MATHEMATICAL ANALYSIS OF PULSATILE FLOW, VORTEX BREAKDOWN AND INSTANTANEOUS BLOW-UP FOR THE AXISYMMETRIC EULER EQUATIONS

TSUYOSHI YONEDA

Abstract. The dynamics along the particle trajectories for the 3D axisymmetric Euler equations are considered. It is shown that if the inflow is rapidly increasing (pushy) in time, the corresponding laminar profile of the incompressible Euler flow is not (in some sense) stable provided that the swirling component is not zero. It is also shown that if the vorticity on the axis is not zero (with some extra assumptions), then there is no steady flow. We can rephrase these instability to an instantaneous blow-up. In the proof, Frenet-Serret formulas and orthonormal moving frame are essentially used.

1. Introduction

We study the dynamics along the particle trajectories for the 3D axisymmetric Euler equations. Such Lagrangian dynamics of the 3D axisymmetric Euler flow (inviscid flow) have already been studied in mathematics (see [6, 7, 8]). For example, in [7], Chae considered a blow-up problem for the axisymmetric 3D incompressible Euler equations with swirl. More precisely, he showed that under some assumption of local minima for the pressure on the axis of symmetry with respect to the radial variations along some particle trajectory, the solution blows up in finite time.

Although the blowup problem of the 3D incompressible Euler equations (also the Navier-Stokes equations) is still an outstanding open problem, in this paper, we focus on a different problem in physics, especially, “pulsatile flow” and “vortex breakdown”. In the pulsatile flow study field, Womersley number is the key. The Womersley number comes from oscillating (in time) solutions to the incompressible Navier-Stokes equations in a tube. Let us explain more precisely. We define a pipe \( \Omega_R \) as

\[
\Omega_R := \{ x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} < R, \ 0 < x_3 < \ell \}
\]

with its side-boundary \( \partial \Omega_R = \{ x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} = R, \ 0 < x_3 < \ell \} \), top and bottom boundaries:

\[
\partial \Omega^{\text{top}}_R := \{ x \in \mathbb{R}^3 : 0 \leq \sqrt{x_1^2 + x_2^2} < R, \ x_3 = \ell \}
\]

and \( \partial \Omega^{\text{bottom}}_R := \{ x \in \mathbb{R}^3 : 0 \leq \sqrt{x_1^2 + x_2^2} < R, \ x_3 = 0 \} \). The incompressible Navier-Stokes equations are described as follows:

\[
\begin{align*}
&\partial_t u + (u \cdot \nabla) u - \nu \Delta u = -\nabla p, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega_R, \\
&\quad u = 0 \quad \text{on} \quad \partial \Omega_R
\end{align*}
\]

with \( u = u(x,t) = (u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t)) \) and \( p = p(x,t) \).

To give the Womersley number, we need to focus on the axisymmetric Navier-Stokes flow without swirl (see [34]). If \( p_1 \) and \( p_2 \) are the pressure at the ends of the pipe \( \Omega_R \), namely, \( \partial \Omega^{\text{top}}_R \) and \( \partial \Omega^{\text{bottom}}_R \), the pressure gradient can be expressed as
(p_1 - p_2)/\ell \ (\text{for the study of pressure boundary conditions on } \partial \Omega_{\text{top}}^R \text{ and } \partial \Omega_{\text{bottom}}^R, \text{ see [21] for example}). If the pressure gradient is time-independent, \((p_1 - p_2)/\ell =: p_s,\) then we can find a stationary Navier-Stokes flow (Poiseuille flow):

\begin{equation}
\begin{aligned}
u\ell (R^2 - r^2)) ,
\end{aligned}
\end{equation}

where \(r = \sqrt{x_1^2 + x_2^2}.\) Note that \(u_s\) is also a solution to the linearized Navier-Stokes equations. Next we consider the oscillating pressure gradient case,

\begin{equation}
\begin{aligned}
u\ell = p_o e^{iNt}
\end{aligned}
\end{equation}

which is periodic in the time. Then its corresponding solution \(u_o = u_o(r,t)\) can be written explicitly by using a Bessel function (see [34 (8)] and [32 (1)]) with \(u_1 = u_2 = 0.\) Thus \(u_o\) is also a solution to the linearized Navier-Stokes equations. Note that \(u_o + u_s\) is a time-periodic solution to the Navier-Stokes equations. In this study field, the following Womersley number \(\alpha\) is the key:

\begin{equation}
\begin{aligned}
\alpha = R \sqrt{\frac{N}{\nu}}.
\end{aligned}
\end{equation}

In [32], they also defined the oscillatory Reynolds number and the mean Reynolds number by using \(u_o\) and \(u_s\) respectively, and they investigated how the transition of pulsatile flow \(u_o + u_s\) from the laminar to the turbulent (critical Reynolds number) is affected by the Womersley number and the oscillatory Reynolds number. According to their experiment, measurement at different Womersley numbers yield similar transition behavior, and variation of the oscillatory Reynolds number also appear to have little effect. Thus they conclude that the transition seems to be determined only by the mean Reynolds number. However it seems they did not investigate the effect of the swirl component (azimuthal component), and our aim here is to show that the non-zero swirl component induces an instability of the laminar profile which is, at a glance, nothing to do with wall turbulence.

On the other hand, in the study of vortex breakdown, determining the possible flow topologies of the steady axisymmetric Navier-Stokes flow in a cylindrical container (such as \(\Omega^R\)) with rotating end-covers (on \(\partial \Omega_{\text{top}}^R\) and \(\partial \Omega_{\text{bottom}}^R\)) has been the main subject (see [5, 16, 19, 30] for example, see also [20]). The flow structures and the stability of the flow turns out to be sensitive to changes in the rotation ratio of the two covers. Using a combination of bifurcation theory for two-dimensional dynamical systems and numerical computations, Brons-Voigt and Sorensen [5] systematically determined the possible flow topologies of the steady vortex breakdown in the axisymmetric flow. Their basic idea is to analyze the streamlines of the ordinary differential equations (c.f. the definition of axis-length streamline \((2.3)\) and axis-length trajectory: Definition \((2.8)\) in this paper). For the detail, see Figure 1 and Section 3 in [5]. Our aim here is to show that non-zero swirl component with laminar profile on the axis (with some extra assumptions) creates unsteady flow.

Remark 1.1. These mathematical analysis must be applicable to a study of reduced cardiovascular 1D model [17, Section 10]. If the blood flow is in large and medium sized vessels, the flow is governed by the usual incompressible Navier-Stokes equations. To obtain the reduced model from the Navier-Stokes equations, we need to assume the flow is always unilateral laminar flow, especially, the axis direction of
the flow \( u_3 \) is assumed to satisfy
\[
(1.4) \quad \int_{\Omega_R} u_3(x_1, x_2, x_3, t)^2 \, dx_1 \, dx_2 = a \left( \int_{\Omega_R} u_3(x_1, x_2, x_3, t) \, dx_1 \, dx_2 \right)^2
\]
for some positive constant \( a > 0 \) (see [17, (10.18)]). However, in this setting, it is unclear whether or not such condition (1.4) is always valid. For example, if the flow is not unilateral, containing the reverse flow (possibly, turbulence), then \( a \) may become infinity.

Since we would not like to take the boundary layer into account (instead, we focus on behavior of the interior flow), it is still valid to consider a simpler model: the inviscid flow in \( \Omega_R \). The incompressible Euler equations (inviscid flow) are expressed as follows:
\[
(1.5) \quad D_t u := \partial_t u + (u \cdot \nabla) u = -\nabla p, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega_R, \\
\quad u|_{t=0} = u_0, \quad u \cdot n = 0 \quad \text{on} \quad \partial \Omega_R,
\]
\[
(1.6) \quad u(x, t)|_{x_3=0} = (0, 0, U_0(t)) \quad \text{with} \quad U_0(r, t) > 0,
\]
where \( r = \sqrt{x_1^2 + x_2^2} \) and \( n \) is a unit normal vector on \( \partial \Omega_R \). Note that the boundary condition here is not important anymore.

