Interpenetration of matter in plate theories obtained as Γ-limits

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Abstract

We consider a potential pathology in the derivation of plate theories as Γ-limits of 3-dimensional nonlinear elasticity [11, 12], which consists in recovery sequences of invertible maps that converge to a non-invertible limit. These pathologies have been noted in [15] in a different context. Using a combination of degree theory, the approximation of Sobolev functions by Lipschitz functions and geometric rigidity we show that the potentially pathological situation in the derivation of plate theories provides sufficient conditions for the self-intersection of the graph of the recovery sequence element, and thus the pathology is excluded.

1 Introduction

In the mathematical theory of nonlinear elasticity, the elastic deformations of an elastic body are identified with (almost-) minimizers of some free elastic energy functional [3]. This identification works as follows: The reference configuration of the elastic body is some domain $\Omega \subset \mathbb{R}^n$, the deformation is a map $y : \Omega \to \mathbb{R}^m$, and the associated energy $I : X \to \mathbb{R}$ has as domain the function space of deformations $X$. Of crucial importance is the right choice for the function space $X$. Unphysical deformations (e.g., non-injective maps, which represent configurations displaying self-penetration of matter) should either be excluded from $X$, or the energy of these configurations should be infinite, signaling that it is not possible to observe them in the “real world”. There exists a large amount of literature on how to choose the function space of elastic deformations in a manner that at the same time excludes unphysical configurations and ensures existence of energy minimizers. We do not attempt to give
In [15], a framework has been introduced that allows for cavitation, i.e., the free energy allows for the formation of holes in the elastic body. Cavity formation can be observed in experiments; the mathematical theory for radially symmetric cavities has been developed in [3]. In [15], the function space $X$ is chosen such that cavities created at one point cannot be filled with matter from elsewhere. Clearly, this is another property that “physical” deformations of an elastic body should fulfill. The mathematical formulation of this condition (called “(INV)” in [15]) is rather technical.

An important question in nonlinear elasticity is the relation between models in three, two and one dimensions. Conceptually and mathematically, the most satisfying approach is the derivation of lower dimensional models from a 3-dimensional one by $\Gamma$-convergence [8]. In [11, 12], a hierarchy of 2-dimensional plate models has been derived from 3-dimensional nonlinear elasticity. These models can be classified by the assumed scaling of the energy per unit thickness $I_h$ in the underlying 3d theory, where $h$ denotes the thickness of the elastic sheet. Assuming $I_h \sim h^\beta$, where $h$ is the thickness of the elastic plate, the $\Gamma$-limit for $\beta = 2$ is nonlinear bending theory [11]. The parameter choice $2 < \beta < 4$ results in “von-Kármán-like” plate theories, see [12].

The 3d models taken as a starting point for this hierarchy of $\Gamma$-limits do not require the condition (INV). In [15] it is shown that in general, if condition (INV) is not imposed, it is possible to construct sequences of (almost everywhere) invertible deformations of finite energy that weakly converge to a non-(a.e.) invertible one. We will give a slightly more detailed presentation of this construction in section 2.1.2 and Figure 1. What matters for us is that such a situation is potentially problematic for the derivation of plate theories by $\Gamma$-convergence: A weakly converging sequence of invertible (a.e.) functions might result in a non-invertible (a.e.) configuration with finite elastic energy in the 2d limit theory. The obvious cure would be, of course, to impose condition (INV) on the 3d theory. In the present contribution, we show that this is not necessary. Thus, in contrast to the existing mathematical literature on interpenetration of matter that mainly focuses on finding sufficient conditions for invertibility of elastic deformations, we here identify sufficient conditions for non-invertibility. Questions related to the image of Sobolev functions are known to be a delicate issue, and these objects may display counterintuitive features, cf. the pathological examples going back to Besicovitch [5, 14]. Here, such pathologies are not problematic, because we want to show that the image of the considered functions is sufficiently large.

This will be achieved in the main theorem of the present paper, Theorem 4. Here, we will assume the typical conditions fulfilled by sequences of elastic deformations of thin films in the derivation of 3d-to-2d $\Gamma$-limits. Additionally, we will assume that the limit configuration is non-invertible in a suitable sense, i.e., that it is “simply interpenetrating”, see Definition 2. This definition is crucial for our method of proof to be workable. The statement of Theorem 4 is that under these assumptions, the considered sequence $y_h$ of elas-
tic deformations must consist of non-invertible functions for \( h \) small enough as \( h \to 0 \). In section 3, we will recall some results from the literature that we will use for its proof. The proof of the theorem itself is based on a reduction to a 2-dimensional domain. The intersection on a sufficiently large set in the 2d-domain is proved by a homotopy argument, and the passage back to the 3-dimensional situation is performed with the help of the geometric rigidity result by Friesecke, James and Müller [11]. In section 5, we recall the derivation of plate theories as \( \Gamma \)-limits of 3d-nonlinear elasticity, and obtain some straightforward corollaries from the application of Theorem 4 to these settings.

2 Statement of results

2.1 Non-invertibility of certain Sobolev functions

2.1.1 Brouwer degree

First we need to recall the definition and some basic properties of the Brouwer degree. For \( U \subset \mathbb{R}^n \) bounded, \( f \in C^\infty(\bar{U}, \mathbb{R}^n) \), and \( y \in \mathbb{R}^n \setminus f(\partial U) \) such that \( \det \nabla f(x) \neq 0 \) for all \( x \in f^{-1}(y) \), the Brouwer degree is defined by

\[
\deg(f, U, y) = \sum_{x \in f^{-1}(y)} \text{sgn}(\det \nabla f(x)).
\]

One can show that for \( \varphi \in C^\infty_0(\mathbb{R}^n) \) with \( \text{supp}\varphi \cap f(\partial U) = \emptyset \), and any \( y \in \mathbb{R}^n \) in the same connected component of \( \mathbb{R}^n \setminus f(\partial U) \) as \( \text{supp}\varphi \),

\[
\deg(f, U, y) \int_{\mathbb{R}^n} \varphi(z) \, dz = \int_U \varphi(f(x)) \det \nabla f(x) \, dx.
\]

By this formula and approximation by smooth functions, one can define the degree for any continuous \( f \in C^0(\bar{U}, \mathbb{R}^n) \) and \( y \notin f(\partial U) \). One basic property of the degree is

\[
\deg(f, U, y) \neq 0 \quad \Rightarrow \quad y \in f(U).
\]

On each connected component of \( \mathbb{R}^n \setminus f(\partial U) \), \( \deg(f, U, \cdot) \) is constant. This fact yields the implication

\[
y_0 \in \partial\{y \in \mathbb{R}^n : \deg(f, U, y) = k\} \quad \Rightarrow \quad y_0 \in f(\partial U),
\]

for any \( k \in \mathbb{N} \). For the details of the definition and the proofs of the properties mentioned here, we refer to [9].

