SHAPOVAlov ELEMENTS FOR $U_q(\mathfrak{sl}(N+1))$

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Abstract. For a simple Lie algebra, Shapovalov elements give rise to highest weight vectors in Verma modules. The usual construction of these elements uses induction on the length of a certain Weyl group element. If $\mathfrak{g} = \mathfrak{sl}(N+1)$ explicit expressions for Shapovalov elements were given in [Mus22a]. Here we adapt the argument to the quantized enveloping algebra of $\mathfrak{g}$.

1. The Quantized Enveloping Algebra

1.1. The PBW Theorem. Let $\mathfrak{f}$ be a field and $k = \mathfrak{f}(q)$ the field of rational functions over $\mathfrak{f}$. Unadorned tensors are taken over $k$. Let $\mathfrak{g} = \mathfrak{sl}(N+1)$ and $A = (a_{ij})$ the Cartan matrix of $\mathfrak{g}$. Let $U_q(\mathfrak{g})$ be the $k$-algebra with generators $e_i, f_i, k_{\pm}^{-1}$ for $i \in [N] := \{1, \ldots, N\}$. As the notation suggests, $k_i$ is a unit in $U_q(\mathfrak{g})$ with two-sided inverse $k_{-1}^{-1}$. Furthermore $U_q(\mathfrak{g})$ satisfies the following relations for all $i$ and $j$:

\begin{align}
(1.1) \quad & k_i k_j = k_j k_i, \\
(1.2) \quad & k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \\
(1.3) \quad & e_i f_j - f_j e_i = \delta_{ij}(k_i^2 - k_{-i}^{-2})/(q^2 - q^{-2}).
\end{align}

as well as the Serre relations, [Yam89], (1.4) and (1.5) which we do not use. They are needed for the proof of Yamane’s PBW theorem, Theorem [1.1] (c). We denote the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i$ (resp. $e_i$) for $i \in [N]$ by $U_q(\mathfrak{n}^-)$ (resp. $U_q(\mathfrak{n}^+)$).

Define $f_{i,i+1} = f_i$. This is a quantum analog of the negative root vector $E_{i+1,i} \in \mathfrak{sl}(N+1)$. For analogs of other negative root vectors, consider the following elements, introduced by Jimbo in [Jim86]. For $j > i + 1$, define

\begin{align}
(1.4) \quad & f_{i,j} = q f_{i,j-1} f_{j-1,j} - q^{-1} f_{j-1,j} f_{i,j-1}.
\end{align}

Define the lexicographic order $<$ on $\mathbb{N} \times \mathbb{N}$ by $(i, j) < (m, n)$ iff $i < m$ or $i = m, j < n$.

**Theorem 1.1.** (a) The group of units $\mathcal{H}$ of $U_q(\mathfrak{g})$ generated by the $k_{i}^\pm 1$ for $i \in [N]$ is free abelian of rank $N$. Denote its group algebra over $k$ by $\mathcal{H}$.

(b) We have a triangular decomposition,

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes \mathcal{H} \otimes U_q(\mathfrak{n}^+).$$

(c) The PBW theorem for $U_q(\mathfrak{n}^-)$: the elements $f_{m_1,n_1} \ldots f_{m_s,n_s}$ with $m_i, n_i \in [N+1], m_i < n_i$ and $(m_1,n_1) \leq \ldots \leq (m_s,n_s)$ form a basis for $U_q(\mathfrak{n}^-)$.

**Proof.** See [Yam89].
If \( \{ \epsilon_i \mid i \in [N + 1] \} \) is the standard basis for the dual of the Cartan subalgebra of \( \mathfrak{gl}(N + 1) \) consisting of diagonal matrices, then the simple roots of \( \mathfrak{g} \) have the form
\[
\alpha_i = \epsilon_i - \epsilon_{i+1},
\]
for \( i \in [N] \). Let \( Q = \sum_{i=1}^{N} \mathbb{Z} \alpha_i \) and \( Q^+ = \sum_{i=1}^{N} \mathbb{N} \alpha_i \) respectively denote the root lattice and the positive root lattice of \( \mathfrak{g} \). In addition set \( \sigma_i = \epsilon_i - \epsilon_{i+1} \). There is an isomorphism \( Q \rightarrow \mathbb{A} \) sending \( \alpha_i \) to \( k_i \). If \( \alpha \in Q \), let \( k_\alpha \in \mathbb{A} \) be the image of \( \alpha \) under this map. Set \( U_q(b^\pm) = U_q(n^\pm) \otimes \mathcal{H} \). This definition does not make it clear that \( U_q(b^\pm) \) is a subalgebra of \( U_q(g) \). However identifying \( x \otimes y \) with \( xy \), we see that \( U_q(b^\pm) \) identifies with the product (not a direct product) of algebras \( U_q(n^\pm) \mathcal{H} \). Since the \( k_\alpha \) with \( \alpha \in \mathbb{A} \) form a \( \mathbb{k} \) basis for \( \mathcal{H} \), every element of \( U_q(n^\pm) \mathcal{H} \) can be written in a unique way as a finite sum \( \sum_{\alpha \in \mathbb{A}} a_\alpha k_\alpha \), where \( a_\alpha \in U_q(n^\pm) \). Equation (1.2) implies that \( U_q(n^\pm) \mathcal{H} = \mathcal{H} U_q(n^\pm) \), and from this it follows that \( U_q(b^\pm) \) is a subalgebra.

The ideal of \( U_q(n^\pm) \) generated by the \( e_i \) is denoted \( I(n^\pm) \). Since \( \mathcal{H} \) is free over \( \mathbb{k} \), \( \mathcal{H} \otimes \mathcal{H} \) is an exact functor from left \( U_q(n^\pm) \)-modules to left \( U_q(b^\pm) \)-modules. Applying this functor to the exact sequence
\[
0 \rightarrow I(n^\pm) \rightarrow U_q(n^\pm) \rightarrow \mathbb{k} \rightarrow 0,
\]
we obtain
\[
0 \rightarrow I(n^\pm) \otimes \mathcal{H} \rightarrow U_q(b^\pm) \rightarrow \mathcal{H} \rightarrow 0.
\]
Because of (1.2), we see that \( I(n^\pm) \otimes \mathcal{H} = \mathcal{H} I(n^\pm) \) is a two-sided ideal of \( U_q(b^\pm) \), so \( U_q(b^\pm)/(I(n^\pm) \otimes \mathcal{H}) \cong \mathcal{H} \) as \( \mathbb{k} \)-algebras. This means that any \( \mathcal{H} \)-module can be regarded as a \( U_q(b^\pm) \)-module with \( \mathcal{H} \) \( I(n^\pm) \) contained in its annihilator.

We mainly use Yamane’s definition because it is easy to work with his PBW basis. However for representation theory we need to compare it to the definition in Jantzen’s book [Jan96] Chapter 4. In Type A (more generally in the simply laced case), there is a bilinear form \( ( , ) \) on \( Q \) defined by \( \langle \alpha_i, \alpha_j \rangle = a_{ij} \) where \( A = (a_{ij}) \) is the Cartan matrix. Since any root \( \alpha \) is conjugate to a simple root under the Weyl group \( W \), and the form \( ( , ) \) is known to be \( W \)-invariant, \( \alpha \) satisfies \( (\alpha, \alpha) = 2 \), so that \( \alpha = \alpha^\vee \) and \( q_\alpha = q \) in [Jan96] 4.2 (1). Then in [Jan96], \( U_q(g) \) is the \( f(v) \) algebra with generators \( E_i, F_i, K_i^{\pm 1} \) for \( i \in [N] \), subject to the relations [Jan96], 4.3 (R2)-(R4).

\[
(1.6) \quad K_i K_j = K_j K_i,
\]
\[
(1.7) \quad K_i E_j K_i^{-1} = v^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = v^{-a_{ij}} F_j,
\]
\[
(1.8) \quad E_i F_j - F_j E_i = \delta_{ij} (K_i - K_i^{-1})/(v - v^{-1}).
\]

and the Serre relations [Jan96], 4.3 (R5) and (R6). For example (R6) says that for \( i \neq j \), we have
\[
(1.9) \quad \sum_{i=0}^{1-a_{ij}} (-1)^i \left[ 1 - a_{ij} \right] F_i^{1-a_{ij}-1} F_j F_i^i,
\]
and (R5) is an analogous relation where \( F_k \) is replaced by \( E_k \). Serre relations are so called because they occur in Serre’s theorem giving generators and relations for a simple Lie algebra, [Hum72] Proposition 18.1.
Now comparing with (1.12) and (1.13), and checking the compatibility of the Serre relations, shows that when $v = q^2$, there is an algebra map $U_v(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ given by

\begin{equation}
E_i \mapsto e_i, \ F_i \mapsto f_i, \ K_i \mapsto k_i^2.
\end{equation}

Many of our computations take place in $U_v(\mathfrak{g})$, so we identify $U_v(\mathfrak{g})$ with its image under the above map.

