SPACES OF POLYNOMIALS WITH CONSTRAINED DIVISORS AS GRASSMANIANS FOR TRAVERSING FLOWS

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Abstract. We study traversing vector flows $v$ on smooth compact manifolds $X$ with boundary. For a given compact manifold $\hat{X}$, equipped with a traversing vector field $\hat{v}$ which is convex with respect to $\partial \hat{X}$, we consider submersions/embeddings $\alpha : X \to \hat{X}$ such that $\dim X = \dim \hat{X}$ and $\alpha(\partial X)$ avoids a priori chosen tangency patterns $\Theta$ to the $\hat{v}$-trajectories. In particular, for each $\hat{v}$-trajectory $\hat{\gamma}$, we restrict the cardinality of $\hat{\gamma} \cap \alpha(\partial X)$ by an even number $d$. We call $(\hat{X}, \hat{v})$ a convex pseudo-envelop/envelop of the pair $(X, v)$. Here the vector field $v = \alpha^\dagger(\hat{v})$ is the $\alpha$-transfer of $\hat{v}$ to $X$.

For a fixed $(\hat{X}, \hat{v})$, we introduce an equivalence relation among convex pseudo-envelops/envelops $\alpha : (X, v) \to (\hat{X}, \hat{v})$, which we call a quasitopy. The notion of quasitopy is a crossover between bordisms of pseudo-envelops and their pseudo-isotopies. In the study of quasitopies $\mathcal{QT}_d(Y, c\Theta)$, the spaces $\mathcal{P}_d^{c\Theta}$ of real univariate polynomials of degree $d$ with real divisors whose combinatorial types avoid the closed poset $\Theta$ play the classical role of Grassmanians.

We compute, in the homotopy-theoretical terms that involve $(\hat{X}, \hat{v})$ and $\mathcal{P}_d^{c\Theta}$, the quasitopies of convex envelopes which avoid the $\Theta$-tangency patterns. We introduce characteristic classes of pseudo-envelops and show that they are invariants of their quasitopy classes. Then we prove that the quasitopies $\mathcal{QT}_d(Y, c\Theta)$ often stabilize, as $d \to \infty$.

1. Introduction

This paper is the second in a series, which is inspired by the works of Arnold [Ar], [Ar1], and Vassiliev [V]. It is a direct continuation of [K9], where many ideas of this article are present. Here we apply these ideas to traversing flows on manifolds with boundary. As [K9], this article relies heavily on computations from [KSW1], [KSW2]. While [K9] is studying immersions of compact $n$-dimensional manifolds into products $\mathbb{R} \times Y$, where $Y$ is a compact $n$-dimensional manifold, the present paper deals with special traversing flows on compact $(n + 1)$-dimensional manifolds $X$, the flows that admit a “virtual global section”.

The existence of such sections allows to establish a transparent correspondence between the universe of immersions into the products and the universe of traversing flows with virtual sections. In fact, with this correspondence in place, many results of the present article are just reformulations of the corresponding results from [K9], thus motivating and justifying them. However, we prove also many propositions with no analogues in [K9] (for example, Proposition 4.1, Theorem 4.1, Corollary 4.4, Proposition 4.2, and Theorem 4.8).
Let us describe here our results informally, in a manner that clarifies their nature, but does not involve their most general forms, which carry the burden of combinatorial decorations.

As we have mentioned above, we study traversing vector flows \( v \) (see Definition 4.1 and [K1], [K2]) on smooth compact \((n+1)\)-dimensional manifolds \( X \) with boundary. Every traversing flow admits a Lyapunov function, and this fact may serve as a working definition of such flows. We are interested in the combinatorial patterns \( \omega \) that describe how the \( v \)-trajectories are tangent to the boundary \( \partial X \). Specifically, we are concerned with the traversing vector flows whose tangency patterns \( \omega \) do not belong to a given closed poset \( \Theta \). Depending on \( \Theta \), the very existence of such flows puts strong restrictions on the topology of \( X \) (see [K1]-[K8], [K10]). For example, in Fig.1 \( \Theta \) includes the cubic and the double tangencies of \( \partial X \) to the family of \( u \)-directed lines. In other words, in this figure, the tangency patterns \( \omega = (3) \) and \( \omega = (22) \) are forbidden.

In order to simplify the investigation of such flows, we “envelop” them into, so called, convex pseudo-envelops \((\hat{X}, \hat{v})\). A convex pseudo-envelop consists of: (i) a compact smooth \((n+1)\)-manifold \( \hat{X} \), (ii) a traversing vector field \( \hat{v} \) on it such that the boundary \( \partial \hat{X} \) is convex (see Definition 4.5) with respect to the \( \hat{v} \)-flow, (iii) a submersion \( \alpha : X \to \hat{X} \) such that \( \alpha(\partial X) \) avoids a forbidden tangency patterns \( \Theta \) to the \( \hat{v} \)-trajectories, and (iv) the pull-back \( \alpha^\dagger(\hat{v}) \) of \( \hat{v} \), coinciding with the given vector field \( v \) (see Fig.1 in which \( \hat{X} \) is a cylinder).

The existence of a convex pseudo-envelop provides the \( v \)-flow on \( X \) with a “virtual global section”, a significantly effective tool. Its existence allows us to transfer the results from [K9] about immersions/embeddings \( \{ \beta : M \to \mathbb{R} \times Y \} \), where \( M \) and \( Y \) are compact smooth \( n \)-manifolds, to the parallel results about convex envelops of traversing flows.

Next, we organize convex pseudo-envelops \( \{ \alpha : (X, v) \to (\hat{X}, \hat{v}) \} \) with the fixed target \((\hat{X}, \hat{v})\) into the quasitopy equivalence classes \( QT_d(\hat{X}, \hat{v}; c\Theta) \), where \( d \) is a given even natural number. For a typical \( \hat{v} \)-trajectory \( \hat{\gamma} \subset \hat{X} \), \( d \) gives an upper bound of the cardinality of the set \( \hat{\gamma} \cap \alpha(\partial X) \). Fig.3 may give some impression of the nature of quasitopy; it depicts a pair of submersions \( \alpha_0, \alpha_1 \), linked by a kind of cobordism with a strict combinatorial control of its tangent patterns.

The quasitopies come in two flavors: \( QT_d^{\text{emb}}(\hat{X}, \hat{v}; c\Theta) \), formed by regular embeddings \( \alpha \), and \( QT_d^{\text{sub}}(\hat{X}, \hat{v}; c\Theta) \), formed by more general submersions \( \alpha \).

Consider the locus \( \partial_1^+ \hat{X}(\hat{v}) \subset \partial \hat{X} \), where \( \hat{v} \) points inside of \( \hat{X} \), and the locus \( \partial_2^- \hat{X}(\hat{v}) \subset \partial_2^- \hat{X}(\hat{v}) \), where \( \hat{v} \) is tangent to \( \partial \hat{X} \). Then we construct two classifying maps

\[
\Phi^{\text{emb}} : QT_d^{\text{emb}}(\hat{X}, \hat{v}; c\Theta) \to \{ (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})), (P_d^{\text{emb}}; pt) \},
\]

\[
\Phi^{\text{sub}} : QT_d^{\text{sub}}(\hat{X}, \hat{v}; c\Theta) \to \{ (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})), (P_d^{\text{sub}}; pt) \},
\]

whose targets are sets of homotopy classes from the quotient space \( \partial_1^+ \hat{X}(\hat{v}) / \partial_2^- \hat{X}(\hat{v}) \) to the universal space \( P_d^{\text{emb}} \)—the “Grassmanian”—, formed by real polynomials of degree \( d \) whose real divisors avoid a given poset \( \Theta \).

Then we prove (see Theorem 4.2) that \( \Phi^{\text{emb}} \) is a bijection and \( \Phi^{\text{sub}} \) is a split surjection.
Figure 1. Convex pseudo-envelops $\alpha : (X, \alpha^\dagger(\partial_u)) \to (\hat{X}, \partial_u)$ of a punctured torus $X$ (on the top), and of a punctured surface $X$ of genus 2 (on the bottom). The envelop $\hat{X}$ is the cylinder $[0,1] \times S^1$, equipped with the constant vector field $\partial_u$. Both submersions $\alpha$ are generic relative to the vertical vector field $\partial_u$ on the cylinder. The dots mark the multiplicity 2 tangencies of $\alpha(\partial X)$ to the $\theta$-fibers; the self-intersections of $\alpha(\partial X)$ are unmarked. In both examples, the cardinality of the fibers $\theta \circ \alpha^\beta : \partial X \to S^1$ does not exceed 6.

For a constant vector field $\hat{v}$ on the standard ball $D^{n+1}$, the sets $\mathcal{QT}_{sub/emb}(D^{n+1}, \hat{v}; c\Theta)$ are groups; the group operation is induced by the boundary connected sum of pseudo-envelops/envelops. These groups are abelian for $n > 1$. The group $\mathcal{QT}_{sub}(D^{n+1}, \hat{v}; c\Theta)$ is a split extension of $\mathcal{QT}_{emb}(D^{n+1}, \hat{v}; c\Theta)$ (see Theorem 4.4). We prove that

$$\mathcal{QT}_{emb}(D^{n+1}, \hat{v}; c\Theta) \approx \pi_n((P_c\Theta, pt)).$$

In the process, in the Subsection 4.2, we notice an interesting relation between the connected components of the of the space of convex traversing flows on the standard $(n+1)$-ball and the monoid $\text{HS}_{n-1}$ of smooth types of integral homology $(n-1)$-spheres, being considered up to connected sums with smooth homotopy $(n-1)$-spheres (see Proposition 4.1).

The groups $\mathcal{QT}^{sub/emb}(D^{n+1}, \hat{v}; c\Theta)$ act on the quasitopies of convex pseudo-envelops $\mathcal{QT}^{sub/emb}(\hat{X}, \hat{v}; c\Theta)$, and the classifying maps $\Phi^{sub/emb}$ in (1.1) are equivariant.

These classifying maps deliver a variety of (co)homotopy and (co)homology invariants of traversing flows which admit convex pseudo-envelops (see Theorem 4.7, Proposition 4.6, and Theorem 4.8).

We also establish several of results (Theorem 4.3, Corollary 4.6) about the stabilization of the sets $\mathcal{QT}^{emb}(\hat{X}, \hat{v}; c\Theta)$ as $d \to \infty$.

Finally, we compute $\mathcal{QT}^{emb}(\hat{X}, \hat{v}; c\Theta)$ for may special cases of $\Theta$ and $(\hat{X}, \hat{v})$. 


2. Spaces of real polynomials with constrained real divisors

For our reader convenience, we state here a number of results from [KSW1] and [KSW2] about the topology of spaces of real monic univariate polynomials with constrained real divisors. These results are crucial for the applications to follow. To make the present paper even partially self-contained, we are basically recycling Section 2 from [K9].

Let \( \mathcal{P}_d \) denote the space of real monic univariate polynomials of degree \( d \). Given a polynomial \( P(u) = u^d + a_{d-1}u^{d-1} + \cdots + a_0 \) with real coefficients, we consider its real divisor \( D_R(P) \). Let \( \omega_i \) denotes the multiplicity of the \( i \)-th real root of \( P \), the real roots being ordered by their magnitude. We call the ordered \( \ell \)-tuple of natural numbers \( \omega = (\omega_1, \ldots, \omega_\ell) \) the real root multiplicity pattern of \( P(u) \), or the multiplicity pattern for short.

Such sequences \( \omega \) form a universal poset \( (\Omega, \succ) \). The partial order "\( \succ \)" in \( \Omega \) is defined in terms of two types of elementary operations: merges \( \{ M_i \} \) and inserts \( \{ I_i \} \). The operation \( M_i \) merges a pair of adjacent entries \( \omega_i, \omega_{i+1} \) of \( \omega = (\omega_1, \ldots, \omega_i, \omega_{i+1}, \ldots, \omega_q) \) into a single component \( \tilde{\omega}_i = \omega_i + \omega_{i+1} \), thus forming a new shorter sequence \( M_i(\omega) = (\omega_1, \ldots, \omega_i, \omega_{i+1}, \ldots, \omega_q) \). The insert operation \( I_i \) either insert 2 in-between \( \omega_i \) and \( \omega_{i+1} \), thus forming a new longer sequence \( I_i(\omega) = (\ldots, \omega_i, 2, \omega_{i+1}, \ldots) \), or, in the case of \( I_0 \), appends 2 before the sequence \( \omega \), or, in the case \( I_q \), appends 2 after the sequence \( \omega \).

We define the order \( \omega \succ \omega' \), where \( \omega, \omega' \in \Omega \), if one can produce \( \omega' \) from \( \omega \) by applying a sequence of these elementary operations.

For a sequence \( \omega = (\omega_1, \omega_2, \ldots, \omega_q) \in \Omega \), we introduce the norm and the reduced norm of \( \omega \) by the formulas:

\[
|\omega| = \text{def} \sum_i \omega_i \quad \text{and} \quad |\omega|' = \text{def} \sum_i (\omega_i - 1).
\]

Note that \( q \), the cardinality of the support of \( \omega \), is equal to \( |\omega| - |\omega|^\prime \).

Let \( \mathcal{R}_d^\Theta \) be of the set of all polynomials with the real root multiplicity pattern \( \omega \), and let \( \mathcal{R}_d^\Theta \) be its closure in \( \mathcal{P}_d \).

For a given collection \( \Theta \) of multiplicity patterns \( \{ \omega \} \), which share the parity of their norms \( |\omega| \), and is closed under the merge and insert operations, we consider the union \( \mathcal{P}_d^\Theta \) of the subspaces \( \mathcal{R}_d^\Theta \), taken over all \( \omega \in \Theta \) such that \( |\omega| \leq d \) and \( |\omega| \equiv d \mod 2 \). We denote by \( \mathcal{P}_d^\Theta \) its complement \( \mathcal{P}_d \setminus \mathcal{P}_d^\Theta \).

Since \( \mathcal{P}_d^\Theta \) is contractible, it makes more sense to consider its one-point compactification \( \bar{\mathcal{P}}_d^\Theta \). If the set \( \mathcal{P}_d^\Theta \) is closed in \( \mathcal{P}_d \), by the Alexander duality on the sphere \( \mathcal{P}_d \equiv S^d \), we get

\[
H^j(\bar{\mathcal{P}}_d^\Theta; \mathbb{Z}) \approx H_{d-j-1}(\partial \bar{\mathcal{P}}_d^\Theta; \mathbb{Z}).
\]

This implies that the spaces \( \mathcal{P}_d^\Theta \) and \( \bar{\mathcal{P}}_d^\Theta \) carry the same (co)homological information. Let us describe it in pure combinatorial terms.

Let us consider the following domain in \( \mathbb{R}^{d+1} \):

\[
\mathcal{E}_d = \text{def} \{ (u, P) \in \mathbb{R} \times \mathcal{P}_d | P(u) \leq 0 \} \quad \text{and its boundary}
\]

\[
\partial \mathcal{E}_d = \text{def} \{ (u, P) \in \mathbb{R} \times \mathcal{P}_d | P(u) = 0 \}.
\]
The pair \((\mathcal{E}_d, \partial \mathcal{E}_d)\) is of a fundamental importance for us.

For a subposet \(\Theta \subset \Omega\) and natural numbers \(d, k\), we introduce the following notations:

\[
\Theta_{[d]} = \{\omega \in \Theta : |\omega| = d\}, \quad \Theta_{(d)} = \{\omega \in \Theta : |\omega| \leq d\}, \\
\Theta_{|\sim'| = k} = \{\omega \in \Theta : |\omega'| = k\}, \quad \Theta_{|\sim'| \geq k} = \{\omega \in \Theta : |\omega'| \geq k\}.
\]

Assuming that \(\Theta \subset \Omega\) is a closed subposet, let

\[
c\Theta = \Omega \setminus \Theta \quad \text{and} \quad c\Theta_{(d)} = \Omega_{(d)} \setminus \Theta_{(d)}.
\]

We denote by \(\mathbb{Z}[\Omega_{(d)}]\) the free \(\mathbb{Z}\)-module, generated by the elements of \(\Omega_{(d)}\). Using the merge operators \(M_k\) and the insert operators \(I_k\) from \([KSW2]\) on \(\Omega\), we define homomorphisms \(\partial_M : \mathbb{Z}[\Omega_{(d)}] \to \mathbb{Z}[\Omega_{(d)}], \partial_I : \mathbb{Z}[\Omega_{(d)}] \to \mathbb{Z}[\Omega_{(d)}]\) by

\[
\partial_M(\omega) = -\sum_{k=1}^{s_\omega-1} (-1)^k M_k(\omega) \quad \text{and} \quad \partial_I(\omega) = \sum_{k=0}^{s_\omega} (-1)^k I_k(\omega),
\]

where \(s_\omega = \text{def} |\omega| - |\omega'|\). In fact, \(\partial_M\) and \(\partial_I\) are anti-commuting differentials \([KSW2]\).

Therefore, the sum

\[
\partial = \text{def} \partial_M + \partial_I : \mathbb{Z}[\Omega_{(d)}] \to \mathbb{Z}[\Omega_{(d)}]
\]

is a differential.

For a closed poset \(\Theta\), the restrictions of the operators \(\partial, \partial_M, \) and \(\partial_I\) to the free \(\mathbb{Z}\)-module \(\mathbb{Z}[\Theta]\) are well-defined. Thus, for any closed subposet \(\Theta_{(d)} \subset \Omega_{(d)}\), we may consider the differential complex \(\partial : \mathbb{Z}[\Theta_{(d)}] \to \mathbb{Z}[\Theta_{(d)}]\), whose \((d - j)\)-grading is defined by the module \(\mathbb{Z}[\Theta_{(d)}] \cap \Theta_{|\sim'| = j}\). We denote by \(\partial^* : \mathbb{Z}[\Theta_{(d)}]^* \to \mathbb{Z}[\Theta_{(d)}]^*\) its dual operator, where \(\mathbb{Z}[\Theta_{(d)}]^* = \text{def} \text{Hom}(\mathbb{Z}[\Theta_{(d)}], \mathbb{Z})\).

Then we consider the quotient set \(\Theta^\# := \Omega_{(d)} / \Theta_{(d)}\). For the closed subposet \(\Theta_{(d)}\), the partial order in \(\Omega_{(d)}\) induces a partial order in the quotient \(\Theta^\#_{(d)}\).

Finally, we introduce a new differential complex \((\mathbb{Z}[\Theta^\#_{(d)}], \partial^\#)\) by including it in the short exact sequence of differential complexes:

\[
0 \to (\mathbb{Z}[\Theta_{(d)}], \partial) \to (\mathbb{Z}[\Omega_{(d)}], \partial) \to (\mathbb{Z}[\Theta^\#_{(d)}], \partial^\#) \to 0.
\]

We will rely on the following result from \([KSW2]\), which reduces the computation of the reduced cohomology \(\check{H}^j(P^\Theta_d ; \mathbb{Z})\) to Algebra and Combinatorics.

**Theorem 2.1.** \([KSW2]\) Let \(\Theta \subset \Omega_{(d)}\) be a closed subposet. Then, for any \(j \in [0, d]\), we get group isomorphisms

\[
\check{H}^j(P^\Theta_d ; \mathbb{Z}) \overset{D}{\approx} H_{d-j}(\check{P}^\Theta_d ; \mathbb{Z}) \approx H_{d-j}(\partial^\# : \mathbb{Z}[\Theta^\#] \to \mathbb{Z}[\Theta^\#]), \\
\check{H}_j(P^\Theta_d ; \mathbb{Z}) \overset{D}{\approx} H^{d-j}(\check{P}^\Theta_d ; \mathbb{Z}) \approx H_{d-j}(\partial^#)^* : (\mathbb{Z}[\Theta^\#])^* \to (\mathbb{Z}[\Theta^\#])^*),
\]

where \(D\) is the Poincaré duality isomorphism. \(\diamondsuit\)
Consider the embedding \( \epsilon_{d,d+2} : \mathcal{P}_d \to \mathcal{P}_{d+2} \), defined by the formula
\[
\epsilon_{d,d+2}(P)(u) = \text{def} \ (u^2 + 1) \cdot P(u).
\]
(2.7)

It preserves the \( \Omega \)-stratifications of the two spaces by the combinatorial types \( \omega \) of real divisors. The embedding \( \epsilon_{d,d+2} \) makes it possible to talk about stabilization of the homology/cohomology of the spaces \( \mathcal{P}_d^{\Theta} \) and \( \mathcal{P}_{d+2}^{\Theta} \), as \( d \to \infty \).

For a closed poset \( \Theta \subset \Omega \), consider the closed finite poset \( \Theta_{[d]} = \Omega_{[d]} \cap \Theta \). It is generated by some maximal elements \( \omega^{(1)}_{\star}, \ldots, \omega^{(\ell)}_{\star} \), where \( \ell = \ell(d) \). We introduce two useful quantities:
\[
\eta_{\Theta}(d) = \text{def} \ \max_{i \in [1, \ell(d)]} \{ (|\omega^{(i)}_{\star}| - 2|\omega^{(i)}_{\star}'|) \},
\]
(2.8)
\[
\psi_{\Theta}(d) = \text{def} \ \frac{1}{2} (d + \eta_{\Theta}(d)).
\]
(2.9)

Note that \( \psi_{\Theta}(d) = \frac{1}{2} \min_{i \in [1, \ell(d)]} \{ (d - |\omega^{(i)}_{\star}'|) + (|\omega^{(i)}_{\star}| - |\omega^{(i)}_{\star}'|) \} < d \),

where both summands, the “codimension” \( (d - |\omega^{(i)}_{\star}'|) \) and the “support” \( (|\omega^{(i)}_{\star}| - |\omega^{(i)}_{\star}'|) \), are positive and each does not exceed \( d \). At the same time, \( \eta_{\Theta}(d) \) may be negative.

