A HECKE ALGEBRA ON THE DOUBLE COVER OF A CHEVALLEY
GROUP OVER $\mathbb{Q}_2$

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Abstract. We prove that a certain genuine Hecke algebra $H$ on the non-linear double
cover of a simple, simply-laced, simply-connected, Chevalley group $G$ over $\mathbb{Q}_2$ admits a
Bernstein presentation. This presentation has two consequences. First, the Bernstein
component containing the genuine unramified principal series is equivalent to $H$-mod.
Second, $H$ is isomorphic to the Iwahori-Hecke algebra of the linear group $G/Z_2$, where
$Z_2$ is the 2-torsion of the center of $G$. This isomorphism of Hecke algebras provides a
correspondence between certain genuine unramified principal series of the double cover of
$G$ and the Iwahori-unramified representations of the group $G/Z_2$.

1. Introduction

Let $G$ be a simple, simply-laced, simply-connected, Chevalley group over a $p$-adic field
$F$, where $F$ contains the $n$-th roots of unity $\mu_n$. Let $T \subset B \subset G$ be a maximal torus and
Borel subgroup, respectively. If $H$ is an algebraic group, we will write $H$ for the $F$-points of $H$. Let $\tilde{G}$ be the $n$-fold cover of $G$. The covering group $\tilde{G}$ fits into a central extension

$$1 \rightarrow \mu_n \rightarrow \tilde{G} \rightarrow G \rightarrow 1.$$  (1)

We fix an embedding $\varepsilon : \mu_n \rightarrow \mathbb{C}^\times$.

When $\text{GCD}(n,p) = 1$, the tame case, Savin [13, 14] studied the genuine Iwahori-Hecke
algebra of $\tilde{G}$ and its relation to the representation theory of $\tilde{G}$. Among his results is a
Bernstein presentation for the $\varepsilon$-genuine Iwahori-Hecke algebra, which yields the following
two consequences. First, the Bernstein component of $\varepsilon$-genuine unramified principal series
is equivalent to the category of modules of the $\varepsilon$-genuine Iwahori-Hecke algebra. Second,
there is an equivalence of categories between the Bernstein component of $\varepsilon$-genuine Iwahori-
unramified representations of $\tilde{G}$ and the Bernstein component of Iwahori-unramified rep-
resentations of $G' = G/Z_n$, where $Z_n$ is the $n$-torsion in the center of $G$. (For details on
Bernstein components see [2, 4].)

When $\text{GCD}(n,p) \neq 1$, the wild case, less is known. Loke-Savin [9] established the
analogous results when $G = \text{SL}(2, \mathbb{Q}_2)$ and $n = 2$. Wood [18] rephrased these results in
terms of the even Weil representation and extended them to the case of $G = \text{Sp}(2r, \mathbb{Q}_2)$ and
$n = 2$. Takeda-Wood [17] reproved the results of Wood for any 2-adic field, and proved an
analogous result for the odd Weil representation.

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At this point, we describe how the GCD\((n, p)\) affects matters. If GCD\((n, p)\) = 1, then \(pr^{-1}(I) \cong I \times \mu_n\), where \(I\) is an Iwahori subgroup of \(G\). Thus \(I\) can be embedded as a subgroup in \(\tilde{G}\). We will say that \(I\) splits sequence (1) and call the embedding a splitting of \(I\) into \(\tilde{G}\). Consequently, \(\tilde{G}\) possesses an \(\varepsilon\)-genuine Iwahori-Hecke algebra. When GCD\((n, p)\) \(\neq 1\), \(I\) need not split sequence (1), leaving no obvious candidate for the analog of the \(\varepsilon\)-genuine Iwahori-Hecke algebra of \(\tilde{G}\). However, Loke-Savin [9] identify a suitable alternative when \(G = SL(2, \mathbb{Q}_2)\) and \(n = 2\).

In this paper we extend the ideas of Loke-Savin [9] to the double cover of higher-rank simple, simply-laced, simply-connected, Chevalley groups over \(\mathbb{Q}_2\). In particular, \(n = 2\) and \(F = \mathbb{Q}_2\). (Since \(n = 2\), \(\varepsilon\) is unique and will be omitted.)

We begin by identifying a Hecke algebra \(H\) to replace the Iwahori-Hecke algebra. The definition of \(H\) involves the following compact open subgroups and representation. Let \(\Gamma_0(4)\) be the pre-image of \(B(\mathbb{Z}/4\mathbb{Z})\) under the mod 4 reduction map \(G(\mathbb{Z}_2) \to G(\mathbb{Z}/4\mathbb{Z})\). Let \(\Gamma_1(4)\) be the pre-image of the unipotent radical of \(B(\mathbb{Z}/4\mathbb{Z})\). It is crucial that \(\Gamma_1(4)\) splits sequence (4) (Theorem 3.3). Fix \((\tau, E)\), an irreducible genuine \(pr^{-1}(\Gamma_0(4))\)-representation that is trivial on the image of the splitting of \(\Gamma_1(4)\). The principal Hecke algebra of the present paper is

\[
H \overset{\text{def}}{=} \{ f \in C_c^\infty(\tilde{G}, \text{End}(E)) | f(k_1 g k_2) = \tau(k_1) f(g) \tau(k_2), \text{ for all } k_1, k_2 \in pr^{-1}(\Gamma_0(4)) \}.
\]

In Theorem 6.15, we prove that \(H\) admits a Bernstein presentation. As in the tame case, this presentation yields two consequences.

**Theorem 1.1.** (See Theorem 7.1.) Let \((\pi, \mathcal{V})\) be a smooth \(\tilde{G}\)-representation that is generated by its \(\tau\)-isotypic vectors

1. Every subquotient of \((\pi, \mathcal{V})\) is generated by its \(\tau\)-isotypic vectors.
2. If \((\pi, \mathcal{V})\) is irreducible, then there is an unramified character \(\chi : T \to \mathbb{C}^\times\) such that \((\pi, \mathcal{V})\) is isomorphic to an irreducible subquotient of \(\text{Ind}_{\tilde{B}}^{\tilde{G}}(i(\chi))\). (The definition of \(i(\chi)\) can be found in Subsection 2.5.)

Let \(I'\) be an Iwahori subgroup of \(G' = G/\mathbb{Z}_2\).

**Theorem 1.2.** (See Theorem 8.1.) The Bernstein presentation of \(H\) induces an algebra isomorphism between \(H\) and the Iwahori-Hecke algebra of \(G'\). This isomorphism induces an equivalence of categories between the category of smooth \(G'\)-representations generated by their \(I'\)-fixed vectors and the category of smooth \(\tilde{G}\)-representations generated by their \(\tau\)-isotypic vectors.

Our approach to achieve these results is indebted to Savin [14] and Loke-Savin [8, 9].

Now we describe the contents of this paper in more detail. In Section 2 we establish notation.

Section 3 contains a proof of the splitting of \(\Gamma_1(4)\) (Theorem 3.3). In addition to existence, we prove an important technical result (Proposition 3.4) required to show that certain functions, which constitute a basis of the Hecke algebra, are well-defined.
In Section 4 we isolate some double coset calculations, which provide a point of departure for our study of \( \mathcal{H} \). The main result in this section (Proposition 4.1) describes a preferred representative of each double coset.

Section 5 introduces \( \mathcal{H} \) a Hecke algebra on the linear group \( G \). This linear Hecke algebra is significant for two reasons. First, multiplication in \( \mathcal{H} \) is related to multiplication in \( \mathcal{H}_\mathbb{C} \), where the existence of a particular algebra homomorphism can simplify computations. This can be seen in Proposition 6.9. Second, we can use \( \mathcal{H} \) to prove Theorem 5.6. This theorem, which was pointed out to me by Gordan Savin, provides a starting point for showing that \( \mathcal{H} \) has an Iwahori-Matsumoto presentation (Theorem 6.12).

Section 6, the principal section of this paper, contains our study of the Hecke algebra \( \mathcal{H} \), which culminates in a Bernstein presentation of \( \mathcal{H} \) (Theorem 6.15). Now we will briefly outline the argument.

First, We begin with support calculations (Proposition 6.4, Proposition 6.7), which ultimately allow us to construct a \( \mathbb{C} \)-basis of the Hecke algebra (Proposition 6.8). This step breaks up into two pieces. First, we must show that certain double cosets do not support any functions in the Hecke algebra. Second, we construct a basis for \( \mathcal{H} \), invoking Proposition 3.4 to prove that the basis functions are well defined.

Second, we show that \( \mathcal{H} \) satisfies the braid relations (Proposition 6.9) and quadratic relations (Proposition 6.11). Using these relations and standard facts about affine Hecke algebras we prove that \( \mathcal{H} \) admits an Iwahori-Matsumoto presentation (Proposition 6.12).

Third, we apply the results of Lusztig [10] to prove that \( \mathcal{H} \) satisfies the Bernstein relations, and use Savin’s trick (Lemma 7.6, [14]) to show that these relations imply all others. This completes the proof of Theorem 6.15.

Section 7 contains a proof that the Bernstein component containing the genuine unramified principal series is equivalent to \( \mathcal{H} \)-mod (Theorem 7.1).

In Section 8, we describe the resulting local Shimura correspondence. The presentation proved in Theorem 6.15 agrees with the Bernstein presentation of an affine Hecke algebra. This leads to an isomorphism between \( \mathcal{H} \) and the Iwahori-Hecke algebra of \( G/\mathbb{Z}_2 \), which yields the desired Shimura correspondence (Theorem 8.1).

The construction of the isomorphism between \( \mathcal{H} \) and the Iwahori-Hecke Algebra depends on several choices. We conclude the present work by enumerating these choices.

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2. Notation

2.1. Root System. Let \((\Phi, \Delta, \mathcal{E})\) be a reduced irreducible simply-laced root system in the Euclidean vector space \( \mathcal{E} \) with roots \( \Phi \) and simple roots \( \Delta \). Let \( r = |\Delta| \). We write \( \Phi^+ \) for the set of positive roots and \( \Phi^- = -\Phi^+ \) for the negative roots. Let \( \varrho = \frac{1}{r} \sum_{\alpha \in \Phi^+} \alpha \).

For \( \alpha, \beta \in \Phi \) we say that \( \alpha \succ \beta \) if \( \alpha - \beta \) is a sum of positive roots. Let \( \alpha_1 \ldots, \alpha_r \) be an
enumeration of the simple roots and let $\alpha_0$ be the lowest root in $\Phi$. Associated with this root system there is a semi-simple complex Lie algebra $\mathfrak{g}$.

### 2.2. Chevalley Group

By choosing a Chevalley basis for $\mathfrak{g}$ we can construct $G$, the associated simply connected Chevalley group over $\mathbb{Z}$ with maximal torus $T$. Let $B \supset T$ denote the Borel subgroup associated to $\Delta$ with unipotent radical $U$; let $B_-$ denote the Borel subgroup opposite to $B$ with unipotent radical $U_-$. Let $N$ denote the normalizer of $T$ in $G$, and let $W$ be the Weyl group of $G$ with respect to $T$.

The torus $T$ has a group of rational characters $X = X^*(T)$ and cocharacters $Y = X_*(T)$. Let $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{Z}$ be the Killing form normalized so that the roots have length 2. Using the Killing Form we can identify $X$ and $Y$ and the root $\alpha$ is identified with the coroot $\alpha^\vee$ (since our root system is simply-laced). This identification induces a bilinear form on $Y$, also denoted by $\langle \cdot, \cdot \rangle$. We define the modified cocharacter lattice to be $\bar{Y} = \{ y \in Y | \langle y, y' \rangle \in 2\mathbb{Z} \text{ for all } y' \in Y \}$.

Let $G = G(\mathbb{Q}_2)$, the $\mathbb{Q}_2$-rational points of $G$. Similarly, we will write $T, U, U_-, B, B_-, \ldots$ for the $\mathbb{Q}_2$-points of $T, U, U_-, B, B_-, \ldots$ respectively.

We will be interested in the topological two-fold cover of $G$, but first we recall some facts about the universal central extension of $G$. Let $\text{St}(\Phi, \mathbb{Q}_2)$ denote the universal central extension of $G$ over $\mathbb{Q}_2$. The group $\text{St}(\Phi, \mathbb{Q}_2)$ is generated by the elements $x'_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathbb{Q}_2$, and subject to the relations

$$ x'_\alpha(t)x'_\beta(u) = x'_\alpha(t + u), $$

$$ [x'_\alpha(t), x'_\beta(u)] \overset{\text{def}}{=} x'_\alpha(t)x'_\beta(u)x'_\alpha(-t)x'_\beta(-u) = \begin{cases} 1, & \text{if } \alpha + \beta \notin \Phi; \\ x'_{\alpha + \beta}(c(\alpha, \beta)tu), & \text{if } \alpha + \beta \in \Phi, \end{cases} $$

where $c(\alpha, \beta) = \pm 1$. (Recall that $G$ is simply-laced.) For more details see [16], where Steinberg writes $G'$ instead of $\text{St}(\Phi, \mathbb{Q}_2)$.

Furthermore, for $\alpha \in \Phi$ and $t \in \mathbb{Q}_2^\times$ we let

$$ w'_\alpha(t) = x'_\alpha(t)x'_{-\alpha}(-t^{-1})x'_\alpha(t), $$

$$ h'_\alpha(t) = w'_\alpha(t)w'_\alpha(-1). $$

The work of Moore [12] and Matsumoto [11] provides a presentation for the kernel of the central extension

$$ 1 \to \text{Ker}(pr') \to \text{St}(\Phi, \mathbb{Q}_2) \xrightarrow{pr'} G \to 1. \quad (2) $$

The elements of the form $\{ t, u \} \overset{\text{def}}{=} h'_\alpha(t)h'_\alpha(u)h'_\alpha(tu)^{-1}$ generate $\text{Ker}(pr')$ and satisfy the relations described in Theorem 12 in [16], where Steinberg writes $f(t, u)$ for $\{ t, u \}$.

