The complete menu of eligible metrics for a family of toy Hamiltonians $H \neq H^\dagger$ with real spectra.

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Abstract

An elementary set of non-Hermitian $N$ by $N$ matrices $H^{(N)}(\lambda) \neq [H^{(N)}(\lambda)]^\dagger$ with real spectra is considered, assuming that each of these matrices represents a selfadjoint quantum Hamiltonian in an ad hoc Hilbert space of states $\mathcal{H}^{(\text{physical})}$. The problem of an explicit specification of all of these spaces (i.e., in essence, of all of the eligible ad hoc inner products and metric operators $\Theta$) is addressed. The problem is shown exactly solvable and, for every size $N = 2, 4, \ldots$ and parameter $\lambda \in (-1, 1)$ in matrix $H^{(N)}(\lambda)$, the complete $N$–parametric set of metrics $\Theta^{(N)}_{\alpha_1,\alpha_2,\ldots,\alpha_N}(\lambda)$ is recurrently defined by closed formula.
1 Introduction

1.1 The concept of the metric $\Theta$ in Hilbert space $\mathcal{H}^{(\text{physical})}$

One-dimensional quantum systems described, in units $\hbar = 2m = 1$, by the ordinary differential Schrödinger equation

$$H \psi(x) = E \psi(x), \quad H = -\frac{d^2}{dx^2} + V(x)$$

(1)

serve as a universal testing ground for the ideas, methods and techniques of quantum mechanics. One works with the standard representation $L^2(\mathbb{R})$ of the Hilbert space of states where the bound states are normalized in usual manner and where the Hamiltonian itself is self-adjoint,

$$\int_{\mathbb{R}} \psi^*(x) \psi(x) \, dx = 1, \quad H = H^\dagger.$$  

(2)

In parallel, the scattering solutions $\psi(x)$ of eq. (1) offer the simplest illustration of the delocalized waves which must remain compatible with the unitarity of the time evolution, etc. (cf., e.g., ref. [1] for numerous illustrations).

The transparency of such an elementary implementation of quantum theory can prove deceptive. People often forget that the requirement of the Hermiticity of $H$ in $L^2(\mathbb{R})$ can be replaced by an alternative, equally acceptable requirement $H = H^\dagger$ of the Hermiticity of the same operator in another Hilbert space $\mathcal{H}^{(\text{physical})} \neq L^2(\mathbb{R})$ where a different definition of the inner product $(\psi, \psi')_\Theta$ is employed,

$$(\psi, \psi')_\Theta = \int_{\mathbb{R}^2} \psi^*(x) \Theta(x, x') \psi'(x') \, dx \, dx', \quad H = H^\dagger \equiv \Theta^{-1} H^\dagger \Theta.$$  

(3)

Although the space $L^2(\mathbb{R})$ remains unchanged as a vector space, the definition of the "correct" linear functionals (i.e., the definition of the mapping $T$ of the "ket-vectors" upon "bra-vectors") is less elementary in $\mathcal{H}^{(\text{physical})}$,

$$T \psi(x) = \int_{\mathbb{R}} \psi^*(x') \Theta(x', x) \, dx'.$$

(4)

Misunderstandings may emerge whenever the metric is nontrivial, $\Theta = \Theta^\dagger \neq I$. Then, our Hamiltonian $H = H^\dagger$ appears non-Hermitian in the conventional Hilbert
space $\mathcal{H}^{\text{Dirac}} = L^2(\mathbb{R})$. The paradox has an elementary resolution since the space $\mathcal{H}^{\text{Dirac}}$ with its standard inner product is not the space of states of the system in question (cf. Appendix A for a brief recollection of a few concrete illustrative examples with this “cryptohermiticity” property).

1.2 The problem of the ambiguity of the metric $\Theta = \Theta(H)$

The deeper study of the similarity (3) between an operator $H$ and its adjoint $H^\dagger$ in a preselected space $\mathcal{H}^{\text{Dirac}}$ dates back to the early sixties [3]. In physics, the first use of such a feature, i.e., of the so called quasi-Hermiticity constraint

$$\Theta H = H^\dagger \Theta$$

imposed upon a sufficiently nontrivial and realistic Hamiltonian $H \neq H^\dagger$ emerged much later [4]. Still, up to now the subject remains full of open questions. One of the most challenging ones concerns the ambiguity of the metric $\Theta$ assigned, via eq. (3), to a given Hamiltonian operator $H$. Indeed, eq. (5) itself defines “too many” alternative physical metric operators $\Theta = \Theta(H)$. An explicit constructive illustration of such an ambiguity of the assignment $H \rightarrow \Theta$ given in section 5 of ref. [5] employs a free Hamiltonian $H = H_0$ which is complemented by a two-parametric family of non-trivial metric operators

$$\Theta_0^{\text{Mostafazadeh}}(F, K) = e^{-F}(\cosh K - \mathcal{P}\sinh K)$$

where $\mathcal{P}$ denotes parity and where $F$ and $K$ are arbitrary real numbers.

The first discussion of the general problem of the ambiguity of $\Theta(H)$ has been published by Scholtz et al [4]. They emphasized that besides the Hamiltonian $H$ itself, any other operator $\mathcal{O} = \mathcal{O}_j$ of an observable quantity in $\mathcal{H}^{\text{physical}}$ must obey the same Hermiticity relation as $H \equiv \mathcal{O}_0$. In an opposite direction, any eligible physical metric operator $\Theta$ must remain compatible with the corresponding set of the quasi-Hermiticity relations

$$\Theta \mathcal{O}_j = \mathcal{O}_j^\dagger \Theta, \quad j = (0), 1, 2, \ldots$$

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These requirements reduce the ambiguity in $\Theta$ at every index $j$. In this sense, the choice of the physical metric $\Theta = \Theta(H)$ can be made, in principle, unique.

We intend to return to the problem of the ambiguity of the metric in what follows. One of our reasons is that the universal strategy represented by requirement (7) is in fact rarely successful in practice. The solution of the complete set of the linear operator relations (7) appears to be hardly feasible. Typically, just $j = 0$ is considered and a particular solution $\Theta(H)$ of eq. (5) is sought for. Appendix B reviews a few alternative proposals of making the Hamiltonian-dependent metric operator $\Theta(H)$ unique in such a case.

