MOMENTUM CONSERVATION AT SMALL $x$

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Abstract

We discuss how momentum conservation is implemented in perturbative computations based on expansions of anomalous dimensions appropriate at small $x$. We show that for any given choice of $F_2$ coefficient functions there always exists a factorization scheme where the gluon is defined in such a way that momentum is conserved at next to leading order.

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An accurate determination of the $x$ and $Q^2$ dependence of structure functions at small $x$ requires solution of next to leading order (NLO) evolution equations appropriate to this kinematical region \cite{1}. The perturbative expansion of the anomalous dimensions or splitting functions which govern perturbative evolution can then be reorganized in order to keep into account the presence of two large scales in the problem ($Q^2$ and $s = (1 - x)Q^2/x$) \cite{2}. The results of this procedure must be consistent with the constraints imposed by conservation laws. Energy-momentum conservation is particularly subtle in this context because energy is now one of the parameters which organize the perturbative expansion: therefore, it can only be imposed by an appropriate definition of the infinite set of contributions which are summed up in the anomalous dimensions. The problem of momentum conservation can thus be tackled only after these contributions have been defined in the most general way, in particular by correctly accounting for the freedom of choosing a factorization scheme. Here we will show that momentum conservation can always be implemented in NLO computations performed in expansion schemes appropriate to small $x$ by a judicious choice of factorization scheme. After defining the appropriate small-$x$ expansions, we will construct the most general transition functions which perform a scheme change. We will then show that given a matrix of anomalous dimensions and a factorization scheme, specified by the coefficient functions which relate $F_2$ to parton distributions (such as, for instance, $\overline{\text{MS}}$ or DIS) there is still enough freedom to perform a further scheme change within the given scheme (i.e. without changing the coefficient functions) such that momentum conservation is then obtained consistently at NLO.

A determination of the evolution of parton distributions by solution of the renormalization group equations corresponds to summing leading (and subleading) logs of the form $\alpha_s^n(\log Q^2)^q(\log \frac{1}{x})^r$. In the usual loop expansion, appropriate to the Bjorken limit, the leading logs are those of $Q^2$. Thus at leading order all terms with $p = q$ are summed, at NLO those with $q < p \leq 2q$, and so forth; it then turns out that both at LO and NLO (and, in fact, at any order) $0 \leq r \leq p$. At small $x$, however, logs of $\frac{1}{x}$ should also be considered leading, and the perturbative expansion reorganized accordingly. It is for example possible \cite{3} to define an expansion scheme appropriate to the Regge limit where the roles of $\ln \frac{1}{x}$ and $\ln Q^2$ are interchanged, so that at LO all logs with $p = r$ are summed, while $1 \leq q \leq p$ (the small $x$ expansion). A scheme where the two logs are treated on the same footing (double leading expansion) can also be defined, in which at LO any power of $\alpha_s$ is accompanied by either of the two logs ($1 \leq q \leq p$, $0 \leq r \leq p$, $1 \leq p \leq q + r$), as well as a number of intermediate schemes. In the sequel we will consider specifically
the small $x$ expansion, which is a theoretically interesting limiting case, and the double
leading expansion, which is most interesting for phenomenology in the HERA region [1].

The constraint imposed by momentum conservation on anomalous dimensions is a
particular case of the general requirement that anomalous dimensions of conserved (or
partially conserved) operators must vanish, which in turn is a consequence of the renor-
malization group equations. This constraint takes the form

$$\gamma_{qq}(1, \alpha) + \gamma_{gq}(1, \alpha) = 0,$$

where the anomalous dimensions are moments of the splitting functions, $\gamma_{ij}(N, \alpha) =
\int_0^1 x^N P_{ij}(x; t)$, and depend on $t = \ln(Q^2/\Lambda^2)$ through the running of the coupling $\alpha$. Momentum conservation [eq. (1)] is imposed order by order in the usual loop expansion in
powers of $\alpha$ of the anomalous dimensions, by suitable choice of normalization of the quark
and gluon distributions. If, however, the order $\alpha^k$ contribution to $\gamma_{ij}$ is further expanded
in powers of $N$ the single terms of this expansion will not, of course, satisfy eq. (1). Now,
the various expansion schemes alluded above are obtained precisely by performing such
expansions, and then including at each order a suitable subset of terms, which then will
not conserve momentum automatically.

