One-loop Correction to the AdS/BCFT Partition Function in the
Three Dimensional Pure Gravity

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We calculate the tree-level partition function of Euclidean BTZ black hole with
the end of the world branes (ETW branes) for arbitrary tension and the one-loop
partition function of Euclidean thermal $AdS_3$ in the presence of a tensionless ETW
brane with the Neumann boundary condition. At the tree level, our results match
with the previous ones in non-rotating cases, but at the one-loop level, our results
contain unexpected terms, which contain the fluctuations of the ghost fields. However
particularly in the case where the chemical potential is zero, we get a similar form
as the original result in AdS/CFT case, but the exponent is slightly changed. In
this case the ghost contribution looks going away and the partition function becomes
physical in a sense that the coefficients of the expansion in powers of $q$ are positive,
which means that the number of states at the level is always positive.

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I. INTRODUCTION

There is a famous work on the calculation of the one-loop partition function in gravity almost half a century ago [1]. In the context of AdS/CFT correspondence [2], there are some works on deriving the full quantum gravity partition function of pure gravity at a finite temperature [3–5]. At the one-loop level, they deduced the results from the CFT path integral [4], and later it was directly proved using the heat kernel method [3]:

\[ Z_{\text{gravity}} = \prod_{m=2}^{\infty} \frac{1}{|1 - q^m|^2}. \]  

(1)

Our goal is to repeat the calculation in the AdS/BCFT case. AdS/BCFT correspondence is an extension of AdS/CFT correspondence to the case where manifolds admit boundaries [6–9]. Roughly speaking, AdS/BCFT identifies the chiral mode with the anti-chiral mode. In this paper firstly we calculate the tree-level partition function of rotating Euclidean BTZ black hole with an end of the world branes (henceforth ETW branes) with arbitrary tension and the result is

\[ Z_{\text{BTZ-tree}} = \exp \left( \frac{\pi R_+}{8G} + \frac{1}{8G} \left( \log \left( \frac{1 + T_x}{1 - T_x} \right) - \log \left( \frac{1 + T_0}{1 - T_0} \right) \right) \right). \]  

(2)

This matches with previous calculations in the non-rotating case [6]. Secondly, we calculate the one-loop partition function of thermal AdS\(_3\) with a tensionless ETW brane and the result is

\[ Z_{\text{gravity}} = \left( \prod_{m=2}^{\infty} \frac{1}{|1 - q^m|} \right) \left( \prod_{l=0}^{\infty} \sqrt{1 - q^l + q^{l+2}} \sqrt{1 - q^{l+1} + q^{l+2}} \right). \]  

(3)

There are two contributions: the first term corresponds to half of the original result and the second term is the new effect due to the ETW brane. The second term explicitly contains anti-chiral mode and represents the contribution from a massive ghost vector field and massless spin-2 field. However, as we consider later if we set \( q = \sqrt{\pi} \), then we have a similar form as [1]. This ghost mode may indicate the breakdown of consistent boundary conditions (henceforth we denote B.C.s) for calculating a one-loop partition function of Euclidean gravity. There are many attempts on this subject, see [10, 11]. Our result indicates one more important thing. At the tree-level, the partition function with tensionless branes becomes a square root of the original result of no brane case. If we consider the Dirichlet boundary condition on the brane [12], the result is the same as the Neumann case in the tensionless case. More clearly in the tensionless case, the tree-level partition function does not depend on the boundary condition. However, at the one-loop level we have additional modes: remnants of gauge symmetry. We expect that these modes represent the fluctuation of ghost fields through the ETW brane. These
are unphysical modes, so we can eliminate these terms if we can impose a consistent boundary condition on gravity. However, we can regard the resulting partition function is physical in a sense that the coefficients of the expansion in powers of $q$ are positive. This means that the number of states at that conformal dimension is positive and can be thought as a summation of the physical setup.

Next, we consider summing over the modular transformation. Boundary CFT lives on the conformal boundary of a half of the solid torus \[13\] as compared to the AdS/CFT case. We expect that it still admits modularity even if we insert branes because from the boundary torus perspective it still admits modular invariance and then we consider how to locate the position of the brane in the bulk. To derive the full partition function, we must sum over then $SL(2,\mathbb{Z})$ transformation of the thermal AdS contribution \[4, 14\]. We discuss briefly in section 4, but we do not give any explicit formula.

This paper is organized as follows. In section 2 we directly calculate the partition function of a rotating BTZ black hole with ETW branes. The calculation is straightforward but seems to be laborious. There are already some works on this calculation for no boundary case \[15–18\]. In section 3 we review the heat kernel method for calculating 1-loop partition function \[19–22\] and then we calculate the 1-loop partition function of the scalar, the vector, and the spin-2 fields using a method of images. In section 4 we discuss the physical interpretation of the partition function and the consistency of the boundary conditions. In section 5 we make conclusions and discussions on the consistency of Neumann B.C. brane at the one-loop level and present some future directions. In appendix A we will give the detailed calculation of derivatives of chordal distance $u$.

II. TREE-LEVEL PARTITION FUNCTION OF BTZ BLACK HOLE WITH ETW BRANES

A. BTZ with tensionless ETW branes

Let us consider an Euclidean rotating BTZ black hole in three dimension \[15\]. The metric is given by

$$ds^2 = \left(\frac{r^2 - R_+^2}{r^2}\right)\left(\frac{r^2 + R_+^2}{r^2} \right)dt^2 + \frac{r^2}{(r^2 - R_+^2)(r^2 + R_+^2)}dr^2 + r^2(d\phi - \frac{R_+ R_-}{r^2}dt)^2.$$ (4)

Here $R_- \text{ is real valued and } R_+ \text{ is the horizon radius. The absence of a conical singularity requires the periodicity of time and rotational angle}$$
where \( \beta = \frac{2\pi R_+}{R_+^2 + R_-^2} \) and \( \theta = \frac{2\pi R_-}{R_+^2 + R_-^2} \). If we define new coordinate as \( \phi' \equiv \phi - \frac{\theta}{\beta} t \), the periodicity can be recast as \( (t, \phi) \sim (t + \beta, \phi') \) \[5\]. We also rewrite the metric in terms of this coordinate as

\[
g_{ab} = \begin{pmatrix}
\frac{(\tau^2 - R_-^2)(\tau^2 + R_+^2)}{r^2} + r^2 \left( \frac{R_- - R_+}{R_+} \right)^2 & 0 & \frac{R_- r^2 - R_+ R_-}{r^2} \\
0 & \frac{(\tau^2 - R_+^2)(\tau^2 + R_-^2)}{r^2} & 0 \\
\frac{R_- r^2 - R_+ R_-}{r^2} & 0 & r^2
\end{pmatrix}.
\] \[6\]

Note that \( \det(g_{ij}) = r^2 \). To identify the location of ETW branes we use the map to the Poincare coordinate, which is given by

\[
\eta = \left( \frac{\tau^2 - R_-^2}{\tau^2 + R_+^2} \right)^{\frac{1}{2}} \cos \left( \frac{R_- + R_+^2}{R_+} \tau + R_- \phi' \right) \exp \left( R_+ \phi' \right), \\
x = \left( \frac{\tau^2 - R_+^2}{\tau^2 + R_-^2} \right)^{\frac{1}{2}} \sin \left( \frac{R_- + R_+^2}{R_+} \tau + R_- \phi' \right) \exp \left( R_+ \phi' \right), \\
z = \left( \frac{R_- + R_+^2}{R_- + R_+^2} \right)^{\frac{1}{2}} \exp \left( R_+ \phi' \right),
\] \[7\]

and the metric becomes

\[
ds^2 = \frac{dz^2 + dx^2 + d\eta^2}{z^2}.
\] \[8\]

In this coordinate the identification \( (t, \phi') \sim (t + \beta, \phi') \) is trivial, but identification \( (t, \phi' + 2\pi) \) is not trivial, so we should identify by hand. If we define the following complexified coordinate \( w = \eta + ix \), then the identification is translated into \( (w, z) \sim (we^{2\pi R_+}, ze^{2\pi R_-}) \).

From this perspective, we can take the fundamental region as \( 1 \leq |w|^2 + z^2 \leq e^{2\pi R_+} \) and the horizon \( r = R_+ \) is mapped to \( x = \eta = 0 \) and \( 1 \leq z \leq e^{2\pi R_+} \).

