Lorentz-covariant deformed algebra with minimal length

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Abstract

The $D$-dimensional two-parameter deformed algebra with minimal length introduced by Kempf is generalized to a Lorentz-covariant algebra describing a $(D+1)$-dimensional quantized space-time. For $D = 3$, it includes Snyder algebra as a special case. The deformed Poincaré transformations leaving the algebra invariant are identified. Uncertainty relations are studied. In the case of $D = 1$ and one nonvanishing parameter, the bound-state energy spectrum and wavefunctions of the Dirac oscillator are exactly obtained.

1 Introduction

Studies in string theory and quantum gravity suggest the existence of a minimal observable length. Quantum mechanically this is described by a nonzero minimal uncertainty in position. Perturbative string theory \cite{1} actually leads to a generalized uncertainty principle (GUP) $\Delta X \geq \frac{\hbar}{2} (\Delta P + \beta \Delta P)$, where $\beta$ is some very small positive parameter. As a consequence $\Delta X \geq \Delta X_0 = \hbar \sqrt{\beta}$. Such a GUP is also in line with the proposed UV/IR mixing.

This generalized uncertainty relation implies some modification of the canonical commutation relations (CCR), for which there have been several suggestions. In this communication, we try to reconcile an old proposal of Snyder, dating back to 1947 \cite{2}, with a more recent one, due to Kempf \cite{3}. In the former, the assumption of continuous space-time is
abandoned, this leading to a Lorentz-covariant quantized space-time, where the existence of a natural unit of length is connected with the non-commutativity of coordinates. In the latter, small quadratic corrections to the CCR are shown to lead to a GUP of the required form, but the resulting algebra is not Lorentz covariant. We plan to show here that it is possible to construct a Lorentz-covariant algebra, reminiscent of that of Kempf and including Snyder algebra as a special case (for more details see [4]).

2 Deformed algebra generalizing that of Kempf

In $D$ dimensions, Kempf considered a deformed algebra, whose commutation relations are given by

$$
\begin{align*}
[X^i, P^j] &= -i\hbar[(1 + \beta P^2)g^{ij} - \beta' P^i P^j], \\
[X^i, X^j] &= i\hbar\frac{2\beta - \beta' + (2\beta + \beta')\beta P^2}{1 + \beta P^2}(P^i X^j - P^j X^i), \\
[P^i, P^j] &= 0,
\end{align*}
$$

(1)

where $X^i, P^i, i = 1, 2, \ldots, D$, denote the position and momentum coordinates, while $\beta$ and $\beta'$ are two very small nonnegative deforming parameters [3]. This algebra gives rise to (isotropic) nonzero minimal uncertainties in the position coordinates $(\Delta X^i)_0 = (\Delta X)_0 = \hbar\sqrt{D\beta + \beta'}$, which for $D = 1$ and $\beta' = 0$ reduce to the value deriving from the GUP mentioned in Sec. 1. In the momentum representation, the deformed position and momentum operators $X^i, P^i$ are represented by

$$
\begin{align*}
X^i &= (1 + \beta p^2)x^i + \beta' p^i(p \cdot x) + i\hbar\gamma p^i, \\
P^i &= p^i,
\end{align*}
$$

(2)

where $x^i = i\hbar\partial/\partial p^i$ and $\gamma$ is an arbitrary real constant, which does not enter the commutation relations (1), but affects the definition of the scalar product in momentum space.

The algebra (1) can be converted into a Lorentz-covariant one, by replacing $p^2$ and $p \cdot x$ in (2) by the Lorentz-invariant expressions $p^2 - (p^0)^2 = -p_\nu p^\nu$ and $p \cdot x - p^0 x^0 = -p_\nu x^\nu$, where greek indices run over $0, 1, 2, \ldots, D$ and $x^\nu, p^\nu$ are contravariant vectors in a $(D + 1)$-dimensional space-time ($[x^\mu, p^\nu] = -i\hbar g^{\mu\nu}$). Instead of (2), we therefore consider the operators

$$
\begin{align*}
X^\mu &= (1 - \beta p_\nu p^\nu)x^\mu - \beta' p^\mu p_\nu x^\nu + i\hbar\gamma p^\mu, \\
P^\mu &= p^\mu,
\end{align*}
$$

(3)
which are \((D + 1)\)-vectors. From (3), we obtain
\[
\begin{align*}
[X^\mu, P^\nu] &= -i\hbar[(1 - \beta P_\rho P^\rho)g^{\mu\nu} - \beta' P^\mu P^\nu], \\
[X^\mu, X^\nu] &= i\hbar\frac{2\beta - \beta' - (2\beta + \beta')\beta P_\rho P^\rho}{1 - \beta P_\rho P^\rho}(P^\mu X^\nu - P^\nu X^\mu), \\
[P^\mu, P^\nu] &= 0,
\end{align*}
\]
(4)
where we still assume \(\beta\) and \(\beta'\) to be two very small nonnegative parameters.

