We introduce the notion of topological entropy of a formal languages as the topological entropy of the minimal topological automaton accepting it. Using a characterization of this notion in terms of approximations of the Myhill-Nerode congruence relation, we are able to compute the topological entropies of certain example languages. Those examples suggest that the notion of a “simple” formal language coincides with the language having zero entropy.

1 Introduction

The Chomsky hierarchy classifies formal languages in levels of growing complexity. At its bottom it puts the class of regular languages, followed by context-free and context-sensitive languages. At the top of the hierarchy it lists the class of all decidable languages. As such, the Chomsky hierarchy gives a method to assign a measure of complexity to formal languages.

However, using the Chomsky hierarchy as a mean to asses the complexity of a language has certain drawbacks. The most severe drawback is that the classification of the Chomsky hierarchy depends on a particular choice of computation models, namely finite automata, non-deterministic pushdown-automata, linear bounded automata, and Turing machines, respectively. It can be argued that this choice results in some contra-intuitive classifications: of course, accepting a language to be “simple” as soon as it is accepted by a finite automaton is reasonable. The converse however is not: not every language that cannot be accepted by a finite automaton is necessarily “complicated”.

An example is the Dyck language $D$ with one sort of parentheses [2]. This is the language of all words of balanced parentheses like $(())$ and $(((())$ but not $(()))$ or $. This language is context-free but not regular, and thus a classification by the Chomsky hierarchy would make this language $D$ appear to be not so “simple”. On the other hand, there is a very simple machine model accepting $D$, namely a two-state automaton with only one counter. It is reasonable to say that this kind of automaton is intuitively simple. The Chomsky hierarchy does not capture this: it puts the Dyck language with one sort of parentheses in the same class as much more complicated languages like palindromes. And there even exist context-sensitive languages that can be accepted by finite automata with only one counter.
To assess the complexity of a language one could now proceed as follows: given a language \( L \), what is the simplest form of computation model that is required to accept \( L \)? It is clear that this approach heavily depends on the notion of “simplest computation model” and the fact that there is such one. Indeed, it requires a hierarchy of all conceivable computation models to make this approach work, an assumption that is hardly realizable.

Instead of considering all possible computation models, we propose another approach, namely to consider one computation model that works for every language. Then given a formal language \( L \) one could ask what the “simplest” instances of this particular computation model is that is required to accept the given language \( L \). This then can be used to assign to \( L \) a measure of complexity that does not depend on a particular a-priori choice of certain computational models.

More precisely, we shall show in this work that we can use the notion of topological automata [15] to assign to every formal languages a notion of entropy that naturally reflects the complexity of the formal languages. As such, we make use of the following facts: for every formal language there exists a topological automaton accepting it. Furthermore, for each topological automaton there exists a natural notion of a smallest automaton accepting the same language. Finally, as topological automata are a particular form of dynamical systems, we can naturally assign a measure of complexity to every topological automaton, namely its entropy. Therefore, we can define the complexity of a formal language \( L \) as the entropy of the minimal topological automaton accepting it. We call this notion the topological entropy of \( L \). Intuitively, the lower the topological entropy of \( L \) the simpler it is. Languages with vanishing entropy are thus the simplest of all formal languages.

An advantage of this approach is that it works for every formal language, and is thus independent of a particular choice of computation models. On the other hand, one could argue that this approach is purely theoretical, as it may not allow us to compute the entropy of formal languages easily. However, we shall show that it is indeed possible to compute the topological entropy for certain examples of languages. For this we use a characterization of the topological entropy in terms of approximations of its Myhill-Nerode congruence relation. Using this, it is not hard to show that all regular languages have entropy 0. Moreover, we shall show that the Dyck language with one sort of parenthesis has also entropy 0. Both of these can thus be called “simple”, and intuitively they are. On the other hand, we shall also show that languages like palindromes or Dyck languages with multiple sorts of parentheses do not have zero entropy.

The paper is structured as follows. We first introduce the notions of topological automata and entropy of semigroup actions and formally define the notion of topological entropy of formal languages. The main part of this paper is then devoted to proof a characterization of topological entropy that allows for a comparably easy way to compute it. This is done in Section 3. We compute the entropy of some example languages in Section 4. We also provide a characterization of the topological entropy in terms of the entropic dimension of suitable pseudo-ultrametric spaces. Finally, we shall summarize the results of this paper and sketch an outline of future work.
2 Topological Entropy of Formal Languages

A variety of notions has been developed to assess different aspects of complexity of formal languages. Most of these notions have been devised with an understanding of complexity in mind that comes with classical complexity theory, and thus these notions are formulated as decision problems. Examples for this are the word problem and the equivalence problem for formal languages, and the complexity of the formal languages is measured by the complexity class for which these problems are complete. Other notions quantify complexity by other means. Examples are the state complexity [18] of a regular language, which gives the complexity of the language as the number of states in its minimal automaton, or the syntactic complexity of a regular language, which instead considers the size of corresponding syntactic semigroup [10].

The core idea of the present article is to expand the methods of measuring a formal language’s complexity by a topological approach in terms of topological entropy, which proved tremendously useful to dynamical systems. Topological entropy was introduced by Adler et al. [1] for single homeomorphisms (or continuous transformations) on a compact Hausdorff space. The literature provides several essentially different extensions of this concept for continuous group and semigroup actions. Among others, there is an approach towards topological entropy for continuous actions of finitely generated (pseudo-)groups due to Ghys et al. [13] (see also [3, 5, 7, 16]), which has also been investigated for continuous semigroup actions in [4, 6, 14].

By a dynamical system we mean a continuous semigroup action on a compact Hausdorff topological space. Topological entropy measures the ability of an observer to distinguish between points of the dynamical system just by recognizing transitions at equal time intervals, i.e., with respect to a fixed generating system of transformations, starting from the initial state. Since the above notion of dynamical system may very well be regarded as the topological counterpart of a finite automaton, it seems natural to utilize the dynamical approach for applications to automata theory.

To link dynamical systems to formal languages we shall use the already mentioned notion of a topological automaton [15]. This notion has been introduced as a topological generalization of the usual notion of a finite automaton by allowing to have an infinite state space. Indeed, topological automata share certain properties with finite automata. For example, for each topological automaton there exists a minimal topological automaton accepting the same language. However, in contrast to finite automata, every formal language is accepted by some topological automaton, and not only regular languages. This allows us to uniformly treat all formal languages with one computation model.

Recall that a (deterministic) automaton over an alphabet $\Sigma$ is a tuple $A = (Q, \Sigma, \delta, q_0, F)$ consisting of a finite set $Q$ of states, a transition function $\delta : Q \times \Sigma \to Q$, a set $F \subseteq Q$ of final states, and an initial state $q_0 \in Q$. The transition function is usually extended to the set of all words over $\Sigma$ by virtue of

\[
\delta^*(q, \varepsilon) := q, \\
\delta^*(q, wa) := \delta(\delta(q, w), a)
\]

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for \( q \in Q, a \in \Sigma, \) and \( w \in \Sigma^* \). The language accepted by \( \mathcal{A} \) is then

\[
\mathcal{L}(\mathcal{A}) := \{ w \in \Sigma^* \mid \delta^*(q_0, w) \in F \}.
\]

It is not hard to see that the function \( \delta^* \) is a monoid action of \( \Sigma^* \) on \( \mathcal{A} \). Indeed it is the unique monoid action of \( \Sigma^* \) on \( \mathcal{A} \).

The notion of deterministic finite automata can now be extended to an infinite state set as follows. Throughout this article a continuous action of a semigroup or monoid \( S \) on a topological space \( X \) is an action \( \alpha \) of \( S \) on \( X \) such that \( \alpha_s: X \to X, \alpha_s(x) = \alpha(x, s) \) is continuous for every \( s \in S \). Note that the latter just means that \( \alpha: X \times S \to X \) is continuous where \( S \) is endowed the discrete topology.

**Definition 2.1** A topological automaton over an alphabet \( \Sigma \) is a tuple \( \mathcal{A} = (X, \Sigma, \alpha, x_0, F) \) consisting of

- a compact Hausdorff space \( X \), called the set of states of \( \mathcal{A} \)
- a continuous action \( \alpha \) of \( \Sigma^* \) on \( X \), called the transition function of \( \mathcal{A} \),
- a point \( x_0 \in X \), called the initial state of \( \mathcal{A} \), and
- a clopen subset \( F \subseteq X \), called the set of final states of \( \mathcal{A} \).

We say that \( \mathcal{A} \) is trim if \( \alpha(x_0, \Sigma^*) \) is dense in \( X \). The language recognized by \( \mathcal{A} \) is defined as

\[
\mathcal{L}(\mathcal{A}) := \{ w \in \Sigma^* \mid \alpha(x_0, w) \in F \}.
\]

Let \( \mathcal{B} = (Y, \Sigma, \beta, y_0, G) \) be another topological automaton. We shall say that \( \mathcal{A} \) and \( \mathcal{B} \) are isomorphic, and write \( \mathcal{A} \cong \mathcal{B} \), if there exists a homeomorphism \( \varphi: X \to Y \) such that

\[
\varphi(\alpha(x, \sigma)) = \beta(\varphi(x), \sigma)
\]

for all \( x \in X, \sigma \in \Sigma, \varphi(x_0) = y_0, \) and \( \varphi(F) = G. \) \( \triangle \)

Evidently, isomorphic automata accept the same language.

Observe that every automaton accepting \( L \) can be turned into an automaton that is trim: if \( \mathcal{A} = (X, \Sigma, \alpha, x_0, F) \) is a topological automaton accepting \( L \), then replacing \( X \) with \( \overline{\alpha(x_0, \Sigma^*)} \) and \( F \) with \( F \cap \overline{\alpha(x_0, \Sigma^*)} \) always yields a trim automaton accepting the same language \( L \).

