Scale anomalies imply violation of the averaged null energy condition

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Considerable interest has recently been expressed regarding the issue of whether or not quantum field theory on a fixed but curved background spacetime satisfies the averaged null energy condition (ANEC). A comment by Wald and Yurtsever [Phys. Rev. D43, 403 (1991)] indicates that in general the answer is no. In this note I explore this issue in more detail, and succeed in characterizing a broad class of spacetimes in which the ANEC is guaranteed to be violated. Finally, I add some comments regarding ANEC violation in Schwarzschild spacetime.

I. INTRODUCTION

Considerable interest has recently been expressed regarding the issue of whether or not quantum field theory on a fixed but curved background spacetime satisfies the averaged null energy condition (ANEC). The ANEC is one of the weakest energy conditions in common use — it is used for instance as a hypothesis in proving sundry focusing theorems for null geodesics [1], in proving certain modifications of the original singularity theorems [2], in proving the topological censorship theorem [3], and in proving a recent version of the positive mass theorem [4]. It is thus a topic of considerable interest to know if the ANEC can itself be proved from more basic postulates of quantum field theory. Promising progress toward such a proof was made by Klinkhammer [5], by Yurtsever [6], and by Wald and Yurtsever [7]. Unfortunately, it is now realised that in general the ANEC must fail. See the “note added in proof” on page 415 of [7]. The Wald-Yurtsever “note added in proof” applies only to generic perturbations around a flat Minkowski spacetime. In this letter I wish to explore this issue in a little more depth and detail.

I show that in a general spacetime the relevant issue is whether or not the renormalized expectation value of the quantum stress–energy tensor possesses a scale anomaly. A scale transformation is defined as a position independent conformal rescaling of the metric. Scale anomalies are related to conformal anomalies but are considerably easier to deal with. I then analyze conditions under which the scale anomaly vanishes: The scale anomaly is zero for Einstein spacetimes (in particular for Schwarzschild spacetime) and is zero for conformally flat spacetimes, but is non–zero generically.

Some more detailed statements about the status of the ANEC in Schwarzschild spacetime can be made by appealing to Howard’s numerical calculations [8,9] of the renormalized vacuum expectation value of the stress–energy tensor.

II. AVERAGED NULL ENERGY CONDITION

Definition 1 The averaged null energy condition is said to hold on a null curve \( \gamma \) if

\[
\int_{\gamma} T_{\mu\nu} k^\mu k^\nu \, d\lambda \geq 0.
\]  

(1)

Here \( \lambda \) is a generalized affine parameter for the null curve, the tangent vector being denoted by \( k^\mu \).

Technical points: (1) If \( \gamma \) is a null geodesic then the generalized affine parameter specializes to the ordinary affine parameter. See for example, [11], page 259. (2) Permitting arbitrary parameterizations would not be useful. If arbitrary parameterizations were to be allowed, the ANEC would be equivalent to the ordinary null energy condition.

To get a physical feel for what is going on, suppose, for the sake of discussion, that the stress–energy tensor is type I. In a suitable orthonormal frame, the energy density and three principal pressures are given by

\[
T^{\hat{\mu}\hat{\nu}} = \begin{bmatrix}
\rho & 0 & 0 & 0 \\
0 & p_1 & 0 & 0 \\
0 & 0 & p_2 & 0 \\
0 & 0 & 0 & p_3
\end{bmatrix}.
\]  

(2)
In this orthonormal frame one can define a function $\xi$, and direction cosines $\cos \psi_j$, by

$$k^\mu = \xi (1; \cos \psi_j).$$

Then

$$\text{ANEC} \iff \int_\gamma \left[ \rho + \sum_j (\cos^2 \psi_j) p_j \right] \xi^2 \, d\lambda \geq 0.$$  

(4)

In applications one typically requires the ANEC to hold on some suitable class $\Gamma = \{ \gamma \}$ of inextendible null geodesics.

III. ANEC VIOLATIONS

The central result of this letter is

**Theorem 1** In any (3 + 1)-dimensional spacetime, for any conformally coupled quantum field, in any conformal quantum state: If the scale anomaly is non-zero, then the renormalization scale $\mu$ can be chosen in such a way that the ANEC is violated.

As background, recall the tremendous amount of technical machinery available for analyzing the behaviour of the stress–energy tensor under conformal deformations [11–13].

