Distance domination of generalized de Bruijn and Kautz digraphs

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Abstract Let \(G = (V, A)\) be a digraph and \(k \geq 1\) an integer. For \(u, v \in V\), we say that the vertex \(u\) distance \(k\)-dominate \(v\) if the distance from \(u\) to \(v\) at most \(k\). A set \(D\) of vertices in \(G\) is a distance \(k\)-dominating set if each vertex of \(V \setminus D\) is distance \(k\)-dominated by some vertex of \(D\). The distance \(k\)-domination number of \(G\), denoted by \(\gamma_k(G)\), is the minimum cardinality of a distance \(k\)-dominating set of \(G\). Generalized de Bruijn digraphs \(G_B(n, d)\) and generalized Kautz digraphs \(G_K(n, d)\) are good candidates for interconnection networks. Denote \(\Delta_k := (\sum_{j=0}^{k} d^j)^{-1}\). F. Tian and J. Xu showed that \([n\Delta_k] \leq \gamma_k(G_B(n, d)) \leq [n/d^k]\) and \([n\Delta_k] \leq \gamma_k(G_K(n, d)) \leq [n/d^k]\). In this paper, we prove that every generalized de Bruijn digraph \(G_B(n, d)\) has the distance \(k\)-domination number \([n\Delta_k]\) or \([n\Delta_k] + 1\), and the distance \(k\)-domination number of every generalized Kautz digraph \(G_K(n, d)\) bounded above by \([n/(d^k-1+d^k)]\). Additionally, we present various sufficient conditions for \(\gamma_k(G_B(n, d)) = [n\Delta_k]\) and \(\gamma_k(G_K(n, d)) = [n\Delta_k]\).

Keywords Combinatorial problems, dominating set, distance dominating set, generalized de Bruijn digraph, generalized Kautz digraph

MSC 05C69, 05C20

1 Introduction

In this paper, we deal with directed graphs (or digraphs) which admit self-loops but no multiple arcs. Unless otherwise defined, we follow [3,9] for terminology and definitions. Let \(G\) be a digraph with vertex set \(V(G)\) and arc set \(A(G)\). If
there is an arc from \( u \) to \( v \), i.e., \((u, v) \in A(G)\), then \( v \) is called an *out-neighbor* of \( u \); we also say that \( u \) *dominates* \( v \). The *out-neighborhood* \( O(u) \) of a vertex \( u \) is the set \( \{v : (u, v) \in A(G)\} \). For \( S \subseteq V(G) \), its *out-neighborhood* \( O(S) \) is the set \( \cup_{u \in S} O(u) \). Set

\[
O_0(u) = \{u\}, \quad O_1(u) = O(u).
\]

We define \( O_i(u) \) recursively, called *i-th out-neighborhood* of \( u \), by \( \{O(O_{i-1}(u))\} \) for \( i \geq 1 \). The *i-th out-neighborhood* of \( S \) is the set

\[
O_i(S) = \bigcup_{u \in S} O_i(u).
\]

The *closed out-neighborhood* \( O[u] \) of \( u \) is the set \( O(u) \cup \{u\} \), and \( O[S] \) and \( O_i[S] \) are defined analogously.

For \( x, y \in V(G) \), the *distance* \( d_G(x, y) \) from \( x \) to \( y \) is the length of an shortest \((x, y)\)-directed path in \( G \). Let \( k \) be a positive integer. A subset \( D \subseteq V(G) \) is called a *distance k-dominating set* of \( G \) if for every vertex \( v \) of \( V(G) \setminus D \), there is a vertex \( u \in D \) such that \( d_G(u, v) \leq k \), i.e.,

\[
\bigcup_{i=0}^{k} O_i(D) = V(G).
\]

The *distance k-domination number* of \( G \), denoted by \( \gamma_k(G) \), is the minimum cardinality of a distance \( k \)-dominating set of \( G \). In particular, the distance 1-dominating set is the ordinary dominating set, which has been well studied [11].

Slater [11] termed a distance \( k \)-dominating set as a \( k \)-basis and also gave an interpretation for \( k \)-basis in terms of communication networks. Since then many researchers paid much attention to this subject, for example, [8,23,24]. The concept of distance domination in graphs has been found applications in many structures and situations which give rise to graphs. A minimum distance \( k \)-dominating set of \( G \) may be used to locate a minimum number of facilities (such as utilities, police stations, hospitals, transmission towers, blood banks, waste disposal dump) such that every intersection is within \( k \) city blocks of a facility. Barkauskas and Host [1] showed that the problem of determining \( \gamma(G) \) is NP-hard for a general graph.

The network topology has a great impact on the system performance and reliability [27]. There are some well-known networks with good properties such as de Bruijn networks, Kautz networks, and their generalizations (see, for example, [2,4,12,27]). Generalized de Bruijn and Kautz networks, denoted by \( G_B(n, d) \) and \( G_K(n, d) \), respectively, were introduced by Imase and Itoh [13]. The generalization removes the restriction on the cardinality of vertex set and make the network more general and valuable as a network model. A lot of features make it suitable for implementation of reliable networks, the most important feature such as small diameter [13], high connectivity [14], easy routing, and high reliability.
The generalized de Bruijn digraph $G_B(n, d)$ is defined by congruence equations as follows:

\[
\begin{align*}
V(G_B(n, d)) &= \{0, 1, \ldots, n-1\}, \\
A(G_B(n, d)) &= \{(x, y) \mid y \equiv dx + i \pmod{n}, 0 \leq i \leq d - 1\}.
\end{align*}
\]

In particular, if $n = d^m$, then $G_B(n, d)$ is the de Bruijn digraph $B(d, m)$. The generalized Kautz digraph $G_K(n, d)$ is defined by following congruence equations:

\[
\begin{align*}
V(G_K(n, d)) &= \{0, 1, \ldots, n-1\}, \\
A(G_K(n, d)) &= \{(x, y) \mid y \equiv -dx - i \pmod{n}, 1 \leq i \leq d\}.
\end{align*}
\]

In particular, if $n = d^m + d^{m-1}$, then $G_K(n, d)$ is the Kautz digraph $K(d, m)$. The graphs $G_B(6, 3)$ and $G_K(9, 2)$ are exhibited in Fig. 1.

