Defect extremal surfaces for entanglement negativity

Debarshi Basu*, Himanshu Parihar†, Vinayak Raj‡, and Gautam Sengupta§

1 Department of Physics,
Indian Institute of Technology,
Kanpur 208 016, India

Abstract

We propose a doubly holographic version of the semi-classical island formula for the entanglement negativity in the framework of the defect AdS/BCFT correspondence where the AdS bulk contains a defect conformal matter theory. In this context, we propose a defect extremal surface (DES) formula for computing the entanglement negativity modified by the contribution from the defect matter theory on the end-of-the-world brane. The equivalence of the DES proposal and the semi-classical island formula for the entanglement negativity is demonstrated in AdS3/BCFT2 framework. Furthermore, in the time-dependent AdS3/BCFT2 scenarios involving eternal black holes in the lower dimensional effective description, we investigate the time evolution of the entanglement negativity through the DES and the island formulae and obtain the analogues of the Page curves.

*E-mail: debarsi@iitk.ac.in
†E-mail: himansp@iitk.ac.in
‡E-mail: vraj@iitk.ac.in
§E-mail: sengupta@iitk.ac.in
# Contents

1 Introduction 3

2 Review of earlier literature 5
   2.1 AdS$_3$/BCFT$_2$ 5
   2.2 Defect extremal surface 5
   2.3 Effective description and boundary island formula 6
   2.4 Entanglement negativity 7

3 Defect extremal surface for entanglement negativity 8

4 Entanglement negativity on a fixed time slice 10
   4.1 Two disjoint intervals 10
      4.1.1 Phase-I 10
      4.1.2 Phase-II 11
      4.1.3 Phase-III 13
   4.2 Two adjacent intervals 16
      4.2.1 Phase-I 16
      4.2.2 Phase-II 17

5 Time dependent entanglement negativity in black holes 18
   5.1 Review of the eternal black hole in AdS/BCFT 18
   5.2 Entanglement negativity between black hole interiors 20
      5.2.1 Connected phase 20
      5.2.2 Disconnected phase 23
      5.2.3 Page curve 24
   5.3 Entanglement negativity between the black hole and the radiation 25
      5.3.1 Connected phase 25
      5.3.2 Disconnected phase 27
      5.3.3 Page curve 29
   5.4 Entanglement negativity between subsystems in the radiation bath 30
      5.4.1 Connected phase 30
      5.4.2 Disconnected phase 32
      5.4.3 Page curve 35

6 Summary 36

7 Acknowledgement 36
1 Introduction

From the past few decades, the study of the black hole information loss paradox has led to several key insights about semi-classical and quantum gravity. Recently, tremendous progress has been made towards a possible resolution of this paradox which involves the appearance of regions termed “islands” in the black hole geometry at late times [1–6]. This leads to the Page curve [7–9], which indicates that the process of black hole formation and evaporation follows a unitary evolution. The appearance of the islands stems from the late time dominance of the replica wormhole saddles in the gravitational path integral for the Rényi entanglement entropy. The resultant island formula was inspired by the advent of quantum extremal surfaces (QES) introduced earlier, to compute the quantum corrections to the holographic entanglement entropy [10–13]. In this connection, in [1, 3, 5, 6] a quantum dot (e.g. SYK model) coupled to a CFT$_2$ on a half-line was regarded as the holographic dual to a 2-dimensional conformal field theory coupled to semi-classical gravity on a hybrid manifold$^1$. For such 2-dimensional conformal field theories coupled to semi-classical gravity, the island formula involves the fine-grained entropy of the Hawking radiation in a region $R$, obtained through the extremization over the entanglement entropy island region $I(R)$ and is expressed as follows

$$S[R] = \min_{I(R)} \left[ \frac{\text{Area}[\partial I(R)]}{4G_N} + S_{\text{eff}}(R \cup I(R)) \right],$$  (1.1)

where $G_N$ is the Newton’s constant and $S_{\text{eff}}(X)$ corresponds to the effective semi-classical entanglement entropy of quantum matter fields located on $X$. For recent related works, see [14–114].

A natural description for the island formulation was provided through a double holographic framework [1] where the $d$-dimensional conformal field theory coupled to semi-classical gravity may be interpreted as a lower dimensional effective description of a bulk $(d + 1)$-dimensional theory of gravity. In this scenario, the $d$-dimensional conformal field theory is considered to possess a dual bulk $(d + 1)$-dimensional gravitational theory in the AdS$_{d+1}$/CFT$_d$ framework. In the double holographic picture the computation of the entanglement entropy through the island formula in the lower dimensional theory reduces to its holographic characterization through the (H)RT formula [10, 11] in the bulk dual AdS$_{d+1}$ geometry. This may be understood as a realization of the ER=EPR proposal [115] where the island region in the black hole interior is contained within the entanglement wedge of the radiation bath through the double holographic perspective.

On a separate note, CFT$_2$s on a manifold with a boundary, termed as boundary conformal field theories (BCFT$_2$s) [116] have received considerable attention in the recent past. The holographic dual of such BCFT$_2$s [117–121] involves an asymptotically AdS$_3$ spacetime truncated by an end-of-the-world (EOW) brane $Q$ with Neumann boundary condition. An extension of this AdS$_3$/BCFT$_2$ duality studied in [43], involved additional defect conformal matter on the EOW brane $Q$ which resulted in the modification of the Neumann boundary condition. The entanglement entropy of an interval in this defect BCFT$_2$ was also computed in [43,92] through a modification of the quantum corrected RT formula. This was termed as the defect extremal surface (DES) formula as it involved contributions from the defect conformal matter fields. Interestingly, this DES formula has been proposed to be the doubly holographic counterpart of the island formula in the context of the defect AdS$_3$/BCFT$_2$ scenario [43]. The authors in [43] compared the entanglement entropy computed through the DES formula in the 3$d$ bulk geometry with that computed through the island formula in the effective 2$d$ description and found an exact agreement. Subsequently, the time dependent AdS$_3$/BCFT$_2$ scenario was studied in [92], where in the effective 2$d$ description, an eternal black hole emerges on the EOW brane. The entanglement entropy for the Hawking radiation from the eternal black hole, obtained through the DES formula reproduced the Page curve and was consistent with the island proposal.

$^1$In such holographic dual theories, the hybrid manifold on which the CFT is defined consists of a flat bath along with a curved geometry with dynamical gravity.
The fine grained entanglement entropy is a viable measure of entanglement for bipartite pure states. For configurations involving bipartite pure states in black hole geometries, the island proposal in the effective picture or the DES formula in the doubly holographic scenario correctly encode the entanglement structure of the Hawking radiation. However, entanglement entropy fails to characterize the structure of entanglement for bipartite mixed states as it receives contributions from irrelevant classical and quantum correlations. For such cases involving bipartite mixed states, it is required to consider alternative mixed state correlation or entanglement measures. Several of such correlation and entanglement measures like the reflected entropy [122,123], the entanglement negativity [124,125], the entanglement of purification [126,127] and the balanced partial entanglement entropy [128,129] have been studied in the literature.

In this context, the crucial issue of characterization of the entanglement structure of bipartite mixed states was addressed in [61] through the computation of the reflected entropy in the time dependent framework involving an eternal black hole in the AdS$_3$/BCFT$_2$ scenario. The authors proposed a 3d bulk DES formula for the reflected entropy and compared their results with the 2d effective field theory computations involving islands. They obtained the analogues of the Page curves for the reflected entropy and demonstrated the appearance of islands at late times.

The above developments bring into sharp focus the crucial issue of the characterization of the mixed state entanglement structure of the Hawking radiation from black holes. In this context, the non-convex entanglement monotone termed the entanglement negativity [124,125] serves as a natural candidate to investigate the entanglement structure of such mixed states. The entanglement negativity has been explored in conformal field theories [130–132] through appropriate replica techniques. Subsequently several holographic constructions for computing the entanglement negativity in the context of the AdS/CFT correspondence was advanced in a series of interesting works in [136–149] which reproduced the field theoretic results in the large central charge limit [134,150,151]. Interestingly, in [152–156], an alternative holographic proposal based on the bulk entanglement wedge cross-section (EWCS) was also investigated. In this connection, an island formulation for the entanglement negativity was recently established in [157] following a similar island construction for the reflected entropy developed in [26,27]. Furthermore, a geometric construction based on the double holographic framework was discussed qualitatively in [157] and subsequently investigated in [158] through a partial dimensional reduction [101] of the 3d bulk space time. In this article, we generalize these doubly holographic scenarios to the framework of AdS/BCFT with defect conformal matter on the EOW brane. We propose a DES formula for computing the bulk entanglement negativity in asymptotically AdS$_3$ geometries truncated by an EOW brane. We propose a DES formula for computing the bulk entanglement negativity in asymptotically AdS$_3$ geometries truncated by an EOW brane. Furthermore, we demonstrate the equivalence of the DES results with the corresponding island computations for the entanglement negativity of bipartite mixed states in both static and time-dependent configurations involving black hole/bath systems in the effective lower dimensional theory.

The rest of the article is organized as follows. In section 2, we recollect various aspects of the DES formula for the entanglement entropy and the corresponding effective lower dimensional picture involving the entanglement islands. In section 3, we provide the island construction for the entanglement negativity [157], and propose the DES formulas for computing the bulk entanglement negativity for disjoint and adjacent subsystems on the conformal boundary of asymptotically AdS$_3$ geometries with defect conformal matter on the EOW brane. In section 4, we compute the entanglement negativity for disjoint and adjacent intervals in a static time slice of the conformal boundary. Beginning with a brief review of the eternal black hole configuration in the 2d effective semi-classical picture, we describe DES and island computations for the entanglement negativity between interior regions of the black hole, between the black hole and radiation in the bath region, and between radiation segments, and demonstrate the equivalence of the two formulations in section 5. Finally, in section 6, we summarize our results and comment on some open problems.
on possible future directions.

2 Review of earlier literature

In this section, we will briefly recall the salient features of the holographic model under consideration. We first review the AdS/BCFT scenario [117] modified through the inclusion of conformal matter on the end-of-the-world (EOW) brane which was proposed in [43, 92]. Following this, we describe the defect extremal surface (DES) formula [43] for computing the entanglement entropy in the bulk AdS geometry truncated by the EOW brane. We will also briefly elucidate the effective 2d description of the model and the semi-classical island formula for computing the entanglement entropy of a subsystem in the effective description.

2.1 AdS$_3$/BCFT$_2$

As described in [117, 118] the bulk dual of a BCFT$_2$ defined on the half line $x \geq 0$ is given by an AdS$_3$ geometry truncated by an EOW brane $\mathcal{Q}$ with Neumann boundary conditions. The gravitational action of the bulk manifold $\mathcal{N}$ is given by

$$I = \int_{\mathcal{N}} \sqrt{-g} (R - 2\Lambda) + 2 \int_{\mathcal{Q}} \sqrt{-h} (K - T),$$

(2.1)

where $h_{ab}$ is the induced metric, $K$ is the trace of the extrinsic curvature $K_{ab}$ on the EOW brane $\mathcal{Q}$ with a tension $T$. The Neumann boundary condition on the EOW brane is given as $K_{ab} = (K - T) h_{ab}$. The 3d bulk geometry may be described by two sets of relevant coordinate charts, $(t, x, z)$ and $(t, \rho, y)$, which are related through

$$x = y \tanh \left( \frac{\rho}{\ell} \right), \quad z = -y \text{sech} \left( \frac{\rho}{\ell} \right).$$

(2.2)

The bulk metric in these coordinates is given by the standard Poincaré slicing, as follows

$$ds^2 = d\rho^2 + \cosh^2 \left( \frac{\rho}{\ell} \right) \frac{-dt^2 + dy^2}{y^2} = \frac{\ell^2}{z^2} (-dt^2 + dx^2 + dz^2),$$

(2.3)

where $\ell$ is the AdS$_3$ radius. In the Poincaré slicing described by the $(t, \rho, y)$ coordinate chart the EOW brane is situated at a constant $\rho = \rho_0$ slice and the induced metric on the brane is given by that of an AdS$_2$ geometry [117].

An extension to this usual AdS$_3$/BCFT$_2$ framework was proposed in [43] where one essentially begins with an orthogonal brane with zero tension and through the addition of conformal matter onto it, turns on a finite tension. The Neumann boundary condition on the EOW brane $\mathcal{Q}$ is modified by the stress tensor of this defect CFT$_2$. The EOW brane $\mathcal{Q}$ is then treated as a defect in the bulk geometry.

2.2 Defect extremal surface

For the modified bulk picture with defect conformal matter on $\mathcal{Q}$, the entanglement entropy of an interval $A$ in the original BCFT$_2$ involves contributions from the defect matter, and the usual RT formula [10] is modified to the defect extremal surface (DES) formula [43, 92] given as

$$S_{\text{DES}}(A) = \min_{A, X} \left[ \frac{\mathcal{A}(\Gamma_A)}{4G_N} + S_{\text{defect}}(D) \right], \quad X = \Gamma_A \cap D,$$

(2.4)

4A convenient choice for a polar coordinate is $\theta = \arccos \left( \text{sech} \left( \frac{\rho}{\ell} \right) \right)$, which determines the angular position of the brane from the vertical as shown in fig. 1.
where $\Gamma_A$ is a co-dimension two extremal surface homologous to the subsystem $A$ and $D$ is the defect region along the EOW brane $Q$ as depicted in fig. 1.

For an interval $A = [0, L]$ in the BCFT$_2$, the generalized entanglement entropy corresponding to a defect $D = [-a, 0]$ on the brane CFT$_2$ may be computed through the DES formula as follows\footnote{Note that we are using the standard geodesic length formula for Poincaré AdS$_3$ instead of the AdS/BCFT techniques employed in [43,92] as both the procedures lead to the same answer and are therefore complementary.} [43]

\[
S_{\text{gen}}(a) = \frac{A(\Gamma_A)}{4G_N} + S_{\text{defect}}([-a, 0]) = \frac{\ell}{4G_N} \cosh^{-1} \left[ \frac{(L + a \sin \theta_0)^2 + (a \cos \theta_0)^2}{2\epsilon a \cos \theta_0} \right] + \frac{c}{6} \log \left( \frac{2\ell}{\epsilon \cos \theta_0} \right), \tag{2.5}
\]

Note that the defect contribution to the generalized entropy is a constant which implies that the defect extremal surface is same as the RT surface for the subsystem $A$. Extremization with respect to the position $a$ of the defect leads to the entanglement entropy of the subsystem $A$ as follows

\[
S_{\text{DES}}([0, L]) = \frac{c}{6} \left[ \log \left( \frac{2L}{\epsilon} \right) + \tanh^{-1}(\sin \theta_0) + \log \left( \frac{2\ell}{\epsilon \cos \theta_0} \right) \right]. \tag{2.6}
\]

where both the central charges of the original BCFT and the defect CFT$_2$ are taken to be equal\footnote{Note that, the equality of the two central charges is essential in order to relate the present bulk description to the effective $2d$ island scenario which involves a CFT$_2$ on the complete hybrid manifold.} to $c$.

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Schematics of the defect extremal surface for the entanglement entropy of a subsystem A. Figure modified from [43].}
\end{figure}\]

\subsection*{2.3 Effective description and boundary island formula}

The lower dimensional effective semi-classical theory for the bulk configuration described above may be obtained through a combination of a partial Randall-Sundrum reduction \cite{159,160} and the usual AdS/BCFT duality \cite{161}. As described in \cite{43,61,92}, this is implemented by dividing the AdS$_3$ bulk into two parts through the insertion of an imaginary co-dimension one surface $Q'$ orthogonal to the asymptotic boundary, with transparent boundary conditions. The portion of the bulk enclosed between $Q$ and $Q'$ is dimensionally reduced along the $\rho$ direction using a partial Randall-Sundrum reduction thereby obtaining an effective $2d$ gravitational theory coupled with the matter CFT$_2$ on $Q$. On the other hand, the rest of the bulk is dual to the original BCFT$_2$ on the half line $x \geq 0$ from the usual AdS/BCFT duality. The transparent boundary conditions along $Q'$ naturally glues the gravity theory on $Q$ and the BCFT$_2$ on the half line.

Note that, the equality of the two central charges is essential in order to relate the present bulk description to the effective $2d$ island scenario which involves a CFT$_2$ on the complete hybrid manifold.
$x \geq 0$, leading to an effective 2d semi-classical theory on a hybrid manifold, similar to that considered in [1,3].

In the effective semi-classical description described above, one may utilize the island formula [1,3] to compute the entanglement entropy. For a subsystem $A = [0, L]$ in the flat CFT$_2$ on the asymptotic boundary, an island region $I_A = [-a, 0]$ appears in the gravitational sector on the EOW brane $Q$. The entanglement entropy is obtained by extremizing the generalized entropy functional as

$$S_{\text{bdy}} = \operatorname{ext}_a S_{\text{gen}}(a) = \operatorname{ext}_a [A(y = -a) + S_{\text{matter}}([-a, L])]$$

$$= \frac{c}{6} \tanh^{-1}(\sin \theta_0) + \frac{c}{6} \log \left( \frac{4L^\ell}{\epsilon \epsilon_y \cos \theta_0} \right).$$

(2.7)

The first term in the above expression is due to the constant area of the quantum extremal surface in the AdS$_3$/BCFT$_2$ framework, given as [43]

$$A(\partial I_A) \equiv \rho_0 = \frac{\ell}{4G_N} \tanh^{-1}(\sin \theta_0),$$

(2.8)

where $\theta_0$ is the angle of the EOW brane with the vertical. It is observed from the above that the island formula leads to the same expression for the entanglement entropy as the DES result in eq. (2.6). In other words, the DES formula may be considered as the doubly holographic counterpart of the island formula in the defect AdS/BCFT framework.

