Abstract Chebyshev coefficients of a coordinate Chebyshev representation can be used to form representations of velocity. One way is to directly apply them to the derivatives of Chebyshev polynomials, another is to compute from them the Chebyshev coefficients of velocity Chebyshev representation. The advantages of the latter over the former ways are illustrated. Also, the approach of generating Chebyshev coefficients developed by Newhall [1] is extended such that coordinate, velocity and acceleration are consistently treated.

Keywords Ephemeris · Chebyshev representation

1 Introduction

Chebyshev expansions, i.e. linear combinations of Chebyshev polynomials, are commonly used to represent high-precision numerical ephemerides of celestial bodies, e.g. the well-known DE, EPM and INPOP planetary and lunar ephemerides. Newhall [1] develops an approach of generating Chebyshev representations of coordinates. Such a representation, the combination coefficients of which are widely distributed and used, is piecewise but continuous up to its first derivative (i.e. the so-called coordinate velocity), e.g. [2]. When applied to the first derivatives of Chebyshev polynomials, the same coefficients are used to compute the velocity.

Here are some simple facts about the above usual practice. Firstly, velocities are not computed using Chebyshev expansions. The advantages of Chebyshev expansion thus become irrelevant. And, as will be obvious later, there are often cases when certain amount of computation can be saved without increasing the size of the distributed ephemeris file. Secondly, the maximum error of a velocity representation, like that of a coordinate one, can be estimated by using the distributed Chebyshev coefficients. Thirdly, though not realized in the standard
interpolation softwares, accelerations can be computed in the same way as velocities. This is of interest because it allows direct comparison between the actually realized force models underlying different ephemerides by using publicly available ephemeris data.

Sect. 2 recalls the basic knowledge of Chebyshev expansions necessary for detailing the brief remarks made in the previous paragraph. And the detailed remarks are presented in Sect. 3. Sect. 4 is devoted to extending the approach developed in [1], so that coordinate, velocity and acceleration are consistently treated.

2 Chebyshev expansion

A Chebyshev expansion of degree $N$ writes

$$p(x) = \sum_{n=0}^{N} p_n T_n(x) \quad (1)$$

where $x$ and $p_n$’s are referred to as Chebyshev time and Chebyshev coefficients, respectively, and the Chebyshev polynomial $T_n(x)$ is a polynomial of degree $n$ defined on $x \in [-1, 1]$ and recursively given by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad (n = 2, ..., N). \quad (2)$$

For all non-negative integers $n$, we have

$$\sup_{x \in [-1, 1]} |T_n(x)| = 1. \quad (3)$$

The first derivative of $p$ with respect to $x$ writes

$$v(x) = \sum_{n=0}^{N} p_n T_n'(x) = \sum_{n=1}^{N} p_n T_n'(x) = \sum_{n=0}^{N-1} v_n T_n(x), \quad (4)$$

where

$$T_0'(x) = 0, \quad T_1'(x) = 1, \quad T_n'(x) = 2xT_{n-1}'(x) + 2T_{n-1}(x) - T_{n-2}'(x) \quad (n = 2, ..., N), \quad (5)$$

and, with $v_N = 0$,

$$v_{N-1} = 2Np_N, \quad v_n = 2(n+1)p_{n+1} + v_{n+2} \quad (n = N-2, ..., 1), \quad v_0 = p_1 + v_2/2. \quad (6)$$

For $n \geq 0$, we have

$$\sup_{x \in [-1, 1]} |T_n'(x)| = n^2. \quad (7)$$

1 Less known than (3), this conclusion can be proved by using another equivalent definition of Chebyshev polynomial, i.e. $T_n(x) = \cos(n \arccos(x))$.
The second derivative of \( p \) with respect to \( x \) writes

\[
a(x) = \sum_{n=0}^{N} p_n T_n''(x) = \sum_{n=2}^{N} p_n T_n''(x) = \sum_{n=0}^{N-2} a_n T_n(x),
\]

