Dual Lie bialgebra structures of the twisted Heisenberg-Virasoro type

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Abstract: In this paper, by studying the maximal good subspaces, we determine the dual Lie coalgebras of the centerless twisted Heisenberg-Virasoro algebra. Based on this, we construct the dual Lie bialgebras structures of the twisted Heisenberg-Virasoro type. As by-products, four new infinite dimensional Lie algebras are obtained.

Key words: twisted Heisenberg-Virasoro algebra, Lie bialgebra, Lie coalgebra, dual Lie bialgebra, maximal good subspace.

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1 Introduction

The twisted Heisenberg-Virasoro algebra, first studied in [1], is an important algebra structure, which has close relations with the Heisenberg algebra and the Virasoro algebra. In recent years, more and more attentions have been paid to this algebra (see, e.g., [2,8,9,11,13,14,21]). Let us first recall the definition here. The twisted Heisenberg-Virasoro algebra is a Lie algebra with the underlining vector space \( \mathcal{HV} = \text{span}_K \{ L_m, I_n, C_L, C_I, C_{LI} \mid m, n \in \mathbb{Z} \} \) over an algebraically closed filed \( K \) of characteristic zero, subject to the following relations:

\[
\begin{align*}
[L_m, L_n] &= (n - m)L_{m+n} + \delta_{m+n,0} \frac{1}{12} (m^3 - m) C_L, \\
[I_m, I_n] &= n \delta_{m+n,0} C_I, \\
[L_m, I_n] &= n I_{m+n} + \delta_{m+n,0} C_{LI}, \\
[\mathcal{L}, C_L] &= [\mathcal{L}, C_I] = [\mathcal{L}, C_{LI}] = 0, \quad m, n \in \mathbb{Z}.
\end{align*}
\]

Obviously, the Virasoro algebra \( \mathcal{V} = \text{span}_K \{ L_m, C_L \mid m \in \mathbb{Z} \} \) and the Heisenberg algebra \( \mathcal{H} = \text{span}_K \{ I_m, C_I \mid m \in \mathbb{Z} \} \) are subalgebras of \( \mathcal{L} \).

In [12], the Lie bialgebra structures of the twisted Heisenberg-Virasoro type were investigated. In the present paper, we will study the dual Lie bialgebra structures of the twisted Heisenberg-Virasoro Lie bialgebra. It is well-known that the notion of Lie bialgebras was

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introduced by Drinfeld in 1983 [6,7] in connection with quantum groups. Since Lie bialgebras as well as their quantizations provide important tools in searching for solutions of quantum Yang-Baxter equations and in producing new quantum groups, a number of papers on Lie bialgebras have appeared (e.g., [3–5, 10, 15–20, 22–24, 29–31]). For instance, the structures of Witt and Virasoro type Lie bialgebras were presented in [16, 22], and a classification of this type Lie bialgebras was given in [23]. All Lie bialgebra structures on the Witt, the one-sided Witt, and the Virasoro algebras were shown to be triangular coboundary, which can be obtained from their nonabelian two dimensional Lie subalgebras (cf. [22]). For the generalized Witt type Lie bialgebras cases, the authors in [24] obtained that all structures of Lie bialgebras on them are coboundary triangular. Similar results also hold for some other kinds of Lie bialgebras (cf., e.g., [30, 31]).

As stated in [25, 26], Lie bialgebra structures of coboundary triangular type may sound simple, but they are not trivial. Indeed, there are many natural problems associated with them remain open. For example, even for the (two-sided) Witt algebra and the Virasoro algebra, a completely classification of coboundary triangular Lie bialgebra structures on them is still open. Nevertheless, rather few is known on representations of infinite dimensional Lie bialgebras. Therefore, it seems to us that more attentions should be paid on this aspect. The authors of [25, 26] studied dual Lie bialgebra structures of the (two-sided) Witt algebra, the Virasoro algebra, the Poisson algebra and the loop and current-Virasoro type algebras. As by-products, some new series of infinite dimensional Lie algebras are obtained. Studying dual Lie bialgebra structures can also provide new approaches to investigate quantum groups, especially in studying Lie bialgebras, and also help us to understand why we state that the coboundary triangular Lie bialgebras are not trivial. We remind that the main problems occurring in the study of dual Lie bialgebras are: (1) the determination of the maximal good spaces, (2) the determination of the Lie algebra structures, especially (1). In the present paper, we have found an efficient way in tackling problem (1) (cf. Theorem 2.5 and (2.14)).

This paper proceeds as follows. Some definitions and preliminary results are briefly recalled in Section 2. Then structures of dual coalgebras of centerless twisted Heisenberg-Virasoro algebra are addressed. In Section 3, structures of dual Lie bialgebras of Heisenberg-Virasoro algebra are investigated. The main results of the present paper are summarized in Theorems 2.5, 3.4, 3.5, 3.7, 3.8, 3.9.

2 Definitions and preliminary results

Let us briefly recall some notions on Lie bialgebras, for details, we refer readers to, e.g., [7,24,26]. Throughout this paper, all vector spaces are assumed to be over an algebraically
closed field \( K \) of characteristic zero, and as usual, \( \mathbb{N} \) denotes the set of nonnegative integers.

A triple \((\mathcal{L}, [\cdot, \cdot], \delta)\) is called a Lie bialgebra, if it satisfies the following conditions:

1. \((\mathcal{L}, [\cdot, \cdot])\) is a Lie algebra and \((\mathcal{L}, \delta)\) is a Lie coalgebra;
2. \(\delta(x, y) = x \cdot \delta(y) - y \cdot \delta(x)\) for all \(x, y \in \mathcal{L}\),

where \(\delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}\) is a derivation and \(x \cdot (y \otimes z) = [x, y] \otimes z + y \otimes [x, z]\) for \(x, y, z \in \mathcal{L}\).

A Lie bialgebra \((\mathcal{L}, [\cdot, \cdot], \delta)\) is coboundary if \(\delta\) is coboundary in the sense that there exists \(r \in \mathcal{L} \otimes \mathcal{L}\) written as \(r = \sum r^{[1]} \otimes r^{[2]}\), such that \(\delta(x) = x \cdot r\) for \(x \in \mathcal{L}\). A coboundary Lie bialgebra \((\mathcal{L}, [\cdot, \cdot], \delta)\) is triangular if \(r\) satisfies the following classical Yang-Baxter equation (CYBE):

\[
C(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{2.1}
\]

where \(r_{12} = \sum r^{[1]} \otimes r^{[2]} \otimes 1, r_{13} = \sum r^{[1]} \otimes 1 \otimes r^{[2]}, r_{23} = \sum 1 \otimes r^{[1]} \otimes r^{[2]}\) are elements in \(U(\mathcal{L})^{\otimes 3}\) and \(U(\mathcal{L})\) is the universal enveloping algebra of \(\mathcal{L}\).

Two Lie bialgebras \((\mathcal{L}, [\cdot, \cdot], \delta)\) and \((\mathcal{L}', [\cdot, \cdot]', \delta')\) are called dually paired if there exists a nondegenerate bilinear form \(\langle \cdot, \cdot \rangle\) on \(\mathcal{L}' \times \mathcal{L}\) (extended uniquely to a bilinear form on \((\mathcal{L} \otimes \mathcal{L}') \times (\mathcal{L} \otimes \mathcal{L}')\)) such that their bialgebra structures are related via

\[
\langle [f, h]'\xi, \eta \rangle = \langle f \otimes h, \delta\xi \rangle, \quad \langle \delta' f \xi \otimes \eta \rangle = \langle f, [\xi, \eta] \rangle \quad \text{for} \quad f, h \in \mathcal{L}', \xi, \eta \in \mathcal{L}. \tag{2.2}
\]

In particular, \(\mathcal{L}\) is called a self-dual Lie bialgebra if \(\mathcal{L}' = \mathcal{L}\) as a vector space.

Note that a finite dimensional Lie bialgebra \((\mathcal{L}, [\cdot, \cdot], \delta)\) is always self-dual as the linear dual space \(\mathcal{L}^*\) is naturally a Lie bialgebra by dualization and there exists a vector space isomorphism \(\mathcal{L} \to \mathcal{L}^*\) which pulls back the bialgebra structure on \(\mathcal{L}^*\) to \(\mathcal{L}\) to obtain another bialgebra structure on \(\mathcal{L}\) to make it to be self-dual. However, infinite dimensional Lie bialgebras have sharp differences, as they are not self-dual in general.

