On the Field Strength Formulation of Effective $QED_3$.

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Abstract

Halpern’s field strength’s formulation of gauge theories is applied to effective $QED_3$, namely, a gauge invariant theory for an Abelian gauge field $A_\mu$ with non-localities and self-interactions. The resulting description in terms of the pseudovector field $\tilde{F}_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda$ is applied to different examples.
The search for gauge-invariant descriptions of gauge theories is a subject with a long-standing history. A natural advantage of gauge-invariant formulations is that, when a calculation can be performed within such a scheme, the result is not obscured by the unphysical features introduced in any non-explicitly gauge-invariant setting. In an explicitly gauge-invariant approach, the proper variables should be first identified, and then the dynamics in terms of those variables reconstructed. In general, this procedure is rendered difficult because gauge-invariant variables may be non-local [1,2], or satisfy extremely complicated equations. An interesting formulation has been proposed by Halpern in the seventies [3–6], the so-called field-strength formalism. There are other interesting gauge-invariant formulations. In the one proposed in references [7,8], a procedure to build the Hilbert space in terms of local gauge-invariant variables is explained. In this letter, we shall apply Halpern’s proposal to a simple case where the gauge-invariant description is easily constructed. The case we consider is the dynamics of an Abelian gauge field in $2 + 1$ Euclidean dimensions, without external matter sources. This does not mean that matter fields are absent, but rather that they could have been integrated out, yielding a contribution to the gauge-field action that can be non-local and non-polynomial in general. Such kind of model has been studied in [9], and from a slightly different point of view in references [10,11], which deal with the cases of compact and non-compact QED$_3$.

The cases where the dynamics of a vector field in $2 + 1$ dimensions is dictated by either a Maxwell or a Yang-Mills action have already been considered by Halpern. Our study is concerned with a situation which is, so to speak, halfway between those cases, since our field will be Abelian, but its action non-quadratic (and generally non-local).

The relevance of this kind of model comes from the many applications $2 + 1$ dimensional theories have, particularly in the realm of Condensed Matter systems [12]. The gauge-invariant variable in this case can be identified as the field strength $F_{\mu\nu}$, or better its dual $\tilde{F}_\mu$, and one constructs a description in terms of this pseudo-vector field. A general action for this field will contain terms involving $\tilde{F}_\mu$ and its derivatives, and the functional integral corresponding to it shall include a delta-functional of the Bianchi condition $\partial \cdot \tilde{F}$. Due to
the property, particular to 3 dimensions, of the dual of $F_{\mu\nu}$ being a pseudovector, we are lead to a theory corresponding to a self-interacting pseudovector field, which is constrained to be transverse.

We begin by reviewing Halpern’s derivation, with some small differences due to the action not necessarily being the Maxwell one. The generating functional for Euclidean Green’s functions of an Abelian gauge field $A_\mu$, with a general (possibly non-local) gauge-invariant action in 3 dimensions is

$$Z(J_\mu) = \int \mathcal{D}A_\mu e^{-S_{\text{inv}}(A)} + \int d^3x J_\mu(x) A_\mu(x)$$

where $S_{\text{inv}}(A)$ satisfies $S_{\text{inv}}(A + \partial \omega) = S_{\text{inv}}(A)$, for any $\omega$ vanishing at infinity. Of the many possible forms for $S_{\text{inv}}$, we can construct, a first classification we make is to distinguish between parity-conserving and parity-violating ones, since this property strongly determines the form of the terms that can be included in $S_{\text{inv}}$. Let us first discuss the parity-conserving case. With this assumption, the most general form for $S_{\text{inv}}$ would be an arbitrary functional of $F_{\mu\nu}$, whose terms involve contractions of different powers of this tensor. We chose to work in terms of $\tilde{F}_\mu$, the dual of $F_{\mu\nu}$, defined by $\tilde{F}_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda$. Thus

$$S_{\text{inv}}(A) = I(\tilde{F}_\mu)$$

where $I$ is an arbitrary functional. We now include into (1) the gauge-fixing factor corresponding to the Landau gauge ($\partial \cdot A = 0$)

$$Z(J_\mu) = \int \mathcal{D}A_\mu \delta(\partial \cdot A) e^{-S_{\text{inv}}(A)} + \int d^3x J_\mu(x) A_\mu(x)$$

where we have omitted the field-independent Faddeev-Popov factor det($-\partial^2$), since in this case it can be absorbed into the normalization of the integration measure and has no effect on the Green’s functions derived from (8). To obtain a formulation in terms of $\tilde{F}_\mu$, we introduce in (8) a ‘1’ written as follows:

