An Inner Convex Approximation Algorithm for BMI Optimization and Applications in Control

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Abstract—In this work, we propose a new local optimization method to solve a class of nonconvex semidefinite programming (SDP) problems. The basic idea is to approximate the feasible set of the nonconvex SDP problem by inner positive semidefinite convex approximations via a parameterization technique. This leads to an iterative procedure to search a local optimum of the nonconvex problem. The convergence of the algorithm is analyzed under mild assumptions. Applications in static output feedback control are benchmarked and numerical tests are implemented based on the data from the COMPLib library.

1. INTRODUCTION

We are interested in the following nonconvex semidefinite programming problem:

\[
\begin{array}{ll}
\text{min} & f(x) \\
\text{s.t.} & F_i(x) \preceq 0, \quad i = 1, \ldots, m, \\
& x \in \Omega,
\end{array}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is convex, \( \Omega \) is a nonempty, closed convex set in \( \mathbb{R}^n \) and \( F_i : \mathbb{R}^n \to \mathbb{R}^p \) \( (i = 1, \ldots, m) \) are nonconvex matrix-valued mappings and smooth. The notation \( A \preceq 0 \) means that \( A \) is a symmetric negative semidefinite matrix.

Optimization problems involving matrix-valued mapping inequality constraints have large number of applications in static output feedback controller design and topology optimization, see, e.g. [4], [10], [13], [18]. Especially, optimization problems with bilinear matrix inequality (BMI) constraints have been known to be nonconvex and NP-hard [3]. Many attempts have been done to solve these problems by employing convex semidefinite programming (in particular, optimization with linear matrix inequality (LMI) constraints) techniques [6], [7], [10], [11], [21]. The methods developed in those papers are based on augmented Lagrangian functions, generalized sequential semidefinite programming and alternating directions. Recently, we proposed a new method based on convex-concave decomposition of the BMI constraints and linearization technique [20]. The method exploits the convex substructure of the problems. It was shown that this method can be applied to solve many problems arising in static output feedback control including spectral abscissa, \( \mathcal{H}_2 \), \( \mathcal{H}_\infty \) and mixed \( \mathcal{H}_2/\mathcal{H}_\infty \) synthesis problems.

Outline. The next section recalls some definitions, notation and properties of matrix operators and defines an inner convex approximation of a BMI constraint. Section 3 proposes the main algorithm and investigates its convergence properties. Section 4 shows the applications in static output feedback control and numerical tests. Some concluding remarks are given in the last section.

2. INNER CONVEX APPROXIMATIONS

In this section, after given an overview on concepts and definitions related to matrix operators, we provide a definition of inner positive semidefinite convex approximation of a nonconvex set.

A. Preliminaries

Let \( \mathcal{P} \) be the set of symmetric matrices of size \( p \times p \), \( \mathcal{P}_+ \), and resp., \( \mathcal{P}_{++} \) be the set of symmetric positive semidefinite, resp., positive definite matrices. For given matrices \( X \) and \( Y \) in \( \mathcal{P} \), the relation \( X \succeq Y \) (resp., \( X \preceq Y \)) means that \( X - Y \in \mathcal{P}_+ \) (resp., \( Y - X \in \mathcal{P}_+ \)) and \( X \succ Y \) (resp., \( X \prec Y \)) is \( X - Y \in \mathcal{P}_{++} \) (resp., \( Y - X \in \mathcal{P}_{++} \)). The quantity \( X \circ Y := \text{trace}(X^TY) \) is an inner product of two matrices \( X \) and \( Y \) defined on \( \mathcal{P} \), where \( \text{trace}(Z) \) is the trace of matrix \( Z \). For a given symmetric matrix \( X \), \( \lambda_{\text{min}}(X) \) denotes the smallest eigenvalue of \( X \).
**Definition 2.1:** [17] A matrix-valued mapping $F : \mathbb{R}^n \to \mathcal{S}^p$ is said to be positive semidefinite convex (psd-convex) on a convex subset $C \subseteq \mathbb{R}^n$ if for all $t \in [0, 1]$ and $x, y \in C$, one has
\[
F(tx + (1-t)y) \preceq tF(x) + (1-t)F(y). \tag{1}
\]
If (1) holds for $\preceq$ instead of $\succeq$ for $t \in (0, 1)$ then $F$ is said to be strictly psd-convex on $C$. In the opposite case, $F$ is said to be psd-nonconvex. Alternatively, if we replace $\preceq$ in (1) by $\succeq$ then $F$ is said to be psd-concave on $C$. It is obvious that any convex function $f : \mathbb{R}^n \to \mathbb{R}$ is psd-convex with $p = 1$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strongly convex with parameter $\rho > 0$ if $f(x) - \frac{\rho}{2}\|x\|^2$ is convex. The notation $\partial f$ denotes the subdifferential of a convex function $f$. For a given convex set $C$, $A_C(x) := \{w | w^T(x - y) \geq 0, \ y \in C\}$ if $x \in C$ and $A_C(x) := \emptyset$ if $x \notin C$ denotes the normal cone of $C$ at $x$.