For convenience, we denote the Lie bracket of Lie algebra \(\mathcal{L} = (\mathcal{L}, [\cdot, \cdot])\) by \(\varphi\), i.e., \([\cdot, 
\]
\[
\varphi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \quad \text{and} \quad \varphi^* : \mathcal{L}^* \to (\mathcal{L} \otimes \mathcal{L})^* \quad \text{to be the dual of} \quad \varphi.
\]

A subspace \(U\) of \(\mathcal{L}^*\) is called a good subspace if \(\varphi^*(U) \subset U \otimes U\). It follows that \(\mathcal{L}^0\) defined below is also a good subspace of \(\mathcal{L}^*\), which is obviously the maximal good subspace of \(\mathcal{L}^*\) [17]:

\[
\mathcal{L}^0 = \sum_{U \in \mathcal{R}} U, \quad \text{where} \quad \mathcal{R} = \{U \mid U \text{ is a good subspace of } \mathcal{L}^*\}. \tag{2.3}
\]

The notion of good subspaces of an associative algebra can be defined analogously. It is proved in [17] that for any good subspace \(U\) of \(\mathcal{L}^*\), the pair \((U, \varphi^*|_U)\) is a Lie coalgebra. In particular, \((\mathcal{L}^0, \varphi^*|_{\mathcal{L}^0})\) is a Lie coalgebra.

For any Lie algebra \(\mathcal{L}\), the dual space \(\mathcal{L}^*\) has a natural right \(\mathcal{L}\)-module structure defined by \((f \cdot x)(y) = f([x, y])\) for \(f \in \mathcal{L}^*,\ x, y \in \mathcal{L}\). We denote \(f : \mathcal{L} = \text{span}\{f \cdot x \mid x \in \mathcal{L}\}\), the space of translates of \(f\) by elements of \(\mathcal{L}\).
We summarize some results of [3–5, 10] as follows.

**Proposition 2.1.** Let $\mathcal{L}$ be a Lie algebra. Then

1. $\mathcal{L}^\circ = \{ f \in \mathcal{L}^* \mid f : \mathcal{L} \text{ is finite dimensional} \}$.
2. $\mathcal{L}^\circ = (\varphi^*)^{-1}(\mathcal{L}^* \otimes \mathcal{L}^*)$, the preimage of $\mathcal{L}^* \otimes \mathcal{L}^*$ in $\mathcal{L}^*$.

For an infinite dimensional Lie algebra $g$, there is no effective approach to determine the good subspace $g^\circ$ of it. However, for an associative commutative algebra $A$, Sweedler [29] gave some approaches to determine $A^\circ$. In the cases of $A = \mathcal{K}[x]$ and $\mathcal{K}[x^{\pm 1}]$, the maximal good subspaces of $A$ were determined (see [18–20, 22, 24–26]). Although the property of the good subspaces of an associative commutative algebra has great difference with the Lie algebra case, if a Lie algebra can be induced from an associative commutative algebra, then the good subspaces of this Lie algebra can be determined through the associative commutative algebra case. Therefore, let us recall some results about associative commutative algebra for later use.

Let $(A, \mu)$ be an associative commutative algebra over a field $\mathcal{K}$. By Proposition 2.1 and [29], we have $A^\circ = (\mu^*)^{-1}(A^* \otimes A^*)$. For $\partial \in \text{Der}(A)$, since

$$\partial \mu = \mu(\text{id} \otimes \partial + \partial \otimes \text{id}), \quad \mu^* \partial^* (f) = (\text{id} \otimes \partial^* + \partial^* \otimes \text{id}) \mu^*(f) \in A^* \otimes A^*,$$

where $f \in A^\circ$, it follows that $\partial^*(A^\circ) \subset A^\circ$.

In the case of $A = \mathcal{K}[x^{\pm 1}]$, let $S = \{ x^n \mid n \in \mathbb{Z} \}$ be the standard basis of $A$ and $S' = \{ \varepsilon^n \mid n \in \mathbb{Z} \}$ be the dual basis of $S$, i.e., $\varepsilon^i(x^j) = \delta_{ij}$ for $i, j \in \mathbb{Z}$. For $f \in A^*$, $f$ can be expressed as $f = \sum_{j \in \mathbb{Z}} f_j \varepsilon^j$. The structures of $A^\circ$ can be found as follows (see [22, 25–27]).

**Lemma 2.2.** Let $f = \sum_{j \in \mathbb{Z}} f_j \varepsilon^j \in A^*$. Then $f \in A^\circ$ if and only if there exist $r \in \mathbb{N}$ and $c_j \in \mathcal{K}$ for $j = 1, 2, \ldots, r$ such that $f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_r f_{n-r}$ for all $n \in \mathbb{Z}$.

Let $(A, \mu_1)$ and $(B, \mu_2)$ be two associative commutative algebras, $\mu_1, \mu_2$ be the multiplications of $A$ and $B$ respectively. Define the direct sum $\mathcal{L} = A \oplus B$ with the multiplication $\mu = (\mu_1, \mu_2)$, i.e. $\mu((a_1, b_1), (a_2, b_2)) = (\mu_1(a_1, a_2), \mu_2(b_1, b_2))$. Then $(\mathcal{L}, \mu)$ is also an associative commutative algebra, and it is easy to prove the following lemma.

**Lemma 2.3.** Let $(\mathcal{L}, \mu)$ be the above associative commutative algebra, then $\mathcal{L}^\circ = (A \oplus B)^\circ = A^\circ \oplus B^\circ$.

The twisted Heisenberg-Virasoro algebra (1.1) is the universal central extension of the algebra with the underlining vector space $\overline{HV} = \text{span}_\mathcal{K}\{ L_m, I_n \mid m, n \in \mathbb{Z} \}$ and the brackets

$$[L_m, L_n] = (n-m)L_{m+n},$$
$$[I_m, I_n] = 0,$$
$$[L_m, I_n] = nI_{m+n}, \quad m, n \in \mathbb{Z}. \quad (2.4)$$
Thus it is called the centerless twisted Heisenberg-Virasoro algebra. This algebra has a polynomial realization as follows. Let \( \mathcal{A} = \mathcal{K}[x^{\pm 1}] \) be the Laurent polynomial over complex field \( \mathcal{K} \), with one variable \( x \) and the usual derivation \( \partial = \frac{d}{dx} \) of \( \mathcal{K}[x^{\pm 1}] \). Denote the direct sum of Laurent polynomial \( \mathcal{K}[x^{\pm 1}] \) by \( \overline{\mathcal{HV}} = \mathcal{A} \oplus \mathcal{A} = \mathcal{K}[x^{\pm 1}] \oplus \mathcal{K}[x^{\pm 1}] \). For \( (f_1(x), f_2(x)), (g_1(x), g_2(x)) \in \overline{\mathcal{HV}} \), we define the multiplication \( \mu = (\mu_1, \mu_2) \) on \( \mathcal{L} \) by

\[
\mu \left( (f_1(x), f_2(x)), (g_1(x), g_2(x)) \right) = \left( \mu_1(f_1(x), g_1(x)), \mu_2(f_2(x), g_2(x)) \right),
\]

where \( \mu_1 \) is the multiplication of the first copy \( \mathcal{A} \) of \( \overline{\mathcal{HV}} \) and \( \mu_2 \) is the multiplication of the second copy \( \mathcal{A} \) of \( \overline{\mathcal{HV}} \) in the usual way. Then \( (\overline{\mathcal{HV}}, \mu) \) is an associative commutative algebra.

Denote

\[
L_m = (x^{m+1}, 0), \quad I_n = (0, x^n) \in \overline{\mathcal{HV}} \text{ for } m, n \in \mathbb{Z}.
\]

Define \( (\partial, \text{id})(x^k, x^l) = (\partial(x^k), \text{id}(x^l)) \) and some other similar operators on \( \overline{\mathcal{HV}} \) are defined in the same way. Now let \( \tau, \text{id} \) and \( T \) be operators on \( \overline{\mathcal{HV}} \) and \( \overline{\mathcal{HV}} \otimes \overline{\mathcal{HV}} \) such that

\[
\tau(x^m, x^n) = (x^m, x^n), \quad \text{id}(x^m, x^n) = (x^m, x^n), \quad T((x^m, x^n), (x^k, x^l)) = ((x^k, x^l), (x^m, x^n)).
\]

Then the brackets \eqref{eq:brackets} can be realized as follows.

\[
[L_m, L_n] = \mu((\text{id}, 0) \otimes (\partial, 0) - (\partial, 0) \otimes (\text{id}, 0))((x^{m+1}, 0), (x^{n+1}, 0))
= (n - m)(x^{m+n+1}, 0),
\]

\[
[I_m, I_n] = \mu((0, \text{id}) \otimes (0, \text{id}) - (0, \text{id}) \otimes (0, \text{id})) \cdot T((0, x^m), (0, x^n))
= 0,
\]

\[
[L_m, I_n] = \mu((0, \text{id}) \otimes (0, \partial)) \cdot (\tau, \text{id})((x^{m+1}, 0), (0, x^n))
= n(0, x^{m+n}), \quad m, n \in \mathbb{Z}.
\]

For the centerless twisted Heisenberg-Virasoro algebra defined by \eqref{eq:brackets}, there are two natural approaches to determine the Lie coalgebras structures on some subspaces of \( \overline{\mathcal{HV}}^\circ \). One is to converse the arrow of the Lie bracket \( \varphi \). That is, let \( \overline{\mathcal{HV}}^\circ \varphi \) be the maximal good subspace of \( \overline{\mathcal{HV}}^\circ \) under \( \varphi^* : \overline{\mathcal{HV}}^* \rightarrow (\overline{\mathcal{HV}} \otimes \overline{\mathcal{HV}})^* \), which is the dual multiplication of the Lie bracket \( \varphi : \overline{\mathcal{HV}} \otimes \overline{\mathcal{HV}} \rightarrow \overline{\mathcal{HV}} \). Take \( \varphi^\circ = \varphi^*|_{\overline{\mathcal{HV}}^\circ} \), then we obtain \( \varphi^\circ(\overline{\mathcal{HV}}^\circ) \subset \overline{\mathcal{HV}}^\circ \otimes \overline{\mathcal{HV}}^\circ \) and \( (\overline{\mathcal{HV}}^\circ, \varphi^\circ) \) is a Lie coalgebra, which we call the dual Lie coalgebra of Lie algebra \( (\overline{\mathcal{HV}}, \varphi) \).

