Eigenvalue bounds for non-self-adjoint Schrödinger operators with non-trapping metrics
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Abstract. We prove weighted uniform estimates for the resolvent of the Laplace operator in Schatten spaces, on non-trapping asymptotically conic manifolds of dimension $n \geq 3$, generalizing a result of Frank and Sabin [16], obtained in the Euclidean setting. As an application of these estimates we establish Lieb–Thirring type bounds for eigenvalues of Schrödinger operators with complex potentials on non-trapping asymptotically conic manifolds, extending those of Frank [13, 14], Frank and Sabin [16], and Frank and Simon [17] proven in the Euclidean setting. In particular, our results are valid for the metric Schrödinger operator in the Euclidean space, with a metric being a sufficiently small compactly supported perturbation of the Euclidean one. To the best of our knowledge, these are the first Lieb–Thirring type bounds for non-self-adjoint elliptic operators, with principal part having variable coefficients.

1. Introduction and statement of results

Recently there have been numerous works devoted to the study of eigenvalues of the Schrödinger operator $\mathcal{P} = \Delta + V$ in $L^2(\mathbb{R}^n)$, with $\Delta$ being the nonnegative Laplace operator and $V$ being a complex-valued potential. Of particular interest here is the problem of obtaining quantitative information concerning the localization and distribution of the eigenvalues of $\mathcal{P}$ under the only assumption that $V \in L^p(\mathbb{R}^n)$, for some $1 \leq p < \infty$. Here we may remark that the spectrum of $\mathcal{P}$ in $\mathbb{C} \setminus [0, \infty)$ consists then of isolated eigenvalues of finite algebraic multiplicity, see [14, Proposition B.2].

The following two types of results are of particular interest for this problem. The first one deals with bounds on the individual eigenvalues of $\mathcal{P}$ in terms of the $L^p$-norm of the potential. If $V$ is real-valued, so that $\mathcal{P}$ admits a natural self-adjoint realization, then the eigenvalues of $\mathcal{P}$ in $\mathbb{C} \setminus [0, \infty)$ are negative and by the variational principle and Sobolev’s inequalities, for any eigenvalue $\lambda < 0$ of $\mathcal{P}$, we have the scale-invariant bounds,

$$|\lambda|^\gamma \leq C_{\gamma,n} \int_{\mathbb{R}^n} |V(x)|^{\gamma + \frac{n}{2}} \, dx$$

(1.1)

for every $\gamma \geq \frac{1}{2}$ if $n = 1$ and every $\gamma > 0$ if $n \geq 2$. Here the constant $C_{\gamma,n} > 0$ depends on $\gamma$ and $n$ only, see [30], [33], [17].

If the potential $V$ is complex-valued, the problem is more involved due to the lack of variational techniques and the absence of a spectral resolution theorem. In dimension
n = 1 the bound (1.1) with $\gamma = \frac{1}{2}$ was proved by Abramov, Aslanyan, and Davies in [1]. In dimensions $n \geq 2$, Frank [13] established the bound (1.1) for all eigenvalues $\lambda \in \mathbb{C} \setminus [0, \infty)$ and for all $0 < \gamma \leq \frac{1}{2}$, see also [17]. The work [14] gives a replacement of the bound (1.1) for all $\gamma > \frac{1}{2}$. We refer to [32], [5], [11], [6], [35], for some other recent works on bounds on the individual eigenvalues for non-self-adjoint operators of Schrödinger type.

The second type of result is concerned with bounds on sums of powers of absolute values of eigenvalues of $P$, generalizing the classical Lieb–Thirring bounds [33] to the non-self-adjoint case. If $V$ is real-valued then the Lieb–Thirring inequality has the following form,

$$
\sum |\lambda|^{\gamma} \leq C_{\gamma,n} \int_{\mathbb{R}^n} V_-(x)^{\gamma+\frac{n}{2}} dx,
$$

(1.2)

where $V_- = \max(-V, 0)$, $\gamma \geq \frac{1}{2}$ if $n = 1$, $\gamma > 0$ if $n = 2$, and $\gamma \geq 0$ if $n \geq 3$. The summation in the left hand side in (1.2) extends over all negative eigenvalues of $P$, counted with their multiplicities. The situation in the non-self-adjoint case is less clear. In particular, Bögli [3] established that for any $p > n$, there exists a non-real potential $V \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that the Schrödinger operator $P$ has infinitely many non-real eigenvalues accumulating at every point of the essential spectrum $[0, \infty)$, thus showing that inequalities like (1.2) cannot hold in the non-self-adjoint case for $p > n$.

A possible modification of Lieb–Thirring’s inequality (1.2) to the non-self-adjoint case was suggested in [8], and is as follows,

$$
\sum \frac{d(\lambda)^{\gamma+\frac{n}{2}}}{|\lambda|^{\frac{n}{2}}} \leq C_{\gamma,n} \int_{\mathbb{R}^n} |V(x)|^{\gamma+\frac{n}{2}} dx,
$$

(1.3)

where

$$
d(\lambda) = \text{dist}(\lambda, [0, \infty)).
$$

(1.4)

We refer to [9], [7], [16], [38], [15] for some of the important contributions to generalizations of Lieb–Thirring’s inequality (1.2) to the setting of complex potentials.

A crucial idea of Frank [13] in establishing bounds (1.1) on the individual eigenvalues of the Schrödinger operator $P$ with a complex-valued potential was to make use of the uniform $L^p$ resolvent estimates for $\Delta$ of Kenig, Ruiz, Sogge [31]. Recently, this approach was extended to the case of non-self-adjoint Schrödinger operators with inverse-square potentials by Mizutani [36], to the case of magnetic Schrödinger and Pauli operators with complex electromagnetic potentials by Cuenin and Kenig [6], and to the case of the Dirac and fractional Schrödinger operators with complex potentials by Cuenin [5].

Developing the idea of Frank [13] further, Frank and Sabin [16] obtained some very interesting uniform weighted bounds for the resolvent of $\Delta$ in suitable Schatten classes, and applied these bounds to derive uniform estimates on the sums of eigenvalues of
non-self-adjoint Schrödinger operators, thus obtaining some results towards proving the conjectured Lieb–Thirring inequality (1.3) in the case of complex potentials. Recently, this approach was extended by Cuenin [5] to the case of the Dirac and fractional Schrödinger operators with complex potentials.

Notice that in all the works described above the principal part of the operators considered has constant coefficients. It is nevertheless of significant interest to extend both types of results to the case of complex potential perturbations of the Laplace–Beltrami operator $\Delta_g$ considered on $\mathbb{R}^n$ or more generally, on a complete non-compact Riemannian manifold, generalizing the Euclidean structure near infinity.

Of particular interest here is the class of asymptotically conic manifolds, introduced by Melrose [34] and defined as follows. We say that $(M, g)$ is asymptotically conic if $M$ is the interior of a smooth compact manifold with boundary $\overline{M}$, $g$ is a smooth metric on $M$ such that there exists a smooth boundary defining function $x$ on $\overline{M}$ with $(M, g)$ isometric outside a compact set to $(0, \epsilon) \times \partial \overline{M}$ with the metric

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2} = \frac{dx^2}{x^4} + \sum h_{jk}(x, y)dy^jdy^k,$$  

where $h$ is a smooth one-parameter family of metrics on the boundary $\partial \overline{M}$. Here $y = (y_1, \ldots, y_{n-1})$ stand for local coordinates on $\partial \overline{M}$ and $(x, y)$ are the corresponding local coordinates on $M$ near $\partial \overline{M}$. Let $z = (z_1, \ldots, z_n)$ be local coordinates away from $\partial \overline{M}$. We say that $M$ is non-trapping if every geodesic $z(s)$ in $M$ reaches $\partial \overline{M}$ as $s \to \pm \infty$. The function $r = 1/x$ near $x = 0$ can be thought of as a “radial” variable near infinity and $y = (y_1, \ldots, y_{n-1})$ can be regarded as $n - 1$ “angular” variables. Rewriting (1.5) in the $(r, y)$ coordinates, we observe that the metric $g$ is asymptotic to the exact conic metric $dr^2 + r^2h(0)$ on $(r_0, \infty) \times \partial \overline{M}$ as $r \to \infty$.

The most important example of an asymptotically conic manifold is the Euclidean space $M = \mathbb{R}^n$, after a radial compactification. It is non-trapping with $\partial \overline{M} = S^{n-1}$ with the standard metric, and with $(r, y)$ being the usual polar coordinates. More generally, any compactly supported perturbation of the Euclidean space is also asymptotically conic, and it is non-trapping provided that it is sufficiently small in $C^2$, see [25].

The purpose of this paper is to extend both types of results on the localization of complex eigenvalues for Schrödinger operators, from the Euclidean setting to that of an asymptotically conic non-trapping manifold. Throughout the paper, we let $M$ be an asymptotically conic non-trapping manifold of dimension $n \geq 3$. From [24], we recall that the Laplace operator $\Delta_g$, associated with the metric $g$, is nonnegative self-adjoint on $L^2(M)$ with the domain $H^2(M)$. The spectrum of $\Delta_g$ is purely absolutely continuous and is given by $\text{Spec}(\Delta_g) = [0, \infty)$.
Our starting point is the following uniform $L^p$ resolvent estimates of the Kenig–Ruiz–Sogge type for the Laplace operator $\Delta_g$ on an asymptotically conic non-trapping manifold, established in the work [22] of the first two authors.

**Theorem 1.** Let $(M, g)$ be an asymptotically conic non-trapping manifold of dimension $n \geq 3$. Then for all $p \in [\frac{2n}{n+2}, \frac{2(n+1)}{n+3}]$, there is a constant $C > 0$ such that for all $z \in \mathbb{C}$ and for all $f \in L^p(M)$, we have

$$\| (\Delta_g - z)^{-1} f \|_{L^{p'}(M)} \leq C |z|^{\frac{n(n-4)}{2}} \| f \|_{L^p(M)}.$$  

Here $\frac{1}{p} + \frac{1}{p'} = 1$.

As explained in [22], when $z \in (0, +\infty)$, the operator in (1.6) may be taken to be either the outgoing or incoming resolvent $(\Delta_g - (z \pm i0))^{-1}$, defined by

$$(\Delta_g - (z \pm i0))^{-1} = \lim_{\delta \to 0^+} (\Delta_g - (z \pm i\delta))^{-1}$$

as a map $x^{1/2+\varepsilon}L^2(M) \to x^{-1/2-\varepsilon}L^2(M)$ for all $\varepsilon > 0$, where $x$ is the boundary defining function, thanks to the limiting absorption principle, see [34], [26] for details.

We shall next recall the definition of the Schatten spaces of operators on $L^2(M)$, see [39]. Let $A$ be a compact operator on $L^2(M)$, and let $\mu_j(A)$ be the singular values of $A$, given by $\mu_j(A) = \lambda_j((A^*A)^{1/2})$. Here $\lambda_j(B)$ denotes the eigenvalues of a positive self-adjoint compact operator $B$, arranged in decreasing order. The Schatten norm of $A$ of order $1 \leq q < \infty$ is defined as follows,

$$\|A\|_{C_q(L^2(M))}^q = \sum_{j=1}^{\infty} \mu_j(A)^q = \text{tr}((A^*A)^{q/2}).$$

The main contribution of the present paper is the following weighted uniform Schatten class estimate for the resolvent of $\Delta_g$, generalizing a result of Frank and Sabin [16, Theorem 12], obtained in the Euclidean setting.

**Theorem 2.** Let $(M, g)$ be an asymptotically conic non-trapping manifold of dimension $n \geq 3$. Let $p \in [\frac{n}{2}, \frac{n+1}{2}]$. Then there exists $C > 0$ such that for all $z \in \mathbb{C} \setminus \{0\}$ and all $W_1, W_2 \in L^{2p}(M)$, we have $W_1(\Delta_g - z)^{-1}W_2 \in C_q(L^2(M))$, $q = \frac{p(n-1)}{n-p} \in [n-1, n+1]$, and

$$\|W_1(\Delta_g - z)^{-1}W_2\|_{C_q(L^2(M))} \leq C |z|^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}.$$  

(1.7)

**Remark.** When $z \in (0, +\infty)$, the operator in (1.7) may be taken to be either the outgoing or incoming resolvent $(\Delta_g - (z \pm i0))^{-1}$.

The proof of Theorem 2 is based on the following weighted Schatten norm estimates on the spectral measure $dE_{\sqrt{\Delta_g}}(\lambda)$ of $\sqrt{\Delta_g}$, which extend the corresponding estimates of Frank and Sabin [16, Theorem 2], obtained in the Euclidean setting. We believe that these estimates may be of some independent interest.
**Theorem 3.** Let \((M, g)\) be an asymptotically conic non-trapping manifold of dimension \(n \geq 3\). Let \(p \in [1, \frac{n}{n-1}]\). Then there exists \(C > 0\) such that for all \(\lambda > 0\) and all \(W_1, W_2 \in L^{2p}(M)\), we have
\[
W_1 dE \sqrt{\Delta_g} (\lambda) W_2 \in C_q(L^2(M)), \quad q = \frac{p(n-1)}{n-p} \in [1, n+1],
\]
and
\[
\|W_1 dE \sqrt{\Delta_g} (\lambda) W_2\|_{C_q(L^2(M))} \leq C \lambda^{-\frac{n}{n-1}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}.
\] (1.8)

Let us now consider the Schrödinger operator \(\Delta_g + V\) with a complex valued potential \(V \in L^p(M), \frac{n}{2} \leq p < \infty\). As explained in Section 6, this operator has a natural \(m\)-sectorial realization on \(L^2(M)\), and the spectrum of \(\Delta_g + V\) in \(\mathbb{C} \setminus [0, \infty)\) consists of isolated eigenvalues of finite algebraic multiplicity.

As an application of Theorem 1, we have the following generalization of the results of Frank [13, 14], and Frank and Simon [17] concerning bounds on the individual eigenvalues of non-self-adjoint Schrödinger operators in the Euclidean setting to that of an asymptotically conic non-trapping manifold, see also [12].

**Theorem 4.** Let \((M, g)\) be an asymptotically conic non-trapping manifold of dimension \(n \geq 3\).

(i) Let \(V \in L^{\gamma + \frac{n}{2}}(M)\) for some \(0 < \gamma \leq \frac{1}{2}\). Then any eigenvalue \(\lambda \in \mathbb{C}\) of the operator \(\Delta_g + V\) satisfies
\[
|\lambda|^\gamma \leq C_{\gamma,n} \|V\|_{L^{\gamma + \frac{n}{2}}(M)},
\] (1.9)
where the constant \(C_{\gamma,n} > 0\) depends on \(\gamma\) and \(n\) only.

(ii) Let \(V \in L^{\frac{n}{2}}(M)\) be such that \(\|V\|_{L^{\frac{n}{2}}(M)}\) is sufficiently small. Then the operator \(\Delta_g + V\) has no eigenvalues.

(iii) Let \(V \in L^{\gamma + \frac{n}{2}}(M)\) for some \(\gamma > \frac{1}{2}\). Then any eigenvalue \(\lambda \in \mathbb{C}\) of the operator \(\Delta_g + V\) satisfies
\[
d(\lambda)^{-\frac{1}{2}} |\lambda|^\frac{\gamma}{2} \leq C_{\gamma,n} \|V\|_{L^{\gamma + \frac{n}{2}}(M)},
\] (1.10)
where \(d(\lambda)\) is given by (1.4) and the constant \(C_{\gamma,n} > 0\) depends on \(\gamma\) and \(n\) only.

**Remark.** Parts (i) and (ii) of Theorem 4 have been established in [22, Proposition 7.2], without specifying the radius of the disk, containing the eigenvalues of \(\Delta_g + V\), in part (i).

As a consequence of Theorem 2, we obtain the following generalization of a result of Frank and Sabin [16, Theorem 16], concerning Lieb-Thirring type inequalities for the sums of eigenvalues of \(\Delta_g + V\) in the case of a short range potential \(V \in L^p(M), \ p = \frac{n}{2} + \gamma, \ 0 \leq \gamma \leq \frac{1}{2}\).
Theorem 5. Let $(M, g)$ be an asymptotically conic non-trapping manifold of dimension $n \geq 3$, and let $V \in L^p(M)$ with $p$ such that
\[ \frac{n}{2} \leq p \leq \frac{n + 1}{2}. \]

Let us denote by $\lambda_j$ the eigenvalues of $\Delta_g + V$ in $\mathbb{C} \setminus [0, \infty)$, repeated according to their algebraic multiplicities. The following estimates then hold:

(i) If $p = \frac{n}{2}$, we have
\[ \sum_j \text{Im} \sqrt{\lambda_j} < \infty, \] (1.11)
where the branch of the square root is chosen to have positive imaginary part.

