HÖLDER CONTINUITY FOR THE $p$-LAPLACE EQUATION
USING A DIFFERENTIAL INEQUALITY

FREDRIK ARBO HØEG

ABSTRACT. We study Hölder continuity for solutions of the $p$-Laplace equation. This is established through a method involving an ordinary differential inequality, in contrast to the classical proof of the De Giorgi-Nash-Moser Theorem which uses iteration of an inequality through concentric balls.

1. Introduction

In this paper, we study solutions of the $p$-Laplace equation,

$$\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u) = 0, \quad 1 < p < \infty.$$  \hspace{1cm} (1.1)

It is a nonlinear second order partial differential equation in divergence form. The equation is degenerate for $p > 2$ and singular for $p < 2$.

Due to the singularity and nonlinearity of the $p$-Laplace operator, we cannot always expect solutions of equation (1.1) to be smooth. We say that $u$ is a weak solution of equation (1.1) in a domain $\Omega \subset \mathbb{R}^n$ provided

$$\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = 0$$

for all smooth test functions $\phi$.

We will show Hölder continuity for weak solutions of the $p$-Laplace equation. This is not a new result, see for example [LU]. However, the method is new and was used by Paulo Tilli in [T] for linear equations in divergence form. We will apply the same strategy here.

Throughout the text, we will restrict the range of values for $p$ to

$$1 < p < n.$$  \hspace{1cm} (1.2)

For $p > n$, in fact all functions in $W^{1,p}_{\text{loc}}(\Omega)$ are continuous, see Theorem 7.17 in [GT]. For $p = n$, the proof is based on Morrey’s Lemma, which can be found in Theorem 7.19 in [GT].

2010 Mathematics Subject Classification. 35J15, 35J92.

Key words and phrases. $p$-Laplacian, Hölder continuity.
A crucial step in proving Hölder continuity is the following Oscillation Theorem. If the Oscillation Theorem is true, the Hölder continuity follows as in page 3 in [T].

**Theorem 1.1. The Oscillation Theorem**

Let $u$ be a bounded weak solution of
\[
\Delta_p u = 0 \quad \text{in } B_4.
\]
If
\[
|\{u \leq 0\} \cap B_1| \geq \frac{1}{2}|B_1|,
\]
then
\[
\sup_{B_1} u^+ \leq C|\{u > 0\} \cap B_2|^{\alpha} \sup_{B_4} u^+
\]
for some $\alpha \in (0, 1)$.

The paper is organized as follows. First, we list some preliminaries for the study. Then, we focus our attention on the Oscillation Theorem. To prove this, we derive an ordinary differential inequality which is used along with a modified Caccioppoli inequality. This is in contrast to the classical proofs in for example [Dg].

## 2. Preliminaries

**Caccioppoli type inequalities.** We give some inequalities needed later for solutions of the $p$-Laplace equation.

**Lemma 2.1. Caccioppoli inequality:** Let $u$ be a weak solution of (1.1) in a domain $\Omega$ and let $r > 0$ with $B_{2r} \subset \Omega$. Then
\[
\int_{B_r} |\nabla u|^p \, dx \leq p^{p-1} \int_{B_{2r}} |u|^p \, dx.
\]

**Proof.** Multiply equation (1.1) with $u\phi^p$, where $\phi$ is a test function, zero outside $B_{2r}$. Then

\[
0 = \int_{\Omega} u\phi^p \div (|\nabla u|^{p-2}\nabla u) \, dx \\
= -\int_{\Omega} \phi^p |\nabla u|^p \, dx - p\int_{\Omega} u\phi^{p-1}|\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx.
\]

By Hölder’s inequality we find
\[
\int_{\Omega} \phi^p |\nabla u|^p \, dx \leq p \int_{\Omega} |u\nabla \phi||\phi \nabla u|^{p-1} \, dx.
\]
HÖLDER CONTINUITY FOR THE $p$-LAPLACE EQUATION USING A DIFFERENTIAL INEQUALITY

\[
\leq p \left( \int_{\Omega} |u \nabla \phi|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\phi \nabla u|^p \, dx \right)^{\frac{1}{p} - 1}.
\]

Rearranging gives

\[
\int_{\Omega} \phi^p |\nabla u|^p \, dx \leq p^p \int_{\Omega} |\nabla \phi|^p u \, dx.
\]

To finish the proof we choose the test function $\phi$ to be radial with

\[
\begin{align*}
\phi &= 1 \quad \text{in } B_r \\
|\nabla \phi| &\leq \frac{1}{r} \quad \text{in } B_{2r} \setminus B_r \\
\phi &= 0 \quad \text{outside } B_{2r}.
\end{align*}
\]

Hence

\[
\int_{B_r} |\nabla u|^p \, dx \leq p^r r^{-p} \int_{B_{2r}} |u|^p \, dx.
\]

□

The Caccioppoli inequality above is often the starting point in the De Giorgi-Nash-Moser approach to achieve Hölder continuity. For our method, we need an inequality relating an interior integral to a boundary integral. In the next Lemma, $(u - k)^+ = \max \{(u - k), 0\}$.

