AN EXTENSION OF A RESULT OF ERDŐS AND ZAREMBA

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Abstract. Erdős and Zaremba showed that $\lim \sup_{n \to \infty} \frac{\Phi(n)}{(\log \log n)^2} = e^\gamma$, $\gamma$ being Euler’s constant, where $\Phi(n) = \sum_{d | n} \frac{\log d}{d}$. We extend this result to the function $\Psi(n) = \sum_{d | n} \frac{(\log d)(\log \log d)}{d}$ and some other functions. We show that $\lim \sup_{n \to \infty} \frac{\Psi(n)}{(\log \log n)^2(\log \log \log n)} = e^\gamma$. The proof requires to develop a new approach. As an application, we prove that for any $\eta > 1$, any finite sequence of reals $\{c_k, k \in K\}$,

$$\sum_{k, \ell \in K} c_k c_\ell \frac{\gcd(k, \ell)^2}{k \ell} \leq C(\eta) \sum_{\nu \in K} c_\nu^2 (\log \log \log \nu)^\eta \Psi(\nu),$$

where $C(\eta)$ depends on $\eta$ only. This improves a recent result obtained by the author.

1. Introduction.

Erdős and Zaremba showed in [4] the following result concerning the arithmetical function $\Phi(n) = \sum_{d | n} \frac{\log d}{d}$,

$$\lim \sup_{n \to \infty} \frac{\Phi(n)}{(\log \log n)^2} = e^\gamma,$$

where $\gamma$ is Euler’s constant. This function appears in the study of good lattice points in numerical integration, see Zaremba [11]. The proof is based on the identity

$$\Phi(n) = \sum_{i=1}^r \sum_{\nu_i=1}^{\alpha_i} \frac{\log p_i^{\nu_i}}{p_i^{\nu_i}} \sum_{d | n p_i^{-\nu_i}} \frac{1}{d} \delta = \prod_{i=1}^r \left( \sum_{\mu_i=0}^{\alpha_i} \frac{1}{p_i^{\nu_i}} \prod_{i=1}^r \left( \sum_{\mu_i=0}^{\alpha_i} \mu_i \log p_i \right) \right).$$

Let $h(n)$ be non-decreasing on integers, $h(n) = o(\log n)$, and consider the slightly larger function

$$\Phi_h(n) = \sum_{d | n} \frac{(\log d) h(d)}{d}.$$

In this case a formula similar to (1.3) no longer hold, the ”log-linearity” being lost due to the extra factor $h(n)$. The study of this function requires to devise a new approach. We study in this work the case $h(n) = \log \log n$, that is the function

$$\Psi(n) = \sum_{d | n} \frac{(\log d)(\log \log d)}{d}.$$

We extend Erdős-Zaremba’s result for this function, as well as for the functions
where \( \Omega(d) \) denotes as usual the number of powers of primes dividing \( d \). These functions are linked to \( \Psi \).

Throughout, \( \log \log x \) (resp. \( \log \log \log x \)) equals 1 if \( 0 \leq x \leq e^e \) (resp. \( 0 \leq x \leq e^{e^e} \)), and equals \( \log \log x \) (resp. \( \log \log \log x \)) in the usual sense if \( x > e^e \) (resp. \( x > e^{e^{e^e}} \)).

One verifies by using standard arguments that

\[
\limsup_{n \to \infty} \frac{\Phi_1(n)}{(\log \log n)^2(\log \log \log n)} \geq e^\gamma,
\]

and in fact that

\[
\limsup_{n \to \infty} \frac{\Phi_1(n)}{(\log \log n)^2(\log \log \log n)} = e^\gamma.
\]

By the observation made after (1.3), the corresponding extension of this result to \( \Psi(n) \) is technically more delicate. It follows from (1.1) that

\[
\limsup_{n \to \infty} \frac{\Psi(n)}{(\log \log n)^3} \leq e^\gamma.
\]

The question thus arises whether the exponent of \( \log \log n \) in (1.8) can be replaced by \( 2 + \varepsilon \), with \( \varepsilon > 0 \) small.

We answer this question affirmatively by establishing the following precise result, which is the main result of this paper.

**Theorem 1.1.**

\[
\limsup_{n \to \infty} \frac{\Psi(n)}{(\log \log n)^2(\log \log \log n)} = e^\gamma.
\]

An application of this result is given in Section 5. The upper bound is obtained, via the inequality

\[
\Psi(n) \leq \Phi_1(n) + \Phi_2(n),
\]

as a combination of an estimate of \( \Phi_1(n) \) and the following estimate of \( \Phi_2(n) \). Recall that Davenport’s function \( w(n) \) is defined by \( w(n) = \sum_{p \mid n} \frac{\log p}{p} \). According to Theorem 4 in [2] we have

\[
\limsup_{n \to \infty} \frac{w(n)}{\log \log n} = 1.
\]

**Theorem 1.2.** For all even numbers \( n \) we have,

\[
\Phi_2(n) \leq C (\log \log \log \omega(n))(\log \omega(n))w(n).
\]

where \( C \) is an absolute constant.

Here and elsewhere \( C \) (resp. \( C(\eta) \)) denotes some positive absolute constant (resp. some positive constant depending only of a parameter \( \eta \)).

The approach used for proving Theorem 1.2 can be adapted with no difficulty to other arithmetical functions of similar type.

The paper is organized as follows. Sections 2 and 3 form the main part of the paper, and consist with the proof of Theorem 1.2 which is long and technical and involves the building of a binary tree (subsection 2.2.1). The proof of Theorem 1.1 is given in section 3. Section 4 contains complementary results and the proofs of (1.6), (1.7). Section 5 concerns the aforementioned application of Theorem 1.1. Additional remarks or results are concluding the paper in Section 6.
2. Proof of Theorem 1.2

We use a chaining argument. We make throughout the convention $0 \log 0 = 0$.

Let $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ be an even number. We will use repeatedly the fact that

$$(2.1) \quad \min_{i=1}^r p_i \geq 3.$$ 

We note that

$$\Phi_2(n) = \sum_{\mu_1=0}^{\alpha_1} \ldots \sum_{\mu_r=0}^{\alpha_r} \frac{1}{p_1^{\mu_1} \ldots p_r^{\mu_r}} \sum_{i=1}^r \mu_i \left( \log p_i \right) \log \left( \sum_{i=1}^r \mu_i \right)$$

$$= \sum_{i=1}^r \sum_{\mu_1=0}^{\alpha_1} \ldots \sum_{\mu_r=0}^{\alpha_r} \frac{1}{p_1^{\mu_1} \ldots p_r^{\mu_r}} \left( \sum_{i=1}^r \mu_i \left( \log p_i \right) \right) \log \left( \sum_{i=1}^r \mu_i \right).$$

As there is no order relation on the sequence $p_1, \ldots, p_r$, it suffices to study the sum

$$\Phi_2(r, n) := \sum_{\mu_1=0}^{\alpha_1} \ldots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \ldots p_{r-1}^{\mu_{r-1}}} \sum_{\mu_r=0}^{\alpha_r} \mu_r \log p_r \log \left[ \sum_{i=1}^{r-1} \mu_i + \mu_r \right].$$

The sub-sums in (2.3) will be estimated by using a recursion argument.

2.1. Preparation. Some technical lemmas are preliminary needed.

**Lemma 2.1.** (i) Let $\varphi_1(x) = x(\log(A + x)) e^{-\alpha x}$, $\varphi_2(x) = (\log(A + x)) e^{-\alpha x}$. Then $\varphi_1(x)$ is non-increasing on $[3, \infty)$ if $A \geq 1$ and $\alpha \geq \log 2$. Further, $\varphi_2(x)$ is non-increasing on $[1, \infty)$, if $A \geq 1$ and $\alpha \geq 1$.

(ii) Assume that $A \geq 1$ and $\alpha \geq \log 2$. For any integer $m \geq 1$,

$$\alpha \int_m^\infty x(\log(A + x)) e^{-\alpha x} \, dx \leq \frac{1}{\alpha^2(A + m)} e^{-\alpha m} + \frac{1}{\alpha} e^{-\alpha m} + \frac{1}{\alpha} (\log(A + m)) e^{-\alpha m}.$$  

(2.4)

(iii) Assume that $A \geq 1$ and $\alpha \geq 1$. Then,

$$\int_1^\infty (\log(A + x)) e^{-\alpha x} \, dx \leq \frac{\log(A + 1)}{\alpha} e^{-\alpha} + \frac{1}{\alpha^2(A + 1)} e^{-\alpha}.$$  

(2.5)

**Proof.** (i) We have $\varphi_1'(x) = (\log(A + x)) e^{-\alpha x} - \alpha(\log(A + x)) e^{-\alpha x} - \alpha' x(\log(A + x)) e^{-\alpha x}$. By assumption and since $\varphi_1'(x) \leq 0 \iff \frac{1}{x} + \frac{1}{(A + x) \log(A + x)} \leq \alpha$, we get

$$\frac{1}{x} + \frac{1}{(A + x) \log(A + x)} \leq \frac{1}{3} + \frac{1}{8 \log 2} \leq \frac{1}{3} + \frac{1}{5} < \log 2 \leq \alpha.$$  

Similarly $\varphi_2'(x) = \frac{1}{x} e^{-\alpha x} - \alpha(\log(A + x)) e^{-\alpha x}$. As $\varphi_2'(x) \leq 0 \iff (A + x) \log(A + x) \geq \frac{1}{\alpha}$, we also get

$$(A + x) \log(A + x) \geq 2 \log 2 > 1 \geq \frac{1}{\alpha}.$$  

(ii) We deduce from (i) that

$$\alpha \left( \log(A + x) \right) e^{-\alpha x} = (\log(A + x)) e^{-\alpha x} + \frac{x}{A + x} e^{-\alpha x} - (x(\log A + x)e^{-\alpha x})'.$$

By integrating,

$$\alpha \int_m^\infty x(\log(A + x)) e^{-\alpha x} \, dx = \int_m^\infty x(\log(A + x)) e^{-\alpha x} \, dx + \int_m^\infty \frac{x}{A + x} e^{-\alpha x} \, dx + m(\log A + m)e^{-\alpha m}.$$  

