Dispersion total photoproduction sum rules for nucleons and few-body nuclei revisited

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Abstract

The questions on the presence and quantitative role of the constant terms in the real part of the high-energy photon-nucleon and photon-nucleus amplitudes representing the contribution of the non-Regge (the fixed $j = 0$-pole) singularities in the finite-energy sum rules (FESR) for the photoabsorption cross sections on nucleons and the lightest atomic nuclei are discussed and new testable relations are presented for relevant combinations of the Compton scattering amplitudes.

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I. INTRODUCTION

In the 1954 seminal paper of Gell-Mann, Goldberger and Thirring (GGT)\[1\] on the use of the causality condition in quantum theory the idea of the "superconvergence" sum rule technique was first suggested and applied to the photonuclear absorption processes. The GGT sum rule follows from the assumption of validity of the unsubtracted dispersion relations for the difference presumably vanishing as \( \nu \to \infty \)

\[
\Delta T = T_{\gamma A}(\nu) - Z T_{\gamma p}(\nu) - N T_{\gamma n}(\nu)
\]

of the forward Compton scattering amplitudes on the nucleus with atomic number \( A = Z + N \) and the sum of amplitudes on the \( Z \) free protons and \( N \) free neutrons. After inclusion of the Thompson value \(-\frac{\alpha Q^2}{M}\) (\( M(Q) \) being the hadron mass and electric charge in the units of the electron charge), for every hadron amplitude \( T(\nu = 0) \) at zero photon energy, the sum rule reads

\[
2\pi^2 \frac{\alpha}{M_n} \left( \frac{-Z^2}{A} + Z \right) + \int_{\nu_{\gamma p}}^{\infty} d\nu [Z\sigma_{\gamma p}(\nu) + N\sigma_{\gamma n}(\nu) - \sigma_{\gamma A}(\nu)] = \int_{\nu_{\gamma p}}^{\nu_{\gamma n}} d\nu \sigma_{\gamma A}(\nu) \tag{2}
\]

The first term in l.h.s. of (2) practically coincides with the "kinetic" part of the long-known Thomas-Reiche-Kuhn sum rule for the electric dipole nuclear photoabsorption

\[
\sigma_0(E1) \equiv \int_{\nu_{\gamma p}}^{\infty} \sigma_{E1}(\nu) d\nu = 4\pi^2 \Sigma_n(E_n - E_0) |\langle n|D_z|A\rangle|^2 = 2\pi^2 \langle A|[D_z[H,D_z]]|A\rangle = (2\pi^2 \frac{\alpha N Z}{AM_n}) + 2\pi^2 \langle A|[D_z[\hat{V}_{NN},D_z]]|A\rangle \tag{3}
\]

where the first term results from the double commutator with the kinetic energy operator of the nuclear hamiltonian \( H \). The present work originates partly from earlier papers of the author \[2, 3, 4\] dealing with sum rules for total photon-hadron cross-sections and aims to present some new experimentally testable and theoretically interesting relations emerging from the dispersion FESR phenomenology.

II. TOWARDS GENERALIZED GGT SUM RULE

It was always tempting and rewarding to combine the power of dispersion relation approach, which is based on very general underlying assumptions and explains many general properties of the scattering amplitudes as well as provides useful relations between them in a rather simple way with particular dynamical ingredients of a given quantum system such as, for instance, implications of the broken chiral symmetry and pionic dynamics dominating peripheral properties and low-energy interactions of hadrons and nuclei.

A. A glance at a possible role of pion degrees of freedom on GGT sum rule

In \[2, 4\] an attempt was maid to introduce corrections to the GGT approach understood as a familiar Impulse Approximation (IA) scheme applied to the \( \gamma A \)-forward scattering...
amplitude. The approximate relevance of IA is seen from the fact that it corresponds to taking into account the singularities closest to the physical region of the peripheral scattering process \((t \leq 0, \ t = (k - k')^2\) is the invariant 4-momentum transfer for elastic scattering). The respective cut in the complex \(t\)-plane is defined by the diagrams schematically represented in Fig. 1a, while the next to the leading ”anomalous” threshold given by Fig. 1a will be the ”normal” \(2\pi\) -exchange diagrams, represented in Fig. 1b, with the cut starting at \(t = 4m_n^2\).

