Compact Q-balls in the complex signum-Gordon model

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Abstract

We discuss Q-balls in the complex signum-Gordon model in $d$-dimensional space for $d = 1, 2, 3$. The Q-balls have strictly finite size. Their total energy is a power-like function of the conserved $U(1)$ charge with the exponent equal to $(d + 2)(d + 3)^{-1}$. In the cases $d = 1$ and $d = 3$ explicit analytic solutions are presented.

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1 Introduction

Q-balls are nontopological solitons which appear in certain nonlinear complex scalar field models [1], see also [2] for a review of early works and applications. Such solitons have been discussed in the context of the dark matter problem, see, e.g., [3], and in condensed matter physics [4]. Their static properties as well as rather complex dynamics have been studied in several papers, see, e.g., [5, 6]. Field potentials in all considered models have a smooth behaviour close to their absolute minima. In consequence, the Q-balls possess exponential tails and interact with each other.

Recently, a new class of scalar field models has been proposed, see [7] for a review. Their common characteristic feature is that the pertinent field potentials are not smooth at the absolute minima – they are V-shaped. It has turned out that models of this kind are quite interesting. First of all, they have a sound physical justification, and they are perfectly well-behaved from the physical viewpoint. Moreover, they can support compact solitons (topological compactons), a whole variety of self-similar solutions [8, 9], as well as time-dependent, finite energy excitations called oscillons [10]. Despite rather unusual shape of the field potentials exact, analytic solutions of all these types have been found. The complex signum-Gordon model also has the V-shaped field potential: it is given by the formula $U(\psi) = \lambda |\psi|$, where $\lambda > 0$ and $\psi$ is a complex scalar field. The plot of $U(\psi)$ has an inverted conical shape. The potential has its minimum right at the bottom of the cone.

In the present paper we show that the complex signum-Gordon model possesses solutions with finite energy which are stationary – the Q-balls. These Q-balls are rather interesting for the following reasons. First, they appear rather surprisingly in the model with the very simple, conical field potential. They are given explicitly as exact solutions of the field equation. Other analytic Q-ball solutions are known in literature, see, e.g., [1, 6], but our solutions seem to be simpler. The Q-balls have a strictly finite size. The scalar field, energy density and charge density vanish exactly outside certain ball of a finite radius. In this sense, our Q-balls are compact (see also the second point in the section Remarks). Another distinguishing feature of our Q-balls is a very simple, power-like dependence of the total energy on the total charge. We also show that there is a whole ladder of higher spherically symmetric Q-balls in the three-dimensional space. Because of the compactness, one can also trivially combine several single Q-ball solutions in order to obtain explicit solutions describing a collection of isolated Q-balls.

The plan of our paper is as follows. In Section 2 we present the field equation,
and we discuss the energy and charge of the Q-balls. Section 3 is devoted to explicit solutions of the field equation. In section 4 we have collected several remarks.

2 The field equation and general properties of the Q-balls

Lagrangian of the complex signum-Gordon model has the following form

\[ L = \partial_{\mu} \psi^* \partial^\mu \psi - \lambda |\psi|, \]  

where \( \psi \) is a complex scalar field in \((d + 1)\)-dimensional Minkowski space-time, \( \lambda > 0 \) is a coupling constant, \( * \) denotes the complex conjugation, \( |\psi| \) is the modulus of \( \psi \). The field \( \psi \), the space-time coordinates \( x^\mu \) and the constant \( \lambda \) are dimensionless. Of course, in physical applications they have to be multiplied by certain dimensional constants. Lagrangian (1) is invariant under the global \( U(1) \) transformations \( \psi(x) \rightarrow \exp(i\alpha) \psi(x) \). The corresponding conserved charge is given by the formula

\[ Q = \frac{1}{2i} \int d^d x (\psi^* \partial_0 \psi - \partial_0 \psi^* \psi). \]  

The energy density

\[ T_{00} = \partial_0 \psi^* \partial_0 \psi + \partial_i \psi^* \partial_i \psi + \lambda |\psi| \]  

has the absolute minimum at \( \psi = 0 \).