As already stated, and in contrast to regular languages, every formal language \( L \subseteq \Sigma^* \) is accepted by a topological automaton, cf. [15, Proposition 2.1].

**Proposition 2.2** Let \( L \subseteq \Sigma^* \) and \( \chi_L \) the characteristic function of \( L \). Equip \( X := \{ 0,1 \} \Sigma^* \) with the product topology, and define the mapping \( \delta: X \times \Sigma^* \to X \) by

\[
\delta(f, u)(v) := f(uv).
\]

Then \( L \) is accepted by the topological automaton \( (X, \Sigma, \delta, \chi_L, T) \) for \( T := \{ f \in X \mid f(\epsilon) = 1 \} \).

With the notation of 2.2, we define the minimal automaton of \( L \) to be

\[
\mathcal{A}_L = (\overline{\chi_L(\Sigma^*)}, \Sigma, \delta, \chi_L, T_L),
\]
where $X(L_\Sigma)$ is the closure of $X(L_\Sigma)$ in $\{0,1\}^\Sigma$, and $T_L = T \cap X(L_\Sigma)$. Clearly, $A_L$ is trim. Indeed we have the following fact that justifies to call $A_L$ minimal, cf. [15, Theorem 2.2].

**Proposition 2.3** Let $L \subseteq \Sigma^*$, and let $A = (X, \Sigma, x_0, \delta, F)$ be a topological automaton accepting $L$. Then $A \cong A_L$ if and only if for every trim automaton $B = (Y, \Sigma, y_0, \lambda, G)$ accepting $L$ there exists a uniquely determined surjective continuous function $\varphi: Y \to X$ satisfying $\varphi(\lambda(y, \sigma)) = \delta(\varphi(y), \sigma)$ and $\varphi(y_0) = x_0$. Moreover, in this case the unique $\varphi$ satisfies $G = \varphi^{-1}(F)$.

Since $A_L \cong A_L$, this proposition immediately yields that the minimal automaton is indeed minimal in the above sense. Moreover, in the case that $L$ is regular, $A_L$ is finite and is the usual minimal automaton of regular languages.

**Example 2.4** Let $\Sigma$ be a finite alphabet and let $a, b \in \Sigma$, $a \neq b$. We consider the Alexandroff compactification $Z_\infty$ of the discrete space of integers $\mathbb{Z}$, that is the set $Z_\infty = \mathbb{Z} \cup \{\infty\}$ equipped with the topology

$$\{M \subseteq \mathbb{Z} \cup \{\infty\} \mid \infty \in M \implies \mathbb{Z} \setminus M \text{ finite}\}.$$ 

We define an action $\alpha$ of $\Sigma^*$ on $Z_\infty$ by setting $\alpha(m, a) = m + 1$, $\alpha(m, b) = m - 1$ and $\alpha(m, c) = m$ for all $m \in \mathbb{Z}_\infty$ and $c \in \Sigma \setminus \{a, b\}$. Then $\alpha$ constitutes a continuous action of $\Sigma^*$ on $Z_\infty$, and for each $n \in \mathbb{N}$ the topological automaton $A = (Z_\infty, \Sigma, \alpha, 0, \{n\})$ accepts the language $L = \{w \in \Sigma^* \mid |w|_a = |w|_b + n\}$. 

We now shall express the complexity of the language $L$ accepted by a topological automaton $A = (Q, \Sigma, \sigma, x_0, F)$ by the topological entropy of the continuous action $\alpha$ of $\Sigma^*$ on $Q$ [1, 8, 17]. To this end, we shall first fix some useful notation and recall some important definitions about continuous actions on compact Hausdorff spaces.

Let $X$ again be a compact Hausdorff space. We shall denote by $C(X)$ the set of all finite open covers of $X$. If $f: X \to X$ is continuous and $U \in C(X)$, then $f^{-1}(U) := \{f^{-1}(U) \mid U \in U\}$ is a finite open cover of $X$ as well. Given $U, V \in C(X)$, we say that $V$ refines $U$ and write $U \preceq V$ if

$$\forall V \in V \exists U \in U: V \subseteq U,$$

and we say that $U$ and $V$ are refinement-equivalent and write $U \equiv V$ if $U \preceq V$ and $V \preceq U$. Furthermore, if $(U_i \mid i \in I)$ is a finite family of finite open covers of $X$, then

$$\forall_{i \in I} U_i := \{ \bigcap_{i \in I} U_i \mid (U_i)_{i \in I} \in \prod_{i \in I} U_i \}.$$ 

is a finite open cover of $X$ as well. For $U \in C(X)$ let

$$N(U) := \inf\{ |V| \mid V \subseteq U, X = \bigcup V \}.$$ 

In preparation for some later considerations, let us recall the following basic observations.

**Remark 2.5** (I1) Let $X$ be a compact Hausdorff space, $U, V \in C(X)$, $I$ be a finite set, $(U_i)_{i \in I}, (V_i)_{i \in I} \in C(X)^I$, and $f: X \to X$ be a continuous map. Then the following statements hold:
1) $\mathcal{U} \subseteq \mathcal{V} \implies N(\mathcal{U}) \leq N(\mathcal{V})$, 

2) $\mathcal{U} \subseteq \mathcal{V} \implies f^{-1}(\mathcal{U}) \leq f^{-1}(\mathcal{V})$, 

3) $(\forall i \in I: \mathcal{U}_i \subseteq \mathcal{V}_i) \implies \bigvee_{i \in I} \mathcal{U}_i \subseteq \bigvee_{i \in I} \mathcal{V}_i$. 

\[ \sqrt{ } \]

Now we come to dynamical systems, i.e., continuous semigroup actions. Let $S$ be a semigroup and consider a continuous action $\alpha$ of $S$ on $X$. For $\mathcal{U} \in C(X)$ we write 

$$s^{-1}(\mathcal{U}) := \alpha_s^{-1}(\mathcal{U}).$$

For every finite $F \subseteq S$ and $\mathcal{U} \in C(X)$ let 

$$(F: \mathcal{U})_\alpha := N(\bigvee_{s \in F} s^{-1}(\mathcal{U})).$$

Assume $F$ to be a finite generating subset of $S$. If $\mathcal{U}$ is a finite open cover of $X$, then we define 

$$\eta(\alpha, F, \mathcal{U}) := \limsup_{n \to \infty} \frac{\log_2(F^n : U)_\alpha}{n}.$$ 

Furthermore, the topological entropy of $\alpha$ with respect to $F$ is defined to be the quantity 

$$\eta(\alpha, F) := \sup \{ \eta(\alpha, F, \mathcal{U}) \mid \mathcal{U} \in C(X) \}.$$ 

Of course, the precise value of this quantity depends on the choice of a finite generating system. However, we observe the following fact.

**Proposition 2.6** Let $S$ be a semigroup and let $\alpha$ be a continuous action of $S$ on some compact Hausdorff space $X$. Suppose $E, F \subseteq S$ to be finite subsets generating $S$. Then 

$$\frac{1}{m} \cdot \eta(\alpha, F) \leq \eta(\alpha, E) \leq n \cdot \eta(\alpha, F),$$

where $m := \inf \{ k \in \mathbb{N} \mid F \subseteq E^k \}$ and $n := \inf \{ k \in \mathbb{N} \mid E \subseteq F^k \}$.

**Proof** Let $\mathcal{U} \in C(X)$. Evidently, $(E^k : \mathcal{U}) \leq (F^{kn} : \mathcal{U})$ for all $k \in \mathbb{N}$, whence 

$$\eta(\alpha, E, \mathcal{U}) = \limsup_{k \to \infty} \frac{\log_2(E^k : \mathcal{U})_\alpha}{k} \leq \limsup_{k \to \infty} \frac{\log_2(F^{kn} : \mathcal{U})_\alpha}{k}$$

$$= n \limsup_{k \to \infty} \frac{\log_2(F^{kn} : \mathcal{U})_\alpha}{kn} \leq n \limsup_{k \to \infty} \frac{\log_2(F^k : \mathcal{U})_\alpha}{k} = n \cdot \eta(\alpha, F, \mathcal{U}).$$

Thus, $\eta(\alpha, E, \mathcal{U}) \leq n \cdot \eta(\alpha, F, \mathcal{U})$. This shows that $\eta(\alpha, E) \leq n\eta(\alpha, F)$. Due to symmetry, it follows that $\eta(\alpha, F) \leq m \cdot \eta(\alpha, E)$ as well. \qed

With all the necessary notions in place we are finally able to define our notion of entropy of formal languages.

**Definition 2.7** Let $L \subseteq \Sigma^*$ and let $A_L = (X, \Sigma, x_0, \delta, F)$ be the minimal automaton of $L$. Then the entropy of $L$ is the entropy $\eta(\delta, \Sigma \cup \{ \epsilon \})$ of $\delta$ with respect to $\Sigma \cup \{ \epsilon \}$. \triangle
3 A Characterization

We claim that the definition of topological entropy is natural. Yet computing using the definition alone may not work very well. It is the purpose of this section to remedy this issue by providing an alternative characterization of the topological entropy of formal languages. For this we exploit another way of considering formal languages as dynamical systems.

To view a formal language \( L \) over an alphabet \( \Sigma \) as some kind of dynamical system we take inspiration from the characterization of regular languages as languages whose Myhill-Nerode congruence relation \( \Theta(L) \) has finite index. Recall that for \( u, v \in \Sigma \) we have

\[
(u, v) \in \Theta(L) \iff \forall w \in \Sigma^* : (uw \in L \iff vw \in L).
\]

The relation \( \Theta(L) \) can be seen as some way of measuring the complexity of \( L \): if \( L \) is regular, the number of equivalence classes is finite and equals the number of states in the minimal automaton of \( L \). Indeed, this is the idea behind the notion of state complexity.