The algebra $U_q(\mathfrak{g})$ is $Q$-graded by setting $\deg e_i = -\deg f_i = \alpha_i$, and $\deg k_i = 0$ for all $i$. For $j > i + 1$ we have

\begin{equation}
\deg f_{i,j} = -\sum_{k=i}^{j-1} \alpha_k.
\end{equation}

If $x \in U_q(\mathfrak{g})$ and $\deg x = \beta$, then

\begin{equation}
k_\alpha x k_\alpha^{-1} = q^{(\alpha, \beta)} x,
\end{equation}

compare [Jan96] 4.7 (1). In particular this gives an easy way to handle commutation relations between the $k_\alpha$ and the $f_{i,j}$. If $x \in U_q(\mathfrak{g})$ has $\deg x = \beta$, we write $x \in U_q(\mathfrak{g})^\beta$. Set $K_\alpha = k_\alpha^2$ and $v = q^2$. Then (1.12) becomes

\begin{equation}
K_\alpha x K_\alpha^{-1} = q^{(\alpha, \beta)} x.
\end{equation}

1.2. Representation Theory. From now on we assume $v = q^2$. Introduce the weight lattice $P$ by

$$P = \{ \lambda \in \mathfrak{h}^* | (\lambda, Q) \subseteq \mathbb{Z} \}.$$ 

Since the form $(,)$ is integer valued on $Q$ we have $Q \subseteq P$. In fact $|P/Q| = N + 1$, the determinant of the Cartan matrix. If $M$ is a $U_v(\mathfrak{g})$-module (or even an $\mathcal{H}$-module) and $\lambda \in P$, we say that $m \in M$ has weight $\lambda$ if

\begin{equation}
K_\mu m = q^{(\lambda, \mu)} m,
\end{equation}

for all $\mu \in Q$, see [Jan96] 5.1 (1). We consider only weight modules of type 1, see [Jan96] 5.2. Because of (1.10), this means that if $M$ is an $U_q(\mathfrak{g})$-module and $\lambda \in P$, we say that $m = m_\lambda \in M$ has weight $\lambda$ if $k_\mu m = q^{(\lambda, \mu)} m$ for all $\mu \in Q$. Let $v_\lambda$ be the one dimensional $\mathcal{H}$-module with weight $\lambda \in P$. By the above remarks, we can regard $v_\lambda$ as an $U_q(\mathfrak{h}^+)$-module annihilated by $I(\mathfrak{n}^+) \otimes \mathcal{H}$. Then define the Verma module $M(\lambda)$ with highest weight $\lambda$, for $U_q(\mathfrak{g})$ by $M(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{n}^+)} v_\lambda$. The set of weights of $M(\lambda)$ is $\lambda - Q^+$. If $r \in \mathbb{N}$ set $[r]_v = (v^r - v^{-r})/(v - v^{-1})$. We use some notation,

$$[K_\mu : a] = (K_\mu v^a - K_\mu^{-1} v^{-a})/(v - v^{-1}),$$

from [Jan96] 1.3 (1). Note that (1.14) implies

\begin{equation}
[K_\mu : a] m_\lambda = [a + (\lambda, \mu)]_v m_\lambda.
\end{equation}
1.3. **Shapovalov elements.** Let \( g \) be a simple Lie algebra with set of simple and positive roots \( \Pi \) and \( \Delta^+ \) respectively. We denote the half sum of positive roots by \( \rho \). The definition and significance of Shapovalov elements is given in \cite{Mus22a} Section 1. We adapt the definitions to the quantum case. For ease of exposition, and because we will only work with Type A in the sequel, we assume that \( g \) is simply laced, and as before all roots \( \alpha \) satisfy \( (\alpha, \alpha) = 2 \), so \( \alpha = \alpha^\vee \).

Fix a positive root \( \eta \) and a positive integer \( m \). Let \( P(\eta m) \) be the set of partitions of \( \eta m \), see \cite{Mus12} Chapter 8 for notation. Then let \( \pi^0 \in P(\eta m) \) be the unique partition of \( \eta m \) such that \( \pi^0(\alpha) = 0 \) if \( \alpha \in \Delta^+ \setminus \Pi \). We say that \( \theta = \theta_{\eta, m} \in U_q(b^-)^{-\eta m} \) is a **Shapovalov element** for the pair \( (\eta, m) \) if it has the form

\[
\theta = \sum_{\pi \in P(\eta m)} e_{-\pi} H_{\pi},
\]

where \( H_{\pi} \in \mathcal{H}, \ H_{\pi^0} = 1 \), and

\[
e_{\pi} \theta \in U_q(g)(\rho_{\eta}k_{\eta} - q^m) + U_q(g)I(n^+), \ \text{for all} \ \alpha \in \Pi.
\]

For a semisimple Lie algebra, the existence of such elements was shown by Shapovalov, \cite{Sha72} Lemma 1. Shapovalov elements for the pair \( (\eta, m) \) are only unique up to the addition of an element from the left ideal in \( U_q(g) \) appearing on the right side of \( (1.17) \), this is the quantum analog of \cite{Mus17} Theorem 2.1. For any element \( \theta \) as in \( (1.16) \) and \( \lambda \in P \), define the evaluation of \( \theta \) at \( \lambda \) in \( P \) to be

\[
\theta(\lambda) = \sum_{\pi \in P(\eta m)} e_{-\pi} H_{\pi}(\lambda).
\]

Note that \( \theta \nu_\lambda = \theta(\lambda) \nu_\lambda \).

Consider the hyperplane in \( h^* \) given by

\[
\mathcal{H}_{\eta, m} = \{ \lambda \in h^* \mid (\lambda + \rho, \eta) = m \}.
\]

From \( (1.17) \), the Shapovalov element \( \theta_{\eta, m} \) has the important property that if \( \lambda \in P \cap \mathcal{H}_{\eta, m} \) then \( \theta_{\eta, m} \nu_{\lambda} \) is a highest weight vector of weight \( \lambda - \eta m \) in \( \mathcal{M}(\lambda) \). The normalization condition \( H_{\pi^0} = 1 \) guarantees that \( \theta_{\eta, m} \nu_\lambda \) is never zero. We give an explicit expression for a Shapovalov elements for \( U_q(\mathfrak{sl}(N + 1)) \). There are two important reductions we can make. First we need only consider the highest root \( \eta = \epsilon_1 - \epsilon_{N + 1} \) of \( \mathfrak{sl}(N + 1) \), because any positive root is the highest root of some special linear subalgebra. The second is somewhat deeper: by a careful consideration of powers of \( \theta_{\eta, 1} \) in Section 3 it is only necessary to consider the case \( m = 1 \). So to simplify notation we set \( \mathcal{H}_\eta = \mathcal{H}_{\eta, 1} \), and denote a Shapovalov element for the pair \( (\eta, 1) \) by \( \theta_\eta \).

2. **Commutation Relations.**

We need several commutation relations for the computation of the Shapovalov elements. These may also be found in \cite{Yam89}. For elements \( a, b \) in an associative algebra, set \( [a, b] = ab - ba \). We use the elementary identity \( [a, bc] = b[a, c] + [a, b]c \), many times without comment. For comparison with the relations in \cite{Yam89}, set \( e_{i,i+1} = e_i \). In particular, we consider the question of when \( [e_\ell, f_{i,j}] \neq 0 \). If \( j = i + 1 \), this occurs if and only if \( i = \ell \) by \( (1.3) \). Now suppose that \( j > i + 1 \).

**Lemma 2.1.** If \( j > i + 1 \) and \( [e_\ell, f_{i,j}] \neq 0 \), then \( \ell = i \) or \( \ell + 1 = j \).

**Proof.** Use induction on \( j - i - 1 \), and Equations \( (1.3) \) and \( (1.4) \). Assume \( \ell \neq i \) and \( \ell \neq j - 1 \), and show \( [e_\ell, f_{i,j}] = 0 \). By definition \( f_{i,j} = qf_{i,j-1}f_{j-1} - q^{-1}f_{j-1}f_{i,j-1} \).
Since \( \ell \neq j - 1 \), \([e_i, f_{i,j-1}] = 0\). If \( j - 1 = i + 1 \), then \([e_i, f_{i,j-1}] = [e_i, f_i]\), and this is zero since \( \ell \neq i \). Otherwise \( j - 1 > i + 1 \) and \([e_i, f_{i,j-1}] = 0\) by induction.

In the exceptional cases \( \ell = i \) (resp. \( \ell + 1 = j \)) the situation is covered by the next two Lemmas, see (2.1), (resp. (2.2)) with some small changes of notation which will be useful later. In both proofs we make use of (1.12). The relations are special cases of the first (resp. sixth) equation in Section 3 (3) of [Yam89]. Since these relations are crucial to our work, but not proved in [Yam89] we give the proofs.

**Lemma 2.2.** When \( i \leq b - 2 \),

\[
[e_i, f_{i,b}] = q f_{i+1,b} k_i^2.
\]

**Proof.** We use induction on \( b \).

Case 1: If \( b = i + 2 \), then \([e_i, f_{i+1,i+2}] = 0\) by (1.3). Therefore using (1.3) again,

\[
[e_i, f_{i,i+2}] = [e_i, q f_{i,i+1} f_{i+1,i+2} - q^{-1} f_{i+1,i+2} f_{i,i+1}]
\]

\[
= (q^2 - q^{-2})^{-1} [(k_i^2 - k_i^{-2}) q f_{i+1,i+2} - q^{-1} f_{i+1,i+2} (k_i^2 - k_i^{-2})]
\]

\[
= (q^2 - q^{-2})^{-1} [q f_{i+1,i+2} + (q^2 k_i^2 - q^{-2} k_i^{-2}) - q^{-1} f_{i+1,i+2} (k_i^2 - k_i^{-2})]
\]

\[
= q f_{i+1,i+2} k_i^2.
\]

Case 2: If \( b > i + 2 \), then \([e_i, f_{b-1,b}] = 0\) by (1.3), and by induction we know the result for \([e_i, f_{b-1,b}]\). Thus,

\[
[e_i, f_i] = [e_i, q f_{i,b-1} f_{b-1,b} - q^{-1} f_{b-1,b} f_{i,b-1}]
\]

\[
= q f_{i+1,b-1} k_i^2 f_{b-1,b} - q^{-1} f_{b-1,b} q f_{i+1,b} k_i^2
\]

\[
= q (q f_{i+1,b-1} f_{b-1,b} - q^{-1} f_{b-1,b} f_{i+1,b}) k_i^2
\]

\[
= q f_{i+1,b} k_i^2.
\]

