Commutators of Sylow subgroups of alternating and symmetric groups, commutator width in the wreath product of groups

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Abstract

This paper investigates bounds of the commutator width \([1]\) of a wreath product of two groups. The commutator width of direct limit of wreath product of cyclic groups are found. For given a permutational wreath product sequence of cyclic groups we investigate its commutator width and some properties of its commutator subgroup. It was proven that the commutator width of an arbitrary element of the wreath product of cyclic groups \(C_{p_i}\), \(p_i \in \mathbb{N}\) equals to 1. As a corollary, it is shown that the commutator width of Sylows \(p\)-subgroups of symmetric and alternating groups \(p \geq 2\) are also equal to 1. The structure of commutator and second commutator of Sylows 2-subgroups of symmetric and alternating groups were investigated. For an arbitrary group \(B\) an upper bound of commutator width of \(C_p \wr B\) was founded.

Key words: wreath product of group, commutator width of wreath product, commutator width of Sylow \(p\)-subgroups, commutator and centralizer subgroup of alternating group.

Mathematics Subject Classification: 20B27, 20E08, 20B22, 20B35, 20F65, 20B07, 20E22, 20E45.

1 Introduction

Let \(G\) be a group. The commutator width of \(G\), \(cw(G)\) is defined to be the least integer \(n\), such that every element of \(G'\) is a product of at most \(n\) commutators if such an integer exists, and \(cw(G) = \infty\) otherwise. The first example of a finite perfect group with \(cw(G) > 1\) was given by Isaacs in [1].

Commutator width of groups, and of elements has proven to be an important property in particular via its connections with "stable commutator length" and bounded cohomology.
A form of commutators of wreath product $A \wr B$ was briefly considered in [4] and presented by us as wreath recursion. For more deep description of this form we take into account the commutator width ($cw(G)$) which was presented in work of Muranov [1].

The form of commutator presentation [4] was presented by us in form of wreath recursion [7] and commutator width of it was studied. We impose more weak condition on the presentation of wreath product commutator then it was imposed by J. Meldrum.

It was known that, the commutator width of iterated wreath products of nonabelian finite simple groups is bounded by an absolute constant [5, 8]. But it was not proven that commutator subgroup of $\bigwedge_{i=1}^{k} C_{p_{i}}$ consists of commutators. We generalize the passive group of this wreath product to any group $B$ instead of only wreath product of cyclic groups and obtain an exact commutator width. Our goal is to improve these estimations and genralize it on a bigger class of passive groups of wreath product. Also we are going to prove that the commutator width of Sylows $p$-subgroups of symmetric and alternating groups $p \geq 2$ is 1.

2 Preliminaries

We call by $g_{ij}$ the state of $g \in AutX^{[h]}$ in vertex $v_{ij}$ as it was called in [7,11].

Denote by $fun(B, A)$ the direct product of isomorphic copies of A indexed by elements of B. Thus, $fun(B, A)$ is a function $B \to A$ with the conventional multiplication and finite supports. The extension of $fun(B, A)$ by B is called the discrete wreath product of A,B. Thus, $A \wr B := fun(B, A) \ltimes B$ moreover, $bfb^{-1} = f^{b}$, $b \in B$, $f \in fun(B, A)$. As well known that a wreath product of permutation groups is associative construction.

Let $G$ be a group acting (from the left) by permutations on a set $X$ and let $H$ be an arbitrary group. Then the (permutational) wreath product $H \wr G$ is the semidirect product $H^{X} \rtimes G$, where $G$ acts on the direct power $H^{X}$ by the respective permutations of the direct factors. The group $C_{p}$ is equipped with a natural action by the left shift on $X = \{1, \ldots, p\}$, $p \in \mathbb{N}$.

The multiplication rule of automorphisms $g, h$ which presented in form of the wreath recursion [2] $g = (g_{(1)}, g_{(2)}, \ldots, g_{(d)})\sigma_{g}$, $h = (h_{(1)}, h_{(2)}, \ldots, h_{(d)})\sigma_{h}$, is given by the formula:

$$g \cdot h = (g_{(1)}h_{(\sigma_{g}(1))}, g_{(2)}h_{(\sigma_{g}(2))}, \ldots, g_{(d)}h_{(\sigma_{g}(d))})\sigma_{g}\sigma_{h}. $$

We define $\sigma$ as $(1,2,\ldots, p)$ where $p$ is defined by context.

We consider $B \wr (C_{p}, X)$, where $X = \{1,\ldots, p\}$, and $B' = \{[f,g] \mid f,g \in B\}$, $p \geq 1$. If we fix some indexing $\{x_{1}, x_{2}, \ldots, x_{m}\}$ of set the X, then an element $h \in H^{X}$ can be
written as \((h_1, ..., h_m)\) for \(h_i \in H\).

The set \(X^*\) is naturally a vertex set of a regular rooted tree, i.e. a connected graph without cycles and a designated vertex \(v_0\) called the root, in which two words are connected by an edge if and only if they are of form \(v\) and \(vx\), where \(v \in X^*\), \(x \in X\). The set \(X^n \subset X^*\) is called the \(n\)-th level of the tree \(X^*\) and \(X^0 = \{v_0\}\). We denote by \(v_{j,i}\) the vertex of \(X^j\), which has the number \(i\). Note that the unique vertex \(v_{k,i}\) corresponds to the unique word \(v\) in alphabet \(X\). For every automorphism \(g \in AutX^*\) and every word \(v \in X^*\) define the section (state) \(g(v) \in AutX^*\) of \(g\) at \(v\) by the rule: \(g_v(x) = y\) for \(x, y \in X^*\) if and only if \(g(vx) = g(v)y\). The subtree of \(X^*\) induced by the set of vertices \(\cup_{k=0}^\infty X^k\) is denoted by \(X^{[k]}\). The restriction of the action of an automorphism \(g \in AutX^*\) to the subtree \(X^{[l]}\) is denoted by \(g|_{X^{[l]}}\). A restriction \(g|_{X^{[l]}}\) is called the vertex permutation (v.p.) of \(g\) in a vertex \(v\). We call the endomorphism \(\alpha|_v\) the restriction of \(g\) in a vertex \(v\) \([2]\). For example, if \(|X| = 2\) then we just have to distinguish active vertices, i.e., the vertices for which \(\alpha|_v\) is non-trivial. As well known if \(X = \{0, 1\}\) then \(AutX^{[k-1]} \cong C_2 \ast \ldots \ast C_2\) \([2]\).

Let us label every vertex of \(X^l\), \(0 \leq l < k\) by sign 0 or 1 in relation to state of v.p. in it. Let us denote state value of \(\alpha\) in \(v_{k,i}\) as \(s_{ki}(\alpha)\) we put that \(s_{ki}(\alpha) = 1\) if \(\alpha|_{v_{ki}}\) is non-trivial, and \(s_{ki}(\alpha) = 0\) if \(\alpha|_{v_{ki}}\) is trivial. Obtained by such way a vertex-labeled regular tree is an element of \(AutX^{[k]}\). All undeclared terms are from \([10][11]\).

Let us make some notations. The commutator of two group elements \(a\) and \(b\), denoted as \([a, b] = aba^{-1}b^{-1}\), conjugation by an element \(b\) as

\[
a^b = bab^{-1},
\]

\(\sigma = (1, 2, ..., p)\). Also \(G_k \simeq Syl_{2^k} A_{2^k}\) that is Sylow 2-subgroup of \(A_{2^k}\), \(B_k = \langle i \rangle_{i=1}^k C_2\). It is convenient do not distinguish \(G_k (B_k)\) from its isomorphic copy in \(AutX^{[k]}\). The structure of \(G_k\) was investigated in \([17]\). For this research we can regard \(G_k\) and \(B_k\) as recursively constructed i.e. \(B_1 = C_2\), \(B_k = B_{k-1} \ast C_2\) for \(k > 1\), \(G_1 = \langle e \rangle\), \(G_k = \{(g_1, g_2)\pi \in B_k | g_1g_2 \in G_{k-1}\}\) for \(k > 1\).

