Condensate fluctuations in finite Bose-Einstein condensates at finite temperature

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A Langevin equation for the complex amplitude of a single-mode Bose-Einstein condensate is derived. The equation is first formulated phenomenologically, defining three transport parameters. It is then also derived microscopically. Expressions for the transport parameters in the form of Green-Kubo formulas are thereby derived and evaluated for simple trap geometries, a cubic box with cyclic boundary conditions and an isotropic parabolic trap. The number fluctuations in the condensate, their correlation time $\tau_c$, and the temperature-dependent collapse-time of the order parameter as well as its phase-diffusion coefficient are calculated.

I. INTRODUCTION

Bose-Einstein condensation in a weakly interacting Bose-gas in three dimensions in the thermodynamic limit of an infinitely extended system is a second order phase transition in which an order parameter, the macroscopic wavefunction, appears spontaneously with a fixed but arbitrary phase, turning the global $U(1)$-gauge-symmetry connected with particle-number conservation into a spontaneously broken or hidden symmetry. The rigidity of the phase of the order parameter against local perturbations and the absence of any phase diffusion gives rise to the Goldstone modes, which take the form of collision-less (zero)-sound or hydrodynamic sound, respectively, depending on whether the sound frequency is in the collision-less mean-field regime or in the collision-dominated regime.

In finite systems, and thus also in all trapped Bose-gases, sharp phase-transitions are impossible and hidden symmetries in a rigorous sense cannot appear. However, a macroscopic wave-function describing a Bose-Einstein condensate still exists. Its phase cannot be stable and must undergo a diffusion process, which restores the $U(1)$ gauge-symmetry over sufficiently long time intervals. This diffusion process is different from the Goldstone modes mentioned before, which are oscillations around a fixed value of the phase and do not restore the symmetry. Rather the Goldstone modes show up either as collision-dominated hydrodynamic phonons or as collision-less phonons, which have also been observed in the finite Bose-Einstein condensates. In the present paper I would like to discuss the dynamics of the complex amplitude of a Bose-Einstein condensate containing a finite number of particles, and in particular analyse the diffusion of its phase. My discussion will extend and correct in several respects the work published in [6].

The stability of the phase-difference between the macroscopic wave functions of two Bose-Einstein condensates in a trap has been measured. In the experimental set-up the relative phase was measured using a time-domain separated-oscillatory-field condensate interferometer. Over the time-interval of 100 ms scanned in the experiment the relative phase was found to be robust. This experimental result demonstrates that the macroscopic wave functions of the condensates cannot be considered as quantum mechanical wave-functions of many particle systems entangled with each other, whose decoherence would indeed be extremely rapid. Rather, the macroscopic wave functions are appropriately viewed as robust classical objects, their quantum mechanical origin (just like magnets, crystals etc) notwithstanding. This does, of course not preclude that there may be quantum effects, for finite condensates, which lead to corrections of the dynamics described by the underlying classical wave-equation, the well-known Gross-Pitaevskii equation. In a number of papers the dispersion of the phase of a trapped Bose-Einstein condensate at zero-temperature was considered, which is due to fluctuations $\delta \mu$ of the chemical potential $\mu$ in a finite system with fixed particle number. An extension of this mechanism to finite temperature has also been proposed. This effect is not an irreversible phase diffusion but corresponds to an effect of inhomogeneous broadening, similar to the de-phasing of precessing spins occurring in spin-systems due to inhomogeneous broadening. Like the decay of the magnetization can be reversed in spin-echos, the decay of the order parameter expectation value in Bose-Einstein condensates due to a finite variance of $\delta \mu$ is in principle reversible in ‘revivals’. Experiments in Bose-Einstein condensation are done at temperature $k_B T \gg \hbar \omega$ and often even at $k_B T \gg \mu$, where $\omega$ is the geometrical mean of the three main trap frequencies. A process of phase-diffusion process should occur in such a regime due to the interaction of the condensate with a thermal bath of collective modes and quasiparticles. An estimate of this phase-diffusion is of interest for the theory of atom lasers, because the fundamental limit of the line-width of an atom laser for a given temperature depends on it similar to the ‘Schawlow-Townes’-formula for the line-width of a laser.

In this paper a theory of dissipation and thermal fluctuations of a trapped Bose-Einstein condensate will be formulated. First a phenomenological framework for the theory in the form of a Langevin equation will be given in which
dissipation appears via a phenomenological parameter and the fluctuation-dissipation relation is invoked to relate it to maximally three intensity-coefficients of the fluctuations. The solution of the Langevin-equation then determines the relaxation of the condensate-number and the diffusion of the phase, quite similar to the dynamics of a laser-amplitude above threshold. Then the Langevin equation is derived from the microscopic theory and formulas for the phenomenological parameters are derived. These are evaluated for a box-like trap and an isotropic harmonic trap-potential as a function of temperature, particle-number and scattering length. The final section contains a discussion of our results and a comparison with earlier related work. The theory presented here makes no claim to apply to the critical regime, nor can we examine here to what extent it covers the regime below but close to $T_c$, where it may be important to take the dynamics of the thermal cloud of non-condensed atoms into account in addition to the excitations from the condensate.

II. MICROSCOPIC EQUATIONS OF MOTION

The weakly interacting Bose-gas in a trap in standard notation is described by the Hamiltonian

$$\hat{H} = \int d^3x \hat{\psi}^\dagger \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \langle \mu \rangle + \frac{U_0}{2} \hat{\psi}^\dagger \hat{\psi} \right\} \hat{\psi}.$$  \hspace{1cm} (2.1)

The total number of atoms $N$ is fixed, i.e. the Hilbert space is the restriction of the Fock-space of $\hat{\psi}$ to the subspace on which $\hat{N} = N$ is satisfied. $\langle \mu \rangle$ is the average of the chemical potential, which is a fluctuating quantity in a system where $N$ is fixed. Later-on we shall denote the fluctuating part of the chemical potential with $\Delta \mu$. The presence of a Bose-Einstein condensate in equilibrium means that many ($N_0 \gg 1$) particles occupy a single mode of a macroscopic classical matter wave, determined as the mode of lowest energy of the classical Hamiltonian corresponding to eq.(2.1).

The latter is obtained by replacing in $H$ the field operator $\psi(x)$ by the classical field $\psi(x) = \sqrt{\langle N_0 \rangle} \exp(i\phi)\tilde{\psi}_0(x)$. We shall restrict our attention to sufficiently low temperatures below the critical temperature $T_c$ so that the interaction of the condensate with the mean field of the thermal cloud of non-condensed particles is negligible. In this way one finds that the condensate mode $\tilde{\psi}_0(x)$, which we take to be normalized to 1, satisfies the Gross-Pitaevskii equation

$$-\left(\frac{\hbar^2}{2m}\right)\nabla^2 \tilde{\psi}_0 + \left(V(x) + U_0 \langle N_0 \rangle |\tilde{\psi}_0(x)|^2 \right) \tilde{\psi}_0 = \langle \mu \rangle \tilde{\psi}_0.$$  \hspace{1cm} (2.2)

For given $\langle N_0 \rangle$ the average value of the chemical potential $\mu$ follows by imposing the normalization condition

$$\int d^3x |\tilde{\psi}_0(x)|^2 = 1.$$  \hspace{1cm} (2.3)

on the solution of the Gross-Pitaevskii equation, and thereby $\langle \mu \rangle$ like $\tilde{\psi}_0(x)$ becomes a function of the mean atom number in the condensate $\langle N_0 \rangle$. As an important consequence of this fact the chemical potential of the system can be expressed as a function of the average number of atoms in the condensate alone. $\langle N_0 \rangle$ differs from $N$, the fixed total number of atoms, by the average number $(N')$ of non-condensed atoms, which needs to be calculated for given $\langle N_0 \rangle$. The condition $N = \langle N_0 \rangle + \langle N' \rangle$ then fixes $\langle N_0 \rangle$ self-consistently. In the experimentally realized Bose-Einstein condensates it is possible to measure $\langle N_0 \rangle$ directly with reasonable accuracy as a function of temperature, and in practice it is therefore reasonable to regard $\langle N_0 \rangle$ as an experimentally given and known function of temperature. The space-dependent mean number density of the condensate is $n_0(x) = \langle N_0 \rangle |\tilde{\psi}_0(x)|^2$. We shall take the mode function $\tilde{\psi}_0(x)$ in the Gross-Pitaevskii equation as real and positive. (This also means we are not considering condensates containing vortices). The physical phase of the condensate is not carried by its mode-function $\tilde{\psi}_0$ but by its complex amplitude denoted as $\alpha_0$, where $\alpha_0 = \sqrt{\langle N_0 \rangle} \exp i\phi$.

If $|\alpha_0|^2$ makes a small fluctuation away from its equilibrium value $\langle N_0 \rangle$ the condensate mode function $\tilde{\psi}_0$ will no longer satisfy eq.(2.2) but will change its form slightly. We shall assume that such fluctuations of $N_0 = |\alpha_0|^2$ occur on a sufficiently large time-scale, that the new form is again determined by the Gross-Pitaevskii equation, but for the changed condensate number $|\alpha_0|^2$ and a correspondingly changed chemical potential $\mu_0$ determined uniquely by $|\alpha_0|^2$, i.e. in eq.(2.2) the replacements $(\tilde{\psi}_0, \langle N_0 \rangle, \langle \mu \rangle) \rightarrow (\psi_0, |\alpha_0|^2, \mu_0)$ have to be made in this case,

$$-\left(\frac{\hbar^2}{2m}\right)\nabla^2 \psi_0 + \left(V(x) + U_0 |\alpha_0|^2 |\psi_0(x)|^2 \right) \psi_0 = \mu_0 \psi_0.$$  \hspace{1cm} (2.4)

We cannot expect, in general, that in any given nonequilibrium state the difference defined by $\Delta_0 \mu = \mu_0 - \langle \mu \rangle$ is the total deviation of the chemical potential from its equilibrium value, because there may obviously be states with
\[ |\alpha_0|^2 = \langle N_0 \rangle \] which differ in other respects from the equilibrium state and may therefore have \( \mu \neq \langle \mu \rangle \). Therefore we use the notation \( \mu_0 \) for the part of the nonequilibrium chemical potential determined by \( |\alpha_0|^2 \).

The presence of the highly occupied condensate mode makes the decomposition of the Heisenberg field-operator

\[ \hat{\psi}(x, t) = (|\alpha_0| \exp(i\phi)\psi_0(x) + \hat{\chi}(x, t)) \exp(-i\langle \mu \rangle t / \hbar) \] (2.5)

useful, where we follow Bogoliubov [11] and describe the condensate classically. \( \hat{\chi}(x, t) \) is taken to be the field operator for the particles outside the condensate. We shall assume that the temporal changes in \( \phi \) can be considered as slow on the time-scales of the dynamics of \( \hat{\chi} \). The phase \( \phi \) and amplitude \( |\alpha_0| \) are additional c-number variables in (2.5). Therefore the taking of expectation values has from now on to include an integration over a distribution of \( |\alpha_0| \) and in addition an integration over all values of \( \phi \). Since the total number \( N \) is fixed \( \langle \hat{\psi} \rangle = 0 \) must hold for all times. However, it will also be useful to consider expectation values in the Fock-space of the operators \( \hat{\chi}, \hat{\chi}^+ \) alone without averaging over \( \phi \). Such expectation values will be denoted as \( \langle \ldots \rangle_\phi \).

Gauge invariance, strictly speaking, is lost by splitting off a c-number term from the field-operator. However, this symmetry is saved by adopting the rule that the phase \( \phi \) of the c-number term in the decomposition also changes under a gauge transformation according to \( \phi \to \phi + \epsilon \). By this device we take into account the fact that the same change of phase would have occurred automatically, if we had not replaced the condensate term by a c-number. The generator of gauge transformations is thus taken as

\[ \hat{N} = i \frac{\partial}{\partial \phi} \bigg|_{\hat{\chi},\hat{\chi}^+} + \int d^3x \hat{\chi}^+ \hat{\chi}, \] (2.6)

from which it is clear (cf. eq. (2.24)) that \( i \frac{\partial}{\partial \phi} \bigg|_{\hat{\chi},\hat{\chi}^+} \) is a representation of \( \hat{N}_0 \). The canonical-conjugate is the phase \( \dot{\phi} \) with

\[ \exp(i\dot{\phi}) = \exp(\partial/\partial N_0)_{\hat{\chi},\hat{\chi}^+}. \] (2.7)

Via (2.5) the Hamiltonian furthermore splits up according to \( \hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4 \) where \( \hat{H}_n \) comprises the terms of \( \hat{H} \) which are of n-th order in \( \hat{\chi}, \hat{\chi}^+ \). Explicitly

\[ \hat{H}_0 = |\alpha_0|^2 \int d^3x \psi_0 \left\{- \frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu_0 + \frac{U_0}{2} |\alpha_0|^2 |\psi_0|^2 \psi_0 + (\mu_0 - \langle \mu \rangle) |\alpha_0|^2 \right\} \] (2.8)

\[ \hat{H}_1 = |\alpha_0| \int d^3x \{ e^{-i\phi} \hat{\chi} \left( - \frac{\hbar^2}{2m} \nabla^2 + V(x) - \langle \mu \rangle + U_0 |\alpha_0|^2 \psi_0^2 \psi_0 + (h.c.) \right) \} \] (2.9)

\[ \hat{H}_2 = \int d^3x \{ \hat{\chi}^+ \left( - \frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu_0 \right) \hat{\chi} \]
\[ + U_0 |\alpha_0|^2 \psi_0^2 \left( e^{-2i\phi} \hat{\chi}^2 + e^{2i\phi} \hat{\chi}^2 + 4 \hat{\chi}^+ \hat{\chi} \right) \]
\[ + (\mu_0 - \langle \mu \rangle) \hat{\chi}^+ \hat{\chi} \} \] (2.10)

\[ \hat{H}_3 = U_0 |\alpha_0| \int d^3x \psi_0 \hat{\chi}^+ (e^{-i\phi} \hat{\chi} + e^{i\phi} \hat{\chi}^+) \hat{\chi} \] (2.11)

\[ \hat{H}_4 = \frac{U_0}{2} \int d^3x \hat{\chi}^+ \hat{\chi} \hat{\chi} \hat{\chi}. \] (2.12)

Using the Gross-Pitaevskii equation (2.4) and its derivative with respect to \( |\alpha_0|^2 \) we derive in the appendix

\[ ^1 \text{This operator with fixed } \hat{\chi}, \hat{\chi}^+ \text{ has to be well distinguished from the unrestricted derivative operator } i(\partial/\partial \phi), \text{ which is a representation of the total particle number } N \text{ and has as formal canonical-conjugate } \dot{\phi} \text{ with } \exp(i\dot{\phi}) = \exp(\partial/\partial N). \]
The term $\hat{H}_1$ can be simplified using the Gross-Pitaevskii equation (2.4) and becomes then

\[
\hat{H}_1 = |\alpha_0| (\mu_0 - \langle \mu \rangle) \int d^3x \{ e^{-i\phi} \hat{\chi} + e^{i\phi} \hat{\chi}^+ \} \psi_0
\]

(2.14)

This expression will be seen to vanish below due to an orthogonality condition.