![Fig. 1](image_url)

In Fig. 1, the solid lines refer to nucleons and nuclei, the wavy (dotted) lines represent photons and pions. Graph (a) represents the impulse approximation (IA), while (b) defines the correction related with the nuclear ”collective” pion cloud and thus is effective due to short-ranged \(NN\)-correlation inside nuclei. Their relative role can qualitatively be characterized by the ratio

\[
\frac{t_0(IA)}{t_0(2\pi)} \simeq \frac{8m_n \varepsilon_b}{(A - 1) \cdot 4m_n^2},
\]

where \(t_0\) refers to the beginning of the respective cut in the complex \(t\)-plane, \(m_n\) is the nucleon mass, \(\varepsilon_b\) is the nuclear binding energy. For instance, this ratio is \(\sim .22\) (.40 and .66) for the \(^3He\) and \(^4He\), respectively. This indicates that, naturally, for \(^3He\) and \(^4He\) the ”pionic” contributions will be significantly more important compared to deuteron. Equation (4) also signals that, in the considered respect, the situation for heavier nuclei is expected to be much alike the \(^4He\) case because of nearly equal binding energy per nucleon.

**B. Towards the measurement and systematization of \(<A|\phi^*\phi|A>\)**

A further step in relevant implementation of pionic d.o.f. into the GGT sum rule was an observation inferred from models providing the convergence of the \(\sigma_0(\text{tot})\)-integral. It was first suggested \(^3\)

\[
\sigma_0 = \int d\nu \sigma_{\text{tot}}(\nu) = 2\pi^2 \langle \phi_1 || [D[H, D]] || \phi_1 \rangle,
\]

where the charged scalar field \(\phi_1\) is locally connected with two scalar fields, \(\phi_2\) being charged one and the other, \(\phi_3\), neutral. The double commutator is then interpreted via the known Schwinger-term, \(i.e.,\) the equal-time commutator of the time- and spatial-component of e.m. current operator. Hence, the generalized, ”GGT”-sum rule, implicitly
including the integrals of the absorptive parts of the amplitudes presented by the diagrams with $2\pi$-exchanges, was written \[4\] in the form

$$
\sigma_0^{\gamma A} - Z\sigma_0^{np} - N\sigma_0^{\gamma n} = 2\pi^2\alpha[Z Am + \int d\mathbf{x} (\langle A|\phi^*\phi|A\rangle - \Sigma_i\langle N_i|\phi^*(x)\phi(x)|N_i\rangle)]. \tag{6}
$$

The photonuclear sum rule including the terms $\langle A|\phi^*\phi|A\rangle$ and $\langle N|\phi^*\phi|N\rangle$, represented by the Feynman diagram in Fig.2

Fig.2

\[\sim\sim\bullet\sim\sim\]

\[\Rightarrow \bigcirc \Rightarrow\]

was later rediscovered \[6\], found to be a useful exploration tool \[7\] and widely discussed (e.g., \[8\] and further references therein) in view of the interesting idea about possible partial restoration of the chiral symmetry in real nuclei.

The matter is that up to constant factors the matrix element corresponding to the seagull graph in Fig.2, which in the forward direction is gauge invariant and may have a direct bearing to measurable quantities, is essentially of the same structure as the matrix element represented by Fig.3

Fig.3

\[\otimes\]

\[\Rightarrow \bigcirc \Rightarrow\]

The symbol $\otimes$ denotes the local, scalar quark-current and therefore it is directly connected with the $\Sigma_\pi$-term, hence with the chiral symmetry breaking and its possible (partial) restoration in nuclear matter.

C. FESR and problem of "Big Circle" contribution

The standard FESR technique enables one to deal with the amplitudes defined in the finite region of the complex energy plane

$$
f(\nu) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - \nu},
$$

where $f(\nu)$ is the spin-averaged, forward Compton scattering amplitude and the integration contour includes both sides of the cuts along the real axes $-R \leq \nu \leq R$ closed by a circle of a "big" radius R. As usual the problem consists in the justified and economical
choice for the representation of amplitudes in the complex energy plane to fulfill the integration over the large but finite-radius circle in the complex plane. We keep the original GGT idea of a relation between the photon-nucleus scattering amplitude and a relevant combination of the photon-nucleon amplitudes at sufficiently large photon energies, but our choice of the ”superconvergent” combination of Compton amplitudes $f_{\gamma A(p,n)}$ is different from GGT. It includes amplitudes of two nuclei with $A_1 = Z_1 + N_1$, $A_2 = Z_2 + N_2$ and is assumed to satisfy the condition:

$$
\lim_{\nu \to R} \left[ \frac{1}{A_1} f_{A_1} - \frac{1}{A_2} f_{A_2} \right] = \frac{Z_1 N_2 - N_1 Z_2}{A_1 A_2} (f_p - f_n)|_{\nu = R} + \frac{S_\pi(A_1)}{A_1} - \frac{S_\pi(A_2)}{A_2} \tag{8}
$$

where

$$
S_\pi(A_i) \simeq \frac{\alpha}{3} \int d^3 x (A_i|\vec{\phi}(x)\vec{\phi}(x)|A_i)
$$

and the scalar product in the integrand is understood to be in the isospin space.

