ISOMINKOWSKIAN UNIFICATION OF THE SPECIAL AND GENERAL RELATIVITIES

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Abstract

We submit a classical unification of the special and general relativities via the new isominkowskian geometry in which the two relativities are differentiated by the basic unit. We then show that the unification admits an operator image in which gravitation verifies the abstract axioms of relativistic quantum mechanics under a universal symmetry which is isomorphic to the Poincaré symmetry. The compliance of the unification with available experimental data is indicated. This study has been permitted by the recent achievement of sufficient mathematical maturity in memoir3f, axiomatic consistency in memoir3t and generalized symmetry principles in memoir4v. More detailed studies are presented in the forthcoming paper3u.

1. Introduction. One of the most majestic achievements of this century for mathematical beauty, axiomatic consistency and experimental verifications has been the special theory of relativity (STR)\(^1\). By comparison, despite equally outstanding achievements, the general theory of relativity (GTR)\(^2\) has remained afflicted by numerous problematic aspects at both classical and quantum levels.

In view of the above, in this note we submit a formulation of the general relativity via the axioms of the special, under the condition of preserving Einstein’s field equations and their experimental verifications. Our study is conducted via axiom-preserving maps of conventional structures, called isotopies, initiated in ref.s\(^3\) and studied by various authors\(^4\)\(^5\).

This study has been permitted by the recent achievement of sufficient mathematical maturity in memoir3f, axiomatic consistency in memoir3t and maturity in generalized symmetry principles in memoir4v. More detailed studies are presented in the forthcoming paper3u.
The property at the foundation of this note is that the transition from the Minkowskian metric $\eta = \text{Diag}(1, 1, 1, -1)$ to a (3+1)-dimensional Riemannian metric $g(x)$ is characterized by a noncanonical transformation $x \to x' = U \times x, U \times U^t \neq I$, for which (by ignoring hereon the dash) $g(x) = U \times \eta \times U^t$.

The above mathematically trivial occurrence has rather serious physical implications at both classical and quantum levels. In fact, at the classical level it implies that gravitational theories constructed over a Riemannian spaces do not possess invariant units of space and time (evidently because they are not preserved by the noncanonical time evolution of the theory by definition), thus implying evident ambiguities in the application of the theory to actual measurements (e.g., because it is not possible to conduct reliable measures, say, of length, with a stationary meter varying in time) and other problematic aspects.

More generally, the noncanonical character of Riemannian theories of gravitation implies that the still unresolved problematic aspects debated during this century on gravitation, quite likely, are not due to Einstein’s field equations, but rather to the lack of proper mathematics used in their treatment, which is the viewpoint adopted in this note.

At the operator level it is easy to see that, for consistency with the classical counterpart, quantum gravity must have a nonunitary time evolution when referred to a conventional Hilbert space over a conventional field. It is now established that theories with nonunitary time evolutions are afflicted by the following shortcomings: 1) They do not have invariant units of space and time, by therefore lacking physically unambiguous applications to experimental measurements; 2) they do not preserve Hermiticity in time, thus lacking physically acceptable observables; and 3) they do not have unique and invariant numerical predictions (because of the lack of uniqueness and invariance of the special functions needed for data elaboration).

In an attempt to resolve the above problematic aspects in due time, in this note we submit a novel formulation of gravity based on the following central assumptions: 1) Preservation unchanged of Einstein’s field equations and related experimental verifications; 2) classical formulation of these equations under the uncompromisable conditions of possessing invariant units of space and time, as it is the case for the special relativity; and 3) operator formulation of gravity based on the axioms of conventional relativistic quantum mechanics.

2. **Isominkowskian geometry.** A study of the above conditions is permitted by a novel form of mathematics called *isomathematics* originally proposed by Santilli and then studies in various works. It is characterized by liftings of conventional mathematics called *isotopies* which map any linear local and canonical structure into its most general possible nonlinear, nonlocal and noncanonical form, which is nevertheless capable of reconstructing linearity, locality and canonicity on certain generalized spaces and fields called *isospaces* and *isofields*.

The isotopies are ideally suited to study assumptions 1), 2) and 3) of Sect. 1 because
they are axiom-preserving by conception and construction, while the new formulations are locally isomorphic to the original ones. This evidently ensures the preservation unchanged of Einstein’s axiom *ab initio*. Jointly, the isotopies do permit the achievement of a theory with a basic invariant unit, as we shall see.

The fundamental isotopy for relativistic theories is the lifting of the unit of conventional theories, the unit \( I = \text{diag.} \ (1, 1, 1, 1) \) of the Minkowski space and of the Poincaré’ symmetry, into a well behaved, nowhere singular, Hermitean and positive-definite \( 4 \times 4 \)-dimensional matrix \( \hat{I} \) whose elements have an arbitrary dependence on local quantities and, therefore, can depend on the space-time coordinates \( x \) and other needed variables, \( I \to \hat{I} = \hat{I}(x) > 0 \), while the conventional associative product \( A \times B \) among generic quantities \( A, B \) is lifted by the inverse amount, \( A \times B \to \hat{A} \times \hat{B} = A \times \hat{T} \times B, \hat{I} = \hat{T}^{-1} \).

Under these assumptions \( \hat{I} \) is the (left and right) generalized unit of the new theory, \( \hat{I} \times A = \hat{A} \equiv A, \forall A \), called the *isounit* and \( \hat{T} \) is called the *isotopic element*. For consistency, the totality of the original theory must then be reconstructed to admit \( \hat{I} \) as the correct (left and right) unit. This implies the isotopies of numbers, angles, fields, spaces, differential calculus, functional analysis, geometries, algebras, symmetries, etc.\(^3\) (see ref.\(^3\) for a recent account).

Let \( M(x, \eta, R) \) be the Minkowski space with space–time coordinates \( x = \{ x^\mu \} = \{ r, x^4 \} \), \( x^4 = c_0 t \) (where \( c_0 \) is the speed of light in vacuum), and metric \( \eta = \text{Diag.}(1, 1, 1, -1) \) over the reals \( R = R(n, +, \times) \). Let \( \Re(x, g(x), R) \) be a \((3+1)\)-dimensional Riemannian space with nowhere singular and symmetric metric \( g = g^i = U \times \eta \times U^t \).

