On-shell renormalization scheme for $\mathcal{N} = 1$ SQED and the NSVZ relation

A. L. Kataev$^{1,2,a}$, A. E. Kazantsev$^{3,b}$, K. V. Stepanyantz$^{4,c}$

$^1$ Institute for Nuclear Research of the Russian Academy of Sciences, 117312 Moscow, Russia
$^2$ Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Moscow Region, Russia
$^3$ Department of Quantum Theory and High Energy Physics, Faculty of Physics, Moscow State University, 119991 Moscow, Russia
$^4$ Department of Theoretical Physics, Faculty of Physics, Moscow State University, 119991 Moscow, Russia

Received: 13 May 2019 / Accepted: 25 May 2019 / Published online: 5 June 2019
© The Author(s) 2019

Abstract In this paper we investigate the renormalization of $\mathcal{N} = 1$ supersymmetric quantum electrodynamics, regularized by higher derivatives, in the on-shell scheme. It is demonstrated that in this scheme the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) equation relating the $\beta$-function to the anomalous dimension of the matter superfields is valid in all orders of the perturbation theory. This implies that the on-shell scheme enters the recently constructed continuous set of NSVZ subtraction schemes. To verify this statement, we compare the anomalous dimension of the matter superfields in the two-loop approximation and the $\beta$-function in the three-loop approximation, which are explicitly calculated in this scheme. The finite renormalizations relating the on-shell scheme to some other NSVZ subtraction schemes formulated previously are obtained.

1 Introduction

Among various renormalization schemes that can be used in quantum electrodynamics the subtraction on the mass shell is one of the most important (for a review, see, e.g., [1]). The reason for this is that in this scheme renormalized quantities such as masses, charges, and anomalous magnetic moments can be subjected to direct experimental measurements. This distinguishes it from the (modified) minimal subtraction or momentum subtraction schemes. The $\mathcal{N} = 1$ supersymmetric generalization of quantum electrodynamics, besides the electron and photon, contains their superpartners, namely, a pair of complex scalar fields and a Majorana spinor. It is most convenient to describe this theory using the superfield formalism with the gauge fixing term respecting supersymmetry. In this case $\mathcal{N} = 1$ supersymmetry appears to be a manifest symmetry of the theory, so that the perturbative calculations can be done in an $\mathcal{N} = 1$ supersymmetric way. This is to be contrasted with the approach, when the gauge superfield is put into the Wess–Zumino gauge, in which only its physical components survive. Although in this case the quantization is performed in terms of physical fields only, the manifest supersymmetry is lost.

An important feature of $\mathcal{N} = 1$ supersymmetric gauge theories is the existence of the relation between the $\beta$-function and the anomalous dimensions of the matter superfields [2–9]. For $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) considered in this paper it can be written as [10,11]

$$\beta(\alpha) = \frac{N_f \alpha^2}{\pi} \left( 1 - \gamma(\alpha) \right).$$

(1)

It is known that the NSVZ relation does not in general hold for an arbitrary renormalization prescription$^{1}$ and is valid only in certain subtraction schemes called the NSVZ schemes. Recently, it was discovered that all NSVZ subtraction schemes in $\mathcal{N} = 1$ SQED form a class that can be parameterized by a single function and a single constant [14]. Various schemes of this class are related by finite renormalizations satisfying a certain condition, which form a subgroup of the general renormalization group transformations [15–18]. This class in particular includes the so-called HD+MSL renormalization prescription (short for Higher Derivatives plus Minimal Subtraction of Logarithms) [19,20]. In this case a theory is regularized by the higher covariant derivative method [21,22] (see also Refs. [23–25] for its various $\mathcal{N} = 1$ supersymmetric versions) and only powers of $\ln \Lambda / \mu$

---

1 The general equations describing how the NSVZ relation changes under finite renormalizations can be found in [12,13].
are included into the renormalization constants. Here \( \Lambda \) is the
dimensionful parameter of the regularized theory, and \( \mu \) is the
subtraction point. Note the HD+MSL prescription gives the
NSVZ-like schemes for various theories, e.g., for
the photon mass in the electrodynamics with softly broken
supersymmetry [26] or for the Adler \( D \)-function in \( N = 1 \)
SQCD [30]. There are indications that the NSVZ scheme in
the non-Abelian case is also given by this prescription in all
orders [36]. This conjecture has been confirmed by explicit
three-loop calculations in [37,38].

The \( \text{DR} \) scheme, most frequently used for practical cal-
culations, does not enter the class of the NSVZ schemes,
as was explicitly demonstrated in the three- [39] and four-
loop [40] approximations. Nevertheless, a finite renormaliza-
tion of the coupling constant, specially tuned in each order
of the perturbation theory, allows constructing the NSVZ
scheme with dimensional reduction [39,41,42]. The differ-
ence between calculations with the higher derivative regulariza-
tion and with dimensional reduction for \( N = 1 \) SQED in
the three-loop approximation has been analyzed in Ref. [43].

Because the subtraction on the mass shell occupies a spe-
cial place in electrodynamics, it would be interesting to find
out whether a relation (1) is satisfied in this scheme and, there-
fore, whether it falls into the class of NSVZ schemes. Using
the results of Ref. [44], the guess was made that the NSVZ
relation in \( N = 1 \) SQED is valid in the on-shell scheme [45].
Note that the explicit calculations in Ref. [44] were done only
in the approximation, where the scheme dependence is not
essential. In this paper we demonstrate that the NSVZ equa-
tion relating the \( \beta \)-function to the mass anomalous dimen-
sion is valid in the on-shell scheme in all orders. This state-
ment is verified by the explicit calculation. Namely, the three-loop
\( \beta \)-function is compared with the two-loop mass anomalous
dimension in the on-shell scheme. This allows to check that
Eq. (1) really holds in this case.

\( 2N = 1 \) SQED: action and the higher derivative
regularization

In the superfield language \( N = 1 \) SQED with \( N_f \) flavors of
massive Dirac fermions and their superpartners is described by
the action

\[
S = \frac{1}{4\epsilon_0^2} \text{Re} \int d^4x d^2\theta \ W^a W_a
\]

\[\uparrow\]\footnote{The NSVZ-like equation describing the renormalization of the gaugino mass has been proposed in [27–29].}

\[\uparrow\]\footnote{This follows from the fact that the renormalization group functions
defined in terms of the bare couplings satisfy the NSVZ and NSVZ-like equation with the higher derivative regularization. At present, it has been rigorously proved in all orders for \( N = 1 \) SQED in [31,32], for the renormalization of the photino mass in softly broken SQED in [33], and for the Adler \( D \)-function in \( N = 1 \) SQCD in [34,35].}

\[
+ \frac{1}{4} \sum_{i=1}^{N_f} \int d^4x d^4\theta \left( \phi_i^* e^{2V} \phi_i + \Phi_i^* e^{-2V} \Phi_i \right)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{N_f} \left( \int d^4x d^2\theta \right) m_0 \Phi_i \phi_i + \text{c.c.}
\]

(2)

where \( m_0 \) is the bare mass of the chiral matter superfields. For
simplicity, and in order not to deal with multiple thresholds,
we assume the masses for different flavors to be equal.