(ii) If $\frac{n}{2} < p \leq \frac{n + 1}{2}$, then
\[ \sum_j \frac{d(\lambda_j)}{|\lambda_j|^{(1-\varepsilon)/2}} \leq C_{\varepsilon, p, n} \|V\|_{L^p(M)}^{(1+\varepsilon)p}, \] (1.12)
for all $\varepsilon$ satisfying
\[ \begin{cases} 
\varepsilon \geq 0, \\
\varepsilon > \frac{p(2n-1) - n^2}{n^2}, \\
\varepsilon > \frac{n^2}{2n-1}, \\
\varepsilon > \frac{n^2}{2n-1}, \\
\varepsilon > \frac{n^2}{2n-1}.
\end{cases} \]

Remark. If $\frac{n}{2} < p \leq \frac{n + 1}{2}$, then by Theorem 4 we know that the eigenvalues of $\Delta_g + V$ are confined to an open disk centered at the origin. Furthermore, it follows from (1.12) that if a sequence of eigenvalues $\mathbb{C} \setminus [0, \infty) \ni \lambda_{j_k} \to E > 0$ then $\text{Im} \lambda_{j_k} \in \ell^1$. In the case $p = \frac{n}{2}$ the bound (1.11) controls a possible accumulation rate of eigenvalues in $\mathbb{C} \setminus [0, \infty)$ at infinity, and it implies in particular with the help of
\[ \text{Im} (\sqrt{\lambda}) = \frac{|\text{Im} \lambda|}{\sqrt{2(|\lambda| + \text{Re} \lambda)}} \]
that if a sequence of eigenvalues $\mathbb{C} \setminus [0, \infty) \ni \lambda_{j_k} \to E > 0$ then $\text{Im} \lambda_{j_k} \in \ell^1$.

As another application of the Schatten class estimates for the resolvent of $\Delta_g$ given in Theorem 2, we get the following generalization of a result of Frank [14, Theorem 1.2], concerning Lieb-Thirring type inequalities for the sums of eigenvalues $\Delta_g + V$ in the case of a long range potential $V \in L^p(M)$, $p = \gamma + \frac{n}{2}$, $\gamma > \frac{1}{2}$.

Theorem 6. Let $(M, g)$ be an asymptotically conic non-trapping manifold of dimension $n \geq 3$, and let $V \in L^p(M)$ with $p = \gamma + \frac{n}{2}$, $\gamma > \frac{1}{2}$. Then the eigenvalues $\lambda_j \in \mathbb{C} \setminus [0, \infty)$ of $\Delta_g + V$, repeated according to their algebraic multiplicities, satisfy the following bounds, for any $\varepsilon > 0$,
\[ \left( \sum_{|\lambda_j| \leq C_{\gamma, n} \int_M |V|^\gamma \frac{d\lambda}{\gamma + \frac{n}{2}}} d(\lambda_j)^{2\gamma + \varepsilon} \right)^{\frac{1}{2(\gamma + \varepsilon)}} \leq L_{\varepsilon, \gamma, n} \int_M |V|^{\gamma + \frac{n}{2}} dx, \]
and for any \( \varepsilon > 0 \), \( 0 < \varepsilon' < \frac{\gamma}{\gamma + \frac{n}{2}} \), and \( \mu \geq 1 \),

\[
\left( \sum_{|\lambda_j|^\gamma \geq \mu C_{\gamma, n} \int_M |V|^\gamma + \frac{n}{2} dx} \frac{d(\lambda_j)^{2\gamma + \varepsilon}}{|\lambda_j|^{2\gamma - \frac{\mu}{\gamma + \frac{n}{2}} + \varepsilon + \varepsilon'}} \right)^{\frac{\gamma(\gamma + \frac{n}{2})}{\gamma - \varepsilon'(\gamma + \frac{n}{2})}} \leq L_{\varepsilon, \varepsilon', \gamma, n, \mu} \int_M |V|^{\gamma + \frac{n}{2}} dx.
\]

Remark. As observed in [14], Theorem 6 has the following consequence: let \( \gamma > 1/2 \) and \( V \in L^{\gamma+\frac{n}{2}}(M) \). If \( (\lambda_j)_{j=1}^\infty \) is a sequence of eigenvalues of \( \Delta_g + V \) with \( \lambda_j \to \lambda_0 \in [0, \infty) \) then \( \text{Im} \lambda_j \in l^p \) for any \( p > 2\gamma \).

The plan of the paper is as follows. In Section 2 we present our strategy for proving Theorem 2, which is the main result of the paper. Section 3 is devoted to the proof of Theorem 3, giving Schatten norm estimates on the spectral measure. In Section 4 we derive some Schatten norm estimates on the resolvent of the Laplacian, as a direct consequence of the Schatten norm estimates on the spectral measure and give their analogues at the endpoint case \( p = \frac{n}{2} \), needed in the proof of Theorem 2. The principal step in the proof of Theorem 2, corresponding to the estimates on the spectrum, is carried out in Section 5. Section 6 contains the proof of Theorem 4, which follows the arguments of [14] and [17] closely, relying on Theorem 1, with some small adjustments due to the fact that we are no longer in the Euclidean setting. Finally, we observe in Section 7 that Theorem 5 and Theorem 6 are direct consequences of Theorem 2 combined with the arguments of [16, Theorem 16] and [14, Theorem 1.2]. Appendix A contains the proof of Lemma 5.5, needed in the main text. Appendix B is concerned with the analysis of the microlocal structure of the spectrally localized outgoing and incoming resolvent, used in the proof of Theorem 2.

2. Strategy of the proof of Theorem 2

2.1. Schatten norm estimates. The basic mechanism for proving Schatten norm estimates of Theorem 2 and Theorem 3 comes from the fact that the Schatten spaces are complex interpolation spaces, see [39, Theorem 2.9], [40, p. 154], and from the following result of Frank and Sabin [16, Proposition 1].

Proposition 2.1. Let \( T_s \) be an analytic family of operators in the sense of Stein, defined on the strip \( \{ s \in \mathbb{C} \mid -\lambda_0 \leq \text{Re}s \leq 0 \} \) for some \( \lambda_0 > 1 \), acting on functions on \( M \). Assume that we have operator norm bounds

\[
\|T_{ir}\|_{L^2(M) \to L^2(M)} \leq M_0 e^{a|r|} \quad \|T_{-\lambda_0 + ir}\|_{L^1(M) \to L^\infty(M)} \leq M_1 e^{a|r|} \quad \forall r \in \mathbb{R}
\]
for some $a \geq 0$ and $M_0, M_1 > 0$. Then for any $W_1, W_2 \in L^{2\lambda_0}(M)$, the operator $W_1 T_{-1} W_2$ belongs to the Schatten class $\mathcal{C}_{2\lambda_0}(L^2(M))$ and we have the estimate
\[
\|W_1 T_{-1} W_2\|_{\mathcal{C}_{2\lambda_0}} \leq M_0^{1-\frac{1}{2\lambda_0}} M_1^{\frac{1}{2\lambda_0}} \|W_1\|_{L^{2\lambda_0}(M)} \|W_2\|_{L^{2\lambda_0}(M)}.
\]

Let us recall briefly the proof of Proposition 2.1. Assuming for convenience that $W_1, W_2$ are non-negative and simple, the result is established by considering the analytic family of operators $S_s = W_1^{-s} T_s W_2^{-s}$. This family has the property that $S_{-1} = W_1 T_{-1} W_2$ and it satisfies the following estimates on the boundary of the strip. For $s = ir$, $r$ real, we have
\[
\|S_{ir}\|_{L^2(M) \to L^2(M)} \leq \|T_{ir}\|_{L^2(M) \to L^2(M)} \leq M_0 e^{a|r|},
\]
and for $s = -\lambda_0 + ir$, we note that $T_s$ has kernel bounded pointwise by $M_1 e^{a|r|}$ and $W_1^{-s}$, $W_2^{-s}$ are $L^2$ functions, hence $S_s$ is a Hilbert-Schmidt operator with the Hilbert-Schmidt norm bounded by $M_1 e^{a|r|} \|W_1\|_{L^{2\lambda_0}(M)} \|W_2\|_{L^{2\lambda_0}(M)}$. Interpolating between the operator norm and the Hilbert-Schmidt norm gives us a bound on the Schatten norms, in particular at $s = -1$, where we obtain the Schatten norm at exponent $2\lambda_0$.

2.2. Strategy. The principal idea of the proof of the Euclidean analog of Theorem 2, which is due to Frank and Sabin [16, Theorem 12], is to establish the following pointwise bound for the Schwartz kernel of the powers of the resolvent $(\Delta - z)^{-\alpha}$,
\[
|[(\Delta - z)^{-\alpha}(x, y)| \leq C e^{C (\text{Im}(\alpha))^2} |z|^{-\frac{n-\alpha-1}{2}} |x - y|^{\text{Re}(\alpha) - n+\frac{1}{2}}, \quad x, y \in \mathbb{R}^n. \tag{2.1}
\]
Here $z \in \mathbb{C} \setminus [0, \infty)$, $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) \in \left[\frac{n-1}{2}, \frac{n+1}{2}\right]$. The desired Schatten bound (1.7) in the Euclidean case is therefore a consequence of (2.1) combined with the Hölder and Hardy–Littlewood–Sobolev inequalities as well as an interpolation argument.

Unfortunately, the natural analog of the pointwise bound (2.1) does not hold in general, for $z$ close to the spectrum of $\Delta_g$, for asymptotically conic manifolds, essentially because there can be conjugate points for the geodesic flow, and to prove the bound (1.7) we have to proceed differently.

Our strategy of the proof of Theorem 2 is to establish the Schatten norm estimate (1.7) for $W_1 (\Delta_g - z)^{-1} W_2$ for $z$ on the negative real axis, and for $z$ just above and below the spectrum, that is, for $W_1 (\Delta_g - (z \pm i0))^{-1} W_2$, for $z > 0$. We then use the Phragmén-Lindelöf theorem to obtain the result on the whole of the complex plane, excluding the origin.

Let us give the proof of Theorem 2, assuming that it has been established for $z < 0$ and for $z \pm i0$, $z > 0$. Let $W_1, W_2 \in L^{2p}(M)$ with $p \in \left[\frac{n}{2}, \frac{n+1}{2}\right]$, and let us consider the following bilinear form for $z \in \mathbb{C} \setminus [0, \infty)$,
\[
B_z(W_1, W_2) := W_1 (\Delta_g - z)^{-1} W_2. \tag{2.2}
\]
When \( z \in (0, \infty) \), we extend the definition of \( B_z \) by taking the outgoing resolvent \((\Delta_g - (z + i0))^{-1}\) in (2.2). Thus, we know that for \( z \in \mathbb{R} \setminus \{0\} \), \( B_z \) is a bounded bilinear form

\[
B_z : L^{2p}(M) \times L^{2p}(M) \to C_q(L^2(M)), \quad p \in \left[ \frac{n}{2}, \frac{n+1}{2} \right], \quad q = \frac{p(n-1)}{n-p},
\]

such that

\[
\|B_z(W_1, W_2)\|_{C_q} \leq C|z|^{-1+\frac{n}{2p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}. \tag{2.3}
\]

We now complete the proof of Theorem 2 by a Phragmén-Lindelöf argument. In doing so, let \( W_1, W_2 \in C^{0}_{0}(M) \). We claim that the function \( H(z) := B_z(W_1, W_2) \) is holomorphic in \( \text{Im} z > 0 \) with values in \( C_q(L^2(M)) \) such that

\[
\|H(z)\|_{C_q} \leq C(|z|^{-1/2} + |z|^{1/2}).
\]

Indeed, for \( \text{Im} z > 0 \), the operator \( W_1(\Delta_g - z)^{-1}W_2 : L^2(M) \to H^2(M) \cap \mathcal{E}'(K) \) is bounded where \( K \) is a compact set containing the support of \( W_1 \). Furthermore, it depends holomorphically on \( z \) with \( \text{Im} z > 0 \), and satisfies the bound

\[
\|W_1(\Delta_g - z)^{-1}W_2\|_{L(L^2(M), H^2(M))} \leq C(|z|^{-1/2} + |z|^{1/2}), \quad \text{Im} z \geq 0, \quad z \neq 0,
\]

see [34] for intermediate values of \( z \), [42] for \( |z| \to \infty \) and [37, Prop. 1.26] for \( |z| \to 0 \). Now the embedding \( H^2(M) \cap \mathcal{E}'(K) \to L^2(M) \) is an operator in \( C^{n/2+\varepsilon} \) for all \( \varepsilon > 0 \) in view of the Weyl law for the Laplacian on a compact manifold. Since \( q > n/2 \), we deduce the claim.

The function \( H(z) \) is continuous for \( \text{Im} z \geq 0, z \neq 0 \), with valued in \( C_q(L^2(M)) \) and to avoid the problem at \( z = 0 \), we consider the map

\[
F(z) := \langle H(e^z), T \rangle e^{(1-\frac{q}{p})z}
\]

for a fixed \( T \in C_q(L^2(M)) \) with norm \( \|T\|_{C_q} = 1 \). Here \( \frac{1}{q} + \frac{1}{p} = 1 \) and the product is the duality pairing between the Banach space \( C_q \) and its dual \( C_{q'} \). Then \( F(z) \) is holomorphic in \( \text{Im} z \in (0, \pi) \), continuous on the closure, and enjoys the bounds

\[
|F(z)| \leq Ce^{C|z|} \text{ for } 0 \leq \text{Im} z \leq \pi,
\]

\[
|F(z)| \leq C\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)} \text{ for } \text{Im} z \in \{0, \pi\}
\]

in view of (2.3). Applying the Phragmén-Lindelöf principle, we deduce that \( |F(z)| \leq C\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)} \) for all \( z \in \mathbb{C} \) such that \( 0 \leq \text{Im} z \leq \pi \), and therefore,

\[
\|H(z)\|_{C_q} \leq C|z|^{-1+\frac{n}{2p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}, \quad \text{Im} z \geq 0, \quad z \neq 0.
\]

By a density argument, we obtain the bound (1.7) for \( \text{Im} z \geq 0, z \neq 0 \). By considering the adjoint of the operator \( B_z \), we complete the proof of Theorem 2.

This argument reduces the problem to proving estimate (1.7) for \( z \in \mathbb{R} \setminus \{0\} \). We find it convenient to first prove the corresponding estimate for the spectral measure.
given in Theorem 3. The proof of Theorem 3 relies crucially on the $TT^*$ structure of the spectral measure.

When $z \in (-\infty, 0)$ and $p \in \left(\frac{n}{2}, \frac{n+1}{2}\right]$, the Schatten norm estimate (1.7) is a direct consequence of Theorem 3, and at the endpoint case $p = \frac{n}{2}$, the Schatten norm estimate (1.7) follows from the heat kernel estimates due to Grigor’yan [20] and Varopoulos [41].

Establishing the Schatten norm estimate (1.7) for $W_1(\Delta_g - (z \pm i0))^{-1}W_2$ with $z > 0$ represents the main difficulty in the proof of Theorem 2. When doing so, following [23], [22] and [28], we use a microlocal partition of the identity $\sum_{i=1}^{N} Q_i(\eta) = \text{Id}$, where $Q_i(\eta)$ are pseudodifferential operators depending on the energy parameter $0 < \eta \sim |z|^{1/2}$, constructed in [23]. Splitting up the operator $W_1(\Delta_g - (z \pm i0))^{-1}W_2$ by means of the partition of the identity, we are led to estimate the individual terms $W_1Q_i(\eta)^*(\Delta_g - (z \pm i0))^{-1}Q_j(\eta)W_2$, and here the most interesting contributions arise when $i = j$. When handling those, we proceed by establishing pointwise bounds for the Schwartz kernel of the operator

$$Q_i(\eta)^* \phi \left(\frac{\Delta_g}{z}\right)(\Delta_g - (z \pm i0))^{-s}Q_j(\eta), \quad \text{Re } s \in \left[\frac{n-1}{2}, \frac{n+1}{2}\right],$$

analogous to the Euclidean estimates (2.1). Here $\phi$ is a cut-off near 1.

3. SCHATTEN NORM ESTIMATES ON THE SPECTRAL MEASURE. PROOF OF THEOREM 3

Our starting point is the operator partition of unity, $\text{Id} = \sum_{i=1}^{N} Q_i(\eta)$, depending on $\eta > 0$, constructed in [23]. This partition of unity enjoys the following estimates, in particular: for all $k = 0, 1, 2, \ldots$, there is $C_k > 0$ such that for all $m, m' \in M$, we have

$$\left|\partial_{\lambda}^k (Q_i(\eta)^* dE_{\Delta_g}(\lambda)Q_i(\eta))(m, m')\right| \leq C_k \lambda^{n-1-k}(1 + \lambda d(m, m'))^{-(\frac{n-1}{2})+k},$$

$$\lambda \in [(1-\delta)\eta, (1+\delta)\eta], \quad (3.1)$$

with $\delta > 0$ sufficiently small but fixed and $d(\cdot, \cdot)$ being the Riemannian distance on $M$.