1.3 Unique metrics in certain matrix models of scattering

A schematic comparison of a few alternative techniques of the removal of the ambiguity of metrics $\Theta(H) \neq I$ has been performed in our two brief comments [6]. We restricted our attention to the mere two-dimensional Hilbert spaces $\mathcal{H}^{(\text{physical})}$. Using an elementary set of two-by-two matrices $H = H^{(2)}$ we compared the merits and shortcomings of various versions of non-equivalent $\Theta$s. Due to the simplicity of the space we were able to base our analysis on an explicit construction of all the solutions $\Theta^{(2)}$ of eq. (5).

In some sense we shall just extend such a study to certain less trivial Hamiltonian matrices $H = H^{(N)}$ in what follows. The practical feasibility of such a project relies on suitable simplifications. It is obvious that for a general matrix $H^{(N)}$ one could hardly consider its size in the range $N > 4$ [7].

One of the most natural, anharmonic-oscillator-inspired choices of the simplified tridiagonal matrices $H^{(N)}$ was proposed in ref. [8]. Even these models with the number of variable matrix elements limited to $N/2$ appeared to be only tractable numerically [9]. In ref. [10] this observation led us to the most drastic reduction of the allowed number $k$ of the variable matrix elements in $H^{(N)}(\lambda_1, \ldots, \lambda_k)$ to the smallest integers $k = 1, 2$ and 3.

It was a pleasant surprise to discover that the latter choice proved extraordinarily successful. Without difficulties we were able to consider all the matrix sizes $N = 2K$.
including even the limiting case of $N = \infty$. The “exceptional” choice of $N = \infty$ made us ready to study and solve certain difficult conceptual problems in scattering theory (cf. the text of ref. [10] for more details). One of the reasons was that already the one-parametric matrix model of dynamics

$$
H = H(\lambda) = \\
\begin{bmatrix}
\ddots & \ddots & 2 & -1 \\
\ddots & 2 & -1 & -1 \\
-1 & 2 & -1 & -1 \\
-1 & 2 & -1 & \ddots \\
\end{bmatrix}
$$

proved compatible with the diagonal matrix solution of eq. (5),

$$
\Theta[H(\lambda)] = \\
\begin{bmatrix}
\ddots & 1 - \lambda \\
\ddots & \ddots & 1 - \lambda \\
\ddots & \ddots & \ddots & 1 - \lambda \\
1 + \lambda & \ddots & \ddots & \ddots \\
\end{bmatrix}
$$

The existence of such a local metric was already considered improbable in the dedicated phenomenological literature [11] and we felt encouraged us to generalize the explicit formula (9) to Hamiltonians with more parameters (cf. [10]) and to some matrices with different structure (cf. [12]).

Here, due to the lack of space, we shall skip all the similar enhancements of sophistication. Rather, we shall return to the bound-state problems where $N < \infty$ in what follows. Before doing so we should add a remark that there exists an amazingly close and direct connection between the two apparently independent sample
Hamiltonians given by eqs. (1) and (8). In Appendices C ad D we shall explain this relationship in more detail.

1.4 Matrix models and bound states

In our present return to model (8) and to the bound states we shall consider all the truncated, finite-dimensional matrix descendants \( H^{(N)} \) of eq. (8) with truncations \( N = 2K \). For this sequence of one-parametric toy Hamiltonians

\[
H^{(2)}(\lambda) = \begin{bmatrix}
2 & -1 - \lambda \\
-1 + \lambda & 2
\end{bmatrix},
\]

\( \text{(10)} \)

\[
H^{(4)}(\lambda) = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 - \lambda & 0 \\
0 & -1 + \lambda & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix},
\]

\( \text{(11)} \)

\[
H^{(6)}(\lambda) = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 - \lambda & 0 & 0 \\
0 & 0 & -1 + \lambda & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}
\]

\( \text{(12)} \)

etc the main result of our present paper will be the explicit construction of the respective complete sets of all the metrics \( \Theta^{(N)}(\lambda) \) in closed form.

Such a project is nontrivial since the bound-state wave functions remain localized so that there is no point in demanding the asymptotic locality constraint which made the scattering metric virtually unique in [10]. One can arrive at a really satisfactory physical interpretation of the bound-state system only via an exhaustive knowledge of all the metrics \( \Theta(H) \) allowed by eq. (5).

In an introductory part of our present paper we shall set \( \lambda = 0 \) in \( H^{(N)}(\lambda) \). In section [2] this simplification will help us to explain our method of construction of all the admissible metrics \( \Theta^{(N)}(0) = \Theta_0^{(N)} \). In essence, we shall combine the brute-force
symbolic-manipulation constructions performed at the first few $N = 2, 4, \ldots$ with the subsequent extrapolation of the resulting closed formulae towards all the even integers $N = 2K$.

In the second half of our paper (cf. sections 3 and 4) we shall return to the nontrivial, asymmetric Hamiltonian matrices $H = H^{(N)}(\lambda)$ with $\lambda \neq 0$. Firstly, in section 3 we shall solve eq. (5) for the first three models (10), (11) and (12). In section 4 we shall then extrapolate the resulting triplet of metrics to all the superscripts $N = 2K$. The closed formula for all of the solutions $\Theta^{(N)}(\lambda)$ of eq. (5) will be obtained as our main result.

In section 5 we shall summarize our message while Appendices A - D will complement it by a few additional remarks and technical notes.

2 The description of the method: $\lambda = 0$

2.1 Trivial starting point: All the metrics at $N = 2$.

In the light of ref. [6] the simplest possible two-dimensional Hamiltonian

$$H_0 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

is easily assigned the real pair of energies $E_{\pm} = 2 \pm 1$ as well as the general real-matrix ansatz for the metric

$$\Theta_0 = \begin{bmatrix} f & b \\ b & f' \end{bmatrix}$$

reflecting its necessary Hermiticity. In an encouraging start of our systematic study we may insert both these matrices in eq. (5) and arrive at the single constraint $f = f'$.