In the stabilization results about quasitopies, the quantity
\[
\xi_{\Theta}(d + 2) = \text{def} \ d + 2 - \psi_{\Theta}(d + 2).
\]
plays the key role.

Now we are in position to state the main stabilization result from [KSW2]:

**Theorem 2.2. (short stabilization: \( \{ d \Rightarrow d + 2 \} \))** Let \( \Theta \) be a closed subposet of \( \Omega \). Let the embedding \( \epsilon_{d,d+2} : \mathcal{P}_d \subset \mathcal{P}_{d+2} \) be as in (2.7).

- Then, for all \( j \geq \psi_{\Theta}(d + 2) - 1 \), we get a homological isomorphism
  \[
  (\epsilon_{d,d+2})_* : H_j(\mathcal{P}_d^{\Theta}; \mathbb{Z}) \approx H_{j+2}(\mathcal{P}_{d+2}^{\Theta}; \mathbb{Z}).
  \]
- and, for all \( j \leq d + 2 - \psi_{\Theta}(d + 2) \), a homological isomorphism
  \[
  (\epsilon_{d,d+2})_* : H_j(\mathcal{P}_d^{\Theta}; \mathbb{Z}) \approx H_j(\mathcal{P}_{d+2}^{\Theta}; \mathbb{Z}).
  \]
  \( \diamond \)

**Definition 2.1.** A closed poset \( \Theta \subseteq \Omega \) is called profinite if, for all integers \( q \geq 0 \), there exist only finitely many elements \( \omega \in \Theta \) such that \( |\omega'| \leq q \).

\( \diamond \)

**Corollary 2.1. (long stabilization: \( \{ d \Rightarrow \infty \} \))** For any closed profinite poset \( \Theta \subset \Omega \), and for each \( j \), the homomorphism
\[
(\epsilon_{d,d'})_* : H_j(\mathcal{P}_d^{\Theta}; \mathbb{Z}) \approx H_j(\mathcal{P}_{d'}^{\Theta}; \mathbb{Z})
\]
is an isomorphism for all sufficiently big \( d \leq d' \), \( d \equiv d' \mod 2 \).

\( \diamond \)
As a result, we may talk about the stable homology $H_j(\mathcal{P}_\infty^{c\Theta};\mathbb{Z})$, the direct limit $\lim_{d \to \infty} H_j(\mathcal{P}^{c\Theta}_d;\mathbb{Z})$.

Let us describe a few special cases of stabilization from [KSW2]. For $k \geq 1$, $q \in [0,d]$, and $q \equiv d \mod 2$, let us consider the closed poset

$$\Omega_{|\sim'| \geq k}^{(q)} \overset{\text{def}}{=} \{ \omega \in \Omega_d \text{ such that } |\omega'| \geq k \text{ and } |\omega| \geq q \}.$$  

(2.11)

Note that, for $\Theta = \Omega_{|\sim'| \geq k}^{(0)} = \Omega_{|\sim'| \geq k}$, the space $\bar{P}_d^\Theta$ is the entire $(d-k)$-skeleton of $\bar{P}_d$.

(2.12) Let $A(d,k,q) = \def\abs{\chi((\mathbb{Z}[\Omega_{|\sim'| \geq k}^{(q)}],\partial))} = \abs{\chi(\bar{P}_d^\Theta_{|\sim'| \geq k}) - 1}$,

the absolute value of the Euler number of the differential complex $(\mathbb{Z}[\Omega_{|\sim'| \geq k}^{(q)}],\partial)$.

**Proposition 2.1.** ([KSW2]) Fix $k \in [1,d]$ and $q \geq 0$ such that $q \equiv d \mod 2$. Let $\Theta = \Omega_{|\sim'| \geq k}^{(q)}$.

Then the one-point compactification $\bar{P}_d^\Theta$ has the homotopy type of a bouquet of $(d-k)$-dimensional spheres. The number of spheres in the bouquet equals $A(d,k,q)$.

**Proposition 2.2.** ([KSW1]) Let $\Theta \subset \Omega_{|\sim'| \geq d+2}$ be a closed poset. For $d' \geq d + 2$ such that $d' \equiv d \mod 2$, let $\hat{\Theta}_{(d')}$ be the smallest closed poset in $\Omega_d$ containing $\Theta$.

Then for $d' \geq d + 2$, we have an isomorphism $\pi_1(P_{d'}^{c\hat{\Theta}(d')}) \cong \pi_1(P_{d+2}^{c\Theta(4)})$ of the fundamental groups.

3. **Submersions & embeddings of manifolds whose boundary has constrained tangency patterns to the product 1-foliations**

This section forms a bridge between the results from [K9] and our main results from Section 4. The reader may choose to surf Section 3 or to proceed directly to Section 4. Since all the results of this section are instant derivations of the similar results from [K9], we provide just an outline of their validations. Although we will not use directly the results of Section 3 in Section 4, this section may induce the “right mindset” for the reader.

Let $Y$ be a smooth compact $n$-manifold. Having in mind applications to traversing vector flows, we move away from immersions and embeddings $\beta : M \to \mathbb{R} \times Y$ of $n$-manifolds $M$, the topic of [K9], to submersions and regular embeddings $\alpha : X \to \mathbb{R} \times Y$ of compact smooth $(n+1)$-manifolds $X$ with boundary into the product $\mathbb{R} \times Y$. When $\dim(X) = \dim(Y) + 1$, the submersions and immersions are the same notion. Moreover, the restriction of a submersion $\alpha$ to $\partial X$ is an immersion. Of course, if $\alpha : X \to \mathbb{R} \times Y$ is an embedding, so is $\alpha^\beta := \alpha| : \partial X \to \mathbb{R} \times Y$.

Therefore many constructions and notions from [K9], with the help of the correspondence $\alpha \sim \alpha^\beta$, apply instantly to submersions $\alpha : X \to \mathbb{R} \times Y$ such that $\dim X = \dim Y + 1$.

**Remark 3.1.** From the viewpoint of this paper, the main difference between immersions $\beta : M \to \mathbb{R} \times Y$ and submersions $\alpha : X \to \mathbb{R} \times Y$, where $\dim X = \dim M + 1$, is that not
any $\beta$ is a boundary $\alpha^0$ of some $\alpha$. For example, the figure $\infty$ in the plane does not bound a submersion of a 2-manifold. See [Pa] for the comprehensive theory of possible extensions of a given immersion $\beta$ to a submersion $\alpha$.

**Example 3.1.** The following simple construction provides *models of submersions* that animate our treatment. Let $W$ be a codimension zero compact submanifold of a given manifold $V$. Consider a covering map $\pi : \tilde{W} \to W$ with a finite fiber. Let $X \subset \tilde{W}$ be a compact codimension zero submanifold. It is possible to isotop the imbedding $X \subset \tilde{W}$ so that $\pi : \partial X \to W$ will be an immersion with all the multiple crossings of $\pi(\partial X)$ being in general position. Of course, each crossing has the multiplicity that does not exceed the cardinality of the $\pi$-fiber. Then $\pi : X \to \tilde{V}$ is the model example of a submersion to keep in mind.

Let us introduce the central to this paper notion of **quasitopy for submersions** $\alpha : X \to \mathbb{R} \times Y$, an analogue of Definition 3.7 from [K9]. We fix a natural number $d$ and consider a closed sub-poset $\Theta \subset \Omega$ such that $(\emptyset) \notin \Theta$ (the, so called, $\Lambda$-condition (3.10) from [K9]). For topological reasons, we will consider only the case $d \equiv 0 \mod 2$.

Let $\mathcal{L}$ be the 1-foliation of $\mathbb{R} \times Y$ by the fibers of the obvious projection $\mathbb{R} \times Y \to Y$, and $\mathcal{L}^*$ be the 1-foliation of $\mathbb{R} \times Y \times [0,1]$ by the fibers of $\mathbb{R} \times Y \times [0,1] \to Y \times [0,1]$.

Let $X$ be a compact smooth $(n+1)$-dimensional manifold with boundary. Consider a smooth map $\alpha : X \to \mathbb{R} \times Y$ such that:

\[ (3.1) \]

- $\alpha$ is a submersion,
- for each $y \in Y$, the total multiplicity $m_\beta(y)$ of $\alpha(\partial X)$ with respect to the foliation $\mathcal{L}$ (see [K9], formula (3.4)) is less than or equal to $d$ and $m_\beta(y) \equiv d \mod 2$,
- for each $y \in Y$, the combinatorial tangency pattern $\omega^\alpha(y) \in \Omega_{\{d\}}$ of $\alpha(\partial X)$ with respect to $\mathcal{L}$ does not belong to $\Theta$,
- for each $y \in \partial Y$, $\omega^\alpha(y) = (\emptyset)$.

Note that the normal bundle $\nu^\alpha$ to $\alpha(\partial X)$ in $\mathbb{R} \times Y$ is trivial.

Let $X_0, X_1$ be two compact smooth (oriented) $(n+1)$-dimensional manifolds with boundary. We consider a compact smooth (oriented) $(n+2)$-manifold $W$ with conners $\partial X_0 \sqcup \partial X_1$ such that $\partial W = (X_0 \sqcup \overline{X_1}) \cup (\partial X_0 \sqcup \partial X_1) \delta W$, where $\delta W$ is a smooth (oriented) cobordism between $\partial X_0$ and $\partial X_1$. Let $Z = \text{def} \mathbb{R} \times Y$ and $Z^\partial = \text{def} \mathbb{R} \times \partial Y$.

Let $A : W \to \mathbb{R} \times Z$, where $\dim(W) = \dim(Z) + 1$, be a submersion. In particular, $A : \delta W \to \mathbb{R} \times Z$, $A|_{\partial X_0} \to \mathbb{R} \times (Y \times \{0\})$, and $A|_{\partial X_1} \to \mathbb{R} \times (Y \times \{1\})$ are immersions.

The next two definitions lay down the foundation for notions of quasitopy of traversing vector fields, the main subject of Section 4 (see Fig. 2).

**Definition 3.1.** Let us fix natural numbers $d' \geq d$, $d' \equiv d \equiv 0 \mod 2$. Consider closed subposets $\Theta' \subset \Theta \subset \Omega$ such that $(\emptyset) \notin \Theta$. 

We say that a two submersions $\alpha_0 : X_0 \to \mathbb{R} \times Y$ and $\alpha_1 : X_1 \to \mathbb{R} \times Y$ are $(d, d'; c\Theta, c\Theta')$-quasitopic, if there exists a compact smooth $(n + 2)$-manifold $W$ as above and a smooth submersion $A : W \to \mathbb{R} \times Z$ so that:

- $A|_{X_0} = \alpha_0$ and $A|_{X_1} = \alpha_1$;
- for each $z \in Z$, the total multiplicity $m_A(z)$ of $A(\delta W)$ with respect to the fiber $\mathcal{L}_z^*$ is such that $m_A(z) \leq d'$, $m_A(z) \equiv d' \mod 2$, and the combinatorial tangency pattern $\omega^A(z)$ of $A(\delta W)$ with respect to $\mathcal{L}_z^*$ belongs to $c\Theta'$;
- for each $z \in Y \times \{0\} \cup \{1\}$, the total multiplicity $m_A(z)$ of $A(\delta W)$ with respect to the fiber $\mathcal{L}_z$ is such that $m_A(z) \leq d$, $m_A(z) \equiv d \mod 2$, and the combinatorial tangency pattern $\omega^A(z)$ of $A(\delta W)$ with respect to $\mathcal{L}$ belongs to $c\Theta$.
- for each $z \in Z^0$, $\omega^A(z) = (\emptyset)$.

We denote by $\overline{QT}_{d,d'}^{\text{sub}/\text{emb}}(Y; c\Theta; c\Theta')$ the set of quasitopy classes of such submersions/embbedings $\alpha : X \to \mathbb{R} \times Y$.

It is easy to check that the quasitopy of submersions is an equivalence relation. Recall that in [K9], Definition 3.7, we have introduced a similar notion of quasitopy for immersions/embbedings $\beta : (M, \partial M) \to (\mathbb{R} \times Y, \mathbb{R} \times \partial Y)$. There, we used the notation

$$QT_{d,d'}^{\text{imm}/\text{emb}}(Y; c\Theta; c\Theta') = \overline{QT}_{d,d'}(Y, \partial Y; c\Theta, (\emptyset); c\Theta')$$

for the set of equivalence classes of immersions/embbedings $\beta$ under the quasitopy relation.

As for immersions $\beta : M \to \mathbb{R} \times Y$, for any choice of connected components $\kappa_1 \in \pi_0(\partial Y_1), \kappa_2 \in \pi_0(\partial Y_2)$, the connected sum operation (see [K9], formula (3.15))

$$\triangleright : QT_{d,d'}(Y_1; c\Theta; c\Theta') \times QT_{d,d'}(Y_2; c\Theta; c\Theta') \to QT_{d,d'}(Y_1 \#_{\partial Y_2} Y_2; c\Theta; c\Theta'),$$

is well-defined for the quasitopies of submersions.

It converts the set $QT_{d,d'}(D^n; c\Theta; c\Theta')$ into a group $\overline{H}_{d,d'}^{\text{sub}/\text{emb}}(n; c\Theta; c\Theta')$ (abelian for $n > 1$). This group acts, via the connected sum operation $\triangleright$, on the set $\overline{QT}_{d,d'}(Y; c\Theta; c\Theta')$, provided that a connected component of $\partial Y$ is chosen. To get a better insight, compare this proposition with Proposition 3.2 from [K9].

For two pairs $X_1 \supset A_1$ and $X_2 \supset A_2$ of topological spaces, we denote by $\{[(X_1, A_1), (X_2, A_2)]\}$ the set of homotopy classes of continuous maps $g : X_1 \to X_2$, where $g(A_1) \subset A_2$.

**Definition 3.2.** Given three pairs of spaces $X_1 \supset A_1$, $X_2 \supset A_2$, $X_3 \supset A_3$, and a fixed continuous map $\epsilon : (X_2, A_2) \to (X_3, A_3)$, we denote by

$$\{[(X_1, A_1), \epsilon : (X_2, A_2) \to (X_3, A_3)]\}$$

the set of homotopy classes $[g]$ of continuous maps $g : (X_1, A_1) \to (X_2, A_2)$, modulo the following equivalence relation: by definition, $[g_0] \sim [g_1]$, where $g_0 : (X_1, A_1) \to (X_2, A_2)$ and $g_1 : (X_1, A_1) \to (X_2, A_2)$ are continuous maps, if the compositions $\epsilon \circ g_0$ and $\epsilon \circ g_1$ are homotopic as maps from $(X_1, A_1)$ to $(X_3, A_3)$.

◊
Following the proof of Proposition 3.3 and Theorem 3.2 from [K9], we get Theorem 3.1, their analogue for submersions. It is crucial that here \( d \equiv 0 \mod 2 \), which implies that any regular embedding \( \beta : M \to \mathbb{R} \times Y \), such that all the multiplicities \( \{m_{\beta}(y)\}_{y \in Y} \) are even, bounds a regular embedding \( \alpha : X \to \mathbb{R} \times Y \), where \( \partial X = M \) and \( \alpha|_{\partial X} = \beta \).

For an \( n \)-dimensional \( Y \), the next lemma reduces the computation of quasitopies \( QT_{d,d'}^{emb}(Y; c\Theta; c\Theta') \), based on regular embeddings \( \alpha : X \to \mathbb{R} \times Y \) of \( (n+1) \)-dimensional manifolds \( X \), to the computation of quasitopies \( QT_{d,d'}^{emb}(Y; c\Theta; c\Theta') \), based on the regular embeddings \( \beta : M \to \mathbb{R} \times Y \) of \( n \)-dimensional manifolds \( M \).

**Lemma 3.1.** For closed posets \( \Theta' \subset \Theta \subset \Omega \) such that \( (\emptyset) \notin \Theta \) and \( d \leq d' \), \( d \equiv d' \equiv 0 \mod 2 \), the map

\[
\Delta : QT_{d,d'}^{emb}(Y; c\Theta; c\Theta') \to QT_{d,d'}^{emb}(Y; c\Theta; c\Theta')
\]

that takes an embedding \( \alpha : X \to \mathbb{R} \times Y \) to the embedding \( \alpha^\beta : \partial X \to \mathbb{R} \times Y \) is a bijection.

As a result, the obvious map \( \mathcal{A}^\bullet : QT_{d,d'}^{emb}(Y; c\Theta; c\Theta') \to QT_{d,d'}^{imm}(Y; c\Theta; c\Theta') \) is injective; in other words, if two embedding are quasitopic via a submersion, they are quasitopic via an embedding.

**Proof.** The main step is contained in the proof of Lemma 3.6 from [K9]. Let us describe its flavor. Since, for \( d \equiv 0 \mod 2 \) and a closed \( n \)-manifold \( M \), any \( (\partial E_d) \)-regular (see Definition 3.4 in [K9]) embedding \( \beta : M \subset \mathbb{R} \times \text{int}(Y) \) bounds a (orientable when \( Y \) is orientable) \( (n+1) \)-manifold \( \alpha : X_\beta \subset \mathbb{R} \times Y \), the map \( \Delta \) is onto. Evidently, the combinatorial tangency types to \( L \) are determined by \( \beta(M) \). Thus, every embedding \( \alpha \in QT_{d,d'}^{emb}(Y; c\Theta; c\Theta') \) produces an element \( \alpha^\beta \in QT_{d,d'}^{emb}(Y; c\Theta; c\Theta') \). By the same token, if a \( (\partial E_d) \)-regular embedding \( \beta : M \subset \mathbb{R} \times \text{int}(Y) \) bounds a \( (\partial E_d) \)-regular embedding \( B : N \subset \mathbb{R} \times Z \) (where \( Z = \text{def} \ Y \times [0,1] \)), whose tangency to \( L^\bullet \) patterns belong to \( c\Theta' \), then \( X_\beta \cup_{\partial M} N \) bounds (an orientable when \( Y \) is orientable) \( (n+2) \)-manifold \( W \subset \mathbb{R} \times Z \), provided \( d' \equiv 0 \mod 2 \). Here \( N \subset \mathbb{R} \times Z \) is a compact \( (n+1) \)-manifold such that \( \partial N = M \) and \( B|_M = \beta \). Thus the map \( \Delta \) is injective. By the previous argument, it is bijective.

By [K9], Proposition 3.5, the map \( \mathcal{A} : QT_{d,d'}^{emb}(Y; c\Theta; c\Theta') \to QT_{d,d'}^{imm}(Y; c\Theta; c\Theta') \) (see formula (3.32) in [K9]) is injective. By the argument above, \( \Delta \) is injective. Chasing the obvious square diagram, formed by the sources and targets of the maps \( \mathcal{A}, \mathcal{A}^\bullet \), we conclude that \( \mathcal{A}^\bullet \) is injective as well.

Combining Lemma 3.1 with Theorem 3.2 from [K9], we get the following results.

**Theorem 3.1.** We fix even natural numbers \( d' \geq d \), \( d' \equiv d \mod 2 \), and closed subposets \( \Theta' \subset \Theta \subset \Omega_{(d)} \) such that \( (\emptyset) \notin \Theta \). Let \( Y \) be a smooth compact \( n \)-manifold.

Then any submersion \( \alpha : X \to \mathbb{R} \times Y \) as in (3.1) gives rise to a map \( \Psi(\alpha) : (Y, \partial Y) \to (P_{d'}^{c\Theta}, P_{d'}^{(0)}) \). Moreover, \( (d,d'; c\Theta, c\Theta') \)-quasitopic submersions/embeddings \( \alpha_0 : X_0 \to \mathbb{R} \times Y \) and \( \alpha_1 : X_1 \to \mathbb{R} \times Y \) produce homotopic maps \( \Psi(\alpha_0) \) and \( \Psi(\alpha_1) \).

\(^1\text{not in a canonical fashion}\)
In this way, we get a map
\[ \Psi_{d,d'}^{\text{sub/emb}} : QT_{d,d'}(\text{emb}) \to [(Y, \partial Y), \epsilon_{d,d'} : (\mathcal{P}^{c\Theta}_d, \mathcal{P}^{(0)}_d) \to (\mathcal{P}^{c\Theta'}_{d'}, \mathcal{P}^{(0)}_{d'})], \]

Conversely, the homotopy class of any continuous map \( G : (Y, \partial Y) \to (\mathcal{P}^{c\Theta}_d, \mathcal{P}^{(0)}_d) \) is realized by a smooth regular embedding \( \alpha : X \hookrightarrow \mathbb{R} \times Y \) which satisfies (3.1); that is, \( G = \Psi_{d,d'}^{\text{emb}}(\alpha) \).

Moreover, \( \Psi_{d,d'}^{\text{emb}} \) is a bijection, and \( \Psi_{d,d'}^{\text{sub}} \) is a surjection, admitting a right inverse. ♦

4. Convex envelopes of traversing flows, their quasitopies, & characteristic classes

4.1. Traversing, generic, and convex vector fields. Morse stratifications. Spaces of convex traversing vector fields.

**Definition 4.1.** A vector field \( v \neq 0 \) on a compact smooth manifold \( X \) is called **traversing**, if each \( v \)-trajectory is homeomorphic either to a closed interval, or to a point. ♦

Let \( v \) be a traversing and boundary generic (see [K1], [K2], and Definition 4.4 below) vector field on a compact smooth \((n+1)\)-manifold \( X \) with boundary. As we will see soon, every trajectory \( \gamma \) of such a vector field \( v \) generates its **tangency divisor** \( D_{\gamma} \), an ordered sequence of points in \( \gamma \), together with their multiplicities (natural numbers).