By [12], the push-out of sequence (2) via the quadratic Hilbert symbol $(\cdot, \cdot)_2 : \text{Ker}(pr') \to \mu_2 = \{ \pm 1 \}$ yields the group $\hat{G}$, the unique nontrivial topological two-fold central extension
of $G$. In particular, we have the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \text{Ker}(pr') & \longrightarrow & \text{St}(\Phi, \mathbb{Q}_2) & \longrightarrow & G & \longrightarrow & 1 \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1.
\end{array}
$$

(3)

For each $\alpha \in \Phi$ and $t \in \mathbb{Q}_2$ let $\tilde{x}_\alpha(t) = pr''(x'_\alpha(t))$. For $\alpha \in \Phi$ and $t \in \mathbb{Q}_2^*$ define $\tilde{w}_\alpha(t) = pr''(w'_\alpha(t))$ and $\tilde{h}_\alpha(t) = pr''(h'_\alpha(t))$. Similarly, let $x_\alpha(t) = pr(\tilde{x}_\alpha(t))$, $w_\alpha(t) = pr(\tilde{w}_\alpha(t))$, and $h_\alpha(t) = pr(\tilde{h}_\alpha(t))$.

For a ring $R$, let $U^*(R)$ be the subgroup of $\tilde{G}$ generated by the elements $\tilde{x}_\alpha(t)$, where $t \in R$ and $\alpha \in \Phi^+$; define $U^*_-(R)$ similarly. Let $T^*_1$ be the subgroup of $\tilde{G}$ generated by $\tilde{h}_\alpha(t)$, where $t \in 1 + 4\mathbb{Z}_2$ and $\alpha \in \Phi$, and let $T_1 = pr(\tilde{T}_1)$.

There are a few subgroups $J$ of $G$ which possess a splitting of the sequence

$$
1 \longrightarrow \mu_2 \longrightarrow \tilde{G} \stackrel{pr}{\longrightarrow} G \longrightarrow 1,
$$

(4)

in other words, a group homomorphism $f : J \rightarrow \tilde{G}$ such that $pr \circ f = \text{id}_H$. The following maps split sequence (4).

1. $U(R) \rightarrow U_1^*(R)$, defined by $x_\alpha(t) \mapsto \tilde{x}_\alpha(t)$, for $\alpha \in \Phi^+$, $t \in R$;
2. $U_-(R) \rightarrow U_1^*(R)$, defined by $x_\alpha(t) \mapsto \tilde{x}_\alpha(t)$, for $\alpha \in \Phi^-$, $t \in R$;
3. $T_1 \rightarrow T_1^*$, defined by $h_\alpha(t) \mapsto \tilde{h}_\alpha(t)$, for $\alpha \in \Phi$, $t \in 1 + 4\mathbb{Z}_2$.

Note that the Steinberg relations and the fact that $(2,2)_2 = 1$ imply that the subgroup of $\tilde{G}$ generated by $h_\alpha(2)$ for all $\alpha \in \Delta$ also splits the exact sequence (4) and is isomorphic to $Y$. For $\lambda = \sum_j c_j \alpha_j \in Y$, let $2^\lambda$ denote $\prod_j h_{\alpha_j}(2)^{c_\lambda}$. Let $\Upsilon : Y \rightarrow \tilde{G}$ be the map defined by $\lambda \mapsto 2^\lambda$. Note that $\Upsilon$ is a group isomorphism.

Let $W$ be the subgroup of $\tilde{G}$ generated by the elements $\tilde{w}_\alpha(1)$, where $\alpha \in \Phi$. Let $N'$ be the subgroup of $\tilde{G}$ generated by the elements $\tilde{w}_\alpha(1)$ and $2^\lambda$, where $\alpha \in \Phi$ and $\lambda \in \tilde{Y}$. Using the Steinberg relations one can show that $N' \cong W \ltimes \tilde{Y}$.

Consider the map $G(\mathbb{Z}_2) \rightarrow G(\mathbb{Z}_2/2^k\mathbb{Z}_2)$ defined by reduction modulo $2^k$. Let $\Gamma(2^k)$ be the kernel of this map. Let $\Gamma_0(2^k)$ be the inverse image of $B(\mathbb{Z}/2^k\mathbb{Z})$ and let $\Gamma_1(2^k)$ to be the inverse image of $U(\mathbb{Z}/2^k\mathbb{Z})$.

Given a subgroup $J \subseteq G$, let $\tilde{J} = pr^{-1}(J)$. A representation of $\tilde{J}$ is said to be genuine if $\mu_2 \subseteq \tilde{J}$ acts nontrivially.

2.3. Affine Weyl Group. Two affine Weyl groups are pertinent to our study. The first is $W_{aff}$, associated to the root system $(\Phi, \mathcal{E})$; the second will be an extended affine Weyl group $\tilde{W}_{aff}$ associated to $(\frac{1}{2}\Phi, \mathcal{E})$. We begin with $W_{aff}$.

Given a root system $(\Phi, \mathcal{E})$, there is an associated affine Weyl group $\tilde{W}_{aff}$ generated by the reflections through the hyperplanes $H_{\alpha, k} \overset{\text{def}}{=} \{ v \in \mathcal{E} | \langle \alpha, v \rangle = k \}$, where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Let $w_{\alpha, k}$ be the reflection fixing $H_{\alpha, k}$ defined by $w_{\alpha, k}(v) = v - \langle \alpha, v \rangle \alpha^\vee$. The
reflections $w_{\alpha_0-1}, w_{\alpha_1,0}, \ldots, w_{\alpha_r,0}$ are a set of Coxeter generators for $W_{\text{aff}}$. We will write $w_{\alpha_i}$ for $w_{\alpha_i,0}$, where $i = 1, \ldots, r$, and $w'_{\alpha_i}$ for $w_{\alpha_0-1}$.

The group $W_{\text{aff}}$ can be decomposed as a semi-direct product $W_{\text{aff}} \cong W \ltimes Y$ (since $Y$ is the lattice of coroots), and it can be realized as the quotient $\mathcal{N}(\mathbb{Q}_2)/\mathcal{T}(\mathbb{Z}_2)$. Furthermore, there is a section $s : W_{\text{aff}} \cong W \times Y \to \mathcal{N}(\mathbb{Q}_2)$ defined as follows. Let $w = w_{\alpha_{i_1}} \ldots w_{\alpha_{i_k}}$, be a minimal expression for $w$ in terms of the generators $w_{\alpha_1}, \ldots, w_{\alpha_r}$. Then we define $s((w, \lambda)) = w_{\alpha_{i_1}}(1) \ldots w_{\alpha_{i_k}}(1)2^\lambda$. This assignment can be shown to be independent of the minimal expression chosen for $w$ as follows. The Steinberg relations imply that the elements $w_{\alpha_i}(1)$ also satisfy the braid relations. Thus independence can be seen by adapting the proof of the Theorem in Section 29.4 in Humphreys [6].

Similarly, there is an (isomorphic) affine Weyl group $\tilde{W}_{\text{aff}}$ associated to the root system $(\frac{1}{2}\Phi, \mathcal{E})$, which can be decomposed as $\tilde{W}_{\text{aff}} \cong W \ltimes \tilde{Y}$. This group has a set of Coxeter generators given by $w_{a_1}, \ldots, w_{a_r}$, and $w_{a_0} \overset{\text{def}}{=} w_{a_0,-2}$.

The group $\tilde{W}_{\text{aff}}$ can be extended by the finite abelian group $\tilde{Y}/2\tilde{Y}$ to give the extended affine Weyl group $\tilde{W}_{\text{aff}} \overset{\text{def}}{=} W \ltimes \tilde{Y} \cong (\tilde{Y}/2\tilde{Y}) \ltimes \tilde{W}_{\text{aff}}$. Let $\mathcal{D}_0 = \{ v \in \mathcal{E} | 0 < \langle \alpha, v \rangle < 1 \text{ for all } \alpha \in \frac{1}{2}\Phi^+ \}$ and let $\Omega = \text{Stab}(\mathcal{D}_0)$.

There is a length function $\ell : \tilde{W}_{\text{aff}} \cong \tilde{Y} \ltimes W \to \mathbb{Z}_{\geq 0}$, which can be computed using the formula of Proposition 1.23 in Iwahori-Matsumoto [7]. Namely,

$$\ell(\lambda s) = \sum_{\alpha \in \frac{1}{2}\Phi^+ \cap s\frac{1}{2}\Phi^+} |\langle \alpha, \lambda \rangle| + \sum_{\alpha \in \frac{1}{2}\Phi^+ \cap s\frac{1}{2}\Phi^-} |\langle \alpha, \lambda \rangle + 1|,$$

where $\lambda \in \tilde{Y}$ and $s \in W$.

We view $\tilde{W}_{\text{aff}}$ as a subgroup of $W_{\text{aff}}$ via the natural inclusion $W \ltimes \tilde{Y} \hookrightarrow W \ltimes Y$.

For an element $(s, \lambda) \in \tilde{W}_{\text{aff}}$, we will sometimes abuse notation and let $s2^\lambda$ denote a representative of $(s, \lambda)$ in either $N$ or $\tilde{N}$.

2.4. Induction and Restriction. If $J$ is a locally compact Hausdorff topological group, let $\delta_J$ be a modular character of $J$.

If $(\sigma, \mathcal{V})$ is a smooth $B$-representation, then the normalized induction functor $i_{\tilde{G}, \tilde{T}} = \text{Ind}_{\tilde{B}}^{\tilde{G}}$ takes the $B$-representation $\sigma$ to the $\tilde{G}$-representation

$$i_{\tilde{G}, \tilde{T}}(\sigma) = \{ f : \tilde{G} \to \mathcal{V} | f \text{ is smooth, and } f(bg) = \delta_{\tilde{B}}(b)^{1/2}\sigma(b)f(g) \text{ for all } b \in \tilde{B} \},$$

where $\tilde{G}$ acts by right translation.

Suppose $(\pi, \mathcal{V})$ is a smooth $\tilde{G}$-representation. Let $\mathcal{V}(U) = \text{span}\{ \pi(u)v - v|u \in U^*, v \in \mathcal{V} \}$. The normalized (Jacquet) restriction functor $r_{\tilde{T}, \tilde{G}}$ takes a $\tilde{G}$-representation $\pi$ to the $\tilde{T}$-representation $\mathcal{V}_U = \mathcal{V}/\mathcal{V}(U)$, where the $\tilde{T}$ action is defined by

$$r_{\tilde{T}, \tilde{G}}(t)(v + \mathcal{V}(U)) = \delta_{\tilde{B}}^{1/2}(t)\pi(t)v + \mathcal{V}(U).$$
2.5. The Covering Torus. In this subsection we recall some facts from Loke-Savin [8] about the structure of $\tilde{T}$ and the classification of its genuine irreducible representations. Let $T^o \overset{\text{def}}{=} T(\mathbb{Z}/4\mathbb{Z}) \cong T(\mathbb{Z}) \cong Y \otimes \mu_2$ and let $\tilde{T}^1(\mathbb{Q}_2) \overset{\text{def}}{=} T^1_1 Y(\mu_2)$. Regarding the structure of $\tilde{T}$, Loke-Savin (page 4908) show that $\tilde{T} \cong (\overline{T^o} \times \overline{T}^1(\mathbb{Q}_2))/\Delta(\mu_2)$, where $\Delta$ embeds $\mu_2$ along the diagonal. Furthermore, they relate $\overline{T}^1(\mathbb{Q}_2)$ to a tame covering torus. For a precise statement see Proposition 4.5 [9].

One consequence of this decomposition is that every genuine representation of $\tilde{T}$ is the tensor product of a genuine representation of $\overline{T^o}$ and a genuine representation of $\overline{T}^1(\mathbb{Q}_2)$. The group $\overline{T^o}$ is a finite two-step nilpotent group and an irreducible genuine $\overline{T^o}$-representation is determined by its central character (Loke-Savin [9], page 4907).

We can also say something about the representations of $\overline{T}^1(\mathbb{Q}_2)$. Let $T^s_{1,8}$ be the subgroup of $\overline{T}$ generated by the elements of the form $\tilde{h}_\alpha(u)$, where $u \in 1 + 8\mathbb{Z}_2$. An irreducible genuine representations of the group $\overline{T}^1(\mathbb{Q}_2)/T^s_{1,8}$ is determined by its central character (Loke-Savin [9], Proposition 4.5 and Proposition 4.3). Thus any irreducible genuine $\overline{T}$-representation in which $T^s_{1,8}$ acts trivially is determined by the action of $Z(\overline{T^o})$ and $Z(\overline{T}^1(\mathbb{Q}_2))$.

For the remainder of this paper, we fix a Weyl group invariant genuine irreducible $\overline{T^o}$-representation $(\tau^o, E)$. For existence see Lemma 4.11 [1], where our $\overline{T^o}$ is an example of the group $M$. In fact each genuine irreducible representation of $\overline{T^o}$ is Weyl group invariant.

