A solution of the Navier–Stokes system of equations in three-dimensions

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A few basic, intuitive, properties of the Navier-Stokes system of equations for incompressible fluid flows are discussed in this paper. We present a rephrased interpretation of the Navier-Stokes equation. A spatially periodic solution of the velocity and pressure fields spanning the entire unbounded domain in three dimensions is then derived for a given smooth solenoidal initial velocity vector field. In this solution all three velocity components depend non-trivially on all three Cartesian coordinate directions.

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1. Introduction

The Navier-Stokes equations are a system of partial differential equations that describe the flow of incompressible fluids: Navier (1827);_Stokes (1845). These equations are nonlinear in nature and therefore, difficult to solve analytically. Necessary background knowledge of the subject can be found in Batchelor (1967), Drazin and Rilev (2007) and the monographs by Ladyzhenskaya (1969).

Ethier and Steinman (1994) presented unsteady analytical solutions involving all three Cartesian velocity components to the Navier-Stokes equations. Recently, Antuono (2020) extended Ethier and Steinman’s work to a class of solutions for the velocity vector field in a three-dimensional torus in terms of a wave number divided into two parts, characterized by positive and negative helicity respectively. In Appendix A, we provide Antuono’s solution with positive helicity.

In a previous article Thambynayagam (2013) presented a method to derive analytical solutions by recasting the Navier-Stokes equation in terms of three distinct terms

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associated, respectively, with the linear viscous forces, inertial forces and the external forces applied to the fluid. In this paper we apply the method to derive a time evolution analytical solution for the velocity vector field and pressure that span the entire unbounded domain: $\mathbb{R}^3 = \{-\infty < x_i < \infty; \ i = 1, 2, 3\}$. Mathematical expressions, particularly equations (4.1)-(4.30), are tedious and exhaustively boring to derive, but they are straightforward.

2. The fundamental problem

In a Cartesian frame of reference, the fundamental equations of momentum and mass conservation for incompressible viscous fluid flow fields are represented as follows:

$$\frac{\partial v_i}{\partial t} + g_i = \kappa \Delta v_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + f_i \quad x \in \mathbb{R}^n, \ t \geq 0 \quad (2.1)$$

where $v_i = v_i(x,t) \ x \in \mathbb{R}^n, \ i = 1, 2, ..., n$, is the solenoidal velocity vector field and $p = p(x,t)$ the pressure of the fluid.

$$g_i = g_i(x,t) = \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} \quad (2.2)$$

is the nonlinear inertial force, $f_i = f_i(x,t)$ are the components of an externally applied force, $\rho$ is the constant density of the fluid, $\kappa$ is the positive coefficient of kinematical viscosity and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables.

The conservation of mass yields the divergence-free (incompressibility) condition

$$\text{div} \ v = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0 \quad x \in \mathbb{R}^n, \ t \geq 0 \quad (2.3)$$

Conservation law also implies that the energy dissipation of a viscous fluid is bounded by the initial kinetic energy. Therefore, if the externally applied force does no net work on the fluid, the kinetic energy of the solution should be finite. There exists a constant $E$ such that

$$\int_{\mathbb{R}^n} |v|^2 \ dx < E \quad t \geq 0 \quad (2.4)$$

The Navier-Stokes equation (2.1) must be solved forward in time $t \geq 0$, starting from an initial divergence-free velocity field

$$v_i(x,0) = v_i^0 \quad x \in \mathbb{R}^n \quad (2.5)$$

with the pressure evolving in time to maintain the incompressibility constraint (2.3). The superscript 0 is used to denote the value of a function at time zero.

$$g_i(x,0) = g_i^0 = \sum_{j=1}^{n} v_j^0 \frac{\partial v_i^0}{\partial x_j} \quad (2.6)$$
3. Recasting the Navier-Stokes equation:

\[ \mathbb{R}^n = \{-\infty < x_i < \infty; \ i = 1, 2, ..., n\} \]

In this section, in order to be perspicuous, we repeat the material from Thambynayagam (2013) relating to the recasting of the Navier-Stokes equation and the resulting lemma. Assuming that the divergence and the linear operator can be commuted, the pressure field can be formally obtained by taking the divergence of (2.1) as a solution of the Poisson equation, which is

\[ \Delta p = \rho \sum_{i=1}^{n} \frac{\partial (f_i - g_i)}{\partial x_i} \quad (3.1) \]

Equation (3.1) is called the pressure Poisson equation. The use of pressure Poisson equation in solving the Navier-Stokes equation is discussed in a paper by Gresho and Sani (1987). It is important to note that while (2.1) and (2.3) lead to the pressure Poisson equation (3.1), the reverse; that is, (2.1) and (3.1), do not always lead to (2.3). In deriving solutions of the Navier-Stokes equations, we therefore ensure that the velocity vector field remains solenoidal at all times.

