A $C^\infty$ CLOSING LEMMA ON TORUS

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ABSTRACT. Asaoka & Irie [3] recently proved a $C^\infty$ closing lemma of Hamiltonian diffeomorphisms of closed surfaces. We reformulated their techniques into a more general perturbation lemma for area-preserving diffeomorphism and proved a $C^\infty$ closing lemma for area-preserving diffeomorphisms on a torus $T^2$ that is isotopic to identity, i.e., we show that the set of periodic orbits is dense for a generic diffeomorphism isotopic to identity area-preserving diffeomorphism on $T^2$. The main tool is the flux vector of area-preserving diffeomorphisms which is, different from Hamiltonian cases, non-zero in general.

1. INTRODUCTION

The $C^r$ closing lemma is one of the fundamental problems in dynamical systems. It goes back to Poincaré in his study of the restricted three body problem. The problem asks whether the set of periodic points of a typical symplectic or volume preserving diffeomorphism is dense on a compact manifold. Smale [16] listed the problem as one of the mathematical problems for the 21st century at the end of the last century.

Let $M$ be a compact manifold, equipped with a symplectic form $\omega$, a closed non-degenerate differential 2-form on $M$. If $M$ is an orientable surface, then we can take $\omega$ to be an area form. Let $\text{Diff}^r(M, \omega)$ be the set of $C^r$ diffeomorphisms on $M$ that preserves $\omega$, where $r = 1, 2, \ldots, \infty$. For $f \in \text{Diff}^r(M, \omega)$, denote $\mathcal{P}(f)$ as the set of periodic points of $f$. A set in a topological space is said to be residual if it is the intersection of countably many open and dense subsets. A property for $C^r$ diffeomorphisms of $M$ is said to be generic if there is a residual subset $R$ in $\text{Diff}^r(M)$ such that the property holds for all $f \in R$. The $C^r$ closing lemma conjectures that for generic $C^r$ area-preserving diffeomorphisms on compact manifolds, the set of all periodic points is dense.

Conjecture 1.1 ($C^r$ closing lemma). There exists a residual subset $R \subset \text{Diff}^r(M, \omega)$ such that for all $g \in R$, the set $\mathcal{P}(g)$ is dense in $M$.

A seemingly weaker, but equivalent statement is the following

Conjecture 1.2. For any open subsets $U \subset M$ and $V \subset \text{Diff}^r(M, \omega)$, there exists $g \in V$ and $x \in U$ such that $x \in \mathcal{P}(g)$.

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To see the equivalence, notice that by the Baire Category Theorem, in a complete metric space, such is $\text{Diff}^r(M, \omega)$, any intersection of countably many open dense sets is again dense, hence the residual set $R$ is dense in $\text{Diff}^r(M, \omega)$. Therefore any open $V \subset \text{Diff}^r(M, \omega)$ contains an element in $R$, therefore the second conjecture follows from the first one.

On the other hand, if the second conjecture is true, to find the residual set $R$ in the first conjecture, let $\{U_i \mid i = 1, 2, \ldots\}$ be a countable basis of $M$. For each $i \in \mathbb{N}$, define

$$H_i = \{\psi \in \text{Diff}^r(M, \omega) \mid \psi \text{ has a non-degenerate periodic point in } U_i\}$$

Then $H_i$ is open. By the second conjecture, $H_i$ is also dense in $\text{Diff}^r(M, \omega)$. So $R = \bigcap_i H_i$ is a residual subset in $\text{Diff}^r(M, \omega)$ with the desired property. □

The $C^r$ closing lemma conjecture was in essence made by Poincaré [12], and it remains open up to now in general cases. However, for $r = 1$, i.e. the $C^1$ closing lemma, it has been completely settled, for symplectic and volume-preserving diffeomorphisms of compact manifolds (cf. Pugh [13], Liao [10], Pugh and Robinson [14]). The $C^r$ closing lemma for $r > 1$ has eluded mathematicians for generations. However, important progress has been made. For example, the Anosov closing lemma for uniformly hyperbolic diffeomorphisms (Anosov [1,2]) and non-uniformly hyperbolic diffeomorphisms (Pesin [11]), a class of partially hyperbolic diffeomorphisms (Xia and Zhang [18]) and other cases. For more details, we refer to Saghin and Xia [15].

Following recent exciting development in contact geometry (cf. Hutchings [7] [8]), an important and remarkable progress has been made for $C^r$ closing lemma for Hamiltonian diffeomorphisms on surfaces. A diffeomorphism is said to be Hamiltonian if it is a time-1 map of a 1-periodic time-dependent Hamiltonian system. Asaoka and Irie [3] showed that a generic $C^1$, $r = 1, 2, \ldots, \infty$, Hamiltonian diffeomorphism of a closed surface, has a dense set of periodic orbits. This result comes as an interesting application of spectral invariants in embedded contact homology, known as $ECH$ invariants.

Our goal is to extend these results to a more general area preserving diffeomorphism that is not coming from periodic Hamiltonian flow. As a first step in this direction, we restrict ourselves to torus $\mathbb{T}^2$ and to area preserving diffeomorphisms that are isotopic to identity. We show that the $C^\infty$ closing lemma holds in this case. More precisely, let $\text{Symp}^\infty_0(\mathbb{T}^2, \omega)$ be the set of all $C^\infty$ area-preserving diffeomorphism on $\mathbb{T}^2$ that is isotopic to identity, we have

**Theorem 1.3.** There exists a residual subset $R \subset \text{Symp}^\infty_0(\mathbb{T}^2, \omega)$ such that for all $g \in R$, the set $\mathcal{P}(g)$ is dense on $\mathbb{T}^2$.