2.1.2 Invertibility almost everywhere and the example by Müller and Spector

Next we introduce appropriate notions of invertibility for Sobolev functions.

**Definition 1** (Invertibility almost everywhere [2, 15]). Let \( U \subset \mathbb{R}^n \), and let \( f \) be (a representative of an equivalence class) in \( W^{1,1}(U, \mathbb{R}^n) \). We say \( f \) is invertible almost everywhere if there is a null set \( N \subset U \) such that \( f|_{U \setminus N} \) is injective.
Figure 1: The pathological example by Müller and Spector. In the upper left frame, the formation of a cavity and subsequent continuous deformation of a quadratic reference configuration is depicted. In the upper right frame, this building block is scaled and periodically continued to a larger square. Two of these larger squares are glued to the ends of a strip. In the lower left frame, the strip is bent so that the material from one end fills the voids from the other. This map is invertible almost everywhere. Letting the scaling in step 2 go to zero, the deformations weakly converge to the one in the lower right frame. This map is 2-to-1 on a set of positive measure.

Note that invertibility almost everywhere only depends on the equivalence class.

In [15], Müller and Spector gave an example of a sequence of a. e. invertible maps that weakly converge to a map that is 2-to-1 on a set of positive measure. Their examples were two-dimensional, but similar (slightly more complicated) constructions can be carried out in higher dimensions too. Crucial for their construction is the assumed regularity. The formation of cavities must be permitted, which is the case if the deformations are $W^{1,p}$ with $p < n$, where $n$ is the dimension of the domain. We do not give the explicit formulas for the examples, but only give a qualitative explanation and refer to Figure 1 where the construction is sketched.

The domain of the example is a strip $\Omega \subset \mathbb{R}^2$. The deformations are in $W^{1,p}(\Omega, \mathbb{R}^2)$ for all $p < 2$, and can be described as follows: The strip is perforated at both its ends by cavities. Then it is bent so that material from one end covers the voids from the other. Letting the period of the perforation tend to 0, the resulting sequence converges weakly in $W^{1,p}(\Omega, \mathbb{R}^2)$, for all $p < 2$, to a deformation that is 2-to-1 on a set of positive measure.
2.1.3 Simple interpenetration

For maps $R^{n-1} \supset U \rightarrow R^n$ as they occur in plate theories, the above definition of invertibility almost everywhere is not suitable. Here, modifications on sets of measure zero will be enough to make deformations with interpenetration of matter injective. We by-pass this problem by restricting ourselves to continuous deformations.

For $U \subset R^{n-1}$, we let $\hat{U} \subset R^n$, denote the cylinder over $U$:

$$\hat{U} := U \times [0, 1].$$

In the following, we will identify $U$ with $U \times \{0\} \subset \hat{U}$.

**Definition 2** (Simple interpenetration). For $i \in \{1, 2\}$, let $U_i \subset R^{n-1}$ be simply connected Lipschitz domains and $u_i \in C^0(U_i, R^n)$. We say that $u_1$ and $u_2$ interpenetrate simply if there exists a continuous extension $\hat{u}_1 : \hat{U}_1 \rightarrow R^n$ of $u_1$ with the following properties:

(i) The sets

$$\{x \in U_2 : u_2(x) \not\in \hat{u}_1(\partial\hat{U}_1), \deg(u_2(x), \hat{U}_1, \hat{u}_1) = k\}, \quad k \in \mathbb{N}$$

have positive $\mathcal{L}^{n-1}$-measure for at least two different $k \in \mathbb{N}$

(ii) The extension satisfies

$$\hat{u}_1(\partial\hat{U}_1 \setminus U_1) \cap u_1(U_1) = \emptyset,$$

$$\hat{u}_1(\partial\hat{U}_1 \setminus U_1) \cap u_2(U_2) = \emptyset. \quad (2)$$

The reader can convince her/himself that this definition indeed covers “simple” cases of intersecting graphs $u_1(U_1)$, $u_2(U_2)$.

**Remark 3.**

(i) Definition is asymmetric with respect to $u_1$, $u_2$. This is done on purpose. In statements such as “$v, w$ interpenetrate simply” it will always be understood that the map mentioned first ($v$) plays the role of $u_1$ and the map mentioned second ($w$) the role of $u_2$. It is always possible to reverse the roles by shrinking the domain of $U_1$, but we are not going to use this fact.

(ii) If $U$ is closed and $u : U \rightarrow R^3$ is an embedding, then there do not exist disjoint subsets $U_1, U_2 \subset U$ such that $u_1 := u|_{U_1}$ and $u_2 := u|_{U_2}$ interpenetrate simply. The converse is not true: there exist non-injective maps $u : U \rightarrow R^3$ such that it is not possible to choose $U_1, U_2$ such that $u_1, u_2$ (defined as before) interpenetrate simply. This is the case, e.g., if the graphs $u(U_1)$ and $u(U_2)$ touch, but do not intersect.

2.1.4 Statement of the main theorem

Let $S \subset R^2$ be open and bounded, and let $\Omega_h = S \times (-h/2, h/2)$. We write $\Omega \equiv \Omega_1$. We will consider sequences of functions $z_h : \Omega_h \rightarrow R^3$. It is convenient to define them on the same domain by introducing $y_h : \Omega \rightarrow R^3$ via $y_h(x_1, x_2, x_3) = z_h(x_1, x_2, hx_3)$. Also, we introduce the scaled gradient

$$\nabla_h y = (\nabla y, \frac{1}{h} \partial_3 y).$$
Theorem 4. Let $S$, $\Omega_h$ and $\Omega$ be as above, and let $U_1, U_2 \subset S$ be disjoint simply connected Lipschitz sets. Further, let $u_1 : U_1 \to \mathbb{R}^3$ and $u_2 : U_2 \to \mathbb{R}^3$ be Lipschitz and simply interpenetrating, $\varepsilon > 0$ and $y_h$ a sequence in $W^{1,2}(\Omega, \mathbb{R}^3)$ such that
\[ \|\text{dist} (\nabla h y_h, \text{SO}(3))\|^2_{L^2(\Omega)} < C h^{1+\varepsilon} \]  
and
\[ \int_{-1/2}^{1/2} y_h(\cdot, x_3) \, dx_3 \to u_i \, \text{ in } W^{1,2}(U, \mathbb{R}^3) \]  
as $h \to 0$ for $i = 1, 2$.