We try to “go around” the fundamental discontinuity of the map \( x \to \gamma_x \to D_{\gamma_x} \), where \( \gamma_x \) stands for the \( v \)-trajectory through \( x \in X \). This requires “to envelop” the pair \((X, v)\) in a **convex envelop/pseudo-envelop** \((\hat{X}, \hat{v})\) (see Definition 4.8). The convex pseudo-envelopes, when available, will greatly simplify our analysis of traversing flows. In the spirit of Section 3, we will apply our results about immersions and submersions (against the background of product 1-foliations) from [K9] to the new environment of convex envelopes of traversing flows.

Following [Mo], for a generic vector field \( v \) on a smooth compact \((n+1)\)-dimensional manifold \( X \), such that \( v \neq 0 \) along \( \partial X \), let us describe an important **Morse stratification** \( \{\partial^j_{\lambda X}(v)\}_{j \in [1, \dim X]} \) of the boundary \( \partial X \). The stratum \( \partial_j X =_{\text{def}} \partial_j X(v) \) has the following description (see [K1]) in terms of an auxiliary function \( z : \hat{X} \to \mathbb{R} \) that satisfies the three properties:

\[
(4.1)
\]

- 0 is a regular value of \( z \),
- \( z^{-1}(0) = \partial X \), and
- \( z^{-1}((-\infty, 0]) = X \).

In terms of \( z \), the locus \( \partial_j X =_{\text{def}} \partial_j X(v) \) is defined by the equations:
\[
\{ z = 0, \mathcal{L}_v z = 0, \ldots, \mathcal{L}_v^{(j-1)} z = 0 \},
\]

where \( \mathcal{L}_v^{(k)} \) stands for the \( k \)-th iteration of the Lie derivative operator \( \mathcal{L}_v \) in the direction of \( v \) (see [K2]). The pure stratum \( \partial_j X^0 \subset \partial_j X \) is defined by the additional constraint
$L^{(j)}_i z \neq 0$. The locus $\partial_j X$ is the union of two loci: (1) $\partial^+_j X$, defined by the constraint $L^{(j)}_i z \geq 0$, and (2) $\partial^-_j X$, defined by the constraint $L^{(j)}_i z \leq 0$. The two loci, $\partial^+_j X$ and $\partial^-_j X$, share a common boundary $\partial_{j+1} X$.

For a generic $v$, all the strata $\partial_j X$ are smooth $(n + 1 - j)$-manifolds. The requirement of $v$ being generic with respect to $\partial X$ may be expressed as the property of the $j$-form

$$ (4.2) \quad dz \wedge d(L_v z) \wedge \ldots \wedge d(L^{(j-1)}_v z) $$

being a nonzero section of the bundle $\bigwedge^j T_X$ along the locus $\partial_j X$ for all $j \in [1, n + 1]$. If $v$ on $X$ is generic to $\partial X$, then each point $x \in \partial X$ belongs to a unique minimal stratum $\partial_j X \subset \partial X$ with a maximal $j = j(x) \leq n + 1$. In the generic case, at each $b \in \partial X$, a flag

$$ \text{Flag}_b(v) = \text{def}\{T_b(\partial X) = F^n \supset F^{n-1} \supset \ldots \supset F^{n-j(b)+1}\} $$

is generated by the tangent spaces at $b$ to all the Morse strata $\{\partial_j X\}_{j \leq j(b)}$ that contain $b$.

Let $\hat{v}$ be a traversing vector field on a compact smooth $(n + 1)$-dimensional manifold $\hat{X}$ with boundary. Consider a submersion $\alpha : X \to \text{int}(\hat{X})$, $\dim X = \dim \hat{X}$, such that the self-intersections of $\alpha(\partial X)$ mutually transversal. Let $v = \alpha^1(\hat{v})$ be the transfer of $\hat{v}$ to $X$.

For general submersions $\alpha$, which are not necessarily embeddings, the situation is more complex: not only one gets multiple self-intersections $\{\Sigma_k\}_{k \in [2, n+1]}$ of various branches of $\alpha(\partial X)$, but such self-intersections may be tangent to the $\hat{v}$-flow in a variety of ways that produce similar Morse-type stratifications of the loci $\Sigma_k$, $k \geq 2$, as well. Prior to Theorem 4.1 we will revisit this complication.

We associate several flags $\{\text{Flag}_b(v)\}_{b \in \alpha^{-1}(a)}$ with each point $a \in \alpha(\partial X)$. Let $\alpha^\partial_a$ denote the differential of the immersion $\alpha^\partial : \partial X \to \hat{X}$.

**Definition 4.2.** We say that several vector subspaces $\{V_i \subset W\}_i$ of a given vector space $W$ are in general position, if the obvious map $W \to \bigoplus_i (W/V_i)$ is onto. Note that this definition allows for any numbers of $V_i$'s to coincide with the ambient $W$.

We say that the flags $\{(\alpha^\partial_a)^{\|\text{Flag}_b(v)\|}_{b \in \alpha^{-1}(a)}$ are in general position in the ambient space $T_a \hat{X}$, if the $\alpha^\partial$-images of the minimal strata $\{F^{n-j(b)+1}(v)\}_{b \in \alpha^{-1}(a)}$ of the flags $\{\text{Flag}_b(v)\}_{b \in \alpha^{-1}(a)}$ are in general position in $T_a \hat{X}$. $\diamondsuit$

**Example 4.1.** Consider the case $n = 2$, depicted in Fig[2].

If $\#(\alpha^{\partial^{-1}(a)} = 1$, then each flag $\{\alpha^\partial_a\}(F^2 \supset F^1 \supset F^0)$, or $\{\alpha^\partial_a\}(F^2 \supset F^1)$, or $\{\alpha^\partial_a\}(F^2)$ is in general position at $a$.

If $\#(\alpha^{\partial^{-1}(a)} = 2$, a pair of flags is in general position, if and only if, it is of the form $\{\alpha^\partial_a\}(F^2 \supset F^1)$ and $\{\alpha^\partial_a\}(G^2)$, or of the form $\{\alpha^\partial_a\}(F^2)$ and $\{\alpha^\partial_a\}(G^2)$.

If $\#(\alpha^{\partial^{-1}(a)} = 3$, a triple of flags is in general position only if the pair is of the form $\{\alpha^\partial_a\}(F^2)$, $\{\alpha^\partial_a\}(G^2)$, $\{\alpha^\partial_a\}(H^2)$. The rest of combinations fail to be generic. $\diamondsuit$

**Definition 4.3.** Let $X, \hat{X}$ be smooth compact $(n + 1)$-manifolds with boundary and $\hat{v}$ a traversing vector field on $\hat{X}$. We assume that a submersion $\alpha : X \to \hat{X}$ has the following
properties: for each point \( a \in \alpha(\partial X) \), there exist a natural number \( k = k(a) \leq n + 1 \), an open neighborhood \( U_a \) of \( a \) in \( \hat{X} \), and smooth functions \( \{ z_1, \ldots, z_k : U_a \rightarrow \mathbb{R} \} \) such that:

1. \( 0 \) is a regular value for each \( z_i \),
2. in \( U_a \), the locus \( \alpha(\partial X) \) is given by the equation \( \{ z_1 \cdot \ldots \cdot z_k = 0 \} \),
3. the differential \( k \)-form \( dz_1 \wedge \ldots \wedge dz_k \mid_{U_a} \neq 0 \).

Let \( \hat{\gamma}_a \) denote the \( \hat{v} \)-trajectory through \( a \).

We say that a point \( a \in \alpha(\partial X) \) has a multiplicity \( j = j(a) \) with respect to \( \hat{v} \), if the jet \( \text{jet}^{j-1}_a(\{z_1 \cdot \ldots \cdot z_k \mid_{\hat{\gamma}_a}\}) = 0 \), but \( \text{jet}^j_a(\{z_1 \cdot \ldots \cdot z_k \mid_{\hat{\gamma}_a}\}) \neq 0 \).

\[ \Diamond \]

**Definition 4.4.** Let \( X, \hat{X} \) be smooth compact \((n + 1)\)-manifolds with boundary and \( \hat{v} \) a traversing vector field on \( \hat{X} \). Let \( U \) be an open \( \hat{v} \)-flow adjusted neighborhood of \( a \) in \( \hat{X} \). For each point \( a \in \text{int}(\hat{X}) \), consider a smooth transversal section \( S \) of the \( \hat{v} \)-flow at \( a \) and the flow-generated local projections \( \pi : U \rightarrow S \).

We say that a submersion \( \alpha : X \rightarrow \text{int}(\hat{X}) \) is locally generic relative to \( \hat{v} \) if, for each point \( a \in \alpha(\partial X) \),

- the images of the flags \( \{\text{Flag}_{b}(v)\}_{b \in \alpha^{-1}(a)} \), under the differentials \( (\alpha^\partial)_* \), are in general position in \( T_a(\hat{X}) \),
- the images of the flags \( \{\text{Flag}_{b}(v)\}_{b \in \alpha^{-1}(a)} \), under the differentials \( (\pi \circ \alpha^\partial)_* \), are in general position in the tangent space \( T_aS \).

\[ \Diamond \]

One may compare the next definition, which utilizes the notion of convexity, with Definition 4.6, introducing the more general notion of \( k \)-convexity.

**Definition 4.5.** Let \( \hat{X} \) be a compact connected smooth manifold with boundary, equipped with a vector field \( \hat{v} \). We say that the pair \((\hat{X}, \hat{v})\) is convex if

\[ \text{Thus } \alpha^\partial \text{ is } k\text{-normal in the sense of Definition 3.3 from [K9].} \]

\[ \text{equivalently, of their minimal strata} \]

---

**Figure 2.** Six locally generic configurations of \( \alpha^\partial(\partial X) \) in 3D. The numbers 1, 2, 3 reflect the local multiplicity of the marked point \( a \) on the trajectory \( \hat{\gamma}_a \) through it.
Lemma 4.1. Any compact connected manifold \( \tilde{X} \) with boundary has a traversing vector field \( [K1] \). However, not any compact manifold with boundary admits a traversing convex vector field! For example, consider any surface \( \tilde{X} \), obtained from a closed oriented connected surface, different from the 2-sphere, by removing an open disk. Such an \( \tilde{X} \) does not admit a traversing convex vector field \( [K5] \).

By \([K1]\), Lemma 4.1, any traversing vector field \( \hat{v} \), admits a Lyapunov function.

If \( Y \) is a closed manifold, then any vector field \( \hat{v} \neq 0 \) that is tangent to the fibers of the obvious projection \( [0, 1] \times Y \to Y \) is evidently convex with respect to \( \partial[0, 1] \times Y \). The obvious function \( f : [0, 1] \times Y \to [0, 1] \) has the desired Lyapunov property \( d\hat{f}(\hat{v}) > 0 \). \( \diamond \)

Example 4.2. Any non-trapping Riemannian metric \( g \) on a compact smooth manifold \( M \) with a convex boundary \( \partial M \) produces the geodesic vector field \( \hat{v}^g \) on the unit spherical bundle \( SM \to M \), which is traversing and convex with respect to \( \partial(SM) \) \([K6]\). The very existence of such a metric \( g \) puts severe restrictions on the topological nature of \( M \). \( \diamond \)

Let \( \text{conv}(\tilde{X}) \) denote the space of traversing vector fields \( \hat{v} \) on \( \tilde{X} \) such that \( (\tilde{X}, \hat{v}) \) is a convex pair, as in Definition \( [4.5] \). The space \( \text{conv}(\tilde{X}) \) is considered in the \( C^{\infty} \)-topology.

Lemma 4.1. For any \( \hat{v}_0, \hat{v}_1 \in \text{conv}(\tilde{X}) \) that belong to the same path-connected component of the space \( \text{conv}(\tilde{X}) \), there exists a smooth diffeomorphism \( \phi : \tilde{X} \to \tilde{X} \) such that \( \phi \) maps \( \hat{v}_0 \)-trajectories to \( \hat{v}_1 \)-trajectories, while preserving their orientations.

If, for a pair \( \hat{v}_0, \hat{v}_1 \in \text{conv}(\tilde{X}) \), there exists a smooth isotopy \( \{\psi^t : \partial\tilde{X} \to \partial\tilde{X} \}_{t \in [0, 1]} \) such that \( \psi^1(\partial_1^+ \tilde{X}(\hat{v}_0)) = \partial_1^+ \tilde{X}(\hat{v}_1) \), then there exists a smooth diffeomorphism \( \phi : \tilde{X} \to \tilde{X} \), an extension of \( \psi^1 \), that maps \( \hat{v}_0 \)-trajectories to \( \hat{v}_1 \)-trajectories, while preserving their orientations.

Proof. If \( \hat{v} \in \text{conv}(\tilde{X}) \) is a convex boundary generic vector field, then by Theorem 6.6 from \([K7]\), the stratification \( \tilde{X} \supset \partial_1^+ \tilde{X}(\hat{v}) \supset \partial_2^+ \tilde{X}(\hat{v}) \) is stable, up to an isotopy of \( \tilde{X} \), under sufficiently small perturbations of \( \hat{v} \). As a result, within a path connected component of \( \hat{v} \) in \( \text{conv}(\tilde{X}) \), the smooth topological type of this Morse stratification is stable via an isotopy. In particular, \( \partial_1^+ \tilde{X}(\hat{v}_0) \) and \( \partial_1^+ \tilde{X}(\hat{v}_1) \) are isotopic in \( \tilde{X} \), provided the \( \hat{v}_0 \) and \( \hat{v}_1 \) are connected by a path in \( \text{conv}(\tilde{X}) \).

Let us denote by \( \{\tilde{\psi}^t : \tilde{X} \to \tilde{X} \}_{t \in [0, 1]} \) the isotopy that transforms the \( \hat{v}_0 \)-induced Morse stratification of \( \partial\tilde{X} \) to the \( \hat{v}_1 \)-induced Morse stratification of \( \partial\tilde{X} \). Let us compare the vector fields \( \tilde{\hat{v}}_0 = \text{def} (\tilde{\psi}^1)_* (\hat{v}_0) \) and \( \hat{v}_1 \). Both vector fields point inside of \( \tilde{X} \) exactly along \( \partial_1^+ \tilde{X}(\hat{v}_1) \). Since \( \tilde{\hat{v}}_0, \hat{v}_1 \) are traversing, they admit some Lyapunov functions \( f_0, f_1 : \tilde{X} \to \mathbb{R} \). Put \( \hat{f}_0 = \text{def} ((\tilde{\psi}^1)^{-1})^* (f_0) \). It serves as Lyapunov’s function for \( \hat{v}_0 \). For \( x \in \partial_1^+ \tilde{X}(\hat{v}_1) \), we denote by \( \gamma_x^{(0)} \) and \( \gamma_x^{(1)} \) the \( \hat{v}_0 \)- and \( \hat{v}_1 \)-trajectories through \( x \). Let \( \text{var}_0(x) \) stands for
the variation of the function \( \tilde{f}_0 \) along \( \gamma^{(0)}_x \) and \( \text{var}_1(x) \) for the variation of the function \( \tilde{f}_1 \) along \( \gamma^{(1)}_x \).

For \( y \in \gamma^{(0)}_x \), consider the unique point \( \tilde{\phi}(y) \in \gamma^{(1)}_x \) such that \( f_1(\tilde{\phi}(y))/\text{var}_1(x) = \tilde{f}_0(y)/\text{var}_0(x) \). Now the diffeomorphism \( \phi := \tilde{\phi} \circ \tilde{\psi} \) takes \( \tilde{v}_0 \)-trajectories to \( \tilde{v}_0 \)-trajectories, while preserving their orientations. Hence, we have shown that the isotopy class of \( \partial^+_1 \hat{X}(\tilde{v}) \) in \( \partial X \) determines the smooth topological type of a convex pair \((\hat{X}, L(\tilde{v}))\), where \( L(\tilde{v}) \) denotes the oriented 1-foliation, determined by a convex traversing \( \tilde{v} \).

**Lemma 4.2.** Let \( \hat{X} \) is a connected compact smooth manifold with boundary. If \( \#(\pi_0(\partial \hat{X})) \geq 3 \), then \( \text{conv}(\hat{X}) = \emptyset \).

**Proof.** For a convex \( \hat{v}, \partial^+_1 \hat{X}(\hat{v}) \) and \( \partial^-_1 \hat{X}(\hat{v}) \), each is a deformation retract of \( \hat{X} \). Therefore, each of these two loci must be connected. Thus each of the loci \( \partial^+_1 \hat{X}(\hat{v}), \partial^-_1 \hat{X}(\hat{v}) \) must be contained in some connected component of \( \partial \hat{X} \). When \( \#(\pi_0(\partial \hat{X})) \geq 3 \), this argument forces at least one component of the boundary to be free from both loci. However, the union of the two loci is the entire boundary. This contradiction proves the claim. \( \square \)

4.2. **Homology spheres and convex flows.** Let \( \text{HS}_{n-1} \) denote the monoid of smooth types of integral homology \((n-1)\)-spheres, being considered up to connected sums with smooth homotopy \((n-1)\)-spheres.

For \( n \neq 4 \), let \( \Theta_n \) denote the group of \( h \)-cobordism classes of smooth homotopy \( n \)-spheres. The operations in \( \text{HS}_{n-1} \) and in \( \Theta_n \) are the connected sums of spheres. We denote the order of the group \( \Theta_n \) by \( |\Theta_n| \).

**Proposition 4.1.** Any convex vector field \( \hat{v} \) on the standard ball \( D^{n+1} \) defines a smooth involution \( \tau_\hat{v} \) on \( S^n = \partial D^{n+1} \), whose fixed point set \( \partial^+_2 D^{n+1}(\hat{v}) \) is an integral homology \((n-1)\)-sphere.

For \( n \geq 6 \), any element of the set \( \text{HS}_{n-1} \) arises as the locus \( \partial^-_2 \hat{X}(\hat{v}) \) for a convex traversing vector field \( \hat{v} \) on a smooth compact contractible \((n+1)\)-dimensional manifold \( \hat{X} \), whose boundary \( \partial \hat{X} \) is a smooth homotopy sphere. Moreover, \( \partial \hat{X} \) admits a smooth involution \( \tau_\hat{v} \) such that \( (\partial \hat{X})^{\tau_\hat{v}} = \partial^+_2 \hat{X}(\hat{v}) \).

For \( n \geq 6 \), the multiple \( |\Theta_n| \cdot [\Sigma] \) of any given element \( [\Sigma] \in \text{HS}_{n-1} \) arises as the locus \( \partial^-_2 D^{n+1}(\hat{v}) \) for a convex traversing vector field \( \hat{v} \) on the ball \( D^{n+1} \). The sphere \( \partial D^{n+1} \) admits a smooth involution \( \tau_\hat{v} \) such that \( (\partial D^{n+1})^{\tau_\hat{v}} = \partial^+_2 D^{n+1}(\hat{v}) \).

**Proof.** Using a convex \( \hat{v} \)-flow, \( \partial^+_1 D^{n+1}(\hat{v}) \) is a deformation retract of \( D^{n+1} \) and thus a contractible manifold. By Poincaré duality, \( \partial^-_2 D^{n+1}(\hat{v}) \), the boundary of \( \partial^+_1 D^{n+1}(\hat{v}) \), is a homology sphere.

By [Ke], Theorem 3, for \( n-1 \geq 5 \), any smooth homology sphere \( \Sigma^{n-1} \), after a connected sum \( \Sigma^{n-1} \# \Sigma_H^{n-1} \) with a unique smooth homotopy \((n-1)\)-sphere \( \Sigma_H^{n-1} \), bounds a contractible smooth manifold \( W^n \).

Consider a smooth metric \( g \) on \( W^n \). We denote by \( d_g(x, \partial W^n) \) the smooth distance function to \( \partial W^n \) on a collar \( U \) of \( \partial W^n \) in \( W^n \). Let \( F : W^n \to \mathbb{R}_+ \) be a function that is
strictly positive and smooth in the interior of $W^n$ and coincides with the function $\tilde{F}(x) := \sqrt{d_g(x, \partial W^n)}$ in the collar $U$.

Consider a smooth manifold $\tilde{X}^{n+1} \subset \mathbb{R} \times W^n$, given by the inequality $\{(t, x) : |t| \leq F(x)\}$. It comes with the vector field $\tilde{v}$ that is tangent to the fibers of the obvious projection $p : \mathbb{R} \times W^n \to W^n$. Since $W^n$ is contractible, the boundary of $\tilde{X}^{n+1}$, the double of $W^n$, is a smooth homotopy $n$-sphere, and $\tilde{X}^{n+1}$ is a homotopy ball. The vector field $\tilde{v}$ on $\tilde{X}^{n+1}$ is convex and defines an involution $\tau_\tilde{v} : \partial \tilde{X}^{n+1} \to \partial \tilde{X}^{n+1}$, whose fixed point set $(\partial \tilde{X}^{n+1})^{\tau_\tilde{v}} = \partial_2 \tilde{X}^{n+1}(\tilde{v}) = \partial W = \Sigma^{n-1} \# \Sigma_H^{n-1}$. This validates the second claim.