We remark that the set of Hamiltonian diffeomorphisms is a proper subset of $\text{Symp}^\infty_0(\mathbb{T}^2, \omega)$, characterized by zero flux, or preservation of center of gravity in
Arnold’s term. There are large classes of area-preserving maps that are not Hamiltonian. We will provide more details on this later, but first,

**Example 1.4.** Using the canonical chart $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ and the standard symplectic form $\omega = dp \wedge dq$. Consider on $(T^2, dp \wedge dq)$ the following simple system:

$$
\begin{align*}
\dot{p} &= \alpha \\
\dot{q} &= \beta
\end{align*}
$$

The flow is area-preserving, but it is Hamiltonian only when $\alpha = \beta = 0$. Similarly, symplectomorphism $f(p, q) = (p + \alpha, q + \beta)$ is Hamiltonian only if $(\alpha, \beta) \in \mathbb{Z}^2$.

To prove our theorem, we need a more general perturbation technique in the context of the area-preserving maps. We state this new perturbation lemma as Theorem 3.1. The advantage of this theorem is that one can directly verify various conditions from the area-preserving maps without going through underlying contact flow. This perturbation lemma can be useful in many other settings.

### 2. A REVIEW OF RECENT RESULTS

In this section we review Asaoka and Irie’s result [3] for $C^\infty$ Hamiltonian diffeomorphisms of closed surfaces. The techniques of their proof is essential for our results.

Consider a compact, oriented surface $M$ and a symplectic form $\omega$ on it. Let $\text{Diff}^\infty(M, \omega)$ denote the group of $C^\infty$ diffeomorphisms on $M$ that preserves $\omega$, equipped with the $C^\infty$ topology. For any time-periodic smooth function on $M$, $H_t : S^1 \times M \to \mathbb{R}$, where $S^1 = \mathbb{R} / \mathbb{Z}$, the associate Hamiltonian vector fields is $X_{H_t}$, satisfying

$$
\iota_{X_{H_t}} \omega = \omega(X_{H_t}, \cdot) = -dH_t.
$$

We denote with

$$
\phi^t_{H_t} : \mathbb{R} \times M \to M, \quad (t, x) \mapsto \phi^t_{H_t}(x)
$$

the flow generated by the vector field $X_{H_t}$. Notice $\phi^t_{H_t}$ is a one-parameter subgroup in the group $\text{Diff}^\infty(M, \omega)$. $\phi^1_{H_t}$ is the Poincaré map on the cross section $\{0\} \times M$. The fixed points of $\phi^1_{H_t}$ correspond exactly to 1-periodic solutions of the vector field $X_{H_t}$.

A symplectic diffeomorphism $f \in \text{Diff}(M, \omega)$ is said to be *Hamiltonian* if there is time-periodic Hamiltonian function $H_t : S^1 \times M \to \mathbb{R}$ such that $f = \phi^1_{H_t}$. We denote the set of all Hamiltonian diffeomorphisms by $\text{Ham}(M, \omega)$. Obviously

$$
\text{Ham}(M, \omega) \subset \text{Diff}(M, \omega),
$$

the inclusion is proper for most manifold $M$. 
For simplicity, we will restrict ourselves to $C^\infty$ diffeomorphisms. Results for $C^r$ topology, $r \in \mathbb{N}$, follows. From now on in our notation, both $\text{Diff}(M, \omega)$ and $\text{Ham}(M, \omega)$ are equipped with $C^\infty$ topology on $\text{Diff}(M)$.

Hamiltonian diffeomorphisms enjoy some fine structures critical to variational techniques in symplectic geometry. These fine structures give rise to many interesting dynamical properties. An important example is the Arnold conjecture for Hamiltonian diffeomorphisms. Another good example is the following Asaoka & Irie’s $C^\infty$ closing lemma for Hamiltonian diffeomorphisms on compact surfaces [3].

**Theorem 2.1 (Asaoka & Irie).** Let $M$ be any closed, oriented surface with an area form $\omega$, $g \in \text{Ham}(M, \omega)$. Let $P(g)$ be the set of all periodic points of $g$. For any nonempty open set $U \subset M$, there exists a sequence $(g_j)_{j \geq 1}$ in $\text{Ham}(M, \omega)$ such that $P(g_j) \cap U \neq \emptyset$ and $\lim_{j \to \infty} g_j = g$.

The key idea in the proof is to embed the area-preserving diffeomorphism on the surface into a three dimensional contact flow, where one can apply Irie’s perturbation lemma for contact flow [9].

Let $(M, \lambda)$ be a contact manifold, where $\lambda$ denotes the contact form with the contact distribution $\xi_\lambda := \ker \lambda$. The Reeb vector field $R_\lambda$ is defined by $\lambda(R_\lambda) = 1$, $d\lambda(R_\lambda, \cdot) \equiv 0$. Let $P(M, \lambda)$ denote the set of periodic orbits of $R_\lambda$: $P(M, \lambda) := \{ \gamma : \mathbb{R}/T_\gamma \mathbb{Z} \to M \mid T_\gamma > 0, \dot{\gamma} = R_\lambda(\gamma) \}$. Here $T_\gamma$ is the period of orbit $\gamma$ of the vector field $R_\lambda$.

**Lemma 2.2 (Irie’s perturbation lemma).** Let $(M, \lambda)$ be a closed contact 3-manifold, For any $h \in C^\infty(M, \mathbb{R} \geq 0) \setminus \{0\}$, there exist $t \in [0, 1]$ and $\gamma \in P(M, (1 + th)\lambda)$ which intersects $\text{supp}(h)$.

