Patterns of Spontaneous Chiral Symmetry Breaking in the Large–$N_c$ Limit of QCD–Like Theories

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Abstract

It is shown that in vector–like gauge theories, such as QCD, and in the limit of a large number of colours $N_c$, the ordering pattern of narrow vector and axial–vector states at low energies is correlated with the size of the possible local order parameters of chiral symmetry breaking. This has implications on the underlying dynamics of spontaneous chiral symmetry breaking in QCD, and also on possible values of the parameter $S$ for oblique electroweak corrections, the analogue of the QCD low energy constant $L_{10}$.

Key-Words: Chiral perturbation theory, Large–$N_c$, Quantum Chromodynamics, Technicolour.
1 Introduction

Sometime ago, Coleman and Witten [1] showed that if QCD with the number of colours $N_c = 3$ confines, and if this property persists in the large–$N_c$ limit [2–4] then, in this limit, the chiral $U(n_f) \times U(n_f)$ invariance of the Lagrangian with $n_f$ flavours of massless quarks is spontaneously broken down to the diagonal $U(n_f)$ subgroup. Their proof assumes in particular that “the breakdown of chiral symmetry is characterized by a nonzero value of some order parameter which is bilinear in the quark fields and which transforms according to the representation $(n_f, \bar{n}_f) + (\bar{n}_f, n_f)$ of the chiral group”. A few years later, the argument was substantiated by a result due to Vafa and Witten [5]. These authors showed that in vector–like theories, the generators associated with the diagonal subgroup always leave the vacuum invariant in the theory with massive quarks. Combined with ’t Hooft’s anomaly matching conditions [6], this leads, in the large–$N_c$ limit, and without further assumptions besides confinement [1], to the expected pattern of spontaneous chiral symmetry breaking (S$\chi$SB), at least for the massless theory defined by the zero mass limit of the massive one.

As further emphasized by the authors of ref. [1], these results do not “give any insight at all into the mechanism of symmetry breakdown”. The purpose of this note is to show how, in the same large–$N_c$ limit, the ordering pattern of possible vector and axial–vector hadronic states in the physical spectrum is correlated to the size of the possible order parameters.

We shall consider a correlation function which is particularly sensitive to properties of S$\chi$SB, namely the two–point function

$$\Pi_{LR}^{\mu\nu}(q) = 2i \int d^4xe^{i q \cdot x} \langle 0 | T \left( L^\mu(x) R^{\nu}(0) \right)^\dagger | 0 \rangle, \quad (1)$$

with currents

$$L^\mu = \bar{u}(x) \gamma^\mu \frac{1}{2} (1 - \gamma_5) u(x) \quad \text{and} \quad R^\mu = \bar{d}(x) \gamma^\mu \frac{1}{2} (1 + \gamma_5) u(x). \quad (2)$$

In the chiral limit where the light quark masses are set to zero, this two–point function only depends on one invariant function ($Q^2 = -q^2 \geq 0$ for $q^2$ spacelike)

$$\Pi_{LR}^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_{LR}(Q^2). \quad (3)$$

We shall be concerned with properties of the self–energy function $\Pi_{LR}(Q^2)$ in the limit of vanishing light quark masses and infinite number of colours. Some of these properties are well known. First, $\Pi_{LR}(Q^2)$ vanishes order by order in perturbation theory and is an order parameter of the spontaneous breakdown of chiral symmetry for all values of the momentum transfer. It also governs the electromagnetic $\pi^+ - \pi^0$ mass difference

$$m_{\pi^+}^2|_{EM} = \frac{\alpha}{\pi} \left[ \frac{3}{8\pi^2} \int_0^\infty dQ^2 Q^2 \Pi_{LR}(Q^2) \right]. \quad (4)$$

This integral converges in the ultraviolet region because, as shown by Shifman, Vainshtein and Zakharov [8] using Wilson’s [9] operator product expansion (OPE),

$$\lim_{Q^2 \to \infty} \Pi_{LR}(Q^2) = \frac{1}{Q^2} \left[ -8\pi^2 \left( \frac{\alpha}{\pi} + \mathcal{O}(\alpha_s^2) \right) \langle \bar{\psi} \psi \rangle^2 \right] + \mathcal{O}\left( \frac{1}{Q^8} \right). \quad (5)$$

\[1\] An additional requirement in [5] is that all vacuum angles vanish.
Witten \cite{10, 11} has furthermore shown that
\[ -Q^2 \Pi_{LR}(Q^2) \geq 0 \quad \text{for} \quad 0 \leq Q^2 \leq \infty , \tag{6} \]
which in particular ensures the positivity of the integral in eq. (6) and thus the stability of the QCD vacuum with respect to small perturbations induced by electromagnetic interactions.