The algebra defined in (4) can be shown to be left invariant under standard Lorentz transformations. In the special case where \(D = 3\) and \(\beta = \gamma = 0\), it reduces to Snyder algebra with \(\beta'\) related to the unit of length contained in the latter. Furthermore, since Kempf algebra, although very similar to (4), cannot be obtained by taking the nonrelativistic limit of the latter, one may say that the algebra proposed here is a truly new one.

The operators \(X^\mu\) and \(P^\mu\) are Hermitian with respect to a new scalar product in momentum space (in agreement with that of Snyder for \(D = 3\) and \(\beta = \gamma = 0\))
\[
\langle \psi | \phi \rangle = \int \frac{d^Dp}{[1 - (\beta + \beta')p_\rho p^\rho]^{\alpha}} \psi^*(p^\mu)\phi(p^\mu),
\]
(5)
where \(\alpha = [2(\beta + \beta')]^{-1}[2\beta + \beta'(D + 2) - 2\gamma]\). The weight function in (5) is free from singularities if physically acceptable states satisfy the condition
\[
(\beta + \beta')(p^0)^2 < 1.
\]
(6)
This means that the energy \(E = cP^0 = cp^0\) is not allowed to take very large values violating condition (6).

3 Deformed Poincaré transformations

As we already mentioned in Sec. 2, the standard infinitesimal proper Lorentz transformations \(X'^\mu = X^\mu + \delta X^\mu, P'^\mu = P^\mu + \delta P^\mu\), where \(\delta X^\mu = \delta \omega^{\mu \nu} X^\nu, \delta P^\mu = \delta \omega^{\mu \nu} P^\nu\) with \(\delta \omega^{\mu \nu} = -\delta \omega^{\nu \mu} \in \mathbb{R}\), leave the algebra (4) invariant. The corresponding generators \(\hat{L}_{\alpha \beta} = (1 - \beta P_\rho P^\rho)^{-1}(X_\alpha P_\beta - X_\beta P_\alpha)\), such that \(\delta X^\mu = [i/(2\hbar)]\delta \omega^{\alpha \beta} [\hat{L}_{\alpha \beta}, X^\mu]\) and \(\delta P^\mu = [i/(2\hbar)]\delta \omega^{\alpha \beta} [\hat{L}_{\alpha \beta}, P^\mu]\), are however deformed, although they satisfy the standard so\((D,1)\) commutation relations
\[
[\hat{L}_{\alpha \beta}, \hat{L}_{\rho \sigma}] = -i\hbar \left( g_{\alpha \rho} \hat{L}_{\beta \sigma} - g_{\alpha \sigma} \hat{L}_{\beta \rho} - g_{\beta \rho} \hat{L}_{\alpha \sigma} + g_{\beta \sigma} \hat{L}_{\alpha \rho} \right).
The invariance of (4) under proper Lorentz transformations can be extended to improper ones as it can be checked from the action of the discrete symmetries, namely parity $P$ and time reversal $T$.

Infinitesimal translations also leave (4) invariant provided they are deformed as $X'^\mu = X^\mu + \delta X^\mu$, $P'^\mu = P^\mu$, where $\delta X^\mu = -\delta a^\mu - g(P_\rho P^\rho)\delta a_\rho P^\mu$ with $\delta a^\mu \in \mathbb{R}$ and $g(P_\rho P^\rho) = (1-\beta P_\rho P^\rho)^{-2}[2\beta - \beta' -(2\beta+\beta')\beta P_\rho P^\rho]$. With these new definitions, the deformed generators $\hat{P}_\alpha = (1-\beta P_\rho P^\rho)^{-1}P_\alpha$, such that $\delta X^\mu = (i/\hbar)\delta a^\alpha [\hat{P}_\alpha, X^\mu]$ and $\delta P^\mu = (i/\hbar)\delta a^\alpha [\hat{P}_\alpha, P^\mu]$, fulfill the standard commutation relations

$$[\hat{P}_\alpha, \hat{P}_\beta] = 0, \quad [\hat{L}_{\alpha\beta}, \hat{P}_\rho] = i\hbar \left( g_{\beta\rho} \hat{P}_\alpha - g_{\alpha\rho} \hat{P}_\beta \right).$$

We conclude that the deformed operators $\hat{L}_{\alpha\beta}$ and $\hat{P}_\alpha$ provide us with a realization of the conventional Poincaré algebra $iso(D,1)$ leaving (4) invariant.