The commutator length of an element \(g\) of the derived subgroup of a group \(G\) is denoted \(cl(G)\), is the minimal \(n\) such that there exist elements \(x_1, ..., x_n, y_1, ..., y_n\) in \(G\) such that \(g = [x_1, y_1]...[x_n, y_n]\). The commutator length of the identity element is 0. The commutator width of a group \(G\), denoted \(cw(G)\), is the maximum of the commutator lengths of the elements of its derived subgroup \([G, G]\). It is also related to solvability of quadratic equations in groups \([3]\).
3 Main result

Let $B \wr C_p$ is a regular wreath product of cyclic group of order $p$ and arbitrary group $B$.

**Lemma 1.** An element of form $(r_1, \ldots, r_{p-1}, r_p) \in W' = (B \wr C_p)'$ iff product of all $r_i$ (in any order) belongs to $B'$, where $B$ is an arbitrary group.

**Proof.** Analogously to the Corollary 4.9 of the Meldrum’s book [4] we can deduce new presentation of commutators in form of wreath recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_p), \]

where $r_i \in B$. If we multiply elements from a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where $r_i = h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}$, $h, g \in B$ and $a, b \in C_p$, then we get a product

\[ x = \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in B', \tag{1} \]

where $x$ is a product of corespondent commutators. Therefore, we can write $r_p = r_{p-1}^{-1} \ldots r_1^{-1} x$. We can rewrite element $x \in B'$ as the product $x = \prod_{j=1}^{m} [f_j, g_j]$, $m \leq cw(B)$.

Note that we impose more weak condition on the product of all $r_i$ to belongs to $B'$ then in Definition 4.5. of form $P(L)$ in [4], where the product of all $r_i$ belongs to a subgroup $L$ of $B$ such that $L > B'$.

In more detail deducing of our representation constructing can be reported in following way. If we multiply elements having form of a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where $r_i = h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}$, $h, g \in B$ and $a, b \in C_p$, then in case $cw(B) = 0$ we obtain a product

\[ \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in B'. \tag{2} \]

Note that if we rearrange elements in (1) as $h_1^{-1} h_2^{-1} g_1^{-1} g_2^{-1} h_2^{-1} g_1^{-1} \ldots h_p^{-1} h_1^{-1} g_p^{-1} g_p^{-1}$ then by the reason of such permutations we obtain a product of corepondent commutators. Therefore, following equality holds true

\[ \prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = \prod_{i=1}^{p} h_i^{-1} g_i^{-1} x \in B', \tag{3} \]

\[ 4 \]
where $x$ is a product of correspondent commutators. Therefore,

$$(r_1, \ldots, r_{p-1}, r_p) \in W' \text{ iff } r_{p-1} \cdot \ldots \cdot r_1 \cdot r_p = x \in B'.$$  \hspace{1cm} (4)

Thus, one element from states of wreath recursion $(r_1, \ldots, r_{p-1}, r_p)$ depends on rest of $r_i$. This dependence contribute that the product $\prod_{j=1}^{p} r_j$ for an arbitrary sequence $\{r_j\}_{j=1}^{p}$ belongs to $B'$. Thus, $r_p$ can be expressed as:

$$r_p = r_1^{-1} \cdot \ldots \cdot r_{p-1}^{-1} x.$$ 

Denote a $j$-th tuple, which consists of a wreath recursion elements, by $(r_{ji_1}, r_{ji_2}, \ldots, r_{ji_p})$. Closedness by multiplication of the set of forms $(r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'$ follows from

$$\prod_{j=1}^{k} (r_{j1} \ldots r_{jp-1} r_{jp}) = \prod_{j=1}^{k} \prod_{i=1}^{p} r_{ji} = R_1 R_2 \ldots R_k \in B',$$  \hspace{1cm} (5)

where $r_{ji}$ is $i$-th element from the tuple number $j$, $R_j = \prod_{i=1}^{p} r_{ji}$, $1 \leq j \leq k$. As it was shown above $R_j = \prod_{i=1}^{p-1} r_{ji} \in B'$. Therefore, the product $R_j$ of $R_j$, $j \in \{1, \ldots, k\}$ which is similar to the product mentioned in [4], has the property $R_1 R_2 \ldots R_k \in B'$ too, because of $B'$ is subgroup. Thus, we get a product of form (1) and the similar reasoning as above are applicable.

Let us prove the sufficiency condition. If the set $K$ of elements satisfying the condition of this theorem, that all products of all $r_i$, where every $i$ occurs in this forms once, belong to $B'$, then using the elements of form

$$(r_1, e, \ldots, e, r_1^{-1}), \ldots, (e, e, \ldots, e, r_i, e, e^{-1}), \ldots, (e, e, \ldots, e, r_{p-1}, r_{p-1}^{-1}), (e, e, \ldots, e, r_1 r_2 \cdot \ldots \cdot r_{p-1})$$

we can express any element of form $(r_1, \ldots, r_{p-1}, r_p) \in W = (C_p \wr B)'$. We need to prove that in such way we can express all element from $W$ and only elements of $W$. The fact that all elements can be generated by elements of $K$ follows from randomness of choice every $r_i$, $i < p$ and the fact that equality (1) holds so construction of $r_p$ is determined.

\begin{lemma}
For any group $B$ and integer $p \geq 2$, $p \in \mathbb{N}$ if $w \in (B \wr C_p)'$ then $w$ can be
represented as the following wreath recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \cdots r_{p-1}^{-1}) \prod_{j=1}^{k} [f_j, g_j], \]

where \( r_1, \ldots, r_{p-1}, f_j, g_j \in B \), and \( k \leq cw(B) \).

**Proof.** According to the Lemma we have the following wreath recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_p), \]

where \( r_i \in B \) and \( r_{p-1}r_{p-2} \cdots r_2r_1r_p = x \in B' \). Therefore, we can write \( r_p = r_1^{-1} \cdots r_{p-1}^{-1}x \).

We also can rewrite element \( x \in B' \) as product of commutators \( x = \prod_{j=1}^{k} [f_j, g_j] \), where \( k \leq cw(B) \).

**Lemma 3.** For any group \( B \) and integer \( p \geq 2 \), \( p \in \mathbb{N} \) if \( w \in (B \wr C_p)' \) is defined by the following wreath recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \cdots r_{p-1}^{-1}[f, g]), \]

where \( r_1, \ldots, r_{p-1}, f_j, g_j \in B \), then \( w \) can be represent as commutator

\[ w = [(a_{1,1}, \ldots, a_{1,p})\sigma, (a_{2,1}, \ldots, a_{2,p})], \]

where

\[
\begin{align*}
a_{1,i} &= e \quad \text{for } 1 \leq i \leq p - 1, \\
a_{2,1} &= (f^{-1})^{r_1^{-1} \cdots r_{p-1}^{-1}}, \\
a_{2,i} &= r_{i-1}a_{2,i-1} \quad \text{for } 2 \leq i \leq p, \\
a_{1,p} &= g^{a_2^{-1}}.
\end{align*}
\]

**Proof.** Let us consider the following commutator

\[
\kappa = (a_{1,1}, \ldots, a_{1,p})\sigma \cdot (a_{2,1}, \ldots, a_{2,p}) \cdot (a_{1,p}^{-1}, a_{1,1}^{-1}, \ldots, a_{1,p-1}^{-1})\sigma^{-1} \cdot (a_{2,1}^{-1}, \ldots, a_{2,p}^{-1}) = (a_{3,1}, \ldots, a_{3,p}),
\]

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where
\[ a_{3,i} = a_{1,i}a_{2,1+(i \mod p)}a_{1,i}^{-1}a_{2,i}^{-1}. \]

At first we compute the following
\[ a_{3,i} = a_{1,i}a_{2,i+1}a_{1,i}^{-1}a_{2,i}^{-1} = a_{2,i+1}a_{2,i}^{-1} = r_i, a_{2,i}a_{2,i}^{-1} = r_i, \text{ for } 1 \leq i \leq p-1. \]

