A NOTE ON THE STATISTICAL MECHANICS OF VIOLENT RELAXATION OF PHASE-SPACE ELEMENTS OF DIFFERENT DENSITIES

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ABSTRACT

The statistical mechanics investigation of violent relaxation of phase-space elements of different densities first derived by Lynden-Bell is reexamined. It is found that the mass independence of the equations of motion of violent relaxation calls for a constraint on the volume of the phase-space elements used to formulate the statistical mechanics description of violent relaxation. In agreement with observations of astrophysical objects believed to have been subject to violent relaxation (e.g., clusters of galaxies), the coarse-grained phase-space distribution \( f \) of the final state in the nondegenerate limit turns into a superposition of Maxwellians, \( f_r \), of a common velocity dispersion. Thus, the velocity dispersion problem present in the investigation of Lynden-Bell is removed.

Subject headings: celestial mechanics, stellar dynamics — cosmology: theory — galaxies: clusters: general — methods: analytical

1. INTRODUCTION

Since the classic paper by Lynden-Bell (1967), the statistical investigation of violent relaxation has been a topic of crucial interest in the discussion of the formation of relaxed astrophysical objects like star clusters, galaxies, and clusters of galaxies. Its reformulation by Shu (1978) in terms of particles instead of phase-space elements represents a change in terminology and, to some extent, a reinterpretation of Liouville’s theorem. However, the basic results of the two attempts are essentially the same and have been a subject of continuous discussion. The debate of Madsen (1987) and Shu (1987) concerning the foundations of a statistical description of violent relaxation provides a summary of resolved and unresolved issues for both particle and phase-space representation.

Starting with Tremaine, Henon, & Lynden-Bell (1986), the form and evolution of the \( H \) function during violent relaxation has become a subject of growing interest. Ever since, one of the main lines along which the statistical mechanics investigation of violent relaxation proceeds deals with this problem (e.g., Stiavelli & Bertin 1987; Kandrup 1987). It has only recently (Soker 1996) become clear that the \( H \) function of collisionless self-gravitating systems does not necessarily increase monotonically during the time evolution of the system.

In the present paper we deal with another issue that is central to both the particle and the phase-space descriptions of violent relaxation. Since the first investigation by Lynden-Bell (1967), one of the most important flaws of the theory has been considered to be the fact that despite the collisionless nature of violent relaxation, the statistical mechanics approach predicts a thermalized final state (Madsen 1987; Shu 1987), i.e., a state in which the square of the velocity dispersion of phase-space elements of different densities is inversely proportional to the density. In order to solve this problem, Shu (1987) applied to his particle approach very stringent assumptions on the initial phase-space density.

Here we reexamine the statistical mechanics of violent relaxation in terms of phase-space elements of different densities. Since the objects on which the statistical mechanics approach of Lynden-Bell (1967) is based are phase-space elements of constant volume but different mass, it is found that the mass independence of the equations of motion of violent relaxation is not fully represented. Considering phase-space elements of different volume but constant mass removes the velocity dispersion problem tendentially present in the investigation of Lynden-Bell (1967).

2. VIOLENT RELAXATION OF PHASE-SPACE ELEMENTS OF DIFFERENT DENSITIES

The collisionless nature of the process of violent relaxation allows us to consider its dynamics in terms of the one-particle phase space \( \mu \). The Vlasov equation

\[ D_t f = \partial_x f \cdot \dot{v} - \nabla_x \Phi \cdot \partial_v f = 0 \]  

(1)

governs the collisionless dynamics of the fine-grained phase-space density \( f \) during violent relaxation. Violent relaxation is essentially a self-gravitating process. Accordingly, the gravitational potential \( \Phi \) in equation (1) is given by

\[ \Phi(x) = -G \int \frac{\int f(v', x') d^3v'}{|x - x'|} d^3x'. \]  

(2)

Equations (1) and (2) describe the dynamics of violent relaxation in terms of the continuum limit without reference to the individual mass of the objects involved. We will restrict ourselves to the continuum limit description and assume its validity. As a consequence, the discussion is limited to the phase-space element approach (Lynden-Bell 1967) and does not refer to its counterpart, the particle approach of Shu (1978; see also Madsen 1987; Shu 1987).