Let us consider inserting the ETW branes. The position of the ETW brane with tension \( T \) is determined by the equation of the motion:

\[
K_{ab} - Kh_{ab} = -Th_{ab}.
\] \[9\]

Firstly let us consider tensionless branes. In the Poincare coordinate, a brane configuration is determined by \[23\]

\[
(z - \alpha)^2 + (x - p)^2 + (\eta - q)^2 = \beta^2,
\] \[10\]

where the tension is given by \( T = \frac{\alpha}{\beta} \). Here ETW branes, where \( \phi' \) is constant, are just a sphere of radius 1 for \( \phi' = 0 \) and \( e^{\pi R_+} \) for \( \phi' = \pi \). We will calculate the action
$$S = -\frac{1}{16\pi G} \int_N \sqrt{g}(R + 2)d^3x - \frac{1}{8\pi G} \int_Q \sqrt{h}(K - T)d^2x - \frac{1}{8\pi G} \int_M \sqrt{h}Kd^2x + \frac{k}{8\pi G}S_{ct}, \quad (11)$$

where $N$ is the three-dimensional bulk region, $Q$ is the ETW brane and $M$ is the conformal boundary placed at $r = R$, which we later take $R \to \infty$. $S_{ct} = \int_M \sqrt{h}$ is a counterterm constructed from only induced geometric quantities [24]. Firstly, let us consider computing the induced metric and the extrinsic curvature at $r = R$ surface. After some computations we get

$$h_{ab} = \begin{pmatrix} \frac{(r^2 - R^2)(r^2 + R^2)}{R^2} & \frac{R^2}{R^+_+} - \frac{R^+R_-}{R^2} & 0 & 0 & \frac{R^+}{R^2}R^2 \, R^+ - \frac{R^+R_-}{R^2} \\ 0 & 0 & 0 & 0 & \frac{R^+}{R^2}R^2 \, R^+ - \frac{R^+R_-}{R^2} \\ \frac{R^+}{R^2}R^2 \, R^+ - \frac{R^+R_-}{R^2} & 0 & 0 & R^2 \end{pmatrix}, \quad (12)$$

and

$$K_{ab} = \begin{pmatrix} \frac{R^2 + R^2}{R^2} & \frac{R^+}{R^2} \, R^+ & 0 \, R^2 \, R^+ & 1 \end{pmatrix} \sqrt{(R^2 - R^2_+)(r^2 + R^2)}. \quad (13)$$

From this we see that

$$\det(h_{ab}) = (R^2 - R^2_+)(R^2 + R^2), \quad K = \frac{2R^2 - R^2_+ + R^2}{\sqrt{(R^2 - R^2_+)(R^2 + R^2)}}. \quad (14)$$

Einstein-Hilbert action is just the volume integral because curvature becomes $R = -6$:

$$-\frac{1}{16\pi G} \int_{R^+}^R dR \int_0^\beta dt \int_0^\pi d\phi'(-4r) = \frac{4\pi \beta}{16\pi G} \left( \frac{R^2 - R^2_+}{2} \right). \quad (15)$$

We can also evaluate the GHY term at $r = R$:

$$-\frac{1}{8\pi G} \int_0^\beta dt \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \frac{1}{8}) d\phi' \sqrt{(R^2 - R^2_+)(R^2 + R^2)} \left( \frac{2R^2 - R^2_+ + R^2}{(R^2 - R^2_+)(R^2 + R^2)} \right) = -\frac{\beta}{8G}(2R^2 - R^2_+ + R^2). \quad (16)$$

Now we determine the counterterm. That should be constructed from a geometric quantity of boundary surface. Here as a most simple one ($k = 1$) we take

$$S_{ct} = \int_M d^2x \sqrt{h} = \beta \pi \sqrt{(R^2 - R^2_+)(R^2 + R^2)}. \quad (17)$$

We combine the above results altogether and we get:

$$S = -\frac{\beta}{8G} \left( R^2 + R^2_+ - \sqrt{(R^2 - R^2_+)(R^2 + R^2)} \right). \quad (18)$$

If we take $R \to \infty$ limit this becomes
This is what we expected because we choose the position of ETWs so that the volume of the bulk space becomes a half of the original volume. In the full AdS case, this is done in [18], which is just twice our calculation.

### B. BTZ with ETW branes for general tension $T$

Next, we consider the general $T$ case. It is natural to assume the rotational symmetry i.e. $p = q = 0$ in $[10]$. Therefore, we will consider the following equation

$$\left(\frac{R_+^2 + R_-^2}{r^2 + R_-^2}\right)^{\frac{1}{2}} \exp(R_+ \phi') - \alpha \right)^2 + \left(\frac{r^2 - R_+^2}{r^2 + R_-^2}\right) \exp(2R_+ \phi') = \beta^2. $$

(20)

Here we have two parameters $\alpha$ and $\beta$, so we must eliminate and represent only by $T$. The missing point is the fact that the brane is anchored at $\phi' = 0, \pi$. If we take $r \to \infty$, then the brane equation becomes

$$\alpha^2 + 1 = \beta^2(\phi' = 0),$$

(21)

$$\alpha^2 + \exp(2\pi R_+) = \beta^2(\phi' = \pi).$$

We note that $T$ has range $-1 < T < 1$ from the above constraint. Using this we can determine the brane configuration: for the case where the brane is anchored at $\phi' = 0$, the equation becomes

$$\phi' = \frac{1}{R_+} \log \left( \frac{T}{\sqrt{1 - T^2}} \sqrt{\frac{R_+^2 + R_-^2}{r^2 + R_-^2}} + \sqrt{\frac{T^2 (R_+^2 + R_-^2)}{1 - T^2 (r^2 + R_-^2)} + 1} \right)$$

(22)

and for the case where the brane is anchored at $\phi' = \pi$, the equation also becomes

$$\phi' = \frac{1}{R_+} \log \left\{ e^{\pi R_+} \left( \frac{T}{\sqrt{1 - T^2}} \sqrt{\frac{R_+^2 + R_-^2}{r^2 + R_-^2}} + \sqrt{\frac{T^2 (R_+^2 + R_-^2)}{1 - T^2 (r^2 + R_-^2)} + 1} \right) \right\}.$$

(23)

We move to the calculation of the partition function. The Einstein-Hilbert action can be calculated directly:
This is the same result as $T=0$ case. It is because the branes are curved in opposite directions in the bulk, the volume is unchanged even in $T \neq 0$ case. Next, we will evaluate the brane action. In three dimension extrinsic curvature $K = 2T$, so what we must do is to calculate the induced metric. We can explicitly do this for the case where the brane is anchored at $\phi' = 0$:

\[
\begin{align*}
S_{EH} &= - \frac{1}{16\pi G} \int_{R_+}^{R_{reg}} dr \int_0^\beta dt ( - 4r ) \frac{1}{R_+} \log \left[ \frac{e^{\pi R_+} \left( \frac{T}{\sqrt{1-T^2}} \sqrt{R^2 + R^2_+} + \frac{T^2}{1-T^2} \left( \frac{R^2 + R^2_+}{R_r^2 + R_{-r}^2} \right) + 1 \right)}{\left( \frac{T}{\sqrt{1-T^2}} \sqrt{R^2 + R^2_+} + \frac{T^2}{1-T^2} \left( \frac{R^2 + R^2_+}{R_r^2 + R_{-r}^2} \right) + 1 \right)} \right] \\
&= \frac{\beta}{8G} ( R^2_{reg} - R^2_+ ). 
\end{align*}
\]

(24)

From this determinant of this induced metric is derived as

\[
\det(h_{ab}) = \frac{r^2 (R^2_+ + R^2_{reg})}{R^2_+ (r^2(1-T^2) + R^2_+ T^2 + R^2 )}. 
\]

(26)

For the case where the brane is anchored at $\phi' = \pi$, we can repeat the same thing and the result is the same as above. The brane action can be written as

\[
S_{brane} = \frac{T}{8\pi G} \int_{R_+}^{R_{reg}} dr \int_0^\beta dt \sqrt{\frac{r^2 (R^2_+ + R^2_{reg})}{R^2_+ (r^2(1-T^2) + R^2_+ T^2 + R^2 )}}. 
\]

(27)

However, the brane contribution for $\phi' = 0, \pi$ is canceled as we note in the calculation of the Einstein-Hilbert action; the brane is curved in the opposite direction with respect to the bulk space. For the other terms on the conformal boundary, we have just the same term as in $T = 0$ case because in $r \to \infty$ limit $\phi'$ goes to 0 and $\pi$, so we do not suffer any modification.

We combine them and we get

\[
S = \frac{\beta}{8G} ( R^2_{reg} - R^2_+ ) - \frac{\beta}{8G} ( 2R^2_{reg} - R^2_+ + R^2 ) + \frac{k^2}{8G} \sqrt{(R^2_{reg} - R^2_+)(R^2_{reg} + R^2)} - \frac{\beta}{16G} (R^2_+ + R^2). 
\]

(28)

This result is trivial, so let us consider a more interesting case, where two branes admit different tensions. If we define the tension of the branes as $T_0, T_\pi$ for the case where the branes
are anchored at $\phi' = 0, \pi$ respectively. For the Einstein-Hilbert part the integral becomes

$$ S_{EH} = -\frac{1}{16\pi G} \int_{R_+}^{R_{reg}} dR \int_0^\beta dt (-4r) \frac{1}{R_+} \log \left\{ e^{\pi R} \left( \frac{R_+}{\sqrt{1-r^2}} \sqrt{\frac{R^2 + R_r^2}{r^2}} + \sqrt{\frac{R_0^2}{1-R_0^2} \frac{R^2 + R_r^2}{r^2}} + \frac{1}{R_+} \right) \right\}. \quad (29) $$

The integral is complicated. We use the following formula:

$$ \int_A^B dr \log(C + \sqrt{D + r^2}) = \left[ \frac{r^2}{2} \log(C + \sqrt{D + r^2}) \right]_A^B - \frac{1}{2} \int_A^B \frac{r^3 dr}{(C - \sqrt{D + r^2}) \sqrt{D + r^2}} $$

$$ = \frac{1}{2} (B^2 - C^2 + D) \log(C + \sqrt{D + B^2}) - \frac{1}{2} (A^2 - C^2 + D) \log(C + \sqrt{D + A^2}) $$

$$ + \frac{1}{4} (A^2 - B^2) + \frac{C}{2} (\sqrt{B^2 + D - \sqrt{A^2 + D}}) $$

$$ \frac{R_0^2}{2} \log(C + \sqrt{D + B^2}) \quad (30) $$

The calculation is straightforward. We choose the counterterm as $k = 1$ and we get

$$ S = -\frac{\beta (R_+^2 + R_r^2)}{16G} - \frac{\beta (R_0^2 + R_r^2)}{16\pi GR_+} \left( \log \left( \frac{1 + T_\pi}{1 - T_\pi} \right) - \log \left( \frac{1 + T_0}{1 - T_0} \right) \right), \quad (31) $$

where we omit the divergent term, which is proportional to $R_{reg}$. This matches with the non-rotating BTZ result derived in [6].

