Embedding the weighted space $Hv_0(G, E)$ of holomorphic functions into the sequence space $c_0(E)$

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Abstract

We embed almost isometrically the generalized weighted space $Hv_0(G, E)$ of holomorphic functions on an open subset $G$ of $\mathbb{C}^N$ with values in a Banach space $E$, into $c_0(E)$, the space of all null sequences in $E$, where $v$ is an operator-valued continuous function on $G$ vanishing nowhere. This extends and generalizes some known results in the literature. We then deduce the non 1-Hyers-Rassias stability of the isometry functional equation in the framework of Banach spaces.

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1. Introduction

A interesting issue when studying Banach spaces is whether such a space can be embedded isometrically into a simpler Banach space. Such a problem has been considered by several authors, especially in weighted spaces of holomorphic functions on an open subset of $\mathbb{C}$ [2,3,9–11].

The first author who dealt with embedding weighted spaces of holomorphic functions on an open subset of $\mathbb{C}$ into sequence spaces seems to be W. Lusky [9]. There, the author showed that, whenever $G$ is the unit open disc $D$ of $\mathbb{C}$ and $v$ is a radial (i.e. $v(z) = v(|z|), z \in D$) strictly positive continuous function on $D$, the Banach space $Hv_0(D)$ of all holomorphic functions $f$ on $D$ such that $v|f|$ vanishes at infinity, endowed with the weighted sup-norm $\| \cdot \|_v$, is always isomorphic to a subspace of $c_0$. He then showed in [10] that there are weights $v$ such that $Hv_0(D)$ is not isomorphic to the whole $c_0$. Actually, as Lusky showed in [11], there are exactly two situations in such a case: either $Hv_0(D)$ is isomorphic to $\ell_\infty$ or it is isomorphic to the Hardy space $H_\infty \subset c_0$. He even gave instances where each situation occurs.

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Concerning the case of several variables, J. Bonet and E. Wolf extended in [2] the result of Lusky to the case where $G$ is an arbitrary open subset of $\mathbb{C}^N$, $N$ being a positive integer, without any further condition on the weight $v$. They showed that if $v$ is any strictly positive continuous function on a nonempty open set $G \subset \mathbb{C}^N$, then $H_{v_0}(G)$ is almost isometrically isomorphic to a subspace of $c_0$. This means that, for every $\varepsilon \in [0,1]$, there is an isomorphism $T$ from $H_{v_0}(G)$ into $c_0$ such that:

$$
(1 - \varepsilon) \|f\|_v \leq \|T(f)\|_{c_0} \leq \|f\|_v, \quad (\forall f \in H_{v_0}(G)).
$$

This seems to be the maximum one can obtain in general since, in [3], C. Boyd and P. Rueda showed that, whenever $G \subset \mathbb{C}^N$ is balanced and $v$ is radial, the isomorphism of $H_{v_0}(G)$ into $c_0$ cannot be an isometry.

Recently, C. Shekhar and B. S. Komal [14] and subsequently M. Klilou and L. Oubbi [7] introduced systems $V$ of weights with values in the set of positive operators on a Hilbert space $H$. They then studied some questions concerning multiplication operators in the corresponding weighted spaces of continuous functions $CV(G, H)$. This study has been enlarged to weights with values in continuous operators on a normed space [8].

In this note, we deal with the question whether for a nonempty open subset $G$ of $\mathbb{C}^N$, a Banach space $E$, and a continuous mapping $v$ from $G$ into the algebra $B(E)$ of bounded operators on $E$, the weighted space $H_{v_0}(G, E)$ can be embedded into the space $c_0(E)$ of all null sequences of $E$. We mainly show that, if $v$ is continuous with respect to the norm topology on $B(E)$ and takes values in the bounded below operators on $E$, then the Banach space $H_{v_0}(G, E)$, endowed with the weighted sup-norm

$$
\|f\|_v := \sup\{\|v(z)(f(z))\|, z \in G\},
$$

is almost isometrically isomorphic to a closed subspace of the space $c_0(E)$. This extends and generalizes the result, alluded to above, of J. Bonet and E. Wolf [2].

