HERMITIAN CURVATURE FLOW ON COMPLEX LOCALLY HOMOGENEOUS SURFACES

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Abstract. We study the Hermitian curvature flow of locally homogeneous non-Kähler metrics on compact complex surfaces. In particular, we characterize the long-time behaviour of the solutions to the flow. Finally, we compute the Gromov-Hausdorff limit of immortal solutions after a suitable normalization. Our results follow by a case-by-case analysis of the flow on each complex model geometry.

1. Introduction

The Hermitian curvature flow (HCF shortly) is a strictly parabolic flow of Hermitian metrics introduced by Streets and Tian in [23]. The flow evolves an initial Hermitian metric in the direction of its second Chern-Ricci curvature tensor modified with some first order terms in the torsion.

More precisely, let \((X, g_0)\) be a Hermitian manifold. The solution to the HCF starting at \(g_0\) is the family of Hermitian metric \(g(t)\) satisfying

\[
\partial_t g(t) = -S(g(t)) + Q(g(t)) , \quad g(0) = g_0 ,
\]

where \(S(g)\) is the second Chern-Ricci curvature tensor and \(Q(g)\) is a \((1, 1)\)-symmetric tensor which is quadratic in the torsion components of the Chern connection (see Section 2). When the starting metric is Kähler the HCF reduces to the Kähler-Ricci flow. Moreover, in the compact case, it is a gradient flow and it is stable near Kähler-Einstein metrics with non-positive Ricci curvature [23].

We stress that different choices of the tensor \(Q\) in (1) give rise to a family of geometric flows. In particular one can choose \(Q\) to preserve different geometric conditions, making each of these new flows well-suited to investigate a certain problem. Among these, one of the most studied is the pluriclosed flow (PCF shortly), which preserves the pluriclosed condition \(\partial \bar{\partial} \omega = 0\) [17, 18, 19, 20, 21, 22, 24, 25].

One of the main reason in studying these new flows is to refine the Enrique-Kodaira classification of compact complex surfaces, as they can be used to detect canonical Hermitian metrics as limit points (see e.g. [21]). Motivated by this, we carry out an analysis of the HCF on compact complex surfaces in the same fashion as Boling did for the PCF in [3].

Our first main result completely characterizes the long-time behavior of locally homogeneous non-Kähler solutions, namely

**Theorem A.** Let \(X\) be a compact complex surface and \(g_0\) a locally homogeneous non-Kähler metric on \(X\). If the solution to the HCF starting from \(g_0\) develops a finite time singularity, then \(X\) is a Hopf surface. Conversely, any locally homogeneous solution to the HCF on a Hopf surface collapses in finite time.

Notice that we restricted our analysis to starting non-Kähler metrics since the behavior of Kähler solutions is already known (see e.g. [5, 16, 26]).

Let us observe that the dynamical systems arising from the PCF and the HCF are rather different. In contrast with Theorem A, locally homogenous non-Kähler solutions to the PCF on compact complex surfaces never develop finite-time singularities [3 Thm 1.1].

Our second main result concerns the Gromov-Hausdorff limits of immortal normalized solutions to the HCF, namely

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Theorem B. Let $X$ be a compact complex surface, $g_0$ a locally homogeneous non-Kähler metric on $X$ and $g(t)$ the solution to the HCF starting from $g_0$.

i) If $X$ is either a hyperelliptic or Kodaira surface, then $(X, (1 + t)^{-1}g(t))$ converges to a point in the Gromov-Hausdorff topology as $t \to \infty$.

ii) If $X$ is a non-Kähler properly elliptic surface, then $(X, (1 + t)^{-1}g(t))$ converges to its base curve $(C, g_{KE})$ in Gromov-Hausdorff topology as $t \to \infty$, where $\text{Ric}(g_{KE}) = -g_{KE}$.

iii) If $X$ is an Inoue surface, then $(X, (1 + t)^{-1}g(t))$ converges to a circle in Gromov-Hausdorff topology as $t \to \infty$.

We point out that the arguments used to prove (i) and (iii) in Theorem B are analogous to those used by Tosatti and Weinkove in [28] for the Chern-Ricci flow (see also [1, 27, 29]), and the limit spaces arising in our context are the same. Finally, we highlight that cohomological aspects of compact complex surfaces along the Chern-Ricci flow were investigated in [1]. It would be interesting to carry out a similar analysis also for the HCF.

Our results can be thought as a first step in the study of the HCF on non-Kähler surfaces. In the same spirit of [3] and [10], we expect the blowdown of any immortal locally homogeneous solution to converge to an expanding soliton. Nonetheless, at the moment we are not able to confirm this statement.

In this direction, in the last few years similar results about blowdown limits have been obtained in different settings. In [2] the second named author, Lafuente and Vezzoni proved that long-time existence of left-invariant solutions to the HCF is always guaranteed on complex unimodular Lie groups and such solutions converge under a suitable normalization to an expanding algebraic soliton (see also [13]). In [2] Arroyo and Lafuente showed that normalized left-invariant solutions to the PCF on 2-step nilmanifolds and almost-abelian Lie groups always converge to expanding solitons (see also [5]).

Recently Ustinovskiy [30, 31] found a new flow in the HCF family which preserves both the Griffiths-positivity and a finite dimensional space of distinguished metrics called almost-abelian Lie groups always converge to expanding solitons (see also [6]). In [2] Arroyo and Lafuente showed that normalized left-invariant solutions to the PCF on 2-step nilmanifolds and almost-abelian Lie groups always converge to expanding solitons (see also [5]).

Recently Ustinovskiy [30, 31] found a new flow in the HCF family which preserves both the Griffiths-positivity and a finite dimensional space of distinguished metrics called induced metrics. We mention that related works have been recently appeared (see e.g. [14]) and it would be interesting to analyze Ustinovskiy’s flow on compact complex surfaces in the same fashion as we did in this paper.

The paper is organized as follows. In Section 2 we recall some basics on HCF, complex model geometries and Gromov-Hausdorff convergence. In Section 3 we explicitly compute the HCF tensor of each compact complex surface in the same fashion as we did in this paper. Finally, in Section 4 we prove Theorem A and Theorem B by a case-by-case analysis of the involved equations.

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2. Preliminaries

2.1. Basics on HCF.

In the sequel, we describe the evolution equation of the Hermitian Curvature Flow on a complex manifold $X = (M, J)$. Given a Hermitian metric $g$ on $X$, we denote by $\nabla$ its Chern connection, by $\Omega$ its Chern curvature tensor $\Omega(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and by $S$ its second Chern-Ricci curvature, i.e.

$$S_{ij} := g^{k\ell} \Omega_{k\ell ij} .$$

Let also $T$ be the torsion of $\nabla$ and consider the tensor $Q = Q_{ij}$ defined by

$$Q := \frac{1}{2}Q^1 - \frac{1}{4}Q^2 - \frac{1}{2}Q^3 + Q^4 ,$$

where

$$Q^1_{ij} := g^{k\ell} g^{m\ell q} T_{i k q} T_{j \ell m} , \quad Q^2_{ij} := g^{k\ell} g^{m\ell q} T_{k m j} T_{\ell q i} ,$$

$$Q^3_{ij} := g^{k\ell} g^{m\ell q} T_{i k q} T_{j \ell m} , \quad Q^4_{ij} := \frac{1}{2} g^{k\ell} g^{m\ell q} (T_{m k j} T_{\ell q i} + T_{q k j} T_{m \ell i}) .$$

Notice that in the formulas above

$$T_{i k j} := g_{k i} T_{j i}^k , \quad T_{i j k} := g_{k i} T_{j i}^k , \quad (g^{ij}) := (g_{ij})^{-1} .$$
Then, given a Hermitian metric $g_0$ on $X$, the evolution equation of the HCF on $X$ starting from $g_0$ is given by

$$\partial_t g(t) = -K(g(t)), \quad g(0) = g_0,$$

where $K := S - Q$. Henceforth, we will refer to $K$ as the HCF tensor.

2.2. HCF tensor on Lie groups.

Let $(G, J, g)$ be a real Lie group $G$ equipped with a left-invariant Riemannian metric $g$ and a left-invariant complex structure $J$ such that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. Let also $\mathfrak{g} := \text{Lie}(G)$ and $\mathfrak{g}(X, Y) := [X, Y]$.

In the following, we compute the components of the HCF tensor in terms of the structure constants of $\mathfrak{g}$.

