Article

A New Parametric Life Family of Distributions: Properties, Copula and Modeling Failure and Service Times

Mansour Shrahili 1,* and Naif Alotaibi 2

1 Department of Statistics and Operations Research, King Saud University, Riyadh 11451, Saudi Arabia
2 Department of Mathematics and Statistics, College of Science, Imam Mohammad ibn Saud Islamic University (IMSIU), P. Box 90950, Riyadh 11623, Saudi Arabia; nmaalotaibi@imamu.edu.sa
* Correspondence: msharahili@ksu.edu.sa

Received: 11 August 2020; Accepted: 2 September 2020; Published: 5 September 2020

Abstract: A new family of probability distributions is defined and applied for modeling symmetric real-life datasets. Some new bivariate type G families using Farlie–Gumbel–Morgenstern copula, modified Farlie–Gumbel–Morgenstern copula, Clayton copula and Renyi’s entropy copula are derived. Moreover, some of its statistical properties are presented and studied. Next, the maximum likelihood estimation method is used. A graphical assessment based on biases and mean squared errors is introduced. Based on this assessment, the maximum likelihood method performs well and can be used for estimating the model parameters. Finally, two symmetric real-life applications to illustrate the importance and flexibility of the new family are proposed. The symmetricity of the real data is proved nonparametrically using the kernel density estimation method.

Keywords: Kumaraswamy family; Burr X family; Farlie–Gumbel–Morgenstern; Clayton copula; simulation; symmetric real data; kernel density estimation

1. Introduction and Genesis

Statistical probability distributions are an important tool in modeling the characteristics of real-life datasets such as “symmetric” or “right” or “left” skewness, “symmetric/asymmetric bi-modality” or “multi-modality” in different applied sciences such as reliability, medicine, engineering and finance, among others. The well-known distributions such as Burr X, normal, Burr XII, Weibull, beta, gamma, Kumaraswamy and Lindley are widely used because of their simple forms and identifiability of their properties. However, in the last few decades, statisticians have focused on the more complex and flexible distributions to increase the modeling ability of these models via adding one or more shape parameters. The well-known family of distributions can be cited as follows: the Marshall–Olkin-G (MO-G) family [1], the beta-G (B-G) family [2], the Kumaraswamy-G (K-G) family [3], the Burr X-G (BX-G) family [4], the Burr XII-G (BXII-G) family [5] and a new Weibull class of distributions [6], among others.

Consider a baseline cumulative distribution function (CDF) \( G_\delta (x) \) and probability density function (PDF) \( g_\delta (x) \) with a parameter vector \( \delta = (\delta_1, \delta_2, \ldots) \). Then due to [4], the CDF BX-G family of distributions is defined by

\[
H_{\lambda, \delta}(x) = \left\{ 1 - \exp \left[ -O^2_{\delta}(x) \right] \right\}^\lambda \quad \text{for } x \in \mathbb{R}^+ \text{ and } \lambda \in \mathbb{R}^+
\]

(1)

where \( O^2_{\delta}(x) = \left( \frac{G_\delta (x)}{1-G_\delta (x)} \right)^2 \). Note that the BX-G family of distributions can be called as the exponentiated Rayleigh-G (ER-G) family of distributions. For \( \lambda = 1 \), the BX-G family of distributions reduces to the
Rayleigh-G (R-G) family. In this paper, we define and study a new family of stochastic models by adding two extra shape parameters in (1) to provide more flexibility to the newly generated family. Based on the Kumaraswamy-G (Kw-G) family pioneered by [3], we define a new G family so called the Kumaraswamy Burr X-G (KBX-G) family and give a comprehensive description of some of its mathematical properties. We hope that this model will attract wider applications in engineering, insurance, reliability and other areas of research. For an arbitrary baseline CDF \( G_\delta(x) \), [3] defined the K-G family by the CDF given by

\[
F_{\alpha,\theta,\delta}(x) = 1 - \left[ 1 - G_\delta(x) \right]^\theta.
\]

In this paper, we define and study a new family of probability distributions based on the Kumaraswamy and Burr X families. The new family is applied to failure and service times of real datasets which are symmetric real data. The symmetricity of the real data is proved nonparametrically using the kernel density estimation (KDE) method. For checking the data “normality”, the quantile–quantile (Q–Q) plot is sketched. On the other hand, concentrating on the Lomax model (as a base line model), the new family contains the “symmetric density” with many useful applied shapes. In applied section, the probability–probability (P–P) plot is presented.

The paper concentrates on the univariate version of the new family. On the other hand, some new bivariate type G families using “copula of Farlie–Gumbel–Morgenstern (FGM)”, modified copula of “Farlie–Gumbel–Morgenstern”, “the well-known Clayton copula” and “copula of Renyi’s entropy” are derived. The authors of [7] gave two major justifications as to why presenting new copulas are of interest to many statisticians. Following [3], the CDF of the KBX-G family can be expressed as

\[
F_{a,\theta,\lambda,\delta}(x) = 1 - \left[ 1 - H_{a,\lambda,\delta}(x) \right]^\theta |a,\theta,\lambda,\delta|_{R^+} \text{ and } x \in R.
\]

For \( \theta = 1 \), the KBX-G reduces to the exponentiated Burr type X G (ExpBX-G) family. For \( \lambda = 1 \), the KBX-G reduces to the Kumaraswamy Rayleigh-G (KR-G) family. The PDF corresponding to (2) can be derived as

\[
f_{a,\theta,\lambda,\delta}(x) = \frac{2a\theta\lambda G_\delta(x)G_\delta(x)\left\{ 1 - \exp\left[ -O^2_\delta(x) \right] \right\}^{\lambda(a+1)-1}}{G^3_\delta(x)\exp\left[ O^2_\delta(x) \right]\left\{ 1 - \left( 1 - \exp\left[ -O^2_\delta(x) \right] \right) \right\}^a} |a,\theta,\lambda,\delta|_{R^+} \text{ and } x \in R.
\]

The rest of the paper is outlined as follows: In Section 2, we derive a useful representation for the KBX-G density function. Five special models of this family are presented in Section 3 corresponding to the baseline exponential, Weibull, Rayleigh, Lomax and Burr XII distributions. We obtain in Section 4 some general mathematical properties of the proposed KBX-G family. In Section 5, simple-type copula using Farlie–Gumbel–Morgenstern (FGM) copula, modified FGM copula, Clayton copula and Renyi’s entropy are presented. Maximum likelihood estimation is defined in Section 6. Section 7 provides a graphical simulation study based on Kumaraswamy Burr X Lomax model. Section 8 provides two applications. Finally, concluding remarks are listed in Section 9.

