The Massless Spectrum of Covariant Superstrings

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We obtain the correct cohomology at any ghost number for the open and closed covariant superstring, quantized by an approach which we recently developed. We define physical states by the usual condition of BRST invariance and a new condition involving a new current which is related to a grading of the underlying affine Lie algebra.
Introduction. Recently, we developed a new approach to the long-standing problem of the covariant quantization of the superstring [1]. The formulation of Berkovits of super-Poincaré covariant superstrings in $9 + 1$ dimensions [2] is based on a free conformal field theory on the world-sheet and a nilpotent BRST charge which defines the physical vertices as representatives of its cohomology. In addition to the conventional variables $x^m$ and $\theta^\alpha$ of the Green-Schwarz formalism, a conjugate momentum $p_\alpha$ for $\theta^\alpha$ and a set of commuting ghost fields $\lambda^\alpha$ are introduced. The latter are complex Weyl spinors satisfying the pure spinor conditions $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ (cf. for example [3]). This equation can be solved by decomposing $\lambda$ with respect to a non-compact $U(5)$ subgroup of $SO(9,1)$ into a singlet 1, a vector 5, and a tensor 10. The vector can be expressed in terms of the singlet and tensor, hence there are 11 independent complex variables in $\lambda^\alpha$.

Since the presence of the non-linear constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ makes the theory unsuitable for a path integral quantization and higher loop computations, we relaxed the pure spinor condition by adding further ghosts. We were naturally led to a finite set of extra fields, but the BRST charge $Q$ of this system was not nilpotent, and the central charge of the conformal field theory did not vanish. The latter problem was solved by adding one more extra ghost system, which we denoted by $\eta^m$ and $\omega^m_z$. The former problem was solved by introducing yet another new ghost pair, $b$ and $c_z$, which we tentatively associated with the central charge generator in the affine superalgebra which plays an essential role in the superstring [4].

The BRST charge is linear in $c_z$, and without further conditions on physical states the theory would be trivial. We proposed that physical states belong not only to the BRST cohomology ($Q |\psi\rangle = 0$, but $|\psi\rangle \neq Q |\phi\rangle$), but also that the deformed stress tensor $T + V^{(0)}$, where $V^{(0)}$ is a vertex operator, satisfies the usual OPE of a conformal spin 2 tensor. (The latter condition is weaker that the requirement that vertex operators be primary fields with conformal spin equal to 1).

In this letter we propose another definition of physical states. We retain the BRST condition, but we replace the stress tensor condition by the requirement that the physical states belong to a subspace $\mathcal{H}'$ of the entire linear space $\mathcal{H}$ of vertex operator. The latter can be decomposed w.r.t. a grading naturally associated with the underlying affine algebra (cf. [5]) as $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$, with negative and non-negative grading, respectively. The BRST charge $Q = \sum_{n \geq 0} Q_n$ contains only terms $Q_n$ with non-negative grading, hence one can consistently consider the action of $Q$ in $\mathcal{H}_+$. The physical space is identified with the cohomology group $H(Q, \mathcal{H}_+)$, namely

$$Q |\psi\rangle = 0, \quad |\psi\rangle \in \mathcal{H}_+, \quad |\psi\rangle \neq Q |\phi\rangle, \quad |\phi\rangle \in \mathcal{H}_+.$$  

We show that with these conditions we obtain in a very simple way the correct massless spectrum of the open and closed superstring. In particular, our results agree with the
work of Berkovits and collaborators [2], who obtains the correct field equations for the massless states $\gamma^m_{\mu \nu \rho \delta} D_\alpha A_\beta = 0$, but defines physical states by only the BRST condition on the unintegrated vertex $U^{(1)} = \lambda^\alpha A_\alpha(x, \theta)$ in terms of pure spinors $\lambda^\alpha$. In their work the superfields $A_m$ and $W^\alpha$ (the latter is denoted by $A^\alpha$ in [1]) of the open string are expressed in term of $A_\alpha$ by the relations of $d = 10$ superspace,

$$A_m = \frac{1}{8} \gamma^\alpha_\beta D_\alpha A_\beta, \quad W^\alpha \equiv A^\alpha = \frac{1}{10} \gamma^m_{\alpha \beta} \left(D_\beta A_m - \partial_m A_\beta\right), \quad (2)$$

whereas in our work these same relations follow from the physical state conditions. We obtain corresponding results for closed (type II) strings, which also agree with [2] and [6].

Then we consider the sectors with ghost number different from one, and obtain in all these sectors the correct results.

We also give a derivation of the $\eta^m, \omega^m_z$ terms in the action, based on a local symmetry which has been found in the constrained spinor approach [7]. This derivation leads us to replace the pair $(\eta^m, \omega^m_z)$ in [1] with $(\eta^m_z, \omega^m)$.

