QUASI-DIAGONALIZATION OF HANKEL OPERATORS

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Abstract. We show that all Hankel operators $H$ realized as integral operators with kernels $h(t + s)$ in $L^2(\mathbb{R}_+)$ can be quasi-diagonalized as $H = L^* \Sigma L$. Here $L$ is the Laplace transform, $\Sigma$ is the operator of multiplication by a function (distribution) $\sigma(\lambda)$, $\lambda \in \mathbb{R}$. We find a scale of spaces of test functions where $L$ acts as an isomorphism. Then $L^*$ is an isomorphism of the corresponding spaces of distributions. We show that $h = L^* \sigma$ which yields a one-to-one correspondence between kernels $h(t)$ and sigma-functions $\sigma(\lambda)$ of Hankel operators. The sigma-function of a self-adjoint Hankel operator $H$ contains substantial information about its spectral properties. Thus we show that the operators $H$ and $\Sigma$ have the same numbers of positive and negatives eigenvalues. In particular, we find necessary and sufficient conditions for sign-definiteness of Hankel operators. These results are illustrated at examples of quasi-Carleman operators generalizing the classical Carleman operator with kernel $h(t) = t^{-1}$ in various directions. The concept of the sigma-function directly leads to a criterion (equivalent of course to the classical Nehari theorem) for boundedness of Hankel operators. Our construction also shows that every Hankel operator is unitarily equivalent by the Mellin transform to a pseudo-differential operator with amplitude which is a product of functions of one variable only (of $x \in \mathbb{R}$ and of its dual variable).

1. Introduction

1.1. Hankel operators can be defined as integral operators

$$ (Hf)(t) = \int_0^\infty h(t + s)f(s)ds $$

(1.1)

in the space $L^2(\mathbb{R}_+)$ with kernels $h$ that depend on the sum of variables only. We refer to the books [14, 15, 16] for basic information on Hankel operators. Of course $H$ is symmetric if $h(t) = h(t)$. There are very few cases when Hankel operators can be explicitly diagonalized. The simplest and most important case $h(t) = t^{-1}$ was considered by T. Carleman in [3].

Our goal here is to show that all Hankel operators can be quasi-diagonalized in the following sense. Let $L$,

$$ (Lf)(\lambda) = \int_0^\infty e^{-t\lambda}f(t)dt, $$

(1.2)

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be the Laplace transform. Under very general assumptions on \( h \), we prove that
\[
H = L^* \Sigma L \quad (1.3)
\]
where \( \Sigma \) is the operator of multiplication by the function \( \sigma(\lambda) \) formally linked to the kernel \( h \) by the relation
\[
h(t) = \int_{-\infty}^{\infty} e^{-t\lambda} \sigma(\lambda) d\lambda \quad (1.4)
\]
(that is, \( h \) is the two-sided Laplace transform of \( \sigma \)). We call \( \sigma(\lambda) \) the sigma-function of the Hankel operator \( H \) or of its kernel \( h(t) \).

It is clear from formula (1.4) that \( \sigma(\lambda) \) can be a regular function only for kernels \( h(t) \) satisfying some specific analytic assumptions. Without such very restrictive assumptions, \( \sigma \) is necessarily a distribution. Even for very good kernels \( h(t) \) (and especially for them), \( \sigma(\lambda) \) may be a highly singular distribution. For example, for \( h(t) = t^k e^{-\alpha t} \) where \( \text{Re} \alpha > 0 \) (\( \alpha \) may be complex) and \( k = 0, 1, \ldots \), the sigma-function \( \sigma(\lambda) = \delta^{(k)}(\lambda - \alpha) \) is a derivative of the delta-function.

Relation (1.3) does not require the condition \( h(t) = \overline{h(t)} \). If however it is satisfied, then under proper assumptions \( H \) can be realized as a self-adjoint operator although \( \Sigma \) is determined by a quadratic form which does not necessarily give rise to a self-adjoint operator.

Let us compare quasi-diagonalization (1.3) of Hankel operators with the standard diagonalization of convolution operators \( B \) with integral kernels \( b(x - y) \) in the space \( L^2(\mathbb{R}) \). Let \( \Phi \) be the Fourier transform, and let \( S \) be the operator of multiplication by the function (the symbol of the convolution operator \( B \)) \( s(\xi) = \sqrt{2\pi}(\Phi b)(\xi), \xi \in \mathbb{R} \). Then
\[
B = \Phi^* S \Phi. \quad (1.5)
\]
Since the operator \( \Phi \) is unitary, formula (1.5) reduces convolution operators to multiplication operators and hence exhibits their complete spectral analysis.

This is not of course the case with Hankel operators because \( L \) is not unitary. Fortunately, in an appropriate sense, \( L \) turns out to be invertible. Therefore it follows from relation (1.3) that, in the self-adjoint case, the total numbers of (strictly) positive \( N_+(H) \) and negative \( N_-(H) \) eigenvalues of a Hankel operator \( H \) equal the same quantities for the operator \( \Sigma \) of multiplication by the function \( \sigma(\lambda) \):
\[
N_\pm(H) = N_\pm(\Sigma). \quad (1.6)
\]
In particular, \( \pm H \geq 0 \) if and only if \( \pm \Sigma \geq 0 \). Moreover, if \( \sigma(\lambda) > 0 \) (or \( \sigma(\lambda) < 0 \)) on a set of positive Lebesgue measure, then the Hankel operator \( H \) has infinite positive (or negative) spectrum. On the other hand, singularities of \( \sigma(\lambda) \) at some isolated points produce finite numbers (depending on the order of the singularity) of positive or negative eigenvalues.

Equality (1.6) can be compared with Sylvester’s inertia theorem which states the same for Hermitian matrices \( H \) and \( \Sigma \) related by equation (1.3) provided the matrix \( L \)
is invertible. In contrast to the linear algebra, in our case the operators $H$ and $\Sigma$ are of a completely different nature and $\Sigma$ (but not $H$) admits an explicit spectral analysis.

Hankel operators can also be realized in the space $l^2(\mathbb{Z}_+)$ of sequences $g = (g_0, g_1, \ldots)$ by the relation

$$(Qg)_n = \sum_{m=0}^{\infty} q_{n+m} g_m, \quad g = (g_0, g_1, \ldots),$$

(1.7)

which is obviously a discrete analogue of continuous definition (1.1). So it is not astonishing that there exists a unitary operator $U : l^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)$ such that the operator

$$H = UQU^{-1}$$

(1.8)

acting in $L^2(\mathbb{R}_+)$ is Hankel if and only if $Q$ is a Hankel operator in $l^2(\mathbb{Z}_+)$. However the construction of the operator $U$ is nontrivial and is given in terms of the Laguerre functions.

Our results can be translated into the space $l^2(\mathbb{Z}_+)$. In particular, it follows from (1.4) that

$$q_n = \int_{-1}^{1} \eta(\mu) \mu^n d\mu$$

(1.9)

where the function $\eta(\mu)$ is linked to the sigma-function $\sigma(\lambda)$ by a simple change of variables. Equations (1.9) for $\eta(\mu)$ are known as the Hausdorff moment problem. Thus the construction of the sigma-function provides an efficient procedure for the solution of this problem.

1.2. The precise sense of formula (1.3) needs of course to be clarified. Actually, instead of (1.3) we prove the identity

$$(H f_1, f_2) = (\Sigma L f_1, L f_2)$$

(1.10)

on a suitable space of test functions $f_1, f_2$. We find a scale of spaces of test functions where $L$ acts as an isomorphism. By duality, the adjoint operator $L^*$ establishes an isomorphism of the corresponding spaces of distributions. Relation (1.4) should also be understood in the sense of distributions and, strictly speaking, it means that $h = L^*\sigma$, that is,

$$\sigma = (L^*)^{-1} h.$$  

(1.11)

Therefore, instead of operators, we consequently work with quadratic forms which is both more general and more convenient. It is natural to also treat $h$ as a distribution. This yields a one-to-one correspondence between kernels $h$ of Hankel operators and their sigma-functions $\sigma$ and makes the theory self-consistent.

To be precise, equality (1.6) is also formulated in terms of the corresponding quadratic forms $(H f, f)$ and $(\Sigma w, w)$. If the form $(H f, f)$ gives rise to the self-adjoint operator $H$, then (1.6) yields an explicit expression for $N_{\pm}(H)$. We emphasize that typically $\Sigma$ cannot be realized as a self-adjoint operator.
Although formula (1.3) does not give the diagonalization of a Hankel operator $H$, it shows that $H$ is unitarily equivalent (the corresponding unitary transformation is essentially the Mellin transform) to a pseudo-differential operator $A$ in the space $L^2(\mathbb{R})$ with the amplitude

$$v(\xi)s(x)v(\eta), \quad x, \xi, \eta \in \mathbb{R},$$

that factorizes into a product of functions depending on one variable only. Here

$$v(\xi) = \frac{\sqrt{\pi}}{\sqrt{\cosh(\pi \xi)}}$$

is quite explicit and $s(x)$ (called the sign-function of a Hankel operator $H$ in [22]) is linked to the sigma-function by the formula

$$s(x) = \sigma(e^{-x}).$$

To put it differently, $A$ is the integral operator in the space $L^2(\mathbb{R})$ with kernel

$$(2\pi)^{-1/2}v(\xi)s(\xi - \eta)v(\eta)$$

where $\hat{s} = \Phi s$ is the Fourier transform of the sign-function $s$.

1.3. We emphasize that the sigma-function $\sigma(\lambda)$ of a Hankel operator $H$ and its symbol $\theta(\xi), \xi \in \mathbb{R}$, are different objects. In some sense they are dual to each other. Let us discuss their link at a formal level for bounded Hankel operators $H$ when $\sigma(\lambda) = 0$ for $\lambda < 0$. It is more convenient for us to work with symbols $\omega(\mu) := \theta(i\mu)$ defined on the imaginary axis $\text{Re} \mu = 0$. Recall that the kernel $h(t)$ and the symbol $\omega(\mu)$ of a Hankel operator are related by the formula

$$h(t) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} e^{t\mu} \omega(\mu) d\mu, \quad t > 0. \quad (1.14)$$

Equality (1.14) does not determine $\omega(\mu)$ uniquely, but it is satisfied if

$$\omega(\mu) = \int_0^\infty e^{-t\mu} h(t) dt. \quad (1.15)$$

This function is analytic in the right half-plane. Substituting here (1.4), we see that $\sigma$ and $\omega$ are linked by the Stieltjes transform:

$$\omega(\mu) = \int_0^\infty (\lambda + \mu)^{-1} \sigma(\lambda) d\lambda. \quad (1.16)$$

The relation between $\sigma$ and $\omega$ can also be expressed in the following way. Let $\mathbb{H}^1_r$ be the Hardy class of functions analytic in the right half-plane, and let $g \in \mathbb{H}^1_r$. Since (see, e.g., the book [10], page 156)

$$g(\lambda) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{g(\mu)}{\lambda - \mu} d\mu, \quad \text{Re} \lambda > 0,$$
it follows from (1.16) that

$$2\pi i \int_0^\infty \sigma(\lambda) \overline{g(\lambda)} d\lambda = \int_{-i\infty}^{i\infty} \omega(\mu) \overline{g(\mu)} d\mu. \quad (1.17)$$

Formula (1.3) is of course consistent with the standard representation of a Hankel operator $H$ in terms of its symbol $\omega$. Indeed, by the Paley-Wiener theorem, the operator $(2\pi)^{-1/2}L$ is a unitary mapping of $L^2(\mathbb{R}_+)$ onto the Hardy class $H^2_\mathbb{R}$. For $f \in L^2(\mathbb{R}_+)$ and $\text{Re} \ \mu \geq 0$, we put $\tilde{f}(\mu) = (2\pi)^{-1/2}(Lf)(\mu)$. Then, by the definition of the symbol, we have

$$(Hf, f) = -i \int_{-i\infty}^{i\infty} \omega(\mu) \overline{\tilde{f}(\mu)} \overline{f(\mu)} d\mu. \quad (1.18)$$

Let us now apply relation (1.17) to the function $g(\mu) = \overline{\tilde{f}(\mu)} \tilde{f}(\mu)$. Then putting together formulas (1.18) and

$$(L^* \Sigma L f, f) = 2\pi \int_0^\infty \sigma(\lambda) |\tilde{f}(\lambda)|^2 d\lambda,$$

we recover representation (1.3).

Notice that, in contrast to the symbol, the sigma-function contains substantial information about spectral properties of $H$. Comparing relations (1.3) and (1.5), we argue that in the theory of Hankel operators, it is rather sigma-functions (and not symbols) that play the role of symbols of convolution operators. We also note that there is the one-to-one correspondence between kernels and sigma-functions and that the notion of the sigma-function does not require the boundedness of $H$.

1.4. A formal proof of the identity (1.3) is quite simple and is actually the same as that of the identity (1.5) for convolutions. Indeed, the integral kernel of the operator in the right-hand side of (1.3) equals

$$\int_\infty^{-\infty} e^{-\lambda t} \sigma(\lambda) e^{-\lambda s} d\lambda = h(t + s)$$

if $\sigma(\lambda)$ and $h(t)$ are linked by formula (1.4). Thus it equals the integral kernel of the Hankel operator $H$.

However a rigorous proof of (1.3) or, more precisely, of (1.10) requires a choice of a suitable set of test functions $f_j(t)$, $j = 1, 2$, and a correct formulation of relation (1.4). The most natural and general choice is to work on functions $f_j \in C_0^\infty(\mathbb{R}_+)$ and to require that $h \in C_0^\infty(\mathbb{R}_+)'$. Note that $L : C_0^\infty(\mathbb{R}_+) \to \mathcal{Y}$ where the set $\mathcal{Y}$ consists of analytic functions $g(\lambda)$ exponentially decaying as $\text{Re} \ \lambda \to +\infty$, exponentially bounded as $\text{Re} \ \lambda \to -\infty$ and decaying faster than any power of $|\lambda|^{-1}$ as $|\text{Im} \ \lambda| \to \infty$. Since $L : C_0^\infty(\mathbb{R}_+) \to \mathcal{Y}$ and hence $L^* : \mathcal{Y}' \to C_0^\infty(\mathbb{R}_+)'$ are isomorphisms, we see that $\sigma \in \mathcal{Y}'$.

It turns out that typically regular kernels (like those of finite rank Hankel operators) yield singular sign-functions. On the contrary, singular kernels (such as $h(t) = t^{-q}$ where $q > 0$ may be arbitrary large) yield smooth sign-functions. Nevertheless the
conditions $h \in C_0^\infty(\mathbb{R}_+)'$ and $\sigma \in \mathcal{Y}'$ are equivalent and $h$ can be recovered from $\sigma$ by formula (1.4). Thus, although singularities of $h$ and $\sigma$ may be quite different, there is the one-to-one correspondence between $h$ and $\sigma$ in the classes of distributions $C_0^\infty(\mathbb{R}_+)'$ and $\mathcal{Y}'$, respectively.

Another possibility is to work on a set of test functions $f(t)$ satisfying certain analyticity assumptions. This approach is more symmetric because for such $f$, functions $(Lf)(\lambda)$ satisfy conditions similar to those on $f(t)$. This leads to the one-to-one correspondence between kernels $h$ and sigma-functions $\sigma$ in the dual spaces of distributions. It is noteworthy that the inversion of the Laplace transform $L$ in the spaces of analytic test functions is quite explicit and relies on its factorization.

In specific examples, the consideration of the form $(\Sigma w, w)$ on analytic functions $w = Lf \in \mathcal{Y}$ is not always convenient. Fortunately under mild additional assumptions on the sigma-function $\sigma$, the set $\mathcal{Y}$ of test functions $w$ can be replaced by functions $w \in C_0^\infty(\mathbb{R}_+)$. The proof of this reduction also relies on the factorization of the Laplace transform $L$.

As was already mentioned, even for very regular kernels $h$, the sigma-function $\sigma$ may be a highly singular distribution. However, we show that, for positive Hankel operators, $\sigma(\lambda)d\lambda$ is given by some positive measure. Thus in the sign-definite case, $\sigma(\lambda)$ cannot be more singular than delta-functions $\delta(\lambda - \alpha)$ where $\alpha > 0$. Note that, for positive Hankel operators, the concept of the sigma-function, or rather of the associated measure $\sigma(\lambda)d\lambda$, goes back at least to Hamburger (see his paper [9] on moment problems or Theorem 2.1.1 in [1]) and to Bernstein (see his theorem on exponentially convex functions in [2] or Theorem 5.5.4 in [1]). Thus, to a certain extent, our results can be considered as an extension of these classical theorems to the non-sign-definite case.

1.5. We illustrate our general results at the example of kernels

$$h(t) = (t + r)^k e^{-\alpha t}, \quad r \geq 0,$$

where $\alpha$ and $k$ are arbitrary real numbers. These kernels give rise to Hankel operators if $\alpha > 0$ or $\alpha = 0$, $k < 0$. If $\alpha = r = 0$ and $k = -1$, then $H$ is the Carleman operator. In the general case we use the term “quasi-Carleman operator” for a Hankel operator with kernel (1.19).