Notations “\( \approx \)” and “\( \lesssim \)” are convenient. The notation “\( a \approx b \)” means there is a positive constant \( C > 0 \) such that
\[
C^{-1} a \leq b \leq C a,
\]
and “\( a \lesssim b \)” means that there is a positive constant \( C > 0 \) such that
\[
a \leq C b.
\]
In the pulsatile flow case, we consider the following inflow setting:
- \( U_{in} = U_s(r) + U_o(r)g(t) \) with rapidly increasing \( g \) (in time) and
\[
|U_o(r, t)| \approx 1, \quad \sup_{1 \leq j, k \leq 2} (|\partial^j_t \partial^k r U_o(r, t)| + |\partial^j_t \partial^k r U_s(r, t)|) \lesssim 1,
\]
Throughout this paper we always assume existence of smooth solutions in \( \Omega_R \times [0, \infty) \) (we can regard nonuniqueness, nonexistence and blowup as some kind of “strong instability”).

Remark 1.2. According to the boundary layer theory, outside the boundary layer the fluid motion is accurately described by the Euler flow. Thus the above simplification seems (more or less) valid. For the recent progress on the mathematical analysis of the boundary layer, see [25].

2. Geometry setting and the main results
To describe the main theorems, we need to give a geometry setting. First we define the particle trajectory.

Definition 2.1. (Particle trajectory \( \Phi_* \).) For given time-dependent smooth vector field \( u = u(x, t) \), the associated Lagrangian flow \( \Phi_*(t) \) is a solution of the following initial value problem
\[
(2.1) \quad \frac{d}{dt} \Phi_* (x, t) = u(\Phi_* (x, t), t), \\
(2.2) \quad \Phi_* (x, 0) = x \in \Omega_R.
\]
Throughout this paper we always assume the vector field $u$ is unilateral, that is, $u_3 > 0$. Also define the axis-length streamline $\tilde{\Phi}(z)$.

**Definition 2.2.** (Axis-length streamline $\Phi$.) Let $Z_3 > 0$. Also define the axis-length streamline $\tilde{\Phi}(z)$.

**Definition 2.3.** (Axis-length trajectory $\Phi$.) Let $Z_3(t) := \Phi_4(t) \cdot e_z$ (with $e_z = (0, 0, 1)$) and since the flow is unilateral, we can define its inverse $Z_3^{-1}(z) = t$. In this case we see $\partial_t Z_3^{-1} = 1/\partial_t Z_3 = 1/u_3$. Let $\Phi$ be such that $\Phi(z) = \Phi_4(x, Z_3^{-1}(z))$.

We restrict our vector field to the axi-symmetric one. Let $e_r := x_h/|x_h|$, $e_\theta := x^*_h/|x^*_h|$ and $e_z = (0, 0, 1)$ with $x_h = (x_1, x_2, 0)$, $x^*_h = (-x_2, x_1, 0)$. The vector valued function $u$ can be rewritten as $u = v_r e_r + v_\theta e_\theta + v_z e_z$, where $v_r = v_r(r, z, t)$, $v_\theta = v_\theta(r, z, t)$, $v_z = v_z(r, z, t)$, $v_{r, 0} = v_r(r, z, 0)$, $v_{\theta, 0} = v_\theta(r, z, 0)$ and $v_{z, 0} = v_z(r, z, 0)$ with $r = |x_h|$ and $z = x_3$.

We define a Lagrangian flow on the meridian plane ($r$-$z$ plane).

**Definition 2.4.** (Lagrangian flow on the meridian plane.) Let $Z_*$ and $R_*$ be such that

(2.4) \[ \frac{d}{dt} Z_* (t) = v_z (R_*(t), Z_*(t), t), \]

and

(2.5) \[ \frac{d}{dt} R_* (t) = v_r (R_*(t), Z_*(t), t), \]

with $Z_*(t) = Z_*(r_0, z_0, t)$ and $R_*(t) = R_*(r_0, z_0, t)$.

Note $Z_*$ is already defined in Definition 2.8.

**Remark 2.5.** We can rephrase $Z_*$ and $R_*$ by using the stream function (see (2.2) in [5] for example).

**Remark 2.6.** The axisymmetric Euler equations can be expressed as follows:

(2.6) \[ \partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r - \frac{v^2_r}{r} + \partial_r p = 0, \]

(2.7) \[ \partial_t v_\theta + v_r \partial_r v_\theta + v_z \partial_z v_\theta + \frac{v_r v_\theta}{r} = 0, \]

(2.8) \[ \partial_t v_z + v_r \partial_r v_z + v_z \partial_z v_z + \partial_z p = 0, \]

(2.9) \[ \frac{\partial_t (r v_r)}{r} + \partial_z v_z = 0. \]

In this paper we use (2.7) which is independent of the pressure term.
Assume $D$ with $\tilde{\Phi}$ and let explicitly expressed as

$$\tilde{\Phi}(\tilde{r}_0, z, t) = \Phi(z) := (\tilde{R}(z) \cos \tilde{\Theta}(z), \tilde{R}(z) \sin \tilde{\Theta}(z), z)$$

with $\tilde{R}(z) = R(\tilde{r}_0, z, t)$, $\tilde{R}(\tilde{r}_0, 0, t) = \tilde{r}_0$, $\tilde{\Theta}(z) = \Theta(z, t)$. We easily see

$$\partial_\gamma \tilde{\Phi} \cdot e_z = 1, \quad \partial_\gamma \tilde{\Phi} \cdot e_r = \partial_\gamma \tilde{R} = \frac{v_r}{v_z} \quad \text{and} \quad \partial_\gamma \tilde{\Phi} \cdot e_\theta = \tilde{R} \partial_\gamma \tilde{\Theta} = \frac{v_\theta}{v_z}.$$ 

Since $\partial_\gamma \tilde{R} > 0$ by the smoothness, we have its inverse $\tilde{r}_0 = \tilde{R}^{-1}(r, z, t)$.

**Remark 2.8.** (Axisymmetric axis-length trajectory.) Also $\Phi$ can be explicitly expressed as

$$\Phi(z) = (R(z) \cos \Theta(z), R(z) \sin \Theta(z), z)$$

with $R(z) = R(r_0, \theta_0, z_0, z)$, $\Theta(z) = \Theta(r_0, \theta_0, z_0, z)$, $R(r_0, \theta_0, z_0, z_0) = r_0$ and $\Theta(r_0, \theta_0, z_0, z_0) = \theta_0$. Note that $\tilde{R}(t)|_{t=Z^{-1}_r(z)} = R(z)$.

In order to show that the non-zero swirl component induces the instability, we need to measure appropriately the rate of disturbing laminar profile of the flow. We now give the key definition.

**Definition 2.9.** (Rate of disturbing laminar profile.) We define “rate of disturbing laminar profile” $L^0$, $L^2$ and $L^4$ as follows: for $(\tilde{r}_0, z) \in [0, R] \times (0, \ell)$,

$$L^0(\tilde{r}_0, z, t) = |\partial_{\tilde{r}_0} \tilde{R}(\tilde{r}_0, z, t)| + |(\partial_{\tilde{r}_0} \tilde{R}^{-1})(\tilde{R}(\tilde{r}_0, z, t), z, t)|$$

$$L^2(\tilde{r}_0, z, t) := \sum_{1 \leq i+j+k \leq 3, (j,k) \neq (0,1)} |\partial^2_{\tilde{r}_0} \tilde{R}(\tilde{r}_0, z, t)| + \sum_{1 \leq i+j+k \leq 3} |(\partial^2_{\tilde{r}_0} \tilde{R})^{-1})(\tilde{R}(\tilde{r}_0, z, t), z, t)|$$

$$L^4(\tilde{r}_0, z, t) = \sum_{2 \leq i+j+k \leq 3} |\partial^4_{\tilde{r}_0} \tilde{R}(\tilde{r}_0, z, t)| + \sum_{1 \leq i+j+k \leq 2} |(\partial^4_{\tilde{r}_0} \tilde{R})^{-1})(\tilde{R}(\tilde{r}_0, z, t), z, t)|$$

Note that $L^0$ and $L^2$ do not include any time derivative, while, $L^4$ includes it. We can see that if $L^4$ is not zero, then the flow cannot be any steady flow.

