 CONDITIONALLY ATOMLESS EXTENSIONS OF SIGMA ALGEBRAS

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Abstract. We give two equivalent definitions of sigma algebras that are atomless conditionally to a smaller sigma algebra.

1. Notation

In this paper\(^1\) we will work with a probability space equipped with three sigma algebras \((\Omega, \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2, \mathbb{P})\). The sigma algebra \(\mathcal{F}_0\) is supposed to be trivial \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) whereas the sigma algebra \(\mathcal{F}_2\) is supposed to express innovations with respect to \(\mathcal{F}_1\). Since we do not put topological properties on the set \(\Omega\) we will make precise definitions later that do not use conditional probability kernels. But essentially we could say that we suppose that conditionally on \(\mathcal{F}_1\) the probability \(\mathbb{P}\) is atomless on \(\mathcal{F}_2\). We will show that such an hypothesis implies that there is an atomless sigma algebra \(\mathcal{B} \subset \mathcal{F}_2\) that is independent of \(\mathcal{F}_1\). In some (under topological hypotheses on \(\Omega, \mathcal{F}_1, \mathcal{F}_2\)) cases the conditional expectation with respect to \(\mathcal{F}_1\) is given by integration with respect to a kernel. We will use the notation \(K\) for such a kernel. More precisely: the mapping \(K\colon \Omega \times \mathcal{F}_2 \to \mathbb{R}_+\) satisfies

1. For almost every \(\omega \in \Omega\), the mapping \(K(\omega, \cdot)\colon \mathcal{F}_2 \to [0, 1]\) is a probability. It is no restriction to suppose that this property holds for every \(\omega \in \Omega\).
2. For each \(A \in \mathcal{F}_2\), the mapping \(K(\cdot, A)\colon \Omega \to [0, 1]\) is \(\mathcal{F}_1\) measurable.
3. For each \(\xi \in L^1(\Omega, \mathcal{F}_2, \mathbb{P})\) we have that almost surely

\[
\mathbb{E}[\xi \mid \mathcal{F}_1](\omega) = \int \xi(\tau) K(\omega, d\tau).
\]

The existence of such a kernel is not always easy to verify. Sometimes it is part of the model that is studied. Applying the property above and integrating with respect to \(\mathbb{P}\) gives

\[
\mathbb{P}[A] = \int \mathbb{P}[d\omega] K(\omega, A).
\]

Or for general \(\xi \in L^1\):

\[
\int_{\Omega} \mathbb{P}[dx] \xi(x) = \int_{\Omega} \mathbb{P}[d\omega] \int_{\Omega} \xi(\tau) K(\omega, d\tau).
\]

Part of the results were developed and used in my paper on commonotonicity, see [1]

\(^{1}\)This paper is to be seen as an exercise in measure theory. It will not be sent to a math. journal.

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2. Atomless Extension of Sigma Algebras

**Definition 1.** We say that \( \mathcal{F}_2 \) is atomless conditionally to \( \mathcal{F}_1 \) if the following holds. For every \( A \in \mathcal{F}_2 \) there exists a set \( B \subset A, B \in \mathcal{F}_2 \), such that \( 0 < E[1_B \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1] \) on the set \( \{E[1_A \mid \mathcal{F}_1] > 0\} \).

In case the conditional expectation could be calculated with a – under extra topological conditions – regular probability kernel, say \( K(\omega, A) \), then the above definition is a measure theoretic way of saying that the probability measure \( K(\omega, \cdot) \) is atomless for almost every \( \omega \in \Omega \). This equivalence will be the topic of the next section.

**Theorem 1.** \( \mathcal{F}_2 \) is atomless conditionally to \( \mathcal{F}_1 \) if for every \( A \in \mathcal{F}_2 \), \( P[A] > 0 \), there is \( B \subset A \) such that

\[
P[0 < E[1_B \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1]] > 0.
\]

**Proof** The proof is a standard exhaustion argument. For completeness we give a proof. Let \( \mathcal{D} \) be the collection of \( \mathcal{F}_1 \)-measurable sets:

\[
\mathcal{D} = \{\{0 < E[1_B \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1]\} \mid B \subset A\}
\]

We show that there is a biggest set in \( \mathcal{D} \) and this set must then be equal to \( \{E[1_A \mid \mathcal{F}_1] > 0\} \).