(9)

where

\[
T_0''(x) = 0, \\
T_1''(x) = 0, \\
T_2''(x) = 4, \\
T_n''(x) = 2xT_{n-1}'(x) + 4T_{n-1}'(x) - T_{n-2}''(x) \quad (n = 3, ..., N),
\]

(11)

and, with \( a_{N-1} = 0 \),

\[
a_{N-2} = 2(N-1)v_{N-1}, \\
a_n = 2(n + 1)v_{n+1} + a_{n+2} \quad (n = N - 3, ..., 1), \\
a_0 = v_1 + a_2/2.
\]

(12)

3 Chebyshev representation of ephemeris

A Chebyshev representation of ephemeris of a celestial body allows users to compute the body’s dynamical state (coordinate, velocity and acceleration) by using a set of piecewise Chebyshev expansions. In practice, this is economically achieved by only providing Chebyshev coefficients of coordinate representations. To be specific, let’s focus on a particular Chebyshev expansion representing a coordinate and being valid on a granule \([t_b, t_e] \) of the ephemeris time \( t \). Introducing the Chebyshev time

\[
x = -1 + 2 \frac{t - t_b}{t_e - t_b} \in [-1, 1],
\]

(13)

one computes the coordinate, velocity and acceleration by using (1), (4) and (9), respectively. To do so, Chebyshev polynomials and their respective first and second derivatives have to be computed for each required value of \( x \).

The dynamical state can also be computed using (1), (5) and (10). An advantage of doing so is time-saving. This is because the \( x \)-dependant parts, i.e. the Chebyshev polynomials, are common to these three expansions, while the \( x \)-independent parts, i.e. the Chebyshev coefficients, need only to be computed once for all \( x \in [-1, 1] \).

As representations in the form of Chebyshev expansion, (4) and (10) allow one to estimate the maximum representation error of velocity and acceleration, respectively, in a way by which [4] estimates that of coordinate. For this, a necessary assumption for \( n \geq N \) is either \( p_n \approx 0 \) or \( |p_{n+1}|/|p_n| < \epsilon \), where \( \epsilon \) is a small positive constant. This assumption is true with \( \epsilon \approx 0.1 \) in the case of distributed planetary and lunar representations, for which the length of granule in \( t \) and the degree \( N \) of representation are conveniently chosen [4]. Together with \(|T_n(x)| \leq 1\),

\[
\frac{dp}{dt} = \frac{2}{t_e - t_b} v \quad \text{and} \quad \frac{d^2p}{dt^2} = \left(\frac{2}{t_e - t_b}\right)^2 a.
\]

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2 Back to \( t \), the velocity writes \( \frac{dp}{dt} = \frac{2}{t_e - t_b} \) and the acceleration \( \frac{d^2p}{dt^2} = \left(\frac{2}{t_e - t_b}\right)^2 a.\)
the above assumed condition implies that the coordinate representation error is bounded from above by

$$\delta_p = \sum_{n=N+1}^{\infty} |p_n| \leq |p_{N+1}| \sum_{i=0}^{\infty} \epsilon^i = \frac{|p_{N+1}|}{1-\epsilon} \leq \frac{|p_N|}{1-\epsilon} \sim \epsilon |p_N|. \quad (14)$$

Checking with an Inpop10e ephemeris file\(^3\) valid on a time span of 2000 years, we find that $|v_N|/|v_{N-1}| < 0.125$, provided that $|p_{N+1}|/|p_N| < \epsilon = 0.1$. This is understandable since $v_n \sim 2(n+1)p_{n+1}$, which means that

$$\frac{|v_n|}{|v_{n-1}|} \sim \frac{n+1}{n} \frac{|p_{n+1}|}{|p_n|} < \epsilon(1 + \frac{1}{n}) \left\{ \begin{aligned} \sim 1.2\epsilon & \leq 0.12 \ (n = \min(N) = 5), \\ \sim \epsilon & = 0.1 \ (n \text{ large}). \end{aligned} \right. \quad (15)$$