All the resulting two-parametric metrics $\Theta_0$ possess eigenvalues $\theta_{\pm}$ expressible in closed form, $\theta_{\pm} = f \pm b$. It is trivial to conclude that our $\Theta_0$ is positive (and can be called a metric) iff $f = f' > 0$ and $f^2 > b^2$.  

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2.2 All the metrics $\Theta_0$ at $N = 4$.

In the first nontrivial step of our analysis let us consider the four-dimensional Hamiltonian $H^{(4)}(\lambda)$ at $\lambda = 0$,

$$H_{0}^{(4)} = H^{(4)}(0) = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

and let us try to deduce the generic form of all of the related matrices $\Theta_0$ directly from the set of $N^2 = 16$ equations (5) for the $N^2 = 16$ unknown (though, presumably, real) matrix elements of $\Theta_0$. These equations are not all linearly independent. No surprise - the general solution contains $N = 4$ real parameters [13, 14].

2.2.1 Construction

As a typical task for Mathematica or Maple we solved our set by the brute force methods of linear algebra and we obtained its complete four-parametric solution

$$\Theta_0^{(4)} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_1 + \alpha_3 & \alpha_2 + \alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_2 + \alpha_4 & \alpha_1 + \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}$$

exhibiting linear dependence on all of its four parameters,

$$\Theta_0^{(4)} = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 .$$

(14)

While $M_1^{(4)}$ is just the four-dimensional unit matrix, the remaining three expansion matrices represent its elementary sparse-matrix generalizations,

$$M_2^{(4)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad M_3^{(4)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \quad M_4^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

This result of the computation indicates the possibility of the existence of a certain friendly extrapolation pattern towards the metrics $\Theta_0^{(N)}$ at any higher $N$. 
2.2.2 Positivity

After we specify the Hamiltonian but before we select any particular solution \( \Theta = \Theta(H) = \Theta^\dagger \) of eq. (5) we have to guarantee that our candidate for the metric is invertible and positive definite. Only then, this operator can consistently specify the corresponding physical Hilbert space \( \mathcal{H}^{(\text{physical})} \) of states of our quantum system [4].

At the larger dimensions \( N \) the proof of the positivity may be difficult. In the four-dimensional matrix example (13) it degenerates to the mere four elementary inequalities

\[
-2 \alpha_4 + 2 \alpha_1 + \alpha_3 - \alpha_2 \pm \sqrt{5} (-\alpha_2 + \alpha_3) > 0,
\]

\[
2 \alpha_4 + 2 \alpha_1 + \alpha_3 + \alpha_2 \pm \sqrt{5} (\alpha_2 + \alpha_3) > 0.
\]

They must be satisfied as a guarantee of the positivity of all the four eigenvalues \( \theta_k \) of the metric \( \Theta_0^{(4)} \).

2.3 Extrapolation to \( N > 4 \)

2.3.1 An ansatz for \( \Theta_0^{(N)} \)

It is natural to expect that formula (14) is just the first special case of the general expansion

\[
\Theta_0^{(N)} = \sum_{j=1}^{N} \alpha_j M_j^{(N)}(0).
\]

Let us activate the experience collected at the smallest \( N \) and assume that all of the matrices \( M = M_j^{(N)}(0) \) are solely composed of the matrix elements 0 or 1. In the \( j \)–th matrix the location of all of the non-vanishing elements may tentatively be selected as follows,

\[
\left( M_j^{(N)} \right)_{ik}(0) = 1 \iff i - k = m, \quad N + 1 - i - k = n,
\]

\[
m = j - 1, j - 3, \ldots, 1 - j, \quad n = N - j, N - j - 2, \ldots, j - N.
\]

Such an educated guess generalizes the above \( N = 2 \) and \( N = 4 \) results to all even dimensions. Its validity has carefully been verified at several higher even integers \( N = 2K \). One should note that the mere insertion of the ansatz followed by the check of the result is quick.
2.3.2 Verification: $N = 6$ etc.

Formulae (15) and (16) determine all the extrapolated $2K$–parametric matrices $\Theta^{(2K)}$. For illustrative purposes let us pick up $N = 2K = 6$. This choice gives the formula

$$
\Theta_6^{(6)} = \begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_2 & \alpha_1 + \alpha_3 & \alpha_2 + \alpha_4 & \alpha_3 + \alpha_5 & \alpha_4 + \alpha_6 & \alpha_5 \\
\alpha_3 & \alpha_2 + \alpha_4 & \alpha_1 + \alpha_3 + \alpha_5 & \alpha_2 + \alpha_4 + \alpha_6 & \alpha_3 + \alpha_5 & \alpha_4 \\
\alpha_4 & \alpha_3 + \alpha_5 & \alpha_2 + \alpha_4 + \alpha_6 & \alpha_1 + \alpha_3 + \alpha_5 & \alpha_2 + \alpha_4 & \alpha_3 \\
\alpha_5 & \alpha_4 + \alpha_6 & \alpha_3 + \alpha_5 & \alpha_2 + \alpha_4 & \alpha_1 + \alpha_3 & \alpha_2 \\
\alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1
\end{bmatrix}.
$$

We may easily verify the validity of the pertaining set of linear equations (5) by the simple-minded and straightforward insertion again.

A different category of verification of the internal consistency of our general $N$–parametric $N < \infty$ result (15) results from a direct study of the continuums limit $N \to \infty$. A few comments on this interesting are collected in Appendix D.

3 Metrics $\Theta^{(N)}$ for $H = H^{(N)}(\lambda)$ at $\lambda > 0$ and $N \leq 6$

3.1 Model with $N = 2$

The key features of bound states in our discrete short-range interaction models $H^{(N)}$ become already well illustrated via their most elementary special case (10). Firstly, this is the simplest model which shares peculiarity of the spectra which remain real in the $N$–independent interval of couplings $\lambda \in (-1, 1)$. For the whole sequence of our Hamiltonians we shall parametrize $\lambda = \cos \varphi \in (-1, 1)$, therefore, with $\varphi \in (0, \pi/2)$.