Then we make some transformations of \( a_{3,p} \):
\[
\begin{align*}
a_{3,p} &= a_{1,p}a_{2,1}a_{1,p}^{-1}a_{2,p}^{-1} \\
&= (a_{2,1}a_{2,1})a_{1,p}a_{2,1}a_{1,p}^{-1}a_{2,p}^{-1} \\
&= a_{2,1}[a_{2,1}, a_{1,p}]a_{2,p}^{-1} \\
&= a_{2,1}a_{2,p}^{-1}a_{2,p}[a_{2,1}, a_{1,p}]a_{2,p}^{-1} \\
&= (a_{2,p}a_{2,1}, a_{1,p}^{-1})^{-1}[(a_{2,1})a_{2,p}, a_{2,1}a_{2,p}^{-1}] \\
&= (a_{2,p}a_{2,1}, a_{1,p}^{-1})^{-1}[(a_{2,1})a_{2,p}, a_{2,1}a_{2,p}^{-1}] \\
&= (a_{2,p}a_{2,1}, a_{1,p}^{-1})^{-1}[(a_{2,1})a_{2,p}, a_{2,1}a_{2,p}^{-1}].
\end{align*}
\]

We transform the commutator \( \kappa \) to a form analogous to that of \( w \):
\[
\begin{cases}
a_{1,i}a_{2,i+1}a_{1,i}^{-1}a_{2,i}^{-1} = r_i, \text{ for } 1 \leq i \leq p-1, \\
(a_{2,p}a_{2,1})^{-1} = r_1^{-1} \cdots r_{p-1}^{-1}, \\
(a_{2,1})a_{2,p}^{-1} = f, \\
\frac{a_{2,p}}{a_{1,p}} = g.
\end{cases}
\]

In order to prove required statement it is sufficient to find at least one solution of equations. We set the following
\[ a_{1,i} = e \text{ for } 1 \leq i \leq p-1. \]

Then we have
\[
\begin{cases}
a_{2,i+1}a_{2,i}^{-1} = r_i, \text{ for } 1 \leq i \leq p-1, \\
(a_{2,p}a_{2,1})^{-1} = r_1^{-1} \cdots r_{p-1}^{-1}, \\
(a_{2,1})a_{2,p}^{-1} = f, \\
\frac{a_{2,p}}{a_{1,p}} = g.
\end{cases}
\]

Now we can see that the form of the commutator \( \kappa \) is analogous to the form of \( w \).
Let us make the following notation
\[ r' = r_{p-1} \ldots r_1. \]

We note that from the definition of \( a_{2,i} \) for \( 2 \leq i \leq p \) it follows that
\[ r_i = a_{2,i+1} a_{2,i}^{-1}, \text{ for } 1 \leq i \leq p - 1. \]

Therefore,
\[ r' = (a_{2,p} a_{2,p-1}^{-1})(a_{2,p-1} a_{2,p-2}^{-1}) \ldots (a_{2,3} a_{2,2}^{-1})(a_{2,2} a_{2,1}^{-1}) \]
\[ = a_{2,p} a_{2,1}^{-1}. \]

And then
\[ (a_{2,p} a_{2,1}^{-1})^{-1} = (r')^{-1} = r_1^{-1} \ldots r_{p-1}^{-1}. \]

Finally let us to compute the following
\[ (a_{2,1}^{-1})^{a_{2,p} a_{2,1}^{-1}} = (((f^{-1})^{r_1^{-1} \ldots r_{p-1}^{-1}})^{-1})^{r'} = (f(r')^{-1})^{r'} = f, \]
\[ a_{2,p}^{a_{2,1}^{-1}} = (g^{a_{2,p}^{-1}})^{a_{2,1}} = g. \]

And now we conclude that
\[ a_{3,p} = r_1^{-1} \ldots r_{p-1}^{-1}[f,g]. \]

Thus, the commutator \( \kappa \) is presented exactly in the similar form as \( w \) has. \( \square \)

For future using we formulate previous lemma for the case \( p = 2 \).

**Corollary 4.** If \( B \) is any group and \( w \in (B \wr C_2)' \) is defined by the following wreath recursion
\[ w = (r_1, r_1^{-1}[f,g]), \]
where \( r_1, f, g \in B \), then \( w \) can be represent as commutator
\[ w = [(e, a_{1,2}) \sigma, (a_{2,1}, a_{2,2})], \]
where
\[ a_{2,1} = (f^{-1})r_1^{-1}, \]
\[ a_{2,2} = r_1a_{2,1}, \]
\[ a_{1,2} = g^{a_{2,1}}. \]

The proof is immediate from Lemma 3 in the case where \( p = 2 \).

**Lemma 5.** For any group \( B \) and integer \( p \geq 2 \) inequality
\[ cw(B \wr C_p) \leq \max(1, cw(B)) \]
holds.

**Proof.** We can represent any \( w \in (B \wr C_p)' \) by Lemma 3 as the following wreath recursion
\[
w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j])
\]
\[
= (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} [f_1, g_1]) \cdot \prod_{j=2}^{k} [(e, \ldots, e, f_j), (e, \ldots, e, g_j)],
\]
where \( r_1, \ldots, r_{p-1}, f_j, g_j \in B, k \leq cw(B) \). Now we can apply Lemma 3 to the element \((r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} [f_1, g_1])\). Lemma 3 implies that \( w \) can be represented as product of \( \max(1, cw(B)) \) commutators.

**Corollary 6.** If \( W = C_{p_k} \ldots \wr C_{p_1} \) then \( cw(W) = 1 \) for \( k \geq 2 \).

**Proof.** If \( B = C_{p_k} \wr C_{p_{k-1}} \) then taking into consideration that \( cw(B) > 0 \) (because \( C_{p_k} \wr C_{p_{k-1}} \) is not commutative group). Since Lemma 3 implies that \( cw(C_{p_k} \wr C_{p_{k-1}}) = 1 \) then according to the inequality \( cw(C_{p_k} \wr C_{p_{k-1}}) \leq \max(1, cw(B)) \) from Lemma 3 we obtain \( cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) = 1 \). Analogously if \( W = C_{p_k} \ldots \wr C_{p_2} \) and supposition of induction for \( C_{p_k} \ldots \wr C_{p_2} \) holds, then using an associativity of a permutational wreath product we obtain from the inequality of Lemma 3 and the equality \( cw(C_{p_k} \ldots \wr C_{p_2}) = 1 \) that \( cw(W) = 1 \).

We define our partial ordered set \( M \) as the set of all finite wreath products of cyclic groups. We make of use directed set \( \mathbb{N} \).

\[
H_k = \bigwedge_{i=1}^{k} C_{p_i}
\]
Moreover, it has already been proved in Corollary 7 that each group of the form \( \prod_{i=1}^{k} C_{p_i} \) has a commutator width equal to 1, i.e. \( \text{cw}(\prod_{i=1}^{k} C_{p_i}) = 1 \). A partial order relation will be a subgroup relationship. Define the injective homomorphism \( f_{k,k+1} \) from the \( \prod_{i=1}^{k} C_{p_i} \) into \( \prod_{i=1}^{k+1} C_{p_i} \) by mapping a generator of active group \( C_{p_i} \) of \( H_k \) in a generator of active group \( C_{p_i} \) of \( H_{k+1} \). In more details the injective homomorphism \( f_{k,k+1} \) is defined as \( g \mapsto (e, ..., e) \), where a generator \( g \in \prod_{i=1}^{k} C_{p_i} \), \( g(e, ..., e) \in \prod_{i=1}^{k+1} C_{p_i} \).

Therefore this is an injective homomorphism of \( H_k \) onto subgroup \( \prod_{i=1}^{k} C_{p_i} \) of \( H_{k+1} \).

**Corollary 7.** The direct limit \( \lim_{\rightarrow} \prod_{i=1}^{k} C_{p_i} \) of direct system \( \left\langle f_{k,j}, \prod_{i=1}^{k} C_{p_i} \right\rangle \) has commutator width 1.

**Proof.** We make the transition to the direct limit in the directed system \( \left\langle f_{k,j}, \prod_{i=1}^{k} C_{p_i} \right\rangle \) of injective mappings from chain \( e \to ... \to \prod_{i=1}^{k} C_{p_i} \to \prod_{i=1}^{k+1} C_{p_i} \to \prod_{i=1}^{k+2} C_{p_i} \to ... \).

Since all mappings in chains were injective homomorphisms, it have a trivial kernel, so the transition to a direct limit boundary preserves the property \( \rho \) that any element of commutator subgroup is commutator, because in each group \( H_k \) from the chain endowed by \( \rho \).

