The Hawaiian earring group is topologically incomplete

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Abstract

If an inverse limit space $X$ is constructed in similar fashion to the Hawaiian earring, then the fundamental group $\pi_1(X, p)$ becomes a topological group with either of two natural but distinct topologies. In either case, despite being uncountable, we show $\pi_1(X, p)$ is not a Baire space. The Hawaiian earring in particular provides a premier example of a Peano continuum whose topological fundamental group, despite regularity and uncountability, admits no complete metric compatible with its topology.

1 Introduction

Endowed with the quotient topology from the space of based loops in $X$, the familiar fundamental group $\pi_1(X, p)$ of a space $X$ becomes a topological group. As a subject in its own right, the theory of topological fundamental groups is in the early stages of development, and the extent to which the theory will prove useful remains to be seen. For example, the theory has no extra value in the context of locally contractible spaces, since if $X$ is locally contractible, then $\pi_1(X, p)$ inherits the discrete topology (11) and encodes no more information than algebraic data.

On the other hand, if we also consider spaces that are not locally contractible then the theory begins to earn its keep. For example both the algebra and topology of $\pi_1(X, p)$ is invariant under the homotopy type of the underlying space (Corollary 3.4 [11]). Consequently, by analyzing the
topology of $\pi_1(X, p)$, the theory has the capacity to distinguish spaces of distinct homotopy types when the familiar algebraic homotopy groups fail to do so (see [13] for examples.)

In any mathematical theory (e.g. the theory of groups, the theory of $n$ manifolds, the theory of Banach spaces, etc...) the investigation of reasonable conjectures tends to work in tandem with the cataloging of properties of the fundamental examples. This paper was written with the latter goal in mind. In particular we investigate the topological fundamental group of one of the simplest nonlocally contractible spaces, the Hawaiian earring $HE$. The space $HE$ is the union of a null sequence of simple closed curves joined at a common point $p$. Hence $HE$ is a one dimensional metric Peano continuum, and the deletion of one point leaves a (disconnected) one manifold.

The algebra of $\pi_1(HE, p)$ has been studied from various perspectives by other authors ([2], [3], [1], [5], [7], [8], [15], [17], [19], [20].) However, relatively little is known about the topology of $\pi_1(HE, p)$. The following tool is indispensable for the investigation of $\pi_1(HE, p)$. Consider $HE = \lim_\leftarrow X_n$ as the inverse limit of a nested sequence of retracts $X_1 \subset X_2, \ldots$ where $X_n$ is the union of $n$ simple closed curves joined at a common point $p$, and consider the canonical homomorphism $\phi : \pi_1(HE, p) \rightarrow \lim_\leftarrow \pi_1(X_n, p)$. Since $\phi$ is continuous (Proposition 3.3 [1]), and since $X_n$ is locally contractible, the question of whether $\pi_1(HE, p)$ is a $T_1$ space (a space whose one point subsets are closed) is precisely equivalent (see [14]) to the question of whether $\phi$ is one to one. A comprehensive affirmative answer to the latter question constitutes the lion’s share of a paper of Morgan and Morrison [17]. Since the $T_1$ space $\pi_1(HE, p)$ is also a topological group, it follows from general properties of topological groups that $\pi_1(HE, p)$ is in fact completely regular. It is an open question whether $\pi_1(HE, p)$ is normal or whether $\pi_1(HE, p)$ is metrizable.

Having said this we should point out that the metrizability (and hence normality) of $\pi_1(HE, p)$ would follow from the published work of another author, work that this author believes to be incorrect. A lack of clarity makes it difficult to pinpoint the location of the first mistake. The paper [1] seems to claim that $\phi : \pi_1(HE, p) \rightarrow \lim_\leftarrow \pi_1(X_n, p)$ is an isomorphism with continuous inverse (Theorem 8.1 [1]). Surjectivity is easily seen to fail since, in $\lim_\leftarrow \pi_1(X_n, p)$, the “infinite word” $x_1, x_2, x_1^{-1}, x_2^{-1}, x_1, x_3, x_1^{-1}, x_3^{-1}, \ldots$ has no counterpart in $\pi_1(HE, p)$. The failure of $\phi$ to be a homeomorphism onto its image is not so easy to see, and an explicit counterexample is contained
in my paper [10].