The connected sum of $|\Theta_n|$ copies of $\partial \tilde{X}^{n+1}$ is a standard $n$-sphere. Consider the boundary connected sum $(\tilde{Y}^{n+1}, \tilde{v})$ of the $|\Theta_n|$ copies of the pair $(\tilde{X}^{n+1}, \tilde{v})$. Here the 1-handles $H = D^n \times [0, 1]$ are attached at pairs of points that belong to different pairs of $\partial W^n$'s so that $|\Theta_n|$ copies of $W^n$ are connected by the chain of 1-handles $D^{n-1}_+ \times [0, 1]$, where $D^{n-1}_+ \subset \partial D^n$ is a hemisphere. The fields $\tilde{v}$ in the different copies extend concavely across the 1-handles. Then $\partial \tilde{Y}^{n+1}$ is the standard sphere $S^n$ which bounds a contractible manifold $\tilde{Y}^{n+1}$. By the h-cobordism theorem (see [Mi]), applied to $\tilde{Y}^{n+1} \setminus D^{n+1}$, we conclude that $\tilde{Y}^{n+1}$ is the standard ball. Thus we managed to build a convex vector field $v^\#$ on $D^{n+1}$ whose locus $\partial_2 D^{n+1}(v^\#)$ is a homology sphere $\tilde{\Sigma}^{n-1}$, the $|\Theta_n|$-multiple of the given class $\Sigma^{n-1} \in \text{HS}_{n-1}$. In particular, $\partial D^{n+1}$ admits a smooth involution $\tau_{v^\#}$ whose fixed point set is $|\Theta_n| \cdot \Sigma^{n-1}$.

Example 4.3. Consider a free action of the icosahedral group $I_{120}$ on $S^3$. Then $I_{120}$ acts freely on $S^7 = \text{join}(S^3, S^3)$. Hence the orbit space $\Sigma^7 = \text{def} S^7 / I_{120}$ is a homology sphere. By Proposition 4.1, there is a convex traversing vector field $\tilde{v}$ on the ball $D^9$, such that its locus $\partial_2 D^9(\tilde{v})$ is a connected sum of 28 copies of $|\Sigma^7| \in \text{HS}_7$, and its locus $\partial_1 D^9(\tilde{v})$ is contractible. Note that $\pi_1(\partial_2 D^9(\tilde{v}))$ is a free product of 28 copies of $I_{120}$.

Corollary 4.1. For $n > 6$, $\pi_0(\text{conv}(D^{n+1}))$ admits a surjection onto a subgroup $G_{n-1}$ of $\text{HS}_{n-1}$ of index $|\Theta_n|$ at most.

In particular, here are a few “clean” surjections:

$$\pi_0(\text{conv}(D^7)) \to \text{HS}_5, \quad \pi_0(\text{conv}(D^{13})) \to \text{HS}_{11}, \quad \pi_0(\text{conv}(D^{62})) \to \text{HS}_{60}.$$ 

Proof. For $n - 1 \geq 5$, let $\Sigma^{n-1}$ be a given smooth homology sphere, and let $\Sigma_H^{n-1}$ be the unique homotopy sphere such that $\Sigma^{n-1} \# \Sigma_H^{n-1}$ bounds a smooth contractible manifold $K_6$. By Proposition 4.1, any convex traversing vector field $\tilde{v}$ produces a homology sphere $\partial_2 D^{n+1}(\tilde{v})$, and the $|\Theta_n|$-multiple of any $\Sigma^{n-1} \# \Sigma_H^{n-1}$ is produced this way. On the other hand, deforming $\tilde{v}$ within the space of boundary generic vector fields does not change the smooth isotopy type of the pair $\partial_1 D^{n+1}(\tilde{v}) \supset \partial_2 D^{n+1}(\tilde{v})$ (see Lemma 4.1 or [K1], [K2]). In particular, the smooth topological type of the pair is preserved along a path in the space $\text{conv}(D^{n+1})$. Therefore, $\pi_0(\text{conv}(D^{n+1}))$ admits a surjection onto a subgroup $G_{n-1}$ of $\text{HS}_{n-1}$ of index $|\Theta_n|$ at most.

The three examples of surjections in the corollary are based on the computations of $|\Theta_n|$ in [KeX], [WaX]: we just use some $n$'s for which $|\Theta_n| = 1$. 

\qed
4.3. Convex pseudo-envelopes of traversing flows. The next key lemma encapsulates a given convex traversing flow $(\hat{X}, \hat{v})$ into the obvious traversing flow $\hat{v}$ in a box $[0, 1] \times Y$ for an appropriate choice of a compact manifold $Y$, $\dim Y = \dim \hat{X}$. The construction that realizes the embedding $(\hat{X}, \hat{v}) \subset ([0, 1] \times Y, \hat{v})$ delivers a global “virtual section” $\{0\} \times Y$ of $\hat{v}$. In turn, this section enables us to apply the key Theorem 3.1 from [K9] to any convex traversing flow $(\hat{X}, \hat{v})$. Therefore, we will be able to transfer many results from [K9] and from Section 3 to the environment of convex envelopes and pseudo-envelops (see Definition 4.5) of traversing boundary generic vector fields.

**Lemma 4.3.** Let $\dim(\hat{X}) = n + 1$. If a pair $(\hat{X}, \hat{v})$ is convex, then there exists a compact smooth $n$-manifold $Y$ such that:

1. $\hat{X} \subset [0, 1] \times Y$,
2. $\hat{v}$ is tangent to the fibers of the projection $p : [0, 1] \times Y \to Y$,
3. the obvious function $h : [0, 1] \times Y \to [0, 1]$ has the property $\partial h(\hat{v}) > 0$,
4. with the help of $p$, the loci $\partial^+_X \hat{v}(\hat{v})$ and $\partial^-_X \hat{v}(\hat{v})$ each is homeomorphic to $Y$.

**Proof.** Since $v$ is a traversing field, it admits a Lyapunov function $f : \hat{X} \to \mathbb{R}$ so that $df(\hat{v}) > 0$ in $\hat{X}$. We add a collar $U$ to $\hat{X}$ along its boundary $\partial \hat{X}$ and denote $\hat{X} \cup_{\partial \hat{X}} U$ by $\hat{X}$. Then, we smoothly extend $\hat{v}$ and $\hat{f}$ in $\hat{X}$ and denote these extensions by $\hat{v}$ and $\hat{f}$. We adjust $U$ so that $\hat{v} \neq 0$ and $\hat{f}(\hat{v}) > 0$ there.

Let $F : \hat{X} \to \mathbb{R}$ be a smooth function such that $0$ is its regular value and $F^{-1}(0) = \partial \hat{X} = \hat{X}$. We add a collar $U$ to $\hat{X}$ along its boundary $\partial \hat{X}$ and denote $\hat{X} \cup_{\partial \hat{X}} U$ by $\hat{X}$. Then, we smoothly extend $\hat{v}$ and $\hat{f}$ in $\hat{X}$ and denote these extensions by $\hat{v}$ and $\hat{f}$. We adjust $U$ so that $\hat{v} \neq 0$ and $\hat{f}(\hat{v}) > 0$ there.

By definition, $\partial^+_X \hat{v}(\hat{v}) := \{x \in \partial \hat{X} | F(x) = 0, \text{ and } (\mathcal{L}_{\hat{v}} F)(x) = 0\}$. The convexty of $\hat{v}$ in $\hat{X}$ means that $\partial^+_X \hat{v}(\hat{v}) = \emptyset$, thus $\partial^+_X \hat{v}(\hat{v}) = \partial^-_X \hat{v}(\hat{v})$. By Morin’s Theorem [Mor] (see [K2] for details), in the vicinity of each point $x \in \partial^-_X \hat{v}(\hat{v})$ there is a system of smooth coordinates $(u, w, y_1, \ldots, y_{n-1})$ in which $F((u, w, y_1, \ldots, y_{n-1})) = u^2 + w$, so that $\partial \hat{X}$ is given by the equation $u^2 + w = 0$, $\hat{X}$ by the inequality $u^2 + w \leq 0$, and each $\hat{v}$-trajectory is produced by freezing the coordinates $w$ and $y_1, \ldots, y_{n-1}$.

Since $df(\hat{v}) > 0$ in $\hat{X}(F, \epsilon)$, the field $\hat{v}$ is traversing in $\hat{X}(F, \epsilon)$. Hence each $\hat{v}$-trajectory $\gamma \subset \hat{X}(F, \epsilon)$ is either transversal to $\partial \hat{X}$ at a pair of points, or it is simply tangent to $\partial \hat{X}$ at a singleton, or does not intersect $\hat{X}$.

Consider the set $Z \subset \hat{X}(F, \epsilon)$ of $\hat{v}$-trajectories through the points of $\hat{X}$ (equivalently, the set of $\hat{v}$-trajectories through $\partial^+_X \hat{v}(\hat{v})$). Every such trajectory $\gamma$ is a closed oriented segment $[a(\gamma), b(\gamma)]$, where $a(\gamma) \neq b(\gamma) \in \partial Z \subset \partial \hat{X}(F, \epsilon)$. Thus, the variation $\text{var}_\gamma(\hat{f})$ of the function $\hat{f}$ along $\gamma$ is strictly positive. Using compactness of $\hat{X}$, we get that $\sup_{\gamma \subset Z} \{\text{var}_\gamma(\hat{f})\} > 0$.

Let $Y := \{a(\gamma) \in \partial Z \cup b(\gamma) \subset \partial Z\}$. Using the local model $\{u^2 + w \leq \epsilon\}$, we see that $\hat{v}$ is transversal to $Y$ for all sufficiently small $\epsilon > 0$.

Now, let us consider a new function on $\gamma$:

$$h_\gamma(x) := \frac{\text{var}_{[a(\gamma), b(\gamma)]}(\hat{f})}{\text{var}_{[a(\gamma), b(\gamma)]}(\hat{f})}.$$
The function \( h : Z \to \mathbb{R} \), defined as a collection of functions \( \{ h_{\gamma} : \gamma \to \mathbb{R} \}_{\gamma \in Z} \), is evidently a new Lyapunov function for \( \tilde{v} \) on \( Z \). It is smooth thanks to the transversality of \( \tilde{v} \) to \( Y \) and the smooth dependence of solutions of ODEs on initial data. In fact, \( h \) gives the product structure \([0,1] \times Y \to Z\). Indeed, any point \( x \in Z \) is determined by the unique trajectory \( \gamma_x \) through \( x \) and by the value at \( x \) of the Lyapunov function \( h \) of the \( \tilde{v} \)-flow.

Finally, the \(( -\tilde{v} )\)-flow defines a smooth map \( p : \partial^+ \tilde{X}(\tilde{v}) \to Y \) which is a homeomorphism. In fact, \( \partial^+ \tilde{X}(\tilde{v}) \) is diffeomorphic to \( Y \) by a small perturbation of \( p \).

Let \( \alpha : X \to \tilde{X} \) be a smooth submersion of a compact smooth manifold \( X \) with boundary into the interior of a compact connected smooth manifold \( \tilde{X} \) of the same dimension, where \( \partial \tilde{X} \neq \emptyset \). Assume that \( \tilde{X} \) is equipped with a traversing vector field \( \tilde{v} \) so that the pair \( (\tilde{X}, \tilde{v}) \) is convex in the sense of Definition 4.4. For each \( \hat{\alpha} \) of Definition 4.4. Let a regular immersion \( \alpha^* : \tilde{X} \to X \) be the tangent space to the minimal stratum \( \partial \alpha^+(\tilde{v}) \). Such \( \alpha^* \) is \( k \)-convex if \( \partial^+ k X(\alpha^+(\tilde{v})) = \emptyset \), \( k \)-concave if \( \partial^- k X(\alpha^+(\tilde{v})) = \emptyset \), and \( k \)-flat if \( \partial k X(\alpha^+(\tilde{v})) = \emptyset \).

Let a regular immersion \( \alpha^\partial : \partial X \to \tilde{X} \) be locally generic relative to \( \tilde{v} \) in the sense of Definition 4.4. For each \( \tilde{v} \)-trajectory \( \hat{\gamma} \subset \tilde{X} \), we pick a point \( a \in \hat{\gamma} \cap \alpha(\partial X) \). Such \( a \) belongs to the intersection of \( k = k(a) \in [1, n + 1] \) local branches of \( \alpha^\partial(\partial X) \), where \( k(a) = \#(\alpha^\partial)^{\ominus 1}(a) \). In particular, as we remarked before, \( a \) belongs to a unique collection of the \( \alpha \)-images of the Morse strata \( \{ \alpha(\partial_{j(b)} X(v)) \}_{b \in \alpha^{-1}(a)} \) with the maximal possible indexes \( j(b) \) (recall that \( v := \alpha^+(\tilde{v}) \) is the \( \alpha \)-transfer of \( \tilde{v} \)). The tangent spaces of these strata are in general position at \( a \), thanks to \( \alpha^\partial \) being locally generic. Recall that this setting includes the cases where some or all the strata \( \{ \alpha(\partial_{j(b)} X(v)) \}_{b \in \alpha^{-1}(a)} \) are \( n \)-dimensional, i.e., \( j(b) = 1 \).

Let \( j(a) \deq \sum_{b \in \alpha^{-1}(a)} j(b) \geq k(a) \) and

\[
(4.3) \quad \partial_{j(a)}(\alpha(X))(\tilde{v}) \deq \bigcap_{b \in \alpha^{-1}(a)} \alpha(\partial_{j(b)} X(v)),
\]

the latter equality being understood as an identity of the two germs at \( a \) of the LHS and RHS loci. By Definition 4.4, the germ of \( \partial_{j(a)}(\alpha(X))(\tilde{v}) \) at \( a \) is a smooth submanifold of \( \tilde{X} \), transversal to the trajectory \( \hat{\gamma}_a \).

Let \( T_a \subset T_{\hat{\gamma}} \tilde{X} \) be the tangent space to the minimal stratum \( \partial_{j(a)}(\alpha(X)) \) at \( a \) (see (4.3)). With the help of the \( \tilde{v} \)-flow, the subspace \( T_a \subset T_{\hat{\gamma}} \tilde{X} \) spreads to form a \( (\dim(\tilde{X}) - j(a)) \)-dimensional subbundle \( T^\bullet_a \) of the tangent bundle \( T \tilde{X} \big|_{\hat{\gamma}} \) along the trajectory \( \hat{\gamma} \). We denote...
by $T_a^\bullet$ the image of $T_a^\star$ under the quotient map $T\hat{X}|_{\hat{\gamma}} \to T\hat{X}|_{\hat{\gamma}}/T\hat{\gamma}$, where $T\hat{\gamma}$ stands for the 1-bundle, tangent to $\hat{\gamma}$.

We introduce a slightly modified version of Definition 3.2 from [K2], a modification that applies to submersions $\alpha : X \to \hat{X}$.

**Definition 4.7.** Let $\hat{v}$ be a traversing vector field on a compact connected smooth manifold $\hat{X}$ with boundary. We say that a submersion $\alpha : X \to \text{int}(\hat{X})$ is traversally generic relative to $\hat{v}$, if:

- $\alpha^\partial$ is locally generic in the sense of Definition 4.4 with respect to $\hat{v}$,
- for each $\hat{v}$-trajectory $\hat{\gamma} \subset \hat{X}$, the subbundles $\{T_a^\bullet\}_{a \in \hat{\gamma} \cap \alpha(\partial X)}$ are in general position in the normal to $\hat{\gamma}$ (trivial) n-bundle $(T\hat{X}|_{\hat{\gamma}})/T\hat{\gamma}$.

**Example 4.4.** The patterns in Fig.2 may be stacked vertically along a trajectory $\hat{\gamma}$. To get a traversally generic pile in the vicinity of $\gamma$, we obey the following rules: (1) to any stack, we may add any number of configuration of type $a$ from Fig.1, as long as the prescribed parity of $m(\hat{\gamma}) \equiv 0 \mod 2$ is not violated; (2) no two configurations of multiplicity 3 (of the types $c, e, f$) reside on $\hat{\gamma}$; (3) at most two configurations of multiplicity 2 (of the types $b, d$) reside on $\hat{\gamma}$, moreover, the ($-\hat{v}$)-directed projections on the transversal section $S$ to $\hat{\gamma}$ of the fold loci (as in $d$) or/and of the simple self-intersections (as in $b$) must be transversal in $S$.

Let $\alpha$ be traversally generic relative to $\hat{v}$. Then, for any $\hat{v}$-trajectory $\hat{\gamma}$, by counting the dimensions of the bundles $\{T_a^\bullet\}_{a \in \hat{\gamma} \cap \alpha(\partial X)}$ and using that they are in general position in the $n$-dimensional bundle, normal to $\hat{\gamma}$, we get that the reduced multiplicity

\[(4.4) \quad m'(\hat{\gamma}) \overset{\text{def}}{=} \sum_{a \in \hat{\gamma} \cap \alpha(\partial X)} (j(a) - 1) \leq n.\]

For a traversally generic $\alpha$ and a trajectory $\hat{\gamma}$, let $\gamma \subset X$ be any segment in $\alpha^{-1}(\hat{\gamma}) \subset X$ which is bounded by a pair of points in $\partial X$. Then, by Theorem 3.5 from [K2], the total multiplicity $m(\gamma) \leq 2n+2$. Although, for a traversally generic $\alpha$, there is no $\alpha$-independent constraint on the total multiplicity $m(\hat{\gamma}) \overset{\text{def}}{=} \sum_{x \in \hat{\gamma} \cap \alpha(\partial X)} j(x)$, there is an universal constraint on the cardinality of the subset $\hat{\gamma}_{[\geq 2]} \subset \hat{\gamma} \cap \alpha(\partial X)$, consisting of points $x$ whose multiplicity $m(x) \geq 2$. Namely, $\#(\hat{\gamma}_{[\geq 2]}) \leq n$, since no more than $n$ proper vector subspaces may be in general position in an ambient $n$-dimensional vector space.

Now we are ready to introduce the central notion of a convex pseudo-envelop.

**Definition 4.8.** Let $\hat{v}$ be a traversing vector field on a compact connected smooth manifold $\hat{X}$ with boundary. We assume that $(\hat{X}, \hat{v})$ is convex in the sense of Definition 4.5.

We call such a pair $(\hat{X}, \hat{v})$ a convex pseudo-envelop of a submersion $\alpha : X \to \text{int}(\hat{X})$, if $\alpha^\partial$ is locally generic relative to $\hat{v}$. We think of $X$ as being equipped with the pull-back vector field $v = \alpha^\partial(\hat{v})$, so that $(X, v)$ is “enveloped” by $(\hat{X}, \hat{v})$.

If $\alpha$ is a locally generic regular embedding, then we call $(\hat{X}, \hat{v})$ a convex envelop of $\alpha$.  

\end{document}
Remark 4.2. Not all traversally generic pairs \((X, v)\) admit convex envelops \((\hat{X}, \hat{v})\). For example, if \(X\) has two closed submanifolds (or singular cycles), \(M\) and \(N\), of complementary dimensions with a nonzero algebraic intersection number \(M \circ N\), then no convex envelop \((\hat{X}, \hat{v})\) of \((X, v)\) exists. Indeed, with the help of the \((-\hat{v})\)-flow, \(M\) is cobordant in \(\hat{X}\) to a cycle \(\hat{M}\) which resides in \(\partial^+\hat{X}(\hat{v})\). Similarly, with the help of the \(\hat{v}\)-flow, \(N\) is cobordant to \(\hat{X}\) to a cycle \(\hat{N}\) which resides in \(\partial^-\hat{X}(\hat{v})\). Note that if \(M \circ N \neq 0\) in \(X\), then evidently the same property holds in any \(\hat{X} \supset X\). Since \(M'\) and \(N'\) are disjoint cycles, their intersection number \(M' \circ N' = 0\), which contradicts to the assumption \(M \circ N \neq 0\). In particular, the non-triviality of \(\text{sign}(X)\), the Wall relative signature of \(X\), obstructs the existence of a convex envelop for any traversing \(v\) on \(X\). As Fig.1 testifies, this argument does not rule out the existence of a convex pseudo-envelop for \((X, v)\). In fact, Fig.1 shows that any compact oriented surface \(X\) can be enveloped.

We denote by \(\text{Sub}(X, \hat{X})\) the space of smooth submersions \(\alpha : X \to \text{int}(\hat{X})\).

When \(\alpha\) is an embedding, by Theorem 3.5 from [K2], in the space \(\mathcal{V}_{\text{trav}}(\hat{X}, \alpha)\) of traversing vector fields \(\hat{v}\) on \(\hat{X}\), there is an open and dense subset \(\mathcal{V}^\perp(\hat{X}, \alpha)\) such that \(\alpha\) is traversally generic (see Definition 4.7) with respect to \(\hat{v} \in \mathcal{V}^\perp(\hat{X}, \alpha)\).

On the other hand, the property of a submersion \(\alpha\) to be traversally generic with respect to a given traversing vector field \(\hat{v}\) (see Definition 4.7) is an “open” property in the \(C^\infty\)-topology on \(\text{Sub}(X, \hat{X})\), since it may be expressed in terms of mutual transversality of the relevant strata in the appropriate jet spaces.

We conjecture that the transversal generality of submersions \(\alpha\) with respect to a fixed convex pair \((\hat{X}, \hat{v})\) is also a “dense” property. Among other things, the next Theorem 4.1 shows that this conjecture is valid for the regular embeddings \(\alpha\) which admit convex envelops. However, for general submersions \(\alpha\), we are able only to prove that a somewhat weaker property “is dense”. That property is described in the second claim of Theorem 4.1. Speaking informally, we can insure by \(\alpha\)-perturbations the general positions of the singularities of the maps \(\alpha^\partial : \partial X \to \hat{X}\) and \(\pi \circ \alpha^\partial : \partial X \to \partial^+_\hat{X}(\hat{v})\) separately, but not mutually. Here the map \(\pi : \hat{X} \to \partial^+_\hat{X}(\hat{v})\) is defined by the \((-\hat{v})\)-directed convex flow.

**Theorem 4.1.** Let \((\hat{X}, \hat{v})\) be a convex pair, and \(X\) a compact smooth manifold with boundary, \(\dim(X) = \dim(\hat{X}) = n + 1\). Assume that \(\text{Sub}(X, \hat{X}) \neq \emptyset\).

