Semi-uniform stability of operator semigroups and energy decay of damped waves

R. Chill, D. Seifert and Y. Tomilov

1 Institut für Analysis, Fakultät für Mathematik, TU Dresden, 01062 Dresden, Germany e-mail: ralph.chill@tu-dresden.de
2 School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne, NE1 7RU United Kingdom e-mail: david.seifert@ncl.ac.uk
3 Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00956 Warsaw, Poland e-mail: ytomilov@impan.pl

Only in the last fifteen years or so has the notion of semi-uniform stability, which lies between exponential stability and strong stability, become part of the asymptotic theory of \( C_0 \)-semigroups. It now lies at the very heart of modern semigroup theory. After briefly reviewing the notions of exponential and strong stability, we present an overview of some of the best known (and often optimal) abstract results on semi-uniform stability. We go on to indicate briefly how these results can be applied to obtain (sometimes optimal) rates of energy decay for certain damped second-order Cauchy problems.

1. Introduction

Exponential and strong stability of \( C_0 \)-semigroups are two classical topics in semigroup theory, and the literature on these topics, through various deep results over the past fifty years, has now reached a reasonably complete state; we refer to \([10,11,115]\) for extensive accounts. Exponential stability is a strong property, and it has a number of natural applications arising from its specific quantitative character and its robustness under perturbations. Meanwhile strong stability, that is to say mere convergence to zero of semigroup orbits, is a rather delicate property which (in the absence of exponential stability) is highly sensitive to perturbations and depends on fine properties of the semigroup generator.

© The Authors. Published by the Royal Society under the terms of the Creative Commons Attribution License http://creativecommons.org/licenses/by/4.0/, which permits unrestricted use, provided the original author and source are credited.
The present survey aims to address recent progress on how the gap between these two extreme kinds of semigroup stability can be bridged. After reviewing some basic aspects of exponential and strong stability and, where appropriate, offering some of our own commentary, we describe an abstract approach to the study of quantified stability for operator semigroups based on Tauberian theorems for Laplace transforms with remainder terms. This allows us to view strong stability as an end-point case of quantified stability theory, with exponential stability at the opposite end. Even though the pioneering work in this area goes back at least fifteen years, certain aspects of the theory may not be well known even among experts. Our main message is simple: many types of quantified asymptotic behaviour of operator semigroups can be characterised, or at least described very precisely, in terms of simple resolvent bounds for semigroup generators and of various related resolvent conditions. Strikingly, the passage from resolvent bounds to decay rates can often be achieved without essential loss, so that optimal decay rates are possible as long as one has sharp resolvent estimates. In fact, sharp resolvent estimates are (almost) equivalent to sharp decay rates. While resolvents have long been used in the study of decay rates, a unified approach leading to optimal rates of decay has emerged only relatively recently. The techniques outlined in the present survey are intended to serve as a partial remedy, although they too have their natural limitations and will not be the optimal tool in all situations.

We begin, in Section 2, by revisiting the classical subjects of exponential and strong stability, paying particular attention to their connections with other topics in the asymptotic theory of operator semigroups such as spectral mapping theorems. Then, in Section 3, we turn to the modern theory of semi-uniform stability of operator semigroups, approached through quantified Tauberian theory before finally, in Section 4, outlining how the abstract theory can be applied to the study of abstract second order problems and thus to energy decay of damped waves. There are many other interesting and important applications of the theoretical results, for instance to decay of correlations of chaotic flows on manifolds, which due to space limitations we are unable to address in our survey. Moreover, even the selection of theoretical topics covered here is necessarily incomplete and rather skewed towards optimal results on Hilbert spaces, which are central to the study of damped waves. Nevertheless, we hope that the survey will shed new light on both classical and modern aspects of the quantified asymptotic theory of $C_0$-semigroups.

2. Exponential and strong stability revisited

(a) Exponential stability and spectral mapping theorems

A $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$ is exponentially stable if for every $x \in X$ there exist $M(x) > 0$ and $\omega(x) > 0$ such that $\|T(t)x\| \leq M(x)e^{-\omega(x)t}$ for all $t \geq 0$. By a simple argument based on the uniform boundedness principle, this is equivalent to the existence of positive constants $M$ and $\omega$ not depending on $x$ such that $\|T(t)\| \leq Me^{-\omega t}$, $t \geq 0$. Thus if one defines the exponential growth bound of $(T(t))_{t \geq 0}$ as

$$\omega(T) := \inf \{ \omega \in \mathbb{R} : \exists M \geq 0 \text{ s.t. } \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0 \},$$

then exponential stability is equivalent to the property $\omega(T) < 0$. Of course, if there exists $t_0 > 0$ such that $\|T(t_0)\| < 1$, then $(T(t))_{t \geq 0}$ is exponentially stable (and vice versa). Thus, any uniform decay rate of the form $\|T(t)x\| = O(r(t))$ for some $r \in C_0(\mathbb{R}_+)$ and every $x \in X$ implies exponential stability of $(T(t))_{t \geq 0}$ in view of the uniform boundedness principle. Once $(T(t))_{t \geq 0}$ is known to be exponentially stable one is often interested in the optimal choices of $M$ and $\omega$ above, and we will address this matter in some detail below.

From the point of view of many applications exponential stability is the “best” type of stability one may hope for. It is robust under small bounded perturbations, indeed even under certain non-linear perturbations, and as a result it forms the basis for the theory of linearised stability. Equilibrium points with exponentially stable linearisations are easily detected, whereas equilibrium points whose linearisations are stable merely in a weaker sense are invisible in
practice; see however [81,82], which deal with an intermediate situation in the spirit of our survey. In many situations the primary task is to characterise exponentially stable or strongly stable semigroups in terms of appropriate a priori information, since the semigroup is rarely known explicitly, and sometimes even when it is the computations quickly become cumbersome. Natural a priori objects, both from the point view of abstract operator theory and also for applications to PDEs, are the spectrum $\sigma(A)$ of the generator $A$ of a $C_0$-semigroup $(T(t))_{t \geq 0}$ and the resolvent $R(z, A) := (z - A)^{-1}$ for $z$ in the resolvent set $\rho(A) := \mathbb{C} \setminus \sigma(A)$ of $A$. If $A$ is bounded, then by the elementary spectral mapping theorem for the Riesz-Dunford functional calculus one has $\sigma(T(t)) = e^{t\sigma(A)}$, $t \geq 0$, and the spectral radius formula implies that

$$\omega(T) = s(A),$$

(2.2)

where $s(A) := \sup \{ \Re z : z \in \sigma(A) \}$ is the spectral bound of $A$. In particular, $s(A) < 0$ is equivalent to $\omega(T) < 0$, and the spectral bound of $A$ alone determines whether or not the semigroup $(T(t))_{t \geq 0}$ is exponentially stable. Unfortunately, (2.2) may fail dramatically if $A$ is an unbounded operator. Recall the fundamental relation from semigroup theory,

$$R(z, A)x = \int_0^\infty e^{-zt}T(t)x \, dt, \quad x \in X, \; \Re z > \omega(T),$$

(2.3)

which says that on suitable half-planes the resolvent of the generator is the Laplace transform of the semigroup (in the strong sense). In particular, the right half-plane $\{ z \in \mathbb{C} : \Re z > \omega(T) \}$ is contained in the resolvent set $\rho(A)$, and moreover $s(A) \leq \omega(T)$. However, one of the main differences between the theory of Laplace transforms and the theory of power series is that differences between the theory of Laplace transforms and the theory of power series is that either $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$, $t \geq 0$, or at least $\sigma(T(t)) = e^{t\sigma(A)}$, $t \geq 0$.

In the first case, one says that the spectral mapping theorem (SMT) holds for $(T(t))_{t \geq 0}$, while in the second case, $(T(t))_{t \geq 0}$ is said to satisfy the weak spectral mapping theorem (WSMT). It has long been known that the SMT holds for eventually norm-continuous semigroups (that is to say, $C_0$-semigroups which are continuous in the uniform operator topology on $(I_0, \infty)$ for some $I_0 \geq 0$); see e.g. [59, Corollary IV.3.12], [73, Theorem 16.4.1], [115, Theorem 2.3.2]. This class is rather large and, in particular, it includes all eventually differentiable semigroups (thus also all analytic semigroups) and all eventually compact semigroups. There are partial generalisations of the SMT based on replacing eventual norm-continuity by norm continuity at infinity and related notions; see e.g. [27], [115, Theorem 2.3.3]. (Note that on Hilbert spaces norm continuity of a $C_0$-semigroup on $(0, \infty)$ can be characterised by the simple resolvent condition $\|R(z, A)\| \to 0$ along an appropriate vertical line.) Unfortunately, these topics are beyond the scope of this survey. For interesting applications of the spectral mapping theorems to the study of concrete linear and non-linear PDEs see e.g. the recent works [56,57,67] and also the slightly less recent papers [50,63].

The WSMT holds for $C_0$-groups $(T(t))_{t \in \mathbb{R}}$ with sufficiently moderate growth as $|t| \to \infty$ [115, Theorem 2.4.4]. However, such (semi)groups cannot be exponentially stable. Note also that by a theorem due to Weis [148] the equality (2.2) holds for positive semigroups on $L^p$-spaces and $C(K)$-spaces. However, as noted in [142], there exists a positive group on the Banach lattice $L^1$ failing to satisfy even the WSMT. Thus it is possible for the SMT to fail even though the crucial property (2.2) still holds. This situation seems to be completely unexplored in the literature. The multiplication semigroup on $\ell^2(\mathbb{N})$ given by $T(t)x = (e^{int}x_n)_{n \geq 1}$ for $x = (x_n)_{n \geq 1}$ provides a simple example of a $C_0$-semigroup satisfying the WSMT but not the SMT. Apart from the one
given in [142], examples of $C_0$-semigroups failing to satisfy the WSMT (with a detailed analysis of the spectrum and resolvent behaviour) may be found in [68,150]. All of these examples are based on a widely cited example of Zabczyk [152] (preceded by [60]) based on a direct sum construction. At the same time, the somewhat neglected example of failure of the WSMT for the Riemann-Liouville $C_0$-group $T(t)_{t \in \mathbb{R}}$ on $L^2(0, 1)$, defined by $(T(t)f)(s) = \frac{1}{\Gamma(t)} \int_0^s (s-r)^{t-1} f(r) \, dr$, can already be found in [73, Section 23.6]. We refer to [70] for an illuminating study of this group and of the resolvent of its generator. Note that in this case one has $\sigma(T(t)) = \{ z \in \mathbb{C} : e^{-|t|\pi/2} \leq |z| \leq e^{t\pi/2} \}$ for $t \in \mathbb{R}$ while $\sigma(A) = \emptyset$. Moreover, the resolvent of $A$ is compact in this example.

Another line of counterexamples to the SMT stems from examples due to Wolff and Greiner, Voigt and Wofit based on the study of translation semigroups like $(T(t)f)(s) = f(e^t s)$ on intersections of weighted $L^p$-spaces or on the Sobolev space $H^1$ over $(1, \infty)$ [10, Example 5.3.2]; see [10] for a thorough discussion and further references. Renardy’s paper [127] is a standard reference for failure of the WSMT in the case of a semigroup originating from the simple hyperbolic equation $u_{tt} - u_{xx} - u_{yy} = e^{2\pi x} u_y$ on $(0, 1) \times (0, 1) \times \mathbb{R}$ with periodic boundary conditions; see also [23,132]. Other important counterexamples from the point of view of applications are given in Lebeau [98], Schenck [135] and Jin [80] in the context of the damped wave equation $u_{tt} + b u_t - u_{xx} - u_{yy} = 0$.

