Multiple Killing horizons

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Abstract
Killing horizons which can be such for two or more linearly independent Killing vectors are studied. We provide a rigorous definition and then show that the set of Killing vectors sharing a Killing horizon is a Lie algebra $\mathcal{A}_H$ of dimension at most the dimension of the spacetime. We prove that one cannot attach different surface gravities to such multiple Killing horizons, as they have an essentially unique nonzero surface gravity (or none). $\mathcal{A}_H$ always contains an Abelian (sub)algebra—whose elements all have vanishing surface gravity—of dimension equal to or one less than $\dim \mathcal{A}_H$. There arise only two inequivalent possibilities, depending on whether or not the nonzero surface gravity exists. We show the connection with near-horizon geometries, and also present a linear system of PDEs, the master equation, for the proportionality function on the horizon between two Killing vectors of a multiple Killing horizon, with its integrability conditions. We provide explicit examples of all possible types of multiple Killing horizons, as well as a full classification of them in maximally symmetric spacetimes.

Keywords: Killing horizon, surface gravity, near horizon geometry, bifurcate horizon, Killing vector

1. Introduction

The notion of a Killing horizon captures the idea that a Killing vector $\xi$ in a spacetime $(M, g)$ may change causal character precisely on a null hypersurface. In more precise terms, a Killing horizon $\mathcal{H}_\xi$ of a Killing vector $\xi$ in a spacetime $(M, g)$ is a null hypersurface where $\xi$ is null,
nonzero and tangent. Killing horizons play a fundamental role in general relativity, in particular in the context of black holes in equilibrium: by Hawking’s rigidity theorem the event horizon of a stationary, asymptotically flat black hole spacetime (supplemented by certain additional assumptions, see [19] for a review), is a Killing horizon. In fact, one often uses the notion of a Killing horizon to provide a quasi-local definition of an equilibrium black hole. Killing horizons are also relevant to particular cases of more general notions such as isolated horizons, weakly isolated horizons or totally geodesic null hypersurfaces, which have been extensively studied in the literature (see [1–4, 14–16, 20, 23, 24, 29] and references therein). Some physically interesting spacetimes, such as pp-waves, can even be foliated by Killing horizons.

Now, it can happen that a null hypersurface, or at least a portion thereof, is simultaneously the Killing horizon of two or more independent Killing vectors. In fact this is a situation known to happen e.g. in Minkowski spacetime where, in standard coordinates, the null hypersurface \( \{ t = x > 0 \} \) is a Killing horizon of the null translational Killing \( \partial_t + \partial_x \), as well as of the boost in the \( x \) direction, \( x \partial_t + t \partial_x \). This article initiates a series of papers where the existence and properties of these multiple Killing horizons (MKHs) are analyzed in detail. From a mathematical viewpoint, this problem turns out to be remarkably rich and elegant. Moreover, it leads to some questions relevant on their own, such as, for instance, whether near-horizon geometries of a multiple degenerate Killing horizon depend on the Killing vector with respect to which the near horizon limit is performed.

MKHs are also interesting from a physical point of view. As mentioned above the event horizon of stationary black holes is a Killing horizon. Given a suitably normalized Killing vector \( \zeta \) with associated horizon \( H_\zeta \) one can introduce a function \( \kappa_\zeta \) which provides a measure for the deviation of the Killing parameter from an affine parameter along the null geodesic generators of \( H_\zeta \). Under suitable asymptotics of the spacetime this function is interpreted as the ‘surface gravity’ of the black hole, as it determines the redshifted force on a near-horizon test body viewed from infinity [37]. It turns out that the surface gravity is constant on \( H_\zeta \) under fairly general circumstances [37], and this establishes the zeroth law of black hole thermodynamics. The interpretation of the surface gravity as a temperature of the black hole is reinforced by the first and second laws of black hole thermodynamics, and turned into a physical certainty by the Hawking emission process and the corresponding Hawking temperature. Thus, when dealing with an MKH an immediate question arises. To an MKH one can ascribe different surface gravities (one for each choice of independent Killing vector) and hence also different temperatures to the black hole. What is the physical meaning of this and what are its physical consequences? As we will see presently, we find a number of interesting properties of MKHs that help in resolving this problem.

In this first paper we focus on the basic concepts and properties of MKHs. In section 2 we provide a rigorous definition of an MKH and prove a first property, namely that all surface gravities are always constant without any further assumptions.

In section 3 we analyze the set of all Killing vectors sharing a null hypersurface \( H \) as MKH, and prove that they constitute a Lie algebra—denoted by \( \mathcal{A}_H \). It is further shown that one merely has to distinguish two cases: either the Lie algebra is Abelian, in which case all Killing vectors are degenerate at the horizon (i.e. have vanishing surface gravities), or it is not Abelian, in which case it contains an Abelian subalgebra of codimension one, and one can find a basis of Killing vectors such that all except one are degenerate. In the first case we call the MKH fully degenerate, in the latter one non-fully degenerate or just nondegenerate. This result states, in particular, that to any MKH one can ascribe a single nonzero surface gravity (or temperature). Another general property obtained in this section is that, letting \( n + 1 \) denote
the spacetime dimension, the maximal dimension $m$ of the Lie algebra $A_H$ is $n$ in the fully degenerate case while it is $n + 1$ in the nondegenerate case.

Section 4 is devoted to explicit examples of spacetimes with MKHs. In particular, we provide an example which shows that MKHs with compact cross-sections exist (which might be regarded as particularly relevant from a physical point of view). Moreover, we show that MKHs exist for any $m \in \{2, \ldots, n\}$ and $m \in \{2, \ldots, n + 1\}$ in the fully degenerate and non-degenerate cases, respectively. In fact, once a spacetime with a fully degenerate MKH has been given for some $m \in \{2, \ldots, n\}$ an associated spacetime with nondegenerate MKH is obtained by computing its near horizon geometry [25]. The reason for that is that when performing the near horizon limit an additional Killing vector, which is nondegenerate, (and possibly others) is added.

Given a spacetime with an MKH $H$ the various Killing vectors are parallel on $H$. In section 5 we derive an equation which is satisfied by the proportionality function between two such Killing vectors. The so-obtained linear PDE system will be called master equation. We also determine its first integrability conditions.

In section 6 we provide a complete classification of MKHs for maximally symmetric spacetimes, i.e. for Minkowski and (Anti-)de Sitter spacetimes. For the convenience of the reader some details of the proof have been shifted to appendix B. In appendix A we recall (and prove, for completeness) a known property of the zeros of a Killing vector.

Let us conclude the introduction with an outlook. In the subsequent papers, [30] and [31], we will face the question raised above concerning the uniqueness of near horizon geometries which arise from an MKH with at least two degenerate Killing vectors. We will further analyze the master equation in more detail. Moreover, we will construct vacuum spacetimes with MKHs via characteristic initial value problems. In this case, and assuming further that the initial surface is arranged to form a bifurcate horizon, the master equation evaluated on the bifurcation surface turns out to be not only necessary but also sufficient for the existence of an MKH in the emerging spacetime.

### 1.1. Notation

$(M, g)$ denotes a connected, oriented and time-oriented $(n + 1)$-dimensional Lorentzian manifold with metric $g$ of signature $(-, +, \ldots, +)$. We sometimes call $(M, g)$ the spacetime. Unless otherwise stated, all submanifolds will be without boundary. The topological closure of a set $A$ is denoted by $\overline{A}$. Given a vector (field) $v$ in $TM$, $v^+\omega$ denotes the corresponding one-form, i.e. the metrically related covector. Similarly, $\omega_v$ denotes the vector obtained by raising indices of a one-form $\omega$. In general, $\mathcal{X}(M)$ denotes the set of smooth vector fields on a differentiable manifold $M$.

We will use index-free as well as index notation. Lowercase Greek letters $\alpha, \beta, \ldots$ are spacetime indices and run from 0 to $n$. Small Latin indices $a, b, \ldots$ are hypersurface indices and take values from 1 to $n$. Capital Latin indices $A, B, \ldots$ are codimension-two submanifold indices running from 2 to $n$. Finally, small Latin indices $i, j, \ldots$ will enumerate the different Killing vectors of multiple Killing horizons and will take values in $\{1, \ldots, m\}$, where $m \leq n + 1$.

### 2. Multiple Killing horizons: basics

We start by recalling the notion of a Killing horizon, which will be the basis of the entire paper. This notion is only relevant when the spacetime dimension is at least two, which we assume from now on.
Definition 1 (Killing horizon of a Killing $\xi$). A smooth null hypersurface $\mathcal{H}_\xi$ embedded in a spacetime $(M, g)$ is a Killing horizon of a Killing vector $\xi$ of $(M, g)$ if and only if $\xi$ is null on $\mathcal{H}_\xi$, nowhere zero on $\mathcal{H}_\xi$ and tangent to $\mathcal{H}_\xi$. Killing horizons can have either one or several connected components, but in the latter case we require that the interior of its closure is a smooth connected hypersurface.

The reason to allow for multiple connected components will become clear later, as this is needed in our main definition 3, and will be illustrated in the examples of section 4.

A more general notion is that of a Killing prehorizon. Its definition is the same as for a Killing horizon except that the condition that $\mathcal{H}_\xi$ is embedded is replaced by injectively immersed. We will also need the related concept of bifurcation at Killing horizons [6, 21, 34].

Definition 2 (Bifurcate Killing horizon). Let $\xi$ be a Killing vector on $(M, g)$ which has a connected and spacelike codimension-two submanifold $S$ of fixed points (i.e. such that $\xi|_S = 0$). Then, the set of points along all null geodesics orthogonal to $S$ comprises what is called a bifurcate Killing horizon with respect to $\xi$.

Observe that the null geodesics orthogonal to $S$ generate two transversal null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The portions $\mathcal{H}_1^+$ and $\mathcal{H}_2^+$ to the future of $S$, as well as the portions $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$ to its past, are all connected Killing horizons. Moreover, $\mathcal{H}_1^+ \cup \mathcal{H}_1^- \subset \mathcal{H}_1$ is also a Killing horizon according to our definition (since its closure is $\mathcal{H}_1$, which is open and connected). The same holds for $\mathcal{H}_2^+ \cup \mathcal{H}_2^-$. Note that $\mathcal{H}_1$, $\mathcal{H}_2$ are not Killing horizons. The union $\mathcal{H}_1^+ \cup \mathcal{H}_2^+ \cup \mathcal{H}_1^- \cup \mathcal{H}_2^- \cup S = \mathcal{H}_1 \cup \mathcal{H}_2$ is the bifurcate Killing horizon.

Our main goal is the study of the following particular class of Killing horizons.

Definition 3 (Multiple Killing horizon (MKH)). A null hypersurface $\mathcal{H}$ embedded in a spacetime $(M, g)$ is a multiple Killing horizon if $(M, g)$ admits Killing horizons $\mathcal{H}_\xi$, $i \in \{1, \ldots, m\}$ with $m \geq 2$, associated to linearly independent Killing vectors $\xi_i$ satisfying

$$\mathcal{H} = \mathcal{H}_{\xi_1} = \cdots = \mathcal{H}_{\xi_m}.$$ 

Note that if $\mathcal{H}$ is an MKH, so it is any open subset of $\mathcal{H}$ whose closure is connected. Observe also that any hypersurface containing $\mathcal{H}$ and contained in $\overline{\mathcal{H}}$ is also an MKH. This stems from the fact that the definition involves $\overline{\mathcal{H}}$. The reason behind taking this closure in the definition is that it is not generally true that, say, $\mathcal{H}_{\xi_1} = \mathcal{H}_{\xi_2}$ and only their closures agree. This behaviour is shown in figure 1 where we represent the null hypersurface $\{t = x\}$ in the two-dimensional Minkowski spacetime. This null hypersurface is a Killing horizon of the null translation $\partial_t + \partial_x$, but only its open subset $\{t = x, x \neq 0\}$ is a Killing horizon of the boost $x\partial_t + t\partial_x$, because this vector is zero at the origin. Our definition of Killing horizon is designed so that the zeros of all Killing generators $\xi_i$ are allowed to belong to the multiple Killing horizon. Other examples of the need (or, at least convenience) to take the closure in the definition of multiple Killing horizon can be found in section 4. Nonetheless, the case when $\mathcal{H}_{\xi_i} = \mathcal{H}_{\xi_j}$ for all $i, j \in \{1, \ldots, m\}$ seems to be still feasible, though it is much rarer.

Killing (pre)horizons of a Killing vector $\xi$ have an associated notion of surface gravity, which is a smooth function $\kappa_\xi : \mathcal{H}_\xi \rightarrow \mathbb{R}$ defined by

$$\nabla_\xi \kappa_\xi = \frac{\kappa_\xi \xi}{\|\xi\|^2} \quad \text{or equivalently} \quad \text{grad}(g(\xi, \xi)) \mathcal{H}_\xi = -2\kappa_\xi \xi.$$ 

If this function vanishes, then $\mathcal{H}_\xi$ is said to be degenerate. It is very easy to check that $\kappa_\xi$ is constant along the null generators of $\mathcal{H}_\xi$, that is

$$\xi(\kappa_\xi) = 0.$$
One can show that $\kappa_\xi$ has the following useful representation [12, 37] (a justification will be provided later in section 5)

$$\kappa^2_{\xi} = \frac{1}{2} \nabla_\mu \xi_\nu \nabla^\mu \xi^\nu$$

(3)

which allows us to prove that $\kappa_\xi$ actually extends as a smooth function to the whole connected $\mathcal{H}_{\xi}$, despite the fact that $\mathcal{H}_{\xi}$ may have several connected components.

We are going to prove that, actually, for any MKH all possible surface gravities are constant. To that end, we need an intermediate basic result. Let $\mathcal{H}$ be an MKH with respect to the Killing vectors $\xi$ and $\eta$. Set

$$\mathcal{H} := \mathcal{H}_{\xi} \cap \mathcal{H}_{\eta}$$

and let $F : \mathcal{H} \rightarrow \mathbb{R}$ be the scalar function defined by

$$\eta \equiv F\xi.$$  

(4)

By construction $F$ is well-defined, smooth and nowhere zero. This function extends smoothly (and uniquely) to all $\mathcal{H}_{\xi}$ but the extension may have zeroes. Furthermore, $F$ cannot be constant on any open subset $\mathcal{U} \subset \mathcal{H}$. This follows from the fact that the set of fixed points of a Killing vector cannot have codimension one (this is known, but we include a proof in appendix A) and the Killing vector $\eta - F_0 \xi$ would vanish on $\mathcal{U}$ if $F|_\mathcal{U} = \text{const} := F_0$.

**Lemma 1.** Let $\mathcal{H}$ be an MKH with respect to the Killing vectors $\xi$ and $\eta$ and denote by $\kappa_\xi$ and $\kappa_\eta$ the surface gravities of $\xi$ on $\mathcal{H}_{\xi}$ and $\eta$ on $\mathcal{H}_{\eta}$ respectively. Then

$$\kappa_{\xi}^{\mathcal{H}} = \xi(F) + F\kappa_\xi.$$  

(5)

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**Figure 1.** Null hypersurface $\mathcal{H} = \{t = x\}$ in the two-dimensional Minkowski spacetime. The null translation $\partial_t + \partial_x$ is null, tangent and nonzero there, so $\mathcal{H}$ is a Killing horizon for it. The boost $x\partial_t + t\partial_x$ is also null and tangent to $\mathcal{H}$, but nonzero only outside the origin. This point needs to be excluded from $\mathcal{H}$ to have a Killing horizon of the boost. A convenient way to deal with this subtlety is to consider closures of Killing horizons provided they are smooth, connected hypersurfaces. This is the motivation behind definition 3.
Proof. A direct calculation using (4) provides
\[
\kappa_\eta \mathcal{H}_\xi = \nabla_\eta \mathcal{H}_\xi = (F^2 \kappa_\xi + F \nabla_\xi F) \xi
\]
from where we deduce
\[
\xi(F) + \kappa_\xi F \mathcal{H}_\xi = \kappa_\eta
\]
which holds even at the fixed points of \( \eta \) (where \( F \) vanishes), because the set of fixed points of \( \eta \) can have at most codimension two, and thus it follows by continuity.

As mentioned above, the surface gravities are constant along the null generators, so the PDE (5) can be explicitly integrated. Let \( \tau : \mathcal{H}_\xi \cap \mathcal{H}_\eta \rightarrow \mathbb{R} \) be a (smooth) scalar function satisfying \( \xi(\tau) = 1 \). Obviously \( \tau \) is not univocally defined, as it is affected by the freedom:
\[
\tau \rightarrow \tau + \tau_0, \quad \xi(\tau_0) = 0.
\]
(6)

This freedom can be fixed by giving initial data on any cut \( S_0 \subset \mathcal{H} \) transversal to \( \xi \) but, for the time being, we leave this free. Now define \( Q_\xi : \mathcal{H}_\xi \rightarrow \mathbb{R} \) by
\[
Q_\xi := \frac{1}{\kappa_\xi} (e^{-\kappa_\xi \tau} - 1) \quad \text{if } \kappa_\xi \neq 0, \\
\tau \quad \text{if } \kappa_\xi = 0.
\]
(7)

This is a smooth function on \( \mathcal{H}_\xi \) irrespectively of whether \( \kappa_\xi \) has zeros or not. Note also that \( \xi(Q_\xi) = e^{-\kappa_\xi \tau} \). Then, the general solution of (5) is given in terms of a smooth nowhere zero function \( f : \mathcal{H}_\xi \cap \mathcal{H}_\eta \rightarrow \mathbb{R} \) satisfying \( \xi(f) = 0 \), by
\[
F = fe^{-\kappa_\xi \tau} + \kappa_\eta Q_\xi.
\]
(8)

Indeed
\[
\xi(F) + F \kappa_\xi = \xi \left( fe^{-\kappa_\xi \tau} + \kappa_\eta Q_\xi \right) + \kappa_\xi \left( fe^{-\kappa_\xi \tau} + \kappa_\eta Q_\xi \right) = -\kappa_\xi fe^{-\kappa_\xi \tau} + \kappa_\eta e^{-\kappa_\xi \tau} + \kappa_\xi \left( fe^{-\kappa_\xi \tau} + \kappa_\eta Q_\xi \right) = \kappa_\eta \left( e^{-\kappa_\xi \tau} + \kappa_\xi Q_\xi \right) = \kappa_\eta.
\]

As before \( f \) extends smoothly to \( \mathcal{H}_\xi \), possibly with zeroes.

We can now prove that in MKHs, all the surface gravities are necessarily constant.

**Theorem 1.** Let \( \mathcal{H} \) be a multiple Killing horizon and \( \mathcal{H}_\xi, \mathcal{H}_\eta \) be Killing horizons satisfying \( \mathcal{H}_\xi = \mathcal{H}_\eta = \mathcal{H} \). Then the respective surface gravities \( \kappa_\xi \) and \( \kappa_\eta \) are constant.

**Remark 1.** Constancy of the surface gravity is known to hold in several circumstances, namely when the Killing generator is integrable \([34]\) (i.e. \( \xi \wedge d\xi = 0 \)), or when the Einstein tensor of \( (\mathcal{M},g) \) satisfies the dominant energy condition \([37, \text{chapter 12}]\), or for bifurcate Killing horizons \([12, 21]\). For multiple Killing horizons the constancy of the surface gravity turns out to be a universal property.

