Multiplicative operators in the spaces of Schwartz families

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Abstract

In this paper we introduce and study the multiplication among smooth functions and Schwartz families. This multiplication is fundamental in the formulation and development of a spectral theory for Schwartz linear operators in distribution spaces, to introduce efficiently the Schwartz eigenfamilies of such operators and to build up a functional calculus for them. The definition of eigenfamily is absolutely natural and this new operation allows us to develop a rigorous and manageable spectral theory for Quantum Mechanics, since it appears in a form extremely similar to the current use in Physics.

1 Introduction

In the Spectral Theory of \( \mathcal{S} \)-linear operators, the eigenvalues corresponding to the elements of certain \( \mathcal{S} \) families have fundamental importance. If \( L \) is an \( \mathcal{S} \)-linear operator and \( v \) is an \( \mathcal{S} \) family, the family \( v \) is defined an eigenfamily of the operator \( L \) if there exists a real or complex function \( l \) - defined on the set of indices of the family \( v \) - such that the relation

\[
L(v_p) = l(p)v_p,
\]

holds for every index \( p \) of the family \( v \). As we already have seen, in the context of \( \mathcal{S} \)-linear operators, it is important how the operator \( L \) acts on the entire family \( v \). Taking into account the above definition, it is natural to consider the image family \( L(v) \) as the product - in pointwise sense - of the family \( v \) by the function \( l \), but:

- is the pointwise multiplication an operation in the space of \( \mathcal{S} \) families?
- what kind of properties are satisfied by this product?

In this chapter we define and study the properties of such product.
2  \( \mathcal{O}_M \) Functions

We recall, for convenience of the reader, some basic notions from theory of distributions.

**Definition (of slowly increasing smooth function).** We denote by \( \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \), or more simply by \( \mathcal{O}_M^{(n)} \), the subspace of all smooth functions \( f \), belonging to the space \( \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{K}) \), such that, for every test function \( \phi \in \mathcal{S}_n \) the product \( \phi f \) lives in \( \mathcal{S}_n \). The space \( \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \) is said to be the \textit{space of smooth functions from} \( \mathbb{R}^n \) \textit{into the field} \( \mathbb{K} \) \textit{slowly increasing at infinity (with all their derivatives)}.

In other terms, the functions \( f \) belonging to the space \( \mathcal{O}_M^{(n)} \) are the only smooth functions which can generate a multiplication operator

\[
M_f : \mathcal{S}_n \to \mathcal{S}_n
\]

of the space \( \mathcal{S}_n \) into the space \( \mathcal{S}_n \) itself, (obviously) by the relation

\[
M_f(g) = fg.
\]

This is the motivation of the importance of these functions in Distribution Theory, and the symbol itself \( \mathcal{O}_M \) depends on this fact (\( \mathcal{O}_M \) stands for multiplicative operators).

Let us see a first characterization.

**Proposition.** Let \( f \in \mathcal{E}_n \) be a smooth function. Then the following conditions are equivalent:

1) for all multi-index \( p \in \mathbb{N}_0^n \) there is a polynomial \( P_p \) such that, for any point \( x \in \mathbb{R}^n \), the following inequality holds

\[
|\partial^p f(x)| \leq |P_p(x)|;
\]

2) for any test function \( \phi \in \mathcal{S}_n \) the product \( \phi f \) lies in \( \mathcal{S}_n \);

3) for every multi-index \( p \in \mathbb{N}_0^n \) and for every test function \( \phi \in \mathcal{S}_n \) the product \( (\partial^p f) \phi \) is bounded in \( \mathbb{R}^n \).
2.1 Topology