The regularization is introduced by adding to the action
the term with higher derivatives

\[
S_\Lambda = \frac{1}{4\epsilon_0^2} \text{Re} \int d^4x d^2\theta \ W^a \left[ R \left( \frac{\beta^2}{\Lambda^2} \right) - 1 \right] W_a,
\]

(3)

where \( R \) is a function which rapidly increases at large values
of the argument and satisfies the condition \( R(0) = 1 \). Moreover,
to regularize divergences in the one-loop approximation,
it is necessary to insert the Pauli–Villars determinants in
the generating functional [46]. Following Refs. [47,48],
let us introduce \( n \) sets of the chiral Pauli–Villars superfields
\( \Phi_{ij} \) with masses \( M_I \), where \( I = 1, \ldots, n \), and include

\[
S_{PV} = \sum_{I=1}^{n} \left( \sum_{i=1}^{N_f} \int d^4x d^4\theta \left( \Phi_i^* e^{2V} \Phi_i + \Phi_i^* e^{-2V} \Phi_i \right) + \frac{1}{2} \sum_{i=1}^{N_f} \left( \int d^4x d^2\theta M M_i \phi_i + \text{c.c.} \right) \right)
\]

(4)

into the total action. Then, to cancel one-loop divergences,
their Grassmannian parities \((-1)^{P_I} \) and masses \( M_I \) should
satisfy the relations

\[
\sum_{I=1}^{n} (-1)^{P_I} + 1 = 0; \quad \sum_{I=1}^{n} (-1)^{P_I} M_I^2 + m_0^2 = 0.
\]

(5)

In the massless case the masses of the Pauli–Villars super-
fields \( M_I \) should be chosen proportional to the parameter \( \Lambda \)
in the higher derivative term. However, in the massive case
it is convenient to present them in the form

\[
M_I^2 = a_I^2 \Lambda^2 + b_I^2 m_0^2,
\]

(6)

where the coefficients \( a_I \) and \( b_I \), independent of the coupling
constant, satisfy the equations

\[
\sum_{I=1}^{n} (-1)^{P_I} a_I^2 = 0; \quad \sum_{I=1}^{n} (-1)^{P_I} b_I^2 + 1 = 0,
\]

(7)

which follow from Eq. (5). It should be noted that the deriva-
tive of \( M_I/\Lambda \) with respect to \( \ln \Lambda \) or \( \ln m_0 \) is of the order
\( m_0^2/\Lambda^2 \) and, therefore, can be neglected in the limit \( \Lambda \to \infty \).

To complete the quantization, the gauge-fixing term

\[
S_{gf} = - \frac{1}{32\epsilon_0^2} \int d^4x d^4\theta D^2 V R \left( \frac{\beta^2}{\Lambda^2} \right) \bar{D}^2 V
\]

(8)
is added to the action. Below we will use the Feynman gauge in which the renormalized gauge fixing parameter is fixed as $\xi = 1$.

### 3 The on-shell subtraction scheme

To construct the on-shell scheme for $\mathcal{N} = 1$ SQED, let us consider the part of the effective action quadratic in the matter superfields. It can be presented in the form

$$
\Gamma^{(2)}_{\phi} = \frac{1}{2} \sum_{i=1}^{N_f} \int \frac{d^4 p}{(2\pi)^4} d^4 \phi \left( \phi_i^* (-p, \theta) \phi_i (p, \theta) \right)
+ \frac{1}{2} \sum_{i=1}^{N_f} \left( \int \frac{d^4 p}{(2\pi)^4} d^2 \theta m_0 \phi_i^*(-p, \theta) \phi_i (p, \theta) \right.
\left. \times J(p/\Lambda, m_0/\Lambda, \alpha_0) + c.c. \right),
$$

where the functions $G$ and $J$ are normalized in such a way that in the tree approximation $G = 1$ and $J = 1$. From the expression (9) it is possible to construct the exact superpropagators for the matter superfields, see Ref. [32] for details. In the coordinate representation they are written as

$$
\left( \frac{\delta^2 \Gamma}{\delta \phi_i \delta \phi_j} \right)^{-1} = \left( \frac{\delta^2 \Gamma}{\delta \phi_i \delta \phi_j} \right)
= \frac{-G D_i^2 D_j^2}{4(\partial^2 G^2 + m_0^2 J^2)} \delta_{ij};
$$

$$
\left( \frac{\delta^2 \Gamma}{\delta \phi_i \delta \phi_j} \right)^{-1} = \frac{-m_0 J D_i^2}{\partial^2 G^2 + m_0^2 J^2} \delta_{ij};
$$

$$
\left( \frac{\delta^2 \Gamma}{\delta \phi_i \delta \phi_j} \right)^{-1} = \frac{-m_0 J D_j^2}{\partial^2 G^2 + m_0^2 J^2} \delta_{ij};
$$

In the momentum representation all these propagators contain the denominator

$$
p^2 G^2(p) - m_0^2 J^2(p),
$$

where all arguments of the functions $G$ and $J$ except for the momentum $p$ were omitted.

The renormalized mass in the on-shell scheme is defined as the pole of the propagators (10),

$$
m = m_0 \frac{J(p)}{G(p)} \bigg|_{p^2 = m^2}.
$$

It is convenient to introduce the mass renormalization constant $Z_m \equiv m_0 / m$, which in the scheme under consideration is given by the expression

$$
Z_m = \frac{G(p)}{J(p)} \bigg|_{p^2 = m^2}.
$$

The matter superfield renormalization constant $Z$ in the on-shell scheme is given by the residue at this pole. For all propagators (10) the result is the same,

$$
Z^{-1} = \frac{G(p)}{J(p)} \left( 1 + 2m^2 \frac{\partial}{\partial p^2} \ln \frac{G(p)}{J(p)} \right) \bigg|_{p^2 = m^2}.
$$

Note that, due to the superpotential non-renormalization in $\mathcal{N} = 1$ supersymmetric theories [49], it is usually assumed that $ZZ_m = 1$. However, in the one-shell scheme it is not so, because

$$
Z^{-1} Z_m^{-1} = J(p) \left( 1 + 2m^2 \frac{\partial}{\partial p^2} \ln \frac{G(p)}{J(p)} \right) \bigg|_{p^2 = m^2}.
$$

Although this expression is not equal to 1, it is finite in the ultraviolet region due to the non-renormalization of the superpotential. This implies that the renormalization constants $Z$ and $Z_m^{-1}$ differ by a finite factor.