The derivative of a matrix-valued mapping $F$ at $x$ is a linear mapping $DF$ from $\mathbb{R}^n$ to $\mathbb{R}^{n \times p}$ which is defined by
\[
DF(x)h := \sum_{i=1}^{n} h_i \frac{\partial F}{\partial x_i}(x), \quad \forall h \in \mathbb{R}^n.
\]

For a given convex set $X \subseteq \mathbb{R}^n$, the matrix-valued mapping $G$ is said to be differentiable on a subset $X$ if its derivative $DF(x)$ exists at every $x \in X$. The definitions of the second order derivatives of matrix-valued mappings can be found, e.g., in [17]. Let $A : \mathbb{R}^n \to \mathcal{S}^p$ be a linear mapping defined as $Ax := \sum_{i=1}^{n} x_i A_i$, where $A_i \in \mathcal{S}^p$ for $i = 1, \ldots, n$. The adjoint operator of $A, A^*$, is defined as $A^*Z := (A_1^* \circ Z, A_2^* \circ Z, \ldots, A_n^* \circ Z)^T$ for any $Z \in \mathcal{S}^p$.

Finally, for simplicity of discussion, throughout this paper, we assume that all the functions and matrix-valued mappings are twice differentiable on their domain.

**B. Psd-convex overestimate of a matrix operator**

Let us first describe the idea of the inner convex approximation for the scalar case. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonconvex function. A convex function $g(y)$ depending on a parameter $y$ is called a convex overestimate of $f(y)$ w.r.t. the parameterization $y := \psi(x)$ if $g(x, \psi(x)) = f(x)$ and $f(\hat{z}) \lesssim g(\hat{z})$ for all $y \hat{z}$, where $\hat{z}$ denotes the parameter value. Let us consider two examples.

**Example 1.** Let $f$ be a continuously differentiable function and its gradient $\nabla f$ is Lipschitz continuous with a Lipschitz constant $L_f > 0$, i.e. $\|\nabla f(y) - \nabla f(x)\| \leq L_f \|y - x\|$ for all $x, y$. Then, it is well-known that $|f(z) - f(x) - \nabla f(x)^T(z - x)| \leq \frac{L_f}{2} \|z - x\|^2$. Therefore, for any $x, y$, we have $f(z) \preceq g(z) := f(x) + \nabla f(x)^T(z - x) + \frac{L_f}{2} \|z - x\|^2$. Moreover, $f(x) = g(x; x)$ for any $x$. We conclude that $g(\cdot, x)$ is a convex overestimate of $f$ w.r.t. the parameterization $y := \psi(x) = x$. Now, if we fix $x = \bar{x}$ and find a point $v$ such that $g(\bar{x}; x) \preceq 0$ then $f(v) \preceq 0$. Consequently if the set $\{x | f(x) < 0\}$ is nonempty, we can find a point $v$ such that $g(\bar{x}; x) \preceq 0$. The convex set $\mathcal{H}(\cdot) := \{z | g(z; x) \preceq 0\}$ is called an inner convex approximation of $\{z | f(z) \preceq 0\}$.

**Example 2.** [2] We consider the function $f(x) = x_1 x_2$ in $\mathbb{R}^2$. The function $g(x, y) = \frac{x^2}{2} + \frac{y^2}{2} x^2$ is a convex overestimate of $f$ w.r.t. the parameterization $y := \psi(x) = x_1/x_2$ provided that $y > 0$. This example shows that the mapping $\psi$ is not always identity.

Let us generalize the convex overestimate concept to matrix-valued mappings.

**Definition 2.2:** Let us consider a psd-nonconvex matrix mapping $F : \mathcal{X} \to \mathcal{S}^p$. A psd-convex matrix mapping $G(\cdot, y)$ is said to be a psd-convex overestimate of $F$ w.r.t. the parameterization $y := \psi(x)$ if $G(x; \psi(x)) = F(x)$ and $F(\hat{z}) \preceq G(\hat{z}; y)$ for all $x, y$ and $\hat{z} \in \mathcal{X}$.

Let us provide two important examples that satisfy Definition 2.2.