On page 4910 [9], Loke-Savin introduce a genuine character $\gamma_2 : Z(\overline{T}^1(\mathbb{Q}_2)) \to \mu_2$ that is the identity on $Z(\overline{T}^1(\mathbb{Q}_2)) \cap T^s_1$ and $\tilde{Y}(\tilde{Y})$. Let $V(\gamma_2)$ be the representation of $\overline{T}^1(\mathbb{Q}_2)$ that is associated with $\gamma_2$. Now for any unramified $\chi : T \to \mathbb{C}^\times$, we define the genuine $\overline{T}$-representation $(\sigma_\chi, i(\chi))$, where $i(\chi) \overset{\text{def}}{=} (\tau^o \otimes V(\gamma_2)) \otimes \chi$. We will also use this notation for the inflation of $i(\chi)$ to $\tilde{B}$. An unramified principal series of $\tilde{G}$ is a representation of the form $i_{\tilde{G},\overline{T}}(\sigma_\chi)$.

One important property of $i(\chi)$ is that for any $w \in W$ we have $i(\chi)^w \cong i(\chi^w)$. This follows from the Weyl group invariance of $\tau^o$ and $V(\gamma_2)$ (Loke-Savin [8], Corollary 5.2).

3. Splitting

This section contains two important results, Theorem 3.3 and Proposition 3.4. Theorem 3.3 states that there is a group homomorphism $S : \Gamma_1(4) \to \tilde{G}$ such that $\text{pr} \circ S = \text{id}_{\Gamma_1(4)}$. (i.e., $S$ splits sequence (4).) This result is necessary to define the Hecke-algebra $\mathcal{H}$. Proposition 3.4 states that $S$ satisfies an important technical property used to construct a basis for $\mathcal{H}$ (Proposition 6.8).

We begin with a few preliminaries.

Lemma 3.1. Let $\alpha \in \Phi^+$ and $u, t \in \mathbb{Q}_2^\times$ such that $1 + tu \neq 0$. Then in $\text{St}(\Phi, \mathbb{Q}_2)$

$$x'_\alpha(t)x'_{-\alpha}(u) = \{1 + tu, \frac{t}{1 + tu}\}^{-1}x'_{-\alpha}(\frac{u}{1 + tu})h'_\alpha(1 + tu)x'_\alpha(\frac{t}{1 + tu}). \quad (9)$$
Proof: This follows from the Steinberg relations. Alternatively, an equivalent identity is a consequence Proposition 2.7 b) in Stein [15].

Corollary 3.2. Let \( \lambda \in \widetilde{Y} \). Let \( u, t \in \mathbb{Q}_2 \) such that \( \text{val}_2(t) \geq \langle \lambda, \alpha \rangle \) and \( \text{val}_2(u) \geq \langle \lambda, -\alpha \rangle + 2 \). Then the following identity holds in \( \tilde{G} \):

\[
\tilde{x}_\alpha(t)\tilde{x}_{-\alpha}(-u) = \tilde{x}_{-\alpha}(u)\tilde{h}_\alpha(1+tu)\tilde{x}_\alpha(t)\tilde{h}_\alpha(1+tu).
\]

Proof: Note that \( 1+tu \in 1+4\mathbb{Z}_2 \), since \( \text{val}_2(t) \geq \langle \lambda, \alpha \rangle \) and \( \text{val}_2(u) \geq \langle \lambda, -\alpha \rangle + 2 \). Thus, \( (1+tu,t)_2 = (1+tu,t)_2 \). Let \( t = 2^{\langle \lambda, \alpha \rangle}t' \), where \( t' \in \mathbb{Z}_2 \). Since \( \lambda \in \widetilde{Y} \) we have \( (1+tu,t)_2 = (1+tu,t')_2 \). Now \( (1+tu,t')_2 = 1 \), because \( 2|t' \) implies \( 1+tu \equiv 1 \pmod{8} \).

Now we can prove that \( \Gamma_1(4) \) splits sequence (4). Let \( \Gamma_1(4)^* \) be the subgroup of \( \tilde{G} \) generated by the elements \( \tilde{x}_\alpha(t) \), \( \tilde{x}_{-\alpha}(u) \), \( \tilde{h}_\alpha(v) \) for all \( \alpha \in \Phi^+ \), \( t \in \mathbb{Z}_2 \) and \( v \in 1+4\mathbb{Z}_2 \).

Theorem 3.3. The group homomorphism \( \text{pr} : \Gamma_1(4)^* \rightarrow \Gamma_1(4) \) is an isomorphism. Moreover, its inverse \( S : \Gamma_1(4) \rightarrow \Gamma_1(4)^* \) splits sequence (4).

Proof: The group \( \Gamma_1(4) \) is generated by the elements \( x_\alpha(t) \), \( x_{-\alpha}(4u) \), and \( h_\alpha(v) \) for all \( \alpha \in \Phi^+ \), \( t \in \mathbb{Z}_2 \) and \( v \in 1+4\mathbb{Z}_2 \). A complete set of relations for \( \Gamma_1(4) \) is given by the Steinberg relations and the identity of Corollary 3.2. The Steinberg relations and Corollary 3.2 also form a complete set of relations for the group \( \Gamma_1(4)^* \). Since the projection map sends the generators of \( \Gamma_1(4)^* \) to the generators of \( \Gamma_1(4) \), this map is an isomorphism. The inverse map \( S \) is a splitting by definition.

Proposition 3.4. Let \( x \in \widetilde{N} \). Then \( \tilde{\Gamma}_0(4) \cap x\Gamma_1(4)^*x^{-1} \subseteq \Gamma_1(4)^* \).

Proof: Let \( \beta_1, \ldots, \beta_\ell \) be an enumeration of the positive roots and consider the element

\[
\gamma = \tilde{x}_{-\beta_\ell}(u_\ell) \cdots \tilde{x}_{-\beta_1}(u_1)\tilde{x}_{\beta_1}(t_1) \cdots \tilde{x}_{\beta_\ell}(t_\ell)h,
\]

where \( t_i, u_i \in \mathbb{Q}_2 \), and \( h \in \tilde{T} \). Then \( \gamma \in \Gamma_1(4)^* \) if and only if \( t_i \in \mathbb{Z}_2 \), \( u_i \in 4\mathbb{Z}_2 \), and \( h \in T_1 \). Furthermore, this factorization is unique. The analogous statement holds if we permute the factors in any order or if we replace \( \Gamma_1(4)^* \) by \( \tilde{\Gamma}_0(4) \). We will use these facts to prove the proposition.

Let \( x = w2^\lambda \), where \( w \in \mathcal{W} \) and \( \lambda \in \widetilde{Y} \). Suppose that \( x\gamma x^{-1} \in \tilde{\Gamma}_0(4) \). Thus

\[
x\gamma x^{-1} = \tilde{x}_{-w^{-\beta}_{\ell}}(\pm2^{\langle \lambda, -\beta_{\ell} \rangle}u_\ell) \cdots \tilde{x}_{-w^{-\beta_1}}(\pm2^{\langle \lambda, -\beta_1 \rangle}u_1) \\
\times \tilde{x}_{w^{\beta_1}}(\pm2^{\langle \lambda, \beta_1 \rangle}t_1) \cdots \tilde{x}_{w^{\beta_\ell}}(\pm2^{\langle \lambda, \beta_\ell \rangle}t_\ell)(xhx^{-1}).
\]

Note that \( xhx^{-1} \in T_1^* \).

By the unique factorization we see that the argument of \( \tilde{x}_{\pm w_{\beta_i}} \) is an element of \( \mathbb{Z}_2 \), if \( \pm w \cdot \beta_i \) is positive, and \( 4\mathbb{Z}_2 \), if \( \pm w \cdot \beta_i \) is negative. Thus each factor in expression (11) is an element of \( \Gamma_1(4)^* \) thus \( x\gamma x^{-1} \in \Gamma_1(4)^* \).

We close this section by identifying an obstruction which prevents the subgroup \( G(\mathbb{Z}_2) \) from splitting the sequence (4). Suppose that \( S' : G(\mathbb{Z}_2) \rightarrow \tilde{G} \) splits sequence (4). Then it follows that for any \( \alpha \in \Phi \), we have \( S'(h_\alpha(-1)) = \pm h_\alpha(-1) \). The element \( h_\alpha(-1) \in G(\mathbb{Z}_2) \)
has order 2; the element \( \pm \bar{h}_\alpha(-1) \in \tilde{G} \) has order 4, since \((-1,-1)_{Q_2} = -1\). Thus \( S' \) cannot exist. More generally, any subgroup of \( \tilde{G} \) which contains \( h_\alpha(-1) \) cannot split the sequence (4). Thus, the Iwahori subgroup \( \Gamma_0(2) \) also does not split the sequence (4).

This obstruction does not appear in the tame case (i.e., \( \text{GCD}(n,p) = 1 \)), because the tame Hilbert Symbol of a local field \( F \) is trivial on \( \mathcal{O}_F^\times \times \mathcal{O}_F^\times \).

4. \( \Gamma_0(4) \) Double Cosets

In this section we compute representatives of the double coset space \( \Gamma_0(4) \backslash G/\Gamma_0(4) \) for the purpose of studying the \( \tilde{\Gamma}_0(4) \)-equivariant Hecke Algebra \( H \). We begin with some notation. For \( \Phi^- \subseteq \Phi \) let

\[ x_A(2) = \prod_{\alpha \in A} x_\alpha(2), \]

where the product is taken with respect to some ordering of the elements of \( A \). (Lemma 4.2 shows that the choice of an order is immaterial to the study of \( \Gamma_0(4) \) double cosets.) Now we can describe representatives of \( \Gamma_0(4) \backslash G/\Gamma_0(4) \).

**Proposition 4.1.** For every \( g \in G \), there exists sets \( A, B \subseteq \Phi^- \), \( w \in W \), and \( \lambda \in Y \) such that

1. \( \Gamma_0(4)g\Gamma_0(4) = \Gamma_0(4)x_A(2)w2^\lambda x_B(2)\Gamma_0(4); \)
2. \( A \cap B = \emptyset; \)
3. \( \langle \lambda, w^{-1}\alpha \rangle > 0 \) for all \( \alpha \in A \);
4. \( \langle \lambda, \beta \rangle \leq 0 \) for all \( \beta \in B \);
5. for all \( \beta \in B \), \( \langle \lambda, \beta \rangle = 0 \) implies that \( w\beta \in \Phi^- \).

We will prove this proposition in a few steps. First observe that the group \( \Gamma_0(4) \) is contained in the Iwahori subgroup \( \Gamma_0(2) \). The Iwahori-Bruhat decomposition states that \( \Gamma_0(2) \backslash G/\Gamma_0(2) \) is in bijection with the affine Weyl group \( \tilde{W} \cong W \rtimes \mathbb{F} \). Thus to find a set of representatives of the doubles cosets of \( G \) with respect to \( \Gamma_0(4) \) we should determine representatives of \( \Gamma_0(4) \backslash \Gamma_0(2) \) and \( \Gamma_0(2)/\Gamma_0(4) \).

**Lemma 4.2.** The set \( \{x_A(2) | A \subseteq \Phi^- \} \) is a complete set of distinct representatives of \( \Gamma_0(4) \backslash \Gamma_0(2) \). The same holds for \( \Gamma_0(2)/\Gamma_0(4) \). Furthermore, this holds for any permutation of the factors of \( x_A(2) = \prod_{\alpha \in A} x_\alpha(2) \).

**Proof:** First we will show that \( \Gamma_0(2)/\Gamma_0(4) \) is in bijection with \( (\Gamma_0(2) \cap U_-)/(\Gamma_0(4) \cap U_-). \) The group \( \Gamma_0(2) \) possesses an Iwahori factorization, thus multiplication defines a bijection

\[ (\Gamma_0(2) \cap U_-) \times (\Gamma_0(2) \cap B) \cong \Gamma_0(2). \]

Since reduction mod 2 takes elements of \( G(\mathbb{Z}_2) \cap B \) to \( B(\mathbb{Z}/2\mathbb{Z}) \), it follows that \( G(\mathbb{Z}_2) \cap B = \Gamma_0(2) \cap B. \) Thus

\[ (\Gamma_0(2) \cap U_-) \times (G(\mathbb{Z}_2) \cap B) \cong \Gamma_0(2). \]

Similarly,

\[ (\Gamma_0(4) \cap U_-) \times (G(\mathbb{Z}_2) \cap B) \cong \Gamma_0(4). \]
Thus there is a bijection
$$\Gamma_0(2)/\Gamma_0(4) \leftrightarrow (\Gamma_0(2) \cap U_-)/(\Gamma_0(4) \cap U_-).$$

Since $U_-$ is a smooth group scheme over $\mathbb{Z}$ it is a smooth group scheme over $\mathbb{Z}_2$. This implies that $(\Gamma_0(2) \cap U_-)/(\Gamma_0(4) \cap U_-)$, which is an abelian group, can be identified with $u_-(\mathbb{F}_2)$, the Lie algebra of $U_-$ over the field with two elements, as abelian groups. In particular, this map sends the Chevalley generators $X_\alpha$ of $u_-(\mathbb{F}_2)$ to $x_\alpha(2)$. Since $u_-(\mathbb{F}_2)$ is abelian we see that the order of the elements in $A$ does not change the coset. The analogous argument proves the result for $\Gamma_0(4)/\Gamma_0(2)$. □

Lemma 4.2 implies that a complete set of representatives of $\Gamma_0(4)/G/\Gamma_0(4)$ is contained among the elements of the set
$$\{x_A(2)w2^\lambda x_B(2)|w \in W, A, B \subseteq \Phi^-, \lambda \in Y\}.$$ The $x_A(2)$ will be referred to as the unipotent elements on the left and the $x_B(2)$ will be referred to as the unipotent elements on the right.

