The game semantics of game theory

Jules Hedges

We use a reformulation of compositional game theory to reunite game theory with game semantics, by viewing an open game as the System and its choice of contexts as the Environment. Specifically, the system is jointly controlled by \( n \geq 0 \) noncooperative players, each independently optimising a real-valued payoff. The goal of the system is to play a Nash equilibrium, and the goal of the environment is to prevent it. The key to this is the realisation that lenses (from functional programming) form a dialectica category, which have an existing game-semantic interpretation.

In the second half of this paper, we apply these ideas to build a compact closed category of ‘computable open games’ by replacing the underlying dialectica category with a wave-style geometry of interaction category, specifically the Int-construction applied to the cartesian monoidal category of directed-complete partial orders.

1 Introduction

Although the mathematics of games shares a common ancestor in Zermelo’s work on backward induction, it split early on into two subjects that are essentially disjoint: game theory and game semantics. Game theory is the applied study of modelling real-world interacting agents, for example in economics or artificial intelligence. Game semantics, by contrast, uses agents to model – this time in the sense of semantics rather than mathematical modelling – situations in which a system interacts with an environment, but neither would usually be thought of as agents in a philosophical sense. On a technical level, game semantics not only restricts to the two-player zero-sum case, but moreover promotes one of the players to be the Player, and demotes the other to mere Opponent. This induces a deep logical duality that pervades game semantics, apparently destroying any hope of bridging the gap to game theory, which typically involves \( n \) players treated symmetrically.

Compositional game theory \[\text{Hed}16\ \text{GHWZ}18\], as its name suggests, is an attempt to introduce the principle of compositionality into game theory, motivated by practical concerns about modelling large (for example economic) systems. It is loosely inspired by game semantics, as well as categorical quantum mechanics \[\text{AC}04\ \text{CK}17\] and much recent work in applied category theory (e.g. \[\text{Fon}16\]). On a technical level, game semantics
involves (typically monoidal) categories in which games are the objects and strategies (with various conditions) are the morphisms, whereas open games form the morphisms of a monoidal category. This means that open games can be denoted by string diagrams, which is invaluable for working with them in practice. As with other categories of open systems, ordinary “closed” games are recovered as scalars, or endomorphisms of the monoidal unit (see [Abr05]), and depicted as string diagrams with trivial boundary.

Central to understanding open games is the concept of a context, which is a compressed representation of a game-theoretic situation in which an open game can be played. Whereas an ordinary game has a set of strategy profiles and a subset of those which are Nash equilibria, in an open game the equilibria depend on the context. This is the key to reuniting game theory and game semantics: we ignore the linguistic coincidence of the term player, and instead view an open game as the System and the choice of contexts as the Environment.

The essence of this idea is already contained in the following quote from the introduction of [Abr97]: “If Tom, Tim and Tony converse in a room, then from Tom’s point of view, he is the System, and Tim and Tony form the Environment; while from Tim’s point of view, he is the System, and Tom and Tony form the Environment.” The view of open games presented in this paper makes this precise when Tom, Tim and Tony are players in a noncooperative game.

In [Hed18] open games were first reformulated in terms of lenses from functional programming [PGW17]. This was extremely useful as a technical trick, but lenses are usually used as destructive update operators on data structures and it is unclear what they have to do with game theory, if anything. The key was a comment by Dusko Pavlovic to the author that the category of lenses \( \mathcal{L} \) is a dialectica category [dP91]; combined with a game-semantic view of dialectica categories [Bla91] we can see open games in their true form: as an interleaving of game theory and game semantics.

Specifically we find that an open game is a dialogue of a particular sort played between a system and its environment. The system is jointly controlled by \( n \geq 0 \) noncooperative players, each independently optimising a real-valued payoff. The winning condition turns out to be Nash equilibrium: the goal of the system is to play an equilibrium, and the goal of the environment is to prevent it. Specifically, an open game consists of three pieces of data: a set \( \Sigma \) of strategy profiles, a labelling function \( \Sigma \to \{ P\text{-strategies} \} \), and a winning (for \( P \)) relation \( E \subseteq \Sigma \times \{ O\text{-strategies} \} \).

In the second half of this paper, we apply these ideas to build a compact closed category of ‘computable open games’ by replacing the underlying dialectica category with a wave-style geometry of interaction category, specifically the Int-construction applied to the cartesian monoidal category of directed-complete partial orders. Ultimately we rely on the following transport of structure result:

**Proposition 1.** Let \( C \) be a compact closed category, \( D \) a symmetric monoidal category and \( F : C \to D \) a strict monoidal functor that is bijective on objects. Then \( D \) can be given a compact closed structure, with duals given by \( F(X) = F(X^*) \), units by \( \eta_{F(X)} = F(\eta_X) \) and counits by \( \varepsilon_{F(X)} = F(\varepsilon_X) \).

**Proof.** The assumption that \( F \) is bijective on objects means that every object of \( D \) is
uniquely assigned a dual, unit and counit. It is simple to check the yanking equations [KL80]:

\[
\begin{align*}
\rho_{F(X)} & \circ (id_{F(X)} \otimes \varepsilon_{F(X)}) \circ a_{F(X),F(X)^*,F(X)} \circ (\eta_{F(X)} \otimes id_{F(X)}) \circ \lambda_{F(X)}^{-1} \\
& = F(\rho_X) \circ (F(id_X) \otimes F(\varepsilon_X)) \circ F(a_{X,X^*,X}) \circ (F(\eta_X) \otimes F(id_X)) \circ F(\lambda_X^{-1}) \\
& = F(\rho_X) \circ (id_X \otimes \varepsilon_X) \circ a_{X,X^*,X} \circ (\eta_X \otimes id_X) \circ \lambda_X^{-1} \\
& = F(id_X) = id_{F(X)}
\end{align*}
\]

and similarly for the other equation. \hfill \Box

The hypotheses of this theorem are already satisfied by a particular functor \( \mathcal{L} \to OG \) that identifies \( \mathcal{L} \) with the subcategory of zero-player open games. Thus it suffices to replace the source category with one that is compact closed, while preserving the hypotheses (and the game-theoretic interpretation).

2 Dialogues

While the name ‘dialectica’ should bring to mind dialogues in the tradition of philosophical logic (for example via Hegel’s dialectics), this is apparently a coincidence. The dialectica interpretation is named after the journal Dialectica, who published Gödel’s paper in their Paul Bernays festschrift. But the dialectica interpretation does have a very dialectical feeling to it.

The game semantic viewpoint on Gödel’s dialectica interpretation [AF98] and de Paiva’s dialectica categories [dP91] was described in Blass’ paper that first introduced game semantics [Bla91]. It has not been considered much since, and is now largely folklore. In this section we recall this viewpoint in detail.

We first introduce a category \( \mathcal{L} \) of dialogues and strategies, which is the dialectica category over an inconsistent (1-valued) logic.

An object of \( \mathcal{L} \) is a 2-stage dialogue \( X^+; S^- \) in which first the System chooses \( x : X \), and then the Environment chooses \( s : S \). This breaks a common requirement in game semantics that Environment moves first. We denote the dialogue \( X^+; S^- \) by \( (X_S) \).

Notice that the set of \( P \)-strategies for \( (X_S) \) is \( X \), and the set of \( O \)-strategies is \( S^X \).

We introduce a monoidal product operator given by synchronous parallel play. Specifically, the parallel play of \( (X_S) \) and \( (Y_R) \) is the 4-stage dialogue \( X^+; Y^+; R^-; S^- \). This peculiar ordering of moves, with the right-hand dialogue being played in the middle of the left-hand dialogue, is characteristic of dialectica. This 4-stage dialogue is strategically equivalent to the 2-stage dialogue \( (X \times Y)^+; (R \times S)^- \), so we set \( (X_S) \otimes (Y_R) = (X \times Y)_{R \times S} \).