**Remark 2.10.** Minimum value of $L^0$ is 2, since $|\partial_{\tilde{r}_0} \tilde{R}^{-1}| = 1/|\partial_{\tilde{r}_0} \tilde{R}|$.

**Remark 2.11.** The typical Euler flow $u(x, t) = (0, 0, g(t))$, namely, a bunch of stationary straight tubes $\bar{R}(\tilde{r}_0, z, t) \equiv \tilde{r}_0$ is the typical laminar flow. In this case

$$L^0 = 2, \quad L^2 = 0 \quad \text{and} \quad L^4 = 0$$

for any $g$.

**Remark 2.12.** Streamlines of outside bubbles which are attaching on the axis (see $B_1, B_2, B_3, C_1, C_2, D, E, F$ in Figure 1 in [5]) may create large $L^2$ and/or $L^0$. Moreover, at a hyperbolic saddle (or stagnation point), they may be infinity.

Now we give the main theorems.

**Theorem 2.13.** (Pulsatile flow case.) Let $\tilde{r}_0(t)$ and $z(t)$ be another expression of particle trajectory such that

$$\tilde{\Phi}(\tilde{r}_0(t), z(t), t) = \Phi_e(x, t)$$

and let $D_\gamma$ be a non-zero swirl region such that $D_\gamma := \{x : |u_0(x) \cdot e_\theta| > \gamma\}$. Assume $D_\gamma \neq \emptyset$ for the corresponding initial data, and assume there is a unique
Remark. (Inflow propagation.) Let \( \rho \) be such that

\[
\rho(\tilde{r}_0, z, t) := \lim_{\epsilon \to 0} \frac{|A(\tilde{r}_0, 0, \epsilon, t)|}{|A(\tilde{r}_0, z, \epsilon, t)|}.
\]

We see that

\[
\rho(\tilde{r}_0, z, t) = \frac{\partial_{\tilde{r}_0} \tilde{R}(\tilde{r}_0, 0, t) \tilde{R}(\tilde{r}_0, 0, t)}{\partial_{\tilde{r}_0} \tilde{R}(\tilde{r}_0, z, t) \tilde{R}(\tilde{r}_0, z, t)} = \frac{\tilde{r}_0}{\partial_{\tilde{r}_0} \tilde{R}(\tilde{r}_0, z, t) \tilde{R}(\tilde{r}_0, z, t)} = \frac{2\tilde{r}_0}{\partial_{\tilde{r}_0} \tilde{R}(\tilde{r}_0, z, t)^2}.
\]

Theorem 2.14. (Vortex breakdown case.) Assume there is a unique smooth solution to the Euler equations \(\text{in } t \in [0, T)\). Then there is a smooth function \(g\) and discrete-time \(\{t_j\}_1\) such that

\[
|g| \approx 1, \quad g'(t_j) \to \infty, \quad g''(t_j) \to \infty \quad \text{and} \quad f(t_j) \to \infty \quad (t_j \to T \text{ as } j \to \infty),
\]

and the following case must occur:

\[
L^*(r_0(t_j), z(t_j), t_j) \geq f(t_j) \quad \text{for } x \in D \quad \text{and} \quad j = 1, 2, \ldots.
\]

Definition 3.1. (Inflow propagation.) Let \(\Omega_1\) in \([3\ (2.1)]\).
Remark 3.2. Since $\tilde{R}(0,z,t) \equiv 0$, we see that
\[
\lim_{\tilde{R}_0 \to 0} \rho = \frac{1}{(\partial_{\tilde{R}_0} R)^2} \quad \text{and} \quad \lim_{\tilde{R}_0 \to 0} \frac{\partial}{\partial_{\tilde{R}_0}} \rho = -\frac{2\partial_{\tilde{R}_0}^2 \tilde{R}}{(\partial_{\tilde{R}_0} R)^3}
\]
on the axis.

Since
\[
2\pi \int_{\tilde{R}(\tilde{R}_0, z, t)} \tilde{\rho} \cdot \tilde{u}_z(r', z, t) r' dr' = 2\pi \int_{\tilde{R}_0} \rho_z(r', 0, t) r' dr'
\]
by divergence-free and Gauss’ divergence theorem, we can figure out $v_z$ by using the inflow propagation $\rho$.

\[
v_z(r, z, t) = \lim_{\epsilon \to 0} \frac{2\pi}{|A(\tilde{R}_0, z, \epsilon, t)|} \int_{\tilde{R}(\tilde{R}_0, z, t)} v_z(r', z, t) r' dr' = \lim_{\epsilon \to 0} \frac{2\pi}{|A(\tilde{R}_0, 0, \epsilon, t)|} \int_{\tilde{R}_0} v_z(r', 0, t) r' dr' = \rho(\tilde{R}_0, 0, t) u_z(\tilde{R}_0, 0, t).
\]

Thus we have the following proposition.

Proposition 3.3. We have the following formulas of $v_z$ and $v_r$:
\[
v_z(r, z, t) = \rho(\tilde{R}^{-1}(r, z, t), z, t) U_{in}(\tilde{R}^{-1}(r, z, t), t)
\]
and
\[
(3.1) \quad v_r(r, z, t) = (\partial_z \tilde{R})(\tilde{R}^{-1}(r, z, t), z, t) v_z(r, z, t).
\]

Remark 3.4. Recall that $e_\theta = (-\sin \Theta(z), \cos \Theta(z), 0)$ and $e_r = (\cos \Theta(z), \sin \Theta(z), 0)$.

We also have the following explicit formulas of $\Theta'$ and $\Theta''$ ($\Theta$ and $R$ already appeared in the axis-length trajectory). See Remark 2.5:

\[
\partial_z \Phi \cdot e_\theta = \frac{\partial \Phi}{v_z} \frac{e_\theta}{v_z} = R(z) \Theta'(z) = \frac{v_\theta(R(z), z, Z^{-1}_{st}(z))}{v_z(R(z), z, Z^{-1}_{st}(z))}
\]

\[
\partial_z \Phi \cdot e_r = \frac{\partial \Phi}{v_z} \frac{e_r}{v_z} = \frac{v_r(R(z), z, Z^{-1}_{st}(z))}{v_z(R(z), z, Z^{-1}_{st}(z))}
\]

\[
= \left( \partial_z \tilde{R} \right) \left( \tilde{R}^{-1}(R(z), z, Z^{-1}_{st}(z)), z, Z^{-1}_{st}(z) \right) = R'(z).
\]

Moreover, along the axis,

\[
\lim_{\tilde{R}_0 \to 0} \Theta'(z) = \frac{(\partial_z v_\theta)(0, z, Z^{-1}_{st}(z))}{v_z(0, z, Z^{-1}_{st}(z))}
\]

Remark 3.5. For the vortex breakdown case, we have the following estimates on $\Theta'$, $\Theta''$, $\Theta'''$, $R'$, $R''$ and $R'''$:

\[
|\Theta'| \approx 1/\epsilon_1, \quad |\Theta''| \lesssim 1/\delta \quad \text{and} \quad |\Theta'''| \lesssim 1/\delta.
\]
Let \( r = R(r_0, z_0, t) \). Moreover we have that

\[
R' = \frac{v_r}{v_z} = C(\delta) r + O(r^2),
\]

\[
R'' = \frac{\partial_z v_r}{v_z} R' + \frac{\partial_t v_r}{v_z} + \frac{\partial_z v_r}{v_z} R' - \frac{v_r}{v_z} \partial_z v_r R' - \frac{v_r}{v_z} \partial_t v_z - \frac{v_r}{v_z} \partial_t Z
\]

\[
= C(\delta) r + O(r^2),
\]

\[
R''' = C(\delta) r + O(r^2),
\]

where \( C(\delta) \) is a positive constant depending only on \( \delta \) (if \( \delta \to 0 \), then \( C(\delta) \to 0 \)).