We obtain as an application, the non 1-Hyers-Rassias stability of the isometry functional equation $\|f(x)\| = \|x\|$ between Banach spaces.

2. Preliminaries

Let $N$ be a positive integer, $G$ a nonempty open subset of $\mathbb{C}^N$, and $(E, \|\|)$ a Banach space. We write $z$ for $z = (z_1, \ldots, z_N) \in G$ and $z_j = x_j + iy_j$ for $j = 1, \ldots, N$. We will denote by $\mathbb{N}$ the set of all non-negative integers, by $\mathbb{N}^*$ the set $\mathbb{N} \setminus \{0\}$, and by $c_0(E)$ the linear space of all null sequences of $E$. The space $c_0(E)$ will be endowed with its natural sup-norm.

We first recall some facts related to holomorphic functions. We refer to [6] and [13] for ample details.

**Definition 2.1** ([13]). A function $f : G \to \mathbb{C}$ is said to be holomorphic in $G$ provided

1. $f$ is continuous (i.e., $f \in C(G)$), and
2. $f$ is holomorphic in each variable separately.

If $f$ is continuously differentiable in the variables $x_j$ and $y_j$, $j = 1, \ldots, N$, it is said to be holomorphic in $G$ in the Cauchy-Riemann sense (see [6, Definition 2.1.1]) if

$$
\frac{\partial f}{\partial \bar{z}_j} = 0, \quad (1 \leq j \leq N),
$$

in $G$, where

$$
\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).
$$

The following theorem is due to Hartogs [6, Theorem 2.2.8].
Theorem 2.2. Let $f$ be a function from $G$ to $\mathbb{C}$. The following properties are equivalent:
1. $f$ is holomorphic in $G$.
2. $f$ is holomorphic in the Cauchy-Riemann sense.

Theorem 2.3. [12, p. 400, Theorem 8] Let $f$ be a function from $G$ into $E$. The following properties are equivalent:
1. The $\mathbb{C}$-valued function $\varphi \circ f$ is holomorphic in $G$ for each $\varphi$ in the topological dual $E'$ of $E$.
2. For every $w \in G$, there exists a neighborhood $U$ of $w$ and elements $x_\alpha \in E$ with $\alpha \in \mathbb{N}^N$ such that $f(z) = \sum_{\alpha \in \mathbb{N}^N} x_\alpha (z-w)^\alpha$.
3. $f$ is holomorphic in each variable separately in the sense described in 1.

We will denote by $H(G, E)$ the linear space of all $E$-valued functions on $G$ satisfying one of (and then all) the assertions in Theorem 2.3, while $C(G, E)$ will denote the space of all continuous functions from $G$ into $E$.

For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. For $w \in \mathbb{C}^N$, we will let $D(w, r)$ denote the polydisc $D(w, r) := \{ z \in \mathbb{C}^N : |z_k - w_k| \leq r, k = 1, \ldots, N \}$. We then have the Cauchy integral formula [12, p. 400]. Let $f \in H(G, E)$, $w \in G$, and $r > 0$ such that $D(w, r) \subset G$. Then

$$f(w) = \frac{1}{(2\pi i)^N} \int_{\partial D(w, r)} \frac{f(z)}{(z_1 - w_1) \cdots (z_N - w_N)} dz_1 \cdots dz_N. \quad (2.1)$$

Therefore, for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ with $|\alpha| = 1$,

$$D^\alpha f(w) = \frac{1}{(2\pi i)^N} \int_{\partial D(w, r)} \frac{f(z)}{(z - w)^{\alpha+1}} dz_1 \cdots dz_N.$$

We will denote by $B(E)$ the Banach algebra of all bounded linear operators from $E$ into itself. The strong operator (resp. the norm) topology on $B(E)$ will be denoted by $\beta$ (resp. $\sigma$).

