On Generalized Lemaitre-Tolman-Bondi Metric
Fractal Matter at the end of Matter-Antimatter Recombination

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Abstract

Many recent researches have investigated the deviations from the Friedmannian cosmological model, as well as their consequences on unexplained cosmological phenomena, such as dark matter and the acceleration of the Universe. On the one hand, a first order perturbative study of matter inhomogeneity returned a partial explanation of dark matter and dark energy, as relativistic effects due to the retarded potentials of far objects. On the other hand, the fractal cosmology, now modeled with a Lemaitre-Tolman-Bondi (LTB) metric, results in distortions of the luminosity distances of SNe Ia, explaining the acceleration as apparent. In this work we extend the LTB metric to ancient times. The origin of the fractal distribution of matter is explained as the matter remnant after the matter-antimatter recombination epoch. We show that the evolution of such an inhomogeneity necessarily requires a dynamical generalization of LTB, and we propose a particular solution.
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1 Introduction

Recent years have witnessed various attempts to explain, at least partially, dark matter and dark energy phenomena as general relativistic effects due to the inhomogeneities in the distribution of matter, at large scales.

1.1 Fractal cosmology

An interesting approach to the study of matter inhomogeneity is based on its description by fractal geometry. The hierarchical structure of galaxies and cluster has allowed for the introduction of the so-called “cosmic fractal”, e.g. in [12]. Remarkably, physical results can be obtained despite the lack of an exact form of the matter density distribution, by focussing rather on its fractal properties, such as the fractal dimension $D < 3$. In [12], the statistical correlation between positions of galaxies has been used to estimate $D \approx 1.2$.

Since the usual Friedmann model assumes an homogeneous distribution of matter, a fractal distribution generates different cosmological laws. The solution of the Einstein Equations with a fractal source $\rho(x)$ would be a formidable mathematical task, due to the absence of any continuous symmetry, as well as for the singularity of the source: $\rho \in D'(\mathbb{R}^3)$. However, one can obtain an approximated solution by considering not the real fractal density $\rho$, but rather the total mass inside any ball of radius $r$. Neglecting the “void bubbles” or the concentration of matter in different directions, one can perform an “homogenization” of the fractal $\bar{\rho}(r)$, which enjoys a quite high symmetry, depending only on $r$ and $t$. Then, the homogenized fractal allows for an exact solution to the Einstein Equation, which is known as the Lemaitre-Tolman-Bondi (LTB) metric, as derived e.g. in [13].

The use of the LTB metric could seemingly violate the Cosmological Principle, as the center of coordinate reference frame plays a key role. But one should bear in mind that the LTB metric is just an approximation of the real metric, which has no center at all. In fact, the homogenization can be performed with respect to any point within the fractal, taken as the center of the balls, and this will result in the LTB metric to have such a point as preferred. Since any point can act as “a center”, the Cosmological Principle is restored. The matter, and the number of galaxies, around a point $x_0$ of the fractal $\mathcal{F}$ must grow approximately as

$$M(r) := M(B(x_0; r)) \approx \Phi r^D,$$

where $D$ is the fractal dimension and $\Phi$ is a “fractal density”. We will assume that $\Phi$ is the same for all points $x_0 \in \mathcal{F}$, i.e. we consider an “homogeneous fractal”. Since $D < 3$, the homogenized density $\bar{\rho}(r) \propto r^{D-3}$ shows a singularity at $r = 0$: this can be traced back to the singular nature of the fractal distribution itself, so that it has an infinite (three-dimensional) density at any point.

It is here worth remarking that the law $\propto r^D$ cannot be true for a point $x_0$ outside the fractal. Inside a void bubble, the mass is zero. Thus, one can appreciate that there is actually no equivalence between all points, but rather the fractal splits the points into two different categories: the material points $x_0 \in \mathcal{F}$, and the void points outside. Indeed, in fractal cosmology the complete Cosmological Principle cannot be assumed, as it gets replaced by the Conditional Cosmological Principle, which does not refer to all observers, but only to the material ones [16].

Within this theoretical framework, various studies have been addressing the differences between the FLRW and LTB cosmologies. E.g. in [7], the luminosity distances of SNe Ia were recalculated, obtaining an alternative explanation for the apparent acceleration of the cosmic expansion. This paved the way to the intriguing possibility that unexplained aspects of the Cosmological Standard Model, such as dark energy, dark matter or the inflation itself, could find a natural explanation within fractal cosmology.

The data fit carried out in [7] yielded to an evaluation $D = 2.9 \pm 0.02$, which is quite different from the previous result [12]. Such a gap might be due to the fact that the spatial extension of the fractal structure is actually limited, since self-similarity disappears beyond a certain “greatness” scale. It is conceivable that the hierarchy of super-super-clusters and void bubbles comes to an end at some scale $L_{EG}$, suggestively named “End of Greatness”. Beyond such a scale length, the smaller fractal figures appear just as juxtaposed, so that

\footnote{The approximation of $\bar{\rho}$ is isotropic, so maybe it should better be called an “isotropization” procedure.}
the mass law approximates \( M(r) \propto r^3 \). An evaluation based on data fit has allowed to estimate \( L_{EG} \cong 100 \) Mpc \(^{18}\).

Therefore, the total homogenized metric is described as a LTB metric below \( L_{EG} \), and as a FLRW one beyond: this is the so-called **Swiss cheese model**, investigated by both \(^{16}\) and \(^{7}\): the spherical fractal region around the center resembles a “Swiss cheese hole”, surrounded by an otherwise homogeneous distribution of “cheese”. The matching between LTB and FLRW metrics at \( L_{EG} \) will be dealt with the so-called **Darmois junction** \(^{9}\).

### 1.2 Perturbative cosmology and retarded potentials

An apparently unrelated approach has been investigated in recent years by a number of papers, such as \(^{6}\), \(^{14}\) and \(^{15}\), and it exploits retarded gravitational potentials, generated by matter concentrations and resulting in distortions of the space-time metric. Such distortions are genuine effects of Einstein’s General Relativity, and they can be interpreted as dark matter and/or dark energy phenomena, whose they may provide an explanation, at least to a certain extent.

Necessarily, within this framework one must deal with an inhomogeneous and anisotropic distribution of matter, such that the gravitational influence of far objects does not cancel out due to Birkhoff Theorem \(^{1}\), as it instead holds for the LTB model. Even if such metric distortions decrease with the distance, one should appreciate a magnification effect due to the very expansion of the Universe: the farther the source, the more it is in the past, and higher the source \( \tilde{\rho} \) itself. Such a magnification has been confirmed by explicit computations e.g. in \(^{14}\) and in \(^{15}\), whose results highlighted a non-negligible effect on the metric, despite the relative smallness of the matter inhomogeneity.

For the time being, an exact solution to the Einstein Equations with a very asymmetrical source is still out of reach, and the search for retarded potentials’ solutions must necessarily rely on perturbation theory. Assuming that the Cosmological Principle is a reasonably good approximation at the visible Universe scale, the matter inhomogeneities are considered as small ones,

\[
\tilde{\rho}(\mathbf{x}; t) := \rho(\mathbf{x}; t) - \bar{\rho}(t),
\]

with respect to a homogeneous background \( \bar{\rho}(t) \). Thus, the inhomogeneities can be treated as a first order perturbation, with the zeroth order approximation returning the Friedmann cosmology: the retarded potentials arise as solutions to the linearization of Einstein Equations around a FLRW background metric.

It can then be appreciated that the details concerning the spacial matter distribution \( \tilde{\rho}(\mathbf{x}) \) are not so relevant, since the crucial quantity is its average amount \( \langle \tilde{\rho} \rangle \), which provides a quantitative estimate of the “total matter inhomogeneity” \(^{14}\) \(^{15}\). In fact, this approach is concerned with the total amount of dark matter and dark energy in the Universe, not with their local distribution. As a consequence, the results obtained in \(^{14}\) \(^{15}\) are insensitive to the fractal nature of the matter distribution, or to any other geometrical property matter can enjoy. However, we should remark that a FLRW zeroth order background is just the simplest starting point, but is quite inaccurate. A further investigation should choose a less trivial background, such that mimics better the shape of matter inhomogeneities.

### 1.3 Retarded potentials and the fractal

A tantalizing possibility is that deeper insights on cosmological matter inhomogeneities may be gained by merging the two approaches presented in the previous Sections. In the present work, we will indeed consider a fractal matter distribution, at least up to the \( L_{EG} \) scale. We will describe the resulting metric, and consider the “Swiss cheese homogenization” as the zeroth order approximation, and we will then deal with a first order perturbative description of the real fractal \( \rho(\mathbf{x}) \). Within the choice of a LTB background, the effects due to retarded potentials will also be effectively dealt with, thus allowing for a more reliable evaluation of cosmological parameters, such as the cosmological constant and the dark matter amount.

Recent computations with retarded potentials have improved the explanation of dark matter as well as of dark energy \(^{15}\). However, after \(^{7}\) it is known that a suitable LTB background can explain the appearance of dark energy. It is thus reasonable to expect that a combination of the above two approaches may result in
considerable advances in the explanation, at least to some non-negligible extent, of both dark matter and dark energy.

Further improvements may also be expected to shed some light on the choice of the homogeneous density $\bar{\rho}$, which turned out to be a tricky feature within the perturbative approach based on retarded potentials. If $\bar{\rho}$ is taken as the average of $\rho$, it returns $\langle \hat{\rho} \rangle = 0$, which means no effects at all from a first order calculation. On the other hand, in [15] it was chosen

$$\bar{\rho} := \min_x \rho(x), \quad (1.3)$$

or

$$\bar{\rho} := \max_x \rho(x), \quad (1.4)$$

but it is not yet clear if these are physically sensible choices. This issue does not arise at all in the fractal approach, because no such a thing as spacial averaging exists for a fractal, which is endowed with a lower and lower average as the space region under consideration widens up; eventually, the average tends to zero because of the void bubbles. Moreover, the growth of void bubbles prevents the determination of a unique real fractal density $\Phi$. In fact, if one tries to define it from $M(r) := \Phi(r)r^D$, this will result in $\Phi(r)$ oscillating indefinitely. Within a fractal, such an issue can be overcome by choosing the minimum $\Phi$ of the oscillations as the reference for the definition of the homogenization $\hat{\rho}(r)$. One can appreciate that this procedure is physically meaningful, because the perturbation $\hat{\rho}(x)$ may have negative and positive values here and there, but its average will certainly be positive, so that first order effects will not vanish.

All in all, in this paper we aim at consistently determining the parameters of our model: $D, \Phi, L_{EG}, \bar{\rho}_0, \rho_{\Lambda 0}, \rho_{R0}$. They will be obtained by fitting the experimental data, such as the dark matter effects and the luminosity distances of SNe Ia. Furthermore, local metric distortions due to retarded effects will be compared to the expected dark matter inside the single galaxy or cluster, thus discerning to what extent they can effectively be explained as relativistic effects.

### 1.4 The origin of the fractal, and the three epochs

The perturbative approach should concern the LTB approximation for ancient times, when most perturbations were generated. The validity of the LTB model over time would actually be an interesting issue per se, because the solution used so far is valid just around the current instant. Back in time, for ancient times, we know that radiation dominates, and the evolution of the metric gets distorted.

