On the Optimal Inflation Rate with Capital Externality

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Abstract
This short note is to show that the strong non-superneutrality of monetary policy in Brunnermeier and Sannikov (2016) does not hold if taking into account the pecuniary externality of capital. Higher money growth rate leads to higher level of capital but not higher growth rate of economy at steady state.
This short note is to show that the strong non-supernatureality of monetary policy in Brunnermeier and Sannikov (2016) (hereafter BK) does not hold if taking into account the pecuniary externality of not only money but also capital. Their paper admits market incompleteness for diversifying idiosyncratic risk and households’ ignorance of pecuniary externality when making portfolio decisions about money. The logic is that when making portfolio decisions, each individual takes the real interest rate as given, which in the aggregate is driven by the economic growth rate, and in turn depends on individual portfolio decisions. Higher inflation due to higher money growth lowers the real interest rate (on money) and tilts the portfolio choice towards physical capital investment. BK concludes that this will boost the overall physical investment and endogenous growth rate. We show that this result is too strong—higher money growth rate may change the level of physical capital at steady state, however, not the growth rate. The key assumption in their paper is constant marginal return to capital at aggregate level. Alternatively, we assume constant marginal return to capital at individual level and decreasing marginal return at aggregate level. When one individual invests in physical capital, she takes her productivity (and marginal value of investment) as constant while does not take into account the fact that her increase of investment leads to over-investment for the whole economy and decreased returns. In other words, there is also pecuniary externality when investing capital, which leads to lower return of capital and less incentive to economic growth. We show that as long as this pecuniary externality is considered, higher inflation would lead to higher level of capital but not higher growth rate of economy at steady state.

We closely follow BK to set up the model, however, in discrete time. The representative household’s problem is,

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)
\]

subject to,

\[
c_t + i_t k_t + q_t k_{t+1} + \frac{m_{t+1}}{p_t} = A(K_t) k_t + z_t + q_t [1 + \Phi(i_t) - \delta] k_t + R_t^M \frac{m_t}{P_{t-1}} + \tau_t w_t
\]

where \(c_t, k_t, i_t, q_t, z_t\) and \(\tau_t\) are individual consumption, capital, investment ratio, price of capital, random dividends to capital and fraction of seignorage transfer over wealth at period \(t\). As in BK, \(z_t = \epsilon_t k_t\) and \(\epsilon_t\) is an iid process.
with standard deviation $\sigma$. $m_t$ is money holdings at the beginning of period $t$, and $R_t^M = P_{t-1}/P_t$ is return of money. $\omega_t = q_{t-1}k_t + m_t/P_{t-1}$ is the initial wealth at period $t$. $\delta$ is depreciation rate of capital, and $\Phi(.)$ investment adjustment cost with $\Phi'(.) > 0$ and $\Phi''(.) > 0$.

The only difference from BK is that we have the TFP of capital measured as $A(K_t) = AK_t^{\alpha-1}$ and $K_t$ is the aggregated capital. In equilibrium, $k_t = K_t$ and we have the economy-wide output given by $AK^\alpha$. Suppose $0 < \alpha < 1$, we have constant marginal return to capital at individual level while decreasing marginal return at aggregate level. This assumption suffices to capture the externality of production. Rewriting the budget constraint as,

$$\omega_{t+1} = R_t^P \omega_t - c_t$$

(2)

where $R_t^P = x_t R_t^K + (1 - x_t) R_t^M + \tau_t$, and $x_t = q_{t-1}k_t/\omega_t$ is the share of capital in the portfolio. Then we have the return of capital as,

$$R_t^K = \frac{A(K_t) - i_t + \epsilon_t + q_t[1 + \Phi(i_t) - \delta]}{q_{t-1}}$$