The low \( Q^2 \) behaviour of this self–energy function \cite{12} is governed by chiral perturbation theory
\[ -Q^2 \Pi_{LR}(Q^2) = f_{\pi}^2 + 4L_{10}Q^2 + \mathcal{O}(Q^4) , \tag{7} \]
where \( L_{10} \) is one of the Gasser–Leutwyler constants \cite{13} of the \( \mathcal{O}(p^4) \) low energy effective chiral Lagrangian, i.e. the Lagrangian formulated in terms of Goldstone degrees of freedom and external local sources only.

In the large–\( N_c \) limit, the spectral function associated to \( \Pi_{LR}(Q^2) \) consists of the difference of an infinite number of narrow vector states and an infinite number of narrow axial–vector states, together with the Goldstone pole of the pion:
\[ \frac{1}{\pi} \text{Im} \Pi_{LR}(t) = \sum_V f_V^2 M_V^2 \delta(t - M_V^2) - \sum_A f_A^2 M_A^2 \delta(t - M_A^2) - f_{\pi}^2 \delta(t) . \tag{8} \]

Since \( \Pi_{LR}(Q^2) \) obeys an unsubtracted dispersion relation, it follows that
\[ -Q^2 \Pi_{LR}(Q^2) = f_{\pi}^2 + \sum_A f_A^2 M_A^2 \frac{Q^2}{M_A^2 + Q^2} - \sum_V f_V^2 M_V^2 \frac{Q^2}{M_V^2 + Q^2} . \tag{9} \]

\section{2 Weinberg Sum Rules}

In the chiral limit, the first Weinberg sum rule \cite{14}
\[ \int_0^\infty dt \text{Im} \Pi_{LR}(t) = 0 , \tag{10} \]
implies that
\[ \sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 = f_{\pi}^2 ; \tag{11} \]
the second Weinberg sum rule \cite{14}
\[ \int_0^\infty dt t \text{Im} \Pi_{LR}(t) = 0 , \tag{12} \]
furthermore implies that
\[ \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0 . \tag{13} \]

In QCD \cite{3}, eq. (11) follows from the fact that there is no local order parameter of dimension \( d = 2 \) and eq. (13) from the absence of a local order parameter of dimension \( d = 4 \). The first possibly non trivial contribution comes from local order parameters of dimension \( d = 6 \), as

\footnote{For a discussion of the Weinberg sum rules in QCD in the presence of explicit chiral symmetry breaking see ref. \cite{15}}
shown in eq. (3). With these constraints incorporated into eq. (4), the self–energy function in the large–$N_c$ limit becomes

$$-Q^2 \Pi_{LR}(Q^2) = \sum_A f_A^2 M_A^6 \frac{1}{Q^2 (M_A^2 + Q^2)} - \sum_V f_V^2 M_V^6 \frac{1}{Q^2 (M_V^2 + Q^2)}. \quad (14)$$

There are further properties of this self–energy function, perhaps not so well known, which we now wish to emphasize:

- The expansion of the hadronic self–energy function in the r.h.s. of eq. (14) in inverse powers of $Q^2$:

$$\Pi_{LR}(Q^2) = \frac{1}{Q^6} (\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6) - \frac{1}{Q^8} (\sum_V f_V^2 M_V^8 - \sum_A f_A^2 M_A^8) + \cdots, \quad (15)$$

has to match the short–distance OPE evaluated in the QCD large–$N_c$ limit $^3$. This means e.g., that in this limit the leading $d = 6$ order parameter in eq. (5) is given by

$$-8\pi^2 \left( \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right) \langle \bar{\psi} \psi \rangle^2 = \sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 \equiv \phi(6). \quad (16)$$

- Witten’s inequality in eq. (6) implies in particular that

$$\lim_{Q^2 \to \infty} -Q^2 \Pi_{LR}(Q^2) \geq 0. \quad (17)$$

From this inequality and eq. (15) there follows the remarkable result that $\phi(6)$ in (16) must be negative (or zero), a fact which, as eq. (5) shows, is indeed confirmed by the explicit calculation to lowest non–trivial order $^3$.