The upper limit $\nu_{\text{max}} \equiv R$ in all integrals should be chosen from the compromise provisions.

The first term derived in the approximation linear in $A_i$, $(i = 1, 2)$ is parameterized through the $a_2(J^P; I^G = 2^+; 1^-)$ -Reggeon exchange in the $t$-channel and includes in addition the real constant term seemingly taking place \[9\] in the $Re f_p$ and referring as the residue of the $j = 0$ fixed-pole in the complex angular momentum plane. Hence one should put $R \geq 1.5 \div 2.0$ GeV to apply the Regge-pole phenomenology with the commonly used parameters \[10\]

$$
Im [f_p(\nu) - f_n(\nu)] = (\nu/4\pi)(\sigma^\text{tot}_p - \sigma^\text{tot}_n) = b_{a_2}\nu^{1/2}
$$

$$
Re [f_p(\nu) - f_n(\nu)] = (1/4\pi)b_{a_2}(-\nu^{1/2}) + C_p - C_n
$$

$$
\sigma^\text{tot}_p(\nu) - \sigma^\text{tot}_n(\nu) = 24.6/\nu^{1/2} \tag{10}
$$

Following \[9\], we accept $C_p \simeq -3.0\mu b \cdot GeV$ and put the $C_n$-value rather arbitrarily to be either $C_n = (2/3)C_p$ or $C_n = 0$ for the sake of further numerical estimations.

Due to the dominant scalar-isoscalar nature of the pionic operators we accept $S_\pi(p) \simeq S_\pi(n)$ while $S_\pi(A_i) \neq 0$ will disclose its essential nonlinear dependence on the atomic number $A$ of real nuclei.

III. NUMERICAL RESULTS AND DISCUSSION

As an example of the generalized nuclear sum rule applications, we choose a pair of lightest nuclei - the deuteron and $^3He$. While in the deuteron case the total photoabsorption cross section is known well above our taken $\nu_{\text{max}} \simeq 1.6$ GeV, the $\sigma_{\text{tot}}(\gamma^3He)$ is known to $0.8$ GeV \[? \] ; hence, in this case, we have to take $\nu_{\text{max}} = 0.8$ GeV. The major purpose of using these new types nuclear sum rules may be the extraction of information about the value of difference of the nuclear matrix elements: $\Delta \sigma_\pi = \int d\vec{x} \frac{m_e^2}{2}|\frac{1}{A_i} <
\[ A_1 |\bar{\phi}(x) \cdot \bar{\phi}(x)|A_1 > - \frac{1}{A_2} < A_2 |\bar{\phi}(x) \cdot \bar{\phi}(x)|A_2 >. \]

The term \( \Delta \sigma_\pi \) can thus be extracted from experimentally measurable quantities to give useful information on the values closely related with the chiral symmetry characteristics in real nuclei. Of special interest is the situation when \( Z_1 N_2 − N_1 Z_2 = 0 \) in (8), as for the deuteron- and \(^4\)He-pair, to mention. The contribution of the \( \alpha_2\)-Reggeon is then absent and the optimal value of \( \nu_{\text{max}} = R \) in dispersion integrals of cross sections could probably be taken at a lower value. Qualitatively, this newly chosen \( R \)-value should provide a reasonable balance between the contribution of the same group of most important nucleon resonances into the real parts of nuclear Compton amplitudes represented by the terms \( S_\pi(A_i) \) and the respective imaginary parts entering into dispersion integrals in the form of the corresponding nuclear photo-pion production cross sections.

For arbitrary \( A_1 = Z_1 + N_1 \) and \( A_2 = Z_2 + N_2 \) our general sum rule reads

\[
\frac{2\pi^2}{A_1} \left[ f_{A_1}(\nu = 0) + S_\pi(A_1) \right] - \frac{f_{A_2}(\nu = 0) + S_\pi(A_2)}{A_2} + \frac{Z_1 N_2 - Z_2 N_1}{A_1 A_2} \left( \frac{2 b_{\alpha_2} \nu_{\text{max}}^{1/2}}{2\pi^2} − C_p + C_n \right) = \frac{\sigma_0^{\nu_{\text{max}}}(\gamma A_1)}{A_1} − \frac{\sigma_0^{\nu_{\text{max}}}(\gamma A_2)}{A_2} \tag{11}
\]

where \( f_{A_i}(\nu = 0) \approx -(\alpha Z_i^2)/(A_i m_n) \) is the Thompson zero-energy amplitude, \( S_\pi(A_i) \) is defined in Eq. (9) and the integration in \( \sigma_0^{\nu_{\text{max}}} \) extends from the photodisintegration threshold to the upper bound \( \nu_{\text{max}} \). In the case of \(^3\)He and deuteron the integration was carried out with the cross-sections tabulated in [11] up to \( \nu_{\text{max}} = .8 \text{ GeV} \). The low-energy integrals up to the pion photoproduction thresholds \( \nu_{\gamma \pi} \) were approximated by