When evolution in the small $x$ region is approached by choosing an appropriate ex-
pansion scheme, momentum conservation should be imposed order by order, just as it is
in the loop expansion. That this is a priori non-trivial is clear from the observation that
the leading order anomalous dimensions in the small $x$ expansion described above violate
eq. (1). Just like in the usual expansion, however, eq. (1) only holds if parton distributions
are defined appropriately, i.e. for suitable choices of the factorization scheme. Before we
discuss the implementation of momentum conservation we must therefore discuss changes
of factorization scheme within various small $x$ expansions.

Changing the factorization scheme amounts to a redefinition of the singlet par-
ton densities $f(N, t) \equiv \left( \frac{q}{g} \right)$, where $q \equiv \sum_i (q_i + \bar{q}_i)$. Letting $f \to f' = UF$, the naive
partonic interpretation of the parton densities will be maintained if we always assume
that $U(N, \alpha) = 1 + O(\alpha)$. The renormalization group equation $\frac{d}{dt} f = \gamma f$ then remains
unchanged provided $\gamma \to \gamma'$, where

$$\gamma' = U \gamma U^{-1} + \left( \frac{d}{dt} U \right) U^{-1}. \quad (2)$$

1 Alternatively, momentum conservation could always be imposed at each order in $\alpha$ by in-
cluding an Ansatz for the (yet unknown) terms which are formally sub-subleading in the small $x$
extractions: several proposals of this kind are discussed in ref. [3].
Because $U$ only depends on $t$ through $\alpha$

\[
\frac{d}{dt} U(N, \alpha) \equiv \beta(\alpha) \frac{\partial}{\partial \alpha} U(N, \alpha),
\]

where $\beta(\alpha)$ is the beta function. Since $\beta(\alpha) = -\beta_0 \alpha^2 + O(\alpha^3)$ it follows that the second term of (2) is subleading compared to the first.

We must now choose a specific expansion scheme. We will eventually prove our result in the physically relevant double leading expansion; however, we consider first the small $x$ expansion, since this will allow us to present the general structure of our results in a somewhat simpler setting. In the small-$x$ scheme the singlet anomalous dimension are given by

\[
\gamma \equiv \gamma^s + \gamma^{ss} + \cdots;
\]

\[
\gamma^s(N, \alpha) \equiv \sum_{n=n_0}^{\infty} \gamma^s_n(\alpha/N)^n,
\]

\[
\gamma^{ss}(N, \alpha) \equiv \alpha \sum_{n=n_0-1}^{\infty} \gamma^{ss}_{n+1}(\alpha/N)^n,
\]

where $\gamma^s(N, \alpha)$ sums the leading singularities, $\gamma^{ss}(N, \alpha)$ the subleading singularities, and so on. The one loop contributions to $\gamma^s$ and $\gamma^{ss}$ [i.e. the contributions with $n = 1$ in eq. (4)] will in general violate the condition (1); they are scheme independent and thus lead inevitably to momentum nonconservation in the usual small $x$ expansion if $n_0 = 1$. We will hence consider here a truncated small $x$ expansion with $n_0 = 2$, i.e. with these one loop contributions suppressed. The results then found will coincide with those of the usual small $x$ scheme in the $x \to 0$ limit; we will use them to prove that momentum conservation holds in the double leading expansion.

At NLO in a typical factorization scheme such as the $Q_0$DIS scheme [5] (or the $\overline{\text{MS}}$ and DIS schemes [6]) $\gamma$ is then of the form

\[
\gamma = \begin{pmatrix} r \gamma_q & \gamma_q \\ r \gamma_g + \hat{\gamma} & \gamma_g \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ r \gamma^s_g + \hat{\gamma}_g & \gamma^{ss}_g \end{pmatrix} + \cdots,
\]

where $r \equiv C_F/C_A$ is the colour-charge factor. It is important for what follows to notice that both of the quark anomalous dimensions vanish at leading order; also, both the gluon anomalous dimensions at leading order, and the quark ones at subleading order obey a colour-charge relation\[2\] At subleading order in the gluon sector the anomalous dimensions

\[2\] Notice that this relation is however not satisfied by the one-loop term $\gamma_1^{ss}$, which has been suppressed.
\( \gamma_{gq} \) and \( \gamma_{gg} \) are as yet unknown beyond two loops so we introduce \( \hat{\gamma}_g \equiv \gamma_{gq} - r \gamma_{gg} \) to allow for possible violations of the colour-charge relation at NLO in the gluon channel.