A study of conditions 1), 2), and 3) of Sect. 1 is then permitted by assuming as basic isounit of our theory the quantity \( \hat{I} = U \times 1 \times U^t = \hat{I} > 0 \) with explicit form derivable form a Riemannian metric via the *isominkowskian factorization* \(^3\footnote{ref.}^{3p}\)

\[
g(x) = \hat{T}(x) \times \eta, \hat{I}(x) = [T(x)]^{-1} = U \times U^t. \tag{1}\]

As an example, for the case of the celebrated Schwarzschild’s metric\(^2\), we have \( U \times U^t = \hat{I} = \text{Diag.}((1 - M/r), (1 - M/r), (1 - M/r), (1 - M/r)^{-1}) \) and similarly for other metrics. Note that the positive-definiteness of \( \hat{I} \) is assured by the locally Minkowskian character of Riemann. Without loss of generality, the isounit can therefore be assumed herein as being diagonal.

To construct the *isotopies of the STR*, also called *isospecial relativity*\(^3h–3t\), we first need the lifting of numbers and fields\(^3p\). For this we note that the conventional multiplicative unit \( I \) is lifted into the isounit, \( I \to U \times 1 \times U^t = \hat{I} \) while the additive unit \( 0 \) remains unchanged, \( 0 \to 0 = U \times 0 \times U^t = 0 \). The numbers are lifted into the so-called *isenumsbers*, \( n \to \hat{n} = U \times n \times U^t = n \times \hat{I} = (n \times m) \times \hat{I} \) with lifting of the product \( n \times m \to \hat{n} \times \hat{m} = \hat{n} \times \hat{T} \times \hat{m} \).
The original field \( R = R(n, +, \times) \) is then lifted into the isofield \( \hat{R} = \hat{R}(\hat{n}, +, \hat{\times}) \) for which all operations are isotopic. It is easy to see that \( \hat{R} \) is locally isomorphic to \( R \) by construction and, thus, the lifting \( R \to \hat{R} \) is an isotopy. Despite its simplicity, the lifting is not trivial, e.g., because the notion of primes and other properties of number theory depend on the assumed unit. For further aspects we refer to Ref. 5 which also includes the isotopies of angles and functions analysis. Note for later needs the identity, \( \hat{n} \hat{\times} A \equiv n \times A \).

Next, we need the lifting of the space \( M \) into the isominkowskian space \( \hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{R}) \) first proposed by Santilli in Ref. 3h which is characterized by the isocoordinates \( x \to \hat{x} = U \times x \times U^t = x \times \hat{I} \), and isometric \( \eta \to \hat{\eta}(x) = U \times \eta \times U^t \equiv g(x) \) although, for consistency, the latter must be defined on \( \hat{R} \), thus having the structure \( \hat{N} = (\hat{N}_{\mu \nu}) = \hat{\eta} \times \hat{I} = (\hat{\eta}_{\mu \nu}) \times \hat{I} \).

The conventional interval on \( M \) is then lifted into the isointerval on \( \hat{M} \) over \( \hat{R} \).

\[
\begin{align*}
(\hat{x} - \hat{y})^2 &= (\hat{x} - \hat{y})^\mu \hat{x} N_{\mu \nu}(\hat{x} - \hat{y})^\nu
= [(x - y)^\mu \times \hat{\eta}_{\mu \nu} \times (x - y)^\nu] \times \hat{I} = \\
&= [(x^1 - y^1) \times \hat{T}_{11} \times (x^1 - y^1) + (x^2 - y^2) \times \hat{T}_{22} \times (x^2 - y^2) +
+ (x^3 - y^3) \times \hat{T}_{33} \times (x^3 - y^3) - (x^4 - y^4) \times \hat{T}_{44} \times (x^4 - y^4) \times \hat{I}].
\end{align*}
\]

As one can see, the above interval coincides with the conventional Riemannian interval byu conception, except for the factor \( \hat{I} \).

It easy to see that \( \hat{M} \) is locally isomorphic to \( M \) and the lifting \( M \to \hat{M} \) is also an isotopy. In particular, the isospace \( \hat{M} \) is isoflat, i.e., it verifies the axiom of flatness in isospace over the isofields, that is, when referred to the generalized unit \( \hat{I} \), otherwise \( \hat{M} \) is evidently curved owing to the dependence \( \hat{\eta} = \hat{\eta}(x) = g(x) \). In other words, assumptions (1) eliminate the curvature while preserving the Riemannian metric. As we shall see, this appears to be essential to achieve a gravitational theory with an invariant basic unit. Note that \( \hat{M} \) and \( \hat{R} \) have the same isounit \( \hat{I} \). Studies of isocontinuity properties on isospaces have been conducted by Kadeisvili\(^{4r} \) and those of the underlying novel isotopology by Tsagas and Sourlas\(^{4s} \).

The isominkowskian geometry\(^{4r} \) is the geometry of isospaces \( \hat{M} \), and incorporates in a symbiotic way both the Minkowskian and Riemannian geometries. In fact, it preserves all geometric properties of the conventional Minkowskian geometry, including the light cone and the maximal causal speed \( c_o \) (see below), while jointly incorporating the machinery of the Riemannian geometry in an isotopic form. As such, it is ideally suited for our objectives.

It should be indicated that this author has studied until the interior gravitational problem via the isotopies of the Riemannian geometry. The use of the isominkowskian geometry for
the characterization of the exterior gravitational problem was briefly indicated in note\textsuperscript{30} and it is studied in more details in this work. Also, the main line of the isominkowskian geometry inclusive of the machinery of the Riemannian geometry are presented in this study for the first time.

To outline the new geometry, one must know that, unexpectedly, the use of the ordinary differential calculus leads to inconsistencies under isotopies (e.g., lack of invariance) because dependent on the assumption of the trivial unit 1 in a hidden way. The central tool of the isominkowskian geometry is therefore the isodifferential calculus on $\dot{M}(\dot{x}, \dot{v}, \dot{R})$, first introduced in\textsuperscript{39}, which is characterized by the isodifferentials, isoderivatives and related properties $\hat{I}_\nu^{\mu} \times dx^\nu, \hat{d}_\mu = \partial/\partial x^\mu = \hat{T}_\mu^\nu \times \partial/\partial x^\nu, \hat{\vartheta}^\mu = \hat{\partial}/\hat{\partial}x_\mu = \hat{I}_\nu^{\mu} \times \partial/\partial x_\nu, \hat{\partial}x^\mu/\partial x^\nu = \delta^\nu_\mu, \hat{\partial}x_\mu/\partial x^\nu = \hat{\eta}_\mu^{\alpha} \hat{\partial}x^{\alpha}/\partial x^\nu = \hat{\eta}_\nu^{\alpha} \hat{\partial}x^\alpha = \hat{\eta}_\mu^{\alpha} \times \hat{\partial}x_\alpha/\hat{\partial}x^\nu = \hat{\eta}^{\mu\nu}$.