We show that for kernels (1.19) the sigma-function defined by relation (1.11) is given by the explicit formula

$$\sigma(\lambda) = \frac{1}{\Gamma(-k)}(\lambda - \alpha)^{-k-1}e^{-r(\lambda-\alpha)}, \quad k \notin \mathbb{Z}_+, \quad \mu_{-k-1}$$

where $\Gamma(\cdot)$ is the gamma function and $\mu_{-k-1}$ is the standard distribution defined below by formula (6.2). If $k \in \mathbb{Z}_+$, then $\sigma(\lambda)$ is expressed in terms of the derivatives of the

\footnote{We always use the term “positive” (“negative”) for a nonnegative (nonpositive) operator or a function. Otherwise we write “strictly positive” (“negative”)}
Dirac function:
\[
\sigma(\lambda) = \delta^{(k)}(\lambda - \alpha)e^{-r(\lambda - \alpha)}.
\]

(1.21)

Distributions (1.20) and (1.21) may be singular at the point \( \lambda = \alpha \), and the order of the singularity is determined by the parameter \( k \). We show that the numbers \( N_{\pm}(H) \) are also determined by the parameter \( k \) only.

If \( k < 0 \), then \( \sigma \in L^1_{\text{loc}}(\mathbb{R}^+) \) and \( \sigma(\lambda) \geq 0 \) so that \( H \geq 0 \).

On the contrary, if \( k > 0 \), then function (1.20) has the singularity at the point \( \lambda = \alpha \) which gets stronger as \( k \) increases. If \( k \not\in \mathbb{Z}^+ \), then the function \( \sigma(\lambda) \) for \( \lambda \neq \alpha \) has the same sign as \( \Gamma(-k) \). Therefore \( H \) has infinite positive (negative) spectrum if the integer part \( [k] \) of \( k \) is odd (even). The analysis of the singularity of the function \( \sigma(\lambda) \) at the point \( \lambda = \alpha \) shows that \( N_{\pm}(H) = \frac{[k]}{2} + 1 \) for even \( [k] \) and \( N_{\pm}(H) = \frac{([k] + 1)}{2} \) for odd \( [k] \). If \( k \in \mathbb{Z}^+ \), then the operator \( H \) has finite rank \( k + 1 \). In this case it follows from formula (1.21) (see [23], for details) that \( N_{+}(H) = N_{-}(H) + 1 = k/2 + 1 \) if \( k \) is even and \( N_{\pm}(H) = (k + 1)/2 \) if \( k \) is odd.

We emphasize that, for example, for \( \alpha > 0 \), \( k > -1 \) and arbitrary \( r \geq 0 \), Hankel operators \( H \) are compact, but \( \Sigma \) are not defined as bounded operators because of the singularity of the function \( \sigma(\lambda) \) at the point \( \lambda = \alpha \).

1.6. Let us briefly describe the structure of the paper. Section 2 plays the central role. Here we give the precise definition of the sigma-function, prove the main identity and discuss its consequences. In Section 2, we work on the space \( C^\infty_0(\mathbb{R}^+) \) of test functions \( f(t) \). Section 3 is specially devoted to bounded Hankel operators. Here we elucidate the relation between symbols and sigma-functions and prove an analogue of the Nehari theorem in terms of sigma-functions. We collect various results relying on the factorization of the Laplace transform in Section 4. In particular, we check here that every Hankel operator \( H \) is unitarily equivalent to a pseudo-differential operator with amplitude (1.12). Then we show that, by a study of the form \( (\Sigma w, w) \), a set of analytic test functions \( w \) can be replaced by the set \( C^\infty_0(\mathbb{R}^+) \). This is technically essentially more convenient. Finally, we carry over here the results of Section 2 to spaces of analytic test functions \( f(t) \). The case of positive Hankel operators when \( \sigma(\lambda) \) is determined by a measure is discussed in Section 5. Hankel operators \( H \) with kernels (1.19) and its various generalizations are studied in Section 6 where we find an explicit formula for the numbers \( N_{\pm}(H) \). Finally, in Section 7 we discuss a translation of our results into the representation of Hankel operators in the space \( l^2(\mathbb{Z}^+) \) of sequences.

Let us introduce some standard notation. We denote by \( \Phi \),
\[
(\Phi u)(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} u(x)e^{-ix\xi}dx,
\]
the Fourier transform and recall that \( \Phi \) is the one-to-one mapping of the Schwartz space \( \mathcal{S} = \mathcal{S}(\mathbb{R}) \) onto itself. Moreover, \( \Phi \) as well as its inverse \( \Phi^{-1} \) are continuous mappings. In such cases we say that a mapping is an isomorphism. The dual class of distributions (continuous antilinear functionals on \( \mathcal{S} \)) is denoted \( \mathcal{S}' \). We use the notation \( \langle \cdot, \cdot \rangle \) and
\langle \cdot, \cdot \rangle$ for the duality symbols in $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R})$, respectively. They are linear in the first argument and antilinear in the second argument.

We often use the same notation for a function and for the operator of multiplication by this function. The letter $C$ (sometimes with indices) denotes various positive constants whose precise values are inessential; $\delta_{n,m}$ is the Kronecker symbol, i.e., $\delta_{n,n} = 1$ and $\delta_{n,m} = 0$ if $n \neq m$.

2. The sigma-function

Here we give the precise definition of the sigma-function $\sigma(\lambda)$ and prove the main identity (1.10).

2.1. We work on test functions $f \in C^\infty_c(\mathbb{R}^+)$ and require that $h$ belong to the dual space $C^\infty_c(\mathbb{R}^+)'$. Let the set $\mathcal{Y}$ consist of entire functions $\varphi(\lambda)$ satisfying, for all $\lambda \in \mathbb{C}$, bounds

$$|\varphi(\lambda)| \leq C_n (1 + |\lambda|)^{-n} e^{r_\pm |\text{Re}\lambda|}, \quad \pm \text{Re} \lambda \geq 0,$$

(2.1)

for all $n$ and some $r_+ = r_+(\varphi) < 0$; the number $r_- = r_-(\varphi)$ may be arbitrary. The space $\mathcal{Y}$ is of course invariant with respect to the complex conjugation $\varphi(\lambda) \mapsto \varphi^*(\lambda) = \varphi(\bar{\lambda})$.

By definition, $\varphi_k(\lambda) \to 0$ as $k \to \infty$ in $\mathcal{Y}$ if all functions $\varphi_k(\lambda)$ satisfy bounds (2.1) with the same constants $r_\pm$, $C_n$ and $\varphi_k(\lambda) \to 0$ as $k \to \infty$ uniformly on all compact subsets of $\mathbb{C}$.

Let the Laplace transform $L$ be defined by formula (1.2). By one of the versions of the Paley-Wiener theorem, $L : C^\infty_c(\mathbb{R}^+) \to \mathcal{Y}$ is the one-to-one continuous mapping of $C^\infty_c(\mathbb{R}^+)$ onto $\mathcal{Y}$ and the inverse mapping $L^{-1} : \mathcal{Y} \to C^\infty_c(\mathbb{R}^+)$ is also continuous. Passing to the dual spaces, we see that the mapping

$$L^* : \mathcal{Y}' \to C^\infty_c(\mathbb{R}^+)'$$

(2.2)

is also an isomorphism. We emphasize that we write $L^*$ here because this operator acts in the spaces of distributions.

Let us construct the sigma-function.

**Definition 2.1.** Assume that

$$h \in C^\infty_c(\mathbb{R}^+).$$

(2.3)

Then the distribution $\sigma \in \mathcal{Y}'$ defined by the formula

$$\sigma = (L^*)^{-1} h$$

(2.4)

is called the sigma-function of the kernel $h$ or of the corresponding Hankel operator $H$.

Since mapping (2.2) is an isomorphism, the kernel $h(t)$ can be recovered from its sigma-function $\sigma(\lambda)$ by the formula $h = L' \sigma$ which gives the precise sense to formal relation (1.4). Thus there is the one-to-one correspondence between kernels $h \in C^\infty_c(\mathbb{R}^+)'$ and their sigma-functions $\sigma \in \mathcal{Y}'$.

2.2. Now we are in a position to check the identity (1.10). The first assertion is quite straightforward. It is a direct consequence of Definition 2.1.
Proposition 2.2. Let assumption (2.3) hold, and let \( \sigma \) be the corresponding sigma-function. Then the identity

\[
\langle h, F \rangle = \langle L^* \sigma, F \rangle = \langle \sigma, LF \rangle
\]

(2.5)

holds for arbitrary \( F \in C_0^\infty(\mathbb{R}_+) \).

Let us introduce the Laplace convolution

\[
(f_1 \ast f_2)(t) = \int_0^t f_1(s)f_2(t-s)ds
\]

(2.6)

of functions \( f_1, f_2 \in C_0^\infty(\mathbb{R}_+) \). Then it formally follows from (1.1) that

\[
(Hf_1, f_2) = \langle h, f_1 \ast f_2 \rangle
\]

(2.7)

where we write \( \langle \cdot, \cdot \rangle \) instead of \( (\cdot, \cdot) \) because \( h \) may be a distribution. Obviously, for arbitrary \( f_1, f_2 \in C_0^\infty(\mathbb{R}_+) \) we have

\[
L(f_1 \ast f_2) = Lf_1Lf_2 = (Lf_1)^*Lf_2 \in \mathcal{Y}.
\]

(2.8)

Now we are in a position to precisely state our main identity.

Theorem 2.3. Let assumption (2.3) be satisfied, and let \( \sigma \in \mathcal{Y}' \) be defined by formula (2.4). Then the identity

\[
\langle h, f_1 \ast f_2 \rangle = \langle \sigma, (Lf_1)^*Lf_2 \rangle
\]

(2.9)

holds for arbitrary \( f_1, f_2 \in C_0^\infty(\mathbb{R}_+) \).

**Proof.** It suffices to apply identity (2.5) to \( F = f_1 \ast f_2 \) and to use relation (2.8). \( \Box \)

The identity (2.9) attributes a precise meaning to (1.3) or (1.10).

2.3. Suppose now that \( h(t) = \overline{h(t)} \) for all \( t > 0 \), or to be more precise \( \langle h, F \rangle = \langle h, \overline{F} \rangle \) for all \( F \in C_0^\infty(\mathbb{R}_+) \). Then it follows from (2.5) that the sigma-function is also real, that is, \( \langle \sigma, w \rangle = \langle \sigma, w^* \rangle \) for all \( w \in \mathcal{Y} \).

Below we use the following natural definition.

**Definition 2.4.** Let \( h[\varphi, \overline{\varphi}] \) be a real quadratic form defined on a linear set \( D \). We denote by \( N_\pm(h) = N_\pm(h; D) \) the maximal dimension of linear sets \( M_\pm \subset D \) such that \( \pm h[\varphi, \overline{\varphi}] > 0 \) for all \( \varphi \in M_\pm, \varphi \neq 0 \).

Definition (2.4) means that there exists a linear set \( M_\pm \subset D \), \( \dim M_\pm = N_\pm(h; D) \), such that \( \pm h[\varphi, \overline{\varphi}] > 0 \) for all \( \varphi \in M_\pm, \varphi \neq 0 \), and for every linear set \( M'_\pm \subset D \) with \( \dim M'_\pm > N_\pm(h; D) \) there exists \( \varphi \in M'_\pm, \varphi \neq 0 \) such that \( \pm h[\varphi, \overline{\varphi}] \leq 0 \).

Of course, if the set \( D \) is dense in a Hilbert space \( \mathcal{H} \) and \( h[\varphi, \overline{\varphi}] \) is semibounded and closed on \( D \), then for the self-adjoint operator \( H \) corresponding to \( h \), we have \( N_\pm(H) = N_\pm(h; D) \). In particular, this is true for bounded operators \( H \).

We apply Definition (2.4) to the forms \( h[f, f] = \langle h, f \ast f \rangle \) on \( f \in C_0^\infty(\mathbb{R}_+) \) and \( \sigma[w, w] = \langle \sigma, w^*w \rangle \) on \( w \in \mathcal{Y} \).

Since \( L : C_0^\infty(\mathbb{R}_+) \to \mathcal{Y} \) is an isomorphism, the following assertion is a direct consequence of Theorem (2.3)
Theorem 2.5. Let $h \in C_0^\infty(\mathbb{R}_+)'$. Then $\sigma = (L^*)^{-1}h \in \mathcal{Y}'$ and

$$N_\pm(h; C_0^\infty(\mathbb{R}_+)) = N_\pm(\sigma; \mathcal{Y}).$$

(2.10)

In particular, the form $\pm \langle h, \bar{f} \ast f \rangle \geq 0$ for all $f \in C_0^\infty(\mathbb{R}_+)$ if and only if the form $\pm \langle \sigma, w^*w \rangle \geq 0$ for all $w \in \mathcal{Y}$.

Thus a Hankel operator $H$ is positive (or negative) if and only if its sigma-function $\sigma(\lambda)$ is positive (or negative).

2.4. In our examples $h(t)$ is a continuous function of $t > 0$. However its behavior as $t \to \infty$ and $t \to 0$ may be arbitrary.

Example 2.6. Let $h(t) = e^{t^2}$. Then representation (1.4) is satisfied with the function $\sigma(\lambda) = 2^{-1/2} \pi^{-1/4} e^{-\lambda^2/4}$. Thus $\langle h, \bar{f} \ast f \rangle \geq 0$ for all $f \in C_0^\infty(\mathbb{R}_+)$. This result can be compared with the fact that the compact Hankel operator $H$ with kernel $h(t) = e^{-t^2}$ has infinite number of both positive and negative eigenvalues (see Proposition B.1 in [22]).

In the case considered, $\text{supp } \sigma = \mathbb{R}$ which is by no means true in the general case. For example, if $h(t) = e^{-\alpha t}$ for some $\alpha \in \mathbb{C}$, then $\langle \sigma, w \rangle = \overline{w(\alpha)}$. Let us mention a particularly simple special case when the relation $h = L^* \sigma$ can be understood in the classical sense.

Proposition 2.7. Let $h \in C_0^\infty(\mathbb{R}_+)'$. With respect to the corresponding sigma-function, assume that

$$\text{supp } \sigma \subset [0, \infty),$$

(2.11)

$$\sigma \in L^1(0, R) \text{ for all } R < \infty \text{ and } \sigma(\lambda) = O(e^{\varepsilon\lambda}) \text{ as } \lambda \to \infty \text{ for all } \varepsilon > 0.$$

Then

$$h(t) = \int_0^\infty e^{-t\lambda} \sigma(\lambda) d\lambda.$$

In particular, the function $h(t)$ is analytic in the right-half plane.

3. Bounded Hankel operators

The main identity (2.9) directly yields a criterion for a Hankel operator to be bounded which provides a new approach to the classical Nehari theorem.

3.1. Let $\mathbb{H}_p^r$, $p \geq 1$, be the Hardy space of functions analytic in the right half-plane. Obviously, $w \in \mathbb{H}_p^r$ if and only if its complex conjugate $w^* \in \mathbb{H}_p^r$. By the Paley-Wiener theorem, the operators

$$(2\pi)^{-1/2} L : L^2(\mathbb{R}_+) \to \mathbb{H}_p^r$$

are unitary. Moreover, $\|Lf\|_{L^2(\mathbb{R}_+)} \leq \sqrt{\pi} \|f\|_{L^2(\mathbb{R}_+)}$ (see, e.g., formulas (1.7) and (4.10) below). Putting $w = Lf$, we see that

$$\sqrt{2}\|w\|_{L^2(\mathbb{R}_+)} \leq \|w\|_{\mathbb{H}_p^r}.$$
Let us state a criterion for boundedness of a Hankel operator \( H \) in terms of its sigma-function. We proceed from the definition of \( H \) by its quadratic form (2.7) where \( f_j \in C_0^\infty(\mathbb{R}_+), j = 1, 2, \) and \( h \in C_0^\infty(\mathbb{R}_+)' \). Recall that, by Definition 2.1, in this case its sigma-function \( \sigma \in \mathcal{Y}' \). Of course \( \mathcal{Y} \subset \mathbb{H}_1^1 \) so that the dual space \((\mathbb{H}_1^1)'\subset \mathcal{Y}'\).

**Theorem 3.1.** A Hankel operator \( H \) is bounded in the space \( L^2(\mathbb{R}_+) \) if and only if its sigma-function \( \sigma \in (\mathbb{H}_1^1)' \).

**Proof.** According to the identity (2.9) \( H \) is bounded if and only if
\[
|\langle \sigma, (Lf_1)^*Lf_2 \rangle| \leq C \|f_1\|_{L^2(\mathbb{R}_+)} \|f_2\|_{L^2(\mathbb{R}_+)}
\]
for all \( f_j \in C_0^\infty(\mathbb{R}_+) \) or, equivalently, all \( f_j \in L^2(\mathbb{R}_+) \). Putting \( w_j = Lf_j \), we rewrite this estimate as
\[
|\langle \sigma, w_1^*w_2 \rangle| \leq C \|w_1\|_{\mathbb{H}_2^1} \|w_2\|_{\mathbb{H}_2^1}, \quad \forall w_j \in \mathbb{H}_2^1.
\]
(3.1)
It is obviously satisfied if \( \sigma \in (\mathbb{H}_1^1)' \).