Consider now parton distributions normalized so as to satisfy the momentum sum rule at some scale \( t = 0 : q(1, 0) + g(1, 0) = 1 \). Momentum is then conserved in the evolution of these distributions to the scale \( t \) if the two conditions (II) are satisfied. With anomalous dimension of the form (5), the momentum sum rule will be violated at LO (and thus in a scheme invariant way) since \( \gamma_{gq}^s (1, \alpha) \neq 0 \). However, the quark distribution only evolves at NLO, and the gluon can only be directly observed at NLO (through measurement of \( F_L \), say); therefore, this LO violation has no physical effects. Momentum conservation starts thus being physically relevant at NLO, where it imposes a condition relating the LO and NLO components of the anomalous dimension, as well as the colour-charge relation in the gluonic sector:

\[
(\gamma_{q}^{ss})_n + (\gamma_{g}^{s})_n + (\gamma_{g}^{gg})_n = 0, \quad (\hat{\gamma}_g)_n = 0, \tag{6}
\]

where \( \gamma_{g}^{s} \equiv \sum (\gamma_{g}^{s})_n (\alpha/N)^n \), etc. The conditions (6) will not in general be satisfied in a given generic scheme; however in what follows we will show that it is always possible to find factorization schemes such that both conditions are satisfied (for \( n > 1 \)), and thus in which momentum is conserved.

Consider first a scheme change \( U \) which is LO in the small-\( x \) expansion, i.e. \( U \equiv 1 + \sum_1^\infty U_n (\alpha/N)^n \). The most general form of \( U \) which retains the identification of \( F_2 \) with the quark density at LO in this expansion is

\[
U = \begin{pmatrix} 1 & \bar{u} \\ \bar{u} & u \end{pmatrix}, \tag{7}
\]

where \( u \equiv 1 + \sum_1^\infty u_n (\alpha/N)^n \) while \( \bar{u} \equiv \sum_1^\infty \bar{u}_n (\alpha/N)^n \) and similarly for \( \bar{u} \). Substitution in (2) gives to LO

\[
\gamma' = \gamma_g (u - \bar{u})^{-1} \begin{pmatrix} \bar{u}(ru - \bar{u}) & \bar{u}(1 - r\bar{u}) \\ ru(\bar{u} - \bar{u}) & u(1 - r\bar{u}) \end{pmatrix} + O(\alpha). \tag{8}
\]

If we now insist that the leading order anomalous dimension is to remain unchanged, we must choose

\[
\bar{u} = 0, \quad \bar{u} = r(u - 1). \tag{9}
\]

We neglect LO transformations proportional to the unit matrix, since they modify the LO relation between \( F_2 \) and \( q \).
A LO scheme change thus amounts essentially to a redefinition of the gluon normalization by the function \(u\). The NLO anomalous dimensions are then correspondingly modified according to

\[
\gamma' = \gamma + (u^{-1} - 1)\gamma_q \begin{pmatrix} \frac{r}{r^2} & 1 \\ -r^2 & -r \end{pmatrix} + (u - 1)\hat{\gamma}_g \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{d}{dt} \ln u \begin{pmatrix} 0 & 0 \\ r & 1 \end{pmatrix},
\]

so that in particular \(\gamma^{gg} \equiv \gamma_q \to \gamma_q / u\). Notice that the colour-charge relation is automatically preserved not only in the LO gluon sector, but also in the NLO quark sector.

The quark distribution is left unaffected by the LO transformations considered so far. However, we may still perform a NLO redefinition \(f' \to f'' = (1 + V)f'\), where

\[
V \equiv \alpha \sum_0^\infty V_{n+1}(\alpha/N)^n.
\]

This induces a corresponding change in the anomalous dimension

\[
\gamma' \to \gamma'' = \gamma' + [V, \gamma'] + O(\alpha^2).
\]

Writing

\[
V \equiv \begin{pmatrix} \tilde{v} & v \\ \tilde{w} & w \end{pmatrix},
\]

and keeping only LO and NLO terms, we find

\[
\gamma'' = \gamma' + \gamma_g' \begin{pmatrix} rv & v \\ (rw - \tilde{w} - r\tilde{v}) & -rv \end{pmatrix} + O(\alpha^2).
\]