Let $\{Z_1, \ldots, Z_n\}$ be a left-invariant frame of $G$. Since the Chern connection is the unique Hermitian connection with vanishing $(1,1)$-part of the torsion, it follows that

$$\nabla_{Z_k}Z_l = \nabla_{Z_l}Z_k + \mu(Z_k, Z_l),$$

or, in terms of the Christoffel symbols of $\nabla$

$$\Gamma^r_{kl} = \mu^{r}_{kl}, \quad \Gamma^r_{k\ell} = \mu^{r}_{k\ell}.$$

On the other hand $\nabla J = \nabla g = 0$ implies

$$g(\nabla_{Z_k}Z_i, Z_j) = g(Z_i, \nabla_{Z_k}Z_j) = -g(Z_i, \mu(Z_k, Z_j))$$

and hence

$$\Gamma^i_{k\ell} = -g^{ij}g_{ip}\mu^{p}_{kj}. \quad (5)$$

By definition, we have

$$\Omega_{k\ell ij} = g(\nabla_{Z_k} \nabla_{Z_i} Z_j, Z_l) - g(\nabla_{Z_i} \nabla_{Z_j} Z_k, Z_l) - g(\nabla_{\mu(Z_k, Z_l)} Z_i, Z_j),$$

with

$$g(\nabla_{Z_k} \nabla_{Z_i} Z_j, Z_l) = g_{ij} \Gamma^r_{kl} \Gamma^p_{kr}, \quad g(\nabla_{Z_i} \nabla_{Z_j} Z_k, Z_l) = g_{ij} \Gamma^r_{kl} \Gamma^p_{kr}, \quad g(\nabla_{\mu(Z_k, Z_l)} Z_i, Z_j) = g_{ij} (\mu^r_{k\ell} \Gamma^p_{ri} + \mu^r_{k\ell} \mu^p_{ri}).$$

Thus, the second Chern-Ricci curvature $S$ takes the form

$$S_{ij} = g^{k\ell}g_{ij}(\mu^r_{k\ell} \Gamma^p_{kr} - \mu^{r}_{k\ell} \Gamma^p_{ri} + \mu^{r}_{k\ell} \mu^p_{ri}). \quad (6)$$

On the other hand, since $T_{ij} = \nabla Z_i Z_j - \nabla Z_j Z_i - \mu(Z_i, Z_j)$, from (5) we have

$$T^k_{ij} = -g^{rs}g_{sj} \mu^{r}_{ik} + g^{rs}g_{si} \mu^{r}_{jk} - \mu^{r}_{ij} \Gamma^p_{ki} \quad (7)$$

and hence

$$T_{ij} = -g_{ij} \mu^m_{ki} + g_{il} \mu^m_{kj} - \mu^m_{lj} \mu^i_{kj}.$$}

Therefore, the explicit expression of the tensor $Q$ can be recovered from (2), (3) and (7).

2.3. Complex model geometries.

In this subsection, we recall some basics about the geometry of locally homogeneous Hermitian manifolds. In particular, we focus on compact locally homogeneous Hermitian surfaces.

A Hermitian manifold $(X, g)$ is said to be locally homogeneous if the pseudogroup of local automorphism of $(X, g)$ acts transitively on $X$, i.e. if for any choice of $x, y \in X$ there exist neighborhoods $U_x, U_y \subset X$ of $x, y$ respectively and a holomorphic local isometry $f : U_x \rightarrow U_y$ such that $f(x) = y$. If in addition $(X, g)$ is compact, then its universal Hermitian covering $(\tilde{X}, g)$ is globally homogeneous (see [15]) and hence it admits a left coset presentation $\tilde{X} = G/H$ for some closed subgroup $G \subset \text{Aut}(\tilde{X}, g)$. Here, with a slight abuse of notation, we denote by $g$ both the Hermitian metric on $X$ and its pullback on the universal cover $\tilde{X}$.

Motivated by this, we recall the following

**Definition 2.1.** A complex model geometry of dimension $n$ is a pair $(\tilde{X}, G)$ given by a connected, simply-connected $n$-dimensional complex manifold $\tilde{X}$ and a real connected Lie group $G$ such that:

- $G$ acts properly, transitively and almost-effectively by biholomorphisms on $\tilde{X}$;
G contains a discrete subgroup \( \Gamma \subset G \) with \( \Gamma \backslash \tilde{X} \) compact. If \( G \) is a minimal group with such properties, then the complex model geometry is said to be minimal.

Let \((\tilde{X}, G)\) be a complex model geometry. A Hermitian manifold \((X, g)\) has geometric structure of type \((\tilde{X}, G)\) if \(\tilde{X}\) is the universal cover of \(X\) and the pulled-back metric \(g\) on \(\tilde{X}\) is invariant under the action of \(G\). Of course, if \((X, g)\) has a geometric structure, then it is locally homogeneous. On the other hand, by the previous observation, any compact locally homogeneous Hermitian manifold has geometric structure of type \((\tilde{X}, G)\) for some minimal complex model geometry \((\tilde{X}, G)\).

By the Riemann Uniformization Theorem, it is known that there exist exactly three minimal complex model geometries of dimension 1, that are

\[(\mathbb{C}, \mathbb{C}) , \quad (\mathbb{CP}^1, \text{SU}(2)) , \quad (\mathbb{CH}^1, \text{SU}(1,1)) .\]

Here, the group \(G\) acts on the respective space \(\tilde{X}\) in the standard way.

Subsequently in \([32, 33]\) Wall classified all complex model geometries of dimension 2. In particular, he proved the following

**Theorem 2.2** \((32, 33)\). If \((\tilde{X}, G)\) is a minimal complex geometry of dimension 2, then one of the following cases occurs:

1. \((\tilde{X}, G) = (\tilde{X}_1 \times \tilde{X}_2, G_1 \times G_2)\) is the product of two complex model geometries of dimension 1.
2. \((\tilde{X}, G) = (\mathbb{CP}^2, \text{SU}(3))\) or \((\tilde{X}, G) = (\mathbb{CH}^2, \text{SU}(2,1))\), both considered endowed with the standard action of \(G\) on \(\tilde{X}\).
3. \(\tilde{X} = (G, J)\) where \(G\) acts on itself by left translations and \(J\) is a left-invariant complex structure.

**Remark 2.3.** If \((\tilde{X}, G)\) is one of the model listed in (i) or (ii) above, then any Hermitian \(G\)-invariant metric on \(\tilde{X}\) is necessarily Kähler-Einstein.

### 2.4. Gromov-Hausdorff convergence.

We collect here some basic facts about Gromov-Hausdorff convergence of compact metric spaces. We refer to \([4]\) Sec. 7.3.2] and \([14]\) for more details.

Let \(Z = (Z, d_Z)\) be a metric space and \(A, B \subset Z\) two compact subsets. The Hausdorff distance between \(A\) and \(B\) is given by

\[d_{\text{H}}^Z(A, B) := \inf \{ \epsilon > 0 : X \subset B_\epsilon(Y), Y \subset B_\epsilon(X) \},\]

where \(B_\epsilon(X) := \{ x \in Z : d_Z(x, X) < \epsilon \}\) is the \(\epsilon\)-tube of \(X\) in \(Z\). The pair

\[\left( \{ \text{compact subsets of } Z \}, d_{\text{H}}^Z \right)\]

is also a metric space and it is compact if and only if \(Z\) is compact as well.

Let now \(X = (X, d_X)\), \(Y = (Y, d_Y)\) be two compact metric spaces. The Gromov-Hausdorff distance between \(X\) and \(Y\) is defined as

\[d_{\text{GH}}(X, Y) := \inf \left\{ d_H^Z(\phi_1(X), \phi_2(Y)) \right\},\]

where the infimum is taken with respect to all metric spaces \(Z\) and all pairs \((\phi_1, \phi_2)\) of isometric embeddings \(\phi_1 : X \rightarrow Z\) and \(\phi_2 : Y \rightarrow Z\). Letting \(X\) denote the set of isometric classes of compact metric spaces, it turns out that \((X, d_{\text{GH}})\) is a complete metric space. Therefore, given a one-parameter family \(\{X_t\}_{t \in [0, T]}\) and an element \(Y\) both in \(X\), whenever \(\lim_{t \rightarrow T^-} d_{\text{GH}}(X_t, Y) = 0\) we write

\[X_t \xrightarrow{\text{GH}} Y \quad \text{as} \quad t \rightarrow T^-\]

and we say that \(X_t\) converges in the Gromov-Hausdorff topology to \(Y\).