Note that in the component formulation of the theory the scalars and the spinors will have the same renormalization constants only if the theory is regularized and quantized in a manifestly supersymmetric way. In the case of using the Wess–Zumino gauge this equality will be lost. This can be seen already in the one-loop approximation, see Ref. [50]. On the other hand, since the relation between the bare and the pole mass must be gauge-independent, the equality between the fermion and the scalar masses must be preserved after renormalization in the on-shell scheme whichever of the two quantization methods is used [51].

Quantum corrections to the two-point Green function of the gauge superfield are encoded in the function $d(k/\Lambda, m_0/\Lambda, \alpha_0)$, which enters the effective action as

$$
\Gamma^{(2)}_V - S_{gf} = -\frac{1}{16\pi} \int \frac{d^4 k}{(2\pi)^4} d^4 \theta \times V(-k, \theta) \delta^2 \Pi_{1/2} V(k, \theta)
$$

$$
\times d^{-1}(k/\Lambda, m_0/\Lambda, \alpha_0),
$$

where $\delta^2 \Pi_{1/2} \equiv -D^a \tilde{D}^2 D_a / 8$, and the normalization constant is chosen in such a way that in the tree approximation $d^{-1} = \alpha_0^{-1}$. The function $d(k/\Lambda, m_0/\Lambda, \alpha_0)$ is the invariant charge [18] of the supersymmetric quantum electrodynamics. In the limit $k \to 0$ it gives the value of the fine-structure constant as a function of $m_0/\Lambda$ and $\alpha_0$ in the supersymmetric case. In the on-shell subtraction scheme this value plays the role of the renormalized coupling constant $\alpha$. The $\beta$-function in this scheme is defined as

$$
\beta(\alpha) = \frac{d\alpha}{d \ln m} \bigg|_{m_0, \Lambda = \text{const}},
$$

where $m$ is the pole mass defined earlier.
4 The three-loop $\beta$-function in the on-shell scheme

An important feature of using the higher covariant derivative regularization in supersymmetric theories is the factorization of the loop integrals contributing to the function $d^{-1}(k/\Lambda, m_0/\Lambda, \alpha_0)$ in the limit $k \to 0$ into integrals of double total derivatives with respect to the momenta. This was first discovered in explicit calculations for $\mathcal{N} = 1$ SQED in [52] (total derivatives) and [44] (double total derivatives). The rigorous all-order proof for the Abelian case has been done in Refs. [31,32]. (The factorization into double total derivatives seems to be a general feature of supersymmetric theories and theories with softly broken supersymmetry regularized by higher covariant derivatives, see, e.g., the calculations of Refs. [33,37,38,53–55].)

In $\mathcal{N} = 1$ SQED the double total derivatives are taken with respect to the momenta of the matter loops to which the external lines of the gauge superfield are attached. If a double total derivative acts on a massless propagator, it produces a delta-function singularity which gives rise to a nonvanishing contribution. However, if a double total derivative acts on massive propagators, the integral of this total derivative vanishes, because all propagators are massive.

Thus, the NSVZ equation similar to Eq. (1) is indeed valid in the on-shell scheme. It relates the $\beta$-function in a given order to the mass anomalous dimension in the previous order. Note that in the on-shell scheme the mass anomalous dimension differs from the anomalous dimension of the matter superfields taken with the opposite sign.\(^5\)

Using Eq. (21) it is possible to construct the three-loop $\beta$-function in the on-shell scheme by calculating the mass anomalous dimension in the two-loop approximation. This is done in this paper for the theory regularized by higher derivatives. Methods of evaluating Feynman integrals with the help of this regularization are not described in the literature in enough detail, while there is some interest in investigating various $D = 4$ techniques for calculating quantum corrections (see the review [58]). That is why in Appendix A we describe in detail how the renormalization constant $Z_m$ is obtained in the two-loop approximation. The result is given by the expression

$$
\ln Z_m = -\frac{\alpha_0}{\pi} \left( \ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{A}{2} \right) + \frac{\alpha_0^2}{\pi^2} \left( \frac{N_f}{2} + N_f \sum_{I=1}^{n} c_I \ln a_I \right) + O(1) 
$$

(23)

In this equation the symbol $O(1)$ denotes finite terms that do not vanish in the limit $\Lambda \to \infty$, and the constant $A$ is defined by

$$
A \equiv 2 \int_0^\infty dK \ln \frac{K}{\Lambda} \frac{d}{dK} \left( \frac{1}{K} \right), 
$$

(24)

Next, following [44], the right-hand side is expressed in terms of the renormalized mass,

$$
\alpha^{-1} - \alpha_0^{-1} = \frac{N_f}{\pi} \left( \ln \frac{\Lambda}{m_0} + \sum_{I=1}^{n} c_I \ln a_I \right). 
$$

(20)

Then differentiating with respect to $\ln m$ gives the NSVZ relation

$$
\beta(\alpha) = \frac{N_f \alpha^2}{\pi} \left( 1 + \gamma_m(\alpha) \right) 
$$

(21)

written in terms of the mass anomalous dimension

$$
\gamma_m(\alpha) = \frac{d \ln Z_m}{d \ln m} \bigg|_{\alpha_0, \Lambda = \text{const}}. 
$$

(22)

\(^4\) In our notation capital letters denote Euclidean momenta.

\(^5\) Exactly as in the case of (non-supersymmetric) QED [57], in the lowest-order approximation the corresponding renormalization constants differ by an ultraviolet finite but infrared divergent term, $Z^{-1}Z_m^{-1} = 1 + \alpha(1 - \ln m/\kappa)/\pi + O(\alpha^2)$, where $\kappa$ is a small photon mass.
where $R_K \equiv R(K^2/\Lambda^2)$. (For the regulator $R(K^2/\Lambda^2) = 1 + (K^2/\Lambda^2)^a$ this integral vanishes, so that $A = 0$.) Differentiating (23) with respect to $m$ and expressing the result in terms of $\alpha$ using (20) we obtain

$$y_m(\alpha) = \frac{d}{d \ln m} \ln Z_m = \frac{\alpha}{\pi} - \frac{\alpha^2(3N_f + 1)}{2\pi^2} + O(\alpha^3).$$  

(25)

As expected in the on-shell scheme, any dependence on the regularization details has disappeared. After substituting the mass anomalous dimension (25) into Eq. (21) the three-loop result for the $\beta$-function takes the form

$$\beta(\alpha) = \frac{N_f\alpha^2}{\pi} \left(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(3N_f + 1)}{2\pi^2} + O(\alpha^3)\right).$$  

(26)

Comparing it with the corresponding result in the $\text{DR}$-scheme (i.e., in the case of using dimensional reduction supplemented by modified minimal subtractions) [39]

$$\beta_{\text{DR}}(\alpha_{\text{DR}}) = \frac{\alpha^2_{\text{DR}}}{\pi} \frac{N_f}{\pi} \times \left(1 + \frac{\alpha_{\text{DR}}^2}{\pi} - \frac{\alpha_{\text{DR}}^2}{4\pi^2} (3N_f + 2) + O(\alpha_{\text{DR}}^3)\right),$$  

(27)

we see that the terms linear in $N_f$ coincide. This follows from the scheme-independence of these terms proved in [20] in all orders, which is related to the so-called conformal symmetry limit of perturbative quenched quantum electrodynamics [59].