The field potential \( U(\psi) = \lambda |\psi| \) can be regarded as a limit of a regularized potential which is smooth at \( \psi = 0 \) and such that its second derivative \( \partial^2 U/\partial|\psi|\partial|\psi| \) taken at \( \psi = 0 \) diverges in that limit. As the example we take \( U_{\kappa}(\psi) = \lambda \sqrt{\kappa + |\psi|^2} \), where \( \kappa \) is a positive constant. The signum-Gordon model is obtained in the limit \( \kappa \rightarrow 0 \). The Euler-Lagrange equation in the regularized model has the form

\[ \partial_\mu \partial^\mu \psi = -\frac{\lambda}{2} \frac{\psi}{\sqrt{\kappa + |\psi|^2}}. \]  

In the \( \kappa \rightarrow 0 \) limit the r.h.s. of this equation is equal to \(-\lambda \psi/(2|\psi|)\) if \( \psi \neq 0 \), and to 0 if \( \psi = 0 \).
We use the standard Ansatz for symmetric Q-balls:

$$\psi = F(r) \exp(i\omega t^0),$$

where $F$ is a real-valued function of $r$, and $\omega > 0$ is a real frequency. $r$ denotes the $d$-dimensional radial coordinate in the cases $d > 1$, and $r = x^1$ for $d = 1$. It is convenient to introduce the variable $y$ and the function $f(y)$:

$$y = \omega r, \quad f(y) = \frac{2\omega^2}{\lambda} F(r).$$

Equation (4) in the case of Q-balls in the limit $\kappa \to 0$ acquires the form

$$f'' + \frac{d-1}{y} f' + f = \text{sign} f,$$

(5)

where $'$ stands for $d/dy$. The sign function has the values $\pm 1$ when $f \neq 0$ and $\text{sign}(0) = 0$. Note that Eq. (5) has the following constant solutions: $f = 0, \pm 1$.

Let us now briefly discuss essential mathematical aspects of Eq. (5). First, it is clear that $f$ has to satisfy the condition $f'(0) = 0$ if $d > 1$. Furthermore, for physical reasons – the finiteness of the energy density (3) – $f$ has to be continuous and differentiable function of $y$. On the other hand, Eq. (5) implies that the second derivative of $f$ has to be discontinuous at those points where $f$ changes its sign. Let $y_0$ be such a point. Integrating both sides of Eq. (5) over a small interval around $y_0$ and shrinking that interval to the point we find that $f'$ is continuous at $y_0$. The method to solve Eq. (5) consists of two steps. First, we find solutions assuming that $f$ has a constant sign. In general, such solutions are valid on certain intervals of the $y$ axis – for this reason we call them partial. In the second step we match such partial solutions using the conditions of continuity of $f$ and $f'$. Differential equations with discontinuous terms are well-known in mathematics and physics. Their general theory is based on the notion of so called weak solutions, see, e.g., [11, 12]. Our solutions fit in that framework.

We show in the next Section that there exist solutions of Eq. (5) for which the charge $Q$ and the energy $E = \int d^d x T_{00}$ are finite. If $f(y)$ is such a solution, then formulas (2), (3) give

$$Q = c_1 \frac{\lambda^2}{\omega^{d+3}}, \quad E = c_2 \frac{\lambda^2}{\omega^{d+2}}$$

(6)

where $c_1, c_2$ are purely numerical constants which do not depend neither on $\omega$ nor $\lambda$. In the $d = 1$ case these constants are computed from the following formulas

$$c_1 = \frac{1}{4} \int_{-\infty}^{\infty} dy \ f^2(y), \quad c_2 = \frac{1}{4} \int_{-\infty}^{\infty} dy \ [f^2 + (\partial_y f)^2 + 2|f|],$$

(7)
while for $d > 1$
\[
c_1 = \frac{\Omega_d}{4} \int_0^\infty dy \, y^{d-1} f^2(y), \quad c_2 = \frac{\Omega_d}{4} \int_0^\infty dy \, y^{d-1} [f^2 + (\partial_y f)^2 + 2|f|],
\]
(8)
where $\Omega_1 = 2\pi$, $\Omega_2 = 4\pi$ ($\Omega_d = \int d\Omega_d$, where $d\Omega_d$ is the $d$-dimensional solid angle element). We shall see in the next Section that the frequency $\omega$ and the radius $r_0$ of the Q-balls are related, $r_0 = y_0/\omega$, where $y_0$ is a numerical constant. Therefore, formulas (6) imply that $Q \sim r_0^{d+3}$, $E \sim r_0^{d+2}$.