Next we construct \( v_\theta \). By (2.7) we see that

\[
\partial_t v_\theta(R_\ast(t), Z_\ast(t), t) = -\frac{v_r(R_\ast(t), Z_\ast(t), t)v_\theta(R_\ast(t), Z_\ast(t), t)}{R_\ast(t)}.
\]

Applying the Gronwall equality, we see

\[
v_\theta(R_\ast(t), Z_\ast(t), t) = v_\theta(r_0, z_0, 0) \exp \left\{ -\int_0^t \frac{v_r(R_\ast(t'), Z_\ast(t'), t')}{R_\ast(t')} dt' \right\}
\]

and then

\[
v_\theta(R(z), z, Z_\ast^{-1}(z)) = v_\theta(r_0, z_0, 0) \exp \left\{ -\int_0^{Z_\ast^{-1}(z)} \frac{v_r(R_\ast(t'), Z_\ast(t'), t')}{R_\ast(t')} dt' \right\}
\]

and

\[
v_\theta(r, z, t) = v_\theta(r_0, z_0, 0) \exp \left\{ -\int_0^t \frac{v_r(R_\ast(r_0, z_0, t'), Z_\ast(r_0, z_0, t'), t')}{R_\ast(r_0, z_0, t')} dt' \right\}
\]

(3.2)

with \( r_0 = R_\ast^{-1}(r, z, t) \) and \( z_0 = Z_\ast^{-1}(r, z, t) \) (distinguish with \( Z_\ast^{-1} \)). In order to estimate spatial derivatives on \( v_\theta \), first we consider a non-incompressible 2D-flow composed by \( R_\ast \) and \( Z_\ast \). Let us denote \( \phi_{2D} = \phi_{2D}(t) = (R_\ast(t), Z_\ast(t)) \), \( \phi_{2D}^{-1} = (R_\ast^{-1}, Z_\ast^{-1}) \) and \( D\phi_{2D} \) be its Lagrangian deformation:

\[
D\phi_{2D} = \begin{pmatrix}
\partial_{r_0} R_\ast & \partial_{z_0} R_\ast \\
\partial_{r_0} Z_\ast & \partial_{z_0} Z_\ast
\end{pmatrix}.
\]

We see \( \det(D\phi_{2D}) = \partial_{r_0} R_\ast \partial_{z_0} Z_\ast - \partial_{z_0} R_\ast \partial_{r_0} Z_\ast \) and thus we have

\[
D(\phi_{2D}^{-1}) = (D\phi_{2D})^{-1} = \frac{1}{\det D\phi_{2D}} \begin{pmatrix}
\partial_{z_0} Z_\ast & -\partial_{z_0} R_\ast \\
-\partial_{r_0} Z_\ast & \partial_{r_0} R_\ast
\end{pmatrix}.
\]

A direct calculation with (2.4), (2.1) and (2.5) yields

\[
\frac{d}{dt}(\det D\phi_{2D}) = (\partial_r v_r + \partial_z v_z)(\det D\phi_{2D}) = -\frac{v_r}{R_\ast(t)}(\det D\phi_{2D}).
\]

Thus

\[
\det D\phi_{2D}(t) = \det D\phi_{2D}(0) \exp \left\{ -\int_0^t \frac{v_r(R_\ast(\tau), Z_\ast(\tau), \tau)}{R_\ast(\tau)} d\tau \right\}.
\]
Combining the Lagrangian deformation on $R$ along the trajectory. In fact, since $|v_r/r| \approx |\partial_r v_r| \lesssim 1$ near the axis, we have

$$\det D\phi_{2D} \approx 1 \quad \text{near the initial time.}$$

Since we have already controlled $\det D\phi_{2D}$, it suffices to estimate $\partial_{r_0} R_*, \partial_{r_0} Z_*$, $\partial_{z_0} R_*$ and $\partial_{z_0} Z_*$ respectively. From Proposition 3.3 We see the following:

$$\partial_t \partial_{z_0} Z_*(t) = \left[ \partial_{z_0} R_* \partial_{r_0} \rho \partial_t \tilde{R}^{-1} + \partial_{z_0} Z_* \partial_{r_0} \rho \partial_t \tilde{R}^{-1} + \partial_{z_0} Z_* \partial_z \rho \right] U_{in}$$

$$\partial_t \partial_{z_0} R_*(t) = \left[ \partial_{z_0} \partial_t \tilde{R} \partial_{r_0} \tilde{R}^{-1} \partial_{z_0} R_* + \partial_{r_0} \partial_t \tilde{R} \partial_{r_0} \tilde{R}^{-1} \partial_{z_0} Z_* + \partial_t^2 \tilde{R} \partial_{z_0} Z_* \right] U_{in} + (v_z \text{ part}).$$

Then we can construct a Gronwall’s inequality of $|\partial_{z_0} Z| + |\partial_{z_0} R|$, that is

$$|\partial_{z_0} Z| + |\partial_{z_0} R| \lesssim e^{Ct},$$

where $C$ is depending on $L^0$, $L^x$ and $L^t$. Again, we just take integration in time, we have

$$\partial_{z_0} Z_* (t) = 1 + \int_0^t \partial_{z_0} v_z dt',$$

and this is the explicit formula of $\partial_{z_0} Z_*$. In a small time interval, we have $\partial_{z_0} Z_* \approx 1$ and by the same calculation, $\partial_{z_0} R_* \approx 0$, $\partial_{r_0} Z_* \approx 0$ and $\partial_{r_0} R_* \approx 1$. By the above estimates, we can estimate derivatives on $v_\theta$.

Now we figure out the explicit formula of $\partial_t |u(\Phi_*, t)|$. Recall that the particle trajectory $\Phi_*(x, t)$ satisfies

$$\Phi_*(x, t) = (R_*(t) \cos \Theta_*(t), R_*(t) \sin \Theta_*(t), Z_*(t)).$$

Then, by $u = v_r e_r + v_\theta e_\theta + v_z e_z$ with $e_\theta = (-\sin \Theta_*(t), \cos \Theta_*(t), 0)$ and $e_r = (\cos \Theta_*(t), \sin \Theta_*(t), 0)$, we see that

$$(3.3) \quad \frac{1}{2} \partial_t |u(\Phi_*(x, t), t)|^2 = \partial_t v_r v_r + \partial_t v_\theta v_\theta + \partial_t v_z v_z$$

along the trajectory. In fact, since

$$\partial_t \Phi_* = (\partial_t R_* \cos \Theta_*, \partial_t R_* \sin \Theta_*, \partial_t Z_*) + \partial_t \Theta_*(-R_* \sin \Theta_*, R_* \cos \Theta_*, 0),$$

and

$$v_\theta = \partial_t \Phi_* \cdot e_\theta = (\partial_t \Theta_*) R_*,$$

we see $\partial_t \Theta_* = v_\theta / R_*$. We multiply $u = v_r e_r + v_\theta e_\theta + v_z e_z$ to

$$\partial_t u = \partial_t v_r e_r + \partial_t v_\theta e_\theta + \partial_t v_z e_z + v_r \partial_t \Theta_* e_\theta - v_\theta \partial_t \Theta_* e_r,$$

then we have $3.3$. Thus we have the following explicit formula:

$$D_t |u| = 2D_t |u|^2 + \frac{\partial_t v_r v_r + \partial_t v_\theta v_\theta + \partial_t v_z v_z}{|u|}.$$

Combining the Lagrangian deformation on $R_*$ and $Z_*$, we also have the explicit formulas of $\partial_t \partial_t |u(\Phi(x, t), t)|$ and $\partial_t \partial_t |u(\Phi(x, t), t)|$. 