Then, for $h$ small enough, $y_h$ is not invertible almost everywhere.

Remark 5. The crucial assumption here is (3). This condition (or, more precisely, its 2-dimensional analog) is not fulfilled by the pathological examples from [15].

3 Preliminaries

For $A \subset \mathbb{R}^n$, we recall the definitions of $m$-dimensional Hausdorff and spherical Hausdorff pre-measures and of the “packing measure”,
\[ \mathcal{H}_\delta^m(A) = \inf \left\{ \omega(m) \sum_j 2^{-m} \text{diam} (A_j) : A \subset \bigcup_j A_j, \, \text{diam} (A_j)/2 \leq \delta \right\} \]
\[ \mathcal{S}_\delta^m(A) = \inf \left\{ \omega(m) \sum_j r_j^m : A \subset \bigcup_j B(x_j, r_j), \, r_j \leq \delta \right\} \]
\[ \mathcal{P}_\delta^m(A) = \omega(m) \delta^m \inf \left\{ \# \{B(x_i, \delta)\} : \bigcup_i B(x_i, \delta) \supset A \right\} \]
where $m \in [0, \infty)$ and $\omega(m) = \Gamma(1/2)^m/\Gamma(m/2 + 1)$; if $m \in \mathbb{N}$, then $\omega(m)$ is the volume of the $m$-dimensional ball. In the above definition, we also allow $\delta = \infty$.

It is well known (see e.g. [11]) that the limits $\lim_{\delta \to 0} \mathcal{H}_\delta^m$, $\lim_{\delta \to 0} \mathcal{S}_\delta^m$ define Borel measures $\mathcal{H}_\delta^m, \mathcal{S}_\delta^m$ on $\mathbb{R}^n$, and that there exists a numerical constant $C = C(n)$ such that
\[ C^{-1} \mathcal{S}_\delta^m(A) \leq \mathcal{H}_\delta^m(A) \leq C \mathcal{S}_\delta^m(A) \]  
and $\mathcal{P}_\delta^m \leq \mathcal{S}_\delta^m(A)$
for every $A \subset \mathbb{R}^n$. Also, we recall the definition of the 1-capacity of a set $A \subset \mathbb{R}^n$,
\[ \text{cap}_1(A) = \inf \{ \text{Per}(E) : E \text{ set of finite perimeter}, A \subset E \} . \]

We cite the relative isoperimetric inequality for sets of finite perimeter. In the following statement, for a set of finite perimeter $E$, $\partial_* E$ denotes the reduced boundary of $E$.

Theorem 6 ([18]). Let $U \subset \mathbb{R}^n$ be open. Then there exists a constant $C = C(U)$ such that for every set $E \subset \mathbb{R}^n$ of finite perimeter,
\[ \min \{ |E \cap U|, |U \setminus E| \}^{n-1/n} \leq C \mathcal{H}^{n-1}(\partial_* E \cap U) . \]
Using the previous theorem, we will now prove a version of the isoperimetric inequality involving capacities instead of the Hausdorff measure. Note that due to $\text{cap}_1 \lesssim \mathcal{H}^{d-1}$ the next lemma is stronger than Theorem 6.

**Lemma 7.** Let $U$ be an open set. Then there exists a constant $C = C(U)$ such that for every bounded set $E$ of finite perimeter,

$$\left(\min (|E \cap U|, |U \setminus E|)\right)^{\frac{n-1}{n}} \leq C \text{cap}_1(\partial_\ast E \cap U).$$

**Proof.** Suppose that the claim of the lemma were not true. Then there exists some $M > 0$ such that for every $\varepsilon > 0$ there exists a set $E$ of finite perimeter with $\min(|E \cap U|, |U \setminus E|) \geq M$ and $\text{cap}_1(\partial_\ast E \cap U) \leq \varepsilon$. By definition there exists an open set of finite perimeter $V$ such that $\text{Per}(V) \leq \text{cap}_1(\partial_\ast E \cap U) + \varepsilon \leq 2\varepsilon$. Using Theorem 6, one has that $|V| \leq C \varepsilon^{n/(n-1)}$. Moreover, using the notation $\tilde{E} = E \cup V$ and noticing that $\partial_\ast \tilde{E} \subset \partial_\ast V$, one has

$$\min(|E|, |U \setminus E|) \leq \min\left(|\tilde{E}|, |U \setminus \tilde{E}|\right) + C \varepsilon^{n/(n-1)} \leq C(U)\left(\mathcal{H}^{n-1}(\partial_\ast \tilde{E} \cap U) + \varepsilon^{n/(n-1)}\right) \leq C(U)\left(\text{Per}(V) + \varepsilon^{n/(n-1)}\right) \leq C(U)\left(\varepsilon + \varepsilon^{n/(n-1)}\right)$$

where $C(U)$ in the previous equation is the constant from the classical relative isoperimetric inequality. By the arbitrariness of $\varepsilon$, we obtain a contradiction.

3.1 Miscellaneous results from the literature

The proof of Theorem 12 is based on the following geometric rigidity theorem, that we shall need in its own right:

**Theorem 8.** (11, Theorem 3.1). Let $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain, with $n \geq 2$. Then there exists a constant $C = C(U)$ for every $v \in W^{1,2}(\mathbb{R}^n)$, there is an associated rotation $R \in SO(n)$ such that,

$$\|\nabla v - R\|_{L^2(U)} \leq C\|\text{dist}(\nabla v, SO(n))\|_{L^2(U)}$$

The constant $C(U)$ is invariant under rescaling of the domain.

We will also use Zhang’s Lemma [17]. An examination of the proof in the latter reference shows that the following (slightly modified) statement holds true as well.