Another approach is induced from the coassociative cocommutative coalgebra \( (\overline{\mathcal{HV}}^\circ, \mu^\circ) \). Let \( (\overline{\mathcal{HV}} = \mathcal{A} \oplus \mathcal{A}, \mu) \) be the associative commutative algebra defined by \eqref{eq:mu} with \( \mu : \overline{\mathcal{HV}} \otimes \overline{\mathcal{HV}} \rightarrow \overline{\mathcal{HV}} \). Then \( \mu^* : \overline{\mathcal{HV}}^* \rightarrow (\overline{\mathcal{HV}} \otimes \overline{\mathcal{HV}})^* \). Denote \( \mu^\circ = \mu^*|_{\overline{\mathcal{HV}}^\circ} \) and \( \mu^\circ : \overline{\mathcal{HV}}^\circ \otimes \overline{\mathcal{HV}}^\circ \rightarrow \overline{\mathcal{HV}}^\circ \otimes \overline{\mathcal{HV}}^\circ \). For \( f \in \overline{\mathcal{HV}}^\circ, f = (f_1, f_2) \in \overline{\mathcal{HV}}^\circ = (\mathcal{A}^\circ \oplus \mathcal{A}^\circ) \) (by Lemma 2.3), we have

\[
\mu^\circ(f) = \mu^\circ((f_1, 0) + (0, f_2)) = \sum_{(f_1)} (f_1^{(1)}, 0) \otimes (f_1^{(2)}, 0) + \sum_{(f_2)} (0, f_2^{(1)}) \otimes (0, f_2^{(2)}), \quad (2.8)
\]
where $\mu^\circ_\phi(f_1) = \sum_{f_1} f_1^{(1)} \otimes f_1^{(2)}$, $\mu_\phi^\circ(f_2) = \sum_{f_2} f_2^{(1)} \otimes f_2^{(2)}$. Let $\partial^\circ = \partial^*|_{A^\circ}$, using (2.7), for $f = (f_1, f_2) = (f_1, 0) + (0, f_2) \in \overline{HV}^\circ_\mu = A^\circ \oplus A^\circ$, we obtain

$$
\Delta_\mu((f_1, 0)) = \left( (\text{id}, 0) \otimes (\partial^\circ, 0) - (\partial^\circ, 0) \otimes (\text{id}, 0) \right) \mu_\phi^\circ(f_1, 0) = \sum_{f_1} \left( (f_1^{(1)}), 0 \right) \otimes (\partial^\circ(f_1^{(2)}), 0) - (\partial^\circ(f_1^{(1)}), 0) \otimes (f_1^{(2)}, 0),
$$

\[
\Delta_\mu((0, f_2)) = \left( (\tau, \text{id}) \cdot ((0, \text{id}) \otimes (0, \partial^\circ)) - (id, \tau) \cdot (0, \partial^\circ) \otimes (0, \text{id})) \right) \mu_\phi^\circ((0, f_2)) = \sum_{f_2} \left( (f_2^{(1)}), 0 \right) \otimes (0, \partial^\circ(f_2^{(2)})) - (0, \partial^\circ(f_2^{(1)})) \otimes (f_2^{(2)}, 0)),
\]

where the second equation is obtained by the following, for $(x^m, x^n), (x^k, x^l) \in S$,

$$
\Delta_\mu(0, f_2)((x^m, x^n) \otimes (x^k, x^l)) = (0, f_2)((x^m, x^n), (x^k, x^l))
\]

$\Delta_\mu(0, f_2)((x^m, x^n) \otimes (x^k, x^l)) = (0, f_2)((x^m, x^n), (x^k, x^l))
\]

$\Delta_\mu(0, f_2)(x^m \cdot \partial(x^k) - \partial(x^m) \cdot x^k - x^m \cdot \partial(x^l) - \partial(x^n) \cdot x^k)
\]

$$
\Delta_\mu(0, f_2)(x^m \cdot \partial(x^l)) - f_2(\partial(x^n) \cdot x^k)
$$

It is easy to verify that $\overline{HV}^\circ_\mu, \Delta_\mu$ is a Lie coalgebra.

We remark that since $(A, \mu_\phi)$ is an associative commutative algebra, the coalgebra $\overline{HV}^\circ_\mu, \Delta_\mu$ is a cocommutative cocommutative algebra. By the second equation of (2.7), we have

$$
\left( (0, \text{id}) \otimes (0, \text{id}) - T \cdot ((0, \text{id}) \otimes (0, \text{id})) \right) \mu^\circ(0, f_2) = 0 \text{ for all } (0, f_2) \in \overline{HV}^\circ_\mu.
$$

We also remark that the difference between $\overline{HV}^\circ_\mu$ and $\overline{HV}^\circ_\varphi$ is that $\overline{HV}^\circ_\mu$ is the maximal good subspace of $\overline{HV}^\circ_\varphi$ under the map $\mu^\circ$, where $\mu$ is the multiplication of associative commutative algebra $(\overline{HV}, \mu)$; while $\overline{HV}^\circ_\varphi$ is the maximal good subspace of $\overline{HV}^\circ_\varphi$ under the map $\varphi^\circ$, where $\varphi$ is the Lie bracket of Lie algebra $(\overline{HV}, \varphi)$.

**Proposition 2.4.** The Lie coalgebra $(\overline{HV}^\circ_\mu, \Delta_\mu)$ is a Lie subcoalgebra of $(\overline{HV}^\circ_\varphi, \Delta_\varphi)$.

**Proof.** For $f \in \overline{HV}^\circ_\mu = A^\circ \oplus A^\circ_\mu$, $f = (f_1, f_2)$, $f_1, f_2 \in A^\circ_\mu$, then

$$
\Delta_\varphi(f) = \Delta_\varphi((f_1, 0) + (0, f_2)) = \left( \left( \mu((\text{id}, 0) \otimes (\partial, 0) - (\partial, 0) \otimes (\text{id}, 0)) \right)^\circ \right) ((f_1, 0)) + \sum_{f_1} \left( (f_1^{(1)}), 0 \right) \otimes (\partial^\circ(f_1^{(2)}), 0) - (\partial^\circ(f_1^{(1)}), 0) \otimes (f_1^{(2)}, 0) + \sum_{f_2} \left( (f_2^{(1)}), 0 \right) \otimes (0, \partial^\circ(f_2^{(2)})) - (0, \partial^\circ(f_2^{(1)})) \otimes (f_2^{(2)}, 0)) = \Delta_\mu(f).
\]
Let \( \mathcal{A} \) be an associative commutative algebra over a field \( \mathcal{K} \), \( \mathcal{L} = \mathcal{A} \oplus \mathcal{A} \) be the direct sum of \( \mathcal{A} \) with itself. Then \( (\mathcal{L}, \mu) \) is also an associative commutative algebra with the multiplication \( \mu((a,b),(c,d)) = (\mu_1(a,c),\mu_2(b,d)) \in \mathcal{L} \) (where \( \mu_1 = \mu_2 \) are multiplications of the first and second copy \( \mathcal{A} \) of \( \mathcal{L} \)). Denote \( \text{Der}(\mathcal{A}) \) the derivation vector space of \( \mathcal{A} \). For \( \partial \in \text{Der}(\mathcal{A}) \), define the bracket \( \varphi \) as following.

\[
\varphi((a,0),(b,0)) = \left( \mu((\text{id},0) \otimes (\partial,0) - (\partial,0) \otimes (\text{id},0)) \right)((a,0),(b,0)),
\]

\[
\varphi((0,c),(0,d)) = \mu\left( (0,\text{id}) \otimes (0,\text{id}) - ((0,\text{id}) \otimes (0,\text{id})) \cdot T \right)((0,c),(0,d)) = 0,
\]

\[
\varphi((a,0),(0,c)) = \mu\left( ((0,\text{id}) \otimes (0,\partial)) \cdot (\tau,\text{id}) \right)((a,0),(0,c)),
\]

\[
\varphi((0,c),(a,0)) = -\varphi((a,0),(0,c)).
\]

Then \( (\mathcal{L}, \varphi) \) is a Lie algebra. For convenience, we denote \( \mathcal{L} = \mathcal{A}_1 \oplus \mathcal{A}_2 \), \( \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A} \).