The general solution of the Poisson equation (3.1) is

\[ p(x, t) = -\frac{\rho}{2\pi} \int_{\mathbb{R}^2} P(y, t) \ln \left( \frac{1}{\sqrt{P_n(x, y)}} \right) \prod_{j=1}^{n} dy_j, \quad n = 2, \]

\[ p(x, t) = -\frac{\rho \Gamma \left( \frac{n}{2} \right)}{2(n-2)\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{P(y, t)}{\{P_n(x, y)\}^{\frac{n-2}{2}}} \prod_{j=1}^{n} dy_j, \quad n \geq 3 \quad (3.2) \]

where \( \Gamma(z) = \int_{0}^{\infty} e^{-u} u^{z-1} du \) \( [\Re z > 0] \), is the Gamma function,

\[ P(x, t) = \sum_{j=1}^{n} \frac{\partial (f_j - g_j)}{\partial x_j} \quad (3.3) \]

and

\[ P_n(x, y) = \sum_{j=1}^{n} (x_j - y_j)^2 \quad (3.4) \]

Differentiating (3.2) with respect to \( x_i \) we obtain

\[ \frac{\partial p}{\partial x_i} = \rho \frac{\Gamma \left( \frac{n}{2} \right)}{2\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{(x_i - y_i) P(y, t)}{\{P_n(x, y)\}^{\frac{n-2}{2}}} \prod_{j=1}^{n} dy_j, \quad n \geq 2 \quad (3.5) \]

Substituting for \( \frac{\partial p}{\partial x_i} \) in (2.1) the following can be formulated:

\[ \frac{\partial v_i}{\partial t} = \kappa \Delta v_i - \frac{\Gamma \left( \frac{n}{2} \right)}{2\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{(x_i - y_i) P(y, t)}{\{P_n(x, y)\}^{\frac{n-2}{2}}} \prod_{j=1}^{n} dy_j + f_i - g_i, \quad n \geq 2, \ x \in \mathbb{R}^n, \ t \geq 0 \quad (3.6) \]
The difficulty in solving the system of equations (2.1) - (2.3) stems from the presence of the nonlinear term $g_i$. We therefore recast the Navier-Stokes equation (3.6) as:

$$\frac{\partial v_i}{\partial t} = \kappa \Delta v_i - U_i + F_i \quad x \in \mathbb{R}^n, \quad t \geq 0 \quad (3.7)$$

where

$$U_i = g_i - \frac{\Gamma (\frac{n}{2})}{2\pi \frac{n}{2}} \int_{\mathbb{R}^n} \frac{(x_i - y_i) \sum_{k=1}^n \frac{\partial g_k(y, t)}{\partial y_k}}{\{P_n(x, y)\}^{\frac{n}{2}}} \prod_{j=1}^n dy_j, \quad x \in \mathbb{R}^n, \quad t \geq 0 \quad (3.8)$$

and

$$F_i = f_i - \frac{\Gamma (\frac{n}{2})}{2\pi \frac{n}{2}} \int_{\mathbb{R}^n} \frac{(x_i - w_i) \sum_{k=1}^n \frac{\partial f_k(w, t)}{\partial w_k}}{\{P_n(x, w)\}^{\frac{n}{2}}} \prod_{j=1}^n dw_j, \quad x \in \mathbb{R}^n, \quad t \geq 0 \quad (3.9)$$

$U_i = U_i(x, t)$ and $F_i = F_i(x, t)$. The three terms on the right hand side of (3.7), $\kappa \Delta v_i$, $U_i$ and $F_i$ are associated, respectively, with the linear viscous force, the nonlinear inertial force and the externally applied force acting on the fluid.

A posteriori we state the following lemma: If $U_i \equiv 0$, that is

$$g_i = \frac{\Gamma (\frac{n}{2})}{2\pi \frac{n}{2}} \int_{\mathbb{R}^n} \frac{(x_i - y_i) \sum_{k=1}^n \frac{\partial g_k(y, t)}{\partial y_k}}{\{P_n(x, y)\}^{\frac{n}{2}}} \prod_{j=1}^n dy_j \quad (3.10)$$

then the solution of the Navier-Stokes equation (2.1) is determined as a solution of the non-homogeneous diffusion equation [Carslaw and Jaeger (1959); Thambynayagam (2011)]:

$$v_i(x, t) = \frac{1}{(2\sqrt{\pi \kappa t})^n} \int_{\mathbb{R}^n} v_i^0(y) e^{-\frac{\sum_{k=1}^n (x_k - y_k)^2}{4\kappa t}} \prod_{j=1}^n dy_j +$$

$$+ \frac{1}{(2\sqrt{\pi \kappa})} \int_{\mathbb{R}^n} \int_0^t \frac{F_i(y, \tau)}{(t - \tau)^{\frac{n}{2}}} d\tau \prod_{j=1}^n dy_j \quad (3.11)$$

If the externally applied force on the fluid $f_i$ is set to zero, then, the second term on the right hand side of equation (3.11) vanishes and the solution of the Navier-Stokes equation (2.1) reduces to that of the Cauchy diffusion equation:

$$v_i(x, t) = \frac{1}{(2\sqrt{\pi \kappa t})^n} \int_{\mathbb{R}^n} v_i^0(y) e^{-\frac{\sum_{k=1}^n (x_k - y_k)^2}{4\kappa t}} \prod_{j=1}^n dy_j \quad (3.12)$$

There are two known solutions that satisfy the relationship (3.10) given in Appendix B: the two-dimensional solution of Taylor (1923) and the unsteady three-dimensional solution of Arnold (1965) derived by Thambynayagam (2013).
4. A solution in $\mathbb{R}^3 = \{-\infty < x_i < \infty; \ i = 1, 2, 3\}$

We consider the velocity vector field $v_i^0(x)$ at time zero of the form

$$v_i^0 = \sin (\alpha x_1 + \xi_1) \cos (\alpha x_2 + \xi_2) \sin (\alpha x_3 + \xi_3) -$$

$$-\sin (\alpha x_1 + \xi_2) \cos (\alpha x_3 + \xi_1) \sin (\alpha x_2 + \xi_3)$$