From (15), the same argument as in the case of coordinate leads to the following apparent estimation of the maximum error of the velocity representation

$$\delta_v = \epsilon |v_{N-1}| \sim 2N \epsilon |p_N| \sim 2N\delta_p. \quad (16)$$

Similarly, the maximum error of the acceleration representation is estimated as

$$\delta_a = \epsilon |a_{N-2}| \sim 2(N-1)\delta_v \sim 4N(N-1)\delta_a. \quad (17)$$

Given sampling data of a function of time, the precision of the best-fit interpolation polynomial depends on its degree, as well as the behavior of the function in the considered time interval. For each given body, \(^1\) chooses the degree of the polynomials representing coordinates, $N$ appearing in (1), and the length of granule, $L = t_e - t_b$ corresponding to 2 Chebyshev time unit $u_{\epsilon}$, such that $\delta_p \sim 0.5\text{mm}$.\(^4\) Table 1 lists the corresponding maximum errors of velocity and acceleration representations, as calculated using (16) and (17), respectively.

From Table 1 we know that the velocity representation errors are extremely small as compared with the most precise doppler velocity measurement used in the present-day spacecraft tracking (several millimeters per second). Therefore, there should be many cases when larger errors are tolerable. In these cases, truncated velocity representations may be used providing that their respective maximum error is known. In the context of truncation, the well known advantage of (5) over (4) can be quantified. For example, neglecting the last term of (5) will induce an error no more than $|v_{N-1}T_{N-1}(x)| \leq |v_{N-1}| = 2|p_N|$, while the error induced by neglecting the last term of (4) can be as large as $|p_N T'_N(x)| \leq N^2 |p_N|$.\(^4\)

### 4 Chebyshev coefficient generation

In \(^1\), an efficient approach is developed to generate the coefficients of a piecewise Chebyshev coordinate representation. What is special of this approach is twofold. Firstly, it uses not only sampling data of coordinate but also of velocity to form a generally overdetermined equation system for the coefficients. Secondly, it uses

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\(^3\) Namely, the ephemeris file inpop10e_TDB_m1000_p1000_littleendian.dat available at http://www.imcce.fr/inpop/

\(^4\) The error in 3-dimensional position is then at the level of sub-millimeter, which is smaller than that of the present-day laser ranging measurement.
Table 1  Estimated maximum errors of velocity and acceleration representations. The estimated values in Chebyshev time unit $u_x$ are computed using (16) and (17), where the maximum error of coordinate representation $\delta_p = 0.5 \text{mm}$ [1]. The corresponding values in ephemeris time $t$ also depend on the ephemeris time length $L$ of granule.

| Body       | $L$ | $N$ | $\delta_v = 2N\delta_p$ mm/day | $\delta_a = 4N(N - 1)\delta_p$ mm/day$^2$ | $\delta_p$ mm |
|------------|-----|-----|---------------------------------|-------------------------------------------|--------------|
| Mercury    | 8   | 13  | 3.2 [14.0]                      | 19.5 [312.0]                              | [1]          |
| Venus      | 16  | 9   | 1.1 [9.0]                       | 2.2 [144.0]                               | [1]          |
| Earth-Moon | 16  | 12  | 1.5 [12.0]                      | 4.1 [264.0]                               | [1]          |
| Mars       | 32  | 10  | 0.6 [10.0]                      | 0.7 [180.0]                               | [1]          |
| Jupiter    | 32  | 7   | 0.4 [7.0]                       | 0.3 [84.0]                                | [1]          |
| Saturn     | 32  | 6   | 0.4 [6.0]                       | 0.2 [60.0]                                | [1]          |
| Uranus     | 32  | 5   | 0.3 [5.0]                       | 0.2 [40.0]                                | [1]          |
| Neptune    | 32  | 5   | 0.3 [5.0]                       | 0.2 [40.0]                                | [1]          |
| Pluto      | 32  | 5   | 0.3 [5.0]                       | 0.2 [40.0]                                | [1]          |
| Moon       | 4   | 12  | 6.0 [12.0]                      | 66.0 [264.0]                              | [1]          |
| Sun        | 16  | 10  | 1.2 [10.0]                      | 2.8 [180.0]                               | [1]          |

exact constraints at both boundary time instants of each representation piece. The advantage of doing so is also twofold: (1) The effect of noises on the derivative of the coordinate representation, used as a velocity representation, is reduced; (2) Both coordinate and velocity representations are continuous.