Recall that a linear mapping $T$ from the Banach space $E$ into another one $F$ is said to be bounded below if there exists $r > 0$ such that $r\|x\|_E \leq \|T(x)\|_F$, for every $x \in E$. We will denote by $\mathcal{L}_{bb}(E)$ the subset of $B(E)$ consisting of all continuous and bounded below operators.

A mapping $\mu : G \to E$ is said to vanish at infinity if for every $\varepsilon > 0$, there is a compact subset $K_\varepsilon \subset G$ such that $\|\mu(z)\| < \varepsilon$ for all $z \notin K_\varepsilon$.

Here we consider generalized Nachbin families consisting of a single weight. Unlike [14] and [7], we no more consider Hilbert spaces but arbitrary Banach spaces.

Definition 2.4. A generalized weight on $G$ is any $\beta$-continuous mapping $v : G \to B(E)$ such that $v(z)$ is injective for every $z$ in some dense subset $G_0$ of $G$. The weight $v$ is said to be equibounded below on a subset $A$ of $G$ if there is $r = r_A > 0$ such that $r\|x\| \leq \|v(z)x\|$ for every $z \in A$ and every $x \in E$.

With a generalized weight $v$ on $G$ are associated the following weighted spaces:

$$Cv_0(G, E) := \{ f \in C(G, E), vf : z \mapsto v(z)(f(z)) \text{ vanishes at infinity on } G \}$$

$$Hv_0(G, E) := \{ f \in H(G, E), vf : z \mapsto v(z)(f(z)) \text{ vanishes at infinity on } G \}.$$
the continuity of \( f \), there exists a neighborhood \( \Omega \) of \( z_0 \) and \( \varepsilon > 0 \) such that \( \| f(z) \| > \varepsilon \) for every \( z \in \Omega \). But the density of \( G_0 \) in \( G \) yields some \( z_1 \in G_0 \) so that \( \| f(z_1) \| > \varepsilon \). Now, \( v(z_1) \) is injective, then \( v(z_1)(f(z_1)) \neq 0 \). Hence \( \| f \|_v \neq 0 \) and \( (Hv_0(G,E), \| \cdot \|_v) \) is a normed space. From now on, \( Hv_0(G,E) \) will be endowed with this norm.

Whenever \( u \) is a strictly positive continuous function on \( G \), if we consider on \( E \) the operator \( T_z : x \mapsto u(z)x \), then the mapping \( v : z \mapsto T_z \) is a generalized weight on \( G \) and the generalized weighted space \( Hv_0(G,E) \) is nothing but the usual weighted space \( H u_0(G,E) \) algebraically and topologically. Obviously, if \( E \) is the complex field, \( Hv_0(G,E) \) is nothing but \( H v_0(G) \). More generally, we have the following.

**Example 2.5.** Let \( u : G \to (0,\infty) \) be a continuous mapping vanishing nowhere on \( G \) and \( T \) \( \in B(E) \). If \( T \) is injective, then the mapping \( v : z \mapsto u(z)T \) is a generalized weight on \( G \). Moreover, if \( T \) is bounded below, then \( v \) is equibounded below on every compact subset of \( G \). Indeed, let \( K \) be such a compact set. Since \( T \) is bounded below, there is \( r > 0 \) such that \( \| T(x) \| \geq r \| x \| \) for all \( x \in E \). Therefore, for each \( z \in K \) and \( x \in E \), we have

\[
\| v(z)x \| = u(z)\| T(x) \| \geq u(z)r\| x \| \geq \inf_{z \in K} u(z)\| z \| \| x \| ,
\]

whence the result.