The first part of $\hat{H}_2$ in (2.10) is diagonalized by introducing quasi-particle operators $\hat{\alpha}_\nu, \hat{\alpha}_\nu^+$ defined by the standard Bogoliubov transformation, with time-dependent $\phi(t)$

\[
\hat{\chi}(x) = e^{i\phi} \sum_\nu (u_\nu(x) \hat{\alpha}_\nu + v_\nu^*(x) \hat{\alpha}_\nu^+).
\]

(2.15)

The $u_\nu, v_\nu$ satisfy the usual Bogoliubov-Fetter equations

\[
\begin{pmatrix}
\frac{-\hbar^2}{2m} \nabla^2 + U_{\text{eff}}(x) - \hbar \omega_\nu & K(x) \\
K(x) & -\frac{-\hbar^2}{2m} \nabla^2 + U_{\text{eff}}(x) + \hbar \omega_\nu
\end{pmatrix}
\begin{pmatrix}
(u_\nu(x)) \\
v_\nu(x)
\end{pmatrix} = 0,
\]

(2.16)

with the abbreviations

\[
U_{\text{eff}}(x) = V(x) - \mu_0 + 2U_0 |\alpha_0|^2 \psi_0(x)^2
\]

\[
K(x) = |\alpha_0|^2 U_0 \psi_0(x)^2.
\]

(2.17)

The Hamiltonian $\hat{H}_2$ now takes the form

\[
\hat{H}_2 = \sum_\nu \hbar \omega_\nu (\hat{\alpha}_\nu^+ \hat{\alpha}_\nu + |v_\nu|^2) + (\mu_0 - \langle \mu \rangle + \hbar \dot{\phi}) \int d^3x \hat{\chi}^+ \hat{\chi}
\]

(2.18)

The coefficients $u_\nu, v_\nu$ and the mode frequencies $\omega_\nu$ become also functions of $|\alpha_0|$ and fluctuate (slowly) with that number. Their equilibrium values will be denoted by $\bar{u}_\nu, \bar{v}_\nu$ and $\bar{\omega}_\nu$ and the corresponding operator $\hat{\chi}$ according to Eq. (2.15) as $\hat{\chi}$. Eq. (2.16) is consistent with the ortho-normality conditions

\[
\int d^3x (u_\mu u_\mu^* - v_\mu v_\mu^*) = \delta_{\nu\mu}
\]

(2.19)

\[
\int d^3r (u_\nu^* v_\mu - u_\mu^* v_\nu) = 0
\]

(2.20)

which guarantee the Bose commutation relations of the $\alpha_\nu, \alpha_\nu^+$. A formal solution of eq. (2.16) at zero energy $\hbar \omega_\nu = 0$ is given by the condensate

\[
u_\nu(x) = -v_\nu^*(x) = \psi_0(x), \quad \omega_\nu = 0.
\]

(2.21)

This solution is obviously not normalizable in the required sense (2.13) to furnish an acceptable solution for the $u_\nu, v_\nu$ and must therefore be excluded from the sum over the terms containing the operators $\hat{\alpha}_\nu, \hat{\alpha}_\nu^+$. The existence of this formal solution implies however that the properly normalizable solutions $u_\nu, v_\nu$ and the condensate mode $\psi_0$ satisfy the important orthogonality relation

\[
\int d^3x \psi_0(u_\nu + v_\nu) = 0.
\]

(2.22)

It follows with (2.13) that

\[
\int d^3x \psi_0(e^{-i\phi} \hat{\chi} + e^{i\phi} \hat{\chi}^+) = 0
\]

(2.23)

which in turn implies that the reduced expression (2.14) for $\hat{H}_1$ vanishes. Using the property (2.23) one can verify that the decomposition (2.5) of $\psi$ implies
\[ N = |\alpha_0|^2 + \hat{N}' \]  

(2.24)

with

\[ \hat{N}' = \int d^3x \hat{\chi}^+(x) \hat{\chi}(x) \]

\[ = \sum_{\nu, \mu} \int d^3x \left( \hat{\alpha}_\nu^+ \hat{\alpha}_\mu (u^*_\nu u^*_\mu + v^*_\nu v^*_\mu) + \frac{1}{2} \hat{\alpha}_\nu \hat{\alpha}_\mu (u^*_\nu v^*_\mu + v^*_\nu u^*_\mu) + \frac{1}{2} \hat{\alpha}_\mu \hat{\alpha}_\nu (u^*_\mu v^*_\nu + v^*_\mu u^*_\nu) + \delta_{\nu\mu} |v_\nu|^2 \right) \]

(2.25)

which serves as a definition of \( N_0 = |\alpha_0|^2 \). The mean thermal density \( n' \) in equilibrium can now be determined via

\[ n'(x) = \langle \hat{\chi}^+(x) \hat{\chi}(x) \rangle = \sum_\nu \left\{ (|\tilde{\nu}_\nu(x)|^2 + |\tilde{v}_\nu(x)|^2) \tilde{n}_\nu + |\tilde{v}_\nu(x)|^2 \right\} \]

(2.26)

with \( \tilde{n}_\nu = (\exp(\beta \tilde{\omega}_\nu) - 1)^{-1} \).

The fluctuations of \( N_0 \) are similarly fixed by

\[ \langle \Delta N_0^2 \rangle = \langle N_0^2 \rangle - \langle N_0 \rangle^2 = \langle \Delta \hat{N}'^2 \rangle \]

(2.27)

\[ = \sum_\nu \sum_\nu' \left\{ \tilde{n}_\nu (\tilde{n}_\nu' + 1) \left| \int d^3x (\tilde{u}_\nu^*(x) \tilde{u}_\nu'(x) + \tilde{v}_\nu'(x) \tilde{v}_\nu(x)) \right|^2 \right. 

\[ + (\tilde{n}_\nu + \tilde{n}_\nu' + 1) \left| \int d^3x (\tilde{u}_\nu(x) \tilde{u}_\nu'(x) + \tilde{v}_\nu'(x) \tilde{v}_\nu(x)) \right|^2 \right\} \]

(2.28)

They have been evaluated in [14] and are also needed below (see eq. (8.20)). For work in the mathematical physics literature on number-fluctuations in the condensate of the ideal Bose-gas and models of the interacting Bose-gas see [12][13] and references given there. For an alternative proposal to define and calculate the number-fluctuations in a Bose-condensate see [15].

After the transformation (2.15) the Hamiltonian is now in the form

\[ \hat{H} = \hat{H}_0 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4 \]

(2.29)

with \( \hat{H}_0, \hat{H}_2, \hat{H}_3, \hat{H}_4 \) given by eqs. (2.13)(2.18)(2.11)(2.12).

### III. Langevin-Equation of the Condensate Amplitude

Neither the Gross-Pitaevskii equation nor the Bogoliubov-Fetter equations furnish an equation for the condensate amplitude \( \alpha_0 = \sqrt{N_0} \exp(i\phi) \). To find such an equation phenomenologically we first turn to a macroscopic quantity like the entropy \( S(|\alpha_0|^2, N) \) for a fixed particle number \( N \), but restricted to a fixed arbitrary value of \( \alpha_0 = \sqrt{N_0} \exp(i\phi) \), where \( N_0 \) is the instantaneous number of particles in the condensate and different from the equilibrium value \( \langle N_0 \rangle \) corresponding to the maximum of \( S(|\alpha_0|^2, N) \). Thus \( \langle N_0 \rangle \) is a function of \( N \). The fluctuations of \( N_0 \) in the closed system formed by the trapped condensate after the evaporative cooling has been switched off are governed by a canonical Boltzmann-Einstein distribution

\[ P(N_0) = \Omega^{-1} \exp \left( S(|\alpha_0|^2, N)/k_B \right). \]

We shall restrict ourselves to temperatures in the condensed regime outside the critical regime, where \( \langle N_0(N) \rangle \) is much larger than its root mean square \( \sqrt{\langle \Delta N_0^2(N) \rangle} = \sqrt{\langle \Delta \hat{N}'^2 \rangle} = (\langle \hat{N}' \rangle - (\langle \hat{N}' \rangle)^2)^{1/2} \), which is also a function of \( N \). Then \( S(|\alpha_0|^2, N) \), expanded to lowest order around its maximum, takes the form

\[ S(|\alpha_0|^2, N) = S^{(eq)}(N) + \Delta S(|\alpha_0|^2, N) \]

with
\[
\Delta S(|\alpha_0|^2, N) = -k_B \frac{(|\alpha_0|^2 - \langle N_0 \rangle)^2}{2\langle N_0^2 \rangle}
\]  

(3.1)

The entropy \( S(|\alpha_0|^2, N) \) not only determines the equilibrium-distribution of the condensate amplitude, but appears also in its equation of motion, both in the conservative part of the dynamics as a conserved quantity, and in the dissipative part as a potential for the irreversible part of the dynamics. Let us first consider both parts separately.

The conservative part of the dynamics of \( \alpha_0 \) is connected with the dynamics of its phase \( \phi \). According to eqs. (2.3-2.12) a change of \( \phi \) changes the total phase of the field-operator \( \hat{\psi} \). For this reason the dynamics of \( \phi \) is given by the equation of motion

\[
\dot{\phi} = -\frac{1}{\hbar} \frac{\partial (\hat{H})}{\partial N} = -\frac{1}{\hbar} \Delta \mu
\]

(3.2)

where \( \Delta \mu \) is the deviation of the chemical potential from its equilibrium value. Such deviations may occur as a result of any fluctuations present in the system and, as discussed already, may in particular occur as a result of fluctuations of the value of \( N_0 \) away from its average \( \langle N_0 \rangle \). This part of the fluctuation of \( \mu \) we shall denote as \( \Delta_0 \mu \). Expanding again to lowest order around the equilibrium \( N_0 = \langle N_0 \rangle \) we can write

\[
\Delta_0 \mu = \frac{\partial (\mu)}{\partial \langle N_0 \rangle} (|\alpha_0|^2 - \langle N_0 \rangle)
\]

(3.3)

The systematic part of the conservative part of the equation of motion of \( \alpha_0 \) can now be written in the form

\[
(i\hbar \hat{a}_0)_{cons} = \Delta_0 \mu \alpha_0.
\]

(3.4)

It is convenient to introduce the fluctuation of the free energy by \( \Delta F = -T \Delta S \). The dynamics (3.4) conserves \( |\alpha_0|^2 \) and \( \Delta F \). In equilibrium the right-hand side of this equation vanishes, because there \( \Delta_0 \mu = 0 \), and the total phase of the condensate \( \phi - \langle \mu \rangle t/\hbar \) changes only with a rate given by the average chemical potential \( \langle \mu \rangle \) in equilibrium.

The dissipative part of the equation of motion of \( \alpha_0 \) near thermal equilibrium is written with the help of \( \Delta F \) in the general form

\[
\hbar (\dot{\hat{a}}_0)_{diss} = -\Gamma_0 \frac{\partial \Delta F(|\alpha_0|^2, N)}{\partial \alpha_0^*}
\]

(3.5)

which contains the positive phenomenological parameter \( \Gamma_0 \) and describes the relaxation of \( N_0 = |\alpha_0|^2 \) to its equilibrium-value \( \langle N_0 \rangle \).

According to general principles of statistical thermodynamics the relaxation process (3.3) must be accompanied by some form of noise. Adding a noise-term the total Langevin-equation of \( \alpha_0 \) can be written in the form

\[
i\hbar \dot{\alpha}_0 = \Delta_0 \mu \alpha_0 - i\Gamma_0 \frac{\partial \Delta F(|\alpha_0|^2, N)}{\partial \alpha_0^*} + \xi(t) \exp(i\phi).
\]

(3.6)

Since the condensate amplitude \( \alpha_0 \) is a collective quantity the noise \( \xi(t) \) can be assumed to be Gaussian due to the central limit theorem. In addition we shall assume \( \xi(t) \) to be a white noise force. This means that the actual correlation time \( \tau_m \) of the noise \( \xi \) is assumed to be much smaller than the time-scale on which the dynamics of \( \alpha_0 \) is observed, an assumption which must be checked for its validity in an any concrete microscopic description. (In the microscopic theory we describe later it is a consistent assumption because the relaxation rate \( \gamma_e \) of \( |\alpha_0|^2 \) turns out to be small compared to the time-scale of motion in the trap.) Thus we assume that \( \langle \xi(t) \rangle = 0 \) and

\[
\langle \xi^*(t) \xi(0) \rangle = \hbar k_B T (2\Gamma_0 + \Gamma') \delta(t)
\]

(3.7)

\[
\langle \xi(t) \xi(0) \rangle = \hbar k_B T (\Gamma'' + i\Gamma''') \delta(t)
\]

(3.8)

where \( \Gamma_0 \) reappears in (3.7) because of the fluctuation-dissipation theorem. The form of the Langevin-equation (3.7) generalizes the work in [6] by taking into account a possible correlation of the phase of the condensate and of the Langevin force, which may exist in condensates with finite particle numbers due to gauge invariance, i.e. particle-number conservation. (However, it will turn out later that the coefficient \( \Gamma'' \) vanishes in condensates with a real condensate mode i.e. without vortices, which can be understood generally as a consequence of time-reversal symmetry.) Gauge-invariance implies that the Langevin-equation for \( \alpha_0 \), including the fluctuating term, must be invariant under the transformation \( \phi \rightarrow \phi + \epsilon \). This makes it useful to write the fluctuating term as \( \exp(i\phi (t)) \xi(t) \).
where $\xi(t)$ is a complex noise source independent of $\phi$, which, physically, describes the scattering of particles in the condensate with those outside. The coefficients $\Gamma', \Gamma''$ describe a possible correlation of the phases of $F_0$ and $\alpha_0$, i.e. the existence of a squeezing in the thermal bath of uncondensed particles, caused by the constraint of total particle-number conservation. We shall see that this effect actually does occur in finite condensates, i.e. the condensate mode imprints its (slowly) fluctuating phase on the non-condensed ‘environment’ due to particle number conservation in such a way that the lowest-lying modes are nearly totally squeezed.

The multiplicative nature of the noise in (3.6) raises the question in which stochastic calculus this equation should be interpreted: in the sense of Ito, or Stratonovich, or in some intermediate sense? This will be specified in a moment. Within the Gauss- and Markoff-assumption the form of the noise-force with the same positive coefficient $\Gamma$ appearing in the dissipative part (3.5) and two further real coefficients $\Gamma', \Gamma''$, is fixed by the requirement that the Langevin-equation must be consistent with the correct equilibrium distribution $\frac{\partial}{\partial N^0} P(\alpha_0, \alpha^*_0) = Z^{-1} \exp(-\Delta F(\vert \alpha_0 \vert^2, N)/k_B T)$ for the condensate. Splitting into real and imaginary parts eqs. (3.7, 3.8) become

\begin{align}
\langle \Re(\xi(t))\Re(\xi(0)) \rangle &= \frac{\hbar k_B T (\Gamma_0 + \Gamma')}{\hbar} \delta(t) \\
\langle \Im(\xi(t))\Im(\xi(0)) \rangle &= \frac{\hbar k_B T \Gamma_0}{\hbar} \delta(t) \\
\langle \Re(\xi(t))\Im(\xi(0)) \rangle &= \frac{1}{2} \frac{\hbar k_B T \Gamma''}{\hbar} \delta(t).
\end{align}

Eq. (3.6) may now be rewritten as

\begin{align}
\frac{\partial N_0}{\partial t} &= -2 \frac{\Gamma_0}{\hbar} \left( N_0 \frac{\partial \Delta F}{\partial N_0} - k_B T \right) + \frac{2}{\hbar} \sqrt{N_0} \Im(\xi(t)) \\
\frac{\partial \phi}{\partial t} &= -\frac{1}{\hbar} \Delta_0 \mu - \frac{1}{\hbar} \sqrt{N_0} \Re(\xi(t))
\end{align}

and must in this form be interpreted as a stochastic differential equation in the sense of Ito.

Eq. (3.13) can be compared with (3.2). This comparison reveals that $\Re(\xi(t))$ must describe the fluctuations of the chemical potential not caused by deviations of $\vert \alpha_0 \vert^2$ from its equilibrium value but by other fluctuations in the system. We shall come back to this point in section VI below.

The three phenomenological coefficients $\Gamma_0, \Gamma', \Gamma''$, are dimensionless, temperature dependent numbers, which must be determined from a microscopic theory. Only one of these coefficients, $\Gamma_0$, is connected with the fluctuations of the number of condensed atoms. If fluctuations of the chemical potential due to other processes are neglected, i.e. $\Re(\xi(t)) = 0$, the remaining two coefficients are fixed at

$$\Gamma' = -\Gamma_0, \quad \Gamma'' = 0$$

This corresponds to the case of maximal squeezing of the noise in the phase-direction.

Linearizing with respect to the small fluctuations $\delta N_0 \ll \langle N_0 \rangle$ we find

\begin{align}
\hbar \delta \dot{N}_0 &= -\frac{2k_B T}{\langle \Delta N^0 \rangle} \langle N_0 \rangle \Gamma_0 \delta N_0 + 2 \sqrt{\langle N_0 \rangle} \Im(\xi(t)) \\
\hbar \dot{\phi} &= -\frac{\partial (\mu)}{\partial \langle N_0 \rangle} \delta N_0 - \frac{1}{\sqrt{\langle N_0 \rangle}} \Re(\xi(t))
\end{align}

A simpler ansatz (see 3.1) ignores the $\phi$-dependence of the Langevin-force in (3.6). Then gauge-invariance of the Fokker-Planck equation which is stochastically equivalent to the Langevin equation implies $\Gamma' = \Gamma'' = 0$, i.e. the complex noise $F_0(t)$ then has random phase-fluctuations which are completely uncorrelated with and equi-distributed with respect to the condensate-phase $\phi$. Note however that this achieves gauge-invariance only in an ensemble sense, not for each individual stochastic physical realization which together form the ensemble. In contrast the form of the Langevin equation considered here does enforce gauge invariance for each stochastic realization.