**Lemma 2.2.** Assume $u$ is a weak solution of equation (1.1) in $B_R$. Then

\[
\int_{B_r} |\nabla (u - k)^+|^p \, dx \leq \int_{\partial B_r} |\nabla (u - k)^+|^{p-1} (u - k)^+ \, ds.
\]

for almost every $r < R$.

**Proof.** For a formal proof, multiply equation (1.1) by $(u - k)^+$, and use the divergence theorem. More carefully, we multiply equation (1.1) with $\eta(u - k)^+$ where $\eta \in H_0^1(B_r)$. This gives

\[
0 = \int_{B_r} \eta(u - k)^+ \text{div} \left( |\nabla u|^{p-2} \nabla u \right) \, dx
\]

\[
= - \int_{B_r} \eta |\nabla u|^{p-2} \langle \nabla (u - k)^+, \nabla u \rangle \, dx - \int_{B_r} (u - k)^+ |\nabla u|^{p-2} \langle \nabla u, \nabla \eta \rangle \, dx,
\]

\[
\int_{B_r} \eta |\nabla (u - k)^+|^p \, dx \leq \int_{B_r} (u - k)^+ |\nabla (u - k)^+|^{p-1} |\nabla \eta| \, dx.
\]
Taking
\[ \eta = \eta_\epsilon(x) = \min\{1, \frac{r-|x|}{\epsilon}\} \]
and sending \( \epsilon \) to zero proves the Lemma.

\[\square\]

A modified Poincaré inequality. The following can be found in Theorem 3.16 in [G].

**Theorem 2.3.** Let \( \Omega \subset \mathbb{R}^n \) be open and connected with a Lipschitz continuous boundary. For every \( u \in W^{1,p}(\Omega) \), \( p < n \), taking the value zero in a set \( A \) with positive measure, we have
\[
||u||_{p^*} \leq c \left( \frac{||\Omega||}{|A|} \right)^\frac{p}{p^*} ||\nabla u||_p,
\]
where \( p^* = \frac{np}{n-p} \).

3. A differential equation

Let
\[
g(\rho) = \int_{B_{2-\rho}} |\nabla (u - k\rho)|^{p-1} (u - k\rho)^+ \, dx, \quad \rho \in (0,1).
\]
Our plan is to find a differential equation for \( g \) and to eventually show that \( g(1) = 0 \). If we can show that \( g(1) = 0 \), the assumptions in Theorem 1.1 can be applied to get the desired bound.

Differentiating we find
\[
-g'(\rho) = \int_{\partial B_{2-\rho}} |\nabla (u - k\rho)|^{p-1} (u - k\rho)^+ \, dx + k \int_{B_{2-\rho}} |\nabla (u - k\rho)|^{p-1} \, dx
\]
\[
\equiv a(\rho) + kb(\rho).
\]
For the differentiation, we used the following rewriting of \( g \).
\[
g(\rho) = \int_{B_{2-\rho}} |\nabla (u - k\rho)|^{p-1} (u - k\rho)^+ \, dx = \int_{B_{2-\rho}} |\nabla u|^{p-1}(u - k\rho)^+ \, dx.
\]

We want to connect \( g' \) and \( g \) to get an ordinary differential equation. By Hölder’s inequality and the modified Caccioppoli inequality, Lemma 2.2, we find
\[
g^p \leq \left\{ \int_{B_{2-\rho}} |\nabla (u - k\rho)|^p \, dx \right\}^{p-1} \int_{B_{2-\rho}} ((u - k\rho)^+)^p \, dx
\]
H"OLDER CONTINUITY FOR THE \( p \)-LAPLACE EQUATION USING A DIFFERENTIAL INEQUALITY

\[
\left\{ \int_{\partial B_{2-\rho}} |\nabla (u - k\rho)^+|^{p-1} (u - k\rho)^+ \, ds \right\}^{p-1} \int_{B_{2-\rho}} \left( (u - k\rho)^+ \right)^p \, dx \\
= (a(\rho))^{p-1} \int_{B_{2-\rho}} \left( (u - k\rho)^+ \right)^p \, dx.
\]

Let

\[ M = \sup_{B_4} u^+ \]

and recall the Sobolev conjugate

\[ p^* = \frac{np}{n - p}. \]