**Theorem 9.** (17, Lemma 3.1). Let $K > 0$. There exist constants $C_1 = C_1(n, m)$, $C_2 = C_2(n, m, K)$ with the following property: If $U \subset \mathbb{R}^n$ is open and bounded, $f \in W^{1,1}(U, \mathbb{R}^m)$ and $\varepsilon > 0$ such that

$$\int_{U \cap \{|
abla f| \geq K\}} |\nabla f| \, dx < \varepsilon,$$
then there exists \( \tilde{f} \in W^{1,\infty}(U, \mathbb{R}^m) \) such that
\[
\begin{align*}
  f = \tilde{f} & \quad \text{on } U \cap \{|\nabla f| \leq K\} \\
  \|\nabla \tilde{f}\|_{L^\infty(U)} & \leq C_1 K \\
  \mathcal{L}^n \left( \{x : f(x) \neq \tilde{f}(x)\} \right) & \leq C_2 \varepsilon
\end{align*}
\]

4 Proof of Theorem 4

Let \( S, \Omega \) and \( \Omega_h \) be as defined in section 2.1.4.

Proposition 10. Let \( y_h \in W^{1,2}(\Omega_h, \mathbb{R}^3) \), let \( \tilde{y}_h : \Omega_h \rightarrow \mathbb{R}^3 \) be Lipschitz, and let \( \varepsilon > 0 \) such that
\[
\left| \{y_h \neq \tilde{y}_h\} \right| \leq C h^{2+\varepsilon},
\]
\[
\|\nabla \tilde{y}_h\| \leq C,
\]
and
\[
\int_{\Omega_h} \text{dist}^2(\nabla y_h, SO(3)) \leq C h^{2+\varepsilon}.
\]

Further, with \( u_h(\cdot) = \tilde{y}_h(\cdot, 0) \), and
\[
F_h := \{x : \text{there exists } \bar{x} \in S \text{ s.t. } u_h(x) = u_h(\bar{x}) \text{ and } |x - \bar{x}| > 2h\}
\]
assume that
\[
\text{cap}_1(F_h) \geq C_1
\]
for some \( C_1 > 0 \). Then there exists \( c = c(C_1) > 0 \) such that for \( h \) small enough,
\[
\mathcal{L}^3 \left( \{x : y_h \text{ is not 1-to-1 at } x\} \right) > ch^2.
\]

Proof. By the assumption on the size of \( |\{y_h \neq \tilde{y}_h\}| \), it is clear that is enough to show the claim for \( \tilde{y}_h \).

For simplicity let us denote
\[
E_h := \int_{\Omega_h} \text{dist}^2(\nabla y_h, SO(3)).
\]

Recalling the relations between capacities and Hausdorff pre-measures we have that \( \text{cap}_1 \leq C_2 \mathcal{H}^1_{\infty} \leq C_2 \mathcal{P}^1_{h/2} \) for some numerical constant \( C_2 \). Hence \( \mathcal{P}^1_{h/2}(F_h) \geq C_1 C_2^{-1} \), and in particular for ever covering of \( F_h \) with balls \( \{B_i\} \) of radius \( h/2 \) we have that
\[
2 \sum_i r(B_i) \geq C_1 C_2^{-1},
\]
where \( r(B) \) denotes the radius of the ball \( B \). Fix a set of points \( X = \{x_i\}_{i \in I} \subset F_h \) such that \( F_h \subset \bigcup_i B(x_i, h/2) \) and \( B(x_i, h/10) \cap B(x_j, h/10) = \emptyset \) for \( i \neq j \). For every \( x \in X \), there exists \( \bar{x} \in F_h \) such that \( u_h(x) = u_h(\bar{x}) \) and \( B(x, h/2) \cap B(\bar{x}, h/2) = \emptyset \).

In the following, we identify the points \( x \in X \subset S \) with the points \( (x, 0) \in \)
$S \times \{0\} \subset \Omega_h$, and whenever we speak of a ball around a point $x \subset X$, it is understood to be three-dimensional.

An element $x \in X$ is said to be “good” if

$$\int_{B(x,h/10) \cup B(x,h/10)} \text{dist}^2(\nabla y_h, SO(3)) \leq \frac{4hc_2}{c_1} E_h$$

and is said to be “bad” otherwise. The subfamily of “good” points will be denoted by $\mathcal{G}$ and the subfamily of “bad” points will be denoted by $\mathcal{B}$. By construction,

$$\#(\mathcal{B} \cup \mathcal{G}) \geq \frac{1}{h} P^1_h(F_h) \geq \frac{c_1}{h c_2}$$

and

$$\# \mathcal{B} \leq \frac{c_1}{2hc_2}.$$

Hence,

$$\# \mathcal{G} \geq \frac{c_1}{2hc_2},$$

and the statement of the proposition will be proven if we can show that there exists some numerical constant $c_3$ such that for every “good” $x \in X$, and $h$ small enough,

$$L^3(\{x' \in B(x,h/2) : y_h \text{ is not 1-to-1 at } x'\}) > c_3 h^3. \quad (7)$$

Let $x_0 \in X$ be a a “good” point. By Theorem 8 and (6), there exists a numerical constant $c = c(c_1, c_2)$ and $R \in SO(3)$ such that

$$\hat{\mathcal{B}}(x_0,h) \left| \nabla y_h - R \right|^2 \leq c h E_h.$$

We set

$$b = \int_{B(x_0,h/2)} (y_h(x) - R x) \, dx.$$

By the Poincaré inequality, there exists $c = c(c_1, c_2)$ such that

$$\int_{B(x_0,h/2)} \left| y_h(x) - Rx - b \right|^2 \, dx \leq c h^3 E_h. \quad (8)$$

Let $\bar{R}, \bar{b}$ be defined analogously to the above when the domain of the integral is $B(\bar{x}_0, h)$, and let $A, \bar{A}$ be the affine maps

$$x \mapsto R x + b, \quad x \mapsto \bar{R} x + \bar{b}$$

respectively. Since $\|y_h - A\|_{W^{1,2}(B(x_0,h/2))} \leq c h E_h$, and $\|y_h - A\|_{W^{1,\infty}(B(x_0,h/2))} \leq c$, we also have

$$\left\| \nabla (y_h - A) \right\|_{L^p(B(x_0,h/2))} \leq c_p h E_h$$

for all $p \in [2, \infty)$.