At $N = 2$ we can easily evaluate not only the closed formula for the energies, $E = E^{(2)}_\pm = 2 \pm \sin \varphi$, but also the norm $= 2 \sin^2 \varphi$ of the related eigenstates $\psi_\pm$. In addition, the cryptohermitian model $H^{(2)}(\cos \varphi)$ nicely illustrates the difference between its right eigenvectors and their left-eigenstate partners. In the spirit and notation of ref. [14] these respective column vectors $|\psi_\pm\rangle$ and their row-vector partners
\[ \langle \psi_\pm \rangle \text{ are different,} \]
\[
|\psi_\pm\rangle \sim \begin{pmatrix} 1 + \cos \varphi \\ \mp \sin \varphi \end{pmatrix}, \quad |\psi_\pm\rangle \sim \begin{pmatrix} 1 - \cos \varphi \\ \mp \sin \varphi \end{pmatrix},
\]
but a biorthogonal basis can be formed of them. Thus, \( H^{(2)}(\cos \varphi) \) is a self-adjoint matrix in an \textit{ad hoc,} Hamiltonian-dependent Hilbert space of states \( \mathcal{H}^{(\text{physical})} \).

In the light of refs. \[ \text{[6]} \] a key merit of the \( N = 2 \) example can be seen in the existence of the explicit spectral definition of the metric,
\[
\Theta = |\psi_+\rangle t_+ \langle \langle \psi_+ | + |\psi_-\rangle t_- \langle \langle \psi_- |. \quad (17)
\]
In this representation the guarantee of the necessary positivity of the metric reads \( t_\pm > 0 \). After the insertion of the eigenvectors we arrive at another explicit formula for the metric,
\[
\Theta \sim \begin{pmatrix} (1 - \cos \varphi)^2(t_+ + t_-) & (1 - \cos \varphi) \sin \varphi(-t_+ + t_-) \\ (1 - \cos \varphi) \sin \varphi(-t_+ + t_-) & \sin^2 \varphi(t_+ + t_-) \end{pmatrix}. \quad (18)
\]
Its inspection reveals that the metric may be re-written as a superposition
\[
\Theta^{(2)}(\lambda) = \alpha_1 M_1^{(2)}(\lambda) + \alpha_2 M_2^{(2)}(\lambda) \quad (19)
\]
with the two \( \lambda \)-dependent matrix coefficients,
\[
M_1^{(2)}(\lambda) = \begin{bmatrix} 1 - \lambda & 0 \\ 0 & 1 + \lambda \end{bmatrix}, \quad M_2^{(2)}(\lambda) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (20)
\]
Such a re-parametrization leaves the positivity criterion entirely transparent,
\[
\alpha_1 > 0, \quad \alpha_1^2(1 - \lambda^2) > \alpha_2^2, \quad N = 2 \quad (21)
\]
so that we may choose any \( \alpha_2 \) from the interval \((-\alpha_1 \sin \varphi, \alpha_1 \sin \varphi)\). The transition to the Hermitian limit \( \lambda \to 0 \) appears facilitated in the new parametrization.

3.2 Shorthand notation

The continuity of the expansion matrices in the free-motion limit as noticed above remains true at all \( N > 2 \). Thus, we may visualize metrics \( \Theta^{(N)}(\lambda) \) as expanded in
terms of the generalized $\lambda$–dependent sparse matrix coefficients obtained as a certain $\lambda$–deformation of their $\lambda = 0$ predecessors defined by closed formula (16). With this perspective in mind let us now define the following infinite sequence of polynomials,

\[ P_0 = 1, \quad P_1^{(\pm)} = 1 \pm \lambda, \quad P_2 = 1 - \lambda^2, \quad P_3^{(\pm)} = (1 \pm \lambda)(1 - \lambda^2), \quad P_4 = (1 - \lambda^2)^2, \quad P_5^{(\pm)} = (1 \pm \lambda)(1 - \lambda^2)^2, \quad P_6 = (1 - \lambda^2)^3, \quad \ldots . \] (22)

In terms of these polynomials the doublet of our sparse expansion matrices $M_{1,2}^{(2)}(\lambda)$ can be characterized by the “incidence” or “indexing” matrices $S_{1,2}^{(2)}$ with certain integer (or empty) entries. In general they will carry all the information about the position and about the degree of polynomial matrix elements $P_n = P_n(\lambda)$ of the respective matrices $M_j^{(N)}(\lambda)$. At $N = 2$ they are defined simply by the following assignment,

\[
M_1^{(2)}(\lambda) = \begin{bmatrix}
P_1^{(-)} & 0 \\
0 & P_1^{(+)}
\end{bmatrix} \quad \iff \quad S_1^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]

\[
M_2^{(2)}(\lambda) = \begin{bmatrix}
0 & P_0 \\
P_0 & 0
\end{bmatrix} \quad \iff \quad S_2^{(2)} = \begin{bmatrix} 0 \\
0 \end{bmatrix}.
\]

In what follows we shall demonstrate, step by step, that the polynomials $P_{2k+1}^{(\pm)}(\lambda)$ with the minus-sign superscript will always sit in the left upper triangle (i.e., above the second diagonal) of the respective expansion coefficient $M_j^{(N)}(\lambda)$ and vice versa. Thus, at any $N$, the expansion-coefficient functions $M_j^{(N)}(\lambda)$ will be unambiguously determined by the mere indexing matrices $S_j^{(N)}$. Of course, up to now we only demonstrated that these observations are valid at $N = 2$.