We say more about this partition of the identity in Section 5.1 below; here, we can use results of [23] and [4] as a ‘black box’. Then for all $\lambda \in [(1-\delta/2)\eta, (1+\delta/2)\eta]$, we use the partition of unity to decompose the spectral measure sandwiched between two $L^{2p}$ functions:

$$W_1 dE_{\Delta_g}(\lambda)W_2 = \sum_{i,j=1}^{N} W_1Q_i(\eta)^* dE_{\Delta_g}(\lambda)Q_j(\eta)W_2. \quad (3.2)$$

Let $p \in \left[1, \frac{n+1}{2}\right]$ and $q = \frac{p(n-1)}{n-p} \in \left[1, n+1\right]$. In the first step, we shall prove microlocalized estimates of the form

$$\|W_1Q_i(\eta)^* dE_{\Delta_g}(\lambda)Q_i(\eta)W_2\|_{L^q(M)} \leq C \lambda^{-1+\frac{2}{p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}, \quad (3.3)$$
for the diagonal \((i = j)\) terms of the decomposition (3.2). In doing so, we shall follow [16, Proof of Theorem 2] and start by showing (3.3) at the endpoints \(p = \frac{n+1}{2}\) and \(p = 1\), i.e.

\[
\|W_1 Q_i(\eta)^* dE \sqrt{\Delta_g}(\lambda) Q_i(\eta) W_2\|_{C_{n+1}} \leq C \lambda^{\frac{n+1}{2}} \|W_1\|_{L^{n+1}(M)} \|W_2\|_{L^{n+1}(M)},
\]

and

\[
\|W_1 Q_i(\eta)^* dE \sqrt{\Delta_g}(\lambda) Q_i(\eta) W_2\|_{C_1} \leq C \lambda^{n-1} \|W_1\|_{L^2(M)} \|W_2\|_{L^2(M)},
\]

respectively. Once the estimates (3.4) and (3.5) have been established, the bound (3.3) follows by a complex interpolation argument applied to the analytic family of operators \(\zeta \mapsto W_1^{\frac{\delta}{n+\delta}} \zeta \frac{1}{n+\delta} Q_i(\eta)^* dE \sqrt{\Delta_g}(\lambda) Q_i(\eta) W_2^{\frac{\delta}{n+\delta}} \zeta \frac{1}{n+\delta}\) in the strip \(0 \leq \text{Re} \zeta \leq 1\), with \(W_j \geq 0\) being simple functions such that \(\|W_j\|_{L^2(M)} = 1\), \(j = 1, 2\), see [39, Theorem 2.9].

Now to prove the estimate (3.4), we shall consider the following family of operators,

\[
T_s := Q_i(\eta)^* \phi \left( \frac{\sqrt{\Delta_g}}{\lambda} \right) \chi_+^s(\lambda - \sqrt{\Delta_g}) Q_i(\eta), \quad -\frac{(n+1)}{2} \leq \text{Re} s \leq 0,
\]

introduced in [23, Definition 3.2] and [4]. Here \(\phi \in C_0^\infty((1 - 4/1, 1 + 4/1))\) is such that \(\phi(t) = 1\) in a neighborhood of \(t = 1\), and \(\chi_+^s\) is the family of distributions on \(\mathbb{R}\), entire analytic in \(s \in \mathbb{C}\) and such that

\[
\chi_+^s(\lambda) = \frac{\lambda^s}{\Gamma(s+1)}, \quad \text{Re} s > -1,
\]

where \(\lambda_+ = \max(\lambda, 0)\), see [29, Section 3.2]. Note that at least formally, we have

\[
\chi_+^0(\lambda - \sqrt{\Delta_g}) = E \sqrt{\Delta_g}(\lambda), \quad \chi_+^k(\lambda - \sqrt{\Delta_g}) = \left( \frac{d}{d\lambda} \right)^{k-1} \lambda^s dE \sqrt{\Delta_g}(\lambda), \quad k = 1, 2, \ldots
\]

Recall from [23, Definition 3.2] that \(T_s\) is the operator whose Schwartz kernel is given by

\[
(Q_i(\eta)^* \phi \left( \frac{\sqrt{\Delta_g}}{\lambda} \right) \chi_+^s(\lambda - \sqrt{\Delta_g}) Q_i(\eta))(m, m')
\]

\[
= \int \chi_+^{k+s}(\lambda - \mu) \partial_\mu^k \left( Q_i(\eta)^* \phi \left( \frac{\mu}{\lambda} \right) \right) dE \sqrt{\Delta_g}(\mu) Q_i(\eta) \right)(m, m') d\mu,
\]

where \(k \in \mathbb{N}\) is such that \(\text{Re} s + k > -1\). As \(\mu \in [\eta(1 - \delta), \eta(1 + \delta)]\) for \(\lambda \in [(1 - \delta)(1 - \delta), (1 + \delta)(1 + \delta)]\) and \(\mu/\lambda \in \text{supp}(\phi)\), thanks to the estimates (3.1) the integral in (3.6) is well defined.

As explained in [23], the family of operators \(T_s\) is analytic in the sense of Stein in the strip \(-\frac{(n+1)}{2} \leq \text{Re} s \leq 0\). When \(\text{Re} s = 0\), we have

\[
\|T_s\|_{L^2(M) \rightarrow L^2(M)} \leq C e^{\frac{\pi}{4}},
\]
and relying on the estimates (3.1) it was shown in [23] and [4] that when \( \text{Re } s = -\frac{(n+1)}{2} \), we have
\[
\|T_s\|_{L^1(M) \to L^\infty(M)} \leq C(1 + |r|)e^{\frac{\pi |r|}{2}} \lambda^{\frac{n+1}{2}}, \quad s = -\frac{(n + 1)}{2} + ir, \ r \in \mathbb{R}.
\]
Applying Proposition 2.1, we get, for any two complex valued functions \( W_1, W_2 \in L^{n+1}(M) \),
\[
W_1 T_{-1} W_2 = W_1 Q_i(\eta)^* \phi \left( \frac{\sqrt{\Delta_g}}{\lambda} \right) \chi_+^{-1}(\lambda - \sqrt{\Delta_g}) Q_i(\eta) W_2
\]
\[
= W_1 Q_i(\eta)^* dE \sqrt{\Delta_g}(\lambda) Q_i(\eta) W_2
\]
is in the Schatten \( \mathcal{C}_{n+1} \) class and (3.4) holds.

To show (3.5), we recall from [23] that we have a pointwise kernel bound on the (microlocalized) spectral measure,
\[
\|Q_i(\eta)^* dE \sqrt{\Delta_g}(\lambda) Q_i(\eta)\|_{L^1(M) \to L^\infty(M)} \leq C \lambda^{n-1}.
\]
(3.7)

Also, we have
\[
dE \sqrt{\Delta_g}(\lambda) = (2\pi)^{-1} P(\lambda) P^*(\lambda),
\]
(3.8)
where \( P(\lambda) : L^2(\partial M) \to L^1(M), \ r \in \left[\frac{2(n+1)}{n-1}, \infty\right], \) is the Poisson operator, see [23]. Using the \( T^* T \) trick, it follows from (3.7) and (3.8) that
\[
\|Q_i(\eta)^* P(\lambda)\|_{L^2(\partial M) \to L^\infty(M)} \leq C \lambda^{\frac{n-1}{2}}.
\]
The Schwartz kernel \( Q_i(\eta)^* P(\lambda)(m, m') \) of the operator \( Q_i(\eta)^* P(\lambda) \) satisfies therefore,
\[
\|Q_i(\eta)^* P(\lambda)(m, \cdot)\|_{L^2(\partial M)} \leq C \lambda^{\frac{n-1}{2}}
\]
for almost all \( m \in M \). Thus, for any \( W_1 \in L^2(M) \), the operator \( W_1 Q_i(\eta)^* P(\lambda) : L^2(\partial M) \to L^2(M) \) is Hilbert-Schmidt with the norm bounded by \( C \lambda^{\frac{n-1}{2}} \|W_1\|_{L^2(M)} \).
Taking adjoints, we find that \( P(\lambda)^* Q_i(\eta) W_2 \) is a Hilbert-Schmidt operator with norm bounded by \( C \lambda^{\frac{n-1}{2}} \|W_2\|_{L^2(M)} \). Therefore, \( (2\pi)^{-1} \) times the composition of these two operators, which is precisely \( W_1 Q_i(\eta)^* dE \sqrt{\Delta_g}(\lambda) Q_i(\eta) W_2 \), is of trace class and (3.5) follows.

In the second step, we shall bound the Schatten norm of the off-diagonal \( (i \neq j) \) terms in the decomposition (3.2), i.e. we shall prove the following estimate,
\[
\|W_1 Q_i(\eta)^* dE \sqrt{\Delta_g}(\lambda) Q_j(\eta) W_2\|_{\mathcal{C}_q} \leq C \lambda^{-1+\frac{q}{2}} \|W_1\|_{L^{2q}(M)} \|W_2\|_{L^{2q}(M)}.
\]
(3.9)
As above, we shall exploit the \( T^* T \) structure of the spectral measure.

Let \( T : L^2(M) \to L^2(\partial M) \) be a compact operator and \( q \geq 1 \). Then \( T^* T \in \mathcal{C}_q(L^2(M)) \) if and only if \( T \in \mathcal{C}_{2q}(L^2(M), L^2(\partial M)) \), and moreover, \( \|T^* T\|_{\mathcal{C}_q} = \|T\|_{\mathcal{C}_{2q}}^2 \).
This is a consequence of the following equality for the singular values,
\[
\mu_k(T^* T) = \mu_k(T)^2.
\]
(3.10)
Moreover, if $T_1, T_2$ are in $C_q(L^2(M), L^2(\partial M))$, then $T_1^*T_2$ is in $C_q(L^2(M))$, and we have

$$
\|T_1^*T_2\|^q_{C_q} \leq \|T_1^*T_1\|^q_{C_q} + \|T_2^*T_2\|^q_{C_q}.
$$

(3.11)

This follows from the Ky Fan inequality for singular values of compact operators $A$ and $B$,

$$
\mu_{m+n-1}(AB) \leq \mu_m(A)\mu_n(B), \quad n, m \geq 1,
$$

see [19, Chapter 2, Section 3], and the fact that $\mu_k(T_1^*) = \mu_k(T_1)$, which combine to give

$$
\mu_{2k}(T_1^*T_2) \leq \mu_{2k-1}(T_1^*T_2) \leq \mu_k(T_1)\mu_k(T_2).
$$

(3.12)

Hence, using (3.10) and (3.12), we get

$$
\sum_{k=1}^{\infty} \mu_k(T_1^*T_2)^q = \sum_{k=1}^{\infty} \mu_{2k}(T_1^*T_2)^q + \sum_{k=1}^{\infty} \mu_{2k-1}(T_1^*T_2)^q
\leq \sum_{k=1}^{\infty} 2(\mu_k(T_1)\mu_k(T_2))^q
\leq \sum_{k=1}^{\infty} \left(\mu_k(T_1)^{2q} + \mu_k(T_2)^{2q}\right)
= \sum_{k=1}^{\infty} \mu_k(T_1^*T_1)^q + \sum_{k=1}^{\infty} \mu_k(T_2^*T_2)^q,
$$

which proves (3.11).

Using (3.8), we write

$$
W_1Q_1(\eta)^*dE\sqrt{\Delta_0}Q_j(\eta)W_2 = (2\pi)^{-1}T_1^*T_2,
$$

(3.13)

where $T_1 = P(\lambda)*Q_1(\eta)W_1$, and $T_2 = P(\lambda)*Q_j(\eta)W_2$. Now it follows from (3.3) that $T_1^*T_1 \in C_q(L^2(M))$, $T_2^*T_2 \in C_q(L^2(M))$, and we have

$$
\|T_1^*T_1\|_{C_q} \leq C\lambda^{-1+\frac{3}{2p}}\|W_1\|_{L^{2p}(M)}^2, \quad \|T_2^*T_2\|_{C_q} \leq C\lambda^{-1+\frac{3}{2p}}\|W_2\|_{L^{2p}(M)}^2.
$$

By the discussion above, this is equivalent to the fact that $T_1 \in C_{2q}(L^2(M), L^2(\partial M))$ and $T_2 \in C_{2q}(L^2(M), L^2(\partial M))$. It follows from (3.13) and discussion above that $W_1Q_1(\eta)^*dE\sqrt{\Delta_0}Q_j(\eta)W_2 \in C_q(L^2(M))$, and using (3.11), we get that

$$
\|W_1Q_1(\eta)^*dE\sqrt{\Delta_0}Q_j(\eta)W_2\|_{C_q} \leq C\lambda^{-1+\frac{3}{2p}}(\|W_1\|_{L^{2p}(M)}^2 + \|W_2\|_{L^{2p}(M)}^2).
$$

Thus, (3.9) follows by bilinearity in $W_1, W_2$. This completes the proof of Theorem 3.
4. Consequences of the spectral measure estimates for \( p \in \left( \frac{n}{2}, \frac{n+1}{2} \right) \) and their analogues at the endpoint \( p = \frac{n}{2} \)

4.1. Consequences of the spectral measure Schatten norm estimate. Using Theorem 3 and Minkowski’s integral inequality, we can deduce some Schatten estimates on the resolvent. In this subsection, we only treat the case \( p > \frac{n}{2} \).

The first result applies for \( z \) in any sector excluding the positive real axis.

**Proposition 4.1.** Let \( p \in \left( \frac{n}{2}, \frac{n+1}{2} \right) \), and suppose \( W_1, W_2 \in L^{2p}(M) \). Let \( \epsilon > 0 \) be arbitrary. Then for \( z \in \mathbb{C} \) such that \( z \neq 0, \arg z \in [\epsilon, 2\pi - \epsilon] \), the sandwiched resolvent \( W_1(\Delta_g - z)^{-1}W_2 \) is in the Schatten class \( \mathcal{C}_q(L^2(M)) \) with \( q = \frac{p(n-1)}{n-p} \in (n-1, n+1) \), and we have

\[
\|W_1(\Delta_g - z)^{-1}W_2\|_{\mathcal{C}_q} \leq C|z|^{-1 + \frac{\epsilon}{2p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)},
\]

where \( C \) depends on \( p, \epsilon \) and \((M, g)\), but not \( z \).

**Proof.** We express the operator \( W_1(\Delta_g - z)^{-1}W_2 \) as

\[
W_1(\Delta_g - z)^{-1}W_2 = \int_0^\infty (\lambda^2 - z)^{-1}W_1dE\sqrt{\Delta_g}(\lambda)W_2d\lambda.
\]

The result follows by estimating the Schatten norm of \( W_1dE\sqrt{\Delta_g}(\lambda)W_2 \) using Theorem 3 and noting that provided \( p > \frac{n}{2} \), we have

\[
\int_0^\infty |\lambda^2 - z|^{-1}\lambda^{-1 + \frac{\epsilon}{2p}} d\lambda \leq C|z|^{-1 + \frac{\epsilon}{2p}},
\]

where \( C \) depends on \( p \) and \( \epsilon \) but does not depend on \( z \) in the given sector. \( \square \)

In a similar manner we obtain ‘elliptic’ estimates on the resolvent, where we remove the singularity in the spectral multiplier. In this way we can obtain estimates on the positive real axis. To state these, we fix a function \( \phi : [0, \infty) \to [0, 1] \) such that \( \phi(t) = 1 \) for \( t \) in a neighbourhood of \( t = 1 \), and has support in a slightly bigger neighborhood of \( t = 1 \).

**Proposition 4.2.** Let \( p \in \left( \frac{n}{2}, \frac{n+1}{2} \right) \), and suppose \( W_1, W_2 \in L^{2p}(M) \). Then for \( z \in \mathbb{C} \setminus \{0\} \), the operator \( W_1(1 - \phi)(\frac{\Delta_g}{|z|})(\Delta_g - z)^{-1}W_2 \) is in the Schatten class \( \mathcal{C}_q(L^2(M)) \) with \( q = \frac{p(n-1)}{n-p} \in (n-1, n+1) \), and we have

\[
\left\|W_1\left(1 - \phi\right)\left(\frac{\Delta_g}{|z|}\right)(\Delta_g - z)^{-1}W_2\right\|_{\mathcal{C}_q} \leq C|z|^{-1 + \frac{\epsilon}{2p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)},
\]

where \( C \) depends on \( p \) and on \((M, g)\), but not \( z \).
Proof. Again we express the operator using an integral over the spectral measure, and estimate the Schatten norm of the spectral measure using Proposition 3 and Minkowski’s integral inequality. This time we obtain the integral
\[
\int_0^\infty |\lambda^2 - z|^{-1} \left( 1 - \phi \left( \frac{\lambda^2}{|z|} \right) \right) \lambda^{-1 + \frac{n}{p}} d\lambda
\]
and it is straightforward to check that this is bounded by \(C|z|^{-1 + \frac{n}{p}}\) uniformly in \(z\). □

4.2. Analogues at the endpoint \(p = \frac{n}{2}\). In the case \(p = \frac{n}{2}\), the arguments used in the proofs of Propositions 4.1 and 4.2 are no longer valid and need to be replaced. In view of the Phragmén–Lindelöf argument, explained in Section 2.2, we only need to do this for \(z\) negative in the case of Proposition 4.1 and \(z\) positive in the case of Proposition 4.2. To this end we prove the following two results.