To show that there is a biggest set in \( \mathcal{D} \) it is sufficient to show that \( \mathcal{D} \) is stable for countable unions. Let \( D_n \) be a sequence in \( \mathcal{D} \) and suppose that for each \( n \) we have a set \( B_n \subset A \) such that \( D_n = \{0 < E[1_{B_n} \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1]\} \). Now take

\[
B = \bigcup_n (B_n \cap (D_n \setminus (\cup_{k<n} D_k)))).
\]

It is easy to check that \( \{0 < E[1_B \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1]\} = \bigcup_n D_n \) and therefore \( \bigcup_n D_n \in \mathcal{D} \).

Let \( D = \{0 < E[1_B \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1]\} \) be a maximum in \( \mathcal{D} \). Suppose now that \( P\{E[1_A \mid \mathcal{F}_1] > 0\} > 0 \). According to the hypothesis of the theorem, there will be a set \( B' \subset (A \setminus D) \) with \( B' \subset \{0 < E[1_{B'} \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1]\} \) and non-negligible. Since \( D \cup D' \in \mathcal{D} \) and \( D \cap D' = \emptyset \), the element \( D \) is not a maximum, a contradiction.

The main result of this section is the following

**Theorem 2.** \( \mathcal{F}_2 \) is atomless conditionally to \( \mathcal{F}_1 \) if and only if there exists an atomless sigma algebra \( \mathcal{B} \subset \mathcal{F}_2 \) that is independent of \( \mathcal{F}_1 \).

The “if” part is easy but requires some continuity argument. Because \( \mathcal{B} \) is atomless, there is a \( \mathcal{B} \)-measurable, \([0, 1]\) uniformly distributed random variable \( U \). The sets \( B_t = \{U \leq t\}, 0 \leq t \leq 1 \) form an increasing family of sets with \( P[B_t] = t \). Let \( A \in \mathcal{F}_2 \) and let \( F = \{0 < E[1_A \mid \mathcal{F}_1]\} \). We may suppose that \( P[F] > 0 \) since otherwise there is nothing to prove. We will show that there is \( t \in [0, 1] \) with \( P[0 < E[1_{A \cap B_t} \mid \mathcal{F}_1] < E[1_A \mid \mathcal{F}_1]] > 0 \). According to the previous theorem, \( \mathcal{F}_2 \) is conditionally atomless with respect to \( \mathcal{F}_1 \). Obviously for \( 0 \leq s \leq t \leq 1 \) we have, by independence of \( \mathcal{B} \) and \( \mathcal{F}_1 \):

\[
\|E[1_A \cap B_t \mid \mathcal{F}_1] - E[1_A \cap B_s \mid \mathcal{F}_1]\|_\infty \leq \|E[1_{B_t \setminus B_s} \mid \mathcal{F}_1]\|_\infty = t - s.
\]

It follows that there is a set of measure 1, say \( \Omega' \), such that for all \( s \leq t \), rational,

\[
|E[1_{A \cap B_t} \mid \mathcal{F}_1] - E[1_{A \cap B_s} \mid \mathcal{F}_1]| \leq t - s
\]

on \( \Omega' \). For \( \omega \in \Omega' \) we can extend the function

\[
\{q \in [0, 1] \mid q \text{ rational}\} \to E[1_{A \cap B_q} \mid \mathcal{F}_1](\omega)
\]
to a continuous function on $[0,1]$. The resulting continuous extension then represents $(\mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1])_t$. For $t = 0$ we have zero and for $t = 1$ we find $\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]$. Because for $\omega \in \Omega'$, the trajectories are continuous, a simple application of Fubini's theorem shows that the real valued function

$$ t \to \mathbb{P} [0 < \mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]] $$