$$1 = \int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F})$$
Note the presence of a delta functional of the Bianchi identity, which is a consistency condition for the equation \( \dot{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda = 0 \), whose solutions are relevant to the first delta-function. The meaning of the inclusion of that factor can be made explicit by means of the following argument: Consider the rhs of Equation \([\text{4}]\), but this time writing both delta-functionals in terms of functional Fourier transforms:

\[
\int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F}) = \int \mathcal{D}\tilde{F}_\mu \mathcal{D}\lambda_\mu \mathcal{D}\theta \exp \left\{ i \int d^3x [\lambda_\mu(\tilde{F}_\mu - \epsilon_{\mu\nu\rho} \partial_\nu A_\rho) + \theta \partial_\mu \tilde{F}_\mu] \right\}
\]

(5)

where \( \lambda_\mu \) and \( \theta \) are Lagrange multipliers. Integrating out \( \tilde{F}_\mu \) in \([\text{5}]\) yields

\[
\int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F})
\]

\[
= \int \mathcal{D}\lambda \mathcal{D}\theta \delta(\lambda_\mu - \partial_\mu \theta) \exp \left( -i \int d^3x \lambda_\mu \epsilon_{\mu\nu\rho} \partial_\nu A_\rho \right)
\]

\[
= \int \mathcal{D}\theta \exp \left( -i \int d^3x \partial_\mu \theta \epsilon_{\mu\nu\rho} \partial_\nu A_\rho \right) = \int \mathcal{D}\theta \exp \left( i \int d^3x \theta \epsilon_{\mu\nu\rho} \partial_\mu \partial_\nu A_\rho \right),
\]

(6)

where the vanishing of \( F_{\mu\nu} \) at infinity was used on the last line, in order to ignore the surface contribution. We conclude, after integrating out \( \theta \) in \([\text{6}]\) that

\[
\int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F})
\]

\[
= \delta(\epsilon_{\mu\nu\rho} \partial_\mu \partial_\nu A_\rho).
\]

(7)

Thus the ‘1’ behaves as a constant factor when inserted into a functional integration over \( A_\mu \) fields whose second partial derivatives commute\(^1\).

After insertion of the ‘1’, the generating functional becomes

\[
\mathcal{Z}(J_\mu) = \int \mathcal{D}A_\mu \mathcal{D}\tilde{F}_\mu \delta(\partial \cdot A) \delta(\partial \cdot \tilde{F}) \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)e^{-I(\tilde{F}_\mu)} + \int d^3x J_\mu(x) A_\mu(x).
\]

(8)

Now we realize that, by using the two delta-functionals \( \delta(\partial \cdot A) \) and \( \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda) \), \( A_\mu \) can be written in terms of \( \tilde{F}_\mu \):

\(^1\)We are ignoring \( \delta(0) \) factors.
\[ A_\mu = -\epsilon_{\mu\nu\lambda} \frac{1}{\partial^2} \partial_\nu \tilde{F}_\lambda, \]  

(9)

and the dependence on \( A_\mu \) (only from the source term) can be completely erased by replacing it by its expression (9) in terms of \( \tilde{F}_\mu \). The \( A_\mu \) field is thus integrated out, yielding for \( Z \) the expression:

\[
Z(J_\mu) = \int \mathcal{D}\tilde{F}_\mu \delta(\partial \cdot \tilde{F}) e^{-I(\tilde{F}_\mu)} - \int d^3 x J_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \partial^{-2} \tilde{F}_\lambda, 
\]

(10)

which contains only \( \tilde{F}_\mu \) as dynamically variable, and may be thought of as the generating functional for a theory describing the dynamics of a pseudovector field \( \tilde{F}_\mu \), with the constraint \( \partial \cdot \tilde{F} = 0 \). We note that, because of the form of the source term in (10), there is a simple relation between Green’s functions for \( \tilde{F}_\mu \) and the ones for \( A_\mu \):

\[
\langle A_{\mu_1} (x_1) A_{\mu_2} (x_2) \cdots A_{\mu_n} (x_n) \rangle 
= \epsilon_{\mu_1\nu_1\lambda_1} \frac{\partial}{\partial x_1} \epsilon_{\mu_2\nu_2\lambda_2} \frac{\partial}{\partial x_2} \cdots \epsilon_{\mu_n\nu_n\lambda_n} \frac{\partial}{\partial x_n} (F_{\lambda_1} (x_1) F_{\lambda_2} (x_2) \cdots F_{\lambda_n} (x_n)). 
\]

(11)

Although a naive look at (10) may suggest that it is tantamount to a gauge fixed version for some gauge-invariant theory, this is not necessarily the case, as the general form of the ‘action’ \( I \) for the pseudovector field is arbitrary.