Conversely, in view of the inner-outer factorization (see, e.g., [14]), every \( g \in \mathbb{H}_1^1 \) admits the representation \( g = w_1^*w_2 \) where \( w_1, w_2 \in \mathbb{H}_2^1 \) and
\[
\|g\|_{\mathbb{H}_1^1} = \|w_1\|_{\mathbb{H}_2^1} \|w_2\|_{\mathbb{H}_2^1}.
\]
Therefore according to (3.1) we have
\[
|\langle \sigma, g \rangle| \leq C \|g\|_{\mathbb{H}_1^1}, \quad \forall g \in \mathbb{H}_1^1,
\]
whence \( \sigma \in (\mathbb{H}_1^1)' \).

\( \square \)

3.2. Theorem 3.1 can equivalently be reformulated in terms of symbols of Hankel operators. This requires the Fefferman duality result (see the original paper [5] or Theorem 4.4 in Chapter VI of the book [6]). We denote by \( \mathbb{B}_r \) the class of analytic in the right half-plane functions which have a bounded mean oscillation on the imaginary axis. We omit standard explanations of the precise meaning of the integral in the right-hand side of (3.2).

**Theorem 3.2** (Fefferman). A functional \( \sigma \in (\mathbb{H}_1^1)' \) if and only if there exists a function \( \omega \in \mathbb{B}_r \) such that
\[
\langle \sigma, g \rangle = -i \int_{-i\infty}^{i\infty} \omega(\mu) \overline{g(\mu)} d\mu
\]
(3.2)
for all \( g \in \mathbb{H}_1^1 \).

Now it is easy to deduce the classical Nehari-Fefferman result from Theorem 3.1. We recall that the symbol \( \omega \) of a Hankel operator \( H \) is defined by formula (1.18) so that
\[
\langle h, \tilde{f}_1 \ast f_2 \rangle = -i \int_{-i\infty}^{i\infty} \omega(\mu) \tilde{f}_1(-\mu) \overline{f_2(\mu)} d\mu, \quad \tilde{f}_j = Lf_j.
\]
(3.3)

**Theorem 3.3.** A Hankel operator \( H \) is bounded in the space \( L^2(\mathbb{R}_+) \) if and only if its symbol \( \omega \in \mathbb{B}_r \).
Proof. If $H$ is bounded, then, by Theorem 3.1 its sigma-function $\sigma \in (H^1_r)'$. By Theorem 3.2 relation (3.2) is satisfied with some $\omega \in B_r$. Now using the main identity (2.9) and applying relation (3.2) to the function $g = (Lf_1)^*Lf_2$, we get (3.3).

Conversely, let $\omega \in B_r$. Then according to Theorem 3.2 it follows from (3.3) that

$$|\langle h, \tilde{f}_1 \ast f_2 \rangle| \leq C \|\tilde{f}_1\|_H^r \|\tilde{f}_2\|_H^r = 2\pi C \|f_1\|_{L^2(R_+)} \|f_2\|_{L^2(R_+)}.$$  

Thus $H$ is bounded.

We emphasize that in contrast to the original proof of the Nehari theorem (see his paper [13] or the book [15]), the proof of Theorem 3.1 does not require either the Hahn-Banach or M. Riesz theorems. Only the inner-outer factorization has been used.

3.3. Let us illustrate the link between $\sigma$ and $\omega$ at the example of the Carleman operator with kernel $h(t) = t^{-1}$. According to (1.4) we have $\sigma(\lambda) = 1$ for $\lambda \in R_+$ and $\sigma(\lambda) = 0$ for $\lambda \notin R_+$.

Let us show that $\omega(\mu) = -\ln \mu$. According to formula (1.17) we only have to check that

$$2\pi i \int_0^\infty g(\lambda)d\lambda = -\int_{-i\infty}^{i\infty} \ln \mu \overline{g(\mu)}d\mu$$  

for $g \in H^1_r$. It suffices to consider the functions $g(\mu) = (\mu + a)^{-n}$ for $n \in Z_+, n \geq 2$, $a > 0$. The right-hand side of (3.4) equals

$$-\int_{-i\infty}^{i\infty} \ln \mu (\mu + a)^{-n}d\mu = 2\pi i(-1)^n \text{Res}_{\mu=a} \left( \ln \mu (\mu - a)^{-n} \right) = 2\pi i(n-1)^{-1}a^{-n+1}$$  

which obviously coincides with the left-hand side of (3.4).

Alternatively, for the calculation of $w(\mu)$, we can proceed from Theorem 8.8 of Chapter 1 in the book [15]. To that end, we first have to extend the distribution $h(t) = t^{-1}$ from $C_0^\infty(R_+)$ onto the Schwartz space $S(R_+)$. This is done by the formula

$$\langle h, \varphi \rangle = \int_0^1 \frac{\varphi(t) - \varphi(0)}{t}dt + \int_1^\infty \frac{\varphi(t)}{t}dt.$$  

Therefore according to formula (1.15) we have

$$\omega(\mu) = \int_0^1 \frac{e^{-\mu t} - 1}{t}dt + \int_1^\infty \frac{e^{-\mu t}}{t}dt = -\ln \mu + \Gamma'(1).$$  

The constant term here can be of course neglected.

4. A factorization of the Laplace transform

In this section we collect various results which rely on a factorization of the Laplace transform.
4.1. For a factorization of the Laplace transform $L$, it is natural to consider more general integral operators 

$$(Af)(t) = \int_0^\infty a(ts)f(s)ds$$

(4.1)

with kernels $a$ depending on the product of the variables only. Such operators can be standardly diagonalized (see, e.g., [20]) by the Mellin transform $M$. Let a unitary operator $U : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R})$ be defined by the formula

$$(Uf)(x) = e^{x/2}f(e^x).$$

(4.2)

Then $M = \Phi U$.

We suppose that the function $a(t)t^{-1/2}$ belongs to $L^1(\mathbb{R}_+)$ (in this case the operator $A$ is bounded in the space $L^2(\mathbb{R}_+)$). Making in (4.1) the change of variables $t = e^x$, $s = e^y$, we see that

$$(UAf)(x) = \int_{-\infty}^{\infty} (Ua)(x+y)(Uf)(y)dy, \quad f \in L^2(\mathbb{R}_+).$$

Passing here to the Fourier transforms, we find that

$$(MAf)(\xi) = \sqrt{2\pi}(Ma)(\xi)(Mf)(-\xi), \quad M = \Phi A.$$  

(4.3)

Let $J$, $(Ju)(\xi) = u(-\xi)$, be the reflection operator and

$$a(\xi) = \sqrt{2\pi}(Ma)(-\xi) = \int_0^\infty a(t)t^{-1/2+i\xi}dt.$$  

(4.4)

It follows from (4.3) that

$$Af = M^{-1}J aMf.$$  

(4.5)

Thus we obtain the following assertion.

**Theorem 4.1.** Let the operator $A$ be defined by formula (4.1) where the function $a(t)t^{-1/2}$ belongs to $L^1(\mathbb{R}_+)$, and let $a(\xi)$ be defined by formula (4.4). Then for all $f \in L^2(\mathbb{R}_+)$ representation (4.5) holds.

Let us apply this result to the case $a(t) = e^{-t}$ when $A = L$ and

$$a(\xi) = \int_0^\infty e^{-t}t^{-1/2+i\xi}dt = \Gamma(1/2 + i\xi)$$

is the gamma function. Recall that the gamma function $\Gamma(z) \neq 0$ for all $z \in \mathbb{C}$. According to the Stirling formula, the function

$$\Gamma(\gamma + i\xi) = e^{\pi i(2\gamma - 1)/4}(2\pi/e)^{1/2}\xi^{\gamma - 1/2}e^{i\xi(\ln \xi - 1)}e^{-\pi \xi/2}(1 + O(\xi^{-1}))$$

(4.6)

tends exponentially to zero as $\xi \rightarrow +\infty$ if $\gamma > 0$ is fixed. Since $\Gamma(\gamma - i\xi) = \overline{\Gamma(\gamma + i\xi)}$, the same is true as $\xi \rightarrow -\infty$. We put

$$(\Gamma \gamma u)(\xi) = \Gamma(\gamma + i\xi)u(\xi).$$

Theorem 4.1 implies the following statement.
Corollary 4.2. For all \( f \in L^2(\mathbb{R}_+) \), the identity
\[
(Lf)(\lambda) = (M^{-1}J_{1/2}Mf)(\lambda), \quad \lambda > 0,
\]
holds.

This formula can be used for the inversion of the Laplace transform:
\[
L^{-1} = M^{-1}J_{1/2}^{-1}JM.
\]

Observe that according to (4.6) the function \( \Gamma(1/2 + i\xi)^{-1} \) exponentially grows as \( |\xi| \to \infty \). This is why, even for very nice kernels \( h(t) \) (for example, for \( h(t) = t^k e^{-\alpha t}, \ \text{Re} \alpha > 0, \ k = 0, 1, \ldots \)), the corresponding sigma-function \( \sigma(\lambda) \) defined by formula (1.11) may be a highly singular distribution.

4.2. Factorization (4.7) allows us to reformulate the main identity (2.9) in a somewhat different form. We suppose for simplicity that conditions (2.11) and (4.8) are satisfied. Then the operators \( \Sigma \) and hence \( H \) are bounded in the space \( L^2(\mathbb{R}_+) \).

Let the function \( s(x) \) be defined for \( x \in \mathbb{R} \) by formula (1.13), and let \( S \) be the operator of multiplication by \( s(x) \) in the space \( L^2(\mathbb{R}) \). Since \( M = \Phi U \), we have
\[
JM\Sigma M^{-1}J = \Phi S\Phi^*.
\]

Let us further observe that
\[
|\Gamma(1/2 + i\xi)| = \frac{\sqrt{\pi}}{\cosh(\pi\xi)} =: v(\xi)
\]
and denote by \( V \) the operator of multiplication by this function in the space \( L^2(\mathbb{R}) \). Set also
\[
(Mf)(\xi) = e^{i\arg \Gamma(1/2 + i\xi)}(Mf)(\xi).
\]

Putting together the identities (1.3), (4.7) and (4.9), we obtain the following result.

Theorem 4.3. Under assumptions (2.11) and (4.8) define \( s(x) \) by formula (1.13) and set
\[
A = V\Phi S\Phi^*V.
\]

Then
\[
H = M^*AM.
\]

Note that according to formula (1.4) under the assumptions of this theorem the kernel \( h(t) \) of \( H \) satisfies the bounds
\[
|h^{(n)}(t)| \leq C_n t^{-1-n}, \quad \forall n \in \mathbb{Z}_+.
\]

Obviously, \( A \) is a pseudo-differential operator in the space \( L^2(\mathbb{R}) \) with amplitude (1.12) which factorizes into a product of functions of one variable only. Assumption (4.8) is by no means necessary. For example, if \( h(t) = P(\ln t)t^{-1} \) where \( P \) is a polynomial, then \( s(x) \) is also a polynomial (see [24], for details). In this case relation (4.11)
holds with a differential operator $A$. Thus Theorem 4.3 shows that, under very general assumptions, Hankel operators are unitarily equivalent to pseudo-differential operators of a very special structure.

4.3. Let us come back to Theorem 2.5. It is usually not convenient to work with analytic test functions. Fortunately under mild additional assumptions on the sigma-function, the set $\mathcal{Y}$ of test functions $w$ in (2.10) can be replaced by the set $C_0^\infty(\mathbb{R}_+)$. Below we sometimes do not distinguish functions $g \in \mathcal{Y}$ and their restrictions on $\mathbb{R}_+$.

Let us introduce the space $S_\gamma$, $\gamma \in \mathbb{R}$, of functions $g \in C_0^\infty(\mathbb{R}_+)$ satisfying estimates

$$|g^{(k)}(\lambda)| \leq C_{\gamma,k} \lambda^{\gamma-1/2-k}(1 + |\ln \lambda|)^{-\kappa}$$

for all $k = 0, 1, 2, \ldots$ and all $\kappa \in \mathbb{R}$. The case $\gamma = 0$ is the most important for us. It is easy to see that $g \in S_0$ if and only if the function $Ug$ belongs to the Schwartz space $S$.

We need the following analytical result.

**Lemma 4.4.** The set $\mathcal{Y}$ is dense in $S_0$.

**Proof.** The result formulated is equivalent to the fact that the set of elements $ULf$ where $f \in C_0^\infty(\mathbb{R}_+)$ is dense in the space $S$. In view of the identity (4.7), it is equivalent to the following assertions: the set of elements $\Phi^{-1}J_{1/2}Mf$ where $f \in C_0^\infty(\mathbb{R}_+)$ is dense in the space $S$ or the set of elements $\Gamma_{1/2}\Phi\psi$ where $\psi = Uf \in C_0(\mathbb{R})$ is dense in the space $S$.

Thus, for an arbitrary $u \in S$, we have to construct a sequence $\psi_k \in C_0^\infty(\mathbb{R})$ such that

$$\Gamma_{1/2}\Phi\psi_k \to u$$

in $S$ as $k \to \infty$. Let $\theta \in C_0^\infty(\mathbb{R})$ and $\theta(\xi) = 1$ in a neighborhood of the point $\xi = 0$. Then $\theta_n(\xi) = \theta(\xi/n)$. Then

$$\theta_n u \to u$$

in $S$ as $n \to \infty$. Next, we put

$$v_n(\xi) = \Gamma(1/2 + i\xi)^{-1}\theta_n(\xi)u(\xi).$$

Obviously, $v_n \in C_0^\infty(\mathbb{R}) \subset S$ and hence, for every $n$, there exists a sequence $\psi_{n,m} \in C_0^\infty(\mathbb{R})$ such that $\Phi\psi_{n,m} \to v_n$ in $S$. It follows that

$$\Gamma_{1/2}\Phi\psi_{n,m} \to \Gamma_{1/2}v_n = \theta_n u$$

in $S$ as $m \to \infty$. Putting together relations (4.14) and (4.15), we can choose a subsequence $\psi_k$ of $\psi_{n,m}$ such that relation (4.13) is true. □

Lemma 4.4 allows us to prove the following assertion.

**Lemma 4.5.** Under assumption (2.11) suppose that $\sigma \in S'_0$. Then

$$N_{\pm}(\sigma; \mathcal{Y}) = N_{\pm}(\sigma; S_0) = N_{\pm}(\sigma; C_0^\infty(\mathbb{R}_+)).$$
Proof. Let us check the first equality (4.16). The inequality $N_+ (\sigma ; \mathcal{Y}) \leq N_+ (\sigma ; \mathcal{S}_0)$ is obvious because $\mathcal{Y} \subset \mathcal{S}_0$.

Let us prove the opposite inequality. Consider for definiteness the sign “+”. Let $\mathcal{L} \subset \mathcal{S}_0$, and let $\sigma [w, w] > 0$ for all $w \in \mathcal{L}$, $w \neq 0$. Suppose first that $N := \dim \mathcal{L} < \infty$ and choose elements $w_1, \ldots, w_N \in \mathcal{L}$ such that $\sigma [w_j, w_k] = \delta_{j,k}$ for all $j, k = 1, \ldots, N$.

Using Lemma 4.4 we can construct elements $w^{(e)}_j \in \mathcal{Y}$ such that $w^{(e)}_j \to w_j$ and hence $w^{(e)}_j \bar{w}^{(e)}_k \to w_j \bar{w}_k$ in $\mathcal{S}_0$ as $\epsilon \to 0$ for all $j, k = 1, \ldots, N$. Since $\sigma \in \mathcal{S}_0^0$, we see that $\sigma [w^{(e)}_j, w^{(e)}_k] \to \delta_{j,k}$ as $\epsilon \to 0$. For an arbitrary $\gamma > 0$, we can choose $\epsilon$ such that $|\sigma [w^{(e)}_j, w^{(e)}_k] - \delta_{j,k}| \leq \gamma$. Then for arbitrary $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, we have

$$\sigma \left[ \sum_{j=1}^{N} \lambda_j w^{(e)}_j, \sum_{j=1}^{N} \lambda_j w^{(e)}_j \right] = \sum_{j=1}^{N} |\lambda_j|^2 \sigma [w^{(e)}_j, w^{(e)}_j] + 2 \Re \sum_{j,k=1; j \neq k}^{N} \lambda_j \bar{\lambda}_k \sigma [w^{(e)}_j, w^{(e)}_k]$$

$$\geq (1 - \gamma) \sum_{j=1}^{N} |\lambda_j|^2 - 2 \gamma \sum_{j,k=1; j \neq k}^{N} \lambda_j \bar{\lambda}_k \geq (1 - (2N - 1) \gamma) \sum_{j=1}^{N} |\lambda_j|^2. \tag{4.17}$$

Thus elements $w^{(e)}_1, \ldots, w^{(e)}_N$ are linearly independent if $(2N - 1) \gamma < 1$. The same inequality (4.17) shows that $\sigma [w, w] > 0$ on all vectors $w \neq 0$ belonging to the space $\mathcal{L}^{(e)}$ spanned by $w^{(e)}_1, \ldots, w^{(e)}_N$.

If $N = \infty$, then the construction above works on every finite dimensional subspace of $\mathcal{L}$ where $\sigma [w, w] > 0$. This yields the space $\mathcal{L}^{(e)} \subset \mathcal{Z}$ of an arbitrary large dimension where $\sigma [w, w] > 0$.

The second equality (4.16) can be proven quite similarly because $C^\infty_0 (\mathbb{R}_+) \subset \mathcal{S}_0$, and it is dense in $\mathcal{S}_0$.

Putting together Theorem 2.5 and Lemma 4.5 we arrive at the following result.