becomes strictly positive for some $t$. For completeness let us now give the details of the application of Fubini's theorem. Suppose on the contrary that for all $\omega \in \Omega'$, the mapping $A \mapsto \mathbb{P}[0 < \mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]]$ has Lebesgue measure zero. However, for $\omega \in \Omega'$, the mapping $A \mapsto \mathbb{P}[0 < \mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]]$ has $m \times \mathbb{P}$ measure zero ($m$ denotes Lebesgue measure). By Fubini's theorem we have that for almost all $\omega \in \Omega'$, the set

$$ \{ t \mid 0 < \mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1](\omega) < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1](\omega) \} $$

must have Lebesgue measure zero. However, for $\omega \in \Omega'$, this contradicts the continuity of the mapping

$$ t \to \mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1](\omega). $$

The proof of the “only if” part is broken down in several steps. We will without further notice, always suppose that $\mathcal{F}_2$ is atomless conditionally to $\mathcal{F}_1$.

**Lemma 1.** Suppose $A \in \mathcal{F}_1$ and $C \subset A$ is such that $\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1] > 0$ on $A$. Then we can construct a decreasing sequence of sets $(B_n)_{n \geq 0}$, $B_n \subset C$, such that $0 < \mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1] \leq 2^{-n}$ on $A$.

**Proof** The statement is obviously true for $n = 0$ since we can take $B_0 = C$. We now proceed by induction and suppose the statement holds for $n$. So the set $B_n \subset A$ satisfies $0 < \mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1] \leq 2^{-n}$ on $A$. Clearly $A \subset \{ \mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1] > 0 \}$. By assumption there is a set $D \subset B_n$ such that on $A \subset \{ \mathbb{E}[\mathbf{1}_{A} \mid \mathcal{F}_1] > 0 \}$ we have

$$ 0 < \mathbb{E}[\mathbf{1}_{D} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1]. $$

We now take

$$ B_{n+1} = \left( D \cap \left\{ \mathbb{E}[\mathbf{1}_{D} \mid \mathcal{F}_1] \leq \frac{1}{2} \mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1] \right\} \right) \cup \left( (B_n \setminus D) \cap \left\{ \mathbb{E}[\mathbf{1}_{D} \mid \mathcal{F}_1] > \frac{1}{2} \mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1] \right\} \right). $$

The set $B_{n+1}$ satisfies the requirements.

**Lemma 2.** Let $C \in \mathcal{F}_2$ and let $h: \Omega \to [0,1]$ be $\mathcal{F}_1$ measurable. Then there is a set $B \subset C$ such that $\mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] = h \mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]$.

**Proof** Let $B_0 = \emptyset$. Inductively we define for $n \geq 1$, classes $\mathcal{B}_n$ and sets $B_n \in \mathcal{B}_n$. For $n \geq 1$ let

$$ \mathcal{B}_n = \{ B_{n-1} \subset B \subset C \mid B \in \mathcal{F}_2, \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] \leq h \mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1] \}. $$

Let $\beta_n = \sup \{ \mathbb{P}[B] \mid B \in \mathcal{B}_n \}$ and take $B_n \in \mathcal{B}_n$ such that $\mathbb{P}[B_n] \geq (1 - 2^{-n})\beta_n$. Clearly $B_n$ is non-decreasing and let $B_{\infty} = \cup_n B_n$. Obviously