However, if \( L \) is not regular this measure is not available anymore. We shall remedy this by not considering the number of equivalence classes of \( \Theta(L) \), but by considering the growth of the number of equivalence classes of a particular approximation of \( \Theta(L) \). Based on this growth we introduce our characterization of topological entropy of \( L \).

Let us first recall some basic notation. Let \( \Theta \) be an equivalence relation on a set \( Y \). For \( y \in Y \) we put \([y]_\Theta := \{ x \in Y \mid (x, y) \in \Theta \}\). Then \( Y/\Theta := \{ [y]_\Theta \mid y \in Y \}\). Furthermore, the index of \( \Theta \) on \( Y \) is defined as \( \text{ind}(\Theta) := |Y/\Theta| \). For a mapping \( f : X \to Y \) we set \( f^{-1}(\Theta) := \{ (s, t) \in X \times X \mid (f(x), f(y)) \in \Theta \}\). Clearly, \( f^{-1}(\Theta) \) then constitutes an equivalence relation on \( X \).

Now let \( \Sigma \) be an alphabet. The Nerode congruence of a language \( L \subseteq \Sigma^* \) is the equivalence relation

\[
\Theta(L) := \{(u, v) \in \Sigma^* \times \Sigma^* \mid \forall w \in \Sigma^* : uw \in L \iff vw \in L\}.
\]

Recall that \( L \) is regular if and only if it is accepted by an automaton. The following characterization of regular languages in terms of the Nerode congruence relation is well-known.

**Theorem 3.1 (Myhill-Nerode)** Let \( \Sigma \) be a finite alphabet. A language \( L \subseteq \Sigma^* \) is regular if and only if \( \Theta(L) \) has finite index.

Starting from this characterization we shall now make precise what we mean by approximating the relation \( \Theta \). For this we introduce another type of equivalence relation.

**Definition 3.2** Let \( \Sigma \) be an alphabet. For \( F \subseteq \Sigma^* \) finite and \( L \subseteq \Sigma^* \) define the function \( \chi_{F,L} : \Sigma^* \to \{0,1\}^F \) by

\[
\chi_{F,L}(u)(w) := \begin{cases} 
1 & \text{if } uw \in L \\
0 & \text{otherwise}
\end{cases}
\]

for \( u \in \Sigma^* \) and \( w \in F \). Now let

\[\Theta(F, L) := \ker(\chi_{F,L}).\]
Now, the equivalence relations $\Theta(F, L)$ constitute an approximation of $\Theta(L)$ in the sense that
\[
\Theta(L) = \bigcap \{ \Theta(F, L) \mid F \subseteq \Sigma^* \text{ finite} \}. \tag{1}
\]
Note that $\text{ind} \Theta(F, L) = [\chi_{F:L}] \leq 2^{|F|}$. In particular, $\Theta(F, L)$ has finite index, and thus the following definition is reasonable.

Therefore, as $(\Theta(\Sigma(n), L) \mid n \in \mathbb{N})$ may be regarded as an approximation of the Myhill-Nerode congruence $\Theta(L)$, it seems natural to consider the exponential growth rate of the corresponding index sequence as a measure of complexity for a given formal language $L$. This motivates the following definition.

**Definition 3.3** Let $\Sigma$ be an alphabet, and denote with $F(\Sigma^*)$ the set of finite subsets of $\Sigma^*$. Define
\[
\gamma : F(\Sigma^*) \times \mathcal{P}(\Sigma^*) \to \mathbb{N}, (F, L) \mapsto \text{ind} \Theta(F, L).
\]

Given $L \subseteq \Sigma^*$, we call
\[
\gamma_L : F(\Sigma^*) \to \mathbb{N}, F \mapsto \gamma(F, L)
\]
the Myhill-Nerode complexity function of $L$. The Myhill-Nerode entropy of a language $L \subseteq \Sigma^*$ is defined to be
\[
h(L) := \limsup_{n \to \infty} \frac{\log_2 \gamma_L(\Sigma(n))}{n},
\]
where $\Sigma(n)$ is the set of all words over $\Sigma$ of length at most $n$. \hfill $\triangle$

The Myhill-Nerode complexity function of $L$ has some immediate properties that we collect in the next proposition.

**Proposition 3.4** Let $\Sigma$ be a finite alphabet, let $E, F \subseteq \Sigma^*$ be finite, and let $L_0, L_1 \subseteq \Sigma^*$. Then
\begin{enumerate}
  \item $\gamma(F, \emptyset) = \gamma(F, \Sigma^*) = 1$, and thus $h(\emptyset) = h(\Sigma^*) = 0$.
  \item $\gamma(E \cup F, L) \leq \gamma(E, L) \cdot \gamma(F, L)$. If $E \subseteq F$, then $\gamma(E, L) \leq \gamma(F, L)$.
  \item $\gamma(F, L) = \gamma(F, \Sigma^* \setminus L)$, and hence $h(L) = h(\Sigma^* \setminus L)$.
  \item $\gamma(F, L_0 \cup L_1) \leq \gamma(F, L_0) \cdot \gamma(F, L_1)$, and thus $h(L_0 \cup L_1) \leq h(L_0) + h(L_1)$.
  \item $\gamma(F, L_0 \cap L_1) \leq \gamma(F, L_0) \cdot \gamma(F, L_1)$, and thus $h(L_0 \cap L_1) \leq h(L_0) + h(L_1)$.
\end{enumerate}

The goal of the rest of this section is to show that the Myhill-Nerode complexity of $L$ coincides with the topological entropy of $L$. We start this endeavor by showing that the Myhill-Nerode entropy of a formal language is bounded from above by the entropy of any topological automaton accepting it. In the case that the automaton is trim, these two notions even coincide.

**Theorem 3.5** Suppose $A = (X, \Sigma, a, x_0, F)$ to be a topological automaton. Consider $S := \Sigma \cup \{ \epsilon \}$ and $\mathcal{U} := \{ F, X \setminus F \}$. Then $h(L(A)) \leq \eta(a, S, \mathcal{U})$. If $A$ is trim, then $h(L(A)) = \eta(a, S, \mathcal{U})$.

We prove this theorem with the following three auxiliary statements.
Lemma 3.6 Let $\mathcal{A} = (X, \Sigma, \alpha, x_0, F)$ be a topological automaton. Let $\Phi : \Sigma^* \to X, w \mapsto \alpha(x_0, w)$ and $\mathcal{U} := \{F, X \setminus F\}$. Consider a finite subset $E \subseteq \Sigma^*$ as well as the equivalence relation

$$\Lambda_E := \{(x, y) \mid \forall w \in E : \alpha(x, w) \in F \iff \alpha(y, w) \in F\}.$$ 

Then the following statements hold:

1) $X/\Lambda_E = (\bigvee_{w \in E} w^{-1}(U)) \setminus \{\emptyset\}.$
2) $\Theta(E, L(\mathcal{A})) = (\Phi \times \Phi)^{-1}(\Lambda_E).$
3) If $\mathcal{A}$ is trim, then $\Phi(\Sigma^*) \cap V \neq \emptyset$ for every $V \in X/\Lambda_E$.

Proof (1): We observe that $V := (\bigvee_{w \in E} w^{-1}(U)) \setminus \{\emptyset\}$ constitutes a finite partition of $X$ into clopen subsets. For any $V \in V$ and $x \in V$, we observe that

$$[x]_{\Lambda_E} = \{y \in Y \mid \forall w \in E : \alpha(x, w) \in F \iff \alpha(y, w) \in F\}$$

$$= \{y \in Y \mid \forall w \in E : x \in w^{-1}(F) \iff y \in w^{-1}(F)\}$$

$$= \{y \in Y \mid \forall w \in E \forall U \in \mathcal{U} : x \in w^{-1}(U) \iff y \in w^{-1}(U)\}$$

$$= \{y \in Y \mid \forall W \in \mathcal{V} : x \in W \iff y \in W\}$$

$$= \{y \in Y \mid y \in V\}$$

$$= V$$

We conclude that $X/\Lambda_E = V$.

(2): Let $L := L(\mathcal{A})$. For any two words $u, v \in \Sigma^*$, it follows that

$$(u, v) \in \Theta(E, L) \iff \forall w \in E : uw \in L \iff vw \in L$$

$$\iff \forall w \in E : \alpha(x_0, uw) \in F \iff \alpha(x_0, vw) \in F$$

$$\iff \forall w \in E : \alpha(\alpha(x_0, u), w) \in F \iff \alpha(\alpha(x_0, v), w) \in F$$

$$\iff \forall w \in E : \alpha(\Phi(u), w) \in F \iff \alpha(\Phi(v), w) \in F$$

$$\iff (\Phi(u), \Phi(v)) \in \Lambda_E.$$

That is, $\Theta(E, L) = (\Phi \times \Phi)^{-1}(\Lambda_E)$.

(3): By (1), the set $X/\Lambda_E$ is a collection of open, non-empty subsets of $X$. If $\mathcal{A}$ is trim, then $\Phi(\Sigma^*)$ is dense in $X$, and thus $\Phi(\Sigma^*) \cap V \neq \emptyset$ for every $V \in X/\Lambda_E$. □

Proposition 3.7 Let $\mathcal{A} = (X, \Sigma, \alpha, x_0, F)$ be a topological automaton and let $\mathcal{U} := \{F, X \setminus F\}$. Consider a finite subset $E \subseteq \Sigma^*$. Then $\gamma_{L(\mathcal{A})}(E) \leq (E : \mathcal{U})_{\alpha}$. Furthermore, if $\mathcal{A}$ is trim, then $\gamma_{L(\mathcal{A})}(E) = (E : \mathcal{U})_{\alpha}$.