There is a price to pay for the first advantage. Indeed, the noises induced by the velocity equations should have negative effects on the coordinate representation. In order to balance between the advantage to gain and the price to pay, [1] uses a universal weighting scheme based on experiments. In this scheme, velocity equations are weighted at 0.4 relative to coordinate ones. Though the scheme may be oversimplified, its ephemeris-independent nature allows the approach to express the searched Chebyshev coefficients as linear combinations of sampling data with ephemeris-independent combination coefficients. This should be considered as another strong point of the approach, because these combination coefficients, depending only on the degree of the coordinate representation and the set of sampling time instants [1], apply to any ephemeris.

In a broad sense, how to assign weights to sampling equations may be considered as a matter of choice. To make a decision, one needs a pre-determined criterion, which can be different for different purposes. Here are two extreme examples: If velocity is not required, then it’s better not to use velocity equations, i.e. they should be assigned with zero weight; And, if acceleration is required, then it’s better to use also acceleration equations, which may be weighted in a way similar to velocity equations, namely at 0.4 relative to velocity equations. Another reasonable and ephemeris-independent weighting scheme is based on the maximum-representation-error relation $\delta_p : \delta_v : \delta_a = 1 : 2N : 4N(N - 1)$ (see section 3). For this we remark that, contrary to sampling by observations, here the sampling data are assumed to be exact, while the noises come from the non-exact representation models. It is then interesting to present an extended approach without specifying a particular weighting scheme.

Let $\{P_m, V_m, A_m\}$, $m = 1, ..., M$, be the instantaneous dynamical states at $x_m \in (-1, 1)$, respectively, which are part of the data carried by a numerical
integration program \([1]\). The extended objective function writes

\[
\chi^2(p_0, ..., p_N) = \sum_{m=1}^{M} \left[ w_p(p(x_m) - P_m) + w_v(v(x_m) - V_m) + w_a(a(x_m) - A_m)\right]^2,
\]

where \(p, v\) and \(a\) are represented as \([1], [3]\) and \([9]\), respectively, and the weights \(w_p, w_v, w_a\) are assumed to be ephemeris-independent.

Let \(\{P_{bl}, V_{bl}, A_{bl}\}, \ l = 1, 2\), be two additional dynamical states sampled at \(x = x_{b1} = -1\) and \(x_{b2} = 1\), respectively. The imposed equality constraints are

\[
\begin{cases}
b_{pl} = p(x_{bl}) - P_{bl} = 0, \\
b_{vl} = v(x_{bl}) - V_{bl} = 0, \\
b_{al} = a(x_{bl}) - A_{bl} = 0,
\end{cases} \quad (l = 1, 2). \tag{19}
\]

Because \(p, v\) and \(a\) are all linear in the searched Chebyshev coefficients \(p_0, ..., p_N\), our optimization problem with equality constraints is linear. The unique solution to this problem can be found by the well-known method of Lagrange multiplier. Let \(\lambda_{pl}, \lambda_{vl}, \lambda_{al} (l = 1, 2)\) be the Lagrange multipliers and

\[
L(p_0, ..., p_N, \lambda_1, ..., \lambda_6) = \chi^2 + \sum_{l=1}^{2} [\lambda_{pl}b_{pl} + \lambda_{vl}b_{vl} + \lambda_{al}b_{al}] \tag{20}
\]

the Lagrange function. The Chebyshev coefficients, together with the 6 multipliers, minimizes \(L\) and solves the normal equations

\[
\begin{cases}
\partial L/\partial p_n = 0 \\
\partial L/\partial \lambda_{pl} = 0, \ \partial L/\partial \lambda_{vl} = 0, \ \partial L/\partial \lambda_{al} = 0
\end{cases} \quad (n = 0, ..., N), \quad (l = 1, 2). \tag{21}
\]