4. Estimates on curvature and torsion along particle trajectory.

Let us define the arc-length trajectory \( \phi(s) := \Phi(z(s)) \) with smooth function \( z(s) \) such that \( z'(s) = |(\partial_z \Phi)(z(s))|^{-1} \). We also define the unit tangent vector \( \tau \) as

\[
\tau(s) = \partial_s \phi(s),
\]

the unit curvature vector \( n \) as \( \kappa n = \partial_s \tau \) with a curvature function \( \kappa(s) > 0 \), the unit torsion vector \( b \) as \( b(s) := \pm \tau(s) \times n(s) \) (\( \times \) is an exterior product) with a torsion function to be positive \( T(s) > 0 \) (once we restrict \( T \) to be positive, then the direction of \( b \) can be uniquely determined). From \( \kappa n \), we can figure out the curvature constant \( \kappa := |\partial_z^2 \phi| \) and corresponding unit normal vector: \( n = \partial_z^2 \phi/|\partial_z^2 \phi| \). Thus, theoretically, we can explicitly figure out \( \kappa \) and \( \partial_s \kappa \) by using \( R \) and \( \Theta \). First, \( \tau \) and \( \kappa n \) are expressed as

\[
\tau = (\partial_z \Phi)z', \quad \kappa n = \partial_z^2 \phi = \partial_z^2 \Phi(z')^2 + \partial_z \Phi z''.
\]

Then direct calculations yield

\[
\partial_z \Phi(x, z) = (-R\Theta' \sin \Theta, R\Theta' \cos \Theta, 1) + (R' \cos \Theta, R' \sin \Theta, 0),
\]

\[
\partial_z^2 \Phi(x, z) = -R(\Theta')^2(\cos \Theta, \sin \Theta, 0) + (-R\Theta'' \sin \Theta, R\Theta'' \cos \Theta, 0) + R''(\cos \Theta, \sin \Theta, 0) + 2R' \Theta'(- \sin \Theta, \cos \Theta, 0),
\]

\[
z'(s) = |\partial_z \Phi|^{-1} = (1 + (R')^2 + (R\Theta')^2)^{-1/2},
\]

\[
z''(s) = -(1 + (R')^2 + (R\Theta')^2)^{-2}(R' R'' + R\Theta'(R' \Theta' + R\Theta'')).
\]

Therefore

\[
\kappa^2 = |\kappa n|^2 = |\partial_z^2 \phi|^2(z')^4 + 2(\partial_z \phi \cdot \partial_z^2 \phi)(z')^2 z'' + |\partial_z \phi|^2(z'')^2
\]

\[
= \left[ R^2(\Theta')^4 - 2R(\Theta')^2 R'' + (R'')^2 + (R\Theta'')^2 + 4R' \Theta' \Theta'' + 4(R' \Theta')^2 \right]
\]

\[
\times (1 + (R')^2 + (R\Theta')^2)^{-2}
\]

\[
+ 2 \left[ -R' R(\Theta')^2 + R' R'' + R^2 \Theta' \Theta' - 2R(\Theta')^2 R' \right]
\]

\[
\times (1 + (R')^2 + (R\Theta')^2)^{-1}(1 + (R')^2 + (R\Theta')^2)^{-2}(R' R'' + R\Theta'(R' \Theta' + R\Theta''))
\]

\[
+ \left[ (R\Theta')^2 + (R')^2 \right] (1 + (R')^2 + (R\Theta')^2)^{-4}(R' R'' + R\Theta'(R' \Theta' + R\Theta'')).
\]

From the above explicit formulas of \( \kappa \), we can figure out the explicit formula of \( \partial_s \kappa \) (omit its detail) which will be important in the proof of the main theorems.

Remark 4.1.  
* (The vortex breakdown case.) If \( \Theta' \) is larger than the other terms, we have

\[
\partial_s \kappa \approx R\Theta' \Theta''
\]

which is a controllable term.

* (Instantaneous blowup case in Appendix.) If \( \Theta'' \) is larger than \( \Theta' \), and \( \Theta'' \) is larger than \( \Theta'' \), then we have

\[
\partial_s \kappa \approx R\Theta''
\]

which will be the dominant term.
5. Rewrite Euler equations by using curvature and torsion

In this section we rewrite the Euler equations by using curvature and torsion. The basic idea comes from Chan-Czubak-Y [9, Section 2.5], more originally, see Ma-Wang [21 (3.7)]. They considered 2D separation phenomena using elementary differential geometry. The key idea here is “local pressure estimate” on a normal coordinate in \( \bar{\theta}, \bar{r} \) and \( \bar{z} \) values. Two derivatives to the scalar function \( p \) on the normal coordinate is commutative, namely, \( \partial_r \partial_{\bar{\theta}} p(\bar{\theta}, \bar{r}, \bar{z}) - \partial_{\bar{\theta}} \partial_r p(\bar{\theta}, \bar{r}, \bar{z}) = 0 \). This fundamental observation is the key to extract the local property of the pressure.

**Remark 5.1.** It should be noticed that Enciso and Peralta-Salas [15] considered the existence of Beltrami fields \( u \) with a nonconstant proportionality factor \( f \):

\[
\nabla \times u = fu, \quad \nabla \cdot u = 0 \quad \text{in} \quad \mathbb{R}^3.
\]

It is well known that a Beltrami field is also a solution of the steady Euler equation in \( \mathbb{R}^3 \). They showed that for a generic function \( f \), the only vector field \( u \) satisfying (5.1) is the trivial one \( u \equiv 0 \). See (2.12), (3.4) and (3.6) in [15] for the specific condition on \( f \). Note that \( g_{ij} \) (induced metric of the level set of \( f \)) is the fundamental component of the condition. It would be also interesting to consider whether we can apply their method to our unsteady flow problem, and compare with our method.

For any point \( x \in \mathbb{R}^3 \) near the arc-length trajectory \( \phi \) is uniquely expressed as \( x = \phi(\bar{\theta}) + \bar{r}n(\bar{\theta}) + \bar{z}b(\bar{\theta}) \) with \( (\bar{\theta}, \bar{r}, \bar{z}) \in \mathbb{R}^3 \) (the meaning of the parameters \( s \) and \( \bar{\theta} \) are the same along the arc-length trajectory). By the Frenet-Serret formulas, we have that

\[
\begin{align*}
\partial_{\bar{\theta}} x &= \tau + \bar{r}(Tb - \kappa n) + \bar{z}kn, \\
\partial_{\bar{r}} x &= n, \\
\partial_{\bar{z}} x &= b.
\end{align*}
\]

This means that

\[
\begin{pmatrix}
\partial_{\bar{\theta}} \\
\partial_{\bar{r}} \\
\partial_{\bar{z}}
\end{pmatrix} = \begin{pmatrix}
1 - \kappa \bar{r} & \bar{z} \kappa & \bar{r} T \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\tau \\
n \\
b
\end{pmatrix}.
\]

**Remark 5.2.** For any smooth scalar function \( f \), we have

\[
\partial_{\bar{\theta}} f(x) = \nabla f \cdot \partial_{\bar{\theta}} x.
\]

\( \nabla f \) itself is essentially independent of any coordinates, thus we can regard a partial derivative as the corresponding vector.

Then we have the following inverse matrix:

\[
\begin{pmatrix}
\tau \\
n \\
b
\end{pmatrix} = \begin{pmatrix}
(1 - \kappa \bar{r})^{-1} & -\bar{z}T(1 - \kappa \bar{r})^{-1} & -\bar{r}T(1 - \kappa \bar{r})^{-1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\partial_{\bar{\theta}} \\
\partial_{\bar{r}} \\
\partial_{\bar{z}}
\end{pmatrix}.
\]

Therefore we have the following orthonormal moving frame: \( \partial_{\bar{r}} = n, \partial_{\bar{z}} = b \) and \((1 - \kappa \bar{r})^{-1} \partial_{\bar{\theta}} - \bar{z}T(1 - \kappa \bar{r})^{-1} \partial_{\bar{r}} - \bar{r}T(1 - \kappa \bar{r})^{-1} \partial_{\bar{z}} = \partial_{\tau} \).

**Lemma 5.3.** We see \( -\nabla p \cdot \tau = D_t |u| := \partial_{\bar{r}} |u(\Phi_+(x, t), t)| \) along the trajectory.
Proof. Let us define a unit tangent vector $\hat{\tau}$ (in time $t$) as follows:

$$\hat{\tau}(t) := \frac{u(\Phi_s(x,t),t)}{|u(\Phi_s(x,t),t)|}.$$ 

Note that there is a re-parametrize factor $s(t)$ such that

$$\tau(s(t)) = \hat{\tau}(t).$$ 

Since $u \cdot \partial_s \tau = 0$, we see that

$$\partial_t |u(\Phi_s(x,t),t)| = \partial_t (u(\Phi_s(x,t),t) \cdot \hat{\tau}(t))$$

$$= \partial_t (u(\Phi_s(x,t),t)) \cdot \hat{\tau}(t) + u(\Phi_s(x,t),t) \cdot \partial_s \tau \partial_t s$$

$$= \partial_t (u(\Phi_s(x,t),t)) \cdot \hat{\tau}(t).$$

By the above calculation we have

$$-\nabla p \cdot \tau = \partial_t (u(\Phi_s(x,t),t)) \cdot \tau = \partial_t (u(\Phi_s(x,t),t)) \cdot \hat{\tau} = D_t |u|.$$ 

\[\blacksquare\]

We now rewrite the Euler equations by using curvature and torsion.