- Inverse moments of the spectral function with the pion pole removed, i.e. integrals like

$$\int_0^\infty dt \frac{1}{tp} \left( \frac{1}{\pi} \text{Im} \Pi_{LR}(t) + f_\pi^2 \delta(t) \right) = \int_0^\infty dt \frac{1}{tp} \left( \frac{1}{\pi} \text{Im} \Pi_V(t) - \frac{1}{\pi} \text{Im} \Pi_A(t) \right), \quad (18)$$

with $p = 1, 2, 3, \cdots$, correspond to non–local order parameters which govern the couplings of local operators of higher and higher dimensions in the low energy effective chiral Lagrangian. For example, the first inverse moment is related to the $O(p^4)$ coupling constant $L_{10}$ as follows $^{16}$

$$-4L_{10} = \int_0^\infty \frac{dt}{t} \left( \frac{1}{\pi} \text{Im} \Pi_V(t) - \frac{1}{\pi} \text{Im} \Pi_A(t) \right) = \sum_V f_V^2 - \sum_A f_A^2. \quad (19)$$

- One of the parameters which characterize possible deviations from the Standard Model predictions in the electroweak sector is the so called $S$–parameter $^{17}$. In the unitary gauge, $S$ measures the strength of an anomalous $W_{\mu\nu}^{(3)} B^{\mu\nu}$ coupling. This is the $SU(2)_L \times SU(2)_R$ coupling analogous to the term proportional to $L_{10}$ in QCD. It has

$^3$Notice that the OPE of this self–energy function is free from renormalon ambiguities, because in perturbation theory it is protected by chiral symmetry, which ensures that $\Pi_{LR}(Q^2) = 0$ order by order in powers of $\alpha_s$. 

3
been argued that if the underlying theory of electroweak breaking is a vector–like gauge theory of the QCD–type, the sign of the \( L_{10} \)–like coupling must be negative, precisely as observed in QCD. In the electroweak sector, this fact is inferred by experimental observation and constitutes at present a serious phenomenological obstacle to technicolour–like formulations of electroweak symmetry breaking. As we shall see, the sign of the \( L_{10} \) coupling depends on the relative size of the local order parameters versus \( f_\pi^2 \).

3 Solving Systems of Weinberg Sum Rule Equations

Given an arbitrary, but finite, number \( n \) of narrow states of spin 1 \( \{ R_1, R_2, \cdots R_n \} \), we want to know which configurations of vector and axial vector narrow states are compatible with large–\( N_c \). We find it convenient to use the following notation: the squared masses and squared couplings will be denoted

\[
X_i \equiv M_{R_i}^2 \quad \text{and} \quad x_i \equiv \pm f_{R_i}^2, \quad i = 1, 2, 3, \cdots n, \quad (20)
\]

with the \( +f_{R_i}^2 \) choice if \( R_i \) is a vector state, and the \( -f_{R_i}^2 \) choice if \( R_i \) is an axial–vector state. The states are ordered according to increasing values of their masses,

\[
X_1 \leq X_2 \leq \cdots X_n. \quad (21)
\]

Cast into this notation, the first and second Weinberg sum rules result then in the following system of two equations:

\[
\begin{aligned}
& x_1 X_1 + x_2 X_2 + x_3 X_3 + \cdots + x_n X_n = f_\pi^2 \\
& x_1 X_1^2 + x_2 X_2^2 + x_3 X_3^2 + \cdots + x_n X_n^2 = 0.
\end{aligned} \quad (22)
\]

Implicit here is the assumption that the sum of the infinite number of narrow vector states and the sum of the infinite number of narrow axial–vector states with masses higher than the mass of the last narrow state \( R_n \) explicitly considered are already dual to their respective perturbative QCD continuum, so that their contributions cancel in the spectral function \( \text{Im}\Pi_{LR}(t) \), and hence in the Weinberg sum rules.

