December 1, 2023

SCHATTEN CLASS HANKEL OPERATORS ON WEIGHTED BERGMAN SPACES INDUCED BY REGULAR WEIGHTS

HAMZEH KESHAVARZI AND FANGLEI WU

Abstract. In this paper, for $1 \leq p < \infty$, we provide several descriptions of Schatten $p$-class Hankel operators $H_f$ and $H_{T'}$ on the weighted Bergman space $A^2_\omega$, in terms of a certain global and local mean oscillation of the symbol $f \in L^2_\omega$, provided $\omega$ is in a class of regular weights. The approaches applied to rely on several classical methods, and simultaneously rely on a novel but more convenient construction associated with the atomic decomposition of $A^2_\omega$.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A non-negative function $\omega \in L^1(\mathbb{D})$ such that $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ is called a radial weight. For $0 < p < \infty$ and such an $\omega$, the Lebesgue space $L^p_\omega$ consists of complex-valued measurable functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{L^p_\omega} = \int_\mathbb{D} |f(z)|^p \omega(z) \, dA(z) < \infty,$$

where $dA(z) = \frac{dx \, dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. The corresponding weighted Bergman space is $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$. As usual, we write $A^p_\omega$ for the classical weighted Bergman spaces induced by the standard weight $\omega(z) = (\eta + 1)(1 - |z|^2)^\eta$ with $-1 < \eta < \infty$.

For a radial weight $\omega$, the norm convergence in $A^2_\omega$ implies the uniform convergence on compact subsets, and therefore the Hilbert space $A^2_\omega$ is a closed subspace of $L^2_\omega$ and the orthogonal Bergman projection $P_\omega : L^2_\omega \to A^2_\omega$ is given by

$$P_\omega(g)(z) = \int_\mathbb{D} g(z) \overline{B^\omega_z(\zeta)} \omega(\zeta) \, dA(\zeta), \quad z \in \mathbb{D},$$

where $B^\omega_z$ is the reproducing kernel of $A^2_\omega$, associated with the point $z \in \mathbb{D}$. In the case of a standard weight, the Bergman reproducing kernels are given by the neat formula $(1-\pi\zeta)^{-(2+\eta)}$. For the convenience, throughout the paper we assume $K^\eta(\zeta) = (1-\pi\zeta)^{-\eta}$, and hence $K^{\eta+2}(\zeta)$ is the kernel of $A^2_\eta$. The most commonly known result on the Bergman projection is due to Bekolle and Bonami [3, 4], see [13, 14] for recent results and the reference therein.

A compact operator $T$ from a Hilbert space $H$ to another Hilbert space $K$ is said to belong to the Schatten class $S_p = S_p(H, K)$ if its sequence of singular numbers $\{\lambda_n\}$ belongs to $\ell^p_p$ ($0 < p < \infty$). It is well-known that for $1 \leq p < \infty$, the class $S_p$ is a Banach space with the norm $\|T\|_{S_p} = (\sum_j |\lambda_j|^p)^{1/p}$. Moreover, $S_p$ is closed under the product of operators, in other words, if $T \in S_p$, $A$ is a bounded operator on $H$ and $B$ a bounded operator on $K$, then $BTA \in S_p$. See [24] for more basic information about the Schatten class.

One important linear operator related to the Bergman projection $P_\omega$ is known as the (big) Hankel operator:

$$H_f(g)(z) = (I - P_\omega)(fg)(z), \quad f \in L^1_\omega, \; z \in \mathbb{D}.$$
The study of the Schatten class Hankel operators \( H_f \) and \( H_{\overline{f}} \) on Hilbert-weighted Bergman spaces is a compelling topic that has attracted considerable attention during the last decades. When the symbol \( f \) is analytic, we refer to [11, 12, 18, 21] for study of the Schatten class \( H_f \) and \( H_{\overline{f}} \) on standard weighted Bergman spaces \( A^2_\eta \) in the unit ball \( B_n \) of \( \mathbb{C}^n \). When \( f \) is considered to be a general integrable function, for the case \( 2 \leq p < \infty \), Zhu [22] characterized the Schatten class \( H_f \) and \( H_{\overline{f}} \) simultaneously on \( A^2_\eta \) in terms of a certain local mean oscillation of the symbol \( f \) associated with Bergman metric; using the same characterization, Xia [19, 20] and Isralowitz [6] solved the question for the case \( \max\{1, \frac{2n}{n+1+\eta}\} < p \leq 2 \) and \( \frac{2n}{n+1+\eta} < p \leq 1 \) respectively; and finally Pau [10] closed the full case \( 0 < p < \infty \) in terms of the same local mean oscillation. Apart from such local mean oscillation, another tool concerning the problem is known as the so-called (global) mean oscillation related to a Berezin-type transform of the symbol \( f \). As a matter of fact, from the aforementioned literature, these two characterizations of \( H_f, H_{\overline{f}} \in S_p \) are equivalent.

The purpose of the paper is to characterize Schatten class \( H_f \) and \( H_{\overline{f}} \) on \( A^2_\omega \), provided \( \omega \) is a class of certain locally smooth weights. We now proceed toward the exact statements via necessary definitions. Throughout this paper we assume \( \widehat{\omega}(z) = \int_{|z|}^1 \omega(s) \, ds > 0 \) for all \( z \in \mathbb{D} \), for otherwise \( A^2_\omega = \mathcal{H}(\mathbb{D}) \). A weight \( \omega \) belongs to the class \( \mathcal{D} \) if there exists a constant \( C = C(\omega) \geq 1 \) such that \( \widehat{\omega}(r) \leq C\widehat{\omega}(\frac{r}{r+1}) \) for all \( 0 \leq r < 1 \). Moreover, if there exist \( K = K(\omega) > 1 \) and \( C = C(\omega) > 1 \) such that \( \widehat{\omega}(r) \geq C\widehat{\omega}(1-\frac{1}{r+1}) \) for all \( 0 \leq r < 1 \), then we write \( \omega \in \mathcal{D} \). In other words, \( \omega \in \mathcal{D} \) if there exists \( K = K(\omega) > 1 \) and \( C' = C'(\omega) > 0 \) such that

\[
\widehat{\omega}(r) \leq C' \int_r^{1-\frac{1}{r+1}} \omega(t) \, dt, \quad 0 \leq r < 1.
\]

The intersection \( \mathcal{D} \cap \mathcal{D} \) is denoted by \( \mathcal{D} \). The class \( \mathcal{R} \subset \mathcal{D} \) of regular weights consists of those radial weights for which \( \widehat{\omega}(r) = \omega(r)(1-r) \) for all \( 0 \leq r < 1 \). We immediately see that all standard weights belong to \( \mathcal{R} \). The true advantage of the class \( \mathcal{R} \) is the local smoothness of its weights. It is clear that if \( \omega \in \mathcal{R} \), then for each \( s \in [0,1] \) there exists a constant \( C = C(s,\omega) > 1 \) such that

\[
C^{-1} \omega(t) \leq \omega(r) \leq C\omega(t), \quad 0 \leq r \leq t \leq r + s(1-r) < 1.
\]