**Lemma 2.3.** When \( a < i \),

\[
[e_i, f_{a,i+1}] = -q^{-1} f_{a,i} k_i^{-2}.
\]

**Proof.** By definition,

\[
f_{a,i+1} = q f_{a,i} f_{i,i+1} - q^{-1} f_{i,i+1} f_{a,i}
\]

and clearly

\[
[e_i, f_{a,i}] = 0,
\]

due to Lemma 2.1 when \( a < i - 1 \) and to (1.3) when \( a = i - 1 \). So by (1.3),

\[
[e_i, f_{a,i+1}] = (q^2 - q^{-2})^{-1} [q f_{a,i} (k_i^2 - k_i^{-2}) - q^{-1} (k_i^2 - k_i^{-2}) f_{a,i}]
\]

\[
= (q^2 - q^{-2})^{-1} [q f_{a,i} (k_i^2 - k_i^{-2}) - q^{-1} f_{a,i} (q^2 k_i^2 - q^{-2} k_i^{-2})]
\]

\[
= (q^2 - q^{-2})^{-1} f_{a,i} (-q + q^{-3}) k_i^{-2}
\]

\[
= -q^{-1} f_{a,i} k_i^{-2}.
\]
Consider a strictly increasing sequence of integers

\[(2.3) \quad I = \{j_0, j_1, \ldots, j_s, j_{s+1}\},\]

with \(1 \leq j_0\) and \(j_{s+1} \leq N + 1\). If \(I\) is a singleton set, then by definition set \(f_I = 1\).

Otherwise define \(f_I \in U_q(n^-)\) by

\[(2.4) \quad f_I = f_{j_0, j_1} f_{j_1, j_2} f_{j_2, j_3} \cdots f_{j_s, j_{s+1}}.\]

The way the factors are ordered in \(2.4\) is consistent with the lexicographic order used in Yamane’s PBW Theorem, Theorem [1.1]. This is because \(j_{k-1} < j_k < j_{k+1}\) implies that \((j_{k-1}, j_k) < (j_k, j_{k+1})\) in the lexicographic order. Let

\[(2.5) \quad \mathbb{I} = \{I \subseteq [N + 1] \mid 1, N + 1 \in I\},\]

and for \(I \in \mathbb{I}\), define

\[(2.6) \quad r(I) = \{s - 1 \mid s \in \bar{I}\}\]

where \(\bar{I}\) be the complement of \(I\) in \(\mathbb{I}\). Note that if \(I \in \mathbb{I}\), then by \(2.4\) and the definition of \(\mathbb{I}\), \(f_I \in U_q(n^-)^{-\eta}\). Clearly elements of \(\mathbb{I}\) correspond bijectively to partitions of \(\eta\), (with a suitable ordering on positive roots) and taking \(J = \mathbb{I}\) we have \(f_J = e_{-\eta^0}\). Thus the \(f_I\) with \(I \in \mathbb{I}\), form a \(k\)-basis for \(U_q(n^-)^{-\eta}\).

Consider the set

\[(2.7) \quad \mathcal{S}_i = \{I \subseteq \mathbb{I} \mid i, i + 1 \in I\}.\]

For \(I \in \mathcal{S}_i\), define

\[(2.8) \quad I^+ = I \setminus \{i\}, \text{ and } I^- = I \setminus \{i + 1\}.\]

If \(i = N\), then \(I^- \notin \mathbb{I}\) and if \(i = 1\), then \(I^+ \notin \mathbb{I}\). We set \(f_{I^-} = 0\) or \(f_{I^+} = 0\) in these cases. It is often useful to write \(I \in \mathcal{S}_i\) in the form

\[(2.9) \quad I = \{1, \ldots, a, i, i + 1, b, \ldots, N + 1\}.\]

Then

\[(2.10) \quad I^+ = \{1, \ldots, a, i + 1, b, \ldots, N + 1\} \text{ and } I^- = \{1, \ldots, a, i, b, \ldots, N + 1\}.\]

The following result summarizes the commutation relations that we need later in the paper.

**Lemma 2.4.** If \(J \in \mathbb{I}\), then \(e_i\) fails to commute with at most one factor of \(f_J\).

Furthermore, for a fixed \(I \in \mathcal{S}_i\), we have:

- (a) \(f_i\) is a factor of \(f_I\), and \([e_i, f_i]\) is given by \((b.3)\).
- (b) \(f_{i,b}\) is a factor of \(f_{I^-}\), and \([e_i, f_{i,b}]\) is given by \((2.1)\).
- (c) \(f_{a,i+1}\) is a factor of \(f_{I^+}\), and \([e_i, f_{a,i+1}]\) is given by \((2.2)\).
- (d) \(e_i\) commutes with all other factors of \(f_I\) and \(f_{I:\pm}\) in parts (a)-(c).
- (e) If \(J\) does not have the form \(I, I^+\) or \(I^-\) for some \(I \in \mathcal{S}_i\), then \(e_i\) commutes with all factors of \(f_J\).

**Proof.** The claims about the factors in (a)-(c) hold by \((2.9)\) and \((2.10)\). The rest follows from Lemma \((2.1)\) \(\square\)
3. Shapovalov elements for the quantized enveloping algebra $U_q(\mathfrak{sl}(N+1))$

For $i \in [N]$ and $J \subseteq \mathfrak{I}$, define

$$h_i = -q^{-1}v^{-(\rho, \sigma_i)^{-1}}K^{-1}_{\sigma_i} : i \in \mathcal{H}, \quad H_J = \prod_{i \in \tau(J)} h_i.$$  

For the rest of the paper, $v_\lambda$ is a highest weight vector of weight $\lambda$ in the Verma module $M(\lambda)$.

**Lemma 3.1.** We have

$$h_i v_\lambda = -q^{-1}v^{-(\lambda+\rho, \sigma_i)^{-1}}[(\lambda+\rho, \sigma_i)]_i v_\lambda$$

for all $\lambda \in \mathbb{Q}$.

**Proof.** Since $\sigma_i$ is the sum of $i$ simple roots, and for a simple root $\alpha$, we have $(\rho, \alpha) = (\rho, \alpha^\vee) = 1$, it follows that $(\rho, \sigma_i) = i$. Thus $(\lambda+\rho, \sigma_i) = (\lambda, \sigma_i) + i$, so by (1.16),

$$[K_{\sigma_i} : i] v_\lambda = [(\lambda+\rho, \sigma_i)]_i v_\lambda.$$ 

Also

$$v^{-(\rho, \sigma_i)^{-1}}K^{-1}_{\sigma_i} v_\lambda = v^{-(\lambda+\rho, \sigma_i)^{-1}}v_\lambda.$$ 

The result follows by combining (3.3) and (3.4). \qed

Our first main result provides an explicit expression for a Shapovalov element $\theta_{\eta}$. 

**Theorem 3.2.** Let $\eta = \epsilon_1 - \epsilon_{N+1}$, and suppose $v_\lambda$ is a highest weight vector in the $\mathfrak{g}$-Verma module $M(\lambda)$ of weight $\lambda$, and set

$$\Theta_{\eta} = \sum_{J \subseteq \mathfrak{I}} f_J H_J.$$ 

Then

(a) $e_{\alpha_k} \Theta_{\eta} v_\lambda = 0$ for $k \in [N-1]$.

(b) If $(\lambda + \rho, \eta) = 1$, then $\Theta_{\eta} v_\lambda$ is a $\mathfrak{g}$-highest weight vector, and so $\Theta_{\eta}$ is a Shapovalov element for the pair $(\eta, 1)$.

By the remarks preceding (2.7) we see that the coefficient of $e_{-\pi_0}$ in (3.5) is $H_{\pi_0} = 1$. Thus (1.10) holds. We show that $\sum_{J \subseteq \mathfrak{I}} f_J H_J$ satisfies Equation (1.17). Given a simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$, to avoid double subscripts we sometimes set $\alpha = \alpha_i$. Then $e_\alpha = e_i$. The goal is to show that if $\lambda \in \mathcal{H}_{\eta}$, then

$$e_i \sum_{J \subseteq \mathfrak{I}} f_J H_J v_\lambda = 0$$

for all $i \in [N]$. This will be shown directly in this Section. In Subsection 4.3 we give a different approach to Shapovalov elements and it is convenient to make some remarks at this point that apply to the second approach. We can rewrite Equations (3.4) as a sequence of equalities in $\mathcal{H}$, depending on $\lambda$, one for each partition of $\eta - \alpha_i$. It is enough to show that the equalities hold for all $\lambda$ in a Zariski dense subset $\Lambda$ of $\mathcal{H}_{\eta}$, with $\Lambda \subseteq P$. The subset $\Lambda$ will be defined in (4.12). The second approach can be used to construct Shapovalov elements $\theta_{\eta,m}$ for $m > 1$. The two approaches are compared in Subsection 4.4.
We conclude the proof of Theorem 3.2 after a series of Lemmas.