**III. ONE-LOOP PARTITION FUNCTION IN THERMAL ADS WITH THE TENSIONLESS ETW BRANE**

In this section, we will use the explicit form of the heat kernel presented in [3], but we will not explicitly write here because it is complicated. Please refer to the original paper if necessary.

**A. A review of the heat kernel method**

The heat kernel method is a convenient way of calculating a one-loop partition function. Let us consider calculating the following partition function

$$ Z = \int D\phi e^{-S(\phi)}. \quad (32) $$

Here we assume that $\phi$ is an arbitrary free field for simplicity. The action can be rewritten as

$$ S(\phi) = \int_M d^3x \sqrt{g} \Delta \phi, \quad (33) $$

where we omit indices of tensorial structure. We will consider a compact space, so $\Delta$ has a discrete set of eigenvalues $\lambda_n$. The one-loop partition function is
If we consider non-compact space, the spectrum becomes continuous and, the one-loop partition function is divergent and proportional to the volume. This divergence can be absorbed by the renormalization of the Newton constant.

The heat kernel is defined as

\[ K(t, x, y) = \sum_n e^{-\lambda_n t} \psi_n(x)\psi_n(y), \]  

which we usually call a propagator. We can normalize the eigenfunctions as

\[ \sum_n \psi_n(x)\psi_n(y) = \delta^3(x - y), \]
\[ \int_M d^3x \sqrt{g} \psi_n(x)\psi_m(x) = \delta_{nm}. \]

The trace of the heat kernel is given by

\[ \int_M d^3x \sum_n \sqrt{g} K(t, x, x) = \sum_n e^{-\lambda_n t}. \]  

Using this we can compute the 1-loop partition function as an integral over \( t \):

\[ S^{(1)} = -\frac{1}{2} \sum_n \log \lambda_n = \frac{1}{2} \int_{+0}^{\infty} dt \int_M d^3x \sum_n \sqrt{g} K(t, x, x). \]

We can show the above equation by differentiating with respect to \( \lambda_n \). Note that it is an identity up to an infinite constant. The point is that \( K \) satisfies the heat conduction equation

\[ (\partial_t + \Delta_x) K(t, x, y) = 0, \]

with a boundary condition at \( t = 0 \)

\[ K(0, x, y) = \delta(x, y). \]

B. One-loop partition function in thermal AdS with ETW brane

In this section, we apply the calculation in [3] to our ETW brane setup. Now consider Poincare \( AdS_3 \) and metric is given by

\[ ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2}. \]
Here an ETW brane is placed at $\Re(z) = 0$ and bulk region is defined as $\Re(z) > 0$. We note that the ETW brane is connected in the bulk. Since the AdS space is maximally isometric, the geodesic distance $r(x, x')$ depends only on the chordal distance $u(x, x')$:

$$r(x, x') = \text{arccosh} \left( 1 + u(x, x') \right), \quad (42)$$

where

$$u(x, x') = \frac{(y - y')^2 + |z - z'|^2}{2yy'}. \quad (43)$$

Thermal AdS can be obtained from AdS using the following identification

$$(y, z) \sim (|q|^{-1} y, q^{-1} z), \quad (44)$$

where $q = e^{2\pi i \tau}$ and $\tau = \tau_1 + i \tau_2$. In the non-zero $\tau_1$ case, the boundary of the BCFT wraps around the torus for many times and in this case the region on which BCFT lives is unclear because the region is not surrounded by the boundaries. To cure this problem later we set $q = \bar{q}$ for simplicity. Now we consider applying the method of images to the heat kernel method. The tensionless ETW brane is inserted at $\Re(z) = 0$, so if we consider a mirror position, $z$ and $\overline{z}$ are mapped to $-\overline{z}$ and $-z$, respectively. Then, one-loop partition function becomes

$$S^{(1)} = \frac{1}{2} \int_{+0}^{\infty} \frac{dt}{t} \int_{\text{thermal AdS}} d^3x \sum_n \sqrt{g} (K^H/Z(t, x, \gamma^nx) + K^H/Z(t, x^\text{mirror}, \gamma^nx)). \quad (45)$$

In the tensionless case the boundary condition for the metric is given by

$$K_{ob} = 0. \quad (46)$$

We note that we treat this condition as a off-shell boundary condition. This means that we must impose the following boundary condition for the perturbation of the metric:

$$\partial_x h_{ij} = 0. \quad (47)$$

The heat kernel on thermal AdS can be obtained using the method of images from that of AdS, so we get

$$S^{(1)} = \frac{1}{2} \int_{+0}^{\infty} \frac{dt}{t} \int_{\text{thermal AdS}} d^3x \sum_n \sqrt{g} (K^H(t, x, \gamma^nx) + K^H(t, x^\text{mirror}, \gamma^nx)). \quad (48)$$

For later convenience we will use different coordinate:
\[ y = \rho \sin \theta, \]
\[ z = \rho \cos \theta e^{i\phi}, \]

where \( 1 \leq \rho \leq e^{2\pi \tau_2} \), \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \). In terms of this coordinate the geodesic distance can be given by

\[
\begin{align*}
\rho(\hat{x}, \gamma^n x) &= \text{arccosh} \left( \frac{\cosh \beta}{\sin^2 \theta} - \frac{\cos \alpha}{\tan^2 \theta} \right), \\
\rho(\hat{x}_{\text{mirror}}, \gamma^n x) &= \text{arccosh} \left( \frac{\cosh \beta}{\sin^2 \theta} + \frac{\cos(2\phi - \alpha)}{\tan^2 \theta} \right),
\end{align*}
\]

(50)

where we define \( \alpha = 2\pi n\tau_1 \) and \( \beta = 2\pi n\tau_2 \).

C. One-loop partition function for a scalar field

The heat kernel of a scalar field on \( \text{AdS}_3 \) is given by

\[
K_{\text{AdS}}^3(t, r(x, x')) = e^{-\frac{(m^2+1)t - r^2}{4t}} \frac{r}{(4\pi t)^{\frac{3}{2}}} \sinh(r). \]

(51)

Let us consider calculating the ordinary (not mirror) part. One-loop determinant can be recast as an integral

\[
- \log \det \Delta = \text{Vol}(H_3/\mathbb{Z}) \int_0^\infty \frac{dt}{t} e^{-\frac{(m^2+1)t}{4t}} \frac{r}{(4\pi t)^{\frac{3}{2}}} \sinh(r),
\]

(52)

where we split into \( n = 0 \) part and non-zero part. The first term can be easily regularized:

\[
\int_0^\infty \frac{dt}{t} e^{-\frac{(m^2+1)t}{4t}} = \frac{(m^2 + 1)^{\frac{3}{2}}}{8\pi^{\frac{3}{2}}} \int_0^\infty dk k^{\frac{3}{2}} e^{-k} = \frac{(m^2 + 1)^{\frac{3}{2}}}{6\pi}. \]

(53)

The second term can be calculated directly. Firstly, we change the variable from \( \theta \) to \( r \) and we get

\[
\sum_{n \neq 0} \int_0^\infty \frac{dt}{t} \int \frac{d\rho d\theta d\phi}{\rho \sin^2 \theta} e^{-\frac{(m^2+1)t - r^2}{4t}} \frac{r}{(4\pi t)^{\frac{3}{2}}} \sinh(r),
\]

(54)

Integrating over \( r \) and \( \phi \) and \( t \) in order, we can reach the final answer:
\[
\sum_{n \neq 0} \int_0^\infty \frac{dt}{t} \int dpdrd\phi \ e^{-\frac{(n^2+1)t}{4\pi} - \frac{r^2}{4t}} \frac{r}{2(\cosh \beta - \cos \alpha)}
\]
\[
= \sum_{n \neq 0} \frac{\sqrt{\pi \tau}}{4(\cosh \beta - \cos \alpha)} \int_0^\infty \frac{dt}{t} e^{-\frac{(n^2+1)t}{4\pi} - \frac{2\pi n \tau^2}{4t}}
\]
\[
= \sum_{n \neq 0} \frac{e^{-2\pi n \tau \sqrt{m^2+1}}}{4n(\cosh \beta - \cos \alpha)}
\]
\[
= \sum_{n=1}^{\infty} \frac{|q|^n (1+\sqrt{m^2+1})}{n(1-|q|^n)^2}.
\]

Now let us consider the mirror part. Firstly, we consider \( n = 0 \) case. The strategy is that we firstly fix \( \phi \) and integrate \( \theta \) or \( r \) and then integrate over other variables. This leads to the divergent result even though we use a regularization by a gamma function:

\[
\int_0^\infty \frac{dt}{t} \int dpdrd\theta \cos \theta \ e^{-\frac{(m^2+1)t}{4\pi} - \frac{r^2}{4t} \sinh(r)}
\]
\[
= -\frac{\tau \sqrt{m^2+1}}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\phi}{1+\cos(2\phi)}.
\]

However, this divergence can be eliminated by the renormalization of Newton constant, so we will ignore this term.