Further linear equations in the couplings \( x_i \), involving higher powers of the squared masses \( X_i \), result from identifying inverse powers of \( Q^2 \) in the OPE of the self–energy function \( \Pi_{LR} \),

\[
\Pi_{LR}(Q^2) = \sum_{p \geq 6} \frac{1}{Q^p} \phi^{(p)}, \quad (23)
\]

with the corresponding expansion of its hadronic counterpart in eq. (23). We can thus write a system of \( n \) linear equations for the \( n \) couplings \( x_i \):

\[
\begin{aligned}
& x_1 X_1^2 + x_2 X_2^2 + x_3 X_3^2 + \cdots + x_n X_n^2 = 0 \\
& x_1 X_1^3 + x_2 X_2^3 + x_3 X_3^3 + \cdots + x_n X_n^3 = \phi^{(6)} \\
& \cdots \\
& x_1 X_1^n + x_2 X_2^n + x_3 X_3^n + \cdots + x_n X_n^n = \phi^{(2n)}.
\end{aligned} \quad (24)
\]

The discriminant of this system is a Vandermonde determinant

\[
\Delta(X_1, X_2, X_3, \cdots X_n) = X_1 X_2 X_3 \cdots X_n \times \prod_{1 \leq i < j \leq n} (X_j - X_i), \quad (25)
\]
and the system has a solution. There are some interesting generic properties which emerge from this system of equations which we next discuss.

- The coupling constant $f^2_\pi$ is a non local order parameter of spontaneous chiral symmetry breaking. Assuming that $f^2_\pi \neq 0$, but setting the chiral condensates $\phi^{(6)} = \phi^{(8)} = \cdots = \phi^{(2n)} = 0$, the system in eq. (24) has a solution with alternate signs for the couplings $x_1, x_2, \cdots, x_n$, and with $x_1 > 0$. Therefore the ordering pattern of states in that case has to be $V_1 - A_2 - V_3 - A_4 - \cdots$, i.e. alternating vector and axial–vector states.

- Once the system in eq. (24) has been solved under the assumption that $\phi^{(6)} = \phi^{(8)} = \cdots = \phi^{(2n)} = 0$, the sizes of the chiral condensates of dimension $d \geq 2n + 2$ are then fixed. In particular,

$$\phi^{(2n+2)} = (-1)^{n+1} f^2_\pi X_1 X_2 X_3 \cdots X_n,$$

which cannot vanish. We then conclude that for QCD in the large–$N_c$ limit spontaneous chiral symmetry breaking à la Nambu–Goldstone with $f^2_\pi \neq 0$ necessarily implies the existence of non–zero local order parameters which transform according to the representation $(n_f, \bar{n}_f) + (\bar{n}_f, n_f)$ of the chiral group. This is in a way the converse of the Coleman–Witten theorem [1] stated in the Introduction.

- The sign of the lowest dimensional non–zero chiral condensate is fixed, and agrees with Witten’s inequality.

Let us next discuss the simplest systems of Weinberg sum rule equations with $n = 2$ and $n = 3$ in further detail.

### 3a The Two–State Case

This is the case originally considered by Weinberg [14], and also in resonance dominance estimates of the low energy constants of the effective chiral Lagrangian [18–20]. With the notation defined previously, the simplest case of two states $\{R_1, R_2\}$ has the solutions:

$$x_1 = f^2_\pi \frac{1}{X_1} \frac{X_2}{X_2 - X_1}, \quad x_2 = -f^2_\pi \frac{1}{X_2} \frac{X_1}{X_2 - X_1}. \quad (27)$$

This results in

$$-4L_{10} \equiv x_1 + x_2 = f^2_\pi \left( \frac{1}{X_1} + \frac{1}{X_2} \right), \quad (28)$$

and therefore the coupling $L_{10}$ in the two–state case has to be negative [21, 22]. Several comments are in order.

- We see that the two–state case is indeed a particular case of what has been stated above. It is the fact that $f^2_\pi \neq 0$ that allows for a non–trivial solution. Since $X_1 < X_2$, we find that $x_2 < 0$. Therefore the second resonance $R_2$ must be an axial–vector state, and the first one a vector state.