Proof Let $L := L(\mathcal{A})$ and $V := \bigvee_{w \in E} w^{-1}(U)$. Since $V \setminus \{\emptyset\}$ constitutes a finite partition of $X$ into clopen subsets, $V \setminus \{\emptyset\}$ does not admit any proper subcover. Consequently, $N(V) = |V \setminus \{\emptyset\}|$. Applying 3.6, we conclude

$$\gamma_L(E) \leq |\Sigma^*/\Theta(E, L)| \leq |X| = |X/\Lambda_E| = |V \setminus \{\emptyset\}| = N(V) = (E : \mathcal{U})_{\alpha}.$$ 

Finally, if $\mathcal{A}$ is trim, then Proposition 3.6 (3) asserts $|\Sigma^*/\Theta(E, L)| = |X/\Lambda_E|$ and therefore $\gamma_L(E) = (E : \mathcal{U})_{\alpha}$. □
The particular choice of the cover $\mathcal{U} = \{ F, X \setminus F \}$ seems arbitrary, but this is not the case. Indeed, if the automaton $A = (Q, \Sigma, \alpha, x_0, F)$ is minimal, then the entropy $\eta(\alpha, \Sigma \cup \{ \varepsilon \})$ of the automaton equals $\eta(\alpha, \Sigma \cup \{ \varepsilon \}, \mathcal{U})$. We shall show this fact in 3.10. As a preparation, we shall first investigate three auxiliary statements.

**Lemma 3.8** Let $X$ be a set, let $S$ be a semigroup, and let $\alpha : S \times X \to X$ be an action of $S$ on $X$. Let $\mathcal{U}$ be a finite cover of $X$ and let $M, N \subseteq S$ be finite. Then

$$\bigvee_{s \in MN} s^{-1}(\mathcal{U}) \equiv \bigvee_{s \in N} s^{-1}(\bigvee_{t \in M} t^{-1}(\mathcal{U})).$$

In particular, the complexities of those two covers coincide.

**Proof** Without loss of generality we may assume that $\mathcal{U}$ is closed under intersection: in fact, $\mathcal{U}$ is refinement-equivalent to the finite cover $\tilde{\mathcal{U}} := \{ \bigcap V \mid V \subseteq \mathcal{U} \}$. Hence, if the desired statement was true for $\tilde{\mathcal{U}}$, then this would imply

$$\bigvee_{s \in MN} s^{-1}(\mathcal{U}) \equiv \bigvee_{s \in MN} s^{-1}(\tilde{\mathcal{U}}) \equiv \bigvee_{s \in N} s^{-1}(\bigvee_{t \in M} t^{-1}(\tilde{\mathcal{U}})) \equiv \bigvee_{s \in N} s^{-1}(\bigvee_{t \in M} t^{-1}(\mathcal{U}))$$

due to the statements (2) and (3) of Remark 2.5.

Henceforth, assume that $\mathcal{U}$ is closed under intersection. We shall show an even stronger claim, namely

$$\bigvee_{s \in MN} s^{-1}(\mathcal{U}) = \bigvee_{s \in N} s^{-1}(\bigvee_{t \in M} t^{-1}(\mathcal{U})). \tag{2}$$

To ease readability, let us denote the left-hand side by $\mathcal{L}$, and the right-hand side by $\mathcal{R}$.

Let $Y \in \mathcal{L}$. Then

$$Y = \bigcap_{s \in MN} s^{-1}(U_s)$$

for some $(U_s \mid s \in MN) \in \prod_{s \in MN} s^{-1}(\mathcal{U})$. For each $s \in MN$ we can choose $\tau_s \in M, \sigma_s \in N$ such that $s = \tau_s \sigma_s$. Then

$$Y = \bigcap_{s \in MN} s^{-1}(U_s)$$

$$= \bigcap_{s \in MN} (\tau_s \sigma_s)^{-1}(U_s)$$

$$= \bigcap_{s \in MN} \sigma_s^{-1}(\tau_s^{-1}(U_{\tau_s \sigma_s}))$$

$$= \bigcap_{s \in MN} \bigcap_{\tau \in M} \sigma^{-1}(\tau^{-1}(U_{\tau \sigma}))$$

$$= \bigcap_{\sigma \in N} \bigcap_{\tau \in M} \sigma^{-1}(\tau^{-1}(U_{\sigma \tau})) \in \mathcal{R}$$

Conversely, let $Y \in \mathcal{R}$. Then

$$Y = \bigcap_{\sigma \in N} \sigma^{-1}(\bigcap_{\tau \in M} \tau^{-1}(U_{\sigma \tau}))$$
for some \( (U_{\sigma, \tau} \mid \sigma \in N, \tau \in M) \in \prod_{(\sigma, \tau) \in M \times N} U^{M \times N} \). Then
\[
Y = \bigcap_{\sigma \in N} \bigcap_{\tau \in M} \sigma^{-1} \left( \tau^{-1}(U_{\sigma, \tau}) \right)
\]
\[
= \bigcap_{\sigma \in N} \bigcap_{\tau \in M} (\tau \sigma)^{-1}(U_{\sigma, \tau})
\]
\[
= \bigcap_{s \in MN} s^{-1} \left( \bigcap \{ U_{\sigma, \tau} \mid \sigma \in N, \tau \in M, s = \tau \sigma \} \right)
\]
Define
\[U_s := \bigcap \{ U_{\sigma, \tau} \mid \sigma \in N, \tau \in M, s = \tau \sigma \}.\]
Then \( U_s \in \mathcal{U} \) for each \( s \in MN \), as \( \mathcal{U} \) is closed under intersections. But then
\[
Y = \bigcap_{s \in MN} s^{-1}(U_s) \in \mathcal{L}
\]
as required.
Finally, Equation 2 and Remark 2.5 (1) yield
\[
N \left( \bigvee_{s \in MN} s^{-1}(\mathcal{U}) \right) = N \left( \bigvee_{s \in N} \bigvee_{t \in M} s^{-1}(t^{-1}(\mathcal{U})) \right),
\]
as it has been claimed. \( \square \)

**Lemma 3.9** Let \( L \subseteq \Sigma^* \) and let \( A = (X, \Sigma, \alpha, x_0, F) \) be the minimal automaton of \( L \). Consider \( S := \Sigma \cup \{ \epsilon \} \) and \( \mathcal{U} := \{ F, X \setminus F \} \). If \( \mathcal{V} \) is a finite open cover of \( X \), then there exists some \( n \in \mathbb{N} \) such that \( \mathcal{V} \subseteq \bigvee_{s \in \Sigma^n} s^{-1}(\mathcal{U}) \).

**Proof** For \( n \in \mathbb{N} \), let us consider the equivalence relation
\[
\Lambda_n := \Lambda_{\Sigma(n)} = \{ (x, y) \in X \times X \mid \forall \omega \in \Sigma^{(n)} : \alpha(\omega, x) \in F \iff \alpha(\omega, y) \in F \}
\]
(cf. Lemma 3.6). We are going to show that
\[
\mathcal{W} := \{ [x]_{\Lambda_n} \mid n \in \mathbb{N}, x \in X, \exists V \in \mathcal{V} : [x]_{\Lambda_n} \subseteq V \}
\]
is an open cover of \( X \). By Lemma 3.6 (1), it follows that \( \mathcal{W} \) is a collection of open subsets of \( X \). Thus, we only need to argue that \( X = \bigcup \mathcal{W} \). To this end, let \( x \in X \). Since \( \mathcal{V} \) is a cover of \( X \), there exists some \( V \in \mathcal{V} \) with \( x \in V \). As \( V \) is open in \( X \) with respect to the subspace topology inherited from \( \{0, 1\} \Sigma^* \), we find a finite set \( E \subseteq \Sigma^* \) such that \( W := \{ y \in X \mid \forall \omega \in E : x(\omega) = y(\omega) \} \subseteq V \). Let \( n \in \mathbb{N} \) where \( E \subseteq S^n \). We observe that
\[
[x]_{\Lambda_n} = \{ y \in X \mid \forall \omega \in S^n : \alpha(\omega, x) \in F \iff \alpha(\omega, y) \in F \}
\]
\[
= \{ y \in X \mid \forall \omega \in S^n : \alpha(\omega, x)(\epsilon) = 1 \iff \alpha(\omega, y)(\epsilon) = 1 \}
\]
\[
= \{ y \in X \mid \forall \omega \in S^n : x(\omega) = 1 \iff y(\omega) = 1 \}
\]
\[
= \{ y \in X \mid \forall \omega \in S^n : x(\omega) = y(\omega) \}
\]
\[
\subseteq W \subseteq V.
\]
Accordingly, \([x]_{\Lambda_n} \in \mathcal{W}\) and hence \(x \in \bigcup \mathcal{W}\). This proves the claim. Now, since \(X\) is compact, there exists a finite subset \(\mathcal{W}_0\) where \(X = \bigcup \mathcal{W}_0\). Due to finiteness of \(\mathcal{W}_0\), there is some \(n \in \mathbb{N}\) such that \(\mathcal{W}_0 \subseteq X / \Lambda_n\). We conclude that

\[
\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{W}_0 \subseteq X / \Lambda_n
\]

which completes the proof.

We finally reached the point where we can show that the Myhill-Nerode complexity and the topological entropy of \(L\) coincide.

**Theorem 3.10** Let \(L \subseteq \Sigma^*\) and let \(\mathcal{A} = (X, \Sigma, \alpha, x_0, F)\) be the minimal automaton of \(L\). Consider \(S := \Sigma \cup \{\epsilon\}\) and \(\mathcal{U} := \{F, X \setminus F\}\). Then \(h(L) = \eta(\alpha, S, \mathcal{U}) = \eta(\alpha, S)\).