(2.6)
Similarly
\[ \alpha \int_{m}^{\infty} (\log(A + x))e^{-\alpha x}dx = \int_{m}^{\infty} \frac{1}{A + x}e^{-\alpha x}dx + (\log(A + m))e^{-\alpha m}. \]

By combining we get,
\[ \alpha \int_{m}^{\infty} x(\log(A + x))e^{-\alpha x}dx = \frac{1}{\alpha} \int_{m}^{\infty} \frac{1}{A + x}e^{-\alpha x}dx + \int_{m}^{\infty} \frac{x}{A + x}e^{-\alpha x}dx + \frac{1}{\alpha} (\log(A + m))e^{-\alpha m} + m(\log A + m)e^{-\alpha m}. \]

Therefore,
\[ \alpha \int_{m}^{\infty} x(\log(A + x))e^{-\alpha x}dx \leq \frac{1}{\alpha^2(A + m)}e^{-\alpha m} + \frac{1}{\alpha}e^{-\alpha m} + \frac{1}{\alpha}(\log(A + m))e^{-\alpha m} + m(\log A + m)e^{-\alpha m}. \]

(iii) We deduce from (i) that
\[ \int_{1}^{N} (\log(A + x))e^{-\alpha x}dx = \frac{1}{\alpha} \int_{1}^{N} \frac{1}{(A + x)}e^{-\alpha x}dx \]
+ \frac{1}{\alpha} \left( (\log(A + 1))e^{-\alpha} - \log(A + N)e^{-\alpha N} \right). \]

As \[ \frac{1}{\alpha} \int_{1}^{N} \frac{1}{A + x}e^{-\alpha x}dx \leq \frac{1}{\alpha^2(A + 1)}e^{-\alpha}, \]
letting \(N\) tend to infinity gives,
\[ \int_{1}^{\infty} (\log(A + x))e^{-\alpha x}dx \leq \frac{\log(A + 1)}{\alpha}e^{-\alpha} + \frac{1}{\alpha^2(A + 1)}e^{-\alpha}. \]

\[ \square \]

**Lemma 2.2.** Assume that \(A \geq 1, \) and \(\alpha \geq 1. \) Then,
\[ \sum_{\mu=0}^{\infty} \alpha \mu (\log(A + \mu))e^{-\alpha \mu} \leq \alpha (\log(A + 1))e^{-\alpha} + 2\alpha (\log(A + 2))e^{-2\alpha} \]
+ \(3\alpha \log(A + 3) + 3\log(A + 3) + \frac{1}{\alpha} \log(A + 3) + \frac{1}{\alpha} + \frac{1}{\alpha^2(A + 3)} \) \(e^{-3\alpha}. \)

**Proof.** As
\[ \sum_{\mu=0}^{\infty} \alpha \mu (\log(A + \mu))e^{-\alpha \mu} = \alpha (\log(A + 1))e^{-\alpha} + 2\alpha (\log(A + 2))e^{-2\alpha} \]
+ \(3\alpha \log(A + 3) + 3\log(A + 3) + \frac{1}{\alpha} \log(A + 3) + \frac{1}{\alpha} + \frac{1}{\alpha^2(A + 3)} \) \(e^{-3\alpha}. \)

by applying Lemma 2.1 (ii), we get
\[ \alpha \sum_{\mu=4}^{\infty} \mu (\log(A + \mu))e^{-\alpha \mu} \leq \alpha \int_{3}^{\infty} x \log(A + x))e^{-\alpha x}dx \]
\leq \( \frac{1}{\alpha^2(A + 3)}e^{-3\alpha} + \frac{1}{\alpha}e^{-3\alpha} + \frac{\log(A + 3)}{\alpha}e^{-3\alpha} + 3(\log A + 3)e^{-3\alpha}. \)

Whence,
\[ \sum_{\mu=0}^{\infty} \alpha \mu (\log(A + \mu))e^{-\alpha \mu} \leq \alpha (\log(A + 1))e^{-\alpha} + 2\alpha (\log(A + 2))e^{-2\alpha} \]
+ \(3\alpha \log(A + 3) + 3\log(A + 3) + \frac{1}{\alpha} \log(A + 3) + \frac{1}{\alpha} + \frac{1}{\alpha^2(A + 3)} \) \(e^{-3\alpha}. \)

\[ \square \]
Lemma 2.3. Under assumption \((2.1)\) we have

\[
\sum_{\mu_s=0}^{\infty} \frac{\log \left( \sum_{i=1}^{s} \mu_i + h \right)}{p_s^e} \leq \log \left( \sum_{i=1}^{s-1} \mu_i + h \right) + \frac{1}{p_s} \left( 1 + \frac{1}{\log p_s} \right) \log \left( \sum_{i=1}^{s-1} \mu_i + h + 1 \right) + \frac{1}{1 + (\sum_{i=1}^{s-1} \mu_i + 2)(\log p_s)^2 p_s}.
\]

In particular,

\[
\sum_{\mu_s=0}^{\infty} \frac{\log \left( \sum_{i=1}^{s-1} \mu_i + h \right)}{p_s^e} \leq \left( 1 + \frac{1}{p_s} \left( 1 + \frac{1}{\log p_s} + \frac{1}{3(\log p_s)^2} \right) \right) \log \left( \sum_{i=1}^{s-1} \mu_i + h + 2 \right).
\]

Proof. As Lemma 2.3. Under assumption \((2.1)\) we have

\[
\sum_{\mu=0}^{\infty} (\log(A + \mu))e^{-\alpha \mu} = \log A + (\log(A + 1))e^{-\alpha} + \sum_{\mu=2}^{\infty} (\log(A + \mu))e^{-\alpha \mu} \leq \log A + (\log(A + 1))e^{-\alpha} + \int_{1}^{\infty} (\log(A + x))e^{-\alpha x} \, dx
\]

we deduce from Lemma \(2.1\)(iii),

\[
(2.6) \sum_{\mu=0}^{\infty} (\log(A + \mu))e^{-\alpha \mu} \leq \log A + e^{-\alpha} \left( \log(A + 1) + \frac{\log(A + 1)}{\alpha} + \frac{1}{\alpha^2(A + 1)} \right).
\]

Consequently,

\[
\sum_{\mu_s=0}^{\infty} \frac{\log \left( \sum_{i=1}^{s} \mu_i + h \right)}{p_s^e} \leq \log \left( \sum_{i=1}^{s-1} \mu_i + h \right) + \frac{1}{p_s} \left( 1 + \frac{1}{\log p_s} \right) \log \left( \sum_{i=1}^{s-1} \mu_i + h + 1 \right) + \frac{1}{1 + (\sum_{i=1}^{s-1} \mu_i + 2)(\log p_s)^2 p_s}.
\]

Finally,

\[
\sum_{\mu_s=0}^{\infty} \frac{\log \left( \sum_{i=1}^{s-1} \mu_i + h \right)}{p_s^e} \leq \left( 1 + \frac{1}{p_s} \left( 1 + \frac{1}{3(\log p_s)^2} \right) \right) \log \left( \sum_{i=1}^{s-1} \mu_i + h + 2 \right).
\]

\[\square\]

Corollary 2.4. Assume that condition \((2.1)\) is satisfied.

(i) If \(\sum_{i=1}^{r-1} \mu_i \geq 1\), then

\[
\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^e} \log \left( \sum_{i=1}^{r-1} \mu_i + \mu_r \right) \leq \frac{\log p_r}{p_r} \log \left( \sum_{i=1}^{r-1} \mu_i + 1 \right) + \frac{2 \log p_r}{p_r^2} \log \left( \sum_{i=1}^{r-1} \mu_i + 2 \right) + \frac{1}{p_r^3 \log p_r} \left( 3 \log p_r + 3 + \frac{1}{\log p_r} \right) \log \left( \sum_{i=1}^{r-1} \mu_i + 3 \right) + \frac{1}{p_r^3 \log p_r} \left( 1 + \frac{1}{(\sum_{i=1}^{r-1} \mu_i + 3) \log p_r} \right).
\]

Further,

\[
\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^e} \log \left( \sum_{i=1}^{r-1} \mu_i + \mu_r \right) \leq 5 \frac{\log p_r}{p_r} \log \left( \sum_{i=1}^{r-1} \mu_i + 3 \right).
\]

(ii) If \(\sum_{i=1}^{r-1} \mu_i = 0\), then

\[
\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^e} \log \left( \sum_{i=1}^{r-1} \mu_i + \mu_r \right) \leq 18 \frac{\log p_r}{p_r}.
\]
Proof. (i) The first inequality follows from Lemma 2.2 with the choice $\alpha = \log p_r$, $A = \sum_{i=1}^{r-1} \mu_i$, noting that by assumption (2.1), $\alpha > 1$. As $p_r \geq 3$, it is also immediate that

$$
\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left( \sum_{i=1}^{r-1} \mu_i + \mu_r \right)
\leq \left\{ 3 \frac{\log p_r}{p_r} + \frac{\log p_r}{9p_r} \left( 3 + 3 \frac{1}{\log p_r^2} \right) \right\} \log \left( \sum_{i=1}^{r-1} \mu_i + 3 \right) + \frac{1}{9p_r \log p_r} \left( 1 + \frac{1}{4 \log p_r} \right)
\leq 5 \frac{\log p_r}{p_r} \log \left( \sum_{i=1}^{r-1} \mu_i + 3 \right).
$$

(ii) If $\sum_{i=1}^{r-1} \mu_i = 0$, the sums relatively to $\mu_i$, $1 \leq i \leq r-1$, do not contribute. Further,

$$
\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left( \sum_{i=1}^{r-1} \mu_i + \mu_r \right) = \sum_{\mu_r=2}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \mu_r = \frac{\alpha_r-1}{p_r^{\mu_r+1}} \log(\mu + 1)
\leq \frac{1}{p_r} \left\{ \sum_{\mu_r=1}^{\infty} \frac{\mu \log p_r}{p_r^{\mu_r}} \log(\mu + 1) + \sum_{\mu_r=1}^{\infty} \frac{\log p_r}{p_r^{\mu_r}} \log(\mu + 1) \right\}.
$$