- There is an open and dense subset \(\mathcal{O} \subset \text{Sub}(X, \hat{X})\) such that, for any \(\alpha \in \mathcal{O}\), the local branches of \(\alpha(\partial X)\) are in general position in \(T_a\hat{X}\) at every point \(a \in \alpha(\partial X)\).

- For any \(\alpha \in \mathcal{O}\) and for each \(\hat{v}\)-trajectory \(\hat{\gamma} \subset \hat{X}\), the \(\hat{v}\)-invariant subbundles \(\{T^\perp_a\}_{a \in \hat{\gamma} \cap \alpha(\partial X)}\), generated by the intersections

\[
\bigcap_{b \in (\alpha^\partial)^{-1}(a), j(b) \geq 2} \alpha_*[T_b(\partial_j(b)X(v))]
\]
of the tangent spaces to the Morse strata \( \{ \partial_j(b)X(v) \}_{b \in (\alpha|) - 1(a), j(b) \geq 2} \) are in general position in the normal to \( \gamma \) (trivial) \( n \)-bundle \( (T\hat{X}|\gamma)/T\gamma \). Here \( v \) is the pullback of \( v \) under \( \alpha \).

**Proof.** We assume that the openness of \( O \) in \( \text{Sub}(X, \hat{X}) \) is clear, due to the compactness of \( X \), and will present the arguments that validate the density of \( O \) in \( \text{Sub}(X, \hat{X}) \). We divide the proof into three steps, marked as (i), (ii), and (iii).

(i) By [LS], the set \( \mathcal{N} \subset C^\infty(\partial X, \hat{X}) \) of smooth maps \( \beta : \partial X \to \hat{X} \), such that \( \beta \) is a \( k \)-normal immersion in the sense of [LS] (see also Definition 3.3 from [K9]) for all \( k \leq n + 1 \), is open and dense.

First we aim to show that, for a given submersion \( \alpha \in \text{Sub}(X, \hat{X}) \), there is an open set \( O_\alpha \subset \text{Sub}(X, \hat{X}) \) such that \( \alpha \in \text{closure}(O_\alpha) \) and, for any \( \tilde{\alpha} \in O_\alpha \), the submersion \( \tilde{\alpha}^\partial \in \mathcal{N} \).

With this goal in mind, we choose an auxiliary metric \( g \) on \( \hat{X} \) such that: (1) the boundary \( \partial \hat{X} \) is convex in \( g \), and (2) there is \( \epsilon_0 > 0 \) such that any two points \( x, y \in \hat{X} \) that are less than \( \epsilon_0 \)-apart are connected by a single geodesic arc. Using the submersion \( \alpha \), we pull-back \( g \) to a Riemannian metric \( g^\partial_\alpha \) on \( X \).

For some \( \epsilon < \epsilon_0 \), the \( \epsilon \)-neighborhood \( C_\epsilon \subset X \) of \( \partial X \) in the metric \( g^\partial_\alpha \) has a product structure \( \psi : \partial X \times [0, \epsilon] \xrightarrow{\cong} C_\epsilon \), so that the curve \( \delta_x = \text{def} \psi(x \times [0, \epsilon]) \) is the unique geodesic in \( g^\partial_\alpha \), normal to \( \partial X \) at \( x \in \partial X \). Using the diffeomorphism \( \psi \), we construct a smooth diffeomotopy \( \{ \psi_t : X \to X \}_{t \in [0, 1]} \) (via the flow inward normal to \( \partial X \)) so that \( \psi_0 = \text{id}_X \) and \( \psi_1(X) = X \setminus \text{int}(C_\epsilon) \).

Using the convexity of \( \partial \hat{X} \) in \( g \) and the choice of \( \epsilon < \epsilon_0 \) we get the following claim: for any \( \beta \) that approximates \( \alpha^\partial \) and each \( x \in \partial X \), there exists the unique geodesic \( \delta_x \subset \hat{X} \) in the metric \( g \) that connects the points \( \beta(x) \) and \( \alpha \circ \psi_1(x) \).

We pick such an approximation \( \beta \in \mathcal{N} \) of \( \alpha^\partial \). Let a smooth map \( \alpha^\partial(\beta, \epsilon) : X \to \hat{X} \) be defined by the two properties: (1) \( \alpha^\partial(\beta, \epsilon)|_{X \setminus C_\epsilon} := \alpha \circ \psi_1 \), (2) for each \( x \in \partial X \), the diffeomorphism \( \alpha^\partial(\beta, \epsilon)| : \delta_x \to \delta_x \) is an isometry with respect to \( g^\partial_\alpha \) and \( g \) along the two geodesic arcs. By picking \( \epsilon \) small enough and \( \beta \in \mathcal{N} \) sufficiently \( C^\infty \)-close to \( \alpha^\partial \), we get that \( \alpha^\partial(\beta, \epsilon) \) is \( C^\infty \)-close to \( \alpha \). Therefore, we may assume that \( \alpha^\partial(\beta, \epsilon) \in \text{Sub}(X, \hat{X}) \) for an appropriate choice of \( \beta \) that approximates \( \alpha^\partial \). By its construction, \( (\alpha^\partial(\beta, \epsilon)) \) \( \mathcal{N} \).

(ii) For a given smooth map \( \beta : \partial X \to \text{int}(\partial^+X(v)) \), let \( \beta_* \) denote the differential of \( \beta \).

Let \( S^{(1)}(\beta) = \text{def} \{ x \in \partial X \mid \text{rk}(\beta_*) \leq n - 1 \} \).

If \( S^{(1)}(\beta) \) is a smooth manifold, then the locus

\[
S^{(1,1)}(\beta) = \text{def} \{ x \in S^{(1)}(\beta) \mid \text{rk}(\beta_*|_{S^{(1)}(\beta)}) \leq n - 2 \}
\]

is well-defined. Continuing this way, the filtration of \( \partial X \) by the loci \( \{ S_{[k]}(\beta) = \text{def} S^{(\omega)}(\beta) \}_{k} \), where \( k \leq n + 1 \) and \( \omega = (1,1,\ldots,1) \) is introduced. Applying Boardman’s Theorems 15.1-15.3, [Bo], the subset \( B \) of maps \( \beta \in C^\infty(\partial X, \partial^+X(v)) \), for which all the strata
\{S_{[k]} = \text{def } S_{[k]}(\beta)\} are smooth manifolds and all the maps
\[\{\beta : S_{[k]}^o = \text{def } S_{[k]} \setminus S_{[k+1]} \to \text{int}(\partial^+_1 X(\hat{v})))\}_k\]
are immersions, is open (\(\partial X\) is compact) and dense. Note that each \(x \in \partial X\) belongs to a unique pure stratum \(S_{[k]}^o\) (see \cite{GG}, Theorem 4.13), the subspace of \(B\), formed maps \(\beta\) for which all the spaces \(\{\beta_s(T_xS_{[k]}(x)))\}_{x \in (\pi \circ \beta)^{-1}(y)}\) are in general position in \(T_y \partial^+_1 X(\hat{v}))\) for all \(k(x) \geq 2\) and all \(y \in \text{int}(\partial^+_1 X(\hat{v}))\), form an open and dense subset \(\mathcal{B}_{\text{NC}} \subset B\) (see \cite{GG}, Theorem 5.2). Here “NC” abbreviates the condition known as “normal crossings”. Note that if \(k(x) = 1\), then \(\beta_s(T_xS_{[k]}(x)))\) coincides with \(T_{\beta(x)}\partial^+_1 X(\hat{v}))\).

We stress that, applying the previous arguments to the map \(\beta = \pi \circ \alpha^\beta\), the strata \(\bigcap_{\beta \in \alpha^{-1}(a)} \alpha(\partial^+_j X(v))\) that involve \(\alpha\)'s with \(j(b) = 1\) become “\(\beta\)-invisible”, thanks to the “erasing” action of \(\pi_\alpha\) on all \(n\)-dimensional spaces \(\alpha(T_a(\partial_1 X(v)))\). Because of this short comming, we cannot claim that the traversally generic \(\alpha\) to \(\hat{v}\) (see Definition 4.7) are dense in \(\text{Sub}(X, \hat{X})\).

(iii) Note that the map \(\pi : \hat{X} \to \text{int}(\partial^+_1 X(\hat{v}))\), defined by the \((-\hat{v})\)-flow, is smooth due to \(\hat{v}\) being convex; moreover, its restriction to \(\text{int}(\hat{X})\) is a submersion.

For a given \(\alpha \in \text{Sub}(X, \hat{X})\), we form the composition \(\alpha^\beta = \text{def } \pi \circ \alpha^\beta\).

By (i), we can approximate \(\alpha\) by a new submersion \(\alpha_1 = \alpha^\beta(\beta, \epsilon)\) such that \(\alpha_1^\beta \in N\).

By (ii), we can approximate \(\alpha_1^\beta = \text{def } \pi \circ \alpha_1^\beta\) by a smooth map \(\beta_1^\beta : \partial X \to \text{int}(\partial^+_1 X(\hat{v}))\) such that \(\beta_1^\beta \in \mathcal{B}_{\text{NC}}\).

Let us fix a Lyapunov function \(\hat{f} : \hat{X} \to \mathbb{R}\) for \(\hat{v}\). In the spirit of Lemma 4.3 the Lyapunov function \(\hat{f}\) for \(\hat{v}\) and the projection \(\pi : \hat{X} \to \partial^+_1 X(\hat{v})\) define global smooth “coordinates” in the interior of \(\hat{X}\). That is, each pair \((t, y)\), where \(y \in \partial^+_1 X(\hat{v})\) and \(t \in \hat{f}(\pi^{-1}(y)))\), determines a unique point \(x \in \hat{X}\) such that \(\hat{f}(x) = t\) and \(\pi(x) = y\). Let \(f_1 = \alpha_1^\beta(\hat{f})\).

For the map \(\beta_1^\beta \in \mathcal{B}_{\text{NC}}\), we define a smooth map \(\beta_1 = \text{def } \beta_1(\beta_1^\beta, \hat{f}) : \partial X \to \hat{X}\) by the formula \(\beta_1(x) = y\), where \(x \in \partial X\) and \(y\) is the unique point on the \(\hat{v}\)-trajectory \(\hat{y}\) through \(\beta_1^\beta(x) \in \partial^+_1 X(\hat{v})\) such that \(\hat{f}(y) = f_1(x)\). Note that by choosing \(\beta_1^\beta\) sufficiently close to \(\alpha_1^\beta\), we insure that \(\beta_1\) is sufficiently close to \(\alpha_1^\beta\). Thus we may assume that \(\beta_1 \in N\).

Recycling the argument that revolves around the construction of \(\alpha_1^\beta(\beta, \epsilon) : X \to \hat{X}\) in part (i) of the proof, we form the submersion \(\alpha_2 = \text{def } \alpha_1^\beta(\beta_1, \epsilon) : X \to \hat{X}\). By the construction of \(\alpha_2\), we have \(\alpha_2^\beta = \beta_1^\beta \in \mathcal{B}_{\text{NC}}\) and \(\beta_1 \in N\).

Therefore, we have shown that any given submersion \(\alpha\) admits an approximation by some \(\alpha_2 \in \text{Sub}(X, \hat{X})\) such that \(\alpha_2^\beta \in N\) and \(\alpha_2^\beta \in \mathcal{B}_{\text{NC}}\). These are exactly the two properties that describe the space \(\mathcal{O}\) in the theorem. \(\square\)

**Corollary 4.2.** Let \((\hat{X}, \hat{v})\) be a convex pair, and \(X\) a compact smooth manifold with boundary, \(\text{dim}(X) = \text{dim}(\hat{X}) = n + 1\).

The regular embeddings \(\alpha : X \subset \hat{X}\) that are traversally generic with respect to \(\hat{v}\) (see Definition 4.7) form an open and dense set in the space of all regular smooth embeddings.
Proof. Since, for a regular embedding \( \alpha : X \hookrightarrow \tilde{X} \), \( \alpha^\partial \) is an embedding, the first claim of Theorem 4.1 is vacuous, and the second claim insures that \( \alpha \) is traversally generic.

**Remark 4.3.** Consider the \( k \)-multiple self-intersection manifolds \( \{ \Sigma_k^{\alpha} \}_k \) of \( k \)-normal (see [LS]) immersions \( \alpha^\partial : \partial X \to \tilde{X} \). By definition, \( \Sigma_k^{\alpha} \) is a submanifold of the \( k \)-fold product \((\partial X)^k\), the preimage of the diagonal \( \Delta \subset (\tilde{X})^k \) under the transversal to it map \((\alpha^\partial)^k\). The projection \( p_1 \) of \( \Sigma_k^{\alpha} \) on the first factor \( \partial X \) of the product \((\partial X)^k\) is an immersion [LS]. By composing \( p_1 \) with \( \alpha^\partial \), we get an immersion of \( \alpha^\partial \circ p_1 : \Sigma_k^{\alpha} \to \tilde{X} \). By using the convex \((-\dot{v})\)-flow, we get a map \( \pi : \tilde{X} \to \partial_1^+ \tilde{X}(\dot{v}) \). Finally, we obtain a smooth composite map \( \pi \circ \alpha^\partial \circ p_1 : \Sigma_k^{\alpha} \to \partial_1^+ \tilde{X}(\dot{v}) \).

We notice that, under the hypotheses and notations of Theorem 4.1, if \( \dot{v} \) is tangent at \( a \) to the intersection \( \bigcap_{b \in (\alpha^\partial)^{-1}(a)} \alpha_*(\partial_{j(b)}X(v)) \), then, evidently, each local branch \( \alpha_*(\partial_{j(b)}X(v)) \), \( b \in (\alpha^\partial)^{-1}(a) \), is tangent to \( \dot{v} \) at \( a \). Therefore, \( j(b) \geq 2 \). In other words, for \( k \geq 2 \), the singular locus of the map \( \pi \circ \alpha^\partial \circ p_1 : \Sigma_k^{\alpha} \to \partial_1^+ \tilde{X}(\dot{v}) \) is always contained in the singular locus of the map \( \pi \circ \alpha^\partial \circ p_1 : (\partial X)^k \to \partial_1^+ \tilde{X}(\dot{v}) \).

4.4. Quasitopologies of convex envelopes and pseudo-envelops. Now, let us modify Definition 3.7 from [K9] and Definition 3.1 from this paper, so that they apply to convex pseudo-envelops of traversing flows (see Fig. 3). This modification is central to our efforts.

**Definition 4.9.** Fix natural even numbers \( d \leq d' \) and consider a closed subposets \( \Theta' \subset \Theta \) of the universal poset \( \Omega \) from Section 2, such that \( (\emptyset) \notin \Theta \).

Let \( \tilde{X} \) be a \((n+1)\)-dimensional compact manifold and \( \dot{v} \) a convex traversing vector field on it. Let \( \tilde{Z} = \{ \tilde{X} \times [0,1] \} \). We denote by \( \dot{v}^\ast \) the vector field on \( \tilde{Z} \) that is tangent to each slice \( \tilde{X} \times \{ t \} \), \( t \in [0,1] \), and is equal to \( \dot{v} \) there.

We say that two convex pseudo-envelops, \( \alpha_0 : X_0 \to \tilde{X} \) and \( \alpha_1 : X_1 \to \tilde{X} \), are \((d,d';\Theta,\Theta')\)-quasitopical in \( \tilde{X} \), if there exists a compact smooth orientable \((n+2)\)-manifold \( W \) whose boundary \( \partial W = (X_0 \coprod X_1) \coprod \{ \partial X_0 \coprod \partial X_1 \} \delta W \), and a smooth submersion \( A : W \to \tilde{Z} \) so that:
- \( A|_{X_0} = \alpha_0 \) and \( A|_{X_1} = \alpha_1 \);
- for each \( z \in \tilde{X} \times \partial [0,1] \), the total multiplicity \( m_A(\hat{\gamma}_z) \) of the \( \dot{v}^\ast \)-trajectory \( \hat{\gamma}_z \) through \( z \), relatively to \( A(\partial X_0 \coprod \partial X_1) \), satisfies the constraints \( m_A(\hat{\gamma}_z) \leq d \), \( m_A(\hat{\gamma}_z) \equiv 0 \) mod 2, and the combinatorial tangency pattern \( \omega_A(\hat{\gamma}_z) \) of \( \hat{\gamma}_z \) with respect to \( A(\partial X_0 \coprod \partial X_1) \) does not belong to \( \Theta \);
- for each \( z \in \tilde{Z} \), the total multiplicity \( m_A(\hat{\gamma}_z) \) of \( \hat{\gamma}_z \) with respect to \( A(\partial W) \) satisfies the constraints \( m_A(\hat{\gamma}_z) \leq d' \), \( m_A(\hat{\gamma}_z) \equiv 0 \) mod 2, and the combinatorial tangency pattern \( \omega_A(\hat{\gamma}_z) \) of \( \hat{\gamma}_z \) with respect to \( A(\partial W) \) does not belong to \( \Theta' \);

We denote by \( QT_{d,d'}^{\mathrm{sub}}(\tilde{X}, \dot{v}; \Theta; \Theta') \) the set of quasitopy classes of such convex pseudo-envelops \( \alpha : (X, \alpha^\partial(\dot{v})) \to (\tilde{X}, \dot{v}) \).
If we insist that $\alpha_0, \alpha_1,$ and $A$ are embeddings, then we get $\mathcal{QT}_{d,d'}^{\text{emb}}(\hat{X}, \hat{v}; c\Theta; c\Theta'),$ the set of quasitopy classes of convex envelops.

It is easy to check that the quasitopy of convex pseudo-envelops (convex envelops) $\alpha : (X, \alpha^1(\hat{v})) \to (\hat{X}, \hat{v})$ is an equivalence relation.

We are in position to state one of the main results of this paper.

**Theorem 4.2.** Let $\Theta' \subset \Theta \subset \Omega$ be closed subposets that do not contain the element $(\emptyset)$. Assume that $d' \geq d$ and $d' \equiv d \equiv 0 \mod 2$. Let $\hat{X}$ be a smooth compact connected $(n + 1)$-dimensional manifold, equipped with a convex traversing vector field $\hat{v}$.

Then, under the notations of Definition 3.2, there is a canonical bijection

$$\Phi^{\text{emb}} : \mathcal{QT}_{d,d'}^{\text{emb}}(\hat{X}, \hat{v}; c\Theta; c\Theta') \xrightarrow{\cong} \left[\left((\partial_1^+ \hat{X}(\hat{v}), \partial_2^+ \hat{X}(\hat{v})), \epsilon_{d,d'} : (P_{d}^{c\Theta}, pt) \to (P_{d'}^{c\Theta'}, pt')\right)\right]$$

and a canonical surjection

$$\Phi^{\text{sub}} : \mathcal{QT}_{d,d'}^{\text{sub}}(\hat{X}, \hat{v}; c\Theta; c\Theta') \xrightarrow{\text{epi}} \left[\left((\partial_1^+ \hat{X}(\hat{v}), \partial_2^+ \hat{X}(\hat{v})), \epsilon_{d,d'} : (P_{d}^{c\Theta}, pt) \to (P_{d'}^{c\Theta'}, pt')\right)\right].$$

The map $\Phi^{\text{sub}}$ admits a right inverse.
Proof. By Lemma 4.3, any convex pseudo-envelope $\alpha : X \to \tilde{X}$ may be incapsulated into a convex pseudo-envelope $\tilde{\alpha} : \tilde{X} \to X \subset [0, 1] \times Y$ of the product type (we called a capsule $[0, 1] \times Y$ "a box").

Conversely, any submersion $\tilde{\alpha} : \tilde{X} \to [0, 1] \times Y$, where $Y$ is homeomorphic to $\partial_1^+\tilde{X}(v)$, and its boundary $\partial Y$ to $\partial_2^-\tilde{X}(v)$. Similarly, the cobordisms $A$ between pairs of quasitopies $\alpha_0, \alpha_1$ can be incapsulated in boxes of the form $([0, 1] \times Y) \times [0, 1]$. Therefore, all the results from Section 3 in [K9] apply to $\alpha^\partial$; in particular, the pivotal Theorem 3.2 from [K9] applies. With its help, the maps $\Phi^{\text{sub}}$ from (4.6) and $\Phi^{\text{emb}}$ from (4.5) are generated as compositions of maps $\pi(v) : (\partial_1^+\tilde{X}(v), \partial_2^-\tilde{X}(v)) \to (Y, \partial Y)$ with the maps $\{\Phi^\partial : Y, \partial Y \to (P_{d, \partial}^{\Theta}, P_{d, \partial}^{(0)})\}$ from Theorem 3.2 in [K9]; the latter map being generated by the locus $\tilde{\alpha}(\partial X) \subset \mathbb{R} \times Y$. Note that the cell $P_{d, \partial}^{(0)} \subset P_d$ contracts to a singleton $pt \in P_{d, \partial}^{(0)}$.

Similarly, for any quasitopy $A : W \to \tilde{X} \times [0, 1]$ as in Definition 4.9, the product $Y \times [0, 1]$ that encapsulates $\tilde{X} \times [0, 1]$ is mapped to $P_{d, \partial}^{\Theta}$ with the help of $A(\delta W)$, while $\partial Y \times [0, 1]$ are mapped to the cell $P_{d, \partial}^{(0)}$, which contracts to the singleton $\epsilon_{d, \partial}(pt) \in P_{d, \partial}^{(0)}$.

As in Theorem 3.2 from [K9], and by similar transversality arguments, $\Phi^{\text{emb}}$ is a 1-to-1 map. Here we need to use Lemma 3.4 from [K9] to conclude that, under a $\partial_2^\delta_{d, \partial}$-regular map (see Definition 3.4 from [K9]), the preimage of the hypersurface $\partial_2^\delta_{d, \partial} \subset \mathbb{R} \times P_d$ (see (2.2)) bounds a compact manifold $X$ in $\tilde{X}$, provided $d \equiv 0 \mod 2$. Again, as in Proposition 3.5 from [K9], the bijective map in (4.5) helps to prove that the map in (4.6) is a split surjective one.

