \mathbb{M}_{n} + \frac{n^{2}}{2} - C_{\text{canon}}(K_{\mathbb{M}}) + p_{\mathbb{M}} - \sum_{i \leq n} m_{i}^{2} + m_{i}} \), representing the dominated localized contribution of \( Y(\Gamma) \) to \( AF_{SW}(\mathbb{M}_{n+1} \times \mathbb{T}(\mathbb{M}) \to \mathbb{M}_{n} \times \mathbb{T}(\mathbb{M})(1, C - \mathbb{M}(E)E), \) can be identified with the mixed family invariant
\[ \text{AFSW}_{M \times M+1} Y(\Gamma) \times T(M) \rightarrow Y(\Gamma) \times T(M) \left( c_{\text{total}}(\tau), C - M(E)E - \sum_{i \leq p} e_i \right) \]

for some \( \tau \in K_0(Y(\Gamma) \times T(M)) \).

Please consult section 6 and page 43 for the notations and the details of the simplifying assumption.

2 A Simple Review of Fintushel-Stern’s Argument

After the discovery of the Seiberg-Witten theory [W], the calculation of Seiberg-Witten invariants has become an important subject as Seiberg-Witten invariants give rise to the smooth diffeomorphism invariants of the four-manifolds. It has been the long term goal of several group of people to understand the behavior of the Donaldson or Seiberg-Witten invariants under several kinds of surgical operation. In the present, we do not attempt to add new ingredients into this beautiful theory. Instead, we would like to generalize a simple formula of R. Fintushel and R. Stern to the family version of Seiberg-Witten invariants. As it will be shown in the second section, the proof of the new formula does not involve any new trick or technical improvement of Seiberg-Witten-Floer theory. Instead, the conjecture raised at the end of the paper strongly indicates the simplicity of the picture while the corresponding Floer theory could be rather complicated.

It was R. Fintushel and R. Stern who first noticed the importance of the switching formula in the context of the original Seiberg-Witten theory.

Let \( C \) be a cohomology class in \( H^2(M, \mathbb{Z}) \) with \( C^2 = -n \), then \( C \) determines a complex line bundle \( E_C \) over \( M \). Suppose the class \( C \) is represented by a \( -n \) two-sphere in the four-manifold \( M \), then one considers the determinant bundles of the following form

\[ \mathcal{L}_k = \mathcal{L}_0 \otimes C^\otimes 2k. \]

To simplify the notation, we will adopt the following alternative additive notation

\[ \mathcal{L}_k = \mathcal{L}_0 + 2kC, k \in \mathbb{Z}. \]

**Question**: How do the Seiberg-Witten invariants of \( \mathcal{L}_0 \) and \( \mathcal{L}_k \) relate to each other?

As the invariant is defined to be zero if the dimension of its moduli space is negative, it is more interesting to consider the case that both the moduli
space dimension $L_0$ and $L_k$ are non-negative. They give explicit lower bounds to $c_1(L_0)^2$ and $c_1(L_k)^2$.

With this convention understood, one may state the following theorem of Fintushel-Stern [FS].

**Theorem 1** (Fintushel-Stern) Let $C \in H^2(M, \mathbb{Z})$ be represented by a $-n$ two-sphere in the smooth four manifold $M$. Let $L_0$ and $L_k = L_0 + 2kC$ be the determinant line bundles of the two spin$^c$ structures related by the tensoring of $2k$ multiple of the complex line bundle $E_C$ associated with $C$.

Suppose that the dimensions of the Seiberg-Witten moduli spaces of $L_0$, $L_k$, $c_1(L_0)^2 - 2\chi - 3\sigma$, $c_1(L_k)^2 - 2\chi - 3\sigma$ are non-negative,

then the Seiberg-Witten invariants of both spin$^c$ structures are equal to each other,

$$SW(L_k) = SW(L_0).$$

As the proof is rather similar to the proof of blowup formula, let us indicate only a few key points.

Let us take a tubular neighborhood of the embedded $S^2$ (poincare dual to $C$) inside $M$ which is diffeomorphic to a two dimensional disk bundle $D$ over the $S^2$. As the self-intersection number is $C^2 = -n$, the normal bundle of the $S^2$ can be identified with a complex rank one bundle of negative first chern class $-n$. In particular, the boundary of the disk bundle $\partial D$, an $S^3$ bundle over the embedded $S^2$, is an oriented three dimensional manifold, denoted by $L_n$. R. Fintushel and R. Stern observe that $L_n$, $n \in \mathbb{N}$ are lens spaces. A key property used in the proof of Stern-Fintushel is the fact that $L_n$ carry positive scalar curvature metrics. This follows from the fact that $S^3$ is the universal covering of all the $L_n$.

Exactly as was used in the proof of the blowup formula, we consider the so-called long neck metric with a neck isometric to $L_n \times \mathbb{R}$ bridging the $S^2$ and $M - D$. The vanishing theorem of Seiberg-Witten invariants on manifolds with positive scalar curvature metrics [W] simplifies the gluing argument dramatically and it implies that the Seiberg-Witten moduli spaces of $L_0$ and $L_k$, $\mathcal{M}_{L_0}$ and $\mathcal{M}_{L_k}$ are diffeomorphic. A simple obstruction bundle calculation leads to the proof of the simple equality.

The reader should consult [FS], [Liu3] for more details.

### 3 The Set Up for Family Switching Formula and a Sketch of its Proof

Let us discuss the set up of our main theorem and fix our notations. Let $\pi : \mathcal{X} \rightarrow B$ be a fiber bundle over $B$ such that the fibers are diffeomorphic to a smooth four manifold $M$ with $b_2^+ > 0$. Let $C \rightarrow B$ be a $S^2$ fiber bundle over
which is embedded into $\mathcal{X} \hookrightarrow B$. Let $N_C \hookrightarrow C$ be the real rank two normal bundle of $C \hookrightarrow B$ in $\mathcal{X}$. We assume that the fiberwise degree of $N_C$ along $C \hookrightarrow B$ is negative $= -n, n \in N$, our goal is to compare the family Seiberg Witten invariants with different multiplicities along $C \hookrightarrow B$.

One assumes additionally that a fiberwise almost complex structure has been equipped upon some tubular neighborhood of $C \subset \mathcal{X}$ such that the fiber bundle $C \hookrightarrow B$ is a relative $\mathbf{CP}^1$ pseudo-holomorphic embedding into $\mathcal{X} \hookrightarrow B$. Then $N_C$ is equipped with a complex structure and is viewed as a complex line bundle over $B$. Let $K$ denote the canonical line bundle of the fiberwise almost complex structure. Then it follows that $K|_{calc} \cong N_C^* \otimes T^*_C/B$.

Let $Q \hookrightarrow C$ be a complex line bundle over the $\mathbf{CP}^1$ bundle $C \hookrightarrow B$. Then there exists a $\partial_Q$ operator on $Q$

$$\bar{\partial}_Q : \Omega^{0,0}_{C/B} \otimes Q \rightarrow \Omega^{0,1}_{C/B} \otimes Q,$$

making $Q$ a relative holomorphic line bundle on $C \hookrightarrow B$. The virtual bundle $\mathbb{R} \pi_c(Q)$ is defined to be the family index bundle $IND(\bar{\partial}_Q) = \text{Ker}(\bar{\partial}_Q) - \text{Coker}(\bar{\partial}_Q)$.

Let $L$ be the determinant line bundle of a fiberwise invariant $spin^c$ structure on $\mathcal{X}$. Let $D(N_C)$ be the disc bundle inside the normal bundle of $C$ in $\mathcal{X}$. Then the boundary of $D(N_C)$ is a lens space bundle over $B$. Let us denote $\partial(D(N_C))$ by $L(n)$. Under this convention $L(1)$ has a structure of an $S^3$ bundle over $B$ and the present set up is reduced to the family blowup setting discussed in [Lin3].

Let $PD(C) \in H^2(\mathcal{X}, \mathbb{Z})$ denote the poincare dual of the sub-manifold $C \hookrightarrow \mathcal{X}$.

Then we may follow the convention in the ordinary Seiberg-Witten theory and denote $L_k = L + 2kPD(C)$ to be the determinant line bundle of a new $spin^c$ structure by tensoring $L$ with $2k$ power of the complex line bundle determined by $PD(C)$. Then we want to relate the family Seiberg Witten invariants of the $spin^c$ structures $L$ and $L_k = L + 2kPD(C)$.

Let the relative degree of $L$ along $C \hookrightarrow B$ be $m \in N$. As $L$ is a characteristic element restricted fiberwise over $B$, $\int_{C/B}(L+C) = m+n$ must be an even integer.

As in [LL1] we introduce the pure and mixed family Seiberg-Witten invariants and denote them by $FSW_B(1, L)$ and $FSW_B(c, L), c \in H^1(B, \mathbb{Z})$, respectively. Then we have the following main theorem on the switching process:

**Theorem 2** Let $\pi : \mathcal{X} \hookrightarrow B$, $\pi : C \hookrightarrow B$, $L(n) \hookrightarrow B$, $N_C \hookrightarrow B$, $L$, $L_k$ and $m, n$ be as defined above and let the family moduli space dimensions

$$\frac{\int_{X/B} c_1(L)^2 - 2\chi - 3\sigma}{4} + \text{dim}_{\mathbb{R}} B, \frac{\int_{X/B} c_1(L_k)^2 - 2\chi - 3\sigma}{4} + \text{dim}_{\mathbb{R}} B$$

be non-negative, then $\exists \mathbb{V} \hookrightarrow B$, a complex virtual vector bundle over $B$ called the relative obstruction virtual bundle and the following family switching formula relates the family invariants of $spin^c$ structures $L_k$ and $L$,.
\[ F_{SW_B}(c, \mathcal{L}_k) = \sum_{i \geq 0} F_{SW_B}(c_i(V) \cup c, \mathcal{L}). \]

(i). \( \text{rank}_C V = \frac{-km + k^2n}{2}. \)

(ii). The virtual vector bundle \( V \mapsto B \) is virtually isomorphic to (in \( K(B) \))

\[- \oplus_{i \leq k} R \pi_* (N^i_C \otimes \mathcal{P}) \in K(B).\]

The complex line bundle \( \mathcal{P} = \sqrt{L} \otimes K|_C \) is well-defined because \( m + n \equiv 0 \) (mod 2).

\textbf{Remark 1} (i). In defining the index bundle \( R \pi_* (N^i_C \otimes \mathcal{P}) \) of \( \bar{\partial}_N \otimes \mathcal{P} \), the bundle \( N_C \) has been given a holomorphic structure. We have assumed that the map \( C \mapsto X \) to be relative pseudo-holomorphic over \( B \). In terms of Gromov theory, the existence locus of a \( -n \) pseudo-holomorphic rational curve within a generic family is of complex codimension \( 1 - n \). To get a relative pseudo-holomorphic embedding like \( C \) over the whole \( B \), one may restrict the family to be above the existence locus of the rational curves. An alternative way to define a complex structure on \( N_C \) is to view \( N_C \) as the complex line bundle induced by its \( S^1 \) circle bundle.

(ii). In the above theorem, we also assume \( C \mapsto B \) to be smooth, i.e. there is no singular fibers. In the algebraic proof of the family switching formula (see section 4), we will drop this assumption and allow \( \pi : C \mapsto B \) to have singular values.

(iii). In the above main theorem, the virtual rank \( \text{rank}_C R \pi_* (N^i_C \otimes \mathcal{P}) \) is equal to \( \left( \frac{-in + \frac{m+n-2}{2} + 1}{2} \right) = \left( \frac{-in + \frac{m+n}{2}}{2} \right) \) may not be positive. It depends on the sign and the absolute value of the number \( m \).

Therefore the virtual rank of the virtual bundle \( V \) is equal to \( \sum_{i \leq k} \left( \frac{-in - \frac{m+n}{2}}{2} \right) = \frac{-km + k^2n}{2} \), which may not be positive, either.

(iv). Whenever the virtual rank \( \text{rank}_C V \) is negative, the previous equality between the different pure and the mixed family Seiberg-Witten invariants could be re-interpreted as

\[ F_{SW}(c, \mathcal{L}) = \sum_j F_{SW}(c_j(W) \cup c, \mathcal{L}_k), \]

where the virtual vector bundle \( W \) is virtually isomorphic to \(-V\) in the \( K \) group \( K(B) \) and is of positive virtual rank.

Or by using the well known relationship \([F]\) between the Segre classes and the Chern classes, we may re-write the previous relationship as

\[ F_{SW}(c, \mathcal{L}) = \sum_j F_{SW}(s_j(V) \cup c, \mathcal{L}_k). \]
Let us discuss several special cases of the family switching formula before giving derivation of the main theorem.

Suppose that $\mathcal{X} \to B$ is the fiber bundle constructed by blowing up a fiber bundle $X_0 \to B$ along a cross section $s : B \to X_0$. Then $\mathcal{C}$ is nothing but the relative $-1$ two-sphere in $\mathcal{X}$ and the disk bundle $D(N_\mathcal{C}) \to \mathcal{C}$ can be identified with the $\overline{\mathbb{CP}}^2(\mathbb{R}^0 \pi_*(N_\mathcal{C}^+)) \to B$ minus the cross section at infinity. The family switching formula reduces to the family blowup formula studied in [Liu3]. The class $PD(\mathcal{C})$ has been denoted by $E$ in this special situation. Let $\mathcal{L} = \mathcal{L}_d + E, \mathcal{L}_d \perp E$ in $H^2(\mathcal{X}, \mathbb{Z})$ and $\mathcal{L} + 2kE = \mathcal{L}_d + (2k + 1)E$ be the two spin$^c$ structures discussed in the switching formula, then the family switching formula asserts that

$$V = - \oplus_{1 \leq j \leq k} R \pi_*(N_B^j \otimes \mathcal{P}).$$

The $\mathbb{P}^1$ bundle $\mathcal{C} \to B$ can be identified with the projectification $\mathbb{CP}^1(\mathbb{R}^0 \pi_*(N_B^+)) \to B$ and one may apply the family index theorem (essentially the Grothendieck-Riemann-Roch formula in the smooth category) to the map $\pi : \mathcal{C} \to B$. Let $K_0$ denote the restriction of the canonical line bundle around the cross section $s : B \to X_0$ to $s$. By the adjunction formula $K_\mathcal{C} = K_0 \otimes N_\mathcal{C}$ and $\mathcal{L} = \mathcal{L}_d \otimes N_\mathcal{C}$, the complex line bundle $\mathcal{P}$ is isomorphic to $N_\mathcal{C} \otimes \sqrt{\mathcal{L}_d \otimes K_0}$. Therefore, $R \pi_*(N_B \otimes \mathcal{P}) = R \pi_*(N_B^{j+1} \otimes \sqrt{\mathcal{L}_d \otimes K_0})$.

Let $N = \mathbb{R}^0 \pi_*(N^*)$ be the complex rank two bundle of $\overline{\partial}_{N-1}$ holomorphic sections along $\mathcal{C} \to B$. Then by the projection formula of $\mathbb{P}^1$ bundles (see exercise 8.3, 8.4 on [Har] page 253 for the corresponding statement in the algebraic category) the virtual vector bundle $R \pi_*(N_B^{j+1} \otimes \sqrt{\mathcal{L}_d \otimes K_0})$ can be identified with $S^{j-1}(N^* \otimes \sqrt{\mathcal{L}_d \otimes K_0})$ when $j > 0$. The resulting family switching formula is consistent with the family blowup formula derived in [Liu3].

Next let us derive one simple corollary of the family switching formula. Let us consider three different spin$^c$ structures which are related from one to another by consecutive switching multiplicities. Let $\mathcal{L}, \mathcal{L}_{k_1} = \mathcal{L} + 2k_1PD(\mathcal{C}), \mathcal{L}_{k_1+k_2} = \mathcal{L} + 2(k_1 + k_2)PD(\mathcal{C})$ be the three fiberwise invariant spin$^c$ structures on $\mathcal{X}$.

Then we can apply the family switching formula from $\mathcal{L}_1$ to $\mathcal{L}_2$, $\mathcal{L}_2$ to $\mathcal{L}_3$ and $\mathcal{L}_1$ to $\mathcal{L}_3$, respectively.

Let $V_{1.2}$, $V_{2.3}$ and $V_{1.3}$ denote the relative virtual obstruction bundles constructed by the main theorem for the three different switching processes. Then we have,

**Corollary 1** The direct sum of the relative virtual obstruction bundles $V_{1.2} \oplus V_{2.3}$ is virtually isomorphic to $V_{1.3}$ in $K(B)$.

Proof of the corollary: To prove the corollary, one shows that the direct factors in $V_{1.3}$ are in one to one correspondence with the direct factors in the direct sums $V_{1.2} \oplus V_{2.3}$.

This can be achieved by showing that $V_{2.3}$ is isomorphic to $- \oplus_{k_1 < j \leq k_1+k_2} R \pi_*(N_B^j \otimes \mathcal{P})$ while $\mathcal{P}_{1.2} \equiv \sqrt{\mathcal{L} \otimes K_\mathcal{C}}, \mathcal{P}_{2.3} \equiv \sqrt{\mathcal{L}_{k_1} \otimes K_\mathcal{C}}, \mathcal{P}_{1.3} \equiv \sqrt{\mathcal{L} \otimes K_\mathcal{C}}.$
It is apparent that $\mathcal{P}_{1,2} = \mathcal{P}_{1,3}$ and $N^+_C \otimes \mathcal{P}_{2,3} = N^{i+k}_C \otimes \mathcal{P}_{1,3}$. Thus $R \pi_*(N^+_C \otimes \mathcal{P}_{2,3}) = R \pi_*(N^{i+k}_C \otimes \mathcal{P}_{1,3})$ and the equality on $\nu_{1,3}$ follows easily. □

Proof of the main theorem:

By tubular neighborhood theorem, one may embed the disk bundle $D(N_C)$ into $\mathcal{X}$. Thus the fiber bundle $\mathcal{X} \to B$ can be separated by the lens space bundle $L_B(n) = \partial D(N_C) \to B$ into two connected components $\mathcal{X} = D(N_C), D(N_C)$.

Consider the fiberwise 'long neck' Riemannian metrics on the fiber bundle $\mathcal{X} \to B$ which has positive scalar curvature on the subset $D(N_C) \cong D \subset \mathcal{X}$ (viewed as a neighborhood of $C$ in $\mathcal{X}$). Let $\mu$ be a fiberwise self-dual two form on $\mathcal{X} \to B$ vanishes on $D(N_C) \subset \mathcal{X}$.

Fix a fiberwise invariant $spin^c$ structure $\mathcal{L}$ along $\mathcal{X} \to B$ and denote the corresponding positive $spin^c$ spinor bundle by $S^+_L$. The family Seiberg-Witten moduli space of $\mathcal{L}$ is defined to be the set of all tuples of $(A, \Psi, b)$ which satisfy

$$P_+ F_A = q(\Psi, \Psi) + i\mu(b),$$

$$D_A(\Psi) = 0,$$

modulo the equivalence relationship of $U(1)$ gauge transformations on $(A, \Psi)$. For the definition of the quadratic symbol $q(\cdot, \cdot): S^+_L \otimes S^+_L \to (\Omega^1_{X/B})_+$, please consult page 55 of [Mor].

Let $\Omega^i_{X/B}, 0 \leq i \leq 2$, denotes the vector bundles of smooth differential $i$-forms of the fibers, then $(\Omega^2_{X/B})_+$ denote the vector bundle of fiberwise self-dual two forms. Let $\Gamma(\mathcal{X}/B, S^+_L)$ denote the infinite dimensional vector bundles over $B$ of the fiberwise sections of positive (negative) $spin^c$ spinors in $S^+_L (S^-_L)$.

Let $d$ and $d^*$ denote the fiberwise deRham operator and its adjoint and let $P_+, \hat{q}, \mu, c(\cdot)$ denote the infinitesimal change of the operator $P_+, \tau$, the self-dual two form $\mu$ and the Clifford multiplication $c(\cdot)$ under the infinitesimal change of the fiberwise Riemannian metrics induced along a given direction $\in T_b B$. Given a Seiberg-Witten solution $(A_0, \Psi_0)$ of a fiber above the point $b \in B$, the infinitesimal deformation complex of the family Seiberg-Witten equations is given by the following Fredholm operator

$$\begin{bmatrix}
0 & d^* & 0 \\
\hat{q}(A_0, \Psi_0) - i\mu & P_+ d & c(\cdot) \cdot \Psi_0 \\
c(A_0) \cdot \Psi_0 & -2\tau(\Psi_0, \cdot) & D_{A_0}
\end{bmatrix}$$

$$T_b B \oplus \Omega^1_{\mathcal{X}/B} \oplus \Gamma(\mathcal{X}/B, S^+_L) \to \Omega^0_{\mathcal{X}/B} \oplus (\Omega^2_{\mathcal{X}/B})_+ \oplus \Gamma(\mathcal{X}/B, S^-_L).$$

The kernel of the above Fredholm operator is the Zariski tangent space of the family moduli space $\mathcal{M}_C$ at the equivalence class of $(A, \Psi, b)$ and the cokernel of the Fredholm operator is the obstruction space of the solution $(A_0, \Psi_0, b)$. 

8
To determine the obstruction semi-bundle of $\mathcal{M}_\mathcal{L}$, it is reduced to calculate the $H^1$ of the deformation complex over all the equivalence classes $[A, \Psi, b] \in \mathcal{M}_\mathcal{L}$.

By our choice of the fiberwise long neck metrics on $\mathcal{X} \to B$, the spinor $\Psi$ of any Seiberg-Witten solutions $[A, \Psi, b] \in \mathcal{M}_\mathcal{L}$ vanishes along the long neck and the whole $D(\mathcal{N}_\mathcal{L}) \times_B \{b\} \cong \mathcal{D} \times_B \{b\} \subset \mathcal{X} \times_B \{b\}$. On the other hand, the connection $A$ is anti-self-dual on $\mathcal{D} \times_B \{b\} \cong D(\mathcal{N}_\mathcal{L}) \times_B \{b\}$, i.e. $P^+ F_A|_{\mathcal{D} \times_B \{b\}} \equiv 0$ and $A$ is gauge equivalent to the trivial connection on the boundary $\partial(\mathcal{D} \times_B \{b\})$. Such a connection is usually called a reducible connection.

The sketch of the gluing argument of the solutions of the Seiberg-Witten equations (please consult [FS] page 226-227 for some details) allows us to conclude,

**Lemma 1** Let $\mathcal{M}_{\mathcal{L}_p}$, $p \in \mathbb{Z}$ be the family Seiberg-Witten moduli space of the spin$^c$ structure $\mathcal{L} + 2pPD(C)$ over $\mathcal{X} \to B$, with the chosen long neck fiberwise Riemannian metric.

Then the restriction map $(A, \Psi, b) \mapsto (A|_{\mathcal{X} \times_B \{b\} - \mathcal{D} \times_B \{b\}}, \Psi|_{\mathcal{X} \times_B \{b\} - \mathcal{D} \times_B \{b\}}, b)$ establishes an inclusion from $\mathcal{M}_{\mathcal{L}_p}$ to the family Seiberg-Witten moduli space $\mathcal{M}$ of the spin$^c$ structure $\mathcal{L}_p|_{\mathcal{X} - \mathcal{D}}$ on the fiber bundle $\mathcal{X} - \mathcal{D}$ of four-manifolds with long cylindrical ends.

A sketch of the argument: The argument of $B = pt$ case (i.e. for a single four-manifold $M$) for rational blowdowns has been sketched in [FS]. The discussion for the general case is parallel. On the fiber bundle of four-manifolds with long cylindrical ends $\mathcal{D} \to B$ there is a unique (up to gauge equivalences) fiberwise anti-self-dual connection $A_{\text{red}}$ of the line bundle $\mathcal{L}_p|_{\mathcal{D}}$.

Given any solution $(A, \Psi, b) \in \mathcal{M}_{\mathcal{L}_p}$, the restriction $A|_{\mathcal{D} \times_B \{b\}}$ to the fiber $\mathcal{D} \times_B \{b\}$ must be gauge equivalent to the restriction of the unique reducible connection $A_{p, \text{red}} \in A^1(\mathcal{D}, \mathcal{L}_p)$ on the specific fiber $\mathcal{D} \times_B \{b\}$.

On the other hand starting from any solution $\in \mathcal{M}$ of the Seiberg-Witten equations on the four-manifold with a long cylindrical end, $\mathcal{X} \times_B \{b\} - \mathcal{D} \times_B \{b\}$, one may glue it with $(A_{p, \text{red}}|_{\mathcal{D} \times_B \{b\}}, 0)$ to get an approximated solution on the closed four-manifold $\mathcal{X} \times_B \{b\}$. By the general perturbation argument to get the exact solution, the obstruction to get the exact solution $\in \mathcal{M}_{\mathcal{L}_p}$ is determined by a finite rank obstruction bundle over $\mathcal{M}$. □

By lemma 1 we may view all the $\mathcal{M}_{\mathcal{L}_p}$, $1 \leq p \leq k$ as subspaces in the moduli space $\mathcal{M}$ over the fiber bundle $\mathcal{X} - \mathcal{D}$ with cylindrical ends. In the following, $\mathcal{M}$ will serve as the ambient space.