Let $w_h = y_h - A$, and $B = B(x_0, h/2)$. Using (8) and Hölder’s inequality, we have

$$\int_B |w_h| \leq \frac{1}{\omega(3) h^3} \left( \int_B |w_h|^2 \right)^{1/2} (\omega(3) h^3)^{1/2} \leq CE_h^{1/2}.$$
We set $p = 3 + \varepsilon/2$. We have for $x \in B$,
\[
\int_B |w_h(x) - w_h(z)| \, dz \leq C \int_B \frac{|\nabla w_h(z)|}{|x - z|^2} \, dz \\
\leq C \left( \int_B |\nabla w_h(z)|^p \right)^{1/p} \left( \int_B \frac{|x - z|^{-2p/(p-1)}}{|x - z|} \right)^{(p-1)/p} \\
\leq C(hE_h)^{1/p}h^{1-3/p} \\
\leq Ch^{1+\varepsilon/(3+\varepsilon/2)}
\]
Thus we get
\[
\sup_{x \in B} |w_h(x)| \leq \int_B |w_h| \, dz + \sup_{x \in B} \int_B |w_h(x) - w_h(z)| \, dz \leq C h^{1+\varepsilon} \quad (9)
\]
where we have set $\bar{\varepsilon} = \varepsilon/(3 + \varepsilon/2)$. The analogous statement holds true for
\[
\sup_{x \in B} \tilde{w}_h = \sup_{x \in B} |y_h - \tilde{A}|,
\]
where $\tilde{B} = B(\tilde{x}_0, h/2)$.

The images of $B$ and $\tilde{B}$ under $A$ and $\tilde{A}$ respectively are balls of radius $h/2$ and centers $A(x_0), \tilde{A}(\tilde{x}_0)$. Eq. (9) implies $|A(x_0) - \tilde{A}(\tilde{x}_0)| = Ch^{1+\varepsilon}$. Note that this holds true uniformly on all balls with good centers. Hence
\[
\lim_{h \to 0} \inf_{x_0 \in G} \frac{\mathcal{L}^3 \left( \{ x \in B : \deg (\tilde{A}, \tilde{B}, A(x) = 1) \} \right)}{\mathcal{L}^3 (B)} = 1
\]
Using the homotopy invariance of the Brouwer degree and again (9), we also get
\[
\lim_{h \to 0} \inf_{x_0 \in G} \frac{\mathcal{L}^3 \left( \{ x \in B : \deg (\tilde{y}_h, \tilde{B}, y_h(x)) = 1) \} \right)}{\mathcal{L}^3 (B)} = 1
\]
This proves (7) and hence the proposition.

Proof of Theorem 4. Let $z_h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$ be defined by
\[
z_h(x', hx_3) = y_h(x', x_3) \quad \text{for all } x' \in S, x_3 \in [-1/2, 1/2].
\]
By (3),
\[
\|\text{dist} (\nabla z_h, \text{SO}(3))\|_{L^2(\Omega_h)}^2 \leq C h^{2+\varepsilon}. \quad (10)
\]
Step 1. Approximation by Lipschitz functions. Let $\bar{K} := 2 \max(1, C_2)$, where $C_2 = C_2(n, m, K)$ is the constant from the statement of Theorem 9 with $n = m = 3, K = 1$. Using (10),
\[
\int_{|\nabla z_h| > \bar{K}} |\nabla z_h| \, dx \leq \frac{1}{\bar{K}} \int_{|\nabla z_h| > \bar{K}} |\nabla z_h|^2 \, dx \\
\leq \frac{4}{\bar{K}} \int_{|\nabla z_h| > \bar{K}} \text{dist}^2 (\nabla z_h, \text{SO}(3)) \, dx \\
\leq C(\bar{K}) h^{2+\varepsilon}
\]
We apply Theorem 9 with $K = \bar{K}, f = z_h, \varepsilon = C(\bar{K}) h^{2+\varepsilon}$ and obtain $\tilde{z}_h \in W^{1,\infty}(\Omega_h, \mathbb{R}^3)$ such that
\[
|\{ z_h \neq \tilde{z}_h \}| \leq C(\bar{K}) h^{2+\varepsilon} \\
\|\nabla \tilde{z}_h\|_{L^{\infty}(\Omega_h)} \leq C\bar{K}
\]
\textbf{Step 2.} Extension to a sphere. By Definition 2, there exists an extension $\hat{u}_1 : \hat{U}_1 \to \mathbb{R}^3$ such that eq. (2) is fulfilled. For $\delta > 0$, let

$$U_{1,\delta} := \{ x \in U_1 : \text{dist} (x, \partial U_1) < \delta \}.$$ 

Now we choose $\delta$ so that $u_1(U_{1,\delta}) \cap u_2(U_2) = \emptyset$. Such a choice of $\delta$ is possible by the fact that $u_1$ and $u_2$ interpenetrate simply, cf. Definition 2. Set $\bar{\delta} := \text{dist} (\hat{u}_1(U_{1,\delta}), u_2(U_2))$. Next let $\chi_{\delta} \in C_0^\infty (U_1)$ with $\chi_{\delta} = 1$ on $U_1 \setminus U_{1,\delta}$ and $\| \nabla \chi_{\delta} \|_{L^\infty} < C\delta^{-1}$. Set

$$\hat{u}_{1,h}(x) = \begin{cases} \tilde{z}_h(x,0) & \text{if } x \in U_1 \setminus U_{1,\delta} \\ \chi_{\delta}(x) (\tilde{z}_h(x,0)) + (1 - \chi_{\delta}(x)) u(x) & \text{if } x \in U_{1,\delta} \\ \hat{u}_1(x) & \text{if } x \in \hat{U}_1 \setminus U_1 \end{cases} \tag{13}$$

and

$$u_{2,h} = \tilde{z}_h(\cdot,0)|_{U_2}. \tag{14}$$

\textbf{Step 3.} Convergence of Brouwer degree in $L^1$.

