Intersection Local Time for two Independent Fractional Brownian Motions

David Nualart
Department of Mathematics, University of Kansas
405 Snow Hall, Lawrence, 66045 KS, USA
nualart@math.ku.edu, http://www.math.ku.edu/~nualart/

Salvador Ortiz-Latorre
Facultat de Matemàtiques, Universitat de Barcelona
Gran Via 585 08007 Barcelona, Spain
sortiz@ub.edu

Abstract

Let $B^H$ and $\tilde{B}^H$ be two independent, $d$-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$. Assume $d \geq 2$. We prove that the intersection local time of $B^H$ and $\tilde{B}^H$

$$I(B^H, \tilde{B}^H) = \int_0^T \int_0^T \delta(B^H_t - \tilde{B}^H_s)dsdt$$

exists in $L^2$ if and only if $Hd < 2$.

Keywords: Fractional Brownian motion. Intersection local time.

Mathematics Subject Classification MSC2000: 60G15, 60F25, 60G18, 60J55.

1 Introduction

We consider two independent fractional Brownian motions on $\mathbb{R}^d, d \geq 2$, with the same Hurst parameter $H \in (0,1)$. This means that we have two $d$-dimensional independent centered Gaussian processes $B^H = \{B^H_t, t \geq 0\}$ and $\tilde{B}^H = \{\tilde{B}^H_t, t \geq 0\}$ with covariance structure given by

$$\mathbb{E}[B^H_t, B^H_s] = \mathbb{E}[\tilde{B}^H_t, \tilde{B}^H_s] = \delta_{ij} R_H(s, t),$$

where $i, j = 1, ..., d$, $s, t \geq 0$ and

$$R_H(s, t) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).$$
The object of study in this paper will be the intersection local time of $B^H$ and $\tilde{B}^H$, which is formally defined as

$$I(B^H, \tilde{B}^H) \equiv \int_0^T \int_0^T \delta_0(B^H_t - \tilde{B}^H_s) ds dt,$$

where $\delta_0(x)$ is the Dirac delta function. It is a measure of the amount of time that the trajectories of the two processes, $B^H$ and $\tilde{B}^H$, intersect on the time interval $[0, T]$. As we pointed out before, this definition is only formal. In order to give a rigorous meaning to $I(B^H, \tilde{B}^H)$ we approximate the Dirac function by the heat kernel

$$p_\varepsilon(x) = (2\pi \varepsilon)^{-d/2} \exp(- |x|^2 / 2\varepsilon),$$

in $\mathbb{R}^d$. Then, we can consider the following family of random variables indexed by $\varepsilon > 0$

$$I_\varepsilon(B^H, \tilde{B}^H) \equiv \int_0^T \int_0^T p_\varepsilon(B^H_t - \tilde{B}^H_s) ds dt,$$

that we will call the approximated intersection local time of $B^H$ and $\tilde{B}^H$. We are interested in the $L^2(\Omega)$ convergence of $I_\varepsilon(B^H, \tilde{B}^H)$ as $\varepsilon$ tends to zero.

For $H = 1/2$, the processes $B^H$ and $\tilde{B}^H$ are classical Brownian motions. The intersection local time of independent Brownian motions has been studied by several authors (see Wolpert [9] and Geman, Horowitz and Rosen [2]). The approach of these papers rely on the fact that the intersection local time of independent Brownian motions can be seen as the local time at zero of some Gaussian vector field. This approach easily allows to consider the intersection of $k$ independent Wiener processes, $k \geq 2$. The applications of the intersection local time theory for Brownian motions range from the construction of relativistic quantum fields, see Wolpert [10], to the construction of the self-intersection local time for the Brownian motion, see LeGall [4]. Further research has been done in order to study such problems for other types of stochastic processes, mainly Lévy processes with a particular structure (strongly symmetric), see Marcus and Rosen [6].

In the general case, that is $H \neq 1/2$, only the self-intersection local time has been studied. Rosen studied in [11] the planar case and a recent paper by Hu and Nualart [3] gives a complete picture for the multidimensional case. On the other hand, Nualart et al. [5] used a weighted version of the 3-dimensional self-intersection local time for the study of probabilistic models for vortex filaments based on the fractional Brownian motion. In recent years the fBm has become an object of intense study. A stochastic calculus with respect to this process has been developed by many authors, see Nualart [7] for an extensive account on this subject. Because of its interesting properties, such as short/long range dependence and selfsimilarity, the fBm it’s being widely used in a variety of areas such finance, hydrology and telecommunications engineering, see [5]. Therefore, it seems interesting to study the intersection local time for this kind of processes.

