Extensions of the Shannon Entropy and the Chaos Game Algorithm to Hyperbolic Numbers Plane

Gamaliel Yafte Téllez Sánchez, Juan Bory Reyes

Abstract

In this paper we provide extensions to hyperbolic numbers plane of the classical chaos game algorithm and the Shannon entropy. Both notions connected with that of probability with values in hyperbolic number, introduced by D. Alpay et al [1]. Within this context, particular attention has been paid to the interpretation of the hyperbolic valued probabilities and the hyperbolic extension of entropy as well.

Keywords: Hyperbolic numbers, Chaos game, Entropy, Probability

2010 MSC: 30G35, 28A80, 28D20

1. Introduction

In 1848 James Cockle [2] introduced the Tessarines, which form an algebra isomorphic to the bicomplex numbers [3]. Hyperbolic numbers, also known as the Lorentz numbers or double numbers are a particular type of the Tessarines. The complex numbers and the hyperbolic numbers are two-dimensional vector spaces over the real numbers, so each can identified with points in the plane \( \mathbb{R}^2 \).

The hyperbolic numbers can be considered as an hybrid between the real and complex numbers, in part because they behave in more similar nature to the...
real numbers, meanwhile having properties related with the complex numbers. Due to the significance of this fact for our purposes, we have compiled some basic notation and terminology in Section 2. For a deeper discussion of this phenomenon we refer the reader to [4, 5, 6, 7].

Given a finite discrete probability distribution, the amount of uncertainty of the distribution, that is, the amount of uncertainty concerning the outcome of an experiment, the possible results of which have the given probabilities is a very important quantity called the entropy of the distribution and plays a key role in many aspects of statistical mechanics and information theory.

The idea of measure the average missing information for a given physical system associated with each possible data value, as the negative logarithm of the probability mass function for the values, was presented by Claude Elwood Shannon in his seminal paper [8]. He proved (original proof goes back to the work of Erdös in [9]) that logarithm is the only function fulfills three basic requisites for defining an entropy-like measure (see below). Section 3 discusses these assumptions in the hyperbolic numbers case.

During the past few years, several extensions of Shannon’s original work have resulted in many alternative measures of entropy. For instance, by relaxing the third of the requirements, that of additive property, Alfréd Rényi [10] was able to extend the Shannon entropy to a continuous parametric family of entropy measures. Another way of stating a refinement of the requirements for entropy measures and so, the systems of postulate characterizing axiomatically these entropies, were given by different authors, see for instance [11, 12, 13, 14, 15, 16, 17].

Since the general acceptance that a mathematical expression of entropy measures are interesting enough, several theoretical attempts have been made to propose different examples of such expression (see for instance [18, 19, 20, 21, 22, 23]). In the vast majority of cases, the key to extending entropies expressions has been the introduction of a revised logarithm.

Proceeding further in this direction, in Section 3, associated with sets of hyperbolic valued probabilities two extensions of the Shannon entropy (called
weak and strong hyperbolic entropy) are introduced by exploiting the hyperbolic logarithm. A generalization of probability space and basic properties in hyperbolic numbers plane is due to D. Alpay et al [1].

The construction of Cantor like sets was generalized into the context of hyperbolic numbers in [24, 25], where the partial order structure of such numbers was utilized. With the notion of iterated function systems (IFS for short) on hyperbolic numbers plane (see [26]) associating probabilities to every function of the IFS (as Barnsley made in [27]) a chaos game algorithm for hyperbolic numbers is proposed in Section 4, where the interpretation of the hyperbolic-valued probabilities meaning play a crucial role.