**Proof.** In the multiple horizon case we work on \( \hat{\mathcal{H}} := \mathcal{H}_\xi \cap \mathcal{H}_\eta \). Since \( \mathcal{H}_\xi \cap \mathcal{H}_\eta = \mathcal{H}_\xi = \mathcal{H}_\eta = \mathcal{H} \), proving constancy on this set also proves it in the respective Killing horizons.
Any Killing horizon has a vanishing second fundamental form relative to the one-form \( \xi \), as follows from the fact that

\[ g(X, \nabla_X \xi) \equiv 0, \quad \forall X \in \mathcal{X}(\mathcal{H}) \]

if \( \xi \) is a Killing vector. This implies the existence of a one-form \( \varphi \in \Lambda(\mathcal{H}) \) such that

\[ \nabla_X \xi \equiv \varphi(X) \xi, \quad \forall X \in \mathcal{X}(\mathcal{H}). \quad (9) \]

Taking the covariant derivative along \( X \) of the first in (1) and using (9)

\[ \varphi(X) \xi^\nu \nabla_\nu \xi^\mu + \xi^\nu \nabla_\sigma \nabla_\rho \xi^\mu \equiv X(\kappa_\xi) \xi^\mu + \kappa_\xi \varphi(X) \xi^\mu. \]

But any Killing vector satisfies [37]

\[ \nabla_\sigma \nabla_\rho \xi^\mu = \xi^\mu R^{\nu}_{\sigma \rho \mu}, \quad (10) \]

where \( R^{\nu}_{\sigma \rho \mu} \) is the Riemann tensor of \((M, g)\), so that using (1) again in the previous expression we arrive at

\[ \xi^\nu X^\sigma R^{\nu}_{\sigma \rho \mu} \xi^\rho \equiv X(\kappa_\xi) \xi^\mu. \quad (11) \]

The same calculation for \( \eta \) leads to

\[ \eta^\nu X^\sigma R^{\nu}_{\sigma \rho \mu} \eta^\rho \equiv X(\kappa_\eta) \eta^\mu \]

so that using here (4) and combining with (11) we get

\[ X(\kappa_\eta) \equiv FX(\kappa_\xi), \quad \forall X \in \mathcal{X}(\mathcal{H}) \]

where \( X \) is any vector field tangent to \( \mathcal{H} \). Now, the combination of this with (2) gives the desired result, as \( F \) given in (8) has \( \tau \)-dependence while the surface gravities do not. To be precise, choose any \( X \in \mathcal{X}(\mathcal{H}) \) such that \([\xi, X] = 0\) and take the directional derivative along \( \xi \) of the previous expression

\[ \xi(X(\kappa_\eta)) \equiv \xi(F)(X(\kappa_\xi)) + F\xi(X(\kappa_\xi)) \quad \Rightarrow \quad X(\xi(\kappa_\eta)) \equiv \xi(F)(X(\kappa_\xi)) + FX(\xi(\kappa_\xi)) \]

and now use (2) and \( \xi(\kappa_\eta) = F^{-1}\eta(\kappa_\eta) = 0 \) to get

\[ \xi(F)(X(\kappa_\xi)) = 0 \]

which holds for arbitrary \( X \in \mathcal{X}(\mathcal{H}) \) as long as it commutes with \( \xi \). If \( X(\kappa_\xi) \neq 0 \) on some open, connected and non-empty subset \( \mathcal{U} \subset \mathcal{H} \), then \( \xi(F) \equiv 0 \) would necessarily follow, so that from (5) \( \kappa_\eta = F\kappa_\xi \) would hold on \( \mathcal{U} \). By restricting \( \mathcal{U} \) if necessary we would then have that \( \kappa_\xi \) vanishes nowhere in this set, and consequently

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4 We use the same symbol \( X \) to denote a vector field \( X \in \mathcal{X}(\mathcal{H}) \) and its image in \( T_q \mathcal{H} \) under the embedding from \( \mathcal{H} \) into \( M \). The precise meaning will be clear from the context.
implying that \( F \) would be a constant on \( \mathcal{U} \), say \( F_0 \). But then the Killing vector \( \eta - F_0 \xi \) would vanish on a codimension one subset of the spacetime, hence everywhere, and \( \eta \) would not be linearly independent of \( \xi \), against hypothesis.

Hence, \( X(\kappa_\xi) \) must vanish on a dense subset of \( \hat{\mathcal{H}} \) —for arbitrary \( X \) subject to \( [\xi, X] = 0 \)—, then also \( X(\kappa_\eta) \) vanishes there, and both \( \kappa_\xi \) and \( \kappa_\eta \) are constant on any connected component of \( \mathcal{H}_\xi \cap \mathcal{H}_\eta \). By continuity of \( \kappa_\xi \) on \( \mathcal{H}_\xi \) it follows that this surface gravity is constant on \( \mathcal{H} \) and the same holds for \( \kappa_\eta \).

\[ FX(\kappa_\xi) = X(\kappa_\eta) \quad \xi \in \mathcal{H} \implies X(F\kappa_\xi) \equiv X(F\kappa_\eta + FX(\kappa_\xi)) \]

\[ \implies X(F\kappa_\xi) = 0 \quad \implies X(F) = 0 \]

3. Multiple Killing horizons: Lie algebra and types

In this section, we start by proving that the set of all Killing vectors in \((\mathcal{M}, g)\) with a common multiple Killing horizon constitute a Lie subalgebra of the Killing Lie algebra, and we also find the possible structure constants and dimensions. This will then allow for distinguishing between different types of MKHs, which will be rigorously defined.

For any spacetime \((\mathcal{M}, g)\) we denote by \( A_\mathcal{M} \) the Lie algebra of Killing vectors. This is a finite dimensional vector space of dimension bounded above by \((n + 1)(n + 2)/2\). Consider a multiple Killing horizon \( \mathcal{H} \) and define \( A_\mathcal{H} \) as the union of the trivial Killing vector and the collection of Killing vectors \( \xi \) which admit a Killing horizon \( \hat{\mathcal{H}}_\xi \) satisfying \( \mathcal{H} = \mathcal{H}_\xi \). It turns out that \( A_\mathcal{H} \) is a Lie subalgebra of \( A_\mathcal{M} \).

**Theorem 2.** Let \( \mathcal{H} \) be a multiple Killing horizon in a spacetime \((\mathcal{M}, g)\) of arbitrary dimension at least two. Then \( A_\mathcal{H} \) is a Lie subalgebra of the Killing algebra \( A_\mathcal{M} \) of \((\mathcal{M}, g)\).

**Proof.** First we prove that \( A_\mathcal{H} \) is a vector subspace of \( A_\mathcal{M} \). Let \( \xi, \eta \in A_\mathcal{H} \). We want to show that \( a_1 \xi + a_2 \eta \in A_\mathcal{H} \), for any \( a_1, a_2 \in \mathbb{R} \). If either \( \xi \) or \( \eta \) is the zero vector, the claim is obvious. Assume both \( \xi \) and \( \eta \) are nontrivial. Then there exists a hypersurface \( \hat{\mathcal{H}} \) which is a Killing horizon with respect to both \( \xi \) and \( \eta \) and \( \hat{\mathcal{H}} \) is a dense subset of \( \mathcal{H} \). We know (4) that \( \xi \) and \( \eta \) are proportional (and null) on \( \hat{\mathcal{H}} \), so the Killing vector \( \zeta := a_1 \xi + a_2 \eta \) is also tangent to \( \hat{\mathcal{H}} \) and null there. Moreover, if it vanishes on a dense subset of \( \hat{\mathcal{H}} \), then by lemma A.1 in appendix A, it vanishes identically, hence belongs to \( A_\mathcal{H} \). Otherwise, there exists an open and dense \( \mathcal{H}_\zeta \subset \mathcal{H} \) where \( \zeta \) does not vanish. In other words, \( \mathcal{H}_\zeta \) is a Killing horizon of \( \zeta \). Given that \( \mathcal{H}_\zeta = \mathcal{H} \), we conclude \( \zeta \in A_\mathcal{H} \), as claimed.

It remains to prove that the commutator of any two Killing vectors \( \xi, \eta \in A_\mathcal{H} \) also belongs to \( A_\mathcal{H} \). Of course, \([\xi, \eta] \in A_\mathcal{M} \) for arbitrary \( \xi, \eta \in A_\mathcal{H} \), so we only need to show that \([\xi, \eta] \) is null and tangent to \( \mathcal{H}_\xi \cap \mathcal{H}_\eta \). But we know that the Killing vectors \( \xi \) and \( \eta \) are related by (4). Given also that they are tangent to \( \mathcal{H} \), we can compute their commutator there

\[ [\xi, \eta] \equiv [\xi, F\xi] \equiv \xi(F\xi) \equiv \xi(\kappa_\xi - F_\kappa_\xi) \]

where in the last step we have used (5). This finishes the proof.
Definition 4 (Lie algebra and order of an MKH). We call \( A_H \) the Lie algebra of the multiple Killing horizon \( H \).

The order \( m \geq 2 \) of an MKH \( H \) is, by definition, the dimension of its Lie algebra \( A_H \).

We shall sometimes loosely speak of double, triple, quadruple, etc., MKHs for \( m = 2, 3, 4, \ldots \).

We can actually say much more about \( A_H \) and its order.

Theorem 3. Let \( A_H \) be the Lie algebra of an MKH \( H \) of order \( m \) in a spacetime \((M, g)\) of arbitrary dimension at least two. Then, \( A_H \) always contains an Abelian subalgebra \( A^\text{deg}_H \) of dimension at least \( m - 1 \) whose elements have vanishing surface gravities, that is to say, they all have (the appropriate dense subset of) \( H \) as a degenerate Killing horizon. If this Abelian subalgebra \( A^\text{deg}_H \) has dimension \( m - 1 \), the remaining independent Killing vector (say \( \xi \)) in \( A_H \setminus A^\text{deg}_H \) has \( \kappa_\xi \neq 0 \) and satisfies

\[
[\xi, \eta] = -\kappa_\xi \eta, \quad \forall \eta \in A^\text{deg}_H.
\]

Proof. Let \( \{\eta_i\} \) be a basis of \( A_H \), and let \( \xi \in A_H \) be nontrivial, otherwise arbitrary. Then

\[
\zeta = b^j \eta_j
\]

where \( b^j \in \mathbb{R} \) are constants. Fix a nonzero element \( \xi \in A_H \) and let \( H_\xi \subset H \) be its corresponding Killing horizon. Expression (4) holds for each \( \eta_i \) with corresponding functions \( F_i \).

From the definition of surface gravity (1) the acceleration of \( \zeta \) on \( H_\xi \) is

\[
\nabla_\zeta \xi = b^j b^i \nabla_{\eta_i} \xi = b^j b^i F_i \nabla_\xi (F_j \xi) = (b^j F_i) b^i (F_j \xi + \xi (F_j)) \xi = (b^j \kappa_\xi) b^i \eta_i \equiv (b^j \kappa_\eta) \zeta
\]

where in the penultimate step we have used the PDE (5) for the functions \( F_j \). This proves that the surface gravity of \( \zeta \) on \( H \) is

\[
\kappa_\xi = b^j \kappa_{\eta_j}.
\]

It follows that every \( \zeta \in A_H \) with

\[
b^j \kappa_{\eta_j} = 0
\]

has a vanishing surface gravity. There are at least \( m - 1 \) linearly independent such degenerate Killing vectors, as follows from the following elementary reasoning: the relation (14) can be seen as the scalar product of the constant vectors \((b^j)\) and \((\kappa_{\eta_j})\) on an \( m \)-dimensional vector space, so that given \((\kappa_{\eta_j})\) as data, there exist \( m - 1 \) linearly independent solutions for \((b^j)\) — if at least one of the \( \kappa_{\eta_j} \) does not vanish. If all the surfaces gravities \( \kappa_{\eta_j} \) vanish then every \( \zeta \in A_H \) has vanishing \( \kappa_\xi \) too.

To end the proof, we use (12). For, if \( \xi \) and \( \eta \) both have vanishing surface gravity, then (12) informs us that \( [\xi, \eta] \nmid H_\xi \equiv 0 \) and therefore the Killing vector \( [\xi, \eta] \) must vanish everywhere. This proves that \( A^\text{deg}_H \) is Abelian. Similarly, if only \( \eta \) has \( \kappa_\eta = 0 \), then (12) implies that \( [\xi, \eta] \nmid H_\xi = -\kappa_\xi \eta \), and thus the Killing vector \( [\xi, \eta] + \kappa_\xi \eta \) must vanish everywhere, finishing the proof.

\[\square\]

5To avoid cumbersome notation we define the surface gravity of the zero vector to be zero.
Remark 2. A precursor of theorem 3 can be found in [27] (theorem IV.5) in the context of nonexpanding horizons satisfying the so-called ‘stronger energy-condition’, where it is found that if such a horizon admits two infinitesimal null symmetries, then the horizon admits an extremal infinitesimal null symmetry. The ‘stronger energy condition’ is a weaker version of the dominant energy condition which still implies that the surface gravity of the horizon with respect to all null symmetries is constant [24].

Remark 3 (Notation). In summary we have proven that, for multiple Killing horizons of order \( m \), there is always a basis of \( \mathcal{A}_H \) with \( m - 1 \) degenerate Killings vectors all of them commuting. Therefore, from now on we will use the following useful notation: \( \{ \eta_i \} \), \( i = 1, \ldots, m \), will always denote a basis of \( \mathcal{A}_H \) with \( \{ \eta_2, \ldots, \eta_m \} \) a basis of \( \mathcal{A}_H^{\text{deg}} \), that is to say,

\[
\kappa_{\eta_2} = \cdots = \kappa_{\eta_m} = 0.
\]

Then, we will also use the name \( \xi = \eta_1 \), and \( \kappa_\xi \) is arbitrary (it may vanish or not).

With this choice of basis we have found all the structure constants of \( \mathcal{A}_H \):

\[
C^{i}_{jk} = 0, \quad C^{j}_{i1} = -\kappa_\xi \delta^j_k, \quad \forall k \neq 1.
\]

Definition 5 (Fully degenerate MKH). A multiple Killing horizon \( \mathcal{H} \) is said to be fully degenerate if \( \mathcal{A}_H = \mathcal{A}_H^{\text{deg}} \), that is to say, if its Lie algebra is Abelian, and all surface gravities vanish.

Observe that non-fully degenerate MKHs possess a unique nonzero surface gravity. To fix the value of this surface gravity requires the use of some normalization for the Killing vector \( \xi \), be it at infinity or in some other appropriate place. This has some physical implications, as one cannot attach two different nonzero surface gravities to a given MKH, despite the fact of being a Killing horizon for multiple Killing vectors.

Corollary 1. The maximum possible dimension of \( \mathcal{A}_H^{\text{deg}} \) is \( n = \text{dim} (M) - 1 \). Therefore, the maximum possible order of an MKH \( \mathcal{H} \) is

1. \( m = n \) for fully degenerate \( \mathcal{H} \),
2. \( m = n + 1 \) for non-fully degenerate \( \mathcal{H} \).

Proof. As \( \mathcal{A}_H^{\text{deg}} \) is Abelian, its dimension can be at most \( n + 1 \). But if it were \( n + 1 \) the spacetime would be homogeneous, and actually locally flat (this follows from the fact that the Riemann tensor on the orbits of a group of motions can be expressed in terms of the structure constants of its Lie algebra, and it vanishes for Abelian groups [36]), in a neighbourhood around \( \mathcal{H} \), and this is not possible, as the Abelian subalgebra is generated by translations, and hence its span is \( n + 1 \) dimensional at every point. Thus, \( \text{dim}(\mathcal{A}_H^{\text{deg}}) \) is at most \( n \).

The bound \( m \leq n + 1 \) is sharp. Examples where the maximal value \( m = n + 1 \) is attained are the maximally symmetric spacetimes \( (M, g) \), see section 4 for explicit examples and section 6, where we present the full classification of MKHs in maximally symmetric spacetimes.

Using the notation fixed in remark 3, the expressions (8) for the elements \( \eta_i \in \mathcal{A}_H^{\text{deg}} \) then reduce simply to

\[
F_i = f_i e^{-\kappa_\xi \tau}, \quad \xi(f_i) = 0 \quad \forall i \in \{2, \ldots, m\} \tag{15}
\]

valid for both cases with \( \kappa_\xi \) zero or not. Then we have the relations
\[ \eta_i \equiv f_i e^{-\kappa \xi}, \quad \xi(f_i) = 0 \quad \forall i \in \{2, \ldots, m\}. \]  

The freedom (6) translates to a simple redefinition \( f_i \rightarrow f_i e^{-\kappa \xi} \) which is consistent given that \( \xi(\tau_0) = 0 \). Note that the zeros of the functions \( f_i \) are fixed points of the corresponding Killing vectors. These fixed points of each \( \eta_i \) are not part of the Killing horizons \( H_{\eta_i} \), but they do belong to their closure and thus to \( \overline{H} \).

Given that \( \kappa_{\eta_i} = 0 \) for all \( i \in \{2, \ldots, m\} \), the vector fields \( \eta_i \) have zero acceleration on their corresponding horizons \( H_{\eta_i} \subset H \), and thus their integral curves are affinely parametrized null geodesics generating \( H_{\eta_i} \). Then, the relations (16) imply that an affine parameter \( \lambda_i \) along the geodesics tangent to \( \eta_i \) in \( H_{\eta_i} \cap H_\xi \) are given, for the non-fully degenerate case \( \kappa_\xi \neq 0 \), by

\[ \lambda_i = \frac{1}{\kappa_{\xi f_i}} e^{\kappa_\xi \tau}, \quad \forall i \in \{2, \ldots, m\} \]

and therefore, the integral curves of \( \xi \) in \( H_\xi \cap H_{\eta_i} \) are incomplete geodesics (the range of the affine parameter \( \lambda_i \) cannot be the whole real line).

Using the results in [6], see also [12, 34], and as \( \kappa_\xi \neq 0 \) is constant, we deduce that these incomplete geodesics do not reach any curvature singularity, and therefore they are only a segment of a larger geodesic in the given spacetime, or the latter is extendable. Actually, the integral curves of \( \eta_i \) are longer geodesics if the given spacetime contains them — otherwise, they could be extended in any proper extension of the spacetime — and along them \( \xi \) vanishes on a codimension-two submanifold \( S \subset \overline{H} \). Therefore, non-fully degenerate multiple Killing horizons can be seen as a branch of a bifurcate Killing horizon with \( \xi \) as the bifurcate Killing vector field and \( S := \{ \xi = 0 \} \cap \overline{H} \) as the bifurcation surface.

4. Examples

In this section we present explicit examples of MKHs with the aim of illustrating the previous results and to gain some insight on their structure. We will also show that all possible types of MKHs exist, fully degenerate or not, and of any possible admissible order.