5 Relations between the on-shell scheme and other NSVZ schemes

In the previous section it was demonstrated that the NSVZ relation (21) is valid in the on-shell scheme in all orders. Therefore, this scheme belongs to the class of NSVZ schemes described in Ref. [14], which also includes the all-order HD+MSL prescription and the NSVZ scheme constructed with dimensional reduction in the three-loop approximation in Refs. [39,43]. According to Ref. [14] any two NSVZ subtraction schemes can be related by a finite renormalization

$$\alpha'(\alpha, \Lambda/\mu) = \alpha'(\alpha_0, \Lambda/\mu) = \alpha(\alpha_0, \Lambda/\mu);$$

$$Z'(\alpha', \Lambda/\mu) = z(\alpha) Z(\Lambda, \Lambda/\mu),$$  

(28)

which is subjected to the constraint

$$\frac{1}{\alpha'(\alpha)} - \frac{1}{\alpha} = \frac{N_f}{\pi} \ln z(\alpha) + B,$$  

(29)

where $B$ is a constant.

First, let us find the finite renormalization relating the on-shell scheme to the HD+MSL scheme. According to the HD+MSL prescription, the calculations are to be carried out with the higher derivative regularization and only powers of $\ln \Lambda/\mu$ are included into renormalization constants, so that in this scheme

$$\frac{1}{\alpha_{\text{HD+MSL}}} = \frac{1}{\alpha_0} + \frac{N_f}{\pi} \ln \frac{\Lambda}{\mu} + \frac{\alpha_0 N_f}{\pi^2} \ln \frac{\Lambda}{\mu} + O\left(\alpha_0^2\right);$$

$$\ln Z_{\text{HD+MSL}} = \frac{\alpha_0}{\pi} \ln \frac{\Lambda}{\mu} + O\left(\alpha_0^2\right).$$  

(30)

Comparing these equations with Eqs. (20) and (23) we derive the required finite renormalization,

$$\alpha_{\text{HD+MSL}}^{-1} \mid_{\mu=m} = \alpha_{\text{OS}}^{-1} - \frac{N_f}{\pi} \sum_{i=1}^n c_i \ln a_i + \frac{N_f \alpha_{\text{OS}}}{\pi^2} \left(-\frac{1}{2} + A\right) + O\left(\alpha_{\text{OS}}^2\right);$$

$$\ln Z_{\text{HD+MSL}} \mid_{\mu=m} + \ln Z_m = \frac{\alpha_{\text{OS}}}{\pi} \left(-\frac{1}{2} + \frac{A}{2}\right) + O\left(\alpha_{\text{OS}}^2\right).$$  

(31)

Evidently, in this case the condition (29) is satisfied with $B = -(N_f/\pi) \sum_i c_i \ln a_i$.

Also it is possible to find the finite renormalization which relates the on-shell scheme to the NSVZ scheme constructed with the dimensional reduction. (For short, we will call this scheme “DR+NSVZ”.) In the case of using the DR+NSVZ scheme the renormalization group functions (RGFs) can be found in [43] and have the form

$$\beta_{\text{DR+NSVZ}}(\alpha_{\text{DR+NSVZ}}) = \frac{N_f}{\pi} \left(1 + \frac{\alpha_{\text{DR+NSVZ}}}{\pi} - \frac{\alpha_{\text{DR+NSVZ}}^2}{2\pi^2} (1 + N_f) + O\left(\alpha_{\text{DR+NSVZ}}^3\right)\right);$$

$$\gamma_{\text{DR+NSVZ}} \mid_{\mu=m} = \alpha_{\text{OS}} - \frac{N_f}{\pi} \left(1 + N_f\right) + O\left(\alpha_{\text{OS}}^2\right);$$

$$\ln Z_{\text{DR+NSVZ}} \mid_{\mu=m} + \ln Z_m = \frac{\alpha_{\text{OS}}}{\pi} \left(-1 + \frac{A}{2}\right) + O\left(\alpha_{\text{OS}}^2\right).$$  

(32)

These expressions should be compared with Eqs. (25) and (26). With the help of the standard equations describing how RGFs transform under finite renormalizations [60] we obtain the finite renormalization after which RGFs in the on-shell scheme are converted into RGFs in the DR+NSVZ scheme,

$$\alpha_{\text{DR+NSVZ}}^{-1} \mid_{\mu=m} = \alpha_{\text{OS}}^{-1} + \frac{N_f}{\pi} z_1 + \frac{N_f \alpha_{\text{OS}}}{\pi^2} (-1 + z_1) + O\left(\alpha_{\text{OS}}^2\right);$$

$$\ln Z_{\text{DR+NSVZ}} \mid_{\mu=m} + \ln Z_m = \frac{\alpha_{\text{OS}}}{\pi} (1 - z_1) + O\left(\alpha_{\text{OS}}^2\right).$$  

(33)

However, these equations contain an undefined constant $z_1$, which reflects the arbitrariness of choosing a renormalization point in the DR+NSVZ scheme. This constant can be found by comparing the one-loop expressions for the renormalized function $d^{-1}$ in the limit $k \to 0$,

$$d_{\text{OS}}^{-1} \mid_{k=0} = d_{\text{DR+NSVZ}}^{-1} \mid_{k=0} = d_{\text{DR+NSVZ}}^{-1} \mid_{k=0} = \frac{N_f}{\pi} \ln \frac{\mu}{m} + O\left(\alpha_{\text{DR+NSVZ}}\right).$$  

(34)
so that \( z_1 = 0 \). This is analogous to the case of (non-supersymmetric) QED in which a similar coefficient also vanishes, \( \alpha^{-1}_{\text{MS}} \bigg|_{\mu=m} = \alpha^{-1}_{\text{OS}} + O(\alpha_{\text{OS}}) \), see Ref. [61].

In the case \( \mu \neq m \) the considered finite renormalization takes the form

\[
\alpha_{\text{DR+NSVZ}}^{-1} = \alpha_{\text{OS}}^{-1} - \frac{N_f}{\pi} \ln \frac{\mu}{m} - \frac{N_f \alpha_{\text{OS}}}{\pi^2} \left(1 + \ln \frac{\mu}{m}\right) + O \left( \alpha_{\text{OS}}^2 \right); \]

\[
\ln Z_{\text{DR+NSVZ}}(\alpha_{\text{DR+NSVZ}}) + \ln Z_{m,\text{OS+DRED}}(\alpha_{\text{OS}}) = -\frac{\alpha_{\text{OS}}}{\pi} \left(1 + \ln \frac{\mu}{m}\right) + O \left( \alpha_{\text{OS}}^2 \right). \tag{35}
\]

One can easily verify that the constraint (29) is also satisfied for the functions (35) with \( B = -N_f \ln(\mu/m)/\pi \).