Then the Fokker-Planck equation corresponding to eqs. (3.12, 3.13) is $\hbar \partial P/\partial t = 2\Gamma_0 \partial /\partial N^0 [N_0 (\partial \Delta F / \partial N^0 + k_B T \partial / \partial N^0) P]$, and has the desired equilibrium distribution $\sim \exp(-\Delta F/k_B T)$. This implies that eq. (3.6) must be interpreted in some intermediate sense, which we need not specify here further. In order to obtain its version in the sense of Ito it is best to bring (3.12, 3.13) into the form (3.4) using the Ito calculus.
Eq. (3.13) describes the relaxation of the condensate to the equilibrium at $\langle N_0 \rangle = |\alpha_0|^2$ on the time-scale

$$\tau_c = \frac{\hbar (\Delta N_0^2)}{2\Gamma_0 (N_0) k_B T}$$  \hspace{1cm} (3.17)

and the thermal fluctuations around it with the correlation function

$$\langle \delta N_0(t) \delta N_0(t') \rangle = \langle \Delta N_0^2 \rangle e^{-|t-t'|/\tau_c}$$  \hspace{1cm} (3.18)

The correlation time $\tau_c$ is an important time-scale of the problem. The noise sources $\Im (\xi(t))$, $\Re (\xi(t))$ must have correlation times short compared to $\tau_c$ in order to be well described by white noise.

On a time-scale much larger than the correlation time $\tau_c$, the fluctuations $\delta N_0(t)$ in the equation for the phase can also be considered as Gaussian white noise with correlation function $2\tau_c \langle \Delta N_0^2 \rangle \delta(t)$. Using this long-time approximation in the equation for the phase $\phi$ and taking the correlation of the effective white noise $\delta N_0(t)$ with $\Re (\xi(t))$ properly into account, $\phi(t)$ is found to satisfy the Langevin equation of a Wiener process

$$d\phi(t) = \sqrt{D_\phi} dw$$

with $(dw)^2 = dt$ and diffusion constant

$$D_\phi = \left(\frac{\Delta N_0^2}{\hbar \langle N_0 \rangle} \frac{\partial (\mu)}{\partial \langle N_0 \rangle} \left( \frac{\Delta N_0^2}{k_B T} \frac{\partial (\mu)}{\partial \langle N_0 \rangle} \Gamma_0 + \Gamma'' \right) \right) + \frac{k_B T}{\hbar \langle N_0 \rangle} (\Gamma_0 + \Gamma)'$$ \hspace{1cm} (3.19)

i.e.

$$\langle (\phi(t) - \phi(0))^2 \rangle = D_\phi |t|.$$ \hspace{1cm} (3.20)

Eq. (3.19) agrees with the result of [6] if we assume as in [6] no squeezing in the noise, i.e. $\Gamma' = \Gamma'' = 0$ and $\partial (\mu) / \partial \langle N_0 \rangle = k_B T / \Delta N_0^2$. Both assumptions will not be made in the present work, however (cf. also the corresponding discussion in the final section).

The expectation value $\langle \alpha_0(t) \rangle$ then decays exponentially according to $\langle \alpha_0(t) \rangle = \langle \alpha_0(0) \rangle e^{-\Delta \nu |t|}$ with the line-width $\Delta \nu$ given by the Schawlow-Townes-type formula $\Delta \nu = \frac{1}{\hbar} D_\phi$.

It is not difficult to solve eqs. (3.15, 3.16) for the phase-fluctuations also on time-scales of the order of $\tau_c$. The result for the mean square of the phase-increment in time is

$$\langle (\phi(t) - \phi(0))^2 \rangle = D_\phi |t| + 2 \frac{\partial (\mu)}{\partial \langle N_0 \rangle} \left( k_B T \Gamma'' + \langle \Delta N_0^2 \rangle \frac{\partial (\mu)}{\partial \langle N_0 \rangle} \right) \tau_c^2 (e^{-|t|/\tau_c} - 1).$$ \hspace{1cm} (3.21)

It interpolates between the diffusive long-time behavior (3.20) for $t >> \tau_c$ and the short-time behavior for $t \ll \tau_c$

$$\langle (\phi(t) - \phi(0))^2 \rangle = k_B T \langle N_0 \rangle (\Gamma_0 + \Gamma') |t| + \frac{1}{\hbar^2} \frac{\partial (\mu)}{\partial \langle N_0 \rangle} \left( \frac{\partial (\mu)}{\partial \langle N_0 \rangle} \langle \Delta N_0^2 \rangle + k_B T \Gamma'' \right) |t|^2.$$ \hspace{1cm} (3.22)

The first term describes phase diffusion due to thermal fluctuations of the chemical potential on time scales much shorter than $\tau_c$. The second term describes a non-diffusive and in principle reversible phase-collapse with the collapse rate \hspace{2cm}

$$\gamma_{\text{collapse}} = \frac{1}{\hbar} \sqrt{\frac{\partial (\mu)}{\partial \langle N_0 \rangle} \left( \frac{\partial (\mu)}{\partial \langle N_0 \rangle} \langle \Delta N_0^2 \rangle + k_B T \Gamma'' \right)}$$ \hspace{1cm} (3.23)

including a contribution from the cross-correlation between both types of fluctuations.

**IV. MICROSCOPIC DERIVATION OF THE LANGEVIN EQUATION**

The microscopic derivation of the equation of motion for the condensate amplitude $\alpha_0$ can be carried out by using the Hamiltonian (2.22). As we did for the phenomenological equations in the preceding section we wish to derive here the microscopic equation of motion only to first order in the deviation $(|\alpha_0| - \sqrt{\langle N_0 \rangle})$ from equilibrium.
$H_0$ given by eq.\(^{(2.13)}\) is the Hamiltonian, in mean field approximation, of the pure condensate. Its free equation of motion is

$$i\hbar\dot{\alpha}_0 = \frac{\partial H_0}{\partial \alpha_0} = (\mu_0 - \langle \mu \rangle)\alpha_0, \quad (4.1)$$

from which

$$\frac{d\phi(t)}{dt} = -\langle \mu_0(|\alpha_0(t)|^2) - \langle \mu \rangle \rangle / \hbar \quad (4.2)$$

follows. Let us first use \((4.2)\) in \((2.18)\) to simplify $\hat{H}$, with respect to it. This changes $\hat{\chi}$, $\hat{\alpha}$ state by $|\alpha\rangle$. We obtain $\hat{\chi}$ in $H_3$, $H_4$ according to

$$\hat{\chi} \rightarrow \hat{\chi}(t) = e^{i\phi(t)} \sum_\nu (\psi_\nu e^{-i\omega_\nu t} + \psi_\nu^* e^{i\omega_\nu t}) \quad (4.3)$$

and its adjoint. The transformed time-dependent Hamiltonians will be denoted as $\hat{H}_3(t)$, $\hat{H}_4(t)$, but $\hat{H}_4(t)$ will not be needed in the following.

The equation of motion of the condensate amplitude $\alpha_0$ now takes the form with the notation $\hat{H}(t) = H_0 + \hat{H}_3(t) + \hat{H}_4(t)$,

$$\frac{\hbar}{\partial t} \frac{d\phi}{dt} = -\left( \frac{\partial \hat{H}(t)}{\partial |\alpha_0|^2} \right)_{\phi, \hat{\chi}, \hat{\chi}^+} \quad (4.4)$$

$$\frac{\hbar}{\partial t} \frac{d|\alpha_0|^2}{dt} = \left( \frac{\partial \hat{H}(t)}{\partial \phi} \right)_{|\alpha_0|, \hat{\chi}, \hat{\chi}^+} \quad (4.5)$$

We obtain

$$\frac{\hbar}{\partial t} \frac{d\phi}{dt} = -\Delta_0 \mu - \frac{1}{\sqrt{\langle N_0 \rangle}} \Re(\hat{\xi}'(t)) - \frac{1}{\sqrt{\langle N_0 \rangle}} \delta \Re(\hat{\xi}'(t)) \quad (4.6)$$

$$\frac{\hbar}{2\partial t} \frac{d|\alpha_0|^2}{dt} = 2\sqrt{\langle N_0 \rangle} \Im(\hat{\xi}(t)) + 2\sqrt{\langle N_0 \rangle} \delta \Im(\hat{\xi}(t)) \quad (4.7)$$

with

$$\Re(\hat{\xi}'(t)) = \frac{1}{2} U_0 \int d^3x (\hat{\psi}^0_0 + 2\langle N_0 \rangle \frac{\partial \hat{\psi}^0_0}{\partial \langle N_0 \rangle}) \hat{\chi}^+(t)(e^{-i\phi} \hat{\chi}(t) + e^{i\phi} \hat{\chi}^+(t)) \hat{\chi}(t) \quad (4.8)$$

$$\Im(\hat{\xi}(t)) = \frac{1}{2i} U_0 \int d^3x (\hat{\psi}^0_0 \hat{\chi}^+(t)(e^{-i\phi} \hat{\chi}(t) - e^{i\phi} \hat{\chi}^+(t)) \hat{\chi}(t) \quad (4.9)$$

The complex noise $\xi(t)$ in eqs.\((3.6)-(3.13)\) should be identified with

$$\hat{\xi}(t) = \Re(\hat{\xi}'(t)) + i\Im(\hat{\xi}(t)). \quad (4.10)$$

It is indeed independent of $\phi$ as required by gauge-invariance of the Langevin-equation, as can be seen from \((1.6)\), \((1.7)\) with \((1.3)\). We shall see in the next section that $\hat{\xi}(t)$ can be replaced by a c-number.

To describe fluctuations around equilibrium we have replaced in the preceding expressions the quantities $|\alpha_0|^2$, $\hat{\psi}_0$, $\hat{\chi}$, $\hat{\chi}^+$ by their equilibrium expressions $\sqrt{\langle N_0 \rangle}$, $\hat{\psi}_0$, $\hat{\chi}$, $\hat{\chi}^+$ and represent the difference in the nonequilibrium state by $\delta \Re(\hat{\xi}'(t))$, $\delta \Im(\hat{\xi}(t))$ in eqs.\((1.6)-(1.7)\). Omitting these differences altogether amounts to neglecting the back-action of the condensate on the thermal reservoir, which describes not only a modification of the fluctuating forces.

\(^4\)For simplicity we disregard here the slow time-dependence of the frequencies $\omega_\nu$.

\(^5\)For the canonically conjugate pair $N_0, \phi$ at fixed $\hat{\chi}$, $\hat{\chi}^+$ cf. eqs.\((2.4)-(2.7)\).
which can indeed be neglected for fluctuations around a stable thermodynamic equilibrium, but also dissipation. To take the latter into account we need to calculate the averages \( \delta \langle \delta \Re(\hat{\xi}(t)) \rangle_\phi, \delta \langle \delta \Im(\hat{\xi}(t)) \rangle_\phi \) to lowest order in the interaction between the condensate and the thermal cloud of atoms. The form which these quantities must take is prescribed completely by the fluctuation-dissipation theorem and symmetry:

For the reversible phase-dynamics the back-action can only lead to a shift in the average chemical potential. Such shifts due to the interaction \( \hat{H}_3 \) will be small and are neglected here. For the irreversible amplitude dynamics the fluctuation-dissipation theorem requires in addition the appearance of a dissipation term. If

\[
S_{JJ}(t - t') = \langle \Im(\hat{\xi}(t))\Im(\hat{\xi}(t')) \rangle_\phi
\]

is the correlation function of the fluctuating force in (4.7), the back action must modify eq.(4.7) to the form

\[
\hat{h}\frac{d|\alpha_0|^2}{dt} = -\frac{4\langle N_0 \rangle}{\hbar k_B T} \int_{-\infty}^{t} dt' S_{JJ}(t - t') \frac{\partial H_0(t')}{\partial |\alpha_0|^2} + 2\sqrt{\langle N_0 \rangle} \Im(\hat{\xi}(t))
\]

(4.12)

The derivation of this equation is given in appendix B.

This stochastic differential equation still differs from the phenomenological equation (3.15) in two respects:

i) The noise still has a finite correlation time \( \tau_{\text{mic}} \). We shall consider these correlation functions in more detail below. Taking the Markovian limit \( \tau_{\text{mic}} \to 0 \) with

\[
S_{JJ}(t - t') = \hbar k_B T \Gamma_0 \delta(t - t')
\]

(4.13)

eq.(4.13) becomes

\[
\frac{d|\alpha_0|^2}{dt} = -2\frac{\Gamma_0}{\hbar} \langle N_0 \rangle \frac{\partial H_0}{\partial |\alpha_0|^2} + \frac{2}{\hbar} \sqrt{\langle N_0 \rangle} \Im(\hat{\xi}(t))
\]

(4.14)

ii) The mean-field Hamiltonian \( H_0(|\alpha_0|^2) \) appears in eqs.(4.13),(4.14) instead of the free energy \( \Delta F(|\alpha_0|^2) \). This is due to the fact that the influence of the thermal excitations on the energy are not yet taken into account. Doing this under isothermal or closed-system boundary conditions we should replace the energy \( H_0(|\alpha_0|^2) \) by the free energy \( \Delta F(|\alpha_0|^2) \) or \(-T\Delta S(|\alpha_0|^2)\), respectively.

This completes our derivation of the Langevin equation for the complex amplitude of the condensate.

V. GREEN-KUBO EXPRESSIONS FOR THE TRANSPORT COEFFICIENTS

Let us now analyse the fluctuating forces in more detail. Inserting the Bogoliubov transformation (4.3) in (4.8, 4.9) the fluctuating forces take the form

\[
\Re(\hat{\xi}(t)) = \frac{1}{4} \sum_{\kappa \mu} \left( (M^{(1)}_{\kappa \mu})^\ast + M^{(2)}_{\mu \kappa} \right) \hat{\alpha}^+_\mu \hat{\alpha}_\mu e^{-i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)t + \text{h.c.}} + \text{(nonresonant terms)}. 
\]

(5.1)

\[
\Im(\hat{\xi}(t)) = \frac{1}{4i} \sum_{\kappa \mu} \left( (M^{(1)}_{\kappa \mu})^\ast - M^{(2)}_{\mu \kappa} \right) \hat{\alpha}^+_\mu \hat{\alpha}_\mu e^{-i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)t - \text{h.c.}} + \text{(nonresonant terms)}. 
\]

(5.2)

Terms are called ‘nonresonant’ if the frequencies of the quasi-particles cannot add up to zero. Such terms have not been written out explicitly because later-on we shall restrict ourselves to the resonance or rotating wave approximation in which they don’t contribute. The relevant matrix elements \( M^{(1)}, M^{(2)} \) are

\[
M^{(1)}_{\kappa \mu} = 2U_0 \int d^3x \tilde{\psi}_\kappa \tilde{\psi}_\mu (\tilde{u}_\kappa^* \tilde{u}_\mu + \frac{1}{2} \tilde{v}_\kappa^* \tilde{v}_\mu) + (\nu \leftrightarrow \mu) 
\]

\[
M^{(2)}_{\mu \kappa} = 2U_0 \int d^3x \tilde{\psi}_\kappa \tilde{\psi}_\mu (\tilde{v}_\mu^* \tilde{v}_\kappa + \frac{1}{2} \tilde{u}_\mu^* \tilde{u}_\kappa) + (\nu \leftrightarrow \mu)
\]

(5.3)
and very similarly

\[
M_{\kappa,\nu,\mu}^{(1)} = 2U_0 \int d^3x (\bar{\psi}_0 + 2\langle N_0 \rangle) \frac{\partial \bar{\psi}_0}{\partial (N_0)} \hat{\nu}_\nu(\hat{u}_\kappa^* \hat{u}_\mu + \frac{1}{2} \hat{v}_\nu^* \hat{v}_\mu) + (\nu \leftrightarrow \mu)
\]

(5.4)

\[
M_{\mu,\nu,\kappa}^{(2)} = 2U_0 \int d^3x (\bar{\psi}_0 + 2\langle N_0 \rangle) \frac{\partial \bar{\psi}_0}{\partial (N_0)} \hat{\nu}_\nu(\hat{\kappa}_\kappa^* \hat{\mu}_\mu + \frac{1}{2} \hat{\nu}_\nu^* \hat{\nu}_\nu) + (\nu \leftrightarrow \mu)
\]

The matrix-elements \(M_{\kappa,\nu,\mu}^{(1)}\) and \(M_{\mu,\nu,\kappa}^{(2)}\) coincide with \(M_{\kappa,\nu,\mu}^{(1)}\) and \(M_{\mu,\nu,\kappa}^{(2)}\) if the dependence of \(\bar{\psi}_0\) on \(\langle N_0 \rangle\) is negligible or vanishes, as e.g. in homogeneous systems.

\(M_{\kappa,\nu,\mu}^{(1)}\) and similarly \(M_{\mu,\nu,\kappa}^{(2)}\) describes a scattering process in which one atom is scattered out of the condensate by the absorption of the two quasiparticles \(\nu, \mu\) from - and the emission of the new quasiparticle \(\kappa\) into - the thermal bath. Likewise \(M_{\mu,\nu,\kappa}^{(2)}\) and similarly \(M_{\mu,\nu,\kappa}^{(2)}\) describes a scattering process where an incoming thermal quasiparticle \(\kappa\) is absorbed, again an atom is kicked out of the condensate, and two quasiparticles \(\nu, \mu\) are emitted into the thermal bath. The scattering amplitudes for both processes are linearly superposed due to the phase-coherence of the condensate which exists on the time-scale of the relaxation process induced by the scattering process even if it is destroyed on a much longer time scale.