The global $S^1$ gauge transformations induces a tautological $S^1$ bundle over $\mathcal{M}$ and we denote by $e$. Let $e \to \mathcal{M}$ denote the complex line bundle $e \times_{S^1} \mathbb{C}$. The circle bundle $e$ and the line bundle $e$ are universal in the sense that their pull-back to the various $\mathcal{M}_{\mathcal{L}_p}$ are the equal to the tautological bundles defined over $\mathcal{M}_{\mathcal{L}_p}$.

Let $D_{A_{p, \text{red}}}: S^+_{\mathcal{L}_p|_D} \to S^-_{\mathcal{L}_p|_D}$ denote the fiberwise Dirac operator induced by the reducible connection $A_{p, \text{red}}$ on $\mathcal{L}_p|_D$. 

It is not too hard to see by using the infinitesimal deformation complex of the Seiberg-Witten equations and the gluing of Seiberg-Witten solutions that the obstruction bundle defining $\mathcal{M}_{L_p} \subset \mathcal{M}$ is isomorphic to $E \otimes \text{Coker}(D_{A_p,\text{red}})$. In the following, we would like to compare the vector bundles $\text{Coker}(D_{A_p,\text{red}}) \rightarrow B$ and $\text{Coker}(D_{A_{p-1},\text{red}}) \rightarrow B$.

In general, there is an equality

$$\text{rank}_C \text{Coker}(D_{A_p,\text{red}}) - \text{rank}_C \text{Coker}(D_{A_{p-1},\text{red}}) = \frac{1}{2} (\dim_R \mathcal{M}_{L_p} - \dim_R \mathcal{M}_{L_{p-1}}).$$

**Proposition 1** Let $s = \dim_R \mathcal{M}_{L_p} - \dim_R \mathcal{M}_{L_{p-1}} < 0$, then the equivalent class $[\text{Coker}(D_{A_p,\text{red}}) - \text{Coker}(D_{A_{p-1},\text{red}})] \in K(B)$ can be realized as a complex rank $-s$ vector bundle $V_p \rightarrow B$.

Let $s = \dim_R \mathcal{M}_{L_p} - \dim_R \mathcal{M}_{L_{p-1}} > 0$, then the equivalent class $[\text{Coker}(D_{A_{p-1},\text{red}}) - \text{Coker}(D_{A_p,\text{red}})]$ can be realized as a complex rank $s$ vector bundle $V_p \rightarrow B$.

The proof of this proposition will be postponed to subsection 3.2 when we identify $V_p$ explicitly.

Based on the existences of such $V_p$ and its dependence on the difference of family Seiberg-Witten expected dimensions, we derive a schematic form of the family switching formula in the next subsection.

### 3.1 The Comparison of Family Seiberg-Witten Moduli Space Expected Dimensions and Family Switching Formula

Consider the $k$-tuple of integers $\int_X (c_1(L) + (2i - 1)PD(C))$ while $i$ running from $i = 1$ to $i = k$. If these numbers do not change signs at all, then one may discuss directly. If it happens that the numbers change signs, there exists a smallest critical $1 \leq k_{\text{crit}} \leq k$ such that $\int_X (c_1(L) + (2k_{\text{crit}} + 1)PD(C)) \in \mathbb{Z}$ and $\text{int}_{C_1}(L) \in \mathbb{Z}$ are of different signs.

In such situation, we may split the original switching process into two cases and prove the family switching formula from $L$ to $L + 2k_{\text{crit}}PD(C)$ and then from $L + 2k_{\text{crit}}PD(C)$ to $L + 2kPD(C)$.

We start from the following lemma on the relative family Seiberg-Witten moduli space dimensions,

**Lemma 2** Suppose $L + 2(p - 1)PD(C)$ and $L + 2pPD(C)$ are two fiberwise spin$^c$ structures of a fiber bundle $X \rightarrow B$ of smooth four-manifolds. Then the difference of their family Seiberg-Witten moduli space expected dimensions is equal to $\int_X (c_1(L) + (2p - 1)PD(C))$. 
Proof of the lemma: To simplify our notation, define $\mathcal{L}' = \mathcal{L} + 2(p - 1)PD(\mathcal{C})$. Then the two spin$^c$ structures are $\mathcal{L}'$ and $\mathcal{L}' + 2PD(\mathcal{C})$, respectively.

The difference of their family dimensions is given by

$$\left\{ \frac{c_1(\mathcal{L}) + 2PD(\mathcal{C})}{4} - 2\chi - 3\sigma \right\} + dim_{\mathbb{R}}B \right\} - \left\{ \frac{c_1(\mathcal{L}')} + 2\chi - 3\sigma \right\} + dim_{\mathbb{R}}B \right\}$$

$$= \int_{\mathcal{C}} \frac{4c_1(\mathcal{L}') + 4PD(\mathcal{C})}{4} = \int_{\mathcal{C}} (c_1(\mathcal{L}') + PD(\mathcal{C})) = \int_{\mathcal{C}} (c_1(\mathcal{L}) + (2p - 1)PD(\mathcal{C})).$$

The lemma is proved. □

If $k_{cr}$ exists such that $\int_{\mathcal{C}} (c_1(\mathcal{L}) + (2i - 1)PD(\mathcal{C}))$ changes signs, the lemma implies that the family Seiberg-Witten moduli space expected dimensions of $\mathcal{L} + 2iPD(\mathcal{C})$, $0 \leq i \leq k_{cr}$, are monotonically decreasing (increasing), while the family Seiberg-Witten moduli space expected dimensions of $\mathcal{L} + 2iPD(\mathcal{C})$, $k_{cr} + 1 \leq i \leq k$ are monotonically increasing (decreasing).

Because the family Seiberg-Witten moduli spaces of all these spin$^c$ structures defined with the positive scalar curvature long neck metrics are all isomorphic, the monotonicity of the family dimensions implies that there is a real rank $\left| \int_{\mathcal{C}} (c_1(\mathcal{L}) + (2p - 1)PD(\mathcal{C})) \right|$ relative obstruction bundle relating the “adjacent” spin$^c$ structures $\mathcal{L} + 2(p - 1)PD(\mathcal{C})$ and $\mathcal{L} + 2(p)PD(\mathcal{C})$ for different $p$.

The proposition implies that the difference virtual bundle of the obstruction bundles defining $\mathcal{M}_{\mathcal{L}_p - 1}$, $\mathcal{M}_{\mathcal{L}_p}$ in $\mathcal{M}$ can be realized as a rank $\left| \int_{\mathcal{C}} (c_1(\mathcal{L}) + (2p - 1)PD(\mathcal{C})) \right|$ complex vector bundle $\mathbf{e} \otimes_{\mathbb{C}} \mathbf{V}_p$ over $\mathcal{M}$.

Given a complex vector bundle, the Euler class of the underlying real vector bundle is equal to its top Chern class.

If $\int_{\mathcal{C}} (c_1(\mathcal{L}) + (2p - 1)PD(\mathcal{C})) < 0$, then $[\mathcal{M}_{\mathcal{L}_p}] \in H_*(\mathcal{M}, \mathbb{Z})$ is homologous to

$$c_{top}(\mathbf{e} \otimes_{\mathbb{C}} \mathbf{V}_p) \cap [\mathcal{M}_{\mathcal{L}_{p - 1}}] = \left( \sum_i c_1(\mathbf{e})^i \cup c_{top - i}(\mathbf{V}_p) \right) \cap [\mathcal{M}_{\mathcal{L}_{p - 1}}]$$

and we expect a family switching formula of the following form,

$$FSW_B(\eta, \mathcal{L} + 2pPD(\mathcal{C})) = \sum_{i \leq rank_{\mathbb{C}} \mathbf{V}_p} FSW_B(\eta \cup c_i(\mathbf{V}_p), \mathcal{L} + (2p - 1)PD(\mathcal{C})),$$

for $\eta \in H^*(B, \mathbb{Z})$.

For $\int_{\mathcal{C}} (c_1(\mathcal{L}) + (2p - 1)PD(\mathcal{C})) \geq 0$, we expect a family switching formula of the following form,

$$FSW_B(\eta, \mathcal{L} + 2(p - 1)PD(\mathcal{C})) = \sum_{i \leq rank_{\mathbb{C}} \mathbf{V}_p} FSW_B(\eta \cup c_i(\mathbf{V}_p), \mathcal{L} + 2pPD(\mathcal{C})),$$
or equivalently, if we choose \( \eta \) to be a multiple of the total Segre class of \( V_p \) by \( \tilde{\eta} \in H^*(B, \mathbb{Z}) \), \( \eta = s_{\text{total}}(V_p) \cup \tilde{\eta} \), then we have

\[
FSW_B(\tilde{\eta}, \mathcal{L} + 2pPD(C)) = \sum_{i \geq 0} FSW_B(\tilde{\eta} \cup s_i(V_p), \mathcal{L} + 2(p - 1)PD(C)),
\]

by using the relationship

\[
c_{\text{total}}(V_p) \cup s_{\text{total}}(V_p) = 1 \in H^0(B, \mathbb{Z}).
\]

We have the following proposition combining the switching formulae for the adjacent spin* structures,

**Proposition 2** Define the virtual bundle \( V_{1\mapsto k} \) by

\[
V_{1\mapsto k} = \bigoplus_p \int_C (c_1(\mathcal{L}) + (2p - 1)PD(C)) < 0 V_p - \bigoplus_p \int_C (c_1(\mathcal{L}) + (2p - 1)PD(C)) > 0 V_p.
\]

(i). Then there is a family switching formula relating the pure (mixed) family invariants of \( \mathcal{L} + 2kPD(C) \) and \( \mathcal{L} \),

\[
FSW_B(\eta, \mathcal{L} + 2kPD(C)) = \sum_{i \geq \infty} FSW_B(\eta \cup c_i(V_{1\mapsto k}), \mathcal{L}).
\]

(ii). The virtual rank of \( V_{1\mapsto k} \) is equal to half of the difference of the family Seiberg-Witten moduli space expected dimensions,

\[
\sum_{p \leq k} \int_C (c_1(\mathcal{L}) + (2p - 1)PD(C)) = \int_C (k \cdot c_1(\mathcal{L}) + k^2PD(C)).
\]

Proof of the proposition:

(i). Suppose that the signs of \( \int_C (c_1(\mathcal{L}) + (2p - 1)PD(C)) \) do not change when the index \( p \in \mathbb{N} \) runs from 1 to \( k \). Then by induction we have

\[
FSW_B(\eta, \mathcal{L} + 2kPD(C)) = FSW_B(\eta \cup c_{\text{total}}(V_k), \mathcal{L} + (2k - 2)PD(C))
\]

\[
= FSW_B(\eta \cup c_{\text{total}}(V_k \oplus V_{k-1}), \mathcal{L} + 2(k - 2)PD(C)) = \cdots = FSW_B(\eta \cup c_{\text{total}}(V_{1\mapsto k}), \mathcal{L}),
\]

by using the multiplicative property of the total Chern class under direct sums;

or

\[
FSW_B(\eta, \mathcal{L} + 2kPD(C)) = FSW_B(\eta \cup s_{\text{total}}(V_k), \mathcal{L} + (2k - 2)PD(C))
\]

\[
= FSW_B(\eta \cup s_{\text{total}}(V_k \oplus V_{k-1}), \mathcal{L} + 2(k - 2)PD(C)) = \cdots = FSW_B(\eta \cup s_{\text{total}}(-V_{1\mapsto k}), \mathcal{L}),
\]
depending on whether the initial value \( \int_C (c_1(L) + PD(C)) \) is negative or positive.

On the other hand, suppose that the signs of \( \int_C (c_1(L) + (2p - 1)PD(C)) \) do change, a similar inductive calculation from \( L_k \) to \( L_{k+1} \) and then from \( L_{k+1} \) to \( L_{k+1} \) shows that

\[
FSW_B(\eta, L + 2kPD(C)) = FSW_B(\eta \cup c_{total}(\oplus_{p \geq k+1} V_p), L + 2kPD(C))
\]

or

\[
FSW_B(\eta, L + 2kPD(C)) = FSW_B(\eta \cup s_{total}(\oplus_{p \geq k+1} V_p), L + 2kPD(C))
\]

depending on the sign of \( \int_C (c_1(L) + PD(C)) \).

In the discussion, we have used the product formula of the total Chern classes

\[
c_{total}(U_1 \oplus U_2) = c_{total}(U_1) \cup c_{total}(U_2),
\]

and the relationship between the total Segre and the total Chern classes of the virtual vector bundles,

\[
s_{total}(U) = c_{total}(-U).
\]

(ii). The formula on the virtual rank of \( V_{1 \rightarrow k} \) is verified by a direct calculation and by a usage of the simple formula \( \sum_{p \leq k} \frac{2p-1}{2} = k^2 \).

This ends the proof of proposition 2. \( \Box \)

In the following subsection, we prove proposition 1 and identify \( V_p \) with the index bundle of the fiberwise \( \bar{\partial} \) operator over \( C \rightarrow B \).

3.2 The Identification of The Bundle \( V_p \)

Consider the fiber bundle of four-manifolds with cylindrical ends \( D \cong D(N_C) \rightarrow B \) and the anti-self-dual connection \( A_{p,red} \) of \( L_p|_D \). The obstruction bundle defining \( \mathcal{M}_{L_p} \subset \mathcal{M} \) is \( e \otimes Coker(D_{A_p,red}) \).

To relate the differential geometrical calculation about the family index \( IND(D_{A_p,red}) \) to some algebraic geometric datum on \( C \), one considers the \( n \)-th Hirzebruch surface fiber bundle over \( B \) constructed from \( C \) canonically.

Recall that \( F_n \), the \( n \)-th Hirzebruch surface, is the rational ruled algebraic surface which has a \( \mathbb{P}^1 \) bundle structure over \( \mathbb{P}^1 \) with two disjoint cross sections with self-intersection numbers \( n \) and \( -n \) respectively. For all \( n \in \mathbb{N} \cup \{0\} \) the surface \( F_n \) can be constructed as toric varieties. To construct the \( F_n \) bundle
over $B$, we consider the projective space bundle over $C$, $\mathbf{P}(N_C \oplus C) \hookrightarrow C$. Moreover, the fiber bundle $\mathcal{X} \hookrightarrow B$ can be viewed as the familywise fiber sum between $C \subset \mathcal{X} \hookrightarrow B$ and $C \cong \mathbf{P}(N_C) \subset \mathcal{F}_n \equiv \mathbf{P}(N_C \oplus C)$ identified along $C$. The embedded $\mathbf{P}^1$ bundle $C \subset \mathcal{X}$ is the relative $-n$ curve over $B$, while $C \cong \mathbf{P}(N_C) \subset \mathbf{P}(N_C \oplus C)$ is the relative (self-intersection number=) $n$ curve over $B$.

Let $D(N_C)$ denote the unit disk bundle of the normal bundle $N_C$, containing a $\mathbf{P}^1$ sub-fiber bundle of self-intersection number $-n$. Then $\overline{D(N_C)}$ can be viewed as the unit disk bundle of $N_C$, $D(N_C^*)$, containing a $\mathbf{P}^1$ sub-fiber bundle with self-intersection number $n$. Then the $n$-th Hirzebruch surface fiber bundle $\mathbf{P}(N_C \oplus C)$ can be decomposed as the union of the $D(N_C)$ and $\overline{D(N_C)}$ gluing along their common boundary $L(n) = \partial(D(N_C))$ and $\overline{L(n)} = \partial(\overline{D(N_C)})$.

Putting in a cylindrical end $\cong L_B(n) \times (-r, r)$ (and let $r \to \infty$) between $D(N_C)$ and $\overline{D(N_C)}$ by stretching the fiberwise Riemannian metrics of $\mathbf{P}(N_C \oplus C)$, the fiberwise Riemannian metrics defined on $\mathcal{F}_n \hookrightarrow B$ forms a long neck between these two $\mathbf{P}^1$ fiber bundles. For simplicity let us denote the $\mathbf{P}^1$ fiber bundles with the self-intersection number $= n$ ($= -n$) by attaching the subscript $\pm$, $\mathcal{C}_+$ and $\mathcal{C}_-$, respectively.

By using the positive scalar curvature Riemannian metrics on the lens space fiber bundle $L_B(n) \hookrightarrow B$, it is well known that the solutions of the Seiberg-Witten equations must decay exponentially and asymptotic to the reducible solutions along the long neck [FS].

Up to the tensor factor $e$, the obstruction bundle of $\mathcal{M}_{\mathcal{C}_p} \subset \mathcal{M}$ is completely determined by $A_{p, red}$ and it only depends on $D$ instead of the whole $\mathcal{X}$. Thus one may glue $D$ into $\mathcal{F}_n \hookrightarrow B$ instead or equivalently, replace the long-necked fiber bundle $\mathcal{X} \hookrightarrow B$ by the long necked fiber bundle $\mathcal{F}_n \hookrightarrow B$.

One extend $\mathcal{L}_D$, $p \in \mathbb{Z}$ simultaneously into fiberwise $spin^c$ structures $\hat{\mathcal{L}}_p$ on $\mathcal{F}_n \hookrightarrow B$ such that they all match on $\mathcal{F}_n - \mathcal{D}$.

Proof of proposition 1:

Let

$$D_p : \Gamma(\mathcal{F}_n/B, S^+_p) \hookrightarrow \Gamma(\mathcal{F}_n/B, S^-_p)$$

and

$$D_{p-1} : \Gamma(\mathcal{F}_n/B, S^+_p) \hookrightarrow \Gamma(\mathcal{F}_n/B, S^-_{p-1})$$

be some fiberwise Dirac operators on $\mathcal{F}_n \hookrightarrow B$ of the fiberwise $spin^c$ structures $\hat{\mathcal{L}}_p$, and $\hat{\mathcal{L}}_{p-1}$. By the excision property of the family index of Dirac operators and the homotopy equivalence of the index bundles under changes of connections, it suffices to show that when $dim R \mathcal{M}_{\mathcal{L}_p} - dim R \mathcal{M}_{\mathcal{L}_{p-1}} < 0 (i, j)$, the class of the virtual bundle $[IND(D_{p-1}) - IND(D_p)] (||IND(D_p) - IND(D_{p-1})||)$ in $K(B)$ be realized as a complex vector bundle of rank $\frac{1}{2}[dim R \mathcal{M}_{\mathcal{L}_p} - dim R \mathcal{M}_{\mathcal{L}_{p-1}}].$

Apparently if we change the fiberwise Riemannian metrics of the fiber bundle $\mathcal{F}_n \hookrightarrow B$ and the connections on the spinor bundles, it does not affect the above class of virtual family index bundles. Thus, we are free to replace the original
fiberwise long necked Riemannian metrics on \( F_n \to B \) to the fiberwise Kahler metrics on \( F_n \to B \) as \( F_n \) is an algebraic surface. Thus, the family Dirac operator \( D_p \) or \( D_{p-1} \) can be identified with the \( \partial + \bar{\partial}^* \) operator.

Let us review briefly about the cohomology ring structure of a Hirzebruch surface \( F_n \). The Hirzebruch surface \( F_n \to \mathbb{P}^1 \) is a rational ruled surface with two cross sections \( C_+ \) and \( C_- \). Let \( F \) denote the divisor class generated by the fibers \( \mathbb{P}^1 \). The middle cohomology of \( F_n \), \( H^2(F_n, \mathbb{Z}) \) is generated by two classes, \([C_-]\) with \([C_-]^2 = -n\), and \([F]\), \( [F] = 0 \). The class \([C_+]\) is related to \([C_-]\) by \([C_+] = [C_-] + n[F]\). The cone of the effective classes in \( H^2(F_n, \mathbb{Z}) \) is generated by the primitive generators of the extremal rays, \([F]\) and \([C_-]\).

The holomorphic sections vanishing at \( F, C_+, C_- \) define holomorphic line bundles on \( F_n \), denoted by \( E_F, E_{C_+} \) and \( E_{C_-} \). Then \( c_1(E_F) = [F] \), \( c_1(E_{C_+}) = [C_+] \) and \( c_1(E_{C_-}) = [C_-] \), respectively.

**Lemma 3** The first Chern class of the canonical line bundle \( K_{F_n} \) is equal to 
\[-2[F] - [C_+] - [C_-] = -(n+2)[F] - 2[C_-].\]

**Proof of the lemma:** This follows from the canonical divisor formula of any toric surface (see page 85 section 4.3. of [F2]),

\[
K_{F_n} = E_F^{-2} \otimes E_{C_+}^{-1} \otimes E_{C_-}^{-1}.
\]

Given a spin\(^c\) determinant line bundle \( \hat{\mathcal{L}} \) on \( F_n \), there is a unique connection (up to gauge equivalence) which gives \( \hat{\mathcal{L}} \) a structure of holomorphic line bundle. Moreover the spinor bundles \( S_\mathcal{L}^+ \cong (\wedge^{0,0} F_n \otimes \wedge^{0,2} F_n) \otimes \sqrt{\hat{\mathcal{L}}} \otimes K_{F_n} \), \( S_\mathcal{L}^- \cong \wedge^{0,1} F_n \otimes \sqrt{\hat{\mathcal{L}}} \otimes K_{F_n} \), and the spin\(^c\) Dirac operator can be identified with the twisted \( \hat{\partial} + \hat{\partial}^* \) operator by the complex line bundle \( \sqrt{\hat{\mathcal{L}}} \otimes K_{F_n} \) (see [Law] page 395-400).

In our case, suppose that \( \text{deg}(\mathcal{L}[C]) = a, a \equiv n(mod 2) \), then the extension \( \hat{\mathcal{L}} \) over \( F_n \to B \) can be identified as \( E_F^a \otimes E_{C_+}^b \otimes \mathcal{L}_0 \) for some yet to be chosen even \( b \in 2\mathbb{Z} \) and a complex line bundle pull-back from the base \( B \), \( \mathcal{L}_0 \). We have made use of the fact \([C_+] \cup [C_-] = 0\).

Accordingly the complex line bundle \( \sqrt{\hat{\mathcal{L}}} \otimes K_{F_n} \) can be re-expressed as \( E_F^{a+1} \otimes E_{C_+}^{-1} \otimes \sqrt{\mathcal{L}_0} \).

By the projection formula, the index bundle of the \( E_F^{a+1} \otimes E_{C_+}^{-1} \otimes \sqrt{\mathcal{L}_0} \) twisted \( \partial + \bar{\partial}^* \) operator can be thought as the index bundle of the \( E_F^{a+2} \otimes E_{C_+}^{-1} \) twisted \( \partial + \bar{\partial}^* \) operator tensoring with \( \sqrt{\mathcal{L}_0} \).

The following lemma implies that for a suitable choice of \( b \in \mathbb{N} \), the kernel of the \( \partial + \bar{\partial}^* \) operator vanishes.
Lemma 4 Let $F_n$ be the Hirzebruch surface with a toric Kalher metric. Then for all $a \in \mathbb{Z}, a+n \in 2\mathbb{Z}$, there exists at least one $b \in 2\mathbb{Z}$ such that the $E_F^{b,n-1} \otimes E_{C_-}^{b-1}$ twisted $\bar{\partial} + \bar{\partial}^*$ operator

$$\bar{\partial} + \bar{\partial}^* : (\Omega^{0,0} \oplus \Omega^{0,2}) \otimes E_F^{b,n-1} \otimes E_{C_-}^{b-1} \mapsto \Omega^{0,1} \otimes E_F^{b,n-1} \otimes E_{C_-}^{b-1}$$

has a trivial kernel.

Proof of the lemma: Given the toric Kahler metric on $F_n$ and the holomorphic structure defined by $bar\partial$ with $\bar{\partial}^2 = 0$, the kernel of the twisted $\bar{\partial} + \bar{\partial}^*$ operator can be identified with

$$H^0_\beta(F_n, E_F^{a,n-1} \otimes E_{C_+}^{b-1}) \oplus H^2_\beta(F_n, E_F^{a,n-1} \otimes E_{C_-}^{b-1}).$$

It suffices to find $b \in 2\mathbb{Z}$ such that both $\beta$ cohomologies vanish. It is well known that the zero-th and the second $\beta$ cohomologies of the holomorphic line bundle $E_F^{a,n} \otimes E_{C_+}^{b-1}$ are isomorphic to the sheaf cohomology $H^2(F_n, \mathcal{O}_{F_n}((a-n+bn) - 1)F + \left(\frac{b}{2} - 1\right)C_-)$ and $H^2(F_n, \mathcal{O}_{F_n}((a-n+bn - 1)F + \left(\frac{b}{2} - 1\right)C_-)$), respectively.