Now fix \( i \) and recall the set \( S_i \) from (2.7). If \( I \in S_i \) then \( e_{-\alpha} = f_i \) is a factor of \( f_I \). With \( I \) as in (2.9), we set
\[
I_1 = \{1, \ldots, a, i\} \quad \text{and} \quad I_2 = \{i + 1, b, \ldots, N + 1\}.
\]
Then we have

**Lemma 3.3.** If \( v_\lambda \) is a highest weight vector of weight \( \lambda \), then
\[
e_i f_I v_\lambda = \begin{cases} (q^2 - q^{-2})^{-1} f_I (k_i^2 - k_i^{-2}) f_I v_\lambda & \text{if } I_2 \text{ is a singleton} \\ (q^2 - q^{-2})^{-1} f_I f_I (q^2 k_i^2 - q^{-2} k_i^{-2}) v_\lambda & \text{otherwise.} \end{cases}
\]

**Proof.** Note that \( f_I = f_I f_I f_I \). We have \( [e_i, f_I] = 0 \) by Lemma 2.4. The first equality follows from this and (1.3). If \( |I_2| > 1 \), the first factor in \( f_{I_2} \) has the form \( f_{i+1,b} \) and by (1.12)
\[
(k_i^2 - k_i^{-2}) f_{i+1,b} = f_{i+1,b}(q^2 k_i^2 - q^{-2} k_i^{-2}).
\]
It is easy to see that \( k_i \) commutes with the other factors of \( f_{I_2} \). \( \square \)

Next with \( I \) as in (2.4), set
\[
I_1^+ = \{1, \ldots, a\} \quad \text{and} \quad I_2^- = \{b, \ldots, N + 1\},
\]
and note the factorizations
\[
f_{I^+} = f_{I_1^+} f_{a,i+1} f_{I_2} \quad \text{and} \quad f_{I^-} = f_{I_1} f_{I_2}.\]
If \( v_\lambda \) is a highest weight vector of weight \( \lambda \), we also need to compute \([e_i, f_{I\pm}] v_\lambda\). Since \( e_i v_\lambda = 0 \), this is equivalent to finding \([e_i, f_{I\pm}]\). Now using Lemma 2.4 and (1.12), we have
\[
[e_i, f_{I^+}] = [e_i, f_{I_1^+} f_{a,i+1} f_{I_2}] = f_{I_1^+} [e_i, f_{a,i+1}] f_{I_2}
\]
\[
= -q^{-1} f_{I_1^+} f_{a,i+1} k_i^{-2} f_{I_2}
\]
\[
= \begin{cases} -q^{-1} f_{I_1} f_{I_2} k_i^{-2} & \text{if } I_2 \text{ is a singleton} \\ -q^{-3} f_{I_1} f_{I_2} k_i^{-2} & \text{otherwise.} \end{cases}
\]

Note that \( i < N \) implies \( |I_2| \neq 1 \) in (3.8) and (3.10). Also by (2.1) and (1.12),
\[
[e_i, f_{I^-}] = [e_i, f_{I_1} f_{I_2}] = f_{I_1} [e_i, f_{I_2}] f_{I_2^-}
\]
\[
= q f_{I_1} f_{I_2} k_i^2 f_{I_2^-}
\]
(3.11)

We record an easy lemma.

**Lemma 3.4.** We have
(a) \( r(I^+) = r(I) \cup \{i - 1\} \) and \( r(I^-) = r(I) \cup \{i\} \).
(b) \( H_{I^+} = h_{i+1} H_I \) and \( H_{I^-} = h_i H_I \).
(c) If \( J \subseteq I \) and \( e_{i} f_{J} v_\lambda \) is a non-zero element of \( f_{I_1} f_{I_2} \mathcal{H} v_\lambda \), then
\[ J = I, I^+ \text{ or } I^- \]

If \( I^+ \) or \( I^- \) is not a subset of \( I \), they should be left out in the above equation.
Proof. (a) follows from the definitions (2.6) and (2.8). Then (b) follows from (3.1).
For (c) note that the pair \((I_1, I_2)\) determines \(I\) (and hence also \(I^\pm\)), since \(I\) is the
disjoint union \(I = I_1 \cup I_2\). By definition \(I\) and \(i\) determine \(I_1\) and \(I_2\). \(\Box\)

We begin with the case \(i = 1\). Recall that \(f_{\ell^+} = 0\) in this case.

**Lemma 3.5.** \(e_i(f_{\ell^+}H_I + f_{\ell^+}H_{-I})v_\lambda = 0\).

**Proof.** If \(i = 1\), apart from some elementary simplifications, we use in order, Lemma [3.4](b), (3.11), (3.2), and Lemma [3.3]

\[
ed_i f_{I^+}H_{I^+}v_\lambda = \begin{cases} f_{I^+} f_{I^+} [\lambda, \alpha_i]vH_Iv_\lambda, & \text{if } i = N, \\ f_{I^+} f_{I^+}[1 + \lambda, \alpha_i]vH_Iv_\lambda, & \text{otherwise.} \end{cases}
\]

(b) If \(1 < i \leq N\), then

\[
ed_i f_{I^+}H_{I^+}v_\lambda = \begin{cases} q^2 f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i]vH_Iv_\lambda, & \text{if } i = N, \\ f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i]vH_Iv_\lambda, & \text{otherwise.} \end{cases}
\]

(c) If \(1 < i < N\), then

\[
ed_i f_{I^+}H_{I^+}v_\lambda = \begin{cases} q^{-4} f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda, & \text{if } i = N, \\ f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda, & \text{otherwise.} \end{cases}
\]

Proof. (a) This follows from Lemma [3.3] and (1.15).

(b) Suppose first that \(1 < i < N\). Then by Lemma [3.4](b), (3.11), and (3.2),

\[
ed_i f_{I^+}H_{I^+}v_\lambda = \begin{aligned}
ed_i f_{I^+}h_{I^+}H_{I^+}v_\lambda & = f_{I^+} f_{I^+}[\lambda, \alpha_i]vH_Iv_\lambda \\ & = q^{-4} f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda \\ & = f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda \\ & = f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda. 
\end{aligned}
\]

The proof is similar if \(i = N\) except that since a different case holds in (3.10), the
first line of the proof becomes

\[
ed_i f_{I^+}H_{I^+}v_\lambda = f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda,
\]

leading to the result stated.

(c) By Lemma [3.4](b), (3.11), and (3.2),

\[
ed_i f_{I^+}H_{I^+}v_\lambda = \begin{aligned}
ed_i f_{I^+}h_{I^+}H_{I^+}v_\lambda & = f_{I^+} f_{I^+}[\lambda, \alpha_i]vH_Iv_\lambda \\ & = q^{-4} f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda \\ & = f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda \\ & = f_{I^+} f_{I^+}[\lambda, \rho, \sigma_i\sigma_i]vH_Iv_\lambda.
\end{aligned}
\]
Until further notice we assume that $1 < i < N$.

Lemma 3.7. \[
[(\lambda + \rho, \alpha_i)]_v + v^{-(\lambda + \rho, \sigma_i)}[(\lambda + \rho, \sigma_{i-1})]_v - v^{-(\lambda + \rho, \sigma_{i-1})}[(\lambda + \rho, \sigma_i)]_v = 0.
\]

Proof. Using $\sigma_{i-1} - \sigma_i = -\alpha_i$ to cancel the second and third terms in the first line below, we have
\[
\frac{(\lambda + \rho, \alpha_i)}{v^{-(\lambda + \rho, \sigma_i)}[(\lambda + \rho, \sigma_{i-1})]_v - v^{-(\lambda + \rho, \sigma_{i-1})}[(\lambda + \rho, \sigma_i)]_v}.
\]

This is equivalent to the statement of the lemma.

Corollary 3.8. For $1 < i < N$, we have
\[
e_i(f_1H_I + f_{I+}H_{I+} + f_{I-}H_{I-})v_{\lambda} = 0.
\]

Proof. We have using (3.13), (3.14) and (3.15),
\[
e_i(f_1H_I + f_{I+}H_{I+} + f_{I-}H_{I-})v_{\lambda} =
\]

and this is zero by Lemma 3.7.

Proof of Theorem 3.2. (a) We first prove the result for $1 < i < N$. By Corollary 3.8 we have
\[
(3.16) e_i(f_1H_I + f_{I+}H_{I+} + f_{I-}H_{I-})v_{\lambda} = 0.
\]

Now if $J$ does not contain $i$ or $i + 1$, then $e_i f_j H_j v_{\lambda} = 0$. Otherwise there exists a set $I \in S_i$ such that $J = I, I+$ or $I^-$. Thus we have using (3.16),
\[
e_a \sum_{I \in S} f_j H_j v_{\lambda} = e_a \sum_{I \in S} (f_1H_I + f_{I+}H_{I+} + f_{I-}H_{I-})v_{\lambda} = 0.
\]

The proof for $i = 1$ is entirely analogous. We use Lemma 3.5 in place of Corollary 3.8. This proves the first statement in the Theorem.

(b) By (3.13) and (3.14) we have $e_N(f_1H_I + f_{I+}H_{I+})v_{\lambda} =
\]

where the last equality follows since $\sigma_N - \sigma_N = \alpha_N$, $(\rho, \alpha_N) = 1$ and $\sigma_N = \eta$. We conclude the proof with the following Lemma.

Lemma 3.9. $v^{(\lambda, \alpha_N)} = v^{(\lambda, \rho, \eta = \alpha_N)}$, if $\lambda \in H_{\eta}$.

Proof. The difference of the two sides is
\[
(3.17) v^{(\lambda, \alpha_N)} - v^{(\lambda, \rho, \eta = \alpha_N)} = v^{(\lambda, \alpha_N)}(1 - v^{1-2(\lambda + \rho, \eta + \rho, \alpha_N)}) = 0.
\]

The last equality follows from $(\lambda + \rho, \eta) = 1$, due to $\lambda \in H_{\eta}$, and $(\rho, \alpha_N) = 1$. \qedsymbol
4. Powers of Shapovalov elements.