Lemma 2.2 applied with $A = 1$ and $\alpha = \log p_r$ provides the bound

$$
\sum_{\mu_r=1}^{\infty} \frac{\mu \log p_r}{p_r^{\mu_r}} \log(\mu + 1) \leq \frac{\log 2 \log p_r}{p_r} + \frac{2(\log 3) \log p_r}{p_r^2} + \frac{1}{p_r} \left\{ (6 \log 2)(\log p_r) + 6 \log 2 + \frac{2 \log 2}{(\log p_r)} + \frac{1}{(\log p_r)^2} \right\}
\leq 8 \left( \frac{\log p_r}{p_r} + \frac{1}{p_r^2} \right).
$$

Next estimate (2.10) applied with $A = 1$ and $\alpha = \log p_r$, further gives,

$$
\sum_{\mu_r=1}^{\infty} \frac{\log p_r}{p_r^{\mu_r}} \log(\mu + 1) \leq \frac{1}{p_r} \left( \log 2 + \frac{\log 2}{\log p_r} + \frac{1}{2(\log p_r)^2} \right) \leq \frac{2}{p_r},
$$

Whence,

$$
\sum_{\mu_r=0}^{\infty} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left( \sum_{i=1}^{r-1} \mu_i + \mu_r \right) \leq 18 \frac{\log p_r}{p_r}.
$$

\square

Remark 2.5. As $\log \left( \sum_{i=1}^{r} \mu_i + h \right) \leq \log (\Omega(n) + 3)$, one can deduce from Corollary 2.4(ii) that

$$
\Phi_2(r, n) \leq 18 \frac{\log p_r}{p_r} \log(\Omega(n) + 3) \prod_{i=1}^{r} \left( \frac{1}{1 - \frac{1}{p_i^{\alpha_i}}} \right).
$$

So that by the observation made at the beginning of section 2

$$
\Phi_2(n) \leq 18 \left( \log(\Omega(n) + 3) \right) \left( \sum_{j=1}^{r} \frac{\log p_j}{p_j} \right) \prod_{i=1}^{r} \left( \frac{1}{1 - \frac{1}{p_i^{\alpha_i}}} \right).
$$

By combining this with the bound for $\Phi_1(n)$ established in Lemma 4.1, next using inequality (1.9), gives

$$
\Psi(n) \leq \left( \prod_{j=1}^{r} \frac{1}{p_j} \right) \left\{ \sum_{i=1}^{r} \frac{\log p_i}{p_i} \left( \frac{\log \log p_i}{p_i - 1} \right) + 18 \left( \sum_{i=1}^{r} \frac{\log p_i}{p_i} \right) \log(\Omega(n) + 3) \right\},
$$

recalling that $r = \omega(n)$. Whence by invoking Proposition 4.3 noticing that $\omega(n) \leq \Omega(n) \leq \log_2 n$,

$$
\Psi(n) \leq e^\gamma (1 + o(1))(\log \log n)^2 \left( \log \log \log n + 18\omega(n) \right).
$$

The finer estimate of $\Psi(n)$ will be derived from a more precise study of the coefficients of $\Psi(r, n)$. This is the object of the next sub-section.
2.2. Estimates of $\Phi_2(r, n)$. We define successively

$$
\begin{aligned}
\mu &= (\mu_1, \ldots, \mu_r), \quad (\mu_1, \ldots, \mu_r) \in \prod_{i=1}^{r} \left( [0, \alpha_i] \cap \mathbb{N} \right), \\
p_{\mu}(s) &= p_{1^{\mu_1}} \cdots p_{s^{\mu_r}}, \quad 1 \leq s \leq r, \\
\Pi_{s} &= \sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_s=0}^{\alpha_s} p_{\mu}(s) = \prod_{i=1}^{s} \left( \frac{1-p_s^{\alpha_i-1}}{1-p_s-1} \right).
\end{aligned}
$$

Next,

$$
\Phi_s(h) = \sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_s=0}^{\alpha_s} p_{\mu}(s) \log \left( \sum_{i=1}^{s} \mu_i + h \right), \quad 1 \leq s \leq r - 1.
$$

We also set

$$
\begin{aligned}
c_1 &= 1, \quad c_2 = \frac{2}{p_r}, \quad c_3 = \frac{1}{p_r} \left( 3 + \frac{3}{\log p_r} + \frac{1}{(\log p_r)^2} \right), \\
c_4 &= \frac{p_r \log p_r}{\log p_r} (1 + \frac{1}{3 \log p_r}), \\
c_0 &= \frac{\log p_r}{p_r}, \\
b_s &= \frac{1}{p_r} \left( 1 + \frac{1}{\log p_r} \right), \\
b_3 &= \frac{1}{2p_r (\log p_r)^2}.
\end{aligned}
$$

2.2.1. Recurrence inequality. We deduce from the first part of Lemma 2.3.

$$
\Phi_s(h) \leq \Phi_{s-1}(h) + \frac{1}{p_s} \left( 1 + \frac{1}{\log p_s} \right) \Phi_{s-1}(h+1) + \frac{1}{2(\log p_s)^2 p_s} \Pi_{s-1}.
$$

Whence with the previous notation,

**Lemma 2.6.** Under assumption (2.1), we have for $s = 2, \ldots, r - 1$,

$$
\Phi_s(h) \leq \Phi_{s-1}(h) + b_s \Phi_{s-1}(h+1) + \beta_s \Pi_{s-1}.
$$

The notation introduced also allows one to rewrite estimate (i) of Lemma 2.4 in a more condensed form. Under assumption (2.1), if $\sum_{i=1}^{r-1} \mu_i \geq 1$, we also have

$$
\Phi_2(r, n) \leq c_0 \sum_{h=1}^{3} \left( c_h \Phi_{r-1}(h) + c_4 \Pi_{r-1} \right).
$$

By applying twice the recurrence inequality, we also obtain

$$
\Phi_2(r, n) \leq \sum_{h=1}^{3} \left( c_h \Phi_{r-3}(h) + c_0 \sum_{h=1}^{3} c_h b_{r-2} \Phi_{r-3}(h+1) + c_0 b_{r-2} \Pi_{r-3} \right) + c_0 \sum_{h=1}^{3} c_h b_{r-1} \Phi_{r-3}(h+1) + c_0 \sum_{h=1}^{3} c_h b_{r-1} b_{r-2} \Phi_{r-3}(h+2) + c_0 b_{r-1} b_{r-2} \Pi_{r-3} + c_0 c_3 \Pi_{r-2} + c_4 \Pi_{r-1}.
$$

One easily verifies (see expressions underlined by (1)) that the coefficient of $\Phi_{r-1}(h)$ is the same as the one of $\Phi_{r-2}(h)$ and $\Phi_{r-3}(h)$. So is also the case for $\Phi_{r-2}(h+1)$, see expressions underlined by (2). New expressions underlined by (3), (4) and linked to $\Phi_{r-3}(h+1), \Phi_{r-3}(h+2)$ appear.

Each new coefficient is kept until the end of the iteration process generated by the recurrence inequality of Lemma 2.6.

We also verify, when applying this inequality, that we pass from a majoration expressed by $\Phi_{r-1}(h), \Pi_{r-1}$, uniquely, to a majoration expressed by $\Phi_{r-2}$ (in $h$ or $h+1$) and $\Pi_{r-2}, \Pi_{r-1}$ uniquely.
This rule is general, and one verifies that when iterating this recurrence relation, we obtain at each step a bound depending on $\Phi_{r-d}$ and the products $\Pi_{r-d}, \Pi_{r-d+1}, \ldots, \Pi_{r-1}$ only.

**Binary tree**: The shift of length $h$ or $h + 1$ generates a binary tree whose branches are at each division (steps corresponding to the preceding iterations), either stationary: $\Phi_{r-d}(h) \rightarrow \Phi_{r-d-1}(h)$, or creating new coefficients: $\Phi_{r-d}(h) \rightarrow \Phi_{r-d-1}(h + 1)$. One can represent this by the diagram below drawn from Lemma 2.6:

\[ \downarrow \text{shift } +1, \text{ new coefficients} \downarrow \]

\[ \Phi_s(h) \leq \Phi_{s-1}(h) + b_s\Phi_{s-1}(h + 1) + \beta_s\Pi_{s-1}. \]

Before continuing, we recall that by (2.6),

\[ \sum_{\mu=0}^{\infty} (\log(A + \mu))e^{-\alpha \mu} \leq \log A + e^{-\alpha} \left( \frac{\log(A + 1)}{\alpha} + \frac{1}{\alpha^2(A + 1)} \right). \]

Thus

\[ \Phi_1(v) \leq \sum_{\mu_1=0}^{\infty} p_\mu(1) \log \left( \sum_{i=1}^{v} \mu_i + 1 \right) = \sum_{\mu_1=0}^{\infty} \frac{\log(v + \mu)}{p_1^\mu} \leq \log v + \frac{1}{p_1} \left( \log(v + 1) + \frac{\log(v + 1)}{\log p_1} + \frac{1}{v(\log p_1)^2} \right) \quad (v \geq 1). \]

Hence,

\[ \Phi_1(h) \leq C \log h. \]

One easily verifies that the $d$-tuples formed with the $b_i$ have all $\Phi_{r-x}(h + d)$ as factor. The terms having $\Phi_{r-x}(h + \cdot)$ as factor are forming the sum

\[ (2.10) \quad c_0 \sum_{d=1}^{r-1} \left( \sum_{1 \leq i_1 < \ldots < i_d < r} b_{i_1} \ldots b_{i_d} \right) \Phi_1(h + d), \]

once the iteration process achieved, that is after having applied $(r - 1)$ times the recurrence inequality of Lemma (2.6).

