Quantum simulations of lattice gauge theories using ultracold atoms in optical lattices

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Abstract
Can high-energy physics be simulated by low-energy, non-relativistic, many-body systems such as ultracold atoms? Such ultracold atomic systems lack the type of symmetries and dynamical properties of high energy physics models: in particular, they manifest neither local gauge invariance nor Lorentz invariance, which are crucial properties of the quantum field theories which are the building blocks of the standard model of elementary particles.

However, it turns out, surprisingly, that there are ways to configure an atomic system to manifest both local gauge invariance and Lorentz invariance. In particular, local gauge invariance can arise either as an effective low-energy symmetry, or as an exact symmetry, following from the conservation laws in atomic interactions. Hence, one could hope that such quantum simulators may lead to a new type of (table-top) experiments which will be used to study various QCD (quantum chromodynamics) phenomena, such as the confinement of dynamical quarks, phase transitions and other effects, which are inaccessible using the currently known computational methods.

In this report, we review the Hamiltonian formulation of lattice gauge theories, and then describe our recent progress in constructing the quantum simulation of Abelian and non-Abelian lattice gauge theories in 1 + 1 and 2 + 1 dimensions using ultracold atoms in optical lattices.

Keywords: quantum simulation, lattice gauge theory, ultracold atoms, optical lattices, high-energy physics, quark confinement

(Some figures may appear in colour only in the online journal)
Thanks to the enormous progress carried out in recent decades on the road towards Universal Quantum Computers, several quantum systems are now highly and precisely controllable in the laboratory, mainly belonging to atomic physics and quantum optics. These are, for example, ultracold atoms in optical lattices [7–11], including Bose–Einstein condensates; trapped ions [12–16]; photonic systems [17]; Rydberg atoms [18] and so on. Thus, although a fully-fledged, universal quantum computer has not been realised yet, one may utilise these systems in order to create analogue quantum simulators, which are indeed non-universal simulators, but nevertheless allow for the quantum simulation of quantum systems. This is done by mapping the degrees of freedom of the simulated systems into those of the controllable simulating system. Such quantum simulators provide simple realisations for quantum models and phenomena which may be otherwise inaccessible, either theoretically, experimentally or numerically, amendable for the direct observation of many-body physics phenomena. Using these simulators, one could study experimentally the spectrum and dynamics of the simulated system, and therefore these simulators may be regarded as analogue quantum computers, specifically tailored to the simulated systems. Quantum simulation also has the possibility of realising physical models which are ‘unreal’, not believed to be found in nature.

One could also propose and realise digital quantum simulators which fulfill the same tasks, but using quantum computation methods of qubits and quantum gates. This work, however, deals with analogue simulations.

Thus, quantum simulation is currently a rapidly growing physical area involving multidisciplinary research, both theoretical and experimental. The theoretical research focuses on mapping the simulated physics into the quantum simulating systems, considering the possible realisations and required approximations of the simulated model, as well as methods to control and get the most out of the simulating systems. Hence the multidisciplinarity: it involves both simulated and simulating physics, which may belong to two totally different physical disciplines, at least from a traditional perspective. The experimental research focuses on improving the capabilities and controllability of the simulating systems, also by developing new techniques and simulating systems, and of course, on realising quantum simulations, serving as table-top experiments with great success.

Having been realised using atomic, molecular and optical physics, models of condensed matter and statistical mechanics are the natural candidates for quantum simulations. These may be Hubbard models, spin chains and Heisenberg models and others, just a few examples are [7, 19–29] and [30]. The effects of artificial, external gauge potentials experienced by cold atoms have also been considered for quantum simulation, among more topological effects such as the Aharonov–Bohm effect, Berry’s phase, topological insulators and the quantum Hall effect [26, 31–41] and [42]. A significant example to the power of quantum simulation in condensed matter physics is the experiment for simulating the phase transition in Hubbard models [20].

However, quantum simulations are not limited only to condensed matter physics, they involve also other physical areas such as gravity, including black hole Hawking [43] and Unruh [44] radiation [45–53]. Relativistic physics has been considered as well including, for example, the Dirac equation (including Zitterbewegung and the Klein paradox) [54–57], the famous Fermi two-atom problem [58, 59], neutrino oscillations [60] and the Majorana equation [61, 62]. The next obvious step is to consider quantum field theories (QFTs) as well.

Quantum simulations of quantum field theories involve those of the vacuum entanglement of a scalar field [63, 64] as well as the interacting scalar–fermionic theories of Thirring and Gross-Neveu [65]. Fermions in lattice gauge theories have been considered as well, for example in the quantum simulations of axions and Wilson fermions [66], Dirac fields in curved space time [67], general simulations of QFT with topological insulators [68] and Schwinger’s pair-creation mechanism [69]. Simulations of nontrivial geometries and topologies and extra dimensions using internal degrees of freedom [70, 71] have been suggested too, as well as simulations of some supersymmetric theories, such as [72–74].

1.2. Quantum simulations of high-energy physics

Another class of quantum simulations is those of high-energy physics (HEP), which involve dynamical gauge fields.

Gauge fields are in the core of the standard model of particle physics [75, 76]. These fields, through their local symmetry, induce the interactions between matter particles, which are fermions or, phrased differently, the excitations of the gauge fields are the so-called gauge bosons, force carriers or interaction mediators. (In the current report, we will not discuss the Brout–Englert–Higgs mechanism [77, 78], which introduces a scalar field, not playing the role of a gauge field.)

Each gauge theory is based on local gauge invariance; invariance to local gauge transformations generated by the so-called gauge group. Such groups may be either Abelian or non-Abelian: for example, QED (quantum electrodynamics) is an Abelian gauge theory based on the group U(1), while the strong interactions are described by the theory of QCD (quantum chromodynamics), which has the non-Abelian local gauge symmetry of SU(3). These symmetries induce the interactions between matter particles, which have different gauge charges; the electric charge in QED, or the non-Abelian colour charge in QCD, for example. In the case of non-Abelian groups, the gauge bosons (e.g. QCD’s gluons) also carry a charge of their own, unlike the Abelian photons of QED.

The standard model of particle physics involves a local gauge symmetry of SU(3) × SU(2) × U(1), corresponding to the strong (QCD), weak and electromagnetic (QED) interactions, respectively. All the gauge groups in this case are continuous groups; Yang–Mills theories [79].

The study of gauge theories, within and beyond the framework of the standard model, employs many theoretical techniques. These include, of course, the perturbative approach of Feynman diagrams [80–84], which has been applied with great success to quantum electrodynamics. However, within QCD such perturbative methods face a problem. In high energies, or short distances (such as the scale of deep inelastic scattering) perturbative techniques work fairly well since, due to asymptotic freedom [85] (and the earlier results of the
parton model and Bjorken scaling [86, 87]), in this regime the quarks behave essentially as free, point-like particles (within the hadrons). On the other hand, at low energies or large distances, the quarks are subject to the so-called quark confinement [88], which forbids the existence of free quarks and rather binds them together into composite particles; hadrons. This limit is nonperturbative, and other techniques must be applied in order to study it.

The most powerful approach that has been developed for nonperturbative QCD effects is the lattice gauge theory [88–90]. It allows the probing and calculating of many important quantities and phenomena of the theory both analytically, as the lattice suggests a regularisation method, and numerically, mostly using Monte-Carlo classical simulation methods. Such studies have led to significant and remarkable results, such as the low-energy spectrum of QCD and the CKM matrix elements [91], several results concerning the quark–gluon plasma [92–95] and the deconfinement phase transition at finite temperature [96–98] to name just a few.

Monte Carlo calculations, nevertheless, are limited in certain important cases. For example, the computationally hard sign problem [99], which limits calculations in regimes with a finite chemical potential for the fermions, which is problematic, for example, for the study of the QCD phases of colour-supercconductivity or quark–gluon plasma [94, 100, 101], or the fact that the Monte-Carlo calculations are of Euclidean correlation functions, and not of real-time dynamics. Indeed a full study of quark confinement with dynamical charges and other nonperturbative phenomena in QCD in 3 + 1 dimensions is still lacking.

Another successful avenue for the exploration of these theories has been the development of toy models, simpler physical theories which capture the essential physics in question. These studies include, for example, confinement in Abelian theories in various dimensions (including on the lattice) [88, 90, 102–109], or QCD, the 1 + 1 dimensional version of quantum chromodynamics, which has been a target of interest in recent decades, as a natural playground for nonperturbative calculations, and as a basis for an analytical understanding of real-world 3 + 1 QCD (QCDS). There, several nonperturbative methods have proven fruitful and useful for the derivation of the spectrum of SU(Nc) theories. First, in the large Nc limit, first discussed (and solved) by ’t Hooft [110, 111], where the mesonic spectrum of the theory is revealed, manifesting the confinement of quarks. Other methods to study the hadronic spectrum of QCD have been DLCQ (discretised light-cone quantisation) [112] and current algebras [113]. Witten’s non-Abelian bosonisation [114] has helped to gain insight into the strong coupling Baryonic spectrum and quark content [115–118]. The relation of the fermion mass with confinement in these theories has been studied, for example, in [119, 120].

Thus, new methods of calculations in gauge theories in particular QCD, could be of great help. Quantum simulation methods may be candidates for this. Once a quantum simulator for a gauge theory has been built, one could use it to observe experimentally the otherwise inaccessible physics of these special, nonperturbative regimes. These quantum simulators may be the quantum computers of HEP calculations: these special, nonperturbative regimes. These quantum simulators may be the quantum computers of HEP calculations: these special, nonperturbative regimes, and current algorithms [113]. Witten’s theories has been studied, for example, in [119, 120].

1.3. Proposals for quantum simulations of high-energy physics

So far, only theoretical proposals have been suggested for quantum simulations of high-energy physics.

The first group of works contains condensed matter systems in which gauge invariance emerges, which may serve as quantum simulators for Abelian gauge theories. These include, for example, an effective emergence of a U(1) spin liquid in Pyrochlore [122]; a ring exchange model using molecular states in optical lattices yielding a Coulomb phase U(1) in the limit of no hopping [123]; the effective emergence of artificial photons in dipolar bosons in an optical lattice [124]; a digital quantum simulation of spin–1/2 U(1) gauge theory as a low energy theory of Rydberg atoms [18]; and the emergence of gauge fields out of time reversal symmetry breaking for spin–1/2 in honeycomb optical lattices [125].

Quantum simulation using ultracold atoms in an optical lattice of continuous QED has been suggested in [126]. However, most of the works so far concentrate on quantum simulations of lattice gauge theories, which are more suited for quantum simulation using optical lattices.

Simulations of pure-gauge compact QED (cQED, see section 2.2.1) include a simulation of the Kogut–Susskind Hamiltonian in 2 + 1 dimensions using Bose–Einstein condensates in optical lattices [127], as well as a simulation of a truncated, spin-gauge model using single atoms in optical lattices [128].

Simulations of U(1) models with matter have been suggested as well. In 1 + 1 dimensions simulation of the lattice link Schwinger model [132], or a generalisation of the spin-gauge approach, to include dynamical fermions in 2 + 1 dimensions [133]. The above proposals for cQED simulations with ultracold atoms introduce
gauge invariance as an effective symmetry, by constraining Gauss’s law in the Hamiltonian, with a large energy penalty. Another approach for simulation, utilising fundamental symmetries of the ultracold atoms, is proposed in [121], for 1 + 1 and 2 + 1 dimensional QED, with or without dynamical matter.

Recent proposals for such simulations also utilise other systems such as superconducting quantum circuits [134, 135] and trapped ions [136], both simulating the Abelian link model. A digital simulation of a U(1) gauge theory is presented in [137], using Rydberg atoms.

Simulations of discrete lattice gauge theories have been suggested as well, including discrete Abelian (ZN) and non-Abelian (DN) models using arrays of Josephson junctions [138], a digital simulation of Z2 with Rydberg atoms [137] and an analogue simulation of ZN, with an explicit construction for Z3 in an optical lattice [121]. A way to truncate the Hilbert space of a U(1) theory in a manner similar to ZN was proposed in [139].

Quantum simulations of non-Abelian gauge theories with continuous groups (SU(2) Yang–Mills models, with possible generalisations to U(N), SU(N)) have been proposed too. Analogue simulations with cold atoms utilising prepotentials [140–143] or rishons in the link model [144, 145] have been suggested in [146] and [147], respectively, where the first utilises hyperfine angular momentum conservation to obtain local gauge invariance, and the latter starts with a larger symmetry group, which must be broken. A digital simulation of an SU(2) gauge magnet [130, 148] has been suggested in [149], while a digital quantum simulation of an SU(2) link model with triangular plaquettes using superconducting qubits has been suggested in [150]. A new way for imposing gauge invariance using dissipation (Zeno effect), with an explicit construction for a non-Abelian model, has been given in [151].

In general, the dimensions of the analogue quantum simulations with optical lattices are restricted only by experimental and technological considerations. Thus, the above suggestions for 2 + 1 dimensional systems using optical lattices should apply, in principle, also to d + 1 systems, given, of course, that the required technological challenge is met. The 1 + 1 quantum simulations mentioned above which use optical lattices may be generalised to further dimensions but, mostly without the magnetic (plaquette) terms, and thus are simply quantum simulations of the strong limit of lattice gauge theories (see section 2 for an explanation of these terms). For other simulating systems and/or digital simulations, the case may be different due to possible dimension-dependent properties of the simulating system.

For a comprehensive review on the recent progress in quantum simulation proposals of link Rishon models, the reader should refer to [152].