The second sheaf cohomology is isomorphic to $H^0(F_n, \mathcal{O}_{F_n}((-a-n+bn - 1)F + \left(-\frac{b}{2} - 1\right)C_-)$, by Serre duality on algebraic surfaces and lemma 3.

We make the following choice on $b$. If $\frac{a-n}{2} - 1$ is negative, we simply set $b = 0$. If $\frac{a-n}{2} - 1 \geq 0$, we choose a non-positive even $b$ such that $0 \leq \frac{a-n+bn}{2} - \frac{n}{2} < n$.

Because $F$ and $C_-$ generate the cone of effective divisors on $F_n$, the dimension $h^0(F_n, \mathcal{O}_{F_n}(AF + BC_-))$ is nonzero only when both $A$ and $B$ are non-negative. It is easy to see that our choices of $b$ make both $E_F^{a-n+bn} \otimes E_{C_+}^{b-1}$ and $E_F^{a-n+bn} \otimes E_{C_-}^{b-1}$ non-effective. Thus, the lemma is proved.

By using lemma 4 to choose the appropriate $b \in 2\mathbb{Z}$, the sheaf of sections of the cokernel bundle $\text{Coker} (\bar{\partial} + \bar{\partial}^*)$ can be identified with the first right derived image sheaf of the sheaf of smooth sections $\mathcal{O}_{F_n}((a-n+bn) + 1)F + \left(\frac{b}{2} - 1\right)C_-$ holomorphic along the fibers of $F_n \to B$.

We apply lemma 4 to $L_p$ and identify $C$ with the relative curve $C_- \to B$ in $F_n \to B$. Consider the following sheaf short exact sequence,

$$0 \to \mathcal{O}_{F_n}((a-n+bn) \cdot \frac{2}{2} - 1)F + (\frac{b}{2} - 2)C_- \to \mathcal{O}_{F_n}((a-n+bn) \cdot \frac{2}{2} - 1)F + (\frac{b}{2} - 1)C_- \to \mathcal{O}_{C_-}((a-n+bn \cdot \frac{2}{2} - 1)F + (\frac{b}{2} - 1)C_-) \to 0.$$ 

Taking the right derived image long exact sequence on the short exact sequence and we find that if

$$\int_C (c_1(L) + 2pPD(C) + c_1(K|_C)) = \int_C (c_1(L) + (2p-1)PD(C)) - 2 = a + n - 2 < 0$$

then one has the following sheaf short exact sequence,
Proposition 3 The bundle $\mathbf{V}_p$ is isomorphic to $R^1\pi_*\left(\sqrt{(L \otimes K)_{|C}} \otimes N_p^C\right)$ if \[\int_{\mathcal{C}}(c_1(L) + (2p - 1)PD(C)) < 0.\] If \[\int_{\mathcal{C}}(c_1(L) + (2p - 1)PD(C)) > 0,\] then $\mathbf{V}_p$ is isomorphic to $R^0\pi_*\left(\sqrt{(L \otimes K)_{|C}} \otimes N_p^C\right)$. 

This implies that the difference

$$[R^1\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right)) - R^1\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right))]$$

is equivalent to the locally free $R^1\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right))$. We denote the vector bundle associated with the locally free sheaf as $\mathbf{V}_p$. Suppose that

$$\int_{\mathcal{C}}(c_1(L) + 2pPD(C) + c_1(K_{|C})) = \int_{\mathcal{C}}(c_1(L) + (2p - 1)PD(C)) - 2 = a + n - 2 \geq 0,$$

there is another sheaf short exact sequence,

$$0 \rightarrow R^0\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right)) \rightarrow R^1\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right)) \rightarrow 0.$$

It implies that

$$[R^1\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right)) - R^1\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right))]$$

is equivalent to the locally free $R^0\pi_*\left(\mathcal{O}_{\mathbb{P}^n}(a - n + bn/2 - 1)F + (b/2 - 1)C\right))$. In this case, we denote $\mathbf{V}_p$ as the vector bundle associated with the locally free sheaf.

This ends the proof of proposition 3 \square
Proof of proposition \[x\] Based on the identification at the end of the proof of proposition \[y\] our task is to identify \( O_{\mathcal{C}}((a-n/{2})b-1)F + (b/2-1)C_{-} \) with the sheaf of smooth sections of \( \sqrt{\mathcal{E} \otimes \mathcal{K}}_{\mathcal{C}} \otimes \mathbb{N}_{\mathcal{C}}^p \) holomorphic along the fibers of \( \mathcal{C}_{-} \mapsto B \).

We notice that the line bundle \( L|_{\mathcal{C}} \) over the curve \( \mathcal{C} \mapsto B \) can be identified with \( (E_{p}^{a+n/bn} \otimes E_{b|_{\mathcal{C}}})|_{\mathcal{C}_{-}} \) on the relative divisor \( C_{-} \mapsto B \) inside the relative Hirzebruch surface \( F_{n} \mapsto B \) for an arbitrary \( b \in 2\mathbb{Z} \).

On the other hand, the restriction of the canonical line bundle, \( K|_{\mathcal{C}} \), is identified with

\[
N_{\mathcal{C}}^c \otimes T_{\mathcal{C}}^c = E_{C|_{-}}^{-1} \otimes E_{F|_{-}}^{-2}|_{-} = E_{p|_{-}}^{-2}|_{-}.
\]

Thus \( \sqrt{\mathcal{L}|_{\mathcal{C}} \otimes \mathcal{K}|_{\mathcal{C}}} \) can be identified with \( \sqrt{\mathcal{L}|_{\mathcal{C}} \otimes E_{F|_{-}}^{a+n/bn} \otimes E_{b|_{\mathcal{C}}}}|_{\mathcal{C}_{-}} \).

By using \( E_{p|_{-}} \approx E_{C|_{-}}^{-1}|_{\mathcal{C}_{-}}, \) the above line bundle is isomorphic to \( \sqrt{\mathcal{L}|_{\mathcal{C}} \otimes E_{F}^{a+n/bn-1} \otimes E_{b|_{\mathcal{C}}}}|_{\mathcal{C}_{-}} \).

Up to the factor \( \sqrt{\mathcal{L}|_{\mathcal{C}}} \) pulled back from the base \( B \), the sheaf of smooth sections of the line bundle \( (E_{F}^{a+n/bn-1} \otimes E_{b|_{\mathcal{C}}})|_{\mathcal{C}_{-}} \) holomorphic along \( \mathcal{C}_{-} \mapsto B \) is equal to \( O_{\mathcal{C}_{-}}((a-{n/{2}})b-1)F + (b/2-1)C_{-} \).

On the other hand, we have \( (\mathcal{L} \otimes \mathcal{L}^{-1})|_{\mathcal{C}} \equiv N_{\mathcal{C}}^p \). So \( \sqrt{\mathcal{L}_p \otimes \mathcal{K}}|_{\mathcal{C}} \) is isomorphic to \( \sqrt{\mathcal{L} \otimes \mathcal{K}}|_{\mathcal{C}} \otimes N_{\mathcal{C}}^p \).

Thus, we may identify \( O_{\mathcal{C}_{-}}((a-{n/{2}})b-1)F + (b/2-1)C_{-} \) as the sheaf of sections of \( \sqrt{(\mathcal{L} \otimes \mathcal{K})}|_{\mathcal{C}} \otimes N_{\mathcal{C}}^p \). This ends the proof of the proposition. \( \square \)

**Corollary 2** Let \( \mathcal{L}|_{\mathcal{C}} \) be of degree \( a \) over the \( \mathbb{P}^1 \) bundle \( \mathcal{C} \). Suppose that the \( \mathbb{P}^1 \) fiber bundle \( \mathcal{C} \mapsto B \) is the projecification of a complex rank two vector bundle \( \mathcal{U} \mapsto B \), i.e. \( \mathcal{C} \equiv \mathbb{P}_B(\mathcal{U}) \), then the relative obstruction virtual bundle \( \mathcal{V}_{1 \rightarrow k} \) of the family switching formula can be identified with

\[
\oplus \int_{c}(\mathcal{L} + (2p-1)PD(\mathcal{L})) > 0 S^{a+n/bn-1-pn-1}(\mathcal{U}) \otimes \sqrt{\mathcal{L}|_{\mathcal{C}}} \otimes \int_{c}(\mathcal{L} + (2p-1)PD(\mathcal{L})) < 0 S^{a+n/bn+pn-1}(U^{*}) \otimes \sqrt{\mathcal{L}|_{\mathcal{C}}},
\]

where \( \mathcal{L}|_{\mathcal{C}} \) is a complex line bundle pulled-back from the base \( B \).

Proof: The corollary is a consequence of proposition \[z\], proposition \[x\] and the projection formula of the \( \mathbb{P}_B(\mathcal{U}) \). When \( \int_{c}(\mathcal{L} + (2p-1)PD(\mathcal{L})) > 0 \), it suffices to prove that

\[
\mathbb{R}^0 \pi_{*}(\sqrt{\mathcal{L} \otimes \mathcal{K}|_{\mathcal{C}} \otimes N_{\mathcal{C}}^p}) \cong S^{a+n/bn-pn-1}(\mathcal{U}) \otimes \sqrt{\mathcal{L}|_{\mathcal{C}}},
\]

and when \( \int_{c}(\mathcal{L} + (2p-1)PD(\mathcal{L})) < 0 \), then

\[
\mathbb{R}^1 \pi_{*}(\sqrt{\mathcal{L} \otimes \mathcal{K}|_{\mathcal{C}} \otimes N_{\mathcal{C}}^p}) \cong S^{a+n/bn+pn-1}(U^{*}) \otimes \sqrt{\mathcal{L}|_{\mathcal{C}}}.
\]

These identities follow from the projection formula (consult [Har], page 253, exercise 8.4.(a), (c) for the formula in the algebraic category) of the \( \mathbb{P}^1 \) bundle \( \mathbb{P}_B(\mathcal{U}) \). \( \square \)
By combining the discussions in subsections 3.1 and 3.2, we have derived the family switching formula of family Seiberg-Witten invariants in the topological category.

### 3.3 A Brief Remark About the Family Switching Formula and Gromov-Taubes theory

At the end of the section, we point out the hidden relationship between the obstruction virtual bundle $V_{1 \rightarrow k}$ and the Gromov-Taubes theory. We restrict to the special case $\deg_C - \deg_B \sqrt{L \otimes K} < 0$.

**Proposition 4** Let $\pi : C \rightarrow B$ denote the projection map (or its restriction to certain subsets) of the $\mathbb{P}^1$ fiber bundle which admits a cross section $s_C : B \rightarrow C$. Let $k$ be a positive integer and let $q = \deg_C - \deg_B \sqrt{L \otimes K} < 0$, then there exists a smooth complex line bundle $Q$ on $C$ pulled back from $B$ such that the obstruction virtual bundle $V_{1 \rightarrow k}$ can be alternatively represented in the $K$ group $K(B)$ as

$$
\sum_{k \geq i \geq 1} \left( R^1 \pi_* (N^i_C) + S^{-q-1}(C \oplus N^{-1}_{s_C}) \otimes N^i_C \mid_{s_C(B)} \otimes \right) \otimes Q.
$$

The symbol $N_{s_C}$ denotes the normal bundle of the cross section $s_C \subset C$.

Proof: Define $\Delta = -q \cdot s_C$ to be the non-reduced relative divisor in the relative $\mathbb{P}^1$ bundle. Then there is a short exact sequence relativizing the $0 \rightarrow \mathcal{I}_{\Delta \times B(b)} \rightarrow \mathcal{O}_{\mathbb{C} \times B(b)} \rightarrow \mathcal{O}_{\Delta \times B(b)} \rightarrow 0$.

$$
0 \rightarrow \mathcal{O}_C(-\Delta) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_\Delta \rightarrow 0.
$$

Because $\deg_C - \deg_B \mathcal{O}_C(-\Delta) = q$ as well, the sheaf of fiberwise $\partial$-holomorphic sections of $P = \sqrt{L \otimes K} \mid_C$ is equivalent to $\mathcal{O}_C(-\Delta) \otimes Q$ for some $\mathbb{C}^\infty$ complex line bundle pulled back from $B$.

By tensoring the short exact sequence with $N^i_C \otimes Q$, $1 \leq i \leq k$, and then push forward along $\pi : C \rightarrow B$, the equality of the $K$ group is a consequence of the equality

$$
R^0 \pi_* (\mathcal{O}_{-q} \mid_{s_C}) = \oplus_{0 \leq i \leq -q-1} N^{-i}_{s_C} = S^{-q-1}(\mathcal{O}_B \oplus N^{-1}_{s_C})
$$

for the normal sheaf $N_{s_C}$. $\square$

Up to the factor $Q$ which depends on $\mathcal{L}$, the direct sum

$$
\sum_{k \geq i \geq 1} R^1 \pi_* (N^i_C)
$$

only depends on $k$ and the local data on $C \rightarrow B$. At the end of the subsection, we make a few remark about its implicit link with Gromov-Taubes theory of pseudo-holomorphic curves.
(i). The first term $R^1\pi_*(N_C)$ is nothing but the obstruction vector bundle of $C \hookrightarrow B$ embedded as a pseudo-holomorphic relative curve in a tubular neighborhood inside $\mathcal{X} \hookrightarrow B$.

In fact, for all $b \in B$, the $P^1, C \times_B \{b\}$ can be viewed as an exceptional curve with self-intersection number $= -n$. Then its obstruction vector space $H^1(C \times_B \{b\}, N_C|_{C \times_B \{b\}})$ in Gromov theory is nothing but the fiber of $R^1\pi_*(N_C)$ at $b \in B$.

(ii). Assuming $X \hookrightarrow B$ to be symplectic and $FSW_B(1, L_k) \neq 0$, Taubes’ hard analysis on $SW^- > Gr \ [T1]$ implies the existence of pseudo-holomorphic curves dual to $c_1(\sqrt{L_k} \otimes K_{X/B})$. When $k > 1$, the multiple terms $R^1\pi_*(N^i_C), 1 \leq i \leq k$ reflect that in enumerating the family Seiberg-Witten invariant of $L_k = L + 2kPD(C)$, a $k$-multiple covering of $C$ contributes to the family invariant.

On the other hand, in Taubes’ gluing argument “Gr-\text{SW}” from two dimensional vortices to solutions of Seiberg-Witten equations (see item 5 on page 241 of [T2]), the index bundle of $\Delta_b (N = N_C|_{C \times_B \{b\}}, C = C \times_B \{b\})$,

$$\Delta_b : \oplus_{1 \leq i \leq k} N^i \mapsto (\oplus_{1 \leq i \leq k} N^i \otimes T^{0,1}C)$$

with $\Delta_b(\cdot) = \partial(\cdot) + \ldots$ equal to $\partial$ up to a zeroth order term, can be identified (up to a homotopy of first order elliptic operators) with our $\oplus_{1 \leq i \leq k} R^1\pi_*(N^i_C)$.

This establishes an implicit link between two different gluing problems.

(iii). Even though in Gromov-Taubes theory Taubes did develop a concept of multiple coverings of exceptional curves. But it differs significantly from the usual Ruan-Tian theory [RT]. Suppose that $C \subset M$ is a $-n$ exceptional rational curve in a symplectic four-manifold and let $N_C$ denote the complex normal bundle $C \subset M$. Then the obstruction vector space at $[g_k]$ for a $k$-multiple covering pseudo-holomorphic map $g_k : P^1 \mapsto C$ is $H^1(P^1, g_k^*N_C) \cong H^1(C, N^i_C)$. This corresponds to the $i = k$ term of the bundle $\oplus_{1 \leq i \leq k} R^1\pi_*(N^i_C)$.

In Taubes’ setting, a $k$-fold multiple covering of an exceptional curve is better understood as a zero locus defined by $k$-fold multiple of the zero locus defining $C \subset M$, through the limiting process of large deformations of symplectic forms [T1], [T2], [T3] and the identification of zero loci of Dirac spinors with pseudo-holomorphic curves. In the algebraic category, this can be thought as putting a non-reduced scheme structure on the corresponding algebraic curve. On the other hand, in the usual Gromov-Witten theory a $k$-fold covering of an exceptional curve is viewed as a map which factors through the $k$-fold covering of $P^1 \mapsto P^1$.

The reader should be aware of the difference between these two theories and the corresponding differences on their dimension formulae.
4 The Algebraic Proof of the Family Switching Formula

In section 3 we derived the family switching formula in the topological category. In this section, we derive the family switching formula of the algebraic family invariants $AFSW$. In the topological category, we have to assume that $C \rightarrow B$ is a smooth $\mathbf{P}^1$ bundle over $B$. In general, the family invariants in the smooth category are difficult to determine directly. The usage of fiberwise long neck metrics allow us to study how do the invariants change under switching of fiberwise spin$_c$ structures. On the other hand, the algebraic family Seiberg-Witten invariants can be read off from the topological datum of the algebraic Kuranishi models. This fact has two consequences, (1). One may derive the family switching formula based on the algebraic family Kuranishi models. (2). One may weaken the smoothness assumption on the relative curve $C \rightarrow B$.

Let us begin by introducing our setup, and point out the difference from the case of smooth topology.

Let $\pi : X \rightarrow B$ be an algebraic fiber bundle over a smooth algebraic base $B$ such that the fibers are smooth algebraic surfaces.

Let $C$ be an $(1,1)$ class of the total space $X$ which restricts to the fiberwise $(1,1)$ class on the fibers. Let $C \subset X$ be a relative curve over $B$ satisfying the following conditions:

(A). The generic fibers of $C \rightarrow B$ are smooth rational curves $\mathbf{P}^1$.

(B). The fiberwise self-intersection number $\pi^*(C \cdot C) \in A_{\text{top}}(B)$ is negative, $= -n, n \in \mathbf{N}$.

Recall the concept of formal excess base dimension, $febd(C, X/B)$, which is an algebraic numerical invariant associated to the class $C$ and the family $X \rightarrow B$, giving the formal base dimension to be thickened.

For the two different $(1,1)$ classes $C$ and $C + kPD(C)$, $k \in \mathbf{Z}$, their $febd(C, X/B)$ may not be the same.

By switching the roles of $C$ and $C + kPD(C)$, we may assume that $k > 0$.

We have the following proposition bounding $febd(C, X/B)$,

**Proposition 5** Given two fiberwise $(1,1)$ classes $C$ and $C + kPD(C)$ with $k > 0$, we always have

$$febd(C, X/B) \leq ffebd(C + kPD(C), X/B)$$

Proof of the proposition: As in [Liu3], denote $T_B(X)$ to be the relative $Pic^0$ variety of $\pi : X \rightarrow B$. Let $E_C$ denote the locally free sheaf over $X \times_B T_B(X)$ whose first Chern class is equal to the image of $C \in H^2(X, \mathbf{Z})$ under

$$H^2(X, \mathbf{Z}) \rightarrow H^2(X \times_B T_B(X), \mathbf{Z})$$

21
Then the coherent sheaf $\mathcal{R}^0 \pi_* (\mathcal{E}_C)$ determines an algebraic cone over $\mathcal{T}_B(\mathcal{X})$ and the projectified cone, denoted by $\mathcal{M}_C$, is the algebraic family moduli space associated to $C$.

Let $D_C$ be the universal effective curve over $\mathcal{M}_C$ poincare dual to $C$. Then we have the following short exact sequence,

$$0 \rightarrow \mathcal{O}_{X \times B \mathcal{M}_C}(D_C) \rightarrow \mathcal{O}_{X \times B \mathcal{M}_C}(D_C + k\mathcal{C}) \rightarrow \mathcal{O}_{kC \times B \mathcal{M}_C}(D_C + k\mathcal{C}) \rightarrow 0.$$

Push forward along $\pi_{\mathcal{M}_C} : \mathcal{X} \times B \mathcal{M}_C \rightarrow \mathcal{M}_C$, and we have the following commutative diagram,

$$
\begin{array}{ccc}
\mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C} & \rightarrow & \mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C}(D_C) \\
\downarrow & & \downarrow \\
\mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C} & \rightarrow & \mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C}(D_C + k\mathcal{C}) \\
& & \downarrow \\
& & 0
\end{array}
$$

Recall that $febd(C, \mathcal{X}/B)$ is the rank of the maximal trivial invertible subsheaf of $\mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C}$ annihilated by the sheaf morphism

$$\mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C} \rightarrow \mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C}(D_C),$$

it is obvious from the commutative diagram that $febd(C, \mathcal{X}/B) \leq febd(C + kPD(C), \mathcal{X}/B)$.$\Box$

**Remark 2** From the surjective morphism

$$\mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C}(D_C) \rightarrow \mathcal{R}^2(\pi_{\mathcal{M}_C})_* \mathcal{O}_{X \times B \mathcal{M}_C}(D_C + k\mathcal{C}) \rightarrow 0,$$

it is apparent that the support of the former coherent sheaf contains the support of the latter coherent sheaf as a subset.

Let us state and prove the main theorem of this section.

Recall the definitions as in [Liu3] that the fiber bundle $\pi : \mathcal{X} \rightarrow B$ to be one-relatively good if there exists an effective relatively ample $D \subset \mathcal{X} \rightarrow B$ such that $D \rightarrow B$ is of relative dimension one. The fiber bundle $\pi : \mathcal{X} \rightarrow B$ is said to be two-relatively good if there exists effective relative ample $D_1, D_2 \rightarrow B$ such that (i). $D_1 \rightarrow B$ is of relative dimension one. (ii). $D_1 \cap D_2 \rightarrow B$ is of relative dimension zero over $B$. The purpose of introducing such condition is to ensure the existence of algebraic family Kuranishi models.

**Theorem 3** Let $\pi : \mathcal{X} \rightarrow B$ be an algebraic fiber bundle of algebraic surfaces over a proper and smooth algebraic manifold $B$. Let $C$ be a $(1,1)$ class of $\mathcal{X}$ which restricts to $(1,1)$ classes of the fibers.

Let $\mathcal{C} \subset \mathcal{X}$ be a relative curve over $B$ which satisfies condition (A) and (B) on page 21.
Suppose the following conditions hold for $C$, $C + kC$, $k \in \mathbb{N}$ and $X \mapsto B$,

\[ AF(i). \quad \frac{C \cdot c_1 \left( \left( K_{X/B} \right) \right) + \dim \mathcal{O}}{2} + \text{febd}(C, X/B) \geq 0. \]

\[ AF(ii). \quad \frac{(C+kPD(C)) \cdot c_1 \left( \left( K_{X/B} + C+kPD(C) \right) \right) + \dim \mathcal{O}}{2} + \text{febd}(C+kPD(C), X/B) \geq 0. \]

AF(iii). If the fiber geometric genus $p_g > 0$, the formal excess dimensions of $C$ and $C+kPD(C)$ are equal. Namely, $\text{febd}(C, X/B) = \text{febd}(C+kPD(C), X/B)$.

If $p_g = 0$, the supports of the coherent sheaves $R^2\pi_* \mathcal{E}_C$, $R^2\pi_* \mathcal{E}_{C+kPD(C)}$ coincide.

AF(iv). Suppose $\text{febd}(C, X/B)$ is equal to the maximum value, the geometric genus of the fibers $\pi^{-1}(b), b \in B$, then the fiber bundle $\pi : X \mapsto B$ is one-relatively good.

Otherwise, when $\text{febd}(C, X/B) < p_g(\pi^{-1}(b)), b \in B$, the fiber bundle $\pi : X \mapsto B$ is required to be two-relatively good.

Under these assumptions AF(i)–AF(iv), the pure algebraic family invariants of $C+kPD(C)$ and the mixed family invariant of $C$ are related by the following formula,

\[ AFSW_{X \mapsto B}(1, C+kPD(C)) = \sum_{0 \leq i < \infty} AFSW_{X \mapsto B}(c_i(V_{1 \mapsto k}), C), \]

where the relative obstruction virtual sheaf $V_{1 \mapsto k}$ can be identified with the virtual sheaf

\[ R^1\pi_* \mathcal{O}_{kC}(D_C + kC) - R^0\pi_* \mathcal{O}_{kC}(D_C + kC). \]

Proof: As in [Liu3], we consider $\mathcal{E}_C$ to be the locally free sheaf over $X_B \times_B \mathcal{T}_B(X)$ (under the convention that $\mathcal{T}_B(X) = B$ if $q(\pi^{-1}(b)) = 0, q \in B$) with first Chern class equal to the image of $C \in H^2(X, \mathbb{Z})$ under $H^2(X, \mathbb{Z}) \mapsto H^2(X \times_B \mathcal{T}_B(X), \mathbb{Z})$. Suppose that $R^0\pi_* \mathcal{E}_C$ is the zero sheaf over $\mathcal{T}_C(X)$, then $C$ is not represented as an effective curve within the algebraic family $X \mapsto B$. If $R^0\pi_* \mathcal{E}_{C+kPD(C)}$ is the zero sheaf as well, then it is easy to see that the algebraic mixed invariants $\mathcal{F}_{\text{ASW}}_{X \mapsto B}(\eta, C) = 0, \mathcal{F}_{\text{ASW}}_{X \mapsto B}(\eta, C+kPD(C)) = 0$, for all $\eta \in \mathcal{A}(B)$. Under this assumption, the family switching formula becomes a trivial identity. Thus, one may assume $R^0\pi_* \mathcal{E}_{C+kPD(C)}$ is non-trivial.