The direct limit of the direct system is denoted by \( \lim_{\rightarrow} \prod_{i=1}^{k} C_{p_i} \) and is defined as disjoint union of the \( H_k \)‘s modulo a certain equivalence relation:

\[
\lim_{\rightarrow} \prod_{i=1}^{k} C_{p_i} = \bigsqcup_{i=1}^{k} C_{p_i} / \sim.
\]

Since every element \( g \) of \( \lim_{\rightarrow} \prod_{i=1}^{k} C_{p_i} \) coincides with some element from one of the groups \( G_m \) of directed system, then by the injectivity of the mappings for \( g \) the property \( \text{cw}(\prod_{i=1}^{k} C_{p_i}) = 1 \) also holds. Thus, it holds for the whole \( \lim_{\rightarrow} \prod_{i=1}^{k} C_{p_i} \).

**Corollary 8.** Commutator width \( \text{cw}(\text{Syl}_p(S_{p^k})) = 1 \) for prime \( p \) and \( k > 1 \) and commutator width \( \text{cw}(\text{Syl}_p(A_{p^k})) = 1 \) for prime \( p > 2 \) and \( k > 1 \).

**Proof.** Since \( \text{Syl}_p(S_{p^k}) \simeq \prod_{i=1}^{k} C_{p} \) (see [12][14]) then \( \text{cw}(\text{Syl}_p(S_{p^k})) = 1 \). As well known \( \text{Syl}_p S_{p^k} \simeq \text{Syl}_p A_{p^k} \) in case \( p > 2 \) (see [23], then \( \text{cw}(\text{Syl}_p(A_{p^k})) = 1 \) if \( p > 2 \).
The following Lemma gives us a criteria of belonging an element from the group $Syl_2S_2^k$ to $(Syl_2S_2^k)'$.

**Lemma 9.** An element $g \in B_k$ belongs to commutator subgroup $B'_k$ iff $g$ has an even index on $X^l$ for all $0 \leq l < k$. Where $B_k \simeq Syl_2S_2^k$.

**Proof.** Let us prove the necessity by induction by a number of level $l$ and index of $g$ on $X^l$. We first show that our statement for base of the induction is true. Actually, if $\alpha, \beta \in B_0$ then $(\alpha\beta\alpha^{-1})\beta^{-1}$ determines a trivial v.p. on $X^0$. If $\alpha, \beta \in B_1$ and $\beta$ has an odd index on $X^1$, then $(\alpha\beta\alpha^{-1})$ and $\beta^{-1}$ have the same index on $X^1$. Consequently, in this case an index of the product $(\alpha\beta\alpha^{-1})\beta^{-1}$ can be 0 or 2. Case where $\alpha, \beta \in B_1$ and has even index on $X^1$, needs no proof, because the product and the sum of even numbers is an even number.

To finish the proof it suffices to assume that for $B_{l-1}$ statement holds and prove that it holds for $B_l$. Let $\alpha, \beta$ be an arbitrary automorphisms from $AutX^{[k]}$ and $\beta$ has index $x$ on $X^l$, $l < k$, where $0 \leq x \leq 2^l$. A conjugation of an automorphism $\beta$ by arbitrary $\alpha \in AutX^{[k]}$ gives us arbitrary permutations of $X^l$ where $\beta$ has active v.p.

Thus following product $(\alpha\beta\alpha^{-1})\beta^{-1}$ admits all possible even indexes on $X^l, l < k$ from 0 to $2^l$.

Let us present $B_k$ as $B_k = B_l \wr B_{l-1}$, so elements $\alpha, \beta$ can be presented in form of wreath recursion $\alpha = (h_1, ..., h_2^m)\pi_1, \beta = (f_1, ..., f_2^m)\pi_2$, $h_i, f_i \in B_{k-l}$, $0 < i \leq 2^l$ and $h_i, f_j$ corresponds to sections of automorphism in vertices of $X^{l+1}$.

Actually, the parity of this index are formed independently of the action of $AutX^{[l]}$ on $X^l$. So this index forms as a result of multiplying of elements of commutator presented as wreath recursion $(\alpha\beta\alpha^{-1})\beta^{-1} = (h_1, ..., h_2^m)\pi_1.(f_1, ..., f_2^m)\pi_2 = (h_1, ..., h_2^m)(f_{\pi_1(1)}, ..., f_{\pi_1(2^m)})\pi_1\pi_2$, where $h_i, f_j \in B_{k-l}$, $l < k$. Let there are $x$ automorphisms, that have active vertex at the level $X^l$, among $h_i$. Analogous automorphisms $h_i$ has number of active v.p. equal to $x$. As a result of multiplication we have automorphism with index $2i : 0 \leq 2i \leq 2x$ on $X^l$. Consequently, commutator $[\alpha, \beta]$ has an arbitrary even indexes on $X^m$, $m < l$ and we showed by induction that it has even index on $X^l$.

Let us prove this Lemma by induction on level $k$. Let us to suppose that we prove current Lemma (both sufficiency and necessity) for $k - 1$. Then we rewrite element $g \in B_k$ with wreath recursion

$$g = (g_1, g_2)\sigma^i,$$

where $i \in \{0, 1\}$.

Now we consider sufficiency.
Let \( g \in B_k \) and \( g \) has all even indexes on \( X^j \) \( 0 \leq j < k \) we need to show that \( g \in B'_k \).

According to condition of this Lemma \( g_1g_2 \) has even indexes. An element \( g \) has form 
\[
g = (g_1, g_2), \quad \text{where } g_1, g_2 \in B_{k-1},
\]
and products \( g_1g_2 = h \in B'_{k-1} \) because \( h \in B_{k-1} \) and for \( B_{k-1} \) induction assumption holds. Therefore, all products of \( g_1g_2 \) indicated in formula (1) belongs to \( B'_{k-1} \). Hence, from Lemma 1 follows that \( g = (g_1, g_2) \in B'_k \).

An automorphisms group of the subgroup \( C_2^{2^{k-1}-1} \) is based on permutations of copies of \( C_2 \). Orders of \( \prod_{i=1}^{k-1} C_2 \) and \( C_2^{2^{k-1}-1} \) are equals. A homomorphism from \( \prod_{i=1}^{k-1} C_2 \) into \( Aut(C_2^{2^{k-1}-1}) \) is injective because a kernel of action \( \prod_{i=1}^{k-1} C_2 \) on \( C_2^{2^{k-1}-1} \) is trivial, action is effective. The group \( G_k \) is a proper subgroup of index 2 in the group \( \prod_{i=1}^{k-1} C_2 \) \([14][17][23]\).

The following theorem can be used for proving structural property of Sylow subgroups.

**Theorem 10.** A maximal 2-subgroup of \( AutX^{[k]} \) acting by even permutations on \( X^k \) has the structure of the semidirect product \( G_k \simeq B_{k-1} \rtimes W_{k-1} \) and is isomorphic to \( Syl_2A_{2k} \). Also \( G_k \leq B_k \).

This theorem is proven by the author in \([17]\).

**Proposition 11.** An element \((g_1, g_2)\sigma^i, i \in \{0, 1\}\) of wreath power \( \prod_{i=1}^{k-1} C_2 \) belongs to its subgroup \( G_k \), iff \( g_1g_2 \in G_{k-1} \).

**Proof.** This fact follows from the structure of elements of \( G_k \) described Theorem 10 in and the construction of wreath recursion. Indeed, due to the structure of elements of \( G_k \) described in Theorem 10 we have an action on \( X^k \) by an even permutations because the subgroup \( W_{k-1} \), containing an even number of transposition, acts on \( X^k \) only by even permutation. The condition \( g_1g_2 \in G_{k-1} \) on states \( g_1, g_2 \) of automorphism \( g = (g_1, g_2)\sigma^i \) is equivalent to conditions that index of \( g \) on \( X^{k-1} \) is even but this condition equivalent to the condition that \( g \) acting on \( X^k \) by even permutation.

**Lemma 12.** The subgroup \( G_k \) has an even index of any level of \( X^{[k]} \).

According to Lemma 9 and Property 11 a number of active vertices index of any level of \( X^{[k]} \) is even. Therefore, equality
\[
\prod_{j=1}^{2^{i-1}} g_{ij} = \prod_{j=2^{i-1}+1}^{2^i} g_{ij} \tag{7}
\]
is true. In case \( i = k - 1 \) there are the even number of transpositions on each of last levels of \( v_{11}X^{k-2} \) and of \( v_{12}X^{k-2} \). Therefore the following condition

\[
\prod_{j=1}^{2^{k-2}} g_{k-1j} = \prod_{j=2^{k-2}+1}^{2^k-1} g_{k-1j} = e
\]

holds.