(4.1)

$$v_2^0 = \sin (\alpha x_2 + \xi_1) \cos (\alpha x_3 + \xi_2) \sin (\alpha x_1 + \xi_3) -$$

$$-\sin (\alpha x_2 + \xi_2) \cos (\alpha x_1 + \xi_1) \sin (\alpha x_3 + \xi_3)$$

(4.2)

$$v_3^0 = \sin (\alpha x_3 + \xi_1) \cos (\alpha x_1 + \xi_2) \sin (\alpha x_2 + \xi_3) -$$

$$-\sin (\alpha x_3 + \xi_2) \cos (\alpha x_2 + \xi_1) \sin (\alpha x_1 + \xi_3)$$

(4.3)

where $\xi_1$, $\xi_2$, and $\xi_3$ are phase angles and $\alpha$ is a real number. Equation (5.10) can be written as follows at time zero in $\mathbb{R}^3$:

$$U_i^0 = g_i^0 - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x_i - y_i) G^0(y_1, y_2, y_3)}{\left\{P_3(x, y)\right\}^2} \prod_{j=1}^{3} dy_j = 0 \quad i = 1, 2, 3$$

(4.4)

where $U_i^0 = U_i(x, 0)$ and $G^0(y_1, y_2, y_3) = \sum_{k=1}^{3} \frac{\partial y_i^k(y, \tau)}{\partial y_k}$. Changing the variable of integration, $u_i = x_i - y_i$, gives

$$U_i^0 = g_i^0 - \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{u_i g_i^0(x_1 - u_1, x_2 - u_2, x_3 - u_3)}{[u_1^2 + u_2^2 + u_3^2]^3} du_1 du_2 du_3 = 0$$

(4.5)

It becomes apparent on closer examination of (4.5) that any term in $g_i^0$, derived from (2.6), that is not a function of $x_i$, cannot be recovered by performing the integrals on the righthand side of (4.5); as a result of the integral identity $\int_{-\infty}^{\infty} \frac{u_i}{u_i^2 + \beta} du = 0$, $\beta$ a constant, these terms vanish entirely. As a consequence, it is a prerequisite for the relationship (4.5) to hold that all of the terms that are not functions of $x_i$ on the righthand side of (2.6) must sum to zero. We therefore divide $g_i^0$ into two parts as follows:

$$g_i^0 = V_i^0 + W_i^0 \quad i = 1, 2, 3$$

(4.6)

$V_i^0$ is the sum of all terms that are not functions of $x_i$, and $W_i^0$ is the sum of all remaining terms. For the relationship (4.5) to hold $V_i^0 = 0$. By substituting $v_1^0$, $v_2^0$ and $v_3^0$ into (2.6) and separating the terms into $V_i^0$ and $W_i^0$ we get the following expressions:

$$g_1^0(x_1, x_2, x_3) = V_1^0(x_2, x_3) + W_1^0(x_1, x_2, x_3)$$

(4.7)

$$g_2^0(x_2, x_3, x_1) = V_2^0(x_3, x_1) + W_2(x_2, x_3, x_1)$$

(4.8)

$$g_3^0(x_3, x_1, x_2) = V_3^0(x_1, x_2) + W_3(x_3, x_1, x_2)$$

(4.9)
where

\[ V_1^0 (x_2, x_3) = -\alpha \frac{1}{4} \cos (\xi_1 - \xi_3) \sin (\xi_2 - \xi_3) \cos (2\alpha x_2 + \xi_1 + \xi_2) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_1 - \xi_3) \cos (\xi_1 - \xi_2) \sin (2\alpha x_3 + \xi_2 + \xi_3) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_1 - \xi_2) \cos (\xi_1 - \xi_2) \sin (2\alpha x_2 + \xi_3 + \xi_1) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_3 - \xi_2) \sin (\xi_1 - \xi_3) \cos (2\alpha x_3 + \xi_1 + \xi_2) + \]
\[ + \alpha \frac{1}{4} \sin (\xi_2 - \xi_1) \sin (\xi_3 - \xi_1) \sin (2\alpha x_2 + \xi_2 + \xi_3) + \]
\[ + \alpha \frac{1}{4} \sin (\xi_2 - \xi_1) \sin (\xi_2 - \xi_3) \sin (2\alpha x_3 + \xi_3 + \xi_1) \]  
(4.10)

\[ V_2^0 (x_3, x_1) = -\alpha \frac{1}{4} \cos (\xi_1 - \xi_3) \sin (\xi_2 - \xi_3) \cos (2\alpha x_3 + \xi_2 + \xi_1) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_1 - \xi_3) \cos (\xi_1 - \xi_2) \sin (2\alpha x_1 + \xi_2 + \xi_3) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_3 - \xi_2) \cos (\xi_1 - \xi_2) \sin (2\alpha x_3 + \xi_3 + \xi_1) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_3 - \xi_2) \sin (\xi_1 - \xi_3) \cos (2\alpha x_1 + \xi_1 + \xi_2) + \]
\[ + \alpha \frac{1}{4} \sin (\xi_2 - \xi_1) \sin (\xi_3 - \xi_1) \sin (2\alpha x_3 + \xi_2 + \xi_3) + \]
\[ + \alpha \frac{1}{4} \sin (\xi_2 - \xi_1) \sin (\xi_2 - \xi_3) \sin (2\alpha x_1 + \xi_3 + \xi_1) \]  
(4.11)