4 Uncertainty relations for position and momentum

Since the algebra (4) is invariant under rotations, we may choose any pair of position and momentum components $X^i, P^i (i \in \{1, 2, \ldots, D\})$ to determine the deformed uncertainty relation:

$$\Delta X^i \Delta P^i \geq \frac{\hbar}{2} \left| 1 - \beta \left\{ \langle (P^0)^2 \rangle - \sum_{j=1}^{D} [(\Delta P^j)^2 + \langle P^j \rangle^2] \right\} + \beta' \left[ (\Delta P^i)^2 + \langle P^i \rangle^2 \right] \right|.$$

If, for simplicity’s sake, we assume isotropic uncertainties $\Delta P^j = \Delta P$, $j = 1, 2, \ldots, D$, we get for each $X^i$ an uncertainty relation similar to the GUP in Sec. 1, from which it results that $\Delta X^i$ has a nonvanishing minimum

$$\Delta X^i_{\min} = \hbar \sqrt{(D\beta + \beta') \left\{ 1 - \beta \left[ \langle (P^0)^2 \rangle - \sum_{j=1}^{D} \langle P^j \rangle^2 \right] + \beta' \langle P^i \rangle^2 \right\}}$$

provided the quantity between curly brackets on the right-hand side is positive. This condition is always satisfied by physically acceptable states due to Eq. (3). We therefore arrive at an isotropic absolutely smallest uncertainty in position given by

$$(\Delta X)_0 = (\Delta X^i)_0 = \hbar \sqrt{(D\beta + \beta') \left[ 1 - \beta \langle (P^0)^2 \rangle \right]}.$$

As compared with Kempf’s result, there is an additional factor $\sqrt{1 - \beta \langle (P^0)^2 \rangle}$ reducing $(\Delta X)_0$. 

4
5 Application to the $(1+1)$-dimensional Dirac oscillator

The $3+1$-dimensional Dirac oscillator (DO) was introduced a long time ago [5], but its name was only coined later on [6]. This system has aroused a lot of interest both because it is one of the few examples of exactly solvable Dirac equation and because it can be applied to a lot of physical problems. As it is the only relativistic problem that has been solved with Kempf algebra [7] (see also [8] for the $1+1$-dimensional case), it is interesting to see what is the effect of the deformed algebra (4) on this system.

For simplicity’s sake, we consider here the simplifying assumptions $D = 1$ and $\beta' = 0$. On taking for the Dirac $2 \times 2$ matrices $\hat{\alpha}_x$ and $\hat{\beta}$ the standard Pauli spin matrices $\sigma_x$ and $\sigma_z$, respectively, and going to dimensionless operators $\tilde{X}^\mu = X^\mu/a$, $\tilde{P}^\mu = (a/\hbar)P^\mu$, $\mu = 0, 1$, where $a = \hbar/(mc)$, the DO equation reads in momentum representation as

$$(\sigma_x \tilde{P} - \tilde{\omega} \sigma_y \tilde{X} + \sigma_z)\psi(\tilde{p}, \tilde{p}^0) = \tilde{P}^0 \psi(\tilde{p}, \tilde{p}^0),$$

where

$\tilde{P}^0 = \tilde{p}^0 \quad \tilde{P} = \tilde{p}$,

$\tilde{X}^0 = -i f(\tilde{p}, \tilde{p}^0) \frac{\partial}{\partial \tilde{p}^0}$,

$\tilde{X} = i f(\tilde{p}, \tilde{p}^0) \frac{\partial}{\partial \tilde{p}}$

and $\tilde{p}^\mu = (a/\hbar)p^\mu$, $\tilde{\omega} = \hbar \omega/(mc^2)$, $\tilde{\beta} = \beta m^2 c^2$.

On separating the wavefunction $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ into large $\psi_1$ and small $\psi_2$ components and defining

$B^\pm = \tilde{P} \pm i \tilde{\omega} \tilde{X} = \tilde{p} \mp \tilde{\omega} f(\tilde{p}, \tilde{p}^0) \frac{\partial}{\partial \tilde{p}}$,

the DO equation (7) can be written as two coupled equations

$$B^+ \psi_2(\tilde{p}, \tilde{p}^0) = (\tilde{p}^0 - 1) \psi_1(\tilde{p}, \tilde{p}^0), \quad B^- \psi_1(\tilde{p}, \tilde{p}^0) = (\tilde{p}^0 + 1) \psi_2(\tilde{p}, \tilde{p}^0).$$

These lead to two factorized equations

$$B^+ B^- \psi_1(\tilde{p}, \tilde{p}^0) = e(\tilde{p}^0) \psi_1(\tilde{p}, \tilde{p}^0), \quad B^- B^+ \psi_2(\tilde{p}, \tilde{p}^0) = e(\tilde{p}^0) \psi_2(\tilde{p}, \tilde{p}^0),$$

for the large and small components, respectively. Here $e(\tilde{p}^0) \equiv (\tilde{p}^0)^2 - 1$.