Next consider \( n \neq 0 \) case. The trick is almost the same, but the situation is slightly changed. The integral over \( \phi \) yields

\[
\sum_{n \neq 0} \int_0^\infty \frac{dt}{t} \int dpdrd\phi \ e^{-\frac{(m^2+1)t}{4\pi} - \frac{2\pi n \tau^2}{4t}} \frac{r}{\sinh(\beta)}
\]
\[
= \sum_{n \neq 0} \int_0^\infty \frac{dt}{t} 2\pi \tau \ e^{-\frac{(m^2+1)t}{4\pi} - \frac{2\pi n \tau^2}{4t}} \frac{r}{\sinh(\beta)}
\]
\[
= \sum_{n=1}^{\infty} \frac{|q|^n (1+\sqrt{m^2+1})}{n(1-|q|^n)^2}.
\]

D. One-loop partition function for a vector field

In this section we consider the vector field contribution to the partition function. For later convenience we define a set of new real coordinates:

\[
x = \Re(z) = \rho \cos \theta \cos \phi,
\]
\[
\eta = \Im(z) = \rho \cos \theta \sin \phi.
\]
The second term can be calculated straightforwardly:

\[ K_{\mu\nu}(t, x, x') = F(t, u) \partial_\mu \partial_\nu u + \partial_\mu \partial_\nu S(t, u), \]  

where

\[ F(t, r) = -\frac{e^{-2r}}{(4\pi t)^{\frac{3}{2}}} \sqrt{\sinh r} \]

\[ S(t, r) = \frac{4}{(4\pi)^{\frac{3}{2}}} \int_0^\infty d\xi e^{-t(1-\xi)^2} \sinh(r\xi). \]

The next step is to calculate the derivatives of \( u \). In Poincare coordinate it can be explicitly given by

\[ \partial_\mu \partial_\nu u = -\frac{1}{z_0 w_0} \left\{ \delta_{\mu\nu} + \frac{(z-w)\delta_{\nu 0} + (w-z)\delta_{\mu 0} - u\delta_{\mu 0}\delta_{\nu 0}}{z_0} \right\}. \]

Here we note that \( z = x \) or \( x^{\text{mirror}} \) and \( w = \gamma^n x \). We will present a detailed calculation of these derivatives in the appendix. Let us consider the ordinary (not mirror) part. Using those results we can compute the kernel:

\[ \int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n g^{\mu\nu} \frac{\partial(x^n x')^{\nu}}{\partial x^\mu} K_{\mu\nu}(t, r, x, \gamma^n x) \]

\[ = \int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n \left\{ \left( (e^\beta - \cosh r) (e^{-\beta} - \cosh r) - 2 \cos \alpha (\cosh r - \cosh \beta) \right) \right\} \]

\[ + \left( F + \left( \frac{\partial}{\partial x^\mu} \right) (\cosh r - 2 \cosh \beta - 2 \cos \alpha) \right) \]

\[ = \text{Vol}(\mathbb{H}/\mathbb{Z}) \int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n \left\{ \frac{2\pi^2 r_3}{(4\pi t)^{\frac{3}{2}}} \frac{e^{-2r_3}}{(\cosh \beta - \cos \alpha)} \frac{\sqrt{t}}{2 \cos \alpha + e^{-t}} \right\}. \]

The first term can be regularized by considering a massless limit of a massive field:

\[ \text{Vol}(\mathbb{H}/\mathbb{Z}) \int_0^\infty \frac{dt}{t} \left( \frac{e^{-t+2+4t}}{(4\pi t)^{\frac{3}{2}}} \right) \]

\[ = \lim_{m \to 0} \text{Vol}(\mathbb{H}/\mathbb{Z}) \int_0^\infty \frac{dt}{t} \left( e^{-t+2+4t} \right) \]

\[ = \lim_{m \to 0} \text{Vol}(\mathbb{H}/\mathbb{Z}) \frac{1}{(4\pi t)^{\frac{3}{2}}} \left\{ \frac{m^2 + 1}{2} \pi \Gamma(-\frac{3}{2}) + 2m \pi \Gamma(-\frac{3}{2}) + 4m^2 \pi \Gamma(-\frac{1}{2}) \right\} \]

\[ = \text{Vol}(\mathbb{H}/\mathbb{Z}) \frac{1}{(4\pi)^{\frac{3}{2}}} \frac{1}{2} \left( \frac{1}{3} \pi - 8 \sqrt{\pi} \right) = -\frac{5}{6\pi} \text{Vol}(\mathbb{H}/\mathbb{Z}). \]

The second term can be calculated straightforwardly:

\[ \sum_{n=1}^\infty \int_0^\infty \frac{dt}{t} \frac{2\pi^2 r_3}{(\cosh \beta - \cos \alpha)} \frac{e^{-2r_3}}{4\pi \sqrt{t}} \frac{\sqrt{t}}{2 \cos \alpha + e^{-t}} \]

\[ = \sum_{n=1}^\infty \frac{2 \cos \alpha e^{-\beta}}{2n(\cosh \beta - \cos \alpha)} \]

\[ = \sum_{n=1}^\infty \frac{q^n 2\pi^2 r_3}{n(1+q^n)^2}. \]

If we omit the term, which is proportional to the volume, the answer is just a half of the original result as we expected.
Next, we consider the mirror part. The brane is inserted at $x = 0$, so $x_{\text{mirror}} = -x$, $\eta_{\text{mirror}} = \eta$ and $y_{\text{mirror}} = y$.

The derivatives of $u$ are also slightly modified and we present the results in the appendix A. We can mimic the previous calculation and the result is

$$
\int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n g^{\mu\nu} \frac{\partial g(x)}{\partial x^\nu} K_{\mu\nu}^H(t, r(x_{\text{mirror}}, \gamma^n x)) \nonumber
$$

$$
= \int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n \left\{ (e^\beta - \cosh r)(e^{-\beta} - \cosh r) - 2 \cos \alpha (\cosh r - \cosh \beta) \right\} + (F + \left( \frac{\partial S}{\partial u} \right))(\cosh r - 2 \cosh \beta - 2 \cos \alpha) \right\}.
$$

(65)

This seems to be surprising because the above form is the same as the previous result, but here we have different geodesic distances:

$$
r = \arccosh \left( \frac{\cosh \beta}{\sin^2 \theta} + \frac{\cos(2\phi - \alpha)}{\tan^2 \theta} \right).
$$

(66)

This distance $r$ depends not only $\theta$, but also $\phi$. To move on we fix $\phi$ and integrate $\theta$ (or $r$) firstly. The integral over $r$ gives the same as the original calculation because it is independent of $\phi$:

$$
\int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n \left\{ (e^\beta - \cosh r)(e^{-\beta} - \cosh r) - 2 \cos \alpha (\cosh r - \cosh \beta) \right\} + (F + \left( \frac{\partial S}{\partial u} \right))(\cosh r - 2 \cosh \beta - 2 \cos \alpha) \right\}
$$

$$
= \int_0^\infty \frac{dt}{t} \sum_n 2\pi n \pi \int_0^{\frac{\pi}{2}} d\phi \frac{1}{2(\cosh \beta + \cos(2\phi - \alpha))} e^{-\frac{\beta^2}{4t}} 2 \cosh \beta + e^{-t}.
$$

(67)

The integral over $\phi$ can also be evaluated:

$$
\int_0^{\frac{\pi}{2}} d\phi \frac{1}{(\cosh \beta + \cos(2\phi - \alpha))}
$$

$$
= \int_0^\pi d\phi \frac{1}{2(\cosh \beta + \cos(\phi - \alpha))}
$$

$$
= \int_0^{2\pi} d\phi \frac{1}{2(\cosh \beta + \cos(\phi))}
$$

$$
= \frac{\pi}{\sinh \beta}.
$$

(68)

The final answer on the mirror part becomes

$$
\int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n g^{\mu\rho} \frac{\partial g(x)}{\partial x^\rho} K_{\mu\nu}^{\gamma H}(t, r(x_{\text{mirror}}, \gamma^n x))
$$

$$
= \sum_{n=1}^\infty \frac{2 \cos \alpha + e^{-\beta}}{2n \sinh \beta}
$$

$$
= \sum_{n=1}^\infty \frac{\gamma^n + \frac{\gamma^n}{|\gamma|^2n}}{n(1 - |\gamma|^2n)}.
$$

(69)
E. One-loop partition function for a symmetric spin-2 field

In the calculation of the vector field one-loop partition function when we calculate the mirror part, we get the same form of the trace of the kernel as that in the ordinary part. This seems to be somewhat miraculous at first sight, but this is natural because the trace of the kernel can only depend on geodesic distance \( r \) although we have different geodesic distances. The kernel of the symmetric spin-2 field can be expanded by the basis of \((2, 2)\) symmetric tensors \([3, 25]\). They are complicated and we do not write explicitly here.