\[ V_3^0 (x_1, x_2) = -\alpha \frac{1}{4} \cos (\xi_1 - \xi_3) \sin (\xi_2 - \xi_3) \cos (2\alpha x_1 + \xi_2 + \xi_1) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_1 - \xi_3) \cos (\xi_1 - \xi_2) \sin (2\alpha x_2 + \xi_2 + \xi_3) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_3 - \xi_2) \cos (\xi_1 - \xi_2) \sin (2\alpha x_1 + \xi_3 + \xi_1) - \]
\[ - \alpha \frac{1}{4} \cos (\xi_3 - \xi_2) \sin (\xi_1 - \xi_3) \cos (2\alpha x_2 + \xi_1 + \xi_2) + \]
\[ + \alpha \frac{1}{4} \sin (\xi_2 - \xi_1) \sin (\xi_3 - \xi_1) \sin (2\alpha x_1 + \xi_2 + \xi_3) + \]
\[ + \alpha \frac{1}{4} \sin (\xi_2 - \xi_1) \sin (\xi_2 - \xi_3) \sin (2\alpha x_2 + \xi_3 + \xi_1) \]  
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\[ W_1^0 (x_1, x_2, x_3) = \frac{\alpha}{4} \sin (2\alpha x_1 + 2\xi_1) - \frac{\alpha}{4} \sin (2\alpha x_1 + 2\xi_1) \cos (2\alpha x_2 + 2\xi_3) + \]
\[ + \frac{\alpha}{4} \sin (2\alpha x_1 + 2\xi_2) - \frac{\alpha}{4} \sin (2\alpha x_1 + 2\xi_2) \cos (2\alpha x_2 + 2\xi_3) - \]
\[ - \frac{\alpha}{2} \sin (\xi_3 - \xi_2) \sin (\xi_3 - \xi_1) \sin (2\alpha x_1 + \xi_1 + \xi_2) - \]
\[ - \frac{\alpha}{4} \sin (\xi_3 - \xi_1) \sin (2\alpha x_1 + \xi_1 + \xi_2) \sin (2\alpha x_2 + \xi_2 + \xi_3) - \]
\[ - \frac{\alpha}{4} \sin (\xi_3 - \xi_2) \sin (2\alpha x_1 + \xi_1 + \xi_2) \sin (2\alpha x_3 + \xi_3 + \xi_1) + \]
\[ + \frac{\alpha}{4} \sin (\xi_2 - \xi_3) \cos (2\alpha x_1 + \xi_3 + \xi_1) \cos (2\alpha x_2 + \xi_1 + \xi_2) + \]
\[ + \frac{\alpha}{4} \cos (\xi_1 - \xi_2) \cos (2\alpha x_1 + \xi_3 + \xi_1) \sin (2\alpha x_2 + \xi_3 + \xi_1) - \]
\[ + \frac{\alpha}{4} \cos (\xi_1 - \xi_2) \cos (2\alpha x_1 + \xi_3 + \xi_1) \sin (2\alpha x_2 + \xi_3 + \xi_1) + \]
\[ + \frac{\alpha}{4} \sin (\xi_1 - \xi_3) \cos (2\alpha x_1 + \xi_3 + \xi_2) \cos (2\alpha x_3 + \xi_1 + \xi_2) \]  
\[ (4.13) \]

\[ W_2^0 (x_2, x_3, x_1) = \frac{\alpha}{4} \sin (2\alpha x_2 + 2\xi_1) - \frac{\alpha}{4} \sin (2\alpha x_2 + 2\xi_1) \cos (2\alpha x_1 + 2\xi_3) + \]
\[ + \frac{\alpha}{4} \sin (2\alpha x_2 + 2\xi_2) - \frac{\alpha}{4} \sin (2\alpha x_2 + 2\xi_2) \cos (2\alpha x_3 + 2\xi_3) - \]
\[ - \frac{\alpha}{2} \sin (\xi_3 - \xi_2) \sin (\xi_3 - \xi_1) \sin (2\alpha x_2 + \xi_1 + \xi_2) - \]
\[ - \frac{\alpha}{4} \sin (\xi_3 - \xi_1) \sin (2\alpha x_2 + \xi_1 + \xi_2) \sin (2\alpha x_3 + \xi_2 + \xi_3) - \]
\[ - \frac{\alpha}{4} \sin (\xi_3 - \xi_2) \sin (2\alpha x_2 + \xi_1 + \xi_2) \sin (2\alpha x_3 + \xi_3 + \xi_1) + \]
\[ + \frac{\alpha}{4} \sin (\xi_2 - \xi_3) \cos (2\alpha x_2 + \xi_3 + \xi_1) \cos (2\alpha x_3 + \xi_1 + \xi_2) + \]
\[ + \frac{\alpha}{4} \cos (\xi_1 - \xi_2) \cos (2\alpha x_2 + \xi_3 + \xi_1) \sin (2\alpha x_1 + \xi_2 + \xi_3) - \]
\[ + \frac{\alpha}{4} \cos (\xi_1 - \xi_2) \cos (2\alpha x_2 + \xi_3 + \xi_1) \sin (2\alpha x_1 + \xi_3 + \xi_1) + \]
\[ + \frac{\alpha}{4} \sin (\xi_1 - \xi_3) \cos (2\alpha x_2 + \xi_3 + \xi_2) \cos (2\alpha x_1 + \xi_1 + \xi_2) \]  
\[ (4.14) \]
In light of the fact that \( g_0 \) is to hold, it is imperative that the following prerequisites are met:

\[
\begin{align*}
\psi_1^0 (x_2, x_3) &= 0 \quad \forall -\infty < x_2 < \infty \text{ and } -\infty < x_3 < \infty \quad (4.16) \\
\psi_2^0 (x_3, x_1) &= 0 \quad \forall -\infty < x_3 < \infty \text{ and } -\infty < x_1 < \infty \quad (4.17) \\
\psi_3^0 (x_1, x_2) &= 0 \quad \forall -\infty < x_1 < \infty \text{ and } -\infty < x_2 < \infty \quad (4.18)
\end{align*}
\]