**Example 3.** Let $\mathcal{Q}_0(X, Y) = X^T Q^{-1} Y + Y^T Q^{-1} X$ be a bilinear form with $Q = Q_1 + Q_2$, $Q_1 > 0$ and $Q_2 > 0$ arbitrarily, where $X$ and $Y$ are two $n \times p$ matrices. We consider the parametric quadratic form:
\[
\mathcal{Q}_0(X; Y; \hat{X}; \hat{Y}) := (X - \bar{X})^T Q_1^{-1} (X - \bar{X}) + (Y - \bar{Y})^T Q_2^{-1} (Y - \bar{Y}) + \bar{X}^T Q_1^{-1} Y + \bar{Y}^T Q_2^{-1} X + \bar{X}^T Q_1^{-1} \bar{Y} - \bar{Y}^T Q_2^{-1} \bar{X} - \bar{X}^T Q_2^{-1} \bar{Y} - \bar{Y}^T Q_1^{-1} \bar{X}
\]

One can show that $\mathcal{Q}_0(X; Y; \hat{X}; \hat{Y})$ is a psd-convex overestimate of $\mathcal{Q}_0(X, Y)$ w.r.t. the parameterization $\psi(\hat{X}; \hat{Y}) = (\hat{X}, \hat{Y})$.

Indeed, it is obvious that $\mathcal{Q}_0(\hat{X}, \hat{Y}; \hat{X}, \hat{Y}) = \mathcal{Q}_0(\hat{X}, \hat{Y})$. We only prove the second condition in Definition 2.2.

Let us consider the expression $\mathcal{Q}_0 := X^T Q_1^{-1} Y + Y^T Q_2^{-1} X + \bar{X}^T Q_1^{-1} \bar{Y} - \bar{Y}^T Q_1^{-1} \bar{X} - \bar{Y}^T Q_2^{-1} \bar{X} + \bar{X}^T Q_2^{-1} \bar{Y} - \bar{Y}^T Q_1^{-1} \bar{X}$. By rearranging this expression, we can easily show that $\mathcal{Q}_0 = -(X - \bar{X})^T Q_1^{-1} (Y - \bar{Y}) - (Y - \bar{Y})^T Q_2^{-1} (X - \bar{X})$. Now, since $Q = Q_1 + Q_2$, by [1], we can write:
\[
\mathcal{Q}_0 = (X - \bar{X})^T (Q_1 + Q_2)^{-1} (Y - \bar{Y}) + (Y - \bar{Y})^T (Q_1 + Q_2)^{-1} (X - \bar{X})
\]

Example 4. Let us consider a psd-nonconvex matrix-valued mapping $F(\cdot) := G_{cvx1}(\cdot) - G_{cvx2}(\cdot)$, where $G_{cvx1}$ and $G_{cvx2}$ are two psd-convex matrix-valued mappings [20]. Now, let $G_{cvx1}$ be differentiable and $\mathcal{L}_j(\cdot) := G_{cvx1}(\cdot) + D G_{cvx1}(\cdot)(\cdot - \hat{x})$ be the linearization of $G_{cvx1}$ at $\hat{x}$. We define $\mathcal{H}_j(\cdot, \hat{x}) := \mathcal{L}_j(\cdot)(\cdot, \hat{x})$. It is not difficult to show that $\mathcal{H}_j(\cdot, \hat{x})$ is a psd-convex overestimate of $\mathcal{H}(\cdot)$ w.r.t. the parameterization $\psi(\hat{x}) = \hat{x}$.

**Remark 2.3:** Example 3 shows that the “Lipschitz coefficient” of the approximating function $g(\cdot)$ is $(Q_1, Q_2)$. Moreover, as indicated by Examples 3 and 4, the psd-convex overestimate of a bilinear form is not unique. In practice, it is important to find appropriate psd-convex overestimates for bilinear forms to make the algorithm perform efficiently. Note that the psd-convex overestimate $\mathcal{Q}_0$ of $\mathcal{Q}_0$ in Example 3 may be less conservative than the convex-concave decomposition in [20] since all the terms in $\mathcal{Q}_0$ are related to $X - \bar{X}$ and $Y - \bar{Y}$ rather than $X$ and $Y$.
3. The Algorithm and its Convergence

Let us recall the nonconvex semidefinite programming problem \(\text{NSDP}\). We denote by
\[
\mathcal{F} := \{x \in \Omega \mid F_i(x) \leq 0, \ i = 1, \ldots, m\},
\]
the feasible set of \(\text{NSDP}\) and
\[
\mathcal{F}^0 := \rho_i(\Omega) \cap \{x \in \mathbb{R}^n \mid F_i(x) < 0, \ i = 1, \ldots, m\},
\]
the relative interior of \(\mathcal{F}\), where \(\rho_i(\Omega)\) is the relative interior of \(\Omega\). First, we need the following fundamental assumption.