### 2.4 Entanglement negativity

In this subsection, we will briefly review the salient features of the mixed state entanglement measure termed the entanglement negativity and its holographic characterization in the context of AdS$_3$/CFT$_2$ scenario. In a seminal work [124], Vidal and Werner introduced the computable mixed state entanglement measure, the entanglement negativity which is defined as the trace norm of the density matrix partially transposed with respect to one of the subsystems. In [130–132], replica techniques were developed to obtain the entanglement negativity for subsystems in CFT$_2$s which involved the even parity $n_e$ of the replica index. The entanglement negativity was obtained through the analytic continuation of the replica index $n_e \to 1$ as follows

$$E = \lim_{n_e \to 1} \log \operatorname{Tr}(\rho_{AB}^T)^{n_e},$$

(2.9)

where the superscript $T_B$ denotes partial transposition with respect to the subsystem $B$. The trace $\operatorname{Tr}(\rho_{AB}^T)^{n_e}$ may be expressed as a twist field correlator in the CFT$_2$, corresponding to the bipartite state under consideration. As an example, we consider the generic bipartite mixed state of two disjoint intervals $A = [u_1, v_1]$ and $B = [u_2, v_2]$ in a CFT$_2$. The trace $\operatorname{Tr}(\rho_{AB}^T)^{n_e}$ is then given by the following four-point correlator of twist fields,

$$\operatorname{Tr}(\rho_{AB}^T)^{n_e} = \langle \mathcal{T}_{n_e}(u_1) \mathcal{T}_{n_e}(v_1) \mathcal{T}_{n_e}(u_2) \mathcal{T}_{n_e}(v_2) \rangle_{\text{CFT}\otimes n_e},$$

(2.10)

where the twist fields $\mathcal{T}_{n_e}$ and $\mathcal{T}_{n_e}$ are primary fields with conformal dimensions

$$\Delta_{n_e} = \frac{c}{12} \left( n_e - \frac{1}{n_e} \right).$$

(2.11)

Subsequently, in a series of works [137–139,144], several holographic proposals for the entanglement negativity were proposed for specific bipartite mixed states. These proposals involved appropriate algebraic sums of the lengths of codimension two bulk static minimal surfaces homologous to various subsystems describing the mixed state. In particular, for two disjoint intervals
$A$ and $B$ sandwiching another interval $C$ in a CFT$_2$, the holographic entanglement negativity may be obtained geometrically in the context of the AdS$_3$/CFT$_2$ correspondence as follows [144]

$$\mathcal{E}(A:B) = \frac{3}{16 G_N} (L_{AC} + L_{BC} - L_C - L_{ABC}),$$

(2.12)

where $L_X$ denotes the length of the extremal curve homologous to subsystem $X$. The configuration of two adjacent intervals $A$ and $B$ may be obtained through the limit $C \to \emptyset$ of the above, and the holographic entanglement negativity is given as [139]

$$\mathcal{E}(A:B) = \frac{3}{16 G_N} (L_A + L_B - L_{AB}).$$

(2.13)

Note that, these proposals have further been extended to various other holographic frameworks including flat holography [134], anomalous AdS/CFT [149] as well as higher dimensional scenarios [140,146–148].

## 3 Defect extremal surface for entanglement negativity

In this section, we propose the defect extremal surface (DES) formula for the entanglement negativity in the AdS/BCFT models which include defect conformal matter on the EOW brane [43,61,92]. To begin with, we recall the semi-classical QES formula for the entanglement negativity involving entanglement islands in the lower dimensional effective picture discussed earlier. As described in [157, 158], the QES proposal for the entanglement negativity between two disjoint intervals in the effective boundary description

$$\mathcal{E}^{\text{bdy}}(A:B) = \min_{\Gamma = \partial I_A \cap \partial I_B} \left[ \frac{3}{16 G_N} \left( A(\partial I_A) + A(\partial I_B) - A(\partial I_{AB}) \right) + \mathcal{E}^{\text{eff}}(A \cup I_A : B \cup I_B) \right],$$

(3.1)

where $I_A$ and $I_B$ are the entanglement negativity islands corresponding to subsystems $A$ and $B$, respectively. The entanglement negativity islands obeys the condition $I_A \cup I_B = I(A \cup B)$, where $I(A \cup B)$ denotes the entanglement entropy island for $A \cup B$, as illustrated in fig. 2. Furthermore, the extremization in the QES formula is performed over the location of the island cross-section $\Gamma \equiv \partial I_A \cap \partial I_B$. In this context, utilizing the constraint $I_A \cup I_B = I(A \cup B)$, the algebraic sum of the area contributions in eq. (3.1) may be reduced to that corresponding to the island cross-section $\Gamma$. Hence, the QES formula may be expressed as [157]

$$\mathcal{E}^{\text{bdy}}(A:B) = \min_{\Gamma} \left[ \frac{3}{8 G_N} A(\Gamma = \partial I_A \cap \partial I_B) + \mathcal{E}^{\text{eff}}(A \cup I_A : B \cup I_B) \right].$$

(3.2)

Inspired by the holographic characterizations for the entanglement negativity described earlier, we now propose DES formulae to obtain the entanglement negativity in the doubly holographic framework of the defect AdS$_3$/BCFT$_2$ scenario. In the presence of the bulk defect theory, the entanglement negativity for a bipartite mixed state $\rho_{AB}$ in the dual BCFT$_2$ involves corrections from the bulk matter fields. Following [12, 13], the effective matter contribution is given by the bulk entanglement negativity between the regions $A$ and $B$ which are obtained by splitting the codimension one region dual to $\rho_{AB}$ via the entanglement wedge cross section. For

---

7Note that, in this article, we use the nomenclature boundary description and lower dimensional effective description interchangeably.

8Note that a defect extremal surface formula for the reflected entropy was developed in [61] utilizing a similar construction. Furthermore, the authors in [61] demonstrated the equivalence of the DES and QES formulae for the reflected entropy in the framework of defect AdS$_3$/BCFT$_2$.
the bipartite mixed state configuration described by two disjoint intervals $A$ and $B$ in the dual CFT$_2$, the 3d bulk dual DES formula for the entanglement negativity is therefore given by

$$\mathcal{E}^{\text{bulk}}(A : B) = \min_{\Gamma} \operatorname{Ext} \left[ \frac{3}{16G_N} \left( \mathcal{L}(\gamma_A) + \mathcal{L}(\gamma_B) - \mathcal{L}(\gamma_{AB}) \right) + \mathcal{E}^{\text{eff}}(A : B) \right], \quad (3.3)$$

where $\mathcal{L}(\gamma_X)$ is the length of the bulk extremal curve homologous to the interval $X$ on the boundary CFT$_2$ and $\mathcal{E}^{\text{eff}}(A : B)$ denotes the effective entanglement negativity between the quantum matter fields inside the bulk regions $A$ and $B$. The bulk effective term in eq. (3.3) reduces to the effective entanglement negativity between the entanglement negativity islands $I_A$ and $I_B$ on the EOW brane as the conformal matter is present only on the EOW brane. Note that if the intervals are far away such that their entanglement wedges are disconnected, the contributions coming from the combination of bulk extremal curves vanishes identically due to phase transitions to other entropy saddles [145].

The DES formula for two adjacent intervals $A$ and $B$ in the bulk description may be obtained from eq. (3.3) through the limit $C \rightarrow \emptyset$ as follows

$$\mathcal{E}^{\text{bulk}}(A : B) = \min_{\Gamma} \operatorname{Ext} \left[ \frac{3}{16G_N} \left( \mathcal{L}(\gamma_A) + \mathcal{L}(\gamma_B) - \mathcal{L}(\gamma_{AB}) \right) + \mathcal{E}^{\text{eff}}(A : B) \right]. \quad (3.4)$$
In the following we will compute the entanglement negativity for various bipartite mixed states in a defect BCFT through the island and the DES formulae and find exact agreement between the bulk and the boundary results.

4 Entanglement negativity on a fixed time slice

4.1 Two disjoint intervals

In this subsection we focus on the computation of the entanglement negativity for the bipartite mixed state of two disjoint intervals \( A = [b_1, b_2] \) and \( B = [b_3, \infty] \) on a static time-slice in the defect AdS\(_3\)/BCFT\(_2\) framework. There are three possible phases for the entanglement negativity for this mixed state configuration based on the subsystem sizes, which we investigate below.

4.1.1 Phase-I

Boundary description

In this phase, the interval \( C \) separating the two disjoint intervals \( A \) and \( B \) is large\(^9\) and the interval \( A \) is small enough such that it does not possess an entanglement entropy island. Consequently, there is no non-trivial island cross-section on the EOW brane as shown in fig. 4. Hence \( \Gamma = \emptyset \), and the area term in the QES formula eq. (3.2) vanishes, namely \( A(\Gamma) = 0 \).

\[
\text{Figure 4: Schematics of the defect extremal surface for the entanglement negativity between two disjoint intervals } A \text{ and } B \text{ in phase-I.}
\]

The effective semi-classical entanglement negativity in this phase may be obtained through a correlation function of twist operators located at the endpoints of the intervals as follows

\[
\mathcal{E}^{\text{eff}}(A : B \cup I_B) = \lim_{n_e \to 1} \log \left[ (\epsilon_y \Omega(-b_3))^{\Delta n_e} \left\langle T_{n_e}(b_1) T_{n_e}(b_2) T_{n_e}(b_3) T_{n_e}(-b_3) \right\rangle_{\text{CFT} \otimes n_e} \right]
\]

\[
\approx \lim_{n_e \to 1} \log \left[ (\epsilon_y \Omega(-b_3))^{\Delta n_e} \left\langle T_{n_e}(b_1) T_{n_e}(b_2) \right\rangle_{n_e} \left\langle T_{n_e}(b_3) T_{n_e}(-b_3) \right\rangle_{n_e} \right]
\]

\[
= 0,
\]

(4.1)

where \( \epsilon_y \) is the UV cut-off on the EOW brane \( Q \) and the warp factor \( \Omega \) is given by [43]

\[
d_s^{2 \text{brane}} = \Omega^{-2(y)} d_s^{2 \text{flat}}, \quad \Omega(-b_3) = \left| b_3 \cos \theta_0 \ell \right|.
\]

In the second equality of eq. (4.1), we have factorized the given four-point function utilizing the corresponding OPE channels. Consequently, in this phase the total entanglement negativity for the two disjoint intervals in the boundary description is vanishing.

\(^9\)Note that, in this phase the interval \( C \) has an entanglement island. In the bulk description, this corresponds to a disconnected entanglement wedge for \( A \cup B \).
**Bulk description**

The dual bulk description for this phase has a disconnected entanglement wedge and hence we have $\Gamma = \emptyset$ similar to the boundary description. Furthermore, as the bulk matter fields are only localized on the EOW brane $Q$ and $A$ has no corresponding island, the effective entanglement negativity between bulk quantum matter fields also vanishes as follows

$$\mathcal{E}_{\text{eff}}(A : B) = \mathcal{E}_{\text{eff}}(\emptyset : I_B) \equiv 0. \quad (4.3)$$

Hence, in the bulk description the holographic entanglement negativity is entirely given by the contribution from the areas of the defect extremal surfaces. The lengths of the bulk DES homologous to various subsystems are given by

$$L_{AC} = L_1 + L_3, \quad L_{BC} = L_2 + L_4, \quad L_C = L_2 + L_3, \quad L_{ABC} = L_1 + L_4. \quad (4.4)$$

Now utilizing the bulk DES formula for the entanglement negativity for two disjoint intervals in eq. (3.3), we obtain

$$\mathcal{E}_{\text{bulk}}(A : B) = \frac{3}{16G_N} (L_{AC} + L_{BC} - L_C - L_{ABC}) = 0. \quad (4.5)$$

Therefore, the boundary and bulk description match trivially, leading to a vanishing entanglement negativity in this phase.

**4.1.2 Phase-II**

**Boundary description**

Next we turn our attention towards the phase where the interval $A$ still does not possess an island, but the interval $C$ sandwiched between $A$ and $B$ is small and is therefore does not lead to an entanglement entropy island as well (cf. footnote 9). In this phase, there is no non-trivial island cross-section as depicted in fig. 5 and hence the area term in eq. (3.2) vanishes identically. On the other hand, the effective semi-classical entanglement negativity is given by

$$\mathcal{E}_{\text{eff}}(A : B \cup I_B) = \lim_{n_e \to 1} \log \left( (\epsilon y \Omega(-b_1))^\Delta_{n_e} \langle \mathcal{T}_{n_e}(-b_1)\mathcal{T}_{n_e}(b_1)\tilde{\mathcal{T}}_{n_e}(b_2)\tilde{\mathcal{T}}_{n_e}(b_3) \rangle_{\text{CFT}} \right)_{\otimes n_e}. \quad (4.6)$$

As described in [144, 145, 150], in the large-central charge limit the above four-point correlation function of the twist operators has the following form

$$\langle \mathcal{T}_{n_e}(-b_1)\mathcal{T}_{n_e}(b_1)\tilde{\mathcal{T}}_{n_e}(b_2)\tilde{\mathcal{T}}_{n_e}(b_3) \rangle_{\text{CFT}} = (1 - x)^{\hat{\Delta}} \quad (4.7)$$

where the conformal dimension $\hat{\Delta}$ corresponding to the dominant Virasoro conformal block, and the cross-ratio $x$ are given as

$$\hat{\Delta} = \frac{c}{6} \left( \frac{n_e}{2} - \frac{2}{n_e} \right), \quad x = \frac{(b_2 - b_1)(b_3 + b_1)}{(b_2 + b_1)(b_3 - b_1)}. \quad (4.8)$$

We may now obtain the the entanglement negativity for this phase in the boundary description by substituting eqs. (4.7) and (4.8) in eq. (4.6) to be

$$\mathcal{E}_{\text{bdy}}(A : B) = \frac{c}{4} \log \left[ \frac{(b_1 + b_2)(b_3 - b_1)}{2b_1(b_3 - b_2)} \right]. \quad (4.9)$$
Bulk description

From the bulk perspective, in this phase the entanglement wedge corresponding to $A \cup B$ is connected. However, as the interval $A$ does not have an island, the minimal entanglement wedge cross-section does not meet the EOW brane $Q$ resulting in a trivial island cross section $\Gamma = \emptyset$. Hence, the effective entanglement negativity between the bulk quantum matter fields vanishes similar to eq. (4.3).

The bulk entanglement negativity consists of the contributions from the combination of the defect extremal surfaces as depicted in fig. 5b. Now utilizing eq. (3.3), we may obtain the entanglement negativity between $A$ and $B$ in this phase as follows

$$E^\text{bulk}(A : B) = \frac{3}{16G_N} [L_3 + (L_2 + L_5) - L_4 - (L_1 + L_5)]$$

$$= \frac{3}{16G_N} (L_2 + L_3 - L_1 - L_4) \quad (4.10)$$

In the framework of defect AdS$_3$/BCFT$_2$ [43, 61, 92], it was observed that the defect extremal surfaces have the same structure as the corresponding RT surfaces since the contribution from the defect matter fields turned out to be constant. The lengths of the defect extremal surfaces $L_3$ and $L_4$ in eq. (4.10) are given by [10, 117]

$$L_3 = 2\ell \log \left(\frac{b_3 - b_1}{\epsilon}\right), \quad L_4 = 2\ell \log \left(\frac{b_3 - b_2}{\epsilon}\right), \quad (4.11)$$

where $\epsilon$ is a UV cut-off in the dual BCFT$_2$. As described in [43, 92], the length of the defect extremal surface $L_1$ ending on the brane $Q$ is given by

$$L_1 = \ell \log \left(\frac{2b_1}{\epsilon}\right) + \ell \tanh^{-1}(\sin \theta_0). \quad (4.12)$$

Furthermore, the length of the defect extremal surface $L_2$ may be obtained as follows [10]

$$L_2 = \ell \cosh^{-1} \left[\frac{(b_2 + b_1 \sin \theta_0)^2 + (b_1 \cos \theta_0)^2}{2\epsilon(b_1 \cos \theta_0)}\right]. \quad (4.13)$$

Note that, in this phase the interval $A$ is very small and therefore we may approximate the above length in the following way

$$L_2 = \ell \cosh^{-1} \left(\frac{b_2^2 + b_1^2 + 2b_1b_2 \sin \theta_0}{b_2^2 - b_1^2}\right) + \ell \log \left(\frac{b_2^2 - b_1^2}{\epsilon b_1}\right). \quad (4.14)$$
The semi-classical effective entanglement negativity may be obtained in terms of the following :

Now utilizing the identity \( \cosh^{-1} x + \cosh^{-1} y = \cosh^{-1} \left( xy + \sqrt{(x^2 - 1)(y^2 - 1)} \right) \) we finally obtain:

\[
\mathcal{L}_2 = \ell \left[ \cosh^{-1} \left( \frac{b_2^2 - b_1^2}{b_2^2 - b_1^2} \right) + \log \left( \frac{b_2^2 - b_1^2}{eb_1} \right) + \cosh^{-1} \left( \frac{1}{\cos \theta_0} \right) \right]
\]

\[
= \ell \log \left( \frac{(b_1 + b_2)^2}{eb_1} \right) + \ell \tanh^{-1} (\sin \theta_0) .
\]  

(4.15)

Substituting eqs. (4.11) and (4.15) in eq. (4.10) we may now obtain the entanglement negativity between \( A \) and \( B \) in the bulk description as follows:

\[
\mathcal{E}_{\text{bulk}}(A : B) = \frac{3\ell}{4G_N} \log \left[ \frac{(b_1 + b_2)(b_3 - b_1)}{2b_1(b_3 - b_2)} \right].
\]  

(4.16)

Upon employing the Brown-Henneaux formula [162], we observe an exact matching with the island result in eq. (4.9).