For what concerns the origin of the fractal distribution of matter, we put forward the conjecture that it arises out as a consequence of the matter-antimatter (M-AM) recombination process. In fact, as a tiny fraction of matter survives the annihilation, it is conceivable that it was not homogeneously distributed, but rather it is scattered only across those regions in which the matter itself turned out to have a slightly larger density. Before the recombination, the inhomogeneity of matter would be mainly due to quantum uncertainty, being very small. However, after recombination only a $10^{-9}$ fraction of the pre-existing matter survives, and thus its inhomogeneity is magnified of a factor $\sim 10^9$. For our purposes, we can suppose that matter was already distributed as a fractal in very ancient times; in fact, this solves also the problem of structure formation: dark matter is not actually needed, if matter was sufficiently concentrated at the very beginning.

In our model, the evolution of the Universe is characterized in terms of three different epochs, as follows.

1. **Before M-AM recombination.** The Universe is well described by FLRW, and quantum uncertainty is the unique source of perturbations.

2. **M-AM recombination.** It generates a matter remnant with fractal distribution, exhibiting a non-negligible inhomogeneity. It generates a large amount of homogeneous radiation, as well.

3. **After M-AM recombination.** At zeroth order, it is approximated by a LTB Universe, starting with the dominance of a homogeneous radiation, progressively fading away into an epoch in which fractal matter gets dominant. The first order perturbations better approximate the actual fractal, and they give rise to retarded distortions. The superposition of these latter for all times effectively results into dark matter phenomena, both globally and locally, as the fractal geometry causes a distortion of the luminosity distances which appears as a Universe acceleration.
1.5 The “Swiss cheese” metric

In this paper we will consider a Swiss cheese metric:

\[ ds^2 = \begin{cases} 
-dt^2 + \frac{A^2}{r^2} dr^2 + A(r; t)^2 d\Omega^2, & L_G \leq r \leq L_{EG}; \\
-dt^2 + a(t)^2 [dx^2 + x^2 d\Omega^2], & x \geq L_{EG}; 
\end{cases} \tag{1.5} \]

where the coordinate arbitrariness is fixed as

\[ A(r; 0) \equiv A_0(r) \equiv r, \quad a(0) \equiv a_0 \equiv 1, \tag{1.6} \]

and we use a prime “′” for \( r \)-derivative and a dot “˙” for \( t \)-derivative.

Today, the matter inside dominates and is homogenized as

\[ \bar{M}_0(r)|_{[L_G; L_{EG}]} = \Phi \left[ \frac{3}{2} \right] \Rightarrow \bar{\rho}_0(r)|_{[L_G; L_{EG}]} = \frac{D}{4\pi} \Phi r^{D-3}. \tag{1.7} \]

The fractal dimension \( D \) can be measured as in [12]. It does not deform the luminosity distances everywhere, but just until \( L_{EG} \), which can be coherent to the different measure in [7].

For \( r < L_G \), the exact metric depends on the distribution of matter in a galaxy. A first simplification is to consider the fractal of matter as made of balls whose minimum radius is \( L_G \); hence, the galaxy would be approximated as a homogeneous sphere, and thus below \( L_G \) another Friedmann metric would arise.

We consider as \( A(r; t) \) is also a FLRW metric during epoch 1, whence it gains an inhomogeneity during epoch 2, and the epoch 3 sees the evolution of fractal. From now on, we will try to describe such \( A(r; t) \), especially during epoch 3.

2 Inadequacy of LTB during epoch 3

2.1 Pure matter

A universe filled with only matter regulates the Friedman Equation outside as

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8}{3} \pi G \bar{\rho}_0 a^{-3}. \tag{2.1} \]

There are no singularities of density, thus the metric must be almost everywhere twice derivable: \( \bar{g}_{\mu\nu} \in C^1 \).

Such a requirement contains the Darmois junction, which defines the dependence \( x(r; t) \). These have especially the consequences

\[ \bar{g}_{rr} \in C^1(L_{EG}) \Rightarrow \dot{x}(L_{EG}; t) \equiv 0 \Rightarrow x(L_{EG}; t) \equiv L_{EG}; \]
\[ \bar{g}_{\Omega\Omega} \in C^0(L_{EG}) \Rightarrow A(L_{EG}; t) = a(t) x(L_{EG}; t) = L_{EG} a(t). \tag{2.2} \]

Within the fractal assumption \( \bar{M}_0(r) := \Phi r^D \), and following [10] and [7] setting \( f := 1 \), we get the functions

\[ A(r; t) = r \left[ 1 + \frac{3}{2} H_0(r) t \right]^\frac{2}{3} = r \left[ 1 + \frac{3}{2} \sqrt{2G\Phi r} \frac{\rho_{03}}{2} t \right]^\frac{2}{3}. \tag{2.3} \]

For Darmois [2,2], it yields to

\[ a(t) = \frac{1}{L_{EG}} A(L_{EG}; t) = \left[ 1 + \frac{3}{2} \sqrt{2G\Phi L_{EG}^2} t \right]^\frac{2}{3}. \tag{2.4} \]
Moreover, by differentiating Darmois, one obtains
\[ h_0 := \frac{\dot{a}_0}{a_0} = \frac{\dot{A}_0(L_{EG})}{A_0(L_{EG})} = H_0(L_{EG}) = \sqrt{2G\Phi L_{EG}^{D-3}}; \] (2.5)\\
\[ a(t) = \left[ 1 + \frac{3}{2} h_0 t \right], \quad H_0(r) = h_0 \left( \frac{r}{L_{EG}} \right)^{\frac{D-3}{2}}. \] (2.6)

Moreover, by imposing the Friedmann equation to hold outside, the following results are achieved:
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8}{3} \pi G \bar{\rho}_{00} a^{-3} = h_0^2 a^{-3} \quad \Rightarrow \quad \dot{a}^2 = h_0^2 a^{-1} \quad \text{s.t.} \quad h_0^2 = \frac{8}{3} \pi G \bar{\rho}_{00}; \] (2.7)\\
\[ a(t) = \left[ \frac{3}{2} h_0 (t - t_I) \right]^\frac{2}{3} \quad \Rightarrow \quad \left[ 1 + \frac{3}{2} h_0 t \right]^\frac{2}{3} = \left[ \frac{3}{2} h_0 (t - t_I) \right]^\frac{2}{3}; \] (2.8)\\
\[ t_I = -\frac{2}{3} h_0^{-1}, \] (2.9)

and
\[ 2G\Phi L_{EG}^{D-3} = H_0(L_{EG})^2 = \frac{8}{3} \pi G \bar{\rho}_{00}; \] (2.10)\\
\[ \bar{M}_0(L_{EG}) = \Phi L_{EG}^D = \frac{4}{3} \pi L_{EG}^3 \bar{\rho}_{00}. \] (2.11)

Thus, the Swiss cheese metric has a time singularity at
\[ t_S = -\frac{2}{3} \left( \frac{L_G}{L_{EG}} \right)^{\frac{3-D}{2}} h_0^{-1} > t_I, \] (2.12)
at which \( A'(L_G; t_S) \) goes to infinity. Here the validity of our pure matter model reaches an end.

**Remark 1.** Usually, the Big Bang is set at the time singularity of the metric. However, for the pure matter model such a singularity depends on \( r \):
\[ t_{BB}(r) = -\frac{2}{3} \left( \frac{r}{L_{EG}} \right)^{\frac{3-D}{2}} h_0^{-1}, \] (2.13)
such that \( t_S \) is just the first instant without singularity: \( t_S := \max_r t_{BB}(r) \). This result makes no sense, since the Big Bang should be the same for all the Universe. Therefore, the pure matter does not provide a satisfactory description, and multi-component model is needed. In particular, a component with a larger \( w \), such as radiation, will do the job: if it dominates in the early Universe, with an initial homogeneous density, it would grant the synchronicity of Big Bang for all \( r \). This reasoning implies that the pure matter Swiss cheese metric \([1.5]\) with \([2.3]\) can be a good approximation only near the current instant, but generally the evolution must concern a multi-component model.

### 2.2 The flat LTB model

A consistent description of the expansion of the Universe would involve many components - namely matter, radiation, and eventually dark energy - and their evolutions.
To this aim, we need to make $\bar{\rho}_{M}(r; t)$ explicit; the functional dependence on time is obtained from \[16\] to be

$$8\pi G \bar{\rho}_{M} = \frac{F'}{2A'}; \quad \text{s.t.} \quad F = 2A\dot{A};$$

(2.14)

$$\Downarrow$$

$$\bar{\rho}_{M}(r; t) = \frac{D}{\pi} \Phi^{D-3} \frac{1}{[2 + 3H_{0}(r)t][2 + DH_{0}(r)t]}.$$  

(2.15)

It should be remarked that $\bar{\rho}_{M}(r; t)$ goes as the inverse of the volume ($\bar{\rho}_{M0}(r) := \bar{\rho}_{M}(r; 0)$):

$$\bar{\rho}_{M}(r; t) = \frac{4}{A^{2}(r; t)A'(r; t)} \bar{\rho}_{M0}(r).$$

(2.16)

as expected, since matter is still. On the other hand, the dark energy does not depend on $t$, so its density reads

$$\bar{\rho}_{\Lambda}(r; t) = \bar{\rho}_{\Lambda0}(r).$$

(2.17)

In case it is a cosmological constant, it should also be independent of $r$.

Analogously to the FLRW model, one would expect that the radiation density goes as

$$\bar{\rho}_{R}(r; t) \propto \left( \frac{r^{2}}{A^{2}(r; t)A'(r; t)} \right)^{\frac{4}{3}},$$

(2.18)

but this should better be confirmed by a more detailed computation (cfr. \[2.15\]) further below).

We will henceforth carry out a detailed treatment of the flat LTB model. The LTB metric returns a diagonal Einstein tensor, with

$$G_{t}^{t} = -\frac{\dot{A}}{A} \left( 2\frac{A'}{A'} + \frac{\dot{A}}{A} \right);$$

(2.19)

$$G_{r}^{r} = -2\frac{\ddot{A}}{A} - \frac{\dot{A}^{2}}{A^{2}};$$

(2.20)

$$G_{\theta}^{\theta} = G_{\phi}^{\phi} = -\frac{\ddot{A}}{A} - \frac{\dot{A}}{A} - \frac{\dot{A}'\dot{A}}{A'A}.$$  

(2.21)

Hence, also $T_{\mu\nu}$ is diagonal, implying still matter. Within the assumption of mostly-plus signature and the symmetries of our system, the energy-momentum tensor of a perfect fluid reads

$$T_{\mu\nu} := (\rho + p)U_{\mu}U_{\nu} + pg_{\mu\nu}, \quad \text{with} \quad U_{\mu} = \delta_{t\mu};$$

(2.22)

$$\Downarrow$$

$$T_{t}^{t} = -(\rho + p) + p = -\rho, \quad T_{r}^{r} = p, \quad T_{\theta}^{\theta} = T_{\phi}^{\phi} = p.$$  

(2.23)

Thus, three independent Einstein equations are obtained, namely:

$$\begin{align*}
-\frac{\dot{A}}{A} \left( 2\frac{A'}{A'} + \frac{\dot{A}}{A} \right) &= -8\pi G\rho; \\
-2\frac{\ddot{A}}{A} - \frac{\dot{A}^{2}}{A^{2}} &= 8\pi Gp; \\
-\frac{\ddot{A}}{A} - \frac{\dot{A}}{A} - \frac{\dot{A}'\dot{A}}{A'A} &= 8\pi Gp.
\end{align*}$$

(2.24)
2.2.1 The Ricci equation as a Riccati equation, and its solutions

With a barotropic equation of state \( \rho = \rho(p) \), one has four equations for the three unknowns \( A, \rho, p \). This should imply some constraint on the form of \( A, \rho, p \). Such a constraint can be obtained from the second and third Einstein equations, as follows:

\[
\frac{2\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} = 8\pi Gp = \frac{\dot{A}'}{A'} + \frac{\dot{A}'A}{A'A},
\]

\[
\Downarrow
\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} = \frac{\dot{A}'}{A'} + \frac{\dot{A}'A}{A'A}.
\]

We can try to solve this non-linear PDE in \( A \), which we will name Ricci equation, and search for a set of self-consistent solutions. Exploiting the definition

\[ H := \frac{\dot{A}}{A} \]

(2.27)

the identity (2.26) can be rewritten in a very simple way,

\[ \dot{H} + 3HH' = 0. \]

(2.28)

For a general Universe, (2.28) constrains the possible matter, radiation and/or dark energy content. It is easy to check that the solution found in [7] satisfies this PDE. Nevertheless, there is no uniqueness proven for the solutions of (2.28), so we can search for other, different solutions.

