Combinatorial Miller-Hagberg Algorithm for Randomization of Dense Networks

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Abstract. We propose a slightly revised Miller-Hagberg (MH) algorithm that efficiently generates a random network from a given expected degree sequence. The revision was to replace the approximated edge probability between a pair of nodes with a combinatorically calculated edge probability that better captures the likelihood of edge presence especially where edges are dense. The computational complexity of this combinatorial MH algorithm is still in the same order as the original one. We evaluated the proposed algorithm through several numerical experiments. The results demonstrated that the proposed algorithm was particularly good at accurately representing high-degree nodes in dense, heterogeneous networks. This algorithm may be a useful alternative of other more established network randomization methods, given that the data are increasingly becoming larger and denser in today’s network science research.

1 Introduction

In network science, there are occasions in which one needs to generate random network samples from a given node degree sequence. A typical context for doing this is to conduct a statistical test of whether empirically observed network properties can be explained by a certain degree distribution or not. Several algorithms have already been developed for this purpose, such as the classic Havel-Hakimi algorithm \cite{1}, the double edge swap method, the configuration model \cite{2,3}, and the Bayati-Kim-Saberi algorithm \cite{4}. However, they come with respective limitations. The Havel-Hakimi algorithm constructs a network using a heuristic, assortativity-inducing procedure, whose outcomes would not be appropriate to be used as fully randomized controls. The double edge swap method is simple but its randomization process is slow and gradual, with no well-defined termination condition. The configuration model is a systematic, well-defined randomization method, but its outcomes often contain parallel edges and self-loops. The Bayati-Kim-Saberi algorithm can be computationally costly and does not guarantee that it can produce a randomized graph as an output.
The Miller-Hagberg algorithm \cite{5} (called the MH algorithm hereafter) addresses those limitations of the other algorithms mentioned above by relaxing the requirement so that it generates a random network from a given expected degree sequence. This relaxation allows for calculation of edge probability independently for each pair of nodes. By sorting the nodes according to their expected degrees and implementing an efficient node-skipping mechanism (see \cite{5} for details), the MH algorithm achieves linear computational complexity $O(N + M)$, where $N$ and $M$ are the numbers of nodes and edges, respectively. This property is highly desirable for large-scale network analysis.

While the MH algorithm can be used with any edge probability functions, its original version uses Chung and Lu’s random graph model \cite{6} that assumes that an edge probability between two nodes with degrees $w_i$ and $w_j$ can be approximated as $\min(1, w_i w_j / \sum_k w_k)$. It is known that this assumption is invalid if the network is dense (i.e., if $w_i$ is not negligible compared to $N$). This issue is typically manifested on high-degree nodes whose degrees generated by this algorithm often deviate greatly from their expected degrees specified in the given degree sequence \cite{5}. This limitation has not been so critical an issue so far because most real-world networks show significant degree heterogeneity and thus they are fundamentally sparse \cite{9}.

With the recent expansion of modeling methodologies and application domains of network science, however, there are now several situations in which one needs to analyze dense networks, such as the ego networks in social media data \cite{10}, the time/layer aggregations of temporal and multilayer networks \cite{11,12}, and the functional connectivity networks of the brain imaging data \cite{13}, to name a few. These networks typically have much higher edge density than other more classical networks, while they still maintain substantial degree heterogeneity. Accurately representing their high-degree nodes in randomized counterparts is thus an important methodological challenge.

In this paper, we aim to address the above challenge by implementing a small yet unique revision in the original MH algorithm, by replacing the Chung-Lu edge probability with a combinatorically calculated edge probability that better captures the likelihood of edge presence especially where edges are dense. In the rest of the paper, we describe technical details of the algorithm revision and then present some results of evaluation of the proposed algorithm through numerical experiments.