2. Hyperbolic numbers

The set of hyperbolic numbers or hyperbolic numbers plane is defined as the ring

\[ \mathbb{D} := \mathbb{R}[k] = \{ a + kb \mid a, b \in \mathbb{R} \}, \]

where \( k \notin \mathbb{R} \) and \( k^2 = 1 \). It is a commutative ring with zero divisor. Two important elements are

\[ e_1 = \frac{1}{2}(1 + k), \quad e_2 = \frac{1}{2}(1 - k). \]

These elements are zero divisors, idempotent, mutually annullable \((e_1 e_2 = 0)\) and can generate the whole hyperbolic plane as

\[ \mathbb{D} = \mathbb{R}e_1 + \mathbb{R}e_2. \]

The set of zero divisors is well located as \( \mathcal{G} = \mathbb{R}e_1 \cup \mathbb{R}e_2 \) and \( \mathcal{G}_0 = \mathcal{G} \cup \{0\} \). The correspondence

\[ a \mapsto ae_1 + ae_2 := \tilde{a} \tag{1} \]

embeds \( \mathbb{R} \) in \( \mathbb{D} \).

The hyperbolic numbers are equipped with a partial order \( \leq \) defined by

\[ \alpha \leq \beta \text{ if and only if } a_1 \leq b_1 \text{ and } a_2 \leq b_2, \]
where $\alpha = a_1e_1 + a_2e_2$, $\beta = b_1e_1 + b_2e_2$. Consequently, an hyperbolic interval may be defined as the set

$$[\alpha, \beta]_k := \{ \xi \in D \mid \alpha \preceq \xi \preceq \beta \}.$$  

Additionally the set of positive hyperbolic numbers is defined as

$$D^+ := \{ \xi \in D \mid \xi \succeq 0 \}.$$  

This set is closed under multiplication.

Every hyperbolic valued function $F : X \to D$ is determined by its components:

$$F(x) = F_1(x)e_1 + F_2(x)e_1, \; x \in X,$$

where $F_1$ and $F_2$ are real valued functions. Formula (2) is called the idempotent representation of $F$.

Alternatively, when we work with the base $\{1, k\}$ for the hyperbolic plane a possible representation of $F$ arise in accordance with the formula

$$F(x) = u(x) + kv(x), \; x \in X,$$

where $u$ and $v$ are real valued functions.

An hyperbolic metric space is a pair $(X, D)$, where $X$ is a no-empty set and $D$ is an hyperbolic positive valued function such that for any $x, y, z \in X$ the following holds:

1. $D(x, y) = 0$ if and only if $x = y$.
2. $D(x, y) = D(y, x)$.
3. $D(x, z) \preceq D(x, y) + D(y, z)$.

The much familiar hyperbolic metric space is $(D, D_k)$ where $D_k$ is defined for all $\alpha, \beta \in D$ with $\alpha = a_1e_1 + a_2e_2$ and $\beta = b_1e_1 + b_2e_2$, such that

$$D_k(\alpha, \beta) := |a_1 - b_1|e_1 + |a_2 - b_2|e_2.$$  

An hyperbolic metric space is called complete if every Cauchy’s sequence of hyperbolic numbers is convergent in the sense of the hyperbolic metric, see [25].
Let \((X,D_1)\) and \((Y,D_2)\) be two hyperbolic metric spaces. We say that a function \(F : (X,D_1) \to (Y,D_2)\) is continuous in \(x_0\), if for every \(\epsilon \in \mathbb{D}^+\), there exists a \(\delta \in \mathbb{D}^+\) such that for every \(x \in X\) with \(D_1(x,x_0) \leq \delta\), then

\[D_2(F(x), F(x_0)) \leq \epsilon.\]

When both spaces \((X,D_1),(Y,D_2)\) are equal to \((\mathbb{D},D_k)\), the hyperbolic continuity implies the real one over \((\mathbb{R}^2,d_u)\), where \(d_u\) is the standard metric.

A function \(F : (X,D_1) \to (Y,D_2)\) is called a contraction function if there exists \(\kappa \in [0,\tilde{1})\) such that for every \(x,y \in X\),

\[D_2(F(x), F(y)) \leq \kappa D_1(x,y).\]

Let \(F : (\mathbb{D},D_k) \to (\mathbb{D},D_k)\), we say that \(F\) is derivable in \(\xi_0 \in \mathbb{D}\), if there exists the limit

\[F'(\xi_0) := \lim_{\sigma \to 0} \frac{F(\xi_0 + \sigma) - F(\xi_0)}{\sigma}. \tag{3}\]