Powers of the Shapovalov element $\theta_{\eta,1}$ for a simple Lie algebra $\mathfrak{g}$ of types A-D were considered in [Mus17]. For $U_q(\mathfrak{g})$ powers of $\theta_{\eta,1}$ were considered in [Mud22], based on previous work by the same author on the R-matrix and inverse Shapovalov form, [Mud16]. In this Section we present an elementary approach to powers of $\theta_{\eta,1}$ in $U_q(sl(N+1))$ by adapting the method of [Mus17]. It is based on the original method of Shapovalov, which is rather different from the method we have used so far in this paper. Thus to tie up any loose ends we first indicate the connection between the two approaches. It is convenient to do this using certain non-commutative determinants which we introduce in the next Subsection.

4.1. Evaluation of Shapovalov element using noncommutative determinants. We define a noncommutative determinant of the $n \times n$ matrix $B = (b_{ij})$, working from left to right, by

\[
\det (B) = \sum_{w \in S_n} \text{sign}(w) b_{w(1),1} \cdots b_{w(n),n},
\]

where $S_n$ is the symmetric group. Cofactor expansions of $\det (B)$ are valid as long as the overall order of the terms is unchanged. Next let $m = N + 1$, and for $i \in [m - 2]$. Define $h_i, H_I$ from (3.1) and (3.2), and let $c_i = h_i(\lambda)$. Then set

\[
D^{N+1}(\lambda) = \begin{bmatrix}
 f_{1,2} & f_{1,3} & \cdots & f_{1,N} & f_{1,N+1} \\
 -c_1 & f_{2,3} & \cdots & f_{2,N} & f_{2,N+1} \\
 0 & -c_2 & \cdots & f_{3,N} & f_{3,N+1} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & -c_{N-1} & f_{N,N+1}
\end{bmatrix}.
\]

**Theorem 4.1.** The Shapovalov element for $\eta$ satisfies

\[
\theta_{\eta}(\lambda) = \det D^{N+1}(\lambda),
\]

for all $\lambda \in \mathcal{H}_\eta$.

**Proof.** We show $\det D^{N+1}(\lambda)$ is the evaluation of the element $\Theta_\eta$ from Theorem 3.2 at $\lambda \in \mathcal{H}_\eta$. In other words, we show

\[
\det D^{N+1}(\lambda) = \sum_{I \subseteq I} f_I H_I(\lambda).
\]

Let $I$ be as in (2.5). We consider the complete expansion of $\det D^{N+1}(\lambda)$. Each term in the expansion is obtained by choosing a non-zero product of elements from each column, with each row occurring exactly once. The product of the chosen elements lying above the subdiagonal has the form $f_I$ for some $I \in I$. The proof of (4.3) is completed by the following Lemma.

**Lemma 4.2.** The product of subdiagonal terms accompanying $f_I$ is

\[
\prod_{i \in r(I)} c_i = \prod_{i \in r(I)} h_i(\lambda) = H_I(\lambda).
\]

**Proof.** It is easy to adapt the proof from [Mus22a] Section 4. \qed
4.2. Some remarks on the adjoint action. We make $U_v(g)$ into a Hopf algebra, such that $U_v(b^\pm)$ are Hopf subalgebras by defining the coproduct $\Delta$ and antipode $S$ on $U_v(b^-)$ as in [Jan96] 9.13 (4)-(6) and (9). In particular for the generators of $U_v(b^-)$, we have

$$\Delta(F_\alpha) = F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha, \quad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha,$$

and

$$S(F_\alpha) = -K_\alpha^{-1}F_\alpha, \quad S(K_\alpha) = K_\alpha^{-1}.$$ 

Hence by (1.10) there is a Hopf algebra structure on $U_q(b^-)$ given by

$$\Delta(f_\alpha) = f_\alpha \otimes 1 + K_\alpha \otimes f_\alpha, \quad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha,$$

and

$$S(f_\alpha) = -K_\alpha^{-1}f_\alpha, \quad S(K_\alpha) = K_\alpha^{-1}.$$ 

Now for any Hopf algebra $H$ and $e \in H$, write $\Delta(e) = \sum e_1 \otimes e_2$ in Sweedler notation. Then define the map $\operatorname{ad}_e : H \rightarrow H$ by $\operatorname{ad}_e(x) = \sum e_1xS(e_2)$. If $x \in H = U_v(b^-)$ and $F = F_\alpha = f_\alpha, K = K_\alpha$ we have

$$\operatorname{ad}_F(x) = Fx - KxK^{-1}F = Fx - \sigma(x)F$$

where $\sigma$ is the automorphism of given by

$$\sigma(x) = KxK^{-1}.$$ 

Note that for $u \in U''$ we have $\sigma(u) = v^{(\alpha,\nu)}u$ by (1.13).

A short calculation (note that $1 - a_{\alpha\beta} \leq 2$) shows that if $\alpha \neq \beta$, then

$$\operatorname{ad}_F^{-1-a_{\alpha\beta}}(F_\beta) = \sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \left[ \begin{array}{c} 1 - a_{\alpha\beta} \\ i \end{array} \right]_v F_\alpha^{1-a_{\alpha\beta}} F_\beta F_\alpha^i,$$

and this is zero by (1.9). Using (1.13) it is easy to check the following

Lemma 4.3.

(a) $\delta = \operatorname{ad}_F$ is a $\sigma$-derivation, that is $\delta(xy) = \delta(x)y + \sigma(x)\delta(y)$.

(b) $v^2(\sigma \circ \delta) = \delta \circ \sigma$.

Adapting the proof of [Good92] Lemma 6.2 which uses a different definition of Gaussian binomial coefficients, $\delta$ satisfies the Leibniz identity.

$$\delta^n(ab) = \sum_{i=0}^{n} v^{i(n-i)} \left[ \begin{array}{c} n \\ i \end{array} \right]_v \sigma^i(\delta^{n-i}(a))\delta^i(b).$$

4.2.1. The induction set-up. Let $\alpha_i$ be the simple roots of $g$ as in (1.5). Then consider the chain of subalgebras

$$\mathfrak{t}_1 \subset \mathfrak{t}_2 \ldots \subset \mathfrak{t}_N = g,$$

such that $\mathfrak{t}_i \cong \mathfrak{sl}(i+1)$. Let $\alpha_1$ be the simple root of $\mathfrak{t}_1$, and for $i > 1$ let $\alpha_i$ be the simple root of $\mathfrak{t}_i$ that is not a root of $\mathfrak{t}_{i-1}$. In the next Subsection, we show how to construct Shapovalov elements (with $m = 1$ in (1.10)) for the highest root of $\mathfrak{t}_i$ by induction on $i$. We only need to show the last step of the induction, so to simplify we use the following notation for the rest of this Subsection. Let $\mathfrak{t} = \mathfrak{t}_{N-1}$. Then if $n^+ \oplus \mathfrak{h} \oplus n^-$ is the standard triangular decomposition of $g$, we set $m = \mathfrak{t} \cap n^-$.

\footnote{For a different Hopf algebra structure on $U_v(g)$, see [Jan96] Lemma 4.8.}
\[ \beta = \alpha_N, \ F = F_\beta \text{ and } \delta = \text{ad}_F. \]

We remark that in [Mus17] (3.8), the induction is set up in an equivalent way using the Weyl group. To explain this, let \( s_i \) be the reflection corresponding to the simple root \( \alpha_i \). If \( w_i = s_i \ldots s_3 s_2 \) for \( i \geq 2 \), then \( w_i \alpha_1 \) is the highest root of \( \mathfrak{f} \). In particular if \( \varpi = w_N \ldots w_1 \), then \( \sigma = 2\alpha_1 \) and \( \eta = w_N \alpha_1 \) are the highest roots of \( \mathfrak{f} \) and \( \mathfrak{g} \) respectively. Let \( \rho \) be the half-sum of the positive roots of \( \mathfrak{g} \), and define the dot action of \( W \) on \( \mathfrak{h}^* \) by \( u \cdot \mu = u(\mu + \rho) - \rho \), for \( u \in W \) and \( \mu \in \mathfrak{h}^* \). Note that

\[
(4.10) \quad \eta = \sigma + \beta,
\]

\[
(4.11) \quad u^{-1} \eta = \eta,
\]

so

\[
(4.12) \quad \Lambda = \{ \nu \in \mathcal{H}_{\alpha_1, m} \mid \text{for } i \in [N], (\nu + \rho, \alpha_i^\vee) \in \mathbb{N}\setminus\{0\} \}.
\]

Note that \( \Lambda \) is Zariski dense in \( \mathcal{H}_{\alpha_1, m} \). If \( \mu \in \varpi \cdot \Lambda \), then

\[
(4.13) \quad p = (\mu + \rho, \beta^\vee) = (u^{-1}(\mu + \rho), (u^{-1} \beta)^\vee)
\]

is a positive integer by \( (4.11) \).

**Lemma 4.4.** For \( x \in U(m) \), there exists \( n \geq 0 \) such that \( \delta^n x = 0 \).

**Proof.** Let \( N = \{ x \in U(m) \mid \delta^n x = 0 \text{ for some } n > 0 \} \). By \( (4.8) \), \( N \) is a subalgebra of \( U(m) \). Thus it suffices to show that \( N \) contains the generators of \( U(m) \). This follows from \( (4.7) \). \( \square \)

The next result follows by an easy computation. Compare [Jos95] 1.2.13 (1).