Formulas (6) imply that for a fixed solution $f$ of the rescaled radial equation (5) the total energy and the charge are related by the very simple formula
\[
E = c_2 \frac{\omega_2}{\lambda_0^{d+2}} \left( \frac{Q}{c_1} \right)^{d+2}. \tag{9}
\]
The power-like dependence of $E$ on $Q$ is the consequence of power-like dependence on $\omega$ in formulas (6). The latter one can be related to a scale invariance of the field equation (4) in the limit $\kappa \to 0$, $[8, 9]$.

Because the exponent on the r.h.s. of formula (9) is smaller that 1, one spherical Q-ball with the total charge $Q = Q_1 + Q_2$ has smaller energy that two smaller spherical Q-balls with the charges $Q_1 > 0$, $Q_2 > 0$. In other words, it is energetically favourable for the two Q-balls to merge.

### 3 The explicit form of the solutions

#### 3.1 The $d=1$ case

In this case Eq. (5) has a family of partial solutions of the form
\[
f_+(y) = 1 + \alpha \sin y,
\]
where $\alpha$ is a real constant. They obey Eq. (5) on (sub)intervals of the $y$-axis determined by the condition $f_+ > 0$. Using the translations $y \to y - a$ one can trivially generate further solutions. Below we will discuss in detail only the simplest solutions, omitting those which can be obtained by the space or time translations, or by Lorentz boosts.

The basic one dimensional Q-ball solution $f_0(y)$ is obtained by combining the trivial solution $f = 0$ with the solution $f_+$. The matching conditions discussed in
the previous Section imply that \( \alpha = -1 \), and that the two solutions match each other at the points \( y = \pi/2, 5\pi/2 \). Hence,

\[
f_0(y) = \begin{cases} 
0 & \text{if } y \leq \pi/2, \\
1 - \sin y & \text{if } \pi/2 < y < 5\pi/2, \\
0 & \text{if } y \geq 5\pi/2.
\end{cases}
\] (10)

Formulas (7) give \( c_1 = 3\pi/4, c_2 = 2\pi \).

The basic solution (10) gives rise to a whole family of multi-Q-ball solutions. They are obtained just by summing translated or Lorentz boosted solutions \( f_0(y) \). The only restriction is that the supports of the \( 1 - \sin(y - a) \) parts should not overlap. Such separate Q-balls do not interact with each other.

The \( d = 1 \) complex signum-Gordon model was considered in [8] in connection with a model of a string in a three dimensional space, pinned to a straight line coinciding with the \( x^1 \) axis. In that model \( \text{Re}\psi, \text{Im}\psi \) are just the two coordinates giving the deviation of the string from the pinning straight line. The Q-ball solutions presented above describe the string rigidly rotating around the \( x^1 \) axis with constant angular velocity equal to \( \omega \).

### 3.2 The d=3 case

Let us start from a particular solution of the form

\[
f_+(y) = 1 + \alpha \frac{\sin y}{y} + \beta \frac{\cos y}{y},
\] (11)

where \( \alpha, \beta \) are real constants. It obeys Eq. (5) in the (sub)intervals determined from the condition \( f_+(y) > 0 \). The simplest Q-ball solution is composed of the trivial solution \( f = 0 \) in the region \( y \geq y_0 \) and \( f_+ \) in the region \( 0 \leq y \leq y_0 \). The condition \( f_+'(0) = 0 \) is satisfied only when \( \beta = 0 \). The matching conditions at \( y = y_0 \) have the form \( f_+(y_0) = 0, f_+'(y_0) = 0 \). They give \( \alpha = -y_0/\sin y_0 \) and the following equation for \( y_0 \)

\[
\tan y_0 = y_0.
\]

This equation has infinitely many solutions, but only for the first one, i.e.,

\[
y_0 \approx 4.4934,
\]

the function \( f_+ \) has positive values in the whole interval \([0, y_0)\). Thus, the basic 3-dimensional Q-ball solution is given by

\[
f_b(y) = \begin{cases} 
1 - \frac{y_0 \sin y}{y \sin y_0} & \text{if } 0 \leq y < y_0, \\
0 & \text{if } y \geq y_0.
\end{cases}
\] (12)
The constants $c_1, c_2$ have the following values $c_1 = \frac{5\pi y_0^3}{6}$, $c_2 = 2\pi y_0^3$.