2. Useful Expansions

Consider the binomial series given by

\[
\left( 1 - \frac{\zeta_1}{\zeta_2} \right)^{\zeta_3} = \sum_{i=0}^{\infty} \left( \frac{\zeta_1}{\zeta_2} \right)^i \left( -1 \right)^i = \sum_{i=0}^{\infty} \left( \frac{\zeta_1}{\zeta_2} \right)^i \frac{(-1)^i}{\Gamma(1+\zeta_3-i)} \Gamma(1+\zeta_3) \left|_{\zeta_3 > 0, |\zeta_1| < 1} \right.
\]

(4)
Then, Equation (3) can be written as

$$f_{\alpha, \theta, \lambda, \delta}(x) = \frac{2\alpha \theta \lambda g_{\delta}(x)}{G_{\delta}(x) \exp\left[\frac{\theta \alpha G_{\delta}(x)}{\lambda^2} \right]} \sum_{i=0}^{\infty} \frac{(-1)^i \binom{1}{i} G_{\delta}(x)}{1 - \exp\left[-\frac{\lambda \alpha(i+1)+1}{\lambda^2}\right]}$$

(A(x))

By expanding $A(x)$, we get

$$A(x) = \sum_{h=0}^{\infty} (-1)^h \left( \frac{\lambda \alpha(i+1)-1}{h} \right) \exp\left[-hO^2_{\delta}(x)\right]$$

(6)

By merging (6) into (5) we have

$$f_{\alpha, \theta, \lambda, \delta}(x) = \frac{2\alpha \theta \lambda g_{\delta}(x)G_{\delta}(x)}{G_{\delta}(x) \exp\left[\frac{\theta \alpha G_{\delta}(x)}{\lambda^2} \right]} \sum_{i,j=0}^{\infty} (-1)^{i+j} \left( \frac{\theta - 1}{i} \right) \left( \frac{\lambda \alpha(i+1)-1}{h} \right) \exp\left[(1+h)O^2_{\delta}(x)\right]$$

(B(x))

By expanding $B(x)$, we get

$$B(x) = \sum_{j=0}^{\infty} \frac{O^2_{\delta}(x)}{j!} (-1)^j (1+h)^j.$$  

(8)

Incorporating (8) into (7), we get

$$f_{\alpha, \theta, \lambda, \delta}(x) = 2\alpha \theta \lambda g_{\delta}(x) \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} (h+1) \left( \frac{\theta - 1}{i} \right) \left( \frac{\lambda \alpha(i+1)-1}{h} \right) \exp\left[(1+h)O^2_{\delta}(x)\right]$$

(9)

Consider the generalized binomial expansion

$$\left(1 - \frac{\zeta_1}{\zeta_2}\right)^{-\zeta_3} = \sum_{k=0}^{\infty} \binom{\zeta_1}{\zeta_2}^k \frac{\Gamma(\zeta_3+k)}{k! \Gamma(\zeta_3)} [\zeta_3 > 0, |\zeta_1| < 1]$$

(10)

Applying (10) to (9) gives

$$f_{\alpha, \theta, \lambda, \delta}(x) = \sum_{k=0}^{\infty} V_{j,k}(x) \pi_{2j+k+2}(x),$$

(11)

where $\pi_{2j+k+2}(x) = (2j+k+2)g_{\delta}(x)G_{\delta}(x)^{2j+k+1}$ refers to the PDF of the exponentiated G (Exp-G) densities with power parameter $2j+k+2$ and

$$V_{j,k} = 2\alpha \theta \lambda \sum_{i,j=0}^{\infty} (-1)^{i+j+k} (h+1)^i \Gamma(2j+3+k) \left( \frac{\theta - 1}{i} \right) \left( \frac{\lambda \alpha(i+1)-1}{h} \right).$$

(12)
The CDF of the KBX-G family can be expressed as

\[ F_{\alpha,\theta,\lambda}(x) = \sum_{j,k=0}^{\infty} V_{j,k} \Pi_{2j+k+2}(x), \]  

(13)

where \( \Pi_{2j+k+2}(x) = G_{\theta}(x)^{2j+k+2} \) denotes the PDF of the exponentiated G (Exp-G) densities with power parameter \( 2j+k+2 \).

3. Special Models

This section presents some special KBX models based on Exponential (E), Weibull (W), Rayleigh (R), Lomax (Lx) and Burr XII (BXII) distributions. Table 1 below presents some new submodels based on the new KBX-G family. For the KBXLx, Figure 1 (right panel) and Figure 1 (left panel) give some plots of its PDF and the hazard rate function (HRF), respectively. Based on Figure 1 (right panel), the PDF of the KBX Lomax (KBXLx) can be “symmetric PDF” and “asymmetric right skewed PDF” with many useful shapes. Based on Figure 1 (left panel), the HRF of the KBXLx can be “decreasing–constant–increasing or U-shape HRF”, “decreasing HRF”, increasing HRF” and “J-HRF”.

Table 1. New Sub Models Based on the New Kumaraswamy Burr X-G (KBX-G) Family.

| No. | Baseline Model | \( O^2_{\delta}(x) \) | The New Model | Author |
|-----|----------------|-----------------|---------------|--------|
| 1   | Exponential (E) | \((e^x - 1)^2\)  | KBXE          | New    |
| 2   | Weibull (W)     | \((e^{x^2} - 1)^2\) | KBXW          | New    |
| 3   | Rayleigh (R)    | \((e^{x^2} - 1)^2\) | KBXR          | New    |
| 4   | Lomax (Lx)      | \((\frac{1}{x} + 1)^b - 1)^2\) | KBXLx  | New    |
| 5   | Burr XII (BXII) | \((ax + 1)^b - 1)^2\) | KBXBXII  | New    |

Figure 1. Probability density function (PDF) plots and HRF plots of the KBX Lomax (Lx) model.

4. Statistical Properties

4.1. Quantile Function

For the KBX quantile function, say \( x = Q(u) \) can be derived via inverting (2) as

\[ F^{-1}(u) = Q_G(u) = G^{-1}\left( \frac{-\gamma u^*\theta_0\alpha}{1 - \frac{\gamma u^*\theta_0\alpha}{\delta}} \right)_{\delta = 1 - u}, \]
where
\[ q_{a}, \theta, \alpha, \lambda = \log\left[1 - \left(1 - u^a b\right)^{\frac{1}{\lambda}}\right] \]

### 4.2. Moments

Let \( Z_{2j+k+2} \) be a nonnegative random variable which has the \( \text{Exp-G} \) with power parameter \( 2j + k + 2 \). The \( r \)th moment of the KBX-G family can be obtained from (11) as
\[ \mu'_r = E(X^r) = \sum_{j=0}^{\infty} V_{j,k} E\left(Z_{2j+k+2}^r\right), \]

where \( E(Z_r^r) = \nu \int_{-\infty}^{\infty} x^r g(x) G(x)^{-1} dx, \nu > 0 \) can be calculated and analyzed numerically in terms of the baseline quantile function \( Q_G(u) \), i.e., \( Q_G(u) = G^{-1}(u) \) as
\[ E(Z_r^r) = \nu \int_{0}^{1} u^{r-1} Q_G(u)' du. \]

For the KBXLx model
\[ \mu'_r = \sum_{j=0}^{\infty} \sum_{w=0}^{r} V_{j,k,w}^r \left(2j + k + 2, \frac{w - r}{b} + 1\right), \]

where \( V_{j,k,w}^r = V_{j,k} (2j + k + 2) a^r (-1)^w \left(\frac{r}{w}\right) \) and \( B(v_1, v_2) = \int_{0}^{1} u^{v_1-1} (1 - u)^{v_2-1} du \) is the complete beta function. Here, we introduce a formula for the moment generating function (MGF) as
\[ M_X(t) = E(e^{tX}) = \sum_{j=0}^{\infty} V_{j,k} M_{2j+k+2}(t), \]

where \( M_{2j+k+2}(t) \) is the moment generating function (MGF) of \( Z_{2j+k+2} \). Consequently, we can easily determine \( M_X(t) \) from the \( \text{Exp-G} \) MGF as
\[ M_X(t)|_{k>0} = \kappa \int_{-\infty}^{\infty} \exp(tX) g(x) G(x)^{k-1} dx = \kappa \int_{0}^{1} u^{k-1} \exp(tQ_G(u)) du \]

which can be calculated numerically from the baseline quantile function, i.e., \( Q_G(u) = G^{-1}(u) \). Thus, for the KBXLx model we have
\[ M_X(t) = \sum_{j,k,r=0}^{\infty} \sum_{w=0}^{r} V_{j,k,w}^r \left(2j + k + 2, \frac{w - r}{b} + 1\right). \]