**Theorem 2.5.** Let \( \mathcal{L} = (\mathcal{A}_1 \oplus \mathcal{A}_2, \mu) \) be the above commutative associative algebra over any field \( \mathcal{K} \) with characteristic different from 2, and \( (\mathcal{L}, \varphi) \) be the Lie algebra defined by (2.11). If there exists \( H = (h,h) \in \mathcal{L} \) such that the idea of \( (\mathcal{L}, \mu) \) which is generated by \( (2\partial(h),\partial(h)) \) has a finite codimension, then \( \mathcal{L}_\mu = \mathcal{L}_\varphi^\circ \). In particular, \( \mathcal{H}\mathcal{V}^\circ = \mathcal{H}\mathcal{V}_\varphi^\circ \).

Proof. Since for \( f \in \mathcal{L}^* = (\mathcal{A}_1 \oplus \mathcal{A}_2)^* \) and \( (a_1,a_2) \in \mathcal{A}_1 \oplus \mathcal{A}_2 \), we have \( f((a_1,a_2)) = f((a_1,0)) + f((0,a_2)) \). Set \( f((a_1,0)) = (f_1(a_1),0) = f_1(a_1), f((0,a_2)) = (0,f_2(a_2)) = f_2(a_2) \). It follows that \( f_i \in \mathcal{A}_i^*, i = 1,2 \), and we get \( \mathcal{L}^* = (\mathcal{A}_1 \oplus \mathcal{A}_2)^* = \mathcal{A}_1^* \oplus \mathcal{A}_2^* \).

Denote by \( \cdot \) and \( * \) the actions of \( (\mathcal{L}, \mu) \) and \( (\mathcal{L}, \varphi) \) on \( \mathcal{L}^* \), respectively. Then for \( f \in \mathcal{L}^* \), \( w,v \in \mathcal{L} \), we have \( (f \cdot w)(v) = f(\mu(w,v)), (f \ast w)(v) = f(\varphi(w,v)) \). Let \( w = (w_1,w_2), v = (v_1,v_2), H = (h,h) \in \mathcal{L} \), we obtain

\[
\varphi(w,\mu(v,H)) - \varphi(\mu(w,H),v) = \varphi((w_1,w_2),(v_1h,v_2h)) - \varphi((w_1h,w_2h),(v_1,v_2))
\]

\[
= (w_1h\partial(v_1) - \partial(w_1)v_1h, w_1\partial(v_2h) - v_1h\partial(w_2))
\]

\[
= (w_1h\partial(v_1) - \partial(w_1)v_1h, w_1h\partial(v_2) - v_1\partial(w_2))
\]

\[
= (2w_1v_1\partial(h), w_1v_2\partial(h) + v_1w_2\partial(h)).
\]

By (2.12) and

\[
((f \ast w) \cdot H - f \ast \mu(w,H))(v_1,0) = f(2w_1v_1\partial(h), v_1w_2\partial(h))
\]

\[
= f\left( \left( f \cdot (2w_1\partial(h),0) + f \cdot (0, w_2\partial(h)) \cdot \tau \right)(v_1,0) \right).
\]

\[
((f \ast w) \cdot H - f \ast \mu(w,H))(0,v_2) = f(0, w_1v_2\partial(h))
\]

\[
= f \cdot (0, w_1\partial(h))(0,v_2),
\]

we obtain

\[
((f \ast w) \cdot H - f \ast \mu(w,H))_1 = f_1 \cdot (2w_1\partial(h)) + \left( f_2 \cdot (w_2\partial(h)) \right) \cdot \tau,
\]

\[
((f \ast w) \cdot H - f \ast \mu(w,H))_2 = f_2 \cdot (w_1\partial(h)),
\]

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where the subscript \( _i \) denotes the \( i \)-th coordinate of an element for \( i = 1, 2 \). If \( f \in \mathcal{L}_\varphi \), i.e., \( f \ast \mathcal{L} \) is finite dimensional, then the left sides of (2.13) are in finite dimensional subspaces. If the idea \( (\mathcal{A} \partial(h)) \) of \( \mathcal{A} \) generalized by \( \partial(h) \) has finite codimension in \( \mathcal{A} \), then the second equation of (2.13) shows that \( f_2 \cdot \mathcal{A} \) is finite dimensional, i.e., \( f_2 \in \mathcal{A}^\circ \). Moreover, if the idea \( (\mathcal{A}(2\partial(h))) \) of \( \mathcal{A} \) which is generalized by \( 2\partial(h) \) also has finite codimension, then the first equation of (2.13) shows that \( f_1 \cdot \mathcal{A} \) is finite dimensional, i.e., \( f_1 \in \mathcal{A}^\circ \). □

By Theorem 2.5, we have \( \overline{HV}_\mu = \overline{HV}_\varphi \), which is now denoted by \( \overline{HV}^\circ \). By Lemmas 2.2 and 2.3, we have

\[
\overline{HV}^\circ = \mathcal{A}^\circ \oplus \mathcal{A}^\circ \quad \text{with} \quad \mathcal{A} = K \left[ x^{\pm 1} \right]
\]

(2.14)

\[\mathcal{A}^\circ = \left\{ f = \sum_{i \in \mathbb{Z}} f_i \varepsilon^i \in \mathcal{A}^* \middle| \exists r \in \mathbb{N}, c_j \in K \text{ such that } f_n = \sum_{j=1}^{r} c_j f_{n-j}, \forall n \in \mathbb{Z} \right\}.
\]

3 Dual Lie bialgebras of the twisted Heisenberg-Virasoro types

As stated in the introduction, the Heisenberg-Virasoro algebra was first studied in [1], it is an important algebra structure which has close relations with the Heisenberg algebra and the Virasoro algebra, and has also some relations with the full-toroidal Lie algebras and conformal algebras (see, e.g., [28]). The representations of the twisted Heisenberg-Virasoro algebra were studied by some authors (see [2, 11, 13, 14]). The authors of [12] investigated the Lie bialgebra structures of the twisted Heisenberg-Virasoro algebra. In this section, we will investigate the dual Lie bialgebra structures of the twisted Heisenberg-Virasoro type. First, we recall some results which are related to the Heisenberg-Virasoro Lie bialgebra (see, e.g., [12, 23]).

**Proposition 3.1.** (1) Let \( \mathcal{W} \) be the classical Witt (or Virasoro) algebra (i.e., \( \mathcal{W} = K[t, t^{-1}] \)) such that \([f, g] = f \frac{dg}{dt} - g \frac{df}{dt}\) for \( f, g \in \mathcal{W} \).

(i) Every Lie bialgebra structure on \( \mathcal{W} \) is coboundary triangular associated to a solution \( r \) of CYBE (2.1) of the form \( r = a \otimes b - b \otimes a \) for some nonzero \( a, b \in \mathcal{W} \) satisfying

\[ [a, b] = kb \quad \text{for some } 0 \neq k \in K. \quad (3.1) \]

(ii) Let \( \mathfrak{g} \) be an infinite dimensional Lie subalgebra of \( \mathcal{W} \) such that \( t \in \mathfrak{g} \) and \( \mathfrak{g} \not\supseteq \mathcal{W} \) as Lie algebras. Denote by \( \mathfrak{g}^{(n)} \) the Lie bialgebra defined on \( \mathfrak{g} \) associated to the solution \( r_n = t \otimes t^n - t^n \otimes t \) of CYBE for any \( t^n \in \mathfrak{g} \). Then every Lie bialgebra structure on \( \mathfrak{g} \) is isomorphic to \( \mathfrak{g}^{(n)} \) for some \( n \) with \( t^n \in \mathfrak{g} \).
Let \((\mathcal{HV}, \varphi)\) be the centerless twisted Heisenberg-Virasoro algebra defined by (2.7), and \((\mathcal{HV}, \varphi, \Delta)\) be a Lie bialgebra. Then \(\Delta = \Delta_r + \delta\) such that \(\Delta_r(x) = x \cdot r \in \mathcal{HV} \otimes \mathcal{HV}\) for some \(r \in \text{Im}(1 - \tau)\), where \(\tau\) is defined by \(\tau(a \otimes b) = b \otimes a\), and \(\delta\) is defined by

\[
\delta(L_n) = (n\alpha + \gamma)(I_0 \otimes I_n - I_n \otimes I_0), \quad \delta(I_n) = \beta(I_0 \otimes I_n - I_n \otimes I_0),
\]

for some fixed \(\alpha, \beta, \gamma \in \mathcal{K}\). Furthermore, \((\mathcal{HV}, \varphi, \delta)\) is a Lie bialgebra.