The one-loop determinant of symmetric traceless tensor is given by

\[-\log \text{det} \Delta = \int_0^{\infty} \frac{dt}{t} \int d^3x \sqrt{\gamma} \sum_n g^{\mu \rho} \frac{\partial (\gamma^n x)^{\mu'}}{\partial x^\rho} g^{\nu \sigma} \frac{\partial (\gamma^n x)^{\nu'}}{\partial x^\sigma} (K_{\mu \nu}^H (t, r(x, \gamma^n x)) + K_{\mu' \nu'}^H (t, r(x_{\text{mirror}}, \gamma^n x))).\]  

(70)

Let us consider the calculation of the first term. The integral over \( r \) gives

\[
\int_0^{\infty} \frac{dt}{t} \int d^3x \sqrt{\gamma} \sum_n g^{\mu \rho} \frac{\partial (\gamma^n x)^{\mu'}}{\partial x^\rho} g^{\nu \sigma} \frac{\partial (\gamma^n x)^{\nu'}}{\partial x^\sigma} K_{\mu \nu}^H (t, r(x, \gamma^n x))
= \sum_n \int_0^{\infty} \frac{dt}{t} \frac{2\gamma^2}{(\cosh \beta - \cos \alpha)} e^{-\beta^2 t} \frac{e^{-t}}{2\pi \sqrt{t}} (e^{-t} \cos 2\alpha + e^{-4t} \cos \alpha + \frac{e^{-5t}}{2})
= \sum_{n=1}^{\infty} \frac{\sinh \beta}{\cos \alpha} (e^{-t} \cos 2\alpha + e^{-2\beta} \cos \alpha + \frac{e^{-5t}}{2}),
\]

where we omit \( n = 0 \) term because it is proportional to the volume. Next, we consider the second term. We can do the integration similarly:

\[
\int_0^{\infty} \frac{dt}{t} \int d^3x \sqrt{\gamma} \sum_n g^{\mu \rho} \frac{\partial (\gamma^n x)^{\mu'}}{\partial x^\rho} g^{\nu \sigma} \frac{\partial (\gamma^n x)^{\nu'}}{\partial x^\sigma} K_{\mu' \nu'}^H (t, r(x_{\text{mirror}}, \gamma^n x))
= \sum_n \int_0^{\infty} \frac{dt}{t} 2\pi \tau_2 \int \frac{d\phi}{2} \frac{1}{2(\cosh \beta + \cos (2\phi - \alpha))} e^{-\beta^2 t} \frac{1}{2\pi \sqrt{t}} (e^{-t} \cos 2\alpha + e^{-4t} \cos \alpha + \frac{e^{-5t}}{2})
= \int_0^{\infty} \frac{dt}{t} 2\pi \tau_2 \int \frac{d\phi}{2} \frac{1}{2(1 + \cos (2\phi))} \frac{1}{2\pi \sqrt{t}} (e^{-t} + e^{-4t} + \frac{e^{-5t}}{2})
+ \sum_n \int_0^{\infty} \frac{dt}{t} 2\pi \tau_2 \frac{e^{-\beta^2 t}}{2\pi \sqrt{t}} \frac{1}{2\pi \sqrt{t}} (e^{-t} \cos 2\alpha + e^{-4t} \cos \alpha + \frac{e^{-5t}}{2}).
\]

(72)

The first term contains a divergent integral and here we will ignore it. The second term can be simplified as the following:

\[
\sum_n \int_0^{\infty} \frac{dt}{t} 2\pi \tau_2 \frac{e^{-\beta^2 t}}{2\pi \sqrt{t}} \frac{1}{2\pi \sqrt{t}} (e^{-t} \cos 2\alpha + e^{-4t} \cos \alpha + \frac{e^{-5t}}{2})
= \sum_{n=1}^{\infty} \frac{1}{\sinh \beta} (e^{-\beta} \cos 2\alpha + e^{-2\beta} \cos \alpha + \frac{e^{-5\beta}}{2}).
\]

F. One-loop partition function for gravity

We will consider a linearized graviton perturbation \( h_{\mu \nu} \) around the AdS background \( g_{\mu \nu} \). The Einstein-Hilbert action in the three dimensions with negative cosmological constant is given by
\[SGR = -\frac{1}{16\pi G} \int d^3x (R + 2) \sqrt{g}. \quad (74)\]

We will use the gauge of [1], where we add the gauge-fixing term to (74):

\[S_{GF} = \frac{1}{32\pi G} \int d^3x \sqrt{g} \nabla^\mu (h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h) \nabla^\nu (h^\rho_\nu - \frac{1}{2} \delta^\rho_\nu h). \quad (75)\]

It is convenient to define the traceless part and the trace part:

\[\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{3} h_{\rho\sigma} h^\rho_\sigma, \quad \phi = h^\rho_\rho. \quad (76)\]

The gauge-fixed action is given by [3]

\[S = -\frac{1}{32\pi G} \int d^3x \sqrt{g} \left\{ \frac{1}{2} \phi_{\mu\nu} (g^{\mu\rho} g^{\nu\sigma} \nabla^2 + 2 R^{\mu\rho\nu\sigma}) \phi_{\rho\sigma} - \frac{1}{12} \phi (\nabla^2 - 4) \phi \right\}. \quad (77)\]

We will wick-rotate \(\phi \to i \phi\) in order to make the kinetic term positive definite. The gauge-fixing term introduces a Fadeev-Popov field, which is a Grassmann-odd vector field:

\[S_{ghost} = \frac{1}{32\pi G} \int d^3x \sqrt{g} \eta_\mu (-g^{\mu\nu} \nabla^2 - R^{\mu\nu}) \eta^\nu. \quad (78)\]

Therefore, the gravity partition function can be obtained by subtracting the contribution of the vector ghost field with \(m^2 = 4\) and the scalar field with \(m^2 = 4\), which corresponds to the trace part of the fluctuation:

\[\log Z_{1-loop}^{gravity} = -\frac{1}{2} \log \det \Delta^{graviton} + \log \det \Delta^{vector} - \frac{1}{2} \log \det \Delta^{scalar}\]

\[= \sum_{n=1}^{\infty} \left( \frac{1}{2n \sinh \beta} + \frac{1}{2n (\cosh \beta - \cos \alpha)} \right) (e^{-\beta} \cos 2\alpha - e^{-2\beta} \cos \alpha) \]

\[+ \sum_{n=1}^{\infty} 2n (\frac{1}{|q^n - 1|^2} + \frac{1}{1 - |q^n|^2}) (q^{2n} (1 - \overline{q}^n) + \overline{q}^{2n} (1 - q^n))\]

\[= \sum_{n=2}^{\infty} \frac{\log |1 - q^n|^2}{2} + \sum_{n=1}^{\infty} \frac{1}{2n} \frac{1}{1 - |q^n|^2} (q^{2n} (1 - \overline{q}^n) + \overline{q}^{2n} (1 - q^n)). \quad (79)\]

The first term is just a half of the contribution in the original theory and the second term comes from the mirror contribution.

**IV. PHYSICAL INTERPRETATION OF THE RESULTS**

**A. BCFT interpretation of the partition function**

In this section we summarize the results of the partition function in the previous section and give the physical interpretation from the BCFT viewpoint. Before we move on let us review our strategy of the calculation. We calculated the partition function using the method of images.
In our calculation we approximate the ETW brane as the hard wall for the background metric, which means that the position of the ETW brane is determined by solving $K_{ab} = 0$. Then, we apply the method of images to the background AdS metric and consider the fluctuation of the metric around the solution. Here we note that any direct contribution to the partition function from the ETW brane. This is because the brane is tensionless and the action is proportional to the tension. Hence, we get the contribution of the ETW brane only through the method of images.

Let us summarize our result. We firstly consider the scalar part. The partition function of the free scalar field with Neumann boundary condition on ETW brane is given by

$$Z_{\text{Scalar}} = \left( \prod_{l=0}^{\infty} \prod_{l'=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+l'+1}}} \right) \left( \prod_{m=0}^{\infty} \frac{1}{\sqrt{1 - q^{2(m+h)+1}}} \right). \tag{80}$$

The first term is the ordinary part and the second one is the mirror part. The first term has a clear interpretation: it is from a primary field with conformal dimension $h$ and summation over its descendants. The square root is because we take the volume of the space to be a half of the original AdS space. The second term is coming from the mirror effect of the original field. However, because in the AdS/BCFT case the rotational symmetry of the torus is broken due the ETW brane, we have only one real parameter $\beta$. Therefore we expect that we should set $q = \overline{q}$ physically. Then we get

$$Z_{\text{Scalar}} = \left( \prod_{l=0}^{\infty} \prod_{l'=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+l'+2h}}} \right) \left( \prod_{m=0}^{\infty} \frac{1}{\sqrt{1 - q^{2(m+h)+2h}}} \right). \tag{81}$$