The structure properties of generalized de Bruijn and Kautz digraphs receive more attention. Du et al. [6] studied the Hamiltonian property of generalized de Bruijn and Kautz networks. Also, several structural objects such as spanning trees, Eulerian tours [16], closed walks [22], and small cycles [10] have been counted. Shan et al. [19–21] studied the absorbants and twin domination of generalized de Bruijn digraphs. Recently, Ducoffe [7] determined when the large generalized de Bruijn cycles, which are Kronecker products of generalized de Bruijn digraphs and decycles, are Hamiltonian. Lien et al. [17] obtained an upper bound on the decycling number of a generalized Kautz digraph. Wang [26] showed that there is an efficient twin dominating set in a generalized de Bruijn digraph $G_B(n, d)$ with $n = c(d + 1)$ if and only if $d$ is even and relatively prime to $c$. Dong et al. [5] completely determined the domination number of generalized de Bruijn digraphs. More studied progress on the generalized de Bruijn and Kautz networks can be found in [27].

In order to make our arguments easier to follow, we introduce the modulo interval so as to represent the out-neighborhood of each vertex in $G_B(n, d)$.\*
and $G_K(n, d)$. Let $I = \{0, 1, \ldots, n-1\}$ denote the vertex set of $G_B(n, d)$ and $G_K(n, d)$. For any integers $i, j$ satisfying $i \not\equiv j \pmod{n}$, a modulo interval $[i, j]$ $(\mod n)$, with respect to modulo $n$, is defined by

$$[i, j] \pmod{n} = \begin{cases} \{i, i+1, \ldots, j\} \pmod{n}, & i \pmod{n} < j \pmod{n}, \\ \{i, \ldots, n-1, 0, \ldots, j\} \pmod{n}, & i \pmod{n} > j \pmod{n}. \end{cases}$$

By the definitions, $I = [0, n-1]$, and for each $j \in [0, n-1]$, clearly,

$$O(j) = \begin{cases} [jd, jd + (d-1)] \pmod{n}, & \text{in } G_B(n, d), \\ [-jd - d, -jd - 1] \pmod{n}, & \text{in } G_K(n, d). \end{cases}$$

Notice that if $d = 1$, then the graph $G_B(n, 1)$ (resp., $G_K(n, 1)$) has $n$ self-loops. Throughout this paper, we always assume $n \geq d \geq 2$, and denote

$$\Delta_k := \left( \sum_{j=0}^{k} d^j \right)^{-1}.$$  

If the set $D = \{x, x + 1, \ldots, x + k\} \pmod{n}$ is a dominating set or a distance $k$-dominating set of $G_B(n, d)$ (resp., $G_K(n, d)$), then $D$ is called a consecutive dominating set or a consecutive distance $k$-dominating set of $G_B(n, d)$ (resp., $G_K(n, d)$). A consecutive minimum dominating set of $G_B(n, d)$ (resp., $G_K(n, d)$) is a consecutive dominating set with cardinality $\gamma(G_B(n, d))$ (resp., $\gamma(G_K(n, d))$) and a consecutive distance $k$-dominating set of $G_B(n, d)$ (resp., $G_K(n, d)$) is a consecutive distance $k$-dominating set with cardinality $\gamma_k(G_B(n, d))$ (resp., $\gamma_k(G_K(n, d))$).

Tian and Xu [25] established the upper and lower bounds on the distance $k$-domination number of $G_B(n, d)$ and $G_K(n, d)$. This paper continues to study distance $k$-domination in generalized de Bruijn and Kautz digraphs. In Subsection 2.1, we show that every generalized de Bruijn digraph $G_B(n, d)$ has the distance $k$-domination number either $\lceil n\Delta_k \rceil$ or $\lceil n\Delta_k \rceil + 1$. In Subsection 2.2, we derive various sufficient conditions for $\gamma_k(G_B(n, d)) = \lceil n\Delta_k \rceil$. In Section 3, we give a sharp upper bound of $\gamma_k(G_K(n, d))$, which improves the previous upper bound of $\gamma_k(G_K(n, d))$ due to Tian and Xu [25]. In closing section, we pose two open problems.

2 Minimum distance $k$-dominating sets in $G_B(n, d)$

In the first subsection of this section, by constructing a distance $k$-dominating set of an arbitrary generalized de Bruijn digraph $G_B(n, d)$, we show that the distance $k$-domination number of $G_B(n, d)$ has exactly two values. In next subsection, we describe various sufficient conditions for the distance $k$-domination number of $G_B(n, d)$ equal to one of two values.
2.1 Distance k-domination number of $G_B(n,d)$

Tian and Xu [25] obtained the upper and lower bounds on $\gamma_k(G_B(n,d))$.

**Lemma 1** [25] *For every generalized de Bruijn digraph $G_B(n,d)$,*

\[
\lceil n \Delta_k \rceil \leq \gamma_k(G_B(n,d)) \leq \lceil \frac{n}{d^k} \rceil.
\]

We are ready to improve the above upper bound on $\gamma_k(G_B(n,d))$ by directly constructing a (consecutive) distance $k$-dominating set of $G_B(n,d)$ with cardinality $\lceil n \Delta_k \rceil + 1$. The following lemma plays a key role in constructing such a distance $k$-dominating set of $G_B(n,d)$.

**Lemma 2** *Every generalized de Bruijn digraph $G_B(n,d)$ contains a vertex $x$ satisfying the following inequality:

\[
x + \lceil n \Delta_k \rceil - (d - 2) \leq dx \leq x + \lceil n \Delta_k \rceil \pmod{n}.
\]

*Proof* We choose an arbitrary vertex $x \in V(G_B(n,d))$. If $x$ satisfies (1), we are done. Otherwise, the vertex $x$ clearly satisfies either

\[
0 \leq dx \leq x + \lceil n \Delta_k \rceil - (d - 1) \pmod{n}
\]

or

\[
x + \lceil n \Delta_k \rceil + 1 \leq dx \leq n - 1 \pmod{n}.
\]

We find the desired vertex by distinguishing the following two cases.