Our next task is to eliminate redundant representatives. However, we will stop short of finding a complete set of distinct representatives.

**Lemma 4.3.** Let $A, B \subseteq \Phi^-$, $\alpha \in A$, $\beta \in B$, $w \in W$, and $\lambda \in Y$. Let $A' = A - \{\alpha\}$ and $B' = B - \{\beta\}$.

1. If $\langle \lambda, -w^{-1}\alpha \rangle \geq 1$, or if $\langle \lambda, w^{-1}\alpha \rangle = 0$ and $w^{-1} \cdot \alpha \in \Phi^+$, then
$$\Gamma_0(4)x_A(2)w2^\lambda x_B(2)\Gamma_0(4) = \Gamma_0(4)x_{A'}(2)w2^\lambda x_B(2)\Gamma_0(4).$$

2. If $\langle \lambda, \beta \rangle \geq 1$, or if $\langle \lambda, \beta \rangle = 0$ and $w \cdot \beta \in \Phi^+$, then
$$\Gamma_0(4)x_A(2)w2^\lambda x_B(2)\Gamma_0(4) = \Gamma_0(4)x_A(2)w2^\lambda x_{B'}(2)\Gamma_0(4).$$

3. If $\langle \lambda, w^{-1}\alpha \rangle = 0$ and $w^{-1} \cdot \alpha \in \Phi^-$, then
$$\Gamma_0(4)x_A(2)w2^\lambda x_B(2)\Gamma_0(4) = \Gamma_0(4)x_A(2)w2^\lambda x_{B \cup \{w^{-1}\alpha\}}(2)\Gamma_0(4).$$

4. If $\langle \lambda, \beta \rangle = 0$ and $w \cdot \beta \in \Phi^-$, then
$$\Gamma_0(4)x_A(2)w2^\lambda x_B(2)\Gamma_0(4) = \Gamma_0(4)x_A(2)w2^\lambda x_{B \cup \{\beta\}}(2)\Gamma_0(4).$$

**Proof:** We will prove statement (1); the proofs of the remaining statements are identical. We have $\Gamma_0(4)x_A(2) = \Gamma_0(4)x_{A'}(2)x_\alpha(2)$, and $x_\alpha(2)w2^\lambda = w2^\lambda x_{w^{-1}\alpha}(\pm 2^{1+\langle \lambda, w^{-1}\alpha \rangle})$. Thus
$$\Gamma_0(4)x_A(2)w2^\lambda x_B(2)\Gamma_0(4) = \Gamma_0(4)x_{A'}(2)x_\alpha(2)w2^\lambda x_B(2)\Gamma_0(4) = \Gamma_0(4)x_{A'}(2)w2^\lambda x_{w^{-1}\alpha}(\pm 2^{1+\langle \lambda, w^{-1}\alpha \rangle})x_B(2)\Gamma_0(4).$$

Since $\langle \lambda, -w^{-1}\alpha \rangle \geq 1$, we have $x_{w^{-1}\alpha}(\pm 2^{1+\langle \lambda, w^{-1}\alpha \rangle}) \in \Gamma_4(4)$. The subgroup $\Gamma_4(4)$ is normal in $G(\mathbb{Z})$ and contained in $\Gamma_0(4)$ so the element $x_{w^{-1}\alpha}(\pm 2^{1+\langle \lambda, w^{-1}\alpha \rangle})$ can be moved right and absorbed into $\Gamma_0(4)$. □

**Proof of Proposition 4.1:** We proved that $\Gamma_0(4)g\Gamma_0(4) = \Gamma_0(4)x_{A'}(2)w2^\lambda x_{B'}(2)\Gamma_0(4)$ for some $A', B' \subseteq \Phi^-$, $w \in W$, and $\lambda \in Y$. Using Lemma 4.3 we can identify a preferred
representative of $\Gamma_0(4)x_{A'}(2)w2^\lambda x_B(2)\Gamma_0(4)$. For each $\alpha \in A'$ we can check to see if $\alpha$ satisfies the hypotheses of item (1) in Lemma 4.3, in which case $\alpha$ can be removed without changing the double coset. Furthermore, if $\alpha$ satisfies item (3), then we can move $\alpha$ to the right-hand side where it becomes $w^{-1}\alpha$. Let $A$ be the set of elements in $A'$ that do not satisfy the hypotheses of items (1), and (3). Similarly, for each $\beta \in B'$ we can check to see if $\beta$ satisfies the hypotheses of item (2) in Lemma 4.3, in which case $\beta$ can be removed. Let $B$ be the the set of elements of $B'$ that do not satisfy (2) and the elements $w^{-1}\alpha$, where $\alpha \in A'$ satisfies (3). Proposition 4.3 above proves that $\Gamma_0(4)x_{A'}(2)w2^\lambda x_B(2)\Gamma_0(4) = \Gamma_0(4)x_{A'}(2)w2^\lambda x_B(2)\Gamma_0(4)$. By construction the pair $(A, B)$ satisfy the conditions of the proposition. □

5. The $\Gamma_0(4)$ Hecke Algebra

In this section we will study $\mathcal{H} \overset{\text{def}}{=} C_c^\infty(\Gamma_0(4)\backslash G/\Gamma_0(4))$, the algebra of $\Gamma_0(4)$-biinvariant compactly supported functions. The multiplication of $\mathcal{H}$ is given by convolution $f_1*f_2(g) = \int_G f_1(h)f_2(h^{-1}g)dh$, where the Haar measure is normalized so that the measure of $\Gamma_0(4)$ is equal to 1.

To begin we introduce the following length function $\ell_4: G \to \mathbb{R}_{\geq 0}$ defined by

$$2^{\ell_4(g)} = |\Gamma_0(4):\Gamma_0(4) \cap g\Gamma_0(4)g^{-1}|.$$ (16)

In fact, $\ell_4(\gamma_1g\gamma_2) = \ell_4(g)$ for any $\gamma_1, \gamma_2 \in \Gamma_0(4)$. This can be seen as follows. The map $\Gamma_0(4)/\Gamma_0(4) \cap g\Gamma_0(4)g^{-1} \to \Gamma_0(4)g\Gamma_0(4)/\Gamma_0(4)$ defined by $\gamma \Gamma_0(4) \cap g\Gamma_0(4)g^{-1} \mapsto \gamma g\Gamma_0(4)$ is a bijection. Moreover, $|\Gamma_0(4):\Gamma_0(4) \cap g\Gamma_0(4)g^{-1}| = |\Gamma_0(4)g\Gamma_0(4)/\Gamma_0(4)|$.

Using the section $s: W_{\text{aff}} \to G$ we can consider the function $\ell_4 \circ s$. We will abuse notation and simply write $\ell_4(w)$ in place of $\ell_4 \circ s(w)$. The main result of this section, Theorem 5.6, states that $\ell_4$ and $\ell$ are proportional when restricted to $\tilde{W}_{\text{aff}}$. We begin with some preliminary results.

**Lemma 5.1.** Let $w_1, w_2 \in W_{\text{aff}}$. Then $\Gamma_0(4)s(w_1w_2)\Gamma_0(4) = \Gamma_0(4)s(w_1)s(w_2)\Gamma_0(4)$.

**Proof:** This follows because $s(w_1w_2)^{-1}s(w_1)s(w_2) \in T(\mathbb{Z}_2) \subset \Gamma_0(4)$.

**Proposition 5.2.** For any $g \in G$, $\ell_4(g) < \infty$.

**Proof:** It suffices to show that the number of cosets $\delta\Gamma_0(4) \subset \Gamma_0(4)\Gamma_0(4)$ is finite. By Proposition 3.1 in Iwahori-Matsumoto [7] we know that the number of cosets $\delta\Gamma_0(2) \subset \Gamma_0(2)\Gamma_0(2)$ is finite. Since $|\Gamma_0(2):\Gamma_0(4)| < \infty$ the result follows. □

Let $\text{ind}: \mathcal{H} \to \mathbb{C}$ be the algebra homomorphism defined by $f \mapsto \int_G f(h)dh$. For the characteristic function $\mathcal{E}_g = 1_{\Gamma_0(4)\Gamma_0(4)}$ we have $\text{ind}(\mathcal{E}_g) = |\Gamma_0(4):\Gamma_0(4)| = 2^{\ell_4(g)}$.

**Proposition 5.3.** Let $g_1, g_2 \in G$. If $\ell_4(g_1g_2) = \ell_4(g_1) + \ell_4(g_2)$, then $\mathcal{E}_{g_1} \ast \mathcal{E}_{g_2} = \mathcal{E}_{g_1g_2}$.

**Proof:** By definition

$$\mathcal{E}_{g_1} \ast \mathcal{E}_{g_2}(g) = \int_G \mathcal{E}_{g_1}(h)\mathcal{E}_{g_2}(h^{-1}g) = \sum_{\delta \in G/\Gamma_0(4)} \mathcal{E}_{g_1}(\delta)\mathcal{E}_{g_2}(\delta^{-1}g).$$ (17)
The right hand side of equation (17) is equal to the number of cosets $\delta \Gamma_0(4)$ satisfying
\[
\delta \Gamma_0(4) \subset \Gamma_0(4)g_1 \Gamma_0(4) \quad \text{and} \quad (\delta \Gamma_0(4))^{-1} g \subseteq \Gamma_0(4)g_2 \Gamma_0(4). \tag{18}
\]

First we consider the case where $g = g_1 g_2$. In this case we can directly check that $g_1 \Gamma_0(4)$ is one such coset. Thus it follows that $\mathbf{c}_g \ast \mathbf{c}_g = \mathbf{c}_g + f$, where $c \in \mathbb{Z}_{\geq 1}$ and $f \in \mathcal{H}$ is a nonnegative function.

We can apply ind to get $2^\ell_4(g_1 g_2) = c^2 \ell_4(g_1 g_2) + \text{ind}(f)$, because $\ell_4(g_1 g_2) = \ell_4(g_1) + \ell_4(g_2)$. Since $\text{ind}(f) \geq 0$ and $c \geq 1$ it follows that $f = 0$ and $c = 1$. Thus $\mathbf{c}_g \ast \mathbf{c}_g = \mathbf{c}_g g_2$.

\textbf{Remark:} The above proof also shows that if $\delta \Gamma_0(4)$ satisfies the coset conditions of (18), then $\Gamma_0(4)g_0 \Gamma_0(4) = \Gamma_0(4)g_1 g_2 \Gamma_0(4)$ and $\delta \Gamma_0(4) = g_1 \Gamma_0(4)$.

\textbf{Proposition 5.4.} Let $w = w_{\alpha_i} \in \tilde{W}_{\text{aff}}$, where $i = 0, \ldots, r$. Then
\[
\text{ind}(w_{\alpha_i}) = 2^2. \tag{19}
\]

\textbf{Proof:} If $i \neq 0$, then the Iwahori factorization implies that
\[
\Gamma_0(4)w_{\alpha_i}(1) \Gamma_0(4)/\Gamma_0(4) \cong U_{\alpha_i}(\mathbb{Z}_2)/U_{\alpha_i}(4\mathbb{Z}_2) \cong \mathbb{Z}/4\mathbb{Z}. \tag{20}
\]
If $i = 0$, then the Iwahori factorization implies that
\[
\Gamma_0(4)w_{\alpha_0}(4) \Gamma_0(4)/\Gamma_0(4) \cong U_{\alpha_0}(4\mathbb{Z}_2)/U_{\alpha_0}(4^2\mathbb{Z}_2) \cong \mathbb{Z}/4\mathbb{Z}. \tag{21}
\]

\textbf{Lemma 5.5.} Let $w_1 \in \Omega$ and $w_2 \in \tilde{W}_{\text{aff}}$. Then $\ell_4(w_1 w_2) = \ell_4(w_2)$.

\textbf{Proof:} First we claim that $s(w_1) \in N_G(\Gamma_0(4))$. By Proposition 1.10. in Iwahori-Matsumoto \cite{7}, $w_1 \in \Omega$ implies that $\ell(w_1) = 0$. Using formula (8), the Iwahori factorization, and the Steinberg relations one can directly check that $s(w_1) \in N_G(\Gamma_0(4))$.

Finally, by Lemma 5.1 and $\Gamma_0(4)s(w_1)s(w_2)\Gamma_0(4) = s(w_1)\Gamma_0(4)s(w_2)\Gamma_0(4)$, we see that $\ell_4(w_1 w_2) = \ell_4(w_2)$.

Now we can prove the main theorem of this section.

\textbf{Theorem 5.6.} Let $w \in \tilde{W}_{\text{aff}}$. Then
\[
\ell_4(w) = 2\ell(w). \tag{22}
\]

\textbf{Proof:} Suppose that $w = w_1 w_2$, where $w_1 \in \Omega$ and $w_2 \in \tilde{W}_{\text{aff}}$. Then $\ell(w_1 w_2) = \ell(w_2)$. By Lemma 5.5, $\ell_4(w_1 w_2) = \ell_4(w_2)$. Thus it suffices to show that $\ell_4(w) = 2\ell(w)$ for $w \in \tilde{W}_{\text{aff}}$.