The standard topology of the space $O^{(n)}_M$ is the locally convex topology defined by the family of seminorms

$$
\gamma_{\phi,p}(\phi) = \sup_{x \in \mathbb{R}^n} |\phi(x)\partial^p f(x)|
$$

with $\phi \in \mathcal{S}_n$ and $p \in \mathbb{N}_0^n$. This topology does not have a countable basis. Also, it can be shown that the space $O^{(n)}_M$ is a complete space. A sequence $(f_j)_{j \in \mathbb{N}}$ converges to zero in $O^{(n)}_M$ if and only if for every test function $\phi \in \mathcal{S}_n$ and for every multi-index $p \in \mathbb{N}_0^n$, the sequence of functions $(\phi \partial^p f_j)_{j \in \mathbb{N}}$ converges to zero uniformly on $\mathbb{R}^n$; or, equivalently, if, for every test function $\phi \in \mathcal{S}_n$, the sequence $(\phi f_j)_{j \in \mathbb{N}}$ converges to zero in $\mathcal{S}_n$. A filter $F$ on $O^{(n)}_M$ converges to zero in $O^{(n)}_M$ if and only if for every test function $\phi \in \mathcal{S}_n$, the filter $\phi F$ converges to zero in $\mathcal{S}_n$.

2.2 Bounded sets in $O^{(n)}_M$

A subset $B$ of $O^{(n)}_M$ is bounded (in the topological vector space $O^{(n)}_M$) if and only if, for all multi-index $p \in \mathbb{N}_0^n$, there is a polynomial $P_p$ such that, for any function $f \in B$, the following inequality holds true

$$
|\partial^p f(x)| \leq P_p(x),
$$

for any point $x \in \mathbb{R}^n$.

2.3 Multiplication in $\mathcal{S}_n$ by $O^{(n)}_M$ functions

The bilinear map

$$
\Phi : O^{(n)}_M \times \mathcal{S}_n \to \mathcal{S}_n : (\phi, f) \mapsto \phi f
$$

is separately continuous with respect to the usual topologies of the spaces $O^{(n)}_M$ and $\mathcal{S}_n$. It follows immediately that the multiplication operator $M_f$, associated with an $O_M$ function $f$, is continuous (with respect to the standard topology of the Schwartz space $\mathcal{S}_n$). Moreover, the transpose of the operator $M_f$ is the operator

$$
^t M_f : \mathcal{S}_n' \to \mathcal{S}_n'
$$

defined by

$$
^t M_f(u)(g) = u(M_f(g)) = u(fg) = fu(g),
$$
for every $u$ in $S'$ and for every $g$ in $S_n$. So that, the transpose of the multiplication $M_f$ is the multiplication on $S'_n$ by the function $f$. Indeed, the multiplication of a tempered distribution by an $O_M$ function is defined by the transpose of $M_f$, since this last operator is self-adjoint with respect to the canonical bilinear form on $S_n \times S_n$. In fact, obviously, we have

$$\langle M_f(g), h \rangle = \langle g, M_f(h) \rangle,$$

for every pair $(g, h)$ in that Cartesian product $S_n \times S_n$. So we can use the standard procedure to extend regular operators (operators admitting an adjoint with respect to the standard bilinear form) from their domain $S_n$ to the entire space $S'_n$.

### 2.4 $S$ Family of the multiplication operator $M_f$

Since the multiplication operator $M_f : S_n \rightarrow S_n$ is continuous, we can associate with it an $S$ family $v$, in the canonical way. We have

$$v_p = (M_f^v)_p =$$

$$= \delta_p \circ M_f =$$

$$= iM_f(\delta_p) =$$

$$= f \delta_p =$$

$$= f(p) \delta_p,$$

for every $p$ in $\mathbb{R}^n$. In the language of Schwartz matrices we can say that to the operator $M_f$ is associated the Schwartz diagonal matrix $f \delta$.

### 3 Product in $L(S_n, S_m)$ by $O_M$ functions

The basic remark is the following.

**Proposition.** Let $A \in L(S_n, S_m)$ be a continuous linear operator and let $f$ be a function of class $O_M^{(m)}$. Then, the mapping

$$fA : S_n \rightarrow S_m : \phi \mapsto fA(\phi)$$

is a linear and continuous operator too; it is indeed the composition

$$M_f \circ A,$$

where $M_f$ is the multiplication operator on $S_m$ by the function $f$. 