- Once the two–state system is solved, all the higher dimension condensates which break spontaneously the chiral symmetry are fully fixed by the masses $X_1$ and $X_2$ only; in particular the $d = 6$ condensate which appears in the OPE, see eq. (5), is now fixed to be

$$\phi^{(6)} \equiv x_1 X_1^3 + x_2 X_2^3 = -f^2_\pi X_1 X_2,$$

and is indeed negative.
• Notice that within the simple two–state pattern $V_1 - A_2$, the possibility that $\langle \bar{\psi} \psi \rangle = 0$, which would imply $\phi(6) = 0$, contradicts eq. (29). Therefore, we conclude that the extreme version of the so called generalized $\chi$PT proposed by J. Stern et al. [23], where $\langle \bar{\psi} \psi \rangle = 0$, and which we shall denote as G$\chi$PT$_0$ in what follows, is incompatible with the simplest realization of an hadronic low energy spectrum with only one $V$–state and only one $A$–state.

3b The Three–State Case

Let us next discuss the case of three narrow states $\{R_1, R_2, R_3\}$ with increasing ordering in masses: $X_1 \leq X_2 \leq X_3$. The corresponding system of Weinberg equations is now

$$
\begin{align*}
& x_1 X_1 + x_2 X_2 + x_3 X_3 = f_\pi^2, \\
& x_1 X_1^2 + x_2 X_2^2 + x_3 X_3^2 = 0, \\
& x_1 X_1^3 + x_2 X_2^3 + x_3 X_3^3 = \phi(6),
\end{align*}
$$

where we know that in the third equation $\phi(6) \leq 0$, but we leave its size as a free parameter. The solution of this system of equations is

$$
\begin{align*}
x_1 &= \frac{f_\pi^2 X_2 X_3 + \phi(6)}{X_1} \frac{1}{(X_2 - X_1)(X_3 - X_1)}, \\
x_2 &= \frac{f_\pi^2 X_3 X_1 + \phi(6)}{X_2} \frac{-1}{(X_2 - X_1)(X_3 - X_2)}, \\
x_3 &= \frac{f_\pi^2 X_1 X_2 + \phi(6)}{X_3} \frac{1}{(X_3 - X_1)(X_3 - X_2)};
\end{align*}
$$

and the sum of these solutions gives the following prediction for $L_{10}$:

$$
-4L_{10} = x_1 + x_2 + x_3 = f_\pi^2 \left( \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} \right) + \frac{\phi(6)}{X_1 X_2 X_3}.
$$

It follows from these equations that the pattern of $V$–states versus $A$–states is now governed by the size of the $d = 6$ order parameter $\phi(6)$. We observe the following generic facts:

• In the large–$N_c$ limit, $\langle \bar{\psi} \psi \rangle = 0$ not only entails $\phi(6) = 0$, but also $\phi(8) = 0$, since the $d = 8$ chiral condensate becomes also proportional to $\langle \bar{\psi} \psi \rangle$. This contradicts the result

$$
\phi(8) = f_\pi^2 X_1 X_2 X_3,
$$

which follows from eqs. (31) to (33). Therefore, we conclude that G$\chi$PT$_0$ is incompatible both with a hadronic low energy spectrum of one vector state and one axial–vector state as well as with a low energy spectrum of two vector states and one axial state.

• In QCD with $\langle \bar{\psi} \psi \rangle \neq 0$, $\phi(6)$ must be negative. From the solutions for the couplings $x_1, x_2, x_3$ above it follows that the ordering pattern will be $V_1 - A_2 - V_3$ provided $f_\pi^2 X_1 X_2 + \phi(6) > 0$. 


• When \( f_\pi^2 X_1 X_2 + \phi^{(6)} = 0 \), \( x_3 = 0 \) and the highest vector state decouples. We are then back to the two–state case \( V_1 - A_2 \) already discussed in the previous subsection 3a. When \( f_\pi^2 X_1 X_2 + \phi^{(6)} < 0 \), the spectrum pattern becomes \( V_1 - A_2 - A_3 \), until \( \phi^{(6)} \) is sufficiently negative, \( \phi^{(6)} = -f_\pi^2 X_1 X_3 \), where the lowest axial state decouples. For \( f_\pi^2 X_1 X_3 + \phi^{(6)} < 0 \), the pattern becomes \( V_1 - V_2 - A_3 \) and remains so until \( f_\pi^2 X_2 X_3 + \phi^{(6)} = 0 \), where \( x_1 = 0 \) and the first vector state decouples. Finally, for \( f_\pi^2 X_2 X_3 + \phi^{(6)} < 0 \), the pattern becomes \( A_1 - V_2 - A_3 \), and it remains like that as \( \phi^{(6)} \) becomes more and more negative.