In [5] was proved that every hyperbolic derivable function \(F(\alpha) = u(\alpha) + k v(\alpha)\) with \(\alpha = a + kb\) in the base \(\{1,k\}\) satisfies the Cauchy-Riemann type system

\[\frac{\partial u}{\partial a_1}(\alpha) = \frac{\partial v}{\partial a_2}(\alpha) \text{ and } \frac{\partial u}{\partial a_2}(\alpha) = \frac{\partial v}{\partial a_1}(\alpha).\]

It is worth noting that the last assertion implies that given a derivable function in the idempotent form \(F = F_1e_1 + F_2e_2\) and \(\alpha = a_1e_1 + a_2e_2\), then

\[F(\alpha) = F_1(a_1)e_1 + F_2(a_2)e_2.\]

On the other hand, in [26] was shown that (3) implies that the derivative of \(F = F_1e_1 + F_2e_2\) is given by

\[F'(\alpha) = \frac{\partial F_1}{\partial a_1}(\alpha)e_1 + \frac{\partial F_2}{\partial a_2}(\alpha)e_2.\]

We follow [5] in assuming that the partial derivatives of \(F\) are indeed total real derivatives and depend only of their respective components, i.e.,

\[F'(\alpha) = \frac{d F_1}{d a_1}(a_1)e_1 + \frac{d F_2}{d a_2}(a_2)e_2.\]
3. Hyperbolic valued probabilities and entropy

Let $\mathcal{P} = (p_1, p_2, \ldots, p_m)$ be a finite discrete probability distribution, that is, suppose that $p_j \geq 0$, $j = 1, \ldots, m$ and $\sum_{j=1}^m p_j = 1$. Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a sample description space of an experiment relates to $\mathcal{P}$ in the way that the probability that occurs $f_j$ is $p_j$.

Entropy is a concept that measures how many information we can get from a probability distribution. An equivalent interpretation is given by Rényi in [10], where he says that entropy is the amount of uncertainty concerning the outcome of an experiment. It is measured by the quantity

$$H(\mathcal{P}) = -\sum_{j=1}^m p_j \log p_j = \sum_{j=1}^m p_j \log \frac{1}{p_j}.$$ 

This formula was introduced by Shannon in [8]. He established that a function to measure the entropy must fulfill with

1. $H$ should be continuous in every $p_j \in \mathcal{P}$.
2. If all $p_j$ are equal, then $H$ should be a monotonic increasing function.
3. Given another set of probabilities $Q$, then $H(\mathcal{P}Q) = H(\mathcal{P}) + H(Q)$.

One function having thses properties is called an entropy function. Later Rényi gave a much more simple proof (see [10]) that just logarithm function has these properties by the next lemma.

Lemma 3.1 ([9]). Let $f : \mathbb{N} \to \mathbb{R}$ an additive number-theoretical function, that is

$$f(nm) = f(n) + f(m),$$

for every $n, m \in \mathbb{N}$, and

$$\lim_{n \to \infty} (f(n+1) - f(n)) = 0.$$

Then we have

$$f(n) = cte \log n$$

3.1. Hyperbolic probability

In [1] was introduced the notion of a probabilistic measure which takes values in hyperbolic numbers. They show that this new measure satisfies the usual properties of a probability. These ideas led to propose the hyperbolic concept of entropy if one consider a finite discrete hyperbolic probability distribution.
Definition 3.2. Let $\mathcal{P} = \{\rho_1, ..., \rho_n\}$ be a finite set of hyperbolic numbers of $[0, \tilde{1}]_k$ given in idempotent representation $\rho_k := p_{k,1}e_1 + p_{k,2}e_2$. We say that $\mathcal{P}$ is an hyperbolic probability distribution, if we have one of these states

I. $\sum_{k=1}^{n} \rho_k = \tilde{1}$,

II. $\sum_{k=1}^{n} \rho_k = 1e_1$,

III. $\sum_{k=1}^{n} \rho_k = 1e_2$.

Remark 3.3. Cases II and III imply that for every $k \in \{1, ..., n\}$, either $\rho_k = pe_1$ or $\rho_k = pe_2$ where $p \in \mathbb{R}$. So $\rho_k$ is a zero divisor.