**Assumption A.1:** The set of interior points \(\mathcal{F}^0\) of \(\mathcal{F}\) is nonempty.

Then, we can write the generalized KKT system of \(\text{NSDP}\) as follows:
\[
\begin{align*}
0 & \in \partial f(x) + \sum_{i=1}^m DF_i(x)^T W_i + \mathcal{M}_2(x), \\
0 & \succeq F_i(x), \ W_i \succeq 0, \ F_i(x) o W_i = 0, \ i = 1, \ldots, m.
\end{align*}
\]
Any point \((x^*, W^*)\) with \(W^* = (W_1^*, \ldots, W_m^*)\) is called a KKT point of \(\text{NSDP}\), where \(x^*\) is called a stationary point and \(W^*\) is called the corresponding Lagrange multiplier.

A. Convex semidefinite programming subproblem

The main step of the algorithm is to solve a convex semidefinite programming problem formed at the iteration \(\bar{x}^k \in \Omega\) by using inner psd-convex approximations. This problem is defined as follows:
\[
\begin{align*}
\min_{x} & \quad f(x) + \frac{1}{2} (x - \bar{x})^T Q_k (x - \bar{x}) \\
\text{s.t.} & \quad G_i(x; \bar{\psi}_i(\bar{x})) \leq 0, \ i = 1, \ldots, m \quad \text{(CSDP)(\bar{x})}
\end{align*}
\]
Here, \(Q_k \in \mathcal{B}_+^n\) is given and the second term in the objective function is referred to as a regularization term; \(\bar{\psi}_i(\bar{x})\) is the parameterization of the convex overestimate \(G_i\) of \(F_i\).

Let us define by \(\mathcal{F}(x_i; Q_i)\) the solution mapping of CSDP\((\bar{x})\) depending on the parameters \((\bar{x}_i, Q_i)\). Note that the problem CSDP\((\bar{x})\) is convex, \(\mathcal{F}(\bar{x}; Q_i)\) is multivalued and convex. The feasible set of CSDP\((\bar{x})\) is written as:
\[
\mathcal{F}(\bar{x}) := \left\{ x \in \Omega \mid G_i(x; \bar{\psi}_i(\bar{x})) \leq 0, \ i = 1, \ldots, m \right\}.
\]

B. The algorithm

The algorithm for solving \(\text{NSDP}\) starts from an initial point \(\bar{x} \in \mathcal{F}^0\) and generates a sequence \(\{\bar{x}^k\}_{k \geq 0}\) by solving a sequence of convex semidefinite programming subproblems CSDP\((\bar{x})\) approximated at \(\bar{x}^k\). More precisely, it is presented in detail as follows.

**Algorithm 1 (Inner Convex Approximation):**

**Initialization.** Determine an initial point \(\bar{x}^0 \in \mathcal{F}^0\). Compute \(\bar{\psi}_i(\bar{x})\) for \(i = 1, \ldots, m\). Choose a regularization matrix \(Q_0 \in \mathcal{B}_+^n\). Set \(k := 0\).

**Iteration** \(k (k = 0, 1, \ldots)\) Perform the following steps:

**Step 1.** For given \(\bar{x}^k\), if a given criterion is satisfied then terminate.

**Step 2.** Solve the convex semidefinite program CSDP\((\bar{x})\) to obtain a solution \(\bar{x}^{k+1}\) and the corresponding Lagrange multiplier \(W^{k+1}\).

**Step 3.** Update \(\bar{\psi}_{i}^{k+1} := \psi_i(\bar{x}^{k+1})\), the regularization matrix \(Q_{k+1} \in \mathcal{B}_+^n\) (if necessary). Increase \(k\) by 1 and go back to Step 1.

End.

The core step of Algorithm 1 is Step 2 where a general convex semidefinite program needs to be solved. In practice, this can be done by either implementing a particular method that exploits problem structures or relying on standard semidefinite programming software tools. Note that the regularization matrix \(Q_k\) can be fixed at \(Q_k = \rho I\), where \(\rho > 0\) is sufficiently small and \(I\) is the identity matrix. Since Algorithm 1 generates a feasible sequence \(\{\bar{x}^k\}_{k \geq 0}\) to the original problem \(\text{NSDP}\) and this sequence is strictly descent w.r.t. the objective function \(f\), no globalization strategy such as line-search or trust-region is needed.

C. Convergence analysis

We first show some properties of the feasible set \(\mathcal{F}(\bar{x})\) defined by \(\mathcal{B}\). For notational simplicity, we use the notation \(\|x\| := (x^T Q(x))^{rac{1}{2}}\).