By considering violent relaxation in the terminology of phase-space elements, the attempt of Lynden-Bell (1967) incorporates the constraint on the time evolution of the \( \mu \)-space provided by the Vlasov equation (eq. [1]), stating the constancy of the phase-space density \( f \) along trajectories in \( \mu \)-space. As a consequence of the Vlasov equation, phase elements that initially do not overlap never overlap. Therefore, a microscopic exclusion principle for phase-space elements in \( \mu \)-space is established. The second characteristic feature of violent relaxation reflected by equation (1) is the independence of the equations of motion from the individ-
ual particle masses. It is incorporated into the statistical mechanics picture by the dimensionality of the \( \mu \)-space that is chosen to yield the mass density in units of mass per spatial and velocity volumes, \( \Delta^3 x \) and \( \Delta^3 v \), respectively.

In order to apply statistical mechanics methods, the \( \mu \)-space is divided into a large number of microcells. With \( \eta \) as the phase-space density of a phase-space element (i.e., an occupied microcell) and \( \omega \) as its volume, the mass associated with the phase element is \( \eta \omega \). A microstate may then be described by the set of occupied microcells \( \omega_j \) and their density \( \eta_j \). While the volume occupied by the microstate in the \( \mu \)-space is

\[
\Delta \mu = \sum_j N_j \omega_j ,
\]

the corresponding volume \( \Delta \Gamma \) in \( \Gamma \)-space is

\[
\Delta \Gamma = \prod_j \omega_j^{N_j} (\eta_j \omega_j)^{3N_j} ,
\]

where \( N_j \) is the total number of phase-space elements of density \( \eta_j \) occupying microcells of volume \( \omega_j \). The first crucial point is that the phase-space volume in \( \Gamma \)-space (eq. [4]) depends not only on the volumes \( \omega_j \) but also on the phase-space density \( \eta_j \). The second point concerns the total energy of the microstate

\[
E = \sum_j \eta_j \omega_j (\frac{1}{2} v_j^2 + \Phi_j) ,
\]

where \( v_j \) is the velocity associated with the \( j \)th microcell and \( \Phi_j \) is the gravitational potential to which the microcell is exposed. What is of importance here is that the total energy \( E \) is related to the \( \Gamma \)-space, i.e., it is not possible to derive \( E \) independently of the absolute value of the phase-space density \( \eta_j \) or, strictly speaking, of the mass \( \eta_j \omega_j \) of the phase-space element. This dependence is recovered in the macroscopic constraints to which the system is subject in the statistical mechanics approach. In this sense, the objects on which the Lynden-Bell (1967) statistic is based are not the (mass-independent) phase-space element volumes \( \omega_j \) in \( \mu \)-space but their mass-dependent counterparts \( \omega_j (\eta_j \omega_j)^3 \) in \( \Gamma \)-space. Therefore, the discussion does not, at least with respect to the energy constraint, reflect the universal mass independence of the equations of motion of violent relaxation. It is indeed easy to see that the energy constraint formally introduces the dependence of the coarse-grained phase-space distribution on the phase-space density \( \eta_j \) in the approach of Lynden-Bell (1967). This means that in order to incorporate or represent the mass independence, it is not enough to just choose the dimensionality of the \( \mu \)-space as discussed above.

We seek a generalized constraint related to the division of the \( \mu \)-space into phase-space elements and microcells. This division of the \( \mu \)-space refers to the intermediate step of the determination of the most probable state in terms of the occupation numbers of microcells. While this step relies on a specific division of the \( \mu \)-space, the final result is, in contrast, independent of it, as will become clear later. With respect to the continuum limit, the volumes of the phase-space elements are arbitrary and have no direct physical significance. Instead, physical significance is attributed to the mass \( \eta_j \omega_j \) (or the corresponding mass differences) of the phase-space elements in \( \mu \)-space, reflecting the properties of the basic counterparts \( \omega_j (\eta_j \omega_j)^3 \) in \( \Gamma \)-space. In this respect, it is natural to incorporate the universal mass independence of the equations of motion of violent relaxation into the statistical mechanics picture by introducing a constraint on the volumes \( \omega_j \) stating that the corresponding phase elements have constant mass, i.e.,

\[
\eta_j \omega_j = \text{const} .
\]

We note that this procedure is in perfect agreement with the nature of the microscopic exclusion principle of Lynden-Bell (1967), which, for the case of only one phase-space density, states that the mass of the phase elements is constant. The approach characterized by equation (6) may thus be considered just its generalization. However, as a result, and in contrast to Lynden-Bell (1967), where the phase-space elements have different mass but constant volume, the phase-space elements considered here differ in volume and have constant mass.