Next, given objects \( (X_S) \) and \( (Y_R) \), we consider the same 4-stage dialogue but with the players interchanged in the former. That is, we consider the dialogue \( X^-; Y^+; R^-; S^+ \). We consider this to be \( (Y_R) \) played relative to \( (X_S) \), and denote it by \( (X_S) \to (Y_R) \).

The set of \( P \)-strategies for \( (X_S) \to (Y_R) \) is \( Y^X \times S^{X \times R} \), or isomorphically \( (Y \times S^R)^X \). The set of \( O \)-strategies is \( X \times R^Y \). We denote the set of \( P \)-strategies for \( (X_S) \to (Y_R) \) by \( \text{hom}_\mathcal{L} ((X_S), (Y_R)) \). As the notation suggests, these are the morphisms of \( \mathcal{L} \).
Given an object \((\frac{X}{S})\), there is a \textit{copycat} \(P\)-strategy for \((\frac{X}{S}) \rightarrow (\frac{X}{S}) = X^-; X^+; S^-; S^+\). As an element of \(X^X \times S^{X \times S}\) it is the pair consisting of the identity and the projection. This is the identity morphism for \((\frac{X}{S})\). Following [Abr97] we denote this strategy by a string diagram:

\[
\begin{array}{cccc}
X^- & X^+ & S^- & S^+ \\
\uparrow & \downarrow & \uparrow & \downarrow
\end{array}
\]

(This syntax is strikingly similar to grammatical reductions in pregroups [CSC10].)

Now suppose we are given \(P\)-strategies \(\lambda : (\frac{X}{S}) \rightarrow (\frac{Y}{R}) = X^-; Y^+; R^-; S^+\) and \(\mu : (\frac{Y}{R}) \rightarrow (\frac{Z}{Q}) = Y^-; Z^+; Q^-; R^+\). There is a way to combine them to produce a \(P\)-strategy \(\mu \circ \lambda : (\frac{X}{S}) \rightarrow (\frac{Z}{Q}) = X^-; Z^+; Q^-; S^+\). Namely, \(P\) \textit{simulates} playing the two together with \(O\) playing a copycat strategy for the middle moves. That is, she simulates the 8-stage dialogue

\[
X^-; Y^+; Y^-; Z^+; Q^-; R^+; R^-; S^+
\]

with the assumption that \(O\) uses a copycat strategy for the moves \(Y^-\) and \(R^-\). By then hiding the \(Y\) and \(R\) moves we get a \(P\)-strategy for the required 4-stage dialogue.

We denote this strategy by the following string diagram:

\[
\begin{array}{cccc}
\lambda & \mu \\
\uparrow & \downarrow & \uparrow & \downarrow \\
X^- & Y^+ & Y^- & Z^+ \\
\downarrow & \uparrow & \downarrow & \uparrow \\
X^- & Y^+ & Z^+ & Q^-\end{array}
\]

Whereas the cap denotes a copycat \(P\)-strategy, the cup denotes a copycat \(O\)-strategy.

A little calculation shows that if \(\lambda\) is given by \(v_\lambda : X \rightarrow Y\) and \(u_\lambda : X \times R \rightarrow S\), and \(\mu\) is given by \(v_\mu : Y \rightarrow Z\) and \(u_\mu : Y \times Q \rightarrow R\), then the composite is given by

\[
v_{\mu \circ \lambda}(x) = v_\mu(v_\lambda(x))
\]

and

\[
u_{\mu \circ \lambda}(x, q) = u_\lambda(x, u_\mu(v_\lambda(x), q))
\]

It is routine to check that this is associative, with identities given by copycat. Thus \(\mathcal{L}\) is indeed a category.
Given this category structure we can also make $\otimes$ into a genuine symmetric monoidal product. Given $P$-strategies $\lambda : (X_1 \times X_2) \to (Y_1 \times Y_2)$ and $\mu : (S_1 \times S_2) \to (R_1 \times R_2)$, we can combine them to produce a winning strategy $\lambda \otimes \mu : (S_1 \times S_2 \times X_1 \times X_2) \to (Y_1 \times Y_2 \times R_1 \times R_2)$.

Finally, we notice that all of the above can be generalised to any base category $C$ with finite products, replacing sets and functions, yielding a category $\mathcal{L}(C)$ whose morphisms are strategies internal to $C$. Specifically, we set

$$\text{hom}_{\mathcal{L}(C)}\left(\left(\frac{X}{S}\right), \left(\frac{Y}{R}\right)\right) = \text{hom}_C(X,Y) \times \text{hom}_C(X \times R, S)$$

(By writing it this way, we do not need to assume that $C$ is cartesian closed.) The category we have been considering so far is $\mathcal{L} = \mathcal{L}(\text{Set})$.

**Proposition 2.** For any category $C$ with finite products, $\mathcal{L}(C)$ is a symmetric monoidal category.

There is a much less obvious generalisation of $\mathcal{L}(C)$ when $C$ is only a monoidal category [Ril18], but we will not need it in this paper.

### 3 Negation and $O$-strategies

To talk about open games, we need to talk explicitly about $O$-strategies in a dialogue. However, the categorical structure of $\mathcal{L}$ is built on $P$-strategies. In turns out, however, that we can use $P$-strategies to talk about $O$-strategies, in a way that respects composition.

Recall that the monoidal unit of $\mathcal{L}$ is the trivial game $I = \left\{\frac{1}{1}\right\} = 1^+; 1^-$. The dialogue $I \to \left(\frac{S}{X}\right)$ is $1^-; X^+; S^-; 1^+$, which is strategically equivalent to $\left(\frac{S}{X}\right)$. Thus the set of $P$-strategies for $I \to \left(\frac{S}{X}\right)$ is $X$.

If we fix a $P$-strategy $h : X$ for $\left(\frac{X}{S}\right)$ and another $P$-strategy $\lambda : \left(\frac{X}{S}\right) \to \left(\frac{Y}{R}\right)$, we can compose them to yield a $P$-strategy $\lambda \circ h$ for $\left(\frac{Y}{R}\right)$, by

$$I \xrightarrow{h} \left(\frac{X}{S}\right) \xrightarrow{\lambda} \left(\frac{Y}{R}\right)$$

Succinctly, there is a functor $V : \mathcal{L} \to \text{Set}$ taking every object to its set of $P$-strategies, namely the covariant functor represented by $I$. Explicitly, $V(\left(\frac{X}{S}\right)) = X$ and $V(\lambda) = v_\lambda$.

On the other hand, the dialogue $\left(\frac{X}{S}\right) \to I$ is $X^-; 1^+; 1^-; S^+$, which is equivalent to $X^-; S^+$. This is not an object, but is $\left(\frac{S}{X}\right)$ with players interchanged. Thus the set of $P$-strategies for $\left(\frac{X}{S}\right) \to I$ is equal to the set of $O$-strategies for $\left(\frac{X}{S}\right)$, namely $S^X$.

Given an $O$-strategy $k$ for $\left(\frac{Y}{R}\right)$ and a $P$-strategy $\lambda : \left(\frac{X}{S}\right) \to \left(\frac{Y}{R}\right)$, we obtain an $O$-strategy $k \circ \lambda$ for $\left(\frac{S}{X}\right)$ by

$$\left(\frac{X}{S}\right) \xrightarrow{\lambda} \left(\frac{Y}{R}\right) \xrightarrow{k} I$$
In this, $O$ ‘hijacks’ $P$’s strategy to produce an element of $S$, since $\lambda$ is a $P$-strategy for a dialogue in which $P$ plays the role of $O$ in $\left(\frac{X}{S}\right)$.

Succinctly, there is a functor $K : \mathcal{L}^{op} \to \text{Set}$ taking every object to its set of $O$-strategies, namely the contravariant functor represented by $I$.