The new current and grading. We based our approach on the following affine superalgebra [4]

$$d_\alpha(z)d_\beta(w) \sim -\frac{\gamma^m_{\alpha \beta} \Pi^m(w)}{z - w}, \quad d_\alpha(z)\Pi^m(w) \sim \frac{\gamma^m_{\alpha \beta} \partial \theta^\beta(w)}{z - w},$$

$$\Pi^m(z)\Pi^n(w) \sim -\frac{1}{(z - w)^2} \eta^m z^n k, \quad d_\alpha(z)\partial_w \theta^\beta(w) \sim \frac{1}{(z - w)^2} \delta_{\alpha \beta} k,$$

$$\Pi^m(z)\partial_w \theta^\beta(w) \sim 0, \quad \partial_z \theta^\alpha(z)\partial_w \theta^\beta(w) \sim 0, \quad (3)$$

where $\sim$ denotes the singular contributions to the OPE’s. This algebra has a natural grading defined as follows: $d_\alpha(z)$ has grading 1/2, $\Pi^m(z)$ has grading 1, $\partial_z \theta^\alpha(z)$ has grading 3/2, and the central charge $k$ (which numerically is equal to unity) has grading 2. The corresponding ghost systems are $(\lambda^\alpha, \beta_2 z^\alpha), (\xi^m, \beta_z m), (\chi_\alpha, \kappa_\alpha^\alpha)$, and $(c_z, b)$. We thus define the following grading for the ghosts and corresponding antighosts

$$\text{gr}(\lambda^\alpha) = \frac{1}{2}, \quad \text{gr}(\xi^m) = 1, \quad \text{gr}(\chi_\alpha) = \frac{3}{2}, \quad \text{gr}(c_z) = 2,$$

$$\text{gr}(\beta_\alpha) = -\frac{1}{2}, \quad \text{gr}(\beta_m) = -1, \quad \text{gr}(\kappa_\alpha^\alpha) = -\frac{3}{2}, \quad \text{gr}(b) = -2. \quad (4)$$

We also need the ghost $\omega^m$ and the antighost $\eta^m_z$, although this pair does not seem to correspond to a generator. We assign the grading $\text{gr}(\eta^m_z) = -2$ and $\text{gr}(\omega^m) = 2$ for the following reason. In [1], we relaxed the pure spinor constraint by successively adding quartets starting from $(\lambda^a, \lambda_{[a \beta]}^{[a \beta]}; \beta^a, \beta^{[a \beta]})$ of [2] (the indices $a, b$ belong to the fundamental representation of the $U(5)$ subgroup of $SO(1,9)$), and adding the fields $(\lambda^\alpha, \beta_\alpha; \xi^\alpha, \beta^\alpha_a$) with grading $(1/2, -1/2, 1, -1)$. This procedure yields the covariant spinors $\lambda_\alpha$ and $\beta_\alpha$,
but now the fields \((\xi^a, \beta'_a)\) are non-covariant w.r.t. \(SO(9,1)\). Thus, we added the quartet \((\xi_a, \beta'_a; \chi_a, \kappa^a)\) with grading \((1, -1, 3/2, -3/2)\). The spinors \((\chi_a, \kappa^a)\) are part of a covariant spinor and the missing parts are introduced by adding the quartets \((\chi^+, \kappa_+, c, b)\) and \((\chi^{[ab]}, \kappa_{[ab]}, \eta^m, \omega^m)\), both with grading \((3/2, -3/2, 2, -2)\). In this way, we obtain the covariant fields \(\lambda^\alpha = (\lambda_+, \lambda^a, \lambda_{(ab)})\); \(\beta_\alpha = (\beta^+, \beta_a, \beta^{[ab]}); \xi^m = (\xi^a, \xi_a); \beta^m = (\beta^a, \beta'_a); \chi_\alpha = (\chi^+, \chi_a, \chi^{[ab]}); \kappa^\alpha = (\kappa_+, \kappa^a, \kappa_{ab}); b, c\) and \(\eta_m, \omega^m\).

As usual for a conformal field theory, it is natural to introduce a current whose OPE’s with the ghost and antighosts reproduce the grading assignments in (4)

\[
j^\text{grad}_z = -\frac{1}{2} \beta_{z,\alpha} \lambda^\alpha - \beta_{z,m} \xi^m - \frac{3}{2} \kappa_z \chi_\alpha - 2 b c_z - 2 \eta^m \omega_m. \tag{5}\]

Independent confirmation that this current might be important comes from the cancellation of the anomaly (namely the terms with \((z - w)^{-3}\)) in the OPE of the stress energy tensor \(T_{zz}(z)\) (cf. eqs. (1-3) of ref. [1]) with \(j^\text{grad}_z\). In fact, one finds

\[
e^\text{grad} = \frac{1}{2} \lambda_\beta + 1 \times (-10) \xi_\beta + \frac{3}{2} \times (+16) \eta_\omega + 0. \tag{6}\]

The requirement that the vertex operators contain only terms with non-negative grading will lead to the correct massless spectrum. It will also severely restrict the contribution of the vertex operators to correlation functions (in the usual RNS approach ghost insertions are needed to compensate the anomaly in the ghost current, whereas here we anticipate to need insertions of fields in \(\mathcal{H}_-\) to compensate the non-negative grading of vertex operators \(\mathcal{U}^{(1)} \in \mathcal{H}_+\).

All the terms in the stress tensor \(T_{zz}(z)\) and in the ghost current

\[
T_{zz} = -\frac{1}{2} \Pi^m_z \Pi_m z - d_{z,\alpha} \partial_\alpha - \beta_{zm} \partial_z \xi^m - \beta_{z,\alpha} \partial_\alpha \xi^m - \kappa_z \partial_z \chi_\alpha + \partial_z b c_z - \eta^m \omega_m, \tag{7}\]

\[
J^b_z = - (\beta_{zm} \xi^m + \kappa_z \chi_\alpha + \beta_{zn} \lambda^\alpha + b c_z + \eta^m \omega_m),
\]

have grading zero, since they are sums of terms of ghost and antighost pairs with opposite grading. On the other hand, the terms in the current \(j^B_z(z)\) (cf. eq. (1.2) in [1]) and the field \(B_{zz}(z)\) have different grading\(^4\). For instance, the BRST current can be decomposed in the following pieces \(j^B_z(z) = \sum_{n=0}^2 j^{B,(n)}_z(z)\)