**Proposition 4.3.** Let \(p = \frac{n}{2}\). There is \(C > 0\) such that for all \(z < 0\) and for all \(W_1, W_2 \in L^n(M)\), the operator \(W_1(\Delta_g - z)^{-1}W_2 \in C_{n-1}(L^2(M))\) and we have
\[
\|W_1(\Delta_g - z)^{-1}W_2\|_{C_{n-1}} \leq C\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)} .
\]

**Proof.** Here we use a slight variation of Proposition 2.1. Let \(W_1, W_2\) be non-negative simple functions and consider the analytic family of operators
\[
S_s = W_1^{-s}(\Delta_g - z)^{-1}W_2^{-s}, \quad -\frac{(n-1)}{2} \leq \text{Re } s \leq 0.
\]
Clearly, when \(\text{Re } s = 0\), we have
\[
\|S_s\|_{L^2(M) \to L^2(M)} \leq C.
\]
Next, we will show that, when \(\text{Re } s = -\frac{(n-1)}{2}\), then \(S_s\) is Hilbert-Schmidt and we have
\[
\|S_s\|_{C_2} \leq Ce^{C|\text{Im } s|}\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)}^{\frac{n-1}{2}} .
\]
This allows us to run the interpolation argument in the proof of Proposition 2.1.

To prove (4.3), on the line \(\text{Re } s = -\frac{(n-1)}{2}\), we express \((\Delta_g - z)^s\) in terms of the heat kernel:
\[
\Gamma(-s)(\Delta_g - z)^s(m, m') = \int_0^\infty t^{-s-1} e^{tz} e^{-t\Delta_g}(m, m')dt.
\]
We now use heat kernel estimates. Due to Varopoulos [41], we have the estimate \(\|e^{-t\Delta_g}\|_{L^1 \to L^\infty} \leq Ct^{-\frac{n}{2}}\) and by a result of Grigor’yan [20], this implies a pointwise upper Gaussian estimate on the heat kernel
\[
|e^{-t\Delta_g}(m, m')| \leq Ct^{-\frac{n}{2}} e^{-\frac{cd(m, m')^2}{t}}, \quad t > 0,
\]
where \(c\) is a constant depending on the geometry of \(M\).
for some $c > 0$. The integral in (4.4) is convergent for all $m \neq m'$ due to (4.5). We thus get for all $m \neq m'$ and $z \in (-\infty, 0)$, and uniformly for all $s$ such that $\Re s = -\frac{(n-1)}{2}$,

$$|\Gamma(-s)(\Delta_g - z)^s(m, m')| \leq C \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{cd(m, m')^2}{t} z t} dt$$

$$\leq Cd(m, m')^{-1} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{c}{t} + \frac{d(m, m')^2}{t} z} dt \quad (4.6)$$

$$\leq Cd(m, m')^{-1}.$$

Using Hölder inequality, the generalized Hardy-Littlewood-Sobolev inequality of [18] and (4.6), we obtain for $\Re s = -\frac{(n-1)}{2}$,

$$\|W_1^{-s}(\Delta_g - z)^sW_2^{-s}\|_{L^2(M)}^2$$

$$\leq C|\Gamma(-s)|^{-1} \int_{M \times M} W_1(m)^{n-1}d(m, m')^{-2}W_2(m')^{n-1}dV_g(m)dV_g(m')$$

$$\leq C|\Gamma(-s)|^{-1}W_1^{n-1}\|_{L^{\frac{n}{n-1}}(M)}\|W_2^{n-1}\|_{L^{\frac{n}{n-1}}(M)} \leq C e^{C|\Im s|}\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)}^{-1}$$

where the factor $e^{C|\Im s|}$ is contributed by the Gamma function. This shows (4.3).

We now interpolate using the family $S_s$ between (4.2) and (4.3), as in the proof of Proposition 2.1, and we obtain at $s = -1$

$$\|W_1(\Delta_g - z)^{-1}W_2\|_{C^{n-1}} \leq C\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)}.$$

(4.7)

which completes the proof for $W_1$ and $W_2$ non-negative and simple. The extension to general $W_1, W_2 \in L^n(M)$ is standard. \qed

We now prove an analogue of Proposition 4.2.

**Proposition 4.4.** Let $p = \frac{n}{2}$ and suppose $W_1, W_2 \in L^n(M)$, and let $\phi$ be as in Proposition 4.2. Then for $z > 0$, the operator $W_1(1 - \phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g - z)^{-1}W_2$ is in the Schatten class $\mathcal{C}_{n-1}(L^2(M))$ and

$$\left\|W_1\left(1 - \phi\right)\left(\frac{\Delta_g}{z}\right)(\Delta_g - z)^{-1}W_2\right\|_{\mathcal{C}_{n-1}} \leq C\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)},$$

uniformly in $z$.

**Proof.** We first note that for $z > 0$, the operator

$$W_1\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1}W_2$$

is in the Schatten class $\mathcal{C}_{n-1}(L^2(M))$, and

$$\left\|W_1\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1}W_2\right\|_{\mathcal{C}_{n-1}} \leq C\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)},$$
uniformly in $z$. This follows from the spectral measure estimate (1.8), since
\[ \int_0^\infty \lambda \phi \left( \frac{\lambda^2}{z} \right) \left( \lambda^2 + z \right)^{-1} d\lambda \]
is bounded uniformly in $z$. Combining this with the result of Proposition 4.3, we see that
\[ W_1 \left( 1 - \phi \right) \left( \frac{\Delta_g}{z} \right) (\Delta_g + z)^{-1} W_2 \text{ is in } C_{n-1}(L^2(M)) \]
and we have
\[ \left\| W_1 \left( 1 - \phi \right) \left( \frac{\Delta_g}{z} \right) (\Delta_g + z)^{-1} W_2 \right\|_{C_{n-1}} \leq C \| W_1 \|_{L^n(M)} \| W_2 \|_{L^n(M)}, \tag{4.8} \]
uniformly in $z$.

Now we write
\[ W_1 \left( 1 - \phi \right) \left( \frac{\Delta_g}{z} \right) (\Delta_g - z)^{-1} W_2 = W_1 \left( 1 - \phi \right) \left( \frac{\Delta_g}{z} \right) (\Delta_g + z)^{-1} W_2 + 2z W_1 \left( 1 - \phi \right) \left( \frac{\Delta_g}{z} \right) (\Delta_g + z)^{-1} (\Delta_g - z)^{-1} W_2. \tag{4.9} \]

The first term in the right hand side of (4.9) has already been shown to lie in $C_{n-1}$ with the bound (4.8). We write the second term on the right hand side of (4.9) in terms of the spectral measure and apply Minkowski’s integral inequality together with the spectral measure estimate (1.8), and find that the norm in $C_{n-1}$ is bounded by
\[ C \left( z \int_0^\infty \left( 1 - \phi \right) \left( \frac{\lambda^2}{z} \right) (\lambda^2 + z)^{-1} (\lambda^2 - z)^{-1} \lambda d\lambda \right) \| W_1 \|_{L^n(M)} \| W_2 \|_{L^n(M)} \]
and a change of variable shows that this integral is convergent and independent of $z$, completing the proof. \( \square \)

5. Resolvent estimates on the spectrum. Completion of the proof of Theorem 2

The key difficulty in proving Theorem 2 is to obtain estimates on the limiting resolvent at the spectrum, $(\Delta_g - (z + i0))^{-1}$, for $z > 0$. Given Proposition 4.2 and Proposition 4.4, we only need to do this localized near the singularity at $z$ of the spectral multiplier $(\lambda^2 - z)^{-1}$. In doing so, following [23], [22] and [28], we shall use a microlocal partition of unity.

5.1. Operator partition of unity. We begin by recalling some results of [22] and [28] on high and low frequency microlocal estimates on the spectral measure and resolvents of $\Delta_g$.

Proposition 5.1. High frequency microlocal estimates. For all high energies $\eta \geq 1/2$, there exists a family of bounded operators $Q_i(\eta) : L^2(M) \to L^2(M)$, $i = 1, \ldots, N_h$, with $N_h$ independent of $\eta$ and with the norm satisfying
\[ \| Q_i(\eta) \|_{L^2(M) \to L^2(M)} \leq C \text{ for some } C \text{ independent of } \eta, \tag{5.1} \]
so that the following properties hold:

1. The operators $Q_i(\eta)$ form an operator partition of unity:

$$\sum_{i=1}^{N_h} Q_i(\eta) = \text{Id.} \quad (5.2)$$

2. Let $\eta \geq 1/2$ and $(i, j) \in \{1, \ldots, N_h\}^2$. There exists $\delta > 0$ small such that for all $z > 0$ such that $\sqrt{z} \in [(1 - \delta)\eta, (1 + \delta)\eta]$, one of the following three alternatives holds:

   (2.i) One has

$$\left( Q_i(\eta)^* (\Delta_g - (z + i0))^{-1} Q_j(\eta) \right)(m, m') \in x(m)^\infty x(m')^\infty z^{-\infty} C^\infty(M \times \overline{M}), \quad (5.3)$$

   for all $m, m' \in M$, where the $C^\infty(M \times \overline{M})$ part depends also on $z$ and is uniformly bounded in $z$ in the smooth topology.

   (2.ii) One has

$$\left( Q_i(\eta)^* (\Delta_g - (z - i0))^{-1} Q_j(\eta) \right)(m, m') \in x(m)^\infty x(m')^\infty z^{-\infty} C^\infty(M \times \overline{M}), \quad (5.4)$$

   for all $m, m' \in M$.

   (2.iii) The spectral measure satisfies, for $\lambda = \sqrt{z} \in [(1 - \delta)\eta, (1 + \delta)\eta]$, the following bounds: for all $k = 0, 1, 2, \ldots$, there is $C_k > 0$ such that for all $m, m' \in M$

$$\left| \partial^k_x (Q_i(\eta)^* dE(\lambda)Q_j(\eta))(m, m') \right| \leq C_k \lambda^{n-1-k} (1 + \lambda d(m, m'))^{-\frac{(n-1)}{2} + k}, \quad (5.5)$$

$$\left( Q_i(\eta)^* dE(\lambda)Q_j(\eta) \right)(m, m') = \lambda^{n-1} \left( \sum_{\pm} e^{\pm i\lambda d(m, m')} a_\pm(\lambda, m, m') + b(\lambda, m, m') \right), \quad (5.6)$$

with $a_\pm, b$ satisfying the estimates for all $k = 0, 1, 2, \ldots$

$$\left| \partial^k_x a_\pm(\lambda, m, m') \right| \leq C_k \lambda^{-k} (1 + \lambda d(m, m'))^{-\frac{(n-1)}{2}}, \quad (5.7)$$

$$\left| \partial^k_x b(\lambda, m, m') \right| \leq C_k \lambda^{-k} (1 + \lambda d(m, m'))^{-K}, \quad \forall K > 1. \quad (5.8)$$

Moreover the alternative (2.iii) always holds if $i = j$.

**Low frequency microlocal estimates.** Similarly, for all low energies $\eta \leq 2$, there exists a family of bounded operators $Q_i(\eta) : L^2(M) \rightarrow L^2(M)$, $i = 0, *, 1, \ldots, N_i$, with $N_i$ independent of $\eta$ satisfying (5.1) and (5.2) (with the sum in this case ranging over $i = 0, *, 1, \ldots, N_i$), satisfying the following:

3. Let $0 < \eta \leq 2$ and $i, j$ range independently in $\{0, *, 1, \ldots, N_i\}$. There exists $\delta > 0$ small such that for all $z > 0$ satisfying $\lambda := \sqrt{z} \in [(1 - \delta)\eta, (1 + \delta)\eta]$, one of the following three alternatives holds:

   (3.i) One has the pointwise kernel bound

$$\left| (Q_i(\eta)^* (\Delta_g - (z + i0))^{-1} Q_j(\eta))(m, m') \right| \leq C \frac{\mu^{\frac{n-1}{2}} (\mu^{\frac{n}{2}} + \mu^{\frac{n}{2}})}{x + x' + \lambda}, \quad (5.9)$$

where $\mu := \sqrt{z} \in [(1 - \delta)\eta, (1 + \delta)\eta]$. The constant $C$ depends only on $\eta$ and $\mu$. The range of the parameter $\lambda := \sqrt{z} \in [(1 - \delta)\eta, (1 + \delta)\eta]$ is included for $\mu := \sqrt{z} \in [(1 - \delta)\eta, (1 + \delta)\eta]$.

Note: The notation and mathematical expressions are complex and involve advanced concepts from differential geometry and spectral theory. The document appears to be discussing properties of operators and their kernels in the context of microlocal analysis, which is a branch of mathematical analysis that focuses on the fine-scale behavior of solutions to differential equations.
where \( x = x(m), x' = x(m') \), and \( \chi \in C_0^\infty((-\varepsilon, \varepsilon), [0, \infty)) \) is such that \( \chi = 1 \) in \([-\varepsilon/2, \varepsilon/2]\). Here \( \varepsilon > 0 \) is small enough.

(3.ii) One has the pointwise kernel bound

\[
| (Q_i(\eta)^*(\Delta_g - (z - i0))^{-1}Q_j(\eta))(m, m') | \leq C \frac{(xx')^{n-1}(\chi(\frac{x}{\lambda}) + \chi(\frac{x'}{\lambda}))}{x + x' + \lambda}.
\]  

(5.10)

(3.iii) For all \( k = 0, 1, 2, \ldots \), there is \( C_k > 0 \) such that \((5.5), (5.6), \) and \((5.8)\) hold.

Moreover if \( i = j \), the alternative (3.iii) holds.

Remark 5.2. The two partitions of the identity do not quite match up in the intermediate energy regime, \( 1/2 \leq \eta \leq 2 \). Because of this, it would be more notationally accurate to label the partitions \( Q_{i}^{\text{high}} \) and \( Q_{j}^{\text{low}} \); to avoid cumbersome notation, we do not do this. We emphasize that in this intermediate regime, either partition can be used.

Remark 5.3. In the low energy case, \( \eta \leq 2 \), let us first point out the meaning of the RHS of \((5.9)\) and \((5.10)\). In \([24]\) it was shown that the resolvent kernel has some Legendrian and polyhomogeneous structure on the low energy space. In the low energy regime, there are 7 boundary hypersurfaces that play a role: \( zf, lb_0, rb_0, lb, rb \) and \( bf \) — see figure 1 of \([24]\).

The resolvent was shown in particular to be polyhomogeneous and vanish to order \( n - 2 \) at the boundary hypersurfaces labelled \( lb_0, rb_0, bf_0 \), and order \( (n - 1)/2 \) at \( lb \) and \( rb \). Cases (3.i) and (3.ii) will apply when there is no wavefront set at \( bf \), meaning there is infinite order vanishing there. Moreover, the cutoff functions vanish in a neighbourhood of \( zf \). On the other hand, \( x \) vanishes to first order at \( lb \), \( lb_0 \) and \( bf_0 \), while \( x' \) vanishes to first order at \( rb \), \( rb_0 \) and \( bf_0 \) and \( x + x' + \lambda \) vanishes to first order at \( bf_0 \), so the product on the RHS of \((5.9)\) and \((5.10)\) precisely encodes the order of vanishing at these boundary hypersurfaces.