(3)

which says the return of capital decreases when each agent invests more due to externality of decreased returns on aggregate capital since $A'(K_t) = (\alpha - 1)K_t^{\alpha-2} < 0$. In constrast, BK assumes $A(K_t) = A$ which is constant and there is no pecuniary externality along with capital. The role of pecuniary externality of capital also shows up in the optimal share of capital after solving the maximization problem of households and taking second-order Taylor expansion,

$$x_t = \frac{q_{t-1}}{\sigma^2} \{ A(K_t) - i_t + q_t[1 + \Phi(i_t) - \delta]$$

$$- q_{t-1} \} - \frac{q_t}{\sigma^2} (R_t^M - 1)$$

in which is clear that agent’s optimal share of capital is attenuated with capital increased in the aggregate level. The maximization problem also delivers,

$$c_t = (1 - \beta) R_t^P \omega_t$$

$$q_t = \frac{1}{\Phi'(i_t)}$$

As in BK, define $p_t = M_t/P_t K_t \geq 0$ the real value of money normalized by the size of aggregate capital stock and $\mu_t = M_t/M_{t-1}$ the gross growth rate
of money supply. Using market clearing conditions of goods and capital, we have similar conditions at steady state as in BK,

\[
q = 1 + \kappa i
\]

(4)

\[
AK^{\alpha-1} = i + \rho(p + q)
\]

(5)

\[
AK^{\alpha-1} + q\mu = \frac{\sigma^2}{p + q}
\]

(6)

where we used the assumption of \( \Phi(i) = \ln(1 + \kappa i)/\kappa \) and defined \( \rho = (1 - \beta)/\beta \). When \( \alpha = 1 \), we have exactly the result of BK, and in particular, that the investment ratio \( i \) is an increasing function of money growth rate \( \mu \). Returning to the law of motion for aggregate capital,

\[
K_{t+1} = (1 + \Phi(i) - \delta)K_t
\]

(7)

BK concludes that the economy is growing with a rate of \( K_{t+1}/K_t = (1 + \Phi(i) - \delta) \), which is affected by money growth rate.

However, if there is pecuniary externality for capital (\( \alpha < 1 \)), then \( i, p, \) and \( q \) would change if \( K \) is increasing, which means the economy is not at steady state. Thus the only possibility is that \( K \) is fixed, which implies that \( \Phi(i) = \delta \) and constant growth rate (\( \equiv 1 \)) of the economy at steady state. Given the form of \( \Phi(.) \), we have \( i = (e^{\delta \kappa} - 1)/\kappa \) and real price of capital \( q = e^{\delta \kappa} \), both of which are constant.\(^1\) Combining the equation for capital share \( x_t \) at steady state and Equations (5) and (6), we have \( x = [(1-1/\mu)q^2 + q\sqrt{q^2(1-1/\mu)^2 + 4\rho \sigma^2}]/2\sigma^2, p = (1-x)q/x, \) and \( K = ((\rho/x + \delta)/A)^{1/(\alpha-1)} \).

We can verify that \( dx/d\mu > 0, dp/d\mu < 0 \) and \( dK/d\mu > 0.\(^2\) \) That is, the share of wealth invested in capital is increasing with inflation. The real value of money is decreasing with inflation. And finally, the steady state value of capital is increasing with inflation. We thus have a weaker version of non-superneutrality as in BK.

**COROLLARY 1 (No Superneutrality):**

When there is also pecuniary externality in investing capital, money is not superneutral since a steady state increase in money supply growth affects the steady state level of the economy, however, not the economic growth rate.

Other propositions in BK still hold.

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\(^1\)In BK, when money growth rate \( \mu \) is higher, investment ratio \( i \) is higher, which leads to higher real price of capital.

\(^2\)We only focus on the monetary equilibrium and assume all necessary conditions are satisfied for such an equilibrium to exist.
References

Brunnermeier, Markus K., and Yuliy Sannikov. 2016. “On the Optimal Inflation Rate.” *American Economic Review Papers and Proceedings*, 106(5): 484–489.