Here $\text{supp}(h) \subset M$ is the support of $h$. The proof of this lemma is based on embedded contact homology (ECH), which is defined in terms of periodic orbits of the Reeb flow. The key property here for ECH and its spectral invariants is that they recover the contact volume (Gardiner, Hutchings and Ramos [6]). This remarkable result implies that one can perturb the structure of periodic orbits by simply perturb contact volume. It is to see that adding $h\lambda$ with $h > 0$ to contact form $\lambda$, one can increase the contact volume.

Irie’s generic density theorem for periodic Reeb orbits for contact flow follows easily from the above perturbation lemma [9].

### 3. A NEW PERTURBATION LEMMA

Next, we would like to introduce a perturbation lemma in the context surface diffeomorphisms, without going through contact flows. We will use quantities and conditions directly verifiable, and computable, in area-preserving maps. We will use this perturbation theorem to prove our main theorem.
Let \( U \) be a simply connected region in \( M \), homeomorphic to a closed disk in \( \mathbb{R}^2 \), with the proper radius such that \( U \) has the same area as the disk. We may assume, for simplicity and without loss of generality, that \( U \) is such a disk. Let \( h \) be a \( C^\infty \) area-preserving diffeomorphism of \( U \) such that \( h \) is identity for all points near the boundary of \( U \). Since \( h \) is area-preserving, the differential 1-form \( h^*(ydx) - ydx \) is exact, therefore, there is a function \( g_h \) on the disk \( U \) such that
\[
dg_h = h^*(ydx) - ydx
\]
The function \( g_h \) is unique up to an integral constant, we fix the constant such that \( g_h = 0 \) on the boundary of \( U \).

The function \( g_h \) is called the action for the diffeomorphism \( h \). We define the total action by integrating \( g_h \) over \( U \):
\[
A(h) = \int_U g_h dxdy.
\]
Let \( \lambda \) be another 1-form such that \( d\lambda = \omega \), then \( \lambda - ydx = dS \) for some real valued function \( S \) on \( U \). Let \( g'_h, A'(h) \) be the action function and total action, respectively, on \( U \) corresponding to \( \lambda \), with the same property that \( g'_h = 0 \) on \( \partial D \).

Then,
\[
g'_h(x) - g_h(x) = S(f(x)) - S(x)
\]
and
\[
A'(h) - A(h) = \int_U S(h(x))\omega - \int_U h(x)\omega = 0.
\]
The last equality is due to the fact that \( h \) preserves \( \omega \). We conclude that the total action \( A(h) \) is independent the choice of 1-form \( ydx \).

Let \( e : U \to M \) be an area-preserving embedding of \( U \) into an closed surface \( M \). We can extend \( h \), by abusing the notation, to an area-preserving diffeomorphism of \( M \), such that \( h = \text{id} \) outside of \( U \). Clearly, \( h \) is Hamiltonian and the action \( A(h) \) is independent of the embedding.

We can now state our perturbation lemma.

**Theorem 3.1 (A perturbation lemma).** Let \( M \) be a closed, oriented surface with an area-form and let \( f \) be Hamiltonian diffeomorphism on \( M \).

- Let \( U_1, U_2, \ldots, U_k \) be finite disjoint closed disks on \( M \).
- For all \( t \in [0, 1] \), Let \( h^t_1, h^t_2, \ldots, h^t_k \) be families of area-preserving diffeomorphisms supported on \( U_1, U_2, \ldots, U_k \), respectively. Assume that \( h^0_1, h^0_2, \ldots, h^0_k \) are all identity maps on respective domains.
- For each \( t \in [0, 1] \), let \( A(h^t_1), A(h^t_2), \ldots, A(h^t_k) \) be their respective total actions.
- Let \( U = U_1 \cup U_2 \cup \ldots \cup U_k \), for any \( t \in [0, 1] \), let
\[
h^t = h^t_1 \circ h^t_2 \circ \ldots \circ h^t_k : U \to U
\]
be a family of area-preserving diffeomorphisms.
Suppose that at \( t = 1 \),
\[
A(h^1_1) + A(h^1_2) + \ldots + A(h^1_k) \neq 0
\]
Then there exists \( t \in [0, 1] \), such that
\[
h^t \circ f : M \to M
\]
has a periodic point in \( U \).

The proof of this theorem follows the same idea as in Asaoka-Irie theorem [3]. One needs to note that the total volume in contact manifold is exactly the total action for area-preserving map. As we mentioned earlier, ECH invariants, defined by periodic orbits of the map, recover the volume of the contact manifold. Any change in contact volume, or the total action for surface diffeomorphisms, forces changes in the set of periodic points.

We need to be more precise on action function. An area-preserving diffeomorphism on a surface compact \( M \) with area form \( \omega \) is said to be exact if there is a 1-form \( \lambda \) with \( d\lambda = \omega \) such that \( f^*\lambda - \lambda = dg \) for some real valued function \( g \) on \( M \). This function \( g \) is called the action. Once \( \lambda \) is fixed, it is defined up to an integration constant. This constant can be fixed by assigning a specific value at any given point. The total action is defined to be
\[
A(f) = \int_M g \omega.
\]

The total action depends on the choice of 1-form \( \lambda \), through the first cohomology of the manifold \( M \).

Fix a 1-form \( \lambda \), a useful fact is that the action is additive for compositions of exact symplectic diffeomorphisms.