An $O$-strategy for $\left(\frac{X}{S}\right) \to \left(\frac{Y}{R}\right)$ consists of a $P$-strategy for $\left(\frac{X}{S}\right)$ and an $O$-strategy for $\left(\frac{Y}{R}\right)$. This defines a functor $\text{cohom}_\mathcal{L} : \mathcal{L} \times \mathcal{L}^{op} \to \text{Set}$, namely

$$\mathcal{L} \times \mathcal{L}^{op} \xrightarrow{\text{YX}} \text{Set} \times \text{Set} \xrightarrow{\times} \text{Set}$$

(where $\text{cohom}$ is intended to stand for ‘cohomomorphism’, not ‘cohomology’!) On objects, it is concretely given by $\text{cohom}_\mathcal{L} \left(\left(\frac{X}{S}\right), \left(\frac{Y}{R}\right)\right) = X \times R^Y$, or more generally over a category $\mathcal{C}$ with finite products, $\text{cohom}_{\mathcal{C}} \left(\left(\frac{X}{S}\right), \left(\frac{Y}{R}\right)\right) = \text{hom}_\mathcal{C}(1, X) \times \text{hom}_\mathcal{C}(Y, R)$.

Given an $O$-strategy $(h, k)$ for $\left(\frac{X}{S_1}\right) \to \left(\frac{Y}{R_1}\right)$, a $P$-strategy $\lambda : \left(\frac{X_1}{S_1}\right) \to \left(\frac{X_2}{S_2}\right)$ and a $P$-strategy $\mu : \left(\frac{Y_2}{R_2}\right) \to \left(\frac{Y_1}{R_1}\right)$, we obtain an $O$-strategy $\text{cohom}_\mathcal{L}(\lambda, \mu)(h, k) = (\lambda \circ h, k \circ \mu)$ for $\left(\frac{X_2}{S_2}\right) \to \left(\frac{Y_1}{R_1}\right)$. This is the $O$-strategy for the dialogue

$$X_1^+ : X_2^+ ; X_1^- ; X_2^- ; Y_1^+ ; Y_2^+ ; Y_1^- ; Y_2^- ; R_1^+ ; R_1^- ; R_2^+ ; R_2^- ; S_2^+ ; S_2^- ; S_1^+$$

with appropriately hidden copycat moves, as given by the string diagram
4 Open games

We can now give an equivalent definition of open games [Hed16, GHWZ18] in terms of dialogues.

An open game \((X_S) \rightarrow (Y_R)\) is in one dimension a dialogue played between a System and an Environment, and in another dimension it is a non-cooperative game in the sense of economics, in which several players jointly control the System while independently optimising payoffs.

An open game \(\mathcal{G} : (X_S) \rightarrow (Y_R)\) is defined by three pieces of data:

- A set \(\Sigma_{\mathcal{G}}\) of strategy profiles
- A labelling function \(\mathcal{G}_\sigma : \Sigma_{\mathcal{G}} \rightarrow \text{hom}_L\left((X_S), (Y_R)\right)\), by which every element \(\sigma : \Sigma_{\mathcal{G}}\) labels a P-strategy \(\mathcal{G}_\sigma\) for the 4-stage dialogue \((X_S) \rightarrow (Y_R)\)
- A winning condition, which is a relation between \(\Sigma_{\mathcal{G}}\) and the set of O-strategies of \((X_S) \rightarrow (Y_R)\), namely \(|\mathcal{G}| \subseteq \Sigma_{\mathcal{G}} \times \text{cohom}_L\left((X_S), (Y_R)\right)\).

We write \(|\mathcal{G}|_\kappa\sigma\) for \((\sigma, \kappa) \in |\mathcal{G}|. We say that \(\sigma\) is a winning strategy profile if \(|\mathcal{G}|_\kappa\sigma\) for all \(\kappa : \text{cohom}_L\left((X_S), (Y_R)\right)\).

We interpret \(|\mathcal{G}|\) as an equilibrium condition. That is, from the dialogue perspective the goal of the System is to reach equilibrium and the goal of the Environment is to prevent equilibrium. In real examples there is rarely a winning strategy profile, and so we focus on \(|\mathcal{G}|\) as a binary relation, or ask about winning strategy profiles for the System against a fixed O-strategy.

From the dialogue perspective, the order of play in an open game \((X_S) \rightarrow (Y_R)\) is:

1. The Environment chooses an initial state of the game from \(X\)
2. The System chooses the final state of the game from \(Y\)
3. The Environment chooses payoffs for the System from \(R\)
4. The System chooses payoffs for the Environment from \(S\)

An O-strategy is a pair \(\kappa = (h, k)\) where \(h : X\) and \(k : Y \rightarrow R\). The history \(h\) determines the initial state of the game. The continuation \(k\) determines the payoffs for System given the final state. The pair \((h, k)\) completely determines the strategic context in which the players that make up System make their choices, reducing the open game to an ordinary normal-form game. For this reason, we also call an O-strategy a context for the open game.

We only need two families of examples of open games to generate a large family of examples, corresponding roughly to extensive-form games, using the sequential and parallel play operators we will define in the next section. These two generating families are the zero-player and the one-player open games.
The zero-player open games \((X) \rightarrow (Y)\) are in bijection with the \(P\)-strategies \(\lambda : (X) \rightarrow (Y)\), and correspond to the situation in which the System has no strategic choices but always follows the strategy \(\lambda\) like an automaton. Specifically, the zero-player open game \(\lambda\) is defined by:

- The set of strategy profiles is the singleton \(\Sigma_\lambda = \{\star\}\), where \(\star\) is a token representing the \(P\)-strategy \(\lambda\).
- The labelling function is \(\lambda : \star \mapsto \lambda\).
- \(\star\) is a winning strategy profile, that is, \(|\lambda\star\|_\kappa\) for all \(O\)-strategies \(\kappa\).

Perhaps the only surprising part of this definition is that \(\star\) is a winning strategy profile. The reason for this ultimately comes down to agreeing with Nash equilibrium on real examples. Nash equilibrium is a negative definition: a strategy profile should fail to be a Nash equilibrium if some particular player has positive incentive to deviate from it. Since there are no players in \(\lambda\), \(\star\) is declared a Nash equilibrium by default.

The second family of examples are the one-player open games. There is one such open game \(D = D_{X,Y} : (\chi) \rightarrow (\chi)\) for every nonempty set \(X\) and \(Y\). In this game:

1. The Environment chooses an initial state from \(X\).
2. The (now unique) Player chooses a final state from \(Y\).
3. The Environment chooses a payoff from \(\mathbb{R}\).

The winning condition of this game is intensional by being a property of the strategies of both Player and Environment, and cannot be written in terms of the play alone. This is because optimality in game theory is a counterfactual: if the System had made a different choice then the resulting payoff would be lower.

Observe that a \(P\)-strategy for this game is a function \(\sigma : X \rightarrow Y\), while an \(O\)-strategy is a pair \((h, k)\) where \(h : X\) and \(k : Y \rightarrow \mathbb{R}\). By definition, the Player wins this game iff \(\sigma(h) \in \arg \max k\), that is to say, if \(k(\sigma(h)) \geq k(y)\) for all \(y : Y\).

This is a small shift in perspective that is quite natural from the perspective of game semantics. In game theory there is no concept of winning, only optimality and equilibrium. Declaring a player to have won if they make an optimal choice may not be meaningful as game theory, but it is appropriate terminology when combining game theory with game semantics.

Writing this out:

- The set of strategy profiles is \(\Sigma_D = Y^X\).
- The labelling function takes \(\sigma : X \rightarrow Y\) to itself considered as a \(P\)-strategy \(D_\sigma : (\chi) \rightarrow (\chi)\), via the bijection \(\text{hom}_2 \left( (\chi), (\chi) \right) \cong Y^X\).
- The winning condition is \(|D|_{h,k}^{\sigma} \iff \sigma(h) \in \arg \max(k)|\)
5 Composing open games

We can make open games into the morphisms of a symmetric monoidal category. The two composition operators, categorical composition and tensor product, correspond to sequential play and simultaneous play.