Now, (2.28) can be rewritten as

\[ 0 = \partial_r \left( \dot{H} + \frac{3}{2}H^2 \right); \]

\[
\Downarrow
\dot{H} + \frac{3}{2}H^2 = c(t),
\]

(2.30)

where the integration constant \( c(t) \) does not depend on \( r \). Eq. (2.30) can be recognized to be a Riccati Equation. For \( c(t) \equiv 0 \), we find again the solution in [7], namely

\[ H(r; t) = \frac{2H_0(r)}{2 + 3H_0(r)t}. \]

(2.31)

But e.g. for a non-zero, constant \( c(t) \equiv c \) we can find different solutions. Calling \( c := \frac{3}{2} \tau^{-2} \), we get

\[ H(r; t) = \frac{1}{\tau} \tanh \frac{3t}{2\tau} + H_0(r), \]

(2.32)

while for negative \( c = -\frac{3}{2} \alpha^2 \) one has the (quite unphysical) solution

\[ H(r; t) = -\alpha \tan \frac{3\alpha t}{2} + H_0(r). \]

(2.33)

On the other hand, we observe that the second Einstein Eq. from (2.24) depends only on \( H \), thus exploiting (2.30) we can obtain the following expression for th pressure \( p \):

\[ 8\pi Gp = -2\frac{\dot{A}}{A} - \frac{\dot{A}^2}{A^2} = -2\dot{H} - 3H^2 = -2\left( \dot{H} + \frac{3}{2}H^2 \right) = -2c(t). \]

(2.34)

Therefore, the integration constant gets related to \( p \) itself: \( c(t) = -4\pi Gp \), which implies that the total pressure must be homogeneous at any time:

\[ p(r; t) = p(t). \]

(2.35)
2.2.2 Conservation of four-momentum and separability of $w$’s

Let us now study the conservation of the four-momentum. One can compute the conservation of energy

$$\dot{\rho} = -\left(\frac{\dot{A}'}{A'} + 2\frac{\dot{A}}{A}\right)(\rho + p),$$

and the conservation of momentum, which turn out to be

$$p' = 0.$$  

This equation is just a confirmation of the result (2.35), expressing the homogeneity of pressure, as it must be for a perfect fluid in a LTB flat Universe.

Let us now consider a particular type of perfect fluid, namely a single-component one, defined by $p := w\rho$. The homogeneity of pressure then immediately implies

$$w\rho' = 0.$$  

We can thus conclude that we no single-component, inhomogeneous flat LTB Universe can exist, unless such a component is matter. It then turns out that the two solutions for the single-component case in a flat LTB Universe were actually already both studied: for $w = 0$, the pure matter flat LTB model, studied in [17], [4], [16] and [7], is retrieved; for $\rho' = 0$, one simply obtained the well-known FLRW model.

The case of a multi-component perfect fluid is more interesting. By setting

$$p = \sum_{w} p_{w} = \sum_{w} w\rho_{w}, \text{ s.t. } \rho = \sum_{w} \rho_{w},$$

the conservation of momentum (2.37) allows for inhomogeneities to exist for any component $\rho_{w}$, but only if the pressure inhomogeneities compensate each other,

$$\sum_{w} w\rho'_{w}(r; t) = 0.$$  

On the other hand, the conservation of energy allows one to study each component separately (i.e., by fixing the corresponding $w$); indeed, (2.36) and (2.39) yield

$$\sum_{w} \dot{\rho}_{w} = -\left(\frac{\dot{A}'}{A'} + 2\frac{\dot{A}}{A}\right)\sum_{w} (1 + w)\rho_{w}. $$

Within the assumption of separation of components$^4$ for each component $w$ we find

$$\partial_{t} \ln \rho_{w} = \frac{\dot{\rho}_{w}}{\rho_{w}} = -(1 + w)\left(\frac{\dot{A}'}{A'} + 2\frac{\dot{A}}{A}\right) = -(1 + w)\partial_{t}(\ln A' + 2 \ln A) = \partial_{t} (A^2 A')^{-1-w}, \forall w;$$

$$\rho_{w}(r; t) = \rho_{w0}(r) \left(\frac{A_{0}(r)^{2}A_{0}'(r)}{A^{2}(r; t)A'(r; t)}\right)^{1+w}, \forall w.$$  

For $w = 0$ (matter), and choosing the radial coordinate s.t. $A_{0}(r) \equiv r$ (cfr. (1.6)), one retrieves (2.16), namely

$$\rho_{M}(r; t) = \rho_{M0}(r) \frac{r^{2}}{A^{2}(r; t)A'(r; t)},$$

$^4$We will see below that such an assumption would not hold during epoch 2 (cfr. Sec. 3).
For $w = 1/3$ (radiation), Eq. (2.43) confirms the conjecture (2.15), namely:

$$\rho_R(r; t) = \rho_{R0}(r) \left( \frac{r^2}{A^2(r; t) A'(r; t)} \right)^{4/3}. \quad (2.45)$$

For $w = -1$ (dark energy) the density is constant, and the previous result is confirmed, namely $\rho_A(r; t) = \rho_{A0}$.

To recap, in a flat LTB Universe with just matter and radiation, the radiation must be homogeneous, and this holds also in presence of a cosmological constant (i.e., of homogeneous dark energy). Note that, while this has been assumed in previous papers (cfr. e.g. [14], [15]), we here deduced it from the conservation of four-momentum.

### 2.3 The “approximation with epochs”

No explicit, exact solutions are known for the Einstein field equations in such a general case, with many components. Thus, we will resort to the so-called “approximation with epochs”: at any $(r; t)$ we will consider as if there were just the dominating component, neglecting the others.

We start by noticing that, even if radiation and dark energy are homogeneous, the matter is not; therefore, it might well be that for some $r$ we could be in an epoch, whereas for some other $r$ we are already in another one. We will consider the case of dominating matter further below (after remark 2), and we will now focus on an evolution dominated by radiation. Moreover, from now on we will not consider the dark energy component in our calculations: they would be just more complicate, without let a better understanding.

An homogeneous distribution of primordial radiation could be assumed, thus giving rise to a Friedmannian expansion during the epoch dominated by radiation:

$$a^2 \approx h_0^2 \Omega_{R0} a^{-2} \Rightarrow a(t) = \left[ 2\sqrt{\Omega_{R0} h_0} (t - t_{BB}) \right]^{1/2}, \quad \text{s.t.} \quad \Omega_{R0} := \frac{8\pi G}{3h_0^2} \rho_{R0}, \quad (2.46)$$

with the radiation evolving as $\Omega_R(t) = \Omega_{R0} a^{-4}(t)$.

**Remark 2.** $\rho_{R0} := \rho_R(r; t = 0)|_{r \geq L_{EG}}$ is the radiation density today beyond the End of Greatness $L_{EG}$, at which it is still uniform. Below $L_{EG}$, one can reasonably assume that $\rho_R(r; 0)$ is not homogeneous, since it developed through an inhomogeneous expansion. This would imply the current measurement of $\Omega_{R0}$ not to be reliable, since they would take place inside our galaxy, and thus in a point of the cosmic fractal: these would be measures of $\rho_R(L_G; 0)$, which could be quite different from the average value $\rho_{R0}$. For instance, inside a void bubble, the density of the cosmic background would undergo a completely different development.

When $\Omega_M(t) \geq \Omega_R(t)$, one would switch to the epoch dominated by matter. $\Omega_M$ must also depend on $r$:

$$\Omega_M(r; t) := \frac{8\pi G}{3h_0^2} \bar{\rho}_M(r; t), \quad \text{s.t.} \quad \bar{\rho}_M(r; t) \propto \begin{cases} \frac{a(t)^{-3}}{A^2}, \quad t < t_{RM}(r); \\ \frac{r^2}{A^2}, \quad t > t_{RM}(r). \end{cases} \quad (2.47)$$

For a fixed $r < L_{EG}$, the “soldering instant” $t_{RM}$ is defined as

$$\Omega_M(r; t_{RM}) := \Omega_R(t_{RM}) \Leftrightarrow \frac{D\Phi r^{D-3}}{\pi [2 + 3H_0(r)t_{RM}] [2 + DH_0(r)t_{RM}]} \bar{\rho}_M(r; 0) = \rho_R(r; t_{RM}). \quad (2.48)$$

#### 2.3.1 “Swiss cheese” with two epochs

Next, we will consider again the Swiss cheese metric, in order to describe a radiation+matter Universe by soldering the corresponding two one-component solutions together.

Let us consider first the outer expansion, which is simpler. The Friedmann Eq. (2.4) at $r > L_{EG}$ reads

$$h^2 := \left( \frac{\dot{a}}{a} \right)^2 = h_0^2(\Omega_{R00} a^{-4} + \Omega_{M00} a^{-3}), \quad \text{s.t.} \quad \Omega_{w00} := \frac{\rho_w}{\rho_0}, \quad \rho_{00} := \frac{3h_0^2}{8\pi G}. \quad (2.49)$$
The outside matter density is related to the inside matter density \( \bar{\rho}_{M0}(r) = \frac{D}{4\pi} \Phi r^{-3} \) by

\[
\Phi = \frac{4}{3} \pi L_{EG}^3 \bar{\rho}_{M00}.
\] (2.50)

If \( \Omega_{M00} > \Omega_{R00} \), it holds that

\[
a(0) := 1 > a_{RM} := \frac{\Omega_{R00}}{\Omega_{M00}} > a_{BB} := 0.
\] (2.51)

Thus, during both epochs, the evolution of the Universe can be approximated as if there were only one component, i.e. the dominating one:

\[
a(t) = \begin{cases} 
(2h_0(t-t_{BB}))^{1/2}, & t_{BB} \leq t \leq t_{RM}; \\
(\frac{4}{3}h_0 t + 1)^{2/3}, & t_{RM} \leq t \leq 0.
\end{cases}
\] (2.52)

Thus, one can compute the “soldering instant” \( t_{RM} \) as follows:

\[
\frac{3}{2} h_0 t_{RM} + 1 = \frac{3}{2} a_{RM} \Rightarrow t_{RM} = \frac{2}{3} h_0^{-1}[a_{RM}^{3/2} - 1].
\] (2.53)

From the continuity of \( a(t) \) at \( t_{RM} \), one obtains also the homogeneity for the Big Bang instant \( t_{BB}(r) \equiv t_{BB} \).