4
Proof. It is absolutely straightforward. First of all we note that the product $fA$ is well defined. In fact, we have
$$(fA)(\phi) = fA(\phi),$$
and the right-hand function lies in the space $S_m$ because the function $f$ lies in the space $O_M^{(m)}$ and the function $A(\phi)$ lies in the space $S_m$. Moreover, the bilinear application
$$\Phi : O_M^{(m)} \times S_m \to S_m : (f, \psi) \mapsto f \psi$$
is separately continuous and we have
$$(fA)(\phi) = fA(\phi) = \Phi(f, A(\phi)) = M_f(A(\phi)),
$$
i.e.,
$$fA = \Phi(f, \cdot) \circ A = M_f \circ A,$$
hence the operator $fA$ is the composition of two linear continuous maps and then it is a linear and continuous operator. ■

Definition. Let $A \in L(S_n, S_m)$ and $f \in O_M^{(m)}$. The operator
$$fA : S_n \to S_m : \phi \mapsto fA(\phi)$$
is called the product of the operator $A$ by the function $f$.

Proposition. Let $A, B \in L(S_n, S_m)$ be two continuous linear operators and $f, g$ be two functions in $O_M^{(m)}$. Then, we have

1) $(f + g)A = fA + gA; \ f(A + B) = fA + fB; \ 1_{R^m}A = A$, where the function $1_{R^m}$ is the constant function of $R^m$ into $K$ with value 1;

2) the map
$$\Phi : O_M^{(m)} \times L(S_n, S_m) \to L(S_n, S_m) : (f, A) \mapsto fA$$
is a bilinear map.

Proof. It’s a straightforward computation. ■

The above bilinear application is called multiplication of operators by $O_M$ functions.
3.1 The algebra $\mathcal{O}_M^{(m)}$

It’s easy to see that the algebraic structure $(\mathcal{O}_M^{(m)}, + \cdot)$ is a commutative ring with identity, with respect to the usual pointwise addition and multiplications. For instance, the multiplication is the operation

$$\cdot : \mathcal{O}_M^{(m)} \times \mathcal{O}_M^{(m)} \to \mathcal{O}_M^{(m)} : (f, g) \mapsto fg,$$

where, obviously, if $f, g \in \mathcal{O}_M^{(m)}$, then the pointwise product $fg$ still lies in $\mathcal{O}_M^{(m)}$. The identity of the ring is the function $1_m := 1_{\mathbb{R}^m}$. Moreover, we have that the subspace $S_m$ of the space $\mathcal{O}_M^{(m)}$ is an ideal of the ring $\mathcal{O}_M^{(m)}$. The subring of $\mathcal{O}_M^{(m)}$ formed by the invertible elements of $\mathcal{O}_M^{(m)}$ is exactly the multiplicative subgroup of those elements $f$ such that the multiplicative inverse $f^{-1}$ belongs to the space $\mathcal{O}_M^{(m)}$ too.

So that, the space $\mathcal{O}_M^{(m)}$ is a locally convex topological algebra with unit element.

3.2 The module $\mathcal{L}(S_n, S_m)$

**Proposition.** Let $\cdot$ be the multiplication by $\mathcal{O}_M^{(m)}$ functions defined in the above theorem. Then, the algebraic structure $(\mathcal{L}(S_n, S_m), +, \cdot)$ is a left module over the ring $(\mathcal{O}_M^{(m)}, +, \cdot)$.

**Proof.** Recalling the preceding theorem, we have to prove only the pseudo-associative law, i.e. we have to prove that for every couple of functions $f, g \in \mathcal{O}_M^{(m)}$ and for every linear continuous operator $A \in \mathcal{L}(S_n, S_m)$, we have

$$(fg)A = f(gA).$$

In fact, for each $\phi \in S_n$, we have

$$[(fg)A](\phi) = (fg)A(\phi) = f(gA(\phi)) = f(gA)(\phi) = [f(gA)](\phi),$$

as we desired. $\blacksquare$
4 Products of $S$ families by $O_M$ functions

The central definition of the chapter is the following.