### 4.3. Conditional Moments

The \( s \)th incomplete moments of \( X \) defined by \( \Omega_s(t) \) for any real \( s > 0 \) can be expressed from (11) as
\[ \Omega_s(t) = \int_{-\infty}^{t} x^s f(x) dx = \sum_{j,k=0}^{\infty} V_{j,k} l_{(-\infty,t)}(x), \]
where \( I_{(-\infty, t)}(x) = \int_{-\infty}^{t} x^j \Omega_{z,j+k+2}(t) dt \) and \( \Omega_{z,j+k+2}(t) = \int_{0}^{G(t)} u^{j+k+1} Q_C(u) du \) and \( \Omega_{z,j+k+2}(t) \) can be evaluated numerically. For the KBXLx

\[
\Omega_s(t) = \sum_{j,k=0}^{\infty} \sum_{w=0}^{j} \frac{\omega^{j+k+2w}}{\Gamma(j+k+2w)} B_{2j+k+2w}(2j+k+2, \frac{w-s}{b} + 1)_{b>0}.
\]

where \( B_{\alpha}(\nu, \mu) = \int_{0}^{\nu} u^{\alpha-1} (1-u)^{\mu-1} du \) is the incomplete beta function.

### 4.4. Mean Deviation

The mean deviations of the mean \( \mu = E(X) \) and the mean deviations of the median \( Med \) are defined by

\[
M_1(x) = E[X - \mu^\prime] = 2\mu^\prime F(\mu^\prime) - 2\Omega_1(\mu^\prime)
\]

and

\[
M_2(x) = E[X - Med] = \mu^\prime - 2\Omega_1(Med)
\]

respectively, where \( \mu^\prime = E(X) \), \( Med = \text{median} (X) = Q_{\frac{1}{2}} \), and \( \Omega_1(t) \) is the first complete moment given by \( \Omega_s(t) \) with \( s = 1 \).

### 5. Simple Type Copula

#### 5.1. BiKBX Type via FGM Copula

Consider the joint CDF of the FGM family \( C_\pi(u, v) = uv(1 + \pi u^\ast v^\ast) \mid u^\ast = 1 - u, \) where the marginal function \( u = F_1, \ v = F_2, \ \pi \in (-1, 1) \) is a dependence parameter and for every \( u, v \in (0, 1), C_\pi(u, 0) = C_\pi(0, v) = 0 \) which is “grounded minimum” and \( u = C_\pi(u, 1) \) and \( v = C_\pi(1, v) \) which is “grounded maximum” (see [7–12]). Then, setting

\[
u^\ast = \left(1 - \left(1 - \exp\left[-O_2^2(x_1)\right]\right)^{a_1} \right)^{\theta_1} \mid_{u \in (0, 1)},
\]

and

\[
u^\ast = \left(1 - \left(1 - \exp\left[-O_2^2(x_2)\right]\right)^{a_2} \right)^{\theta_2} \mid_{v \in (0, 1)},
\]

we have

\[
C(F_1, F_2) = F(x_1, x_2) = \left[1 - \left(1 - \exp\left[-O_2^2(x_1)\right]\right)^{a_1} \right]^{\theta_1} \times \left[1 - \left(1 - \exp\left[-O_2^2(x_2)\right]\right)^{a_2} \right]^{\theta_2} \times \left[1 + \pi \left(1 - \left(1 - \exp\left[-O_2^2(x_1)\right]\right)^{a_1} \right)^{\theta_1} \times \left[1 - \left(1 - \exp\left[-O_2^2(x_2)\right]\right)^{a_2} \right]^{\theta_2} \right].
\]

The joint PDF can then derived from \( c_\pi(u, v) = 1 + \pi u^\ast v^\ast \mid_{u^\ast = 1 - 2u} \) and \( v^\ast = 1 - 2v \) or from \( f(x_1, x_2) = f_{1,2}(F_1, F_2) \).

#### 5.2. BiKBX Type via Modified FGM Copula

Consider the modified version of FGM copula defined as (see [11]) \( C_\pi(u, v) = uv[1 + \pi \delta(u) \omega(v)]_{u \in (-1, 1)} \) or \( C_\pi(u, v) = uv + \pi \delta^\ast_u \omega^\ast_v \mid_{u \in (-1, 1)} \) where \( \delta^\ast_u = u \delta(u), \) and \( \omega^\ast_v = v \omega(v). \) Where \( \delta(u) \) and \( \omega(v) \) are two functions on \( (0, 1) \) with the following conditions:
I The boundary condition:
\[ \delta(0) = \delta(1) = \omega'(0) = \omega'(1) = 0. \]

II Let
\[
\begin{align*}
\tau_1 &= \inf \left\{ \delta_u^* : \frac{\partial}{\partial u} \delta_u^* |_{K_1} < 0, \right\} \\
\tau_2 &= \sup \left\{ \delta_u^* : \frac{\partial}{\partial u} \delta_u^* |_{K_1} < 0, \right\} \\
\xi_1 &= \inf \left\{ \omega_0^* : \frac{\partial}{\partial v} \omega_0^* |_{K_2} > 0, \right\} \\
\xi_2 &= \sup \left\{ \omega_0^* : \frac{\partial}{\partial v} \omega_0^* |_{K_2} > 0. \right\}
\end{align*}
\]
Then,
\[ 1 \leq \min(\tau_1 \tau_2, \xi_1 \xi_2) < \infty, \]
where
\[ \frac{\partial}{\partial u} \delta_u^* = \delta(u) + u \frac{\partial}{\partial u} \delta(u), \]
\[ K_1 = \left\{ u : u \in (0,1) \left| \frac{\partial}{\partial u} \delta_u^* \text{ exists} \right. \right\}, \]
and
\[ K_2 = \left\{ v : v \in (0,1) \left| \frac{\partial}{\partial v} \omega_0^* \text{ exists} \right. \right\}. \]

BvKBX-FGM (Type I) model
Consider \( C_{\pi}(u,v) = uv + \pi \delta_u^* \omega_0^* |_{\pi \in (-1,1)} \), then we get
\[
C_{\pi}(u,v) = \pi \delta_u^* \omega_0^* + \begin{bmatrix} 1 - \left( 1 - \left( 1 - \exp\left[ -O_2^2(u) \right] \right)^{a_1 \lambda_1} \right) \theta_1 \\
\times \left[ 1 - \left( 1 - \exp\left[ -O_2^2(v) \right] \right)^{a_2 \lambda_2} \right]^{a_2 \lambda_2} \theta_2 \end{bmatrix}.
\]

BvKBX-FGM (Type II) model
Let \( \delta(u) \) and \( \omega(v) \) be two functional forms to satisfy all the conditions stated earlier where
\[ \delta(u)^* = u^{n_1} (1 - u)^{1 - n_1} |_{n_1 > 0} \] and \( \omega(v)^* = v^{n_2} (1 - v)^{1 - n_2} |_{n_2 > 0} \).
The corresponding BvKBX-FGM (Type II) can be derived from
\[ C_{\pi,n_1,n_2}(u,v) = uv[1 + \pi \delta(u)^* \omega(v)^*]. \]

BvKBX-FGM (Type III) model
Consider the following functional form for both \( \Theta(u) \) and \( \phi(v) \) which satisfy all the conditions stated earlier where
\[ \Theta(u) = u|\log(1 + u)||\bar{v}| = 1 - u \] and \( \phi(v) = v|\log(1 + v)||\bar{u}| = 1 - v. \]
In this case, one can also derive a closed form expression for the associated CDF of the BivKBX-FGM (Type III).