\[
j^{B,(0)}_z(z) = - \xi^m \kappa_z \gamma_{\alpha \beta \lambda} \beta_{z,m} - \frac{1}{2} \lambda^\alpha \gamma_{\alpha \beta \lambda} \beta_{z,m} + \frac{1}{2} \beta_{z,\alpha} \partial_\alpha \xi^m - \frac{3}{2} \chi_\alpha \partial_z \lambda^\alpha + \frac{1}{2} \partial_z \kappa_z \chi_\alpha + \partial_z (b c_z - \eta^m \omega_m), \tag{8}\]

\[
j^{B,(1)}_z(z) = - \xi^m \Pi_m z, \]

\[
j^{B,(2)}_z(z) = - \chi_\alpha \partial_\alpha \theta^\alpha, \quad j^{B,(2)}_z(z) = c_z.
\]

\(^4\) In [1] we presented four different solutions \(B^i\) of the the equation \(T_{zz}(z) = \{Q, B^i_{zz}(z)\}\). None of the solutions \(B^i\) have definite grading except \(B^{1V}_zz(z) = b \hat{T}_{zz}(z) + b \partial_z b c_z - \frac{1}{2} \partial_z b\) which has grading equal to \(-2\) carried by the antighost \(b\).
It is clear that all terms in $j^B_z(z)$ have non-negative grading.

**Massless Spectrum of the Open Superstring.** We now turn to the determination of the massless cohomology for the open superstring, namely we will compute $H(Q, \mathcal{H}_+)$ for any ghost number.

Following the RNS approach, the cohomology $H^{(1)}(Q, \mathcal{H}_+)$ at ghost number 1 should be identified with the physical fields of the open superstring. In particular, the massless spectrum coincides with spin zero world-sheet fields $\mathcal{U}^{(1)}(z)$. The most general scalar expression in $H^{+}$ is

$$
\mathcal{U}^{(1)}(z) = \lambda^\alpha A_\alpha + \xi^m A_m + \chi_\alpha W^\alpha + \omega^m B_m \\
+ b \left( \xi^m \xi^n F_{mn} + \lambda^\alpha \lambda^\beta F_{\alpha \beta} + \lambda^\alpha \xi^m F^{\alpha m} + \chi_\alpha \chi_\beta F_{\alpha \beta} \right) \\
+ b \omega^m \left( \lambda^\alpha G_{\alpha m} + \xi^n G_{mn} + \chi_\alpha G_\alpha^m \right) + b \omega^m \omega^n K_{mn},
$$

where $A_\alpha, \ldots, K_{mn}$ are arbitrary superfields of $x_m, \theta^\alpha$. It clearly makes a big difference that $\omega^m$ no longer is a 1-form. Notice that the requirement that $\mathcal{U}^{(1)}(z) \in \mathcal{H}_+$ forbids the terms $b \left( \lambda^\alpha \lambda^\beta F_{\alpha \beta} + \lambda^\alpha \xi^m F_{\alpha m} \right)$ in the vertex (these terms are indeed present in eq. (6.4) of [1]).

The condition $\{Q, \mathcal{U}^{(1)}(z)\} = 0$ implies the following equations

$$
D_{(\alpha A_\beta)} - \frac{1}{2} \gamma_{m}^{\alpha \beta} A_m = 0, \\
\partial_m A_\alpha - D_\alpha A_m + \gamma_{m \alpha \beta} W^\beta = 0, \\
\partial_m [A_n] + F_{mn} = 0, \quad D_\beta W^\alpha + F_{\alpha}^{\ \beta} = 0, \\
\partial_m W^\alpha + F_{m}^{\ \alpha} = 0, \quad F^{\alpha \beta} = 0, \\
D_\alpha B_m - G_m = 0, \quad \partial_n B_m - G_{mn} = 0, \\
G_\alpha^m = 0, \quad K_{mn} = 0,
$$

where $D_\alpha \equiv \partial/\partial \theta^\alpha + \frac{1}{2} \theta^\beta \gamma_{\alpha \beta}^{m} \partial/\partial x^m$. The terms in $\{Q, \mathcal{U}^{(1)}(z)\}$ which contain the field $b$ yield equations which are the Bianchi identities [1]. From the first two equations of (10) one gets the field equations for $N = 1, d = (9, 1)$ SQED [8]

$$
\gamma_{[mnpqr]}^{\alpha \beta} D_\alpha A_\beta = 0, \\
$$

as well as the definition of the vector potential $A_m$ and the spinorial field strength $W^\alpha$ in terms of $A_\alpha$

$$
A_m = \frac{1}{8} \gamma_{m}^{\alpha \beta} D_\alpha A_\beta, \quad W^\alpha = \frac{1}{10} \gamma_{m}^{\alpha \beta} (D_\beta A_m - \partial_m A_\beta).
$$

The normalization is chosen such that $D_\alpha D_\beta + D_\beta D_\alpha = \gamma_{m}^{\alpha \beta} \partial_m$. We define $D_{(\alpha A_\beta)} = \frac{1}{2} (D_\alpha A_\beta + D_\beta A_\alpha)$ and $\partial_m [A_n] = \frac{1}{2} (\partial_m A_n - \partial_n A_m)$.
Moreover, the remaining equations in (10) imply that the curvatures $F_{mn}, F^\alpha_m,$ and $F^\beta \alpha$ are expressed in terms of the spinor potential $A_\alpha$, and similarly the superfields $G_{m\alpha}, G_{mn}$ are solely functions of $B_m$.