$$ \mathbb{P}[B_{\infty}] \geq \limsup \beta_n \geq \liminf \beta_n \geq \lim \mathbb{P}[B_n] = \mathbb{P}[B_{\infty}]. $$
We claim that \( \mathbb{E}[1_{B_\infty} \mid \mathcal{F}_1] = h \mathbb{E}[1_C \mid \mathcal{F}_1] \). Obviously we already have that \( \mathbb{E}[1_{B_\infty} \mid \mathcal{F}_1] \leq h \mathbb{E}[1_C \mid \mathcal{F}_1] \). If \( \mathbb{P}[\mathbb{E}[1_{B_\infty} \mid \mathcal{F}_1] < h \mathbb{E}[1_C \mid \mathcal{F}_1]] > 0 \) then \( \mathbb{P}[B] < \mathbb{P}[C] \) and there must be \( m \geq 1 \) such that \( \mathbb{P}[\mathbb{E}[1_{B_m} \mid \mathcal{F}_1] < h \mathbb{E}[1_C \mid \mathcal{F}_1] - 2^{-m}] > 0 \). The previous lemma allows to find \( D \subset C \setminus B_\infty, \mathbb{P}[D] = \eta > 0 \), such that \( 0 < \mathbb{E}[1_D \mid \mathcal{F}_1] \leq 2^{-m} \) on the set \( \{\mathbb{E}[1_B \mid \mathcal{F}_1] < h \mathbb{E}[1_C \mid \mathcal{F}_1] - 2^{-m}\} \) and zero elsewhere. The set \( D \cup B_\infty \) is in all classes \( B_n \) and for \( n \) big enough:

\[
\beta_n \geq \mathbb{P}[D \cup B_\infty] \geq \mathbb{P}[B_n] + \eta \geq (1 - 2^{-n}) \beta_n + \eta \geq \beta_n + \eta - 2^{-n} > \beta_n,
\]
yielding a contradiction. So we must have \( \mathbb{E}[1_{B_\infty} \mid \mathcal{F}_1] = h \mathbb{E}[1_C \mid \mathcal{F}_1] \).

**Remark 1.** The lemma above is a variant of Sierpiński’s theorem, [3]. This theorem states that in an atomless probability space \((\Omega, \mathcal{E}, \mathbb{P})\), for every set \( A \in \mathcal{E} \) and every \( 0 < t < 1 \), there is a set \( B \subset A \) with \( \mathbb{P}[B] = t \mathbb{P}[A] \). The usual proof — presented in many probability courses — uses the Axiom of Choice (AC). A referee of [1] pointed out that for many people AC — or Zorn’s lemma — is an extra assumption. To prove Sierpiński’s theorem we only need the Axiom of Countable Dependent Choice, which is a countable form of the axiom of choice. In analysis this is the axiom that is usually needed and used. The proof above follows the approach given by Lorenc and Witula, [2].

**Lemma 3.** There is an increasing family of sets \((B_t)_{t \in [0,1]}\) such that \( \mathbb{E}[1_{B_t} \mid \mathcal{F}_1] = t \). The sigma algebra \( \mathcal{B} \), generated by the family \((B_t)_{t \in [0,1]} \), is independent of \( \mathcal{F}_1 \). The system \((B_t)_{t \in [0,1]}\) can also be described as \( B_t = \{U \leq t\} \) where \( U \) is a random variable that is independent of \( \mathcal{F}_1 \) and uniformly distributed on \([0,1]\).

**Proof** The proof is a repeated use of the previous lemma where we take \( h = 1/2 \). We start with \( B_0 = \emptyset, B_1 = \Omega \). Suppose that for the diadic numbers \( k2^{-n}, k = 0, \ldots, 2^n \) the sets are already defined. Then we consider the set \( B_{(k+1)2^{-n}} \setminus B_{k2^{-n}} \) and apply the previous lemma with \( h = 1/2 \). We get a set \( D \subset B_{(k+1)2^{-n}} \setminus B_{k2^{-n}} \) with \( \mathbb{E}[1_D \mid \mathcal{F}_1] = 2^{-(n+1)} \). We then define \( B_{(2k+1)2^{-(n+1)}} = B_{k2^{-n}} \cup D \). For non-diadic numbers \( t \) we find a sequence of diadic numbers \( d_n \) such that \( d_n \uparrow t \). Then we define \( B_t = \bigcup_n B_{d_n} \). This completes the construction. Since the system \((B_t)_{t \in [0,1]}\) is trivially stable for intersection, the relation \( \mathbb{E}[1_{B_t} \mid \mathcal{F}_1] = t \) shows that the sigma algebra \( \mathcal{B} \) generated by \((B_t)_{t \in [0,1]}\) is independent of \( \mathcal{F}_1 \). The construction of \( U \) is standard. At level \( n \) we put \( U_n = \sum_{k=1}^{2^n} k2^{-n} 1_{B_{k2^{-n}} \setminus B_{(k-1)2^{-n}}} \). \( U_n \) then decreases to a random variable \( U \) that satisfies the needed properties.