Suppose we have given open games \( G : (X^S) \to (Y^R) \) and \( H : (Y^R) \to (Z^Q) \). The sequential composition \( H \circ G : (X^S) \to (Z^Q) \) has set of strategy profiles \( \Sigma_{H \circ G} = \Sigma_G \times \Sigma_H \). The idea is that \( G \) and \( H \) are each associated with sets \( G, H \) of decisions. Each decision \( g \in G, h \in H \) has an associated set \( \Sigma_g, \Sigma_h \) of strategies, and the set of strategy profiles in each case should be thought of as the set of tuples of strategies, one for each decision: \( \Sigma_g = \prod_{g \in G} \Sigma_g \) and \( \Sigma_h = \prod_{h \in H} \Sigma_h \). The set of decisions made in a composite game is the disjoint union of the decisions made in the components, and so \( \Sigma_{H \circ G} = \prod_{g \in G} \Sigma_g \times \prod_{h \in H} \Sigma_h = \Sigma_G \times \Sigma_H \).

The labelling function for a sequential composition can be defined using the underlying composition in \( \Sigma : (H \circ G)_{\sigma, \tau} = H_\tau \circ G_\sigma \).

In order to define the winning condition of \( H \circ G \), we must modify a context for \( H \circ G \) into contexts for \( G \) and \( H \). We can do this using the fact that cohom is a functor, together with the fact that we have strategy profiles for \( G \) and \( H \) available. A strategy profile \((\sigma, \tau)\) for \( H \circ G \) is winning (that is to say, a Nash equilibrium) against the \( O \)-strategy \( \kappa \) iff \( \sigma \) is winning in \( G \) against the \( O \)-strategy cohom(id, \( H_\tau \))(\( \kappa \)), and \( \tau \) is winning in \( H \) against the \( O \)-strategy cohom(\( G_\sigma \), id)(\( \kappa \)). That is to say,

\[
|H \circ G|_{\sigma, \tau}^{(\sigma, \tau)} \iff |G|_{\text{cohom(id, } H_\tau)}^{\sigma} \land |H|_\text{cohom(id, id)}^{\tau}
\]

This makes open games into the morphisms of a category. (More properly it is a bicategory, and only a category after quotienting out compatible bijections of strategy profiles. This is a purely technical complication that we will ignore in this paper.)

Next we consider simultaneous play. Given open games \( G : (X^S) \to (Y^R) \) and \( H : (X^T) \to (Y^Q) \), we combine them to form an open game

\[
G \otimes H : \left( \begin{array}{c} X^1 \times X^2 \\ S^2 \times S^1 \\ S_2 \times S_1 \end{array} \right) \to \left( \begin{array}{c} Y^1 \times Y^2 \\ R^1 \times R_2 \\ R_2 \times R_1 \end{array} \right)
\]

As before the strategy profiles of \( G \otimes H \) are pairs, \( \Sigma_{G \otimes H} = \Sigma_G \times \Sigma_H \), for the same reason as before: we take the disjoint union of the set of decisions. The strategy profile \((\sigma, \tau)\) labels the synchronous parallel play of \( G_\sigma \) and \( H_\tau \), that is, \((G \otimes H)_{\sigma, \tau} = G_\sigma \otimes H_\tau \).

In order to define the winning condition for \( G \otimes H \) we need to do some more work.

Given strategy profiles \( \sigma : \Sigma_G \) and \( \tau : \Sigma_H \), and an \( O \)-strategy \( \kappa \) for \( (X^1 \times X^2) \to (Y^1 \times Y^2) \), \( (S^1 \times S_2) \to (R^1 \times R_2) \), we need to ‘project’ \( \kappa \) to \( G \) and \( H \)’s view of it, as \( O \)-strategies for \( (X^1) \to (Y^1) \) and \( (S_2) \to (R_2) \).

We can indeed do this. To produce an \( O \)-strategy for \( (X^1) \to (Y^1) \), consider the dialogue

\[
X^-; Y^+; X^+; Y^-; R^-; S^+; R^+; S^-;
\]

with the strategy
We call this O-strategy $\mathcal{H}_\tau/\kappa$. When $\kappa = (h, k)$ we write $\mathcal{H}_\tau/\kappa = (h_1, k_2^{\mathcal{H}_\tau})$. Concretely, the new continuation is $k_2^{\mathcal{H}_\tau}(y_1) = k(y_1, v_{\mathcal{H}_\tau}(h_2))_1$.

Similarly, we can produce an O-strategy $\mathcal{G}_\sigma \setminus \kappa$ for $(\frac{X_2}{S_2}) \rightarrow (\frac{Y_2}{R_2})$ by considering the same dialogue with the strategy $\mathcal{G}_\sigma$.

When $\kappa = (h, k)$ we write $\mathcal{G}_\sigma \setminus \kappa = (h_2, k_1^{\mathcal{G}_\sigma})$, where $k_1^{\mathcal{G}_\sigma}(y_2) = k(v_{\mathcal{G}_\sigma}(h_1), y_2)_2$.

With this, we can finally define the winning condition for $\mathcal{G} \otimes \mathcal{H}$: The strategy profile $(\sigma, \tau)$ is winning against $\kappa$ in $\mathcal{G} \otimes \mathcal{H}$ iff $\sigma$ is winning against $\mathcal{H}_\tau/\kappa$ in $\mathcal{G}$ and $\tau$ is winning against $\mathcal{G}_\sigma \setminus \kappa$ in $\mathcal{H}$, that is to say,

$$|\mathcal{G} \otimes \mathcal{H}|_{\kappa}^{(\sigma, \tau)} \iff |\mathcal{G}|_{\mathcal{H}_\tau/\kappa}^{\sigma} \land |\mathcal{H}|_{\mathcal{G}_\sigma \setminus \kappa}^{\tau}$$
Proposition 3. There is a symmetric monoidal (bi)category $\text{OG}$ whose objects are pairs of sets and morphisms are open games.

What cannot be stated easily is that this categorical structure moreover does the right thing in game theory. We can draw string diagrams in this monoidal category that reflect the information flow in a game, and interpret them in the category $\text{OG}$, and find that the equilibrium condition is the same as the one given by a standard game-theoretic analysis. The reader is referred to [GHWZ18] for details, with proofs in [Hed16] and a formal analysis of the relevant string diagrams in [Hed17].

6 Dialogues and wave-style geometry of interaction

In order to obtain a connection between the dialectica and $\text{Int}$ constructions, we need to apply the $\text{Int}$ construction to categories that are traced cartesian monoidal. This is wave-style geometry of interaction, so-called because every point in our string diagrams is consistently assigned a value [Abr96]. (It is contrasted with particle-style GoI, which applies to monoidal categories built on a coproduct and in which we imagine a token moving around the diagram.)

Game-semantic interpretations of wave-style GoI have not been widely considered. In this section we suggest such an interpretation that will be suitable for our purposes.

The $\text{Int}$-construction can be defined over any traced monoidal category $\mathcal{C}$, but we restrict to traced cartesian categories. These are equivalent to Conway cartesian categories, or cartesian categories with a natural family of fixpoint operators [Has99]. A canonical example is the category $\text{DCPO}$ of directed-complete partial orders and continuous maps.

By definition, an object of the category $\text{Int}(\mathcal{C})$ is a pair $(\frac{X}{S})$ of objects of $\mathcal{C}$, and a morphism $(\frac{X}{S}) \to (\frac{Y}{R})$ in $\text{Int}(\mathcal{C})$ is a morphism $X \times R \to Y \times S$ in $\mathcal{C}$. Since $\mathcal{C}$ is cartesian monoidal, a morphism $(\frac{X}{S}) \to (\frac{Y}{R})$ is equivalently a pair of morphisms $X \times R \to Y$ and $X \times R \to S$.