Suppose that $w = w_{\alpha_1} \ldots w_{\alpha_k}$ is a minimal expression of $w$ with respect to the Coxeter generators $w_{\alpha_0}, \ldots, w_{\alpha_r}$ of $\tilde{W}_{\text{aff}}$. By Lemma 5.1 and Proposition 5.3
\[
2^\ell(w) = \text{ind}(w_{\alpha_1}) \ldots \text{ind}(w_{\alpha_k}) = 2^{2k} = 2^{2\ell(w)}. \tag{23}
\]
Thus $\ell_4(w) = 2\ell(w)$ for $w \in \tilde{W}_{\text{aff}}$. 
\qed
6. The $\tilde{G}_0(4)$ Hecke Algebra

Let $(\tau^\circ, E)$ be a finite dimensional irreducible genuine Weyl group-invariant representation of $T^\circ$. Since $\tilde{G}_0(4) \cong T^\circ \rtimes S(\Gamma_1(0))$, $\tau^\circ$ inflates to a representation of $\tilde{G}_0(4)$ which we call $\tau$. Let $\tau^\vee$ be the contragredient of $\tau$. The $\tau$-spherical Hecke algebra of $\tilde{G}$ is

$$\mathcal{H} \overset{\text{def}}{=} \mathcal{H}(\tilde{G}, \tau^\vee) = \{ f \in C_c^\infty(\tilde{G}, \text{End}(E)) | f(k_1 g k_2) = \tau(k_1) f(g) \tau(k_2), \text{ for all } k_1, k_2 \in \tilde{G}_0(4) \}.$$

For $f_1, f_2 \in \mathcal{H}$, the multiplication is defined by $f_1 \ast f_2(g) = \int_{\tilde{G}} f_1(h) f_2(h^{-1} g) dh$, where the Haar measure on $\tilde{G}$ is normalized so that $\tilde{G}_0(4)$ has measure 1.

The main result of this section, Theorem 6.15, describes a Bernstein presentation of $\mathcal{H}$.

Now we outline our approach to Theorem 6.15. First, we construct a $\mathbb{C}$-basis for $\mathcal{H}$, Proposition 6.8. Second, we identify some multiplicative relations among these basis elements, propositions 6.9 and 6.11. Third, we use propositions 6.9 and 6.11 to prove that $\mathcal{H}$ admits an Iwahori-Matsumoto presentation, Proposition 6.12.

Finally, using results of Lusztig [10], the Iwahori-Matsumoto presentation implies that $\mathcal{H}$ satisfies the Bernstein relations, and Savin’s trick (Lemma 7.6, [14]) shows that these relations imply all others. This results in Theorem 6.15, a Bernstein presentation of $\mathcal{H}$.

Our first step is to construct a $\mathbb{C}$-basis for $\mathcal{H}$. After a few technical preliminaries, we show that the nontrivial action of $\mu_2$ prohibits certain double cosets from supporting any functions in $\mathcal{H}$, propositions 6.4 and 6.7. After identifying these constraints on support, we can construct a basis for $\mathcal{H}$, Proposition 6.8.

**Proposition 6.1** (Stein [15], Corollary 2.9). Let $\alpha \in \Phi$. Then $[\tilde{x}_\alpha(2), \tilde{x}_{-\alpha}(2)] = (-1)\gamma$, where $-1 \in \mu_2$ and $\gamma \in S(\Gamma(4))$.

**Proof:** This follows by applying Lemma 3.1 and the Steinberg relations. \[\square\]

**Lemma 6.2.** Let $m \in \mathbb{Z}_{\geq 1}$, and let $\beta, \beta_j \in \Phi^-$. Then

$$\tilde{x}_\beta(\pm 2^m) \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_k}(2) \mathcal{S}(\Gamma(4)) = \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_k}(2) \tilde{x}_\beta(\pm 2^m) \Gamma(4)^*,$$

and

$$\Gamma(4)^* \tilde{x}_\beta(\pm 2^m) \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_k}(2) = \Gamma(4)^* \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_k}(2) \tilde{x}_\beta(\pm 2^m).$$

**Proof:** The result follows from induction on $\ell$ and the Steinberg relations. \[\square\]

**Lemma 6.3.** Let $m \in \mathbb{Z}_{\geq 1}$, $\eta \in \Phi, \beta_j \in \Phi^-, w \in W$, and $\lambda \in X_+(T)$, such that $-w \cdot \eta \in \Phi^+$, $\langle \lambda, w^{-1} \cdot \beta_j \rangle > 0$ for all $j$, and $\langle \lambda, -\eta \rangle \geq 0$. Then

$$\Gamma(4)^* \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_k}(2) \tilde{x}_{-w \cdot \eta}(\pm 2^m) = \Gamma(4)^* \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_k}(2).$$

**Proof:** We prove this proposition using induction on $k$. When $k = 0$ the result holds since $\tilde{x}_{-w \cdot \eta}(\pm 2^m) \in \Gamma(4)^*$. Now suppose that the result holds for $k - 1$.

Consider the identity $\tilde{x}_{\beta_k}(2) \tilde{x}_{-w \cdot \eta}(\pm 2^m) = [\tilde{x}_{\beta_k}(2), \tilde{x}_{-w \cdot \eta}(\pm 2^m)] \tilde{x}_{-w \cdot \eta}(\pm 2^m) \tilde{x}_{\beta_k}(2)$. To compute the commutator we prove that $-\beta_k \neq -w \cdot \eta$. Suppose $-\beta_k = -w \cdot \eta$, then $0 > \langle \lambda, w^{-1} \cdot \beta_k \rangle = \langle \lambda, -\eta \rangle \geq 0$. This is a contradiction so $-\beta_k \neq -w \cdot \eta$.

Thus the commutator can be computed using the Steinberg relations. Since our root system is simply laced we see that if $-w \cdot \eta + \beta_k \notin \Phi$, then $[\tilde{x}_{\beta_k}(2), \tilde{x}_{-w \cdot \eta}(\pm 2^m)] = 1$; if
Proposition 6.4. Let $A, B \subseteq \Phi^-$ and let $w \in W$ and $\lambda \in Y$. Suppose $(A, B)$ satisfy the conditions (2), (3), (4), (5) of Proposition 4.1 and either $A$ or $B$ is nonempty. If $f \in H$, then

$$f(\tilde{\Gamma}_0(4)\tilde{x}_A(2)w2^\lambda \tilde{x}_B(2)\tilde{\Gamma}_0(4)) = 0.$$  

Proof: Let us suppose that $B = \{\beta_1, \ldots, \beta_t\}$ is nonempty and 

$$\tilde{x}_B(2) = \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_t}(2).$$

Recall from Proposition 4.1 that for all $j$

$$\langle \lambda, \beta_j \rangle \leq 0.$$  

It will be convenient to single out $\beta_\ell$, so we will let $\eta = \beta_\ell$. The set $A$ may or may not be empty. If $A$ is not empty, Corollary 4.1 implies for $\alpha \in A$

$$\langle \lambda, -w^{-1} \cdot \alpha \rangle < 0.$$  

Let $f \in H$. We will prove that $f(\tilde{x}_A(2)w2^\lambda \tilde{x}_B(2)) = -f(\tilde{x}_A(2)w2^\lambda \tilde{x}_B(2))$.

First note that $\tilde{x}_-\eta(2) \in S(\Gamma_1(4))$, so

$$f(\tilde{x}_A(2)w2^\lambda \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_t}(2)) = f(\tilde{x}_A(2)w2^\lambda \tilde{x}_{\beta_1}(2) \ldots \tilde{x}_{\beta_t}(2)\tilde{x}_{-\eta}(2)). (24)$$

By Proposition 6.1 it follows that $\tilde{x}_{\beta_1}(2)\tilde{x}_{-\eta}(2) = \tilde{x}_{-\eta}(2)\tilde{x}_{\beta_1}(2)(-1)M$, where $M \in \Gamma_1(4)^*$. Therefore,

$$\langle \lambda, -w^{-1} \cdot \alpha \rangle < 0.$$  

Now for any $j$ we have $[\tilde{x}_{\beta_1}(-2), \tilde{x}_{-\eta}(-2)]$ is equal to 1 or $\tilde{x}_{\beta_1}(-2, \tilde{x}_{-\eta}(-2)) \in S(\Gamma_4(4))$. In either case $[\tilde{x}_{\beta_1}(-2), \tilde{x}_{-\eta}(-2)] \in S(\Gamma_4(4))$. These elements can be moved to the right since $S(\Gamma_4(4))$ is normalized by elements of the form $\tilde{x}_{\beta}(2)$, where $\beta \in \Phi$. By using induction on $\ell$ in conjunction with the Steinberg relations and Lemma 6.2 it follows that

$$[\tilde{x}_{\beta_1}(-2), \tilde{x}_{-\eta}(2)\tilde{x}_{\beta_1}(2)]. (25)$$

We continue to push the element $\tilde{x}_{-\eta}(2)$ to the left using the identities $2^\lambda \tilde{x}_{-\eta}(2) = \tilde{x}_{-\eta}(2^{1+(\lambda, -\eta)})2^\lambda$ and $w\tilde{x}_{-\eta}(2^{1+(\lambda, -\eta)}) = \tilde{x}_{-w\eta}(2^{1+(\lambda, -\eta)})w$. Therefore,

$$[\tilde{x}_{\beta_1}(-2), \tilde{x}_{-\eta}(2)\tilde{x}_{\beta_1}(2)]. (26)$$

If $-w \cdot \eta \in \Phi^-$ and $\langle \lambda, \eta \rangle < 0$, then we can apply Lemma 6.2 to prove that

$$f(\tilde{x}_A(2)w2^\lambda \tilde{x}_B(2)) = -f(\tilde{x}_A(2)w2^\lambda \tilde{x}_B(2)). (27)$$

If $-w \cdot \eta \in \Phi^-$ and $\langle \lambda, \eta \rangle = 0$, then Proposition 4.1 implies that $w\eta \in \Phi^-$, a contradiction.
If \(-w \cdot \eta \in \Phi^+\), then we can apply Lemma \ref{lem:Weyl_group} to remove \(\tilde{x} - w \cdot \eta (\pm 2^\lambda (\lambda - \eta))\). Thus we see that
\[
f(\tilde{x}_A(2) wt(2^\lambda) \tilde{x}_B(2)) = -f(\tilde{x}_A(2) wt(2^\lambda) \tilde{x}_B(2)),
\]
as desired.

One can apply a similar argument when \(B\) is empty. \(\square\)

**Remark:** The proof of Proposition \ref{prop:support} above does not utilize the full \(\tilde{\Gamma}_0(4)\) transformation law; it only requires \(\Gamma^*_1(4)^*\)-invariance.

To continue with the support calculations we will utilize the following lemma. Before we begin we introduce some notation. If \(J\) is a subgroup of a group \(K\) and \(k \in K\), then let \(kJk^{-1}\). If \(\mathcal{Y}\) is a representation of \(J\) and \(k \in K\), then we will write \(k \mathcal{Y}\) for the representation of the group \(kJk^{-1}\) acting on \(\mathcal{Y}\) via \((kj^{-1}) \cdot v = j \cdot v\).

**Lemma 6.5** (Bushnell-Kutzko \cite{BushnellKutzko}, Proposition (4.1.1)). Let \(g \in \tilde{G}\). The following are equivalent:

(i) \(\text{Hom}_{\Gamma_0(4) \cap \tilde{\Gamma}_0(4)}(g \mathcal{Y}, \mathcal{Y}) \neq 0\);

(ii) there exists \(\Phi \in \mathcal{H}\) with \(\Phi(g) \neq 0\).

If the element \(g\) satisfies these conditions, then we have a canonical vector space isomorphism between \(\text{Hom}_{\Gamma_0(4) \cap \tilde{\Gamma}_0(4)}(g \mathcal{Y}, \mathcal{Y})\) and the space of functions \(\Phi \in \mathcal{H}\) which vanish outside the double coset \(\tilde{\Gamma}_0(4) g \tilde{\Gamma}_0(4)\).

Next we show that any double coset can support at most a one dimensional space of functions in \(\mathcal{H}\).

**Proposition 6.6.** Let \(w \in \mathcal{W}\), \(\lambda \in \mathcal{Y}\), and \(g = w2^\lambda\), then
\[
\dim(\text{Hom}_{\Gamma_0(4) \cap \tilde{\Gamma}_0(4)}(g E, E)) \leq 1.
\]

**Proof:** Since \(\tilde{T}^\circ \subseteq \tilde{\Gamma}_0(4) \cap \tilde{\Gamma}_0(4)\) we have \(\text{Hom}_{\Gamma_0(4) \cap \tilde{\Gamma}_0(4)}(g E, E) \subseteq \text{Hom}_{\tilde{T}^\circ}(g E, E)\). Recall that \((\tau^\circ, E)\) is Weyl group-invariant and irreducible, and note that the action of \(2^\lambda\) on \(\tilde{T}^\circ\) by conjugation is trivial. Therefore Schur’s lemma implies that \(\text{Hom}_{\Gamma_0(4) \cap \tilde{\Gamma}_0(4)}(g E, E)\) has dimension at most 1.

Now we can prove our second constraint on the support of functions in \(\mathcal{H}\).

**Proposition 6.7.** Let \(f \in \mathcal{H}\), \(w \in \mathcal{W}\), and \(\lambda \in \mathcal{Y} - \tilde{Y}\), then \(f(w2^\lambda) = 0\).

**Proof:** From Lemma \ref{lem:Weyl_group} it suffices to show that \(\text{Hom}_{\Gamma_0(4) \cap \tilde{\Gamma}_0(4)}(g E, E) = 0\), where \(g = w2^\lambda\). We begin with a few preliminary remarks. By assumption \(\lambda \notin \tilde{Y}\), so there exists \(\alpha \in \Phi\) such that \(\langle \lambda, \alpha \rangle\) is odd. Let \(t \in \mathbb{Z}_2^\times\) such that \(t \equiv 5 \mod (8)\). This implies that \((2, t)_2 = -1, (-1, t)_2 = 1, and (t, t)_2 = 1\). Thus \(h_\alpha(t)2^\lambda h_\alpha(t)^{-1} = -2^\lambda\), since \(\langle \lambda, \alpha \rangle\) is odd; \(wh_\alpha(t)w^{-1} = h_{w\alpha}(t)\), since \(t \equiv 1 \mod (4)\).