**Remark 2.** After the first version was made available, I got the remark that the paper [4] of Shen, J., Shen, Y., Wang, B., and Wang, R. contains similar concepts and results.2 In their notation they work with a measurable space \((\Omega, \mathcal{A})\) on which they have a finite number of probability measures \( Q_1, \ldots, Q_n \). They introduce

**Definition 2.** The set \((Q_1, \ldots, Q_n)\) is conditionally atomless if there exists a dominating measure \( Q \) (i.e. \( Q_k \ll Q \) for each \( k \leq n \)) as well as a continuously distributed random variable \( X \) (for the measure \( Q \)) such that the vector of Radon-Nikodym derivatives \( \left( \frac{dQ_k}{dQ} \right)_k \) is independent of \( X \).

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1I thank Ruodu Wang for pointing out these relations and for the subsequent discussions we had on the topic.

2Their paper also considers an infinite number of measures but to clarify the relation between their paper and my approach, I only consider a finite number of measures.
They then prove the following

**Proposition 1.** Are equivalent

1. \((Q_1, \ldots, Q_n)\) is conditionally atomless
2. in the definition we can take \(Q = \frac{1}{n}(Q_1 + \ldots + Q_n)\)
3. \(X\) can be taken as uniformly distributed over \([0, 1]\).

There are several differences with my approach. There is the technical difference that they suppose the existence of a continuously distributed random variable \(X\). In doing so they avoid the technical points between the more conceptual definition using conditional expectations and the construction of a suitable sigma-algebra with a uniformly distributed random variable. A further difference is that they use a dominating measure that later can be taken as the mean of \((Q_1, \ldots, Q_n)\). Of course their result together with the results here show that the definition of \((Q_1, \ldots, Q_n)\) being conditionally atomless, is equivalent to the statement that for the measure \(Q_0 = \frac{1}{n}(Q_1 + \ldots + Q_n)\), the sigma-algebra \(A\) is conditionally atomless with respect to the sigma-algebra generated by the Radon-Nikodym derivatives \(\left(\frac{dQ_k}{dQ_0}\right)_{k}\). In [4] it is also shown that one can take any strictly positive convex combination of the measures \((Q_1, \ldots, Q_n)\). Below we will show that this sigma-algebra in some sense has a minimal property, a result that clarifies the relation between the two approaches. Before doing so, let us recall two easy results from introductory probability theory.

**Result 1.** For a given probability space \((\Omega, A, Q)\) let us denote \(N = \{N \in A \mid Q[N] = 0\}\). Suppose that a sub sigma-algebra \(F \subset A\) is given and that \(G, F \subset G\), is another sub sigma-algebra which is included in the sigma-algebra generated by \(F\) and \(N\). Then for each \(\xi \in L^1(\Omega, A, Q)\)

\[E_Q[\xi | F] = E_Q[\xi | G] \text{ a.s.}\]

**Result 2.** With the notation in the previous exercise let \(F : \Omega \rightarrow \mathbb{R}^n\) and \(F' : \Omega \rightarrow \mathbb{R}^n\) be two vectors that are equal a.s. Let \(F\) be generated by \(F\) and \(G\) be generated by \(F'\). Then \(F\) and \(G\) are equal up to sets in \(N\). More precisely \(G\) is included in the sigma-algebra generated by \(F\) and \(N\) (and of course conversely), i.e. \(\sigma(F, N) = \sigma(G, N)\).