2 Revising the MH Algorithm with Combinatorial Edge Probability

We revise the MH algorithm by replacing the approximated edge probability with a combinatorially calculated edge probability. This calculation is done by counting the number of all network configurations in each of the two scenarios:

\footnote{There have been a couple of modifications of edge probability calculation proposed to address this issue \cite{7,8}, mostly using statistical physics approaches.}
the presence or the absence of an edge between two focal nodes. Let \( w_i \) and \( w_j \) be the degrees of two nodes, \( i \) and \( j \), for which the edge probability between them is to be calculated. Also let \( N \) and \( M \) be the numbers of nodes and edges in the network, respectively. Assuming that each network configuration occurs randomly with equal probability, the edge probability between the two nodes can be written as

\[
p(N, M, w_i, w_j) = \frac{C_c(N, M, w_i, w_j)}{C_c(N, M, w_i, w_j) + C_d(N, M, w_i, w_j)},
\]

where \( C_c(N, M, w_i, w_j) \) is the number of network configurations in which the two nodes \( i \) and \( j \) are connected directly, and \( C_d(N, M, w_i, w_j) \) the number of network configurations in which those nodes are not connected directly (Fig. 1). Eq. (1) can be rewritten as

\[
p(N, M, w_i, w_j) = \left(1 + \frac{C_d(N, M, w_i, w_j)}{C_c(N, M, w_i, w_j)}\right)^{-1}
\]

if \( C_c(N, M, w_i, w_j) \neq 0 \).

Both \( C_c \) and \( C_d \) can be calculated as the product of the following three combinatorial quantities (Fig. 1):

- Number of possibilities of placing the edges that emanate from node \( i \) to the rest of the network
• For $C_c$: \( \binom{N-2}{w_j-1} \)
  - Number of possibilities of placing the edges that emanate from node $j$ to the rest of the network

• For $C_c$: \( \binom{N-2}{w_i} \)
  - Number of possibilities of placing the edges not adjacent to the two nodes among the rest of nodes in the network

• For $C_c$: \( \binom{N-2}{w_i-w_j+1} \)
  - Number of possibilities of placing the edges that emanate from node $j$ to the rest of the network

By multiplying these three quantities, we obtain

\[
C_c(N, M, w_i, w_j) = \left(1 + \frac{N - w_i - 1}{w_i} \frac{N - w_j - 1}{w_j} \frac{M - w_i - w_j + 1}{\binom{N-2}{w_i-w_j+1}}\right)^{-1},
\]

and

\[
C_d(N, M, w_i, w_j) = \left(1 + \frac{2M^*(N - w_i - 1)(N - w_j - 1)}{w_iw_j(N^2 - 5N + 8 - 2M^*)}\right)^{-1},
\]

where $M^* = M - w_i - w_j + 1$. In actual computation of $p$, we use the following more straightforward formula that does not involve inversion:

\[
p(N, M, w_i, w_j) = \frac{X}{X+Y},
\]

\[
X = w_iw_j(N^2 - 5N + 8 - 2M^*)
\]

\[
Y = 2M^*(N - w_i - 1)(N - w_j - 1)
\]

This correctly gives $p = 0$ if $w_i$ or $w_j = 0$, which is convenient for practical purposes.

The formula obtained above is surprisingly simple, involving only a finite, constant number of basic arithmetic operations. Therefore, the revised MH algorithm with this combinatorial edge probability (called the combinatorial MH algorithm hereafter) still maintains the original computational complexity $O(N + M)$. Also note that Eqs. (7)–(9) recovers the original Chung-Lu formula $w_iw_j/(2M) = w_iw_j/\sum_k w_k$, if $N \to \infty$ and $w_i, w_j \ll M \ll N^2$.

Eqs. (7)–(9) capture the edge probability more accurately where edge density is high. Considering some extreme cases helps illustrate this benefit. For example, in a complete graph made of $N$ nodes, each node has $N - 1$ as its degree,
and the total number of edges is $N(N-1)/2$. Letting $w_i = w_j = N-1$ and $M = N(N-1)/2$ (i.e., $M^* = N(N-1)/2 - (N-2) - (N-2) + 1$) in Eqs. (7)–(9) produces $p = 1$, correctly indicating that any pair of nodes must be connected directly. However, the Chung-Lu model gives $p = (n-1)/n < 1$ in the same situation. A more extreme case is a star graph made of $N$ nodes and $N-1$ edges. The edge probability between the central node (with $w_i = N-1$) and a peripheral node (with $w_j = 1$) is correctly calculated to be $p = 1$ by Eqs. (7)–(9), while the Chung-Lu model gives $p = 1/2$, which is far off the actual probability 1. Finally, another example that shows the opposite way of deviation is a disconnected graph made of two 6-node star graphs ($N = 12$, $M = 10$). In this graph, the edge probability between the two central nodes ($w_i = w_j = 5$) is calculated to be $p = 125/129$ by Eqs. (7)–(9), which correctly captures the small possibility that those two central nodes do not have a direct connection to each other. In the meantime, the Chung-Lu model gives $p = \min(1, 5/4) = 1$, which forces the two central nodes to always be connected in randomized networks. These examples demonstrate the accuracy of the combinatorial edge probability proposed in this study.

