Modeling of solar sail surface oscillations during interplanetary flight

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Abstract. The purpose of this work is to study the influence of solar sail surface oscillations on the motion of a spacecraft performing an interplanetary flight. The solar sail is a propulsion system, creating thrust due to the pressure of sunlight. The article consists of three parts. The first part focuses on the wave equation applied to this problem. The equation for oscillations is derived for various cases of sail loading, and various types of its design. Also, we derived a formula for the thrust produced by the solar sail. In the second part, the problem of oscillations of the sail structure is solved and its thrust is estimated. Solar sail thrust depends both on the shape of the sail, and on its distance from the Sun. In the third part, the interplanetary flight is modeled taking into account these factors.

1. Introduction
We consider the modeling of forced vibrations of the solar sail surface due to sunlight pressure on it. The solar sail is a propulsion system creating thrust due to the pressure of sunlight on its surface [1]. Thrust produced by solar sail depends on the distance to the Sun, its orientation relative to the Sun, its reflectivity and the form of its surface. Our focus will be on the study of the effect of latter factor.

The possibility of interplanetary flights of a spacecraft using a solar sail (SSS) has been studied by many authors, including [2, 3, 4]. In these papers the solar sail is assumed to be a perfectly flat surface. The main purpose of this paper is to derive analytical expressions for determining the shape of solar sail oscillations and to assess their possible influence on its motion during interplanetary flights.

Our study is focused on Solar Sail Spacecraft (SSS) “Sunjammer”, developed by L’Garde, Inc. The solar sail, mounted on it, is rectangular, 38×38 m with thickness 5 nm [5]. The influence of Sunjammer SS oscillations on the thrust produced by it will be assessed.

2. A mathematical model of the solar sail surface
To solve the problem of solar sail (SS) oscillations we make the following assumptions [6]:
- The solar sail is a membrane fixed at the edges;
- Oscillations of the SS are assumed to be transverse and small;
- The change in in sail surface area during oscillations are ignored;
- The sail is assumed to be made of homogenous material.
We introduce rectangular reference frame $Oxyu$. The base plane $xOy$ coincides with undeformed solar sail surface. The base direction $Ou$ is perpendicular to base plane and directed against the line of action of external forces.

2.1. Derivation of the equation for free oscillation of a rectangular sail
Let the membrane at rest has the form of a rectangle bounded by straight lines $x = 0, x = l, y = 0, y = m$. Following [7], the problem of membrane free oscillations reduces to solving the following homogeneous equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

with initial conditions

$$u|_{t=0} = f(x, y), \quad \frac{\partial u}{\partial t}|_{t=0} = F(x, y)$$

and boundary conditions at the boundaries of the rectangle

$$u|_{x=0} = 0, u|_{x=l} = 0, u|_{y=0} = 0, u|_{y=m} = 0.$$  

In equation (1) coefficient $a^2$ is the ratio of the tensile force to the density of SS material:

$$a^2 = \frac{T}{\rho}.$$  

The solutions to equation (1) with the abovementioned initial and boundary conditions is found using the Fourier method as a product of three functions, each of which depends only on one argument:

$$u(x, y, t) = X(x)Y(y)T(t).$$  

The solution satisfying the initial conditions (2) will be sought in form of series. Each partial solution depends on two indices, $k$ and $n$, leading to the double sum:

$$u(x, y, t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left( a_{k,n} \cos \omega_k t + b_{k,n} \sin \omega_k t \right) \sin \lambda_k x \sin \mu_n y.$$  