Combining this NLO transformation with the LO transformation \(U\), we thus have altogether at NLO

\[
\begin{align*}
\gamma''_{qq} &= \gamma_{qq} + r(u^{-1} - 1)\gamma_{ss} + rv\gamma_g, \\
\gamma''_{qg} &= \gamma_{qg} + (u^{-1} - 1)\gamma_{ss} + v\gamma_g, \\
\gamma''_{gq} &= \gamma_{gq} + (u - 1)(\hat{\gamma}_g + r\hat{\gamma}_q) - r^2(u^{-1} - 1)\gamma_{ss} + (rw - \tilde{w} - r\tilde{v})\gamma_g + r\frac{d}{dt} \ln u, \\
\gamma''_{gg} &= \gamma_{gg} - r(u^{-1} - 1)\gamma_{ss} - rv\gamma_g + \frac{d}{dt} \ln u.
\end{align*}
\]

The colour-charge relation in the quark sector is thus again preserved automatically. Starting for definiteness in a parton scheme (such as, say the Q_0DIS scheme), where \(F_2(x, t) = \langle e^2 \rangle xq(x, t)\) (and \(\langle e^2 \rangle\) is a numerical factor) the functions \(v\) and \(\tilde{v}\) then give the
$F_2$ coefficient functions, since $q = (1 - \tilde{v})q'' - vg''$. To enforce the colour-charge relation in the coefficient functions we should take

$$\tilde{v} = rv + \hat{v}; \quad \hat{v} = -v_1 \alpha$$

(as is the case in $\overline{\text{MS}}$ scheme, for example [6]). Similarly, the NLO terms violating the colour-charge relation at NLO in the gluon sector can be removed by choosing $rw - \tilde{w}$ appropriately.

The most general NLO scheme change is thus parameterized by two parameters $u$ and $v$ (or more properly a two-fold infinity of parameters, $u_n$ and $v_n$), plus a parameter $\hat{v}$ which is fixed requiring the coefficient functions to satisfy the colour charge relation, and a parameter $rw - \tilde{w}$ which can be used to impose the colour-charge relation in NLO anomalous dimensions, while the orthogonal combination $w + rw$ has no effect at all at NLO. Given then parton distributions in a particular parton scheme (such as $Q_0$DIS) the parameter $u$ redefines the normalization of the gluon, without affecting $F_2$ directly, thus takes to different parton schemes (such as DIS or SDIS) while $v$ moves to a different (generally non partonic) scheme (such as $\overline{\text{MS}}$).

We can now extend these results to the physically relevant double leading expansion, which treats the two large scales on an equal footing (other expansion schemes [4] can be handled in a similar way). In this expansion, the matrix of singlet anomalous dimensions consists of a large $\gamma$ and a small $\gamma$ contribution, $\gamma^L$ and $\gamma^S$, respectively:

$$\gamma = \gamma^L + \gamma^S; \quad \gamma^L = \gamma^1 + \gamma^2, \quad \gamma^S = \gamma^s + \gamma^{ss},$$

(17)

where $\gamma^1$ and $\gamma^2$ are the usual one and two loop anomalous dimensions, while the leading and subleading singularities have the form (4), but now with $n_0 = 3$, so that both the one and the two loop terms are suppressed to avoid double counting. Scheme changes at LO are still effected by the matrix $U$ eq. (7) (no nontrivial LO scheme change of $\gamma^L$ is possible). These transform the singular anomalous dimensions according to eq. (8), (10) and are thus subject to the constraint eq. (9). In addition, further singular contributions are produced by the action of $U$ on $\gamma^L$, i.e. $U \gamma^L U^{-1}$. These may be combined with the LO transformation eq. (10) of $\gamma^S$ to give

$$\gamma'_{qq} = \gamma_{qq} + r(u^{-1} - 1)\gamma_{qq},$$

$$\gamma'_{gq} = \gamma_{gq} + (u^{-1} - 1)\gamma_{gq},$$

$$\gamma'_{gq} = \gamma_{gq} + (u - 1)[(\gamma_{gq} - r\gamma_{gg} + r\gamma_{qq}) + r^2(u^{-1} - 1)\gamma_{qq}] + r \frac{d}{dt} \ln u,$$

$$\gamma'_{gg} = \gamma_{gg} - r(u^{-1} - 1)\gamma_{qq} + \frac{d}{dt} \ln u,$$

(18)
independently of the split of $\gamma$ into $\gamma^L$ and $\gamma^S$. Notice that all terms on the right hand side beyond the first are NLO (because in particular $u\gamma^1 = O(\alpha(\alpha/N)^n)$), and that neither $\gamma^2$ nor $\gamma^s$ make any contribution to them.