**BvKBX-FGM (Type IV) model**

According to [13], the CDF of the BvKBX-FGM (Type IV) model can be derived from

\[ C(x, y) = uF^{-1}(y) + vF^{-1}(u) - F^{-1}(u)F^{-1}(v), \]

Then,

\[ F^{-1}(u) = G^{-1}\left(\frac{-\frac{1}{2}q_u,\theta_1,\alpha,\lambda_1}{1 - \frac{1}{2}q_u,\theta_1,\alpha,\lambda_1}\right), \]

\[ F^{-1}(v) = G^{-1}\left(\frac{-\frac{1}{2}q_v,\theta_2,\alpha,\lambda_2}{1 - \frac{1}{2}q_v,\theta_2,\alpha,\lambda_2}\right), \]

where \( q_u,\theta_1,\alpha,\lambda_1 = \log\left[1 - \frac{1}{\sqrt{\alpha}}\frac{1}{1 - v}\right] \) and \( q_v,\theta_2,\alpha,\lambda_2 = \log\left[1 - \frac{1}{\sqrt{\alpha}}\frac{1}{1 - u}\right] \).

5.3. BvKBX Type via Clayton Copula

The Clayton copula can be considered as

\[ C(v_1, v_2) = [(1/v_1) + (1/v_2) - 1]^{-1}_{[0,\infty)}. \]

Let us assume that \( T \sim \text{KBX}(\alpha_1, \theta_1, \lambda_1, \delta) \) and \( W \sim \text{KBX}(\alpha_2, \theta_2, \lambda_2, \delta) \). Then, setting

\[ v_1 = v(t) = \left[1 - \left(1 - \{1 - \exp[-O_2(t)]\}^{\alpha_1,\lambda_1}\right)^{\theta_1}\right], \]

and

\[ v_2 = v(w) = \left[1 - \left(1 - \{1 - \exp[-O_2(w)]\}^{\alpha_2,\lambda_2}\right)^{\theta_2}\right]. \]

Then, the BvKBX type distribution can be derived as

\[ C(v_1, v_2) = C(F(t), F(w)) = \left(\left[1 - \left(1 - \{1 - \exp[-O_2(t)]\}^{\alpha_1,\lambda_1}\right)^{\theta_1}\right]^{-1} + \left[1 - \left(1 - \{1 - \exp[-O_2(w)]\}^{\alpha_2,\lambda_2}\right)^{\theta_2}\right]^{-1}\right)^{-1}. \]

5.4. BvKBX Type via Renyi’s Entropy

Consider the theorem of [14] where

\[ R(u, v) = x_2u + x_1v - x_1x_2, \]

then, the associated BvKBX will be

\[ R(u, v) = R(F(x_1), F(x_2)) = -x_1x_2 + x_2\left[1 - \left(1 - \{1 - \exp[-O_2(x_1)]\}^{\alpha_1,\lambda_1}\right)^{\theta_1}\right] + x_1\left[1 - \left(1 - \{1 - \exp[-O_2(x_2)]\}^{\alpha_2,\lambda_2}\right)^{\theta_2}\right]. \]

By fixing \( a \) and \( b \) we then have a five-dimension parameter BvKBX-type distribution.
5.5. MvKBX Extension via Clayton Copula

The $m$-dimensional extension from the above will be

$$C(v_i) = \left[ \sum_{i=1}^{m} v_i + 1 - m \right]^{-\frac{1}{p}}.$$

Then, the MvKBX extension can be expressed as

$$C(X) = \left( \sum_{i=1}^{m} \left( 1 - \left( 1 - \exp\left[ -O_2^2(x_i) \right] \right)^{x,\lambda} \right)^{\frac{1}{p}} \right) + 1 - m \right]^{-\frac{1}{p}}.$$

where $X = x_1, x_2, \cdots, x_m$.

6. Maximum Likelihood Estimation

Let $\psi = (\alpha, \theta, \lambda, \delta)^T$ be the vector of parameters, then the log-likelihood function for $\psi$ is given by

$$L_n(\psi) = n \log(2\alpha \theta \lambda) + \sum_{i=0}^{n} \log g_2(x_i) + \sum_{i=0}^{n} \log G_2(x_i) - 3 \sum_{i=0}^{n} \log \overline{G}_2(x_i) + (\lambda 1 - 1) \sum_{i=0}^{n} \log \left( 1 - \exp\left[ -O_2^2(x_i) \right] \right) - \sum_{i=0}^{n} O_2^2(x_i)$$

The components of the score function $U_n(\psi) = \left( U_n(\alpha), U_n(\theta), U_n(\lambda), U_n(\delta) \right)$. Setting the nonlinear system of equations $U_n(\alpha), U_n(\theta), U_n(\lambda), U_n(\delta)$ equal to zero and solving the equations simultaneously yields the maximum likelihood estimation (MLE) of $\psi$, where these equations cannot be solved analytically; thus, we use any statistical software to solve these equations.

7. Simulations

To assess of the finite sample behavior of the MaxLEs under the KBXLx model, we will consider and apply the following algorithm:

1—Use

$$x_a = a \left\{ 1 - \left( 1 - \left( 1 - \frac{\overline{q}_{2, a, \alpha, \lambda}}{\overline{q}_{2, a, \alpha, \lambda}} \right)^{\frac{1}{p}} \right) \right\}^{-\frac{1}{p}}$$

to generate N = 1000 samples of size $n$ from the KBXLx distribution;

2—Compute the MaxLEs for the N = 1000 samples.

3—Compute the standard errors (StEs) of the MaxLEs for the N = 1000 samples.

4—Compute the biases ($B_h$) and mean squared errors (MSEs) given for $h = \alpha, \theta, \lambda, a, b$. We repeated these steps for $n = 100, \ldots, 500$ with $\alpha = \theta = \lambda = a = b = 1$ so computing biases, mean squared errors (MSEs) for $h = \alpha, \theta, \lambda, a, b$ and $n = 100, \ldots, 500$.

Figures 2–6 (left panels) prove that the five biases decrease as $n \to \infty$. Figures 2–6 (right panels) prove that the five MSEs decrease as $n \to \infty$. The red line in Figure 4 means that the bias reached zero. From Figures 2–6 (left panels), the biases for each parameter decrease to zero as $n \to \infty$. From Figures 2–6 (right panels), the MSEs decrease to zero as $n \to \infty$. 

Figure 2. Bias and MSE for parameter $\alpha$.

Figure 3. Bias and MSE for parameter $\theta$. 