Recall that the essential spectrum $\sigma_{\text{ess}}(T)$ of a bounded linear operator $T$ can be defined as $\sigma_{\text{ess}}(T) := \{ z \in \mathbb{C} : z - T \text{ is not Fredholm} \}$. It is instructive to observe that the set $\sigma(T) \setminus \sigma_{\text{ess}}(T)$ is at most countable and consists of isolated eigenvalues of $T$ of finite multiplicity, accumulating only at points in $\sigma_{\text{ess}}(T)$. Thus $\sigma_{\text{ess}}(T)$ in a natural sense contains “most” of the spectrum of $T$. The essential spectrum of Hilbert space $C_0$-semigroups $(T(t))_{t \geq 0}$ originating from a large class of hyperbolic equations was described in non-resolvent, geometric terms in the deep paper [89] by Koch and Tataru. More specifically, it was shown in [89, Theorem 3] that $(T(t))_{t \geq 0}$ governs a hyperbolic initial boundary value problem on a smooth, compact and connected Riemannian manifold with smooth boundary, then, under some mild technical assumptions, $\sigma_{\text{ess}}(T(t))$ has rotational symmetry, being either a disk, an annulus or the union of an annulus and the origin for all times $t$ outside a countable exceptional set. In particular, if $(T(t))_{t \geq 0}$ is the semigroup arising in Renardy’s example, then $\sigma_{\text{ess}}(T(t)) = \{ z \in \mathbb{C} : e^{-t\pi/2} \leq |z| \leq e^{t\pi/2} \}$, while the spectrum of the generator lies on the imaginary axis; see [89, Example 2]. It would be instructive to develop a more complete theory involving the properties of the resolvent of the semigroup generator. We point out that the spectral description due to Koch and Tataru is related to the famous “circles conjecture” from semigroup theory; see e.g. [150, Theorem 4.1] and the comments following it. In [72], Herbst proved that if the SMT fails in the Hilbert space setting then it fails dramatically. Specifically, Herbst showed that if $(T(t))_{t \geq 0}$ is a $C_0$-semigroup on a Hilbert space $X$, with generator $A$, then $e^{t\alpha} \in \sigma(T(t)) \setminus \sigma^{\text{res}}(A)$ implies that $e^{t\alpha} \mathbb{T} \subseteq \sigma(T(t))$ for almost all $t \geq 0$, where $\mathbb{T}$ denotes the unit circle. The “circles conjecture” is that the result holds more generally for $C_0$-semigroups on Banach spaces, but this remains an open problem. What is known, however, is that if $(T(t))_{t \geq 0}$ is a Banach space $C_0$-semigroup, with generator $A$, and if $\sup \| R(a + ib, A) \| : b \in \mathbb{R} = \infty$, then $e^{t\alpha} \mathbb{T} \subseteq \sigma(T(t))_{t \geq 0}$ for almost all $t \geq 0$; see [150, Cor. 4.4]. If $\Re z < s(A)$ for all $z \in \sigma(A)$, then one may set $a = s(A)$ in the inclusion above. It is plausible that the exceptional set here is not merely a null set but necessarily at most countable, as in the Koch-Tataru result. As an example of an application of Herbst’s theorem we mention [28, Section 4], where it was used to clarify the spectral structure of Lax-Phillips semigroups arising in the study of local energy decay.

When $s(A) < \omega(T)$ the identity (2.3) suggests looking more closely at the resolvent of $A$ rather than the spectral bound $s(A)$. This appears to be the right way to proceed on Hilbert spaces and, more generally, on Banach space with good geometry. If $\omega > \omega(T)$ then necessarily $\{ z \in \mathbb{C} : \Re z \geq \omega \} \subseteq \rho(A)$ and $\sup_{\Re z \geq \omega} \| R(z, A) \| < \infty$. By applying the vector-valued version of Plancherel’s theorem on Hilbert spaces and the resolvent identity, one can show that the latter property already characterises the exponential growth bound of $C_0$-semigroups on Hilbert spaces.
Theorem 2.1. If \((T(t))_{t \geq 0}\) is \(C_0\)-semigroup on a Hilbert space, with generator \(A\), then

\[
\omega(T) = \inf \left\{ \omega > s(A) : \sup_{Re \, z \geq \omega} \| R(z, A) \| < \infty \right\} =: s_0(A).
\]

(2.4)

In particular, if \((T(t))_{t \geq 0}\) is bounded, then the following assertions are equivalent:

(i) \((T(t))_{t \geq 0}\) is exponentially stable;

(ii) \(i \mathbb{R} \subseteq \rho(A)\) and \(\sup_{s \in \mathbb{R}} \| R(is, A) \| < \infty\).

This result is usually referred to as the Gearhart–Prüss theorem, although it was also obtained independently by Huang [76] and Monauni [113]; see also [112]. It is also a direct consequence of the description of the resolvent of \(T(t)\) by means of the resolvent of \(A\). Indeed, \(\mu \in \rho(T(t)) \setminus \{0\}\) if and only if the set \(\{z \in \mathbb{C} : e^{zt} = \mu\} \subseteq \rho(A)\) and \(R(z, A)\) is bounded on this set; see e.g. [120]. (There is also a Banach space version of the latter result, in which one replaces boundedness of the resolvent by convergence of certain Cesàro averages or by an appropriate Fourier multiplier condition; see e.g. [93].) Note that the same description of the spectrum of \(T(t)\) was obtained, independently and almost simultaneously, by Herbst [71] and Howland [75]. However, these authors did not explore its applications to stability theory.

It is instructive to observe that Theorem 2.1 does not generalize to \(L^p\)-spaces for \(p \neq 2\); see for instance [10, Examples 5.11 and 5.2.2]. Moreover, returning to the Hilbert space setting, the local version of Theorem 2.1 does not hold. Indeed, let \((T(t))_{t \geq 0}\) be the (positive) \(C_0\)-semigroup on \(X = L^2(1, \infty)\) given by \((T(t)f)(s) = f(e^t s)\), with generator \(A\), and let \(f \in X\) be the characteristic function of the set \(\bigcup_{n \in \mathbb{N}} (e^n, e^n + n^{-2})\). Then the map \(z \mapsto R(z, A)f\) extends to a bounded holomorphic function on \(\mathbb{C}_{-1}\), but the map \(t \mapsto \| T(t)f \|\) grows exponentially; see e.g. [10, Example 5.2.3] for further details. There is a partial remedy for \(x \in D((-A)^\alpha)\), where \(\alpha > 0\) is fixed. The exponential growth bound for \(\| T(t)x \|\), uniform in \(x \in D((-A)^\alpha)\), is given by

\[
\omega_\alpha(T) := \inf \{ \omega \in \mathbb{R} : e^{-\omega t} \| T(t)R(\mu, A)^\alpha \| < \infty \},
\]

where \(\mu \in \rho(A)\) is fixed, and \(\omega_\alpha(T)\) can be characterised in terms of polynomial bounds on \(R(z, A)\) on an appropriate right half-plane, namely

\[
\omega_\alpha(T) = \inf \left\{ \omega > s(A) : \sup_{Re \, z \geq \omega} (1 + |z|)^{-\alpha} \| R(z, A) \| < \infty \right\} =: s_\alpha(A);
\]

(2.5)

see [147]. Moreover, the function \(\alpha \mapsto \omega_\alpha(T)\) is convex on \([0, \infty)\). It remains an open question whether it is possible to say more about exponential stability of individual orbits for special classes of semigroups such as \(C_0\)-semigroups of contractions on Hilbert spaces.

There are important situations in which the spectrum of the generator determines the exponential growth bound of a \(C_0\)-semigroup, and there exists an intricate abstract theory relating to this question. One such situation arises when the system of root vectors of the generator forms a Riesz basis, which occurs frequently in applications. To our knowledge, the following result by Miloslavskii [111, Theorem 1] is one of the most general results of its kind.

Theorem 2.2. Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup on a Hilbert space \(X\), with generator \(A\) satisfying

\[
\sigma(A) \subseteq \{ z \in \mathbb{C} : Re \, z < 0 \}, \quad \sigma(A) = \bigcup_{k=1}^\infty \Omega_k, \quad \Omega_l \cap \Omega_m = \emptyset, \quad l \neq m,
\]

where \(\Omega_k\) is a finite set for each \(k\). Assume further that the spectral projections \(P_k\) corresponding to \(\Omega_k\) have finite rank and that the subspaces \(X_k := P_k(X)\), \(k \geq 1\), form a Riesz basis of subspaces in \(X\). If there
exists $\gamma > 0$ such that
\[
\sup_{z \in R_k} |\operatorname{Re} z| \geq \gamma \dim X_k \log(\dim X_k),
\]
then $(T(t))_{t \geq 0}$ is exponentially stable. If, moreover, $N := \sup_{k \geq 1} \dim X_k$ is finite, then we have the explicit estimate
\[
\|T(t)\| \leq C(1 + t)^{N-1} e^{s(A)t}, \quad t \geq 0,
\]
for some constant $C > 0$.

Zabczyk’s example mentioned above shows that both parts of this result become false if the relevant assumptions on the dimensions of the spaces $X_k$ are omitted. Note also that $s(A) = \omega(T)$ whenever $N$ is finite. It is plausible that in the setting of Theorem 2.2 the WSMT holds for $(T(t))_{t \geq 0}$ if and only if $N$ is finite, but this does not appear to be known. Note that there exist many statements in the literature related to Theorem 2.2; we mention only the recent papers [5,6], where the equality $s(A) = \omega(T)$ is proved for generators $A$ whose eigenvectors form a Riesz basis.

In many applications, for example those dealing with linearisations of non-linear PDEs, one encounters the problem of finding sharp bounds for the constant $M = M(\omega)$ in the estimate $\|T(t)\| \leq M e^{\omega t}, t \geq 0$, for a fixed $\omega \in \mathbb{R}$, especially when $(T(t))_{t \geq 0}$ is known to be exponentially stable. One thus arrives at the task of obtaining an explicit estimate for $\sup \{e^{-\omega t}\|T(t)\| : t \geq 0\}$.

It is usually straightforward to obtain a rough exponential estimate of the form $\|T(t)\| \leq L e^{Mt}$, $t \geq 0$. Helffer and Sjöstrand in [69, Proposition 2.1] obtained the following useful bound which, incidentally, leads to yet another proof of Theorem 2.1.

**Theorem 2.3.** Let $(T(t))_{t \geq 0}$ be a $C_0$-semigroup on a Hilbert space, with generator $A$. Let $L > 0$ and $\lambda \in \mathbb{R}$ be as above and suppose that $\omega < \lambda$ is such that $N(\omega) := \sup_{\operatorname{Re} z \geq \omega} \|R(z, A)\|$ is finite. Then
\[
\|T(t)\| \leq L(1 + 2LN(\omega)(\lambda - \omega))e^{\omega t}, \quad t \geq 0.
\] (2.6)

It is natural to ask what happens in the limit as $\omega \to \omega(T)$. The answer depends on the rate of blow-up of $R(z, A)$ or, more precisely, of $N(\omega)$ near $\omega(T)$. Assuming that $\omega(T) > -\infty$, $N(\omega)$ blows up as $\omega \to \omega(T)$. If there are $C, k > 0$ such that $N(\omega) \leq C (\omega - \omega(T))^{-k}$ for $\omega - \omega(T) > 0$ sufficiently small, then, as shown in [69], one gets $\sup_{t \geq 1} t^{-k} e^{-\omega(T)t}\|T(t)\| < \infty$. In fact, one may associate with every blow-up rate of the resolvent an appropriate correction of exponential growth of the semigroup; see [132] for details. Recently, using the techniques of [69], the following interesting result was proved by Wei in [146, Theorem 1.3].

**Theorem 2.4.** Let $(T(t))_{t \geq 0}$ be a $C_0$-semigroup of contractions on a Hilbert space $X$. If $i\mathbb{R} \subseteq \rho(A)$ and $\omega_0 := \sup_{\sigma \in \mathbb{R}} \|R(is, A)\| < \infty$, then
\[
\|T(t)\| \leq e^{\pi/2} e^{-\omega_0 t}, \quad t \geq 0.
\] (2.7)

Some Banach space counterparts of Theorem 2.3 (as well as additional references) may be found in [94,132], while several interesting and relevant examples of applications are considered e.g. in [69,146] and [77, Remark 3.13].

It is clear that if a $C_0$-semigroup $(T(t))_{t \geq 0}$ is exponentially stable then its orbits $T(\cdot)x$ ($x \in X$) and its weak orbits $(T(\cdot)x, y)$ ($x \in X$, $y \in X^\ast$) belong to (vector-valued) $L^p$ and many other classical Banach function spaces on $\mathbb{R}^+$. A line of theorems originating from Datko’s famous result (which are of importance in some applications, e.g. in control theory) show that certain converse statements hold, too. We thus have a phenomenon which is Tauberian in character. Moreover, it is possible to estimate the exponential growth bound of a semigroup, as the following result shows. The first part of the result is due to Datko and Pazy (see e.g. [115]), and the second part was proved by Weiss [149].
2.5 Stability

Theorem 2.5. Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup on a Banach space \(X\). Assume either that there exist \(p \in [1, \infty)\) and \(C > 0\) such that
\[
\int_0^\infty \|T(t)x\|^p \, dt \leq C\|x\|^p, \quad x \in X, \tag{2.8}
\]
or, if \(X\) is a Hilbert space, that there exist \(p \in [1, \infty)\) and \(C > 0\) such that
\[
\int_0^\infty \|(T(t)x, y)\|^p \, dt \leq C\|x\|^p\|y\|^p, \quad x, y \in X. \tag{2.9}
\]
Then \(\omega(T) \leq -(pC)^{-1}\).