Throughout this work, ℏ = c = 1, and the Einstein summation convention is assumed.

2. Lattice gauge theory: a brief review

Lattice gauge theories [89, 90, 153–156] are formulations of gauge theories on a discretised space or spacetime. They were originally invented by Wilson as a tool for the study of the quark confinement problem [88]. It is an important tool in high-energy physics, and especially in QCD (quantum chromodynamics - the theory of the strong interactions) as it allows one to perform nonperturbative calculations (numerically, using Monte-Carlo methods) and thus provides insights into the perturbative-inaccessible regions of QCD, due to the running coupling.

Besides addressing the confinement of quarks (or charges in analogous Abelian theories, as will be later discussed), lattice gauge theories have enabled the computation of other important quantities in QCD, such as the hadronic structure and spectrum, as may be read in the reviews [90, 91, 101] and many others.

In the following review of lattice gauge theory, we follow the Hamiltonian (canonical), Kogut–Susskind formulation of lattice gauge theories [89, 90, 157] (rather than the more conventional Euclidean approach). This is since the Hamiltonian language is much more natural in the context of atomic and optical physics used for the simulating systems.

2.1. Hamiltonian formulation

2.1.1. Hopping fermions and global gauge symmetry. Consider fermions on a $d$-dimensional spatial lattice, with nearest-neighbour hopping. The Hamiltonian takes the form

$$H = \sum_n \sum_{\mathbf{k}} \psi_n^\dagger \psi_n + \epsilon \sum_{n, \mathbf{k}} \psi_n^\dagger \psi_{n+\mathbf{k}} + \text{h.c.}$$

(1)

where $n \in \mathbb{Z}^d$ label the lattice’s vertices and $\mathbf{k} \in \{1, ..., d\}$ labels the lattice’s directions; $\mathbf{k}$ is a unit vector (where the lattice spacing is $a = 1$) in the $k$th direction. The Hamiltonian consists of local ‘mass’ terms and hopping terms (see figure 1).

$\psi_n$ are spinors of some unitary gauge group $G$, in some representation $r$. For example, $\psi_n$ are merely fermionic annihilation operators in the case $G = U(1)$. For $G = SU(N)$, in the fundamental representation, these are $N$-component spinors, containing $N$ annihilation operators:

\[\psi_n\]

Note that we only consider here the gauge degrees of freedom; each component of these gauge spinors may also be a spinor in terms of the spin or flavour degrees of freedom, but we disregarded it here.
where in Hamiltonian (1), and in what follows, summation on the group indices is assumed, e.g.

\[ \psi_{n}^{\dagger} \psi_{n} = \sum_{i} \psi_{n,i}^{\dagger} \psi_{n,i} \]  

(3)

The Hamiltonian (1) is invariant under global gauge transformations of the gauge group G. This means that if we pick some group element \( V \in G \) (which is unitary) and perform the transformation

\[ \psi_{n} \rightarrow V_{n} \psi_{n}, \quad \psi_{n}^{\dagger} \rightarrow V_{n}^{\dagger} \psi_{n}^{\dagger} \]  

(4)

the Hamiltonian is left intact. This corresponds to a conservation of the total number of fermions,

\[ N_{\text{tot}} = \sum_{n} \psi_{n}^{\dagger} \psi_{n} \]  

(5)

which is obvious, as all the terms of the Hamiltonian annihilate one fermion and create another one.

2.1.2. Local gauge symmetry. Next, let us lift the symmetry to be local by generalising the transformation to

\[ \psi_{n} \rightarrow V_{n} \psi_{n}, \quad \psi_{n}^{\dagger} \rightarrow V_{n}^{\dagger} \psi_{n}^{\dagger} \]  

(6)

i.e. each vertex is assigned locally a unitary element of the gauge group. The mass terms of (1) are local and hence remain intact, but the hopping terms vary under the transformation, for example

\[ \psi_{n}^{\dagger} \psi_{n+k} \rightarrow V_{n}^{\dagger} V_{n+k} \psi_{n}^{\dagger} \psi_{n+k} \]  

(7)

The gauge symmetry will be restored if we introduce a connection \( U_{n,k} \) on the link emanating from the vertex \( n \) in the \( k \)th direction. \( U_{n,k} \) is an element of the gauge group in the appropriate representation (corresponding to the spinors to which it is coupled), a unitary operator undergoing the gauge transformation as

\[ U_{n,k} \rightarrow V_{n} U_{n,k} V_{n+k}^{\dagger} \]  

(8)

then, if the Hamiltonian is changed to

\[ H_{\text{GM}} = \sum_{n} M_{n} \psi_{n}^{\dagger} \psi_{n} + \epsilon \sum_{n,k} (\psi_{n}^{\dagger} U_{n,k} \psi_{n+k} + \text{h.c.}) \]  

(9)

the gauge invariance is indeed restored, and the gauge transformation leaves it intact. The name \( H_{\text{GM}} \) stands for the gauge–matter interactions described by it.

We denote the local gauge transformation generators by \( G_{n}^{i} \). Due to the gauge symmetry, they commute with the Hamiltonian,

\[ [H, G_{n}^{i}] = 0 \]  

(10)

and thus the physical Hilbert space \( \mathcal{H} \) is also gauge-invariant, i.e. divided into sectors of eigenvalues of \( G_{n}^{i} \) (see figure 2):

\[ \mathcal{H} = \bigoplus \mathcal{H}(\{q_{n}^{i}\}) \]  

(11)

Each such sector \( \mathcal{H}(\{q_{n}^{i}\}) \) has a set of eigenvalues \( \{q_{n}^{i}\} \), called static charges, such that for every \( \psi(\{q_{n}^{i}\}) \in \mathcal{H}(\{q_{n}^{i}\}) \),

\[ G_{n}^{i} \psi(\{q_{n}^{i}\}) = q_{n}^{i} \psi(\{q_{n}^{i}\}) \]  

(12)

Note that as the generators do not commute for a non-Abelian group (this will be clear in the following section, where they are explicitly defined). The static charges can thus only be defined by the eigenvalues of a subset of some commuting operators. The above equation is the generalised Gauss’s Law, and the dynamics does not mix these sectors, i.e.

\[ \{\psi(\{p_{n}^{i}\})|H|\psi(\{q_{n}^{i}\})\} \propto \prod_{n} \delta_{p_{n}^{i},q_{n}^{i}} \]  

(13)

2.1.3. Dynamical gauge fields. We will identify the connections \( U_{n,k} \) with the gauge group elements, i.e. introduce the group parameters \( \theta_{n,k} \) – Hermitian operators, such that in a given representation \( r \), with the matrix representation of the group generators \( T_{r}^{ij} \),

\[ U_{n,k}^{(r)} = e^{i \theta_{n,k} T_{r}^{ij}} \]  

(14)

where a summation on the group indices \( i \) is implicit. We will assume next working in the fundamental representation (generalisations to other representations are straightforward), and disregard the representation indices in this case: \( U_{n,k} = e^{i \theta_{n,k} T} \).

One could then define operators which are conjugate to the group parameters to serve as generalised momenta. In the case of the Abelian group \( U(1) \), where \( U_{n,k} = e^{i \theta_{n,k}} \), one could define the conjugate momentum to be proportional simply to \( \theta_{n,k} \). But generally, when \( G \) may be non-Abelian, one must pay attention to the non-commutativity of the group in doing that. Thus, we could reformulate the Abelian case and define the conjugate momentum there as

\[ L_{n,k} = -i U_{n,k} U_{n,k}^{\dagger} = -i U_{n,k}^{\dagger} U_{n,k} \]  

(15)
The differential representation would be
\[ L_{n,k} = -\frac{i}{\hbar} \frac{\partial}{\partial \theta_{n,k}} \] (16)
from which the canonical commutation relation
\[ [\theta_{n,k}, L_{n,k}] = i \] (17)
results, and from which, as it is the generator of angular translations,
\[ [L_{n,k}, U_{n,k}] = U_{n,k} \] (18)
Thus, a reasonable choice for the dynamic part of the gauge field would be
\[ H^{ab}_E \propto \sum_{n,k} L^2_{n,k} \] (19)
in agreement with the Abelian version of the Kogut–Susskind Hamiltonian (see below).

Let us now generalise it for non-Abelian groups [158]. Due to the non-Abelian nature of the group, one could define two sets of such operators, left and right
\[ L_{n,k} = -iU_{n,k}l_{n,k}^i ; \quad R_{n,k} = -iU_{n,k}^i_{\cdot n,k} \] (20)
(for an Abelian group \( L = R \), but this does not hold in general). One can expand these operators in terms of the group’s representation matrices,
\[ L_{n,k} = L^i_{n,k} T_i ; \quad R_{n,k} = R_{n,k}^i T_i \] (21)
and then construct the differential forms of the operators \( \{L^i_{n,k}\}, \{R^i_{n,k}\} \). These operators are called left and right generators of the group, since they satisfy the group’s algebra\(^4\),
\[ [L^i_{n,k}, U_{n,k}] = T_i U_{n,k} \] (22)
\[ [R^i_{n,k}, U_{n,k}] = U_{n,k} T_i \] (23)
\[ [L^i_{n,k}, l_{n,k}^j] = -if_{ijl}^k l^j_{n,k} \] (24)
\[ [R^i_{n,k}, l_{n,k}^j] = if_{ijl}^k l^j_{n,k} \] (25)
\[ [L^i_{n,k}, R^j_{n,k}] = 0 \] (26)
where \( T_i \) are the group’s generators and \( f_{ijl}^k \) are the group’s structure constants\(^5\). Generators and group elements from different links commute, of course.

Note that due to the definition (20), one could prove that
\[ R_{n,k} = U_{n,k}^i U_{n,k} \] (27)

\(^4\) Note that the right and left operators may be defined in other conventions as well, resulting in different signs in the group’s algebra.

\(^5\) In a general representation \( r \), these equations would be generalised in a straightforward manner, e.g. \( [U_{n,k}, U_{n,k}^r] = T_i^{(r)} U_i^{(r)}_{n,k} \), where \( T_i^{(r)} \) stands for the matrix representation of the generator in \( r \), and \( U_i^{(r)}_{n,k} \) stands for the matrix representation of the group element. If one wishes to explicitly write the group’s indices, the equations will look like, for example, \( [U_{n,k}^r, U_{n,k}^s] = (T_i^{(r,s)})_{ijl} \).

and thus if we calculate the trace in group space we find out that
\[ \text{tr}(R_{n,k}^i) = \text{tr}(L_{n,k}^2) \] (28)
This can be reformulated using the generators,
\[ \sum_i (R_{n,k}^i)^2 = \sum_i (L_{n,k}^i)^2 \equiv L^2_{n,k} \] (29)
giving us the electric part of the non-Abelian Kogut–Susskind Hamiltonian,
\[ H_E \propto \sum_{n,k} L^2_{n,k} \] (30)
We can refer to \( \{L^i_{n,k}\}, \{R^i_{n,k}\} \) as the left and right electric fields, conjugate (in some sense) to the vector potential. Their difference along a link corresponds to the charge carried by it: no charge in the Abelian case, and the colour charge in the \( SU(N) \) case.

The gauge transformation generators are then given by [157]
\[ G_n = \sum_k (L_{n,k}^i - R_{n,k}^j_{\cdot -k}) - Q_n^i \] (31)
where \( Q_n^i \) is the dynamical charge, which will be introduced later.

The Abelian version of the generator is merely
\[ G_n = \sum_k (L_{n,k}^i - R_{n,k}^j_{\cdot -k}) - Q_n \equiv \text{div}_n L_{n,k} - Q_n \] (32)
in which the discrete divergence is defined, and from which the relation to Gauss’s law is obvious.

One would also like to consider a gauge-field self interaction. Gauge invariance forces us to consider only group traces of products of group elements along close paths. The smallest of such interactions are the so-called plaquette terms, involving unit squares of the lattice, from which we obtain the magnetic part of the Kogut–Susskind Hamiltonian [157]
\[ H_0 \propto \sum_{\text{plaquettes}} (\text{tr}(U_1 U_2 U_3^i U_4^j) + \text{h.c.}) \] (33)
with the group elements usually in the fundamental representation (even if there are charges in another representation which requires the use of other representations when coupling them to the gauge field) which simplifies in the Abelian case to [89]
\[ H_0^{ab} \propto \sum_{\text{plaquettes}} \cos(\theta_1 + \theta_2 - \theta_3 - \theta_4) \] (34)
where the numbers in both the equations are according to the plaquette convention presented in figure 3. Note that this agrees with the terms of the Wilson action [88], and thus with the ‘usual’ procedure of obtaining the Hamiltonian (by a Legendre transformation of the Lagrangian).

We have obtained the terms of the Kogut–Susskind Hamiltonian only up to some proportion factors. One can find them by taking the (classical) continuum limit. Doing that (or using a Legendre transformation of the Lagrangian) one obtains the Kogut–Susskind Hamiltonian

\[ H_{KS} = \frac{1}{2} g^2 \sum_{n,k} L_{n,k}^2 - \frac{1}{2g^2} \sum_{\text{plaquettes}} \left( \text{tr} (U_{12} U_{34}^\dagger U_{14}^\dagger) + \text{h.c.} \right) \]  

(35)

where \( g \) is the coupling constant.

2.2. Abelian theories

2.2.1. Compact QED (cQED). The first theory we will consider is compact quantum electrodynamics—cQED [89]. This theory’s continuum limit is quantum electrodynamics (QED). Its lattice version is compact, involving angular variables.