It will turn out that the global mean oscillations defined by a certain Berezin-type transform and local mean oscillations related to the Bergman metric are both efficient tools for depicting \( H_f, H_{\overline{f}} \in S_p \). Note that for any radial weight \( \omega \), there exists a sufficiently large \( \eta = \eta(\omega) > 0 \) such that the inclusion \( A^2_\omega \subset A^2_\eta \), which makes \( k_{\omega,z}^{\eta+2} := K_2^{\eta+2}\|K_2^{\eta+2}\|_{A^2_\omega} \) well-defined. Now, for \( g \in L^1_\omega \) and such \( \eta \), the Berezin-type transform \( B_{\omega,\eta}(g) \) is defined as

\[
B_{\omega,\eta}(g)(z) = \langle g k_{\omega,z}^{\eta+2}, k_{\omega,z}^{\eta+2} \rangle_{L^2_\omega}.
\]

Then we define

\[
MO_{\omega,\eta}(f)(z) = (B_{\omega,\eta}(|f|^2)(z) - |B_{\omega,\eta}(f)(z)|^2)^{\frac{1}{2}},
\]

which, in some senses, can be treated as a certain global mean oscillations because

\[
MO_{\omega,\eta}(f)(z) = \|f k_{\omega,z}^{\eta+2} - B_{\omega,\eta}(f)(z) k_{\omega,z}^{\eta+2}\|_{A^2_\omega},
\]

and

\[
MO_{\omega,\eta}(f)(z) = \left( \int_{\mathbb{D}} \left| f(u) - f(\zeta) \right|^2 |k_{\omega,z}^{\eta+2}(u)|^2 |k_{\omega,z}^{\eta+2}(\zeta)|^2 \omega(u) \omega(\zeta) \, dA(u) \, dA(\zeta) \right)^{\frac{1}{2}}.
\]
Before defining the local mean oscillation of a locally square integrable function on $\mathbb{D}$ in the Bergman metric, recall that the Bergman metric $\beta$ on $\mathbb{D}$ is defined by

$$\beta(z, \zeta) = \frac{1}{2} \log \frac{1 + |\varphi_z(\zeta)|}{1 - |\varphi_z(\zeta)|}, \quad z, w \in \mathbb{D},$$

where $\varphi_z$ is the automorphism of $\mathbb{D}$, i.e. $\varphi_z(\zeta) = \frac{z - \zeta}{1 - \overline{z}\zeta}$. For a fixed $r > 0$, the Bergman disc $D(z, r)$ centered at $z$ with the radius of $r$ is defined by $D(z, r) = \{\zeta \in \mathbb{D} : \beta(z, \zeta) < r\}$. It is well-known that $D(z, r)$ is the Euclidean disc centered at $(1 - \tanh^2 r/2) \zeta/(1 - \tanh^2 r/2)$ and of radius $(1 - |\zeta|^2) \tanh^2 r/(1 - \tanh^2 r|\zeta|^2)$. Then the local mean oscillation of $f \in L^2_\omega$ in the Bergman metric is defined to be

$$MO_{\omega, r}(f)(z) = \left(\frac{1}{\omega(D(z, r))} \int_{D(z, r)} |f(\zeta) - \widehat{f}_{\omega, r}(z)|^2 \omega(\zeta) \, dA(\zeta)\right)^{1/2},$$

where $\omega(D(z, r)) = \int_{D(z, r)} \omega \, dA$ and the averaging function $\widehat{f}_{\omega, r}$ is given by

$$\widehat{f}_{\omega, r}(z) = \frac{1}{\omega(D(z, r))} \int_{D(z, r)} f(\zeta) \omega(\zeta) \, dA(\zeta).$$

It is easy to check that for any $z \in \mathbb{D}$ and $r > 0$, one has

$$MO_{\omega, r}(f)(z) = \left(\frac{1}{2\omega(D(z, r))} \int_{D(z, r)} \int_{D(z, r)} |f(u) - f(\zeta)|^2 \omega(u) \omega(\zeta) \, dA(u) \, dA(\zeta)\right)^{1/2}. \quad (1.4)$$

Let $d\lambda$ be the Möbius invariant measure on $\mathbb{D}$. That is, $d\lambda(z) = dA(z)/(1 - |z|^2)^2$. Our main result can be stated as follows.

**Theorem 1.** Let $1 \leq p < \infty$, $\omega \in \mathcal{R}$, and $f \in L^2_\omega$. Then the following statements are equivalent:

(i) $H_f$ and $H_T$ are in $S_p(A^p_\omega, L^2_\omega)$;

(ii) There exists an $\eta_0 = \eta_0(\omega) > 0$ such that $MO_{\omega, \eta}(f) \in L^p(\mathbb{D}, \omega \, d\lambda)$ for some (equivalently for all) $\eta > \eta_0$;

(iii) $MO_{\omega, r}(f) \in L^p(\mathbb{D}, \omega \, d\lambda)$ for some (equivalently for all) $r = r(\omega) > 0$.

It is worth mentioning that (iii) in the theorem does actually imply the compactness of both $H_f$ and $H_T$, provided $\omega \in \mathcal{R}$. Being precise, since $\omega \in \mathcal{R}$, it follows from [12, (1.2)] that for any fixed $r > 0$ and for any $u, \zeta \in D(z, r), \omega(u) \propto \omega(\zeta) \propto \omega(z)$, $\omega(D(z, r)) \propto \omega(z)(1 - |z|^2) \propto \omega(z)(1 - |z|^2)^2$, $z \in \mathbb{D}$, and hence (1.4) is comparable to $MO_{2, r}$ in [3]. Therefore (iii) implies $\lim_{|z| \to 1} MO_{\omega, r}(f)(z) = 0$ and hence it follows from [5, Theorem 4.5] with the special case $p = q$ that both $H_f$ and $H_T$ are compact on $A^2_\omega$. Alternatively, the similar proof of [15, Theorem 1] yields the compactness of both $H_f$ and $H_T$ directly. Apparently, the result covers partially of [10] Theorem 1 in the case of $1 \leq p < \infty$ and in the setting of the unit disc.

The proof of the theorem will be done by verifying that (i) $\Rightarrow$ (iii), (ii) $\Leftrightarrow$ (iii), and (ii) $\Rightarrow$ (i) respectively. The method used to prove that (i) implies (iii) is inherited from [6], which depends on an efficient estimate of the local mean oscillation of the symbol $f$. The proof of (ii) $\Leftrightarrow$ (iii) is proved with the aid of Lemma [3] which can be set up by the approach used in [10, Lemma 3.1]. Meanwhile, we can prove the equivalence between (ii) and (iii) for the full case $0 < p < \infty$ and for the involved weight $\omega \in \mathcal{D}$. To prove that (ii) implies (i), for the full range $0 < p < \infty$, instead of using some classical techniques, we construct a linear bounded operator on $A^2_\omega$, which makes the proof easier and hence avoid lots of laborious calculations.
To be more concrete, suppose \( \{e_j\} \) is an orthonormal basis for \( A^2_\omega \). Note that if \( \omega \in \mathcal{D} \), then for a large enough \( \eta = \eta(\omega) \)
\[
|k_{\omega,z}^{\eta+2}(\zeta)| \approx \frac{(1-|z|)^{\eta+3/2}\hat{\omega}(z)^{-1/2}}{|1-\overline{z}\zeta|^\eta}, \quad z, \zeta \in \mathcal{D}.
\] (1.5)

This together with the atomic decomposition (see [17, Theorem 2] with the special case \( p = q = 2 \)) enable us to define a certain linear operator \( A: A^2_\omega \to A^2_\omega \) satisfying \( Ae_j = k_{\omega,j}^{\eta+2} \) and \( A \) is bounded and onto, where \( \{a_j\} \) is \( r \)-lattice of \( \mathbb{D} \). Then an application of [24, Theorem 1.27 and Proposition 1.31] gives everything we aim for.