• In all cases where \( f_\pi^2 X_2 X_3 + \phi^{(6)} \geq 0 \), we find that \(-4L_{10} > 0\). However, as \( \phi^{(6)} \) becomes more and more negative, there is a critical value

\[
\hat{\phi}^{(6)} = -f_\pi^2 (X_2 X_3 + X_1 X_3 + X_1 X_2) ,
\]

at which \(-4L_{10} = 0\). Beyond that critical value \(-4L_{10} < 0\). The pattern for positive values of \( L_{10} \) has to be \( A_1 - V_2 - A_3 \).

![Fig. 1](image-url)  
**Fig. 1** *The three normalized couplings \( x_1, x_2, x_3 \) versus the parameter \( \delta \) of eq. (37).*

The evolution of patterns as a function of \( \phi^{(6)} \) is illustrated in Fig. 1 for a choice of the mass squared values \( X_{1,2,3} \) as given by the central values of the masses of the observed lowest hadronic vector and axial–vector resonances, \( X_1 = 0.6 \text{ GeV}^2, X_2 = 1.5 \text{ GeV}^2, X_3 = 2.1 \text{ GeV}^2 \).
In this figure we plot the predicted couplings \( x_1, x_2, \) and \( x_3, \) normalized to their overall maximum, as functions of the dimensionless parameter \( \delta \) defined as follows

\[
\phi^{(6)} = -f_\pi^2 X_1 X_2 \delta.
\] (37)

Positive values of \( x_i \) correspond to vector states and negative values of \( x_i \) to axial–vector states. For \( \delta = 0.5 \), for instance, the pattern is \( V_1 - A_2 - V_3 \) in that order of increasing masses, while for \( \delta = 5 \) the pattern is \( A_1 - V_2 - A_3 \) also in that order of increasing masses. With this choice of masses, the constant \( L_{10} \) becomes positive for \( \delta \gtrsim 6 \).

4 Phenomenological Implications

There are several phenomenological implications which follow from the analyses reported in the previous sections, and which we now wish to discuss.

4a QCD Low Energy Phenomenology and the Two–State Case

The phenomenological ansatz that the hadronic spectrum in the vector and axial-vector channels can be well approximated by a single dominant low energy narrow state plus a perturbative QCD continuum with an onset adjusted by global duality arguments \(^{[24]} \) has been shown to be rather successful in many instances. It is at the basis of the QCD sum rule approach pioneered by the ITEP group \(^{[8]} \). Moreover, as already mentioned, it is the underlying assumption in the resonance dominance estimates of the low energy constants of the effective chiral Lagrangian \(^{[18–20]} \), estimates which are rather successful. The phenomenological successes of models, like the extended Nambu–Jona-Lasinio model \(^{[24]} \) are also correlated with this ansatz. It can be shown \(^{[26]} \) that the large–\( N_c \) limit of QCD and the OPE provide a logical support to these successes. One can see a posteriori that the approximation works because of the relatively small size of the perturbative \( \Lambda_{QCD} \)–scale. The onset of the dual continuum, in the vector and axial–vector channels, happens at values where perturbative QCD can already be trusted. The analysis of section 3a above shows that in the large–\( N_c \) limit the two–state pattern is necessarily correlated with the usual picture, where spontaneous breakdown of chiral symmetry is triggered through the formation of a strong \( \langle \bar{\psi} \psi \rangle \) condensate. Indeed, from eqs. \(^{[16]} \) and \(^{[23]} \) it follows that in this case

\[
8\pi^2 \frac{\alpha_s}{\pi} \langle \bar{\psi} \psi \rangle^2 \simeq f_\pi^2 X_1 X_2.
\] (38)

Using the experimental central values of \( f_\pi, M_\rho \) and \( M_{A_1} \), and with \( \alpha_s \) evaluated at the corresponding scale \( s_0 \) of global duality \((s_0 \simeq 1.6 \text{ GeV}^2)\), this equation predicts a quark condensate \( \langle \bar{\psi} \psi \rangle (1 \text{ GeV}) \simeq -(270 \text{ MeV})^3 \) which, within the expected errors of the approximations involved, is compatible with the values which can be obtained from other phenomenological estimates \(^{[27]} \).

