Generalized golden mean and the efficiency of thermal machines

Ramandeep S Johal

Department of Physical Sciences, Indian Institute of Science Education and Research Mohali, Sector 81, S.A.S. Nagar, Manauli PO 140306, Punjab, India

E-mail: rsjohal@iisermohali.ac.in

Received 16 June 2020, revised 20 July 2020
Accepted for publication 28 July 2020
Published 14 October 2020

Abstract

We investigate generic heat engines and refrigerators operating between two heat reservoirs, for the condition when their efficiencies are equal to each other. It is shown that the corresponding value of efficiency is given as the inverse of the generalized golden mean, $\phi_p$, where the parameter $p$ depends on the degrees of irreversibility of both engine and refrigerator. The reversible case ($p = 1$) yields the efficiency in terms of the standard golden mean. We also extend the analysis to a three-heat-reservoir setup.

Keywords: thermal machines, golden ratio, thermal efficiency

(Some figures may appear in colour only in the online journal)

1. Introduction

Although the golden mean (golden ratio) has engaged artists, mathematicians and philosophers since antiquity [1–4], it has become appreciated more recently that it is not a unique number as far as many of its algebraic and geometric properties are concerned [5, 6]. In fact, one of the simplest generalizations of the golden mean may be defined by the positive solution, $\phi_p$, of the following quadratic equation:

$$\phi^2 - px - 1 = 0,$$

(1)

where $p$ is a given positive number. The number $\phi_p$ is given by

$$\phi_p = \sqrt{p^2 + 4} + p,$$

(2)

also referred as the $p$th order extreme mean (POEM) [5]. More specifically, when $p$ is a positive integer $n$, it is addressed as the $n$th order extreme mean (NOEM) [6] or a member of the family...
of metallic means. For example, $p = 1$ gives the golden mean $\phi_1 = (\sqrt{5} + 1)/2$; $p = 2$ yields the silver mean, $\phi_2 = \sqrt{2} + 1$, and so on. Among other mathematical constructions, the ratio of successive terms in the generalized Fibonacci sequence $F_n = pF_{n-1} + F_{n-2}$ (with $F_0 = 0$ and $F_1 = 1$), tend to this number:

$$
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi_p.
$$

(3)

It also relates the lengths of diagonals of a regular odd $n$-gon ($n \geq 5$). For other properties and identities related to $\phi_p$, see reference [6].

The generalized golden mean $(\phi_p)$ appears as the optimal solution determining the shape of Newton’s frustum that faces the least resistance while moving through a rare medium [7]. The metallic means family has been related to quasiperiodic dynamics [8]. Another simple physical example is a semi-infinite resistor network [9, 10] as shown in figure 1(a), whose equivalent resistance $(r')$ between points A and B satisfies the quadratic equation: $(r')^2 - rr' - 1 = 0$, and therefore, $r' = \phi_r$. In this article, we describe an occurrence of the generalized golden mean in the context of thermodynamics by relating this number to the efficiencies of a heat engine and refrigerator. The analysis is accessible to undergraduate students of physics and engineering sciences. It exposes them to generic expressions for the thermal efficiencies of irreversible machines and how the generalized golden mean connects the performance measures of engines and refrigerators.

2. Two-heat-reservoir setup

A heat engine and a refrigerator constitute basic mechanisms of a thermal machine operating, say, between two heat reservoirs at unequal temperatures $T_h$ and $T_c$ ($T_c < T_h$) [11]. An irreversible machine always leads to a net increase in the entropy of the Universe, while the entropy of the Universe is conserved by an ideal or reversible machine. The efficiency of such a machine achieves the upper bound given by the Carnot value $1 - T_c/T_h$ for a heat engine,
and $T_c/(T_h - T_c)$ for a refrigerator. On the other hand, the efficiency of an irreversible thermal machine is always less than the Carnot bound.

**Heat engine.** We first consider an irreversible heat engine working in a cycle, and suppose that $W$ amount of work is extracted per cycle when $Q_h$ amount of heat is absorbed from the hot reservoir. The total entropy generated per cycle is

$$\Delta_{tot}S = -\frac{Q_h}{T_h} + \frac{Q_c}{T_c} > 0.$$  

(4)

where $Q_c = Q_h - W$, is the heat rejected to the cold reservoir. All quantities defined above are positive. From equation (4), we can express the work output of an irreversible cycle as

$$W = (1 - \theta)Q_h - T_c\Delta_{tot}S.$$  

(5)

where $\theta = T_c/T_h$. Let $\Delta S = Q_h/T_h$ be the entropy transferred from the hot reservoir to the working medium, in the reversible case. Then, the efficiency, $E = W/Q_h$, can be written as

$$E = 1 - \left(1 + \frac{\Delta_{tot}S}{\Delta S}\right)\theta.$$  

(6)

Here, we define the ‘irreversibility’ parameter, $z = 1 + \Delta_{tot}S/\Delta S > 1$, so that the efficiency of the engine can be expressed as $E = 1 - z\theta$. The reversible case corresponds to $\Delta_{tot}S = 0$, or $z = 1$, yielding Carnot efficiency equal to $1 - \theta$. In other words, $0 \leq E \leq 1 - \theta$ implies $1 \leq z \leq 1/\theta$.