Theorem 4.6. Let $h \in C^\infty_0 (\mathbb{R}_+)$’. Suppose that condition (2.11) is satisfied and that $\sigma \in \mathcal{S}_0$. Then

$$N_{\pm} (h; C^\infty_0 (\mathbb{R}_+)) = N_{\pm} (\sigma; C^\infty_0 (\mathbb{R}_+)). \tag{4.18}$$

The following consequence of Theorem 4.6 is very convenient for applications to concrete Hankel operators.

Theorem 4.7. Let the assumptions of Theorem 4.6 hold.

1. If $\pm \sigma \geq 0$, then $\pm \langle h, \check{f} * f \rangle \geq 0$ for all $f \in C^\infty_0 (\mathbb{R}_+)$.

2. If $\sigma \in L^1_{\text{loc}}(\Delta)$ for some interval $\Delta \subset \mathbb{R}$ and $\pm \sigma (\lambda) \geq \sigma_0 > 0$ for almost all $\lambda \in \Delta$, then $N_{\pm} (h; C^\infty_0 (\mathbb{R}_+)) = \infty$.

Proof. The first assertion is a direct consequence of relation (4.18).

For the proof of the second assertion, choose some number $N$ and a function $\varphi \in C^\infty_0 (\mathbb{R}_+)$ such that $\varphi (\lambda) = 1$ for $\lambda \in [-\delta, \delta]$ and $\varphi (\lambda) = 0$ for $\lambda \notin [-2\delta, 2\delta]$ where $\delta = \delta_N$ is a sufficiently small number. Let points $\alpha_j \in \Delta$, $j = 1, \ldots, N$, be such
that $\alpha_{j+1} - \alpha_j = \alpha_j - \alpha_{j-1}$ for $j = 2, \ldots, N - 1$. Set $\Delta_j = (\alpha_j - \delta, \alpha_j + \delta)$, $\tilde{\Delta}_j = (\alpha_j - 2\delta, \alpha_j + 2\delta)$. For a sufficiently small $\delta$, we may suppose that $\tilde{\Delta}_j \subset \Delta$ for all $j = 1, \ldots, N$ and that $\tilde{\Delta}_j \cap \tilde{\Delta}_{j+1} = \emptyset$ for $j = 1, \ldots, N - 1$. We set $\varphi_j(\lambda) = \varphi(\lambda - \alpha_j)$. Since $\pm \sigma(\lambda) \geq \sigma_0 > 0$ for $\lambda \in \Delta$, we have

$$\pm \sigma[\varphi_j, \varphi_j] = \pm \int_0^\infty \sigma(\lambda) |\varphi_j(\lambda)|^2 d\lambda \geq 2\delta \sigma_0 > 0.$$  

The functions $\varphi_1, \ldots, \varphi_N$ have disjoint supports and hence $\pm \sigma[w, w] > 0$ for an arbitrary nontrivial linear combination $w$ of the functions $\varphi_j$. Therefore $N_\pm(\sigma; C_0^\infty(\mathbb{R}^+)) \geq N$. Since $N$ is arbitrary, it remains to use relation (4.18).

4.4. Here we define a scale of spaces of analytic functions where the Laplace transform acts as an isomorphism. This extends the one-to-one correspondence of Section 2 between kernels of Hankel operators and their sigma-functions to new spaces of distributions.

Let the space $Z = Z(\mathbb{R})$ of test functions be defined as the subset of the Schwartz space $S = S(\mathbb{R})$ which consists of functions $\varphi(x)$ admitting the analytic continuation to entire functions in the complex plane $\mathbb{C}$ and satisfying, for all $z \in \mathbb{C}$, bounds

$$|\varphi(z)| \leq C_\varphi(1 + |z|)^{-\kappa} e^{r|\text{Im} z|}$$  

(4.19)

for some $r = r(\varphi) > 0$ and all $\kappa$. The space $Z$ is of course invariant with respect to the complex conjugation, that is, $\varphi^*(z) = \overline{\varphi(\bar{z})}$ belongs to $Z$ together with $\varphi$. By definition, $\varphi_k(z) \to 0$ as $k \to \infty$ in $Z$ if all functions $\varphi_k(z)$ satisfy bounds (4.19) with the same constants $r, C_\varphi$ and $\varphi_k(z) \to 0$ as $k \to \infty$ uniformly on all compact subsets of $\mathbb{C}$. Recall (see, e.g., the book [7]) that the Fourier transform $\Phi$ is a one-to-one mapping of $Z$ onto $C_0^\infty(\mathbb{R})$. Moreover, $\Phi$ as well as its inverse $\Phi^{-1}$ are continuous mappings so that $\Phi : Z \to C_0^\infty(\mathbb{R})$ is an isomorphism.

Let $U$ be operator (1.2). We define the set $Z_0$ of test functions $f(t)$ by the condition

$$f \in Z_0 \iff U f \in Z.$$  

The set $Z_0 \subset L^2(\mathbb{R}^+)$, and it is dense in $L^2(\mathbb{R}^+)$ because $Z$ is dense in $L^2(\mathbb{R})$. Define also the set $Z_\gamma$ for an arbitrary $\gamma \in \mathbb{R}$ by the condition $f \in Z_\gamma$ if and only if the function $t^{-\gamma} f(t)$ belongs to $Z_0$. Thus $f \in Z_\gamma$ if and only if the function $F(x) = e^{(1/2-\gamma)x} f(e^x)$ belongs to $Z$, that is,

$$f(t) = t^{-\gamma - 1/2} F(\ln t)$$  

where $F \in Z$. Functions $f(t)$ admit analytic continuation $f(\zeta)$ onto the Riemann surface of the logarithmic function, and they satisfy the bounds

$$|f(\zeta)| \leq C_\varphi |\zeta|^{-\gamma - 1/2} (1 + |\ln |\zeta||)^{-\kappa} e^{r|\text{arg } \zeta|}$$  

(4.20)

with some constant $r = r(f) > 0$ for all $\kappa \in \mathbb{R}$. Note that $Z_\gamma \subset S_\gamma$ where the set $S_\gamma$ is defined by conditions (4.12). The sets $Z_\gamma$ are invariant with respect to the complex conjugation because $Z$ is. The topology on $Z_\gamma$ is of course induced by that on $Z$. 

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Clearly, for all $\gamma, \beta \in \mathbb{R}$, a function $f \in Z_\gamma$ if and only if the function $t^\beta f(t)$ belongs to $Z_{\gamma+\beta}$. Note that there is no ordering between different spaces $Z_\gamma$. If $\gamma_2 > \gamma_1$, then functions $f \in Z_{\gamma_2}$ are better than those in $Z_{\gamma_1}$ as $t \to 0$ but worse as $t \to \infty$. Of course neither of the inclusions $Z_\gamma \subset C_0^\infty(\mathbb{R}_+)$ nor $C_0^\infty(\mathbb{R}_+) \subset Z_\gamma$ (for any $\gamma$) is true.

Since the product of two functions in $Z$ also belongs to this space, the statement below is a direct consequence of the definition of $Z_\gamma$.

**Lemma 4.8.** If $f \in Z_\gamma$ and $g \in Z_\beta$ for some $\gamma, \beta \in \mathbb{R}$, then $fg \in Z_{\gamma+\beta-1/2}$.

Applying now Theorem 4.1 to the kernel $a(t) = e^{-t\gamma-1/2}$, we obtain the following generalization of Corollary 4.2.

**Lemma 4.9.** Let $\Omega$ be the operator of multiplication by $t$ or $\lambda$. Then, for all $\gamma > 0$, the representation

$$L = \Omega^{1/2-\gamma} M^{-1} JT_\gamma M \Omega^{1/2-\gamma}$$

holds.

Observe now that $U : Z_0 \to Z$, $\Phi : Z \to C^\infty_0(\mathbb{R})$ and hence $M : Z_0 \to C^\infty_0(\mathbb{R})$ are isomorphisms. Moreover, $JT_\gamma : C^\infty_0(\mathbb{R}) \to C^\infty_0(\mathbb{R})$ is an isomorphism because $\Gamma(\gamma + i\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Therefore Lemma 4.9 yields

**Corollary 4.10.** For all $\gamma > 0$, the mapping

$$L : Z_{\gamma-1/2} \to Z_{1/2-\gamma}$$

(4.21)

is an isomorphism.

Note that functions $f(t)$ in $Z_{1/2-\gamma}$ are better (worse) than those in $Z_{\gamma-1/2}$ at infinity (at zero) if $\gamma \leq 1/2$. It is the opposite if $\gamma \geq 1/2$. For $\gamma = 1/2$, the mapping $L : Z_0 \to Z_0$ is an automorphism.

Let $Z'_\gamma$ be the space dual to $Z_\gamma$. Obviously, $h \in Z'_\gamma$ if and only if the function $e^{(\gamma+1/2)x}h(e^x)$ belongs to the space $Z'$ dual to $Z$. Since $Z_{\gamma+\beta} = \Omega^* Z_\beta$, we have $Z'_{\gamma+\beta} = \Omega^{-\gamma} Z'_\beta$ for all $\gamma, \beta \in \mathbb{R}$. Note that for all $\gamma, \beta \in \mathbb{R}$, a distribution $h \in Z'_\gamma$ if and only if the distribution $t^\beta h(t)$ belongs to $Z'_{\gamma+\beta}$.

It follows from (4.21) that the mapping

$$L^* : Z'_{1/2-\gamma} \to Z'_{\gamma-1/2}, \quad \forall \gamma > 0,$$  

(4.22)

is an isomorphism which according to Definition 2.1 yields the following result.

**Proposition 4.11.** For all $\gamma > 0$, there is the one-to-one correspondence between kernels $h \in Z'_{\gamma-1/2}$ and their sigma-functions $\sigma \in Z'_{1/2-\gamma}$, that is,

$$h \in Z'_{\gamma-1/2} \iff \sigma = (L^*)^{-1} h \in Z'_{1/2-\gamma}.$$

It follows from condition (4.12) that $h \in S'_{\gamma-1/2} \subset Z'_{\gamma-1/2}$ if $h \in L^1_{\text{loc}}(\mathbb{R}_+)$ and the integral

$$\int_0^\infty |h(t)| t^{\gamma-1} (1 + |\ln t|)^{-\infty} dt < \infty.$$
converges for some $\varkappa_0 \in \mathbb{R}$. In particular, the estimate
\[
|h(t)| \leq Ct^{-\gamma}(1 + |\ln t|)^\varkappa
\]
for some $\varkappa \in \mathbb{R}$ guarantees that $h \in S'_{\gamma-1/2}$.

The case $\gamma = 1$ when $h \in Z'_{1/2}$ and hence $\sigma \in Z'_{-1/2}$ is most important. It is shown in [22] that for all bounded Hankel operators $H$, their kernels $h \in S'_{1/2} \subset Z'_{1/2}$. The converse is false. For instance, the kernels $h(t) = t^{-1} \ln^k t$ where $k = 0, 1, 2, \ldots$ satisfy the condition $h \in S'_{1/2}$, but the corresponding Hankel operators are unbounded if $k \geq 1$ (see [24]).

Note also that the inclusion $h \in S'_{\gamma-1/2}$ does not imply that $\sigma \in S'_{1/2-\gamma}$. For example, if $h(t) = e^{-\alpha t}$, $\text{Re} \alpha > 0$ (the Hankel operator $H$ with such kernel has rank 1), then $h \in S'_{\gamma-1/2}$ for all $\gamma > 0$, but if $\text{Im} \alpha \neq 0$ the corresponding function $\sigma(\lambda) = \delta(\lambda - \alpha)$ does not belong to $S'_{1/2-\gamma}$ for any $\gamma > 0$.

The proof of the main identity (1.10) in classes of analytic functions is quite similar to that in Section 2. First we note an analogue of Proposition 2.2.

**Proposition 4.12.** Let $h \in Z'_{\gamma-1/2}$ for some $\gamma > 0$, and let $\sigma \in Z'_{1/2-\gamma}$ be the corresponding sigma-function. Then the identity (2.5) holds for arbitrary $F \in Z_{\gamma-1/2}$.

An analogue of relation (2.8) requires a short proof.

**Lemma 4.13.** If $f_1, f_2 \in Z_{(\gamma-1)/2}$ where $\gamma > 0$, then
\[
L(\bar{f}_1 \ast f_2) = (Lf_1)^*Lf_2 \in Z_{1/2-\gamma}.
\]

**Proof.** Since according to (4.20)
\[
|f_j(t)| \leq C_\varkappa t^{-1+\gamma/2}(1 + |\ln t|)^{-\varkappa}, \quad \forall \varkappa,
\]
the integrals $(Lf_j)(\lambda)$, $j = 1, 2$, converge absolutely for all $\lambda > 0$. Therefore using the Fubini theorem and making the change of variables $s + t = \tau$, we find that
\[
(Lf_1)^*(\lambda)(Lf_2)(\lambda) = \int_0^\infty \int_0^\infty e^{-\lambda(t+s)}\bar{f}_1(t)f_2(s)dtds = \int_0^\infty d\tau e^{-\lambda\tau} \int_0^\tau \bar{f}_1(t)f_2(\tau - t)dt
\]
which yields the left-hand side of (4.23). By Corollary 4.10 we have $Lf_j \in Z_{(1-\gamma)/2}$, $j = 1, 2$. Thus the inclusion in (4.23) follows from Lemma 4.8.

The role of Theorem 2.3 is now played by the following result.

**Theorem 4.14.** Let $h \in Z'_{\gamma-1/2}$ for some $\gamma > 0$, and let $\sigma \in Z'_{1/2-\gamma}$ be defined by formula (2.4). Then the identity (2.9) holds for arbitrary $f_1, f_2 \in Z_{(\gamma-1)/2}$.

**Proof.** It suffices to apply identity (2.5) to $F = \bar{f}_1 \ast f_2$ and to use Lemma 4.13.

Thus under the assumption $h \in Z'_{\gamma-1/2}$ where $\gamma > 0$, the results of Section 2 remain true.
5. Positive Hankel operators

As we have seen, the sigma-function $\sigma$ may be a highly singular distribution. However it cannot be too singular for nonnegative Hankel forms.

5.1. An important necessary condition of positivity of a Hankel operator $H$ is imposed by Bernstein’s theorem. Actually, we need its extension to distributions. We consider the problem in a very general setting regarding quadratic forms instead of operators.

**Theorem 5.1.** Let $h \in C^\infty_0(\mathbb{R}_+)^\prime$ and

$$\langle h, \bar{f} \ast f \rangle \geq 0$$

(5.1)

for all $f \in C^\infty_0(\mathbb{R}_+)$. Then there exists a positive measure $M$ on $\mathbb{R}$ such that

$$h(t) = \int_{-\infty}^{\infty} e^{-t\lambda} dM(\lambda)$$

(5.2)

where the integral converges for all $t > 0$.

We emphasize that the measure $dM(\lambda)$ may grow almost exponentially as $\lambda \to +\infty$ and it tends to zero super-exponentially as $\lambda \to -\infty$, that is,

$$\int_{0}^{\infty} e^{-t\lambda} dM(\lambda) < \infty \quad \text{and} \quad \int_{0}^{\infty} e^{t\lambda} dM(-\lambda) < \infty$$

(5.3)

for an arbitrary small $t > 0$ and for an arbitrary large $t > 0$, respectively.

Theorem 5.1 can be viewed as a continuous version of the Hamburger moment problem (see [9] or Theorem 2.1.1 in [1]).

Observe that if the function $h(t)$ is a priori supposed to be continuous, then Theorem 5.1 is exactly the Bernstein theorem on exponentially convex functions (see [2] or Theorem 5.5.4 in [1]). It is also noted in [1] that due to the theorem of Sierpinski [18], the condition $h \in C(\mathbb{R}_+)$ in the Bernstein theorem can be significantly relaxed.

The representation (5.2) is of course a particular case of (1.4). It is much more precise than (1.4) but requires the positivity of $\langle h, \bar{f} \ast f \rangle$. Theorem 5.1 shows that the positivity of $\langle h, \bar{f} \ast f \rangle$ imposes very strong conditions on $h(t)$. In particular, representation (5.2) implies that the distribution $h(t)$ is actually a $C^\infty$ function. It admits the analytic continuation in the half-plane $\text{Re} \, t > 0$ and

$$h(\tau + i\sigma) = \int_{-\infty}^{\infty} e^{-i\sigma\lambda} e^{-\tau\lambda} dM(\lambda), \quad \tau > 0.$$  

This allows us to state the following result.

**Corollary 5.2.** Under the assumptions of Theorem 5.1 the function $h \in C^\infty(\mathbb{R}_+)$. Moreover, it admits the analytic continuation in the right-half plane $\text{Re} \, t > 0$ and is uniformly bounded in every strip $\text{Re} \, t \in (t_1, t_2)$ where $0 < t_1 < t_2 < \infty$.  


Observe that representation (5.2) can equivalently be rewritten as

\[ \langle h, F \rangle = \int_{-\infty}^{\infty} (LF)(\lambda) dM(\lambda) \]  

where the operator \( L \) is defined by equality (1.2) and \( F \in C_0^\infty(\mathbb{R}_+) \) is arbitrary. Similarly, the Hankel quadratic form admits the representation

\[ \langle h, \bar{f} \ast f \rangle = \int_{-\infty}^{\infty} |(Lf)(\lambda)|^2 dM(\lambda), \quad \forall f \in C_0^\infty(\mathbb{R}_+). \]  

Of course these representations are consistent with formulas (2.5) and (2.9).