4.1.3 Phase-III

Boundary description

In the final phase both the intervals \( A \) and \( B \) are large enough to possess entanglement islands \( I_A \equiv [-a, -a'] \) and \( I_B \equiv [-a', -\infty) \) respectively. They are also considered to be in proximity such that they have a connected entanglement wedge as shown in fig. 6. The area term in eq. (3.2) for the island cross-section \( \Gamma \equiv \partial I_A \cap \partial I_B \) is then given as [43, 61, 92]

\[
\mathcal{A}(\Gamma) = \frac{\ell}{4G_N} \tanh^{-1} (\sin \theta_0) ,
\]  

(4.17)

The semi-classical effective entanglement negativity may be obtained in terms of the following five-point twist correlator

\[
\mathcal{E}_{\text{eff}}^\text{(A \cup I_A : B \cup I_B)} = \lim_{n_e \to 1} \log \left[ (\epsilon_y \Omega(-a))^\Delta_{n_e} (\epsilon_y \Omega(-a'))^\Delta_{n_e} \right] \times \langle \mathcal{T}_{n_e}(b_1) \mathcal{T}_{n_e}(-a) \mathcal{T}_{n_e}(b_2) \mathcal{T}_{n_e}^2(-a') \mathcal{T}_{n_e}(b_3) \rangle_{\text{CFT} \otimes \hat{n}_e},
\]  

(4.18)

where \( \epsilon_y \) is a UV regulator on the AdS_2 brane \( \mathbb{Q} \) and the warp factor \( \Omega(-a') \) is given in eq. (4.2). The five-point twist correlator in eq. (4.18) have the following factorization [157] in the corresponding OPE channel

\[
\langle \mathcal{T}_{n_e}(b_1) \mathcal{T}_{n_e}(-a) \mathcal{T}_{n_e}(b_2) \mathcal{T}_{n_e}^2(-a') \mathcal{T}_{n_e}(b_3) \rangle \approx \langle \mathcal{T}_{n_e}(b_1) \mathcal{T}_{n_e}(-a) \rangle \langle \mathcal{T}_{n_e}(b_2) \mathcal{T}_{n_e}^2(-a') \mathcal{T}_{n_e}(b_3) \rangle \]

\[
= \frac{1}{(a + b_1)^{2\Delta_{\hat{n}_e}} (b_3 + a')^{\Delta_{\hat{n}_e}} (b_2 + a')^{\Delta_{\hat{n}_e}} (b_2 - b_2)^{2\Delta_{\hat{n}_e} - \Delta_{\hat{n}_e}^{(2)}}}
\]  

(4.19)

where \( \Delta_{\hat{n}_e} \) and \( \Delta_{\hat{n}_e}^{(2)} \) are the conformal dimensions of the twist operators \( \mathcal{T}_{n_e} \) and \( \mathcal{T}_{n_e}^2 \) respectively and are given as [130, 131]

\[
\Delta_{n_e} = \frac{c}{12} \left( 1 - \frac{1}{n_e} \right) , \quad \Delta_{n_e}^{(2)} = \frac{c}{6} \left( \frac{n_e}{2} - \frac{2}{n_e} \right) .
\]  

(4.20)

Note that the point \( a \) on the brane is determined by the DES for the subsystem \( A \) to be \( a = b_1 \) [43]. Therefore, by utilizing the contractions in (4.19) along with the areas term in
Figure 6: Schematics of the quantum extremal surface for the entanglement negativity between two disjoint intervals $A$ and $B$ in phase-III. In this phase, we have a non-trivial island cross-section on the brane at coordinate $a'$. 

According to eq. (4.17), we may obtain the generalized negativity in the boundary description from eq. (3.1) to be

$$
E_{\text{bdy}}^{\text{gen}} (A : B) = \frac{c}{4} \left[ \tanh^{-1} (\sin \theta_0) + \log \frac{\ell (b_2 + a') (b_3 + a')}{a' (b_3 - b_2) \epsilon_y \cos \theta_0} \right].
$$

(4.21)

The extremization with respect to the island cross-section $\Gamma$ with the coordinate $a'$ on the brane leads to

$$
\partial_{a'} E_{\text{bdy}}^{\text{gen}} = 0 \implies a' = \sqrt{b_2 b_3}.
$$

(4.22)

Substituting this into eq. (4.21), we may obtain the total entanglement negativity between $A$ and $B$ in phase-III from the boundary description to be

$$
E_{\text{bdy}} (A : B) = \frac{c}{4} \left[ \tanh^{-1} (\sin \theta_0) + \log \left( \frac{\sqrt{b_3} + \sqrt{b_2}}{\sqrt{b_3} - \sqrt{b_2}} \right) + \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) \right].
$$

(4.23)

**Bulk description**

The bulk description in phase-III consists of a connected entanglement wedge and the minimal cross-section ends on the EOW brane. The configuration is sketched in fig. 7. Since the bulk quantum matter is entirely situated on the EOW brane, the effective entanglement negativity between the bulk quantum matter fields in the bulk regions $A$ and $B$ reduces to the effective matter negativity between the corresponding island regions $I_A$ and $I_B$,

$$
\mathcal{E}^{\text{eff}} (A : B) \equiv \mathcal{E}^{\text{eff}} (I_A : I_B) = \lim_{n_e \to 1} \log \left( \epsilon_y \Omega (-a') \Delta^{(2)}_{n_e} \langle T_{n_e} (-b_1) T_{n_e}^2 (-a') \rangle_{\text{BCFT} \otimes n_e} \right),
$$

where $\epsilon_y$ is the UV cut-off on the EOW brane and $\Omega$ is the conformal factor as given in eq. (4.2). Utilizing the doubling trick $[116,120]$ the above two-point function in the defect BCFT$_2$ may be reduced to a four-point correlator of chiral twist fields in a CFT$_2$ defined on the whole complex plane. As described in $[120]$, the four-point correlator in the chiral CFT$_2$ has two dominant channels depending on the cross-ratio as follows.

**I. BOE channel:** In this channel the two point correlator factorizes into two one-point functions in the BCFT$_2$ as follows

$$
\langle T_{n_e} (b_1) T_{n_e}^2 (-a') \rangle_{\text{BCFT} \otimes n_e} = \langle T_{n_e} (-b_1) \rangle_{\text{BCFT} \otimes n_e} \langle T_{n_e}^2 (-a') \rangle_{\text{BCFT} \otimes n_e} = \frac{\epsilon_y \Delta_{n_e} + \Delta^{(2)}_{n_e}}{(2b_1) \Delta_{n_e} (2a') \Delta^{(2)}_{n_e}}.
$$

(4.25)
Therefore, the effective bulk entanglement negativity in this phase is given by

$$\mathcal{E}^{\text{eff}}(I_A : I_B) = \frac{c}{4} \log \frac{2\ell}{\epsilon_y \cos \theta_0}.$$  

(4.26)

Note that this effective entanglement negativity is equal to the Rényi entropy of order half for the interval $I_A$ (or $I_B$) which is consistent with the expectations from quantum information theory.

II. OPE channel: In this channel, the two-point correlator of twist fields on the BCFT$_2$ reduces to a three-point correlator of chiral twist fields on the full complex plane as follows [61, 116, 120]

$$\langle \mathcal{T}_{n_e}(-b_1) \bar{T}_{n_e}^{2}(-a') \rangle_{\text{BCFT} \otimes \bar{n_e}} = \langle \mathcal{T}_{n_e}(-b_1) \mathcal{T}_{n_e}(b_1) \bar{T}_{n_e}^{2}(-a') \rangle_{\text{CFT} \otimes \bar{n_e}} = \frac{C_{\mathcal{T}_{n_e} \mathcal{T}_{n_e} \bar{T}_{n_e}}}{(a'^2 - b_1^2)\Delta_{n_e}^{(2)} (2b_1)\Delta_{n_e}^{(2)} - 2\Delta_{n_e}}.$$  

(4.27)

Therefore, the effective bulk entanglement negativity in this channel is given by

$$\mathcal{E}^{\text{eff}}(I_A : I_B) = \frac{c}{4} \log \left[ \frac{\ell (a'^2 - b_1^2)}{a'b_1\epsilon_y \cos \theta_0} \right].$$  

(4.28)

Figure 7: Schematics of the defect extremal surface for the entanglement negativity between two disjoint intervals $A$ and $B$ in phase-III. In this phase, the EWCS ends on the island cross-section $\Gamma$ on the EOW brane.

As shown in fig. 7, the contribution to the bulk entanglement negativity from the defect extremal surfaces homologous to different combinations of subsystems is given by

$$\frac{3}{16G_N} (\mathcal{L}_2 + \mathcal{L}_4 - \mathcal{L}_3) = \frac{3\ell}{16G_N} \left[ \cosh^{-1} \left \{ \frac{(b_2 + a' \sin \theta_0)^2 + (a' \cos \theta_0)^2}{2\epsilon(a' \cos \theta_0)} \right \} + \cosh^{-1} \left \{ \frac{(b_3 + a' \sin \theta_0)^2 + (a' \cos \theta_0)^2}{2\epsilon(a' \cos \theta_0)} \right \} ight. \left. - 2\log \left( \frac{b_3 - b_2}{\epsilon} \right) \right]$$  

(4.29)

The entanglement negativity between the disjoint intervals $A$ and $B$ is obtained by extremizing the generalized negativity over the position of the island cross-section $\Gamma$. For the OPE channel of the effective bulk entanglement negativity there is no extremal solution while for the BOE channel we obtain

$$\partial_{a'} \mathcal{E}^{\text{bulk}} = 0 \quad \implies \quad a' = \sqrt{b_2 b_3}$$  

(4.30)
Substituting this and utilizing the proximity limit \( b_3 \to b_2 \) in the intermediate step, we obtain the entanglement negativity between \( A \) and \( B \) in the bulk description as follows

\[
\mathcal{E}^{\text{bulk}} = \frac{3\ell}{16G_N} \left[ \cosh^{-1} \left( \frac{b_2 + b_3 + 2\sqrt{b_2 b_3} \sin \theta_0}{(b_3 - b_2) \cos \theta_0} \right) \right] + \frac{c}{4} \log \left( \frac{2\ell}{\epsilon_y \cos \theta_0} \right)
\]

The above expression for the holographic entanglement negativity matches exactly with the QES result in eq. (4.23) obtained through the island formula eq. (3.1). This provides yet another consistency check of our holographic construction for the entanglement negativity in the defect AdS\(_3\)/BCFT\(_2\) scenario.

### 4.2 Two adjacent intervals

Having computed the entanglement negativity for configurations involving two disjoint intervals, we now turn our attention to the mixed state of two adjacent intervals \( A = [0, b_1] \) and \( B = [b_1, b_2] \) on a fixed time-slice in the AdS\(_3\)/BCFT\(_2\) model. The interval \( A \) in this case always possess an entanglement island as it starts from the interface between the EOW brane and the asymptotic boundary. We however, have two possible phases for this case based on the size of the interval \( B \) which are described below.

#### 4.2.1 Phase-I

**Boundary description**

For this phase, we consider that the interval \( B \) is large enough to possess an entanglement island described as \( I_B \) in fig. 8. The area term in eq. (3.2) for the point \( \Gamma = \partial I_A \cap \partial I_B \) is as given in eq. (2.8). The effective semi-classical entanglement negativity is given by the following two point twist correlator

\[
\mathcal{E}^{\text{eff}}(A \cup I_A : B \cup I_B) = \lim_{n_e \to 1} \log \left[ \epsilon_y \Omega(-a) \langle T_{n_e}^2(b_1)\bar{T}_{n_e}^2(-a) \rangle_{\text{CFT}^{\otimes n_e}} \right] = \frac{c}{4} \log \left( \frac{\ell(b_1 + a)^2}{\epsilon \epsilon_y a \cos \theta_0} \right),
\]

where \( \epsilon \) and \( \epsilon_y \) are the UV cut-offs on the asymptotic boundary and the EOW brane \( \mathcal{Q} \) respectively, and the warp factor \( \Omega(a) \) is as given in eq. (4.2). The point \( a = b_1 \) on the brane is determined through the entanglement entropy computation of the interval \( A \). Using this in eq. (4.32) along with the area term, we may obtain the total entanglement negativity in the boundary description to be

\[
\mathcal{E}^{\text{bdy}}(A : B) = \frac{c}{4} \left[ \log \left( \frac{2b_1}{\epsilon} \right) + \log \left( \frac{2\ell}{\epsilon_y \cos \theta_0} \right) + \tanh^{-1}(\sin \theta_0) \right],
\]

where we have used the Brown-Henneaux formula in the area term [162].

**Bulk description**

In the double holographic description, the entanglement wedge corresponding to the subsystem \( A \cup B \) is connected in the bulk. The contribution to the effective entanglement negativity between the bulk matter fields in regions \( A \) and \( B \) arises solely from the quantum matter fields situated
Figure 8: Schematics of the defect extremal surface for the entanglement negativity between two adjacent intervals $A$ and $B$ in phase-I. $I_A$ and $I_B$ on the EOW brane describe the entanglement island corresponding to intervals $A$ and $B$ respectively.

on the EOW brane as follows

$$
\mathcal{E}^{\text{eff}}(A : B) = \mathcal{E}^{\text{eff}}(I_A : I_B) = \lim_{n_e \to 1} \log \left[ (\epsilon y \Omega(-a))^\Delta \langle T_{n_e}(-a')\overline{T}_{n_e}(-a) \rangle_{\text{BCFT} \otimes n_e} \right] 
= \frac{c}{4} \log \left( \frac{2\ell}{\epsilon y \cos \theta_0} \right). 
$$

(4.34)

Utilizing eq. (3.4), the total entanglement negativity for this case including the contribution from the combinations of the bulk extremal curves is obtained to be

$$
\mathcal{E}^{\text{bulk}}(A : B) = \mathcal{E}^{\text{eff}}(A : B) + \frac{3}{16G_N} 2\mathcal{L}_1 
= \frac{c}{4} \log \left( \frac{2\ell}{\epsilon y \cos \theta_0} \right) + \frac{3\ell}{8G_N} \left[ \log \left( \frac{2b_1}{\epsilon} \right) + \tanh^{-1}(\sin \theta_0) \right],
$$

(4.35)

where we have used the fact that the entanglement entropy computation for the interval $A$ fixes $a = b_1$. On utilization of the Brown-Henneaux formula [162], the above expression matches exactly with the result obtained from the boundary perspective in eq. (4.33).

4.2.2 Phase-II

Boundary description

For this phase, we now consider the case where the interval $B$ is small such that it lacks an entanglement entropy island as shown in fig. 9. This implies that the island cross-section $\Gamma$ is a null set. The remaining effective semi-classical entanglement negativity is obtained through the following three point twist correlator

$$
\mathcal{E}^{\text{eff}}(A \cup I_A : B \cup I_B) = \lim_{n_e \to 1} \log \left[ (\epsilon y \Omega(-a))^\Delta \langle T_{n_e}(-a)\overline{T}_{n_e}^2(b_1)T_{n_e}(b_2) \rangle_{\text{CFT} \otimes n_e} \right] 
= \frac{c}{4} \log \left[ \frac{(b_1 + a)(b_2 - b_1)}{(b_2 + a)\epsilon} \right].
$$

(4.36)

Again, the point on the AdS$_2$ brane $\mathcal{Q}$ is fixed to be $a = b_1$ through the entanglement entropy computation of the interval $A$. Utilizing this value of $a$, we may obtain the total entanglement negativity for this phase in the boundary description to be

$$
\mathcal{E}^{\text{bdy}}(A : B) = \frac{c}{4} \log \left[ \frac{2b_1(b_2 - b_1)}{(b_2 + b_1)\epsilon} \right].
$$

(4.37)
Figure 9: Schematics of the defect extremal surface for the entanglement negativity between two adjacent intervals $A$ and $B$ in phase-II. In this phase, the island of the interval $B$ is an empty set.

Bulk description

For the bulk description of this phase, we observe in fig. 9 that the entanglement wedge for the subsystem $A \cup B$ is connected. However, since the interval $B$ does not have an entanglement island, the effective entanglement negativity term in eq. (3.4) vanishes. The only contribution to the total entanglement negativity comes from the lengths of the extremal curves labelled as $\mathcal{L}_i$ ($i = 1, 2, 3$) in fig. 9. To this end, we note that the lengths of the extremal curves $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same form as given in eqs. (4.11) and (4.12) respectively. Using similar approximations as were employed for the bulk description in subsection 4.1.2, the length of the extremal curve $\mathcal{L}_3$ may be computed to be

$$\mathcal{L}_3 = \ell \log \left( \frac{(b_1 + b_2)^2}{eb_1} \right) + \ell \tanh^{-1}(\sin \theta_0), \quad (4.38)$$

where we have used $a = b_1$. We may now obtain the total entanglement negativity for this phase using eq. (3.4) to be

$$\mathcal{E}^{\text{bdy}}(A : B) = \frac{3\ell}{8G_N} \log \left[ \frac{2b_1(b_2 - b_1)}{(b_2 + b_1)e} \right], \quad (4.39)$$

which on utilization of the usual Brown-Henneaux formula [162] matches exactly with the result obtained through the boundary description in eq. (4.37).