### 3. STATISTICAL MECHANICS OF VIOLENT RELAXATION

In this section we attempt to derive the coarse-grained phase-space distribution, \( f_\Gamma \), of the final state of a system consisting of phase-space elements of \( J \) different phase-space densities, \( \eta_j \), subject to violent relaxation. The densities \( \eta_j \) are obtained from the initial fine-grained phase-space density \( f \). To determine the final state, we apply the same maximum entropy principle used by Lynden-Bell (1967). The state that maximizes the entropy under the constraint of conserved total energy and mass is considered the most probable final state attained by the system. As has been pointed out by several authors (e.g., Lynden-Bell 1967; Shu 1978, 1987; Madsen 1987), this final state is rather an idealized final state since it implicitly relies on the assumption of complete mixing in phase space, which does not eventually occur in real systems because of limited timescales.

Let us start with determination of the volume \( \omega_j \) of a phase element of density \( \eta_j \). According to equation (6), this volume is given by

\[
\omega_j = m/[\eta_j] ,
\]

where \( m \) is considered to be constant and is interpreted as the individual mass of all phase-space elements under consideration. In order to simplify the discussion without introducing further restrictions, we now assume that the different phase-space volumes \( \omega_j \) are all multiples of some smallest volume

\[
\omega = \min \{ \omega_j \} .
\]

The volume \( \omega_j \) occupied by a phase-space element of density \( \eta_j \) becomes

\[
\omega_j = g_j \omega ,
\]

where \( g_j \) is the factor by which \( \omega_j \) is larger than the smallest volume, \( \omega \); i.e., the microcell \( \omega_j \) is made up of \( g_j \) microcells of elementary volume \( \omega \).

The \( \mu \)-space is assumed to be divided into a large number of microcells of volume \( \omega \) each. At the macroscopic level, these microcells are grouped into macrocells containing a large number, \( v \), of microcells. The corresponding volume of the macrocell is \( v \omega \). Suppose that there is a microstate in which the \( a \)th macrocell contains \( n_a \) phase-space elements of different densities \( \eta_j \). Because of the collisionless nature of the interaction in the violent relaxation process described
by the Vlasov equation (eq. [1]), there is no cohabitation of microcells (Lynden-Bell 1967).

We now calculate the number of ways of assigning \( n_{aj} \) microcells of volume \( \omega_j \) and density \( \eta_j \), without cohabitation to the \( a \)th macrocell. Under the assumption that the phase space is resolved to the scale of the volume of the smallest microcell, \( \omega_j \), we find that the number of ways of assigning \( n_{aj} \) microcells of volume \( \omega_j \) without cohabitation to the \( a \)th macrocell is, according to equation (9), approximated by

\[
\frac{v!}{(v - g_j n_{aj})!(v - g_k n_{ak})!} \times \ldots \times \frac{(v - g_j n_{aj})!}{(v - g_k n_{ak})!} = \frac{v!}{(v - g_j n_{aj})!} \times \ldots \times \frac{v!}{(v - g_k n_{ak})!}.
\]

By the ordering of \( g_j \) and the exclusion principle, expression (eq. [10]) does not exclude every situation corresponding to partially overlapping phase-space elements. Consider, for instance, a situation in which a phase-space element of volume \( \omega_j \) is assigned a position between regions of the macrocell that have been already occupied. If the volume \( \omega_l \) is larger than the volume between the occupied regions, then a situation is recovered in which phase-space elements may partially overlap. However, as is the case for phase-space elements overlapping the macrocell, situations representing overlapping phase-space elements are also rare. This applies in particular to cases characterized by \( n_{aj} \omega_j \ll v, v_j \). We will return to this question in a subsequent section.