This is an Abelian continuous gauge theory with the gauge group \( U(1) \). Thus, the group elements are simply unitary phases

\[ U_{n,k} = e^{i\theta_{n,k}} \]  

(36)

as we have already discussed, \( \theta_{n,k} \) are analogous to the vector-potential, and the conjugate momenta are \( L_{n,k} = -i \frac{\partial}{\partial \theta_{n,k}} \) - \( U(1) \) (planar) angular momentum operators. The canonical commutation relation \([\theta_{n,k}, L_{n,k}] = i \) ensures that \( U_{n,k} \) is a raising operator of the electric field \([L_{n,k}, U_{n,k}] = U_{n,k} \), and \( U_{n,k}^\dagger \) is the lowering operator, both are unitary.

The Kogut–Susskind Hamiltonian takes the form

\[ H_{KS} = \frac{g^2}{2} \sum_{n,k} \sum_{n,k'} \cos(\theta_{n,k} - \theta_{n,k'}) \]  

(37)

where the indices inside the plaquette term are with respect to the convention of figure 3. Also note that the argument of the cosine corresponds to the lattice curl, manifesting that the continuum limit of this term gives rise to the magnetic energy.

Let us consider the local Hilbert space on each link. One could work on the basis of electric (flux) eigenstates, which is a good basis in the strong coupling limit \( g^2 \gg 1 \), as will be discussed when we consider confinement. The eigenstates of \( L \) on a single link satisfy

\[ L|m\rangle = m|m\rangle \]  

(38)

where \( m \) takes any integer value, either positive or negative. In coordinate (angle) representation, the wavefunctions are

\[ \phi_m(\theta) = \langle \theta|m\rangle = \frac{1}{\sqrt{2\pi}} e^{im\theta} \]  

(39)

The Abelian Gauss’s law takes the form

\[ G_n = \sum_{k} (L_{n,k} - L_{n-k,k}) - Q_n \]  

\[ \equiv \text{di}v_n L_{n,k} - Q_n \]  

(40)

and thus, on this basis, the gauge-invariant states satisfy (without dynamical matter)

\[ G_n |\psi\rangle = \sum_{k} (m_{n,k} - m_{n-k,k}) |\psi\rangle = q_n |\psi\rangle \]  

(41)

where \( q_n \) is the (C-number) static charge.

What are the interactions?

(i) The plaquette interaction, in this language, changes the electric flux on a plaquette, such that in two links the flux increases by a single unit, and in the other two it is lowered. This is done only in two possible orientations, which are the only possibilities which leave the eigenvalue of \( G_n \) intact in all the four vertices of each plaquette—thus, this is the ‘physical interpretation’ of the commutativity of the plaquette interactions with Gauss’s law.

(ii) If we wish to introduce dynamical charges, staggered [157, 159], for example\(^8\), with a Hamiltonian of the form

\[ H_d = \epsilon \sum_{n,k} (\psi_{n+1,k}^\dagger \psi_{n,k} + H.c.) + m \sum_n (\sum_{k} n_{n,k} \psi_{n+1,k}^\dagger \psi_{n,k} - n_{n,k} \psi_{n,k}^\dagger \psi_{n+1,k} ) \]  

(42)

where \( \{n_k\} \) are the indices of the vertex \( n \). We may interpret the gauge invariant interactions as ones which raise the dynamical charge on one edge of the link, lower it on the other side and raise/lower the electric flux on the link in a way that Gauss’s law, now including the dynamical charges, will hold in both the edges. In this staggered representation, even vertices represent ‟particles”, with a positive mass \( m \), and odd ones ‟holes”, with a negative mass \(-m\). In fact, the negative vertices may be interpreted as the Dirac sea: suppose we shift the energy by a constant and measure the mass of the odd vertices with respect to \(-m\). Then an occupied odd vertex corresponds to a state with zero mass, no particle at all, and a vacant odd vertex corresponds to an anti-particle with mass \( m \). This corresponds to the canonical transformation of second quantised Dirac fields, where the holes’ creation operators are replaced with anti-particles’ annihilation operators. We can also define the charges of the particles: the particle would have charge 1 (in units of \( 160\)) and the anti-particle charge \(-1\). This means, in the language of fermionic operators, that

\(^8\)Note that staggered fermions are not the only possible recipe for discretisation of fermionic fields, required by the doubling problem [90, 153–156, 159, 160]. The different approaches, which are beyond the scope of this paper, vary in both quantitative and qualitative details, and have their own advantages and disadvantages. Yet, when a quantum simulator is designed, one should consider which type of lattice fermions to use. We mostly use the staggered fermions formulation, but other approaches should be possible as well.
\[ Q_n = \psi_n^\dagger \gamma^8 \psi_n - \frac{1}{2} \left( 1 - (-1)^{\sum n} \right) \]  

and this completes the particle–anti-particle picture.

These two possible interactions may be understood as the meaning of local gauge invariance. Note that the only two possibilities of contracting gauge invariant operators are traces of products of group elements along closed loops, or along open loops bounded by two fermionic operators on the edges. These two interactions are the shortest such interactions possible, and thus the most local ones. The same considerations apply as well to other lattice gauge theories.

2.2.2. Z_n. Another Abelian gauge theory of relevance is \( Z_n \). This is a discrete group, and thus this is not a Yang–Mills theory, and its structure is a little different. Nevertheless one can use this group to formulate a lattice gauge theory and observe interesting physics. This group is highly relevant for the confinement in QCD, as \( Z_3 \) is the centre of this group, and the centre is responsible for confinement (large-distance phenomena) [161].

On every link of the lattice, we define two unitary operators, \( P \) and \( Q \) [162], such that

\[ P^i P = Q^i Q = 1 \]  

which satisfy the \( Z_n \) algebra,

\[ P^N = Q^N = 1 ; \quad P^i Q P = e^{i\delta} Q \]  

where \( \delta = \frac{2\pi}{N} \).

Let us define the eigenstates of \( P \) as

\[ P|m\rangle = e^{im\delta}|m\rangle \]  

there are \( N \) such states,

\[ m \in \left\{ \frac{-N-1}{2}, \ldots, \frac{-N+1}{2} \right\} \]  

for an odd \( N \) (the generalisation for an even \( N \) is straightforward). \( Q \) is a unitary raising operator,

\[ Q|m\rangle = |m-1\rangle \]  

with the cyclic property

\[ Q \left| \frac{N-1}{2} \right\rangle = \left| \frac{N-1}{2} \right\rangle \]  

alternatively, one may use the eigenstates of \( Q \), and then \( P \) will be a unitary raising operator, also cyclic.

One can define Hermitian operators \( E, A \) such that

\[ P = e^{i\delta E} \quad Q = e^{iA} \]  

and in the \( N \to \infty \) limit they will correspond to \( \epsilon \)QED’s conjugate electric field and vector potential.

The Hamiltonian of this theory takes the form [162]

\[ H = -\frac{1}{2} \sum_{n,k} (P_{n,k} + P^*_{n,k}) - \frac{1}{2} \sum_{\text{plaq}} (Q_{n}Q^*_{n+1} + \text{H.c.}) \]  

with local terms and plaquette interactions, which tend to the Abelian Kogut–Susskind Hamiltonian (37) as \( N \to \infty \). The plaquette indexing convention is, again, according to figure 3.

One can define static modular charges the vertices

\[ q_n = e^{-in\varphi} \]  

and then the Gauss’s law is given by

\[ G_n|\phi\rangle = q_n|\phi\rangle \]  

where

\[ G_n = \prod_{l=1}^{l_n} P^+_{l} \prod_{l=-l_n}^{-1} P_{l} \]  

with \( l_+ \) being links which start from \( n \) (positive links) and \( l_- \) ending there (negative ones).

A description of gauge theories with finite groups (both Abelian and non-Abelian), with dynamical, fermionic matter as well, is found in [163].

2.3. Non-Abelian Yang–Mills theories: SU(2) as an example

As an example of a continuous non-Abelian gauge theory, we examine next the SU(2) gauge theory [90, 157]. SU(2) (the rotation group) has three generators, with the structure constants

\[ f_{ijk} = \epsilon_{ijk} \]  

which implies for the left and right algebras

\[ [L^i_{n,k}, L^j_{n,k}] = -i\epsilon_{ijl} L^l_{n,k} \]  

\[ [R^i_{n,k}, R^j_{n,k}] = i\epsilon_{ijl} R^l_{n,k} \]  

The pure-gauge Hamiltonian is (35). One may choose the representation of the \( U \) matrices, and let us do that in the fundamental representation of \( SU(2) \) (\( j = \frac{1}{2} \)). In this representation we work with the Pauli matrices,

\[ T^i = \frac{1}{2} \sigma^i \]  

and thus

\[ U_{n,k} = e^{i\delta \sigma^0} \]  

which are \( 2 \times 2 \) matrices of operators. The reader may find an explicit expression for this operators, using the formulation [163] when we discuss the quantum simulation of this model.

The links’ local Hilbert space, in the flux basis, is now described using three quantum numbers \( |jmn\rangle \). These are eigenstates of the operators

\[ L^i_{e} |jmn\rangle = R^i_{e} |jmn\rangle = j(j+1) |jmn\rangle \]  

\[ L^i_{e} |jmn\rangle = m |jmn\rangle \]  

\[ R^i_{e} |jmn\rangle = m |jmn\rangle \]  

A mechanical interpretation of this Hilbert space may be a rigid body, described in both body and space coordinates.
In both of them the total angular momentum is equal, but in these two frames of reference the components of the angular momentum will be measured differently, as they form two separate, commuting algebras; the body and the space algebras corresponding to the two algebras we have here [157, 164].

By acting with the group elements, which are rotation matrices, on the $|JMM\rangle$ states, an amount of angular momentum is given to the state (or taken from it). Unlike in the $U(1)$ case, where the angular momentum is described by a single quantum number, here the representation is also a dynamical quantity: for example, acting with $U_{\eta,h}$ in the fundamental representation on a $|JMM\rangle$ state means adding a $\frac{1}{2}$ quanta of angular momentum to the state, resulting in a superposition of $J + \frac{1}{2}$ and $J - \frac{1}{2}$ states (ordinary addition of angular momentum). Quantitatively this is expressed by [165]

$$U_{\eta,h}|JMM\rangle = C_+|J + \frac{1}{2}, M + m, M' + m\rangle + C_-|J - \frac{1}{2}, M + m, M' + m\rangle$$

where $C_\pm$ depends on the appropriate Clebsch–Gordan coefficients [165, 166]. Thus, the representation of the state, the quantum number $J$, is a dynamical observable here, unlike in continuum field theories, where each field has a constant, well-defined, static representation: this is due to the fact that lattice gauge fields are not Fock-space bosons [163].

If one wishes to include dynamical fermions as well, the interaction which will be described by a Hamiltonian of the form (9), where $\psi_n$ are group spinors. Thus, if we pick $U$ to be in the fundamental representation, this will also be the case for the spinors, which will contain now two components; each Lorentz component will be such a spinor. With staggered fermions we will have such a single-colour spinor on each vertex. In this case the charges will simply be [90, 157]

$$Q_n^a = \frac{1}{2} \sum_j \bar{\psi}_n^j (\sigma^a)_{ij} \psi_n^j$$

satisfying the group’s algebra. Note that the ‘Dirac sea’ state can be obtained here as well (as a no-interactions vacuum). Again, we measure the mass of each particle with respect to $-m$, and a fully occupied vertex, due to the fermionic nature of the matter particles, corresponds to zero charge. Thus, the Dirac sea is the case with empty even vertices and fully occupied (by two fermions) odd vertices [167].

The description of other $SU(N)$ groups is similar; changing the group means changing the local Hilbert spaces, but the Hamiltonian has the same form. A description of Hamiltonian lattice compact $QCD$ ($SU(3)$) is given in [100].

2.4. Physical content of the models

Gauge theories in general, including ones on the lattice, exhibit an interesting phase structure. These include, for example, $QCD$’s phases of colour-supercor conductivity or quark–gluon plasma [94, 100, 101], whose treatment using the usual Monte-Carlo methods utilised in lattice gauge theory is rather problematic, due to the computationally hard sign problem [99] that limits calculations in regimes with a finite chemical potential for the fermions.

Another interesting and important property of gauge theories is the existence of confining phases, dependent on the coupling constant and other parameters. We will review quark confinement in pure gauge theories in detail below.

2.4.1. Quark confinement. Free quarks are not found in nature, but rather bind together and form hadrons such as baryons and mesons. The reason for that is the physical phenomenon of quark confinement, manifested in QCD but also in other gauge theories: in large distances, or low energies, the static potential between quarks is linear in the distance, and thus it is energetically unfavourable (and practically impractical) to separate bound quarks. It is a highly intriguing concept, forming a fundamental aspect in the understanding of the Hadronic spectrum, involving nontrivial low energy physics, which cannot be addressed perturbatively.

Schwinger showed, by exactly solving $1 + 1$ dimensional $QED$ with massless charges [102, 103], that vector mesons can be massive if massless fermions are screened by the vacuum polarisation, obtaining massive photons. Further studies [104, 105] have shown that only massive photons, not electrons, exist in the asymptotic states of this model. The latter work has also shown, in accordance with the expected confining behaviour, that electrons arise in deepinelastic-scattering scenarios; i.e. short distances and high energies, and they behave like free point-like particles (agreeing with the ‘parton model’, or the asymptotic freedom of $QCD$, where the confined quarks are asymptotically free within the hadrons at short ranges and high energies). The existence of massive photons is in accordance with the fact that in terms of statistical mechanics, confinement is a gapped, disordered phase, while the ordered phases do not confine [158, 168].