Again the LO transformation redefines the normalization of the gluon distribution without affecting the quark, which only transforms under a NLO transformations. This now contains two components: a small $x$ NLO transformation of the form (11), (13) considered previously, plus a standard scheme changing $O(\alpha)$ function $V^L$, which vanishes as $N \to 0$:

$$V = V^L + V^S; \quad V^S = \alpha \sum_{n=0}^{\infty} V^s_{n+1}(\alpha/N)^n. \quad (19)$$

Since $V^L$ has no effect on $\gamma^S$ at NLO, producing only terms of $O(\alpha^2(\alpha/N)^n)$, we will not consider it any further here: its only effect is to change the two loop anomalous dimension $\gamma^2$ in the usual way. The effect of the remaining singular transformation $V^S$ is then very similar to that already discussed in the small $x$ expansion: $\gamma^S$ transforms according to the small $x$ eq. (14), while the $O(\alpha)$ contribution to $V^S$ (which could in fact equivalently be viewed as a contribution to $V^L$) also produces a NLO change of $\gamma^L$. The general scheme transformation at NLO in the double leading expansion is thus

$$\gamma'' = \gamma' + \gamma^s_g \left( (r w - \bar{w} - r \bar{v}) \frac{v}{-rv} \right) + \alpha [V^s_{n}, \gamma^1], \quad (20)$$

where here $\gamma^s_g$ is given by (11) with $n_0 = 1$ (i.e. including the one-loop contribution) $\bar{v}$ was defined in eq. (16), and $\gamma'$ is the anomalous dimension transformed at LO according to eq. (18).

We thus find again that, starting with a parton scheme, the parameter $v$ takes us to nonpartonic schemes, while the further parameters $rw - \bar{w}$ and $\bar{v}$ may be fixed by requiring the color-charge relations in the anomalous dimensions $\gamma^S$ and in the corresponding small $x$ coefficient functions, respectively. In comparison to standard scheme change at large $x$ [4] there seems thus to be an additional freedom, parametrized by $u$, of redefining the normalization of the gluon. This redefinition by a LO function of $\alpha/N$ modifies the singular part of the NLO anomalous dimension (and appears thus to be peculiar of small $x$ expansion schemes). The effect of this transformation can in practice be rather important. Two (in some sense extremal) choices have been considered in the literature. Taking $u(N, \alpha) = R(N, \alpha)^{-1}$, the singular normalization factor for the gluon in $\overline{\text{MS}}$ [3], takes one from the $Q_0\text{DIS}$ scheme to the more singular DIS scheme [3]. Because of the singular
behavior of $R(N, \alpha)$ the anomalous dimensions $\gamma_q^{ss}$ in this scheme are substantially larger. Conversely, one may take $u = \gamma_{qq}/\gamma_q^1$: this removes the singular terms in the quark sector altogether, factoring them in the initial gluon distribution (the ‘SDIS’ scheme$^4$).

The effect on the leading order relation between $F_L$ and the gluon distribution is equally significant. In the double leading expansion and $Q_0$DIS factorization scheme$^5$ $F_L = (C_1^L + h_L)g + O(\alpha^2)$, where $C_1^L$ is the usual two loop longitudinal coefficient function, and $h_L = \alpha \sum_1^\infty h_{n+1}(\alpha/N)^n$; after the (LO) change of scheme

$$F_L = (C_1^L + h_L)(u^{-1}g' + r(u^{-1} - 1)q') + O(\alpha^2).$$

(21)