**Refrigerator.** Now, let us consider the machine in the refrigerator mode. Suppose, an input work $\mathcal{W}$ is required to transport $\mathcal{Q}_c$ amount of heat against the temperature gradient. Let $Q_h = Q_c + \mathcal{W}$, be the amount of heat deposited in the hot reservoir. The efficiency of a refrigerator is defined as $R = Q_c/\mathcal{W}$. The total entropy generated per cycle [11] is

$$\Delta_{tot}\mathcal{S} = \frac{Q_h}{T_h} - \frac{Q_c}{T_c} > 0.$$  

(7)

Let $\Delta\mathcal{S}$ be the amount of heat transferred reversibly, from the cold reservoir to the working medium. Then we have $\mathcal{Q}_c = T_c\Delta\mathcal{S}$, and so, we can write

$$\mathcal{W} = \frac{1 - \theta}{\theta} \mathcal{Q}_c + T_h\Delta_{tot}\mathcal{S}.$$  

(8)

The efficiency is then given by

$$R = \theta\left(1 + \frac{\Delta_{tot}\mathcal{S}}{\Delta\mathcal{S}}\right)^{-1}.$$  

(9)

Analogous to the case of engine, we define the parameter $z' = 1 + \Delta_{tot}\mathcal{S}/\Delta\mathcal{S} > 1$, and express the efficiency of the refrigerator as $R = \theta(z' - \theta)^{-1}$. In the reversible case, $z' = 1$, and so $R = \theta(1 - \theta)^{-1}$. Note that, unlike the parameter $z$, $z'$ is not bounded from above.

In the above, we have identified the expressions for efficiencies as functions of the ratio of temperatures as well as the irreversibility parameter $z$ or $z'$. These expressions refer to any irreversible thermal cycle between the two reservoirs, with given values of $\theta$, $z$ and $z'$. We may now ask, for what ratio of temperatures, a heat engine and a refrigerator with the given values of their respective irreversibility parameters, have the same efficiency, and what is its value?
Thereby, setting $E = R$, and solving for $0 < \theta < 1$, we obtain

$$\theta = \frac{zz' + 2 - \sqrt{(zz')^2 + 4}}{2z}. \quad (10)$$

Then, the efficiency, $E = 1 - z\theta$, at the above condition is given by

$$E = R = \frac{\sqrt{(zz')^2 + 4 - zz'}}{2}. \quad (11)$$

Interestingly, the above expression for the efficiency depends only on the product of the irreversibility parameters. Thus, we may define $zz' \equiv p$, and rewrite equation (11) as follows:

$$E = \frac{\sqrt{p^2 + 4} - p}{2} = \frac{1}{\phi_p}, \quad (12)$$

where $E$ is expressed in terms of $\phi_p$ from equation (2). Note that, in the above $p \geq 1$, where $p = 1$ implies the reversible case, with $z = z' = 1$, yielding $E = R = (\sqrt{5} - 1)/2 = 1/\phi_1$, which was earlier noted in reference [12]. Correspondingly, the ratio of temperatures in the reversible case is required to be: $\theta = (3 - \sqrt{5})/2 = 2 - \phi_1$.

### 3. Three-heat-reservoir setup

Next, we show the occurrence of the generalized golden mean in a slightly different setting. Let us consider three heat reservoirs with temperatures ordered as $T_l < T_c < T_h$. Assume that we can operate an engine between the reservoirs at $T_h$ and $T_l$, and a refrigerator between the reservoirs at $T_c$ and $T_l$ (see figure 2). Further, let these be ideal or reversible machines, so that the respective thermal efficiencies are given as

$$E_{hl} = \frac{T_h - T_l}{T_h}, \quad R_{cl} = \frac{T_c - T_l}{T_c - T_l}. \quad (13)$$

Now, for given values of $T_h$ and $T_c$, we look for the temperature $T_l$ such that these two efficiencies become equal. Setting $E_{hl} = R_{cl}$, we obtain a quadratic equation for $T_l$:

$$T_l^2 - (2T_h + T_c)T_l + T_hT_l = 0, \quad (14)$$

whose solutions are

$$T_l = \frac{T_h}{2} \left[ \theta + 2 \pm \sqrt{\theta^2 + 4} \right], \quad (15)$$

where $\theta = T_c/T_h$. Only the negative root above yields an allowed value of the efficiency, given by

$$E_{hl} = R_{cl} = \frac{\sqrt{\theta^2 + 4} - \theta}{2} = \frac{1}{\phi_\theta}. \quad (16)$$

Note that the generalized golden mean appears above within a reversible setup. Secondly, due to $0 < \theta < 1$, it covers a range of parameters complementary to the expression (12), where $p \geq 1$. It is interesting to note that the three-reservoir condition may be reduced to a two-reservoir condition by taking $\theta \rightarrow 1$ or $T_c \rightarrow T_h$ (see figure 2). The corresponding value of the efficiency, from equation (16), reduces to $1/\phi_1$. 

