A Summary on Two Types of Real Integrals Using the Residue Theorem

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Abstract. It’s not easy, even impossible, to write anti-derivatives of most real functions in closed forms, which bring technical difficulty in using the fundamental theorem of calculus to compute real integrals. However, the residue theorem can be utilized to make calculations of some real integrals possible. In this survey, general methods of using the residue theorem are summarized to compute real integrals, and two specific real integrations are discussed.

1. Introduction

The residue theorem, one of the fundamental theorems in Complex Analysis, has a lot of applications in other branches of mathematics, such as mathematical analysis, linear algebra, and analytic number’s theory. One can refer to [1-10] and references therein for more details.

Theorem. (Residue theorem) Suppose $f$ is complex analytic in a simply connected region $G$, except for finitely many isolated singularities $a_1, a_2, \cdots, a_n$, as shown in Figure 1. Let $C$ be a positively oriented Jordan contour that does not run through any singularities, then we have

$$\int_C f\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}[f(z), z_j]$$

where $\text{Res}[f(z), z_j]$ means the residue of $f(z)$ at $z_j$.

Figure 1. Residue Theorem
By using the residue theorem, we can make possible computations of some real integrations which could not be calculated by the fundamental theorem of Calculus. The policy of computing real integration via the residue theorem can be outlined as the following steps:

1) Firstly, we select a holomorphic function according to the integrand.
2) Secondly, we design a closed contour that contains a part of or the whole of the real axis.
3) Thirdly, we use the residue theorem to compute the integration along the closed contour.

In the survey, we mainly discuss integrands that decay like or faster than $\frac{1}{z}$ through proving the following two useful theorems.

**Theorem.** $f(z)$ is a holomorphic function on the upper complex plane except for finite points $a_1, \cdots, a_n$. If there exist a constant $M \in \mathbb{R}^+$ satisfying:

$$|f(z)| \cdot |z| < M$$

then we have: $\lambda < 0$

$$\lim_{x_a \to -\infty, x_b \to \infty} \int_C f(z) e^{i\lambda z} dz = 0 \quad (2)$$

where

$$C = C_a \cup C_b \cup C_c.$$  

$C_a(t) = \{(x_a, i\cdot t)|0 < t < x_a + x_b\}$;

$C_b(t) = \{(t, i(x_a + x_b))| -x_b < t < x_a\}$;

$C_c(t) = \{(-x_b, i\cdot t)| t \text{ goes from } -x_a - x_b \text{ to } 0\}$.

![Figure 2. Rectangular paths of height (a) and width (b) $x_a + x_b$.](image)

Similarly, $f(z)$ is a holomorphic function on the upper complex plane except for finite points $a_1, \cdots, a_n$. If there exist a constant $M \in \mathbb{R}^+$ satisfying:

$$|f(z)| \cdot |z|^a < M$$

then we have: $\lambda = 0$

$$\lim_{R \to \infty} \int_C f(z) dz = 0 \quad (4)$$

**Theorem.** $f(z)$ is a holomorphic function on the upper complex plane except for finite points $a_1, \cdots, a_n$. If $a > 1$ and $M \in \mathbb{R}^+$ satisfying

$$|f(z)| \cdot |z|^a < M$$

then we have:

$$\lim_{R \to \infty} \int_C f(z) dz = 0 \quad (4)$$
where \( C = Re^{i\theta}, 0 < \theta < \pi \), as shown in Figure 3.

![Diagram of semicircle path for \( C = Re^{i\theta} \)](image)

Figure 3. Semicircle path for \( C = Re^{i\theta} \)

### 2. Main work

#### 2.1 Integrands that decay faster than \( \frac{1}{z} \)

**Definition 1.** \( f(z) \) is a holomorphic function on the upper or lower half complex plane except for finitely many isolated points, which are not on the real axis. If there exists a constant \( a > 1 \) and a positive number \( M > 0 \), satisfying

\[
|f(z)| \cdot |z|^a < M
\]

Then we say \( f \) decay faster than \( \frac{1}{z} \).

**Theorem 2.** If \( f(z) \) decay faster than \( \frac{1}{z} \), then

\[
\lim_{R \to \infty} \int_{C_R} f(z) \, dz = 0
\]  

(5)

**Proof.** Without loss of generality, we suppose \( f(z) \) is complex analytic on upper half complex plane except isolated points \( a_1, \cdots, a_n \).