Note that the original axioms must be preserved for an isotopy. Thus, $\hat{\partial}_\nu \hat{\partial}_\beta = \hat{\partial}_\beta \hat{\partial}_\alpha$ and, therefore, $\hat{\partial}_\alpha \hat{\partial}_\beta = T_{\alpha}^{\mu} \times T_{\beta}^{\nu} \times \partial_\mu \partial_\nu$.

Note also the hidden isoquotient\textsuperscript{30} $A/B = (A/B) \times \dot{I}$ and isoproduct $\hat{\partial} \times \hat{\partial}$. Thus, by including the isoquotient, the quantity $\hat{\partial} \hat{\partial}$ should be more rigorously written $\hat{\partial} \times \hat{\partial}$. This results in an inessential final multiplication of the expression considered -by $\dot{I}$ and, as such, it will be ignored hereon for simplicity.

The entire formalism of the Riemannian geometry can then be formulated on the isominkowskian space via the isodifferential calculus. This aspect is studied in details elsewhere\textsuperscript{31}. We here mention: isochristoffel’s symbols $\hat{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2} \times (\hat{\partial}_\alpha \hat{\eta}_{\beta\gamma} + \hat{\partial}_\gamma \hat{\eta}_{\alpha\beta} - \hat{\partial}_\beta \hat{\eta}_{\alpha\gamma}) \times \dot{I}$, isocovariant differential $\hat{\dot{D}} \hat{X}^\beta = \hat{d} \hat{X}^\beta + \hat{\Gamma}^{\beta}_\alpha \hat{X}^\alpha \times \hat{d} \hat{x}^\gamma$, isocovariant derivative $\hat{\dot{X}}^\beta_{\mu} = \hat{T}_{\mu}^\nu \hat{X}^\beta + \hat{\Gamma}^\beta_\alpha \hat{X}^\alpha \times \hat{\partial}_\mu \hat{X}^\gamma$, isocurvature tensor $\hat{\dot{R}}^{\beta}_{\alpha\gamma\delta} = \hat{\partial}_\beta \hat{\Gamma}^{\beta}_{\alpha\gamma} - \hat{\partial}_\gamma \hat{\Gamma}^{\beta}_{\alpha\delta} + \hat{\Gamma}^{\beta}_{\rho\gamma} \hat{x}^{\rho} \hat{x}^{\alpha} + \hat{\Gamma}^{\beta}_{\rho\delta} \hat{x}^{\rho} \hat{x}^{\alpha}$, etc.

The verification, this time, of the Riemannian properties is shown by the fact that (under the assumed conditions) the isocovariant derivatives of all isometrics on $\dot{M}$ over $\dot{R}$ are identically null, $\hat{\eta}_{\alpha\beta\gamma} \equiv 0, \alpha, \beta, \gamma = 1, 2, 3, 4$. This illustrates that the Ricci Lemma also holds under the Minkowskian axioms.

A similar occurrence holds for all other axioms of the Riemannian geometry (including the forgotten Freud identity, as we shall study in more detail elsewhere\textsuperscript{31}).

3. Classical unification of the special and general relativities. We are now equipped to present, apparently for the first time, the classical equations of our isominkowskian formulation of gravity, here called isoeinstein equations on $\dot{M}$ over $\dot{R}$, which can then be written

$$\hat{G}_{\mu\nu} = \hat{\dot{R}}_{\mu\nu} - \frac{1}{2} \times \hat{N}_{\mu\nu} \times \dot{R} = \hat{k} \times \hat{\tau}_{\mu\nu},$$

(3)
where $\hat{\tau}_{\mu\nu}$ is the source isotensor on $\hat{M}$, $\frac{1}{2} = \frac{1}{2} \times \hat{I}$, $\hat{N}_{\mu\nu} = \hat{\tau}_{\mu\nu} \times \hat{I}$, $\hat{I}, \hat{k} = k \times \hat{I}$ and $k$ is the usual constant.

Despite apparent differences, it should be indicated that Eq.s (3) coincide numerically with Einstein’s equations both in isospace as well as in their projection in ordinary spaces for all diagonal Riemannian metrics. In fact, in which case $\hat{\tau}$ is also diagonal with $\hat{\eta} \equiv g(x)$.

In isospace, the isoderivative $\hat{\partial}_{\mu} = \hat{T}_{\mu}^\alpha \partial_{\alpha}$ deviates from the conventional derivative $\partial_{\mu}$ by the isotopic factor $\hat{I}$. But its numerical value must be referred to $\hat{I} = \hat{T}^{-1}$, rather than $I$. This implies the preservation in isospace of the original value of $\partial_{\mu}$ and, consequently, of the original field equations.

For the case of the projection of in ordinary spaces, the isequations are reducible to the conventional equations multiplied by common isotopic factors which, as such, are inessential and can be eliminated. In fact, the isochristoffel’s symbols deviate from the conventional symbols by the same factor $\hat{I}$ (again, because $\hat{\eta} \equiv g$), and the same happens with other terms, except for possible redefinition of the source when needed, thus preserving again the conventional field equations and related experimental verifications.

A more detailed study of Eq.s (3) and related isominkowskian geometry is presented in the forthcoming Ref.\textsuperscript{3u}, including the use of the forgotten Freud identity of the Riemannian geometry in its isominkowskian realization.

In summary, the isominkowskian formulation of gravity permits a geometric unification of the special and general relativity into one single relativity, the isospecial relativity\textsuperscript{3s} where for $\hat{I} = I = \text{Diag.}(1,1,1,1)$ we have the special and for $\hat{I} = I(x) = U \times U^t$ we have the general. The invariance of the isounit is illustrated below. More detailed studies are available in the forthcoming paper\textsuperscript{3t}.

4. Operator unification of the special and general relativities. We now indicate that the above classical unification admit a step–by–step operator counterpart, here called operator isogravity (OIG). It should be indicated from the outset that OIG is structurally different than the conventional quantum gravity (QG)\textsuperscript{6} on numerous grounds, e.g., because OIG and QM have different units, Hilbert spaces, etc. In particular, the word ”operator” in OIG is suggested to keep in mind the differences with ”quantum” mechanics (as it should also be for QG).