The aim of this paper is to prove the existence of the intersection local time of $B^H$ and $\tilde{B}^H$, for an $H \neq 1/2$ and $d \geq 2$. We have obtained the following result.
Theorem 1  
(i) If \( Hd < 2 \), then the family of random variables \( I_\varepsilon(B^H, \tilde{B}^H) \) converges in \( L^2(\Omega) \). We will denote this limit by \( I(B^H, \tilde{B}^H) \).

(ii) If \( Hd \geq 2 \), then

\[
\lim_{\varepsilon \downarrow 0} \mathbb{E}[I_\varepsilon(B^H, \tilde{B}^H)] = +\infty
\]

and

\[
\lim_{\varepsilon \downarrow 0} \text{Var}[I_\varepsilon(B^H, \tilde{B}^H)] = +\infty.
\]

If \( \{B^{1/2}_t, t \geq 0\} \) is a planar Brownian motion, then

\[
I_\varepsilon = \int_0^T \int_0^T e^{i(\xi, B^H_t - \tilde{B}^H_s)} e^{-\varepsilon |\xi|^2} ds dt
\]

diverges almost sure, when \( \varepsilon \) tends to zero. Varadhan, in [12], proved that the renormalized self-intersection local time defined as \( \lim_{\varepsilon \rightarrow 0} (I_\varepsilon - \mathbb{E}[I_\varepsilon]) \), exists in \( L^2(\Omega) \). Condition (ii) implies that Varadhan renormalization does not converge in this case.

For \( Hd \geq 2 \), according to the previous theorem, \( I_\varepsilon(B^H, \tilde{B}^H) \) doesn’t converge in \( L^2(\Omega) \) and therefore \( I(B^H, \tilde{B}^H) \), the intersection local time of \( B^H \) and \( \tilde{B}^H \), doesn’t exist. The proof of Theorem 1.1 rest on Lemma 4, which deals with the integral of a negative power of the determinant of some covariance matrix.

The paper is organized as follows. In Section 2 we prove Theorem 1.1. In order to clarify the exposition, some technical lemmas needed in the proof are stated and proved in the Appendix.

2 Intersection Local Time of \( B^H \) and \( \tilde{B}^H \), Case \( Hd < 2 \)

Let \( B^H \) and \( \tilde{B}^H \) two independent fractional Brownian motions on \( \mathbb{R}^d \) with the same Hurst parameter \( H \in (0, 1) \).

Using the following classical equality

\[
p_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_\mathbb{R}^d e^{i(\xi, x)} e^{-\varepsilon |\xi|^2} d\xi
\]

from Fourier analysis, and the definition of \( I_\varepsilon(B^H, \tilde{B}^H) \), we obtain

\[
I_\varepsilon(B^H, \tilde{B}^H) = \frac{1}{(2\pi)^d} \int_0^T \int_0^T \int_\mathbb{R}^d e^{i(\xi, B^H_t - \tilde{B}^H_s)} e^{-\varepsilon |\xi|^2} d\xi ds dt.
\]

Therefore,
where we have used that \( \langle \xi, B_t^H - \tilde{B}_t^H \rangle \sim N(0, |\xi|^2(s^{2H} + t^{2H})) \), so

\[
\mathbb{E}[e^{i\langle \xi, B_t^H - \tilde{B}_t^H \rangle}] = e^{-(s^{2H} + t^{2H})|\xi|^2/2},
\]

and the fact that

\[
\int_{\mathbb{R}^d} e^{-(s^{2H} + t^{2H})|\xi|^2/2} d\xi = \left( \frac{2\pi}{s^{2H} + t^{2H}} \right)^{d/2}.
\]