**Proof** Define \(\mathcal{U} := \{F, X \setminus F\}\). Since \(\mathcal{A}\) is trim, we know that \(h(L) = \eta(\alpha, S, \mathcal{U})\) by Theorem 3.5 and hence \(h(L) \leq \eta(\alpha, S)\). To show the converse inequality, let \(\mathcal{V}\) be a finite open cover of \(X\). We show that \(\eta(\alpha, S, \mathcal{V}) \leq \eta(\alpha, S, \mathcal{U})\). According to 3.9, there exists some finite \(m \in \mathbb{N}\) such that \(\mathcal{V}\) is refined by \(\bigvee_{s \in S^n} s^{-1}(\mathcal{U})\). Then

\[
N\left(\bigvee_{s \in S^n} s^{-1}(\mathcal{V})\right) \leq N\left(\bigvee_{s \in S^n} s^{-1}(\bigvee_{t \in S^m} t^{-1}(\mathcal{U}))\right) = N\left(\bigvee_{s \in S^{n+m}} s^{-1}(\mathcal{U})\right)
\]

by 3.8. Now we obtain

\[
\eta(\alpha, S, \mathcal{V}) = \limsup_{n \to \infty} \frac{\log_2 (S^n : \mathcal{V})^\alpha}{n} \overset{2.5(1)}{=} \limsup_{n \to \infty} \frac{\log_2 (S^{n+m} : \mathcal{U})^\alpha}{n} = \eta(\alpha, S, \mathcal{U}).
\]

Therefore, \(\eta(\alpha, S) \leq \eta(\alpha, S, \mathcal{U})\) and hence \(\eta(\alpha, S) = \eta(\alpha, S, \mathcal{U}) = h(L)\) by 3.5.

4 **Examples**

3.10 allows us to easily compute the topological entropy of certain classes of languages. To begin with, we show that all regular languages have zero entropy.

**Theorem 4.1** Let \(\Sigma\) be an alphabet and \(L \subseteq \Sigma^*\). The following are equivalent:

1) \(L\) is regular,  
2) \(\gamma_L\) is bounded, and  
3) there exists some finite subset \(F \subseteq \Sigma^*\) such that \(\Theta(F, L) = \Theta(L)\).

**Proof** 1) \(\implies\) 2). Due to 3.1, \(\Theta(L)\) has finite index. Note that \(\Theta(L) \subseteq \Theta(F, L)\) and hence \(\gamma_L(F) \leq \text{ind} \Theta(L)\) for all \(F \subseteq \Sigma^*\) finite. Thus, \(\gamma_L\) is bounded.

2) \(\implies\) 3). Suppose that \(\gamma_L\) is bounded. Then there exists some finite \(F_0 \subseteq \Sigma^*\) such that \(\gamma_L(F_0) = \sup \{ \gamma_L(F) \mid F \subseteq \Sigma^*\ \text{finite}\}\). We shall show that \(\Theta(F_0, L) = \Theta(L)\). Of course, \(\Theta(L) \subseteq \Theta(F_0, L)\). Let \((u, v) \in (\Sigma^* \times \Sigma^*) \setminus \Theta(L)\). By (1) there exists some finite \(F_1 \subseteq \Sigma^*\) such
that $(u,v) \notin \Theta(F_1,L)$. Obviously, $F_0 \cup F_1 \subseteq \Sigma^*$ is finite and $\Theta(F_0 \cup F_1, L) \subseteq \Theta(F_0, L)$. By assumption, $\gamma_L(F_0 \cup F_1) \leq \gamma_L(F_0)$. Consequently, $\Theta(F_0 \cup F_1, L) = \Theta(F_0, L)$ and therefore $(u,v) \in (\Sigma^* \times \Sigma^*) \setminus \Theta(F_1, L) \subseteq (\Sigma^* \times \Sigma^*) \setminus \Theta(F_0 \cup F_1, L) = (\Sigma^* \times \Sigma^*) \setminus \Theta(F_0, L)$. This substantiates that $\Theta(F_0, L) = \Theta(L)$.

3) $\implies$ 1). By assumption $\Theta(L) = \Theta(F, L)$, and since $\Theta(F, L)$ has finite index, $\Theta(L)$ has finite index as well. Hence, $L$ is regular due to 3.1.

**Corollary 4.2** Let $\Sigma$ be an alphabet. If $L \subseteq \Sigma^*$ is regular, then $h(L) = 0$.

The converse of this corollary does not hold, i.e., there are non-regular languages with vanishing topological entropy. To see this we shall show that *Dyck languages* always have zero entropy (cf. 4.8). We shall put the corresponding argumentation in a more general framework, by estimating the entropy of languages defined by groups. For this purpose, we recall the concept of *growth* in groups. Consider a finitely generated group $G$. Let $S$ be a finite symmetric generating subset of $G$ containing the neutral element. The *exponential growth rate* of $G$ with respect to $S$ is defined to be

$$\text{egr}(G, S) := \limsup_{n \to \infty} \frac{\log_2 |S^n|}{n}.$$  

Note that this quantity is finite as $|S^n| \leq |S|^n$ for every $n \in \mathbb{N}$. Furthermore,

$$\text{egr}(G, S) = \lim_{n \to \infty} \frac{\log_2 |S^n|}{n}$$

due to a well-known result by Fekete [12]. Of course, the precise value of the exponential growth rate depends upon the particular choice of a generating set.

However, if $T$ is another finite symmetric generating subset of $G$ containing the neutral element, then

$$\frac{1}{k} \cdot \text{egr}(G, T) \leq \text{egr}(G, S) \leq l \cdot \text{egr}(G, T)$$

where $k := \inf\{m \in \mathbb{N} \setminus \{0\} \mid T \subseteq S^m\}$ and $l := \inf\{m \in \mathbb{N} \setminus \{0\} \mid S \subseteq T^m\}$. This justifies the following definition: $G$ is said to have *sub-exponential growth* if $\text{egr}(G, S) = 0$ for some (and thus any) symmetric generating set $S$ of $G$ containing the neutral element. The class of finitely generated groups with sub-exponential growth encompasses all finitely generated abelian groups. In fact, if $G$ is abelian, then

$$S^n \subseteq \left\{ \prod_{s \in S} s^{a(s)} \mid a: S \to \{0, \ldots, n\} \right\}$$

and thus $|S^n| \leq (n+1)^{|S|}$ for all $n \in \mathbb{N}$. Now let us return to formal languages.

**Theorem 4.3** Let $\Sigma$ be an alphabet. Let $G$ be a group, $\varphi: \Sigma^* \to G$ a homomorphism, $H \subseteq G$, and $E \subseteq G$ finite. Define

$$P_\varphi(H) := \{w \in \Sigma^* \mid \forall u \text{ prefix of } w: \varphi(u) \in H\},$$

$$L_\varphi(H, E) := P_\varphi(H) \cap \varphi^{-1}(E).$$
Then $\gamma(F, L\varphi(H, E)) \leq |E| \cdot |\varphi(F)| + 1$ for all finite $F \subseteq \Sigma^*$. In particular,

$$h(L\varphi(H, E)) \leq \lim_{n \to \infty} \frac{\log_2 (|\varphi(\Sigma^n)|)}{n} \leq \log_2 |\Sigma|.$$  

Furthermore, if $S$ is a finite symmetric generating subset of $G$ containing the neutral element and $k := \inf\{m \in \mathbb{N} \setminus \{0\} \mid \varphi(\Sigma) \subseteq S^m\}$, then

$$h(L\varphi(H, E)) \leq k \cdot \text{egr}(G, S).$$

**Proof** We abbreviate $P := P\varphi(H)$ and $L := L\varphi(H, E)$. Consider a finite subset $F \subseteq \Sigma^*$. Then $Q := E\varphi(F)^{-1}$ is a finite subset of $G$. Fix any object $\infty \not\in Q$ and define $Q_{\infty} := Q \cup \{\infty\}$. Let us consider the map $\psi: \Sigma^* \to Q_{\infty}$ given by

$$\psi(u) := \begin{cases} \varphi(u) & \text{if } u \in P \cap \varphi^{-1}(Q), \\ \infty & \text{otherwise} \end{cases} \quad (u \in \Sigma^*).$$

We show $\ker \psi \subseteq \Theta(F, L)$. To this end, let $(u, v) \in \ker \psi$. We proceed by case analysis.

First case: $\psi(u) = \psi(v) \neq \infty$. Now, $u, v \in P \cap \varphi^{-1}(Q)$ and $\varphi(u) = \varphi(v) = \varphi(w)$. Let $w \in F$ and suppose that $uw \in L$. We show $vw \in L$. We observe that

$$\varphi(vw) = \varphi(v)\varphi(w) = \varphi(u)\varphi(w) = \varphi(uw) \in E,$$

i.e., $vw \in \varphi^{-1}(E)$. In order to prove that $vw \in P$, let $x$ be a prefix of $vw$. If $x$ is a prefix of $v$, then $\varphi(x) \in H$ as $v \in P$. Otherwise, there exists a prefix $y$ of $w$ such that $x = vy$, and so we conclude that $\varphi(x) = \varphi(vy) = \varphi(v)\varphi(y) = \varphi(u)\varphi(y) = \varphi(uy) \in H$, because $uw \in P$ and $uy$ is a prefix of $uw$. Hence, $vw \in L$. On account of symmetry, it follows that $(u, v) \in \Theta(F, L)$.

Second case: $\psi(u) = \psi(v) = \infty$. Let $x \in \{u, v\}$. If $x \not\in \varphi^{-1}(Q)$, then we conclude that $\varphi(xw) = \varphi(x)\varphi(w) \not\in E$ and thus $xw \not\in L$ for any $w \in F$. If $x \not\in P$, then $xw \not\in P$ and hence $xw \not\in L$ for any $w \in F$. This proves that $\{uw, vw\} \cap L = \emptyset$ for all $w \in F$. Consequently, $(u, v) \in \Theta(F, L)$.