6 Conclusion

We have explicitly demonstrated that the NSVZ equation in \( \mathcal{N} = 1 \) SQED is valid in the on-shell scheme in all orders. In this case it relates the \( \beta \)-function to the mass anomalous dimension. The NSVZ relation appears in the on-shell scheme due to the fact that quantum corrections to the photon polarization operator in the limit of zero momentum are given by integrals of double total derivatives with the higher derivative regularization. In the massive case these total derivatives act on nonsingular expressions in all orders beyond the one-loop approximation. The remaining one-loop contribution produces the NSVZ relation between the \( \beta \)-function and the mass anomalous dimension. This implies that the \( \beta \)-function in a given order can be found by calculating the mass anomalous dimension in the previous order. In this paper, having calculated the latter to the two-loop order, we obtained the \( \beta \)-function in the on-shell scheme to the three-loop order.

It was also investigated how the on-shell scheme in \( \mathcal{N} = 1 \) SQED is related to other known NSVZ schemes, namely HD+MSL and the NSVZ scheme based on dimensional reduction. Finite renormalizations relating the on-shell scheme to these two schemes have been constructed. They were shown to satisfy the constraint (29) derived in [14].

Acknowledgements The work of ALK and AEK was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics 'BASIS', Grant No 17-11-120.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: The paper is entirely based on analytical calculations which are described in detail in it.]

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP3.

A The renormalization constant \( Z_m \) in the two-loop order

This appendix is devoted to the calculation of the two-loop renormalization constant \( Z_m \) in the on-shell scheme. In particular, we describe the technique of evaluating the \( D = 4 \) loop integrals appearing with the higher derivative regularization.

A.1 \( Z_m \) as a sum of loop integrals

In the considered approximation the logarithm of the renormalization constant \( Z_m \) is written as

\[
\ln Z_m = \ln G(m) - \ln J(m) = \Delta G(m) - \Delta J(m) - \frac{(\Delta G(m))^2}{2} + \frac{(\Delta J(m))^2}{2} + \cdots, \tag{36}
\]

where \( \Delta G \equiv G - 1 \) and \( \Delta J \equiv J - 1 \). Using the results of Refs. [47,48], after the Wick rotation it is possible to present this expression as a sum of Euclidean loop integrals

\[
\ln Z_m = \alpha_0 I_{\text{one-loop}} + \alpha_0^2 \left( I_1 + I_2 + I_3 + I_4 + I_5 \right) - \frac{1}{2} \left( I_{\text{one-loop}} \right)^2 + O \left( \alpha_0^3 \right). \tag{37}
\]

In explicit expressions for these integrals (presented below), Euclidean momenta will be denoted by capital letters. Due to the higher derivative regularization, denominators of the integrands contain the function \( R_K \). In the simplest case it can be chosen as \( R_K = 1 + K^{2n}/\Lambda^{2n} \). However, in the general case considered here it is sufficient to require that \( R_K(0) = 1 \) and (due to the presence of higher powers of the momentum) \( R_K \rightarrow \infty \) in the limit \( K \rightarrow \infty \). In Eq. (37)

\[
\alpha_0 I_{\text{one-loop}} = -8\pi \alpha_0 \int \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 R_K \left( (P + K)^2 + m^2 \right)} \bigg|_{p^2 = -m^2} \tag{38}
\]

is the one-loop contribution, while the remaining integrals

\[6\] With the higher derivative regularization the superdiagrams contributing to the function \( G \) have first been calculated in Refs. [62,63].
correspond to the two-loop approximation. Note that in these integrals terms proportional to $\alpha_0^2 (P^2 + m^2)$ were omitted, because they evidently vanish due to the condition $P^2 = -m^2$. Also all these integrals were expressed in terms of the renormalized mass $m$. Therefore, the one-loop superdiagrams give both the integral $I_{\text{one-loop}}$ and the integral $I_1$. (The latter one is produced by the one-loop superdiagrams containing an insertion of the one-loop mass counterterm.)

### A.2 One-loop contribution

The one-loop contribution is given by the expression

\[
I_1 = \frac{m_0^2 - m^2}{\alpha_0} \int \frac{d^4 K}{(2\pi)^4} \frac{8\pi}{K^2 R_K ((P + K)^2 + m^2)} \bigg|_{P^2 = -m^2}; \tag{39}
\]

\[
I_2 = -128\pi^2 \int \frac{d^4 \mu}{(2\pi)^4} \frac{k^4 L^4}{(2\pi)^4} \frac{P_{\mu}(K + L)^4 - m^2}{K^2 R_K L^2 R_L ((P + K)^2 + m^2)((P + L)^2 + m^2)} \bigg|_{P^2 = -m^2}; \tag{40}
\]

\[
I_3 = 128\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{m^2 ((P + K)^2 + m^2)((P + K + L)^2 + m^2)} \bigg|_{P^2 = -m^2}; \tag{41}
\]

\[
I_4 = 64\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{k^2 R_K L^2 R_L ((P + K)^2 + m^2)} \bigg|_{P^2 = -m^2}; \tag{42}
\]

\[
I_5 = 64\pi^2 N_f \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{k^2 R_K^2 ((P + K)^2 + m^2)} \bigg|_{P^2 = -m^2}; \tag{43}
\]

where the bare mass $m_0$ was replaced by the renormalized mass $m$, because their difference $\Delta m^2 \equiv m^2 - m_0^2$ is proportional to $\alpha_0$ and is essential in the next order. The integral in Eq. (44) can be calculated in four-dimensional spherical coordinates using the method of Refs. [62,63]. Introducing the variable $x \equiv \cos \theta_3$, where $\theta_3$ is the angle between the vectors $K^\mu$ and $P^\mu$, it can be rewritten in the form

\[
\ln Z_m = -\alpha_0 \int_0^\infty dK \frac{1}{\pi^2 R_K} \int dx \frac{\sqrt{1 - x^2}}{x} K + 2i\eta x + O(\alpha_0^2). \tag{45}
\]

The contour $C$ is presented in Fig. 1. To calculate this integral it is necessary to find the residues at the points $x = \infty$ and $x = i K/2m$. The result written as an integral over $z \equiv K^2/\Lambda^2$ has the form

\[
\ln Z_m = \frac{\alpha_0}{\pi} \int_0^\infty dz \frac{\Lambda^2}{4m^2 R(z)} \left(1 - \frac{1}{\sqrt{1 + \frac{4m^2}{\Lambda^2}}}ight) + O(\alpha_0^2). \tag{46}
\]