Under the assumptions in the Oscillation Theorem, we apply Theorem 2.3 to get

\[
g^p \leq a^{p-1} \int_{B_{2-\rho}} \left( (u - k\rho)^+ \right)^p \, dx \\
\leq a^{p-1} M \frac{n + p - p^2}{n - (p-1)} \int_{B_{2-\rho}} \left( (u - k\rho)^+ \right)^{(p-1)^*} \, dx \\
\leq CM \frac{n + p - p^2}{n - (p-1)} a^{p-1} \left\{ \int_{B_{2-\rho}} \left| (\nabla (u - k\rho)^+) \right|^{p-1} \, dx \right\}^{(p-1)^*} \\
= CM \frac{n + p - p^2}{n - (p-1)} a^{p-1} b^{\frac{n}{n - (p-1)}}.
\]

To relate \( g \) to \( g' = -(a + kb) \), we let \( q > 1 \) and use Young’s inequality with an \( \epsilon \),

\[
a^{p-1} b^{\frac{n}{n - (p-1)}} \leq \frac{1}{q} \left( a a^{p-1} \right)^q + \frac{q - 1}{q} \left( b^{\frac{n}{n - (p-1)}} \right)^{\frac{q}{q-1}}.
\]

To match the exponents for \( a \) and \( b \), we take \( q = 1 + \frac{n}{(p-1)(n - (p-1))} \). This gives

\[
a^{p-1} b^{\frac{n}{n - (p-1)}} \leq \frac{1}{q} \epsilon^q \left( a^{(p-1)q} + (q - 1) \epsilon^{\frac{q^2}{q-1}} \right)^{(p-1)q}.
\]

We now choose \( \epsilon \) from the following equality to match the derivative of \( g \),

\[
(q - 1) \epsilon^{\frac{q^2}{q-1}} = k^{(p-1)q}.
\]

Then

\[
a^{p-1} b^{\frac{n}{n - (p-1)}} \leq \frac{1}{q} \epsilon^q \left( a^{(p-1)q} + (kb)^{(p-1)q} \right) \leq \frac{Ce^q}{q} \left( a + kb \right)^{(p-1)q} \\
= C_2 \epsilon^q \left( -g'(\rho) \right)^{(p-1)q}.
\]
The differential inequality can then be written
\[
\frac{g^p \epsilon^{-q}}{C_3 M^{\frac{n + p - 2}{n + (p-1)}}} \leq (-g')^{(p-1)q}.
\]

We can now insert the values for \( \epsilon \) and \( q \). For convenience, put
\[
\gamma = \gamma(n, p) = np - (p - 1)^2 > 0.
\]

Then
\[
\frac{g'}{g^{p-1}} \leq -\frac{k^n}{C_4 M^{\frac{n + p - 2}{\gamma}}}.
\]

4. Proof of the Oscillation Theorem

We are now ready to prove Theorem 1.1 using the differential inequality (3.1). We calculate
\[
\frac{d}{d\rho} \left( g(\rho) \right)^{\frac{p-1}{\gamma}} = \frac{p-1}{\gamma} g^{\frac{p-1}{\gamma} - 1} g' \leq -\frac{k^n}{C_5 M^{\frac{n + p - 2}{\gamma}}}.
\]

Now, we claim that \( g(1) = 0 \). The above differential inequality can be integrated from \( \rho = 0 \) to \( \rho = 1 \), provided \( g(\rho) \neq 0 \) for \( \rho \) in this range. If \( g(\rho) = 0 \) for some \( \rho < 1 \), the claim is true, since \( g' \leq 0 \). If not, we integrate from 0 to 1 to get
\[
\left( g(1) \right)^{\frac{p-1}{\gamma}} - \left( g(0) \right)^{\frac{p-1}{\gamma}} \leq -\frac{k^n}{C_6 M^{\frac{n + p - 2}{\gamma}}}.
\]

By definition, \( g(1) \geq 0 \), so we want to exclude the case \( g(1) > 0 \). If this is the case, we have
\[
0 < \left( g(1) \right)^{\frac{p-1}{\gamma}} \leq \left( g(0) \right)^{\frac{p-1}{\gamma}} - \frac{k^n}{C_6 M^{\frac{n + p - 2}{\gamma}}}.
\]

Now, we choose \( k \) such that the above is zero,
\[
k = C_0 M^{\frac{n + p - 2}{n}} g(0)^{\frac{p-1}{n}} = C_0 \left( \sup_{B_1} u^+ \right)^{\frac{n - p (p-1)}{n}} \left\{ \int_{B_1} |\nabla u^+|^p u^+ \, dx \right\}^{\frac{p-1}{n}}.
\]

This gives the contradiction \( 0 < g(1) \leq 0 \), so we must have \( g(1) = 0 \), i.e.
\[
\int_{B_1} |\nabla (u - k)^+|^p (u - k)^+ \, dx = 0.
\]
We can rewrite this to
\[
\int_{B_1} \left| \nabla \left( \left( (u - k)^+ \right)^{\frac{p}{p-1}} \right) \right|^{p-1} \, dx = 0.
\]