**Proposition 3.2.** Let \( f_1 \) and \( f_2 \) be exact symplectic diffeomorphisms of \( M \). Let \( g_1 \), \( g_2 \) and \( g_{12} \) be action functions for \( f_1 \), \( f_2 \) and \( f_2 \circ f_1 \) respectively. Suppose there is a point \( x_0 \in M \) such that
\[
 g_{12}(x_0) = g_1(x_0) + g_2(f_1(x_0)).
\]

Then
\[
 A(f_2 \circ f_1) = A(f_1) + A(f_2).
\]

This lemma shows that change in action with a perturbation can be easily calculated.

**Proof.** The action function \( g_{12} \) is defined by
\[
dg_{12} = (f_2 \circ f_1)^*\lambda - \lambda,
\]
hence,
\[
dg_{12} = \{f_1^*f_2^*\lambda - f_2^*\lambda\} + \{f_2^*\lambda - \lambda\}
= dg_1 + dg_2.
\]
for some function $\tilde{g}_1$. In fact
\[
d\tilde{g}_1 = f_1^*(f_2^*\lambda) - f_2^*\lambda = \{f_1^* (f_2^* \lambda - \lambda) - (f_2^* \lambda - \lambda)\} + \{f_1^* \lambda - \lambda\}
\]
\[
= \{f_1^* dg_2 - dg_2\} + dg_1 = d(g_2 \circ f_1 - g_2) + dg_1.
\]
Obviously, such $\tilde{g}_1$ exists, we may choose
\[
\tilde{g}_1 = (g_2 \circ f_1 - g_2) + g_1,
\]
and then we may choose
\[
g_{12} = \tilde{g}_1 + g_2 = (g_2 \circ f_1 - g_2) + g_1 + g_2 = g_2 \circ f_1 + g_1,
\]
i.e., we may choose the action function for $f_2 \circ f_1$ by adding the action function for $f_1$ and the $f_1$-shifted action function for $f_2$. Under this choice, we have
\[
g_{12}(x_0) = g_1(x_0) + g_2(f_1(x_0)). \tag{3.1}
\]
Now
\[
A(f_2 \circ f_1) = \int_M g_{12}\omega
\]
\[
= \int_M (g_1 + g_2 \circ f_1)\omega
\]
\[
= \int_M g_1\omega + \int_M (g_2 \circ f_1)\omega
\]
\[
= \int_M g_1\omega + \int_M g_2(f_1^*\omega)
\]
\[
= \int_M g_1\omega + \int_M g_2\omega
\]
\[
= A(f_1) + A(f_2).
\]
We remark that the above equality is independent of the choice of 1-form $\lambda$ and therefore the choices of the action functions, as long as condition (3.1) holds.

This proves the proposition. \qed

There is one issue concerning the action function: it is not well defined for symplectic diffeomorphisms on compact manifold without boundary. Hamiltonian diffeomorphisms are exact provided that $\omega$ has a primitive 1-form $\lambda$. It is well-known that no such 1-form exists for compact symplectic manifold without boundary.

To overcome this difficulty, Asaoka-Irie’s idea is to blow-up a fixed point on compact surface. The existence of such fixed point is guaranteed by Floer’s proof of Arnold conjecture. Moreover, in order to fit into the contact manifold framework, one needs to construct the so-called open book decomposition.

Theorem 3.1 follows from Asaoka-Irie’s open book decomposition, Proposition 3.2, and, of course, the volume theorem of Gardiner, Hutchings and Ramos [6].

We conclude this section by showing how a local perturbation can change action.
Example 3.3. Let $D^2$ be a closed unit disk in $\mathbb{R}^2$, we construct a family of $C^\infty$ area-preserving map $h_t$ on $D^2$, such that

- $h_0 = id$;
- The action function $g_t \equiv 0$ on $\partial D^2$ for all $t \in [0,1]$;
- $A(h_1) \neq 0$.

Let $b : [0,1] \to \mathbb{R}$ be a $C^\infty$ bump function, such that

$$b(r) = \begin{cases} 
1, & 0 \leq r \leq \frac{1}{3} \\
0, & \frac{1}{3} \leq r \leq 1
\end{cases}$$

and $b(r) \geq 0, b'(r) \leq 0$ for all $r \in [0,1]$.

Take the standard area form $\omega$. Under the polar coordinate $(r, \theta)$ on $D^2$, $\omega = rdrd\theta$, we take $\lambda = \frac{r^2}{2}d\theta$. Define a family of diffeomorphisms $h_t(r, \theta) = (r, \theta + tb(r))$. Then we have

$$h_t^* \beta - \beta = \frac{r^2}{2}d(\theta + tb(r)) - \frac{r^2}{2}d\theta = \frac{r^2}{2}tb'(r)dr = dg_t.$$

Hence the action function is a path independent line integral

$$g_t = g_t(r, \theta) = \int_{(1,0)}^{(r,\theta)} \frac{r^2}{2}tb'(r)dr + g(1,0) = t \int_{1}^{r} \frac{r^2}{2}b'(r)dr.$$

Here we set $g(1,0) = 0$. Finally we have $g(r, \theta) \geq 0$ for all $(r, \theta)$ and the total action

$$A(h_t) = t \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \left( \int_{0}^{r} \frac{r^2}{2}b'(s)ds \right) r d\theta dr \geq 0.$$

In particular, $A(h_0) = 0$ and $A(h_t) > 0$ for any $t > 0$.

4. Flux

Our goal is to generalize the $C^\infty$ closing lemma to certain non-Hamiltonian area-preserving diffeomorphisms. The key property that distinguishes a Hamiltonian diffeomorphism from a general symplectic diffeomorphism involves the concept of flux. We first give a general definition of flux, even though the flux in our specific application is much simpler.