Our proof of Theorem 5.1 relies on a reduction to the case of continuous functions \( h(t) \). This is similar in spirit to the extension by L. Schwartz to distributions of the Bochner theorem on continuous functions of positive type. To be more precise, we follow closely the scheme of §3, Chapter II of the book [8]. The difference is that now the Laplace transform plays the role of the Fourier transform and the Laplace convolution defined by (2.6) plays the role of the usual convolution. Since the proof of Theorem 5.1 is quite far from the mainstream of the present paper, it will be given in the Appendix.

Note that the assertion converse to Theorem 5.1 is trivially correct: if a function \( h(t) \) admits representation (5.2), then the corresponding Hankel quadratic form is given by relation (5.5), and hence it is positive.

### 5.2. Under assumptions of subs. 4.4 representation (5.2) also holds. In this case one can obtain essentially more detailed information on the measure \( dM(\lambda) \). For the proof of a such result, we combine Theorem 4.14 with the Bochner-Schwarz theorem (see, e.g., Theorem 3 in §3, Chapter II of the book [8]).

It can be stated as follows. Let a distribution \( s \in \mathcal{Z}' \) satisfy the condition

\[ \langle s, u^* u \rangle \geq 0, \quad \forall u \in \mathcal{Z}, \]  

(such \( s \) are sometimes called distributions of positive type). Then there exists a non-negative measure \( dM(x) \) satisfying the condition

\[ \int_{-\infty}^{\infty} (1 + |x|)^{-\kappa} dM(x) < \infty \]  

for some \( \kappa \in \mathbb{R} \) (that is, of at most polynomial growth at infinity) and such that

\[ \langle s, \varphi \rangle = \int_{-\infty}^{\infty} \overline{\varphi(x)} dM(x), \quad \forall \varphi \in \mathcal{Z}. \]  

In particular, the distribution \( s \) can be extended by continuity to the whole Schwartz space \( \mathcal{S}' \).

Our goal is to prove the following result.
Theorem 5.3. Let \( h \in Z'_{\gamma-1/2} \) for some \( \gamma > 0 \) and let condition (5.1) be satisfied for all \( f \in Z_{(\gamma-1)/2} \). Then the representation

\[ h(t) = \int_0^\infty e^{-\lambda t} dM(\lambda), \quad \forall t > 0, \]

holds with a positive measure \( dM(\lambda) \) on \( \mathbb{R}_+ \) satisfying for some \( \nu \in \mathbb{R} \) the condition

\[ \int_0^\infty (1 + |\ln \lambda|)^{-\nu} \lambda^{-\gamma} dM(\lambda) < \infty. \]  

Proof. Put

\[ u(x) = e^{-\gamma x/2}(Lf)(e^{-x}) \]  

and

\[ s(x) = e^{(\gamma-1)x} \sigma(e^{-x}). \]

It follows from Corollary 4.10 that \( Lf \in Z_{(1-\gamma)/2} \) and hence \( u \in Z \). Moreover, since \( L : Z_{(\gamma-1)/2} \to Z_{(1-\gamma)/2} \) is an isomorphism, for every \( u \in Z \), we can find \( f \in Z_{(\gamma-1)/2} \) such that (5.11) holds. According to (4.22) we have \( \sigma = (L^*)^{-1}h \in Z'_{1/2-\gamma} \) and hence \( s \in Z' \). Making the change of variable \( \lambda = e^{-x} \), we see that

\[ \langle \sigma, (L^*f)L^* \rangle = \langle s, u^*u \rangle. \]

Therefore using the main identity (2.9) and assumption (5.1), we obtain condition (5.6) on the distribution \( s(x) \).

The Bochner-Schwartz theorem implies that there exists a positive measure \( dM(x) \) satisfying condition (5.7) and such that representation (5.8) holds. Let us now make in (5.8) the inverse change of variables \( x = -\ln \lambda \) and put \( \varphi(x) = e^{-\gamma x} \psi(e^{-x}) \),

\[ e^{-\gamma x} dM(x) = dM(e^{-x}) \]

The measure \( dM(\lambda) \) satisfies condition (5.10) and

\[ \langle \sigma, \psi \rangle = \langle s, \varphi \rangle = \int_0^\infty \psi(\lambda) dM(\lambda). \]

Since \( \varphi \in Z \) is arbitrary, \( \psi \in Z_{1/2-\gamma} \) is also arbitrary. Now the identity (2.5) with \( F = L^{-1} \psi \) implies the relation

\[ \langle h, F \rangle = \int_0^\infty (LF)(\lambda) dM(\lambda). \]  

(5.12)

Here \( F \in Z_{\gamma-1/2} \) is arbitrary because \( L : Z_{\gamma-1/2} \to Z_{1/2-\gamma} \) is an isomorphism. Relations (5.9) and (5.12) are equivalent. \( \square \)

Remark 5.4. If \( h \in Z'_{\gamma_1-1/2} \cap Z'_{\gamma_2-1/2} \) for some \( 0 < \gamma_1 < \gamma_2 < \infty \), then the representation (5.9) holds with a measure \( dM(\lambda) \) satisfying instead of (5.10) the stronger condition

\[ \int_1^\infty (1 + |\ln \lambda|)^{-\gamma_1} \lambda^{-\gamma} dM(\lambda) + \int_0^1 (1 + |\ln \lambda|)^{-\gamma_2} \lambda^{-\gamma} dM(\lambda) < \infty \]
for some $\kappa_1, \kappa_1 \in \mathbb{R}$.

Under the assumptions of Theorem 5.3 $h(t)$ satisfies the conclusions of Corollary 5.2. Furthermore, we have

**Corollary 5.5.** Under the assumptions of Theorem 5.3 for all $t > 0$ and all $n = 0, 1, 2, \ldots$, inequalities

$$( -1)^n h^{(n)}(t) \geq 0 \tag{5.13}$$

hold (such functions $h(t)$ are called completely monotonic). Moreover, for some $\kappa \in \mathbb{R}$ and $C > 0$ we have the estimate

$$h(t) \leq C t^{-\gamma} (1 + |\ln t|)^{\kappa}, \quad t > 0. \tag{5.14}$$

All these assertions are direct consequences of the representation (5.9). In particular, under condition (5.10) we have

$$h(t) \leq C \max_{\lambda \geq 0} \left( e^{-t \lambda} \lambda^\gamma (1 + |\ln \lambda|)^{\kappa} \right)$$

which yields (5.14).

Under the assumptions of Theorem 5.3 we have the representation

$$\langle h, f \star f \rangle = \int_0^{\infty} \left| (Lf)(\lambda) \right|^2 dM(\lambda), \quad \forall f \in \mathcal{Z}_{(\gamma-1)/2}.$$

In contrast to (5.3) the integral here is taken over the positive half-line only. This is of course due to stronger assumptions on $h(t)$.

Note that according to the Bernstein theorem (see the original paper [2] or Theorems 5.5.1 and 5.5.2 in the book [1]) condition (5.13) implies that the function $h(t)$ admits the representation (5.9) with some measure $dM(\lambda)$. Of course, condition (5.13) does not impose any restrictions on the measure $dM(\lambda)$ (except that the integral (5.9) is convergent for all $t > 0$). In contrast to the Bernstein theorem we deduce the representation (5.9) from the positivity of the Hankel form. In this context condition (5.10) is due to the assumption $h \in \mathcal{Z}_{\gamma-1/2}$.

We also mention that H. Widom considered in [19] Hankel operators $H$ with kernels $h(t)$ admitting the representation (5.9). He showed that $H$ is bounded if and only if $M([0, \lambda)) = O(\lambda)$ as $\lambda \to 0$ and as $\lambda \to \infty$. In this case $h(t) \leq Ct^{-1}$ for some $C > 0$. To a certain extent, Theorem 5.3 and estimate (5.14) can be regarded as an extension of Widom’s results to unbounded operators.

### 6. Quasi-Carleman operators

#### 6.1. Here we consider Hankel operators $H$ (we call them “quasi-Carleman” operators) with kernels (1.19) that belong to the set $C^\infty_0(\mathbb{R}_+, \gamma)$ for all $\alpha \in \mathbb{R}$, $r \geq 0$ and $k \in \mathbb{R}$. To be more precise, we study the corresponding quadratic forms. It can be shown that these forms give rise to self-adjoint operators $H$ in the space $L^2(\mathbb{R}_+)$ if and only if $\alpha > 0$ or $\alpha = 0$, $k > 0$ (in these cases $h(t) \to 0$ as $t \to \infty$). Moreover, if $\alpha > 0$, $r > 0$, then $H$ is compact for all $k$. If $\alpha > 0$, $r = 0$, then it is compact for $k > -1$ and
is bounded for \( k = -1 \). If \( \alpha = 0, r > 0 \), then it is compact for \( k < -1 \) and is bounded for \( k = -1 \). Finally, if \( \alpha = r = 0 \), then \( H \) is bounded if and only if \( k = -1 \). In all these cases we have the equality \( N_\pm(H) = N_\pm(h) \).

There are probably no chances to explicitly find the spectrum and eigenfunctions of quasi-Carleman operators. The only exceptions are the cases \( k = -1, \alpha = 0 \) (if in addition \( r = 0 \), then \( H \) is the Carleman operator) and \( k = -1, r > 0, \alpha > 0 \) considered by F. G. Mehler [12] and W. Magnus [11], respectively (see also §3.14 of the book [4] and the papers [17], [21]).

Our first goal is to prove formula (1.20) for the sigma-functions. We consider all \( k \in \mathbb{C} \) and start with the case \( \text{Re} \, k < 0 \) when distribution (1.20) does not have a strong singularity at the point \( \lambda = \alpha \). Formally, the proof is quite simple. Indeed, for \( h(t) = t^k \), we apply the relation
\[
\int_0^\infty \lambda^{-k-1}e^{-\lambda t}d\lambda = \Gamma(-k)\, t^k.
\]

To pass to the general case, one can use the following observation. If
\[
h_{r,\alpha}(t) = h(t + r)e^{-\alpha t}, \quad r \geq 0,
\]
(that is, a kernel \( h(t) \) is shifted and multiplied by an exponential), then according to (1.4) the corresponding sigma-function equals
\[
\sigma_{r,\alpha}(\lambda) = e^{-r(\lambda-\alpha)}\sigma(\lambda - \alpha).
\]

Let us now give the precise proof of (1.20).

**Lemma 6.1.** If \( \alpha \in \mathbb{R}, r \geq 0 \) and \( \text{Re} \, k < 0 \), then for all \( F \in C_0^\infty(\mathbb{R}_+) \) the identity
\[
\int_0^\infty (t + r)^k e^{-\alpha t}F(t) dt = \frac{e^{\alpha r}}{\Gamma(-k)} \int_0^\infty (\lambda - \alpha)^{-k-1} e^{-r\lambda}(L^\lambda F)(\lambda)d\lambda \tag{6.1}
\]
holds.

**Proof.** We use definition (1.2) of the operator \( L \) and according to the Fubini theorem interchange the order of integrations in the right-hand side of (6.1). Thus it equals
\[
\frac{e^{\alpha r}}{\Gamma(-k)} \int_0^\infty dtF(t) \int_0^\infty (\lambda - \alpha)^{-k-1} e^{-(t+\lambda)}d\lambda.
\]
Since the integral over \( \lambda \) equals \( \Gamma(-k)(t + r)^k e^{-\alpha(t + r)} \), this yields the left-hand side of (6.1). \( \square \)

Our next goal is to extend formula (6.1) to \( k \) in the right-half plane. The left-hand side of (6.1) is obviously an analytic function of \( k \in \mathbb{C} \). As is well known, the analytic continuation of the integral in the right-hand side of (6.1) to the strip \( n < \text{Re} \, k < n + 1 \) where \( n \in \mathbb{Z}_+ \) is given by the integral
\[
\int_0^\infty (\lambda - \alpha)^{-k-1}(\omega(\lambda) - \sum_{p=0}^n \frac{1}{p!}\omega^{(p)}(\alpha)(\lambda - \alpha)^p)d\lambda =: \int_0^\infty (\lambda - \alpha)^{-k-1}\omega(\lambda)d\lambda \tag{6.2}
\]
where $\omega(\lambda) = e^{-r\lambda}(LF)(\lambda)$. Here we use the standard notation $(\lambda - \alpha)^{k-1}_{+}$ for the distribution determined by this formula (we refer, for example, to the book [7] for a discussion of such distributions). This distribution is also well defined, although by a slightly different formula, on the lines $\text{Re} k \in \mathbb{Z}_+$. This concludes the proof of relation (1.20). Let us formulate the result obtained.

Lemma 6.2. Let $h(t)$ be given by formula (1.19) where $\alpha \in \mathbb{R}$, $r \geq 0$. If $k \in \mathbb{R} \setminus \mathbb{Z}_+$, then the sigma-function is given by equality (1.20). If $k \in \mathbb{Z}_+$, it is given by equality (1.21).

Putting together this result with Theorem 2.3, we get the following assertion.

Proposition 6.3. Let $h(t)$ be given by formula (1.19), and let the function $\sigma(\lambda)$ be given by equalities (1.20) or (1.21). Then the identity (2.9) holds for all $f_1, f_2 \in C_0^\infty(\mathbb{R}_+)$. Since for the sigma-function (1.20), the function

$$s(x) := \sigma(e^{-x}) = \frac{e^{ar}}{\Gamma(-k)}(e^{-x} - \alpha)^{k-1}_{+}e^{-re^{-x}}$$

belongs to the Schwarz class $S'$, we see that $\sigma \in S'_0$. The same is true for the sigma-function (1.21).

6.2. Our next goal is to calculate the numbers $N_\pm(h) := N_\pm(h; C_0^\infty(\mathbb{R}_+))$. Observe that the number $N_\pm(h)$ does not depend on $\alpha \in \mathbb{R}$ in definition (1.19). Indeed, if $\pm\langle h, f * f \rangle > 0$ for all $f \neq 0$ in some linear space $D \subset C_0^\infty(\mathbb{R}_+)$ and $h_\gamma(t) = e^{\gamma t}h(t)$ for some $\gamma \in \mathbb{R}$, then $\pm\langle h_\gamma, f * f \rangle > 0$ for all $f \neq 0$ in the linear space $D_\gamma$ consisting of functions $e^{-\gamma t}f(t)$ where $f \in D$. The spaces $D$ and $D_\gamma$ have of course the same dimension.

The case of Hankel operators of finite rank was treated in [23]. If $k \in \mathbb{Z}_+$ and $\alpha > 0$, then the form $\langle h, f * f \rangle$ gives rise to a Hankel operator of rank $k + 1$ and $N_\pm(H) = N_\pm(h)$. The above remark allows us to extend the result of [23] to all $\alpha \in \mathbb{R}$. Let us state the corresponding assertion.

Theorem 6.4. Let $h(t)$ be given by formula (1.19) where $\alpha \in \mathbb{R}$, $r \geq 0$ and $k \in \mathbb{Z}_+$. Then $N_\pm(h) = N_\pm(h) + 1 = k/2 + 1$ if $k$ is even and $N_\pm(h) = (k + 1)/2$ if $k$ is odd.

Our goal here is to prove the following result.

Theorem 6.5. Let $h(t)$ be given by formula (1.19) where $\alpha \in \mathbb{R}$ and $r \geq 0$. If $k < 0$, then $\langle h, f * f \rangle \geq 0$ for all $f \in C_0^\infty(\mathbb{R}_+)$. If $k > 0$ but $k \notin \mathbb{Z}_+$, then $N_\pm(h) = [k]/2 + 1$, $N_-(h) = \infty$ for even $[k]$ and $N_-(h) = ([k] + 1)/2$, $N_+(h) = \infty$ for odd $[k]$.

If $k < 0$, then, by formula (1.20), $\sigma \in L^1_{\text{loc}}$ and $\sigma(\lambda) \geq 0$. Therefore it suffices to use the identity (2.9).