Note that the existence of constants \(C\) such that (2.8) and (2.9) hold would already follow from the fact that all orbits (or weak orbits) of \((T(t))_{t \geq 0}\) lie in an \(L^p\)-space, by a simple application of the closed graph theorem. Observe also that the estimate \(\omega(T) \leq -(pC)^{-1}\) is optimal in the sense that \(\omega(T)\) equals the infimum of the numbers \(- (pC)^{-1}\), with \(C > 0\) running over all constants for which (2.8) holds with respect to an equivalent norm; see [115, p. 82].

For other results of this type and relevant ideas we refer the reader to [115, Chapter 3], [10], and to [116]. There are many more statements along the lines of Theorem 2.5 in the literature (although most of them are primarily of theoretical interest).

While there are many perturbation results for the exponential stability, most of them concern perturbations which are small in a metric sense; in this case exponential stability of the perturbed semigroup comes as no surprise. However, there are some nice exceptions of this rule, as the following recent result by Prüss [121] shows.

Theorem 2.6. If \(A\) generates an exponentially stable \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\), and if \(B \in \mathcal{L}(X)\) is such that \(\lim_{t \to 0^+} \|(T(t) - I)B\| = 0\) (for example, if \(B\) is compact) and \(\lambda - A - B\) is invertible for \(\lambda \in \mathbb{C}^+_+\), then \(A + B\), too, generates an exponentially stable \(C_0\)-semigroup.

2.6 Strong stability

(b) Strong stability

Strong stability is another basic stability concept in the theory of operator semigroups. Unlike exponential stability, it is distinctly qualitative in character. Nevertheless, it is precisely this notion of stability that lies behind many of the more recent developments in the quantified asymptotic behaviour of operator semigroups.

Recall that a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\) is said to be strongly stable, or simply stable, if for every \(x \in X\),
\[
\lim_{t \to \infty} \|T(t)x\| = 0.
\]

By the uniform boundedness principle, every stable semigroup is necessarily bounded and, by the spectral inclusion theorem, the generator \(A\) of a stable semigroup satisfies \(\sigma(A) \subseteq \{ z \in \mathbb{C} : \text{Re}\, z \leq 0 \}\). The example of the left-shift semigroup on \(L^2(\mathbb{R}^+\) shows that a semigroup may be stable and yet have a generator \(A\) whose spectrum is “maximal” in the sense that \(\sigma(A) = \{ z \in \mathbb{C} : \text{Re}\, z \leq 0 \}\). Thus the location of the spectrum of the generator alone does not tell us a great deal about stability of the semigroup. The initial intuition behind the spectral approach is that smaller the spectrum of the generator on the imaginary axis better the stability properties of the associated semigroup. A particularly revealing illustration of this is the following famous result due to Arendt and Batty, and Lyubich and Vũ [10, Theorem 5.5.5].

Theorem 2.7 (Arendt-Batty-Lyubich-Vũ). Let \((T(t))_{t \geq 0}\) be a bounded \(C_0\)-semigroup on a Banach space \(A\), and suppose that the generator \(A\) is such that the boundary spectrum \(\sigma(A) \cap i\mathbb{R}\) is at most countable and contains no eigenvalues of the adjoint of \(A\). Then \((T(t))_{t \geq 0}\) is stable.

The result is best possible as far as the spectral assumptions on \(A\) are concerned; see e.g. [10]. Subsequently, more sophisticated results of this kind, applicable in the case when the
spectrum contains arbitrary large sets on the imaginary axis, were proved in the series of papers [29,42,43,144]; see also the survey [46]. Here the above “spectral smallness” principle was replaced by a more general one: the more the resolvent of the generator can be extended to or across the imaginary axis, the better the stability properties of the semigroup orbits. To convey the flavour of the relevant results we formulate a statement which turned out to be particularly useful in some applications to PDEs; see [32, Theorem 5], [144, p. 75-76] and [46] for further details.

**Theorem 2.8.** Let \((T(t))_{t \geq 0}\) be a completely non-unitary \(C_0\)-semigroup of contractions on a Hilbert space \(X\), with generator \(A\). Then \((T(t))_{t \geq 0}\) is stable if and only if the set
\[
\{ x \in X : \lim_{a \to 0^+} \sqrt{a} R(a + ib, A)x = 0 \text{ for almost all } b \in \mathbb{R} \}
\]
is dense in \(X\). If \((T(t))_{t \geq 0}\) is merely assumed to be bounded then it is stable provided the set
\[
\{ x \in X : \lim_{a \to 0^+} \sqrt{a} R(a + ib, A)x = 0 \text{ for all } b \in \mathbb{R} \}
\]
is dense in \(X\).

Note that it is possible to formulate necessary and sufficient (integral) conditions for stability of a bounded \(C_0\)-semigroup \((T(t))_{t \geq 0}\) or its individual orbits. For an account of these results concerning strong stability, including stability of semigroups on Banach spaces, characterisations of stability in non-resolvent terms, and even stability of (non-autonomous) evolution families, we refer the reader to the survey [46] and the references therein.

One may also define a notion of weak stability for \(C_0\)-semigroups. Indeed, one says that \((T(t))_{t \geq 0}\) is weakly stable if all of the orbits of \((T(t))_{t \geq 0}\) converge to zero in the weak topology as \(t \to \infty\). The notion of weak stability corresponds to the concept of mixing in ergodic theory, where one deals with weakly stable unitary \(C_0\)-groups of Koopman operators on \(L^2\)-spaces. The general theory of weak stability of \(C_0\)-semigroups is not particularly well developed, and in particular weakly mixing unitary groups are still studied in ergodic theory largely on a case-by-case basis. On the other hand, weak stability can often serve as a Tauberian condition for strong stability, as the following statement shows.

**Theorem 2.9.** Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup of contractions on a Hilbert space \(X\), with generator \(A\). If \((T(t))_{t \geq 0}\) is completely non-unitary then it is weakly stable. Moreover, if \(A\) has compact resolvent then the following conditions are equivalent:

(i) \((T(t))_{t \geq 0}\) is weakly stable;
(ii) \((T(t))_{t \geq 0}\) is strongly stable;
(iii) \((T^*(t))_{t \geq 0}\) is strongly stable;
(iv) \((T(t))_{t \geq 0}\) is completely non-unitary.

If \(A = A_0 - BB^*\), where \(A_0\) is skew-adjoint, has compact resolvent, and \(B\) is bounded, then each of the conditions (i)-(iv) is equivalent to the property that \(B^*x \neq 0\) for every non-zero \(x \in X\) such that \(A_0x = x\) for some \(s \in \mathbb{R}\).

This result has appeared, in one form or another, in a number of papers. It is used implicitly in the paper [79], which studies energy decay for a class of hyperbolic equations including the damped wave equation; see [15] for a discussion. Proofs of the equivalence of (i)-(iv) in Theorem 2.9 may be found in [26,99,103], while the last statement of the theorem is contained in [92, Theorem 5] and [18, Theorem 14]. As another statement similar in flavour to Theorem 2.9, recall that weak stability of a bounded semigroup \((T(t))_{t \geq 0}\) on an \(L^1\)-space implies strong stability (see [46]), and the same holds on \(C(K)\)-spaces, where \(K\) is a compact Hausdorff space, if \((T(t))_{t \geq 0}\), in addition, is irreducible; see [108]. For further results in which the stability (or, more
generally, convergence) of a $C_0$-semigroup is a consequence of mild assumptions on the geometry of the underlying Banach space, see [64] and the references therein.

3. Quantified Tauberian theorems and semi-uniform stability of operator semigroups

In many cases, stability results for $C_0$-semigroups or their individual orbits are consequences of general Tauberian theorems for Laplace transforms. The latter often serve as an inspiration for the former, and in modern treatments both are recognised as being two sides of the same coin. Nowadays, the role of Tauberian theory in the theory of $C_0$-semigroups is well understood. The relevant theory was developed in a number of papers, arguably starting with the seminal paper [9]. Later, in [41], a finer distributional approach was developed, which has recently been extended to include quantitative aspects; see [20, 45, 53, 54]. A good introduction to modern Tauberian theory may be found in [91], while applications to operator semigroups are discussed thoroughly in [10].

The following result is usually attributed to Ingham [78] and Karamata [84], although a version was in fact first discovered by M. Riesz [128, Satz II]. It is a classical Tauberian theorem which underpins just about all of what follows here; we refer to [46] for a short proof.

**Theorem 3.1.** Let $X$ be a Banach space and let $f : \mathbb{R}_+ \to X$ be a bounded and uniformly continuous function whose Laplace transform $\hat{f}$ admits a holomorphic extension to each point of $i\mathbb{R}$. Then $\lim_{t \to \infty} \|f(t)\| = 0$.

Theorem 3.1 is the starting point for numerous results in the asymptotic theory of $C_0$-semigroups, and it has been generalised in various directions. It is natural in analysis, and also in Tauberian theory, to equip a convergence result with a rate whenever this is possible. In the case of Theorem 3.1 this can be done in a way which has a number of interesting consequences, not least for operator semigroups. Observe, however, that Theorem 3.1 is best possible in the sense that one cannot expect any rate of decay for $f$ if no further assumptions are imposed on the growth of $\hat{f}$, even if $\hat{f}$ extends to an entire function; see e.g. [33, 54, 55]. In order to obtain quantitative results one assumes that $\hat{f}$ extends analytically beyond $i\mathbb{R}$ to some precise domain and that this analytic extension satisfies an appropriate bound in this domain. Given continuous and increasing functions $M, K : \mathbb{R}_+ \to (0, \infty)$ we define

$$\Omega_M := \left\{ z \in \mathbb{C} : \text{Re} z > -\frac{1}{M(|\text{Im} z|)} \right\}$$

and

$$M_K(s) := M(s)(\log(1 + K(s)) + \log(1 + s))$$

for $s \geq 0$. Note that the function $M_K$ is continuous, strictly increasing and unbounded, and hence it has an inverse function $M_K^{-1}$ defined on $[a, \infty)$ for some $a > 0$. The function $M$ itself may not be strictly increasing, and we denote by $M^{-1}$ any right-inverse of $M$. Without further comment we extend both inverses by $0$ to $\mathbb{R}_+$. The following result is a rather general Tauberian theorem.

**Theorem 3.2.** Let $f \in L^p_{\text{loc}}(\mathbb{R}_+; X)$ be a function with weak derivative $f' \in L^p(\mathbb{R}_+; X)$, where $p \in [1, \infty]$. Let $M, K : \mathbb{R}_+ \to (0, \infty)$ be continuous and increasing functions satisfying, for some $\varepsilon \in (0, 1)$, $C \geq 0$,

$$M(s) \leq K(s) \leq Ce^{e^{cM(s)^{1-\varepsilon}}}, \quad s \in \mathbb{R}_+.$$  

Assume that $\hat{f}$ extends analytically to $\Omega_M$ and that

$$\|\hat{f}(z)\| \leq K(|\text{Im} z|), \quad z \in \Omega_M.$$  

Then there exists $c > 0$ such that

$$(t \mapsto M_K^{-1}(ct)\|f(t)\|) \in L^p(\mathbb{R}_+).$$
This Tauberian theorem is the result of developments over a period of almost ten years. In the special case $p = \infty$ and $K = M$ it was first obtained by Batt y and Duyckaerts in [22]. In that case, the function $M_K(s) = M(s)\log(1 + M(s)) + \log(1 + s)$ is often denoted by $M_{\log}$. It is the achievement of Batty, Borichev and Tomilov [20] to extend the basic result from [22] and the classical Ingham-Karamata-Riesz theorem to general $p$ and to allow functions $K$ of the form $(1 + s)^\alpha M(s)^\beta$ (for some $\alpha, \beta \geq 0$) instead of just $K = M$. In this case $M_K$ and $M_{\log}$ do not differ essentially. There is also an extensive discussion in [20, Section 4] of the fact that more general functions $K$ are possible, too. In fact, there are sometimes non-trivial relations between the functions $M$ and $K$, and the interplay between the shape of $\Omega_M$ and the growth of $f$ in $\Omega_M$ can be analysed by means of more or less standard techniques of complex analysis, such as the theory of harmonic measure. For instance, if $M(s) = C(1 + s)^p$ and if $f$ grows polynomially in $\Omega_M$, then, as was noted in [30, Lemma 3.4], one necessarily has $|f(z)| \leq C_t(1 + |z|^{n+\epsilon})$ for all $z \in \Omega_M$ for an appropriate constant $c > 1$. Theorem 3.2 in its full generality was finally proved in an unpublished manuscript by Stahn [138, Theorem 1.1]; see also [139].