**Case 1** $0 \leq dx \leq x + \lceil n \Delta_k \rceil - (d - 1) \pmod{n}$.

Note that if $x$ increases by integer $i$, then the value of $dx$ is increased to $d(x + i) = dx + di$. In this case, we find the desired vertex by increasing the value of $x$. Since $dx \leq x + \lceil n \Delta_k \rceil - (d - 1) \pmod{n}$, there exists an integer $i$ ($\geq 0$) such that $x$ and $i$ satisfy the following inequality:

\[
d(x + i) \leq (x + i) + \lceil n \Delta_k \rceil - 2(d - 2) \pmod{n},
\]

since $i = 0$ satisfies the inequality. Let $i$ be the maximal integer satisfying (2). We claim that

\[
d(x + i) \geq (x + i) + \lceil n \Delta_k \rceil - 2(d - 2) \pmod{n}.
\]

Indeed, if

\[
d(x + i) \leq (x + i) + \lceil n \Delta_k \rceil - 2(d - 2) - 1 \pmod{n},
\]

then

\[
d(x + i + 1) \leq (x + i + 1) + \lceil n \Delta_k \rceil - (d - 2) \pmod{n}.
\]

So $i + 1$ also satisfies (2), which contradicts the maximality of $i$. Hence, (3) follows. If the equality holds in (2), that is,

\[
d(x + i) \equiv x + \lceil n \Delta_k \rceil - (d - 2) \pmod{n},
\]
then \( x + i \) satisfies (1). So we replace \( x \) by \( x + i \), and obtain the desired vertex. Otherwise, by (3), we have

\[
(x + i) + \lceil n\Delta_k \rceil - 2(d - 2) \leq d(x + i) \leq (x + i) + \lceil n\Delta_k \rceil - (d - 1) \quad (\text{mod } n).
\]

Hence,

\[
(x + i + 1) + \lceil n\Delta_k \rceil - (d - 3) \leq d(x + i + 1) \leq (x + i + 1) + \lceil n\Delta_k \rceil \quad (\text{mod } n).
\]

Clearly, \( x + i + 1 \) satisfies (1). Thus, we replace \( x \) by \( x + i + 1 \) and obtain the desired vertex.

**Case 2** \( x + \lceil n\Delta_k \rceil + 1 \leq dx \leq n - 1 \text{ (mod } n) \).

We can obtain the desired vertex by decreasing the value of \( x \). Clearly, there exists an integer \( i \) \((\geq 0)\) such that \( x \) and \( i \) satisfy the following inequality:

\[
d(x - i) \geq (x - i) + \lceil n\Delta_k \rceil \quad (\text{mod } n),
\]

(4) since the inequality \( dx \geq x + \lceil n\Delta_k \rceil + 1 \) implies that \( i = 0 \) satisfies (4). Let \( i \) be the maximal integer satisfying (4). We claim that

\[
d(x - i) \leq (x - i) + \lceil n\Delta_k \rceil + d - 2 \quad (\text{mod } n).
\]

(5) Suppose, to the contrary, that

\[
d(x - i) \geq (x - i) + \lceil n\Delta_k \rceil + d - 1 \quad (\text{mod } n).
\]

Equivalently,

\[
d(x - (i + 1)) \geq (x - (i + 1)) + \lceil n\Delta_k \rceil \quad (\text{mod } n).
\]

But then \( i + 1 \) satisfies (4). This contradicts the maximality of \( i \). Thus, (5) holds. If the equality holds in (4), then the vertex \( x - i \) satisfies (1). So we obtain the desired vertex by replacing \( x \) by \( x - i \). Otherwise, by (5), we have

\[
(x - i) + \lceil n\Delta_k \rceil + 1 \leq d(x - i) \leq (x - i) + \lceil n\Delta_k \rceil + d - 2 \quad (\text{mod } n).
\]

Hence,

\[
(x - (i + 1)) + \lceil n\Delta_k \rceil - (d - 2) \leq d(x - (i + 1))
\]

\[
\leq (x - (i + 1)) + \lceil n\Delta_k \rceil - 1 \quad (\text{mod } n).
\]

Hence, \( x - (i + 1) \) satisfies (1). We obtain the desired vertex by replacing \( x \) by \( x - (i + 1) \). \( \square \)

**Theorem 1** For every generalized de Bruijn digraph \( G_B(n, d) \),

\[
\gamma_k(G_B(n, d)) = \lceil n\Delta_k \rceil \text{ or } \lceil n\Delta_k \rceil + 1.
\]
Proof. By Lemma 1, it suffices to show that
\[ \gamma(G_B(n,d)) \leq \lceil n \Delta_k \rceil + 1. \]
The proof is obtained by directly constructing a (consecutive) distance \(k\)-dominating set of \(G_B(n,d)\) with cardinality \(\lceil n \Delta_k \rceil + 1\). By Lemma 2, there is a vertex \(x\) in \(G_B(n,d)\) satisfying (1). Let
\[ D = \{x, x + 1, \ldots, x + \lceil n \Delta_k \rceil\}. \]
We show that \(D\) is a distance \(k\)-dominating set of \(G_B(n,d)\). By the definition, we need to prove that
\[ \bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d)). \]
First, we show that the vertices of \(O_{i-1}(D) \cup O_i(D)\) are consecutive for all \(i, 1 \leq i \leq k\). The out-neighborhoods of vertices in \(D\) are given as follows:
\[ O(x) = \{dx, dx + 1, \ldots, dx + d-1\} \pmod{n}, \]
\[ O(x + 1) = \{d(x + 1), d(x + 1) + 1, \ldots, d(x + 1) + d-1\} \pmod{n}, \]
\[ \cdots, \]
\[ O(x + \lceil n \Delta_k \rceil) = \{d(x + \lceil n \Delta_k \rceil), d(x + \lceil n \Delta_k \rceil) + 1, \ldots, d(x + \lceil n \Delta_k \rceil) + d-1\} \pmod{n}. \]
Then
\[ O(D) = \left[dx, d(x + \lceil n \Delta_k \rceil) + d - 1\right] \pmod{n}. \]
Similarly, the \(i\)-th out-neighborhoods
\[ O_i(D) = \left[d^{i}x, d^{i}(x + \lceil n \Delta_k \rceil) + (d - 1) \sum_{j=0}^{i} d^{j}\right] \pmod{n}, \quad i = 1, 2, \ldots, k. \]
Since \(x\) satisfying inequality (1), there exists an integer \(h, 0 \leq h \leq d - 2\), such that
\[ dx \equiv x + \lceil n \Delta_k \rceil - h \pmod{n}, \]
and so we have
\[ d^{2}x \equiv d(x + \lceil n \Delta_k \rceil) - dh \pmod{n}, \]
\[ d^{3}x \equiv d^{2}(x + \lceil n \Delta_k \rceil) - d^{2}h \pmod{n}, \]
\[ \cdots, \]
\[ d^{k}x \equiv d^{k-1}(x + \lceil n \Delta_k \rceil) - d^{k-1}h \pmod{n}. \]
Thus,
\[ O_{i-1}(D) \cap O_i(D) \neq \emptyset, \quad i = 1, 2, \ldots, k. \]
This implies that the vertices of $O_{i-1}(D) \cup O_i(D)$ are consecutive, since the vertices of $O_i(D)$ are consecutive for each $i$, $0 \leq i \leq k$. Therefore, the vertices of $\bigcup_{i=0}^{k} O_i(D)$ are consecutive.

