Quantum spectrum for a Kerr-Newman black hole

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Abstract

In this paper, we consider the quantum area spectrum for a rotating and charged (Kerr-Newman) black hole. Generalizing a recent study on Kerr black holes (which was inspired by the static-black hole formalism of Barvinsky, Das and Kunstatter), we show that the quantized area operator can be expressed in terms of three quantum numbers (roughly related to the mass, charge and spin sectors). More precisely, we find that

\[ A = 8\pi\hbar\left[ n + \frac{1}{2} + \frac{p_1}{2} + p_2 \right] \]

where \( n, p_1 \) and \( p_2 \) are strictly non-negative integers. In this way, we are able to confirm a uniformly spaced spectrum even for a fully general Kerr-Newman black hole. Along the way, we derive certain selection rules and use these to demonstrate that, in spite of appearances, the charge and spin spectra are not completely independent.

1. INTRODUCTION

There has been significant interest, over the last decade, in the subject known as black hole spectroscopy. (For an overview, see [1].) The general concept is that a black hole observable, in particular the surface area of the horizon, should be quantized in terms of a discretely changing quantum number (or numbers). That this should be the case was first advocated by Bekenstein [2], not long after the realization of black holes as thermodynamic systems [3,4]. The main point of Bekenstein’s argument is that, for a slowly evolving black hole, the horizon area (\( A \)) behaves as an adiabatic invariant [5]. The significance of this property follows from Ehrenfest’s principle, which indicates that a classical adiabatic invariant corresponds to a quantum observable with a discrete spectrum.

Bekenstein went on to suggest that the quantum area spectrum should be evenly spaced with increments of size \( \epsilon\hbar \) [2,6], where \( \epsilon \) is a numerical factor of the order unity and \( \hbar \) is the Planck constant. (Here and throughout, we restrict the discussion to a four-dimensional spacetime and set all other fundamental constants to unity.) This discreteness can be viewed

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as a consequence of the uncertainty principle, which tells us that a quantum point particle can, at best, only be localized to a single Compton length.

Much of the subsequent interest in black hole spectroscopy has centered around the idea that such heuristic arguments can somehow be substantiated by more rigorous means. There has, indeed, been significant progress in studies along this line (see [8] for references); for instance, the algebraic approach to black hole quantization, as developed by Bekenstein and one of the current authors [7,8,9]. In this approach, starting with very elementary assumptions and then exploiting symmetries, one can obtain the algebra of the relevant operators of the black hole. Bekenstein has used this methodology to provide a more rigorous proof [7] for the equally spaced area spectrum. In the subsequent analysis [8,9], the neutral, non-rotating black hole observables have been constructed by subjecting a pair of creation operators, or “building blocks”, to a simple algebra. This construction reproduces the evenly spaced area spectrum, provides a rigorous proof that the $n$-th area eigenvalue is exponentially degenerate and predicts that a logarithmic correction term, $-3/2 \log A$, should be added to the Bekenstein-Hawking entropy [3,4].

Another such program, which is particularly relevant to the current paper, was initiated by Barvinsky and Kunstatter [10]. To summarize, these authors expressed the black hole dynamics in terms of a reduced phase space (the reduction comes by way of a minisuperspace type of approximation), which they were then able to quantize. For a static and uncharged black hole (which was the focus of this seminal paper), the reduced phase space consists of only the black hole mass observable and its canonical conjugate [11,12]. With one justifiable assumption - namely, the conjugate to the mass is identified with a periodicity which coincides with that of Euclidean time [15] - the authors reproduced a uniformly spaced area spectrum such that $\epsilon = 8\pi \bar{h}$ (They also found a zero-point contribution of $4\pi \bar{h}$, which in no way undermines the original Bekenstein proposal.)