By a theorem of Weinstein in [17], the Lie group $\text{Symp}(M, \omega)$, for compact symplectic manifold $M$, is locally path-connected. Let $\text{Symp}_0(M, \omega)$ denote the path component of the Lie group of symplectomorphisms of $(M, \omega)$ containing the identity, for every $\psi \in \text{Symp}_0(M, \omega)$ there exists a smooth family of symplectomorphisms $\psi_t \in \text{Symp}(M, \omega)$ such that $\psi_0 = id$ and $\psi_1 = \psi$. For such a family of symplectomorphisms there exists a unique family of vector fields $X_t : M \to TM$ such that

$$\frac{d}{dt} \psi_t = X_t \circ \psi_t.$$

Since $\psi_t$ preserving the symplectic structure we have $L_{X_t} \omega = 0$ hence $\iota(X_t) \omega = \lambda_t$ is a family of closed 1-forms.
First consider a closed loop \( \{ \phi_t \} \) in \( \text{Symp}_0(\text{M},\omega) \) with \( \phi_0 = \phi_1 = \text{id} \), let \( \{ \lambda_t \} \) be the family of closed 1-forms generating this loop.

**Definition 4.1.** The flux of this loop is given by

\[
flux(\{ \phi_t \}) = \int_0^1 [\lambda_t] dt \in H^1(\text{M},\mathbb{R}).
\]

Under the usual identification of \( H^1(\text{M};\mathbb{R}) \) with \( \text{Hom}(\pi_1(\text{M});\mathbb{R}) \), the above cohomology class corresponds to the homomorphism \( \pi_1(\text{M}) \to \mathbb{R} \) defined by

\[
l \mapsto \int_0^1 \int_0^1 \omega(X_t(l(s)), \dot{l}(s)) ds dt
\]

for \( l : S^1 = \mathbb{R}/\mathbb{Z} \to \text{M} \in \pi_1(\text{M}) \). Use this fact we can prove that the right hand side only depends on the homotopy class of the path \( \phi_t \) in \( \pi_1(\text{Symp}_0(\text{M},\omega)) \). Thus we get a homomorphism

\[
flux : \pi_1(\text{Symp}_0(\text{M},\omega)) \to H^1(\text{M},\mathbb{R})
\]

The image of this homomorphism \( \Gamma \subset H^1(\text{M},\mathbb{R}) \) is called the **Calabi group**, or **flux group**.

Next, given any \( \psi \in \text{Symp}_0(\text{M},\omega) \), any path \( \{ \psi_t \} \in \text{Symp}_0(\text{M},\omega) \) that connected \( \psi \) and identity also determined a family of vector fields and a family of closed 1-forms. Then we can define its flux in the same way

\[
flux(\{ \psi_t \}) = \int_0^1 [\lambda_t] dt \in H^1(\text{M},\mathbb{R}).
\]

notice that the result will depend on the choice of homotopy type of the path connecting \( \text{id} \) with \( \psi \), but the difference between two choices belongs to \( \Gamma \). Hence we have

\[
flux : \text{Symp}_0(\text{M},\omega) \to H^1(\text{M},\mathbb{R})/\Gamma.
\]

We give a geometric description of the flux homomorphism. The value of \( flux(\{ \psi_t \}) \) on a 1-cycle \( l \) on \( \text{M} \) is the symplectic area swept out by the path of \( l \) under the isotopy \( \{ \psi_t \} \). More precisely, let \( \partial[l] = \cup_t \psi_t(l) \) be a 2-cycle which is the image of \( l \) under \( \psi_t \), then \( (flux(\{ \psi_t \})), [l] = ([\omega], \partial[l]) \) for all \( l \in H_1(\text{M},\mathbb{Z}) \).

Hence the flux group is a subgroup of \( H^1(M;P_\omega) \), where \( P_\omega := ([\omega],H_2(M;\mathbb{Z})) \) is the countable group of periods of \( \omega \), i.e., the set of values taken by \( \omega \) on the countable group \( H_2(M;\mathbb{Z}) \) of integral 2-cycles.

The flux homomorphism for a symplectic manifold \( (M,\omega) \) was first introduced by Calabi in [5] in 1970, and studied further by Banyaga in [4]. Banyaga showed that \( \ker(F) \) is exactly \( \text{Ham}(M,\omega) \), thus we have an exact sequence

\[
0 \to \text{Ham}(M,\omega) \to \text{Symp}_0(M,\omega) \to H^1(M,\omega)/\Gamma \to 0.
\]

That is, we have the following lemma:
Lemma 4.2. For $\psi \in \text{Symp}_0(M,\omega)$, it is Hamiltonian if and only if there exists a symplectic isotopy

$$[0,1] \to \text{Symp}_0(M,\omega) : t \mapsto \psi_t$$

such that $\psi_0 = \text{id}$, $\psi_1 = \psi$, and $\text{flux} \{\{\psi_t\}\} = 0$ in $H^1(M,\mathbb{R})$.

Moreover, if $\text{flux} \{\{\psi_t\}\} = 0$ then $\psi_t$ is isotopic with fixed endpoints to a Hamiltonian isotopy.

This lemma states that zero flux is necessary, but not sufficient, for a symplectic diffeomorphism to be Hamiltonian. The sufficient condition is that there is a zero flux isotopy. For a proof of this lemma, see Banyaga [4].

We now consider two dimensional case, where the group of symplectomorphisms are those area-preserving diffeomorphisms that preserve the orientation. We normalize the area form $\omega$ by assume that $\int_M \omega = 1$. In this case the flux group is exactly $\mathbb{Z}$, since $P_\omega$, the set of values taken by $\omega$ on the countable group $H_2(M;\mathbb{Z})$ of integral 2-cycles is exactly $\mathbb{Z}$. Hence we can simply define the flux of a diffeomorphism $f \in \text{Diff}_0(M,\omega)$ in the following way.