Proof. This is a combination of several results from \([23]\) and \([22]\). In the high energy case, \( \eta \geq 1/2 \), Lemma 5.3 of \([22]\) tells us that the pairs \((i, j)\) split into four cases. In the first two cases, \( Q_i(\eta)^* \) is either not-incoming or not-outgoing related to \( Q_j(\eta) \), and then Proposition 6.7 of \([22]\) applies; note that the estimates in (2.i) and (2.ii) above appear in the proof, rather than the statement, of Proposition 6.7. In the third and fourth cases, Theorem 1.12 of \([23]\) applies and shows that estimates \((5.5)\) hold, see also Proposition 6.4 of \([22]\). Also in the third and fourth cases, Proposition 1.5 of \([28]\) holds and gives the estimates \((5.6), (5.7)\) and \((5.8)\). Note that \([28, \text{Proposition 1.5}]\) is written in the case when \( i = j \) but the proof of that proposition shows that it remains valid more generally when \( i \neq j \) but the microsupports are close enough.
In the low energy case, as shown in Section 6 of [22], case (3.iii) applies to the pairs (0, 0), (*, *), and (i, j) where i, j ≥ 1 and |i − j| ≤ 1. Moreover, case (3.iii) also applies to any pair where either i = * or j = *. That is because in these cases, the operator \( Q_\ast(\eta) \) annihilates all the wavefront set of the spectral measure at bf, with the consequence that the spectral measure estimates

\[
\left| \partial^k \left( (Q_i(\eta))^* dE \delta_{\Lambda_y} \right) (\lambda Q_j(\eta))(m, m') \right| \leq C_k \lambda^{-1-k} (1 + \lambda d(m, m'))^{-\frac{\alpha - 1}{2} + k}, \tag{5.11}
\]

hold if either i = * or j = *, and this leads to estimates (5.5) as in the high energy case. For (3.iii) with i, j ≥ 1, the estimates (5.6), (5.7) and (5.8) are proven in [28, Proposition 1.5] in the case when i = j but the proof shows that it remains valid more generally when i ≠ j but the microsupports are close enough. The case i, j ∈ \{0, *\} in (3.iii) is also shown in [28, Proposition 1.5].

The cases i = 0 and j ≥ 1, or i ≥ 1 and j = 0, fit any one of the cases (3.i), (3.ii), (3.iii) above. This is because here the wavefront set at bf is wiped out by \( Q_0(\eta) \), while the wavefront set at fibre-infinity is wiped out by \( Q_j(\eta) \) for j ≥ 1.

The final case remaining, where i, j ≥ 1 and |i − j| ≥ 2, fit into cases (3.i) or (3.ii) according to whether \( Q_i(\eta)^* \) is not incoming-related or not outgoing-related to \( Q_j(\eta) \), as shown in Proposition 6.9 of [22]. □

Cases (3.i) and (3.ii) will be treated using the following lemma.

**Lemma 5.4.** Let \((M, g)\) be an asymptotically conic manifold of dimension \( n \geq 3 \). Then if an integral operator \( K \) has kernel \( K(m, m') \) bounded pointwise by

\[
C \left( \frac{(x(x'))^{\frac{n}{2}}(\chi(\frac{x}{\Lambda}) + \chi(\frac{x'}{\Lambda}))}{x + x' + \lambda} \right), \quad 0 < \lambda \leq 3,
\]

then for \( W_1, W_2 \in L^{2p}(M), p \in \left[\frac{n}{2}, \frac{n + 1}{2}\right] \), the operator \( W_1 KW_2 \) is Hilbert Schmidt and we have

\[
\|W_1 KW_2\|_{C_2} \leq C \lambda^{-2 + \frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{5.12}
\]

**Proof.** Using Hölder’s inequality with 1/p' + 1/p = 1 and \( p' \in \left[\frac{n+1}{n-1}, \frac{n}{n-2}\right] \), we get

\[
\|W_1 KW_2\|_{C_2} \leq \|W_1\|_{L^{2p}} \|W_2\|_{L^{2p}} \left( \int_{M \times M} \frac{(x(m)x(m'))^{(n-1)p'}}{(x(m) + x(m') + \lambda)^{2p'}} dV_g(m) dV_g(m') \right)^{1/2p'}.
\]

We use the coordinates \( m = (x, y), m' = (x', y') \) near the boundary, where the measure \( dV_g(m) \) is comparable to \( \frac{dxdy}{x^{n-1}} \). Let us introduce the polar coordinates \((x, x') = (R \sin(\theta), R \cos(\theta))\) with \( \theta \in [0, \pi/2] \), near \( x = x' = 0 \). Using that \((n-1)p' - (n+1) \geq 0\)
and \( x + x' \sim R \), we get
\[
\left( \int_{\mathcal{M} \times M} \frac{(xx')(n-1)p'}{(x+x'+\lambda)^{2p'}} dV_g dV_g' \right)^{\frac{1}{2p'}} \leq C \left( \int_{0 < x < 2\lambda} \frac{(xx')(n-1)p'-(n+1)}{(x+x'+\lambda)^{2p'}} dxx' \right)^{\frac{1}{2p'}}
\]
\[
\leq C \left( \int_{0}^{\infty} \int_{0 < \sin \theta < 2\lambda/R} \frac{R^{2(n-1)p'-2n-1}}{(R + \lambda)^{2p'}} dR \right)^{\frac{1}{2p'}}
\]
\[
\leq C \lambda^{\frac{2n}{p}-2} + C \lambda^{\frac{2n}{p}} \left( \int_{2\lambda}^{\infty} R^{2(n-2)p'-2n-2} dR \right)^{\frac{1}{2p'}} \leq C \lambda^{\frac{2n}{p}}.
\]

Here we used that \((n - 1)p' > n\) and \(2(n - 2)p' - 2n - 1 < 0\). The same argument works with the term involving \(\chi(x'/\lambda)\) and the estimate (5.12) follows. \(\square\)

5.2. Analytic family of operators. In this section we closely follow Section 4 of [22], especially Remark 4.2 (which is substantially due to Adam Sikora). Let \(\phi \in C_0^\infty((1-\delta/4)^2, (1+\delta/4)^2)\) be such that \(\phi(t) = 1\) in a neighborhood of \(t = 1\), where \(\delta > 0\) is small, and consider the analytic family of operators in Re \((s) \leq 0)\,

\[
H_{s,z,\varepsilon}(\Delta_g) = \phi \left( \frac{\Delta_g}{z} \right) (\Delta_g - (z + i\varepsilon))^s, \quad z > 0, \quad \varepsilon > 0.
\]

By spectral theorem, we have
\[
H_{s,z,\varepsilon}(\Delta_g) = z^{s+\frac{1}{2}} \int_{0}^{\infty} \left( \lambda - \left( 1 + i\frac{\varepsilon}{z} \right) \right)^s \frac{\phi(\lambda)}{2\sqrt{\lambda}} dE \sqrt{\Delta_g}(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}) d\lambda.
\]

(5.13)

Let \(\eta > 0\) be such that \(z^{1/2} \in [(1-\delta/2)\eta, (1+\delta/2)\eta]\) and let \(Q_i(\eta)\) and \(Q_j(\eta)\) be such that the condition (2.iii) or (3.iii) of Proposition 5.1 holds, in the high energy, respectively, low energy case. Then using (5.13), we have on the level of Schwartz kernels, for \(m, m' \in M\),
\[
(Q_i(\eta)^* H_{s,z,\varepsilon}(\Delta_g) Q_j(\eta))(m, m') = z^{s+\frac{1}{2}} \int_{0}^{\infty} \left( \lambda - \left( 1 + i\frac{\varepsilon}{z} \right) \right)^s \psi(\lambda) d\lambda,
\]

(5.14)

where
\[
\psi(\lambda) = \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_g}(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}) Q_j(\eta)(m, m').
\]

Here, as \(\delta > 0\) is small, we have \(z^{1/2}\lambda^{1/2} \in [(1-\delta)\eta, (1+\delta)\eta]\) when \(z^{1/2} \in [(1-\delta/2)\eta, (1+\delta/2)\eta]\) and \(\lambda \in \text{supp}(\phi)\), and therefore, in view of (5.5), we have \(\psi(\lambda) \in C_0^\infty(\mathbb{R})\).
Letting $\varepsilon \to 0$ in (5.14), we define the operators \( Q_i(\eta) H_{s,z,0}(\Delta_0) Q_j(\eta) \), when \( z^{1/2} \in [(1-\delta/2)\eta, (1+\delta/2)\eta] \), as operators whose Schwartz kernels are given by

\[
(Q_i(\eta) H_{s,z,0}(\Delta_0) Q_j(\eta))(m, m') = z^{s+\frac{1}{2}} \int_0^\infty (\lambda - (1+i0))^s \psi(\lambda) d\lambda = z^{s+\frac{1}{2}} (\lambda - i0)^s \psi(\lambda)(1). \tag{5.15}
\]

We are interested in pointwise estimates for the kernel of \( Q_i(\eta) H_{s,z,0}(\Delta_0) Q_j(\eta) \) and to this end, we shall need the following result of Lemma 5.5. Let

\[
\text{Lemma 5.5. Let } a < b < c \leq 0 \text{ and let us write } b = \theta a + (1 - \theta)c, \ 0 < \theta < 1. \text{ Then there is } C > 0 \text{ such that for all } f \in C^\infty_0(\mathbb{R}), \text{ all } t \in \mathbb{R}, \text{ and all } 0 < \varepsilon \ll 1, \text{ we have}
\]

\[
\| (\lambda \pm i\varepsilon)^{b+it} * f \|_{L^\infty} \leq C(1 + |t|) e^{\frac{3|\varepsilon| t}{2}} \| \chi^a_+ * f \|_{L^\infty} \| \chi^c_+ * f \|_{L^\infty}. \tag{5.16}
\]

We have the following result.

**Proposition 5.6.** Suppose that \((i, j)\) are such that the condition (2.iii) or (3.iii) holds, in the high energy, respectively, low energy case. Then there is \( C > 0 \) such that the kernel of the operator \( Q_i(\eta) H_{s,z,0}(\Delta_0) Q_j(\eta) \) with \( z > 0 \) and \( z^\frac{1}{2} \in [(1-\delta/2)\eta, (1+\delta/2)\eta] \) has the following pointwise estimates,

(i) For \( \text{Re}(s) = -\frac{(n+1)}{2} \), we have

\[
|Q_i(\eta) H_{s,z,0}(\Delta_0) Q_j(\eta)(m, m')| \leq C |\text{Re}(s)|^\frac{1}{2} \tag{5.17}
\]

for all \( m, m' \in M \), uniformly in \( z \) and \( \eta \).

(ii) For \( \text{Re}(s) = -\frac{(n-1)}{2} \), we have

\[
|Q_i(\eta) H_{s,z,0}(\Delta_0) Q_j(\eta)(m, m')| \leq C |\text{Re}(s)| d(m, m')^{-1}. \tag{5.18}
\]

for all \( m, m' \in M \), uniformly in \( z \) and \( \eta \).

**Proof.** Estimate (5.17) is proved in [22, Remark 4.2]. Estimate (5.18) is proved in the same way, except for the case \( n = 3 \), relying on the estimates (5.5) only. Indeed, in the case \( n \geq 5 \) is odd, we take \( a = -\frac{(n+1)}{2} \) and \( c = -\frac{(n-3)}{2} \) in Lemma 5.5 and using that

\[
\chi^k_+ = \delta^{(k-1)}_0, \quad k = 1, 2, \ldots,
\]
we get
\[
\left| Q_i(\eta)^* H_{s, z, 0}(\Delta_y) Q_j(\eta)(m, m') \right| \leq C z^{2-\frac{n}{2}} (1 + |\text{Im}(s)|)^{\frac{n}{2}} e^{\frac{2\pi |\text{Im}(s)|}{z^2}}
\times \left\| \partial_{\lambda}^{\frac{n-1}{2}} \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_y}(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}) Q_j(\eta)(m, m') \right) \right\|_{L^\infty}^{1/2}
\times \left\| \partial_{\lambda}^{\frac{n-1}{2}} \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_y}(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}) Q_j(\eta)(m, m') \right) \right\|_{L^\infty}^{1/2},
\]
and therefore, using (5.5), we obtain that
\[
\left| Q_i(\eta)^* H_{s, z, 0}(\Delta_y) Q_j(\eta)(m, m') \right| \leq C e^{C |\text{Im}(s)|} z^{\frac{1}{2}} (1 + z^{\frac{1}{2}} d(m, m'))^{-1}
\leq C e^{C |\text{Im}(s)|} d(m, m')^{-1}.
\tag{5.19}
\]

For \( n \geq 4 \) even, taking \( a = -\frac{n}{2} \), \( c = -\frac{(n-2)}{2} \) in Lemma 5.5 and using (5.5), we also get (5.19). We have therefore established (5.18) for all \( n \geq 4 \).

When \( n = 3 \), using Lemma 5.5 with \( a = -2 \) and \( c = 0 \), and the fact that \( \chi_0(\lambda) = H(\lambda) \) is the Heaviside function, we obtain that
\[
\left| Q_i(\eta)^* H_{s, z, 0}(\Delta_y) Q_j(\eta)(m, m') \right| \leq C z^{\frac{1}{2}} (1 + |\text{Im}(s)|)^{\frac{3}{2}} e^{\frac{3\pi |\text{Im}(s)|}{z^2}}
\times \left\| \partial_{\lambda} \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_y}(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}) Q_j(\eta)(m, m') \right) \right\|_{L^\infty}^{1/2}
\times \left\| H * \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_y}(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}) Q_j(\eta)(m, m') \right) \right\|_{L^\infty}^{1/2}.
\tag{5.20}
\]

By (5.5), we get
\[
\left\| \partial_{\lambda} \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_y}(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}) Q_j(\eta)(m, m') \right) \right\|_{L^\infty} \leq C z.
\tag{5.21}
\]

Now if we show that
\[
\left\| H * \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_y}(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}) Q_j(\eta)(m, m') \right) \right\|_{L^\infty} \leq C d(m, m')^{-2}.
\tag{5.22}
\]
then the estimate (5.18) will follow from (5.20), (5.21) and (5.22). To prove (5.22), using (5.6), we write

\[ H \ast \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE \sqrt{\Delta_\eta} (z^{\frac{1}{2}} \lambda^{\frac{1}{2}}) Q_j(\eta)(m, m') \right) (\lambda) \]

\[ = \int_0^{\lambda^{\frac{1}{2}}} \phi(\mu^2)Q_i(\eta)^* dE \sqrt{\Delta_\eta} (z^{\frac{1}{2}} \mu) Q_j(\eta)(m, m') d\mu \]

\[ = \int_0^{\lambda^{\frac{1}{2}}} \phi(\mu^2)z\mu^2 \left[ \sum_{\pm} e^{\pm iz^{\frac{1}{2}} \mu d(m, m')} a_{\pm}(z^{\frac{1}{2}} \mu, m, m') + b(z^{\frac{1}{2}} \mu, m, m') \right] d\mu. \]

The terms involving \( a_{\pm} \) in (5.23) can be treated similarly and in what follows we shall only consider the term involving \( a_{+} \) and drop the sign +. To estimate this term, we integrate by parts and get

\[ \int_0^{\lambda^{\frac{1}{2}}} \phi(\mu^2)z\mu^2 e^{iz^{\frac{1}{2}} \mu d(m, m')} a(z^{\frac{1}{2}} \mu, m, m') d\mu \]

\[ = \frac{1}{iz^{\frac{1}{2}} d(m, m')} \left[ \phi(\mu^2)z\mu^2 e^{iz^{\frac{1}{2}} \mu d(m, m')} a(z^{\frac{1}{2}} \mu, m, m') \big|_{\mu=0}^{\mu=\lambda^{\frac{1}{2}}} \right. \]

\[ - \int_0^{\lambda^{\frac{1}{2}}} \partial_\mu (\phi(\mu^2)z\mu^2 a(z^{\frac{1}{2}} \mu, m, m')) e^{iz^{\frac{1}{2}} \mu d(m, m')} d\mu \].

Estimating the terms in the left hand side of (5.24) with the help of (5.7), we obtain that

\[ \left| \int_0^{\lambda^{\frac{1}{2}}} \phi(\mu^2)z\mu^2 e^{iz^{\frac{1}{2}} \mu d(m, m')} a(z^{\frac{1}{2}} \mu, m, m') d\mu \right| \leq C\lambda^{\frac{1}{2}} d(m, m')^{-2}, \]

uniformly in \( z \). To estimate the term involving the remainder \( b \) in (5.23), we use (5.8) with \( K = 2 \) and get

\[ \int_0^{\lambda^{\frac{1}{2}}} \phi(\mu^2)z\mu^2 |b(z^{\frac{1}{2}} \mu, m, m')| d\mu \leq C \int_0^{\lambda^{\frac{1}{2}}} \phi(\mu^2)z\mu^2 (1 + z^{\frac{1}{2}} \mu d(m, m'))^{-2} d\mu \]

\[ \leq C d(m, m')^{-2}. \]

Now (5.22) follows from (5.23), (5.25) and (5.26). This completes the proof of estimate (5.18).

When proving the Schatten bound on the resolvent on the spectrum in Section 5.3 below, the cases (2.iii) and (3.iii) of Proposition 5.1 will be treated using the following result.