Next, we move to the vector field partition function. It is given by

$$Z_{\text{vector}} = \left( \prod_{l=0}^{\infty} \prod_{l'=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+l'+1}}} \frac{1}{\sqrt{1 - q^{l+l'+1}}} \frac{1}{\sqrt{1 - q^{2l+1}}} \frac{1}{\sqrt{1 - q^{2l+1}}} \right) \cdot \left( \prod_{l=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+l'+1}}} \frac{1}{\sqrt{1 - q^{l+l'+1}}} \frac{1}{\sqrt{1 - q^{2l+1}}} \frac{1}{\sqrt{1 - q^{2l+1}}} \right). \tag{82}$$

As is the case before the first term is the ordinary part and the second one is the mirror part. In both lines, the second term is the contribution from the longitudinal scalar mode, so we do not treat it here. The rest is coming from the transverse vector mode. The summation is over $L_{-1}$ and $\overline{L}_{-1}$ plus descendants contribution, which represents massless spin-1 particle. For a massive vector field we can replace the powers of $q + l$ with $1 + l + h$. If we identify $q$ and $\overline{q}$, then the partition function yields

$$Z_{\text{vector}} = \left( \prod_{l=0}^{\infty} \prod_{l'=0}^{\infty} \frac{1}{1 - q^{l+l'+1}} \frac{1}{1 - q^{l+l'+1}} \right) \left( \prod_{l=0}^{\infty} \frac{1}{1 - q^{2l+1}} \frac{1}{1 - q^{2l+1}} \right). \tag{83}$$

Finally, let us consider gravity partition function. After some calculation we get
The first part gives the summation over vacuum and its chiral Virasoro descendants. The second term looks complicated, but it has important physical meaning: the numerator represents the massive vector field with $h = 1$ and the denominator represents a massless spin-2 field. From $AdS_3/CFT_2$ case the bulk field with spin $l$ and mass $M$ can be related to a conformal dimension $2h$ via

\[ M^2 = \Delta(\Delta - 2) + l^2 \]
\[ \Delta = 2h + l. \]  

(85)

Therefore, the $h = 1$ massive vector field has a mass $M^2 = 4$ in the bulk, which exactly matches what appears as a ghost vector field when we fix the gauge redundancy of gravity. On this point we revisit in sub-section 4-C.

From the viewpoint of open string or chiral mode, it is natural to identify left and right-moving modes, so if we set $q = \overline{q}$, we get

\[ Z_\text{gravity} = \left( \prod_{m=2}^{\infty} \frac{1}{|1 - q^m|} \right) \left( \prod_{l=0}^{\infty} \frac{\sqrt{1 - q^l + 2q^{l+1}} \sqrt{1 - q^{l+1} + 2q^l}}{\sqrt{1 - q^{l+2}} \sqrt{1 - q^{l+2}}} \right). \]

(84)

(86)

There appears to be unusual exponent $2l + 2$ from the original result $[3]$. We note that even though taking into the ghost contribution this partition function seems to be physical in a sense that the coefficients of the expansion in powers of $q$ are positive. This means that the number of states at that conformal dimension is always positive. It is surprising that the ghost contribution goes away.

B. One-loop partition function with the Dirichlet boundary condition

So far we impose the Neumann boundary condition on the ETW brane. However, in general we must consider mixed boundary conditions. Therefore, in this section we repeat the calculation as we did in the previous section, but with the Dirichlet boundary condition on the ETW brane. We note that in $AdS/BCFT$ we usually impose the Neumann boundary condition on the ETW brane, but it may be interesting to impose the Dirichlet boundary condition as a generalization of holography, see $[12]$. In this section we write only the results briefly. We attach the detailed calculation in the appendix.

The procedure is simple: just flip the sign of the mirror part. We can easily deduce the result:
\[ Z_{\text{Scalar}} = \left( \prod_{m=0}^{\infty} \frac{1}{\sqrt{1 - q^m + h q^m}} \right) \left( \prod_{l=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+2} + h q^{l+2}}} \right) \] (87)

\[ Z_{\text{Vector}} = \left( \prod_{l=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+1} q' + 1 - q^{l+1} q'}} \right) \left( \prod_{l=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+1} q' + 1 - q^{l+1} q'}} \right) \] (88)

\[ Z_{\text{Gravity}} = \left( \prod_{m=2}^{\infty} \frac{1}{1 - q^m} \right) \left( \prod_{l=0}^{\infty} \frac{1}{\sqrt{1 - q^{l+2} q' + 1 - q^{l+2} q'}} \right) \] (89)

If we set \( q = \bar{q} \), we get

\[ Z_{\text{Scalar}} = \prod_{l \neq l'} \frac{1}{\sqrt{1 - q^{l+l'} + h q^{l+l'}}} \] (90)

\[ Z_{\text{Transverse Vector}} = \prod_{l \neq l'} \frac{1}{1 - q^{l+l'+1}} \] (91)

\[ Z_{\text{Gravity}} = \prod_{l=0}^{\infty} \frac{1}{(1 - q^{2l+3})^2} \] (92)

where we omit the longitudinal modes in the vector field partition function. The crucial point is that the result is the multiples of non-diagonal, which means that only the \( l \neq l' \) terms survive in the vector and the scalar part and for the gravity part factor is an odd integer unlike Neumann case, which is a factor of even integer.

### C. Consistency of the boundary condition

In this section we will be more careful about boundary conditions. Usually, in gravity we impose Neumann boundary condition (henceforth Neumann B.C.) or Dirichlet boundary condition (Dirichlet B.C.) on the ETW brane. At the tree-level, these conditions harm nothing in calculating physical quantities such as the partition function and the extrinsic curvature. However, the situation changes when we consider at the one-loop level. At the linearized level of the metric, we need a gauge-fixing term and a ghost field. The boundary conditions for these fields affect the one-loop partition function and what is more, the gauge transformation for the metric may not be compatible with the boundary condition. That is, the boundary operator \( B \) defines the gauge-invariant boundary condition

\[ B \phi \big|_{\partial M} = 0, \] (93)
if and only if there exist boundary conditions for the corresponding gauge parameter $\xi$

$$B \xi = 0,$$  \hspace{1cm} (94)

such that

$$B \delta \xi \phi = 0.$$ \hspace{1cm} (95)

This condition ensures the validity of the one-loop calculation of the gauge invariance with the Faddeev-Popov trick. Now we revisit our problem. In our calculation we impose Neumann or Dirichlet conditions for all fields including ghost fields. Let us check that this is reasonable or not. Firstly, we consider the Neumann case. Note that we now split the metric into the background AdS Poincare metric $g_{\mu\nu}$ and $h_{\mu\nu}$ representing the metric fluctuation. Of course, we should impose Neumann B.C. on $h_{ij}$

$$\partial_x h_{ij} \big|_{x=0} = 0; \hspace{1cm} (96)$$

where $i, j$ represents the tangential direction along the brane. This comes from the original condition $K_{ab} = 0$. The symmetric traceless tensor $\phi_{\mu\nu}$ and trace part $\phi$ are given by

$$\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \phi,$$

$$\phi = h^\rho_{\rho}.$$ \hspace{1cm} (97)

Therefore, we should also impose Neumann B.C. for these fields:

$$\partial_x \phi_{ij} \big|_{x=0} = 0,$$

$$\partial_x \phi \big|_{x=0} = 0.$$ \hspace{1cm} (98)

To see the condition for ghost fields we remember that the ghost generates gauge transformation for the metric:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_\mu \eta_\nu + \nabla_\nu \eta_\mu.$$ \hspace{1cm} (99)

Let us check the gauge invariance of Neumann B.C. The condition for the ghost field is

$$\partial_x (\nabla_i \eta_j + \nabla_j \eta_i) = 0.$$ \hspace{1cm} (100)

Christoffel symbols are functions of $y$, so we can satisfy this equation if we impose Neumann B.C. for the ghost vector field $\eta_i$ and the anti-ghost vector field $\overline{\eta}_i$. Next, we move to the components in the normal direction. If we allow the boundary to fluctuate infinitesimally along the $x$-direction, then we should impose Neumann boundary condition for $\eta_x$ and $\overline{\eta}_x$: 
\[ \partial_x \eta_x \big|_{x=0} = 0, \]
\[ \partial_x \eta_x \big|_{x=0} = 0. \]  
(101)

Additionally, the BRST variation for \( \eta_{\mu} \) is given by

\[ \delta \eta_{\mu} = \nabla^\nu h_{\mu \nu} - \frac{1}{2} \partial_{\mu} \phi. \]  
(102)

This variation should also satisfy 101 therefore we can get the boundary condition of the metric along the normal direction:

\[ \partial_x \nabla^\nu h^{\mu \nu} \big|_{x=0} = 0. \]  
(103)

Christoffel symbols are only functions of \( y \), so we can satisfy the above equation if we impose

\[ \partial_x h_{\mu \nu} \big|_{x=0} = 0. \]  
(104)

This is what we explicitly calculated in section 3. However, we can consider one more possible boundary condition. If we strictly fix the boundary to be on \( x = 0 \), then we should impose the Dirichlet boundary condition for \( \eta_x \) and \( \overline{\eta}_x \):

\[ \eta_x \big|_{x=0} = 0, \]
\[ \overline{\eta}_x \big|_{x=0} = 0. \]  
(105)

Correspondingly BRST variation of the ghost field changes as

\[ \delta \eta_{\mu} = \nabla^\nu h_{\mu \nu} - \frac{1}{2} \partial_{\mu} \phi = 0, \]  
(106)

at the boundary \( x = 0 \). This equation determines the boundary condition for the metric along \( x \) direction.