**Lemma 4.5.** For \( 1 \leq i \leq \ell - 1 \), we have

\[
v^i \left[ \begin{array}{c} \ell - 1 \\ i \end{array} \right]_{v} + v^{-(\ell - i)} \left[ \begin{array}{c} \ell - 1 \\ i - 1 \end{array} \right]_{v} = \left[ \begin{array}{c} \ell \\ i \end{array} \right]_{v}.
\]

Equivalently,

\[
v^{-i(\ell - 1 - i)} \left[ \begin{array}{c} \ell - 1 \\ i \end{array} \right]_{v} + v^{-(i+1)(\ell - i)} \left[ \begin{array}{c} \ell - 1 \\ i - 1 \end{array} \right]_{v} = v^{-i(\ell - i)} \left[ \begin{array}{c} \ell \\ i \end{array} \right]_{v}.
\]

**Lemma 4.6.** For \( k \geq 0 \) and \( z \in U \), we have

\[
\sigma(\text{ad}_F^k(z)) = v^{-2k}\text{ad}_F^k(\sigma(z)).
\]

**Proof.** We can assume that \( z \in U^n \). Then \( \text{ad}_F^k(z) \in U^{n-k\beta} \). Hence the result follows from \( (4.13) \) and the fact that \( (\beta, -\beta) = -2 \). \( \square \)

**Lemma 4.7.** For \( u \in U^\nu \), we have

\[
F^\ell u = \sum_{i=0}^{\ell} v^{-i(\ell - i)} \left[ \begin{array}{c} \ell \\ i \end{array} \right]_{v} v^{i(\beta, \nu)} \text{ad}_F^{\ell - i}(u) F^i.
\]
Proof. Since $\sigma^i(u) = v^{i(\beta, v)}u$, the given identity is the same as the identity

\begin{equation}
F^\ell u = \sum_{i=0}^{\ell} v^{-i(\ell-1)} \left[ \begin{array}{c} \ell \\ i \end{array} \right] \mathrm{ad}_{F}^{\ell-i}((\sigma^i(u))) F^i,
\end{equation}

and we prove this by induction on $\ell$. When $\ell = 1$, this follows by writing Equation (4.15) in the form

\begin{equation}
F u = \mathrm{ad}_{F}(u) + \sigma(u) F.
\end{equation}

Suppose by induction that

\begin{equation}
y := F^{\ell-1} u = \sum_{i=0}^{\ell-1} v^{-i(\ell-1-i)} \left[ \begin{array}{c} \ell-1 \\ i \end{array} \right] \mathrm{ad}_{F}^{\ell-1-i}((\sigma^i(u))) F^i.
\end{equation}

Then

\[
F^\ell u = F y = \sum_{i=0}^{\ell-1} v^{-i(\ell-1-i)} \left[ \begin{array}{c} \ell-1 \\ i \end{array} \right] F \cdot \mathrm{ad}_{F}^{\ell-1-i}((\sigma^i(u))) F^i 
\]

\[
= \sum_{i=0}^{\ell-1} v^{-i(\ell-1-i)} \left[ \begin{array}{c} \ell-1 \\ i \end{array} \right] \{ \mathrm{ad}_{F}^{\ell-i}((\sigma^i(u))) + \sigma(\mathrm{ad}_{F}^{\ell-1-i}((\sigma^i(u))) F) \} F^i 
\]

\[
= \sum_{i=0}^{\ell-1} v^{-i(\ell-1-i)} \left[ \begin{array}{c} \ell-1 \\ i \end{array} \right] \mathrm{ad}_{F}^{\ell-i}(\sigma^i(u)) F + \sum_{i=0}^{\ell-1} v^{-i(i+1)(\ell-1-i)} \left[ \begin{array}{c} \ell-1 \\ i-1 \end{array} \right] \mathrm{ad}_{F}^{\ell-i}(\sigma^i(u)) F^{i+1} 
\]

\[
= \mathrm{ad}_{F}(u) + \sum_{i=1}^{\ell-1} \{ v^{-i(\ell-1-i)} \left[ \begin{array}{c} \ell-1 \\ i \end{array} \right] + v^{-i(i+1)(\ell-1-i)} \left[ \begin{array}{c} \ell-1 \\ i-1 \end{array} \right] \} \mathrm{ad}_{F}^{\ell-i}(\sigma^i(u)) F^i + \sigma(u) F^\ell 
\]

\[
= \sum_{i=0}^{\ell} v^{-i(\ell-1)} \left[ \begin{array}{c} \ell \\ i \end{array} \right] \mathrm{ad}_{F}^{\ell-i}(\sigma^i(u)) F^i,
\]

where the third equality is due to (4.15) with $u$ replaced by $\mathrm{ad}_{F}^{\ell-1-i}(\sigma^i(u))$, the fourth follows from Lemma 4.6, the fifth follows a shift in the summation index for the second sum, the sixth is a rearrangement of terms, and the last equality follows from (4.13). \qed

**Lemma 4.8.** Use the same notation as Lemma 4.4. For $u \in U(m)^\nu$ there exists $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}$,

\[ F^\ell u = \sum_{i=0}^{k} v^{-i(\ell-1)\nu(\beta, v)} \left[ \begin{array}{c} \ell \\ i \end{array} \right] \mathrm{ad}_{F}(u) F^{\ell-i}. \]

**Proof.** By Lemma 4.3 there exists $k \in \mathbb{N}$ such that $\delta^{k+1}(u) = 0$. The rest follows from Lemma 4.7 where the summation index $i$ is replaced by $\ell - i$. \qed

**Corollary 4.9.** $S = \{ F^n \mid n \geq 0 \}$ is an Ore set in $U(n^-)$.

**Proof.** It is enough to prove the Ore condition for $F$ and an element $u \in U(m)$. To do this use Lemma 4.8 to adapt the proof of Lemma 3.1 in [Mus17]. \qed
4.3. The approach of Shapovalov. We need a basic fact about Verma modules for $U_q(\mathfrak{g})$.

**Theorem 4.10.** For $\lambda, \mu \in P$, every nonzero element of $\text{Hom}_{U_q(\mathfrak{g})}(M(\mu), M(\lambda))$ is injective.

**Proof.** The key point is that $U_q(\mathfrak{g})$ is a domain. [JL92] Proposition 4.10. Using this we can adapt the proof in [Mus12] Theorem 9.3.1. The result is implicit in [Jos95] 3.4.9.

Shapovalov elements in $U_q(\mathfrak{g})$ may be constructed inductively using the next Lemma, compare [Hum08] Section 4.13 or [Mus12] Lemma 9.4.3 for the non-quantum case. We adapt the notation to be consistent with our induction set-up.

**Lemma 4.11.** With the same notation as Subsection 4.2.7, set
\[
\mu = s_\beta \cdot \lambda, \quad \kappa = m\eta, \quad \kappa' = m\sigma.
\]
Assume that
\begin{itemize}
  \item[(a)] $p = (\mu + \rho, \beta^\vee) \in \mathbb{N}\setminus\{0\}$.
  \item[(b)] $\theta' \in U(\mathfrak{m})^{-\kappa'}$ is such that $\omega = \theta'v_\mu \in M(\mu)$ is a highest weight vector.
\end{itemize}
Then there is a unique $\theta \in U(\mathfrak{n}^-)^{-\kappa}$ such that
\begin{equation}
F^{p+m}\theta'v_\mu = \theta F^p v_\mu.
\end{equation}
(c) $\theta F^p v_\mu$ is a highest weight vector in $U_q(\mathfrak{g})F^p v_\mu = M(\lambda)$.

**Proof.** The proof in the classical case is based on the representation theory of $\mathfrak{sl}(2)$. Thus in the quantum case we use the representation theory of $U_v(\mathfrak{sl}(2))$. Suppose that $E, F, K \in U_v(\mathfrak{g})$ satisfy the relations of $U_v(\mathfrak{sl}(2))$, [Jan96] 1.1. By [Jos95], Lemma 4.2.6, $v_\lambda = F^p v_\mu$ is a highest weight vector with weight $\lambda$ in $M(\mu)$. Thus $U_q(\mathfrak{g})F^p v_\mu \cong M(\lambda)$ by Theorem 4.10. This accounts for the identification in (c).

Set $\omega = \theta'v_\mu$. We claim that
\begin{equation}
F^{p+m}\omega \in M(\lambda).
\end{equation}
By Lemma 4.8, there is a positive integer $\ell$ such that $F^\ell\omega \in U(\mathfrak{n}^-)F^p$, and hence $F^\ell\omega \in M(\lambda)$. We may assume that $\ell \geq (p+m)$. Note that $\omega$ has weight $\mu - \kappa'$ and
\begin{equation}
(\mu - \kappa', \beta^\vee) = p + m - 1.
\end{equation}
The first equality below is [Jan96] 1.3 (5).
\begin{align*}
EF^\ell\omega &= (F^\ell E + [\ell]_v F^{\ell-1}[K;1-\ell])\omega \\
&= [\ell]_v [1 - \ell + (\mu - \kappa', \beta)]_v F^{\ell-1}\omega \\
&= [\ell]_v [p + m - \ell]_v F^{\ell-1}\omega.
\end{align*}
The second equality follows from (4.18) and the third from (4.19). If $\ell > p + m$, then the coefficient of $F^{\ell-1}\omega$ in (4.20) is a non-zero Laurent polynomial in $v$. Thus $M(\lambda)$ contains $F^{\ell-1}\omega$. Repeating this argument gives (4.18). Since $M(\lambda) = U(\mathfrak{n}^-)v_\lambda$ is a free $U(\mathfrak{n}^-)$-module, there exists an element $\theta \in U(\mathfrak{n}^-)^{-\kappa}$ such that (4.17) holds. Since $F$ is not a zero divisor in $U(\mathfrak{n}^-)$, the element $\theta$ satisfying (4.17) is unique.