Such factorized equations have the form of energy-eigenvalue equations in SUSYQM [9]. It should be stressed, however, that the starting Hamiltonian $H = B^+ B^-$ depends on $(\tilde{p}^0)^2$.
which determines the eigenvalues $e(\tilde{p}^0)$. Hence, in contrast with conventional SUSYQM, $H$ is energy dependent. We have however shown that provided one proceeds with care, the shape-invariance method can be used to determine the possible values of $e(\tilde{p}^0)$. The result reads $e_n(\tilde{p}^0) = \tilde{\omega} n (2 + \tilde{\beta} \tilde{\omega} n) [1 - \tilde{\beta}(\tilde{p}^0)^2]$.

For such an expression to be compatible with the definition of $e(\tilde{p}^0)$, given below (9), $\tilde{p}^0$ must be quantized and its allowed values given by

$$\tilde{p}^0_{n,\tau} = \tau \left( \frac{1 + \tilde{\omega} n (2 + \tilde{\beta} \tilde{\omega} n)}{1 + \tilde{\beta} \tilde{\omega} n (2 + \tilde{\beta} \tilde{\omega} n)} \right)^{1/2} = \frac{\tau}{\sqrt{\beta}} \left( 1 + \frac{\beta - 1}{(1 + \beta \tilde{\omega} n)^2} \right)^{1/2},$$

where $\tau = \pm 1$ and $n$ may, in principle, run over $n = 0, 1, 2, \ldots$. Since $E = cp^0 = mc^2 \tilde{p}^0$, the Dirac oscillator energy spectrum is given by

$$E_{n,\tau} = \frac{\tau c}{\sqrt{\beta}} \left( 1 + \frac{\beta m^2 c^2 - 1}{(1 + \beta m \omega n)^2} \right)^{1/2},$$

(10)

where the values of $(n, \tau)$ depend on the existence of normalizable solutions to Eq. (8). It can actually be proved that $n = 0, 1, 2, \ldots$, or $n = 1, 2, \ldots$, according to whether $\tau = +1$ or $\tau = -1$.

Eq. (10) has three important consequences. Firstly, it shows that we have to restrict ourselves to deformations such that $\beta < 1/(m^2 c^2)$, because otherwise we would get decreasing values of $|E_{n,\tau}|$ when $n$ increases — an unphysical behaviour for an energy spectrum. This is also confirmed by the existence of well-behaved wavefunctions in such a range of $\beta$ values. Secondly, the energy spectrum turns out to be bounded ($mc^2 \leq |E_{n,\tau}| < c/\sqrt{\beta}$) in contrast with what happens for the conventional DO. It can actually be shown that in the limit where $\beta$ vanishes, the usual unbounded energy spectrum is retrieved. Thirdly, in the nonrelativistic limit $\hbar \omega/(mc^2) \ll 1$, we get the spectrum obtained with Kempf algebra [8], except for the presence of the additional factor $(1 + \beta m \hbar \omega n)^{-1}$ in the energies. This is another evidence of the novelty of the Lorentz-covariant deformed algebra considered here, as compared with Kempf one.

The wavefunctions associated with the spectrum (10) can be found in two steps. First, SUSYQM techniques are used to determine the eigenfunctions of $H$ and of its SUSY partner, satisfying Eq. (9). From them, the large and small components, $\psi_1^{(n,\tau)}(\tilde{p}) \equiv \psi_1(\tilde{p}, \tilde{p}^0_{n,\tau})$ and $\psi_2^{(n,\tau)}(\tilde{p}) \equiv \psi_2(\tilde{p}, \tilde{p}^0_{n,\tau})$, fulfilling Eq. (8), are then obtained. Note that the latter step is not trivial because though the solutions of (8) provide us with solutions to (9), the converse
is not necessarily true. As a result of the calculations, we get wavefunctions normalized with respect to the scalar product \([5]\), but since the weight function \(1/f(\tilde{\rho}, \tilde{\rho}_n, \tau)\) in (5) is energy dependent, the standard orthogonality relation between separate bound states is lost. This problem is one of the many known puzzles inherent in the use of energy-dependent Hamiltonians in quantum mechanics (for a recent review see [10]).

In conclusion, we would like to mention a few remaining problems for future investigations: (i) a thorough analysis of the time-energy uncertainty, required by its special status in quantum mechanics; (ii) an attempt to restore some of the properties of ordinary quantum mechanics that are spoilt by the energy dependence of the Dirac oscillator Hamiltonian; and (iii) a search for a physical interpretation of the deforming parameter \(\beta\), taking into account that \(\beta'\), being related to Snyder natural unit of length, has already received one [11].

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