5 Time dependent entanglement negativity in black holes

In this section we investigate the nature of mixed state entanglement through the entanglement negativity in a time-dependent defect AdS$_3$/BCFT$_2$ scenario involving an eternal black hole in the effective two-dimensional description [61,92]. The lower dimensional effective model involves the appearance of entanglement islands during the emission of the Hawking radiation from the eternal black hole.

5.1 Review of the eternal black hole in AdS/BCFT

As described in [61,92], we consider a BCFT$_2$ defined on the half-plane ($x, \tau \geq 0$). The corresponding bulk dual is described by the Poincaré AdS$_3$ geometry truncated by an end-of-the-world (EOW) brane located at the hypersurface $\tau = -z \tan \theta_0$. Here $\theta_0$ is the angle made by the EOW brane with the vertical, and $\tau$ and $z$ are the timelike and holographic coordinates respectively.

Note that in the Euclidean signature, there is no essential difference between the timelike and spacelike coordinates and the present parametrization is a convenient choice adapted in [61,92].
Utilizing a global conformal map, the boundary of the BCFT$_2$ is then mapped to a circle
\[ x'^2 + \tau'^2 = 1. \tag{5.1} \]
The bulk dual of such a global conformal transformation is given by the following Banados map \[61,92,117]\n
\[
\begin{align*}
\tau' &= 1 + \frac{\tau - \frac{1}{2} \left( \tau^2 + x^2 + z^2 \right)}{1 - \tau + \frac{1}{4} \left( \tau^2 + x^2 + z^2 \right)}, \\
x' &= \frac{x}{1 - \tau + \frac{1}{4} \left( \tau^2 + x^2 + z^2 \right)}, \\
z' &= \frac{z}{1 - \tau + \frac{1}{4} \left( \tau^2 + x^2 + z^2 \right)}. \tag{5.2}
\end{align*}
\]
The EOW brane is mapped to a portion of a sphere under these bulk transformations,
\[ x'^2 + \tau'^2 + (z' + \tan \theta_0)^2 = \sec^2 \theta_0. \tag{5.3} \]
Note that, as the above transformation is a global conformal map, the metric in the bulk dual spacetime as well as the metric induced on the EOW brane are preserved under the Banados map eq. (5.2). The schematics of this time-dependent AdS/BCFT scenario is depicted in fig. 10a.

Finally employing the partial Randall-Sundrum reduction combined with the AdS$_3$/BCFT$_2$ correspondence discussed in \[43, 61, 92\], one obtains a two-sided (1 + 1)-dimensional eternal black hole on the EOW brane which is coupled to the BCFT$_2$ outside the circle eq. (5.1) in the 2d effective description. The schematics of the configuration is depicted in fig. 10b. The hybrid manifold consisting of a 2d eternal black hole with a fluctuating geometry coupled to the flat BCFT$_2$ may conveniently be described in terms of the Rindler coordinates \((X, T)\) defined through
\[
\begin{align*}
x' &= e^X \cosh T, \\
t' &\equiv -i\tau' = e^X \sinh T. \tag{5.4}
\end{align*}
\]
These Rindler coordinates naturally capture the near-horizon geometry of the 2d black hole \[92\].

In the following, we will compute the entanglement negativity for various bipartite states involving two disjoint and two adjacent intervals in the time-dependent defect AdS/BCFT scenario discussed above. In this regard, we will employ the semi-classical island formula eq. (3.2) in the lower dimensional effective description as well as the doubly holographic defect extremal surface proposals in eqs. (3.3) and (3.4) and find exact agreement between the two.

(a) Euclidean AdS/BCFT with the BCFT defined outside the circle. (b) 2d eternal black hole in Lorentzian signature.

Figure 10
5.2 Entanglement negativity between black hole interiors

In this subsection, we compute the time-dependent entanglement negativity between different regions of the black hole interior. As described in [61, 92] the black hole region $B$ is defined as the space-like interval from $Q \equiv (t_0', x_0')$ to $P \equiv (t_0'', x_0'')$ as shown in fig. 11. We perform the computations in the Euclidean signature with $\tau_0' = it_0'$ and subsequently obtain the final result in Lorentzian signature through an analytic continuation. Depending on the configuration of the extremal surface for the entanglement entropy of $B$, there are two possible phases for the extremal surfaces corresponding to the entanglement negativity between the black hole subsystems $B_L$ and $B_R$.

5.2.1 Connected phase

The connected phase corresponds to the scenario where there is no entanglement island for the radiation bath in the effective boundary description as shown in fig. 11. From the bulk perspective, this corresponds to a connected extremal surface for $B_L \cup B_R$. In this phase, it is required to compute the entanglement negativity between the two adjacent intervals $B_L \equiv |O'Q|$ and $B_R \equiv |O'P|$, where the point $O'$ is dynamical as it resides on the EOW brane with a gravitational theory.

![Figure 11: Schematics of the quantum extremal surface for the entanglement negativity between black hole interiors in the connected phase at a constant time slice.](image)

Boundary description

In the 2d boundary description, the effective semi-classical entanglement negativity between $B_L$ and $B_R$ may be computed through the three-point correlation function of twist operators as follows

$$E_{\text{eff}}(B_L : B_R) = \lim_{n_e \to 1} \log \left[ (\epsilon g \Omega_{O'})^{\Delta_{n_e}^{(2)}} \langle T_{n_e}(Q) T_{n_e}^2(O') T_{n_e}(P) \rangle \right]. \quad (5.5)$$

It is convenient to perform the computations in the un-primed coordinates $(y, x)$ given in eq. (2.2), where $y$ measures the distance along the EOW brane and $x$ is the spatial coordinate describing the BCFT2. In these coordinates, the conformal factor associated with the dynamical point $O'$ on the EOW brane $Q$ is given by [43, 61, 92]

$$\Omega_{O'}(y) = \left| \frac{y \cos \theta_0}{\ell} \right|. \quad (5.6)$$

where $\ell$ is the AdS3 radius inherited from the bulk geometry. The form of the CFT2 three-point function in eq. (5.5) is given by

$$\langle T_{n_e}(Q) T_{n_e}^2(O') T_{n_e}(P) \rangle = C_{T_{n_e} T_{n_e}^2 T_{n_e}} \left| O'P \right|^{-\Delta_{n_e}^{(2)}} \left| O'Q \right|^{-\Delta_{n_e}^{(2)}} \left| PQ \right|^{-2\Delta_{n_e} + \Delta_{n_e}^{(2)}}. \quad (5.7)$$

where $C_{T_{n_e} T_{n_e}^2 T_{n_e}}$ is the constant OPE coefficient which is neglected henceforth. Substituting eqs. (5.6) and (5.7) in eq. (5.5), we may obtain the following expression for the generalized entanglement negativity between $B_L$ and $B_R$ in the boundary description

$$E_{\text{gen}}(B_L : B_R) = \frac{c}{4} \log \left[ \left( \frac{(y_0 + y)^2 + x_0^2}{2x_0} \right) \right] + \frac{c}{4} \log \left( \frac{\ell}{\epsilon g y \cos \theta_0} \right) + \frac{c}{4} \tanh^{-1}(\sin \theta_0). \quad (5.8)$$
where we have added the area term eq. (2.8) corresponding to the point $O'$ on the EOW brane in the QES formula. The above expression is extremized over the position $y$ of the dynamical point $O'$ to obtain

$$y_0 = \sqrt{\tau_0^2 + x_0^2}. \quad (5.9)$$

Substituting the above expression in eq. (5.8), the semi-classical entanglement negativity in the $2d$ effective boundary description is obtained as follows

$$E_{\text{bdy}}(B_L : B_R) = \frac{c}{4} \log \left( \frac{\tau + \sqrt{\tau^2 + x^2}}{x} \right) + \frac{c}{4} \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) + \frac{c}{4} \tanh^{-1}(\sin \theta_0). \quad (5.10)$$

Now transforming back to the primed coordinates using eq. (5.2) and analytically continuing to the Lorentzian signature, the above expression reduces to

$$E_{\text{bdy}}(B_L : B_R) = \frac{c}{4} \log \left( \frac{x'^2 - t'^2 - 1 + \sqrt{4x'^2 + (x'^2 - t'^2 - 1)^2}}{2x_0'} \right) + \frac{c}{4} \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) + \frac{c}{4} \tanh^{-1}(\sin \theta_0). \quad (5.11)$$

In terms of the Rindler coordinates $(X, T)$, the final result for the entanglement negativity between the black hole interiors becomes

$$E_{\text{bdy}}(B_L : B_R) = \frac{c}{4} \log \left( \frac{e^{2X_0} - 1 + \sqrt{4e^{2X_0} \cosh^2 T + (e^{2X_0} - 1)^2}}{2e^{X_0} \cosh T} \right) + \frac{c}{4} \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) + \frac{c}{4} \log \left( \frac{\cos \theta_0}{1 - \sin \theta_0} \right), \quad (5.12)$$

where $X_0$ describes the boundary of the black hole region at a fixed Rindler time $T$. Note that the above expression for the entanglement negativity between $B_L$ and $B_R$ is a decreasing function of the Rindler time $T$ in this phase.

**Bulk description**

Next we focus on the three-dimensional bulk description for the connected phase of the entanglement negativity between the black hole interiors. To compute the holographic entanglement negativity, we note that the mixed state configuration described by $B_L$ and $B_R$ corresponds to the case of two adjacent intervals $|O'P|$ and $|O'Q|$. The configuration of the bulk extremal curves homologous to various subsystems under consideration is depicted in fig. 12. Now employing the DES formula given in eq. (3.4), we may obtain

$$E_{\text{gen}}^\text{bulk}(B_L : B_R) = \frac{3}{16G_N} (\mathcal{L}_1 + \mathcal{L}_2 - \mathcal{L}_3) + E_{\text{eff}}(B_L : B_R), \quad (5.13)$$

where $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are the lengths of the bulk extremal curves homologous to $|O'P|$, $|O'Q|$ and $|PQ|$ respectively and $E_{\text{eff}}(B_L : B_R)$ denotes the effective entanglement negativity between bulk quantum matter fields residing on the EOW brane.

In the un-primed coordinates, the Cauchy slice on the EOW brane is in a pure state as described in [61]. Hence, the effective entanglement negativity in eq. (5.13) may be obtained through the Rényi entropy of order half for a part of matter fields on the EOW brane. Consequently, similar to eq. (4.26), the effective entanglement negativity is a constant given by

$$E_{\text{eff}}(B_L : B_R) = \frac{c}{4} \log \frac{2\ell}{\epsilon_y \cos \theta_0}. \quad (5.14)$$
As the effective entanglement negativity turns out to be a constant, the entanglement negativity in this phase is determined entirely through the algebraic sum of the lengths of the extremal curves in eq. (5.13). To obtain the lengths of these extremal curve, we employ the unprimed coordinate system with the Poincaré AdS$_3$ metric [61]. Under the bulk map in eq. (5.2), the coordinates of $P$ and $Q$ may be mapped to $(\tau_0, x_0, 0)$ and $(\tau_0, -x_0, 0)$ where

$$
\tau_0 = \frac{2(x_0^2 + \tau_0^2 - 1)}{(\tau_0^2 + 1)^2 + x_0^2} , \quad x_0 = \frac{4x_0'}{(\tau_0^2 + 1)^2 + x_0'^2} .
$$

(5.15)

Utilizing the left-right $\mathbb{Z}_2$ symmetry of the configuration, we may set the coordinates of the dynamical point $O'$ on the brane as $O': (-z \tan \theta_0, 0, z)$ where $z$ is determined through the extremization of the generalized negativity functional in eq. (5.13). The lengths of the extremal curves may now be obtained in the un-primed coordinates through the standard Poincaré AdS$_3$ result as follows [10, 11]

$$
L_1 = \ell \cosh^{-1} \left[ \frac{(\tau_0 + z \tan \theta_0)^2 + x_0^2 + z^2}{2z} \right] - \ell \log \left[ \frac{4z'}{(\tau_0^2 + 1)^2 + x_0'^2} \right] = L_2 ,
$$

$$
L_3 = 2\ell \log(2x_0) - 2\ell \log \left[ \frac{4\epsilon}{(\tau_0^2 + 1)^2 + x_0^2} \right] .
$$

(5.16)

In the above expression, $\epsilon$ is the UV cut-off for the original BCFT$_2$ in the primed coordinates and the second logarithmic term arises due to the cut-off in the un-primed coordinates (cf. the Banados map in eq. (5.2)). Now extremizing the generalized negativity with respect to $z$ we may obtain the position of $O'$ to be

$$
\partial_z \mathcal{E}_{\text{gen}}^{\text{bulk}} = 0 \implies z = \sqrt{x_0^2 + \tau_0^2} \cos \theta_0 .
$$

(5.17)

Substituting the above value of $z$ in eq. (5.13), we may obtain the bulk entanglement negativity between $B_L$ and $B_R$ as follows

$$
\mathcal{E}_{\text{bulk}}(B_L : B_R) = \frac{c}{4} \left[ \cosh^{-1} \left( \frac{\sqrt{x_0^2 + \tau_0^2} + \tau_0 \sin \theta_0}{x_0 \cos \theta_0} \right) + \log \left( \frac{\ell}{\epsilon \gamma \cos \theta_0} \right) \right] ,
$$

(5.18)

where the effective contribution from the quantum matter fields given in eq. (5.14) has been included. Now utilizing the hyperbolic identity

$$
\cosh^{-1} \left( \frac{\sqrt{x_0^2 + \tau_0^2} + \tau_0 \sin \theta_0}{x_0 \cos \theta_0} \right) = \log \left( \frac{\tau_0 + \sqrt{x_0^2 + \tau_0^2}}{x_0} \right) + \cosh^{-1}(\sec \theta_0) ,
$$

(5.19)
eq. (5.18) may be expressed as

$$E_{\text{bulk}}(B_L : B_R) = \frac{c}{4} \left[ \log \left( \frac{\tau_0 + \sqrt{\tau_0^2 + \tau_0^2}}{x_0} \right) + \log \frac{\ell}{\epsilon_y \cos \theta_0} + \log \frac{\cos \theta_0}{1 - \sin \theta_0} \right]$$

(5.20)

Transforming to the primed coordinates using eq. (5.2) and subsequently to the Rindler coordinates eq. (5.4) via the analytic continuation \(\tau' = it'\), we may obtain the bulk entanglement negativity between \(B_L\) and \(B_R\) to be

$$E_{\text{bulk}}(B_L : B_R) = \frac{c}{4} \log \left[ \frac{\epsilon^2 x_0 - 1 + \sqrt{4\epsilon^2 x_0 \cosh^2 T + (\epsilon^2 x_0 - 1)^2}}{2\epsilon x_0 \cosh T} \right] + \frac{c}{4} \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right)
+ \frac{c}{4} \log \left( \frac{\cos \theta_0}{1 - \sin \theta_0} \right).$$

(5.21)

The above expression matches identically with the boundary QES result in eq. (5.12) which provides a strong consistency check of our holographic construction.

5.2.2 Disconnected phase

In this subsection, we concentrate on the disconnected phase for the extremal surface for \(B_L \cup B_R\), depicted in fig. 13. In this case, there are entanglement islands corresponding to the radiation bath on the EOW brane, and a part of the entanglement wedge for the radiation bath is subtended on the brane. This splits the black hole regions into two disjoint subsystems, namely \(B_R \equiv |PP'|\) and \(B_L \equiv |QQ'|\), where the points \(P'\) and \(Q'\) are determined by the extremal surface for \(B_L \cup B_R\).

![Figure 13: Schematics of the quantum extremal surface for the entanglement negativity between black hole interiors in the disconnected phase](image)

Boundary description

In the two-dimensional effective boundary description, the area term for the generalized entanglement negativity vanishes since there is no non-trivial island cross-section, \(\partial B_L \cap \partial B_R = \emptyset\). The effective semi-classical entanglement negativity between \(B_L\) and \(B_R\) may be computed through the following four-point correlator of twist operators placed at the endpoints of the intervals,

$$E_{\text{eff}}(B_L : B_R) = \lim_{n_e \to 1} \log \left[ (\epsilon_y \Omega_P)^{\Delta_{n_e}} (\epsilon_y \Omega_Q)^{\Delta_{n_e}} \langle T_{n_e}(Q) \bar{T}_{n_e}(Q') \bar{T}_{n_e}(P') T_{n_e}(P) \rangle \right]$$

(5.22)

As indicated by the disconnected extremal surfaces shown in fig. 13, the above four-point correlator factorizes into the product of two 2-point correlators as follows

$$\langle T_{n_e}(Q) \bar{T}_{n_e}(Q') \bar{T}_{n_e}(P') T_{n_e}(P) \rangle \approx \langle T_{n_e}(Q) \bar{T}_{n_e}(Q') \rangle \langle T_{n_e}(P) \bar{T}_{n_e}(P') \rangle.$$

(5.23)

Now utilizing eq. (4.20), we may observe that, in the replica limit \(n_e \to 1\), the above correlation function vanishes identically. Hence, in this phase the total entanglement negativity between the black hole interiors is also vanishing.
**Bulk description**

As depicted in fig. 13, the entanglement wedges corresponding to the subsystems $B_L$ and $B_R$ are naturally disconnected and hence, the configuration corresponds to two disjoint intervals on the boundary which are far away from each other. In this case, the area contribution to the bulk entanglement negativity vanishes [144,145]. The effective entanglement negativity between portions of bulk quantum matter on the EOW brane is given by the BCFT correlation function of twist fields inserted at $P'$ and $Q'$ as follows

$$
E_{\text{eff}} = \lim_{n_c \to 1} \log \langle \mathcal{T}_{n_c}(P') \bar{\mathcal{T}}_{n_c}(Q') \rangle_{\text{BCFT} \otimes n_c}.
$$

The coordinates of $P'$ and $Q'$ are obtained via extremizing the generalized entropy functional for $B_L \cup B_R$ which, in the $(y,x)$ coordinates, are given by $(\tau_0, x_0)$ and $(\tau_0, -x_0)$ respectively [92]. Now employing the doubling trick [116, 120], the correlation function in eq. (5.24) may be expressed as a chiral four-point function on the full complex plane as

$$
\langle \mathcal{T}_{n_c}(P') \bar{\mathcal{T}}_{n_c}(Q') \rangle_{\text{BCFT} \otimes n_c} = \langle \mathcal{T}_{n_c}(P') \bar{\mathcal{T}}_{n_c}(Q') \bar{\mathcal{T}}_{n_c}(Q'') \mathcal{T}_{n_c}(P'') \rangle_{\text{CFT} \otimes n_c}
$$

where $P'' : (\tau_0, x_0)$ and $Q'' : (\tau_0, -x_0)$ are the image points of $P'$ and $Q'$ upon reflection through the boundary at $\tau = 0$. The above four point correlator is again factorized into two two-point functions in the dominant channel and similar to the previous subsection, the effective semi-classical entanglement negativity vanishes. Hence, the boundary QES result is reproduced through the bulk computations.