**Example 2.6.** Let \( v \) be a generalized weight on \( G \) and \( q \) be the real function assigning to any \( z \in G \) the minimum modulus \( \mu(v(z)) \) of \( v(z) \) \([5]\), where

\[
\mu(v(z)) := \inf \{ \| v(z)x \|, \| x \| = 1 \}.
\]

If \( q \) is lower semi-continuous and does not vanish on \( G \), then \( v \) is equibounded below on each compact subset \( K \) of \( G \). Indeed, if \( r_K = \inf \{ q(z), z \in K \} \), then \( \| v(z)x \| \geq r_K \) for all \( z \in K \) and all \( x \in E \), hence \( \| v(z)x \| \geq r_K \| x \| \) for all \( x \in E \).

**Proposition 2.7.** Let \( v : G \to B(E) \) be a generalized weight on \( G \). If \( v \) is equibounded below on the compact subsets of \( G \), then the space \( Hv_0(G,E) \), endowed with the norm \( \| \cdot \|_v \), is a Banach space.

**Proof.** The space \( Cv_0(G,E) \) is a Banach space by Theorem 3.3 of \([7]\). Then it is sufficient to prove that \( Hv_0(G,E) \) is a closed subspace of \( Cv_0(G,E) \). Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence in \( Hv_0(G,E) \) converging to some function \( f \) in \( Cv_0(G,E) \). Fix a compact \( K \subset G \) and \( \varepsilon > 0 \). Since \( v \) is equibounded below on \( K \), there exists \( r_K > 0 \), such that \( \| x \| < r_K \| v(z)(x) \| \), \( z \in K \), \( x \in E \). But also, there exists \( N \in \mathbb{N} \), such that, for all \( n \geq N \), \( \| f_n - f \|_v < r_K^{-1} \varepsilon \). Therefore, for every \( z \in G \), we have

\[
\| f_n(z) - f(z) \| < r_K \| v(z)(f_n(z) - f(z)) \| \leq r_K \| f_n - f \|_v < \varepsilon .
\]

Hence \( (f_n)_{n \in \mathbb{N}} \) converges to \( f \) uniformly on every \( K \). Then \( f \) is holomorphic. Since \( f \in Cv_0(G,E) \), \( f \in Hv_0(G,E) \).

3. Embedding \( Hv_0(G,E) \) into \( c_0(E) \)

Our main result gives instances where \( Hv_0(G,E) \) is almost isometrically isomorphic to a subspace of \( c_0(E) \), extending and generalizing a result of \([2]\). From now on, let us denote by \( d \) the sup-norm metric on \( \mathbb{C}^N \). This is \( d(z,w) := \max_{i=1,...,N} |z_i - w_i| \), \( z,w \in \mathbb{C}^N \).

**Theorem 3.1.** Let \( E \) be a Banach space, \( G \) a nonempty open subset of \( \mathbb{C}^N \), and \( v : G \to B(E) \) a generalized weight. If \( v \) is \( \sigma \)-continuous and maps \( G \) into \( \mathcal{L}_0(E) \), then the space \( Hv_0(G,E) \) is almost isometrically isomorphic to a closed subspace of \( c_0(E) \).
Proof. Fix $\varepsilon \in ]0, 1[$ and consider an exhaustion of $G$ by an increasing sequence $(K_k)_{k \in \mathbb{N}}$ of compact subsets of $G$. Since $v$ is continuous, $M_k := \sup \{ \| v(z) \|_{B(E)} : z \in K_k \} < +\infty$. Set

$$a_k := \min \left( \frac{1}{2} d(K_k, C^N \setminus K_{k+1}) \right).$$

As $v(G) \subseteq \mathcal{L}_{bb}(E)$, for all $z \in K_k$, there exists $r_z > 0$ such that $r_z \| x \| \leq \| v(z) x \|$, for all $x \in E$. By Theorem 2.1 of [1], $\mathcal{L}_{bb}(E)$ is open in $(B(E), \sigma)$. Then, by the $\sigma$-continuity of $v$, there exists a neighborhood $U_z$ of $z$ in $G$, such that $v(U_z) \subseteq B(v(z), \frac{r_z}{2}) \cap \mathcal{L}_{bb}(E)$, $B(v(z), \frac{r_z}{2})$ being the ball in $(B(E), \sigma)$ centered at $v(z)$ with radius $\frac{r_z}{2}$. Then for all $w \in U_z$, we have