**Definition (product of Schwartz families by smooth functions).** Let $v \in S(\mathbb{R}^m, S'_n)$ be an $S$ family of distributions and let $f \in C^\infty(\mathbb{R}^m, \mathbb{K})$ be a smooth function. The **product of the family** $v$ **by the function** $f$ is the family

$$fv := (f(p)v_p)_{p \in \mathbb{R}^m}.$$

**Theorem.** Let $v \in S(\mathbb{R}^m, S'_n)$ be an $S$ family and $f \in O^{(m)}_M$. Then, the family $fv$ lies in $S(\mathbb{R}^m, S'_n)$. Moreover, we have

$$(fv)^\wedge = f\tilde{\tilde{v}}.$$ 

Consequently, concerning the superposition operator of the family $fv$, since $f\tilde{\tilde{v}} = M_f \circ \tilde{\tilde{v}}$, we have

$$t(fv)^\wedge = t\tilde{\tilde{v}} \circ tM_f,$$

or equivalently, in superposition form

$$\int_{\mathbb{R}^m} a(fv) = \int_{\mathbb{R}^m} (fa)v,$$

for every coefficient distribution $a$ in $S'_n$.

**Proof.** Let $\phi \in S_n$ be a test function, we have

$$(fv)(\phi)(p) = (fv)_p(\phi) = (f(p)v_p)(\phi) = f(p)v_p(\phi) = f(p)\tilde{\tilde{v}}(\phi)(p)$$

and hence the function $(fv)(\phi)$ equals $f\tilde{\tilde{v}}(\phi)$, which lies in $S'_n$. Thus, the product $fv$ lies in the space of Schwartz families $S(\mathbb{R}^m, S'_n)$. For any test function $\phi \in S_n$, by the above consideration, we deduce

$$(fv)^\wedge(\phi) = f\tilde{\tilde{v}}(\phi),$$

that is, the equality of operators

$$(fv)^\wedge = f\tilde{\tilde{v}},$$
where \( f\hat{v} \) is the product of the operator \( \hat{v} \) by the function \( f \), product which belongs to the space \( L(S_m, S_n) \). Moreover, concerning the superposition operator of the family \( fv \), we obtain

\[
\int_{\mathbb{R}^m} a(fv) = \left( t(fv) \right)(a) = \left( t(f\hat{v}) \right)(a) = \left( t(M_f \circ \hat{v}) \right)(a) = \left( t(\hat{v} \circ \hat{t}M_f) \right)(a) = \left( t\hat{v}(tM_f(a)) \right) = \left( t\hat{v}(fa) \right) = \int_{\mathbb{R}^m} (fa)v,
\]
for every distribution \( a \) in \( S_m' \). ■

**Theorem.** Let \( f, g \) two functions in the space \( \mathcal{O}_M^{(m)} \) and \( v, w \) two Schwartz families in the space \( S(\mathbb{R}^m, S_n') \). Then, we have:

1) \((f + g)v = fv + gv\), \(f(v + w) = fv + fw\) and \(1_m v = v\);

2) the map

\[
\Phi : \mathcal{O}_M^{(m)} \times S(\mathbb{R}^m, S_n') \rightarrow S(\mathbb{R}^m, S_n') : (f, v) \mapsto fv
\]

is a bilinear map.

**Proof.** 1) For all \( p \in \mathbb{R}^m \), we have

\[
[(f + g)v](p) = (f + g)(p)v_p = (f(p) + g(p))v_p = f(p)v_p + g(p)v_p = (fv)_p + (gv)_p,
\]
i.e. \((f + g)v = fv + gv\). For all \( p \in \mathbb{R}^m \), we have

\[
[f(v + w)](p) = f(p)(v + w)_p = f(p)v_p + f(p)w_p = (fv)_p + (fw)_p,
\]
i.e. \(f(v + w) = fv + fw\). For all \( p \in \mathbb{R}^m \), we have

\[
(1_{\mathbb{R}^m} v)(p) = 1_{\mathbb{R}^m} (p)v_p = v_p;
\]
i.e. \( 1_g v = v \). 2) follows immediately by 1). \( \square \)

The bilinear application of the point 2) of the preceding theorem is called multiplication of Schwartz families by \( \mathcal{O}_M \) functions.