The main difference from the case of real probability distribution is that in general a set of hyperbolic probabilities can not be totally ordered, it is necessary to make additional assumptions, as was mentioned in [24].

Following the notion of product space probability, there exists a simple interpretation of hyperbolic valued probabilities. Indeed, an hyperbolic valued probabilities $\rho_k = p_{k,1}e_1 + p_{k,2}e_2$. could be identify with

$$p_k := \frac{p_{k,1} + p_{k,2}}{2},$$

which is the usual accumulate probability in the real sense. An alternative interpretation will be explain in Section 4.

3.2. Hyperbolic entropy

We begin with a weak hyperbolic extension of the concept of the Shannon entropy using the hyperbolic correspondence (1) of real logarithm and the accumulated probabilities (4).

Definition 3.4. Let $\mathcal{P} = \{\rho_1, ..., \rho_n\}$ be a hyperbolic probability distribution. The weak hyperbolic entropy associated to $\mathcal{P}$ is defined as

$$H_w(\mathcal{P}) := \sum_{k=1}^{n} -\rho_k \tilde{\log} \left( \frac{p_{k,1} + p_{k,2}}{2} \right) =$$

$$\sum_{k=1}^{n} -\rho_k \left( \log \left( \frac{p_{k,1} + p_{k,2}}{2} \right) e_1 + \log \left( \frac{p_{k,1} + p_{k,2}}{2} \right) e_2 \right).$$
Although weak Shannon entropy can be clearly seen as a generalization, we realize that the form of the weak entropy expression resembles the Shannon entropy and does not change the measure in essence.

We therefore propose to define a strong hyperbolic extension of the Shannon entropy, where the role played in the traditional theory by the logarithm, is now played by the well-defined holomorphic logarithm function in the hyperbolic theory, to be denoted in this work by $\text{Log}_\mathbb{D}$. See [1, Subsection 6.5] for more details.

Firstly, let us state the hyperbolic version of Lemma 3.1, whose proof follows by making appeal to the idempotent representation of the hyperbolic logarithm function.

**Lemma 3.5.** Let the hyperbolic valued function $F : \mathbb{N} \to \mathbb{D}$, such that

$$F(mn) = F(m) + F(n)$$

and

$$\lim_{n \to \infty} (F(n + 1) - F(n)) = 0,$$

for every $n, m \in \mathbb{N}$. Then there exists $\kappa \in \mathbb{D}$ so that $F$ has the form

$$F(N) = \kappa \text{Log}_\mathbb{D}(N).$$

In trying to define a strong extension of the Shannon entropy to hyperbolic numbers plane, it is to be expected that the considered hyperbolic function satisfies related hyperbolic versions of the aforementioned axioms of the Shannon entropy. To this aim, an alternative axiomatization of the concept of strong hyperbolic entropy, which relaxed the continuity property and assuming the holomorphic condition to the hyperbolic entropy measure is addressed.

**Definition 3.6.** The strong hyperbolic entropy associated to $\mathcal{P}$ is defined as

$$\mathbf{H}_s(\mathcal{P}) := \sum_{k=1}^{n} -\rho_k \text{Log}_\mathbb{D}(\rho_k) = \sum_{k=1}^{n} -\rho_k (\log(p_{k,1})e_1 + \log(p_{k,2})e_2).$$

**Remark 3.7.** As the Shannon entropy measure, both of these hyperbolic extensions have fundamental properties which legitimate it as reasonable measure of choice. For instance:

- Only when we are certain of the outcome does the measures vanish. Otherwise they are positive.
Both measures $H_w$ and $H_s$ are maximum and equal to one of these values $\log(n), \log(n)e_1, \log(n)e_2$ or $\log_D(n), \log_D(n)e_1, \log_D(n)e_2$ respectively, when all the $\rho_k$ are equal (e.g., $\frac{1}{n}, \frac{1}{n}e_1, \frac{1}{n}e_2$), which is again the most uncertain situation.