**Proposition 5.7.** Suppose that \((i, j)\) are such that the condition (2.iii) or (3.iii) holds, in the high energy, respectively low energy case. Let \( p \in \left[ \frac{n}{2}, \frac{n+1}{2} \right] \). Then there is \( C > 0 \)
such that for all \( z \in (0, \infty), \) \( z^{-\frac{1}{2}} \in [(1 - \delta/2)\eta, (1 + \delta/2)\eta], \) and all \( W_1, W_2 \in L^2^p(M), \) we have \( W_1 Q_1(\eta)^* H_{-1,0}(\Delta_g)Q_j(\eta)W_2 \in C_q(L^2(M)), \) \( q = \frac{p(n-1)}{n-p}, \) and

\[
\| W_1 Q_1(\eta)^* H_{-1,0}(\Delta_g)Q_j(\eta)W_2 \|_{C_q} \leq C z^{-1+\frac{1}{p}} \|W_1\|_{L^2^p(M)} \|W_2\|_{L^2^p(M)}. \tag{5.27}
\]

**Proof.** First thanks to Proposition 5.6, case (i), we know that for Re \( s = -\frac{(n+1)}{2}, \)

\[
\| Q_1(\eta)^* H_{s,0}(\Delta_g)Q_j(\eta) \|_{L^1(M) \to L^\infty(M)} \leq CE^{C|\text{Im}(s)|} z^{-\frac{1}{2}},
\]

By spectral theorem, we also know that for Re \( s = 0, \)

\[
\| Q_1(\eta)^* H_{s,0}(\Delta_g)Q_j(\eta) \|_{L^2(M) \to L^2(M)} \leq CE^{C|\text{Im}(s)|}.
\]

Hence, Proposition 2.1 implies that \( W_1 Q_1(\eta)^* H_{-1,0}(\Delta_g)Q_j(\eta)W_2 \in C_{n+1}(L^2(M)) \) and moreover,

\[
\| W_1 Q_1(\eta)^* H_{-1,0}(\Delta_g)Q_j(\eta)W_2 \|_{C_{n+1}} \leq C z^{-\frac{1}{n+1}} \|W_1\|_{L^{n+1}(M)} \|W_2\|_{L^{n+1}(M)}. \tag{5.28}
\]

Now when Re \( s = -\frac{(n-1)}{2}, \) thanks to Proposition 5.6 (ii), the kernel of the operator \( Q_1(\eta)^* H_{s,0}(\Delta_g)Q_j(\eta) \) has the bound (5.18), which is the same as the bound (4.6) in the proof of Proposition 4.3. Proceeding exactly as in the proof of Proposition 4.3, we get

\[
\| W_1 Q_1(\eta)^* H_{-1,0}(\Delta_g)Q_j(\eta)W_2 \|_{C_{n-1}} \leq C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}. \tag{5.29}
\]

In view of (5.28) and (5.29), the bound (5.27) follows by a complex interpolation argument applied to the analytic family of operators

\[
\zeta \mapsto W_1^{-\frac{1}{n+1}} W_2^{-\frac{1}{n+1}} Q_1(\eta)^* H_{-1,0}(\Delta_g)Q_j(\eta)W_2 \]

in the strip \( 0 \leq \text{Re} \zeta \leq 1, \) with \( W_j \geq 0 \) being simple functions such that \( \|W_j\|_{L^2(M)} = 1, \) \( j = 1, 2, \) see [40, p. 154]. \( \square \)

### 5.3. Resolvent estimates on the spectrum

The final ingredient in the proof of Theorem 2 is the following result.

**Proposition 5.8.** Let \( \phi \in C^\infty_c(((1 - \delta/4)^2, (1 + \delta/4)^2)) \) be such that \( \phi(t) = 1 \) in a neighborhood of \( t = 1, \) where \( \delta > 0 \) is small, and let \( p \in \left[ \frac{n}{2}, \frac{n+1}{2} \right]. \) Then there is \( C > 0 \) such that for all \( z \in (0, \infty) \) and all \( W_1, W_2 \in L^2^p(M), \) then for \( q = \frac{p(n-1)}{n-p} \) we have \( W_1 \phi(\Delta_g)(\Delta_g - (z + i0))^{-1}W_2 \in C_q(L^2(M)) \) and

\[
\left\| W_1 \phi \left( \frac{\Delta_g}{z} \right) (\Delta_g - (z + i0))^{-1}W_2 \right\|_{C_q} \leq C z^{-1+\frac{1}{p}} \|W_1\|_{L^2^p(M)} \|W_2\|_{L^2^p(M)}. \tag{5.30}
\]
Proof. Let us first take the high energy case \( z \geq 1 \) and let \( \eta \geq 1 \) be such that \( \sqrt{z} \in [(1 - \delta/2)\eta, (1 + \delta/2)\eta] \). We decompose the spectrally localized outgoing resolvent \( \phi(\frac{\Delta_g}{z})(\Delta_g - (z + i0))^{-1} \) into microlocalized pieces

\[
W_1\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i0))^{-1}W_2 = \sum_{i,j=1}^{N_h} W_1Q_i(\eta)\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i0))^{-1}Q_j(\eta)W_2.
\]

The bound (5.30) will follow if we show that for all \((i, j)\), we have

\[
\left\|W_1Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i0))^{-1}Q_j(\eta)W_2\right\|_{C_q} \leq C z^{-1+\frac{q}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}.
\]

(5.31)

To that end, the pairs \((i, j)\) will be divided into three cases as in Proposition 5.1.

In the first case, (2.i), in view of (5.3) and Corollary B.5, we know that the Schwartz kernel of the operator \( Q_i(\eta)^*\phi(\frac{\Delta_g}{z})(\Delta_g - z - i0))^{-1}Q_j(\eta) \) is \( \mathcal{O}(z^{-N}) \) in \( L^{2p'}(M \times M) \) with \( 1/p' + 1/p = 1 \). Using this together with the fact that \( q \geq 2 \) and Hölder's inequality, we get

\[
\left\|W_1Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i0))^{-1}Q_j(\eta)W_2\right\|_{C_q} \leq \left\|W_1Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i0))^{-1}Q_j(\eta)W_2\right\|_{C_2} \leq \mathcal{O}(z^{-N})\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)},
\]

for any \( N \in \mathbb{N} \), showing (5.31).

In the second case, (2.ii), using Stone’s formula, we write

\[
W_1Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i0))^{-1}Q_j(\eta)W_2 = W_1Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z - i0))^{-1}Q_j(\eta)W_2
\]

(5.32)

\[
+ \frac{\pi i}{\lambda} W_1Q_i(\eta)^*dE\sqrt{\Delta_g}(\lambda)Q_j(\eta)W_2, \quad \lambda = \sqrt{z}.
\]

Then the estimate for the term involving the incoming resolvent in (5.32) follows exactly as in case (2.i). On the other hand, we have already proved the corresponding estimate (3.9) for the spectral measure, which leads to the estimate (5.31) in this case.

In the third case, (2.iii), we get

\[
W_1Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i0))^{-1}Q_j(\eta)W_2 = W_1Q_i(\eta)^*H_{-1,z,0}(\Delta_g)Q_j(\eta)W_2,
\]

(5.33)

where the operator \( Q_i(\eta)^*H_{-1,z,0}(\Delta_g)Q_j(\eta) \) is defined in (5.15). The required estimate for this term therefore is a consequence of Proposition 5.7.
In the low energy case, $0 < z \leq 1$, the argument is similar. In cases (3.i) and (3.ii) we use Corollary B.5 together with Lemma 5.4 and the bound (3.9) for the spectral measure to deduce the Schatten norm estimate. In case (3.iii), the argument is the same as for case (2.iii). This concludes the proof of the proposition.

6. Bounds on individual eigenvalues. Proof of Theorem 4

In this section we shall follow some of the arguments of [14] and [17], making some necessary changes due to the fact that we are no longer in the Euclidean setting.

Let us recall that $n = \dim(M) \geq 3$. We have the following result which is a generalization of [14, Lemma 4.2] to the case of the Laplace operator on asymptotically conic manifolds.

**Proposition 6.1.** Let $V \in L^p(M)$ with $\frac{n}{2} \leq p < \infty$. The operator $\sqrt{|V|}(\Delta_g + 1)^{-\frac{1}{2}}$ is compact on $L^2(M)$.

**Proof.** We follow [14, Lemma 4.2]. First we shall show that

$$\|W(\Delta_g + 1)^{-\frac{1}{2}}\|_{L^2(M)} \leq C\|W\|_{L^{2p}(M)}, \quad W \in L^{2p}(M). \tag{6.1}$$

Indeed, we have

$$(\Delta_g + 1)^{-\frac{1}{2}} : L^2(M) \to H^1(M), \tag{6.2}$$

is bounded, and therefore, by Sobolev’s embedding $H^1(M) \subset L^{\frac{2n}{n-2}}(M)$, which is valid on an asymptotically conic manifold of dimension $n \geq 3$, see [22, Proposition 2.1], we get

$$(\Delta_g + 1)^{-\frac{1}{2}} : L^2(M) \to L^{\frac{2n}{n-2}}(M) \tag{6.3}$$

is also bounded. Using Hölder’s inequality, the logarithmic convexity of $L^p$ norms, and (6.2), (6.3), we obtain that

$$\|W(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^2(M)} \leq \|W\|_{L^{2p}(M)} \|(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^{\frac{2p}{p-1}}(M)}$$

$$\leq \|W\|_{L^{2p}(M)} \|(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^2(M)} \|(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^{\frac{2np}{2np-n}}(M)}$$

$$\leq C\|W\|_{L^{2p}(M)}\|f\|_{L^2(M)},$$

showing (6.1).

Let $W_j \in C_0^\infty(M)$ be such that $W_j \to \sqrt{|V|}$ in $L^{2p}(M)$. By Rellich’s compactness theorem, the operator $W_j(\Delta_g + 1)^{-\frac{1}{2}}$ is compact on $L^2(M)$, and it follows from (6.1) that $W_j(\Delta_g + 1)^{-\frac{1}{2}} \to \sqrt{|V|}(\Delta_g + 1)^{-\frac{1}{2}}$ in $L(L^2(M), L^2(M))$. The proof is complete. 

Setting

$$\sqrt{V(x)} = \begin{cases} \frac{V(x)}{\sqrt{|V(x)|}}, & V(x) \neq 0, \\ 0, & V(x) = 0, \end{cases}$$
and combining Proposition 6.1 with [14, Lemma B.1], we get that the quadratic form
\[ \|(\Delta_g)^{1/2}u\|_{L^2(M)}^2 + (\sqrt{V}u, \sqrt{|V|}u)_{L^2(M)}, \]
equipped with the domain \( H^1(M) \), is closed and sectorial. Associated to the quadratic form is an \( m \)-sectorial operator with domain \( \subseteq H^1(M) \), which we shall denote by \( \Delta_g + V \). The spectrum of \( \Delta_g + V \) in \( \mathbb{C} \setminus [0, \infty) \) consists of isolated eigenvalues of finite algebraic multiplicity, see [14, Proposition B. 2].

Now interpolating between the estimate, valid for \( z \in \mathbb{C} \setminus [0, \infty) \),
\[ \|(\Delta_g - z)^{-1}\|_{L^2(M) \to L^2(M)} = \frac{1}{d(z)}, \]
and the uniform estimate (1.6), with \( p = \frac{2(n+1)}{n+3} \), we obtain the following result.

**Corollary 6.2.** Let \((M, g)\) be an asymptotically conic non-trapping manifold of dimension \( n \geq 3 \). Then for all \( p \in [\frac{2(n+1)}{n+3}, 2] \), there is a constant \( C > 0 \) such that for all \( z \in \mathbb{C} \setminus [0, \infty) \),
\[ \|(\Delta_g - z)^{-1}\|_{L^p(M) \to L^{p'}(M)} \leq C d(z)^{(n+1)\left(\frac{1}{p} - \frac{1}{2}\right) - 1}\left|z\right|^{\frac{1}{2} - \frac{1}{p}}. \tag{6.4} \]

We shall now proceed to prove Theorem 4. In doing so we shall follow [17, Theorem 3.2]. Let \( \lambda \in \mathbb{C} \) be an eigenvalue and \( \psi \in H^1(M) \) be the corresponding eigenfunction of \( \Delta_g + V \),
\[ (\Delta_g + V)\psi = \lambda \psi. \]

(i) Let \( 0 < \gamma \leq \frac{1}{2} \). Assume first that \( \lambda \in \mathbb{C} \setminus [0, \infty) \). Let us choose \( p > 1 \) such that
\[ \gamma + \frac{n}{2} = \frac{p}{2 - p}, \tag{6.5} \]
and notice that then \( \frac{2n}{n+2} < p \leq \frac{2(n+1)}{n+3} \) and \( \frac{2(n+1)}{n-1} \leq p' < \frac{2n}{n-2} \).

By Sobolev’s embedding, we have \( \psi \in L^\frac{2n}{n+2}(M) \), and thus, \( \psi \in L^r(M) \) for \( r \in [2, \frac{2n}{n-2}] \), by interpolation. In particular, \( \psi \in L^{p'}(M) \), and by Hölder’s inequality, we get
\[ \|V\psi\|_{L^p(M)} \leq \|V\|_{L^\frac{2n}{n+2}(M)} \|\psi\|_{L^{p'}(M)} = \|V\|_{L^{\gamma + \frac{n}{2}}(M)} \|\psi\|_{L^{p'}(M)}. \]

We have
\[ \psi = (\Delta_g - \lambda)^{-1}(\Delta_g - \lambda)\psi = -(\Delta_g - \lambda)^{-1}(V\psi). \]
Hence, using (1.6), we get
\[ \|\psi\|_{L^{p'}(M)} \leq \|(\Delta_g - \lambda)^{-1}\|_{L^p(M) \to L^{p'}(M)} \|V\psi\|_{L^p(\mathbb{R}^n)} \leq C |\lambda|^{\frac{n}{2}(\frac{2}{p} - 1) - 1} \|V\|_{L^{\gamma + \frac{n}{2}}(M)} \|\psi\|_{L^{p'}(M)}, \tag{6.6} \]
which implies (1.9) in view of
\[ \frac{n}{2} \left(\frac{2}{p} - 1\right) - 1 = -\gamma + \frac{n}{2}. \]
Assume now that $\lambda \in (0, \infty)$. Then for $\varepsilon > 0$, we set

$$
\psi_\varepsilon = (\Delta_g - \lambda - i\varepsilon)^{-1}(\Delta_g - \lambda)\psi = f_\varepsilon(\Delta_g)\psi,
$$

where

$$
f_\varepsilon(t) = \frac{t - \lambda}{t - \lambda - i\varepsilon}, \quad t \in \mathbb{R}.
$$

By the spectral theorem, we have

$$
\|\psi_\varepsilon - \psi\|^2_{L^2(M)} = \|f_\varepsilon(\Delta_g)\psi - \psi\|^2_{L^2(M)} = \int |f_\varepsilon(t) - 1|^2d(E_{\Delta_g}(t)\psi, \psi)_{L^2(M)},
$$

where $dE_{\Delta_g}(t)$ is the spectral measure of $\Delta_g$. Using the dominated convergence theorem together with the fact that $f_\varepsilon(t) \to 1$ as $\varepsilon \to 0$ for all $t \neq \lambda$, and that $E_\lambda = 0$ as $\lambda$ is not an eigenvalue of $\Delta_g$, we conclude that $\psi_\varepsilon \to \psi$ in $L^2(M)$.

On the other hand, we have

$$
\psi_\varepsilon = -(\Delta_g - \lambda - i\varepsilon)^{-1}(V\psi).
$$

Choosing $p > 1$ satisfying (6.5) and using (1.6), we obtain that

$$
\|\psi_\varepsilon\|_{L^p(M)} \leq C|\lambda|^\frac{n}{2}(|\frac{\varepsilon}{\lambda}|^{-1})\|V\|_{L^{\gamma+\frac{2}{n}}(M)}\|\psi\|_{L^p(M)},
$$

i.e. $\psi_\varepsilon$ is uniformly bounded in $L^p(M)$. Passing to a subsequence, we may assume that there exists $\tilde{\psi} \in L^p(M)$ such that $\psi_\varepsilon \to \tilde{\psi}$ in the weak * topology of $L^p(M)$. It follows that $\psi = \tilde{\psi} \in L^p(M)$. By the lower semi-continuity of the norm and (6.7), we get

$$
\|\psi\|_{L^p(M)} \leq \liminf_{\varepsilon \to 0} \|\psi_\varepsilon\|_{L^p(M)} \leq C|\lambda|^\frac{n}{2}(|\frac{\varepsilon}{\lambda}|^{-1})\|V\|_{L^{\gamma+\frac{2}{n}}(M)}\|\psi\|_{L^p(M)},
$$

which shows (1.9) when $\lambda \in (0, \infty)$.