Then, by the assumptions of Theorem 1.1 we can use Theorem 2.3 to get
\[
\int_{B_1} \left( ((u - k)^+)^{\frac{n p}{n - (p - 1)}} \right) \, dx = \int_{B_1} \left( (u - k)^+ \right)^{\frac{p}{p-1}} \, dx 
\leq C \left\{ \int_{B_1} \left| \nabla \left( ((u - k)^+)^{\frac{p}{p-1}} \right) \right|^{p-1} \, dx \right\}^{\frac{(p-1)^*}{p-1}} = 0,
\]
which means that
\[ (u - k)^+ \equiv 0 \text{ in } B_1. \]

Hence, \( \sup_{B_1} u^+ \leq k \). This gives
\[
\sup_{B_1} u^+ \leq k = C_0 \left( \sup_{B_4} u^+ \right)^{\frac{n - p (p - 1)}{n}} \left\{ \int_{B_1} \left| \nabla u^+ \right|^{p-1} u^+ \, dx \right\}^{\frac{p-1}{n}}
\leq C_0 \left( \sup_{B_4} u^+ \right)^{\frac{n - p (p - 1)}{n}} \left( \sup_{B_4} u^+ \right)^{\frac{p-1}{n}} \left\{ \int_{B_1} \left| \nabla u^+ \right|^{p-1} \, dx \right\}^{\frac{p-1}{n}}
= C_0 \left( \sup_{B_4} u^+ \right)^{\frac{n - p (p - 1)^2}{n}} \left\{ \int_{B_1} \left| \nabla u^+ \right|^{p-1} \, dx \right\}^{\frac{p-1}{n}}.
\]

By Hölder’s inequality we find
\[
\int_{B_2} \left| \nabla u^+ \right|^{p-1} \, dx = \int_{B_2} \left| \nabla u^+ \right|^{p-1} \chi_{\{u > 0\} \cap B_2} \, dx = \left\{ \int_{B_2} \left| \nabla u^+ \right|^p \, dx \right\}^{\frac{p-1}{p}} \left\{ \int_{B_2} \chi_{\{u > 0\} \cap B_2} \, dx \right\}^{\frac{1}{p}}
\leq \left\{ \int_{B_2} \left| \nabla u^+ \right|^p \, dx \right\}^{\frac{p-1}{p}} \left\{ \int_{B_2} \chi_{\{u > 0\} \cap B_2} \, dx \right\}^{\frac{1}{p}}
= \left\{ \int_{B_2} \left| \nabla u^+ \right|^p \, dx \right\}^{\frac{p-1}{p}} \left| \{u > 0\} \cap B_2 \right|^{\frac{1}{p}},
\]
which gives
\[
\sup_{B_1} u^+ \leq C_0 \left| \{u > 0\} \cap B_2 \right|^{\frac{1}{np}} \left( \sup_{B_4} u^+ \right)^{\frac{n - p (p - 1)^2}{np}} \left\{ \int_{B_2} \left| \nabla u^+ \right|^p \, dx \right\}^{\frac{(p-1)^2}{np}}.
\]

We now use the ordinary Caccioppoli inequality, Lemma 2.1, with \( r = 2 \). Hence,
\[
\sup_{B_1} u^+ \leq C_1 \{ \{ u > 0 \} \cap B_2 \}^{\frac{p-1}{np}} \left( \sup_{B_4} u^+ \right)^{\frac{n-(p-1)^2}{n}} \left\{ \int_{B_4} (u^+)^p \, dx \right\}^{\frac{(p-1)^2}{np}} \\
\leq C_2 \{ \{ u > 0 \} \cap B_2 \}^{\frac{p-1}{np}} \sup_{B_4} u^+,
\]

which proves the Oscillation Theorem.

REFERENCES

[Dg] E. De Giorgi: Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli reali. Mem. Accad. Sci. Torino. Cl. Sci. Mat. Nat. 3: 25–43(1957).

[G] E. Giusti: Direct methods in the calculus of variations. World Scientific, 2003.

[GT] D. Gilbarg and N. Trudinger: Elliptic Partial Differential Equations of Second Order. 2nd Edition, Springer, Berlin, 1983.

[LU] O. Ladyzhenskaya and N. Uraltseva: Linear and Quasilinear Elliptic Equations. Academic Press, New York, 1968. Academic Press, New York 1968.

[T] P. Tilli: Remarks on the Hlder continuity of solutions to elliptic equations in divergence form. Calculus of Variations and Partial Differential Equations, 25: 395–401, 2006.

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

E-mail address: fredrik.hoeg@ntnu.no