A careful reader may have already realized that the proof of (i) \( \Rightarrow \) (iii) is just dealt with in the case of \( \omega \in \mathcal{R} \) instead of \( \omega \in \mathcal{D} \). However, a substantial obstacle will appear in the proof if one tries to use a similar method for a general weight in \( \mathcal{D} \). Indeed, if \( \omega \) is only assumed to belong to \( \mathcal{D} \), then by [13, Theorem 1], we may find a large enough \( r_0 = r_0(\omega) \) such that for all \( r > r_0 \),
\[
\|B^2_z\|_{A^2_\omega} \approx \omega(D(z, r))^{-1} \approx 1 \omega(z)|1-|z||^{\omega}, \quad z \in \mathbb{D}.
\] (1.6)

Nevertheless, we are not in a position to obtain the same estimate as in Lemma [3] because \( r \) is supposed to be sufficiently small. This obstacle does not happen if \( \omega \in \mathcal{R} \), since the last (1.6) is valid for all \( r > 0 \). Therefore, some new techniques should be developed in this case.

We finish the introduction by a couple of words about the notation used in this paper. Throughout the paper \( \frac{1}{p} + \frac{1}{q} = 1 \) for \( p < \infty \). Further, the letter \( C = C(\cdot) \) will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and which may change from one occurrence to another. If there exists a constant \( C = C(\cdot) > 0 \) such that \( a \leq Cb \), then it is written either \( a \lesssim b \) or \( b \gtrsim a \). In particular, if \( a \lesssim b \) and \( a \gtrsim b \), then it is denoted by \( a \asymp b \) and said that \( a \) and \( b \) are comparable.

2. Auxiliary results

In this section, we are going to present several auxiliary lemmas that are useful for proving Theorem [1]. Some of them are proved not only for regular weights but for a wider class of weights.

We begin with a simple but important result for \( \omega \in \mathcal{D} \).

Lemma 2. Let \( \omega \in \mathcal{D} \). Then for any \( c \geq 0 \) there exists an \( \eta_0 = \eta_0(\omega, c) > 0 \) such that for all \( \eta > \eta_0 \)
\[
\int_{\mathcal{D}} |K^2_{\omega}(\zeta)|^{1/2} \omega(z, \zeta)^{c} dA(\zeta) \lesssim \frac{\hat{\omega}(z)}{(1-|z|)^{\eta-c}}, \quad z \in \mathbb{D}.
\] (2.1)

Proof. The definition of the Bergman distance implies that \( \beta \) grows logarithmically, and hence for any \( \varepsilon > 0 \), we have
\[
\beta(z, \zeta) \lesssim \left( \frac{1-|z|}{1-|z|} \right)^{-\varepsilon}, \quad z, \zeta \in \mathbb{D}.
\] (2.2)

Since \( \omega \in \mathcal{D} \), by [16, Lemma B], we see that there exists a \( \beta = \beta(\omega) > 0 \) such that the function \( \frac{\hat{\omega}(r)}{(1-r)^c} \) is essentially decreasing on \((0, 1)\). For a sufficiently small \( \varepsilon = \varepsilon(\omega, c) > 0 \) such that \( c \varepsilon < \beta \), we have \( \omega_{[-c\varepsilon]}(z) = \omega(z)(1-|z|)^{-c\varepsilon} \in \mathcal{D} \). Indeed, on one hand, since \( \omega \in \mathcal{D} \subset \hat{\mathcal{D}} \),
\[
\omega_{[-c\varepsilon]}(\frac{1+r}{2}) = \int_{\frac{1+r}{2}}^{1} \omega(t) \frac{dt}{(1-t)^c} \approx (1-\frac{1+r}{2})^{-c\varepsilon} \hat{\omega}(\frac{1+r}{2}) \approx (1-r)^{-c\varepsilon} \hat{\omega}(r), \quad 0 < r < 1.
\]
On the other hand, using an integration by parts, the fact that \( \omega \in \mathcal{D} \subset \hat{\mathcal{D}} \) and [16] Lemma B, we have

\[
\hat{\omega}(t)(1-t)^{-ce} = \int_1^1 (1-t)^{-ce} \, dt = -\int_1^1 (1-t)^{-ce} \, d\hat{\omega}(t)
\]

Therefore, \( \omega(z)(1-|z|)^{-ce} \in \hat{\mathcal{D}} \). Now, for any \( \eta > \eta_0 = \eta_0(\omega, c) \) with \( \eta_0 > 2ce + 1 \), [22], [11] Lemma 2.1 (vii) and (2.3) yield

\[
\int_{\mathbb{D}} |K^\eta_z(\zeta)| \beta(\zeta, \zeta)^c \omega(\zeta) dA(\zeta) = \int_{\mathbb{D}} \frac{1}{|1-\zeta|^\eta} \beta(\zeta, \zeta)^c \omega(\zeta) dA(\zeta)
\]

with

\[
\approx (1-|z|)^{-ce} \int_{\mathbb{D}} \frac{1}{|1-\zeta|^\eta} (1-|\zeta|)^{-ce} \omega(\zeta) dA(\zeta)
\]

The following lemma plays a critical role in the proof, while the corresponding result can not be generalized to the case \( \omega \in \mathcal{D} \) by the same method.

**Lemma 3.** Let \( \omega \in \mathcal{R} \). Then there exists an \( r = r_0(\omega) > 0 \) such that

\[
MO_{\omega, r}(f)(z)^2 \lesssim \frac{1}{\omega(D(z, r))} \int_{D(z, r)} \left( |f(u) - f(\zeta)| B^\omega_u(\zeta) \omega(\zeta) dA(\zeta) \right)^2 \omega(u) dA(u), \quad z \in \mathbb{D}.
\]

**Proof.** Using the same method of proving [9] p.129], we see that for any \( 0 < p < \infty \) and \( 0 < \sigma < r \)

\[
|f(u) - f(z)|^p \lesssim \frac{\beta(z, u)^p}{(1-|z|)^2} \int_{D(z, r)} |f(\zeta)|^p dA(\zeta), \quad u \in D(z, \sigma), \quad f \in \mathcal{H}(\mathbb{D}).
\]

This together with [16] Lemma 6 and (2.4)] yields for any fixed \( r > 0 \)

\[
|B^\omega_u(\zeta) - B^\omega_z(\zeta)| \lesssim |B^\omega_u(\zeta) - B^\omega_z(u)| + |B^\omega_z(u) - B^\omega_z(z)|
\]

where the last estimate holds due to the fact that \( B^\omega_z(z) = \|B^\omega_z(z)\|_{A^2_z} \). That is, there exists a constant \( C = C(\omega) \) such that

\[
\frac{|B^\omega_u(\zeta)|}{\|B^\omega_z\|^2} - 1 \leq C(\beta(\zeta, z) + \beta(z, u)) \leq C, \quad z \in \mathbb{D}.
\]
Now, if we take \( L_u(\zeta) = \left( \frac{B^\omega_u(\zeta)}{B_{\infty}^\omega} - 1 \right) \), then \( |L_u(\zeta)| \leq C \) for all \( u, \zeta \in D(z, r) \). Since \( \omega \in \mathcal{R} \), it follows from \([4, 6]\) that \( \|B_x^\omega\|_{A_2^\omega}^2 = \omega(D(z, r))^{-1} = \frac{1}{\omega(z)(1-|z|)} \) for all \( r > 0 \). Therefore, we have

\[
\frac{1}{\omega(D(z, r))} \int_{D(z, r)} \left| \int_{D(z, r)} \left( f(u) - f(\zeta) \right) B^\omega_u(\zeta) \omega(\zeta) \, dA(\zeta) \right|^2 \omega(u) \, dA(u) = \frac{1}{\omega(D(z, r))} \int_{D(z, r)} \left| \int_{D(z, r)} (f(u) - f(\zeta)) (1 + L_u(\zeta)) \|B_x^\omega\|_{A_2^\omega}^2 \omega(\zeta) \, dA(\zeta) \right|^2 \omega(u) \, dA(u) \quad (2.4)
\]