Let \(\psi \in \text{Hom}_{\Gamma_0(4) \cap \tilde{\Gamma}_0(4)}(g E, E)\). We will show that \(\psi = -\psi\). For any \(v \in E\), consider \(\psi(w2^\lambda h_\alpha(t)(w2^\lambda)^{-1}v)\). Since \(\Gamma_1(4)^*\) acts trivially on \(E\) and \(t \equiv 1 \mod (4)\) it follow that that \(h_{w\alpha}(t)\) acts trivially on \(E\). Thus, \(\psi(w2^\lambda h_\alpha(t)(w2^\lambda)^{-1}v) = -\psi(h_{w\alpha}(t)v) = -\psi(v)\). On the other hand, since \(\psi\) is an intertwining operator and \(t \equiv 1 \mod (4)\)
we have $\psi(w2^\lambda h_\alpha(t)(w2^\lambda)^{-1}v) = h_\alpha(t)\psi(v) = \psi(v)$. Thus $\psi(v) = -\psi(v)$. Therefore, $\text{Hom}_{\overline{\Gamma}_0(4) \cap \Gamma_0(4)}(gE, E) = 0$. 

Now we can construct a basis for $\mathcal{H}$. By Proposition 5.2 in Adams-Barbash-Paul-Trapa-Vogan [1], the representation $\tau^o$ can be extended from $\mathcal{T}^o$ to $\mathcal{W}$. (Since the Hilbert symbol of $\mathbb{R}$ and $\mathbb{Q}_2$ agree on $\{\pm 1\} \times \{\pm 1\}$, the group $\mathcal{W}$ is a subgroup of the group $\overline{K}$ appearing in Proposition 5.2 of [1].) We will call this extension $\tau_\mathcal{W}$. Recall that $\overline{N}' \cong \mathcal{W} \times \mathcal{Y}$. Thus $\tau_\mathcal{W}$ inflates to a representation of $\overline{N}'$, which we call $\tau_{\overline{N}'}$.

Proposition 6.4 and Proposition 6.7 together state that the support of a function in $\mathcal{H}$ is contained in the double cosets of the form $\overline{\Gamma}_0(4)\overline{\Gamma}_0(4)$ where $x \in \overline{N}'$. For each $w \in \overline{W}_{\text{aff}}$ we define a function $e_w$. Let $x$ be any element of $\overline{N}'$ that maps to $w$ under the natural map $\overline{N}' \cong \mathcal{W} \times \mathcal{Y} \to \mathcal{W} \times \mathcal{Y} \cong \overline{W}_{\text{aff}}$. (Note that $\mathcal{T}^o \subset \mathcal{W}$ is a normal subgroup, $\mathcal{W}/\mathcal{T}^o \cong \mathcal{W}$, and $\mathcal{T}^o$ commutes with $\mathcal{Y}(\mathcal{Y})$.) Define $e_w$ to be the unique function in $\mathcal{H}$ supported on $\overline{\Gamma}_0(4)x\overline{\Gamma}_0(4)$ such that

$$e_w(\Gamma_1(4)^*x\Gamma_1(4)^*) = \tau_{\overline{N}'}(x).$$

We must show that $e_w$ is well-defined and that the definition of $e_w$ is independent of our choice of $x$. Once we show that $e_w$ is well-defined, it is straightforward to show independence since every preimage of $w$ is of the form $xt$, where $t \in \mathcal{T}^o$.

**Proposition 6.8.** The functions $e_w \in \mathcal{H}$, where $w \in \overline{W}_{\text{aff}}$, are well-defined and form a $\mathbb{C}$-basis for $\mathcal{H}$.

**Proof:** (Well-defined) Suppose that $\gamma_i \in \overline{\Gamma}_0(4)$ and $x = \gamma_1 x \gamma_2$. To prove that $e_w$ is well-defined we must prove that $\tau_{\overline{N}'}(x) = \tau(\gamma_1)\tau_{\overline{N}'}(x)\tau(\gamma_2)$. Let $\gamma_1 = t_1 u_1$ and $\gamma_2 = u_2 t_2$, where $t_i \in \mathcal{T}^o$ and $u_j \in \Gamma_1(4)^*$. Thus, $\tau(\gamma_1)\tau_{\overline{N}'}(x)\tau(\gamma_2) = \tau(t_1)\tau_{\overline{N}'}(x)\tau(t_2) = \tau_{\overline{N}'}(t_1)\tau_{\overline{N}'}(x)\tau_{\overline{N}'}(t_2) = \tau_{\overline{N}'}(t_1 t_2 x)$. Proposition 3.4 implies that $x = t_1 t_2 x$. Thus $\tau_{\overline{N}'}(x) = \tau_{\overline{N}'}(t_1 t_2 x)$, as desired.

(Basis) Proposition 6.6 implies that $e_w$ generates the space of functions in $\mathcal{H}$ supported on the double coset $\overline{\Gamma}_0(4)x\overline{\Gamma}_0(4)$. Thus the functions $e_w$ form a basis of $\mathcal{H}$, as $w$ varies over elements of $\overline{W}_{\text{aff}}$. 

Our second step is to prove the multiplicative relations described in propositions 6.9 and 6.11. Using these relations we prove that $\mathcal{H}$ admits an Iwahori-Matsumoto presentation in Proposition 6.12.

**Proposition 6.9.** If $w_1, w_2 \in \overline{W}_{\text{aff}}$ and $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, then $e_{w_1} * e_{w_2} = e_{w_1 w_2}$.

**Proof:** The proof is similar to the proof of Proposition 6.2 in Savin [14]. Let $x_1, x_2 \in \overline{N}'$ represent $w_1$ and $w_2$ respectively. For each $x \in \overline{N}'$, to compute $e_{w_1} * e_{w_2}(x)$ we must find all cosets $\delta\overline{\Gamma}_0(4)$ such that

$$\delta\overline{\Gamma}_0(4) \subseteq \overline{\Gamma}_0(4)x_1\overline{\Gamma}_0(4) \quad \text{and} \quad \overline{\Gamma}_0(4)\delta^{-1}x \subseteq \overline{\Gamma}_0(4)x_2\overline{\Gamma}_0(4).$$ (30)
These containments only depend on $G$. So we may focus on finding all cosets $\text{pr}(\delta)\Gamma_0(4)$ such that

$$\text{pr}(\delta)\Gamma_0(4) \subseteq \Gamma_0(4)\text{pr}(x_1)\Gamma_0(4) \quad \text{and} \quad \Gamma_0(4)\text{pr}(\delta)^{-1}\text{pr}(x) \subseteq \Gamma_0(4)\text{pr}(x_2)\Gamma_0(4).$$

(31)

By Theorem 5.6 and the remark after Proposition 5.3, we know that $\Gamma_0(4)\text{pr}(x)\Gamma_0(4) = \Gamma_0(4)\text{pr}(x_1)\Gamma_0(4)$ and $\text{pr}(\delta)\Gamma_0(4) = \text{pr}(x_1)\Gamma_0(4)$. Thus it is enough for us to compute $e_{w_1} * e_{w_2}(x_1x_2)$. By definition

$$e_{w_1} * e_{w_2}(x_1x_2) = \int_G e_{w_1}(h)e_{w_2}(h^{-1}x_1x_2)dh = \sum_{\delta \in G/\Gamma_0(4)} e_{w_1}(\delta)e_{w_2}(\delta^{-1}x_1x_2).$$

(32)

We have seen that the only coset that satisfies (30) is $x_1\tilde{\Gamma}_0(4)$. Thus

$$e_{w_1}(x_1)e_{w_2}(x_2) = e_{w_1w_2}(x_1x_2),$$

(33)

as desired. The last equality follows because $\tau_{\tilde{N}'}$ is a representation of $\tilde{N}'$.

For $\alpha \in \Delta \cup \{\alpha_0\}$ we will write $e_\alpha \overset{\text{def}}{=} e_{w_\alpha}$. Next we will prove that the elements $e_\alpha$ satisfy a quadratic relation. We begin with the following lemma.

Lemma 6.10. Let $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Then

$$\tau_{\tilde{N}'}(w_\alpha(2^k)) + \tau_{\tilde{N}'}(w_\alpha(2^{k-1}))^{-1} = \epsilon\sqrt{2}I_E,$$

(34)

where $I_E$ is the identity map of $E$ and $\epsilon = \pm 1$. Moreover, $\epsilon$ is independent of $\alpha$ and $k$.

Proof: In this proof we will write $w_\alpha$ for $\tilde{w}_\alpha(1)$. First, we show that $\tau_{\tilde{N}'}(w_\alpha) + \tau_{\tilde{N}'}(w_\alpha)^{-1}$ is a scalar endomorphism. Specifically, for any $\beta \in \Delta$ we have $\tau^\circ(h_\beta(-1))[\tau_{\tilde{N}'}(w_\alpha) + \tau_{\tilde{N}'}(w_\alpha)^{-1}]\tau^\circ(h_\beta(-1))^{-1} = \tau_{\tilde{N}'}(w_\alpha) + \tau_{\tilde{N}'}(w_\alpha)^{-1}$, which follows from the Steinberg relations and because $\tau^\circ(h_\alpha(-1)) = \tau_{\tilde{N}'}(h_\alpha(-1))$. Since $(\tau^\circ, E)$ is irreducible, $\tau_{\tilde{N}'}(w_\alpha) + \tau_{\tilde{N}'}(w_\alpha)^{-1}$ is a scalar endomorphism, by Schur’s Lemma.

So $\tau_{\tilde{N}'}(w_\alpha) + \tau_{\tilde{N}'}(w_\alpha)^{-1} = cI_E$ for some scalar $c$. We square this equation to get $\tau_{\tilde{N}'}(h_\alpha(-1)) + \tau_{\tilde{N}'}(h_\alpha(-1))^{-1} + 2I_E = c^2I_E$. However, $\tau_{\tilde{N}'}(h_\alpha(-1)) + \tau_{\tilde{N}'}(h_\alpha(-1))^{-1} = \tau_{\tilde{N}'}(h_\alpha(-1))[I_E + \tau_{\tilde{N}'}(h_\alpha(-1))^{-2}] = \tau_{\tilde{N}'}(h_\alpha(-1))[I_E - I_E] = 0$. Thus $c = \pm\sqrt{2}$.

Now consider $\tau_{\tilde{N}'}(\tilde{w}_\alpha(2^k)) + \tau_{\tilde{N}'}(\tilde{w}_\alpha(2^{k-1}))$. Since $\tilde{w}_\alpha(2^k) = \tilde{h}_\alpha(2^k)\tilde{w}_\alpha$ and $\tilde{w}_\alpha(-2^k) = \tilde{h}_\alpha(2^k)\tilde{w}_\alpha(1)$ we see that

$$\tau_{\tilde{N}'}(\tilde{w}_\alpha(2^k)) + \tau_{\tilde{N}'}(\tilde{w}_\alpha(2^{k-1}))^{-1} = \tau_{\tilde{N}'}(\tilde{h}_\alpha(2^k))[\epsilon\sqrt{2}I_E].$$

(35)

But $\tau_{\tilde{N}'}(\tilde{h}_\alpha(2^k)) = I_E$ so the result follows.

We have already seen that $\epsilon$ is independent of $k$. Now we will show that it is independent of $\alpha$. Since we are considering simply-laced root systems, the action of $W$ on $\Phi$ has a single orbit. Let $\alpha, \beta \in \Phi$ and let $w \in W$ such that $w(\alpha) = \beta$. Then by the Steinberg relations we have that

$$\tau_{\tilde{N}'}(w)[\tau_{\tilde{N}'}(w_\alpha) + \tau_{\tilde{N}'}(w_\alpha)^{-1}]\tau_{\tilde{N}'}(w)^{-1} = \tau_{\tilde{N}'}(w_{w(\alpha)}) + \tau_{\tilde{N}'}(w_{w(\alpha)})^{-1} = \tau_{\tilde{N}'}(w_{\beta}) + \tau_{\tilde{N}'}(w_{\beta})^{-1}.$$

(36)
But since $\tau_{\tilde{\Delta}_I, \alpha}(w_\alpha) + \tau_{\tilde{\Delta}_I, \alpha}(w_\alpha)^{-1}$ is a scaler we also have
\[ \tau_{\tilde{\Delta}_I, \alpha}(w)\tau_{\tilde{\Delta}_I, \alpha}(w_\alpha) + \tau_{\tilde{\Delta}_I, \alpha}(w_\alpha)^{-1} \tau_{\tilde{\Delta}_I, \alpha}(w)^{-1} = \tau_{\tilde{\Delta}_I, \alpha}(w_\alpha) + \tau_{\tilde{\Delta}_I, \alpha}(w_\alpha)^{-1}. \] (37)

Thus $\epsilon$ is independent of $\alpha$.

**Proposition 6.11.** Let $\alpha \in \Delta \cup \{\alpha_0\}$. Then
\[ e_\alpha^2 = \epsilon \sqrt{2} e_\alpha + 4 \mathbb{I}, \]
where $\mathbb{I}$ is the function in $\mathcal{H}$ supported on $\tilde{\Gamma}_0(4)$ such that $\mathbb{I}(\gamma) = \tau(\gamma)$.