A significant step in the study of confinement was done by Wilson, who introduced lattice gauge theory for this purpose [88]. Using loops in lattice gauge theory, which were later called ‘Wilson Loops’, he showed that the $3 + 1$ dimensional compact $QED$ has a confining phase (for strong coupling, $g^2 \gg 1$), which binds charges together and forbids the existence of free electrons and positrons, as well as a Coulomb phase (for weak coupling, $g^2 \ll 1$), as in continuous $QED$. It was shown later (see section 2.4.3) that $2 + 1$ dimensional $cQED$ confines for all values of the coupling constant [106]: confinement in compact $QED$ was identified as a topological effect, having to do with instantons and magnetic monopoles. This result was supported by other works as well [107–109].

Similar to the ‘charge confinement’ in Abelian theories, confinement also takes place in non-Abelian gauge theories, such as non-Abelian Yang–Mills theories (as $QCD$) as well as the discrete theory of $Z_N$ presented earlier. It was argued [161] that large-distance phenomena, involving the confinement of quarks in $QCD$, is related to the group’s centre; $Z_N$, which makes the phase structure of $Z_N$ interesting [162, 169, 170]: The theory confines in the strong coupling limit ($\lambda \to \infty$). In
where \( T^a_i \) is the matrix representation of the group’s generator in representation \( r \), and \( C_2(r) = \sum_j T^a_i \) is just a C-number; the eigenvalue of the Casimir operator in this representation.

Thus, for example, in the case of \( cQED \), we get [89]

\[
L^2 U |0\rangle = U |0\rangle
\]

(68)

and in the case of \( SU(2) \), for charges in the fundamental representation \( j = \frac{1}{2} [90], \)

\[
L^2 U |0\rangle = \left( \frac{1}{4} \sum_a \sigma^a \right) |0\rangle = \frac{3}{4} U |0\rangle
\]

(69)
or in any other representation \( j \),

\[
L^2 U' |0\rangle = \phi_j (j_j + 1) U' |0\rangle
\]

(70)

This means that if we have \( N \) links with a single excitation, the energy of the state (in zeroth order) will be \( \frac{N^2}{2} C_2(r) \).

In order to respect Gauss’s law (and fulfill gauge invariance), the eigenstates of \( H_0 \) in this charge configuration will contain a flux line of \( U |0\rangle \) states, ranging between the charges. The minimal energy is obtained for the direct line, and thus the ground state is:

\[
| \Omega_{R0, mn} \rangle = \left( \prod_{n=n}^{n+\beta(r)-1} u_{n, x} \right) | \Omega \rangle
\]

(71)

See footnote\(^9\). This flux-line is the so-called electric flux tube, manifesting confinement.

The zeroth order energy of this state is

\[
E_0 | \Omega_{R0, mn} \rangle = H_0 | \Omega_{R0, mn} \rangle = \frac{g^2}{2} C_2(r) | \Omega_{R0, mn} \rangle
\]

(72)

and thus we get that the strong coupling limit confines, with a string tension (the ratio between the static potential and the distance) of

\[
\sigma_{\text{strong}} = C_2(r)
\]

(73)

Such a flux-tube connecting the appropriate pair of them is nothing but a meson. One could also create a long flux tube connecting static charges and watch it dynamically breaking into shorter, energy favourable tubes by dynamic pair creation. Also, as mentioned before, outside the strong limit, but within the confining phase, the state will be modified such that the flux-tube is thicker (wider than just a thin, single lattice line).

2.4.3. The weak limit of \( cQED \). As \( g \) decreases, perturbative corrections must be taken into account, until perturbation theory breaks down and nonperturbative methods of the weak

\[^9\text{It is worth commenting that strong coupling calculations were initially performed in the Euclidean formalism. The first to do that was Wilson [88].}\]

\[^{10}\text{The indices in the state are another manifestation of the fact we disregard the charge degrees of freedom in this context. If one introduces the matter operators \( \psi_n \), the operator creating the state out of the full vacuum would be \( \sum_{n=\beta}^{\beta-1} \sum_{x} u_{n, x} v_{n, x} \psi_n \psi^\dagger \). We easily identify as a gauge invariant string.}\]
regime must be utilised. As an example, we will comment about the weak limit of $cQED$.

In $3 + 1$ dimensions, one can use continuum limit Wilson loop arguments \cite{89}, as the weak limit corresponds to the continuum limit, and obtain the Coulomb potential.

In $1 + 1$ dimensions, on the other hand, the theory confines for all values of the coupling constant, as it is solvable and has no dependence on the coupling constant $g$. This was shown (for the massless case) in the continuum by Schwinger \cite{102, 103}, but similar results apply also in the lattice compact case in \cite{171} (as well as in other, more recent, lattice studies, such as using tensor network states \cite{172-178}).

In $2 + 1$ dimensions interesting physics takes place, involving instantons and topological effects. Due to magnetic monopoles, there is a mass gap in the weak limit corresponding to quark confinement. This was shown by Polyakov \cite{106} using Euclidean partition functions, by Drell, Quinn, Svetitsky and Weinstein, \cite{108} using the variational Hamiltonian approach, and by Banks, Myerson and Kogut \cite{107} using the Villain approximation. The three approaches were compared by Ben-Menahem \cite{109} who also demonstrated that compactness is necessary for confinement in lattice $QED$.

In some sense, the confinement mechanism of $2 + 1$ dimensional $cQED$ is more related to the confinement of quarks in $QCD$ as both of them are topological effects.

The discussion above applies for zero temperature. For a finite, nonzero temperature, a Coulomb phase arises for weak values of the coupling constant, with a phase transition to the Coulomb phase at some value of $T, g$ \cite{93, 95}.

3. Ultracold atoms in optical lattices: a brief review

3.1. Generation of an optical potential

Optical lattices are created by ‘external’, classical lasers, and virtual processes to an excited atomic state \cite{7, 9, 11, 26}.

Consider a two-level atom (disregard the other levels), whose levels are $|\alpha\rangle$; some meta-stable level (in which the atom is initially prepared), and $|e\rangle$, an excited one (see figure 4). We may write its Hamiltonian in the form

$$H_{\text{atom}} = \frac{p^2}{2m} + \omega\delta |e\rangle \langle e|$$  \hspace{1cm} (74)

The atoms are usually alkaline, therefore they have a single valence electron, which simplifies their treatment by many aspects.

The atom experiences an external, classical laser field, of the form

$$E(x, t) = E(x)e^{-i\omega t}$$  \hspace{1cm} (75)

Switching to a system rotating with the laser’s frequency $\omega$, and defining the detuning $\delta = \omega - \omega$ (see figure 4), we get

$$H_{\text{atom}} = \frac{p^2}{2m} + \delta |e\rangle \langle e|$$  \hspace{1cm} (76)

The interaction between the laser and the atom is given in the dipole approximation, which is valid if we assume that the electric field’s amplitude varies slowly compared to the atomic size (in space) and $1/\omega$ (in time; note that here, for simplicity, the amplitude was not even considered as having a time dependence, but one may generalise to a time-dependent amplitude, of the form $E(x, t)$). This interaction has the form

$$H_{\text{dipole}} = -\mathbf{d} \cdot \mathbf{E}(x, t)$$  \hspace{1cm} (77)

where $d$ is the dipole moment of the operator; after transforming to the rotating frame, performing a rotating wave approximation (neglecting the fast-rotating terms) and plugging $d_{ij} = \delta \cdot \langle i|j\rangle$, the dipole moment matrix elements (projected on the laser’s polarisation), one obtains the interaction Hamiltonian

$$H_{\text{dipole}} = -d_{ee}E(x)|e\rangle \langle e| + \text{h.c.} \equiv \frac{\Omega(x)}{2} |e\rangle \langle e| + \text{h.c.}$$  \hspace{1cm} (78)

where

$$\Omega(x) = -2E(x)d_{ee}$$  \hspace{1cm} (79)

is the Rabi frequency ($d_{ee} = d_{\text{net}}$, as the dipole operator is Hermitian).

We assume that the detuning $\delta$ is sufficiently large, the coupling is non-resonant, and that transitions between the atomic levels are practically impossible. However, virtual second-order transitions to the excited level and back are possible, and thus we get the effective Hamiltonian (for the $|\alpha\rangle$ subspace)

$$H = \frac{p^2}{2m} + V_{\text{op}}(x)$$  \hspace{1cm} (80)

where

$$V_{\text{op}}(x) = -\frac{\Omega(x)^2}{4\delta}$$  \hspace{1cm} (81)

is called the optical trapping potential, and it is an effective potential experienced by the atom in the $|\alpha\rangle$ state. In fact, this is merely the AC Stark effect. Note that we have actually ‘traced out’ the internal levels, due to the energy restriction (large detuning) and what we now have is a configuration space Hamiltonian. Choosing the effective field properly, one could generate an optical lattice: a periodic trapping potential allowing the trapped atoms to fill its minima. Since $V_{\text{op}}(x) \propto |E(x)|^2$, a standing wave of the form $E(x) = E_0 \cos(kx)$ would result, for example, in an optical potential of the form $V_{\text{op}}(x) = V_0 \cos^2(kx)$; one could add lasers in more than one

![Figure 4. The energy levels of the trapped atom.](400x702 to 458x774)
direction to obtain a two and three-dimensional optical lattices of various geometries.

Next we will consider an optical lattice with many atoms. This will be done in the context of second quantisation, for which we are interested in the single-particle eigenfunctions of the trapping Hamiltonian.

Due to the lattice symmetry, the single particle wavefunctions are merely Bloch waves. These states are not localised, therefore if one wishes to work on the number basis of local minima (Fock basis), a transformation to local single-particle wavefunctions is required. This is fulfilled by the orthonormal basis of Wannier functions, \( w_n(x - x_0) \), which are the discrete Fourier transform of the Bloch wavefunctions \( u^{\alpha}(x) \) over the lattice [9, 11, 26]:

\[
    w_n(x - x_0) \propto \sum_{q} e^{-i q \cdot (x - x_0)} u^{\alpha}(x)
\]

These states are labelled by \( n \), which is the energy band, and are each spatially centered around a different local minimum \( x_0 \) of the optical potential. These are the desired local site (minimum) wavefunctions.

Working in low enough temperatures, it is reasonable to include only the lowest energy band, \( w_n(x - x_0) \).

One can expand the second-quantisation ‘wavefunction’ of the condensates in terms of the localised Wannier functions \( \phi_n(x) \equiv w_n(x - x_0) \) and single-particle annihilation operators \( a_n \) (bosonic or fermionic, depending on the statistics of the atoms), both defined at the site (minimum of the potential) \( n \), as

\[
    \Phi(x) = \sum_n a_n \phi_n(x)
\]

and construct from which a usual second quantisation Hamiltonian density:

\[
    \mathcal{H} = \Phi^\dagger(x) \left( -\frac{\hbar^2}{2m} + V_{\text{op}}(x) + V_T(x) \right) \Phi(x)
    + \int d^3x \int d^3x' \Phi^\dagger(x') \Phi^\dagger(x) V(x - x') \Phi(x) \Phi(x')
\]

where \( V_T(x) \) is an external trapping potential, varying slowly compared to the optical lattice potential \( V_{\text{op}}(x) \), and \( V(x - x') \) is some two-body interaction, which will be specified later.

### 3.2. Multi-species lattices and spinor condensates

It is also possible to utilise more atomic levels a species-dependent lattice: i.e. with different minima for different atomic species, which may be, for example, different atomic hyperfine levels [8, 11, 26, 179, 180].

The atomic hyperfine structure is given by an SU(2) Hilbert space, characterised by two quantum numbers \( F, m_F \). Just like ‘regular’ angular momentum,

\[
    F^2 |F, m_F\rangle = F(F + 1) |F, m_F\rangle
\]

and

\[
    F^z |F, m_F\rangle = m_F |F, m_F\rangle
\]

\( F \) is defined as

\[
    F = I + L + S
\]

where \( I \) is the nuclear spin, and \( L, S \) stand for the valence electron’s orbital angular momentum and spin, respectively (recall that we are dealing with alkaline atoms, with a single valence electron), and thus the Lie algebra

\[
    [F_i, F_j] = i\epsilon_{ijk} F_k
\]

is satisfied. Each species \( |\alpha \rangle \) can then experience a different optical potential \( V^{\alpha}_{\text{op}}(x) \), and these potentials form a superlattice.

Such lattices/condensates are sometimes referred to as spinor lattices / condensates.

As an example, consider \( ^87\text{Rb} \) and \( ^{23}\text{Na} \) have a nuclear spin \( I = \frac{3}{2} \). Thus, in the S-wave hyperfine manifold (which is relevant for ultracold atoms, as explained in section 3.3), \( F = 1 \) or 2.

We denote the internal levels by Greek indices, and the second-quantisation expansion generalises to

\[
    \Phi_\alpha (x) = \sum_n a_{n,\alpha} \phi_{n,\alpha}(x)
\]

for each species, where the Wannier functions may be species dependent or independent, depending on the lattice structure and symmetry and the choice of these levels.

The Hamiltonian density is now

\[
    \mathcal{H} = \sum_{\alpha,\beta} \Phi_\alpha^\dagger(x) \left( -\frac{\hbar^2}{2m} + V^{\alpha}_{\text{op}}(x) + V_T(x) \right) + \Omega^{\alpha\beta}(x) \Phi_\alpha(x) + \sum_{\alpha,\beta,\gamma,\delta} \int d^3x \phi_\alpha^\dagger(x') \phi_\beta^\dagger(x) \phi_\gamma(x) \phi_\delta(x')
\]

where \( \Omega^{\alpha\beta}(x) \) corresponds to a Rabi coupling of atomic levels by more lasers.