To identify OIG, we note that the original noncanonical transform $U \times U^t = \hat{I} = I$ is mapped into a nonunitary transform on a conventional Hilbert space $\mathcal{H}$ over the complex field $C(c,+,\times)$. The isounit of the operator theory is therefore $\hat{I} = U \times U^t = \hat{I}, \hat{T} = (U \times U^t)^{-1} = \hat{T}^t = \hat{I}^{-1}$, where the representation of gravity occurs as per Eq.s (1). Then, OIG requires the isotopies of the totality of relativistic quantum mechanics (RQM) resulting in a formulation known as relativistic hadronic mechanics (RHM)\textsuperscript{3t}.

Besides the preceding isotopies $R \to \hat{R}$ and $M \to \hat{M}$, RHM is based on the lifting of the Hilbert space $\mathcal{H}$ with states $|\Psi>, |\Phi>,...$ and inner product $<\Phi|\Psi> \in C(c,+,\times)$ into the
written, in terms of the isodifferential calculus of ref. 3

isohermitian equations

The (nonrelativistic) isohermitian equations 3 and isoschrödinger equations 3 can be written, in terms of the isodifferential calculus of ref. 3

\[
\hat{x}dA/\hat{t} = i \times \hat{I}_t \times dA/dt = [A, H] = A \times \hat{T}_s \times H - H \times \hat{T}_s \times A, \quad \hat{I} = \hat{I}_s \times \hat{I}_t,
\]

\[
\hat{\hat{x}}dA/\hat{d}t = \hat{i} \times \hat{\vec{r}}_t | \hat{\vec{\Psi}} >= \hat{i} \hat{\vec{I}}_t \times \hat{\vec{\partial}} | \hat{\vec{\Psi}} >= H \times \hat{\vec{T}}_s | \hat{\vec{\Psi}} >= \hat{\vec{E}} \times \hat{\vec{\partial}}_s | \hat{\vec{\Psi}} >= \hat{\vec{E}} \times \hat{\vec{I}}_s \times \hat{\vec{T}}_s \times | \hat{\vec{\Psi}} >= \hat{\vec{E}} \times | \hat{\vec{\Psi}} >.
\]

(4)

Note that the final numbers of the theory are conventional. We also have the lifting of expectation values into the form \(<A> = <\hat{\vec{\Psi}} \times \hat{\vec{T}} \times A \times \hat{\vec{T}} \times | \hat{\vec{\Psi}} > / < | \hat{\vec{\Psi}} \times \hat{\vec{T}} \times | \hat{\vec{\Psi}} > >

It is easy to prove that RHM preserves all conventional properties of RQM. In particular: isohermiticity coincides with conventional hermiticity, \(H^\dagger = H\) (all quantities which are originally observables remain, therefore, so under isotopies); the iso-eigenvalues of iso-hermitian operators are isoreal (thus conventional because of the identity \(\hat{E} \times | \hat{\vec{\Psi}} > = E \times | \hat{\vec{\Psi}} >

RHM is form invariant under isounitary transforms \(\hat{\vec{U}} \times \hat{\vec{U}}^\dagger = \hat{\vec{U}}^\dagger \times \hat{\vec{U}} = \hat{I}\). In fact, we have the invariance of the isounit \(\hat{I} \rightarrow \hat{I}' = \hat{\vec{U}} \times \hat{\vec{I}} \times \hat{\vec{U}}^\dagger \equiv \hat{\vec{I}}\), of the isoassociative product \(\hat{\vec{U}} \times (\hat{\vec{A}} \times \hat{\vec{B}}) \times \hat{\vec{U}}^\dagger = \hat{\vec{A}} \times \hat{\vec{B}}\); etc.; and the same occurs for all other properties (including causality). Note that nonunitary transforms on \(\hat{H}\) can always be identically rewritten as isounitary transforms on \(\hat{H}, U = \hat{\vec{U}} \times \hat{\vec{T}}^{1/2}, U \times U^\dagger \equiv \hat{\vec{U}} \times \hat{\vec{U}}^\dagger = \hat{\vec{U}}^\dagger \times \hat{\vec{U}} = \hat{I}\), under which RHM is invariant.

It should be stressed that RHM is not a new theory, but merely a new realization of the abstract axioms of RQM. In fact, RHM and RQM coincide at the abstract, realization-free level where all distinctions are lost between \(\hat{I}\) and \(\hat{\vec{I}}\), \(\hat{R}\) and \(\hat{\vec{R}}\), \(\hat{M}\) and \(\hat{\vec{M}}\), \(\hat{\vec{H}}\) and \(\hat{H}\), etc. Yet, RHM is broader than RQM, it recovers the latter identically for \(\hat{I} = \hat{I}\) and can approximate the latter as close as desired for \(\hat{I} \approx \hat{I}\).

On summary, the entire formulation of RHM of memoir can be consistently specialized for the gravitational isounit \(\hat{I}(x)\) yielding the proposed OIG.

5. The universal isopoincaré symmetry of gravitation. An important property
of the isominkowskian formulation of gravity, which is lacking for conventional formulations, is that of admitting a universal, classical and operator symmetry for all possible Riemannian formulations of gravitation first identified by Santilli$^{3h-3t}$ under the name of

isopoincaré symmetry $\hat{P}(3.1)$, which results to be locally isomorphic to the conventional symmetry $P(3.1)^{3h-3t}$.

The isosymmetry can be easily constructed via the isotopies of Lie’s theory$^{3a,3d}$ called Lie–Santilli isorelation$^5$ which essentially consists in the reconstruction of all branches of Lie’s theory

(ordinary enveloping algebras, Lie algebras, Lie group, transformation and representation theories, etc.) for the generalized unit $\hat{I} = [\hat{T}(x)]^{-1}$. Since $\hat{I} > 0$, one can see from the incertion that the isopoincaré symmetry is isomorphic to the conventional one, $\hat{P}(3.1) \approx P(3.1)$ (see ref.$^{4t}$ for a recent accounts).