Figure 4. Bias and MSE for parameter $\lambda$.

Figure 5. Bias and MSE for parameter $b$. 
8. Applications and Comparing Models

In this section, we provide two real-life applications to illustrate the importance, potentiality and flexibility of the KBXLx model. We compare the fit of the KBXLx with some well-known competitive models (see Table 2).

Dataset I (84 Aircraft Windshield): Times of Failure: The first real-life dataset represents the data on times of failure for 84 aircraft windshields (see [15]). The data observations are: 3.7790, 0.0400, 1.248, 2.154, 2.9640, 4.278, 4.449, 1.6190, 2.0100, 2.688, 3.9240, 1.2810, 2.038, 2.224, 3.1170, 1.506, 1.866, 2.0850, 2.890, 4.121, 1.3030, 2.089, 2.902, 4.167, 1.4320, 2.097, 2.934, 4.2400, 0.943, 1.9120, 2.632, 3.5950, 1.0700, 1.914, 2.6460, 3.699, 1.1240, 4.376, 2.3850, 3.443, 0.3010, 1.876, 2.820, 3, 4.035, 1.281, 1.480, 2.135, 2.962, 4.2550, 1.505, 2.4810, 3.467, 0.309, 1.8990, 2.610, 3.4780, 0.557, 1.981, 2.661, 2.190, 3.000, 4.3050, 1.568, 2.1940, 3.103, 1.9110, 2.625, 3.5780, 1.615, 2.2230, 3.114, 4.485, 1.652, 2.2290, 3.166, 4.570, 1.652, 2.3000, 3.344, 4.602, 1.7570, 2.324, 3.3760, 4.663. Dataset II (63 Aircraft Windshield): Times of Service: The second real-life dataset represents the times of service for 63 aircraft windshields (see [15]).

The data observations are: 0.046, 0.622, 1.978, 3.0030, 0.9000, 2.053, 0.2800, 1.794, 3.483, 1.492, 2.600, 0.150, 3.3040, 0.9960, 3.1020, 0.952, 2.065, 0.487, 2.2400, 4.015, 1.183, 2.3410, 4.167, 1.4320, 2.097, 2.934, 4.2400, 0.943, 1.9120, 2.632, 3.5950, 1.0700, 1.914, 2.6460, 3.699, 1.1240, 4.376, 2.3850, 3.443, 0.3010, 1.876, 2.820, 3, 4.035, 1.281, 1.480, 2.135, 2.962, 4.2550, 1.505, 2.4810, 3.467, 0.309, 1.8990, 2.610, 3.4780, 0.557, 1.981, 2.661, 2.190, 3.000, 4.3050, 1.568, 2.1940, 3.103, 1.9110, 2.625, 3.5780, 1.615, 2.2230, 3.114, 4.485, 1.652, 2.2290, 3.166, 4.570, 1.652, 2.3000, 3.344, 4.602, 1.7570, 2.324, 3.3760, 4.663. See supplementary for detailed data.

Many other useful real-life datasets can be found in [16–27]. For exploring the outliers, the box plot is plotted in Figure 7. Based on Figure 7, we note that no outliers were found. For checking the data normality, the Q–Q plot is sketched in Figure 8. Based on Figure 8, we note that normality nearly exists. For exploring the shape of the HRF for real data, the total time test (TTT) plot (see [28]) is provided (see Figure 9). Based on Figure 9, we note that the HRF is “increasing monotonically” for the two real-life datasets. For exploring the initial shape of real data nonparametrically, the KDE is provided in Figure 10. Figures 11 and 12 give the estimated Kaplan–Meier survival (EKMS) plot, estimated PDF (EPDF), P–P plot and estimated HRF (EHRF) for dataset I and II, respectively.
Table 2. The Competitive Model.

| No. | Model                                   | Abbreviation | Author |
|-----|-----------------------------------------|--------------|--------|
| 1   | Special generalized mixture Lomax       | SGMLx        | [29]   |
| 2   | Odd log-logistic Lx                     | OLLLx        | [30]   |
| 3   | Reduced OLLLx                           | ROLLlx       | [30]   |
| 4   | Reduced Burr–Hatke Lx                   | RBHLx        | [6]    |
| 5   | Transmuted Topp–Leone Lx                | TTLlx        | [31]   |
| 6   | Reduced TTLLx                           | RTTLLx       | [31]   |
| 7   | Gamma Lx                                | GamLx        | [32]   |
| 8   | Kumaraswamy Lx                          | KumLx        | [33]   |
| 9   | McDonald Lx                             | McLx         | [33]   |
| 10  | Beta Lx                                 | BLx          | [33]   |
| 11  | Exponentiated Lx                        | ExpLx        | [34]   |
| 12  | Lomax                                   | Lx           | [35]   |
| 13  | Proportional reversed hazard rate Lx    | PRHRLx       | New    |

Figure 7. Box plots.

Figure 8. Normal quantile–quantile (Q–Q) plots.

Figure 9. Total time test (TTT) plots.
Failure times data

Service times data

Figure 9. Total time test (TTT) plots.

Failure times data

Service times data

Figure 10. Nonparametric kernel density estimation (KDE).

The following goodness-of-fit (GOF) statistics are used: Akaike information criterion (AIC), Bayesian IC (BIC), consistent AIC (CAIC), Hannan–Quinn IC (HQIC), Cramér-von Mises ($W^*$) and Anderson–Darling ($A^*$) for comparing the competitive models. For failure time data, the results are listed in Tables 3 and 4. Table 3 gives the MaxLEs and StErs for failure time data. Table 4 gives the $-\hat{\ell}$ and GOF statistics for failure time data. For service time data, the analysis results are listed in Tables 5 and 6. Table 5 gives the MaxLEs and StErs for service time data, and Table 6 gives the $-\hat{\ell}$ and GOFs statistics for the service time data. Based on Tables 4 and 6, we note that the KBXLx model gives the lowest values for the AIC, CAIC, BIC, HQIC, $A^*$ and $W^*$ among all the fitted models. Hence, it could be chosen as the best model under these criteria.
Table 4. -ℓ and Goodness-of-Fit (GOF) Statistics for Failure Time Data.

| Model   | -ℓ  | AIC    | CAIC   | BIC    | HQIC   | AIC• | W•   |
|---------|-----|--------|--------|--------|--------|------|------|
| KBXLx   | 127.099 | 264.197 | 264.966 | 276.351 | 269.083 | 0.512 | 0.065 |
| McLx    | 129.802 | 269.605 | 270.364 | 281.818 | 274.517 | 0.667 | 0.086 |
| OLLx    | 134.424 | 274.847 | 275.147 | 282.139 | 277.779 | 0.941 | 0.101 |
| TTLLx   | 135.570 | 279.140 | 279.646 | 288.863 | 283.049 | 1.126 | 0.127 |
| GamLx   | 138.404 | 282.808 | 283.105 | 290.136 | 285.756 | 1.367 | 0.162 |
| BLx     | 138.718 | 285.435 | 285.935 | 295.206 | 289.365 | 1.408 | 0.168 |
| ExpLx   | 141.400 | 288.799 | 289.096 | 296.127 | 291.747 | 1.744 | 0.219 |
| ROLLLx  | 142.845 | 289.690 | 289.839 | 294.552 | 291.645 | 1.957 | 0.255 |
| SGMLx   | 143.087 | 292.175 | 292.475 | 299.467 | 295.106 | 1.347 | 0.158 |
| RTTLLx  | 153.981 | 313.962 | 314.262 | 321.254 | 316.893 | 3.753 | 0.559 |
| PRHRLx  | 162.877 | 331.754 | 332.054 | 339.046 | 334.686 | 1.367 | 0.161 |
| Lx      | 164.988 | 333.977 | 334.123 | 338.862 | 335.942 | 1.398 | 0.167 |
| RBHLx   | 168.604 | 341.208 | 341.356 | 346.070 | 343.162 | 1.671 | 0.207 |

It is worth mentioning that the presented class of stochastic distributions can have also an important utilization in the insurance industry, risks, reliability, medicine and engineering (see [36–47]).