The methodology of [10] was later extended for the highly non-trivial inclusions of non-vanishing charge (by Barvinsky, Das and Kunstatter [16,17]) and spin (by the current authors [18]). In the former, charged case, the reduced phase space consists of the two relevant observables (the mass, $M$, and the charge, $Q$) and their respective conjugates [19]. Imposing the same periodicity condition as discussed above, Barvinsky et al found an area spectrum of the following form [16]:

$$A - A_{\text{ext}}(Q) = 8\pi \bar{h} \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots,$$

where $A_{\text{ext}} = A_{\text{ext}}(Q)$ is the extremal value of the horizon area. (It should be kept in mind that $A_{\text{ext}}$ represents a lower bound on the horizon area of a classical black hole. Note, however, that the zero-point term in Eq.(1) prohibits the quantum black hole from actually

1 Although the simplicity of this picture may appear counter-intuitive, it actually follows quite naturally from either Birkhoff’s theorem [13] or the “no-hair” principles of black holes [14].

2 A charged and/or rotating black hole typically has a pair of distinct horizons, with the point of coincidence defining the extremal horizon. Note that, in this paper, an unqualified $A$ will always signify the area of the outermost horizon.
approaching this extremal value.) After quantizing the charge sector of the theory, Barvinsky et al. finally obtained

\[ A = 8\pi\hbar \left(n + \frac{p}{2} + \frac{1}{2} \right), \quad n, p = 0, 1, 2, \ldots, \]  

(2)

where \( p \) is related to the black hole charge via \( Q^2 = \hbar p \).

The latter, rotating case was complicated by the lack of a concrete example for the reduced phase of a spinning black hole. Nevertheless, we argued for the existence of such a reduced space by appealing to the “no-hair” principles of black holes [14]. With this assumption and the usual periodicity constraint, we were able to deduce an area spectrum of the following form [18]:

\[ A - A_{\text{ext}}(J_{cl}) = 8\pi\hbar \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots, \]  

(3)

where \( J_{cl} \) is a rotation operator that is related to but distinct from the angular momentum of the black hole. After quantizing the spin sector, we found that the spectrum (3) could be re-expressed as follows [18]:

\[ A = 8\pi\hbar \left(n + m + \frac{1}{2} \right), \quad n, m = 0, 1, 2, \ldots, \]  

(4)

with the rotation operator having been quantized according to \( J_{cl} = \hbar m \). It should be noted that, in the limit of very large spin or \( m >> 1, m \sim j \) where \( j \) is the angular-momentum eigenvalue of the black hole.

In our prior paper, we purposefully neglected any consideration of charge so as to stress the technical issues that are indigenous to the case of a rotating black hole. In the current paper, we rectify this omission and extend the methodology to a black hole with both charge or spin; that is, we determine the area spectrum of a fully general Kerr-Newman black hole. Note that, to avoid superfluous repetition, we will go over some steps rather quickly. For a better understanding of the subtleties of this procedure, the reader is referred to the earlier paper [18]. Also, let us take this moment to emphasize that only two (well-motivated) assumptions go into our quantization procedure: a rotating black hole can be described, in analogy to studies on static black holes [11,12,19], by a relatively simple form of reduced phase space and the conjugate to the mass is periodic in accordance with Euclidean considerations [12,13,10].

It should be noted that, in a recent study of interest, Makela et al. have similarly considered the area spectrum of the Kerr-Newman black hole [20]. The approach of these authors (also see [21]) is based on formulating a Schrodinger-like equation for the black hole observables and quantizing this equation by way of WKB techniques. Their form of the area spectrum differs somewhat with what we eventually derive here; however, a direct comparison is highly non-trivial due to a fundamental distinction in what quantity is precisely being

3To emphasize this distinction, we always use, in both the prior [18] and current paper, the subscript \( cl \) for classical. This is, however, somewhat misleading, as \( J_{cl} \) is ultimately elevated to the status of a quantum operator at some point in the analysis.
quantized. More specifically, Makela et al quantized $A + A_-$ (where $A_-$ is the area of the inner black hole horizon) as opposed to $A - A_{\text{ext}}$ (cf, Eqs. (1,3)).