**Proof:** We will address the case of $\alpha \in \Delta$ completely. When $\alpha = \alpha_0$ the computation is similar (because equation (21) holds) and will be omitted.

Suppose that $\alpha \in \Delta$. We claim that $\text{supp}(e_\alpha^2) \subseteq \tilde{\Gamma}_0(4) \cup \tilde{\Gamma}_0(4) w_\alpha \tilde{\Gamma}_0(4)$. Using the Iwahori factorization and the Steinberg relations we can show that
\[ \tilde{\Gamma}_0(4) w_\alpha(1) \tilde{\Gamma}_0(4) w_\alpha(1) \tilde{\Gamma}_0(4) = \cup_{t \in \mathbb{Z}/4\mathbb{Z}} \tilde{\Gamma}_0(4)x_\alpha(t) \tilde{\Gamma}_0(4). \] (38)

For $t = 0$, $\tilde{\Gamma}_0(4)x_\alpha(t) \tilde{\Gamma}_0(4) = \tilde{\Gamma}_0(4)$. For $t = \pm 1$, $\tilde{\Gamma}_0(4)x_\alpha(t) \tilde{\Gamma}_0(4) = \tilde{\Gamma}_0(4)w_\alpha(1)\tilde{\Gamma}_0(4)$. For $t = 2$, $\tilde{\Gamma}_0(4)x_\alpha(t) \tilde{\Gamma}_0(4) = \tilde{\Gamma}_0(4)\tilde{\Gamma}_0\tilde{\Gamma}_0(4)$. Thus Proposition 6.4 implies $\text{supp}(e_\alpha^2) \subseteq \tilde{\Gamma}_0(4) \cup \tilde{\Gamma}_0(4) w_\alpha \tilde{\Gamma}_0(4)$ and we can write $e_\alpha^2 = ae_\alpha + b \mathbb{I}$, where $a, b \in \mathbb{C}$. To determine $a$ and $b$ it suffices to compute $e_\alpha^2(1)$ and $e_\alpha^2(w_\alpha)$.

We begin with a preliminary computation.

\[ e_\alpha^2(g) = \int_\tilde{G} e_\alpha(y)e_\alpha(y^{-1}g)dy \]

\[ = \sum_{y \in \tilde{\Gamma}_0(4)w_\alpha \tilde{\Gamma}_0(4)/\tilde{\Gamma}_0(4)} e_\alpha(y)e_\alpha(y^{-1}g) \]

\[ = \sum_{\delta \in \tilde{\Gamma}_0(4)/\tilde{\Gamma}_0(4) \cap w_\alpha \tilde{\Gamma}_0(4)w_\alpha^{-1}} e_\alpha(\delta w_\alpha)e_\alpha(w_\alpha^{-1}\delta^{-1}g) \]

\[ = \sum_{u \in \mathbb{Z}/4\mathbb{Z}} e_\alpha(x_\alpha(u)w_\alpha)e_\alpha(w_\alpha^{-1}x_\alpha(-u)g) \]

\[ = e_\alpha(w_\alpha) \sum_{u \in \mathbb{Z}/4\mathbb{Z}} e_\alpha(w_\alpha^{-1}x_\alpha(u)g). \]

It remains to compute $e_\alpha^2(1)$ and $e_\alpha^2(w_\alpha)$. The computation of $e_\alpha^2(1)$ is straightforward and will be omitted. One finds that $e_\alpha^2(1) = 4I = 4 \mathbb{I}(1)$.

Finally we will compute $e_\alpha^2(w_\alpha)$.

\[ e_\alpha^2(w_\alpha) = e_\alpha(w_\alpha) \sum_{u \in \mathbb{Z}/4\mathbb{Z}} e_\alpha(w_\alpha^{-1}x_\alpha(u)w_\alpha) \]

\[ = e_\alpha(w_\alpha) \sum_{u = \pm 1} e_\alpha(x_\alpha(u)) \]
= \epsilon_\alpha(w_\alpha)(e_\alpha(w_\alpha) + e_\alpha(w_\alpha^{-1}))
= \epsilon \sqrt{2} e_\alpha(w_\alpha).

The last equality follows from Lemma 6.10.

Remark: The previous result implies that \((\epsilon \sqrt{2} e_\alpha - 2)(\epsilon \sqrt{2} e_\alpha + 1) = 0\).

Now we will prove that \(\mathcal{H}\) has an Iwahori-Matsumoto presentation. Let \(H\) be the unital \(\mathbb{C}\)-algebra generated by the symbols \(T_w\), where \(w \in \tilde{W}_{\text{aff}}\), subject to the relations

\[ T_{w_1}T_{w_2} = T_{w_1w_2}, \quad \text{for } w_j \in \tilde{W}_{\text{aff}} \text{ such that } \ell(w_1w_2) = \ell(w_1) + \ell(w_2); \quad (39) \]
\[ (T_\alpha + 1)(T_\alpha - 2) = 0, \quad \text{for any } \alpha \in \Delta \cup \{\alpha_0\}. \quad (40) \]

This is the algebra studied in Section 3 of [10], where \(v\) specialized to \(\sqrt{2}\) and Lusztig’s \(L\) is equal to our \(\ell\).

Proposition 6.12. The linear map \(\Psi : H \to \mathcal{H}\) defined by \(\Psi(T_w) = (\epsilon \sqrt{2})^{\ell(w)} e_w\) is an isomorphism of \(\mathbb{C}\)-algebras.

Proof: By Proposition 6.9 and Proposition 6.11 we know that \(\Psi\) is a \(\mathbb{C}\)-algebra homomorphism. As \(w\) varies over \(\tilde{W}_{\text{aff}}\) the elements \(T_w\) and \((\epsilon \sqrt{2})^{\ell(w)} e_w\) form a basis for \(H\) and \(\mathcal{H}\) respectively. Thus \(\Psi\) is an isomorphism of \(\mathbb{C}\)-algebras.

Proposition 6.13. For all \(w \in \tilde{W}_{\text{aff}}\), the element \(e_w \in H\) is invertible.

Proof: Let \(w = \sigma w' \in \tilde{W}_{\text{aff}} \cong \Omega \times (W \ltimes 2Y)\), where \(\sigma \in \Omega\) and \(w' \in W \ltimes 2Y\). The group \(W \ltimes 2Y\) is an affine Weyl group with generators \(w_{\alpha_0}, \ldots, w_{\alpha_r}\). Suppose that \(w' = w_{\alpha_{i_1}} \ldots w_{\alpha_{i_k}}\) is a minimal expression for \(w'\) with respect to these generators. Then Proposition 6.9 implies that \(e_w = e_{\sigma}e_{\alpha_{i_1}} \ldots e_{\alpha_{i_k}}\). By Proposition 6.11, \(e_{\alpha_{i_j}}\) is invertible. Since \(e_{\sigma}e_{\sigma^{-1}} = 1\), \(e_{\sigma}\) is invertible. Thus \(e_w\) is invertible.

Remark: It is this step where we deviate from Savin [14]. To establish the analog of Proposition 6.13 in [14], Savin studies the unramified principal series and invokes the theory of the Bernstein center. In the present paper, we can circumvent these issues using Theorem 5.6. Using this theorem we can prove an Iwahori-Matsumoto presentation, from which the invertibility is a simple consequence. In the tame case considered by Savin, the difficulty stems from the failure of the analog of equation (21).

Let \(\mathcal{H}_0\) be the subalgebra of \(\mathcal{H}\) consisting of functions supported on \(\tilde{G}(\mathbb{Z}_2)\). This subalgebra is generated by the elements \(e_\alpha\), where \(\alpha \in \Delta\). Let \(\mathcal{A}\) be the subalgebra of \(\mathcal{H}\) generated by \(e_\lambda\), where \(\lambda \in \tilde{Y}\) and \(\lambda\) is dominant.

Proposition 6.14. We have \(\mathcal{H} = \mathcal{H}_0 \cdot \mathcal{A} \cdot \mathcal{H}_0\).

Proof: The proof of Proposition 6.3 in Savin [14] directly adapts to this case. □

For \(\lambda \in \tilde{Y}\) let

\[ t_\lambda = q^{-\langle \lambda, \psi \rangle} e^{\lambda_1} e_{\lambda_2}^{-1}, \]
where $\lambda_1, \lambda_2 \in \tilde{Y}$ are dominant and $\lambda = \lambda_1 - \lambda_2$. Note that the definition of $t_\lambda$ does not depend on the choice of $\lambda_1$ and $\lambda_2$.

Now we prove our main theorem. The Hecke algebra $\mathcal{H}$ has a Bernstein presentation.

**Theorem 6.15.** Let $\mathcal{H}$ be the $\mathbb{C}$-algebra generated by $f_\alpha$, for all $\alpha \in \Delta$, and $u_\lambda$, for all $\lambda \in \tilde{Y}$ modulo the relations

1. $(f_\alpha - 2)(f_\alpha + 1) = 0$;
2. $f_\alpha \cdot f_\beta = f_\beta \cdot f_\alpha$, if $\langle \alpha, \beta \rangle = 0$;
3. $u_\lambda \cdot u_\lambda' = u_{\lambda + \lambda'}$.

Let $2m = \langle \alpha, \lambda \rangle$, where $\alpha$ is a simple root.

4. $f_\alpha \cdot u_\lambda = u_{\lambda + \lambda'}$.

The map $A : \mathcal{H} \to \mathcal{H}$ defined by

$$
\begin{cases}
A(u_\lambda) = t_\lambda \\
A(f_\alpha) = \epsilon(2)e_{w_\alpha}
\end{cases}
$$

is an isomorphism of $\mathbb{C}$-algebras.

**Proof:** For the Hecke algebra $\mathcal{H}$, we established the analog identity (1), in Proposition 6.11, and the analogs of identities (2) and (3), in Proposition 6.9. Identity (4) in $\mathcal{H}$ is exactly Proposition 3.6 in [10], which is applicable by Proposition 6.12.

We have shown that $\mathcal{H}$ satisfies identities (1), (2), (3), and (4). Thus the map $A : \mathcal{H} \to \mathcal{H}$ is a well-defined homomorphism of algebras. Proposition 6.14 implies that $A$ is surjective, and to prove that $A$ is injective one can apply the trick of Lemma 7.6 and the discussion immediately follow it in Savin [14]. Thus the map $A : \mathcal{H} \to \mathcal{H}$ is is an isomorphism of algebras.

**Corollary 6.16.** The subalgebra generated by all of the $t_\lambda$ is isomorphic to the group algebra $\mathbb{C}[\tilde{Y}]$ and the multiplication map defines an isomorphism of vector spaces

$$
\mathbb{C}[\tilde{Y}] \otimes \mathcal{H}_0 \cong \mathcal{H}.
$$

### 7. Representations Generated By $\tau$-isotypic Vectors

Let $\mathcal{R}(\tilde{G})$ be the category of smooth $\tilde{G}$-representations, and let $\mathcal{R}(\tilde{G}, \tau)$ be the full subcategory of representations that are generated by their $\tau$-isotypic vectors.

Now we prove that the Bernstein component of the genuine unramified principal series is $\mathcal{R}(\tilde{G}, \tau)$.

**Theorem 7.1.**

1. The category $\mathcal{R}(\tilde{G}, \tau)$ is closed relative to subquotients in $\mathcal{R}(\tilde{G})$. 

(2) For every irreducible object \((\pi, \mathcal{V})\) in \(\mathcal{R}G, \tau\), there is an unramified character \(\chi : T \to \mathbb{C}^\times\) such that \((\pi, \mathcal{V})\) is isomorphic to an irreducible subquotient of \(\text{Ind}_B^G(i(\chi))\).

We begin with a few preliminary results. Recall that \(\tau\) is a genuine irreducible representation of \(\widetilde{T}\), and \(\tau\) is the inflation of \(\tau\) to \(\Gamma_0(4)\). We will write \(\tau_{\tilde{T}}\) for the restriction \(\tau\) to \(\widetilde{T} \cap \Gamma_0(4)\).

**Lemma 7.2.** Let \(\pi\) be a smooth irreducible \(\widetilde{T}\)-representation. Then the \(\tau_{\tilde{T}}\)-isotypic subspace \(\pi^{\tau_{\tilde{T}}} \neq 0\) if and only if \(\pi \cong i(\chi)\) as \(\tilde{T}\)-representations, for some unramified character \(\chi : T \to \mathbb{C}^\times\).

**Proof:** Suppose that \(\pi^{\tau_{\tilde{T}}} \neq 0\). Since \(\widetilde{T} \cap \Gamma_0(4) \cong \mathbb{T} \times T_1^*\) and \(\tau_{\tilde{T}} \cong \tau \otimes 1\) under this isomorphism, the central character of \(\pi\) is equal to the central character of \(i(\chi)\) for some unramified \(\chi\). Since an irreducible representation of \(\widetilde{T}\) in which \(T_1^*\) acts by the identity is determined by its character, we have \(\pi \cong i(\chi)\).