This sums can thus be bounded from above by (recalling that $h = 1, 2$ or $3$)

\[ c_0 \sum_{d=1}^{r-1} \left( \log d \right) \left( \sum_{1 \leq i_1 < \ldots < i_d < r} b_{i_1} \ldots b_{i_d} \right). \]

But, for all positive integers $a_1, \ldots, a_r$ and $1 \leq d \leq r$, we have,

\[ \left( \sum_{i=1}^{r} a_i \right)^d \geq d! \sum_{1 \leq i_1 < \ldots < i_d \leq r} a_{i_1} \ldots a_{i_d}. \]

Thus

\[ \sum_{d=1}^{r-1} \left( \log d \right) \left( \sum_{1 \leq i_1 < \ldots < i_d < r} b_{i_1} \ldots b_{i_d} \right) \leq \sum_{d=1}^{r-1} \left( \frac{\log d}{d!} \right) \left( \sum_{i=1}^{r} b_i \right)^d. \]

As moreover,

\[ b_i = \frac{1}{p_i} \left( 1 + \frac{1}{p_i + 1} \right) \leq \frac{1}{p(i)} + \frac{1}{p(i) \log p(i)}, \]

one has by means of (1.2),

\[ \sum_{i=1}^{r} b_i \leq \sum_{i=1}^{r} \left( \frac{1}{i \log i} + \frac{1}{i(\log i)^2} \right) \leq \log \log r + C. \]
By (2.12) it is easy to check that the coefficients \( \Pi \). Moreover, (2.13) factor successively generates (2.3). Coefficients related to \( \Pi \). We now note that by definition of \( \Pi \) we deduce that (4.2), one has

\[
\sum_{d=1}^{r-1} \frac{(\log d)(\log \log r + C)^d}{d!} \leq (1 + \varepsilon + \log \log \log r) \sum_{d=1}^{r-1} \frac{(\log \log r + C)^d}{d!} \leq C(1 + \varepsilon + \log \log \log r) \log r.
\]

On the other, utilizing the classical estimate \( d! \geq C\sqrt{d}d^d e^{-d} \), one has

\[
\sum_{d>1} \frac{(\log d)^d}{d!} (\log \log r)^d \leq \sum_{d>1} \frac{(\log d)^d}{\sqrt{d}} e^{-d(\log d - 1 - \log \log r)} \leq \sum_{d>1} \frac{(\log d)^d}{\sqrt{d}} e^{-c d} < \infty.
\]

One thus deduces, concerning the sum in (2.10) that,

\[
(2.11) \quad c_0 \sum_{d=1}^{r-1} \left( \sum_{1 \leq i_1 < \ldots < i_d < r} b_{i_1} \ldots b_{i_d} \right) \Phi_1(h + d) \leq C \frac{\log p_r}{p_r} (1 + \log \log \log r) \log r;
\]

2.3. Coefficients related to \( \Pi \). By applying the recurrence inequality (Lemma 2.6), one successively generates

\[
c_4 \Pi_{r-1} + c_0 c \beta_{r-1} \Pi_{r-2} + c_4 \Pi_{r-1} + c_0 c \beta_{r-1} \Pi_{r-2} = c_0 c \beta_{r-2}(1 + b_{r-1} b_{r-2}) \Pi_{r-3} + c_4 \Pi_{r-1} + c_0 c \beta_{r-1} \Pi_{r-2} + c_0 c \beta_{r-2}(1 + b_{r-1} b_{r-2}) \Pi_{r-3} + c_0 c \beta_{r-3}(1 + b_{r-2} + b_{r-1} + + b_{r-1} b_{r-2}) \Pi_{r-4}.
\]

Coefficients:

\[
\Pi_{r-1} : c_4, \quad \Pi_{r-2} : c_0 c \beta_{r-1}, \quad \Pi_{r-3} : c_0 c \beta_{r-2}(1 + b_{r-1}), \quad \Pi_{r-4} : c_0 c \beta_{r-3}(1 + b_{r-2} + b_{r-1} + b_{r-1} b_{r-2}).
\]

It is easy to check that the coefficients \( \Pi_{r-x} \) are exactly those of \( \Phi_{r-x+1}(\cdot) \) affected with the factor \( c_0 c \beta_{r-x+1} \). The products form the sum

\[
(2.12) \quad c_0 c \sum_{d=0}^{r-2} \beta_{r-d} \left( 1 + \sum_{1 \leq i_1 < \ldots < i_d < r} b_{i_1} \ldots b_{i_d} \right) \Pi_{r-d-1}.
\]

By (4.2), one has

\[
(2.13) \quad \beta_j = \frac{1}{2p_j(\log p_j)^2} \leq \frac{1}{2p(j)(\log p(j))^2} \leq \frac{1}{2j(\log j)^3}, \quad \text{si } j \geq 2,
\]

Moreover, (4.2) and (4.4) imply that

\[
(2.14) \quad \Pi_j = \prod_{\ell=1}^{j} \left( 1 - \frac{r}{p^\ell} \right) \leq \prod_{\ell=1}^{j} \left( 1 - \frac{1}{p^\ell} \right) \leq \prod_{p \leq j(\log j + \log \log j)} \left( 1 - \frac{1}{p} \right) \leq C(\log j).
\]

We now note that by definition of \( \Pi_j \), we also have

\[
(2.15) \quad \Pi_j \leq \max_{\ell \leq 5} \prod_{p \leq p(\ell)} \frac{1}{1 - \frac{1}{p}} = C_0.
\]

We deduce that

\[
(2.16) \quad \Pi_j \leq C(\log j), \quad \text{si } j \geq 2.
\]
Consequently, (2.14) and (2.13) imply that

\[ \beta_{j+1} \Pi_j \leq \frac{C}{j(\log j)^2}, \quad \text{if } j \geq 2. \tag{2.15} \]

It is resulting from it that the sum in (2.12) can be bounded as follows:

\[
\begin{align*}
&c_0 \sum_{r-2}^{r-1} \beta_{r-d} \left(1 + \sum_{d=0}^{r-2} b_{r-i_1} \cdots b_{r-i_d} \right) \Pi_{r-d-1} \\
&\leq c_0 \prod_{i=1}^{r-1} (1 + b_{r-i}) \cdot \sum_{d=0}^{r-2} \beta_{r-d} \Pi_{r-d-1} = c_0 \prod_{j=2}^{r-1} (1 + b_j) \cdot \sum_{d=0}^{r-2} \beta_{r-d} \Pi_{r-d-1} \\
&\leq c_0 C \prod_{j=2}^{r-1} (1 + b_j) \cdot \sum_{d=0}^{r-2} \frac{1}{(r-d)(\log(r-d))^2} \\
&\leq c_0 C \prod_{j=2}^{r-1} (1 + b_j). \tag{2.16}
\end{align*}
\]

We recall that

\[ \sum_{p \leq x} \frac{1}{p} \leq \log \log x + C. \]

See for instance [8], inequality (3.20). Thus,

\[ \prod_{i=1}^{r} (1 + b_i) \leq C \log r. \tag{2.17} \]

Now estimate (2.17) implies that

\[ c_0 \sum_{r-2}^{r-1} \beta_{r-d} \left(1 + \sum_{1 \leq i_1 < \cdots < i_d < r} b_{r-i_1} \cdots b_{r-i_d} \right) \Pi_{r-d-1} \leq c_0 C \log r \leq C \frac{\log p_r}{p_r} \log r. \tag{2.18} \]

We thus deduce from (2.11) and (2.11) that

\[ \Phi_2(r, n) \leq C \frac{\log p_r}{p_r} (1 + \log \log \log r) \log r + C \frac{\log p_r}{p_r} \log r \leq C \frac{\log p_r}{p_r} (\log r)(\log \log r). \tag{2.19} \]

As a result, by taking account of the observation made at the beginning of section 2, we obtain

\[ \Phi_2(n) \leq C (\log \log r)(\log r) \sum_{i=1}^{r} \frac{\log p_i}{p_i} = C (\log \log r)(\log r)w(n). \tag{2.20} \]

By combining (2.20) with the upper estimate \( \Phi_1(n) \) established at Lemma 4.1 and using inequality (1.9), we arrive to

\[ \Psi(n) \leq \left( \prod_{j=1}^{r} \frac{1}{1 - p_j} \right) \sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1} + C (\log \log r)(\log r)w(n), \tag{2.21} \]

recalling that \( p_j \geq 3 \) by assumption (2.1).
3. Proof of Theorem 1.1

First we prove inequality (1.9). We recall the convention $0 \log 0 = 0$. Inequality (1.9) is an immediate consequence of the following convexity lemma.

**Lemma 3.1.** For any integers $\mu_i \geq 0$, $p_i \geq 2$, we have

$$\sum_{i=1}^{r} (\mu_i \log p_i) \log \left( \sum_{i=1}^{r} \mu_i \log p_i \right) \leq \sum_{i=1}^{r} \mu_i (\log p_i) \left( \log \log p_i \right)$$

$$+ \sum_{i=1}^{r} \mu_i (\log p_i) \log \left( \sum_{i=1}^{r} \mu_i \right).$$

**Proof.** We may restrict to the case $\sum_{i=1}^{r} \mu_i \geq 1$, since otherwise the inequality is trivial. Let $M = \sum_{i=1}^{r} \mu_i$ and write that

$$\sum_{i=1}^{r} \mu_i (\log p_i) \log \left( \sum_{i=1}^{r} \mu_i \log p_i \right) = M \left\{ \sum_{i=1}^{r} \frac{\mu_i}{M} (\log p_i) \log \left( \sum_{i=1}^{r} \frac{\mu_i}{M} \log p_i \right) \right\}$$

$$+ \sum_{i=1}^{r} \frac{\mu_i}{M} (\log p_i) (\log M).$$

By using convexity of $\psi(x) = x \log x$ on $\mathbb{R}^+$, we get

$$\sum_{i=1}^{r} \frac{\mu_i}{M} (\log p_i) \log \left( \sum_{i=1}^{r} \frac{\mu_i}{M} \log p_i \right) \leq \sum_{i=1}^{r} \frac{\mu_i}{M} (\log p_i) \left( \log \log p_i \right).$$

Thus

$$\sum_{i=1}^{r} (\mu_i \log p_i) \log \left( \sum_{i=1}^{r} \mu_i \log p_i \right) \leq \sum_{i=1}^{r} \mu_i (\log p_i) \left( \log \log p_i \right) + \sum_{i=1}^{r} \mu_i (\log p_i) \log \left( \sum_{i=1}^{r} \mu_i \right).$$

\[ \square \]

The odd case (i.e. condition (2.1) is satisfied) is obtained by combining (2.2) with Corollary 4.2 and utilizing inequality (1.9). Since $r \leq \log n$, by taking account of estimate of $w(n)$ given in (1.10), we get

$$\Psi(n) \leq e^\gamma (1 + o(1)) (\log \log n)^2 (\log \log \log n) + C (\log \log \log \log n) (\log \log n)^2$$

$$(3.1) = e^\gamma (1 + o(1)) (\log \log n)^2 (\log \log n).$$

To pass from the odd case to the general case is not easy. This step will necessitate an extra analysis of some other properties of $\Psi(n)$.