Let $l$ be an oriented closed curve in $M$, then there is an oriented disk $D \subset M$ such that the boundary $\partial D = f(l) - l$. The flux of $f$ across $l$ can be calculated as

$$F_f(l) = \int_D \omega \mod 1.$$ 

Note that the choice of $D$ is up to the second homology class of $M$, the above quantity is defined up to mod 1, therefore independent of the choice of $D$. When $f$ is isotopic to identity, $D$ can be uniquely defined by the isotopy.

As we see before, the flux depends only on the homology class of $l$, hence $F_f : H_1(M,\mathbb{R}) \to \mathbb{R} \mod 1$ can be represented by a cohomology vector.

Consider an orientable compact surface $M$ of genus $g$ with $g \geq 1$. Let $(a_i, b_i), i = 1,2,...g$ be the canonical generators of its first homology $H_1(M,g;\mathbb{R})$, let $V_f = (F_f(a_1), F_f(b_1), F_f(a_2), F_f(b_2),...F_f(a_g), F_f(b_g))$ be the associate vector in $H^1(M,\mathbb{R})$. We call $V_f$ the flux vector of $f$. The flux vector has a nice additive property under the iterations of $f$, i.e., $F_{f^k} = kF_f$.

The key fact about flux vector is the following lemma:

Lemma 4.3. Let $f \in \text{Diff}_0(M,\omega)$ be an area preserving diffeomorphism that isotopic to identity, then for any neighborhood $V$ of $f$ in there is a map $g$ in $V$ such that the flux vector $v_g$ is rational.

Proof. Let $(a_i, b_i)$ be the canonical generators of the first homology $H_1(M,\mathbb{Z})$. For any $i = 1,2,...g$, we may assume that $a_i$ and $b_i$ are simple closed curves such that these curves don’t intersect each other except $a_i$ and $b_i$; $a_i$ and $b_i$ intersect at only one point and the intersection is transversal.
Fix $i$, we choose a small tubular neighborhood of $b_i$, without loss of generality, we may parametrize this tubular neighborhood by an area-preserving parametrization:

$$\delta_{b_i} : S^1 \times [-\delta, \delta] \to M$$

Let $\beta : [-\delta, \delta] \to \mathbb{R}$ be a $C^\infty$ function such that $\beta(-\delta) = \beta(\delta) = 0$, $\beta(t) > 0$ for all $-\delta < t < \delta$, and all the derivatives of $\beta(t)$ at $\pm\delta$ are zero. Let

$$T_\epsilon : S^1 \times [-\delta, \delta] \to M, \quad (\theta, t) \mapsto (\theta + \epsilon \beta(t))$$

be a small perturbation on the tubular neighborhood.

Let $h_\epsilon : M \to M$ be a diffeomorphism defined as

$$h_\epsilon(z) = \begin{cases} 
   z & z \notin \delta_{b_i} \\
   \delta_{b_i} \circ T_\epsilon \circ (\delta_{b_i})^{-1}(z) & z \in \delta_{b_i}
\end{cases}$$

Then $h_\epsilon \to Id_M$ in $C^\infty$ topology as $\epsilon \to 0$, hence $f \circ h_\epsilon \to f$ as $\epsilon \to 0$.

It is easy to see that the flux of $h_\epsilon$ across $a_j$ and $b_j$ are all zero for $j \neq i$ and the flux across $b_i$ is also zero. We have infinitely many choices of $\epsilon$ in $F_{h_\epsilon}(a_i) \neq 0$ such that $F_{h_\epsilon}(a_i) \neq 0$, hence $F_{f \circ h_\epsilon}(a_i) = F_f(a_i) + F_{h_\epsilon}(a_i)$ could be choice to be rational.

We can do similar perturbations to every closed curves $a_i$ and $b_i$ so that the flux across $a_i$ and $b_i$ are all rational. $\square$

5. PROOF OF THE MAIN RESULT

In this section we consider $f \in \text{Diff}^\infty_0(\mathbb{T}^2, \omega)$, the area-preserving diffeomorphism of $\mathbb{T}^2$ that homotopic to the identity. Identify $\mathbb{T}^2$ with the quotient $\mathbb{R}^2/\mathbb{Z}^2$, and let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the associated covering map. For each continuous map $f : \mathbb{T}^2 \to \mathbb{T}^2$ we can lift it to $\mathbb{R}^2$, to a continuous map $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \circ \pi = \pi \circ \tilde{f}$. If $\tilde{f}_1, \tilde{f}_2$ are two lifts of $f$ then there exists $z \in \mathbb{Z}^2$ such that $\tilde{f}_1(v) = \tilde{f}_2(v) + z$ for all $v \in \mathbb{R}^2$.

Any diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ that is isotopic to identity can be written in the form

$$f(x, y) = (x + a(x, y), y + \beta(x, y))$$

where the functions $a$ and $\beta$ are of period 1 in both variables $x, y \in \mathbb{R}$. For this type of diffeomorphisms there exist a lift

$$\tilde{f}(x, y) = (x + \tilde{a}(x, y), y + \tilde{\beta}(x, y)).$$

such that $\tilde{f}(x + z_1, y + z_2) = \tilde{f}(x, y) + (z_1, z_2)$ for $z = (z_1, z_2) \in \mathbb{Z}^2$.