4.1. Flat spacetime

In \((n+1)\)-dimensional Minkowski spacetime \((\mathbb{R}^{n+1}, g^b)\), where \( g^b \) is the flat metric (with vanishing curvature tensor), any null hyperplane is an MKH of maximal order \( m = n + 1 \) (and therefore, non-fully degenerate). To check this, choose a global Cartesian coordinate system \( \{t, x^a\} \) such that

\[ g^b = -dt^2 + \sum_{a=1}^{n} (dx^a)^2. \]  

and select, for instance, the null hyperplane \( H := \{t = x^1 \} \). Let \( A \in \{2, \ldots, n\} \) and consider the following collection of \( n+1 \) linearly independent Killing vectors of \((\mathbb{R}^{n+1}, g^b)\):

\[ \eta_1 = \xi = x^1 \partial_t + t \partial_{x^1}, \]  

\[ \eta_2 = \partial_t + \partial_{x^1}, \]  

\[ \eta_{A+1} = x^A \partial_t + t \partial_{x^A} + x^A \partial_{x^1} - x^1 \partial_{x^A} = x^A (\partial_t + \partial_{x^1}) + (t - x^1) \partial_{x^A}. \]
These are all obviously null, and proportional to \( \eta_2 \), at \( \mathcal{H} \). \( \eta_2 \) is nonzero everywhere, and thus the entire \( \mathcal{H} \) is a Killing horizon for \( \eta_2 \). On \( \mathcal{H} \) we also have \( \eta_{A+1} \equiv x^A (\partial_t + \partial_A) \), so that each \( \eta_{A+1} \) vanishes on the codimension two surface \( \mathcal{H} \cap \{ x^A = 0 \} \). Thus the Killing horizon for each \( \eta_{A+1} \) is given by \( \mathcal{H}_{\eta_{A+1}} = \mathcal{H} \setminus \{ x^A = 0 \} \), has two connected components given by \( x^A > 0 \) and \( x^A < 0 \), but also \( \mathcal{H}_{\eta_{A+1}} = \mathcal{H} = \mathcal{H} \). Concerning \( \eta_1 = \xi \), we have \( \xi = t (\partial_t + \partial_1) \) and thus \( \mathcal{S} := \mathcal{H} \cap \{ t = 0 \} = \{ t = x = 0 \} \) is a codimension-two spacelike surface of fixed points for \( \xi \). The Killing horizon \( \mathcal{H}_\xi \) of \( \xi \) has thus two connected components defined by \( t > 0 \), say \( \mathcal{H}_1^+ \), and by \( t < 0 \), say \( \mathcal{H}_1^- \), but again \( \mathcal{H}_\xi = \mathcal{H} \). Therefore, \( \mathcal{H} \) is a multiple Killing horizon of maximal order \( m = n + 1 \).

All the Killing vectors shown above except \( \eta_1 = \xi \) are affinely parametrized geodesic vector fields on \( \mathcal{H} \), and thus their surface gravities vanish. Also, \( \nabla_\xi \xi = \xi \) so that \( \kappa_\xi = 1 \).

Observe that \( \mathcal{H}_\xi \), \( \xi \) having a set of fixed points at \( x \), is a branch of a bifurcate Killing horizon, the second branch being given by the hyperplane \( \{ t + x^1 = 0 \} \) which provides the future and past connected components \( \mathcal{H}_1^+ \) and \( \mathcal{H}_1^- \) for \( t > 0 \) and \( t < 0 \), respectively. This hyperplane is itself an MKH of maximal order.

The full classification of MKHs in flat spacetime, as well as (anti)-de Sitter spacetimes, is presented in section 6.

**4.2. A double Killing horizon with compact sections**

Consider the two-dimensional de Sitter space \( dS_2 \) of constant curvature \( \kappa^2 \) and the two-dimensional sphere \( S^2 \) with the round metric of radius \( 1/\kappa \). The Nariai spacetime is the product manifold \( dS_2 \times S^2 \) endowed with the product metric (a generalization to arbitrary dimension also exists [7] and one can easily extend the discussion below to that situation). This spacetime is a solution of the \( \Lambda \)-vacuum Einstein equations with cosmological constant \( \Lambda = \kappa^2 \). It is straightforward to check that the Killing algebra is six dimensional with a basis consisting on three linearly independent Killings vectors of \( S^2 \) and three independent Killing vectors on the sphere. In standard global coordinates of \( S^2 \) the Nariai metric takes the form

\[
g_N = -dt^2 + \cosh^2(\kappa t^2) dx^2 + \frac{1}{\kappa^2} \gamma_{S^2}
\]

where \( \gamma_{S^2} \) is the standard unit metric on the sphere. The most general Killing vector of this metric is given by

\[
\zeta = (A \cos(\kappa x) + B \sin(\kappa x)) \partial_t + [\beta + (B \cos(\kappa x) - A \sin(\kappa x)) \tanh(\kappa t)] \partial_x + \hat{\zeta}
\]

where \( \hat{\zeta} \) is a Killing vector on \( (S^2, \gamma_{S^2}) \). We consider the null hypersurface \( \mathcal{H} \) defined as the connected component of \( \tanh(\kappa t) - \sin(\kappa x) = 0 \) containing \( t = x = 0 \). Observe that the range of \( x \) is given by

\[
x \in \left( -\frac{\pi}{2\kappa}, \frac{\pi}{2\kappa} \right).
\]

By construction \( \mathcal{H} \) contains the sphere at \( \{ t = x = 0 \} \). Topologically \( \mathcal{H} \cong \mathbb{R} \times S^2 \). The null generator of \( \mathcal{H} \) is

\[
k = \partial_t + \frac{1}{\cos(\kappa x)} \partial_x.
\]

It is immediate to check that the most general Killing vector that is proportional to \( k \) on \( \mathcal{H} \) is given by
\[ \zeta = (A \cos(\kappa x) + B \sin(\kappa x)) \partial_t + [A + (B \cos(\kappa x) - A \sin(\kappa x)) \tanh(\kappa t)] \partial_x. \]  

(22)

On the sphere \( S_0: = \{ t = x = 0 \} \), the Killing vector (22) evaluates to

\[ \zeta|_{S_0} = A (\partial_t + \partial_x). \]

Thus

\[ \xi := \sin(\kappa x) \partial_t + \cos(\kappa x) \tanh(\kappa t) \partial_x \]

is a Killing vector for which \( H \setminus S_0 \) is a nondegenerate Killing horizon with bifurcation surface at \( S_0 \). The linearly independent Killing vector

\[ \eta := \cos(\kappa x) \partial_t + [1 - \sin(\kappa x) \tanh(\kappa t)] \partial_x \]

vanishes nowhere in the spacetime, in particular on \( H \). The corresponding surface gravity vanishes. This follows immediately from the fact that the square norm of \( \eta \) can be written as

\[ g(\eta, \eta) = \cosh^2(\kappa t) [\tanh(\kappa t) - \sin(\kappa x)]^2 \]

which has a zero of order two at \( H \). Thus \( H \) is a degenerate Killing horizon of \( \eta \) and given that the closure of \( H \setminus S_0 \) is \( H \), \( H \) is an MKH of order two—a double Killing horizon.

A direct calculation gives \( [\xi, \eta] = -\kappa \eta \) and thus, according to theorem 3, the surface gravity of \( H \) is \( \kappa_\xi = -\kappa = -\sqrt{\Lambda} \).

### 4.3. Fully degenerate MKHs of any order

We want to ascertain if fully degenerate MKHs exist, and which orders are feasible for them. In this section we provide explicit examples for fully degenerate MKHs of any admissible order \( m \).

To that end, we use the following construction. In section 4.1 we found MKHs of maximal order \( m = n + 1 \). The idea is then to try to retain (part or all of) the Abelian subgroup \( A^\text{deg}_H \) which is generated by \( \{ \eta_i \} \) with \( i = 2, \ldots, n + 1 \) in (19) and (20), but removing the nondegenerate Killing vector (18) that generates the bifurcate Killing horizon. To accomplish this, we perform a conformal transformation of the flat metric (17), that is

\[ g = \Omega g^\flat, \]  

(23)

where \( \Omega : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) is a smooth nonvanishing function. To keep \( \eta_2 \) as a Killing vector of \( g \) we require

\[ \mathcal{L}_{\eta_2} g = \mathcal{L}_{\eta_2}(\Omega g^\flat) = g^\flat (\partial_t \Omega + \partial_x \Omega) = 0 \quad \Rightarrow \quad \Omega(t - x^1, x^A). \]  

(24)

Similarly, to keep any of the \( \eta_{n+1} \) as Killing vectors of the metric (23) we demand

\[ \mathcal{L}_{\eta_{n+1}} g = \mathcal{L}_{\eta_{n+1}}(\Omega g^\flat) = g^\flat (\partial_\ell \Omega + \partial_x \Omega) \]

and using here (24)

\[ \partial_x \Omega = 0. \]  

(25)

Hence, by allowing \( \Omega \) in (24) to be independent of a number \( q \leq n - 1 \) of the variables \( \{ x^A \} \) we have that the corresponding \( q \) vector fields \( \eta_{n+1} \) are Killing vectors of the new metric \( g \). As null hypersurfaces and null vectors are preserved by conformal transformations (23), we know that all these ‘surviving’ Killing vectors together with \( \eta_2 \) are tangent to and null on
H := \{ t = x^1 \}. On the other hand, the remaining \( \eta_1 \) in (18) is not a Killing vector in general, because using (24)

\[ \mathcal{L}_{\eta_1} g = \mathcal{L}_{\eta_1} (\Omega g^b) = g^b \eta_1 (\Omega) = g^b \left( x^1 \partial_t \Omega + t \partial_1 \Omega \right) = (x^1 - t) \partial_t \Omega g^b \neq 0 \]

which is nonvanishing in general —as long as \( \Omega \) has nontrivial dependence on \( t - x^1 \).

We still need to check that the kept Killing vectors have vanishing surface gravity on \( H \), but this must be the case due to theorem 3 because they all commute. To check it explicitly though, simply notice that for every \( \eta_i \)

\[ g(\eta_i, \eta_i) = \Omega g^b(\eta_i, \eta_i) \implies \text{grad}(g(\eta_i, \eta_i)) = \text{grad}\Omega g^b(\eta_i, \eta_i) + \Omega \text{grad}(g^b(\eta_i, \eta_i)) \mid_{t=x^1} = 0. \]

The case of maximal order, that is with \( m = n \) (so \( q = n - 1 \)), has a conformal factor

\[ \Omega(t - x^1), \]

describe conformally flat plane waves, known to be solutions of the Einstein–Maxwell equations [36] for a null electromagnetic field

\[ F = (dt - dx^1) \wedge \nu_A dx^A \]

where \( \nu_A \) are functions of \( t - x^1 \)—and more generally these are solutions of the Einstein-\( p \)-form equations for a null \( p \)-form, arising in higher dimensional theories such as supergravity. By using null coordinates

\[ U = t - x^1, \quad V = t + x^1 \]

the metric can be written in the forms

\[ g = \Omega(U) \left( -du dv + \sum_{A=2}^{n} (dx^A)^2 \right) = -du dv + \Omega(u) \sum_{A=2}^{n} (dx^A)^2 \quad (27) \]

where \( \Omega(U) du := du \). The last expression is the canonical Einstein–Rosen form of the (conformally flat) plane wave. Every null hypersurface \( u = \text{const.} \) is a fully degenerate MKH of maximal order \( m = n \) in these spacetimes.

As is well known, plane waves such as (27) can be cast (and actually extended through removable singularities arising at the zeros of \( \Omega(u) \)) in Kerr–Schild form, where the space-time is geodesically complete. The extension is given by the new set of coordinates \( \{ u, v, \xi^A \} \) defined by (an overdot means derivative with respect to \( u \))

\[ V = -2v - \frac{\dot{\Omega}}{2 \Omega} \sum_{A=2}^{n} (\xi^A)^2, \quad x^A = \frac{1}{\Omega^{1/2}} \xi^A, \]

so that (27) becomes

\[ g = 2 du dv + \Psi(u) \delta_{AB} \xi^A \xi^B du^2 + \sum_{A=2}^{n} (dx^A)^2 \]

(28) with

\[ \Psi(u) := \frac{\ddot{\Omega}}{2 \Omega} - \frac{\dot{\Omega}^2}{4 \Omega^2} \]
4.4. Ricci-flat metrics with fully degenerate MKHs

Now that we know that fully degenerate MKHs exist and can have any order, we wish to present an example of a spacetime which contains a fully degenerate MKH and solves the vacuum Einstein field equations, that is, its Ricci tensor vanishes. The previous subsections showed us that perhaps plane waves are good candidates for this purpose. Therefore, let us consider the most general vacuum (i.e. Ricci flat) plane wave, given by

\[ g = 2du dv + M_{AB}(u)z^Az^B du^2 + \sum_{A=2}^{n} (dz^A)^2, \quad \delta^{AB}M_{AB} = 0 \]

where \( M_{AB}(u) \) is a trace-free symmetric matrix of functions of \( u \). To exclude the Minkowski case we assume that \( \text{rank} (M_{AB}) \geq 1 \).

The most general Killing vector field \( \zeta \) for (29) reads

\[ \zeta = (a_0 + a_1u)\partial_u + (b - a_1\bar{v} - \dot{c}_A\bar{z}^A)\partial_v + (c_A(u) + \epsilon_{AB}\dot{z}^B)\partial_A \]

where \( a_0, a_1, b \) and \( \epsilon_{AB} = -\epsilon_{BA} \) are real constants, and \( c_A(u) \) are functions satisfying

\[ (a_0 + a_1 u)\dot{M}_{AB} + 2a_1 M_{AB} = \epsilon_{AC}M^C_B + \epsilon_{BC}M^C_A, \]

\[ \ddot{c}_A = M_{AB}\dot{c}^B \]

where \( A, B \) indices are raised with \( \delta^{AB} \). Hence, the spacetime has at least \( 2(n-1) + 1 = 2n - 1 \) Killing vectors which are determined by the parameters \( c_A(0) \) and \( \dot{c}_A(0) \), which are the initial data for the second-order ODEs (32), plus \( b \). There might be additional Killing vectors depending on whether or not \( M_{AB}(u) \) is such that (31) admits a nontrivial solution for the constants \( (a_0, a_1, \epsilon_{AB}) \).

The candidates to an MKH are the hypersurfaces \( u = \text{const} \). Without loss of generality, let us consider the null hypersurface \( \mathcal{H} := \{ u = 0 \} \), and we are interested in those Killing vectors for which this is a horizon. This will be the case if and only if \( a_0 = 0, \epsilon_{AB} = 0 \) and \( \dot{c}_A = 0 \). In that case, \( a_1 = 0 \) is also required as otherwise \( M_{AB} \propto u^{-2} \), which would be singular at \( \mathcal{H} \).

Thus the most general Killing vector in \( \mathcal{A}_\mathcal{H} \) is given by

\[ \eta = (b - \dot{c}_A\bar{z}^A)\partial_v + c_A(u)\partial_A \]

where all the \( c_A \) vanish at \( u = 0 \). In particular

\[ \eta|_\mathcal{H} = (b - \dot{c}_A\bar{z}^A)\partial_v. \]

Notice that \( g(\eta, \eta) = c_A(u)c^A(u) \) whose gradient vanishes at \( u = 0 \), and thus all the surface gravities are zero, so that \( \mathcal{A}_\mathcal{H} = \mathcal{A}_{\mathcal{H}}^{\text{deg}} \) and \( \mathcal{H} \) is a fully degenerate MKH of order \( n \).

4.5. Near horizon geometry: double (or higher) Killing horizons

Observe that in the previous example with conformally flat metric (23) we could have also kept the nondegenerate Killing \( \eta_1 \) in (18) had we also requested that \( \Omega_{\eta_1} = 0 \), and in that way we would obtain MKHs of any order and non-fully degenerate. This is actually a completely general property of fully degenerate Killing horizons of any order \( m \), in the sense that they can be promoted to non-fully degenerate MKHs of order at least \( m + 1 \). This general construction will be discussed in the next subsection, as it involves the so-called near horizon geometry which we analyze next.
The metric of any Near Horizon geometry [25], which can be thought of as the ‘focused’ local geometry near any degenerate Killing horizon, actually possesses a non-fully degenerate MKH. This can be seen from the explicit expression of the metric in local coordinates \( \{u, v, x^A\} \)

\[
g_{\text{NHG}} = 2dv \left( du + 2u s_A dx^A + \frac{1}{2} u^2 H dv\right) + \gamma_{AB} dx^A dx^B
\]  

(33)

where \( \gamma_{AB} \) is the metric on any cut \( S \subset \mathcal{H} \), \( s_A \) is a one-form on \( S \), and \( H \) a smooth function on \( S \), while the degenerate Killing horizon is given by \( \mathcal{H} = \mathcal{H}_\eta := \{u = 0\} \), where the Killing vector is \( \eta = \partial_v \). Observe that \( g(\eta, \eta) = u^2 H \) so that \( \eta \) is null on \( \mathcal{H}_\eta \) and obviously \( \kappa_\eta = 0 \).

As noted in [25], see also [28, 33], the metric (33) always has another Killing vector given by

\[
\xi = v \partial_u - u \partial_v.
\]

Obviously this Killing vector is null on \( \mathcal{H}_\eta \) and tangent to it, except at \( S := \{u = v = 0\} \) where it vanishes. Thus, \( \mathcal{H}_\xi = \mathcal{H}_\eta \setminus \{v = 0\} \) is also a Killing horizon for \( \xi \), with two connected components and \( \mathcal{H}_\xi = \mathcal{H}_\eta \), hence \( \{u = 0\} \) is (at least) a double Killing horizon. A direct calculation provides

\[
[\xi, \eta] = -\eta
\]

so that, from theorem 3 follows that \( \{u = 0\} \) is non-fully degenerate and that \( \kappa_\xi = 1 \), while \( \tau = \ln |v| \). The Killing \( \xi \) generates a bifurcate Killing horizon based on \( S \) with branches given by \( \{u = 0\} \) and \( \{v = 0\} \). Actually, a bifurcate Killing horizon is defined by any cut \( \{u = 0, v = v_0\} \) on \( \mathcal{H}_\eta \), with bifurcation Killing vector \( \xi - v_0 \eta \).

4.6. From fully to non-fully degenerate MKHs

The results of the previous subsection provide a method to generate a non-fully degenerate MKH starting from a fully degenerate one. Moreover, combining this method with the results of section 4.3 we can construct non-fully degenerate MKHs of any order explicitly.

The idea consists in taking a fully degenerate MKH of order \( m \), and then computing its near horizon geometry (33). This always provides a non-fully degenerate MKH as seen in section 4.5, but to ensure that the construction works we need to check that none of the multiple Killing vectors of the original MKH is lost in the process. And this follows from a classical and very elegant argument by Geroch [13] concerning hereditary properties when taking limits of one-parameter families of spacetimes. Geroch proved that, given a family \( \{H_{\lambda}, s_{\lambda}\}_{\lambda > 0} \) of spacetimes depending on a continuous parameter \( \lambda \) and all of them having \( g \) linearly independent Killing vectors, then the limit spacetime defined by taking the limit when \( \lambda \to 0 \) (when this limit exists) also has, at least, \( q \) linearly independent Killing vectors. This result applies to our construction because the near horizon geometry (33) is actually defined as follows: nearby a degenerate Killing horizon, there exist local Gaussian null coordinates such that the metric takes the form

\[
g = 2dv \left( du + 2u \dot{s}_A dx^A + \frac{1}{2} u^2 \dot{H} dv\right) + \ddot{\gamma}_{AB} dx^A dx^B
\]

where now \( \dot{H}, \dot{s}_A, \text{ and } \ddot{\gamma}_{AB} \) may all depend on \( u \) too: they are functions depending on \( u \) and \( x^A \). Defining the one-parameter family of metrics \( \{g_{\lambda}\}_{\lambda > 0} \) by replacing \( v \to v/\lambda \) and \( u \to u\lambda \) and taking the limit \( \lambda \to 0 \) leads to the metric (33) where \( H = \dot{H}|_{u=0}, s_A = \dot{s}_A|_{u=0} \) and \( \gamma_{AB} = \ddot{\gamma}_{AB}|_{u=0} \).
Of course, it could still happen that one of the Killing vectors was lost in the limiting process, and ‘replaced’ by the new one that the near horizon limit always adds. But this is not possible in the case where the original group of motions is Abelian, as the only possibility is a contraction of the Lie algebra in the sense of [18, 35], see [5]: a higher (or equal) number of structure constants vanish after the limit. Thus, due to theorem 3, if we start with a fully degenerate MKH of order \( m \), its near horizon geometry necessarily will have an MKH of order (at least) \( m + 1 \). This line of reasoning also proves that, for any non-fully degenerate MKH of order \( m \geq 3 \), its near horizon geometry has an MKH of order at least \( m \). Summarizing, we have established the following result.