In the following we assume $\mathcal{M}_{C+kPD(C)}$, the projectified cone formed by $R^0\pi_* \mathcal{E}_{C+kPD(C)}$ to be non-empty. According to the discussion in [Liu3], one may separate into three cases,

(i). Either $p_g(\pi^{-1}(b)) = 0, b \in B$ and $R^2\pi_* \mathcal{E}_{C+kPD(C)}$ supports over a Zariski closed $Z \subset \mathcal{T}_B(X), Z \neq \emptyset$. Then there exists an algebraic family Kuranishi model of $C+kPD(C)$ over the Zariski open set $U = Z^c$, $(\mathcal{V}, \mathcal{W}, \Phi_{\mathcal{VW}}), \mathcal{V}, \mathcal{W}$ locally free over $U$,

\[ \Phi_{\mathcal{VW}} : \mathcal{V} \mapsto \mathcal{W}, \]

23
a sheaf morphism with
\[ \ker(\Phi_{VW}) \cong R^0(\pi)_*(\mathcal{E}_{C+kP(D,C)})|_U \] and \( \text{Coker}(\Phi_{VW}) \cong R^1(\pi)_*(\mathcal{E}_{C+kP(D,C)})|_U \).

(ii). Or \( p_g(\pi^{-1}(b)) \geq 0, b \in B \) and \( R^2(\pi)_*(\mathcal{E}_{C+kP(D,C)}) \) vanishes. Then there exists an algebraic family Kuranshi model \((V, W, \Phi_{VW}) \) over \( T_B(X) \), \( V \), \( W \) locally free and the sheaf morphism \( \Phi_{VW} : V \to W \).

(iii). Or \( p_g(\pi^{-1}(b)) > 0, b \in B \), and the coherent sheaf \( R^2(\pi)_*(\mathcal{E}_{C+kP(D,C)}) \) is non-vanishing. In such situation, there exists an algebraic family Kuranshi model \((V, W, \Phi_{VW})\), with algebraic locally free \( V, W \) over \( T_B(X) \). The locally free \( W \) is constructed by a pair of locally free \( V, W \).

For the details, please consult [Liu3] section 3.

**Proposition 6** Let \( \underline{C} \) be a \((1,1)\) class of \( \pi : X \to B \) which restricts to fiberwise \((1,1)\) class. In the case (i) above, let \( U \subset T_B(X) \) be the Zariski open set which is the complement of the support of \( R^2(\pi)_*(\mathcal{E}_{\underline{C}}) \). Otherwise, take \( U = T_B(X) \). Then the pure algebraic family Seiberg-Witten invariant \( \mathcal{AFSW}_{X \to B}(1, \underline{C}) \) is equal to \( c_{\dim B+q}(W-V) \in A_0(U) \). Likewise, the mixed family invariant \( \mathcal{AFSW}_{X \to B}(\eta, \underline{C}), \eta \in A_*(B) \) can be identified with \( \eta \cap c_{\dim B+q-deg(\eta)}(W-V) \in A_0(U) \).

Proof: As in [Liu3], we follow the convention that the bold characters \( V, W \) denote the algebraic vector bundle associated with the locally free sheaves \( V, W \), etc. Recall that the family invariant \( \mathcal{AFSW}_{X \to B}(1, \underline{C}) \) is defined to be

\[
c_1^2 - \frac{c_1(K_{X/B})}{2} \cap \dim B + p_g(\eta) \cap \mathcal{C}_{\text{rank} C} W(H \otimes \pi^*_P(V) W) \in A_0(P_U(V))
\]

in both the (i) and (ii) cases, with \( U = Z^c, Z = \text{supp}(R^2(\pi)_*(\mathcal{E}_{C+kP(D,C)})) \), \( p_g = p_g(\pi^{-1}(b)) = 0, b \in B \) in the case (i), \( U = T_B(X) \) in the case (ii).

In case (iii), the algebraic family Seiberg-Witten invariant is defined to be

\[
\mathcal{AFSW}_{X \to B}(1, \underline{C}) = c_1^2 - \frac{c_1(K_{X/B})}{2} \cap \dim B + \text{frbd}(C, X/B)(H) \cap \mathcal{C}_{\text{rank} C} W(H \otimes \pi^*_P(T_B(X))(V) W) \in A_0(P_{T_B(X)}(V)).
\]

In this case, we may take \( U = T_B(X) \).

Recall from [F] page 71 chap. 4 that for a cone \( C_X \) over a scheme \( X \), \( q : P(C_X \oplus 1) \to X \) denotes the projection map, then the Segre class of \( C_X \) is defined by \( \sum_{i \geq 0} c_1(O(1))^i \cap P(C_X \oplus 1) \in A_*(X) \).

Applying to our situation, we take \( C_X \) to be the vector bundle cone formed by \( V \) and take \( q = \pi_{P_U(V)} \circ P_{T_B(X)}(V) \to U \). We also have \( c_1(O(1)) = c_1(H) \).

By using the top Chern class identity

\[
c_{\text{top}}(H \otimes q^* W) = \sum_{j \geq 0} c_j(H)^j \cap \mathcal{C}_{\text{rank} C} W^{-j}(q^* W)
\]

\[
24
\]
Then in case (i) and (ii), we have $AFSW_{X/B}(1, C) =$

$$q_*(c_1(H) \frac{c^2 - c_1(K_{X/B})}{2}^{dim \mathcal{C} + B + p_g} \cap c_{rank \mathcal{C}}w(H \otimes q^*W) \cap [P_U(V)]) \in A_0(U)$$

$$= \sum_{j \geq 0} q_*(c_1(H) \frac{c^2 - c_1(K_{X/B})}{2}^{dim \mathcal{C} + B + p_g} \cap c_1(H)^j \cap c_{rank \mathcal{C}}w(q^*W) \cap [P_U(V)])$$

$$= \sum_{k \geq \frac{c^2 - c_1(K_{X/B})}{2}^{dim \mathcal{C} + B + p_g}} q_*(c_1(H)^k \cap [P_U(V)]) \cap c_{rank \mathcal{C}}w + \frac{c^2 - c_1(K_{X/B})}{2}^{dim \mathcal{C} + B + p_g - k} \cdot (W) \cap \eta, C^{V+1}$$

By using the identity

$$\frac{C^2 - C \cdot c_1(K_{X/B})}{2} + p_g - q + 1 = rank \mathcal{C}V - rank \mathcal{C}W,$$

we may simplify the above summation into

$$\sum_{l \geq \frac{c^2 - c_1(K_{X/B})}{2}^{dim \mathcal{C} + B + q - rank \mathcal{C}W}} s_l(V) \cap c_{dim \mathcal{C}B + q - l}(W) = \sum_{m \leq rank \mathcal{C}W} s_{dim \mathcal{C}B + q - m}(V)c_m(W)$$

$$= c_{dim \mathcal{C}B + q}(W - V) = c_{dim \mathcal{C}B + q}(W - V) \in A_0(U).$$

Likewise, the enumeration of the mixed invariant $AFSW_{X/B}((\eta, C))$ is almost identical. The insertion of $\eta$ effectively drops $dim \mathcal{C}B$ to $dim \mathcal{C}B - deg(\eta)$ and the answer becomes $\eta \cap c_{dim \mathcal{C}B + q - deg(\eta)}(W - V) \in A_0(U)$.

This finishes the proof of the (i). and (ii). cases.

For (iii)., the calculation of the pure invariant or the $\eta$ inserted mixed invariant is almost identical. The major difference is that we replace $p_g$ by the formal excess base dimension $febd(C, X/B)$ and we use the identity

$$\frac{C^2 - C \cdot c_1(K_{X/B})}{2} + febd(C, X/B) - q + 1 = rank \mathcal{C}V - rank \mathcal{C}W$$

in reducing the intersection number. $\square$

Consider the invertible sheaves $E_C$, $E_{C + kPD(C)}$ with first Chern class $C$, $C + kPD(C)$. By tensoring $E_{C + kPD(C)}$ to the short exact sequence
one gets the following short exact sequence

$$0 \to \mathcal{O}_{X \times B T_B(X)}(-kC) \to \mathcal{O}_{X \times B T_B(X)} \to \mathcal{O}_{kC \times B T_B(X)} \to 0,$$

By taking the direct images along $\pi : X \times B T_B(X) \to T_B(X)$, one gets the following identities in the $K_0$ group of coherent sheaves on $T_B(X)$,

$$[\pi_* (\mathcal{E}_{C+kPD(C)})] = [\pi_* (\mathcal{E}_C)] + [\pi_* (\mathcal{O}_{kC \times B T_B(X)} \otimes \mathcal{E}_{C+kPD(C)})],$$

where $\pi_* (\mathcal{E}_C)$ denotes

$$\sum_{0 \leq i \leq 2} (-1)^i R^i \pi_* (\mathcal{E}_C),$$

etc.

**Proposition 7** Let $C$ be a fiberwise monodromy invariant $(1,1)$ class of $X \hookrightarrow B$ and let $(\mathcal{V}, \mathcal{W}, \Phi_{\mathcal{VW}})$ be an algebraic family Kuranishi model of $M_C$ over a Zariski open set $U \subset T_B(X)$, then there exists an equality in the $K$ group of coherent sheaves on $U$ when $p_g = 0$ or $R^2 \pi_* (\mathcal{E}_C) = 0$,

$$[\mathcal{V} - \mathcal{W}] = [\pi_* (\mathcal{E}_C)|_U].$$

There exists an equality in the reduced $K$ group of coherent sheaves on $U$ when $p_g > 0$ and $R^2 \pi_* (\mathcal{E}_C) \neq 0$,

$$[\mathcal{V} - \mathcal{W}] = [\pi_* (\mathcal{E}_C)|_U] - [R^2 \pi_* (\mathcal{O}_{X \times B T_B(X)})|_U].$$

Proof of the proposition: We discuss the (i), (ii). cases first. Recall that the open set $U \subset T_B(X)$ is defined to be the complement of the support of the coherent sheaf $R^2 \pi_* (\mathcal{E}_C)$, then $\pi_* (\mathcal{E}_C)$ is reduced to

$$R^0 \pi_* (\mathcal{E}_C) - R^1 \pi_* (\mathcal{E}_C).$$

On the other hand, for both (i). and (ii)., we have

$$0 \to R^0 \pi_* (\mathcal{E}_C)|_U \mathcal{V} \to \mathcal{W} \to R^1 \pi_* (\mathcal{E}_C)|_U \to 0,$$

by the defining property of the algebraic family Kuranishi model, then we have the identity

$$[\mathcal{V} - \mathcal{W}] = [\pi_* (\mathcal{E}_C)|_C],$$

in the $K_0(U)$.

For the case when $R^2 \pi_* (\mathcal{E}_C) \neq 0$ and $p_g > 0$, we have the following equalities instead,
\[|\mathcal{V}| + |\tilde{\mathcal{V}}| - |\tilde{\mathcal{W}}| = [\pi_* (\mathcal{E}_C)].\]
\[|\mathcal{V} - \mathcal{W}| = |\mathcal{V} + \tilde{\mathcal{V}}| - |\tilde{\mathcal{W}}| - |\mathcal{F}|.\]

where \([\mathcal{F}] = [\sum 2\pi_* (\mathcal{O}_{X \times_B \mathcal{T}_B(\mathcal{X})}) - \mathcal{O}_{\mathcal{T}_B(\mathcal{X})}^{febd}].\) For the details, please consult [Liu3], section 4.

Then up to the trivial factor \(\mathcal{O}_{\mathcal{T}_B(\mathcal{X})}^{febd},\) we have the following equality
\[|\mathcal{V} - \mathcal{W}| = [\pi_* (\mathcal{E}_C)] - [\sum 2\pi_* (\mathcal{O}_{X \times_B \mathcal{T}_B(\mathcal{X})})]\]
in the reduced \(K\) group of \(\mathcal{T}_B(\mathcal{X}).\) This ends the proof of proposition \(\Box\)

Apply the proposition \(6\) and proposition \(7\) to \(C = C + kPD(\mathcal{C}),\) find that
\[\mathcal{AFSW}_{X \rightarrow B}(1, C + kPD(C))\]
is equal to
\[= c_{q + \dim C B} ([\pi_* (\mathcal{E}_{C+kPD(C)})] - [\mathcal{R}^2 \pi_* (\mathcal{O}_{X \times_B \mathcal{T}_B(\mathcal{X})})])\]
when \(febd(C + kPD(C), \mathcal{X}/B) < p_g.\)

The above chern class can be re-written as
\[c_{\dim C B + q} ([\pi_* (\mathcal{E}_C)] + [\pi_* (\mathcal{O}_{kC \times_B X} \times \mathcal{E}_{C+kPD(C)})] - [\mathcal{R}^2 \pi_* (\mathcal{O}_{X \times_B \mathcal{T}_B(\mathcal{X})})]).\]

\[\sum_{r \geq 0} c_r ([\pi_* (\mathcal{O}_{kC \times_B X} \times \mathcal{E}_{C+kPD(C)})]) \cap c_{\dim C B + q - r} ([\pi_* (\mathcal{E}_C)] \cap - [\mathcal{R}^2 \pi_* (\mathcal{O}_{X \times_B \mathcal{T}_B(\mathcal{X})})]).\]

According to the assumption \(\mathcal{AF}(iii)\), we know that \(0 \leq febd(C, \mathcal{X}/B) = febd(C + kPD(\mathcal{C}), \mathcal{X}/B) < p_g\), then apply proposition \(6\) and proposition \(7\) to \(C = C,\) we find that for \(\eta_r = c_r ([\pi_* (\mathcal{O}_{kC \times_B X} \times \mathcal{E}_{C+kPD(\mathcal{C})})]),\) the expression
\[c_r ([\pi_* (\mathcal{O}_{kC \times_B X} \times \mathcal{E}_{C+kPD(\mathcal{C})})]) \cap c_{\dim C B + q - r} ([\pi_* (\mathcal{E}_C)] \cap - [\mathcal{R}^2 \pi_* (\mathcal{O}_{X \times_B \mathcal{T}_B(\mathcal{X})})])\]
is nothing but the mixed invariant \(\mathcal{AFSW}_{X \rightarrow B}(\eta_r, C).\) By combining all the identities, we get
\[\mathcal{AFSW}_{X \rightarrow B}(1, C + kPD(C)) = \sum_{r \geq 0} \mathcal{AFSW}_{X \rightarrow B}(c_r ([\pi_* (\mathcal{O}_{kC \times_B X} \times \mathcal{E}_{C+kPD(\mathcal{C})})]), C).\]

This proves the family switching formula when \(0 \leq febd(C + kPD(\mathcal{C})) < p_g.\)

On the other hand, when \(febd(C + kPD(\mathcal{C}), \mathcal{X}/B) = p_g,\) either \(\mathcal{R}^2 \pi_* (\mathcal{E}_{C+kPD(\mathcal{C})}) = 0\) or \(\mathcal{R}^2 \pi_* (\mathcal{O}_{X \times_B \mathcal{T}_B(\mathcal{X})}) \cong \mathcal{O}_{\mathcal{T}_B(\mathcal{X})}^{p_g}\) and \(\mathcal{F} = 0.\) When \(p_g = 0,\) either \(\mathcal{R}^2 \pi_* (\mathcal{E}_{C+kPD(\mathcal{C})}) = 0\) or the algebraic Kuranishi model of \(C + kPD(\mathcal{C})\) is built above the complement of the coherent sheaf \(\mathcal{R}^2 \pi_* (\mathcal{E}_{C+kPD(\mathcal{C})}) = 0.\)

In all these cases, \(\mathcal{AFSW}_{X \rightarrow B}(1, C + kPD(C))\) has the following schematic form,
the singular values of the map and the ranks $R$

not smooth, neither of $C$

construction bundle appearing in the sheaf $Witten$ invariants, the various Chern classes of the relative obstruction virtual $\pi$

family switching formula. The condition on the formal excess base dimension when the relative curve $C \hookrightarrow p$

$\leq 2$ are locally free $C \mapsto B$ is smooth (as was assumed in the $C^\infty$ version of family switching formula), one can show that the supports of $R^0\pi_*(\mathcal{E}^{+kPD(C)})$ and of $R^2\pi_*(\mathcal{E}^-C)$ do not overlap. By shrinking the open set $U$ to avoid the support of $R^2\pi_*(\mathcal{E}^-C)$, one may prove the family switching formula in the case $p_0 = 0$, $R^2\pi_*(\mathcal{E}^-C) \neq 0$, without assuming the supports of $R^2\pi_*(\mathcal{E}^-C)$ and $R^2\pi_*(\mathcal{E}^{+kPD(C)})$ coincide.

In the statement of the family switching formula of algebraic family Seiberg-Witten invariants, the various Chern classes of the relative obstruction virtual sheaf $\pi_*(\mathcal{O}_{bB}(X) \otimes \mathcal{E}_{C^{+kPD(C)}})$

relates the algebraic family invariants. When the relative curve $C \mapsto B$ is not smooth, neither of $R^i\pi_*(\mathcal{O}_{kC \times B}(X) \otimes \mathcal{E}_{C^{+kPD(C)}})$, $i = 1, 2$ are locally free and the ranks $R^i\pi_*(\mathcal{O}_{kC \times B}(X) \otimes \mathcal{E}_{C^{+kPD(C)}}) \otimes k(b)$ jump as $b$ range through the singular values of the map $C \mapsto B$.

At the end of the section, we discuss its relationship with the relative obstruction bundle appearing in the $C^\infty$ version of the family switching formula.

Consider the short exact sequences

\[
= c_q + dim C B([\pi_*(\mathcal{E}_{C^{+kPD(C)}})]|U])
\]

\[
= c_q + dim C B([\pi_*(\mathcal{E}_C)]|U) + [\pi_* (\mathcal{O}_{kC \times B} \times \mathcal{E}_{C^{+kPD(C)}})]|U])
\]

\[
= \sum_{r \geq 0} c_r([\pi_*(\mathcal{O}_{kC \times B} \times \mathcal{E}_{C^{+kPD(C)}})]|U]) \cap c_q + dim C B - r([\pi_*(\mathcal{E}_C)]|U)),
\]

When $p_0 > 0$, $U$ is taken to be the whole space $T_B(X)$. When $p_0 = 0$, $U$ is taken to be the complement of the support of the sheaf $R^2\pi_*(\mathcal{E}_{C^{+kPD(C)}})$ in $T_B(X)$.

By the assumption $AF(iii)$, the family switching formula $febd(C, X/B) = febd(C^{+kPD(C)}, X/B) = p_0 > 0$ or the support of $R^2\pi_*(\mathcal{E}_C) = U^c \subset T_B(X)$, the above summation can be recasted into

\[
= \sum_{r \geq 0} AFSW_{X \mapsto B}(c_r([\pi_*(\mathcal{O}_{kC \times B} \times \mathcal{E}_{C^{+kPD(C)}})]), C).
\]

This finishes the proof of the theorem. $\square$

**Remark 3** The condition $AF(iii)$ has no counterpart in the $C^\infty$ proof of the family switching formula. The condition on the formal excess base dimension when $p_0 > 0$ is needed as AF$SW$ differ from the usual FSW for families of $p_0 > 0$ algebraic surfaces. The condition on the supports of $R^2\pi_*(\mathcal{E})$ is needed when the relative curve $C \mapsto B$ has singular fibers. When $C \mapsto B$ is smooth (as was assumed in the $C^\infty$ version of family switching formula), one can show that the supports of $R^0\pi_*(\mathcal{E}^{+kPD(C)})$ and of $R^2\pi_*(\mathcal{E}_C)$ do not overlap. By shrinking the open set $U$ to avoid the support of $R^2\pi_*(\mathcal{E}_C)$, one may prove the family switching formula in the case $p_0 = 0$, $R^2\pi_*(\mathcal{E}_C) \neq 0$, without assuming the supports of $R^2\pi_*(\mathcal{E}_C)$ and $R^2\pi_*(\mathcal{E}^{+kPD(C)})$ coincide.
Proof of the proposition: For $n = 2, \cdots, k$, which comes from tensoring $E_{C+nPD(C)}$ to $0 \mapsto \mathcal{O}_{(n-1)C}(-C) \otimes E_{C+nPD(C)} \mapsto \mathcal{O}_{nC} \otimes E_{C+nPD(C)} \mapsto \mathcal{O}_C \otimes E_{C+nPD(C)} \mapsto 0$

with $A = C$, $B = (n-1)C$ being the relative divisors in $\pi : X \mapsto B$.

**Proposition 8** In the $K$ group of coherent sheaves on $B$, there is an equality

$$[\pi_*(\mathcal{O}_{kC \times_B T_B(x)} \otimes E_{C+kPD(C)})] = \sum_{1 \leq p \leq k} [\pi_*(\mathcal{O}_C(pC) \otimes E_C)].$$

Proof of the proposition: For $k = 1$, the formula is an identity once we realize $E_{C+PD(C)} = \mathcal{O}(C) \otimes E_C$.

Assume that by induction hypothesis the equality has been proved for $k-1$, namely,

$$[\pi_*(\mathcal{O}_{(k-1)C \times_B T_B(x)} \otimes E_{C+(k-1)PD(C)})] = \sum_{1 \leq p \leq k-1} [\pi_*(\mathcal{O}_C(pC) \otimes E_C)].$$

By using the above short exact sequences and $E_{C+pPD(C)} = E_C \otimes \mathcal{O}(pC)$, we get

$$[\pi_*(\mathcal{O}_{kC \times_B T_B(x)} \otimes E_{C+kPD(C)})] = [\pi_*(\mathcal{O}_{(k-1)C \times_B T_B(x)} \otimes E_{C+(k-1)PD(C)})] + [\pi_*(\mathcal{O}_{C \times_B T_B(x)}(kC) \otimes E_C)]$$

$$= \sum_{1 \leq p \leq k-1} [\pi_*(\mathcal{O}_C(pC) \otimes E_C)] + [\pi_*(\mathcal{O}_{C \times_B T_B(x)}(kC) \otimes E_C)]$$

$$= \sum_{1 \leq p \leq k} [\pi_*(\mathcal{O}_C(pC) \otimes E_C)].$$

Because equivalent elements in the $K$ group have identical Total Chern classes, we may use $\sum_{1 \leq p \leq k-1} [\pi_*(\mathcal{O}_C(pC) \otimes E_C)]$ in the family switching formula.

If $C \mapsto B$ is smooth, then $\pi_*(\mathcal{O}_C(pC) \otimes E_C)$ is equal to $\mathcal{R}^0 \pi_* (\mathcal{O}_C(pC) \otimes E_C)$
if $\int_C (C + pPD(C)) \geq 0$; it is equal to 0 if $\int_C (C + pPD(C)) = -1$; it is equal to $-\mathcal{R}^1 \pi_* (\mathcal{O}_C(pC) \otimes E_C)$ if $\int_C (C + pPD(C)) < -1$.

Then under the identifications:

$$\mathcal{O}_C(C) \cong \mathcal{N}_C$$

$$\mathcal{E}_C^2 \otimes K^{-1}_{X/B} \cong \mathcal{L}$$

each term $\pi_*(\mathcal{O}_C(pC) \otimes E_C)$ is identified with the locally free sheaf $\mathcal{V}_p$ associated to the relative obstruction sheaf (see prop. 3).
5 The Switching of the Canonical Obstruction Bundles

As in the previous section let $\mathcal{X} \hookrightarrow B$ be an algebraic fiber bundle and let $\mathcal{X} \supset C \hookrightarrow B$ be a $\mathbb{P}^1$ fibration with negative fiberwise self-intersection number $\int_{\mathcal{X}/B} PD(C)^2 < 0$.

The algebraic construction of the family switching formula suggests that when the $(1, 1)$ classes $C$ and $C+kPD(C)$ differ by the $k$-multiple of the Poincare dual class, the algebraic family Seiberg-Witten invariants of $C$ and $C+kPD(C)$ are related to each other through the topological data determined by $C$, the multiplicity $k$ and $C$.

In fact, this is one of the key observations used in the proof of the Göttsche-Yau-Zaslow conjecture [Liu1].

In this section, we construct the switching long exact sequence for the algebraic canonical obstruction bundles of the universal families.