4. Hyperbolic chaos game

To make our exposition self-contained we give a brief sketch of the chaos game algorithm of creating a fractal in euclidean spaces (see [27]). The algorithm consists in the iteratively creation of a sequence of points, using a polygon and starting with a base random point inside it. Each point in the sequence is a given fraction of the distance between the previous point and one of the vertices of the polygon, which is chosen at random in each iteration.

In addition, to illustrating how the algorithm work in practice, let us provide a well-known example

Example 4.1. Let a triangle with vertices at $a_1 = (0, 1)$, $a_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $a_3 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and let $a_0 = (0, 0)$ be the basic point. Fig. 1 illustrates the algorithm procedure for performing the fractal and some of its iterations. The result is an approximation of the well known Sierpinski Triangle.

![Figure 1](image)

Figure 1: (a) Process to select the new base point. (b) Outcome of five thousand of iterations. (c) Outcome of one million of iterations.

To deals with general metric spaces, Barnslet [27] appealed to the concept
of IFS with probabilities on metric spaces getting a generalization of the chaos game algorithm. A discussion of a more general case can be found in [28].

Here we set up some notation and terminology for completeness. An IFS on a complete metric space \((X,d)\) is a finite collection \(\{f_1, \ldots, f_m\}\) of contractions over \(X\) (see for instance [27]).

Let \(\mathcal{H}(X)\) the collection of every nonempty and compact subsets of \(X\), so the Hutchinson operator \(J : \mathcal{H}(X) \to \mathcal{H}(X)\) is defined over subsets \(A \in \mathcal{H}(X)\) as
\[
J(A) = \bigcup_{j=1}^{m} f_j(A).
\]

Due that \((\mathcal{H}(X), h)\) is a complete metric space, when \(h\) is the Hausdorff metric induced by the metric \(d\), see [27, Sec. 2.6], the operator \(J\) is a contraction and by Banach’s fixed point theorem there exists a set \(A \in \mathcal{H}(X)\) that is the unique fixed point of \(J\). This fixed point will be called the attractor of the IFS.

The chaos game algorithm for generating a point-set approximation of the IFS attractor works as follow: Let an IFS on a complete metric space \((X,d)\) with probabilities, i.e., an IFS \(\{f_1, \ldots, f_m\}\) and a finite discrete probability distribution \(\{p_1, \ldots, p_m\}\) such that each contraction \(f_j\) is associated with \(p_j\). Choose a base point \(x_0 \in X\) and select contraction \(f_j\) at random according to the probabilities \(p_j\) and compute \(x_{i+1} = f_j(x_i)\). Return to the second step and repeat the process iteratively.

To exemplify this algorithm, let us consider again the Sierpinski Triangle. Let \(X = \mathbb{R}^2\) and \(d\) the usual metric. The contractions are
\[
\begin{align*}
f_1(x) &= \frac{1}{2}x, \quad f_2(x) = \frac{1}{2}x + \left(\frac{1}{2}, 0\right), \\
f_3(x) &= \frac{1}{2}x + \left(\frac{1}{4}, \frac{1}{2}\right),
\end{align*}
\]
and all the probabilities \(p_j, \ j = 1, 2, 3\) must be equal to \(\frac{1}{3}\).

4.1. Hyperbolic chaos game

It has already been remarked that the standard chaos game for a complete metric space only require an IFS with probabilities, and because both concepts
were extended to hyperbolic metric spaces so it appear feasible to adapt the algorithm to the hyperbolic frame.

**Definition 4.2 (Hyperbolic chaos game).** Let be \((X, D)\) a complete hyperbolic metric space and let \(\mathcal{F} = \{F_1, ..., F_n\}\) be a finite set of hyperbolic contractions. Set \(\mathfrak{P}\) an hyperbolic probability distribution (see Definition 3.2). We associate to every \(F_k\) the real probability \(p_k\) given by (4). The algorithm consists in taking a starting point \(x_0 \in X\) and then choose randomly the contraction \(F_k \in \mathcal{F}\) on each iteration accordingly to the assigned probability \(p_k\). The rest of the algorithm runs as before.

If the hyperbolic probabilities are zero divisors, then we only have to consider their natural identification with \(\mathbb{R}\). See Remark 3.3.