**Theorem 4.** Let \( \mathcal{H} \) be a multiple Killing horizon of order \( m \) and \((\mathcal{M}_{\text{NHG}}, g_{\text{NHG}})\) be the nearhorizon geometry of a degenerate Killing vector \( \eta \) of \( \mathcal{H} \). Then

(i) If \( \mathcal{H} \) is fully degenerate, \((\mathcal{M}_{\text{NHG}}, g_{\text{NHG}})\) admits a multiple Killing horizon of order at least \( m + 1 \).

(ii) If \( \mathcal{H} \) is non-fully degenerate and \( m \geq 3 \), then \((\mathcal{M}_{\text{NHG}}, g_{\text{NHG}})\) has a multiple Killing horizon of order at least \( m \).

**Remark 4.** Item (ii) implies, in particular, that if \( \mathcal{H} \) is of maximal order, then any near horizon geometry that one may construct from it must also be of maximal order.

**Remark 5.** A natural question is whether the NHG spacetime \((\mathcal{M}_{\text{NHG}}, g_{\text{NHG}})\) arising from a multiple Killing horizon is independent of the choice of degenerate Killing vector \( \eta \). This question will be addressed in [30].

To illustrate the points in theorem 4, let us carry over the construction explicitly for the fully degenerate MKHs of section 4.3. Starting with the metric (23), and assuming that \( \Omega(t - x^1, x^A) \) is independent of a number \( q \geq 1 \) of the coordinates \( \{x^i\} \) so that \( \mathcal{H} := \{t = x^1\} \) is a fully degenerate MKH of order \( q + 1 \), we need to construct its near horizon geometry (33). To be explicit, we split the set of coordinates \( \{x^i\} \) into two subsets \( \{x^i\} := \{x^{A'}, y^\Upsilon\} \) and use the notation

\[
\{x^{A'}\} = \{x^i\}_{A = 2, \ldots, n - q} \quad (\implies A', B', \ldots \in \{2, \ldots, n - q\}),
\]

\[
\{y^\Upsilon\} = \{x^i\}_{A = n + 1 - q, \ldots, n} \quad (\implies \Upsilon \in \{n + 1 - q, \ldots, n\}).
\]

For the construction, we choose \( \eta = \eta_2 \) as the degenerate Killing vector (because \( \mathcal{H} = \mathcal{H} = \mathcal{H}_{\eta_2} \)) and, instead of looking for Gaussian null coordinates around the MKH, we use the fact that \( F, s_\ell \) and \( \gamma_{AB} \) have a clear geometric interpretation as follows: let \( S_0 \subset \mathcal{H} \) be the codimension-two submanifold defined by \( \{u = 0, v = \tau_0\} \) in the metric (33). Then [25]

- \( \gamma_{AB} \) is the inherited metric on \( S_0 \)
- \( s_\ell \) is the torsion one-form on \( S_0 \) defined by \( s(V) := \ell(\nabla_V \eta_2) \) for any \( V \in \mathfrak{X}(S_0) \), where \( \ell \) is the unique null vector field orthogonal to \( S_0 \) satisfying \( g(\ell, \eta_2) = -1 \).
- \( H = 2\gamma(s, s) - \text{div} s + \frac{1}{2}R|_{S_0} - \frac{1}{2}\gamma_{AB}R_{AB}|_{S_0} \)

where \( \text{div} \) is the divergence on \( S_0 \), \( R \) is the scalar curvature and \( R_{AB} \) are the \( AB \)-components of the Ricci tensor of \((\mathcal{M}, g)\), both evaluated at \( S_0 \).

Set by definition \( \Omega_0(x'') := \Omega|_{\mathcal{H}} = \Omega|_{t=x^1} \). Then the metric on \( S_0 \) reads simply

\[
\gamma = \Omega_0 \gamma^\Upsilon
\]

(34)
where $\gamma^b$ is the flat $(n-1)$-dimensional Euclidean metric. Noting that $\ell = -\frac{1}{2}(dt + dx^i)$, a straightforward calculation shows that

$$s = -\frac{1}{2}d\ln \Omega_0$$

and therefore only the components $s_{A'}$ are nonidentically vanishing. Finally, to compute $H$ we use the fact that $\eta_2$ is null everywhere for the metric (23), and this must be kept for its near horizon geometry, so that $\eta = \eta_2$ is null everywhere. But $g_{\text{NHG}}(\eta, \eta) = u^2 H$, and thus $H$ necessarily vanishes.

Hence, the near-horizon limit of (23) with $\Omega(t-x^i, x^{A'})$ leads us to the metric

$$g_{\text{NHG}} = 2dv(du - ud\ln \Omega_0) + \Omega_0 \sum_{A=2}^n (dx^A)^2,$$

with $\Omega_0(x^A)$ any arbitrary positive function independent of the $q$ coordinates $\{y^\Upsilon\}$ among the $\{x^i\}$. This metric has a non-fully degenerate MKH $\mathcal{H} := \{u = 0\}$ of order $q + 2$.

To crosscheck that the construction works fine, we can exhibit the Killing vectors generating $\mathcal{A}_H$, which are given by

$$\zeta = (av + cy^\Upsilon + b)\partial_v - au\partial_u - \frac{1}{\Omega_0} uc^\Upsilon \partial_\Upsilon$$

where $a$, $b$ and $c_\Upsilon = c^\Upsilon$ are $(q + 2)$ arbitrary constants. At $u = 0$ all of them are proportional to $\eta_2 = \partial_v$, and setting $a = 0$ we get $\mathcal{A}_H^{\text{deg}}$.

The case with $\Omega_0 = \text{const.}$ is flat spacetime, which thus arises as the near-horizon geometry of the maximal fully degenerate MKH in the conformally flat plane waves (27).

5. The master equation for MKHs

In this section we look for an equation that the proportionality function between different Killing vectors of a given MKH must satisfy. Let $\mathcal{H}$ be a multiple Killing horizon, and using the notation of remark 3 let $\eta \in \mathcal{A}_H^{\text{deg}}$ and $\xi \in \mathcal{A}_H$, so that $\kappa_\eta = 0$ and $\kappa_\xi$ can be zero or not. Given that $\xi$ is normal and non-zero on its corresponding Killing horizon $\mathcal{H}_\xi \subset \mathcal{H}$, we know that on $\mathcal{H}_\xi$ there exists a one-form $\Phi \in T_{\mathcal{H}_\xi}M$ such that (for a proof, see [12])

$$d\xi \supseteq 2\Phi \wedge \xi \tag{36}$$

or equivalently

$$\nabla_\mu \xi_\nu \supseteq \Phi_\mu \xi_\nu - \Phi_\nu \xi_\mu. \tag{37}$$

The one-form $\Phi$ cannot stay bounded at the zeros of $\xi$ in $\mathcal{H}$ (if any) because $d\xi$ and $\xi$ cannot vanish simultaneously at any point. $\Phi$ is not univocally defined, as there is the gauge freedom

$$\Phi \rightarrow \Phi + B\xi, \tag{38}$$

for an arbitrary smooth function $B : \mathcal{H}_\xi \rightarrow \mathbb{R}$. Contracting (36) with any $X \in \mathfrak{X}(\mathcal{H}_\xi)$ (and thus fulfilling the condition $\xi(X) = 0$) one obtains
\[ \nabla_X \xi \overset{\mathcal{H}_\xi}{=} \Phi(X) \xi, \quad \forall X \in \mathfrak{X}(\mathcal{H}_\xi) \]  
(39)

and, in particular, for \( X = \xi \)

\[ \nabla_\xi \xi \overset{\mathcal{H}_\xi}{=} \Phi(\xi) \xi, \quad \implies \Phi(\xi) = \kappa_\xi. \]  
(40)

Incidentally, this provides a proof of expression (3) by just squaring (37). Comparing (9) with (39) we observe that

\[ \Phi(X) = \varphi(X), \quad \forall X \in \mathfrak{X}(\mathcal{H}_\xi). \]

Furthermore, from (39) and (40)

\[ \Phi(\nabla_X \xi) = \kappa_\xi \Phi(X) = \kappa_\xi \varphi(X), \quad \forall X \in \mathfrak{X}(\mathcal{H}_\xi) \]

which, upon using the constancy of \( \Phi(\xi) \) on \( \mathcal{H}_\xi \), can be rewritten as

\[ \xi(\nabla_X \Phi + \Phi(X) \Phi) = 0 \]  
(41)

so that the vector fields \( \nabla_X \Phi + \Phi(X) \Phi \) are tangent to \( \mathcal{H}_\xi \) for arbitrary \( X \in \mathfrak{X}(\mathcal{H}_\xi) \). Another consequence of (39) is the well-known fact that \( \mathcal{H}_\xi \) is totally geodesic, i.e. that given any pair of (spacetime) vector fields \( X, Y \) tangent to \( \mathcal{H}_\xi \), the vector \( \nabla_X Y \) is also tangent to \( \mathcal{H}_\xi \). This means, in particular, that \( \nabla_X Y \) makes sense as a map

\[ \nabla : \mathfrak{X}(\mathcal{H}_\xi) \times \mathfrak{X}(\mathcal{H}_\xi) \longrightarrow \mathfrak{X}(\mathcal{H}_\xi). \]  
(42)

Similarly, for \( \eta \in \mathfrak{A}^{\text{def}}(\mathcal{H}_\eta) \) we have, on its corresponding Killing horizon,

\[ d\eta \overset{\mathcal{H}_\eta}{=} 2w \wedge \eta, \]  
(43)

for a one-form \( w \in T^* M \). As before, this one-form diverges at fixed points of \( \eta \) on \( \mathcal{H}_\eta \) (in particular, at the points \( \{ f = 0 \} \subset \mathcal{H}_\xi \)), and is defined up to the addition of an arbitrary multiple of \( \xi \)

\[ w \rightarrow w + G\xi, \quad G \in C^\infty(\mathcal{H}_\eta). \]  
(44)

A similar calculation as above leads to

\[ \nabla_X \eta \overset{\mathcal{H}_\eta}{=} w(X) \eta, \quad \forall X \in \mathfrak{X}(\mathcal{H}_\eta) \]  
(45)

and in particular, using \( \kappa_\eta = 0 \),

\[ \eta(w) = 0 \quad \implies \xi(w) = 0. \]

Hence, \( w \) is tangent to \( \mathcal{H}_\eta \) everywhere. Using (16) together with (39) and (45) the following equation follows on \( \hat{\mathcal{H}} := \mathcal{H}_\xi \cap \mathcal{H}_\eta \)

\[ \nabla_X (fe^{-\kappa_\xi \tau}) + fe^{-\kappa_\xi \tau} (\Phi(X) - w(X)) \overset{\hat{\mathcal{H}}}{=} 0, \quad \forall X \in \mathfrak{X}(\hat{\mathcal{H}}) \]

or equivalently

\[ \nabla_X f \overset{\hat{\mathcal{H}}}{=} f (w(X) + \kappa_\xi X(\tau) - \varphi(X)) \]

which provides no new information for \( X = \xi \), and it fully determines the pullback of the one-form \( w \) to \( \hat{\mathcal{H}} \); let \( \iota : \hat{\mathcal{H}} \rightarrow M \) be the inclusion of \( \hat{\mathcal{H}} \) into the manifold \( M \), and let \( \iota^* \) be its pullback, then the previous expression can be rewritten as

\[ \iota^* w = d \ln f - \kappa_\xi d\tau + \varphi \]  
(46)
where the exterior derivative should be understood as the one in \( \hat{\mathcal{H}} \) as a manifold. Observe that this relation is unaffected by the gauge (44), as \( \iota^* \xi = 0 \), and that \( \iota^* \Phi = \varphi \). Expression (46) recovers the previous result that \( w \) is ill-defined at the fixed points of \( \eta \) in \( \mathcal{H}_\xi \), where \( f \) has zeros.

Contracting (10) with \( X \in \mathfrak{X}(\mathcal{H}_\xi) \) while using (36) and (39) one gets on \( \mathcal{H}_\xi \)

\[
\langle \nabla_X \Phi + \Phi(X) \Phi \rangle \wedge \xi \overset{\mathcal{H}_\xi}{=} X^\lambda \xi_\sigma \Omega^\sigma_{\lambda} \quad \forall X \in \mathfrak{X}(\mathcal{H}_\xi)
\]

where \( \Omega^\sigma_{\lambda} := \frac{1}{2} R^\sigma_{\lambda \mu \nu} dx^\mu \wedge dx^\nu \) are the 2-forms of curvature. With indices

\[
(X^\sigma \nabla_\sigma \Phi_\mu + \Phi_\sigma X^\sigma \Phi_\mu) \xi_\nu - (X^\sigma \nabla_\sigma \Phi_\nu + \Phi_\sigma X^\sigma \Phi_\nu) \xi_\mu \overset{\mathcal{H}_\xi}{=} X^\lambda \xi_\mu \Omega^\sigma_{\mu \lambda \nu}.
\]

(47)

This implies, on the one hand (by (41))

\[
X^\lambda \xi_\mu \xi^\nu \Omega^\rho_{\mu \lambda \nu} \overset{\mathcal{H}_\xi}{=} 0, \quad \forall X \in \mathfrak{X}(\mathcal{H}_\xi)
\]

(48)

which is nothing else than (11)—as \( \kappa_\xi \) is constant—, and on the other hand, for \( X = \xi \), the existence of a function \( \mathcal{G} : \mathcal{H}_\xi \rightarrow \mathbb{R} \) such that

\[
\nabla_\xi \Phi + \kappa_\xi \Phi = \mathcal{G} \xi,
\]

which implies \( 2\mathcal{G} \kappa_\xi = 2\kappa_\xi (\Phi_\rho \Phi^\rho) + \nabla_\xi (\Phi_\rho \Phi^\rho) \). An analogous calculation starting from

\[
\nabla_\xi \nabla_\mu \eta_\nu = \eta_\mu \Omega^\nu_{\lambda \mu \nu}
\]

leads \( \forall X \in \mathfrak{X}(\mathcal{H}_\eta) \) to

\[
(X^\sigma \nabla_\sigma w_\mu + w_\sigma X^\sigma w_\mu) \eta_\nu - (X^\sigma \nabla_\sigma w_\nu + w_\sigma X^\sigma w_\nu) \eta_\mu \overset{\mathcal{H}_\xi}{=} X^\lambda \xi_\mu \Omega^\rho_{\mu \lambda \nu}.
\]

Introducing here (16) this becomes

\[
(X^\sigma \nabla_\sigma w_\mu + w_\sigma X^\sigma w_\mu) \xi_\nu - (X^\sigma \nabla_\sigma w_\nu + w_\sigma X^\sigma w_\nu) \xi_\mu \overset{\mathcal{H}_\xi}{=} X^\lambda \xi_\mu \Omega^\rho_{\mu \lambda \nu}.
\]

(49)

which is an alternative expression for the righthand side of (47). For \( X = \xi \) this gives

\[
\nabla_\xi w = \mathcal{G} \xi \quad \Rightarrow \quad \nabla_\xi (w_\mu w^\mu) = 0.
\]

Combining the two expressions (47) and (49) we get, on \( \hat{\mathcal{H}} = \mathcal{H}_\xi \cap \mathcal{H}_\eta \)

\[
Y^\mu (X^\sigma \nabla_\sigma \Phi_\mu + \Phi_\sigma X^\sigma \Phi_\mu - X^\sigma \nabla_\sigma w_\mu - w_\sigma X^\sigma w_\mu) = 0, \quad \forall X, Y \in \mathfrak{X}(\hat{\mathcal{H}}).
\]

(50)

This expression together with (46) provides a second order PDE for the function \( f \) which is the basic fundamental equation of MKHs. We call it the master equation.

### 5.1. The master equation as a PDE on any cut of \( \mathcal{H} \)

The contraction of either (47) or (49) with \( \xi^\nu \) gives no information due to (48). Similarly, contraction with two vector fields tangent to \( \mathcal{H} \) gives known information, namely that certain components of the Riemann tensor vanish on \( \mathcal{H} \). Thus, the relevant information contained in either (47) or (49) is given by contraction with a vector field transversal to \( \mathcal{H} \) everywhere, and a vector field tangent to \( \mathcal{H} \) but different from \( \xi \). Concerning the master equation (50), both vectors \( X \) and \( Y \) must be different from \( \xi \) for it to yield a non-trivial equation. To extract this information we work on \( \hat{\mathcal{H}} \), and select a scalar function \( \tau : \hat{\mathcal{H}} \rightarrow \mathbb{R} \) as in section 2, i.e. satisfying \( \xi(\tau) = 1 \). The level sets of this function defines a foliation \( \{ S_\tau \} \) of \( \hat{\mathcal{H}} \) by spacelike codimension-two surfaces. By restricting \( \hat{\mathcal{H}} \) if necessary we may assume that it admits a cross
section, i.e. a spacelike codimension-two surface crossed precisely once by each inextendable null generator. Under this assumption, the freedom (6) implies that one of the leaves of the foliation can be selected arbitrarily, and then the whole foliation is uniquely fixed. Everything that follows is valid for any such choice of τ.

Define the set of vector fields associated to the foliation \{S_\tau\}

\[ \mathcal{X}(\{S_\tau\}) := \{ V \in \mathcal{X}(\mathcal{H}), \ [\xi, V] = 0, \ V(\tau) = 0 \}. \]

Note that any vector field \(X_0\) in a given leaf \(S_\tau_0\), \(X_0 \in \mathcal{X}(\mathcal{H})\) gives rise to an element \(X \in \mathcal{X}(\{S_\tau\})\) by simply solving \([\xi, X] = 0\) with initial data \(X_0\). Conversely, any \(X \in \mathcal{X}(\{S_\tau\})\) defines a vector field \(X_0\) tangent to \(S_\tau_0\) by simply \(X_0 := X|_{S_\tau_0}\). It is immediate to check that this is an isomorphism (see [32] for further details). An easy consequence of this isomorphism is that \(\omega \in \Lambda(\mathcal{H})\) vanishes if and only if it vanishes on \(\xi\) and all \(V \in \mathcal{X}(\{S_\tau\})\). We make the statement explicit for later use

For \(\omega \in \Lambda(\mathcal{H})\):

\[ \begin{align*}
\omega(V) & = 0 \quad \forall V \in \mathcal{X}(\{S_\tau\}) \\
\omega(\xi) & = 0
\end{align*} \]

\[ \implies \omega = 0. \tag{51} \]

The following fact will also be needed. Let \(\ell \in T_M\mathcal{H}\) be a vector field in \(M\) along \(\mathcal{H}\), uniquely defined by the conditions of being null, orthogonal to \(S_\tau\) \(\forall \tau\) and satisfying \(g(\ell, \xi) = -1\). It follows immediately that

\[ \iota_\tau(\ell) = -d\tau := \hat{\ell}. \]

The Lie derivative along \(\xi\) commutes with the spacetime covariant derivative \(\nabla\), and this property descends to \(\nabla_\tau W\) when this operation is viewed as in (42). Hence \([\xi, \nabla_\tau W] = 0\) for any \(V, W \in \mathcal{X}(\{S_\tau\})\). This allows one to define a torsion-free covariant derivative \(D\) on \(\mathcal{X}(\{S_\tau\})\) by means of

\[ D_\ell W := \nabla_\tau W - K(V, W)\xi, \quad \forall V, W \in \mathcal{X}(\{S_\tau\}) \tag{52} \]

where \(K\) is the second fundamental form of \(S_\tau\) along the transverse normal \(\ell\), that is

\[ K(V, W) := -\hat{\ell}(\nabla_\tau W), \quad \forall V, W \in \mathcal{X}(\{S_\tau\}). \tag{53} \]

One has \(D_\ell W \in \mathcal{X}(\{S_\tau\})\) because (i) \(L_\xi(\nabla_\tau W) = L_\xi\hat{\ell} = L_\xi\xi = 0\), hence \(L_\xi(D_\ell W) = 0\) and (ii) \((D_\ell W)(\tau) = -\hat{\ell}(D_\ell W) = -\hat{\ell}(\nabla_\tau W) - K(V, W) = 0\). By the isomorphism above, \(D\) can also viewed as a covariant derivative of any of the submanifolds \(S_\tau\). It is immediate to check that this \(D\) is actually the Levi-Civita connection associated to the induced metric

\[ h(V, W) = g(V, W), \quad \forall V, W \in \mathcal{X}(S_\tau). \]

Note that all \((S_\tau, h)\) are isometric for Killing horizons.