We can now calculate the correlation functions of the fluctuating forces. Their averages over the bath of quasiparticles vanish, \(\langle \Re(\xi(t)) \rangle = 0 = \langle \Im(\xi(t)) \rangle\). Their second-order correlation functions are obtained as

\[
\langle \Re(\hat{\xi}(t))\Re(\hat{\xi}(t')) \rangle_\phi = \frac{1}{8} \sum_{\kappa,\nu,\mu} |(M_{\kappa,\nu,\mu}^{(1)})^* + M_{\mu,\nu,\kappa}^{(2)}|^2
\]

\[
\left\{ \begin{array}{l}
\bar{n}_\kappa(\bar{n}_\nu + 1)(\bar{n}_\mu + 1)e^{i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)(t-t')} \\
+\bar{n}_\nu\bar{n}_\mu(\bar{n}_\kappa + 1)e^{-i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)(t-t')}
\end{array} \right\}
\]

(5.5)

\[
\langle \Im(\hat{\xi}(t))\Im(\hat{\xi}(t')) \rangle_\phi = \frac{1}{8} \sum_{\kappa,\nu,\mu} |(M_{\kappa,\nu,\mu}^{(1)})^* - M_{\mu,\nu,\kappa}^{(2)}|^2
\]

\[
\left\{ \begin{array}{l}
\bar{n}_\kappa(\bar{n}_\nu + 1)(\bar{n}_\mu + 1)e^{i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)(t-t')} \\
+\bar{n}_\nu\bar{n}_\mu(\bar{n}_\kappa + 1)e^{-i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)(t-t')}
\end{array} \right\}
\]

(5.6)

\[
\langle \Re(\hat{\xi}(t))\Im(\hat{\xi}(t')) \rangle_\phi = \frac{1}{8i} \sum_{\kappa,\nu,\mu} \left\{ \begin{array}{l}
(M_{\kappa,\nu,\mu}^{(1)})^* - M_{\mu,\nu,\kappa}^{(2)} \left( M_{\kappa,\nu,\mu}^{(1)} + (M_{\mu,\nu,\kappa}^{(2)})^* \right) \\
(\bar{n}_\nu + 1)(\bar{n}_\mu + 1)\bar{n}_\kappa e^{i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)(t-t')} \\
- \left( M_{\kappa,\nu,\mu}^{(1)} - (M_{\mu,\nu,\kappa}^{(2)})^* \right) \left( (M_{\kappa,\nu,\mu}^{(1)})^* + M_{\mu,\nu,\kappa}^{(2)} \right) \\
(\bar{n}_\mu + 1)\bar{n}_\nu e^{-i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)(t-t')} \\
(\bar{n}_\kappa + 1)\bar{n}_\nu\bar{n}_\mu e^{-i(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu)(t-t')}
\end{array} \right\}
\]

(5.7)

These correlation functions can be replaced by delta-functions provided that the frequency-sums contain a flat quasi-continuum of nearly resonant terms in a neighborhood of the resonance \(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu = 0\) which is broad compared to the damping rates we calculate here. This assumption will be satisfied in sufficiently large condensates. The strengths of the delta-functions can then be extracted from the expressions (5.3), (5.6), (5.7) by taking the time-averages \(\int_{-\infty}^{\infty} dt (t-t') \langle \hat{\xi}(t)\hat{\xi}(t') \rangle_\phi\) and \(\int_{-\infty}^{\infty} dt (t-t') \langle \hat{\xi}^+(t)\hat{\xi}(t') \rangle_\phi\).

\(\Re(\hat{\xi}(t))\), \(\Im(\hat{\xi}(t))\) are here given as expressions involving operators. Provided the Markovian approximation is satisfied the average of their commutators over the quasi-particle bath are again given by delta functions in time. Explicitly we obtain for the coefficients of the delta-functions
These expressions constitute our general results for the three transport parameters. They have to be evaluated implied by eqs. (3.7, 3.8) we obtain the representations which leads to the formulas

\[ \langle \hat{\xi}(t) \rangle = 0 \]

from the fact that the fluctuating force \( \Re \) necessarily thermal fluctuations of the amplitudes of the excited states. In our microscopic results this can be seen possible to treat the condensate classically, even after taking its interaction with the quasi-particles into account.

Let us now proceed to derive formulas for the three transport parameters \( \Gamma_0, \Gamma' \) and \( \Gamma'' \). From

\[ 2\Gamma_0 + \Gamma' = \frac{1}{\hbar k_B T} \int_{-\infty}^{\infty} dt \langle \xi(t) \rangle \langle \xi(0) \rangle \]

implied by eqs. (3.7, 3.8) we obtain the representations

\[ \Gamma_0 = \frac{1}{\hbar k_B T} \int_{-\infty}^{\infty} d(t-t') \langle \hat{\xi}(t) \rangle \langle \hat{\xi}(t) \rangle \]

which have the form of Green-Kubo relations for the transport coefficients. Using the explicit form (5.1, 5.2) of the fluctuating forces the thermal averages can be taken and the time-integrals in eqs. (5.11, 5.10, 5.12) can be carried out which leads to the formulas

\[ \Gamma_0 = \frac{\pi}{2\hbar k_B T} \sum_{\kappa,\nu,\mu} |(M_{\kappa,\nu,\mu}^{(1)})^* - M_{\nu,\mu,\kappa}^{(2)}|^2 \tilde{n}_n \tilde{n}_\mu (\tilde{n}_\mu + 1) \delta(\tilde{\omega}_n - \tilde{\omega}_\mu - \tilde{\omega}_\nu) \]

\[ \Gamma_0 + \Gamma' = \frac{\pi}{2\hbar k_B T} \sum_{\kappa,\nu,\mu} |(M_{\kappa,\nu,\mu}^{(1)})^* + M_{\nu,\mu,\kappa}^{(2)}|^2 \tilde{n}_n \tilde{n}_\mu (\tilde{n}_\mu + 1) \delta(\tilde{\omega}_n - \tilde{\omega}_\mu - \tilde{\omega}_\nu) \]

\[ \Gamma'' = -\frac{i\pi}{2\hbar k_B T} \sum_{\kappa,\nu,\mu} \left\{ \left( (M_{\kappa,\nu,\mu}^{(1)})^* - M_{\nu,\mu,\kappa}^{(2)} \right) \left( M_{\kappa,\nu,\mu}^{(1)} + M_{\nu,\mu,\kappa}^{(2)} \right)^* \right. \]

\[ \left. - \left( (M_{\kappa,\nu,\mu}^{(1)})^* - M_{\nu,\mu,\kappa}^{(2)} \right)^* \left( M_{\kappa,\nu,\mu}^{(1)} + M_{\nu,\mu,\kappa}^{(2)} \right) \right\} \tilde{n}_n \tilde{n}_\mu (\tilde{n}_\mu + 1) \delta(\tilde{\omega}_n - \tilde{\omega}_\mu - \tilde{\omega}_\nu) \]

These expressions constitute our general results for the three transport parameters. They have to be evaluated separately for each individual trap geometry.

VI. RELATION TO THE FLUCTUATION AND DISSIPATION OF THE EXCITATIONS

As was pointed out after eq. (5.13) by phenomenological arguments, the noise term \( \Re(\xi(t)) \) is not connected with the fluctuations of the number of particles in the condensate, but must be due to other fluctuations, which are then necessarily thermal fluctuations of the amplitudes of the excited states. In our microscopic results this can be seen from the fact that the fluctuating force \( \Re(\xi(t)) \) according to eq. (5.13) contains precisely the same operator which also appears in \( \hat{H}_3(t) \) and couples the atoms in the thermal cloud to the condensate.

In the special case where the difference between the coupling matrix-elements \( M_{\ell}^{(1,2)} \) and \( M_{\ell}^{(1,2)} \) is negligible (which is exactly satisfied in box-like traps, cf. section VII) the intensity \( \Gamma_0 + \Gamma' \) of the noise source can be expressed entirely
as a property of the excitations, as we shall now demonstrate.\footnote{\label{footnote:condensate}
For their amplitudes $\hat{\alpha}_\nu(t), \hat{\alpha}_\nu^\dagger(t)$ a quantum-Langevin equation can be derived microscopically along the same lines employed here for the condensate amplitude. We have done this elsewhere \cite{21}, see also \cite{21}, with the result, in the Markovian limit,
\begin{equation}
\frac{d\hat{\alpha}_\nu(t)}{dt} = -\iota \omega_\nu \hat{\alpha}_\nu(t) - \gamma_\nu \hat{\alpha}_\nu(t) + \xi(t)
\end{equation}
with Gaussian fluctuating force-operators with vanishing average and
\begin{align}
\langle \xi_\nu^+ (t) \xi_\mu (t') \rangle &= 2\gamma_\nu \bar{n}_\nu \delta(t - t') \delta_{\nu\mu} \\
\langle [\dot{\xi}_\nu (t), \dot{\xi}_\mu (t')] \rangle &= 2\gamma_\nu \delta(t - t') \delta_{\nu\mu}
\end{align}
where the damping rates $\gamma_\nu$ are given by
\begin{equation}
\gamma_\nu = \frac{\pi \langle N_0 \rangle}{\hbar^2} \sum_{\kappa,\mu} \{ |(M_{\kappa,\nu\mu}^{(1)})^* + M_{\nu\mu,\kappa}^{(2)}|^2 (\bar{n}_\mu - \bar{n}_\kappa) \delta(\omega_\kappa - \omega_\mu - \omega_\nu) \\
+ |M_{\nu\kappa,\mu}^{(1)} + (M_{\nu\mu,\kappa}^{(2)})^*|^2 (\bar{n}_\kappa + \frac{1}{2}) \delta(\omega_\kappa + \omega_\mu - \omega_\nu) \},
\end{equation}
The first term describes Landau-damping of the mode $\nu$ by scattering a quasi-particle from mode $\mu$ to mode $\kappa$ and is equivalent to a result derived in \cite{19} by the golden rule. The second term in eq.(6.3) describes Beliaev damping, where the mode $\nu$ decays into two modes $\kappa, \mu$. It survives even for $T \to 0$ where $\bar{n}_\kappa \to 0$ for all modes.

Let us now establish the connection between $\Gamma_0 + \Gamma'$ and the damping rates $\gamma_\nu$ as given by (6.3). We shall show that the simple sum-rule
\begin{equation}
\Gamma_0 + \Gamma' = \frac{\hbar}{3\langle N_0 \rangle k_B T} \sum_\nu \bar{n}_\nu (\bar{n}_\nu + 1) \gamma_\nu
\end{equation}
holds. To see this we need to consider
\begin{equation}
\sum_\nu \bar{n}_\nu (\bar{n}_\nu + 1) \gamma_\nu = \frac{\pi \langle N_0 \rangle}{\hbar^2} \sum_{\kappa,\mu} \{ |(M_{\kappa,\nu\mu}^{(1)})^* + M_{\nu\mu,\kappa}^{(2)}|^2 (\bar{n}_\mu - \bar{n}_\kappa) \delta(\omega_\kappa - \omega_\nu - \omega_\mu) \\
+ |M_{\nu\kappa,\mu}^{(1)} + (M_{\nu\mu,\kappa}^{(2)})^*|^2 (\bar{n}_\kappa + \frac{1}{2}) \delta(\omega_\kappa + \omega_\mu - \omega_\nu) \}
\end{equation}
The second term in the curly bracket arises from the second term in (6.3) by first exchanging the notations for the summation indices $\nu$ and $\kappa$ and then symmetrizing in $\nu$ and $\mu$, because the matrix-elements are already symmetric in these indices. The remainder of the proof then consists simply in noting that for $\omega_\kappa = \omega_\nu + \omega_\mu$ the identities
\begin{align}
(\bar{n}_\mu - \bar{n}_\kappa)\bar{n}_\nu (\bar{n}_\nu + 1) &= \bar{n}_\mu \bar{n}_\nu (\bar{n}_\kappa + 1) \\
(\bar{n}_\mu + \bar{n}_\nu + 1)\bar{n}_\kappa (\bar{n}_\kappa + 1) &= \bar{n}_\mu \bar{n}_\nu (\bar{n}_\kappa + 1)
\end{align}
hold. Using this in eq.(6.5) and then comparing with (5.14) establishes the sum-rule. We can also note that the processes due to Landau scattering contribute to the sum-rule with precisely twice the strength of those due to Beliaev scattering.

Thus we see that, in general, the noise-amplitudes proportional to the combination of matrix-elements $(M_{\kappa,\nu\mu}^{(1)})^* - M_{\nu\mu,\kappa}^{(2)}$ drive the number-fluctuations in the condensate, while those proportional to $(M_{\nu\kappa,\mu}^{(1)})^* + M_{\nu\mu,\kappa}^{(2)}$ are due to fluctuations of the occupation numbers in the excited states, couple in the Hamiltonian to the particle number in the condensate and therefore drive the phase-fluctuations in the condensate.

\footnote{In the general case the coupling of the condensate to the non-condensate modes differs from the coupling between the non-condensate modes and the relation between $\Gamma_0 + \Gamma'$ and the $\gamma_\nu$ is less direct.}
VII. EVALUATION OF THE TRANSPORT PARAMETERS FOR A BOX-LIKE TRAP

For simplicity we consider now a trap consisting of a cube of length $L$ with cyclic boundary conditions. In the following equilibrium values of all parameters are implied, but we shall omit in this section the tilde and write $\mu$ for $(\mu)$ to simplify our notation. The normalized $u$ and $v$ coefficients are in this case

$$u_\nu = \frac{E_\nu + p_\nu^2/2m}{\sqrt{2E_\nu p_\nu^2/m}} \frac{1}{\sqrt{V}} e^{i\vec{p}_\nu \cdot \vec{x}/\hbar}$$  \hspace{1cm} (7.1)$$

$$v_\nu = -\frac{E_\nu - p_\nu^2/2m}{\sqrt{2E_\nu p_\nu^2/m}} \frac{1}{\sqrt{V}} e^{i\vec{p}_\nu \cdot \vec{x}/\hbar}$$  \hspace{1cm} (7.2)$$

with

$$E_\nu = \sqrt{\left(\frac{p_\nu^2}{2m} + \mu\right)^2 - \mu^2}$$  \hspace{1cm} (7.3)$$

and $\vec{p}_\nu = \hbar \frac{2\pi}{L} \vec{n}_\nu$ with integer vector $\vec{n}_\nu$.

A. Transport coefficients

The squares of the relevant matrix elements for $E_\nu = E_\nu + E_\mu$ become

$$\left|M^{(1)}_{\nu\mu,\nu} - M^{(2)}_{\nu\mu,\nu}\right|^2 = \left(U_0 \frac{V}{E_\nu E_\mu E_\nu E_\mu} \sigma_{\vec{n}_\nu,\vec{n}_\nu + \vec{n}_\mu}\right)^2$$  \hspace{1cm} (7.4)$$

with

$$G(x, y, \alpha) = \sqrt{\alpha^2 + (x + y)^2} \left(3\sqrt{\alpha^2 + x^2}\sqrt{\alpha^2 + y^2} - xy + 2\alpha(\sqrt{\alpha^2 + x^2} + \sqrt{\alpha^2 + y^2} + \alpha)\right) + (x + y) \left(x(\sqrt{\alpha^2 + y^2} + \alpha) + y(\sqrt{\alpha^2 + x^2} + \alpha)\right) + 2\alpha(\sqrt{\alpha^2 + x^2} + \alpha) + \alpha(\sqrt{\alpha^2 + y^2} + \alpha) + \alpha^2$$  \hspace{1cm} (7.5)$$

The transport coefficients are then expressed as the simple result

$$\Gamma'' = 0$$  \hspace{1cm} (7.7)$$

and

$$\Gamma_0 = \frac{2}{\pi} \frac{a}{L}\left(\frac{a}{L}\right)^2 \sum_{\vec{n}_\nu,\vec{n}_\mu} \frac{\delta(\varepsilon_\nu + \varepsilon_\mu - \varepsilon_{\nu + \mu})}{\varepsilon_\nu \varepsilon_\mu (\varepsilon_\nu + \varepsilon_\mu)} G(\varepsilon_\nu, \varepsilon_\mu, \alpha) F(\varepsilon_\nu, \varepsilon_\mu, \frac{k_BT}{\hbar\omega_0})$$  \hspace{1cm} (7.8)$$

$$\Gamma_0 + \Gamma' = \frac{2}{\pi} \frac{a}{L}\left(\frac{a}{L}\right)^2 \sum_{\vec{n}_\nu,\vec{n}_\mu} \frac{\delta(\varepsilon_\nu + \varepsilon_\mu - \varepsilon_{\nu + \mu})}{\varepsilon_\nu \varepsilon_\mu (\varepsilon_\nu + \varepsilon_\mu)} G(\varepsilon_\nu, \varepsilon_\mu, -\alpha) F(\varepsilon_\nu, \varepsilon_\mu, \frac{k_BT}{\hbar\omega_0})$$  \hspace{1cm} (7.9)$$

with

$$F(\varepsilon_\nu, \varepsilon_\mu, \frac{k_BT}{\hbar\omega_0}) = \frac{\hbar\omega_0}{k_BT} \left(e^{\beta\hbar\omega_0(\varepsilon_\nu + \varepsilon_\mu)} - 1\right) \frac{e^{\beta\hbar\omega_0\varepsilon_\nu} - 1}{(e^{\beta\hbar\omega_0\varepsilon_\nu} - 1)(e^{\beta\hbar\omega_0\varepsilon_\mu} - 1)}$$  \hspace{1cm} (7.10)$$

Here we scaled the scattering length $a = mU_0/4\pi\hbar^2$ with $L$ and the energies $E_\nu, E_\mu$ and $k_BT$ with the energy $\hbar\omega_0 = (2\pi\hbar)^2/2mL^2$, defining

14
\[ \varepsilon_\nu = \sqrt{(n_\nu^2 + \alpha)^2 - \alpha^2} \] (7.11)

\[ \varepsilon_{\nu+\mu} = \sqrt{((\bar{n}_\nu + \bar{n}_\mu)^2 + \alpha)^2 - \alpha^2} \] (7.12)

with \( \alpha = \mu/\hbar \omega_0 \).