Finally, a \(GH\) \(\epsilon\)-approximation between two metric spaces \(X, Y \in X\), with \(\epsilon > 0\), is a pair of non-necessarily continuous maps \(\varphi : X \rightarrow Y\) and \(\psi : Y \rightarrow X\) satisfying for any \(x, x' \in X\) and \(y, y' \in Y\)

\[
\begin{align*}
|d_X(x, x') - d_Y(\varphi(x), \varphi(x'))| &< \epsilon, \\
|d_X(x, (\psi \circ \varphi)(x)) &< \epsilon, \\
|d_Y(y, y') - d_X(\psi(y), \psi(y'))| &< \epsilon, \\
|d_Y(y, (\varphi \circ \psi)(y)) &< \epsilon. 
\end{align*}
\]

Remarkably, if there exists a \(GH\) \(\epsilon\)-approximation \((\varphi, \psi)\) between \(X\) and \(Y\), then \(d_{\text{GH}}(X, Y) \leq \frac{3}{2} \epsilon\) (see e.g. \([14]\) Lemma 1.3.3]).
3. HCF Tensor on Complex Model Geometries

The aim of this section is to compute the HCF tensor $K$ of any 2-dimensional complex model geometry $(\tilde{X}, J)$ endowed with an invariant metric $g$. By means of Remark 2.3, we will restrict our discussion to those minimal complex model geometries arising from (iii) in Theorem 2.2. Hence, following [3, Sec. 2.2], we list below all the connected, simply-connected 4-dimensional Lie groups which admits a left-invariant complex structure, their compact quotients according to Enriques-Kodaira classification and their HCF tensors. We mention here that all the computations were made with the help of the software Maple.

In the following, given a connected, simply connected 4-dimensional real Lie groups $(G, J)$ equipped with a left-invariant complex structure, we will consider a fixed left-invariant $(1,0)$-frame $\{Z_1, Z_2\}$ and we will denote by $\{\zeta^1, \zeta^2\}$ its dual frame. This allows us to write any left-invariant Hermitian metric $g$ on $(G, J)$ in the form

$$g = x \zeta^1 \circ \zeta^1 + y \zeta^2 \circ \zeta^2 + z \zeta^1 \circ \zeta^2 + \bar{z} \zeta^2 \circ \zeta^1,$$

with $x, y \in \mathbb{R}_{>0}$, $z \in \mathbb{C}$ and $xy - |z|^2 > 0$.

**Complex tori.**

The Lie group is $G = \mathbb{R}^4$, which is abelian and admits a unique left-invariant complex structure $J_{st}$. In this case, the HCF tensor of any left-invariant metric on $\mathbb{C}^2 = (\mathbb{R}^4, J_{st})$ is just $K = 0$. Compact quotients of $\mathbb{C}^2$ are complex tori.

**Hyperelliptic surfaces.**

The Lie group is $G = \tilde{\text{SE}}(2) \times \mathbb{R}$, where $\tilde{\text{SE}}(2)$ is the universal cover of the special Euclidean group $\text{SE}(2) := \text{SO}(2) \rtimes \mathbb{R}^2$. It admits a unique left-invariant complex structure $J$ and the structure constants $\mu$ of its complexified Lie algebra are

$$\mu(Z_1, Z_2) = Z_1, \quad \mu(Z_1, \bar{Z}_2) = -Z_1.$$

The HCF tensor of a left-invariant Hermitian metric on $(\tilde{\text{SE}}(2) \times \mathbb{R}, J)$ is given by

$$K_{11} = \frac{x^2 |z|^2}{(xy - |z|^2)^2}, \quad K_{22} = \frac{|z|^4}{(xy - |z|^2)^2}, \quad K_{12} = \frac{x^2 yz}{(xy - |z|^2)^2}.$$

Compact quotients of $(\tilde{\text{SE}}(2) \times \mathbb{R}, J)$ are hyperelliptic surfaces, which admit Kähler metrics.

**Hopf surfaces.**

The Lie group is $G = \text{SU}(2) \times \mathbb{R}$. It admits a one-parameter family $J_\lambda$ of left-invariant complex structures, where $\lambda \in \mathbb{R}$, and with respect to $J_\lambda$ the structure constants $\mu = \mu_\lambda$ of its complexified Lie algebra are

$$\mu(Z_1, Z_2) = Z_2, \quad \mu(Z_1, \bar{Z}_2) = -\bar{Z}_2, \quad \mu(Z_2, \bar{Z}_2) = (-1 + \sqrt{-1}\lambda)Z_1 + (1 + \sqrt{-1}\lambda)\bar{Z}_1.$$

The HCF tensor of a left-invariant Hermitian metric on $(\text{SU}(2) \times \mathbb{R}, J_\lambda)$ is given by

$$K_{11} = \frac{x^4 (1 + \lambda^2) + |z|^2 (2x^2 + |z|^2)}{(xy - |z|^2)^2},$$

$$K_{22} = \frac{(1 + \lambda^2) x^2 |z|^2 + 2xy - |z|^2)^2 + |z|^2 (y^2 + 2z^2) - 2(1 + \lambda^2) x^2(y^2 - |z|^2)}{(xy - |z|^2)^2},$$

$$K_{12} = \frac{xyz (\lambda^2 x^2 + (x + y)^2)}{(xy - |z|^2)^2}.$$

Compact quotients of $(\text{SU}(2) \times \mathbb{R}, J_\lambda)$ are Hopf surfaces, which are non-Kähler.

**Non-Kähler properly elliptic surfaces.**

The Lie group is $G = \tilde{\text{SL}}(2, \mathbb{R}) \rtimes \mathbb{R}$, where $\tilde{\text{SL}}(2, \mathbb{R})$ is the universal cover of $\text{SL}(2, \mathbb{R})$. It admits a one-parameter family $J_\lambda$ of left-invariant complex structure, with $\lambda \in \mathbb{R}$, with respect to which the structure constants $\mu = \mu_\lambda$ of its complexified Lie algebra are

$$\mu(Z_1, Z_2) = \sqrt{-1}Z_1, \quad \mu(Z_1, \bar{Z}_2) = \sqrt{-1}\bar{Z}_1, \quad \mu(Z_1, \bar{Z}_1) = (-\lambda + \sqrt{-1})Z_2 + (\lambda + \sqrt{-1})\bar{Z}_2.$$

The HCF tensor of a left-invariant Hermitian metric on \( (\text{SL}(2, \mathbb{R}) \times \mathbb{R}, J_\lambda) \) is given by
\[
K_{11} = \frac{(1 + \lambda^2)y^2z^2 - 2(xy - |z|^2)^2 + |z|^2(x^2 - 2|z|^2) - 2(1 + \lambda^2)y^2(xy - |z|^2)}{(xy - |z|^2)^2},
\]
\[
K_{22} = \frac{\lambda^2y^4 + (y^2 - |z|^2)^2}{(xy - |z|^2)^2},
\]
\[
K_{12} = \frac{yz(\lambda^2y^2 + (x - y)^2)}{(xy - |z|^2)^2}.
\]
Compact quotients of \( (\text{SL}(2, \mathbb{R}) \times \mathbb{R}, J_\lambda) \) are non-Kähler properly elliptic surfaces.

**Primary Kodaira surfaces.**

The Lie group is \( G = \mathbb{R} \times H_3(\mathbb{R}) \), where \( H_3(\mathbb{R}) \) is the three-dimensional real Heisenberg group. It admits a unique left-invariant complex structure \( J \) and the structure constants \( \mu \) of its complexified Lie algebra are

\[
\mu(Z_1, Z_1) = \sqrt{-1}(Z_2 + \bar{Z}_2).
\]

The HCF tensor of a left-invariant Hermitian metric on \( (\mathbb{R} \times H_3(\mathbb{R}), J) \) is
\[
K_{11} = \frac{-2y^2(xy - |z|^2) + y^2|z|^2}{(xy - |z|^2)^2}, \quad K_{22} = \frac{y^4}{(xy - |z|^2)^2}, \quad K_{12} = \frac{y^3z}{(xy - |z|^2)^2}.
\]
Compact quotients of \( (\mathbb{R} \times H_3(\mathbb{R}), J) \) are primary Kodaira surfaces, which are non-Kähler.