According to the representation (1) for \( I_{\epsilon}(B^H, \tilde{B}^H) \), we have that

\[
\mathbb{E}[I_{\epsilon}^2(B^H, \tilde{B}^H)] = \frac{1}{(2\pi)^{2d}} \int_{[0,T]^4} \int_{\mathbb{R}^{2d}} \mathbb{E}[e^{i\langle \xi, B_t^H - \tilde{B}_t^H \rangle + \langle \eta, B_v^H - \tilde{B}_u^H \rangle}] 
\times e^{-\epsilon^2 \int_0^T \int_0^T \frac{1}{2} d\xi d\eta d\eta d\ell} d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell.
\]

Let introduce some notation that we will use throughout this paper,

\[
\lambda = \lambda(s, t) = s^{2H} + t^{2H},
\]
\[
\rho = \rho(u, v) = u^{2H} + v^{2H},
\]

and

\[
\mu = \mu(s, t, u, v) = \frac{1}{2} \left( s^{2H} + t^{2H} + u^{2H} + v^{2H} - |t - v|^{2H} - |s - u|^{2H} \right).
\]

Notice that \( \lambda \) is the variance of \( B_{v,1}^H - B_{u,1}^H \), \( \rho \) is the variance of \( B_{v,2}^H - B_{u,2}^H \)
and \( \mu \) is the covariance between \( B_{v,1}^H - B_{u,1}^H \) and \( B_{v,2}^H - B_{u,2}^H \), where \( B_{v,1}^H \) and \( B_{v,2}^H \) are independent one-dimensional fractional Brownian motions with Hurst parameter \( H \).

Using that \( \langle \xi, B_t^H - \tilde{B}_t^H \rangle + \langle \eta, B_v^H - \tilde{B}_u^H \rangle \sim N(0, \lambda|\xi|^2 + \rho|\eta|^2 + 2\mu(\xi, \eta)) \)
and (3) we can write for all \( \epsilon > 0 \)

\[
\mathbb{E}[I_{\epsilon}^2(B^H, \tilde{B}^H)] = \frac{1}{(2\pi)^{2d}} \int_{[0,T]^4} \int_{\mathbb{R}^{2d}} e^{-\frac{1}{4} \left( \lambda(\lambda + \epsilon)|\xi|^2 + \rho(\rho + \epsilon)|\eta|^2 + 2\mu(\xi, \eta) \right)} d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell.
\]

\[
= \frac{1}{(2\pi)^{d/2}} \int_{[0,T]^4} ((\lambda + \epsilon)(\rho + \epsilon) - \mu^2)^{-d/2} d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell d\xi d\eta d\ell.
\]
The last equality follows from the well known fact that
\[
\int_{\mathbb{R}^d} e^{-\frac{1}{2}(x,Ax)} dx = \frac{(2\pi)^d}{(\det A)^{1/2}},
\]
with
\[
A = \text{Id}_d \otimes \begin{pmatrix} \lambda + \varepsilon & \mu \\ \mu & \rho + \varepsilon \end{pmatrix},
\]
where \(\text{Id}_d\) is the \(d\)-dimensional identity matrix and \(\otimes\) denotes the Kronecker product of matrices. We also have that
\[
\det A = \det \left( \text{Id}_d \otimes \begin{pmatrix} \lambda + \varepsilon & \mu \\ \mu & \rho + \varepsilon \end{pmatrix} \right)
= (\det (\text{Id}_d))^2 \cdot \det \left( \begin{pmatrix} \lambda + \varepsilon & \mu \\ \mu & \rho + \varepsilon \end{pmatrix} \right)^d
= ((\lambda + \varepsilon)(\rho + \varepsilon) - \mu^2)^d.
\]

**Proof of Theorem 1.** Suppose first that \(Hd < 2\). A slight extension of (4) yields
\[
\mathbb{E}[I_\varepsilon(B^H, \tilde{B}^H)I_\eta(B^H, \tilde{B}^H)] = \int_{[0,T]^4} ((\lambda + \varepsilon)(\rho + \eta) - \mu^2)^{-d/2} dsdtdu dv.
\]
Consequently, a necessary and sufficient condition for the convergence in \(L^2(\Omega)\) of \(I_\varepsilon(B^H, \tilde{B}^H)\) is that
\[
\int_{[0,T]^4} (\lambda \rho - \mu^2)^{-d/2} dsdtdu dv < +\infty.
\]
Then the result follows from Lemma 4.