To present explicitly this equation system in matrix form, we introduce three groups of matrices, namely: The parameters to be optimized

\[
s = [p_0, ..., p_N]^t \quad \text{(order \((N + 1)\times 1\))},
\lambda = [\lambda_{p1}, \lambda_{p2}, \lambda_{v1}, \lambda_{v2}, \lambda_{a1}, \lambda_{a2}]^t \quad \text{(order \(6 \times 1\))}, \tag{22}
\]

where and in the following the superscript \(t\) stands for transposition; The dynamical state variables sampled at inner and boundary time instants

\[
S = [P_1, ..., P_M, V_1, ..., V_M, A_1, ..., A_M]^t \quad \text{(order \(3M \times 1\))},
S_b = [P_{b1}, P_{b2}, V_{b1}, V_{b2}, A_{b1}, A_{b2}]^t \quad \text{(order \(6 \times 1\))}; \tag{23}
\]

And the Chebyshev polynomials and their first and second derivatives evaluated at inner and boundary instants

\[
T = [T_n(x_m)], \ T' = [T'_n(x_m)], \ T'' = [T''_n(x_m)] \quad \text{(order \((N + 1) \times M\))},
T_b = [T_n(x_{b1})], \ T'_b = [T'_n(x_{b1})], \ T''_b = [T''_n(x_{b1})] \quad \text{(order \((N + 1) \times 2\))}. \tag{24}
\]

As building blocks, the third group matrices form the following matrices,

\[
F = F^t = 2(w_p^2 T T^t + w_v^2 T' T'^t + w_a^2 T'' T''^t) \quad \text{(order \((N + 1) \times (N + 1)\))},
G = 2(w_p^2 T, w_v^2 T', w_a^2 T'') \quad \text{(order \((N + 1) \times 3M\))}, \tag{25}
H = (T_b, T'_b, T''_b) \quad \text{(order \((N + 1) \times 6\))}.
\]
Remarks on Chebyshev representation of ephemeris

Straightforward calculation leads us to the following matrix form of (21)

\[ \mathcal{M}_L \begin{pmatrix} s \\ \lambda \end{pmatrix} = \mathcal{M}_R \begin{pmatrix} S \\ S_b \end{pmatrix}, \]  
(26)

where, with \( 0_{m \times n} \) and \( I_{m \times n} \) standing respectively for zero and unit matrices of order \( m \times n \),

\[ \mathcal{M}_L = \begin{pmatrix} F & H \\ H^t & 0_{6 \times 6} \end{pmatrix} \quad \text{(order \( (N+7) \times (N+7) \))}, \]
\[ \mathcal{M}_R = \begin{pmatrix} G & 0_{(N+1) \times 6} \\ 0_{6 \times 3M} & I_{6 \times 6} \end{pmatrix} \quad \text{(order \( (N+7) \times (3M+6) \))}. \]
(27)

Scaling weight-dependent matrices, \( F \) and \( G \), with the same non-zero constant factor \( \alpha \), one obtains a new matrix equation, which can be derived from the objective function \( \alpha \chi^2 \) having the same conditional extreme point as our \( \chi^2 \). To be explicit, we remark that, if \( [s, \lambda]^t \) solves (26), then \( [s, \lambda/\alpha]^t \) solves the new equation. This means that, as expected, \( w_p, w_v \) and \( w_a \) can be simultaneously scaled.

The solution of (26) can be written as \( [s, \lambda]^t = \mathcal{M}[S, S_b]^t \), where \( \mathcal{M} = \mathcal{M}_L^{-1} \mathcal{M}_R \) is a matrix of order \( (N+7) \times (3M+6) \). Let \( \mathcal{M}_n \) be the \( n \)th row vector of \( \mathcal{M} \), then the searched Chebyshev coefficients write

\[ p_n = \mathcal{M}_{n+1} \begin{pmatrix} S \\ S_b \end{pmatrix} \quad (n = 0, ..., N). \]
(28)

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