Then applying the triangle inequality that \( |a + b|^2 \geq \frac{|a|^2}{2} - |b|^2 \) to the inner integral of the last formular above, we have

\[
\frac{1}{2} \left| \int_{D(z, r)} (f(u) - f(\zeta)) (1 + L_u(\zeta)) \omega(\zeta) \, dA(\zeta) \right|^2 - \frac{1}{2} \left| \int_{D(z, r)} (f(u) - f(\zeta)) L_u(\zeta) \omega(\zeta) \, dA(\zeta) \right|^2 \quad (2.5)
\]

Therefore, \([2.4], (2.5)\) and the above estimate of \( |L_u(\zeta)| \) yield

\[
\frac{1}{\omega(D(z, r))} \int_{D(z, r)} \left| \int_{D(z, r)} (f(u) - f(\zeta)) B^\omega_u(\zeta) \omega(\zeta) \, dA(\zeta) \right|^2 \omega(u) \, dA(u) \geq \frac{1}{2 \omega(D(z, r))} \int_{D(z, r)} \frac{1}{\omega(D(z, r))} \int_{D(z, r)} (f(u) - f(\zeta)) \omega(\zeta) \, dA(\zeta) \right|^2 \omega(u) \, dA(u) \quad (2.6)
\]

On one hand, we have

\[
\frac{1}{\omega(D(z, r))} \int_{D(z, r)} (f(u) - f(\zeta)) \omega(\zeta) \, dA(\zeta) = f(u) - \widehat{\int \omega_r}(z). \quad (2.7)
\]
On the other hand, Cauchy-Schwarz inequality yields

\[
\int_{D(z,r)} \left( \int_{D(z,r)} |f(u) - f(\zeta)| \omega(\zeta) \, dA(\zeta) \right)^2 \omega(u) \, dA(u) \\
\leq \omega(D(z,r)) \int_{D(z,r)} \int_{D(z,r)} |f(u) - f(\zeta)|^2 \omega(\zeta) \omega(u) \, dA(\zeta) \, dA(u) \\
= \omega(D(z,r))^3 \frac{1}{\omega(D(z,r))^2} \int_{D(z,r)} \int_{D(z,r)} |f(u) - f(\zeta)|^2 \omega(\zeta) \omega(u) \, dA(\zeta) \, dA(u) \\
= 2\omega(D(z,r))^3 MO_{\omega, r}(f)(z)^2, \quad z \in \mathbb{D}.
\]  

Then, combining (2.6), (2.7) and (2.8), we deduce

\[
\frac{1}{\omega(D(z,r))} \int_{D(z,r)} \left( \int_{D(z,r)} (f(u) - f(\zeta)) B_\omega(\zeta) \omega(\zeta) \, dA(\zeta) \right)^2 \omega(u) \, dA(u) \\
\geq \left( \frac{1}{2} - 2C^2r^2 \right) MO_{\omega, r}(f)(z)^2, \quad z \in \mathbb{D}.
\]

Finally, by choosing \( r \) so that \( 0 < C^2r^2 < \frac{1}{4} \), we arrive at the desired result. \( \square \)

The following lemma is critical to the proof of the lemma 5.

**Lemma 4.** Let \( \omega \in \mathcal{D} \) and \( f \in L^2_\omega \). Then there exist an \( r_0 = r_0(\omega) > 0 \) such that for any \( r \geq r_0 \) and \( z, \zeta \in \mathbb{D} \) with \( \beta(z, \zeta) < r \) we have

\[
|\widehat{f}_{\omega, r}(z) - \widehat{f}_{\omega, r}(\zeta)| \lesssim MO_{\omega, 2r}(f)(z), \quad |\widehat{g}_{\omega, r}(z)| \lesssim MO_{\omega, 2r}(f)(z)^2,
\]

where \( g = f - \widehat{f}_{\omega, r} \).

**Proof.** We first observe that since \( \omega \in \mathcal{D} \) by the hypothesis, there exists an \( r_0 = r_0(\omega) > 0 \) such that \( \omega(D(z, r)) \sim \omega(z)(1 - |z|) \) as \( |z| \to 1^- \), provided \( r > r_0 \). This asymptotic equality together with Fubini’s theorem and Hölder inequality indicates that for \( z, \zeta \in \mathbb{D} \) with \( \beta(z, \zeta) < r \)

\[
|\widehat{f}_{\omega}(z) - \widehat{f}_{\omega}(\zeta)|^2 = \left\{ \frac{1}{\omega(D(z, r)) \omega(D(\zeta, r))} \int_{D(\zeta, r)} \int_{D(z, r)} f(u) \omega(u) \, dA(u) \omega(v) \, dA(v) \right. \\
- \frac{1}{\omega(D(z, r)) \omega(D(\zeta, r))} \int_{D(z, r)} \int_{D(\zeta, r)} f(v) \omega(v) \, dA(v) \omega(u) \, dA(u) \left\}^2 \\
\leq \frac{1}{\omega(D(\zeta, r)) \omega(D(z, r))} \left( \int_{D(z, r)} \int_{D(\zeta, r)} |f(u) - f(v)| \omega(u) \omega(v) \, dA(u) \, dA(v) \right)^2 \\
\leq \frac{1}{\omega(D(\zeta, r)) \omega(D(z, r))} \int_{D(z, r)} \int_{D(\zeta, r)} |f(u) - f(v)|^2 \omega(u) \omega(v) \, dA(u) \, dA(v) \\
\lesssim \frac{1}{\omega(D(z, 2r))^2} \int_{D(z, 2r)} \int_{D(z, 2r)} |f(u) - f(v)|^2 \omega(u) \omega(v) \, dA(u) \, dA(v) = MO_{\omega, 2r}(f)(z).
\]
Next, to see the second inequality, the triangle inequality and the first inequality yield
\[
\left( \left| g \right|_r^2(z) \right)^{\frac{1}{2}} = \left( \frac{1}{\omega(D(z, r))} \int_{D(z, r)} \left| f(\zeta) - \widehat{f}_{\omega, r}(\zeta) \right|^2 dA_{\omega}(\zeta) \right)^{\frac{1}{2}} \\
\leq \left( \frac{1}{\omega(D(z, r))} \int_{D(z, r)} \left| f(\zeta) - \widehat{f}_{\omega, r}(\zeta) \right|^2 \omega(\zeta) dA(\zeta) \right)^{\frac{1}{2}} \\
+ \left( \frac{1}{\omega(D(z, r))} \int_{D(z, r)} \left| \widehat{f}_{\omega, r}(z) - \widehat{f}_{\omega, r}(\zeta) \right|^2 \omega(\zeta) dA(\zeta) \right)^{\frac{1}{2}} \\
\leq \left( \frac{1}{\omega(D(z, 2r))} \int_{D(z, 2r)} \left| f(\zeta) - \widehat{f}_{\omega, r}(\zeta) \right|^2 \omega(\zeta) dA(\zeta) \right)^{\frac{1}{2}} \\
+ MO_{\omega, 2r}(f)(z) \left( \frac{1}{\omega(D(z, r))} \int_{D(z, r)} \omega(\zeta) dA(\zeta) \right)^{\frac{1}{2}} \approx MO_{\omega, 2r}(f)(z).
\]
The proof is complete. \(\square\)

Minor modifications in the proof of \cite{10} Lemma 3.1 together with Lemma \ref{lemma4} yield the following result, which plays a key role in the proof of the main theorem.

