HOMOTHETIC COVERING OF CONVEX HULLS OF COMPACT CONVEX SETS

SENLIN WU, KEKE ZHANG, AND CHAN HE

Abstract. Let $K$ be a compact convex set and $m$ be a positive integer. The covering functional of $K$ with respect to $m$ is the smallest $\lambda \in [0, 1]$ such that $K$ can be covered by $m$ translates of $\lambda K$. Estimations of the covering functionals of convex hulls of two or more compact convex sets are presented. It is proved that, if a three-dimensional convex body $K$ is the convex hull of two compact convex sets having no interior points, then the least number $c(K)$ of smaller homothetic copies of $K$ needed to cover $K$ is not greater than 8 and $c(K) = 8$ if and only if $K$ is a parallelepiped.

1. Introduction

Let $K$ be a compact convex subset of $\mathbb{R}^n$ that contains distinct points. We denote by relint $K$, relbd $K$, int $K$, bd $K$, and ext $K$ the relative interior, relative boundary, interior, boundary, and the set of extreme points of $K$, respectively. For each $x \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, the set

$$x + \lambda K := \{x + \lambda y \mid y \in K\}$$

is called a smaller homothetic copy of $K$. We denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$, i.e., the set of compact convex sets in $\mathbb{R}^n$ having interior points.

For each compact convex set $K$, we denote by $c(K)$ the least number of translates of relint $K$ needed to cover $K$. Concerning the least upper bound of $c(K)$ in $\mathcal{K}^n$, there is a long standing conjecture (see [6], [3], and [2] for more information about this conjecture):

Conjecture 1 (Hadwiger’s covering conjecture). For each $K \in \mathcal{K}^n$, $c(K)$ is bounded from the above by $2^n$, and this upper bound is attained only by parallelopipeds.

The assertion “$c(K) \leq 2^n$, $\forall K \in \mathcal{K}^n$” will be referred to as the “inequality part” of Conjecture 1. This conjecture has been completely verified for several classes of convex bodies including: all planar convex bodies
bodies (cf. [10]), zonotopes, zonoids, belt bodies (cf. [3, §34]), and convex hulls of a pair of compact convex sets contained in two parallel hyperplanes in $\mathbb{R}^3$ (cf. [15]). And the inequality part of Conjecture 1 has been verified for centrally symmetric convex bodies in $\mathbb{R}^3$ (cf. [7]), convex polyhedron in $\mathbb{R}^3$ having an affine symmetry (cf. [1]), convex bodies in $\mathbb{R}^3$ symmetric about a plane (cf. [5]).

For each $m \in \mathbb{Z}^+$, we use the short-hand notation

$$[m] = \{ t \in \mathbb{Z}^+ \mid 1 \leq t \leq m \}.$$  

Note that, for each compact convex set $K$, $c(K)$ equals the least number of smaller homothetic copies of $K$ needed to cover $K$ (see, e.g., [3, p. 262, Theorem 34.3]). Therefore, $c(K) \leq m$ for some $m \in \mathbb{Z}^+$ if and only if $\Gamma_m(K) < 1$, where $\Gamma_m(K)$ is defined by

$$\Gamma_m(K) := \min \left\{ \gamma > 0 \mid \exists \{x_i \mid i \in [m]\} \subseteq \mathbb{R}^n \text{ s.t. } K \subseteq \bigcup_{i=1}^{m} (x_i + \gamma K) \right\},$$

and is called the covering functional of $K$ with respect to $m$ (cf. [8], where $\Gamma_m(K)$ is called the $m$-covering number of $K$, and [16]).

In this paper, we extend the results in [15] by studying the homothetic covering problem for compact convex sets that can be expressed as convex hulls of two or more compact convex sets. In Section 2, we provide an estimation of covering functionals for this class of convex bodies in $\mathcal{K}^n$. In Section 3, we solve Hadwiger’s conjecture with respect to convex bodies in $\mathcal{K}^3$ that are convex hulls of two compact convex sets having empty interiors.