We first exclude the trivial case when $n$ is a pure power of 2, since $\Psi(2^k) \leq C$ uniformly over $k$, and $C$ is a finite constant.

Now if 2 divides $n$, writing $n = 2^m m$, $2 \not| m$, we have

$$\Psi(n) = \sum_{d|n} \frac{\log d}{d} \frac{(\log \log d)}{\log \log d} = \sum_{k=0}^{v} \sum_{d|m} \frac{(\log(2^k d))(\log \log(2^k d))}{2^k d}.$$

As the function $x \mapsto \frac{\log x}{x} \frac{(\log \log x)}{x}$ decreases on $[x_0, \infty)$ for some positive real $x_0$, we can write

$$\sum_{k=0}^{v} \frac{(\log(2^k d))(\log \log(2^k d))}{2^k d} \leq \sum_{k=0}^{k_0-1} \frac{(\log(2^k d))(\log \log(2^k d))}{2^k d} + \sum_{k=k_0+1}^{v} \frac{(\log(2^k d))(\log \log(2^k d))}{2^k d}$$

$$\leq \sum_{k=0}^{k_0-1} \frac{(\log(2^k d))(\log \log(2^k d))}{2^k d} + \int_{2^{k_0} d}^{\infty} \frac{(\log u)(\log \log u)}{u^2} du,$$
where \( k_0 \) is depending on \( x_0 \) only. Moreover

\[
\left( \frac{(\log u)(\log \log u)}{u} \right)' \geq -\frac{(\log u)(\log \log u)}{u^2}.
\]

Thus

\[
\sum_{k=0}^{v} \left( \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} \right)
\leq \sum_{k=0}^{k_0-1} \left( \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} \right) + \left( \frac{(\log(2^{k_0} \delta))(\log \log(2^{k_0} \delta))}{2^{k_0} \delta} \right),
\]

whence

\[
(3.2) \quad \Psi(n) \leq \sum_{k=0}^{k_0} \sum_{|m|} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta}.
\]

Let \( m = p_1^{b_1} \ldots p_n^{b_n} \). We have by (2.21)

\[
\Psi(m) \leq \left( \prod_{j=1}^{\mu} \frac{1}{1 - p_j^{-1}} \right) \sum_{i=1}^{\mu} \frac{\log p(i)}{p(i) - 1} - C (\log \log \log \mu)(\log \mu)w(m)
\leq \left( \prod_{j=2}^{\mu} \frac{1}{1 - p(j)^{-1}} \right) \sum_{i=1}^{\mu} \frac{\log p(i)}{p(i) - 1} + C (\log \log \log \mu)(\log \mu)w(m)
= \frac{1}{2} \left( \prod_{j=1}^{\mu} 1 - p(j)^{-1} \right) \sum_{i=1}^{\mu} \frac{\log p(i)}{p(i) - 1} + C (\log \log \log \mu)(\log \mu)w(m)
\leq \frac{e^\gamma}{2} \left( \log \mu + \mathcal{O}(1) \right) \sum_{i=1}^{\mu} \frac{\log p(i)}{p(i) - 1} + C (\log \log \log \mu)(\log \mu)w(m),
\]

by using Mertens’ estimate (4.6) and since \( p(\mu) \sim \mu \log \mu \). Furthermore by using estimate (4.5), and since \( 2^u \leq m \) we get

\[
\Psi(m) \leq \frac{e^\gamma}{2} \left( \log \mu + \mathcal{O}(1) \right) (1 + \varepsilon)(\log \mu)(\log \log \mu) + C (\log \log \log \mu)(\log \mu)w(m)
\leq \frac{e^\gamma}{2} \left( \log \frac{\log m}{\log 2} + \mathcal{O}(1) \right) (1 + \varepsilon)(\log \frac{\log m}{\log 2})(\log \frac{\log m}{\log 2})
+ C (\log \log \log \frac{\log m}{\log 2})(\log \frac{\log m}{\log 2})(1 + o(1)) \log \log m
\leq \frac{e^\gamma}{2} (1 + 2\varepsilon)(\log \log m)^2(\log \log \log m),
\]

for \( m \) large.

Now let \( \psi(2^k m) = \sum_{|m|} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{\delta} \), \( 1 \leq k \leq k_0. \) If \( n \) is not a pure power of 2, then its odd component \( m \) tends to infinity with \( n \). Thus with (3.2),

\[
(3.4) \quad \frac{\Psi(n)}{(\log \log n)^2(\log \log \log n)} \leq \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{|\delta|} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{(\log \log m)^2(\log \log \log m)}.
\]

But

\[
\frac{(\log(2^k \delta))(\log \log(2^k \delta))}{\delta} = \frac{(k(\log 2))(\log \log(2^k \delta)) + (\log \delta)(\log \log(2^k \delta))}{\delta}
\leq k_0(\log 2) \frac{\log(k_0(\log 2) + \log \delta)}{\delta} + \frac{\log \delta \log \log(2^{k_0} \delta)}{\delta}.
\]

Now we have the inequality: \( \log \log(a + x) \leq \log(b \log x) \) where \( b \geq (a + \varepsilon) \) and \( a \geq 1 \), which is valid for \( x \geq \varepsilon \). Thus

\[
(3.6) \quad \log \left( k_0(\log 2) + \log \delta \right) \leq \log(k_0 \log 2 + \varepsilon) + \log \log \delta.
\]
Consequently
\[
\sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} k_0 \frac{(\log 2)^{\log (2k_0 \log 2 + \log \delta)}}{(\log \log m)^2(\log \log \log m)} 
\leq \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} k_0 \frac{(\log (k_0 \log 2 + \epsilon))}{(\log \log m)^2(\log \log \log m)}
\]
\[
+ \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} k_0 \frac{(\log \log \delta)}{\delta(\log \log m)^2(\log \log \log m)}
\leq 2k_0 \frac{(\log (k_0 \log 2 + \epsilon))}{(\log \log m)^2(\log \log \log m)}
\]
\[
+ 2k_0 \frac{(\log 2)}{(\log \log m)^2(\log \log \log m)} \sum_{\delta|m} \frac{\log \log \delta}{\delta} \frac{\sigma_{-1}(m)}{(\log \log m)(\log \log \log m)}
\leq C(k_0) \left\{ \frac{1}{\log \log m(\log \log \log m)} + \frac{\sigma_{-1}(m)}{(\log \log m)(\log \log \log m)} \right\}
\leq \frac{C(k_0)}{\log \log \log m} \to 0 \quad \text{as } m \text{ tends to infinity.}
\]
(3.7)

Further
\[
\sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{(\log \delta)(\log(2^k \log \delta))}{(\log \log m)^2(\log \log \log m)} \leq \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{\log \log \delta}{\delta(\log \log m)^2(\log \log \log m)} \Psi(m)
\]
\[
\leq 2 \log (k_0 \log 2 + \epsilon) \frac{\sigma_{-1}(m)}{(\log \log m)(\log \log \log m)} \Psi(m)
\leq \frac{C(k_0)}{\log \log m} \frac{1 + 2^\gamma (1 + 2\epsilon)}{2} \frac{(\log \log m)^2(\log \log \log m)}{C(k_0)}
\leq \frac{C(k_0)}{\log \log m} + \frac{C}{2} (1 + 2\epsilon),
\]
(3.8)

for \( m \) large, where we used estimate (3.3).

Plugging estimates (3.7) and (3.8) into (3.4) finally leads, in view of (3.5), to
\[
\frac{\Psi(n)}{(\log \log n)^2(\log \log \log n)} \leq \frac{C}{\log \log m} + e^\gamma (1 + 2\epsilon)
\]
(3.9)

for \( m \) large, where \( C \) depends on \( k_0 \) only. As \( \epsilon \) can be arbitrary small, we finally obtain
\[
\limsup_{n \to \infty} \frac{\Psi(n)}{(\log \log n)^2(\log \log \log n)} \leq e^\gamma.
\]
(3.10)

This establishes Theorem 1.1.

4. Complementary results.

In this section we prove complementary estimates \( \Phi_1 \), \( \Phi_2 \) and \( \Psi \), notably estimates (1.6) and (1.7).
4.1. Upper estimates.

Lemma 4.1. We have the following estimate,
\[
\Phi_1(n) \leq \left( \prod_{j=1}^{r} \frac{1}{1-p_j} \right) \sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1}.
\]

Proof. We have
\[
\Phi_1(n) \leq \left( \prod_{j=1}^{r} \frac{1}{1-p_j} \right) \sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1}
= \sum_{i=1}^{r} \sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_r=0}^{\alpha_r} \frac{1}{p_i^{\mu_1} \cdots p_i^{\mu_r}} \left( \sum_{\mu_i=0}^{\alpha_i} \frac{\mu_i \log p_i}{p_i^{\mu_i}} \right)
\]
the sum relatively to \(\mu_i\) is excluded
\[
= \sum_{i=1}^{r} \prod_{j=1}^{r} \left( \frac{1-p_j^{-\alpha_j-1}}{1-p_j^{-1}} \right) \left[ \sum_{\mu_i=0}^{\alpha_i} \frac{\mu_i \log p_i}{p_i^{\mu_i}} \right].
\]

Now as
\[
\sum_{\mu=0}^{\alpha} \frac{\mu}{p_i^\mu} \leq \sum_{j=1}^{\infty} \frac{j}{p_i^j} = \frac{1}{(p_i-1)(1-p_i^{-1})},
\]
we obtain
\[
\Phi_1(n) \leq \sum_{i=1}^{r} \prod_{j=1}^{r} \left( \frac{1-p_j^{-\alpha_j-1}}{1-p_j^{-1}} \right) \frac{(\log p_i)(\log \log p_i)}{(p_i-1)(1-p_i^{-1})}
\leq \left( \prod_{j=1}^{r} \frac{1}{1-p_j^{-1}} \right) \sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1}.
\]