Fortunately the main results of this paper do not depend on these contestable claims. We prove that if $X$ is the Hawaiian earring, or if $X$ is constructed in similar fashion to the Hawaiian earring, then $\pi_1(X, p)$, despite being uncountable, is not a Baire space.

An immediate application is the following: Ignoring the question of whether the regular space $\pi_1(X, p)$ is metrizable, we can conclude by the Baire category theorem that $\pi_1(X, p)$ admits no complete metric compatible with its topology. This in turn yields a ‘no retraction theorem’ whose roots depend on the topological rather than the algebraic properties of the groups in question: Because $\pi_1(X, p)$ is not a Baire space, the main result of [12] shows that $X$ cannot be embedded as retract of any space $Y$ such that $\pi_1(Y, p)$ admits a complete metric.

Finally, in the process of proving $\pi_1(X, p)$ is not a Baire space we also prove that $\text{im}(\phi)$ is not a Baire space in $\lim_{\leftarrow} \pi_1(X_n, p)$. Thus this paper seems to contain the premier exhibition of the following phenomenon: There exists a Peano continuum $X$ such that with either of two natural but distinct $T_1$ topological group structures, the fundamental group of $X$ admits no complete metric compatible with its topology.

It is hoped that uncovering the properties of the simplest examples such as $HE$ will serve as a useful benchmark when pursuing further developments in the theory of topological fundamental groups.

\section{Preliminaries}

Given a set $S$ a \textbf{word} in $S$ is either the empty set or a finite sequence in $S$. The entries of a word are called \textbf{letters}. A \textbf{subword} $v$ of the word $w$ is the word obtained by deleting finitely many letters from $w$.

Suppose $H_1, H_2, \ldots$ is a sequence (either finite or infinite) of pairwise disjoint groups. Let $W$ denote the set of words in the set $H_1 \cup H_2 \cup \ldots$.

If $a$ is a letter of the word $w \in W$ then $a$ is of \textbf{type} $i$ if $a \in H_i$.

Define $K : W \to W$ as follows. Suppose $w$ is a word in $W$. To define $K(w)$ first partition $w$ into a minimal number of cells such that each cell is a subword of $w$ consisting of consecutive letters of $w$, such that each letter is of the same type. Delete each cell whose multiply to make $id_{H_i}$. Replace the remaining cells by the product of its letters. Call the resulting subword $K(w)$. Notice $K(w)$ has no more letters than $w$, and that $K(w)$ and $w$ have
the same number of letters if and only if \( K(w) = w \). Thus the sequence \( w, K(w), K^2(w), \ldots \) is eventually constant. If \( K(w) = w \) we say \( w \) is **reduced**. Moreover a word \( w \) is reduced if and only if \( w \) is empty word or \( w \) is a nonempty word in the set \( (H_1 \cup H_2 \cup \ldots) \setminus \{id_{H_1}, id_{H_2}, \ldots\} \) such that no two consecutive letters are of the same type.

Define \( RD : W \to W \) such that \( RD(w) = \lim_{n \to \infty} K^n(w) \). Declare two words \( w \) and \( v \) in \( W \) equivalent if \( RD(v) = RD(w) \). By construction each equivalence class contains a unique reduced word.

The group **free product** \( H_1 \ast H_2 \ast \ldots \) consists of the set of all reduced words in \( H_1 \cup H_2 \cup \ldots \) with multiplication determined by the rule \( (a_1, a_2, \ldots a_n) \ast (b_1, b_2, \ldots b_m) = RD(a_1, a_2, \ldots a_n, b_1, \ldots b_m) \). Thus \( \emptyset \) serves as identity, and inverses are determined such that if \( w = a_1, a_2, \ldots a_n \) then \( w^{-1} = a_n^{-1}, \ldots, a_1^{-1} \). See [16] for more details on free products of groups.

A surjective map \( q : Y \to X \) is a **quotient map** provided \( U \) is open in \( X \) if and only if \( f^{-1}(U) \) is open in \( Y \).