2. COVERING FUNCTIONAL OF CONVEX HULLS OF COMPACT

CONVEX SETS

The following estimation of the covering functionals of convex hulls of compact convex sets $K_1, \ldots, K_p$ uses only information about the covering functionals of each $K_i$.

**Theorem 1.** Suppose that $K \in \mathcal{K}^n$ is the convex hull of convex compact sets $K_1, \ldots, K_p$ and $m_1, \ldots, m_p \in \mathbb{Z}^+$.

1. If $p \leq n + 1$, then
   $$\Gamma_{m_1+\cdots+m_p}(K) \leq \max \left\{ \frac{p - 1 + \Gamma_{m_i}(K_i)}{p} \mid i \in [p] \right\}.$$  

2. If $p > n + 1$, then
   $$\Gamma_{m_1+\cdots+m_p}(K) \leq \max \left\{ \frac{n + \Gamma_{m_i}(K_i)}{n + 1} \mid i \in [p] \right\}.$$
Proof. Without loss of generality, we may assume that $o \in \text{relint } K$. For each $i \in [p]$, put $\gamma_i = \Gamma_{m_i}(K_i)$. Then, for each $i \in [p]$, there exists a set $\{ y^i_j \mid j \in [m_i] \}$ of $m_i$ points such that

$$K_i \subseteq \bigcup_{j=1}^{m_i} (y^i_j + \gamma_i K_i) \subseteq \bigcup_{j=1}^{m_i} (y^i_j + \gamma_i K).$$

Let $x$ be an arbitrary point in $K$.

**Case 1.** $p \leq n + 1$. By Theorem 3.13 in [13], there exist $p$ points $x_1, \ldots, x_p$, $p$ numbers $\lambda_1, \ldots, \lambda_p \in [0, 1]$ such that

$$x_i \in K_i, \forall i \in [p], \sum_{i \in [p]} \lambda_i = 1, \text{ and } x = \sum_{i \in [p]} \lambda_i x_i.$$

We may assume, without loss of generality, that $\lambda_1 \geq \frac{1}{p}$ and $x_1 \in y^1_1 + \gamma_1 K_1 \subseteq y^1_1 + \gamma_1 K$.

Then

$$x = \lambda_1 x_1 + \sum_{i=2}^{p} \lambda_i x_i = \frac{1}{p} x_1 + \left( \lambda_1 - \frac{1}{p} \right) x_1 + \sum_{i=2}^{p} \lambda_i x_i$$

$$\leq \frac{1}{p} y^1_1 + \frac{p-1+\gamma_1}{p} K$$

$$\subseteq \frac{1}{p} y^1_1 + \max \left\{ \frac{p-1+\gamma_i}{p} \mid i \in [p] \right\} K.$$

It follows that

$$K \subseteq \bigcup_{i \in [p]} \left\{ \frac{1}{p} y^i_j \mid j \in [m_i] \right\} + \max \left\{ \frac{p-1+\gamma_i}{p} \mid i \in [p] \right\} K.$$

**Case 2.** $p > n + 1$. By the Carathéodory’s theorem, there exist $n + 1$ points $x_1, \ldots, x_{n+1} \in \bigcup_{i=1}^{p} K_i$ and $n + 1$ numbers $\lambda_1, \ldots, \lambda_{n+1} \in [0, 1]$ such that

$$x = \sum_{i=1}^{n+1} \lambda_i x_i \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

We may assume, without loss of generality, that $\lambda_1 \geq \frac{1}{n+1}$ and $x_1 \in y^1_1 + \gamma_1 K_1 \subseteq y^1_1 + \gamma_1 K$.