Next, we show that $\bigcup_{i=0}^{k} O_i(D)$ contains all the vertices of $G_B(n,d)$. Note that $O_1(D) \cap D \neq \emptyset$. Thus, it suffices to show that $O_k(D) \cap D \neq \emptyset$. For the last vertex in $O_k(D)$, since $x$ satisfies (1), we have

\[ d^k(x + \lceil n\Delta_k \rceil) + (d - 1)\Delta_k^{-1} = d^{k-1}(x + \lceil n\Delta_k \rceil - h) + d^k \lceil n\Delta_k \rceil + (d - 1)\Delta_k^{-1} = d^{k-1}x + (d^k + d^{k-1}) \lceil n\Delta_k \rceil + (d - 1)d^k - hd^{k-1} + (d - 1)\Delta_k^{-1} = \ldots = x + \lceil n\Delta_k \rceil \Delta_k^{-1} - h\Delta_{k-1}^{-1} + (d - 1)\Delta_k^{-1} = x + (d - 1) + \lceil n\Delta_k \rceil \Delta_k^{-1} + (d(d - 1) - h)\Delta_{k-1}^{-1} \geq x \pmod{n}. \]

The last inequality holds, since $d \geq 2$ and $0 \leq h \leq d - 2$. Hence, $O_k(D) \cap D \neq \emptyset$, and so

\[ \bigcup_{i=1}^{k} O_i(D) \supseteq \{ x + \lceil n\Delta_k \rceil, x + \lceil n\Delta_k \rceil + 1, \ldots, n - 1, 0, 1, \ldots, x \}. \]

This implies that

\[ \bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d)), \]

that is, $D$ is a (consecutive) distance $k$-dominating set of $G_B(n,d)$. Consequently,

\[ \gamma_k(G_B(n,d)) \leq |D| = \lceil n\Delta_k \rceil + 1. \]

For distance $k = 1$, we obtain the following result.

**Corollary 1** [5] For every generalized de Bruijn digraph $G_B(n,d)$, either

\[ \gamma(G_B(n,d)) = \left\lfloor \frac{n}{d+1} \right\rfloor \]

or

\[ \gamma(G_B(n,d)) = \left\lfloor \frac{n}{d+1} \right\rfloor + 1. \]

### 2.2 Generalized de Bruijn digraphs $G_B(n,d)$ with $\gamma(G_B(n,d)) = \left\lfloor \frac{n}{d+1} \right\rfloor$

In this subsection, we derive various sufficient conditions for the distance $k$-domination number to achieve the value $\lceil n\Delta_k \rceil$ in a generalized de Bruijn digraph $G_B(n,d)$.

**Theorem 2** If there exists a vertex $x \in V(G_B(n,d))$ satisfying the congruence equation

\[ (d - 1)x \equiv \lceil n\Delta_k \rceil - h \pmod{n} \] (6)
for some \( h \), where

\[
0 \leq \Delta_k^{-1} h \leq \Delta_k^{-1} \lfloor n \Delta_k \rfloor - n, \tag{7}
\]

then

\[
\gamma_k(G_B(n, d)) = \lfloor n \Delta_k \rfloor,
\]

and

\[
D = \{ x, x + 1, \ldots, x + \lfloor n \Delta_k \rfloor - 1 \}
\]

is a consecutive minimum distance \( k \)-dominating set of \( G_B(n, d) \).

**Proof** Let \( x \) be a vertex of \( G_B(n, d) \) satisfying (6). Note that \( |D| = \lfloor n \Delta_k \rfloor \).

By Theorem 1, it is sufficient to show that

\[
D = \{ x, x + 1, \ldots, x + \lfloor n \Delta_k \rfloor - 1 \}
\]

is a distance \( k \)-dominating set of \( G_B(n, d) \). For this purpose, we show that

\[
\bigcup_{i=1}^{k} O_i(D) = V(G_B(n, d)).
\]

We first prove that the vertices of \( O_{i-1}(D) \cup O_i(D) \) are consecutive for all \( i, 1 \leq i \leq k \). By the definition of \( G_B(n, d) \), the out-neighborhoods \( O(D) \) of \( D \) are given as follows:

\[
O(x) = \{ dx, dx + 1, \ldots, dx + d - 1 \} \pmod{n},
\]

\[
O(x + 1) = \{ d(x + 1), d(x + 1) + 1, \ldots, d(x + 1) + d - 1 \} \pmod{n},
\]

\[
\ldots,
\]

\[
O(x + \lfloor n \Delta_k \rfloor - 1) = \{ d(x + \lfloor n \Delta_k \rfloor) - d, d(x + \lfloor n \Delta_k \rfloor) - d + 1, \ldots, d(x + \lfloor n \Delta_k \rfloor) - 1 \} \pmod{n}.
\]

Then

\[
O(D) = \lfloor dx, dx + d \lfloor n \Delta_k \rfloor - 1 \rfloor \pmod{n}.
\]