**Lemma 5.** Let \(0 < p, d, \delta < \infty\) and \(\omega \in D\). Then there exist an \(r_0 = r_0(\omega) > 0\) such that for any \(r \geq r_0\) and any \(r\)-lattice \(\{a_j\}\),
\[
\left| \widehat{f}_{\omega, r}(z) - \widehat{f}_{\omega, r}(\zeta) \right| \leq N_p(f, \zeta)^{1/p} \left| 1 - \zeta \right|^d(1 + \beta(\zeta, z))(\min \min \beta(z, \zeta, (1 - |z|, 1 - |\zeta|))^{-\delta} z, \zeta \in \mathbb{D},
\]
with
\[
N_p(f, \zeta) = \sum_{j} MO_{\omega, 2r}(f)(a_j)^p \frac{(1 - |a_j|^2)^{\delta p}}{|1 - \zeta a_j|^{pd}}.
\]

**Proof.** The argument is similar as the proof of \cite{10} Lemma 3.1. For the convenience of the reader, we provide the proof. For any \(z, \zeta \in \mathbb{D}\), let \(\gamma(t)\), \(0 \leq t \leq 1\) be the geodesic in the Bergman metric from \(z\) to \(\zeta\). Let \(N = \lfloor 3\beta(z, \zeta)/r \rfloor + 1\) (where \(\lfloor x \rfloor\) denotes the largest integer no greater than \(x\)) and \(t_j = j/N, j = 0, 1, \cdots, N\). Setting \(z_j = \gamma(t_j)\), we have
\[
\beta(z_j, z_{j+1}) = \frac{\beta(z_j, \zeta)}{N} \geq \frac{r}{3}.
\]
For each \(j\), choose a point \(a_j\) in the lattice such that \(\beta(z_j, a_j) \leq \frac{r}{5}\) and hence \(\beta(z_{j-1}, a_j) \leq \frac{5r}{6} < r\). Then the above estimate, triangle inequality and Lemma \ref{lemma4} yield
\[
\left| \widehat{f}_{\omega, r}(z) - \widehat{f}_{\omega, r}(\zeta) \right| \leq \sum_{j=1}^{N} \left| \widehat{f}_{\omega, r}(z) - \widehat{f}_{\omega, r}(z_{j-1}) \right| \\
\leq \sum_{j=1}^{N} \left| \widehat{f}_{\omega, r}(z_{j-1}) - \widehat{f}_{\omega, r}(a_j) \right| + \left| \widehat{f}_{\omega, r}(z_j) - \widehat{f}_{\omega, r}(a_j) \right| (2.9)
\]
\[
\leq \sum_{j=1}^{N} MO_{\omega, 2r}(f)(a_j), z, \zeta \in \mathbb{D}.
\]
It follows from the proof of [10] Lemma 3.1] that
\[
\frac{|1 - \zeta a_j|}{|1 - \zeta z|} \leq 2, \quad z, \zeta \in \mathbb{D}.
\]

Applying this estimate into (2.9), for any 0 < d < \infty we deduce
\[
|\widehat{f}_z(z) - \widehat{f}_z(\zeta)| \lesssim \sum_{j=1}^{N} \frac{MO_{\omega,2r}(f)(a_j)}{|1 - \zeta a_j|^{d}} |1 - \zeta z|^d, \quad z, \zeta \in \mathbb{D}.
\]

Finally, by repeating the same steps as they were used in the proof of [10] Lemma 3.1, we are in a position to get the result we are aiming for.

To prove the main result, we also need the following result, which is dealt with for a wider class of weights.

**Lemma 6.** Let \( \omega \in \hat{\mathcal{D}} \) and \( f \in L^2_\omega \). Then, there exists an \( \eta_0 = \eta_0(\omega) > 0 \) such that for all \( \eta > \eta_0 \)
\[
\|H_f k^{\eta+2}_{\omega,z}\|_{A^2_\omega} + \|H_{\bar{f}} k^{\eta+2}_{\omega,z}\|_{A^2_\omega} \lesssim MO_{\omega,\eta}(f)(z), \quad z \in \mathbb{D}.
\]

**Proof.** Since \( \omega \in \hat{\mathcal{D}} \), by [11] Lemma 2.1, there exists an \( \eta_0 = \eta_0(\omega) > 0 \) such that for all \( \eta > \eta_0 \), \( k^{\eta+2}_{\omega,z} \in A^2_\omega \). Therefore, Cauchy-Schwarz’s inequality yields
\[
|B_{\omega,\eta}(f)(z)| = |\langle f k^{\eta+2}_{\omega,z}, \bar{k}_{\omega,z}^{\eta+2} \rangle_{A^2_\omega}| = |\langle P_{\omega}(f k^{\eta+2}_{\omega,z}), \bar{k}_{\omega,z}^{\eta+2} \rangle_{A^2_\omega}| \leq \|P_{\omega}(f k^{\eta+2}_{\omega,z})\|_{A^2_\omega}.
\]

This together with the Pythagorean theorem implies
\[
\|H_f k^{\eta+2}_{\omega,z}\|_{A^2_\omega} = \|f k^{\eta+2}_{\omega,z}\|_{\mathcal{S}(\omega)} - \|P_{\omega}(f k^{\eta+2}_{\omega,z})\|_{\mathcal{S}(\omega)} \leq \left( \|f k^{\eta+2}_{\omega,z}\|_{\mathcal{S}(\omega)} - \|B_{\omega,\eta}(f)(z)\|_{\mathcal{S}(\omega)} \right)^{\frac{1}{2}} = \left( B_{\omega,\eta}(f)(z) \right)^{\frac{1}{2}} = MO_{\omega,\eta}(f)(z).
\]

Likewise, we obtain \( \|H_{\bar{f}} k^{\eta+2}_{\omega,z}\|_{A^2_\omega} \lesssim MO_{\omega,\eta}(f)(z) \). Therefore,
\[
\|H_f k^{\eta+2}_{\omega,z}\|_{A^2_\omega} + \|H_{\bar{f}} k^{\eta+2}_{\omega,z}\|_{A^2_\omega} \lesssim MO_{\omega,\eta}(f)(z).
\]

\[\square\]

### 3. Proof of Theorem [11]

Now, we are ready to prove our main result. The proof will be finished by verifying that (ii) \( \implies \) (i), (i) \( \implies \) (iii) and (ii) \( \iff \) (iii) respectively. A sequence \( \{\alpha_j\} \) of distinct points in \( \mathbb{D} \) is called \( r \)-separated if \( \inf_{i \neq j} \beta(a_i, a_j) > r > 0 \). \( \{\alpha_j\} \) is called \( r \)-lattice if it is \( r \)-separated and satisfies \( \mathbb{D} = \bigcup_{j} D(a_j, r) \). Here and from now on, write simply \( D(a_j, r) \) by \( D_j \).