This substantiates that $\ker \psi \subseteq \Theta(F, L)$. Therefore

$$\gamma(F, L) = \text{ind} \Theta(F, L) \leq \text{ind}(\ker \psi) \leq |Q_{\infty}| \leq |Q| + 1 \leq |E| \cdot |\varphi(F)| + 1.$$  

In particular, it follows that

$$h(L) = \lim_{n \to \infty} \frac{\log_2 (|\varphi(\Sigma^n)|)}{n} \leq \lim_{n \to \infty} \frac{\log_2 (|E| \cdot |\varphi(\Sigma^n)| + 1)}{n}$$

$$= \lim_{n \to \infty} \frac{\log_2 (|\varphi(\Sigma^n)|)}{n} \leq \lim_{n \to \infty} \frac{\log_2 (|\Sigma^n|)}{n} = \log_2 |\Sigma|.$$  

Finally, suppose $S$ to be a finite symmetric generating subset of $G$ containing the neutral element. Since $\Sigma$ is finite, $M := \{m \in \mathbb{N} \setminus \{0\} \mid \varphi(\Sigma) \subseteq S^m\}$ is not empty. Let $k := \inf M$. Our considerations above now readily imply that

$$h(L\varphi(S, E)) \leq \lim_{n \to \infty} \frac{\log_2 (|\varphi(\Sigma^n)|)}{n} \leq k \cdot \lim_{n \to \infty} \frac{\log_2 |S^n|}{n} = k \cdot \text{egr}(G, S). \quad \square$$
For groups whose growth is sub-exponential the previous theorem yields that the corresponding languages $L_\varphi(S, E)$ have zero entropy.

**Corollary 4.4** Let $\Sigma$ be an alphabet, let $G$ be a group with sub-exponential growth, and $\varphi: \Sigma^* \to G$ a homomorphism. Then for each $S \subseteq G$ and finite $E \subseteq G$, it is true that $h(L_\varphi(S, E)) = 0$.

We immediately obtain the following statement.

**Corollary 4.5** Let $\Sigma$ be an alphabet, let $G$ be a finitely generated abelian group, and $\varphi: \Sigma^* \to G$ a homomorphism. Then for each $S \subseteq G$ and finite $E \subseteq G$, it is true that $h(L_\varphi(S, E)) = 0$.

The following corollaries are immediate consequences of Theorem 4.3 for $S = G$.

**Corollary 4.6** Let $\Sigma$ be a finite alphabet and $L \subseteq \Sigma^*$. Let $G$ be a group, $\varphi: \Sigma^* \to G$ a homomorphism and $E \subseteq G$ finite such that $L = \varphi^{-1}(E)$. Then $\gamma(F, L) \leq |E| \cdot |\varphi(F)| + 1$ for all finite $F \subseteq \Sigma^*$. In particular,

$$h(L) \leq \limsup_{n \to \infty} \log_2 \frac{\log |\varphi(\Sigma^{(n)})|}{n} \leq \log_2 |\Sigma|.$$

**Corollary 4.7** Let $\Sigma$ be a finite alphabet, $L \subseteq \Sigma^*$. Let $G$ be an abelian group, $\varphi: \Sigma^* \to G$ a homomorphism and $E \subseteq G$ finite such that $L = \varphi^{-1}(E)$. Then $h(L) = 0$.

With the previous results in place, we are now able to argue that Dyck languages have finite entropy. Recall that the Dyck language with $k$ sorts of parentheses consists of all balanced strings over $\{ (1), \ldots, (k) \}$. Alternatively, we can view the Dyck language with $k$ sorts of parentheses as the set of all strings that can be reduced to the empty word by successively eliminating matching pairs of parentheses.

We can formalize this as follows. Let $\Sigma, \overline{\Sigma}$ be two alphabets, $\Delta := \Sigma \cup \overline{\Sigma}$, and let $\kappa: \Sigma \to \overline{\Sigma}$ be a bijection. Consider the free group $F(\Sigma)$ with generator set $\Sigma$, and denote with $\varphi: \Delta^* \to F(\Sigma)$ the unique homomorphism satisfying $\varphi(a) = a$ and $\varphi(\kappa(a)) = a^{-1}$ for all $a \in \Sigma$. Define

$$D(\kappa) := \{ w \in \Delta^* \mid \varphi(w) = e \land (\forall u \text{ prefix of } w : |u|_a \geq |u|_{\kappa(a)}) \}.$$

If $\Sigma = \{ (1, \ldots, (k) \}, \overline{\Sigma} = \{ )1, \ldots, )k \}$, and $\kappa((i) = )i$, then the set $D(\kappa)$ coincides with the Dyck language with $k$ sorts of parentheses.

**Theorem 4.8** Let $\kappa: \Sigma \to \overline{\Sigma}$ be a bijection between finite sets. Then

$$\log_2 |\Sigma| \leq h(D(\kappa)) \leq \log_2 (2|\Sigma| - 1)$$

for $S := \Sigma \cup \Sigma^{-1} \cup \{ e \}$, where $e$ denotes the neutral element of $F(\Sigma)$.

**Proof** Let $L := D(\kappa)$. We first show the inequality $\ind \Theta(\Sigma^{(n)}, L) \geq |\Sigma^n|$, since this implies $\log_2 |\Sigma| \leq h(D(\kappa))$. For this let $u, v \in \Sigma^n, u \neq v$. Define $\kappa(u) := \kappa(u_{|u|}) \ldots \kappa(u_1)$, where
Then $h$ is a homomorphism satisfying
\[ \psi(b)(a) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases} \quad (a, b \in \Sigma). \]

We observe $D(\kappa) = L_\varphi(\psi^{-1}(\mathbb{N}_\Sigma), \{e\})$, where the mapping $\varphi$ is as above. Hence, we have $h(D(\kappa)) = \text{egr}(D(\kappa), S)$ by 4.5. As it is known that $\text{egr}(D(\kappa), S) = \log_2(2|\Sigma| - 1)$ we obtain the claim. \hfill \Box

Note that for $|\Sigma| = 1$ we have $h(D(\kappa)) = 0$. Thus $D(\kappa)$ is an example of a non-regular language with zero entropy. For $|\Sigma| > 1$ the exact value of $h(D(\kappa))$ is unknown to the authors.

The reason that Dyck languages with more than one type of parentheses have non-zero positive entropy is the following: the different types of parentheses occurring in a word $w \in D(\kappa)$ need to be mutually balanced, i.e., $\varphi(w) = e$. In other words, if we replace this requirement by the weaker condition that each opening parenthesis has to be closed eventually, then we obtain a class of languages with zero entropy.

**Theorem 4.9** Let $\kappa: \Sigma \rightarrow \Sigma$ be a bijection between finite sets, let $\Delta := \Sigma \cup \Sigma$, and consider the language
\[ D'(\kappa) := \{ w \in \Delta^* \mid \forall a \in \Sigma: (|w|_a = |w|_{\kappa(a)}) \wedge (\forall u \text{ prefix of } w : |u|_a \geq |u|_{\kappa(a)}) \}. \]

Then $h(D'(\kappa)) = 0$.

**Proof** Let us consider the homomorphism $\varphi: \Delta^* \rightarrow \mathbb{Z}_\Sigma$ given by
\[ \varphi(w)(a) = |w|_a - |w|_{\kappa(a)} \quad (w \in \Delta^*, a \in \Sigma). \]

We observe that $D(\kappa) = L_\varphi(\mathbb{N}_\Sigma, \{0\})$, wherefore $h(D(\kappa)) = 0$ by 4.3. \hfill \Box

Other non-regular languages with vanishing entropy are discussed in the following examples.

**Example 4.10** Let $\Sigma$ be an alphabet.

1) Let $m \in \mathbb{N}$ and $a, b \in \Sigma$, $a \neq b$. Then $L := \{ w \in \Sigma^* \mid |w|_a = |w|_b + m \}$ is not regular. However, $h(L) = 0$ by Corollary 4.7. To see this, note that the mapping $\varphi: \Sigma^* \rightarrow \mathbb{Z}$, $w \mapsto |w|_a - |w|_b$ constitutes a homomorphism where $L = \varphi^{-1}(\{m\})$.

2) Suppose $\Sigma = \{ a, b, c \}$. Then $L := \{ a^m b^m c^m \mid m \in \mathbb{N} \}$ is not context-free, but $h(L) = 0$. To see this we show that for every $n$ the relation $\Theta = \Theta(\Sigma^n, L)$ has the equivalence classes
\[
\begin{align*}
[a^k]_{\Theta}, & \quad k \leq n/2 \\
[a^\ell b^k]_{\Theta}, & \quad 1 \leq \ell \leq k, 2k - \ell \leq n \\
[a^\ell b^k c^k]_{\Theta}, & \quad 1 \leq \ell \leq k, k - \ell \leq n \\
[b]_{\Theta}.
\end{align*}
\]
From this it follows \( \text{ind } \Theta(\Sigma(n), L) \in \mathcal{O}(n^2) \), and thus \( h(L) = 0 \).

To see that the sets in (3) are indeed all equivalence classes of \( \Theta(\Sigma(n), L) \), let \( u \in \Sigma^* \) such that \( u \) is not an element of the first three types of classes in (3). We need to show that then \( u \in [b]_{\Theta(\Sigma(n), L)} \). We do this by showing that there is no \( w \in \Sigma(n) \) such that \( uw \in L \).

Assume by contradiction that such a word \( w \) exists. Then \( w \) must be of one of the following forms

\[
\begin{align*}
    w &= a^\ell b^k c^k, \quad 2k + \ell \leq n, \quad 0 \leq \ell \leq k, \\
    w &= b^\ell c^k, \quad k + \ell \leq n, \quad 0 \leq \ell < k, \\
    w &= c^\ell, \quad 0 \leq \ell < n.
\end{align*}
\]

If \( w = a^\ell b^k c^k, 2k + \ell \leq n, 0 \leq \ell \leq k \), then \( u = a^{k-\ell}, k-\ell \leq n/2 \), and therefore \( u \in [a^{k-\ell}]_{\Theta(\Sigma(n), L)} \), a contradiction. If \( w = b^\ell c^k, k + \ell \leq n, 0 \leq \ell < k \), then \( u = a^k b^{k-\ell} \), and \( k - \ell > 0, 2k - (k - \ell) \leq n \), thus \( u \in [a^k b^{k-\ell}]_{\Theta(\Sigma(n), L)} \), again a contradiction.