**Theorem (of structure).** Let \( \cdot \) the operation defined above. Then, the algebraic structure \((\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)\) is a left module over the ring \((\mathcal{O}_M^{(m)}, +, \cdot)\).

**Proof.** It’s analogous to the proof of the corresponding proposition for operators. \( \square \)

**Theorem (of isomorphism).** The application
\[
(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)
\]
is a module isomorphism.

**Proof.** It follows easily from the above theorem. \( \square \)

## 5 \( \mathcal{O}_M \) Functions and Schwartz basis

In this section we study some important relations among a Schwartz family \( w \) and its multiples \( fw \).

**Theorem.** Let \( w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \) be a Schwartz family and let \( f \in \mathcal{O}_M^{(m)} \). Then, the hull \( \mathcal{S}\text{span}(w) \) of the family \( w \) contains the hull \( \mathcal{S}\text{span}(fw) \) of the multiple family \( fw \). Moreover, if a distribution \( a \) represents the distribution \( u \) with respect to the family \( fw \) (that is, if \( u = a.(fw) \)) then the distribution \( fa \) represents the distribution \( u \) with respect to the family \( w \) (that is, if \( u = (fa).w \)).

**Proof.** 1) Let \( u \) be a vector of the \( \mathcal{S} \)linear hull \( \mathcal{S}\text{span}(fw) \). Then, there exists a coefficient distribution \( a \in \mathcal{S}'_m \) such that
\[
u = \int_{\mathbb{R}^m} a(fw),
\]
and this is equivalent (as we already have seen) to the equality
\[
u = \int_{\mathbb{R}^m} (fa)w;
\]

hence the vector \( u \) belongs also to the \( \mathcal{S} \)linear hull \( \mathcal{S}\text{span}(w) \). Hence the \( \mathcal{S} \)linear hull \( \mathcal{S}\text{span}(fw) \) is contained in the \( \mathcal{S} \)linear hull \( \mathcal{S}\text{span}(w) \). \( \square \)
Theorem. Let \( w \in S(\mathbb{R}^m, S'_n) \) be a Schwartz family and let \( f \in \mathcal{O}_M^{(m)} \) be a function different from 0 at every point of its domain. Then, the following assertions hold true:

1) if the family \( w \) is \( S \)-linearly independent, the family \( fw \) is \( S \)-linearly independent too;

2) the Schwartz linear hull \( S \text{span}(w) \) contains the hull \( S \text{span}(fw) \);

3) if the family \( w \) is \( S \)-linearly independent, for each vector \( u \) in the hull \( S \text{span}(fw) \), we have
\[
[u \mid w] = f[u \mid fw],
\]
where, as usual, by \([u \mid v]\) we denote the Schwartz coordinate system of a distribution \( u \) (in the Schwartz linear hull of \( v \)) with respect to a Schwartz linear independent family \( v \);

4) if the family \( w \) is an \( S \)-basis of a subspace \( V \), then \( fw \) is an \( S \)-basis of its \( S \)linear hull \( S \text{span}(fw) \) (that in general is a proper subspace of the hull \( S \text{span}(w) \)).

Proof. 1) Let \( a \in S'_m \) be such that
\[
\int_{\mathbb{R}^m} a(fw) = 0_{S'_n},
\]
we have
\[
0_{S'_n} = \int_{\mathbb{R}^m} a(fw) = \int_{\mathbb{R}^m} (fa)w,
\]
thus, because the family \( w \) is \( S \)-linearly independent we have \( fa = 0_{S'_n} \). Since \( f \) is different from 0 at every point, we can conclude \( a = 0_{S'_m} \).