Let us introduce the ring of functions \(\mathcal{F}(\{S_\tau\}) := \{ h \in \mathcal{F}(\mathcal{H}) : \xi(h) = 0 \}\). It is clear that \(\mathcal{X}(\{S_\tau\})\) is a module over \(\mathcal{F}(\{S_\tau\})\). Consider its dual module \(\mathcal{X}^*(\{S_\tau\})\), i.e. the set of \(\mathcal{F}(\{S_\tau\})\)-linear maps \(\omega : \mathcal{X}(\{S_\tau\}) \rightarrow \mathcal{F}(\{S_\tau\})\). It is a simple exercise to show that this module is isomorphic to

\[ \mathcal{X}^*(\{S_\tau\}) := \{ \omega \in \Lambda(\mathcal{H}), \ L_\xi\omega = 0, \ \omega(\xi) = 0 \}, \]

and we shall use this representation in the following. The covariant derivative \(D\) extends to the dual \(\mathcal{X}^*(\{S_\tau\})\) by the standard Leibniz rule \((D_\ell \omega)(W) := V(\omega(W)) - \omega(D_\ell W)\), where \(V, W \in \mathcal{X}(\{S_\tau\})\).
Let $\psi \in \mathfrak{T}^*_s M$ be any one-form in $M$ along $\mathcal{H}$ such that $\mathcal{L}_\xi (\iota^*_s \psi) = 0$. It follows that $\psi(\xi) \in \mathcal{F}(\{S_r\})$ because

$$\mathcal{L}_\xi (\psi(\xi)) = \mathcal{L}_\xi (\iota^*_s (\psi(\xi))) = \mathcal{L}_\xi (\iota^*_s (\psi)(\xi)) = 0.$$ 

Define a $\sim$ operation on such one-forms $\psi$ by $\tilde{\psi} := \iota^*_s \psi - \psi(\xi) d\tau$. The property $\mathcal{L}_\xi \tilde{\psi} = 0$ is immediate and, in addition,

$$\tilde{\psi}(\xi) = (\iota^*_s \psi)(\xi) - \psi(\xi)\xi(\tau) = \psi(\iota^*_s (\xi)) - \psi(\xi) = 0,$$

so $\tilde{\psi} \in \mathcal{X}^*(\{S_r\})$. A similar argument establishes

$$\tilde{\psi}(V) = \psi(V) \quad \forall V \in \mathcal{X}(\{S_r\}). \quad (54)$$

For any pair $V, W \in \mathcal{X}(\{S_r\})$ we compute

$$(\nabla \psi)(V, W) = \nabla_v (\psi(W)) - \psi(\nabla_v W) = V(\tilde{\psi}(W)) - \psi(D_v W + K(V, W)\xi)$$

$$= V(\tilde{\psi}(W)) - \tilde{\psi}(D_v W) - K(V, W)\psi(\xi)$$

$$= (D_v \tilde{\psi})(W) - K(V, W)\psi(\xi). \quad (55)$$

We want to use this construction applied to $\Phi$. Observe that $\Phi$ satisfies $\mathcal{L}_\xi (\iota^*_s (\Phi)) = 0$ because

$$\forall V \in \mathcal{X}(\{S_r\}) \quad \mathcal{L}_\xi (\iota^*_s (\Phi))(V) = \mathcal{L}_\xi (\iota^*_s (\Phi)(V)) = \mathcal{L}_\xi (\nabla_v \xi) = 0 \quad \mathcal{L}_\xi (\iota^*_s (\Phi))(\xi) = \mathcal{L}_\xi (\kappa_\xi) = 0 \quad \implies \quad \mathcal{L}_\xi (\iota^*_s (\Phi)) = 0,$$

where we used that $\mathcal{L}_\xi$ commutes with $\nabla$ and the implication is a consequence of (51). Thus $\Phi$ makes sense and in fact

$$\Phi = \varphi - \kappa_\xi d\tau := -s$$

and using (46)

$$\iota^*_s w = d \ln f - s. \quad (56)$$

By the isomorphism above, $s$ is actually the torsion one-form of each leaf $S_r$:

$$s(V) = -\varphi(V) = -d\tau(\nabla_v \xi) = \iota^*_s (\ell)(\nabla_v \xi) = \ell(\nabla_v \xi) \quad \forall V \in \mathcal{X}(\{S_r\}).$$

Now we can get the essential information contained in (47) as well as in (50). Let $\{e_A\}$ be a basis of $\mathcal{X}(\{S_r\})$. Then, contraction of (47) with $\ell^\mu e^\nu_B$ and letting $X = e_A$ we get, on using (55) and (40),

$$-D_A s_B - \kappa_\xi K_{AB} + s_A s_B = e^\lambda_A e^\nu_B R^\rho_{\lambda \mu \nu} \ell^\mu e^\nu_B.$$ 

(57)

This is actually an identity valid for any Killing horizon, be it multiple or not. Analogously, setting $X = e_A$ and $Y = e_B$ in (50) we arrive at

$$D_A D_B \ln f + D_A \ln f D_B \ln f - s_A D_B \ln f - s_B D_A \ln f + \kappa_\xi K_{AB} \tilde{H} \equiv 0 \quad (58)$$

which is a PDE nonlinear in $\ln f$. An alternative form of this PDE, linear in $f$ reads

$$D_A D_B \ln f - s_A D_B \ln f - s_B D_A \ln f + \kappa_\xi K_{AB} \tilde{H} \equiv 0. \quad (59)$$

Note that equation (58) holds in $\tilde{H}$, where $f$ is nonzero by construction. In the form (59), $f$ is allowed to vanish, so the equation extends to the whole of $\mathcal{H}_\xi$, as indicated in the equality sign.
Equation (59) is the master equation in neat form. Given that all terms are Lie constant along $\xi$, this equation is satisfied if and only if it is satisfied on any cross section $S$ of $\mathcal{H}_\xi$. Moreover, by using the Gauss identity on $S$ one easily finds that it can also be rewritten as the following equation on $S$.

$$D_A D_B f - s_A D_B f - s_B D_A f + \frac{1}{2} f \left( D_A s_B + D_B s_A + 2 s_A s_B + R_{AB} - h^{AB} R \right) \leq 0 \quad (60)$$

where $h_{AB}$ refers to the Ricci tensor of the induced metric $h$ on $S$. Given the importance of this equation we summarize the findings of this section in the following theorem.

**Theorem 5.** Let $\mathcal{H}$ be a multiple Killing horizon and select a Killing generator $\xi$ with corresponding Killing horizon $\mathcal{H}_\xi$ satisfying $\mathcal{H}_\xi = \mathcal{H}$ and surface gravity $\kappa_\xi$. Let $\eta$ be any element of $\mathcal{A}_{\text{deg}}^\mathcal{H}$, i.e. a degenerate Killing generator of $\mathcal{H}$. Pick up any cross section $S$ of $\mathcal{H}_\xi$ and define the torsion one-form $s_A$ and the transverse second fundamental form $K_{AB}$ as

$$s_A = g(\ell, \nabla e_A \xi), \quad K_{AB} = -g(\ell, \nabla e_A e_B), \quad \{e_A\} \text{ local basis of } \mathcal{X}(S)$$

where $\ell$ is the unique null normal to $S$ satisfying $g(\ell, \xi) = -1$. Then the proportionality function $f : S \to \mathbb{R}$ defined by $\eta|_S = f \xi|_S$ satisfies the master equation (59) (or equivalently (60)) where $D$, $R_{AB}$ are respectively the Levi-Civita derivative and Ricci tensor of the induced metric of $S$.

By using initial-value formulation techniques on null hypersurfaces and bifurcate horizon properties [8–11, 17] one can actually prove that, conversely, given a solution $f$ of the above equation on any cut of $\mathcal{H}_\xi$—and the appropriate initial conditions for the existence of $\xi$ and $\mathcal{H}_\xi$—, there exists a spacetime with a (non-fully degenerate) MKH for $\xi$ and (16). This will be analyzed in [31].

Expression (59) can thus be seen as a linear system of PDEs for $f$—and its trace gives an elliptic PDE on $f$. Given that it is written in normal form, any solution is determined by the values of $f$ and $D_f$ at any point $p \in \mathcal{H}$. Therefore, (59) has, at most, $(n - 1) + 1 = n$ independent solutions, which gives the maximum possible dimension for $\mathcal{A}_{\text{deg}}^\mathcal{H}$ in accordance with corollary 1. Observe that if $\kappa_\xi = 0$ then $f = 1$ is one of the solutions and $\xi$ itself is degenerate.

The precise number of independent solutions that the master equation (59) can have depends on the properties of the ambient spacetime $(M, g)$ and on the intrinsic and extrinsic geometry of the foliation $\{S_\tau\}$ for $\mathcal{H}$ via its integrability conditions. These are briefly derived in the next subsection, and the complete analysis will be presented in a subsequent paper.

### 5.2. Integrability conditions

The integrability conditions of (59) are given by the Ricci identity

$$(D_C D_A - D_A D_C) D_B f = -D_A f h^{BC} R^{DCA}$$

where $h$ is the curvature tensor of the connection $D$, which coincides with the Riemann tensor of any of the cuts $(S_\tau, h)$. A straightforward calculation using (59) leads to
\[ D_{df} \left[ h_{BCA} R_{D} - \delta_{B}^{D} (D_{A} s_{C} - D_{C} s_{A}) + s_{B} (s_{A} \delta_{C}^{D} - s_{C} \delta_{A}^{D}) - \delta_{C}^{D} D_{A} s_{B} + \delta_{A}^{D} D_{C} s_{B} \right] \\
+ \kappa \xi \left( \delta_{K}^{D} K_{CB} - \delta_{K}^{D} K_{AB} \right) \right] + \kappa \xi f \left( D_{A} K_{CB} - D_{C} K_{AB} - s_{A} K_{CB} + s_{C} K_{AB} \right) = 0 \]  
(61)

which can be rewritten, on using (57), as

\[ D_{df} \left[ h_{BCA} R_{D} - \delta_{B}^{D} (D_{A} s_{C} - D_{C} s_{A}) + (\delta_{C}^{D} e_{A}^{\lambda} - \delta_{A}^{D} e_{C}^{\lambda}) \xi_{B} R_{\lambda \rho \mu \nu} e_{\mu} e_{\nu} \right] \\
+ \kappa \xi f \left( D_{A} K_{CB} - D_{C} K_{AB} - s_{A} K_{CB} + s_{C} K_{AB} \right) \right] \right] = 0. \]  
(62)

Using here for the last term in brackets the Codazzi equation for the foliation \( \{ S_{\tau} \} \) we can still write

\[ D_{df} \left[ h_{BCA} R_{D} - \delta_{B}^{D} (D_{A} s_{C} - D_{C} s_{A}) + (\delta_{C}^{D} e_{A}^{\lambda} - \delta_{A}^{D} e_{C}^{\lambda}) \xi_{B} R_{\lambda \rho \mu \nu} e_{\mu} e_{\nu} \right] \\
+ \kappa \xi f \left( D_{A} K_{CB} - D_{C} K_{AB} - s_{A} K_{CB} + s_{C} K_{AB} \right) \right] \right] = 0. \]  

Every MKH lives in a spacetime such that this is satisfied by the function \( f \) defined by (4) and (8) for \( \eta \in A_{deg}^{H} \) (or, equivalently, by the proportionality function \( \eta_{S}^{f} \xi_{S}^{f} \) on the cross section \( S = \{ \tau = 0 \} \)). In particular, the maximum dimension of \( A_{deg}^{H} \) is attained whenever the previous condition holds identically, that is, for any values of \( f \) and \( D_{df} \). In other words, when the term in brackets vanishes and the factor multiplying \( f \) does too. This allows us to analyze in detail the spacetimes with (fully degenerate or not) MKHs of maximal order, while (62) still contains useful information in the cases with other values of the order \( m \).

6. Classification of MKHs in maximally symmetric spacetimes

In this section we study the multiple Killing horizons in the (A)-de Sitter and Minkowski spacetimes of arbitrary dimension \( n + 1 \) at least two. Among other things, we show that any point \( p \) in these spacetimes is contained in a multiple Killing horizon of maximal order \( n + 1 \).

We start with the (A)dS case, which requires a machinery that can then be applied to the Minkowski case.

6.1. The (A)dS case

The (anti)-de Sitter space of curvature radius \( a > 0 \), denoted by (A)dS\(_{a}^{n+1} \), is the maximally extended and simply connected \( (n+1) \)-dimensional \((n \geq 1)\) Lorentzian manifold of constant curvature \( K = \frac{\epsilon}{a^2} \) where \( \epsilon = 1 \) in the de Sitter case and \( \epsilon = -1 \) in the anti-de Sitter case. We intend to give the full classification of MKHs in these spaces. From theorem 3 we know that any such MKH has \( dim A_{deg}^{H} \geq m - 1 \), where \( m \geq 2 \) is the order of the MKH, so that to classify the MKHs it suffices to determine all degenerate Killing horizons, and then find which of those are multiple.

To that aim, it is convenient to view (A)dS\(_{a} \) as an embedded hypersurface in a higher-dimensional flat space. More specifically, let \( M^{p,q} \) be the simply connected, complete pseudo-Riemannian manifold of vanishing curvature and signature \((p,q)\). We assume \( p + q = n + 2 \) and select a Cartesian coordinate system \( \{ x^{\alpha'} \} \) \((\alpha', \beta' \cdots = 0, \cdots, n + 1)\)
which will stay fixed from now on. The components of the flat metric $g^\flat$ in these coordinates are $g^\flat_{\alpha\beta} = \text{diag}\{-1, \ldots, -1, +1, \ldots, +1\}$.

We shall consider the two cases at the same time. Recall that $\epsilon := \pm 1$, and fix $2p = 3 - \epsilon$, i.e. when $\epsilon = 1$ we work with the $(n + 2)$-dimensional Minkowski space $M^{1,n+1}_\epsilon$ and when $\epsilon = -1$ we have $M^{2,n}$. Denote them collectively by $M^{n+2}_\epsilon$. There exists an isometric immersion of $(A)dS^{n+1}_u$ into $M^{n+2}_\epsilon$ whose image is

$\Sigma_o := \{ x \in M^{n+2}_\epsilon, \quad \langle x, x \rangle = \epsilon a^2 \}$,

where $(\cdot, \cdot)$ denotes scalar product with $g^\flat$ and we are making use of the affine structure of $M^{n+2}_\epsilon$, which makes it into a vector space with inner product $g^\flat$. When $\epsilon = 1$ the immersion is in fact a proper embedding. When $\epsilon = -1$, there is a covering map $\pi : \text{AdS}^{n+1}_u \rightarrow \text{AdS}^{n+1}_u$ onto a space which is diffeomorphic to $\Sigma_o$. Thus, the MKHs in $(A)dS^{n+1}_u$ can be studied by considering their images in $\Sigma_o$.

The algebra of Killing vectors of $(A)dS^{n+1}_u$ can be obtained by restriction in $\Sigma_o$ of the set of Killing vectors in $M^{n+2}_\epsilon$ which leave the origin $o \in M^{n+2}_\epsilon$ invariant, given by

$\zeta_F|_x = F^\flat(x)$

where $F^\flat := M^{n+2}_\epsilon \rightarrow M^{n+2}_\epsilon$ is a skew symmetric linear map, i.e. satisfying for all $x, y \in M^{n+2}_\epsilon$

$\langle F^\flat(x), y \rangle = -\langle x, F^\flat(y) \rangle$.

Given a nonzero vector $Z \in M^{n+2}_\epsilon$ we define $\langle Z \rangle^\perp$ to be the hyperplane orthogonal to $Z$ passing through the origin, i.e. the set of points $\{ x \in M^{n+2}_\epsilon : \langle Z, x \rangle = 0 \}$. A point $Z \in M^{n+2}_\epsilon$ is called respectively timelike, null or spacelike if $\langle Z, Z \rangle$ is negative, zero or positive.

We can now state our main result concerning degenerate Killing horizons in $(A)dS^{n}_u$.

**Theorem 6.** Let $\mathcal{H}$ be a degenerate Killing horizon of $(A)dS^{n+1}_u$. Then there exists a null, nonzero vector $k \in M^{n+2}_\epsilon$ such that $\mathcal{H}$ is a subset of the intersection of $\Sigma_o$ with the hyperplane $k^\perp \subset M^{n+2}_\epsilon$. Moreover, the set of Killing vectors which respect to which an open and dense subset of $\mathcal{H}$ is a degenerate Killing horizon is given by the restriction to $\Sigma_o$ of $\zeta_F$ with

$F^\flat = k \otimes w - w \otimes k$

and $w \in M^{n+2}_\epsilon$ is a vector linearly independent of $k$ and satisfying $\langle k, w \rangle = 0$. Conversely, for any pair $\{ k, w \}$ as before, the Killing vector $\zeta := \zeta_F|_\Sigma_o$ admits a degenerate Killing horizon given by the hypersurface

$\mathcal{H}_\zeta := \{ x \in \Sigma_o \cap \langle k \rangle^\perp \text{ such that } \langle w, x \rangle \neq 0 \}$

or any open subset thereof.

**Remark 6.** When $\epsilon = 1$, $w$ must be spacelike because a causal vector perpendicular to $k$ cannot be linearly independent of $k$. When $\epsilon = -1$, there is no such restriction and $w$ is allowed to have any norm (including null).

The proof of this theorem is somewhat long, and requires several results on skew symmetric linear maps on pseudo-Riemannian vector spaces. We devote appendix B to establishing the necessary lemmas and give the proof.
With this theorem above at hand, it is easy to determine the MKHs in $(\mathcal{A})dS^{n+1}_{a}$.  