According to the triangle inequality for integrals theorem, we have

\[
\left| \int_{C_R} f(z) \, dz \right| \leq \int_{C_R} |f(z)||dz|
\]

Notice that there exist a positive number \( M > 0 \) and a constant \( a > 1 \), satisfying

\[
|f(z)| \leq \frac{M}{|z|^a}
\]

For \( |z| \) large enough.

Thus,

\[
\left| \int_{C_R} f(z) \, dz \right| \leq \int_{C_R} |f(z)||dz| \leq \int_{C_R} \frac{M}{|z|^a} |dz|
\]

Put \( Re^{i\theta} \) into \( z \). We get:

\[
\int_{C_R} |f(z)||dz| = \int_{R^{i\theta}} |f(Re^{i\theta})||dRe^{i\theta}| = \int_{0}^{\pi} \frac{M}{R^a} R \, d(\theta) = \frac{M\pi}{R^{a-1}}
\]

Let \( R \to \infty \), then we get

\[
\frac{M\pi}{R^{a-1}} \to 0
\]

Thus,
\[
\lim_{R \to \infty} \int_{C_R} f(z) \, dz = 0
\]

**Theorem 3.** \( f(z) \) decays faster than \( \frac{1}{z} \), then we have:

\[
\int_{-\infty}^{\infty} f(x) \, dx = 2\pi i \sum_{j=1}^{n} \text{Res} \left[ f(z), z_j \right]
\]

**Proof.** Consider the closed contour \( C =: [R, R] \cup C_R \), where \( C_R = \text{Re}^i \theta \), \( \theta \in (0, \pi) \) or \( \theta \in (\pi, 2\pi) \). By using the Residue theorem, we obtain:

\[
\int_C f(z) \, dz = \int_{-R}^{R} f(x) \, dx + \int_{C_R} f(z) \, dz
\]

Let \( R \to \infty \), by theorem 2, we know that

\[
\lim_{R \to \infty} \int_{C_R} f(z) \, dz = 0
\]

It follows that

\[
\lim_{R \to \infty} \int_{C_R} f(z) \, dz = \int_{-\infty}^{\infty} f(x) \, dx
\]

Thus,

\[
\int_{-\infty}^{\infty} f(x) \, dx = 2\pi i \sum_{j=1}^{n} \text{Res} \left[ f(z), z_j \right]
\]

### 2.1.1 Example 1

\[
\int_{-\infty}^{\infty} \frac{dx}{x^2 - 4x + 5}
\]

First, we let \( f(z) = \frac{1}{z^2 - 4z + 5} \)

\( f(z) \) is complex analytic on the upper half plane except for the points \( 2 + i \).

Notice that

\[
\lim_{|z| \to \infty} \frac{|z|^2}{|z^2 - 4z + 5|} = 0
\]

Thus, \( f(z) \) decay faster than \( \frac{1}{z} \).

By Theorem 3, we know that:

\[
\int_{-\infty}^{\infty} \frac{dx}{x^2 - 4x + 5} = 2\pi \text{Res}[f(z), 2 + i] = \pi
\]

### 2.2 Integrands that decay like \( \frac{1}{z} \)

**Definition 2.** \( f(z) \) is a holomorphic function on the upper or lower half complex plane except for finitely many isolated points, which are not on the real axis.

If there exists a positive number \( M > 0 \), satisfying when \( |z| \) large enough

\[
|f(z)| \cdot |z| < M
\]

Then we say \( f \) decay like \( \frac{1}{z} \).

**Theorem 4.** Suppose \( f \) is complex analytic on the upper half plane except for finitely many isolated points. \( f(z) \) decays like \( \frac{1}{z} \). Then we have:
\[ \lim_{x_a \to -\infty, x_b \to \infty} \int_C f(z)e^{\lambda z} \, dz = 0 \] (7)

where

\[ C = C_a \cup C_b \cup C_c, \lambda > 0 \]
\[ C_a(t) = \{(x_a, i-t)|0 < t < x_a+x_b\}; \]
\[ C_b(t) = \{(t, i(x_a+x_b))|x_b < t < x_a\}; \]
\[ C_c(t) = \{-x_b, i-t\}|t \text{ goes from } -x_a-x_b \text{ to } 0\]

**Proof.** We estimate the integration along \( C_a \) by triangle inequality, we get:

\[ \int_{C_a} |f(z)e^{i\lambda z}| \, dz \leq \int_{C_a} |f(z)|e^{\lambda |z|} \, dz \]

