Critical exponents of ferromagnetic Ising model on fractal lattices

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January 17, 2022

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Abstract

We review the value of the critical exponents $\nu^{-1}$, $\beta/\nu$, and $\gamma/\nu$ of ferromagnetic Ising model on fractal lattices of Hausdorff dimension between one and three. They are obtained by Monte Carlo simulation with the help of Wolff algorithm. The results are accurate enough to show that the hyperscaling law $d_f = 2\beta/\nu + \gamma/\nu$ is satisfied in non-integer dimension. Nevertheless, the discrepancy between the simulation results and the $\epsilon$-expansion studies suggests that the strong universality should be adapted for the fractal lattices.

Introduction

Fractal lattices have self-similar character. Any small part of them is magnified; the structure similar to the entire one is observed. They are good candidates to investigate the physical properties in non-integer dimensions. At the early of 1980’s, Gefen and his coworkers did the pioneer works on phase transitions in fractal networks [1, 2, 3, 4]. They studied the ferromagnetic Ising model on fractal lattices by Migdal-Kadanoff bond-moving real space renormalization group method and found that the Hausdorff dimension should replace the space dimension in the description of the critical phenomena. They pointed out that topological effects play an important role at criticality. For example, fractals of finite ramification order can be solved exactly and show no phase transition at finite temperature; contrarily, fractals of infinite ramification order exhibit a second order one. Latter, Bhanot et al. [5, 6] executed Metropolis Monte Carlo simulations and suggested the average number of bonds per spin should be used instead. Bonnier et al. [7, 8] used alternative bond-moving real space renormalization approach and high temperature expansion as research tools but could not close the problems. Few years ago, two groups [9, 10] performed Monte Carlo simulations equipped with high efficient Wolff algorithm and obtained precisely the critical exponents on Sierpinski carpets. They
showed that the hyperscaling law, \( d_f = 2\beta/\nu + \gamma/\nu \), is satisfied for two different fractals of Hausdorff dimension \( d_f \) equal to 1.8927 and 1.7925. Recently, Hsiao, Monceau, and Perreau [11] extended their probes to the fractals of dimension between 2 and 3, called Sierpinski sponge, and gave the conclusion reinforcing the previous studies.

**Ferromagnetic Ising model on Sierpinski lattices**

We denote the Sierpinski lattice at \( k \)th iteration step by the symbol SP(\( \ell^d \), N\(_{\text{occ}}\), \( k \)) where \( \ell \) is the size of the generating cell, \( d \) the dimension of embedding space, and N\(_{\text{occ}}\) the number of occupied sites in the generating cell. The lattice is generated by enlarging the \((k-1)\)th iteration one by replacing each occupied site by the whole generating cell. Strictly speaking, a Sierpinski lattice is not a true fractal except \( k \) tends to infinity. When \( d \) is equal to 2, Sierpinski lattice is called Sierpinski carpet; and when \( d = 3 \), it is named Sierpinski sponge. In fig. 1 we give two generators of fractal lattices SP(\( 4^2 \), 12) and SP(\( 3^3 \), 18), whose Hausdorff dimension \( d_f \) = log \( N_{\text{occ}}/\log \ell \) are equal to 1.792 and 2.631, respectively. We will put a spin on the center of an occupied site unlike what Gefan et al. or Bonnier et al. did to put a spin on each vertex of an occupied site. We should notice that putting spins on vertices will cause the ambiguity how to define the Hausdorff dimension because the total number of spins is not equal to the total number of occupied sites. The Hamiltonian of Ising model on a Sierpinski lattice is defined as usual case: \( H(\{s_i\}) = -\sum_{<i,j>} s_i s_j \) where \( <i,j> \) runs over all the nearest neighbor bonds on SP(\( \ell^d \), N\(_{\text{occ}}\), \( k \)) and the coupling constant has been set to 1. We will study fractals with infinite ramification order to insure it exhibits a second order phase transition.

**Homogeneity hypothesis and finite size scaling**

The generalized homogeneity of the free energy per spin in integer space dimension \( d \) was first proposed by Widom [2] and later by Wilson renormalisation group approach [3]. It states that near the critical point, the free energy per spin behaves as \( f(t, h, L) = b^{-d} f(t b^\alpha, h b^\gamma, L/b) \) under the change of length scale from 1 to \( b \), where \( t \) is the reduced
temperature \((T - T_c)/T_c\), \(h\) the external magnetic field, and \(L\) the size of the system. Due to the scale invariant character of fractal lattices, one can extend the above hypothesis to fractals by directly replacing the space dimension \(d\) by the Hausdorff one \(d_f\). The immediate consequence is that the critical exponents should satisfy the following two relations: \(\alpha + 2\beta + \gamma = 2\), \(d_f\nu = 2\beta + \gamma\). According to Fisher’s finite size scaling theory \[14\], for a lattice of finite size \(L\), the specific heat per spin \(C_L(t)\), the thermal average of the absolute value of magnetization per spin \(m_L(t)\), the magnetic susceptibility per spin \(\chi_L(t)\), and the logarithmic derivative of magnetization \(\phi_L(t) = \partial \ln m_L(t)/\partial \beta_B\) should behave like following:

\[
\Omega_L(t) = L^{\Delta_\Omega} Q_\Omega(t L^{1/\nu})
\]

where \(\Omega\) stands for \(C, m, \chi, \) and \(\phi\) and \(\Delta_\Omega\) equal to \(\alpha/\nu, -\beta/\nu, \gamma/\nu, \) and \(\nu^{-1}\), respectively. We can see that the maximum value of physical quantity \(\Omega\) is determined by the function \(Q_\Omega(x)\) with \(x = t L^{1/\nu}\). The critical exponents \(\alpha/\nu, \gamma/\nu, \) and \(\nu^{-1}\) can be, hence, extracted by finding the associated maximum values \(\Omega_L^{\max}\) at different lattices sizes. Moreover, if the maximum occurs at \(T_\Omega(L)\) for a system of size \(L\), the critical temperature \(T_c\) for a lattice of infinite size can be extrapolated with the help of the equation \(T_\Omega(L) = T_c(1 + x_\Omega^* L^{-1/\nu})\). Therefore we can execute Monte Carlo simulations at criticality \(T_c\) and extract the exponents \(\alpha/\nu, \beta/\nu, \gamma/\nu, \) and \(\nu^{-1}\) directly.

Monte Carlo simulation

We setup periodic boundary condition on the studied lattices and perform Monte Carlo simulation. Wolff algorithm \[15\] is applied at some temperature for producing spin configurations. Histogram method is then used to extract all possible information near this temperature. We repeat the whole above processes several times to get a sufficient number of samples for doing statistical analysis.

Results and discussions

In table 1, we list the critical temperatures and the critical exponents \(\nu^{-1}, \beta/\nu, \) and \(\gamma/\nu\) of the five different fractal lattices. The results of SP(4\(^2,12\)) and SP(3\(^2,8\)) are taken from the paper of Carmona et al. \[10\] in which the two lattices are denoted by fractal B and fractal A, respectively. The three fractals of dimension between 2 and 3 are reported in our recent article \[11\]. The effective dimensions \(2\beta/\nu + \gamma/\nu\) of the five fractals can be calculated to be 1.808(40), 1.890(3), 2.640(18), 2.900(23), and 2.955(41), respectively; they are consistent with their Hausdorff dimensions. It proves the validity of homogeneity hypothesis for fractals.

The effect of correction-to-scaling can be observed in the studies. It is found to be weaker in the case of dimension between 2 and 3 than that between 1 and 2. This can be understood by considering the convergence speed of the deviation ratio \(\rho(\ell^d, N_{\text{occ}}, k)\) which is defined as the percentage of the difference between the mean number of neighbors
Table 1: The critical temperatures and exponents of five different fractals. \(^a\) Data reproduced from the article of Carmona et al. [10]. \(^b\) Results reported in [11].

| SP       | \((4^2, 12)^a\) | \((3^2, 8)^a\) | \((3^3, 18)^b\) | \((4^3, 56)^b\) | \((3^3, 26)^b\) |
|----------|----------------|----------------|----------------|----------------|----------------|
| \(d_f\)  | 1.792          | 1.893          | 2.631          | 2.904          | 2.966          |
| \(T_c\)  | 1.078(3)       | 1.4813(2)      | 2.35090(9)     | 3.99893(10)    | 4.21701(6)     |
| \(1/\nu\) | 0.309(8)       | 0.59(1)        | 1.185(27)      | 1.410(36)      | 1.503(53)      |
| \(\beta/\nu\) | 0.069(10) | 0.080(1)      | 0.3224(14)     | 0.493(5)       | 0.506(11)      |
| \(\gamma/\nu\) | 1.67(2)   | 1.730(1)      | 1.995(15)      | 1.914 (13)     | 1.943(19)      |

at step \(k\) and that at infinite step. \(\rho(\ell^d, N_{occ}, k)\) measures the discrepancy between the *pseudo*-fractal and the true one and can be shown to be:

\[
\left[ \frac{(N_{occ} - N_S)d}{N_I} - 1 \right] \left( \frac{N_S}{N_{occ}} \right)^k
\]

(2)

where \(N_S\) is the number of occupied sites on each surface (boundary) of the generating cell and \(N_I\) the number of its internal bonds. As \(k\) increases, \(\rho(\ell^d, N_{occ}, k)\) tends to zero faster than the fractals embedded in a two dimensional space.

Physicists believe that the critical properties do not depend on the structure of the lattice but only on the dimension of system, the range of couplings, and the symmetry of order parameter. This is called the strong universality. According to it, the critical exponents of a fractal lattice should fall in the same class like the hypercubic one of the same dimension. But comparing with the values calculated by \(\epsilon\)-expansion [16], we find that they are not consistent with each other. The simulation results reinforce the argument of Gefen et al. that the topology effects of a fractal play an indispensable role in determining its critical properties. Strong universality should be revised for self-similar lattices.

**Acknowledgments**

The author acknowledges the financial support from the ICSC World Laboratory in Switzerland.

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