\( \psi_1^0 (x_2, x_3), \psi_2^0 (x_3, x_1) \) and \( \psi_3^0 (x_1, x_2) \) are, respectively, given by (4.10), (4.11) and (4.12).

It is therefore necessary to choose the phase angles \( \xi_1, \xi_2 \) and \( \xi_3 \) such that they satisfy the prerequisites (4.16), (4.17) and (4.18). In this particular case, as can be seen by substituting into (4.10), (4.11) and (4.12), the phase angles \( \xi_1 = -\frac{\pi}{\alpha}, \xi_2 = \frac{\pi}{\alpha}, \) and \( \xi_3 = \frac{\pi}{\alpha} \) satisfy (4.16), (4.17) and (4.18).

In light of the fact that \( g_0^i \) equals \( \psi_0^i \), the following expressions can be derived for \( g_0^i \) and \( \psi_0^0 (x_1, x_2, x_3) = \sum_{k=1}^{3} \frac{\partial \psi_0^i}{\partial x_k} \):

\[
\begin{align*}
g_0^i (x_1, x_2, x_3) &= -\frac{3\alpha}{8} \sin (2\alpha x_1) - \frac{3\alpha}{32} \sin (2\alpha x_1) \cos (2\alpha x_2) - \frac{3\alpha}{32} \sin (2\alpha x_1) \cos (2\alpha x_3) - \\
&\quad - \frac{3\sqrt{3}\alpha}{32} \cos (2\alpha x_1) \cos (2\alpha x_3) + \frac{3\sqrt{3}\alpha}{32} \cos (2\alpha x_1) \cos (2\alpha x_2) + \\
&\quad + \frac{3\sqrt{3}\alpha}{32} \sin (2\alpha x_1) \sin (2\alpha x_2) - \frac{3\sqrt{3}\alpha}{32} \sin (2\alpha x_1) \sin (2\alpha x_3) + \\
&\quad + \frac{3\alpha}{32} \cos (2\alpha x_1) \sin (2\alpha x_3) + \frac{3\alpha}{32} \cos (2\alpha x_1) \sin (2\alpha x_2)
\end{align*}
\]
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\[ g^0_2(x_2, x_3, x_1) = -\frac{3\alpha}{8} \sin (2\alpha x_2) - \frac{3\alpha}{32} \sin (2\alpha x_2) \cos (2\alpha x_3) - \frac{3\alpha}{32} \sin (2\alpha x_2) \cos (2\alpha x_1) - \frac{3\sqrt{3}\alpha}{32} \cos (2\alpha x_2) \cos (2\alpha x_1) + \frac{3\sqrt{3}\alpha}{32} \cos (2\alpha x_2) \cos (2\alpha x_3) + \frac{3\sqrt{3}\alpha}{32} \sin (2\alpha x_2) \sin (2\alpha x_3) - \frac{3\sqrt{3}\alpha}{32} \sin (2\alpha x_2) \sin (2\alpha x_1) + \frac{3\alpha}{32} \cos (2\alpha x_2) \sin (2\alpha x_1) + \frac{3\alpha}{32} \cos (2\alpha x_2) \sin (2\alpha x_3) \]

\[ g^0_3(x_3, x_1, x_2) = -\frac{3\alpha}{8} \sin (2\alpha x_3) - \frac{3\alpha}{32} \sin (2\alpha x_3) \cos (2\alpha x_1) - \frac{3\alpha}{32} \sin (2\alpha x_3) \cos (2\alpha x_2) - \frac{3\sqrt{3}\alpha}{32} \cos (2\alpha x_3) \cos (2\alpha x_1) + \frac{3\sqrt{3}\alpha}{32} \cos (2\alpha x_3) \cos (2\alpha x_2) + \frac{3\sqrt{3}\alpha}{32} \sin (2\alpha x_3) \sin (2\alpha x_1) - \frac{3\sqrt{3}\alpha}{32} \sin (2\alpha x_3) \sin (2\alpha x_2) + \frac{3\alpha}{32} \cos (2\alpha x_3) \sin (2\alpha x_2) + \frac{3\alpha}{32} \cos (2\alpha x_3) \sin (2\alpha x_1) \]