4b The Generalized Chiral Perturbation Theory Alternative

The formation of a large \( \langle \bar{\psi} \psi \rangle \) condensate is usually accepted as being the actual mechanism of \( S\chi SB \). This assumption is dropped in G\( \chi PT \) \(^{[23]} \), which considers the possibility that the \( \langle \bar{\psi} \psi \rangle \) condensate may not be the dominant order parameter of \( S\chi SB \). It turns out that it is

\(^{[2]} \)See e.g. ref. \(^{[25]} \) for a recent review article where many other references can be found.
not easy to distinguish phenomenologically this possibility from the standard one with the present precision of low energy measurements. As we have demonstrated, in the combined large–\(N_c\) and chiral limit, the extreme version of G\(\chi\)PT, where the quark condensate would vanish exactly \(\langle \bar{\psi} \psi \rangle = 0\), requires a rather complex structure of the large–\(N_c\) spectrum before asymptotic freedom sets in. The minimum structure compatible with a vanishing quark condensate, but with a non–zero mixed condensate:

\[
\langle \bar{\psi} \psi \rangle = 0, \quad \langle \bar{\psi} \sigma_{\mu\nu} G^{\mu\nu} \psi \rangle \neq 0, \quad (39)
\]

is two vector states and two axial–vector states with the ordering pattern: \(V_1 - A_2 - V_3 - A_4\). It is the mixed condensate \(\langle \bar{\psi} \sigma_{\mu\nu} G^{\mu\nu} \psi \rangle\) which, even in the absence of a quark condensate, can give a term with dimension \(d = 10\) in the OPE.

It is interesting to ask how the \(V_1 - A_2 - V_3 - A_4\) pattern compares with what is presently known from experiment [28]. The answer is shown in Fig. 2 where we display the central mass values of the observed vector and axial–vector resonances in the isospin \(I = 1\) (Fig. 2a) and \(I = 1/2\) (Fig. 2b) channels. In the \(I = 1/2\) channel, the situation is not far from reproducing the required pattern. On the other hand, there is no established second axial–vector state in the \(I = 1\) channel. Hadronic \(\tau\) decays are at present the best place to observe a possible second \(I = 1\) axial–vector state, although the phase space limitations are severe. From a phenomenological point of view and with \(\Lambda_{QCD} \lesssim 300\) MeV (in the \(\overline{MS}\)–scheme), there is no compulsory need to fill the hadronic low energy spectrum with more than a prominent narrow state in the vector and axial–vector channels in order to obtain duality with perturbative QCD. In that sense, the option of S\(\chi\)SB with a vanishing quark condensate, although not rigorously ruled out, seems unnatural. On the other hand, the present analysis cannot exclude large \(1/N_c\)–corrections in the factorization of the vacuum expectation values of four–quark operators, and hence the possibility that in the real world the quark condensate might be smaller, say \(\langle \bar{\psi} \psi \rangle \sim -f_\pi^2\), than what is found in present estimates and than what is usually believed.

### 4c Implications for Models of Electroweak Symmetry Breaking

Perhaps the most interesting observation which emerges from the analyses reported in the previous sections is the possible existence of low energy particle spectra in vector–like gauge theories with a rather different structure than the one observed in the QCD hadronic spectrum. The simple three-state case discussed in section 3b shows already an explicit example of how the ordering pattern in the spectrum and the value of the electroweak \(S\) parameter are explicitly correlated to the size of the \(d = 6\) condensate. From this example, it seems plausible to consider technicolour–like models with a sufficiently large number of fermions so that the \(\beta\)–function enhances the value of the running coupling at the “pertinent” duality scale and makes the \(d = 6\) condensate strongly negative, and hence \(S\) negative as well.

We hope that these observations may open new paths in model building of electroweak symmetry breaking.

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Fig. 2a Low lying vector and axial–vector resonances in the isospin $I = 1$ channel.

Fig. 2b Low lying vector and axial–vector resonances in the isospin $I = 1/2$ channel.
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