**Lemma 5.4.** Along the arc-length trajectory, we have

$$3\kappa D_t |u| + \partial_s \kappa |u|^2 = \partial_t D_t |u|$$

and

$$T\kappa |u|^2 = \partial_s D_t |u|.$$ 

Proof. Let us re-define $\phi(s) = \Phi_s(x,t(s))$ with smooth function $t(s)$ satisfying $\partial_s t = |u|^{-1}$. We see that

$$\partial_s \phi \cdot \tau = 1.$$ 

By the unit normal vector with the curvature constant, we see

$$\kappa n = \partial_t^2 \phi = \partial_s (\partial_t \Phi_s \partial_s t) = \partial_t^2 \Phi_s (\partial_s t)^2 + \partial_t \Phi_s \partial_t^2 t.$$ 

Recall the Euler equation: $\partial_t^2 \Phi_s = -\nabla p$. Then we have

$$-(\nabla p \cdot n) = (\partial_t^2 \Phi_s \cdot n) = \kappa |u|^2,$$

$$-\partial_s (\nabla p \cdot n) = \partial_s (\kappa (\partial_s t)^{-2}) = \partial_s \kappa (\partial_s t)^{-2} - 2\kappa (\partial_s t)^{-3}(\partial_t^2 t),$$

$$-\nabla p \cdot \tau = -|u|^3 \partial_t^2 t,$$

$$-\nabla p \cdot b = 0.$$ 

Note that $\partial_t^2 t$ is unknown, so we now figure out it by Lemma 5.3 and the above third equality:

$$\partial_t^2 t = -|u|^{-3} \partial_s |u|.$$ 

Along the arc-length trajectory, we have (recall $\partial_\theta = \partial_s$)

$$-\partial_\theta (\nabla p \cdot \tau) = -\partial_\theta \partial_s p$$

$$= -\kappa \partial_\theta p - \partial_\theta \partial_\phi p - T \partial_\theta p$$

(commute $\partial_\tau$ and $\partial_\theta$)

$$= -\kappa (\nabla p \cdot \tau) - \partial_\theta (\nabla p \cdot n) - T (\nabla p \cdot b)$$

$$= -\kappa |u|^3 \partial_t^2 t + \partial_\theta \kappa (\partial_s t)^{-2} - 2\kappa (\partial_s t)^{-3}(\partial_t^2 t)$$

$$= 3\kappa \partial_t |u| + \partial_s \kappa |u|^2.$$
Since $\nabla p \cdot b = \partial_z p \equiv 0$ along the trajectory, then

\[-\partial_z(\nabla \cdot \tau)|_{\bar{r}, \bar{z} = 0} = -\partial_z \partial_z p|_{\bar{r}, \bar{z} = 0} = -\partial_z \partial_g p - T \partial \tau p = -T(\nabla \cdot \tau) = T\kappa |u|^2.

By Lemma 5.3 along the arc-length trajectory $\phi$, we have

\[3\kappa \partial_t |u| + \partial_x \kappa |u|^2 = -\partial_x (\nabla \cdot \tau)|_{\bar{r}, \bar{z} = 0} = \partial_x D_x |u|
\]

and

\[T\kappa |u|^2 = -\partial_z (\nabla \cdot \tau)|_{\bar{r}, \bar{z} = 0} = \partial_z D_z |u|.
\]

6. PROOF OF THE MAIN THEOREM (THE PULSATILE FLOW CASE).

To prove the main theorem, it is enough to show the following lemma:

**Lemma 6.1.** Let $t_j > 0$ ($j = 1, 2, \ldots$) be fixed. For any $x \in \Phi(D_\gamma, t_j)$, there is $\beta > 0$ such that $\beta \leq u(x, t_j) \cdot e_\gamma \leq \beta^{-1}$, $x \cdot e_r > 2\beta$ and $|L^0(\tilde{r}_0(t_j), z(t_j), t_j)| + |L^1(\tilde{r}_0(t_j), z(t_j), t_j)| \leq 1/(2\beta)$. For any $\epsilon > 0$, then there is $\delta > 0$ such that for any small time interval $I$ with initial time $t_j$, at least one of the following four cases must happen:

- $L^0(\tilde{r}_0(t), z(t), t), L^0(\tilde{r}_0(t), z(t), t) > 1/\beta$,
- $L^1(\tilde{r}_0(t), z(t), t) \gtrsim 1/\epsilon$,
- $|\Phi_\ast(x, t) \cdot e_r| < \beta$,
- $\tilde{r}_0(t) < \beta$,

for some $t \in I$, with any inflow $g(t)$ satisfying

\[g(t) \approx 1, \quad |g'(t)| < 1/\epsilon \quad \text{and} \quad 1/\delta \approx |g''(t)| \quad \text{in} \quad t \in I,
\]

where $\tilde{r}_0(t)$ and $z(t)$ are determined by $\tilde{R}(\tilde{r}_0(t), z(t), t) = \Phi_\ast(x, t)$ (in this case $\Phi_\ast(x, t_j) = x$). Since $\Phi(D_\gamma, t)$ is always compact and the solution is always smooth, $\delta$ can be independent of the choice of $x \in \Phi(D_\gamma, t)$.

Since the time interval $I$ is arbitrary, we see that $L^0$ or $L^\ast$ or $\Phi_\ast \cdot e_r$ or $\tilde{r}_0$ is not continuous at $t_j$, or $L^1 \gtrsim 1/\epsilon$ for some $t \in I$. The discontinuity contradicts the smoothness property, thus

\[L^1 \gtrsim 1/\epsilon\]

only occurs.

**Proof.** In what follows, we prove the above lemma. For any small time interval $I$, assume that the axisymmetric smooth Euler flow satisfies the following conditions:

- $L^0(\tilde{r}_0(t), z(t), t), L^0(\tilde{r}_0(t), z(t), t) \leq 1/\beta$ and $L^1(\tilde{r}_0(t), z(t), t) \lesssim 1/\epsilon$
- $|\Phi_\ast(x, t) \cdot e_r| \geq \beta$ and $\tilde{r}_0(t) \geq \beta$

for any $t \in I$, where $(\tilde{r}_0(t), z(t)) = (\tilde{\Phi}^{-1} \circ \Phi_\ast)(x, t)$, and we employ a contradiction argument. By the second assumption: $|\Phi_\ast(x, t) \cdot e_r| \geq \beta$, $R$ satisfies the following:

\[R(Z_\ast(t)) = R_\ast(t) \geq \beta \quad \text{for} \quad t \in I.
\]

By the explicit formulas in Section 3, we have the following lemma (these are direct calculations, thus we omit its proof).
Lemma 6.2. For $t = Z_{s_{\ell}}^{-1} \in I$, we have the following estimates along the axis-length trajectory:

\[
\begin{align*}
|\partial_z \nu_z(R(z), z, Z_{s_{\ell}}^{-1}(z))| &\lesssim 1/\varepsilon, \\
|\partial_z^2 \nu_z(R(z), z, Z_{s_{\ell}}^{-1}(z))| &\approx 1/\delta,
\end{align*}
\]

(6.1)

Moreover, we have

\[
\begin{align*}
\beta &\lesssim |\nu_\theta(R_\ast(z), z, Z_{s_{\ell}}^{-1}(z))| \lesssim 1/\beta, \\
|\partial_z \nu_\theta(R(z), z, Z_{s_{\ell}}^{-1}(z))| &\lesssim 1/\beta, \\
|\partial_z^2 \nu_\theta(R(z), z, Z_{s_{\ell}}^{-1}(z))| &\lesssim 1/\epsilon,
\end{align*}
\]

(6.2) (6.3) (6.4)

\[
\partial_t |u(\Phi(x, t), t)| \lesssim 1/\epsilon,
\]

(6.5)

By the above lemma with Remark 3.4 we immediately have $|\Theta''| \lesssim 1/\epsilon$ and

$\Theta'' \approx 1/\delta$ (for sufficiently small $\delta$ compare with $\epsilon$) in $t \in I$.