The method applied here to prove (ii) \( \implies \) (i) is quite different from the approaches used to prove the case of the standard weight, which relies on a construction of a linear bounded operator on \( A^2_\omega \) and hence avoids lots of tedious calculations. Moreover, we will also prove (ii) \( \implies \) (i) for the full range \( 0 < p < \infty \).

**Proof of (ii) \( \implies \) (i):** Assume (ii) Holds. Again, since \( \omega \in \mathcal{D} \), there exists an \( \eta_0 = \eta_0(\omega) > 0 \) such that for all \( \eta > \eta_0 \), \( k^{\eta+2}_{\omega,z} \in A^2_\omega \). Then using the atomic decomposition of \( A^2_\omega \) (see [17], Theorem 2] with the special case \( p = q = 2 \), we see that for any \( f \in A^2_\omega \) there exists a sequence \( \{\lambda_j\} \in \ell^2 \) such that
\[
f(z) = \sum_{j=0}^{\infty} \lambda_j k^{\eta+2}_{\omega,z}(z), \quad z \in \mathbb{D}.
\]

for a \( r = r(\omega) \)-lattice \( \{\alpha_j\} \). Conversely, each \( f \) with the form of (3.1) and \( \{\lambda_j\} \in \ell^2 \) must belong to \( A^2_\omega \). Those facts enable us to define a linear bounded surjective operator on \( A^2_\omega \),
which plays a critical role in the proof. To be concrete, let \( \{ e_j \}_j \) be an orthonormal basis of \( \mathbb{A}_\omega^2 \). Then for each \( f \in \mathbb{A}_\omega^2 \), it can be written as
\[
f(z) = \sum_{j=0}^{\infty} \langle f, e_j \rangle_{\mathbb{A}_\omega^2} e_j(z), \quad z \in \mathbb{D},
\]
where \( \langle \cdot, \cdot \rangle_{\mathbb{A}_\omega^2} \) is the inner product of \( \mathbb{A}_\omega^2 \). Moreover, we have \( \| \{ \langle f, e_j \rangle_{\mathbb{A}_\omega^2} \}_j \|_{l^2} = \| f \|_{\mathbb{A}_\omega^2} \). Now, define the linear operator \( A : \mathbb{A}_\omega^2 \to \mathbb{A}_\omega^2 \) as follows:
\[
A(f)(z) = \sum_{j=0}^{\infty} \langle f, e_j \rangle_{\mathbb{A}_\omega^2} k_{\omega,a_j}^{n+2}(z), \quad z \in \mathbb{D}.
\]
It is easy to see that \( A \) is bounded and onto and
\[
A e_j = k_{\omega,a_j}^{n+2}, \quad j = 0, 1, \ldots.
\]
We will see that (ii) implies (iii) trivially and hence both \( H_f \) and \( H_T \) are compact on \( \mathbb{A}_\omega^2 \) by [57, Theorem 4.5]. Therefore, to show (i), it suffices to show that both \( A^* H_f^* H_f A \), \( A^* H_T^* H_T A \in \mathcal{S}_{\mathbb{A}_\omega^2} \) by [24, Proposition 1.30], which can be done if we show
\[
\sum_j \langle (A^* H_f^* H_f A)^\frac{p}{2} e_j, e_j \rangle_{\mathbb{A}_\omega^2} + \langle (A^* H_T^* H_T A)^\frac{p}{2} e_j, e_j \rangle_{\mathbb{A}_\omega^2} < \infty.
\]
If \( 0 < p \leq 2 \), then by [24, Proposition 1.31], Lemma [6] and the hypothesis that \( \omega \in \mathcal{D} \), we deduce
\[
\sum_j \langle (A^* H_f^* H_f A)^\frac{p}{2} e_j, e_j \rangle_{\mathbb{A}_\omega^2} + \langle (A^* H_T^* H_T A)^\frac{p}{2} e_j, e_j \rangle_{\mathbb{A}_\omega^2}
\leq \sum_j \| H_f k_{\omega,a_j}^{n+2} \|_{L^2}^p + \| H_T k_{\omega,a_j}^{n+2} \|_{L^2}^p
\leq \sum_j MO_{\omega,\eta}(f)(a_j)^p \sum_j \int_{D_j} MO_{\omega,\eta}(f)(z)^p \, d\lambda(z).
\]
Now, it follows from [23, (2.20)] that \( |1 - \omega_0 \zeta| \approx |1 - \zeta| \) for all \( \zeta \in \mathbb{D} \) and \( z \in D_j \). This estimate together with [15] and the fact that \( \omega(a_j) \approx \omega(z) \) as long as \( z \in D_j \), we get that there exists an \( \eta_1 = \eta_1(\omega) > \eta_0 \) such that for all \( \eta \geq \eta_1 \),
\[
k_{\omega,a_j}^{n+2}(\zeta) \approx k_{\omega,z}^{n+2}(\zeta), \quad z \in D_j, \ \zeta \in \mathbb{D}.
\]
Applying this estimate into (3.2) and noticing the definition of \( MO_{\omega,\eta} \), we deduce
\[
\sum_j \langle (A^* H_f^* H_f A)^\frac{p}{2} e_j, e_j \rangle_{\mathbb{A}_\omega^2} + \langle (A^* H_T^* H_T A)^\frac{p}{2} e_j, e_j \rangle_{\mathbb{A}_\omega^2}
\leq \sum_j \int_{D_j} MO_{\omega,\eta}(f)(a_j)^p \, d\lambda(z) = \sum_j \int_{D_j} MO_{\omega,\eta}(f)(z)^p \, d\lambda(z)
\leq \int_{\mathbb{D}} MO_{\omega,\eta}(f)(z)^p \, d\lambda(z) = \| MO_{\omega,\eta}(f) \|_{L^p(\mathbb{D},d\lambda)}^p < \infty,
\]
which is the desired result we are aiming for.

If \( 2 < p < \infty \), then we may reach the same result by following the above steps by applying [24, Theorem 1.27] instead of [24, Proposition 1.31].
We are in a position to prove that (ii) ⇔ (iii) for the full range \(0 < p < \infty\) and for the weight belongs to \(\mathcal{D}\).

**Proof of (ii) ⇔ (iii):** Since \(\omega \in \mathcal{D}\), by the double integration representations (3.3) and (3.4) of \(\text{MO}_{\omega, \eta}\) and \(\text{MO}_{\omega, r}\), we may easily see that \(\text{MO}_{\omega, r}(f)(z) \lesssim \text{MO}_{\omega, \eta}(f)(z)\) for a sufficiently large \(r > 0\) and \(\eta > 0\) depending on \(\omega\), which gives (ii)⇒(iii).