Let $M$ be an algebraic surface over $C$. Let $M_n$ denote the $n$-th universal space constructed in [Y],[Liu1]. Let $\Gamma$ be an $n$-vertexes admissible graph. Following the notation of [Liu1], let $Y(\Gamma) = \prod_{Y \subset \Gamma} Y_\Gamma$. $\prod_{Y \subset \Gamma}$ denote the closure of the locally closed subset $Y_\Gamma \subset M_n$ called the admissible strata. Then $B = Y(\Gamma) \subset M_n$ is the base space of the fiber bundle $\pi = f_n|_{Y(\Gamma)} : Y(\Gamma) \times M_n \rightarrow Y(\Gamma)$.

As a fiber product of $M_{n+1} \rightarrow M_n$ and $Y(\Gamma) \subset M_n$, the space $Y(\Gamma) \times M_n \rightarrow M_{n+1}$ can be constructed from $Y(\Gamma) \times M_n$ by blowing up $n$ consecutive times. Let $E_i$, $1 \leq i \leq n$ denote the $i$-th exceptional divisor of the blowing ups. Then $E_i$ induces a unique cohomology class in $H^{1,1}(Y(\Gamma) \times M_n, M_{n+1}, \mathbb{Z})$. On the other hand, any given $C \in H^{1,1}(M, \mathbb{Z})$ also induces a class in $H^{1,1}(Y(\Gamma) \times M_n, M_{n+1}, \mathbb{Z})$. We slightly abuse the notation and denote them by the same symbols $E_i$, $1 \leq i \leq n$ and $C$.

Consider a topological type of algebraic curve singularity, let $m_i \in \mathbb{N}$ be the multiplicities of the minimal resolution of the curve singularities.

Consider $\Sigma = C - \sum_{1 \leq i \leq n} m_iE_i$, also denoted as $C - \mathcal{M}(E)E$, following the convention in [Liu1].

Over the generic strata $Y_\Gamma$ of the smooth space $Y(\Gamma)$, there are a finite number of irreducible smooth type $I$ exceptional curves in the fibers of $Y_\Gamma \times M_n$, $M_{n+1} \rightarrow Y_\Gamma$ whose cohomology classes are the combinations of the various $E_i$, $e_i = E_i - \sum_j E_j$, where in the summation $\sum_j$ the $j$-th vertexes run through all the direct descendents of the $i$-th vertex in the graph $\Gamma$. All such $e_i$ generate a simplicial cone in $H^{1,1}(Y(\Gamma) \times M_n, M_{n+1}, \mathbb{Z})$.

According to the general curve enumeration scheme in section 3 of [Liu4], there are a finite number of type $I$ exceptional classes $e_k = \sum E_k - \sum_j E_j$, $1 \leq i \leq p$, characterized by the property that $(C - \mathcal{M}(E)E) \cdot e_k < 0$ if and only if $1 \leq i \leq p$. Such $e_k$, for $1 \leq i \leq p$, generate a sub-simplicial cone, called the type $I$ exceptional cone of $\Sigma = C - \mathcal{M}(E)E$ over $Y_\Gamma$.
The complex codimension of $Y(\Gamma) \subset M_n$ is equal to $-\sum_{1 \leq i \leq n} \frac{e_i^2 - e_i c_1(K_{M_n+1/M_n})}{2}$, which is also equal to the number of edges of the admissible graph $\Gamma$.

Each $e_{k_i}$ is represented by holomorphic curves above $Y(\Gamma)$ and it corresponds to a $\mathbb{P}^1$ fibration (may contain singular fibers) $\Xi_{k_i} \rightarrow B$ such that

(i). The generic fibers of $\Xi_{k_i} \rightarrow B$ are smooth $\mathbb{P}^1$.

(ii). The fiberwise self intersection number of $\Xi_{k_i}$ is equal to $e_{k_i} \cdot e_{k_i} < 0$.

Therefore, each of the curve fibrations $\Xi_{k_i}$, $1 \leq i \leq p$ are qualified to be the $C$ in the discussion of the algebraic family Seiberg-Witten switching formula in section 4.

The family switching formula concludes that the family invariants of $\Xi = C - M(E)E$ and of $\Xi - e_{k_1}$, $\Xi - e_{k_1} - e_{k_2}$, $\cdots$, or $\Xi - \sum_{1 \leq i \leq p} e_{k_i}$ are related to each other through the insertion of Chern classes of certain relative obstruction bundles.

In the following, we assume additionally that the class $C$ is raised to a higher multiple such that $\text{deg}_{\omega_M}(C - c_1(K_M)) > 0$. As a consequence of Serre duality, the sheaf $\mathcal{R}^2\pi_* (\mathcal{E}_C) = 0$.

**Lemma 5** Let $X = Y(\Gamma) \times_{M_n} M_{n+1} \rightarrow Y(\Gamma) = B$ be the algebraic fiber bundle. Suppose that

$$\text{deg}_{\omega_B}(C - c_1(K_{X/B})) > 0,$$

then $\mathcal{R}^2\pi_* (\mathcal{E}_{C-M(E)E} - \sum_{1 \leq i \leq p} e_{k_i}) = 0$, $\mathcal{R}^2\pi_* (\mathcal{E}_{C-M(E)E}) = 0$.

**Proof of lemma** Both $C - M(E)E$ and $C - ME(E) - \sum_{1 \leq i \leq p} d_i E_i$ can be schematically written as $C + \sum_{1 \leq i \leq n} d_i E_i$ for some tuples of $d_i \in \mathbb{Z}$.

If either of the sheaves $\mathcal{R}^2\pi_* (\mathcal{E}_{C-M(E)E} - \sum_{1 \leq i \leq p} e_{k_i})$, $\mathcal{R}^2\pi_* (\mathcal{E}_{C-M(E)E})$ is not zero, then there exists a $b \in T(M) \times B$ such that the base change $\mathcal{R}^2\pi_* (\mathcal{E}_C + \sum_{1 \leq i \leq n} d_i E_i) \otimes k(b)$ is nonzero. Then by base change theorem [Ha], the second sheaf cohomology $H^2(X_b, \mathcal{E}_C + \sum_{1 \leq i \leq n} d_i E_i)|_b$ is nonzero. By Serre duality, it implies that $\mathcal{K}_{X_b} \otimes \mathcal{E}_C + \sum_{1 \leq i \leq n} d_i E_i$ has a non-trivial holomorphic section over $X_b$.

Thus by adjunction formula of the canonical class, the class $c_1(K_M) + \sum_{1 \leq i \leq n} (1 - d_i) E_i - C$ is represented by a holomorphic curve in $X_b$.

Let $\omega_0$ be an ample polarization on $M$. Because the smooth fiber algebraic surface $X_b$ is blown up from $(M, \omega_0)$ by $n$-consecutive blowing-ups (with the blowing up centers determined by the image of $b$ in $M_n$), the polarization class on $X_b$ can be chosen to be $\omega_0 - \sum_{1 \leq i \leq n} \epsilon_i E_i$ for some sequence of sufficiently (and arbitrarily) small $\epsilon_i$, $1 \leq i \leq n$, $0 < \epsilon_i \rightarrow 0$.

Then the degree of the effective class $c_1(K_M) + \sum_{1 \leq i \leq n} (1 - d_i) E_i - C$,

$$(\omega_0 - \sum_{1 \leq i \leq n} \epsilon_i E_i) \cdot (c_1(K_M) + \sum_{1 \leq i \leq n} (1 - d_i) E_i - C) = (c_1(K_M) - C) \cdot \omega_0 + \sum_{1 \leq i \leq n} \epsilon_i (d_i - 1) > 0.$$
However this implies that $\text{deg}_{e_\omega}(c_1(K_M) - C) \geq \lim \sum_{1 \leq i \leq n} \varepsilon_i(d_i - 1) = 0$ when we take $\varepsilon_i \to 0$. Contradicting to the assumption $\text{deg}_{e_\omega}(c_1(K_M) - C) = -\text{deg}_{e_\omega}(C - c_1(K_M)) < 0$. □

In the long paper [Liu1], the relationship among the family invariants was read off by using the long exact sequence

$$0 \to R^0\pi_*\left(\mathcal{E}_{C-M(E)}E - \sum_{i\leq p} e_{k_i}\right) \to R^0\pi_*\left(\mathcal{E}_{C-M(E)}E\right) \to R^0\pi_*\left(O_{\sum_{1 \leq i \leq p} e_{k_i}} \otimes \mathcal{E}_{C-M(E)}E\right) \to 0.$$

Following the discussion in section 5.1 of [Liu3], canonical algebraic family Kuranishi models $(V_{\text{canon}}, W_{\text{canon}}, \Phi_{V_{\text{canon}}, W_{\text{canon}}})$ and $(V_{\text{canon}}, W_{\text{canon}}, \Phi_{V_{\text{canon}}, W_{\text{canon}}})$ can be constructed for $C-M(E)$ and $C-M(E) - \sum_{1 \leq i \leq p} e_{k_i}$, respectively.

The following lemma characterizes the $V_{\text{canon}}, V_{\text{canon}}^0$ for both $C-M(E)$ and $C-M(E) - \sum_{1 \leq i \leq p} e_{k_i}$.

**Lemma 6** Let $C$ be a $(1, 1)$ class on $M$ such that $\mathcal{E}_{C-c_1(K_M)}$ is ample on $M$.

Let $(V_{\text{canon}}, W_{\text{canon}}, \Phi_{V_{\text{canon}}, W_{\text{canon}}})$ and $(V_{\text{canon}}, W_{\text{canon}}, \Phi_{V_{\text{canon}}, W_{\text{canon}}})$ denote the canonical algebraic family Kuranishi models of $C = C-M(E)$ and $C = C-M(E) - \sum_{1 \leq i \leq p} e_{k_i}$. Then $V_{\text{canon}} = V_{\text{canon}}^0$ and the bundle $V_{\text{canon}}$ on $T(M) \times M_n$ is the pullback from an algebraic vector bundle over $T(M)$ of rank $\frac{c_2-c_1(K_{M_{n+1}/M_n})}{2} - q(M) + p_g + 1$ by the projection map $T(M) \times M_n \to T(M)$.

Proof of the lemma: Assuming that $\mathcal{E}_{C-c_1(K_M)}$ is ample on $M$, then by Kodaira vanishing theorem the higher derived image sheaves of $\mathcal{E}_{C-c_1(K_M)} \otimes K_M \cong \mathcal{E}_C$ along the projection map $\pi_{T(M)} : M \times T(M) \to T(M)$ vanish.

Thus, $R^0(\pi_{T(M)})_*\mathcal{E}_C$ is locally free of rank $\frac{c_2-c_1(K_M)}{2} - q(M) + p_g(M) + 1$. (The rank is determined by surface Riemann Roch formula)

On the other hand, in the definition 5.3 of [Liu3], both of the bundles $V_{canon}, V_{canon}^0$ are defined to be the algebraic bundles associated to the locally free sheaf $R^0\pi_*\mathcal{E}_C$, where the sheaf $\mathcal{E}_C$ is pulled back from $M \times T(M)$ to $M_{n+1} \times T(M)$ through the composition map

$$M_{n+1} \times T(M) \mapsto M_n \times M \times T(M) \mapsto M \times T(M).$$

Therefore there is a commutative diagram

$$\begin{array}{ccc}
M_{n+1} \times T(M) & \to & M \times T(M) \\
\downarrow & & \downarrow \\
M_n \times T(M) & \to & T(M)
\end{array}$$

Thus, $R^0\pi_*\mathcal{E}_C$ is pulled back from $T(M)$. □
The coherent sheaves $\mathcal{R}^0 \pi_* (\mathcal{E}_{C-M(E) E-\sum e_i})$ and $\mathcal{R}^0 \pi_* (\mathcal{E}_{C-M(E) E})$ determine algebraic cones in the total space of $V_{canon}^o = V_{canon}$ and their projectifications are the algebraic family moduli spaces $\mathcal{M}_{C-M(E) E-\sum e_i} \to M_n \times T(M)$, $\mathcal{M}_{C-M(E) E} \to M_n \times T(M)$.

According to the general construction of algebraic family Kuranishi models, $\mathcal{M}_{C-M(E) E-\sum e_i}$, $\mathcal{M}_{C-M(E) E}$ are sub-schemes of $P_{M_n \times T(M)}(V_{canon})$ which are the zero loci of the canonical sections of the obstruction bundles $H \otimes \pi_{(V_{canon})}^* W_{canon}^o$ and $H \otimes \pi_{(V_{canon})}^* W_{canon}$.

Let $C = C-M(E)E$. Suppose that $\Phi_{V_{canon} W_{canon}^o} : V_{canon} \to W_{canon}^o$ denote the canonical algebraic Kuranishi map, then the canonical section of $H \otimes W_{canon}$ on $P_{M_n \times T(M)}(V_{canon})$, $s_{canon}$, is induced by the bundle map $\Phi_{V_{canon} W_{canon}^o}$ as the following. Each ray in $V_{canon}$ determines a unique image ray in $W_{canon}$ through $\Phi_{V_{canon} W_{canon}^o}$ and induces a map

$$H^* \to \pi_{(V_{canon})}^* W_{canon}^o,$$

or equivalently a map

$$\text{HOM}(C, (H^*)^* \otimes \pi_{(V_{canon})}^* W_{canon}^o) \cong H \otimes \pi_{(V_{canon})}^* W_{canon}.$$

This map can be viewed as the canonical section $s_{canon}$ of $H \otimes \pi_{(V_{canon})}^* W_{canon}$ defining the algebraic family moduli space $\mathcal{M}_C$ as a sub-scheme in $P(V_{canon})$.

The discussion for $C = C-M(E)E-\sum e_i$ and $\Phi_{V_{canon} W_{canon}^o} : V_{canon} \to W_{canon}^o$ is parallel and the corresponding canonical section is denoted by $s_{canon}^o$.

When we restrict to the subspace $Y(\Gamma) \subset M_n$ of the universal space, the classes $e_i$, $e_i \cdot (C-M(E)E) < 0$, $1 \leq i \leq p$ become effective exceptional curve classes. We would like to compare the canonical algebraic obstruction bundles $W_{canon}$ and $W_{canon}^o$.

The main conclusion of the section is the following proposition,

**Proposition 9** Let $(\mathcal{V}_{canon}^o, W_{canon}^o, \Phi_{V_{canon} W_{canon}^o})$ and $(\mathcal{V}_{canon}, W_{canon}, \Phi_{V_{canon} W_{canon}})$ denote the sheaf theoretic version of algebraic canonical family Kuranishi models of $C-M(E)E-\sum e_i$ and $C-M(E)E$, respectively.

While restricting to the smooth subspace $Y(\Gamma) \times T(M) \subset M_n \times T(M)$, there is a commutative diagram of sheaf morphisms between the two algebraic family Kuranishi models.
Proof of the proposition: The isomorphism of $\mathcal{V}_\text{canon}^\circ$ and $\mathcal{V}_\text{canon}$ has been addressed in lemma 11. The rows are portions of the four term exact sequences characterizing the algebraic family Kuranishi model maps. The last (third) column is a portion of the long exact sequence induced from the short exact sequence

$$0 \to \mathcal{E}_{C-M(E)} E - \sum_i e_i \to \mathcal{E}_{C-M(E)} E \to \mathcal{O}_{\sum_i e_i} \otimes \mathcal{E}_{C-M(E)} E \to 0,$$

as was mentioned earlier.

The second column in the commutative diagram is a four term exact sequence on $\mathcal{W}_\text{canon}^\circ$ and $\mathcal{W}_\text{canon}$. By [Liu3], $\mathcal{W}_\text{canon}^\circ$ and $\mathcal{W}_\text{canon}$ are by definition $\mathcal{R}^0 \pi_* (\mathcal{O}_{\sum_{i \leq n} m_i E_i + \sum_{i \leq p} \Xi_{k_i}} \otimes \mathcal{E}_C)$ and $\mathcal{R}^0 \pi_* (\mathcal{O}_{\sum_{i \leq n} m_i E_i} \otimes \mathcal{E}_C)$, respectively.

We start from the following exact sequence

$$0 \to \mathcal{O}_{\sum_{i \leq p} \Xi_{k_i}} (\sum_{i \leq n} m_i E_i) \to \mathcal{O}_{\sum_{i \leq n} m_i E_i + \sum_{i \leq p} \Xi_{k_i}} \to \mathcal{O}_{\sum_{i \leq n} m_i E_i} \to 0$$

of divisors on $M_{n+1} \times M_n \times \mathbb{G}(\Gamma) \times T(M)$ and tensor it by $\mathcal{E}_C$. By using $\mathcal{E}_C \otimes \mathcal{O} (\sum_{i \leq n} m_i E_i) = \mathcal{E}_{C-M(E)} E$, we take its derived image long exact sequence.

Because $\mathcal{R}^1 \pi_* (\mathcal{O}_{\sum_{i \leq n} m_i E_i + \sum_{i \leq p} \Xi_{k_i}} \otimes \mathcal{E}_C) = 0$, the long exact sequence is truncated into a four-term exact sequence. The commutativity of the diagram follows from the naturality of all the sheaf morphisms involved and the compatibilities of the connecting homomorphisms. □

**Proposition 10** Let $H$ denote the (restriction of the) hyperplane bundle of $\mathbb{P}(\mathcal{V}_\text{canon})$. Let $s^\circ_{\text{canon}} \in \Gamma(\mathbb{P}(\mathcal{V}^\circ_{\text{canon}}), H \otimes \pi^*_{\mathbb{P}(\mathcal{V}_{\text{canon}})} \mathcal{W}^\circ_{\text{canon}})$ and $s_{\text{canon}} \in \Gamma(\mathbb{P}(\mathcal{V}_{\text{canon}}), H \otimes \pi^*_{\mathbb{P}(\mathcal{V}_{\text{canon}})} \mathcal{W}_{\text{canon}})$ denote the canonical sections defining $\mathcal{M}_{C-M(E)} E - \sum e_i$, $\mathcal{M}_{C-M(E)} E$.

Then the bundle map

$$H \otimes \pi^*_{\mathbb{P}(\mathcal{V}_{\text{canon}})} \mathcal{W}^\circ_{\text{canon}}|_{\mathbb{G}(\Gamma) \times T(M)} \to H \otimes \pi^*_{\mathbb{P}(\mathcal{V}_{\text{canon}})} \mathcal{W}_{\text{canon}}|_{\mathbb{G}(\Gamma) \times T(M)}$$
induced by the bundle map

\[ W^\circ_{\text{canon}}|Y(\Gamma)\times T(M) \mapsto W_{\text{canon}}|Y(\Gamma)\times T(M) \]

(introduced in the proof of prop. 4) maps \( s^\circ_{\text{canon}} \) to \( s_{\text{canon}} \).

Proof of the proposition: The commutativity of the diagram in prop. 4 implies the following commutative square on the corresponding algebraic vector bundles,

\[
\begin{array}{ccc}
W^\circ_{\text{canon}}|Y(\Gamma)\times T(M) & \xrightarrow{\Phi^\circ_{\text{canon}}W^\circ_{\text{canon}}} & W^\circ_{\text{canon}}|Y(\Gamma)\times T(M) \\
\| & & \| \\
V^\circ_{\text{canon}}|Y(\Gamma)\times T(M) & \xrightarrow{\Phi^\circ_{\text{canon}}W^\circ_{\text{canon}}} & V^\circ_{\text{canon}}|Y(\Gamma)\times T(M)
\end{array}
\]

Then it implies the following commutative squares of bundle maps,

\[
\begin{array}{ccc}
H^* & \rightarrow & \pi^*_{\mathcal{P}(V_{\text{canon}})} W^\circ_{\text{canon}}|Y(\Gamma)\times T(M) \\
\| & & \| \\
H^* & \rightarrow & \pi^*_{\mathcal{P}(V_{\text{canon}})} W_{\text{canon}}|Y(\Gamma)\times T(M)
\end{array}
\]

From this commutative square one derives the conclusion of the proposition immediately. \( \square \)

If \( R^0 \pi_* \left( \mathcal{O}_{\sum \xi_k} \otimes \mathcal{E}_{C-M(E)E} \right) \) is the zero sheaf over \( Y(\Gamma) \times T(M) \), then \( R^1 \pi_* \left( \mathcal{O}_{\sum \xi_k} \right) \) is locally free and \( W^\circ_{\text{canon}}|Y(\Gamma)\times T(M) \mapsto W_{\text{canon}}|Y(\Gamma)\times T(M) \) will be injective.

In this case the zero loci of \( s^\circ_{\text{canon}}|Y(\Gamma)\times T(M) \), \( M_{C-M(E)E-\sum \xi_k} \times M_{\pi_{\text{canon}}} Y(\Gamma) \), and of \( s_{\text{canon}}|Y(\Gamma)\times T(M) \), \( M_{C-M(E)E} \times M_{\pi_{\text{canon}}} Y(\Gamma) \), coincide as subschemes of \( \mathcal{P}(V_{\text{canon}}) \).

In general the kernel sheaf \( R^0 \pi_* \left( \mathcal{O}_{\sum \xi_k} \otimes \mathcal{E}_{C-M(E)E} \right) \) may be non-zero.

To analyze the difference of \( M_{C-M(E)E-\sum \xi_k} \times M_{\pi_{\text{canon}}} Y(\Gamma) \) and \( M_{C-M(E)E} \times M_{\pi_{\text{canon}}} Y(\Gamma) \), one factorizes the map

\[ R^0 \pi_* \left( \mathcal{O}_{\sum_{i \leq n} m_i E_i + \sum_{j \geq p} \xi_k} \otimes \mathcal{E}_C \right) \]

into a sequence of sheaf morphisms,

\[ R^0 \pi_* \left( \mathcal{O}_{\sum m_i E_i + \sum_{i \leq p-r} \xi_k} \otimes \mathcal{E}_C \right) \rightarrow R^0 \pi_* \left( \mathcal{O}_{\sum m_i E_i + \sum_{i \leq p-r-1} \xi_k} \otimes \mathcal{E}_C \right), 0 \leq r \leq p-1, \]

imitating the switching process \( C-M(E)E \mapsto C-M(E)E-e_k \mapsto \cdots \mapsto C-M(E)E-\sum_{i \leq p} e_k \) on the cohomology classes.

The kernel of each of the above morphisms is isomorphic to \( R^0 \pi_* \left( \mathcal{O}_{\sum_{i \leq p-r} \xi_k} \otimes \mathcal{E}_{C-M(E)E-\sum_{i \leq p-r-1} e_k} \right), 0 \leq r \leq p-1. \)

35
Lemma 7  For $0 \leq r \leq p-1$, the sheaf $R^0 \pi_*(\mathcal{O}_{\sum_{i=r+1}^{p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i})$

fits into the following exact sequence,

$$0 \rightarrow R^0 \pi_*(\mathcal{O}_{\sum_{i=r+1}^{p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i}) \rightarrow R^0 \pi_*(\mathcal{O}_{\sum_{i=r+1}^{p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r+1} e_{k_i}) \rightarrow \cdots$$

And for each $r \in \mathbb{Z}$, $0 \leq r \leq p-1$, there exists an equality in the $K$ group of coherent sheaves on $Y(\Gamma) \times T(M)$,

$$[\pi_*(\mathcal{O}_{\sum_{i=r+1}^{p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i})] = [\pi_*(\mathcal{O}_{\sum_{i=r+1}^{p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r+1} e_{k_i})] + [\pi_*(\mathcal{O}_{\Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i})].$$

(A sketch of) the Proof: The proof of the lemma follows from taking the derived long exact sequence of the short exact sequence,

$$0 \rightarrow \mathcal{O}_{\sum_{i=r+1}^{p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r+1} e_{k_i} \rightarrow \mathcal{O}_{\sum_{i=r+1}^{p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i} \rightarrow \mathcal{O}_{\Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i} \rightarrow 0.$$

We omit the details. □

The rational curve fibrations dual to $e_{k_i}$, $\Xi_{k_i}$, $1 \leq i \leq r$ form $\mathbb{P}^1$ fibrations on $Y(\Gamma)$. Over the open strata $Y_T \subset Y(\Gamma)$ all the fibers of these $\Xi_{k_i}$ are smooth and irreducible. When the point specializes to be in $Y(\Gamma) - Y_T$, the fibers of some of the $\Xi_{k_i}$ may break up into more than one irreducible component and different components smooth normal crossing $\mathbb{P}^1$.

The following proposition constraints the support of the kernel sheaf $R^0 \pi_*(\mathcal{O}_{\sum_{i\leq p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E)$ of $W^0_{canon}|Y(\Gamma) \times T(M) \rightarrow W_{canon}|Y(\Gamma) \times T(M)$.

**Proposition 11** Let $Z_{k_i} \subset Y(\Gamma) - Y_T \subset M_{\mathbb{A}}$ be the closed subset consisting of the singular values of $\Xi_{k_i} \rightarrow Y(\Gamma)$ (equivalently, the set over which the fibers of $\Xi_{k_i}$ fail to be irreducible). Then the support of the sheaf $R^0 \pi_*(\mathcal{O}_{\sum_{i\leq p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E)$ is contained in the subset $(\cup_{1 \leq i \leq p} Z_{k_i}) \times T(M)$.