**Lemma 3.1:** Let \(\{\bar{x}^k\}_{k \geq 0}\) be a sequence generated by Algorithm 1 Then:

a) The feasible set \(\mathcal{F}(\bar{x}^k) \subseteq \mathcal{F}\) for all \(k \geq 0\).

b) It is a feasible sequence, i.e. \(\{\bar{x}^k\}_{k \geq 0} \subseteq \mathcal{F}\).

c) \(\bar{x}^{k+1} \in \mathcal{F}(\bar{x}^k) \cap \mathcal{F}(\bar{x}^{k+1})\).

d) For any \(k \geq 0\), it holds that:
\[
f(\bar{x}^{k+1}) \leq f(\bar{x}^k) - \frac{1}{2} \|\bar{x}^k - \bar{x}\|_Q + \frac{\rho}{2} \|\bar{x}^k - \bar{x}\|_Q,
\]
where \(\rho \geq 0\) is the strong convexity parameter of \(f\).

**Proof:** For a given \(\bar{x}\), we have \(\bar{x}^k = \bar{\psi}_i(\bar{x})\) and \(F_i(x) \succeq G_i(x; \bar{\psi}_i(\bar{x})) \leq 0\) for \(i = 1, \ldots, m\). Thus if \(x \in \mathcal{F}(\bar{x})\) then \(x \in \mathcal{F}\), the statement a) holds. Consequently, the sequence \(\{\bar{x}^k\}\) is feasible to \(\text{NSDP}\) which is indeed the statement b). Since \(\bar{x}^{k+1}\) is a solution of CSDP\((\bar{x})\) it shows that \(\bar{x}^{k+1} \in \mathcal{F}(\bar{x}^k)\). Now, we have to show it belongs to \(\mathcal{F}(\bar{x}^{k+1})\). Indeed, since \(G_i(\bar{x}^{k+1}; \bar{\psi}_i(\bar{x})) = F_i(\bar{x}^{k+1}) \leq 0\) by Definition 2.2 for all \(i = 1, \ldots, m\), we conclude \(\bar{x}^{k+1} \in \mathcal{F}(\bar{x}^{k+1})\). The statement c) is proved. Finally, we prove d). Since \(\bar{x}^{k+1}\) is the optimal solution of CSDP\((\bar{x})\) we have \(f(\bar{x}^{k+1}) + \frac{1}{2} \|\bar{x}^{k+1} - \bar{x}\|_Q^2 \leq f(x) + \frac{1}{2} (x - \bar{x})^T Q(x - \bar{x}) - \frac{\rho}{2} \|x - \bar{x}\|_Q^2\) for all \(x \in \mathcal{F}(\bar{x})\). However, we have \(\bar{x} \in \mathcal{F}(\bar{x})\) due to c). By substituting \(x = \bar{x}^k\) in the previous inequality we obtain the estimate d).

Now, we denote by \(\mathcal{F}(\alpha) := \{x \in \mathcal{F} \mid f(x) \leq \alpha\}\) the lower level set of the objective function. Let us assume that \(G_i(\cdot; y)\) is continuously differentiable in \(\mathcal{F}(\bar{f}(\bar{x}))\) for any \(y\). We say that the Robinson qualification condition for CSDP\((\bar{x})\) holds at \(\bar{x}\) if \(0 \in \text{int}G_i(\bar{x}; \bar{\psi}_i(\bar{x}))+D_iG_i(\bar{x}; \bar{\psi}_i(\bar{x}))\Omega - \bar{x} + \mathcal{F}_+^2\) for \(i = 1, \ldots, m\). In order to prove the convergence of Algorithm 1 we require the following assumption.

**Assumption A.2:** The set of KKT points of \(\text{NSDP}\) is nonempty. For a given \(y\), the matrix-valued mappings \(G_i(\cdot; y)\) are continuously differentiable on \(\mathcal{F}(\bar{f}(\bar{x}))\). The convex problem CSDP\((\bar{x})\) is solvable and the Robinson qualification condition holds at its solutions.

We note that if Algorithm 1 is terminated at the iteration \(k\) such that \(\bar{x}^k = \bar{x}^{k+1}\) then \(\bar{x}^k\) is a stationary point of \(\text{NSDP}\).
Theorem 3.2: Suppose that Assumptions A[1] and A[2] are satisfied. Suppose further that the lower level set \( \mathcal{L}_f(f(\tilde{x}^k)) \) is bounded. Let \( \{ (\tilde{x}^k, \tilde{W}^k) \}_{k \geq 1} \) be an infinite sequence generated by Algorithm [1] starting from \( \tilde{x}^0 \in \mathcal{F}^0 \). Assume that \( \lambda_{\text{max}}(Q_4) \leq M < +\infty \). Then if either \( f \) is strongly convex or \( \lambda_{\text{min}}(Q_4) > \rho > 0 \) for \( k \geq 1 \) then every accumulation point \( (\tilde{x}^k, \tilde{W}^k) \) of \( \{ (\tilde{x}^k, \tilde{W}^k) \} \) is a KKT point of (NSDP). Moreover, if the set of the KKT points of (NSDP) is finite then the whole sequence \( \{ (\tilde{x}^k, \tilde{W}^k) \} \) converges to a KKT point of (NSDP).