Next, we consider the Dirichlet case. For the tangential direction we should impose Dirichlet B.C.

\[ h_{ij} = 0, \]
\[ \phi_{ij} = \phi = 0, \]
\[ \eta_i = \overline{\eta}_i = 0, \]  
(107)

at the boundary \( x = 0 \). If we require that the gauge transformation should vanish at \( x = 0 \), then we get

\[ \eta_x \big|_{x=0} = 0, \]
\[ \overline{\eta}_x \big|_{x=0} = 0. \]  
(108)
The BRST variation gives additional constraints on the metric

$$\delta \eta_{\mu} = \nabla^{\nu} h_{\mu \nu} - \frac{1}{2} \partial_{\mu} \phi = 0.$$  \hspace{1cm} (109)

This result is previously discussed in [26] and is somewhat remarkable; in the Neumann case we show that we can impose the Neumann condition for all fields along the $x$-direction at the boundary, but in the Dirichlet case we must impose the Neumann like boundary condition for the metric as in [109]. Therefore, our calculation in section 3 for the Dirichlet case is not BRST invariant because we previously imposed the Dirichlet boundary condition for all fields and all components at the boundary. The equation [109] states that the fluctuation of the metric for the $x$-direction is not zero, but this seems to be inconsistent with the previous assumption that the gauge transformation should vanish at the boundary. For the completion of classifying the boundary condition we consider one more case. If we allow the boundary to fluctuate along the $x$ direction, we get

$$\partial_x \eta_x = \partial_x \eta_x = 0,$$  \hspace{1cm} (110)

at the boundary. Then, BRST variation gives

$$\partial_x \left( \nabla^{\nu} h^{\mu \nu} - \frac{1}{2} \partial^{\nu} \phi \right) = 0,$$  \hspace{1cm} (111)

at the boundary. However, in the Dirichlet case we should consider one more problem: the ellipticity of the differential operators [26].

The authors discuss that Euclidean linearized gravity with purely Dirichlet B.C. is not elliptic and hence perturbatively ill-defined. We note that in the AdS/CFT case we can allow the Dirichlet B.C. up to the Weyl transformation of the metric and this B.C. is elliptic as in [26]. As a consistency check let us consider the ellipticity of the Neumann B.C. at the level of [26]. Now we impose the Neumann B.C. on all the components of the metric, so the metric can be written as

$$h_{\mu \nu} = \zeta_{\mu \nu} \cos (k_x x) \exp (ik_1 x^1).$$  \hspace{1cm} (112)

If we differentiate with respect to the $x$, then at the boundary it vanishes automatically.

In the Dirichlet case we seem that there are no pathologies in our calculation of the partition function. However, we already know that this B.C. is not elliptic. It seems that this is inconsistent, but we can see the breakdown of the ellipticity in a similar way to the above case. We can take the Fourier transformation of the kernel and $x$ dependence comes from $\exp (ikx)$. Then the above argument shows that in the Dirichlet case the B.C. is not elliptic. We expect...
that in our calculation the infinitely many zero-modes are hidden in our regularization by the zeta function.

Our problem also comes from the fact that we have yet to find a suitable set of boundary conditions Euclidean gravity at one-loop level with manifold respecting BRST invariance. In this sense we can say that gravity dual of the BCFT may not exist at the one-loop level in the pure gravity. This problem has a long history: a small selection being \[10, 11, 22, 26\].

Previously we derive

$$Z_{\text{gravity}} = \left( \prod_{m=2}^{\infty} \frac{1}{|1 - q^m|} \right) \left( \prod_{l=0}^{\infty} \frac{\sqrt{1 - q^{l+2}q^{l+1}}}{\sqrt{1 - q^{l+2}q} \sqrt{1 - q^l q^{l+2}}} \right),$$  \hspace{1cm} (113)

and we see that for a vector field with $h = 1$ has mass $M^2 = 4$ in the bulk.

We summarize the above discussion. Firstly in both the Neumann and the Dirichlet case we have two sets of boundary conditions for the normal component of the vector fields like in 101 and 105. Therefore, our calculation in section 3 will be consistent with BRST invariance for the Neumann boundary condition on all the fields and all the components, but for the Dirichlet case, it is inconsistent. Secondly we consider the case where the Dirichlet B.C. is imposed on all the components of the metric. In this case the differential operator is not elliptic, so it is perturbatively ill-defined. However in the Neumann case we can check that the differential operator is elliptic at the level of 26.

D. $SL(2, \mathbb{Z})$ summation of the partition function

We derive the one-loop partition function in section 3 and consider taking $SL(2, \mathbb{Z})$ summation as in 4, 14. At first sight this seems to be hard because the first term in 84 cannot be expanded as a polynomial of $q$ and $\overline{q}$. However, physically we remember that we can set $q = \overline{q}$, so we will use 80 and 92.

Firstly, let us consider the Neumann case. In section 1 we calculated the tree-level partition function of the BTZ black hole. To derive the tree-level partition function of the thermal AdS, we can change the modular parameter as $\tau \to -\frac{1}{\tau}$. Therefore, partition function of the thermal AdS is given by

$$Z_{0,1}(\tau) = |q\overline{q}|^{-\frac{1}{2}} \left( \prod_{m=2}^{\infty} \frac{1}{|1 - q^m|} \right) \left( \prod_{l=0}^{\infty} \frac{\sqrt{1 - q^{l+2}q^{l+1}}}{\sqrt{1 - q^{l+2}q} \sqrt{1 - q^l q^{l+2}}} \right) = q^{-k} \prod_{l=0}^{\infty} \frac{1}{(1-q^{l+2})^2},$$  \hspace{1cm} (114)

where $k = \frac{1}{16G}$. The whole classical Einstein solution with the boundary torus is obtained by implementing modular transformation:
\[ \tau \to \gamma \tau = \frac{a \tau + b}{c \tau + d}, \quad (115) \]

where \( \gamma \in \text{SL}(2, \mathbb{Z})/\{\pm 1\} \). Hence, here we can take \( c > 0 \) and sum over \((c, d)\), which are relatively prime integers. The full partition function can be written as

\[ Z(\tau) = \sum_{(c,d)} q^{-k} \prod_{l=0}^{\infty} \frac{1}{(1 - q^{2l+2})} \]

\[ = \sum_{(c,d)} q^{-k+\frac{1}{6}} \]

\[ = \frac{1}{\sqrt{3(2\tau)\eta(2\tau)^2}} \sum_{(c,d)} \left( \sqrt{3(2\tau)} q^{-k+\frac{1}{6}} \right) |\gamma \]

\[ = E(\tau; \frac{1}{2} - \frac{1}{6}, 0) \]

\[ = \sqrt{3(2\tau)\eta(2\tau)^2}, \quad (116) \]

where we define \( E(\tau; n, m) = \sum_{(c,d)} \left( \sqrt{3(2\tau)} q^n q^m \right) \). Here we use the modular invariance of \( \sqrt{3(2\tau)\eta(2\tau)^2} \).

Next, we consider the Dirichlet case. As in the Neumann case, the partition function becomes

\[ Z(\tau) = \sum_{(c,d)} q^{-k-\frac{1}{12}} \eta(2\tau)^2 (1 - q)^2 \]

\[ = \frac{\eta(2\tau)^2}{\eta(\tau)^2 \sqrt{3(2\tau)}} \sum_{(c,d)} \left( q^{-k+\frac{1}{12}} (1 - q)^2 \sqrt{3(\tau)} \right) |\gamma \]. \quad (117) \]

We can continue the calculation as in [4], but we stop here and briefly discuss how to treat this partition function. Firstly, this Poincare series is divergent, so we need some regularization. One possible way is that we consider the following convergent series

\[ \sum_{(c,d)} (\sqrt{3(2\tau)})^q q^{-k+\frac{1}{6}}. \quad (118) \]

This series is convergent for \( \Re(s) > 1 \). However, as is presented in [4], we can take the analytic continuation to \( \Re(s) \leq 1 \) and especially at \( s = \frac{1}{2} \) this series is regular. We expect that the spectrum has negative densities of states as is the case in pure gravity [27, 28]. It will be an interesting direction to specify the black-hole microstates using the modularity of the theory. Note that here we have naturally assumed that we still admit modular invariance even inserting the ETW brane because from the boundary torus viewpoint it admits modularity and we just consider how to locate the position of the ETW brane in the solid torus.

E. One-loop exactness of the partition function

One-loop exactness of the partition function is an important problem as in any other theory. Pure gravity in three dimensions is one-loop exact because the bulk diffeomorphism is governed
by Virasoro symmetry, hence the partition function does not suffer any quantum correction other than Virasoro descendants and is one-loop exact. We expect the same thing in this study, though we have not explicitly shown it. In BCFT, we can use the double trick as explained in [29], so we have

\[ L_n = L_{-n}. \]  

(119)

Therefore, a half of the Virasoro symmetry or only the chiral mode survives. This also guarantees that if we can properly calculate respecting BRST invariance, then the partition function of our case will be one-loop exact.