Proof of the proposition: By using lemma and by using induction, the support of $R^0 \pi_*(\mathcal{O}_{\sum_{i\leq p} \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E)$, is contained in the union of the supports

$$\cup_{1 \leq i \leq p} supp(R^0 \pi_*(\mathcal{O}_{\Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i})).$$

It suffices to show that the support of $R^0 \pi_*(\mathcal{O}_{\Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)} E - \sum_{1 \leq i \leq r} e_{k_i}) \subset Z_{k_i} \times T(M)$ and we argue by contradiction.
If not, there exists at least an $i$ with $1 \leq i \leq p$ and a $t \not\in Z_{k_i} \times T(M)$ such that after a base change to $t$ the sheaf cohomology $H^0(\Xi_{k_i}|_t, \mathcal{O}_{\Xi_{k_i}}|_t \otimes \mathcal{E}_{C-M(E)E - \sum_1 \leq j \leq i-1} e_{k_j}) \neq 0$.

Because each pair of distinct $1 \leq a, b \leq p$, $\Xi_{k_a}|_{Y_\Gamma}, \Xi_{k_b}|_{Y_\Gamma}$ co-exist as $\mathbb{P}^1$ fiber bundles over $Y_\Gamma$, then for all $t \in Y_\Gamma$,

$$e_{k_a} \cdot e_{k_b} = PD(\Xi_{k_a}|_t) \cdot PD(\Xi_{k_b}|_t) \geq 0, a \neq b.$$

But this implies that for $\Xi_{k_i}|_t \cong \mathbb{P}^1$,

$$0 \leq \text{deg} \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{E}_{C-M(E)E - \sum_1 \leq j \leq i-1} e_{k_j} = e_{k_i} \cdot (C - M(E)E - \sum_1 \leq j \leq i-1) e_{k_j}$$

$$= e_{k_i} \cdot (C - M(E)E) - e_{k_i} \cdot (\sum_1 \leq j \leq i-1) e_{k_j} \leq e_{k_i} \cdot (C - M(E)E) < 0.$$

Contradiction! $\square$

The proposition singles out the geometric obstruction for $s_{\text{canon}}^o$ and $s_{\text{canon}}$ to define the same zero locus over $Y(\Gamma) \times T(M)$, i.e. $\mathcal{M}_{C-M(E)E - \sum_1 \leq p} e_{k_i} | Y(\Gamma) \times T(M) \equiv \mathcal{M}_{C-M(E)E - \sum_1 \leq p} e_{k_i} | Y(\Gamma) \times T(M)$, to the possibility that the fibers of $\Xi_{k_i}, 1 \leq i \leq p$ can break into more than one irreducible component. This is exactly when the so-called higher level admissible decompositions pop up within the family $\mathcal{X} = M_{n+1} \times M_n \rightarrow Y(\Gamma) \Rightarrow Y(\Gamma) = B$.

When this happens, the restriction of the algebraic family Kuranishi model of $C - M(E)E - \sum_1 \leq p e_{k_i}, (\mathcal{V}_{\text{canon}}^o, \mathcal{W}_{\text{canon}}^o, \Phi_{\mathcal{V}_{\text{canon}}, \mathcal{W}_{\text{canon}}})$, to $Y(\Gamma') \times T(M)$ ($\Gamma' < \Gamma$) is not an accurate approximation of the original algebraic family Kuranishi model ($\mathcal{V}_{\text{canon}}^o, \mathcal{W}_{\text{canon}}^o, \Phi_{\mathcal{V}_{\text{canon}}, \mathcal{W}_{\text{canon}}}$) any more and we have to choose a better approximation over $Y_{\Gamma'} \times T(M), \Gamma' < \Gamma$, through replacing each $e_{k_i}, 1 \leq i \leq p$ by the effective type $I$ exceptional classes $e'_{k_i}, e'_j, (C - M(E)E) < 0, 1 \leq j \leq p'$ irreducible on $Y_{\Gamma'}$. (Consult the definition of $>$ on page 55).

Over $Y(\Gamma)$ the exceptional classes $e_{k_i}, 1 \leq i \leq p$ are all effective. Then we may adjoin any effective curve dual to the class $C - M(E)E - \sum_1 \leq p e_{k_i}$ with the union of type $I$ exceptional curves dual to $\sum_1 \leq i \leq p e_{k_i}$ to produce a curve dual to $C - M(E)E$. This induces an inclusion

$$\mathcal{M}_{C-M(E)E - \sum_1 \leq p} e_{k_i} | Y(\Gamma) \times T(M) \subset \mathcal{M}_{C-M(E)E - \sum_1 \leq p} e_{k_i} | Y(\Gamma) \times T(M).$$

The following proposition characterizes the difference of $\mathcal{M}_{C-M(E)E - \sum_1 \leq p} e_{k_i}$ and $\mathcal{M}_{C-M(E)E}$ in terms of the intersection of $s_{\text{canon}}^o$ with the algebraic cone associated with $\mathbb{R}^0 \pi_*(\mathcal{O}_{\sum \Xi_{k_i}} \otimes \mathcal{E}_{C-M(E)E})$.

**Proposition 12** Let $\mathcal{H}$ denote the invertible sheaf associated with the hyperplane line bundle on the projectification of $\mathcal{V}_{\text{canon}}^o = \mathcal{V}_{\text{canon}}^o, \mathcal{P}_{Y(\Gamma) \times T(M)}(\mathcal{V}_{\text{canon}}^o)$,
Then the locally free sheaf \( \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}} \) determines a vector bundle cone over \( \mathcal{P}_Y(\Gamma) \times T(M)(\mathcal{V}_{\text{canon}}) \), the total space of \( \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times T(M)\).

The sheaf injection

\[
0 \mapsto \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{R}_0^0 \pi_\ast(\mathcal{O}_X \oplus \mathcal{E}C - \mathcal{M}(E)) \mapsto \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times T(M) \mapsto \mathcal{P} \mapsto 0.
\]

Proof of the proposition: To simplify our notations, we will drop the subscript \( Y(\Gamma) \times T(M) \) and denote \( \mathcal{P}_Y(\Gamma) \times T(M)(\mathcal{V}_{\text{canon}}) \equiv \mathcal{P}_Y(\Gamma) \times T(M)(\mathcal{V}_{\text{canon}}) \) by \( \mathcal{P}(\mathcal{V}_{\text{canon}}) \).

Let us recall some basic facts about the construction of algebraic cones. By [F] B.5.5, page 434, the algebraic vector bundle cone determined by a locally free sheaf \( D \) over a scheme \( X \) is \( \text{Spec}(\text{Sym}(D^*)), \) where \( D^* = \mathcal{HOM}_X(D, \mathcal{O}_X) \).

Moreover, for a coherent sheaf \( \mathcal{R} \) over a scheme \( X \), one may construct an algebraic cone over \( X \) by the recipe \( \mathcal{C}(\mathcal{R}) = \text{Spec}(\mathcal{S}^\bullet(\mathcal{R})), \) where \( \mathcal{S}^\bullet = \text{Sym}(\mathcal{R}) \) is the sheaf of graded \( \mathcal{O}_X \) algebra generated by \( \mathcal{S}^1 = \mathcal{R} \) (of grade one).

In the current context, we take \( X = \mathcal{P}_Y(\Gamma) \times T(M)(\mathcal{V}_{\text{canon}}) \). Take \( \mathcal{P} \) to be the image of the sheaf morphism \( g : \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times Y(\Gamma) \mapsto \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times Y(\Gamma) \). Then there is a short exact sequence

\[
0 \mapsto \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{R}_0^0 \pi_\ast(\mathcal{O}_X \oplus \mathcal{E}C - \mathcal{M}(E)) \mapsto \mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times T(M) \mapsto \mathcal{P} \mapsto 0.
\]

Because \( \mathcal{HOM}_X(\cdot, \mathcal{O}_X) \) is a contra-variant left exact functor on the category of sheaves of \( \mathcal{O}_X \) modules, there is a short exact sequence over \( X = \mathcal{P}(\mathcal{V}_{\text{canon}}) \);

\[
0 \mapsto \mathcal{P}^\ast \mapsto (\mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times T(M))^\ast \mapsto \mathcal{R} \mapsto 0,
\]

where \( \mathcal{R} \) is defined to be the cokernel of the injective morphism \( \mathcal{P}^\ast \mapsto (\mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times T(M))^\ast \).

We define the cone \( \mathcal{C}_\rho = \mathcal{C}(\mathcal{R}) \) by the above recipe in [F]. We may take \( \mathcal{D} \) to be the locally free \( (\mathcal{H} \otimes \pi_\ast^\circ \mathcal{P}(\mathcal{V}_{\text{canon}}) \mathcal{W}_{\text{canon}}|Y(\Gamma) \times T(M))^\ast \) and then \( \mathcal{C}(\mathcal{D}) \) is the associated algebraic vector bundle cone.
Proof: By dualizing the exact sequence we get therefore coincides with the original factors through $H \otimes \pi$

$H \otimes \pi$ canonically isomorphic to $H \otimes \pi$

Lemma 8 The dual of the coherent sheaf $R$ satisfies $R^* = \mathcal{H} \otimes \pi^*_t (\mathcal{O}_{\sum_{i \leq p} \Xi_i}) \otimes \mathcal{E}_{C-M(E)E}$.

For all $t \in \mathcal{P}(V_{canon})$, the fiber of the cone, $C_t = C \times \mathcal{P}(V_{canon}) \{ t \}$ at $t$ is canonically isomorphic to $\mathcal{H} \otimes \pi^*_t (\mathcal{O}_{\sum_{i \leq p} \Xi_i}) \otimes \mathcal{E}_{C-M(E)E}$ for all $t$.

The next lemma relates the algebraic sub-cone $C_\rho$ with the coherent sheaf $\mathcal{H} \otimes \pi^*_t (\mathcal{O}_{\sum_{i \leq p} \Xi_i}) \otimes \mathcal{E}_{C-M(E)E}$.

$C_\rho \mid t = \text{Spec}(\text{Sym}(\mathcal{R} \otimes \mathcal{O}_X k(t))) = \text{Spec}(\text{Sym}(\mathcal{R} \otimes \mathcal{O}_X k(t))) = HOM_k(t)(\mathcal{R} \otimes \mathcal{O}_X k(t), k(t))$

Therefore $C_\rho \mid t = \mathcal{H} \otimes \pi^*_t \mathcal{P}(V_{canon}) \mathcal{O}_{\sum_{i \leq p} \Xi_i} \otimes \mathcal{E}_{C-M(E)E}$.

Let $s_{canon}$ and $O$ denote the sub-schemes of the total space of $H \otimes \pi^*_t (\mathcal{O}_{\sum_{i \leq p} \Xi_i}) \otimes \mathcal{E}_{C-M(E)E}$ defined by the sections $s_{canon}$ and the zero section. Likewise set $O^0$ to be the
zero section sub-scheme of the total space of $H \otimes \pi^*_{P(V_{\text{canon}})} W_{\text{canon}} | Y(\Gamma) \times T(M)$.

We notice that the algebraic family moduli space $M_{C-\text{M}(E)E \times M_n} Y(\Gamma)$, defined to be $s_{\text{canon}}^{-1}(\emptyset)$, can be viewed as the intersection $s_{\text{canon}} \cap O$.

Define $g$ to be the algebraic map on the total spaces of the vector bundles induced by $H \otimes \pi^*_{P(V_{\text{canon}})} W_{\text{canon}} | Y(\Gamma) \times T(M) \mapsto H \otimes \pi^*_{P(V_{\text{canon}})} W_{\text{canon}} | Y(\Gamma) \times T(M)$. The proposition $\text{[10]}$ implies that $g(s_{\text{canon}}^\circ) = s_{\text{canon}}$.

Moreover, the vector bundles projection maps induce the following commutative diagram of isomorphisms.

\[
\begin{array}{ccc}
\text{s}_{\text{canon}}^\circ & \xrightarrow{g} & \text{s}_{\text{canon}} \\
\downarrow & & \downarrow \\
\text{O}_{\text{canon}}^\circ & \xrightarrow{g} & \text{O}_{\text{canon}}
\end{array}
\]

On the other hand, lemma $\text{[8]}$ implies that $C_\rho = g^{-1}(O)$. Thus,

\[g(s_{\text{canon}}^\circ \cap C_\rho) = g(s_{\text{canon}}^\circ) \cap g(C_\rho) = s_{\text{canon}} \cap O.
\]

On the scheme theoretic level, the ideal sheaf $I$ defines $s_{\text{canon}}^\circ \cap C_\rho \subset s_{\text{canon}}^\circ$ is the inverse image ideal sheaf $g^{-1}I \cdot O_{s_{\text{canon}}}$ (see [Ha] page 163) of the ideal sheaf $I$ defining $s_{\text{canon}} \cap O \subset s_{\text{canon}}$ under the map $g : s_{\text{canon}}^\circ \mapsto s_{\text{canon}}$.

Since the commutative diagram of isomorphisms imply that $g$ induces an isomorphism between $s_{\text{canon}}^\circ$ and $s_{\text{canon}}$, and therefore $g^{-1}O_{s_{\text{canon}}^\circ} = O_{s_{\text{canon}}}$. Thus, $g^{-1}I \subset g^{-1}O_{s_{\text{canon}}} = O_{s_{\text{canon}}^\circ}$ and $I' = g^{-1}I \cdot O_{s_{\text{canon}}} = g^{-1}I \cong I$.

Thus, the scheme $s_{\text{canon}} \cap O$ and $s_{\text{canon}}^\circ \cap C_\rho$ are isomorphic.

This ends the proof of proposition $\text{[12]}$. \Box

In general a cone can be decomposed into its irreducible components. Therefore, we may write $C_\rho$ as $\sum \rho_i C_{\rho_i}$, where each $C_{\rho_i}$ is an irreducible sub-cone of $C_\rho$. By lemma $\text{[8]}$ the cone $C_{\rho_i}$ always contains an irreducible sub-cone $O^\circ$, the zero section cone of the vector bundle. Then there is a distinguished irreducible component among $C_{\rho_i}$, called $C_{\rho_0} = O^\circ$.

We have the following immediate corollary of proposition $\text{[12]}$

**Corollary 3** Over the closure of the admissible strata, $Y(\Gamma)$, the algebraic family moduli space of curves in $C - \text{M}(E)E$, $M_{C-\text{M}(E)E \times M_n} Y(\Gamma)$ admits a decomposition into

\[
M_{C-\text{M}(E)E \times M_n} Y(\Gamma) = M_{C-\text{M}(E)E} - \sum e_i \times M_n Y(\Gamma) \cup \bigcup_{i \neq 0} \pi_{H \otimes W_{\text{canon}}}(s_{\text{canon}}^\circ \cap C_{\rho_i}).
\]

Each of the term $\pi_{H \otimes W_{\text{canon}}}(s_{\text{canon}}^\circ \cap C_{\rho_i})$ is a closed sub-scheme of $X = P(V_{\text{canon}}^\circ)$. 

40
6 The Localized Top Chern Class and The Contribution to the Family Invariants

At the end of this paper, we would like to discuss some implication of the discussion presented above. Let us begin by reviewing the concept of localized top Chern class of a vector bundle (see [F] p244 for the details).

Let \( E \to X \) be a rank \( e \) vector bundle on a purely \( m \)-dimensional scheme \( X \) and \( s : X \to E \) be a section with zero scheme \( Z(s) \). Let \( s_E \) denote the zero section. Define \( Z(s) = s_E^*(|X|) \in \mathcal{A}_{m-e}(Z(s)) \). It is called the localized top Chern class of \( E \) with respect to \( s \). The most important property of \( Z(s) \) is \( i^*Z(s) = c_e(E) \cap [X] \in \mathcal{A}_{m-e} \) under the inclusion map \( i : Z(s) \to X \).

Namely it is mapped to the (global) top Chern class of \( E \) under the inclusion map \( i \).

If one goes through the definition of \( s_E^* \), one may write \( s_E^*(|X|) \) in an alternative way.

\begin{proposition}
Let \( C_{Z(s)} \) denote the normal cone of \( Z(s) \) in \( X \) and let \( s(Z(s), X) = s(C_{Z(s)}) \) denote the total Segre class of the normal cone. Then the localized top Chern class \( Z(s) \) is equal to \( \{c(i^*E) \cap s(Z(s), X)\}_{m-e} \).

Moreover, for all \( r \in \mathbb{N} \) the cycle class \( \{c(i^*E) \cap s(Z(s), X)\}_{m-e-r} = 0 \).
\end{proposition}

Proof: This is the consequence of the commutative diagram on [F] page 244 and proposition 6.1.(a) [F] page 94. The object \( \{c(i^*E) \cap s(Z(s), X)\}_{m-e-r} \) vanishes in the cycle class group because in the proof of proposition 6.1.(a) [F] \( c_{d+r}(\xi) = 0 \) for \( d = \text{rank}_{C\xi}, r \in \mathbb{N} \). \( \square \)

The constraint on the grading will be used extensively in the following discussion.

Instead of taking \( Z(s) \), we consider \( Y \subset Z(s) \) to be a closed sub-scheme and denote the inclusion \( Y \subset X \) by \( i_Y \). Then the expression \( \{c(i_Y^*E) \cap s(Y, X)\}_{m-e} \) defines a cycle class localized in \( Y \).

\begin{definition}
Define the cycle class \( Z_Y(s) = \{c(i_Y^*E) \cap s(Y, X)\}_{m-e} \) to be the localized top Chern class contribution of \( Y \subset Z(s) \).
\end{definition}

The geometric meaning of this definition is clarified by the following proposition.

\begin{proposition}
Suppose that the zero locus \( Z(s) \) can be decomposed into \( \bigsqcup q Y_q \) such that \( Y_{q_1} \cap Y_{q_2} = \emptyset \), whenever \( q_1 \neq q_2 \) and let \( j_Y : Y_q \to Z(s) \) denote the inclusion map.

Then
\[
\sum_q j_{Y_q}^*Z_Y(s) = Z(s).
\]
\end{proposition}
Proof: When \( Z(s) = \prod_q Y_q \), \( C_Z(s) = \prod_q C_{Y_q} \). Thus \( \mathcal{A}_s(Z(s)) \ni s(Z(s), X) = \sum_q j_{Y_q} \ast s(Y_q, X) \).

Then by projection formula (theorem 3.2(c) on [F] page 50)

\[
Z(s) = \{ c(i^*E) \cap s(Z(s), X) \}_{m-e} = \sum_q \{ c(i^*E) \cap \{ j_{Y_q} \ast s(Y_q, X) \} \}_{m-e}
\]

\[
= \sum_q j_{Y_q} \ast \{ c(i_{Y_q}^*i^*E) \cap s(Y_q, X) \} \}_{m-e} = \sum_q \{ j_{Y_q} \ast (c(i_{Y_q}^*E) \cap s(Y_q, X)) \}_{m-e} = \sum_q j_{Y_q} \ast Z_{Y_q}(s).
\]

\( \square \)

We apply this set up to the family invariant of \( C - M(E), \mathcal{A}_{FSW_{M_n+1 \times T(M)}} \rightarrow M_n \times T(M) \) \( (1, C - M(E)/E) \).

6.1 An Identification Upon the Local Contribution

In our setting, we take \( X = P_{M_n \times T(M)}(V_{canon}) \) and \( E = H \otimes \pi_X^* \text{W}_{canon}. \)

Then the algebraic family moduli space \( \mathcal{M}_{C - M(E)} = Z(s_{canon}) \) and the family invariant \( \mathcal{A}_{FSW_{M_n+1 \times T(M)} \rightarrow M_n \times T(M)} \) \( (1, C - M(E)/E) \) is defined to be

\[
Z \cong \mathcal{A}_0(X) \ni c_{top}(H \otimes \pi_X^* \text{W}_{canon}) \cap c_1(H)^{\text{dim} M_n + \frac{c^2 - C \cdot c_1(K_M)}{2} + p_E - \sum_i \frac{m_i^2 + m_i}{2}}.
\]

As in the earlier sections let \( \Gamma \neq \gamma_n \) be an \( n \)-vertex admissible graph such that all the type I exceptional classes \( e_i, 1 \leq i \leq n \) over \( Y(\Gamma) \) satisfy the following condition,

**Special Condition:**

either

(i). \((C - M(E)/E) \cdot e_i < 0\), i.e. \( M(E)/E \cdot e_i > 0. \)

or

(ii). the condition \( e_i^2 = -1 \), i.e. \( e_i \) is a \(-1\) type I exceptional class.

Let \( k_i, 1 \leq i \leq p \) be the subscripts in \( \{1, 2, \cdots, n\} \) such that \((C - M(E)/E) \cdot e_{k_i} < 0. \) By permuting the indexes we may assume \( k_i = i \) for \( 1 \leq i \leq p. \) From now on we adopt this simplified notation.

Because \( Y(\Gamma) \subset M_n \) is a closed inclusion, the restriction \( \mathcal{M}_{C - M(E)/E} \times M_n \)

\( Y(\Gamma) \) is a closed sub-scheme of \( \mathcal{M}_{C - M(E)/E} = Z(s_{canon}). \) Then we may take

\( Y = \mathcal{M}_{C - M(E)/E} \times M_n \)

\( Y(\Gamma) \) and definition \( [\square] \) determines a localized top Chern class contribution of \( Y \subset \mathcal{M}_{C - M(E)/E}. \)

Then

\[
Z_{Y}(s) \cap c_1(i_Y^*H)^{\text{dim} M_n + \frac{c^2 - C \cdot c_1(K_M)}{2} + p_E - \sum_i \frac{m_i^2 + m_i}{2}} = \{ c(i_Y^*H \otimes \pi_X^* \text{W}_{canon}) \cap s(Y, X) \}^{\text{dim} M_n + \frac{c^2 - C \cdot c_1(K_M)}{2} + p_E - \sum_i \frac{m_i^2 + m_i}{2}} \in \mathcal{A}_0(\mathcal{M}_{C - M(E)/E} \times M_n Y(\Gamma)) \cong Z
\]

gives a localized contribution of the algebraic family invariant over \( Y(\Gamma). \)

42
**Question 1:** Can we express (enumerate) the localized contribution of the (algebraic) family invariant of $C - M(E)E$ over $Y(\Gamma)$ in terms of the family invariant of some other classes?

The complete answer to this question in terms of differential topology has been presented in [Liu1] by using the concept of modified family invariants, the complete solution by a purely algebraic approach is beyond the scope of the present paper. Instead, we try to motivate the readers by providing a light-weighted version which answers the following questions.

**Question 2:** What is the explicit form of the typical family invariant that we express the localized contributions of the family invariant over $Y(\Gamma)$?

**Question 3:** Why does the procedure of enumerating the local contributions of the algebraic family invariants involve the so-called higher level admissible decomposition classes (defined in [Liu1]), i.e. the local contributions from the $Y(\Gamma')$, $\Gamma' < \Gamma$?

The conceptual understanding of both **Question 2** and **Question 3** are essential to understand the solution of **Question 1**.

In corollary 3 on page 40, we have shown that $M_{C-M(E)E} \times M_n Y(\Gamma)$ can be decomposed into $M_{C-M(E)E} - \sum C_i \times M_n Y(\Gamma)$ and $\cup \pi_1 H \otimes \pi_1 W_{\text{canon}} E_{\text{canon}} (C_i \cap s^2_{\text{canon}})$. As our goal is to illustrate the patterns and difficulties involved, we make an additional assumption to simplify the discussion while the general situation without imposing this assumption will be treated elsewhere [Liu5] by using the residual intersection theory.

**Simplifying Assumption:** The space $C_{C-M(E)E} - \sum C_i \times M_n Y(\Gamma)$ and $\cup \pi_1 H \otimes \pi_1 W_{\text{canon}} E_{\text{canon}} (C_i \cap s^2_{\text{canon}})$ are disjoint, i.e. $C_{C-M(E)E} - \sum C_i \times M_n Y(\Gamma)$ and $\cup \pi_1 H \otimes \pi_1 W_{\text{canon}} E_{\text{canon}} (C_i \cap s^2_{\text{canon}}) = \emptyset$.

As a direct consequence of this assumption we may name $Y_1 = C_{C-M(E)E} - \sum C_i \times M_n Y(\Gamma)$, $Y_2 = \cup \pi_1 H \otimes \pi_1 W_{\text{canon}} E_{\text{canon}} (C_i \cap s^2_{\text{canon}})$, and $Y = M_{C-M(E)E} = Y_1 \coprod Y_2$.