Let us now consider the inner expansion; for a fixed \( r < L_{EG} \), since the radiation epoch must be homogeneous, we know that the evolution is

\[
A(r; t) = \begin{cases} 
 r (2h_0(t-t_{BB}))^{1/2}, & t_{BB} \leq t \leq t_{RM}(r); \\
 r (\frac{2}{3}h_0 r + 1)^{2/3}, & t_{RM}(r) \leq t \leq 0,
\end{cases}
\] (2.54)

where we recalled the result (2.6). When we try to compute the “soldering instant” \( t_{RM}(r) \) within this regime, we can appreciate the inadequacy of the framework under consideration in order to describe a Universe with radiation and matter; indeed, we should impose the continuity of \( A(r; t) \), and thus solve

\[
(2h_0(t_{RM}(r)-t_{BB}))^3 = \left(\frac{3}{2} H_0(r)t_{RM}(r) + 1\right)^4,
\] (2.55)

which is a fourth degree algebraic equation. Surely, for \( r \to L_{EG}^- \), we will retrieve the expression of \( t_{RM} \) computed within the outside expansion, because \( H_0(L_{EG}) = h_0 \). Nevertheless, let us consider the definition of \( t_{RM}(r) \) as the instant when the matter density and the radiation density are equal; by defining \( x := \frac{r}{L_{EG}} \), one can write

\[
\bar{\rho}_{R00}[2h_0(t-t_{BB})]^{-2} = \bar{\rho}_{R00} a^{-4} = \bar{\rho}_R(r; t) = \bar{\rho}_M(r; t)
\]

\[
= \bar{\rho}_{M0}(r)\left(\frac{r^2}{A^2 A'}\right) = \frac{D}{3} x^{-3} \bar{\rho}_{M00} \left(1 + \frac{3}{2} x \frac{D}{3} h_0 t\right)^{-1} \left(1 + \frac{2}{3} x \frac{D}{3} h_0 t\right)^{-1}
\]

\[
\downarrow
\]

\[
a_{RM} \left(1 + \frac{3}{2} x \frac{D}{3} h_0 t\right) \left(1 + \frac{D}{3} x \frac{D}{3} h_0 t\right) = \frac{D}{3} x^{-3} \left[2h_0(t-t_{BB})\right]^2 = \frac{D}{3} x^{-3} \left(1 + \frac{3}{2} x \frac{D}{3} h_0 t\right)^{8/3},
\] (2.57)

where we used (2.55) in the last step of (2.57). As mentioned, for \( x \to 1^- \) one should find again \( t = t_{RM} \), thus obtaining

\[
\left(\frac{3}{D} a_{RM}\right)^3 \left(1 + \frac{D}{3} (a_{RM}^{3/2} - 1)\right)^3 = (1 + a_{RM}^{3/2} - 1)^5;
\] (2.58)

\[
\downarrow
\]

\[
\left(\frac{3}{D} a_{RM} + a_{RM}(a_{RM}^{3/2} - 1)\right)^3 = a_{RM}^{15/2} \leftrightarrow \left(\frac{3}{D} - 1\right) a_{RM} = 0.
\] (2.59)
Thus, we obtain that only trivial solutions are allowed for consistency, namely, the trivial FLRW solution $D = 3$, or the pure matter solution $a_{RM} = 0$.

2.4 Inadequacy of the flat LTB model

We have found that the Swiss cheese metric, with inhomogeneous matter and non-zero radiation, cannot be self-consistent when assuming a spatially flat metric and still energy-matter. In other words, a spatially flat, inhomogeneous LTB solution with still energy-matter must necessarily contain only matter, and possibly some dark energy, whose evolution $\propto Vol^0$ allows to preserve the homogeneity (however, dark energy cannot dominate near the Big Bang, which will necessarily be inhomogeneous in any such model; cfr. remark 1 above).

By setting to zero the velocity field, the conservation of momentum implies the homogeneity of pressure ($p' = 0$) at any instant, so that there are no forces. Within this framework, one can appreciate that the inconsistency between inhomogeneous matter and non-zero radiation can be traced back to the homogeneity of pressure. Indeed, since the matter has vanishing pressure, the conservation of momentum yields homogeneous radiation density, at any instant. But the expansion itself is inhomogeneous, due to the matter inhomogeneity; as a consequence, even if the radiation is homogeneous at a given instant, it will evolve inhomogeneously with the expansion, thus breaking the conservation of momentum.

In a Universe undergoing a two-epochs evolution (as we are assuming in this Section), the conservation of momentum approximately holds during both epochs: as for the homogeneous expansion during the radiation-dominated epoch, so for the zero pressure expansion during the matter-dominated epoch. However, the “two-epochs approximation” fails in proximity of the “soldering instant” $t_{RM}$, namely when radiation and matter are about to be equal. In such an intermediate period of time, the pressure is no more negligible, but the expansion is still inhomogeneous. The inconsistency arises because the conservation of momentum prevents the determination of a well-defined “soldering instant” $t_{RM}$.

It is here worth remarking that that this inconsistency cannot be solved by adding other components, possibly aiming at compensating the inhomogeneity of the pressure of radiation. Indeed, even if some other $\rho_w$ allows to set $w\rho_w + \frac{1}{3}p_R' = p' = 0$ for a given instant, this cannot hold for other instants, because the $w$ component evolves as $\propto (Vol)^{-1-w}$ with $w \neq \frac{1}{2}$, whereas the radiation evolves $\propto Vol^{-4/3}$.

The above clashing of volumetric expansions implies that the consistent way to add the radiation, or any other component with $w \neq 0, -1$, to the LTB model, is at most two-fold, as one could consider a non-vanishing velocity field $v$ (yielding a fourth Einstein equation, the one sourced by the component $T_{ir}$ of energy-momentum tensor), and/or a non-vanishing spatial curvature.

2.5 The non-flat LTB model: $k = k(r;t)$ and $v \neq 0$

Let us generalize the LTB metric by adding a non-vanishing spatial curvature $k := k(r)$,

$$ds^2 = -dt^2 + \frac{(A')^2}{f^2}dr^2 + A^2d\Omega^2, \quad \text{s.t.} \quad f(r)^2 = 1 - k(r)^2. \quad (2.60)$$

The treatment of this metric given in \[\text{LTB}\] yields the Einstein tensor to be diagonal again; in particular, $G^t_t = 0$. In turn, this implies a diagonal energy-momentum tensor, and for a perfect fluid the conservation of momentum yields the following result:

$$0 = \partial_r T^r_r + 2\Gamma^r_{r\theta}(T^r_r - T^\theta_\theta) = p'. \quad (2.61)$$

However, the aforementioned inconsistency plaguing the flat LTB Universe is not (yet) resolved in such a non-flat Universe. In fact, a still energy-matter evolves as $\propto (Vol)^{-1-w}$, with some dependence on $f$ in the formula of the volume $Vol$; consequently, the conservation of momentum still allows only matter and dark energy within an inhomogeneous Universe, still exhibiting an inhomogeneous Big Bang (cfr. remark 1 above).

Thus, one must necessarily consider a non-vanishing velocity field ($v \neq 0$) within a non-flat LTB Universe. Since (compare with §6.1)

$$T^t_r = -\rho v - p\sqrt{1 + \left(\frac{f}{A'}\right)^2} \neq 0, \quad (2.62)$$
this would imply a non-vanishing $Gt$, again forbidden by [17]. The only way out is to consider a moving energymatter ($v \neq 0$) within a Universe with the most general type of (non-vanishing) spatial curvature (although thespherical symmetry is required nevertheless), namely $k = k(r,t)$, thus implying $f^2 = 1 - k(r,t)^2 = f(r,t)^2$.Indeed, the (tr)-component of the Einstein Eqs. results to be

$$\frac{A'}{A} \frac{f}{f} = -4\pi G(\rho + p)v \sqrt{1 + \left(\frac{f}{A}v\right)^2}. \quad (2.63)$$

Thus, we have four variables $A, \rho, v, f$ for four Einstein Eqs. (namely, the three diagonal components $(tt)$, $(rr)$, $(\theta\theta)$, and the non-diagonal component $(tr)$).

3 Expansion during M-AM recombination

3.1 Inseparability of components

Let us consider again (2.41). In the treatment given above, we have assumed the separation of (2.41) into each of its $w$-components, and we have obtained that an inhomogeneous LTB Universe with non-vanishing radiation requires a non-zero velocity field and a spatial curvature $k$ depending both on $t$ and $r$. However, during the M-AM recombination (corresponding to the epoch 2; cfr. Sec. 1.4), the separation of Eq. (2.41) into its $w$-components is a sufficient but not necessary condition for the solution of (2.41) itself. In general, some mixing terms among the different components $w$'s can occur, as a consequence of the recombination between matter and antimatter, in which a huge quantity of $w = 0$ (matter) component gets transformed into the $w = 1/3$ (radiation) component.

For simplicity’s sake, let us consider now the case with matter ($w = 0$) and radiation ($w = 1/3$) only. Eq.s (2.39) and (2.37) imply

$$\rho_R = \rho_R(t) = 3p(t). \quad (3.1)$$

Then, the equation of the conservation of the energy (2.41) can be written as

$$\dot{\rho}_M + \dot{\rho}_R = -\left(\frac{A'}{A} + 2\frac{\dot{A}}{A}\right) \left(\rho_M + \frac{4}{3}\rho_R\right); \quad (3.2)$$

$$\dot{\rho}_M = -\frac{\partial_t(A^2A')}{A^2A'} \rho_M - \left[3\dot{p} + 4\frac{\partial_t(A^2A')}{A^2A'} p\right]. \quad (3.3)$$

It can be integrated as

$$\rho_M(r;t) = \frac{K_M(r) + \int_0^t \dot{\rho}(\tau)A^2(r;\tau)A'(r;\tau)d\tau}{A^2(r;t)A'(r;t)} - 4p(t), \quad (3.4)$$

where $K_M(r) = r^2[\rho_M0(r) + 4p_0]$.

3.2 Einstein equations (flat LTB without dark energy)

Having obtained the explicit functional dependence of the matter density and its relation with the pressure, let us now try to solve the Einstein equations (2.24) within the flat LTB model within the flat LTB model. By specifying only matter and

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3 This may result into an overly simplified physical picture during M-AM recombination, but nevertheless we search for a solution within this framework.

4 From now on, we will not put any dark energy as a component, since we want to explain the luminosity distance observations as a consequence of the fractal metric. It is possible an analogous model with a cosmological constant, but it would further complicate the equations.

5 A detailed treatment of the Einstein equations for the non-flat LTB model with $k = k(r,t)$ and $v \neq 0$ will be given in Sec. 4.
radiation, and recalling (3.1), Einstein equations read

\[
\begin{align*}
\dot{A}^2 + 2\frac{\dot{A}^2}{A} &= 8\pi G [\rho_M + 3p(t)]; \\
4\ddot{A} + \dot{A}^2 &= -8\pi G p(t); \\
\dddot{A}' + \dot{A}' + \frac{\dot{A}^2}{A} &= -8\pi G p(t),
\end{align*}
\]

where we stressed the fact that the pressure depends only on time, as expressed by (3.1), which in turn guarantees the conservation of momentum. From the treatment given in the previous Section, the conservation of energy is given by (3.3), whereas the equation of state for matter and radiation has been taken into account by specifying \( \rho = \rho_M + \rho_R = \rho_M + 3p \).