It is convenient to introduce the new variable

\[
\eta \equiv \frac{a}{4} \exp \left\{ -\frac{2z}{a} + \frac{2z}{a} \sqrt{1 + \frac{a}{z} - 1}ight.\right. \left. + \ln \left(1 + 2\frac{2z}{a} + 2\sqrt{\frac{z}{a} + \frac{z^2}{a^2}}\right) \right\}, \tag{47}
\]

where $a \equiv 4m^2/\Lambda^2$, such that

\[
\frac{d\eta}{2\eta} = \frac{dz}{a} \left(-1 + \sqrt{1 + \frac{a}{z}}\right). \tag{48}
\]

Note that $z = 0$ and $z \to \infty$ correspond to $\eta = \exp(-1)a/4$ and $\eta \to \infty$, respectively. Therefore the integral under consideration can be rewritten as

\[
\ln Z_m = -\frac{\alpha_0}{2\pi} \int_{\exp(-1)a/4}^\infty \frac{d\eta}{\eta R(z(\eta))} + O(\alpha_0^2). \tag{49}
\]

This integral diverges in the limit $a \to 0$, when $z(\eta) = \eta + O(a)$. However, if the function $R^{-1}(z)$ is expanded in powers of $a$, then only the leading term will produce a divergent integral, while the other terms are given by convergent integrals, which vanish in the limit $a \to 0$. Therefore, omitting the terms suppressed by powers of $m^2/\Lambda^2$ we obtain

\[
\ln Z_m = -\frac{\alpha_0}{2\pi} \int_{\exp(-1)a/4}^\infty \frac{d\eta}{\eta R(\eta)} + O(\alpha_0^2). \tag{50}
\]
Integrating by parts, it is possible to extract the divergent part of the remaining integral,
\[
\int_{\text{exp}(-1)m^2/\Lambda^2}^{\infty} d\eta \frac{1}{\eta R(\eta)} = \frac{1}{R(\eta)} \ln |\exp(-1)m^2/\Lambda^2|
\]
\[- \int_{\text{exp}(-1)m^2/\Lambda^2}^{\infty} d\eta \ln \eta \frac{d}{d\eta} \left( \frac{1}{R(\eta)} \right)
\]
\[= 2 \ln \frac{\Lambda}{m} + 1 - A + O(m^2/\Lambda^2), \quad (51)
\]
where the constant \( A \) is given by the equation
\[
A \equiv \int_{0}^{\infty} d\eta \ln \eta \frac{d}{d\eta} \left( \frac{1}{R(\eta)} \right), \quad (52)
\]
which is equivalent to Eq. (24). Thus, in the one-loop approximation
\[
\ln Z_m = -\frac{\alpha_0}{\pi} \left( \ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{A}{2} \right) + O(\alpha_0^2). \quad (53)
\]

A.3 Two-loop contribution

To find the two-loop contribution to the renormalization constant \( Z_m \), it is necessary to calculate the integrals \( I_1 - I_2 \) in Eq. (37).

The integral \( I_1 \) is convergent in the ultraviolet region, but diverges in the infrared one. That is why it is necessary to regularize it by introducing a small photon mass \( \kappa \),
\[
I_1 = \frac{m_0^2 - m^2}{\alpha_0} \int_{0}^{\infty} dK \pi^2 (K^2 + \kappa^2) \int_{C} dx \frac{\sqrt{1 - x^2}}{(K + 2i \kappa x)^2}, \quad (55)
\]
where the contour \( C \) is presented in Fig. 1. The integral over \( x \) can be found by calculating the residues at the points \( x = \infty \) and \( x = iK/2m \),
\[
I_1 = \frac{m_0^2 - m^2}{2m^2 \alpha_0} \int_{0}^{\infty} dK \pi \left( \frac{K}{K^2 + \kappa^2} - \frac{1}{\sqrt{K^2 + 4m^2}} \right)
\]
\[+ \frac{\pi}{2(\kappa^2 + 4m^2)} \kappa^2 (K^2 + \kappa^2) \sqrt{K^2 + 4m^2}. \quad (56)
\]
Taking into account that \( m_0 = Z_m m \) and omitting the last term in the round brackets (which gives a convergent integral proportional to \( \kappa \to 0 \)) this expression can be written as
\[
I_1 = \frac{Z_m^2 - 1}{4\pi \alpha_0} \left( \ln(1 + K^2/\kappa^2) - 2 \arcsinh(K/2m) \right) \bigg|_{0}^{\infty}
\]
\[= \frac{Z_m^2 - 1}{2\pi \alpha_0} \ln \frac{m}{\kappa}. \quad (57)
\]
After substituting the one-loop result for \( Z_m \) from Eq. (53) with the considered accuracy the integral \( I_1 \) takes the form
\[
I_1 = -\frac{1}{\pi^2} \left( \ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{A}{2} \right) \ln \frac{m}{\kappa}. \quad (58)
\]

The integral \( I_2 \) is convergent and does not contain infrared divergences. This implies that (due to the condition \( p^2 = -m^2 \)) this integral is equal to a finite number and does not contribute to \( \gamma_m \) in the considered approximation. Below we will omit such terms.

The integral \( I_3 \) diverges in the infrared region and should be regularized by introducing the small photon mass \( \kappa \),
\[
I_3 = 128\pi^2 \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{1}{(K^2 + \kappa^2)R_K(L^2 + \kappa^2)R_L}
\times \frac{m^2}{((P + K)^2 + m^2)((P + K + L)^2 + m^2)} \bigg|_{p^2 = -m^2}. \quad (59)
\]
This expression can be equivalently rewritten as
\[
I_3 = 128\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \times \frac{1}{m^2} \frac{1}{(K^2 + \kappa^2)R_K R_L}
\times \left( (L^2 + \kappa^2)R_L((P + K)^2 + m^2)^2((P + L)^2 + m^2)^2 \right)
\times \left( 1 + \frac{(P + L)^2 - (P + K + L)^2}{(P + K + L)^2} \right) \bigg|_{p^2 = -m^2} = I_3' + I_3''.
\]

(60)

The integral \(I_3'\) (corresponding to 1 in the round brackets) can be presented as a product of two integrals which have already been calculated above,
\[
I_3' = 128\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{m^2}{(K^2 + \kappa^2)R_K ((P + K)^2 + m^2)^2}
\times \int \frac{d^4 L}{(2\pi)^4} \frac{1}{(L^2 + \kappa^2)R_L ((P + L)^2 + m^2)^2}
\times \frac{1}{(P + K + L)^2} \bigg|_{p^2 = -m^2}.
\]

(61)