Notice that \( |x_a + it| \geq x_a \), thus,

\[ \int_{C_a} e^{-\lambda t} \, dt \leq \frac{M}{x_a} \int_0^{x_a+x_b} e^{-\lambda t} \, dt = \frac{M}{x_a} \cdot \frac{1-e^{-\lambda(-x_a-x_b)}}{\lambda} \]

Let \( x_a \) goes to infinity, we get:

\[ \left| \int_{C_a} f(z)e^{i\lambda z} \, dz \right| \to 0 \]

By using similar techniques to the path \( C_b, C_c \), we can get

\[ \left| \int_{C_b} f(z)e^{i\lambda z} \, dz \right| \to 0 \]

and

\[ \left| \int_{C_c} f(z)e^{i\lambda z} \, dz \right| \to 0 \]

**Theorem 5.** Suppose \( f(z) \) defined on the upper half complex plane with isolated points \( \{z_i\}_{i=1}^n \)

\( f(z) \) decays like \( \frac{1}{z} \), then we have: \( \forall \lambda > 0 \)

\[ \int_{-\infty}^{+\infty} \cos(\lambda x)f(x) \, dx = \text{Re} \{2\pi i \sum_{i=1}^n \text{Res}[e^{i\lambda z}f(z), z_i]\} \] (8)

and

\[ \int_{-\infty}^{+\infty} \sin(\lambda x)f(x) \, dx = \text{Im} \{2\pi i \sum_{i=1}^n \text{Res}[e^{i\lambda z}f(z), z_i]\} \] (9)

**Proof.** Consider the contour \( C =: [-R, R] \cup C_a \cup C_b \cup C_c \).

By using the Residue theorem, we obtain:

\[ \int_C e^{i\lambda z}f(z) \, dz = 2\pi i \sum_{j=1}^n \text{Res}[e^{i\lambda z}f(z), z_j] \]

Notice that:

\[ \int_C f(z) \, dz = \int_{-R}^R e^{i\lambda z} f(x) \, dx + \int_{C_a \cup C_b \cup C_c} e^{i\lambda z} f(z) \, dz \]

Let \( R \to \infty \), by theorem 6, we know that

\[ \lim_{R \to \infty} \int_{C_a \cup C_b \cup C_c} f(z) \, dz = 0 \]

It follows that

\[ \lim_{R \to \infty} \int_{C_a \cup C_b \cup C_c} e^{i\lambda z} f(z) \, dz = \int_{-\infty}^{+\infty} e^{i\lambda z} f(x) \, dx \]

Thus

\[ \int_{-\infty}^{+\infty} e^{i\lambda z} f(x) \, dx = 2\pi i \sum_{j=1}^n \text{Res}[e^{i\lambda z}f(z), z_j] \]
Then, we take the real part and imagine part on both sides respectively, we get:

\[
\int_{-\infty}^{+\infty} \cos(\lambda x) f(x) \, dx = \text{Re} \left\{ 2\pi i \sum_{j=1}^{n} \text{Res}(e^{i\lambda z} f(z), z_j) \right\}
\]

\[
\int_{-\infty}^{+\infty} \sin(\lambda x) f(x) \, dx = \text{Im} \left\{ 2\pi i \sum_{j=1}^{n} \text{Res}(e^{i\lambda z} f(z), z_j) \right\}
\]

2.2.1 Example 2. \( \int_{-\infty}^{\infty} \frac{\sin \alpha x}{b^2 + x^2} \, dx \)

First, we let \( f(z) = \frac{1}{z^2 + b^2} \)

\( f(z) \) is complex analytic on the upper half plane except for the points \( b \). Notice that

\[
\lim_{|z| \to \infty} \frac{|z|}{|z^2 + b^2|} = 0
\]

Thus, \( f(z) \) decay likes \( \frac{1}{z} \). By Theorem 5, we know that:

\[
\int_{-\infty}^{+\infty} \frac{\cos \alpha x}{x^2 + b^2} = 2\pi \text{Re} [e^{i\alpha z} f(z), b_j] = \frac{\pi}{b} e^{-ab}
\]

3. Conclusion

With the help of the residue theorem, we are able to calculate the more complex and difficult integrals in an easier way, especially for Integrals that decay faster than \( 1/z \) and those that decay like \( 1/z \). In the future, we’d like do more analysis on this fundamental theorem of complex analysis and employ it to solve more integrals.

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