\[ \mathcal{G}^0(x_1, x_2, x_3) = -\frac{3\alpha^2}{4} \cos (2\alpha x_1) - \frac{3\alpha^2}{4} \cos (2\alpha x_2) - \frac{3\alpha^2}{4} \cos (2\alpha x_3) - \frac{3\alpha^2}{8} \cos (2\alpha x_1) \cos (2\alpha x_2) - \frac{3\alpha^2}{8} \cos (2\alpha x_1) \cos (2\alpha x_3) + \frac{3\sqrt{3}\alpha^2}{8} \sin (2\alpha x_1) \cos (2\alpha x_2) - \frac{3\sqrt{3}\alpha^2}{8} \sin (2\alpha x_1) \cos (2\alpha x_3) + \frac{3\sqrt{3}\alpha^2}{8} \cos (2\alpha x_1) \sin (2\alpha x_2) - \frac{3\sqrt{3}\alpha^2}{8} \cos (2\alpha x_1) \sin (2\alpha x_3) - \frac{3\alpha^2}{8} \sin (2\alpha x_1) \sin (2\alpha x_2) - \frac{3\alpha^2}{8} \sin (2\alpha x_1) \sin (2\alpha x_3) - \frac{3\alpha^2}{8} \cos (2\alpha x_2) \cos (2\alpha x_3) - \frac{3\alpha^2}{8} \sin (2\alpha x_2) \cos (2\alpha x_3) + \frac{3\sqrt{3}\alpha^2}{8} \cos (2\alpha x_2) \sin (2\alpha x_3) - \frac{3\sqrt{3}\alpha^2}{8} \sin (2\alpha x_2) \sin (2\alpha x_3) \]

By substituting \( \mathcal{G}^0 \) into \( \mathcal{U}^0 \) and performing the integrations term by term, we can see that \( \mathcal{U}^0 \equiv 0 \), \( i = 1, 2, 3 \). We have used the integral identities \([C4] - [C8]\) given in Appendix C to perform the integrations.

Substituting for the initial condition \( v^0_i \), \( i = 1, 2, 3 \) from \([4.11], [4.12] \) and \([4.13]\) into \([5.12]\) with phase angles \( \xi_1, \xi_2 \) and \( \xi_3 \) set, respectively, to \( -\frac{\pi}{2}, \frac{\pi}{2} \), and \( \frac{\pi}{2} \) and performing the integrations, we arrive at the solution of the Navier-Stokes equation \([2.1]\):
\[ v_1 = \left[ \sin \left( \alpha x_1 - \frac{\pi}{3} \right) \cos \left( \alpha x_2 + \frac{\pi}{3} \right) \sin \left( \alpha x_3 + \frac{\pi}{2} \right) - \sin \left( \alpha x_1 + \frac{\pi}{3} \right) \cos \left( \alpha x_3 - \frac{\pi}{3} \right) \sin \left( \alpha x_2 + \frac{\pi}{2} \right) \right] e^{-3a^2\kappa t} \] (4.23)

\[ v_2 = \left[ \sin \left( \alpha x_2 - \frac{\pi}{3} \right) \cos \left( \alpha x_3 + \frac{\pi}{3} \right) \sin \left( \alpha x_1 + \frac{\pi}{2} \right) - \sin \left( \alpha x_2 + \frac{\pi}{3} \right) \cos \left( \alpha x_1 - \frac{\pi}{3} \right) \sin \left( \alpha x_3 + \frac{\pi}{2} \right) \right] e^{-3a^2\kappa t} \] (4.24)

\[ v_3 = \left[ \sin \left( \alpha x_3 - \frac{\pi}{3} \right) \cos \left( \alpha x_1 + \frac{\pi}{3} \right) \sin \left( \alpha x_2 + \frac{\pi}{2} \right) - \sin \left( \alpha x_3 + \frac{\pi}{3} \right) \cos \left( \alpha x_2 - \frac{\pi}{3} \right) \sin \left( \alpha x_1 + \frac{\pi}{2} \right) \right] e^{-3a^2\kappa t} \] (4.25)

We have used the integral identities \([C1]-[C3]\) given in Appendix C to perform the integrations.

We may also express (4.23) – (4.25) as:

\[ v_1 = \left[ \frac{3}{4} \cos(\alpha x_1) \sin(\alpha x_2 - \alpha x_3) - \frac{1}{\sqrt{3}} \sin(\alpha x_1) \sin(\alpha x_2 + \alpha x_3) - \frac{2}{\sqrt{3}} \cos(\alpha x_1) \cos(\alpha x_2) \cos(\alpha x_3) \right] e^{-3a^2\kappa t} \] (4.26)

\[ v_2 = \left[ \frac{3}{4} \cos(\alpha x_2) \sin(\alpha x_3 - \alpha x_1) - \frac{1}{\sqrt{3}} \sin(\alpha x_2) \sin(\alpha x_3 + \alpha x_1) - \frac{2}{\sqrt{3}} \cos(\alpha x_2) \cos(\alpha x_3) \cos(\alpha x_1) \right] e^{-3a^2\kappa t} \] (4.27)

\[ v_3 = \left[ \frac{3}{4} \cos(\alpha x_3) \sin(\alpha x_1 - \alpha x_2) - \frac{1}{\sqrt{3}} \sin(\alpha x_3) \sin(\alpha x_1 + \alpha x_2) - \frac{2}{\sqrt{3}} \cos(\alpha x_3) \cos(\alpha x_1) \cos(\alpha x_2) \right] e^{-3a^2\kappa t} \] (4.28)