The gauge transformations of the vertex $\mathcal{U}^{(1)}(z)$ are generated by the BRST variation of spin zero field $\Omega^{(0)}(z) \in \mathcal{H}_+$ with ghost number zero, whose most general expression is given by $\Omega^{(0)}(z) = C + b \omega^m M_m$, with $C$ and $M_m$ arbitrary superfields. The BRST variation of $\Omega^{(0)}$ is

$$\delta \mathcal{U}^{(1)}(z) = \left[ Q, \Omega^{(0)}(z) \right] = \lambda^\alpha D_\alpha C + \xi^m \partial_m C + \omega^m M_m + b \omega^m (\lambda^\alpha D_\alpha M_m + \xi^n \partial_n M_m). \quad (13)$$

One can easily check that the field $M_m$ can be used to gauge away $B_m$, while $C$ is the usual parameter of the gauge transformations on the SQED potentials: $\delta A_\alpha = D_\alpha C, \quad \delta A_m = \partial_m C$. Thus, the only independent superfield is $A_\alpha$, and it satisfies (11) which is gauge invariant.

In order to exhibit explicitly the physical degrees of freedom it is convenient to work in the gauge $\theta^\alpha A_\alpha = 0$. The photon $a_m$ and the photino $u^\alpha$ are identified with the first coefficients in the expansion of $A_m$ and $W^\alpha$ in powers of $\theta$

$$A_m(x, \theta) = a_m(x) + \theta^\alpha \gamma_{m\alpha\beta} u^\beta(x) + \ldots,$$

$$W^\alpha(x, \theta) = u^\alpha(x) + \gamma^m \gamma_{m\alpha\beta} \theta^\beta \partial_m a_n(x) + \ldots. \quad (14)$$

The ellipses denote terms which contain derivatives of $a_m$ and $u^\alpha$. From the Bianchi identities and (11) one derives $\partial_m F^{mn} = 0$ and $\gamma_{m\alpha\beta} \partial_m W^\beta = 0$, while $\theta^\alpha A_\alpha = 0$ implies that $A_\alpha = \frac{1}{2} a^m (\gamma_m \theta)_\alpha + a^{m_1 \ldots m_5} (\gamma_{m_1 \ldots m_5} \theta)_\alpha + (\gamma^m \theta)_\alpha (u_m \gamma_\theta) + (\gamma^{m_1 \ldots m_5} \theta)_\alpha v_\beta^{m_1 \ldots m_5} \theta^\beta + \ldots$. From (11) it follows that $a^{m_1 \ldots m_5} = v_\beta^{m_1 \ldots m_5} = 0$ and in this way one has obtained the linearized field equations for the gauge field $a_m$ and the gaugino $u^\alpha$.

At other ghost numbers the cohomology groups $H^{(n)}(Q, \mathcal{H}_+)$ vanish, except the group $H^{(2)}(Q, \mathcal{H}_+)$ which contains the antifields $a^*_m, u^*_\alpha$ of the gauge field $a_m$ and of the gaugino $u^\alpha$ [7]. The analysis can be performed along the lines of the previous discussion and one can see that all the $\omega$-dependent terms are cohomologically trivial, and therefore can be reabsorbed by a gauge transformation. For the $\omega$-independent terms one has the following general decomposition

$$\mathcal{U}^{(2)} = \lambda^\alpha \lambda^\beta A^*_\alpha\beta + \lambda^\alpha \xi^m A^*_\alpha m + \ldots + \chi_\alpha \chi_\beta A^*_{\alpha\beta} + b \left( \lambda^\alpha \lambda^\beta \xi^m F^*_\alpha\beta m + \lambda^\alpha \xi^m \xi^n F^*_\alpha mn + \lambda^\alpha \lambda^\beta \chi_\gamma F^*_{\alpha\beta\gamma} + \ldots + \chi_\alpha \chi_\beta \chi_\gamma F^*_{\alpha\beta\gamma} \right). \quad (15)$$

The condition $\mathcal{U}^{(2)} \in \mathcal{H}_+$ does not allow the term $\lambda^\alpha \lambda^\beta \lambda^\gamma F^*_{\alpha\beta\gamma}$, which is permitted by ghost number counting. This implies that the superfield $A^*_\alpha\beta$ should satisfy the field equation $D_\alpha A^*_\alpha\beta - \frac{1}{2} \gamma_{(\alpha\beta\gamma)} A^*_\gamma m = 0$. Furthermore, the gauge transformations are generated by the
BRST variation of a ghost-number 1 superfield $\Omega^{(1)}(z)$ (whose decomposition is given in eq. (9)), namely

$$\delta U(z) \equiv \{Q, \Omega^{(1)}(z)\}.$$  

One obtains that the gauge transformations of the super-antifield $A^*_\alpha\beta$ are the equations of motion of the potential $A_\alpha$

$$\delta A^*_\alpha\beta = D_{(\alpha}A_{\beta)} - \frac{1}{2} \gamma^{m}_{\alpha\beta}A_m. \quad (16)$$

As shown in [2] the only states in the BRST cohomology at non-zero momentum are the on-shell gauge field $a_m$ and the gluino $u^{\alpha}$ at ghost number +1 and the corresponding antifields $a^*_m, u^*_\alpha$ at ghost number +2. The latter are the coefficients of the superfield $A^*_\alpha\beta$. In fact, since $A^*_\alpha\beta$ is a symmetric bispinor, it can be decomposed into a vector part and a self-dual 5-form one:

$$A^*_\alpha\beta = \gamma^m_{\alpha\beta}A_m + \gamma^{[mnrpq]}_{\alpha\beta}A^*_{[mnrpq]}. \quad \text{The gauge transformations (16) remove the vector component } A^*_m, \text{ and the first coefficients of the } \theta \text{-expansion of } A^*_{[mnrpq]} \text{ exhibit the on-shell fields}$$

$$A^*_{[mnpqr]} = (\theta \gamma^{[mnp}\theta)(\theta \gamma_{qr}\theta) A^*_m(x) + (\theta \gamma^{[mnp}\theta)(\theta \gamma_{qr}\theta) A^*_a(x) + \ldots. \quad (17)$$

The ellipses involve terms with derivatives of $a^*_m$ and $u^*_\alpha$.