$$\| v(z)x \| - \| v(w)x \| \leq \| v(w)x \|, \quad \forall x \in E.$$ 

Hence

$$\| v(z)x \| - \| v(z) - v(w) \| \| x \| \leq \| v(w)x \|, \quad \forall x \in E.$$ 

Therefore

$$r_z \| x \| - \frac{r_z}{2} \| x \| = \frac{r_z}{2} \| x \| \leq \| v(w)x \|, \quad \forall x \in E.$$ 

Since $K_k$ is compact, we can find a finite set $\{ z_1, \ldots, z_n \} \subseteq K_k$ such that $K_k \subseteq \bigcup_{i=1}^n U_{z_i}$. Now, for $r_k = \inf_{i=1, \ldots, n} \frac{r_{z_i}}{2}$, we have

$$r_k \| x \| \leq \| v(z)x \|, \quad \forall z \in K_k, \quad \forall x \in E.$$ 

For arbitrary $f \in Hv_0(G, E)$ and $k \in \mathbb{N}$, with $\| f \|_v = 1$, we have

$$1 = \sup_{z \in G} \| v(z)f(z) \| \leq r_k \sup_{z \in K_k} \| f(z) \|.$$ 

Hence, for each $z \in K_k$, the following inequality holds:

$$\| f(\zeta) \| \leq \frac{1}{r_{k+1} a_k}, \quad \forall \zeta \in D(z, a_k).$$

(3.1)

For $\alpha \in \mathbb{N}^N$ with $|\alpha| = 1$, we have

$$\| D^\alpha f(z) \| = \left\| \frac{1}{(2\pi i)^N} \int_{D(z, a_k)} \frac{f(\zeta)}{(\zeta - z)^{a_1 + 1}} d\zeta_1 \ldots d\zeta_N \right\| \leq \frac{1}{r_{k+1} a_k}.$$ 

(3.2)

Whereby

$$\| D^\alpha f(z) \| \leq \frac{1}{r_{k+1} a_k}, \quad \forall z \in K_k.$$ 

(3.3)

If $A_k := K_k \setminus K_{k-1}$ and $\delta_k > 0$ satisfy

$$\delta_k < \min \left( a_k, \varepsilon \left( \frac{1}{r_k a_k} + \frac{M_{k+1} N}{a_k r_{k+1} k+2} \right)^{-1} \right),$$

(3.4)

then

$$A_k \subseteq \bigcup_{z \in A_k} \{ z' \in G, d(z', z) < \delta_k \text{ and } ||v(z') - v(z)|| < \delta_k \}. $$

By the compactness of $A_k$, there is a finite subset $F_k$ of $A_k$ such that

$$A_k \subseteq \bigcup_{z \in F_k} \{ z' \in G, d(z', z) < \delta_k \text{ and } ||v(z') - v(z)|| < \delta_k \}. $$

Consequently, for each $z \in A_k$, there is $w \in F_k$ with $d(w, z) < \delta_k$ and $\| v(w) - v(z) \| < \delta_k$. On the other hand, $D(z, \delta_k) \subseteq D(z, a_k) \subseteq K_{k+1}$. This implies

$$\| v(z)f(z) \| \leq \| v(z)f(z) - v(w)f(z) \| + \| v(w)f(z) \|$$

$$\leq \| v(z) - v(w) \| || f(z) || + \| v(w)(f(z) - f(w)) \| + \| v(w)f(w) \|.$$ 