\(\square\)

Corollary 4.2. We have the following estimate,
\[
\limsup_{n \to \infty} \frac{\Phi_1(n)}{(\log \log n)^2(\log \log \log n)} \leq e^\gamma.
\]

Proof. Let \(p(j)\) denote the \(j\)-th consecutive prime number, and recall that [8 (3.12-13)],

\[
p(i) \geq \max(i\log i, 2), \quad i \geq 1,
\]
\[
p(i) \leq i(\log i + \log \log i), \quad i \geq 6.
\]

Let \(\varepsilon > 0\) and an integer \(r_0 \geq 4\). If \(r \leq r_0\), then
\[
\sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1} \leq \delta r_0, \quad \delta = \sup_{p \geq 3} \frac{(\log p)(\log \log p)}{p - 1} < \infty.
\]

If \(r > r_0\), then
\[
\sum_{i=r_0+1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1} \leq \left( \max_{i \geq r_0} \frac{p(i)}{p(i)-1} \right) \sum_{i=r_0+1}^{r} \frac{(\log p(i))(\log \log p(i))}{p(i)}
\leq \left( \max_{i \geq r_0} \frac{p(i)}{p(i)-1} \right) \sum_{i=r_0+1}^{r} \frac{(\log(i \log i))(\log \log(i \log i))}{i \log i}
\]

We choose \(r_0 = r_0(\varepsilon)\) so that \(\log r_0 \geq 1/\varepsilon\) and the preceding expression is bounded from above by
\[
(1 + \varepsilon) \sum_{i=r_0+1}^{r} \frac{\log \log i}{i}.
\]
We thus have
\[ \sum_{i=r_0+1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1} \leq (1 + \varepsilon) \int_{r_0}^{r} \log \log t \, dt \]
(4.4)
\[ \leq (1 + \varepsilon)(\log r)(\log \log r). \]
Consequently, for some \( r(\varepsilon) \),
\[ \sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i - 1} \leq (1 + \varepsilon)(\log r)(\log \log r), \quad r \geq r(\varepsilon). \]
(4.5)
By using Mertens' estimate
\[ \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) = e^{\gamma} \log x + O(1) \quad x \geq 2, \]
(4.6)
we further have
\[ \prod_{\ell=1}^{r} \left( 1 - \frac{1}{\ell \log \ell} \right) \leq \prod_{\ell=1}^{r} \left( 1 - \frac{1}{\ell \log \ell} \right) \leq \prod_{p \leq r(\log r + \log \log r)} \left( 1 - \frac{1}{p} \right) \leq e^{\gamma}(\log r) + C, \]
(4.7)
if \( r \geq 6 \), and so for any \( r \geq 1 \), modifying \( C \) if necessary. As \( r = \omega(n) \) and \( 2^{\omega(n)} \leq n \), we consequently have,
\[ \Phi_1(n) \leq e^{\gamma}(1 + C \varepsilon)^2(\log \log n)^2(\log \log \log n), \]
si \( r > r_0 \). If \( r \leq r_0 \), we have
\[ \Phi_1(n) \leq \delta e^{\gamma}(1 + \varepsilon)((\log r_0) + C) := C(\varepsilon). \]
Whence,
\[ \Phi_1(n) \leq e^{\gamma}(1 + \varepsilon)^2(\log \log n)^2(\log \log \log n) + C(\varepsilon). \]
As \( \varepsilon \) can be arbitrary small, the result follows. \( \square \)

The following lemma is nothing but the upper bound part of (4.3). We omit the proof.

**Lemma 4.3.** We have the following estimate,
\[ \sum_{d|\omega(n)} \log \frac{d}{\omega(n)} \leq \prod_{p|\omega(n)} \left( 1 - \frac{1}{p-1} \right) \sum_{p|\omega(n)} \log p \]
\[ \leq \prod_{p|\omega(n)} \left( 1 - \frac{1}{p-1} \right) \sum_{p|\omega(n)} \log p - 1. \]
Moreover,
\[ \limsup_{n \to \infty} \frac{1}{(\log \log n)(\log \omega(n))} \sum_{d|\omega(n)} \frac{\log d}{d} \leq e^{\gamma}. \]

**4.2. Lower estimates.** We recall that the smallest prime divisor of an integer \( n \) is noted by \( P^-(n) \).

**Lemma 4.4.** Let \( n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}, \ r \geq 1, \ \alpha_i \geq 1 \). Then,
\[ \Phi_1(n) \geq \left( 1 - \frac{1}{P^-(n)} \right) \prod_{j=1}^{r} \left( 1 + \frac{1}{p_j - 1} \right) \left[ \sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i} \right] \]

**Proof.** By (4.3),
\[ \Phi_1(n) = \sum_{i=1}^{r} \prod_{j=1}^{r} \left( 1 - \frac{1}{p_j} \right) \frac{\alpha_i}{1 - p_j - 1} \left[ \sum_{\mu_i=0}^{\alpha_i} \mu_i \frac{(\log p_i)(\log \log p_i)}{p_i^{\mu_i}} \right] \]
\[ \geq \sum_{i=1}^{r} \prod_{j=1}^{r} \left( 1 - \frac{1}{p_j} \right) \frac{\alpha_i}{1 - p_j - 1} \left[ \frac{(\log p_i)(\log \log p_i)}{p_i} \right] \]
\[ \geq \prod_{j=1}^{r} (1 + p_j^{-1}) \left[ \sum_{i=1}^{r} \frac{(1 - p_i^{-1})(\log p_i)(\log \log p_i)}{p_i} \right]. \]

Thus
\[ \Phi_1(n) \geq \left(1 - \frac{1}{P^{-1}(n)}\right)^{\prod_{j=1}^{r} (1 + p_j^{-1})} \left[ \sum_{i=1}^{r} \frac{(\log p_i)(\log \log p_i)}{p_i} \right]. \]

We easily deduce from Lemma 4.1 and Lemma 4.4 the following corollary.

**Corollary 4.5.** Let \( n = p_1^{a_1} \cdots p_r^{a_r}, \ r \geq 1, \ a_i \geq 1. \) Then,
\[ (1 - \frac{1}{P^{-1}(n)}) \prod_{j=1}^{r} (1 + p_j^{-1}) \leq \frac{\Phi_1(n)}{\sum_{i=1}^{r} \frac{\log p_i}{p_i}} \leq 2 \prod_{j=1}^{r} (1 - p_j^{-1}). \]

**Proposition 4.6.** We have the following estimates

a) \( \limsup_{n \to \infty} \frac{1}{(\log \log n)} \sum_{d|n} \frac{(\log d)}{d} \geq e^\gamma \)

b) \( \limsup_{n \to \infty} \frac{\Phi_1(n)}{(\log \log n)^2(\log \log \log n)} \geq e^\gamma, \)

c) \( \limsup_{n \to \infty} \frac{\Psi(n)}{(\log \log n)^2(\log \log \log n)} \geq e^\gamma. \)

**Proof.** Case a) is Erdős-Zaremba’s lower bound of function \( \Phi(n). \) Since it is used in the proof of b) and c), we provide a detailed proof for the sake of completeness.

a) Let \( n_j = \prod_{p \leq e^{j}} p^j. \) Recall that \( p(i) \geq \max(i \log i, 2) \) if \( i \geq 1. \) Let \( r(j) \) be the integer defined by the condition \( p(r(j)) < e^j < p(r(j) + 1). \)

By using (1.2) and following Gronwall’s proof \([6]\), we have,
\[ \sum_{d|n_j} \frac{\log d}{d} = \sum_{i=1}^{r(j)} \prod_{\ell \leq i} \left( \frac{1 - p(\ell)^{-j-1}}{1 - p(\ell)^{-1}} \right) \left[ \sum_{\mu=0}^{j} \frac{\mu \log p(i)}{p(i)^{\mu}} \right] \]
\[ \geq \frac{1}{\zeta(j+1)} \prod_{\ell=1}^{r(j)} \left( 1 - p(\ell)^{-1} \right) \left( 1 - p(i)^{-1} \right) \frac{\log p(i)}{p(i)} \left[ 1 + \frac{1}{p(i)} + \ldots + \frac{1}{p(i)^{j-1}} \right] \]
\[ = \frac{1}{\zeta(j+1)} \prod_{\ell=1}^{r(j)} \left( 1 - p(\ell)^{-1} \right) \sum_{i=1}^{r(j)} \frac{\log p(i)}{p(i)} (1 - p(i)^{-j}). \]

Recall that \( \vartheta(x) = \sum_{p \leq x} \log p \) is Chebycheff’s function and that \( \vartheta(x) \geq (1 - \varepsilon(x))x, \ x \geq 2, \) where \( \varepsilon(x) \to 0 \) as \( x \) tends to infinity. Thus, \( \log n_j = j \vartheta(e^j) = je^j(1 + o(1)), \) and thus \( \log \log n_j = j(1 + o(1)). \)

On the one hand, by (4.6),
\[ (4.8) \quad \prod_{\ell=1}^{r(j)} (1 - p(\ell)^{-1}) = \prod_{p \leq e^j} (1 - p^{-1}) = \frac{e^{-\gamma}}{j} (1 + O(\frac{1}{j})). \]

And on the other, by Mertens’ estimate
\[ (4.9) \quad \sum_{p \leq e^j} \frac{\log p}{p} = j + O(1) \geq (1 + o(1)) \log \log n_j. \]

Thus
\[ (4.10) \quad \sum_{d|n_j} \frac{\log d}{d} \geq (1 + o(1))e^\gamma(\log \log n_j)^2 \quad j \to \infty. \]
since \(\zeta(j+1) \to 1\) as \(j \to \infty\).