**Secondary Kodaira surfaces.**

The Lie group is \( G = \mathbb{R} \times H_3(\mathbb{R}) \). It admits two different left-invariant complex structure \( J_\pm \) and the structure constants \( \mu = \mu_\pm \) of its complexified Lie algebra are

\[
\mu(Z_1, Z_2) = \varepsilon Z_1, \quad \mu(Z_1, \bar{Z}_2) = -\varepsilon Z_1, \quad \mu(Z_1, \bar{Z}_1) = -\sqrt{-1}\varepsilon(Z_2 + \bar{Z}_2), \quad \text{with} \ \varepsilon = \pm 1.
\]

The HCF tensor of a left-invariant Hermitian metric on \( (\mathbb{R} \times H_3(\mathbb{R}), J_\pm) \) is given by
\[
K_{11} = \frac{|z|^2(x^2 + y^2) - 2y^2(xy - |z|^2)}{(xy - |z|^2)^2}, \quad K_{22} = \frac{y^4 + |z|^4}{(xy - |z|^2)^2}, \quad K_{12} = \frac{yz(x^2 + y^2)}{(xy - |z|^2)^2}.
\]
Compact quotients of \( (\mathbb{R} \times H_3(\mathbb{R}), J_\pm) \) are secondary Kodaira surfaces, which are non-Kähler.

**Inoue surfaces of type \( S^0 \).**

The group \( G = \text{Sol}_0^4 \) is a solvable 4-dimensional real Lie group which admits a two-parameter family \( J_{a,b} \) of left-invariant complex structures, where \( a, b \in \mathbb{R} \), and with respect to \( J_{a,b} \) the structure constants \( \mu = \mu_{a,b} \) of its complexified Lie algebra are

\[
\mu(Z_1, Z_2) = -(b + \sqrt{-1}a)Z_1, \quad \mu(Z_1, \bar{Z}_2) = (b + \sqrt{-1}a)Z_1, \quad \mu(Z_2, \bar{Z}_2) = -2\sqrt{-1}a(Z_2 + \bar{Z}_2).
\]

The HCF tensor of a left-invariant Hermitian metric on \( (\text{Sol}_0^4, J_{a,b}) \) is given by
\[
K_{11} = \frac{x^2|z|^2(b^2 + 9a^2)}{(xy - |z|^2)^2},
\]
\[
K_{22} = \frac{|z|^4(a^2 + b^2) + 16|z|^2a^2xy - 8a^2x^2y^2}{(xy - |z|^2)^2},
\]
\[
K_{12} = \frac{x^2yz(b^2 + 9a^2)}{(xy - |z|^2)^2}.
\]
Notice that \( (\text{Sol}_0^4, J_{a,b}) \) does not always admit a co-compact lattice. When such a lattice does exist, the quotient is an Inoue surfaces of type \( S^0 \), which is non-Kähler.
Inoue surfaces of type $S^\pm$.

The group $G = \text{Sol}_4^1$ is a solvable 4-dimensional real Lie group which admits two different left-invariant complex structure $J_1, J_2$. The structure constants $\mu = \mu_1$ of the complexified Lie algebra of $(\text{Sol}_4^1, J_1)$ are

$$
\mu(Z_1, Z_2) = -Z_2, \quad \mu(\bar{Z}_1, Z_2) = -Z_2, \quad \mu(Z_1, \bar{Z}_1) = -Z_1 + \bar{Z}_1
$$

and the HCF tensor of a left-invariant Hermitian metric on $(\text{Sol}_4^1, J_1)$ is given by

$$
K_{11} = -3 - \frac{|z|^2(z - \bar{z})^2}{(xy - |z|^2)^2}, \quad K_{22} = -\frac{y^2(z - \bar{z})^2}{(xy - |z|^2)^2}, \quad K_{12} = \frac{y(z(\bar{z}^2 - z^2) - 2xy(\bar{z} - z))}{(xy - |z|^2)^2}.
$$

On the other hand, the structure constants $\mu = \mu_2$ of the complexified Lie algebra of $(\text{Sol}_4^1, J_2)$ are

$$
\mu(Z_1, Z_2) = -Z_2, \quad \mu(\bar{Z}_1, Z_2) = -Z_2, \quad \mu(Z_1, \bar{Z}_1) = -Z_1 + \bar{Z}_1 + Z_2 - \bar{Z}_2
$$

and the HCF tensor of a left-invariant Hermitian metric on $(\text{Sol}_4^1, J_2)$ is given by

$$
K_{11} = -3 - \frac{|z|^2(z + \bar{z})^2 + 2y^2(xy - |z|^2) - y^2|z|^2}{(xy - |z|^2)^2},
$$

$$
K_{22} = \frac{y^2((z - \bar{z})^2 - y^2)}{(xy - |z|^2)^2},
$$

$$
K_{12} = \frac{y(z(\bar{z}^2 - z^2) - 2xy(\bar{z} - z) + y^2z)}{(xy - |z|^2)^2}.
$$

Compact quotients of $(\text{Sol}_4^1, J_1)$ are Inoue surfaces of type $S^\pm$, while compact quotient of $(\text{Sol}_4^1, J_2)$ are Inoue surfaces of type $S^\mp$. In both cases, these surfaces are non-Kähler.

4. HCF on locally homogeneous surfaces

In this section we study the behavior of locally homogeneous solutions to the HCF on the family of compact complex surfaces we listed in Section 3. Furthermore, whenever a solution to the HCF is immortal, we determine the Gromov-Hausdorff limit of its normalization $(1 + t)^{-1}g(t)$ as $t \to +\infty$.

Let $X$ be a compact complex surface covered by a connected, simply-connected 4-dimensional real Lie group $G$ and $G \subset G$ a co-compact lattice such that $X = \Gamma \backslash G$. By construction, all left-invariant tensor fields on $G$ factorizes through $X$. This yields a one-to-one correspondence between locally homogeneous solutions to the HCF on $X$ and solutions to the corresponding ODE on $G$

$$\frac{d}{dt}g(t) = -K(g(t)), \quad g(0) = g_0,$$

where $g_0$ denotes the pull-back of the starting metric on $G$. Nonetheless, since the standard left-action of $G$ on itself does not always factorize through $X = \Gamma \backslash G$, the quotient $\Gamma \backslash G$ is not globally $G$-homogeneous in general.

Notation. Any left-invariant Hermitian metric $g$ on $(G, J)$ will be considered in the form of $(S)$. For the sake of shortness, we set $D := xy - |z|^2$ and $u := |z|^2$.

4.1. Hyperelliptic surfaces.

The HCF on $(\text{SE}(2) \times \mathbb{R}, J)$ reduces to the following ODEs system:

$$
\dot{x} = -\frac{x^2u}{D^2}, \quad \dot{y} = \frac{u^2}{D^2}, \quad \dot{u} = -2\frac{x^2yu}{D^2}.
$$

Proposition 4.1. Let $g_0$ be a locally homogeneous Hermitian metric on a hyperelliptic surface $X$. Then, the solution $g(t)$ to the HCF starting from $g_0$ exists for all $t \geq 0$. Moreover

$$
(X, (1 + t)^{-1}g(t)) \xrightarrow{\text{GH}} \{\text{point}\} \quad \text{as } t \to \infty.
$$

Proof. A direct computation yields that

$$
\dot{D} = \frac{xu}{D} \geq 0,
$$
i.e. the determinant of \( g(t) \) is always increasing. On the other hand, since all \( x, y, u \) decrease, the first claim follows. The last claim follows directly from the fact that
\[
(1+t)^{-1}x(t), (1+t)^{-1}y(t), (1+t)^{-1}u(t) \to 0
\]
as \( t \to +\infty \).

It is easy to show that a left-invariant metric \( g \) on \( (SE(2) \times \mathbb{R}, J) \) is Kähler if and only if \( z = 0 \) and in that case it is also flat. Hence, it comes the following

**Corollary 4.2.** Any locally homogeneous solution \( g(t) \) to the HCF on a hyperelliptic surface \( X \) converges exponentially fast to a flat Kähler metric \( g_\infty \).

**Proof.** We recall that \( g(t) \) is immortal and \( \dot{D}(t) > 0 \), \( x(t) < x_0 \), \( y(t) < y_0 \), \( u(t) < u_0 \) for any \( t \geq 0 \). Notice that
\[
\dot{u} \leq -\frac{2}{y_0} u,
\]
which implies \( u(t) \leq u_0 e^{-\frac{2}{y_0} t} \) for all \( t \geq 0 \). Finally, since
\[
\lim_{t \to +\infty} D(t) = D_\infty \in (D_0, +\infty),
\]
it comes that \( x(t) \to x_\infty \in (0, x_0) \) and \( y(t) \to y_\infty \in (0, y_0) \) as \( t \to +\infty \).