The double sums over \( \bar{n}_\nu, \bar{n}_\mu \) start with \( \bar{n} \)-values with \( |\bar{n}| = 1 \). They are approximated by integrals according to

\[ \sum_{\bar{n}_\nu, \bar{n}_\mu} \delta (\varepsilon_\nu + \varepsilon_\mu - \varepsilon_{\nu+\mu}) (\ldots) = \pi^2 \int \int_{\sqrt{1+2\alpha} \sqrt{1+2\alpha}} \frac{\varepsilon_\nu \varepsilon_\mu (\varepsilon_\nu + \varepsilon_\mu) (\ldots) d\varepsilon_\mu d\varepsilon_\mu}{\sqrt{(\varepsilon_\nu^2 + \alpha^2)(\varepsilon_\mu^2 + \alpha^2)((\varepsilon_\nu + \varepsilon_\mu)^2 + \alpha^2)}} \] (7.13)

Here \( (\ldots) \) is any smooth function of \( \varepsilon_\nu, \varepsilon_\mu \). In all experiments so far \( \alpha \gg 1 \) is satisfied, i.e. we can replace \( \sqrt{1+2\alpha} \rightarrow \sqrt{2\alpha} \). This leaves us with the integral expressions

\[ \Gamma_0 = 2\pi \left( \frac{a}{L} \right)^2 \int \int_{\sqrt{2\alpha} \sqrt{2\alpha}} G(\varepsilon_\nu, \varepsilon_\mu, \alpha) F(\varepsilon_\nu, \varepsilon_\mu, \frac{k_B T}{\hbar \omega_0}) d\varepsilon_\mu d\varepsilon_\mu \] (7.14)

\[ \Gamma_0 + \Gamma' = 2\pi \left( \frac{a}{L} \right)^2 \int \int_{\sqrt{2\alpha} \sqrt{2\alpha}} G(\varepsilon_\nu, \varepsilon_\mu, -\alpha) F(\varepsilon_\nu, \varepsilon_\mu, \frac{k_B T}{\hbar \omega_0}) d\varepsilon_\mu d\varepsilon_\mu \] (7.15)

The expression for \( \Gamma_0 \) and the asymptotic behavior for \( \varepsilon_\nu, \varepsilon_\mu \rightarrow 0 \): \( G(\varepsilon_\nu, \varepsilon_\mu, |\alpha|) F(\varepsilon_\nu, \varepsilon_\mu, k_B T/\hbar \omega_0) \rightarrow 18(k_B T/\hbar \omega_0)^2/|\alpha| \varepsilon_\nu \varepsilon_\mu (\varepsilon_\nu + \varepsilon_\mu) \) make it amply clear that the states with the smallest energies \( \varepsilon_\nu \ll \alpha \) make a large contribution to \( \Gamma_0 \) (but not to \( \Gamma_0 + \Gamma' \)). To calculate this contribution it is permitted \(^7\) to use \( \beta \hbar \omega_0 \varepsilon_\nu \ll 1 \), \( \beta \hbar \omega_0 \varepsilon_\mu \ll 1 \) under the integral and to approximate

\[ F(\varepsilon_\nu, \varepsilon_\mu, \frac{k_B T}{\hbar \omega_0}) = \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \frac{1}{\varepsilon_\nu + \varepsilon_\mu} \] (7.16)

in addition to approximating \( \sqrt{\varepsilon_\nu^2 + \alpha^2} \sqrt{\varepsilon_\mu^2 + \alpha^2} (\varepsilon_\nu + \varepsilon_\mu)^2 + \alpha^2 \cong \alpha^3 \), and neglecting terms of order \( \varepsilon_\nu^2/\alpha^2 \).

This contribution to \( \Gamma_0 \), which we shall denote as \( \Gamma_{00} \), then reduces to

\[ \Gamma_{00} = 36\pi \left( \frac{a}{L} \right)^2 \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \int \frac{d\varepsilon_\nu}{\sqrt{2\alpha}} \int \frac{d\varepsilon_\mu}{\sqrt{2\alpha}} \frac{1}{\varepsilon_\nu \varepsilon_\mu (\varepsilon_\nu + \varepsilon_\mu)} \] (7.17)

The double-integral can be evaluated as \( \sqrt{2/\alpha} \ln 2 \) which yields the final result

\[ \Gamma_{00} = 36\pi \sqrt{2} \ln 2 \left( \frac{a}{T} \right)^2 \left( \frac{\hbar \omega_0}{\mu} \right)^{1/2} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 = 1.59.. \left( \frac{T}{T_c} \right)^2 \left( \frac{N}{(N_0)} \right)^{1/2} N^{1/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{3/4} \] (7.18)

The second form of this expression is obtained by eliminating \( V = L^3 \) in favor of the critical temperature of the equivalent ideal Bose-gas at the same density via

\[ V = (N/\zeta(3/2))(2\pi \hbar^2/k_B T_c m)^{3/2}. \] (7.19)

There is yet another particularly important contribution to \( \Gamma_0 \) due to the infrared-singularity of the integrand, we shall denote it as \( \Gamma_{01} \), where only one of the two excitation frequencies, say \( E_\nu \), is small compared to \( \mu \), while the other is larger, of order \( \mu \) or even \( k_B T \). With respect to \( \epsilon_\nu \) the low-energy asymptotics may then still be used. We then obtain

\[ \ldots \]
\[
\Gamma_{01} = \frac{4\pi}{\alpha} \left( \frac{a}{L} \right)^2 \int_0^{q_0} d\varepsilon_\nu \int_0^{q_0} d\varepsilon_\mu \frac{G(0, \varepsilon_\mu, \alpha) - G(0, 0, \alpha) \varepsilon_\nu (\varepsilon_\mu + \alpha^2)}{e^{\beta \hbar \omega_0 \varepsilon_\mu} - 1} \left( e^{\beta \hbar \omega_0 \varepsilon_\mu} - 1 \right)^2.
\] (7.20)

where we have set an upper cut-off for the small energy at a fraction \( q \) of the chemical potential. Most of the \( \varepsilon_\mu \)-integral comes from a range around \( \alpha \) and we may therefore replace the thermal function by its asymptotics for \( \beta \hbar \omega_0 \varepsilon_\mu \to 0 \), which is \( (k_B T/\hbar \omega_0 \varepsilon_\mu)^2 \). The integrals can then be performed with the result

\[
\Gamma_{01} = 16\pi \left( \frac{a}{L} \right)^2 \ln \left( \frac{q^2 \mu}{2\hbar \omega_0} \right) \frac{(k_B T)^2}{\hbar \omega_0 \mu} = 1.22.. \left( \frac{T}{T_c} \right)^2 \frac{N(N_0)}{(N_0^2)} \frac{(k_B T_a^2 \alpha^2)}{\hbar^2} \frac{1}{2} \ln \left( \frac{q^2 \mu}{2\hbar \omega_0} \right).
\] (7.21)

We conclude that this contribution to \( \Gamma_0 \) is smaller than the leading term \( \Gamma_{00} \) by the order of magnitude \( (\hbar \omega_0 / \mu)^{1/2} \ln(q^2 \mu / 2\hbar \omega_0) \).

Let us now turn to the expressions for \( \Gamma_0 - \Gamma_{00} - \Gamma_{01} \) and \( \Gamma_0 + \Gamma' \). They can be simplified for \( \alpha \gg 1 \) by rescaling \( \varepsilon_\nu, \varepsilon_\mu \) by \( \alpha \) and taking the limit \( \sqrt{2/\alpha} \to 0 \) for the lower boundaries of the rescaled integrals. We find in this way

\[
\Gamma_0 - \Gamma_{00} - \Gamma_{01} = 2\pi \left( \frac{a}{L} \right)^2 \left( \frac{\mu}{\hbar \omega_0} \right) \int_0^\infty dx \int_0^\infty dy \left[ -\frac{G(x, y, 1) F(x, y, \frac{k_B T}{\mu})}{\sqrt{(x^2 + 1)(y^2 + 1)(x + y)^2 + 1}} + \frac{G(x, y, 1) F(x, y, \frac{k_B T}{\mu})}{\sqrt{(x^2 + 1)(y^2 + 1)(x + y)^2 + 1}} \right] \]

\[
+ 8 \frac{x(x^2 + 1)(y^2 + \sqrt{y^2 + 1} - 1) + y(y^2 + 1)(x^2 + \sqrt{x^2 + 1} - 1)}{x^2 y(x^2 + 1)(y^2 + 1)}
\]

\[
\Gamma_0 + \Gamma' = 2\pi \left( \frac{a}{L} \right)^2 \left( \frac{\mu}{\hbar \omega_0} \right) \int_0^\infty dx \int_0^\infty dy \frac{G(x, y, -1) F(x, y, \frac{k_B T}{\mu})}{\sqrt{(x^2 + 1)(y^2 + 1)(x + y)^2 + 1}}
\]

where we used the scaling property

\[
F(\varepsilon_\nu, \varepsilon_\mu, \frac{k_B T}{\hbar \omega_0}) = \frac{1}{\alpha} F(\frac{\varepsilon_\nu}{\alpha}, \frac{\varepsilon_\mu}{\alpha}, \frac{k_B T}{\hbar \omega_0 \alpha})
\] (7.24)

Eqs(7.22, 7.23) are the complete result for the temperature dependent transport parameters of the condensate for box-like traps. In general the integrals have to be done numerically. We shall here consider some asymptotic results only.

First we consider these expressions asymptotically for \( k_B T / \mu \gg 1 \). Then the integrals receive important contributions from \( x, y \) of the order of \( 1 \), i.e. from quasi-particle-energies of the order of the chemical potential, and also from values of \( x, y \) large compared to \( 1 \), i.e. quasi-particle energies of order \( k_B T \). The contributions \( \Gamma_0^{(>)} \) and \( \Gamma'^{(>)} \) from large energies can be determined in leading power in \( (k_B T / \mu) \) by approximating

\[
F(x, y, \frac{k_B T}{\mu}) \approx \frac{\mu}{k_B T} e^{-\frac{k_B T}{\mu}(x+y)}
\] (7.25)

and rescaling \( x \) and \( y \) by \( k_B T / \mu \). In the integrals for \( \Gamma_0 \) and \( \Gamma_0 + \Gamma' \) we can then let \( \mu / k_B T \to 0 \) without any problem, using the property \( G(x, y, 0) = 4xy(x+y) \), whereupon they are easily evaluated with the asymptotic results

\[
\Gamma_0^{(>)} + \Gamma'^{(>)} \approx \Gamma_0^{(>)} \approx 8\pi \left( \frac{a}{L} \right)^2 \frac{k_B T}{\hbar \omega_0} = 1.27.. \frac{T_c}{k_B T_a^2 a^2 m \hbar^2}
\] (7.26)

We see that the result for \( \Gamma'^{(>)} \) vanishes to this order.

For the contributions \( \Gamma_0^{(\mu)}, \Gamma^{(\mu)} \) from quasi-particles with energies around \( \mu \) we can approximate \( F(x, y, k_B T / \mu) \) according to eq.(7.18) and find

---

*In the high-energy regime the subtractions of the infrared-divergent terms in the integrand of [7.24] are of no importance.*
\[ \Gamma_0^{(\mu)} \simeq B_0^{(\mu)} \left( \frac{a}{L} \right)^2 \frac{k_B T^2}{\hbar \omega_0 \mu} = 0.0243 \cdot B_0^{(\mu)} \left( \frac{T}{T_c} \right)^2 \frac{N}{\langle N_0 \rangle} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/2} \] (7.27)

\[ \Gamma_0^{(\mu)} + \Gamma'^{(\mu)} \simeq B'^{(\mu)} \left( \frac{a}{L} \right)^2 \frac{k_B T^2}{\hbar \omega_0 \mu} = 0.0243 \cdot B'^{(\mu)} \left( \frac{T}{T_c} \right)^2 \frac{N}{\langle N_0 \rangle} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/2} \] (7.28)

with the numbers \( B_0^{(\mu)}, B'^{(\mu)} \) defined by the integrals

\[ B_0^{(\mu)} = 2\pi \int_0^\infty dx \int_0^\infty dy \frac{G(x, y, 1)}{\sqrt{(x^2 + 1)(y^2 + 1)((x + y)^2 + 1)xy(x + y)}} \]

\[- \frac{18}{xy(x + y)} \]

\[-8 \frac{x^2 + 1)(y^2 + \sqrt{y^2 + 1} - 1) + y(y^2 + 1)(x^2 + \sqrt{x^2 + 1} - 1)}{x^2y^2(x^2 + 1)(y^2 + 1)} \] (7.29)

\[ B'^{(\mu)} = 2\pi \int_0^\infty dx \int_0^\infty dy \frac{G(x, y, -1)}{\sqrt{(x^2 + 1)(y^2 + 1)((x + y)^2 + 1)xy(x + y)}} \] (7.30)

We can conclude that the contribution from quasi-particles at energies of order \( \mu \) is larger (for \( \Gamma_0 \) by an order of magnitude \( k_B T/\mu \)) than the contribution from energies of order \( k_B T \), but \( \Gamma_0^{(\mu)} \) is, in large condensates, still subdominant to \( \Gamma_0 \) by the order of magnitude \( \sqrt{\hbar \omega_0/\mu} \).

Now let us consider also the low-temperature limit, namely \( k_B T/\mu \ll 1 \) or, equivalently, \( T/T_c \ll \left( k_B T_c a^2 m/\hbar^2 \right)^{1/2} \).