Similarly, we have

\[
O_i(D) = \lfloor d^i x, d^i (x + \lfloor n \Delta_k \rfloor) - 1 \rfloor \pmod{n}.
\]

Clearly,

\[
|O_i(D)| = d^i \lfloor n \Delta_k \rfloor, \quad i = 0, 1, \ldots, k.
\]

Since \( x \) satisfies (6), we have

\[
O(D) = \lfloor x + \lfloor n \Delta_k \rfloor - h, d(x + \lfloor n \Delta_k \rfloor) - 1 \rfloor \pmod{n},
\]

\[
O_2(D) = \lfloor d(x + \lfloor n \Delta_k \rfloor) - dh, d^2(x + \lfloor n \Delta_k \rfloor) - 1 \rfloor \pmod{n},
\]

\[
\ldots,
\]

\[
O_k(D) = \lfloor d^{k-1}(x + \lfloor n \Delta_k \rfloor) - dh, d^k(x + \lfloor n \Delta_k \rfloor) - 1 \rfloor \pmod{n}.
\]

Hence, it can be seen that

\[
|O_{i-1}(D) \cap O_i(D)| = d^{i-1}h, \quad i = 1, 2, \ldots, k.
\]
Note that the vertices of each \( O_i(D) \) \((i \geq 0)\) are consecutive. By the above observations, if \( h = 0 \), then the last vertex of \( O_{i-1}(D) \) and the first vertex of \( O_i(D) \) are consecutive; while if \( h > 0 \), then
\[
O_{i-1}(D) \cap O_i(D) \neq \emptyset.
\]
Thus, the vertices of \( O_{i-1}(D) \cup O_i(D) \) are consecutive for all \( i, 1 \leq i \leq k \).

We next show that
\[
\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d)).
\]
As observed above, we see that the vertices of \( \bigcup_{i=0}^{k} O_i(D) \) are consecutive. In particular, the vertices of \( D \cup O_1(D) \) are consecutive. Thus, it suffices to show that the vertices \( O_k(D) \cup D \) are consecutive. For the last vertex in \( O_k(D) \), since
\[
0 \leq \Delta_{k-1}^{-1} h \leq \Delta_{k}^{-1}[n\Delta_k] - n,
\]
we have
\[
d^k(x + [n\Delta_k]) - 1 \equiv x + \Delta_{k}^{-1}[n\Delta_k] - \Delta_{k-1}^{-1} h - 1 \pmod{n} \quad (\text{by } (6))
\]
\[
\geq x - 1 \pmod{n}.
\]
This implies that the vertices of \( O_k(D) \cup D \) are consecutive, so
\[
\bigcup_{i=1}^{k} O_i(D) \supseteq \{x + [n\Delta_k], x + [n\Delta_k] + 1, \ldots, n - 1, 0, 1, \ldots, x - 1\}.
\]
This implies that
\[
\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d)),
\]
and hence, \( D \) is a distance \( k \)-dominating set of \( G_B(n,d) \). This completes the proof of Theorem 2.

As a special case of Theorem 2, we immediately have the following corollary.

**Corollary 2** Let \( \Delta_{k}^{-1} \mid n \). If there is a vertex \( x \in V(G_B(n,d)) \) satisfying congruence equation
\[
(d - 1)x \equiv n\Delta_k \pmod{n},
\]
then
\[
\gamma_k(G_B(n,d)) = n\Delta_k
\]
and
\[
D = \{x, x + 1, \ldots, x + n\Delta_k - 1\}
\]
is a consecutive minimum distance \( k \)-dominating set of \( G_B(n,d) \).
Remark 1 When $G_B(n, d)$ contains no vertex $x$ satisfying (6) in Theorem 2, it is possible to encounter 

$$\gamma_k(G_B(n, d)) = [n\Delta_k] + 1.$$ 

For example, let $G_B(40, 3)$ and $k = 3$. The congruence equation 

$$(d - 1)x \equiv [n\Delta_k] - h \pmod{n}$$ 

is 

$$2x \equiv 1 \pmod{40},$$ 

where $h = 0$, since $40/\sum_{j=0}^{3} 3^j = 1$. Clearly, there is no vertex satisfying (9). We can deduce that 

$$\gamma_3(G_B(40, 3)) = \left\lfloor \frac{40}{\sum_{j=0}^{3} 3^j} \right\rfloor + 1 = 2.$$ 

Indeed, for each $x$ of $G_B(40, 3)$, it can be verified that $\{x\}$ is not a distance 3-dominating set of $G_B(40, 3)$ by simple enumeration.

Recall that $G_B(d^m, d) = B(d, m)$ when $n = d^m$. For cases $k = 1$ and $k = 2$, the distance $k$-domination numbers of a de Bruijn digraph $B(d, m)$ were proved by Araki [1] and Tian [25], respectively. As an application of Theorem 2, we can determine the distance $k$-domination number of a de Bruijn digraph for all $k \geq 1$.

Corollary 3 For $d \geq 2$, $\gamma_k(B(d, m)) = [d^m\Delta_k]$.

Proof If $m \leq k$, then, by Theorem 2, clearly, 

$$\gamma_k(B(d, m)) = \gamma_k(G_B(d^m, d)) = 1 = [d^m\Delta_k],$$

and so the assertion holds. We may therefore assume $m > k$. Let $m = ik + l$, where $i \geq 1$ and $0 \leq l \leq k - 1$. Note that 

$$d^m = \Delta_k^{-1}(d^{m-k} - d^{m-k-1}) + d^{m-k-1},$$ 

$$d^{m-k-1} = \Delta_k^{-1}(d^{m-2k-1} - d^{m-2k-2}) + d^{m-2k-2}, \ldots.$$ 

Then we have 

$$d^m = \begin{cases} 
\Delta_k^{-1}(d^{m-k} - d^{m-k-1}) + (d^{m-2k-1} - d^{m-2k-2}) + \ldots \\
+ (d^{m-(i-1)k-(i-2)} - d^{m-(i-1)k-(i-1)}) + d^{m-(i-1)k-(i-1)}, & l < i, \\
\Delta_k^{-1}(d^{m-k} - d^{m-k-1}) + (d^{m-2k-1} - d^{m-2k-2}) + \ldots \\
+ (d^{m-ik-(i-1)} - d^{m-ik-i}) + d^{m-ik-i}, & l \geq i. 
\end{cases}$$ 