Corollary 4.3. The sets $\mathcal{Q}^{\text{emb}}_{d, \partial}(\tilde{X}, \tilde{v}; c\Theta; c\Theta')$ are trivial when $\dim(\tilde{X}) < \max_{\omega \in \Theta}\{|\omega|\}$.

Proof. Since $\text{codim}(P_{d, \partial}^{\Theta}, P_d) = \max_{\omega \in \Theta}\{|\omega|\}$ and $P_d$ is contractible, by Theorem 4.2, the claim follows by a general position argument, applied to homotopies of maps from the pair $(\partial_1^+\tilde{X}(v), \partial_2^-\tilde{X}(v))$ to the pair $(P_{d, \partial}^{\Theta}, pt)$.

Corollary 4.4. The sets $\mathcal{Q}^{\text{emb}}_{d, \partial}(\tilde{X}, \tilde{v}; c\Theta; c\Theta'), \mathcal{Q}^{\text{sub}}_{d, \partial}(\tilde{X}, \tilde{v}; c\Theta; c\Theta')$ are invariants of the path-connected component of the vector field $\tilde{v}$ in the space conv($\tilde{X}$) of convex traversing vector fields.

Proof. By their definitions, the sets $\mathcal{Q}^{\text{emb}}_{d, \partial}(\tilde{X}, \tilde{v}; c\Theta; c\Theta')$ and $\mathcal{Q}^{\text{sub}}_{d, \partial}(\tilde{X}, \tilde{v}; c\Theta; c\Theta')$ depend only of the smooth topological types of the oriented 1-dimensional foliations $\mathcal{L}(\tilde{v})$, and $\mathcal{L}^*(\tilde{v}^*)$ on $\tilde{X}$ and $\tilde{X} \times [0, 1]$, respectively. By Lemma 4.1, the smooth topological types of these foliations do not change along any path in the space conv($\tilde{X}$), that contains the point $\tilde{v}$.

Corollary 4.5. The set $\mathcal{Q}^{\text{emb}}_{d, \partial}(\tilde{X}, \tilde{v}; c\Theta; c\Theta')$ depends only on the homotopy type of the pair $(\partial_1^+\tilde{X}(\tilde{v}), \partial_2^-\tilde{X}(\tilde{v}))$.

Proof. The claim follows instantly from the bijection in formula (4.5).
The next theorem is our main result about the stability of $\mathcal{QT}^{\text{emb}}_{d,d+2}(\hat{X}, \hat{v}; c\Theta; c\Theta)$ as a function of $d$ in terms of the function $\eta_{\Theta}(d+2)$ from \cite{KSW2}.

**Theorem 4.3. (short stabilization: $\{d \Rightarrow d+2\}$)** Let $\Theta$ be a closed subposet of $\Omega$. If

\[ \dim(\hat{X}) < d + 2 - \psi_{\Theta}(d+2) \]

and $|\omega'| > 2$ for all $\omega \in \Theta_{(d+2)}$, then there exists a bijection

\[ \Phi^{\text{emb}} : \mathcal{QT}^{\text{emb}}_{d,d+2}(\hat{X}, \hat{v}; c\Theta; c\Theta) \xrightarrow{\approx} \left[ (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})), (\mathcal{P}_{d+2}^{c\Theta}, pt) \right] \]

**Proof.** Let $k = \text{def} d + 2 - \psi_{\Theta}(d+2)$. If $|\omega'| > 2$ for all $\omega \in \Theta_{(d+2)}$, then both $\mathcal{P}_{d+2}^{c\Theta}$ and $\mathcal{P}_{d+2}^{c\Theta}$ are simply-connected, since $\dim(\mathcal{P}_{d+2}^{c\Theta}, \mathcal{P}_{d+2}) > 2$ and $\dim(\mathcal{P}_{d}^{c\Theta}, \mathcal{P}_{d}) > 2$. By combining Theorem 2.2 with the Alexander duality as in \cite{Hurewicz}, we get an isomorphism $\epsilon : H_{j}(\mathcal{P}_{d+2}^{c\Theta}; \mathbb{Z}) \approx H_{j}(\mathcal{P}_{d+2}^{c\Theta}; \mathbb{Z})$ for all $j < k$. By the Whitehead Theorem (the inverse Hurewicz Theorem) (see Theorem (7.13) in \cite{Wh} or Theorem 10.1 in \cite{Hu}), the map $\epsilon : \mathcal{P}_{d+2}^{c\Theta} \to \mathcal{P}_{d+2}^{c\Theta}$ is $(k-1)$-connected. Thus, if $n = \dim(\partial_1^+ \hat{X}(\hat{v})) \leq k-1$, then, by the standard application of the obstruction theory, no map $\Phi : (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})) \to (\mathcal{P}_{d+2}^{c\Theta}, pt)$, which is not null-homotopic, becomes null-homotopic in $(\mathcal{P}_{d+2}^{c\Theta}, pt')$. Now the claim follows from Theorem 4.2  

Theorem 4.3 leads to the following straightforward, but important implication.

**Corollary 4.6. (long stabilization $d \Rightarrow \infty$)** Let $\Theta$ be a closed profinite (see Definition 2.1) poset such that $|\omega'| > 2$ for all $\omega \in \Theta$.

Then, given a convex pair $(\hat{X}, \hat{v})$, the quasiisotopy set $\mathcal{QT}^{\text{emb}}_{d,d}(\hat{X}, \hat{v}; c\Theta; c\Theta)$ stabilizes towards the set of homotopy classes $\left[ (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})) \right]$ for all sufficiently big $d$ relative to $\dim(\hat{X})$. 

\[ \Diamond \]

### 4.5. Group structure on the ball-shaped convex pseudo-envelops

The connected sum $\omega_1 \cup \omega_2$ of convex pseudo-envelops $\omega_1 : X_1 \to \hat{X}_1, \omega_2 : X_2 \to \hat{X}_2$ can be introduced in a fashion, similar to the operation $\cup$ in formulae (3.14) and (3.15) from \cite{K9} and formula (3.2) above.

Let $D^{n-1}$ denote the Southern hemisphere in $\partial D^n$. As with the connected sums of submersions from Section 3, there is an ambiguity about how to attach a 1-handle $D^1_+ \times [0,1]$ to $\partial_1^+ \hat{X}_1(\hat{v}_1) \bigsqcup \partial_1^+ \hat{X}_2(\hat{v}_2)$ and a 1-handle $H = D^n \times [0,1]$ to $\hat{X}_1 \bigsqcup \hat{X}_2$ to form the “coordinated” connected sums $\partial_1^+ \hat{X}_1(\hat{v}_1) \#_{\partial} \partial_1^+ \hat{X}_2(\hat{v}_2)$ and $\hat{X}_1 \#_{\partial} \hat{X}_2$. The ambiguity arises if $\partial_2^- \hat{X}_1(\hat{v}_1)$ or $\partial_2^- \hat{X}_2(\hat{v}_2)$ has more than a single connected component. To avoid it, as in \cite{K9}, formula (3.14), we need to pick a preferred connected component of $\partial_2^- \hat{X}_1(\hat{v}_1)$ and $\partial_2^- \hat{X}_2(\hat{v}_2)$. Using the local models of convex vector fields (as in the proof of Lemma 4.3), the vector fields $\hat{v}_1$ and $\hat{v}_2$ extend across $H$ so that the convexity of the extended traversing vector field $\hat{v} \#_{\partial}$ is enforced. For example, the lower diagram in Fig.2 is the connected sum of the upper diagram with itself.
Theorem 4.4. Let $\Theta' \subset \Theta \subset \Omega$ be closed subposets which do not contain the element ($\emptyset$). Assume that $d' \geq d$ and $d' \equiv d \equiv 0 \mod 2$. Let $\hat{v}$ be a convex traversing vector field on the standard ball $D^{n+1}$ such that $\partial_1^+ D^{n+1}(\hat{v})$ is diffeomorphic to the standard ball $D^n$.\footnote{The constant vector field will do.}

The group operation in the sources of the maps \((4.7)\) and \((4.8)\) below is the connected sum $\#$ of convex envelopes/convex pseudo-envelops.

- There is a group isomorphism
  \[
  \Phi_{\text{emb}} : QT_{d,d'}^{\text{emb}}(D^{n+1}, \hat{v}; c\Theta; c\Theta') \xrightarrow{\approx} \langle (D^n, S^{n-1}), \epsilon_{d,d'} : (P_{c\Theta}^c, pt) \to (P_{c\Theta'}^c, pt') \rangle.
  \]

The group homomorphism
\[
\Phi_{\text{sub}} : QT_{d,d'}^{\text{sub}}(D^{n+1}, \hat{v}; c\Theta; c\Theta') \to \langle (D^n, S^{n-1}), \epsilon_{d,d'} : (P_{c\Theta}^c, pt) \to (P_{c\Theta'}^c, pt') \rangle.
\]

is an split epimorphism. Moreover, we have a split group extension
\[
(4.9) \quad 1 \to \ker(\Phi_{\text{sub}}) \to QT_{d,d'}^{\text{sub}}(D^{n+1}, \hat{v}; c\Theta; c\Theta') \xrightarrow{\approx} QT_{d,d'}^{\text{emb}}(D^{n+1}, \hat{v}; c\Theta; c\Theta') \to 1.
\]

- For $n > 1$ all these groups are abelian.
- For $n < \max_{\omega \in \Theta, \Theta'} \{ |\omega'| \}$, the groups $QT_{d,d'}^{\text{emb}}(D^{n+1}, \hat{v}; c\Theta; c\Theta')$ are trivial.

Proof. The main observation is that the construction of the boxes $(\hat{X}, \hat{v}) \subset ([0,1] \times Y, \tilde{v})$ in Lemma 4.3 is amenable to the connected sum operation for convex pseudo-envelops of traversing flows. That is, given two boxes $([0,1] \times Y_1, \tilde{v}_1) \supset (\hat{X}_1, \hat{v}_1)$ and $([0,1] \times Y_2, \tilde{v}_2) \supset (\hat{X}_2, \hat{v}_2)$ as in Lemma 4.3, we get that
\[
(\hat{X}_1 \#_{\hat{\partial}} \hat{X}_2, \hat{v}_1 \#_{\hat{\partial}} \hat{v}_2) \subset ([0,1] \times (Y_1 \#_{\hat{\partial}} Y_2), \tilde{v}_1 \#_{\hat{\partial}} \tilde{v}_2)
\]
is also a box as in that lemma. Therefore, all the constructions and arguments from Section 3 in [K9] (like Proposition 3.4, Corollary 3.3, and Theorem 3.2) apply to the convex pseudo-envelopes/envelopes $\alpha : (X, \alpha^+(\hat{v})) \to (D^{n+1}, \hat{v})$, where the traversing vector field $\hat{v}$ is convex with respect to $\partial D^{n+1}$. Thus, as in formula (3.2) above (see formula (3.15) and Proposition 3.2 from [K9]), for a convex vector field $\hat{v}$ with $\partial_1^+ D^{n+1}(\hat{v})$ being a smooth $n$-ball, the quasitopies $QT_{d,d'}^{\text{emb}}(D^{n+1}, \hat{v}; c\Theta; c\Theta')$ and $QT_{d,d'}^{\text{sub}}(D^{n+1}, \hat{v}; c\Theta; c\Theta')$ are groups. For $n > 1$ they are commutative by arguments as in Proposition 3.2 from [K9].

The last claim follows by the general position argument. \hfill \Box

Recall that, by Corollary 4.5, the set $QT_{d,d'}^{\text{emb}}(\hat{X}, \hat{v}; c\Theta; c\Theta')$ depends only on the homotopy type of the pair $\langle \partial_1^+ \hat{X}(\hat{v}), \partial_2^+ \hat{X}(\hat{v}) \rangle$; however, the corollary does not make any claims about $QT_{d,d'}^{\text{sub}}(\hat{X}, \hat{v}; c\Theta; c\Theta')$. The next proposition, in line with Proposition 4.1 (which deals with the homology $n$-spheres, $n \geq 6$), is a hint that $QT_{d,d'}^{\text{sub}}(\hat{X}, \hat{v}; c\Theta; c\Theta')$ may also depend only on the homotopy type of the pair $\langle \partial_1^+ \hat{X}(\hat{v}), \partial_2^+ \hat{X}(\hat{v}) \rangle$. 

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Proposition 4.2. For $n \geq 5$, assuming that the locus $\partial_2^+ D^{n+1}(\hat{v})$ is simply-connected, the groups $\mathcal{QT}_{d,d'}^{\text{sub/emb}}(D^{n+1}, \hat{v}; c\Theta; c\Theta')$ do not depend on the choice of the convex traversing vector field $\hat{v}$ on $D^{n+1}$.

For $n \leq 3$, the groups $\mathcal{QT}_{d,d'}^{\text{sub/emb}}(D^{n+1}, \hat{v}; c\Theta; c\Theta')$ also do not depend on the convex $\hat{v}$.

Proof. For a convex $\hat{v}$, the locus $\partial_1^+ \hat{X}(\hat{v})$ is a deformation retract of $D^{n+1}$ and thus has a homotopy type of a point. By the Poincaré duality, $\partial_2^+ D^{n+1}(\hat{v}) = \partial(\partial_1^+ \hat{X}(\hat{v}))$ is a homology $(n-1)$-sphere $\Sigma^{n-1} \subset \partial D^{n+1}$. Let us delete a small smooth ball $B^n$ from the interior of $\partial_1^+ \hat{X}(\hat{v})$. We denote its complement by $W^n$. For $n \geq 5$, assuming that $\pi_1(\partial_2^+ D^{n+1}(\hat{v})) = 1$, we may apply the smooth h-cobordism theorem (see [Mi]) to $W$ to conclude that it is diffeomorphic to the product $S^{n-1} \times [0,1]$. Therefore, $\partial_1^+ D^{n+1}(\hat{v})$ is a smooth ball $D^n \subset S^n$. Any two regular embeddings $D^n \hookrightarrow S^n$ are isotopic. For $n = 3$, if the domain $\partial_1^+ D^4(\hat{v}) \subset S^3$ has a smooth boundary that is homology 1-sphere, then the domain is the 2-ball, and any 2-balls in $S^2$ are isotopic. For $n = 3$, if the domain $\partial_1^+ D^4(\hat{v}) \subset S^3$ has a smooth boundary that is a homology 2-sphere. By the classification of 2-surfaces, it follows that $\partial_2^+ D^{n+1}(\hat{v})$ is the standard 2-sphere. By the solution of the 3-dimensional Poincaré Conjecture [P1], [P3], the contractible domain $\partial_1^+ D^4(\hat{v})$ is with the spherical boundary is the 3-ball, and any two 3-balls in $S^3$ are isotopic. Thus, by Lemma 4.1, the smooth isotopy type of the locus $\partial_1^+ D^{n+1}(\hat{v}) \subset S^n$ determines the smooth topological type of the foliation $\mathcal{L}(\hat{v})$. Since, any two standard n-balls $\partial_1^+ \hat{X}(\hat{v}_1)$ and $\partial_1^+ \hat{X}(\hat{v}_2)$ are isotopic in $S^n$, the two statements of the proposition are validated. The difficult case $n = 4$ is wide open. \[ \Box \]

Combining Theorem 4.2 and Theorem 4.4, we get the following claim.

Corollary 4.7. Let $\hat{v}^{||}$ be a constant vector field on the ball $D^{n+1} \subset \mathbb{R}^{n+1}$, and let $\hat{v}$ be a convex vector field on $\hat{X}$. Put $r = \#(\pi_0(\partial_2^+ \hat{X}(\hat{v})))$. With the help of the maps from (4.7) and (4.8), the groups

$$G^{\text{emb}} := (\mathcal{QT}_{d,d'}^{\text{emb}}(D^{n+1}; \hat{v}^{||}; c\Theta; c\Theta'))^r$$

and

$$G^{\text{sub}} := (\mathcal{QT}_{d,d'}^{\text{sub}}(D^{n+1}; \hat{v}^{||}; c\Theta; c\Theta'))^r$$

are represented in the group

$$(\pi_n(\mathcal{P}^{\Theta}_{d,d}, pt)/\ker \{ (\epsilon_{d,d}^*)^* : \pi_n(\mathcal{P}^{\Theta}_{d,d}, pt) \to \pi_n(\mathcal{P}^{\Theta'}_{d,d}, pt') \})^r.$$ 

We denote by $\Psi^{\text{emb}}$ and $\Psi^{\text{sub}}$ these two representations (the first one is an isomorphism).

Then the maps $\Phi^{\text{sub}}(\hat{X}, \hat{v})$ in (4.6) and $\Phi^{\text{emb}}(\hat{X}, \hat{v})$ in (4.5) from Theorem 4.2 are equivariant with respect to the $G^{\text{sub}}$- and $G^{\text{emb}}$-actions on their source sets and the $\Psi^{\text{sub}}(G^{\text{sub}})$- and $\Psi^{\text{emb}}(G^{\text{emb}})$-actions on their target sets. \[ \Box \]
4.6. Quasitopies of envelops with generic combinatorics $\Theta$ and $d' = d$. From now and until Subsection 4.9, each result about quasitopies $Q\mathcal{T}_{d,d'}^{\text{sub/emb}}(\hat{X}, \hat{v}; c\Theta; c\Theta')$ of convex envelops is a recognizable imagery of a similar result from [K9] about the quasitopies $Q\mathcal{T}_{d,d'}^{\text{imm/emb}}(Y, \partial Y; c\Theta; (\emptyset), c\Theta')$ of immersions/embedding into the products $\mathbb{R} \times Y$.

**Theorem 4.5.** Let $(\hat{X}, \hat{v})$ be a convex pair. Let $d, k$ be natural numbers such that $2 < k < d$ and $d \equiv 0 \mod 2$. Put $\Omega_{|\sim|' \leq k-1} = \text{def } \mathcal{C} \Omega_{|\sim|' \geq k}$. Then we get a bijection

$$\Phi : Q\mathcal{T}_{d,d'}^{\text{emb}}(\hat{X}, \hat{v}; \Omega_{|\sim|' \leq k-1} ; \Omega_{|\sim|' \leq k-1}) \approx \left\{ (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})), \left( \bigvee_{\ell=1} A_k \right) \right\}.$$  

**Proof.** The theorem is based on Proposition 3.10 from [K9]. For $d > k > 2$, Proposition 2.1 being combined with the Alexander duality, describes the homotopy type of $\mathcal{P}_d^{\Omega_{|\sim|' \leq k-1}} = \mathcal{P}_d^{\mathcal{C} \Omega_{|\sim|' \leq k}}$ as a bouquet $\bigvee_{\ell=1}^{A(d,k)} S^{k-1}_\ell$ of $(k-1)$-spheres. The space $\mathcal{P}_d^{\emptyset}$ is contractible to the point $\star$. Thus, by Theorem 4.2 (based on Theorem 3.1 from [K9]), the claim follows.

We consider now the “combinatorially generic” case of convex pseudo-envelops $\alpha : (X, \alpha^+(\hat{v})) \to (\hat{X}, \hat{v})$ for which $\hat{v}$ is traversally generic with respect to $\alpha(\partial X)$. By Definition 4.7, any such $\alpha$ has tangency patterns that belong to the poset $\Omega_{|\sim|' \leq n}$, where $n = \dim X - 1$.

**Corollary 4.8.** Let $(\hat{X}, \hat{v})$ be a convex pair, $\dim \hat{X} = n + 1$. For $d > n > 2$, we get a bijection

$$\Phi : Q\mathcal{T}_{d,d'}^{\text{emb}}(\hat{X}, \hat{v}; \Omega_{|\sim|' \leq n} ; \Omega_{|\sim|' \leq n}) \approx \mathbb{Z}^{A(d,n+1)}.$$  

These $\mathbb{Z}$-valued invariants of convex envelopes are delivered by the degrees of the maps

$$\{ \Phi_\ell : \partial_1^+ \hat{X}(\hat{v}) / \partial_2^- \hat{X}(\hat{v}) \to S^{\emptyset}_\ell \}_{\ell \in [1, A(d,n+1)]},$$

of the individual spheres, induced by the map $\Phi$ from Theorem 4.3. In particular, the $(d,d'; \Omega_{|\sim|' \leq n} ; \Omega_{|\sim|' \leq n})$-quasitopy class of any traversally generic convex envelop $\alpha : (X, \alpha^+(\hat{v})) \to (\hat{X}, \hat{v})$ is determined by the collection of such degrees.

**Proof.** Since $\hat{X}$ is connected and $\hat{v}$ is convex, the locus $\partial_1^+ \hat{X}$ is connected as well.

We repeat the arguments from Proposition 3.6 in [K9]. For $n > 2$, the homotopy classes of the classifying maps $\Phi^\alpha : (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})) \to (\mathcal{P}_d^{\Omega_{|\sim|' \leq n}}, \mathcal{P}_d^{\emptyset})$ are in 1-to-1 correspondence with homotopy classes of the corresponding maps $\Phi_\ell^\alpha : (\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v})) \to \left( \bigvee_{\ell=1}^{A(d,n+1)} S^{\emptyset}_\ell, \star \right)$. For $n \geq 2$, the latter ones are detected by the degrees of the maps $\{ \Phi_\ell^\alpha \}$ to the individual spheres $\{ S^{\emptyset}_\ell \}$. \hfill $\Box$

**Example 4.5.** Take $d = 6$ and $n = 3$. Then $\mathcal{P}_d^{\Omega_{|\sim|' \geq 4}}$ consist of a single 0-dimensional cell (the “$\infty$” in $\mathcal{P}_6$), one 1-dimensional cell, labelled by $\omega = (6)$, and five 2-dimensional cells, labelled by $\omega = (51), (15), (42), (24), (33)$. Hence, $A(6,4) = \chi(\mathcal{P}_6^{\Omega_{|\sim|' \geq 4}}) - 1 = 4$. The space

\footnote{see (2.12) for the definition of the number $A(d,k) := A(d,k,0)$}
\( P_6^{\Omega|\sim|'\geq 4} \) has a homotopy type of a bouquet \( \bigvee_{\ell=1}^4 S^2 \) of four 2-spheres. By the Alexander duality, \( P_6^{\Omega|\sim|'\geq 4} = P_6^{\Omega|\sim|'\leq 3} \) has a homotopy type of a bouquet \( \bigvee_{\ell=1}^4 S^3 \) of four 3-spheres.