**Proposition 2.** Let \(Q_1, \ldots, Q_n\) be probability measures on a measurable space \((\Omega, A)\). Let \(Q_0\) denote a convex combination of these measures \(Q_0 = \sum_k \lambda_k Q_k\) where each \(\lambda_k > 0\). Let \(f_k\) denote an \(A\) measurable version \(\frac{dQ_k}{dQ_0}\). Let \(Q\) be another dominating measure with \(g_k\) an \(A\) measurable version of \(\frac{dQ_k}{dQ}\). Let \(N = \{N \in A \mid Q_0[N] = 0\}\). Let \(F\) be generated by \(f_k, k = 1 \ldots n\) and let \(G\) be generated by \(g_k, k = 1 \ldots n\). Then \(F \subset \sigma(G, N)\)

**Proof** Clearly \(Q_0 \ll Q\) so let \(h = \frac{dQ_0}{dQ}\). It is now immediate that \(g_k = f_k h \cdot Q\) a.s. To see this, observe that the values of \(f_k\) on \{\(h = 0\}\) do not matter. The functions \(g_k\) and \(h\) are \(G\) measurable since \(h\) can be taken as \(h = \sum_k \lambda_k g_k\). Then we define \(f'_k = \frac{g_k}{h}\) on \{\(h > 0\}\) and \(f'_k = 0\) on \{\(h = 0\}\). This choice shows that the \(f'_k\) are \(G\) measurable. It is immediate that \(f_k = f'_k Q_0\) a.s. The result now follows.

From the theorem it follows that the sigma-algebra augmented with the class \(N\) is the same for all strictly positive convex combinations. The theorem shows that in the definition of conditionally atomless with respect to \(F\), we can also add the null sets \(N\) to \(F\). To check that \(A\) is conditionally atomless with respect to a sigma-algebra \(F\) it is clear that the smaller
Theorem 3. The probability measure $K(\omega,.)$ is atomless on $\mathcal{F}_2$. In case the sigma algebra $\mathcal{F}_2$ is generated by a countable family of sets, the converse holds, i.e. if for almost every $\omega \in \Omega$, the probability $K(\omega,.)$ is atomless on $\mathcal{F}_2$, then $\mathcal{F}_2$ is conditionally atomless with respect to $\mathcal{F}_1$.

Proof We first suppose that $\mathcal{F}_2$ is atomless with respect to $\mathcal{F}_1$. According to the previous section there is an atomless sub sigma algebra $\mathcal{B} \subset \mathcal{F}_2$ that is independent of $\mathcal{F}_1$. There is also a random variable $U$ which is independent of $\mathcal{F}_1$ and is uniformly distributed on $[0, 1]$. Let $C[0, 1]$ be the space of real valued continuous functions on $[0, 1]$, equipped with the sup norm. This space is separable and so we can take a (sup-norm) dense sequence $(g_n)_{n \geq 1}$ in $C[0, 1]$. For each $n \geq 1$ we have a.s. :

$$E[g_n(U) | \mathcal{F}_1](\omega) = E[g_n(U)] = \int_0^1 g_n(t) \, dt.$$ 

So we have a.s. , say on $\Omega_n, P[\Omega_n] = 1$;

$$\int K(\omega, d\tau) g_n(U(\tau)) = \int_0^1 g_n(t) \, dt.$$ 

For $\omega \in \cap_{n \geq 1} \Omega_n$ we have by density of the sequence $(g_n)_n$, for all $g \in C[0, 1]$:

$$\int K(\omega, d\tau) g(U(\tau)) = \int_0^1 g(t) \, dt.$$ 

This proves that a.s. the random variable $U$ is for $K(\omega,.)$ uniformly $[0, 1]$ distributed. That can only happen when $K(\omega,.)$ is atomless on $\mathcal{F}_2$.

We now prove the converse. Suppose that $\mathcal{F}_2$ is not conditionally atomless with respect to $\mathcal{F}_1$. In this case there is a set $A$ with $P[A] > 0$ such that for all $B \subset A$:

$$P \{0 < E[1_B | \mathcal{F}_1] < E[1_A | \mathcal{F}_1] \} = 0.$$ 

In order words, if $B \subset A$ then a.s. either $E[1_B | \mathcal{F}_1] = 0$ or $E[1_B | \mathcal{F}_1] = E[1_A | \mathcal{F}_1]$. By definition of the kernel $K$, this means $K(\omega, B) = 0$ or $K(\omega, B) = K(\omega, A)$ a.s. . In other words for $B \subset A$:

$$P \{\omega \mid K(\omega, B) = 0 \text{ or } K(\omega, B) = K(\omega, A)\} = 1.$$ 