Now suppose that \(Hd \geq 2\), then from (2) and using monotone convergence theorem
\[
\lim_{\varepsilon \downarrow 0} \mathbb{E}[I_\varepsilon(B^H, \tilde{B}^H)] = \int_0^T \int_0^T (s^{2H} + t^{2H})^{-d/2} dsdt,
\]
and this integral is divergent by Lemma 3. According to the expression (2) for \(\mathbb{E}[I_\varepsilon(B^H, \tilde{B}^H)]\) and the expression (4) for \(\mathbb{E}[I_\varepsilon^2(B^H, \tilde{B}^H)]\) we obtain
\[
\lim_{\varepsilon \downarrow 0} \text{Var}[I_\varepsilon(B^H, \tilde{B}^H)] = \lim_{\varepsilon \downarrow 0} \left\{ \mathbb{E}[I_\varepsilon(B^H, \tilde{B}^H)^2] - \left( \mathbb{E}[I_\varepsilon(B^H, \tilde{B}^H)] \right)^2 \right\}
= \int_{[0,T]^4} (\lambda \rho - \mu^2)^{-d/2} - (\lambda \rho)^{-d/2} dsdtdu dv.
\]
Set
\[
D_\varepsilon := \{(s, t, u, v) \in \mathbb{R}_+^4 \mid s^2 + t^2 + u^2 + v^2 \leq \varepsilon^2\}. \quad (5)
\]
We can find $\varepsilon > 0$ such that $D_{\varepsilon} \subset [0, T]^d$. Making a change to spherical coordinates, as the integrand is always positive, we have

$$\int_{[0,T]^d} (\lambda \rho - \mu^2)^{-d/2} - (\lambda \rho)^{-d/2} \, dsdtduv$$

$$\geq \int_{D_{\varepsilon}} (\lambda \rho - \mu^2)^{-d/2} - (\lambda \rho)^{-d/2} \, dsdtduv = \int_0^{\varepsilon} \int_{[0, T]} r^{3-2Hd} dr \int_{\Theta, \Psi} (\theta) \, d\theta,$$

where the integral in $r$ is convergent if and only if $Hd < 2$, and the angular integral is different from zero thanks to the positivity of the integrand. Therefore, if $Hd \geq 2$, then

$$\lim_{\varepsilon \downarrow 0} \text{Var}[I_{\varepsilon}(B^H, \tilde{B}^H)] = +\infty.$$

3 Appendix

For clarity of exposition, we state and prove some technical lemmas in this appendix.

**Lemma 2** Let $\alpha > 0$, and let

$$\gamma(\alpha, x) \equiv \int_0^x e^{-y}y^{\alpha-1}dy \quad (6)$$

be the lower incomplete gamma function. Then for all $\varepsilon < \alpha$ and $x > 0$,

$$\gamma(\alpha, x) \leq K(\alpha) x^\varepsilon,$$

where $K(\alpha) \equiv \frac{1}{\alpha} \vee \Gamma(\alpha)$ and $\Gamma(\alpha) = \gamma(\alpha, +\infty)$.

**Proof.** If $x \geq 1$,

$$\gamma(\alpha, x) \leq \Gamma(\alpha) x^\varepsilon,$$

for all $\varepsilon > 0$. On the other hand, if $x < 1$,

$$\gamma(\alpha, x) \leq \int_0^x y^{\alpha-1}dy = \frac{x^\alpha}{\alpha} = \frac{1}{\alpha} x^\varepsilon,$$

if $\varepsilon < \alpha$. ■

**Lemma 3** The following integral

$$\int_0^T \int_0^T (s^{2H} + t^{2H})^{-d/2} \, dsdt,$$

is finite if and only if $Hd < 2$.

**Proof.** It easily follows from a polar change of coordinates. ■
Lemma 4 Let
\[ A_T \equiv \int_{[0,T]^4} (\lambda \rho - \mu^2)^{-d/2} dsdtduv, \]
then \( A_T < +\infty \) if and only if \( Hd < 2 \).

Proof. The necessary condition follows from a spherical change of coordinates. We can find \( \varepsilon > 0 \) such that \( D_\varepsilon \subset [0,T]^4 \), where \( D_\varepsilon \) is given in (5). As the integrand in \( A_T \) is always positive we have
\[ A_T \geq \int_{D_\varepsilon} (\lambda \rho - \mu^2)^{-d/2} dsdtduv = \int_0^\varepsilon r^{3-2Hd} dr \int_0^\Theta \phi(\theta) d\theta, \]
where the integral in \( r \) is convergent if and only if \( Hd < 2 \), and the angular integral is different from zero thanks to the positivity of the integrand. Therefore, if \( Hd \geq 2 \), then \( A_T = +\infty \).