2) Let \( u \) be a vector of the Schwartz linear hull \( S \text{span}(fw) \). Then, there exists a coefficient distribution \( a \in S'_m \) such that
\[
u = \int_{\mathbb{R}^m} a(fw),
\]
or equivalently such that
\[
u = \int_{\mathbb{R}^m} (fa)w,
\]
and hence the vector \( u \) belongs also to the hull \( S \text{span}(w) \). Hence the Schwartz linear hull \( S \text{span}(fw) \) is contained in the hull \( S \text{span}(w) \).
3) If the family \( w \) is \( S \)-linearly independent, from the above two equalities, we deduce \((u)_{fw} = a\) and \((u)_w = fa\), from which

\[
(u)_w = fa = f(u)_w,
\]

as we claimed.

4) is an obvious consequence of the preceding properties. ■

6 \( O_M \) Invertible functions and \( S \) basis

We recall that an invertible element of \( O_M^{(m)} \) is any function \( f \) everywhere different from 0 and such that its multiplicative inverse \( f^{-1} \) lives in \( O_M^{(m)} \) too. The set of the invertible elements of the space \( O_M^{(m)} \) is a group with respect to the pointwise multiplication, and we will denote it by \( G_M^{(m)} \).

**Theorem.** Let \( w \in S(\mathbb{R}^m, S'_n) \) be a Schwartz family and let \( f \in G_M^{(m)} \) be an invertible element of the ring \( O_M^{(m)} \) (in particular, it must be a function different from 0 at every point). Then, the following assertions hold true:

1) the family \( w \) is \( S \)-linearly independent if and only if the multiple family \( fw \) is \( S \)-linearly independent;

2) the hull \( S \text{span}(w) \) coincides with the hull \( S \text{span}(fw) \);

3) if the family \( w \) is \( S \)-linearly independent, then, for each vector \( u \) in the hull \( S \text{span}(w) \), we have

\[
[u \mid fw] = (1/f)[u \mid w],
\]

where, as usual, by \([u \mid v]\) we denote the Schwartz coordinate system of a distribution \( u \) (in the Schwartz linear hull of \( v \)) with respect to a Schwartz linear independent family \( v \);

4) the family \( w \) is an \( S \) basis of a subspace \( V \) if and only if its multiple \( fw \) is an \( S \) basis of the \( S \) linear hull \( S \text{span}(fw) \) (that in this case coincides with \( S \text{span}(w) \)).

**Proof.** 1) Let \( a \in S'_m \) be a distribution such that

\[
\int_{\mathbb{R}^m} aw = 0_{S'_n},
\]
we have
\[
0_{S'_n} = \int_{\mathbb{R}^m} aw = \\
= \int_{\mathbb{R}^m} (f^{-1}a)(fw),
\]
thus, because \(fw\) is \(S\) linearly independent we have \(f^{-1}a = 0_{S'_n}\). Since \(f^{-1}\) is different form 0 at every point we can conclude \(a = 0_{S'_n}\).

2) Let \(u\) be in \(\mathcal{S}\text{span}(w)\). Then, there exists a distribution \(a \in S'_m\) such that
\[
u = \int_{\mathbb{R}^m} aw.
\]
Now, we have
\[
u = \int_{\mathbb{R}^m} (f^{-1}a)(fw),
\]
so the distribution \(u\) lies in \(\mathcal{S}\text{span}(fw)\), and hence \(\mathcal{S}\text{span}(w)\) is contained in \(\mathcal{S}\text{span}(fw)\). Vice versa, let \(u\) be in \(\mathcal{S}\text{span}(fw)\). Then, there exists a distribution \(a \in S'_m\) such that
\[
u = \int_{\mathbb{R}^m} a(fw).
\]
Now, we have (equivalently)
\[
u = \int_{\mathbb{R}^m} (fa)w,
\]
and hence \(u\) lies also in \(\mathcal{S}\text{span}(w)\), hence \(\mathcal{S}\text{span}(fw)\) is contained in \(\mathcal{S}\text{span}(w)\) (as we already have seen in the general case). Concluding
\[
\mathcal{S}\text{span}(w) = \mathcal{S}\text{span}(fw).
\]

3) For any distribution \(u\) in the Schwartz linear hull of the family \(w\), we have
\[
u = \int_{\mathbb{R}^m} [u \mid w] w,
\]
hence
\[
u = \int_{\mathbb{R}^m} (f^{-1}[u|w])(fw),
\]
as we desired.