Lemma 6.3. For any $\epsilon > 0$, we have

\[|u|^2 |\partial_s \kappa| \gg \kappa D_\ell |u|
\]

for sufficiently small $\delta > 0$.

Proof. From Section 4, we see

\[
\begin{align*}
\partial_s (\kappa^2) &= 2(\partial_s \kappa) = 2 R \Theta''(R \Theta''')(1 + (R')^2)^{-5/2} + \text{remainder}, \\
\kappa &= |R \Theta''(1 + (R')^2)^{-1} + \text{remainder}, \\
\partial_s \kappa &= \frac{R \Theta''(R \Theta''')(1 + (R')^2)^{-5/2}}{\kappa} + \text{remainder} \\
&= -R \Theta'''(1 + (R')^2)^{-3/2} + \text{remainder} \\
&\approx 1/\delta.
\end{align*}
\]

in $t \in I$. “remainder” is small compare with the main terms provided by small $\epsilon, \delta > 0$. Thus we immediately obtain $|u|^2 |\partial_s \kappa| \gg \kappa D_\ell |u|$ for sufficiently small $\delta > 0$. 

By Lemma 5.4 we see

\[
0 \geq |\partial_s \kappa| |u|^2 - |\partial_t D_\ell |u| - |\partial_z D_\ell |u| - |3 \kappa D_\ell |u|
\]

and it is in contradiction, since $\partial_s \kappa$ is sufficiently large compare with the other terms.
7. Proof of the main theorem (the vortex breakdown case)

Assume
\[ |\partial_r v_z(0, z, 0)| + |\partial_z v_z(0, z, 0)| + |\partial_r v_z(0, 0)| + |\partial_t v_z(0, 0)| + |\partial_r^2 v_z(0, 0)| \leq 1/\epsilon_2 \]
and employ a contradiction argument. Recall that, by Remark \[ 55 |\Theta'| \approx 1/\epsilon_1, \]
and \[ |\Theta''| \lesssim 1/\delta \] and \[ |\Theta'''| \lesssim 1/\delta \] in some small time interval. From Section \[ 4 \] near the axis, we have \[ (r = R(r_0, z_0, t)) \]
\[ \kappa = r(\Theta')^2 + O(r^2) \quad \text{and} \quad \partial_s \kappa = C(\delta)r + O(r^2). \]

Thus near the axis, we have
\[ 3\kappa D_t|u| + \partial_s \kappa|u|^2 \approx (\Theta')^2 r + O(r^2). \]
Since \[ \partial_r D_t|u| = 3\kappa D_t|u| + \partial_s \kappa|u|^2 \] and \[ \kappa = \partial_s \kappa = 0 \] along the axis, we have \[ \partial_r D_t|u| = 0 \] along the axis. By the mean value theorem, we have
\[ \partial_r^2 D_t|u| \approx (\Theta')^2. \]
along the axis (note that \[ \partial_r \rightarrow \partial_s \] if the corresponding point approaches the axis). However it is in contradiction, since the right hand side is large, while the left hand side is not large.

8. Appendix: Instantaneous blow-up

In this section we show instantaneous blow-up. Let us consider the Euler equations in the whole space \[ \mathbb{R}^3 \]:
\begin{align*}
\partial_t u + (u \cdot \nabla)u &= -\nabla p, \quad \nabla \cdot u = 0 \quad \text{in} \quad \mathbb{R}^3, \\
|u|_{t=0} &= u_0.
\end{align*}

The first existence results for (8.1) were proved in the framework of Hölder spaces by Gyunter 13, Lichtenstein 23, and Wolibner 33. More refined results were obtained subsequently by Kato 22, Bardos and Frisch 1, Ebin 13, Chemin 10, Constantin 12, and Majda and Bertozzi 26 among others. On the other hand, Bardos and Titi 2 found examples of solutions in Hölder spaces \[ C^{\alpha} \] and the Zygmund space \[ B^{\infty, \infty}_p \] which exhibit an instantaneous loss of smoothness in the spatial variable for any \[ 0 < \alpha < 1 \] (see also 11, 27). Similar examples in logarithmic Lipschitz spaces \[ \text{logLip}^{\alpha} \] were given by the authors in 27. In another direction Cheskidov and Shvydkoy 11 constructed periodic solutions that are discontinuous in time at \[ t = 0 \] in the Besov spaces \[ B^{s}_{p, \infty} \] where \[ s > 0 \] and \[ 2 < p \leq \infty \]. After their work, in a series of papers Bourgain and Li 8, 4 constructed smooth solutions which exhibit instantaneous blowup in borderline spaces such as \[ W^{n/p+1, p} \] for any \[ 1 \leq p < \infty \] and \[ B^{n/p+1}_{q,q} \] for any \[ 1 \leq p < \infty \] and \[ 1 < q \leq \infty \] as well as in the standard spaces \[ C^k \] and \[ C^{k-1, 1} \] for any integer \[ k \geq 1 \]; see also Elgindi and Masmoudi 14 and 28. As observed in 4 the cases \[ C^k \] and \[ C^{k-1, 1} \] are particularly intriguing in view of the classical existence and uniqueness results mentioned above.

In 29 (see also 28), they revisited the picture of local well-posedness in the sense of Hadamard for the Euler equations in Hölder spaces. They present a simple example based on a DiPerna-Majda type shear flow which shows that in general the data-to-solution map of \[ 1, 25 \] is not continuous into the space \[ L^{\infty}([0, T), C^{1, \alpha}) \] for any \[ 0 < \alpha < 1 \]. On the other hand, continuity of this map is restored (in the
strong sense) if the Cauchy problem is restricted to the so called little Hölder space $c^{1,\alpha}$.

**Remark 8.1.** For $u_0 \in c^{2,\alpha}$, we can also show that there exists a unique solution $u$ which is in (see [24, Section 4.4] and [29] for example)

$$C([0, T] : c^{2,\alpha}(\mathbb{R}^3)) \cap C^4([0, T] : c^{1,\alpha}(\mathbb{R}^3)) \cap C^2([0, T] : c^{0,\alpha}(\mathbb{R}^3)).$$

Therefore, if the solution $u$ is axi-symmetric, then the corresponding components $v_r$ and $v_z$ satisfy

$$|\partial_t v_r(0, z_j, t)| + |\partial_t v_z(0, z_j, t)| \lesssim 1,$$

$$|\partial_r v_r(0, z_j, t)| + |\partial_r v_z(0, z_j, t)| + |\partial_z v_z(0, z_j, t)| \lesssim 1$$

and

$$|\partial_t^2 v_r(0, z_j, t)| + |\partial_t^2 v_z(0, z_j, t)| \lesssim 1 \text{ for } t \in [0, T].$$

In this appendix, we show that even if the solution to the Euler equations is wellposed, such as, in $c^{2,\alpha}$, it may blow up (in some norm) instantaneously.