Proof: First, we show that the solution mapping \( \mathcal{L}(\tilde{x}^k, Q_k) \) is closed. Indeed, by Assumption A[2] CSDP(\( \tilde{F}^k \)) is feasible. Moreover, it is strongly convex. Hence, \( \mathcal{L}(\tilde{x}^k, Q_k) = \{ \tilde{x}^{k+1} \} \), which is obviously closed. The remaining conclusions of the theorem can be proved similarly as [20, Theorem 3.2] by using Zangwill’s convergence theorem [22, p. 91] of which we omit the details here.

Remark 3.3: Note that the assumptions used in the proof of the closedness of the solution mapping \( \mathcal{L}(\cdot) \) in Theorem 3.2 are weaker than the ones used in [20, Theorem 3.2].

4. APPLICATIONS TO ROBUST CONTROLLER DESIGN

In this section, we present some applications of Algorithm [1] for solving several classes of optimization problems arising in static output feedback controller design. Typically, these problems are related to the following linear, time-invariant (LTI) system of the form:

\[
\begin{align*}
\dot{x} &= Ax + B_1w + Bu, \\
z &= C_1x + D_{11}w + D_{12}u, \\
y &= Cx + D_{21}w,
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state vector, \( w \in \mathbb{R}^{n_w} \) is the performance input, \( u \in \mathbb{R}^{n_u} \) is the input vector, \( z \in \mathbb{R}^m \) is the performance output, \( y \in \mathbb{R}^{n_y} \) is the physical output vector, \( A \in \mathbb{R}^{n \times n} \) is state matrix, \( B \in \mathbb{R}^{n \times n_w} \) is input matrix and \( C \in \mathbb{R}^{n_y \times n} \) is the output matrix. By using a static feedback controller of the form \( u = Fy \) with \( F \in \mathbb{R}^{n_u \times n_y} \), we can write the closed-loop system as follows:

\[
\begin{align*}
\dot{x}_F &= A_F x_F + B_F w, \\
z &= C_F x_F + D_F w.
\end{align*}
\]

The stabilization, \( \mathcal{H}_2 \), \( \mathcal{H}_\infty \) optimization and other control problems of the LTI system can be formulated as an optimization problem with BMI constraints. We only use the psd-convex overestimate of a bilinear form in Example 3 to show that Algorithm [1] can be applied to solving many problems in static state/output feedback controller design such as:

1. Sparse linear static output feedback controller design;
2. Spectral abscissa and pseudospectral abscissa optimization;
3. \( \mathcal{H}_2 \) optimization;
4. \( \mathcal{H}_\infty \) optimization;
5. and mixed \( \mathcal{H}_2/\mathcal{H}_\infty \) synthesis.

These problems possess at least one BMI constraint of the from \( B_1(x, Y, Z) \leq 0 \), where \( B_1(x, Y, Z) := X^T Y + Y^T X + \mathcal{A}(Z) \), where \( X, Y \) and \( Z \) are matrix variables and \( \mathcal{A} \) is an affine operator of matrix variable \( Z \). By means of Example 3, we can approximate the bilinear term \( X^T Y + Y^T X \) by its psd-convex overestimate. Then using Schur’s complement to transform the constraint \( G_i(x, \tilde{x}^k) \leq 0 \) of the subproblem CSDP(\( \tilde{F}^k \)) into an LMI constraint [20]. Note that Algorithm [1] requires an interior starting point \( \tilde{x}^0 \in \mathcal{F}^0 \). In this work, we apply the procedures proposed in [20] to find such a point. Now, we summarize the whole procedure applying to solve the optimization problems with BMI constraints as follows:

**Scheme A.1:**

**Step 1.** Find a psd-convex overestimate \( G_i(x, y) \) of \( F_i(x) \) w.r.t. the parameterization \( y = \psi_i(x) \) for \( i = 1, \ldots, m \) (see Example 1).

**Step 2.** Find a starting point \( \tilde{x}^0 \in \mathcal{F}^0 \) (see [20]).

**Step 3.** For a given \( \tilde{x}^k \), form the convex semidefinite programming problem CSDP(\( \tilde{F}^k \)) and reformulate it as an optimization with LMI constraints.