9. Conclusions

A new family of probability distributions based on the Kumaraswamy and Burr X families is derived and studied. Some new bivariate type G families using Farlie–Gumbel–Morgenstern copula, modified Farlie–Gumbel–Morgenstern copula, Clayton copula and Renyi’s entropy copula are derived. Some special models based on exponential, Weibull, Rayleigh, Lomax and Burr XII distributions are presented. Then, special attention is devoted to the Lomax case. Some of its statistical properties, including the quantile function, moments, incomplete moments and mean deviation, are presented. Simulation based on biases and mean squared errors is introduced. Based on this simulation, the maximum likelihood method performs well and can be used for estimating the model parameters. Finally, two real-life applications to illustrate the importance of the new family are proposed.

Supplementary Materials:
The first and second datasets are given by [15].

https://onlinelibrary.wiley.com/doi/book/10.1002/047147326X

Author Contributions:
All authors contributed equally for this work. All authors have read and agreed to the published version of the manuscript.

Funding:
King Saud University.
Table 3. Maximum Likelihood Estimation (MLEs) and SEs for Failure Time Data.

| Model          | Estimates               |
|----------------|-------------------------|
| KBXLx(θ,α,λ,b,a) | 0.08869 (0.0097)        |
|                | 0.141769 (0.00216)      |
|                | 4.47434 (171.22)        |
|                | 1.6099 (20.845)         |
|                | 1.98809 (38.960)        |
| McLx(θ,α,λ,b,a)  | 2.1875 (0.5211)         |
|                | 119.1751 (140.297)      |
|                | 12.4171 (120.486)       |
|                | 19.9243 (9.8116)        |
|                | 75.6606 (186.01)        |
| KLx(θ,β,b,a)    | 2.6150 (0.3822)         |
|                | 100.2756 (15.608)       |
|                | 5.27710 (120.486)       |
|                | 78.6774 (9.8116)        |
| TTTLLx(θ,β,b,a) | −0.8075 (0.1396)        |
|                | 2.47663 (0.5418)        |
|                | (15,608) (160.4.4)      |
|                | (38.628) (123.94)       |
| BLx(θ,β,b,a)    | 3.60360 (0.6187)        |
|                | 33.63870 (63.715)       |
|                | 4.83070 (9.2382)        |
|                | 118.837 (1.02382)       |
| PRHRLx(β,b,a)   | 3.73 \times 10^6 (1.01 \times 10^6) |
|                | 4.707 \times 10^{-1} (0.00001) |
|                | 4.49 \times 10^6 (37.14684) |
| SGMLx(θ,b,a)    | −1.04 \times 10^{-1} (0.1223) |
|                | 9.83 \times 10^6 (4843.3) |
|                | 1.18 \times e^7 (501.04) |
| RTTLLx(θ,β,a)   | −0.84732 (0.1001)       |
|                | 5.52057 (1.1848)        |
|                | 1.15678 (0.09588)       |
| OLLLx(θ,b,a)    | 2.32636 (2.14 \times 10^{-1}) |
|                | (7.17 \times e^5) (1.19 \times e^4) |
| ExpLx(θ,b,a)    | 3.62610 (0.6236)        |
|                | 20.074.5 (2041.8)       |
|                | 26.257.7 (99.74)        |
| GamLx(θ,b,a)    | 3.58760 (0.5133)        |
|                | 52.001.4 (7955)         |
|                | 37.029.7 (81.16)        |
| ROLLl(θ,a)      | 3.890564 (0.36524)      |
|                | 0.57316 (0.01946)       |
| RBHLx(b,a)      | 10,801,754 (983,309)    |
|                | 51,367,189 (232,312)    |
| Lx(b,a)         | 51,425.4 (5933.5)       |
|                | 131,790 (296,12)        |

Table 4. -ℓ and Goodness-of-Fit (GOF) Statistics for Failure Time Data.

| Model          | -ℓ | AIC | CAIC | BIC | HQIC | A* | W* |
|----------------|----|-----|------|-----|------|----|----|
| KBXLx          | 127.099 | 264.197 | 264.966 | 276.351 | 269.083 | 0.512 | 0.065 |
| McLx           | 129.802 | 269.605 | 270.364 | 281.818 | 274.517 | 0.667 | 0.086 |
| OLLLx          | 134.424 | 274.847 | 275.147 | 282.139 | 277.779 | 0.941 | 0.101 |
| TTTLLx         | 135.570 | 279.140 | 279.646 | 288.863 | 283.049 | 1.126 | 0.127 |
| GamLx          | 138.404 | 282.808 | 283.105 | 290.136 | 285.756 | 1.367 | 0.162 |
| BLx            | 138.718 | 285.435 | 285.935 | 295.206 | 289.365 | 1.408 | 0.168 |
| ExpLx          | 141.400 | 288.799 | 289.096 | 296.127 | 291.747 | 1.744 | 0.219 |
| ROLLLLx        | 142.845 | 289.690 | 289.839 | 294.552 | 291.645 | 1.957 | 0.255 |
| SGMLx          | 143.087 | 292.175 | 292.475 | 299.467 | 295.106 | 1.347 | 0.158 |
| ROTTLLx        | 153.981 | 313.975 | 314.275 | 321.254 | 316.893 | 3.753 | 0.559 |
| PRHRLx         | 162.877 | 331.754 | 332.054 | 339.046 | 334.688 | 1.367 | 0.161 |
| Lx             | 164.988 | 333.977 | 334.123 | 338.862 | 335.942 | 1.398 | 0.167 |
| RBHLx          | 168.604 | 341.208 | 341.356 | 346.070 | 343.162 | 1.671 | 0.207 |
### Table 5. MLEs and SEs for Service Time Data.