**Remark 3.2.** It is proved in [16] that if two elements \(a, b\) in a Lie algebra \((\mathcal{L}, [\cdot, \cdot])\) satisfy (3.1), then \(r = a \otimes b - b \otimes a\) is a solution of CYBE, and one obtains a coboundary triangular Lie bialgebra \((\mathcal{L}, [\cdot, \cdot], \Delta_r)\) by defining \(\Delta_r : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}\) as \(\Delta_r(z) = z \cdot r\) for \(z \in \mathcal{L}\). Proposition 3.3 below shows that for \(a = (x, 0), b = (x, qx^m) \in \mathcal{HV}\) with \(0 \neq q \in \mathcal{K}, m \in \mathbb{Z}\), even though (3.1) does not hold, we still have a solution of CYBE \(r = a \otimes b - b \otimes a\). Thus, (3.1) is not the necessary condition for \(r = a \otimes b - b \otimes a\) to be a solution of CYBE.

**Proposition 3.3.** Let \((\mathcal{HV}, \varphi)\) be the centerless twisted Heisenberg-Virasoro algebra defined by (2.7). Then \(r = (x, 0) \otimes (x^{m+1}, qx^n) - (x^{m+1}, qx^n) \otimes (x, 0)\) with \(m, n \in \mathbb{Z}, q \in \mathcal{K}\) is a solution of CYBE if and only if one of the following holds: (1) \(m = n\); (2) \(m = 0\); (3) \(q = 0\). Furthermore, \(r = (x, 0) \otimes (0, qx^n) - (0, qx^n) \otimes (x, 0)\) is a solution of CYBE.

**Proof.** By computation, we can get

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]
= (n - m)(x^{m+1}, 0) \otimes (0, qx^n) \otimes (x, 0) + (m - n)(x^{m+1}, 0) \otimes (x, 0) \otimes (0, qx^n)
+ (m - n)(0, qx^n) \otimes (x^{m+1}, 0) \otimes (x, 0) - (m - n)(0, qx^n) \otimes (x, 0) \otimes (x^{m+1}, 0)
- (m - n)(x, 0) \otimes (x^{m+1}, 0) \otimes (0, qx^n) + (m - n)(x, 0) \otimes (0, qx^n) \otimes (x^{m+1}, 0).
\]

Then \([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\) if and only if one of the four cases in Proposition 3.3 occurs. 

The authors of [25, 26] constructed the dual Lie bialgebra structures of Witt (Virasoro) and Poisson type Lie bialgebras. The dual structures of loop type and current type Lie bialgebras were considered in [27]. Now, we start to investigate the dual Lie bialgebra structures of the twisted Heisenberg-Virasoro type. Let \(\mathcal{HV} = A_1 \oplus A_2, A_1 = A_2 = \mathcal{K}[x^{\pm 1}]\) and \(S = \{(x^m, x^n) \mid m, n \in \mathbb{Z}\}\) be the standard basis of \(\mathcal{HV}\). Denote by \(S^* = \{(\varepsilon_i, \varepsilon_j) \mid i, j \in \mathbb{Z}\}\) the set of dual basis of \(S\), i.e., \langle (\varepsilon_i, 0), (x^j, 0) \rangle = \langle (0, \varepsilon_i), (0, x^j) \rangle = \delta_{i,j} \) for \(i, j \in \mathbb{Z}\), and \langle (\varepsilon_i, 0), (0, x^j) \rangle = 0, \langle (0, \varepsilon_i), (x^j, 0) \rangle = 0.\]
Theorem 3.4. Let \((HV, \varphi)\) be the twisted Heisenberg-Virasoro algebra defined by (2.7), and \((HV^o, \Delta)\) be a Lie coalgebra, where \(HV^o\) is determined by Lemmas 2.2, 2.3 and Theorem 2.5, then the cobracket \(\Delta\) is uniquely determined, for \(m \in \mathbb{Z}\), by

1. \(\Delta((\varepsilon^m, 0)) = \sum_{i,j \in \mathbb{Z}, i+j=m+1} (j-i)(\varepsilon^i, 0) \otimes (\varepsilon^j, 0),\)
2. \(\Delta((0, \varepsilon^m)) = \sum_{i,j \in \mathbb{Z}, i+j=m+1} (j(\varepsilon^i, 0) \otimes (0, \varepsilon^j) - i(0, \varepsilon^i) \otimes (\varepsilon^j, 0)).\)

Proof. For \(\varepsilon^m \in A^*\), assume \(\mu_1(\varepsilon^m) = \mu_2(\varepsilon^m) = \sum_{i,j} c_{i,j} \varepsilon^i \otimes \varepsilon^j\) for some \(c_{i,j} \in \mathbb{K}\), then \(c_{i,j} = \sum_{i,j} c_{i,j} \varepsilon^i \otimes \varepsilon^j(x^i \otimes x^j) = \mu_k^*(\varepsilon^m)(x^i \otimes x^j) = \varepsilon^m(x^{i+j}) = \delta_{m,i+j}.\) Therefore \(\mu_k^*(\varepsilon^m) = \sum_{i+j=m} \varepsilon^i \otimes \varepsilon^j, i, j \in \mathbb{Z}\) and \(k = 1, 2.\) Assume \(\partial^*(\varepsilon^m) = \sum_{i \in \mathbb{Z}} c_i \varepsilon^i\) for some \(c_i \in \mathbb{K}.\) Then \(c_i = \sum_{i \in \mathbb{Z}} c_i \varepsilon^i(x^i) = \partial^*(\varepsilon^m)(x^i) = \varepsilon^m(\partial(x^i)) = i\delta_{m,i-1}.\) From this, we have \(\partial^*(\varepsilon^m) = (m+1)\varepsilon^{m+1}.\)

By (2.8) and (2.9), for \(i, j \in \mathbb{Z}\), we obtain

\[
\Delta((\varepsilon^m, 0)) = \sum_{i+j=m} (\varepsilon^i, 0) \otimes (\partial^*(\varepsilon^j), 0) - (\partial^*(\varepsilon^i), 0) \otimes (\varepsilon^j, 0) \\
= \sum_{i+j=m+1} (j-i)((\varepsilon^i, 0) \otimes (\varepsilon^j, 0)),
\]

\[
\Delta((0, \varepsilon^m)) = \sum_{i+j=m} ((\varepsilon^i, 0) \otimes (0, \partial^*(\varepsilon^j)) - (0, \partial^*(\varepsilon^i)) \otimes (\varepsilon^j, 0)) \\
= \sum_{i+j=m} ((j+1)(\varepsilon^i, 0) \otimes (0, \varepsilon^{j+1}) - (i+1)(0, \varepsilon^{i+1}) \otimes (\varepsilon^j, 0)) \\
= \sum_{i+j=m+1} (j(\varepsilon^i, 0) \otimes (0, \varepsilon^j) - i(0, \varepsilon^i) \otimes (\varepsilon^j, 0)).
\]

\[\square\]

Theorem 3.5. Let \((HV, \varphi, \Delta_r)\) be a coboundary triangular Lie bialgebra related to the solution of CYBE \(r = (x, 0) \otimes (x^{m+1}, 0) - (x^{m+1}, 0) \otimes (x, 0)\) with \(m \neq 0.\) Then the dual Lie bialgebra of \((HV, \varphi, \Delta_r)\) is \((HV^o, [\cdot, \cdot], \Delta),\) where the underline vector space \(HV^o\) is determined by (2.14), the cobracket \(\Delta\) is determined by Theorem 3.4 and the bracket is uniquely determined by

1. \([\varepsilon^i, 0], (\varepsilon^j, 0)] = \begin{cases} (2m-j+1)(\varepsilon^{j-m}, 0) & \text{if } i = 1, j \neq 1, \\ (j-1)(\varepsilon^j, 0) & \text{if } i = m+1, j \neq 1, m+1, \\ 0 & \text{if } i, j \notin \{1, m+1\}. \end{cases}\)
2. \(([\varepsilon^i, 0], (0, \varepsilon^j)] = \begin{cases} (m-j)(0, \varepsilon^{j-m}) & \text{if } i = 1, \\ j(0, \varepsilon^j) & \text{if } i = m+1, \\ 0 & \text{if } i \notin \{1, m+1\}. \end{cases}\)
3. \(([0, \varepsilon^i], (0, \varepsilon^j)] = 0 \text{ for } i, j \in \mathbb{Z}.\)

Convention 3.6. (1) In the dual Lie bialgebra \((HV^o, [\cdot, \cdot], \Delta),\) we always use \([\cdot, \cdot]\) to denote its Lie bracket and \(\Delta\) to denote its Lie cobracket, i.e., \([\cdot, \cdot] = \Delta^o, \Delta = \varphi^o.\)
(2) For \( f \in \mathcal{H}^0 \cong \mathcal{A}^0 \oplus \mathcal{A}^e \), \( f = (f_1, f_2) \) and \((x^k, x^l) \in \mathcal{A} \oplus \mathcal{A} \), we always write \( f(x^k, x^l) = (f_1(x^k), f_2(x^l)) \).