If \( w = c^\ell \), then \( u = a^k b^{k-\ell} c^\ell \), and \( k - (k-\ell) \leq n \), so \( u \in [a^k b^\ell c^\ell]_{\Theta(\Sigma(n), L)} \), another contradiction.

Thus, our assumption that \( w \) exists is false. The same is true for the word \( b \), and thus \( u \in [b]_{\Theta(\Sigma(n), L)} \) as required. \( \diamond \)

Next we are looking for an example of a language with non-zero entropy. Of course, by what we have already shown, a suitable candidate for this has to be non-regular. But do not have to require much more: the following example shows that there are context-free languages with non-zero entropy.

**Example 4.11** Suppose \( |\Sigma| \geq 2 \). Then the *palindrome language*

\[
L := \{ wv^R \mid w \in \Sigma^* \}
\]

is not regular, but context-free, and \( h(L) \in (0, \infty) \).

To see \( h(L) > 0 \), observe that for each \( n \in \mathbb{N} \) and all \( u, v \in \Sigma^n \), if \( (u, v) \in \Theta(\Sigma(n), L) \), then \( u = v \). This is because if \( uv^R \in L \), we also have \( uv^R \in L \), and hence \( u = v \). Thus

\[
[u]_{\Theta(\Sigma(n), L)} \neq [v]_{\Theta(\Sigma(n), L)} \quad (u \neq v)
\]

Thus \( \text{ind } \Theta(\Sigma(n), L) \geq |\Sigma^n| = |\Sigma|^n \), and we obtain

\[
h(L) = \limsup_{n \to \infty} \frac{\log_2 |\Sigma|^n}{n} = \log_2 |\Sigma| > 0.
\]

To see \( h(L) < \infty \) we shall consider the relation \( \Theta^* \) defined by

\[
(u, v) \in \Theta^* \iff (u, v) \in \Theta(\Sigma(n), L) \text{ and } (|u| \leq n \iff |v| \leq n).
\]

Then \( \text{ind } \Theta(\Sigma(n), L) \leq \text{ind } \Theta^* \). We shall show

\[
\limsup_{n \to \infty} \frac{\log_2 (\text{ind } \Theta^*)}{n} < \infty.
\]
There are at most \(|\Sigma(n)|\) many equivalence classes \(|u|\Theta^*\) for \(u \in \Sigma^*, |u| < n\). To count the number of equivalence classes for \(|u| \geq n\) we define
\[
\ell_n(u) := \{a_1 \ldots a_i | 1 \leq i \leq n, a_1, \ldots, a_i \in \Sigma, u = a_1 \ldots a_i u', u' \in L\}.
\]
Then for \(u, v \in \Sigma^* \setminus \Sigma^{(n)}\) we have
\[
(u, v) \in \Theta^* \iff (u, v) \in \Theta(\Sigma^{(n)}, L) \iff \ell_n(u) = \ell_n(v).
\]
(4)
The first equivalence is clear. To see the second equivalence let \((u, v) \in \Theta(\Sigma^{(n)}, L)\), and let \(a_1 \ldots a_i \in \ell_n(u)\). By definition of \(\ell_n(u)\) it is then true that \(u(a_1 \ldots a_i)^R \in L\). Because \((u, v) \in \Theta(\Sigma^{(n)}, L)\) we therefore obtain \(v(a_1 \ldots a_i)^R \in L\), i.e., \(v\) is of the form \(v = a_1 \ldots a_i v'\) for some \(v' \in L\). This yields \(a_1 \ldots a_i \in \ell_n(v)\). By symmetry we obtain \(\ell_n(u) = \ell_n(v)\) as required.
Conversely, assume \(\ell_n(u) = \ell_n(v)\), and let \(w \in \Sigma^{(n)}\) be such that \(uw \in L\). Because \(|u| \geq n\), there exists \(u' \in L\) with \(uw = w^k u w\). Then \(w^k \in \ell_n(u) = \ell_n(v)\), and therefore \(v = w^k v'\) for some \(v' \in L\). But then \(v w \in L\). By symmetry \(v w \in L \implies uw \in L\) for each \(w \in \Sigma^{(n)}\), and therefore \((u, v) \in \Theta(\Sigma^{(n)}, L)\), as required.
Using the characterization from Equation (4) we have
\[
\left|\Sigma^* \setminus \Sigma^{(n)} / \Theta^*\right| = \left|\{\ell_n(u) | u \in \Sigma^* \setminus \Sigma^{(n)}\}\right|.
\]
Now every set \(\ell_n(u)\) with \(u = u_1 \ldots u_k, k \geq n\), can be represented by the prefix \(u_1 \ldots u_n\) of \(u\) of length \(n\) together with a tuple \(t \in \{0, 1\}^n\) defined by
\[
l_i = 1 \iff u_1 \ldots u_i \in \ell_n(u).
\]
Therefore,
\[
\left|\Sigma^* \setminus \Sigma^{(n)} / \Theta^*\right| = \left|\{\ell_n(u) | u \in \Sigma^* \setminus \Sigma^{(n)}\}\right| \leq |\Sigma|^n \cdot 2^n.
\]
This yields
\[
\text{ind} \Theta^* = \left|\Sigma^{(n)} / \Theta^*\right| + \left|\Sigma^* \setminus \Sigma^{(n)} / \Theta^*\right| \leq |\Sigma^{(n)}| + |\Sigma|^n \cdot 2^n,
\]
and thus
\[
\limsup_{n \to \infty} \frac{\log_2(\text{ind} \Theta^*)}{n} \leq \log_2(2|\Sigma|) < \infty.
\]
It is unclear to the authors whether the upper bound obtained in the proof of 4.11 is related to the one in 4.8.
It is not hard to see that the entropy of a formal language can very well be infinite. This is illustrated by the following example.

**Example 4.12** Let \(|\Sigma| \geq 2\), and choose mappings \(\varphi_n: \Sigma^{2^n} \to \mathcal{P}(\Sigma^n)\) for each \(n \in \mathbb{N}\) such that \(|\text{im}(\varphi_n)| = |\Sigma|^{2^n} = 2^{2^n}\). Then define a language \(L \subseteq \Sigma^*\) by
\[
L \cap \Sigma^m := \begin{cases} \{uv \mid u \in \Sigma^m, v \in \varphi_n(u)\} & \text{if } m = 2^n + n \text{ for some } n \in \mathbb{N}, \\ \emptyset & \text{otherwise.} \end{cases}
\]

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Then \(2^{2^n} \leq \gamma_L(n),\) i.e.,

\[
2^{2^n} \leq \text{ind } \Theta(\Sigma^n, L).
\]

To see this we shall show that each word \(\varphi_n(u)\) defines its own equivalence class, i.e., for words \(u_0, u_1 \in \Sigma^2\) with \(\varphi_n(u_0) \neq \varphi_n(u_1)\) we have \((u_0, u_1) \notin \Theta(\Sigma^n, L)\). This is because if \(\varphi_n(u_0) \neq \varphi_n(u_1)\) we can assume without loss of generality that there exists some word \(v \in \varphi_n(u_0) \setminus \varphi_n(u_1)\). By definition of \(L\) we then have \(u_0v \in L\), but since \(|u_1v| = 2^n + n\) and \(v \notin \varphi_n(u_1)\) we also get \(u_1v \notin L\). Thus \((u_0, u_1) \notin \Theta(\Sigma^n, L)\).

But then (5) implies

\[
\limsup_{n \to \infty} \frac{\log_2 \gamma_L(n)}{n} \geq \limsup_{n \to \infty} \frac{\log_2 2^{2^n}}{n} = \infty,
\]

and thus \(h(L) = \infty\). \(\diamondsuit\)

5 Topological entropy and entropic dimension

Another interesting characterization of the entropy of formal languages is in terms of the entropic dimension of a suitable precompact pseudo-ultrametric space. For this recall that a pseudo-metric space \((X, d)\) is called precompact if for each \(r \in (0, \infty)\) there exists some finite set \(F \subseteq X\) such that

\[X = \bigcup \{ B_d(x, r) \mid x \in F \} .\]

If \((X, d)\) is a precompact pseudo-metric space, then define

\[\gamma_{(X, d)}(r) := \inf \{ |F| \mid F \subseteq X \text{ finite}, X = \bigcup \{ B_d(x, r) \mid x \in F \} \} .\]

Then the entropic dimension \(\text{dim}(X, d)\) of the precompact pseudo-metric space \((X, d)\) is defined as [11]

\[\text{dim}(X, d) := \limsup_{r \to 0+} \frac{\log_2(\gamma_{(X, d)}(r))}{\log_2(1/r)} .\]

To now obtain a precompact pseudo-metric space \((X, d)\) whose entropic dimension is the same as the topological entropy of a given language \(L\), we shall first start with a general observation. Let \(X\) be a non-empty set and let \(\Theta = (\Theta_n \mid n \in \mathbb{N})\) be a descending sequence of equivalence relations on \(X\). Define \(d_\Theta : X \times X \to [0, \infty)\) as

\[d_\Theta(x, y) := 2^{-\inf \{ n \in \mathbb{N} \mid (x, y) \notin \Theta_n \}} \quad (x, y \in X) .\]