Suppose now that \( Hd < 2 \). By symmetry we have that
\[ A_T = 4 \int_T (\lambda \rho - \mu^2)^{-d/2} dsdtduv, \]
where
\[ T \equiv \{(s,t,u,v) : 0 < v < t, 0 < t \leq T, 0 < u < s, 0 < s \leq T\}. \]

Notice that \( \lambda \rho - \mu^2 = \text{det} \Var(Z) \),
where \( Z \equiv (B_t^{H,1} - \overline{B}_s^{H,1}, B_v^{H,1} - \overline{B}_u^{H,1}) \). Due to the independence of \( B^H \) and \( \overline{B}^H \), we have that
\[ \Var(Z) = \Var(B_t^{H,1}, B_v^{H,1}) + \Var(\overline{B}_s^{H,1}, \overline{B}_u^{H,1}), \]
and
\[ \lambda \rho - \mu^2 \geq \det(\Var(B_t^{H,1}, B_v^{H,1})) + \det(\Var(\overline{B}_s^{H,1}, \overline{B}_u^{H,1})), \]
because the matrices \( \Var(B_t^{H,1}, B_v^{H,1}) \) and \( \Var(\overline{B}_s^{H,1}, \overline{B}_u^{H,1}) \) are strictly positive definite (see A8, (viii) in [5]). Then
\[ A_T \leq 4 \int_T (\varphi(t,v) + \varphi(s,u))^{-d/2} dsdtduv, \]
where
\[ \varphi(t,v) \equiv \det(\Var(B_t^{H,1}, B_v^{H,1})) = t^{2H} v^{2H} - \frac{1}{4} \left( t^{2H} + v^{2H} - |t-v|^{2H} \right)^2. \]

Using Fubini’s Theorem and
\[ \lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-\lambda z} z^{\alpha-1} dz, \]

7
for all $\lambda, \alpha > 0$, we obtain
\[
\int_T (\varphi(t, v) + \varphi(s, u))^{-d/2} ds dt dv = \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty e^{-(\varphi(t, v) + \varphi(s, u))z} z^{\frac{d}{2} - 1} dz ds dt dv
\]
\[
= \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty z^{\frac{d}{2} - 1} A^2(z) \, dz,
\]
where
\[
A(z) = \int_0^T \int_t^T e^{-\varphi(t, v)z} dv dt.
\]
As $A(z) < +\infty$, for all $z \in [0, 1]$, the integral is convergent in a neighborhood of zero. Hence, we have to study the convergence of
\[
\int_1^\infty z^{\frac{d}{2} - 1} A^2(z) \, dz.
\]
Due to the homogeneity of order $4H$ of $\varphi(t, v)$, if we make the change of coordinates $t = z^{-\frac{1}{4H}} x, v = z^{-\frac{1}{4H}} y$, we obtain
\[
\int_1^\infty z^{\frac{d}{2} - 1} A^2(z) \, dz = \int_1^\infty z^{\frac{d}{2} - 1} \left( \int_0^{T z^{-\frac{1}{4H}}} \int_0^x e^{-\varphi(x, y)dy} \right)^2 dz.
\]
Now, using that $\{(x, y) : 0 < x < T z^{-\frac{1}{4H}}, 0 < y < x\} \subset \{(x, y) : x^2 + y^2 \leq 2T^2 z^{-\frac{1}{4H}}\} \equiv S$, and making a polar change of coordinates we have
\[
\int_0^{T z^{-\frac{1}{4H}}} \int_0^x e^{-\varphi(x, y)dy} \, dx \leq \int_S e^{-\varphi(x, y)dy} \, dx = \int_0^{\pi/4} \int_0^{\sqrt{2T z^{-\frac{1}{4H}}}} re^{-r^{4H}\varphi(\theta)} \, dr \, d\theta,
\]
where $\varphi(\theta) \equiv \varphi(\cos \theta, \sin \theta)$. After the new change of variable $x = r^{4H} \varphi(\theta)$, the last integral is equal to
\[
\int_0^{\pi/4} \varphi(\theta)^{-\frac{1}{4H}} \int_0^{2H r^{4H} T^{4H} \varphi(\theta)} \frac{1}{4H} e^{-x \frac{1}{4H} - 1} dx \, d\theta
\]
\[
= \frac{1}{4H} \int_0^{\pi/4} \varphi(\theta)^{-\frac{1}{4H}} \gamma \left((2H)^{-1}, 2^{2H} T^{4H} \varphi(\theta)\right) \, d\theta,
\]
where $\gamma(\alpha, x)$ is given by (8). Applying Lemma
\[
\int_1^\infty z^{\frac{d}{2} - 1} A^2(z) \, dz
\]
\[
\leq \frac{1}{16H^2} \int_1^\infty z^{\frac{d}{2} - 1} \left( \int_0^{\pi/4} \varphi(\theta)^{-\frac{1}{4H}} \gamma \left((2H)^{-1}, 2^{2H} T^{4H} \varphi(\theta)\right) \, d\theta \right)^2 dz
\]
\[
\leq \frac{24H \varepsilon T^{4H}}{16H^2} K^2 \left( \frac{1}{2H} \right) \int_1^\infty z^{\frac{d}{2} - 1 + 2\varepsilon} \left( \int_0^{\pi/4} \varphi(\theta)^{-\frac{1}{4H}} \, d\theta \right)^2.
\]
The integral in \( z \) is convergent provided \( \varepsilon < \frac{2 - H}{4H} \). It’s an exercise of computation of limits to prove that \( \varphi(\theta) \sim \theta^{2H} \) as \( \theta \downarrow 0 \) and \( \varphi(\theta) \sim (\pi/4 - \theta)^{2H} \) as \( \theta \uparrow \pi/4 \), the main tool is to substitute the trigonometric functions by their first order approximations at the respective points. As a consequence, the integral