Proof of Theorem 3.5. By computation, we can get

\[
\langle [(\varepsilon^i, 0), (\varepsilon^j, 0)], (x^k, x^l) \rangle
\]
\[
= \langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), \Delta_r((x^k, x^l)) \rangle
\]
\[
= \langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), \delta_{i,1,0}((x^k, x^l)) \rangle
\]
\[
= \langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), ((1 - k)x^k, -lx^l) \rangle \otimes (x^{m+1}, 0) - (x^{m+1}, 0) \otimes ((1 - k)x^k, -lx^l)
\]
\[
+ (x, 0) \otimes ((m + 1 - k)x^{m+k}, -lx^{m+l}) - ((m + 1 - k)x^{m+k}, -lx^{m+l}) \otimes (x, 0)
\]
\[
= ((1 - k)\delta_{i,k,0})(\delta_{j,m+1,0}) + (\delta_{i,1,0})(m + 1 - k)\delta_{m,k,j,0}
\]
\[
- (\delta_{i,m+1,0})(1 - k)\delta_{j,k,0} - (\delta_{j,1,0})(m + 1 - k)\delta_{m,k,i,0}
\]
\[
= ((1 - i)\delta_{i,k,0})(\delta_{j,m+1,0}) + (\delta_{i,1,0})(2m + 1 - j)\delta_{j,m-k,0}
\]
\[
- (\delta_{i,m+1,0})(1 - j)\delta_{j,k,0} - (\delta_{j,1,0})(2m + 1 - i)\delta_{j,k,i,0}
\]
\[
= \langle \delta_{j,m+1}(1 - i)\varepsilon^i, 0, \delta_{i,1}(m + 1 - j)\varepsilon^j - m, 0 \rangle
\]
\[
- \delta_{i,m+1}(1 - j)\varepsilon^j, 0 - \delta_{j,1}(m + 1 - i)\varepsilon^j - m, 0, (x^k, x^l) \rangle.
\]

We obtain

\[
[(\varepsilon^i, 0), (\varepsilon^j, 0)] = \delta_{j,m+1}(1 - i)\varepsilon^i, 0, \delta_{i,1}(m + 1 - j)\varepsilon^j - m, 0
\]
\[
- \delta_{i,m+1}(1 - j)\varepsilon^j, 0 - \delta_{j,1}(m + 1 - i)\varepsilon^j - m, 0. \tag{3.3}
\]

From this, we obtain \([(\varepsilon^i, 0), (\varepsilon^j, 0)] = 0 \) if \( i, j \not\in \{1, m + 1\} \). If \( i = 1 \neq j \), by noting that \( m \neq 0 \), we have \([(\varepsilon^1, 0), (\varepsilon^j, 0)] = ((2m + 1 - j)\varepsilon^j - m, 0) \) by (3.3) for \( j \in \mathbb{Z} \). If \( i = m + 1 \) and \( j \neq 1, m + 1 \), then \([(\varepsilon^{m+1}, 0), (\varepsilon^j, 0)] = ((j - 1)\varepsilon^j, 0) \). The case (1) of Theorem 3.5 holds.

For the case (2), since

\[
\langle [(\varepsilon^i, 0), (0, \varepsilon^j)], (x^k, x^l) \rangle = \langle (\varepsilon^i, 0) \otimes (0, \varepsilon^j), \Delta_r((x^k, x^l)) \rangle
\]
\[
= (\delta_{i,1,0})(0, -l\delta_{j,m+1}) - (\delta_{i,m+1,0})(0, -l\delta_{j,l})
\]
\[
= (\delta_{i,1}(0, (m - j)\varepsilon^j - m) - \delta_{i,m+1}(0, -j\varepsilon^j), (x^k, x^l)),
\]

we have

\[
[(\varepsilon^i, 0), (0, \varepsilon^j)] = \delta_{i,1}(0, (m - j)\varepsilon^j - m) - \delta_{i,m+1}(0, -j\varepsilon^j). \tag{3.4}
\]

From this, it follows that \([(\varepsilon^i, 0), (0, \varepsilon^j)] = 0 \) if \( i \not\in \{1, m + 1\} \). If \( i = 1 \), we have \([(\varepsilon, 0), (0, \varepsilon^j)] = (0, (m - j)\varepsilon^j - m) \); if \( i = m + 1 \), we have \([(\varepsilon^{m+1}, 0), (0, \varepsilon^j)] = (0, j\varepsilon^j) \). The case (2) of the theorem is proved.

Finally, for the case (3), since \( \langle [(0, \varepsilon^i), (0, \varepsilon^j)], (x^k, x^l) \rangle = \langle (0, \varepsilon^i) \otimes (0, \varepsilon^j), \Delta_r((x^k, x^l)) \rangle \)
\[
= 0 \text{ for } i, j, k, l \in \mathbb{Z}, \text{ we have } [(0, \varepsilon^i), (0, \varepsilon^j)] = 0. \]
Theorem 3.7. Let $(\mathcal{HV}, \varphi, \Delta_r)$ be the coboundary triangular Lie bialgebra which is related to the solution of CYBE $r = (x, 0) \otimes (x^{m+1}, qx^m) - (x^{m+1}, qx^m) \otimes (x, 0)$, then its dual Lie bialgebra is $(\mathcal{HV}^\circ, [\cdot, \cdot], \Delta)$, where $\mathcal{HV}^\circ$ is determined by (2.14), the cobracket $\Delta$ is determined by Theorem 3.4, and the bracket is uniquely determined by

$$
(1) \quad [(\varepsilon^i, 0), (\varepsilon^j, 0)] = \begin{cases} 
(2m - j + 1)(\varepsilon^{j-m}, 0) & \text{if } i = 1, j \neq 1, \\
(j - 1)(\varepsilon^j, 0) & \text{if } i = m + 1, j \neq 1, m + 1, \\
0 & \text{if } i, j \notin \{1, m + 1\}.
\end{cases}
$$

$$
(2) \quad [(\varepsilon^i, 0), (0, \varepsilon^j)] = \begin{cases} 
-mq(\varepsilon^{j-m+1}, 0) & \text{if } i = 1, m \neq 0, \\
mq(\varepsilon^{j-m+1}, 0) - (j - m)(0, \varepsilon^{j-m}) & \text{if } i = m + 1, j = m, \\
j(0, \varepsilon^j) & \text{if } i = m + 1, j \neq m, \\
(1 - i)q(\varepsilon^i, 0) & \text{if } i \neq 1, m + 1, j = m, \\
0 & \text{otherwise}.
\end{cases}
$$

$$
(3) \quad [(0, \varepsilon^i), (0, \varepsilon^j)] = \begin{cases} 
jq(0, \varepsilon^j) & \text{if } i = m, j \neq m, \\
0 & \text{if } i \neq m, j \neq m.
\end{cases}
$$

Proof. We can compute that

$$
\left\langle [(\varepsilon^i, 0), (\varepsilon^j, 0)], (x^k, x^l) \right\rangle
= \left\langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), \Delta_r((x^k, x^l)) \right\rangle
= \left\langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), (x^k, x^l) \cdot ((x, 0) \otimes (x^{m+1}, qx^m) - (x^{m+1}, qx^m) \otimes (x, 0)) \right\rangle
= \left\langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), ((1 - k)x^k, -lx^l) \otimes (x^{m+1}, qx^m) - (x^{m+1}, qx^m) \otimes (1 - k)x^k, -lx^l) \\
+ (x, 0) \otimes ((m + 1 - k)x^{m+k}, mqx^{m-k-1} - lx^{m+l}) - ((m + 1 - k)x^{m+k}, mqx^{m-k-1} - lx^{m+l}) \otimes (x, 0) \right\rangle
= \left\langle (1 - k)\delta_{i,k}, 0 \right\rangle \delta_{j,m+1} + \delta_{i,1}((m + 1 - k)\delta_{j,m+k}, 0) - ((1 - k)\delta_{i,k}, 0)\delta_{i,m+1} \\
- \delta_{j,1}((m + 1 - k)\delta_{i,m+k}, 0)
= \left\langle \delta_{j,m+1}((1 - i)\varepsilon^i, 0) + \delta_{i,1}((2m + 1 - j)\varepsilon^{j-m}, 0) - \delta_{i,m+1}((1 - j)\varepsilon^j, 0) \\
- \delta_{j,1}((2m + 1 - i)\varepsilon^{i-m}, 0), (x^k, x^l) \right\rangle.
$$