Regardless of $p$, Theorem 3.2 is optimal in the sense that one cannot obtain any better weights in (3.3) than $M_K^{\frac{1}{p}}$. Optimality for $p = \infty$ and polynomial $M = K$ was proved in [30, Theorem 3.8] by an explicit and fairly involved construction. By developing further the ideas from [30], this approach was then extended to all $p$ and to some other functions $M$ (and also to individual orbits of Hilbert space semigroups) in [20, Sections 5 and 7], and in [138, Theorem 4.1] for an even larger class of functions $M$ and $K$; see also [54] and [53] for an alternative abstract approach to optimality.

In order to make the link between general Tauberian theory and $C_0$-semigroups, it suffices to observe that if $(T(t))_{t \geq 0}$ is a bounded $C_0$-semigroup on a Banach space $X$ with generator $A$, and if $A$ has no spectrum on the imaginary axis, then applying Theorem 3.1 to individual orbits shows that the semigroup is stable. This is of course a very special case of Theorem 2.7. Applying Theorem 3.1 to the bounded, Lipschitz continuous, operator-valued function $T(\cdot)A^{-1}$ yields

$$\lim_{t \to \infty} \|T(t)A^{-1}\| = 0.$$  \hspace{1cm} (3.4)

To distinguish this kind of asymptotic behaviour of $(T(t))_{t \geq 0}$, we call a bounded $C_0$-semigroup $(T(t))_{t \geq 0}$ satisfying (3.4) semi-uniformly stable, noting that in this case the stability of $(T(t))_{t \geq 0}$ is uniform with respect to initial values from the unit ball of $D(A)$ endowed with the graph norm. We emphasise that our terminology is not particularly common in the literature. However, the terminology is at least natural from the point of view that every exponentially stable semigroup is semi-uniformly stable, and every semi-uniformly stable semigroup is (strongly) stable. It is straightforward to see that the converses of these statements are both false.

A natural question, both from an operator-theoretic perspective and from the point of view of applications to PDEs, is whether semi-uniform stability can be quantified. To clarify the quantification problem note first that convergence of $\|T(\cdot)A^{-1}\|$ to zero can formally be translated into an estimate of the form

$$\|T(t)x\| = O(r(t)), \quad t \to \infty,$$  \hspace{1cm} (3.5)

for some function $r \in C_0(\mathbb{R}_+)$ and for every $x \in D(A)$, that is, for every classical solution of the Cauchy problem $\dot{u} = Au$. If the semigroup $(T(t))_{t \geq 0}$ is stable but not exponentially stable, then one can never expect such an estimate to hold for every initial value $x \in X$. Indeed, having (3.5) for every $x \in X$ is equivalent to exponential stability by a simple application of the uniform boundedness principle, as has already been observed. In fact, for strongly stable semigroups which are not exponentially stable we obtain the following statement; see [114,115].

**Theorem 3.3.**

(a) Let $(T(t))_{t \geq 0}$ be a strongly stable but not exponentially stable $C_0$-semigroup on a Banach space $X$. Then for every $r \in C_0(\mathbb{R}_+)$ there exists $x \in X$ such that $\|T(t)x\| \geq r(t)$ for all $t \geq 0$.

(b) Let $(T(t))_{t \geq 0}$ be a weakly stable but not exponentially stable $C_0$-semigroup on a Hilbert space $X$. Then for every $r \in C_0(\mathbb{R}_+)$ there exists $x \in X$ such that $\|T(t)x, x\| \geq r(t)$ for all $t \geq 0$. 
Note that for non-exponentially stable semigroups the above statements also rule out the possibility of various integral conditions being satisfied for all semigroup orbits. This is in accordance with Theorem 2.5 stated above. For more results on lower bounds for weak orbits of $C_0$-semigroups on mostly reflexive Banach spaces, see [114] and [141].

While the decay of a stable semigroup can be arbitrarily slow, the decay of sufficiently regular orbits admits a rate whenever the semigroup is semi-uniformly stable. This claim is made precise by the following theorem, a substantial part of which can be deduced from the Tauberian theorem 3.2. The statement was obtained in [22] by Batty and Duyckaerts, as a culmination of several notable preceding results originating from both abstract operator theory and its applications, e.g. in [16,35,98,102]. The paper [22] sparked a considerable amount of further research on quantitative aspects of semi-uniform stability for $C_0$-semigroups, and it is a precursor of many of the quantitative stability results discussed below.

Theorem 3.4. Let $(T(t))_{t\geq0}$ be a bounded $C_0$-semigroup on a Banach space $X$, with generator $A$, and let $\mu \in \rho(A)$. Then the following are equivalent:

(i) $\sigma(A) \cap i\mathbb{R}$ is empty;
(ii) $\lim_{t \to \infty} \|T(t)R(\mu, A)\| = 0$.

Moreover, if either (i) or (ii) hold, let

$$M(s) := \sup \{|R(ir, A)| : |r| \leq s\}, \quad s \geq 0,$$

define the function $M_{\log}$ by $M_{\log} = M_K$ where $K = M$ and $M_K$ is as in (3.1), and let $M^{-1}$ denote any right-inverse of $M$. Then there exist positive constants $C, C', c, c'$ such that

$$\frac{C'}{M^{-1}(c't)} \leq \|T(t)A^{-1}\| \leq \frac{C}{M_{\log}^{-1}(ct)} \tag{3.6}$$

for all sufficiently large $t$.

The implication (i) $\implies$ (ii) is proved for instance in [17, p. 803] and [145, Corollary 3.3], although it was already contained implicitly in [9]; it may be viewed as a consequence of the Ingham-Karamata-Riesz theorem. In order to prove the converse implication, assume that (ii) holds and that $s \in \mathbb{R}$ is such that $i s \in \sigma(A)$. Then $e^{ist}(1 - is)^{-1} \in \sigma(T(t)R(1, A))$ by the spectral inclusion theorem [73, Theorem 16.3.5] for the Hille-Phillips functional calculus, and in particular $|1 - is|^{-1} \leq \|T(t)R(1, A)\|$ for all $t \geq 0$. Letting $t \to \infty$ yields the desired contradiction, since (ii) in particular holds for $\mu = 1$ by a simple application of the resolvent identity.

Observe that if $\|R(is, A)\| \leq M(|s|)$ for $s \in \mathbb{R}$, then the Neumann series expansion shows that, for any $c > 1$, the region $\Omega_{cM}$ is contained in the resolvent set of $A$ and $\|R(z, A)\| \leq \frac{1}{1-\tau} M(|\text{Im } z|)$ for $z \in \Omega_{cM}$. Hence the upper bound in (3.6) is a direct consequence of Theorem 3.2. In contrast to the setting of Theorem 3.2, however, in this case it is natural to describe the region of holomorphic extension of $R(z, A)$ and the growth of the extension by the same function $M$, up to constant multiples. The lower bound in (3.6) arises as consequence of simple operator-theoretic considerations based on the semigroup analogue of the fundamental theorem of calculus.

Note that if $M(s) = Ce^{as}$, then $M^{-1}$ and $M_{\log}^{-1}$ are asymptotically of the same order, and Theorem 3.4 yields the optimal logarithmic rate of decay for $\|T(t)A^{-1}\|$. This result, which goes back to [98] and, with an optimal rate, to Burq [35], is of great value for applications, for example to the study of damped wave equations. By going to the opposite end of the scale and taking a constant function $M$ one recovers part of the Weis-Wrobel relation (2.5) for bounded $C_0$-semigroups in the case $\alpha = 1$. A general local version of this statement is due to van Neerven [115]; see also [19] the references therein. Thus it is natural and important when the upper bound in (3.6) matches the lower bound. It can be easily checked that for semigroups of normal operators and for sufficiently regular $M$ one can indeed replace the upper bound $1/M_{\log}^{-1}$ by the lower
bound \(1/M^{-1}\); see e.g. [21, Section 5.1]. However, it is known that the function \(M^{-1}\) in (3.6) cannot in general be replaced by \(M^{-1}\) for bounded Banach space \(C_0\)-semigroups. As in the case of Theorem 3.2, Theorem 3.4 is optimal in the sense that one cannot obtain better upper estimates for the semigroup orbits than the one given by (3.6). In the case of polynomial resolvent growth this goes back once again to [30], but more recently an elegant abstract approach for obtaining optimal lower bounds for semigroup orbits was found by Debruyne and Seifert in [53,54]. Note, however, that the lower bounds contained in these optimality results hold only along a subsequence of \(\mathbb{R}_+\). It is not known whether one may prove stronger optimality results in the spirit of Theorem 3.3.

Concerning the closely related situation of so-called cut-off (or Lax-Phillips) semigroups it is important to note that around the same time as Batty and Duyckaerts in [22], Christiannson proved a very similar decay rate result for functions of the form \(f(t) = \chi_1 R(t,x)\chi_2\), where \((T(t))_{t \geq 0}\) is a \(C_0\)-semigroup of contractions on a Hilbert space, \(\chi_1\) and \(\chi_2\) are bounded linear operators, the Laplace transform \(\tilde{f}\) extends to a domain \(\mathcal{D}_M\) where it satisfies a polynomial growth estimate with \(K(s) = C(1 + s)^N\), and \(k \geq N + 2\) [47, Theorem 3]. The result states that \(\|f(t)\| = O(1/\tilde{M}_\log (ct)^{k/2})\), where \(\tilde{M}_\log (s) = M(s) \log (1 + s)\). This result has been applied to the study of concrete equations, in particular damped wave equations, for instance in [47,48,80,129]. It is worth pointing out that, unlike Batty and Duyckaerts, Christiannson in his assumptions already uses separate functions \(M\) and (a polynomial) \(K\) for the description of the region of analytic continuation and for the estimate of the cut-off resolvent \(\chi_1 R(z,A)\chi_2\) in that region. However, improved versions of the Tauberian Theorem 3.2 (see [22, Theorem 4.2], [138, Theorem 1.1]) yield the decay rate \(\|f(t)\| = O(1/\tilde{M}_\log (ct)^{k})\); see also [22, Corollary 4.2] which is especially designed for the situation of cut-off semigroups. Note that \(M_\log^{-1}\) and \(\tilde{M}_\log^{-1}\) are comparable for polynomial \(M\), and therefore the result from [22] is stronger in this case.

Of course, from the point of view of applications to PDEs, one’s primary interest is in the situation where \((T(t))_{t \geq 0}\) is a bounded \(C_0\)-semigroup on a Hilbert space. It turns out that for such semigroups Theorem 3.4 can be substantially sharpened, by replacing \(M_\log\) with \(M\) for a very large class of functions \(M\), thus answering an open question in [22] for polynomial \(M\). As in the situation of the Gearhart-Prüss theorem, the availability of the vector-valued Plancherel theorem plays a crucial role here. The following result was first obtained in [30, Theorem 2.4]. It shows that if, in Theorem 3.2, \(X\) is a Hilbert space and \(M\) is of polynomial form, then the upper bound in (3.6) can be sharpened to match the lower bound, and in particular the sharpened upper bound is optimal.

**Theorem 3.5.** Let \((T(t))_{t \geq 0}\) be a bounded \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on a Hilbert space, with generator \(A\), and let \(\alpha > 0\) be a constant. The following assertions are equivalent:

1. \(\|T(t)(I - A)^{-1}\| = O(t^{-1/\alpha})\) as \(t \to \infty\);
2. \(iR \subseteq \rho(A)\) and \(\|R(is,A)\| = O(|s|^\alpha)\).

Note that polynomial decay rates as in Theorem 3 are stable with respect to certain classes of factored perturbations \(BC\) of \(A\), where the operators \(B\) and \(C\) are subordinated to appropriate fractional powers of \(\lambda - A\) and \(\lambda - A^*\); see e.g. [118] for details.