| Model         | Estimates                  |                   |                   |
|---------------|---------------------------|-------------------|-------------------|
| KBXLx(θ, α, λ, b, a) | 0.1831                     | 1.6937            | 0.25705           |
|               | (3.5 × 10²)                | (2.4124)          | (0.3912)          |
| BLx(β, b, a)  | 1.9218                     | 31.2594           | 4.9684            |
|               | (0.3184)                   | (316.84)          | (50.528)          |
| KLx(β, b, a)  | 1.6691                     | 60.5673           | 2.56490           |
|               | (0.257)                    | (86.013)          | (4.7589)          |
| TTLLx(β, b, a) | (-0.607)                   | 1.78578           | 2123.391          |
|               | (0.2137)                   | (0.4152)          | (163.915)         |
| RTTLx(θ, β, α) | -0.6715                   | 2.74496           | 1.01238           |
|               | (0.18746)                  | (0.6696)          | (0.1141)          |
| PRHRLx(β, b, a) | 1.59 × 10⁶                 | 3.93 × 10⁻¹       | 1.30 × 10⁶        |
|               | 2.01 × 10³                 | 0.0004 × 10⁻¹     | 0.95 × 10⁶        |
| SGMLx(θ, b, a) | -1.04 × 10⁻¹               | 6.45 × 10⁶        | 6.33 × 10⁶        |
|               | (4.1 × 10⁻¹⁰)              | (3.21 × 10⁶)      | (3.8573)          |
| GamLx(β, b, a) | 1.9073                     | 35,842.433        | 39,197.57         |
|               | (0.3213)                   | (6945.074)        | (151.653)         |
| OLLLx(θ, b, a) | 1.66419                    | 6.340 × 10⁵       | 2.01 × 10⁶        |
|               | (1.79 × 10⁻¹¹)             | (1.68 × 10⁴)      | (7.22 × 10⁶)      |
| ExpLx(θ, b, a) | 1.9145                     | 22,971.15         | 32,882            |
|               | (0.348)                    | (3209.53)         | (162.2)           |
| RBHLx(b, a)   | 14,055,522                 | 53,203,423        |                   |
|               | (422.01)                   | (28,523.2)        |                   |
| ROLLx(θ, a)   | 2.37233                    | 0.69109           |                   |
|               | (0.2683)                   | (0.0449)          |                   |
| Lx(b, a)      | 99,269.8                   | 207,019.4         |                   |
|               | (11,864)                   | (301.237)         |                   |

### Table 6. -ℓ and GOFs Statistics for Service Time Data.

| Model    | -ℓ  | AIC  | CAIC | BIC   | HQIC | A*  | W*  |
|----------|-----|------|------|-------|------|-----|-----|
| KBXLx    | 98.0851 | 206.170 | 207.223 | 216.886 | 210.385 | 0.219 | 0.032 |
| KLx      | 100.866 | 209.735 | 210.425 | 218.308 | 213.107 | 0.739 | 0.122 |
| TTLLx    | 102.449 | 212.900 | 213.589 | 221.472 | 216.271 | 0.943 | 0.155 |
| GamLx    | 102.833 | 211.666 | 212.073 | 218.096 | 214.195 | 1.112 | 0.184 |
| SGMLx    | 102.894 | 211.788 | 212.195 | 218.218 | 214.317 | 1.113 | 0.184 |
| BLx      | 102.961 | 213.922 | 214.612 | 222.495 | 217.294 | 1.134 | 0.187 |
| ExpLx    | 103.550 | 213.099 | 213.506 | 219.529 | 215.628 | 1.233 | 0.204 |
| OLLLx    | 104.904 | 215.808 | 216.215 | 222.238 | 218.337 | 0.942 | 0.155 |
| PRHRRLx  | 109.299 | 224.597 | 225.004 | 231.027 | 227.126 | 1.126 | 0.186 |
| Lx       | 109.299 | 222.598 | 222.798 | 226.884 | 224.283 | 1.127 | 0.186 |
| ROLLx    | 110.729 | 225.457 | 225.657 | 229.744 | 227.143 | 2.347 | 0.391 |
| RTTLx    | 112.186 | 230.371 | 230.778 | 236.800 | 232.900 | 2.688 | 0.453 |
| RBHLx    | 112.601 | 229.201 | 229.401 | 233.487 | 230.877 | 1.398 | 0.232 |

It is worth mentioning that the presented class of stochastic distributions can have also an important utilization in the insurance industry, risks, reliability, medicine and engineering (see [36–47]).

### 9. Conclusions

A new family of probability distributions based on the Kumaraswamy and Burr X families is derived and studied. Some new bivariate type G families using Farlie–Gumbel–Morgenstern copula, modified Farlie–Gumbel–Morgenstern copula, Clayton copula and Renyi’s entropy copula are derived. Some special models based on exponential, Weibull, Rayleigh, Lomax and Burr XII distributions are presented. Then, special attention is devoted to the Lomax case. Some of its statistical properties, including the quantile function, moments, incomplete moments and mean deviation, are...
presented. Simulation based on biases and mean squared errors is introduced. Based on this simulation, the maximum likelihood method performs well and can be used for estimating the model parameters. Finally, two real-life applications to illustrate the importance of the new family are proposed.

**Supplementary Materials:** The first and second datasets are given by [15]. https://onlinelibrary.wiley.com/doi/book/10.1002/047147326X.

**Author Contributions:** All authors contributed equally for this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** King Saud University.

**Acknowledgments:** The authors extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group No (RG-1438-086).

**Conflicts of Interest:** The authors declare no conflict of interest

**References**

1. Marshall, A.W.; Olkin, I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika* 1997, 84, 641–652. [CrossRef]
2. Eugene, N.; Lee, C.; Famoye, F. Beta-normal distribution and its applications. *Commun. Stat. Theory Methods* 2002, 31, 497–512. [CrossRef]
3. Cordeiro, G.M.; de Castro, M. A new family of generalized distributions. *J. Stat. Comput. Simul.* 2011, 81, 883–893. [CrossRef]
4. Yousof, H.M.; Afify, A.Z.; Hamedani, G.G.; Aryal, G. The Burr X generator of distributions for lifetime data. *J. Stat. Theory Appl.* 2017, 16, 288–305. [CrossRef]
5. Cordeiro, G.M.; Yousof, H.M.; Ramires, T.G.; Ortega, E.M.M. The Burr XII system of densities: Properties, regression model and applications. *J. Stat. Comput. Simul.* 2018, 88, 432–456. [CrossRef]
6. Yousof, H.M.; Majumder, M.; Jahanshahi, S.M.A.; Ali, M.M.; Hamedani, G.G. A new Weibull class of distributions: Theory, characterizations and applications. *J. Stat. Res. Iran* 2018, 15, 45–83. [CrossRef]
7. Fisher, N.I. Copulas. In *Encyclopedia of Statistical Sciences, Updated Volume 1*; Kotz, S., Read, C.B., Banks, D.L., Eds.; John Wiley and Sons: New York, NY, USA, 1997; pp. 159–164.
8. Farlie, D.J.G. The performance of some correlation coefficients for a general bivariate distribution. *Biometrika* 1960, 47, 307–323. [CrossRef]
9. Morgenstern, D. Einfache beispiele zweidimensionaler verteilungen. *Mitteilungsblatt Math. Stat.* 1965, 8, 234–235.
10. Gumbel, E.J. Bivariate exponential distributions. *J. Am. Stat. Assoc.* 1960, 55, 698–707. [CrossRef]
11. Gumbel, E.J. Bivariate logistic distributions. *J. Am. Stat. Assoc.* 1961, 56, 335–349. [CrossRef]
12. Rodríguez-Lallena, J.A.; Ubeda-Flores, M. A new class of bivariate copulas. *Stat. Probab. Lett.* 2004, 66, 315–325. [CrossRef]
13. Ghosh, I.; Ray, S. Some alternative bivariate Kumaraswamy type distributions via copula with application in risk management. *J. Stat. Theory Pract.* 2016, 10, 693–706. [CrossRef]
14. Pougaza, D.B.; Djafari, M.A. Maximum Entropies Copulas. In Proceedings of the 30th International Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering, Chamonix, France, 4–9 July 2010; AIP Press: College Park, MD, USA, 2011; pp. 329–336.
15. Murthy, D.N.P.; Xie, M.; Jiang, R. *Weibull Models*; Wiley: Hoboken, NJ, USA, 2004.
16. Aryal, G.R.; Ortega, E.M.; Hamedani, G.G.; Yousof, H.M. The Topp Leone Generated Weibull distribution: Regression model, characteristics and applications. *Int. J. Stat. Probab.* 2017, 6, 126–141. [CrossRef]
17. Yousof, H.M.; Altun, E.; Ramires, T.G.; Alizadeh, M.; Rasekhi, M. A new family of distributions with properties, regression models and applications. *J. Stat. Manag. Syst.* 2018, 21, 163–188. [CrossRef]
18. Elbiely, M.M.; Yousof, H.M. A New Extension of the Lomax Distribution and its Applications. *J. Stat. Appl.* 2018, 2, 18–34.
19. Altun, E.; Yousof, H.M.; Chakraborty, S.; Handique, L. Zografos-Balakrishnan Burr XII distribution: Regression modeling and applications. *Int. J. Math. Stat.* 2018, 19, 46–70.
20. Ibrahim, M.; Yousof, H.M. A new generalized Lomax model: Statistical properties and applications. *J. Data Sci.* 2020, 18, 190–217.
21. Ibrahim, M. A new extended Fréchet distribution: Properties and estimation. Pak. J. Stat. Oper. Res. 2019, 15, 773–796.