Hence we have

$$
[(\varepsilon^i, 0), (\varepsilon^j, 0)] = \delta_{j,m+1}((1 - i)\varepsilon^i, 0) + \delta_{i,1}((2m + 1 - j)\varepsilon^{j-m}, 0) \\
- \delta_{i,m+1}((1 - j)\varepsilon^j, 0) - \delta_{j,1}((2m + 1 - i)\varepsilon^{i-m}, 0).
$$

Just as the proof of Theorem 3.5, we obtain case (1) of Theorem 3.7.
For case (2), by computation, we have

\[
\left\langle [(\epsilon^i, 0), (0, \epsilon^j)], (x^k, x^l) \right\rangle
= \left\langle (\epsilon^i, 0) \otimes (0, \epsilon^j), \Delta_r((x^k, x^l)) \right\rangle
= \left\langle (\epsilon^i, 0) \otimes (0, \epsilon^j), (x^k, x^l) \cdot \left( (x, 0) \otimes (x^{m+1}, qx^m) - (x^{m+1}, qx^m) \otimes (x, 0) \right) \right\rangle
\]

Thus

\[
[(\epsilon^i, 0), (0, \epsilon^j)] = \delta_{j,m}((1 - i)q\epsilon^i, 0) + \delta_{i,1}(mq\epsilon^{j-m+1}, 0) - (0, (j - m)\epsilon^{j-m}) + \delta_{i,m+1}(0, j\epsilon^j).
\]

From this, if \( i \neq 1, m + 1 \) and \( j \neq m \), we obtain \( [(\epsilon^i, 0), (0, \epsilon^j)] = 0 \). If \( i = 1 \), then

\[
[(\epsilon^1, 0), (0, \epsilon^j)] = (mq\epsilon^{j-m+1}, 0) - (0, (j - m)\epsilon^{j-m}) + \delta_{1,m+1}(0, j\epsilon^j).
\]

Thus \( [(\epsilon^1, 0), (0, \epsilon^j)] = 0 \) if \( m = 0 \), and \( [(\epsilon^1, 0), (0, \epsilon^j)] = (mq\epsilon^{j-m+1}, 0) - (0, (j - m)\epsilon^{j-m}) \) if \( m \neq 0 \). Assume \( i = m + 1 \neq 1 \). Then (3.5) gives

\[
[(\epsilon^{m+1}, 0), (0, \epsilon^j)] = \delta_{j,m}(-mq\epsilon^{m+1}, 0) + (0, j\epsilon^j) = \begin{cases} 
(\text{if } j = m), \\
(0, j\epsilon^j) 
\end{cases}
\]

Assume \( i \neq 1, m + 1 \). Then

\[
[(\epsilon^i, 0), (0, \epsilon^j)] = \delta_{j,m}((1 - i)q\epsilon^i, 0) = \begin{cases} 
(1 - i)q\epsilon^i, 0) \text{ if } j = m, \\
0 \text{ if } j \neq m.
\end{cases}
\]

Therefore case (2) is obtained.

For case (3) of Theorem 3.7, since

\[
\left\langle [(0, \epsilon^i), (0, \epsilon^j)], (x^k, x^l) \right\rangle
= \left\langle (0, \epsilon^i) \otimes (0, \epsilon^j), \Delta_r((x^k, x^l)) \right\rangle
= \left\langle (0, \epsilon^i) \otimes (0, \epsilon^j), (x^k, x^l) \cdot \left( (x, 0) \otimes (x^{m+1}, qx^m) - (x^{m+1}, qx^m) \otimes (x, 0) \right) \right\rangle
\]
\[ \langle (0, \varepsilon^i) \otimes (0, \varepsilon^j), ((1 - k)x^k, -lx^l) \otimes (x^{m+1}, qx^m) \]
\[ - (x^{m+1}, qx^m) \otimes ((1 - k)x^k, -lx^l) \]
\[ + (x, 0) \otimes ((m + 1 - k)x^{m+k} - mqx^{m+k-1} - lx^{m+l}) \]
\[ - ((m + 1 - k)x^{m+k} - mqx^{m+k-1} - lx^{m+l}) \otimes (x, 0) \]
\[ = (0, -l\delta_{i,l})q\delta_{j,m} - q\delta_{i,m}(0, -l\delta_{j,l}) \]
\[ = \left\{ \begin{array}{ll}
(0, jq\varepsilon^j) & \text{if } i = m, j \neq m, \\
0 & \text{if } i \neq m, j \neq m.
\end{array} \right. \]  
(3.6)

we have

\[ [(0, \varepsilon^i), (0, \varepsilon^j)] = \delta_{i,m}(0, jq\varepsilon^j) - \delta_{j,m}(0, iq\varepsilon^i) = \left\{ \begin{array}{ll}
(0, jq\varepsilon^j) & \text{if } i = m, j \neq m, \\
0 & \text{if } i \neq m, j \neq m.
\end{array} \right. \]

The theorem is proved.

\[ \square \]

**Theorem 3.8.**  
(1) Let \((\mathcal{HV}, \varphi, \Delta_r)\) be the coboundary triangular Lie bialgebra related to the solution of CYBE \(r = (x, 0) \otimes (x, qx^m) - (x, qx^m) \otimes (x, 0)\), then its dual Lie bialgebra is \((\mathcal{HV}^\circ, [\cdot, \cdot], \Delta)\), where \(\mathcal{HV}^\circ\) is defined by (2.14), the cobracket \(\Delta\) is determined by Theorem 3.4, and the bracket is uniquely determined by

(a) \([([\varepsilon^i, 0], (\varepsilon^j, 0)] = 0; \]

(b) \([([\varepsilon^i, 0], 0, \varepsilon^j)]) = \left\{ \begin{array}{ll}
mq(\varepsilon^{i-m+1}, 0) & \text{if } i = 1, \\
(1 - i)q(\varepsilon^i, 0) & \text{if } i \neq 1, j = m, \\
0 & \text{if } i \neq 1, j \neq m; \end{array} \right. \]

(c) \([([0, \varepsilon^i], 0, \varepsilon^j)]) = \left\{ \begin{array}{ll}
jq(0, \varepsilon^j) & \text{if } i = m, j \neq m, \\
0 & \text{if } i \neq m, j \neq m. \end{array} \right. \]

(2) Let \((\mathcal{HV}, \varphi, \Delta'_r)\) be the coboundary triangular Lie bialgebra related to the solution of CYBE \(r' = (x, 0) \otimes (0, qx^m) - (0, qx^m) \otimes (x, 0)\), then its dual Lie bialgebra is \((\mathcal{HV}^\circ, [\cdot, \cdot], \Delta)\), where \(\mathcal{HV}^\circ\) is defined by (2.14), the cobracket \(\Delta\) is determined by Theorem 3.4, and the bracket is uniquely determined by

(a) \([([\varepsilon^i, 0], (\varepsilon^j, 0)] = 0; \]

(b) \([([\varepsilon^i, 0], 0, \varepsilon^j)]) = \left\{ \begin{array}{ll}
mq(\varepsilon^{i-m+1}, 0) & \text{if } i = 1, \\
(1 - i)q(\varepsilon^i, 0) & \text{if } i \neq 1, j = m, \\
0 & \text{if } i \neq 1, j \neq m; \end{array} \right. \]

(c) \([([0, \varepsilon^i], 0, \varepsilon^j)]) = \left\{ \begin{array}{ll}
jq(0, \varepsilon^j) & \text{if } i = m, j \neq m, \\
0 & \text{if } i \neq m, j \neq m. \end{array} \right. \]
Proof. For any $k, l \in \mathbb{Z}$,
\[
\left\langle [(\varepsilon^i, 0), (\varepsilon^j, 0)], (x^k, x^l) \right\rangle \\
= \left\langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), \Delta_r((x^k, x^l)) \right\rangle \\
= \left\langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), ((1 - k)x^k, -lx^l) \otimes (x, qx^m) - (x, qx^m) \otimes ((1 - k)x^k, -lx^l) \\
+ (x, 0) \otimes ((1 - k)x^k, mqx^{m+k-1} - lx^l) - ((1 - k)x^k, mqx^{m+k-1} - lx^l) \otimes (x, 0) \right\rangle \\
= \delta_{j,1}((1 - k)\delta_{i,k}, 0) + \delta_{i,1}((1 - k)\delta_{j,k}, 0) - \delta_{i,1}((1 - k)\delta_{j,k}, 0) - \delta_{j,1}((1 - k)\delta_{i,k}, 0) \\
= 0,
\]

it gives that $[(\varepsilon^i, 0), (\varepsilon^j, 0)] = 0$, so the first case of (1) is obtained.