We note that Eqs. (7)–(9) may not provide accurate edge probabilities for low-degree nodes. For example, they give a non-zero (positive) edge probability between two peripheral nodes in a star graph, since their mandatory connections to the central node are ignored when the edge probability between them is calculated. In general, the proposed algorithm tends to produce slightly higher-than-expected degrees for peripheral nodes in heterogeneous networks (which will be seen in numerical results later). Also, Eqs. (7)–(9) may malfunction if a graphically impossible input is given, because the formula was derived using combinatorial enumerations under the assumption that the given parameters ($N$, $M$, $w_i$, $w_j$) are graphically possible. For example, $(N, M, w_i, w_j) = (5, 10, 1, 1)$ (which is graphically impossible) gives a meaningless value $p = -5/76$. However, such a problem will not arise as long as the formula is used for randomizing the topology of an existing network. In what follows, we exclusively consider cases in which the expected degree sequence is always obtained from the degree sequence of another existing network.

3 Evaluations

We first tested the proposed combinatorial MH algorithm by applying it to several illustrative dense networks. The following three networks were used:

- Zachary’s Karate Club network [14] (34 nodes; 78 edges; density: 0.139)
- Ego network of an arbitrarily chosen user (user ‘3000’ for this example) in Leskovec-McAuley Facebook dataset [10] (92 nodes; 2,044 edges; density: 0.488)
- Dense heterogeneous network constructed using the Barabási-Albert model [15] (300 nodes; 20,000 edges; density: 0.446)

Figure 2 shows the results in which the degree sequences among the given original network and two randomized ones (by the original and combinatorial
Fig. 2. Comparison of degree sequences among the original network (black, solid lines) and two randomized ones (green, dotted lines: original MH algorithm; red, dashed lines: combinatorial MH algorithm). Top: Zachary’s Karate Club network [14]. Middle: Ego network in Leskovec-McAuley Facebook dataset [10]. Bottom: Dense heterogeneous network constructed using the Barabási-Albert model [15]. For each randomization algorithm, the average result of 500 independent randomization trials is shown. Nodes are sorted in descending order of their degrees in the original network. A clear difference between the original and combinatorial MH algorithms is seen on high-degree nodes (highlighted with red circles).
MH algorithms) were compared. For each randomization algorithm, the average result of 500 independent randomization trials is shown. It is clearly seen in these plots that the combinatorial MH algorithm (red, dashed lines) was able to represent high-degree nodes more accurately than the original MH algorithm (green, dotted lines).

We also evaluated the effect of edge density on the performance of randomization algorithms. Figure 3 shows the results of a numerical experiment in which the edge density was systematically varied on Erdős-Rényi and Barabási-Albert networks. The performance of the algorithms was measured by the difference in average node degrees between given and randomized networks. The combinatorial MH algorithm successfully reproduced average node degrees that were closer to the given ones, especially for high edge density cases.

4 Conclusions

In this paper, we presented the combinatorial MH algorithm in which the edge probability between a pair of nodes was combinatorically calculated. The derived edge probability formula involved only a constant number of basic arithmetic operations, keeping the linear computational complexity of the original MH algorithm. Numerical experiments demonstrated that the proposed algorithm was particularly good at accurately representing high-degree nodes in dense, heterogeneous networks. This algorithm may be a useful alternative of other more established network randomization methods, given that the data are increasingly becoming larger and denser in today’s network science research.
What is particularly unique about the proposed algorithm is that it captures, in some sense, certain non-local topological dependencies in calculating edge probability (this helps accuracy), even though the probability itself is still calculated independently for each node pair (this helps computational efficiency). In the meantime, such independent calculation of edge probability may also be a limitation of the algorithm because, as noted earlier, it may produce inaccurate results where edges are sparse. This limitation should be taken into account when one decides which network randomization algorithm should be used for a specific network dataset. Proper handling of such interdependency of edge probabilities will require more careful mathematical analysis and algorithm design, which is among our future work.

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