From the treatment of Sec. 2.2.1 we know that equating the second and third Einstein equations, one obtains a Riccati equation (2.30) for the Hubble parameter \( H \) (2.27):

\[
\dot{H} + \frac{3}{2} H^2 + 4\pi G p(t) = 0,
\]

where Eq. (2.34) has been recalled. We have discussed above the solutions for \( p(t) = 0 \) (vanishing pressure) and for \( p(t) = p \neq 0 \) (non-vanishing, constant pressure), respectively given by Eqs. (2.31) (obtained in [7]) and (2.32). Following the usual method to solve such a class of differential equations (cfr. e.g. [13]), we define the auxiliary variable \( y(r; t) \) as follows:

\[
H = \frac{2}{3} \dot{y}, \quad (3.7)
\]

in terms of which the Riccati equation (3.3) becomes linear:

\[
\ddot{y} + 6\pi G p(t)y = 0. \quad (3.8)
\]

It can be appreciated that \( y \) provides an alternative description of the expansion of Universe, in place of the coefficient \( A(r; t) \); indeed, by recalling (2.27) and (3.7), one gets \( A^3 = y^2 \), and \( A^2 A' \propto y' y \).

Hence, one can rewrite the Eq. (3.3) of conservation of energy as

\[
\dot{\rho}_M = -\frac{\partial_t (yy')}{{yy'}} \rho_M - \left[ 3\dot{\rho} + 4\frac{\partial_t (yy')}{{yy'}} \rho \right]. \quad (3.9)
\]

Analogously, one can rewrite the other Einstein equations

\[
6\pi G [\rho_M + 3p(t)] = \frac{y' y'}{{yy}}, \quad (3.10)
\]

\[
-6\pi G p(t) = \frac{\bar{y}}{y}. \quad (3.11)
\]

By construction, the third Einstein equation from (3.5) is equivalent to the second one via (3.8); both of them are (3.8) again. Thus, the Einstein system (3.5) can be rewritten in a simpler way in terms of the \( y \) function (3.7) as follows:

\[
\begin{cases}
\frac{\dot{y}'^2}{y}\ddot{y} = 6\pi G [\rho_M + 3p(t)]; \\
\ddot{y} = -6\pi G p(t)y.
\end{cases} \quad (3.12)
\]

We observe that it is useless to substitute \( \rho_M \) from the first Einstein equation inside (3.3), since it gives again (3.8).

Thus, we end up with the system (3.12) composed by two independent PDE’s in terms of the functions \( y(r; t) \) (3.7) and \( \rho_M (r; t) \) (3.3), but the 1-variable function \( p(t) \) remains here undetermined. It is then evident that some other condition is needed in order to obtain a consistent evolution of the Universe; it is easy to realize that such a missing condition should be provided by the law of transformation from matter to radiation as resulting from the M-AM recombination, which we did not consider yet. FIN QUI
3.3 New variables

We observe that the linear Riccati equation (3.8) does not actually depend on \( r \); thus, since it is a second order equation, its general solution \( y(r; t) \) will be given by a linear combination of two purely \( t \)-dependent functions \( y_1(t) \) and \( y_2(t) \), with \( r \)-dependent coefficients,

\[
y(r; t) := c_1(r)y_1(t) + c_2(r)y_2(t), \tag{3.13}
\]

where

\[
\ddot{y}_{1,2}(t) = -6\pi Gp(t)y_{1,2}(t). \tag{3.14}
\]

The conditions at \( t = 0 \) can be fixed e.g. by setting

\[
\begin{aligned}
  y_1(0) &= 0 = \dot{y}_2(0); \\
  \dot{y}_1(0) &= 1 = y_2(0),
\end{aligned} \tag{3.15}
\]

which yields

\[
y(r; t) = A_0(r)^{3/2} \left[ \frac{3}{2} H_0(r)y_1(t) + y_2(t) \right]. \tag{3.16}
\]

Next, we notice the importance of the variable

\[
V := y^2 = A^3 \Rightarrow yy' = \frac{1}{2}V', \tag{3.17}
\]

which represents the volume of the sphere centred in \( \vec{0} \) with radius \( r \). By exploiting the definition (3.17), the first Einstein equation of (3.12) can be recast in the following form (where \( \rho = \rho_M + 3p \)) :

\[
\dot{y}'y' = 3\pi G\rho V', \tag{3.18}
\]

whereas the equation of energy conservation (3.9) and the formula of \( \rho_M(r; t) \) (3.4) respectively acquire the following forms:

\[
\dot{\rho} = -\frac{\dot{V}'}{V'} (\rho + p), \tag{3.19}
\]

and

\[
\rho = \frac{1}{V'} \left[ K_M(r) - \int^r p(\tau)V'(r; \tau) d\tau \right]. \tag{3.20}
\]

By inspecting Eq. (3.19), one can appreciate that an even better variable to be used would be the total energy inside the sphere of radius \( r \),

\[
E(r; t) := \int^r \rho(s; t)dV(s; t) = M(r; t) + 3p(t)V(r; t), \tag{3.21}
\]

where

\[
M(r; t) := \int_0^r \rho_M(s; t)V(s; t)ds. \tag{3.22}
\]

By virtue of the fact that definition (3.21) implies

\[
E' = \rho V', \tag{3.23}
\]

the first Einstein equation (3.18) boils down to

\[
\dot{y}'y' = 3\pi GE'. \tag{3.24}
\]
It can be integrated in \( r \), obtaining
\[
\dot{y}^2 = 6\pi G E \Rightarrow \dot{E} = \frac{1}{3\pi G} \dot{y}\dot{y} = -2\dot{y}\dot{p} = -p\dot{V},
\]
(3.25)
where the second Einstein equation (3.12) was used and definition (3.17) recalled. Finally, by integrating further in \( t \), one obtains
\[
E(r; t) = E_0(r) - \int_0^t p(\tau)V(\tau)d\tau,
\]
(3.26)
where \( E_0(r) = M_0(r) + 3p_0 \).

The evaluation of (3.26) today (i.e. for \( t = 0 \)) and the use of the time derivative of (3.16) yields the relation between \( E_0(r) \equiv E(r; 0) \) and \( H_0(r) \) (by recalling the conditions (3.15)),
\[
\dot{y}^2(r; 0) = 6\pi G E_0(r) \Leftrightarrow H_0^2(r) = \frac{8}{3}\pi G \frac{E_0(r)}{A_0(r)^3} = \frac{8}{3}\pi G \overline{\rho}_0(r),
\]
(3.27)
where we defined \( \overline{\rho} := E/V = \dot{\rho}_M + 3\rho \) as the average density of energy inside the ball of radius \( r \). The equation on the r.h.s. of (3.27) is a well known relation for FRW model, but in the framework under consideration it could get a similar expression for \( \rho \). Analogously to FRW, we can define the \( \Omega \) parameters as follows :
\[
\Omega_M(r; t) := \frac{8\pi G M \overline{\rho}_M}{3H^2}, \quad \Omega_R(r; t) := \frac{8\pi G p}{H^2} \quad \text{s.t.} \quad \Omega_M(0) + \Omega_R(0) := 1,
\]
(3.28)
where \( \Omega_M(0) \equiv \Omega_M(r; 0) \) and \( \Omega_R(0) \equiv \Omega_R(r; 0) \).

### 3.4 General form

By plugging the time derivative of \( y(r; t) \) (3.16) into (3.25), one obtains
\[
6\pi G E(r; t) = \dot{y}^2(r; t) = A_0(r)^3 \left[ \frac{3}{2} H_0(r)\dot{y}_1(t) + \dot{y}_2(t) \right]^2 = A_0(r)^3[H_0(r)\dot{T}_2(t) + H_0(r)T_1(t) + T_0(t)],
\]
(3.29)
where
\[
T_2(t) := \frac{9}{4}\dot{y}_1^2(t), \quad T_1(t) := 3\dot{y}_1(t)\dot{y}_2(t), \quad T_0(t) := \dot{y}_2^2(t).
\]
(3.30)

On the other hand, by exploiting the first Einstein equation of (3.12) an recalling the definition (3.17), one could get a similar expression for \( \rho \), but it turns out to be non-polynomial,
\[
6\pi G \rho = \frac{\dot{y}'\dot{y}}{\dot{y}'\ddot{y}} = \frac{[3H_0(r)\dot{y}_1(t) + 2\dot{y}_2(t)][3H_0(r)A_0'(r)\dot{y}_1(t) + 2A_0'(r)\dot{y}_2(t) + 2A_0(r)H_0'(r)\dot{y}_1(t)]}{[3H_0(r)\dot{y}_1(t) + 2\dot{y}_2(t)][3H_0(r)A_0'(r)\dot{y}_1(t) + 2A_0'(r)\dot{y}_2(t) + 2A_0(r)H_0'(r)\dot{y}_1(t)]}.
\]
(3.31)

#### 3.4.1 Pure matter

Let us consider the pure matter case : \( p = 0 \). From the second Einstein equation of (3.12), one obtains
\[
y_1(t) = t, \quad y_2 = 1
\]
(3.32)
yielding to
\[
A(r; t) = y^{2/3}(r; t) = A_0(r) \left[ \frac{3}{2} H_0(r)t + 1 \right]^{2/3}.
\]
(3.33)

Since \( p = 0 \), Eqs. (3.21) and (3.26) imply that
\[
\dot{M} = \dot{E} = 0,
\]
(3.34)
and the first Einstein equation of the system (3.12) simplifies down to
\[ \rho_M (r; t) = \rho_{M0} (r) \frac{A_0' (r)}{\frac{1}{2} H_0 (r) t + 1 + A_0 (r) H_0' (r)/t} \left[ \frac{2}{3} H_0 (r) t + 1 \right], \] (3.35)
such that
\[ \rho_{M0} (r) = \frac{1}{4\pi G} \frac{H_0 (r) \left[ \frac{4}{3} A_0' (r) H_0 (r) + A_0 (r) H_0' (r) \right]}{A_0 (r)}. \] (3.36)
This is consistent with what we already know. The \( T \) functions (3.30) read
\[ T_2 = \frac{\sqrt{3}}{4}, \quad T_1 = 0, \quad T_0 = 0. \]