The approach of [30] was refined and extended in [21,105]. Here the study of uniform stability of \(T(\cdot)x\) for \(x \in D(A)\) was extended to orbits with \(x \in \text{Im}(A)\) and also, as a combination of the two, to uniform stability of orbits with \(x \in \text{Im}(A) \cap D(A)\). Uniform stability of \(T(\cdot)x\) for \(x \in \text{Im}(A)\) corresponds in operator-theoretic terms to \(\|T(t)(I - A)^{-1}\| \to 0\), and the latter condition is characterised (in the Hilbert space setting) by the spectral condition \(\sigma(A) \cap i\mathbb{R} \subseteq \{0\}\) together with the resolvent condition \(\sup_{\lambda \geq 1} \|R(is,A)\| < \infty\). The spectral condition on its own characterises (even on Banach spaces) the decay \(\|T(t)(I - A)^{-1}\| \to 0\), which is the operator-theoretic formulation of uniform stability of orbits \(T(\cdot)x\) for \(x \in \text{Im}(A) \cap D(A)\). Apart from providing a general framework within which to analyse these three types of semi-uniform
functions (of positive index) considered in \[ \]. This is a large class of functions, which in particular contains the class of very nice paper by Anantharaman and Léautaud dealing mostly with damped wave equations. This subsection is based on the presentation in a of illustrative applications of our techniques, to begin with a discussion of an abstract model and books dealing with asymptotic behaviour of hyperbolic equations and operator matrices, but then to illustrate briefly, in the concluding remarks, how it relates to several concrete situations. Since this survey emphasises abstract aspects we found it instructive, in passing to descriptions of multiplication semigroups in, for details. In particular, the paper contains versions of Theorem for unbounded semigroups; see \[ \text{Theorem 3.2}. \]

\[ \text{Theorem 3.6 (Rozendaal-Seifert-Stahn). Let } (T(t))_{t \geq 0} \text{ be a bounded } C_0\text{-semigroup on a Hilbert space } X, \text{ with generator } A, \text{ and let } M : \mathbb{R}_+ \to (0, \infty) \text{ be a function of positive increase. The following assertions are equivalent:} \]

\begin{enumerate}
  \item \[ |T(t)|^{-1} = O(1/M^{-1}(t)) \text{ as } t \to \infty; \]
  \item \[ i \mathbb{R} \subseteq \rho(A) \text{ and } \|R(is, A)\| = O(M(|s|)) \text{ as } |s| \to \infty. \]
\end{enumerate}

Note that, as the analysis of multiplication semigroups in, Section 5.1 reveals, the class of functions of positive increase is the largest possible class for which the resolvent estimate \( \|R(is, A)\| \leq M(|s|), \ s \in \mathbb{R}, \ implies the decay rate \( |T(t)A^{-1}| = O(1/M^{-1}(t)) \text{ as } t \to \infty; \) see also \[ \text{[131].} \]

It sometimes happens in applications that a resolvent estimate on \( i \mathbb{R} \) is known only along a subsequence. To clarify this situation and complement Theorem 3.4, observe that if \( M : \mathbb{R}_+ \to (0, \infty) \) is a continuous increasing function such that \( M(s) \to \infty \text{ as } s \to \infty \) and \( \lim_{|s| \to \infty} M(|s|)^{-1} \|R(is, A)\| > 0 \), then there exists \( c > 0 \) such that \( \lim_{t \to \infty} M^{-1}(ct)\|T(t)(I-A)^{-1}\| > 0 \), and if \( M \) has positive increase then the latter inequality holds for all \( c > 0 \); see \[ \text{[44, Proposition 5.4].} \] In fact, the same applies more generally to bounded \( C_0\)-semigroups on Banach spaces.

Exponential and semi-uniform stability with polynomial rates can also be approached by means of Fourier multiplier techniques. We are unable to discuss this approach in our short survey, and we instead refer to the interesting recent paper \[ \text{[133] (and the references therein) for details. In particular, the paper contains versions of Theorem for unbounded semigroups; see e. g. [133, Section 4].} \]

4. Applications to second-order Cauchy problems

Since this survey emphasises abstract aspects we found it instructive, in passing to descriptions of illustrative applications of our techniques, to begin with a discussion of an abstract model and then to illustrate briefly, in the concluding remarks, how it relates to several concrete situations dealing mostly with damped wave equations. This subsection is based on the presentation in a very nice paper by Anantharaman and Léautaud \[ \text{[8, Part III], which in fact considers a slightly more general setup than ours. Note that similar arguments can be found in a number of papers and books dealing with asymptotic behaviour of hyperbolic equations and operator matrices, but we refrain from mentioning all of them here.} \]

For the remainder of this section, let \( A \) and \( B \) be self-adjoint positive semi-definite operators on a Hilbert space \( X \). Assume that \( A \) has compact resolvent, and that \( B \) is bounded. Assume
further that
\[ \|Bx\| > 0 \text{ for every (non-zero) eigenvector } x \text{ of } A. \quad (4.1) \]
The general approach given below in fact works under much milder assumptions on \( A \) and \( B \), but in view of our applications to damped wave equations it is reasonable to restrict our attention to this particular setting. Consider then the following second-order Cauchy problem in \( X \):
\[
\ddot{u} + B\dot{u} + Au = 0, \quad u(0) = u_0, \quad \dot{u}(0) = u_1, \tag{4.2}
\]
for certain initial values \( u_0, u_1 \in X \). Setting \( U = (u, \dot{u}) \), this second-order Cauchy problem can be rewritten on the space \( \mathcal{X} := D(A^{1/2}) \times X \) as a first-order Cauchy problem
\[
\dot{U} = AU, \quad U(0) = (u_0, u_1) \in \mathcal{X}, \tag{4.3}
\]
where
\[
A = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}, \quad \text{with } D(A) = D(A) \times D(A^{1/2}). \tag{4.4}
\]
The operator \( A \) generates a \( C_0 \)-semigroup of contractions \( (T(t))_{t \geq 0} \) on \( \mathcal{X} \) by the Lumer-Phillips theorem and a simple perturbation argument. Hence for every pair of initial values \((u_0, u_1) \in D(A) \times D(A^{1/2})\) the second order problem (4.2) admits a unique classical solution \( u \in C([0, \infty); D(A)) \cap C^1([0, \infty); D(A^{1/2})) \cap C^2([0, \infty); X) \).

Since \( A \) has compact resolvent, the embeddings \( D(A) \hookrightarrow D(A^{1/2}) \hookrightarrow X \) are compact, and boundedness of \( B \) implies that \( A \) too has compact resolvent. The spectrum of \( A \) contains therefore only isolated eigenvalues of finite multiplicity. It was proved in [8] that the spectrum \( \sigma(A) \) is contained in the union of the open strip \( \{ z \in \mathbb{C} : -\frac{1}{2} \| B \| < \text{Re } z < 0 \} \) and the interval \([ -\| B \|, 0 ]\). Moreover, \( \text{Ker } A = \text{Ker } A \times \{ 0 \} \), so that \( 0 \in \sigma(A) \) if and only if \( 0 \in \sigma(A) \). Note that assumption (4.1) plays a crucial role in showing that \( \sigma(A) \cap i\mathbb{R} \subseteq \{ 0 \} \).

If \( 0 \in \sigma(A) \) we let \( P_0 \) be the Riesz projection corresponding to the eigenvalue \( 0 \) (noting that \( P_0 \) need not be orthogonal), and if \( A \) is invertible we set \( P_0 = 0 \). Let \( \mathcal{X}_0 := \{ I - P_0 \} \mathcal{X} \) and \( \mathcal{A}_0 := A|_{\mathcal{X}_0} \).

Then \( \mathcal{A}_0 \) is the generator of the \( C_0 \)-semigroup \((T_0(t))_{t \geq 0}\) of contractions (namely the restriction of the semigroup \((T(t))_{t \geq 0}\) to the \( T \)-invariant subspace \( \mathcal{X}_0 \)) and \( \sigma(\mathcal{A}_0) = \sigma(A) \setminus \{ 0 \} \). In particular, it follows from Theorem 3.4 that the semigroup \((T_0(t))_{t \geq 0}\) is semi-uniformly stable.

One would typically like to determine whether \((T_0(t))_{t \geq 0}\) is exponentially stable or, if not, to obtain uniform decay rates for classical solutions. The decay of \((T_0(t))_{t \geq 0}\) is intimately related to the decay of the physically more relevant energy of solutions \( u \) of the second-order problem (4.2):
\[
E(u, t) := \frac{1}{2} \left( \|A^{1/2}u\|^2 + \|u\|^2 \right). \tag{4.5}
\]
In fact, the decay rate of the energy \( E(u, t) \) for all pairs of initial values \((u_0, u_1) \in D(A) \times D(A^{1/2})\) (no restriction to \( \mathcal{X}_0 \)) is the same as the square of the decay rate of \( \|T_0(t)A_0^{-1}\| \) and, as a result, we may apply the results in Section 3. It is then natural to say that the energy of (4.2) decays semi-uniformly with rate \( r \in C_0(\mathbb{R}_+) \) if \( E(u, t) \leq Cr^t \| A(u_0, u_1) \|^2 \) for all \((u_0, u_1) \in D(A) \times D(A^{1/2})\). Note that \( E(u, t) = O(r(t)^2) \) for some \( r \in C_0(\mathbb{R}_+) \) and for all initial values \((u_0, u_1) \in \mathcal{X}\).

Recall from Section 2 the discussion of circles contained in the spectrum of \( C_0 \)-semigroups. This discussion implies in particular that the well-known alternative – either \( \|T_0(t)\| = 1 \) for all \( t \geq 0 \) or \((T_0(t))_{t \geq 0}\) is exponentially stable – holds here in a much stronger form: either \((T_0(t))_{t \geq 0}\) is exponentially stable or \( \mathcal{T} \subseteq \sigma(T_0(t)) \) for almost every \( t \geq 0 \). Thus, unless \((T_0(t))_{t \geq 0}\) is exponentially stable, its peripheral spectrum (in fact, even the peripheral essential spectrum) is as large as possible for almost all values of \( t \).

In order to link the study of \((T_0(t))_{t \geq 0}\) with the results from the preceding sections, define the quadratic operator pencil \( P(z) := z^2 + zB + A, z \in \mathbb{C} \), with \( D(P(z)) = D(A) \). The next statement in combination with the abstract decay criteria from Section 3 tells us that semi-uniform decay
rates for the energy $E$ can be rewritten as operator norm estimates for $P(z)^{-1}$ with $z \in i\mathbb{R}$; see [8, Lemma 4.6].

**Theorem 4.1.** There exists $C > 1$ such that for all $s \in \mathbb{R}$ with $|s| \geq 1$,
\[
\|R(is, A_0)\| - C|s|^{-1} \leq \|R(is, A)\| \leq \|R(is, A_0)\| + C|s|^{-1}, \quad \text{and}
\]
\[
C^{-1}|s|\|P(is)^{-1}\| \leq \|R(is, A)\| \leq C(1 + |s|)\|P(is)^{-1}\|.
\]

Note that for $s \in \mathbb{R}$ one has $\|P(is)^{-1}\| = \|P(-is)^{-1}\|$ and, similarly, $\|R(is, A_0)\| = \|R(-is, A_0)\|$, so one may restrict attention in the above estimates to large positive values of $s$. Thus Theorem 4.1 reduces the problem of estimating the decay rate of $(T_0(t))_{t \geq 0}$ to the stationary problem of obtaining norm estimates for $P(is)^{-1}$ for large $s > 0$. The latter, however, is a very delicate matter and is intimately related to fine structure of $A$ and $B$. The problem of obtaining estimates for $P(is)^{-1}$ as $s \to \infty$ is far from being fully understood, even for comparatively simple models such as the damped wave equation discussed below.

On the other hand, resolvent estimates for $A$, and hence norm estimates for $P(\cdot)^{-1}$, can often be deduced from control-theoretic properties of the system governed by (4.2) and/or joint spectral properties of $A$ and $B$. The first approach relies, in particular, on the study of various observability properties including Hautus-type tests, the second approach concerns so-called wave-packet conditions. Their advantages become apparent when dealing with “rough” dampings, where there is less to be gained from employing microlocal techniques. Among the papers following this (often rather effective) ideology we mention, for example, [8, 40, 82, 109, 110, 124]. To give a flavour of the relevant results, we formulate several resolvent estimates proved (in a more general context allowing also for unbounded operators $B$) in the recent paper [44]. Similar (and sometimes stronger) results can be found in the papers mentioned above.