Then, in a similar way as above, we can show that

$$x \in \frac{1}{n+1} y^1_1 + \max \left\{ \frac{n+\gamma_i}{n+1} \mid i \in [p] \right\} K.$$
It follows that
\[ K \subseteq \bigcup_{i \in [p]} \left\{ \frac{1}{n+1} y_j^i \mid j \in [m_i] \right\} + \max \left\{ \frac{n+\gamma_i}{n+1} \mid i \in [p] \right\} K. \tag*{\square} \]

In particular, we have the following:

**Corollary 2.** Suppose that \( K \) is the convex hull of two non-empty compact convex sets \( L \) and \( M \), and that \( m_1, m_2 \in \mathbb{Z}^+ \). Then
\[ \Gamma_{m_1+m_2}(K) \leq \max \left\{ \frac{1+\Gamma_{m_1}(L)}{2}, \frac{1+\Gamma_{m_2}(M)}{2} \right\}. \]

**Corollary 3.** If \( K \) is the convex hull of segments \( K_1, \ldots, K_p \), then
\[ \Gamma_{2p}(K) \leq \frac{2p-1}{2p}. \]

**Remark 4.** When applying Theorem 1 to get a good estimation of \( \Gamma_m(K) \), a suitable representation of \( K \) as the convex hull of compact convex sets is necessary. For example, let \( K \) be a three-dimensional simplex with \( a, b, c, d \) as vertices. If we use the representation \( K = \text{conv}(\{a\} \cup \text{conv} \{b, c, d\}) \) then, by Theorem 1, we have \( \Gamma_4(K) \leq \frac{5}{6} \). But, if we use \( K = \text{conv}(\text{conv} \{a, b\} \cup \text{conv} \{c, d\}) \), we will have the estimation \( \Gamma_4(K) \leq \frac{3}{4} \), which is much better.

When \( n \) is odd and \( K \) is an \( n \)-dimensional simplex, it is not difficult to verify that \( \Gamma_{n+1}(K) = \frac{n}{n+1} \). By Corollary 3, we have \( \Gamma_{n+1}(K) \leq \frac{n}{n+1} \). This shows that the estimation in Theorem 1 is tight in general. However, it can be improved in many other cases by taking the extremal structure of \( K \) into consideration.

### 3. The Three-Dimensional Case

Let \( K \subset \mathbb{R}^n \) be a compact convex set, \( x \in \text{relbd} K \), and \( u \in \mathbb{R}^n \) be a non-zero vector. If there exists a scalar \( \lambda > 0 \) such that \( x + \lambda u \in \text{relint} K \), then we say that \( u \) illuminates \( x \). It is not difficult to see that, a set \( D \) of directions illuminates \( \text{relbd} K \) if and only if \( D \) illuminates all extreme points of \( K \). Moreover (cf. Theorem 34.3 in [3]), \( c(K) \) equals to the minimal cardinality of a set of directions that can illuminate \( \text{relbd} K \).

A pair of points \( a, b \) in a set \( X \subseteq \mathbb{R}^n \) is called antipodal provided there are distinct parallel hyperplanes \( H_a \) and \( H_b \) through \( a \) and \( b \), respectively, such that \( X \) lies in the slab between \( H_a \) and \( H_b \).

**Lemma 5.** Let \( K \in \mathcal{K}^n \) and \( x, y \in \text{bd} K \). If \( x \) and \( y \) are not antipodal, then there is a direction that illuminates both \( x \) and \( y \).
Proof. We only need to consider the case when \( x \neq y \). Since \( x \) and \( y \) are not antipodal, the segment \([x, y]\) is not an affine diameter (cf. [12] for the definition and basic properties of affine diameters) of \( K \). Let \([u, v]\) be an affine diameter of \( K \) parallel to \([x, y]\) and \( c \) be an interior point of \( K \). Without loss of generality, we may assume that

\[
\frac{x - y}{\|x - y\|} = \frac{u - v}{\|u - v\|}.
\]

Then there exists a number \( \lambda \in (0, 1) \) such that

\[
s - t = x - y, \quad s = \lambda c + (1 - \lambda)u, \quad t = \lambda c + (1 - \lambda)v.
\]

Clearly, both \( s \) and \( t \) are interior points of \( K \). Let \( d = s + t = x - y \). Then

\[
x + d = \frac{x - y}{2} + \frac{s + t}{2} = s, \quad \text{and} \quad y + d = \frac{y - x}{2} + \frac{s + t}{2} = t.
\]

I.e., \( x \) and \( y \) are both illuminated by \( d \neq 0 \). □

Theorem 6. Let \( K \in K^3 \) be a convex body. If there exist two compact convex sets \( L \) and \( M \) with empty interior such that \( K = \text{conv}(L \cup M) \), then \( c(K) \leq 8 \) and the equality holds if and only if \( K \) is a parallelepiped.