The operator version of the isopoincaré symmetry is characterized by the conventional

generators and parameter opnly lifted into isospaces over isofields $X = \{X_k\} = \{M_{\mu\nu} = x_{\mu}p_\nu - x_{\mu}p_\nu, p_\alpha\} \to \hat{X} = \{M_{\mu\nu} = \hat{x}_{\mu} \times \hat{p}_\nu - \hat{x}_{\mu} \times \hat{p}_\nu, \hat{p}_\alpha\}, k = 1, 2, \ldots, 10, \mu, \nu = 1, 2, 3, 4$, and $w = \{w_k\} = \{(\theta, v), a\} \in R \to \hat{w} = w \times \hat{I} \in \hat{R}(\hat{n}, +, \hat{x})$. The isotopies preserve the original connectivity properties$^{3r}$. The connected component of $P(3.1)$ is then given by

$\hat{P}_0(3.1) = S\hat{O}(3.1)\hat{\times}\hat{T}(3.1)$, where $S\hat{O}(3.1)$ is the connected isorelation group$^{3h}$ and $\hat{T}(3.1)$ is the group of isotranslations$^{3k}$. $\hat{P}_0(3.1)$ can be written via the isorelation as $\hat{e}^A = \hat{I} + A/1! + A\hat{x}A/2! + \ldots = (e^{A\times\hat{T}}) \times \hat{I}$ characterized by the isotopic Poincaré–Birkhoff–Witt theorem$^{3a,3d,5}$ of the underlying isorelation associative algebra

$$\hat{P}_0(3.1) : \hat{A}(\hat{w}) = \Pi_k e^{ix \times \hat{x}} = (\Pi_k e^{i \times X \times \hat{T} \times w}) \times \hat{I} = \hat{A}(w) \times \hat{I}. \quad (5)$$

Note the appearance of the gravitational isotopic element $\hat{T}(x)$ in the exponent of the group structure. This illustrates the nontriviality of the lifting and its nonlinear character, as evidently necessary for any symmetry of gravitation. What is intriguing is that the isopoincaré symmetry recovers linearity on $\hat{M}$ over $\hat{R}$, a property called isolinearity$^{3u}$.

Conventional linear transforms on $M$ violate isolinearity on $\hat{M}$ and must then be replaced with the isotransforms $\hat{x}' = \hat{A}(\hat{w}) \hat{x} = \hat{A}(\hat{w}) \hat{x} \hat{T}(x) \times \hat{x}$ which can be written from (5) for computational purposes (only) $\hat{x}' = \hat{A}(w) \times \hat{x}$. The preservation of the original dimension is ensured by the isotopic Baker–Campbell–Hausdorff Theorem$^{3a,3d,5}$. Structure (5) then forms a connected Lie–Santilli isogroup$^5$ with laws $\hat{A}(\hat{w}) \times \hat{A}(\hat{w}') = \hat{A}(\hat{w}') \times \hat{A}(\hat{w}) = \hat{A}(\hat{w} + \hat{w}')$, $\hat{A}(\hat{w}) \hat{x} \hat{A}(\hat{w}) = \hat{A}(\hat{w}) \hat{x} \hat{A}(\hat{w}) = \hat{A}(\hat{w}) \hat{x} \hat{A}(\hat{w}) = \hat{A}(\hat{w}) \times \hat{I}(x) = [\hat{T}(x)]^{-1}$.

As one can see, $\hat{P}_0(3.1)$ is noncanonical on $M$ over $\hat{R}$ (e.g., because it does not preserve the conventional unit $I$), but it is canonical on $\hat{M}$ over $\hat{R}$, a property called isocanonicity (because it leaves invariant by construction the isounit). This confirms the achievement, apparently for the first time, of an operator theory of gravity verifying the fundamental
invariance of its unit. The invariance at the classical level is consequential.

The *isodiscrete transforms* are given by \( \hat{\pi} \times x = \pi \times x = (-r, x^4), \hat{\tau} \times x = \tau \times x = (r, -x^4), \)
where \( \hat{\pi} = \pi \times \hat{I}, \hat{\tau} = \tau \times \hat{I}, \) and \( \pi, \tau \) are the conventional inversion operators. Despite such a simplicity, the physical implications are nontrivial because of the possibility of reconstructing as exact discrete symmetries when believed to be broken, which is studied by embedding all symmetry breaking terms in the isounit. One should be aware that this is a rather general property of the Lie–Santilli isothory, thus holding also for continuous symmetries. In fact, contrary to a popular beliefs, this note shows that the Lorentz and Poincaré symmetries are exact for gravitation.

The use isodifferential calculus on \( \hat{M} \) then yields the Lie–Santilli isoalgebra \( \hat{p}_0(3.1) \)

\[
[\hat{M}_{\mu\nu}, \hat{M}_{\alpha\beta}] = i \times (\hat{\eta}_{\alpha\nu} \times \hat{M}_{\mu\beta} - \hat{\eta}_{\mu\alpha} \times \hat{M}_{\nu\beta} - \hat{\eta}_{\nu\beta} \times \hat{M}_{\mu\alpha} + \hat{\eta}_{\mu\beta} \times \hat{M}_{\nu\alpha}), \\
[\hat{M}_{\mu\nu}, \hat{p}_\alpha] = i \times (\hat{\eta}_{\alpha\nu} \times \hat{p}_\mu - \hat{\eta}_{\mu\alpha} \times \hat{p}_\nu), [\hat{p}_\alpha, \hat{p}_\beta] = 0, \hat{\eta}_{\mu\nu} = g_{\mu\nu}(x),
\]

where \([A, B] = A \times \hat{T}(x) \times B - B \times \hat{T}(x) \times A\) is the *isoproduct* (originally proposed in \(3^b\)), which does indeed satisfy the Lie axioms in isospace, as one can verify. Note the appearance of the Riemannian metric \( \hat{\eta}_{\mu\nu} = g_{\mu\nu}(x) \), this time, as the ”structure functions” \( \hat{\eta}_{\mu\nu} \) of the isoalgebra. Note also that the *momentum components isocommute* (while they are notoriously non–commutative for QG). This confirms the achievement of an isoflat representation of gravity.

The local isomorphism \( \hat{p}_0(3.1) \approx p_0(3.1) \) is ensured by the positive–definiteness of \( \hat{T} \). In fact, the use of the generators in the form \( \hat{M}_{\mu} = \hat{x}^\mu \times \hat{p}_\nu - \hat{\xi}^{\nu} \times \hat{p}_\mu \) would yield the *conventional* structure constants under a *generalized* Lie product, as one can verify. The above local isomorphism is sufficient, per se’, to guarantee the axiomatic consistency of OIG.