### 3.3. Scattering of ultracold atoms

How cold is ultracold? The ultracold regime is defined as the one in which the lowest partial wave governs the scattering. It is a dilute and cold regime.

We are dealing with the so-called weak interaction regime, involving the ground state of a gas of either bosonic or fermionic atoms. For ideal bosons, we assume that they all occupy the lowest energy band (Bose–Einstein condensate). A phase transition from Bose–Einstein condensate takes place in a finite temperature (in three dimensions), when the thermal de-Broglie wavelength \( \lambda_T = \frac{2\pi}{\sqrt{n}} \) (in \( \hbar = 1 \) units) approaches \( n^{-1/3} \), the average inter-particle distance (\( n \) is the number density). For fermions, the states are full up to the Fermi energy, \( \epsilon_F \), for temperatures satisfying \( k_B T \ll \epsilon_F \) the gas is considered degenerate [9].

When the energy is low enough, i.e. the de-Broglie wavelength of the atoms is much larger than the effective range of the potential [11], the interaction potential in (84) is accurately described by the pseudopotential:
\[ V(x - x') = \frac{2\pi a}{m} \delta^{(3)}(x - x') \equiv \frac{g}{2} \delta^{(3)}(x - x') \]  
(91)

For bosons the scattering is governed by the S-wave channel. For identical fermions, on the other hand, due to Pauli’s exclusion principle, there is no S-wave scattering [9]; this is not a problem for the scattering of two different fermions (non-identical).

The density of the atoms, \( n \), is typically \( 10^{12} - 10^{15} \text{ cm}^{-3} \). This gives a typical interparticle distance \( n^{-1/3} \) of the order of 0.1–10 \( \mu \text{m} \), and the scattering length is in the range of few nanometers [9]. The scattering length of \( ^{87}\text{Rb} \), for example, has an order of magnitude of 100 times the Bohr radius, varying slightly according to the different internal states [180]. The gas is sufficiently dilute for \( n^{1/3}a \ll 1 \), in the case of bosonic repulsive interactions (\( a > 0 \)), \( \sqrt{na} \) is the small parameter of the Bogolyubov theory, describing weakly interacting Bose gases [9].

We thus obtain a simplified version for the single-species case Hamiltonian density:

\[ \mathcal{H} = \Phi(x) \left( \frac{\nabla^2}{2m} + V_{\text{op}}(x) + V_{\text{T}}(x) \right) \Phi(x) + \frac{g}{2} \Phi(x) \Phi(x) \Phi(x) \Phi(x) \]  
(92)

after plugging in the Wannier functions and creation/annihilation operators (83) one may integrate over the Wannier functions. Using the overlap integrals

\[ \epsilon_n = \int d^3x \phi_n^*(x) \left( \frac{\nabla^2}{2m} + V_{\text{op}}(x) + V_{\text{T}}(x) \right) \phi_n(x) \]  
(93)

\[ J_{nn} = \int d^3x \phi_n^*(x) \left( \frac{-\nabla^2}{2m} + V_{\text{op}}(x) + V_{\text{T}}(x) \right) \phi_n(x) \]  
(94)

\[ U_{nmkl} = g \int d^3x \phi_n^*(x) \phi_m^*(x) \phi_k(x) \phi_l(x) \]  
(95)

where \( m, n, k, l \) label the lattice sites. The most general Hamiltonian, taking into account all possible overlap integrals, is

\[ H = \sum_{m,n} J_{nn} \hat{a}_m^\dagger \hat{a}_n + \sum_{m,n,k,l} U_{nm,k,l} \hat{a}_m^\dagger \hat{a}_n^\dagger \hat{a}_k \hat{a}_l \]  
(96)

### 3.4. Multiple species scattering

In the case of multiple species, the most general form of the Hamiltonian is

\[ H = \sum_{m,n,\alpha,\beta} J_{\alpha\beta} \hat{a}_m^\dagger \hat{a}_n^\dagger + \sum_{m,n,k,l,\alpha,\beta,\gamma,\delta} U_{\alpha\beta\gamma\delta} \hat{a}_m^\dagger \hat{a}_n^\dagger \hat{a}_k^\dagger \hat{a}_l \]  
(97)

however, if we remember that the species are different hyperfine states, and take into account the hyperfine rotational symmetry (conservation of the hyperfine angular momentum in atomic collisions) we can simplify this Hamiltonian further, as we will do next [179, 180].

Suppose we wish to consider the scattering of two atoms with distinct \( F, m_F \) values; for example, let us scatter an \( F_1 \) atom on an \( F_2 \) atom. As a result, we will get a pair of two atoms with their own \( F, m_F \) values. However, taking into account the rules of addition of angular momenta, one should be aware that the scattering may take place ‘through’ an ‘intermediate state’ with several values of total angular momentum \( F_T \). Thus, the effective scattering potential will take the form

\[ V_{a_0,\beta,\gamma,\delta}(x - x') = \frac{2\pi}{m} \delta^{(3)}(x - x') \sum_{F_T} a_{F_T}(P_{F_T})_{a_0,\beta,\gamma,\delta} \]  
(98)

where we sum over the possible values of total angular momentum \( F_T \), and consider, for each value, its corresponding scattering length \( a_{F_T} \). \( P_{F_T} \) is the projection operator onto the subspace of total angular momentum \( F_T \). In the case of different atomic masses, the appropriate reduced masses should replace the mass \( m \).

The projection operators can be represented using functions of \( F_1 \cdot F_2 \). Physically, that may be understood as a consequence of rotational invariance: terms which depend on the \( F \) values of two atoms and conserve the hyperfine angular momentum must be constructed out of powers of these terms. Mathematically,

\[ F_1 \cdot F_2 = \frac{1}{2} (F_1^2 + F_2^2 - F_{12}^2) \]  
(99)

and as \( F_{12}^2 \) are practically constants for our purposes, a function of \( F_1 \cdot F_2 \) is in fact a function of \( F_{12}^2 \), and thus one can construct the projection operators out of this scalar product. If there are \( n \) possible values of total hyperfine angular momentum, the projection operators are polynomials of the form

\[ P_{F_T} = \sum_{k=0}^{n-1} G_{F_T,k}(F_{1} \cdot F_{2})^k \]  
(100)

and eventually obtain

\[ V_{a_0,\beta,\gamma,\delta}(x - x') = \delta^{(3)}(x - x') \sum_{k=0}^{n-1} \langle a_{F_T} G_{F_T,k} \rangle_{a_0,\beta,\gamma,\delta} \]  
(101)

where the \( \langle \rangle \) coefficients depend simply on the scattering lengths \( \{ a_{F_T} \} \) and the coefficients in the projection operators’ expansion, \( \{ G_{F_T,k} \} \). The scattering lengths are tunable using Feshbach resonances, either magnetic or optical [9, 180–188].

We can also express the projection operators, and thus the scattering potential, in terms of the Clebsch–Gordan coefficients:

\[ V_{a_0,\beta,\gamma,\delta}(x - x') = \delta^{(3)}(x - x') \times \sum_{F_T} \langle C_{F_T} \rangle_{a_0,F_1,m_{F_1} = \alpha; F_2, m_{F_2} = \beta} |F_T, M_T \rangle \times \{ C_{F_T} \}_{a_0, \beta, \gamma, \delta} \]  
(102)

which explicitly manifests, as a consequence of hyperfine angular momentum conservation, that \( \alpha + \beta = \gamma + \delta \), and the \( \{ C_{F_T} \} \) coefficients depend on the scattering lengths.

As an example [179, 180], one may consider a spinor condensate of \( F = 1 \). The possible values of total angular
momentum in collisions are naively $F_T = 0, 1, 2$, but since the atoms are bosons, and the states must be symmetric, we remain with $F_T = 0, 2$. Thus, the scattering lengths we need are $a_0$ and $a_2$, and the highest power of $F_1 \cdot F_2$ we need is 1. The projection operators are

$$P_0 = \frac{1}{3}(1 - F_1 \cdot F_2)$$

and

$$P_2 = \frac{1}{3}(F_1 \cdot F_2 + 2)$$

One can define

$$g_0 = \frac{4\pi}{3m}(2a_2 + a_0)$$

and

$$g_2 = \frac{4\pi}{3m}(a_2 - a_0)$$

and obtain the scattering part of the Hamiltonian density

$$\frac{g_0}{2} \sum_{\alpha,\beta} \Phi_{\alpha}^\dagger(x)\Phi_{\beta}^\dagger(x)\Phi_{\beta}(x)\Phi_{\alpha}(x) + \frac{g_2}{2} \sum_{\alpha,\beta,\gamma,\delta} (\Phi_{\alpha}^\dagger(x)\Phi_{\beta}(x)) (\Phi_{\gamma}^\dagger(x)\Phi_{\delta}(x))$$

where $F$ are the $F = 1$ representation matrices of $SU(2)$.

3.5. Summary of the available interactions and possibilities

To conclude, in optical lattices one could use either bosonic or fermionic ultracold atoms, of species $\{\alpha\}$, with single particle local wavefunctions at sites $n$, $\phi_{n,\alpha}(x)$ and $\psi_{n,\alpha}(x)$ for bosons and fermions, respectively, and annihilation operators $a_{n,\alpha}$ and $c_{n,\alpha}$ for bosons and fermions, respectively, satisfying the canonical commutation relations

$$[a_{n,\alpha}, a_{m,\beta}^\dagger] = \delta_{nm}\delta_{\alpha\beta}$$

$$[c_{n,\alpha}, c_{m,\beta}^\dagger] = \delta_{nm}\delta_{\alpha\beta}$$

$$[a_{n,\alpha}, c_{m,\beta}] = [c_{n,\alpha}, a_{m,\beta}] = [a_{n,\alpha}, c_{m,\beta}] = 0$$

Out of these operators and wavefunctions, the second-quantisation field operators are constructed,

$$\Phi_{\alpha}(x) = \sum_n a_{n,\alpha}\phi_{n,\alpha}(x)$$

$$\Psi_{\alpha}(x) = \sum_n c_{n,\alpha}\psi_{n,\alpha}(x)$$

and they are the ingredients of the atomic Hamiltonian, consisting of three main parts:

(i) The single-particle terms,

$$H_0 = \sum_{\alpha} \int d^3x (\Psi_{\alpha}(x)H_0^F\Psi_{\alpha}(x) + \Phi_{\alpha}(x)H_0^\Psi\Phi_{\alpha}(x))$$

where $H_0^F, H_0^\Psi$ are the fermionic and bosonic single-particle Hamiltonians, including the kinetic energy and the optical and trapping potentials. They depend on (or, from another perspective, affect) the shape of the lattice (through the optical potential) and thus the single-particle wavefunctions, $\phi_{n,\alpha}(x)$ and $\psi_{n,\alpha}(x)$.

(ii) The scattering terms,

$$H_{sc} = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \int d^3x \Phi_{\alpha}^\dagger(x)\Phi_{\gamma}(x)\Phi_{\beta}^\dagger(x)\Phi_{\delta}(x)$$

$$+ \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \int d^3x \phi_{\alpha}(x)\phi_{\beta}(x)\phi_{\gamma}(x)\phi_{\delta}(x)$$

$$+ \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \int d^3x \psi_{\alpha}(x)\psi_{\beta}(x)\psi_{\gamma}(x)\psi_{\delta}(x)$$

which govern the two-body S-wave scattering (atomic collisions), and depend on the scattering coefficients $g$. They depend on the scattering lengths, and of course on the nature of collisions; fermion–fermion (FF), boson–boson (BB) or boson–fermion (BF) and the atomic species involved. These coefficients are not independent of each other, thanks to the conservation of hyperfine angular momentum, and are tunable using Feshbach resonances.

(iii) Rabi (laser) terms,

$$H_R = \sum_{\alpha,\beta} \int d^3x \Phi_{\alpha}^\dagger(x)\Phi_{\beta}(x)$$

$$+ \sum_{\alpha,\beta} \int d^3x \phi_{\alpha}(x)\phi_{\beta}(x)$$

which are governed by external lasers and may induce the desired hopping processes.

One could also include transitions to and from molecular states, but these are not of relevance for this report.

Current experimental technologies allow the creation of complex and controllable optical lattices. The range of parameters is wide, as exemplified by the typical numbers presented above: the ratio $V_0/E_R$ is tunable. Optical lattices may be created in various geometries and dimensionalities using holographic techniques, for example. The lattices may be very large (for example, 150 000 lattice sites in [20]) and contain many atoms. The atoms may be distributed across the lattice in various ways: lattice sites may occupy a few atoms, even single ones [20, 30, 189–191] or much more (for example, 80 in [192], using tube-shaped sites).

Common measurements are the so-called ‘time-of-flight’ measurements, in which the lattice is switched off (suddenly or adiabatically) and the atoms are released, in a way that their spatial density is proportional to their momentum distribution, which is measurable using standard imaging methods [9]. Contemporary experiments also demonstrate with great success the single-addressability of lattice sites and atoms [189–192], which opens the way for various possibilities, both in measurements and initial state preparation.

Further detailed information on the topic may be found in [11].
4. A framework for the quantum simulation of lattice gauge theories with ultracold atoms in optical lattices

We will review several methods for quantum simulation, i.e. physical realisation using ultracold atoms in optical lattices of three different lattice gauge theories: compact QED, SU(N) and \( \mathbb{Z}_N \).