Let $E := \{ x \in U_2 : u_2(x) \in \hat{u}_1(\partial \hat{U}_1) \}$. We claim that

$$\deg (\hat{u}_{1,h} \hat{U}_1, u_{2,h} (\cdot)) \to \deg (\hat{u}_1 \hat{U}_1, u_2 (\cdot)) \text{ in } L^1(U_2 \setminus E) \text{ as } h \to 0. \tag{15}$$

We prove this claim by a homotopy argument. By definition of $z_h$ and $\tilde{z}_h$,

$$\int_{[-h/2,h/2]} dx_3 z_h(\cdot, x_3) \rightharpoonup u_i \text{ in } W^{1,2}(U_1, \mathbb{R}^3).$$

By definition of $\tilde{z}_h$, this holds also true if $z_h$ is replaced by $\tilde{z}_h$. By the uniform Lipschitz bound on $\tilde{z}_h$, we also have

$$\tilde{z}_h(\cdot,0) \rightharpoonup u_i \text{ in } W^{1,2}(U_i, \mathbb{R}^3).$$

By the definitions of $\hat{u}_{1,h}, u_{2,h}$ in (13) and (14), we get

$$\hat{u}_{1,h} \rightharpoonup \hat{u}_1 \text{ in } W^{1,2}(\hat{U}_1, \mathbb{R}^3) \text{ and } u_{2,h} \rightharpoonup u_2 \text{ in } W^{1,2}(U_2, \mathbb{R}^3). \tag{16}$$

Since the uniform Lipschitz bound holds for $\tilde{z}_h$, it also holds for $\hat{u}_{1,h}$ and $u_{2,h}$ by definition of the latter two. Hence the weak convergence in (16) is also true in $W^{1,p}$ for every $1 < p < \infty$. By the compact Sobolev embedding, we have $\hat{u}_{1,h} \to \hat{u}_1$ and $u_{2,h} \to u_2$ in $C^{0,\alpha}$ for every $0 < \alpha < 1$, and in particular, we have uniform convergence. The claim (15) follows from the continuity of the degree function in the first and the third argument with respect to uniform convergence.
**Step 4.** Application of isocapacitary inequality and passage back to 3d. By the definition of simple interpenetration (Definition 2), there exist \(k_1, k_2 \in \mathbb{N}, k_1 \neq k_2\) and some \(C > 0\) such that
\[
\left\{ x \in U_2 : \deg (\hat{u}_1, \hat{U}_1, u_2(x)) = k_i \right\} > C \text{ for } i = 1, 2.
\]
Hence by step 3, there exists \(h_0 > 0\) such that
\[
\left\{ x \in U_2 : \deg (\hat{u}_{1,h}, \hat{U}_1, u_{2,h}(x)) = k_i \right\} > C \text{ for } i = 1, 2 \quad (17)
\]
for \(h < h_0\) (which we assume from now on). Let
\[
A_h := \{ x \in U_2 : \deg (\hat{u}_{1,h}, \hat{U}_1, u_{2,h}(x)) = k_1 \}
\]
and let \(U_2^o\) denote the interior of \(U_2\). Then by (17),\(\min(|A_h \cap U_2^o|, |U_2^o \setminus A_h|) > C\). We apply Lemma 7 and obtain
\[
\text{cap}_1(\partial A \cap U_2^o) > C.
\]
On the other hand, \(x \in \partial A \cap U_2^o\) implies
\[
u_{2,h}(x) \in \partial\{ y \in \mathbb{R}^3 : \deg (\hat{u}_{1,h}, \hat{U}_1, y) = k_1 \}
\]
and hence by (1),
\[
\partial A \cap U_2^o \subset \{ x \in U_2 : u_{2,h}(x) \in \hat{u}_{1,h}(\partial \hat{U}_1) \}.
\]
By the definition of \(\hat{u}_{1,h}\) in (13) and the uniform convergence \(\hat{u}_{1,h} \to \hat{u}_1, u_{2,h} \to u_2\), we may assume that \(\text{dist}(x, \partial U_2) > \delta\) whenever \(u_{2,h}(x) \in \hat{u}_{1,h}(\partial \hat{U}_1)\), whence \(u_{2,h}(x) = u_h(x)\) for \(x \in \partial A_h \cap U_2^o\) and
\[
\partial A \cap U_2^o \subset F_h := \{ x \in U_2 : \text{there exists } \bar{x} \text{ s.t. } u_h(x) = u_h(\bar{x}) \text{ and } |x-\bar{x}| > 2h \}.
\]
Thus we have proved
\[
\text{cap}_1(F_h) > C.
\]
By this last inequality and the results from step 1, the conditions of Proposition 18 are fulfilled, and the claim of the present theorem follows.

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**5 Application to plate theories derived as \(\Gamma\)-limits**

As before, let \(S \subset \mathbb{R}^2\) be open and bounded and \(\Omega = S \times [-1/2, 1/2]\). We define the elastic energy of a 3-dimensional body. Let the inhomogeneous stored energy \(W : \Omega \times \mathbb{R}^{3 \times 3} \to [0, \infty)\) satisfy
\[
\begin{align*}
(i) & \quad W(x, FR) = W(x, F) \text{ for all } R \in SO(n) \\
(ii) & \quad W(x, \text{Id}_{3 \times 3}) = 0 \\
(iii) & \quad W(x, F) \geq c \text{dist}^2(F, SO(3)) \text{ for some uniform constant } c
\end{align*}
\]
(iv) \( W \in C^2(S, \mathcal{T}(x)) \) where \( \mathcal{T}(x) \) is an \( \epsilon \)-neighbourhood of \( SO(3) \), with \( \epsilon \) independent of \( x \).

(v) \( W(x, F) = W(z, F) \) if \( (x - z)\|e_3 \).

We introduce the quadratic forms \( Q_3 : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R} \), \( Q_2 : S \times \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R} \) by

\[
Q_3(\bar{x}, \bar{F}) = \frac{D^2W(x, F)}{DF^2}|_{x=\bar{x}, F=\text{Id}}(\bar{F}, \bar{F})
\]

\[
Q_2(\bar{x}', \bar{F}') = \min \left\{ Q_3(\bar{x}, F' + a \otimes e_3 + e_3 \otimes a) : a \in \mathbb{R}^3 \right\}
\]

The integral of \( W \) satisfying properties (i) through (v) above is the (rescaled) elastic energy functional

\[
I_h : W^{1,2}(\Omega, \mathbb{R}^3) \rightarrow \mathbb{R}
\]

\[
y \mapsto \int_{\Omega} W(x, \nabla_h y(x)) \, dx .
\]

The penalization of interpenetration of matter is expressed in a modification of the 3d energy functional \( I_h \), assigning infinite energy to non-physical deformations. We define \( \bar{I}_h : W^{1,2}(\Omega, \mathbb{R}^3) \rightarrow \mathbb{R} \cup \{+\infty\} \) by

\[
\bar{I}_h(y) = \begin{cases} 
\int_{\Omega} W(x, \nabla_h y(x)) \, dx & \text{if } y \text{ is invertible a.e.} \\
+\infty & \text{else.}
\end{cases}
\]

### 5.1 Contractive maps

In [7], the \( \Gamma \)-limit of the functional \( h^{-\beta}I_h \) for the scaling regime \( 0 < \beta < 5/3 \) has been derived (using results from [13]). The result can be stated as follows:

We say \( y_h \in W^{1,2}(\Omega, \mathbb{R}^3) \) converges uniformly to \( u \in W^{1,2}(S, \mathbb{R}^3) \) as \( h \rightarrow 0 \) if

\[
\lim_{h \rightarrow 0} \text{ess sup}_{(x,y,z) \in \Omega_h} |y_h(x,y,z) - u(x,y)| = 0 .
\]

Further, we say that \( u \in W^{1,\infty}(S, \mathbb{R}^3) \) is short if

\[
\nabla u^T \nabla u \leq \text{id} \quad \text{a.e.}
\]

i.e., \( \text{id} - \nabla u^T \nabla u \) is positive semidefinite almost everywhere. The \( \Gamma \)-convergence result from [7] can be stated as saying that for \( 0 < \beta < 5/3 \)

\[
\left( \Gamma - \lim_{h \rightarrow 0} h^{-\beta}I_h \right)(u) = \begin{cases} 
0 & \text{if } u \text{ is short} \\
+\infty & \text{else,}
\end{cases}
\]

where the \( \Gamma \)-limit is taken with respect to uniform convergence. In fact, it could just as well have been formulated for weak convergence in \( W^{1,2}(\Omega, \mathbb{R}^3) \) (see the discussion in [7]). This result includes the trivial lower bound

\[
\liminf_{h \rightarrow 0} h^{-\beta}I_h(y_h) \geq 0
\]

for sequences \( y_h \) that converge towards a short map \( u \). The application of Theorem [4] immediately yields the following corollary, that is a sharper lower bound for \( h^{-\beta}I_h \) for \( 1 < \beta < 5/3 \).
Corollary 11 (to Theorem 4). Let $1 < \beta$, $u \in W^{1,\infty}(S, \mathbb{R}^3)$, and let $U_1, U_2 \subset S$ be disjoint simply connected Lipschitz domains such that with $u_1 := u|_{U_1}$, $u_2 := u|_{U_2}$, $u_1$ and $u_2$ interpenetrate simply. Further let $y_h \in W^{1,2}(\Omega_h)$ converge uniformly to $u$. Then
\[
\liminf_{h \to 0} h^{-\beta} \bar{I}_h(y_h) = +\infty.
\]

5.2 Nonlinear bending theory

In [11], the nonlinear Kirchhoff plate theory was obtained as the $\Gamma$-limit of the scaled functional $h^{-2} I_h$. Nonlinear plate theory can be defined as follows:

Let the set of $W^{2,2}$-isometries of $S$ into $\mathbb{R}^3$ be denoted by
\[
\mathcal{A} = \{ u \in W^{2,2}(S, \mathbb{R}^3) : \nabla u^T \nabla u = \text{id} \}.
\]
Further, the second fundamental form is given by
\[
\Pi_{[u]} = \nabla u^T \cdot \nabla \nu
\]
where $\nu = u_1 \wedge u_2$ is the normal of the isometry $u$. Nonlinear plate theory may be defined via the energy functional
\[
I_{\text{Kh}} : W^{2,2}(S, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}
\]
\[
u \mapsto \begin{cases}
\frac{1}{2} \int_S Q_2(x', \Pi_{[u]}) \, dx' & \text{if } u \in \mathcal{A} \\
+\infty & \text{else}
\end{cases}
\]
The limiting deformations with finite bending energy will be the set of $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ such that there exists $u \in \mathcal{A}$ with
\[
y(x', x_3) = u(x') \quad \text{for a.e. } x' \in S, x_3 \in [-1/2, 1/2]. \quad (19)
\]
We define the auxiliary functional $I_{\text{Kh}}^{3d} : W^{1,2}(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by
\[
I_{\text{Kh}}^{3d}(y) = \begin{cases}
I_{\text{Kh}}(u) & \text{if } \exists u \in \mathcal{A} \text{ such that eq. (19) holds} \\
\infty & \text{else.}
\end{cases}
\]

Theorem 12 ([11], $\Gamma$ - lim inf-inequality). Let $y_h, y \in W^{1,2}(\Omega, \mathbb{R}^3)$, $y_h \rightharpoonup y$ in $W^{1,2}(\Omega, \mathbb{R}^3)$. Then
\[
\liminf_{h \to 0} h^{-2} I_h(y_h) \geq I_{\text{Kh}}^{3d}(y).
\]
The application of Theorem 4 to nonlinear bending theory yields the following sharper version of the lower bound for $h^{-2} I_h$.

Corollary 13 (to Theorem 4). Let $u \in \mathcal{A}$, and let $U_1, U_2 \subset S$ be disjoint simply connected Lipschitz domains such that with $u_1 := u|_{U_1}$, $u_2 := u|_{U_2}$, $u_1$ and $u_2$ interpenetrate simply. Further, let $y_h \rightharpoonup y$ in $W^{1,2}(\Omega, \mathbb{R}^3)$, with
\[
\limsup_{h \to 0} h^{-2} \| \text{dist} (\nabla y_h, SO(3)) \|_{L^2(\Omega)}^2 < \infty
\]
and
\[
y(x', x_3) = u(x') \quad \text{for a.e. } x' \in S, x_3 \in [-1/2, 1/2].
\]
Then
\[
\liminf_{h \to 0} h^{-2} \bar{I}_h(y_h) = +\infty.
\]
5.3 Von-Kármán-like plate theories

In [12], a hierarchy of plate models has been derived from 3d elasticity, classified by the energy scaling in the 3d theory. Here we cite the result that identifies the Γ-limits of the functionals $h^{-\beta}I_h$ for $2 < \beta < 4$ as $h \to 0$. The limiting plate theories are bending theories with a linearized isometry constraint. More precisely, we say that a pair $(u, v) \in W^{1,2}(S, \mathbb{R}^2) \times W^{1,2}(S)$ satisfies the linear isometry constraint if

$$\text{sym} \nabla'u + \frac{1}{2} \nabla'v \otimes \nabla'v = 0. \quad (21)$$

It follows from this equation that

$$v \in W^{2,2}(S) \cap W^{1,\infty}(S) \quad (22)$$

and $\det \nabla^2v = 0$, see [12]. The corresponding plate theory is defined by the functional $I_{\text{lim}}^\text{Klin} : W^{1,2}(S, \mathbb{R}^2) \times W^{1,2}(S) \to \mathbb{R} \cup \{+\infty\}$, given by

$$I_{\text{lim}}^\text{Klin}(u, v) = \begin{cases} \frac{1}{24} \int_S Q_2((\nabla')^2v) \, dx & \text{if } (u, v) \text{ satisfies } (21) \\ +\infty & \text{else.} \end{cases}$$

We cite the result that identifies the above functional as a Γ-limit of 3d elasticity.