**4.2. Hopf surfaces.**

The HCF on \( (SU(2) \times \mathbb{R}, J_\lambda) \) reduces to the ODEs system
\[
\begin{align*}
\dot{x} &= -\frac{cx^4 + u(2x^2 + u)}{D^2}, \\
\dot{y} &= -2 + \frac{2cx^2D - cx^2u - u(y^2 + 2u)}{D^2}, \\
\dot{u} &= -2 \frac{xu(cx^2 + 2xy + y^2)}{D^2}
\end{align*}
\]
with \( c := 1 + \lambda^2 \).

**Proposition 4.3.** Let \( g_0 \) be a locally homogeneous Hermitian metric on a Hopf surface \( X \). Then, the solution \( g(t) \) to the HCF starting from \( g_0 \) develops a finite extinction time \( T < \infty \) and \( (X, g(t)) \) collapses as \( t \to T^- \).

**Proof.** Let \( T \in (0, +\infty) \) be the maximal existence time of the flow. Then for any \( t \in [0, T) \) we have
\[
\dot{D} = \frac{cx^3 - 2x^2 y + (4x + y) u}{D},
\]
\[
\dot{x} < 0, \quad \dot{u} < 0 \quad \Rightarrow \quad x(t) \leq x_0, \quad u(t) \leq u_0.
\]
Let us suppose by contradiction that \( T = +\infty \). Then it necessarily holds
\[
\begin{align*}
\lim_{t \to +\infty} \dot{x}(t) &= 0 \quad \Rightarrow \quad \lim_{t \to +\infty} (c - 1) \left( \frac{x^2}{D} \right)^2 = \lim_{t \to +\infty} \frac{x^2 + u}{D} = 0, \\
\lim_{t \to +\infty} \dot{u}(t) &= 0 \quad \Rightarrow \quad \lim_{t \to +\infty} \frac{x}{u} (c - 1) \left( \frac{x^2}{D} \right)^2 \leq \lim_{t \to +\infty} x u \left( \frac{x + y}{D} \right)^2 = 0.
\end{align*}
\]
On the other hand
\[
\dot{y} + 2 = \frac{2cx^2D - cx^2u - u(y^2 + 2u)}{D^2} \leq 2c \frac{x^2}{D} \leq 2c \frac{x^2 + u}{D},
\]
and so by means of 11
\[
\lim_{t \to +\infty} \dot{y}(t) \leq -2
\]
which is absurd. Thus \( g(t) \) develops a finite time singularity \( T < \infty \). In order to prove the last claim, let us suppose by contradiction that \( D \to \infty \) as \( t \to T^- \). Then
\[
\lim_{t \to T^-} \dot{x}(t) = 0 \quad \text{and} \quad \lim_{t \to T^-} \dot{y}(t) < -2,
\]
this in turn imply \( \lim_{t \to T^-} D \neq \infty \), which is not possible. On the other hand, since the solution cannot be extended over \( t = T \), the limit \( \lim_{t \to T^-} D \) cannot be positive and finite. Therefore, \( \lim_{t \to T^-} D = 0 \) and the thesis follows. \( \square \)

Next, we exhibit an explicit solution to the HCF starting from a diagonal metric on \((\text{SU}(2) \times \mathbb{R}, J_\lambda)\).

**Example 4.4.** Let \( g_0 \) be a left-invariant diagonal Hermitian metric on \((\text{SU}(2) \times \mathbb{R}, J_\lambda)\). Then, the ODEs system (9) reduces to

\[
\dot{x} = -\frac{c x^2}{y^2}, \quad \dot{y} = -2\frac{y - cx}{y}.
\]

It is worth noting that

\[
\dot{x} = -4c \frac{x^2}{y^2} \left( y - \frac{3}{2} cx \right), \quad \dot{y} = +4c \frac{x}{y^2} \left( y - \frac{3}{2} cx \right).
\]

Now suppose that \( y_0 = \frac{3}{2} cx_0 \) and that the solution to (12) starting from \( y_0 \) satisfies

\[
y(t) = \frac{3}{2} x(t) \quad \text{for all } t \in [0, T).
\]

Then by (13) we would get

\[
\dot{x}(t) = \dot{y}(t) = 0,
\]

which in turn implies

\[
x(t) = x_0 + kt, \quad y(t) = \frac{3}{2} cx_0 + \frac{3}{2} c kt
\]

for some \( k \in \mathbb{R} \). A direct computation yields that (14) solves (12) if and only if \( k = -\frac{4}{3c} \). Notice that the maximal existence time for this explicit solution is \( T = \frac{4}{3c} x_0 \).

### 4.3. Non-Kähler properly elliptic surfaces.

The HCF on \((\text{SL}(2, \mathbb{R}) \times \mathbb{R}, J_\lambda)\) reduces to the ODEs system

\[
\dot{x} = 2 + \frac{2cy^2 D - cy^2 u - ux^2 + 2u^2}{D^2},
\]

\[
\dot{y} = -\frac{cy^4 - 2y^2 u + u^2}{D^2},
\]

\[
\dot{u} = -\frac{yu(x^2 - 2xy + cy^2)}{D^2}
\]

with \( c := 1 + \lambda^2 \).

**Proposition 4.5.** Let \( g_0 \) be a locally homogeneous Hermitian metric on a non-Kähler properly elliptic surface \( X \). Then, the solution \( g(t) \) to the HCF starting from \( g_0 \) exists for all \( t \geq 0 \). In particular, \( x(t) \sim 2t \) and \( y(t) < y_0 \), \( u(t) < u_0 \) for any \( t > 0 \).

**Proof.** Let \( T \in (0, +\infty) \) be the maximal existence time of the flow. Then, for any \( t \in [0, T) \), we have

\[
\dot{D} = \frac{cy^3 + 2y(D - u) + xu}{D},
\]

\[
\dot{y} < 0, \quad \dot{u} < 0 \implies y(t) \leq y_0, \quad u(t) \leq u_0.
\]

We prove now that \( \dot{D}(t) > 0 \) for any \( t \in [0, T) \). Let us suppose by contradiction that there exists \( t_* \in [0, T) \) such that \( \dot{D}(t_*) \leq 0 \). Then using (16) we get

\[
-x(t_*) u(t_*) \geq cy(t_*)^3 - 2y(t_*) (u(t_*) - D(t_*)) \quad (17)
\]

On the other hand, since \( D(t) = x(t)y(t) - u(t) \) and \( \dot{u}(t_*) < 0 \), it necessarily holds

\[
\dot{x}(t_*) y(t_*) + x(t_*) \dot{y}(t_*) \leq 0. \quad (18)
\]

Moreover, by (17) and a straightforward computation we get

\[
D(t_*)^2 \dot{x}(t_*) y(t_*) \geq 4D(t_*)^2 y(t_*) + 3cy(t_*)^3 D(t_*) \quad (19)
\]

and

\[
D(t_*)^2 x(t_*) \dot{y}(t_*) \geq 4y(t_*) u(t_*) D(t_*) - cy(t_*)^3 D(t_*) \quad (20).
\]
Finally, (18), (19) and (20) imply
\[ 4D(t_*)y(t_*) + 2cy(t_*)^2 + 4y(t_*)u(t_*) \leq D(t_*)(\dot{x}(t_*)y(t_*) + x(t_*)\dot{y}(t_*)) \leq 0 \]
which is not possible. Hence the determinant \( D \) satisfies
\[ \dot{D} > 0 \implies D(t) \geq D_0 \quad \text{for all } t \in [0, T). \tag{21} \]
On the other hand, it holds
\[ \dot{x} \leq 2 + \frac{2cy^2D + 2u^2}{D^2} \leq 2\left(1 + c\frac{y_0^2}{D_0} + \frac{u_0^2}{D_0}\right) \implies x(t) \leq 2\left(1 + c\frac{y_0^2}{D_0} + \frac{u_0^2}{D_0}\right)t + x_0 \tag{22} \]
and hence (16), (21) and (22) imply \( T = +\infty \).