Before closing this section, we point out that the choice of the subspace $\mathcal{H}_+$ is one of the possible choices. Another interesting situation is the restriction to the subspace $\mathcal{H}'_+$ with strictly positive grading. It is straightforward to see that, at ghost number 1, the cohomology $H^{(1)}(Q, \mathcal{H}'_+)$ is identified with the gauge potentials which satisfy $F_{mn} = D_{\alpha}W^\beta = 0$. This corresponds to a topological gauge model in the target space.

The Massless Spectrum of the Closed Superstring. According to our formalism [1], the BRST charge is the sum of the BRST charge for the left and right movers denoted by $Q_L$ and $Q_R$, respectively. In the super-Poincaré covariant formulation of closed superstring, one can choose the spinors $\theta^\alpha_L$ and $\theta^{\tilde{\alpha}}_R$ to be Weyl or anti-Weyl in the left- or in the right-mover sector: the same chirality for both spinors leads to Type IIB strings, opposite chiralities lead to Type IIA strings. In the following we will not distinguish between the two cases.

As in [2], also in our formalism the holomorphic and anti-holomorphic sector are decoupled. Each operator in the left-mover sector has its counterpart in the right-mover one. Therefore, the assignments of ghost number and grading, as well as the central charge cancellations (cf. eq. (6)), are exactly the same in both sectors.

Denoting by $\mathcal{H}^L_+$ and $\mathcal{H}^R_+$ the left- and right-mover subspaces with non-negative gradings, the physical state condition is expressed by

$$Q_L|\psi\rangle = Q_R|\psi\rangle = 0, \quad |\psi\rangle \in \mathcal{H}^L_+ \otimes \mathcal{H}^R_+, \quad |\phi\rangle, \quad |\chi\rangle \in \mathcal{H}^L_+ \otimes \mathcal{H}^R_+, \quad (18)$$

$$Q_R|\chi\rangle = Q_L|\phi\rangle = 0.$$
The cohomology group $H^{(1,1)}(Q, \mathcal{H}_L^L \otimes \mathcal{H}_R^R)$ is identified with the physical degrees of freedom. In particular, the spin zero vertex operator $U^{(1,1)}$ contains the massless fields of the closed superstring. To determine the linearized field equations, we first analyze the $\omega, \hat{\omega}$-dependent terms of the vertex. Since $\omega$ and $\hat{\omega}$ are BRST inert, one can analyze the sectors of $H^{(1,1)}(Q, \mathcal{H}_L^L \otimes \mathcal{H}_R^R)$ with different powers of $\omega$ and $\hat{\omega}$ separately. We prove that all the sectors with positive powers of $\omega$ and $\hat{\omega}$ vanish. Consider the generic decomposition with one power of $\omega$, $U^{(1,1)} = \omega^m (B_m(\hat{C}) + b C^A F_{mA}(\hat{C}))$, where $C^A$ and $\hat{C}^A$ denote collectively the ghost fields ($\lambda^\alpha, \xi^m, \chi_\alpha, b$) of left- and right-moving sectors. The functions $B_m$ and $F_{mA}$ are polynomials in the right-moving ghost fields $\hat{C}^A$ with superfield coefficients.

The conditions $[Q_L, U^{(1,1)}] = [Q_R, U^{(1,1)}] = 0$ imply that

$$Q_L B_m(\hat{C}) - C^A F_{mA}(\hat{C}) = 0, \quad Q_R B_m(\hat{C}) = Q_R F_{mA}(\hat{C}) = 0. \quad (19)$$

Thus, the vertex $U^{(1,1)} = \{Q_L, \Lambda\}$ is BRST trivial where $\Lambda = \left(b \omega^m B_m(\hat{C}) \right)$ and $\{Q_R, \Lambda\} = 0$. The same argument can be easily repeated for any positive power of $\omega$ or $\hat{\omega}$. Therefore, we have to analyze only the $\omega, \hat{\omega}$-independent terms.

The vertex $U$ can be decomposed into the following terms

$$U(z, \hat{z}) = U^{(1,1)} + b U^{(2,1)} + \hat{b} U^{(1,2)} + b \hat{b} U^{(2,2)},$$

$$U^{(1,1)} = \left(\lambda^\alpha_L \lambda^\beta_R A_{\alpha\beta} + \ldots + \chi_{\lambda} \chi_{\hat{\lambda}} A^{\alpha\hat{\alpha}}\right),$$

$$U^{(2,1)} = \left(\xi^m \xi^n \lambda^\beta_R B_{mn\hat{\alpha}} + \ldots + \chi_{\lambda} \chi_{\lambda} B^{\alpha\beta\hat{\alpha}}\right),$$

$$U^{(1,2)} = \left(\lambda^\alpha_L \xi^n \lambda^\beta_R C_{mn\hat{\alpha}} + \ldots + \chi_{\lambda} \chi_{\hat{\lambda}} B^{\alpha\beta\hat{\alpha}}\right),$$