We now deal with the parity-violating case. The crucial difference with the previous discussion is that, when parity is violated, (2) is no longer valid. The reason is that now we are allowed to include Chern-Simons like terms, which are functions not only of \( \tilde{F}_\mu \), but also of \( A_\mu \), namely

\[
S_{\text{inv}}(A) = I(\tilde{F}_\mu, A). 
\]

(12)

However, an analogous procedure to the one carried out for the parity-conserving case can be followed here, since \( A_\mu \) can also be expressed in terms of \( \tilde{F}_\mu \) as in (11). This expression for \( A \) in terms of \( \tilde{F}_\mu \) is then inserted into (12), and the generating functional for the parity-violating case becomes:

\[
Z(J_\mu) = \int \mathcal{D}\tilde{F}_\mu \delta(\partial \cdot \tilde{F}) e^{-I(\tilde{F}_\mu)} - \int d^3 x J_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \partial^{-2} \tilde{F}_\lambda. 
\]

(13)
Thus, to calculate correlation functions of $\tilde{F}_\mu$, both for the parity-conserving and parity-violating cases, one has a generating functional corresponding to an ‘action’ $I$ which is a functional of $\tilde{F}_\mu$, with the constraint $\partial \cdot \tilde{F} = 0$.

In order to do actual calculations with the theory defined in terms of $\tilde{F}_\mu$, a set of Feynman rules should be defined. It is convenient to introduce a Lagrange multiplier field $\theta$ in order to deal with the delta-functional $\partial \cdot \tilde{F}$, and also to add a source term for $\theta$, since $\tilde{F}_\mu$ and $\theta$ are coupled. We add a source term for $\tilde{F}_\mu$ (not to be confused with the source for $A_\mu$), since the Green’s functions for $A$ may be obtained by applying (11) to the $\tilde{F}_\mu$’s Green’s functions.

Thus the generating functional we define is

$$Z = \int \mathcal{D}\tilde{F}_\mu \mathcal{D}\theta \exp \left\{ -\int d^3x [I(\tilde{F}_\mu) - i\theta \partial \cdot \tilde{F} - J_\mu \tilde{F}_\mu - j_\theta \theta] \right\}$$

(14)

and Euclidean correlation functions are simply obtained by functional differentiation. Free propagators are obtained from evaluation of the Gaussian integral corresponding to a quadratic action, which in the parity-conserving case becomes

$$I(\tilde{F}_\mu) \equiv I_0(\tilde{F}_\mu) = \int d^3x \frac{1}{2} \tilde{F}_\mu D(-\partial^2) \tilde{F}_\mu$$

(15)

with $D$ a given function without real poles. It is immediate to extract the (momentum space) free propagators that follow from (14) with the action (15)

$$\langle \tilde{F}_\mu \tilde{F}_\nu \rangle = D^{-1}(k^2)(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})$$

$$\langle \theta \theta \rangle = \frac{D(k^2)}{k^2}$$

$$\langle \tilde{F}_\mu \theta \rangle = \frac{k_\mu}{k^2}$$

(16)

We shall now consider some examples of application of the general recipe to different models.

A simple example of an application would be to consider a model defined by the parity-conserving functional

$$I(\tilde{F}_\mu) = \int d^3x \left[ \frac{1}{2} \tilde{F}_\mu D(-\partial^2) \tilde{F}_\mu + \frac{g}{4!} (\tilde{F}_\mu \tilde{F}_\mu)^2 \right]$$

(17)
where $D$ can be a complicated function of $\partial^2$, and $g$ is a coupling constant. The quartic term induces of course vertices with four $\bar{F}_\mu$ lines, and the theory is in that respect quite simple. One should however be a bit careful due to the presence of the Lagrange multiplier field $\theta$. Due to the quadratic mixing it is better to regard $\bar{F}_\mu$ and $\theta$ as two components in some ‘internal space’ of some field, and assign to the propagator the corresponding matrix structure. This is useful when trying to find the expression for Green’s functions in terms of one-particle irreducible ones. The application of this procedure to the full $\bar{F}_\mu$ propagator yields

$$
\langle \bar{F}_\mu \bar{F}_\nu \rangle = \frac{1}{D(k^2) + \Pi^\perp(k^2)} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad (18)
$$

where $\Pi^\perp$ is the transverse component of the irreducible two-point function for the field $\bar{F}_\mu$

$$
\Pi_{\mu\nu}(k^2) = \Pi^\perp(k^2)(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + \Pi^\parallel(k^2) \frac{k_\mu k_\nu}{k^2} \quad (19)
$$