The expression for \(I_3''\) (which is obtained from the second term in the round brackets in Eq. (60)) is not divergent in the infrared region, so that it is possible to set \(\kappa = 0\),
\[
I_3'' \bigg|_{\kappa = 0} = -128\pi^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} \frac{1}{K^2 R_K L^2 R_L}
\times \frac{m^2 K^\mu (K + 2P + 2L)_\mu}{((P + K)^2 + m^2)^2((P + L)^2 + m^2)^2}
\times \frac{1}{(P + K + L)^2} \bigg|_{p^2 = -m^2}.
\]

(62)

It is also not divergent in the ultraviolet region. Therefore, it is a finite number, which does not contribute to the two-loop mass anomalous dimension. This implies that
\[
I_3 = \frac{1}{2\pi^2} \ln \frac{m}{\kappa} \left( 2\ln \frac{\Lambda}{m} + 1 - A \right) + \text{a finite constant.} \tag{63}
\]

After some transformations the expression \(I_4 = \frac{1}{2} (I_{\text{one-loop}})^2 / 2\) can be rewritten as
\[
I_4 = \frac{1}{2} (I_{\text{one-loop}})^2
\times \frac{d}{d\ln \Lambda} \left[ \frac{I_4}{2} \right] \bigg|_{m = 0} = -64\pi^2 \int \frac{d^4 L}{(2\pi)^4} \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 R_K L^2 R_L ((P + K)^2 + m^2)^2}
\times \frac{L_\mu K^\mu}{((P + L)^2 + m^2)((P + K + L)^2 + m^2)} \bigg|_{p^2 = -m^2}.
\]

(64)

Let us calculate the derivative of the integral (64) with respect to \(\ln \Lambda\) in the case \(m = 0\),
\[
\frac{d}{d\ln \Lambda} \left[ \frac{I_4}{2} \right] \bigg|_{m = 0} = -64\pi^2 \int \frac{d^4 L}{(2\pi)^4} \frac{d^4 K}{(2\pi)^4} \frac{1}{K^2 R_K L^2 R_L (K + L)^2}.
\]

(66)

Following Ref. [52], this integral can be presented as
\[
\frac{1}{2\pi^2} \int \frac{d K}{K^3} \int \frac{d L}{L^3} \frac{d}{d\ln \Lambda} \left( \frac{1}{K R_L} \right)
+ \frac{1}{2\pi^2} \int \frac{d K}{K^3} \int \frac{d L}{L^3} \frac{d}{d\ln \Lambda} \left( \frac{1}{R_K R_L} \right).
\]

(67)

After the substitution \(L = \rho K\) in the last integral, this expression can be rewritten in the form\(^7\)
\[
\frac{1}{2\pi^2} \int \frac{d K}{K} \int \frac{d \rho}{\rho} \frac{d}{d\ln \Lambda} \left( \frac{1}{K R_K} \right)
- \frac{1}{2\pi^2} \int \frac{d \rho}{\rho} \int \frac{d K}{K} \frac{d}{d\ln \Lambda} \left( \frac{1}{R_K R_K} \right) = \frac{1}{2\pi^2}.
\]

(68)

This implies that in Eq. (65) \(f_2 = 0\) and \(f_1 = 1/2\pi^2\). Therefore, omitting terms proportional to \(m/\Lambda\), we obtain
\[
I_4 = -\frac{1}{2} (I_{\text{one-loop}})^2 = \frac{1}{2\pi^2} \ln \frac{\Lambda}{m} + \text{a finite constant.} \tag{69}
\]

The remaining integral \(I_5\) can be calculated using the equation
\[
\int \frac{d^4 L}{(2\pi)^4} \frac{1}{(L^2 + m^2)((K + L)^2 + m^2)}
+ \sum_{i=1}^n (-1)^{P_i} \int \frac{d^4 L}{(2\pi)^4} \frac{1}{(L^2 + M_i^2)((K + L)^2 + M_i^2)}
= J(K/m) + \sum_{i=1}^n (-1)^{P_i} J(K/M_i),
\]

(70)

where\(^7\)
\[
J(K/M) = -\frac{1}{8\pi^2} \left( \ln \frac{M}{K} + \sqrt{1 + \frac{4M^2}{K^2}} \arctan \left( \frac{K^2}{K^2 + 4M^2} \right) \right).
\]

This result agrees with the calculation of Ref. [52] carried out for the particular case \(R(x) = 1 + x^n\).
To find the integral $J$ the Pauli–Villars masses is convergent in both ultraviolet and infrared regions. Therefore, it is equal to a finite constant, and only the terms with the Pauli–Villars masses $M_I$ nontrivially contribute to the divergent part of the integral $I_5$. To calculate them, let us consider the expression

$$N_f \left[ \frac{1}{2} \ln \frac{\Lambda}{m} \ln \left( \frac{1}{2} \right) + \ln \frac{\Lambda}{m} \ln \frac{3}{2} \right] + O(1).$$

This implies that the expression (72) can be written as

$$\frac{N_f}{\pi^2} \left[ -\frac{1}{2} \ln \frac{\Lambda}{m} - \ln \frac{\Lambda}{m} \left( \ln a + \frac{3}{2} \right) \right] + O(1).$$

Consequently, the integral $I_5$ takes the form

$$I_5 = \frac{N_f}{\pi^2} \left[ -\frac{1}{2} \ln \frac{\Lambda}{m} + \ln \frac{\Lambda}{m} \left( \sum_{i=1}^{n} c_I \ln a^i + \frac{3}{2} \right) \right] + a \text{ finite constant},$$

where $c_I = (-1)^{Pr+1}$ and terms vanishing in the limit $\Lambda \to \infty$ were omitted.

Collecting the results (53), (58), (63), (69), and (80) we obtain that the two-loop mass renormalization constant is given by the expression (23),

$$\ln Z_m = -\frac{a_0}{\pi} \left( \ln \frac{\Lambda}{m} + \frac{1}{2} - \frac{\Lambda}{2} \right) - \frac{a_0^2}{\pi^2} \left[ \frac{N_f}{\pi^2} \ln \frac{\Lambda}{m} \right]$$

$$+ \ln \frac{\Lambda}{m} \left( \sum_{i=1}^{n} c_I \ln a^i + \frac{3N_f + 1}{2} \right) + a \text{ finite constant} + O \left( \frac{a_0}{\pi} \right),$$

and does not contain infrared divergences.