**Figure 14: Schematics of the defect extremal surface for the entanglement negativity between black hole interiors in the disconnected phase**

### 5.2.3 Page curve

From the results of the last two subsections, we may infer that the time evolution of the entanglement negativity between the black hole interiors is governed by the two phases of the extremal surfaces corresponding to the entanglement entropy of $B_L \cup B_R$. It is well known that the unitary time evolution of the entanglement entropy for a subsystem in the Hawking radiation flux from a black hole is governed by the *Page curve* [7–9]. Hence the transition between the two different phases of the entanglement negativity between $B_L$ and $B_R$ occurs precisely at the Page time $T_P$, given by [61,92]

$$
T_P = \cosh^{-1} \left( \sinh X_0 \epsilon \tanh^{-1}(\sin \theta_0) \frac{2 \ell}{\epsilon y \cos \theta_0} \right).
$$

In the first phase the entanglement negativity is a decreasing function of the Rindler time given by eq. (5.12). At the Page time $T_P$ the extremal surface for the entanglement entropy transits to the disconnected phase and an entanglement entropy island corresponding to the radiation bath
appears inside the gravitational regions on the EOW brane $Q$. At this time, the entanglement negativity also transits to the corresponding disconnected phase and vanishes identically. The variation of the entanglement negativity between black hole interiors with the Rindler time $T$ is plotted in fig. 15.

$$\theta_0 = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$$

**Figure 15:** The Page curve for entanglement negativity between black hole interiors for three different values of the EOW brane angle $\theta_0$. Here the variation of the entanglement negativity with respect to the Rindler time $T$ is shown in units of $\frac{c}{4}$ with $X_0 = 1$, $\epsilon_\gamma = 0.1$, $\ell = 1$ and $\theta_0 = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.

### 5.3 Entanglement negativity between the black hole and the radiation

In this subsection, we now proceed to the computation of the entanglement negativity between the black hole region and the radiation region in the time-dependent defect AdS$_3$/BCFT$_2$ scenario. To this end, we consider the black hole region to be described by a space-like interval $B_L$ on the left-half of the two sided eternal black hole and the radiation region to be described by a semi-infinite interval $R_L$ adjacent to $B_L$ as shown in fig. 16. Similar to the previous case, there are two phases possible in this case which are investigated below.

#### 5.3.1 Connected phase

The connected phase corresponds to the case where $R_L$ does not posses an entanglement island and thus $B_L$ covers the complete left black hole region on the EOW brane as shown in fig. 16. From the doubly holographic perspective, this corresponds to an extremal surface for $R_L \cup B_L$ extending from the dynamical endpoint $O'$ of $B_L$ on the EOW brane to spatial infinity. We compute the entanglement negativity between the two adjacent intervals $B_L \equiv |O'Q|$ and $R_L \equiv |QA|$ in this phase where we have regularized the semi-infinite interval $R_L$ to end at some point $A : (\tau'_0, -x'_\infty)$ which is later taken to infinity.

**Figure 16:** Schematics of the defect extremal surface for the entanglement negativity between the black hole and the radiation in the connected phase.
Boundary description

For the connected phase, the absence of the entanglement island for the radiation region \( R_L \) implies that the generalized entanglement negativity doesn’t receive any area contribution in the 2d effective boundary description. The remaining effective semi-classical entanglement negativity between \( B_L \) and \( R_L \) may be computed through the following three-point twist correlator

\[
\mathcal{E}^{\text{eff}}(B_L : R_L) = \lim_{n_e \to 1} \log \left[ (\epsilon_y \Omega_{O'})^{\Delta_{n_e}} \langle \mathcal{T}_{n_e}(O') \mathcal{T}_{n_e}^2(Q) \mathcal{T}_{n_e}(A) \rangle \right],
\]

(5.27)

where \( \epsilon_y \) is a UV cut off on the dynamical EOW \( Q \) and \( \Omega_{O'} \) is the warp factor as given in eq. (5.6). In the un-primed coordinates, the points \( O' \) and \( Q \) are located at \((-y, 0)\) and \( (\tau_0, -x_0)\) respectively. Using eq. (5.2), we may locate the spatial infinity \( A \) in the un-primed coordinates at \((\tau, x) = (2, 0)\). Now, by utilizing the usual form of a CFT\(_2\) three-point twist correlator given in eq. (5.7), we may obtain the total generalized entanglement negativity in the boundary description for this case to be

\[
\mathcal{E}_{\text{gen}}^{\text{bdy}} = c \left\{ \frac{\log \left[ (\tau_0 + y)^2 + x_0^2 \right]}{(y + 2)^2} - 2 \log \left[ \frac{4\epsilon}{(\tau_0 + 1)^2 + x_0^2} \right] \right\},
\]

(5.28)

where \( \epsilon \) is the UV cut-off in the primed coordinates. The above expression is then extremized over the position \( y \) of the dynamical point \( O' \) on the EOW brane to obtain

\[
y = \frac{x_0^2}{2 - \tau_0} - \tau_0.
\]

(5.29)

It may be checked through eqs. (5.2) and (5.4) that \( \tau < 2 \) for the Rindler time \( T > 0 \) which guarantees the non-negativity of \( y \) for large \( x_0 \). We may now compute the entanglement negativity by substituting the above value of \( y \) in eq. (5.28) to be

\[
\mathcal{E}^{\text{bdy}} = \frac{c}{4} \log x_0 - \frac{c}{4} \log \left[ \frac{4\epsilon}{(\tau_0 + 1)^2 + x_0^2} \right].
\]

(5.30)

Transforming this result to the Rindler coordinates \((X, T)\) through eqs. (5.2) and (5.4), we may obtain the final expression for the entanglement negativity between the black hole region \( B_L \) and the radiation region \( R_L \) in the boundary description to be

\[
\mathcal{E}^{\text{bdy}}(B_L : R_L) = \frac{c}{4} \log \frac{\cosh T}{\epsilon} + X_0,
\]

(5.31)

where \( X_0 \) corresponds to the endpoint \( Q \) of the black hole region \( B_L \) at the fixed Rindler time \( T \). We note here that the entanglement negativity in the above expression is an increasing function of the Rindler time \( T \) in this connected phase.

Bulk description

In the bulk description for this phase as depicted in fig. 17, the generalized entanglement negativity between the black hole region \( B_L \equiv |O'Q| \) and the radiation region \( R_L \equiv |QA| \) is computed by employing the following formula

\[
\mathcal{E}_{\text{gen}}^{\text{bulk}}(B_L : R_L) = \frac{3}{16G_N} (\mathcal{L}_{R_L} + \mathcal{L}_{B_L} - \mathcal{L}_{B_L \cup R_L})
\]

\[
= -\frac{c}{8} \log \left[ \frac{(\tau_0 + z \tan \theta_0)^2 + x_0^2 + z^2}{(2 + z \tan \theta_0)^2 + z^2} \right] - 2 \log \left( \frac{4\epsilon}{(\tau_0 + 1)^2 + x_0^2} \right),
\]

(5.32)

where \( O' : (-z \tan \theta_0, 0, z) \) and \( Q : (\tau_0, -x_0, 0) \) are the endpoints of \( B_L \), and \( A : (2, 0, 0) \) is the regularized endpoint of the semi infinite radiation region \( R_L \) in the un-primed coordinates. We
have also used the Brown-Henneaux formula [162] in the second equality of the above expression. Note that the semi-classical effective entanglement negativity appearing as the second term in eq. (3.4) vanishes in this case as $R_L$ does not possess an entanglement island. We may extremize eq. (5.32) over the position of the dynamical point $O'$ to obtain

$$
\partial_z c_{\text{gen}}^{\text{bulk}} = 0 \implies z = \frac{x_0^2 \cot \theta_0}{2 - \tau_0},
$$

(5.33)

where we have used the approximation that $x_0$ is large. The total entanglement negativity between the black hole region $B_L$ and the radiation region $R_L$ in the Rindler coordinates ($X,T$) for this phase may now be obtained by utilizing eqs. (5.2), (5.4), (5.32) and (5.33) to be

$$
E_b(B_L : R_L) = \frac{c}{4} \log \frac{\cosh T}{\epsilon} + X_0,
$$

(5.34)

where $X_0$ corresponds to the point $Q$ at the fixed Rindler time $T$. Remarkably, the above expression for the entanglement negativity matches exactly with the boundary description result in eq. (5.31).

5.3.2 Disconnected phase

The disconnected phase is described by the case where the semi-infinite radiation region $R_L$ has an entanglement island labelled as $I_L \equiv |O'Q'|$ as depicted in fig. 18. From the bulk perspective, this corresponds to an extremal surface for $R_L$ to end on some point $Q'$ on the EOW brane. The entanglement negativity between the black hole region $B_L$ and the radiation region $R_L$ for this case will thus receive contribution from the island region $I_L$ on the EOW brane.

**Figure 17: Schematics of the defect extremal surface for the entanglement negativity between the black hole and the radiation in the connected phase**

**Figure 18: Schematics of the defect extremal surface for the entanglement negativity between the black hole and the radiation in the disconnected phase**

**Boundary description**

In the boundary description, the area contribution to the entanglement negativity corresponding to the point $Q' = \partial B_L \cap \partial I_L$ is as given in eq. (2.8). The remaining effective semi-classical
entanglement negativity between \( B_L \) and \( R_L \) may be computed through the following four-point twist correlator,

\[
\mathcal{E}^{\text{eff}}(B_L : R_L \cup I_L) = \lim_{n_\text{e} \to 1} \log \left( \epsilon_y \Omega_{Q'} \right)^{\Delta_{n_\text{e}}^{(2)}} \left( \epsilon_y \Omega_Q \right)^{\Delta_{n_\text{e}}} \langle \mathcal{T}_{n_\text{e}}(A) \mathcal{T}_{n_\text{e}}^2(Q) \mathcal{T}_{n_\text{e}}^2(Q') \mathcal{T}_{n_\text{e}}(O') \rangle,
\]

(5.35)

where \( \Omega \) are the warp factors as given in eq. (5.6), \( A \) is the regularized endpoint of the semi-infinite interval \( R_L \) and the point \( Q' \) and \( Q \) are at position \((-y, -x)\) and \((\tau_0, -x_0)\), respectively in the un-primed coordinates. For the bipartite configuration under consideration, the above four-point twist correlator factorizes into two two-point twist correlators in the following way,

\[
\langle \mathcal{T}_{n_\text{e}}(A) \mathcal{T}_{n_\text{e}}^2(Q) \mathcal{T}_{n_\text{e}}^2(Q') \mathcal{T}_{n_\text{e}}(O') \rangle \approx \langle \mathcal{T}_{n_\text{e}}(A) \mathcal{T}_{n_\text{e}}(O') \rangle \langle \mathcal{T}_{n_\text{e}}^2(Q) \mathcal{T}_{n_\text{e}}^2(Q') \rangle.
\]

(5.36)

Utilizing the above factorization in eq. (5.35) along with the area term in eq. (2.8), the generalized entanglement negativity for this case may be expressed as

\[
\mathcal{E}_{\text{gen}}^{\text{bdy}} = \frac{c}{4} \left( \frac{\cosh^{-1}(\tau_0)}{\epsilon_y \cosh \tau_0} + \log \left( \frac{\epsilon^2 x_0 - 1}{\epsilon} + \log \frac{2 \ell}{\epsilon} \right) \right),
\]

(5.37)

where again \( \epsilon \) is the UV cut-off in the primed coordinates. Interestingly, the regularized point \( A \) does not enter the computation in this case. We may now extremize the above generalized entanglement negativity over the position of the dynamical point \( Q' \) i.e., \( \partial_y \mathcal{E}_{\text{gen}}^{\text{bdy}} = 0 \) and \( \partial_x \mathcal{E}_{\text{gen}}^{\text{bdy}} = 0 \) to obtain

\[
y = \tau_0, \quad x = x_0.
\]

(5.38)

Using the above values of the coordinates \( y \) and \( x \) in eq. (5.37) and transforming the result to the primed coordinates eq. (5.2) and subsequently to the Rindler coordinates (5.4) via the analytic continuation \( \tau = i \ell' \), we may obtain the total entanglement negativity between the black hole region \( B_L \) and the radiation region \( R_L \) to be

\[
\mathcal{E}^{\text{bdy}}(B_L : R_L) = \frac{c}{4} \left( \frac{\cosh^{-1}(\tau_0)}{\epsilon_y \cosh \tau_0} + \log \left( \frac{\epsilon^2 x_0 - 1}{\epsilon} + \log \frac{2 \ell}{\epsilon} \right) \right).
\]

(5.39)

Here \( x_0 \) corresponds to the endpoint \( Q \) of the black hole region \( B_L \). Note that the above expression for the entanglement negativity is independent of the Rindler time \( T \) and only depends on the position of the point \( Q \).

**Bulk description**

In the 3d bulk description for the disconnected phase, the entanglement negativity between the black hole region \( B_L \) and the radiation region \( R_L \) is computed by employing the DES formula in eq. (3.4) as follows

\[
\mathcal{E}_{\text{gen}}^{\text{bulk}}(B_L : \mathcal{R}_L) = \frac{3}{16G_N} \left( \mathcal{L}_{B_L} + \mathcal{L}_{R_L} - \mathcal{L}_{B_L \cup R_L} \right) + \mathcal{E}^{\text{eff}}(B_L : \mathcal{R}_L)
\]

(5.40)

where \( \mathcal{L}_2 \) is the extremal curve between points \( Q' : (-z \tan \theta_0, -x_1, z) \) and \( Q : (\tau_0, -x_0, 0) \) and \( I_L \) is the island region corresponding to the radiation region \( R_L \) as depicted in fig. 19. In the second term of the above expression we have also utilized the fact that bulk matter fields are only localized on the EOW brane \( Q \). We note here that, consistent with the boundary description, the regularized point \( A \) does not enter the computation in this phase. The length of the extremal curve \( \mathcal{L}_2 \) in the un-primed coordinates may be expressed as [10,11]

\[
\mathcal{L}_2 = \mathcal{L}_{QQ'} = \ell \log \left( \frac{(\tau_0 + z \tan \theta_0)^2 + (x_0 - x_1)^2 + z^2}{z} \right) - \ell \log \left( \frac{4 \epsilon}{(\tau_0^2 + 1)^2 + x_0^2} \right).
\]

(5.41)
The semi-classical effective entanglement negativity appearing as the last term in eq. (5.40) may be obtained to be

$$E_{\text{eff}}(B_L : I_L) = \frac{c}{4} \log \frac{2\ell}{\epsilon_y \cos \theta_0},$$

(5.42)

where $\epsilon_y$ is the UV cut-off on the dynamical EOW brane. Extremizing the generalized entanglement negativity obtained by substituting eqs. (5.41) and (5.42) in eq. (5.40), with respect to the position of $Q'$ i.e., $\partial_2 E_{\text{gen}} = 0$ and $\partial_{x_1} E_{\text{gen}} = 0$, we may obtain

$$z = \tau_0 \cos \theta_0, \quad x_1 = x_0.$$  

(5.43)

The total entanglement negativity between the black hole region $B_L$ and the radiation region $R_L$ in the primed coordinates may then be obtained through eqs. (5.2) and (5.43) to be

$$E_{\text{bulk}}(B_L : R_L) = \frac{c}{4} \left[ \log \left( \frac{x'^2_0 + x'^2_1 - 1}{\ell} \right) + \tanh^{-1}(\sin \theta_0) + \log \left( \frac{2\ell}{\epsilon_y \cos \theta_0} \right) \right],$$

(5.44)

where the Brown-Henneaux formula [162] has been used. On transformation to the Rindler coordinates eq. (5.4) via the analytic continuation $\tau' = i t'$, the above expression matches exactly with the boundary perspective result in eq. (5.39) which serves as a strong consistency check for our proposals.

![Figure 19: Schematics of the defect extremal surface for the entanglement negativity between the black hole and the radiation in the disconnected phase](image)

5.3.3 Page curve

We now analyse the results of the last two subsections where we have computed the entanglement negativity between the black hole region $B_L$ and the radiation region $R_L$ for the two possible phases. In the connected phase, the entanglement negativity computed in eq. (5.31) is an increasing function of the Rindler time $T$. In contrast, for the disconnected phase, the entanglement negativity given in eq. (5.39) is independent of $T$ and only depends on the size of the black hole region $B_L$. A transition from the connected phase to the disconnected phase is observed at the Page time for the entanglement entropy given in eq. (5.26). In fig. 20, we show the variation of the time dependent entanglement negativity between the black hole region $B_L$ and the radiation $R_L$ with the Rindler time $T$ for three different values of the EOW brane angle $\theta_0$. 