### 3.4.2 Pure radiation
On the other hand, in the case of pure radiation \( \rho_M \equiv 0 \), Eqs. (3.21) and (3.25) imply
\[ E = 3pV \Rightarrow \dot{E} = 3(\dot{p}V + p\dot{V}) = -p\dot{V} \Leftrightarrow \frac{\dot{V}}{V} = -\frac{3}{4} \frac{\dot{p}}{p} \Leftrightarrow V (r; t) = A_0 (r)^3 p(t)^{-3/4}, \] (3.37)
and by recalling (3.17) one obtains
\[ A_0 (r)^3 p(t)^{-3/4} = V := y^2 = A_0 (r)^3 \left( \frac{3}{2} H_0 (r) y_1 (t) + y_2 (t) \right)^2. \] (3.38)
And therefore, in this case it holds that
\[ \frac{3}{2} H_0 y_1 (t) + y_2 (t) = p(t)^{-3/8}. \] (3.39)
From
\[ H_0^2 = \frac{8}{3} \pi G E_0 (r) / A_0 (r)^3 = 8\pi G p_0 \Leftrightarrow p_0 = \frac{H_0^2}{8\pi G}, \] (3.41)
one gets
\[ p(t) = \frac{1}{32\pi G} \left( t + \frac{1}{2H_0} \right)^{-2}. \] (3.42)
This result allows to explicitly solve the second Einstein equation of (3.12) yielding that
\[ y_1 (t) = \frac{(2H_0 t + 1)^{3/4} - (2H_0 t + 1)^{1/4}}{H_0}, \quad y_2 (t) = \frac{3(2H_0 t + 1)^{1/4} - (2H_0 t + 1)^{3/4}}{2}, \] (3.43)
finally leading to the following expression:
\[ y = A_0 (r)^{3/2} \left[ \frac{3}{2} \frac{H_0}{H_0^2} \left( (2H_0 t + 1)^{3/4} - (2H_0 t + 1)^{1/4} \right) + \frac{1}{2} \left( 3(2H_0 t + 1)^{1/4} - (2H_0 t + 1)^{3/4} \right) \right] \]
\[ = A_0 (r)^{3/2} (2H_0 t + 1)^{3/4}, \] (3.44)
implying
\[ A_0 (r; t) = y (r; t)^2 / 3 = A_0 (r) \sqrt{2H_0 t + 1}, \] (3.45)
in which we recognize a feature of the FLRW model with pure radiation.
3.4.3 Beyond pure models

The two functional forms (3.32) and (3.44), respectively concerning the cases of pure matter and pure radiation can be recognized to belong to a more general family of solutions of the form

\[ y \propto (t + \theta)^{\frac{1+a}{2}}, \quad \text{s.t.} \quad \begin{cases} a = 1 \text{ for pure matter;} \\ a = \frac{1}{2} \text{ for pure radiation.} \end{cases} \] (3.46)

Again, the second Einstein equation of (3.12) implies

\[ p(t) = \frac{1 - a^2}{24\pi G} (t + \theta)^{-2}. \] (3.47)

By recalling the definition (3.28) of \( \Omega_R(r; t) \), one then obtains

\[ \frac{1 - a^2}{24\pi G \theta^2} = p(0) = \frac{\Omega_{R0}(r) H_0(r)^2}{8\pi G} \iff \theta^{-1}(r; a) = H_0(r) \sqrt{\frac{3}{1 - a^2 \Omega_{R0}(r)}}. \] (3.48)

Note that such a result implies that in general \( \theta \) does depend on \( r \) (as well as on the parameter \( a \)).

We should now remember that we are considering the expansion of the Universe during M-AM recombination only in presence of matter and radiation (namely, we are disregarding the contribution of dark energy, for simplicity’s sake). Thus, \( p_M \geq 0 \) and \( p \geq 0 \) always, which imply \(|a| \leq 1\). Furthermore, it is reasonable to assume \( \theta > 0 \), so that the Big Bang happened in some past instant \( t_{BB} = -\theta \). Within these assumptions, the expression of \( y(r; t) \) for the family of solutions under consideration reads

\[ y_1(t; a) = \frac{\theta}{a} \left[ \left( \frac{t}{\theta} + 1 \right)^{\frac{1+a}{2}} - \left( \frac{t}{\theta} + 1 \right)^{\frac{1-a}{2}} \right]; \] (3.49)

\[ y_2(t; a) = \frac{1}{2a} \left[ (a - 1) \left( \frac{t}{\theta} + 1 \right)^{\frac{1+a}{2}} + (a + 1) \left( \frac{t}{\theta} + 1 \right)^{\frac{1-a}{2}} \right], \] (3.50)

which finally allows one to explicitly write down the functional form of the \( a \)-parametrized family of solutions under consideration:

\[ y(r; t; a) = \frac{A_0^2(r)}{2a} \left[ \left( a + \sqrt{3 - a^2 \Omega_{R0}(r)} - 1 \right) \left( \frac{t}{\theta} + 1 \right)^{\frac{1+a}{2}} + \left( a - \sqrt{3 - a^2 \Omega_{R0}(r)} + 1 \right) \left( \frac{t}{\theta} + 1 \right)^{\frac{1-a}{2}} \right], \] (3.51)

where \( \theta = \theta(r; a) \) given by (3.44). In turn, this implies the formula

\[ A(r; t) = y^{2/3}(r; t) = \frac{A_0^2(r)}{(2a)^{2/3}} \left[ \left( a + \sqrt{3 - a^2 \Omega_{R0}(r)} - 1 \right) \left( \frac{t}{\theta} + 1 \right)^{\frac{1+a}{2}} + \left( a - \sqrt{3 - a^2 \Omega_{R0}(r)} + 1 \right) \left( \frac{t}{\theta} + 1 \right)^{\frac{1-a}{2}} \right]^{2/3}. \] (3.52)

4 The Lemaître model

Now we consider again the epoch 3, for which we saw the LTB solution is not general enough. Thus, in this section we will use its generalization, called the Lemaître model. It was described e.g. in [3]. We choose the coordinates which diagonalize the metric tensor, and we redefine \( t \) with respect to eq. (7) of [3] in order to get \( g_{tt} := -1 \), which will mean that the energy-matter has some radial velocity \( U_\phi \). Hence, in our gauge the metric results to be

\[ ds^2 = -dt^2 + \left( \frac{A'}{f} \right)^2 dr^2 + A^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (4.1)

where the spatial curvature is \( k(r; t) = \sqrt{1 - f(r; t)^2} \), as (2.60).

\(^{a}\)We should bear in mind that, physically, the “right” \( p(t) \) depends on the M-AM recombination law.
4.1 Einstein equations

We will now adopt the tetrad formalism, in which $ds^2 = \eta_{ab}e^a \otimes e^b$, and which allows us to compute the Vielbein as

$$e^0 = dt, \quad e^1 = \frac{A'}{f} dr, \quad e^2 = Ad\theta, \quad e^3 = A \sin \theta d\phi.$$  \hspace{1cm} (4.2)

We can compute the Einstein tensor,

$$G^0_0 = -2\frac{A'}{A^2} - \frac{A^2}{A^2} - \frac{k^2}{A^2} + 2\frac{\dot{A}'A}{A} + 2\left(\frac{4}{A} - \frac{f}{f}\right)\frac{f}{f};$$

$$G^1_0 = -2\frac{\dot{f}}{A};$$

$$G^1_1 = -2\frac{\dot{A}}{A} - \frac{A^2}{A^2} - \frac{k^2}{A^2} + 2\frac{\dot{f}}{f};$$

$$G^2_2 = G^0_0 = -\frac{\dot{A}}{A} - \frac{A^2}{A^2} + \frac{\dot{f}}{f} + \left(\frac{2A'}{A} + \frac{A}{A}\right)\frac{f}{f}. \hspace{1cm} (4.3)$$

On the other hand, in presence of a non-vanishing velocity field, the energy-momentum tensor reads

$$T^a_b = (\rho + p)U^a U_b + \rho \delta^a_b \quad \text{s.t.} \quad U_a = \sqrt{v^2 + 1}e^0 + ve^1 = \sqrt{v^2 + 1}dt + v\frac{A'}{f}dr,$$  \hspace{1cm} (4.4)

namely

$$T^0_0 = -(\rho + p)(v^2 + 1) + p = -\rho - v^2(\rho + p);$$

$$T^0_1 = v\sqrt{v^2 + 1}(\rho + p);$$

$$T^1_1 = p + v^2(\rho + p);$$

$$T^2_2 = T^3_3 = p. \hspace{1cm} (4.5)$$

Thus, we can finally write the Einstein equations for the Lemaître model metric in §4.1 with $k = k(r,t)$ and $v \neq 0$:

$$\begin{cases} 2\frac{A'}{A} + \frac{4}{A^2} + \frac{\dot{A}}{A} - 2\frac{\dot{f}}{\dot{A}} - 2\left(\frac{4}{A} - \frac{f}{f}\right)\frac{f}{f} = 8\pi G\rho + 8\pi Gv^2(\rho + p); \\ \frac{\dot{f}}{f} = -4\pi Gv\sqrt{v^2 + 1}(\rho + p); \\
2\frac{\dot{A}}{A} + \frac{\dot{A}'}{A} + \frac{\dot{f}}{f} - 2\frac{\dot{f}'}{\dot{A}} - \frac{\dot{f}}{\dot{A}} - \frac{\dot{f}'}{\dot{A}} - \dot{f} - \left(\frac{2A'}{A} + \frac{A}{A}\right)\frac{f}{f} = -8\pi Gp. \end{cases} \hspace{1cm} (4.6)$$

Notice that we don’t express them in terms of the $M$ variable, defined in eq. (10) of §3. We stress that $M$ is not the ‘empirical amount of mass’ we defined in §3.3 and we will use again in §4.2.

The velocity $v$ represents the matter which falls on itself. We can assume that it will be small almost always, w.r.t. $c = 1$. Thus, we can approximate $\pi G$ up to the first order in $v$.

$$\begin{cases} 2\frac{A'}{A} + \frac{4}{A^2} + \frac{\dot{A}}{A} - 2\frac{\dot{f}}{\dot{A}} - 2\frac{\dot{f}'}{\dot{A}} = 8\pi G\rho + o(v); \\ \frac{\dot{f}}{f} = -4\pi GvA(\rho + p) + o(v); \\
2\frac{\dot{A}}{A} + \frac{\dot{A}'}{A} + \frac{\dot{f}}{f} = -8\pi Gp + o(v); \\
\frac{\dot{A}}{A} + \frac{\dot{A}'}{A} - \frac{\dot{f}'}{\dot{A}} - \dot{f} - \left(\frac{2A'}{A} + \frac{A}{A}\right)\frac{f}{f} = -8\pi Gp. \end{cases} \hspace{1cm} (4.7)$$
where we used than $\dot{f} = O(v)$ from the second equation. Moreover, the first equation can be rewritten in the more compact form
\[
\partial_t [A(\dot{A}^2 + k^2)] = 8\pi G \frac{A^2}{f} \left[ (f - \dot{A}v) \rho - \dot{A}vp \right].
\]

4.2 Conservation laws

In order to write the energy-momentum conservation, by recalling the energy-momentum tensor (4.4)-(4.5), one can approximate it as follows:
\[
U_\mu = \sqrt{v^2 + 1} dt + v \frac{A'}{f} dr \Rightarrow U^\mu = -\sqrt{v^2 + 1} \partial_t + v \frac{f}{A'} \partial_r,
\]
which implies
\[
T^t_t = -\rho + o(v);
\]
\[
T^t_r = -v \frac{A'}{f} \rho + o(v);
\]
\[
T^r_t = v \frac{f}{A'} \rho + o(v);
\]
\[
T^r_r = p + o(v);
\]
\[
T^\phi_\phi = p = T^\phi_\phi.
\]

Hence, the conservation of energy reads
\[
\dot{\rho} = \left[ -\partial_t (A'A^2/f) \right] + v \frac{f}{A'} \partial_r (A'A^2/f) + \partial_r \left( v \frac{f}{A'} \right) \rho + v \frac{f}{A'} \rho' + o(v),
\]
whereas the conservation of momentum is
\[
p' = \left[ 2 \frac{v}{A^2} \partial_t \left( \frac{A'A^2}{f} \right) + \frac{A'A^2}{f} \partial_t \left( \frac{v}{A^2} \right) \right] \rho + v \frac{A'}{f} \rho + o(v).
\]