Because $m = ik + l$ and $0 \leq l \leq k - 1$, if $l < i$, then 

$$d^{m-(i-1)k-(i-1)} = d^{l+k-(i-1)} \leq d^{k}.$$
and if \( l \geq i \), then
\[
d^{m-ik-i} = d^{l-i} < d^k.
\]
Thus,
\[
[d^m \Delta_k] = 1 + (d - 1) \begin{cases} 
d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-(i-1)k-(i-1)}, & l < i, \\
d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-ik-i}, & l \geq i.
\end{cases}
\]
Hence, either
\[
x = d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-(i-1)k-(i-1)}
\]
or
\[
x = d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-ik-i}
\]
in \( B(d, m) \) satisfies the congruence equation
\[
(d - 1)x \equiv [d^m \Delta_k] - h \pmod{n},
\]
where \( h = 1 \) and
\[
0 \leq h \Delta_k^{-1} - \Delta_k^{-1} [d^m \Delta_k] - d^m.
\]
Therefore,
\[
\gamma_k(B(d, m)) = [d^m \Delta_k]
\]
by Theorem 2.

As an application of Corollary 2, we provide a new sufficient condition for \( \gamma_k(G_B(n, d)) \) equal to \( [n \Delta_k] \). For this purpose, we need the following result in elementary number theory.

For notational convenience, \( m \parallel n \) means that \( m \) divides \( n \) and \( m \nmid n \) means that \( m \) does not divide \( n \), where \( m, n \) are integers. For integers \( a_1, a_2, \ldots, a_n \), the greatest common divisor of \( a_1, a_2, \ldots, a_n \) is denoted by \( (a_1, a_2, \ldots, a_n) \).

**Lemma 3** [18]  For integers \( a_1, a_2, \ldots, a_m \ (m \geq 1), b, \) and \( n, \) the congruence equation
\[
\sum_{i=1}^{m} a_i x_i \equiv b \pmod{n}
\]
has at least a solution if and only if \((a_1, a_2, \ldots, a_m, n) \mid b\).

**Theorem 3**  For every generalized de Bruijn digraph \( G_B(n, d) \), if both \( n \) and \( d \) satisfy one of the following conditions:
\begin{enumerate}
  \item \( \Delta_k^{-1} \mid n \) and \((d - 1, n) \mid n \Delta_k;\)
  \item \([n \Delta_k] \equiv q \pmod{(d - 1, n)},\) where \( q \) satisfies the inequality
    \[
    0 \leq q \Delta_k^{-1} - \Delta_k^{-1} [n \Delta_k] - n,
    \]
\end{enumerate}
then \( \gamma_k(G_B(n, d)) = [n \Delta_k] \) and there is a vertex \( x \in V(G_B(n, d)) \) such that \( D = \{x, x + 1, \ldots, x + [n \Delta_k] - 1\} \) is a consecutive minimum distance \( k \)-dominating set of \( G_B(n, d) \).
Proof Let $n$ and $d$ satisfy one of conditions (i) and (ii). We show that $G_B(n, d)$ contains a vertex $x$ such that $D = \{x, x+1, \ldots, x+\lceil n\Delta_k \rceil - 1\}$ is a consecutive minimum distance $k$-dominating set of $G_B(n, d)$. By Theorem 2, it suffices to show that there exists a vertex $x \in V(G_B(n, d))$ satisfies (6) for some $h$ satisfying (7).

For $n$ and $d$ satisfying (i), by Lemma 3, there is a vertex $x \in V(G_B(n, d))$ satisfying

$$(d-1)x \equiv n\Delta_k \pmod{n},$$

and so the assertion follows directly from Corollary 2.

For $n$ and $d$ satisfying (ii), let

$$(d-1,n)=r, \quad [n\Delta_k]=pr+q,$$

where

$$p \geq 0, \quad 0 \leq q \leq r-1.$$ 

Set $q = h$. Since $(d-1,n) \mid pr$, the equation

$$(d-1)x \equiv pr \pmod{n}$$

has a solution by Lemma 3. Hence, there exists a vertex $x \in V(G_B(n, d))$ satisfying

$$(d-1)x \equiv [n\Delta_k] - h \pmod{n},$$

as desired. $\square$

By applying Theorems 1 and 2, we obtain the following sufficient condition for $\gamma_k(G_B(n, d))$ equal to $[n\Delta_k]$.

**Theorem 4** If $n = p\Delta_k^{-1} + q$, where

$$p \geq 1, \quad 1 \leq q \leq \min\{1 + 2\Delta_{k-1}^{-1}, \Delta_k^{-1} - 1\},$$

then $\gamma_k(G_B(n, d)) = [n\Delta_k]$.

**Proof** By Theorem 1, we have known that $G_B(n, d)$ contains a vertex satisfying (1). Let $x$ be such a vertex and let $D = \{x, x+1, \ldots, x+\lceil n\Delta_k \rceil - 1\}$. We claim that $D$ is a distance $k$-dominating set of $G_B(n, d)$. By the definition, it suffices to show that $\bigcup_{i=0}^k O_i(D) = V(G_B(n, d))$.

As before, we first show the vertices of $O_{i-1}(D) \cup O_i(D)$ are consecutive for all $i, 1 \leq i \leq k$. As already observed in Theorem 2, for $i = 0, 1, \ldots, k$, we have

$$O_i(D) = [d^i x, d^i (x+\lceil n\Delta_k \rceil) - 1] \pmod{n}, \quad |O_i(D)| = d^i [n\Delta_k].$$

Since $x$ satisfies inequality (1), there exists an integer $h, 0 \leq h \leq d-2$, such that

$$d x \equiv x + [n\Delta_k] - h \pmod{n},$$

$$d^2 x \equiv d(x + \lceil n\Delta_k \rceil) - dh \pmod{n},$$

$$d^3 x \equiv d^2(x + \lceil n\Delta_k \rceil) - d^2h \pmod{n},$$

$$\ldots,$$

$$d^k x \equiv d^{k-1}(x + \lceil n\Delta_k \rceil) - d^{k-1}h \pmod{n}.$$
Since
\[ O_i(D) = [d^i x, d^i (x + \lfloor n \Delta_k \rfloor) - 1] \pmod{n}, \quad i = 0, 1, \ldots, k, \]
the vertices of \( O_{i-1}(D) \cap O_i(D) \neq \emptyset \) are consecutive for all \( i, 1 \leq i \leq k \).