(3.5)
We then have, denoting \( \alpha_i = (0, \ldots, 1, 0, \ldots) \), where 1 is in the \( i \)th place:

\[
\|f(z) - f(w)\| = \|f(z_1, \ldots, z_N) - f(w_1, \ldots, w_N)\|
\leq \|f(z_1, \ldots, z_N) - f(w_1, z_2, \ldots, z_N) + f(w_1, z_2, \ldots, z_N) - f(w_1, w_2, \ldots, w_{N-1}, z_N) - f(w_1, \ldots, w_N)\|
\leq \sup_{z \in \mathcal{D}(z, \delta_k)} \|D^N f(\zeta)\| |z_1 - w_1| + \cdots + \sup_{z \in \mathcal{D}(z, \delta_k)} \|D^N f(\zeta)\| |z_N - w_N|.
\]

Taking (3.3) into consideration, it follows that

\[
\|f(z) - f(w)\| \leq \frac{N \delta_k}{a_{k+1}^r k+2}.
\]

(3.6)

Now, since \( w \) belongs to \( K_{k+1} \), it follows from (3.1), (3.4), (3.5), and (3.6) that

\[
\sup_{z \in A_k} \|v(z) f(z)\| \leq \varepsilon + \max_{w \in F_k} \|v(w) f(w)\|.
\]

Setting \( F := \cup \{F_k, k \in \mathbb{N}\} \), we conclude

\[
1 \leq \varepsilon + \sup_{w \in F} \|v(w) f(w)\|.
\]

Denote the elements of \( F \) as a sequence \( (z_n)_{n \in \mathbb{N}} \subset G \). Then \( z_n \) tends to the boundary \( \partial G \) of \( G \), i.e., for each \( k \in \mathbb{N} \), there is \( n_0 \in \mathbb{N} \) such that \( z_n \notin K_k \) for every \( n > n_0 \). Since \( F \) does not depend on the function \( f \), the correspondence \( g \mapsto (v(z_n) g(z_n))_{n \in \mathbb{N}} \) defines an operator \( T \) from \( Hv_0(G,E) \) into \( c_0(E) \). Now, if \( g \in Hv_0(G,E) \) with \( g \neq 0 \), we have

\[
\left\| \frac{g}{\|g\|_v} \right\|_v = 1 \leq \varepsilon + \left\| T\left(\frac{g}{\|g\|_v}\right) \right\|_{c_0(E)}.
\]

Thus

\[
(1 - \varepsilon) \|g\|_v \leq \|T(g)\|_{c_0(E)}.
\]

Since \( \|T(f)\|_{c_0(E)} = \sup\{\|v(w) f(w)\|, w \in F\} \leq \|f\|_v \), we obtain

\[
(1 - \varepsilon) \|f\|_v \leq \|T(f)\|_{c_0(E)} \leq \|f\|_v, \quad \forall f \in Hv_0(G,E)
\]

showing that \( T \) is an almost isometry. \( \square \)

**Remark 3.2.** It comes out from the proof of Theorem 3.1 that \( v \) is equibounded below on compact subsets of \( G \) if and only if its range lies in \( \mathcal{L}_{bb}(E) \).

If \( u : G \to (0, +\infty) \) is continuous, \( T \) is the identity of \( E \), and \( v := uT \), as in Example 2.5, we get, as a corollary, the vector-valued version of J. Bonet and E. Wolf’s theorem.

**Corollary 3.3.** Let \( E \) be a Banach space, \( G \) a nonempty open subset of \( \mathbb{C}^N \), and \( u \) a strictly positive and continuous weight on \( G \). Then the space \( Hu_0(G,E) \) is isomorphic to a closed subspace of \( c_0(E) \). Actually, \( Hu_0(G,E) \) embeds almost isometrically into \( c_0(E) \).

In case \( E = \mathbb{C} \), Corollary 3.3 is nothing but the result of J. Bonet and E. Wolf [2].