We can rewrite the conservation laws by calling $\frac{A'A^2}{f} := V'$, where $V(r; t)$ is the volume inside the sphere of radius $r$. The conservation of energy simplifies
\[
\partial_t (V' \rho) = \partial_r \left[ A^2 v(\rho + p) \right] - \dot{V}' \rho.
\]
The l.h.s. of (4.13) is related to the total energy inside the sphere, defined by $E(r; t) := \int_0^r \rho(s; t) V'(s; t) ds$, such that (4.23) holds, namely $E' = V' \rho$. Within the assumption of separation of (4.13) into its $w$-components (which holds with a good approximation after the M-AM recombination, when the transformations occur only in the stars), one can write
\[
\dot{E}' = \partial_r \left[ A^2 v(\rho + p) \right] - \dot{V}' p \Rightarrow \dot{E}'_w = (1 + w) \partial_r \left[ v \frac{f}{A'} E'_w \right] - w \frac{\dot{V}'}{V'} E'_w , \forall w.
\]
Indeed, for the static case $v = 0$ we have just the volume deformation $E'_w \propto (V')^{-w}$. For the general case, the matter component has a particularly simple law,
\[
\dot{M}' = \partial_r \left[ v \frac{f}{A'} M' \right] \Rightarrow \dot{M} = v \frac{f}{A'} M'.
\]

Moreover, the conservation of momentum (4.12) becomes
\[
V' p' = \partial_t \left[ v \frac{A'}{f} V'(\rho + p) \right].
\]
4.3 General system

The Lemaître model with \( k = k(r; t) \) and \( v \neq 0 \), filled by matter and radiation only, is described by five independent PDEs, which one can write at the first order in \( v \) as follows:

\[
\begin{align*}
\dot{f} &= -4\pi G v A (\rho_M + \frac{3}{2} \rho_R); \\
2\dot{A} + \ddot{A}^2 + 1 - f^2 &= -\frac{8}{3} \pi G A^2 \rho_R; \\
\partial_t (V' \rho_M) &= \partial_v \left( v \frac{A'}{A} \cdot V' \rho_M \right); \\
\partial_t ([V'^{1/3} \rho_R] = \frac{4}{3} \sqrt{\rho} \partial_v (v \frac{A'}{A} \cdot V' \rho_R); \\
\rho'_R &= 3\dot{A} \left( v \frac{A'}{A} \right) \rho_M + 4 \sqrt{\rho} \partial_t (v \frac{A'}{A}) \rho_R.
\end{align*}
\]

The independent variables are \( A, f, v, \rho_M, \rho_R \), and the quantity \( V := \int_0^r A'^2 \) has been defined. Moreover, the last three conservation laws of \( 4.17 \) can alternatively be expressed as

\[
\begin{align*}
\dot{M} &= v \frac{A'}{A} M' ; \\
\partial_t \left[ 4v \frac{A'}{A} E_R - 3 \dot{E}_R \right] &= \frac{V''}{V} E_R'; \\
E_R'' &= \left[ \frac{V''}{V} + 4 \sqrt{\rho} \partial_t (v \frac{A'}{A}) \right] E_R' + 3 \partial_t (v \frac{A'}{A}) M',
\end{align*}
\]

where are defined the total quantities \( E_w := \int_0^r V' \rho_w \).

4.4 Approximated models

The PDE system in Sec. 4.3 is quite difficult to solve. The searched solution fulfills a condition at the initial time \( t_R \) (\( A = ra(t), 1 - f^2 = Kv^2, \rho_R = \rho_R(t) \)) and some other constraints at the final instant (\( A = r, f = 1, M = \Phi R^D \)). Hence, we cannot even exploit a numerical approach, at least at the first step, because it would require a complete set of conditions at a certain instant, e.g. the initial or the final one; if we fix the initial condition, it is then not ensured that we will find an acceptable final state (it is very improbable, indeed), and \textit{vice versa}.

What we need is a model, at least an approximated one, which satisfies some conditions both at the start and at the end. If it does not solve exactly the PDEs, we can nevertheless take it as a zeroth order, perturbing to the right version, even numerically.

4.4.1 \( r \) as a label

The crucial observation is that, for an only matter Universe, the evolution law results to be \( A(r; t) = r \left[ 1 + \frac{2}{3} H_0(t) \right]^{2/3} \).

A FLRW Universe with pure matter has analogously \( a(t) = \left[ 1 + \frac{2}{3} H_0 t \right]^{2/3} \). Hence, the inhomogeneous Universe has, at radius \( r \), a metric \( A(r; t) = ra_r(t) \) s.t. \( \frac{\dot{a}_r^2}{a_r^2} = H_0(r)^2 a_r^{-3} \). If one considers only the spherical region until \( r \), and the total matter inside is regarded as if it were homogeneous, the subsequent evolution law in \( t \) depends on the label \( r \) exactly as the LTB solution.

This observation works exactly only for pure matter. We see this by setting \( v := 0 \) and \( f := 1 \). The conservations of matter and radiation respectively yield \( \rho_M \propto (A^2)^{-1} \) and \( \rho_R \propto (A^2)^{-4/3} \). By plugging these into the first Einstein equation, one obtains

\[
\partial_r (\dot{A}^2 A) = 8\pi G \left( \rho_{M0} + \frac{\rho_{R0}}{\sqrt{A_0^2}} \right) A_0^2 A_0^2.
\]
One can realize that this is exactly integrable for pure matter, while the radiation returns a non trivial term. Henceforth, we should bear in mind that the ‘radius as label’ method is just an approximation: it works fairly well near M-AM, when the universe is almost FLRW, and near today, when the matter dominates, whereas it gets worse during the intermediate period. With this caveat, we attempt at writing

\[ A(r; t) := r a_r(t) \quad \text{s.t.} \quad \frac{a_r^2}{H_0(r)^2} = \Omega_M(0)r^{-1} + \Omega_R(0)r^{-2}. \]  

(4.20)

Since we consider here just the final components, \( \Omega_{K0} = 0 \). They are defined as usual, s.t. \( \Omega_M + \Omega_R = 1 \). We can solve this by means of exact integrations, as follows:

\[ \frac{da}{dt} = \frac{H_0}{a} \sqrt{\Omega_M a + \Omega_R}; \]  

(4.21)

\[ \int_{t_{BB}}^t H_0 dt = \int_{t_{BB}}^t \frac{ada}{\sqrt{\Omega_M a + \Omega_R}} = \left[ 2\Omega_M a \sqrt{\Omega_M a + \Omega_R} \right]_{t_{BB}}^t - 2\left[ \frac{3}{3}(\Omega_M a + \Omega_R)^{3/2} \right]_{t_{BB}}^t; \]  

(4.22)

\[ \frac{3}{2} H_0 (t - t_{BB}) = (\Omega_M a - 2\Omega_R) \sqrt{\Omega_M a + \Omega_R} + 2\Omega_R^{3/2}, \]  

(4.23)

where we used the fact \( a(t_{BB}) := 0 \). Moreover, by setting \( a(0) := 1 \), one obtains

\[ -\frac{3}{2} H_0 t_{BB} = (\Omega_M - 2\Omega_R) \sqrt{\Omega_M a + \Omega_R} + 2\Omega_R^{3/2} = \Omega_M + 2\Omega_R(\Omega_R^{1/2} - 1); \]  

(4.24)

\[ \frac{3}{2} H_0 t = (\Omega_M a - 2\Omega_R) \sqrt{\Omega_M a + \Omega_R} + (2\Omega_R - \Omega_M). \]  

(4.25)

This is an exact evolution law \( a = a(t) \), albeit implicitly expressed; the explicit dependence can be obtained by exploiting Cardano’s formula.

Next, we proceed and set the parameter of the real Universe. First of all, the time singularity \( t_{BB}(r) \) must be spatially homogeneous; since \( t = 0 \) is today, we can call \( T \) the age of the Universe, so that \( -t_{BB} = T \). Therefore, we set the fractal \( \rho_{M0} := \frac{\rho}{a^D} \). The evolution law evaluated at \( t = -T \) yields the last constraint,

\[ \frac{3}{2} H_0 T = \Omega_M + 2\Omega_R(\Omega_R^{1/2} - 1) \Rightarrow 2(1 - \Omega_M)^{3/2} + 3\Omega_M - 2 = T \frac{3}{2} \frac{\sqrt{DG\Phi}}{r^3} \Omega_M^{1/2}. \]  

(4.26)

Notice that \( \Omega_M(0) \) can be expressed as the solution to an high order algebraic equation, and this fixes also \( H_0, \Omega_R \) and \( \rho_{00} \).

It is difficult to solve exactly the algebraic equation of \( \Omega_M(0) \). Here, we confine ourselves to provide an approximated solution, relying on the fact that \( \Omega_R \ll \Omega_M \). Indeed, the evolution equation at \( -T \) becomes

\[ \frac{3}{2} H_0 T = \Omega_M + 2\Omega_R(\Omega_R^{1/2} - 1) \cong \Omega_M - 2\Omega_R \cong \Omega_M. \]  

(4.27)

Substituting \( \rho_{M0} \), one reaches the following result:

\[ \frac{3}{2} H_0 T \cong \frac{3}{2} \frac{\sqrt{DG\Phi}}{r^3} \frac{\Omega_M^{1/2}}{9T} \Rightarrow H_0(r) \cong \frac{3}{2} \frac{\sqrt{DG\Phi}}{9T} \frac{r^{3/2}}{T}; \]  

(4.28)

\[ \Omega_M(r) \cong \frac{3}{2} T \frac{\sqrt{DG\Phi}}{9T} \frac{r^{3/2}}{T} = \frac{3}{2} \frac{\sqrt{DG\Phi}}{9T} T^2 \frac{r^{3/2}}{T}; \]  

(4.29)

\[ \rho_{00}(r) = \frac{3H_0^2}{8\pi G} - \rho_{M0} \cong \frac{1}{4\pi} \left[ \frac{\sqrt{D\Phi^2}}{3GT^2} \right] \frac{r^{3/2}}{T} + D\Phi \frac{r^{3/2}}{T}; \]  

(4.30)
where the numerical parameters $D$, $\Phi$ and $T$ can be deduced from astronomical measurements.

Notwithstanding the fact that the above formulæ are quite simple, this model has a major drawback: the expansion is not homogeneous near $-T$, because also the radiation is not homogeneous; in fact, it goes as

$$a^2 \sim -T H_0^2 \Omega_{R0} a^{-2} \Rightarrow a_r(t) \sim \rho_{R0}(r)^{1/4} \sqrt{4 \left(\frac{2}{3} \pi G\right)^{1/2} (t + T)},$$

which expands faster for bigger $r$.

### 4.4.2 Step functions

An improvement can be achieved by admitting an evolution of the $\Omega$’s. Indeed, we know that the matter and radiation densities do not change just because of the expansion, but they also move through $-T$ by the PDEs. That is why the radiation can be homogeneous near $-T$ and inhomogeneous at $t = 0$: $\Omega_{R0}$ changes with time.

This fact can be roughly described inserting initial and final values, $H_I, \Omega_{MI}, \Omega_{RI}$ resp. $H_F, \Omega_{MF}, \Omega_{RF}$. In other words, $H_0 = H_0(t)$ is a step function that jumps from $H_I$ to $H_F$, and the same holds for the others. The jumps take place at some middle instant $t_m$, at which we assume all the changes to be concentrated.