Since $\mathcal{F}_2$ is countably generated there is a countable Boolean algebra $\mathcal{A} \subset \mathcal{F}_2$ that generates $\mathcal{F}_2$. For each set $B \in \mathcal{A}$ we have that

$$\Omega_B = \{\omega \mid K(\omega, B \cap A)^2 = K(\omega, A) K(\omega, B \cap A)\},$$ 

has measure 1. The set $\Omega' = \cap_{B \in \mathcal{A}} \Omega_B$ still has probability 1. We claim that for each $\omega \in \Omega'$ and each $B \in \mathcal{F}_2$ we have that either $K(\omega, B \cap A) = 0$ or $= K(\omega, A)$. This means that for

$\mathcal{F}$, the easier it is to satisfy the condition. In my opinion the above clarifies the relation between this paper and [4].

3. An Equivalent Definition

As already mentioned in the previous section, the definition of being conditionally atomless is related to a similar statement for the kernel $K$. We suppose that the conditional expectation with respect to $\mathcal{F}_1$ is given by the kernel $K$. We have the following

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3. An Equivalent Definition

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**Theorem 3.** If $\mathcal{F}_2$ is conditionally atomless with respect to $\mathcal{F}_1$ then for almost every $\omega \in \Omega$ the probability measure $K(\omega,.)$ is atomless on $\mathcal{F}_2$. In case the sigma algebra $\mathcal{F}_2$ is generated by a countable family of sets, the converse holds, i.e. if for almost every $\omega \in \Omega$, the probability $K(\omega,.)$ is atomless on $\mathcal{F}_2$, then $\mathcal{F}_2$ is conditionally atomless with respect to $\mathcal{F}_1$.

**Proof** We first suppose that $\mathcal{F}_2$ is atomless with respect to $\mathcal{F}_1$. According to the previous section there is an atomless sub sigma algebra $\mathcal{B} \subset \mathcal{F}_2$ that is independent of $\mathcal{F}_1$. There is also a random variable $U$ which is independent of $\mathcal{F}_1$ and is uniformly distributed on $[0, 1]$. Let $C[0, 1]$ be the space of real valued continuous functions on $[0, 1]$, equipped with the sup norm. This space is separable and so we can take a (sup-norm) dense sequence $(g_n)_{n \geq 1}$ in $C[0, 1]$. For each $n \geq 1$ we have a.s. :

$$E[g_n(U) | \mathcal{F}_1](\omega) = E[g_n(U)] = \int_0^1 g_n(t) \, dt.$$ 

So we have a.s. , say on $\Omega_n, P[\Omega_n] = 1$;

$$\int K(\omega, d\tau) g_n(U(\tau)) = \int_0^1 g_n(t) \, dt.$$ 

For $\omega \in \cap_{n \geq 1} \Omega_n$ we have by density of the sequence $(g_n)_n$, for all $g \in C[0, 1]$:

$$\int K(\omega, d\tau) g(U(\tau)) = \int_0^1 g(t) \, dt.$$ 

This proves that a.s. the random variable $U$ is for $K(\omega,.)$ uniformly $[0, 1]$ distributed. That can only happen when $K(\omega,.)$ is atomless on $\mathcal{F}_2$.

We now prove the converse. Suppose that $\mathcal{F}_2$ is not conditionally atomless with respect to $\mathcal{F}_1$. In this case there is a set $A$ with $P[A] > 0$ such that for all $B \subset A$:

$$P \{0 < E[1_B | \mathcal{F}_1] < E[1_A | \mathcal{F}_1] \} = 0.$$ 

In order words, if $B \subset A$ then a.s. either $E[1_B | \mathcal{F}_1] = 0$ or $E[1_B | \mathcal{F}_1] = E[1_A | \mathcal{F}_1]$. By definition of the kernel $K$, this means $K(\omega, B) = 0$ or $K(\omega, B) = K(\omega, A)$ a.s. . In other words for $B \subset A$:

$$P \{\omega \mid K(\omega, B) = 0 \text{ or } K(\omega, B) = K(\omega, A)\} = 1.$$ 