We are now ready to prove the second part of the proposition. To do this, we use again a contradiction argument. Let us denote with
\[ u_\infty := \lim_{t \to +\infty} u(t), \quad y_\infty := \lim_{t \to +\infty} y(t), \]
and suppose by contradiction that \( u_\infty > 0 \). Since
\[ \lim_{t \to +\infty} \dot{y}(t) = 0 \implies \lim_{t \to +\infty} \left(c - 1\right)\left(\frac{y^2}{D}\right)^2 = \lim_{t \to +\infty} \frac{y^2 - u}{D} = 0, \]
\[ \lim_{t \to +\infty} \dot{u}(t) = 0 \implies \lim_{t \to +\infty} \frac{u}{y} \left(c - 1\right)\left(\frac{y^2}{D}\right)^2 = \lim_{t \to +\infty} yu \left(\frac{x-y}{D}\right)^2 = 0, \]
we have by means of (10)
\[ \lim_{t \to +\infty} \frac{y(x-y)}{D} = \lim_{t \to +\infty} \frac{y^2 - u}{D} = 0 \implies \lim_{t \to +\infty} \frac{x - y}{y} = \lim_{t \to +\infty} \frac{1 - y}{y} = 0. \tag{23} \]
In view of (23), we have two cases depending on whether \( \lim_{t \to +\infty} |1 - \frac{y}{y_\infty}| \) is bounded or not. If we suppose that \( \lim_{t \to +\infty} |1 - \frac{y}{y_\infty}| < \infty \), then
\[ \lim_{t \to +\infty} xy = y_\infty \quad \text{and} \quad \lim_{t \to +\infty} D = 0. \]
On the other hand, if \( \lim_{t \to +\infty} |1 - \frac{y}{y_\infty}| = \infty \), then
\[ \lim_{t \to +\infty} xy = 0 \quad \text{and} \quad \lim_{t \to +\infty} D = -u_\infty. \]
Since both cases lead to an absurd, it comes
\[ u_\infty = 0. \tag{24} \]
Finally, we use (24) to prove that \( x(t) \sim 2t \) as \( t \to \infty \). Let us suppose by contradiction that \( x(t) \to x_\infty < +\infty \) as \( t \to +\infty \). Then \( D(t) \to D_\infty = x_\infty y_\infty \in (D_0, +\infty) \) as \( t \to +\infty \) and therefore it must holds \( x_\infty > 0 \). By means of (10)
\[ \lim_{t \to +\infty} \dot{D}(t) = 0 \implies cy_\infty^3 + 2y_\infty D_\infty = 0 \implies y_\infty = 0 \implies D_\infty = 0 \]
which is not possible. Therefore \( x(t) \to \infty \) as \( t \to \infty \). On the other hand, we have
\[ \dot{x} = 2 + 2\frac{y^2}{D} - cu \left(\frac{y}{D}\right)^2 - \frac{ux^2}{D^2} + \frac{2u^2}{D^2} \]
and, since
\[ \frac{y^2}{D} \to 0, \quad u \left(\frac{y}{D}\right)^2 \to 0, \quad \frac{ux^2}{D^2} \to 0, \quad \frac{u^2}{D^2} \to 0, \]
the thesis follows. \( \square \)

In view of this result it comes the following

**Proposition 4.6.** Let \( X \) be a non-Kähler properly elliptic surface and \( g(t) \) be a locally homogeneous solution to the HCF on \( X \). Then
\[ (X,(1+t)^{-1}g(t)) \overset{GH}{\to} (C,g_{KE}) \quad \text{as } t \to \infty, \]
where \( C \) is the base curve of \( X \) and \( g_{KE} \) is the Kähler-Einstein metric on \( C \) with \( \text{Ric}(g_{KE}) = -g_{KE} \).
The proof of this statement follows the same arguments used in [28, Thm 1.6 (c)]. For this reason, we just recall the main points.

Proof. By definition, a properly elliptic surface is a compact complex surface $X$ with Kodaira dimension $\kappa(X) = 1$ and first Betti number $b_1(X)$ odd admitting an elliptic fibration $\pi : X \to C$ over a compact complex curve $C$ of genus $g(C) \geq 2$. Moreover, by the Riemann Uniformization Theorem, $C$ admits a unique Kähler-Einstein metric $g_{KE}$ with $\text{Ric}(g_{KE}) = -g_{KE}$. Note that, this metric also satisfies $\pi^* g_{KE} = 2\zeta^1 \otimes \zeta^1$.

On the other hand, the fibers of the elliptic fibration $\pi : X \to C$ are spanned by the real and imaginary parts of $Z_2$, which shrinks to zero along $(1+t)^{-1}g(t)$ as $t \to \infty$. Therefore, if we consider a not necessarily continuous function $f : C \to S$ satisfying $\pi \circ f = \text{id}$, then for any $\epsilon > 0$ there exists $t_*(\epsilon) > 0$ such that $(\pi, f)$ is a GH $\epsilon$-approximation between $(X, (1+t)^{-1}g(t))$ and $(\mathcal{C}, g_{KE})$ for any $t > t_*(\epsilon)$. This concludes the proof. \qed

4.4. Primary Kodaira surfaces.

The HCF on $(\mathbb{R} \times H_3(\mathbb{R}), J)$ reduces to the ODEs system

$$
\dot{x} = \frac{2y^2 D - y^2 u}{D^2}, \quad \dot{y} = -\frac{y^4}{D}, \quad \dot{u} = -\frac{2y^3 u}{D^2}.
$$

(25)

Proposition 4.7. Let $g_0$ be a locally homogeneous Hermitian metric on a primary Kodaira surface $X$. Then, the solution $g(t)$ to the HCF starting from $g_0$ exists for all $t \geq 0$. Moreover,

$$(X, (1+t)^{-1}g(t)) \xrightarrow{\text{GH}} \{\text{point}\} \quad \text{as } t \to \infty.
$$

Proof. Let $T \in (0, +\infty]$ denote the maximal existence time of the flow. Then, for any $t \in [0, T)$, it holds that

$$
\dot{D} = \frac{y^3}{D} > 0 \implies \quad D(t) \geq D_0,
$$

$$
\dot{y} < 0, \quad \dot{u} < 0 \implies \quad y(t) \leq y_0, \quad u(t) \leq u_0
$$

and, on the other hand

$$
\dot{D} \leq \frac{y_0^3}{D_0} \implies \quad D(t) \leq \sqrt{2y_0^3 + D_0^2},
$$

$$
\dot{x} \leq \frac{2y_0^2}{D_0} \implies \quad x(t) \leq \left(\frac{2y_0^2}{D_0}\right)t + x_0.
$$

(26) \quad (27)

Therefore, the long-time existence of the solution follows from (20) and (27). For the second claim, we notice that

$$
\lim_{t \to +\infty} \dot{y}(t) = 0 \implies \lim_{t \to +\infty} \frac{y^2}{D} = 0,
$$

$$
\lim_{t \to +\infty} \dot{u}(t) = 0 \implies \lim_{t \to +\infty} \frac{y^3 u}{D^2} = 0.
$$

(28)

Now, let us suppose by contradiction that $\frac{\dot{x}}{x \dot{t}} \to \delta > 0$, as $t \to +\infty$. From this and (28) it comes that

$$
\dot{x} \sim -\frac{y^2 u}{D^2} \quad \text{as } t \to \infty
$$

and hence there exist $0 < \delta' < \delta$ and $t_0 > 0$ such that, for any $t \in [t_0, +\infty)$, it holds

$$
\dot{x} \leq -\delta' \implies \quad x(t) \leq \delta't + x(t_0),
$$

which is not possible. As a consequence, we have that $\dot{x}(t) \to 0$ as $t \to +\infty$. From this last claim, arguing again by contradiction, we also get $(1+t)^{-1}x(t) \to 0$ as $t \to +\infty$. \qed
4.5. Secondary Kodaira surfaces.