The remainder of the paper is organized as follows. In Section 2, we consider a Kerr-Newman black hole at the classical level, with particular emphasis on the (conjectured) reduced phase space. The quantization of this phase space, as detailed in Section 3, leads to an area spectrum that is formally analogous to Eqs. (1,3). In Section 4, we focus on the charge and spin sector, and develop a means by which this can be suitably quantized. After implementing certain selection rules, we are able to show that the area spectrum is, indeed, uniformly spaced, even for a fully general Kerr-Newman solution. The final section contains a brief summary and some further discussion on the selection rules.

II. CLASSICAL ANALYSIS

We begin by considering the Kerr-Newman black hole, which may be regarded as the most general solution of the vacuum Einstein equations. Because of the “no-hair” principles [14], it is expected that an external observer can describe this system in terms of a few macroscopic parameters: the black hole mass, $M$, charge, $Q$, and an angular momentum, $\vec{J}_{\text{cl}}$. Furthermore, the first law of black hole mechanics [3,4] allows us to relate these quantities in the following manner:

$$dM = \frac{\kappa}{8\pi} dA + \Phi dQ + \Omega dJ_{\text{cl}}.$$  \hspace{1cm} (5)

Here, $A$ is the (outermost) horizon area, $\kappa$ is the surface gravity at this horizon, $\Phi$ is the electrostatic potential, $\Omega$ is the angular velocity, and $J_{\text{cl}} = |\vec{J}_{\text{cl}}|$ is the magnitude of the angular-momentum vector.

The thermodynamic properties of a Kerr-Newman black hole are explicitly known [13] and expressible as follows:

$$A = 8\pi M \left[ M - \frac{Q^2}{2M} + \sqrt{M^2 - Q^2 - \frac{J_{\text{cl}}^2}{M^2}} \right]$$  \hspace{1cm} (6)

or equivalently

$$M^2 = \frac{A}{16\pi} + 4\pi \frac{J_{\text{cl}}^2}{A} + \frac{Q^2}{2} + \frac{1}{\pi} \frac{Q^4}{A},$$  \hspace{1cm} (7)

and

$$\kappa = \frac{1}{4M} - 16\pi^2 \frac{J_{\text{cl}}^2}{MA^2} - 4 \frac{Q^4}{MA^2},$$  \hspace{1cm} (8)

$$\Phi = \frac{Q}{2M} \left[ 1 + \frac{4Q^2}{\pi A} \right],$$  \hspace{1cm} (9)

$$\Omega = 4\pi \frac{J_{\text{cl}}}{MA}.$$  \hspace{1cm} (10)
Generalizing the philosophy of \cite{18}, we will assume that the Kerr-Newman black hole can be dynamically expressed, at the classical level, in terms of a reduced phase space consisting of the physical observables and their canonical conjugates. (For further justification, also see \cite{20}.) More specifically, we propose that the phase space can be described by the following set of observables:

\begin{equation}
A, Q, J_\alpha, J_\beta, J_\gamma,
\end{equation}

along with their respective conjugates

\begin{equation}
P_A, P_Q, \alpha, \beta, \gamma.
\end{equation}

In this formulation, \(J_\alpha\), \(J_\beta\), and \(J_\gamma\) represent the Euler components of the angular momentum \cite{22} and their conjugates, \(\alpha\), \(\beta\) and \(\gamma\), are the corresponding Euler angles. For future reference, note that \cite{22}

\begin{equation}
J^x = -\cos \alpha \cot \beta J_\alpha - \sin \alpha J_\beta + \frac{\cos \alpha}{\sin \beta} J_\gamma,
\end{equation}

\begin{equation}
J^y = -\sin \alpha \cot \beta J_\alpha + \cos \alpha J_\beta + \frac{\sin \alpha}{\sin \beta} J_\gamma,
\end{equation}

\begin{equation}
J^z = J_\gamma,
\end{equation}

where \(J^x\), etc. are the Cartesian components of the angular momentum.