22. Ibrahim, M. The compound Poisson Rayleigh Burr XII distribution: Properties and applications. J. Appl. Probab. Stat. 2020, 15, 73–97.

23. Ibrahim, M. The generalized odd Log-logistic Nadarajah Haghighi distribution: Statistical properties and different methods of estimation. J. Appl. Probab. Stat. 2020, 15, 61–84.

24. Yadav, A.S.; Goual, H.; Alotaibi, R.M.; Rezk, H.; Ali, M.M.; Yousof, H.M. Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson GOF test with different methods of estimation. Symmetry 2020, 12, 57. [CrossRef]

25. Mansour, M.; Yousof, H.M.; Shehata, W.A.M.; Ibrahim, M. A new two parameter Burr XII distribution: Properties, copula, different estimation methods and modeling acute bone cancer data. J. Nonlinear Sci. Appl. 2020, 13, 223–238. [CrossRef]

26. Goual, H.; Yousof, H.M. Validation of Burr XII inverse Rayleigh model via a modified chi squared GOF test. J. Appl. Stat. 2019, 47, 1–32.

27. Goual, H.; Yousof, H.M.; Ali, M.M. Lomax inverse Weibull model: Properties, applications and a modified Chi-squared GOF test for validation. J. Nonlinear Sci. Appl. 2020, 13, 330–353. [CrossRef]

28. Aarset, M.V. How to identify a bathtub hazard rate. IEEE Trans. Reliab. 1987, 36, 106–108. [CrossRef]

29. Khalil, M.G.; Hamedani, G.G.; Yousof, H.M. The Burr X exponentiated Weibull model: Characterizations, mathematical properties and applications to failure and survival times data. Pak. J. Stat. Oper. Res. 2019, XV, 141–160. [CrossRef]

30. Altun, E.; Yousof, H.M.; Hamedani, G.G. A new log-location regression model with influence diagnostics and residual analysis. Facta Univ. Ser. Math.Inform. 2018, 33, 417–449.

31. Yousof, H.M.; Alizadeh, M.; Jahanshahiand, S.M.A.; Ramires, T.G.; Ghosh, I.; Hamedani, G.G. The transmuted Topp-Leone G family of distributions: Theory, characterizations and applications. J. Data Sci. 2017, 15, 723–740.

32. Cordeiro, G.M.; Ortega, E.M.; Popovic, B.V. The gamma-Lomax distribution. J. Stat. Comput. Simul. 2015, 85, 305–319. [CrossRef]

33. Lemonte, A.J.; Cordeiro, G.M. An extended Lomax distribution. Statistics 2013, 47, 800–816. [CrossRef]

34. Gupta, R.C.; Gupta, P.L.; Gupta, R.D. Modeling failure time data by Lehman alternatives. Commun. Stat. Theory Methods 1998, 27, 887–904. [CrossRef]

35. Lomax, K.S. Business failures: Another example of the analysis of failure data. J. Am. Stat. Assoc. 1954, 49, 847–852. [CrossRef]

36. Pešta, M.; Okhrin, O. Conditional least squares and copulae in claims reserving for a single line of business. Insur. Math. Econ. 2014, 56, 28–37. [CrossRef]

37. Peštová, B.; Peštová, M. Change point estimation in panel data without boundary issue. Risks 2017, 5, 7. [CrossRef]

38. Hamedani, G.G.; Yousof, H.M.; Rasekhi, M.; Alizadeh, M.; Najibi, S.M. Type I general exponential class of distributions. Pak. J. Stat. Oper. Res. 2017, 14, 39–55. [CrossRef]

39. Mansour, M.; Korkmaz, M.C.; Ali, M.M.; Yousof, H.M.; Ansari, S.I.; Ibrahim, M. A generalization of the exponentiated Weibull model with properties, Copula and application. Eurasian Bull. Math. 2020, 3, 84–102.

40. Korkmaz, M.C.; Yousof, H.M.; Hamedani, G.G.; Ali, M.M. The Marshall-Olkin generalized G Poisson family of distributions. Pak. J. Stat. Oper. Res. 2018, 34, 251–267.

41. Hamedani, G.G.; Altun, E.; Korkmaz, M.C.; Yousof, H.M.; Butt, N.S. A new extended G family of continuous distributions with mathematical properties, characterizations and regression modeling. Pak. J. Stat. Oper. Res. 2018, 14, 737–758. [CrossRef]

42. Hamedani, G.G.; Rasekhi, M.; Najibi, S.M.; Yousof, H.M.; Alizadeh, M. Type II general exponential class of distributions. Pak. J. Stat. Oper. Res. 2019, 15, 503–523. [CrossRef]

43. Korkmaz, M.C.; Altun, E.; Yousof, H.M.; Hamedani, G.G. The odd power Lindley generator of probability distributions: Properties, characterizations and regression modeling. Int. J. Stat. Probab. 2019, 8, 70–89.

44. Nascimento, A.D.C.; Silva, K.F.; Cordeiro, G.M.; Alizadeh, M.; Yousof, H.M. The odd Nadarajah-Haghighi family of distributions: Properties and applications. Stud. Sci. Math. Hung. 2019, 56, 1–26. [CrossRef]
45. Elsayed, H.A.H.; Yousof, H.M. The generalized odd generalized exponential Fréchet model: univariate, bivariate and multivariate extensions with properties and applications to the univariate version. *Pak. J. Stat. Oper. Res.* 2020, 16, 529–544. [CrossRef]

46. Ibrahim, M.; Yousof, H.M. Transmuted Topp-Leone Weibull Lifetime Distribution: Statistical Properties and Different Method of Estimation. *Pak. J. Stat. Oper. Res.* 2020, 16, 501–515. [CrossRef]

47. Ibrahim, M.; Mohammed, W.S.; Yousof, H.M. Bayesian and Classical Estimation for the One Parameter Double Lindley Model. *Pak. J. Stat. Oper. Res.* 2020, 16, 409–420.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).