**Theorem 7.** Let $(\mathcal{A})dS^{n+1}_{a}$ be the $(\mathcal{A})$-de Sitter spacetime of dimension $n + 1 \geq 2$ and view this as a hypersurface in $M^{n+2}_{a}$ as described above. A null hypersurface $\mathcal{H}$ embedded in $(\mathcal{A})dS^{n+1}_{a}$ is a multiple Killing horizon if and only if $\mathcal{H}$ is an open subset of the hypersurface $(k) \perp \cap \Sigma_{a}$ where $k \in M^{n+2}_{a}$ is nonzero and null. Moreover, $A_{\mathcal{H}}$ is generated by the restriction to $\Sigma_{a}$ of the Killing vectors in $M^{n+2}_{a}$ $\zeta_{F_{k},z}(x) = F^{\xi}_{k,z}(x), x \in M^{n+2}_{a}$ with $F^{\xi}_{k,z} := M^{n+2}_{a} \rightarrow M^{n+2}_{a}$ given by

$$F^{\xi}_{k,z} := k \otimes Z - Z \otimes k, \quad Z \in M^{n+2}_{a}. \quad (63)$$

**Remark 7.** The collection of vectors $\zeta_{F_{k},z}$ is obviously a vector subspace of all Killing vectors in $M^{n+2}_{a}$ leaving invariant the origin of $M^{n+2}_{a}$, in agreement with theorem 2. Define the equivalence relation, $Z \sim Z' \Leftrightarrow Z = Z' \in \text{span}(k)$. The quotient space, denoted $M^{n+2}_{a}/k$, is clearly an $(n + 1)$-dimensional vector space. It turns out that $A_{\mathcal{H}}$ is isomorphic to $M^{n+2}_{a}/k$. Indeed, define the map

$$\Psi : M^{n+2}_{a}/k \rightarrow A_{\mathcal{H}}$$

$$Z \rightarrow \zeta_{F_{k},z}|_{\Sigma_{a}}$$

where $Z$ is any representative in the equivalence class $Z$. This map is well defined (i.e. independent of the representative chosen in the class) because for $Z' = Z + ck, c \in \mathbb{R},$

$$F^{\xi}_{k,z} = k \otimes Z' - Z' \otimes k = k \otimes (Z + ck) - (Z + ck) \otimes k = k \otimes Z - Z \otimes k = F^{\xi}_{k,z}$$

and the Killing vector $\zeta_{F_{k},z'} = \zeta_{F_{k},z}$. The map is obviously linear. It is also a bijection because $\zeta_{F_{k},z} = \zeta_{F_{k},z'}$ agree on $\Sigma_{a}$ if and only if they agree everywhere, i.e. $F^{\xi}_{k,z} = F^{\xi}_{k,z'}$, or, explicitly,

$$k \otimes (Z' - Z) = (Z' - Z) \otimes k = 0. \quad (64)$$

This clearly holds if and only if $Z' - Z$ proportional to $k$. We therefore conclude that the dimension of $A_{\mathcal{H}}$ is $n + 1$.

**Proof.** $\mathcal{H}$ has an open and dense subset $\mathcal{H}_{z}$ which is a degenerate Killing horizon of $(\mathcal{A})dS^{n+1}_{a}$ associated to the Killing vector $\zeta$. By theorem 6, this occurs if and only if there exists $k \in M^{n+2}_{a}$ null and nonzero such that $\mathcal{H}$ is an open subset of $\Sigma_{a} \cap (k) \perp$. This proves the first part of the theorem.

In order to identify $A_{\mathcal{H}}$, let $\mathcal{H}_{z}$ be a Killing horizon (not necessarily degenerate) such that $\overline{\mathcal{H}}_{z} = \overline{\mathcal{H}}_{\zeta} = \mathcal{H}$. Since $\Sigma_{a} \cap (k) \perp$ is closed, we also have $\overline{\mathcal{H}}_{\zeta} \subset \Sigma_{a} \cap (k) \perp$. Let $F^{\xi}_{z}$ be the endomorphism in $M^{n+2}_{a}$ such that $\xi_{\mid x} = F^{\xi}_{z}(x), \forall x \in \Sigma_{a}$. Up to scaling, $k$ is the only normal to $\mathcal{H}_{z} \subset \Sigma_{a}$. Thus, it must be that at any point $x \in \mathcal{H}_{z}, F^{\xi}_{z}(x) = Z_{\mid x}k$ holds, where $Z_{\mid x}$ is a nonzero real number (it may depend on $x \in \mathcal{H}_{z}$). Since $\mathcal{H}_{z}$ is an open subset of $\Sigma_{a} \cap (k) \perp$, it follows that $\text{span}(\mathcal{H}_{z}) = (k) \perp$. By linearity of $F^{\xi}_{z}$ it follows

$$F^{\xi}_{z}(w) = Z_{\mid x}k, \quad \forall w \in (k) \perp. \quad (65)$$
We may apply lemma B.7 to conclude that $F^\#:\xi$ is given as in (63) and hence any $\xi \in \mathcal{A}_H \setminus \{0\}$ must be the restriction to $\Sigma_\alpha$ of $\zeta_{F^\#,\xi}$, as claimed in the theorem. Conversely, any $F^\#:\xi$ of this form with $Z$ and $k$ linearly independent defines a Killing vector in $\mathbb{M}^{p+2}$ which, when restricted to $\Sigma_\alpha \cap (k)\perp$ gives a null, tangent vector. Combined with the fact that when $Z = \alpha k$, $\alpha \in \mathbb{R}$ we have $F^\#:\xi = 0$ and therefore $\zeta_{F^\#,\xi} = 0$ we conclude that

$$\mathcal{A}_H = \{\zeta_{F^\#,\xi}\}$$

and the theorem is proved.

\[\Box\]

**Remark 8.** The Killing horizon of $\zeta_{F^\#,\xi}|_{\Sigma_\alpha}$ is (any open subset of) $H_{k,Z} = \{x \in \Sigma_\alpha \cap (k)\perp \text{such that } \langle Z, x \rangle \neq 0\}$.

To compute the surface gravity we first note that the square norm of $\zeta_{F^\#,\xi}$ is

$$\langle Z, Z \rangle \langle k, x \rangle^2 - 2\langle k, Z \rangle \langle k, x \rangle \langle Z, x \rangle$$

whose gradient evaluated at $x \in H_{k,Z}$ reads

$$2\langle k, Z \rangle \langle Z, x \rangle k.$$

Given that (at such $x$) $\zeta_{F^\#,\xi}|_x = k(Z, x)$ we conclude from (1)

$$\kappa_{H_{k,Z}} = \langle k, Z \rangle.$$

Note that when $Z$ and $k$ are orthogonal, the surface gravity is zero and we recover the degenerate Killing horizon of theorem 6.

### 6.2. The Minkowski case

Using the same notation as above, the general Killing vector $\zeta$ of $\mathbb{M}^{1,n}$ is

$$\zeta_{z,F^\#}|_x = z + F^\#(x)$$

where $z \in \mathbb{M}^{1,n}$, $F^\# : \mathbb{M}^{1,n} \to \mathbb{M}^{1,n}$ is a skew-symmetric endomorphism. As in the previous subsection, we start with the classification of degenerate Killing horizons.

**Theorem 8.** Let $\mathcal{H}$ be a degenerate Killing horizon of a Killing vector $\zeta$ in $\mathbb{M}^{1,n}$. Then, and only then, one of the two following possibilities hold:

(a) There exists $z, z' \in \mathbb{M}^{1,n}$ with $z$ null and nonzero such that $\zeta = z$ and $\mathcal{H}$ is an open subset of the hyperplane $H_{z,z'} := z' + (z)\perp$.

(b) There exist $A \in \mathbb{R}$ and $k, w, z' \in \mathbb{M}^{1,n}$ with $\{k, w\}$ linearly independent, $k$ null, $w$ space-like and orthogonal to $k$ and $z'$ arbitrary, such that

$$\zeta|_x = Ak + k\langle w, x \rangle - w\langle k, x - z' \rangle$$

and $\mathcal{H}$ is an open subset of the hypersurface

$$z' + (k)^\perp \setminus S_w$$

27
where \( S_w \) is the closed, codimension-two null plane defined by
\[
S_w := -A \langle w, w \rangle^{-1} w + \text{span}(k, w)^\perp .
\]  

\textbf{Proof.} Let \( \lambda_\zeta := -g^b(\zeta, \zeta) \). The degenerate Killing horizon \( \mathcal{H} \) must be a subset of \( \{ \lambda_\zeta = 0 \} \cap \{ \text{grad}(\lambda_\zeta) = 0 \} \). Let \( z, F^z \) be such that \( \zeta = \zeta_{F^z} \). The square norm of \( \zeta \) is
\[
-\lambda_\zeta = \langle z, z \rangle + 2 \langle x, F^z(x) \rangle (F^z(x), F^z(x))
\]
and the gradient
\[
-\text{grad}(\lambda_\zeta) = -2 F^z(z + F^z(x)).
\]

This implies that for any \( x_1, x_2 \in \mathcal{H} \)
\[
(F^z \circ F^z)(x_1 - x_2) = 0
\]
holds, so \( x_1 - x_2 \) belongs to the kernel of \( F^z \circ F^z \). In particular, the tangent space \( T_x \mathcal{H} \) at any \( x \in \mathcal{H} \) must satisfy
\[
T_x \mathcal{H} \subset \text{Ker}(F^z \circ F^z).
\]

Since \( T_x \mathcal{H} \) is \( n \)-dimensional it must be that \( \dim(\text{Ker}(F^z \circ F^z)) \) is either \( n \) or \( n + 1 \). In the latter case, called (a) lemma B.5 implies \( F^z = 0 \), so that \( \zeta = z \) with \( \langle z, z \rangle = 0 \). Thus, \( \zeta \) is null and nonzero everywhere and \( \mathbb{M}^{1,n} \) is foliated by Killing horizons of \( \zeta \) defined as the hypersurfaces orthogonal to \( z \), i.e. the hyperplanes \( z' + (\zeta)^\perp \), \( z' \in \mathbb{M}^{1,n} \). This proves case (a) of the theorem.

Consider next case (b), defined by the condition that \( F^z \circ F^z \) has rank one, or equivalently, there is \( k \in \mathbb{M}^{1,n} \) nonzero such that \( F^z \circ F^z = \mu k \otimes k, \mu \neq 0 \). The kernel of \( F^z \circ F^z \) (namely \( (k)^\perp \)) must contain the null hyperplane \( T_x \mathcal{H}, x \in \mathcal{H} \), so \( k \) must be null and \( T_x \mathcal{H} = \langle k \rangle^\perp \) for all \( x \in \mathcal{H} \). Thus, \( \mathcal{H} \) must be a subset of one of the hyperplanes normal to \( k \). In other words, there is \( z' \in \mathbb{M}^{1,n} \) such that \( \mathcal{H} \) is an open subset of \( \mathcal{H}_{\ell, k} := z' + \langle k \rangle^\perp \). To impose the condition that \( \zeta \) is null and tangent to \( \mathcal{H} \), we need the form of \( F^z \). We apply lemma B.6 and find that there exists \( w \in \mathbb{M}^{1,n} \), orthogonal to, and linearly independent of, \( k \) such that
\[
F^z = k \otimes w - w \otimes k.
\]

Note that in Lorentzian signature \( w \) is necessarily spacelike, so \( \mu = -\langle w, w \rangle < 0 \). Evaluating \( \zeta \) at \( x \in \mathcal{H} \subset \mathcal{H}_{\ell, k} \) one finds
\[
\zeta|_x = z + F^z(x) = z + k\langle w, x \rangle - w\langle k, x \rangle = z + k\langle w, x \rangle - w\langle k, z' \rangle.
\]

This vector is proportional to the normal of \( \mathcal{H} \) (i.e. to \( k \)) if and only if \( z = w\langle k, z' \rangle + Ak \) for some \( A \in \mathbb{R} \). This shows (67). To prove (68) we simply note that \( \zeta|_x \), as given in (67) vanishes at \( x \in \mathcal{H}_{\ell, k} \) if and only if \( A + \langle w, x \rangle = 0 \). Write \( x = -A\langle w, w \rangle^{-1}w + y \) and this condition becomes \( \langle w, y \rangle = 0 \), as claimed in the proposition. The ‘only then’ part in case (b) is immediately checked.

We can now classify the MKHs in the \((n + 1)\)-dimensional Minkowski spacetime.
Theorem 9. Let $\mathbb{M}^{1,n}$ be the Minkowski spacetime of dimension $n + 1 \geq 2$. A null hypersurface $\mathcal{H}$ embedded in $\mathbb{M}^{1,n}$ is a multiple Killing horizon if and only if $\mathcal{H}$ is an open subset of a hyperplane $z' + \{k\}$ with $z', k \in \mathbb{M}^{1,n}$ and $k$ is null and nonzero. Moreover, $\mathcal{A}_H$ is given by

$$\mathcal{A}_H = \{ \zeta_{AZ}|_x = Ak + Z(k, z') + k\langle Z, x \rangle - Z\langle k, x \rangle, \quad A \in \mathbb{R}, Z \in \mathbb{M}^{1,n} \}.$$  

Proof. $\mathcal{H}$ has an open and dense subset $\mathcal{H}_\xi$ which is a degenerate Killing horizon of $\mathbb{M}^{1,n}$. By theorem 8 we know that $\mathcal{H}_\xi$ is an open subset of a hyperplane $\mathcal{H}_\xi := z' + \{k\}$ where $k \neq 0$ is null. To show that $\mathcal{H}$ is a multiple horizon (and also to determine $\mathcal{A}_H$) we need to find the most general Killing vector $\zeta_{FS}^{\prime}$ admitting a Killing horizon, denoted by $\mathcal{H}_{\xi,FS}$ such that $\mathcal{H}_{\xi,FS} = \mathcal{H}_{\xi}$ and $\mathcal{H}_{\xi,FS}$ is an open subset of $z' + \{k\}$, so the condition that $\zeta$ is null and tangent to $\mathcal{H}_{\xi,FS}$ on $\mathcal{H}_{\xi,FS}$, namely

$$\zeta_{FS}|_x = z + F^a(x) = f|_k k, \quad \forall x \in \mathcal{H}_{\xi,FS}$$

must hold. By linearity this relation extends to all $c + \{k\}$. Thus, for all $X \in \{k\}$

$$F^a(X) = f|_k k - F^a(z') - Z$$

holds. This applies, in particular to $X = 0$ from which $z = f|_0 k - F^a(z')$ and thus

$$F^a(X) = (f|_X - f|_0) k, \quad \forall X \in \{k\}$$

which allows us to conclude that there is $Z \in \mathbb{M}^{1,n}$ such that

$$F^a = k \otimes Z - Z \otimes k.$$  

Note that this implies $z = f|_0 k - F^a(z') = f|_0 k - k\langle Z, z' \rangle + Z\langle k, z' \rangle = Ak + Z\langle k, z' \rangle$, after redefining $A := f|_0 - \langle Z, z' \rangle$. We have proved the inclusion

$$\mathcal{A}_H \subset \{ \zeta_{k, x} = Ak + Z\langle k, z' \rangle + k\langle Z, x \rangle - Z\langle k, x \rangle, \quad A \in \mathbb{R}, Z \in \mathbb{M}^{1,n} \}.$$  

The reverse inclusion (and hence equality) is immediate, since the Killing vector $\zeta_{AZ}$ (with obvious notation) is tangent and null at the hyperplane $z' + \{k\}$ and vanishes only on the lower dimensional subset

$$S_{AZ} := \{ Y \in z' + \{k\}; A + \langle Z, Y \rangle = 0 \}. \quad \Box$$

Remark 9. Two Killing vectors $\zeta_{AZ}$ and $\zeta_{A'}Z'$ agree if and only if $Z' - Z = ak$ and $A' = A - a\langle k, z' \rangle$, for some arbitrary constant $a$. Thus, the dimension of $\mathcal{A}_H$ is $a + 1$.

The surface gravity of the Killing horizon associated to $\zeta_{AZ}$ is computed easily as follows

$$-\lambda_{AZ} := g^a(\zeta_{AZ}, \zeta_{AZ}) = \langle Z, Z \rangle (k, z' - x)^2 + 2\langle k, Z \rangle (A + \langle Z, x \rangle)(k, z' - x)$$

so its gradient is

$$\text{grad}(\lambda_{AZ}) = 2\langle Z, Z \rangle (k, z' - x)k - 2\langle k, Z \rangle (k, z' - x)Z + 2\langle k, Z \rangle (A + \langle Z, x \rangle)k,$$

which evaluated on $z' + \{k\}$ gives
\[ \text{grad}(\lambda_{A,Z})\big|_{\mathcal{C}^+(k)^\perp} = 2\langle k, Z \rangle (A + \langle Z, x \rangle) k = 2\langle k, Z \rangle \zeta_{A,Z}\big|_{\mathcal{C}^+(k)^\perp} \]

and the surface gravity is \( \kappa_{A,Z} = \langle k, Z \rangle \).

7. Summary, conclusions and outlook

In this paper we have introduced the notion of a multiple Killing horizon and have investigated its main general properties. It turns out that to any MKH \( \mathcal{H} \) in a spacetime \( (M, g) \) one can attach a Lie-subalgebra \( \mathcal{A}_H \) of the Killing algebra of \( (M, g) \) of dimension between two and the dimension of \( M \). Moreover, this Lie subalgebra always includes an Abelian subalgebra \( \mathcal{A}^{\text{deg}}_H \) of codimension at most one. All nontrivial elements in \( \mathcal{A}_H \) admit a Killing horizon whose closure agrees with \( \mathcal{H} \). The surface gravity of each one of these horizons is constant irrespectively of energy conditions, or any other restrictions on the spacetime or the horizon. The surface gravity of any element in \( \mathcal{A}^{\text{deg}}_H \) is necessarily zero, so one can attach at most one nonzero surface gravity to any multiple Killing horizon. This may have implications for the thermodynamic properties of these objects. By means of examples we have shown that all allowed dimensions of \( \mathcal{A}_H \) and of \( \mathcal{A}^{\text{deg}}_H \) are feasible. As examples where the maximal dimension is realized we have the Minkowski, de Sitter and anti-de Sitter spacetimes and a complete classification of their multiple Killing horizons has been provided. A particularly relevant example of multiple Killing horizons are the horizons in ‘near horizon geometry’ spacetimes. In particular, we have shown that if one starts with any multiple Killing horizon and performs the limiting process that yields its near horizon geometry with respect to any of its degenerate generators, the resulting spacetime has a degenerate Killing subalgebra \( \mathcal{A}^{\text{deg}}_H \) of at least the same dimension than the original spacetime. In addition, the ‘symmetry enhancement’ characteristic of this limiting process yields a nondegenerate generator \([25, 28, 33]\). This behaviour opens up a number of interesting questions concerning the potential dependence of the near horizon geometry with respect to the choice of degenerate generator, when the original Killing horizon is multiple with \( \mathcal{A}^{\text{deg}}_H \) of dimension at least two.

One of the main results is the so-called master equation. This is an overdetermined linear PDE that must be necessarily satisfied by the proportionality function \( \eta|_S = f \xi|_S \) between any degenerate generator \( \eta \) and an (arbitrarily) preselected element \( \xi \) of \( \mathcal{A}_H \) (degenerate or not). The master equation is a PDE on \( S \), which is any cross section of the Killing horizon \( \mathcal{H}_\xi \) of \( \xi \) satisfying \( \mathcal{H}_\xi = \mathcal{H} \). The master equation gives necessary conditions for a Killing horizon \( \mathcal{H}_\xi \) to be multiple. A task that needs to be done is to analyze in detail the consequences of this equation. In this paper we have only given a preliminary analysis of the first integrability conditions. We plan to expand this in a future paper.

A main issue concerning the master equation is to find conditions under which it is also sufficient. As already mentioned, one can use \([31]\) a characteristic initial value formulation of the Einstein vacuum field equations with cosmological constant (including zero) to show that, under suitable circumstances, the master equation is also sufficient for the existence of multiple Killing horizons.

Multiple Killing horizons may have interesting implications in the physics of black holes in higher dimensions. For instance a natural question is whether there exist any asymptotically flat (or (anti)-de Sitter) black hole in equilibrium in dimension higher than four for which its event horizon is a multiple Killing horizon. The plethora of examples in dimension five, and the fact that even more cases are expected in higher dimensions makes it possible, at least \emph{a priori}, that such examples may exist. If that were the case, studying issues like Hawking...
radiation in those spacetimes would be highly interesting, particularly if examples exist with non-fully degenerate multiple Killing horizons, where the surface gravity is nonzero for one generator and zero for all others, in a suitable chosen basis.