Since $\mathcal{F}_2$ is countably generated there is a countable Boolean algebra $\mathcal{A} \subset \mathcal{F}_2$ that generates $\mathcal{F}_2$. For each set $B \in \mathcal{A}$ we have that

$$\Omega_B = \{\omega \mid K(\omega, B \cap A)^2 = K(\omega, A) K(\omega, B \cap A)\},$$ 

has measure 1. The set $\Omega' = \cap_{B \in \mathcal{A}} \Omega_B$ still has probability 1. We claim that for each $\omega \in \Omega'$ and each $B \in \mathcal{F}_2$ we have that either $K(\omega, B \cap A) = 0$ or $= K(\omega, A)$. This means that for
each \( \omega \in \Omega' \) with \( K(\omega, A) > 0 \), the measure \( K(\omega, \cdot) \) has \( A \) as an atom, a contradiction to the hypothesis. To show the claim we use a monotone class argument. Let

\[
\mathcal{M} = \{ B \in \mathcal{F}_2 \mid \text{for each } \omega \in \Omega' : K(\omega, B \cap A)^2 = K(\omega, A) K(\omega, B \cap A) \},
\]

Clearly \( \mathcal{A} \subset \mathcal{M} \) and it is obvious that \( \mathcal{M} \) is a monotone class, meaning that it is stable for increasing countable unions and for decreasing countable intersections. It is well known that this implies \( \mathcal{M} = \mathcal{F}_2 \), completing the proof of the theorem.

4. A Counterexample

We now give a counterexample when \( \mathcal{F}_2 \) is not countably generated. The basic ingredient is the interval \([0,1]\) with its Borel sigma algebra \( \mathcal{B} \) and the Lebesgue measure \( m \). We define \( S = [0,1] \) and \( \Omega = [0,1] \times (S \times [0,1]) \). The sigma algebra \( \mathcal{F}_1 \) is generated by the first coordinate and the Borel sigma algebra, \( \mathcal{B} \), on \([0,1]\). On \( S \times [0,1] \) we put the sigma algebra defined as follows:

\[
\mathcal{A} = \{ B \mid \text{there is a countable set } D \text{ and for } s \in D : B_s \in \mathcal{B} \text{ otherwise } B_s = \emptyset \text{ or } B_s = [0,1] \}.
\]

The sigma algebra \( \mathcal{F}_2 \) is the product sigma algebra \( \mathcal{B} \otimes \mathcal{A} \). For each \( x \in [0,1] \) we define the transition probability \( k(x, B) \) for \( B \in \mathcal{A} \):

\[
k(x, B) = \sum_{s \in S} \mathbf{1}_{\{x\}}(s) m(B_s).
\]

Then we define the kernel (defined on \( \Omega \)) as \( K(\omega, \cdot) = \delta_x \otimes k(x, \cdot) \), where \( \omega = (x, s, y) \) and where \( \delta_x \) is the Dirac measure concentrated on the point \( x \). The probability measure on \( \Omega \) is constructed with \( m \) and the transition kernel \( k \):

\[
E \in \mathcal{B}, B \in \mathcal{A} \quad \mathbb{P}[E \times B] = \int_E m(dx) m(B_x) = m(E \cap \{ x \mid B_x = [0,1] \}).
\]

For each \( x \in [0,1] \) the kernel \( K \) is atomless. For \( A \in \mathcal{F}_2 \) we find putting \( A_x = \{(s, z) \mid (x, s, z) \in A\} \) and \( A_{x,x} = \{y \mid (x, x, y) \in A\} \):

\[
\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1](x) = k(x, A_x) = m(A_{x,x}),
\]

which is almost surely 0 or 1. This makes it impossible that \( \mathcal{F}_2 \) is conditionally atomless with respect to \( \mathcal{F}_1 \).

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