In order to simulate a gauge theory, three requirements must be fulfilled [121]:

(i) The theory must contain both fermions and bosons. The fermions will represent matter fields and the bosons gauge fields. This makes ultracold atoms natural candidates, as bosonic and fermionic atoms are practically free resources in these systems. Ultracold atoms in optical lattices are currently highly controllable systems. Of course, the nature of lattice fermions should be chosen carefully, based on both the experimental considerations and the desired theoretical approach for lattice fermions, as discussed above.

(ii) The theory must be Lorentz invariant, i.e. have a causal structure, as a field theory. This may seem problematic, but one can simulate the lattice gauge theories described above, which obtain this symmetry in the continuum limit. This goes along quite well with the structure of the simulating systems, which are optical lattices.

(iii) The theory must have local gauge invariance, which is the symmetry responsible for gauge–matter interactions. This seems the most problematic demand, and the biggest challenge, as this does not seem to be a fundamental symmetry in such atomic systems, and has been one of the most important research steps.

Thus, after considering these three requirements and recognising that ultracold atoms in optical lattices should enable their fulfillment, such systems have been chosen as the simulating systems; building blocks for optical realisations of lattice gauge theories, opening the way for quantum simulations of lattice gauge theories. The main idea is to obtain the gauge invariant interactions which are not fundamental for ultracold atoms. We present two methods for that; one in which gauge invariance is imposed as a constraint, the effective method, in which the gauge invariance is only an emerging, low-energy sector symmetry, and the other one, in which gauge invariance is mapped into a fundamental symmetry of the atoms, the conservation of hyperfine angular momentum in atomic collisions.

In all the simulation approaches presented below, regardless of the specific implementation and representation of the gauge degrees of freedom, they are represented by bosons populating the links of an optical lattice. Fermions in proposals involving dynamical matter are always located in minima coinciding with the vertices (see figure 5(b)).

5. Effective gauge invariance emerging at low energies

First we will describe a class of models, in which the gauge symmetry is not fundamental, but rather arises in a low-energy sector, as an emerging, effective symmetry. In this method, the atomic Hamiltonian is not gauge invariant, but rather contains a large penalty term demanding, as a low-energy constraint, that Gauss’s law will be satisfied. The constraint Hamiltonian takes the form

\[
H_G = \lambda \sum_n G_n^2
\]  

(115)

where \( \lambda \) is the largest energy scale in the system. This automatically divides the physical Hilbert space into sectors, corresponding to different eigenvalues of \( G_n \) (31); different static charge configurations, but the Hamiltonian interactions involve transitions between them. These are practically impossible, due to energy conservation. Thus, an effective Hamiltonian
for the ground sector has been introduced in any of these proposals, giving rise to a gauge-invariant effective theory, which includes the desired interaction (see figure 6).

5.1 cQED in $2 + 1$ dimensions—link interactions

It is shown in [127], as a proof of principle, that a quantum simulation of the Kogut–Susskind Hamiltonian for cQED is, indeed, possible, and this was the first work arguing that. This first proposal discussed only the pure-gauge theory, suggesting a method to observe confinement of static charges, by probing electric flux tubes.

In this work, every link of a two-dimensional spatial lattice (labelled by each vertex it emanates from $(m,n)$ and its direction $k$) is occupied by a Bose–Einstein condensate, with a mean number of atoms $N_0$ (uniform all over the lattice). We expand the local number operators $N^{k}_{m,n}$

$$N^{k}_{m,n} = N_0 + \delta^{k}_{m,n}$$

where $N_0$ is a C-number and $\delta^{k}_{m,n}$ an operator, measuring the excess of the atomic population over (or below) $N_0$, which can take negative values as well, and represents the electric field. These operators are conjugate to the local condensates’ phases, $\theta^{k}_{m,n}$, playing the role of the compact vector potential.

In a limit where the expectation value, as well as the uncertainty of $\delta^{k}_{m,n}$, are much smaller than $N_0$, a quantum-rotor approximation may be introduced, allowing the expansion of local annihilation (or creation) operators as

$$a \approx \sqrt{N_0} \exp(i\theta)$$

Neighbouring links are populated by condensates of different atomic species and thus direct, gauge-variant tunneling is eliminated. This is possible by the use of holographic masks, for example, for the creation of the optical lattice. Local scattering processes are responsible for the electric Hamiltonian, as well as along with neighbouring links scattering, Gauss’s law constraint. Using external lasers, the desired hopping processes between neighbouring links (diagonally) are introduced (see figure 7). These terms are gauge-variant as well, but they induce effective gauge invariance: these hopping processes correspond to the violation of the constraint, but second-order perturbation theory ties

Figure 6. Effective gauge invariance. The original atomic Hamiltonian (a) mixes among the Hilbert space sectors of different static charge configurations, as it is not gauge invariant: the generators of gauge transformations (31) do not commute with the Hamiltonian. Using the Gauss’s law constraint (115), one effectively eliminates the transitions among these sectors, resulting in the desired gauge invariant interactions, and the restoration of gauge invariance (10) by constructing an effective Hamiltonian to such a sector of interest.

Figure 7. The lattice structure of [127]. Bosons of four different species (1–4) occupy the links such that neighbouring links are occupied by different species, and thus nearest-neighbour hopping is avoided. Instead, diagonal hopping processes are induced by external lasers (denoted by the black arrows on the right): $1 \leftrightarrow 3, 3 \leftrightarrow 2, 2 \leftrightarrow 4, 4 \leftrightarrow 1$. Scattering processes take place among the four species meeting in (and overlapping on) a vertex, and must be tailored properly.
pairs of them together, forming the plaquette interactions (see figure 8).

Experimental considerations (homogeneity of the scattering lengths) result in a sum-Gauss’s law, rather than with a divergence, i.e.

\[ G_{m,n} = \delta_{m,n}^1 + \delta_{m,n}^2 + \delta_{m-1,n}^1 + \delta_{m,n-1}^2 - \Delta_{m,n} \]  

and thus a (theoretical) canonical transformation, involving inversion of signs, must be carried out in order to obtain the ‘right’ cQED degrees of freedom,

\[ E_{m,n}^k = (-1)^{m+n} \delta_{m,n}^k \partial_{m,n}^k \rightarrow (-1)^{m+n} \partial_{m,n}^k; \]

\[ q_{m,n} = (-1)^{m+n} \Delta_{m,n} \]  

(119)

Being theoretical, this transformation should be carried out on the results of the measurements: a strong coupling electric flux-tube will thus be manifest by a line of alternating values of \( b_{m,n}^k \) between the charges.

This proposal also allows us to leave the perturbative (in terms of the simulated theory) strong-coupling regime, but to a limited extent, since the atomic excess must be small comparing to the mean population, which cannot be too large due to experimental consequences. However, in \( 2 + 1 \) dimensions, where confinement holds also in the weak limit, the simulation of the weak limit should work, at least qualitatively, even if not describing the exact weak limit of the full Kogut–Susskind Hamiltonian. This can be understood from the weak limit analysis of [128].

5.2. Truncated cQED model—link interactions

The second proposal [128] was a first step in the direction of a simpler implementation. Instead of Bose–Einstein condensates, each link is occupied by a single atom belonging to each of \( 2\ell' + 1 \) possible atomic species, representing \( 2\ell' + 1 \) possible values of the electric field on a link: here we have introduced the spin-gauge model, in which the electric field is truncated (taking values between \( -\ell' \) and \( \ell' \)), but the gauge symmetry is unaffected, which should lead, at least qualitatively, to similar effects as those of the Kogut–Susskind Hamiltonian. The \( U(1) \) angular momenta are replaced by \( SU(2) \) angular momenta in an integer representation \( \ell' \) with \( 2\ell' + 1 \) levels, with the \( L_\pm \) operators of the electric field replaced by the truncated \( L_\pm \), and the unitary raising/lowering phase operators replaced by the non-unitary \( SU(2) \) ladder operators \( L_{\pm} \):

\[ U \rightarrow \frac{1}{\sqrt{\ell' (\ell' + 1)}} L_+ \]  

in a confining phase, the electric field is supposed to take small values \( m \), and thus with a large enough \( \ell' \), as well as the \( m \ll \ell' \) of confinement, the unitarity of the ladder operators is approximately restored, at least in the sense that when one acts with them on the \( |m\ell'\rangle \) states, no prefactor is obtained:

\[ \frac{1}{\sqrt{\ell' (\ell' + 1)}} L_+ |m\ell'\rangle = \frac{1 - m(m + 1)}{\ell' (\ell' + 1)} |\ell', m + 1\rangle \approx |\ell', m + 1\rangle \]  

(121)

In this representation, one obtains the Spin-gauge Hamiltonian [128],

\[ H_{SG} = \frac{g^2}{2} \sum_{n,k} L_{n,n+k}^2 - \frac{1}{2g^2 \ell' (\ell' + 1)^2} \times \sum_{\text{plaquettes}} (L_+ n n + 1 L_+ n + 2 L_+ n + 3 + H.c.) \]  

(122)

We have compared this model to the Kogut–Susskind Hamiltonian (37). First, we have shown the more obvious result, that in the strong limit, as the states are eigenstates of electric flux (see section 2.4.2), one may use a truncated theory which suffices to describe the state of the system in the presence of static charges. Conditions on \( \ell' \), depending on the magnitude of the static charges have been discussed using
perturbation theory considerations. It was shown what is the highest order in strong-limit perturbation theory, in which the spin–gauge states will correspond to the Kogut–Susskind ones, for a given charge configuration.

Furthermore, we have addressed the nonperturbative, non-trivial weak limit as well, and compared numerically the single-plaquette results for a truncated Kogut–Susskind theory as well as the spin–gauge theory to the analytical results of [108]. The results converge quantitatively to the analytical ones very quickly, for small values of $\ell$ (already for $\ell \sim 2, 3$).

An implementation for $\ell = 1$, three bosonic internal levels, has been introduced. It involves a three-scale hierarchy; the first, strongest constraint is responsible for the effective angular-momentum interactions [195] and the second one to gauge invariance, in a similar manner to [127]. Working with $\ell = 1$ limits the possibility of going into the weak limit. Nevertheless, the rapid convergence of the results in the weak limit to the ones of the Kogut–Susskind model, do show that implementations of the spin–gauge model for higher values of $\ell$ will serve as an accurate weak-limit simulation. Confinement and flux tubes take place here in the form of alternating atomic species along the line connecting the static charges, in the strong limit.

Due to the non-unitarity of the $L_+^i$ operators, the effective constructions involves some undesired terms. However, they are gauge-invariant as well (as they satisfy Gauss’s law constraint) and their effect is discussed in [128].

Finally, in [133] we have introduced dynamical fermions as well. Still considering $cQED$, in the spin–gauge model presented in paper [128], but regardless of the implementation the gauge degrees of freedom, we have introduced naive fermions into the system, in $2 + 1$ dimensions.

Working with naive fermions in two spatial dimensions, each vertex contains two fermionic species, simulating a two-component Dirac spinor,

$$\psi_n = \begin{pmatrix} c_n \\ d_n \end{pmatrix}$$

(123)

with the local charge

$$Q_n = \psi_n^\dagger \psi_n - 1 = N_n^C + N_n^D - 1$$

(124)

(similar to the charge in the second quantised Dirac field. There, for the lower entry of the spinor, we swap the fermionic creation and annihilation operators, and thus obtain that this charge transforms to the well-known $N_n^C - N_n^D$ form).

The simulated Hamiltonian (besides the pure-gauge part, (122)) in $2 + 1$ dimensions, is

$$H_{\text{naive}} = \frac{ie}{\sqrt{\ell(\ell + 1)}} \sum_{n,k}(\psi_n^\dagger n \psi_{n+1,k}^{\dagger} L_{+,n,k} - \text{H.c.}) + M \sum_{n,k} \psi_n^\dagger \sigma \psi_n$$

(125)

with the Dirac matrices being Pauli matrices, $\alpha_{1,2} = \sigma_1$, and $\beta = \sigma_2$. The fermionic mass terms manifest explicitly that the particles and antiparticles are not divided among the vertices as in the staggered formulation.

Gauge invariance is still effective here. The inclusion of dynamical charges into the Gauss’s law constraints made it harder to implement, also requiring alternating signs to the dynamical charges. A detailed prescription for the implementation, based on fermion–boson scattering was given, enabling the generation of this required constraint; the interaction between the gauge field and the matter arises as an effective second order term as well: in order to satisfy the constraint, fermions can only hop along with a change of the electric field on the link.

The generalisation of this method to other types of lattice fermions is straightforward. One simply has to change the fermionic ingredients and tailor the desired hopping. The effective construction will make sure that only the gauge-invariant interactions are obtained.

5.3. Effective gauge invariance—some general remarks

The three proposals [127, 128, 133] involved gauge invariance as an effective symmetry. This allows the introduction of gauge invariance to the simulating systems, but it is not the only possible approach, as shown in the subsequent papers [121, 146].

The Gauss’s law constraint involves the scattering of atoms, representing the gauge field degrees of freedom on several neighbouring links. All the interactions must be with the same amplitude, which is hard to implement. This may lead to practical impossibilities to satisfy the Gauss’s law, for example, or to the unintended inclusion of fractional charges.

The robustness of this method was considered by Kasamatsu et al [196, 197], who showed, mainly based on [127], but also on other works by us and other groups, that one may relax the fine-tuning demands of the simulating Hamiltonian parameters, and still get a gauge-invariant theory, involving Higgs fields.