The *isocasimir invariants* of \( \hat{p}_0(3.1) \) are the simple isotopic image of the conventional ones

\[
C^0 = \hat{I} = [\hat{T}(x)]^{-1}, C^{(2)} = \hat{p}_\mu \times \hat{p}^\mu = \hat{\eta}^{\mu\nu} \hat{p}_\mu \times \hat{p}_\nu, C^{(4)} = \hat{W}_\mu \times \hat{W}_\nu, \hat{W}_\mu = \epsilon_{\mu\alpha\beta\pi} \hat{M}^{\alpha\beta} \times \hat{p}^\pi.
\]

From them, one can construct any needed *gravitational relativistic equation*, such as the

\[
(\hat{\gamma}^\mu \times \hat{p}_\mu + i \times \hat{m}) | \hat{\pi} \hat{\gamma}^\mu(x) \times \hat{T}(x) \times \hat{p}_\nu - i \times m \times \hat{I}(x) \times \hat{T}(x) \times | >= 0,
\]

\[
\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = \hat{\gamma}^\mu \times \hat{T} \times \hat{\gamma}^\nu + \hat{\gamma}^\nu \times \hat{T} \times \hat{\gamma}^\mu = 2 \times \hat{\eta}^{\mu\nu} \equiv 2 \times g^{\mu\nu}, \hat{\gamma}^\mu = \hat{T}_{\mu\nu}^{1/2} \times \hat{\gamma}^\mu \times \hat{T} \text{ (no sum),}
\]

Where \( \gamma^\mu \) are the conventional gammas and \( \hat{\gamma}^\mu \) are the *isogamma matrices*. Note that the *anti-iso-commutators of the isogamma matrices yield* (twice) the Riemannian metric \( g(x) \), thus confirming the representation of Einstein’s (or other) gravitation in the structure of
Dirac’s equation. As an illustration, we have the Dirac–Schwarzschild equation given by Eq.s (7) with \( \dot{\gamma}_k = (1 - 2M/r)^{-1/2} \times \gamma_k \times \hat{I} \) and \( \dot{\gamma}_4 = (1 - 2M/r)^{1/2} \times \gamma^4 \times \hat{I} \). Similarly one can construct the isogravitational version of all other equations of RQM.

These equations are not a mere mathematical curiosity because they establish the compatibility of OIG with experimental data in particle physics in view of the much lower character of gravitational over electromagnetic, weak and strong contributions. Our unification of the special and general relativities is, therefore, compatible with experimental evidence at both classical and operator levels.

The space components \( SO(3) \), called isorotations\(^3i\), can be computed from isoexponentiations (5) with the explicit form in the \((x,y)\)-plane (were we ignore again the factorization of \( \hat{I} \) for simplicity)

\[
x' = x \times \cos(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3) - y \times \hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \sin(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3),
\]
\[
y' = x \times \hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \sin(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3) + y \times \cos(\hat{T}_{11}^{\frac{1}{2}} \times \hat{T}_{22}^{\frac{1}{2}} \times \theta_3),
\]

(see\(^3s\) for general isorotations in all there Euler angles). Isotransforms (8) leave invariant all ellipsoidal deformations \( x \times \hat{T}_{11} \times y \times \hat{T}_{22} \times y + z \times \hat{T}_{33} \times z = R \) of the sphere \( x \times x + y \times y + z \times z = r \). Such ellipsoid become perfect spheres \( \hat{r}^2 = (\hat{r}^t \times \hat{\delta} \times \hat{r}) \times \hat{I}_s \) in isoeuclidean spaces\(^3h\,3r\) \( \hat{E}(\hat{r}, \hat{\delta}, \hat{R}), \hat{r} = \{r_k\} \times \hat{I}_s, \hat{\delta} = \hat{T}_s \times \delta, \delta = Diag. (1, 1, 1), \hat{T}_s = Diag.(\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}), \hat{I}_s = \hat{T}_{s}^{-1}, \) called isospheres.

In fact, the deformation of the semi-axes \( 1_k \rightarrow \hat{T}_{kk} \) while the related units are deformed of the inverse amounts \( 1_k \rightarrow \hat{T}_{kk}^{-1} \) preserves the perfect spheridicity (because the invariant in isospace is \((Length)^2 \times (Unit)^2\)). Note that this perfect sphericity in \( \hat{E} \) is the geometric origin of the isomorphism \( \hat{O}(3) \equiv O(3) \), with consequential preservation of the exact rotational symmetry for the space–components \( q(r) \) of all possible Riemannian metrics.

The connected isolorentz symmetry \( SO(3,1) \) is characterized by the isorotations and the isolorentz boosts\(^3h\) which can be written in the \((3,4)\)-plane

\[
x^{3'} = x^3 \times \sinh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) - x^4 \times \hat{T}_{33}^{-\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times \cosh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44} \times v) = \nonumber \\
= \tilde{\gamma} \times (x^3 - \hat{T}_{33}^{-\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times \beta \times x^4) 
\]
\[
x^{4'} = -x^3 \times \hat{T}_{33} \times c_o^{-1} \times \hat{T}_{44}^{\frac{1}{2}} \times \sinh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44} \times v) + x^4 \times \cosh(\hat{T}_{33}^{\frac{1}{2}} \times \hat{T}_{44}^{\frac{1}{2}} \times v) = \nonumber 
\]
\[
\begin{align*}
\tilde{\gamma} &= (x^4 - \hat{T}_{43} x^{3} \hat{T}_{44}^{-1} x + \tilde{\beta}) x^3 \\
\tilde{\beta} &= v_k \times \hat{T}_{44} / c_0 \times \hat{T}_{44}^{1}, \quad \gamma = (1 - \tilde{\beta}^2)^{-\frac{1}{2}}. 
\end{align*}
\] (9)

Note that the above isotransforms are formally similar to the Lorentz transforms, as expected from their isotopic character. Isotransforms (9) characterize the light isocone\(^{36}\), i.e., the perfect cone in isospace \(\hat{M}\). In a way similar to the isosphere, we have the deformation of the light cone axes \(1_\mu \rightarrow \hat{T}_\mu\) while the corresponding units are deformed of the inverse amount \(1_\mu \rightarrow \hat{T}_\mu^{-1}\).