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Appendix A. Lower bound on codimension of fixed-point sets for Killing vectors

Here we recall the following well-known fact, which we nevertheless prove for completeness. A Killing vector is nontrivial if it is not the zero vector field.

Lemma A.1. Let $(M, g)$ be an $(n+1)$-dimensional spacetime and $\xi$ a nontrivial Killing vector. Then the set of zeros of $\xi$ has codimension at least two.

Proof. We consider the relevant case $n \geq 1$. We know that the zeros of a Killing vector form a finite collection of smooth embedded submanifolds $S_i$ [22]. Let $p$ be a point in one of them, say $S_1$ and assume that $\text{dim}(S_1) \geq n$. Let $G^2$ be the endomorphism $T_pM \rightarrow T_pM$ defined by $g(G^2(Z), Z') = d\xi[Z, Z']$, $\forall Z, Z' \in T_pM$. Since $d\xi[p]$ is a two-form in $T_pM$, $G^2$ is skew symmetric with respect to $g[p]$. The tangent plane $T_pS_1$ lies in the kernel of $G^2$, so its dimension is at least $n$, or equivalently $\text{rank}(G^2) \in \{0, 1\}$. If $\text{rank}(G^2) = 1$ then $G^2 = k \otimes a$ for some vector $k \in T_pM$ and some one-form $a \in \Lambda^1_pM$, which is clearly incompatible with the skew-symmetry of $G^2$ —as $g$ is nondegenerate— unless $a = 0$. Thus, $G^2 = 0$, i.e. $d\xi[p] = 0$. This immediately implies that $\xi$ is a trivial Killing vector. 

The previous theorem can be considered to hold for $n = 0$ too if the statement is understood as saying that $\xi$ cannot have zeros. For assume $p \in M$ were a fixed point of $\xi$ and select a coordinate chart $\{x\}$ containing $p$, with $x_p = x(p)$. The metric can be written as $g = j(x)dx^2$, with $j$ nonzero in the domain of the chart. The Killing could be written as $\xi = l(x)\partial_x$ with $l(x_p) = 0$. The condition of being a Killing vector is

$$\mathcal{L}_\xi(g) = 0 \quad \iff \quad iljdx - 2jdlx = 0.$$ 

Since $l(x_p) = 0$, uniqueness of solutions of ODE would imply $l(x) = 0$ everywhere, so the Killing would be trivial.

Appendix B. Proof of theorem 6

In order to prove theorem 6 we need several algebraic lemmas on skew symmetric linear maps. Several of these results are likely to be known in the mathematics literature, but they are not standard knowledge in the relativity community (given that they involve various signatures). So we provide a proof for completeness.
Lemma B.2. Let \((V, g^0)\) be an \(n\)-dimensional vector space and \(g^b\) a pseudo-Riemannian inner product of signature \(\{p, q\}\). Let \(\Pi\) be a linear subspace with the property that \(g^b\) restricted to \(\Pi\) is identically zero (we call such spaces totally degenerate). Then the dimension of \(\Pi\) is bounded above by \(\min(p, q)\), and this bound is sharp.

Proof. By interchanging \(g^b\) with \(-g^b\), we may assume without loss of generality that \(p \leq q\). Let \(\{e_i\}\) be an orthonormal basis of \((V, g^b)\) and consider the vector space \(\Pi_0 = \text{span}(e_1 + e_{p+1}, e_2 + e_{p+2}, \ldots, e_p + e_{2p})\), which has dimension \(p\). Since 
\[
\langle e_i + e_{p+i}, e_j + e_{p+j} \rangle = \langle e_i, e_j \rangle + \langle e_{p+i}, e_{p+j} \rangle = \delta_{ij} + \delta_{ij} = 0,
\]
the restriction \(g^b|_{\Pi_0}\) is identically zero. Thus, the upper bound claimed in the lemma is attained.

It remains to show that any totally degenerate vector subspace \(\Pi\) satisfies \(\dim(\Pi) \leq p\). We argue by contradiction, so let \(\Pi\) by a totally degenerate space of dimension \(p + 1\) and \(\{v_1, \ldots, v_{p+1}\}\) a basis of \(\Pi\). The orthogonal decomposition \(V = \text{span}\{e_1, \ldots, e_p\} \oplus \{e_{p+1}, \ldots, e_{p+q}\}\) allows us to decompose any \(v \in V\) as \(v = v^\parallel + v^\perp\). It is clear that \(\{v_1^\parallel, \ldots, v_{p+1}^\parallel\}\) is a linearly dependent subset. By reordering vectors if necessary we may assume that \(v_{p+1}^\parallel = \sum_{i=1}^p a_i v_i^\parallel\). The fact that \(g^b|_{\Pi} = 0\) implies, for all \(a, b, 1, \ldots, p + 1\),
\[
0 = \langle v_a, v_b \rangle = \langle v_a^\parallel + v_a^\perp, v_b^\parallel + v_b^\perp \rangle = \langle v_a^\parallel, v_b^\parallel \rangle + \langle v_a^\perp, v_b^\perp \rangle \iff \langle v_a^\perp, v_b^\perp \rangle = -\langle v_a^\parallel, v_b^\parallel \rangle. (B.1)
\]

Let us compute
\[
\langle v_{p+1}^\parallel, v_{p+1}^\perp - \sum_{i=0}^p a_i v_i^\perp, v_{p+1}^\parallel - \sum_{i=0}^p a_i v_i^\perp \rangle = \langle v_{p+1}^\parallel, v_{p+1}^\perp \rangle - 2 \sum_{i=1}^p a_i \langle v_{p+1}^\parallel, v_i^\perp \rangle + \sum_{i=1}^p \sum_{j=1}^p a_i a_j \langle v_i^\perp, v_j^\perp \rangle
\]
\[
= -\langle v_{p+1}^\parallel, v_{p+1}^\perp \rangle + 2 \sum_{i=1}^p a_i \langle v_{p+1}^\parallel, v_i^\perp \rangle - \sum_{i=1}^p \sum_{j=1}^p a_i a_j \langle v_i^\perp, v_j^\perp \rangle
\]
\[
= -\langle v_{p+1}^\parallel, v_{p+1}^\perp \rangle - \sum_{i=0}^p a_i v_i^\perp, v_{p+1}^\parallel - \sum_{i=0}^p a_i v_i^\perp \rangle = 0,
\]
where in the third equality we used (B.1). Since \(v_{p+1}^\parallel - \sum_{i=0}^p a_i v_i^\perp\) lies in a \(q\)-dimensional vector subspace where \(g^b\) is positive definite it must be \(\sum_{i=0}^p a_i v_i^\perp = 0\), but then also \(v_{p+1}^\parallel = \sum_{i=0}^p a_i v_i^\perp\), which is a contradiction.

We shall also need the following property of totally degenerate subspaces of maximal dimension.

Lemma B.3. Let \((V, g^0)\) satisfy the same assumptions as in lemma B.2. Let \(\Pi\) be a totally degenerate vector subspace of maximal dimension \(r := \min(p, q)\) and \(\{k_1, \ldots, k_r\}\) a basis of \(\Pi\). Select any \(r\)-dimensional vector subspace \(T\) with the property that \(g^b|_T\) is negative definite (if \(p \leq q\)) or positive definite (if \(p \geq q\)) and for any \(v \in V\) write \(v = v^\parallel + v^\perp\) according to the direct sum decomposition \(V = T \oplus T^\perp\). Then the following properties hold:

(i) The set \(\{k_1, \ldots, k_r\}\) is linearly independent.

(ii) The set \(\{k_1^\perp, \ldots, k_r^\perp\}\) is linearly independent.

(iii) The vector space \(\Pi_T := \text{span}\{k_1^\parallel, \ldots, k_r^\parallel, k_1^\perp, \ldots, k_r^\perp\}\) is \(2r\)-dimensional and \(g^b|_{\Pi_T}\) has signature \(\{r, r\}\). Moreover, there exists an orthonormal basis \(\{e_1, \ldots, e_2r\}\) of \(\Pi_T\) with the properties
(a) \( \text{span}\{e_1, \ldots, e_r\} = \text{span}\{k_1, \ldots, k_r\} \).
(b) \( \text{span}\{e_{r+1}, \ldots, e_2\} = \text{span}\{k_{r+1}, \ldots, k_2\} \).
(c) \( \Pi = \text{span}\{e_i + e_{r+i}, \ldots, e_r + e_{2r}\} \).

(iv) A vector \( v \in V \) is orthogonal to \( \Pi \) if and only if there exists \( \vartheta \in \Pi_\perp \) such that \( v - \vartheta \in \Pi \).

**Proof.** Item (i) uses a similar argument as in the previous proof. Indeed, if \( \{k_1, \ldots, k_r\} \) were linearly independent, say \( k_i = \sum_{j=1}^{r-1} a_j k_j \), by the argument in the proof of B.2 we would have that \( k_i = \sum_{j=1}^{r-1} a_j k_j \) has zero norm and belongs to a space (namely \( T_{\perp} \)) where the metric is positive or negative definite. Hence, this vector is zero and we conclude that \( k_r = \sum_{j=1}^{r-1} k_j \), which is a contradiction. The proof of item (ii) follows the same steps.

To show (iii), we first note that the orthogonal decomposition \( V = T \oplus T_{\perp} \) implies that \( \Pi_T := \text{span}\{k_1, \ldots, k_r\} \oplus \text{span}\{k_{r+1}, \ldots, k_2\} \). The dimension of \( \Pi_T \) is 2 \( r \) as a consequence of (i) and (ii) and the signature of \( g^T |\Pi_T \) is clearly \( \{r, r\} \) because \( g^T |T \) and \( g^T |T_{\perp} \) are positive and negative definite, or vice versa.

Given that \( \text{span}\{k_1, \ldots, k_r\} \) endowed with the restriction of \( g^T \) defines a Riemannian vector space, we can apply the Gram–Schmidt orthonormalization procedure to define an adapted orthonormal basis \( \{e_1, \ldots, e_r\} \). It follows that \( e_i = \sum_{j=1}^{r-1} a_j k_j \). We claim that the vectors \( e_{r+i} := \sum_{j=1}^{r} a_j k_j \), \( i = 1, \ldots, p \), define an orthonormal basis of \( \text{span}\{k_{r+1}, \ldots, k_2\} \). Indeed, the conditions \( \langle k_i, k_j \rangle = 0 \) are equivalent to \( \langle k_i, k_j \rangle = -\langle k_j, k_i \rangle \), and then
\[
\langle e_{r+i}, e_{r+j} \rangle = \sum_{l=1}^{r} \sum_{m=1}^{r} a_l a_m \delta_{ij} = -\sum_{l=1}^{r} \sum_{m=1}^{r} a_l a_m \delta_{ij} = -\langle e_i, e_j \rangle = -\sigma \delta_{ij}
\]
where \( \sigma := g^T(e_1, e_1) \). If we denote by \( (b_i^j) \) the inverse matrix of \( (a_i^j) \) it follows that
\[
k_i = k_i + k_i = \sum_{j=1}^{r} b_j e_j + \sum_{j=1}^{r} b_j e_{r+j} = \sum_{j=1}^{r} b_j (e_j + e_{r+j}),
\]
which in particular implies that \( \Pi = \text{span}\{k_1, \ldots, k_r\} \) is also \( \Pi = \text{span}\{e_i + e_{r+i}\} \). This proves (iii).

To establish (iv), decompose \( v = \sum_{i=1}^{r} a_i e_i + b_i e_{r+i} + \vartheta \) according to the orthogonal decomposition \( V = \Pi_T \oplus \Pi_{\perp} \). The condition that \( v \) is orthogonal to all \( \{k_i\} \), i.e. to all \( \{e_i + e_{r+i}\} \) imposes \( \sigma (a_i - b_i) = 0 \) and we conclude
\[
v - \vartheta = \sum_{i=1}^{r} a_i (e_i + e_{r+i}) \in \Pi
\]
as claimed. \( \square \)

**Lemma B.4.** Let \( F^T \) be a skew-symmetric endomorphism in an \( n \)-dimensional vector space \( V \) endowed with an inner product \( g^T \) of signature \( \{ p, q \} \). If \( \dim(\text{Ker}(F^T)) \geq n - 1 \) then \( F^T = 0 \) and conversely.
Proof. If \( \dim(\ker(F^2)) = n \) there is nothing to prove, so let us assume that the kernel has dimension \( n - 1 \), i.e. \( \text{rank}(F^2) = 1 \) or, equivalently that there exists a nonzero vector \( k \in V \) such that \( F^2(u) = a(u)k \), for all \( u \in V \). By linearity \( a(u) \) is a one-form, hence a continuous linear map. By skew-symmetry

\[
0 = \langle u, F^2(u) \rangle = a(u)\langle u, k \rangle.
\]

Thus, \( a(u) = 0 \) on all vectors not lying in \( k^\perp := \{v \in V, \langle k, v \rangle = 0\} \). The inner product being nondegenerate, \( k^\perp \) has dimension at most \( n - 1 \) and hence its complementary is dense in \( V \). The one-form \( a \) vanishes on this set and hence everywhere by continuity. \( \square \)

Lemma B.5. Let \( F^2 \) be a skew-symmetric endomorphism in a vector space \( V \) endowed with an inner product \( g^0 \) of signature \( \{p, q\} \). The condition \( F^2 \circ F^4 = 0 \) is equivalent to

(i) If \( p = 1 \) or \( q = 1 \): \( F^4 = 0 \)

(ii) If \( p = 2 \) and \( q \geq 2 \): \( F^4 = k \otimes \ell - \ell \otimes k \), where \( \{k, \ell\} \) is a basis of a two-dimensional totally degenerate linear subspace.

Proof. By skew-symmetry

\[
\langle F^2 \circ F^4(u), v \rangle = -\langle F^2(u), F^4(v) \rangle, \quad \forall u, v \in V
\]

so the condition \( F^2 \circ F^4 = 0 \) is equivalent to the linear space \( \Pi := \text{Image}(F^4) \) being totally degenerate. By lemma B.2 the dimension of \( \Pi \) is at most one when \( p = 1 \) and at most two when \( p = 2, q \geq 2 \). \( \text{rank}(F^4) \leq 1 \) is equivalent to \( \dim(\ker(F^4)) \geq n - 1 \) and by lemma B.4 this happens if and only if \( F^2 = 0 \).

It remains to consider the case \( p = 2, q \geq 2 \) with \( \Pi \) two-dimensional. Let \( \{k_1, k_2\} \) be a basis and fix a two-dimensional linear subspace \( T \in V \) with negative definite induced inner product. As before the orthogonal decomposition \( V = T \oplus T^\perp \) allows us to write \( v = v^\parallel + v^\perp \) for any vector \( v \). By lemma B.3 we know that \( \Pi = \text{span}\{e_1 + e_3, e_2 + e_4\} \) where \( \{e_i\} \) is an orthonormal basis of \( \Pi_T := \text{span}\{k_1^\parallel, k_2^\parallel\} \oplus \text{span}\{k_1^\perp, k_2^\perp\} \) which is adapted to the direct sum decomposition \( T \oplus T^\perp \). As \( \Pi \) has been defined as the image of \( F^2 \), there exist two nonzero one-forms \( a, b \) such that

\[
F^2(u) = a(u)(e_1 + e_3) + b(u)(e_2 + e_4).
\]

By skew symmetry, any \( u \in \Pi_T^\parallel \) must satisfy \( F^2(u) \in \Pi_T^\parallel \). Thus \( a|_{\Pi_T^\parallel} = b|_{\Pi_T^\parallel} = 0 \). Also by skew symmetry \( F^2(e_i) (i = 1, 2, 3, 4) \) is perpendicular to \( e_i \), so \( a(e_1) = a(e_3) = b(e_2) = b(e_4) = 0 \) and, in addition,

\[
0 = \langle F^2(e_1), e_2 \rangle + \langle F^2(e_2), e_1 \rangle = b(e_1) - a(e_2) \iff b(e_1) = a(e_2).
\]

Applying \( F^2 \) to (B.2) we find

\[
0 = F^2 \circ F^2(u) = a(u)(b(e_1) + b(e_3))(e_2 + e_4) + b(u)(a(e_2) + a(e_4))(e_1 + e_3)
\]

\[
\iff b(e_1) + b(e_3) = 0 \quad \text{and} \quad a(e_2) + a(e_4) = 0.
\]
where we used the fact that neither \( \mathbf{a} \) nor \( \mathbf{b} \) can vanish identically (otherwise the rank of \( F^\sharp \) would not be two). Putting things together, there exist a nonzero constant \( \alpha := \mathbf{a}(e_4) \) such that \( \mathbf{a} = \alpha (e_2 + e_4) \) and \( \mathbf{b} = -\alpha(e_1 + e_3) \). We conclude that

\[
F^\sharp = \alpha (e_2 + e_4) \otimes (e_1 + e_3) - \alpha (e_1 + e_3) \otimes (e_2 + e_4),
\]

which is \( F^\sharp = \mathbf{k} \otimes \mathbf{\ell} - \mathbf{\ell} \otimes \mathbf{k} \) after defining \( \mathbf{k} = e_1 + e_3 \) and \( \mathbf{\ell} = \alpha(e_2 + e_4) \). Since \( \{ \mathbf{k}, \mathbf{\ell} \} \) is a basis of \( \Pi \) the lemma is proved.

\[ \square \]

**Lemma B.6.** Let \( F^\sharp \) be a skew-symmetric endomorphism in a vector space \( V \) endowed with an inner product \( g^\sharp \) of signature \( \{ p, q \} \) with either \( p \) or \( q \) different from zero. Assume that \( F^\sharp \circ F^\sharp = \mu \mathbf{k} \otimes \mathbf{k} \) with \( \mathbf{k} \in V \) nonzero and null and \( \mu \neq 0 \). Suppose, moreover, that

(i) \( V \) is of Lorentzian signature, or
(ii) Image\( (F^\sharp)_{\{\mathbf{k}\perp\}^\perp} \subset \text{span}(\mathbf{k}) \).

Then, and only then, there exists \( \mathbf{w} \in V \), linearly independent and orthogonal to \( \mathbf{k} \) such that

\[
F^\sharp = \mathbf{k} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{k}, \quad \mu = -\langle \mathbf{w}, \mathbf{w} \rangle.
\]

(B.3)

**Proof.** Let \( \mathbf{\ell} \in V \) be transverse to the codimension-one vector subspace \( \{ \mathbf{k} \}^\perp \). Since \( \{ \mathbf{k}, \mathbf{\ell} \} \neq 0 \) we may (after scaling \( \mathbf{\ell} \) if necessary) assume that \( \langle \mathbf{\ell}, \mathbf{k} \rangle = 1 \). Define \( \mathbf{w} := -F^\sharp(\mathbf{\ell}) \) and observe

\[
F^\sharp(\mathbf{w}) = -F^\sharp \circ F^\sharp(\mathbf{\ell}) = -\mu \mathbf{k}.
\]

Thus, for all \( \mathbf{u} \in V \),

\[
\langle \mathbf{k}, F^\sharp(\mathbf{u}) \rangle = -\frac{1}{\mu} \langle F^\sharp(\mathbf{w}), F^\sharp(\mathbf{u}) \rangle = \frac{1}{\mu} \langle \mathbf{w}, F^\sharp \circ F^\sharp(\mathbf{u}) \rangle = \langle \mathbf{w}, \mathbf{k} \rangle \langle \mathbf{k}, \mathbf{u} \rangle,
\]

\[
\langle F^\sharp(\mathbf{u}), F^\sharp(\mathbf{u}) \rangle = -\langle \mathbf{u}, F^\sharp \circ F^\sharp(\mathbf{u}) \rangle = -\mu(\mathbf{k}, \mathbf{u})^2.
\]

In particular, for \( \mathbf{u} \in \{ \mathbf{k} \}^\perp \), \( F^\sharp(\mathbf{u}) \) is null and orthogonal to \( \mathbf{k} \). In Lorentzian signature, this can only occur if and only if \( F^\sharp(\mathbf{u}) \) is proportional to \( \mathbf{k} \) and we fall into case (ii). We may thus assume (ii) irrespectively of the signature.