### 3.3 Model with $N = 4$

Hamiltonian $H^{(N)}(\lambda)$ of eq. (11) is the simplest model which is purely kinetic near its “distant” lattice points and which is dynamically nontrivial just in the vicinity of the origin. The $\lambda$–dependent coupling merely connects the two points $x_{N/2}$ and $x_{N/2+1}$ in the middle of the lattice. The four eigenvalues of matrix $H^{(4)}(\lambda)$ remain
real in the same interval of couplings $\lambda \in (-1, 1)$ as above (cf. Figure 1),

$$E_{\pm \pm} = 2 \pm \frac{1}{2} \sqrt{6 - 2 \lambda^2} \pm 2 \sqrt{5 - 6 \lambda^2 + \lambda^4}.$$  

Symbolic manipulations on the computer enable us to find all the corresponding matrices of the metric $\Theta^{(4)}(\lambda)$,

$$
\begin{bmatrix}
\alpha_1 (1 - \lambda) & \alpha_2 (1 - \lambda) & \alpha_3 & \alpha_4 \\
\alpha_2 (1 - \lambda) & \alpha_1 (1 - \lambda) + \alpha_3 (1 - \lambda) & \alpha_4 + \alpha_2 (1 - \lambda^2) & \alpha_3 \\
\alpha_3 & \alpha_4 + \alpha_2 (1 - \lambda^2) & \alpha_1 (1 + \lambda) + \alpha_3 (1 + \lambda) & \alpha_2 (1 + \lambda) \\
\alpha_4 & \alpha_3 & \alpha_2 (1 + \lambda) & \alpha_1 (1 + \lambda)
\end{bmatrix}
$$

They may again be interpreted as the sums

$$\Theta^{(4)}(\lambda) = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4$$  \hspace{1cm} (23)

where

$$M_1 = \begin{bmatrix}
1 - \lambda & 0 & 0 & 0 \\
0 & 1 - \lambda & 0 & 0 \\
0 & 0 & 1 + \lambda & 0 \\
0 & 0 & 0 & 1 + \lambda
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
0 & 1 - \lambda & 0 & 0 \\
1 - \lambda & 0 & 1 - \lambda^2 & 0 \\
0 & 1 - \lambda^2 & 0 & 1 + \lambda \\
0 & 0 & 1 + \lambda & 0
\end{bmatrix},$$
$$M_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1+\lambda & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (24)$$

In the shorthand notation of our previous paragraph the following four incidence matrices $S_j^{(4)}$ will carry again all the necessary information about the respective four matrix polynomial functions of $\lambda$. In computations, these incidence matrices $S_j^{(4)}$, i.e.,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

will be used for the encoding and/or efficient reconstruction of the respective expansion matrices $M_j^{(4)}(\lambda)$.

It is worth noticing that even the simplest metric with $\alpha_2 = \alpha_3 = \alpha_4$ which is proportional to the first coefficient $M_1$ and which remains diagonal (i.e., in the language of coordinates on the lattice, “local”) ceases to be proportional to the unit matrix so that our model resides in a nontrivial Hilbert space where $\Theta \neq I$.

### 3.4 Model with $N = 6$

Although all the six eigenvalues of the matrix $H^{(6)}(\lambda)$ may be expressed in closed form in principle, we shall only graphically confirm that all of them remain real in the same interval as above, with $\lambda \in (-1, 1)$ (cf. Figure 2). Inside this interval the metric $\Theta^{(6)}(\lambda)$ exists and its general form is obtainable from eq. (5) by its straightforward computer-assisted solution. The resulting matrices $\Theta^{(6)}(\lambda)$ are displayed here in the
The use of the shorthand symbols $S_j^{(6)}$ becomes indispensable for the sufficiently efficient and compact encoding of the $N = 6$ expansion formula

$$\Theta^{(N)}(\lambda) = \sum_{j=1}^{N} \alpha_j M_j^{(N)}(\lambda).$$

(25)
At $j = 1, 2, \ldots, 6$ the six respective compact incidence or indexing matrices $S_j^{(6)}$ are

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

These matrices form an inseparable part of any practical computer-assisted application of the formalism at $N \geq 4$. At $N = 6$, moreover, their explicit form also offers an interesting insight in their $N$–dependence forming a sufficiently inspiring starting point of our extrapolation programme.

4 Extrapolation

Starting from $N = 8$, the explicit form of the indexing matrices $S_j^{(N)}$ becomes also rather large for being printed. Still, their computer-assisted use remains as easy and straightforward as at $N = 10$ etc. Thus, in the main part of our homework we simply formulated and tested the alternative extrapolation hypotheses. Now, it remains for us just to summarize the results.
4.1 Recurrences for the incidence matrices, with the central ones exempted.

It is easy to reconstruct the \( \lambda \)-dependent matrices \( M_j^{(2K)}(\lambda) \) from the knowledge of the respective shorthand symbols \( S_j^{(2K)} \). What remains for us to do is to define all the set of the shorthand symbols \( S_j^{(2K)} \) representing all our expansion matrices. In such a context, the trial and error method enabled us to collect a sufficiently extensive set of these symbols. In the next step, their inspection revealed that at any given \( N = 2K \) the first \( K - 1 \) matrices \( S_j^{(2K)} \) with \( j = 1, 2, \ldots, K - 1 \) would be easily constructed from their predecessors \( S_j^{(2K-2)} \).

Purely empirically, the latter recurrent construction has been found in an enlargement of the dimension followed by a symmetric attachment of the two \( j \)-plets of units “1” in the empty parts of the left upper corner and of the right lower corner.

In an entirely similar manner, the last \( K \) matrices \( S_{2K+1-j}^{(2K)} \) with \( j = 1, 2, \ldots, K \) become also formed in the similar manner. Explicitly, their \( K \) predecessors \( S_{2K-1-j}^{(2K-2)} \) must be modified by attaching \( j \) zeros “0” in the right upper corner and in the left lower corner.

In both of the “leftmost-subsequence” and “rightmost-subsequence” scenarios, the results displayed in section 3 offer a sufficiently instructive illustration of such a recipe. An explicit algebraic reformulation of such a doublet of two-dimensional recurrences would be also as straightforward and compact as their above, purely verbal description. An algorithmic version of these recurrences was, after all, needed also during our practical computer-assisted evaluation of the matrix elements of \( \Theta^{(N)}(\lambda) \) at all the \( N \) which we considered. Of course, all these descriptions of our two-dimensional recurrences are equivalent and amenable to the rigorous proof by mathematical induction. Due to the lack of space, the routine details of algebra as well as of its rigorous proofs are skipped here and left to interested readers.
4.2 Recurrences for the central incidence matrices $S_k^{(2K)}$

At the remaining subscript $j = K$ the construction of the most complicated missing member $S_k^{(2K)}$ of the family must be discussed separately. In both the “leftmost-subsequence” and “rightmost-subsequence” scenarios its $(2K - 2)$ by $(2K - 2)$ predecessor proves, rather unexpectedly, different from the naively expected matrix $S_k^{(2K-2)}$.