A *scale transformation* is just the special case of a conformal transformation when the conformal rescaling factor is position independent

$$g(x) \rightarrow \bar{g}(x) = \Omega^2 g(x).$$

(5)

For a conformally coupled quantum field, consider the renormalized vacuum expectation value of the quantum stress–energy tensor evaluated on a conformal quantum state. In (3 + 1)-dimensions this object undergoes an anomalous scaling transformation

$$T^\mu_\nu \left( \bar{g} \right) = \Omega^{-4} \left( T^\mu_\nu \left( g \right) - 8a \ln \Omega \left[ \nabla_\alpha \nabla^\beta + \frac{1}{2} R_\alpha^\beta \right] C^{\alpha \mu}_{\beta \nu} \right).$$

(6)

This is a standard result. For example, this is a special case of equation (66) of Page [13]. For future convenience it is useful to define

$$Z^\mu_\nu \equiv \left[ \nabla_\alpha \nabla^\beta + \frac{1}{2} R_\alpha^\beta \right] C^{\alpha \mu}_{\beta \nu}.$$  

(7)

The anomalous scaling of the stress–energy under a rescaling of the metric is a consequence of the fact that in regularizing the conformal quantum field one has had to introduce a cutoff. This cutoff breaks the conformal invariance and, after proper renormalization to remove explicit cutoff dependence, results in a dimensional transmutation effect whereby the expectation value depends on a so–called renormalization scale $\mu$. (See, for example, [13,14]). One may make this renormalization scale dependence explicit by writing

$$T^\mu_\nu \left( \Omega g; \mu \right) = T^\mu_\nu \left( g; \mu/\Omega \right).$$

(8)

Thus, the scale anomaly may be recast as

$$T^\mu_\nu \left( g; \mu/\Omega \right) = \Omega^{-4} \left[ T^\mu_\nu \left( g; \mu \right) - 8a \ln \Omega \left( Z^\mu_\nu \right) \right].$$

(9)

The coefficient $a$ is exactly the same as that arising in the perhaps more familiar conformal trace anomaly

$$T^\mu_\mu = a \left( C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \right) + b \left( * R_{\mu\nu\lambda\rho} * R^{\mu\nu\lambda\rho} \right) + c \Box R + d R^2.$$  

(10)

Now consider the effect of the scale anomaly on the ANEC integral. Let $\gamma$ be any null curve of the metric $g$ parameterized by a generalized affine parameter $\lambda$. Then
\[
I_\gamma (\mu/\Omega) \equiv \int_\gamma T_{\mu \nu} (g; \mu/\Omega) k_\mu k_\nu \, d\lambda,
\]
(11)
\[
= \int_\gamma \Omega^{-4} \left[ T_{\mu \nu} (g; \mu) - 8a \ln \Omega Z^{\mu \nu} \right] k_\mu k_\nu \, d\lambda,
\]
\[
= \Omega^{-4} I_\gamma (\mu) - 8a \Omega^{-4} \ln \Omega \int_\gamma Z^{\mu \nu} k_\mu k_\nu \, d\lambda.
\]
(12)
For convenience define
\[
J_\gamma = \int_\gamma Z^{\mu \nu} k_\mu k_\nu \, d\lambda.
\]
(13)
Now if \( J_\gamma \neq 0 \) it is always possible to choose \( \ln \Omega \) sufficiently large (either positive or negative) to force \( I_\gamma < 0 \). This means that ANEC is guaranteed to be violated for the null curve \( \gamma \) for a suitable choice of the renormalization scale \( \mu \). [Technical points: (1) For simplicity I have assumed that all the integrals converge. The analysis of Wald and Yurtsever [7], pages 404–405, can be adapted to deal with the case where the ANEC integral does not converge. (2) If \( Z^{\mu \nu} \neq 0 \) then a generic null curve which intersects the support of \( Z^{\mu \nu} \) will have \( J_\gamma \neq 0 \).] This completes the proof.

An alternative formulation is

**Theorem 2** In \((3 + 1)\)-dimensional spacetime the ANEC is guaranteed to be violated whenever

\[
Z^{\mu \nu} \equiv \left[ \nabla_\alpha \nabla^\beta + \frac{1}{2} R^\alpha_{\beta} \right] C^{\alpha \mu \beta \nu} \neq 0.
\]
(14)

Comments:

- If \( Z^{\mu \nu} = 0 \) we cannot deduce that the ANEC must be satisfied. We can only deduce that it is possible for the ANEC to be satisfied. Cases are known where \( Z^{\mu \nu} = 0 \), with the ANEC being satisfied on null geodesics, but with the ANEC violated along certain classes of non–geodesic null curves [3]. For another example, see the discussion of the Schwarzschild spacetime given below.