29
5.4 Entanglement negativity between subsystems in the radiation bath

In this subsection, we compute the entanglement negativity between the right subsystem $R_R$ and the left subsystem $R_L$ in the radiation region as shown in fig. 21. In the Rindler coordinates $(X, T)$, the right radiation subsystem $R_R$ extends from $(X_0, T)$ to $(X_1, T)$ and the left radiation subsystem $R_L$ extends from $(X_0, -T + i\pi)$ to $(X_1, -T + i\pi)$. In the primed coordinates, the subsystems $R_R \equiv |NQ|$ and $R_L \equiv |PM|$ are mapped to the intervals $[(\tau'_0, x'_0), (\tau'_1, x'_1)]$ and $[(\tau'_1, -x'_1), (\tau'_0, -x'_0)]$ respectively. Similar to the earlier subsections, we perform the computation in the Euclidean signature and subsequently transform the results to Rindler coordinates in the Lorentzian signature. Depending on the phase transition of the extremal surfaces corresponding to $R_L \cup R_R$, the DES corresponding to the entanglement negativity between them crosses from a connected phase to a disconnected phase. In the following, we investigate the time evolution of the entanglement negativity between $R_L$ and $R_R$ from both the bulk and the boundary perspective.

5.4.1 Connected phase

In the connected phase, there are no entanglement entropy islands corresponding to $R_L$ and $R_R$ in the effective boundary description as illustrated in fig. 21. In this phase, we compute the entanglement negativity between the disjoint radiation subsystems $R_L$ and $R_R$.

![Figure 21: Schematics of the quantum extremal surface for the entanglement negativity between intervals in the radiation region in the connected phase.](image)

**Boundary description**

As there are no island contributions in this phase, we observe from eq. (3.2) that the entanglement negativity between the radiation subsystems in 2d effective boundary description reduces
to the effective entanglement negativity between two disjoint intervals as follows
\[
\mathcal{E}^{\text{bdy}}(R_L : R_R) = \mathcal{E}^{\text{eff}}(R_L : R_R) = \lim_{n \to 1} \log \left( \langle \mathcal{T}_{n_c}(P) \mathcal{T}_{n_c}(M) \mathcal{T}_{n_c}(N) \mathcal{T}_{n_c}(Q) \rangle_{\text{CFT} \otimes \sigma_{n_c}} \right). \tag{5.45}
\]
As described in section 4.1.2, for the two disjoint subsystems in the \(t\)-channel, the above four point twist correlator may be computed in the large central charge limit as follows [144]
\[
\mathcal{E}^{\text{bdy}}(R_L : R_R) = \frac{c}{4} \log \left( \frac{|PN||MQ|}{|MN||PQ|} \right) = \frac{c}{4} \log \left( \frac{(e^{X_0} + e^{X_1})^2 - (e^{X_0} - e^{X_1})^2 \tanh^2 T}{4e^{X_0+X_1}} \right). \tag{5.46}
\]
Note that the entanglement negativity between the radiation subsystems \(R_L\) and \(R_R\) is a monotonically decreasing function of the Rindler time \(T\) in this phase.

**Bulk description**

In the 3d bulk description, the effective entanglement negativity in eq. (3.3) vanishes as the corresponding entanglement wedges contain no quantum matter fields. The entanglement negativity between \(R_L\) and \(R_R\) is then given entirely by the combination of the lengths of the defect extremal surfaces as follows
\[
\mathcal{E}^{\text{bulk}}(R_L : R_R) = \frac{3}{16G_N} (\mathcal{L}_{PN} + \mathcal{L}_{MQ} - \mathcal{L}_{MN} - \mathcal{L}_{PQ})
= \frac{3\ell}{8G_N} \left[ \log \left( \frac{(x'_a + x'_b)^2 + (\tau'_1 - \tau'_0)^2}{\epsilon^2} \right) - \log \left( \frac{2x'_0}{\epsilon} \right) - \log \left( \frac{2x'_1}{\epsilon} \right) \right], \tag{5.47}
\]
where we have used the fact that the length of an extremal curve \(\mathcal{L}_{ab}\) connecting two points \((\tau'_a, x'_a)\) and \((\tau'_b, x'_b)\) on the boundary is given by [11]
\[
\mathcal{L}_{ab} = \ell \log \left[ \frac{(x'_a - x'_b)^2 + (\tau'_a - \tau'_b)^2}{\epsilon^2} \right]. \tag{5.48}
\]
Now analytically continuing to the Lorentzian signature and transforming to the Rindler coordinates in eq. (5.4), we obtain the entanglement negativity between \(R_L\) and \(R_R\) in the bulk description to be
\[
\mathcal{E}^{\text{bulk}}(R_L : R_R) = \frac{c}{4} \log \left( \frac{(e^{X_0} + e^{X_1})^2 - (e^{X_0} - e^{X_1})^2 \tanh^2 T}{4e^{X_0+X_1}} \right), \tag{5.49}
\]
which matches exactly with the result from the boundary description, given in eq. (5.46).

![Figure 22: Schematics of the defect extremal surface for the entanglement negativity between intervals in the radiation region in the connected phase.](image)

31
5.4.2 Disconnected phase

For the disconnected phase, the entanglement entropy corresponding to the radiation subsystems receives island contributions as depicted in fig. 23. The entanglement negativity islands corresponding to the radiation subsystems \( R_L \) and \( R_R \), located on the EOW brane, are denoted as \( I_L \equiv |M'O'| \) and \( I_R \equiv |O'N'| \) respectively.\(^{11}\) We now proceed to compute the entanglement negativity between the radiation subsystems in the boundary and bulk descriptions in this phase.

\[ \text{Boundary description} \]

In the 2d boundary perspective, the area term corresponding to the point \( O' = \partial I_L \cap \partial I_R \) is a constant given by eq. (2.8). The remaining effective semi-classical entanglement negativity in eq. (3.2) may be expressed as

\[
\mathcal{E}^{\text{eff}}(R_L \cup I_L : R_R \cup I_R) = \lim_{n_e \to 1} \log \left[ (\epsilon_y \Omega_{M'})^{\Delta_{n_e}} (\epsilon_y \Omega_{N'})^{\Delta_{n_e}} (\epsilon_y \Omega_{O'})^{\Delta^{(2)}_{n_e}} \times \langle \mathcal{T}_{t_{n_e}}(P) \mathcal{T}_{t_{n_e}}(M') \mathcal{T}_{t_{n_e}}^2(O') \mathcal{T}_{t_{n_e}}(N') \mathcal{T}_{t_{n_e}}(N) \mathcal{T}_{t_{n_e}}(Q) \rangle \right]_{n_e}.
\]

(5.50)

In the large central charge limit, the above twist correlator may be factorized in the dominant channel as follows\(^{12}\)

\[
\langle \mathcal{T}_{t_{n_e}}(M') \mathcal{T}_{t_{n_e}}(N') \rangle \langle \mathcal{T}_{t_{n_e}}(P) \mathcal{T}_{t_{n_e}}^2(O') \mathcal{T}_{t_{n_e}}(Q) \rangle \langle \mathcal{T}_{t_{n_e}}(N') \mathcal{T}_{t_{n_e}}(N) \rangle
\]

(5.51)

Now, employing the replica limit \( n_e \to 1 \), the effective semi-classical entanglement negativity eq. (5.50) in the disconnected phase reduces to

\[
\mathcal{E}^{\text{eff}}(R_L \cup I_L : R_R \cup I_R) \approx \lim_{n_e \to 1} \log \left[ (\epsilon_y \Omega_{O'})^{\Delta^{(2)}_{n_e}} \langle \mathcal{T}_{t_{n_e}}(P) \mathcal{T}_{t_{n_e}}^2(O') \mathcal{T}_{t_{n_e}}(Q) \rangle \right] = \frac{c}{4} \log \left( \frac{\ell}{\epsilon_y y \cos \theta_0} \right) + \frac{c}{4} \log \left( \frac{(y + \tau_1)^2 + x_1^2}{2x_1} \right),
\]

(5.52)

where the coordinates for \( P, Q \) and \( O' \) are given by \((\tau_1, -x_1), (\tau_1, x_1)\) and \((-y, 0)\) respectively. The generalized entanglement negativity between \( R_L \) and \( R_R \) in the 2d boundary description may now be obtained using eqs. (3.2) and (5.52) as follows

\[
\mathcal{E}^{\text{bdy}}_{\text{gen}}(R_L : R_R) = \frac{c}{4} \log \left( \frac{\ell}{\epsilon_y y \cos \theta_0} \right) + \frac{c}{4} \log \left( \frac{(y + \tau_1)^2 + x_1^2}{2x_1} \right) + \frac{c}{4} \tanh^{-1}(\sin \theta_0).
\]

(5.53)

\(^{11}\)Note that the entanglement negativity islands together constitute the entanglement entropy island for \( R_L \cup R_R \).

\(^{12}\)The correlators are factorized into their respective contractions as depicted by the choice of the extremal surfaces in fig. 24.
Extremizing the above generalized entanglement negativity with respect to $y$ we obtain

$$\partial_y E_{\text{gen}}^{\text{bdy}}(R_L : R_R) = 0 \implies y = \sqrt{\tau_1^2 + x_1^2}. \tag{5.54}$$

Now, substituting the value of $y$ in eq. (5.53), the entanglement negativity between the radiation subsystems for the disconnected phase in the boundary description is given by

$$E_{\text{bdy}}^{\text{gen}}(R_L : R_R) = \frac{c}{4} \left[ \log \frac{\sqrt{\tau_1^2 + x_1^2} + \tau_1}{x_1} + \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) + \tanh^{-1}(\sin \theta_0) \right]. \tag{5.55}$$

Finally, transforming to the primed coordinates in eq. (5.2), performing the Lorentzian continuation and utilizing eq. (5.4) we obtain the entanglement negativity between the radiation subsystems in terms of the Rindler coordinates $(X, T)$ in the 2d effective boundary description as follows

$$E_{\text{bdy}}^{\text{gen}}(R_L : R_R) = \frac{c}{4} \left[ \log \frac{\sqrt{\tau_1^2 + x_1^2} + \tau_1}{x_1} + \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) + \tanh^{-1}(\sin \theta_0) \right]. \tag{5.55}$$

Finally, transforming to the primed coordinates in eq. (5.2), performing the Lorentzian continuation and utilizing eq. (5.4) we obtain the entanglement negativity between the radiation subsystems in terms of the Rindler coordinates $(X, T)$ in the 2d effective boundary description as follows

$$E_{\text{bdy}}^{\text{gen}}(R_L : R_R) = \frac{c}{4} \left[ \log \frac{\sqrt{\tau_1^2 + x_1^2} + \tau_1}{x_1} + \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) + \tanh^{-1}(\sin \theta_0) \right]. \tag{5.55}$$

Bulk description

In the disconnected phase, due to the presence of entanglement islands, a portion of the EOW brane $Q$ is contained within the entanglement wedge of the radiation in the 3d bulk description. As depicted in fig. 24, $|M'N'|$ denotes the entanglement entropy island corresponding to $R_L \cup R_R$. The bulk EWCS ends on the EOW brane at the point $O'$ and splits the entanglement wedge corresponding to $R_L \cup R_R$ into two codimension one regions $R_L$ and $R_R$ respectively. For this phase, the entanglement negativity between $R_L$ and $R_R$ corresponds to the configuration of disjoint subsystems $|PM| \cup |O'M'|$ and $|NQ| \cup |O'N'|$, sandwiching the region $|MM'| \cup |NN'|$ in between. Therefore, we may employ the DES formula in eq. (3.3) to obtain

$$E_{\text{gen}}^{\text{bulk}}(R_L : R_R) = \frac{3}{16G_N} \left[ (\mathcal{L}_1 + \mathcal{L}_4) + (\mathcal{L}_2 + \mathcal{L}_3) - (\mathcal{L}_3 + \mathcal{L}_4) - \mathcal{L}_5 \right] + \mathcal{E}^{\text{eff}}(R_L : R_R) \tag{5.57}$$

where as earlier, the effective entanglement negativity between bulk matter fields reduces to that between the adjacent intervals $I_L \equiv |O'M'|$ and $I_R \equiv |O'N'|$ on the EOW brane. The lengths of the extremal surfaces in eq. (5.57) are given by \[10, 11, 92\]

$$\mathcal{L}_1 = \ell \log \frac{\tau_1 + z \tan \theta_0}{z} + x_1^2 + z^2 \right) - \ell \log \frac{4\epsilon}{\tau_1^2 + x_1^2} = \mathcal{L}_2, \tag{5.58}$$

$$\mathcal{L}_5 = 2\ell \log(2x_1) - 2\ell \log \left( \frac{4\epsilon}{(\tau_1^2 + x_1^2)} \right), \tag{5.58}$$

where the second logarithmic term corresponds to the UV cut-off in the unprimed coordinates \[61, 92\].
The effective entanglement negativity in eq. (5.57) between the island regions $I_L$ and $I_R$ may be computed through a three-point correlator of twist fields inserted at the endpoints of the intervals as follows

$$\mathcal{E}^{\text{eff}}(I_L : I_R) = \lim_{n_e \to 1} \log \left[ \left( \epsilon^{2} \Omega_{M'} \Omega_{N'} \right)^{\Delta_{n_e}^{c_e}} \langle T_{n_e}(M') \bar{T}_{n_e}^{2}(O') T_{n_e}(N') \rangle_{\text{BCFT} \otimes n_e} \right].$$  

The above three-point twist correlator on the half plane describing the BCFT may be expressed as a six-point correlator of chiral twist fields on the whole complex plane using the doubling trick [116,120] as follows

$$\langle \bar{T}_{n_e}(M') T_{n_e}(M) \bar{T}_{n_e}^{2}(O') T_{n_e}^{2}(O) T_{n_e}(N') \bar{T}_{n_e}(N) \rangle_{\text{CFT} \otimes n_e},$$  

where $M$, $N$ and $O$ are the image of the points $M'$, $N'$ and $O'$ on the EOW brane respectively. In the large central charge limit, the six-point correlator may be further factorized in the dominant channel similar to eq. (5.51) as follows

$$\langle \bar{T}_{n_e}(M') T_{n_e}(M) \bar{T}_{n_e}^{2}(O') T_{n_e}^{2}(O) T_{n_e}(N') \bar{T}_{n_e}(N) \rangle_{\text{CFT} \otimes n_e}.$$  

Now, reversing the doubling trick and subsequently employing the replica limit $n_e \to 1$, we may obtain the effective entanglement negativity between $I_L$ and $I_R$ as

$$\mathcal{E}^{\text{eff}}(I_L : I_R) = \lim_{n_e \to 1} \log \left[ \left( \epsilon^{2} \Omega_{O'} \right)^{\Delta_{n_e}^{c_e}} \langle \bar{T}_{n_e}^{2}(O') \rangle_{\text{BCFT} \otimes n_e} \right] = \frac{c}{4} \log \frac{2 \ell}{\epsilon y \cos \theta_0},$$  

We may obtain the generalized entanglement negativity by substituting eqs. (5.58) and (5.62) in eq. (5.57) to be

$$\mathcal{E}^{\text{bulk}}_{\text{gen}}(\mathcal{R}_L : \mathcal{R}_R) = \frac{c}{4} \log \left( \frac{(\tau_1 + z \tan \theta_0)^2 + x_1^2 + z^2}{2 x_1 z} \right) + \log \left( \frac{2 \ell}{\epsilon y \cos \theta_0} \right),$$  

where we have used the Brown-Henneaux formula [162] in the first term. The location of the dynamical point $O'$ in the above expression is fixed through extremization at

$$z = \sqrt{\tau_1^2 + x_1^2 \cos \theta_0}.$$  

Substituting the above value of $z$ in eq. (5.63) and subsequently transforming the result to the Rindler coordinates using eqs. (5.2) and (5.4), we may obtain the entanglement negativity
between the radiation subsystems $R_L$ and $R_R$ to be

$$
E_{\text{bulk}}^{(R_L : R_R)} = \frac{c}{4} \left[ \log \frac{e^{2X_1} - 1 + \sqrt{4e^{2X_1} \cosh^2 T + (e^{2X_1} - 1)^2}}{2e^{X_1} \cosh T} + \log \left( \frac{\ell}{\epsilon_y \cos \theta_0} \right) 
+ \log \left( \frac{\cos \theta_0}{1 - \sin \theta_0} \right) \right].
$$

(5.65)

We again observe that the above result in the bulk description matches exactly with the entanglement negativity from the boundary description in eq. (5.56) for the disconnected phase.