$$U^{(2,2)} = \left(\xi^m \xi^n \xi^r \lambda^\beta_R C_{mn\hat{r}} + \ldots + \chi_{\lambda} \chi_{\hat{r}} B^{\alpha\beta\hat{r}}\right),$$

where all the coefficients $A_{\alpha\beta}, \ldots C^{\alpha\beta\hat{r}}$ are arbitrary superfields. Notice that the condition $U(z, \hat{z}) \in \mathcal{H}_L^L \otimes \mathcal{H}_R^R$ forbids the terms $b \lambda^\alpha_L \lambda^\beta_R, b \lambda^\alpha_L \xi^m$, and the corresponding ones for right-movers. Finally, also the terms $b \hat{b} \lambda^\alpha_L \lambda^\beta_R \lambda^\gamma_R, \ldots, b \hat{b} \lambda^\alpha_L \xi^m \lambda^\beta_R \xi^r$ are ruled out.

As in the case of the open superstring, the absence of these terms combined with the BRST invariance yields the following equations

$$D_{(\alpha A_{\beta})\hat{\alpha}} - \frac{1}{2} \gamma_{\alpha\beta} A_{m\hat{\alpha}} = 0, \quad D_{(\hat{\alpha} A_{\alpha})\hat{\gamma}} - \frac{1}{2} \gamma_{\alpha\hat{\beta}} A_{m\hat{\gamma}} = 0,$$

$$\partial_m A_{\alpha\hat{\alpha}} - D_{\alpha} A_{m\hat{\alpha}} + \gamma_{m\alpha\beta} A_{\beta\hat{\alpha}} = 0, \quad \partial_m A_{\alpha\hat{\alpha}} - D_{\hat{\alpha}} A_{m\hat{\alpha}} + \gamma_{m\alpha\beta} A_{\alpha\hat{\beta}} = 0,$$

$$D_{(\alpha A_{\beta})m} - \frac{1}{2} \gamma_{\alpha\beta} A_{mn} = 0, \quad D_{(\hat{\alpha} A_{\alpha})m} - \frac{1}{2} \gamma_{\alpha\hat{\beta}} A_{m\hat{\gamma}} = 0,$$

$$D_{(\hat{\alpha} A_{\alpha})\hat{\alpha}} - \frac{1}{2} \gamma_{\alpha\hat{\beta}} A_{m\hat{\alpha}} = 0, \quad D_{(\hat{\alpha} A_{\alpha})\hat{\gamma}} - \frac{1}{2} \gamma_{\alpha\hat{\beta}} A_{m\hat{\gamma}} = 0,$$

$$D_{(\hat{\alpha} A_{\alpha})m} - \frac{1}{2} \gamma_{\alpha\hat{\beta}} A_{mn} = 0,$$

**6** The N=2 D=9,1 supersymmetric derivatives are defined by $D_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} + \frac{1}{2} \gamma_{\alpha\beta} \theta^{\beta} \partial_m$ and $\hat{D}_{\hat{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\hat{\alpha}}} + \frac{1}{2} \gamma_{\hat{\alpha}\hat{\beta}} \bar{\theta}^{\hat{\beta}} \partial_m$. 

7
\[
\begin{align*}
\partial_m A_{\alpha n} - D_\alpha A_{mn} + \gamma_{m\alpha\beta} A_\beta^n &= 0, & \partial_m A_{\alpha \hat{n}} - D_{\hat{\alpha}} A_{m\hat{n}} + \gamma_{m\hat{\alpha}\hat{\gamma}} A_n^{\hat{\gamma}} &= 0, \\
D_{(\alpha} A_{\beta)} - \frac{1}{2} \gamma^{m}_{\alpha\beta} A_m^{\hat{\gamma}} &= 0, & D_{(\hat{\alpha}} A^\alpha_{\hat{n})} - \frac{1}{2} \gamma^m_{\hat{\alpha}\hat{\gamma}} A_m^\alpha &= 0, \\
\partial_m A_\alpha^{\hat{\gamma}} - D_\alpha A_m^{\hat{\gamma}} + \gamma_{m\alpha\beta} A_\beta^{\hat{\gamma}} &= 0, & \partial_m A_\alpha^{\hat{\gamma}} - D_{\hat{\alpha}} A_m^{\alpha} + \gamma_{m\hat{\alpha}\hat{\gamma}} A^{\hat{\gamma}}_n &= 0.
\end{align*}
\]

The other equations coming from \([Q_L + Q_R, U] = 0\) will not impose any further constraint on the superfields, but they define consistently the curvature in terms of \(A_{m\hat{n}}, \ldots A^{\hat{\alpha}\hat{\gamma}}\). Notice that the superfields \(A_{m\hat{n}}, A_{\alpha m}, \ldots A^{\hat{\alpha}\hat{\gamma}}\) are solved in terms of the fundamental field \(A_{\alpha\hat{\alpha}}\) as follows

\[
\begin{align*}
A_{m\hat{n}} &= \frac{1}{8} D_{\alpha} \gamma_m^\alpha A_{\alpha\hat{n}}, & A_{\alpha m} &= \frac{1}{8} D_{\hat{\alpha}} \gamma_m^\hat{\alpha} A_{\alpha\hat{n}}, \\
A^{\alpha}_{\hat{n}} &= \frac{1}{10} \gamma_{m\alpha\beta} \left( D_\beta A_{m\hat{n}} - \partial_m A_{\beta\hat{n}} \right), & A^{\hat{\alpha}}_m &= \frac{1}{10} \gamma_{m\hat{\alpha}\hat{\gamma}} \left( D_{\hat{\gamma}} A_{\alpha m} - \partial_m A_{\alpha\hat{\gamma}} \right), \\
A_\alpha^{\hat{\gamma}} &= \frac{1}{8} \gamma_m^\alpha D_\alpha A_m^{\hat{\gamma}}, & A^{\hat{\alpha}}_\alpha &= \frac{1}{10} \gamma_{m\alpha\beta} \left( D_\beta A_m^{\hat{\alpha}} - \partial_m A_{\alpha\hat{n}} \right).
\end{align*}
\]