Let $A$ be a skew-adjoint operator and $B$ a bounded self-adjoint positive-definite operator on some Hilbert space. The pair $(A, B)$ is said to satisfy the non-uniform Hautus test if there exist increasing functions $p, q : \mathbb{R}^+ \to (0, \infty)$ with $r_0 > 0$ such that
\[
\|x\|^2 \leq p(|s|)\|(is - A)x\|^2 + q(|s|)\|Bx\|^2, \quad x \in D(A), s \in \mathbb{R}. \tag{4.6}
\]

Note that we may express our operator $A$ in the form $A = A - B$ where
\[
A = \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}
\]
are of the requisite type. In this situation, the non-uniform Hautus test (4.6) can be rewritten (without essential loss) in terms of $A$ and $B$. Indeed, by [44, Proposition 3.10], if $\tilde{p}, \tilde{q} : \mathbb{R}^+ \to (0, \infty)$ are increasing functions such that
\[
\|x\|^2 \leq \tilde{p}(s)\|(-A^{1/2})x\|^2 + \tilde{q}(s)\|B^{1/2}x\|^2, \quad x \in D(A^{1/2}), s \geq 0, \tag{4.7}
\]
then $(A, B)$ satisfies (4.6) with $q(s) = 4\tilde{q}(s)$ and $p(s) = C(\tilde{p}(s) + \tilde{q}(s))$ for some $C > 0$. Conversely, if (4.6) holds then (4.7) is satisfied for $\tilde{p}(s) = p(s)$ and $\tilde{q}(s) = q(s)/2$. This allows us to obtain the following resolvent estimate; see [44, Proposition 3.10].

**Theorem 4.2.** If the pair $(A, B)$ satisfies (4.7), then $i\mathbb{R} \subseteq \rho(A)$ and $\|R(is, A)\| \leq C(\tilde{p}(|s|) + \tilde{q}(|s|))$ for all $s \in \mathbb{R}$ and some $C > 0$.

Let $A$ be self-adjoint and let $\delta : \mathbb{R} \to (0, \infty)$ be a bounded function. Define the $(s, \delta(s))$-wave-packet WP$_{s, \delta(s)}(A)$ as the spectral subspace of $A$ associated with the segment $(s - \delta(s), s + \delta(s))$. The next statement provides an estimate for $\|R(is, A)\|$ (which is proved via an intermediate estimate for $\|(-s^2 + isB + A)^{-1}\|$) in terms of joint spectral properties of $A$ and $B$; see [44, Theorem 3.8].
Theorem 4.3. If $\gamma : [0, \infty) \to (0, \infty)$ is a bounded function such that
\[
\|B^{1/2} x\| \geq \gamma(s) \|x\|, \quad x \in WP_{s,\delta(s)}(A^{1/2}),
\]
for all $s \geq 0$, then $iR \subseteq \rho(A)$ and $\|R(is, A)\| \leq C \gamma(|s|)^{-2} \delta(|s|)^{-2}$ for all $s \in \mathbb{R}$ and some $C > 0$.

The wave-packet estimate (4.8) can in turn be used to pass from a slightly different Hautus-type condition for $A$ and $B$ to a resolvent estimate for $A$; see [44, Proposition 3.9].

Theorem 4.4. If $p_S : \mathbb{R}_+ \to (0, \infty)$ and $q_S : \mathbb{R}_+ \to [0, \infty)$ with $r_0 > 0$ and $\eta := \inf_{s \geq 0} p_S(s)(1 + s^2)^2 > 0$ are such that
\[
\|x\|^2 \leq p_S(s)(\|x^2 - A\| + q_S(s)\|B^{1/2} x\|^2), \quad x \in D(A), s \geq 0,
\]
then $iR \subseteq \rho(A)$ and $\|R(is, A)\| \leq C(1 + s^2)p_S(|s|)q_S(|s|)$ for all $s \in \mathbb{R}$ and some $C > 0$.

Indeed, one may prove that if (4.9) holds then the conditions of Theorem 4.3 are satisfied for $\delta(s) = c_0(2p_S(|s|)(1 + s^2))^{-1/2}$ and $\gamma(s) = (2q_S(|s|))^{-1/2}$, with $c_0 = \min(\sqrt{\eta}, 1/2)$.

Yet another “non-uniform” observability condition is given by the estimate
\[
C_\tau\|T_A(t)\|^2 \leq \int_0^\tau \|BT_A(t)\|^2 dt, \quad x \in D(A),
\]
for some fixed non-negative constants $\beta$, $\tau$, and $C_\tau$, where $(T_A(t))_{t \in \mathbb{R}}$ is the unitary $C_0$-group generated by $A$. This condition, which is used widely and often in a context which is similar to ours (see e.g. [58,110]), also leads to resolvent estimates, for example by relating it to wave-packets conditions of the form (4.8). For more on the asymptotic consequences of (4.10), including polynomial (and more general) rates of energy decay for (4.2), we refer to [44, Section 4] and the references therein.

5. Concluding remarks

The aim of this survey has been to present recently developed abstract methods for the study of rates of decay of semigroup orbits. Unfortunately, we are unable in this short text to give a full account of the many exciting applications of semigroup methods and the theory of resolvent estimates. In closing, however, we turn briefly to one particularly rich example, namely the damped wave equation
\[
\begin{align*}
    u_{tt} + b(x)u_t - \Delta u &= 0 \text{ in } (0, \infty) \times M, \quad u = 0 \text{ on } (0, \infty) \times \partial M, \\
    u(0, \cdot) &= u_0 \text{ in } M, \quad u_t(0, \cdot) = u_1 \text{ in } M,
\end{align*}
\]
on a compact, connected, smooth Riemannian manifold $(M, g)$ with (smooth) boundary $\partial M$ (which might be empty). Here $b \in L^\infty(M)$ is a nonnegative function such that $b > 0$ on a subset of $M$ with positive measure, and $\Delta = \Delta_g$ is the Laplace-Beltrami operator on $M$. Of course, the damped wave equation is an example of a partial differential equation which can be rewritten as an abstract second order problem of the form (4.2) in $X = L^2(M)$. In this case $A = -\Delta$ is the (negative) Laplace-Beltrami operator with Dirichlet boundary conditions on $L^2(M)$ and $B = b$ is a multiplication operator. Both are self-adjoint and positive semi-definite. Since $M$ is connected, and by Calderón’s unique continuation principle, non-zero eigenfunctions of $\Delta$ cannot vanish on an open set. Thus, if $b > 0$ almost everywhere on an open set, then $bu \neq 0$ for every non-zero eigenfunction of $\Delta$. It follows that the operator matrix
\[
    A = \begin{pmatrix} 0 & 1 \\ \Delta & -b \end{pmatrix}
\]
on the space $X = H^1_0(M) \times L^2(M)$ (or more precisely, its restriction $A_0$ to an appropriate subspace $X_0$) has no spectrum on the imaginary axis and that $A_0$ generates a semi-uniformly
stable $C_0$-semigroup $(T_0(t))_{t \geq 0}$. Note that $A = A_0$ if and only if the boundary of the manifold $\mathcal{M}$ is nonempty.

For a solution $u$ of the damped wave equation we define the physical energy (the sum of the potential and the kinetic energy) by

$$E(u, t) = \frac{1}{2} \int_{\mathcal{M}} |\nabla u|^2 + \frac{1}{2} \int_{\mathcal{M}} |ux|^2,$$

which corresponds precisely to the energy defined in the previous section. In particular, the decay rate of the energy $E(u, t)$ of classical solutions is the same as the decay rate of $\|T_0(t), A_0^{-1}\|^2$.

By using Carleman estimates (a somewhat quantified version of Calderón’s unique continuation principle) one may show that the resolvent of $A$ grows at most exponentially along the imaginary axis [98, Théorème 1 (i)], and therefore the energy decays at least logarithmically [35] (see also Theorem 3.4). This exponential resolvent bound is optimal in general: take $\mathcal{M}$ a 2-sphere (hence $\mathcal{M}$ has strictly positive curvature) and $b$ a damping localised near a pole; see [98, Théorème 1 (ii)]. Consequently, one cannot in general hope to obtain faster than logarithmic energy decay. Depending on the geometry of the manifold and/or the regularity and the localisation of the support of the damping function $b$ one may, however, obtain better decay rates in some cases.

If $\mathcal{M} = \mathbb{T}^2$ is the 2-torus and $b \in L^\infty(\mathcal{M})$, then the energy decays with the rate $O(t^{-\frac{1}{2}})$, and the decay improves to $O(t^{-(1-\epsilon)})$ for certain $b \in W^{k, \infty}(\mathcal{M})$, no matter how small the support of $b$; see Anantharaman and Léautaud [8, Theorems 2.3 and 2.6]. On the other hand, for simple characteristic functions $b$, the optimal rate of decay is $O(t^{-\frac{3}{2}})$ [8,137], no matter how large the support of $b$. The fact that the regularity of $b$ may play a more important role than its support is also nicely illustrated in [52,87,88,97].

If the manifold $\mathcal{M}$ has strictly negative curvature and no boundary, and if $\gamma$ is a hyperbolic closed geodesic in $\mathcal{M}$ and the damping $b$ vanishes in an $\varepsilon$-neighbourhood of $\gamma$ but is positive everywhere else, then the energy decays semi-uniformly with the rate $O(e^{-\varepsilon\sqrt{t}})$ by [37, Theorem 1]; see [48, Section 5], [129, Corollary A2], [134], [135, Theorem 1] and [80, Theorem 1.1] for related results. The results in [98], [135, Theorem 1] and [80] are particularly interesting since they show that the energy of classical solutions may decay exponentially without the semigroup $(T_0(t))_{t \geq 0}$ being exponentially stable; this renders another counterexample to the equality of the exponential growth bound and the spectral bound.

Nevertheless, exponential stability of the semigroup $(T_0(t))_{t \geq 0}$ does occur, for instance in the simple situations where $\mathcal{M}$ is a compact interval [49] or when $b \geq b_0$ for some constant $b_0 > 0$, that is, the damping is active on the whole manifold. Building on the idea of propagation of singularities (see for instance Hörmander [74]),Ralston [123] (see also [122]) showed that the energy of solutions to wave-type equations may localise along a single generalised geodesic (or, in different terminology, along a single ray of geometric optics). As a result, it turns out to be possible to characterise exponential stability of $(T_0(t))_{t \geq 0}$ in terms of the relationship between the support of $b \in C(\mathcal{M})$ and the generalised geodesics or, more precisely, in terms of the so-called geometric control condition, which very roughly means that every generalised geodesic enters the set $\{b > 0\}$ in finite time; we do not go into details here. Sufficiency of the geometric control condition was proved for smooth dampings and manifolds without boundaries by Rauch and Taylor [125]; see also [123,126]. As a culmination of a series of preceding works [13], [100] and [15], the result was extended in [14] by Bardos, Lebeau and Rauch to the case of manifolds with boundaries in the sense that they formulated a sufficient geometric control condition which is also close to being necessary. Later on the approach of [14] was simplified and the result was generalised to continuous dampings in [38] by means of control-theoretical arguments (in the formally less general setting of smooth domains). Note also that a detailed treatment of geodesic flows from the point of view of fine observability estimates for wave equations may be found in the recent paper [95], which moreover improves on several similar arguments given in [14].

Finally, we mention that rates of decay for the derivatives of the energy in the general setting of (5.1) were obtained by means of the abstract $L^p$-Tauberian Theorem 3.2 in [20, Corollary 6.5].
Many of the results above rely on abstract semigroup theory, often on resolvent bounds. However, we emphasize that in special geometries and for special damping functions it is sometimes possible to derive so-called integral inequalities for the energy. These integral inequalities, which are typically based on sophisticated multiplier methods and clever estimates of various integrals using monotonicity or convexity assumptions, lead directly to decay rate estimates for the associated energies without the passage through the resolvent estimates. There is a vast literature literature on this approach; we restrict ourselves to mentioning just a few illustrative examples, such as J.-L. Lions [101], Komornik [90], Martinez [106,107] and Alabau [2]. This approach has the considerable advantage of being applicable also to non-linear equations. A comparatively recent and comprehensive survey of applications of multiplier methods to hyperbolic equations may be found in [3].

The classical damped wave equation discussed here is, of course, just one (prototypical) example of how abstract semigroup results may be applied to concrete PDEs. We conclude by mentioning a small selection of other interesting applications of semigroup methods to differential equations, along with sample references: damped wave equations on unbounded domains/manifolds [4,39,51,83,85,104,117]; local energy decay for damped wave equations [31,130]; Klein-Gordon and Kelvin-Voigt type equations [7,36,151]; energy decay for non-linear damped wave equations [24,81,82,119]; vectorial damped wave equations [86]; damped wave equations with unbounded and/or indefinite dampings [1,12,25,61,62]; viscoelastic boundary dampings [140]; wave equations with periodic (or even general non-stationary) dampings [83,96]; fractional damped wave equations [65,66]; damped wave equations on manifolds with rough metrics [143]. Even though the subject of damped wave equations is already vast, we hope and expect that the stream of substantial advances in this area, whether obtained by abstract techniques or otherwise, will continue for many years to come.

Acknowledgements. The authors kindly thank the two anonymous referees for their numerous helpful comments and suggestions.