Intuitive considerations suggest that the horizon area is invariant under rotations of its spin and under gauge transformations. It is thus follows that Eqs.(11,12) form a set of generalized commuting coordinates and their conjugates. (In the algebraic approach \cite{1}, one similarly starts with the area, charge and angular momentum as the initial commuting observables.) On the other hand, we ultimately want to work with the mass, \(M\), rather than the area, \(A\) (this allows us to exploit the periodicity of the mass conjugate, as noted in Section 1), and it can be shown that \cite{18}

\begin{equation}
\{M, J_\beta\} \neq 0.
\end{equation}

(Here, \(\{\ , \ ; \ \}\) denotes a commutator or Poisson bracket in the Dirac sense \cite{23}.) We can, however, circumvent this awkward situation by replacing \(J_\beta\) with \(J_{\text{cl}}\).

One can see the relevance of the proposed “switch” by inspecting the explicit form of \(J_{\text{cl}}\):

\begin{equation}
J_{\text{cl}}^2 = J_x^2 + J_y^2 + J_z^2
= \frac{1}{\sin^2 \beta} \left[ J_\alpha^2 + J_\gamma^2 - 2 \cos \beta J_\alpha J_\beta \right] + J_\beta^2,
\end{equation}

where we have applied Eqs.(13,14) and treated \(J_\alpha\), \(J_\beta\), \(J_\gamma\) as classical or non-operating quantities. It is, in fact, the presence of \(\beta\) (but neither \(\alpha\) nor \(\gamma\)) in the above expression that makes \(J_\beta\) a rather poor choice in constructing the phase space.
With prompting from the above discussion, we now consider a new set of observables:

\[ M = M(A, Q, J_{cl}), Q, J_{cl}, J_\alpha, J_\gamma \]

and denote their respective conjugates as follows:

\[ \Pi_M, \Pi_Q, \Pi_{cl}, \Pi_\alpha, \Pi_\gamma. \]

The transformation from Eqs.(11,12) into Eqs.(18,19) is canonical if

\[ \{ M, \Pi_M \} = \{ Q, \Pi_Q \} = \{ J_{cl}, \Pi_{cl} \} = \{ J_\alpha, \Pi_\alpha \} = \{ J_\gamma, \Pi_\gamma \} = 1, \]

\[ \{ all \ other \ combinations \} = 0, \]

where the derivatives are taken with respect to the original set of generalized coordinates (11,12).

Applying some straightforward but lengthy calculations, one can readily verify that the proposed transformation is canonical provided that

\[ \Pi_M = \frac{8\pi}{\kappa} P_A, \]

\[ \Pi_Q = P_Q - \frac{8\pi}{\kappa} \Phi P_A, \]

\[ \Pi_{cl} = P_{cl} - \frac{8\pi}{\kappa} \Omega P_A, \]

\[ \Pi_\alpha = P_\alpha, \]

\[ \Pi_\gamma = P_\gamma. \]

Note that \( P_{cl} \) is defined by first making a canonical transformation from Eq.(11,12) to the same set but with \( J_{cl} (P_{cl}) \) replacing \( J_\beta (\beta) \).

**III. QUANTUM ANALYSIS**

To proceed with a suitable process of quantization, in the manner originally advocated by Barvinsky and Kunstatter [10], the following condition of periodicity is required:

\[ \Pi_M \sim \Pi_M + \frac{2\pi}{\kappa}. \]

Although technically an assumption, this condition has a well-justified pedigree that follows from the known periodicity of Euclidean time [13] and the identification of \( \Pi_M \) with a measure of Schwarzschild-like time [12]. (Consult [10,16,17,18] for further discussion.)
Following the prescribed program, we now proceed by introducing a pair of variables that directly incorporate the periodic nature of $\Pi_M$:

$$X = \sqrt{\frac{\hbar B(M, Q, J_{cl}, J_\alpha, J_\gamma)}{\pi}} \cos(\kappa \Pi_M), \quad (28)$$

$$P_X = \sqrt{\frac{\hbar B(M, Q, J_{cl}, J_\alpha, J_\gamma)}{\pi}} \sin(\kappa \Pi_M). \quad (29)$$