**Lemma 5.1.** Let $f_t \in \text{Symp}_0(\mathbb{T}^2, \omega)$ be a symplectic isotopy with a lift $\tilde{f}_t : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\tilde{f}_t(x + z_1, y + z_2) = \tilde{f}_t(x, y) + (z_1, z_2)$$

for $(x, y) \in \mathbb{R}^2$ and $(z_1, z_2) \in \mathbb{Z}^2$. Then

$$\text{Flux}(\{\tilde{f}_t\}) = a_1 dx + a_2 dy, (a_1, a_2) = \int_0^1 \int_{\mathbb{T}^2} (\tilde{f}_t(x, y) - (x, y)) dx dy$$
Here $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the Poisson matrix.

Hence the flux homomorphism for the torus $(\mathbb{T}^2, \omega)$ descends to a homomorphism $\rho : \text{Symp}_0(\mathbb{T}^2, \omega) \to H^1(\mathbb{T}^2; \mathbb{R})/H^1(\mathbb{T}^2; \mathbb{Z})$ and $\Gamma = H^1(\mathbb{T}^2; \mathbb{Z})$.

**Proof.** Let $H_t$ be a smooth family of generating Hamiltonians for $f_t$, so that
\[ \frac{d}{dt} \tilde{f}_t = -J_0 \nabla H_t \circ \tilde{f}_t. \]
We claim that there are functions $h_j(t)$, $j = 1, 2$ such that
\[ H_t(x + z_1, y + z_2) = H_t(x, y) + h_1(t)z_1 + h_2(t)z_2 \]
for $0 \leq t \leq 1$. To see this, define $h_j(t)$ so that the equation holds when $(z_1, z_2)$ is the 1,2 th vector of the standard basis, and then use linearity with respect to $(z_1, z_2)$. Hence
\[
\begin{align*}
a &= J_0 \int_{\mathbb{T}^2} (\tilde{f}_1(x, y) - (x, y))dxdy \\
&= J_0 \int_{0}^{1} \int_{\mathbb{T}^2} \frac{d}{dt} \tilde{f}_1(x, y)dxdydt \\
&= \int_{0}^{1} \int_{\mathbb{T}^2} \nabla H_t \circ \tilde{f}_1(x, y)dxdydt \\
&= \int_{0}^{1} \int_{\mathbb{T}^2} \nabla H_t(x, y)dxdydt \\
&= \int_{0}^{1} h(t)dt
\end{align*}
\]
Furthermore, although the functions $H_t$ are not defined on $\mathbb{T}^2$, both $dH_t$ and $X_{H_t}$ descend to $\mathbb{T}^2$. Thus we find
\[
\begin{align*}
\text{Flux}(\{f_t\}) &= \int_{0}^{1} [i(X_{H_t})]dxdy dt \\
&= \int_{0}^{1} [dH_t]dt \\
&= \int_{0}^{1} h_1(t)dtdx + \int_{0}^{1} h_2(t)dtdy \\
&= a_1 dx + a_2 dy
\end{align*}
\]

**Definition 5.2.** A symplectomorphism $f$ of the torus is called exact if it admits a lift $\tilde{f}$ such that
\[ \tilde{f}(x + z_1, y + z_2) = \tilde{f}(x, y) + (z_1, z_2), \int_{\mathbb{T}^2} (\tilde{f}(x, y) - (x, y))dxdy = 0. \]

We remark that, the condition on the lift of $f$ on $\mathbb{T}^2$ implies that $f$ is isotopic to identity.

Now we are ready to prove the main result.

**Proof of theorem 1.3**
Let $f$ be a symplectic diffeomorphism on $\mathbb{T}^2$, isotopic to identity. Hamiltonian diffeomorphism is characterized by the fact that there exists a lift $\tilde{f}$ such that

$$\int_{\mathbb{T}^2} (\tilde{f}(x,y) - (x,y)) dxdy = 0.$$

According to lemma 3.2, we may assume, by making a small perturbation, that $f$ has a rational flux vector. That is,

$$V_f = (F_f(a_1), F_f(b_1)) = (\frac{p_1}{q}, \frac{p_2}{q})$$

for some $p_1, p_2, q \in \mathbb{Z}$.

The flux vector of $f$ is computed in terms of a lift $\tilde{f}$:

$$V_f = (\int_{\mathbb{T}^2} \tilde{a}(x,y) dxdy, \int_{\mathbb{T}^2} \tilde{b}(x,y) dxdy)$$
and $V_f = V_{\tilde{f}} \mod 1$.

Hence we can write $V_{\tilde{f}} = (\frac{p_1}{q}, \frac{p_2}{q})$. Recall that the flux vector has an additive property, $F_f = kF_{\tilde{f}}$ for all $k \in \mathbb{Z}$, so we have $V_f = qV_{\tilde{f}} = (p_1, p_2) \in \mathbb{Z}^2$.

Define $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto \tilde{f}(x,y) - (p_1, p_2)$. Then we have

$$\int_{\mathbb{T}^2} (\tilde{g}(x,y) - (x,y)) dxdy = 0.$$

It is easy to see that $\tilde{g}$ is a lift of diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ and $g$ is a Hamiltonian diffeomorphism. Notice that $f^q = g$ on $\mathbb{T}^2$. According to Asaoka-Irie’s result (Theorem 2.1), for any nonempty open set $U$, there exists a sequence $(\tilde{g}_j)_{j \geq 1} \to g$ such that $\mathcal{P}(g_j) \cap U \neq \emptyset$ and $\supp(g^{-1} \circ g_j) \subset U$.