In particular, the isolight cone also has the conventional characteristic angle, as a necessary condition for an isotopy (the proof of the latter property requires the use of isotrigonometric and isohyperbolic functions). Thus, the maximal causal speed in isominkowski space is the conventional speed \(c\), which is the maximal causal speed in isominkowski space, \(S_1 = \hat{S}_1\). The identity of the light cone and isocones is the geometric origin of the isomorphism \(SO(3,1) \approx SO(3,1)\) and, thus, of the exact validity of the Lorentz symmetry for all possible Riemannian metrics \(g(x)\).

The isotranslations can be written \(x' = (\hat{e}^{x} \times \hat{p} \times a) \times \hat{x} = [x + a \times A(x)] \times \hat{I}, \hat{p}' = (\hat{e}^{x} \times \hat{p} \times a) \times \hat{p} = \hat{p}\), where \(A_\mu = \hat{T}_\mu^{1/2} + a^\alpha \times [\hat{T}_\mu^{1/2} \hat{p}_\alpha]/1! + ...\) and they are also nonlinear, as expected.

Intriguingly, the isotopies identify one additional symmetry which is absent in the conventional case. It is here called isoselfscalar invariance and it is given by the rescaling of the unit \(\hat{I} \rightarrow \hat{I}' = n^2 \times \hat{I}\), where \(n\) is an 11-th parameter, under which the interval remains invariant, \(x'^2 = (x^\mu \times \hat{T}_\mu^a \times \eta_{\alpha\nu} \times x^\nu) \times \hat{I} \equiv [x^\mu \times (n^{-2} \times \hat{T}_\mu^a) \times \eta_{\alpha\nu} \times x^\nu] \times (n^2 \times \hat{I})\). Note that, even though \(n^2\) is factorizable, the corresponding isosymmetry is not trivial, e.g., because \(n^2\) enters into the argument of the isolorentz transforms (9).

The same symmetry also holds for the isoinner product (whenever \(n\) does not depend on the integration variable), \(<\hat{\Phi} \times \hat{T} \times \hat{\Psi} > \times \hat{I} \equiv <\hat{\Phi} \times (n^{-2} \times \hat{T}) \times \hat{\Psi} > (n^2 \times \hat{I})\). Note finally that the latter symmetries have remained undetected throughout this century because they required the prior discovery of new numbers, those with an arbitrary unit\(^{39}\).

6. Inclusion of interior gravitation. The attentive reader may have noted that the isotopies leave unrestricted the functional dependence of the isometric. Its sole dependence on the coordinates is therefore a restriction which has been used so far for a representation of exterior gravitation in vacuum.

In the general case we have isometrics with an unrestricted functional dependence, \(\hat{\eta} = \hat{T}(x, x, \partial \Psi, ...) \times \eta, \hat{T} > 0\), which, as such, can represent interior gravitation problems with an unrestricted nonlinearity in the velocities, wave functions and their derivatives, as expected in realistic interior models, e.g., of neutron stars, quasars, black holes and all that.

Note also that the isometric can also contain nonlocal–integral terms, e.g., representing wave–overlappings\(^{38}\). Nevertheless, the theory verifies the condition of locality in isospace,
called *isolocality*, because its topology is everywhere local except at the unit\textsuperscript{3,4r}.

The *general isopoincaré symmetry* is here defined as the 11–dimensional set of *isorotations, isoboosts, isotranslations, isoinversions and isoselfscalar transforms*. The *restricted isopoincaré transforms* are those in which the isounit is averaged into constants. The results of this note therefore imply the following:

**Theorem 1.** The 11-dimensional, general isopoincaré symmetry on isominkowski spaces over real isofields for well behaved and nowhere null isounits constitues the largest possible isolinear, isolocal and isocanonical invariance of isoseparation (2) for nonsingular isometrics with positive-definite isounits, thus constituting the universal invariance of exterior and interior gravitations.

The verification of the invariant under the isopoincaré transforms of all possible separation (2) is instructive. The maximal character of the isosymmetry can be proved as in the conventional case. Note that for any arbitrarily given (diagonal) Riemannian metric $g(x)$ (such as Schwarzschild, Krasner, etc.) *there is nothing to compute* because one merely plots the $T_{\mu\nu}$ terms in the decomposition $g_{\mu\nu} = T_{\mu\nu} \times \eta_{\mu\nu}$ (no sum) in the above given isotransforms. The invariance of the separation $x^t \times g \times x$ is then ensured. The $(2 + 2)$–de Sitter or other cases can be derived from the theorem via mere changes of signature or dimension of the isounit.

7. **Concluding remarks.** In summary, in this note we have presented, apparently for the first time, a geometric unification of the special and general relativities in both classical and operator mechanics, as well as for both exterior and interior problems. The results are centralluy dependent on the use of *isominkowskian geometry* as introduced in this note, rather than the use of the *isoriemannian form* as studied in Ref.\textsuperscript{3s}.

The classical and operator geometric unification of the special and general relativities for the exterior problem in vacuum is centrally dependent on the achievement of a universal symmetry for gravitation which, by conception and construction, is locally isomorphic to the Poincaré symmetr of the special relativity. This eliminates the historical difference between the special and general relativities whereby the former admits a universal symmetry, while the latter does not\textsuperscript{1,2}. Note the *necessity* of the representation of gravity in *isominkovski* space for the very formulation of its universal isopoincaré symmetry. In fact, no isosymmetry can be constructed in the Riemannian space, to our best knowledge.

The above occurrence has a number of implications. First, it allow to illustrate the viewpoint expressed in Sect. 1 to the effect that some of controversies in gravitation debated over this century are not due to Einstein’s field equations, but rather to insufficiencies in the mathematics used for their treatment.
A typical case is the controversy whether the total conservation laws of general relativity are compatible with those of the special relativity. Our representation of Einstein’s equations via the novel isomathematics permits a resolution of this old controversy via a mere visual examination.

Recall that the generators of all space-time symmetries characterize total conserved quantities. The compatibility of the total conservation laws of the general and special relativities is therefore established by the visual observation that the generators of the Poincare’ and isopoincare’ symmetries coincide. In fact, only the mathematical operations on them are changed in the transition from the relativistic to the gravitational case.

The isominkowskian treatment of gravity also permits a resolution of some of the limitations of conventional gravitational models, such as their insufficiency to provide an effective representation of interior gravitational problems. In fact, conventional formulations of gravity admit only a limited dependence on the velocities, while being strictly local-differential and derivable from a first-order Lagrangians (variationally self-adjoint\(^3\)), characteristics which are evidently exact for exterior problems in vacuum.