The yet-to-be-specified function $B$ will be partially fixed via the constraint that Eqs. (18,19) transform canonically into the set of observables

$$X, Q, J_{cl}, J_\alpha, J_\gamma \quad (30)$$

and respective conjugates

$$P_X, P_Q, P_{cl}, P_\alpha, P_\gamma \quad (31)$$

Let us therefore consider the following necessary and sufficient condition for a canonical transformation:

$$P_X \delta X + P_Q \delta Q + P_{cl} \delta J_{cl} + P_\alpha \delta J_\alpha + P_\gamma \delta J_\gamma = \Pi_M \delta M + \Pi_Q \delta Q + \Pi_{cl} \delta J_{cl} + \Pi_\alpha \delta J_\alpha + \Pi_\gamma \delta J_\gamma. \quad (32)$$

Up to a total variation, one finds that

$$P_X \delta X = \frac{\hbar \kappa}{2\pi} \left[ \frac{\partial B}{\partial M} \delta M + \frac{\partial B}{\partial Q} \delta Q + \frac{\partial B}{\partial J_{cl}} \delta J_{cl} + \frac{\partial B}{\partial J_\alpha} \delta J_\alpha + \frac{\partial B}{\partial J_\gamma} \delta J_\gamma \right]. \quad (33)$$

Substituting Eq. (33) into Eq. (32), we promptly obtain the following pivotal result:

$$\frac{\partial B}{\partial M} = \frac{2\pi}{\hbar \kappa}. \quad (34)$$

For future reference, we also have

$$\frac{\partial B}{\partial Q} = \frac{2\pi}{\hbar \kappa \Pi_M} (\Pi_Q - P_Q), \quad (35)$$

$$\frac{\partial B}{\partial J_{cl}} = \frac{2\pi}{\hbar \kappa \Pi_M} (\Pi_{cl} - P_{cl}) \quad (36)$$

and analogous expressions involving $P_\alpha$ and $P_\gamma$.

It is instructive to compare Eq. (34) with the first law, Eq. (5), indicating that $\partial A/\partial M = 4\hbar \partial B/\partial M$. This suggestive result directly implies the following:

$$B(M, Q, J_{cl}, J_\alpha, J_\gamma) = \frac{1}{4\hbar} A(M, Q, J_{cl}) + F(Q, J_{cl}, J_\alpha, J_\gamma), \quad (37)$$
where $F$ is an essentially arbitrary function of the charge and angular momentum. That is to say, for any well-behaved choice of $F$, one will always be able to find expressions for $\mathcal{P}_Q$, $\mathcal{P}_{cl}$, $\mathcal{P}_\alpha$ and $\mathcal{P}_\gamma$ that maintain a canonical transformation.

In spite of this apparent freedom in $F$, we are able to fix this function by way of the following argument. Let us first point out that the (outer) horizon area of a Kerr-Newman black hole is bounded from below - at least classically - by its extremal value \[13\]; that is,

$$A \geq A_{ext} = 4\pi \sqrt{Q^4 + 4J^2_{cl}}. \quad (38)$$

As elaborated on in the related works \[10,16,17,18\], it is most convenient if $F$ is chosen so that Eq.(38) translates into the bound $B \geq 0$. On this basis, we can unambiguously set $F = -A_{ext}/4\hbar$ and obtain

$$B = \frac{1}{4\hbar} \left[ A(M, Q, J_{cl}) - 4\pi \sqrt{Q^4 + 4J^2_{cl}} \right]. \quad (39)$$

Let us now reconsider Eqs.(28,29), which can be squared and summed to yield $\hbar B = \pi(X^2 + \mathcal{P}^2_X)$. Incorporating this finding into Eq.(39), we have

$$X^2 + \mathcal{P}^2_X = \frac{1}{4\pi} \left[ A(M, Q, J_{cl}) - 4\pi \sqrt{Q^4 + 4J^2_{cl}} \right] \geq 0. \quad (40)$$

It is especially relevant that the mass and its conjugate, $M$ and $\Pi_M$, have been mapped into a complete two-dimensional plane, $X$ and $\mathcal{P}_X$. Any other choice of $F$ would have left a “hole” in this plane and unnecessarily complicated the impending process of quantization.