b) Let \(\sigma_1'(n) = \sum_{d|n, d \geq 3} 1/d\). Let also \(X\) be a discrete random variable equal to \(\log d\) if \(d|n\) and \(d \geq 3\), with probability \(1/(d\sigma_1'(n))\). By using convexity of the function \(x \log x\) on \([1, \infty)\), we get

\[
\mathbb{E} X \log X = \sum_{\substack{d|n \atop d \geq 3}} \frac{(\log d)(\log \log d)}{\sigma_1'(n)} \geq (\mathbb{E} X) \log (\mathbb{E} X)
\]

\[
= \left( \sum_{\substack{d|n \atop d \geq 3}} \frac{(\log d)}{\sigma_1'(n)} \right) \log \left( \sum_{\substack{d|n \atop d \geq 3}} \frac{(\log d)}{\sigma_1'(n)} \right)
\]

\[
\geq \left( \sum_{\substack{d|n \atop d \geq 1}} \frac{(\log d)}{\sigma_1'(n)} - C \right) \left( \log \left( \sum_{\substack{d|n \atop d \geq 3}} \frac{(\log d)}{d} - C \right) - \log \sigma_1(n) \right).
\]

Whence

\[
\sum_{\substack{d|n \atop d \geq 3}} \frac{(\log d)(\log \log d)}{d} \geq \left( \sum_{\substack{d|n \atop d \geq 3}} \frac{(\log d)}{d} - C \sigma_1(n) \right) \left( \log \left( \sum_{\substack{d|n \atop d \geq 3}} \frac{(\log d)}{d} - C \right) - \log \sigma_1(n) \right)
\]

Letting \(n = n_j\), we deduce from (4.10) that

\[
\Psi(n) \geq \sum_{\substack{d|n_j \atop d \geq 3}} \frac{(\log d)(\log \log d)}{d} \geq \left( 1 + o(1) \right) e^\gamma (\log \log n_j)^2 - C \log \log n_j
\]

\[
\times \left\{ \log \left\{ (1 + o(1)) e^\gamma (\log \log n_j)^2 - C \right\} - \log C \log \log n_j \right\}
\]

\[
\geq (1 + o(1)) e^\gamma (\log \log n_j)^2 \log \log \log n_j.
\]

Consequently,

\[
\limsup_{n \to \infty} \frac{\Psi(n)}{(\log \log n)^2 \log \log \log n} \geq e^\gamma.
\]

c) We have

\[
\Phi_1(n_j) = \sum_{i=1}^{r(j)} \prod_{\ell=1}^{r(j)} \left( \frac{1 - p(\ell)^{-1}}{1 - p(\ell)^{-1}} \right) \sum_{\ell=1}^{r(j)} \frac{\mu(p(i)) (\log \log p(i))}{p(i)^{\mu}}
\]

\[
\geq \frac{1}{\zeta(j+1)} \prod_{\ell=1}^{r(j)} \left( 1 - \frac{1}{p(\ell)^{-1}} \right)
\]

\[
\times \sum_{i=1}^{r(j)} (1 - p(i)^{-1}) \frac{\log p(i) (\log \log p(i))}{p(i)} \left[ 1 + \frac{1}{p(i)} + \ldots + \frac{1}{p(i)^{-1}} \right]
\]

\[
\geq \frac{1}{\zeta(j+1)} (e^\gamma j (1 + O(1/j)) \sum_{i=1}^{r(j)} \frac{(\log p(i)) (\log \log p(i))}{p(i)} (1 - p(i)^{-j}).
\]

by (4.8). Let \(0 < \varepsilon < 1\). By using (4.9), we also have for all \(j\) large enough,

\[
\sum_{p < e^\varepsilon} \frac{(\log p)(\log \log p)}{p} \geq \sum_{\varepsilon^j \leq p < e^j} \frac{(\log p)(\log \log p)}{p}
\]

\[
\geq (1 + o(1)) \left( \log(e^\varepsilon) \right) \sum_{\varepsilon^j \leq p < e^j} \frac{(\log p)}{p}
\]

\[
\geq (1 + o(1)) (1 - \varepsilon) j \left( \log(e^\varepsilon) \right) \left( 1 + O(1/j) \right)
\]

\[
\geq (1 + o(1)) (1 - \varepsilon) (\log \log n_j) \left( \log(e \log \log n_j) \right).
\]
As \( \log(\varepsilon \log \log n_j) \sim \log \log \log n_j, \ j \to \infty \), we have
\[
\limsup_{j \to \infty} \frac{\Phi_1(n_j)}{(\log \log n_j)^2(\log \log \log n_j)} \geq e^\gamma (1 - \varepsilon).
\]
As \( \varepsilon \) can be arbitrarily small, this proves (c).

\[\square\]

**Lemma 4.7.** We have the following estimate
\[
\Phi_2(n) \geq (\log 2) \left( \frac{P^-(n)}{P^-(n) + 1} \right) \left( \prod_{i=1}^r (1 + \frac{1}{p_i}) \right) \sum_{j=1}^r \left( \frac{\log p_j}{p_j} \right).
\]

**Proof.** We observe from (2.3) that
\[
\Phi_2(r, n) \geq \Phi_2(n) \geq \sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \cdots p_{r-1}^{\mu_{r-1}}} \log p_r \log \left[ \sum_{i=1}^{r-1} \mu_i + 1 \right].
\]
It is clear that the above multiple sum can contribute (is not null) only if \( \max_{i=1}^{r-1} \mu_i \geq 1 \), in which case \( \log \left[ \sum_{i=1}^{r-1} \mu_i + 1 \right] \geq \log 2 \). We thus have
\[
\Phi_2(r, n) \geq (\log 2) \frac{\log p_r}{p_r} \sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \cdots p_{r-1}^{\mu_{r-1}}} \geq (\log 2) \frac{\log p_r}{p_r} \prod_{i=1}^{r-1} \left( 1 + \frac{1}{p_i} \right).
\]
(4.11)
Consequently,
\[
\Phi_2(n) \geq (\log 2) \frac{\log p_r}{p_r} \sum_{j=1}^r \left( \frac{\log p_j}{p_j} \right) \prod_{i=1}^r \left( 1 + \frac{1}{p_i} \right).
\]
(4.12)
\[\square\]

## 5. An application.

We deduce from of Theorem 1.1 the following result.

**Theorem 5.1.** Let \( \eta > 1 \). There exists a constant \( C(\eta) \) depending on \( \eta \) only, such that for any finite set \( K \) of distinct integers, and any sequence of reals \( \{c_k, k \in K\} \), we have
\[
\sum_{k, \ell \in K} c_k c_\ell (k, \ell)^2 \leq C(\eta) \sum_{\nu \in K} \left( \log \log \log n \right)^{\eta} \Psi(\nu).
\]
Further,
\[
\sum_{k, \ell \in K} c_k c_\ell (k, \ell)^2 \leq C(\eta) \sum_{\nu \in K} \left( \log \log \log \log n \right)^{2(\log \log \log n)^{1+\eta}}.
\]
This much improves Theorem 2.5 in [9] where a specific question related to Gál’s inequality was investigated, see [9] for details. The interest of inequality (5.1), is naturally that the bound obtained tightly depends on the arithmetical structure of the support \( K \) of the coefficient sequence, while being close to the optimal order of magnitude \( (\log \log \log n)^2 \).

Theorem 5.1 is obtained as a combination of Theorem 1.1 with a slightly more general and sharper formulation of Theorem 2.5 in [9].
Theorem 5.2. Let \( \eta > 1 \). Then, for any real \( s \) such that \( 0 < s \leq 1 \), for any sequence of reals \( \{c_k, k \in K\} \), we have
\[
(5.3) \quad \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} \leq C(\eta) \sum_{\nu \in K} c_\nu^2 \frac{(\log \log \log \nu)^\eta}{\log \delta} \sum_{d | \nu} \frac{(\log \delta)(\log \log \delta)}{\delta^{2s-1}}.
\]

The constant \( C(\eta) \) depends on \( \eta \) only.

Remark 5.3. From Theorem 2.5-(i) in [3], follows that for every \( s > 1/2 \),
\[
(5.4) \quad \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} \leq \zeta(2s) \inf_{0 < \varepsilon \leq 2s-1} \frac{1 + \varepsilon}{\varepsilon} \sum_{\nu \in K} c_\nu^2 \sigma_{1+\varepsilon-2s}(\nu),
\]
\( \sigma_u(\nu) \) being the sum of \( u \)-th powers of divisors of \( \nu \), for any real \( u \). As
\[
\sum_{d | \nu} \frac{1}{d^s} \ll \sum_{d | \nu} \frac{1}{d^{2s-1}} = \sigma_{1+\varepsilon-2s}(\nu),
\]
estimate \( (5.3) \) is much better than the one given \( (5.4) \).

Proof of Theorem 5.2. The proof is similar to that one of Theorem 2.5 in [3] and shorter. Let \( \varepsilon > 0 \) and let \( J_\varepsilon \) denote the generalized Euler function. We recall that
\[
(5.5) \quad J_\varepsilon(n) = \sum_{d | n} d^\varepsilon \mu(\frac{n}{d}).
\]

We extend the sequence \( \{c_k, k \in K\} \) to all \( \mathbb{N} \) by putting \( c_k = 0 \) if \( k \notin K \). By Möbius’ formula, we have \( n^\varepsilon = \sum_{d | n} J_\varepsilon(d) \). By using Cauchy-Schwarz’s inequality, we successively obtain
\[
L := \sum_{k, \ell = 1}^n c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} = \sum_{k, \ell \in K} c_k c_\ell \sum_{d \in F(K)} J_2s(d) 1_{d \mid k} 1_{d \mid \ell}
\]
\[
(k = ud, \ell = vd) \leq \sum_{u, v \in F(K)} \frac{1}{u^s v^s} \left( \sum_{d \in F(K)} \frac{J_2s(d)}{d^{2s}} c_u c_v \right) \leq \sum_{u, v \in F(K)} \frac{1}{u^s} \left( \sum_{d \in F(K)} \frac{J_2s(d)}{d^{2s}} c_u \right)^{1/2} \left( \sum_{d \in F(K)} \frac{J_2s(d)}{d^{2s}} c_v^2 \right)^{1/2}
\]
\[
= \left[ \sum_{u \in F(K)} \frac{1}{u^s} \left( \sum_{d \in F(K)} \frac{J_2s(d)}{d^{2s}} c_u \right)^{1/2} \right]^2\]
\[
(5.6) \quad \leq \left( \sum_{u \in F(K)} \frac{1}{u^s \psi(u)} \right) \left( \sum_{\nu \in K} \frac{c_\nu^2}{\nu^{2s}} \right) \sum_{u \in F(K)} \sum_{u | \nu} J_2s(\frac{\nu}{u}) u^s \psi(u),
\]
where \( \psi(u) > 0 \) is a non-decreasing function on \( \mathbb{R}^+ \). We then choose
\[
\psi(u) = u^{-s} \psi_1(u) \sum_{t | u} t(\log t)(\log \log t), \quad \psi_1(u) = (\log \log \log u)^\eta.
\]