By comparison, interior gravitational problems, such as all forms of gravitational collapse, are constituted by extended and hyperdense hadrons in conditions of total mutual penetration in large numbers into small regions of space. It is well known that these conditions imply effects which are \textit{arbitrarily nonlinear in the velocities as well as in the wavefunctions, nonlocal-integral on various quantities and variationally nonselfadjoint}\(^3\), \((i.e.\) not representable via first-order Lagrangians). It is evident that the latter conditions are beyond any scientific expectation of quantitative treatment via conventional gravitational theories.

The isominkowskian formulation of gravity resolve this limitation too and shows that it is equally due to insufficiencies in the underlying mathematics. In fact, isogravitation extends the applicability of Einstein’s axioms to a form which is ”directly universal for exterior and interior gravitations”, namely, capable of representing all exterior and interior conditions considered (universality), directly in the x-frame of the experimenter (direct universality), thus extending the above unification to interior conditions.

As indicated earlier, this extension is due to the fact that the functional dependence of the metric in Riemannian treatments is restricted to the sole dependence on the local coordinates, \(g = g(x)\), while under isotopies the same dependence becomes unrestricted, \(g = g(x, v, \phi, \partial \psi,...)\) without altering the original geometric axioms. This results in \textit{geometric unification of exterior and interior problems}, despite their sizable structural differences of topological, analytic and other characters. The latter unification was studied in details in ref.\(^3\) under the isoriemannian geometry and it is studied with the isominkowskian geometry in this note for the reasons indicated earlier.

A first illustration of the extension of the axioms to realistic interior conditions is offered by the isoselfscalar transforms \(\hat{T}_{\mu\mu} \rightarrow n^{-2} \times \hat{T}_{\mu\mu}\) which permit the \textit{representation of electromagnetic waves propagating within physical media with local varying speed} \(c = c_0/n\). This
allows the construction, apparently for the first time, of Schwarzschild’s and other gravitational models for the interior of atmospheres and chromospheres with a locally varying velocity of light. Applications to specific cases, such as gravitational horizons, is then expected to permit refinements of current studies evidently due to deviations from the valid in vacuum of the speed of light in the hyperdense chromospheres outside gravitational horizons.

Except for being well behaved (and non–null), the parameter n remains unrestricted by the isotopies and, therefore, $n = 1$ in vacuum but otherwise it can be $n > 1$ or $< 1$. As a result, the isopoincaré symmetry is a natural invariance for arbitrary causal speeds, whether equal, smaller or bigger than the speed of light. The latter have been predicted since some time in interior problems only, but experimentally detected only recently, e.g., for the speed of photons traveling in certain guides or for the speed of matter in astrophysical explosions. The recent Ref. 8f has identified solutions of conventional relativistic equations with arbitrary speeds in vacuum of which $\hat{P}(3.1)$ is evidently the natural invariance).

Despite the local variation of $c$, the maximal causal speed on $\hat{M}$ over $\hat{R}$ remain $c_0$ again, because the change $c \rightarrow c_0/n$ is compensated by an inverse change of the unit. By recalling that the STR is evidently inapplicable (and not “violated”) for arbitrary causal speeds, we can therefore say that the isotopies render the STR universally applicable, not only for classical and operator gravitation, but also for arbitrary causal speeds.

Also the light isocone remains applicable for interior gravitational cases with arbitrary $c$. As such, the light isocone appears to be more appropriate than the conventional light cone for calculations, e.g., outside gravitational horizons which, being composed of hyperdense chromospheres, do not admit the conventional speed of light in vacuum $c_0$.

The indication of a number of developments currently under study appears recommendable. First, we note that the the zeros of the space (time) component of the isounit represent gravitation horizons (singularities). This representation is trivially equivalent to the conventional one for the exterior case in vacuum. However, gravitational collapse is a typical interior case for which the isotopic representation becomes nontrivial, e.g., because it permits the inclusion of the nonlinear, nonlocal and noncanonical effects indicated earlier.

Note that the zeros of the isounit have been excluded from Theorem 1 because of their yet unknown topological structure.

Another aspect which is under study is the iso–grand–unification in which the isounit is no longer Hermitean. This broader class geometries in a natural way the interior irreversibility and it has been used, e.g., for the black hole
model of ref. The third class of methods is given by the (multi–valued) hyperstructures, in which the generalized unit is constituted by a set of non–Hermitean quantities. The latter most general known class appears to be particularly significant for quantitative studies of biological structures in which the conventional RQM is manifestly inapplicable due to its reversibility.

Also, the isotopies, genotopies and hyperstructures admit antiautomorphic images, called isodualities, and characterized by the map $\hat{I} \rightarrow -\hat{I}^d = -\hat{I}^\dagger$ which are currently under study for antimatter. In this case the energy–momentum tensor of antimatter becomes negative–definite, thus removing a problem of compatibility between the current representations of antimatter in classical and particle physics. The gravitational treatment of antimatter via the isodualities of the isominkowskian geometry will be studied elsewhere.

On historical grounds, we note that, as studied in detail in memoir for the general case of RHM, our OIG can be interpreted as a nonunitary completion of RQM considerably along the historical $E - P - R$ argument for which von Neumann theorem and Bell’s inequalities do not apply evidently because of its nonunitary structure.

Moreover, from the abstract identity of the right modular associative action $H \times |\Psi\rangle$ and its isotopic image $\hat{H} \hat{x} |\hat{\Psi}\rangle$, one can see that the iso-eigenvalue equation $\hat{H} \hat{x} |\hat{\Psi}\rangle = \hat{E}_\lambda \times |\hat{\Psi}\rangle$ characterizes an explicit and concrete operator realization of the "hidden variable" $\lambda = \lambda(x, ...)$ $\equiv \hat{T}$. Our isotopic formulation of gravity can therefore be interpreted as a realization of the theory of hidden variables. After all, the "hidden" character of gravitation in our theory is illustrated by the recovering of the conventional unit under the isoexpectation value $\langle \hat{I} \rangle = I$.

In conclusion, the viewpoint we have attempted to convey in this note is that an axiomatic consistent operator formulation of gravity always existed. It did creep in un–noticed until now because embedded where nobody looked for, in the unit of RQM.

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