Pressure is obtained from the solution of the Poisson equation (3.2). Because (3.10) is satisfied, we can express (3.5) as follows:

\[ \frac{\partial p}{\partial x_i} = \rho g_i = \rho g_i^0 e^{-6a^2\kappa t}, \quad i = 1, 2, 3 \] (4.29)
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This leads to straightforward integration for pressure:

\[
p = \frac{3\rho}{16} \left[ \cos (2\alpha x_1) + \cos (2\alpha x_2) + \cos (2\alpha x_3) \right] e^{-6\alpha^2 \kappa t} + \\
+ \frac{3\rho}{64} \left[ \cos (2\alpha x_1 - 2\alpha x_2) + \cos (2\alpha x_3 - 2\alpha x_1) \right] e^{-6\alpha^2 \kappa t} + \\
+ \frac{3\sqrt{3}\rho}{64} \left[ \sin (2\alpha x_1 - 2\alpha x_2) + \sin (2\alpha x_3 - 2\alpha x_1) \right] \\
+ \sin (2\alpha x_2 - 2\alpha x_3) + \cos (2\alpha x_3 - 2\alpha x_2) \right] e^{-6\alpha^2 \kappa t}
\] (4.30)

5. Concluding Remarks

At the outset, the solution we have presented for the velocity vector field appears to be the same as that of Antuono (2020): \((A_1)-(A_3)\) except for the augmenting coefficient \((\frac{4\sqrt{2}}{3\sqrt{3}}) U_0\). However, they are not the same.

Antuono (2020) describes the velocity vector field in a tours \(T^3 = [0, L]^3\), in terms of a wave number divided into two parts, characterized by positive and negative helicity respectively. In his work, a singularity that arose from the derivation was eliminated by imposing the bounded energy constraint \((2.5)\). The velocity components were normalized by a reference velocity \(U_0\) so that the average kinetic energy per unit of mass is \((\frac{U_0^2}{2})\) at \(t = 0\). The pressure field was obtained in terms of a reference pressure and reference density from the velocity field.

In three dimensions, we have four unknowns, three components of the velocity vector field and pressure, and four equations. It is important to note that defining pressure at the initial time independent of velocity would render the problem ill-posed; therefore, only a solenoidal initial velocity vector field is prescribed. In a Cartesian frame of reference, we have solved the fundamental equations of momentum, the Navier-Stokes equation and mass, the continuity equation for incompressible fluids. The method we have presented does not necessitate the imposition of the bounded energy constraint \((2.5)\). The solutions \((1.24)-(1.26)\) and \((1.30)\) for velocity and pressure, respectively, are valid in the whole unbounded domain \(\mathbb{R}^3 = \{ -\infty < x_i < \infty; \ i = 1, 2, 3\}\). We have shown that the solutions of the Navier-Stokes system of equations for incompressible fluids for the velocity field and pressure are given by the Cauchy diffusion equation and the Poisson equation, respectively, provided the condition \((3.10)\) is satisfied.
Appendix A. Solution with positive helicity given by Antuono (2020)

\[ u_1 = \left( \frac{4\sqrt{2}}{3\sqrt{3}} \right) U_0 \left[ \sin \left( \omega x_1 - \frac{5\pi}{6} \right) \cos \left( \omega x_2 - \frac{\pi}{6} \right) \sin \left( \omega x_3 \right) - \right. \\
\left. \sin \left( \omega x_1 - \frac{\pi}{6} \right) \cos \left( \omega x_3 - \frac{5\pi}{6} \right) \sin \left( \omega x_2 \right) \right] e^{-3\omega^2 \kappa t} \] (A 1)

\[ u_2 = \left( \frac{4\sqrt{2}}{3\sqrt{3}} \right) U_0 \left[ \sin \left( \omega x_2 - \frac{5\pi}{6} \right) \cos \left( \omega x_3 - \frac{\pi}{6} \right) \sin \left( \omega x_1 \right) - \right. \\
\left. \sin \left( \omega x_2 - \frac{\pi}{6} \right) \cos \left( \omega x_1 - \frac{5\pi}{6} \right) \sin \left( \omega x_3 \right) \right] e^{-3\omega^2 \kappa t} \] (A 2)

\[ u_3 = \left( \frac{4\sqrt{2}}{3\sqrt{3}} \right) U_0 \left[ \sin \left( \omega x_3 - \frac{5\pi}{6} \right) \cos \left( \omega x_1 - \frac{\pi}{6} \right) \sin \left( \omega x_2 \right) - \right. \\
\left. \sin \left( \omega x_3 - \frac{\pi}{6} \right) \cos \left( \omega x_2 - \frac{5\pi}{6} \right) \sin \left( \omega x_1 \right) \right] e^{-3\omega^2 \kappa t} \] (A 3)

where \( u_1, u_2 \) and \( u_3 \) are the components of the velocity vector field, \( U_0 \) is the reference velocity, \( \kappa \) is the kinematical viscosity and \( \omega \) is the wave number.

Appendix B. Derivations of two known solutions satisfying equation (3.10)

B.1. The Taylor solution in \( \mathbb{R}^2 = \left\{ -\infty < x_i < \infty; i = 1, 2 \right\} \)

\[ v_1^0 = \sin (\pi x_1) \cos (\pi x_2), \ v_2^0 = -\cos (\pi x_1) \sin (\pi x_2) \] and \( f_1 = f_2 = 0. \)

Substituting for \( v_1^0 \) and \( v_2^0 \) in (2.3) and (3.10) we obtain

\[ g_i^0 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x_i - y_i) \sum_{k=1}^2 \frac{\partial g_i^0(y,t)}{\partial y_k}}{P_2(x,y)} \prod_{j=1}^n dy_j = \frac{\pi}{2} \sin \left( 2\pi x_i \right) \] (B 1)

resulting in \( U_i(x,t) \equiv 0. \ v(x,t), p(x,t) \) are obtained from (3.11) and (3.2):

\[ v_1 = \sin (\pi x_1) \cos (\pi x_2) e^{-2\pi^2 \kappa t} \] (B 2)

\[ v_2 = -\cos (\pi x_1) \sin (\pi x_2) e^{-2\pi^2 \kappa t} \] (B 3)
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\[ p = -\rho e^{-\pi^2 \kappa t} \frac{1}{4} \left[ \cos(2\pi x_1) + \cos(2\pi x_2) \right] \] (B 4)