Suppose \( X \) is a topological space and \( p \in X \). Endowed with the compact open topology, let \( C_p(X) = \{f : [0, 1] \to X \mid f(0) = f(1) = p\} \). Then the **topological fundamental group** \( \pi_1(X, p) \) is the quotient space of \( C_p(X) \) obtained by considering the path components of \( C_p(X) \) as points. Endowed with the quotient topology, \( \pi_1(X, p) \) is a topological group (Proposition 3.1 [1]). Moreover a map \( f : X \to Y \) determines a continuous homomorphism \( f^* : \pi_1(X, p) \to \pi_1(Y, f(p)) \) (Proposition 3.3 [1]).

If \( A_1, A_2, \ldots \) are topological spaces and \( f_n : A_{n+1} \to A_n \) is a continuous surjection then, (endowing \( A_1 \times A_2 \ldots \) with the product topology), the inverse limit space \( \lim_{\leftarrow} A_n = \{(a_1, a_2, \ldots) \in (A_1 \times A_2 \ldots) \mid f_n(a_{n+1}) = a_n\} \).

A topological space \( Z \) is a **Baire** space provided \( \cap_{n=1}^{\infty} U_n \) is dense in \( Z \) whenever each of \( U_1, U_2, \ldots \) is an open dense subspace of \( Z \). It is a classical fact that every complete metric space is a Baire space (Theorem 7.2 [18]).

### 3 Definitions

For the remainder of the paper we make the following assumptions.

All topological fundamental groups are endowed with the quotient topology.

Suppose \( X^- = \cup_{n=1}^{\infty} X_n \) is a metrizable topological space. Assume \( X_1 \subset X_2 \subset X_3 \ldots \) is a nested sequence of locally simply connected subspaces with \( p \in X_1 \).
Assume $X_{n+1} = X_n \cup Y_n$ with $X_n \cap Y_n = p$ and assume $Y_n$ is not simply connected.

Let $r_n : X_{n+1} \to X_n$ denote the retraction collapsing $Y_{n+1}$ to the point $p$.

Assume $X$ admits a topologically compatible metric such that $diam(Y_n) \to 0$. This assumption serves, for example, to distinguish the (compact) Hawaiian earring from the (noncompact) union of countably many circles of radius 1 joined at the common point $p$.

Let $\kappa_{\infty} : \bigcup_{n=1}^{\infty} X_n \to X$ denote the inverse limit space under the maps $r_n$.

Let $\psi_n : G_{n+1} \to G_n$ denote the canonical epimorphism $\psi_n = RD(\kappa_n)$. Thus, to compute $\psi_n$ of the reduced word $w_{n+1}$, first delete from $w_{n+1}$ all letters of type $n+1$, and then reduce the resulting subword as much as possible.

Let $G$ denote the subgroup of $\lim_{\to} G_n$ consisting of all sequences $(w_1, w_2, ...)$ such that for each $m \geq 1$ the following sequence is eventually constant: $\kappa_m(w_1), \kappa_m(w_2), ...$

Define $\sigma : G \to \{1, 2, 3, ...\}$ such that $\sigma(\{w_n\}) = N$ if $N$ is minimal such that $\kappa_1(w_N), \kappa_1(w_{N+1}), ...$ is constant.

4 Facts about the preliminary data

**Proposition 1** $X^\ast$ is canonically homeomorphic to $X$ under the map $h : \bigcup_{n=1}^{\infty} X_n \to X$ satisfying $h(x_n) = (p, p, ..., x_n, x_n, ...)$. Moreover $h_{X_n} \hookrightarrow X$ is an embedding. Henceforth we consider $X^\ast$ and $X$ to be the same space.

**Proposition 2** The van Kampen theorem determines a canonical isomorphism $j_n^* : \pi_1(X_n, p) \to G_n$. Henceforth we will consider these groups to be the same. See [16] for details.
Most of [17] is devoted to Theorem 4.1, which, translated into the notation of this paper, becomes Proposition 3.

**Proposition 3** The canonical map \( \phi : \pi_1(X,p) \to \lim \rightarrow G_n \) is a monomorphism. Moreover \( \text{im}(\phi) = G \).

The following proposition is perhaps surprising and is the subject of [10].