To show (c), write $v_\lambda = F^p v_\mu$. By Theorem 4.10, the submodule of $M(\mu)$ generated by $v_\lambda$ is isomorphic to $M(\lambda)$. We must show that $E_\alpha v_\lambda = 0$ for every simple root $\alpha$. 
If $\alpha \neq \beta$ this follows since $[E_\alpha, F_\beta] = 0$. If $\alpha = \beta$ we have by (4.17), $F^{p+m} \omega = \theta \nu \lambda$ and taking $\ell = p + m$ in (4.20), this is zero because $[p + m - \ell] \nu = 0$. □

4.4. Comparison of two approaches to Shapovalov elements. Recall that the subdiagonal entries in the matrix (4.2) are $-c_i$, where $c_i = h_i(\lambda)$, see (3.2).

**Lemma 4.12.** Assume $(\lambda + \rho, \eta) = 1$ and $(\lambda + \rho, \beta) = -p$ as in Lemma (4.11). Then

(a) $c_{N-1} = h_{N-1}(\lambda) = -q^{-1}v^{-p}[p + 1]v$.

(b) For $i \in [N - 2]$, we have $c_i = h_i(\mu)$.

**Proof.** From (4.10) and the hypotheses, $(\lambda + \rho, \sigma) = p + 1$. Therefore using (3.1) and (3.2) we obtain (a).

If $i \in [N - 2]$, $c_i$ in depends only on the value of $(\lambda + \rho, \sigma_i)$. Since $(\beta, \sigma_i) = 0$, we have $(\mu + \rho, \sigma_i) = (\lambda + \rho, \sigma_i)$. Hence $h_i(\mu) = h_i(\lambda)$. □

We need an analog of (2.2) for $U_q(sl(N))$. Thus define

\[
\mathbb{J} = \{J \subseteq [N] \mid 1, N \in J\}.
\]

For $J \in \mathbb{J}$, set $f_J$ as in (2.24). If $J$ is as in (2.3), with $j_0 = 1$ and $j_{s+1} = N$, define $J_1, J_2 \in \mathbb{J}$ by

\[
J_1 = \{j_0, j_1, \ldots, j_s, N, N + 1\} \quad \text{and} \quad J_2 = \{j_0, j_1, \ldots, j_s, N + 1\}.
\]

Note that by (4.11),

\[
\deg f_J = c_N - c_1.
\]

By (1.4), (4.13) and (4.5), with $F = f_{N,N+1}$,

\[
\text{ad}_F(f_i, N) = f_{N,N+1}f_i, N - v^i_f f_{N,N+1} = -q^i f_{i,N+1}.
\]

More generally, we have the part (a) of the following Lemma. Part (b) is a similar relation.

**Lemma 4.13.**

(a) $\text{ad}_F(f_J) = -q^i f_{J_2}$.

(b) $f_J F = f_{J_1}$.

**Proof.** (a) Induct on $s = |J|$. The case $s = 1$ follows from (4.23). Suppose $s > 1$ and write $f_J = f_{j_0, j_1} f_{j', J'}$ and $f_{J_2} = f_{j_0, j_1} f_{j', J'_2}$. By induction, $\text{ad}_F(f_{j'}) = -q^i f_{J_2}$. Left multiplication of it by $f_{j_0, j_1}$ clearly is $f_{j_0, j_1} \text{ad}_F(f_{j'}) = -q^i f_{J_2}$. The left side here is $f_{j_0, j_1} \text{ad}_F(f_{j'}) = f_{j_0, j_1}(F f_{j'} - K f_{j'} K^{-1} F)$, where $K = K_{N,N+1}$. Since $\{j_0, j_1\}$ and $\{N, N + 1\}$ are disjoint, $f_{j_0, j_1}$ commutes with both $F$ and $K$, and the last expression becomes $F f_{j_0, j_1} f_{j'} - K f_{j_0, j_1} f_{j'} K^{-1} F = F f_{j} - K f_{j} K^{-1} F = \text{ad}_F(f_J)$, as needed. Part (b) is proved in a similar way. The case $s = 1$ of the induction holds by the definition of $f_{J_1}$. □

If $u = f_J$, we can take $k = 1$ in Lemma 4.8. Note that $(\beta, \deg f_J) = 1$ by (4.22). Hence by Lemmas 4.13 and 4.12,

\[
F^{p+1} f_J = \sum_{i=0}^{1} v^{-i(p+1-i)} v^{(p+1-i)(\beta,v)} \left[ \begin{array}{c} p + 1 \\ i \end{array} \right] v^i_{\text{ad}_F(f_J)} F^{p+1-i}.
\]

\[
= v^{p+1} f_J F + [p + 1] \nu \text{ad}_F(f_J)) F^p
\]

\[
= v^{p+1} (f_J - q^{-1} v^{-p}[p + 1] v f_{J_2}) F^p
\]

\[
(4.24) \quad = v^{p+1} (f_J + c_{N-1} f_{J_2}) F^p.
\]
Thus from (4.24), (4.25) and (4.26) follows that

$$ \sum_{J \in \mathfrak{H}} H_J(\mu) \mathcal{f}_{J} = \operatorname{det} \mathcal{D}_1 = \operatorname{det} \mathcal{D}^N(\mu). $$

Now $f_{J_2}$ is obtained from $f_J$ by replacing the last factor $f_{J_s,N}$ by $f_{J_s,N+1}$. Since $D_2$ is obtained from $D_1$ by replacing the entries $f_{i,N}$ in the last column by $f_{i,N+1}$, it follows that

$$ \sum_{J \in \mathfrak{H}} H_J(\mu) f_{J_2} = \operatorname{det} D_2. $$

Thus from (4.24), (4.25) and (4.26),

$$ F^{p+1} \operatorname{det} \mathcal{D}^N(\mu) = F^{p+1} \sum_{J \in \mathfrak{H}} H_J(\mu) \mathcal{f}_J $$

$$ = v^{p+1} \sum_{J \in \mathfrak{H}} H_J(\mu)(f_J F + c_{N-1} f_{J_2}) F^p $$

$$ = v^{p+1}(\operatorname{det} D_1 f_{N,N+1} + c_{N-1} \operatorname{det} D_2) F^p $$

$$ = v^{p+1} \operatorname{det} \mathcal{D}^{N+1}(\lambda) F^p, $$

where the last equality uses cofactor expansion along the last row of $\mathcal{D}^{N+1}(\lambda)$. \qed

For a Shapovalov element, there is a normalization condition that a certain coefficient in $\mathfrak{H}$ is equal to one. No such condition is made on the elements $\theta'$ and $\theta$ in Lemma 4.11. From (4.24), we can easily see what these coefficients are in Theorem 4.14. In (4.21), take $J = [N] \in \mathfrak{H}$. Now $f_J = f_{[N]}$ and $f_{J_1} = f_{[N+1]}$ occur with coefficient 1, in the Shapovalov elements $\theta_\sigma$ and $\eta_\eta$ respectively and we have,

$$ F^{p+1} f_J = v^{p+1} f_{J_1} F^p. $$

This means that if we take $m = 1$ in (4.17) and define $\theta = v^{-p-1} \theta$, we have

$$ F^{p+1} \theta \mathcal{v}_\mu = v^{p+1} \theta F^p \mathcal{v}_\mu = v^{p+1} \theta \mathcal{v}_\lambda. $$

This means that if $\theta'(\mu)$ is the evaluation of the Shapovalov element for $\sigma$, then $\theta(\lambda)$ is the evaluation of the Shapovalov element for $\eta$, in accordance with Theorems 4.13 and 4.14.

4.5. **A uniform construction for Shapovalov elements in $U(\mathfrak{b}^-)$**. Up to this point we have only evaluated a Shapovalov element $\theta_{\gamma,m}$ at points $\lambda \in \mathcal{H}_{m,\gamma}$. However, to study the behavior of powers of Shapovalov elements, we need to evaluate at arbitrary points, $\lambda \in \mathfrak{b}^*$. However some care must be taken since the Shapovalov element $\theta_{\gamma,m}$ is only defined modulo the ideal $U(\mathfrak{b}^-)\mathfrak{I}(\mathcal{H}_{\gamma,m})$.

To explain the problem we briefly review the construction of Shapovalov elements from [Mus12] 9.4, adapted to the quantum case. First write the root $\gamma$ in the form $\gamma = w\beta$ with $\beta$ a simple root and $w \in W$. Next by induction on the length of

\[ \theta(\lambda) = \sum_{J \in \mathfrak{H}} H_J(\mu) f_{J_2}. \]
the $w$, we construct $\theta_{\gamma,m} \in U_q(n^-)^{-m\gamma}$ such that $\theta_{\gamma,m}v_\lambda$ is a highest weight vector for $M(\lambda)$ whenever $\lambda \in \mathcal{H}_{\gamma,m}$, see the remarks after (1.17). Then we fix a (non-unique) lifting of $\theta_{\gamma,m}$ to an element $\Theta_{\gamma,m} \in U_q(b^-)^{-m\gamma} = U_q(n^-)^{-m\gamma} \otimes \mathcal{H}$ such that $\Theta_{\gamma,m}v_\lambda = \Theta_{\gamma,m}(\lambda)v_\lambda = \theta_{\gamma,m}v_\lambda$, for all $\lambda \in \mathcal{H}_{\gamma,m}$.

The problem is that if $\lambda \in \mathcal{H}_{\gamma,m}$, then $\Theta_{\gamma,m}v_\lambda$ is a highest weight vector of weight $\lambda - m\gamma$, but $\lambda - m\gamma \notin \mathcal{H}_{\gamma,m}$, and thus we need to evaluate $\Theta_{\gamma,m}$ at points that are not in $\mathcal{H}_{\gamma,m}$. Thus the non-uniqueness of the lifting of $\theta_{\gamma,m}$ presents a potential problem. To resolve this issue, we give a uniform inductive construction of $\Theta_{\gamma,m}$, following [Mus17] 5.1.2 or [Mus22b]. This depends on a specific choice of $\beta$ and a shortest length expression for $w$.