Now we prove (b) of case (1). Since
\[
\left\langle [(\varepsilon^i, 0), (0, \varepsilon^j)], (x^k, x^l) \right\rangle \\
= \left\langle (\varepsilon^i, 0) \otimes (0, \varepsilon^j), \Delta_r((x^k, x^l)) \right\rangle \\
= q\delta_{j,m}((1 - k)\delta_{i,k}, 0) + \delta_{i,1}(0, mq\delta_{j,m+k-1} - l\delta_{j,l}) - \delta_{i,1}(0, -l\delta_{j,l}) \\
= \langle \delta_{j,m}((1 - i)q\varepsilon^i, 0) + \delta_{i,1}(mq\varepsilon^{j-m+1}, 0), (x^k, x^l) \rangle,
\]
we obtain
\[
[(\varepsilon^i, 0), (0, \varepsilon^j)] = \delta_{j,m}((1 - i)q\varepsilon^i, 0) + \delta_{i,1}(mq\varepsilon^{j-m+1}, 0). \tag{3.7}
\]

From this, we have
\[
[(\varepsilon^i, 0), (0, \varepsilon^j)] = \begin{cases} 
(mq\varepsilon^{j-m+1}, 0) & \text{if } i = 1, \\
((1 - i)q\varepsilon^i, 0) & \text{if } i \neq 1, j = m, \\
0 & \text{if } i \neq 1, j \neq m.
\end{cases}
\]

Similarly, since
\[
\left\langle [(0, \varepsilon^i), (0, \varepsilon^j)], (x^k, x^l) \right\rangle = \left\langle (0, \varepsilon^i) \otimes (0, \varepsilon^j), (x^k, x^l) \cdot r \right\rangle \\
= (0, -l\delta_{i,l})q\delta_{j,m} - q\delta_{i,m}(0, -l\delta_{j,l}) \\
= \langle \delta_{j,m}(0, -iq\varepsilon^i) - \delta_{i,m}(0, -jq\varepsilon^j), (x^k, x^l) \rangle,
\]
we have
\[
[(0, \varepsilon^i), (0, \varepsilon^j)] = \delta_{i,m}(0, jq\varepsilon^j) - \delta_{j,m}(0, iq\varepsilon^i) = \begin{cases} 
(0, jq\varepsilon^j) & \text{if } i = m, j \neq m, \\
(0, iq\varepsilon^i) & \text{if } i \neq m, j \neq m.
\end{cases}
\]
The third case of (1) holds.
Now we prove case (2). For \((x^k, x^l) \in \mathcal{HV}\) with \(k, l \in \mathbb{Z}\),
\[
(x^k, x^l) \cdot r' = (x^k, x^l) \cdot ((x, 0) \otimes (0, qx^m) - (0, qx^m) \otimes (x, 0))
\]
\[
= ((1 - k)x^k, -lx^l) \otimes (0, qx^m) + (x, 0) \otimes (0, mqx^{m+k-1})
\]
\[
- (0, mqx^{m+k-1}) \otimes (x, 0) - (0, qx^m) \otimes ((1 - k)x^k, -lx^l),
\]
we obtain \(\langle [(\varepsilon^i, 0), (\varepsilon^j, 0)], (x^k, x^l) \rangle = 0\). Therefore, \(\langle (\varepsilon^i, 0), (\varepsilon^j, 0) \rangle = 0\), case (a) is obtained.

Similarly, from
\[
\langle [(\varepsilon^i, 0), (0, \varepsilon^j)], (x^k, x^l) \rangle = ((1 - k)\delta_{i,k}, 0)q\delta_{j,m} + \delta_{i,1}(0, mq\delta_{j,k+m-1})
\]
\[
= \langle \delta_{j,m}((1 - i)q\varepsilon^i, 0) + \delta_{i,1}(mq\varepsilon^{j-m+1}, 0), (x^k, x^l) \rangle,
\]
we obtain
\[
\langle (\varepsilon^i, 0), (0, \varepsilon^j) \rangle = \delta_{j,m}((1 - i)q\varepsilon^i) + \delta_{i,1}(mq\varepsilon^{j-m+1}, 0)
\]
\[
= \begin{cases} 
(mq\varepsilon^{j-m+1}, 0) & \text{if } i = 1, \\
((1 - i)q\varepsilon^i, 0) & \text{if } i \neq 1, j = m, \\
0 & \text{if } i \neq 1, j \neq m.
\end{cases}
\]

From the relation
\[
\langle [(0, \varepsilon^i), (0, \varepsilon^j)], (x^k, x^l) \rangle = (0, -l\delta_{i,l})q\delta_{j,m} - q\delta_{i,m}(0, -l\delta_{j,l})
\]
\[
= \langle \delta_{i,m}(0, jq\varepsilon^j) - \delta_{j,m}(0, iq\varepsilon^i), (x^k, x^l) \rangle,
\]
it follows that
\[
\langle (0, \varepsilon^i), (0, \varepsilon^j) \rangle = \delta_{i,m}(0, jq\varepsilon^j) - \delta_{j,m}(0, iq\varepsilon^i) = \begin{cases} 
(0, jq\varepsilon^j) & \text{if } i = m, j \neq m, \\
0 & \text{if } i \neq m, j \neq m.
\end{cases}
\]

\[\Box\]

**Theorem 3.9.** Let \((\mathcal{HV}, \varphi, \delta)\) be the Lie bialgebra with \(\delta\) defined by (3.2). Then its dual Lie bialgebra is \((\mathcal{HV}^\circ, [\cdot, \cdot], \Delta)\), where \(\mathcal{HV}^\circ\) is determined by (2.14), the cobracket \(\Delta\) is determined by Theorem 3.4, and the bracket is uniquely determined by

1. \(\langle (\varepsilon^i, 0), (\varepsilon^j, 0) \rangle = 0\),
2. \(\langle (\varepsilon^i, 0), (0, \varepsilon^j) \rangle = 0\),
3. \(\langle (0, \varepsilon^i), (0, \varepsilon^j) \rangle = \begin{cases} 
(j\alpha + \gamma)(\varepsilon^{j+1}, 0) + \beta(0, \varepsilon^j) & \text{if } i = 1, j \neq 1, \\
0 & \text{if } i \neq 1, j \neq 1.
\end{cases}\)

**Proof.** For \(k, l \in \mathbb{Z}\),
\[
\delta((x^k, x^l)) = \delta((x^k, 0) + (0, x^l))
\]
\[
= ((k - 1)\alpha + \gamma)((0, 1) \otimes (0, x^{k-1}) - (0, x^{k-1}) \otimes (0, 1))
\]
\[
+ \beta((0, 1) \otimes (0, x^l) - (0, x^l) \otimes (0, 1)),
\]

we obtain \( \langle [\varepsilon^i, 0], (\varepsilon^j, 0) \rangle, (x^k, x^l) \rangle = \langle (\varepsilon^i, 0) \otimes (\varepsilon^j, 0), \delta((x^k, x^l)) \rangle = 0 \). Thus \([\varepsilon^i, 0], (\varepsilon^j, 0)] = 0\). Similarly, we have \([\varepsilon^i, 0], (0, \varepsilon^j)] = 0\). Finally, by computation, we have

\[
\begin{align*}
\langle [0, \varepsilon^i], (0, \varepsilon^j) \rangle, (x^k, x^l) \rangle
&= \langle (0, \varepsilon^i) \otimes (0, \varepsilon^j), \delta((x^k, x^l)) \rangle \\
&= \langle (j \alpha + \gamma) \delta_{i,0}(\varepsilon^{j,1}, 0) - (i \alpha + \gamma) \delta_{j,0}(\varepsilon^{i,1}, 0) + \beta \delta_{i,0}(0, \varepsilon^j) - \delta_{j,0}(0, \varepsilon^i), (x^k, x^l) \rangle,
\end{align*}
\]

Further, we can get

\[
[0, \varepsilon^i], (0, \varepsilon^j) = (j \alpha + \gamma) \delta_{i,0}(\varepsilon^{j,1}, 0) - (i \alpha + \gamma) \delta_{j,0}(\varepsilon^{i,1}, 0) + \beta \delta_{i,0}(0, \varepsilon^j) - \delta_{j,0}(0, \varepsilon^i).
\]

From this, we obtain

\[
[0, \varepsilon^i], (0, \varepsilon^j) = \begin{cases} 
(j \alpha + \gamma)(\varepsilon^{j,1}, 0) + \beta(0, \varepsilon^j) & \text{if } i = 0, j \neq 0, \\
0 & \text{if } i \neq 0, j \neq 0.
\end{cases}
\]

Obviously, the Lie algebra structures defined in Theorems 3.5, 3.7, 3.8 and 3.9 can be applied to the underlying space \( \mathcal{L} = K[\varepsilon, \varepsilon^{-1}] \oplus K[\varepsilon, \varepsilon^{-1}] \). Thus, as by-products, we obtain four new classes of infinite dimensional Lie algebras (\( \mathcal{L}, [\cdot, \cdot] \)).

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