By the above fact, we show that \( \bigcup_{i=1}^{k} O_i(D) \) contains all the vertices of \( G_{B}(n,d) \setminus D \) by showing the vertices of \( O_k(D) \cup D \) are consecutive. We consider the last vertex in \( O_k(D) \).

Since \( n = p \Delta_{k}^{-1} + q \), we have
\[ \lfloor n \Delta_k \rfloor \Delta_k^{-1} = n - q + \Delta_k^{-1}. \]
Hence, by
\[ dx \equiv x + \lfloor n \Delta_k \rfloor - h \pmod{n}, \quad 0 \leq h \leq d - 2, \]
we have
\[
\begin{align*}
d^k x + d^k \lfloor n \Delta_k \rfloor - 1 &= d^{k-1} (x + \lfloor n \Delta_k \rfloor - h) + d^k \lfloor n \Delta_k \rfloor - 1 \\
&= d^{k-1} x + (d^k + d^{k-1}) \lfloor n \Delta_k \rfloor - d^{k-1} h - 1 \\
&= \cdots \\
&\equiv (x - 1) + \lfloor n \Delta_k \rfloor \Delta_k^{-1} - h \Delta_k^{-1} \pmod{n} \\
&\equiv (x - 1) + 1 + (d - h) \Delta_k^{-1} - q \pmod{n} \\
&\geq (x - 1) + 1 + 2 \Delta_k^{-1} - q \pmod{n} \\
&\geq x - 1.
\end{align*}
\]
The last inequality holds, since
\[ 1 \leq q \leq \min\{1 + 2 \Delta_{k-1}^{-1}, \Delta_k^{-1} - 1\}. \]
Note that the vertices of \( O_i(D) \) are consecutive for all \( i, 0 \leq i \leq k \). Then
\[ \bigcup_{i=1}^{k} O_i(D) \supseteq \{ x + \lfloor n \Delta_k \rfloor, x + \lfloor n \Delta_k \rfloor + 1, \ldots, n - 1, 0, 1, \ldots, x - 1 \}. \]
This implies that
\[ \bigcup_{i=1}^{k} O_i(D) \supseteq V(G_{B}(n,d)) \setminus D, \]
and hence, \( D = \{ x, x + 1, \ldots, x + \lfloor n \Delta_k \rfloor - 1 \} \) is a distance \( k \)-dominating set of \( G_{B}(n,d) \). Thus,
\[ \gamma_k(G_{B}(n,d)) \leq |D| = \lfloor n \Delta_k \rfloor. \]
By Theorem 1,
\[ \gamma_k(G_{B}(n,d)) = \lfloor n \Delta_k \rfloor. \]
3 Minimum distance \( k \)-dominating sets in \( G_K(n, d) \)

Tian and Xu [25] observed the following upper and lower bounds on \( \gamma_k(G_K(n, d)) \).

**Lemma 4** [25] *For any generalized Kautz digraph* \( G_K(n, d) \),

\[
\lceil n\Delta_k \rceil \leq \gamma_k(G_K(n, d)) \leq \left\lfloor \frac{n}{d^k} \right\rfloor.
\]

In this section, we shall improve the above upper bound on \( \gamma_k(G_K(n, d)) \) by constructing a consecutive distance \( k \)-dominating set of \( G_K(n, d) \).

**Theorem 5** Let \( G_K(n, d) \) be a generalized Kautz digraph. Then

\[
D = \left\{ 0, 1, \ldots, \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor - 1 \right\}
\]

is a distance \( k \)-dominating set of \( G_K(n, d) \), and

\[
\gamma_k(G_K(n, d)) \leq \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor.
\]

**Proof** We show that \( D \) is a distance \( k \)-dominating set of \( G_K(n, d) \). By the definitions of \( G_K(n, d) \) and \( i \)-th out-neighborhood, we have

\[
O_1(D) = \left\{ n - 1, n - 2, \ldots, n - d \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor \right\};
\]

\[
O_2(D) = \left\{ 0, 1, \ldots, d^2 \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor - 1 \right\},
\]

\[
O_3(D) = \left\{ n - 1, n - 2, \ldots, n - d^3 \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor \right\},
\]

\[
O_4(D) = \left\{ 0, 1, \ldots, d^3 \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor - 1 \right\},
\]

\[
\ldots,
\]

If \( k \) is odd, then we obtain

\[
O_{k-1}(D) = \left\{ 0, 1, \ldots, d^{k-1} \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor - 1 \right\},
\]

\[
O_k(D) = \left\{ n - 1, n - 2, \ldots, n - d^k \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor \right\};
\]

and if \( k \) is even, then

\[
O_{k-1}(D) = \left\{ n - 1, n - 2, \ldots, n - d^{k-1} \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor \right\},
\]

\[
O_k(D) = \left\{ 0, 1, \ldots, d^k \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor - 1 \right\}.
\]
In both cases, we have
\[ |O_{k-1}(D)| = d^{k-1} \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor, \quad |O_k(D)| = d^k \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor. \]
Note that the vertices of \( O_{k-1}(D) \) and \( O_k(D) \) are consecutive, and
\[ (d^k + d^{k-1}) \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor \geq n. \]
Then
\[ O_{k-1}(D) \cup O_k(D) = V(G_K(n, d)). \]
Hence, \( D \) is a distance \( k \)-dominating set for \( G_K(n, d) \). Therefore,
\[ \gamma_k(G_K(n, d)) \leq |D| = \left\lfloor \frac{n}{d^k + d^{k-1}} \right\rfloor. \]
\( \square \)