Here we assume that expression (10) yields a sufficient approximation to the total number of ways of assigning the \( \sum_j n_{aj} \) phase-space elements to the \( a \)th macrocell. The total number of microstates \( W((n_{aj})) \) corresponding to a given set of occupation numbers \( \{n_{aj}\} \) is found by multiplying the expressions in equation (10) and taking into account the number of ways of splitting the total of \( N \) distinguishable elements into groups \( n_{aj} \). This yields

\[
W((n_{aj})) = \prod_{j} N_j! \prod_{a} w(n_{aj}) = \prod_{j} N_j! \prod_{a} \left( \frac{v!}{(v - g_j n_{aj})!} \right).
\]

Since the numbers \( w(n_{aj}) \) are approximations, expression (11) also has to be considered an approximation.

The macroscopic constraints to which the system is subject are the \( J \) constraints related to conservation of the total number \( N_j \) (or total mass \( M_j \)) of phase-space elements of densities \( \eta_j \):

\[
\sum_a \eta_j \omega_j n_{aj} = \eta_j \omega_j N_j = M_j.
\]

The energy constraint reads

\[
\sum_j \sum_a \eta_j \omega_j n_{aj} \left( \frac{1}{2} v_j^2 + \frac{1}{2} \Phi_a \right) = E,
\]

where the gravitational potential is defined by

\[
\Phi_a = \Phi(x_a) = -\sum_j \sum_{b=1, b \neq a} G \eta_j \omega_j n_{bk} |x_a - x_b|.
\]

Note that the volumes \( \omega_j \) or the associated masses \( \omega_j \eta_j \) enter the calculation through these macroscopic constraints.

The most probable state is found by the standard procedure of maximizing \( \log W \) subject to the constraints of constant total energy and constant mass. Introducing the Lagrangian multipliers \( \gamma_j \) and \( \beta \), which are related respectively to the constraints (12) and (13), the expression to be maximized is

\[
\ln [W((n_{aj}))] = \alpha_j \sum_j \sum_a \eta_j \omega_j n_{aj} - \beta \sum_j \sum_a \eta_j \omega_j n_{aj} \left( \frac{1}{2} v_j^2 + \frac{1}{2} \Phi_a \right).
\]

The variation of this expression leads to

\[
\sum_j \left[ -\ln(n_{aj}) + \ln \left( v - \sum_j g_j n_{aj} \right) \right] \delta n_{aj} - \alpha_j \sum_j \sum_a \eta_j \omega_j \left( \delta n_{aj} + \frac{1}{2} n_{aj} \delta \Phi_a \right) = 0,
\]

and by taking into account the independence of the \( \delta n_{aj} \)'s, equation (16) splits into the \( J \) equations

\[
\ln \left( v - \sum_j g_j n_{aj} \right) = \gamma_j \eta_j \omega_j + \beta \eta_j \omega_j \epsilon_a \ (j = 1, \ldots, J),
\]

where \( \epsilon_a = \frac{1}{2} v_j^2 + \Phi_a \) stands for the total energy per unit mass related to the \( a \)th microcell. The corresponding occupation numbers \( \{n_{aj}\} \) are

\[
n_{aj} = (v - \sum_j g_j n_{aj}) \exp \left[ -\gamma_j \eta_j \omega_j - \beta \eta_j \omega_j \epsilon_a \right].
\]

By multiplying equation (18) with \( g_j \) and summing over \( j \) and subsequent algebraic manipulations, we find that

\[
v - \sum_j g_j n_{aj} = \sum_j g_j \exp \left[ -\gamma_j \eta_j \omega_j - \beta \eta_j \omega_j \epsilon_a \right] + 1.
\]

According to equation (18), with \( \mu_j = -\gamma_j / \beta \) and with the constant \( \omega_j \eta_j = m \) from equation (7), the most probable occupation numbers \( \{n_{aj}\} \) become

\[
n_{aj} = \frac{v \exp \left[ -\beta m \epsilon_a - \mu_j \right]}{\sum_j g_j \exp \left[ -\beta m \epsilon_a - \mu_j \right]} + 1.
\]

The coarse-grained phase-space distribution \( f \) is defined as the sum of the \( J \) phase-space distributions \( f_j \), where

\[
f_j(v, x) \approx \tilde{f}_j(v_a, x_a) = \frac{g_j n_{aj} \eta_j}{v}.
\]