Another disadvantage of this approach applies to the simulation of non-Abelian gauge theories. There, the generators of gauge transformations required for the Gauss’s law constraint, are much more complicated (several non-commuting generators (31)), which makes the constraint Hamiltonian (115) very difficult to realise.

6. Local gauge invariance arising from many-body interaction symmetries

The basic idea of the second simulation approach we describe here, is that one could use local interactions of the many-body system, atomic collisions, to achieve local gauge invariance. In particular, we have utilised the conservation of the hyperfine angular momentum in atomic collisions. This enables us to obtain, as fundamental ingredients of the Hamiltonian, the on-link gauge–matter (boson–fermion) interactions of (9), without any constraints or perturbation theory [121, 146].

As a second step, in $2 + 1$ (and more) dimensional setups, one may introduce the loop method [121] presented below, which allows the construction of the plaquette interactions effectively, but out of the already gauge invariant on-link interactions.
The basic, on-link interactions (9) are obtained from boson–fermion collisions, utilising the overlap of four Wannier functions on a link (two bosonic on the links and two fermionic on its edges). The only allowed scattering channels are those which conserve the hyperfine angular momentum. Thus, one can specifically choose the hyperfine levels representing the bosons and the fermions, such that the $F$ conserving processes will be mapped to the gauge invariant ones. This gives us the desired interactions, along with other ones, gauge invariant as well, which may be tolerated or tamed.

6.1. Compact QED—on-link interactions

We will show this procedure in detail for compact QED in $1+1$ dimensions—the lattice Schwinger model. Besides its advantage as a toy model which allows a clear and simple presentation of the method, the quantum simulation of $1+1$ dimensional QED with matter, in the manner presented below, has several advantages:

(i) As gauge invariance is fundamental here, no use of effective interactions is required, i.e. the simulating Hamiltonian produces the simulated theory without any the use of perturbation theory. This implies that the energy scales are larger, and thus the simulation should be faster and more robust to noise and decoherence.

(ii) It is a possibly simpler implementation (compared with [127] and [128]).

(iii) The $1+1$ dimensional Schwinger model is exactly solvable [102, 103]. Lattice results are available as well ([171] and much more, including new calculations involving matrix product states [172]). Along with 3, this make this model a possibly preferred model for a first physical realisation, as one has analytical and numerical results to compare with.

While the fermions are simply fermionic atoms in this simulation (staggered ones, to be specific, but they may be generalised to other lattice fermion representations), the bosonic fields are constructed out of the so-called Schwinger bosons, utilising Schwinger’s representation of SU(2) [198, 199]: in this cQED simulation, each link may be populated by two different bosonic species, $a$ and $b$, from which the algebra is constructed in the following way:

$$ L_z = \frac{1}{2} (a^\dagger a - b^\dagger b) \quad ; \quad \ell = \frac{1}{2} (a^\dagger a + b^\dagger b) \quad (126) $$

$$ L_+ = a^\dagger b \quad ; \quad L_- = b^\dagger a \quad (127) $$

using this representation, along with the conservation of hyperfine angular momentum, simulations of the spin–gauge Hamiltonian may be obtained, but much more simply than in the effective approach. Assuming that the bosons do not interact between links, $\ell$ is fixed and governed by the number of on-link bosons. If we consider a single link, whose left edge may be occupied only by the fermionic species $c$, and its right edge by $d$, we wish to obtain the interactions

$$ c^\dagger L_+ d + d^\dagger L_- c = c^\dagger a^\dagger b d + d^\dagger b^\dagger a c \quad (128) $$

and this is possible by selecting

$$ m_\ell(a) + m_\ell(c) = m_\ell(b) + m_\ell(d) \quad (129) $$
(see figure 9). Even two atoms per link, equivalent to \( \ell = 1 \), give rise to the required gauge symmetry. But enlarging \( \ell \) here involves many atoms of the same two species, condensates, rather than many degrees of freedom, as in [128]. The electric energy term of \( L_z^2 \) is obtained by other ‘gauge invariant’ scattering processes, between the bosons of a single link.

This simulation proposal has been treated numerically using matrix product states [174].

6.2. SU(2)—Yang–Mills theory—on-link interactions

6.2.1. Left and right Schwinger representations. One can utilise this type of interactions for other gauge theories as well, such as the SU(2) lattice gauge theory. In [146], we have suggested a way for realising an SU(2) lattice gauge theory, in \( 1 + 1 \) dimensions, utilising the prepotential method [140–143] for representing the gauge degrees of freedom (which generalises the Schwinger representation). In this method, every link is decomposed into two parts—left and right, corresponding to the left and right degrees of freedom. The Hilbert space of each edge of the link consists of two harmonic oscillators, generating a separate Schwinger algebra: \( a_{1,2} \) on the left side, with the left SU(2) algebra (24) generated by

\[
L_{\alpha} = \frac{1}{2} \sum_{y} a_{y}^{\dagger}(\sigma_{y})_{\alpha} a_{y}
\]

and \( b_{1,2} \) on the right side, with the right SU(2) (25) algebra generated by

\[
R_{\alpha} = \frac{1}{2} \sum_{y} b_{y}^{\dagger}(\sigma_{y})_{\alpha} b_{y}
\]

The left and right couples of oscillators have the same total number of excitations

\[
N_{L} \equiv a_{1}^{\dagger}a_{1} + a_{2}^{\dagger}a_{2} = b_{1}^{\dagger}b_{1} + b_{2}^{\dagger}b_{2} \equiv N_{R}
\]

(132)

(which is proportional to the total angular momentum in the Schwinger representation (126)), and thus, the relation (60) is satisfied and the correct Hilbert space with three degrees of freedom and left and right SU(2) algebras is achieved. The rotation matrices’ elements are constructed out of the creation and annihilation operators of these oscillators. One can decompose them to \( U = U_{L}U_{R} \), with the matrices

\[
U_{L} = \frac{1}{\sqrt{N_{L} + 1}} \begin{pmatrix} a_{1}^{\dagger} & -a_{2}^{\dagger} \\
 a_{2}^{\dagger} & a_{1} \end{pmatrix} ; \quad U_{R} = \begin{pmatrix} b_{1}^{\dagger} & b_{2}^{\dagger} \\
 -b_{2} & b_{1} \end{pmatrix} \frac{1}{\sqrt{N_{R} + 1}}
\]

(133)
defined on both sides of the link.

The implementation presented in [146] involves bosons on the links and fermions on the vertices as before, but now, due to the new, exact gauge invariance method, the hyperfine levels of the atoms must be chosen carefully, such that only the desired scattering processes which correspond to gauge invariant interaction will be allowed by hyperfine angular momentum conservation. The links were not established as first principle interactions, but rather glued out of two building blocks, using auxiliary, ancillary fermions in second-order perturbation theory, forming effective link interactions: each of the building blocks is obtained using the above prescription, involving transition to the auxiliary fermionic site in the middle, whose energy is too high, and in order to satisfy this constraint one obtains, in second order, the full SU(2) link (see figure 10). Nevertheless, gauge invariance is not effective in this proposal in the sense there is no Gauss’s law constraint, and this is the achievement of this work. The link bosons involved contain four species playing the role of the harmonic oscillators of the prepotential method, from which the algebras are constructed, and the other four auxiliary species, serving as reference baths required for the number-conserving scattering processes.

However, this is not a simulation of the full SU(2) Kogut–Susskind theory using the exact prepotential Hamiltonian; much like some of the cQED simulations (in the spin–gauge method), in which the electric field was truncated, here it was restricted to the representations \( j = 0, \frac{1}{2} \). This, along with the square roots in the denominators of prepotential representation of the group elements (rotation matrices), which were not achieved in the simulation, have generated some inaccuracies compared to the Kogut–Susskind theory, which result in the dynamics being accurate only in a regime where one may consider the gauge–matter interactions as a small perturbation (to the sixth order in it).

6.2.2. Truncation scheme for the Hilbert space of non-Abelian groups. One can overcome the problem of the square roots of operators in the denominator, as well as the need for auxiliary bosonic species and the decomposition of the links, if the truncation scheme of [163] is utilised. In this method, we still work with a truncated theory, but the truncation is done in a gauge invariant way, which does not involve decomposition of the link into two (left and right) pieces.

The bosonic degrees of freedom are represented by single bosonic atoms residing on each link of the lattice. The state of each such atom may be one of \( N = \sum_{j=0}^{j_{max}} (2j + 1)^{2} \) internal levels, labelled by \( |jm\rangle \). These are generated from the local atomic vacuum state \( |0\rangle \) by the creation operators \( a_{j}^{\dagger} \), with

\[
|jm\rangle = a_{j}^{\dagger}|0\rangle
\]

(134)

Out of these creation operators and their conjugate annihilation operators, one can construct the SU(2) algebras

\[
L_{\alpha} = \sum_{j} a_{j}^{\dagger}(T_{\alpha})_{j} a_{j} \quad \quad \quad R_{\alpha} = \sum_{j} a_{j}^{\dagger}(T_{\alpha})_{j} a_{j}^{\dagger}
\]

(135)

Figure 10. Structure of the non-Abelian lattice atomic simulator in the prepotential method: the simulated link is decomposed into two pieces, tied together by an auxiliary fermion.
where a summation is assumed on double indices. \( T^{j}_{\alpha} \) are the \( j \)th representation matrices of \( SU(2) \) (for example, \( T^{1/2}_{\alpha} = \sigma_{\alpha} / 2 \)). These operators satisfy the left and right \( SU(2) \) algebras presented above. Ladder operators may be defined as \( L_{\pm} = L_{x} \mp iL_{y} = \sum a_{\alpha m}^{j}(T^{j}_{\pm})_{\alpha mn}a_{\beta mm'}^{j} \) and \( R_{\pm} = R_{x} \mp iR_{y} = \sum a_{\alpha mm'}^{j}(T^{j}_{\pm})_{mn\beta}a_{\beta mm'}^{j} \). All these operators satisfy the algebras

\[
\hat{L}_{\alpha}a_{\alpha m}^{j}a_{\beta mm'}^{j} = (T^{j}_{\alpha})_{mn}a_{\alpha mn}^{j}a_{\beta mm'}^{j},
\]

\[
[R_{\alpha}a_{\alpha mm}^{j}]_{\beta mm'}^{j} = a_{\alpha mm}^{j}(T^{j}_{\alpha})_{\beta mn}a_{\beta mn}^{j},
\]

implying that they undergo left and right rotations within the \( j \)th representation. Thus, if we perform a rotation \( V \in SU(2) \) on the left and \( W^\dagger \in SU(2) \) on the right, these operators will transform according to

\[
a_{\alpha mn}^{j} \rightarrow V_{mn}a_{\alpha mn}^{j}W_{mn}^\dagger,
\]

with \( V, W^\dagger \) the \( j \)th matrix representation of the group elements \( V, W^\dagger \).

Next, one would like to exploit these transformation properties for the construction of rotation matrices, i.e. \( SU(2) \) group elements. This is done, as explained for general groups in [163], using the Clebsch–Gordan series. We denote the Clebsch–Gordan coefficients by \( \alpha_{\alpha \beta m}^{mn} \), and define

\[
u_{mn}^{j}(J, K) = \alpha_{m}^{\beta \alpha}^{MN}(J, K)\alpha_{\beta m}^{MN}(J, K)\alpha_{\alpha m}^{KM}(J, K)\alpha_{\beta mm}^{JM}a_{\alpha mn}^{j}
\]

The \( u_{mn}^{j}(J, K) \) operators undergo rotations according to the \( j \)th representation, and thus the corresponding rotation matrix elements [165] may be constructed out of them:

\[
U_{mn}^{j} = \sum_{J=0}^{J_{\max}} \frac{2J+1}{2K+1} u_{mn}^{j}(J, K)
\]

However, as the \( u_{mn}^{j}(J, K) \) operators undergo rotations independently of each other, we can also construct objects with the desired transformation properties without including an infinite number of levels, i.e. with \( J_{\max} < \infty \). This is the result of truncating the sums in the generators and the rotation matrices, limiting them to \( J \leq J_{\max} \). Thus, also for a finite \( J_{\max} \) one can obtain rotation matrices operating on the \( j < J_{\max} \) sectors, and this allows the construction of a truncated but yet gauge invariant lattice gauge theory, as we will describe next. The unitarity is lost, and now

\[
\text{tr}(U^\dagger U) = \text{tr}(U U^\dagger) = 2J + 1 - f_J(J_{\max})P_{J_{\max}}
\]

where \( P_{j} = \sum_{m,n} a_{\alpha mn}^{j}a_{\beta mm'}^{j} \) is a projection operator to the \( j \) sector (recall that we are dealing with the Hilbert subspace of a single atom).