4) It follows immediately from the above properties.

\textbf{Theorem.} Let \(e \in \mathcal{B}(\mathbb{R}^m, S'_n)\) be an \(S\) basis of the space \(S'_n\) and let \(f \in \mathcal{O}_M^{(m)}\). Then the multiple \(fe\) is an \(S\) basis of the space \(S'_n\) if and only if the factor \(f\) is an invertible element of the ring \(\mathcal{O}_M^{(m)}\).
Proof. We must prove that, if \( fe \) is an \( S \)-basis of \( S'_n \), then \( f \) is an invertible element of the ring \( \mathcal{O}_M^{(m)} \). First of all observe that, since \( fe \) is a basis, then \( fe \) is \( S \)-linearly independent and consequently linearly independent in the ordinary algebraic sense; consequently every distribution \( f(p)e_p \) must be a non zero distribution and this implies that any value \( f(p) \) must be different from 0, so we can consider the multiplicative inverse \( f^{-1} \). We now have to prove that the multiplicative inverse \( f^{-1} \) lives in \( \mathcal{O}_M^{(m)} \), or equivalently that, for every test function \( g \) in \( S_m \), the product \( f^{-1}g \) lives in \( S_m \). For, let \( g \) be in \( S_m \), since \( fe \) is a basis, its associated operator from \( S_n \) into \( S_m \) is surjective, then there is a function \( h \) in \( S_n \) such that \((fe)\wedge(h) = g\), the last equality is equivalent to

\[ fe(h) = g, \]

that is

\[ f^{-1}g = e(h), \]

so that \( f^{-1}g \) actually lives in the space \( S_m \). ■

We can generalize the above result as it follows.

Theorem. Let \( e \in \mathcal{B}(\mathbb{R}^m, V) \) be an \( S \)-basis of a (weakly*) closed subspace \( V \) of the space \( S'_n \) and let \( f \in \mathcal{O}_M^{(m)} \). Then the multiple family \( fe \) is an \( S \)-basis of the subspace \( V \) if and only if the factor \( f \) is an invertible element of the ring \( \mathcal{O}_M^{(m)} \).

Proof. We must prove that, if \( fe \) is an \( S \)-basis of the subspace \( V \), then \( f \) is an invertible element of the ring \( \mathcal{O}_M^{(m)} \). First of all observe that, since \( fe \) is a basis, then \( fe \) is \( S \)-linearly independent and consequently linearly independent in the ordinary algebraic sense; consequently every distribution \( f(p)e_p \) must be a non zero distribution and this implies that any value \( f(p) \) must be different from 0. So we can consider its multiplicative inverse \( f^{-1} \). We now have to prove that the multiplicative inverse \( f^{-1} \) lives in \( \mathcal{O}_M^{(m)} \), or equivalently that, for every test function \( g \) in \( S_m \), the product \( f^{-1}g \) lives in \( S_m \). For, let \( g \) be in \( S_m \), since \( fe \) is an \( S \)-basis of the topologically closed subspace \( V \), its associated operator \((fe)\wedge \) from \( S_n \) into \( S_m \) is surjective (this follows, by the closedness of \( V \), from the Dieudonné-Schwartz theorem, since the transpose of the operator \((fe)\wedge \) is the superposition operator of \( fe \), which is injective since the family \( fe \) is Schwartz linearly independent). Hence, by surjectivity, there is a function \( h \) in \( S_n \) such that \((fe)\wedge(h) = g\), the last equality is equivalent to the following one

\[ fe(h) = g, \]

that is

\[ f^{-1}g = e(h), \]

so that the function \( f^{-1}g \) actually lives in the space \( S_m \). ■
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