(ii) Let $V \in L^{\frac{2}{n}}(M)$. Setting $p = \frac{2n}{n+2}$, and arguing as in the case (i) above, for $\lambda \in \mathbb{C} \setminus \{0\}$, we obtain that

$$
\|\psi\|_{L^p(M)} \leq C\|V\|_{L^{\frac{2}{n}}(M)}\|\psi\|_{L^p(M)}.
$$

The case $\lambda = 0$ is handled similarly using that

$$
\|(\Delta_g - i\varepsilon)^{-1}\|_{L^p(M) \to L^p(M)} \leq \mathcal{O}(1),
$$

in view of (1.6). The claim (ii) follows.

(iii) Let $\gamma > \frac{1}{2}$, and let $\lambda \in \mathbb{C} \setminus [0, \infty)$ be an eigenvalue of $\Delta_g + V$, and $\psi \in H^1(M)$ be the corresponding eigenfunction. Choosing $p > 1$ satisfying (6.5), we have $\frac{2(n+1)}{n+3} < p < 2$ and $2 < p' < \frac{2(n+1)}{n-1}$. Using that $\psi \in L^{p'}(M)$ and (6.4), similarly to above, we obtain that

$$
\|\psi\|_{L^{p'}(M)} \leq \|(\Delta_g - \lambda)^{-1}\|_{L^p(M) \to L^p(M)}\|V\psi\|_{L^p(M)}
$$

$$
\leq C\delta(\lambda)^{(n+1)(\frac{1}{2} - \frac{1}{p'})^{-1}}\|\frac{1}{\lambda - \frac{\varepsilon}{\lambda}}\|_{L^{\gamma+\frac{2}{n}}(M)}\|\psi\|_{L^{p'}(M)},
$$

where $\delta(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. The claim (iii) follows.
which implies (1.10) in view of the fact that \( \frac{1}{p} = \frac{1+\gamma+\frac{3}{2}}{2(\gamma+\frac{3}{2})} \). This completes the proof of Theorem 4.

7. Bounds on sums of eigenvalues for Schrödinger operators with complex potentials

7.1. Short range potentials. Proof of Theorem 5. Let \( V \in L^p(M), \frac{n}{2} \leq p \leq \frac{n+1}{2} \), and let \( q = \frac{p(n-1)}{n-p} \). Then Theorem 2 implies that for \( z \in \mathbb{C} \setminus [0, \infty) \), we have

\[
\sqrt{V}(\Delta_g - z)^{-1} \sqrt{|V|} \in C_q(L^2(M))
\]

and notice that

\[
\sqrt{V}(\Delta_g - z)^{-1} \sqrt{|V|} \in C_q(L^2(M)) \leq C|z|^{-1+\frac{3}{2p}}\|V\|_{L^p(M)}. \tag{7.1}
\]

We claim that the map

\[
\mathbb{C} \setminus [0, \infty) \ni z \mapsto \sqrt{V}(\Delta_g - z)^{-1} \sqrt{|V|} \tag{7.2}
\]

is holomorphic with values in \( C_q(L^2(M)) \). First let us check that (7.2) is holomorphic with values in \( L^2(M), L^2(M) \). Indeed, letting \( z_0 \in \mathbb{C} \setminus [0, \infty) \), we write

\[
\sqrt{V}(\Delta_g - z)^{-1} \sqrt{|V|} = \sqrt{V} \sum_{j=0}^{\infty} (z - z_0)^j (\Delta_g - z_0)^{-j-1} \sqrt{|V|} \tag{7.3}
\]

and notice that

\[
\|\sqrt{V}(\Delta_g - z_0)^{-j-1} \sqrt{|V|}\|_{L^2(M), L^2(M)} \leq \|\sqrt{V}(\Delta_g - z_0)^{-1} \sqrt{|V|}\|_{L^2(M), L^2(M)}
\]

\[
\|(-\Delta - z_0)^{-1} \sqrt{|V|}\|_{L^2(M), L^2(M)} \|((\Delta_g - z_0)^{-1} \sqrt{|V|})^j \|_{L^2(M), L^2(M)} \leq C^{j+1},
\]

for some \( C > 0 \). Here we have used that the operators \( \sqrt{V}(-\Delta - z_0)^{-1}, (\Delta_g - z_0)^{-1} \sqrt{|V|} \)

are bounded on \( L^2(M) \) as seen by arguing as in the proof of (6.1). This shows that the series (7.3) converges in \( L^2(M), L^2(M) \) for \( |z - z_0| \) small, and therefore, the map (7.2) is holomorphic with values in \( L^2(M), L^2(M) \). In particular, if \( T \in C(B^2(M)) \), i.e. of trace class, the map

\[
\mathbb{C} \setminus [0, \infty) \ni z \mapsto \langle \sqrt{V}(\Delta_g - z)^{-1} \sqrt{|V|}, T \rangle \tag{7.4}
\]

is holomorphic. Using the density of \( C_1(L^2(M)) \) in \( C_q(L^2(M)) \), the bound (7.1), and Hölder’s inequality in Schatten classes, we conclude that the map (7.4) is holomorphic for all \( T \in C_q(L^2(M)) \), establishing the claim.

Consider the holomorphic function

\[
h(z) := \det_{[q]}(1 + \sqrt{V}(\Delta_g - z)^{-1} \sqrt{|V|}), \quad z \in \mathbb{C} \setminus [0, \infty),
\]

where \([q]\) is the smallest integer \( \geq q \), and \( \det_{[q]} \) is the regularized determinant, see [39, Chapter 9]. As explained in [16, proof of Theorem 16], using (7.1), we get

\[
\log |h(z)| \leq C\|\sqrt{V}(\Delta_g - z)^{-1} \sqrt{|V|}\|^q_{C_q} \leq C|z|^{-(1+\frac{3}{2p})q}\|V\|^q_{L^p(M)}, \tag{7.5}
\]
Combining Proposition 6.1 and Lemma B.1 of [14], we conclude that the following version of the Birman–Schwinger principle holds: $z \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of $\Delta_g + V$ if and only if

$$\ker (1 + \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}) \neq \{0\}. \quad (7.6)$$

An application of Lemma 3.2 of [14] gives that (7.6) is equivalent to the fact that $h(z) = 0$ and that the order of vanishing of $h$ at $z$ agrees with the algebraic multiplicity of $z$ as an eigenvalue of $\Delta_g + V$.

At this point we are exactly in the same situation as in [16, Theorem 16]. Here we may remark that the proof of Theorem 16 in [16] is based on a result of Borichev, Golinskii and Kupin [2], concerning the distribution of zeros of a holomorphic function in the unit disc, growing rapidly at a boundary point. The proof of Theorem 5 is therefore complete.

7.2. Long range potentials. Proof of Theorem 6. First we have the following result: let $\gamma \geq 1/2$. Then there exists a constant $C > 0$ such that for all $W \in L^2(\gamma + \frac{\alpha}{2})(M)$ and all $z \in \mathbb{C} \setminus [0, \infty)$,

$$\|W(\Delta_g - z)^{-1}W\|_{L^2(\gamma + \frac{\alpha}{2})(M)} \leq C d(z)^{-1+\frac{1}{\gamma+\frac{\beta}{2}}} |z|^{-\frac{1}{\gamma+\frac{\beta}{2}}} \|W\|^2_{L^2(\gamma + \frac{\alpha}{2})(M)}. \quad (7.7)$$

Indeed this follows as in [14, Proposition 2.1] by interpolation between (1.7) with $p = \frac{\alpha + 1}{2}$ and the standard bound

$$\|W(\Delta_g - z)^{-1}W\|_{L^2(M) \to L^2(M)} \leq d(z)^{-1} \|W\|^2_{L^\infty(M)}.$$ 

Now an application of [14, Theorem 3.1] to the holomorphic family $K(z) = \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}$ completes the proof of Theorem 6 exactly in the same way as in [14, Theorem 1.2].

Appendix A. Proof of Lemma 5.5

We shall follow the proof of Lemma 3.3 in [23] closely. Let $a < b < c \leq 0$ and let $\alpha := a - c - 1 < -1$ and $\beta := b - c - 1 < -1$. We shall show the estimate (5.16) for $\|(\lambda - i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda}$, as the bound (5.16) for $\|(\lambda + i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda}$ can be proved similarly.

To that end, let $\chi_z^\varepsilon$ be the family of distributions on $\mathbb{R}$ holomorphic in $z \in \mathbb{C}$ given by

$$\chi_z^\varepsilon(\lambda) = \frac{\lambda^z}{\Gamma(z + 1)}, \quad \text{Re } z > -1,$$
where
\[ \lambda^z = \begin{cases} 0 & \text{if } \lambda > 0, \\ |\lambda|^z & \text{if } \lambda < 0. \end{cases} \]

We have \( \chi^z(-\lambda) = \chi^z_+(\lambda) \). Recall from [29, Section 3.2] that when \( \text{Re } z > -1 \), we have
\[ (\lambda - i0)^z = \lambda^z_+ + e^{-i\pi z} \lambda^z_- , \tag{A.1} \]
and from [29, Example 7.1.17] that for \( \varepsilon > 0 \) and \( z \in \mathbb{C} \), we have
\[ \mathcal{F}((\lambda - i\varepsilon)^{-z}) (\xi) = 2\pi e^{i\pi/2} e^{i\varepsilon} \chi^{-z-1} (\xi), \tag{A.2} \]
and
\[ \mathcal{F}(\chi^z_+) (\xi) = e^{-i(z+1)\pi/2} (\xi-i0)^{-z-1}. \tag{A.3} \]

Consider the family of operators \( A_t \) for \( t \in \mathbb{R} \) given by
\[ A_t : C_0^\infty(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}), \quad A_t f := \hat{\eta}_t \ast f, \tag{A.4} \]
where
\[ \hat{\eta}_t (\xi) = \frac{2\pi e^{i(-\beta-it)\pi/2-i\pi(c+1)} e^{i\varepsilon} \xi^{-\beta-1-it}}{\Gamma(-b-it)(\sigma + e^{-i(\alpha+1)\pi/2}(\xi - i0)^{-\alpha-1})}, \tag{A.5} \]
when \( c < 0 \), and
\[ \hat{\eta}_t (\xi) = \frac{2\pi e^{-i(b+1+it)\pi/2} e^{i\varepsilon} \xi^{-b-it}}{\Gamma(-b-it)(\sigma - e^{-i\pi a/2}(\xi - i0)^{-a})}, \tag{A.6} \]
when \( c = 0 \), and \( \sigma \in \mathbb{C}, |\sigma| = 1 \) and \( \sigma \notin \{ i e^{-i\pi/2}, -i e^{i\pi/2}, e^{i\pi/2} \} \). In view of (A.1), we see that \( \hat{\eta}_t \in \mathcal{S}'(\mathbb{R}) \).

We notice that for all \( t \in \mathbb{R} \), \( \hat{\eta}_t \in L^1_{\text{loc}}(\mathbb{R}) \). Furthermore, using that \( \left| \frac{1}{\Gamma(-b-it)} \right| \leq C e^{\pi|t|} \), we have, for \( |\xi| \geq 1 \),
\[ |\partial_\xi \hat{\eta}_t (\xi)| \leq C e^{\frac{3\pi/2}{2}} (1 + |t||\xi|^{-\beta+a-1}}, \tag{A.7} \]
and for \( |\xi| \leq 1 \), we get
\[ |\partial_\xi \hat{\eta}_t (\xi)| \leq C e^{\frac{3\pi/2}{2}} (1 + |t||\xi|^{-\beta-2}}, \tag{A.8} \]
and therefore,
\[ \partial_\xi \hat{\eta}_t \in L^p(\mathbb{R}) \cap L^1(\mathbb{R}, (\xi)^{\delta} d\xi) \text{ for some } p \in (1, 2), \delta > 0. \]

By Hausdorff–Young’s inequality, we see that \( u(\lambda) := \lambda \hat{\eta}_t (\lambda) \in L^{p'}(\mathbb{R}) \) with \( p' \in (2, \infty) \) being the dual exponent to \( p \). We also have
\[ |u(\lambda) - u(\lambda')| \leq (2\pi)^{-1} \int |e^{i\xi \lambda} - e^{i\xi \lambda'}||\hat{u}(\xi)| d\xi \leq C \int |\xi|^{\delta} |\lambda - \lambda'|^{\delta} |\hat{u}(\xi)| d\xi, \tag{A.9} \]
where
showing that $u = \lambda \eta_t \in C^\delta(\mathbb{R})$. Thus, by Hölder inequality, we get
\[
\int_\mathbb{R} |\eta_t(\lambda)| d\lambda \leq C \left( \int_{|\lambda|>1} |\lambda \eta_t|^\alpha d\lambda \right)^{\frac{1}{\alpha}} + ||\lambda \eta_t||_{C^\delta} \int_{|\lambda|<1} |\lambda|^{-1+\delta} d\lambda < \infty. \tag{A.10}
\]

It follows from (A.10) combined with Hausdorff–Young’s inequality, (A.7), (A.8) and (A.9) that
\[
||\eta_t||_{L^1(\mathbb{R})} \leq C(1 + |t|) e^{\frac{3|\xi|}{2}},
\]
and therefore, $A_t$ extends as a bounded operator on $L^\infty$ with norm
\[
||A_t||_{L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})} \leq C(1 + |t|) e^{\frac{3|\xi|}{2}},
\]
where the constant $C > 0$ is independent of $\varepsilon$ and $t$.

Next let $B$ be the operator
\[
B : C^\infty_0(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad Bf := (\sigma \chi^c_+ + \chi^a_+) * f
\]
which is also equal to
\[
B = \mathcal{F}^{-1} \mu \mathcal{F}
\]
with
\[
\mu(\xi) := \sigma e^{-i(c+1)/2}(\xi - i0)^c e^{-i(\alpha+1)/2}(\xi - i0)^\alpha - 1,
\]
in view of (A.3).

If $c < 0$ then $\mu \in L^1_{\text{loc}}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$. Using also the fact that the distribution $(\xi - i0)^z$ is of polynomial growth when $\text{Re} \ z > -1$, we have $\mu \tilde{f} \in L^1(\mathbb{R})$ for any $f \in C^\infty_0(\mathbb{R})$. Thus, the operator $B : C^\infty_0(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded.

Now if $c = 0$ then $Bf := \sigma H * f + \chi^a_+ * f$, where $H$ is the Heaviside function. The fact that the convolution with the Heaviside function maps $C^\infty_0$ functions into $L^\infty$ functions implies that the operator $B : C^\infty_0(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded also in the case $c = 0$.

Thus, the composition $A_t B : C^\infty_0(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded in all cases $c \leq 0$. We claim that
\[
A_t Bf = (\lambda - \varepsilon)^{b+it} * f, \quad f \in C^\infty_0(\mathbb{R}). \tag{A.13}
\]

Indeed, (A.13) follows from (A.4), (A.11), and the equality
\[
\hat{\eta}_t \mu = \mathcal{F}((\lambda - \varepsilon)^{b+it})
\]
obtained from (A.5), (A.6) (A.12), and (A.2). In the case $c = 0$, we also use that
\[
\xi^b - b-it(\xi - i0)^{-1} = \xi^{b-1-it}, \quad b < 0.
\]

We thus get for all $\varepsilon > 0$ and $t \in \mathbb{R}$
\[
\|((\lambda - \varepsilon)^{b+it} * f\|_{L^\infty} \leq C(1 + |t|) e^{\frac{3|\xi|}{2}} (\|\chi^c_+ * f\|_{L^\infty} + \|\chi^a * f\|_{L^\infty}). \tag{A.14}
\]
Now a scaling argument as in the proof of Lemma 3.3 of [23] finishes the proof. Indeed, letting \( f_\tau(\lambda) = f(\tau \lambda) \), we have
\[
\chi_+^n * f_\tau(\lambda) = \tau^{-z-1}((\chi_+^n * f)(\lambda)), \quad (\lambda - i\varepsilon)^z * f_\tau(\lambda) = \tau^{-z-1}((\lambda - i\tau \varepsilon)^z * f)(\tau \lambda) \quad (A.15)
\]
for all \( \tau > 0 \) and \( z \in \mathbb{C} \). It follows from (A.14) and (A.14) that for each \( \tau > 0 \)
\[
\tau^{-b ||(\lambda - i\tau \varepsilon)^b + it * f||_L^\infty \leq C(1 + |t|)e^{\frac{3\pi|t|}{2}} ||\chi_+^n * f||_L^\infty + \tau^{-a} ||\chi_+^n * f||_L^\infty \)
\]
and choosing \( \tau := ||\chi_+^n * f||_L^\infty^{1/(a-c)} ||\chi_+^n * f||_L^\infty^{-1/(a-c)} \), we obtain the desired estimate (5.16). The proof of Lemma 5.5 is complete.