The case $k > 0$ is essentially more complicated. Without loss of generality, we suppose that $\alpha > 0$. Then the operators $H$ are compact and $N_\pm(H) = N_\pm(h)$. We
Proof. Put \( \varphi(\lambda) = e^{-r\lambda}|w(\lambda)|^2 \). It follows from definition (6.2) that
\[
\Gamma(-k)e^{-\alpha r}\sigma[w,w] = \int_0^{\infty} (\lambda - \alpha)^{-k-1}(\varphi(\lambda) - \sum_{p=0}^{n} \frac{1}{p!}\varphi^{(p)}(\lambda)(\lambda - \alpha)^p) d\lambda.
\]
Under assumptions (6.4), (6.5) we have
\[
\varphi(\alpha) = \varphi'(\alpha) = \cdots = \varphi^{(n)}(\alpha) = 0
\]
so that the right-hand side of (6.7) is nonnegative. Since \( \Gamma(-k) < 0 \) for even and \( \Gamma(-k) > 0 \) for odd, equality (6.7) implies (6.6). \( \square \)

Lemma 6.8. Suppose that \( \alpha > 0 \) and \( k \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \). Let a function \( \psi = \tilde{\psi} \in C^\infty_0(\mathbb{R}_+) \) satisfy the conditions
\[
\psi(\alpha) = 1 \quad \text{and} \quad \psi'(\alpha) = \cdots = \psi^{(n)}(\alpha) = 0 \quad \text{if} \quad n = [k] \geq 1,
\]
and let \( Q(\lambda) \) be a polynomial of \( \deg Q \leq n \). Then
\[
\int_0^{\infty} (\lambda - \alpha)^{-k-1}Q(\lambda)|\psi^2(\lambda)| d\lambda = \int_0^{\infty} Q(\lambda)(\psi^2(\lambda) - 1) d\lambda.
\]
Proof. Put \( \omega(\lambda) = Q(\lambda)|\psi^2(\lambda) \). According to (6.9) we have \( \omega^{(p)}(\alpha) = Q^{(p)}(\alpha) \) for all \( p = 0, \ldots, n \), whence
\[
\sum_{p=0}^{n} \frac{1}{p!}\omega^{(p)}(\alpha)(\lambda - \alpha)^p = \sum_{p=0}^{n} \frac{1}{p!}Q^{(p)}(\alpha)(\lambda - \alpha)^p = Q(\lambda)
\]
if \( n \geq \deg Q \). Therefore relation (6.10) is a direct consequence of definition (6.2). \( \square \)
Now we are in a position to prove Theorem 6.5. In view of Lemma 6.6 to that end we only have to calculate the numbers $N_\pm(\sigma)$. Let us consider $N_+(\sigma)$ for even $n$ and $N_-(\sigma)$ for odd $n$. First we show that $N_+(\sigma) \leq \ell$ with $\ell$ defined by (6.5). Suppose the contrary. Then there exist linearly independent functions $w_j \in C^\infty_0(\mathbb{R}_+)$, $j = 1, \ldots, \ell + 1$, such that

$$(-1)^{n+1}\sigma[w, w] < 0$$

(6.11)
on all their nontrivial linear combinations

$$w(\lambda) = \sum_{j=1}^{\ell+1} c_j w_j(\lambda).$$

(6.12)

Substituting this expression into $\ell$ equations (6.4), we find a nontrivial solution of this system for the coefficients $c_1, \ldots, c_{\ell+1}$. According to Lemma 6.7 for the corresponding function (6.12) we have inequality (6.6). Clearly, $w \neq 0$ because the functions $w_1, \ldots, w_{\ell+1}$ are linearly independent. Therefore inequalities (6.6) and (6.11) are incompatible.

Let us prove the opposite estimates $N_+(\sigma) \geq \ell$. We choose a function $\psi = \tilde{\psi} \in C^\infty_0(\mathbb{R}_+)$ satisfying conditions (6.9) and such that $0 \leq \psi(\lambda) \leq 1$. Let us calculate form (6.3) on functions

$$w(\lambda) = P(\lambda)\psi(\lambda)e^{r\lambda/2}$$

(6.13)

where $P(\lambda)$ is an arbitrary polynomial of deg $P \leq [n/2]$. Applying Lemma 6.8 to $Q(\lambda) = |P(\lambda)|^2$, we see that $\Gamma(-k)e^{-ar}\sigma[w, w]$ equals expression (6.10). This yields a linear subspace of functions (6.13) of dimension $[n/2] + 1$ where $\Gamma(-k)\sigma[w, w] < 0$ for all $P \neq 0$.

Thus we have proven that $N_+(h) = N_+(\sigma) = n/2 + 1$ for $n$ even and $N_-(h) = N_-(\sigma) = (n + 1)/2$ for $n$ odd. Since $\Gamma(-k)\sigma(\lambda) > 0$ for all $\lambda > \alpha$, it follows from part 2º of Theorem 6.7 that $N_-(h) = \infty$ for $n$ even and $N_+(h) = \infty$ for $n$ odd (this result also follows from the fact that the rank of $H$ is infinite). The proof of Theorem 6.5 is complete.

6.3. The proof of Theorem 6.5 actually relies only on the study of the singularity of the sigma-function at the point $\alpha > 0$. To emphasize this idea, we obtain here more general results where conditions are formulated in terms of the sigma-function $\sigma(\lambda)$ of Hankel operators without making specific assumptions on their kernels $h(t)$. To obtain an upper bound on numbers (1.18), we require that the singularity of $\sigma(\lambda)$ at $\lambda = \alpha$ is not too strong.

**Lemma 6.9.** Suppose that $h \in C^\infty_0(\mathbb{R}_+)'$ and that the corresponding sigma-function $\sigma(\lambda)$ is continuous away from the point $\lambda = \alpha$, bounded as $\lambda \to 0$ and $\lambda \to \infty$ and, for some $n \in \mathbb{Z}_+$, the function $(\lambda - \alpha)^{n+1}\sigma(\lambda)$ belongs to $L^1_{\text{loc}}(\mathbb{R}_+)$. Assume also that

$$(-1)^{n+1}\sigma(\lambda) \geq 0, \quad \lambda \neq \alpha.$$  

(6.14)

Then $N_+(h) \leq n/2 + 1$ for $n$ even and $N_-(h) \leq (n + 1)/2$ for $n$ odd.
Proof. If a function \( w \in C_0^\infty(\mathbb{R}_+) \) satisfies conditions (6.4) where \( \ell \) is defined by (6.5), then the function \( \varphi(\lambda) = |w(\lambda)|^2 \) satisfies conditions (6.8) so that \( \varphi(\lambda) = O(|\lambda - \alpha|^{n+1}) \). It follows that

\[
(-1)^{n+1}\sigma[w, w] = (-1)^{n+1} \int_0^\infty \sigma(\lambda)|w(\lambda)|^2 d\lambda
\]

where the integral converges (at the point \( \lambda = \alpha \)). By condition (6.14) this expression is nonnegative. So it remains to repeat the proof of Theorem 6.5 of the upper bounds on the numbers \( N_\pm(\sigma) \).

To obtain a lower bound on the numbers \( N_\pm(h) \), we assume that \( \sigma(\lambda) = \sigma_0(\lambda) + \tilde{\sigma}(\lambda) \) (6.15) where \( \sigma_0(\lambda) \) is given by formula (1.20) and the singularity of \( \tilde{\sigma}(\lambda) \) at the point \( \alpha \) is weaker than that of \( \sigma_0(\lambda) \). Namely, we accept the following

Assumption 6.10. Set \( \varphi_\varepsilon(\lambda) = \omega((\lambda - \alpha)/\varepsilon) \) where \( \omega \in C_0^\infty(\mathbb{R}) \). Then

\[
\langle \tilde{\sigma}, \varphi_\varepsilon \rangle = o(\varepsilon^{-k}) \quad \text{as} \quad \varepsilon \to 0.
\]

Moreover, it is supposed that this relation holds uniformly for functions \( \omega \) having common support in \( \mathbb{R} \) and uniformly bounded in \( C^n \)-norm (as usual \( n = \lfloor k \rfloor \)).

The following result generalizes Theorem 6.5.

Theorem 6.11. In addition to the conditions of Lemma 6.9 suppose that representation (6.15) holds with \( \sigma_0 \) given by formula (1.20) where \( k \in (n, n+1) \) and \( \tilde{\sigma} \) satisfying Assumption 6.10. Then \( N_+(h) = n/2 + 1 \) for \( n \) even and \( N_-(h) = (n+1)/2 \) for \( n \) odd.

Proof. The upper estimate on the numbers \( N_\pm(h) \) is given by Lemma 6.9. To prove the lower estimate, we use again test functions (6.13) but introducing a small parameter \( \varepsilon \) we put

\[
w_{\varepsilon}(\lambda) = P((\lambda - \alpha)/\varepsilon)\psi(\alpha + (\lambda - \alpha)/\varepsilon) e^{r\lambda/2}.
\]

Here \( \psi = \bar{\psi} \in C_0^\infty(\mathbb{R}_+) \) satisfies conditions (6.9), \( 0 \leq \psi(\mu) \leq 1 \) and \( P(\mu) \) is an arbitrary polynomial of \( \deg P \leq \lfloor n/2 \rfloor \). Similarly to the proof of (6.10), we now find that

\[
\Gamma(-k)e^{-\alpha r} \sigma_0[w_{\varepsilon}, w_{\varepsilon}] = \varepsilon^{-k} \int_0^\infty \mu^{k-1}|P(\mu)|^2(\psi(\alpha + \mu)^2 - 1) d\mu.
\]

Put \( \|P\|^2 = |p_0|^2 + \cdots + |p_{[n/2]}|^2 \) where \( p_0, p_1, \ldots, p_{[n/2]} \) are the coefficients of \( P(\mu) \). Since

\[
\max_{\|P\|=1} \int_0^\infty \mu^{k-1}|P(\mu)|^2(\psi(\alpha + \mu)^2 - 1) d\mu < 0,
\]

we see that

\[
\Gamma(-k)\sigma_0[w_{\varepsilon}, w_{\varepsilon}] \leq -c\varepsilon^{-k}\|P\|^2
\] (6.17)
for some constant $c > 0$. Applying Assumption 6.10 to the function $\omega(\mu) = |P(\mu)|^2 \psi^2(\alpha + \mu)e^{\mu \varepsilon}$, we see that

$$\bar{\sigma}[w_\varepsilon, w_\varepsilon] = o(\varepsilon^{-k})$$

(6.18)

where the limit is uniform for all polynomials with $\|P\| \leq 1$. Combining estimates (6.17) and (6.18), we see that

$$\Gamma(-k)\sigma[w_\varepsilon, w_\varepsilon] \leq -c\|P\|\varepsilon^{-k}(1 + o(1)), \quad c > 0.$$

Since the right-hand side here is negative for sufficiently small $\varepsilon$, this yields us a space of dimension $[n/2] + 1$ where the form $\Gamma(-k)\sigma$ is negative. □

We note that if the function $\sigma(\lambda)$ changes the sign for $\lambda \neq \alpha$, then $N_\pm(h) = \infty$ according to Theorem 4.7.

**Example 6.12.** Let

$$h(t) = ((t + r)^k + \sum_{j=1}^{j_0} a_j (t + r)^{k_j})e^{-\alpha t}, \quad a_j = \bar{a}_j, \quad r \geq 0, \quad (6.19)$$

where $k_j \in [0, k)$ for all $j = 1, \ldots, k_{j_0}$. According to formula (1.20) representation (6.15) is satisfied with

$$\bar{\sigma}(\lambda) = e^{-r(\lambda - \alpha)} \sum_{j=1}^{j_0} \frac{a_j}{\Gamma(-k_j)} (\lambda - \alpha)^{-k_j} + 1$$

(6.20)

(if $k_j \notin \mathbb{Z}_+$ for all $j = 1, \ldots, k_{j_0}$). Assumption 6.10 holds true because all $k_j < k$. Now condition (6.14) is fulfilled if

$$1 + \sum_{j=1}^{j_0} a_j \frac{\Gamma(-k)}{\Gamma(-k_j)} (\lambda - \alpha)^{-k_j} \geq 0, \quad \forall \mu > 0.$$  

(6.21)

In particular, it suffices to require that $(-1)^{n_j} a_j \geq 0$ where $n_j = [k_j]$ for all $j$. Then all conclusions of Theorem 6.11 hold.

We emphasize that it is allowed in (6.19) that $k_j \in \mathbb{Z}_+$. According to Lemma 6.2 the sigma-functions $\sigma_j(\lambda)$ of such kernels $t^{k_j}e^{-\alpha t}$ are combinations of delta functions $\delta(\lambda - \alpha)$ and their derivatives so that $\sigma_j(\lambda) = 0$ for $\lambda \neq \alpha$. Therefore the corresponding term in (6.21) should be omitted.

**6.4.** Finally, we consider the case when the sigma-function has singularities at several points. It turns out that the contributions of different singularities to the numbers $N_\pm(h)$ are independent of each other.

**Theorem 6.13.** Let

$$h(t) = \sum_{m=1}^{M} (-1)^{n_m+1} b_m(t + r_m)^{k_m} e^{-\alpha_m t}, \quad b_m = \bar{b}_m, \quad r_m \geq 0, \quad (6.22)$$
where \( k_m > 0 \), \( k_m \not\in \mathbb{Z}_+ \) and \( n_m = [k_m] \). Then:

1° If \( b_m > 0 \) for some \( m = 1, \ldots, M \), then \( N_+(h) = \infty \). If \( b_m < 0 \) for some \( m = 1, \ldots, M \), then \( N_-(h) = \infty \).

2° Put

\[
N = \sum_{m=1}^{M} \lceil n_m/2 \rceil + M. \tag{6.23}
\]

If all \( b_m < 0 \), then \( N_+(h) = N \). If all \( b_m > 0 \), then \( N_-(h) = N \).

Proof. It follows from formula (1.20) that the sigma-function of kernel (6.22) equals

\[
\sigma(\lambda) = \sum_{m=1}^{M} \sigma_m(\lambda) \quad \text{where} \quad \sigma_m(\lambda) = \frac{(-1)^{n_m+1}b_m}{\Gamma(-k_m)}(\lambda - \alpha_m)^{-k_m-1}e^{-\epsilon m(\lambda - \alpha_m)}.
\]

Obviously, the function \( \sigma(\lambda) \) is continuous away from the points \( \alpha_1, \ldots, \alpha_M \). Clearly, \( \sigma \in \mathcal{S}_0 \) if \( \alpha_m > 0 \) for all \( m = 1, \ldots, M \) which we can always suppose. Note that \((-1)^{n_m+1}\Gamma(-k_m) > 0 \). Therefore if \( b_m > 0 \) (\( b_m < 0 \)) for some \( m \), then this function tends to \( +\infty \) (\(-\infty \)) as \( \lambda \to \alpha_m+0 \). In this case, by Theorem 4.7 the positive (negative) spectrum of the operator \( H \) is infinite.

Let us prove part 2°. Suppose, for example, that \( b_m > 0 \) for all \( m = 1, \ldots, M \); then \( \sigma(\lambda) \geq 0 \) for all \( \lambda \not\in \{\alpha_1, \ldots, \alpha_M\} \). Let a function \( w \in C_0^\infty(\mathbb{R}_+) \) satisfy the conditions \( w^{(p)}(\alpha_m) = 0 \) for \( p = 0, 1, \ldots, \lceil n_m/2 \rceil \) and all \( m = 1, \ldots, M \). Then according to Lemma 6.7 we have \( \sigma_m[w, w] \geq 0 \) for all \( m = 1, \ldots, M \) and hence \( \sigma[w, w] \geq 0 \). Quite similarly to the proof of the upper bound on \( N_-(h) \) in Theorem 6.5, this implies that \( N_-(h) \leq N \).

To prove the opposite inequality, we consider trial functions \( w_{\varepsilon,m}, m = 1, \ldots, M \), defined by formula (6.16) where \( \alpha = \alpha_m, r = r_m \) and \( P_m \) is a polynomial of degree \( \lceil n_m/2 \rceil \). Instead of (6.17), we now have the estimate

\[
\sigma_m[w_{\varepsilon,m}, w_{\varepsilon,m}] \leq -c\varepsilon^{-k_m}\|P_m\|^2, \quad c > 0.
\]

Since the functions \( \sigma_p(\lambda) \) where \( p \neq m \) are continuous at the point \( \alpha_m \), this implies the same estimate on \( \sigma[w_{\varepsilon,m}, w_{\varepsilon,m}] \). Taking linear combinations of functions \( w_{\varepsilon,m} \), we obtain a space of dimension \( N \) where the form \( \sigma \) is negative. \hfill \Box

Remark 6.14. Suppose that some of \( k_m \) in (6.22) are integers. Then \( N_+(h) = \infty \) if \( \pm b_m > 0 \) for some \( m \) such that \( k_m \not\in \mathbb{Z}_+ \). Assertion 2° of Theorem 6.13 remains unchanged.

Remark 6.15. Let \( h(t) \) be given by formula (6.22). Then \( \pm \langle h, \tilde{f} * f \rangle \geq 0 \) for all \( f \in C_0^\infty(\mathbb{R}_+) \) if and only if \( k_m \leq 0 \) and \( \mp(-1)^{n_m}b_m \geq 0 \) for all \( m = 1, \ldots, M \).

7. DISCRETE REPRESENTATION

Here we consider Hankel operators \( Q \) defined by equality (1.7) in the space \( l^2(\mathbb{Z}_+) \) of sequences \( g = (g_0, g_1, \ldots) \) and discuss their relation by formula (1.8) to integral Hankel
operators $H$ in the space $L^2(\mathbb{R}_+)$. It turns out that the concept of the sigma-function is also very convenient for finding a link between matrix elements $q_n$ of $Q$ and kernels $h(t)$ of $H$.