It is thus given as

\[
f(v, x) \approx \tilde{f}(v_a, x_a) = \sum_j \tilde{f}_j(v_a, x_a) = \sum_j \frac{g_j n_{aj} \eta_j}{v}.
\]
Substituting for $\eta_{aj}$ from equation (20), the coarse-grained phase-space distribution $f$ becomes finally

$$f(v, x) = \sum_j g_j \eta_j \exp \left\{ - \beta m [e(v, x) - \mu_j] \right\} + 1, \quad (23)$$

where $e(v, x) = \frac{1}{2} v^2 + \Phi(x)$ is the total energy per unit mass. According to the initial assumptions, the state characterized by the occupation numbers (eq. [20]) or the coarse-grained phase-space distribution (eq. [23]) is identified with the (idealized) final state attained by a system that consists of phase-space elements of different densities $\eta_j$ and is subject to violent relaxation. The Fermi-like functional form of the phase-space elements of different densities and is subject to violent relaxation. The Fermi-like functional form of the phase-space elements of different densities and is subject to violent relaxation. The Fermi-like functional form of the phase-space elements of different densities and is subject to violent relaxation.

One should keep in mind that distributions in equations (20) and (23) have been derived under the approximations that led to worsen. It is clear, equation (11) of the original treatment of violent relaxation (Lynden-Bell 1967) corresponds to a given set of occupation numbers $\{n_{aj}\}$. However, for the two limiting cases $f_j \ll \eta_j$, $\forall j$ the approximated equation made in equation (10) is very good since, for both situations, the overlap of phase-space elements of different volumes $\omega_j$ does not occur.

In finishing this section, let us return to the question of the explicit determination of the microcell and macrocell volumes $\omega_j$ and $\omega_0$, respectively. While the volumes $\omega_j$ are absorbed in the common constant mass factor $m = \omega_j \eta_j$ because of equation (7), equations (20)–(23) illustrate by the canceling of $\nu$ the arbitrariness of the volume $\omega_0$ of the macrocell.

The only constraint thus remaining is related to the applicability of statistical methods, i.e., the constraint $\nu \gg 1$.

4. DISCUSSION AND CONCLUSIONS

In order to discuss the result (eq. [23]), we compare the equilibrium distribution $f$ with the coarse-grained phase-space distribution $f_{LB}$, which was originally derived by Lynden-Bell (1967):

$$f_{LB}(v, x) = \sum_j \eta_j \exp \left\{ - \beta [e(v, x) - \mu_j] \right\} + 1. \quad (24)$$

In the nondegenerate limit characterized by $f_j \ll \eta_j$, $\forall j$, the coarse-grained phase-space distribution (eq. [24]) turns into a sum of Maxwellians of different velocity dispersions $\sigma_j^2 = \beta_j^{-1}$:

$$f_{LB}(v, x) = \sum_j \eta_j \exp \left\{ - \beta [e(v, x) - \mu_j] \right\}, \quad (25)$$

where $\beta_j \propto \eta_j$. In the nondegenerate limit, the final state of the process of violent relaxation is therefore identified with a thermalized state. The velocity dispersions of the individual components are inversely proportional to their phase-space densities, and, as a consequence, the components have the same temperature, a fact one would expect for a system subject to a two-body relaxation process. This is not only counterintuitive but also in disagreement with observations of astrophysical objects that are believed to have been subject to violent relaxation.

In contrast, the corresponding result derived from equation (23) turns out to reflect the independence of the violent relaxation process on the phase-space densities $\eta_j$ of the different components involved. In the nondegenerate limit ($f_j \ll \eta_j, \forall j$), the coarse-grained phase-space distribution (23) again becomes a sum of Maxwellians. But the Maxwellians are all characterized by a single velocity dispersion $\sigma^2 = \beta^{-1}$. The coarse-grained phase-space distribution in the nondegenerate limit is

$$f(v, x) = \sum_j g_j \eta_j \exp \left\{ - \beta m [e(v, x) - \mu_j] \right\}. \quad (26)$$

We note, that according to § 3, the approximation of $W(n_{aj})$ on which equation (26) is based is very good because the condition $n_{aj} \omega_j \ll \nu_j$ is equivalent to the condition $f_j \ll \eta_j, \forall j$. In the nondegenerate limit, the final state of the violent relaxation process defined by equation (26) is the superposition of Maxwellians of a common velocity dispersion. This is in agreement with the common velocity dispersion of the different components observed, for instance, in clusters of galaxies (see, e.g., Lubin & Bahcall 1993). The common velocity dispersion is equivalent to an equipartition of energy per mass. As a consequence, there is no mass segregation present in systems subject to violent relaxation, as has been observed.