With the notation of Lemma 4.4, let $S$ be the multiplicative subset $S = \{F^n|n \geq 0\}$ as in Corollary 4.3 and let $U_S$ be the corresponding Ore localization. We define generalized conjugation operators for non-isotropic roots. Lemma 4.8 leads to a formula for conjugation by $F^r$ which extends to the 1-parameter family of automorphism $\Psi_r, r \in \mathfrak{k}$ of $U_S$, given by

\begin{equation}
\Psi_r(u) = \sum_{i \geq 0} e^{-i(r-i)} \left[ \sum_{i} e^{i(\beta,\nu)} \text{ad}_F^i(u) F^{r-i} \right],
\end{equation}

for $u \in U^\nu$. Note that for $r \in \mathbb{N}$ and $u \in U$ we have $\Psi_r(u) = F^r u F^{-r}$ by Lemma 4.7.

We extend $\Psi_r$ to an automorphism of $U = U(n_1)$ by setting $\Psi_r(F) = F$. Then,

\begin{equation}
\Psi_r(FuF^{-1}) = \Psi_{r+1}(u).
\end{equation}

As noted earlier, the value of $\Psi_r(u)$ for $r \in \mathfrak{k}$ is determined by its values for $r \in \mathbb{N}$. In particular the two previous equations hold for all $r \in \mathfrak{k}$. To stress the dependence of $\Psi_r$ on the root $\beta$ we sometimes denote it by $\Psi_r^\beta$.

Because of the polynomial nature of the coefficients in (4.29), this implies that $\Psi_r$ is an automorphism of $U_S$ for all $r \in \mathfrak{k}$. Now (4.17) implies that $\Psi_r(e_{-\alpha} \theta_{\gamma,m}) = \theta_{\gamma,m}$ if $\mu \in \mathcal{H}_{\gamma,m}$ and $r = (\mu + \rho, \alpha^\vee)$.

**Lemma 4.15.** Suppose that $\mu \in \mathcal{H}_{\gamma',m}$, $\alpha$ is a simple root, $\gamma = s_\alpha \gamma'$, $(\gamma, \alpha^\vee) = 1$ and $\lambda = s_\alpha \cdot \mu$. Assume $r = (\mu + \rho, \alpha^\vee)$. Then we have

\begin{equation}
\theta_{\gamma,m}(\lambda) = \Psi_r(F^{-m} \theta_{\gamma',m}(\mu)).
\end{equation}

**Proof.** If $r$ is a positive integer, this follows from Lemma 4.11. Since both sides depend polynomially on $\mu$ we have the result.

We propose (4.31) as the definition of $\theta_{\gamma,m}(\lambda)$ for arbitrary points, $\lambda = s_\alpha \cdot \mu \in \mathfrak{b}^*$, without the assumption that $\mu \in \mathcal{H}_{\gamma',m}$. Thus $\theta_{\gamma,m}$ is now a function from $\mathfrak{h}^*$ to $U(n^-)^{-m\gamma}$ which evaluates correctly on $\lambda \in \mathcal{H}_{\gamma,m}$ and has $H_{\pi^0} = 1$ as the coefficient of $e_{-\pi^0}$. Thus the Shapovalov elements $\Theta_{\gamma,m} \in U(\mathfrak{b})^{-m\gamma}$, are defined inductively by

\begin{equation}
\Theta_{\gamma,m}(\lambda) = \Psi_r(e_{-\alpha} \Theta_{\gamma',m}(\mu)),
\end{equation}

where $r = (\mu + \rho, \alpha^\vee)$. Now we have a uniform construction of such elements. In the case when $\beta$ is a simple root, we set $\Theta_{\beta,m} = F_{-\beta}^{-m} \in U(\mathfrak{b})^{-m\beta}$. 
At first glance it appears that we have only $\Theta_{\gamma,m}(\lambda) \in U_S$, where $S$ is the Ore set from Lemma 4.9. However we have

**Lemma 4.16.** For all $\lambda \in \mathfrak{h}^*$, and $j > 0$, we have $\Theta_{\gamma,j}(\lambda) \in U$.

**Proof.** It is easy to adapt the proof from [Mus17] Corollary 5.6.\hfill \square

### 4.6. Powers of Shapovalov elements.

**Theorem 4.17.** If $\lambda \in \mathcal{H}_{\gamma,m}$, then

\[(4.33)\quad \Theta_{\gamma,m}(\lambda) = \Theta_{\gamma,1}(\lambda - (m-1)\gamma) \cdots \Theta_{\gamma,1}(\lambda - \gamma)\Theta_{\gamma,1}(\lambda).\]

Equivalently if $v_\lambda$ is a highest weight vector of weight $\lambda$ in the Verma module $M(\lambda)$ and $\lambda \in \mathcal{H}_{\gamma,m}$, we have $\Theta_{\gamma,m}v_\lambda = \Theta_{\gamma,1}^{\mu_\lambda}v_\lambda$. If $\lambda \in \mathcal{H}_{\gamma,m}$ this is a highest weight vector in $M(\lambda)$.

**Proof.** Clearly (4.33) holds if $\gamma$ is a simple root. Suppose that

\[(4.34)\quad \lambda = s_\alpha \cdot \mu, \quad \gamma = s_\alpha \gamma', \quad p = (\mu + \rho, \alpha^\vee), \quad q = (\gamma, \alpha^\vee).

For $i = 0, \ldots, m - 1$ we have

\[(\mu + \rho - i\gamma', \alpha^\vee) = p + i,

so by the inductive definition (4.30) and (4.32),

\[(4.35)\quad \Theta_{\gamma,1}(\lambda - i\gamma) = \Psi_{p+i}(F\Theta_{\gamma',1}(\mu - i\gamma')) = \Psi_p(F^{(i+1)}\Theta_{\gamma',1}(\mu - i\gamma')F^{-i}).\]

Now using the corresponding result for $\Theta_{\gamma',m}(\mu)$ we have

\[
F^m\Theta_{\gamma',m}(\mu) = F^m\Theta_{\gamma',1}(\mu - (m-1)\gamma') \cdots \Theta_{\gamma',1}(\mu - \gamma')\Theta_{\gamma',1}(\mu)
= F^{(m-1)}(F\Theta_{\gamma',1}(\mu - (m-1)\gamma'))F^{-(m-1)} \times \times F^{(m-2)}(F\Theta_{\gamma',1}(\mu - (m-2)\gamma'))F^{-(m-2)} \times \times \times \times F^{(i+1)}\Theta_{\gamma',1}(\mu - i\gamma')F^{-i} \times \times \times \times F(F\Theta_{\gamma',1}(\mu - \gamma'))F^{-1} \times F\Theta_{\gamma',1}(\mu).
\]

The result follows by applying the automorphism $\Psi_p$ to both sides and using (4.32) and (4.35).\hfill \square

**References**

[Good92] Goodearl, K. R., *Prime ideals in skew polynomial rings and quantized Weyl algebras*, J. Algebra **150** (1992), no. 2, 324–377. MR 1176901, https://doi.org/10.1016/S0021-8693(92)80036-5

[Hum72] Humphreys, James E., *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York, 1972. MR 0323842

[Hum08] ———, *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, Graduate Studies in Mathematics, Vol. **94**, American Mathematical Society, Providence, RI, 2008. ISBN 978-0-8218-4678-0, MR 2428237

[Jan96] Jantzen, J. C., *Lectures on Quantum Groups*, Graduate Studies in Mathematics, Vol. **6**, American Mathematical Society, 1996. ISBN 0-8218-0478-2, MR 1359532

[Jim86] Jimbo, M., *A $q$-analogue of $U(gl(N+1))$, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), no. 3, 247–252. MR 0841713
[JL92] Joseph, A., Letzter, G., *Local finiteness of the adjoint action for quantized enveloping algebras*, J. Algebra 153 (1992), no. 2, 289–318. MR 1198203, https://doi.org/10.1016/0021-8693(92)90157-H

[Jos95] Joseph, A., *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol. 29, Springer-Verlag, Berlin, 1995. ISBN 3-540-57057-8, MR 1315966, https://doi.org/10.1007/978-3-642-78400-2

[Mud16] Mudrov, A., *R-matrix and inverse Shapovalov form*, J. Math. Phys. 57 (2016), no. 5, 051706, 10pp. MR 3503949, https://doi.org/10.1063/1.4950894

[Mud22] Mudrov, A., *Factorization of Shapovalov elements*, preprint, https://arxiv.org/abs/2203.02813

[Mus12] Musson, I. M., *Lie Superalgebras and Enveloping Algebras*, Graduate Studies in Mathematics, Vol. 131, American Mathematical Society, Providence, RI, 2012. ISBN 978-0-8218-6867-6, MR 2906817

[Mus17] ________, *Shapovalov elements and the Jantzen sum formula for contragredient Lie superalgebras*, preprint. https://arxiv.org/abs/1710.10528

[Mus22a] ________, *Explicit expressions for Shapovalov elements in Type A*, preprint, https://arxiv.org/abs/2203.02813

[Mus22b] ________, *The construction of modules with prescribed characters for contragredient Lie superalgebras*, in preparation, 2022.

[Sha72] Shapovalov, N. N., *A certain bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra* (Russian), Funkcional. Anal. i Priložen 6 (1972), no. 4, 65–70. MR0320103

[Yam89] Yamane, H., *A Poincaré-Birkhoff-Witt theorem for quantized universal enveloping algebras of type AN*, Publ. RIMS. Kyoto Univ. 25 (1989), 503–520. MR 1018513

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