We first prove \( F^\sharp(\mathbf{k}) = 0 \). Indeed, under (ii), there is \( \nu \in \mathbb{R} \) such that \( F^\sharp(\mathbf{k}) = \nu \mathbf{k} \). Since

\[
F^\sharp \circ F^\sharp(\mathbf{k}) = \mu \mathbf{k} \langle \mathbf{k}, \mathbf{k} \rangle = 0 \implies F^\sharp(\nu \mathbf{k}) = \nu^2 \mathbf{k} = 0
\]

we conclude that \( \nu \) must vanish, i.e. \( F^\sharp(\mathbf{k}) = 0 \). From \( V = \text{span}(\mathbf{\ell}) \oplus \{ \mathbf{k} \}^\perp \), it follows that \( \text{Image}(F^\sharp) = \text{span}(\mathbf{k}, \mathbf{w}) \) and, moreover, that \( \{ \mathbf{k}, \mathbf{w} \} \) are linearly independent (otherwise \( \text{Image}(F^\sharp) = \text{span}(\mathbf{k}) \subset \text{Ker} F^\sharp \) and \( F^\sharp \circ F^\sharp \) would be zero, contradicting the assumptions). There exists two one-forms \( \mathbf{a}, \mathbf{b} \in V^* \) such that

\[
F^\sharp(\mathbf{u}) = \mathbf{a}(\mathbf{u}) \mathbf{k} + \mathbf{b}(\mathbf{u}) \mathbf{w}, \quad \forall \mathbf{u} \in V.
\]

Applying \( F^\sharp \) yields

\[
\mu \langle \mathbf{k}, \mathbf{u} \rangle = F^\sharp \circ F^\sharp(\mathbf{u}) = F^\sharp(\mathbf{a}(\mathbf{u}) \mathbf{k} + \mathbf{b}(\mathbf{u}) \mathbf{w}) = -\mathbf{b}(\mathbf{u}) \mu \mathbf{k} \iff \mathbf{b} = -\mathbf{k}.
\]
Skew symmetry then forces $a = w$ so that
\[ F^\sharp = k \otimes w - w \otimes k \]
and we still need to impose
\[ -\mu k = F^\sharp(w) = k\langle w, w \rangle - w\langle k, w \rangle \quad \iff \quad \langle k, w \rangle = 0, \; \langle w, w \rangle = -\mu. \]

This proves the ‘then’ part of the lemma. The ‘only then’ is immediate since an $F^\sharp$ given by (B.3) with $k$ null and $w$ perpendicular to $k$ immediately satisfies $F^\sharp \circ F^\sharp = -\langle w, w \rangle k \otimes k$. \qed

**Lemma B.7.** Let $F^\sharp$ be a skew-symmetric endomorphism in a vector space $V$ endowed with an inner product $g^\sharp$ of signature $\{ p, q \}$ with either $p$ or $q$ different from zero. Assume that there exists $k \in V$, nonzero and satisfying $\langle k, k \rangle = 0$ such that $F^\sharp|_{\langle k \rangle^\perp}$ takes values in $\text{span}(k)$. Then there exists $v \in V$ such that
\[ F^\sharp = k \otimes v - v \otimes k. \tag{B.4} \]

**Proof.** We follow a similar path as in the proof of lemma B.6. Let $\ell \in V$ be a vector transverse to $\langle k \rangle^\perp$ and define $s := F^\sharp(\ell)$. Since $\text{span}(\ell) \oplus \langle k \rangle^\perp = V$, the hypothesis of the lemma implies that $\text{Image}(F^\sharp) = \text{span}(k, s)$, if $s$ is proportional to $k$, then the rank of $F^\sharp$ is at most one and lemma B.4 implies that $F^\sharp = 0$ which is of the form (B.4) with $v$ proportional to $k$. Thus, we may assume that $v$ and $k$ are linearly independent. There exist two one-forms $a, b$ in the dual space $V^\ast$ such that
\[ F^\sharp = k \otimes a + s \otimes b. \]
The condition $F^\sharp(u)$ proportional to $k$ for all $u \in \langle k \rangle^\perp$ requires $b|_{\langle k \rangle^\perp} = 0$ (here we use that $k, s$ are linearly independent), or equivalently $b = -ck$ for some nonzero constant $c$ (if it were zero, then $s = F^\sharp(\ell)$ would not be linearly independent of $k$). By skew-symmetry, we conclude
\[ F^\sharp = c (k \otimes s - s \otimes k) \]
which is (B.4) after defining $v = cs$. \qed

All the ingredients to prove the theorem are already in place.

**Proof of theorem 6.** Let $\zeta$ be the Killing vector of $(\Lambda)dS_n$ for which $\mathcal{H}$ is a degenerate Killing horizon. We view $\mathcal{H}$ as a codimension submanifold of $M^{n+2}_n$ and we know there is a skew-symmetric $F^\sharp : M^{n+2}_n \rightarrow M^{n+2}_n$ such that $\zeta = F^\sharp|_{\Sigma_n}$. Let $\lambda := -g_{(\Lambda)dS_n}(\zeta, \zeta)$ be (minus) the square norm of $\zeta$ in the $(\Lambda)$-de Sitter space. By definition of degenerate Killing horizon, $\lambda|_{x} = 0$ and $\text{grad}(\lambda)|_{x} = 0$ at all points $x \in \mathcal{H}$. The function $\lambda$ is the restriction to $\Sigma_n$ of (minus) the square norm of $\zeta_{F^\sharp}$, which is
\[ (\lambda)(x) := -\langle F^\sharp(x), F^\sharp(x) \rangle = \langle x, F^\sharp \circ F^\sharp(x) \rangle. \]
The gradient of $\lambda$ vanishes at and only if $\text{grad}(\lambda)(x)$ is normal to $\Sigma_n$ at $x$. $\Sigma_n$ admits $\langle x, x \rangle = \epsilon a^2$ as defining function, so the normal vector to this hypersurface is $n = x$. The gradient is $\text{grad}(\lambda) = 2F^\sharp \circ F^\sharp(x)$, so at every point $x \in \mathcal{H}$, there must exist a real number $b\epsilon$, such that
\[ F^\#: \circ F^\#(x) = b|_x. \]

In addition it must be that \( \tilde{\lambda}|_x = 0 \), i.e. \( \langle x, F^\# \circ F^\#(x) \rangle = 0 \) and we conclude, taking into account that \( x \) is non-null,

\[ F^\# \circ F^\#(x) = 0, \quad \forall x \in \mathcal{H}. \]

This condition is linear in \( x \) and \( \mathcal{H} \) is everywhere transversal to the rays \( \sigma x, \sigma \in \mathbb{R} \). In addition, the dimension of \( \mathcal{H} \) is \( n - 1 \). Thus, the kernel \( F^\# \circ F^\# \) must be at least of dimension \( n \) (equivalently, the rank of \( F^\# \circ F^\# \) is at most one). We now distinguish two cases (a) \( \text{rank}(F^\# \circ F^\#) = 1 \) or (b) \( \text{rank}(F^\# \circ F^\#) = 0 \).

We start with (a). Let \( k \) be a generator of \( \text{Image}(F^\# \circ F^\#) \). Since \( F^\# \circ F^\# \) is symmetric, there is \( \mu \in \mathbb{R} \setminus \{0\} \) such that

\[ F^\# \circ F^\# = \mu k \otimes k. \tag{B.5} \]

The kernel of \( F^\# \circ F^\# \) is therefore \( \Pi_k := (k) \) which is a codimension one hyperplane of \( M_{n+2}^\epsilon \). As shown above, a necessary condition for \( x \in \Sigma_a \) to lie in a Killing horizon of \( \zeta_F \) is that \( x \in \Pi_k \). Since \( \Pi_k \) is transverse to \( \Sigma_a \), the intersection \( \Sigma_a \cap \Pi_k \) is a smooth codimension one submanifold in \( \Sigma_a \), and \( \mathcal{H} \) must be an open subset thereof. For any fixed \( x \in \mathcal{H} \), the tangent plane \( T_x \mathcal{H} \) is a codimension-two vector subspace of \( M_{n+2}^\epsilon \) (we make the usual identification of \( M_{n+2}^\epsilon \) and \( T_x M_{n+2}^\epsilon \) induced by the affine structure). Moreover, \( T_x \mathcal{H} \) satisfies

\[ T_x \mathcal{H} \subset \Pi_k, \quad T_x \mathcal{H} \subset \langle x \rangle^\perp, \tag{B.6} \]

the first because \( \mathcal{H} \) is a hypersurface of the linear space \( \Pi_k \) and the second because \( T_x \Sigma_a \subset \langle x \rangle^\perp \). The property \( x \in \mathcal{H} \subset \Pi_k \), i.e. \( x \) normal to \( k \) also says says that \( k \) is tangent to \( T_x \Sigma_a \). By \( (B.6) \) \( k \) is a normal vector of the null hyperplane \( T_x \mathcal{H} \) within the Lorentzian vector space \( T_x \Sigma_a \). This can only occur if \( k \) has zero norm \( \langle k, k \rangle = 0 \). Moreover, \( \mathcal{H} \) being a Killing horizon of \( \zeta \) requires that the Killing vector at \( x \) is proportional to \( k \), i.e.

\[ F^\#(x) = q|_x k, \tag{B.7} \]

with \( q|_x \) nonzero given that \( \zeta|_x \) does not vanish anywhere on its Killing horizon. Applying \( F^\# \) to \( (B.7) \) one finds

\[ F^\# \circ F^\#(x) = q|_x F^\#(k) = \mu k \otimes k = 0 \quad \Rightarrow \quad F^\#(k) = 0 \tag{B.8} \]

where in the second equality we used \( (B.5) \). Take any vector \( s \) tangent to \( T_x \mathcal{H} \). Given that \( \langle s, x \rangle = \langle s, k \rangle = 0 \), skew symmetry implies

\[ \langle F^\#(s), k \rangle = \langle F^\#(s), x \rangle = 0, \]

so \( F^\#(s) \) is also tangent to \( T_x \mathcal{H} \). Moreover,

\[ 0 = -\mu \langle k, s \rangle = -\langle F^\# \circ F^\#(s), s \rangle = \langle F^\#(s), F^\#(s) \rangle \]

so \( F^\#(s) \) has zero norm. It must therefore be that \( F^\#(s) \in \text{span}(k) \) for all vectors in \( T_x \mathcal{H} \). Using \( \langle x \rangle \subset T_x \Sigma_a \subset (k) \) we conclude that \( F^\# \) maps \( (k) \) into \( \text{span}(k) \). Thus, by lemma \( B.6 \) there exists \( w \in M_{n+2}^\epsilon \) linearly independent to \( k \), orthogonal to \( k \) and satisfying \( \langle w, w \rangle = -\mu \neq 0 \) such that \( F^\# = k \otimes w - w \otimes k \). This proves the ‘if’ part of the theorem in case (a). For the
converse, we check that, given such \( k \) and \( w \), the Killing vector \( \zeta_{\mathcal{F}} \) admits as degenerate Killing horizon the hypersurface

\[
\mathcal{H}_\zeta := (\Sigma_a \cap \{ k \}) \setminus \{ \langle w, x \rangle = 0 \}.
\]

Indeed, the square norm of \( \zeta := \zeta_\mathcal{F}|_{\Sigma_a} \) is \( \lambda = -g_{(\mathcal{A})\mathcal{D}^S}(\zeta, \zeta) = -\mu(k, x)^2 \) which vanishes on \( \Sigma_a \cap \{ k \} \). This is a smooth null embedded hypersurface of \( (\mathcal{A})\mathcal{D}^S_a \) and \( \zeta \) restricted to this hypersurface takes the form \( \zeta = k \langle w, x \rangle \), so it is tangent, null, and nonzero exactly on \( \mathcal{H}_\zeta \).

Moreover, this Killing horizon is degenerate because \( d\lambda \leq 0 \). This concludes the proof of the theorem in case (a).

We now consider case (b), i.e. we assume \( F^\sharp \circ F^\sharp = 0 \). By lemma B.5 (item (i)) we see that this can only happen in the anti-de Sitter case (i.e. \( \epsilon = -1 \)) and in dimension \( n \geq 2 \). Applying item (ii) in the same lemma, there is \( \{ \ell_0, k_0 \} \) basis of a two-dimensional totally degenerate linear subspace such that \( F^\sharp = k_0 \otimes \ell_0 - \ell_0 \otimes k_0 \). The Killing vector \( \zeta \) is null everywhere which opens up the possibility that \( \text{AdS} \) is foliated by Killing horizons of \( \zeta \). To confirm this we need to check first that \( \zeta \) vanishes nowhere. Assume, on the contrary that there is \( x \in \Sigma_a \) where \( \zeta|_x = 0 \). Then

\[
0 = \zeta|_x = \zeta_\mathcal{F}|_x = F^\sharp(x) = k_0 \langle \ell_0, x \rangle - \ell_0 \langle k_0, x \rangle.
\]

By linear independence this can only happen if \( \langle \ell_0, x \rangle = \langle k_0, x \rangle = 0 \). Applying item (iv) of lemma B.3 we conclude that \( x = \pi + a_1 k_0 + a_2 \ell_0 \) for some constants \( a_1, a_2 \). Moreover \( x \in \Sigma_a \) so

\[
-\alpha^2 \equiv \langle x, x \rangle = \langle \pi + a_1 k_0 + a_2 \ell_0, \pi + a_1 k_0 + a_2 \ell_0 \rangle = \langle \pi, \pi \rangle
\]

which is impossible since \( \pi \) lies in a space with positive definite inner product. Thus \( \zeta \) has no zeros, and the Frobenius theorem (see [26]) implies immediately that \( \text{AdS}_a \) is foliated by Killing horizons. We want to show that, in fact, the foliation is by Killing horizons, i.e. that the leaves are embedded submanifolds (and identify them explicitly). Consider the collection of hyperplanes \( \Pi_\alpha := \{ \cos \alpha \ell_0 + \sin \alpha \ell_0 \} \subset \mathbb{R}^n_{\epsilon = -1} \), where \( \alpha \in \mathbb{S}^1 \) and define \( \mathcal{H}_\alpha := \Sigma_a \cap \Pi_\alpha \). The hyperplane \( \Pi_\alpha \) is transverse to \( \Sigma_a \). Indeed, being both submanifolds of codimension one, they can fail to be transverse only at points \( x \in \text{AdS}_a \) where \( T_x \Sigma_a = \Pi_\alpha \). This coincidence occurs iff the corresponding normal vectors are parallel, i.e. iff \( x = v(\cos \alpha k_0 + \sin \alpha \ell_0) \) for some nonzero \( v \). But this immediately contradicts \( \langle x, x \rangle = -\alpha^2 \neq 0 \).

Transversality of \( \Pi_\alpha \) and \( \Sigma_a \) implies that \( \mathcal{H}_\alpha \) is an embedded submanifold of \( \text{AdS}_a \). We claim that \( \mathcal{H}_\alpha \) is a Killing horizon of \( \zeta \). Note first that the two vectors

\[
k_1 := \cos \alpha k_0 + \sin \alpha \ell_0,
\]

\[
\ell_1 := -\sin \alpha k_0 + \cos \alpha \ell_0
\]

are linearly independent (hence a basis of the totally degenerate plane \( \Pi \)) and satisfy

\[
F^\sharp := k_0 \otimes \ell_0 - \ell_0 \otimes k_0 = k_1 \otimes \ell_1 - \ell_1 \otimes k_1.
\]

Moreover by construction \( k_1 \) is tangent to \( \Pi_\alpha \) (because \( k_1 \) is orthogonal to itself). At any point \( x \in \mathcal{H}_\alpha \subset \Pi_\alpha \) the Killing vector \( \zeta \) takes the form

\[
\zeta|_x = k_1 \langle \ell_1, x \rangle - \ell_1 \langle k_1, x \rangle = k_1 \langle \ell_1, x \rangle.
\]
Thus $\zeta|_x$ is null, nonzero and tangent to $\mathcal{H}_\alpha$. Moreover $\mathcal{H}_\alpha$ a null hypersurface of $\text{AdS}_a$ because $k_1$ is normal to $\mathcal{H}_\alpha$ (any vector $v \in T_p \mathcal{H}_\alpha$ must also belong to $\Pi_\alpha$, which requires $\langle v, k_1 \rangle = 0$). We conclude that $\mathcal{H}_\alpha$ is a Killing horizon of $\zeta$. Note that $\Pi_\alpha = \Pi_{\alpha + \pi}$, so we may restrict $\alpha$ to lie in $(-\pi/2, \pi/2)$. We claim that the collection of such $\{ \mathcal{H}_\alpha \}$ defines a foliation of $\text{AdS}_a$ by embedded null hypersurfaces. Indeed, assume $\alpha \neq \beta$ then $\Pi_\alpha \cap \Pi_\beta$ is the collection of points $x_0 \in M^{a+2}_{c=1}$ orthogonal to both $k_0$ and $\ell_0$, which are characterized in item (iv) of lemma B.3. We have shown above that none of these points belongs to $\Sigma_\alpha$. Thus $\mathcal{H}_\alpha \cap \mathcal{H}_\beta = \emptyset$. The collection $\{ \mathcal{H}_\alpha \}$ defines a foliation provided for any $x \in \Sigma_\alpha$, there is $\alpha \in (-\pi/2, \pi/2)$ such that $x \in \mathcal{H}_\alpha$. But this is clear because the union $\bigcup_{\alpha \in \mathbb{R}} \Pi_\alpha = M^{a+2}_c$, since for any $x \in M^{a+2}_{c=1}$, the equation

$$
\cos \alpha \langle x, k_0 \rangle + \sin \alpha \langle x, \ell_0 \rangle = 0
$$

always admits solutions for $\alpha$ in this interval.

We can now finish the proof of the theorem in case (b). Let $\mathcal{H}$ be the degenerate Killing horizon in the statement of the theorem, $\zeta$ any Killing vector of $\text{AdS}_a$ for which either $\mathcal{H}$ or an open and dense subset thereof is a degenerate Killing horizon of $\zeta$, and assume that $\zeta = \zeta_{F^2}$ with $F^1 \circ F^3 = 0$. Let $k_0, \ell_0$ be such that $F^2 = k_0 \otimes k_0 - \ell_0 \otimes \ell_0$. Fix $x \in \mathcal{H}$ and solve (B.10). Since $x \in \mathcal{H} \subset \Sigma_\alpha$ not both $\langle x, k_0 \rangle$ and $\langle x, \ell_0 \rangle = 0$ vanish, and the equation admits precisely one solution $\alpha_0 \in (\pi/2, \pi/2]$. The hypersurface $\mathcal{H}_{\alpha_0} := \Pi_{\alpha_0} \cap \Sigma_\alpha$ is a maximal Killing horizon of $\zeta$. Thus $\mathcal{H}$ is a subset of $\mathcal{H}_{\alpha_0}$. Setting $k = k_1$ and $w = \ell_1$ the direct part of the theorem follows. The converse is clear from the results above.

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