- In Minkowski spacetime \( Z^{\mu \nu} = 0 \). Thus Minkowski spacetime evades the no–go theorem. This result is compatible with the theorems proved by Klinkhammer [5].

- In conformally flat spacetimes \( Z^{\mu \nu} = 0 \). All conformally flat spacetimes evade the no–go theorem.

- In \((1 + 1)\)-dimensions the scale anomaly vanishes. [For example, take equation (6.134) of Birrell and Davies [11] and set \( \nabla \Omega = 0 \).] This happy accident is due to the fact that all two–dimensional manifolds are conformally flat. Thus all \((1 + 1)\)-dimensional spacetimes evade the no–go theorem. This result is compatible with the theorems proved by Yurtsever [8] and Wald and Yurtsever [7].

- In any Einstein spacetime \( Z^{\mu \nu} = 0 \). To see this, recall that an Einstein spacetime is defined by

\[
R_{\mu \nu} = \Lambda g_{\mu \nu}.
\]
(15)
Then note

\[
R^\alpha_{\beta} C^{\alpha \mu \beta \nu} = \Lambda g^\alpha_{\beta} C^{\alpha \mu \beta \nu} = 0.
\]
(16)
Furthermore, starting from the Bianchi identities, a single contraction yields (See, for example, [10], equation (2.32) page 43.)

\[
\nabla_\mu C^{\mu \nu \sigma \rho} = -R_{\mu [\nu | \sigma | \rho]} + \frac{1}{6} g_{\mu [\sigma} R_{\rho]} \]
(17)
In view of the definition of an Einstein spacetime this implies

\[
\nabla_\mu C^{\mu \nu \sigma \rho} = 0.
\]
(18)
Therefore \( Z^{\mu \nu} = 0 \); all Einstein spacetimes have zero scale anomaly and evade the no–go theorem. In particular, Schwarzschild spacetime has \( Z^{\mu \nu} = 0 \) and so evades the no-go theorem.
• On the other hand, generic perturbations of any spacetime with zero scale anomaly will lead to a non–zero scale anomaly. So generic perturbations of manifolds satisfying the ANEC lead to manifolds where the ANEC is violated by a suitable choice of renormalization scale. In this sense violations of the ANEC are generic. The observation generalizes the case of linearized perturbations around flat Minkowski space which was addressed in [7].

• If one chooses to work with massless quantum fields that are not conformally coupled the analysis is more complicated — this is why the Wald–Yurtsever involves two independent fourth order curvature tensors [7]. (Conformal coupling implies that their coefficient $b$ is zero.)

IV. SCHWARZSCHILD SPACETIME

In the particular case of Schwarzschild spacetime the explicit numerical calculations of Howard [8], and Howard and Candelas [9] permit one to make some more detailed statements about the ANEC. By spherical symmetry one knows that

$$\langle 0_H|T^{\hat{\mu}\hat{\nu}}|0_H \rangle \equiv \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \quad (19)$$

For example, at the horizon, a conformally coupled scalar field in the Hartle–Hawking vacuum state has stress–energy

$$\langle 0_H|T^{\hat{\mu}\hat{\nu}}|0_H \rangle \approx \pi^2 \left(\frac{1}{8\pi m}\right)^4 \begin{bmatrix} -37.728 & 0 & 0 & 0 \\ 0 & +37.728 & 0 & 0 \\ 0 & 0 & +10.29 & 0 \\ 0 & 0 & 0 & +10.29 \end{bmatrix}. \quad (20)$$

This immediately implies, from the negativity of $\rho$, violation of the weak and dominant energy conditions. Checking that $\rho + p < 0$, one also sees that the null and strong energy conditions are violated. (For definitions, see for instance [10]). Warning: The paper by Fawcett [15] unfortunately contains an error. See [8,9].

The status of the ANEC is considerably more complicated: rewrite the ANEC integral as

$$I_\gamma = \int_\Gamma (\rho - \tau \cos^2 \psi + p \sin^2 \psi) \xi^2 \, d\lambda, \quad (21)$$

$$= \int_\Gamma (|\rho - \tau| + |\tau + p| \sin^2 \psi) \xi^2 \, d\lambda. \quad (22)$$

Comments:

• The numerical data of Howard [8] indicate that outside the horizon $\rho - \tau > 0$. ($\rho = \tau$ at the horizon.) The numerical data covers the range from $r = 2m$ to $r = 6.8m$. For larger radial distances one resorts to Page’s analytic approximation, which becomes increasingly good as one moves further away from the horizon [8,13]. This implies that the ANEC is satisfied for radial null geodesics. Caveat: The ANEC integral has to be modified to a one–sided integral by cutting it off at the event horizon.