Conversely, suppose that \(\pi \cong i(\chi)\). Consider the subspace \(\pi^{T_{\tilde{T}}}\) of \(T_{\tilde{T}}\)-invariants. It is a \(\mathbb{T} \times T_1^*\)-subrepresentation because \(\tilde{T} \cap \Gamma_0(4) \cong \mathbb{T} \times T_1^*\). We can see that \(\pi^{T_{\tilde{T}}} \neq 0\) by showing that \(V(\gamma_{\tilde{T}})T_{\tilde{T}} \neq 0\). This can be seen by viewing \(V(\gamma_{\tilde{T}})\) as an induced representation. Specifically, let \(\gamma_2\) be the character of the maximal abelian subgroup \(T_1^*YO(\tilde{T})\mu_2\) extending \(\gamma_2\) such that \(\gamma_2|T_1^* = 1\). Then \(V(\gamma_{\tilde{T}}) \cong \text{Ind}_{T_1^*YO(\tilde{T})\mu_2}^{T_{\tilde{T}}} (\gamma_{\tilde{T}})\) and the functions supported on \(T_1^*YO(\tilde{T})\mu_2\) are fixed by \(T_1^*\).

**Proposition 7.3.** Let \((\pi, \mathcal{V})\) be a smooth \(G\)-representation. We will write \((\pi_U, \mathcal{V}(\mathcal{U}))\) for the normalized Jacquet module with respect to \(U^*\), and \(q : \mathcal{V} \to \mathcal{V}(\mathcal{U})\) for the canonical quotient map. Then the induced map

\[ q : \mathcal{V} \to \mathcal{V}(\mathcal{U}) \]

is an isomorphism.

**Proof:** This follows from Theorem 7.9 in Bushnell-Kutzko [4]. Hypothesis a) in Theorem 7.9 holds because \(\Gamma_0(4)\) possess an Iwahori-factorization; hypothesis b) holds because of our Proposition 6.13.

**Proposition 7.4.** Let \(\pi\) be a smooth irreducible \(G\)-representation. Then the \(\tau\)-isotypic space \(\pi^\tau \neq 0\) if and only if \(\text{Hom}_G(\pi, \text{Ind}_B^G(i(\chi)) \neq 0\) for some unramified character \(\chi : T \to \mathbb{C}^\times\).

**Proof:** Suppose that the \(\pi^\tau \neq 0\). By Proposition 7.3, \(\pi^{T_{\tilde{T}}} \cong \pi^\tau \neq 0\). Since \(\pi\) is irreducible, \(\pi_U\) is finitely generated. Because \(\pi_U^{T_{\tilde{T}}} \neq 0\) we can apply Lemma 7.2 and the Bernstein decomposition to see that \(\pi_U\) has an irreducible quotient isomorphic to \(i(\chi)\) for some unramified character \(\chi : T \to \mathbb{C}^\times\). Thus by Frobenius reciprocity \(0 \neq \text{Hom}_F(\pi_U, i(\chi)) \cong \text{Hom}_G(\pi, \text{Ind}_B^G(i(\chi)))\).

Conversely, suppose that \(\text{Hom}_G(\pi, \text{Ind}_B^G(i(\chi)) \neq 0\). By Frobenius reciprocity there is a surjective map \(\psi : \pi_U \to i(\chi)\) of \(\tilde{T}\)-representations, since \(i(\chi)\) is irreducible. By Lemma
7.2. \( \tau_F \) is a \( \tilde{T} \cap \Gamma_0(4) \)-subrepresentation of \( i(\chi) \). Since \( \tau_F \) is an irreducible representation of the compact group \( \tilde{T} \cap \Gamma_0(4) \), \( \tau_F \) must be a \( \tilde{T} \cap \Gamma_0(4) \)-subrepresentation of \( \pi_U \). Thus \( \pi^{\tau_F}_U \neq 0 \). Finally, by Proposition 7.3, \( \pi^\tau \cong \pi^{\tau_F}_U \neq 0 \).

**Proof of Theorem 7.1:** We begin with item (1). Let \((\pi, \mathcal{V})\) be smooth \( \widetilde{G} \)-representation generated by \( \mathcal{V}^\tau \). Using the Bernstein decomposition of \( \mathcal{R}(\widetilde{G}) \) it suffices to assume that \((\pi, \mathcal{V})\) is contained in a single Bernstein component. Under this assumption, we claim that \( \mathcal{V} \) is in the block associated to the \( \tilde{T} \)-representation \( i(1) \). To see this we consider \( \mathcal{V} \) a \( \widetilde{G} \)-representation generated by some \( v \in \mathcal{V}^\tau - \{0\} \). Since \( \mathcal{W} \) is finitely generated it has an irreducible quotient \( \mathcal{W}' \). Since the image of \( v \) is nonzero in \( \mathcal{W}' \), it follows that \( (\mathcal{W}')^\tau \neq 0 \). Thus by Proposition 7.4, \( \mathcal{V} \) has an irreducible subquotient in the block associated to the \( \tilde{T} \)-representation \( i(1) \), and it follows that \( \mathcal{V} \) must be in the same block.

Now suppose that \( \mathcal{W} \) is any subrepresentation of \( \mathcal{V} \) such that \( \mathcal{W}^\tau \) does not generate \( \mathcal{W} \). It suffices to assume that \( \mathcal{W}^\tau = 0 \), since we can take the quotient of \( \mathcal{V} \) and \( \mathcal{W} \) by the \( \widetilde{G} \)-subrepresentation generated by \( \mathcal{W}^\tau \). Under this assumption, we claim that \( \mathcal{W} \) has an irreducible subquotient that does not live in the block associated to the \( \tilde{T} \)-representation \( i(1) \). This follows from Proposition 7.4, since the \( \tau \)-isotypic subspace of any irreducible quotient of \( \mathcal{W} \) must be trivial. This implies that \( \mathcal{W} \) and \( \mathcal{V} \) are in different blocks. This contradiction prove item (1).

Item (2) is exactly Proposition 7.4.

**Remark:** In the language of Bushnell-Kutzko [4], Theorem 7.1 states that \( \mathcal{R}(\widetilde{G}, \tau) \) is the Bernstein component associated to the inertial support \([\tilde{T}, i(1)]_{\widetilde{G}}\); Proposition 7.4 states that \((\Gamma_0(4), \tau)\) is a \([\tilde{T}, i(1)]_{\widetilde{G}}\)-type. Moreover, Theorem 4.3 in Bushnell-Kutzko [4] states that the functor \( \mathcal{V} \mapsto \text{Hom}_{\Gamma_0(4)}(\tau, \mathcal{V}) \) defines an equivalence of categories between \( \mathcal{R}(\widetilde{G}, \tau) \) and \( \mathcal{H}\text{-mod} \). Thus the Bernstein component associated to \([\tilde{T}, i(1)]_{\widetilde{G}}\) is equivalent to \( \mathcal{H}\text{-mod} \).

8. **Local Shimura Correspondence**

Consider the split algebraic group \( G' = G/Z_2 \), where \( Z_2 \) is the 2-torsion of the center of \( G \). The possible 2-groups which can arise as \( Z_2 \) are described in Table 1. Let \( I' \) denote an Iwahori subgroup of \( G' \). The Bernstein Presentation of the Iwahori-Hecke algebra \( \mathcal{H}(G', I') \) of \( G' \) implies the vector space isomorphism \( \mathcal{H}(G', I') \cong \mathbb{C}[Y^* \cap \frac{1}{2}Y] \otimes \mathcal{H} \), since \( Y^* \cap \frac{1}{2}Y \) is the co-weight lattice of \( G' \). However, \( 2(Y^* \cap \frac{1}{2}Y) = \tilde{Y} \), so \( \lambda \mapsto 2\lambda \) defines an algebra isomorphism \( \mathbb{C}[Y^* \cap \frac{1}{2}Y] \to \mathbb{C}[\tilde{Y}] \), which extends to an algebra isomorphism \( \mathcal{H}(G', I') \cong \mathcal{H} \), by Theorem 6.15 and Corollary 6.16.

| \( \Phi \) | \( A_{2n} \) | \( A_{2n+1} \) | \( D_{2n} \) | \( D_{2n+1} \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |
|---|---|---|---|---|---|---|---|
| \( Z_2 \) | \{1\} | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{C} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \{1\} | \( \mathbb{Z}/2\mathbb{Z} \) | \{1\} |

Table 1. Central 2-torsion of simple, simply-laced, simply-connected Chevalley groups.
Now we can describe the Shimura correspondence. Let $\mathcal{R}(G', I')$ be the category of smooth representations of $G' = G/Z_2$ that are generated by their $I'$ fixed vectors. Recall that $\mathcal{R}(\tilde{G}, \tau)$ is the category of smooth representations of $\tilde{G}$ that are generated by their $\tau$-isotypic vectors.

**Theorem 8.1.** The isomorphism $\mathcal{H}(G', I') \xrightarrow{\phi} \mathcal{H}$ induces an equivalence of categories between $\mathcal{R}(G', I')$ and $\mathcal{R}(\tilde{G}, \tau)$.

**Proof:** The algebra isomorphism $\mathcal{H}(G', I') \xrightarrow{\phi} \mathcal{H}$ induces an equivalence of categories between $\mathcal{H}(G', I')$-mod and $\mathcal{H}$-mod.

By results of Bushnell-Kutzko [4] (Theorem 4.3), the functor $V \mapsto \text{Hom}_\Gamma(1, V) = V'$ defines an equivalence of categories between $\mathcal{R}(G', I')$ and $\mathcal{H}(G', I')$-mod. Thus, $\mathcal{R}(G', I')$ is equivalent to $\mathcal{H}$-mod.

Again, by results of Bushnell-Kutzko [4] (Theorem 4.3), the functor $V \mapsto \text{Hom}_{\tilde{\Gamma}_0(4)}(\tau, V)$ defines an equivalence of categories between $\mathcal{R}(\tilde{G}, \tau)$ and $\mathcal{H}$-mod. Theorem 4.3 is applicable because our Proposition 7.4 implies that $(\tilde{\Gamma}_0(4), \tau)$ is a $[\tilde{T}, i(1)]_{\tilde{G}}$-type. Thus $\mathcal{R}(G', I')$ is equivalent to $\mathcal{R}(\tilde{G}, \tau)$.

**Remark:** Finally, we enumerate the choices we made in constructing $\mathcal{H}(G', I') \xrightarrow{\phi} \mathcal{H}$. Ostensibly we have made two choices. However, only one of these choices is material, the choice of a genuine character of $Z(\tilde{T}^\circ) \cong \text{Hom}(\tilde{Y}/2Y, \mathbb{C}^\times)$.

First, the Hecke algebra $\mathcal{H}$ depends on the choice of an irreducible genuine Weyl group invariant $\tilde{T}^\circ$-representation $(\tau^\circ, E)$. Since each irreducible genuine $\tilde{T}^\circ$-representation is Weyl group invariant and determined by its central character, we can count the number of genuine central characters of $\tilde{T}^\circ$. Note that

$$1 \to \mu_2 \to \tilde{T}^\circ \to Y \otimes \mu_2 \cong \tilde{Y}/2Y \to 1. \quad (43)$$

One can show that the center of $\tilde{T}^\circ$ is the preimage of $\tilde{Y}/2Y$. So we get the split exact sequence of $\mathbb{F}_2$-vector spaces

$$1 \to \mu_2 \to Z(\tilde{T}^\circ) \to \tilde{Y}/2Y \to 1. \quad (44)$$

Thus we see that genuine characters of $Z(\tilde{T}^\circ)$ are in bijection with $\text{Hom}(\tilde{Y}/2Y, \mathbb{C}^\times)$. An explicit description of $\tilde{Y}/2Y$ can be found in Section 16.1 in Gan-Gao [5]. We note that $Z_2 \cong \tilde{Y}/2Y$.

Second, we chose a particular normalization for the basis elements $e_w \in \mathcal{H}$. (Recall the discussion before Proposition 6.8.) This normalization depends on a choice of an extension of $\tau^\circ$ to $W$, which we called $\tau_W$. The number of such extensions is in bijection with the set $\text{Hom}(W/\tilde{T}^\circ, \mathbb{C}^\times) \cong \text{Hom}(W, \mathbb{C}^\times)$. We claim that $\text{Hom}(W, \mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$, where the nontrivial character is defined by $w \mapsto (-1)^{\ell(w)}$. This is the result of the following standard facts about Weyl groups: the group $W$ is generated by simple reflections; the Weyl group acts transitively on roots of the same length; for any $\alpha, \beta \in \Phi$, $w_{\alpha}w_{\beta}w_{\alpha}^{-1} = w_{w_{\alpha}(\beta)}$. Furthermore, we see that if $\tau_W$ is one of the extensions, then the other must be given by
\( \gamma_W \otimes \text{sign} \), where now we write \( \text{sign} \) for the inflation of the sign representation of \( \mathcal{W} \) to \( \mathcal{W} \).

If we replace \( \tau_W \) by \( \tau_W \otimes \text{sign} \), this has the effect of changing \( \epsilon \) to \(-\epsilon\); \( e_\alpha \) to \(-e_\alpha \), where \( \alpha \in \Delta \); and \( t_\lambda \) remains invariant for \( \lambda \in \bar{\Delta} \). Thus \( \sqrt{2} e_\alpha \) and \( t_\lambda \) remain invariant for \( \alpha \in \Delta \) and \( \lambda \in \bar{\Delta} \). This implies that the isomorphism of Theorem 6.15, and thus the isomorphism \( \mathcal{H}(G', I') \overset{\phi}{\rightarrow} \mathcal{H} \), does not depend on our choice of extension of \( \tau^\circ \) from \( \bar{T}^\circ \) to \( \mathcal{W} \).

In summary, the isomorphism \( \mathcal{H}(G', I') \overset{\phi}{\rightarrow} \mathcal{H} \) only depends on the choice of a genuine character of \( Z(\bar{T}^\circ) \cong \text{Hom}(\bar{\Delta}/2\bar{\Delta}, \mathbb{C}^\times) \).

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