The electric Hamiltonian will simply consist of \( \mu \) projections with the right prefactors, corresponding to linear terms in the bosonic number operators. The gauge–matter interactions, however, more complicated, are

\[
H_{G} = \frac{1}{2} \sum_{\nu} (\Omega_{\nu} - \Omega_{\nu}^{2}(\chi_{\nu}^{\dagger} \chi_{\nu} - \chi_{\nu} \chi_{\nu}^{\dagger}))
\]

where \( L \) is the set of links, \( V_{\nu} \) of the even vertices and \( V_{\nu} \) of the odd ones, and \( \Omega_{\nu} = \Omega_{\nu}^{2}, \Omega_{\nu}^{2} = \Omega_{\nu} \) (see figure 12).
and energy conservation considerations, we further reduce the number of possible scattering processes (assume $\Omega_{1,2}$ are large enough compared to the other energy scales, and perform a rotating wave approximation with respect to $H_0$). In first quantisation terms, the S-wave scattering is governed by a pseudo-potential $V_S$, whose matrix elements are

$$\langle 2, m'_b, \frac{3}{2}, m'_f | V_S | 2, m_b, \frac{3}{2}, m_f \rangle = \delta_{m'_b+m_f, m_b+m_f} \sum_{F=\frac{1}{2}}^\frac{3}{2} C_F \langle 2, m'_b, \frac{3}{2}, m'_f | F, m'_b + m'_f \rangle \times \langle F, m_b + m_f | 2, m_b, \frac{3}{2}, m_f \rangle$$

(142)

(The Kronecker delta is redundant due to the Clebsch–Gordan coefficients, but is explicitly written to manifest angular momentum conservation.)

By properly choosing the $C_F$ coefficients and the overlapping integrals (by manipulating the shape of the optical potential and hence the Wannier functions) one may obtain the required link interactions as in (139) and the electric energy as in equation (30) (which is simply a sum of bosonic number operators, using the projectors $P_j$ defined above), as well as avoiding extra interaction terms which conserve hyperfine angular momentum, but yet are not part of the desired Hamiltonian. Unfortunately, however, we have not been able, so far, either to satisfy all the required symmetries by the simulating system, or to prove it is impossible. An experimental realisation might be possible for other atomic configurations, with less stringent symmetries (i.e. a different choice of atomic levels).

For the simplest configuration described above (which could be generalised to more complicated atomic systems if required), with a theoretical simulating system with the right Hamiltonian parameters, one could proceed as follows: For even links we obtain the interaction Hamiltonian

$$\epsilon \sum_j \psi_j^\dagger M_j \chi_j + \text{H.c.}$$

(143)

and for odd ones

$$\epsilon \sum_j \chi_j^\dagger M_j^\dagger \psi_j + \text{H.c.}$$

(144)

with the bosonic matrix

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} b_2^* b_0 + b_0^* b_{-2} & -b_1^* b_0 + b_1^* b_{-1} \\ -b_1^* b_0 + b_1^* b_{-1} & b_2^* b_0 + b_0^* b_{-2} \end{pmatrix}$$

(145)

The next step is to map between the atomic degrees of freedom $b_a$ to the ones of the simulated theory; $a_{mn}$. The mapping is as follows:

(i) For even links,

$$b_{\pm 2}^\dagger = \alpha_{1/2,\pm 1/2}$$

$$b_{\pm 1}^\dagger = -\alpha_{1/2,\mp 1/2}$$

$$b_0^\dagger = a_{00}^{10}$$

(146)

(ii) For odd links,

$$b_{\pm 2}^\dagger = \alpha_{1/2,\pm 1/2}$$

$$b_{\pm 1}^\dagger = \alpha_{1/2,\mp 1/2}$$

$$b_0^\dagger = a_{00}^{10}$$

(147)

Using this mapping, one obtains that $M = U$ for the even links, and $M^\dagger = U$ for the odd ones (with $U$ being the truncated rotation matrix, acting only within the five-dimensional Hilbert space of $j = 0, \frac{1}{2}$). Thus, if we relabel all the fermions $\Psi$, we obtain the desired interaction Hamiltonian,

$$H_{\text{int}} = \frac{\epsilon}{\sqrt{2}} \sum_n \langle \psi_{n+1}^\dagger U_{n,n+1} \psi_{n+1} + \text{H.c.} \rangle$$

(148)

6.3. Plaquette interactions: $d + 1$ dimensional Abelian and non-Abelian theories

The non-effective simulations may be extended to further dimensions. The nontrivial generalisation is from $1 + 1$ to $2 + 1$ dimensions, as it involves the inclusion of the plaquette interactions. In the previous methods, which involved effective gauge invariance, these were effectively obtained out of the gauge-variant interactions of the original Hamiltonian, using the gauge invariance constraint. Here they are obtained effectively as well, but out of already gauge invariant building
blocks; the on-link gauge-matter interactions, which are not effective here, which is a great advantage.

In [121], we have introduced the loop method, which uses heavy ancillary fermions constrained to populate only some special vertices that may move only virtually, closing loops along plaquettes, carrying the gauge degrees of freedom along the way (utilising the on-link interactions (128)) and thus effectively forming the plaquette interactions (see figure 13). Although these interactions are obtained as fourth-order processes and might seem too weak, they are practically only second-order interactions, since the perturbative series of the gauge–matter Hamiltonian contains only even orders (fermions must hop virtually across an even distance, in order to finish in a vertex in which they may stay) and thus the fourth order is the second-leading one.

One can introduce yet other fermionic species (non-constrained) to serve as regular, dynamical fermions, on top of the ancillary static ones.

We have utilised the loop method for a $2 + 1$ dimensional compact $QED$ in the spin-gauge approach; plaquettes are constructed from the $1 + 1$ dimensional link interactions, generated due to hyperfine angular momentum conservation. This included a numerical proof of principle, considering some gauge invariant but non-Kogut–Susskind corrections to the Hamiltonian, due to the effective calculation in the truncated case. It was shown that the effective Hamiltonian still gives rise to the expected spectrum regardless of these corrections in several regimes of the Hamiltonian parameter, also outside the strong coupling limit.

The loop method has also been formulated and utilised for ideal simulations, rather than the above described realistic condition $cQED$ in $2 + 1$ dimensions. This includes ideal (not truncated) $cQED$, as well as $SU(N)$ gauge theory with unitary matrices (which is not realistically achieved in the current simulation method).

Finally, we will comment on the simulation of the $\mathbb{Z}_N$ lattice gauge theory, whose relevance to $QCD$ confinement has been stated above. Since this theory involves finite Hilbert spaces on the links, one could, in principle, simulate the exact model, with no truncations and approximations. An explicit example for $\mathbb{Z}_3$ was given. Since the local Hilbert space is finite, we use different bosonic levels on each link, playing the role of the different eigenstates of the $\mathbb{Z}_N$ $P$ operator. The on-link interactions are with auxiliary fermions (which are not real $\mathbb{Z}_N$, and are eliminated in the loop method), which can thus be also hard-core bosons (the statistics is irrelevant). The gauge invariant operators are obtained using the conservation of hyperfine angular momentum in collisions as before, but the challenge here is to realise the cyclic Escher staircase nature of the group. This is done by the hybridisation of atomic levels, as discussed in [121].

Dynamical fermions may be included in these simulations as well, following the formulation of [163].

6.4. Gauge invariance from atomic interaction symmetries—some general remarks

To conclude, the approach of exact gauge invariance seems to be better than the previous effective approach, as it is applicable to a larger class of models (with the loop method for the plaquette interaction, for example).

This is since it does not have to include the Gauss’s law constraints, which are hard to realise (and even harder for non-Abelian gauge groups, due to the non-commuting and complicated generators) and thus this approach seems the preferred one to proceed with.

Regarding the simulated theories: the simulation of $cQED$ in the non-effective approach seems closer to a physical realisation than the previous, effective ones. A better physical realisation of the non-Abelian models is required in order to

(i) Allow for a better $1 + 1$ simulation, with no need to use perturbation theory at all (although the current realisation does not rely on perturbation theory for gauge invariance, only to the generation of links).

(ii) Include further nonperturbative regimes of $1 + 1$ dimensional theories.

(iii) Extend the simulations to $2 + 1$ dimensions, in a way that plaquettes will be generated using the loop method even out of gauge invariant building blocks which contain non-unitary matrices; this should be possible using the truncation scheme of [163], but as explained above, a concrete physical realisation is still lacking.

(iv) Allow for simple generalisations to $SU(N)$ with $N > 2$ and further gauge groups.

7. Discussion

The class of Abelian and non-Abelian lattice gauge theories that has been considered in the present work constitutes the building blocks of the standard model of elementary particles, that
currently provides us with the best known description of high-energy physics phenomena. The standard model has been so far extensively tested, and currently agrees with the experimental results from particle accelerators and cosmological observations up to energy scales of \(10^{16} \text{ eV}\). On the other hand, the physics which is used here to describe the many-body system of ultracold atoms is well understood and has been tested with high accuracy down to energy scales of \(10^{-7} \text{ eV}\). So the problem at hand seems conceptually not obvious: can high-energy field theories be simulated by the physics of ultra low-energy systems?

It turns out, however, and somewhat surprisingly, that there are several ways to manifest both Lorentz (as an effective long wavelength property) and local-gauge invariance. The latter can be obtained in two ways. It may arise either as a property of a low-energy sector of the atomic system, hence as an effective symmetry, or, alternatively, by re-arranging the interactions such that the given low-energy symmetries of the atomic collisions are converted into a form which is equivalent to local (Abelian or non-Abelian) gauge symmetries. The first case of effectively emerging gauge symmetry seems to provide yet another framework to study models in which local gauge invariance is not a fundamental property of the complete theory, but rather emerges as a low-energy phenomenon as suggested in [201]. The alternative exact mapping might, on the other hand, be more suitable for mimicking theories wherein local gauge invariance is a fundamental property.

In this report, we have reviewed recent progress on the quantum simulations of lattice gauge theories using ultracold atoms, focusing on some of the works and approaches. These proposals serve as proofs of principle of the possibility to simulate high-energy physics, and even gauge theories using ultracold atoms.

Our current results already suggest that simple enough models, such as compact-QED \((U(1))\), and possibly also more involved, non-Abelian \(SU(2)\) models, manifesting exotic QCD effects, (such as quark confinement) could, indeed, be experimentally studied using current table-top cold atoms experimental methods.

7.1. Advantages of quantum simulation over other approaches

(i) As fermions come for free in these atomic scenarios, one may avoid the sign problem of Grassman integration and consider regimes of finite chemical potential, such as the exotic colour-superconductivity and quark–gluon plasma phases of quantum chromodynamics [94, 100, 101].

(ii) Real-time dynamics is possible here, rather than in the statistical Monte-Carlo simulations (and this applies already for the current proposals). This allows, or will allow,

(a) The observation of real-time dynamical phenomena, including the ones described in the papers: deforming and breaking of flux tubes and loops, pair creation etc.

(b) Changing adiabatically the Hamiltonian parameters; finding the ground states of systems by switching from ‘trivial’ regimes where the ground state is known (e.g. no-interactions, strong limit) to other regimes; also probing for phase transitions and creating phase diagrams of theories, depending on the theory’s constants. This will be highly relevant for QCD once feasible simulation systems are proposed and realised.

(iii) The possibility of considering finite temperature models as well [93, 95, 100], including real-time dynamics.

(iv) The possibility of realising otherwise only gedanken experiments. For example, a realisation of the gedanken experiment proposed in [200], which brings the idea of Ramsey spectroscopy [202] into high-energy physics, suggesting a way to probe for the confining phase and measuring the string tension related with confinement (see figure 14).

7.2. Simulated models

As described in this report, a large class of Abelian and non-Abelian physical models can be simulated, including, explicitly, the lattice gauge theories of

(i) Compact QED Kogut–Susskind and spin-gauge (truncated) models, in \(1 + 1\) and \(2 + 1\) dimensions, with both effective and exact gauge invariance, with or without dynamical matter. In \(1 + 1\) dimensions, the simulation is possible with no perturbation theory at all.

(ii) \(Z_3\)—so far, \(Z_3\) has been extensively studied, \(2 + 1\) pure gauge or with dynamical matter (following [163]), an exact quantum simulation of a non-truncated Hilbert space.

(iii) \(SU(N)\)—so far \(SU(2)\) in \(1 + 1\) dimensions, with limited applicability to more dimensions has been studied in detail, while other \(SU(N)\) groups still await extended study.

The simulatable theories, as presented in this report, are summarised in the table below, where KS—Kogut–Susskind, trunc.—truncated, st.c.—Strong coupling, YM—Yang–Mills.

|          | 1 + 1 with matter | 2 + 1 Pure | 2 + 1 with matter |
|----------|-------------------|------------|-------------------|
| \(U(1)\) | Full KS + trunc.   | Full KS + trunc. | Full KS + trunc.   |
| \(Z_3\)  | Full              | Full       | Full              |
| \(SU(2)\) | YM + trunc. (st. c.) | YM + trunc. (st. c.) | YM + trunc. (st. c.) |
models and for increasing the number of atoms which may populate a single site.

(ii) Improving the controllability of parameters; new ways to control scattering coefficients, for example; also, strengthening the possible interactions is important, for example for the realisation of effective, \( \sim J/U \) interactions, or the second \( SU(2) \) realisation scheme presented above, as well as, potentially, the use of the truncation scheme of [163] for the simulation of other non-Abelian theories (with either compact Lie or finite gauge groups) including, of course \( SU(3) \) for QCD.

(iii) Methods to create longer-living and more robust lattices, cooling techniques, ways to deal with atomic losses and more.

(iv) Measurement techniques; the current single-addressing techniques [189–191] require very sophisticated experimental methods, limited to a small number of experimental groups. The development of such, and other techniques, will help boost the quantum simulations of complicated physical models.

7.4. Summary

Atomic physics and high-energy physics have, indeed, something to offer each another, suggesting a very interesting and exciting physical research focusing at their interface. Reconstructing high-energy physics from atomic components offers a great lesson in understanding the interactions unfolding within the standard model of particle physics. Also, in the broader sense, the continuation of these works on the quantum simulations of high-energy physics and the growth of the community working in this direction may lead to useful new ways, methods and results in understanding the most fundamental degrees of freedom in our physical universe.

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