In this region it is not necessary to distinguish \( N \) and \( \langle N_0 \rangle \). The integrals now receive their contributions for \( x, y \) both small compared to 1, but we can still use the approximation (7.24) for small \( x, y \) we can expand

\[ G(x, y, 1) \simeq 18 + 3(x^2 + y^2 + xy) \quad G(x, y, -1) \simeq \frac{9}{32} x^2y^2(x + y)^2 \] (7.31)

To obtain the leading term it is enough to keep only the smallest powers of \( x, y \) in the integrands. The integrals are easily evaluated with the asymptotic low-temperature results

\[ \Gamma_0 - \Gamma_0 = 36\pi \left( \frac{a}{L} \right)^2 \frac{k_B T}{\hbar \omega_0} = 5.72 \cdot \frac{T}{T_c} \frac{k_B T_c a^2 m}{\hbar^2} \] (7.32)

\[ \Gamma_0 + \Gamma' = \frac{189}{2} \pi \left( \frac{a}{L} \right)^2 \frac{\mu}{\hbar \omega_0} \left( \frac{k_B T}{\mu} \right)^7 = 0.366 \cdot \left( \frac{T}{T_c} \right)^7 \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{-2} \] (7.33)

As long as the temperature is high enough to satisfy \( k_B T \gg \sqrt{\mu \hbar \omega_0} \) the part \( \Gamma_0 \) still dominates the value of \( \Gamma_0 \).

\[ \langle \Delta N_0^2 \rangle = \sum_\nu \frac{2 \bar{n}_\nu (\bar{n}_\nu + 1) (E_\nu^2 + 2 \mu^2) + \mu^2}{2 E_\nu^2} \] (7.34)

Approximated by an integral this becomes

\[ \langle \Delta N_0^2 \rangle = \pi \int_{\sqrt{1+2\alpha}}^\infty d\varepsilon \sqrt{\varepsilon^2 + \alpha^2 - \alpha} \left( \alpha^2 + \frac{\varepsilon^2 + 2\alpha^2}{2 \sin \frac{\sqrt{\varepsilon^2 + \alpha^2}}{2}} \right) \] (7.35)
The dominant contribution comes from the lower boundary of the integration [14] which contributes, for \( \alpha \gg 1 \)

\[
\langle \Delta N_0^2 \rangle \simeq 2\pi \left( \frac{k_B T}{\hbar \omega_0} \right)^2 A' \left( \frac{m k_B T}{\hbar^2} \right)^2 V^{4/3}
\]

(7.36)

with

\[
A' = \frac{1}{2\pi^3} = 0.0161..
\]

(7.37)

More precisely the dominant contribution to \( \langle \Delta N_0^2 \rangle \) is given by the discrete sum [14]

\[
\langle \Delta N_0^2 \rangle = 2 \mu^2 (k_B T)^2 \sum_{\nu} \frac{1}{E^2_{\nu}}
\]

(7.38)

which gives the same expression as (7.36) but with the prefactor

\[
A = \frac{2}{(2\pi)^4} \sum_{\vec{n}_\nu \neq 0} \frac{1}{n^4_{\nu}} = 0.021..
\]

(7.39)

If we eliminate the volume in favour of the critical temperature of the ideal Bose-gas of the same density via eq.(7.19) we get

\[
\langle \Delta N_0^2 \rangle = A\frac{(2\pi)^2}{\zeta(3/2)^{4/3}} \left( \frac{T}{T_c} \right)^2 N^{4/3}.
\]

(7.40)

At temperature \( T = 0 \) a similar evaluation of (2.28) gives

\[
\langle \Delta N_0^2 \rangle|_{T=0} = 2\sqrt{\pi} (\alpha N)^{3/2} V^{-1/2}.
\]

(7.41)

C. Particle-number relaxation rate

We are now in a position to evaluate the rate \( \tau_c^{-1} \) from eq.(3.17) using the results for \( \langle \Delta N_0^2 \rangle \) (numbers are calculated with the prefactor \( A' \)) and \( \Gamma_0 \simeq \Gamma_{00} \). We obtain

\[
\gamma_c = \frac{1}{\tau_c} = 18.0 \frac{T}{T_c} \left( \frac{\langle N_0 \rangle}{\langle N \rangle} \right) \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{3/4} \frac{k_B T_c}{\hbar}
\]

(7.42)

This result applies also in the low-temperature region, because it makes use only of the results for \( \langle \Delta N_0^2 \rangle \) and \( \Gamma_0 \) which also hold in that region.

To get an idea of order of magnitudes we compare this and the following results with the damping rate \( \gamma_0 \) of the lowest lying modes, which is given by [19]

\[
\gamma_0 = \frac{3\pi^2}{4} \left( \frac{a}{L} \right) \frac{k_B T}{\hbar} = 4.06.. \frac{T}{T_c} N^{-1/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/2} \frac{k_B T_c}{\hbar}
\]

(7.43)

We see that the relaxation rate \( \gamma_c \) is of the order

\[
\gamma_c \sim N^{1/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/4} \gamma_0
\]

(7.44)

The proportionality factor is of the order of \( \sqrt{\mu/\hbar \omega_0} \) and is large in large and strongly interacting condensates. I.e. then the relaxation of the condensate to its equilibrium is faster than the relaxation of the low-lying collective modes, but slower than the frequency of the lowest lying modes, which is \( \sqrt{2\omega_0 M/\hbar} \).

\[\text{9The formula (7.39) differs from the one given in [14] by a factor 2}^{-4} \text{ whereas the numerical result differs by yet another factor; the formula (7.41) differs from the one in [14] by a factor 2.}\]
D. Phase collapse rate

The phase collapse rate is given by eq. (3.23) and requires only the result for \( \langle \Delta N_0^2 \rangle \) and \( \Gamma'' = 0 \). At zero temperature it reduces to

\[
\gamma_{\text{collapse}}|_{T=0} = \frac{1}{\hbar} \frac{\partial \mu}{\partial \langle N_0 \rangle} \sqrt{\langle \Delta N_0^2 \rangle}|_{T=0}
\] (7.45)

from which we get

\[
\gamma_{\text{collapse}}|_{T=0} = 23.6 \cdot \frac{\hbar c T c}{\hbar} \frac{k_B T c a^2 m}{h^2} \left( \frac{\hbar}{k_B T c a^2 m} \right)^{7/8} \frac{1}{\sqrt{N}}
\] (7.46)

For finite temperature we obtain

\[
\gamma_{\text{collapse}} = 0.876 \cdot \frac{k_B T}{\hbar} N^{-1/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/2}
\] (7.47)

By comparison with (7.43) we see that \( \gamma_{\text{collapse}} \) is of the order of \( \gamma_0 \) and is therefore in large condensates smaller than \( \gamma_c \).

In summary, the phase-collapse is not effective in large condensates because it occurs with a rate \( \gamma_{\text{collapse}} < \gamma_c \) and is at the same time restricted to a time interval \( \Delta t < \frac{1}{\gamma_c} \) since for larger times phase-diffusion takes over.

E. Phase-diffusion rate

The phase-diffusion coefficient is a somewhat complicated quantity because it receives contributions from several processes, which are physically distinct. We consider the different contributions separately and also distinguish the two temperature regimes of high temperature \( k_B T > \mu \), for which we give the result first, and \( k_B T < \mu \) (low temperature).

1. Low frequency condensate number fluctuations

From eq. (3.19) we infer with \( \Gamma_0 = \Gamma_{00} \)

\[
D^{(\alpha)} = \frac{1}{h k_B T (N_0) \Gamma_{00}} \left( \langle \Delta N_0^2 \rangle \frac{\partial \mu}{\partial \langle N_0 \rangle} \right)^2
\] (7.48)

which is evaluated as

\[
D^{(\alpha)} = 0.0853 \cdot \frac{T}{T_c} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/4} \frac{1}{(N_0)^{1/2}} \frac{k_B T_c}{h}. \] (7.49)

The same result holds in the low temperature regime \( k_B T < \mu \). In comparison with \( \gamma_0 \) (7.43) it is of the order

\[
D^{(\alpha)} \sim N^{-1/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{-1/4} \gamma_0 \sim \frac{\hbar c_0 a^2 m}{\mu} \gamma_0
\] (7.50)

and is much smaller in large and strongly interacting condensates. Still this contribution to the phase-diffusion rate is always the dominant one at low temperatures and may dominate even at higher temperatures (see below).

2. Condensate number fluctuations due to quasi-particles around energies \( \mu \)

Splitting \( \Gamma_0 = \Gamma_{00} + (\Gamma_0 - \Gamma_{00}) \) and expanding to first order

\[
\frac{1}{\Gamma_0} = \frac{1}{\Gamma_{00}} - \frac{\Gamma_0 - \Gamma_{00}}{\Gamma_{00}^2} \] (7.51)
we estimate as contribution $D_\phi^{(\beta)}$ from the higher frequency condensate number fluctuations described by $\Gamma_0 - \Gamma_{00}$ as given by (7.26)

$$D_\phi^{(\beta)} \sim -T \frac{1}{T_c} \frac{k_B T_c}{\hbar}$$

(7.52)

which is in absolute value smaller than the contribution $D_\phi^{(\alpha)}$ from low-energy excitations by the order of magnitude factor $\sqrt{\hbar \omega_0 / \mu}$. This contribution is therefore negligible in very large condensates. In not so large condensates the complete integral in the result for $\Gamma_0$ needs to be evaluated.

In the low-temperature regime $k_B T < \mu$ we get instead

$$D_\phi^{(\beta)} = -0.306 \cdots \frac{1}{N} \left( \frac{k_B T_c a^2 m}{\hbar} \right)^{1/2} \frac{k_B T_c}{\hbar}$$

(7.53)

This contribution differs from $D_\phi^{(\alpha)}$ by the order of magnitude factor $(\mu/k_B T_c)^{1/2}/N^{1/3}$.

3. Fluctuations in the thermal cloud at energies of order $\mu$

By eq.(3.19) this contribution is given by

$$D_\phi^{(\gamma)} = \frac{k_B T}{\hbar(N_0)} (\Gamma_0 + \Gamma')$$

(7.54)

which is evaluated to

$$D_\phi^{(\gamma)} = 0.0243 \cdots B^{(\omega)} \left( \frac{T}{T_c} \right)^3 \frac{N}{\langle N_0 \rangle^2} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/2} \frac{k_B T_c}{\hbar}.$$

(7.55)

This contribution differs from $D_\phi^{(\alpha)}$ by the order of magnitude factor $(T/T_c)^2 \sqrt{\mu \hbar \omega_0 / k_B T_c}$ and is therefore much smaller.

For temperatures $k_B T < \mu$ we find instead

$$D_\phi^{(\gamma)} = 0.366 \cdots \left( \frac{T}{T_c} \right)^8 \frac{1}{\langle N_0 \rangle} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{-2} \frac{k_B T_c}{\hbar}$$

(7.56)

which is again negligibly small compared to $D_\phi^{(\alpha)}$.

In summary, the phase-diffusion is caused dominantly by the low-frequency particle number fluctuations in the condensate and the phase-diffusion constant is given by eq.(7.49). It is proportional to temperature, and scales proportional to $N^{-2/3}$ for fixed $T_c$ or proportional to $N^{-1/2}$ for fixed volume of the trap.

VIII. EVALUATION OF THE TRANSPORT PARAMETERS FOR AN ISOTROPIC HARMONIC TRAP

In this section we consider the more realistic case of condensates in a parabolic trapping potential $m \omega_0^2 x^2/2$, which we assume to be isotropic for simplicity. In order to analyse the noise $\Im(\xi(t))$ driving the fluctuations of $|\alpha_0|^2$ we must consider in detail the relevant linear combination of matrix elements

$$(M^{(1)}_{\kappa \nu \mu \rho})^* - M^{(2)}_{\nu \mu \kappa} = 2 U_0 \int d^3 x \psi_0 \left\{ (\tilde{u}_\kappa - \tilde{v}_\kappa)(\tilde{u}^*_\nu \tilde{u}^*_\mu + \tilde{v}^*_\nu \tilde{v}^*_\mu) - \tilde{u}_\kappa \tilde{u}^*_\mu \tilde{u}^*_\nu + \tilde{v}_\kappa \tilde{v}^*_\mu \tilde{v}^*_\nu \right\}$$

(8.1)

In the following we shall make use of the local density and Thomas-Fermi approximation, restricting ourselves to large condensates. For the high-lying states we can then use the local energies in Thomas-Fermi approximation

$$E(p, x) = \sqrt{\frac{p^2}{2m} + |U_0 n_0(x)|^2} - U_0^2 n_0^3(x) \Theta(\mu - V(x))$$

(8.2)

with the condensate density

$$n_0(x) = \frac{2}{\hbar^3} \int \frac{d^3 k}{(2\pi)^3} \left| \frac{\omega_k}{|\kappa|} \right|^2 f_k$$

(8.3)

where $f_k$ is the Fermi distribution function.
The high-lying quasi-particle modes can be represented similarly to the spatially homogeneous case as

\[ u_\kappa(x) = \frac{E_\kappa + p_\kappa^2/2m}{\sqrt{2E_\kappa p_\kappa^2/m}} e^{ip_\kappa \cdot x/h} \]  

\[ v_\kappa(x) = -\frac{E_\kappa - p_\kappa^2/2m}{\sqrt{2E_\kappa p_\kappa^2/m}} e^{ip_\kappa \cdot x/h} \]  

The low-lying collective modes can be represented as

\[ u_\nu(x) = \left( \sqrt{\frac{U_0 n_0(x)}{2\hbar \omega_\nu}} + \frac{1}{2} \sqrt{\frac{\hbar \omega_\nu}{2U_0 n_0(x)}} \right) \chi_\nu(x) \]

\[ v_\nu(x) = \left( -\sqrt{\frac{U_0 n_0(x)}{2\hbar \omega_\nu}} + \frac{1}{2} \sqrt{\frac{\hbar \omega_\nu}{2U_0 n_0(x)}} \right) \chi_\nu(x) \]  

with

\[ \int d^3 x |\chi_\nu(x)|^2 = 1 \]

The mode-functions \( \chi_\nu(x) \) are known in the hydrodynamic (long-wavelength) and Thomas-Fermi approximation \[23,24,26,25,27\] by analytic solutions of the Bogoliubov-equations. In spatially isotropic parabolic traps they have the form \[23\]

\[ \chi_\nu(x) = \frac{1}{r_{TF}^{\nu}} P_{\ell_\nu}^{(2n_\nu)}(x/r_{TF})(x/r_{TF})^\nu Y_{\ell_\nu}(\theta, \varphi) \Theta(1 - x/r_{TF}) \]  

The polynomials \( P_{\ell}^{(2n)}(x) \) of degree \( 2n \) are the normalized solutions of the radial part of the Bogoliubov-Fetter equations in the Thomas-Fermi and long-wavelength limit \[23,24\] given by \[24\]

\[ P_{\ell}^{(2n)}(x) = \frac{\sqrt{4\ell + 2\ell + 3}}{n!} x^{-2\ell-1} \frac{d^n}{dx^n} \left[ x^{2\ell+2n+1} (1 - x^2)^n \right] \]

with the normalization

\[ \int_0^1 dx x^{2\ell+2} \left| P_{\ell}^{(2n)}(x) \right|^2 = 1 \]

In the phonon part of the excitation spectrum we have \( \omega_\nu \simeq -v_\nu \sim \omega_\lambda^{-1/2} \). Furthermore, in that low-energy region the statistical factor in eqs.\([5.13]\), \([5.14]\), \([5.15]\) is well approximated by \( \tilde{n}_\kappa \tilde{n}_\mu \tilde{n}_\nu \approx (k_B T)^3 \hbar^2 \omega_\kappa \omega_\mu \omega_\nu \). Just as in the case of box-like traps the frequency factors in the denominator, together with similar further factors in the denominator coming from the matrix elements, make the phonon contribution to the sums in \([5.13]\) the dominant one, at least in large condensates, and we shall therefore concentrate on this contribution in the following. This frequency range has a natural upper cut-off at \( \langle \mu \rangle / \hbar \), where the collective phonons go over smoothly into particle-like excitations.

For \( E_\kappa, E_\nu, E_\mu \ll \langle \mu \rangle \) the matrix elements \( (M_{\kappa,\nu,\mu})^*, M_{\ell_\nu,\ell_\mu,\ell_\kappa}^{(2)} \) are given by the integral
(M^{(1)}_{\nu,\mu})^* \approx -M^{(2)}_{\nu,\mu,\kappa} \tag{8.11}

\approx -\left(\frac{15}{8\pi r_F^3} \frac{3U_0(\mu)^{3/2}}{2E_\mu E_\nu(E_\nu + E_\mu)}\right)J(n_\kappa, n_\nu, n_\mu; \ell_\kappa, \ell_\nu, \ell_\mu)C(\ell_\kappa|\ell_\mu m_\mu, \ell_\nu m_\nu) \tag{8.12}

where \ J denotes the integral

\[ J(n_\kappa, n_\nu, n_\mu; \ell_\kappa, \ell_\nu, \ell_\mu) = \int_0^1 dx x^2 (1 - x^2)^2 x^{\ell_\kappa + \ell_\nu + \ell_\mu} P^2_{\ell_\kappa}(x) P^2_{\ell_\nu}(x) P^2_{\ell_\mu}(x) \]

and the Clebsch-Gordon coefficients \ C(\ell_\kappa|\ell_\mu m_\mu, \ell_\nu m_\nu) \ are given by the angle-integral

\[ C(\ell_\kappa|\ell_\mu m_\mu, \ell_\nu m_\nu) = \int d\Omega Y_{\ell_\kappa,m_\mu}^*(\theta,\phi)Y_{\ell_\mu,m_\mu}(\theta,\phi)Y_{\ell_\nu,m_\nu}(\theta,\phi) \]

if \ |\ell_\mu - \ell_\nu| \leq \ell_\kappa \leq \ell_\mu + \ell_\nu, otherwise they vanish. Later-on we shall have to calculate e.g. \ \sum_{m_\nu, m_\mu} |(M^{(1)}_{\nu,\mu})^* - M^{(2)}_{\nu,\mu,\kappa}|^2 \]

where we can make use of the sum-rule, for \ |\ell_\nu - \ell_\mu| \leq \ell_\kappa \leq \ell_\mu + \ell_\nu

\[ \sum_{m_\nu, m_\mu} |C(\ell_\kappa|\ell_\mu m_\mu, \ell_\nu m_\nu)|^2 = 1 \]

so that the Clebsch-Gordon coefficients need actually not be used explicitly.