**Theorem 14** ([12]). *Let the stored energy function $W$ satisfy conditions (i)-(v), $2 < \beta < 4$, and assume that $S$ is connected.*

**(Compactness.)** Let $\tilde{y}_h$ be a sequence in $W^{1,2}(\Omega, \mathbb{R}^3)$ with

$$\limsup_{h \to 0} h^{-\beta}I_h(\tilde{y}_h) < \infty.$$

Then there exists a subsequence (denoted by $\tilde{y}_h$ again), rotations $R^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$, $u \in W^{1,2}(S, \mathbb{R}^2)$, $v \in W^{2,2}(S)$ such that with

$$y_h := R^h\tilde{y}_h - c^h$$

$$u_h(x') := h^{2-\beta} \left( \int_I \frac{(y_h)_1(x',x_3)}{(y_h)_2(x',x_3)} \, dx_3 - x' \right) \quad (23)$$

$$v_h(x') := h^{1-\beta/2} \left( \int_I (y_h)_3(x',x_3) \, dx_3 \right) \quad (24)$$

we have

$$u^h \rightharpoonup u \text{ in } W^{1,2}(S, \mathbb{R}^2)$$

$$v^h \rightharpoonup v \text{ in } W^{1,2}(S).$$

**(Lower bound.)** If $y_h$ is a sequence in $W^{1,2}(\Omega, \mathbb{R}^3)$ such that $u^h$ and $v^h$ as defined in (23), (24) converge weakly to $u, v$ in $W^{1,2}(S, \mathbb{R}^2)$ and $W^{1,2}(S)$ respectively, then

$$\liminf_{h \to 0} h^{-\beta}I_h(y_h) \geq I_{\text{lim}}^\text{Klin}(u, v)$$
(Upper bound.) If \((u, v) \in W^{1,2}(S, \mathbb{R}^2) \times W^{1,2}(S)\), then there exists a sequence \(y_h\) such that for the sequences \(u^h, v^h\) as defined in \((23)\) and \((24)\), \(u^h \to u\) in \(W^{1,2}(S, \mathbb{R}^2)\), \(v^h \to v\) in \(W^{1,2}(S)\), and
\[
\lim_{h \to 0} h^{-\beta} I^h(y_h) = I_{\text{lin}}^* (u, v).
\]

**Remark 15.** One obvious consequence of the theorem is that whenever
\[
\limsup_{h \to 0} h^{-\beta} I^h(y_h) < \infty \quad \text{and} \quad y_h \to y \quad \text{in} \quad W^{1,2}(\Omega, \mathbb{R}^3)
\]
then \(y\) is (up to a rigid motion) just the projection onto the first two components, \(y(x) = x'\). This indicates that if \(S\) is connected, and \(U_1, U_2 \subset S\) are disjoint subsets, it is impossible to state sufficient conditions for the “limits” \(u, v\) that assure that \(y_h |_{(U_1 \cup U_2) \times [-1/2, 1/2]}\) is 2-to-1 on a set of positive measure.

The remark serves as a motivation to consider \(S\) with more than one connected component. (Note that such an assumption was not necessary in sections 5.1 and 5.2)

Let \(U_1, U_2 \subset \mathbb{R}^2\) be disjoint simply connected Lipschitz sets, and for \(i \in \{1, 2\}\), let \(\Omega^{(i)} = U_i \times [-1/2, 1/2]\), and \(\Omega = \Omega^{(1)} \cup \Omega^{(2)}\). Further, let \(\iota : \mathbb{R}^3 \to \mathbb{R}^2\) be defined by \(\iota(x', x_3) = x'\).

**Theorem 16.** Let \(y_h\) be a sequence in \(W^{1,2}(\Omega, \mathbb{R}^3)\), \(2 < \beta < 4\), and let \(v_h\) be defined by \((24)\). Assume that
\[
\begin{align*}
\limsup_{h \to 0} h^{-\beta} I^h(y_h) &< \infty \\
v_h &\to v \quad \text{in} \quad W^{1,2}(U_1 \cup U_2) \\
((y_h)_1, (y_h)_2) &\to \iota + c_i \quad \text{in} \quad W^{1,2}(\Omega^{(i)}, \mathbb{R}^2) \quad \text{for some} \quad c_i \in \mathbb{R}^2 \quad \text{for} \quad i = 1, 2
\end{align*}
\]
Further let \(u_i : U_i \to \mathbb{R}^3\) be defined by
\[
u_i(x') = (x' + c_i, v(x'))
\]
for \(i = 1, 2\) and assume that \(u_1, u_2\) interpenetrate simply. Then
\[
\limsup_{h \to 0} h^{-\beta} I^h(y_h) = \infty.
\]

**Proof.** By the compactness and lower bound part of Theorem 14 applied to \(y_h|_{\Omega^{(i)}}\) for \(i = 1, 2\), there must exist \(u \in W^{1,2}(U_1 \cup U_2, \mathbb{R}^2)\) such that the pair \((u, v)\) fulfills the linear isometry constraint \((21)\). Hence by \((22)\), \(u_i \in W^{2,2}(U_i, \mathbb{R}^3) \cap W^{1,\infty}(U_i, \mathbb{R}^3)\) for \(i = 1, 2\). (Thus the requirement that they interpenetrate simply makes sense.) By assumption we have
\[
\lim_{h \to 0} \int_{-1/2}^{1/2} dx_3 \begin{pmatrix} id_{2 \times 2} & 0 \\ 0 & h^{1-\beta/2} \end{pmatrix} y_h(x') = u_i(x') \quad \text{for} \quad x' \in U_i, \ i = 1, 2,
\]
where the limit is taken in weak-\(W^{1,2}(U_i, \mathbb{R}^3)\). We now apply Theorem 4 to the sequence
\[
\begin{pmatrix} id_{2 \times 2} & 0 \\ 0 & h^{1-\beta/2} \end{pmatrix} y_h
\]
The latter is injective almost everywhere if and only if \(y_h\) is, and the claim of the theorem follows.
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