Assuming that indices $k$ and $n$ are positive, we will take into account the contributions of all the partial solutions. For $t = 0$ we obtain

$$u|_{t=0} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{k,n} \frac{k\pi x}{l} \sin \frac{n\pi y}{m} = f(x, y),$$

$$\frac{\partial u}{\partial t}|_{t=0} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \omega_k a_{k,n} \frac{k\pi x}{l} \sin \frac{n\pi y}{m} = F(x, y).$$  

where

$$a_{k,n} = \frac{4}{lm} \int_{0}^{l} \int_{0}^{m} f(x, y) \sin \frac{k\pi x}{l} \sin \frac{n\pi y}{m} \, dx \, dy,$$

$$b_{k,n} = \frac{4}{l\omega_k} \int_{0}^{l} \int_{0}^{m} F(x, y) \sin \frac{k\pi x}{l} \sin \frac{n\pi y}{m} \, dx \, dy.$$  

Substituting expressions (8) and (9) into (5), we complete the solution to the problem.

2.2. Derivation of the equation for forced oscillations of a rectangular sail
The problem of forced oscillations of a membrane is reduced to the solution of following inhomogeneous equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + G(x, y, t).$$

The solution to this equation is obtained as the sum of two functions:
\[ u(x, y, t) = v(x, y, t) + w(x, y, t). \]  

(11)

where \( v(x, y, t) \) satisfies the homogeneous equation \( \frac{\partial^2 v}{\partial t^2} = a^2 \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \) and initial and boundary conditions \( v_{|t=0} = f(x, y), \frac{\partial v}{\partial t}_{|t=0} = F(x, y), v_{|x=0} = v_{|x=l} = 0 \). This function describes free oscillations caused by an initial sail displacement and/or velocity. This function was found in Section 2.1. In case of free oscillations \( v(x, y, t) = 0 \). Function \( w(x, y, t) \) should satisfy the inhomogeneous equation

\[
\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + G(x, y, t)
\]

(12)

and zero initial and boundary conditions \( w_{|t=0} = w_{|t=0} = 0, w_{|x=0} = w_{|x=m} = 0 \). This function describes forced oscillations under the influence of external forces in the absence of initial perturbations. This function is obtained in the form of a series of eigenfunctions \( \sin \frac{k \pi x}{l} \) and \( \sin \frac{n \pi y}{m} \) of the homogenous problem:

\[
w(x, y, t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{k,n}(t) \sin \frac{k \pi x}{l} \sin \frac{n \pi y}{m},
\]

(13)

where \( \gamma_{k,n}(t) \) are unknown functions of \( t \). Function \( w(x, y, t) \) satisfies boundary conditions \( w_{|x=0} = w_{|x=l} = 0, w_{|y=0} = w_{|y=m} = 0 \) as all eigenfunctions are equal zero at \( x=0, x=l, y=0, x=m \). For \( \gamma_{k,n}(t) = 0 \) and \( \gamma_{k,n}(t) = 0 \) function \( w(x, y, t) \) satisfies the initial zero conditions.

Let us expand function \( G(x, y, t) \) in double Fourier series in intervals \([0, l] \) и \([0, m] \) for eigenfunctions \( \sin \frac{k \pi x}{l} \) and \( \sin \frac{n \pi y}{m} \):

\[
G(x, y, t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} g_{k,n}(t) \sin \frac{k \pi x}{l} \sin \frac{n \pi y}{m},
\]

(14)

where

\[
g_{k,n}(t) = \frac{4}{lm} \int_{0}^{l} \int_{0}^{m} G(x, y, t) \sin \frac{k \pi x}{l} \sin \frac{n \pi y}{m} \text{d}x \text{d}y.
\]

(15)

Substituting (14) into (12), we obtain the differential equation for finding unknown functions \( \gamma_{k,n}(t) \)

\[
\gamma_{k,n}(t) + \omega^2 \gamma_{k,n}(t) = g_{k,n}(t),
\]

(16)

with initial conditions

\[
\gamma_{k,n}(0) = 0, \quad \gamma_{k,n}'(0) = 0,
\]

(17)

where \( \omega^2 = a^2 (\lambda^2 + \mu^2) \).