Another limiting case for which the approximation made in evaluating the coarse-grained phase-space distribution (23) yields an exact result is found for $f_j \approx \eta_j, \forall j$ This situation corresponds to a dense coverage of the phase space by phase-space elements with one specific density, $\eta_j$.

For the case of dense coverage of the phase space by phase-space elements of different volumes $\omega_j$ (i.e., $f_j \approx \eta_j, \forall j$), the situation becomes more complicated because the approximations that led to equation (11) worsening. It is clear, however, that the effective total number of microstates corresponding to a given set of occupation numbers $\{n_{aj}\}$ is less than $W(n_{aj})$ since, as discussed in § 3, the number $W(n_{aj})$ includes cases that correspond to partially overlapping phase-space elements. The question remains how the correction of this approximation would influence our result. An investigation of this point is left for subsequent work.

In summary, we have performed a statistical mechanics investigation of violent relaxation of phase-space elements of different mass densities. A well-motivated constraint on the volume of the phase-space elements has been introduced: all phase-space elements independent of their density are of constant mass. As a consequence, statistical mechanics accounts for the independence of the equations of motion in the process of violent relaxation on the mass density of the phase elements. As in the case of Lynden-Bell (1967), the final state reached by a system that is subject to violent relaxation is characterized by a Fermi-like coarse-grained phase-space distribution $f$, which reflects the microscopic exclusion principle (Lynden-Bell 1967) that has been applied in order to account for the collisionless nature of violent relaxation. However, the functional form of this coarse-grained phase-space distribution is independent of the different phase-space densities of the interacting components. In the nondegenerate limit, the coarse-grained phase-space distribution $f$ becomes a superposition of Maxwellians, $f_j$, all possessing the same velocity dispersion. This is in agreement with the observation of a common velocity dispersion of different components of astrophysical objects that are believed to have undergone violent relaxation (e.g., the different components of matter in clusters of galaxies). Thus, the velocity dispersion problem that was present in the original treatment of violent relaxation (Lynden-Bell 1967) has been removed.
As has been pointed out for a long time (e.g., Lynden-Bell 1967; Shu 1978; Madsen 1987) and confirmed by N-body simulations (e.g., Melott 1983; McGlynn 1984; White 1996), violent relaxation eventually fades before the final state becomes established throughout real astrophysical systems. However, expression (23) is likely to hold in the central regions where violent relaxation occurs most violently. In comparing the results obtained here with results from N-body simulations, one should probably take into account recently expressed skepticism concerning the reliability of N-body simulation. As has been shown (see, e.g., Kuhlman, Melott, & Shandarin 1996; Melott et al. 1997), high-resolution N-body simulations may pose a problem because of collision and discreteness errors. When dealing with the violent relaxation of nonbaryonic dark matter as, e.g., massive neutrinos (Kull, Treumann, & Böhringer 1996), this problem becomes worse since massive neutrinos behave like an almost continuous medium.

The assumptions and approximations on which the present investigation relies are (1) that the process of violent relaxation can be described in the continuum limit, (2) that violent relaxation of a multicomponent system is essentially a collisionless process leading to a final equilibrium state, (3) that the characteristics of the Vlasov equation in a statistical mechanics picture of violent relaxation have to be represented not only by a microscopic exclusion principle but also by considering phase-space elements of constant mass, and (4) that the approximations made in deriving the occupation numbers are valid, at least in the nondegenerate case. As a consequence of condition 1, the present investigation, conducted in terms of phase-space elements, does not refer to its counterpart formulated in terms of particles (Shu 1978). The main difference with Lynden-Bell (1967) is the generalized constraint on the mass of the phase elements.

The assumptions and approximations made here do not allow us to consider the important case of incomplete violent relaxation. They also do not apply to the fact that the final state reached by violent relaxation is, in a global picture, a transient state since two-body collisions will eventually start driving the system toward a thermalized state. In addition, the present investigation does not apply, as discussed, to all cases of dense coverage of the phase space by phase-space elements. However, we believe that a more sophisticated treatment of the problem, which corrects for these approximations, will not change our basic result.

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