**Corollary 13.** The rooted permutation in \( v_0 \) of any automorphism from \( G_k' \) is always trivial also \( g_{11} = g_{12} \).

**Proof.** The proof is immediate follows from Lemma 12 in particular \( g_{11} = g_{12} \) immediately follows from the equality (7). The triviality of v.p. in \( v_0 \) implies from the fact of level order permutation parity in commutant which implies from parity of level index. \( \square \)

Recall that a subgroup \( A \) of direct product is called subdirect product of groups \( H_i \) if projection of \( A \) on any subgroup \( H_i \) coincides with the subgroup \( H_i \). We denote by \( \perp \) operation of even subdirect product which admits all possible tuples of active vertices from multipliers of form \( G_{k-1} \) with condition of parity of transpostions number in the product \( G_{k-1} \perp G_{k-1} \) on any level \( X^l, l > k - 1 \).

As it was said above the subgroup \( G_{k-1} \) is isomorphic copy of \( Syl_2A_{2k-1} \) in the group \( AutX^{k-1} \). 

**Theorem 14.** The group \( G_k' \) has a structure \( G_k' \simeq G_{k-1} \perp G_{k-1} \).

**Proof.** Relation (7) from Lemma 12 implies the parity of the level index \( Inv(g) \). Conversely the operation \( \perp \) admits all possible tuples of transpositions from multipliers of form \( G_{k-1} \) with condition of parity of transpostions number in the product \( G_{k-1} \perp G_{k-1} \) on any level \( X^l, l > k - 1 \). Since parity of transpositions in both factors is the same and so condition (7) is satisfied. \( \square \)

Let us denote by \( s_{ij} \) state of automorphism \( G'' \) in vertex \( v_{ij} \). Recall that any automorphism of \( g \in AutX^{[n]} \) can be uniquely represented as \( g = (g_x, x \in X) \pi \). We find the commutator \([g, h] \) in the form of the wreath recursion \((g_{21}, g_{22}, g_{23}, g_{24})\pi \), where \( \pi \) is a rooted permutation of first level states.

The portraits of automorphisms \( \theta \) on level \( X^{k-1} \) can be characterized by the sequence \( (s_1, s_2, \ldots, s_{2^{k-1}}) \), \( s_i \in 0, 1 \). The set of vertices of \( X^{k-1} \) can be disjoint into subsets \( X_1 \) and \( X_2 \). Let \( X_1 = \{v_{k-1,1}, v_{k-1,2}, \ldots, v_{k-1,2^{k-2}}\} \) and \( X_2 = \{v_{k-1,2^{k-2}+1}, \ldots, v_{k-1,2^{k-1}}\} \).

We call a distance structure \( \rho_1(\theta) \) of \( \theta \) a tuple of distances between its active vertices from \( X^l \). Let group \( Syl_2A_{2k} \) acts on \( X^{[k]} \).
Corollary 15. If $g, h \in G'$ then states of $s = [g, h] \in G'' |_{X^3}$ in vertices of $X^2$ comply with the equalities

$$s_{21} = s_{22}, \quad s_{23} = s_{24}.$$  

Proof. Since if $\alpha$ is a vertex permutation of $g$ an $v_{2j}$, then we do not distinguish $\alpha$ from the section $g_{2j}$ defined by it, i.e., we can write $g_{2j} = \alpha$. Note that permutations of $g \in G''$ in vertices of $X^0$ and $X^1$ are trivial, conversely $g_{21}$ is coincide with v.p. in $v_{21}$. The proof is based on Lemma 12. Taking in the consideration that $G_{k'} \simeq H_{k-1} \sqcup H_{k-1}$ it is enough to prove that $s_{21} = s_{22}$ the prove of $s_{23} = s_{24}$ is analogous. Therefore we consider projection $g_1, h_1$ of $g, h \in G'$ on $H_{k-1}$ such that $g_1 \in H_{k-1} \triangleleft Autv_{11}X^{k-1}, h_1 \in H_{k-1} \triangleleft Autv_{11}X^{k-1}$. Thus it can be presented in form $g_1 = (g_{21}, g_{22})\sigma, h_1 = (h_{21}, h_{22})\pi \in G'$, where $\sigma, \pi \in S_2$. We find the commutator $[g_1, h_1]$ in form of wreath recursion $(s_{21}, s_{22})\pi_1, g = (g_{21}, g_{22}, g_{23}, g_{24})\sigma = ((g_{21}, g_{22})\sigma_1, (g_{23}, g_{24})\sigma_2),$ where $g_{2i}, 1 \leq i \leq 4$ are states of second level, $\sigma$ are rooted permutation of states of first level, because v.p. above are trivial, which permutes vertices of second level with its subtrees and $\sigma_1, \sigma_2$ rooted permutations of subtrees with roots in $v_{21}, v_{22}$ and $v_{23}, v_{24}$. Therefore $\sigma = (\sigma_1, \sigma_2)$. We shall consider $g = (g_{21}, g_{22})\sigma, h = (h_{21}, h_{22})\pi \in G'$, where

$$\sigma = (1, 2), \quad \pi = e.$$  

Therefore $g \in H_{k-1} \triangleleft Autv_{11}X^{k-1}, h \in H_{k-1} \triangleleft Autv_{11}X^{k-1}$. This case is possible because we take 2 different elements $g, h \in G'$. Then we have $[(g_{21}, g_{22})\sigma, (h_{21}, h_{22})\pi] = (g_{21}h_{22}g_{21}h_{21}, g_{22}h_{21}g_{22}h_{22})$ we conclude that it is sufficient to prove equality $s_{21} = s_{22}$. The prove that $s_{23} = s_{24}$ is the same. Elements from first factor $H_{k-1}$ of $G_{k'}$ has form $g = (g_{21}, g_{22})\sigma, h = (h_{21}, h_{22})\pi \in G_{k-1} \triangleleft Autv_{12}X^{k-1}$. To find $[g, h]$ in rest of cases besides $\sigma = (1, 2), \quad \pi = e$, we have to shortly consider the feasible cases:

1. $\sigma = \widetilde{g}_{11} = e, \quad \pi = \widetilde{g}_{12} = e,$

Therefore we obtain a commutator $[(g_{21}, g_{22})\sigma, (h_{21}, h_{22})\pi] = (g_{21}h_{21}g_{21}h_{21}, g_{22}h_{22}g_{22}h_{22}).$

2. $\sigma = \pi = \widetilde{g}_{11} = \widetilde{g}_{12} = (1, 2),$

then $[(g_{21}, g_{22})\sigma, (h_{21}, h_{22})\pi] = (g_{21}h_{22}g_{22}h_{21}, g_{22}h_{21}g_{21}h_{22}).$ In view of the fact of commutativity of $g_{21}, g_{22}, g_{23}, g_{24} \in S_2$ we have

$$g_{21}h_{22}g_{22}h_{21} = g_{22}h_{21}g_{21}h_{22},$$

viz $s_{21} = s_{22}$.

3. Case

$$\sigma = e, \quad \pi = (1, 2).$$
Then we have

\[ [(g_{21}, g_{22})\sigma, (h_{21}, h_{22})\pi] = (g_{21} h_{21} g_{22} h_{21}, g_{22} h_{22} g_{21} h_{22}). \]

Consider states of this commutator

\[ g_{21} h_{21} g_{22} h_{21} = g_{21} g_{22} h_{21} h_{21} = g_{21} g_{22} = g_{22} g_{21} h_{22} h_{22} = g_{22} h_{22} g_{21} h_{22}. \]

Thus, we see that the states are equal again, i.e. \( s_{21} = s_{22} \).

4. And recall the case \( \sigma = (1, 2), \pi = e \)

is completely analogous to case 3 because

\[ [(g_{21}, g_{22})\sigma, (h_{21}, h_{22})\pi] = (g_{21} h_{22} g_{21} h_{21}, g_{22} h_{21} g_{22} h_{22}). \]

thus \( s_{21} = s_{22} \).