### 5.4.3 Page curve

We now analyse the behaviour of the time dependent entanglement negativity between radiation subsystems as discussed in last two subsections. We observe that the entanglement negativity decreases with the Rindler time $T$ in both phases and eventually plateaus out. In the limit when both the radiation subsystems $R_L$ and $R_R$ extends to spatial infinity, namely $X_1 \to \infty$, the asymptotic behaviour of the entanglement negativity is given as

$$
E_{\text{bulk}}^{(R_L : R_R)} = \begin{cases} 
\frac{c}{4} \left( X_1 - X_0 - 2 \log(2 \cosh T) \right) & T < T_P \\
\frac{c}{4} \left( X_1 - \log \cosh T + \log \frac{\cos \theta_0}{1 - \sin \theta_0} + \log \frac{\epsilon_y}{\cos \theta_0} \right) & T > T_P,
\end{cases}
$$

(5.66)

where $T_P$ is the Page time for the entanglement entropy as given in eq. (5.26). Hence, the entanglement negativity shows a sudden jump at $T_P$, and in the limit $X_1 \to \infty$ this jump is given by

$$
\Delta E = \frac{c}{4} \left[ 2 \log \left( \frac{2\ell}{\epsilon_y \cos \theta_0} \right) + 2 \log \left( \frac{\cos \theta_0}{1 - \sin \theta_0} \right) + \log \left( e^{2X_0} - 1 \right) \right].
$$

(5.67)

The analogue of the Page curve for the entanglement negativity for this bipartite configuration is shown in fig. 25.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{page_curve.png}
\caption{The Page curve for entanglement negativity between the radiation and the radiation with respect to the Rindler time $T$ in units of $\frac{c}{4}$. Here we choose $X_0 = 1$, $X_1 = 30$, $\epsilon_y = 0.1$, $\ell = 1$ and $\theta_0 = \frac{\pi}{3}, \frac{\pi}{6}$.}
\end{figure}
6 Summary

To summarize, in this article, we have proposed a defect extremal surface (DES) prescription for the entanglement negativity of bipartite mixed state configurations in the AdS$_3$/BCFT$_2$ scenarios which include defect conformal matter on the EOW brane. Furthermore, we have extended the island formula for the entanglement negativity to the framework of the defect AdS$_3$/BCFT$_2$, utilizing the lower dimensional effective description involving a CFT$_2$ coupled to semiclassical gravity. Interestingly, the bulk DES formula may be understood as the doubly holographic counterpart of the island formula for the entanglement negativity.

To begin with, we computed the entanglement negativity in the time independent scenarios involving adjacent and disjoint intervals on a static time slice of the conformal boundary of the 3d braneworld. To this end, we have demonstrated that the entanglement negativity for various bipartite states obtained through the DES formula matches exactly with the results from the corresponding QES prescription involving entanglement negativity islands. Subsequently we obtained the entanglement negativity in various time-dependent scenarios involving an eternal black hole coupled to a radiation bath in the effective lower dimensional picture. In such time-dependent scenarios, we have obtained the entanglement negativity between subsystems in the black hole interior, between subsystems involving black hole and the radiation bath, and between subsystems in the radiation bath utilizing the island as well as the bulk DES formulae. In this connection, we have studied the time evolution of the entanglement negativity for the above configurations and obtained the analogues of the Page curves. Interestingly, the transitions between different phases of the defect extremal surfaces corresponding to the entanglement negativity for the above configurations occur precisely at the Page time for the corresponding entanglement entropy. Remarkably, it was observed that the entanglement negativity from the boundary and bulk proposals are in perfect agreement for the time-dependent cases, thus demonstrating the equivalence of both formulations. This serves as a strong consistency check for our proposals. We would like to emphasize that our results might lead to several insights about the structure of quantum information about the black hole interior encoded in the Hawking radiation.

There are several possible future directions to investigate. One such issue would be the extension of our proposals to higher dimensional defect AdS/BCFT scenarios. One may also generalize our doubly holographic formulation for the entanglement negativity with the defect brane at a constant tension to arbitrary embeddings of the brane in the 3d bulk geometry. Furthermore, it would also be interesting to derive the bulk DES formula for the entanglement negativity through the gravitational path integral techniques utilizing the replica symmetry breaking wormhole saddles. We leave these open issues for future investigations.

7 Acknowledgement

The work of GS is partially supported by the Jag Mohan Chair Professor position at the Indian Institute of Technology, Kanpur.

References

[1] A. Almheiri, R. Mahajan, J. Maldacena, and Y. Zhao, “The Page curve of Hawking radiation from semiclassical geometry,” JHEP 03 (2020) 149, arXiv:1908.10996 [hep-th].

[2] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, “The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole,” JHEP 12 (2019) 063, arXiv:1905.08762 [hep-th].
[3] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, “Replica Wormholes and the Entropy of Hawking Radiation,” *JHEP* 05 (2020) 013, arXiv:1911.12333 [hep-th].

[4] A. Almheiri, R. Mahajan, and J. E. Santos, “Entanglement islands in higher dimensions,” *SciPost Phys.* 9 no. 1, (2020) 001, arXiv:1911.09666 [hep-th].

[5] A. Almheiri, R. Mahajan, and J. Maldacena, “Islands outside the horizon,” arXiv:1910.11077 [hep-th].

[6] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, “The entropy of Hawking radiation,” arXiv:2006.06872 [hep-th].

[7] D. N. Page, “Information in black hole radiation,” *Phys. Rev. Lett.* 71 (1993) 3743–3746, arXiv:hep-th/9306083.

[8] D. N. Page, “Average entropy of a subsystem,” *Phys. Rev. Lett.* 71 (1993) 1291–1294, arXiv:gr-qc/9305007.

[9] D. N. Page, “Time Dependence of Hawking Radiation Entropy,” *JCAP* 09 (2013) 028, arXiv:1301.4995 [hep-th].

[10] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” *Phys. Rev. Lett.* 96 (2006) 181602, arXiv:hep-th/0603001.

[11] V. E. Hubeny, M. Rangamani, and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” *JHEP* 07 (2007) 062, arXiv:0705.0016 [hep-th].

[12] T. Faulkner, A. Lewkowycz, and J. Maldacena, “Quantum corrections to holographic entanglement entropy,” *JHEP* 11 (2013) 074, arXiv:1307.2892 [hep-th].

[13] N. Engelhardt and A. C. Wall, “Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime,” *JHEP* 01 (2015) 073, arXiv:1408.3203 [hep-th].

[14] L. Anderson, O. Parrikar, and R. M. Soni, “Islands with gravitating baths: towards ER = EPR,” *JHEP* 21 (2020) 226, arXiv:2103.14746 [hep-th].

[15] Y. Chen, “Pulling Out the Island with Modular Flow,” *JHEP* 03 (2020) 033, arXiv:1912.02210 [hep-th].

[16] V. Balasubramanian, A. Kar, O. Parrikar, G. Sárosi, and T. Ugajin, “Geometric secret sharing in a model of Hawking radiation,” *JHEP* 01 (2021) 177, arXiv:2003.05448 [hep-th].

[17] Y. Chen, X.-L. Qi, and P. Zhang, “Replica wormhole and information retrieval in the SYK model coupled to Majorana chains,” *JHEP* 06 (2020) 121, arXiv:2003.13147 [hep-th].

[18] F. F. Gautason, L. Schneiderbauer, W. Sybesma, and L. Thorlacius, “Page Curve for an Evaporating Black Hole,” *JHEP* 05 (2020) 091, arXiv:2004.00598 [hep-th].

[19] A. Bhattacharya, “Multipartite purification, multiboundary wormholes, and islands in $AdS_3/CFT_2$,” *Phys. Rev. D* 102 no. 4, (2020) 046013, arXiv:2003.11870 [hep-th].

[20] T. Anegawa and N. Iizuka, “Notes on islands in asymptotically flat 2d dilaton black holes,” *JHEP* 07 (2020) 036, arXiv:2004.01601 [hep-th].
[21] K. Hashimoto, N. Iizuka, and Y. Matsuo, “Islands in Schwarzschild black holes,” *JHEP* **06** (2020) 085, arXiv:2004.05863 [hep-th].

[22] T. Hartman, E. Shaghoulian, and A. Strominger, “Islands in Asymptotically Flat 2D Gravity,” *JHEP* **07** (2020) 022, arXiv:2004.13857 [hep-th].

[23] C. Krishnan, V. Patil, and J. Pereira, “Page Curve and the Information Paradox in Flat Space,” arXiv:2005.02993 [hep-th].

[24] M. Alishahiha, A. Faraji Astaneh, and A. Nasch, “Island in the presence of higher derivative terms,” *JHEP* **02** (2021) 035, arXiv:2005.08715 [hep-th].

[25] H. Geng and A. Karch, “Massive islands,” *JHEP* **09** (2020) 121, arXiv:2006.02438 [hep-th].

[26] T. Li, J. Chu, and Y. Zhou, “Reflected Entropy for an Evaporating Black Hole,” *JHEP* **11** (2020) 155, arXiv:2006.10846 [hep-th].

[27] V. Chandrasekaran, M. Miyaji, and P. Rath, “Including contributions from entanglement islands to the reflected entropy,” *Phys. Rev. D* **102** no. 8, (2020) 086009, arXiv:2006.10754 [hep-th].

[28] D. Bak, C. Kim, S.-H. Yi, and J. Yoon, “Unitarity of entanglement and islands in two-sided Janus black holes,” *JHEP* **01** (2021) 155, arXiv:2006.11717 [hep-th].

[29] C. Krishnan, “Critical Islands,” *JHEP* **01** (2021) 179, arXiv:2007.06551 [hep-th].

[30] A. Karlsson, “Replica wormhole and island incompatibility with monogamy of entanglement,” arXiv:2007.10523 [hep-th].

[31] T. Hartman, Y. Jiang, and E. Shaghoulian, “Islands in cosmology,” *JHEP* **11** (2020) 111, arXiv:2008.01022 [hep-th].

[32] V. Balasubramanian, A. Kar, and T. Ugajin, “Entanglement between two disjoint universes,” *JHEP* **02** (2021) 136, arXiv:2008.05274 [hep-th].

[33] V. Balasubramanian, A. Kar, and T. Ugajin, “Islands in de Sitter space,” *JHEP* **02** (2021) 072, arXiv:2008.05275 [hep-th].

[34] W. Sybesma, “Pure de Sitter space and the island moving back in time,” *Class. Quant. Grav.* **38** no. 14, (2021) 145012, arXiv:2008.07994 [hep-th].

[35] H. Z. Chen, R. C. Myers, D. Neuenfeld, I. A. Reyes, and J. Sandor, “Quantum Extremal Islands Made Easy, Part II: Black Holes on the Brane,” *JHEP* **12** (2020) 025, arXiv:2010.00018 [hep-th].

[36] Y. Ling, Y. Liu, and Z.-Y. Xian, “Island in Charged Black Holes,” *JHEP* **03** (2021) 251, arXiv:2010.00037 [hep-th].

[37] J. Hernandez, R. C. Myers, and S.-M. Ruan, “Quantum extremal islands made easy. Part III. Complexity on the brane,” *JHEP* **02** (2021) 173, arXiv:2010.16398 [hep-th].

[38] D. Marolf and H. Maxfield, “Observations of Hawking radiation: the Page curve and baby universes,” *JHEP* **04** (2021) 272, arXiv:2010.06602 [hep-th].

[39] Y. Matsuo, “Islands and stretched horizon,” *JHEP* **07** (2021) 051, arXiv:2011.08814 [hep-th].
[40] I. Akal, Y. Kusuki, N. Shiba, T. Takayanagi, and Z. Wei, “Entanglement Entropy in a Holographic Moving Mirror and the Page Curve,” Phys. Rev. Lett. 126 no. 6, (2021) 061604, arXiv:2011.12005 [hep-th].

[41] E. Caceres, A. Kundu, A. K. Patra, and S. Shashi, “Warped information and entanglement islands in AdS/WCFT,” JHEP 07 (2021) 004, arXiv:2012.05425 [hep-th].

[42] S. Raju, “Lessons from the information paradox,” Phys. Rept. 943 (2022) 2187, arXiv:2012.05770 [hep-th].

[43] F. Deng, J. Chu, and Y. Zhou, “Defect extremal surface as the holographic counterpart of Island formula,” JHEP 03 (2021) 008, arXiv:2012.07612 [hep-th].

[44] T. Anous, M. Meineri, P. Pelliconi, and J. Sonner, “Sailing past the End of the World and discovering the Island,” arXiv:2202.11718 [hep-th].

[45] R. Bousso and E. Wildenhain, “Islands in Closed and Open Universes,” arXiv:2202.05278 [hep-th].

[46] Q.-L. Hu, D. Li, R.-X. Miao, and Y.-Q. Zeng, “AdS/BCFT and Island for curvature-squared gravity,” arXiv:2202.03304 [hep-th].

[47] G. Grimaldi, J. Hernandez, and R. C. Myers, “Quantum Extremal Islands Made Easy, Part IV: Massive Black Holes on the Brane,” arXiv:2202.00679 [hep-th].

[48] C. Akers, T. Faulkner, S. Lin, and P. Rath, “The Page Curve for Reflected Entropy,” arXiv:2201.11730 [hep-th].

[49] M.-H. Yu, C.-Y. Lu, X.-H. Ge, and S.-J. Sin, “Island, Page Curve and Superradiance of Rotating BTZ Black Holes,” arXiv:2112.14361 [hep-th].

[50] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, “Entanglement Phase Structure of a Holographic BCFT in a Black Hole Background,” arXiv:2112.09132 [hep-th].

[51] C.-J. Chou, H. B. Lao, and Y. Yang, “Page Curve of Effective Hawking Radiation,” arXiv:2111.14551 [hep-th].

[52] T. J. Hollowood, S. P. Kumar, A. Legramandi, and N. Talwar, “Grey-body Factors, Irreversibility and Multiple Island Saddles,” arXiv:2111.02248 [hep-th].

[53] S. He, Y. Sun, L. Zhao, and Y.-X. Zhang, “The universality of islands outside the horizon,” arXiv:2110.07598 [hep-th].

[54] I. Aref’eva and I. Volovich, “A Note on Islands in Schwarzschild Black Holes,” arXiv:2110.04233 [hep-th].

[55] Y. Ling, P. Liu, Y. Liu, C. Niu, Z.-Y. Xian, and C.-Y. Zhang, “Reflected entropy in double holography,” JHEP 02 (2022) 037, arXiv:2109.09243 [hep-th].

[56] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, “Partial islands and subregion complexity in geometric secret-sharing model,” JHEP 12 (2021) 091, arXiv:2109.07842 [hep-th].

[57] S. Azarnia, R. Fareghbal, A. Naseh, and H. Zolfi, “Islands in flat-space cosmology,” Phys. Rev. D 104 no. 12, (2021) 126017, arXiv:2109.04795 [hep-th].
[58] A. Saha, S. Gangopadhyay, and J. P. Saha, “Mutual information, islands in black holes and the Page curve,” arXiv:2109.02996 [hep-th].

[59] T. J. Hollowood, S. P. Kumar, A. Legramandi, and N. Talwar, “Ephemeral islands, plunging quantum extremal surfaces and BCFT channels,” JHEP 01 (2022) 078, arXiv:2109.01895 [hep-th].

[60] P.-C. Sun, “Entanglement Islands from Holographic Thermalization of Rotating Charged Black Hole,” arXiv:2108.12557 [hep-th].

[61] T. Li, M.-K. Yuan, and Y. Zhou, “Defect extremal surface for reflected entropy,” JHEP 01 (2022) 018, arXiv:2108.08544 [hep-th].

[62] S. E. Aguilar-Gutierrez, A. Chatwin-Davies, T. Hertog, N. Pinzani-Fokeeva, and B. Robinson, “Islands in multiverse models,” JHEP 11 (2021) 212, arXiv:2108.01278 [hep-th].

[63] B. Ahn, S.-E. Bak, H.-S. Jeong, K.-Y. Kim, and Y.-W. Sun, “Islands in charged linear dilaton black holes,” Phys. Rev. D 105 no. 4, (2022) 046012, arXiv:2107.07444 [hep-th].

[64] M.-H. Yu and X.-H. Ge, “Islands and Page curves in charged dilaton black holes,” Eur. Phys. J. C 82 no. 1, (2022) 14, arXiv:2107.03031 [hep-th].

[65] Y. Lu and J. Lin, “Islands in Kaluza–Klein black holes,” Eur. Phys. J. C 82 no. 2, (2022) 132, arXiv:2106.07845 [hep-th].

[66] E. Caceres, A. Kundu, A. K. Patra, and S. Shashi, “Page Curves and Bath Deformations,” arXiv:2107.00022 [hep-th].

[67] I. Akal, Y. Kusuki, N. Shiba, T. Takayanagi, and Z. Wei, “Holographic moving mirrors,” Class. Quant. Grav. 38 no. 22, (2021) 224001, arXiv:2106.11179 [hep-th].

[68] I. Aref’eva, T. Rusalev, and I. Volovich, “Entanglement entropy of near-extremal black hole,” arXiv:2202.10259 [hep-th].

[69] I. Aref’eva and I. Volovich, “Complete evaporation of black holes and Page curves,” arXiv:2202.00548 [hep-th].

[70] R. Bousso, X. Dong, N. Engelhardt, T. Faulkner, T. Hartman, S. H. Shenker, and D. Stanford, “Snowmass White Paper: Quantum Aspects of Black Holes and the Emergence of Spacetime,” arXiv:2201.03096 [hep-th].

[71] C. Krishnan and V. Mohan, “Interpreting the Bulk Page Curve: A Vestige of Locality on Holographic Screens,” arXiv:2112.13783 [hep-th].

[72] D.-f. Zeng, “Spontaneous Radiation of Black Holes,” arXiv:2112.12531 [hep-th].

[73] D. Teresi, “Islands and the de Sitter entropy bound,” arXiv:2112.03922 [hep-th].

[74] K. Okuyama and K. Sakai, “Page curve from dynamical branes in JT gravity,” JHEP 02 (2022) 087, arXiv:2111.09551 [hep-th].

[75] P. Chen, M. Sasaki, D.-h. Yeom, and J. Yoon, “Solving information loss paradox via Euclidean path integral,” arXiv:2111.01005 [hep-th].