B.2. Unsteady solution of Arnold-Beltrami-Childress derived by Thambanayagam (2013) in \( \mathbb{R}^3 = \{ -\infty < x_i < \infty; \; i = 1, 2, 3 \} \)

\[ v_1^0 = a \sin \pi x_3 - c \cos \pi x_2, \quad v_2^0 = b \sin \pi x_1 - a \cos \pi x_3, \quad v_3^0 = c \sin \pi x_2 - b \cos \pi x_1 \] and \( f_1 = f_2 = f_3 = 0. \) \( a, b \) and \( c \) are real constants.

Substituting for \( v_1^0, v_2^0 \) and \( v_3^0 \) in (2.3) and (3.10) we obtain

\[ g_1^0 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x_1 - y_1)}{\{ P_3(x, y) \}^{\frac{3}{2}}} \prod_{j=1}^3 dy_j = \pi \left\{ bc \sin(\pi x_1) \sin(\pi x_2) - ab \cos(\pi x_1) \cos(\pi x_3) \right\} \] (B 5)

\[ g_2^0 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x_2 - y_2)}{\{ P_3(x, y) \}^{\frac{3}{2}}} \prod_{j=1}^3 dy_j = \pi \left\{ ac \sin(\pi x_2) \sin(\pi x_3) - bc \cos(\pi x_1) \cos(\pi x_2) \right\} \] (B 6)

\[ g_3^0 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x_3 - y_3)}{\{ P_3(x, y) \}^{\frac{3}{2}}} \prod_{j=1}^3 dy_j = \pi \left\{ ab \sin(\pi x_1) \sin(\pi x_3) - ac \cos(\pi x_2) \cos(\pi x_3) \right\} \] (B 7)

\( v(x, t), p(x, t) \) are obtained from (3.11) and (3.2):

\[ v_1 = \{ a \sin \pi x_3 - c \cos \pi x_2 \} e^{-\pi^2 \kappa t} \] (B 8)

\[ v_2 = \{ b \sin \pi x_1 - a \cos \pi x_3 \} e^{-\pi^2 \kappa t} \] (B 9)

\[ v_3 = \{ c \sin \pi x_2 - b \cos \pi x_1 \} e^{-\pi^2 \kappa t} \] (B 10)

and

\[ p = -\rho e^{-2\pi^2 \kappa t} \left[ bc \cos(\pi x_1) \sin(\pi x_2) + ab \cos(\pi x_3) \sin(\pi x_1) + ac \cos(\pi x_2) \sin(\pi x_3) \right] \] (B 11)
Appendix C. Integral identities used in this paper

\[
\int_{-\infty}^{\infty} e^{-\frac{(x-u)^2}{4\tau}} \, du = 2\sqrt{\pi\tau} \tag{C 1}
\]

\[
\int_{-\infty}^{\infty} \sin(\alpha u) e^{-\frac{(x-u)^2}{4\tau}} \, du = 2\sqrt{\pi\tau} e^{-\alpha^2\tau}\sin(\alpha x) \tag{C 2}
\]

\[
\int_{-\infty}^{\infty} \cos(\alpha u) e^{-\frac{(x-u)^2}{4\tau}} \, du = 2\sqrt{\pi\tau} e^{-\alpha^2\tau}\cos(\alpha x) \tag{C 3}
\]

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{u_1 \cos\left\{ \beta (x_1 - u_1) \right\}}{\sqrt{u_1^2 + u_2^2 + u_3^2}} \, du_1 du_2 du_3 = \frac{4\pi}{\beta} \sin(\beta x_1) \tag{C 4}
\]

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{u_1 \cos\left\{ \beta (x_1 - u_1) \right\} \cos\left\{ \beta (x_2 - u_2) \right\}}{\sqrt{u_1^2 + u_2^2 + u_3^2}} \, du_1 du_2 du_3 = \frac{2\pi}{\beta} \sin(\beta x_1) \cos(\beta x_2) \tag{C 5}
\]

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{u_1 \sin\left\{ \beta (x_1 - u_1) \right\} \cos\left\{ \beta (x_2 - u_2) \right\}}{\sqrt{u_1^2 + u_2^2 + u_3^2}} \, du_1 du_2 du_3 = -\frac{2\pi}{\beta} \cos(\beta x_1) \cos(\beta x_2) \tag{C 6}
\]

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{u_1 \cos\left\{ \beta (x_1 - u_1) \right\} \sin\left\{ \beta (x_2 - u_2) \right\}}{\sqrt{u_1^2 + u_2^2 + u_3^2}} \, du_1 du_2 du_3 = \frac{2\pi}{\beta} \sin(\beta x_1) \sin(\beta x_2) \tag{C 7}
\]

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{u_1 \sin\left\{ \beta (x_1 - u_1) \right\} \sin\left\{ \beta (x_2 - u_2) \right\}}{\sqrt{u_1^2 + u_2^2 + u_3^2}} \, du_1 du_2 du_3 = -\frac{2\pi}{\beta} \cos(\beta x_1) \sin(\beta x_2) \tag{C 8}
\]

where \( \alpha \) and \( \beta \) are real constants.

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