Equation (5) can formally be regarded as Newton’s equation of motion for a fictitious particle in the effective potential $V_{\text{eff}} = \frac{f^2}{2} - |f|$ and with the friction force equal to $-(d-1)f'/y$. The role of time is played by $y$, while $f$ gives the position of the particle. Such reinterpretation of radial equation was used already by, e. g., Lee and Coleman \[1\]. In the case of Q-ball solution, the particle starts at the time $y = 0$ from a certain point $f > 0$ with vanishing velocity $f'(0) = 0$, reaches the point $f = 0$ at the time $y_0$, again with vanishing velocity $f'(y_0) = 0$, and rests at that point for later times ($y > y_0$). It is clear that also other trajectories are possible. In particular, the particle can have an excess of energy when arriving to the point $f = 0$, hence it continues to move with the negative velocity $f'(y)$ to the negative values of $f$, bounces back and finally arrives at the point $f = 0$ with vanishing velocity where it stops. This is possible because the particle looses its energy due to the friction. This trajectory corresponds to a higher Q-ball solution which is composed of the following three partial solutions

$$
f_+(y) = 1 - \alpha_0 \frac{\sin y}{y}, \quad f_-(y) = -1 + \alpha_1 \frac{\sin y}{y} + \beta_1 \frac{\cos y}{y}, \quad f = 0.
$$

The matching conditions $f_+ = f_-$, $f_+ = f_-'$ at $y_{10}$, and $f_- = 0$, $f_- = 0$ at $y_{11} > y_{10}$ give

$$
\alpha_0 = \frac{y_{10}}{\sin y_{10}}, \quad \alpha_1 = y_{11} \sin y_{11} + \cos y_{11}, \quad \beta_1 = y_{11} \cos y_{11} - \sin y_{11},
$$

$$
y_{10} \approx 3.4826, \quad y_{11} \approx 8.4970.
$$

This solution has single isolated zero at $y = y_{10}$, and it merges with the $f = 0$ solution at $y_{11}$. The $f_+$ part is for $y \in [0, y_{10}]$ and the $f_-$ part for $y \in [y_{10}, y_{11}]$.

It is clear that the fictitious particle can also bounce back $n > 1$ times – such a trajectory corresponds to the Q-ball solution with $n$ isolated zeros located at $y_{n0}, \ldots, y_{nn}$.

Similarly as in the $d = 1$ case, single Q-balls can be translated and boosted. One can also simply add solutions with non overlapping supports of the $f_+, f_-$ parts in order to form noninteracting multi-Q-ball solutions.

4 Remarks

1. We have omitted a discussion of the case $d = 2$ because Q-ball solutions in two spatial dimensions are completely analogous to the ones from the $d = 3$
case. The only difference is that the functions $\sin y/y, \cos y/y$ are now replaced by the Bessel functions $J_0(y), Y_0(y)$, respectively, with very similar plots \cite{13}. Therefore, the structure of solutions is essentially the same as in the $d = 3$ case. This can also be seen from the mechanical interpretation of the radial equation (5).

2. Our Q-ball solutions approach the vacuum field $\psi = 0$ exactly at the radius $r_0 = y_0/\omega$. For $r \rightarrow r_0$-- the profile function $F$ has a parabolic shape, as can be seen from formulas (10) and (12). Such behaviour is typical for field models with V-shaped interaction terms \cite{7}. The energy of the Q-balls is strictly localized inside the ball of radius $r_0$ -- there is no exponential tail.

It should be noted that a compact Q-ball was found already by Werle \cite{1}. In that paper a rather peculiar field potential which involves two fractional powers of $|\psi|$ was considered. Moreover, the first derivative of the potential becomes infinite for the vacuum field.

3. In general, Q-ball dynamics is very complex already on the level of purely classical theory, see, e.g., \cite{5}, and still richer when one includes quantum effects, see, e.g., \cite{14} and references therein. The simplicity of the Q-balls of the complex signum-Gordon model can perhaps facilitate the theoretical studies, and make them interesting for applications. Nevertheless, we do not expect that their dynamics will be much simpler. To illustrate the point, let us take the problem of stability of the Q-balls. We have checked that the basic Q-balls are stable against just radial shrinking or expanding. However, a slightly perturbed non symmetric Q-ball can apriori behave in various ways. For example, it can perhaps decay by emitting a number of small Q-balls, or by emitting waves of the scalar field. Undoubtedly, there are many interesting problems in the dynamics of Q-balls to be investigated.

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