Then the argument of proposition 13 implies that (after replacing $Z(s)$ by $Z(s) \times X Y$)

$$Z_Y (s_{\text{canon}}) = jY_1 \ast Z_{Y_1} (s_{\text{canon}}) + jY_2 \ast Z_{Y_2} (s_{\text{canon}}).$$

This implies that the localized contribution of the algebraic family invariant of $C - M(E)E$ over $Y(\Gamma)$ can be decomposed into two parts, one from $jY_1 \ast Z_{Y_1} (s_{\text{canon}}) \cap c_1 (i_1^* H)^{\dim_{C-M_n} + \frac{c^2 - c_1^2 K_m}{2} + p_y - \sum_{i \leq n} \frac{m^2 + m}{2}} \in Z$ and another from $jY_2 \ast Z_{Y_2} (s_{\text{canon}}) \cap c_1 (i_1^* H)^{\frac{c^2 - c_1^2 K_m}{2} + p_y - \sum_{i \leq n} \frac{m^2 + m}{2}} \in Z$.

To answer **Question 1**, we have the following theorem:
Theorem 4 Under the Simplifying Assumption above, the integer \( j_{Y_1}Z_{Y_1}(s_{\text{canon}}) \cap e_1(\gamma^\ast H)^{\dim_C M_n + \frac{C^2 - C,E_1(K_M)}{2} + p_g - \sum_{i \leq n} m_i^2 + m_i} \) can be identified with the mixed family invariant

\[
AFSW_{M_n+1 \times M_n Y(\Gamma) \times T(M)} \to Y(\Gamma) \times T(M) (c_{\text{total}}(\tau), C - M(E) - \sum_{i \leq p} e_i) \]

for some \( \tau \in \mathcal{K}_0(\mathbb{Y}(\Gamma) \times T(M)) \) represented by a locally free sheaf on some Zariski open subset of \( Y(\Gamma) \times T(M) \).

The mixed family invariant is identically zero when there exists an exceptional class \( e_i \) with \( 0 > (C - M(E)) \cdot e_i > e_i^2 \).

Remark 4 The statement in the theorem still holds without the Simplifying Assumption, but it is beyond the reach of the current discussion. The element \( \tau \in \mathcal{K}_0(\mathbb{Y}(\Gamma) \times T(M)) \) admits a locally free representative and its associated vector bundle was called \( \kappa \), the residual relative obstruction bundle on page 443 of [Liu1].

Proof of Theorem Because \( Y_1 = \mathcal{M}_{C - M(E)} \rightarrow \mathbb{Y}(\Gamma) \) and \( Y(\Gamma) \), our goal is to identify \( j_{Y_1}Z_{Y_1}(s_{\text{canon}}) \cap (i_{Y_1}^\ast H) \)

with some mixed family invariant of \( C - M(E) \rightarrow \mathbb{Y}(\Gamma) \) through a nine steps process.

Step I: By definitions of \( m = \dim_C \mathbb{P}(V_{\text{canon}}^\circ) = \dim_C B + q(M) + \text{rank}_C V_{\text{canon}}^\circ - 1 \) and \( e = \text{rank}_C W_{\text{canon}}, Z_{Y_1}(s_{\text{canon}}) = \{ e(i_{Y_1}^\ast (H \otimes \pi_P(V_{\text{canon}})^\ast W_{\text{canon}})) \cap s(Y_1, \mathbb{P}(V_{\text{canon}})) \} \).

Denote the projection map \( Y_1 \rightarrow \mathbb{Y}(\Gamma) \) by \( \pi_{Y_1} \). Firstly, the inclusion \( Y_1 \subset X = \mathbb{P}(V_{\text{canon}}) \) factors through \( Y_1 \subset X \times M_n Y(\Gamma) \subset X \) and it induces the short exact sequence of cones (see page 72, example 4.1.6. in [F] for the definition)

\[
0 \to \pi_{Y_1}^\ast N_{Y(\Gamma)M_n} \to C_{Y_1}X \to C_{Y_1}X \times M_n \mathbb{Y}(\Gamma) \to 0
\]

and it implies the equality

\[
s(Y_1, X) = s(\pi_{Y_1}^\ast N_{Y(\Gamma)M_n} \cap s(Y_1, X \times M_n \mathbb{Y}(\Gamma))).
\]

We plug this identity into the defining formula of \( Z_{Y_1}(s_{\text{canon}}) \) and get

\[
Z_{Y_1}(s_{\text{canon}}) = \{ e(i_{Y_1}^\ast (H \otimes \pi_P(V_{\text{canon}})^\ast W_{\text{canon}})) \cap s(\pi_{Y_1}^\ast N_{Y(\Gamma)M_n}) \cap s(Y_1, X \times M_n \mathbb{Y}(\Gamma)) \} \).
\]

Step II: By lemmas the cone \( U_{\gamma \neq 0} C_{\gamma} \) (excluding the irreducible component \( O^\circ \)) is the locus over which the map \( g \) from the total space of \( H \otimes \pi_P(V_{\text{canon}})^\ast W_{\text{canon}}|Y(\Gamma) \times T(M) \) to the total space of \( H \otimes \pi_P(V_{\text{canon}})^\ast W_{\text{canon}}|Y(\Gamma) \times T(M) \) fails to be injective.
Denote the projection map from $Y_1$ to $Y(\Gamma) \times T(M) \subset M_n \times T(M)$ by $k_{Y_1}$.

The assumption that $Y_1 \cap Y_2 = \emptyset$ implies the injection,

$$0 \mapsto i^*_Y H \otimes k^*_Y W^\infty \mapsto i^*_Y H \otimes k^*_Y W_{cannon}$$

because $Y_2 \subset s^\circ_{cannon}$ the locus over which the map $g|_{s^\circ_{cannon}}$ fails to be injective, is disjoint from $Y_1$.

By using $\pi_{p(V^\infty_{cannon})} \circ i_{Y_1} = k_{Y_1}$, we may rewrite $c(i^*_Y (H \otimes \pi_{p(V^\infty_{cannon})} W_{cannon} /))$

as $c(i^*_Y H \otimes k^*_Y W^\infty_{cannon}) \cap c(i^*_Y H \otimes k^*_Y W_{cannon} /i^*_Y H \otimes k^*_Y W^\infty_{cannon})$.

Step III: As a direct consequence of our Simplifying Assumption on page 43 that the cone $\cup_{i \neq 0} C_{p_i}$ is disjoint from $Y_1$, the sheaf $i^*_Y R^0 \pi_* (\mathcal{O}_{\pi^{-1}(Y^\circ)} (-M(E) \otimes \mathcal{E}_C))$ is trivial and the sheaf $i^*_Y R^1 \pi_* (\mathcal{O}_{\pi^{-1}(Y^\circ)} (-M(E) \otimes \mathcal{E}_C))$ is locally free on $Y_1 = \mathcal{M}C_{-M(E)} E - \sum_{\mathcal{S} \leq \mathcal{P}} \mathcal{S} \times M_n Y(\Gamma)$. We may denote the algebraic vector bundle associated with the first derived image sheaf by $G \mapsto Y_1$. By the commutative diagram in the statement of proposition 18, $i^*_Y (H \otimes G) \cong i^*_Y H \otimes k^*_Y W_{cannon} /i^*_Y H \otimes k^*_Y W^\infty_{cannon}$.

We notice that $Y_1 : Y_1 \subset X$ factors through $\tilde{i}_Y : Y_1 \subset X \times M_n Y(\Gamma) = X'$.

Combining the conclusion of step II and the above identification, we may rewrite

$$\tilde{i}_Y Z_{Y_1} (s_{cannon}) = \tilde{i}_Y s \{ c(i^*_Y H \otimes k^*_Y W^\infty_{cannon}) \cap s(Y_1, X') \cap c(i^*_Y H \otimes G) \cap s(\pi_{Y_1 N Y(\Gamma) M_n}) \}_{m-e}$$

$$= \sum_{-\infty < r \leq \text{dim}_C X'} \tilde{i}_Y s \{ c(i^*_Y H \otimes k^*_Y W^\infty_{cannon}) \cap s(Y_1, X') \}_{m-e+r} \cap \tilde{i}_Y s \{ c(i^*_Y H \otimes G) \cap s(\pi_{Y_1 N Y(\Gamma) M_n}) \}_{\text{dim}_C X'-r}.$$
6.2 The Canonical Algebraic Kuranishi Model of Type I Exceptional Curves

Step V: We construct the canonical algebraic Kuranishi model of a type I exceptional class \( e_i = E_i - \sum_{j_i} E_{j_i} \) as the following.

Let \( \Gamma_{e_i} \) denote the \( n \)-vertex admissible graph such that

(i). the direct descendents of the \( i \)-th vertex are exactly all the \( j_i \)-th vertexes.

(ii). the \( i \)-th vertex is the unique vertex among the \( n \) vertexes which has any direct descendent.

See figure 1 for an example. Then \( Y(\Gamma_{e_i}) \subset M_n \) is the locus over which the class \( e_i \) becomes effective and \( \text{codim}_C Y(\Gamma_{e_i}) \) is equal to the number of 1-edges in the graph \( \Gamma_{e_i} \).

Lemma 9 There exists a canonical sheaf isomorphism on the normal sheaf
\( \mathcal{N}_{Y(\Gamma_{e_i})} M_n \cong R^1 \pi_*(\mathcal{O}_{\Xi}(E_i - \sum_{j_i} E_{j_i})) \).

Proof: The sheaf short exact sequence

\[
0 \rightarrow \mathcal{O}_{M_{n+1} \times M_n Y(\Gamma_{e_i})} \rightarrow \mathcal{O}_{M_{n+1} \times M_n Y(\Gamma_{e_i})}(E_i - \sum_{j_i} E_{j_i}) \rightarrow \mathcal{O}_{\Xi}(E_i - \sum_{j_i} E_{j_i}) \rightarrow 0
\]

implies the following short exact sequence

\[
0 \rightarrow R^1 \pi_*(\mathcal{O}_{M_{n+1} \times M_n Y(\Gamma_{e_i})}) \rightarrow R^1 \pi_*(\mathcal{O}_{M_{n+1} \times M_n Y(\Gamma_{e_i})}(E_i - \sum_{j_i} E_{j_i})) \rightarrow R^1 \pi_*(\mathcal{O}_{\Xi}(E_i - \sum_{j_i} E_{j_i})) \rightarrow 0.
\]

We notice \( R^1 \pi_*(\mathcal{O}_{M_{n+1} \times M_n Y(\Gamma_{e_i})}) \cong H^1(M, \mathcal{O}_M) \otimes \mathcal{O}_{Y(\Gamma_{e_i})} \).

Consider the sheaf short exact sequence

\[
0 \rightarrow \mathcal{O}_{M_{n+1}}(E_i - \sum_{j_i} E_{j_i}) \rightarrow \mathcal{O}_{M_{n+1}}(E_i) \rightarrow \mathcal{O}_{\sum_{j_i} E_{j_i}}(E_i) \rightarrow 0
\]
and take the right derived long exact sequence along $π = f_n : M_{n+1} \mapsto M_n$, we get the following five-term sheaf exact sequence over $M_n$,

$$0 \mapsto R^0 π_* (O_{M_{n+1}}(E_i - \sum_{j_i} E_{j_i})) \mapsto O_{M_{n}} \mapsto R^0 π_* (O_{\sum_{j_i} E_{j_i}}(E_i)) \mapsto R^1 π_* (O_{M_{n+1}}(E_i - \sum_{j_i} E_{j_i})) \mapsto R^1 π_* (O_{M_{n+1}}(E_i)) \mapsto 0.$$

Similar to the discussion in proposition 5.3 of [Liu3], the sheaf $R^0 π_* (O_{\sum_{j_i} E_{j_i}}(E_i))$ is locally free with its rank equal to the number of 1-edges in $Γ_{e_i}$. It is the canonical obstruction bundle of the type I class $e_i$.

By using

$$O_Y(Γ_{e_i}) \otimes H^1(M, O_M) \cong R^1 π_* (O_{M_{n+1} \times M_n} Y(Γ_{e_i})) \cong R^1 π_* (O_{M_{n+1}}(E_i))|_{Y(Γ_{e_i})},$$

it is easy to modify the above five-term exact sequence to the following canonical algebraic Kuranishi model of $e_i = E_i - \sum_{j_i} E_{j_i}$,

$$0 \mapsto R^0 π_* (O_{M_{n+1}}(E_i - \sum_{j_i} E_{j_i})) \mapsto O_{M_{n}} \mapsto R^0 π_* (O_{\sum_{j_i} E_{j_i}}(E_i)) \mapsto R^1 π_* (O_{M_{n+1}}(E_i - \sum_{j_i} E_{j_i}))/R^1 π_* (O_{M_{n+1}}) \mapsto 0.$$

The morphism $O_{M_{n}} \cong R^0 π_* (O_{M_{n+1}}(E_i)) \mapsto R^0 π_* (O_{\sum_{j_i} E_{j_i}}(E_i))$ defines a canonical section whose zero locus $= Y(Γ_{e_i})$. By proposition 4.3. in [Liu1], the space $Y(Γ_{e_i}) \subset M_n$ is smooth and its codimension in $M_n$ matches with the rank of the locally free sheaf $R^0 π_* (O_{\sum_{j_i} E_{j_i}}(E_i))$. Thus the algebraic section defining $Y(Γ_{e_i})$ is regular. An immediate consequence of the regularity of the section is the following identification of the normal sheaf

$$N_{Y(Γ_{e_i})M_n} \cong R^0 π_* (O_{\sum_{j_i} E_{j_i}}(E_i))|_{Y(Γ_{e_i})} \cong R^1 π_* (O_{M_{n+1}}(E_i - \sum_{j_i} E_{j_i}))/R^1 π_* (O_{M_{n+1}})|_{Y(Γ_{e_i})} \cong R^1 π_* (O_{E_i} (E_i - \sum_{j_i} E_{j_i})).$$

The lemma is proved. □

6.3 Short Exact Sequences on $N_{Y(Γ_{e_i})M_n}$

Step VI: Consider the $\mathbb{P}^1$ fibration $Ξ_i \mapsto Y(Γ_{e_i})$ and the projection map $π : T(M) \times Ξ_i \mapsto T(M) \times Y(Γ_{e_i})$ induced from $π : M_{n+1} \times M_n Y(Γ_{e_i}) \mapsto Y(Γ_{e_i})$. Then we have the following proposition,
Proposition 15 Let $e_i$ be a type I exceptional class satisfying $e_i \cdot (C - M(E)E) < 0$. Suppose that the intersection pairing $e_i \cdot (C - M(E)E) \leq e_i^2 < 0$, then there exists a relative effective divisor $\Delta_i \subset \Xi_i \mapsto Y(\Gamma_{e_i})$ of relative degree $(M(E)E + e_i) \cdot e_i \geq 0$, an invertible sheaf $\pi^* Q_i$ on $\Xi_i$ pulled-back from $Y(\Gamma_{e_i})$ and a short exact sequence of locally free sheaves,

$$0 \mapsto R^0 \pi_* (O_{\Delta_i}(E_i - \sum_{j_i} E_{j_i})) \otimes Q_i \mapsto R^1 \pi_* (O_{\Xi_i}(-M(E)E)) \mapsto R^1 \pi_* (O_{\Xi_i}(E_i - \sum_{j_i} E_{j_i})) \otimes Q_i \mapsto 0$$

exact on a Zariski open subset of $Y(\Gamma_{e_i})$.

Suppose that the intersection pairing satisfies $0 > e_i \cdot (C - M(E)E) > e_i^2$, then there exists a relative effective divisor $\Delta_i \subset \Xi_i \mapsto Y(\Gamma_{e_i})$ of relative degree $-(M(E)E + e_i) \cdot e_i > 0$, an invertible sheaf $\pi^* Q_i$ on $\Xi_i$ pulled-back from $Y(\Gamma_{e_i})$ and a short exact sequence of locally free sheaves,

$$0 \mapsto R^0 \pi_* (O_{\Delta_i}(E_i - \sum_{j_i} E_{j_i})) \otimes Q_i \mapsto R^1 \pi_* (O_{\Xi_i}(E_i - \sum_{j_i} E_{j_i})) \otimes Q_i \mapsto R^1 \pi_* (O_{\Xi_i}(-M(E)E)) \mapsto 0$$

exact on a Zariski open subset of $Y(\Gamma_{e_i})$.

The reader should notice the similarity between proposition 15 and proposition 4 in subsection 3.3. By lemma 9 in the previous subsection, these exact sequences can be viewed as exact sequences about $N_{Y(\Gamma_{e_i})} M_n$.

Proof of the proposition: Denote the set of all descendent indexes of $i$ by $J_i$. Firstly, notice that the effective divisors $E_{j_i} \subset M_{n+1}, j_i \in J_i$ restrict to cross sections on $\Xi_i \mapsto Y(\Gamma_{e_i})$. As $\Xi_i$ is a $P^1$ fibration, the invertible sheaves $O_{\Xi_i}(E_{j_i}|_{\Xi_i})$ for different $j_i \in J_i$ are equivalent after tensoring invertible sheaves pulled back from the base $Y(\Gamma_{e_i})$. Define $q = (M(E)E + e_i) \cdot e_i \in \mathbb{Z}$. Suppose that $q \geq 0$, fix one $i \in J_i$ and define $\Delta_i = q E_i|_{\Xi_i} + \sum_{j \in J_i \cup \{i\}} m_j E_j$. Then $O_{\Xi_i}(-M(E)E)$ and $O_{\Xi_i}(-\Delta_i + (E_i - \sum_{j \in J_i \cup \{i\}} E_{j_i}))$ have the same relative degrees along $\Xi_i \mapsto Y(\Gamma_{e_i})$ and the former is equivalent to the latter after tensoring the latter sheaf by some invertible sheaf $\pi^* Q_i$ pulled-back from the base $Y(\Gamma_{e_i})$.

Then by tensoring the defining short exact sequence of $\Delta_i$ by $O_{\Xi_i}(E_i - \sum_{j \in J_i} E_{j_i}) \otimes \pi^* Q_i$, one gets

$$0 \mapsto O_{\Xi_i}(-M(E)E) \mapsto O_{\Xi_i}(E_i - \sum_{j \in J_i} E_{j_i}) \otimes \pi^* Q_i \mapsto O_{\Delta_i}(E_i - \sum_{j \in J_i} E_{j_i}) \otimes \pi^* Q_i \mapsto 0.$$

By taking the derived long exact sequence along $\pi : \Xi_i \mapsto Y(\Gamma_{e_i})$ and restrict to the complement of the support of $R^0 \pi_* (O_{\Xi_i}(-M(E)E))$, we get the desired sheaf short exact sequence stated in the proposition.

Notice that the choices of $\Delta_i, Q_i$ are not unique.

The proof of the $q < 0$ case is rather similar and we leave it to the reader. □
6.4 Some Vanishing Results about the Local Contribution

Step VII: In this step, we derive a vanishing lemma which will be used in the next few steps.

Lemma 10 Let $V \to X$ be a vector bundle of rank $v$ and let $s : X \to V$ be a regular section of $V$ with $i : Z(s) \subset X$, $\text{codim}_X Z(s) = v$. Let $Q$ be a line bundle over $X$, then $0 = c_r(i^*(Q \otimes V) - i^*V) = \{c(i^*(Q \otimes V)) \cap s(i^*V)\}_{\text{dim}_Z Z(s) - r} \in A_{\text{dim}_Z Z(s) - r}(Z(s))$ for all $r \in \mathbb{N}$.

A sketch of the proof: Firstly, assume that $Q$ is effective and consider a fixed section $s_{DQ}$ defining the effective divisor $D_Q$. Then $s \otimes s_{DQ}$ is a section of $V \otimes Q$ and $Z(s) \subset Z(s \otimes s_{DQ})$.

By the regularity condition on $s$, $N_{Z(s)}X \cong i^*V$. Blowing up $Z(s) \subset X$ into a smooth divisor $D$.

By a direct computation following [F] page 161-162, equations (1), (2), (3), one finds that

$$c(i^*(Q \otimes V) \cap s(Z(s), X) = c(Q \otimes V) \cap s(Z(s \otimes s_{DQ}), X) - c(O(-D) \otimes Q \otimes V) \cap s(D_Q, X).$$

By the grading constraint from proposition, the degree $\text{dim}_X X - v - r = \text{dim}_X Z(s) - r$ pieces of both terms on the right hand side vanish for all $r \in \mathbb{N}$.

Thus $c_r(i^*(Q \otimes V) - i^*V) = 0$ for all $r \in \mathbb{N}$.

Secondly when $Q$ is not effective, write $Q = Q_1 \otimes Q_2^{-1}$, where $Q_1, Q_2$ are both effective. This is always possible as we can twist $Q$ by a high power of ample line bundle $D$ to make both $Q_1 = Q \otimes D^l$ and $Q_2 = D^l$ effective for large enough $l \gg 0$.

We know that $f_r(m, n) = c_r(i^*(Q_1^m \otimes Q_2^{-1} \otimes V) - i^*V) = 0$ for all $m, n, r \in \mathbb{N}$ because $Q_1^m \otimes Q_2^{-1}$ is effective.

On the other hand, $c_1(Q_1^m \otimes Q_2^{-1}) = m \cdot c_1(Q_1) + n \cdot c_1(Q_2)$ and $f_r(m, n)$, being an algebraic combination of Chern classes of $i^*Q$ and $i^*V$, must be a polynomial expression in terms of $m$ and $n$. Then the polynomial in $n, f_r(1, n)$ has an infinite number of roots, $f_r(1, n) \equiv 0$ for all $n \in \mathbb{Z}$. In particular, $f_r(1, -1) = 0$ and thus

$$c_r(i^*(Q \otimes V) - i^*V) = c_r(i^*(Q_1 \otimes Q_2^{-1} \otimes V) - i^*V) = 0. \quad \Box$$

Step VIII: After the preparation in step VI and step VII, we continue the discussion from step IV and show that

Proposition 16 Suppose that $c_i \cdot (C - M(E)E) > c_i^2$ for some $i$, $1 \leq i \leq p$, then local contribution of $\mathcal{M}_{C - M(E)E} - \sum_{i \leq p} c_i \times M_x Y(\Gamma)$ to the family invariant $\mathcal{A} \mathcal{F} \mathcal{S} \mathcal{W}_{M_{n+1} \times T(M) \to M_n \times T(M)}(1, C - M(E)E)$ defined by (consult page 26)

$$j_{Y_1} Z_{Y_1}(s_{\text{canon}}) \cap c_1(i^*_{Y_1} \mathcal{H})_{\text{dim}_Z M_n + \frac{c^2 - C \cdot c_1(K_y)}{2} + p_y - \sum_{i \leq n} \frac{m^2 + m}{2}} \in \mathbb{Z}$$

vanishes.
Proof: For notational simplicity, we may assume that \( e_1 \cdot (\mathcal{C} - \mathbf{M}(E)E) > e_1^2 \) by permuting the indexes \( \{1, 2, \ldots, p\} \) (if it is necessary).

Firstly, we notice that for \( \mathcal{C} = \mathcal{C} - \mathbf{M}(E)E \), we have an equality,

\[
0 < e_1 \cdot (\mathcal{C} - \mathbf{M}(E)E - e_1) = e_1 \cdot (\mathcal{C} - e_i) = \left( \frac{C^2 - c_1(K_{\mathcal{M}_{M+1}/M_n}) \cdot C}{2} + p_g \right) - \left( \frac{(C - e_1)^2 - c_1(K_{\mathcal{M}_{M+1}/M_n}) \cdot (C - e_1)}{2} + p_g + \frac{e_1^2 - c_1(K_{\mathcal{M}_{M+1}/M_n}) \cdot e_1}{2} \right).
\]

Thus the family dimension of the class \( \mathcal{C} \) is strictly larger than the family dimension of the splitting \( \mathcal{C} - e_1 \) and \( e_1 \). We will argue the vanishing of the intersection number by the negativity of the dimension count.

Firstly, take \( \mathcal{W} \) to be the algebraic vector bundle associated to \( \mathcal{O}_{\mathcal{M}(E)E+E_1 - \sum_{j_i} E_{j_i}} \otimes \mathcal{E}_C \).

Then we argue

\[
\{c(i_1^* H \otimes \pi_1^*(V_{\text{canon}}) \mathcal{W}) \cap s(Y_1, X') \} \cap \mathcal{C} \cdot \mathcal{X} - \text{codim} \mathcal{X} - \text{dim} \mathcal{C} \mathcal{W} \cap c_1(i_1^* H) \frac{C^2 - c_1(K_{\mathcal{M}_{M+1}/M_n}) \cdot C}{2} + p_g + \text{dim} \mathcal{C} \mathcal{M}_n
\]

vanishes.