Next, let us elevate the classically defined quantities to quantum operators (denoted by “hats”). Eq.(40) then takes the following form:

$$\frac{\hbar}{2\pi} \hat{B} \equiv \frac{1}{8\pi} \left[ \hat{A} - 4\pi \sqrt{\hat{Q}^4 + 4\hat{J}^2_{cl}} \right] = \frac{\hat{X}^2}{2} + \frac{\hat{P}^2_X}{2}. \quad (41)$$

Given that the domain of $\hat{X}$ and $\hat{P}_X$ is a complete two-dimensional plane, the spectrum of these operators is trivially that of a harmonic oscillator. Hence, we can write

$$B_n = 2\pi \left[ n + \frac{1}{2} \right], \quad n = 0, 1, 2, .... \quad (42)$$

where $B_n$ are the eigenstates of the operator $\hat{B}$. Keep in mind that $B_n$ is essentially a measure of the deviation of the horizon area from extremality.

It is interesting to note that, by virtue of the zero-point term in Eq.(12), quantum fluctuations will inhibit the black hole from ever reaching a precise state of extremality. A similar observation has been made for both charged (but non-rotating) \[16\] and rotating (but uncharged) \[18\] black holes.

\(^4\text{This extremal area can be obtained by constraining the mass observable so that the square-root argument in Eq.}\(6\) is exactly vanishing.\)
IV. CHARGE AND SPIN SECTOR

Our task is not yet complete, as we still require that the spectra for $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}_{\text{ext}} = 4\pi\sqrt{\hat{Q}^4 + \hat{J}_d^2}$ be explicitly separated. (Note that, for this purpose, a complete separation of the spectra for $\hat{J}_d$ and $\hat{Q}$ is not necessarily required.) It turns out that this objective can readily be accomplished by way of some straightforward arguments.

At a first glance, the quantization of $\hat{\mathbf{A}}_{\text{ext}}$ appears to be a trivial process; inasmuch as $\hat{Q}$ is the generator of $U(1)$ gauge transformations, its spectrum of eigenvalues ($Q$) must certainly be of the form

$$Q = em_1, \quad m_1 = 0, \pm 1, \pm 2, \ldots,$$

where $e$ is the fundamental unit of electrostatic charge. Furthermore, the quantum limit of the angular-momentum operator ($\hat{\mathbf{J}}_{\text{cl}}$, cf. Eq.(17)),

$$\hat{J}_d^2 = \frac{1}{\sin^2 \beta} \left[ \hat{J}_d^2 + \hat{J}_d^2 - 2 \cos \beta \hat{J}_d \hat{J}_\beta \right] + \hat{J}_\beta^2,$$

has been shown [18] to have the following discrete set of eigenvalues ($J_d$):

$$J_d = \hbar m_2, \quad m_2 = 0, 1, 2, \ldots.$$

Therefore, by substituting Eqs.(43,45) (and also Eq.(42)) into Eq.(41), we find that the area spectrum can be expressed as follows:

$$A_{n,m_1,m_2} = 8\pi \hbar \left[ n + \frac{1}{2} \right] + \frac{1}{2} e^4 m_1^2 + 4\hbar^2 m_2^2.$$  

Given this form of the spectrum, it is not at all obvious that the levels could, in general, be evenly spaced. Nonetheless, by utilizing the periodicity of $\Pi_M$ to impose selection rules on $m_1$ and $m_2$, we will demonstrate below that a uniformly spaced spectrum is, indeed, consistently realized.

With the above in mind, let us first consider the quasi-charge sector. It is useful to recall Eq.(35):

$$P_Q = \Pi_Q + \Phi \Pi_M + \frac{\kappa}{8\pi} \Pi_M \frac{\partial A_{\text{ext}}}{\partial Q},$$

where the first law of black hole mechanics (5) and the precise forms of $B$ (37) and $F = -A_{\text{ext}}/4\hbar$ have also been applied. For sake of clarity, let us further re-express this relation as follows:

$$P_Q = \chi_1 + \frac{\theta}{8\pi} \frac{\partial A_{\text{ext}}}{\partial Q},$$

We use this “quasi” terminology because the charge and spin sectors do not, in general, separate completely.
where \( \chi_1 \equiv \Pi_Q + \Phi \Pi_M \) and \( \theta \equiv \kappa \Pi_M \). It should be kept in mind that \( \theta \) is an angle (\( i.e. \), has a periodicity of exactly \( 2\pi \)); \( c.f. \) Eq.(27). Note that, for a non-rotating black hole, it has been shown [10] that the variable \( \chi_1 \) is, up to a dimensional factor, also an angular quantity. However, this need not be the case when the spin is “turned on”, and so we will not apply (nor require) this result.