This means that the sequence of the “middle” or “central” matrices $S_k^{(2K)}$ should be treated as exceptional. Fortunately, they remain created by a straightforward recurrent recipe. Its idea relies on the use of certain specific predecessor matrices $L^{(2K-2)}$. With a freedom in their specification let us decide to proceed in the “rightmost-subsequence” manner. This means we shall enlarge the dimension of $L$ (whatever it is) and we shall fill $K$ “neighboring” units “1” in the left upper corner and in the right lower corner.

We are now ready to define the specific predecessors $L^{(2K-2)}$. They appear to be constructed from the old “middle” matrices $S_{K-1}^{(2K-2)}$ via a specific two-step recipe. Firstly we replace each “old” numerical element in $S_{K-1}^{(2K-2)}$ by its successor, i.e., we replace “old 0” by “1”, “old 1” by “2”, etc. In the second step we form a left-right reflection of the resulting matrix and arrive at the final form of the necessary predecessor $L^{(2K-2)}$ as a result. Thus, at $N = 4$ we have the sequence

$$S_1^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \rightarrow L^{(2)} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \rightarrow S_2^{(4)}.$$  

Similarly, the recurrent construction of the matrix $S_{N/2}^{(N)}$ at $N = 6$ will result from adding six units “1” to the auxiliary predecessor matrix $L^{(4)}$ in the formula

$$S_2^{(4)} \rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} \rightarrow L^{(4)} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \\ 2 & 2 \end{bmatrix} \rightarrow S_3^{(6)}.$$  

etc. We may conclude that the central matrices $S_k^{(2K)}$ at the respective $K = 1, 2, 3, 4$
(etc) form the sequence

\[
\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}, \begin{bmatrix}
1 & 2 \\
1 & 3 & 2 \\
2 & 1 \\
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 1 \\
2 & 3 & 1 \\
1 & 1 \\
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 & 1 \\
1 & 3 & 4 & 2 \\
2 & 3 & 1 \\
1 \\
\end{bmatrix}
\]

(etc). The general pattern of their recurrent construction is obvious.

### 4.3 Verifications

The full-fledged formulae for the eight-parametric \( \Theta^{(8)}(\lambda) \) already cease to be easily printable but their characterization using the incidence matrices remains fully transparent and compact. All of the individual expansion matrices entering the general series \([25]\) for metrics \( \Theta \) exhibit the same simultaneous change of the sign of \( \lambda \) after the reflection with respect to their second diagonal. This is well visible in our last illustrative equation

\[
\begin{bmatrix}
1 - \lambda \\
1 - \lambda \\
1 - \lambda^2 \\
1 - \lambda \\
(1 - \lambda) (1 - \lambda^2) \\
(1 - \lambda^2) \\
1 - \lambda^2 \\
1 - \lambda^2 \\
1 - \lambda^2 \\
\end{bmatrix}
\begin{bmatrix}
1 - \lambda \\
(1 - \lambda) (1 - \lambda^2) \\
(1 - \lambda^2) \\
(1 + \lambda) (1 - \lambda^2) \\
1 - \lambda^2 \\
\end{bmatrix}
\begin{bmatrix}
1 - \lambda^2 \\
(1 - \lambda^2)^2 \\
(1 + \lambda) (1 - \lambda^2) \\
1 + \lambda \\
\end{bmatrix}
\]

where a part of the real and symmetric matrix \( M_{K}^{(2K)}(\lambda) \) at \( K = 4 \) is displayed.
5 Summary

In quantum theory an operator $H$ represents an observable provided only that it is self-adjoint in a Hilbert space equipped with a metric $\Theta$. For a given $H$, equation $H^\dagger \Theta = \Theta H$ specifies a complete menu of all the eligible $\Theta = \Theta(H)$ needed to determine the inner product. We illustrated the feasibility of the construction of all of these $\Theta$s for an infinite sequence of certain one-parametric $N$ by $N$ matrices $H^{(N)}(\lambda)$ with $\lambda \in (-1, 1)$.

A recurrent method of the construction of all the admissible metric matrices $\Theta^{(N)}(\lambda)$ has been proposed and tested at $\lambda = 0$. For $\lambda \in (-1, 1)$, the straightforward construction of the individual $N$–parametric $\Theta^{(N)}(\lambda) = \Theta^{(N)}_{\alpha_1, \alpha_2, \ldots, \alpha_N}(\lambda)$ was based on the computer-assisted symbolic manipulations at the smallest $N \leq N_{(\text{minimal})}$. Next we extrapolated these formulae for the metric to all the subsequent larger $N > N_{(\text{minimal})}$. We carefully verified the validity of our extrapolations via tests performed at a number of sample $N > N_{(\text{minimal})}$.

We revealed that the recurrences which are needed for the reconstruction of all the set of all the $N$–parametric matrices $\Theta^{(N)}(\lambda) = \Theta^{(N)}_{\alpha_1, \alpha_2, \ldots, \alpha_N}(\lambda)$ from the set of their $N - 1$ predecessors $\Theta^{(N-1)}(\lambda) = \Theta^{(N-1)}_{\alpha_1, \alpha_2, \ldots, \alpha_{N-1}}(\lambda)$ degenerate to the recurrences needed for the reconstruction of the $N$–plets of matrix coefficients $M^{(N)}_j(\lambda)$. The ultimate simplification of our recurrent recipe has been achieved when all the matrices $M^{(N)}_j(\lambda)$ proved easily constructed from the knowledge of the related elementary indexing matrices $S^{(N)}_j$ with integer or empty entries.