References

1. F. Abdallah, D. Mercier, and S. Nicaise. Spectral analysis and exponential or polynomial stability of some indefinite sign damped problems. *Evol. Equ. Control Theory*, 2(1):1–33, 2013.
2. F. Alabau-Boussouira. Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. *Appl. Math. Optim.*, 51:61–105, 2005.
3. F. Alabau-Boussouira. On some recent advances on stabilization for hyperbolic equations. In *Control of partial differential equations*, volume 2048 of *Lecture Notes in Math.*, pages 1–100. Springer, Heidelberg, 2012.
4. L. Aloui and M. Khenissi. Stabilisation pour l’équation des ondes dans un domaine extérieur. *Rev. Mat. Iberoamericana*, 18(1):1–16, 2002.
5. K. Ammari, M. Dimassi, and M. Zerzeri. The rate at which energy decays in a viscously damped hinged Euler-Bernoulli beam. *J. Differential Equations*, 257(9):3501–3520, 2014.
6. K. Ammari, M. Dimassi, and M. Zerzeri. Rate of decay of some abstract Petrowsky-like dissipative semi-groups. *Semigroup Forum*, 93(1):1–16, 2016.
7. K. Ammari, F. Hassine, and L. Robbiano. Stabilization for the wave equation with singular Kelvin–Voigt damping. *Arch. Ration. Mech. Anal.*, 236(2):577–601, 2020.
8. N. Anantharaman and M. Léautaud. Sharp polynomial decay rates for the damped wave equation on the torus. *Anal. PDE*, 7(1):159–214, 2014. With an appendix by Stéphane Nonnenmacher.
9. W. Arendt and C. J. K. Batty. Tauberian theorems and stability of one-parameter semigroups. *Trans. Amer. Math. Soc.*, 306:837–852, 1988.
10. W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.
11. W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.
12. A. Arifoski and P. Siegl. Pseudospectra of the damped wave equation with unbounded damping. *SIAM J. Math. Anal.*, 52(2):1343–1362, 2020.
13. C. Bardos, G. Lebeau, and J. Rauch. Un exemple d’utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques. *Rend. Sem. Mat. Univ. Politec. Torino*, Special Issue: Nonlinear hyperbolic equations in applied sciences (1988), 11–31, 1989.
14. C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optimization*, 30:1024–1065, 1992.
15. C. Bardos, G. Lebeau, and J. Rauch. Microlocal ideas in control and stabilization. In *Control of boundaries and stabilization (Clermont-Ferrand, 1988)*, volume 125 of *Lect. Notes Control Inf. Sci.*, pages 14–30. Springer, Berlin, 1989.
16. A. Bátkai, K.-J. Engel, J. Prüss, and R. Schnaubelt. Polynomial stability of operator semigroups. *Math. Nachr.*, 279(13-14):1425–1440, 2006.
17. C. J. K. Batty. Tauberian theorems for the Laplace-Stieltjes transform. *Trans. Amer. Math. Soc.*, 322(2):783–804, 1990.
18. C. J. K. Batty and Quôc Phong Vû. Stability of individual elements under one-parameter semigroups. *Trans. Amer. Math. Soc.*, 322:805–818, 1990.
19. C. J. K. Batty. Bounded Laplace transforms, primitives and semigroup orbits. *Arch. Math. (Basel)*, 81(1):72–81, 2003.
20. C. J. K. Batty, A. Borichev, and Y. Tomilov. $L^p$-tauberian theorems and $L^p$-rates for energy decay. *J. Funct. Anal.*, 270(3):1153–1201, 2016.
21. C. J. K. Batty, R. Chill, and Y. Tomilov. Fine scales of decay of operator semigroups. *J. Eur. Math. Soc. (JEMS)*, 18(4):853–929, 2016.
22. C. J. K. Batty and Th. Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Ecol. Equ.*, 8(4):765–780, 2008.
23. C. J. K. Batty, Jin Liang, and Ti-Jun Xiao. On the spectral and growth bound of semigroups associated with hyperbolic equations. *Adv. Math.*, 191(1):1–10, 2005.
24. M. Bellassoued. Decay of solutions of the wave equation with arbitrary localized nonlinear damping. *J. Differential Equations*, 211(2):303–332, 2005.
25. A. Benaddi and B. Rao. Energy decay rate of wave equations with indefinite damping. *J. Differential Equations*, 161(2):337–357, 2000.
26. C. D. Benchimol. A note on weak stabilizability of contraction semigroups. *SIAM J. Control Optim.*, 16(3):373–379, 1978.
27. M. Blake. A spectral bound for asymptotically norm-continuous semigroups. *J. Operator Theory*, 45:111–130, 2001.
28. J.-F. Bony and V. Petkov. Resolvent estimates and local energy decay for hyperbolic equations. *Ann. Univ. Ferrara Sci. VII Sci. Mat.*, 52(2):233–246, 2006.
29. A. Borichev, R. Chill, and Y. Tomilov. Uniqueness theorems for (sub-)harmonic functions with applications to operator theory. *Proc. Lond. Math. Soc. (3)*, 95(3):687–708, 2007.
30. A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. *Math. Ann.*, 347(2):455–478, 2010.
31. I.-M. Bouclet and J. Royer. Local energy decay for the damped wave equation. *J. Funct. Anal.*, 266(7):4538–4615, 2014.
32. K. N. Boyadzhiev and N. Levan. Strong stability of Hilbert space contraction semigroups. *Stud. Sci. Math. Hung.*, 30:165–182, 1995.
33. F. Broucke, G. Debruyne, and J. Vindas. On the absence of remainders in the Wiener-Ikehara and Ingham-Karamata theorems: a constructive approach. Preprint, 2020. https://arxiv.org/abs/2001.01635
34. Nicolas Burq. Contrôlabilité exacte des ondes dans des ouverts peu réguliers. *Asymptot. Anal.*, 14:157–191, 1997.
35. N. Burq. Décroissance de l’énergie locale de l’équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.*, 180(1):1–29, 1998.
36. N. Burq. Decays for Kelvin-Voigt damped wave equations 1: the black box perturbative method. Preprint, 2019. https://arxiv.org/abs/1904.08318
37. N. Burq and H. Christianson. Imperfect geometric control and overdamping for the damped wave equation. *Comm. Math. Phys.*, 336(1):101–136, 2015.
38. N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(7):749–752, 1997.
39. N. Burq and R. Joly. Exponential decay for the damped wave equation in unbounded domains. *Commun. Contemp. Math.*, 18(6):1650012, 27, 2016.
40. N. Burq and M. Zworski. Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, 17(2):443–471 (electronic), 2004.
41. R. Chill. Tauberian theorems for vector-valued Fourier and Laplace transforms. *Studia Math.*, 128:55–69, 1998.
42. R. Chill and Y. Tomilov. Stability of $\mathrm{C}_0$-semigroups and geometry of Banach spaces. *Math. Proc. Cambridge Phil. Soc.*, 135:493–511, 2003.
43. R. Chill and Y. Tomilov. Analytic continuation and stability of operator semigroups. *J. Analyse Math.*, 93:331–358, 2004.
44. R. Chill, L. Paunonen, D. Seifert, R. Stahn, and Y. Tomilov. Non-uniform stability of damped contraction semigroups. Preprint, 2019. [arxiv.org/abs/1911.04804](https://arxiv.org/abs/1911.04804)
45. R. Chill and D. Seifert. Quantified versions of Ingham’s theorem. *Bull. Lond. Math. Soc.*, 48(3):519–532, 2016.
46. R. Chill and Y. Tomilov. Stability of operator semigroups: ideas and results. In *Perspectives in operator theory*, vol. 75 of *Banach Center Publ.*, pages 71–109. Polish Acad. Sci., Warsaw, 2007.
47. H. Christianson. Applications of cutoff resolvent estimates to the wave equation. *Math. Res. Lett.*, 16(4):577–590, 2009.
48. H. Christianson, E. Schenck, A. Vasy, and J. Wunsch. From resolvent estimates to damped waves. *J. Differential Equations*, 264(12):7023–7054, 2018.
49. S. Cox and E. Zuazua. The rate at which energy decays in a damped string. *Comm. Partial Differential Equations*, 19(1-2):213–243, 1994.
50. D. Cramer and Y. Latushkin. Gearhart-Prüss theorem in stability for wave equations: a survey. In *Evolution equations*, volume 234 of *Lecture Notes in Pure and Appl. Math.*, pages 105–119. Dekker, New York, 2003.
51. M. Daoulatli. Energy decay rates for solutions of the wave equation with linear damping in exterior domain. *Evol. Equ. Control Theory*, 5(1):37–59, 2016.
52. K. Datchev and P. Kleinhenz. Sharp polynomial decay rates for the damped wave equation with Hölder-like damping. *Proc. Amer. Math. Soc.*, 2020. [https://doi.org/10.1090/proc/15018](https://doi.org/10.1090/proc/15018).
53. G. Debruyne and D. Seifert. An abstract approach to optimal decay of functions and operator semigroups. *Israel J. Math.*, 233(1):439–451, 2019.
54. G. Debruyne and D. Seifert. Optimality of the quantified Ingham-Karamata theorem for operator semigroups with general resolvent growth. *Arch. Math. (Basel)*, 113(6):617–627, 2019.
55. G. Debruyne and J. Vindas. Note on the absence of remainders in the Wiener-Ikehara theorem. *Proc. Amer. Math. Soc.*, 146(12):5097–5103, 2018.
56. R. Donninger and B. Schörkhuber. Stable blow up dynamics for energy supercritical wave equations. *Trans. Amer. Math. Soc.*, 366(4):2167–2189, 2014.
57. R. Donninger and B. Schörkhuber. A spectral mapping theorem for perturbed Ornstein-Uhlenbeck operators on $L^2(\mathbb{R}^d)$. *J. Funct. Anal.*, 268(9):2479–2524, 2015.
58. Th. Duyckaerts. Optimal decay rates of the energy of a hyperbolic-parabolic system coupled by an interface. *Asymptot. Anal.*, 51(1):17–45, 2007.
59. K. J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer Verlag, Heidelberg, Berlin, New-York, 1999.
60. C. Foiaş. Sur une question de M. Reghiş. *An. Univ. Timişoara Ser. Şti. Mat.*, 11:111–114, 1973.
61. P. Freitas, P. Siegl, and Ch. Tretter. The damped wave equation with unbounded damping. *J. Differential Equations*, 264(12):7023–7054, 2018.
62. P. Freitas and E. Zuazua. Stability results for the wave equation with indefinite damping. *J. Differential Equations*, 132(2):338–352, 1996.
63. F. Gesztesy, C. K. R. T. Jones, Y. Latushkin, and M. Stanislawova. A spectral mapping theorem and invariant manifolds for nonlinear Schrödinger equations. *Indiana Univ. Math. J.*, 49(1):221–243, 2000.
64. J. Glück. Spectral and asymptotic properties of contractive semigroups on non-Hilbert spaces. *J. Operator Theory*, 76(1):3–31, 2016.
65. W. Green. On the energy decay rate of the fractional wave equation on $\mathbb{R}$ with relatively dense damping. Preprint, 2019. [https://arxiv.org/abs/1904.1094](https://arxiv.org/abs/1904.1094)
66. W. Green, B. Jaye, and M. Mitkovski. Uncertainty principles associated to sets satisfying the geometric control condition. Preprint, 2019. [https://arxiv.org/abs/1912.05077](https://arxiv.org/abs/1912.05077)
67. M. P. Gualdani, S. Mischler, and C. Mouhot. Factorization of non-symmetric operators and exponential $H$-theorem. *Mém. Soc. Math. Fr. (N.S.)*, (153):137, 2017.
68. J. Hejtmanek and H. G. Kaper. Counterexample to the spectral mapping theorem for the exponential function. *Proc. Amer. Math. Soc.*, 96(4):563–568, 1986.
69. B. Helffer and J. Sjöstrand. From resolvent bounds to semigroup bounds. Actes du colloque d’Evian, 2009. https://arxiv.org/abs/1001.4171v1
70. D. B. Henry. Topics in analysis. *Publ. Soc. Mat. Univ. Autónoma Barcelona*, 31(1):29–84, 1987.
71. I. Herbst. The spectrum of Hilbert space semigroups. *J. Operator Theory*, 10(1):87–94, 1983.
72. I. W. Herbst. Contraction semigroups and the spectrum of $A_1 \otimes I + I \otimes A_2$. *J. Operator Theory*, 7(1):61–78, 1982.
73. E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957.
74. L. Hörmander. On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseign. Math. (2)*, 17:99–163, 1971.
75. J. S. Howland. On a theorem of Gearhart. *Int. Equ. Operator Theory*, 7(1):138–142, 1984.
76. F. L. Huang. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. *Ann. Differential Equations*, 1(1):43–56, 1985.
77. S. Ibrahim, Y. Maekawa, and N. Masmoudi. On pseudospectral bound for non-selfadjoint operators and its application to stability of Kolmogorov flows. *Ann. PDE*, 5(2):Paper No. 14, 84, 2019.
78. A. E. Ingham. On Wiener’s method in Tauberian theorems. *Proc. London Math. Soc.*, 38:458–480, 1935.
79. N. Iwasaki. Local decay of solutions for symmetric hyperbolic systems with dissipative and coercive boundary conditions in exterior domains. *Publ. Res. Inst. Math. Sci.*, 5:193–218, 1969.
80. L. Jin. Damped wave equations on compact hyperbolic surfaces. *Comm. Math. Phys.*, 373(3):771–794, 2020.
81. R. Joly and C. Laurent. Stabilization for the semilinear wave equation with geometric control condition. *Anal. PDE*, 6(5):1089–1119, 2013.
82. R. Joly and C. Laurent. Decay of semilinear damped wave equations: cases without geometric control condition. Annales Henri Lebesgue, to appear, 2020.
83. R. Joly and J. Royer. Energy decay and diffusion phenomenon for the asymptotically periodic damped wave equation. *J. Math. Soc. Japan*, 70(4):1375–1418, 2018.
84. J. Karamata. Weiterführung der N. Wienerschen Methode. *Math. Z.*, 38(1):701–708, 1934.
85. M. Khenissi. Équation des ondes amorties dans un domaine extérieur. *Bull. Soc. Math. France*, 131(2):211–228, 2003.
86. G. Klein. Best exponential decay rate of energy for the vectorial damped wave equation. *SIAM J. Control Optim.*., 56(5):3432–3453, 2018.
87. P. Kleinhenz. Decay rates for the damped wave equation with finite regularity damping. Preprint, 2019. https://arxiv.org/abs/1910.06372
88. P. Kleinhenz. Stabilization rates for the damped wave equation with Hölder-regular damping. *Comm. Math. Phys.*, 369(3):1187–1205, 2019.
89. H. Koch and D. Tataru. On the spectrum of hyperbolic semigroups. *Comm. Partial Differential Equations*, 20(5-6):901–937, 1995.
90. V. Komornik. *Exact controllability and stabilization*. RAM: Research in Applied Mathematics. Masson, Paris, 1994. The multiplier method.
91. J. Korevaar. *Tauberian theory*, volume 329 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2004. A century of developments.
92. V. I. Korobov and G. M. Sklyar. On the question of the strong stabilizability of contracting systems in Hilbert space. *Differentsial’nye Uravneniya*, 20(11):1862–1869, 1984.
93. Y. Latushkin and F. Räbiger. Operator valued Fourier multipliers and stability of strongly continuous semigroups. *Integral Equations Operator Theory*, 51(3):375–394, 2005.
94. Y. Latushkin and V. Yurov. Stability estimates for semigroups on Banach spaces. *Discrete Contin. Dyn. Syst.*, 33(11-12):5203–5216, 2013.
95. C. Laurent and M. Léautaud. Uniform observability estimates for linear waves. *ESAIM Control Optim. Calc. Var.*, 22(4):1097–1136, 2016.
96. J. Le Rousseau, G. Lebeau, P. Terpolilli, and E. Trélat. Geometric control condition for the wave equation with a time-dependent observation domain. *Anal. PDE*, 10(4):983–1015, 2017.
97. M. Léautaud and N. Lerner. Energy decay for a locally undamped wave equation. *Ann. Fac. Sci. Toulouse Math. (6)*, 26(1):157–205, 2017.
98. G. Lebeau. Équation des ondes amorties. In Algebraic and geometric methods in mathematical physics (Kaciveli, 1993), volume 19 of Math. Phys. Stud., pages 73–109. Kluwer Acad. Publ., Dordrecht, 1996.