**Remark 4.3.** Attractors of contractive hyperbolic IFS on complete hyperbolic metric spaces may not be unique because the respective \(\mathbb{D}\)-Hausdorff distance is not a metric, see [26]. Consequently, the hyperbolic chaos game represents a method of generating a representative element of the attractor class.

4.2. \(\mathbb{D}\)-chaos game

Now, we focus our study to modify the chaos game on the hyperbolic metric space \((\mathbb{D}, D_k)\). More precisely, treating the hyperbolic probability distribution as true hyperbolic numbers instead of their real identifications.

**Algorithm:** Let \(\mathcal{F} = \{F_1, ..., F_n\}\) be a finite set of contractions over \((\mathbb{D}, D_k)\) where \(F_k = F_{k,1}e_1 + F_{k,2}e_2\) and let \(\mathfrak{P}\) be an hyperbolic probability distribution associated to \(\mathcal{F}\). The sets of idempotent components of \(\mathcal{F}\) will be denoted by \(\Phi_1 = \{F_{1,1}, ..., F_{n,1}\}\) and \(\Phi_2 = \{F_{1,2}, ..., F_{n,2}\}\) respectively.

Step 1. Chose a base point \(\xi_0 \in \mathbb{D}\).

Step 2. Randomly select two function \(F_{s,1} \in \Phi_1\) and \(F_{t,2} \in \Phi_2\), according to the real idempotent components \(p_{s,1}\) and \(p_{t,2}\) of the probabilities \(\rho_s, \rho_t \in \mathfrak{P}\) respectively and define the function \(G_{s,t} = F_{s,1}e_1 + F_{t,2}e_2\).

Step 3. Subsequently compute the next point as \(\xi_{i+1} = G_{s,t}(\xi_i)\).

Step 4. The process is repeated from the step two iteratively a finite set of times.

A natural question to ask is whether functions \(G_{s,t}\) are also contractions. The answer is given by the next lemma.
Lemma 4.4. Let $F_1 = F_{1,1}e_1 + F_{1,2}e_2$ and $F_2 = F_{2,1}e_1 + F_{2,2}e_2$ be two hyperbolic contractions over $(\mathbb{D}, D_k)$ with contractive factors $\kappa_1 = c_{1,1}e_1 + c_{1,2}e_2$ and $\kappa_2 = c_{2,1}e_1 + c_{2,2}e_2$ respectively. Then function $G_{s,t} = F_{s,1}e_1 + F_{t,2}e_2$ is again a contraction over $(\mathbb{D}, D_k)$ and its contractible factor is $\kappa = c_{s,1}e_1 + c_{t,2}e_2$.

Proof. The proof follows by the observation that every contraction may be viewed in a point-wise way, making use of the hyperbolic partial order (see [26, Def. 2.8]). \hfill \Box

Example 4.5. Let the IFS generates the Sierpinski Triangle in the Fig. 1 and identify every contraction with its respective hyperbolic version, which are affine hyperbolic transformation (see [26, Sec. 2.3]). Then we have the hyperbolic IFS with probabilities \{$(D, D_k)$, $F_1$, $F_2$, $F_3$\} where

\begin{align*}
F_1(\xi) &= \frac{1}{2}\xi, \\
F_2(\xi) &= \frac{1}{2}\xi + \frac{1}{4}e_1 + \frac{1}{2}e_2, \\
F_3(\xi) &= \frac{1}{2}\xi + \frac{1}{2}e_1,
\end{align*}

and for every $k \in \{1, 2, 3\}$ the probability $\rho_k$ is equal to $\frac{1}{3}$.

One approximation of this hyperbolic IFS with the hyperbolic chaos game is shown in the Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{Representation of the result to apply the hyperbolic chaos game to IFS 5 with $\rho_k = \frac{1}{3}$. (a) 10000 iterations. (b) 100000 iterations, (c) 1000000 iterations.}
\end{figure}

Example 4.6. Now we change the hyperbolic probability distribution in the
hypothesis of the Example 4.5 by

\[ \rho_1 = \frac{1}{10}e_1 + \frac{1}{4}e_2, \]
\[ \rho_2 = \frac{3}{10}e_1 + \frac{1}{5}e_2, \]
\[ \rho_3 = \frac{3}{5}e_1 + \frac{11}{20}e_2, \]  

(6)

Note that we get a similar figure when we repeat the process a big quantity of times, but the agglomeration of points in \([0, 1]_k\) is different. See Fig. 3.