Proof. We denote by \( \text{aff} L \) and \( \text{aff} M \) the affine dimensions of \( L \) and \( M \), respectively. We distinguish four cases.

Case 1. \( 0 \in \{\text{aff} L, \text{aff} M\} \). Assume without loss of generality that \( \text{aff} L = 0 \). Then \( \text{aff} M = 2 \). By Theorem 4 in [14] and the fact that \( \Gamma_7(M) \leq \frac{1}{2} \) holds for each planar convex body (cf. [9]), we have

\[
\Gamma_8(K) \leq \frac{1}{2 - \Gamma_7(M)} \leq \frac{2}{3}.
\]

By Corollary 2 and the fact that \( \Gamma_4(M) \leq \frac{\sqrt{2}}{2} \) holds for each planar convex body \( M \) (cf. [8]), we have

\[
\Gamma_5(K) \leq \frac{1 + \sqrt{2}}{2} \approx 0.854.
\]

Case 2. \( \text{aff} L = \text{aff} M = 1 \). In this situation, \( K \) is a three-dimensional simplex. We have (cf. [16])

\[
\Gamma_8(K) \leq \Gamma_5(K) = \frac{9}{13}.
\]

Case 3. \( \{\text{aff} L, \text{aff} M\} = \{1, 2\} \). Assume without loss of generality that \( \text{aff} L = 1 \) and \( \text{aff} M = 2 \). Then \( \Gamma_2(L) = \frac{1}{2} \) and \( \Gamma_6(M) \leq \sin^2 \frac{3\pi}{10} \) (cf. [9]). Then, Corollary 2 shows that

\[
\Gamma_8(K) \leq \frac{1 + \Gamma_6(M)}{2} \leq \frac{1 + \sin^2 \frac{3\pi}{10}}{2} \approx 0.827.
\]
In a similar way as in Case 1, we have
\[
\Gamma_6(K) \leq \frac{1 + \sqrt{2}}{2} \approx 0.854.
\]

**Case 4.** \(\text{aff } L = \text{aff } M = 2\). In this case we have
\[
\Gamma_8(K) \leq \frac{1 + \sqrt{2}}{2} \approx 0.854.
\]

In the rest, we characterize the case when \(c(K) = 8\). The foregoing statements imply that \(\text{aff } L = \text{aff } M = 2\). If one of \(L\) and \(M\), say \(L\), is not a parallelogram, then \(\Gamma_3(L) < 1\) and \(\Gamma_4(M) < 1\). It follows from Corollary 2 that
\[
\Gamma_7(K) \leq \max \left\{ \frac{1 + \Gamma_3(L)}{2}, \frac{1 + \Gamma_4(M)}{2} \right\} < 1.
\]
Thus, \(c(K) \leq 7\), a contradiction. In the following we assume that both \(L\) and \(M\) are parallelograms. Since \(c(K) = 8\), \(\text{ext } K = \text{ext } L \cup \text{ext } M\) consisting of 8 points. Lemma 5 shows that the points in \(\text{ext } L \cup \text{ext } M\) are pairwise antipodal. By the main result in [11] (see also p. 225 in [1]), \(\text{ext } K\) is the set of vertices of a parallelepiped. When \(K\) is a parallelepiped, it is clear that \(c(K) = 8\). \(\square\)

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Email address: wusenlin@nuc.edu.cn

Email address: S1908029@st.nuc.edu.cn

Email address: hechan@nuc.edu.cn

Department of Mathematics, North University of China, 030051 Taiyuan, China