The superfield \(A_{\alpha\hat{\alpha}}\) satisfies the field equations

\[
\gamma_{mnpqr}^{\alpha\beta} D_\alpha A_{\beta\hat{\gamma}} = 0, \quad \gamma_{mnpqr}^{\hat{\alpha}\hat{\gamma}} D_{\hat{\alpha}} A_{\hat{\gamma}} = 0,
\]

which are the linearized \(N = 2\) supergravity equations of motion. The gauge transformations are generated by the BRST variations of two generic superfields \(\Omega^{(0,1)}\) and \(\Omega^{(1,0)}\) by \(\delta U(z, \bar{z}) = \{Q_L, \Omega^{(0,1)}\} + \{Q_R, \Omega^{(1,0)}\}\), with \(\{Q_R, \Omega^{(0,1)}\} = \{Q_L, \Omega^{(1,0)}\} = 0\) and \(\Omega^{(0,1)} \in \mathcal{H}_+, \Omega^{(1,0)} \in \mathcal{H}_+\). Assuming for \(\Omega^{(0,1)}\) the general decomposition

\[
\Omega^{(0,1)}(z) = \lambda_{\hat{\alpha}}^\alpha \hat{\Lambda}_\hat{\alpha} + \xi_{\hat{\alpha}}^m \hat{\Lambda}_m + \chi_{R\hat{\alpha}} \hat{\Lambda}_\hat{\alpha}
\]

\[
+ b_R \left( \xi_{\hat{\alpha}}^m \xi_{R\hat{\alpha}} \hat{\Xi}_m + \lambda_{\hat{\alpha}}^\alpha \lambda_{R\hat{\alpha}} \hat{\Xi}_m + \chi_{R\hat{\alpha}} \xi_{R\hat{\alpha}} \hat{\Xi}_m + \chi_{R\hat{\alpha}} \chi_{R\hat{\alpha}} \hat{\Xi}_m \right),
\]

and analogously for \(\Omega^{(1,0)}\) exchanging \(R \rightarrow L\) and \(\hat{\Lambda}_\hat{\alpha}, \ldots, \hat{\Xi}_m \rightarrow \Lambda_\alpha, \ldots, \Xi_\alpha\), the relevant gauge transformations are

\[
\delta A_{\alpha\hat{\alpha}} = D_{\alpha} \hat{\Lambda}_\hat{\alpha} + D_{\hat{\alpha}} \Lambda_\alpha.
\]

The conditions \(\{Q_R, \Omega^{(0,1)}\} = \{Q_L, \Omega^{(1,0)}\} = 0\) imply that the gauge parameters \(\Lambda_\beta\) and \(\hat{\Lambda}_\hat{\gamma}\) satisfy the following equations

\[
\gamma_{[mnpqr]}^{\alpha\beta} D_\alpha \Lambda_\beta = 0, \quad \gamma_{[mnpqr]}^{\hat{\alpha}\hat{\gamma}} D_{\hat{\alpha}} \hat{\Lambda}_{\hat{\gamma}} = 0.
\]

The gauge transformations for the other superfields \(A_{m\hat{n}}, \ldots, A^{\alpha\hat{\alpha}}\) can be easily derived from eq. (26) using their definitions (22). The field equations (23) and the gauge transformations (25) derived for the closed superstring are in agreement with [2] and [6]. The
physical degrees of freedom can be easily read from the first components of the $\theta^\alpha_L$ and $\theta^\alpha_R$ of the superfields $A_{mn}(x, \theta_L, \theta_R), A^\gamma_m(x, \theta_L, \theta_R), A^\alpha_m(x, \theta_L, \theta_R)$ and $A^{\alpha\hat{\alpha}}(x, \theta_L, \theta_R)$. The first component of $A_{mn} = (g_{mn} + b_{mn} + \eta_{mn}\phi) + \mathcal{O}(\theta_L, \theta_R)$ describe the graviton, the NS-NS two-form and the dilaton. The first components of $A^\alpha_m = \psi^\alpha_m + \mathcal{O}(\theta_L, \theta_R)$ contain the N=2 gravitinos $\psi^\alpha_m$, and the chirality of the right-mover spinor $\theta_R$ determines if the string is Type IIA or IIB. Finally, $A^{\alpha\hat{\alpha}} = F^{\alpha\hat{\alpha}} + \mathcal{O}(\theta_L, \theta_R)$ where $F^{\alpha\hat{\alpha}}$ are the R-R field strenghts. The linearized field equations for $g_{mn}, b_{mn}, \phi, \psi^\alpha_m, \psi^\gamma_m$ and $F^{\alpha\hat{\alpha}}$ are discussed in great detail in [6].

Of course the closed superstring massless spectrum can be understood as the tensor product of the cohomologies for open superstring of left- and right-movers. Mathematically, this is encoded in the well-known Künneth formula $H^{(1,1)}(Q_L + Q_R, \mathcal{H}_L \otimes \mathcal{H}_R^*) = H^{(1)}(Q_L, \mathcal{H}_L^*) \otimes H^{(1)}(Q_R, \mathcal{H}_R^*)$.