Figure 3: Representation of the result to apply the D-chaos game to IFS 5 with the hyperbolic probability distribution (6). (a)- 10000 iterations. (b)- 100000 iterations, (c)- 1000000 iterations.

The similitude is due that in the hyperbolic case the attractor associated to an IFS is an equivalence class of no empty compact sets (see [26, Sec. 3.1]).

Given \( \mathcal{F} = \{F_1, \ldots, F_n\} \) an hyperbolic IFS over \((\mathbb{D}, D_k)\) with \( \mathfrak{P} \) an hyperbolic probability distribution associated to \( \mathcal{F} \), the number of possible combination functions \( \{G_{s,t}\} \) generated by the \( \mathbb{D} \)-chaos game is \( \{1, \ldots, n^2\} \) and the real probabilities considering the selection of the idempotent components of the contractions like independent events are \( \pi_m = p_{s,1} p_{t,2} \) with \( m \in \{1, \ldots, n^2\} \). Let \( \tilde{\Omega} \) denotes the hyperbolic probability distribution \( \{\tilde{\pi}_1, \ldots, \tilde{\pi}_{n^2}\} \).

In an alternative way, we will use the symbol \( \mathfrak{K} \) to denote the hyperbolic probability distribution \( \{\omega_1, \ldots, \omega_{n^2}\} \), where \( \omega_m = \frac{1}{n} (p_{s,1} e_1 + p_{t,2} e_2) \).

Because the strong hyperbolic entropy measure the information in every component of the system the following relations hold.
\[ H_s(\mathcal{P}) \preceq \overline{H(\mathcal{Q})}, \]

\[ H_s(\mathcal{P}) \preceq H_s(\mathcal{R}). \]

5. Further remarks

Repeating the \( \mathbb{D} \)-chaos game to an IFS, a little number of times, it is created a sequence of points in the interior of the hyperbolic interval \([0, \overline{1}]_k\), which may be viewed as a rectangle in the euclidean plane. If we take now an image (looking it as the rectangle) and we deform, by an affine transformation, the original rectangle in order to overlap it to the image, we can see in this overlapping a patter of noise in the image. This suggests a natural question of whether the noise in an image could be removed by using the \( \mathbb{D} \)-chaos game. Some partial evidence support the conjecture that the use of an analogous of the collage theorem (see [20]), together with some technique of colored, introduced in [22], would be used to get an image noise reduction.

References

[1] D. Alpay, M. E. Luna-Elizarrarás, M. Shpairo, Kolmogorov’s axioms for probabilities with values in hyperbolic numbers, Adv. Appl. Clifford Algebras 27 (2) (2017) 913–929. doi:10.1007/s00006-016-0706-6.

[2] J. Cockle, On certain functions resembling quaternions and on a new imaginary in algebra, London-Dublin-Edinburgh Philosophical Magazine and Science Journal 32 (3) (1848) 435–439. doi:10.1080/14786444808646139.

[3] C. Segre, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, Math. Ann. 40 (3) (1892) 413–467. doi:10.1007/BF01443559.
[4] M. E. Luna-Elizarrarás, M. Shapiro, D. C. Stuppa, A. Vajiac, Bicomplex Holomorphic Functions: The Algebra, Geometry and Analysis of Bicomplex Numbers, 1st Edition, Birkhäuser/Springer, Switzerland, 2015. doi:10.1007/978-3-319-24868-4.

[5] J. C. Vignaux, A. Durañona, On the theory of functions of an hyperbolic complex variable, Univ Nac. La Plata. Publ. Fac. Ci. Fisicomat. Contrib. 104 (1935) 139–183.

[6] G. Sobczyk, The hyperbolic number plane, The College Mathematic Journal 26 (4) (1995) 268–280. doi:10.1080/07468342.1995.11973712.