99. N. Levan. On some relationships between the LaSalle invariance principle and the Nagy-Foiaş canonical decomposition. J. Math. Anal. Appl., 77(2):493–504, 1980.

100. J.-L. Lions. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, volume 8 of Recherches en Mathématiques Appliquées. Masson, Paris, 1988. Contrôlabilité exacte. With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

101. J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. SIAM Rev., 30(1):1–68, 1988.

102. Zh. Liu and B. Rao. Characterization of polynomial decay rate for the solution of linear evolution equation. Z. Angew. Math. Phys., 56(4):630–644, 2005.

103. Zh.-H. Luo, B.-Zh. Guo, and O. Morgul. Stability and stabilization of infinite dimensional systems with applications. Communications and Control Engineering Series. Springer-Verlag London, Ltd., London, 1999.

104. M. Malloug and J. Royer. Energy decay in a wave guide with dissipation at infinity. ESAIM Control Optim. Calc. Var., 24(2):519–549, 2018.

105. M. M. Martinez. Decay estimates of functions through singular extensions of vector-valued Laplace transforms. J. Math. Anal. Appl., 375(1):196–206, 2011.

106. P. Martinez. A new method to obtain decay rate estimates for dissipative systems. ESAIM Control Optim. Calc. Var., 4:419–444, 1999.

107. P. Martinez. A new method to obtain decay rate estimates for dissipative systems with localized damping. Rev. Mat. Complut., 12(1):251–283, 1999.

108. E. Ménard and J.-P. Troallic. Semigroupes de contractions linéaires sur C(X): irréductibilité et théorèmes de convergence. Semigroup Forum, 47(1):87–95, 1993.

109. L. Miller. Controllability cost of conservative systems: resolvent condition and transmutation. J. Funct. Anal., 218(2):425–444, 2005.

110. L. Miller. Resolvent conditions for the control of unitary groups and their approximations. J. Spectr. Theory, 2(1):1–55, 2012.

111. A. I. Miloslavski˘ı. Stability of certain classes of evolution equations. Sibirsk. Mat. Zh., 26(5):118–132, 206, 1985.

112. L. Monauni. Linear dynamical systems with abstract state-spaces. PhD thesis, MIT, Cambridge, MA, 1978.

113. L. Monauni. On the abstract Cauchy problem and the generation problem for semigroups of bounded operators. Control Theory Centre Report No. 90, Warwick, 1980.

114. V. Müller and Y. Tomilov. “Large” weak orbits of C0-semigroups. Acta Sci. Math. (Szeged), 79(3-4):475–505, 2013.

115. J. M. A. M. van Neerven. The Asymptotic Behaviour of Semigroups of Linear Operators. Operator Theory: Advances and Applications 88. Birkhäuser Verlag, 1996.

116. N. Nikolski. Estimates of the spectral radius and the semigroup growth bound in terms of the resolvent and weak asymptotics. Algebra i Analiz, 14(4):141–157, 2002.

117. J. Prüss. Remarks on the asymptotic behavior of the solution to damped wave equations. J. Differential Equations, 261(7):3893–3940, 2016.

118. L. Paunonen. Robustness of strongly and polynomially stable semigroups. J. Funct. Anal., 263(9):2555–2583, 2012.

119. K. D. Phung. Decay of solutions of the wave equation with localized nonlinear damping and trapped rays. Math. Control Relat. Fields, 1(2):251–265, 2011.

120. J. Prüss. On the spectrum of C0-semigroups. Trans. Amer. Math. Soc., 284:847–857, 1984.

121. Jan Prüss. Perturbations of exponential dichotomies for hyperbolic evolution equations. In Operator semigroups meet complex analysis, harmonic analysis and mathematical physics, volume 250 of Oper. Theory Adv. Appl., pages 453–461. Birkhäuser/Springer, Cham, 2015.

122. J. Prüss. Gaussian beams and the propagation of singularities. In Studies in partial differential equations, volume 23 of MAA Stud. Math., pages 206–248. Math. Assoc. America, Washington, DC, 1982.

123. J. V. Ralston. Solutions of the wave equation with localized energy. Comm. Pure Appl. Math., 22:807–823, 1969.

124. K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak. A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator. J. Funct. Anal., 226(1):193–229, 2005.
125. J. Rauch and M. Taylor. Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana Univ. Math. J.*, 24:79–86, 1974.

126. J. Rauch and M. Taylor. Decay of solutions to nondissipative hyperbolic systems on compact manifolds. *Comm. Pure Appl. Math.*, 28(4):501–523, 1975.

127. M. Renardy. On the type of certain $C_0$-semigroups. *Comm. Partial Differential Equations*, 18:1299–1307, 1993.

128. M. Riesz. Über die Summierbarkeit durch typische Mittel. *Acta Litt. Sci. Szeged*, 2:18–31, 1924.

129. G. Rivière. Eigenmodes of the damped wave equation and small hyperbolic subsets. *Ann. Inst. Fourier (Grenoble)*, 64(3):1229–1267, 2014. With an appendix by Stéphane Nonnenmacher and Rivière.

130. J. Royer. Local decay for the damped wave equation in the energy space. *J. Inst. Math. Jussieu*, 17(3):509–540, 2018.

131. J. Rozendaal, D. Seifert, and R. Stahn. Optimal rates of decay for operator semigroups on Hilbert spaces. *Adv. Math.*, 346:359–388, 2019.

132. J. Rozendaal and M. Veraar. Sharp growth rates for semigroups using resolvent bounds. *J. Evol. Equ.*, 18(4):1721–1744, 2018.

133. J. Rozendaal and M. Veraar. Stability theory for semigroups using $(L^p, L^q)$ Fourier multipliers. *J. Funct. Anal.*, 275(10):2845–2894, 2018.

134. E. Schenck. Energy decay for the damped wave equation under a pressure condition. *Comm. Math. Phys.*, 300(2):375–410, 2010.

135. E. Schenck. Exponential stabilization without geometric control. *Math. Res. Lett.*, 18(2):379–388, 2011.

136. D. Seifert. A Katznelson-Tzafriri theorem for measures. *Integral Eq. Operator Theory*, 81(2):255–270, 2015.

137. R. Stahn. Optimal decay rate for the wave equation on a square with constant damping on a strip. *Z. Angew. Math. Phys.*, 68(2):Art. 36, 10, 2017.

138. R. Stahn. A quantified Tauberian theorem and local decay of $C_0$-semigroups. Preprint, 2017. https://arxiv.org/abs/1705.03641

139. R. Stahn. Decay of $C_0$-semigroups and local decay of waves on even (and odd) dimensional exterior domains. *J. Evol. Equ.*, 18(4):1633–1674, 2018.

140. R. Stahn. On the decay rate for the wave equation with viscoelastic boundary damping. *J. Differential Equations*, 265(6):2793–2824, 2018.

141. K. V. Storozhuk. Obstructions to the uniform stability of a $C_0$-semigroup. *Sibirsk. Mat. Zh.*, 51(2):410–419, 2010.

142. P. Takac. Two counterexamples to the spectral mapping theorem for semigroups of positive operators. *Integral Eq. Operator Theory*, 9:460–467, 1986.

143. M. Taylor. Wave decay on manifolds with bounded Ricci tensor, and related estimates. *J. Geom. Anal.*, 25(2):1018–1044, 2015.

144. Y. Tomilov. A resolvent approach to stability of operator semigroups. *J. Operator Theory*, 46:63–98, 2001.

145. V. ˜U. Quôc Phóng. Theorems of Katznelson-Tzafriri type for semigroups of operators. *J. Funct. Anal.*, 103(1):74–84, 1992.

146. D. Wei. Diffusion and mixing in fluid flow via the resolvent estimate. Sci. China Math., 2019. https://doi.org/10.1007/s11425-018-9461-8

147. L. Weis and V. Wrobel. Asymptotic behaviour of $C_0$-semigroups in Banach spaces. *Proc. Am. Math. Soc.*, 124:3663–3671, 1996.

148. L. Weis. The stability of positive semigroups on $L_p$ spaces. *Proc. Amer. Math. Soc.*, 123(10):3089–3094, 1995.

149. G. Weiss. Weak $L^p$-stability of a linear semigroup on a Hilbert space implies exponential stability. *J. Differential Equations*, 76(2):269–285, 1988.

150. V. Wrobel. Stability and spectra of $C_0$-semigroups. *Math. Ann.*, 285:201–219, 1989.

151. J. Wunsch. Periodic damping gives polynomial energy decay. *Math. Res. Lett.*, 24(2):571–580, 2017.

152. J. Zabczyk. A note on $C_0$-semigroups. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 23(8):895–898, 1975.