**Theorem 8.2.** There is an axisymmetric initial data $u_0 \in c^{2,\alpha}(\mathbb{R}^3)$ such that the corresponding unique solution $u$ is not in $C^4([0, T] : C^2(\mathbb{R}^3))$ for any $T > 0$. More precisely, we choose an axisymmetric initial data as the following: there is sufficiently small $\beta > 0$ such that for any $\{\epsilon_j\}_j (\epsilon_j \to 0)$ and $\{z_j\}_j (z_j \to z)$, there is $\{\delta_j\}_j (\delta_j \to 0$ as $j \to \infty$) such that

$$|v_z(0, z_j, 0)| \approx 1,$$

$$|\partial_z v_0(0, z_j, 0)| \approx 1/\beta,$$

$$|\partial_z^2 v_0(0, z_j, 0)| \approx 1/\delta_j,$$

$$\sum_{0 \leq j + k \leq 2} |\partial_j^2 \partial_k v_0(0, z_j, 0)| \lesssim 1,$$

$$\sum_{0 \leq j + k \leq 3} |\partial_j^2 \partial_k v_r(0, z_j, 0)| \lesssim 1,$$

$$\sum_{1 \leq j + k \leq 3} |\partial_j^2 \partial_k v_z(0, z_j, 0)| \lesssim 1.$$

Then we have

$$|\partial_t^2 \partial_r v_r(0, z_j, 0)| + |\partial_t^2 \partial_r v_z(0, z_j, 0)| + |\partial_t^2 \partial_z v_z(0, z_j, 0)| + |\partial_t^2 \partial_z v_r(0, z_j, 0)| > 1/\epsilon_j.$$

**Proof.** The proof is similar to the “vortex breakdown” case. By Remark 3.4, we can figure out that $\Theta^{\prime\prime\prime}|_{r=0}$ and $\Theta^{\prime\prime\prime\prime}|_{r=0}$ are not large. Due to Remark 3.1, we see $R'$, $R''$ and $R'''$ are all small. Let $r = R(r_0, z_0, t)$. By Lemma 5.3 near the axis, we see

$$\kappa = r\Theta^{\prime\prime\prime} + O(r^2), \quad \partial_r \kappa = \Theta^{\prime\prime\prime\prime} + O(r^2).$$

Thus near the axis, we have

$$3\kappa D_t |u| + \partial_r \kappa |u|^2 \approx \Theta^{\prime\prime\prime\prime} + O(r^2).$$

By the same argument as in the previous section, we have

$$\partial_t^2 D_t |u| \approx \Theta^{\prime\prime\prime\prime\prime}.$$

along the axis. This estimate tells us that $|\partial_t^2 v_r(0, z_j, 0)| \approx 1/\delta_j$.
Acknowledgments. The author would like to thank Professor Norikazu Saito for letting me know the book [17], Professor Hiroshi Suito for letting me know “Womersley number”, and also Doctor Kento Yamada for letting me know the articles [5] [10] [19] [30]. The author was partially supported by Grant-in-Aid for Young Scientists A (17H04825), Japan Society for the Promotion of Science (JSPS), and also partially supported by JST CREST.

References
1. C. Bardos and U. Frisch, Finite-time regularity for bounded and unbounded ideal incompressible fluids using Hölder estimates, Turbulence and Navier-Stokes equations (Proc. Conf., Univ. Paris-Sud, Orsay, 1975), Lecture Notes in Math., vol. 565, Springer, Berlin 1976
2. C. Bardos and E. Titi, Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations, Discrete Cont. Dyn. Syst. ser. S3 (2010), 185-197.
3. J. Bourgain and D. Li, Strong ill-posedness of the incompressible Euler equations in borderline Sobolev spaces, Invent. math. 201, (2015), 97-157; preprint [arXiv:1307.7090 [math.AP]].
4. J. Bourgain and D. Li, Strong ill-posedness of the incompressible Euler equation in integer $C^m$ spaces, Geom. funct. anal. 25 (2015), 1-86; preprint [arXiv:1405.2817 [math.AP]].
5. M. Brons, L.K. Voigt and J. N. Sorensen, Streamline topology of steady axisymmetric vortex breakdown in a cylinder with co- and counter-rotating end-covers, J. Fluid Mech. 401, (1999), 275-292.
6. D. Chae, On the Lagrangian dynamics for the 3D incompressible Euler equations, Comm. Math. Phys., 269, (2007), 557-569.
7. D. Chae, On the blow-up problem for the axisymmetric 3D Euler equations, Nonlinearity, 21, (2008), 2053-2060.
8. D. Chae, On the Lagrangian dynamics of the axisymmetric 3D Euler equations, J. Diff. Eq., 249 (2010), 571-577.
9. C-H. Chan, M. Czubak and T. Yoneda, An ODE for boundary layer separation on a sphere and a hyperbolic space, Physica D, 282 (2014), 34-38.
10. J. Chemin, Perfect Incompressible Fluids, Clarendon Press, Oxford 1998.
11. A. Cheskidov and R. Shvydkoy, Ill-posedness of basic equations of fluid dynamics in Besov spaces, Proc. A.M.S. 138, (2010), 1059-1067.
12. P. Constantin, An Eulerian-Lagrangian approach for incompressible fluids: local theory, J. Amer. Math. Soc. 14 (2001), 263-278.
13. D. Ebih, A concise presentation of the Euler equations of hydrodynamics, Comm. Partial Differential Equations 9 (1984), 539-559.
14. T. Elgindi and N. Masmoudi, $L^\infty$ ill-posedness for a class of equations arising in hydrodynamics, preprint [arXiv:1405.2478 [math.AP]].
15. A. Enciso and D. Peralta-Salas, Beltrami fields with a nonconstant proportionality factor are rare, Arch. Rational Mech. Anal., 220 (2016), 243-260.
16. M.P. Escudier, Observations of the flow produced in a cylindrical container by a rotating endwall, Experiments in Fluids, 2 (1984), 189-196.
17. L. Formaggia, A. Quarteroni and A. Veneziani, Cardiovascular mathematics, modeling and simulation of the circulatory system, Springer-Verlag, Italia, Milano, 2009.
18. N. Gyunter, On the motion of a fluid contained in a given moving vessel, (Russian), Izvestia Akad. Nauk USSR, Ser. Phys. Math. 20 (1926), 1323-1348, 1503-1532; 21 (1927), 621-556, 735-756, 1139-1162; 22 (1928), 9-30.
19. M.G. Hall, Vortex breakdown, Annu. Rev. Fluid Mech., 4 (1972), 195-218.
20. P-Y. Hsu, H. Notsu, T. Yoneda, A local analysis of the axisymmetric Navier-Stokes flow near a saddle point and no-slip flat boundary, J. Fluid Mech., 794 (2016) 444-459.
21. J. G. Heywood, R. Rannacher and S. Turek, Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations, Int. J. Numerical Methods in Fluid, 22 (1996), 325-352.
22. T. Kato, On classical solutions of the two-dimensional non-stationary Euler equation, Arch. Ration. Mech. Anal. 25 (1967), 188-200.
23. L. Lichtenstein, Über einige Existenzprobleme der Hydrodynamik, Math. Zeit. 23 (1925), 89-154, 309-316; 26 (1927), 196-323; 28 (1928), 387-415; 32 (1930), 608-640.
24. T. Ma and S. Wang, Boundary layer separation and structural bifurcation for 2-D incompressible fluid flows. Partial differential equations and applications, Discrete Contin. Dyn. Syst., 10 (2004), 459-472.
25. Y. Maekawa and A. Mazzucato, Inviscid limit and boundary layers for Navier-Stokes flows, to appear in Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Y. Giga and A. Novotný Ed., Springer; arXiv:1610.05372.
26. A. Majda and A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge 2002.
27. G. Misiolek and T. Yoneda, Ill-posedness examples for the quasi-geostrophic and the Euler equations, Analysis, geometry and quantum field theory, Contemp. Math. 584, Amer. Math. Soc., Providence, RI, 2012, 251-258.
28. G. Misiolek and T. Yoneda, Local ill-posedness of the incompressible Euler equations in $C^1$ and $B^1_{\infty,1}$, Math. Ann. 364 (2016), 243-268; Erratum, 363 (2015), 1399-1400.
29. G. Misiolek and T. Yoneda, Continuity of the solution map of the Euler equations in Hölder spaces and weak norm inflation in Besov spaces, to appear in Trans. Amer. Math. Soc.
30. T. Sarpkaya, On stationary and travelling vortex breakdowns, J. Fluid Mech., 45 (1971), 545-559.
31. H. Swann, The existence and uniqueness of nonstationary ideal incompressible flow in bounded domains in $\mathbb{R}^3$, Trans. Amer. Math. Soc. 179 (1973), 167-180.
32. R. Trip, D.J. Kuik, J. Westerweel and C. Poelma, An experimental study of transitional pulsatile pipe flow, Phys. Fluids, 24 (2012), 014103.
33. W. Wolibner, Un théorème sur l’existence du mouvement plan d’un fluide parfait, homogène, incompressible, pendant un temps infiniment long, Math. Z. 37 (1933), 698-726.
34. J. R. Womersley, Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known, J. Physiol., 127 (1955), 553-563.

Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1 Meguro, Tokyo 153-8914, Japan
E-mail address: yoneda@ms.u-tokyo.ac.jp