The identity on $(\frac{X}{S})$ in $\text{Int}(\mathcal{C})$ is the identity on $X \times S$ in $\mathcal{C}$. The composition of $\lambda : (\frac{X}{S}) \to (\frac{Y}{R})$ and $\mu : (\frac{Y}{R}) \to (\frac{Z}{Q})$ in $\text{Int}(\mathcal{C})$ is given by

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda} & Y \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
Q & \xrightarrow{\mu} & R \\
\end{array}
\]

in $\mathcal{C}$, using the string diagram language for traced monoidal categories [Sel11, section 5.7].
The monoidal product of $\textbf{Int}(\mathcal{C})$ is defined on objects by $(X_1, S_1) \otimes (X_2, S_2) = (X_1 \otimes X_2, S_2 \otimes S_1)$, with the obvious definition on morphisms. As is well known, $\textbf{Int}(\mathcal{C})$ can be equipped with the structure of a compact closed category, which satisfies the universal property of being the free compact closed category on the traced monoidal category $\mathcal{C}$. Note that there are two different conventions in use: we follow [JSV96], which defines $\otimes$ with a twist in the contravariant place, rather than [Abr96] which does not.

The idea of interpreting objects and morphisms of $\textbf{Int}(\mathcal{C})$ as dialogues is to view them as repeated play of the corresponding dialogues for $\Sigma(\mathcal{C})$, starting from $\perp$ and converging to a fixpoint, after which the play terminates and all moves except the final ones are hidden.

We view the object $(X, S)$ as a dialogue

$$X^+; S^-; X^+; S^-; \cdots$$

We do not allow arbitrary strategies, but restrict the allowed $P$-strategies to $\mathcal{C}$-morphisms $S \to X$, and the allowed $O$-strategies to the $\mathcal{C}$-morphisms $X \to S$. Given such a pair of strategies $(h, k)$, the play that results is by definition

$$\perp_X; \perp_S; h(\perp_S); k(h(\perp_X)); h(h(\perp_S)); \cdots$$

When $\mathcal{C}$ is DCPO or another suitable category, this play stabilises after finitely many stages to the $(x, s)$ that is the least fixpoint of the recursion $x = h(s), s = k(x)$. By hiding the approximating moves, we consider the play resulting from $(h, k)$ to be $(x, s)$.

We can succinctly represent this repetition and hiding by extending our string diagram language for strategies with a trace.

Given objects $(X, S)$ and $(Y, R)$, the dialogue $(X, S) \to (Y, R)$ is

$$X^-; Y^+; R^-; S^+; X^-; Y^+; R^-; S^+; \cdots$$

We restrict the allowed $P$-strategies to $\mathcal{C}$-morphisms $X \times R \to Y \times S$ and the allowed $O$-strategies to $\mathcal{C}$-morphisms $Y \times S \to X \times R$. Given a $P$-strategy $\lambda = \langle v, u \rangle$ and an $O$-strategy $\kappa = \langle h, k \rangle$, the resulting play is

$$x_0 = \perp_X; \quad y_0 = \perp_Y; \quad r_0 = \perp_R; \quad s_0 = u(x_0, r_0);$$

$$x_{n+1} = h(y_n, s_n); \quad y_{n+1} = v(x_{n+1}, r_n); \quad r_{n+1} = k(y_{n+1}, s_n); \quad s_{n+1} = u(x_{n+1}, r_{n+1})$$

This stabilises after finitely many stages to $(x, y, r, s)$ which is the least fixpoint of $(x, r) = \kappa(y, s), (y, s) = \lambda(x, r)$. Again we hide the approximating moves so that $(x, y, r, s)$ is the visible play.

### 7 From dialectica to geometry of interaction

The previous section suggests that every $P$-strategy in $\text{hom}_{\Sigma(\mathcal{C})}(\langle X, S \rangle, \langle Y, R \rangle)$ can also be viewed as a $P$-strategy in $\text{hom}_{\textbf{Int}(\mathcal{C})}(\langle X, S \rangle, \langle Y, R \rangle)$. We could also discover this fact simply by inspecting the definitions, without thinking in terms of dialogues.
Incidentally, [HH17] refers to the Int-construction as “bidirectional computation”, a technical term that usually refers to lenses and related constructions (e.g. [GIS16]).

In this section we use string diagrams in the underlying category $C$. This is the language of traced symmetric monoidal categories [Sel11 section 5.7] which are cartesian monoidal [Sel11 section 6.1]. Implicitly, string diagrams for cartesian monoidal categories use the fact that a monoidal product is cartesian iff every object can be compatibly equipped with a commutative comonoid structure making every morphism into a comonoid homomorphism [Fox76].

**Proposition 4.** Let $C$ be a traced cartesian category. Then there is a strict monoidal functor $-^* : \mathcal{L}(C) \to \text{Int}(C)$, which is identity on objects and takes the strategy $(v,u)$ to

![String Diagram](image)

**Proof.** The identity morphism $(X_S) \to (X_S)$ of $\mathcal{L}(C)$ is sent to

![String Diagram](image)

which is equal to the identity on $X \times S$ since the black structure is a comonoid. This is the identity morphism $(X_S) \to (X_S)$ of $\text{Int}(C)$.

Next, consider morphisms $\lambda : (X_S) \to (Y_R)$ and $\mu : (Y_R) \to (Z_Q)$ in $\mathcal{L}(C)$. If we compose them in $\text{Int}(C)$ we obtain the morphism $\mu^* \circ \lambda^*$ with string diagram

![String Diagram](image)
Using the fact that $v_\lambda$ is a comonoid homomorphism, followed by coassociativity and symmetry of the black structure, we can transform this to

On the other hand, if we compose in $\mathcal{L}(\mathcal{C})$, we obtain $(\mu \circ \lambda)^*$ with string diagram

By inspection, we see that these string diagrams are equivalent. Equality of the two morphisms then follows from the coherence theorem for traced symmetric monoidal categories [Sel11, theorem 5.22].

Finally, it can be seen by inspection that the functor is strict monoidal, since $\mathcal{L}(\mathcal{C})$ and $\text{Int}(\mathcal{C})$ have the same objects and the monoidal product is defined in the same way.

The author conjectures that when $\mathcal{C}$ is only a monoidal category there is still an identity-on-objects strict monoidal category $\mathcal{L}(\mathcal{C}) \to \text{Int}(\mathcal{C})$, where $\mathcal{L}(\mathcal{C})$ is now taken to be the more general category of optics [Rill18].

We also note that the functor $-^*$ takes the counit lens $\varepsilon_X : (\overset{X}{X}) \to I$ to the morphism $\varepsilon_X : (\overset{X}{X}) \to I$ that is the counit of the compact closed structure of $\text{Int}(\mathcal{C})$.

8 Abstracting open games

Inspecting the definition of open games, it appears that we can define open games over any symmetric monoidal category $\mathcal{C}$ with a chosen functor $\text{cohom} : \mathcal{C} \times \mathcal{C}^{\text{op}} \to \text{Set}$. This does indeed give us a category of open games, but defining a monoidal product of open
games requires an additional piece of structure, namely the ability to project individual
$P$-strategies out of a $O$-strategy for a composite. This is axiomatised by the following
definition.

**Definition 1.** A context for a symmetric monoidal category $C$ is a symmetric monoidal
functor $\text{cohom}_C : C \times C^{\text{op}} \to \text{Set}$ together with a natural family of functions

$$/ : \text{hom}_C(X_2, Y_2) \to (\text{cohom}_C(X_1 \otimes X_2, Y_1 \otimes Y_2) \to \text{cohom}_C(X_1, Y_1))$$

The naturality condition required is that for all morphisms $W_1 \xrightarrow{\lambda_1} X_1$, $Y_2 \xrightarrow{\nu_1} Z_1$ and
$W_2 \xrightarrow{\lambda_2} X_2$, $Y_2 \xrightarrow{\nu_2} Z_2$, the diagram

$$\begin{align*}
\text{cohom}_C(Z_1 \otimes Z_2, W_1 \otimes W_2) & \xrightarrow{(\nu_2 \circ \mu_2 \circ \lambda_2)/-} \text{cohom}_C(Z_1, W_1) \\
\text{cohom}_C(Y_1 \otimes Y_2, X_1 \otimes X_2) & \xrightarrow{\mu_2/-} \text{cohom}_C(Y_1, X_1)
\end{align*}$$
in $\text{Set}$ commutes.