**Corollary 3.4.** Let \( G \) be an open subset of \( \mathbb{C}^N \) and \( v \) be a strictly positive and continuous weight on \( G \). Then the space \( Hv_0(G) \) embeds almost isometrically into \( c_0 \).

Notice that, for every nonzero \( x \in E \) and every generalized weight \( v \) on \( G \), the mapping \( v_x : z \mapsto \|v(z)(x)\| \) is continuous. Therefore the normed weighted space \( Hv_{v_0}(G) := Hv(v_0(G)) \) is complete provided \( v(z) \) is injective for every \( z \in G \). Since the correspondence \( f \mapsto x \otimes f \) is an isometry from \( Hv_{v_0}(G) \) into \( Hv_0(G,E) \), the space \( Hv_{v_0}(G) \), identified with \( x \otimes Hv_{v_0}(G) := \{x \otimes f, f \in Hv_{v_0}(G)\} \), is a closed subspace of \( Hv_0(G,E) \), where \( (x \otimes f)(z) := f(z)x \) for every \( z \in G \) and every \( f \in Hv_{v_0}(G) \).

Now, recall the following result.
Lemma 3.5. [2, Corollary 2] Let $G$ be an open subset of $\mathbb{C}^N$, and let $v$ be a strictly positive and continuous weight on $G$. If the space $Hv_0(G)$ is infinite dimensional, then $Hv_0(G)$ is not reflexive.

We then obtain the following theorem extending the lemma above.

Theorem 3.6. Let $G$ be an open subset of $\mathbb{C}^N$, and let $v$ be a generalized weight on $G$ such that $v(z)$ is injective for every $z \in G$. If the space $Hv_0(G)$ is infinite dimensional for some $x \in E$, then $Hv_0(G,E)$ is not reflexive.

Proof. Since $x \otimes Hv_0(G)$ is a closed subspace of $Hv_0(G,E)$ and, by Lemma 3.5, $x \otimes Hv_0(G)$ is not reflexive, then $Hv_0(G,E)$ is not reflexive as well. □

As an application, we will show that the isometry equation $\|f(x)\| = \|x\|$ is not 1-Hyers-Rassias-stable. It is known that the Cauchy equation $f(x+y) = f(x) + f(y)$ satisfies $\|f\| = 1$ for every $p \neq 1$, see [4]. This means that, for every such $p$ and every real $\theta > 0$, if a function $f : X \to Y$ between Banach spaces $X$ and $Y$ satisfies $\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$, $x, y \in X$, then there exists a unique additive function $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\theta \epsilon_p}{2p-2} \|x\|^p$, $x \in X$, where $\epsilon_p = \text{sign}(p-1)$ is the sign of $p-1$. The same equation fails to be stable for $p = 1$, as shown in [4].

Here, we will show that the isometry functional equation $\|T(f)\| = \|f\|$, where $T$ is a (even linear) mapping from the Banach space $Hv_0(G)$ into $c_0$ is not 1-Hyers-Rassias stable as well. Indeed, let $\theta > 0$ be arbitrary; using Corollary 3.3, there exists a linear mapping $T_{\theta} : Hv_0(G) \to c_0$ such that

$$(1 - \theta)\|f\| \leq \|T_{\theta}(f)\| \leq \|f\|,$$

where $f \in Hv_0(G)$. It follows from this that $T_{\theta}$ is an approximate isometry, this is $\|T(f)\| - \|f\| \leq \theta(\|T_0(f)\| - \|f\|)$, $f \in Hv_0(G)$. However, for $G = \Delta$, the unit disc of $\mathbb{C}$, and a positive continuous and radial weight $v$ (i.e., $v(\lambda) = v(\lambda z)$ for every $\lambda \in \mathbb{T}$), there exists no isometry, at all, from $Hv_0(\Delta)$ into $c_0$ as shown in [3], Corollary 17. Hence there exists no isometry approximating $T_{\theta}$. Therefore the isometry equation fails to be 1-Hyers-Rassias stable.

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