The HCF on \((\mathbb{R} \times H_3(\mathbb{R}), J)\) reduces to the ODEs system
\[
\dot{x} = \frac{2y^2 D - u(x^2 + y^2)}{D^2}, \quad \dot{y} = \frac{-y^2 + u^2}{D^2}, \quad \dot{u} = -2yu(x^2 + y^2).
\]  \tag{29}

**Proposition 4.8.** Let \(g_0\) be a locally homogeneous Hermitian metric on a secondary Kodaira surface \(X\). Then, the solution \(g(t)\) to the HCF starting from \(g_0\) exists for all \(t \geq 0\). Moreover
\[
(X, (1+\varepsilon)^{-1}g(t)) \xrightarrow{\text{GH}} \{\text{point}\} \quad \text{as} \ t \to \infty.
\]

**Proof.** Let \(T \in (0, +\infty)\) be the maximal existence time of the solution. Then, for any \(t \in [0, T)\) it holds
\[
\dot{D} = \frac{y^2 + xu}{D} > 0 \implies D(t) \geq D_0,
\]
\[
\dot{y} < 0, \quad \dot{u} < 0 \implies y(t) \leq y_0, \quad u(t) \leq u_0.
\]
Moreover, since
\[
\dot{x} < \frac{2y^2}{D} \leq \frac{2y_0^2}{D_0} \implies x(t) \leq \frac{2y_0^2}{D_0} t + x_0,
\]
we have
\[
\lim_{t \to +\infty} \dot{y}(t) = 0 \implies \lim_{t \to +\infty} \frac{y}{D} = \lim_{t \to +\infty} \frac{u}{D} = 0,
\]
\[
\lim_{t \to +\infty} \dot{u}(t) = 0 \implies \lim_{t \to +\infty} \frac{x^2 y u}{D^2} = 0.
\]  \tag{30}

On the other hand, it follows by (30) that
\[
\lim_{t \to +\infty} D(t) = +\infty \implies \lim_{t \to +\infty} x(t)y(t) = +\infty.
\]

Finally, let us assume by contradiction that \(\frac{x^2 y u}{D^2} \to \delta > 0\) as \(t \to +\infty\). Then we get
\[
\dot{\delta} > 0 \implies \delta(t) < -\delta t + x(t_*),
\]
which is absurd. Consequently it comes \(\dot{x}(t) \to 0\) as \(t \to +\infty\). Arguing again by contradiction, we finally get \((1+\varepsilon)^{-1}x(t) \to 0\) as \(t \to +\infty\). \qed

4.6. Inoue surfaces of type \(S^0\).

The HCF on \((\text{Sol}_2^t, J_{a,b})\) reduces to the ODEs system
\[
\dot{x} = -(9a^2 + b^2) \frac{x^2 u}{D^2},
\]
\[
\dot{y} = 8a^2 - (9a^2 + b^2) \left(\frac{u}{D}\right)^2,
\]
\[
\dot{u} = -2(9a^2 + b^2) \frac{x^2 y u}{D^2}.
\]  \tag{31}

**Proposition 4.9.** Let \(g_0\) be a locally homogeneous Hermitian metric on an Inoue surfaces \(X\) of type \(S^0\). Then, the solution \(g(t)\) to the HCF starting from \(g_0\) exists for all \(t \geq 0\). In particular, \(y(t) \sim 8a^2 t\) and \(x(t) < x_0, u(t) < u_0\) for any \(t > 0\).
Moreover, since

\[ \text{Proposition 4.10. Let } \{a, b\} \text{ be an Inoue surface of type } (32), \text{ with } K := (1 - \sqrt{2}a, 0) \text{ and } a, b \in \mathbb{R}, \text{ with } a > 0 \text{ and } b \neq 0, \text{ and } \lambda \in \mathfrak{gl}(3, \mathbb{Z}) \text{ be a matrix with eigenvalues} \]

\[ \lambda := \left( \begin{array}{ccc} e^{2\sqrt{2}a} & 0 & 0 \\ 0 & e^{-2\sqrt{2}a} & 0 \\ 0 & 0 & 1 \end{array} \right) \text{ and } p, q, r, s \in \mathbb{R}, \]

\[ \text{Indeed, let } \{E^i_3\} \text{ denote the standard basis of } \mathfrak{gl}(3, \mathbb{C}). \text{ Then the Lie algebra } \mathfrak{g}_{a,b} := \text{Lie}(G_{a,b}) \subset \mathfrak{gl}(3, \mathbb{C}) \text{ is the } \mathbb{R}-\text{span of} \]

\[ X_1 := (1 - \sqrt{2}a)E^1_3, \quad X_2 := (1 + \sqrt{2}a)E^1_3, \quad X_3 := E^2_3, \quad X_4 := \sqrt{2}(a - \sqrt{2}b)E^1_3 + 2\sqrt{2}aE^2_3. \]

Since the structure constants of \( \mathfrak{g}_{a,b} \) with respect to \( \{X_i\} \) are given by

\[ [X_1, X_4] = \sqrt{2}aX_1 - \sqrt{2}bX_2, \quad [X_2, X_4] = \sqrt{2}bX_1 + \sqrt{2}aX_2, \quad [X_3, X_4] = -2\sqrt{2}aX_3, \]
setting
\[ Z_1 := \frac{X_1 - \sqrt{2}X_2}{\sqrt{2}} \quad \text{and} \quad Z_2 := \frac{X_4 - \sqrt{2}X_4}{\sqrt{2}} , \]
one obtains the structure constants given in Section 3. Let now \((v_1, v_2, v_3) \in \mathbb{R}^3\) and \((w_1, w_2, w_3) \in \mathbb{C}^3\) be the eigenvectors of \(e^{2\sqrt{2}a}\) and \(e^{-2\sqrt{2}b}\), respectively, and consider the lattice \(\Gamma_{a,b} \subset G_{a,b}\) generated by
\[ h_0 := \begin{pmatrix} e^{\sqrt{2}(a+b)} & 0 & 0 \\ 0 & e^{\sqrt{2}a} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad h_i := \begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix} , \quad i = 1, 2, 3 . \]
Then the left action of \(\Gamma_{a,b}\) on \(G_{a,b}\) is explicitly given by
\[ h_0 \cdot M(p, q, r, s) = M(e^{-\sqrt{2}a}(\cos(\sqrt{2}b)p - \sin(\sqrt{2}b)q), e^{-\sqrt{2}a}(\sin(\sqrt{2}b)p + \cos(\sqrt{2}b)q), e^{2\sqrt{2}a}r, s + 1) \]
\[ h_i \cdot M(p, q, r, s) = M(p + \text{Re}(w_i), q + \text{Im}(w_i), r + v_i, s) \]
and the quotient \(X = \Gamma_{a,b}\backslash G_{a,b}\) is an Inoue surface of type \(S^0\).

Proof of Proposition 4.10. Let \(X = \Gamma_{a,b}\backslash G_{a,b}\) be an Inoue surface of type \(S^0\) and \(g(t)\) a locally homogeneous solution to the HCF on \(X\). By (33), the projection \(G_{a,b} \to \mathbb{R}\), \(M(p, q, r, s) \mapsto s\)
factorizes to a map \(\pi : X \to S^1 = \mathbb{R}/\mathbb{Z}\), which is a fibration with standard fiber \(T^3\) (see [5]). On the other hand, the path
\[ \mathbb{R} \to G_{a,b} , \quad s \mapsto M(0, 0, 0, s) \]
factorizes to a section \(\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to X\) whose length with respect to \(g(t)\) is
\[ \ell_{g(t)}(\gamma) = \sqrt{g(t)} \ . \]
Notice also that by Proposition 4.9
\[ (1+t)^{-1}g(t) \to \tilde{g}_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 8a^2 \end{pmatrix} \quad \text{as} \quad t \to \infty . \]
Moreover, in analogy with [28] Lemma 5.2, the kernel of \(\tilde{g}_\infty\) is the integrable distribution \(\mathcal{D}\) spanned by \(X_1, X_2\), which is dense inside any fiber of \(\pi\). Finally, the claim follows by \(4.1\) and this last observation (see e.g. [3] Cor 3.18).

4.7. Inoue surfaces of type \(S^\pm\).

The HCF on \((\text{Sol}^1, J_1)\) reduces to the ODEs system
\[ \dot{x} = 3 - \frac{u|z - \bar{z}|^2}{D^2} , \quad \dot{y} = -\frac{y^2|z - \bar{z}|^2}{D^2} , \quad \dot{u} = -\frac{2xy|z - \bar{z}|^2}{D^2} . \]

Proposition 4.11. Let \(g_0\) be a locally homogeneous Hermitian metric on an Inoue surfaces \(X\) of type \(S^\pm\) obtained by \((\text{Sol}^1, J_1)\). Then, the solution \(g(t)\) to the HCF starting from \(g_0\) exists for all \(t \geq 0\). In particular, \(x(t) \sim 3t\) and \(y(t) < y_0\), \(u(t) < u_0\) for any \(t > 0\).