**Appendix B. Microlocal structure of the spectrally localized resolvent**

In this appendix, we analyze the microlocal structure of the spectrally localized resolvent \( \phi(\frac{\Delta}{\mu})(\Delta_g - (z \pm i0))^{-1} \), where \( z > 0 \) and \( \phi \in C_0^\infty(((1 - \delta/4)^2, (1 + \delta/4)^2)) \) is such that \( \phi(t) = 1 \) for \( t \in ((1 - \delta/8)^2, (1 + \delta/8)^2) \), for \( \delta > 0 \) small. In doing so, we use the notation and results established in the works [23], [24], and [27].

**Proposition B.1.** Let \( \phi \) be as above. For all \( \mu > 0 \), the operator \( \phi(\frac{\Delta}{\mu}) \) is a pseudodifferential operator in the following senses:

(i) High energy case. For \( h = \mu^{-1} \leq 2 \), the operator \( \phi(h^2 \Delta_g) \) is a semiclassical scattering pseudodifferential operator with microsupport in \( \{ (z, \zeta) \mid |\zeta|_g \in ((1 - \delta/4)^2, (1 + \delta/4)^2) \} \) where \( \zeta \) is the semiclassically-rescaled cotangent variable, i.e. \( \zeta_i \) is the symbol of \( -ih\partial_{z_i} \).

(ii) Low energy case. For \( \mu \in (0, 2) \), the operator \( \phi(\frac{\Delta}{\mu}) \) is a pseudodifferential operator in the class \( \Psi_k^\infty(M, \Omega_k^{1/2}) + \mathcal{A}^\mathcal{E}(M_k^2, \Omega_k^{1/2}) \) where \( \mathcal{E} \) is an index family for the boundary hypersurfaces of \( M_k^2 \) satisfying \( \mathcal{E}_{\text{bdy}} = 0, \mathcal{E}_{\text{sf}} = n, \mathcal{E}_{\text{lb}} = \mathcal{E}_{\text{rb}} = n/2, \mathcal{E}_{\text{lb}} = \mathcal{E}_{\text{rb}} = \mathcal{E}_{\text{bdy}} = \infty \). That is, it is the sum of a pseudodifferential operator in the class defined in [24, Section 5] and a conormal function which is smooth across the diagonal, but has nontrivial behaviour at the boundary hypersurfaces \( \text{lb}_0 \) and \( \text{rb}_0 \).

**Proof.** (i) This follows by expressing the operator \( \phi(h^2 \Delta_g) \) using the Helffer-Sjöstrand formula for the self-adjoint functional calculus,
\[
\phi(h^2 \Delta_g) = \frac{1}{2\pi i} \int \partial \tilde{\phi}(z)(h^2 \Delta_g - z)^{-1} \bar{dz} \wedge dz,
\]
where \( \tilde{\phi} \) is an almost holomorphic extension of \( \phi \), see [10, Theorem 8.1]. In terms of the notation for the spaces of semiclassical scattering pseudodifferential operators used in [42], we have \( \phi(h^2 \Delta_g) \in \Psi_{sc,h}^{-\infty,0,0}(M) \).

(ii) The same argument applies to show that the operator \( \phi(\frac{\Delta}{\mu}) \) is pseudodifferential in a neighbourhood of the diagonal on the space \( M_k^{2,sc} \). We also need to understand
the behaviour of the kernel of this operator away from the diagonal. Here, we recall from [24] that the spectral measure is conormal away and vanishes to order \( n - 1 \) at \( \text{zf} \), order \( n/2 - 1 \) at \( \text{lb}_0 \) and \( \text{rb}_0 \) and order \(-1\) at \( \text{bf}_0 \) as a b-half-density on \( M^2_{k,b} \), while it is Legendrian (oscillatory) at \( \text{lb}, \text{rb} \) and \( \text{bf} \). As a result, the integral

\[
\phi\left(\frac{\Delta_g}{\mu^2}\right) = \int \phi\left(\frac{\lambda^2}{\mu^2}\right) d\overline{E}\sqrt{\Delta_g}(\lambda) d\lambda
\]  

(\text{B.1})

is conormal on \( M^2_{k,b} \) and vanishes to order \( n \) at \( \text{zf} \), order \( n/2 \) at \( \text{lb}_0 \) and \( \text{rb}_0 \), order 0 at \( \text{bf}_0 \) and to order \( \infty \) at \( \text{lb}, \text{rb} \) and \( \text{bf} \).

Remark B.2. The pseudodifferential nature of \( \phi(h^2\Delta_g) \) can also be proved via the spectral measure using the results of [24]. Recall from this article that the spectral measure \( d\overline{E}\sqrt{\Delta_g}(\lambda) \) for \( \lambda \geq 1 \) is a Legendre distribution associated to a pair of Legendre submanifolds \((L, L^\#)\), where \( L \) is the flowout by (left) bicharacteristic flow starting from \( N^*\text{Diag}_b \cap \Sigma ) \) where \( N^*\text{Diag}_b \) is the conormal bundle to the diagonal in \( M^2_b \). Here \( \Sigma \) denotes the ‘left’ characteristic variety of the operator \( h^2\Delta_g - 1 \), that is, the set \( \{(z, \zeta, z', \zeta') \mid |\zeta|_g = 1\} \) where the semiclassical symbol of \( h^2\Delta_g - 1 \), acting in the left variable \( z \), vanishes. Being a Legendre distribution, the spectral measure may be expressed (up to a trivial kernel, that is, one that is smooth and rapidly vanishing both as \( h \to 0 \) and as one approaches the boundary of \( M^2_{k,b} \)) as a finite sum of oscillatory integrals associated to neighbourhoods of the submanifold \( L \). The phase function for this oscillatory integral takes the form \( \lambda \Phi \), where \( \Phi \) is independent of \( \lambda \). If we then integrate in the \( \lambda \) variable as in (\text{B.1}) (with \( h = \mu^{-1} \) in the high-energy case), then it is straightforward to check that the phase function \( \lambda \Phi \) parametrizes the conormal bundle to the diagonal, and the result is a semiclassical scattering pseudodifferential operator of order 0.

Remark B.3. It is not hard to see that the operator \( \phi\left(\frac{\Delta_g}{\mu^2}\right) \) is microlocally equal to the identity for \( |\zeta|_g \in \{(1 - \delta/8)^2, (1 + \delta/8)^2\} \), where \( \zeta \) is the rescaled cotangent variable. First, the operator \( \phi\left(\frac{\Delta_g}{\mu^2}\right) \) is elliptic in this region. Next, choose a function \( \phi_1 \) supported in the interior of the region where \( \phi = 1 \). Then by functional calculus, \( \phi_1\left(\frac{\Delta_g}{\mu^2}\right) = \phi\left(\frac{\Delta_g}{\mu^2}\right)\phi_1\left(\frac{\Delta_g}{\mu^2}\right) \), from which it follows that \( \phi\left(\frac{\Delta_g}{\mu^2}\right) \) is microlocally equal to the identity on the elliptic set of \( \phi_1\left(\frac{\Delta_g}{\mu^2}\right) \), which is an arbitrary subset of \( \{(z, \zeta) \mid |\zeta|_g \in \{(1 - \delta/8)^2, (1 + \delta/8)^2\}\} \).

We next consider the microlocal structure of the spectrally localized resolvent.

Proposition B.4. The microlocal structure of the operator \( \phi\left(\frac{\Delta_g}{\mu^2}\right)(\Delta_g - (z \pm i0))^{-1} \), \( z > 0 \), is as follows:

(i) High energy case. Here we use semiclassical notation and write \( z = h^{-2} \). The operator \( \phi(h^2\Delta_g)(h^2\Delta_g - (1 \pm i0))^{-1} \), acting on half-densities, lies in the same microlocal
space as the semiclassical resolvent as detailed in [27, Theorem 1.1], indeed in a ‘better’ space as the differential order is $-\infty$ rather than $-2$. That is, the spectrally localized resolvent is a sum of three terms $S_1 + S_2 + S_3$, where

- $S_1$ is a semiclassical pseudodifferential operator of differential order $-\infty$ and semiclassical order 0,
- $S_2$ is an intersecting Legendre distribution associated to the conormal bundle $N^*\text{Diag}_b$ and to the propagating Legendrian $L$, and
- $S_3$ is a conic Legendre pair associated to $L$ and to the outgoing Legendrian $L^+_2$.

Moreover, $S_2 + S_3$ are microlocally identical to the full resolvent in a neighbourhood of the characteristic variety $\Sigma_l$ of $h^2\Delta_g - 1$.

(ii) Low energy case. Let $z \in (0,2)$. The operator $\phi(\frac{\Delta_g}{z})(\Delta_g - (z \pm i0))^{-1}$, acting on half-densities, lies in the same microlocal space as the resolvent as detailed in [24, Theorem 3.9], indeed in a better space as the differential order is $-\infty$ rather than $-2$. In detail, the operator $\phi(\frac{\Delta_g}{z})(\Delta_g - (z \pm i0))^{-1}$ can be decomposed as $S_1 + S_2 + S_3 + S_4$ (with $\sqrt{z}$ playing the role of the spectral parameter on $M^{2,b}_{k,b}$) where

- $S_1 \in \Psi^{-\infty}(M,\Omega^{1/2}_{k,b})$ is a pseudodifferential operator of order $-\infty$ in the calculus of operators defined in [23];
- $S_2 \in I^{-1/2,B}(M^{2,b}_{k,b},(scN^*\text{Diag}_b,L^b_+);\Omega^{1/2}_{k,b})$ is an intersecting Legendre distribution on $M^{2,b}_{k,b}$, microsupported close to $scN^*\text{Diag}_b$;
- $S_3 \in I^{-1/2,(n-2)/2,(n-1)/2;\Omega^{1/2}_{k,b}}(M^{2,b}_{k,b},(L^b_+,L^#_+);\Omega^{1/2}_{k,b})$ is a Legendre distribution on $M^{2,b}_{k,b}$ associated to the intersecting pair of Legendre submanifolds with conic points $(L^b_+,L^#_+)$, microsupported away from $scN^*\text{Diag}_b$;
- $S_4$ is supported away from $bf$ and is such that $e^{\pm i\lambda r}e^{\pm i\lambda'}R_4$ is polyhomogeneous conormal on $M^{2,b}_{k,b}$.

Here $B = (B_{bf_0},B_{b_0},B_{rb_0},B_{zd})$ is an index family with minimal exponents (i.e. order of vanishing) $\min B_{bf_0} = -2$, $\min B_{b_0} = \min B_{rb_0} = n/2 - 2$, $\min B_{zd} = 0$. In addition $S_4$ vanishes to order $\infty$ at $lb$ and $bf$ and to order $(n - 1)/2$ at $rb$.

**Corollary B.5.** The estimates (5.3), (5.4), (5.9) and (5.10) hold if the resolvent $(\Delta_g - (z \pm i0))^{-1}$ is replaced by the spectrally localized resolvent $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$.

**Proof of Corollary B.5.** The proofs of these estimates only used the location of the wavefront set of the resolvent kernel, together with the vanishing orders of the resolvent on the boundary hypersurfaces of $M^{2,b}_{k,b}$ at $z = 0$. In view of Proposition B.4, the same proof applies verbatim to the spectrally localized resolvent.

**Proof of Proposition B.4.** (i) We study the composition of the operator $\phi(h^2\Delta_g)$ with the incoming or outgoing resolvent, $(h^2\Delta_g - (1 \pm i0))^{-1}$. We know from [27, Theorem 1.1] that the actual resolvent can be decomposed into a sum of three terms $R_1 + R_2 + R_3$...
as in the proposition (except that $R_1$ will have differential order $-2$). We may assume that $R_2$ and $R_3$ are microsupported in the region where $|\zeta_g| \in \{(1 - \delta/8)^2, (1 + \delta/8)^2\}$, and $R_1$ is microsupported in the region where $|\zeta_g| \notin \{(1 - \delta/16)^2, (1 + \delta/16)^2\}$. The composition $S_1 := \phi(h^2\Delta_g)R_1$ is another semiclassical pseudodifferential operator, of semiclassical order 0 and differential order $-\infty$. On the other hand, the operator $\phi(h^2\Delta_g)$ is microlocally equal to the identity on the microsupport of $R_2$ and $R_3$, so using [23, Section 7], we find that the composition of $\phi(h^2\Delta_g)$ with $R_2 + R_3$ is equal to $R_2 + R_3$ up to an operator that is residual in all senses, that is, a smooth kernel that vanishes rapidly as $h \to 0$ or upon approach to the boundary of $M_b^2$. So we can take $S_2 = R_2$ and $S_3 = R_3$ up to a residual kernel.

(ii) Similarly, in the low energy case the actual resolvent has a decomposition into $R_1 + R_2 + R_3 + R_4$ having properties as in the proposition (with $R_1$ of differential order $-2$). We also need to decompose the operator $\phi(h^2\Delta_g) = B_1 + B_2$ into two parts, where $B_1$ is supported close to the diagonal on the space $M_{k,b}^2$, and $B_2$ has empty wavefront set. This second piece $B_2$ can be taken to vanish to infinite order at $bf$, $lb$ and $rb$, and to be polyhomogeneous conormal to $bf_0$, $lb_0$, $rb_0$ and $zf$ vanishing to order 0 at $bf_0$, $n/2$ at $lb_0$ and $rb_0$ and order $n$ at $zf$. When we apply $B_1$ to the resolvent, the argument is just as in the high energy case, using [23, Section 5] instead of [23, Section 7].

To understand what happens when we apply $B_2$ to the resolvent, we view the composition of operators as pushforward of the product of the Schwartz kernels on a ‘triple space’ $M_{k,b}^2$ down to $M_{k,b}^2$, as was done in the appendix of [21]. As a multiple of a non-vanishing $b$-half-density on $M_{k,b}^2$ we find that $B_2$ (multiplied by $|dk/k|^{1/2}$, $k = \sqrt{\zeta}$, which is a purely formal factor) is polyhomogeneous conormal, with no log terms at leading order, and vanishes to order $n$ at $zf$, 0 at $bf_0$ and $n/2$ at $lb_0$ and $rb_0$. On the other hand, we can decompose the resolvent kernel as the sum of $R_1 + R_2$, supported near the diagonal, and $R_3 + R_4$, which is microsupported in the set where $|\zeta_g| \in \{(1 - \delta/8)^2, (1 + \delta/8)^2\}$, where $\zeta$ is the cotangent variable rescaled by a factor $\sqrt{\zeta}$.

The composition of $B_2$ with $R_1 + R_2$ can be treated by lifting both kernels to the space $M_{k,b}^2$ and pushing forward. Since $B_2$ has no wavefront set, the composition has no wavefront set, so it is polyhomogeneous conormal, and the order of vanishing can be read off as $n$ at $zf$, $n/2$ at $lb_0$, $n/2 - 2$ at $rb_0$, $-2$ at $bf_0$, and $\infty$ at $lb, rb$ and $bf$. This lies in a better space than claimed in the proposition.

The composition of $B_3$ with $R_3 + R_4$ can also be analyzed by lifting both kernels to $M_{k,b}^2$ and then pushing forward. Although $R_3 + R_4$ is not polyhomogeneous conormal at the boundary hypersurfaces $bf$, $lb$ and $rb$, when lifted to $M_{k,b}^2$ and multiplied by the lift of $B_2$, the rapid vanishing of $B_2$ at $bf$ and $rb$ means that the product of the two kernels is rapidly decreasing as the ‘middle variable’ (the right variable of $B_2$ and the left variable of $R_3 + R_4$) tends to the boundary. As for the right variable of
$R_3 + R_4$, after multiplying the kernel of $R_3 + R_4$ by $e^{\mp i\lambda r'}$ (where $r' = 1/x'$ is the right radial variable) it becomes polyhomogeneous conormal also at $rb$. So the product of the kernels $B_2$ (in the left and middle variables) and $(R_3 + R_4)e^{\mp i\lambda r'}$ (in the middle and right variables) on $M^3_{k,b}$ is polyhomogeneous conormal. After pushing forward to $M^2_{k,b}$ a calculation similar to that done in [21, Appendix] shows that the result is $e^{\mp i\lambda r'}$ times a polyhomogeneous kernel which vanishes to order $n - 2$ at $zf$, $-2$ at $bf_0$, $\min(n/2, n - 2)$ at $lb_0$, $n/2 - 2$ at $rb_0$, $(n - 1)/2$ at $rb$ and $\infty$ at $lb$ and $bf$, with no log terms to leading order except possibly at $lb_0$ in the case $n = 4$. Again this is in a better space than is claimed in the proposition. This completes the proof.

\[\square\]

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