Conversely, suppose (iii) holds. We will prove that (ii) is also valid by applying the technique used in [10]. Since \(\omega \in \mathcal{D}\) by the hypothesis, \(\omega(D(z, r)) \simeq \widehat{\omega}(z)(1 - |z|)\) for a sufficiently large \(r = r_0\), say \(r > r_0 = r_0(\omega)\). Let now \(0 < p < \infty\) and \(\{a_j\}\) be an \(r\)-lattice of \(\mathcal{D}\). Then the fact that the number of discs \(D_j\) to which each \(z\) may belong is uniformly bounded yields

\[
\sum_j \text{MO}_{\omega, r}(f)(a_j)^p = \int_\mathcal{D} \text{MO}_{\omega, r}(f)(z)^p d\lambda(z).
\]

(3.3)

Then for a sufficiently large \(\eta\) depending on \(\omega\), say \(\eta > \eta_0(\omega)\), by (3.4) and triangle inequality, we deduce

\[
\text{MO}_{\omega, \eta}(f)(z)^2 \leq \sum_j \sum_k \int_{D_j} \int_{D_k} |f(u) - f(\zeta)|^2 |k_{\omega, z}^{\eta + 2}(u)|^2 \omega(u) \omega(\zeta) dA(u) dA(\zeta)
\]

\[
\lesssim \sum_j |k_{\omega, z}^{\eta + 2}(a_j)|^2 \int_{D_j} |f(u) - \widehat{f}_{\omega, r}(z)|^2 \omega(u) dA(u) \lesssim A_1(f, z) + A_2(f, z), \quad z \in \mathcal{D},
\]

(3.4)

where

\[
A_1(f, z) = \sum_j \text{MO}_{\omega, r}(f)(a_j)^2 \omega(D_j) |k_{\omega, z}^{\eta + 2}(a_j)|^2
\]

and

\[
A_2(f, z) = \sum_j |k_{\omega, z}^{\eta + 2}(a_j)|^2 |\widehat{f}_{\omega, r}(a_j) - \widehat{f}_{\omega, r}(z)|^2 \omega(D_j).
\]

Therefore, to show (ii), by (3.3) and (3.4), it suffices to show that

\[
\int_\mathcal{D} A_i(f, z) \frac{dA(z)}{2} \lesssim \sum_j \text{MO}_{\omega, r}(f)(a_j)^p, \quad i = 1, 2.
\]

(3.5)

First, let us estimate the above integral with integrant \(A_1(f, z)\). Since \(\omega \in \mathcal{D}\), there exists an \(\eta_1 = \eta_1(\omega) > \eta_0\) such that for all \(\eta > \eta_1\), \(\frac{(1 - |z|^2)^p(\eta + 3/2)^{-2}}{\omega(\eta)^p/2} \in \mathcal{D}\).

If \(0 < p \leq 2\), then [13, Theorem 1] yields

\[
\int_\mathcal{D} A_1(f, z)^{p/2} dA(z) \lesssim \sum_j \text{MO}_{\omega, r}(f)(a_j)^p \omega(D_j)^{p/2} \int_\mathcal{D} \frac{\widehat{\omega}(z)^{-p/2}(1 - |z|^2)^p(\eta + 3/2)^{-2}}{|1 - \omega(z)^p(\eta + 2)|} d\lambda(z)
\]

\[
\lesssim \sum_j \text{MO}_{\omega, r}(f)(a_j)^p \omega(D_j)^{p/2} \int_\mathcal{D} \frac{(1 - |z|^2)^p(\eta + 3/2)^{-2}}{|1 - \omega(z)^p(\eta + 2)\widehat{\omega}(z)^p|} dA(z)
\]

\[
\lesssim \sum_j \text{MO}_{\omega, r}(f)(a_j)^p \omega(D_j)^{p/2} \left( \int_0^{|a_j|} \frac{(1 - s)^p(\eta + 2)^{-1}}{(1 - s)^p(\eta + 2)\widehat{\omega}(s)^p} + 1 \right)
\]

(3.6)

\[
\approx \sum_j \text{MO}_{\omega, r}(f)(a_j)^p.
\]

If \(2 < p < \infty\), then for a sufficiently small \(\varepsilon = \varepsilon(\omega) > 0\), [8, (14)] yields \(\omega(D_j) \lesssim \widehat{\omega}(a_j)(1 - |a_j|)\) and hence Hölder’s inequality, [24, Lemma 10], Fubini’s theorem and [13, Theorem 1]
yield

\[ \int_D A_1(f, z)^\frac{n}{p} dA(z) \approx \int_D \left( \sum_j MO_{\omega, r}(f)(a_j)^2 \omega(D_j) \frac{\hat{\omega}(z)^{-1} (1 - |z|^2)^{2(\eta+3/2)}}{|1 - \alpha_j z|^{2\eta+4}} \right)^{\frac{n}{2}} dA(z) \]

\[ \lesssim \int_D \left( \sum_j MO_{\omega, r}(f)(a_j)^p \frac{\hat{\omega}(z)^{-p/2} (1 - |z|^2)^{\frac{p}{2}(\eta+3/2)}}{|1 - \alpha_j z|^{\frac{p}{2}(2\eta+3-\varepsilon)}} \right) \cdot \left( \sum_j \frac{(1 - |a_j|) \frac{r}{p}}{|1 - \alpha_j z|^{(1+\varepsilon) \frac{p}{2}}} \right)^{\frac{n+p-2}{2}} dA(z) \]

\[ \lesssim \int_D \sum_j MO_{\omega, r}(f)(a_j)^p \frac{\hat{\omega}(z)^{-p/2} (1 - |z|^2)^{\frac{p}{2}(\eta+3/2)}}{|1 - \alpha_j z|^{\frac{p}{2}(2\eta+3-\varepsilon)}} \sum_i \hat{\omega}(a_j)^\frac{p}{2} \int_{D_i} \frac{\hat{\omega}(z)^{-p/2} (1 - |z|^2)^{\frac{p}{2}(\eta+3/2-\varepsilon)}}{|1 - \alpha_j z|^{\frac{p}{2}(2\eta+3-\varepsilon)}} dA(z) \]

\[ \lesssim \sum_j MO_{\omega, r}(f)(a_j)^p. \]

(3.7)

Next, we proceed to estimate the second integral in (3.5). Lemma 2 and Lemma 5 yield

\[ \int_D A_2(f, z)^\frac{n}{p} dA(z) \approx \int_D \left( \sum_j |\hat{f}_{\omega, r}(a_j) - \hat{f}_{\omega, r}(z)| \right)^{\frac{n}{2}} dA(z) \]

\[ \lesssim \int_D \left( \hat{\omega}(z)^{-1} (1 - |z|^2)^{2\eta+3-2\delta} N_p(f, z)^{\frac{n}{2}} \int_{D_D} \frac{(1 + \beta(z, \zeta))^2 \hat{\omega}(\zeta)}{|1 - \zeta z|^{2(\eta-d+2)} 1 - |\zeta|^2} dA(\zeta) \right)^{\frac{n}{2}} dA(z) \]

\[ \lesssim \int_D \left( (1 - |z|^2)^{2d-2\delta} N_p(f, z)^{2/p} \right)^{\frac{n}{2}} dA(z) \]

\[ \lesssim \sum_j MO_{\omega, r}(f)(a_j)^p (1 - |a_j|^2)^{\delta p} \int_D \frac{(1 - |z|^2)^{p(d-\delta)-2}}{|1 - \alpha_j z|^{\rho d}} dA(z) \lesssim \sum_j MO_{\omega, r}(f)(a_j)^p. \]

(3.8)

Finally, combining (3.6), (3.7), and (3.8), we prove that (ii) holds.