During our study of our matrix toy-model Hamiltonians we persuaded ourselves that the underlying mathematics is friendly and that the recurrences generate the matrix indices $S^{(N)}_j$ in a really transparent form. Although this fact should partially be attributed to the mere one-parametric choice of our class of Hamiltonians $H^{(N)}(\lambda)$, we firmly believe that our method which mixed the low–$N$ evaluations with subsequent all–$N$ extrapolations might keep its efficiency for a number of more realistic and, in particular, more-parametric sparse-matrix toy-model Hamiltonians.
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Acknowledgement

Work supported by the MŠMT “Doppler Institute” project Nr. LC06002, by the Institutional Research Plan AV0Z10480505 and by the GAČR grant Nr. 202/07/1307.

Figure captions

Figure 1. Spectrum of $H^{(4)}(\lambda)$.

Figure 2. Spectrum of $H^{(6)}(\lambda)$.
Appendix A: Toy-model potentials in eq. (11)

Among the oldest explicit examples of the phenomenological Hamiltonian operators $H$ exhibiting the cryptohermiticity property (3) one may cite, e.g., the imaginary cubic anharmonic oscillators studied by Caliceti et al [15], the “wrong-sign” quartic anharmonic oscillators described by Buslaev and Grecchi and others [16] or a family of the more general, “non-Hermitian” but stable bound-state models as proposed by Bender and Boettcher [17]. In all of these one-dimensional schematic models of bound states the reconstruction of any suitable metric $\Theta$ from a given oscillator Hamiltonian $H$ proves rather difficult. *Pars pro toto* we may recall the perturbation-series construction of $\Theta(H)$ for the imaginary cubic $H$ [5], with many further relevant references cited therein.

A perceivably better picture of the properties of the physics-determining metrics $\Theta(H)$ has been obtained for certain exactly solvable potentials of bound states (cf. the semi-numerical construction of $\Theta(H)$ for a square-well model $H \neq H^\dagger$ [18] as an example).

The situation appeared perceivably worsened after transition to the scattering regime where even the combination of a sophisticated perturbation construction with the choice of the really most elementary exactly solvable interaction potentials in eq. (11) did leave many conceptual questions unanswered (cf., e.g., refs. [11, 19]).

Appendix B: Conventional choices of $\Theta = \Theta(H)$ in quantum theory

In the majority of applications of quantum theory using non-Dirac Hilbert spaces $\mathcal{H}^{(\text{physical})}$ the construction of the metric remains almost prohibitively difficult even when one restricts attention to the *single* observable, i.e., to $j = 0$ in eq. (7). One may feel forced to work just with a drastically simplified and/or highly schematic class, say, of point-interaction Hamiltonians [19, 20]. The construction of $\Theta(H)$ may remain feasible only when one employs an approximate (e.g., perturbation [5, 18])
method. Still, even in many papers which define a quantized system in an unusual Hilbert space $\mathcal{H}^{(\text{physical})}$ which is equipped with a non-Dirac metric $\Theta \neq I$ their authors usually avoid the overcomplicated requirement $(7)$ and select just one of particular solutions $\Theta(H)$ of eq. $(5)$ instead.

As one of the the simplest illustrations of such a strategy we might recall even the most common model $(1)$ with a most common real potential. Evidently, just the trivial solution $\Theta^{(\text{Dirac})} = I$ of eq. $(5)$ is being assigned to $H = H^\dagger$. In particular, out of all of the above-cited eligible metrics $\Theta_0^{(\text{Mostafazadeh})}(F, K)$ just the most trivial particular solution with $F = K = 0$ is being used in connection with the free-motion version of eq. $(1)$.

In $\mathcal{PT}$–symmetric quantum mechanics admitting $\Theta \neq \Theta^{(\text{Dirac})}$ (cf. its thorough recent review written by Carl Bender [21]), the problem of the ambiguity of the choice of the metric $\Theta = \Theta(H)$ has been circumvented as well. Although this less traditional formalism admits various nonstandard, apparently non-Hermitian models (including even field models with real spectra [22] etc), the current choice of the space $\mathcal{H}^{(\text{physical})}$ is equally restrictive, preferring special metrics $\Theta^{(\text{Bender})} = \mathcal{CP}$ where $\mathcal{C} = \mathcal{C}(H)$ represents a unique “charge” while $\mathcal{P}$ is the usual parity.

The most natural generalization of the $\mathcal{PT}$–symmetric theories with $\Theta \neq \Theta^{(\text{Dirac})}$ has been described by Mostafazadeh [13]. He re-attracted the attention of the international scientific community to the abstract quantization rule $(7)$ of ref. [4] and to the related ambiguity of the reconstruction of the correct Hilbert space $\mathcal{H}^{(\text{physical})}$ form a given Hamiltonian. He worked out some illustrative examples (cf. [5]) and, together with Batal [18], he emphasized the possible physical relevance of metrics $\Theta \neq \Theta^{(\text{Bender})}$. 
Appendix C: Discretized Runge-Kutta version of eq. (1)

One of the key simplifications of some of the technical aspects of solving differential Schrödinger eq. (1) is commonly sought in its replacement by its Runge-Kutta difference-equation approximation

\[-\frac{\psi(x_{k+1}) - 2\psi(x_{k}) + \psi(x_{k-1})}{h^2} + V(x_k)\psi(x_k) = E\psi(x_k)\]  

(cf., e.g., ref. [23] for more details). In place of the real line of coordinates \(x \in \mathbb{R}\) the equidistant lattice of points \(x_k\) may be conveniently defined by the formula \(x_k = -1 + kh, k \in \mathbb{Z}\) in terms of a suitable (i.e., usually, sufficiently small) real constant \(h > 0\). For the purposes of the description of scattering the lattice remains infinite so that the kinetic energy operator \(-\frac{d^2}{dx^2}\) may be visualized as the following tridiagonal matrix

\[
H_0 = \begin{bmatrix}
\ddots & \ddots & \ddots & & \\
\ddots & 2 & -1 & & \\
& -1 & 2 & -1 & \\
& -1 & 2 & -1 & \\
& -1 & 2 & -1 & \\
& -1 & 2 & -1 & \\
& \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}.
\]  

(27)

In the bound-state context with the Dirichlet boundary conditions \(\psi(-1) = \psi(1) = 0\) one usually considers just the finite set of the lattice points, \(k = 1, 2, \ldots, N\). With \(x_{N+1} = +1\) we, in effect, fix an elementary length \(h = 2/(N + 1)\) which would vanish in the continuum limit \(N \to \infty\).