Proof. Let \(T \in (0, +\infty)\) be the maximal existence time of the flow. Then, for any \(t \in [0, T)\), we have
\[ \dot{D} = 3y + \frac{y|z - \bar{z}|^2}{D} \geq 0 , \]
\[ \dot{y} < 0 , \quad \dot{u} < 0 \Rightarrow y(t) \leq y_0 , \quad u(t) \leq u_0 . \]
On the other hand
\[ \dot{x} = 3 - \frac{u|z - \bar{z}|^2}{D^2} \leq 3 \Rightarrow x(t) \leq 3t + x_0 \]
and the long-time existence follows, i.e. \(T = +\infty\). Finally, to conclude the proof it is enough to show
\[ \lim_{t \to \infty} \frac{|z - \bar{z}|}{D} = 0 . \]
Let us assume by contradiction that $\frac{|z| - \bar{z}|}{D} \rightarrow \epsilon > 0$. Then, by the means of (35) and (36), there exists $t_* > 0$ and a constant $k_1 > 1$ such that
\[-k_1 y(t)^2 \leq \dot{y}(t) \leq -\frac{1}{k_1} y(t)^2 \quad \text{for any } t \geq t_* .\]
This in turn implies, for any $t \geq t_*$,
\[\frac{1}{k_1(t - t_*) + \frac{1}{y(t_*)}} \leq y(t) \leq \frac{1}{k_1(t - t_*) + \frac{1}{y(t_*)}} .\] (38)
Besides, up to enlarge $t_*$, there also exists a constat $k_2 > 1$ such that
\[-k_2 y(t) \leq \dot{u}(t) \leq -\frac{1}{k_2} y(t) \quad \text{for any } t \geq t_* .\]
Therefore, since (38) holds, for any $t \geq t_*$ we have
\[-k_2 \frac{1}{k_1(t - t_*) + \frac{1}{y(t_*)}} \leq \dot{u}(t) \leq -\frac{1}{k_1k_2(t - t_*) + \frac{1}{y(t_*)}} \]
and
\[u(t_*) - k_1k_2 \log \left( \frac{y(t_*)}{k_1(t - t_*) + 1} \right) \leq u(t) \leq u(t_*) - \frac{1}{k_1k_2 \log (k_1y(t_*)(t - t_*) + 1) .\]
Nonetheless, this would imply $\lim_{t \rightarrow +\infty} u(t) = -\infty$, which is not possible. Hence, (37) holds and $x \sim 3t$ follows. \qed

The HCF on $(\text{Sol}_1^4, J_2)$ reduces to the ODEs system
\[
\begin{align*}
\dot{x} &= 3 + \frac{|z| + \bar{z}|^2 + 2y^2D - y^2u}{D^2} , \\
\dot{y} &= -\frac{y^2(|z| - \bar{z}|^2 + y^2)}{D^2} , \\
\dot{u} &= -\frac{2y^2(x|z - \bar{z}|^2 + yu)}{D^2} .
\end{align*}
\] (39)

**Proposition 4.12.** Let $g_0$ be a locally homogeneous Hermitian metric on an Inoue surfaces $X$ of type $S^+$ obtained by $(\text{Sol}_1^4, J_2)$. Then, the solution $g(t)$ to the HCF starting from $g_0$ exists for all $t \geq 0$. In particular, $x(t) \sim \alpha t$ for some $\alpha \geq 3$ and $y(t) < y_0$, $u(t) < u_0$ for any $t > 0$.

**Proof.** Let $T \in (0, +\infty]$ denote the maximal existence time of the solution. Then, a direct computation yields that
\[
\begin{align*}
\dot{D} &= 3y + \frac{y(|z| - \bar{z}|^2 + y^2)}{D^2} \geq 0 , \\
\dot{\bar{D}} &< 0 , \quad \dot{u} < 0 \quad \Rightarrow \quad y(t) \leq y_0 , \quad u(t) \leq u_0 .
\end{align*}
\] (40)
On the other hand, since
\[
\begin{align*}
\dot{x} \leq 3 + \frac{4u^2}{D^2} + \frac{2y^2}{D} \leq 3 + \frac{4u_0^2}{D_0} + \frac{2y_0^2}{D_0} ,
\end{align*}
\] we have $T = +\infty$ and the first part of the claim follows. To conclude the proof it is enough to show that
\[
\lim_{t \rightarrow +\infty} \frac{|z| + \bar{z}|^2 + 2y^2D - y^2u}{D^2} = k ,
\]
is necessarily non negative. By the means of (40), we can have either
\[
\lim_{t \rightarrow +\infty} D(t) = +\infty \quad \text{or} \quad \lim_{t \rightarrow +\infty} D(t) = +\infty ,
\]
but the former case directly implies $k = 0$, while the latter implies $y(t) \rightarrow 0$ for $t \rightarrow \infty$. Thus $k \geq 0$ and the claim follows. \qed

In view of the above results, we have
Proposition 4.13. Let $X$ be an Inoue surface of type $S^\pm$ and $g(t)$ be a locally homogeneous solution to the HCF on $X$. Then

$$(X,(1+t)^{-1}g(t)) \xrightarrow{\text{GH}} S^1(\rho) \quad \text{as } t \to \infty,$$

where $S^1(\rho) = \{ z \in \mathbb{C} : |z| = \rho \}$ is the circle of length $2\pi \rho$, for some $\rho \geq \frac{\sqrt{3}}{2}$.

We briefly recall the construction of Inoue surfaces of type $S^+$. Let $N \in SU(2,\mathbb{Z})$ be a unimodular matrix with real positive eigenvalues given by $\lambda$ and $\lambda^{-1}$, with $\lambda > 1$. It is well known that any $S^+$ surface can be realized as the quotient of the group

$$G_+ := \left\{ M_+(r,q,v,u) := \begin{pmatrix} 1 & u & v \\ 0 & q & r \\ 0 & 0 & 1 \end{pmatrix} : \ r,v,u \in \mathbb{R}, \quad q \in \mathbb{R}_{>0} \right\}.$$

by a lattice $\Gamma_+ := \langle f_0, f_1, f_2, f_3 \rangle$, where $f_i \in G_+$ are defined starting from $N$ (see [8]). Notice that Inoue surfaces of type $S^+$ enjoy nearly the same properties of surfaces of type $S^0$ (see [8]). In particular, they do not contain complex curves and any $S^+$ surface is diffeomorphic to a bundle over $S^2$. Moreover, since any $S^-$ surface admits an unramified double cover given by a $S^+$ surface, it is enough to prove the statement for Inoue surfaces of type $S^+$.

Proof of Proposition 4.13. Let $X = \Gamma_+ \setminus G_+$ be an Inoue surface of type $S^+$ and $g(t)$ a locally homogeneous solution to the HCF on $X$. The application

$$G_+ \to \mathbb{R}, \quad M_+(r,q,v,u) \mapsto \log \frac{\rho}{\log g}$$

factorizes to a map $\pi : X \to S^1$, which is a locally trivial fibration (see [8]). On the other hand, the path

$$\mathbb{R} \to G_+ , \quad s \mapsto M_+(0,\lambda^s,0,0)$$

factorizes to a section $\gamma : S^1 \to X$ whose length with respect to $g(t)$ is

$$\ell_{g(t)}(\gamma) = \sqrt{x(t)}.$$

Now, in view of the above results

$$(1+t)^{-1}g(t) \to \tilde{g}_\infty := \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as } t \to \infty,$$

for some $\alpha \geq 3$. Again, the kernel of $\tilde{g}_\infty$ is the integrable distribution $\mathcal{D}$ spanned by the real and imaginary part of $Z_2$, which is dense inside any fiber of $\pi$ (see [28, Lemma 6.2]). In analogy with the case of $S^0$ surfaces, the claim follows by setting $\rho := \frac{\sqrt{3}}{2}$.

We are now in position to prove Theorem A and Theorem B.

Proof of Theorem A and Theorem B. Let $X$ be a compact complex surface and $g_0$ a locally homogeneous non-Kähler metric on $X$. By Theorem 2.2 and Remark 2.3 $X$ is a quotient $\Gamma \setminus G$, where $G$ is one of the Lie groups listed in Section 3 i.e.

$$SE(2) \times \mathbb{R}, \quad SU(2) \times \mathbb{R}, \quad SL(2,\mathbb{R}) \times \mathbb{R}, \quad \mathbb{R} \times H_3(\mathbb{R}), \quad \mathbb{R} \times H_3^+(\mathbb{R}), \quad SO_4^+, \quad SO_4^{-},$$

and $\Gamma \subset G$ is a co-compact lattice.

Let also $T \in (0,\infty]$ be the extinction time of the HCF solution starting from $g_0$. Then, by means of Proposition 4.11 Prop. 4.13 Prop. 4.5 Prop. 4.1 Prop. 4.9 Prop. 4.11 and Prop. 4.12 we have $T < \infty$ if and only $G = SU(2) \times \mathbb{R}$. This implies Theorem A.

Finally, Theorem B comes from Prop. 4.11 Prop. 4.9 Prop. 4.7 Prop. 4.5 Prop. 4.11 and Prop. 4.13.
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