To finish the proof of the main theorem, it remains to prove that (i) implies (iii). The method used here originates from a technical construction in [6]. Before presenting the proof, let us recall the definition of the commutator on \( L^2_\omega \). For an \( f \in L^2_\omega \), the commutator \([M_f, P_\omega] := M_f P_\omega - P_\omega M_f\). It is well-known that the study of \([M_f, P_\omega]\) is equivalent to the simultaneous study of \(H_f\) and \(H_\tau\), which can be partially explained by the identity

\[ [M_f, P_\omega] = H_f P_\omega - (H_\tau P_\omega)^*. \]  

(3.9)

**Proof of (i) \Rightarrow (iii):** Let \( 1 \leq p < \infty \). Suppose \( H_f \) and \( H_\tau \) are both in \( S_p \). Then the identity (3.9) is also in \( S_p \) of \( L^2_\omega \) due to the boundedness of \( P_\omega \) on \( L^2_\omega \). Let now \( \{\varepsilon_j\}^\infty_{j=1} \) be an
orthonormal basis for $L^2_\omega$, and let $\{a_j\}$ be a $r$-lattice of $\mathbb{D}$ for a certain $r = r(\omega) > 0$. Set
\[ h_j(z) = \omega(D_j)^{-1/2} \chi_{D_j}(z) \quad \text{and} \quad g_j(z) = \chi_{D_j}(z)[Mf, P_\omega]h_j(z)/\|\chi_{D_j}[Mf, P_\omega]h_j\|_{L^2_\omega}. \]
Then it is easy to see that the following two linear operators
\[ A(e_j)(z) = g_j(z) \quad \text{and} \quad B(e_j)(z) = h_j(z), \quad j = 1, 2, \ldots, \quad z \in \mathbb{D}. \]
are bounded on $L^2_\omega$. Therefore for an $f \in L^2_\omega$, $T := A^*[Mf, P_\omega]B \in S_p(L^2_\omega)$ and furthermore
\[ \|T\|_{S_p}^p \lesssim \|[Mf, P_\omega]\|_{S_p}^p, \quad f \in L^2_\omega. \] (3.10)
Moreover, a simple calculation gives
\[
\langle Te_j, e_j \rangle_{L^2_\omega} = \langle [Mf, P_\omega]h_j, g_j \rangle_{L^2_\omega} \\
= \int_{\mathbb{D}} \frac{1}{\|\chi_{D_j}[Mf, P_\omega]h_j\|_{L^2_\omega}} \int_{D_j} \frac{f(z) - f(\zeta)}{\omega(D_j)^{1/2}} B^\omega_z(\zeta) \omega(\zeta) dA(\zeta) dA(z) \\
= \|\chi_{D_j}[Mf, P_\omega]h_j\|_{L^2_\omega}, \quad j = 1, 2, \ldots.
\]
This means that $T$ is a positive operator on $L^2_\omega$, and hence $T^p \in S_1$ with $\|T\|_{S_p}^p = \|T^p\|_{S_1}$ by [24] Lemma 1.25. This together with [24] Theorem 1.27 and Proposition 1.31 and (3.10) yields
\[ \sum_j \langle [Mf, P_\omega]h_j, g_j \rangle_{L^2_\omega}^p = \sum_j \langle Te_j, e_j \rangle_{L^2_\omega}^p \leq \sum_j \langle T^p e_j, e_j \rangle_{L^2_\omega} < \infty. \]
Nevertheless, it is elementary to deduce
\[ \|\chi_{D_j}[Mf, P_\omega]h_j\|_{L^2_\omega} = \left( \int_{D_j} \left| \int_{D_j} \frac{f(z) - f(\zeta)}{\omega(D_j)^{1/2}} B^\omega_z(\zeta) \omega(\zeta) dA(\zeta) \right|^2 \omega(z) dA(z) \right)^{1/2}, \]
which together with Lemma 3 implies that $\sum_{j=1}^\infty MO_\omega, r(f)(a_j)^p < \infty$, and hence the assertion follows due to (3.3).

References

[1] M. Abate and A. Saracco, Carleson measures and uniformly discrete sequences in strongly pseudoconvex domains. (English summary) J. Lond. Math. Soc. (2) 83 (2011), 587–605.
[2] J. Araby, S. Fisher, J. Peetre, Hankel operators on weighted Bergman spaces. Amer. J. Math. 110 (1988), 989–1053.
[3] D. Bekollé, Inégalités à poids pour le projecteur de Bergman dans la boule unité de $C^n$, Weighted inequalities for the Bergman projection in the unit ball of $C^n$. Studia Math. 71 (1981/82), 305–323.
[4] D. Bekollé and A. Bonami, Inégalités à poids pour le noyau de Bergman. (French) C. R. Acad. Sci. Paris Sér. A-B 286 (1978), 775–778.
[5] Z. Hu, J. Lu, Hankel operators on Bergman spaces with regular weights. J. Geom. Anal. 29 (2019), 3494–3519.
[6] J. Isralowitz, Schatten $p$ class commutators on the weighted Bergman space $L^2_\omega(B_n, dv_\gamma)$ for $2n/(n+1+\gamma) < p < \infty$, Indiana Univ. Math. J. 62 (2013), 201–233.
[7] S. Janson, Hankel operators between weighted Bergman spaces. Ark. Mat. 26 (1988), 205–219.
[8] B. Liu and J. Rättyä, Compact differences of weighted composition operators. Collect. Math. 73 (2022), 89–105.
[9] D.H. Luecking, Multipliers of Bergman spaces into Lebesgue spaces. Proc. Edinb. Math. Soc. 29 (1986) 125–131.
[10] J. Pau, Characterization of Schatten-class Hankel operators on weighted Bergman spaces. Duke Math. J. 165 (2016), 2771–2791.
[11] J. A. Peláez, Small weighted Bergman spaces. In: Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics (2016).
[12] J. A. Peláez and J. Rättyä: Weighted Bergman spaces induced by rapidly increasing weights. Mem. Am. Math. Soc. 227(1066) (2014)
[13] J. A. Peláez and J. Rättyä: Two weight inequality for Bergman projection. J. Math. Pures Appl. (9) 105 (2016), 102–130.
[14] J. A. Peláez and J. Rättyä, Bergman projection induced by radial weight. Adv. Math. 391 (2021), Paper No. 107950, 70 pp.
[15] J. A. Peláez, A. Perälä, and J. Rättyä, Hankel operators induced by radial Bekollé-Bonami weights on Bergman spaces. Math. Z. 296 (2020), 211–238.
[16] J. A. Peláez and J. Rättyä, and K. Sierra, Berezin transform and Toeplitz operators on Bergman spaces induced by regular weights. J. Geom. Anal. 28 (2018), 656–687.
[17] J. A. Peláez and J. Rättyä, and K. Sierra, Atomic decomposition and Carleson measures for weighted mixed norm spaces. J. Geom. Anal. 31 (1) (2021), 715–747.
[18] R. Wallstén, Hankel operators between weighted Bergman spaces in the ball. Ark. Mat. 28 (1990), 183–192.
[19] J. Xia, Hankel operators in the Bergman spaces and Schatten p-classes: the case 1 < p < 2. Proc. Amer. Math. Soc. 129 (2001), 3559–3567.
[20] J. Xia, On the Schatten class membership of Hankel operators on the unit ball. Illinois J. Math. 46 (2002), 913–928.
[21] K. Zhu, Hilbert-Schmidt Hankel operators on the Bergman space. Proc. Amer. Math. Soc. 109 (1990), 721–730.
[22] K. Zhu, Schatten class Hankel operators on the Bergman space of the unit ball. Amer. J. Math. 113 (1991), 147–167.
[23] K. Zhu, Spaces of holomorphic functions in the unit ball (Springer-Verlag, New York, 2005).
[24] K. Zhu, Operator Theory in Function Spaces. American Mathematical Society, Providence, RI (2007)

Department of Mathematics, College of Sciences, Shiraz University, Shiraz, Iran
Email address: amzehkeshavarzi67@gmail.com

University of Eastern Finland, P.O.Box 111, 80101 Joensuu, Finland
Email address: fanglei.wu@uef.fi
Email address: fangleiwu1992@gmail.com