Now consider that, in the coordinate representation with \( \hat{Q} = -i\hbar \partial / \partial \Pi_Q \), the wavefunctions for the charge eigenstates take the form

\[
\Psi_Q(P_Q) \sim \exp \left[ iQ \frac{\partial A_{\text{ext}}}{\partial Q} \right]
\]

and we can, therefore, make the following identification:

\[
\frac{QP_Q}{\hbar} \sim \frac{QP_Q}{\hbar} + 2\pi n_1,
\]

where \( n_1 \) is an arbitrary integer.

Let us next consider the implication of Eqs.(48) and (50) when taken together. Holding \( \chi_1 \) constant, we are able to deduce that

\[
\theta \frac{Q}{8\pi \hbar} \frac{\partial A_{\text{ext}}}{\partial Q} \sim \theta \frac{Q}{8\pi \hbar} \frac{\partial A_{\text{ext}}}{\partial Q} + 2\pi n_2,
\]

where \( n_2 \) is another arbitrary integer. That is, the quantity on the left-hand side must necessarily be an angle. However, \( \theta \) is, by hypothesis, itself an angle, and so we can write

\[
\frac{Q}{8\pi \hbar} \frac{\partial A_{\text{ext}}}{\partial Q} = p_1,
\]

where \( p_1 \) is yet another integer (which is manifestly non-negative, as can be seen by inspecting the left-hand side).

We can make the above expression more explicit by utilizing Eq.(38) to obtain

\[
\frac{1}{\hbar} \frac{Q^4}{\sqrt{Q^4 + 4J_{cl}^2}} = p_1, \quad p_1 = 0, 1, 2, ....
\]

This quantization condition will be referred to as selection rule no.1.

Let us next examine the quasi-spin sector, beginning with the appropriately revised form of Eq.(36):

\[
P_{cl} = \chi_2 + \frac{\theta}{8\pi} \frac{\partial A_{\text{ext}}}{\partial J_{cl}},
\]

where \( \chi_2 \equiv \Pi_{cl} + \Omega \Pi_M \). Because of the obvious symmetry between this relation and its quasi-charge sector analogue (48), it is clear that the current quantization procedure will

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6Here, we are treating \( \chi_1 \) and \( \theta \) as independent variables. One might be concerned that both depend on the conjugate \( \Pi_M \); however, \( \chi_1 \) also depends on a variable, \( \Pi_Q \), that is clearly independent of \( \Pi_M \). Hence, we argue that \( \chi_1 \) can be held constant without loss of generality.
closely follow the prior analysis. Repeating the steps, as outlined above, for the current case, we find that

$$\frac{J_{cl}}{8\pi \hbar} \frac{\partial A_{ext}}{\partial J_{cl}} = p_2,$$

(55)

where $p_2$ is strictly a non-negative integer.

Substituting Eq.(38) into the above relation, we obtain selection rule no.2:

$$\frac{2J_{cl}^2}{\hbar \sqrt{Q^4 + 4J_{cl}^2}} = p_2, \quad p_2 = 0, 1, 2, ....$$

(56)

Next, we consider an appropriate linear combination of the two selection rules (53,56). Multiplying no.2 by two and adding this to no.1, we have (after some trivial manipulations)

$$\sqrt{Q^4 + 4J_{cl}^2} = \hbar (p_1 + 2p_2), \quad p_1, p_2 = 0, 1, 2, ....$$

(57)

Let us now recall and appropriately rearrange Eq.(39):

$$A = 4\hbar B + 4\pi \sqrt{Q^4 + 4J_{cl}^2}.$$ 

(58)

Quantizing this relation and then incorporating Eqs.(57,12), we can now write the area spectrum in the following elegant form:

$$A_{n,p_1,p_2} = 8\pi \hbar \left[ n + \frac{1}{2} + \frac{p_1}{2} + p_2 \right], \quad n, p_1, p_2 = 0, 1, 2, ....$$

(59)

Hence, we have realized an evenly spaced spectrum for the area of a fully general Kerr-Newman black hole.