\[
\int_{0}^{\pi/4} \varphi(\theta)^{\varepsilon - \frac{1}{2H}} d\theta
\]

is always convergent. ■

References

[1] Doukhan P., Oppenheim G., Taqqu M.S. (2003). Theory and Applications of Long Range Dependence. Birkhäuser, Boston.

[2] Geman, D., Horowitz, J., Rosen, J. (1984). A Local Time Analysis of Intersections of Brownian Paths in the Plane. *Annals of Probability*. **12** 86-107.

[3] Hu, Y., Nualart, D. (2005). Renormalized Self-Intersection Local Time for Fractional Brownian Motion. *Annals of Probability*. **33** 948-983.

[4] LeGall, J.F. (1985). Sur le Temps Local d'Intersection du Mouvement Brownien Plan et la Méthode de Renormalisation de Varadhan. *Séminaire de Probabilités XIX. Lecture Notes in Math.* **1123**, 314-331. Springer, Berlin.

[5] Muirhead, R.J. (1982) Aspects of Multivariate Statistical Theory. *Wiley Series in Probability and Mathematical Statistics*. John Wiley & Sons, Inc., New York.

[6] Marcus, M.B., Rosen, J. (1999). Additive Functionals of Several Levy Processes and Intersection Local Times. *Annals of Probability*. **27** 1643-1678.

[7] Nualart, D. (2003) Stochastic Integration with Respect to Fractional Brownian Motion and Applications. *Contemporary Mathematics*. **336** 3-39.

[8] Nualart, D., Rovira, C. and Tindel, S. (2003) Probabilistic Models for Vortex Filaments Based on Fractional Brownian Motion. *Annals of Probability*. **31** 1862-1899.

[9] Wolpert, R. (1978). Wiener Path Intersections and Local Time. *Journal of Functional Analysis*. **30** 329-340.

[10] Wolpert, R. (1978). Local Time and a Particle Picture for Euclidean Field Theory. *Journal of Functional Analysis*. **30** 341-357.

[11] Rosen, J. (1987). The intersection local time of fractional brownian motion in the plane. *Journal of Multivariate Analysis*. **23** 37-46.
[12] Varadhan, S.R.S. (1969) Appendix to "Euclidian quantum field theory". By K. Symanzik, in: "Local Quantum Theory". Jost, R. (ed). Academic Press, New York.