In the latter scenario the explicit specification (3) of the non-Dirac Hilbert space \(\mathcal{H}_{\text{physical}}\) must be slightly modified,

\[(\psi, \psi')_\Theta = \sum_{n,n' \in \mathbb{Z}} \psi^*(x_n) \Theta(x_n, x_{n'}) \psi'(x'_{n'}) , \quad H = H^\dagger.\]  

(28)
On this background the numerical use of the approximation (1) \(\rightarrow\) (26) finds numerous applications in cryptohermitian quantum mechanics. In papers [24], for example, the exact, analytic solvability of the differential eq. (1) with a certain class of sufficiently simple complex potentials \(V(x) \neq V^*(x)\) has been shown paralleled by the exact solvability of the discrete partner eq. (26). Whenever the potential \(V(x)\) remains sufficiently smooth, the role of the discretization errors may be expected negligible, in the domain of the sufficiently large \(N \gg 1\) at least. Difficulties may arise for point interactions. They proved popular [25] and found applications in relativistic equations [26] and in manybody systems [27]. This, as we already mentioned, motivated our interest in in the new field of cryptounitary scattering [12] and, in particular, in its simplest discrete model (8).

Appendix D: Models \(H^{(N)}(\lambda)\) at large \(N \gg 1\)

At a sufficiently large \(N\) our one-parametric Hamiltonian \(H^{(N)}(\lambda)\) represents in fact one of the discrete versions of eq. (1) with a certain point interaction \(V(x)\) localized in the origin. For a deeper understanding of such a correspondence let us ascept that \(x_0 = -1\) and \(x_{2K+1} = 1\) and let us abbreviate \(2 - \hbar^2 E = \cos \epsilon\) as usual [12]. [24]. We may then treat \(\epsilon \in (0, \pi)\) as a new energy variable and we may visualize the wave functions \(\psi(x)\) with \(x \neq 0\) as satisfying the free-motion eq. (26), complemented by the boundary conditions \(\psi(x_0) = \psi(x_{2K+1}) = 0\). This picture of the \(N \gg 1\) system must be completed by the doublet of the \(\lambda\)-dependent relations

\[
(1 + \lambda) \psi(x_{K+1}) - 2 \cos \epsilon \psi(x_K) + \psi(x_{K-1}) = 0,
\]

\[
\psi(x_{K+2}) - 2 \cos \epsilon \psi(x_{K+1}) + (1 - \lambda) \psi(x_K) = 0
\]

so that we may assume the emergence of a discontinuity of \(\psi(x)\) in the origin.

At the sufficiently large \(N \sim 1/\hbar \gg 1\) the wave functions near \(x = 0\) remain well represented by their respective one-sided Taylor series so that eqs. (29) and (30) may be interpreted simply as a matching condition. We return to the original energy
variable $h^2E = 2 - 2\cos\epsilon \equiv F$ and insert the truncated expansions

$$
\psi(x_{K-1}) = \psi_L(0) - \frac{3}{2}h\psi'_L(0) + \mathcal{O}(h^2), \quad \psi(x_K) = \psi_L(0) - \frac{1}{2}h\psi'_L(0) + \mathcal{O}(h^2),
$$

$$
\psi(x_{K+1}) = \psi_R(0) + \frac{1}{2}h\psi'_R(0) + \mathcal{O}(h^2), \quad \psi(x_{K+2}) = \psi_R(0) + \frac{3}{2}h\psi'_R(0) + \mathcal{O}(h^2)
$$

in eqs. (29) and (30). A straightforward algebra leads to the following elementary condition

$$
\frac{h}{2} \begin{pmatrix} -(1 + \lambda) & F + 1 \\ -(F + 1) & 1 - \lambda \end{pmatrix} \begin{pmatrix} \psi'_R(0) \\ \psi'_L(0) \end{pmatrix} = \begin{pmatrix} 1 + \lambda & F - 1 \\ F - 1 & 1 - \lambda \end{pmatrix} \begin{pmatrix} \psi_R(0) \\ \psi_L(0) \end{pmatrix}
$$

(31)

which matches the wave functions and their derivatives in the origin. In the domain of sufficiently small $h > 0$ this relation is equivalent to the original constraints (29) and (30). We may conclude that in the continuum limit $N \to \infty$ our sequence of the matrix Hamiltonians $H^{(N)}(\lambda)$ can be reinterpreted as a series of dynamical models which converge to a specific differential eq. (1) which is split in two halves. Indeed, the $h \to 0$ limit of eq. (31) produces the elementary opaque-wall constraint $\psi_R(0) = \psi_L(0) = 0$.

At all the nonvanishing small “elementary lengths” $h > 0$, our rigorous definition (31) leaves the point-interaction potential term $V(x)$ translucent and manifestly energy-dependent. Its definition by the mixed boundary conditions is nonlinear in the coupling $\lambda$. Various special cases of this $N \gg 1$ bound-state model may be studied noticing, for example, that the energy-dependence disappears in the low-excitation regime where the quantity $F = h^2E$ remains negligible.

Another interesting special case is encountered when the interaction is completely switched off, $\lambda \to 0$. Then we may recollect our $N-$parametric free-motion formula (15) for the metric $\Theta_0$ and we may compare it with its two-parametric differential-operator counterpart (2). This comparison confirms that a linear combination of the matrices $M_1^{(N)}(0)$ and $M_N^{(N)}(0)$ survives the limiting transition $N \to \infty$ while, in contrast, all the other terms in (15) disappear due to the build-in requirement of the absence of an elementary length in the Mostafazadeh’s theory of ref. [5].