We pick a convex and traversing vector field \( \hat{v} \) on \( D^4 \). Recall that \( \partial_1^+ D^4(\hat{v}) \) is contractible and thus \( \partial_2^- D^4(\hat{v}) \) a homology 2-sphere, which implies that \( \partial_2^- D^4(\hat{v}) \) is diffeomorphic to \( S^2 \subset S^3 \). Thus \( \partial_1^+ D^4(\hat{v}) \), by [P1], [P2], is diffeomorphic to the ball \( D^3 \). Therefore, for any convex \( \hat{v} \), we get the group isomorphism

\[
\mathcal{Q}\mathcal{T}_{6,6}^{\text{emb}}(D^4, \hat{v}; \Omega_{|\sim|'\leq 3}; \Omega_{|\sim|'\leq 3}) \cong \pi_3(\bigvee_{\ell=1}^4 S^3_\ell) \approx \mathbb{Z}^4.
\]

As a result, any element \([\alpha] \in \mathcal{Q}\mathcal{T}_{6,6}^{\text{emb}}(D^4, \hat{v}; \Omega_{|\sim|'\leq 3}; \Omega_{|\sim|'\leq 3})\) generates four integer-valued characteristic invariants. For embeddings \( \alpha \), they determine \([\alpha]\).

At the same time, by Corollary 4.9 below, \( \mathcal{Q}\mathcal{T}_{6,8}^{\text{emb}}(D^4, \hat{v}; \Omega_{|\sim|'\leq 3}; \Omega_{|\sim|'\leq 4}) = 0 \).

The following claim contrasts Corollary 4.8

**Corollary 4.9.** Under the hypotheses of Corollary 4.8, including \( d > n > 2 \), the set \( \mathcal{Q}\mathcal{T}_{d, d+2}^{\text{emb}}(\hat{X}, \hat{v}; \Omega_{|\sim|'\leq n}; \Omega_{|\sim|'\leq n+1}) \) consists of a single element, represented by \( X = \emptyset \).

In particular, the \((d, d+2; \Omega_{|\sim|'\leq n}, \Omega_{|\sim|'\leq n+1})\)-quasitopy class of any traversally generic convex envelop \( \alpha: (X, \alpha^1(\hat{v})) \to (\hat{X}, \hat{v}) \) is trivial.

**Proof.** Consider a convex pseudo-envelop \( \alpha: (X, \alpha^1(\hat{v})) \to (\hat{X}, \hat{v}) \) as in Theorem 4.5.

The claim is based on the observation that any map from the \( n \)-dimensional CW-complex \( \partial_1^+ \hat{X}(\hat{v})/\partial_2^- \hat{X}(\hat{v}) \) to \( P_d^{\Omega|\sim|'\geq n} \)—homotopically a bouquet of \( n \)-spheres—is null-homotopic in \( P_{d+2}^{\Omega|\sim|'\geq n+1} \), since the latter space has the homotopy type of a bouquet of \((n+1)\)-spheres. Again, by Theorem 4.2 the pseudo-envelop \( \alpha \) is null-quasitopic. \( \Box \)

4.7. From inner framed cobordisms of \( \partial_1^+ \hat{X}(\hat{v}) \) to quasitopies of \( k \)-flat convex envelops. Theorem 4.5 suggests a somewhat unexpected connection between quasitopies of certain convex envelops \( \alpha: X \to \hat{X} \) and inner framed cobordisms of the manifold \( \partial \hat{X} \).

Let \( Y \) be a compact smooth \( n \)-manifold. For \( k - 1 \leq n \), we associate with \( Y \) the set of inner framed cobordisms \( \mathcal{F}\mathcal{B}_n^{k-1}(Y) \). These cobordisms are based on codimension \( k - 1 \) smooth closed submanifolds \( Z \) of \( Y \) of the form \( Z = \bigsqcup_{\sigma \in A(d,k,q)} Z_\sigma \), where the number \( A(d,k,q) \) is introduced in (2.12). The normal \((k - 1)\)-bundle \( \nu(Z, Y) \) is required to be framed. The disjoint “components” \( \{Z_\sigma\} \) of \( Z \) are marked with different colors \( \sigma \) from a pallet \( \Xi \) of cardinality \( A(d,k,q) \).

We have seen in [K1], Proposition 3.11, that the inner framed \( \Xi \)-colored codimension \( k - 1 \) bordisms of the \( n \)-dimensional manifold \( Y \) produce, via the Thom construction, quite special \( k \)-flat embeddings \( \beta: M \to \mathbb{R} \times Y \), where \( \dim M = \dim Y \). The analogous mechanism, with the help of Lemma 4.3 and Theorem 4.5, generates special \( k \)-flat envelops (see Definition 4.6).
Proposition 4.3. Let $\hat{X}$ be a smooth compact connected $(n+1)$-dimensional manifold, equipped with a convex traversing vector field $\hat{v}$. Let $k \in [3, n+1]$, $k < d$, and $d \equiv 0 \mod 2$. Then the Thom construction delivers a bijection

$$Th : FB_{d,d}^k(\partial_1^+ \hat{X}(\hat{v})) \approx QT_{d,d}^{emb}(\hat{X}, \hat{v}; c\Omega_{|\sim|\geq k}; c\Omega_{|\sim|\geq k}),$$

where $FB_{d,d}^k(\partial_1^+ \hat{X}(\hat{v}))$ denotes the set of inner framed $\Xi$-colored $(n-k+1)$-dimensional bordisms of the space $\partial_1^+ \hat{X}(\hat{v})$. Here $\#\Xi = A(d,k)$ (see [2.12]).

Example 4.6. Let us recycle Example 3.5 from [K9]: take $d = 6$, $q = 0$, and $n = 3$. Then $\#\Xi = 10$. So we get a homotopy equivalence $Th : FB_{d,d}^k(\partial_1^+ \hat{X}(\hat{v})) \approx QT_{d,d}^{emb}(\hat{X}, \hat{v}; c\Theta_{|\sim|\geq 3}; c\Theta_{|\sim|\geq 3}),$ any framed link $Z \subset \partial_1^+ \hat{X}(\hat{v})$, colored with 10 distinct colors at most, produces a quasitopy class of a convex envelop $\alpha : (X, \alpha^1(\hat{v})) \to (\hat{X}, \hat{v})$. Its tangency patterns $\omega$, except for $\omega = (\emptyset)$, have only entries from the list $\{1, 2, 3\}$, so that no more than one 3 is present in $\omega$, and no more than two 2’s are present, while $|\omega| \leq 6$.

Now, let us assume that $D^4$ carries a constant vector field $v^\parallel$ and that $\partial_2^+ \hat{X}(\hat{v}) \neq \emptyset$. Note that any element of $H_1(\partial_1^+ \hat{X}(\hat{v}); \mathbb{Z})$ may be realized by a disjoint union of framed oriented loops. Then, like in Example 3.5 from [K9], the orbit-space of the $G_6^{emb}(D^4; c\Theta_{|\sim|\geq 3}; c\Theta_{|\sim|\geq 3})$-action on $QT_{6,6}^{emb}(\hat{X}, \hat{v}; c\Theta_{|\sim|\geq 3}; c\Theta_{|\sim|\geq 3})$ admits a surjection on the group $(H_1(\partial_1^+ \hat{X}(\hat{v}); \mathbb{Z}) \otimes)_{10} \approx (H_1(\hat{X}; \mathbb{Z}) \otimes)_{10}^1$, provided that $\partial \hat{X}$ is orientable.

Let us give a couple of specific examples of this fact. Consider the box $T_0^3 \times [0, 1]$, where $T_0^3$ is the compliment in the 3-torus $T^3$ to a ball $D^3$. By rounding the corners of the box, we get a convex pair $(\hat{X}, \hat{v})$, where $\hat{X}$ is homeomorphic to $T_0^3 \times [0, 1]$. Then the orbit-space of the $G_6^{emb}(D^4; c\Theta_{|\sim|\geq 3}; c\Theta_{|\sim|\geq 3})$-action on $QT_{6,6}^{emb}(\hat{X}, \hat{v}; c\Theta_{|\sim|\geq 3}; c\Theta_{|\sim|\geq 3})$ is mapped onto the lattice $\mathbb{Z}^{30}$.

Let $M = \mathbb{H}^3/\Gamma$ be a compact hyperbolic 3-manifold and $M_0 := M \setminus D^3$. By rounding the corners of the box $M_0 \times [0, 1]$, we get a convex pair $(\hat{X}, \hat{v})$. Then the orbit-space of the $G_6^{emb}(D^4; c\Theta_{|\sim|\geq 3}; c\Theta_{|\sim|\geq 3})$-action on $QT_{6,6}^{emb}(\hat{X}, \hat{v}; c\Theta_{|\sim|\geq 3}; c\Theta_{|\sim|\geq 3})$ is mapped onto the abelian group $(\Gamma/\Gamma)_{10}$, where $[\Gamma, \Gamma]$ denotes the commutator.

By Proposition 4.3 and repeating the (obviously modified) arguments in the proof of Corollary 3.13 from [K9], we get the following corollary.

Corollary 4.10. Let $d > k$, and $d \equiv 0 \mod 2$. Let $(\hat{X}, \hat{v})$ be a convex pair.

- For $\dim \hat{X} = k \geq 3$ and any choice of $k \in \pi_0(\partial_2\hat{X}(\hat{v}))$, the group $QT_{d,d}^{emb}(D^k; \hat{v}^\parallel; c\Omega_{|\sim|\geq k}; c\Omega_{|\sim|\geq k})$ acts freely and transitively on the set $QT_{d,d}^{emb}(\hat{X}, \hat{v}; c\Omega_{|\sim|\geq k}; c\Omega_{|\sim|\geq k})$. Thus both sets are in a 1-to-1 correspondence.
• For dim $\hat{X} = k + 1 > 5$, a simply-connected $\hat{X}$, and any choice of $\kappa \in \pi_0(\partial_2^+ \hat{X}(\hat{v}))$, the group $Q^\text{emb}_{d,d}(D^{k+1}, \hat{1}; c\Omega_{|\sim|\geq k}; c\Omega_{|\sim|\geq k})$ acts freely and transitively on the set $Q^\text{emb}_{d,d}(\hat{X}, \hat{v}; c\Omega_{|\sim|\geq k}; c\Omega_{|\sim|\geq k})$. Again, both sets are in a 1-to-1 correspondence.

\begin{example}
Let $\hat{X} = \mathbb{CP}^2 \times D^2$ with $\hat{v}$ on $\mathbb{CP}^2 \times D^2$ being generated by a constant vector field on $D^2$. Then $Q^\text{emb}_{d,d}(D^6, \hat{v}; c\Omega_{|\sim|\geq 5}; c\Omega_{|\sim|\geq 5})$ acts freely and transitively on the set $Q^\text{emb}_{d,d}(D^6, \hat{v}; c\Omega_{|\sim|\geq 5}; c\Omega_{|\sim|\geq 5})$ for all even $d \geq 6$. Thus both sets are in 1-to-1 correspondence. As a result, $Q^\text{emb}_{d,d}(\mathbb{CP}^2 \times D^2, \hat{v}; c\Omega_{|\sim|\geq 5}; c\Omega_{|\sim|\geq 5})$ acquires the structure of the abelian group $\pi_5(\bigvee_{i=1}^{A(d,5)} S^4, \ast)$.
\end{example}

4.8. Convex envelops with special combinatorics $\Theta$. For a given $\omega \in \Omega$, we denote by $\langle \omega \rangle$ the minimal closed poset, generated by $\omega$. Combining Theorem 4.2 and Theorem 3.5 from [K9] with Proposition 4.2, we get the following result, in which the constraint $d \leq 12$ reflects only the scope of the numerical experiments in [KSW2].

\begin{theorem}
Assume that $d \leq 12$ and $d \equiv 0 \mod 2$. Let $\omega \in \Omega_{\leq d}$ be such that $|\omega|' > 2$. Let $\hat{X}$ be a smooth compact connected $(n+1)$-dimensional manifold, equipped with a convex traversing vector field $\hat{v}$.

Then the quasitopy set $Q^\text{emb}_{d,d}(\hat{X}, \hat{v}; c\langle \omega \rangle; c\langle \omega \rangle)$ either consists of single element (is trivial), or is isomorphic to the cohomotopy set $\pi^k(\partial_1^+ \hat{X}(\hat{v})/\partial_2^+ \hat{X}(\hat{v}))$, where $k = k(\omega) \in [|\omega|' - 1, d - 1]$.

In particular, for a constant vector field $\hat{v} = \hat{v}_1$, $d \leq 12$, and $\omega$ and $k = k(\omega)$ from the table in [K9, Appendix], the group $Q^\text{emb}_{d,d}(D^{n+1}, \hat{v}; c\langle \omega \rangle, c\langle \omega \rangle) \approx \pi_n(S^k)$. A similar claim is valid for any convex $\hat{v}$ on $D^{n+1}$, provided that either $n \leq 3$ or $n \geq 5$ and $\partial_1^+ \hat{X}(\hat{v})$ is simply-connected.

For $d \leq 13$, the table from Appendix in [K9] lists all $\omega$’s and the corresponding $k = k(\omega)$’s for which $P_d^\Theta(\omega)$ is homologically nontrivial. In fact, $P_d^\Theta(\omega)$ is a homology $k$-sphere and even a homotopy $k$-sphere, at least when $|\omega|' > 2$ [K9], a quite mysterious phenomenon...

Let us recycle Example 3.4 from [K9], while adapting it to convex envelops.

\begin{example}
Let $d = 8$ and $\omega = (4)$. Then, for any convex pair $(\hat{X}, \hat{v})$, using the list in [K9, Appendix], we get a bijection

\[Q^\text{emb}_{8,8}(\hat{X}, \hat{v}; c\langle (4) \rangle, c\langle (4) \rangle) \approx \pi^4(\partial_1^+ \hat{X}(\hat{v})/\partial_2^+ \hat{X}(\hat{v}))\]

where $\pi^4(\sim)$ stands for the 4-cohomotopy set. In particular, using Proposition 4.2 for any convex $\hat{v}$ on $D^8$ such that the locus $\partial_2^+ D^8 (\hat{v})$ is simply-connected (the constant $\hat{v}$ will do), we get a group isomorphism:

\[Q^\text{emb}_{8,8}(D^8, \hat{v}; c\langle (4) \rangle, c\langle (4) \rangle) \approx \pi_7(S^4) \approx \mathbb{Z} \times \mathbb{Z}_{12}\]

Let $d = 12$ and $\omega = (11213)$. Then, for any for any convex pair $(\hat{X}, \hat{v})$, by the same list from [K9], we get a bijection

\[Q^\text{emb}_{12,12}(\hat{X}, \hat{v}; c\langle (11213) \rangle, c\langle (11213) \rangle) \approx \pi^6(\partial_1^+ \hat{X}(\hat{v})/\partial_2^+ \hat{X}(\hat{v}))\]

\end{example}
Again, using Proposition 4.2, for any convex \( \hat{v} \) on \( D^d \), such that the locus \( \partial_2 D^d(\hat{v}) \) is simply-connected (again, the constant \( \hat{v} \) will do), we get a group isomorphism

\[
\text{QT}_{12,12}^\text{emb}(D^d, \hat{v}; c((11213)), c((11213))) \approx \pi_0(S^d) \approx \mathbb{Z}_{24}. \quad \diamond
\]

Proposition 4.4 below deals with special \( \Theta \)'s for which \( P^\Theta_d \) are \( K(\pi,1) \)-spaces and \( \pi \) is a free group.

Let \( H, G \) be two groups, and \( \text{Hom}(H, G) \) be the group of their homomorphisms. Then \( G \) acts on \( \text{Hom}(H, G) \) by the conjugation: for any homomorphism \( \phi : H \to G \), \( h \in H \), and \( g \in G \), we define \( (Ad_g \phi)(h) \) by the formula \( g^{-1} \phi(h) g \). We denote by \( \text{Hom}^*(H, G) \) the quotient \( \text{Hom}(H, G)/Ad_G \).

**Proposition 4.4.** Let \( c \Theta \) consist of all \( \omega \)'s with entries 1 and 2 only and no more than a single entry 2. Put \( \kappa(d) = \frac{d(d-2)}{4} \) for \( d \equiv 0 \mod 2 \). We denote by \( F_\kappa(d) \) the free group in \( \kappa(d) \) generators.

Consider a convex pair \( (\hat{X}, \hat{v}) \), where \( \hat{X} \) is connected. If \( \partial_2^+ \hat{X}(\hat{v}) \neq \emptyset \), then there is a bijection

\[
\Phi^{\text{emb}} : \text{QT}_{d,d}^\text{emb}(\hat{X}, \hat{v}; c \Theta; c \Theta) \xrightarrow{\simeq} \text{Hom}(\pi_1(\hat{X}), F_{\kappa(d)}(d))
\]

and a surjection

\[
\Phi^{\text{sub}} : \text{QT}_{d,d}^\text{sub}(\hat{X}, \hat{v}; c \Theta; c \Theta) \twoheadrightarrow \text{Hom}(\pi_1(\hat{X}), F_{\kappa(d)}(d)).
\]

When \( \partial_2^+ \hat{X}(\hat{v}) = \emptyset \), then similar claims hold with the target of \( \Phi^{\text{emb}} \) and \( \Phi^{\text{sub}} \) being replaced by the set \( \text{Hom}^*(\pi_1(\hat{X}), F_{\kappa(d)}(d)) \).

In particular, \( \Phi^{\text{emb}} : \text{QT}_{d,d}^\text{emb}(S^1 \times [0, 1], \hat{v}; c \Theta, c \Theta) \xrightarrow{\simeq} F_{\kappa(d)}(d)/Ad_F_{\kappa(d)}(d) \), the free group of cyclic words in \( \kappa(d) \) letters (see Fig.3 from \( [K9] \)). Here \( \hat{v} \) is tangent to the fibers of the obvious projection \( S^1 \times [0, 1] \to S^1 \).

If \( \pi_1(\hat{X}) \) has no free images, then the group \( \text{QT}_{d,d}^\text{emb}(\hat{X}, \hat{v}; c \Theta; c \Theta) \) is trivial.

**Proof.** We observe that \( \pi_1(\partial_1^+ \hat{X}(\hat{v}) \approx \pi_1(\hat{X}) \) since the two spaces are homotopy equivalent due to the convexity of \( \hat{v} \). By Lemma 4.3 we can incapsulate \( \hat{X}, \hat{v} \) in a box \( \mathbb{R} \times Y \), where \( Y \) is homeomorphic to \( \partial_2^+ \hat{X}(\hat{v}) \). Using this \( Y \), the claim follows from Corollary 3.12 in \( [K9] \). \( \square \)

The next proposition is a stabilization result, by the increasing \( d' \geq d \), for the convex envelops with (using the terminology of \( [Ar] \)), \( k \)-moderate tangent patterns \( \omega \in c \Theta_{max \geq k} \).

The entries of such \( \omega \)'s are all less than \( k \). Proposition 4.5 below follows instantly from \( [K9] \), Proposition 3.8, by combining it with Lemma 4.3.

**Proposition 4.5.** Let \( k \geq 4 \). If \( \dim Y \leq (k-2)(\lceil d/k \rceil + 1) - 2 \), then the classifying map

\[
\Phi^{\text{emb}} : \text{QT}_{d,d}^\text{emb}(\hat{X}, \hat{v}; c \Theta_{max \geq k}; c \Theta_{max \geq k}) \xrightarrow{\simeq} [(\partial_1^+ \hat{X}(\hat{v}), \partial_2^+ \hat{X}(\hat{v})), (P_d^{c \Theta_{max \geq k}}, pt)]
\]

is a bijection, and the classifying map

\[
\Phi^{\text{sub}} : \text{QT}_{d,d}^\text{sub}(\hat{X}, \hat{v}; c \Theta_{max \geq k}; c \Theta_{max \geq k}) \xrightarrow{\simeq} [(\partial_1^+ \hat{X}(\hat{v}), \partial_2^+ \hat{X}(\hat{v})), (P_d^{c \Theta_{max \geq k}}, pt)]
\]
is a surjection for any \( d' \geq d, \ d' \equiv d \mod 2 \).