[76] J. F. Pedraza, A. Svesko, W. Sybesma, and M. R. Visser, “Microcanonical Action and the Entropy of Hawking Radiation,” arXiv:2111.06912 [hep-th].
[77] B. Guo, M. R. R. Hughes, S. D. Mathur, and M. Mehta, “Contrasting the fuzzball and wormhole paradigms for black holes,” Turk. J. Phys. 45 no. 6, (2021) 281–365, arXiv:2111.05295 [hep-th].

[78] T. Kibe, P. Mandayam, and A. Mukhopadhyay, “Holographic spacetime, black holes and quantum error correcting codes: A review,” arXiv:2110.14669 [hep-th].

[79] R. Renner and J. Wang, “The black hole information puzzle and the quantum de Finetti theorem,” arXiv:2110.14653 [hep-th].

[80] X. Dong, S. McBride, and W. W. Weng, “Replica Wormholes and Holographic Entanglement Negativity,” arXiv:2110.11947 [hep-th].

[81] S. Raju, “Failure of the split property in gravity and the information paradox,” arXiv:2110.05470 [hep-th].

[82] C. H. Nam, “Entanglement entropy and Page curve of black holes with island in massive gravity,” arXiv:2108.10144 [hep-th].

[83] J. Kames-King, E. Verheijden, and E. Verlinde, “No Page Curves for the de Sitter Horizon,” arXiv:2108.09318 [hep-th].

[84] B. Chen, B. Czech, and Z.-z. Wang, “Quantum Information in Holographic Duality,” arXiv:2108.09188 [hep-th].

[85] Y. Sato, “Complexity in a moving mirror model,” arXiv:2108.04637 [hep-th].

[86] J. Kudler-Flam, V. Narovlansky, and S. Ryu, “Distinguishing Random and Black Hole Microstates,” PRX Quantum 2 no. 4, (2021) 040340, arXiv:2108.00011 [hep-th].

[87] X. Wang, K. Zhang, and J. Wang, “What can we learn about islands and state paradox from quantum information theory?,” arXiv:2107.09228 [hep-th].

[88] D. S. Ageev, “Shaping contours of entanglement islands in BCFT,” arXiv:2107.09083 [hep-th].

[89] L. Buoninfante, F. Di Filippo, and S. Mukohyama, “On the assumptions leading to the information loss paradox,” JHEP 10 (2021) 081, arXiv:2107.05662 [hep-th].

[90] M. Cadoni and A. P. Sanna, “Unitarity and Page Curve for Evaporation of 2D AdS Black Holes,” arXiv:2106.14738 [hep-th].

[91] D. Marolf and H. Maxfield, “The page curve and baby universes,” Int. J. Mod. Phys. D 30 no. 14, (2021) 2142027, arXiv:2105.12211 [hep-th].

[92] J. Chu, F. Deng, and Y. Zhou, “Page curve from defect extremal surface and island in higher dimensions,” JHEP 10 (2021) 149, arXiv:2105.09106 [hep-th].

[93] E. Y. Urbach, “The entanglement entropy of typical pure states and replica wormholes,” JHEP 12 (2021) 125, arXiv:2105.15059 [hep-th].

[94] R. Li, X. Wang, and J. Wang, “Island may not save the information paradox of Liouville black holes,” Phys. Rev. D 104 no. 10, (2021) 106015, arXiv:2105.03271 [hep-th].

[95] D. Neuenfeld, “Homology Conditions for RT Surfaces in Double Holography,” arXiv:2105.01130 [hep-th].

[96] L. Aalsma and W. Sybesma, “The Price of Curiosity: Information Recovery in de Sitter Space,” JHEP 05 (2021) 291, arXiv:2104.00006 [hep-th].
[97] K. Ghosh and C. Krishnan, “Dirichlet baths and the not-so-fine-grained Page curve,” *JHEP* **08** (2021) 119, arXiv:2103.17253 [hep-th].

[98] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, “Islands and complexity of eternal black hole and radiation subsystems for a doubly holographic model,” *JHEP* **05** (2021) 135, arXiv:2103.15852 [hep-th].

[99] H. Geng, Y. Nomura, and H.-Y. Sun, “Information paradox and its resolution in de Sitter holography,” *Phys. Rev. D* **103** no. 12, (2021) 126004, arXiv:2103.07477 [hep-th].

[100] C. Krishnan and V. Mohan, “Hints of gravitational ergodicity: Berry’s ensemble and the universality of the semi-classical Page curve,” *JHEP* **05** (2021) 126, arXiv:2102.07703 [hep-th].

[101] E. Verheijden and E. Verlinde, “From the BTZ black hole to JT gravity: geometrizing the island,” *JHEP* **11** (2021) 092, arXiv:2102.00922 [hep-th].

[102] R. Bousso and A. Shahbazi-Moghaddam, “Island Finder and Entropy Bound,” *Phys. Rev. D* **103** no. 10, (2021) 106005, arXiv:2101.11648 [hep-th].

[103] G. K. Karananas, A. Kehagias, and J. Taskas, “Islands in linear dilaton black holes,” *JHEP* **03** (2021) 253, arXiv:2101.00024 [hep-th].

[104] K. Goto, T. Hartman, and A. Tajdini, “Replica wormholes for an evaporating 2D black hole,” *JHEP* **04** (2021) 289, arXiv:2011.09043 [hep-th].

[105] A. Bhattacharya, A. Chanda, S. Maulik, C. Northe, and S. Roy, “Topological shadows and complexity of islands in multiboundary wormholes,” *JHEP* **02** (2021) 152, arXiv:2010.04134 [hep-th].

[106] H. Z. Chen, Z. Fisher, J. Hernandez, R. C. Myers, and S.-M. Ruan, “Evaporating Black Holes Coupled to a Thermal Bath,” *JHEP* **01** (2021) 065, arXiv:2007.11658 [hep-th].

[107] C. A. Agón, S. F. Lokhande, and J. F. Pedraza, “Local quenches, bulk entanglement entropy and a unitary Page curve,” *JHEP* **08** (2020) 152, arXiv:2004.15010 [hep-th].

[108] A. Laddha, S. G. Prabhu, S. Raju, and P. Shrivastava, “The Holographic Nature of Null Infinity,” *SciPost Phys.* **10** no. 2, (2021) 041, arXiv:2002.02448 [hep-th].

[109] C. Akers, N. Engelhardt, and D. Harlow, “Simple holographic models of black hole evaporation,” *JHEP* **08** (2020) 032, arXiv:1910.00972 [hep-th].

[110] H. Z. Chen, Z. Fisher, J. Hernandez, R. C. Myers, and S.-M. Ruan, “Information Flow in Black Hole Evaporation,” *JHEP* **03** (2020) 152, arXiv:1911.03402 [hep-th].

[111] J. K. Basak, D. Basu, V. Malvimat, H. Parihar, and G. Sengupta, “Reflected Entropy and Entanglement Negativity for Holographic Moving Mirrors,” arXiv:2204.06015 [hep-th].

[112] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, “Information Transfer with a Gravitating Bath,” *SciPost Phys.* **10** no. 5, (2021) 103, arXiv:2012.04671 [hep-th].

[113] H. Geng, S. Lüst, R. K. Mishra, and D. Wakeham, “Holographic BCFTs and Communicating Black Holes,” *jhep* **08** (2021) 003, arXiv:2104.07039 [hep-th].
[114] M. Afrasiar, J. K. Basak, A. Chandra, and G. Sengupta, “Islands for Entanglement Negativity in Communicating Black Holes (Appearing together with this paper),” arXiv:2205.xxxxx [hep-th].

[115] J. Maldacena and L. Susskind, “Cool horizons for entangled black holes,” Fortsch. Phys. 61 (2013) 781–811, arXiv:1306.0533 [hep-th].

[116] J. L. Cardy, “Boundary conformal field theory,” arXiv:hep-th/0411189.

[117] T. Takayanagi, “Holographic Dual of BCFT,” Phys. Rev. Lett. 107 (2011) 101602, arXiv:1105.5165 [hep-th].

[118] M. Fujita, T. Takayanagi, and E. Tonni, “Aspects of AdS/BCFT,” JHEP 11 (2011) 043, arXiv:1108.5152 [hep-th].

[119] M. Rozali, J. Sully, M. Van Raamsdonk, C. Waddell, and D. Wakeham, “Information radiation in BCFT models of black holes,” JHEP 05 (2020) 004, arXiv:1910.12836 [hep-th].

[120] J. Sully, M. V. Raamsdonk, and D. Wakeham, “BCFT entanglement entropy at large central charge and the black hole interior,” JHEP 03 (2021) 167, arXiv:2004.13088 [hep-th].

[121] J. Kastikainen and S. Shashi, “Structure of holographic BCFT correlators from geodesics,” Phys. Rev. D 105 no. 4, (2022) 046007, arXiv:2109.00079 [hep-th].

[122] S. Dutta and T. Faulkner, “A canonical purification for the entanglement wedge cross-section,” JHEP 03 (2021) 178, arXiv:1905.00577 [hep-th].

[123] C. Akers, T. Faulkner, S. Lin, and P. Rath, “Reflected entropy in random tensor networks,” arXiv:2112.09122 [hep-th].

[124] G. Vidal and R. F. Werner, “Computable measure of entanglement,” Phys. Rev. A 65 (2002) 032314, arXiv:quant-ph/0102117.

[125] M. B. Plenio, “Logarithmic Negativity: A Full Entanglement Monotone That is not Convex,” Phys. Rev. Lett. 95 no. 9, (2005) 090503, arXiv:quant-ph/0505071.

[126] T. Takayanagi and K. Umemoto, “Entanglement of purification through holographic duality,” Nature Phys. 14 no. 6, (2018) 573–577, arXiv:1708.09393 [hep-th].

[127] P. Nguyen, T. Devakul, M. G. Halbasch, M. P. Zaletel, and B. Swingle, “Entanglement of purification: from spin chains to holography,” JHEP 01 (2018) 098, arXiv:1709.07424 [hep-th].

[128] Q. Wen, “Balanced Partial Entanglement and the Entanglement Wedge Cross Section,” JHEP 04 (2021) 301, arXiv:2103.00415 [hep-th].

[129] H. A. Camargo, P. Nandy, Q. Wen, and H. Zhong, “Balanced Partial Entanglement and Mixed State Correlations,” arXiv:2201.13362 [hep-th].

[130] P. Calabrese, J. Cardy, and E. Tonni, “Entanglement negativity in quantum field theory,” Phys. Rev. Lett. 109 (2012) 130502, arXiv:1206.3092 [cond-mat.stat-mech].

[131] P. Calabrese, J. Cardy, and E. Tonni, “Entanglement negativity in extended systems: A field theoretical approach,” J. Stat. Mech. 1302 (2013) P02008, arXiv:1210.5359 [cond-mat.stat-mech].
[132] P. Calabrese, J. Cardy, and E. Tonni, “Finite temperature entanglement negativity in conformal field theory,” *J. Phys. A* 48 no. 1, (2015) 015006, arXiv:1408.3043 [cond-mat.stat-mech].

[133] V. Malvimat, H. Parihar, B. Paul, and G. Sengupta, “Entanglement Negativity in Galilean Conformal Field Theories,” *Phys. Rev. D* 100 no. 2, (2019) 026001, arXiv:1810.08162 [hep-th].

[134] D. Basu, A. Chandra, H. Parihar, and G. Sengupta, “Entanglement Negativity in Flat Holography,” arXiv:2102.05685 [hep-th].

[135] M. R. Setare and M. Koochgard, “Holographic entanglement negativity in flat space generalized minimal massive gravity,” *Int. J. Mod. Phys. A* 37 no. 05, (2022) 2250024, arXiv:2105.00167 [hep-th].

[136] M. Rangamani and M. Rota, “Comments on Entanglement Negativity in Holographic Field Theories,” *JHEP* 10 (2014) 060, arXiv:1406.6989 [hep-th].

[137] P. Chaturvedi, V. Malvimat, and G. Sengupta, “Entanglement negativity, Holography and Black holes,” *Eur. Phys. J. C* 78 no. 6, (2018) 499, arXiv:1602.01147 [hep-th].

[138] P. Chaturvedi, V. Malvimat, and G. Sengupta, “Holographic Quantum Entanglement Negativity,” *JHEP* 05 (2018) 172, arXiv:1609.06609 [hep-th].

[139] P. Jain, V. Malvimat, S. Mondal, and G. Sengupta, “Holographic entanglement negativity conjecture for adjacent intervals in AdS/CFT,” *Phys. Lett. B* 793 (2019) 104–109, arXiv:1707.08293 [hep-th].

[140] P. Jain, V. Malvimat, S. Mondal, and G. Sengupta, “Holographic entanglement negativity for adjacent subsystems in AdS_{d+1}/CFT_d,” *Eur. Phys. J. Plus* 133 no. 8, (2018) 300, arXiv:1708.00612 [hep-th].

[141] P. Chaturvedi, V. Malvimat, and G. Sengupta, “Covariant holographic entanglement negativity,” *Eur. Phys. J. C* 78 no. 9, (2018) 776, arXiv:1611.00593 [hep-th].

[142] P. Jain, V. Malvimat, S. Mondal, and G. Sengupta, “Covariant holographic entanglement negativity for adjacent subsystems in AdS_3 /CFT_2,” *Nucl. Phys. B* 945 (2019) 114683, arXiv:1710.06138 [hep-th].

[143] P. Jain, V. Malvimat, S. Mondal, and G. Sengupta, “Holographic Entanglement Negativity for Conformal Field Theories with a Conserved Charge,” *Eur. Phys. J. C* 78 no. 11, (2018) 908, arXiv:1804.09078 [hep-th].

[144] V. Malvimat, S. Mondal, B. Paul, and G. Sengupta, “Holographic entanglement negativity for disjoint intervals in AdS_3/CFT_2,” *Eur. Phys. J. C* 79 no. 3, (2019) 191, arXiv:1810.08015 [hep-th].

[145] V. Malvimat, S. Mondal, B. Paul, and G. Sengupta, “Covariant holographic entanglement negativity for disjoint intervals in AdS_3/CFT_2,” *Eur. Phys. J. C* 79 no. 6, (2019) 514, arXiv:1812.03117 [hep-th].

[146] J. Kumar Basak, H. Parihar, B. Paul, and G. Sengupta, “Holographic entanglement negativity for disjoint subsystems in AdS_{d+1}/CFT_d,” arXiv:2001.10534 [hep-th].

[147] S. Mondal, B. Paul, G. Sengupta, and P. Sharma, “Holographic entanglement negativity for a single subsystem in conformal field theories with a conserved charge,” arXiv:2102.05848 [hep-th].
[148] M. Afrasiar, J. K. Basak, V. Raj, and G. Sengupta, “Holographic Entanglement Negativity for Disjoint Subsystems in Conformal Field Theories with a Conserved Charge,” arXiv:2106.14918 [hep-th].

[149] D. Basu, H. Parihar, V. Raj, and G. Sengupta, “Entanglement negativity, reflected entropy, and anomalous gravitation,” Phys. Rev. D 105 no. 8, (2022) 086013, arXiv:2202.00683 [hep-th].

[150] M. Kulaxizi, A. Parnachev, and G. Policastro, “Conformal Blocks and Negativity at Large Central Charge,” JHEP 09 (2014) 010, arXiv:1407.0324 [hep-th].

[151] V. Malvimat and G. Sengupta, “Entanglement negativity at large central charge,” Phys. Rev. D 105 no. 8, (2022) 086013, arXiv:2202.00683 [hep-th].

[152] J. Kudler-Flam and S. Ryu, “Entanglement negativity and minimal entanglement wedge cross sections in holographic theories,” Phys. Rev. D 99 no. 10, (2019) 106014, arXiv:1808.00446 [hep-th].

[153] Y. Kusuki, J. Kudler-Flam, and S. Ryu, “Derivation of Holographic Negativity in AdS3/CFT2,” Phys. Rev. Lett. 123 no. 13, (2019) 131603, arXiv:1907.07824 [hep-th].

[154] J. Kumar Basak, V. Malvimat, H. Parihar, B. Paul, and G. Sengupta, “On minimal entanglement wedge cross section for holographic entanglement negativity,” arXiv:2002.10272 [hep-th].

[155] J. Kumar Basak, H. Parihar, B. Paul, and G. Sengupta, “Covariant holographic negativity from the entanglement wedge in AdS3/CFT2,” arXiv:2102.05676 [hep-th].

[156] D. Basu, A. Chandra, V. Raj, and G. Sengupta, “Entanglement wedge in flat holography and entanglement negativity,” SciPost Phys. Core 5 (2022) 013, arXiv:2106.14896 [hep-th].

[157] J. Kumar Basak, D. Basu, V. Malvimat, H. Parihar, and G. Sengupta, “Islands for entanglement negativity,” SciPost Phys. 12 no. 1, (2022) 003, arXiv:2012.03983 [hep-th].

[158] J. Kumar Basak, D. Basu, V. Malvimat, H. Parihar, and G. Sengupta, “Page curve for entanglement negativity through geometric evaporation,” SciPost Phys. 12 no. 1, (2022) 004, arXiv:2106.12593 [hep-th].

[159] A. Karch and L. Randall, “Locally localized gravity,” JHEP 05 (2001) 008, arXiv:hep-th/0011156.

[160] L. Randall and R. Sundrum, “An Alternative to compactification,” Phys. Rev. Lett. 83 (1999) 4690–4693, arXiv:hep-th/9906064.

[161] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231–252, arXiv:hep-th/9711200.

[162] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” Commun. Math. Phys. 104 (1986) 207–226.