[7] G. Sobczyk, Complex and hyperbolic numbers, in: New Foundations in Mathematics, Birkhäuser, Boston, 2013, pp. 23–42. doi:10.1007/978-0-8176-8385-6_2.

[8] C. E. Shannon, A mathematical theory of communication, Bell System Tech. J. 27 (3) (1948) 379–423. doi:10.1002/j.1538-7305.1948.tb01338.x.

[9] P. Erdős, On the distribution function of additive functions, Ann. of Math. 47 (1) (1946) 1–20. doi:10.2307/1969031.

[10] A. Rényi, A mathematical theory of communication, in: Proc. Fourth Berkeley Symp. on Math. Statist. and Prob, University California Press, Berkeley, Calif, USA, 1961, pp. 547–561.

[11] D. K. Fadeev, Zum begriff der entropie einer endlichen wahrscheinlichkeitsschernas, in: Arbeiten zur Informationstheorie I, Deutscher Verlag der Wissenschaften, 1957, pp. 85–90.

[12] K. K. Nambiar, P. K. Varma, V. Saroch, An axiomatic definition of shannon’s entropy, Appl. Math. Lett. 5 (4) (1992) 45–46. doi:10.1016/0893-9659(92)90084-M.
[13] J. M. Velimir, M. S. Stanković, An axiomatic characterization of generalized entropies under analyticity condition, Appl. Math. Inf. Sci 9 (2) (2015) 609–613.

[14] J. Aczél, Z. Daróczy, On measures of information and their characterizations, 1st Edition, Academic Press, New York, 1975.

[15] B. Ebanks, P. Sahoo, W. Sander, Characterizations of information measures, World Scientific, Singapore, 1998.

[16] P. M. Arora, On characterizing some generalizations of shannon’s entropy, Inform. Sci. 21 (1) (1980) 13–22. doi:10.1016/0020-0255(80)90007-9.

[17] P. M. Arora, On the shannon measure of entropy, Inform. Sci. 23 (1) (1981) 1–9. doi:10.1016/0020-0255(81)90036-0.

[18] M. R. Ubriaco, Entropies based on fractional calculus, Phys. Lett. A 373 (30) (2009) 25162519. doi:10.1016/j.physleta.2009.05.026.

[19] A. Karci, Fractional order entropy: New perspective, Optik 127 (20) (2016) 9172–9177. doi:10.1016/j.ijleo.2016.06.119.

[20] C. Tsallis, Possible generalization of boltzmann-gibbs statistics, J. Statist. Phys. 52 (1-2) (1988) 479487. doi:10.1007/BF01016429.

[21] M. D. Esteban, A general class of entropy statistics, Appl. Math. 42 (3) (1997) 161–169. doi:10.1023/A:1022447020419.

[22] M. D. Esteban, D. Morales, A summary on entropy statistics, Kybernetika 31 (4) (1995) 337–346.

[23] S. Furuichi, Characterizations of generalized entropy functions by functional equations, Adv. Math. Phys. (2011) 12 pp. doi:10.1155/2011/126108.

[24] A. S. Balankin, J. Bory-Reyes, M. E. Luna-Elizarrarás, M. Shapiro, Cantor-type sets in hyperbolic numbers, Fractals 24 (4) (2016) 1650051. doi:10.1142/S0218348X16500511.
[25] G. Y. Téllez-Sánchez, J. Bory-Reyes, More about cantor like sets in hyperbolic numbers, Fractals 25 (5) (2017) 1750046. doi:10.1142/S0218348X17500463.

[26] G. Y. Téllez-Sánchez, J. Bory-Reyes, Generalized iterated function systems on hyperbolic number plane, Fractals 27 (4) (2019) 1950045. doi:10.1142/S0218348X19500452.

[27] M. F. Barnsley, Fractals Everywhere, 1st Edition, Academic Press, London, 1988.

[28] M. F. Barnsley, A. Vince, The chaos game on a general iterated function system, Ergod. Th. & Dynam. Sys. 31 (4) (2011) 1073–1079. doi:10.1017/S0143385710000428.