7.1. Similarly to the continuous case, the most general definition of Hankel operators $Q$ is given in terms of quadratic forms

$$q[g, g] = \sum_{n,m=0}^{\infty} q_{n+m} g_m \bar{g}_n$$

considered on the set $\ell_0 \subset l^2(\mathbb{Z}_+)$ of elements $g$ with only finite number of non-zero components. This definition does not require any assumptions on elements $q_n$, but it does not guarantee that $Q$ is correctly defined as an operator (even unbounded) in $l^2(\mathbb{Z}_+)$. Let us construct a unitary operator $U : l^2(\mathbb{Z}_+) \rightarrow L^2(\mathbb{R}_+)$ such that relation (1.8) is satisfied. To be precise, we consider the relation

$$\langle h, U g \star U g \rangle = q[g, g], \quad g \in \ell_0,$$

between the corresponding quadratic forms. Recall that, for an arbitrary value of the parameter $\kappa > -1$, the Laguerre polynomial (see [4], Chapter 10.12) of degree $n$ is defined by the formula

$$L^\kappa_n(t) = n!^{-1} t^{-\kappa} d^n(e^{-t} t^{n+\kappa})/dt^n, \quad t > 0,$$

and the functions

$$u^\kappa_n(t) = \sqrt{\frac{n!}{\Gamma(n + 1 + \kappa)}} t^{\kappa/2} e^{-t/2} L^\kappa_n(t), \quad n = 0, 1, \ldots,$$

form an orthonormal basis in the space $L^2(\mathbb{R}_+)$. Therefore the operator $U_\kappa : l^2(\mathbb{Z}_+) \rightarrow L^2(\mathbb{R}_+)$ defined by the formula

$$(U_\kappa g)(t) = \sum_{n=0}^{\infty} g_n u^\kappa_n(t)$$

is unitary and hence

$$(U_\kappa^{-1} f)_n = \int_0^\infty f(t) u^\kappa_n(t) dt.$$

Observe that if $g \in \ell_0$, then according to (7.2) and (7.3) we have

$$(U_0 g \star U_0 g)(t) = \sum_{n,m=0}^{\infty} g_m \bar{g}_n e^{-t/2} \int_0^t L^0_m(s)L^0_n(t-s) ds. \quad (7.4)$$

Putting together formulas (10.12.23) and (10.12.31) in [4], we see that

$$\int_0^t L^0_m(s)L^0_n(t-s) ds = (n + m + 1)^{-1} t L^1_{n+m}(t). \quad (7.5)$$
Now to get relation (7.1) with $U = U_0$ and $q_n = 1 \quad (7.6)$

we only have to multiply (7.3) by $h(t)$ and integrate it in $t \in \mathbb{R}_+$. Since the operator $U_1$ is unitary, the last relation can formally be rewritten as

$$h(t) = \sum_{n=0}^{\infty} q_n L_n(t) e^{-t/2}. \quad (7.7)$$

As usual, we consider $h(t)$ as a distribution. The problem is that the functions $(U_0 g)(t)$ and $tL_n(t)e^{-t/2}$ do not belong to the class $C_0^\infty(\mathbb{R}_+)$, and hence the assumption $h \in C_0^\infty(\mathbb{R}_+)'$ does not allow us to give a precise sense to formulas (7.1) and (7.6). Therefore we introduce the set $X \subset C_0^\infty(\mathbb{R}_+)$ that consists of functions $\varphi(t)$ satisfying estimates

$$|\varphi^{(n)}(t)| \leq C_n t \quad \text{and} \quad |\varphi^{(n)}(t)| \leq C_n e^{-\gamma t} \quad (7.8)$$

for some $\gamma < 1/2$ and all $n$. Since all functions $(U_0 g \ast U_0 g)(t)$ and $tL_n(t)e^{-t/2}$ belong to $X$, relations (7.4) and (7.5) imply the following result.

**Theorem 7.1.** Let $h \in X'$, and let

$$q_n = \langle h, \tilde{u}_n^1 \rangle \quad \text{where} \quad \tilde{u}_n^1(t) = (n+1)^{-1} tL_n(t)e^{-t/2}. \quad (7.10)$$

Then for all elements $g \in \ell_0$ identity (7.1) holds with $U = U_0$.

Since the operator $U_1$ is unitary, it follows from (7.6) that

$$\sum_{n=0}^{\infty} (n+1)|q_n|^2 = \int_0^{\infty} |h(t)|^2 t dt. \quad (7.11)$$

This relation simply means that the Hilbert-Schmidt norms of the operators $H$ and $Q$ related by formula (1.8) are the same.

We note that as shown in [22] (Theorem 3.8), the condition $h \in X'$ is satisfied for all bounded operators. For bounded operators $H$ and $Q$, identity (7.1) extends to all $g \in l^2(\mathbb{Z}_+)$ which yields (1.8). Note that $h \in X'$ if

$$\int_0^{1} t|h(t)| dt + \int_1^{\infty} e^{-\gamma t}|h(t)| dt < \infty \quad (7.9)$$

for some $\gamma < 1/2$. This assumption is by no means optimal although it even admits an exponential growth of $h(t)$ as $t \to \infty$. Even the condition $h \in L_1^{\text{loc}}(\mathbb{R}_+)$ is not required.

**Example 7.2.** Let $h(t) = \delta^{(k)}(t-t_0)$ for some $k \in \mathbb{Z}_+$ and $t_0 > 0$; then $h \in X'$. It follows from formula (7.10) that the matrix elements of the corresponding Hankel operator $Q^{(k)}$ are given by the equality

$$q_n^{(k)} = \frac{1}{n+1} (-1)^k (tL_n(t)e^{-t/2})^{(k)}|_{t=t_0}. \quad (7.10)$$
If \( k = 0 \), then, as shown in [22], the spectrum of the operator \( H \) consists of the eigenvalues \(-1, 0, 1\) of infinite multiplicity. According to Theorem 7.1, the spectrum of the operator \( Q^{(0)} \) is the same. Note that formula (10.15.1) in [3] shows that
\[
L_n^*(t_0) = \pi^{-1/2} t_0^{-3/4} e^{\gamma/2} n^{1/4} \cos(2\sqrt{n!} t_0 - 3\pi/4) + O(n^{-1/4})
\]
so that even the boundedness of \( Q^{(0)} \) does not look obvious. If \( k \geq 1 \), then (see [24]) the operators \( H \) are unbounded and their spectra consist of eigenvalues accumulating both at \(+\infty\) and \(-\infty\). According to Theorem 7.1, the spectra of the Hankel operators \( Q^{(k)} \) with matrix elements (7.10) possess the same properties.

Recall that in the class of bounded operators, Hankel operators in \( L^2(\mathbb{Z}_+) \) can be characterized by the commutation relation
\[
QT = T^*Q \tag{7.11}
\]
where \( T \) is the shift defined by the formula \((Tg)_n = g_{n-1}\) (with \( g_{-1} = 0 \)). Similarly, Hankel operators in \( L^2(\mathbb{R}_+) \) can be characterized (see [22], subs. 3.2, for details) by the commutation relation
\[
HT(\tau) = T(\tau)^*H, \quad \forall \tau \geq 0, \tag{7.12}
\]
where \((T(\tau)f)(t) = f(t - \tau)\) for \( t \geq \tau \) and \((T(\tau)f)(t) = 0\) for \( t < \tau \).

**Proposition 7.3.** A bounded operator \( H \) satisfies (7.12) if and only if the operator \( Q = U_0^{-1}HU_0 \) satisfies (7.11).

**Proof.** As shown in [22] (see Corollary 3.5), (7.12) is equivalent to the relation \( H\Sigma = \Sigma^*H \) where
\[
(\Sigma f)(t) = e^{-t/2} \int_0^t e^{s/2} f(s) ds.
\]
Therefore we only have to verify that \( U_0T = (I - \Sigma)U_0 \). By definition (7.3), to that end we have to check the identity
\[
u^0_{n+1}(t) = u^0_n(t) - e^{-t/2} \int_0^t e^{s/2} u^0_n(s) ds \tag{7.13}
\]
where \( u^0_n(t) \) are functions (7.2). Both sides of (7.13) equal 1 for \( t = 0 \). The equality of their derivatives follows from the identity
\[
d(L^0_n(t) - L^0_{n+1}(t))/dt = L^0_n(t)
\]
(see formula (10.12.16) in [4]).

It is possible to indicate the general form of unitary operators \( U : L^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+) \) such that operator (1.3) is a (bounded) Hankel operator in \( L^2(\mathbb{R}_+) \) if and only if \( Q \) is a Hankel operator in \( L^2(\mathbb{Z}_+) \). Let the dilation \( D_\rho, \rho > 0 \), be defined in the space \( L^2(\mathbb{R}_+) \) by the formula \((D_\rho f)(t) = \rho^{-1/2} f(\rho^{-1}t)\), and let the involution \( J \) be defined in the space \( L^2(\mathbb{Z}_+) \) by the formula \((Jg)_n = (-1)^n g_n\). It is shown in the Appendix to [23] that if an operator \( V \) is unitary in \( L^2(\mathbb{R}_+) \) and \( VHV^{-1} \) are Hankel operators for
all Hankel operators $H$, then necessarily either $V = D_\rho$ or $V = D_\rho U_0 J U_0^{-1}$ for some $\rho > 0$. It follows that all unitary operators $U : l^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)$ possessing property (1.8) admit one of the two following forms: $U = D_\rho U_0$ or $U = D_\rho U_0 J$ where $\rho > 0$.

Using this observation, we can choose an arbitrary large $\gamma > 0$ in definition (7.8) of the class $\mathcal{X}$. Then Theorem 7.1 remains true with $U = D_\rho U_0$ for a suitable $\rho > 0$.

7.2. Let us find a relation between matrix elements $q_n$ of a Hankel operator $Q$ and the sigma-function $\sigma(\lambda)$ of the corresponding Hankel operator (1.8). Let us suppose that supp $\sigma \subset [0, \infty)$ and substitute expression (1.4) into formula (7.6):

$$q_n = \frac{1}{n+1} \int_0^\infty d\lambda \sigma(\lambda) \int_0^\infty t L_n^1(t) e^{-(1/2+\lambda)t} dt. \quad (7.14)$$

According to formula (10.12.32) in [4] we have

$$\int_0^\infty t L_n^1(t) e^{-(1/2+\lambda)t} dt = \frac{n+1}{(\lambda + 1/2)^2} \left( \frac{\lambda - 1/2}{\lambda + 1/2} \right)^n,$$

and hence it follows from (7.14) that

$$q_n = \int_0^\infty \frac{\sigma(\lambda)}{(\lambda + 1/2)^2} \left( \frac{\lambda - 1/2}{\lambda + 1/2} \right)^n d\lambda.$$

Introducing now the function

$$\eta(\mu) = \sigma(\lambda) \quad \text{where} \quad \mu = \frac{\lambda - 1/2}{\lambda + 1/2} \in (-1, 1), \quad (7.15)$$

we obtain the representation (1.9).

For the precise proof of (1.9), we need only to justify the change of order of integrations in (7.14). It can be done by the Fubini theorem. We state only the simplest result which is however sufficient in many specific applications.

**Proposition 7.4.** Let the sigma-function $\sigma(\lambda)$ of a Hankel operator $H$ satisfy assumptions (2.11) and (4.8), and let the function $\eta(\mu)$ be defined by formula (7.15). Then $Q = U_0^{-1} H U_0$ is the Hankel operator in the space $l^2(\mathbb{Z}_+)$ with matrix elements (1.9).

Of course under the assumptions of this theorem $|h(t)| \leq Ct^{-1}$, $\eta \in L^\infty(-1, 1)$, $q_n = O(n^{-1})$ and the operators $H$ and $Q$ are bounded.

7.3. The method presented here gives, in principle, a constructive approach to the solution of the Hausdorff moment problem (1.9). We describe it in this subsection at a formal level.

Given a sequence $q_n$, we first construct the kernel $h(t)$ by formula (7.7). Then we find its sigma-function $\sigma(\lambda)$ by the inversion of the Laplace transform and, finally, we make the change of variables (7.15). The function $\eta(\mu)$ obtained (we also call it the sigma-function of the Hankel operator $Q$) yields the solution of the moment problem (1.9). In general, $\eta$ is a distribution obtained from $\sigma \in \mathcal{Y}'$ by the change of variables (7.15), but $\eta(\mu)d\mu$ is a positive measure if $Q \geq 0$. We note (see Theorem 2.6.4 in [1])
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that original conditions for the solvability of the moment problem (1.9) were formulated in rather different terms.

Alternatively, we can exhibit an expression for the function \( \eta(\mu) \), or rather for the Mellin transform of \( \lambda^{-1/2} \sigma(\lambda) \), directly in terms of the coefficients \( q_n \), avoiding the construction of the kernel \( h(t) \):

\[
\int_0^\infty \lambda^{-1+\xi} \sigma(\lambda) d\lambda = 2^{1-i\xi} \sum_{n=0}^\infty i^{-n} q_n P_n(-\xi).
\]

(7.16)

Here

\[
P_n(\xi) = i^n \sum_{m=0}^n \frac{(-1)^m m^{2m}}{m!} C_{n+1}^{m+1} (1+i\xi) \cdots (m+i\xi)
\]

is the polynomial of degree \( n \) known as the Meixner-Pollaczek polynomial; note that the term corresponding to \( m = 0 \) in this sum is 1 and \( C_{n+1}^{m+1} \) are binomial coefficients. Recall that \( P_n(\xi) \), denoted also \( P_\alpha^1(n; \pi/2) \) in §10.21 of [4], are orthogonal polynomials in the space \( L^2(\mathbb{R}; |\Gamma(1-i\xi)|^2 d\xi) \) related to the hypergeometric function \( F(-n, 1-i\xi, 2; 2) \) by formula (10.21.10). We give only a formal proof of relation (7.16). According to Lemma 4.9 for \( \gamma = 1 \) we have

\[
\Gamma(1-i\xi)(M\Omega^{-1/2}\sigma)(-\xi) = (M\Omega^{1/2}h)(\xi).
\]

Applying the operator \( M\Omega^{1/2} \) to both sides of equality (7.7), we see that

\[
(M\Omega h)(\xi) = \sum_{n=0}^\infty q_n \int_0^\infty t^{-i\xi} L_n^1(t) e^{-t/2} dt
\]

where

\[
\int_0^\infty t^{-i\xi} L_n^1(t) e^{-t/2} dt = i^{-n} 2^{1-i\xi} \Gamma(1-i\xi) P_n(-\xi)
\]

according to formula (10.12.33) and expression (2.1.4) for \( F(-n, 1-i\xi, 2; 2) \) in [4]. Combining the formulas obtained, we get relation (7.16).

7.4. Let us come back to quasi-Carleman operators \( H \) with kernels (1.19) where we now suppose that \( \alpha \geq 0 \). Our goal is to calculate matrix elements \( q_n = q_n(\alpha, k, r) \) of Hankel operators \( Q = U_0^{-1} H U_0 \). Let us proceed from formula (1.9) for \( q_n \) in terms of the sigma-function. We suppose that \( k \not\in \mathbb{Z}_+ \) since for \( k \in \mathbb{Z}_+ \) the operators \( Q \) have finite rank and the well-known formula for \( q_n \) is, for example, a direct consequence of (1.21). Putting together relations (1.20) and (7.15), we obtain the following result.

Proposition 7.5. Let \( k \not\in \mathbb{Z}_+ \). In the case \( r = 0 \) assume additionally that \( k > -2 \).

Then the matrix elements \( q_n \) of the operator \( Q = U_0^{-1} H U_0 \) are determined by relation (1.9) where

\[
\eta(\mu) = (\alpha + 1/2)^{-k-1} \Gamma(-k)^{-1} (1 - \mu)^{k+1} (\mu - \gamma)^{-k-1} \exp \left( -r (\alpha + 1/2) \frac{\mu - \gamma}{1 - \mu} \right)
\]

(7.17)
The case \( r = 0 \) is particularly simple. If \( \alpha = 1/2 \), then it follows from (1.9), (7.17) that

\[
q_n = \Gamma(k+2)\Gamma(n-k) / \Gamma(-k)\Gamma(n+2).
\]

If \( k = -1 \), then

\[
q_n = \int_{\gamma}^{1} \mu^n d\mu = \frac{1 - \gamma^{n+1}}{n+1}.
\] (7.18)

If \( \alpha = 0 \), then \( \gamma = -1 \) and we recover of course the matrix elements of the Carleman operator. If \( \alpha > 0 \), then \( \gamma \in (-1, 1) \) and we obtain the generalized (but different from those considered by M. Rosenblum in [17]) Hilbert matrices. The y reduce to the standard Hilbert matrix for \( \gamma = 0 \).

Alternatively, we could proceed from formula (7.6) for \( q_n \) in terms of the kernel \( h(t) \). In the particular case \( r = 0 \), using formula (10.12.33) in [4], we can express the coefficients \( q_n \) via the hypergeometric function \( F \):

\[
q_n = q_n(\beta,k) = \Gamma(2+k)\beta^{2+k}F(-n,2+k,2;\beta), \quad \beta = (\alpha + 1/2)^{-1} \in (0, 2].
\] (7.19)

Observe that if \( r = 0 \), then formulas (1.9) as well as (7.6) make sense for \( k > -2 \) only while the Hankel quadratic form \( \langle\langle\langle h, f \star f \rangle\rangle\rangle \) is well defined on \( f \in C_0^\infty(\mathbb{R}_+) \) for all kernels \( h(t) = t^k e^{-\alpha t} \). Thus the example of quasi-Carleman operators shows that considerations of Hankel operators in the spaces \( L^2(\mathbb{R}_+) \) and \( l^2(\mathbb{Z}_+) \) are not always equivalent.

Note that, for all \( \rho > 0 \), Hankel operators \( H \) and \( H_\rho \) with kernels \( h(t) \) and \( h_\rho(t) = \rho h(\rho t) \) are unitarily equivalent (by the dilation transformation). In particular, all Hankel operators \( H_\rho \) with kernels \( h_\rho(t) = \rho^{1+k}t^k e^{-\rho t/2} \) are unitarily equivalent to each other for all \( \rho > 0 \). This implies the following assertion.