References

1. A. Grozin, “Lectures on QED and QCD,” Lectures at 3rd Dubna International Advanced School of Theoretical Physics 29 Jan–6 Feb 2005. Dubna. arXiv:hep-ph/0508242
2. V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B 229, 381 (1983)
3. D.R.T. Jones, Phys. Lett. 123B, 45 (1983)
4. V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Phys. Lett. 166B, 329 (1986)
5. V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Sov. J. Nucl. Phys. 43, 294 (1986)
6. V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Yad. Fiz. 43, 451 (1986)
7. M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 277, 456 (1986)
8. M.A. Shifman, A.I. Vainshtein, Sov. Phys. JETP 64, 428 (1986)
9. M.A. Shifman, A.I. Vainshtein, Z. Eksp. Teor. Fiz. 91, 723 (1986)
10. A.I. Vainshtein, V.I. Zakharov, M.A. Shifman, JETP Lett. 22, 244 (1975) [Pisma Zh. Eksp. Teor. Fiz. 42 (1985) 182]
11. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Phys. Lett. 166B, 334 (1986)
12. D. Kutasov, A. Schwimmer, Nucl. Phys. B 702, 369 (2004)
13. A.L. Kataev, K.V. Stepanyantz, Theor. Math. Phys. 181, 1531 (2014)
14. I.O. Goriachuk, A.L. Kataev, K.V. Stepanyantz, Phys. Lett. B 785, 561 (2018)
15. E.G. Steuckelberg, A. Petermann, Helv. Phys. Acta 26, 499 (1953)
16. M. Gell-Mann, F.E. Low, Phys. Rev. 95, 1300 (1954)
17. N.N. Bogolyubov, D.V. Shirkov, Nuovo Cim. 3, 845 (1956)
18. N.N. Bogolyubov, D.V. Shirkov, Introduction To the theory of quantized fields. Intersci. Monogr. Phys. Astron. 3, 1 (1959) [Moscow: Nauka, the fourth edition, (1984) 416 p, In Russian]
19. A.L. Kataev, K.V. Stepanyantz, Nucl. Phys. B 875, 459 (2013)
20. A.L. Kataev, K.V. Stepanyantz, Phys. Lett. B 730, 184 (2014)
21. A.A. Slavnov, Nucl. Phys. B 31, 301 (1971)
22. A.A. Slavnov, Theor. Math. Phys. 13, 1064 (1972) [Teor. Mat. Fiz. 13 (1972) 174]
23. V.K. Krivoshchekov, Theor. Math. Phys. 36, 745 (1978) [Teor. Mat. Fiz. 36 (1978) 291]
24. P.C. West, Nucl. Phys. B 268, 113 (1986)
25. S.S. Aleshin, A.E. Kazantsev, M.B. Skoptsov, K.V. Stepanyantz, JHEP 1605, 014 (2016)
26. I.V. Nartsev, K.V. Stepanyantz, JETP Lett. 105 (2), 69 (2017)
27. J. Hisano, M.A. Shifman, Phys. Rev. D 56, 5475 (1997)
28. I. Jack, D.R.T. Jones, Phys. Lett. B 415, 383 (1997)
29. L.V. Avdeev, D.I. Kazakov, I.N. Kondrashuk, Nucl. Phys. B 510, 289 (1998)
30. A.L. Kataev, A.E. Kazantsev, K.V. Stepanyantz, Nucl. Phys. B 926, 295 (2018)
31. K.V. Stepanyantz, Nucl. Phys. B 852, 71 (2011)
32. K.V. Stepanyantz, JHEP 1408, 096 (2014)
33. I.V. Nartsev, K.V. Stepanyantz, JHEP 1704, 047 (2017)
34. M. Shifman, K. Stepanyantz, Phys. Rev. Lett. 114(5), 051601 (2015)
35. M. Shifman, K.V. Stepanyantz, Phys. Rev. D 91, 105008 (2015)
36. K.V. Stepanyantz, Nucl. Phys. B 909, 316 (2016)
37. V.V. Shakhmanov, K.V. Stepanyantz, Nucl. Phys. B 920, 345 (2017)
38. A.E. Kazantsev, V.V. Shakhmanov, K.V. Stepanyantz, JHEP 1804, 130 (2018)
39. I. Jack, D.R.T. Jones, C.G. North, Phys. Lett. B 386, 138 (1996)
40. R.V. Harlander, D.R.T. Jones, P. Kant, L. Mihaila, M. Steinhauser, JHEP 0612, 024 (2006)
41. I. Jack, D.R.T. Jones, C.G. North, Nucl. Phys. B 486, 479 (1997)
42. I. Jack, D.R.T. Jones, A. Pickering, Phys. Lett. B 435, 61 (1998)
43. S.S. Aleshin, I.O. Goriachuk, A.L. Kataev, K.V. Stepanyantz, Phys. Lett. B 764, 222 (2017)
44. A.V. Smilga, A. Vainshtein, Nucl. Phys. B 704, 445 (2005)
45. A.V. Smilga, Private communication (2017)
46. A.A. Slavnov, Theor. Math. Phys. 33, 977 (1977) [Teor. Mat. Fiz. 33 (1977) 210]
47. A.E. Kazantsev, K.V. Stepanyantz, J. Exp. Theor. Phys. 120(4), 618 (2015)
48. A.E. Kazantsev, K.V. Stepanyantz, Z. Eksp. Teor. Fiz. 147(4), 714 (2015)
49. M.T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B 159, 429 (1979)
50. J. Wess, B. Zumino, Nucl. Phys. B 78, 1 (1974)
51. J. Goity, T. Kugo, R.D. Peccei, Phys. Rev. D 29, 2412 (1984)
52. A.A. Soloshenko, K.V. Stepanyantz, Theor. Math. Phys. 140, 1264 (2004) [Teor. Mat. Fiz. 140 (2004) 437]
53. A.B. Pimenov, E.S. Shevtsova, K.V. Stepanyantz, Phys. Lett. B 686, 293 (2010)
54. K.V. Stepanyantz, Factorization of integrals defining the two-loop $\beta$-function for the general renormalizable $N=1$ SYM theory, regularized by the higher covariant derivatives, into integrals of double total derivatives. arXiv:1108.1491 [hep-th]
55. I.L. Buchbinder, N.G. Pletnev, K.V. Stepanyantz, Phys. Lett. B 751, 434 (2015)
56. K.V. Stepanyantz, J. Phys. Conf. Ser. 343, 012115 (2012)
57. D.J. Broadhurst, N. Gray, K. Schilcher, Z. Phys. C 52, 111 (1991)
58. C. Gnendiger et al., Eur. Phys. J. C 77(7), 471 (2017)
59. A.L. Kataev, JHEP 1402, 092 (2014)
60. A.A. Slavnov, Sov. J. Nucl. Phys. 31, 558 (1980) [Yad. Fiz. 31 (1980) 1083]
61. D.J. Broadhurst, A.L. Kataev, O.V. Tarasov, Phys. Lett. B 298, 445 (1993)
62. A. Soloshenko, K. Stepanyantz, Two loop renormalization of $N = 1$ supersymmetric electrodynamics, regularized by higher derivatives. arXiv:hep-th/0203118
63. A.A. Soloshenko, K.V. Stepanyantz, Theor. Math. Phys. 134, 377 (2003) [Teor. Mat. Fiz. 134 (2003) 430]