The generalized solution to equation (16) for \( g_{k,n}(t) = \text{const} \) can be presented as:

\[
A_{k,n} \cos \omega t + B_{k,n} \sin \omega t + \frac{g_{k,n}}{\omega^2},
\]

(18)

where \( A_{k,n} \) and \( B_{k,n} \) are arbitrary constants. These constants can be found via substituting initial conditions (17) into solution (18).

2.3. Solar sails of triangular shapes

Most of solar sails are rectangular, sail petals are triangular. Therefore, to solve the problem of solar sail oscillations, we need to transform the original rectangular model into triangular. This can be achieved by adding a nodal line equation to solution (13):
\[ b_{1,1} \cos(\pi x) + b_{1,3} \cos(\pi y) = 0. \]  

For \( b_{3,1} = -b_{1,3} \), this equation gives \( \cos(\pi x) = \cos(\pi y) \), i.e. \( x = y \) is the nodal line in the diagonal of a rectangle. For \( b_{3,1} = b_{1,3} \), we have \( \cos(\pi y) = -\cos(\pi x) \). In this case, the nodal line is another diagonal.

Substituting the nodal line equation with \( b_{3,1} = b_{1,3} \) into the rectangle sail oscillation equations new oscillation equations for the triangular sail shapes are obtained.

2.4. Evaluation of thrust produced by solar sail

Let the solar sail under the influence of sunlight produces pressure parallel to axis \( Ou \). The thrust created by an infinitely-small surface area of the solar sail can be estimated as [8, 9]:

\[ dF = pd\sigma \cos \vartheta, \]

where \( p \) is sunlight pressure at some distance from the Sun, Pa; \( \vartheta \) is the angle between the normal to solar sail surface and axis \( Ou \).

Since the oscillations of the sail are transverse, the normal to its surface can be found as:

\[ n = \left\{ -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}, 1 \right\}. \]

In this case, the thrust produced by the solar sail can be estimated as:

\[ F = p \iint_D \frac{1}{\sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + 1}} d\sigma. \]  

The surface integral of first kind in this formula can be reduced to a double integral by projecting the surface on \( xOy \) plane:

\[ ds = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + 1} dxdy. \]

Thus, expression (21) takes the form

\[ F = p \iint_{x_0}^{x_m} \iint_{y_0}^{y_m} \frac{1}{\sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + 1}} dxdy = p \iint_{x_0}^{x_m} \iint_{y_0}^{y_m} \frac{1}{\sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + 1}} dxdy. \]  

Substituting the expressions for the derivatives of \( u \) into equation (22) and taking the integrals, the thrust force produced by the solar sail is obtained.

3. Solution to the problem of the solar sail surface oscillations

3.1. Forced oscillations

Let the sunlight pressure \( P \) acts on the surface of SSS “Sunjammer” at time instant \( t = 0 \). In this case in equation (10) \( G(x, y, t) = -P \). Remembering (15), coefficient \( g_{k,n}(t) \) can be found as:

\[ g_{k,n}(t) = -\frac{4P}{lm} \int_0^l \sin \frac{k\pi x}{l} \sin \frac{n\pi y}{m} dxdy = -\frac{4P}{\pi^2 kn} (1 - \cos k\pi)(1 - \cos n\pi), \]

Hence,

\[ g_{2r,2s} = 0, \quad g_{2r+1,2s+1} = -\frac{16P}{\pi^2 (2r + 1)(2s + 1)}. \]

Functions \( \gamma_{2r,2s}(t) \) are zero for all times, since they satisfy the homogeneous equation
\[ \gamma_{2r,2s}(t) + a^2 \pi^2 \left( \left( \frac{2r}{l} \right)^2 + \left( \frac{2s}{m} \right)^2 \right) \gamma_{2r,2s}(t) = 0 \]

with zero initial conditions \( \gamma_{2r,2s}(t=0) = 0 \).