The evolution law can be written with differentials as

$$H dt = \frac{ada}{\sqrt{\Omega_M a + \Omega_R}} = d \left[2\Omega_M a \sqrt{\Omega_M a \Omega_R}\right] - d \left[\frac{4}{3}(\Omega_M a + \Omega_R)^{3/2}\right] = \frac{2}{3} d[(\Omega_M a - 2\Omega_R)\sqrt{\Omega_M a + \Omega_R}].$$

Setting $a(-T) := 0$ and $a(0) := 1$, it respectively holds that

\[
\begin{aligned}
H_I(t + T) &= (\Omega_{MI} a - 2\Omega_{RI})\sqrt{\Omega_{MI} a + \Omega_{RI}} + 2\Omega_{RI}^{3/2}, & -T \leq t \leq t_m; \\
H_F t &= (\Omega_{MF} a - 2\Omega_{RF})\sqrt{\Omega_{MF} a + \Omega_{RF}} - (\Omega_{MF} - 2\Omega_{RF}), & t_m \leq t \leq 0.
\end{aligned}
\]

(4.33)

Since the Einstein equations are of second order, it must be $a(t) \in C^1(t_m)$. Calling $a_m := a(t_m)$, we can write such request as

\[
\begin{aligned}
\int_{r_I}^{r_m} \left[(\Omega_{MI} a_m - 2\Omega_{RI})\sqrt{\Omega_{MI} a_m + \Omega_{RI}} + 2\Omega_{RI}^{3/2}\right] = \frac{2}{3}(t_m + T) = \\
\int_{r_F}^{r_m} \left[(\Omega_{MF} a_m - 2\Omega_{RF})\sqrt{\Omega_{MF} a_m + \Omega_{RF}} - (\Omega_{MF} - 2\Omega_{RF})\right] + \frac{3}{2}T;
\end{aligned}
\]

(4.34)

Moreover, we can set the initial and final states as

\[
\begin{aligned}
\Omega_{MI}(r) + \Omega_{RI}(r) &= 1 = \Omega_{MF}(r) + \Omega_{RF}(r); \\
H_I(r)^2\Omega_{RI}(r) &= \frac{8}{3}\pi G \rho_{RI}; & H_F(r)^2\Omega_{MF}(r) &= \frac{2}{3} DG \Phi D^{-3}.
\end{aligned}
\]

(4.35)

These are overall 6 conditions involving 7 functions: $\Omega_{MI}(r), \Omega_{RI}(r), \Omega_{MF}(r), \Omega_{RF}(r), H_I(r), H_F(r)$ and $a_m(r)$ (which is equivalent to $t_m$).

In order to obtain the seventh condition, we recall that both the matter and radiation densities depend on $v$, according to the following conservation laws:

\[
\begin{aligned}
\partial_t (V' \rho_M) &= \partial_r (A^2 v \rho_M); \\
\partial_t (V'^{3/2} \rho_R) &= \frac{4}{3} V^{3/2} \partial_r (A^2 v \rho_R).
\end{aligned}
\]

(4.36)
Since here the $\rho$'s jump at $t_m$, all these derivatives have a Dirac delta peak. For this reason, we can neglect the variations $\partial_t(V')$, $\partial_r(\rho_M)$ and $\partial_r(\rho_R)$, and take them approximately constant w.r.t. the jumps. Consequently, the conservation laws become

\[
\begin{align*}
V'\Delta(\rho_M) &\cong \rho_M \Delta(A^2 v); \\
(V^{4/3} \Delta(\rho_R)) &\cong \frac{4}{3} \sqrt[V]{\rho_R} \Delta(A^2 v); \\
\end{align*}
\]

where the $\Delta$'s on the $\rho$'s are intended as $\Delta(f) := \lim_{t \to t_m^-} f(t) - \lim_{t \to t_m^+} f(t)$. Thus, one can rewrite

\[
\frac{4}{3} \Delta \ln \rho_M \cong \Delta \ln \rho_R, \tag{4.37}
\]

and therefore

\[
\Delta \ln \rho_M \cong \Delta \ln \rho_R. \tag{4.38}
\]

Thence, one is able to determine completely the functions involved into the model, with the remaining parameters being just numbers: $T$, $\rho_R$, $D$ and $\Phi$.

In order to obtain a manageable set of algebraic equations, we enforce the approximations $\rho_R \gg \rho_M$, $\rho_M \gg \rho_R$, and $a_m \gg 0$, then reaching the following results:

\[
\begin{align*}
\Omega_{RI} &\cong 1, \quad \Omega_{MI} \cong 0, \quad \Omega_{MF} \cong 1, \quad \Omega_{RF} \cong 0; \\
\end{align*}
\]

\[
H_I(r) \cong \sqrt{\frac{8}{3} \pi G \rho_{RI}}, \quad H_F(r) \cong \sqrt{\frac{8}{3} \pi G \rho_{MF}} = \left(\frac{2}{3} DG\Phi r^{-D-3}\right), \tag{4.40}
\]

and

\[
0 \cong \frac{1}{H_I} \left[\Omega_{MI} a_m - 2\Omega_{RI}\sqrt{\Omega_{MI} a_m + \Omega_{RI}} + 2\Omega_{RI}^{3/2}\right] = \frac{1}{H_F} \left[\Omega_{MF} a_m - 2\Omega_{RF}\sqrt{\Omega_{MF} a_m + \Omega_{RF}} - (\Omega_{MF} - 2\Omega_{RF}) + \frac{3}{2} T \cong \frac{a_m^2}{H_F} - \frac{3}{2} T, \tag{4.41}
\]

yielding

\[
a_m(r) \cong \left[1 - \frac{3}{2} TH_F\right]^{2/3} \cong \left[1 - T \sqrt{\frac{3}{2} DG\Phi r^{-D-3}}\right]^{2/3}. \tag{4.42}
\]

Then, we find from the other constraints that

\[
\begin{align*}
\frac{8}{3} \pi G \rho_{RI} &\cong H_I^2 (\Omega_{MI} a_m + \Omega_{RI}) = H_F^2 (\Omega_{MF} a_m + \Omega_{RF}) = \frac{8}{3} \pi G (\rho_{MF} a_m + \rho_{RF}); \\
\end{align*}
\]

\[
(\frac{\rho_{MF}}{\rho_{MI}})^{4/3} \cong \frac{\rho_{RF}}{\rho_{RI}} \Rightarrow \rho_{MI}(r) \cong \rho_{MF}\left(\frac{\rho_{RI}}{\rho_{RF}}\right)^{3/4} \cong \rho_{MF}\left(\frac{\rho_{RI}}{\rho_{RI} - \rho_{MF} a_m}\right)^{3/4}. \tag{4.44}
\]

The four parameters of our simplified model can be empirically fixed, in order to quantitatively compare the theoretical predictions with the observational measurements. Following [12], we evaluate $D \cong 1.2$, between the magnitude orders $L_G \cong 10^9$ly and $L_{EG} \cong 3 \times 10^9$ly. The fractal density can thus be obtained from the amount of observed matter,

\[
\Phi L_{EG}^D \cong M(L_{EG}) \cong \frac{4}{3} \pi L_{EG}^3 \rho_{B0} = \frac{H_0^2}{2G} L_{EG}^3 \Omega_{B0}; \tag{4.46}
\]

\[
\Phi \cong \frac{H_0^2}{2G} L_{EG}^{3-D} \Omega_{B0} \cong 9.974 \times 10^{24} \text{kg/ly}^D, \tag{4.47}
\]

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where $H_0 \cong 6.867 \times 10^{-11}y^{-1}$ and $\Omega_{B0} \cong 0.044$ are the parameters of the Cosmological Concordance Model (CCM). Analogously, we can evaluate the amount of radiation, 

$$\frac{H_0^2}{3G}L^3_{EG}\Omega_{R0} \cong \int_0^{L_{EG}} \rho_{RF} 4\pi r^2 dr \cong \frac{4}{3}\pi L^3_{EG}\rho_{RI} + \int_0^{L_{EG}} \left[ \frac{4\pi r^2 \rho_{RI} - D\Phi r^{D-1}}{\Phi r} \left(1 - T \sqrt{\frac{3}{2} DG\Phi r^{D-1}} \right) \right]^{2/3} dr 

= \frac{4}{3}\pi L^3_{EG}\rho_{RI} - \left[ \Phi r^{D-2}F_1 \left(-\frac{2}{3}, \frac{2D}{3} - D - 1; \frac{3}{2}D - 3; T \sqrt{\frac{3}{2} DG\Phi r^{D-1}} \right) \right]^{L_{EG}}_{L_G} \downarrow 

\rho_{RI} \cong \rho_{R0} + 2.5\rho_{MF}(L_{EG})_2F_1 \left(-\frac{2}{3}, \frac{4}{3}; -\frac{1}{3}; T \sqrt{1.8G\Phi L_{EG}^{-0.9}} \right) 

-2.5 \left(\frac{L_{G}}{L_{EG}}\right)^3 \rho_{MF}(L_{G})_2F_1 \left(-\frac{2}{3}, \frac{4}{3}; -\frac{1}{3}; T \sqrt{1.8G\Phi L_{G}^{-0.9}} \right), \quad (4.48)$$

where $\frac{85G\rho_{R0}}{3H_0^2} := \Omega_{R0} \cong 8.24 \times 10^{-5}$ are CCM parameters again, and $2F_1$ denotes the hypergeometric function.

The remaining parameter $T$ can be evaluated by fitting further cosmological observations.

## 5 Conclusions

In this paper we have started a systematic development of the framework focussed on the analysis of the consequences of fractal cosmology on the evolution of the Universe. We have proposed a genesis of the cosmic fractal, as well as a partition in epochs in Sec. 1.4, both suitable to obtain quantitative results. Only the first epoch can consistently be described with the usual FLRW solution; on the other hand, the LTB solution was exploited for the second epoch in Sec. 3, and we proved in Sec. 2 that an even more general Lemaître solution is necessary for the third epoch, because of general restrictions arising from the momentum conservation in the LTB metric.

Of course, our calculations admit further improvements, for instance provided by a more precise solution to the evolution equations of the Lemaître model, as discussed in Sec. 4. After our analysis, we may reasonably wonder that a more detailed analysis would describe the fall of the matter fractal onto itself, thus providing self-consistency and stability within fractal cosmology, while the homogeneous FLRW would just be an unstable solution. Future works might also improve the description of the second epoch, e.g. implementing the transformation law of matter into radiation.

It is worth pointing out here that the whole theoretical framework dealing with LTB and Lemaître models provides a smooth approximation to the actual fractal dynamics. Indeed, a more realistic model for fractal cosmology should make use of distributional General Relativity, which is a quite formidable task, or at least of a first order perturbative approximation towards the anisotropic distribution. These latter perturbative methods, applied to an LTB or Lemaître background, should expectedly provide some amount of effective dark matter, due to retarded potentials [14], [15]. Since the fractal approach is able to explain dark energy phenomena [7], it is conceivable that a combined framework will be able to overcome many of the drawbacks of the Cosmological Concordance Model.

Finally, we would like to remark that a deeper quantitative analysis of the LTB metric is of potential relevance also in other frameworks, such as the IR-completion of gravity [10] [11]. or within the attempts to explain the tension of the Hubble parameter [8].

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