It is because

\[
\text{dim} \mathcal{C} \cdot \mathcal{X} - \text{codim} \mathcal{C} \mathcal{Y} - \text{rank} \mathcal{C} \mathcal{W} = \frac{(C - e_1)^2 - c_1(K_{\mathcal{M}_{M+1}/M_n}) \cdot (C - e_1)}{2} + p_g + \text{dim} \mathcal{C} \mathcal{M}_n + \sum_{1 \leq i \leq p} \frac{e_i^2 - c_1(K_{\mathcal{M}_{M+1}/M_n}) \cdot e_i}{2}
\]

Secondly, we would like to transform \( \alpha = \{c(i_1^* H \otimes \mathcal{W}) \cap s(Y_1, X') \} \cap \mathcal{C} \cdot \mathcal{X} - \text{codim} \mathcal{X} - \text{rank} \mathcal{C} \mathcal{W} \in \mathcal{A}_{\mathcal{X} - \text{codim} \mathcal{X} - \text{rank} \mathcal{C} \mathcal{W}}(Y_1) \) to the original local expression \( \{c(i_1^* \pi_1^*(V_{\text{canon}}) \mathcal{W}) \cap s(Y_1, X \times_{M_n} Y(\Gamma)) \} \) \( \text{m-e} \).

By proposition and the fact \( e_1 \cdot (\mathcal{C} - \mathbf{M}(E)E) > e_1^2 \), there exists a short exact sequence on \( \mathcal{R}^1 \pi_*(\mathcal{O}_{\Xi_1}(E_1 - \sum_{j_i} E_{j_i})) \). We may tensor it with \( \mathcal{E}_C \) and get

\[
0 \rightarrow \mathcal{R}^0 \pi_*(\mathcal{O}_{\mathcal{U}_2}(E_1 - \sum_{j_i} E_{j_i})) \otimes \mathcal{Q}_1 \otimes \mathcal{E}_C \rightarrow \mathcal{R}^1 \pi_*(\mathcal{O}_{\Xi_1}(E_1 - \sum_{j_i} E_{j_i})) \otimes \mathcal{Q}_1 \otimes \mathcal{E}_C \rightarrow \mathcal{R}^1 \pi_*(\mathcal{O}_{\Xi_1}(-\mathbf{M}(E)E) \otimes \mathcal{E}_C) \rightarrow 0.
\]

Let us set a few notations before moving forward.

By proposition the sheaf \( \mathcal{R}^0 \pi_*(\mathcal{O}_{\Delta_2}(E_1 - \sum_{j_i} E_{j_i})) \otimes \mathcal{Q}_1 \otimes \mathcal{E}_C \) is locally free on \( \mathcal{U}_i \), the complement of the support of \( \mathcal{R}^0 \pi_*(\mathcal{O}_{\Xi_1}(-\mathbf{M}(E)E)) \). Denote the corresponding vector bundle by \( \mathcal{V}_i \rightarrow \mathcal{U}_i \). Denote the normal bundle of \( Y(\Gamma_{e_i}) \subset M_n \) by \( \mathcal{N}_{e_i}, 1 \leq i \leq p \). Denote the inclusion \( Y(\Gamma) \subset Y(\Gamma_{e_i}) \) by \( h_i \).
Because $Y(\Gamma)$ is the locus over which all $e_i, 1 \leq i \leq p$ are effective, $Y(\Gamma) = \cap_{1 \leq i \leq p} Y(\Gamma_{e_i})$ and $N_{Y(\Gamma)} = \oplus_{1 \leq i \leq p} h_i^* N_i$.

Define $U = U_1 \cap Y(\Gamma) \subset Y(\Gamma)$ and set $\pi_F : F = \pi_X, h_i^* \nu \otimes \pi^* Y \otimes X' \times Y(\Gamma) U \to X' \times Y(\Gamma)$ U to be the projection map of the vector bundle $F$. Set $s_F : X' \times Y(\Gamma) U \to F$ to be the zero section map.

Because $Y_1 \subset X' \times Y(\Gamma) U$, $\alpha \in A_{\dim C X - \codim C Y(\Gamma) \to \rank C W}(Y_1)$ defines a class in $A_{\dim C X - \codim C Y(\Gamma) \to \rank C W}(X' \times Y(\Gamma))$ and we abuse the notation slightly and denote it by the same symbol. Then by page 67, example 3.3.2. of [F], $c(F) \cap \alpha = s_F s_F \alpha$ and then

$$\alpha = \{ \alpha \}_{\dim C X - \codim C Y(\Gamma) \to \rank C W} = s_F^* s_F \{ s(F) \cap \alpha \}_{\dim C X - \codim C Y(\Gamma) \to \rank C W + \rank C F}.$$ 

By the short exact sequence of $\nu_1$ in proposition [10] and by lemma [11] we have

$$s(\pi_X^* \nu_1 \otimes H) \cap s(\pi_X^* (h_i^* R^1 \pi_* (O_{\Xi}(-M(E)) \otimes E)) \otimes H|_{X' \times Y(\Gamma) U}) = s(\pi_X^* (h_i^* (N_1 \otimes Q_1 \otimes E)) \otimes H|_{X' \times Y(\Gamma) U})$$

$$= s(\pi_X^* (h_i^* N_1)|_{X' \times Y(\Gamma) U} \cap c(\pi_X^* (h_i^* N_1|_{X' \times Y(\Gamma) U} - \pi_X^* h_i^* N_1 \otimes Q_1 \otimes E \otimes H|_{X' \times Y(\Gamma) U}) = s(\pi_X^* (h_i^* N_1)|_{X' \times Y(\Gamma) U}).$$

$$c(\pi_X^* (h_i^* N_1)|_{X' \times Y(\Gamma) U} - \pi_X^* h_i^* N_1 \otimes Q_1 \otimes E \otimes H|_{X' \times Y(\Gamma) U}) = s(\pi_X^* (h_i^* N_1)|_{X' \times Y(\Gamma) U}) = 1$$

because $X \times M_\mu Y(\Gamma_{e_i})$ is the regular zero locus of the global regular section of $\pi_X^* R^0 \pi_* (O_{\sum_{i=1}^p E_i})$ on $X = P_{M_\mu \times T(M)} (V_{\text{canon}})$, lemma [10] (we take $Q = \pi_X^* (h_i^* Q \otimes E) \otimes H$) and the fact $U \subset Y(\Gamma) \subset Y(\Gamma_{e_i})$.

Thus,

$$\{ s(i_1^* F) \cap \alpha \}_{\dim C X + \codim C Y(\Gamma) \to \rank C W + \rank C F} = \{ s(i_1^* \pi_X^* N_1|_{Y(\Gamma)}) \cap c(i_1^* H \otimes R^1 \pi_* (O_{\Xi}(-M(E)) \otimes E))$$

$$\cap c(i_1^* (H \otimes \pi_X^* W)) \cap s(Y_1, X') \}_{\dim C X + \codim C Y(\Gamma) \to \rank C W + \rank C F}$$

$$= \{ s(N_1|_{Y(\Gamma)}) \cap c(i_1^* H \otimes \pi_{\text{canon}}^* W_{\text{canon}}) \cap s(Y_1, X') \}_{\dim C X - \codim C Y(\Gamma) \to \rank C W + \rank C F} \in A_*(Y_1),$$

due to the Chern classes identity

$$c(W_{\text{canon}}) = c(W) \cap c(R^1 \pi_* (O_{\Xi}(-M(E)) \otimes E))$$

derived from a short exact sequence similar to the one in proposition [11] relating $W_{\text{canon}}$ and $W_{\text{canon}}$.

Thirdly, the integral grading of the last expression above is equal to

$$\dim C X + \codim C Y(\Gamma) \to \rank C W + \rank C F = \dim C X + \sum_{1 \leq i \leq p} \frac{e_i^2 - c_1(K_{M_{i+1}/M_i}) \cdot e_i}{2} - \rank C W - \sum_{2 \leq i \leq p} \frac{e_i^2 - c_1(K_{M_{i+1}/M_i}) \cdot e_i}{2}$$

51
\[ + \text{rank}_C \nu_1 = \dim_C X + \frac{e_1^2 - c_1(K_{M_{n+1}/M_n}) \cdot e_1}{2} - \text{rank}_C W + \text{rank}_C \nu_1 = \dim_C X - \text{rank}_C W_{\text{canon}} \]

\[ + \left\{ \frac{e_1^2 - c_1(K_{M_{n+1}/M_n}) \cdot e_1}{2} + \text{rank}_C \nu_1 + \text{rank}_C R^1 \pi_* \left( \mathcal{O}_{\mathbb{Z}_1}(-M(E)\otimes\mathcal{E}_C) \right) \right\} = \text{rank}_C X - \text{rank}_C W_{\text{canon}} = m - e, \]

and the final expression

\[ \{ c(i_{Y_1}^* H \otimes \pi^*_P(V_{\text{canon}}^0) W_{\text{canon}}) \cap s(Y_1, X \times_{M_n} Y(\Gamma)) \cap s(\pi_{Y_1}^* N_{Y(\Gamma)}) \}_{m-e} \]

matches with the final expression in step I. Therefore, its cap product with

\[ c_1(i_{Y_1}^* H) \overset{\sum_{i \leq p} h^i}{\longrightarrow} \text{dim}_C Y \text{ rank}_C \tau_{Y_1} \oplus \text{rank}_C \tau_{Y_1} \oplus \text{dim}_C M_n \]

must be zero as well. □

Step IX: In this final step, we work with the situation that \( 0 > e_i^2 \geq e_i \cdot (C - M(E)E), 1 \leq i \leq p \). Define the class \( \tau \) and identify the local family invariant contribution on \( Y_1 \) with \( \mathcal{AFSW}_{M_{n+1} \times_{M_n} Y(\Gamma) \times T(M) \rightarrow Y(\Gamma) \times T(M)}(\text{total}(\tau), C - M(E)E - \sum_{1 \leq i \leq p} e_i) \).

Define \( \tau \) to be the equivalence class represented by the sheaf

\[ \oplus_{1 \leq i \leq p} h^i \left( R^0 \pi_* \left( \mathcal{O}_{\mathbb{Z}_1, t} \otimes \mathcal{E}_C \right) \right) \oplus R^0 \pi_* \left( \mathcal{O}_{\Delta, s}(-M(E)E) \otimes \mathcal{E}_C \right), \]

locally free on \( \cap_{1 \leq i \leq p} U_i \times T(M) \), where \( U_i \subset Y(\Gamma) \) stands for the Zariski open subset defined on page 50 in step VIII. Because \( \pi_{Y_1} : Y_1 \rightarrow Y(\Gamma) \) factors through \( U \subset Y(\Gamma) \), \( c_{\text{rank}_C \tau + r}(\pi_{Y_1}^* \tau) = 0 \) for \( r \in \mathbb{N} \).

The family moduli space \( \mathcal{M}_{C - M(E)E - \sum_{1 \leq i \leq p} e_i} \quad Y(\Gamma) = Y_1 \subset X' \) above \( Y(\Gamma) \) is defined to be the zero locus \( Z(s_{\text{canon}}^0) \times_{M_n} Y(\Gamma) \). By applying proposition 13, the mixed family invariant \( \mathcal{AFSW}_{M_{n+1} \times_{M_n} Y(\Gamma) \times T(M) \rightarrow Y(\Gamma) \times T(M)}(\text{total}(\tau), C - M(E)E - \sum_{1 \leq i \leq p} e_i) \) can be identified with the following expression involving localized top Chern class,

\[ \bar{i}_{Y_1*} \left( \sum_{0 \leq r \leq \text{rank}_C \tau} c_r(i_{Y_1}^* \tau) \right) \cap \left( c(i_{Y_1}^* H \otimes \pi^*_P(V_{\text{canon}}^0) W_{\text{canon}}^0) \cap s(Y_1, X') \right) \text{dim}_C X' - e' \]

\[ \left. \cap_c(i_{Y_1}^* H) \left( \sum_{i \leq p} s_i \cdot (K_{M_{n+1}/M_n}) \cap s(Y_1, X') \right) \text{dim}_C Y(\Gamma) + \text{rank}_C \tau - r \right) \]

\[ = \bar{i}_{Y_1*} \left( c_{\text{rank}_C \tau}(i_{Y_1}^* H \otimes \pi^*_P(W_{\text{canon}}^0)) \right) \cap \left( c(i_{Y_1}^* H \otimes \pi^*_P(V_{\text{canon}}^0) W_{\text{canon}}^0) \cap s(Y_1, X') \right) \text{dim}_C X' - e' \]

\[ \cap_c(i_{Y_1}^* H) \left( \sum_{i \leq p} s_i \cdot (K_{M_{n+1}/M_n}) \cap s(Y_1, X') \right) \text{dim}_C Y(\Gamma) \]

Thus, it suffices to identify
At the end, let us address Question 3 under the Simplifying Assumption over the whole $Y$.

Some Partial Orderings Among $Y$

We may repeat our discussion upon $c$ and are related to the local contributions upon $(G \otimes \pi_Y^* \tau)$ and argue the vanishing of $(c(i_Y^* H \otimes \pi_Y^* \tau))(\cap_{i=1}^p h_i^* N_i)$ from the definitions of $\pi_Y^* \tau$, $\pi_Y^* H$, $\pi_Y^* E$, and $\pi_Y^* C$ to denote the locally free sheaf associated with $G$, $N_i$, $Q_i$, and $E_C$.

This equality follows from the definitions of $\pi_Y^* \tau$ and $G$, the short exact sequences of $N_i$ stated in proposition 15 and the following short exact sequences

$$0 \rightarrow R^0\pi_*(O_{\mathcal{E}_i}(-M(E)E)\otimes E_C) \rightarrow R^1\pi_*(O_{\mathcal{E}_i}(-M(E)E)\otimes E_C) \rightarrow 0$$

on $U_i \times T(M)$. This ends the proof of theorem 4. □

6.5 Some Partial Orderings Among $(\Gamma, \sum_{\epsilon_i}(C-M(E))<\epsilon_i)$

At the end, let us address Question 3 on page 53 briefly. Theorem 4 has answered the question of identifying the local contributions of the family invariant under the Simplifying Assumption.

On the other hand, it is clear from proposition 12 the local contributions over the whole $Y(\Gamma)$ (not only from $Y_1$) involve more terms yet to be identified and are related to the local contributions upon $(C_{\rho} - O^\rho) \cap s_{canon}$. In principle we may repeat our discussion upon $Y(\Gamma)$ to some other admissible $Y(\Gamma')$, $\Gamma' < \Gamma$. (compare with □ on page 54.)
Our discussion in theorem 4 makes us believe that the mixed family invariants similar to the expression in theorem 4 could have appeared while we enumerate these unknown contributions.

On the other hand, if we consider the local contributions of the family invariants upon the closures of two admissible strata $Y(\Gamma_1), Y(\Gamma_2), Y(\Gamma_1) \cap Y(\Gamma_2) \neq \emptyset$ simultaneously, it leads to potential over-counting as the local contributions from $Y(\Gamma_1) \cap Y(\Gamma_2)$ are counted twice altogether. The phenomenon becomes more complicated when more than two different $Y(\Gamma)$ are involved and some combinatorial partial ordering (see page 54 below) upon these admissible graphs $\Gamma \in adm(n)$ have to be imposed in order to get a consistent enumeration on $\mathcal{FASW}$ without over-counting.

A few partial orderings among the admissible strata $Y_\Gamma, \Gamma \in adm(n)$ can been introduced as below:

Given a fixed $M(E)E = \sum_{1 \leq i \leq n} m_i E_i$ encoding the singular multiplicities of curve singularities, consider all the admissible strata $Y_\Gamma, \Gamma \in adm(n)$ satisfying the special condition on page 42.

One may introduce three partial orderings $>, \sqsupseteq, \gg$ among all such $(\Gamma, \sum_{e_i, (C-M(E))E < 0} e_i)$ by the following conditions based on the degenerations of type I exceptional classes:

Let $\Gamma$ and $\Gamma'$ be two admissible graphs. If the following condition (i) holds, (i). $\Gamma > \Gamma'$, i.e. $Y_{\Gamma'} \subset Y(\Gamma) - Y_\Gamma$ then $(\Gamma, \sum_{e_i, M(E)E > 0} e_i)$ is said to be larger than $(\Gamma', \sum_{e'_i, M(E)E > 0} e'_i)$ under a partial ordering $>$,

$$(\Gamma, \sum_{e_i, M(E)E > 0} e_i) > (\Gamma', \sum_{e'_i, M(E)E > 0} e'_i).$$

We say that $(\Gamma, \sum_{e_i, M(E)E > 0} e_i) \sqsupseteq (\Gamma', \sum_{e'_i, M(E)E > 0} e'_i)$ if additional to (i), the following condition (ii). is satisfied,

(ii). The class $\sum_{e_i, (C-M(E))E < 0} e_i - \sum_{e'_i, (C-M(E))E < 0} e'_i$ is effective over $Y_{\Gamma'}$. In other words for all $b \in Y_{\Gamma'}$, the exceptional curve above $b$ dual to each $e'_j$ with $e'_j \cdot (C-M(E)) < 0$ is an irreducible component of the tree of $P^1$'s dual to an $e_i$ with $e_i \cdot (C-M(E)) < 0$.

We say that $(\Gamma, \sum_{e_i, M(E)E > 0} e_i) \gg (\Gamma', \sum_{e'_i, M(E)E > 0} e'_i)$ under $\gg$ if (i). and the following condition (iii). hold,

(iii). For all $e_i$ with $e_i \cdot (C-M(E)) < 0$, the corresponding $e'_i = e_i$. There exists at least one $1 \leq j \leq n$ such that $e_j^2 = -1$ but $e'_j \cdot (C-M(E)) < 0$. (compare with page 409-410 definition 4.5 of [Liu1])

If the equality may hold, we replace the symbols $>, \sqsupseteq, \gg$ by $\geq, \succeq, \gg$.

We end the current paper by the following observation,
Proposition 17 Let $\Gamma$ satisfies the special condition on page 42 let $(\Gamma, \sum e_i (C-M(E)_E<0 \epsilon_i) > (\Gamma', \sum e'_i (C-M(E)_E<0 \epsilon'_i))$, then there exists an intermediate pair $(\Gamma'', \sum e''_i (C-M(E)_E<0 e''_i))$ such that

$$(\Gamma, \sum_{e_i (C-M(E)_E<0 \epsilon_i))} \geq (\Gamma'', \sum_{e''_i (C-M(E)_E<0 e''_i))} \geq (\Gamma', \sum_{e'_i (C-M(E)_E<0 e'_i)).}$$

Proof: Consider the cohomology class $\sum e_i (C-M(E)_E<0 \epsilon_i) - \sum e'_i (C-M(E)_E<0 \epsilon'_i)$ is not effective on $Y_{\Gamma'}$. one defines the index sets $I = \{i|1 \leq i \leq n, e_i \cdot (C-M(E)_E<0)\}$ and $J = \{j|1 \leq j \leq n, e'_j \cdot (C-M(E)_E<0)\}$ as subsets of the universal set $\{1, 2, \ldots, n\}$. Then there exists a non-empty subset $J_0 \subset J - I$ such that

(a) $\sum_{i \in I} e_i - \sum_{j \in J - J_0} e'_j$ is effective on $Y_{\Gamma'}$.

(b) Consider any proper subset $J_1 \subset J_0$, then $\sum_{i \in I} e_i - \sum_{j \in J - J_1} e'_j$ is non-effective on $Y_{\Gamma'}$.

By a direct calculation the intersection pairing $e_i \cdot e'_j \in \mathbb{Z}$ is always negative. Thus for all points in $Y_{\Gamma'}$, the irreducible $\mathbb{P}^1$ dual to $e'_j$ is always an irreducible component of the tree of $\mathbb{P}^1$ dual to $e_i$. Therefore, $e_i - e'_j$, $i \in I$, is effective over $Y_{\Gamma'}$. So is their sum $\sum_{i \in I} e_i - e'_j = \sum_{i \in I} e_i - \sum_{j \in J} e'_j$.

As $\sum_{i \in I} e_i - \sum_{j \in J} e'_j$ is effective but $\sum_{i \in I} e_i - \sum_{j \in J} e'_j$ is not, there exists at least one minimal subset $J_0 \subset J - I$ satisfying (a) and (b).

Then the co-existence of the type I exceptional classes $e'_j, j \in (J-J_0) \cup J'$ as irreducible rational curves defines an admissible stratum encoded by some admissible graph $\Gamma''$ with the properties $e''_j \equiv e_j$ for $j \in (J-J_0) \cup J'$, $e''_j = e_j$, i.e. $(e''_j)^2 = -1$, for $j \in J_0$.

Recall that the closed set $Y(\Gamma)$ is the locus in $M_n$ over which the type I exceptional classes $e_i, 1 \leq i \leq n$ are effective and it is apparent that all $e_i$, as effective combinations of $e''_j = e_j$ for $j \in (J-J_0) \cup J'$ and some $-1$ classes $e''_j, j \in J_0$ (effective over the whole $M_n$), are effective over $Y_{\Gamma''}$. Thus $Y(\Gamma) \supset Y_{\Gamma'}$ and $\Gamma > \Gamma''$ accordingly. For a similar reason $\Gamma'' > \Gamma'$ as well.

The non-emptyness of $J_0$ implies $(\Gamma'', \sum e''_j (C-M(E)_E<0 e''_j)) \gg (\Gamma', \sum e'_j (C-M(E)_E<0 e'_j)), (\Gamma', \sum e''_j (C-M(E)_E<0 e''_j)).$

On the other hand, to show that $\sum e_i (C-M(E)_E<0 \epsilon_i) - \sum e''_j (C-M(E)_E<0 e''_j)$ is effective over $Y_{\Gamma''}$ it suffices as required by the partial ordering $\sqsupset$, we show that the difference must be an effective combination of $e''_j, j \in (J-J_0) \cup J'$ and $e''_j, j \in J_0$.

Firstly, we know that

$$\sum_{e_i (C-M(E)_E<0 \epsilon_i) - \sum e''_j (C-M(E)_E<0 e''_j)} = \sum_{j \in J - J_0} e_j - \sum_{j \in J} e'_j$$

is effective over $Y_{\Gamma'}$. Thus, it must be an effective combination of $e''_j, 1 \leq j \leq n$, say $\sum_{1 \leq j \leq n} e_j e''_j, e''_j \geq 0$. 

55
We argue that for all \( j \in J_0 \), \( c_j = 0 \). Otherwise, we may take \( J_1 = \{ j | j \in J_0, c_j = 0 \} \subset J_0 \) and make \( \sum_{i \in I} e_i - \sum_{j \in J - J_1} e'_j \) effective over \( Y_{\Gamma'} \), violating the minimality condition (b). That \( J_0 \) satisfies.

Thus, \( \sum_{j \in n} c_j e'_j = \sum_{j \in J - J_0} c_j e'_j + \sum_{j \notin J} c_j e'_j \).

By the defining properties of \( \Gamma' \) and \( \Gamma'' \), \( e'_j = e''_j \), \( j \in (J - J_0) \cup J^c \). Thus all these type I classes \( e'_j, j \in (J - J_0) \cup J^c \) are effective over \( Y_{\Gamma''} \) and so is their effective combination

\[
\sum_{j \notin J_0} c_j e'_j = \sum_{e_i \cdot (C - M(E)E) < 0} e_i - \sum_{e_j'' \cdot (C - M(E)E) < 0} e''_j.
\]

The proposition is proved. \( \square \)

We point out the geometric origins of these partial orderings.

**Remark 5** The partial ordering \( \succ \) indicates that the admissible stratum \( Y_{\Gamma'} \) can be degenerated from \( Y_{\Gamma} \). It implies that \( M_{C - M(E)E} \times_{M_n} Y(\Gamma') \) is contained inside \( M_{C - M(E)E} \times_{M_n} Y(\Gamma) \).

The partial ordering \( \sqsubseteq \) indicates that every type I classes \( e'_j \) with \( e'_j \cdot (C - M(E)E) < 0 \) are degenerated from some \( e_i \) as one of its irreducible components. The \( \sqsubseteq \) implies that family moduli space \( M_{C - M(E)E} - \sum_{e_i \cdot (C - M(E)E) < 0} e_i \times M_n \), \( Y(\Gamma') \) is contained inside \( M_{C - M(E)E} - \sum_{e_i \cdot (C - M(E)E) < 0} e_i \times M_n \), \( Y(\Gamma) \).

The partial ordering \( \gg \) indicates that when one degenerates from \( Y_{\Gamma} \) to \( Y_{\Gamma''} \), some new type I class \( e'_j \) with \( e'_j \cdot (C - M(E)E) < 0 \) appears while the remaining \( e_i \), \( e_i \cdot (C - M(E)E) < 0 \) are preserved. The \( \gg \) implies that \( M_{C - M(E)E} - \sum_{e_i \cdot (C - M(E)E) < 0} e_i \times M_n \), \( Y(\Gamma) \) is contained in \( M_{C - M(E)E} - \sum_{e_i \cdot (C - M(E)E) < 0} e_i \times M_n \), \( Y(\Gamma') \).

Notice that the \( \Gamma' \) or \( \Gamma'' \) in proposition 17 may not always satisfies the special condition on page 54. This indicates that there is some other admissible strata \( Y_{\Gamma} \) in \( M_n \) satisfying the special condition and \( Y_{\Gamma'} \) or \( Y_{\Gamma''} \) is in the intersection of \( Y(\Gamma) \) and \( Y(\det \Gamma) \).

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