**Step 4.** Apply Algorithm [1] with an SDP solver to solve the given problem.

Now, we test Algorithm [1] for three problems via numerical examples by using the data from the COMPlib library [12]. All the implementations are done in Matlab 7.8.0 (R2009a) running on a Laptop Intel(R) Core(TM)i7 Q740 1.73GHz and 4Gb RAM. We use the YALMIP package [14] as a modeling language and SeDuMi 1.1 as a SDP solver [19] to solve the LMI optimization problems arising in Algorithm [1] at the initial phase (Phase 1) and the subproblem CSDP(\( \tilde{F}^k \)). The code is available at http://www.kuleuven.be/optec/software/BMIsolver.

We also compare the performance of Algorithm [1] and the convex-concave decomposition method (CCDM) proposed in [20] in the first example, i.e. the spectral absissa optimization problem. In the second example, we compare the \( \mathcal{H}_\infty \)-norm computed by Algorithm [1] and the one provided by HIFOO [8] and PENBMI [9]. The last example is the mixed \( \mathcal{H}_2/\mathcal{H}_\infty \) synthesis optimization problem which we compare between two values of the \( \mathcal{H}_2 \)-norm level.

A. Spectral abscissa optimization

We consider an optimization problem with BMI constraint by optimizing the spectral absissa of the closed-loop system \( \dot{x} = (A + BFC)x \) as [5], [13]:

\[
\begin{align*}
\max_{P,F,B} & \quad \beta \\
\text{s.t.} \quad & (A + BFC)^T P + P(A + BFC) + 2\beta P < 0, \quad P = P^T, \quad P \succ 0.
\end{align*}
\]

Here, matrices \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times n_w} \) and \( C \in \mathbb{R}^{n_y \times n} \) are given. Matrices \( P \in \mathbb{R}^{n \times n} \) and \( F \in \mathbb{R}^{n_u \times n_y} \) and the scalar \( \beta \) are considered as variables. If the optimal value of (10) is strictly positive then the closed-loop feedback controller \( u = Fy \) stabilizes the linear system \( \dot{x} = (A + BFC)x \).

By introducing an intermediate variable \( A_F := A + BFC + \beta I \), the BMI constraint in the second line of (10) can be written \( A_F P + P^T A_F \prec 0 \). Now, by applying Scheme [1] one can solve the problem (10) by exploiting the Sedumi SDP...
solv...m, respectively; \( \alpha \) is the number of iterations, \( \text{time}[s] \) is the CPU time in seconds. Both methods, Algorithm I and CFCDM fail or make only slow progress after a finite number of iterations, i.e. \( f^{k+1} - f^k \leq 10^{-4}(1 + |f^k|) \) for some \( k = k \) and \( \hat{k} = k + 1 \), where \( f^k := f(\hat{x}^k) \).

We test Algorithm I for several problems in COMPLib and compare our results with the ones reported by the convex-concave decomposition method (CCDM) in [20].

**TABLE I**

| Problem | Convex-Concave Decom. | Inner Convex App. |
|---------|-----------------------|--------------------|
|          |                       |                    |

The numerical results and the performances of two algorithms are reported in Table I. Here, we initialize both algorithms with the same initial guess \( F^0 = 0 \).

The notation in Table I consists of: Name is the name of problems, \( \alpha_0(A), \alpha_0(AF) \) are the maximum real part of the eigenvalues of the open-loop and closed-loop matrices \( A, AF \), respectively; \( \text{iter} \) is the number of iterations, \( \text{time}[s] \) is the CPU time in seconds. Both methods, Algorithm I and CFCDM fail or make only slow progress towards a local solution with 6 problems: AC18, DIS5, PAS, NN6, NN7, NN12 in COMPLib. Problems AC5 and NN5 are initialized with a different matrix \( P \) and \( \bar{P} \) as in [20]. The notation in Table I consists of: Name is the name of problems, \( \alpha_0(A), \alpha_0(AF) \) are the maximum real part of the eigenvalues of the open-loop and closed-loop matrices \( A, AF \), respectively; \( \text{iter} \) is the number of iterations, \( \text{time}[s] \) is the CPU time in seconds. Both methods, Algorithm I and CFCDM fail or make only slow progress towards a local solution with 6 problems: AC18, DIS5, PAS, NN6, NN7, NN12 in COMPLib. Problems AC5 and NN5 are initialized with a different matrix \( P \) and \( \bar{P} \) as in [20].

To determine a starting point, we perform the heuristic procedure called Phase 1 proposed in [20] which is terminated after a finite number of iterations. In this example, we also test Algorithm I for several problems in COMPLib using the same parameters and the stopping criterion as in the previous subsection. The computational results are shown in Table I. The numerical results computed by HIFOO and PENBMI are also included in Table III.