Consider the product of commutators \([g, h] \cdot [g_1, h_1] \) it already proven that \( In_2(g), In_2(g_1), In_2(h), In_2(h_1) \) are even its states satisfy \( s_{21} = s_{22}, s_{23} = s_{24} \). Recall that elements of \( G'' \) have form \((s_{21}, s_{22})\pi_1, (s_{23}, s_{24})\pi_2 \), where \( \pi_1, \pi_2 \) are trivial then elements \([g, h], [g_1, h_1] \) multiply directly. Consequently, the equality of states of product \([g, h] \cdot [g_1, h_1] \) holds.

\[ \square \]

**Lemma 16.** An element \( g \) belongs to \( G'_k \simeq Syl_2 A_{2k} \) iff \( g \) is an arbitrary element from \( G_k \) which has all even indexes on \( X^l, l < k-1 \) of \( X^{[k]} \) and on \( X^{k-2} \) of subtrees \( v_{11} X^{[k-1]} \) and \( v_{12} X^{[k-1]} \).

**Proof.** Let us prove the ampleness by induction on a number of level \( l \) and index of automorphism \( g \) on \( X^l \). Conjugation by automorphism \( \alpha = \alpha_0 \) from \( Aut v_{11} X^{[k-1]} \) of automorphism \( \theta \), that has index \( x : 1 \leq x \leq 2^{k-2} \) on \( X_1 \) does not change \( x \). Also automorphism \( \theta^{-1} \) has the same number \( x \) of v. p. on \( X_{k-1} \) as \( \theta \) has. If \( \alpha \) from \( Aut v_{11} X^{[k-1]} \) and \( \alpha \notin v_{12} Aut X^{[k]} \) then conjugation \((\alpha \theta \alpha^{-1})\) permutes on \( X^{k-1} \) vertices which of \( X_1 \).

Thus, \( \alpha \theta \alpha^{-1} \) and \( \theta \) have the same parities of number of active v.p. on \( X_1 (X_2) \). Hence, a product \( \alpha \theta \alpha^{-1} \theta^{-1} \) has an even number of active v.p. on \( X_1 (X_2) \) in this case. More over a coordinate-wise sum by \( \mod 2 \) of active v. p. from \((\alpha \theta \alpha^{-1}) \) and \( \theta^{-1} \) on \( X_1 (X_2) \) is even and equal to \( y : 0 \leq y \leq 2x \).

If conjugation by \( \alpha \) permutes sets \( X_1 \) and \( X_2 \) then there are coordinate-wise sums of no trivial v.p. from \( \alpha \theta \alpha^{-1} \theta^{-1} \) on \( X_1 \) (analogously on \( X_2 \)) have form:
\[(s_{k-1,1}(\alpha \theta \alpha^{-1}), \ldots, s_{k-1,2^{k-2}}(\alpha \theta \alpha^{-1})) \oplus (s_{k-1,1}(\theta^{-1}), \ldots, s_{k-1,2^{k-2}}(\theta^{-1}))\]. This sum has even number of v.p. on \(X_1\) and \(X_2\) because \((\alpha \theta \alpha^{-1})\) and \(\theta^{-1}\) have a same parity of no trivial v.p. on \(X_1\) \((X_2)\). Hence, \((\alpha \theta \alpha^{-1})\theta^{-1}\) has even number of v.p. on \(X_1\) as well as on \(X_2\).

An automorphism \(\theta\) from \(G_k\) was arbitrary so number of active v.p. \(x\) on \(X_1\) is an arbitrary \(0 \leq x \leq 2^l\). And \(\alpha\) is an arbitrary from \(\text{Aut}X^{[k-1]}\) so vertices can be permuted in such way that the commutator \([\alpha, \theta]\) has arbitrary even number \(y\) of active v.p. on \(X_1\), \(0 \leq y \leq 2^l\).

A conjugation of an automorphism \(\theta\) having index \(x, 1 \leq x \leq 2^l\) on \(X^l\) by different \(\alpha \in \text{Aut}X^{[k]}\) gives us all tuples of active v.p. with the same \(\rho_l(\theta)\) that \(\theta\) has on \(X^l\), by which \(\text{Aut}X^{[k]}\) acts on \(X^l\). Let supposition of induction for element \(g\) with index \(2k - 2\) on \(X^l\) holds so \(g = (\alpha \theta \alpha^{-1})\theta^{-1}\), where \(In_l(\theta) = x\). To make a induction step we complete \(\theta\) by such vertex permutation in \(v_{l,x}\) too \(\theta\) has suitable distance structure for \(g = (\alpha \theta \alpha^{-1})\theta^{-1}\), also if \(g\) has rather different distance structure \(\rho_l(g)\) from \(\rho_l(\theta)\) then have to change \(\theta\). In case when we complete \(\theta\) by \(v_{l,x}\) it has too satisfy a condition \((\alpha \theta \alpha^{-1})(v_{x+1}) = v_{l,y}\), where \(v_{l,y}\) is a new active vertex of \(g\) on \(X^l\). Note that \(v(x + 1)\) always can be chosen such that acts in such way \(\alpha(v(x + 1)) = v(2k + 2)\) because action of \(\alpha\) is 1-transitive. Second vertex arise when we multiply \((\alpha \theta \alpha^{-1})\) on \(\theta^{-1}\). Hence \(In_l(\alpha \theta \alpha^{-1}) = 2k + 2\) and coordinates of new vertices \(v_{2k+1}, v_{2k+2}\) are arbitrary from 1 to \(2^l\).

So multiplication \((\alpha \theta \alpha^{-1})\theta\) generates a commutator having index \(y\) equal to coordinate-wise sum by mod2 of no trivial v.p. from vectors \((s_{l1}(\alpha \theta \alpha^{-1}), s_{l2}(\alpha \theta \alpha^{-1}), \ldots, s_{l2^l}(\alpha \theta \alpha^{-1})) \oplus (s_{l1}(\theta), s_{l2}(\theta), \ldots, s_{l2^l}(\theta))\) on \(X^l\). A indexes parities of \(\alpha \theta \alpha^{-1}\) and \(\theta^{-1}\) are same so their sum by mod2 are even. Choosing \(\theta\) we can choose an arbitrary index \(x\) of \(\theta\) also we can choose arbitrary \(\alpha\) to make a permutation of active v.p. on \(X^l\). Thus, we obtain an element with arbitrary even index on \(X^l\) and arbitrary location of active v.p. on \(X^l\).

Check that property of number parity of v.p. on \(X_1\) and on \(X_2\) is closed with respect to conjugation. We know that numbers of active v.p. on \(X_1\) as well as on \(X_2\) have the same parities. So action by conjugation only can permutes it, hence, we again get the same structure of element. Conjugation by automorphism \(\alpha\) from \(\text{Aut}v_{11}X^{[k-1]}\) automorphism \(\theta\), that has odd number of active v.p. on \(X_1\) does not change its parity. Choosing the \(\theta\) we can choose arbitrary index \(x\) of \(\theta\) on \(X^{k-1}\) and number of active v.p. on \(X_1\) and \(X_2\) also we can choose arbitrary \(\alpha\) to make a permutation active v.p. on \(X_1\) and \(X_2\). Thus, we can generate all possible elements from a commutant. Also this result can be deduced due to Lemma \([\text{?}]\).

Let \(\kappa_1, \kappa_2 \in K\) and each of which has even index on \(X^l\) and \(2^l\)-tuples of v.p.
(s_{l,1}(\kappa_1), ..., s_{k-1,2}(\kappa_1)), (s_{l,\kappa_1(1)}(\kappa_2), ..., s_{l,\kappa_1(2)}(\kappa_2))$ corresponds to portrait of $\kappa_1, \kappa_2$ on $X_l$. Then a number of non-trivial coordinates in a coordinate-wise sum 
$(s_{l,1}(\kappa_1), ..., s_{k-1,2}(\kappa_1)) \oplus (s_{l,\kappa_1(1)}(\kappa_2), ..., s_{l,\kappa_1(2)}(\kappa_2))$ is even.

Let us check that the set of all commutators $K$ from $Syl_2A_{2k}$ is closed with respect to multiplication of commutators. Note that conjugation of $\kappa$ can permute sets $X_1$ and $X_2$ so parities of $x_1$ and $X_2$ coincide. It is obviously that the parity of index of $\alpha\kappa\alpha^{-1}$ is the same as index of $\kappa$.