In order to have well-defined expressions for the rate-coefficients we again need to smoothen the delta-function expressing energy-conservation, which is done physically by experimental imperfections or limitations in resolution. Here this can be done by replacing the discrete sum over the ‘quantum-number’ \ \ell_\kappa \ by an integral

\[ \sum_{\ell_\kappa} \delta(E_\kappa - E_\nu - E_\mu)(...) \approx \int dE_\kappa \frac{1}{dE_\kappa/d\ell_\kappa} \delta(E_\kappa - E_\nu - E_\mu)(...) \]

= \int dE_\kappa \frac{E_\nu + E_\mu}{(\hbar \omega_0)^2(n_\kappa + 1/2)} \delta(E_\kappa - E_\nu - E_\mu)(...) \tag{8.13}

where we used the expression for the excitation-energies \ [23]

\[ E_\kappa = \hbar \omega_0 e_\kappa \]

\[ e_\kappa = \sqrt{2n_\kappa^2 + 2n_\kappa \ell_\kappa + 3n_\kappa + \ell_\kappa} \tag{8.14} \]

We introduced the dimensionless eigenvalues \ e_{\nu,\mu,\kappa} \ which will appear in the ensuing expressions from now on. The integration over \ E_\kappa \ with the delta-function then picks out the energy-value \ E_\kappa = E_\nu + E_\mu \ so that \ \ell_\kappa \ becomes a function \ \ell_\kappa^{(0)} \ of the other ‘quantum-numbers’

\[ \ell_\kappa^{(0)} = \frac{(e_\nu + e_\mu)^2 - 2n_\kappa^2 - 3n_\kappa}{2n_\kappa + 1} \]

The inequalities

\[ |\ell_\nu - \ell_\mu| \leq \ell_\kappa^{(0)} \leq \ell_\nu + \ell_\mu \]

then imply that \ n_\kappa \ must lie in the interval

\[ n_{\kappa-} \leq n_\kappa \leq n_{\kappa+} \]

with

\[ n_{\kappa\pm} = \frac{1}{2} \left( |\ell_\nu + \ell_\mu|^2 + |\ell_\nu \pm \ell_\mu| + 9/4 + 2(e_\nu + e_\mu)^2 - |\ell_\nu \pm \ell_\mu| - 3/2 \right) \]

Using all this we obtain from eq. [5.13]

\[ \Gamma_0 = B_00 \left( \frac{a}{d_0} \frac{k_B T}{\hbar \omega_0} \right)^2 = B_00 \left( \frac{T^2 k_B T_a a^2 m}{\hbar^2} \right)^1 \frac{N}{\zeta(3)} \] \tag{8.15}
where $\hbar \omega_0$ is now eliminated in favour of $k_B T_c$ via the relation $\hbar \omega_0 = k_B T_c (\zeta(3)/N)^{1/3}$, and where the temperature- and particle-number-independent positive real number $B_{00}$ is defined by the multiple sums

$$B_{00} = \frac{135 \pi^2}{2} \sum_{\kappa} \sum_{\mu} \sum_{\ell_\mu} \sum_{\kappa \mu \mu} \frac{(2 e_\kappa^{(0)} + 1) J^2(n_\kappa, n_\mu, \mu; \kappa_\mu, \ell_\mu, \ell_\mu)}{e_\kappa^2 e_\mu^2 (e_\mu + e_\kappa)(2n_\kappa + 1)}$$

(8.16)

The result for $\Gamma_0$ agrees, except for the numerical prefactor, with the result of [1] which was evaluated there using the local-density approximation and imposing a lower cut-off for the excitation-frequencies at the geometrical mean trap-frequency $\bar{\omega}$. It can also be compared with the corresponding result [21] for the box-like trap, which shows the same dependence on temperature and particle-number (if we stipulate $\langle N_0 \rangle \sim N$), but the comparison of the prefactor is problematic because the condensate in the parabolic trap has two length-scales $d_0$ and $r_T$. Whereas in the box-like trap only the length-scale $L$ is relevant.

The property [8.11] of the matrix-elements implies that the low-lying excitations don’t contribute to $\Gamma_0 + \Gamma'$. The reality of the matrix-elements furthermore implies that $\Gamma''$ vanishes. These remarkably simple results mean that the noise-source $\xi(t)$ introduced in eq. (8.4) is purely imaginary, corresponding to total squeezing in the direction of the phase $\phi$. In other words, the coupling of the condensate to the collective excitations introduces a direct Langevin noise-source only for the number-fluctuations $\delta N_0$, not the phase-variable $\phi$.

The fact that $\Gamma_0 + \Gamma' = 0$ for the contribution from the low-lying states implies that the contributions from the higher lying states must also be considered in order to evaluate the small (compared to $\Gamma_0$) but finite value of this quantity. For this purpose we need to consider the matrix-element $(M_{\kappa, \mu, \nu, \ell})^* + M_{\kappa, \mu, \nu, \ell}^{(2)}$. It differs from the matrix-elements we have considered so far by the replacement $\bar{\psi}_0(x) \rightarrow \bar{\psi}_0(x) + 2\langle N_0 \rangle \partial \bar{\psi}_0(x)/\partial (N_0)$ in the matrix-element. In Thomas-Fermi approximation this is tantamount to the replacement

$$\bar{\psi}_0(x) \rightarrow (2/5) \bar{\psi}_0(x)/(1 - x^2/r_T^2).$$

(8.17)

Physically this implies a reduced coupling of the thermal fluctuations with the center of the condensate and a strongly enhanced coupling at its boundary, as one would expect for fluctuations located in the thermal cloud. A mathematical consequence is the fact that the integrals defining these matrix-elements diverge at the boundary in the Thomas-Fermi approximation, meaning that we encounter here the limitations of that approximation. Instead of a full-fledged extension of the theory beyond the Thomas-Fermi approximation it will be sufficient for our purposes here to cure its deficiencies by substituting as a cut-off the finite thickness of the boundary-layer given by [22]

$$d = \frac{1}{2} r_T F \left( \frac{\hbar \omega_0}{\langle \mu \rangle} \right)^{2/3}$$

The matrix-element itself is then evaluated in the local-density approximation [22], where we can make use to good purpose of the analysis already performed in the preceding section. The finite volume $V = L^3$ (and the associated $\hbar \omega_0 = (2m)^{1/2}/2mL^3$ which is not to be confused with the trap frequency called $\omega_0$ in the present section) is then an arbitrary local subvolume of the condensate, introduced merely as a technical device like a quantization-volume. It must be sufficiently small so that the condensate within it can be treated as homogeneous, and sufficiently large that we can replace sums over local momenta by integrals. At the end we have to check for consistency whether the result is indeed independent of the choice of this volume. The result obtained in this way is the local average of the result (7.28) for the homogenous case, which now becomes space-dependent, because we have to substitute a space-dependent chemical potential $\mu \rightarrow \langle \mu \rangle (1 - x^2/r_T^2)$. This local result can be written as

$$\Gamma_0(x) + \Gamma'(x) = \frac{B'(\mu)}{2\pi^2} \frac{(k_B T)^2 a^2 m}{\hbar^2 \langle \mu \rangle (1 - x^2/r_T^2)}$$

and is indeed independent of the choice of $V$. The local average has to be performed with the weight $(\bar{\psi}_0(x) + 2\langle N_0 \rangle \partial \bar{\psi}_0(x)/\partial (N_0))^2$ determined from (8.17). Doing the average and regulating the divergency of the integral at the boundary of the condensate by the physical cut-off we obtain

\[\text{10The latter is of course affected by the noise-source indirectly, because the fluctuations of } \delta N_0 \text{ driven by the latter cause fluctuations in the chemical potential.}\]
\[ \Gamma_0 + \Gamma' = \frac{3}{10} \frac{2^{1/3}}{\sqrt[5]{15^{-2}15^{-2}}} B^{(\mu)} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \langle N_0 \rangle^{-2/15} \left( \frac{a}{d_0} \right)^{28/15} = 0.024 \ldots B^{(\mu)} \left( \frac{T}{T_c} \right)^2 \left( \frac{N}{\langle N_0 \rangle} \right)^{2/15} N^{2/9} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{14/15}. \]  
(8.18)

In order to extract results for the relaxation-rate of the condensate number and the phase-diffusion rate it is necessary to know also the mean square of the number-fluctuations \( \langle \Delta N_0^2 \rangle \). This can be evaluated from eq. (2.28), using the fact that these fluctuations are also dominated by the low-lying modes \([4]\). The result of this calculation to leading order in \( (\hbar \omega_0/k_B T) \) is

\[ \langle \Delta N_0^2 \rangle = A \left( \frac{\langle N_0 \rangle a}{d_0} \right)^{4/5} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \]  
(8.19)

\[ = \frac{A}{(\zeta(3))^{8/15}} \left( \frac{T}{T_c} \right)^2 \left( \frac{\langle N_0 \rangle}{N} \right)^{4/5} N^{4/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{2/5} \]  
(8.20)

with the number \( A \) given by the multiple sums

\[ A = \frac{(15)^{4/5}}{2} \sum_n \sum_{n'} \sum_{\ell} \frac{2\ell + 1}{(e(n, \ell) e(n', \ell))^2} \int_0^1 dx (1 - x^2)^{2(\ell + 1)} P_\ell^{(2n)}(x) P_\ell^{(2n')}(x) \right| \]  
(8.21)

In order to find the scaling of \( \langle \Delta N_0^2 \rangle \) in the thermodynamic limit \( N \to \infty, \omega_0 \to 0, k_B T_c = \hbar \omega_0 (N/\zeta(3))^{1/3} \) fixed it is necessary to use the form of the preceding results in which \( \hbar \omega_0 \) is eliminated in favor of \( k_B T_c \) and to use \( \langle N_0 \rangle \approx N \). Then the scaling \( \langle \Delta N_0^2 \rangle \sim N^{4/3} \) derived in \([4]\) is recovered. The particle-number relaxation rate now follows from eqs. (8.17) and (8.18) as

\[ \gamma_c = \frac{2B_00}{A} \langle N_0 \rangle^{-1/5} \left( \frac{a}{d_0} \right)^{6/5} \frac{k_B T}{\hbar} = \frac{2(\zeta(3))^{1/5} B_00}{A} \left( \frac{T}{T_c} \right)^{1/5} \left( \frac{\langle N_0 \rangle}{N} \right)^{1/5} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{3/5} \frac{k_B T_c}{\hbar}. \]  
(8.22)

It is the largest of the various rates we calculate here but is still small compared to \( \omega_0 \), the inverse time-scale of motion in the trap, by the order of magnitude \( N^{-2/3}(N a/d_0)^{6/5} \).

The phase-collapse rate is obtained from (8.24). At \( T \neq 0 \) (more precisely above a cross-over temperature of order \( \hbar \omega_0 \)) we find

\[ \gamma_{\text{collapse}} = \frac{15^{2/5} A^{1/2}}{5} \langle N_0 \rangle^{-1/5} \left( \frac{a}{d_0} \right)^{1/5} \frac{k_B T}{\hbar} = \frac{15^{2/5}(\zeta(3))^{2/15} A^{1/2}}{5} \left( \frac{T}{T_c} \right) \left( \frac{N}{\langle N_0 \rangle} \right)^{1/5} N^{-1/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{2/5} \frac{k_B T_c}{\hbar}. \]  
(8.23)

Apart from the numerical prefactor this is the same asymptotic expression as obtained for the damping-rate \( \gamma_c \) of the low-lying collective modes (see e.g. [21]). It is smaller than \( \gamma_c \) by the order of magnitude \( (\langle N_0 \rangle a/d_0)^{-2/5} \), i.e. the phase-collapse remains inefficient before phase-diffusion takes over.

The phase-diffusion constant \( D_\phi^{(\alpha)} \) due to the exchange of particles between the condensate and low-lying excitations is gotten by inserting the results for \( \langle \Delta N_0^2 \rangle \) and \( \Gamma_0 \) in eq. (5.19):

\[ D_\phi^{(\alpha)} = \frac{(15)^{4/5} A^{2}}{25 B_00} \langle N_0 \rangle^{-3/5} \left( \frac{a}{d_0} \right)^{2/5} \frac{k_B T}{\hbar} = \frac{15^{4/5}(\zeta(3))^{1/15} A^{2}}{25 B_00} \left( \frac{T}{T_c} \right) \left( \frac{N}{\langle N_0 \rangle} \right)^{3/5} N^{-2/3} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/5} \frac{k_B T_c}{\hbar}. \]  
(8.24)

It is smaller than \( \gamma_{\text{collapse}} \), again by the order of magnitude \( (\langle N_0 \rangle a/d_0)^{-2/5} \).

Finally, the contribution of the fluctuations in the thermal cloud to the phase-diffusion is also obtained from (3.19) by inserting the result (8.18) for \( \Gamma_0 + \Gamma' \):

\[ D_\phi^{(\gamma)} = \frac{3}{10} \frac{2^{13/5} B^{(\mu)}}{15^{2/15} \pi^2} \langle N_0 \rangle^{-17/15} \left( \frac{a}{d_0} \right)^{28/15} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \left( \frac{T}{T_c} \right)^3 \left( \frac{N}{\langle N_0 \rangle} \right)^{17/15} N^{-7/9} \left( \frac{k_B T_c a^2 m}{\hbar^2} \right)^{14/15} \frac{k_B T_c}{\hbar}. \]  
(8.25)

It differs from the previous rates, which were all proportional to temperature, by the stronger temperature-dependence \( \sim T^3 \). However, this contribution to \( D_\phi \) remains smaller than \( D_\phi^{(\alpha)} \) by the order of magnitude \( N^{-1/9}(k_B T_c a^2 m/\hbar^2)^{11/15}(T/T_c)^2 \).
IX. DISCUSSION AND CONCLUSION

In this paper we have put forward a detailed theory of fluctuations and relaxation processes of the condensate in thermal equilibrium with the cloud of its excitations. For a given number of particles \( N_0 \) in the condensate, we have defined the condensate mode as the corresponding normalized solution of the Gross-Pitaevskii equation, defining at the same stroke the \( N_0 \)-dependent part of the chemical potential. The equilibrium value of \( \langle N_0 \rangle \) is distinguished as the value of \( N_0 \) for which the number of particles in the thermal cloud in equilibrium with the condensate plus \( N_0 \) equals \( N \). We have calculated the fluctuations of \( N_0 \) around its equilibrium value and also the fluctuations of the phase of the complex amplitude \( \alpha_0 \) of the condensate with \( |\alpha_0|^2 = N_0 \). In a general phenomenological framework presented in the first part of this paper we were able to separate the fluctuations of the complex condensate amplitude into several contributions, which have different physical origin:

- The fluctuation of the atom-number in the condensate, which are driven by the exchange of atoms between the condensate and the thermal cloud.
- The fluctuation of the chemical potential with two different contributions, namely the fluctuations of \( \mu \) due to number-fluctuations in the condensate, and the faster fluctuations of \( \mu \) at constant \( N_0 \) caused by number-fluctuations in the excitations.

The importance of number-fluctuations in the condensate, assumed at first in the phenomenological approach due to the importance of \( N_0 \) for the value of the chemical potential, but later born out by the microscopic calculations, leads to the appearance of the linear relaxation-rates \( \gamma_c \) of the condensate-number as an important characteristic inverse time-scale of the problem. At times much shorter than \( \gamma_c^{-1} \) phase-diffusion of the condensate-phase due to the fast number-fluctuations in the excitations can occur. In the same regime may also occur the process of collapse due to the reversible spreading of the phase caused by the static uncertainty in \( N_0 \) and the associated chemical potential.

At times large compared to \( \gamma_c^{-1} \) the number-fluctuations in the condensate are dynamical and irreversible, and lead to the replacement of the reversible collapse by an irreversible phase-diffusion with a larger diffusion-rate than in the short-time regime.

The second and larger part of this paper was devoted to microscopic theory. First we have provided a microscopic derivation of the phenomenological Langevin equation, established microscopic formulas for all phenomenological parameters and also exhibited the relation between the short-time diffusion rate and fluctuation rates of the population numbers of excitations via a sum-rule. Then the microscopic theory was used to evaluate the transport-parameters and the various rates as a function of temperature, particle-number and the scattering length of the interaction potential. The evaluation was done for two simple cases – the cubic box-like trap, where the form of the condensate mode does not depend on \( N_0 \) and the thermal cloud penetrates the condensate homogeneously, and the isotropic harmonic trap, where the form of the condensate-mode changes with \( N_0 \) and the thermal cloud is located preferentially near the boundaries of the condensate. The physically important results for both kinds of traps are similar, even though they have to differ, obviously, in the details of the scalings with the atom-numbers and the scattering length.