Hence,
\[
L \leq \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u)} \sum_{t | u} t(\log t)(\log \log t) \right) \left( \sum_{\nu \in K} \frac{c_\nu^2}{\nu^{2s}} \sum_{u | \nu} J_2s(\frac{\nu}{u}) \psi_1(u) \sum_{t | u} t(\log t)(\log \log t) \right)
\]
\[
\leq \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u)} \sum_{t | u} t(\log t)(\log \log t) \right) \left( \sum_{\nu \in K} \frac{c_\nu^2 \psi_1(\nu)}{\nu^{2s}} \sum_{u | \nu} J_2s(\frac{\nu}{u}) \sum_{t | u} t(\log t)(\log \log t) \right).
\]

As \( \nu \in K \), we can write
\[
\sum_{u \in F(K)} \frac{J_2s(\frac{\nu}{u})}{u | \nu} \sum_{t | u} t(\log t)(\log \log t) = \sum_{u | \nu} \sum_{d | \frac{\nu}{u}} \mu(\frac{\nu}{ud}) \sum_{t | u} t(\log t)(\log \log t)
\]
Proof of Theorem 5.1. Letting \( s \) satisfying the condition \( \sum \) partial estimates. By using a convexity argument one shows that \( (5.1) \)

\[
\sum_{d|\nu} \mu\left(\frac{\nu}{d}\right) \sum_{\ell|\nu} t(\log t)(\log \log t)
\]

(\text{writing } u = tx) \quad \Rightarrow \quad \sum_{d|\nu} \sum_{\ell|\nu} t(\log t)(\log \log t) \sum_{x|\nu} \mu\left(\frac{\nu}{d}\right)

(\text{writing } \frac{\nu}{d} = x\theta) \quad \Rightarrow \quad \sum_{d|\nu} \sum_{\ell|\nu} t(\log t)(\log \log t) \sum_{\theta|\nu} \mu\left(\frac{\nu}{d}\right)

\[
(5.7) \quad \Rightarrow \quad \sum_{d|\nu} d^{2s} \left(\frac{\nu}{d}\right) (\log \log \left(\frac{\nu}{d}\right)) (\log \log \left(\frac{\nu}{d}\right)),
\]

where in the last inequality we used the fact that \( \sum_{d|n} \mu(d) \) equals 1 or 0 according to \( n = 1 \) or \( n > 1 \).

Consequently,

\[
L \leq \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u) \sum_{d|u} t(\log t)(\log \log t)} \right) \left( \sum_{\nu \in K} \frac{c_1^2 \nu \sum_{d|\nu} d^{2s} \left(\frac{\nu}{d}\right) (\log \log \left(\frac{\nu}{d}\right)) (\log \log \left(\frac{\nu}{d}\right))}{\nu} \right)
\]

\[
= \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u) \sum_{d|u} t(\log t)(\log \log t)} \right) \left( \sum_{\nu \in K} c_1^2 \nu \sum_{d|\nu} \frac{1}{\delta} (\log \delta) (\log \log \delta) \right)
\]

From the trivial estimate \( \sum_{d|u} t(\log t)(\log \log t) \geq u(\log u)(\log \log u) \), it is resulting that

\[
\sum_{k,\ell=1}^{n} c_k c_\ell \frac{(k,\ell)^{2s}}{k^s \ell^s} \leq \left( \sum_{u \geq 1} \frac{1}{u(\log u)(\log \log u)(\log \log \log u)^\eta} \right) \times \left( \sum_{\nu \in K} c_1^2 (\log \log \log \nu)^\eta \sum_{d|\nu} \frac{(\log \delta) (\log \log \delta)}{\delta^{2s-1}} \right)
\]

\[
(5.8) \quad = \ C(\eta) \sum_{\nu \in K} c_1^2 (\log \log \log \nu)^\eta \sum_{d|\nu} \frac{(\log \delta) (\log \log \delta)}{\delta^{2s-1}}.
\]

\( \square \)

**Proof of Theorem 5.7.** Letting \( s = 1 \) in Theorem 5.2 and using Theorem 1.1, we obtain

\[
(5.9) \quad \sum_{k,\ell=1}^{n} c_k c_\ell \frac{(k,\ell)^2}{k \ell} \leq \ C(\eta) \sum_{\nu \in K} c_1^2 (\log \log \log \nu)^\eta \Phi(\nu),
\]

which proves Theorem 5.1 \( \square \)

6. **Concluding Remarks.**

The proof of Theorem 1.2 can be adapted with no difficulty to similar arithmetical functions. However, a possible extension of Erdős-Zaremba’s result to the function

\[
\Phi_\eta(n) = \sum_{d|n} \frac{(\log d)^\eta}{d}, \quad \eta > 1,
\]

is a more delicate task. In particular, the application of the chaining argument used in the proof of Theorem 1.2 to \( \Phi_\eta(n) \), raises serious technical complications. We only indicate partial estimates. By using a convexity argument one shows that

\[
(6.1) \quad \limsup_{n \to \infty} \frac{\Phi_\eta(n)}{(\log \log n)^{1+\eta}} \geq e^\gamma.
\]

For integers \( n \) with distant prime divisors, this lower bound is optimal. More precisely, there exists a constant \( C(\eta) \) depending on \( \eta \) only, such that for any integer \( n = \prod_{i=1}^{r} p_i^{\alpha_i} \) satisfying the condition \( \sum_{i=1}^{r} \frac{1}{p_i - 1} \leq 2^{1-\eta} \), one has

\[
(6.2) \quad \Phi_\eta(n) \leq C(\eta)(\log \log n)^\eta \sigma_1(n).
\]
As \( \sigma_{-1}(n) \leq C \log \log n \), it follows that \( \Phi_n(n) \leq C(n)(\log \log n)^{1+\eta} \).

We conclude with some remarks concerning Davenport's function \( w(n) \). At first, if \( p_1, \ldots, p_r \) are the \( r \) first consecutive prime numbers and \( n = p_1 \ldots p_r \), then \( w(n) \sim \log \omega(n) \). Next, the obvious bound \( w(n) \leq \log \log n \) holds true when the prime divisors of \( n \) are large, for instance when these ones, write them \( p_1, \ldots, p_r \), verify for some given positive number \( B \), that

\[
\sum_{j=1}^{r} \frac{\log p_j}{p_j} \leq B \quad \text{and} \quad p_1 \ldots p_r \gg e^B.
\]

More generally, one can establish the following result. Let \( \{p_i, i \geq 1\} \) be an increasing sequence of prime numbers enjoying the following property

\[
\sum_{j=1}^{s} \frac{\log p_j}{p_j} \leq p_{s+1} \quad s = 1, 2, \ldots
\]

Numbers of the form \( n = p_1 \ldots p_r \) with \( p_1 \ldots p_{i-1} \leq p_i, 2 \leq i \leq \nu, \nu = 1, 2, \ldots \) appear as extremal numbers in some divisors questions, see Erdős and Hall [3].

Lemma 6.1. Let \( \{p_i, i \geq 1\} \) be an increasing sequence of prime numbers satisfying condition \( 6.4 \). There exists a constant \( C \), such that if \( p_1 \geq C \), then for any integer \( n = p_1^{\alpha_1} \ldots p_r^{\alpha_r} \) such that \( \alpha_i \geq 1 \) for each \( i \), we have \( w(n) \leq \log \log \log n \).

Proof. We use the following inequality. Let \( 0 < \theta < 1 \). There exists a number \( h_\theta \) such that for any \( h \geq h_\theta \) and any \( H \) such that \( e^{(1-\theta)\log H} \leq H \leq h \), we have

\[
h \leq e^{h \log \frac{\log(H+h)}{\log H}}.
\]

Indeed, note that \( \log((1+x) \geq \theta x \) if \( 0 \leq x \leq (1-\theta)/\theta \). Let \( h_\theta \) be such that if \( h \geq h_\theta \), then \( h \log h \leq \theta (\log 2)e^h \). Thus

\[
h \leq e^{h \log \frac{\log 2}{\log H}} \leq e^{h \log \frac{2}{\log H}} \leq e^{h \log \left(1 + \frac{\log 2}{\log H}\right)} = e^{h \log \left(\frac{\log 2H}{\log H}\right)} \]

We shall show by a recurrence on \( r \) that

\[
\sum_{i=1}^{r} \frac{\log p_i}{p_i} \leq \log \log \log (p_1 \ldots p_r).
\]

This is trivially true if \( r = 1 \) by the notation made in the Introduction, and since \( p \geq 2 \). Assume that \( 6.6 \) is fulfilled for \( s = 1, \ldots, r-1 \). Then, by the recurrence assumption,

\[
\sum_{i=1}^{r} \frac{\log p_i}{p_i} \leq \log \log \log (p_1 \ldots p_{r-1}) + \frac{\log p_r}{p_r}.
\]

Put \( H = \sum_{i=1}^{r-1} \log p_i, h = \log p_r \). It suffices to show that

\[
\frac{\log p_r}{p_r} = \frac{h}{e^h} \leq \log \frac{\log \sum_{i=1}^{r} \log p_i}{\log \sum_{i=1}^{r-1} \log p_i} = \log \frac{\log H + h}{\log H},
\]

But \( H \leq h \), by assumption \( 6.4 \). Choose \( C = e^{(1-\theta)\log H} \). Then \( H \geq \log p_1 \geq e^{(1-\theta)\log H} \). The searched inequality thus follows from \( 6.5 \).

Let \( n = p_1^{\alpha_1} \ldots p_r^{\alpha_r} \), where \( \alpha_i \geq 1 \) for each \( i \). We have \( w(n) \leq \log \log \log (p_1 \ldots p_r) \leq \log \log \log n \).

\[\square\]
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