4
Finally, we extend our exploration on the three-heat-reservoir case also into the irreversible regime. Consider an irreversible engine between $T_h$ and $T_l$, specified by the irreversibility parameter $z$. Likewise, let the corresponding parameter for the refrigerator operating between $T_c$ and $T_l$ be $z'$. Then, following section 2, the efficiencies of these machines can be written as

$$\bar{E}_{hl} = 1 - \frac{z T_l}{T_h}, \quad \bar{R}_{cl} = \frac{T_l}{z T_c - T_l}. \quad (17)$$

Now, setting the condition $\bar{E}_{hl} = \bar{R}_{cl}$, we can solve for $T_l$:

$$T_l = T_h \frac{2 \theta}{z} \left[ p \theta + 2 \pm \sqrt{(p \theta)^2 + 4} \right], \quad (18)$$

where $p = zz'$. Consequently, the allowed solution for efficiency is given by

$$\bar{E}_{hl} = \frac{\sqrt{(p \theta)^2 + 4} - p \theta}{2} = \frac{1}{\phi p \theta}. \quad (19)$$

For $p = 1$, we obtain the reversible case discussed above (equation (16)).

We note that the two-reservoir irreversible case may be considered as a special case of the three-reservoir irreversible case. For that purpose, we again let $\theta \to 1$ in the latter case above. Then, the setup reduces to two reservoirs at $T_h$ and $T_l$. Thus, equation (17) simplifies to $\bar{E}_{hl} = 1 - \frac{z T_l}{T_h}$ and $\bar{R}_{cl} = T_l / (z T_c - T_l)$, for which the condition $\bar{E}_{hl} = \bar{R}_{cl}$ implies that each efficiency is equal to $1 / \phi_p$, as obtained in equation (12).

4. Conclusions

In the above, we have observed that the generalized golden mean determines the efficiencies of a heat engine and a refrigerator when the latter are set equal to each other. When both the machines are reversible, the efficiency is related to the standard golden mean. In order to generalize to irreversible domain, we have first recast the efficiency of a generic heat engine and a refrigerator in terms of an irreversibility parameter and the ratio of the reservoir temperatures. Then, the efficiency depends only on the product of the two irreversibility parameters. The importance of this step can be appreciated by noting that only for the reversible case, we have $R = Q_c / W_h$ along with $E = W / Q_h$, where $Q_c = Q_h - W$, i.e. the same amounts of heat and
work appear for a reversible heat engine as well as for a refrigerator. So, in this case, we can express the condition $E = R$, as follows:

$$\frac{W}{Q_h} = \frac{Q_c}{W}. \quad (20)$$

From this, we obtain the equation: $E = E^{-1} - 1$, whose solution is $E = R = 1/\phi_1$ [12]. However, the efficiencies in the irreversible case are $E = W/Q_h$ and $R = Q_c/W'$, which are not useful for applying the $E = R$ condition. Thereby, the forms (6) and (9) have been employed. Further, we have also extended this condition to three-reservoir scenario which also includes the limiting case of two-reservoir setup. It will be interesting to identify physical situations leading to the equality of the efficiencies of engine and refrigerator under the given conditions. Here, we have discussed one possible setup using three heat reservoirs. The interested reader can identify other engine-refrigerator pairs in the three-reservoir setup which lead to equal efficiencies, expressed in terms of suitable generalized golden means.

**References**

[1] Coxeter H S M 1969 *Introduction to Geometry* 2nd edn (New York: Wiley)
[2] Ogilvy C S 1990 *Excursions in Geometry* (New York: Dover) pp 122–5
[3] Markowsky G 1992 *Coll. Math. J.* 23 2
[4] Livio M 2002 *The Golden Ratio: The Story of Phi, The World’s Most Astonishing Number* (New York: Broadway Books)
[5] Falbo C 2005 *Coll. Math. J.* 36 123
[6] Fowler D H 1982 *Fibonacci Q.* 20 146
[7] Cruz-Sampedro J and Tetlalmatzi-Montiel M 2010 *Coll. Math. J.* 41 145
[8] Spindel V W D 1997 *Chaos Solitons Fractals* 8 1631
[9] Srinivasan T P 1992 *Am. J. Phys.* 60 461
[10] Kasperski M and Kloubus W 2013 *Eur. J. Phys.* 35 015008
[11] Zemansky M and Dittman R 1997 *Heat and Thermodynamics: An Intermediate Textbook (International Series in Pure and Applied Physics)* (New York: McGraw-Hill)
[12] Popkov V V and Shipitsyn E V 2000 *Phys. Usp.* 43 1155