**Proposition 4** The continuous isomorphism \( \phi : \pi_1(X,p) \to G \) is not a homeomorphism.

# 5 Main result: \( \pi_1(X,p) \) is not a Baire space

**Lemma 5** For each \( N \geq 1 \), the set \( \sigma^{-1}(N) \) is closed in \( \lim \rightarrow G_n \).

**Proof.** Recall an element \( f \in \lim \rightarrow G_n \) is a function \( f : \{1,2,3,...\} \to \bigcup_{n=1}^{\infty} G_n \) such that \( w_n = f(n) \in G_n \) and \( w_n \) is a reduced word in \( H_1 \ast \ast H_n \) and \( \psi_n(w_{n+1}) = w_n \). Suppose the sequence \( \{f_m\} \in \sigma^{-1}(N) \) converges to \( f \in \lim \rightarrow G_n \). Suppose \( N \leq i \). Since \( f_m \) converges to \( f \) pointwise, and since both \( G_i \) and \( G_{i+1} \) have the discrete topology, there exists \( M \) such that \( f(i) = f_M(i) \) and \( f(i + 1) = f_M(i + 1) \). Since \( f_M \in \sigma^{-1}(N) \) we know \( \kappa_1(f_M(i)) = \kappa_1(f_M(i + 1)) \). Thus, if \( i \geq N \) \( \kappa_1(f(i)) = \kappa_1(f(i + 1)) \). In similar fashion we may choose \( K \) such that \( f(N - 1) = f_K(N - 1) \) and \( f(N) = f_K(N) \). It follows that \( \kappa_1(f(N - 1)) \neq \kappa_1(f(N)) \). Thus \( \sigma(f) = N \). Hence \( \sigma^{-1}(N) \) is closed in \( \lim \rightarrow G_n \). 

**Lemma 6** For each \( K \geq 1 \) the set \( \sigma^{-1}(K) \) contains no open subset of \( G \).

**Proof.** Suppose \( \sigma(f) = K \). For each \( i \geq 1 \), choose \( h_i \in H_i \setminus \text{id}_{H_i} \). For \( n \geq 2 \) define \( g_n \in \lim \rightarrow G_n \) such that if \( i \geq n \) then \( g_n(i) = h_1, h_n, h_1^{-1}, h_n^{-1} \ldots \)

In similar fashion define \( g^n \in \lim \rightarrow G_n \) such that if \( i \geq n \) then \( g^n(i) = h_n, h_1, h_n^{-1}, h_1^{-1} \ldots \)

Note \( g_n \to (0,0,\ldots) \) and \( g^n \to (0,0,\ldots) \). Let \( f_n = g_n \ast f \) and \( f^n = g^n \ast f \).

Since \( \lim \rightarrow G_n \) is a topological group, \( f_n \to f \) and \( f^n \to f \). Let \( \kappa_1(f(K)) = x_k, x_{k2}, \ldots x_{ks} \). Suppose \( n \geq K + 1 \). Since \( \sigma(f) < n \) we conclude

\[
\kappa_1(f(n)) = x_{k1}, x_{k2}, \ldots x_{ks} = \kappa_1(f(n - 1)) = \kappa_1(f_n(n - 1)) = \kappa_1(f^n(n - 1)).
\]
Next we show that at least one of the following inequalities hold: \( \kappa_1(f_n(n)) \neq x_{k_1}, x_{k_2}, \ldots, x_{k_s} \) or \( \kappa_1(f^n(n)) \neq x_{k_1}, x_{k_2}, \ldots, x_{k_s} \).

Let \( f(n) = x_1, x_2, \ldots, x_m \in H_1 \ast \ast H_n \).

If \( x_1 \notin H_n \) then \( f_n(n) = h_1, h_n, h_n^{-1}, h_n^{-1}, x_1, x_2, \ldots, x_m \) and \( \kappa_1(f_n(n)) = h_1, h_n^{-1}, x_1, x_2, \ldots, x_{k_s} \neq x_{k_1}, x_{k_2}, \ldots, x_{k_s} \).