• Null geodesics that come in from infinity and return to infinity never get closer to the origin than $r = 3m$. See, for instance, [14], pages 672–678. Inspection of Howard’s numerical data indicates that the integrand of the ANEC integral is strictly positive for $r \geq 3m$. (Inspection shows that $\rho, -\tau,$ and $p$ are all positive for $r \geq 3m$.) Thus the ANEC is satisfied along all null geodesics that come from, and return to, infinity.

• For the unstable circular photon orbit at $r = 3m$ the ANEC integral is proportional to $\rho + p$. Inspection of Howard’s numerical data indicates that $\rho + p > 0$ at $r = 3m$, so the ANEC is satisfied along this particular null geodesic.

• For incoming null geodesics with smaller than critical impact parameter, the null geodesic may circle the black hole a large number of times, but is guaranteed to ultimately plunge into the event horizon [16, pages 672–678]. This makes the analysis a little more subtle. Inspect the ANEC integrand. Inspecting the data shows that $\rho - \tau$ is always positive, while $\tau + p$ is always negative. (Where the numerical data is available, check this using the
numerical data. At large radius, check this using Page’s approximation.) Now use the fact that for an infalling null geodesic

$$\sin^2 \psi < 27 \frac{m^2}{r^2} (1 - 2m/r).$$

(23)

Since $\tau + p$ is negative, this implies a lower bound on the ANEC integral

$$I_\gamma \geq \int \{ [\rho - \tau] + 27 [\tau + p] \frac{m^2}{r^2} (1 - 2m/r) \} \xi^2 \, d\lambda.$$

(24)

Inspection, either of the numerical data, or of Page’s approximation, indicates that the integrand of this lower bound is strictly positive (zero at the horizon). Thus the ANEC holds on all infalling null geodesics.

- By time reversal, the ANEC holds on all outgoing null geodesics that reach infinity.
- There remains the issue of those null geodesics that emerge from the horizon, circle the black hole, and plunge back in. For such trapped null geodesics one has at all times

$$\sin^2 \psi > 27 \frac{m^2}{r^2} (1 - 2m/r).$$

(25)

Unfortunately, this observation does not permit one to derive a useful bound on the ANEC integral. (By working in terms of the “impact parameter” it is relatively easy to convince oneself that the ANEC integrand is permitted to be negative for regions sufficiently close to the event horizon.)

- For any circular null curve at fixed $r$ (not a geodesic except in the case of $r = 3m$) the ANEC integral is still proportional to $\rho + p$. Inspection of Howard’s numerical data indicates that $\rho + p < 0$ for $r \lesssim 2.25 m$. (Page’s analytic approximation gives a slightly different result, $\rho + p < 0$ for $r < 2.18994 m$.) Thus the ANEC is not satisfied for this particular class of non-geodesic null curves.

Collecting these comments, I have:

**Theorem 3** Consider the renormalized vacuum expectation value of the stress–energy tensor for a conformally coupled scalar field in Schwarzschild spacetime. Then (1) the ANEC is satisfied for all null geodesics that reach spatial infinity. (2) there are non-geodesic null curves along which the ANEC is violated.

It should be possible to generalize these observations. For instance: (1) The behaviour of the ANEC on trapped null geodesics is still somewhat obscure. Presumably the ANEC depends on the “impact parameter” in an interesting way. (2) By using Page’s analytic approximation one could write down analytic estimates for the ANEC integral. I have not attempted to do so in this letter because I felt it to be instructive to first develop simple physical bounds on the ANEC integral. (3) It would be nice to go beyond the numerics; to develop some exact analytic arguments that go beyond the Page approximation.

**V. DISCUSSION**

Investigation of the properties of the averaged null energy condition is of considerable interest to diverse applications in both classical and semiclassical quantum gravity. It is now clear that, in general, semiclassical quantum fields do not satisfy the ANEC. In this letter, I have related ANEC violations to the existence of the scale anomaly — If the scale anomaly is non–zero then the ANEC is guaranteed to be violated. Even if the scale anomaly vanishes, this does not necessarily imply that the ANEC is satisfied: one has to do a case by case analysis. As an example, the situation in Schwarzschild spacetime was investigated.

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