In particular, for a given \((\hat{X}, \hat{\varphi})\), the quasitopy \(QT_{d,d'}^{emb}(\hat{X}, \hat{\varphi}; c\Theta_{\max \geq k}; c\Theta_{\max \geq k})\) stabilizes for all \( d' \geq d \geq \frac{k}{k-2}(\dim \hat{X} + 3 - k)\), a linear function in \( \dim \hat{X} \).

\[\quad\]

\textbf{Example 4.9.} Let \( k = 4 \). By Proposition 4.5 for any compact connected 3-dimensional convex pair \((\hat{X}, \hat{\varphi})\), we get bijections:

\[
\begin{align*}
QT_{4,4}^{emb}(\hat{X}, \hat{\varphi}; c\Theta_{\max \geq 4}; c\Theta_{\max \geq 4}) & \cong \pi^2(\partial^+_{\hat{1}} \hat{X}(\hat{\varphi})/\partial^+_{\hat{2}} \hat{X}(\hat{\varphi})), \\
QT_{4,6}^{emb}(\hat{X}, \hat{\varphi}; c\Theta_{\max \geq 4}; c\Theta_{\max \geq 4}) & \cong \pi^2(\partial^+_{\hat{1}} \hat{X}(\hat{\varphi})/\partial^+_{\hat{2}} \hat{X}(\hat{\varphi})), \\
QT_{6,6}^{emb}(\hat{X}, \hat{\varphi}; c\Theta_{\max \geq 4}; c\Theta_{\max \geq 4}) & \cong \pi^2(\partial^+_{\hat{1}} \hat{X}(\hat{\varphi})/\partial^+_{\hat{2}} \hat{X}(\hat{\varphi})),
\end{align*}
\]

whose target is the second cohomotopy group \( \pi^2(\partial^+_{\hat{1}} \hat{X}(\hat{\varphi})/\partial^+_{\hat{2}} \hat{X}(\hat{\varphi})) \) of the singular connected surface \( \partial^+_{\hat{1}} \hat{X}(\hat{\varphi})/\partial^+_{\hat{2}} \hat{X}(\hat{\varphi}) \). This cohomotopy group is isomorphic to \( \mathbb{Z} \) via the degree invariant. Thus, all the three types of quasitopy classes are determined by this degree. Assuming \( \partial^+_{\hat{1}} \hat{X}(\hat{\varphi}) = \emptyset \), we may replace \( \pi^2(\partial^+_{\hat{1}} \hat{X}(\hat{\varphi})/\partial^+_{\hat{2}} \hat{X}(\hat{\varphi})) \) by \( \pi^2(\hat{X}) \).

\[\quad\]

\textbf{4.9. Characteristic classes of convex pseudo-envelops.} In this subsection, we will use the cohomology \( H^*(\mathcal{P}^\Theta_d) \) of the classifying space \( \mathcal{P}^\Theta_d \) (see [KSW2] and Section 2) to produce a variety of characteristic classes of convex pseudo-envelops.

Since any convex pseudo-envelop \( \alpha : (X, \alpha(\hat{\varphi})) \to (\hat{X}, \hat{\varphi}) \) produces an immersion \( \alpha^\partial : \partial X \to \hat{X} \), the following theorem follows directly from Theorem 3.3, [K9].

\[\quad\]

\textbf{Theorem 4.7.} Let \((\hat{X}, \hat{\varphi})\) be a convex pair. Pick \( d' = d, \ d' \equiv 0 \mod 2, \) and \( \Theta' = \Theta \).

Then any convex pseudo-envelop \( \alpha : (X, \alpha(\hat{\varphi})) \to (\hat{X}, \hat{\varphi}) \) induces a characteristic homomorphism \( (\Phi^\partial)^* \) from the \( * \)-homology of the differential complex

\[
\{(\partial^\#)^* : \mathbb{Z}[\Theta^\#_{\{d\}}]^* \to \mathbb{Z}[\Theta^\#_{\{d\}}]^* \},
\]

dual to the differential complex in [4.5], to the cohomology \( H^*(\hat{X}; \mathbb{Z}) \), and, via \( \alpha^* \), further to the cohomology \( H^*(X; \mathbb{Z}) \).

The \((d, d; c\Theta, c\Theta)\)-quasitopic pseudo-envelops/envelops induce the same characteristic homomorphisms.

\[\quad\]

Let us revisit the Arnold-Vassiliev case [Ar], [V] of real polynomials with moderate singularities. Let \( \Theta_{\max \geq k} \subset \Omega_{\{d\}} \) be the closed poset, consisting of \( \omega \)'s with the maximal entry \( \geq k \). For \( k \geq 3 \), the cohomology \( H^j(\mathcal{P}^{e\Theta_{\max \geq k}}, pt; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \) in each dimension \( j \) of the form \( (k - 2)m \), where the integer \( m \leq d/k \), and vanishes otherwise [Ar]. The cohomology ring \( H^*(\mathcal{P}^{e\Theta_{\max \geq k}}, pt; \mathbb{Z}) \) was computed by Vassiliev in [V], Theorem 1 on page 87. Here is the summary of his result: consider the graded ring \( \mathcal{V}ass_{d,k} \), multiplicatively generated over \( \mathbb{Z} \) by the elements \( \{e_m\}_{m \leq d/k} \) of the degrees \( \deg(e_m) = m(k - 2) \), subject to the relations

\[
e_1 \cdot e_m = \frac{(l + m)!}{l! \cdot m!} e_{l+m} \quad \text{for} \quad k \equiv 0 \mod 2, \quad \text{and the relations} \quad e_1 \cdot e_1 = 0, \quad e_1 \cdot e_{2m} = e_{2m+1},
\]
(4.11) \[ e_{2l} \cdot e_{2m} = \frac{(l + m)!}{l! \cdot m!} \cdot e_{2l+2m} \text{ for } k \equiv 1 \mod 2. \]

Combining Lemma 4.3 with Proposition 3.7 from [K9], we get the following assertion.

**Proposition 4.6.** Let \( k \geq 3 \). Consider a convex pseudo-envelop \( \alpha : (X, v) \to (\hat{X}, \hat{v}) \) whose tangency patterns to the \( \hat{v} \)-flow belong to \( c\Theta_{\max \geq k} \subset \Omega_{[d]} \) (the tangencies are \( k \)-moderate).

Then \( \alpha \) generates a characteristic ring homomorphism

\[ (\Phi^\alpha)^* : \text{Vass}_{d,k} \to H^*(\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v}); \mathbb{Z}), \]

which is an invariant of the quasitopy class of \( \alpha \). In other words, we get a map

\[ \Phi_{d,k} : \mathcal{Q}'_{d,k}(\hat{X}, \hat{v}; c\Theta_{\max \geq k}) \to \text{Hom}_{\text{ring}}(\text{Vass}_{d,k}, H^*(\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v}); \mathbb{Z})). \]

In particular, for any generator \( e_1 \in \text{Vass}_{d,k} \), we get a characteristic element \( (\Phi^\alpha)^*(e_1) \in H^l(\hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v}); \mathbb{Z}) \) which is an invariant of the quasitopy class of \( \alpha \).

If \( \partial_2^- \hat{X}(\hat{v}) = \emptyset \), then \( (\Phi^\alpha)^*(e_1) \) lives in \( H^{l(k-2)}(\hat{X}; \mathbb{Z}) \approx H^{l(k-2)}(\partial_2^- \hat{X}(\hat{v}); \mathbb{Z}) \).

If \( \hat{X} \) is oriented and \((n + 1)\)-dimensional, using the Poincaré duality in \( \partial_1^+ \hat{X}(\hat{v}) \), we produce a homology class \( \mathcal{D}(\Phi^\alpha)^*(e_1) \in H_{n-l(k-2)}(\partial_1^+ \hat{X}(\hat{v}); \mathbb{Z}) \approx H_{n-l(k-2)}(\hat{X}; \mathbb{Z}) \) which is again an invariant of the quasitopy class of \( \alpha \).

\( \diamond \)

**Remark 4.4.** We notice that if \( (\Phi^\alpha)^*(e_1) = 0 \), then, by (4.10) and (4.11), all \( \{(\Phi^\alpha)^*(e_q)\}_{q \geq l} \) must be torsion elements in \( H^{q(k-2)}(\partial_1^+ \hat{X}(\hat{v}), \partial_2^- \hat{X}(\hat{v}); \mathbb{Z}) \). For an orientable \( \hat{X} \), they may be viewed as elements of \( H_{n-q(k-2)}(\hat{X}; \mathbb{Z}) \).

\( \diamond \)

### 4.10. How to manufacture convex envelopes with desired combinatorial tangency patterns

Let us recall one classical construction, leading to the Alexander duality. Let \( K \subset S^d \) be a CW-subcomplex of the \( d \)-sphere. For an element \( a \in H_p(K; \mathbb{Z}) \), we denote by \( c_a \in H_{p+1}(S^d, K; \mathbb{Z}) \) the unique element such that \( \partial_a(c_a) = a \). The Alexander duality operator \( \mathcal{A} \) is defined by the formula \( \mathcal{A}(a) := \{ D(c_a) \in H^{d-p-1}(S^d \setminus K; \mathbb{Z}) \} \), where \( D \) is the Poincaré duality operator, the inverse of the operator \([S^d] \cap \). Pick \( b \in H_{d-p-1}(S^d \setminus K; \mathbb{Z}) \). Then linking number of \( a, b \) is defined by the formula \( \text{lk}(a, b) := \langle \mathcal{A}(a), b \rangle \), where \( \langle \cdot, \cdot \rangle \) is the natural pairing between cohomology and homology of dimension \( n - p - 1 \).

The next lemma is instrumental in producing examples of convex envelopes with prescribed combinatorial patterns of their trajectories.

**Lemma 4.4.** For any element \( \omega \in \Omega_{[d]} \), with the help of the differential \( \partial(\omega) \), given by formula (2.4), the Alexander duality \( \mathcal{A} \) produces a cohomology class

\[ \theta_\omega = \text{def} \mathcal{A}([\partial R^\omega]) \in H^{[\omega']}(\mathcal{P}^{c\omega'}_{d}; \mathbb{Z}), \]

where \( \omega_\prec \) denotes the set of elements of \( \Omega_{[d]} \) that are smaller than \( \omega \) (so, \( \omega \in c\{\omega_\prec\} \)), and \( \partial R^\omega \) denotes the algebraic boundary of the cell \( R^\omega \).

**Proof.** For any \( \omega \in \Omega_{[d]} \), take the closed poset \( \omega_\prec = \{ \omega' < \omega \} \subset \Omega_{[d]} \) for the role of \( \Theta \) in Theorem 2.1. We denote by \( R^\omega_d \) the one-point compactification of the closed cell \( R^\omega_d \subset \mathcal{P}_d \) (the interior \( (R^\omega_d)^{\circ} \) of \( R^\omega_d \) is an open \( (d - |\omega'|)\)-ball). Then the differential \( \partial(\omega) \), given by
the formula (2.4), represents the $(d - |\omega|' - 1)$-cycle $\partial R^\omega_d$ in the chain complex $C_*(\overline{P}_d^{\omega_\infty}; \mathbb{Z})$ (note that $\partial R^\omega_d$ is a boundary in $\overline{P}_d^{\omega_\infty}$, but not in $\overline{P}_d^{\omega_\infty}$!) and thus defines a nontrivial element $[\partial R^\omega_d] \in H_{d - |\omega|'}(\overline{P}_d^{\omega_\infty}; \mathbb{Z})$. By the Alexander duality, this element produces a cohomology class $\theta_\omega \overset{\text{def}}{=} \mathcal{A}(\partial R^\omega_d) \in H^{\omega'}(\mathcal{P}_d^{\omega_\infty}; \mathbb{Z})$.

\begin{example}
If $d = 6$, we get the following cohomology classes:
\begin{itemize}
  \item $\theta_{(121)} = \mathcal{A}(R_6^{(31)} - R_6^{(13)} - R_6^{(2121)} + R_6^{(1212)}) \in H^1(\mathcal{P}_6^{c(\{121\})}; \mathbb{Z})$,
  \item $\theta_{(3111)} = \mathcal{A}(R_6^{(411)} - R_6^{(321)} + R_6^{(312)}) \in H^2(\mathcal{P}_6^{c(\{3111\})}; \mathbb{Z})$,
  \item $\theta_{(31)} = \mathcal{A}(R_6^{(4)} - R_6^{(231)} + R_6^{(321)} - R_6^{(312)}) \in H^2(\mathcal{P}_6^{c(\{31\})}; \mathbb{Z})$,
  \item $\theta_{(1221)} = \mathcal{A}(R_6^{(321)} - R_6^{(141)} + R_6^{(123)}) \in H^2(\mathcal{P}_6^{c(\{1221\})}; \mathbb{Z})$.
\end{itemize}

\end{example}

Theorem 4.8 below gives a simple recipe for manufacturing traversing flows with the desired number of $v$-trajectories of a given combinatorial type $\omega$ on compact smooth manifolds $X$ with boundary. Although the resulting construction is explicit, the topological nature of the pull-back $X$ is open due to, paraphrasing Thom [Th], [Th1], “the mysterios nature of transversality”.

**Theorem 4.8.** For any $\omega \in \Omega_{(d)}$, let $\theta_\omega \in H^{\omega'}(\mathcal{P}_d^{c(\omega_\infty)}; \mathbb{Z})$ be the cocycle that takes the value $\mathfrak{v}(Z, \partial R^\omega)$ on each $|\omega|'$-dimensional cycle $Z$ from $H_{|\omega|'}(\mathcal{P}_d^{c(\omega_\infty)}; \mathbb{Z})$. Let $Y$ be an oriented closed and smooth $|\omega|'$-dimensional manifold. We denote by $\hat{v}$ a non-vanishing vector field that is tangent to the fibers of the projection $\mathbb{R} \times Y \to Y$. Let a map $\Phi : Y \to \mathcal{P}_d^{c(\omega_\infty)}$ be $(\mathcal{E}_d)$-regular in the sense of Definition 3.4 from [K9] (see also (2.2))

- Then the natural coupling 
  \[
  \langle \Phi^* (\theta_\omega), [Y] \rangle = \langle \Phi^*(\mathcal{A}(\partial R^\omega_d)), [Y] \rangle
  \]
  gives the oriented count of the $\hat{v}$-trajectories of the combinatorial type $\omega$ in the $(|\omega|' + 1)$-dimensional convex envelop $(\mathbb{R} \times Y, \hat{v})$ of the pair 
  \[
  (X_\Phi = \overset{\text{def}}{=} \{(u, y) \mid \Phi(y)(u) \leq 0, \hat{v} \} \subset (\mathbb{R} \times Y, \hat{v})).
  \]
  The combinatorial types of the $\hat{v}$-trajectories relative to $\partial X_\Phi$ belong to the poset 
  \[
  c(\omega_\infty) = \omega_\infty = \overset{\text{def}}{=} \{ \omega' \in \Omega_{(d)} \mid \omega' \geq \omega \}.
  \]
- If $\Phi$ is transversal to the cell $(R_d^\omega)^o$ in $\mathcal{P}_d^{c(\omega_\infty)}$, then the number of $\hat{v}$-trajectories in $\mathbb{R} \times Y$, whose intersection with $\partial X_\Phi$ has the combinatorial type $\omega$, equals the cardinality of the intersection $\Phi(Y) \cap (R_d^\omega)^o$. Thus the number of such trajectories is greater than or equal to the absolute value $|\langle \Phi^* (\theta_\omega), [Y] \rangle|$.

\begin{proof}
The argument is similar to the proof of Lemma 4.4. By Theorem 2.1, the homology class of the cycle $\partial R^\omega_d$ in $H_{d - |\omega|'}(\overline{P}_d^{\omega_\infty}; \mathbb{Z})$, represented by formula (2.4), is nontrivial since $d - |\omega|' - 1$ is the top grading of the differential complex $(\mathbb{Z}[\omega_\infty], \partial)$.

\end{proof}

\textsuperscript{7}By [K9], Corollary 3.2, such maps form and open and dense set in the space of all smooth maps.
The proof amounts to spelling out the nature of Alexander duality $\mathcal{A}$. By its definition, 
\( \langle Z, \mathcal{A}(\partial R_d^\omega) \rangle = \text{lk}(Z, \partial R_d^\omega) \) for any cycle $Z$ in $\mathcal{P}_d^{c(\omega)}$ of dimension $|\omega|$. Thus
\[
\langle \Phi^*(\theta_\omega), [Y] \rangle = \langle \theta_\omega, \Phi_*([Y]) \rangle = \text{lk}(\partial R_d^\omega, \Phi(Y)) = R_d^\omega \circ \Phi(Y).
\]

Examining the construction of the space $X_\Phi \subset Y \times \mathbb{R}$, given by $(id \times \Phi)^{-1}(E_d)$, where the hypersurface $E_d$ was introduced in (2.2), the latter intersection number gives an oriented count of the $\hat{v}$-trajectories in $\mathbb{R} \times Y$ of the combinatorial type $\omega$ relative to the boundary $\partial X_\Phi$.

If $\Phi : Y \to \mathcal{P}_d^{c(\omega)}$ is transversal to the open cell $(R_d^\omega)^\circ$, then the cardinality of the geometric intersection $\Phi(Y) \cap (R_d^\omega)^\circ$ is exactly the total number of $\hat{v}$-trajectories of the combinatorial type $\omega$ with respect to $\partial X_\Phi$. The intersection points from $\Phi(Y) \cap (R_d^\omega)^\circ$ (equivalently, the trajectories of the type $\omega$ with respect to $\partial X_\Phi$) come in two flavors, $\{+, \ominus\}$, depending on whether the canonical normal orientation of the cell $(R_d^\omega)^\circ$ in $\mathcal{P}_d \Phi$ agrees or disagrees with the preferred orientation of the cycle $\Phi(Y)$.

The next corollary follows directly from Theorem 4.8.

**Corollary 4.11.** Take $\omega = (1, 2, \ldots, 2, 1)$. Let $\Phi : Y^n \to \mathcal{P}_d^{c(\omega)}$ be as in Theorem 4.8.

Then the oriented number of trajectories of the combinatorial type $\omega$ in the pull-back $X^{n+1}_\Phi \subset \mathbb{R} \times Y^n$ equals the linking number of the cycle $\Phi(Y^n)$ with the $(n + 1)$-cycle
\[
\partial R^{(12\ldots21)}_{2n+2} = R^{(32\ldots21)}_{2n+2} - R^{(142\ldots21)}_{2n+2} + \cdots + (-1)^{n-1} R^{(12\ldots241)}_{2n+2} + (-1)^n R^{(12\ldots23)}_{2n+2}.
\]

In particular, the number of $\omega$-tangent trajectories in $X^{n+1}_\Phi$ is at least
\[
\text{lk}(\Phi(Y^n), \partial R^{(12\ldots21)}_{2n+2}) = \langle h^*(\theta_{(12\ldots21)}), [Y] \rangle.
\]

4.11. Comments and questions. The following basic, however non-trivial question animates many of our previous investigations:

"For a given closed poset $\Theta \subset \Omega(d)$, what are the restrictions on the topology of compact manifolds $X$ that admit traversing $\nu$-flows (or their pseudo-envelops/envelops $(\bar{X}, \nu)$) whose tangency patterns avoid $\Theta$?"

In particular, what smooth topological types $X$ may arise via classifying maps $\Phi$ (say, as in Theorem 4.8 or Corollary 4.11)? We do not have any problems with manufacturing $\partial E_d$-regular maps from a given compact $n$-manifold $Y$ to $\mathcal{P}_d^{c\Theta}$, $d \equiv 0 \mod 2$, and using such maps to produce embeddings $\alpha : X \subset \mathbb{R} \times Y$ and traversing flows on $X$ that avoid the $\Theta$-patterns. However, the topological types of such $X$’s are beyond our control: we get what we get... The resulting $X$ is subject to many restrictions, whose nature we do not understand conceptually. Let us sketch just a couple examples which indicate that the restrictions on $X$ by $\Theta$ can be severe.

In [K10], and [K7], Chapters 2-4, we proposed an answer this question for 3-folds $X$. For them, the minimal number $gc(X)$ of $\nu$-trajectories of the combinatorial type $\omega = (1221)$ was introduced as a measure of complexity of a 3-fold $X$ and was linked directly to the combinatorial complexity $c(X)$ of their 2-spines. Via this link, $gc(X)$ is related to the...
classification of 3-folds. For instance, a connected compact oriented 3-fold with a simply-connected boundary, which admits a traversing flow that avoids $\Theta = (1221)$, is a connected sum of several 3-balls and products $S^1 \times S^2$ (K10, Theorem 3.14). Therefore, no other 3-folds with a simply-connected boundary can admit a convex pseudo-envelop or even a traversing flow that avoids the tangency pattern (1221).

In a quite different setting, consider a closed hyperbolic $(n + 1)$-manifold $Z$. Let $X$ be obtained from $Z$ by deleting a number $(n + 1)$-balls. Then no such $X$ can support a traversing $v$-flow that avoids the set of isolated $\hat{v}$-trajectories of combinatorial types $\{\omega \in \Omega_{|\omega| = n}\}$. Indeed, the positivity of Gromov’s simplicial semi-norm $\|Z\|_\Delta$ rules out the possibility of such an avoidance (see [AK], Theorem 1). In fact, the positivity of simplicial semi-norms of various homology classes $h \in H_*(X; \mathbb{R})$ is the only general mechanism known to us that imposes constraints on the combinatorial tangency types of any generic traversing flow on $X$ [K11].

Let us conclude with the following remark. Unfortunately, we do not know examples of invariants that distinguish between the quasitopies $QT_{d,d'}(\hat{X}, \hat{v}; c\Theta; c\Theta')$ and $QT_{d,d'}^{sub}(\hat{X}, \hat{v}; c\Theta; c\Theta')$ of convex envelops and pseudo-envelops. It seems natural to think about invariants that utilize the $k$-multiple self-intersection manifolds $\{\Sigma^a_{\alpha} \}_{k \geq 2}$ of $\alpha^\partial$ (see Remark 4.3). However, for the convex pseudo-envelops, the analogue of the distinguishing map (3.3) in [K9] is trivial: in fact, for a submersion $\alpha : X \to \hat{X}$, the bordism class of the map $\pi \circ \alpha^\partial \circ p_1 : \Sigma^a_{\alpha} \to \hat{X}$ vanishes, due to arguments as in [K9], Corollary 3.1. As a result, a direct analogue of Proposition 3.6 from [K9] is vacuous in the environment of convex envelops.

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