Check that a set $K$ is a set closed with respect to conjugation.

Let $\kappa \in K$, then $\alpha\kappa\alpha^{-1}$ also belongs to $K$, it is so because conjugation does not change index of an automorphism on a level. Conjugation only permutes vertices on a level because elements of $AutX^{[l-1]}$ acts on vertices of $X^l$. But as it was proved above elements of $K$ have all possible indexes on $X^l$, so as a result of conjugation $\alpha\kappa\alpha^{-1}$ we obtain an element from $K$.

Check that the set of commutators is closed with respect to multiplication of commutators. Let $\kappa_1, \kappa_2$ be an arbitrary commutators of $G_k$. The parity of the number of vertex permutations on $X^l$ in the product $\kappa_1\kappa_2$ is determined exceptionally by the parity of the numbers of active v.p. on $X^l$ in $\kappa_1$ and $\kappa_2$ (independently from the action of v.p. from the higher levels). Thus $\kappa_1\kappa_2$ has an even index on $X^l$.

Hence, a normal closure of the set $K$ coincides with $K$. It means that commutator subgroup of $Syl_2A_{2k}$ consists of commutators.

Proposition 17. An element $(g_1, g_2)\sigma^i \in G'_k$ iff $g_1, g_2 \in G_{k-1}$ and $g_1g_2 \in B'_{k-1}$.

Proof. Since, if $(g_1, g_2) \in G'_k$ then indexes of $g_1$ and $g_2$ on $X^{k-1}$ are even according to Lemma 16, thus, $g_1, g_2 \in G_{k-1}$. A sum of indexes of $g_1$ and $g_2$ on $X^l$, $l < k - 1$ are even according to Lemma 18 too, so index of product $g_1g_2$ on $X^l$ is even. Thus, $g_1g_2 \in B'_{k-1}$.

Hence, necessity is proved.

Let us prove the sufficiency via Lemma 16. Wise versa, if $g_1, g_2 \in G_{k-1}$ then indexes of these automorphisms on $X^{k-2}$ of subtrees $v_1X^{[k-1]}$ and $v_2X^{[k-1]}$ are even as elements from $G'_k$ have. The product $g_1g_2$ belongs to $B'_{k-1}$ by condition of this Lemma and so sum of indexes of $g_1, g_2$ on any level $X^l$, $0 \leq l < k - 1$ is even. Thus, the characteristic property of $G'_k$ described in Lemma 16 holds.

Lemma 18. An element $g = (g_1, g_2)\sigma^i$ of $G_k$, $i \in \{0, 1\}$ belongs to $G'_k$ iff $g$ has even index on $X^l$ for all $l < k - 1$ and elements $g_1, g_2$ have even indexes on $X^{k-1}$, that is equally matched to $g_1, g_2 \in G_{k-1}$.

Proof. The proof immediately follows from Lemma 16 and Proposition 17.
Recall the Lemma on the commutators structure for an embedded commutator [?].

Lemma 19. Suppose $G$ is a group with a subgroup $H$ such that $H \triangleright G'$ and $G = \langle H, x \rangle$. If $w$ is a commutator in $G$, then $w = [axe, b]$ for some $a, b \in H$ and $e \in Z$.

If we assume that $B_k = G$, $H = G_k$ and since $G_k \supseteq B'_k$, then according to Lemma [19] any element $w \in B'_k$ can be presented as commutator of an element from $B_k$ and an extending element. In our case as an extending element for maximal subgroup $H = G_k$ to $B_k$ we could take the generator $\alpha_{k-1}$ [17]. Thus, any element $w \in B'_k$ can be presented as $w = [axe, b]$. Where $x$ is an extending element for $G_k$ to $B_k$. Also as an extending element it can be chosen an arbitrary element with an odd index on $k-1$ level and zero indexes on rest of levels.

Proposition 20. The following inclusion $B'_k < G_k$ holds.

Proof. Since $B'_k = \langle \prod_{i=1}^{k-1} C_2 = B_{k-1} \rangle$ and $G_k \simeq B_{k-1} \rtimes W_{k-1}$ we have $B'_k < G_k$. \hfill $\square$

Proposition 21. The group $G_k$ is normal in wreath product $\prod_{i=1}^{k} C_2$ i.e. $G_k \triangleleft B_k$.

Proof. The commutator of $B_k$ is $B'_k < B_{k-1}$. In other hand $B_{k-1} < G_k$ because $G_k \simeq B_{k-1} \rtimes W_{k-1}$ consequently $B'_k < G_k$. Thus, $G_k \triangleleft B_k$. \hfill $\square$

There exists a normal embedding (normal injective monomorphism) $\varphi : G_k \to B_k$ [25] i.e. $G_k \triangleleft B_k$. Actually, it implies from Proposition 21. Also according to [17] the index $|B_k : G_k| = 2$ so $G_k$ is a normal subgroup that is a factor subgroup $B_k/C_2 \simeq G_k$.

Theorem 22. Elements of $B'_k$ have the following form $B'_k = \{[f, l] \mid f \in B_k, l \in G_k\} = \{[l, f] \mid f \in B_k, l \in G_k\}$.

Proof. It is enough to show either $B'_k = \{[f, l] \mid f \in B_k, l \in G_k\}$ or $B'_k = \{[l, f] \mid f \in B_k, l \in G_k\}$ because if $f = [g, h]$ then $f^{-1} = [h, g]$.

We prove this Theorem by induction on $k$. Since $B'_1 = \langle e \rangle$ then base of induction is verified.

Due Lemma 1 we already know that every element $w \in B'_k$ can be represent as $w = (r_1, r_1^{-1}[f, g])$

for some $r_1, f \in B_{k-1}$ and $g \in G_{k-1}$ (by induction hypothesis). By the Corollary 4 we can represent $w$ as commutator of $(e, a_{1,2}) \sigma \in B_k$ and $(a_{2,1}, a_{2,2}) \in B_k$, 18
where
\[
\begin{align*}
a_{2,1} &= (f^{-1})^r_1, \\
a_{2,2} &= r_1 a_{2,1}, \\
a_{1,2} &= g a_2.
\end{align*}
\]

We note that \( g \in G_{k-1} \) then by Proposition 11 we obtain \((e, a_{1,2}) \sigma \in G_k\). \( \square \)

Directly from this Proposition follows next Remark, that needs no proof.

**Remark 23.** Let us to note that Theorem 22 improve Corollary 8 for the case \( p = 2 \).

**Proposition 24.** If \( g \) is an element of wreath power \( \wr_{i=1}^k C_2 \cong B_k \) then \( g^2 \in B'_k \).

**Proof.** As it was proved in Lemma 9 commutator \([\alpha, \beta]\) from \( B_k \) has arbitrary even indexes on \( X^m, m < k \). Let us show that elements of \( B'_{2k} \) have the same structure.

Let \( \alpha, \beta \in B_k \) an indexes of the automorphisms \( \alpha^2, (\alpha\beta)^2 \) on \( X^l, l < k - 1 \) are always even. In more detail the indexes of \( \alpha^2, (\alpha\beta)^2 \) and \( \alpha^{-2} \) on \( X^l \) are determined exceptionally by the parity of indexes of \( \alpha \) and \( \beta \) on \( X^l \). Actually, the parity of this index are formed independently of the action of \( \text{Aut}X^l \) on \( X^l \). So this index forms as a result of multiplying of elements \( \alpha \in B_k \) presented as wreath recursion \( \alpha^2 = (h_1, ..., h_{2l}) \pi_1 \cdot (h_1, ..., h_{2l}) \pi_1 = (h_1, ..., h_{2l}) (h_{\pi_1(1)}, ..., h_{\pi_1(2l)}) \pi^2_1 \), where \( h_i, h_j \in B_{k-l}, \pi_1 \in B_l, l < k \). Let there are \( x \) automorphisms, that have an active vertex at level \( X^l \), among \( h_i \). Analogous automorphisms \( h_i \) has number of active v.p. equal to \( x \). As a result of multiplication we have automorphism with index \( 2i : 0 \leq 2i \leq 2x \).