Using the symmetry, we can derive from this a natural family of functions

$$\backslash : \text{hom}_C(X_1, Y_1) \to (\text{cohom}_C(X_1 \otimes X_2, Y_1 \otimes Y_2) \to \text{cohom}_C(X_2, Y_2))$$

and vice versa.

The structures we defined earlier do indeed give a context on $\mathfrak{L}(C)$, namely

$$\text{cohom}_{\mathfrak{L}(C)} \left( \begin{pmatrix} X \\ S \end{pmatrix}, \begin{pmatrix} Y \\ R \end{pmatrix} \right) = \text{hom}_C(1, X) \times \text{hom}_C(Y, R)$$

There are trivial examples of contexts that carry no game-theoretic information, which
we will ignore. For example, we can always take $\text{cohom}_C$ to be a constant functor. We
give a second family of nontrivial examples, which we will use later.

**Proposition 5.** Every traced monoidal category $C$ can be equipped with the context
$\text{cohom}_C(X, Y) = \text{hom}_C(Y, X)$, with $\lambda/\kappa$ defined by

```
Y_1
  ↓
Y_2
  ↓
X_2
  ↓
X_2
  ↑
Y_2
```

K
Proof. Let $\kappa : \text{hom}_C(Z \otimes Z', W \otimes W')$, $\lambda_1 : W_1 \to X_1$, $\nu_1 : Y_2 \to Z_1$ and $W_2 \xrightarrow{\lambda_2} X_2 \xrightarrow{\mu_2} Y_2 \xrightarrow{\nu_2} Z_2$. We chase the context $\kappa$ around the commuting diagram in definition 1. By the upper route we obtain

\[
\begin{array}{c}
Y \\
\nu_1 \\
\kappa \\
\mu_2 \\
\lambda_2 \\
X \\
\lambda_1 \\
\nu_2
\end{array}
\]

and by the lower route we obtain

\[
\begin{array}{c}
Y \\
\nu_1 \\
\kappa \\
\nu_2 \\
\lambda_1 \\
X \\
\lambda_2 \\
\mu_2
\end{array}
\]

By the coherence theorem for traced monoidal categories, these denote equal morphisms.

**Definition 2.** Let $\mathcal{C}$ be a symmetric monoidal category with a context $\text{cohom}$, and let $X, Y$ be objects of $\mathcal{C}$. An open game $\mathcal{G} : X \to Y$ over $\mathcal{C}$ consists of

- A set $\Sigma_\mathcal{G}$ of strategy profiles
- A labelling function $\mathcal{G} : \Sigma_\mathcal{G} \to \text{hom}_\mathcal{C}(X, Y)$
- A winning condition $\mathcal{G} \subseteq \Sigma_\mathcal{G} \times \text{cohom}_\mathcal{C}(X, Y)$

Given open games $\mathcal{G} : X \to Y$ and $\mathcal{H} : Y \to Z$ over $\mathcal{C}$, their sequential composition $\mathcal{H} \circ \mathcal{G} : X \to Z$ is defined by $\Sigma_{\mathcal{H} \circ \mathcal{G}} = \Sigma_\mathcal{G} \times \Sigma_\mathcal{H}$, $(\mathcal{H} \circ \mathcal{G})_{(\sigma, \tau)} = \mathcal{H}_\tau \circ \mathcal{G}_\sigma$ and

\[
|\mathcal{H} \circ \mathcal{G}^\kappa|_{(\sigma, \tau)} \iff |\mathcal{G}^\kappa|_{\text{cohom}_\mathcal{C}(X, \mathcal{H}_\tau)(\kappa)} \land |\mathcal{H}^\tau|_{\text{cohom}_\mathcal{C}(\mathcal{G}_\sigma, Z)(\kappa)}
\]
Given open games $\mathcal{G} : X_1 \to Y_1$ and $\mathcal{H} : X_2 \to Y_2$ over $\mathcal{C}$, their simultaneous composition $\mathcal{G} \otimes \mathcal{H}$ is defined by $\Sigma_{\mathcal{G} \otimes \mathcal{H}} = \Sigma_{\mathcal{G}} \times \Sigma_{\mathcal{H}}$, $(\mathcal{G} \otimes \mathcal{H})_{(\sigma, \tau)} = \mathcal{G}_\sigma \otimes \mathcal{H}_\tau$ and

$$|\mathcal{G} \otimes \mathcal{H}|_{(\sigma, \tau)} \iff |\mathcal{G}|_{\mathcal{G}_\sigma, /\kappa} \wedge |\mathcal{H}|_{\mathcal{H}_\tau, \kappa}$$

**Proposition 6.** For any symmetric monoidal category $\mathcal{C}$ with a context, there is a symmetric monoidal category $\text{OG}(\mathcal{C})$ of open games over $\mathcal{C}$. When $\mathcal{C} = \mathcal{C}(\text{Set})$ with the usual context, we obtain the original category of open games.

The proof of this proposition formally follows the proof that $\text{OG}$ is a symmetric monoidal category. (The clearest presentation is in [Hed18, section 5 & appendix].) The definition of a context contains precisely the conditions needed for this proof to work.

Given a morphism $\lambda : X \to Y$ of $\mathcal{C}$, we define an open game $\lambda : X \to Y$ over $\mathcal{C}$ by $\Sigma_\lambda = \{\ast\}$, $\lambda_\ast = \lambda$ and $|\lambda|_\ast$ holding for all $\kappa$.

**Proposition 7.** This defines a faithful identity-on-objects symmetric monoidal functor $\mathcal{C} \to \text{OG}(\mathcal{C})$.

### 9 Morphisms of contexts

In order to relate open games over different categories, we need a notion of functors that respect contexts. Even after doing this, $\text{OG}(\cdot)$ is not functorial, due to mixed variance in the definition of open games.

**Definition 3.** Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories with contexts $\text{cohom}_\mathcal{C}$ and $\text{cohom}_\mathcal{D}$. A strict morphism of contexts is a strict symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ together with a monoidally natural family of functions

$$\text{cohom}_F(X, Y) : \text{cohom}_\mathcal{C}(X, Y) \to \text{cohom}_\mathcal{D}(F(X), F(Y))$$

such that

$$\text{cohom}_{\mathcal{C}}(X_1 \otimes X_2, Y_1 \otimes Y_2) \xrightarrow{f/-} \text{cohom}_{\mathcal{C}}(X_1, Y_1)$$

$$\text{cohom}_{\mathcal{D}}(F(X_1 \otimes X_2), F(Y_1 \otimes Y_2)) \xrightarrow{F(f)/-} \text{cohom}_{\mathcal{D}}(F(X_1), F(Y_1))$$

commutes for all $f : X_2 \to Y_2$.

Defining non-strict morphisms of contexts takes a bit more care, but is not necessary for our purposes.