For functions with odd indices \( \gamma_{2r+1,2s+1}(t) \) Equation (16) takes the form

\[ \gamma_{2r+1,2s+1}(t) + a^2 \pi^2 \left( \left( \frac{2r+1}{l} \right)^2 + \left( \frac{2s+1}{m} \right)^2 \right) \gamma_{2r+1,2s+1}(t) = 0. \]

Solving this differential equation, we obtain:

\[ \gamma_{2r+1,2s+1}(t) = \frac{16P}{\pi^4 a^2 (2r+1)(2s+1)} \left( 1 - \cos \left( a \pi \sqrt{\frac{(2r+1)^2}{l} + \frac{(2s+1)^2}{m}} t \right) \right). \]

Substituting this function into (13), we obtain the equation for solar sail oscillations:

\[ w(x, y, t) = -\frac{16P}{\pi^4 a^2} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{1 - \cos \left( a \pi \sqrt{\frac{(2r+1)^2}{l} + \frac{(2s+1)^2}{m}} t \right)}{(2r+1)(2s+1)} \left( \frac{2r+1}{l} \right)^2 \sin \left( \pi \frac{2r+1}{l} x \right) \sin \left( \pi \frac{2s+1}{m} y \right). \]

Maximal displacement of surface \( w \) is shown on figure 1.

**Figure 1.** Forced oscillations of a rectangle solar sail.

Now we find the equation for oscillations of a triangle solar sail. We add the nodal line equation to the previously obtained equation. As a result we obtain the new equation

\[ w(x, y, t) = -\frac{16P}{\pi^4 a^2} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{1 - \cos \left( a \pi \sqrt{\frac{(2r+1)^2}{l} + \frac{(2s+1)^2}{m}} t \right)}{(2r+1)(2s+1)} \left( \frac{2r+1}{l} \right)^2 \sin \left( \pi \frac{2r+1}{l} x \right) \sin \left( \pi \frac{2s+1}{m} y \right). \]

The maximal displacement \( w \) is presented on Figure 2.

The series in this equation converges quickly, so there is no need in considering a large number of its terms. By experimental way the optimal number of terms was found to be 5-7.
3.2. The variations of the solar sail thrust.
Using Formula (22), we calculate the thrust produced by the solar sail, taking into account its oscillations. Figure 3 shows how this thrust changes with time at various distances from the Sun.

It can be concluded that solar sail oscillations cause small losses of thrust. The amplitude of oscillations and thrust loss increase when we get closer to the Sun. For solar sail presented in this paper thrust loss at 1AU equal 2.45%.

4. Simulation of an interplanetary flight with a solar sail taking into account its oscillations
For modeling of the heliocentric motion of the chosen SSS we use software package “Modeling of heliocentric motion of solar sail spacecraft” developed at the Department of Space Engineering of Samara State Aerospace University. We have chosen a flight to the orbit of Venus. The flight consists of two stages:
- Firstly, the SSS begins deceleration and the focal parameter of the orbit decreases when the spacecraft approaches the orbit of Venus.
- Secondly, upon reaching the orbit of Venus, the eccentricity of the SSS orbit SSS starts to decrease. At the end of this stage the SSS orbit will coincide with the orbit of Venus.
The simulation results and the final position of SSS are shown in Figure 4.
Figure 4. The flight of SSS taking into account losses of thrust due solar sail oscillations.

Flight to the orbit of Venus with thrust losses takes 5 days more than the flight without them taken into account. As a result, long periods of time thrust losses begin to have an effect on SSS motion and must be taken into account.

5. Conclusion
Having analyzed the influence of the solar sail surface oscillations on the motion of a spacecraft, we can conclude that:

- Oscillations of the solar sail surface decrease the thrust produced by this sail. The magnitude of losses increases with Sun is approached.
- Ratio of average thrust losses of a oscillating sail to the flat solar sail thrust for a given design remains constant regardless of the distance to the Sun and is approximately equal to 2.45%.
- During long periods of time, these losses significantly affect the motion of SS, and should be taken into account when calculating flight times.

6. References
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