**Proposition 7.6.** Let \( Q(\beta,k) \) be the Hankel operator in the space \( l^2(\mathbb{Z}_+) \) with the matrix elements (7.19) where \( k > -2 \). Then the operators \((-1+2/\beta)^{1+k}Q(\beta,k)\) are unitarily equivalent to each other for all \( \beta \in (0, 2). \) In particular (for \( k = -1 \)), the generalized Hilbert matrices determined by formula (7.18) are unitarily equivalent to each other for all \( \gamma \in (-1, 1) \). Thus their spectra are absolutely continuous, simple and coincide with the interval \([0, \pi].\)

This result does not look obvious in the discrete representation \( l^2(\mathbb{Z}_+) \), but it becomes quite transparent after the transformation of the problem into the space \( L^2(\mathbb{R}_+) \).

**7.5.** Our next goal is to find the asymptotics of matrix elements \( q_n \) of the quasi-Carleman operators as \( n \to \infty \). It easily follows from formula (7.17) that for any
\( a \in (0, 1) \)
\[
\int_{-a}^{a} \eta(\mu)\mu^n d\mu = O(b^n), \quad \forall b > a,
\]
so that the asymptotics of \( q_n \) is determined by neighborhoods of the points \( \mu = \pm 1 \) in the integral representation (1.9).

Consider first the point \( \mu = -1 \). If \( \alpha > 0 \), that is \( \gamma > -1 \), then function (7.17) equals zero in a neighborhood of the point \(-1\). So the contribution of this point to the asymptotics of \( q_n \) is also zero. If \( \alpha = 0 \), that is \( \gamma = -1 \), then it follows from formula (7.17) that
\[
\eta(\mu) = 4^{k+1}\Gamma(-k)^{-1}(\mu + 1)^{-k-1}(1 + O(\mu + 1))
\]
as \( \mu \to -1 \). So we have
\[
q_n^{(-)} := \int_{-1}^{0} \eta(\mu)\mu^n d\mu
\]
\[
= 4^{k+1}\Gamma(-k)^{-1} \int_{-1}^{0} (\mu + 1)^{-k-1}\mu^n d\mu + O\left( \int_{-1}^{0} (\mu + 1)^{-k-1}|\mu|^n d\mu \right).
\]
(7.20)

Note that
\[
\int_{-1}^{0} (\mu + 1)^{-k-1}\mu^n d\mu = (-1)^n \frac{\Gamma(-k)\Gamma(n + 1)}{\Gamma(n - k + 1)} = (-1)^n \Gamma(-k)n^k(1 + O(n^{-1}))
\]
where we have used the asymptotic formula (1.18.4) in [4] for the ratio of the gamma functions. Thus according to (7.20) we have
\[
q_n^{(-)} = (-1)^n 4^{k+1}n^k(1 + O(n^{-1})).
\]
(7.21)

Next, we consider a neighborhood of the point \( \mu = 1 \). If \( r > 0 \), then function (7.17) exponentially tends to zero as \( \mu \to 1 \) so that the contribution \( q_n^{(+)} \) of this point to \( q_n \) is negligible. If \( r = 0 \), then it follows from formula (7.17) that
\[
\eta(\mu) = \Gamma(-k)^{-1}(1 - \mu)^{k+1}(1 + O(1 - \mu))
\]
as \( \mu \to 1 \). So we have
\[
q_n^{(+)} := \int_{0}^{1} \eta(\mu)\mu^n d\mu = \Gamma(-k)^{-1} \int_{0}^{1} (1 - \mu)^{k+1}\mu^n d\mu + O\left( \int_{0}^{1} (1 - \mu)^{k+2}\mu^n d\mu \right)
\]
Calculating again the integrals here in terms of the beta function and using formula (1.18.4) in [4], we find that
\[
q_n^{(+)} = \Gamma(-k)^{-1}\Gamma(k + 2)n^{-k-2}(1 + O(n^{-1})).
\]
(7.23)

Let us put together the results obtained.
Proposition 7.7. Let the assumptions of Proposition 7.5 hold. If \( \alpha > 0 \) and \( r > 0 \), then the matrix elements \( q_n \) of the Hankel operator \( Q \) decay faster than any power of \( n^{-1} \) as \( n \to \infty \). If \( \alpha = 0 \) but \( r > 0 \), then the asymptotics of \( q_n \) is given by formula (7.21) where \( q_n = q_n^{(-)} \). If \( r = 0 \) but \( \alpha > 0 \), then the asymptotics of \( q_n \) is given by formula (7.23) where \( q_n = q_n^{(+)} \). Finally, if \( \alpha = r = 0 \), then
\[
q_n = (-1)^n 4^{k+1} n^k (1 + O(n^{-1})) + \Gamma(-k)^{-1} \Gamma(k+2) n^{-k-2} (1 + O(n^{-1})).
\] (7.24)

Remark 7.8. Of course if \( k \in (-2, -1) \) (if \( k > -1 \)), then the first (the second) term in (7.24) can be neglected. If \( k = -1 \), then both terms in (7.24) have the same order.

We emphasize that under the assumptions of Proposition 7.7 the sequence \( q_n \) does not necessarily tend to 0 and the operator \( Q \) may be unbounded.

Appendix A. A generalization of the Bernstein theorem

Our proof of Theorem 5.1 will be divided in a series of simple lemmas. For an arbitrary \( \varphi \in C_0^\infty(\mathbb{R}_+) \), we set
\[
\eta = \varphi \star \bar{\varphi}
\] (A.1)
and define the distribution
\[
h_\varphi(t) = \int_0^\infty h(s) \eta(s-t) ds = \langle h, \eta(\cdot - t) \rangle
\] (A.2)
which is actually a continuous function of \( t > 0 \). It follows from (A.1) and (A.2) that
\[
\langle h_\varphi, f \rangle = \langle h, \varphi \star \bar{\varphi} \star f \rangle
\] (A.3)
for an arbitrary \( f \in C_0^\infty(\mathbb{R}_+) \).

Let us check that
\[
\int_0^\infty \int_0^\infty h_\varphi(\tau + \sigma) g(\sigma) \bar{g(\tau)} d\tau d\sigma \geq 0
\] (A.4)
for all \( g \in C_0^\infty(\mathbb{R}_+) \). Using (A.1) and (A.2), we can rewrite the last integral as
\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty h(t + s + \tau + \sigma) \varphi(s) \bar{\varphi(t)} g(\sigma) \bar{g(\tau)} dtds d\tau d\sigma.
\]
Making here the changes of variables \( x = t + \tau, y = s + \sigma \), we see that this expression equals
\[
\int_0^\infty \int_0^\infty h(x+y) \psi(y) \bar{\psi(x)} dxdy
\] (A.5)
where
\[
\psi(x) = \int_0^x g(x-t) \varphi(t) dt.
\]
Since \( \psi \in C_0^\infty(\mathbb{R}_+) \), expression (A.5) is positive by the condition (5.1). This proves (A.4).

Thus applying the Bernstein theorem on exponentially convex functions to the function \( h_\varphi(t) \), we obtain the following intermediary result.
Lemma A.1. For an arbitrary $\varphi \in C_0^\infty(\mathbb{R}_+)$, let the function $h_\varphi(t)$ be defined by formulas (A.1), (A.2). Then under the assumption (5.1), there exists a positive measure $M_\varphi$ on $\mathbb{R}$ such that

$$h_\varphi(t) = \int_{-\infty}^{\infty} e^{-t\lambda} dM_\varphi(\lambda)$$

where the integral is convergent for all $t > 0$.

Let us now compare the measures $M_\varphi$ corresponding to different functions $\varphi$.

Lemma A.2. For arbitrary functions $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}_+)$, $j = 1, 2$, and all $\lambda \in \mathbb{R}$ the relation

$$|(L\varphi_2)(\lambda)|^2 dM_{\varphi_1}(\lambda) = |(L\varphi_1)(\lambda)|^2 dM_{\varphi_2}(\lambda)$$

(A.7)

holds.

Proof. Let us proceed from definition (A.3) which, for an arbitrary $f \in C_0^\infty(\mathbb{R}_+)$, yields relations

$$\langle h_{\varphi_1}, \eta_2 * f \rangle = \langle h, \eta_1 * (\eta_2 * f) \rangle, \quad \langle h_{\varphi_2}, \eta_1 * f \rangle = \langle h, \eta_2 * (\eta_1 * f) \rangle$$

where notation (A.1) has been used. Since $\eta_1 * (\eta_2 * f) = \eta_2 * (\eta_1 * f)$, it follows that

$$\langle h_{\varphi_1}, \eta_2 * f \rangle = \langle h_{\varphi_2}, \eta_1 * f \rangle.$$  

(A.8)

According to Lemma A.1 we have

$$\langle h_{\varphi_1}, \eta_2 * f \rangle = \int_{-\infty}^{\infty} \frac{(L(\eta_2 * f))(\lambda)}{(L(\eta_2)(\lambda))} dM_{\varphi_1}(\lambda) = \int_{-\infty}^{\infty} \frac{L(f)(\lambda)}{(L(\eta_2)(\lambda))} dM_{\varphi_1}(\lambda)$$

$$= \int_0^{\infty} dt \bar{f}(t) \int_{-\infty}^{\infty} e^{-t\lambda} L(\eta_2)(\lambda) dM_{\varphi_1}(\lambda).$$

The exactly similar identity is true for $\langle h_{\varphi_2}, \eta_1 * f \rangle$. Hence it follows from equality (A.8) that

$$\int_0^{\infty} dt \bar{f}(t) \int_{-\infty}^{\infty} e^{-t\lambda} L(\eta_2)(\lambda) dM_{\varphi_1}(\lambda) = \int_0^{\infty} dt \bar{f}(t) \int_{-\infty}^{\infty} e^{-t\lambda} L(\eta_1)(\lambda) dM_{\varphi_2}(\lambda).$$

This ensures that for all $t > 0$

$$\int_{-\infty}^{\infty} e^{-t\lambda} L(\eta_2)(\lambda) dM_{\varphi_1}(\lambda) = \int_{-\infty}^{\infty} e^{-t\lambda} L(\eta_1)(\lambda) dM_{\varphi_2}(\lambda)$$

because $f \in C_0^\infty(\mathbb{R}_+)$ is arbitrary. Therefore by the uniqueness theorem for Laplace integrals we have

$$L(\eta_2)(\lambda) dM_{\varphi_1}(\lambda) = L(\eta_1)(\lambda) dM_{\varphi_2}(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

Since $(L\eta_j)(\lambda) = |(L\varphi_j)(\lambda)|^2$, $j = 1, 2$, this is equivalent to identity (A.7). □
Now we define the measure \( dM(\lambda) \) on \( \mathbb{R} \) by the relation
\[
dM(\lambda) = |(L\varphi)(\lambda)|^{-2}dM(\lambda) . \tag{A.9}
\]
In view of Lemma A.2 this definition does not depend on the choice of the function \( \varphi \in C_0^\infty(\mathbb{R}_+) \).

**Lemma A.3.** The measure \((A.9)\) satisfies estimates \((5.3)\).

**Proof.** We proceed from estimates \((5.3)\) on the measures \(dM(\lambda)\). Observe that if \( \varphi \in C_0^\infty(\mathbb{R}_+) \), \( \varphi \neq 0 \), \( \varphi(t) \geq 0 \) for \( t < t_0/2 \) and \( \varphi(t) = 0 \) for \( t \geq t_0/2 \), then
\[
(L\varphi)(\lambda) \geq c(\varphi)e^{-t_0\lambda/2}, \quad c(\varphi) = \int_0^\infty \varphi(t)dt > 0. \tag{A.10}
\]
It follows from \((A.9)\) that
\[
\int_0^\infty e^{-t\lambda}dM(\lambda) \leq c(\varphi)^{-2} \int_0^\infty e^{-(t-t_0)\lambda}dM(\lambda).
\]
For an arbitrary \( t > 0 \), we can choose \( t_0 \) so small that \( t-t_0 > 0 \), and hence the integral on the right is convergent. This yields the first estimate \((5.3)\) for the measure \(dM(\lambda)\).

The second estimate \((5.3)\) is even simpler because if \( \varphi \in C_0^\infty(\mathbb{R}_+) \), \( \varphi \neq 0 \) and \( \varphi(t) \geq 0 \), then \((L\varphi)(-\lambda) \geq c(\varphi)\) with \(c(\varphi)\) defined in \((A.10)\) for all \( \lambda \geq 0 \). It follows again from \((A.9)\) that
\[
\int_0^\infty e^{t\lambda}dM(-\lambda) \leq c(\varphi)^{-2} \int_0^\infty e^{t\lambda}dM(\lambda)
\]
where the integral on the right is convergent for an arbitrary large \( t > 0 \).

It remains to verify representation \((5.4)\), or equivalently \((5.4)\), for the measure \(dM(\lambda)\) defined by relation \((A.9)\).

**Lemma A.4.** Representation \((5.4)\) is true for functions \( F = \eta \ast f \) where \( \eta \) is function \((A.1)\) and \( f \in C_0^\infty(\mathbb{R}_+) \) is arbitrary.

**Proof.** It follows from relations \((A.3)\), \((A.6)\) and definition \((A.9)\) of the measure \(dM(\lambda)\) that
\[
\langle h, F \rangle = \langle h_{\varphi}, f \rangle = \int_{-\infty}^\infty (Lf)(\lambda) dM_{\varphi}(\lambda) = \int_{-\infty}^\infty (Lf)(\lambda)|L\varphi(\lambda)|^2dM(\lambda). \tag{A.11}
\]
According to property \((2.8)\) of the Laplace transform we have
\[
(LF)(\lambda) = (Lf)(\lambda)|L\varphi(\lambda)|^2.
\]
Therefore equality \((A.11)\) yields \((5.4)\).

Finally, we extend representation \((5.4)\) to all functions \( F \in C_0^\infty(\mathbb{R}_+) \). Let us choose \( \omega = \bar{\omega} \in C_0^\infty(\mathbb{R}_+) \) such that \( \int_0^\infty \omega(t)dt = 1 \) and set
\[
\varphi_\varepsilon(t) = \varepsilon^{-1}\omega(\varepsilon^{-1}t). \tag{A.12}
\]
It follows from Lemma [A.4] that
\[ \langle h, \varphi_\varepsilon \ast \varphi_\varepsilon \ast F \rangle = \int_{-\infty}^{\infty} (LF)(\lambda) |(L\varphi_\varepsilon)(\lambda)|^2 dM(\lambda). \] (A.13)

Let us pass here to the limit \( \varepsilon \to 0 \).

Lemma A.5. Let \( F \in C_0^\infty(\mathbb{R}_+) \). Then
\[ \varphi_\varepsilon \ast \varphi_\varepsilon \ast F \to F \] (A.14)
as \( \varepsilon \to 0 \) in the space \( C_0^\infty(\mathbb{R}_+) \).

Proof. Since
\[ (\varphi_\varepsilon \ast \varphi_\varepsilon)(t) = \varepsilon^{-1} \zeta(\varepsilon^{-1} t) \]
where \( \zeta = \omega \ast \omega \), we have
\[(\varphi_\varepsilon \ast \varphi_\varepsilon \ast F)(t) = \varepsilon^{-1} \int_0^t \zeta(\varepsilon^{-1} s) F(t-s) ds \]
\[ = \int_0^{t/\varepsilon} \zeta(\sigma) F(t-\varepsilon \sigma) d\sigma \to F(t) \int_0^{\infty} \zeta(\sigma) d\sigma = F(t) \]
as \( \varepsilon \to 0 \). Similar relations are of course also true for all derivatives in \( t \). Since the supports of \( \varphi_\varepsilon \) are small, the supports of all functions \( \varphi_\varepsilon \ast \varphi_\varepsilon \ast F \) are contained in a common interval \([t_1, t_2] \in \mathbb{R}_+ \). This leads to (A.14). \( \Box \)

Thus the left-hand side of (A.13) converges to the left-hand side of (5.4).

Lemma A.6. Let \( F \in C_0^\infty(\mathbb{R}_+) \). Then the right-hand side of (A.13) converges to the right-hand side of (5.4).

Proof. It follows from (A.12) that
\[ (L\varphi_\varepsilon)(\lambda) = \int_0^{\infty} e^{-\varepsilon \lambda s} \omega(s) ds, \] (A.15)
and hence \( (L\varphi_\varepsilon)(\lambda) \to 1 \) as \( \varepsilon \to 0 \) for all \( \lambda \in \mathbb{R} \). Moreover, if \( \text{supp} \omega \in [\omega_1, \omega_2] \), then function (A.15) is bounded by \( C e^{-\varepsilon \lambda \omega_1} \) for \( \lambda \geq 0 \) and by \( C e^{\varepsilon \lambda |\omega_2|} \) for \( \lambda \leq 0 \). Recall also that the measure \( dM(\lambda) \) satisfies estimates (5.2). Thus by the dominated convergence theorem, the right-hand side of (A.13) converges as \( \varepsilon \to 0 \) to the right-hand side of (5.4). \( \Box \)

Putting together relation (A.13) with Lemmas A.5 and A.6, we conclude the proof of Theorem 5.1.
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