Given a traced monoidal category $\mathcal{C}$, the category $\text{Int}(\mathcal{C})$ is compact closed, and hence in particular traced monoidal. We consider it to have the context defined for traced monoidal categories. That is,

$$\text{cohom}_{\text{Int}(\mathcal{C})} \left( \begin{pmatrix} X \\ S \end{pmatrix}, \begin{pmatrix} Y \\ R \end{pmatrix} \right) = \text{hom}_{\text{Int}(\mathcal{C})} \left( \begin{pmatrix} Y \\ R \end{pmatrix}, \begin{pmatrix} X \\ S \end{pmatrix} \right) = \text{hom}_\mathcal{C}(Y \otimes S, X \otimes R)$$

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Proposition 8. Let \( C \) be a traced cartesian category. Then \( -^* : \mathcal{L}(C) \to \mathbf{Int}(C) \) can be made into a strict morphism of contexts, by defining

\[
\text{cohom}_*: \hom_C(1, X) \times \hom_C(Y, R) \to \hom_C(Y \times S, X \times R)
\]

to take \((h, k)\) to

\[
\begin{array}{c}
\text{Y} \\
\downarrow h \\
\text{X} \\
\downarrow k \\
\text{S} \\
\end{array}
\]

Proof. We already checked that \(-^*\) is strict symmetric monoidal. We get naturality for free by noting that \(\text{cohom}_*\) can be equivalently defined by

\[
\hom_C(I, X) \times \hom_C(Y, R) \xrightarrow{\cong} \hom_{\mathcal{L}(C)} \left( I, \left( \begin{array}{c} X \\ S \end{array} \right) \right) \times \hom_{\mathcal{L}(C)} \left( \left( \begin{array}{c} Y \\ R \end{array} \right), I \right)
\]

\[
\xrightarrow{\circ} \hom_{\mathcal{L}(C)} \left( \left( \begin{array}{c} Y \\ R \end{array} \right), \left( \begin{array}{c} X \\ S \end{array} \right) \right)
\]

\[
\xrightarrow{-^*} \hom_{\mathbf{Int}(C)} \left( \left( \begin{array}{c} Y \\ R \end{array} \right), \left( \begin{array}{c} X \\ S \end{array} \right) \right)
\]

Suppose we have a \(P\)-strategy \(\lambda : (\frac{X_2}{S_2}) \to (\frac{Y_2}{R_2}).\) We must verify that the square

\[
\begin{array}{c}
\hom_C(1, X_1 \times X_2) \times \hom_C(Y_1 \times Y_2, R_2 \times R_1) \\
\downarrow \\
\hom_C(Y_1 \times Y_2 \times S_2 \times S_1, X_1 \times X_2 \times R_2 \times R_1) \\
\end{array}
\]

\[
\xrightarrow{\lambda/^-} \hom_C(1, X_1) \times \hom_C(Y_1, R_1)
\]

\[
\xrightarrow{-^*/^-} \hom_C(Y_1 \times S_1, X_1 \times R_1)
\]

commutes. Chasing\(^1\) a context \((h, k)\) around the top yields

\(^1\)The author has named this proof technique ‘string diagram chasing’, i.e. chasing an element around a commuting diagram whose nodes are all formed from homsets in a monoidal category.
As a useful intermediate point, since \( h \) is a comonoid homomorphism this is equivalent to

On the other hand, chasing \((h, k)\) around the bottom yields
This diagram is equivalent to the previous one modulo traced cartesian categories. This completes the proof. □

10 Relating open games over different categories

The motivation for studying morphisms of contexts is that if we have such a morphism \( C \to D \) then we can relate open games in \( \text{OG}(C) \) and \( \text{OG}(D) \), despite \( \text{OG}(\quad) \) not being functorial.

**Definition 4.** Let \( F : C \to D \) be a morphism of categories with contexts. Let \( G : X \to Y \) be an open game in \( \text{OG}(C) \), and let \( H : F(X) \to F(Y) \) be an open game in \( \text{OG}(D) \). We say that \( H \) is conservative over \( G \) if the following conditions hold:

- \( \Sigma_H = \Sigma_G \)
- \( H_\sigma = F(G_\sigma) \) for all \( \sigma \in \Sigma_G \)
- \( |G'_\kappa| \iff |H'_\kappa|_{\text{cohom}_F(X,Y)(\kappa)} \) for all \( \sigma \in \Sigma \) and \( \kappa : \text{cohom}_C(X,Y) \)

**Proposition 9.** Let \( F : C \to D \) be a strict morphism of categories with contexts. Then \( F(f) \) is conservative over \( f \) for all morphisms \( f \) of \( C \).

**Proof.** Trivial. □

**Proposition 10.** Let \( F : C \to D \) be a strict morphism of categories with contexts. Let \( G' \) be conservative over \( G : X \to Y \) and \( H' \) be conservative over \( H : Y \to Z \). Then \( H' \circ G' \) is conservative over \( H \circ G \).

**Proof.** First note that \( \Sigma_{H' \circ G'} = \Sigma_{G'} \times \Sigma_{H'} = \Sigma_G \times \Sigma_H = \Sigma_{H \circ G} \) and that

\[
(H' \circ G')_{\sigma,\tau} = H'_\tau \circ G'_\sigma = F(H_\tau) \circ F(G_\sigma) = F(H_\tau \circ G_\sigma) = F((H \circ G)_{\sigma,\tau})
\]

Let \( \kappa : \text{cohom}_C(X,Z) \), then

\[
|H \circ G|^{\sigma,\tau}_\kappa \iff |G'|_{\text{cohom}_C(X,H_\sigma)(\kappa)} \land |H'|_{\text{cohom}_C(G_\sigma,Z)(\kappa)} \\
\iff |G'|_{\text{cohom}_F(X,Y)(\kappa)} \land |H'|_{\text{cohom}_F(Y,Z)(\kappa)} \\
\iff |G'|_{\text{cohom}_F(F(X),F(H_\sigma))(\kappa)} \land |H'|_{\text{cohom}_F(F(G_\sigma),F(Z))(\kappa)} \\
\iff |G'|_{\text{cohom}_F(F(X),H'_\sigma)(\kappa)} \land |H'|_{\text{cohom}_F(G'_\sigma,F(Z))(\kappa)} \\
\iff |H|_{\text{cohom}_D(X,Z)(\kappa)}
\]

**Proposition 11.** Let \( F : C \to D \) be a strict morphism of categories with contexts. Let \( G' \) be conservative over \( G \) and \( H' \) be conservative over \( H \). Then \( G' \otimes H' \) is conservative over \( G \otimes H \).
Proof. We first note that the statement type-checks because \( F \) is strict monoidal, that is, \( F(X \otimes Y) = F(X) \otimes F(Y) \). The sets of strategy profiles are equal as before, and the play functions similarly have

\[
(G' \otimes H')_{\sigma,\tau} = G'_\sigma \otimes H'_\tau = F(G'_\sigma) \otimes F(H'_\tau) = F(G_{\sigma} \otimes H_{\tau}) = F((G \otimes H)_{\sigma,\tau})
\]

Supposing that the games have types \( G : X_1 \to Y_1 \) and \( H : X_2 \to Y_2 \), for a context \( \kappa : \text{cohom}_C(X_1 \otimes X_2, Y_1 \otimes Y_2) \) we have

\[
|G \otimes H|_{\kappa,\tau} \iff |G|_{\text{cohom}_F(X_1,Y_1)\otimes \kappa} \wedge |H|_{\text{cohom}_F(X_2,Y_2)\otimes \kappa} \wedge |G'|_{\text{cohom}_F(X_1,Y_1)\otimes \kappa} \wedge |H'|_{\text{cohom}_F(X_2,Y_2)\otimes \kappa} \wedge |G'|_{\text{cohom}_F(X_1,Y_1)\otimes \kappa} \wedge |H'|_{\text{cohom}_F(X_2,Y_2)\otimes \kappa}
\]

The previous proposition would be delicate to state for a non-strict monoidal functor. Unfortunately, the obvious way to relate open games over \( L \) and \( L(DCPO) \) is by lifting the flat domain functor \( Set \to DCPO \) using functoriality of \( L(-) \). The flat domain functor is not even strong monoidal, although it is both lax and colax monoidal (c.f. [AM10]). This means we must leave a game-theoretic understanding of open games over \( L(DCPO) \) informal.

Usually one considers a finitely generated sub-monoidal category of \( OG = OG(L(Set)) \) generated by zero-player games \( \overline{X} \) and one-player games \( D_{X,Y} \). Equivalently, this is the image in \( OG \) of the interpretation of all teleological string diagrams [Hed17] labelled by these generators. The game-theoretic meaning of this sub-category is well understood. (Other generating morphisms or operators can be added on a case-by-case basis.) As a result of the results in this section, given a strict morphism of contexts \( L(Set) \to C \), it suffices to find open games in \( OG(C) \) that are conservative over the one-player open games \( D_{X,Y} \), in order to obtain a ‘conervative extension’ of open games, i.e. one that agrees with \( OG \) when both are defined.