The result is especially clear in the small $x$ expansion, where the quark distribution is subleading compared to the gluon$^2$: the dominant contribution to $F_L$ is then equal to $F_L = h_L Rg' + O(\alpha^2)$ in DIS$^5$, while in SDIS $F_L = (h_L/h_2)c_q \alpha g' + O(\alpha^2)$ (since $\gamma_q^1 = c_q \alpha + O(\alpha^2)$). Because $(h_L/h_2) = (1 - \gamma_q^s)/(1 + \frac{3}{2}\gamma_q^s(1 - \gamma_q^s)) = 1 + O(\alpha/N)$, at small $x$ $F_L$ will then differ from the gluon distribution by a large factor. The leading order identification of $F_L$ with the gluon may however be restored by choosing a ‘GDIS’ scheme in which $u = 1 + h_L/C_1^L$ so that $F_L = C_1^L \alpha g'$, so that in the small $x$ expansion $F_L = c_q \alpha g' + O(\alpha^2)$ since $C_1^L = c_q \alpha + O(\alpha^2)$). In the GDIS scheme $\gamma_q = (h_2/h_L)c_q \alpha + O(\alpha^2)$, so the quark anomalous dimensions are less singular than in $Q_0$DIS or DIS, but more singular than in SDIS.

This extra freedom in the choice of factorization scheme at small $x$ can however be constrained by requiring momentum conservation. Combining the two transformations (18) and (20) and imposing the constraint of momentum conservation (1) we get at $N = 1$ the two conditions

$$-\beta_0 \alpha^2 \frac{du}{d\alpha} + (\gamma_g^s + \gamma_q^{ss} + r\gamma_q^{ss} - (r - 1)v\tilde{\gamma}_q^s)u = (r - 1)\gamma_q^{ss},$$

(22)

$$\hat{\gamma}_g + (u - 1)\hat{\gamma}_g + (rw - \tilde{w} - r\tilde{v})\tilde{\gamma}_g = 0,$$

(23)

where we used the crucial fact that momentum conservation at one and two loops, i.e. $\gamma^1(1, \alpha) = \gamma^2(1, \alpha) = 0$ naturally eliminates all non-singular terms. Notice that Eq. (23) is equivalent to the color-charge relation (in the subleading gluon sector), which then does not have to be imposed separately, but rather follows automatically.

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$^4$ One could also consider a SDIS’ scheme$^5$, where $u = \gamma_{qq}/(\gamma_q^3 + \gamma_{qq}^2)$, so that $\gamma_q$ reduces to its two-loop expression.
Now, for each choice of \( u \) eq. (22) has a unique solution for \( v[u] \) [i.e. for the infinite set of coefficients \( v_n(u_m) \)], while similarly (23) has a unique solution for \((rw - \tilde{w})[u]\). Conversely, since for a given \( v \) eq. (22) is a first order differential equation for the function \( u(1, \alpha) \) with boundary condition \( u(1, 0) = 1 \) (compatible with (22) because by construction \( \gamma^s \) and \( \gamma^{ss} \) contain no one loop terms), it always has a unique solution \( u[v] \), so the functional \( v[u] \) is invertible. It follows that eq. (22) defines a monotonic curve \( u[v] \) in the two dimensional space of schemes \((u, v)\) along which momentum is conserved: imposing momentum conservation at NLO the gluon normalization is fixed uniquely by the choice of \( F_2 \) coefficient functions.

In order to actually determine the curve \( u[v] \), however, one needs full knowledge of the (unknown) subleading gluon anomalous dimension, i.e. of \( \gamma^{ss}_g \) and \( \hat{\gamma}_g \) in a given scheme, say \( Q_0 \)DIS. Indeed, the same argument which shows that for a given \( \gamma^S(1, \alpha) \) eq. (22) determines \( v[u] \) or \( u[v] \) also implies that for a given \( v \) eq. (22) determines \( u[\gamma^{ss}_g] \) (or \( \gamma^{ss}_g[u] \)): for every \( v \) and \( \gamma^{ss}_g \) there exists a \( u \) which conserves momentum. Conversely, for every pair of \( u \) and \( v \) momentum conservation determines a unique two dimensional surface \( \gamma^{ss}_g[u, v] \), whose intersection with the plane \( \gamma^{ss}_g = (\gamma^{ss}_g)_{Q_0 \text{DIS}} \) gives back the curve \( u[v] \). Moving on the plane parametrized by \( u \) and \( v \) for fixed \( \gamma^{ss}_g \) yields anomalous dimensions which are equivalent up to a NLO change of scheme (and only conserve momentum along the \( u[v] \) curve); thus, it explores the uncertainty related to the ignorance of NNLO corrections. Different choices of \( \gamma^{ss}_g \) then explore the uncertainty related to the ignorance of the NLO gluon anomalous dimension.