The calculation of the transport-parameters reveals some interesting physical results:

- The fluctuations driving the absolute value \( |\alpha_0| \) and the phase \( \phi \) of \( \alpha_0 \) are quite different in strength, those driving \( |\alpha_0| \) being the much stronger ones. The reason for this is a pronounced squeezing of the bath of thermal excitations with respect to the instantaneous phase of the condensate. This squeezing reaches nearly 100% for the lowest-lying modes, which is the reason that fluctuations of \( \phi \) are practically not driven by such modes. On the other hand, the contribution of the high-lying modes to the fluctuating forces driving \( |\alpha_0| \) and \( \phi \) is nearly the same (after the obvious normalization with \( |\alpha_0| \)), i.e. there is no squeezing in this (much weaker) contribution to the noise.
- The cross-correlation between the fluctuations driving \( |\alpha_0| \) and \( \phi \) are found to vanish exactly in a real condensate, where both the Gross-Pitaevskii equation and the Bogoliubov-Fetter equations are real and all solutions can (but need not) be taken real. This can also be understood as a general consequence of time-reversal symmetry: \( \phi \) is a velocity potential and therefore odd under time-reversal while \( |\alpha_0| \) is even under time-reversal. Their fluctuating forces therefore transform oppositely. In a time-reversal symmetric condensate (no vortices) the cross-correlation between an even and an odd quantity under time-reversal must vanish.

It turns out that the relaxation rate \( \gamma_c \) of the atom-number in the condensate is the largest of the calculated rates. In particular it is larger than the collapse-rate and the phase-diffusion rate which, like \( \gamma_c \), are proportional to temperature in the regime \( k_B T > \mu \). It is also larger than the decay rates of the lowest-lying collective modes \( \gamma_0 \), which might look surprising because at the same time the theory tells us that \( \gamma_c \) is dominated by the particle transfer-rates between the condensate and the low-lying modes. However, it is clear that \( \gamma_c \) ought to be larger than \( \gamma_0 \) because the condensate couples to all low-lying modes in parallel which increases the number of decay channels by a factor proportional to the ratio of the chemical potential and the lowest-lying mode frequency.

The next largest rate we find is the thermal phase-collapse rate \( \gamma_{\text{collapse}} \). It turns out to have the same functional dependence on \( T, a, \langle N_0 \rangle \) and \( N \) as the decay-rate of the lowest-lying collective modes. I cannot see any
fundamental reason for this coincidence and have to count it just as that. Physically the smallness of \( \gamma_{\text{collapse}}/\gamma_c \) means that the phase-collapse will not be observable at finite temperature because it can only lead to a decay-factor \( \exp(-\frac{1}{2}(\gamma_{\text{collapse}}/\gamma_c)^2) \) very close to 1 before phase-diffusion takes over.

Finally, the phase-diffusion rate \( D_\phi \) is the smallest of the rates calculated here. We find the simple nice result that the ratios \( \gamma_c/\gamma_0 \) and \( \gamma_0/D_\phi \) are of about equal order of magnitude, given by the ratio of \( \mu \) to the smallest excitation energy, which is \( \hbar \omega_c \) for the harmonic and \( \sqrt{\mu/m(2\pi\hbar/L)} \) for the box-like trap. Instead of \( \gamma_0 \) we may also take \( \gamma_c \) in these ratios with the same conclusion. \( D_\phi \) like the rate \( \gamma_c \) is found to be dominated by the atom-number exchange between the condensate and the low-lying modes.

This observation actually explains the coincidence of the two ratios we have just indicated and turns them into a precise relation: In (3.19) for \( D_\phi \) we put \( \Gamma c = 0 \) which is exact for real condensate-modes and neglect \( \Gamma_{\text{c}} + \Gamma' \), which comes from high-lying excitations. Then multiplying the resulting expression for \( D_\phi \), with \( \gamma_c = \tau_{\text{c}}^{-1} \) from (3.17), we readily find

\[
\frac{1}{2} D_\phi \cdot \gamma_c = \gamma_{\text{collapse}}^2
\]

with \( \gamma_{\text{collapse}} \) from (3.23) again with \( \Gamma' = 0 \).

Let us now compare our results with related ones found in the literature. Most closely related to the present work in goal and scope is a paper by Jakobsch et al \[28\] on the intensity and amplitude fluctuations of a Bose-Einstein condensate at finite temperature, which builds on extensive earlier work by Gardiner and Zoller with collaborators (cf. the references given in [28]). Unlike the present paper it also takes into account trap losses. The theory presented in [28] is based on a conceptual division of the Bose gas into two energy regions called the condensate band and the noncondensate band. In this construction the boundary between the two regions is chosen in such a way that the noncondensate band is not significantly affected by the mean field of the condensate, while the influence of excitations in the condensate band is neglected. Thus the main physical difference of [28] to the present work is that it neglects fluctuations of particles from the condensate mode to quasiparticle modes as well as to very low-lying one-particle excitations.

By contrast in the present work we avoid the division of the energy region into two parts. We find, as we have discussed, that the exchange of particles between the condensate and the low-lying modes makes not only an important but in fact the dominant contribution to the relaxation rate of the condensate-number and the phase-diffusion rate, determining their dependence on temperature, atom number, population of the condensate and scattering length.

The importance of the particle-exchange between the low-lying excitations for the phase-diffusion of the condensate and the number-relaxation rate \( \gamma_c \) was first pointed out in \[14\], while for the intensity of the number-fluctuations in the condensate this had already been shown in \[13\]. The theory put forward in \[14\] already proceeded along essentially the same lines we follow here, but it had some short-comings which we overcome and correct in the present work: The squeezing of the noise from the thermal cloud with respect to the phase of the condensate was briefly remarked upon in \[14\], but was not taken into account in the calculation of the transport coefficients presented there and in the formula for phase-diffusion. Moreover, in the conservative part of the Langevin-equation (3.6) \( \Delta \mu \) was replaced by \( \partial \Delta F(\alpha^2)/\partial |\alpha|^2 \) in \[14\], which, on scrutiny, appears questionable when used in conjunction with the fluctuation formula (2.27,2.28) for \( \langle \Delta N_0^2 \rangle \). After all, neither \( \Delta \mu \) nor \( \Delta F \) are equilibrium quantities. The use of the aforementioned relation between them is therefore avoided here.

Even though in the present paper I have opted for the use of the fluctuation-formulas (2.27,2.28), which in my opinion have a firm basis, it is only fair to mention that they are still under debate in the current literature, see e.g. \[13\]. In another recent paper with some bearing on this topic Bergman et al. \[22\] use as equilibrium distribution for the condensate number \( P(N_0) \sim \exp[(\mu)N_0 - \frac{5}{12}(15N_0a/d_0)^{2/5}N_0]/k_B T \), (cf. the discussion after their eq.(21)), which implies \( \langle \Delta N_0^2 \rangle \sim T(N_0)^{3/5} \), a result which is rather different, both in the temperature-dependence and in the scaling with the particle-number, from the result (8.19) on which our present calculations have been based. It is clear that not the method but the details of our results on the dynamics of the fluctuations of the condensate would change, if the results on the statics would be changed. Needless to say that a resolution of the theoretical debate concerning the correct approach to the statics seems urgent and would be highly welcome. Vice versa experimental results on the dynamics (i.e. on \( \gamma_c \) and \( D_\phi \)) would also help to decide, by applying the theory presented here, which of the approaches to the statics of the number-fluctuations in the condensate proposed in the literature describes the physics correctly.

A quantum kinetic theory of trapped atomic gases has also been formulated by Stoof \[30\]. In \[30\] the general coupled Fokker-Planck equations of the condensate and the excited modes are presented and applied to the kinetics of the formation of a condensate. This problem has also been studied by Gardiner and coworkers \[31\] and Kagan and Svistunov \[32\] as well as Kagan and Svistunov \[32\], where also earlier work by further authors is quoted.
By contrast the present work has focussed on the fluctuations around the equilibrium state of the condensate, after it has been formed. However, the application of our approach to the kinetics of the formation of a condensate would be an interesting goal for future work.

Experimentally the rates $\gamma_c$ and $D\phi$ we have calculated should be measurable. The rate $\gamma_c$ may be observable as the relaxation rate of the condensate back to its equilibrium state after creating a non-equilibrium state by a sudden small change of temperature via evaporative cooling. The sum of the phase-diffusion rates of two condensates could be measured by monitoring the phase-difference between them after it was initially fixed by measurement or preparation at a reference-time $t = 0$. Methods for measuring phase-differences in Bose-Einstein condensates have recently been demonstrated \[3–5\]. It is to be hoped therefore that the phase-diffusion in Bose-Einstein condensates - a fundamental process intimately linked to the spontaneously broken gauge symmetry in a finite system - will be measured in the near future.

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APPENDIX A:

Here we wish to derive the expression \((2.13)\) for $H_0$. Using the Gross-Pitaevskii equation \((2.4)\) we put \((2.8)\) in the form

$$H_0 = (\mu_0 - \langle \mu \rangle) |\alpha_0|^2 - \frac{U_0}{2} |\alpha_0|^4 \int d^3 x \psi_0^4 \tag{A1}$$

Taking the derivative with respect to $|\alpha_0|^2$ we get

$$\frac{\partial H_0}{\partial |\alpha_0|^2} = \mu_0 - \langle \mu \rangle + |\alpha_0|^2 \left( \frac{\partial \mu_0}{\partial |\alpha_0|^2} - U_0 \int d^3 x \psi_0^4 - |\alpha_0|^2 \int d^3 x \psi_0^2 \frac{\partial \psi_0^2}{\partial |\alpha_0|^2} \right) \tag{A2}$$

To evaluate this further we use eq.\((2.4)\) and its derivative with respect to $|\alpha_0|^2$

$$\left(- \frac{\hbar^2}{2m} \nabla^2 + V - \mu_0 + 3U_0 |\alpha_0|^2 \psi_0^2 \right) \frac{\partial \psi_0}{\partial |\alpha_0|^2} = \left( \frac{\partial \mu_0}{\partial |\alpha_0|^2} - U_0 \psi_0^2 \right) \psi_0 \tag{A3}$$

Multiplying eq.\((A3)\) with $\psi_0$ and integrating over space, using the Gross-Pitaevskii equation \((2.4)\) after partial integration, we derive the identity

$$U_0 |\alpha_0|^2 \int d^3 x \psi_0^2 \frac{\partial \psi_0^2}{\partial |\alpha_0|^2} = \frac{\partial \mu_0}{\partial |\alpha_0|^2} - U_0 \int \psi_0^2 d^3 x \tag{A4}$$

which is used in \((A2)\) to yield $\partial H_0 / \partial |\alpha_0|^2 = \mu_0 - \langle \mu \rangle$, and upon integration results in \((2.13)\).

APPENDIX B:

Here we wish to derive eq.\((4.14)\). This is achieved if we succeed to show that the coupling of the condensate and the thermal cloud via

$$\hat{H}_3 = U_0 \sqrt{\langle N_0 \rangle} \int d^3 x \tilde{\psi}_0 \tilde{\chi}^+ (e^{-i\phi} \tilde{\chi}^+ + e^{i\phi} \tilde{\chi}^+) \tag{B1}$$

gives rise to the systematic change of $\Im(\tilde{\xi}(t))$, to first order in the interaction, of

$$\delta \langle \Im(\tilde{\xi}(t)) \rangle = \frac{2 \sqrt{\langle N_0 \rangle}}{\hbar k_B T} \int_{-\infty}^t dt' S_{JJ}(t - t') \frac{\partial H_0(t')}{\partial |\alpha_0|^2} \tag{B2}$$
because this can then be used in (4.7) to yield (4.14). In (B1) we could put $|\alpha_0| = \sqrt{\langle N_0 \rangle}$ since we linearize around equilibrium and only wish to calculate the dissipation in $|\alpha_0|^2$ which is conjugate to $\phi$, the variable we kept in (B1). Standard first order perturbation theory with adiabatic switch-on of the interaction gives, with $\epsilon \to +0$,

$$\delta \langle \mathcal{S}(\bar{\xi}(t)) \rangle_\phi = -\frac{i}{\hbar} \int_{-\infty}^{t} dt' \langle [\mathcal{S}(\bar{\xi}(t)), \mathcal{H}_3(t')] \rangle_\phi e^{\epsilon t'}.$$  

(B3)

We can rewrite this as

$$\delta \langle \mathcal{S}(\bar{\xi}(t)) \rangle_\phi = -2i \sqrt{\langle N_0 \rangle} \int_{-\infty}^{t} dt' \left( \chi''_\xi(t, t') e^{-i\phi(t')} + \chi''_{J\xi}(t, t') e^{i\phi(t')} \right) e^{\epsilon t'}$$  

(B4)

where we introduced the response functions

$$\chi''_\xi(t, t') = \frac{1}{2\hbar} \langle [\mathcal{S}(\bar{\xi}(t)), \bar{\xi}(t')] \rangle_\phi \Theta(t - t')$$

(B5)

$$\chi''_{J\xi}(t, t') = \frac{1}{2\hbar} \langle [\mathcal{S}(\bar{\xi}(t)), \bar{\xi}^+(t')] \rangle_\phi \Theta(t - t')$$

with

$$\dot{\bar{\xi}}(t) = U_0 \int d^3x \psi_0 \bar{\xi}^+(t) \hat{\chi}(t) \hat{\chi}(t).$$  

(B6)

Here $\Theta(t - t')$ is the Heaviside step-function. We shall define $\Theta(0) = 0$ without loss of generality. The fluctuation-dissipation theorem (in the classical frequency domain $\hbar \omega \ll k_B T$) ensures the relations

$$\chi''_\xi(t, t') = \frac{-i \Theta(t - t')}{{2k_B T}} \frac{\partial}{\partial t} S_{J\xi}(t, t')$$

(B7)

$$\chi''_{J\xi}(t, t') = \frac{-i \Theta(t - t')}{{2k_B T}} \frac{\partial}{\partial t} S_{J\xi}(t, t')$$

with the correlation functions$^{11}$

$$S_{J\xi}(t, t') = \langle \mathcal{S}(\bar{\xi}(t)) \bar{\xi}(t') \rangle_\phi$$

(B8)

$$S_{J\xi^+}(t, t') = \langle \mathcal{S}(\bar{\xi}(t)) \bar{\xi}^+(t') \rangle_\phi.$$  

We can use (B7) in (B4) and apply a partial integration in $t'$ to obtain

$$\delta \langle \mathcal{S}(\bar{\xi}(t)) \rangle_\phi = \frac{\sqrt{\langle N_0 \rangle}}{ik_B T} \int_{-\infty}^{t} dt' \left( S_{J\xi}(t, t') e^{-i\phi(t')} - S_{J\xi^+}(t, t') e^{i\phi(t')} \right) \frac{d\phi(t')}{dt'} e^{\epsilon t'}$$

$$- \frac{\sqrt{\langle N_0 \rangle}}{k_B T} \left( S_{J\xi}(t, t) e^{-i\phi(t)} + S_{J\xi^+}(t, t) e^{i\phi(t)} \right)$$  

(B9)

which can be rewritten as

$$\delta \langle \mathcal{S}(\bar{\xi}(t)) \rangle_\phi = \frac{2\sqrt{\langle N_0 \rangle}}{k_B T} \int_{-\infty}^{t} dt' S_{J\xi}(t, t') \frac{d\phi(t')}{dt'} - \frac{2\sqrt{\langle N_0 \rangle}}{k_B T} S_{J\xi^+}(t, t)$$  

(B10)

$^{11}$\langle \mathcal{S}(\bar{\xi}(t)) \rangle$ according to eq.(4.9) contains an explicit external time-dependence via $\phi(t)$, in addition to the internal time-dependence of $\hat{\chi}(t), \hat{\chi}^+(t)$ via their Heisenberg equations of motion. This explicit time-dependence has to be taken into account when applying the fluctuation-dissipation theorem. We avoid this additional step by applying the time-derivative in the fluctuation-dissipation relation (B3) directly to the second time argument $t'$, of course with the appropriate extra minus-sign.
with $S_{JR}(t',t)$ defined by eqs. (4.9,4.11) and

$$ S_{JR}(t',t) = \frac{1}{4i} \langle (\hat{\xi}(t)e^{-i\phi(t)} - \hat{\xi}^+(t)e^{i\phi(t)})(\hat{\xi}(t')e^{-i\phi(t')} + \hat{\xi}^+(t')e^{i\phi(t')}) \rangle_\phi. $$

(B11)

The constant term with $S_{JR}(t',t)$ amounts to a small shift of the equilibrium value of $\mu$ in the final result which we shall neglect like other terms contributing to such shifts. Then using eq. (4.2) we put $\hbar d\phi(t')/dt' = -\partial H_0(t')/\partial |\alpha_0|^2$ in eq. (B10) which establishes (B1) and hence eq. (4.14).

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