The mixed propagator $\langle \bar{F}_\mu \theta \rangle$ does not renormalizes, and for the $\langle \theta \theta \rangle$ we obtain

$$
\langle \theta \theta \rangle = \frac{D(k^2) + \Pi^\parallel(k^2)}{k^2} \quad (20)
$$

The one-loop correction to the effective action is easily computed within this scheme, and it is even quite straightforward to obtain calculate, in the same approximation, the effective action in the presence of an external ‘monopole’ source $\rho$, introduced by modifying the Bianchi identity in the following way:

$$
\partial \cdot \bar{F} \rightarrow \partial \cdot \bar{F} - \rho(x) \quad (21)
$$

An interesting example of an application is the calculation of the static interaction energy between two (static) monopoles, defined as the part of the effective action depending on the distance between two localized static sources of strengths $\phi_1$ and $\phi_2$ located at $\bar{x}_1$ and $\bar{x}_2$. The corresponding $\rho$ is defined by:

$$
\rho(x) = \phi_1 \delta(x_3) \delta(\bar{x} - \bar{x}_1) + \phi_2 \delta(x_3) \delta(\bar{x} - \bar{x}_2) \quad (22)
$$
The static energy density $\mathcal{E}(\vec{x}_1 - \vec{x}_2)$ is given by

$$\mathcal{E}(\vec{x}_1 - \vec{x}_2) = \lim_{L \to \infty} \frac{1}{L^3} \int \mathcal{D}\tilde{F}_\mu \mathcal{D}\theta \exp \left\{ -I(\tilde{F}) + i \int d^3x \theta (\partial \cdot \tilde{F} - \rho) \right\} \exp \left\{ -I(\tilde{F}) + i \int d^3x \theta (\partial \cdot \tilde{F}) \right\}$$

(23)

where $L$ is the length of the Euclidean box where the theory is defined.

In the one-loop approximation, $\mathcal{E}$ becomes

$$\mathcal{E}(\vec{x}_1 - \vec{x}_2) = \phi_1 \phi_2 \gamma(r)$$

(24)

where $r = |\vec{r}| = \vec{x}_1 - \vec{x}_2$, and

$$\gamma(r) = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot \vec{r}} \frac{1}{I^{-1}(k)} k_\mu k_\nu$$

(25)

with $I_{\mu\nu} = \frac{\delta^2 I}{\delta \tilde{F}_\mu(x_1) \delta \tilde{F}_\nu(x_2)}$. This formula yields the interaction potential $\gamma$ as a complicated functional of the inputs of the effective theory.

For the particular case of a static $F_{\mu\nu}$, and generalizing from the quartic potential to a general one $V(\tilde{F}^2)$, the form of $\gamma$ can be further simplified to

$$\gamma(r) = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot \vec{r}} \frac{1}{k^2[D(k^2) + 2V'(\tilde{F}^2)]}$$

(26)

As another example, we note that the situation, particular to $2 + 1$ dimensions, of $\tilde{F}_\mu$ being a one-form field, allows us to construct action functionals $I$ depending only on the ‘field strength’ $W_{\mu\nu} = \partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu$. That is to say, one can consider models where $\tilde{F}$ plays the role of a connection. Any such functional $I$ will be invariant under a new set of gauge transformations, defined as

$$\tilde{F}_\mu \rightarrow \tilde{F}_\mu + \partial_\mu \omega$$

(27)

This gauge invariance of $I$ allows us to regard now the constraint $\partial \cdot \tilde{F}$ as a particular gauge fixing for this symmetry, and thus to use a different gauge fixing without affecting the physics. For example, one ends up with a generating functional of the form:
\[ Z(J_\mu) = \int D\tilde{F}_\mu e^{-I(\partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu) - \int d^3x \frac{1}{2\alpha} (\partial_\mu \tilde{F}_\mu)^2 + \int d^3x J_\mu \tilde{F}_\mu}. \]  

(28)

when the family of covariant \( \alpha \)-gauges is used. Of course, physical results should be independent of \( \alpha \). The physical meaning of this independence of physical results on \( \alpha \) would be at first sight surprising, since it means that one can modify the Bianchi identity quite arbitrarily, introducing monopoles into the play without altering the physics. The reason is that, in the original variables, this kind of model depends on \( A_\mu \) only through the combination \( \partial_\mu F_{\mu\nu} \), namely

\[ S_{\text{inv}}(A) = \mathcal{F}(\partial_\mu F_{\mu\nu}) \]  

(29)

where \( \mathcal{F} \). This automatically imposes the existence of second derivatives for \( A_\mu \), forbidding the existence of monopoles.

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