**Conclusions.** We conclude by giving an argument for the presence of the term $-\eta^m_\bar{z}\bar{\partial}\omega_m$ in the action, with $(\eta^m_\bar{z}, \omega_m)$ a ghost-charge $(-1, 1)$ spin $(1, 0)$ system. We start from the observation that in the approach with the pure spinor constraint $\lambda^\alpha \gamma^m_\beta \lambda^\beta = 0$ the action with $\mathcal{L} = -\beta_{z,\alpha} \bar{\partial}\lambda^\alpha$ has the local gauge invariance

$$\delta\beta_{z,\alpha} = \Lambda_{z\bar{m}} \gamma^m_{\alpha\beta} \lambda^\beta, \quad \delta\lambda^\alpha = 0.$$  \hspace{1cm} (27)

The gauge parameter has ghost number $-2$. To remove the pure spinor constraint we introduce the Lagrange multiplier field $\alpha_{z\bar{m}}$ with the ghost number $-2$

$$\mathcal{L} = -\beta_{z,\alpha} \bar{\partial}\lambda^\alpha + \frac{1}{2} \alpha_{z\bar{m}} \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta.$$  \hspace{1cm} (28)

The action with unconstrained $\lambda^\alpha$ is still gauge invariant if $\alpha_{z\bar{m}}$ transforms as $\delta\alpha_{z\bar{m}} = -\bar{\partial}\Lambda_{z\bar{m}}$. We fix this local gauge symmetry as usual by adding a BRST exact term. The gauge parameter $\Lambda_{z\bar{m}}$ becomes a field $\eta_{z\bar{m}}$ with ghost number $-1$, which is an antighost. As the BRST exact term we take

$$s \left( \omega^m_{z\bar{m}} \right) = d^m \alpha_{z\bar{m}} - \omega^m \bar{\partial}\eta_{z\bar{m}}.$$  \hspace{1cm} (29)

The field $\omega^m$ has ghost number $+1$, and is a ghost field; it transforms into a BRST auxiliary field $d^m$ with ghost number $+2$. Integrating over $d^m$ and $\alpha_{z\bar{m}}$ sets these fields to zero, and one is indeed left with the term $\bar{\partial}\eta_{z\bar{m}} \omega^m = -\eta_{z\bar{m}} \bar{\partial}\omega^m$ in the action. We end up by recalling that the previous argument is a strong indication in favour of the presence of the fields $\eta^m_\bar{z}$ and $\omega_m$ in our formalism, nevertheless a complete picture can be only achieved by starting from a gauge invariant action and using the Batalin-Vilkovisky framework for its quantization.

Before concluding, we would like to point out the relation between the cohomology in the Berkovits’ approach and in our formalism which also indicates the path toward the proof of the equivalence of the two approaches.
The group $H(Q_B|\mathcal{H}_{p.s.})$, which represents the cohomology in the Berkovits’ formalism is an example of a constrained BRST cohomology, or equivalently, of equivariant cohomology [9]. In the latter case, the BRST cohomology is computed on the supermanifold $M$ with coordinates $x^m, \theta^\alpha$ on which the space-time translations $x^m \to x^m + \frac{1}{2} \lambda \gamma^m \lambda$, generated by unconstrained spinors $\lambda^\alpha$, act freely. One finds that $Q_B^2 = -\mathcal{L}_V$ where $V^m = \frac{1}{2} \lambda \gamma^m \lambda$. Notice that the r.h.s. can be also written in terms of the Lie derivative $\mathcal{L}_V = d \iota_V + \iota_V d$, where $\iota_V$ is the contraction of a form with the vector $V^m$. One can represent $\iota_V$ by the operator $\oint dz V^m \beta_m$; its action on (parity reversed forms) $\xi^m$ is then given by the OPE of $\beta_m(z)$ with $\xi^m(z)$. The exterior differential $d$ is $\xi^m \partial_m$ where $\xi^m$ are the parity-reversed coordinates of the cotangent bundle $\Pi T^*M$. The usual exterior derivative $d = dx^m \partial_m$ has been replaced by $-\oint dz \xi^m \Pi_{zm}$. Since $\Pi_z^m(z) \partial^l x^n(w) \sim (z - w)^{-l-1}$, the operator $-\oint dz \xi^m \Pi_{zm}$ represents the exterior derivative on the jet bundle \{ $x^m, \partial x^m, \partial^2 x^m, \ldots$ \}. Following the approach of equivariant cohomology [9], one can define a new BRST operator $Q'$ by

$$Q' = Q_B + d + \iota_V = Q - \oint \xi^m \Pi_{zm} - \oint \frac{1}{2} \lambda^\alpha \gamma^m_{\alpha \beta} \lambda^\beta \beta_m.$$  

Unfortunately, this operator fails to be nilpotent and the solution of this problem has been discussed in [1]. Moreover, it turns out that the BRST cohomology computed in the space of zero conformal-weight vertex operators with non-negative gradings coincides with the massless spectrum of the superstring, and this suggests a complete equivalence of the two approaches. It has been also pointed out in similar examples discussed in [10] that one needs further conditions on the functional space to identify the correct physical observables as elements of the BRST cohomology.

In the present paper, we identify the massless spectrum for the open and closed superstrings by the cohomology at ghost number +1 of the BRST operator restricted to a subspace of the entire linear space of vertex operators. The subspace is selected by means of a natural grading which is assigned to the ghosts and antighosts.

We are involved in the computation of amplitudes using the present formalism. The preliminary results are very encouraging and we hope to report on this soon.

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