This implies that given a symplectic diffeomorphism $f$ on $\mathbb{T}^2$, with rotation vector $(\frac{p_1}{q}, \frac{p_2}{q})$, there is a sequence of symplectic diffeomorphisms $(g_j)_{j \geq 1}$, such that $(\tilde{g}_j)_{j \geq 1} \to f^q - (p_1, p_2)$, and $(\tilde{g}_j)_{j \geq 1} \to f^q$ such that there are periodic points in $U$. In other words, we can perturb $f^q$ to create periodic points in any prescribed open set $U$.

However, the above argument does not prove our theorem. We need to perturb $f_j$ instead of $f^q$, to create periodic points. The problem is that there may not be any map, say $f_j$, such that $f_j^q = g$. To solve this problem, we apply the more general perturbation lemma, Theorem 3.1 instead of the original theorem of Asaoka-Irie.

For any $U_0 \subset M$, suppose $f$ no periodic points in $U_0$. Denote $U_1 = f^{-1}(U_0)$, ..., $U_{q-1} = f^{-1}(U_{q-2})$. By shrinking $U_0$, we may assume that $U_0, U_1, \ldots, U_{q-1}$ are disjoint, $U_i \cap U_j = \emptyset$ for $i \neq j$. Let

$$U = U_0 \cup U_1 \cup \ldots \cup U_{q-1}$$

For $t \in [0, 1]$, let $h_t : U_0 \to U_0$ be a family of area-preserving diffeomorphism such that $h_0 = \text{id}$ and total action is nonzero at $t = 1$, or, $\mathcal{A}(h_1) \neq 0$, as the example
we constructed in Section 3. Consider the map 
\[ h_t \circ f : M \to M. \]

We have

- For all \( t \in [0, 1] \), \((h_t \circ f)^q \equiv f \) for all \( f \in M \setminus (U_0 \cup U_1 \cup \ldots \cup U_{q-1}) = M \setminus U \).
- \((h_0 \circ f)^q = f^q\)
- The total action of map 
\[ (h_1 \circ f)^q \circ f^{-q} : U \to U \]

is 
\[ A((h_1 \circ f)^q \circ f^{-q}) = qA(h_1) \neq 0. \]

By Theorem 3.1 there is a \( t \in [0, 1] \) such that 
\[ (h_t \circ f)^q : M \to M. \]

has a periodic point in \( U \), which implies that 
\[ h_t \circ f : M \to M. \]

has a periodic point in \( U_0 \).

This proves the theorem.

REFERENCES

[1] D. V. Anosov, Roughness of geodesic flows on compact Riemannian manifolds of negative curvature, Dokl. Akad. Nauk SSSR 145 (1962), 707–709. MR0143156
[2] D. V. Anosov and E. V. Zhuzhoma, Closing lemmas, Differ. Equ. 48 (2012), no. 13, 1653–1699. MR3188568
[3] Masayuki Asaoka and Kei Irie, A C∞ closing lemma for Hamiltonian diffeomorphisms of closed surfaces, Geom. Funct. Anal. 26 (2016), no. 5, 1245–1254. MR3568031
[4] Augustin Banyaga, Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, Comment. Math. Helv. 53 (1978), no. 2, 174–227. MR490874
[5] Eugenio Calabi, On the group of automorphisms of a symplectic manifold, Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N. J., 1969), 1970, pp. 1–26. MR0350776
[6] Daniel Cristofaro-Gardiner, Michael Hutchings, and Vinicius Gripp Barros Ramos, The asymptotics of ECH capacities, Invent. Math. 199 (2015), no. 1, 187–214. MR3294959
[7] Michael Hutchings, Lecture notes on embedded contact homology, Contact and symplectic topology, 2014, pp. 389–484. MR3220947
[8] ———, Mean action and the Calabi invariant, J. Mod. Dyn. 10 (2016), 511–539. MR3570996
[9] Kei Irie, Dense existence of periodic Reeb orbits and ECH spectral invariants, J. Mod. Dyn. 9 (2015), 357–363. MR3436746
[10] Shan Tao Liao, An extension of the C1 closing lemma, Kexue Tongbao 24 (1979), no. 19, 865–868. MR556294
[11] Ja. B. Pesin, Characteristic Lyapunov exponents, and smooth ergodic theory, Uspehi Mat. Nauk 32 (1977), no. 4 (196), 55–112. MR0466791
[12] Henri Poincaré, New methods of celestial mechanics. Vol. I, History of Modern Physics and Astronomy, vol. 13, American Institute of Physics, New York, 1993. Periodic and asymptotic solutions, Translated from the French, Revised reprint of the 1967 English translation, With endnotes by V. I. Arnol’d, Edited and with an introduction by Daniel L. Goroff. MR1194622
[13] Charles C. Pugh, The closing lemma, Amer. J. Math. 89 (1967), 956–1009. MR226669
[14] Charles C. Pugh and Clark Robinson, *The $C^1$ closing lemma, including Hamiltonians*, Ergodic Theory Dynam. Systems 3 (1983), no. 2, 261–313. MR742228

[15] Radu Saghin and Zhihong Xia, *Generic properties of symplectic diffeomorphisms*, Nonlinear dynamics and evolution equations, 2006, pp. 247–255. MR2223355

[16] Steve Smale, *Mathematical problems for the next century*, Math. Intelligencer 20 (1998), no. 2, 7–15. MR1631413

[17] Alan Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Advances in Math. 6 (1971), 329–346 (1971). MR286137

[18] Zhihong Xia and Hua Zhang, *A $C^r$ closing lemma for a class of symplectic diffeomorphisms*, Nonlinearity 19 (2006), no. 2, 511–516. MR2199401

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