Here, three last columns are the results and the performances of our method, the columns HIFOO and PENBMI indicate the \( H_{\infty} \)-norm of the closed-loop system for the static output feedback controller given by HIFOO and PENBMI, respectively. We can see from Table III that the optimal values reported by Algorithm I and HIFOO are almost similar for many problems whereas in general PENBMI has difficulties in finding a feasible solution.

### C. \( H_2 \) or \( H_{\infty} \) optimization: BMI formulation

Motivated from the \( H_{\infty} \) optimization problem, in this example we consider the mixed \( H_2 \) or \( H_{\infty} \) synthesis optimization problem. Let us assume that \( D_{11} = 0, D_{21} = 0 \) and the performance output \( z \) is divided in two components, \( z_1 \) and \( z_2 \). Then the linear system \( (8) \) becomes:

\[
\begin{align*}
\min_{F,X,Y} & \quad \gamma \\
\text{s.t.} & \quad \begin{bmatrix} A_F^T X + X A_F & X B_1 & C_F^T \\ B_1^T X & -\gamma I & D_F^T \\ C_F & D_11 & -\gamma I \\ \end{bmatrix} < 0, \\
& \quad X > 0, \quad \gamma > 0.
\end{align*}
\]

Here, as before, we define \( A_F := A + BFC \) and \( C_F := C_1 + D_{12}FC \). The bilinear matrix term \( A_F^T X + X A_F \) at the top-left corner of the first constraint can be approximated by the form of \( \mathcal{Q} \) defined in (2). Therefore, we can use this psd-convex overestimate to approximate the problem (11) by a sequence of the convex subproblems of the form \( \text{CSDP}(x^k) \).

Then we transform the subproblem into a standard SDP problem that can be solve by a standard SDP solver thanks to Schür’s complement [1], [20].

The mixed \( H_2 \) or \( H_{\infty} \) control problem is to find a static output feedback gain \( F \) such that, for \( u = Fv \), the \( H_2 \)-norm of the closed loop from \( w \) to \( z_2 \) is minimized, while the \( H_{\infty} \)-norm from \( w \) to \( z_1 \) is less than some imposed level \( \gamma \) [4], [13], [16]. This problem leads to the following optimization problem

\[
\begin{align*}
\min_{F,X,Y} & \quad \gamma \\
\text{s.t.} & \quad \begin{bmatrix} A_F^T X + X A_F & X B_1 & C_F^T \\ B_1^T X & -\gamma I & D_F^T \\ C_F & D_11 & -\gamma I \\ \end{bmatrix} < 0, \\
& \quad X > 0, \quad \gamma > 0.
\end{align*}
\]
with BMI constraints [16]:

\[
\begin{align*}
\min_{F, A_1, Z} & \quad \text{trace}(Z) \\
\text{s.t.} & \quad A_1^T P_1 + P_1 A_1 + (C_1^T F C_1 + C_2^T F C_2) P_1 B_1 - \gamma^2 I < 0, \\
& \quad A_1^T P_2 + P_2 A_1 + P_2 B_1 - I < 0, \\
& \quad P_1 > 0, P_2 > 0, \\
& \quad 0 < P_2 C_2 F Z, \end{align*}
\]

where \(A_F := A + B F C \), \(C_1 := C_1 + D_{11}^T F C \) and \(C_2 := C_2 + D_{12}^T F C \). Note that if \(C = I_n \), the identity matrix, then this problem becomes a mixed \(H_2/H_\infty\) of static state feedback design problem considered in [16].

Now, we implement Algorithm [I] for solving the problem [I]. As before, we use a procedure proposed in [20] to determine a starting point for Algorithm [I]. We test the algorithm described above for several problems in COMPLib with the level values \(\gamma = 4\) and \(\gamma = 10\). In this test, we assume that the output signals \(z_1 \equiv z_2\). Thus we have \(C_1 = C_2 = 1\) and \(D_{11} = D_{12} = D_{12} = 1\). The parameters and the stopping criterion of the algorithm are chosen as in the \(H_\infty\) problem. The computational results are reported in Table III with \(\gamma = 4\) and \(\gamma = 10\). Here, \(H_2\) and \(H_\infty\) are the \(H_2\) and \(H_\infty\) norms of the closed-loop systems for the static output feedback controller, respectively. With \(\gamma = 10\), the computational results show that Algorithm [I] satisfies the condition \(|P_2(s)| \leq 10\) for all the test problems. The problems AC11 and AC12 encounter a numerical problems that Algorithm [I] cannot solve. While, with \(\gamma = 4\), there are 6 problems reported infeasible, which are denoted by “-”. The \(H_\infty\)-constraint of three problems AC11 and NN8 is active with respect to \(\gamma = 4\).
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