\[
\sigma_0^{\nu_{\gamma \pi}} = 60 \frac{N Z}{A} (1 + \kappa_A^{\text{exp}}) \left[ \mu b \cdot \text{GeV} \right] \tag{12}
\]

where \( \kappa_A^{\text{exp}} = .75 \pm .10 \ (,37 \pm .11) \), following [12]. To have an idea about the scale of \( S_\pi \) for the nuclei considered, we confronted the calculated values of

\[
(2\pi^2 \alpha)/(3) [(1/3)S_\pi(^3\text{He}) − (1/2)S_\pi(d)] \simeq 7.75 \ (1.17) \left[ \mu b \cdot \text{GeV} \right]
\]

for \( C_p = −3, C_n = −2 \ (0) \) with the value

\[
60 \cdot [(2/9) \kappa_3\text{He} − (1/4) \kappa_d] = (1/3) \cdot 40 \cdot (75 \pm .10) − (1/2) \cdot 30 \cdot (37 \pm .11) \simeq 4.4 \pm 2.1 \left[ \mu b \cdot \text{GeV} \right]
\]

representing the ”potential parts” in the difference of non-relativistic TRK sum rules

\[
2\pi^2 \alpha[(1/3) <^3\text{He}|D,[V_{NN}]|^3\text{He}] > − (1/2) < d|[D,[V_{NN}]]|d >
\]

The correspondence looks reasonable because the non-relativistic value is in between two values following from a more general sum rule with the differing values of \( C_n \). We also draw attention to a strong dependence of the mentioned estimations on two chosen numerical values of \( C_n \), which emphasizes the significance of sum rule as a source of new interesting information.
IV. CONCLUSION

The pion-nucleon sigma-term

\[ \sigma = \frac{\hat{m}}{2m_p} \langle p|\bar{u}u + \bar{d}d|p \rangle, \quad \hat{m} = \frac{1}{2}(m_u + m_d), \]

and, generally, sigma-terms of a given hadron are proportional to the scalar quark currents

\[ \langle A|m_q\bar{q}q|A \rangle ; q = u, d, s ; A = \pi, K, N, Z A_N - nuclei. \]

These are of great physical significance because they are related to the hadron masses, to the meson scattering amplitudes [13], to the strangeness content of \( A \), and to the properties of nuclear [14] and dark [15] matter. Our derived and discussed photoabsorption sum rules are focused on the comparison of the scalar pion densities, hence, in part, of the pionic \( \sigma \)-terms for different nuclei to trace their dependence on the atomic number. The deuteron sum rule provides thereupon the situation most close to free nucleons while the helium-4 plays the role of a drop of the real nuclear matter.

In view of the above discussion the following looks to be practically important:

1. To extend measurements of the total photoabsorption on the \( ^3He \) and \( ^4He \)-nuclei at least up to energy of photons \( 1.5 \div 2.0 \text{ GeV} \).
2. To complete calculation of \( < A|[D[H, D]|A >, A = ^3(^4)He, \) with best modern potentials and respective wave functions as well as with estimation of relativistic corrections.

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[1] M.Gell-Mann, M.L.Goldberger and W.Thirring, Phys.Rev.95(1954) 1612.
[2] S.B.Gerasimov, Phys.Lett., 5(1963) 259.
[3] S.B.Gerasimov, Phys.Lett. 13(1964) 240.
[4] S.B.Gerasimov, in Proc.Int.Conf. on Electromagnetic Interactions at Low and Medium Energies, AN USSR, Moscow, 1972, v.3, p.382.
[5] S.B.Gerasimov and J.Moulin, Nucl.Phys. B98 (1975) 349.
[6] S.A.Kulagin, TRK 91-39, Uni. Regensburg, 1991.
[7] M.Ericson, M.Rosa-Clot, and S.A.Kulagin, Nuovo.Cim. A111(1998)75.
[8] M.Ericson, G.Chanfray, J. Delorm, and M.Rosa-Clot, Nucl.Phys. A663&664 (2000) 369c.
[9] M.Damashek and F.J.Gilman, Phys.Rev. D1 (1970)1319.
[10] W.P. Hesse, et al., Phys.Rev.Lett. 25 (1970) 613.
[11] M.MacCormick, et al., Phys.Rev.C 53 (1996) 41.
[12] D.Drechsel and Y.E.Kim, Phys.Rev.Lett. 40 (1978) 531.
[13] M.E. Sainio, PiN Newslett. 16 (2001) 138.
[14] J.N. Ginocchio, Phys.Rep. 414 (2005) 165.
[15] A. Bottino, et al., Astropart. Phys., 18 (2002) 205.