V. CONCLUSIONS AND DISCUSSIONS

We calculated the tree-level partition function of the BTZ black hole in the presence of the end of the world brane with arbitrary tension and the result matches with non-rotating previous results. At the one-loop level we calculated the thermal AdS partition function with a tensionless brane and the result depends heavily on the choice of sets of boundary conditions. We explicitly calculated the partition function in the case where all the components of all the fields satisfy the Neumann boundary condition and in another case where all the components of all the fields satisfy the Dirichlet boundary condition. For the Neumann case, we expect that the system will be consistent with BRST quantization, but the partition function actually contains remnants of ghost fields. We expect that they represent the fluctuation of ghost fields. We note that this ghost mode does not arise from wrong sign of kinetic terms. Therefore this ghost mode is irrelevant to the stability of the ETW brane. However if we consider the physical set up where the BCFT region if surrounded by the boundaries, we can set \( q = 7 \) and we get a similar form as 1. Even though we take into the ghost contribution, the resulting partition function seems to be physical in a sense that it can be regarded as a summation over the positive number of states at any level of the conformal dimension. For the Dirichlet case we also encounter unphysical modes. There are other possible sets of boundary conditions, which are sets of mixed boundary conditions for each component, but in this paper we have not calculated due to its computational difficulty. This will be one of our future problems. Of course there are some possible models so that we can restore the consistency by adding a suitable set of matter fields. For example, in the supergravity case, we can determine using supersymmetry transformations [1]. Therefore, it will be interesting to specify the minimal set of boundary conditions and matter fields for well-defined gravity. For another future direction it will be the most important to specify what kind of mode localizes on the brane or decouple
from the bulk. As a bypath for formulating as gravity, it may be interesting to understand our problem in Chern-Simons formulation. We expect that in this setup we can understand the edge mode more clearly because we only consider the boundary conditions for gauge fields.

Toward deriving full partition function we should sum over all possible geometries with a given boundary condition \[30\]. In this study the bulk geometry is a half of the solid torus, therefore we can sum over torus moduli \(SL(2,\mathbb{Z})\). As we discussed briefly in section 4, the computation is similar as the one in \[4\]. We do not explicitly calculate, hence it will be an interesting direction to analyze the spectrum using modular Bootstrap in BCFT case as in pure gravity case \[27, 28\].

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APPENDIX A THE DERIVATIVES OF A GEODESIC DISTANCE

Here we present the calculation of the derivatives of $u$. For ordinary part (not mirror) we get

\[
\begin{align*}
\partial_{y} \partial_{x'} u &= -\frac{\cos(\theta)(e^\beta \cos(\phi - \alpha) - \cos(\phi))}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_{y} \partial_{\eta'} u &= -\frac{\cos(\theta)(e^\beta \sin(\phi - \alpha) - \sin(\phi))}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_{\phi} \partial_{\eta} u &= -\frac{\cos(\theta)(-e^\beta \cos(\phi - \alpha) + \cos(\phi))}{\rho^2 e^{2\beta} \sin^2(\theta)}, \\
\partial_{\phi} \partial_{\phi'} u &= -\frac{\cos(\theta)(-e^\beta \sin(\phi - \alpha) + \sin(\phi))}{\rho^2 e^{2\beta} \sin^2(\theta)}, \\
\partial_{y} \partial_{y'} u &= -\frac{(2\cosh(\beta) - \cosh r)}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_{x} \partial_{x'} u &= -\frac{1}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_{y} \partial_{\eta} u &= -\frac{1}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_{x} \partial_{\eta} u &= \partial_{\eta} \partial_{x'} u = 0, \\
\partial_{y} u &= \frac{1}{\rho \sin \theta} (e^{-\beta} - \cosh r), \\
\partial_{x} u &= -\frac{\cos(\theta)(e^\beta \cos(\phi - \alpha) - \cos(\phi))}{\rho e^\beta \sin^2(\theta)}, \\
\partial_{q} u &= -\frac{\cos(\theta)(e^\beta \sin(\phi - \alpha) - \sin(\phi))}{\rho e^\beta \sin^2(\theta)}, \\
\partial_{y'} u &= \frac{1}{\rho \sin \theta} (e^\beta - \cosh r), \\
\partial_{x'} u &= \frac{\cos(\theta)(e^\beta \cos(\phi - \alpha) - \cos(\phi))}{\rho e^\beta \sin^2(\theta)}, \\
\partial_{\eta'} u &= \frac{\cos(\theta)(e^\beta \sin(\phi - \alpha) - \sin(\phi))}{\rho e^\beta \sin^2(\theta)}. 
\end{align*}
\]

(120)

We also present those of mirror part
\[
\begin{align*}
\partial_y \partial_{x'} u &= \frac{\cos(\theta)(e^\beta \cos(\phi - \alpha) + \cos(\phi))}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_y \partial_{y'} u &= \frac{-\cos(\theta)e^\beta \sin(\phi - \alpha) - \sin(\phi)}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_x \partial_{y'} u &= \frac{\cos(\theta)(e^\beta \cos(\phi - \alpha) + \cos(\phi))}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_y \partial_{y'} u &= -\frac{\cos(\theta)\left(-e^\beta \sin(\phi - \alpha) + \sin(\phi)\right)}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_y \partial_{y'} u &= -\frac{(2 \cosh \beta - \cosh r)}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_x \partial_{x'} u &= -\frac{1}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_y \partial_{y'} u &= -\frac{1}{\rho^2 e^\beta \sin^2(\theta)}, \\
\partial_x \partial_{y'} u &= \partial_y \partial_{x'} u = 0,
\end{align*}
\]

\[(121)\]

**APPENDIX B THE CALCULATION OF THE DIRICHLET CASE**

Let us consider the case where we impose the Dirichlet B.C. on all the components of all the fields. In this case we can flip the sign of the mirror part in the calculation of the heat kernel. For the scalar part we can repeat the calculation and we get

\[
\begin{align*}
S^{(1)}_{\text{scalar}} &= \frac{1}{2} \int_{+0}^{\infty} \frac{dt}{t} \int_{\text{thermal AdS}} d^3x \sum_{\mu} \sqrt{g} \left(K^H(t, x, \gamma^n x) - K^H(t, x^{\text{mirror}}, \gamma^n x)\right) \\
&= \sum_{n=1}^{\infty} \frac{|q|^n (1 + \sqrt{m^2 + 1})}{n |1 - q^n|^2} - \sum_{n=1}^{\infty} \frac{|q|^n (1 + \sqrt{m^2 + 1})}{n (1 - |q|^{2n})}.
\end{align*}
\]

\[(122)\]

The vector and symmetric traceless part can also be calculated

\[
\begin{align*}
S^{(1)}_{\text{vector}} &= \int_{0}^{\infty} \frac{dt}{t} \int d^3x \sqrt{g} \sum_{\mu} g^{\mu \rho} \frac{\partial (\gamma^n x)_{\nu}}{\partial x^\rho} \left(K^H_{\mu \nu}(t, r(x, \gamma^n x)) - K^H_{\mu \nu}(t, r(x^{\text{mirror}}, \gamma^n x))\right), \\
&= \sum_{n=1}^{\infty} \frac{q^n + \overline{q}^n + |q|^{2n}}{n |1 - q^n|^2} - \sum_{n=1}^{\infty} \frac{q^n + \overline{q}^n + |q|^{2n}}{n (1 - |q|^{2n})},
\end{align*}
\]

\[(123)\]
\[ S_{\text{spin-2}}^{(1)} = \int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} \sum_n g^{\mu\rho} \frac{\partial (\gamma^n x)^{\mu'}}{\partial x^\rho} g^{\nu\sigma} \frac{\partial (\gamma^n x)^{\nu'}}{\partial x^\sigma} (K_{\mu\nu'}^{\text{H}}(t, r(x, \gamma^n x)) - K_{\mu\nu'}^{\text{mir}}(t, r(x_{\text{mirror}}, \gamma^n x)), \]
\[
= \sum_{n=1}^\infty \left( \frac{1}{n (\cosh \beta - \cos \alpha)} - \frac{1}{n \sinh \beta} \right) (e^{-\beta} \cos 2\alpha + e^{-2\beta} \cos \alpha + \frac{e^{-\sqrt{5}\beta}}{2}). \tag{124}
\]

Finally the gravity partition function becomes
\[
\log Z_{\text{gravity}}^{1\text{-loop}} = -\frac{1}{2} \log \det \Delta^{\text{graviton}} + \log \det \Delta^{\text{vector}} - \frac{1}{2} \log \det \Delta^{\text{scalar}},
\]
\[
= \sum_{n=1}^\infty \left( \frac{1}{2n \sinh \beta} - \frac{1}{2n (\cosh \beta - \cos \alpha)} \right) (e^{-\beta} \cos 2\alpha - e^{-2\beta} \cos \alpha),
\]
\[
\sum_{n=1}^\infty \frac{1}{2n} \left( \frac{1}{|q^n - 1|^2} - \frac{1}{1 - |q^n|^2} \right) (q^{2n}(1 - \bar{q}^n) + \bar{q}^{2n}(1 - q^n)), \tag{125}
\]
\[
= \sum_{n=2}^\infty \frac{\log |1 - q^n|^2}{2} - \sum_{n=1}^\infty \frac{1}{2n} \frac{1}{1 - |q^n|^2} (q^{2n}(1 - \bar{q}^n) + \bar{q}^{2n}(1 - q^n)).
\]