It is easy to see that \(d_\Theta(x, x) = 0\) and \(d_\Theta(x, y) = d_\Theta(y, x)\) is true for all \(x, y \in X\). Moreover, as

\[
\{ n \in \mathbb{N} \mid (x, z) \notin \Theta_n \} \subseteq \{ n \in \mathbb{N} \mid (x, y) \notin \Theta_n \lor (y, z) \notin \Theta_n \}
= \{ n \in \mathbb{N} \mid (x, y) \notin \Theta_n \} \cup \{ n \in \mathbb{N} \mid (y, z) \notin \Theta_n \},
\]

we also have \(d_\Theta(x, z) \leq \max \{ d_\Theta(x, y), d_\Theta(y, z) \}\) for all \(x, y, z \in X\). Because of this \((X, d_\Theta)\) is a pseudo-ultrametric space.
Proposition 5.1 Let $X$ be a non-empty set and let $\Theta = (\Theta_n \mid n \in \mathbb{N})$ be a descending sequence of equivalence relations on $X$ such that each $\Theta_n$ has finite index in $X$. Then $(X, d_\Theta)$ is precompact and
\[
\dim(X, d_\Theta) = \limsup_{n \to \infty} \frac{\log_2 |X/\Theta_n|}{n}.
\]

Proof We first observe that for all $x, y \in X$ and $n \in \mathbb{N}$
\[
d_\Theta(x, y) < 2^{-n} \iff n < \inf \{ m \in \mathbb{N} \mid (x, y) \notin \Theta_m \} \iff (x, y) \in \Theta_n.
\]
Therefore, $X/\Theta_n = \{ B_{d_\Theta}(x, 2^{-n}) \mid x \in X \}$. Since $X/\Theta_n$ is finite, $(X, d_\Theta)$ is precompact and
\[
\gamma(x, d_\Theta)(2^{-n}) = |X/\Theta_n|.
\]
Consequently,
\[
\dim(X, d_\Theta) = \limsup_{r \to 0+} \frac{\log_2 (\gamma(x, d_\Theta)(r))}{\log_2 (1/r)}
= \limsup_{n \to \infty} \frac{\log_2 (\gamma(x, d_\Theta)(2^{-n}))}{n}
= \limsup_{n \to \infty} \frac{\log_2 |X/\Theta_n|}{n}
\]
as required. \(\square\)

A straightforward application of this lemma is the following theorem.

Corollary 5.2 Let $\Sigma$ be an alphabet and let $L \subseteq \Sigma^*$. Then with $\Theta := (\Theta(\Sigma^{(n)}), L) \mid n \in \mathbb{N}$
\[
\dim(\Sigma^*, d_\Theta) = h(L).
\]

In the case that the language $L$ is represented by a topological automaton we obtain the following result.

Theorem 5.3 Let $\mathcal{A} = (X, \Sigma, \alpha, x_0, F)$ be a topological automaton. Let $\Lambda = (\Lambda_n \mid n \in \mathbb{N})$ where
\[
\Lambda_n := \Lambda_{\Sigma^{(n)}} = \{ (x, y) \in X \times X \mid \forall w \in \Sigma^{(n)} : \alpha(w, x) \in F \iff \alpha(w, y) \in F \}
\]
whenever $n \in \mathbb{N}$ (cf. 3.6). Then $h(L(\mathcal{A})) \leq \dim(X, d_\Lambda)$. Furthermore, if $\mathcal{A}$ is trim, then $h(L(\mathcal{A})) = \dim(X, d_\Lambda)$.

Proof Let $L := L(\mathcal{A})$ and $n \in \mathbb{N}$. We observe that $\gamma_L(\Sigma^{(n)}) = |\Sigma^* / \Theta(\Sigma^{(n)}), L| \leq X / \Lambda_n$ by 3.6 (2). Moreover, if $\mathcal{A}$ is trim, then $\gamma_L(\Sigma^{(n)}) = X / \Lambda_n$ due to 3.6 (3). Hence, 5.1 yields the desired statements. \(\square\)
The pseudo-metric considered in the theorem above does not necessarily generate the topology of the respective automaton. In fact, this happens to be true if and only if the automaton is minimal, i.e., isomorphic to the minimal automaton of the accepted language. Furthermore, this case can be characterized in terms of a separation property: a topological automaton is minimal if and only if the induced pseudo-metric is a metric.

**Proposition 5.4** Let $A = (X, \Sigma, \alpha, x_0, F)$ be a topological automaton and $L := L(A)$. Then the topology generated by $d_\Lambda$ is contained in the topology of $X$. Furthermore, the following statements are equivalent:

1) $A \cong A(L)$.
2) $d_\Lambda$ is a metric.
3) $d_\Lambda$ generates the topology of $X$.

**Proof** By 3.6 (1), the subset $B_{d_\Lambda}(x, \epsilon) = [x]_{\Lambda-\lfloor \log_2 \epsilon \rfloor}$ is open in $X$ for all $x \in X$ and $\epsilon \in (0, \infty)$. Hence, the topology generated by $d_\Lambda$ is contained in the original topology of $X$. Now let us prove the claimed equivalences:

(2)$\Rightarrow$(3): Suppose that $d_\Lambda$ is a metric. Then the topology generated by $d_\Lambda$ is a Hausdorff topology. Since this topology is contained in the compact topology of $X$, both topologies coincide due to a basic result from set-theoretic topology (see [9, §9.4, Corollary 3]).

(3)$\Rightarrow$(1): Assume that $d_\Lambda$ generates the topology of $X$. This clearly implies $d_\Lambda$ to be a metric. Consider the unique surjective continuous homomorphism $\phi: A \to A(L)$. We are going to show that $\phi$ is injective. To this end, let $x, y \in X$ such that $\phi(x) = \phi(y)$. We argue that $d_\Lambda(x, y) = 0$. Let $n \in \mathbb{N}$. For every $w \in \Sigma^n$, we observe that

$$a(x, w) \in F \iff \phi(a(x, w)) \in T_L \iff \delta(\phi(x), w) \in T_L \iff \delta(\phi(y), w) \in T_L \iff \phi(a(y, w)) \in T_L \iff a(y, w) \in F.$$ 

Thus, $(x, y) \in \Lambda_n$. It follows that $(x, y) \in \bigcap_{n \in \mathbb{N}} \Lambda_n$ and hence $d_\Lambda(x, y) = 0$. Since $d_\Lambda$ is a metric, we conclude that $x = y$. Accordingly, $\phi$ is a bijective continuous map between compact Hausdorff spaces and therefore a homeomorphism. This again is due to an elementary result from set-theoretic topology (see [9, §9.4, Corollary 2]).

(1)$\Rightarrow$(2): Suppose $\phi: A \to A(L)$ to be the necessarily unique isomorphism. Concerning any two points $x, y \in X$, we observe that

$$(x, y) \in \Lambda_n(A) \iff \forall w \in \Sigma^n: a(x, w) \in F \Leftrightarrow a(y, w) \in F \iff \forall w \in \Sigma^n: \phi(a(x, w)) \in T_L \Leftrightarrow \phi(a(y, w)) \in T_L \iff \forall w \in \Sigma^n: \delta(\phi(x), w) \in T_L \Leftrightarrow \delta(\phi(y), w) \in T_L \Leftrightarrow (\phi(x), \phi(y)) \in \Lambda_n(A(L))$$

for every $n \in \mathbb{N}$. Hence, $d_{A(A)}(x, y) = d_{A(A(L))}((\phi(x), \phi(y)))$ for all $x, y \in X$. Accordingly, it suffices to show that $d_{A(A(L))}$ is a metric. To this end, let $f, g \in \overline{\chi_l(\Sigma^*)}$ such that $d_{A(A(L))}(f, g) = 0$. We argue that $f = g$. Let $w \in \Sigma^*$. Then there exists $n \in \mathbb{N}$ where $w \in \Sigma^n$. Since $d_{A(A(L))}(f, g) = 0$, we conclude that $(f, g) \in \Lambda_n(A(L))$ and thus

$$f(w) = 1 \iff \delta(f, w)(\epsilon) = 1 \iff \delta(f, w) \in T_L \iff \delta(g, w)(\epsilon) = 1 \iff g(w) = 1.$$
Therefore, $f(w) = g(w)$. It follows that $f = g$. This shows that $d_{A(L)}$ is a metric and hence completes the proof. □

6 Conclusions

In this paper we have introduced the notion of topological entropy of formal languages as the topological entropy of the minimal topological automaton accepting it. We have shown that this notion is equal to the Myhill-Nerode complexity of the language, and can also be characterized in terms of the entropic dimension of suitable pseudo-ultrametric spaces. Using these characterizations, we were able to compute the topological entropy of certain types of languages.

The main motivation of this work was the goal to uniformly assess the complexity of formal languages independent of a particular collection of computation models. We believe that the examples we have provided in this work show that the notion of topological entropy of formal languages is a suitable candidate for such a complexity measure. In particular, we have shown that some languages intuitively considered to be simple all have zero entropy: regular languages, Dyck languages with one sort of parentheses, our “commutative” version of Dyck languages with arbitrary sorts of parentheses, and the language \{ $a^n b^n c^n$ | $n \in \mathbb{N}$ \}. Indeed, all of these languages are accepted by simple models of computation, e.g., one-way finite automata with a fixed number of counters.

On the other hand, we have presented examples of languages that have non-zero entropy that can hardly be considered as simple, namely Dyck languages with more than one sort of parentheses as well as the palindrome languages. Indeed, palindromes cannot be accepted by deterministic pushdown automata, and Dyck languages with more than one sort of parentheses give rise to the hardest context-free languages [2].

A natural next step in investigating the notion of topological entropy is to provide more examples that test the suitability of this notion as a measure of complexity of formal languages. For example, we have already shown that all languages accepted by finite automata have zero entropy. A natural question is now to ask for which classes of computation models the topological entropy of the accepted languages is also zero. We suspect that this is the case for one-way finite automata equipped with a fixed number of counters and an acceptance condition that does only require to check local conditions, including the current values of the counters.

Conversely, one could ask what properties languages with non-zero entropy possess. What form of non-locality in a suitable machine model is necessary to accept such languages, given that they are decidable? And what properties do languages have if their topological entropy is infinite? Are there context-free languages with infinite entropy?

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