If \( x_1 \in H_n \) then \( f^n(n) = h_n, h_1, h_n^{-1}, h_n^{-1}, x_1, x_2, \ldots, x_m \) and \( \kappa_1(f^n(n)) = h_1, h_n^{-1}, x_1, x_2, \ldots, x_{k_s} \neq x_{k_1}, x_{k_2}, \ldots, x_{k_s} \).

Thus, for each \( n \) at least one of \( f_n \) or \( f^n \) does not belong to \( \sigma^{-1}(K) \). Hence \( \sigma^{-1}(K) \) contains no open subset of \( G \). ■

**Example 7** Replace the usual topology of the rational numbers \( Q \) with the discrete topology creating the space \( Q' \). Then \( \text{id} : Q' \rightarrow Q \) is continuous and one to one but not an embedding. In particular the set \( \{0\} \subset Q \) contains no open set. However \( \{0\} \) is open in \( Q' \).

**Remark 8** It is shown in [III] that \( \phi \) is not an embedding. Hence, considering example [7] Lemma [9] is not an immediate consequence of Lemma [8]. However the same idea drives both proofs.

**Lemma 9** For each \( K \geq 1 \) the set \( \phi^{-1}(\sigma^{-1}(K)) \) contains no open subset of \( \pi_1(X, \{p\}) \).

**Proof.** Since \( Y_n \) is not simply connected for each \( n \geq 1 \) select \( h_n \in C_p(X_n) \) such that \( [h_n] \in \pi_1(Y_n,p) \backslash \text{id}_n \). Let \( h_\infty \) denote the constant map \( p \). Let \( U = \phi^{-1}(\sigma^{-1}(K)) \). Let \( F : C_p(X) \rightarrow \pi_1(X, p) \) denote the canonical surjective quotient map. Let \( A \subset U \). Let \( B = F^{-1}(U) \). Since \( F \) is a quotient map, in order to prove \( A \) is not open it suffices to prove \( B \) is not open in \( C_p(X) \).

Suppose \( [\alpha] \in A \). For each \( n \in \{\infty\} \cup \{1, 2, 3, \ldots\} \) let \( g_n = h_1 \ast h_n \ast h_n^{-1} \ast h_n^{-1} \ast [\alpha] \) denote the path in \( C_p(X) \) obtained by concatenation and compatible with the partition \( \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\} \).

In similar fashion define \( g_n = h_\infty \ast h_1 \ast h_\infty^{-1} \ast h_\infty^{-1} \ast [\alpha] \). Note \( g_n \rightarrow g_\infty \) and \( g_n \rightarrow g_\infty \) uniformly and \( \{g_\infty, g_\infty\} \subset B \) since \( g_\infty \) and \( g_\infty \) are path homotopic in \( X \) to \( \alpha \). For \( n \geq 2 \) let \( f^n = \phi F(g^n) \), let \( f_n = \phi F(g_n) \) and let \( f = \phi F(\alpha) \).

Since \( \phi F \) is continuous, we conclude \( f_n \rightarrow f \) and \( f^n \rightarrow f \). Now we may appeal to the proof of Lemma [3] to conclude that if \( n \geq K + 1 \) then at least one \( f^n \) or \( f_n \) is not an element of \( \sigma^{-1}(K) \). Thus at least one of \( g^n \) or \( g_n \) does not belong to \( B \). Hence \( B \) is not open and this proves Lemma [9]. ■

**Theorem 10** The topological groups \( G \) and \( \pi_1(X, p) \) are not Baire spaces.
Proof. To prove that a topological space $Z$ is not a Baire space it suffices to prove $Z$ is the countable union of closed subspaces $Z_1 \cup Z_2 \ldots$ such that $Z_n$ has empty interior for each $n$. Taking $Z_n = \sigma^{-1}(n)$ it follows directly from Lemmas 5 and 6 that $G$ is not a Baire space. To prove $\pi_1(X, p)$ is not a Baire space consider the sets $A_n = \phi^{-1}(Z_n)$. Since $\phi$ is continuous, $A_n$ is closed by Lemma 9. Lemma 9 shows $A_n$ has empty interior. ■

Corollary 11 Endowed with either the quotient topology or the inverse limit space topology, $\pi_1(X, p)$ does not admit a complete metric.

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