### 11 A compact closed category of computable open games

We can finally put all the pieces together, by considering the category \( OG(Int(DCPO)) \). Concretely, for DCPOs \( X, S, Y, R \), such an open game \( G : (X_S) \to (Y_R) \) consists of:

1. A set \( \Sigma \) of strategy profiles
2. A family of continuous play functions \( \mathbf{P}_G(\sigma) : X \times R \to Y \)
3. A family of continuous coplay functions \( \mathbf{C}_G(\sigma) : X \times R \to S \)
4. An equilibrium set \( \mathbf{E}_G(h, k) \subseteq \Sigma \) for each continuous history \( h : Y \times S \to X \) and continuation \( k : Y \times S \to R \)

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This definition is written in the style of the original concrete definition of open games in [GHWZ18] for ease of comparison. Specifically, besides changing the base category from \( \text{Set} \) to \( \text{DCPO} \) this definition differs by making \( C_0 \) additionally a function of \( R \), \( h \) a function of \( Y \) and \( S \), and \( k \) a function of \( S \).

This category is compact closed, as a result of applying proposition 1 to the zero-player functor \( \text{Int} (\text{DCPO}) \rightarrow \text{OG} (\text{Int} (\text{DCPO})) \).

Recall that for the open game \( \mathcal{D} : \left( \begin{array}{c} X \\ 1 \end{array} \right) \rightarrow \left( \begin{array}{c} Y \\ R \end{array} \right) \) over \( \mathcal{L} (\text{Set}) \), a context is a pair \( h : X \rightarrow \mathbb{R} \) and \( k : Y \rightarrow \mathbb{R} \), and the equilibrium condition for a strategy \( \sigma : X \rightarrow Y \) is that \( \sigma(h) \in \text{arg max}(k) \). We define \( \mathcal{D} \) in exactly the same way over \( \mathcal{L} (\text{DCPO}) \), with \( \Sigma_{\mathcal{D}} = \text{hom}_{\text{DCPO}} (X, Y) \). (These are the strategies by which a player can choose from \( Y \) given an observation from \( X \), under the usual interpretation of elements of a domain as partial observations of some true value.) We assume that there is still a suitable operator \( \text{arg max} : (Y \rightarrow \mathbb{R}) \rightarrow \mathcal{P} (Y) \) in \( \text{DCPO} \), where \( \mathbb{R} \) is a suitable domain of reals and \( \mathcal{P} \) is a suitable powerdomain operator. There are many options for defining these, and we remain agnostic between them.

Over \( \text{Int} (\text{DCPO}) \), the context for a decision \( \mathcal{D}_{X,Y} \) has the form

\[
\kappa = (h, k) : \text{cohom}_{\text{Int} (\text{DCPO})} \left( \left( \begin{array}{c} X \\ 1 \end{array} \right), \left( \begin{array}{c} Y \\ R \end{array} \right) \right) \cong \text{hom}_{\text{DCPO}} (Y, X) \times \text{hom}_{\text{DCPO}} (Y, \mathbb{R})
\]

Notice that since the coutility type \( S = 1 \) is terminal, the only difference from a context over \( \mathcal{L} (C) \) is that the history \( h \) may depend on the move from \( Y \). (That is, the future action may affect the past observation.)

We define the decision \( \mathcal{D}' : \left( \begin{array}{c} X \\ 1 \end{array} \right) \rightarrow \left( \begin{array}{c} Y \\ R \end{array} \right) \) over \( \text{Int} (\text{DCPO}) \) again to have \( \Sigma_{\mathcal{D}'} = \text{hom}_{\text{DCPO}} (X, Y) \), and the play function given in the obvious way.

A natural definition for equilibrium is that the least fixpoint \( y = \sigma(h(y)) \) is in \( \text{arg max}(k) \), which we write

\[
|\mathcal{D}'|_{h,k}^{\sigma} \iff \mu y.\sigma(h(y)) \in \text{arg max}(k)
\]

**Proposition 12.** \( \mathcal{D}' \) is conservative over \( \mathcal{D} : \left( \begin{array}{c} X \\ 1 \end{array} \right) \rightarrow \left( \begin{array}{c} Y \\ R \end{array} \right) \).

*Proof.* Let \( \sigma : \text{hom}_{\text{DCPO}} (X, Y) \) be a strategy and \( (h, k) \) a context for \( \mathcal{D} \), where \( h : \text{hom}_{\text{DCPO}} (1, X) \) and \( k : \text{hom}_{\text{DCPO}} (Y, \mathbb{R}) \). This is sent by \( \text{cohom} \) to the context \( (h', k) \) for \( \mathcal{D}' \), where \( h' : Y \rightarrow X \) is given by \( h'(y) = h \). The least (in fact unique) fixpoint of \( y = \sigma(h'(y)) \) is \( y = \sigma(h) \), and so \( |\mathcal{D}'|_{h',k}^{\sigma} \) iff \( \sigma(h) \in \text{arg max}(k) \). This is precisely the definition of \( |\mathcal{D}|_{h,k}^{\sigma} \), as required. \( \square \)

As an exercise, we work out the transpose \( \mathcal{G}^* : \left( \begin{array}{c} Y \\ 1 \end{array} \right) \rightarrow \left( \begin{array}{c} X \\ R \end{array} \right) \) of a general open game \( \mathcal{G} : \left( \begin{array}{c} X \\ 1 \end{array} \right) \rightarrow \left( \begin{array}{c} Y \\ R \end{array} \right) \) over \( \text{Int} (C) \). The set of strategy profiles stays the same up to isomorphism, \( \Sigma_{\mathcal{G}^*} \cong \Sigma_{\mathcal{G}} \) (because the transpose is defined by composition with various open games whose set of strategy profiles is 1). The play function is modified by taking the transpose in \( \text{Int} (C) \), which in the end simply exchanges the play and coplay functions \( X \times R \rightarrow Y \), \( X \times R \rightarrow S \) (and swaps their inputs). A context \( (h, k) \), for \( h : S \times Y \rightarrow R \) and \( k : S \times Y \rightarrow \)
$X$ is again swapped to give a context $h' = k : Y \times S \to X$ and $k' = h : Y \times S \to R$ for $\mathcal{G}$, so equilibrium is defined by $|\mathcal{G}|_h^k \iff |\mathcal{G}|_{k,h}^\sigma$.

Given a morphism $f : \text{hom}_C(X, Y)$, the open games $f : (X_1) \to (Y_1)$ and $f^{-1} : (Y_1) \to (X_1)$ are transposes of each other. Thus we are conservatively extending the notion of duality that already exists in categories of open games.

As a genuinely new concrete example, we work out the transpose $D^* : (\emptyset Y) \to (\emptyset X)$ of a decision $D$. The set of strategy profiles is still $\Sigma_{D^*} = \text{hom}_{\text{DCPO}}(X, Y)$. Given a strategy profile $\sigma : X \to Y$, the play function $P_{D^*}(\sigma) : \mathbb{R} \times X \to 1$ is given by $P_{D^*}(\sigma)(u, x) = \sigma(x, u) = \star$, and the coplay function $C_{D^*}(\sigma) : \mathbb{R} \times X \to Y$ by $C_{D^*}(\sigma)(u, x) = P_D(\sigma)(x, u) = \sigma(x)$. Given a context $h : Y \to \mathbb{R}$ and $k : Y \to X$, the equilibrium condition $|D^*|_{h,k}^\sigma$ is defined by $|D|_{k,h}^\sigma$, i.e. $\mu_y.\sigma(k(y)) \in \arg\max(h)$. Thus we see that this is ultimately made possible by the dependence of $h$ on $Y$.

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