5 While always true in QCD with \( N_c \) colours, since then \( r = \frac{1}{2}(1 - 1/N_c^2) \), in supersymmetric Yang-Mills \( r = 1 \), so (22) fixes \( u \) independently of \( v \).

6 One may, of course, leave the line \( v[u] \) while preserving momentum conservation at NNLO: for example one could perform a scheme change of the conventional momentum conserving form \( V = \left( -\frac{r \hat{v}}{r \hat{v}} - \frac{\hat{v}}{\hat{v}} \right) \) (where \( \hat{v} \) is NLO). Under this change of scheme \( \gamma \to \gamma' \), where, if \( \gamma \) satisfies the colour-charge relation in both quark and gluon sectors

\[
\gamma' = \gamma - \hat{v}(\gamma_g + \gamma_q) \begin{pmatrix} r & 1 \\ -r^2 & -r \end{pmatrix} + O(\alpha^2).
\]

(24)

If \( \gamma \) satisfies momentum conservation, \((\gamma_q + \gamma_g)_{N=1} = 0 \) and the second term vanishes; however \( v\gamma_q \), which must be kept in order to get momentum conservation, is actually NNLO. One would then be forced to include some NNLO contributions in a NLO computations in order to preserve the momentum sum rule.
Momentum conservation constrains the uncertainty in that it fixes one of the parameters in terms of the other two; the overall uncertainty however is still necessarily NLO. Now, given $\gamma_g^s$ and $\gamma_q^{ss}$ (from eq. (18) for a certain choice of $u$ and $v$) $\gamma_g^{ss}$ is determined algebraically (by eq. (1) for $n > 2$) without the need for an explicit computation [1]; it is thus convenient to fix it thus, and vary $u$ for given $v$ in order to estimate the corresponding NLO uncertainty [1]. In practice, this uncertainty should only have minor effects on $F_2$, because since $\gamma_q^s$ vanishes, $q$ evolves only at NLO, while although $g$ evolves at LO it only affects structure functions at NLO; the effect of $\gamma_g^{ss}$ is then subleading when compared to $\gamma_g^s$, and thus effectively NNLO.

In conclusion, we have shown that momentum conservation can be enforced in QCD evolution equations at NLO even when anomalous dimensions are computed within an expansion scheme which sums up all leading and subleading logs of both $1/x$ and $Q^2$, such as the double leading expansion [2]. Within these expansion schemes there is a wider freedom of choice of factorization scheme than in the usual (large $x$) loop expansion of anomalous dimensions [4]: besides the usual freedom to perform a NLO scheme change which modifies the $F_2$ coefficient functions, there is now also the possibility of performing a LO scheme change which does not affect the $F_2$ coefficient functions or the LO anomalous dimensions, but changes the definition of the gluon distribution (or, equivalently, the $F_L$ coefficient function). The latter freedom is however fixed by the requirement of momentum conservation. This is analogous to what happens in large $x$ computations, where the momentum sum rule fixes the normalization of the gluon; however at small $x$ the normalization is actually given by a general LO function of $\alpha/N$, rather than being just a number. Because the NLO $\gamma^{gg}$ and $\gamma^{gq}$ gluon anomalous dimensions are still unknown, this freedom cannot in practice be pinned down, so that it is still necessary to vary the gluon normalization within a given NLO factorization scheme, thereby introducing a NLO uncertainty in the solution of the evolution equations [1]. This uncertainty has fortunately only sub-subleading effects on the structure function $F_2$, however, due to the vanishing of the LO singularities in the quark anomalous dimensions.

If $\gamma_g^{ss}$ were to be determined explicitly (for instance by computing subleading corrections to the BFKL kernel in a particular factorization scheme) it would then be possible use momentum conservation to fix the definition of the gluon distribution, thereby reducing the uncertainty in the solution of evolution equations to a purely NNLO one. This could have significant phenomenological consequences in that it might substantially reduce the large
scheme dependence which is at present an intrinsic feature of perturbative computations at small $x$ and relatively low $Q^2$.

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