Since \( g^2 \) admits only an even index on \( X^l \) of \( \text{Aut}X[k] \), \( 0 < l < k \), then \( g^2 \in B'_k \) according to Lemma 9. \( \square \)

Since as well known a group \( G^2_k \) contains the subgroup \( G' \) then a product \( G^2 G' \) contains all elements from the commutant. Therefore, we obtain that \( G^2_k \cong G'_k \).

**Proposition 25.** For arbitrary \( g \in G_k \) following inclusion \( g^2 \in G'_k \) holds.

**Proof.** We make the prove by the induction on positive integer \( k \). Elements of \( G^2_1 \) have form \(((e, e) \sigma)^2 = e\), where \( \sigma = (1, 2) \) so statement holds. In general case when \( k > 1 \) elements of \( G_k \) have the following form
\[
g = (g_1, g_2) \sigma^i, \quad g_1, g_2 \in B_{k-1}, \quad i \in \{0, 1\}.
\]

Hence we have two possibilities
\[
g^2 = (g_1^2, g_2^2) \quad \text{or} \quad g^2 = (g_1 g_2, g_2 g_1).
\]

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We first show that

\[ g_1^2 \in B'_{k-1}, g_2^2 \in B'_{k-1} \]

after we will prove

\[ g_1 g_2 \cdot g_2 g_1 \in B'_{k-1}, \]

actually, according to Proposition 14, \( g_1^2, g_2^2 \in B'_{k-1} \) which implies \( g_1^2 g_2^2 \in B'_{k-1} \) and \( g_1^2, g_2^2 \in G_{k-1} \) by Proposition 20 also \( g_1^2, g_2^2 \in G_{k-1} \) by induction assumption. From Proposition 11 it follows that \( g_1 g_2 \in G_{k-1} \).

Note that \( B'_{k-1} < B_{k-2} \). In other hand \( B_{k-2} < G_{k-1} \) because \( G_{k-1} \simeq B_{k-2} \times W_{k-2} \) consequently \( B'_{k-1} < G_{k-1} \). Besides we have \( g_1^2 \in B'_{k-1} \) hence \( g_1^2 \in G_{k-1} \).

Thus, we can use Proposition 17 (about \( G'_k \)) from which yields \( g^2 = (g_1^2, g_2^2) \in G'_k \).

Consider the second case \( g^2 = (g_1 g_2, g_2 g_1) \), then \( g_1 g_2 \in G_{k-1} \) by proposition 11. Also second coordinate \( g_2 g_1 = g_1 g_2 g_2^{-1} g_1^{-1} g_2 g_1 = g_1 g_2 [g_2^{-1}, g_1^{-1}] \in G_{k-1} \) by Propositions 20 and 11. Analyze the coordinate product of \( g^2 = (g_1 g_2, g_2 g_1) \):

\[ g_1 g_2 \cdot g_2 g_1 = g_1 g_2^2 g_1 = g_1^2 [g_2^{-2}, g_1^{-1}]. \]

According to Proposition 21 \( g_1^2, g_2^2 \in B'_{k-1} \) so we obtain \( g_1^2 [g_2^{-2}, g_1^{-1}] \in B'_{k-1} \). Thus, \( (g_1 g_2, g_2 g_1) \in G'_k \) by Proposition 17.

\[ \square \]

Using this structural property of \( (Syl_2 A_{2k})' \) we deduce a following result.

**Theorem 26.** The commutator subgroup \( G'_k \) coincides with set of all commutators, other words \( G'_k = \{ [f_1, f_2] \mid f_1, f_2 \in G_k \} \).

**Proof.** For the case \( k = 1 \) we have \( G'_1 = (e) \), if \( k = 2 \) then \( (Syl_2 A_{2k})' \simeq G'_2 \simeq K'_4(e) \). So, further we consider case \( k \geq 2 \). In order to prove this Theorem we fix an arbitrary element \( w \in G'_k \) and then we represent this element as commutator of elements from \( G_k \).

We already know by Lemma 2 that every element \( w \in G'_k \) we can represent as follow

\[ w = (r_1, r_1^{-1} x), \]

where \( r_1 \in G_{k-1} \) and \( x \in B'_{k-1} \).

By proposition 22 we have \( x = [f, g] \) for some \( f \in B_{k-1} \) and \( g \in G_{k-1} \). Therefore,

\[ w = (r_1, r_1^{-1} [f, g]). \]
By the Corollary 3 we can represent $w$ as commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where $a_{2,1} = (f^{-1})r_1^{-1}, a_{2,2} = r_1 a_{2,1}, a_{1,2} = g^{a_{2,2}^{-1}}.$

Let us consider the commutator of this elements

$$\kappa = (e, a_{1,2})\sigma(a_{2,1}, a_{2,2})(a_{1,2}^{-1}, e)\sigma^{-1}(a_{2,1}^{-1}, a_{2,2}^{-1}) = (a_{1,1}a_{2,2}a_{1,1}^{-1}a_{2,2}^{-1}, a_{1,2}a_{2,1}a_{1,2}^{-1}a_{2,2}^{-1})$$

$$= (a_{2,2}a_{2,1}^{-1}, r_1^{-1}[(a_{2,1}^{-1})r_1^{-1}, (a_{1,2})]).$$

Since we transform commutator $\kappa$ in form which is similar to form for $w$. This implies the equations for elements of $\kappa$: $r_1 = a_{2,2}a_{2,1}^{-1}, f = (a_{2,1}^{-1})r_1, g = (a_{1,2})^{a_{2,2}}$.

Let us make sure that this commutator is arbitrary element of form $w = (r_1, r_1^{-1}x)$. For this goal it is only left to show that $(e, a_{1,2})\sigma, (a_{2,1,2,2}) \in G_k$. Since $x = [f, g]$ then we have a correspondence $[f, g]$ to $(a_{2,2}a_{2,1}^{-1}, r_1^{-1}[(a_{2,1}^{-1})r_1^{-1}, (a_{1,2})]).$ As a result we obtain:

$$a_{2,1} = (f^{-1})r_1^{-1},$$
$$a_{2,2} = r_1 a_{2,1},$$
$$a_{1,2} = g^{a_{2,2}^{-1}} \in G_{k-1}, \text{ by Proposition 21 } G_{k-1} \triangleleft B_{k-1} \text{ so } g^{a_{2,2}^{-1}} \in G_{k-1},$$
$$a_{1,2}^{a_{2,2}} \in G_{k-1},$$
$$a_{2,2}a_{2,1} = r_1 a_{2,1}^{2} \in G_{k-1},$$
$$a_{2,1}a_{2,2} = a_2, r_1 a_{2,1} = r_1[r_1, a_{2,1}]a_{2,1}^{-1} \in G_{k-1} \text{ because } [r_1, a_{2,1}] \in B_{k-1}' \text{ by Theorem 22.}$$

In order to use Proposition 11 we note that $a_{1,2} = g^{a_{2,2}^{-1}} \in G_{k-1}$ by Proposition 21. Also $a_{2,1}a_{2,2} = a_2 r_1 a_{2,1} = r_1[r_1, a_{2,1}]a_{2,1}^{-1} \in G_{k-1}$ by Proposition 20 and Proposition 23.

So we have $(e, a_{1,2})\sigma \in G_k$ and $(a_{2,1}, a_{2,2}) \in G_k.$ \hfill $\square$

Thus, as it was stated by us in abstract 21 we obtain the following result.

Corollary 27. Commutator width of the group $\text{Syl}_2 A_8$ equal to 1 for $k \geq 2$.

Example 28. A commutator of $\text{Syl}_2(A_8)$ consist of elements: $\{e, (13)(24)(57)(68), (12)(34), (14)(23)(57)(68), (56)(78), (13)(24)(58)(67), (12)(34)(56)(78), (14)(23)(58)(67)\}$. The commutator $\text{Syl}_2(A_8) \cong C_2^3$ that is an elementary abelian $2$-group of order 8.

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4 Conclusion

The commutator width of the wreath product $C_p \wr B$ was founded. The commutator width of Sylow 2-subgroups of alternating group $A_{2k}$, permutation group $S_{2k}$ and Sylow $p$-subgroups of $Syl_2 A_p^k (Syl_2 S_p^k)$ is equal to 1. Commutator width of permutational wreath product $B \wr C_n$, were $B$ is an arbitrary group, was researched.

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