**Remark 2** The upper bound on the distance \( k \)-domination number given in Theorem 5 is sharp. For example, we consider the digraph \( G_K(7, 2) \). We claim that
\[ \gamma_2(G_K(7, 2)) = 2 = \left\lfloor \frac{7}{2 + 4} \right\rfloor. \]
Suppose not, we have \( \gamma_2(G_K(7, 2)) = 1 \) by Lemma 4. Let \( \{x_0\} \) be a minimum distance 2-dominating set of \( G_K(7, 2) \). Since \( |O_i(x)| = d = 2 \) for each \( x \in V(G_K(7, 2)) \), we have \( O_i(x_0) \cap O_j(x_0) = \emptyset \) for all \( 0 \leq i \neq j \leq 2 \). On the other hand, it can be verified that for each \( x \in V(G_K(7, 2)) \), there exist integers \( i, j, \) \( 0 \leq i \neq j \leq 2 \), such that \( O_i(x) \cap O_j(x) \neq \emptyset \) by the simple enumeration. Thus, each vertex \( x \) of \( G_K(7, 2) \) cannot form a distance 2-dominating set of \( G_K(7, 2) \), as claimed. By Theorem 5, \( D = \{0, 1\} \) must be a minimum distance 2-dominating set of \( G_K(7, 2) \).

The following result on the domination number of \( G_K(n, d) \), due to Kikuchi and Shibata [15], is an immediate consequence of Lemma 4 and Theorem 5.

**Corollary 4** [15] For every generalized Kautz digraph \( G_K(n, d) \), \( \gamma(G_K(n, d)) = \lceil n/(d + 1) \rceil \).

It seems difficult to determine the minimum distance \( k \)-dominating set for general generalized Kautz digraphs \( G_K(n, d) \). Now, we present a sufficient condition for the distance \( k \)-domination number of \( G_K(n, d) \) to be the lower bound \( \lceil n\Delta_k \rceil \) in Theorem 5.

**Theorem 6** For every generalized Kautz digraph \( G_K(n, d) \), if
\[ (d^{k-1} + d^k)\lceil n\Delta_k \rceil \geq n \quad \text{or} \quad d^k \lceil n\Delta_k \rceil \geq \left\lceil \frac{n}{d + 1} \right\rceil, \]
then \( \gamma_k(G_K(n, d)) = \lceil n\Delta_k \rceil \).

**Proof** The proof is by directly constructing a (consecutive) distance \( k \)-dominating set of \( G_K(n, d) \) with cardinality \( \lceil n\Delta_k \rceil \). Let \( D = \{0, 1, \ldots, \lceil n\Delta_k \rceil \} - \{0\} \) and...
1). We claim that $D$ is a distance $k$-dominating set of $G_K(n,d)$. As we have observed, if $k$ is odd, then
\[
O_{k-1}(D) = \{0,1,\ldots,d^{k-1}[n\Delta_k] - 1\},
\]
\[
O_k(D) = \{n-1,n-2,\ldots,n - d^k[n\Delta_k]\};
\]
and if $k$ is even, then
\[
O_{k-1}(D) = \{n-1,n-2,\ldots,n - d^{k-1}[n\Delta_k]\},
\]
\[
O_k(D) = \{0,1,\ldots,d^k[n\Delta_k] - 1\}.
\]
Clearly,
\[
|O_{k-1}(D)| = d^{k-1}[n\Delta_k], \quad |O_k(D)| = d^k[n\Delta_k].
\]
Suppose that
\[
(d^{k-1} + d^k)[n\Delta_k] \geq n.
\]
Note that the vertices of $O_{k-1}(D)$ and $O_k(D)$ are consecutive. Then
\[
O_{k-1}(D) \cup O_k(D) = V(G_K(n,d)).
\]
Thus, $D = \{0,1,\ldots,[n\Delta_k] - 1\}$ is a distance $k$-dominating set of $G_K(n,d)$.

Suppose that
\[
d^{k-1}[n\Delta_k] \geq \left\lceil \frac{n}{d+1} \right\rceil.
\]
By Lemma 4 and Theorem 5, $D_1 = \{0,1,\ldots,\lceil n/(d+1) \rceil - 1\}$ is a minimum dominating set of $G_K(n,d)$. Let $D'_1 = \{n-1,n-2,\ldots,n - \lceil n/(d+1) \rceil\}$. By the definition of $G_K(n,d)$, we have $O(D'_1) = \{0,1,\ldots,d\lceil n/(d+1) \rceil - 1\}$. Since
\[
|D'_1 \cup O(D'_1)| = (d+1)\left\lceil \frac{n}{d+1} \right\rceil \geq n,
\]
$D'_1$ is also a minimum dominating set of $G_K(n,d)$. Since the vertices of $D$ are consecutive and $d^{k-1}[n\Delta_k] \geq \lceil n/(d+1) \rceil$, we have either $O_{k-1}(D) \supseteq D_1$ or $O_{k-1}(D) \supseteq D'_1$. Hence, $D = \{0,1,\ldots,[n\Delta_k] - 1\}$ is a distance $k$-dominating set of $G_K(n,d)$.

\section{Closing remarks}

In this paper, we prove that the distance $k$-domination number of $G_B(n,d)$ takes on exactly one of two values $\lceil n\Delta_k \rceil$ and $\lceil n\Delta_k \rceil + 1$. In Theorems 2–4, we provide various sufficient conditions for $\gamma_k(G_B(n,d))$ equal to $\lceil n\Delta_k \rceil$. It is of interest to determine the necessary and sufficient condition for $\gamma_k(G_B(n,d))$ equal to $\lceil n\Delta_k \rceil$. In Theorem 5, we establish the sharp upper bound on $\gamma_k(G_B(n,d))$. Furthermore, we provide a sufficient conditions for $\gamma_k(G_K(n,d))$ equal to $\lceil n\Delta_k \rceil$ in Theorem 6. We propose the following open problems.
Problem 1  Determine whether the sufficient condition in Theorem 3 is also necessary for $\gamma_k(G_B(n, d))$ equal to $\lceil n\Delta_k \rceil$.

For Problem 1, Dong et al. [5] proved that the assertion is true for the case when $k = 1$.

Problem 2  For any generalized Kautz digraph $G_K(n, d)$, $\gamma_k(G_K(n, d)) = \lceil n/(d^{k-1} + d^k) \rceil$.

For Problem 2, if $k = 1$, Corollary 4, due to Kikuchi and Shibata [15], implies that the assertion is true.

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