It should be kept in mind that, strictly speaking, the quantum numbers $p_1$ and $p_2$ are not independent parameters. Rather, these integers are related by way of the selection rules; so that, once $p_1$ has been fixed, $p_2$ will be restricted to a limited set of allowable values and vice versa. We will elaborate on this point in the section to follow.

V. CONCLUDING DISCUSSION: SELECTION RULES

In summary, we have considered the quantum area spectrum of a Kerr-Newman (rotating and charged) black hole. Extending the methodology of Barvinsky et al for a static system [10,16], as well as a recent treatment on Kerr black holes [18], we have obtained an explicit form for the area spectrum in terms of three integer-valued quantum numbers. Moreover, the spectrum was shown to be uniformly spaced, in direct compliance with the heuristic arguments of Bekenstein [2,6,1] and the algebraic approach [7,8,9].

Let us re-emphasize that some conjectural, although well-motivated inputs were used in attaining this result. First of all, it was assumed that a reduced phase space description exists for a fully general Kerr-Newman black hole. Secondly, we utilized a periodicity condition on the conjugate to the mass that was first proposed by Barvinsky and Kunstatter [10].
Nonetheless, we feel that the elegance of the resulting spectrum only strengthens our convictions with regard to the use of such inputs. We expect, however, to more rigorously address these issues at a future time.

Finally, let us more closely examine the implications of the selection rules on the charge and spin spectra. For illustrative purposes, we will first focus on the case of vanishing spin. Then Eqs. (43,53) require

$$\eta \equiv \frac{e^2}{\hbar} = \frac{p_1}{m_1^2};$$  \hspace{1cm} (60)

that is, this ratio of fundamental constants, $\eta$, is constrained to be a rational, constant number. It follows that $\eta$ can always be expressed in the following manner:

$$\eta = \frac{a}{b},$$  \hspace{1cm} (61)

where $a$ and $b$ are mutually prime (i.e., non-divisible) integers. Therefore, Eq. (60) rules out many eigenvalues of $\hat{Q}$ and only the following are allowed:

$$Q = em_1, \quad \text{where} \quad m_1^2 \mod b = 0.$$  \hspace{1cm} (62)

That is to say, $m_1^2$ must be divisible by $b$.

For the case of vanishing charge but non-vanishing spin, the results are substantially different, as Eqs. (45,56) do not provide any new information. Rather, we only have the trivial outcome of $m_2 = p_2$ for a neutral black hole.

Next, moving on to the fully general case, we find (after some manipulations) that the selection rules (53,56) now imply a pair of constraints:

$$\eta^2 m_1^4 = p_1(p_1 + 2p_2) \quad \text{and} \quad 2m_2^2 = p_2(p_1 + 2p_2).$$  \hspace{1cm} (63)

Since $\eta$ is the same rational number given by Eq. (61), the charge sector is still quantized according to Eq. (62). Furthermore, Eq. (63) shows that, for any definite value of $m_1$, $m_2$ can only take on a limited set of allowable values and vice versa. To put it another way: as a consequence of the selection rules, the charge and angular-momentum spectra are not completely independent.

It has been suggested by Barvinsky et al [16] that the above type of analysis can also be used to fix the fundamental constants of nature, $\hbar$ and $e$, by a Coleman-like “big-fix” mechanism [24]. Certainly, the ratio $e^2/\hbar$ must necessarily be a rational number (cf, Eq. (60)) independently of any other considerations. However, it remains unclear if this outcome is an artifact of an intrinsically semi-classical framework or a manifestation of some deep, fundamental principle of quantum gravity. We can only hope that future investigations can shed some light on this intriguing question.

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